Topology

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Part I

General Topology I – Basics

In this part of the lecture notes we first introduce the notion of topological spaces and continuous maps between topological spaces. We discuss several properties of topological spaces, with a particular focus on compactness, connectedness and the Hausdorff property. Afterwards we take a short break from topological spaces and in two chapters turn to the study of categories. First we introduce categories, functors and natural transformations. Afterwards we discuss the notions of (co-) products and (co-) limits in categories. These general concepts guide serve us as a guide in our further exploration of topological spaces, when we introduce products, disjoint unions and quotients of topological spaces. These chapters are the foundation of all other chapters of this book.

This part of the lecture notes is organizes as follows:

- In the preliminary Chapter 1 we state several results from set theory and elementary mathematics, which we will use in these lecture notes. This chapter is not meant to be readable, it is meant to be a reservoir of statements on which we can draw.
- In the second preliminary Chapter 2 we state several definitions and results from linear algebra which we will use in these lecture notes. Again this chapter is not meant to be readable, it is just meant to be a reservoir of statements on which we can draw.
- In Chapter 3 we will introduce the notion of a topological space. We give many examples and we discuss important properties, like the Hausdorff property and compactness.
- In Chapter 4 we will define what it means for a map between topological spaces to be continuous. Among many other notions we discuss and compare the notions of being connected and being path-connected.
- In Chapter 5 we will introduce the notion of a category and the related notion of a functor and a natural transformation.
- In Chapter 6 we will continue with our discussion of concepts from category theory, namely we will introduce products and coproducts, and more generally, limits and colimits. These notions will play an important guiding role throughout a big chunk of these lecture notes.
- In Chapter 7 we will study the categorical notions of products and coproducts in the topological category. In other words, we will introduce the product and the disjoint union of a family of topological spaces.
- In Chapter 8 we will introduce the notion of the quotient of a topological space by an equivalence relation or a subspace. This construction allows us in particular to give a clean definition of the surface of genus g and it allows us to introduce the real and complex projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

CHAPTER 1

Preliminaries from set theory

In this preliminary chapter we will introduce some concepts and results from set theory which we will use throughout these notes, but which otherwise have little to do with topology itself. The reader might want to skip this preliminary chapter and directly move to Chapter 3, where we actually start with topology.

This chapter is organized as follows:

- In Section 1.1 we will recall some basic concepts and results from elementary set theory.
- In Section 1.2 we will recall the notion of relations and equivalence relations on sets.
- In Section 1.3 we will study various notions of orders on sets.
- In Section 1.4 we will state Zorn's Lemma and the Well-Ordering Theorem.
- In Section 1.5 we will study the notion of maximal and minimal elements in ordered sets.
- In Section 1.6 we will discuss the cardinality of sets.
- In Section 1.7 we will discuss more specifically the cardinality of finite sets.
- In Section 1.8 we will provide exercises for Chapter 1.

In the next Chapter 2 we will recall some concepts from linear algebra.

1.1. Basic set theory.

Topology builds on top of set theory. We will not go into proper set theory, but whatever we are doing is covered by ZFC, the Zermelo-Frankel set theory with the Axiom of Choice, which was developed by Ernst Zermelo and Adolf Fraenkel [**Zer1908, Frae1922, Zer1930**]. We refer the reader to [**Cie1997**], [**FPr1985**] and [**Halm1960**] for a detailed introduction to set theory for the "working mathematician". Also, at this stage reading "Logicomix" [**DoP2009**], see the figure below, is almost a must.

In these notes we mostly employ the language of "naive set theory". In particular, when we describe sets we sometimes use creative notation and language. The hope is that the language is chosen in such a way that any good-willed reader will understand what is meant.

In this section we recall a few basic definitions and results from set theory which will frequently use throughout these notes. Usually we will not refer to the statements explicitly, unless it helps to clarify an argument.

First we introduce the following notation which should mostly be standard.

Notation 1.1.

(1) We write $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$.
(2) We denote by $\varnothing$ the empty set.
(3) Given a set X we write $A \subseteq X$ if A is a subset of X and we write $A \subsetneq X$ if A is a subset of X with $A \neq X$.



(4) Given a set X and a subset $A \subseteq X$ we denote by $X \setminus A := \{x \in X \mid x \notin A\}$ the complement of A in X.
(5) Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

Definition 1.2. A family of subsets of a set X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$.

Remark.
(1) When it comes to families of subsets we often use suggestive notation, e.g. we might write "let $\{U_i\}_{i\in I}$ be a family of subsets of X", which just means that I is a set and the map $I \to \mathcal{P}(X)$ is denoted by $i \mapsto U_i$.
(2) When we write "let $U_1, \dots, U_k$ be subsets" or "let A, B be subsets", then often we will think of these as families of subsets with index sets $\{1, \dots, k\}$ respectively $\{1, 2\}$. □

Convention 1.3. Let X be a set. We will view any subset $\mathcal{U} \subseteq \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$. This convention might sound silly, but it will be useful on many occasions.

We continue with two of the most important definitions of set theory.

Definition 1.4. Let X be a set.
(1) Let $\{U_i\}_{i\in I}$ be a family of subsets of X. We write

$$\bigcup_{i\in I} U_i = \{x \in X \mid \text{there exists an } i \in I \text{ with } x \in U_i\} \quad \text{(Union)}$$
$$\bigcap_{i\in I} U_i = \{x \in X \mid \text{for every } i \in I \text{ we have } x \in U_i\} \quad \text{(Intersection)}.$$

In particular, for $I = \varnothing$ we have $\bigcup_{i\in\varnothing} U_i = \varnothing$ and $\bigcap_{i\in\varnothing} U_i = X$.[1]

(2) As we mentioned above, we view subsets $U_1, \ldots, U_k$ of X as a family of subsets. We write $U_1 \cup \cdots \cup U_k := \bigcup_{i \in \{1,\ldots,k\}} U_i$ and the same way for the intersection.
(3) Let A and B be two subsets of X.
  (a) We say that the subsets A and B of X are disjoint if $A \cap B = \varnothing$.
  (b) We say A intersects B if $A \cap B \neq \varnothing$.

The very first lemma of these lecture notes says that intersection is distributive with respect to union.

Lemma 1.5. (Intersection–Distributive Lemma) Let X be a set and let $\{U_i\}_{i \in I}$ be a family of subsets of X. Given any subset A of X we have

$$A \cap \left(\bigcup_{i \in I} U_i \right) = \bigcup_{i \in I} (A \cap U_i).$$

Proof. The lemma follows easily from the definitions and elementary logic. ■

Notation 1.6. Let $\{U_i\}_{i \in I}$ be a family of subsets of a given set X. If the U_i are pairwise disjoint, i.e. if for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$, then at times we write

$$\bigsqcup_{i \in I} U_i := \bigcup_{i \in I} U_i.$$

We move on to the famous de Morgan's Laws from elementary set theory.

Lemma 1.7. (De Morgan's Laws) Let X be a set and let $\{A_i\}_{i \in I}$ be a family of subsets of X. If given a subset $B \subseteq X$ we denote its complement by $B^c := X \setminus B$, then

$$\left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c \quad \text{and} \quad \left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i) \quad \text{and} \quad X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i).$$

Proof. The lemma follows again easily from the definitions and elementary logic. ■

Next we study maps between sets and their effect on subsets.

Notation 1.8. Let $f \colon X \to Y$ be a map between sets.
(1) Given a subset $A \subseteq X$ we write $f(A) := \{f(x) \mid x \in A\}$.
(2) Given a subset $A \subseteq Y$ we write $f^{-1}(A) := \{x \in X \mid f(x) \in A\}$. Furthermore, given $y \in Y$ we write $f^{-1}(y) := f^{-1}(\{y\})$.[2]

The next lemma concerns the interaction between operations on sets and maps between sets.

[1]This simple observation will be useful at times.
[2]Purists might decry this notation, but it does make notation much more readable.

Lemma 1.9. (Image-Preimage Lemma) Let $f\colon X \to Y$ be a map between two sets X and Y. Then the following statements hold:

(0)	(a)			X	$=$	$f^{-1}(Y)$
	(b)			$\varnothing$	$=$	$f^{-1}(\varnothing)$
(1)	(a)	for every subset $U \subseteq X$ we have		U	$\subseteq$	$f^{-1}(f(U))$
	(b)	furthermore, if f is injective[3], then we have		U	$=$	$f^{-1}(f(U))$
(2)	(a)	for every subset $V \subseteq Y$ we have		V	$\supseteq$	$f(f^{-1}(V))$
	(b)	furthermore, if f is surjective, then we have		V	$=$	$f(f^{-1}(V))$
(3)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcup_{i\in I} U_i\Big)$		$=$	$\bigcup_{i\in I} f(U_i)$
(4)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcap_{i\in I} U_i\Big)$		$\subseteq$	$\bigcap_{i\in I} f(U_i)$
(5)	(a)	for every subset $U \subseteq X$ we have	$f(X \setminus U)$		$\supseteq$	$f(X) \setminus f(U)$
	(b)	furthermore, if $f^{-1}(f(U)) = U$, then we have	$f(X \setminus U)$		$=$	$f(X) \setminus f(U)$
(6)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcup_{i\in I} V_i\Big)$		$=$	$\bigcup_{i\in I} f^{-1}(V_i)$
(7)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcap_{i\in I} V_i\Big)$		$=$	$\bigcap_{i\in I} f^{-1}(V_i)$
(8)		for every subset $V \subseteq Y$ we have	$f^{-1}(Y \setminus V)$		$=$	$X \setminus f^{-1}(V)$.

Proof. The lemma follows again from the definitions and elementary logic. ■

Definition 1.10. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of the family $\{X_i\}_{i\in I}$ is defined as the set[4] [5]

$$\prod_{i\in I} X_i \coloneqq \Big\{ f\colon I \to \bigcup_{i\in I} X_i \;\Big|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \Big\}.$$

Furthermore, given a set X and a set I we write

$$X^I \coloneqq \prod_{i\in I} X.$$

Example. Let $\{X_i\}_{i\in\varnothing}$ be an empty family of sets. In this case the product is precisely the map $\varnothing \to \varnothing$. In other words, the product of the empty family of sets is a set with precisely one element. □

We can now state the Axiom of Choice, which is discussed in greater detail in [**Cie1997**, p. 15] and in [**Je1973**].

Axiom of Choice 1.1. For every family of sets $\{X_i\}_{i\in I}$ and such that each X_i is non-empty, the product $\prod_{i\in I} X_i$ is non-empty.

[3] As a totally irrelevant side remark we would like to point out that the fundamental terms "surjective", "injective" and "bijective" are actually fairly recent additions to the mathematical language. They were first introduced on page 80 of Nicolas Bourbaki's "Théorie des ensembles" first published in 1954, see [**Bou1954**]. The reader who has not read the biography of the great mathematician Nicolas Bourbaki is strongly encouraged to do so.

[5] This definition is now possibly beyond the realm of "naive set theory", since it is perhaps a little unclear what the union of arbitrary sets is supposed to mean. But the Zermelo-Fraenkel axioms allow for unions of arbitrary sets.

As we mentioned in the beginning, in these lecture notes we will work with ZFC theory, in particular we will use the Axiom of Choice, even though for the most part it is for our purposes rather irrelevant.

For finitely many sets we adopt the following convenient notation.

Notation 1.11.

(1) Given finitely many sets $X_1, \dots, X_n$ we write

$$X_1 \times \cdots \times X_n \ := \ \{(x_1, \dots, x_n) \,|\, \text{ for } i = 1, \dots, n \text{ we have } x_i \in X_i\}.$$

This set has an obvious bijection to the product $\prod_{i=1}^{n} X_i$, and we will freely go back and forth between these two notations.

(2) Given a set X and given $n \in \mathbb{N}$ we write

$$X^n \ := \ \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

We conclude this section with one last piece of notation.

Notation 1.12.

(1) If $\{A_i\}_{i \in I}$ is a family of sets, then we define the disjoint union as the set[6] [7]

$$\bigsqcup_{i \in I} A_i \ := \ \bigcup_{i \in I} (A_i \times \{i\}).$$

(2) If $I = \{1, \dots, k\}$, then we write

$$A_1 \sqcup \cdots \sqcup A_k \ := \ \bigsqcup_{i=1}^{k} A_i \ := \ \bigsqcup_{i \in \{1, \dots, k\}} A_i.$$

(3) We use obvious variations on (1) and (2), for example given sets A and B we refer to

$$A \sqcup B \ := \ (A \times \{1\}) \cup (B \times \{2\})$$

as the disjoint union of A and B

Remark. Let $\{U_i\}_{i \in I}$ be a family of subsets of a given set X such that for every $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$. The expression $\bigsqcup_{i \in I} U_i$ now describes two different sets, namely it describes the subset of X from Notation 1.6 and it describes the disjoint union as defined above. There is a natural bijection between these two sets and thus we will freely go back and forth between these notations. In practice should not pose any problems. □

1.2. Relations and equivalence relations on sets.

We start out with recalling the notion of a relation.

Definition 1.13. A relation on a set S is a subset of $S \times S$.

The definition of a relation in its full generality is pretty useless. It becomes more interesting once we introduce several adjectives.

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[6] Note that the $A_i \times \{i\}$ are disjoint subsets of the product $\prod_{i \in I} X_i$. Thus it makes sense to take their union, in the sense of Definition 1.4. Furthermore, since these subsets are disjoint, we see that the notation is consistent with Notation 1.6.

[7] For aesthetic or psychological reasons we sometimes also write $\bigsqcup_{i \in I} A_i \ := \ \bigcup_{i \in I} (\{i\} \times A_i)$. This slight discrepancy should not pose any problems.

Definition 1.14. Let S be a set and let $\mathrm{R} \subseteq S \times S$ be a relation on S.
(1) The relation is called reflexive, if for every $s \in S$ we have $(s, s) \in \mathrm{R}$.
(2) The relation is called transitive, if for every $(s, t), (t, u) \in \mathrm{R}$ we also have $(s, u) \in \mathrm{R}$.
(3) The relation is called symmetric, if for every (s, t) in R we also have $(t, s) \in \mathrm{R}$.

We adopt the following notation:

Notation 1.15. Let S be a set and let $\mathrm{R} \subseteq S \times S$ be a relation on S. Given $x, y \in S$ we write $x \,\mathrm{R}\, y$ if and only if $(x, y) \in \mathrm{R}$.[8]

Example. We consider the set $\mathbb{R}$ together with the relation

$$\leq \;:=\; \{(x, y) \in \mathbb{R}^2 \,|\, y - x \text{ is not negative}\}.$$

One can easily verify that this relation $\leq$ is reflexive and transitive, but that it is not symmetric. Clearly given $x, y \in \mathbb{R}$ the above expression "$x \leq y$" has the usual meaning that we are all used to. □

Most of the relations that we will consider are going to be reflexive and transitive. There is a big difference though in the roles of symmetric and (possibly) non-symmetric relations. Let us consider the symmetric ones first. We will turn to the (possibly) non-symmetric relations in the next section.

Definition 1.16. An equivalence relation on a set S is a relation that is reflexive, transitive and symmetric.

Convention 1.17. Equivalence relations on a set S are almost invariably denoted by "$\sim$" or some variation thereof.

To practice our conventions and notations, the three properties of an equivalence relation "$\sim$" on a set S are precisely the following three statements:
(1) for any $s \in S$ we have $s \sim s$ (reflexivity),
(2) for any $s, t, u \in S$ with $s \sim t$ and $t \sim u$ we have $s \sim u$ (transitivity),
(3) for any $s, t \in S$ with $s \sim t$ we also have $t \sim s$ (symmetry).

We continue with the following definition:

Definition 1.18. Let S be a set together with an equivalence relation "$\sim$". An equivalence class of $(S, \sim)$ is a subset C of S with the following three properties:
(a) the subset is non-empty,
(b) given any two $s, t \in C$ we have $s \sim t$, and
(c) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

The following lemma summarizes the key properties of equivalence classes.

Lemma 1.19. (Equivalence Class Lemma) Let S be a set together with an equivalence relation "$\sim$".
(1) Every element of S is contained in a unique equivalence class, namely

$$[s] \;:=\; \{t \in S \,|\, t \sim s\}.$$

(2) Two equivalence classes are either disjoint or the same.

[8]Most relations that we are working with are denoted by symbols like "$\sim$", "$\leq$", "$\cong$" or "$\simeq$".

Proof. Surely the reader will have seen the proof of the lemma before. Nonetheless we give the proof, in particular to highlight the fact that we do indeed need the symmetry condition.

(1) Let $s \in S$. Since $\sim$ is reflexive we see that $s \in [s]$. Thus it remains to show that $[s] \coloneqq \{t \in S \,|\, t \sim s\}$ is indeed an equivalence class.
   (a) We already showed that $s \in [s]$, in particular we see that $[s]$ is non-empty.
   (b) Let $t, t' \in [s]$. By definition this means that $t \sim s$ and $t' \sim s$. By symmetry we obtain that $s \sim t'$. By transitivity we deduce from $t \sim s$ and $s \sim t'$ that $t \sim t'$.
   (c) Let $t \in [s]$ and suppose that $t \sim u$. Since by definition we have $s \sim t$ we obtain from transitivity that $s \sim u$, i.e. we see that $u \in [s]$.

(2) Let C and D be two equivalence classes. We suppose that $C \cap D \neq \varnothing$. Thus there exists an $x \in C \cap D$. We need to show that $C = D$. Thus let $d \in D$. By Condition (2) of an equivalence class applied to D we know that $x \sim d$. By condition (1) of an equivalence class, applied to C, we see that $d \in C$. Thus we have shown that $D \subseteq C$. Evidently the same argument shows that $C \subseteq D$. ■

Throughout these notes we adopt the following standard notation.

Notation 1.20. Let S be a set together with an equivalence relation "$\sim$".

(1) We denote by $S/\sim$ the set of equivalence classes.
(2) Given $s \in S$ we denote by $[s]$ the unique equivalence class that contains s.
(3) We refer to the map
$$\begin{array}{rccc} p\colon & S & \to & S/\sim \\ & s & \mapsto & [s] \end{array}$$
as the natural projection.

Example. Let $n \in \mathbb{N}$. We consider the set $\mathbb{Z}$ together with the equivalence relation that is given by
$$x \sim y \quad :\Longleftrightarrow (x - y) \text{ is divisible by } n.$$
One can easily show that there are exactly n equivalence classes, namely
$$\begin{aligned} [0] &= \{\dots, -2n, -n, 0, n, 2n, 3n, \dots\} \\ [1] &= \{\dots, 1-2n, 1-n, 1, 1+n, 1+2n, 1+3n, \dots\} \\ &\vdots \\ [n-1] &= \{\dots, -2n-1, -n-1, -1, n-1, 2n-1, 3n-1, \dots\}. \end{aligned}$$
We write $\mathbb{Z}/n\mathbb{Z} \coloneqq \mathbb{Z}/\sim$. □

The following lemma shows that the set of equivalence classes has a certain universal property:

Lemma 1.21. (Quotient-Factorization Lemma) Let $\sim$ be an equivalence relation on a set X. We denote by $p\colon X \to X/\sim$ the natural projection. Let $f\colon X \to Y$ be a map to some set Y with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map
$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! g} \\ & & Y. \end{array}$$

Proof. Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\sim$. Since equivalences classes are by definition non-empty there exists an $x \in X$ with $a = [x]$. We set

$$g(a) \;:=\; f(x).$$

Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\sim$ is surjective we see that g is uniquely determined by this property. ■

We will now see that every relation gives rise to an equivalence relation:

Definition 1.22. Let S be a set and let $\mathrm{R} \subseteq S \times S$ be a relation on S. The equivalence relation generated by the relation R is defined as the intersection of all equivalence relations on S that contain R.[9]

Example. We consider the set $S = \mathbb{R}$ and the relation that is defined by $x\,\mathrm{R}\,y$ if and only if $y = x + 1$. Clearly the equivalence relation generated by R is the equivalence relation that is given by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. □

The following strengthening of the above Quotient-Factorization Lemma 1.21 gets used all the time (mostly without realizing):

Lemma 1.23. (Quotient-Factorization Lemma) Let R be a relation on a set X. We denote by $\sim$ the equivalence relation generated by R. We denote by $p\colon X \to X/\sim$ the natural projection. Let $f\colon X \to Y$ be a map to some set Y with the property that $f(x) = f(y)$ whenever $x\,\mathrm{R}\,y$. Then the map

$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! g} \\ & & Y. \end{array}$$

Proof. Note that $x\mathrm{S}x' :\Leftrightarrow f(x) = f(x')$ defines an equivalence relation on X that contains R. By Definition 1.22 we know that $\sim$ is contained in S. Thus we see that $f(x) = f(x')$ whenever $x \sim x$'. The desired statement now follows from the Quotient-Factorization Lemma 1.21. ■

[9]For practical purposes one can give the following concrete description: We say that $x, y \in S$ are *equivalent*, if there exists a sequence $x_0, \ldots, x_k$ of elements in S such that $x = x_0$, $x_k = y$ and such that for each $i \in \{0, \ldots, k-1\}$ we have $x_i\,\mathrm{R}\,x_{i+1}$ or $x_{i+1}\,\mathrm{R}\,x_i$. It is straightforward to verify that "$\sim$" defines an equivalence relation on S and that it is equivalence relation generated by the relation R.

We conclude this discussion with the following convention:

Convention 1.24. At times we are a little sloppy in our notation and we do not distinguish in our notation between a relation and the equivalence relation it generates.

1.3. **Orders on sets.** In this section we turn our gaze towards relations which are not required to be symmetric. We have the following harmless definition.

Definition 1.25.

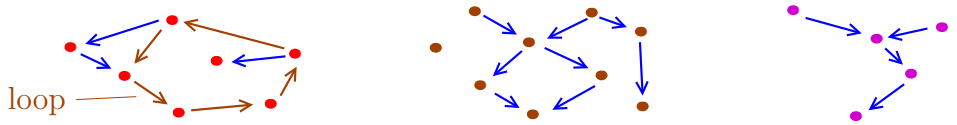
(1) A relation "$\leq$" on a set S is called antisymmetric if $a \leq b$ and $b \leq a$ implies $a = b$.
(2) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(3) A set together with a partial order is called a partially ordered set.
(4) Let "$\leq$" be a partial order on a set A. Given $a, b \in A$ we write $a < b$ if $a \leq b$ and if $a \neq b$.

Example.

(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a partially ordered set.
(2) We equip $\mathbb{R} \cup \{\pm\infty\}$ with the relation $\leq$ that is given by the usual relation on $\mathbb{R}$ together with $-\infty \leq a \leq \infty$ for any $a \in \mathbb{R}$. This turns $\mathbb{R} \cup \{\pm\infty\}$ into a partially ordered set. For $a, b \in \mathbb{R} \cup \{\pm\infty\}$ we adopt the following notation:

$$\begin{aligned} [a,b] &:= \{x \in \mathbb{R} \cup \{\pm\infty\} \,|\, a \leq x \leq b\}, \\ (a,b] &:= \{x \in \mathbb{R} \cup \{\pm\infty\} \,|\, a < x \leq b\}, \\ [a,b) &:= \{x \in \mathbb{R} \cup \{\pm\infty\} \,|\, a \leq x < b\}, \\ (a,b) &:= \{x \in \mathbb{R} \cup \{\pm\infty\} \,|\, a < x < b\}. \end{aligned}$$

Any of these sets, which happens to be a subset of $\mathbb{R}$, will be called an interval.
(3) Let M be a set. Recall that we denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subseteq$", is a partial order on $\mathcal{P}(M)$.
(4) Let X be a set. The trivial partial order $\leq_{\text{tr}}$ is the partial order where $x \leq_{\text{tr}} y$ if and only if $x = y$.
(5) We consider the three reflexive transitive relations that are illustrated in the figure below: Here it is understood that $P \leq Q$ if there exists a "directed path" from Q to P.



Example (a) Example (b) Example (c)

(a) The example on the left has a "loop". Given any two points P, Q on the loop we have $P \leq Q$ and $Q \leq P$, hence this example is not a partial order.
(b) This example is a partial order.
(c) This example is also a partial order. □

The following construction lets us build new examples of partially ordered sets out of given ones.

Definition 1.26. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define[10]

$$(a, b) < (a', b') \quad :\Longleftrightarrow \quad \text{either } (a < a') \text{ or } (a = a' \text{ and } b < b')$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. One can easily verify that "$\leq$" is a partial order on $A \times B$. We refer to this partial order on $A \times B$ as the lexicographic order on $A \times B$.

We conclude this introduction to partially ordered sets with a short discussion of maps between partially ordered sets.

Definition 1.27.

(1) Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A map $f\colon A \to B$ is called order-preserving, if for every two elements $a_1, a_2 \in A$ with $a_1 \leq a_2$ we also have $f(a_1) \leq f(a_2)$.[11]

(2) We say that two partially ordered sets $(A, \leq)$ and $(B, \leq)$ have the same order type if there exists a bijection $f\colon A \to B$ which is order-preserving and such that the inverse $f^{-1}\colon B \to A$ is also order-preserving.

Examples.

(1) We consider the half-open interval $[0, 1)$, the set $\mathbb{R}_{\geq 0}$ and the set $\mathbb{N}_0$, each equipped with the usual partial order $\leq$. In Exercise 1.10 we will see that the lexicographically ordered set $[0, 1) \times \mathbb{N}_0$ has the same order type as $\mathbb{R}_{\geq 0}$.

(2) There is an obvious order-preserving map $(\mathbb{N}_0, \leq) \to (\mathbb{Q}, \leq)$. Note though that the two partially ordered sets $(\mathbb{N}_0, \leq)$ and $(\mathbb{Q}, \leq)$ have different order types, for example in the first case given any element $x \in \mathbb{N}_0$ there exist only finitely many elements in $\mathbb{N}_0$ less than x, but the analogous statement for $\mathbb{Q}$ is of course wrong. □

1.4. **Zorn's Lemma and the Well-Ordering Theorem.** We need a few more definitions before we can state the first non-trivial results on partially ordered sets.

Definition 1.28. Let $(P, \leq)$ be a partially ordered set.

(1) A chain in P is a subset Q such that for every two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.

(2) An upper bound of a subset Q is an element $p \in P$ such that $q \leq p$ for all $q \in Q$.

(3) A maximal element of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.

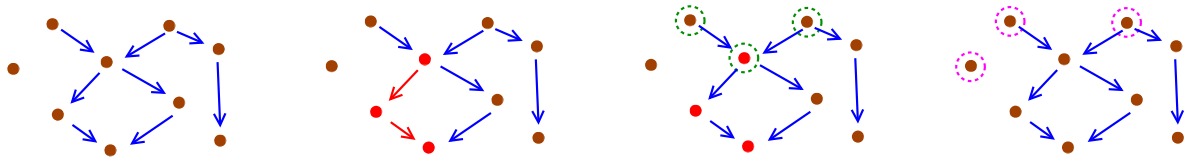
We define the notion of a lower bound and a minimal element in the obvious analogous way.

Example.

(1) In the figure below we once again consider the partially ordered set in the middle of the previous figure. We show a chain, we show upper bounds for the chain and we show maximal elements for P.

[10] Note that if $a < a'$, then we do not demand that $b \leq b'$.

[11] Note that "order-preserving" does not mean that $a_1 < a_2$ implies $f(a_1) < f(a_2)$.



partially ordered set P | chain Q | upper bounds for the chain Q | maximal elements of the set P

(2) We consider the interval $P = [0, \infty)$ with the usual partial ordering. Every subset of P is a chain, but not every subset has an upper bound. For example $P = [0\,\infty)$ itself does not have an upper bound. Also note that $P = [0, \infty)$ does not have a maximal element.
(3) We consider the interval $P = (-\infty, 0]$ with the usual partial ordering. Every subset of P is a chain and 0 is an upper bound for every subset. Furthermore 0 is a maximal element of $P = (-\infty, 0]$. □

Next we formulate Zorn's Lemma. Its modest name conceals its true importance in mathematics.

Lemma 1.29. (Zorn's Lemma) Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. If P is non-empty, then the set P contains at least one maximal element.

Remark.

(1) Zorn's Lemma even made it into "The Simpsons", see

`https://getyarn.io/yarn-clip/cb6407a6-ea52-47f1-828f-f45a48a9ac1d`



(2) The lemma is named after Max Zorn [**Zo1935**] who published a proof in 1935. In fact the same statement was already proved by Casimir Kuratowski [**Kura1922**] in 1922. A nice discussion of Zorn's Lemma and its historical context is given in [**Sti2013**, Section 7.2]. □

Proof. A proof of Zorn's Lemma, assuming the Axiom of Choice 1.1, is given in [**Cie1997**, Theorem 4.3.4] and [**Je1973**, p. 9]. In fact, as is shown in [**Sti2013**, p. 154] or alternatively [**Je1973**, Theorem 2.1], Zorn's Lemma is equivalent to the Axiom of Choice 1.1. ■

We continue with the next adjective that we can associate to a relation.

Definition 1.30.
(1) A relation "$\leq$" on a set S is called connex if for every $a, b \in S$ we have $a \leq b$ or $b \leq a$.
(2) A partial order that is connex is called a total order.
(3) A set together with a total order is called a totally ordered set.



Examples.

(1) Evidently the usual "$\leq$"-relation on any subset of $\mathbb{R}$ is connex.
(2) Given two totally ordered sets $(A, \leq)$ and $(B, \leq)$ one can show easily that the lexicographic order on $A \times B$ is again a total order. □

Finally we introduce one last type of partial orders.

Definition 1.31.

(1) A total order $\leq$ on a set S is called a well-order if every non-empty subset T has a minimal element.
(2) A set together with a well-order is called a well-ordered set.

Examples.

(1) The usual order on $\mathbb{Q}$ is a total order but it is not a well-order, for example the subset $T := \{x \in \mathbb{Q} \,|\, x > 2\}$ does not have a minimal element. But since $\mathbb{Q}$ is in bijection to $\mathbb{N}$ we see that $\mathbb{Q}$ admits a well-order.
(2) The usual order on $\mathbb{R}$ is a total order but it is not a well-order. It is a priori not so clear how one can define a well-order on $\mathbb{R}$.
(3) Given two well-ordered sets $(A, \leq)$ and $(B, \leq)$ one can again show easily that the lexicographic order on $A \times B$ is a well-order. □

Perhaps somewhat shockingly the following theorem, due to Ernst Zermelo [**Zer1904**], says that every set admits a well-order.

Theorem 1.32. (Well-Ordering Theorem) Every set admits a well-order.

Proof. We refer to [**Cie1997**, Theorem 4.3.3] or alternatively to [**Je1973**, p. 9] for a proof assuming the Axiom of Choice 1.1. In fact, as for Zorn's Lemma it is shown in [**Je1973**, Theorem 2.1] that the Well-Ordering Theorem is equivalent to the Axiom of Choice 1.1. ■

1.5. **Maximal and minimal elements.** We recall the following definition from the last section:

Definition 1.33. Let $(P, \leq)$ be a partially ordered set and let $Q \subseteq P$ be a subset.

(1) A maximal element of Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.
(2) A minimal element of Q is an element $q \in Q$ such that there is no $r \in Q$ with $q > r$.

In important cases, maximal elements and minimal elements, if they exist, are unique. This leads us to the following notation:

Notation 1.34. If $(P, \leq)$ is a partially ordered set and let $Q \subseteq P$ be a subset. If the relation is antisymmetric and total, then if a maximum element exists, it is unique, and we denote it by $\max(Q)$. Similarly we define $\min(Q)$, if it exists.

It is clear that any bounded subset $Q \subseteq \mathbb{Z}$ has a maximal element and a minimal element. For some purposes the following notation is useful:

Notation 1.35. Let $Q \subseteq \mathbb{Z}$ be a subset.

(1) (a) If Q has no upper bound, then we write $\max(Q) := \infty$.
(b) We write $\max(\varnothing) := -\infty$.
(2) (a) If Q has no lower bound, then we write $\min(Q) := -\infty$.
(b) We write $\min(\varnothing) := +\infty$.

1.6. **The cardinality of sets.** In this section we introduce the notion of the cardinality of a set.

Definition 1.36.

(1) We say that two given sets S and T are equinumerous if there exists a bijection $\varphi\colon S \to T$.[12]
(2) Given a set S we define the cardinality $\#S$ of S as the equinumerosity equivalence class of S in the class of all sets[13]. Put differently we have

$$\#S \;:=\; \text{class of all sets } T \text{ that are equinumerous with } S.$$

(3) We say that a set is finite if there exists an $n \in \mathbb{N}_0$ such that $\#S = \#\{1, \dots, n\}$, otherwise we say that the set is infinite
(4) If a set S is finite, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ with $\#S = \#\{1, \dots, n\}$. If S infinite, then sometimes we write $\#S = \infty$[14]. At times we also write $|S| \in \mathbb{N}_0 \cup \{\infty\}$ instead of $\#S \in \mathbb{N}_0 \cup \{\infty\}$.
(5) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or if $\#S = \#\mathbb{N}$, then we say that S is countable, otherwise we say that S is uncountable.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by

$$\begin{array}{rcl} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}, \\ x - 1, & \text{if } x \in \mathbb{N}. \end{cases} \end{array}$$

□

Remark. Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T. □

The following lemma gets used on many occasions in these lecture notes.

Lemma 1.37. (Countability Lemma)

(1) Every subset of a countable set is again countable.

[12] It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.
[13] We defined the notion of an equivalence relation on a set, but not on a class. You can ask any of your professors, as long as it is not me, what this now means.
[14] This notation is a little risque. The sets $\mathbb{N}$ and $\mathbb{R}$ are both infinite, but in the $\mathbb{R}$-Uncountable Corollary 1.40 we will see that $\mathbb{N}$ and $\mathbb{R}$ are not equinumerous. Thus we write $\#\mathbb{N} = \infty$ and $\#\mathbb{R} = \infty$, but $\#\mathbb{N} \neq \#\mathbb{R}$.

(2) Let S be a countable set and let $f\colon S \to T$ be a map. The image $f(S)$ is again countable.
(3) The product of finitely many countable sets is again countable.
(4) The disjoint union of countably many countable sets is again countable.
(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. Then S is countable if and only if T is countable.
(6) If S is a finite set and if T is a countable set, then the set of maps from S to T is also countable.

Example. We consider the map

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{N} & \to & \mathbb{Q} \\ (m,n) & \mapsto & \frac{m}{n}. \end{array}$$

Evidently this map is surjective. It follows from the Countability Lemma 1.37 (2) and (3) that $\mathbb{Q}$ is countable. □

Sketch of proof.

(1) Let T be a subset of a countable set S. After excluding the trivial case that S is finite and the case that T is finite one sees that one has to prove the following: Given any infinite subset T of $S = \mathbb{N}$ there exists a bijection $\varphi\colon \mathbb{N} \to T$. Such a bijection is given by defining $\varphi(i)$ as the i-th smallest element of T.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. We only really need to consider the case that $S = \mathbb{N}$ and that $f(S)$ is infinite. In this case the bijection $\varphi\colon \mathbb{N} \to f(S)$ is given by defining $\varphi(1) = f(1)$ and by defining iteratively

$$\varphi(i+1) \;:=\; \left\{ f(k) \;\middle|\; \begin{array}{c} k \in \mathbb{N} \text{ is the smallest number such} \\ \text{that } f(k) \notin \{\varphi(1), \ldots, \varphi(i)\} \end{array} \right\}.$$

(3) By induction and by ignoring the rather dull case of finite sets it remains to show that $\mathbb{N} \times \mathbb{N}$ is countable. This can be shown by the well-known argument sketched in the figure below.



An alternative, more formal argument would be to consider the map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is given by $(k,l) \mapsto (2^k \cdot 3^l)$. This map is injective, thus it follows from (1) that $\mathbb{N} \times \mathbb{N}$ is countable.



(4) Let $\{S_i\}_{i \in I}$ be a countable family of countable subsets. It follows from the hypothesis that each S_i is countable together with the Axiom of Choice 1.1 that for each $i \in I$ we

can pick an injective map $\varphi_i\colon S_i \to \mathbb{N}$. Furthermore, note that by the countability of I we can pick an injective map $\gamma\colon I \to \mathbb{N}$. One can now easily verify that the map

$$\begin{array}{rcl} \bigsqcup_{i\in I} S_i & \to & \mathbb{N}\times\mathbb{N} \\ (s,i) & \mapsto & (\varphi_i(s),\gamma(i)) \end{array}$$

is injective. Thus it follows from (1) and (3) that the disjoint union $\bigsqcup_{i\in I} S_i$ is countable.

(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. If S is countable, then it follows from (4) that $T = (T \setminus S) \cup S$ is countable. On the other hand if T is countable, then it follows from (1) that S is countable.

(6) Let $S = \{v_1, \ldots, v_n\}$ be a finite set and let T be a countable set. We have an obvious bijection between T^n and the set of maps from S to T. Thus it follows from (3) that the set of maps from S to T is countable. ■

We move on to the next definition.

Definition 1.38. Let S and T be two sets.

(1) We write $\#S \leq \#T$ if there exists an injection $\varphi\colon S \to T$.
(2) We write $\#S < \#T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Example. It follows from the Countability Lemma 1.37 (1) that a set S is countable if and only if $\#S \leq \#\mathbb{N}$. □

The following theorem was first proved by Georg Cantor [**Cant1892**]. The ratio of significance of the statement to length of the proof seems to be unbeatable.

Theorem 1.39. (Cantor's Theorem) Recall that given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. the set of all subsets of X. For every set X we have $\#X < \#\mathcal{P}(X)$.

Proof. First note that the map $\begin{array}{rcl} X & \to & \mathcal{P}(X) \\ x & \mapsto & \{x\} \end{array}$ is evidently injective, hence $\#X \leq \#\mathcal{P}(X)$. Now let $f\colon X \to \mathcal{P}(X)$ be a map. We have to show that it cannot be surjective. We consider the set

$$Y \;:=\; \{x \in X \,|\, x \notin f(x)\}.$$

We claim that Y is not in the image of f. Indeed, if there did exist a $z \in X$ with $f(z) = Y$, then we would have

$$z \in Y \quad \underset{\text{by definition of } Y}{\Longleftrightarrow} \quad z \notin f(z) \quad \underset{\text{since } f(z)=Y}{\Longleftrightarrow} \quad z \notin Y.$$

We have thus obtained a contradiction. ■

Corollary 1.40. ($\mathbb{R}$-Uncountable Corollary) The set $\mathbb{R}$ of real numbers is uncountable.

Proof. [15] We consider the maps

$$\begin{array}{rcl} f\colon \mathcal{P}(\mathbb{N}) & \to & \{\mathbb{Z}/2\mathbb{Z}\}^{\mathbb{N}} \\ X & \mapsto & \left(\begin{array}{rcl} \mathbb{N} & \to & \mathbb{Z}/2\mathbb{Z} \\ k & \mapsto & \begin{cases} 1, & \text{if } k \in X \\ 0, & \text{otherwise} \end{cases} \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \{\mathbb{Z}/2\mathbb{Z}\}^{\mathbb{N}} & \to & \mathbb{R} \\ f & \mapsto & \sum_{k=1}^{\infty} \frac{f(k)}{3^k}. \end{array}$$

[15] There are many proofs that the real numbers are uncountable. Perhaps the most beautiful and astounding proof that $\mathbb{Q}$ is countable is given via the amazing Farey sequence, see [**Hat2022**, Chapter 3].

One can easily verify that $f\colon \mathcal{P}(\mathbb{N}) \to \{\mathbb{Z}/2\mathbb{Z}\}^{\mathbb{N}}$ is a bijection and that $g\colon \{\mathbb{Z}/2\mathbb{Z}\}^{\mathbb{N}} \to \mathbb{R}$ is injective. Thus it follows from Cantor's Theorem 1.39 together with the Countability Lemma 1.37 (1) that $\mathbb{R}$ is uncountable. ■

Now we can formulate the final result of this section.

Theorem 1.41. (Bernstein-Schröder Theorem)[16] Let S and T be two sets. Then

$$\#S \leq \#T \quad \text{and} \quad \#T \leq \#S \implies \#S = \#T.$$

Example. Let U be a non-empty subset of $\mathbb{R}$ which contains an open interval (a, b) with $a < b$. Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \to \mathbb{R}$ is of course injective, thus we see that $\#U \leq \#\mathbb{R}$. Furthermore the map $\varphi\colon (a, b) \to \mathbb{R}$ given by $x \mapsto \frac{x}{(x-a)(x-b)}$ defines a bijection between (a, b) and $\mathbb{R}$. Thus $\mathbb{R} \xrightarrow{\varphi^{-1}} (a, b) \to U$ is also an injection. This means that $\#\mathbb{R} \leq \#U$. Therefore it follows from the Bernstein-Schröder-Theorem 1.41 that there exists a bijection from U to $\mathbb{R}$. □

Proof. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint.[17] We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $b \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of g there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

Proof. So let I be an equivalence class and let $a \in A \cap I$.[18] The subset $I \subseteq A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:[19][20]

			A	B	A	B	A	B		
(1)	a',	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	with $a' \in A \setminus g(B)$
(2)	b',	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	with $b' \in B \setminus f(A)$
(3)	$\dots$	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$.⊞

Recall that by the Equivalence Class Lemma 1.19 we know that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of

[16] The theorem was initially stated by Georg Cantor [**Cant1887**] in 1887 without a proof and it was first proved by Felix Bernstein [**Bern1905**] and Ernst Schröder [**Schröd1896**] in 1896. Sometimes it is also called the Cantor-Bernstein-Schröder Theorem. The proof we provide is attributed to Julius König [**König1906**].

[17] If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

[18] There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in $b \in B$, but then it also contains $g(b) \in A$, since $b \sim g(b)$.

[19] To clarify the description: In (1) and (2) we look at sets which are described by "half-infinite" sequences starting at a' respectively b'. In (3) we look at a sequence which is infinite in "both directions".

[20] Note that we do not claim that the elements in the given sequences are distinct, for example $f\colon A \to B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j\in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j\in J}(A \cap I_j) \text{ and } B = \bigsqcup_{j\in J}(B \cap I_j).$$

Since for each $j \in J$ we have a bijection $A \cap I_j \to B \cap I_j$ we get a bijection $A \to B$. ■

Remark. The *continuum hypothesis*, first formulated by Georg Cantor [**Cant1878**] in 1878, states that there exists no set S with $\#\mathbb{Q} < \#S < \#\mathbb{R}$. Any attempt at proving this statement relies on first of all making precise what "sets" are supposed to be or what axioms they are supposed to satisfy. As wikipedia writes "the answer to this problem is independent of ZFC set theory (that is, Zermelo-Fraenkel set theory with the Axiom of Choice 1.1 included), so that either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent." This independence was proved in 1963 by Paul Cohen [**CoheP1963**] for which he obtained a Fields medal in 1966. More useful information can as always be found on wikipedia:

`https://en.wikipedia.org/wiki/Continuum_hypothesis` □

1.7. **The cardinality of finite sets.** We conclude this chapter with a few elementary observations about the cardinality of some finite sets.

Lemma 1.42. (Power Set–Cardinality Lemma) Let X be a finite set with n elements. The power set $\mathcal{P}(X)$ has 2^n elements.

Proof. We enumerate the elements of X by $x_1, \ldots, x_n$ Similar to the proof of the $\mathbb{R}$-Uncountable Corollary 1.40 we consider the maps

$$\begin{array}{rcl} f\colon \mathcal{P}(X) & \to & \{\mathbb{Z}/2\mathbb{Z}\}^n \\ A & \mapsto & \left(\begin{array}{rcl} \{1,\ldots,n\} & \to & \{0,1\} \\ k & \mapsto & \begin{cases} 1, & \text{if } x_k \in A \\ 0, & \text{otherwise} \end{cases} \end{array} \right). \end{array}$$

One can easily verify that $f\colon \mathcal{P}(X) \to \{0,1\}^n$ is a bijection. Clearly the latter set has 2^n elements. ■

Finally we want to consider subsets of a fixed cardinality. This discussion will make use of the following definition:

Definition 1.43. Let $n \in \mathbb{N}_0$ and let $k \in \mathbb{N}_0$. We define n choose k as

$$\binom{n}{k} := \begin{cases} \dfrac{n!}{k! \cdot (n-k)!}, & \text{if } k \in \{0,\ldots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Example. In Exercise 1.13 we will see that for any $n \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have the equality

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This equality is sometimes called the *recurrence relation*. □

Lemma 1.44. (Subsets-of-Fixed Cardinality Lemma) Let X be a finite set with n elements and let $k \in \mathbb{N}_0$. The number of subsets of X with precisely k elements is given by $\binom{n}{k}$.

Proof. This lemma can be proved easily by induction on k, by making use of the above recurrence relation. The details will be provided in Exercise 1.14. ■

1.8. Exercises for Chapter 1.

Exercise 1.1. Show that for statements (1a), (2a), (4) and (5a) of the Image-Preimage Lemma 1.9 there are indeed examples such that the stated inclusion is indeed a strict inclusion. More precisely:

(a) Give an example of a map $f\colon X \to Y$ and a subset $U \subseteq X$ such that $U \subsetneq f^{-1}(f(U))$.
(b) Give an example of a map $f\colon X \to Y$ and a subset $V \subseteq Y$ such that $f(f^{-1}(V)) \subsetneq V$.
(c) Give an example of a map $f\colon X \to Y$ and a family $\{U_i\}_{i\in I}$ of subsets of X such that
$$f\Big(\bigcap_{i\in I} U_i\Big) \subsetneq \bigcap_{i\in I} f(U_i)$$
(d) Give an example of a map $f\colon X \to Y$ between two sets and a subset $U \subseteq X$ such that we have $f(X) \setminus f(U) \subsetneq f(X \setminus U)$.

Exercise 1.2. Let $f\colon X \to Y$ be a map between two sets.

(a) Let $A \subseteq X$ and let $B \subseteq Y$ be two subsets. Prove the following equivalence of statements:
$$f(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq f^{-1}(B).$$
(b) Let $A \subseteq X$ and let $B \subseteq Y$. Show that
$$f(A \cap f^{-1}(B)) \;=\; f(A) \cap B.$$

Exercise 1.3. Let S be a set and let $\{S_i\}_{i\in I}$ be a family of subsets of S. Furthermore suppose that for each $i, j \in I$ we are given an injective map $\varphi_{ij}\colon S_i \cap S_j \to S_i \cap S_j$. We assume that the following condition is satisfied:

(∗) For all $i, j, k \in I$ we have $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}\colon S_i \cap S_j \cap S_k \to S_k$.

Prove the following two statements:

(a)
- For each $i \in I$ we have $\varphi_{ii} = \mathrm{id}_{S_i}$.
- For each $i, j \in I$ we have $\varphi_{ji} = \varphi_{ij}^{-1}\colon S_i \cap S_j \to S_i \cap S_j$.

(b) We consider the set
$$T \;:=\; \Big(\bigsqcup_{i\in I} S_i\Big)/\sim$$
where $\sim$ is the equivalence relation that is generated (in the sense of Definition 1.22) by the relation that for all $i, j \in I$ and all $a \in S_i \cap S_j$ we have $(a, i) \sim (\varphi_{ij}(a), j)$. Show that for each $i \in I$ the map
$$\begin{aligned} \gamma_i\colon S_i &\to T \\ s &\mapsto [(s, i)] \end{aligned}$$
is injective.

Exercise 1.4. Let X be a set and let $\sim$ be an equivalence relation on X. Use the Axiom of Choice 1.1 to show that there exists a subset $S \subseteq X$ which contains precisely one element of each equivalence class.

Exercise 1.5. Let $(V, \leq)$ be a totally ordered set of cardinality $n \in \mathbb{N}$. We equip the set $\{1, \ldots, n\}$ with the usual total order $\leq$. Show that there exists a *unique* order-preserving bijection $\{1, \ldots, n\} \to V$.

Exercise 1.6. Let $f\colon X \to Y$ be a map between two sets. Show that there exist total orders "$\leq_X$" on X and "$\leq_Y$" on Y such that the map f is order-preserving.

Exercise 1.7. Let $(P, \leq)$ be a partially ordered set and let Q be a subset.

(a) Is an upper bound necessarily a maximal element?
(b) Conversely, is a maximal element necessarily an upper bound?

Exercise 1.8. Show that there exists an uncountable well-ordered set $(X, \leq)$ such that for every $y \in X$ the set $\{x \in X \,|\, x \leq y\}$ is countable.

Exercise 1.9. We consider $\mathbb{N}$ and $\mathbb{Z}$ with the usual partial order $\leq$.

(a) Does the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$ have the same order type as $\mathbb{N}$?
(b) Does the lexicographically ordered set $\mathbb{Z} \times \mathbb{Z}$ have the same order type as $\mathbb{Z}$?

Exercise 1.10. We consider the half-open interval $[0, 1)$, the set $\mathbb{R}_{\geq 0}$ and the set $\mathbb{N}_0$, each equipped with the usual partial order $\leq$.

(a) Show that the lexicographically ordered set $\mathbb{N}_0 \times [0, 1)$ has the same order type as $\mathbb{R}_{\geq 0}$.
(b) Show that the lexicographically ordered set $[0, 1) \times \mathbb{N}_0$ does not have the same order type as $\mathbb{R}_{\geq 0}$.

Exercise 1.11.

(a) Let $n \in \mathbb{N}$. Show that $\mathbb{R}$ and $\mathbb{R}^n$ have the same cardinality.
(b) Show that $\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}$ have the same cardinality.

Exercise 1.12. Show that $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality.
Hint. Use the Bernstein-Schröder Theorem 1.41.

Exercise 1.13. Show that for any $n \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have the equality

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Exercise 1.14. Let X be a finite set with n elements and let $k \in \mathbb{N}_0$. Show that the number of subsets of X with precisely k elements is given by $\binom{n}{k}$.

CHAPTER 2

Preliminaries from linear algebra

In this second preliminary chapter we will introduce some concepts and results from linear algebra which we will use throughout these notes, but which otherwise have little to do with topology itself. Again the reader might want to skip this preliminary chapter and directly move to Chapter 3, where we actually start with topology.

This chapter is organized as follows:

- In Section 2.1 we will recall basic definitions in group theory.
- In Section 2.2 we will state some conventions for (linear) algebra.
- In Section 2.3 we will recall the definition of the determinant of a square matrix over a commutative ring.
- In Section 2.4 we will introduce bilinear maps.
- In Section 2.5 we will state and prove the Cauchy-Schwarz Inequality.
- In Section 2.6 we will recall the Gram-Schmidt Orthogonalization Process.
- In Section 2.7 we will recall the Spectral Theorem.
- In Section 2.8 we will recall the concept of a group action on a set.
- In Section 2.9 we will introduce normed vector spaces.
- In Section 2.10 we will introduce metric spaces. We will also see that normed vector spaces are an important source of metric spaces.
- In Section 2.11 we will recall some concepts from elementary geometry.
- In Section 2.12 we will provide exercises for Chapter 2.

In the next Chapter 3 we will introduce topological spaces.

2.1. Basic definitions in group theory.

We recall a few basic definitions from the theory of monoids and groups.

Definition 2.1. A monoid is a *set* S together with a map

$$\begin{array}{ccc} S \times S & \to & S \\ (x, y) & \mapsto & x \cdot y \end{array}$$

such that the following two conditions are satisfied:[21]

(1) There exists a neutral element $e \in S$, i.e. an element which satisfies $e \cdot x = x \cdot e = x$ for all $x \in S$.

(2) The operation "$\cdot$" is associative.[22]

Furthermore:

(3) The monoid is called a group if given any $x \in S$ there exists a $y \in S$ with $x \cdot y = y \cdot x = e$.

(4) The monoid is called abelian if for all $x, y \in S$ we have $x \cdot y = y \cdot x$.

Example. The group $\{e\}$ consisting of a single element e is called the trivial group. Sometimes, especially when subconsciously we are thinking about abelian groups, then we also denote the trivial group by 0. □

For completeness' sake we also give the definition of a monoid homomorphism:

Definition 2.2. Let S and T be monoids. A map $\varphi\colon S \to T$ is called a monoid homomorphism if the following two conditions hold:

- For any $s, s' \in S$ we have $\varphi(s \cdot s') = \varphi(s) \cdot \varphi(s')$.
- The image of the neutral element of S is the neutral element of T.[23]

For the time being we will not make use of monoids. Thus we will now concentrate on groups:

Definition 2.3. Let G be a group.

(1) Given a subgroup $H \subseteq G$ we write $gH \coloneqq \{gh \mid h \in H\}$.
(2) A subgroup H is called normal if for every $g \in G$ we have $g^{-1}Hg = H$. If $H \subseteq G$ is normal, then we often write $H \lhd G$.
(3) Let H be a subgroup of G.
   (a) The subgroup H defines an equivalence relation on G by setting
   $$g \sim \widetilde{g} \quad :\Longleftrightarrow \quad \text{there exists an } h \in H \text{ with } gh = \widetilde{g}.$$
   Put differently, we have $g \sim \widetilde{g}$ if $\widetilde{g}^{-1}g \in H$.
   (b) We denote the set of equivalence classes by G/H. (Note that the equivalence classes are precisely the sets of the form gH for some $g \in G$.)

Let G be a group and let H be a normal subgroup of G. The next lemma shows that the group structure on G descends to a group structure on G/H.

Lemma 2.4. (Quotient Group Lemma) Let G be a group and let $H \subseteq G$ be a normal subgroup.[24]

(0) Given aH and bH in G/H we set
$$aH \cdot bH \;\coloneqq\; (a \cdot b)H.$$
This product on G/H is well-defined and it defines a group structure on G/H.
(1) (a) The obvious map $G \to G/H$ is a group homomorphism.
   (b) Let $\varphi\colon G \to \Gamma$ be a group homomorphism with the property that $H \subseteq \ker(\varphi)$. There exists a unique homomorphism $\psi\colon G/H \to \Gamma$ such that the following diagram commutes:
$$\begin{array}{ccc} G & \longrightarrow & G/H \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \psi} \\ & & \Gamma. \end{array}$$

[21] A *semi-group* is defined similarly, except that we do not demand the existence of a neutral element.
[22] Recall that "associative" means that for any $a, b, c \in S$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. It is well-known that this means that we can unambiguously define the product of finitely many elements using any choice of parentheses. Even though this statement is completely familiar, if one thinks about it, one realizes that this requires a proof. Such a proof is for example provided in [**DF2004**, p. 19].
[23] Note that in contrast to a group homomorphism we need this condition on the neutral elements.

Proof. Let G be a group and let H be a normal subgroup of G.

(0) The following claim shows that the product on G/H is well-defined.

Claim. Given any $a, b \in G$ and $h \in H$ we have $(a \cdot b \cdot h)H = (a \cdot b)H = (ah \cdot b)H$.

Proof. The following argument shows that the first equality holds:

$$(a \cdot b \cdot h)H \;=\; (a \cdot b) \cdot \underbrace{hH}_{=H} \;=\; (a \cdot b)H.$$

For the second equality note that

$$(a \cdot h \cdot b)H \;=\; (a \cdot b \cdot \underbrace{b^{-1} \cdot h \cdot b}_{\text{in } H \text{ since } H \text{ is normal}})H \;=\; (a \cdot b)H.$$

It is now straightforward to verify that this product turns G/H into a group. ⊞

(1) (a) This statement follows immediately from the definitions.

(b) For $gH \in G/H$ we put $\psi(gH) \coloneqq \varphi(g)$. Since $H \subseteq \ker(\varphi)$ this definition is independent of the choice of the representatives g.[25] It is straightforward to verify that ψ is a homomorphism and that it is uniquely determined by the desired property. ■

The Quotient Group Lemma 2.4 leads us to the following definition:

Definition 2.5. Let G be a group and let $H \subseteq G$ be a normal subgroup. We equip G/H with the product from the Quotient Group Lemma 2.4 (0). We refer to G/H as the quotient of G by H.

2.2. **Conventions for (linear) algebra.** Next we want to introduce the notion of rings. Since the definition of a "ring" in the literature can differ considerably let us state precisely what we mean by a ring in these notes.

Definition 2.6.

(1) A ring is a triple $(R, +, \cdot)$ consisting of a set R and two maps $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$) such that $(R, +)$ is an abelian group and such that the following conditions are satisfied:
   (a) The multiplication "$\cdot$" is associative.
   (b) The multiplication "$\cdot$" is distributive.
   (c) There exists a multiplicatively neutral element $1 = 1_R$.

(2) A ring R is called commutative, if given any $r, s \in R$ we have $r \cdot s = s \cdot r$.

(3) A ring homomorphism between two rings $(R, +, \cdot)$ and $(S, +, \cdot)$ is a map $R \to S$ that preserves the additive and multiplicative structures and that sends 1_R to 1_S.

(4) A field is a commutative ring with $0 \neq 1$ such that every non-zero element has a multiplicative inverse.

(5) Given a ring R we denote by $R^\times$ the multiplicative group of multiplicative units in R.

Example. Let A be an abelian group. Given $n \in \mathbb{N}$ and $a \in A$ we define

$$n \cdot a \;\coloneqq\; \underbrace{a + \cdots + a}_{n \text{ times}} \qquad \text{and} \qquad -n \cdot a \;\coloneqq\; \underbrace{-a - \cdots - a}_{n \text{ times}}.$$

[24] The weird numbering stems from the fact that this lemma is in a long family of future lemmas where we introduce objects that are characterized by a "universal property".

[25] Indeed, if $gH = \widetilde{g}H$, then $g = \widetilde{g}h$ for some $h \in H$, therefore $\varphi(g) = \varphi(\widetilde{g}h) = \varphi(\widetilde{g}) \cdot \varphi(h) = \varphi(\widetilde{g})$.

Clearly this defines a $\mathbb{Z}$-module structure on A. We will thus view any abelian group also as a $\mathbb{Z}$-module. □

In the following notation we introduce some of the most important fields.

Notation 2.7. Given a prime p we denote by $\mathbb{F}_p$ the field where the underlying group is given by $\mathbb{Z}/p\mathbb{Z}$ and where the multiplication is the usual one coming from $\mathbb{Z}$.

We move on to the next piece of notation.

Notation 2.8. Let R be a ring and let $m, n \in \mathbb{N}_0$. We denote by $\mathrm{M}(m \times n, R)$ the set of all $(m \times n)$-matrices with entries in R. Note that entrywise addition turns $\mathrm{M}(m \times n, R)$ into an abelian group.

Definition 2.9. Let R be a commutative ring.

(1) Let M be an R-module. A basis for M is a subset $\mathcal{B} \subseteq M$ such that every element in M can be written uniquely as a linear combination of elements in $\mathcal{B}$. Note that a finite ordered basis $\mathcal{B} = (b_1, \ldots, b_m)$ admits a natural R-module isomorphism

$$\begin{array}{rcl} \Xi_{\mathcal{B}} \colon R^m & \to & M \\ (r_1, \ldots, r_m) & \mapsto & \sum\limits_{i=1}^{m} r_i \cdot b_i. \end{array}$$

(2) An R-module is called free if it admits a basis.

We state one key example as another definition:

Definition 2.10. Let R be a commutative ring and let $m \in \mathbb{N}_0$. Given $i \in \{1, \ldots, m\}$ we denote by

$$\mathfrak{e}_i \;:=\; (0, \ldots, 0, \underset{\substack{\uparrow \\ \text{the } i\text{-th entry}}}{1}, 0, \ldots, 0) \;\in\; R^m$$

as the i-th standard basis vector. Clearly $\mathfrak{e}_1, \ldots, \mathfrak{e}_m$ forms a basis of R^m. It is called the standard basis of R^m.

The previous definition is the standard definition well-known from earlier courses in linear algebra. Towards the end of the notes we will work with module homomorphisms over non-commutative rings. At that point we will see that at times it is convenient to switch to a different convention.

Definition 2.11. Let R be a commutative ring. Let V and W be two free R-modules. We assume that V and W are equipped with finite ordered bases $\mathcal{B} = (b_1, \ldots, b_m)$ and $\mathcal{C} = (c_1, \ldots, c_n)$. Let $\varphi \colon V \to W$ be a homomorphism of R-modules. We refer to the unique $(n \times m)$-matrix $[\varphi]_{\mathcal{B}}^{\mathcal{C}} = (p_{ij}) \in \mathrm{M}(n \times m, R)$ with

$$\varphi(b_i) \;=\; \sum_{j=1}^{n} p_{ji} \cdot c_j$$

as the matrix representing φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. Note that this is the unique matrix that makes the following diagram commute:

$$\begin{array}{ccc} R^m & \xrightarrow{\xi \mapsto [\varphi]^{\mathcal{C}}_{\mathcal{B}} \cdot \xi} & R^n \\ \cong \downarrow \Xi_{\mathcal{B}} & & \cong \downarrow \Xi_{\mathcal{C}} \\ V & \xrightarrow{\varphi} & W. \end{array}$$

For completeness sake we provide the following lemma from linear algebra:

Lemma 2.12. (Representing Matrix Lemma) Let R be a commutative ring.

(1) Let V be a free R-module of finite rank. If $\mathcal{B}$ is a basis for V, then $\mathrm{id}^{\mathcal{B}}_{\mathcal{B}} = \mathrm{id}$.

(2) Let $\varphi\colon U \to V$ and $\psi\colon V \to W$ be two homomorphisms between free R-left modules that are equipped with bases $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$. Then

$$[\psi \circ \varphi]^{\mathcal{D}}_{\mathcal{B}} \;=\; [\psi]^{\mathcal{D}}_{\mathcal{C}} \cdot [\varphi]^{\mathcal{C}}_{\mathcal{B}}.$$

Proof.

(1) This statement follows immediately from the definitions.

(2) This statement follows from the observation that the following diagram commutes:

$$\begin{array}{ccccc} & & \xi \mapsto [\psi\circ\varphi]^{\mathcal{D}}_{\mathcal{B}} \cdot \xi & & \\ & \overset{\frown}{} & & \overset{\frown}{} & \\ R^{1\times l} & \xrightarrow[\xi \mapsto [\varphi]^{\mathcal{C}}_{\mathcal{B}} \cdot \xi]{} & R^{1\times m} & \xrightarrow[\xi \mapsto [\psi]^{\mathcal{D}}_{\mathcal{C}} \cdot \xi]{} & R^{1\times n} \\ \cong \downarrow \Xi_{\mathcal{B}} & & \cong \downarrow \Xi_{\mathcal{C}} & & \cong \downarrow \Xi_{\mathcal{D}} \\ U & \xrightarrow{\varphi} & V & \xrightarrow{\psi} & W. \\ & \underset{\smile}{} & & \underset{\smile}{} & \\ & & \psi\circ\varphi & & \end{array}$$

■

We continue with the following definition:

Definition 2.13. Let R be a commutative domain. Given an R-module M we define

$$\mathrm{rank}(M) \;:=\; \dim_{K(R)}(M \otimes_R K(R))$$

($\otimes_R$: tensor product over R as defined in Section 312.2)

where $K(R)$ denotes the quotient field of R.

Next we fix our notation for matrices:

Notation 2.14. Let R be a ring.

(1) Let X, Y be sets. We refer to[26]

$$\mathrm{M}(X \times Y, R) := \left\{ f \in R^{X\times Y} \;\middle|\; \begin{array}{l} \text{for each } x \in X \text{ the set } \{y \in Y \mid f(x,y) \neq 0\} \text{ is finite} \\ \text{for each } y \in Y \text{ the set } \{x \in X \mid f(x,y) \neq 0\} \text{ is finite} \end{array} \right\}$$

as the set of $(X \times Y)$-matrices with entries in R. Given $f \in \mathrm{M}(X \times Y, R)$ and given $x \in X, y \in Y$ we refer to $f(x,y)$ as the (x,y)-entry of the matrix φ.

(2) Given $m, n \in \mathbb{N}_0$ we write, like everybody else

$$\mathrm{M}(m \times n, R) \;:=\; \mathrm{M}(\{1,\ldots,m\} \times \{1,\ldots,n\}, R).$$

Example. Let R be a ring

(1) Let X be a set. The map $(x,y) \mapsto \delta_{x,y} \cdot 1_R$ defines an element of $\mathrm{M}(X \times Y, R)$, which we denote by id_X. This is called the identity matrix.
(2) Let X, Y be sets. The map $(x,y) \mapsto 0$ defines an element of $\mathrm{M}(X \times Y, R)$. It is called the zero matrix.
(3) Note that if $X = Y = \varnothing$ is the empty set, then the discussion on page 98 shows that $\mathrm{M}(\varnothing \times \varnothing, R) = \mathrm{M}(0 \times 0, R)$ consists of a single matrix, which is the identity matrix and also the zero matrix. □

Lemma 2.15. (Matrix Multiplication Lemma) Let R be a ring and let X, Y, Z be sets. If Y is finite, then the map

$$\begin{array}{rcl} \mathrm{M}(X \times Y, R) \times \mathrm{M}(Y \times Z) & \to & \mathrm{M}(X \times Z) \\ (f\colon X \times Y \to R), (g\colon Y \times Z \to R) & \mapsto & \begin{pmatrix} X \times Z & \to & R \\ (x,z) & \mapsto & \sum\limits_{y \in Y} f(x,y) \cdot g(y,z) \end{pmatrix} \end{array}$$

is well-defined[27], distributive and associative.

Proof. The lemma follow easily from the definitions. ■

Definition 2.16. Let R be a ring.

(1) Let X be a set.
  (a) We say a matrix $A \in \mathrm{M}(X \times X, R)$ is invertible if there there exists a $B \in \mathrm{M}(X \times X, R)$ with $A \cdot B = B \cdot A = \mathrm{id}$.
  (b) We write
$$\mathrm{GL}(X,R) := \{A \in \mathrm{M}(X \times X, R) \mid A \text{ is invertible}\}.$$
  We write
$$\mathrm{GL}(X,R) := \left\{ A \in \mathrm{M}(X \times X, R) \,\middle|\, \begin{array}{l} \text{there exists } B \in \mathrm{M}(X \times X, R) \\ \text{with } A \cdot B = B \cdot A = \mathrm{id} \} \end{array} \right\}.$$
(2) Given $n \in \mathbb{N}_0$ we write $\mathrm{GL}(n,R) := \mathrm{GL}(\{1, \dots, n\}, R)$.

Example. Let R be a ring. It follows easily from the definitions and the discussion on page 98 that $\mathrm{GL}(0,R) = \{1\}$ is the trivial group. □

2.3. The determinant.

Next we move on to the all important notion of the determinant:

Definition 2.17. Let R be a commutative ring, let V and W be R-modules and let

$$\begin{array}{rcl} \Xi\colon V^k = \overbrace{V \times \cdots \times V}^{k\text{-times}} & \to & W \\ (v_1, \dots, v_k) & \mapsto & \Xi(v_1, \dots, v_k) \end{array}$$

[26] In some cases we will indeed study the case where X and Y are infinite, e.g. we might consider the case that $X = Y = \mathbb{N}$. In this setting our condition says that each "row" and "column" has only finitely many non-zero entries.

[27] Here by "well-defined" we mean that the sum has only finitely many non-zero summands and that we do indeed up with an element of $\mathrm{M}(X \times Z, R)$.

be a map.

(i) We say that Ξ is multilinear if Ξ is R-linear in every of its k arguments. More precisely, we require that for all $v, v' \in V$ and all $\lambda \in R$ we have[28]

$$\begin{aligned} \Xi(\dots, v + v', \dots) &= \Xi(\dots, v, \dots) + \Xi(\dots, v', \dots) \\ \Xi(\dots, \lambda \cdot v, \dots) &= \lambda \cdot \Xi(\dots, v, \dots). \end{aligned}$$

(ii) We say that Ξ is alternating if for all $v, v' \in V$ we have

$$\Xi(\dots, v, \dots, v', \dots) = -\Xi(\dots, v', \dots, v, \dots).$$

We recall the following characterization of determinants that is proved in every proper course on linear algebra:

Proposition 2.18. (Determinant–Characterization Proposition) Let R be a commutative ring and let $n \in \mathbb{N}_0$. There exists a *unique* map, called the determinant,

$$\det\colon \mathrm{M}(n \times n, R) = (R^n)^n \to R$$

with the following properties:

(i) The map is multilinear and alternating.

(ii) The map is normed, i.e. if $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ denotes the standard basis of R^n, then we have $\det(\mathfrak{e}_1 \dots \mathfrak{e}_n) = 1$. In other words, we have $\det(\mathrm{id}_n) = 1$. [29]

The determinant has furthermore the following two properties:

(iii) Given any $n \in \mathbb{N}$ and given any $A = (a_{ij})_{i,j=1,\dots,n} \in \mathrm{M}(n \times n, R)$ we have

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot a_{1,\sigma(1)} \cdot \ldots \cdot a_{n,\sigma(n)} \qquad \text{(Leibniz Formula)}$$

where $\operatorname{sign}(\sigma) \in \{\pm 1\}$ denotes the sign of the permutation. It follows in particular that the map $\det\colon \mathrm{M}(n \times n, R) \to R$ is smooth.

(iv) The determinant is multilinear and alternating in the rows.

Sketch of proof. It is straightforward to show that the Leibniz Formula gives a map $\det\colon \mathrm{M}(n \times n, R) = (R^n)^n \to R$ which has the desired properties (i) and (ii). It is also not hard to show that there exists at most one map that satisfies (i) and (iii). Finally Statement (iv) can be verified easily using the Leibniz Formula. We refer the reader to [**Lang1993**, Chapter XIII.4] or any other linear algebra book for details. ∎

Proposition 2.19. (Determinant–Multiplicative Proposition) Let R be a commutative ring and let $n \in \mathbb{N}_0$. Given any two $A, B \in \mathrm{M}(n \times n, R)$ we have

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

Proof. If R is a domain, then the lemma follows quite easily from the uniqueness statement of the Determinant–Characterization Proposition 2.18. The details will be filled in Exercise 2.5. The general case is dealt with in [**Lang1993**, Proposition XIII.4.15]. ∎

Note that it follows from the Determinant–Multiplicative Proposition 2.19 that the determinant of an invertible matrix is a unit in the ring R. We will now see that the converse to this statement holds. More precisely, if the determinant is a unit, then we can even write

[28] The non-specified arguments on the left and right are of course understood to be the same.

[29] In particular, for $n = 0$ this implies that the determinant of the 0×0-matrix equals 1_R.

down an explicit inverse of our given matrix. To formulate this result we need to introduce the following notion:

Definition 2.20. Let R be a ring and let $A \in \mathrm{M}(m \times m, R)$.

(1) Given $i, j \in \{1, \dots, m\}$ we denote by $A_{\hat{i},\hat{j}}$ the matrix that is obtained from A by removing the i-th row and the j-th column.

(2) If R is commutative, then we define the adjugate of A as the matrix

$$\mathrm{adj}(A) \;:=\; \text{matrix whose } (i,j)\text{-entry is given by } (-1)^{i+j} \cdot \det(A_{\hat{j},\hat{i}}).$$

Now we are fully equipped to formulate the promised result:

Lemma 2.21. (Adjugate Matrix Lemma) Let R be a commutative ring, let $n \in \mathbb{N}_0$ and let $A \in \mathrm{M}(n \times n, R)$. If $\det(A) \neq 0$, then an inverse to A is given by the matrix

$$\frac{1}{\det(A)} \cdot \mathrm{adj}(A).$$

Example. Let R be a commutative ring and let $a, b, c, d \in R$ with $ad - bc \neq 0$. Then the Adjugate Matrix Lemma 2.21 tells us that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \;=\; \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \qquad \square$$

Proof. This lemma is proved in most books on linear algebra, see e.g. [**HJa2013**, Chapter 0.8.2] or [**SerD2010**, Chapter 3.2] or [**Lang1993**, Proposition XIII.4.16]. ■

The following lemma gets used surprisingly frequently:

Lemma 2.22. (Determinant–Ring Homomorphism Lemma) Let $n \in \mathbb{N}_0$ and let $\varphi\colon R \to S$ be a homomorphism between two commutative rings. For every square matrix $A \in \mathrm{M}(n \times n, R)$ we have

$$\det(\varphi(A)) \;=\; \varphi(\det(A)).$$

Proof. The lemma follows easily from the Leibniz Formula in the Determinant–Characterization Proposition 2.18 (iii). ■

Finally we will frequently use the following notation:

Notation 2.23. Given a commutative ring R we write[30]

$$\begin{aligned} \mathrm{GL}(n,R) &:= \{A \in \mathrm{M}(n \times n, R) \mid \det(A) \text{ is a multiplicative unit in } R\} \\ \mathrm{SL}(n,R) &:= \{A \in \mathrm{M}(n \times n, R) \mid \det(A) = 1\}. \end{aligned}$$

For $\mathbb{K} = \mathbb{R}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{GL}_+(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid \det(A) > 0\} \\ \mathrm{GL}_-(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid \det(A) < 0\}. \end{aligned}$$

2.4. **Bilinear maps.** In this section we will provide a few definitions, which will appear in many different contexts throughout these notes:

[30]Note that it follows from the Determinant–Multiplicative Proposition 2.19 together with the Adjugate Matrix Lemma 2.21 that this definition of $\mathrm{GL}(n,R)$ is consistent with Definition 2.16.

Definition 2.24. Let A, B, C be abelian groups.

(1) A map $\varphi\colon A \times B \to C$ is called bilinear if for all $a, a' \in A$ and for all $b, b' \in B$ we have the equalities $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$ and $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$.
(2) A bilinear map $\varphi\colon A \times B \to C$ is called a pairing on $A \times B$.
(3) A pairing $\varphi\colon A \times A \to C$ is called symmetric if for all $a, b \in A$ we have the equality $\varphi(a, b) = \varphi(b, c)$.

Definition 2.25. Let R be a commutative ring and let V and W be R-modules. A pairing $\varphi\colon V \times W \to R$ is called R-bilinear if for all $\lambda \in R$ and all $v \in V$ and $w \in W$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \lambda \cdot \varphi(v, w)$.

Example. Let A, B and C be abelian groups and let $f\colon A \times B \to C$ be a bilinear map. In Exercise 2.9 we will show that for all $a \in A$, $b \in B$ and $m \in \mathbb{Z}$ we have the equality

$$f(m \cdot a, b) \;=\; f(a, m \cdot b) \;=\; m \cdot f(a, b).$$

We will use this basic fact frequently. This observation implies in particular that a bilinear map of abelian groups is also a $\mathbb{Z}$-bilinear pairing between the corresponding $\mathbb{Z}$-modules from page 116. □

As for homomorphisms it is popular to express pairings in terms of matrices. This leads us to the following definition:

Definition 2.26. Let R be a commutative ring and let V and W be two finitely generated free R-modules with bases $v_1, \ldots, v_k$ and $w_1, \ldots, w_l$. Furthermore let $\varphi\colon V \times W \to R$ be an R-bilinear pairing. We refer to the $(k \times l)$–matrix

$$(\varphi(v_i, v_j))_{i=1,\ldots,k, j=1,\ldots,l} \;\in\; \mathrm{M}(k \times l, R)$$

as a matrix representing the R-bilinear pairing φ.

2.5. **The Cauchy-Schwarz Inequality.** We now specialize to the case that the ring R are the real numbers:

Definition 2.27. Let V be a real vector space.

(3) (a) We say that a pairing φ is positive-definite if $\varphi(v, v) > 0$ for all $v \in V \setminus \{0\}$.
(b) We say that a pairing φ is positive-semidefinite if $\varphi(v, v) \geq 0$ for all $v \in V \setminus \{0\}$.
(4) An inner product on V is a positive-definite, symmetric $\mathbb{R}$-bilinear pairing.

Example. Let $n \in \mathbb{N}_0$. The standard inner product on $\mathbb{R}^n$ is given by

$$\begin{array}{rcl} \langle -, - \rangle_{\mathrm{Eucl}} \;=\; \langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ (v, w) & \mapsto & \langle v, w \rangle_{\mathbb{R}^n} := \sum_{i=1}^{n} v_i \cdot w_i. \end{array}$$

Here we use the standard convention that for $v \in \mathbb{R}^n$ we denote by $v_1, \ldots, v_n$ its coordinates. This standard pairing on $\mathbb{R}^n$ is evidently $\mathbb{R}$-bilinear, symmetric and positive-definite, i.e. it is indeed an inner product. Sometimes we also write $v \cdot w := \langle v, w \rangle_{\mathbb{R}^n}$. □

Proposition 2.28. (Cauchy-Schwarz Inequality) Let V be a real vector space and let

$$\begin{array}{rcl} \varphi\colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \varphi(v, w) \end{array}$$

be an $\mathbb{R}$-bilinear form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have
$$\varphi(v,w)^2 \;\leq\; \varphi(v,v)\cdot\varphi(w,w).$$
(2) Now suppose that φ is positive-definite. In this equality in (1) holds if and only if v and w are linearly dependent.

Example. Let $v_1,\dots,v_n \in \mathbb{R}^n$ and let $w_1,\dots,w_n \in \mathbb{R}$. It follows immediately from the Cauchy-Schwarz Inequality 2.28, applied to the standard inner product on $\mathbb{R}^n$, that
$$(v_1\cdot w_1+\cdots+v_n\cdot w_n)^2 \;\leq\; (v_1^2+\cdots+v_n^2)\cdot(w_1^2+\cdots+w_n^2). \qquad \square$$

Proof. First we just assume that φ is positive semidefinite. Let $v, w \in V$. Given $t \in \mathbb{R}$ we consider
$$f(t) \;:=\; \varphi(v+t\cdot w, v+t\cdot w) \underset{\uparrow}{=} \varphi(v,v)+2t\cdot\varphi(v,w)+t^2\cdot\varphi(w,w).$$
since φ is $\mathbb{R}$-bilinear and symmetric

Since φ is positive semidefinite we see that f assumes only non-negative values. Since f is a polynomial function we obtain from elementary analysis that f assumes a minimum, which in turn implies that there exists a $t_0 \in \mathbb{R}$ with $f'(t_0)=0$. It follows from the second description of $f(t)$ that
$$f'(t) \;=\; 2\cdot\varphi(v,w)+2t\cdot\varphi(w,w).$$
This implies that $t_0\cdot\varphi(w,w) = -\varphi(v,w)$. Now note that
$$\begin{aligned}\overbrace{\varphi(v+t_0\cdot w, v+t_0\cdot w)}^{\geq 0}\cdot\overbrace{\varphi(w,w)}^{\geq 0} &= \varphi(w,w)\cdot\varphi(v,v)+2\overbrace{t_0\cdot\varphi(w,w)}^{=-\varphi(v,w)}\cdot\varphi(v,w)+\overbrace{t_0^2\cdot\varphi(w,w)\cdot\varphi(w,w)}^{=\varphi(v,w)^2}\\ &\underset{\uparrow}{=} \varphi(w,w)\cdot\varphi(v,v)-\varphi(v,w)\cdot\varphi(v,w).\end{aligned}$$
follows from $t_0\cdot\varphi(w,w) = -\varphi(v,w)$

Since φ is positive semidefinite we see that $\varphi(v+t_0\cdot w, v+t_0\cdot w)\cdot\varphi(w,w) \geq 0$. Thus follows from the above little calculation above that $\varphi(v,w)^2 \leq \varphi(v,v)\cdot\varphi(w,w)$.

Now suppose that $\varphi(v,w)^2 = \varphi(v,v)\cdot\varphi(w,w)$. This implies, by the above calculation, that $\varphi(v+t_0\cdot w, v+t_0\cdot w)\cdot\varphi(w,w) = 0$. Since φ is positive-definite this means that $v+t_0\cdot w = 0$ or $w = 0$. In either case, v, w are linearly dependent. ■

Finally on some occasions we will need a variation of the Cauchy-Schwarz Inequality 2.28 where we work with pairings over $\mathbb{C}$. This leads us to the following definitions.

Definition 2.29. Let V be a complex vector space.

(1) A pairing $\varphi\colon V\times V\to\mathbb{C}$ is called $\mathbb{C}$-sesquilinear if for all $\lambda\in\mathbb{C}$ and all $v,w\in V$ we have $\varphi(\lambda\cdot v,w) = \lambda\cdot\varphi(v,w)$ and $\varphi(v,\lambda\cdot w) = \varphi(v,w)\cdot\overline{\lambda}$.
(2) A pairing $\varphi\colon V\times V\to\mathbb{C}$ is called Hermitian if $\varphi(v,w) = \overline{\varphi(w,v)}$ for all $v,w\in V$.
(3) (a) We say that a pairing φ is positive-definite if $\varphi(v,v)\in\mathbb{R}_{>0}$ for all $v\in V\setminus\{0\}$.
(b) We say that a pairing φ is positive-semidefinite if $\varphi(v,v)\in\mathbb{R}_{\geq 0}$ for all $v\in V\setminus\{0\}$.
(5) An inner product on V is a positive-definite, Hermitian $\mathbb{C}$-sesquilinear pairing.

Example. Let $n\in\mathbb{N}_0$. The standard inner product on $\mathbb{C}^n$ is given by
$$\begin{aligned}\langle -,-\rangle\colon \mathbb{C}^n\times\mathbb{C}^n &\to \mathbb{R}\\ (v,w) &\mapsto \langle v,w\rangle_{\mathbb{C}^n} := \sum_{i=1}^n v_i\cdot\overline{w}_i.\end{aligned}$$

This pairing on $\mathbb{C}^n$ is evidently $\mathbb{C}$-sesquilinear, Hermitian and positive-definite, in other words, it is indeed an inner product. □

Notation 2.30. Given $z = x + y\mathrm{i} \in \mathbb{C}$ with $x, y \in \mathbb{R}$ we write $|z| := \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$.

Now we can formulate the following "complex analog" of the Cauchy-Schwarz Inequality 2.28:

Proposition 2.31. (Cauchy-Schwarz Inequality) Let V be a complex vector space and let

$$\begin{array}{rccc} \varphi\colon & V \times V & \to & \mathbb{R}_{\geq 0} \\ & (v, w) & \mapsto & \varphi(v, w) \end{array}$$

be a Hermitian form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have

$$|\varphi(v, w)|^2 \ \leq \ \varphi(v, v) \cdot \varphi(w, w).$$

(2) Now suppose that φ is positive-definite. In this case equality in (1) holds if and only if v and w are linearly dependent.

Proof. The proof is very similar to the proof of the above Cauchy-Schwarz Inequality 2.28. We leave it to the reader to make the necessary minor modifications. ■

2.6. **The Gram-Schmidt Orthogonalization Process.** The following proposition (which is hopefully familiar from an earlier linear algebra course) gives us one of the main tools from linear algebra, which we will use frequently, and sometimes in surprising settings, throughout these notes:

Proposition 2.32. (Gram-Schmidt Orthogonalization Process) Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$. Let V be a $\mathbb{K}$-vector space and let $\langle\ ,\ \rangle\colon V \times V \to \mathbb{K}$ be an inner product, i.e. a positive-definite symmetric $\mathbb{K}$-sesquilinear form. Given $v \in V$ we write $\|v\| := \sqrt{\langle v, v\rangle}$. Let $v_1, \ldots, v_n$ be a basis of V. We consider the following sequence of n-tuples of vectors in V:[31]

$$\begin{array}{ll}
u_{0,1}, .., u_{0,n} := v_1, .., v_n & \\
u_{1,1}, .., u_{1,n} := \frac{v_1}{\|v_1\|}, \|v_1\| \cdot v_2, v_3, .., v_n & \\
\quad\vdots \qquad\qquad \vdots & \\
u_{i,1}, .., u_{i,n} := u_{i-1,1}, .., u_{i-1,i-1}, \frac{z}{\|z\|}, \|z\| \cdot v_{i+1}, v_{i+2}, .., v_n & \text{with } z := v_i - \sum_{j=1}^{i-1} \langle v_i, u_{i-1,j}\rangle \cdot u_{i-1,j} \\
\quad\vdots \qquad\qquad \vdots & \\
u_{n,1}, .., u_{n,n} := u_{n-1,1}, .., u_{n-1,n-1}, \frac{z}{\|z\|} & \text{with } z := v_n - \sum_{j=1}^{n-1} \langle v_n, u_{n-1,j}\rangle \cdot u_{n-1,j}.
\end{array}$$

This sequence has the following properties:

(1) For each $i \in \{1, \ldots, n\}$ the vectors $\{u_{i,1}, \ldots, u_{i,n}\}$ form a basis of V.

(2) For each $i \in \{1, \ldots, n-1\}$ the base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{i,1}, \ldots, u_{i,n}$ has determinant 1.

(3) The base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{n,1}, \ldots, u_{n,n}$ has positive determinant.

(4) For each $i \in \{1, \ldots, n\}$ the vectors $u_{i,1}, \ldots, u_{i,i}$ are orthonormal.

(5) Let $m \in \{0, \dots, n\}$.
  (a) We have $\operatorname{span}_{\mathbb{K}}\{v_1, \dots, v_m\} = \operatorname{span}_{\mathbb{K}}\{u_1, \dots, u_m\}$.
  (b) If $v_1, \dots, v_m$ are orthonormal, then the first m vectors are unchanged, i.e. for any $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, m\}$ we have $u_{i,j} = v_j$.
(6) If $\varphi\colon V \to V$ is a linear isometry, then the above process commutes with application of φ.

The analogous statement also holds if we are dealing with a positive-definite Hermitian $\mathbb{C}$-sesquilinear form over a complex vector space.

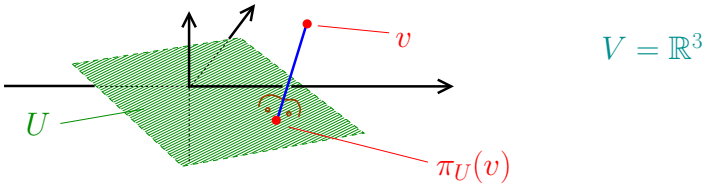
Proof. The lemma follows from elementary arguments which we leave to the readers who suffer from linear algebra withdrawal syndrome. ■

Remark. In the Matrix Groups Deformation Retract Proposition 19.4 we will study further properties of the Gram-Schmidt Orthogonalization Process 2.32. □

As an application to the Gram-Schmidt Orthogonalization Process 2.32 we will prove the following lemma:

Lemma 2.33. (Orthogonal Projection Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let V be a $\mathbb{K}$-vector space and let $\langle\ ,\ \rangle\colon V \times V \to \mathbb{K}$ be an inner product. Furthermore let $U \subseteq V$ be a vector subspace.

(1) Given any vector $v \in V$ there exists a unique vector $\pi_U(v) \in V$ with the following properties:
  (a) $\pi_U(v) \in U$.
  (b) $v - \pi_U(v) \in U^{\perp}$.
  We refer to $\pi_U(v)$ as the orthogonal projection of v onto U.
(2) If $u_1, \dots, u_n$ is an orthonormal basis of U, then
$$\pi_U(v) \ = \ \sum_{i=1}^{n} \langle u_i, v\rangle \cdot v.$$
(3) (a) The map $\pi_V\colon V \to U$ is a homomorphism.
  (b) Let $W \subseteq V$ be a vector subspace. If $W \cap U^{\perp} = \{0\}$, then $\pi_U\colon W \to U$ is a monomorphism.



Proof. First note that it follows from the Gram-Schmidt Orthogonalization Process 2.32 that we can pick an orthonormal basis $u_1, \dots, u_n$ of U. Now let $v \in V$. We consider
$$\pi_U(v) \ := \ \sum_{i=1}^{n} \langle u_i, v\rangle \cdot u_i.$$

[31] Our formulation is more complicated than the standard formulation of the Gram-Schmidt Orthogonalization Process. The reason is that we want to have control over the base change matrices. This is important when we show, in the proof of the Matrix Group-π_0-Proposition 4.54 (2), that $\mathrm{SL}(n, \mathbb{R})$ is path-connected.

Clearly $\pi_U(V) \subseteq U$. Using the fact the u_i form an orthonormal basis one can easily verify that $\langle u_i, v - \pi_U(v)\rangle = 0$ for $i = 1, \dots, n$. This shows that $v - \pi_U(v) \in U^\perp$. This gives us the existence in (1). The uniqueness statement of (1) is clear, since the only vector in $U \cap U^\perp$ is the 0-vector. In summary, we have shown Statements (1) and (2).

It follows either directly from Statement (1), or the explicit formula in Statement (2), that $\pi_U \colon V \to U$ is a homomorphism.

Finally note that Statement (3b) follows from the observation that if for some $v \in V$ we have $\pi_U(v) = 0 \in U$, then we know by Statement (1b) that $\pi_U(v) \in U^\perp$. ■

2.7. **The Spectral Theorem.** We start out by recalling the following standard notation:

Notation 2.34. Given $n \in \mathbb{N}_0$ we write

$$\begin{aligned} \mathrm{O}(n) &:= \{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid A^{\mathrm{T}} \cdot A = \mathrm{id}\} \\ \mathrm{SO}(n) &:= \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid A^{\mathrm{T}} \cdot A = \mathrm{id} \text{ and } \det(A) = 1\}, \\ \mathrm{U}(n) &:= \{A \in \mathrm{M}(n \times n, \mathbb{C}) \mid \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id}\} \\ \mathrm{SU}(n) &:= \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{M}(n \times n, \mathbb{C}) \mid \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

We can now formulate the following theorem:

Theorem 2.35. (Spectral Theorem)

(1) If $B \in \mathrm{M}(n \times n, \mathbb{R})$ is a symmetric matrix, then there exists an orthogonal matrix $P \in \mathrm{O}(n)$ such that $P^{\mathrm{T}} \cdot B \cdot P$ is a diagonal matrix where all diagonal entries are real numbers. If B is positive-definite, then the diagonal entries are positive.

(2) (a) If $B \in \mathrm{M}(n \times n, \mathbb{C})$ is a Hermitian matrix, then there exists a unitary matrix $P \in \mathrm{U}(n)$ such that $\overline{P}^{\mathrm{T}} \cdot B \cdot P$ is a diagonal matrix where all diagonal entries are real numbers. If B is positive-definite, then the diagonal entries are positive.

(b) If $B \in \mathrm{U}(n)$ is a unitary matrix, then there exists a unitary matrix $P \in \mathrm{U}(n)$ such that $\overline{P}^{\mathrm{T}} \cdot B \cdot P$ is a diagonal matrix where all diagonal entries are complex numbers of length 1.

Proof. This theorem is proved in any reasonable book on linear algebra, see e.g. [**Ax2024**, Theorem 7.31], and also in any reasonable set of lecture notes on linear algebra. ■

2.8. **Group actions.** The following definition might also have appeared in an earlier math course.

Definition 2.36. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

with the following properties:

$$\begin{aligned} e \cdot x &= x, && \text{for all } x \in X, \\ g \cdot (h \cdot x) &= (g \cdot_G h) \cdot x, && \text{for all } x \in X \text{ and } g, h \in G. \end{aligned}$$

($g \cdot (h \cdot x)$: the two $\cdot$ are group actions; $g \cdot_G h$: multiplication in the group G)

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

(3) We say that G acts transitively, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.

Remark. Sometimes what we call an "action" is more precisely referred to as a "left action". A "right action" is exactly the same as a left action except that the second property gets replaced by $g \cdot (h \cdot x) = (h \cdot_G g) \cdot x$. Or, more intuitively, a right-action is a map $X \times G \to X$ such that we have $(x \cdot h) \cdot g = x \cdot (h \cdot_G g)$. Basically all the statements and definitions that we give for left actions carry over, with minuscule modifications, to the setting of right actions. □

Examples.

(1) Given any group G the map
$$\begin{array}{rcl} G \times G & \to & G \\ (g,h) & \mapsto & ghg^{-1} \end{array}$$
defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

(2) Let $\mathbb{F}$ be a field and let V be a k-dimensional vector space over $\mathbb{F}$. We denote by B the set of all ordered bases of V. The map
$$\begin{array}{rcl} \mathrm{GL}(k,\mathbb{F}) \times B & \to & B \\ \big((a_{ij})_{i,j=1,\dots,k}, (v_1,\dots,v_k)\big) & \mapsto & \Big(\sum_{i=1}^{k} a_{i1} \cdot v_i, \dots, \sum_{i=1}^{k} a_{ik} \cdot v_i\Big) \end{array}$$
is a transitive action. It is an amusing linear algebra exercise to figure out under what circumstances this action is free.

(3) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free but it is not transitive. □

We formulate our final example as a lemma.

Lemma 2.37. (Orthogonal Action Lemma) Let $n \in \mathbb{N}_0$.

(1) The maps
$$\begin{array}{rcl} \mathrm{O}(n+1) \times S^n & \to & S^n \\ (A,v) & \mapsto & A \cdot v \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{O}(n+1) \times \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ (A,v) & \mapsto & A \cdot v \end{array}$$
define actions of $\mathrm{O}(n+1)$ on S^n and on $\overline{B}^{n+1}$.

(2) The action of $\mathrm{SO}(n+1)$ on S^n is free if and only if $n = 1$.

(3) The action of $\mathrm{O}(n+1)$ on S^n is transitive.

(4) If $n \geq 1$, then the action of $\mathrm{SO}(n+1)$ on S^n is also transitive.

(5) Let $k \in \{0,\dots,n\}$. The obvious action of $\mathrm{SO}(n)$ on the set of k-dimensional vector subspaces of $\mathbb{R}^n$ is transitive.

Proof. Statement (1) is clear. Statement (2) can be verified easily. We turn to the proof of Statements (3) and (4). The statements fall in the realm of linear algebra, but for the reader's convenience we provide the argument. First we prove Statement (3). It suffices to show that given any $v \in S^n$ there exists an $A \in \mathrm{O}(n+1)$ with $A \cdot \mathfrak{e}_1 = v$. To produce such a matrix we find vectors $v_2,\dots,v_{n+1}$ such that $v, v_2, \dots, v_{n+1}$ form a basis for $\mathbb{R}^{n+1}$. We apply the Gram-Schmidt Orthonormalization Process 2.32 to the ordered basis $(v, v_2, \dots, v_{n+1})$. Since $\|v\| = 1$ this process leaves the first vector invariant and we obtain an orthonormal

basis $(v, \widetilde{v}_2, \dots, \widetilde{v}_{n+1})$. We denote by A the orthogonal matrix with columns given by $v, \widetilde{v}_2, \dots, \widetilde{v}_{n+1}$. Clearly $A \cdot \mathfrak{e}_1 = v$.

We turn to the proof of Statement (4). We argue as in Statement (3). The only difference is, that if $\det(A) = -1$ and if $n \geq 1$, then we replace $\widetilde{v}_2$ by $-\widetilde{v}_2$, and now we are happy.

We move on to the proof of Statement (5). Now let $k \in \{0, \dots, n\}$ and let $V, W \subseteq \mathbb{R}^n$ be two k-dimensional vector subspaces of $\mathbb{R}^n$. The case $k = n$ is trivial. So we can assume that $k \in \{0, \dots, n-1\}$. By the Gram-Schmidt Orthogonalization Process 2.32 we can pick orthogonal bases $v_1, \dots, v_k$ and $w_1, \dots, w_k$. It suffices to show that for $i = 1, \dots, k$ there exists an $A_i \in \mathrm{SO}(n)$ such that $A \cdot v_j = \pm w_j$ for $j = 1, \dots, i$. These matrices $A_1, \dots, A_k$ can be found iteratively by applying (4) to the vectors v_j and w_j in the orthogonal complement of $v_1, \dots, v_{j-1}$, which has dimension $n - (j-1) \geq n - (k-1) \geq n - ((n-1)-1) \geq 2$. ■

We conclude this section on group actions with the following elementary lemma:

Lemma 2.38. (Group Action-Equivalence Relation Lemma) Let X be a set and let G be a group that acts on X. Then

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X.

Proof. The statement follows easily from the definitions. ■

The Group Action-Equivalence Relation Lemma 2.38 leads us to the following definition:

Definition 2.39. Let X be a set and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 2.38 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We denote by[32]

$$X/G \;:=\; X/\sim_G .$$

the set of equivalence classes.

We conclude this discussion of actions with a convention:

Convention 2.40. Let G be a group that acts on a set A. If the set A is equipped with an extra structure, e.g. if A is an abelian group, then we demand that the group action preserves this structure, e.g. we demand that the map

$$\begin{array}{ccc} A & \to & A \\ a & \mapsto & g \cdot a \end{array}$$

is a homomorphism. We will apply this convention to many settings, e.g. A could be a vector space or (later) a topological space. In these cases we demand that A acts via vector space homomorphisms or continuous maps.

2.9. **Normed vector spaces.** We continue with another definition from linear algebra.

[32]It is arguably more logical to write $G\backslash X$ since we are dealing with a left action and to reserve the notation X/G for right-actions. We stick to X/G since it is closer to our language "X mod G" and since we will not deal with right actions at all.

Definition 2.41. Let V be a real vector space. A norm on V is a map $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ that has the following three properties:

(1) For every $v \in V$ we have $\|v\| = 0 \iff v = 0$.
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$.
(3) For every $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

Lemma 2.42. (Pairing–Norm Lemma) Let V be a real vector space. Suppose that $\langle -, - \rangle\colon V \times V \to \mathbb{R}_{\geq 0}$ is a positive-definite symmetric$\mathbb{R}$-linear pairing, then the map

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \coloneqq \sqrt{\langle v, v \rangle} \end{array}$$

is a norm.

Proof. It is straightforward to verify that the first two properties of a norm are satisfied. It remains to prove the third property. Given $v, w \in \mathbb{R}^n$ we see that

$$\begin{array}{rcl} \|v+w\|^2 & = & \langle v+w, v+w \rangle \overset{\text{follows from the } \mathbb{R}\text{-bilinearity of } \langle\ ,\ \rangle}{=} \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \overset{\text{since } \langle w, v \rangle = \langle v, w \rangle}{=} \langle v, v \rangle + 2 \cdot \langle v, w \rangle + \langle w, w \rangle \\ & \leq & \langle v, v \rangle + 2 \cdot \sqrt{\langle v, v \rangle \cdot \langle w, w \rangle} + \langle w, w \rangle = \left(\sqrt{\langle v, v \rangle} + \sqrt{\langle w, w \rangle} \right)^2 = (\|v\| + \|w\|)^2. \end{array}$$

$\uparrow$ by the Cauchy-Schwarz Inequality 2.28 we have $\langle v, w \rangle^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle$

By taking square roots on both sides we obtain that $\|v + w\| \leq \|v| + \|w\|$. ■

Next we define one of the most important norms:

Definition 2.43. The standard inner product $\langle -, - \rangle_{\text{Eucl}}$ on $\mathbb{R}^n$ from page 122 gives rise to the Euclidean norm

$$\begin{array}{rcl} \|-\|_{\text{Eucl}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ v = (v_1, \ldots, v_n) & \mapsto & \sqrt{\langle v, v \rangle} = \sqrt{v_1^2 + \cdots + v_n^2}. \end{array}$$

Usually, when there is no source of confusion, we just write $\|v\|$ instead of $\|v\|_{\text{Eucl}}$.

Example. We will now give more examples of norms $\mathbb{R}^n$. More precisely, in Exercise 2.6 we will see that the following maps are norms and we will see that for $n \geq 2$ they do *not* arise from a positive-definite $\mathbb{R}$-bilinear pairing:

$$\underbrace{\begin{array}{rcl} \|-\|_{\max}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \max\{|x_1|, \ldots, |x_n|\} \end{array}}_{\text{maximum norm}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \|-\|_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \sum_{i=1}^{n} |x_i|. \end{array}}_{\text{Manhattan norm}}$$

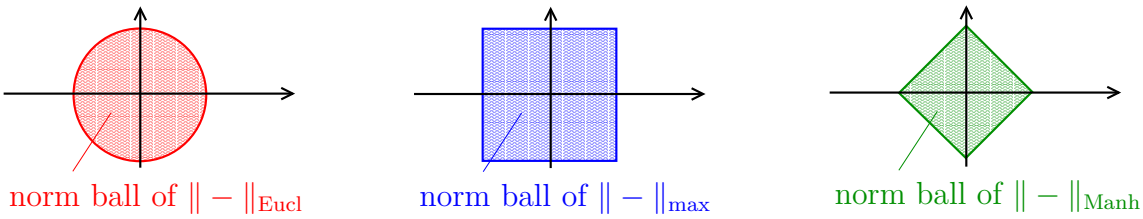
□

Since the human brain is equipped with a great image processor it is usually helpful to turn an abstract definition into an image. In the present context the key to doing so is given by the following definition:

Definition 2.44. Given a norm $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ we define

$$\text{norm ball of } \|-\| \ \coloneqq\ \{v \in V \mid \|v\| \leq 1\}.$$

Example. In the figure below we show the norm balls of the Euclidean norm, the maximum norm and the Manhattan norm.



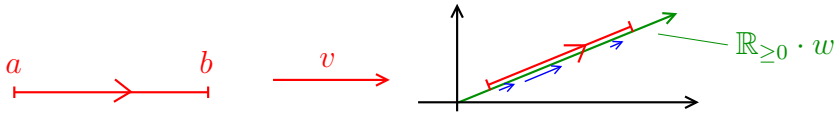
Note that the norm ball is each of these examples, and in fact also in general, convex. □

We conclude our sections on norms with the following lemma which we will use on several occasions.

Lemma 2.45. (Integral Inequality Lemma) For every continuous map $v\colon [a,b] \to \mathbb{R}^n$ the following inequality of real numbers holds:

$$\left\| \int_a^b v(t)\,dt \right\|_{\text{Eucl}} \leq \int_a^b \|v(t)\|_{\text{Eucl}}\,dt.$$

Furthermore equality holds if and only if there exists a $w \in \mathbb{R}^n$ and a continuous map $\lambda\colon [a,b] \to \mathbb{R}_{\geq 0}$ such that for every $t \in [a,b]$ we have $v(t) = \lambda(t) \cdot w$.



Proof. We first show that the inequality holds. We set

$$u \coloneqq \int_a^b v(t)\,dt \quad \in \mathbb{R}^n.$$

If $\|u\| = 0$, then there is nothing to show. Thus we might as well assume that $\|u\| > 0$. We obtain the following (in-) equalities:

$$\begin{aligned}
\|u\| \cdot \left\| \int_a^b v(t)\,dt \right\| = \|u\|^2 = \langle u,u\rangle &= \left\langle u, \int_a^b v(t)\,dt \right\rangle \\
&= \left\langle (u_1,\dots,u_n), \left(\int_a^b v_1(t)\,dt, \dots, \int_a^b v_n(t)\,dt \right) \right\rangle \\
&= \sum_{i=1}^n u_i \cdot \int_a^b v_i(t)\,dt = \int_a^b \sum_{i=1}^n u_i \cdot v_i(t)\,dt \\
&= \int_a^b \langle u, v(t)\rangle\,dt \leq \int_a^b \|u\| \cdot \|v(t)\|\,dt = \|u\| \cdot \int_a^b \|v(t)\|\,dt.
\end{aligned}$$

by the Cauchy-Schwarz Inequality 2.28 we know that for any $x, y \in \mathbb{R}^n$ we have $\langle x,y\rangle \leq \|x\| \cdot \|y\|$

The desired inequality follows from canceling $\|u\|$.

Now we turn to the "furthermore statement". In fact this statement follows from the above considerations together with the "furthermore statement" of the Cauchy-Schwarz Proposition 2.28. We leave it to the reader to fill in the details. ■

2.10. **Metric spaces.** Finally we recall the notion of a metric space which is most likely familiar from an earlier course on real analysis.

Definition 2.46. A metric space is a pair (X, d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \coloneqq \{x \in \mathbb{R} \,|\, x \geq 0\} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

with the following properties:

(1) $d(x,y) = 0 \iff x = y$

(2) for all $x, y \in X$ we have $d(x,y) = d(y,x)$ (symmetry)

(3) for all $x, y, z \in X$ we have $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

Sometimes we refer to $d(x, y)$ as the distance between x and y.

The following lemma shows that any norm on a vector space V give rise to a metric on V.

Lemma 2.47. (Norm–Metric Lemma) Let V be a real vector space. Furthermore let $\| - \| \colon V \to \mathbb{R}_{\geq 0}$ be a norm. Then

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & d(v, w) \coloneqq \|v - w\| \end{array}$$

is a metric on V.

Proof. The three properties of a metric follow easily from the three properties of a norm. For example, given $v, w \in V$ we see that

$$d(v,w) = \|v - w\| = \|(-1) \cdot (w - v)\| \underset{\uparrow}{=} |-1| \cdot \|w - v\| = \|w - v\| = d(w,v).$$

property (2) of a norm

The other two properties of a metric are verified in a similar fashion. ∎

It follows from the Norm–Metric Lemma 2.47 that the Euclidean norm $\| - \|_{\text{Eucl}}$ gives rise to a metric, called the Euclidean metric, which is perhaps the most frequently used example of a metric:

Definition 2.48. Given $x, y \in \mathbb{R}^n$ we set

$$d(x,y) \coloneqq \|x - y\|_{\text{Eucl}} \quad \text{where} \quad \|v\|_{\text{Eucl}} = \|(v_1, \ldots, v_n)\|_{\text{Eucl}} \coloneqq \sqrt{v_1^2 + \cdots + v_n^2}\,.$$

We refer to the map

$$\begin{array}{rcl} d \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

as the Euclidean metric on $\mathbb{R}^n$.[33] Usually we consider $\mathbb{R}^n$ as a metric space with respect to this Euclidean metric, unless we say something else.

Of course there are many other examples of metrics on $\mathbb{R}^n$. In the following we will see two examples, the first one corresponds to the Manhattan norm and the other one is an example of a metric that does not come from a norm.

[33] It follows from the Pairing–Norm Lemma 2.42 together with the Norm–Metric Lemma 2.47 that this is indeed a metric on $\mathbb{R}^n$.

Example. Let $n \in \mathbb{R}^n$.

(1) By the Norm–Metric Lemma 2.47 the Manhattan norm from page 129 gives rise to the metric

$$\begin{array}{rccl} d_{\text{Manh}} \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array}$$

The metric d_{Manh} is called the Manhattan metric since it measures the distance between two points if one can go only vertically (avenues) and horizontally (streets).

(2) We fix a point $P \in \mathbb{R}^n$. We consider the following map:

$$\begin{array}{rccl} d_{\text{SNCF}} \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} & \\ (x, y) & \mapsto & \begin{cases} \|x - y\|, & \text{if } x, y, P \text{ lie on a line,} \\ \|x - P\| + \|y - P\|, & \text{otherwise.} \end{cases} \end{array}$$

In Exercise 2.10 we will see that this map is a metric.

Manhattan metric



SNCF-metric



The metric d_{SNCF} is called the SNCF metric since it models the distance in France where every train between two points necessarily has to go through Paris. In Exercise 2.10 we will see that the SNCF-metric does not arise from a norm. □

Not surprisingly there are many other examples of metric spaces. The last definition shows that we can define a metric on any set.

Definition 2.49. Let X be a set. The map

$$\begin{array}{rccl} X \times X & \to & \mathbb{R}_{\geq 0} & \\ (x, y) & \mapsto & \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases} \end{array}$$

defines a metric on X, it is called the trivial metric.

2.11. **Concepts in elementary geometry.** In this final section we will just provide two basic definitions, which hopefully will help us with staying consistent later on:

Definition 2.50. Let $n \in \mathbb{N}_0$ and let $P, Q \in \mathbb{R}$.

(1) The subset $\overline{PQ} := \{(1-t) \cdot P + t \cdot Q \,|\, t \in [0, 1]\}$ is called the Euclidean segment from P to Q.

(2) A Euclidean segment as above is called trivial, if it consists of a single point. Otherwise it is called non-trivial.

Sometimes a Euclidean segment is also just called a segment or a line segment.

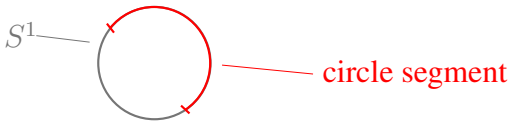
Euclidean segment



Definition 2.51. We consider the unit circle $S^1 = \{x + \mathrm{i}y \in \mathbb{C} \,|\, x^2 + y^2 = 1\}$. Let $z \in S^1$ and let $t \in [0, 2\pi)$.

(1) The subset $\{z \cdot \exp(2\pi \mathrm{i} s) \,|\, s \in [0, t]\}$ is called a circle segment.
(2) A circle segment as above is called trivial, if it consists of a single point. Otherwise it is called non-trivial.

We apply the same language to circles of other radii and center.



2.12. Exercises for Chapter 2.

Exercise 2.1. Let X be a set and let $\mathcal{P}(X)$ be the power set. Given $A, B \in \mathcal{P}(X)$ we define the symmetric difference

$$A \triangle B \;:=\; (A \setminus B) \cup (B \setminus A).$$

What axioms of a ring does $(\mathcal{P}(X), \triangle, \cap)$ satisfy?

Exercise 2.2. Let $(M, +)$ be an abelian monoid with neutral element 0 and let $S \subseteq M$ be a submonoid. Given $m, m' \in M$ we write

$$m \sim_S m' \quad :\Longleftrightarrow \quad \text{there exist } s, s' \in S \text{ with } m + s = m' + s'.$$

Show that the following statements hold:

(1) The above relation $\sim_S$ is an equivalence relation on M.
(2) The set $M/S := M/\sim_S$ of equivalence classes inherits the structure of an abelian monoid. It is called the *quotient monoid*.
(3) If for every $m \in M$ there exists an $m' \in M$ such that $m + m' \in S$, then M/S is an abelian group.

Exercise 2.3. Let R be a ring and let $x \in R$. Show that if we are given $a, b \in R$ with $a \cdot x = b \cdot x = 1$, then $a = b$.
Remark. The exercise says that in a ring left inverses are, if they exist, unique. Of course the same statements also hold with "left' and "right" reversed.

Exercise 2.4. Let R be a ring and let $x \in R$. Suppose that we are given $a \in R$ with $a \cdot x = 1$. Does it follow that $x \cdot a = 1$?

Exercise 2.5. Let R be a commutative domain and let $n \in \mathbb{N}_0$.

(a) Let $C \in \mathrm{M}(n \times n, R)$. Show that $\det(C) = 0$ if and only if there exists a $v \in R^n$ with $C \cdot v = 0$.
(b) Use (a) and the uniqueness statement of the Determinant–Characterization Proposition 2.18 to show given any two $A, B \in \mathrm{M}(n \times n, R)$ we have the equality

$$\det(A \cdot B) \;=\; \det(A) \cdot \det(B).$$

Exercise 2.6. We consider the following two maps:

$$\underbrace{\begin{array}{rcl} \|-\|_{\max}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\dots,x_n) & \mapsto & \max\{|x_1|,\dots,|x_n|\} \end{array}}_{\text{maximum norm}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \|-\|_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\dots,x_n) & \mapsto & \sum_{i=1}^{n} |x_i|. \end{array}}_{\text{Manhattan norm}}$$

(a) Show that both maps define norms on $\mathbb{R}$.
(b) We now suppose that $n \geq 2$. Show that the norms do not arise from a positive-definite $\mathbb{R}$-bilinear pairing, i.e. show that there is no positive-definite $\mathbb{R}$-bilinear pairing $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for all $v \in \mathbb{R}^n$ we have $\|v\|_{\max} = \sqrt{\varphi(v,v)}$ and similarly for $\|-\|_{\text{Manh}}$.

Exercise 2.7. If we consider $\mathbb{R}^n$ with the standard inner product, then the Gram-Schmidt Orthogonalization Process 2.32 defines a map

$$\begin{array}{rcl} \Gamma\colon \mathrm{GL}(n,\mathbb{R}) & \to & \mathrm{O}(n) \\ (v_1 \ \dots \ v_n) & \mapsto & \text{matrix whose columns are obtained from } v_1,\dots,v_n \text{ via} \\ & & \text{the Gram-Schmidt Orthogonalization Process 2.32} \end{array}$$

(a) Let $A \in \mathrm{GL}(n,\mathbb{R})$ and let $P \in \mathrm{O}(n)$. Does the equality $\Gamma(P \cdot A) = P \cdot \Gamma(A)$ hold?
(b) Let $A \in \mathrm{GL}(n,\mathbb{R})$ and let $P \in \mathrm{O}(n)$. Does the equality $\Gamma(A \cdot P) = \Gamma(A) \cdot P$ hold?

Exercise 2.8. Let V be a real vector space. Recall that given a norm $n\colon V \to \mathbb{R}_{\geq 0}$ its *norm ball* is defined as $B(n) \coloneqq \{v \in V \mid n(v) \leq 1\}$. Show that every norm is determined by its norm ball.

Exercise 2.9. Let A, B and C be abelian groups and let $f\colon A \times B \to C$ be a bilinear map. Show that for all $a \in A$, $b \in B$ and all $m \in \mathbb{Z}$ we have the equality

$$f(m \cdot a, b) \ = \ f(a, m \cdot b) \ = \ m \cdot f(a,b).$$

Exercise 2.10. Let $n \in \mathbb{N}$. We fix a point $P \in \mathbb{R}^n$ and we consider the map

$$\begin{array}{rcll} d_{\text{SNCF}}\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} & \\ (x,y) & \mapsto & \begin{cases} \|x-y\|, & \text{if } x,y,P \text{ lie on a line,} \\ \|x-P\| + \|y-P\|, & \text{otherwise.} \end{cases} & \end{array}$$

which on page 132 we called the SNCF metric.

(a) Show that this map is indeed a metric on $\mathbb{R}^n$.
(b) Show that there is no norm $\|-\|$ on $\mathbb{R}^n$ such that $d_{\text{SNCF}}(v,w) = \|v-w\|$ for all $v, w \in \mathbb{R}^n$.

CHAPTER 3

Topological spaces: basic definitions and properties

In this chapter we will give the definition of topological spaces, we provide many examples and we discuss a long list of basic properties. Many of these results might well be familiar to the reader. Nonetheless we attempt to give a fairly complete list of basic statements so that later on we have a firm common foundation to work with.

This chapter is organized as follows:

- In Section 3.1 we will give the definition of topological spaces.
- In Section 3.2 we will see that metric spaces have a natural topology.
- In Section 3.3 we will introduce balls and spheres in metric spaces and we will see that they define closed and/or open subsets.
- In Section 3.4 we will endow each subset of a topological space with the subspace topology.
- In Section 3.5 we will study covers of topological spaces.
- In Section 3.6 we will introduce the interior and the closure of a subset of a topological space.
- In Section 3.7 we will introduce the notion of a dense subset of a topological space.
- In Section 3.8 we will introduce what it means for a topological space to be Hausdorff.
- In Section 3.9 will study the convergence of sequences of points in a topological space. We will see that in a Hausdorff space the limit is, if it exists, unique.
- In Section 3.10 we will introduce the notion of a compact topological spaces.
- In Section 3.11 we will prove the Heine-Borel Theorem 3.44 which gives a convenient characterization of compact subspaces of $\mathbb{R}^n$.
- In Section 3.12 we will define what it means for a topological space to be normal. For Hausdorff spaces being normal is a strengthening of being compact.
- In Section 3.13 we will introduce the notion of a (sub-) basis of a topology. These make it easy to study topological spaces and to define new topological spaces.
- In Section 3.14 we will provide exercises for Chapter 3.

In the next Chapter 4 we will introduce the notion of a continuous map between two topological spaces.

3.1. **Topological spaces.** In this section we give the definition of a topological space and we discuss some very basic definitions and statements. Presumably most readers will have seen this material. Throughout these lecture notes, in the proofs, we strive for giving precise references to earlier results. The main exceptions are some of the basic lemmas from this and the following chapter, which we will use without referring to them explicitly. The following definition will keep us occupied for the rest of these notes.

Definition 3.1.

(1) Let X be a set. A topology on X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$[34] with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
   The sets in $\mathcal{T}$ are called open with respect to $\mathcal{T}$. If $\mathcal{T}$ is understood from the context, then we just say open.
(2) A topological space is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

Remark.

(1) Usually we suppress the topology from the notation, e.g. usually we just write "let X be a topological space".
(2) If X is a topological space, then the properties of a topology imply immediately the following three statements:
   (a) the empty set and the entire set X are open,
   (b) the intersection of *finitely many* open subsets is again open,
   (c) the union of *arbitrarily many* open subsets is again open.
(3) It took mathematicians several decades to settle on the above Definition 3.1, the tortuous history of the definition of a topological space is presented in [**BP2024**, Chapter 7.6.2] and [**MooG2008**]. Some of the key steps in the development are given by [**Hausd1914**, p. 213] and [**Alexf1925**, p. 98]. □

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the trivial topology on X.[35]
(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as the discrete topology on X. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.
(3) Let X be a set. We define $\mathcal{T} \subseteq \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \ :\Longleftrightarrow \text{ either } U = \varnothing \text{ or } U \text{ is the complement of a finite subset of } X.$$

   It follows easily from de Morgan's Laws 1.7, that $\mathcal{T}$ is a topology on X. We refer to it as the cofinite topology. Sometimes this topology is also called the finite-complement topology. In a very similar fashion one can define the cocountable topology.[36]
(4) Let X be the empty set. Together with $\mathcal{T} = \{\varnothing\}$ this is a topological space, called the empty topological space. It is a never ending source of counterexamples to carelessly formulated statements.
(5) (a) Let $X = \{*\}$ be a set with a single element $*$. It follows immediately from the definition of a topology there is a unique topology on $X = \{*\}$, namely the topology given by $\mathcal{T} = \{\varnothing, \{*\}\}$. Together with the empty set this is the only case where

[34] In more humble words, $\mathcal{T}$ is a set of subsets of X.
[35] Other names that are used are the *indiscrete topology* and the *chaotic topology*.
[36] To verify that the axioms are satisfied one will need the Countability Lemma 1.37.

the topology is already determined by the set. We refer to any topological space consisting of a single point as a singleton.

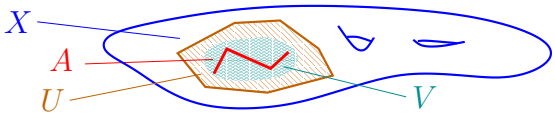
(b) Given a set $X = \{a, b\}$ with two elements it is straightforward to see that there exist four distinct topologies on X. More precisely, there is the trivial and the discrete topology, and there are the two topologies where precisely one of the subsets $\{a\}$ and $\{b\}$ is open. The latter two topologies correspond to the Sierpiński space which we will discuss in greater detail in Exercise 4.26.

Not surprisingly the number of topologies on a finite set grows rapidly with the cardinality of the set, we refer to [**KR1970, BMcMu2002**] for details. □

Definition 3.2. Let X be a topological space.

(1) Let $A \subseteq X$ be a subset. We say that $U \subseteq X$ is a neighborhood of A if there exists an open subset V such that $A \subseteq V \subseteq U$. We say U is an open neighborhood of A, if U is furthermore open. This definition is illustrated in the figure below.

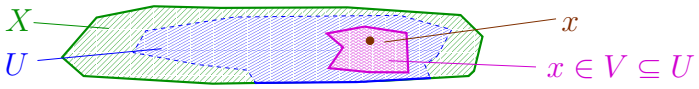
(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.



Example. If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X. □

Using the previous definition we can now state the following lemma which gives a useful criterion for showing that a given subset is open.

Lemma 3.3. (JH-Openness Criterion[37]) Let X be a topological space and let $U \subseteq X$ be a subset. If for every $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.

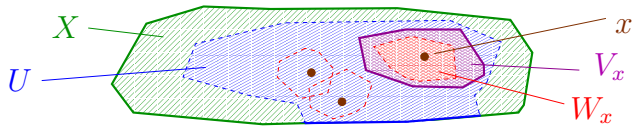


Proof. Let X be a topological space and let $U \subseteq X$ be a subset. Suppose that for every $x \in U$ there exists a neighborhood V_x of x that is contained in U. It follows from the definition of a neighborhood and the Axiom of Choice 1.1 that for each $x \in U$ we can find an open subset W_x with $x \in W_x \subseteq V_x$.[38] We see that

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} W_x \underset{\substack{\uparrow \\ \text{since each } W_x \text{ is} \\ \text{contained in } U}}{\subseteq} U.$$

[37] A student, let's call him JH, pointed out to me in an earlier version of these notes, that I was implicitly frequently using the statement of this lemma.

[38] We could also define W_x to be the union of all open subsets containing x. This approach would avoid the use of the Axiom of Choice 1.1. But in these lecture notes we will take the Axiom of Choice 1.1 for granted and we will not try to minimize its use.



It follows that $U = \bigcup_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. ■

Definition 3.4. Let X be a topological space. We say that a subset A of X is closed if the complement $X \setminus A$ is open.

Example. Recall that by the discussion on page 103 an *interval* is defined as any subset of $\mathbb{R}$ of the form

$$\begin{array}{rcll} (a,b) & \coloneqq & \{x \in \mathbb{R} \,|\, a < x < b\} & \text{where } a, b \in \mathbb{R} \cup \{\pm\infty\} \\ (a,b] & \coloneqq & \{x \in \mathbb{R} \,|\, a < x \leq b\} & \text{where } a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \\ [a,b) & \coloneqq & \{x \in \mathbb{R} \,|\, a \leq x < b\} & \text{where } a \in \mathbb{R} \text{ and } b \in \mathbb{R} \cup \{\infty\} \\ [a,b] & \coloneqq & \{x \in \mathbb{R} \,|\, a \leq x \leq b\} & \text{where } a, b \in \mathbb{R} \text{ with } a \leq b. \end{array}$$

One can easily verify that the intervals of the form (a,b), $(-\infty,b)$ and (a,∞) with $a,b \in \mathbb{R}$ are open subsets of $\mathbb{R}$ and that the intervals of the form $[a,b]$, $(-\infty,b]$ and $[a,\infty)$ with $a,b \in \mathbb{R}$ are closed subsets of $\mathbb{R}$. Note that $\varnothing$ and $\mathbb{R}$ are also intervals, they are both open and closed in $\mathbb{R}$. Finally note that the intervals of the form $(a,b]$ and $[a,b)$ with $a < b$ are neither open nor closed. □

The following lemma mirrors the remark on page 136.

Lemma 3.5. (Closed Subset-Lemma) Let X be a topological space.
(1) The empty set and the entire set X are closed.
(2) The intersection of *arbitrarily many* closed subsets is again closed.
(3) The union of *finitely many* closed subsets is again closed.

Proof. The lemma is an immediate consequence of the definition of a topology and de Morgan's Laws 1.7. ■

Remark. Let X be a set. If we specify what we mean by closed subsets such that the three properties of the Closed Subset-Lemma 3.5 are satisfied, then it follows again from de Morgan's Laws 1.7 that the complements of the closed subsets define a topology. □

3.2. **The metric topology.** Recall that in Definition 2.46 we introduced the notion of a metric space. We will now see that metric spaces give rise to topological spaces. To do so we will need the following notation:

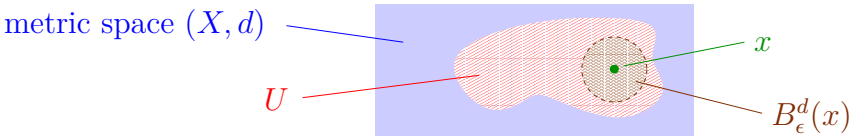
Notation 3.6. Let (X,d) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$ we write

$$B_r^d(x) \coloneqq \{y \in X \,|\, d(x,y) < r\}$$

If the metric d is understood from the context, then sometimes we drop it from the notation.

The following lemma shows that metric spaces give rise to topological spaces.

Lemma 3.7. (Metric-Topology Lemma) Let (X, d) be a metric space. A subset U of X is called open if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon^d(x)$ is contained in U. These open subsets define a topology on X.



Proof.

(1) It is clear that the empty set and X itself are open.

(2) Suppose that $U_1, \dots, U_k$ are open subsets of X. Recall that we need to show that the intersection $U_1 \cap U_2 \cap \dots \cap U_k$ is also open. Thus let $x \in U_1 \cap U_2 \cap \dots \cap U_k$. For each $i \in \{1, \dots, k\}$ there exists an $\epsilon_i > 0$ with $B_{\epsilon_i}(x) \subseteq U_i$. We set $\epsilon := \min\{\epsilon_1, \dots, \epsilon_k\}$. Then
$$B_\epsilon(x) \underset{\substack{\uparrow \\ \text{since } \epsilon \leq \epsilon_i}}{\subseteq} B_{\epsilon_1}(x) \cap \dots \cap B_{\epsilon_k}(x) \underset{\substack{\uparrow \\ \text{since } B_{\epsilon_i}(x) \subseteq U_i}}{\subseteq} U_1 \cap U_2 \cap \dots \cap U_k.$$

(3) Let $\{U_i\}_{i \in I}$ be a family of open subsets. We want to show that $\bigcup_{i \in I} U_i$ is also open. Thus let $x \in \bigcup_{i \in I} U_i$. This means that there exists an $i \in I$ with $x \in U_i$. Since U_i is open there exists an $\epsilon > 0$ with $B_\epsilon(x) \subseteq U_i$. Thus we have
$$B_\epsilon(x) \subseteq U_i \subseteq \bigcup_{i \in I} U_i.$$
■

The Metric-Topology Lemma 3.7 leads us to the following definition:

Definition 3.8. Let (X, d) be a metric space.

(1) We refer to the topology introduced in the Metric-Topology Lemma 3.7 as the metric topology.

(2) Unless we say something else we always equip (X, d) with the metric topology.

Example. Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be the trivial metric that we introduced in Definition 2.49. It is straightforward to see that the corresponding metric topology is the discrete topology. □

Convention 3.9. Let $n \in \mathbb{N}$.

(1) Unless we say something else we consider $\mathbb{R}^n$ as a metric space with the Euclidean metric given by
$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \dots, z_n)\| := \sqrt{z_1^2 + \dots + z_n^2}.$$
Furthermore, unless we say something else we equip $\mathbb{C}^n$ with the metric given by[39]
$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \dots, z_n)\| := \sqrt{z_1 \cdot \overline{z_1} + \dots + z_n \cdot \overline{z_n}}.$$

(2) In particular, unless we say explicitly otherwise, we equip $\mathbb{R}^n$ with the topology arising from the Euclidean metric and we equip $\mathbb{C}^n$ with the topology arising from the above metric. We call these the Euclidean topologies on $\mathbb{R}^n$ and $\mathbb{C}^n$.

[39] It follows from the observation that $(x + \mathrm{i}y) \cdot \overline{x + \mathrm{i}y} = (x + \mathrm{i}y) \cdot (x - \mathrm{i}y) = x^2 + y^2$ that the metric on $\mathbb{C}^n$ is the Euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$ in slight disguise. In particular it is indeed a metric.

We have now seen that metrics give rise to topologies. In general different metrics will give rise to different topologies, for example the Euclidean metric and the trivial metric on $\mathbb{R}^n$ give rise to wildly different topologies. But different metrics might also give rise to the same topology. This leads us to the following definition:

Definition 3.10. Let X be a set. We say that two metrics d and e on X are equivalent if they give rise to the same topology.

Lemma 3.11. (Equivalent Metric Lemma) Let X be a set and suppose that we have two metrics d and e on X such that the following two conditions are satisfied:
(1) For each $x \in X$ and each $\mu > 0$ there exists a $\nu > 0$ with $B^e_\nu(x) \subseteq B^d_\mu(x)$.
(2) For each $x \in X$ and each $\nu > 0$ there exists a $\mu > 0$ with $B^d_\mu(x) \subseteq B^e_\nu(x)$.
Then the two metrics are equivalent.

Example. Let $n \in \mathbb{R}^n$ and let $P \in \mathbb{R}^n$. On page 132 we introduced the following metrics:

$$\begin{aligned} d_{\text{Manh}} \colon \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x,y) &\mapsto \sum_{i=1}^{n} |x_i - y_i| \end{aligned} \quad \& \quad \begin{aligned} d_{\text{SNCF}} \colon \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x,y) &\mapsto \begin{cases} \|x-y\|, & \text{if } x,y,P \text{ lie on a line,} \\ \|x-P\|+\|y-P\|, & \text{otherwise.} \end{cases} \end{aligned}$$

In Exercise 3.7 we will use the Equivalent Metric Lemma 3.11 to show that the Manhattan metric is equivalent to the Euclidean metric. Furthermore we will see in Exercise 3.7 that the SNCF-metric is not equivalent to the Euclidean metric. □

Remark. Later, in the proof of the Norm Equivalence Proposition 4.19, we will use the Equivalent Metric Lemma 3.11 to show that on a given finite-dimensional real vector space, any two norms define the same topology. □

Proof. We need to show that a set $U \subseteq X$ which is open with respect to d is open with respect to e and vice versa. So suppose that U is open with respect to d. We need to show that U is open with respect to e, i.e. we need to show that for each $x \in U$ there exists a $\nu > 0$ with $B^e_\nu(x) \subseteq U$. Thus let $x \in U$. Since U is open with respect to d there exists a $\mu > 0$ such that $B^d_\mu(x) \subseteq U$. By hypothesis there exists a $\nu > 0$ such that $B^e_\nu(x) \subseteq B^d_\mu(x)$. This ν has the desired property. The same argument evidently also works with the roles of d and e reversed. ■

Metric spaces are a popular source of examples of topological spaces. This leads us to the following definition:

Definition 3.12. A topological space X is called metrizable if there exists a metric d on X such that the given topology agrees with the topology coming from d.

Even though many interesting topological spaces are indeed metrizable, we will see in Exercise 3.2 that a finite topological space is metrizable if and only if the topology is discrete.

3.3. **Balls and spheres.** The following proposition gives us important examples of closed and open subsets.

Proposition 3.13. (Ball–Sphere Proposition) Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.

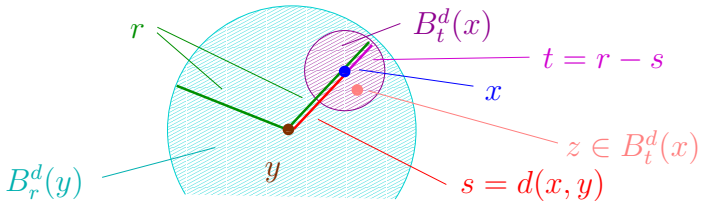
(1) The set $B_r^d(y) \coloneqq \{x \in X \mid d(x, y) < r\}$ is open in X.
(2) The set $\overline{B}_r^d(y) \coloneqq \{x \in X \mid d(x, y) \leq r\}$ is closed in X.
(3) The set $S_r^d(y) \coloneqq \{x \in X \mid d(x, y) = r\}$ is closed in X.

Proof. Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.

(1) Let $x \in B_r^d(y)$. We set $s \coloneqq d(x, y)$. By definition of $B_r^d(y)$ we have $s < r$. We set $t \coloneqq r - s$. Since $t > 0$ it suffices to show that $B_t^d(x) \subseteq B_r^d(y)$. Thus let $z \in B_t^d(x)$. We see that

$$d(y, z) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} d(x, y) + d(x, z) < s + t = r.$$

Thus we see that $z \in B_r^d(y)$, which is what we needed to show.



(2) Almost the same argument as in (1) shows that $X \setminus \overline{B}_r^d(y)$ is open, which implies that $\overline{B}_r^d(y)$ is closed.
(3) We have $X \setminus S_r^d(y) = B_r^d(y) \cup (X \setminus \overline{B}_r^d(y))$. By (1) and (2) we see that $X \setminus S_r^d(y)$ is the union of two open subsets, thus by the third property of a topological space we see that $X \setminus S_r^d(y)$ is open, thus $S_r^d(y)$ is indeed closed. ■

The most frequent application of the Ball–Sphere Proposition 3.13 is to the metric space $\mathbb{R}^n$. Thus we introduce the following notation:

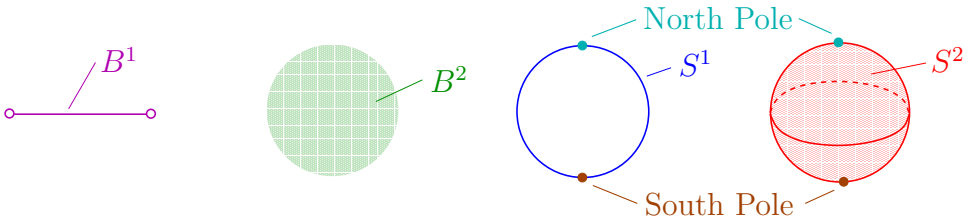
Notation 3.14. Let $n \in \mathbb{N}_0$. Given $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we write[40] [41]

$$\begin{array}{rcll} B_r^n(y) & \coloneqq & \{x \in \mathbb{R}^n \mid \|x - y\| < r\} & \text{called open } n\text{-ball} \\ \overline{B}_r^n(y) & \coloneqq & \{x \in \mathbb{R}^n \mid \|x - y\| \leq r\} & \text{called closed } n\text{-ball} \\ S_r^{n-1}(y) & \coloneqq & \{x \in \mathbb{R}^n \mid \|x - y\| = r\} & \text{called } (n-1)\text{-sphere.} \end{array}$$

We adopt the following conventions:

(1) A 2-ball is often called a disk and a 1-sphere is often referred to as a circle.
(2) If $y = 0$, then we drop y from the notation and if $r = 1$, then we drop r from the notation.
(3) Sometimes we write rB^n instead of $B_r^n(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}_r^n(0)$.
(4) We refer to the point $(0, \ldots, 0, 1) \in S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as the North Pole of S^n and we refer to the point $(0, \ldots, 0, -1) \in S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as the South Pole of S^n.

[41] These definitions also make sense for $r < 0$, in that case all of the sets are the empty set. Also note that for $n = 0$ we obtain that $S_r^{n-1}(y) = S_r^{-1}(y) = \varnothing$.



3.4. **The subspace topology.** The next definition gives us a rich source of new examples of topological spaces.

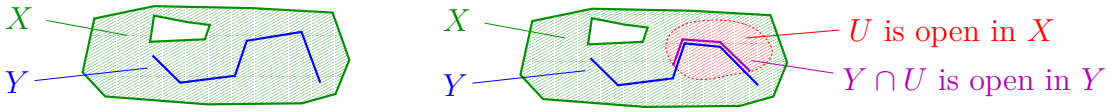
Definition 3.15. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ be a subset.

(1) It follows from elementary set theory, see the Intersection–Distributive Lemma 1.5, that

$$\mathcal{S} \coloneqq \{Y \cap U \,|\, U \in \mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the subspace topology on Y induced by Y.

(2) If Y is indeed equipped with the subspace topology, then sometimes we refer to Y as a subspace of X.



Convention 3.16. Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.

We continue with the following useful lemma.

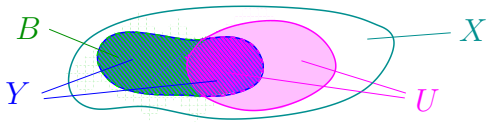
Lemma 3.17. (Subspace-Characterization Lemma) Let X be a topological space and let $Y \subseteq X$ be a subset. We equip Y with the subspace topology.

(1) $V \subseteq Y$ is open in Y $\iff$ there exists an open subset $U \subseteq X$ with $V = U \cap Y$

(2) $B \subseteq Y$ is closed in Y $\iff$ there exists a closed subset $A \subseteq X$ with $B = A \cap Y$.

Proof. Statement (1) is just a reformulation of the definition of the subspace topology. We turn to the proof of Statement (2). Now let $B \subseteq Y$. Elementary set theory shows that for any subset U of X we have

$$(*) \qquad U \cap Y = Y \setminus B \iff Y \setminus (U \cap Y) = B \iff (X \setminus U) \cap Y = B.$$



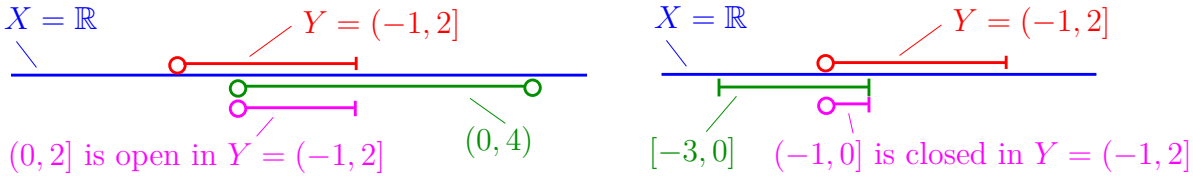
Now Statement (2) is basically obvious. Indeed, we have

$$\begin{aligned} B \text{ is closed in } Y &\iff Y \setminus B \text{ is open in } Y \\ &\iff \text{there exists } U \subseteq Y \text{ open with } U \cap Y = Y \setminus B \\ &\underset{\text{by } (*)}{\iff} \text{there exists } U \subseteq Y \text{ open with } (X \setminus U) \cap Y = B \\ &\iff \text{there exists } A \subseteq Y \text{ closed with } A \cap Y = B. \end{aligned}$$

■

Example. We consider $Y = (-1, 2]$ equipped with the subspace topology coming from $\mathbb{R}$.

(1) The Subspace-Characterization Lemma 3.17 (1) implies that $V = (0, 2]$ is an open subset of Y, since we can write $V = (0, 4) \cap Y$ and $(0, 4)$ is an open subset of $\mathbb{R}$.
(2) The Subspace-Characterization Lemma 3.17 (2) implies that $B = (-1, 0]$ is a closed subset of Y, since we can write $B = [-3, 0] \cap Y$ and $[-3, 0]$ is a closed subset of $\mathbb{R}$.



This short discussion shows that the subspace topology can at times be a little counterintuitive. □

We conclude this short discussion of the subspace topology with the following lemma which subconsciously gets used all the time.

Lemma 3.18. (Iterated Subspace Lemma) Let X be a topological space. Furthermore let $Z \subseteq Y \subseteq X$ be two subsets. The following two topologies are the same:

(1) The subspace topology on Z induced by X.
(2) The subspace topology on Z induced by the subspace topology on Y, which is in turn induced by X.

Proof. This lemma follows almost immediately from the definitions. ■

3.5. **Covers of topological spaces.** On many occasions it will be useful to "break" a topological space into smaller pieces. This leads us to the following definition:

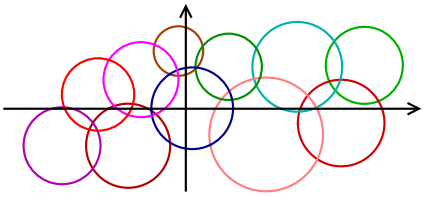
Definition 3.19. Let X be a topological space.

(1) A cover[42] of a subset $A \subseteq X$ is a family[43] $\{U_i\}_{i \in I}$ of subsets of X with $A \subseteq \bigcup_{i \in I} U_i$.
(2) We say that a cover $\{U_i\}_{i \in I}$ is open if each U_i is open. Similarly we define a closed cover.
(3) We say that a cover $\{U_i\}_{i \in I}$ is finite if I is finite. Otherwise we say that the cover is infinite.
(4) We say that a family $\{U_i\}_{i \in I}$ of subsets of X is locally finite if each $x \in X$ admits an open neighborhood W such that $\{i \in I \mid U_i \cap W \neq \varnothing\}$ is finite.

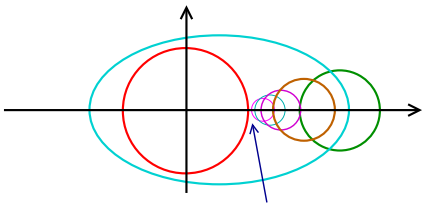
Example. In the figure below we sketch two closed covers of $X = \mathbb{R}^2$:
The closed cover to the left is locally finite. On the other hand the closed cover $\{U_i\}_{i \in I}$ to the right has the property that each $x \in X$ is contained in only finitely many U_i, but nonetheless it fails to be locally finite. □

[43]What we call a "cover" is often also called a "covering". We reserve the name "covering" for a completely different concept that we will introduce in Definition 91.8.

[43]Recall that according to Definition 1.2 a family of subsets of a given set X consists of a set I and a map $f : I \to \mathcal{P}(X)$. We also recall the Convention 1.3 which says that we will view any subset $\mathcal{U} \subseteq \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$.



locally finite closed cover of $X = \mathbb{R}^2$



not locally finite

The following lemma is crucial for many arguments.

Lemma 3.20. (Cover-Open Subset Lemma) Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be an *arbitrary* open cover of X. The following two statements hold:
   (a) M is open if and only if each intersection $M \cap U_i$ is open in U_i,
   (b) M is closed if and only if each intersection $M \cap U_i$ is closed in U_i.

(2) Let $\{A_i\}_{i\in I}$ be a *locally finite* closed cover of X. (In most applications we will deal with a finite closed cover.) The following two statements hold:
   (a) M is open if and only if each intersection $M \cap A_i$ is open in A_i,
   (b) M is closed if and only if each intersection $M \cap A_i$ is closed in A_i.

Example. Let X be a topological space and let $\{A_i\}_{i\in I}$ be a family of closed subsets of X. If the family is locally finite, then it follows from the Cover-Open Subset Lemma 3.20 (2b) together with the Closed Subset-Lemma 3.5 (2) that $\bigcup_{i\in I} A_i$ is also a closed subset of X. □

Remark. Note that in the Cover-Open Subset Lemma 3.20 we allow an arbitrary family of open subsets but we only allow a locally finite family of closed subsets. One can easily see that this restriction is necessary. For example, consider $X = \mathbb{R}$ and the infinite family $\{A_i\}_{i\in I}$ of closed subsets of $X = \mathbb{R}$ given by all subsets consisting of a single point. Let $M \subseteq \mathbb{R}$ be a subset. For each $i \in I$ the intersection $M \cap A_i$ is either the empty set or all of A_i, in particular the intersection $M \cap A_i$ is closed in A_i. But of course there is no reason why M should be a closed subset of $\mathbb{R}$. □

Proof. Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of X. The "only if"-directions of (a) and (b) are an immediate consequence of the Subspace-Characterization Lemma 3.17 (1). It remains to prove the "if"-directions.
   (a) Suppose that for every $i \in I$ the intersection $M \cap U_i$ is an open subset of U_i. By definition of the subspace topology this means that for each $i \in I$ there exists an open subset Z_i of X with $Z_i \cap U_i = M \cap U_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } \bigcup_{i\in I} U_i = X}}{=} \bigcup_{i\in I} (M \cap U_i) = \bigcup_{i\in I} \overbrace{(Z_i \cap U_i)}^{\substack{\text{union of open subsets of } X, \\ \text{hence open in } X}} \underset{\substack{\uparrow \\ \text{union of open subsets,} \\ \text{hence open}}}{=} \text{open subset of } X.$$

(b) Now suppose that each intersection $M \cap U_i$ is closed in U_i. In other words, each set $U_i \setminus (M \cap U_i)$ is open in U_i. Since $(X \setminus M) \cap U_i = U_i \setminus (M \cap U_i)$ we obtain from (a) that $X \setminus M$ is open in X. In other words, M is closed in X.

(2) First note that, as in (1) it remains to prove the "if"-directions.

(b) First we deal with the case that we are given a finite closed cover $\{A_i\}_{i=1,\dots,m}$ of X such that for each i the intersection $M \cap A_i$ is a closed subset of A_i. It follows from the Subspace-Characterization Lemma 3.17 (2) that for each $i \in \{1, \dots, m\}$ there exists a closed subset Z_i of X with $Z_i \cap A_i = M \cap A_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } A_1 \cup \dots \cup A_m = X}}{=} \bigcup_{i=1}^{m} (M \cap A_i) = \bigcup_{i=1}^{m} \overbrace{(Z_i \cap A_i)}^{\substack{\text{intersection of} \\ \text{closed subsets of } X, \\ \text{hence closed in } X}} \underset{\substack{\uparrow \\ \text{union of finitely} \\ \text{many closed subsets,} \\ \text{hence closed}}}{=} \text{closed subset of } X$$

Now suppose that we are given a locally finite closed cover $\{B_i\}_{i \in I}$ of X such that for each $i \in I$ the intersection $M \cap B_i$ is a closed subset of B_i. Since the cover is locally finite there exists for each $x \in X$ an open neighborhood W_x such that $I_x := \{i \in I \mid W_x \cap B_i \neq \varnothing\}$ is finite. Since $\{W_x \cap B_i\}_{i \in I_x}$ is a finite closed cover of W_x we obtain from the above that $M \cap W_x$ is closed. But since $\{W_x\}_{x \in X}$ is an open cover of X we obtain from (1b) that M is a closed subset of X.

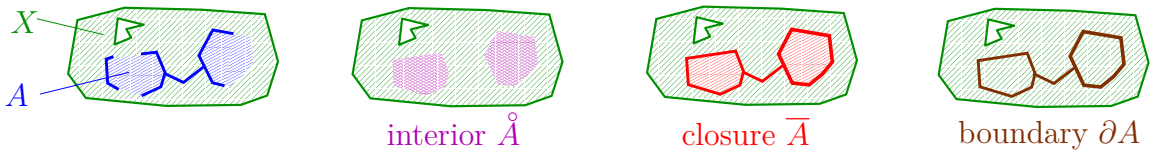
(a) As in (1) we can reduce the proof of (a) to the proof of (b). ■

3.6. **The interior and the closure of a subset.** In this section we will introduce the interior and the closure of a subset of a topological space. We start out with the elegant, but not necessarily intuitive definition:

Definition 3.21. Let X be a topological space and let A be a subset of X. We make the following two definitions:[44]

$$\text{interior } \mathring{A} := \begin{array}{c}\text{union of all open subsets of } X \\ \text{that are contained in } A\end{array} \qquad \text{closure } \overline{A} := \begin{array}{c}\text{intersection of all closed} \\ \text{subsets of } X \text{ that contain } A\end{array}$$

Sometimes, to avoid confusion, we will also write $\operatorname{int}(A)$ for the interior of a subset A. Furthermore we define the boundary of A in X as $\partial A := \overline{A} \setminus \mathring{A}$.



In the following we will state general properties of the interior, the closure and the boundary of a subset. Evidently the point is not to memorize the following lemmas. Instead they are meant as a resource to turn to, when one is dealing with these objects. Therefore it is warmly recommended to move on to the next section and to return to the present section when the need arises.

Lemma 3.22. (Interior-Closure Lemma) Let X be a topological space and let A be a subset of X.

(1) The interior $\mathring{A}$ is an open subset of X.
(2) The closure $\overline{A}$ of A is a closed subset of X.
(3) We have $\mathring{A} \subseteq A \subseteq \overline{A}$.
(4) A is open $\iff \mathring{A} = A$.
(5) A is closed $\iff \overline{A} = A$.
(6) We have $\mathring{A} = A \setminus \partial A$ and $\overline{A} = A \cup \partial A$.
(7) If we set $B := X \setminus A$, then the following statements hold:
   (a) $\overline{B} = X \setminus \mathring{A}$,
   (b) $\mathring{B} = X \setminus \overline{A}$,
   (c) $\partial B = \partial A$.

Proof.

(1) The union of arbitrarily many open subsets is, by definition of a topology, again an open subset. Thus the interior $\mathring{A}$ is an open subset of X.
(2) In the Closed Subset-Lemma 3.5 we saw that the intersection of arbitrarily many closed subsets is a closed subset. It follows immediately that the closure $\overline{A}$ is a closed subset of X.
(3) This statement follows immediately from the definitions.
(4) The"$\Leftarrow$"-statement follows immediately from (1). We turn the proof of the"$\Rightarrow$"-statement. Given any subset $A \subseteq X$ we always have $\mathring{A} \subseteq A$. If A is open, then by definition we also have $A \subseteq \mathring{A}$. Thus we see that $\mathring{A} = A$.
(5) The"$\Leftarrow$"-statement follows immediately from (2). We turn the proof of the"$\Rightarrow$"-statement. Given any subset $A \subseteq X$ we always have $A \subseteq \overline{A}$. If A is closed, then by definition we also have $\overline{A} \subseteq A$. Thus we see that $\overline{A} = A$.
(6) We note that:

$$A \setminus \partial A \underset{\text{Definition 3.21 of } \partial A}{=} A \setminus (\overline{A} \setminus \mathring{A}) \underset{\text{since } \mathring{A} \subseteq A \subseteq \overline{A}}{=} A \setminus (A \setminus \mathring{A}) = \mathring{A}.$$

$$\text{and} \quad A \cup \partial A = A \cup (\overline{A} \setminus \mathring{A}) = \overline{A}.$$

(7) We set $B := X \setminus A$. First we prove (a):

$$\overline{B} \overset{\text{Definition 3.21 of the closure}}{=} \bigcap_{\substack{Y \subseteq X \text{ closed} \\ \text{with } B \subseteq Y}} Y \overset{\text{substitution } U = X \setminus Y}{=} \bigcap_{\substack{U \subseteq X \text{ open} \\ \text{with } U \subseteq A}} (X \setminus U) \overset{\text{de Morgan's Laws 1.7}}{=} X \setminus \bigcup_{\substack{U \subseteq X \text{ open} \\ \text{with } U \subseteq A}} U \overset{\text{Definition 3.21 of the interior}}{=} X \setminus \text{interior of } A.$$

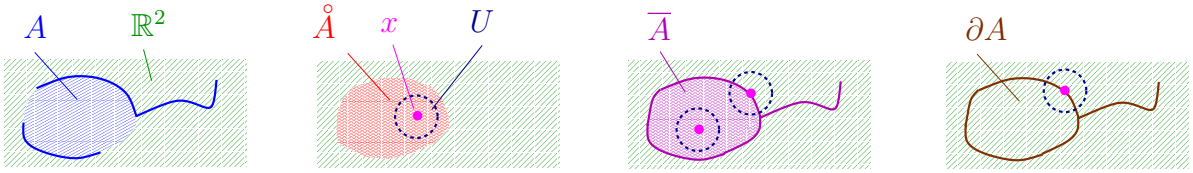
Next note that (b) is an immediate consequence of (a). Finally note that (c) is an immediate consequence of (a) and (b). ■

The above Definition 3.21 of the interior and the closure of a subset is succinct and precise. But just by using the definition it is perhaps difficult to find the interior and the closure of a given subset of a topological space. In many situations the following lemma takes care of that difficulty.

Lemma 3.23. (Interior-Closure Characterization Lemma) Let X be a topological space and let A be a subset of X. We have the following three equalities:

(1) $\mathring{A} = \{x \in A \mid \text{there exists a neighborhood } U \text{ of } x \text{ that is contained in } A\}$
(2) $\overline{A} = \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A\}$
(3) $\partial A = \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A \text{ and one point that does not lie in } A\}$

The same statements also hold with "neighborhood" replaced by "open neighborhood".



Proof. The proof of the lemma is Exercise 3.6. ■

Examples.

(1) We consider the topological space $X = \mathbb{R}$ and the subset $A = [-1, 2)$. Using the Interior-Closure Characterization Lemma 3.23 one can easily see that the interior of A is the open interval $(-1, 2)$ and that the closure of A is the closed interval $[-1, 2]$. Furthermore $\partial A = \{-1, 2\}$.

(2) Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We view $B_r^n(x)$, $\overline{B}_r^n(x)$ and $S_r^{n-1}(x)$ as subsets of $\mathbb{R}^n$. Using the Interior-Closure Characterization Lemma 3.23 one can show fairly easily that

$$\overline{B_r^n(x)} = \overline{B}_r^n(x), \quad \operatorname{int}(\overline{B}_r^n(x)) = B_n^r(x), \quad \partial\overline{B}_r^n(x) = \partial B_r^n(x) = \partial S_r^{n-1}(x) = S_r^{n-1}(x).$$

(3) It is rather unfortunate that the notations $\mathring{A}$, $\overline{A}$ and ∂A hide the role of X. For example let us consider $A = [0, 2]$. If we view it as a subset of $X = \mathbb{R}$, then $\partial A = \partial[0, 2] = \{0, 2\}$. On the other hand, if we view $A = [0, 2]$ as a subset of $X = \mathbb{R}_{\geq 0}$, then we have $\partial A = \partial[0, 2] = \{2\}$. This ambiguity in the notation can sometimes be a source of confusion. Thus it would be more appropriate to specify the role of X in the above notation. But mathematicians, as all humans, tend to be lazy and thus tend to simplify the notation.

(4) We consider $X = \mathbb{R}$ and the subset $A = \mathbb{Q}$. Using the Interior-Closure Characterization Lemma 3.23 one can easily show, using standard facts about real numbers, that $\mathring{\mathbb{Q}} = \varnothing$ and that $\overline{\mathbb{Q}} = \mathbb{R}$. The same conclusion also holds for $\mathbb{R} \setminus \mathbb{Q}$. □

Finally we consider the relationship between closures and interiors under taking unions and intersections. As the following examples show, this is a tricky topic.

Example. Throughout we work with the topological space $X = \mathbb{R}$.

(1) Let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. We see that

$$\text{interior of } \underbrace{\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})}_{=\mathbb{R}} = \mathbb{R} \neq \varnothing = \underbrace{\text{interior of } \mathbb{Q}}_{=\varnothing} \cup \underbrace{\text{interior of } \mathbb{R} \setminus \mathbb{Q}}_{=\varnothing}$$

$$\text{closure of } \underbrace{\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})}_{=\mathbb{R}} = \varnothing \neq \mathbb{R} = \underbrace{\text{closure of } \mathbb{Q}}_{=\mathbb{R}} \cap \underbrace{\text{closure of } \mathbb{R} \setminus \mathbb{Q}}_{=\mathbb{R}}.$$

This discussion shows that taking interiors does not commute even with just taking *finite* unions and analogously that taking closures does not even commute with taking *finite* intersections.

(2) We consider the subsets $[0, \frac{1}{n})$ and $[0, 1 - \frac{1}{n})$ of $\mathbb{R}$. Now we see that

$$\text{interior of } \bigcap_{n\in\mathbb{N}} [0, \tfrac{1}{n}) = \text{interior of } \{0\} = \varnothing \subsetneq \{0\} = \bigcap_{n\in\mathbb{N}} \underbrace{\text{interior of } [0, \tfrac{1}{n})}_{=[0,\frac{1}{n})}$$

$$\text{closure of } \bigcup_{n\in\mathbb{N}} [0, 1-\tfrac{1}{n}) = \text{closure of } [0, 1) = [0, 1] \supsetneq [0, 1) = \bigcup_{n\in\mathbb{N}} \underbrace{\text{closure of } [0, 1-\tfrac{1}{n})}_{=[0,1-\frac{1}{n})}$$

These examples show that taking interiors does not commute even with taking infinite intersections and analogously that taking closures does not commute with taking infinite unions. □

After this exhaustive list of counterexamples the reader will be relieved to see that one can in fact formulate a lemma regarding the interaction of interiors and closures with unions and intersections.

Lemma 3.24. (Interior-Closure-Cup-Cap Lemma) Let X be a topological space and let $\{U_i\}_{i\in I}$ be a family of subsets. The following inclusions hold:

(1a) interior of $\bigcup_{i\in I} U_i \supseteq \bigcup_{i\in I} \mathring{U}_i$ and (1b) closure of $\bigcap_{i\in I} U_i \subseteq \bigcap_{i\in I} \overline{U_i}$

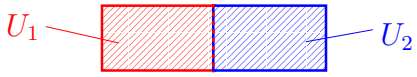
(2a) interior of $\bigcap_{i\in I} U_i \subseteq \bigcap_{i\in I} \mathring{U}_i$ and (2b) closure of $\bigcup_{i\in I} U_i \supseteq \bigcup_{i\in I} \overline{U_i}$.

Furthermore, if the family $\{U_i\}_{i\in I}$ is locally finite (e.g. if it is finite), then the following equalities hold:

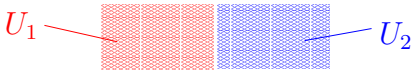
(2a′) interior of $\bigcap_{i\in I} U_i = \bigcap_{i\in I} \mathring{U}_i$ and (2b′) closure of $\bigcup_{i\in I} U_i = \bigcup_{i\in I} \overline{U_i}$.

Example.

(1) In the figure below we show two subsets U_1, U_2 of $\mathbb{R}^2$ such that the inclusions in the Interior-Closure-Cup-Cap Lemma 3.24 (1a) and (1b) are actually proper inclusions.



interior of $U_1 \cup U_2 \supsetneq \mathring{U}_1 \cup \mathring{U}_2$



closure of $U_1 \cap U_2 \subsetneq \overline{U}_1 \cup \overline{U}_2$

(2) In Exercise 3.20 we will see that there are cases where in (2a) and (2b) we do not have equalities. □

Proof.

(1a) By the Interior-Closure Lemma 3.22 we know that each $\mathring{U}_i$ is open. Therefore $\bigcup_{i\in I} \mathring{U}_i$ is open. Evidently this set is contained in $\bigcup_{i\in I} U_i$. By Definition 3.21 of the interior the set $\bigcap_{i\in I} \mathring{U}_i$ is now also contained in the interior of $\bigcup_{i\in I} U_i$.

(1b) It is straightforward to prove (1b) directly. In the following we show alternatively, that (1b) is in fact a formal consequence of (1a). Indeed, we have

$$\begin{array}{rcl} \text{closure of } \bigcap_{i\in I} U_i & \overset{\text{by the Interior-Closure Lemma 3.22 (7)}}{=} & X \setminus \text{interior of } \left(X \setminus \bigcap_{i\in I} U_i\right) \\ & \overset{\text{follows from (1a)}}{\subseteq} & X \setminus \bigcup_{i\in I} \text{interior of } (X \setminus U_i) \\ & \overset{\text{by de Morgan's Laws 1.7}}{=} & \bigcap_{i\in I} X \setminus \text{interior of } (X \setminus U_i) \\ & \overset{\text{by de Morgan's Laws 1.7}}{=} & X \setminus \text{interior of } \left(\bigcup_{i\in I} (X \setminus U_i)\right) \\ & \overset{\text{by the Interior-Closure Lemma 3.22 (7)}}{=} & \bigcap_{i\in I} \overline{U}_i. \end{array}$$

(2a) The interior V of $\bigcap_{i\in I} U_i$ is an open subset which is contained in each U_i. By Definition 3.21 of the interior of U_i we see that V is contained in each $\mathring{U}_i$. The promised inclusion is an immediate consequence of this observation.

(2b) The closure A of $\bigcup_{i\in I} U_i$ is a closed subset that contains each U_i. By Definition 3.21 of the closure of U_i we see that A contains each $\overline{U}_i$. The promised inclusion is an immediate consequence of this observation. observation.

We now assume that the family $\{U_i\}_{i\in I}$ is locally finite.

(2a′) By (2a) we already know that the "$\subseteq$"-inclusion holds. Thus it suffices to prove the "$\supseteq$"-inclusion. By Definition 3.21 of the interior it suffices to show that $\bigcap_{i\in I} \mathring{U}_i$ is open. By the JH-Openness Criterion 3.3 it remains to prove the following claim:

Claim. For each $x \in \bigcap_{i\in I} \mathring{U}_i$ there exists an open neighborhood W of $x \in X$ such that $W \subseteq \bigcap_{i\in I} \mathring{U}_i$.

Proof. Let $x \in \bigcap_{i\in I} \mathring{U}_i$. Since $\{U_i\}_{i\in I}$ is a locally finite family of subsets there exists by Definition 3.19 an open neighborhood V of x such that $J := \{i \in I \,|\, V \cap U_i \neq \varnothing\}$ is finite. We see that

$$W := \underbrace{V \cap \bigcap_{i\in J} \mathring{U}_i}_{\substack{\text{intersection of finitely} \\ \text{many open subsets}}} = \bigcap_{i\in J} (V \cap \mathring{U}_i) \underset{\text{by definition of } J}{\subseteq} \bigcap_{i\in I} (V \cap \mathring{U}_i) \subseteq \bigcap_{i\in I} \mathring{U}_i.$$

Since the intersection of finitely many open subsets is open we see that W is an open neighborhood of $x \in X$. Thus we are done. ⊞

(2b′) By (2b) we already know that the "$\supseteq$"-inclusion holds. Thus it suffices to prove the "$\subseteq$"-inclusion. Before we delve into the proof, we prove the following claim which is perhaps interesting in its own right.

Claim 1. Let $\{B_i\}_{i\in I}$ be a family of subset of X. If this family is locally finite, then the family $\{\overline{B}_i\}_{i\in I}$ of closures is also locally finite.

Proof. This claim follows easily from Definition 3.19 and the following observation: If for an open subset V and a subset $B \subseteq X$ we have $V \cap B = \varnothing$, then B is contained in the closed subset $X \setminus V$, which by Definition 3.21 means that $\overline{B} \subseteq X \setminus V$, which in turn implies that $V \cap \overline{B} = \varnothing$. ⊞

Now we turn to the proof of the "$\supseteq$"-inclusion. Thus let $\{U_i\}_{i\in I}$ be a locally finite family of subsets of X. By Definition 3.21 of the closure it suffices to show that $\bigcup_{i\in I}\overline{U}_i$ is closed. In other words, we have to show that the complement is open. By the JH-Openness Criterion 3.3 it remains to prove the following claim:

Claim 2. For each $x \in X \setminus \bigcup_{i\in I}\overline{U}_i$ there exists an open neighborhood W of $x \in X$ such that $W \subseteq X \setminus \bigcup_{i\in I}\overline{U}_i$.

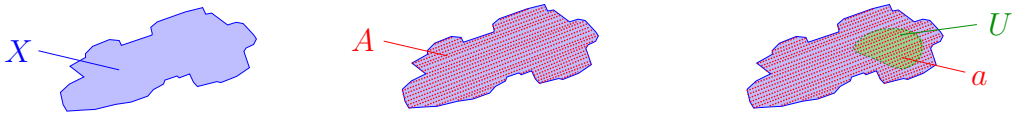
Proof. Let $x \in X \setminus \bigcup_{i\in I}\overline{U}_i$. By hypothesis and Claim (1) we know that $\{\overline{U}_i\}_{i\in I}$ is a locally finite family of subsets. Thus by Definition 3.19 there exists an open neighborhood V of x such that $J \coloneqq \{i \in I \,|\, V \cap \overline{U}_i \neq \varnothing\}$ is finite. We now see that

$$W \coloneqq \underbrace{V \cap \bigcap_{i\in J}(X \setminus \overline{U}_i)}_{\text{by the above discussion this is the intersection of finitely many open subsets}} = \bigcap_{i\in J}(V \cap (X \setminus \overline{U}_i)) \overset{\text{note that } J=\{i\in I\,|\,V\not\subseteq X\setminus\overline{U}_i\}}{\subseteq} \bigcap_{i\in I}(V \cap (X \setminus \overline{U}_i))$$
$$\subseteq \bigcap_{i\in I}(X \setminus \overline{U}_i) \overset{\text{de Morgan's Laws 1.7}}{=} X \setminus \bigcup_{i\in I}\overline{U}_i.$$

Since the intersection of finitely many open subsets is open we see that W is an open neighborhood of $x \in X$. Thus we are done. ∎

3.7. **Dense subsets.** We conclude this section with the discussion of the following definition, which will show up every now and then.

Definition 3.25. Let X be a topological space. We say that a subset A of X is dense in X if every open non-empty subset U of X contains a point $a \in A$.



Example.

(1) Let $n \in \mathbb{N}$. It follows from basic facts in real analysis that $A = \mathbb{Q}^n$ is a dense subset of $X = \mathbb{R}^n$.

(2) It is very tempting to think that dense *open* subsets of say $X = [0,1]$ must be "almost the whole set". But the following example might dent one's optimism. First we pick a bijection $f\colon \mathbb{N} \to \mathbb{Q} \cap [0,1]$. For $i \in \mathbb{N}$ we write $x_i \coloneqq f(i)$. We consider the set

$$U \coloneqq \bigcup_{i\in\mathbb{N}} \underbrace{(x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i}) \cap [0,1]}_{\text{open in } [0,1]} \subseteq [0,1].$$

The same argument as in (1) shows that U is a dense subset of $[0,1]$. Since U is the union of open subsets we see that U is also an open subset of $[0,1]$. On the other hand its Lebesgue measure (as defined in [**Fol1999**]) satisfies

$$\operatorname{vol}(U) = \operatorname{vol}\Big(\bigcup_{i\in\mathbb{N}}(x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i}) \cap [0,1]\Big) \leq \sum_{i\in\mathbb{N}} \operatorname{vol}\big((x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i})\big) \leq \sum_{i\in\mathbb{N}} \frac{2}{3^i} = \frac{2}{3}.$$

Thus we have shown that the Lebesgue measure of U is less than the Lebesgue measure of $[0,1]$. So by no means can one reasonably claim that the dense open U subset is "almost all of $[0,1]$".

(3) Here is an immediate, perhaps not entirely serious application of topology to the eternal problem for how to capture a lion in the desert [**Barri1981**]: We give the desert the *leonine topology*, in which a subset is closed if it is the whole desert or if it contains no lions.[45] The set of lions is now a dense subset.[46] Now put an open cage in the desert. By density it contains a lion. Once we shut the cage we have caught the lion![47] □

In the following we relate the notion of dense subset to the notions of closure and interiors that we introduced in the previous section.

Lemma 3.26. (Density-Interior-Closure Lemma) Let X be a topological space and let $A \subseteq X$ be a subset. The following three statements are equivalent:

(1) A is dense in X.
(2) We have $\overline{A} = X$.
(3) The interior of $X \setminus A$ is empty.

Proof. Let $A \subseteq X$. The equivalence of (2) and (3) is a consequence of the Interior-Closure Lemma 3.22. Thus it remains to show that (1) and (3) are equivalent. In fact we have

$$\begin{aligned} A \text{ is dense} &\iff \text{for every open non-empty subset } U \text{ of } X \text{ we have } U \cap A \neq \varnothing \\ &\iff \text{for every open non-empty subset } U \text{ of } X \text{ we have } U \subsetneq X \setminus A \\ &\iff \text{there is no open non-empty subset } U \text{ of } X \text{ with } U \subseteq X \setminus A \\ &\iff \text{the interior of } X \setminus A \text{ is empty.} \end{aligned}$$

■

3.8. The Hausdorff property.

We start out with the following essential definition:

Definition 3.27. We say that a topological space X is Hausdorff, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.

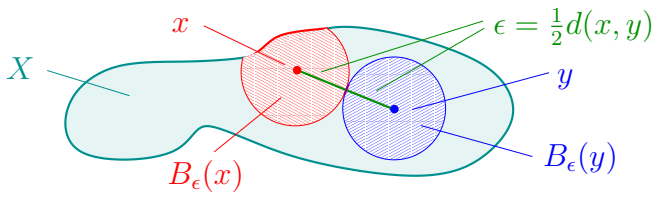


The following proposition, which very likely is well-known to most readers, says that topological spaces arising from metric spaces are Hausdorff.

Proposition 3.28. (Metric Space-Hausdorff Proposition) Let (X,d) be a metric space. The corresponding topological space is Hausdorff.

Proof. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x,y)$.
It follows from the Ball–Sphere Proposition 3.13 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. ■

[45] Why is this a topology?
[46] Why is that?
[47] This and many other mathematical approaches to hunting lions can be found in [**Boas1995**].

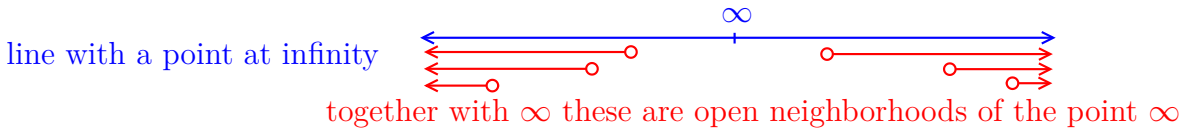


To get some practice for the notion of the Hausdorff property it is convenient to introduce two more examples of topological spaces, which do not come from a metric space. Both new examples are of the type "$\mathbb{R}^n$ together with an extra point".

Definition 3.29. Let $n \in \mathbb{N}$. We consider the set $X := \mathbb{R}^n \cup \{\infty\}$, i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say that $U \subseteq X$ is open, if both of the following two conditions are satisfied:

(1) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$,
(2) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subseteq U$.

It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as $\mathbb{R}^n$ with a point at infinity. For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the line with a point at infinity.[48]

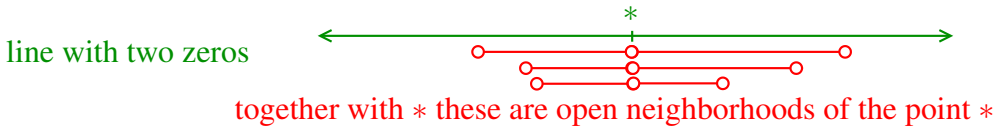


The above example has a mischievous twin brother, which is often a counterexample to carelessly formulated statements.

Definition 3.30. We consider the set $X := \mathbb{R} \cup \{*\}$, i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say that $U \subseteq X$ is open, if the following two conditions are satisfied:

(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$,
(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subseteq U$.

One can easily verify that this is indeed a topology on X. We refer to this topological space as the line with two zeros.[49]



Example. Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity", they are strikingly different topological spaces. For example in Exercise 3.15 we will see that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that on the other hand "the line with two zeros" is not Hausdorff. □

[48]It is straightforward to see that the subspace topology on $\mathbb{R}^n \subseteq \mathbb{R}^n \cup \{\infty\}$ is the usual topology on $\mathbb{R}^n$.
[49]It is straightforward to see that the subspace topology on $\mathbb{R}^n \subseteq \mathbb{R}^n \cup \{*\}$ is the usual topology on $\mathbb{R}^n$.

The following lemma is basically trivial, but since it gets used on many occasions it is still worth spelling it out.

Lemma 3.31. (Subspace-Hausdorff Lemma) Let X be a topological space and let A be a subspace of X. If X is Hausdorff, then A is also Hausdorff.

Proof. The lemma follows almost immediately from the definitions. ■

The following lemma gives us a modest but still quite useful consequence of being Hausdorff.

Lemma 3.32. (Points-in-Hausdorff Lemma) Let X be a topological space.

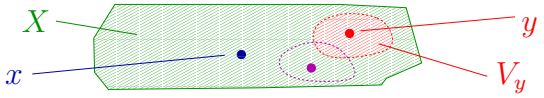
(1) If X is Hausdorff, then for every $x \in X$ the corresponding subset $\{x\}$ is closed.
(2) If X is Hausdorff, then for every $x \neq y \in X$ there exists an open neighborhood U of x such that $y \notin \overline{U}$.

Proof. Let X be a topological space which is Hausdorff.

(1) Let $x \in X$. Since X is Hausdorff we know in particular that for every $y \neq x$ there exists an open neighborhoods V_y of y that is disjoint from x. Note that

$$X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} \{y\} \underset{\substack{\uparrow \\ \text{since } y \in V_y}}{\subseteq} \bigcup_{y \in X \setminus \{x\}} V_y \underset{\substack{\uparrow \\ \text{since } x \notin V_y}}{\subseteq} X \setminus \{x\}.$$

Thus we see that the inclusions are equalities. Since each V_y is open we see that $X \setminus \{x\}$ is the union of open subsets, in particular $X \setminus \{x\}$ is open. In other words, $\{x\}$ is a closed subset of X.

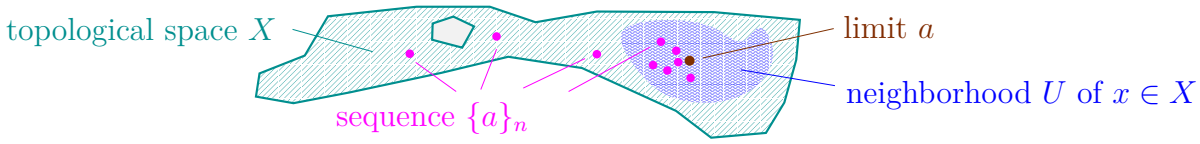


(2) Let $x \neq y \in X$. Since X is Hausdorff there exist disjoint open neighborhoods U of x and V of y. Since $U \subseteq X \setminus V$ and since $X \setminus V$ is closed we obtain from the Definition 3.21 of the closure that $\overline{U} \subseteq X \setminus V$. In particular $y \notin \overline{U}$. ■

3.9. **Convergence of sequences.** In this short section we will introduce the notion of convergence of a sequence of points in a topological space and we will see how the Hausdorff property comes in handy. We will barely make use of the results of this section. For a reader who is used to analysis it is often quite natural to argue with limits. But when it comes to topological spaces it is in most cases more elegant to avoid arguing with limits.

After this preamble, let us finally introduce the notion of a convergent sequence in a topological space.

Definition 3.33. Let X be a topological space. We say that a sequence $\{a_n\}_{n \in \mathbb{N}}$ converges if there exists an $a \in X$ such that for every neighborhood U of a there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \in U$. We refer to a as a limit of the sequence.

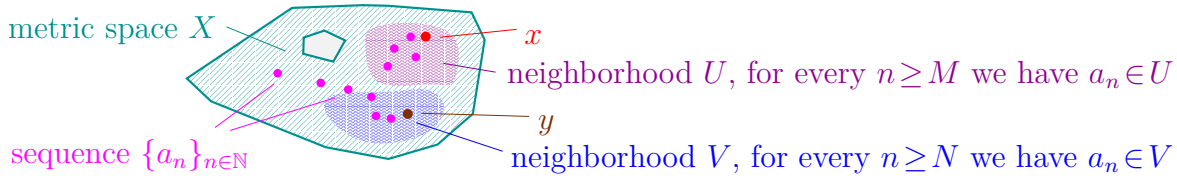


Example. We consider the line with two zeros, i.e. the topological space $X = \mathbb{R} \cup \{*\}$ defined above in Definition 3.30. The sequence $a_n = \frac{1}{n}$ converges, more or less evidently with limit 0. But strikingly enough, it follows easily from the definitions that the sequence also converges to $*$. Thus we see that the limit is not unique. □

Lemma 3.34. (Unique Limit Lemma) Let X be a topological space. If X is Hausdorff, then every convergent sequence in X has a unique limit.

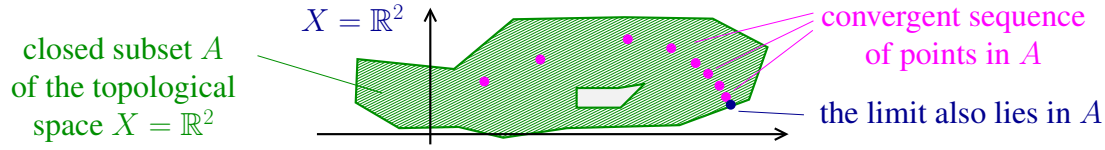
Proof. Assume there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X that converges to two different points x and y. Since X is Hausdorff there exist disjoint neighborhoods U and V of x and y. By our hypothesis we know the following:

- There exists an $M \in \mathbb{N}$ such that for every $n \geq M$ we have $a_n \in U$.
- There exists an $N \in \mathbb{N}$ such that for every $n \geq N$ we have $a_n \in V$.



But this means that for $n \geq \max\{M, N\}$ have to lie in U and V at the same time. But that is not possible and we have thus obtained a contradiction. ■

Lemma 3.35. (Closed Subset Limit Lemma) Let X be a topological space, let $A \subseteq X$ be a closed subset and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of points in A. Every limit of the sequence lies in A.



Proof. We suppose that the sequence has a limit $x \in X \setminus A$. Since A is closed we know that $U := X \setminus A$ is open. But if we now apply the definition of a limit to this open neighborhood $U = X \setminus A$ of x we see that almost all of the a_n have to lie in $U = X \setminus A$. We have thus obtained a contradiction. ■

3.10. **Compact topological spaces.** In this section we introduce the notion of compactness. Usually it takes a while till one appreciates the brilliance of the definition.

Definition 3.36. We say that a topological space X is compact if for each open cover $\{U_i\}_{i\in I}$ of X there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$X = U_{i_1} \cup \cdots \cup U_{i_k}.$$

Remark. The introduction of the notion of compact space by Pavel Alexandrov and Pavel Urysohn [**AU1929**] in 1929 shows that often in mathematics the really clever thing is not necessarily the proofs, but coming up with just the right definition. It is difficult to exaggerate the role of the notion of compactness in topology and analysis. □

Example.

(1) Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.
(2) We consider the topological space $X = \mathbb{R}$. The sets $\{(-n, n)\}_{n\in\mathbb{N}}$ form an open cover of $X = \mathbb{R}$, but evidently we cannot cover $X = \mathbb{R}$ by finitely many of these sets. This shows that $X = \mathbb{R}$ is not compact. □

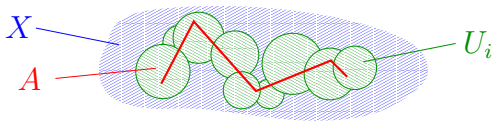
In the following we will often use the following two naming conventions.

Convention 3.37. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact).
(1) If X is a topological space that satisfies P we might just say that "X is a P-space". For example, a Hausdorff space is a topological space that is Hausdorff.
(2) We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

Frequently and often subconsciously the following elementary lemma is employed.

Lemma 3.38. (Compact Subspace Covering Lemma) Let X be a topological space, let $A \subseteq X$ be a subset and let $\{U_i\}_{i\in I}$ be an open cover of the subset $A \subseteq X$. If A is compact, then there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$A \subseteq U_{i_1} \cup \cdots \cup U_{i_k}.$$



Proof. Given $i \in I$ we set $V_i := U_i \cap A$. By definition of the subspace topology each V_i is an open subset of A. Thus $\{V_i\}_{i\in I}$ is an open cover of A. Since A is compact there exist $i_1, \ldots, i_k \in I$ with $A = V_{i_1} \cup \cdots \cup V_{i_k}$. But this implies that $A \subseteq U_{i_1} \cup \cdots \cup U_{i_k}$. ■

Before we give some more interesting examples of compact spaces we want to discuss a few properties of compact spaces and compact subsets. The following lemma might be the lemma that gets most often cited in these notes.

Lemma 3.39. (Compact-Closed Lemma) Let X be a topological space.
(1) If X is compact, then any closed subset of X is also compact.
(2) Let $A \subseteq X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.

Example.
(1) Let X be a topological space. As we pointed out on page 155, any finite subset of X is compact. Thus if X is Hausdorff, then it follows from the Compact-Closed Lemma 3.39 (2) that any finite subset of X, in particular any point in X, is a closed subset of X. Thus we have found a new proof for the Points-in-Hausdorff Lemma 3.32 (1).
(2) Let X be a topological space, let A be a closed subset and let K be a compact subset of X. It follows from the Compact-Closed Lemma 3.39 (1) that $A \cap K$ is also compact. This simple observation will get used frequently.
(3) We consider the topological space $X = \{0, 1\}$ that is equipped with the trivial topology $\{\varnothing, X\}$. The subset $A = \{0\}$ is evidently compact, but it is not a closed subset of X.

This shows that in the formulation of Compact-Closed Lemma 3.39 (2) we cannot drop the hypothesis that X is Hausdorff. □

Proof.

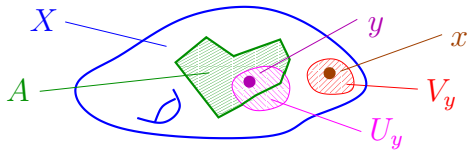
(1) Let X be a compact space and let A be a closed subset. Let $\{U_i\}_{i\in I}$ be an open cover of A. By definition of the subset topology there exists for every $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is closed the complement $X \setminus A$ is open. The open subsets $\{V_i\}_{i\in I}$ together with $X \setminus A$ are an open cover of X. Since X is compact, we can cover X by finitely many of these open subsets. But then we can also cover A by finitely many of the U_i.



(2) Let X be a Hausdorff space and let $A \subseteq X$ be a compact subset. We want to show that $X \setminus A$ is open. By the JH-Openness Criterion 3.3 it suffices to prove the following claim:

Claim. For every $x \in X \setminus A$ there exists an open neighborhood V of x that is contained in $X \setminus A$.

Proof. We apply the Hausdorff property to x and every $y \in A$. For every $y \in A$ we obtain disjoint open neighborhoods U_y of y and V_y of x.



Evidently we have

$$A = \bigcup_{y\in A} \{y\} \subseteq \bigcup_{y\in A} (U_y \cap A) \subseteq A.$$

Thus we see that $\{U_y \cap A\}_{y\in A}$ is an open cover of A. Since A is compact there exist $y_1, \ldots, y_k \in A$ such that

$$A = \bigcup_{i=1}^{k} (U_{y_i} \cap A).$$

Now we consider

$$V := \bigcap_{i=1}^{k} V_{y_i}.$$

Since V is the intersection of finitely many open subsets, it is open itself. Furthermore V does not intersect any of the U_{y_i}, $i = 1, \ldots, k$. This implies that V is disjoint from $A \subseteq U_{y_1} \cup \cdots \cup U_{y_k}$. ■

The following little lemma turns out to be surprisingly useful:

Lemma 3.40. (Compact-Discrete Lemma)

(1) Every discrete space that is compact is finite.
(2) Let X be a compact space and let A be a closed subset. If A is discrete, then A is finite.

Remark. Note that the conclusion of the Compact-Discrete Lemma 3.40 (2) does not necessarily hold, if we drop the hypothesis that A is closed. For example, consider the

topological space $X = [0,1]$. The subset $A := \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is a discrete subset, but evidently it is not finite. □

Proof.

(1) Let X be a discrete space. Each point thus defines an open subset. Evidently X is covered by all the open subsets given by the points in X. Since X is compact, we can cover X by finitely many of those open subsets. But this means that X contains only finitely many points.

(2) Let X be a compact space and let A be a closed subset. It follows from the Compact-Closed Lemma 3.39 (1) that A is also compact. So if A is discrete, then it follows from (1) that A is finite. ■

We conclude this section with the following frequently employed lemma.

Lemma 3.41. (Compact–Union-Intersection Lemma) Let X be a topological space.

(1) The union of finitely many compact subsets of X is again a compact subset.

(2) If X is Hausdorff, then the intersection of arbitrarily many compact subsets of X is again a compact subset.

Remark. Note that in the Compact–Union-Intersection Lemma 3.41 we cannot drop the hypothesis that X is Hausdorff. For example consider the set $X = \mathbb{N} \cup \{x_0, x_1\}$ where we equip $\mathbb{N}$ with the discrete topology and where the only other open subsets are $\mathbb{N} \cup \{x_0\}$, $\mathbb{N} \cup \{x_1\}$ and $\mathbb{N} \cup \{x_0, x_1\}$. The subspaces $A_0 = \mathbb{N} \cup \{x_0\}$ and $A_1 = \mathbb{N} \cup \{x_1\}$ are easily seen to be compact, but the intersection $A_0 \cap A_1 = \mathbb{N}$ is certainly not compact. □

Proof.

(1) This statement follows easily from the definitions, we leave it to the reader to fill in the details.

(2) Let $\{A_i\}_{i \in I}$ be a family of compact subsets of X. We pick $j \in I$. By the Compact-Closed Lemma 3.39 (2) we know that each $A_i \cap A_j$ is a closed subset of A_i. Thus $\bigcap_{i \in I} (A_i \cap A_j)$ is a closed subset of the compact space A_j. Hence $\bigcap_{j \in I} A_i$ is compact by the Compact-Closed Lemma 3.39 (1). ■

3.11. **The Heine-Borel Theorem.** We start out this section with the following almost self-explanatory definition.

Definition 3.42. Let (X, d) be a metric space. A subset A of X is called bounded if there exists a $C \geq 0$ such that for all $x_0, x_1 \in A$ we have $d(x_0, x_1) \leq C$.

Example. Let (X, d) be a metric space, let $y \in X$ and let $r \geq 0$. It follows easily from the triangle inequality that the balls $B_r^d(y)$, $\overline{B}_r^d(y)$ and the sphere $S_r^d(y)$ are bounded with $C = 2r$. □

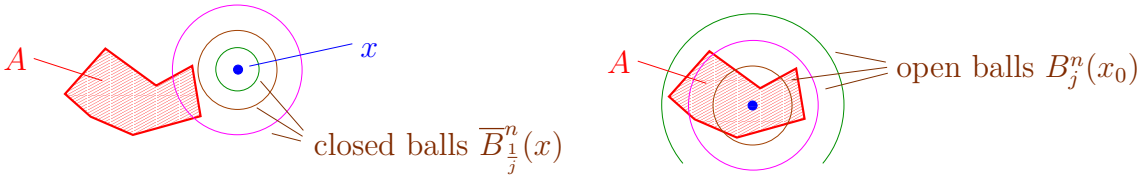
Lemma 3.43. (Compact–Closed+Bounded Lemma) Let (X, d) be a metric space and let $A \subseteq X$ be a subset. If A is compact, then A is closed and it is bounded.

Proof. Let (X, d) be a metric space and let $A \subseteq X$ be a subset.

(1) We want to show that $X \setminus A$ is open. Let $x \in X \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B_\epsilon^d(x) \subseteq X \setminus A$. Given $j \in \mathbb{N}$ we consider the subset $U_j = X \setminus \overline{B}_{\frac{1}{j}}^d(x)$.

By the Ball–Sphere Proposition 3.13 we know that the U_j are open. Evidently these open subsets cover A. Since A is compact we obtain from the Compact Subspace Covering Lemma 3.38 that there exist $j_1, \dots, j_k \in \mathbb{N}$ with $A \subseteq U_{j_1} \cup \dots \cup U_{j_k}$. We set $\epsilon := \min\{\frac{1}{j_1}, \dots, \frac{1}{j_k}\}$. We obtain that $A \subseteq X \setminus \overline{B}^d_\epsilon(x)$. Thus we have found the desired ϵ.

(2) We need to show that A is bounded. If A is empty, then there is nothing to show. Now suppose that A is non-empty. We pick $x_0 \in A$. It follows from the triangle inequality that it suffices to show that there exists an $R \geq 0$ with $A \subseteq \overline{B}^d_R(x_0)$. We consider the open subsets $U_j = B^d_j(x_0)$ with $j \in \mathbb{N}$.



Evidently these sets cover A. Since A is compact we obtain again from the Compact Subspace Covering Lemma 3.38 that there exist $j_1, \dots, j_k \in \mathbb{N}$ such that $A \subseteq U_{j_1} \cup \dots \cup U_{j_k}$. We set $R := \max\{j_1, \dots, j_k\}$. We obtain that $A \subseteq B^d_R(x_0) \subseteq \overline{B}^d_R(x_0)$. We have thus found the desired R. ■

The ever-popular Heine-Borel Theorem says that for the metric space $X = \mathbb{R}^n$ mercifully the converse to the Compact–Closed+Bounded Lemma 3.43 holds.

Theorem 3.44. (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

In the following we first provide several examples before we turn to the chore of proving the Heine-Borel Theorem.

Example.

(1) Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 3.44 and the Ball–Sphere Proposition 3.13 that the closed ball $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are compact.

(2) It is an amusing exercise, see Exercise 3.15, to show, using the Heine-Borel Theorem 3.44, that "$\mathbb{R}^n$ with a point at infinity" is compact.

(3) It is an equally entertaining exercise, see Exercise 3.15, to show that the "line with two zeros" is not compact.

(4) Since the concepts of being bounded and closed are initially much more intuitive than being compact it is not uncommon that many, especially students, like to replace "compact" by "bounded and closed". But this equality only holds for subsets of $\mathbb{R}^n$, it does not hold for other metric spaces. For example, the set $[0,1] \cap \mathbb{Q}$ is a bounded and closed subset of the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$, but as we will see in Exercise 3.24, it is not compact. Furthermore in Exercise 3.25 we will see that in general the conclusion of the Heine-Borel Theorem 3.44 does not hold for the infinite-dimensional real vector space

$$\mathbb{R}^\infty := \{(x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} \mid \text{there exists an } i \in \mathbb{N} \text{ such that } x_j = 0 \text{ for } j \geq i\}$$

equipped with the obvious generalization of the Euclidean metric. □

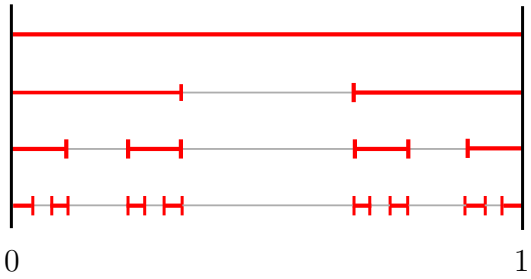
Our final example deserves to be introduced as a proper definition.

Definition 3.45. We set $C_0 := [0,1]$. Iteratively we now define sets $C_2, C_3, \ldots$ as follows:

$$C_i := \underbrace{\{\tfrac{1}{3}\cdot x \,|\, x \in C_{i-1}\}}_{=\frac{1}{3}\cdot C_{i-1}} \cup \underbrace{\{\tfrac{2}{3}+\tfrac{1}{3}\cdot x \,|\, x \in C_{i-1}\}}_{=\frac{2}{3}+\frac{1}{3}\cdot C_{i-1}}.$$

Note that each C_i consists of 2^i closed intervals of length $\frac{1}{3^i}$ and that we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. We define

$$\text{Cantor set } C := \bigcap_{i\in\mathbb{N}_0} C_i.$$



$C_0 = [0,1]$
$C_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$
$C_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{3}{9}\right] \cup \left[\frac{6}{9},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$
C_3

Lemma 3.46. (Cantor Set–Compact Lemma) The Cantor set is compact.

Proof. Each C_i is the union of 2^i closed intervals, hence each C_i is closed. It follows from the Closed Subset-Lemma 3.5 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 3.44 implies that the Cantor set is compact. ■

The following proposition is the key ingredient in the proof of the Heine-Borel Theorem 3.44.

Proposition 3.47. (Hyperrectangle-Compact Proposition) Let $n \in \mathbb{N}$. Furthermore let $a_i < b_i$, $i = 1, \ldots, n$ be real numbers. The hyperrectangle

$$\underbrace{[a_1,b_1] \times \ldots \times [a_n,b_n]}_{:=\{(x_1,\ldots,x_n)\in\mathbb{R}^n \,|\, x_i\in[a_i,b_i] \text{ for } i=1,\ldots,n\}} \subseteq \mathbb{R}^n$$

is compact.

Proof. To simplify the notation we only treat the case $n = 1$. The general case is treated in a very similar fashion.[50] To simplify the notation further we assume that $a = 0$ and $b = 1$. Thus we will now show that $I := [0,1]$ is compact. In an attempt at a proof by contradiction we suppose that I is not compact. This means that there exists an open cover $\{U_j\}_{j\in J}$ such that I cannot be covered by finitely many of the U_j.

Claim 1. There exists a sequence of closed intervals

$$I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$$

such that for every $m \in \mathbb{N}_0$ the following two conditions hold:
(1) I_m cannot be covered by finitely many of the U_j,
(2) the length of I_m equals 2^{-m}.

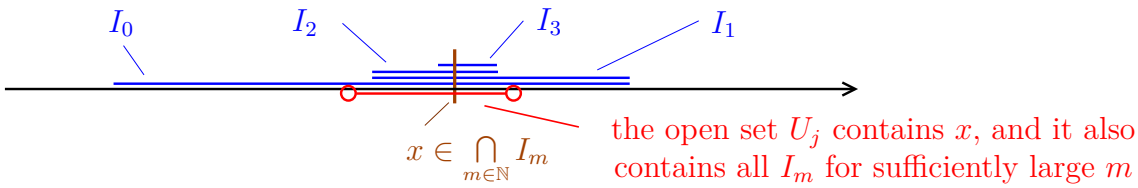
[50] The full details are for example provided in [**RudiW1976**, Theorem 2.4.1]. Alternatively one can also use the Product Topology-Properties Proposition 7.9 (1).

Proof. We prove the claim by induction. We set $I_0 := I$. Suppose we have already found $I_0, I_1, \ldots, I_m$. We divide the interval I_m into two closed intervals I' and I'' of half the length. Since I_m cannot be covered by finitely many U_j it follows that at least one of the intervals I' and I'' cannot be covered by finitely many U_j either. We declare such interval to be I_{m+1}. ⊞

Claim 2. There exists an $x \in I$ with $x \in \bigcap_{m\in\mathbb{N}} I_m$.

Proof. For every $m \in \mathbb{N}$ we pick an $a_m \in I_m$. For any $m \geq n$ we know that a_m and a_n both lie in I_n. In particular $$|a_m - a_n| \;\leq\; \text{length of } I_n \;=\; \frac{1}{2^n}.$$ It follows that $(a_m)_{m\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete we know that the sequence converges. We denote by x the limit of this Cauchy sequence. We want to show that $x \in \bigcap_{m\in\mathbb{N}} I_m$. Put differently, we want to show that for every $m \in \mathbb{N}$ we have $x \in I_m$. Thus let $m \in \mathbb{N}$. For all $n \geq m$ we have $a_n \in I_n \subseteq I_m$. In particular, $(a_n)_{n\geq m}$ is a sequence that converges in I_m. Since I_m is closed we know by the Closed Subset Limit Lemma 3.35 that $\lim_{n\to\infty} a_n \in I_m$. ⊞

Since $\{U_j\}_{j\in J}$ is an open cover of I there exists a $j \in J$ with $x \in U_j$. Since U_j is open there exists an $\epsilon > 0$ with $(x-\epsilon, x+\epsilon) \cap I \subseteq U_j$.



We pick $m \in \mathbb{N}$ with $2^{-m} < \epsilon$. Now note that

$$I_m \underset{\uparrow}{\subseteq} [x-2^{-m}, x+2^{-m}] \cap I \underset{\uparrow}{\subseteq} (x-\epsilon, x+\epsilon) \cap I \underset{\uparrow}{\subseteq} U_j$$

since $x \in I_m$ and since the length of I_m equals 2^{-m} — choice of m — choice of U_j

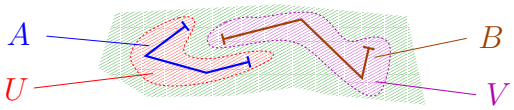
i.e. I_m is contained in a unique U_j. But this contradicts property (1) of the intervals $\{I_m\}_{m\in\mathbb{N}}$. ■

Now we can finally give the proof of the Heine-Borel Theorem 3.44.

Proof of the Heine-Borel Theorem 3.44. In light of the Compact–Closed+Bounded Lemma 3.43 we only need to show that if A is a bounded and closed subset of $\mathbb{R}^n$, then it is compact. Since A is bounded there exists a $C \geq 0$ such that $A \subseteq [-C, C]^n$. By the Hyperrectangle-Compact Proposition 3.47 we know that $[-C, C]^n$ is compact. Since A is closed in $\mathbb{R}^n$ it follows from the Subspace-Characterization Lemma 3.17 that A is also a closed subset of $[-C, C]^n$. It follows from the Compact-Closed Lemma 3.39 that A itself is compact. ■

3.12. **Normal topological spaces.** There are many variations on the definition of the Hausdorff property. We will discuss some of these in Chapters 12 and 13, and especially in Section 13.4. For the time being we just want to introduce one variation on being Hausdorff:

Definition 3.48. A topological space X is called normal if any two disjoint closed subsets A and B of X can be separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.

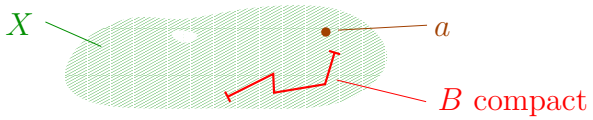


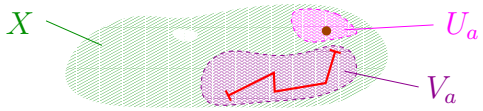
We will discuss this notion of begin normal in greater detail in Section 12.1. For the time being we only want to prove the following lemma, which can be quite useful.

Lemma 3.49. (Compact Hausdorff-Normal Lemma) Every compact Hausdorff space is normal.

Proof. Let X be a Hausdorff space. We start out with the following claim:

Claim. Let B be a compact subspace of X. Given any $a \in X \setminus B$ there exists an open neighborhood U_a of $a \in X$ and an open neighborhood V_a of B that are disjoint.





Proof. Let $a \in A$. Since X is Hausdorff we can find for any $b \in B$ disjoint open neighborhoods R_b and S_b of a and b. Furthermore, since B is compact there exist $b_1, \dots, b_n \in B$ such that $B \subseteq S_{b_1} \cup \dots \cup S_{b_n}$. The open subsets

$$U_a \coloneqq R_{b_1} \cap \dots \cap R_{b_n} \quad \text{and} \quad V_a \coloneqq S_{b_1} \cup \dots \cup S_{b_n}$$

have the desired properties. ⊞

Now assume that X is also compact. We need to show that X is normal. Thus let A and B be two disjoint closed subsets of X. Note that the Compact-Closed Lemma 3.39 (1) implies that A and B are also compact.

Since B is compact we can apply the above claim to every $a \in A \subseteq X \setminus B$ and B. In the following we use the notation of the above claim. Since A is compact we can find $a_1, \dots, a_m \in A$ such that $A \subseteq U_{a_1} \cup \dots \cup U_{a_m}$. The open subsets

$$U \coloneqq U_{a_1} \cup \dots \cup U_{a_m} \quad \text{and} \quad V \coloneqq V_{a_1} \cap \dots \cap V_{a_m}$$

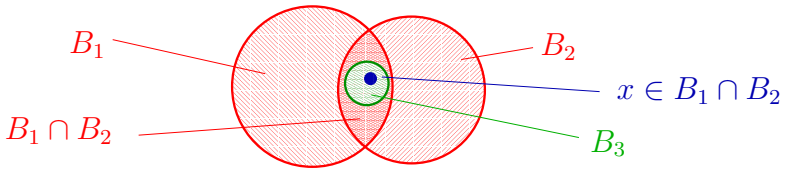
have the desired properties. ■

3.13. **Bases of a topology.** In this section we will introduce the notion of a (sub-) basis of a topology. These notions make it possible to define topologies and to describe a given topology. Nonetheless, the topic is somewhat dry and it is best not to spend too much time on this section.

Definition 3.50. Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a subset of the power set of X. We say that $\mathcal{B}$ has the basis property if the following two conditions are satisfied:

(B1) For every $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.



Example. Let (X, d) be a metric space. It follows from an elementary argument, see e.g. the proof of the Ball–Sphere Proposition 3.13 (1), that the set of all open balls

$$\mathcal{B} \;=\; \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \;\subseteq\; \mathcal{P}(X)$$

satisfies (B1) and (B2). □

The following lemma justifies our sudden interest in this new definition.

Lemma 3.51. (Topology-from-Basis Lemma) Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$. We set

$$\mathcal{T}(\mathcal{B}) \;:=\; \{V \subseteq X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subseteq V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Definition 3.52. Let X be a set.

(1) If $\mathcal{B} \subseteq \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the topology on X generated by $\mathcal{B}$.

(2) Conversely, if $\mathcal{T}$ is a given topology on X, if $\mathcal{B} \subseteq \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a basis of the topology $\mathcal{T}$ of X.

Examples.

(1) If (X, d) is a metric space, then the topology generated by the set of all open balls, is by definition precisely the metric topology we already introduced on page 139.

(2) Let X be a set and let $\mathcal{T}$ be a topology on X. It follows easily from the definitions that $\mathcal{T}$ has the basis property. Furthermore it follows from the JH-Openness Criterion 3.3 that the topology generated by $\mathcal{T}$ is in fact $\mathcal{T}$.

(3) Let $n \in \mathbb{N}$ and let $\mathbb{R}^n \cup \{\infty\}$ be the topological space defined on page 152. It follows immediately from the definition of the topology that a basis for the topological space $\mathbb{R}^n \cup \{\infty\}$ is given by

$$\mathcal{B} \;=\; \left\{B_r^n(x) \,\middle|\, r > 0 \text{ and } x \in \mathbb{R}^n\right\} \cup \left\{\{\infty\} \cup (\mathbb{R}^n \setminus \overline{B}_C^n(0)) \,\middle|\, C \in \mathbb{R}\right\}. \qquad \square$$

For completeness we provide the elementary proof of the Topology-from-Basis Lemma 3.51, which turns out to be quite similar to the proof of the Metric-Topology Lemma 3.7.

Proof. Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$. We write

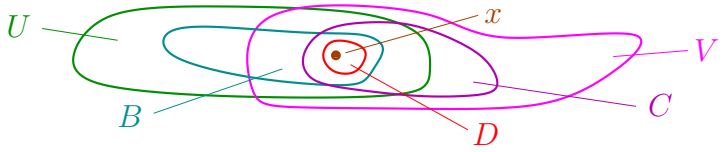
$$\mathcal{T} \;:=\; \mathcal{T}(\mathcal{B}) \;:=\; \{V \subseteq X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subseteq V\}.$$

We need to show that $\mathcal{T}$ satisfies the three properties (1), (2) and (3) in the definition of a topological space, see Definition 3.1.

(1) First we need to show that $\varnothing, X \in \mathcal{T}$. It is clear that the empty set $\varnothing$ lies in $\mathcal{T}$. Furthermore given any $x \in X$ there exists by (B1) a $B \in \mathcal{B}$ with $x \in B$. Thus it follows from the definitions that $X \in \mathcal{T}$.

(2) We need to show that the intersection of finitely many sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

First we consider the case of two sets in $\mathcal{T}$. Thus let $U, V \in \mathcal{T}$. We need to show that $U \cap V \in \mathcal{T}$. Let $x \in U \cap V$. Since U and V lie in $\mathcal{T}$ there exist $B, C \in \mathcal{B}$ with $x \in B \subseteq U$ and $x \in C \subseteq V$. By (B2) there exists a $D \in \mathcal{B}$ with $x \in D$ and $D \subseteq B \cap C$. In particular we have $x \in D \subseteq U \cap V$. This shows that $U \cap V \in \mathcal{T}$.



Now let $U_1, \ldots, U_k$ be sets in $\mathcal{T}$. Using the above statement and an elementary induction argument we see that $U_1 \cap \cdots \cap U_k$ lies in $\mathcal{T}$ as well.

(3) Let $\{U_j\}_{j \in J}$ be a family of sets in $\mathcal{T}$. We need to show that the union $\bigcup_{j \in J} U_j$ lies in $\mathcal{T}$. Thus let $x \in \bigcup_{j \in J} U_j$. This means that there exists a $k \in J$ with $x \in U_k$. Since $U_k \in \mathcal{T}$ there exists a $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U_k$. But then we also have

$$x \in B_x \subseteq U_k \subseteq \bigcup_{j \in J} U_j. \qquad \blacksquare$$

Sometimes the following two lemmas can be useful.

Lemma 3.53. (Union-of-Basis Sets Lemma) Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{B}$ has the basis property, then

$$V \subseteq X \text{ is open with respect to } \mathcal{T}(\mathcal{B}) \iff V \text{ is the union of sets in } \mathcal{B}.$$

Proof. We first prove the"$\Leftarrow$"-direction. Thus let V be the union of sets in $\mathcal{B}$. Since sets in $\mathcal{B}$ are open with respect to $\mathcal{T}(\mathcal{B})$ and since $\mathcal{T}(\mathcal{B})$ is a topology we see that $V \in \mathcal{T}(\mathcal{B})$.

Now we prove the"$\Rightarrow$"-direction. Thus we suppose that $V \subseteq X$ is open with respect to $\mathcal{T}(\mathcal{B})$. For each $x \in V$ there exists, by definition of $\mathcal{T}(\mathcal{B})$, a set $B_x \in \mathcal{B}$ with $x \in B_x \subseteq V$. We see that

$$V = \bigcup_{x \in V} \{x\} \subseteq \bigcup_{x \in V} B_x \subseteq V, \quad \text{thus it follows that } V = \bigcup_{x \in V} B_x. \qquad \blacksquare$$

Lemma 3.54. (Small Basis Lemma) Let $\mathcal{B}$ be a basis for the topology of a topological space X. Furthermore let $\{U_j\}_{j \in J}$ be an open cover of X, i.e. $\{U_j\}_{j \in J}$ is a family of open subsets with $\bigcup_{j \in J} U_j = X$. Then

$$\{B \in \mathcal{B} \mid \text{ there exists some } j \in J \text{ with } B \subseteq U_j\}$$

is also a basis for the topology of X.

Example. We consider the topological space $\mathbb{R}^n$. As we pointed out on page 162, the set of open balls $\mathcal{B} = \{B_r^n(x) \mid r > 0 \text{ and } x \in \mathbb{R}^n\}$ is a basis for the topology of $\mathbb{R}^n$. We denote by $\mathcal{U}$ the family of all open subsets that are contained in an open ball of radius 1. Then the open subsets in $\mathcal{U}$ cover $\mathbb{R}^n$. It follows from the Small Basis Lemma 3.54 that

$$\{B_r^n(x) \mid r \in (0,1) \text{ and } x \in \mathbb{R}^n\}$$

is also a basis for $\mathbb{R}^n$. $\square$

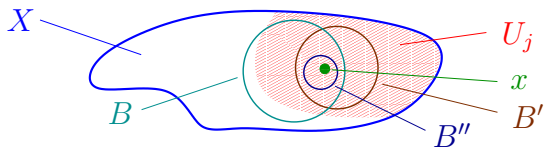
Proof. We set

$$\mathcal{C} := \{B \in \mathcal{B} \mid \text{ there exists some } j \in J \text{ with } B \subseteq U_j\}.$$

Since $\mathcal{C} \subseteq \mathcal{B}$ we have $\mathcal{T}(\mathcal{C}) \subseteq \mathcal{T}(\mathcal{B})$. It remains to show the reverse inclusion $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}(\mathcal{C})$. It follows easily from the JH-Openness Criterion 3.3 that it suffices to show that $\mathcal{B} \subseteq \mathcal{T}(\mathcal{C})$. In other words, it suffices to prove the following claim:

Claim. Given any $x \in X$ and any $B \in \mathcal{B}$ with $x \in B$ there exists a $C \in \mathcal{C}$ such that $x \in C \subseteq B$.

Proof. So let $x \in X$ and let $B \in \mathcal{B}$ with $x \in B$. By hypothesis there exists a j such that $x \in U_j$. Since U_j is open and since $\mathcal{B}$ is a basis for the topology of X there exists a $B' \in \mathcal{B}$ with $x \in B' \subseteq U_j$. By (B2) there exists a $B'' \subseteq B$ with $x \in B''$ and $B'' \subseteq B \cap B' \subseteq U_j$.



So we are done since by definition we have $B'' \in \mathcal{C}$. ■

The following lemma gives us in particular another source of topologies.

Lemma 3.55. (Produce Basis Lemma) Let X be a set. For any $\mathcal{C} \subseteq \mathcal{P}(X)$ the set

$$\mathcal{B}(\mathcal{C}) \quad := \quad \begin{array}{c}\text{all subsets of } X \text{ that can be written as} \\ \text{intersections of a finite family of sets in } \mathcal{C}\end{array}$$

has the basis property.

Proof.

(1) In Definition 1.4 we pointed out that if we take the empty family of subsets of X, then the intersection is all of X. In other words, we see that $X \in \mathcal{B}(\mathcal{C})$. It follows that $\mathcal{B}(\mathcal{C})$ satisfies (B1).
(2) It is easy to verify that $\mathcal{B}(\mathcal{C})$ satisfies (B2). ■

The previous lemma leads us to the following definition:

Definition 3.56. Let X be a set and let $\mathcal{C} \subseteq \mathcal{P}(X)$.

(1) We define $\mathcal{B} = \mathcal{B}(\mathcal{C})$ as in the Produce Basis Lemma 3.55 and we refer to $\mathcal{T}(\mathcal{B})$ as the topology generated by $\mathcal{C}$.
(2) Conversely, if $\mathcal{T}$ is a topology on X and if the topology generated by $\mathcal{C}$ agrees with $\mathcal{T}$, then we say that $\mathcal{C}$ is a subbasis for $\mathcal{T}$.

Example.

(1) Let $\mathcal{C}$ be the set of all intervals in $\mathbb{R}$ of length one. One can easily verify that $\mathcal{C}$ generates the usual topology on $\mathbb{R}$.
(2) It follows easily from the definitions that a basis of a given topology is also a subbasis of the topology. □

The following lemma gives us a convenient characterization of (sub-) bases for a given topology.

Lemma 3.57. (Basis Characterization Lemma) Let X be a topological space and let $\mathcal{C}$ be a family of subsets of X.

(1) The following two statements are equivalent:
   (a) Every $C \in \mathcal{C}$ is open and given any open subset U and given any $x \in U$ there exists a $C \in \mathcal{C}$ with $x \in C \subseteq U$.
   (b) $\mathcal{C}$ is a basis for the topology of X.

(2) Suppose that $\mathcal{B}$ is a basis for the topology of X. The following two statements are equivalent:
   (a) Every $C \in \mathcal{C}$ is open and given any $x \in X$ and given any $B \in \mathcal{B}$ there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $x \in C_1 \cap \cdots \cap C_m \subseteq B$.
   (b) $\mathcal{C}$ is a subbasis for the topology of X.

Example. In Exercise 3.8 we will show, using the Basis Characterization Lemma 3.57, that the set of open *rational* intervals forms a basis for the usual topology on $\mathbb{R}$. In particular $\mathbb{R}$ has a countable basis for its topology. □

Proof.

(1) We denote by $\mathcal{T}$ the topology of X. The implication (b)⇒(a) follows immediately from the definitions. So let us turn to the proof of(a)⇒(b). First we show that $\mathcal{C}$ has the basis property:
   (B1) If we apply the property of $\mathcal{C}$ to $U = X$ then we obtain immediately that given any $x \in X$ there exists a $C \in \mathcal{C}$ with $x \in C$.
   (B2) Let $x \in C_1 \cap C_2$ with $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open the intersection $U = C_1 \cap C_2$ is also open. It follows from the hypothesis on $\mathcal{C}$, applied to U, that there exists a $C \in \mathcal{C}$ with $x \in C \subseteq C_1 \cap C_2$.

   Let $\mathcal{S}$ be the topology on X that is generated by $\mathcal{C}$. We need to show that $\mathcal{S} = \mathcal{T}$.

   First we show the inclusion $\mathcal{S} \subseteq \mathcal{T}$. Since all sets in $\mathcal{C}$ are open with respect to $\mathcal{T}$ it follows from the Union-of-Basis Sets Lemma 3.53 that $\mathcal{S} \subseteq \mathcal{T}$.

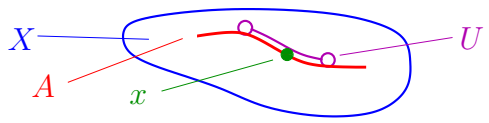
   Finally we show the reverse inclusion $\mathcal{T} \subseteq \mathcal{S}$. Thus let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{S}$. Thus let $x \in U$. By our hypothesis there exists a $C \in \mathcal{C}$ with $x \in C \subseteq U$. Therefore by definition we have $U \in \mathcal{S}$. We have thus shown that $\mathcal{T} \subseteq \mathcal{S}$.

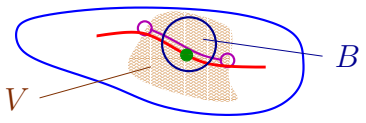
(2) The verification of this statement is almost as thrilling as the proof of (1). We leave this pleasurable task to the reader. ■

We conclude this section with the following not terribly inspiring lemma.

Lemma 3.58. (Subspace-Basis Lemma) Let X be a topological space and let $A \subseteq X$ be a subset. If $\mathcal{B}$ is a basis for the topology of X, then $\{B \cap A \,|\, B \in \mathcal{B}\}$ is a basis for the topology of A, equipped with the subspace topology.

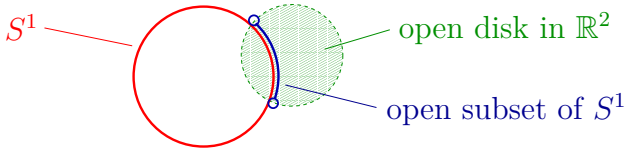
Proof. We prove the lemma using the Basis Characterization Lemma 3.57 (1). Thus let $U \subseteq A$ be an open subset in A and let $x \in U$. By definition of the subspace topology there exists an open subset $V \subseteq X$ of X with $U = V \cap A$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $B \in \mathcal{B}$ with $x \in B \subseteq V$.





This implies that $x \in B \cap A \subseteq V \cap A = U$. It follows from the Basis Characterization Lemma 3.57 (1) that $\{B \cap A \,|\, B \in \mathcal{B}\}$ is indeed a basis for the topology of A. ■

Example. We consider $$S^1 \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} \;\subseteq\; \mathbb{R}^2.$$
As we saw on page 162, a basis for the topology on $\mathbb{R}^2$ is given by
$$\mathcal{B} \;=\; \{B_r(x,y) \,|\, (x,y) \in \mathbb{R}^2 \text{ and } r > 0\}.$$
The intersection of such an open ball with S^1 is an "open interval on S^1", i.e. it is a subset of the form $\{\exp(\mathrm{i}\varphi) \,|\, \varphi \in (a,b)\}$.



It follows from the Subspace-Basis Lemma 3.58 that these "open intervals" form a basis for the topology on S^1. □

3.14. Exercises for Chapter 3.

Exercise 3.1. Recall that a topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. We say that a metric space (X,d) is *complete* if every Cauchy sequence in (X,d) converges. Furthermore we say that a topological space X is *completely metrizable* if there exists a complete metric d on X such that the given topology agrees with the topology coming from d.

(a) Is $(0,1)$ completely metrizable?
(b) Is $\mathbb{Q}$ completely metrizable?

Exercise 3.2. Let X be a finite set. Show that a topology $\mathcal{T}$ on X is metrizable if and only if it is the discrete topology.

Exercise 3.3. Let (X,d) be a metric space and let $A \subseteq X$ be a subset. The restriction of the metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ to the map $d\colon A \times A \to \mathbb{R}_{\geq 0}$ defines a metric on A. We equip the metric spaces (X,d) and (A,d) with the corresponding topologies $\mathcal{X}$ and $\mathcal{A}$ as defined in the Metric-Topology Lemma 3.7. Show that the topology $\mathcal{A}$ on A is the subspace topology of $A \subseteq X$ from Definition 3.15.

Exercise 3.4. Let X be a set and let $\mathcal{S}, \mathcal{T}$ be two topologies on X. We assume that the following statements hold:

(1) For each $U \in \mathcal{S}$ there exists a $V \in \mathcal{T}$ with $V \subseteq U$.
(2) For each $V \in \mathcal{T}$ there exists a $U \in \mathcal{S}$ with $U \subseteq V$.

Does it follow that $\mathcal{S} = \mathcal{T}$?

Exercise 3.5. Let $X \subseteq \mathbb{R}^n$.

(a) Show that if the subspace topology on X is the discrete topology, then X is countable.
(b) Show that the converse to (a) does not hold.

Exercise 3.6. Let X be a topological space and let A be a subset of X. Show that the following three equalities hold:

(a) $\mathring{A} = \{x \in A \mid \text{there exists a neighborhood } U \text{ of } x \text{ that is contained in } A\}$
(b) $\overline{A} = \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A\}$
(c) $\partial A = \{x \in X \mid \text{every neighborhood of } x \text{ contains at least one point in } A \text{ and one point that does not lie in } A\}$

Exercise 3.7. Let $n \in \mathbb{N}$. We fix a point $P \in \mathbb{R}^n$ and we consider the maps

$$d_{\text{Manh}} \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \quad (x,y) \mapsto \sum_{i=1}^{n} |x_i - y_i| \qquad \& \qquad d_{\text{SNCF}} \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \quad (x,y) \mapsto \begin{cases} \|x-y\|, & \text{if } x,y,P \text{ lie on a line,} \\ \|x-P\|+\|y-P\|, & \text{otherwise.} \end{cases}$$

which page 132 we called the Manhattan metric respectively the SNCF metric.

(a) Show that the Manhattan metric is equivalent to the Euclidean metric, i.e. show that they define the same topology on $\mathbb{R}^n$.
(b) Show that the SNCF metric is not equivalent to the Euclidean metric.

Exercise 3.8.

(a) Show that the open *rational* intervals, i.e. the open intervals (a,b) with $a,b \in \mathbb{Q}$, form a basis for the usual topology on $\mathbb{R}$.
(b) Let $n \in \mathbb{N}_0$. Show that the set of open balls $B_r(x)$ with $r \in \mathbb{Q}_{>0}$ and $x \in \mathbb{Q}^n$ form a basis for the usual topology on $\mathbb{R}^n$.

Exercise 3.9. Let U be an open subset of $\mathbb{R}^2$. Is it true that the interior of the closure $\overline{U}$ equals U?

Exercise 3.10. Let $\{U_s\}_{s \in S}$ be an open cover of the interval $[0,1]$. Show that there exist $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ such that each interval $[t_i, t_{i+1}]$ is contained in one of the U_s.
Remark. Note that we do not assume that the U_s are connected.

Exercise 3.11. Let (X,d) be a compact metric space. A rather surprising theorem, see [**Gros1964**] or alternatively [**Ris2006**], says that there exists a unique $r \in \mathbb{R}_{\geq 0}$ such that for every finite sequence of points $x_1, \ldots, x_n \in X$ there exists an $x' \in X$ with $\frac{1}{n}\sum_{i=1}^{n} d(x_i, x') = r$. This number is called the *rendezvous value of* (X,d).

(a) Determine the rendezvous value of the interval $X = [0,1]$ equipped with the metric $d(x,y) = |x-y|$.
(b) Determine the rendezvous value of the circle $X = S^1$ equipped with the Euclidean metric of $\mathbb{R}^2$.

Exercise 3.12. Let X be a Hausdorff space and let $x_1, \ldots, x_k$ be pairwise distinct points in X. Show that there exist pairwise disjoint open neighborhoods $V_1, \ldots, V_k$ of $x_1, \ldots, x_k$

Exercise 3.13. Let X be a topological space that is Hausdorff.

(a) Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ be a sequence of compact non-empty subsets. Show that $\bigcap_{n \in \mathbb{N}} K_n$ is also non-empty.
(b) Does the conclusion of (a) hold, if we only assume that the K_n are closed?

Exercise 3.14. Let X be a topological space that is Hausdorff and let $x, y \in X$ be two distinct points. Can you always find open neighborhoods U of x and V of y such that the closures are disjoint, i.e. such that $\overline{U} \cap \overline{V} = \varnothing$?

Exercise 3.15.

(a) Show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and compact.
(b) Show that "the line with two zeros" is neither Hausdorff nor compact.

Exercise 3.16. Given $a, b \in \mathbb{N}$ we define

$$U_a(b) \;:=\; \{b + n \cdot a \in \,|\, n \in \mathbb{N}\} \;\subseteq\; \mathbb{N}.$$

Furthermore we set

$$\mathcal{C} \;:=\; \{U_p(b) \,|\, p, b \in \mathbb{N} \text{ and } p \text{ prime}\} \;\subseteq\; \mathcal{P}(\mathbb{N}).$$

The topology $\mathcal{T}$ on $\mathbb{N}$ generated by $\mathcal{C}$ is called the *prime integer topology*.

(a) Show that $(\mathbb{N}, \mathcal{T})$ is Hausdorff.
(b) A topological space X is called *regular* if given any $P \in X$ and given any closed subset $A \subseteq X$ with $x \notin A$ there exist disjoint open subsets U and V with $P \in U$ and $A \subseteq V$. Show that $(\mathbb{N}, \mathcal{T})$ is not regular.

Exercise 3.17. Given $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z}$ we define

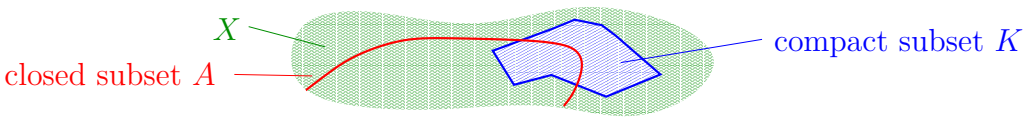
$$S(a, b) \;:=\; \{a \cdot n + b \,|\, n \in \mathbb{Z}\} \;=\; a \cdot \mathbb{Z} + b.$$

We say that $U \subseteq \mathbb{Z}$ is open if for every $x \in U$ there exists a subset $S(a, b)$ as above with $x \in S(a, b) \subseteq U$. We denote by $\mathcal{T}$ the set of all open subsets of $\mathbb{Z}$.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$. (This topology is called the *evenly spaced integer topology* on $\mathbb{Z}$).
(b) Show that each set of the form $S(a, b)$ is also closed.
(c) Let P be the set of prime numbers. Show that $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in P} S(p, 0)$.
(d) Show that there exist infinitely many primes.

Remark. This proof that there exist infinitely many primes was found by Harry Furstenberg [**Furs1955**] in 1955.

Exercise 3.18. Let X be a topological space, let $A \subseteq X$ be a closed subset and let $K \subseteq X$ be a compact subset. Show that $A \cap K$ is also compact.



Exercise 3.19. Let X be a topological space and let $A \subseteq X$ be a subset.

(a) We suppose that X is Hausdorff. Show that $\partial A = \partial \overline{A}$.
(b) Does the conclusion in (a) also hold if X is not Hausdorff?

Exercise 3.20. Show that in general in the Interior-Closure-Cup-Cap Lemma 3.24 (2a) and (2b) we do not have equalities. More precisely:

(a) Give an example of a topological space X and a family of subsets $\{U_i\}_{i \in I}$ such that

$$\text{interior of } \bigcap_{i \in I} U_i \;\subsetneq\; \bigcap_{i \in I} \mathring{U}_i$$

(b) Give an example of a topological space X and a family of subsets $\{U_i\}_{i\in I}$ such that

$$\text{closure of } \bigcup_{i\in I} U_i \supsetneq \bigcup_{i\in I} \overline{U_i}.$$

Remark. By the Interior-Closure-Cup-Cap Lemma 3.24 we know that equalities hold if $\{U_i\}_{i\in I}$ is locally finite.

Exercise 3.21. Let X be a topological space and let A and B be subsets of M such that $\overline{A}$ and $\overline{B}$ are compact.

(a) Show that if X is Hausdorff, then $\overline{A\cap B}$ and $\overline{A\cup B}$ are also compact.
(b) Does the conclusion of (a) also always hold if we drop the hypothesis that X is Hausdorff?

Exercise 3.22. Let X be a topological space.

(a) Show that the intersection of finitely many open and dense subsets is again open and dense.
(b) Show that the intersection of countably many open and dense subsets is in general not dense.

Exercise 3.23. Let X be a topological space. We say that a subset $U\subseteq X$ is *large* if U is open and dense.

(a) Show that the intersection of finitely many large subsets of X is again large.
(b) Is it true that the intersection of countably many large subsets of X is necessarily large?

Exercise 3.24. Show that the topological space $[0,1]\cap\mathbb{Q}$, equipped with the subspace topology coming from $\mathbb{R}$, is not compact.

Exercise 3.25. We consider

$$\mathbb{R}^\infty := \{(x_1,x_2,\dots)\in\mathbb{R}^{\mathbb{N}} \mid \text{there exists an } i\in\mathbb{N} \text{ such that } x_j=0 \text{ for } j\geq i\}.$$

We equip $\mathbb{R}^\infty$ with the Euclidean metric

$$\begin{aligned} d\colon \mathbb{R}^\infty\times\mathbb{R}^\infty &\to \mathbb{R}^\infty \\ ((x_1,x_2,\dots),(y_1,y_2,\dots)) &\mapsto \sqrt{\sum_{i\in\mathbb{N}} |x_i-y_i|^2}. \end{aligned}$$

Show that the closed unit ball in $(\mathbb{R}^\infty,d)$ is not compact.

Exercise 3.26. Let X be a topological space. Recall that a subset $A\subseteq X$ is called *precompact in* X if the closure $\overline{A}$ is compact. Let X be a subset of some topological space Y and let $A\subseteq X$ be a subset that is precompact in X.

(a) Show that A is not necessarily precompact in Y.
(b) Show that if Y is Hausdorff, then A is in fact precompact in Y.

Remark. Dealing with closures can be quite subtle. For example, it seems to me that for (b) you want to use the Subspace-Characterization Lemma 3.17 (2) at some point.

Exercise 3.27. Let X be a topological space. We say that a family $\{A_i\}_{i\in I}$ of subsets has the *finite intersection property* if given any finite collection of indices $i_1,\dots,i_r\in I$ the corresponding intersection $A_{i_1}\cap\dots\cap A_{i_r}$ is non-empty.

(a) Let X be a compact space and let $\{A_i\}_{i\in I}$ be a family of closed subsets that has the finite intersection property. Show that $\bigcap_{i\in I} A_i\neq\varnothing$.

(b) Show that in (a) we cannot drop the hypothesis that X is compact.

Exercise 3.28. Show that every open subset of $\mathbb{R}^n$ is the union of countably many open balls.

Exercise 3.29. Let X be a Hausdorff space, let A be a subset of X and let B be a compact subset of X. Show that if $A \cap B = \varnothing$, then there exists an open subset U of X with $A \subseteq U$ and $U \cap B = \varnothing$.
Hint. Look at the (proof of) the Compact Hausdorff-Normal Lemma 3.49.



CHAPTER 4

Continuity

In the previous Chapter 3 we introduced the concept of topological spaces and we discussed various examples and properties. Nonetheless, as it is the theory is rather dull. The theory of topological spaces comes alive with the right notion of suitable maps between topological spaces, namely the notion of continuous maps. The notion of continuity is the main topic of this chapter.

This chapter is organized as follows:

- In Section 4.1 we will introduce the all important notion of a continuous map between two topological spaces.
- In Section 4.2 we will introduce what it means for a map between two topological spaces to be "open" or "closed".
- In Section 4.3 we will prove the Compact Image Lemma 4.15 which says that image of a compact topological space under a continuous map is again compact.
- In Section 4.4 we will introduce the notion of an embedding and of a homeomorphism. We will prove the best result in topology, namely that a continuous bijective map from a compact space to a Hausdorff space is in fact a homeomorphism.
- In Section 4.5 we will prove that every open non-empty convex subset of $\mathbb{R}^n$ is homeomorphic to the open n-ball B^n.
- In Section 4.6 we will introduce two notions of what it means for a topological space to be (path-) connected.
- In Section 4.7 we will introduce the (path-) components of a topological space.
- In Section 4.8 we will determine the path-components of matrix groups.
- In Section 4.9 we will introduce the concept of local and regional properties of a topological space. We will see that for locally path-connected topological spaces the notions "connected" and "path-connected" conveniently agree.
- In Section 4.10 we will introduce the notion of proper maps and closed embeddings between topological spaces.
- In Section 4.11 we will provide exercises for Chapter 4.

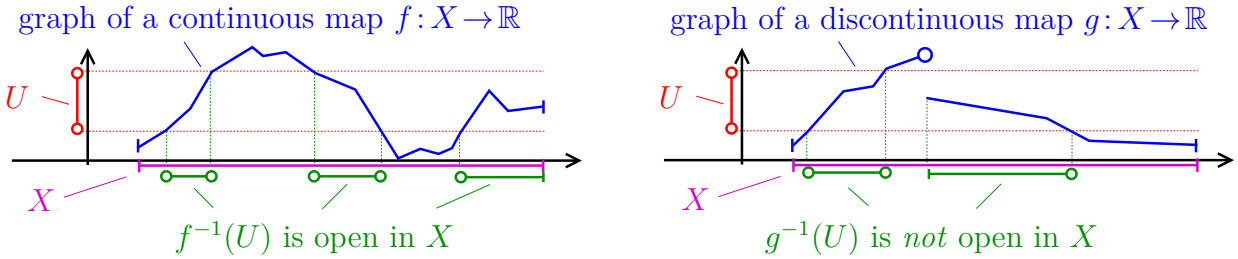
In the next Chapter 5 we will introduce categories, which provide the right language for most of topology.

4.1. **Continuous maps.** The following is the key definition in the context of topological spaces:

Definition 4.1. We say that a map $f\colon X \to Y$ between two topological spaces X and Y is continuous, if for each open subset U in Y the preimage $f^{-1}(U)$ is open in X. Otherwise the map is called discontinuous.

Examples.

(1) For subspaces $X \subseteq \mathbb{R}$ and maps $\mathbb{R} \to \mathbb{R}$ we see in the figure below that the above Definition 4.1 of continuous matches our intuition.



(2) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then every map $f: X \to Y$ is continuous.

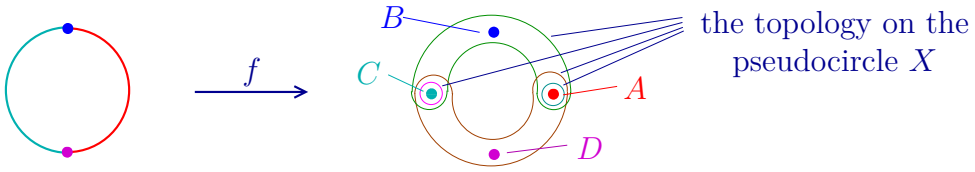
(3) We consider the set $X = \{A, B, C, D\}$ where the topology is given by the set

$$\mathcal{T} \coloneqq \{\varnothing, \{A\}, \{C\}, \{A, C\}, \{A, B, C\}, \{A, C, D\}, X\}.$$

We refer to this to topological spaces as the *pseudocircle*. It follows easily from the definitions that the map

$$\begin{array}{rcl} f\colon S^1 & \to & X \\ \exp(it) & \mapsto & \begin{cases} A, & \text{if } t \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B, & \text{if } t = \frac{\pi}{2}, \\ C & \text{if } t \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ D & \text{if } t = \frac{3\pi}{2}. \end{cases} \end{array}$$

is continuous.



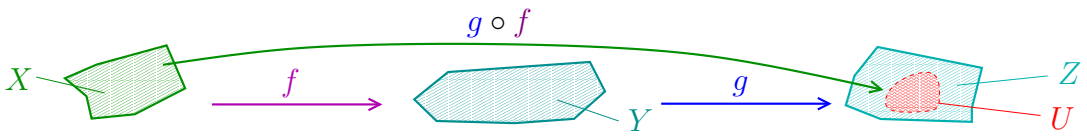
(4) We say that a map $f: X \to Y$ between two sets is *constant* if for any $x_1, x_2 \in X$ we have $f(x_1) = f(x_2)$. Any constant map $f: X \to Y$ between two topological spaces is continuous since the preimage of an open subset of Y is either the empty set or X. This simple example will come in handy on many occasions. □

The following lemma is almost certainly the lemma on continuous maps that gets used most frequently:

Lemma 4.2. (Continuous Composition Lemma) If $f: X \to Y$ and $g: Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f: X \to Z$ is also continuous.

Since the lemma is so absolutely essential to every discussion of continuous maps we will just take it for granted, and usually we will not acknowledge that we make use of it.

Proof. Let $f: X \to Y$ and $g: Y \to Z$ be two continuous maps between topological spaces. We need to show that the composition $g \circ f: X \to Z$ is also continuous.



Let $U \subseteq Z$ be an open subset. Then

$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)) \;\underset{\text{since } g \text{ is continuous and since } U \text{ is open}}{\overset{\uparrow}{=}}\; f^{-1}(\text{the open subset } g^{-1}(U)) \;\underset{\text{since } f \text{ is continuous}}{\overset{\uparrow}{=}}\; \text{open.}$$

■

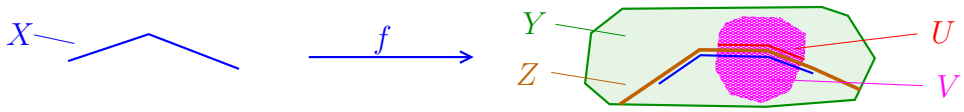
We move on to another elementary lemma which explains the relationship between continuity and subspaces:

Lemma 4.3. (Subspace Continuity Lemma)

(1) Let X be a topological space and let A be a subset of X. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.
(2) Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset of X. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.
(3) Let $f\colon X \to Y$ be a continuous map. If $Z \subseteq Y$ is a subset with $f(X) \subseteq Z$, then the map $f\colon X \to Z$ is also continuous.

Proof.

(1) This statement follows immediately from the definitions.
(2) This statement is an immediate consequence of (1) and the Continuous Composition Lemma 4.2 since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.
(3) For clarity we denote by $g\colon X \to Z$ the map that is given by $x \mapsto f(x)$. Let $U \subseteq Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subseteq Y$ with $U = Z \cap V$.



Thus we see that

$$g^{-1}(U) \;=\; g^{-1}(Z \cap V) \;\underset{\text{definition of } g}{\overset{\uparrow}{=}}\; f^{-1}(Z \cap V) \;\underset{\text{since } f(X) \subseteq Z}{\overset{\uparrow}{=}}\; f^{-1}(V) \;\underset{\text{since } f \text{ is continuous}}{\overset{\uparrow}{=}}\; \text{open subset of } X.$$

■

The following proposition makes it much easier to show that a given map is in fact continuous.

Proposition 4.4. (Subbasis-Continuity Proposition) Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of Y (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds

$$f \text{ is continuous} \iff \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

Proof. Let $f\colon X \to Y$ be a map between topological spaces and let $\mathcal{C}$ be a subbasis for the topology of Y. As in the Produce Basis Lemma 3.55 we denote by $\mathcal{B} := \mathcal{B}(\mathcal{C})$ all subsets of Y that can be obtained by intersecting finitely many sets of $\mathcal{C}$.

The"$\Rightarrow$"-direction of the proposition follows from the observation that elements of a subbasis are in fact open. Thus let us turn to the"$\Leftarrow$"-direction. Now we assume that for each $C \in \mathcal{C}$ the preimage $f^{-1}(C)$ is open in X. We start out with a little observation: If we are given $C_1, \ldots, C_k \in \mathcal{C}$, then

$$f^{-1}(C_1 \cap \cdots \cap C_k) \overset{\text{by the Image-Preimage Lemma 1.9}}{=} \underbrace{f^{-1}(C_1)}_{\text{open in } X} \cap \cdots \cap \underbrace{f^{-1}(C_k)}_{\text{open in } X}$$

is also open in X. This shows, by definition, that for each $B \in \mathcal{B} = \mathcal{B}(\mathcal{C})$ the preimage $f^{-1}(B)$ is open in X.

Now we turn to the actual proof that f is continuous. Thus let $U \subseteq Y$ be an open subset. By the Union-of-Basis Sets Lemma 3.53 there exists a family $\{B_j\}_{j\in J}$ of sets in $\mathcal{B}$ with

$$U = \bigcup_{j\in J} B_j.$$

Then we see that

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{j\in J} B_j\Big) \overset{\text{by the Image-Preimage Lemma 1.9}}{=} \underbrace{\bigcup_{j\in J} \underbrace{f^{-1}(B_j)}_{\text{open in } X}}_{\substack{\text{union of open} \\ \text{sets, thus open in } X}}.$$

■

At times it is also useful to have a characterization of continuous maps in terms of closed subsets.

Lemma 4.5. (Continuity-via-Closed Subsets Lemma) A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed subset A in Y the preimage $f^{-1}(A)$ is closed in X.

Proof. This lemma is a straightforward consequence of the Image-Preimage Lemma 1.9 (8) and the definitions. We leave it to the reader to fill in the details. ■

For completeness' sake we also state the following characterization of continuous maps.

Lemma 4.6. (Closure-Interior Mapping Lemma) Let $f\colon X \to Y$ be a map between topological spaces.

(1) The following statements are equivalent:
   (a) The map is continuous.
   (b) For every subset $B \subseteq Y$ we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
   (c) For every subset $B \subseteq Y$ we have $f^{-1}(\mathring{B}) \subseteq$ interior of $f^{-1}(B)$.

(2) If f is continuous, then the following statements hold:
   (a) For every subset $A \subseteq X$ we have $f(\overline{A}) = \overline{f(A)}$.
   (b) For every subset $A \subseteq X$ we have $\operatorname{int}(f(\operatorname{int} A)) = \operatorname{int}(f(A))$.

Proof.

(1) We prove (a)⇒(b). Thus we assume that $f\colon X \to Y$ is continuous. Let $B \subseteq Y$. By the Continuity-via-Closed Subsets Lemma 4.5 we know that $f^{-1}(\overline{B})$ is a closed subset of X that contains $f^{-1}(B)$. By the Definition 3.21 of the closure this means that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$. We will prove the remaining statements in Exercise 4.14.

(2) (a) Evidently we have $f(A) \subseteq f(\overline{A})$. Thus we have $\overline{f(A)} \subseteq f(\overline{A})$. We now show the reverse inclusion $f(\overline{A}) \subseteq \overline{f(A)}$. Since $\overline{f(A)}$ is a closed subset it suffices to show that $f(\overline{A}) \subseteq \overline{f(A)}$. Let Z be a closed subset of Y that contains $f(A)$. We need to show that $f(\overline{A}) \subseteq Z$. First note that we have $f(A) \subseteq Z$. Thus we see that $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(Z)$. Since Z is closed and f is continuous we obtain from the Continuity-via-Closed Subsets Lemma 4.5 that $f^{-1}(Z)$ is closed. Thus we have $\overline{A} \subseteq f^{-1}(Z)$. This finally implies that $f(\overline{A}) \subseteq f(f^{-1}(Z)) \subseteq Z$.

(b) Evidently we have $f(\operatorname{int}(A)) \subseteq f(A)$. Thus we have $\operatorname{int}(f(\operatorname{int}(A))) \subseteq \operatorname{int}(f(A))$. We now show the reverse inclusion $\operatorname{int}(f(A)) \subseteq \operatorname{int}(f(\operatorname{int}(A)))$. Since $\operatorname{int}(f(\operatorname{int} A))$ is an open subset it suffices to show that $\operatorname{int}(f(A)) \subseteq f(\operatorname{int}(A))$. Let U be an open subset of Y that is contained in $f(A)$. We need to show that $U \subseteq f(\operatorname{int}(A))$. We have $f^{-1}(U) \subseteq f^{-1}(f(A)) \subseteq A$. Since f is continuous we know that $f^{-1}(U)$ is an open subset of X. Thus we see that $f^{-1}(U) \subseteq \operatorname{int}(A)$. Since U is contained in the image of X we see that $U = f(f^{-1}(U))$. Thus we see that $U = f(f^{-1}(U)) \subseteq f(\operatorname{int}(A))$. ■

By the Subspace Continuity Lemma 4.3 (2) we know that the restriction of a continuous map to any subspace is still continuous. The following lemma can be viewed as a converse: If "enough" restrictions of a given map are continuous, then the original map is also continuous.

Proposition 4.7. (Pasting Proposition) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If there exists an open cover $\{W_i\}_{i\in I}$ of X such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.

(2) If there exists a *locally finite* closed cover $\{A_i\}_{i\in I}$ of X such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

In practice often the following variation on (2) gets used:

(2′) If X is Hausdorff and if there exists a locally finite cover $\{A_i\}_{i\in I}$ of X such that each A_i is compact and such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

Remark.

(1) In practice the Pasting Proposition 4.7 (2) often gets used in the following way: Suppose we are given two topological spaces X and Y and closed subsets $A_1, \ldots, A_m$ of X such that $X = A_1 \cup \cdots \cup A_m$. Furthermore we are given continuous maps $f_i\colon A_i \to Y$, $i = 1, \ldots, m$. If the f_i agree on the overlaps, then the map

$$\begin{array}{ccll} X & \to & Y & \\ P & \mapsto & f_i(P) & \text{if } P \in A_i \end{array}$$

is well-defined and it follows from the Pasting Proposition 4.7 (2) that it is continuous.

(2) Note that in the Pasting Proposition 4.7 (1) we are allowed to deal with arbitrarily many open subsets whereas in the Pasting Proposition 4.7 (2) we are restricted to finitely many closed subsets. It is clear that the Pasting Proposition 4.7 (2) cannot

be generalized to arbitrarily many closed subsets, after all, the restriction of a map $f\colon \mathbb{R} \to \mathbb{R}$ to any one-point subset $\{x\} \subseteq \mathbb{R}$ is continuous, but clearly not every map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. □

The proof of the Pasting Proposition 4.7 rests on the shoulders of the following lemma, which will at times also be of independent interest.

Auxiliary Lemma 4.8. (Topology-Defined-by-Family Lemma) Let X be a topological space and let $\{X_i\}_{i\in I}$ be a family of subsets. The following two statements are equivalent:

(1) For every $U \subseteq X$ we have: U is open if each $U \cap X_i$ is open in X_i.
(2) For every $U \subseteq X$ we have: U is closed if each $U \cap X_i$ is closed in X_i.

Furthermore, if (1) or equivalently (2) holds, then also the following statement holds:

(3) A map $f\colon X \to Y$ to another topological space Y is continuous if each restriction $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof of Lemma 4.8.

(1)⇒(2) We assume that (1) holds. Now let $U \subseteq X$ be a subset. We have

$$\begin{array}{l} \text{for every } i \in I \text{ the set} \\ X_i \cap U \text{ is closed in } X_i. \end{array} \quad \begin{array}{c} \text{since } (X \setminus U) \cap X_i = X_i \setminus (X_i \cap U) \\ \downarrow \\ \Longrightarrow \\ \uparrow \\ \text{since (1) holds} \end{array} \quad \begin{array}{l} \text{for every } i \in I \text{ the set} \\ (X \setminus U) \cap X_i \text{ is open in } X_i \\ \Longrightarrow X \setminus U \text{ is open in } X \Longrightarrow U \text{ is closed in } X. \end{array}$$

(2)⇒(1) This proof is basically the same as the above argument.

(1)⇒(3) Let $f\colon X \to Y$ be a map to some topological space Y. We suppose that each $f|_{X_i}$ is continuous. Let $U \subseteq Y$ be an open subset. We need to show that $f^{-1}(U)$ is an open subset of X. By (1) it suffices to show that each intersection $f^{-1}(U) \cap X_i$ is an open subset of X_i. But this follows from the observation that $f^{-1}(U) \cap X_i = (f|_{X_i})^{-1}(U)$ and the hypothesis that $f|_{X_i}$ is continuous. ■

Proof of the Pasting Proposition 4.7. Let $f\colon X \to Y$ be a map between topological spaces.

(1) This statement follows immediately from the combination of the Cover-Open Subset Lemma 3.20 (1a) and the Topology-Defined-by-Family Lemma 4.8 (1)⇒(3).
(2) Similar to (1) this statement follows immediately from the combination of the Cover-Open Subset Lemma 3.20 (2a) and the Topology-Defined-by-Family Lemma 4.8 (2)⇒(3).
(2′) This statement follows from (2) and the Compact-Closed Lemma 3.39 (2). ■

The notion of continuity is of course already familiar from earlier analysis courses. The following proposition shows that for maps between metric spaces the earlier "$\epsilon-\delta$"–definition of continuity agrees with the above Definition 4.1 of continuity.

Proposition 4.9. (Metric Continuity Proposition) Let $f\colon X \to Y$ be a map between metric spaces. Then the following holds:

$$f \text{ is continuous} \iff \forall_{x\in X}\ \forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{x'\in B_\delta(x)}\ d(f(x), f(x')) < \epsilon.$$

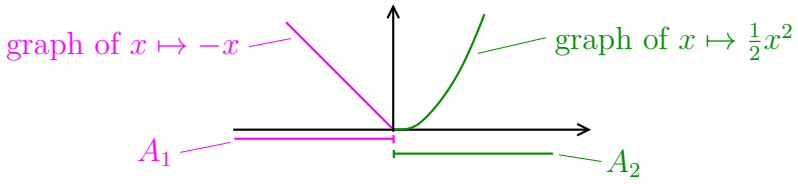
Example. Let $n \in \mathbb{N}$. For each $i \in \{1, \dots, n\}$ the projection

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \dots, x_n) & \mapsto & x_i \end{array}$$

is continuous since the "$\epsilon - \delta$"-condition is satisfied with $\delta = \epsilon$. □

Remark. We use the Metric Continuity Proposition 4.9 to import from real analysis the various continuous functions on $\mathbb{R}$ and $\mathbb{C}$, e.g. polynomials, trigonometric functions and the exponential function. The replacement of the classic "$\epsilon - \delta$"–definition of continuity by the definition in terms of open subsets can even be helpful in the context of real analysis. For example, consider the function

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases} \end{array}$$



With the "$\epsilon - \delta$"–definition it is slightly annoying to prove continuity at 0. But applying the Pasting Proposition 4.7 (2) to the closed subsets $A_1 = (-\infty, 0]$ and $A_2 = [0, \infty)$ it is totally evident that this function is continuous. □

The previous remark leads us straight to the following convention.

Convention 4.10. The purist will correctly complain that the definition of the map in the previous remark is irregular, since the two cases treated in the definition of the map f are not disjoint. Nonetheless, on numerous occasions we will use such irregular definitions, usually to highlight the fact that we can use the Pasting Proposition 4.7 to show that the map is continuous.

Proof of the Metric Continuity Proposition 4.9. Let $f\colon X \to Y$ be a map between metric spaces. We need to show that

$$f \text{ is continuous} \iff \underset{x \in X}{\forall} \underset{\epsilon > 0}{\forall} \underset{\delta > 0}{\exists} \underset{y \in B_\delta(x)}{\forall} d(f(x), f(y)) < \epsilon.$$

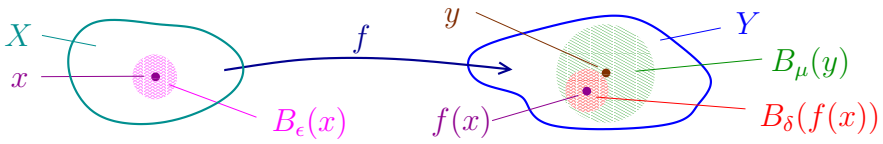
We prove the two directions separately:

"⇒" Let $x \in X$ and let $\epsilon > 0$. We consider $V := f^{-1}(B_\epsilon(f(x))) \subseteq X$. Since f is continuous we know that V is open. By the definition of the topology on a metric space there exists a $\delta > 0$ such that $B_\delta(x) \subseteq V$. This delta has the desired property.

"⇐" On page 162 we saw that the open balls form a basis for the topology of metric spaces. Thus by the Subbasis-Continuity Proposition 4.4 it suffices to show that if $y \in Y$ and $\mu > 0$, then $f^{-1}(B_\mu(y))$ is an open subset of X. By the definition of the topology on X it suffices to prove the following claim:

Claim. Given any $x \in f^{-1}(B_\mu(y))$ there exists an $\epsilon > 0$ with $B_\epsilon(x) \subseteq f^{-1}(B_\mu(y))$.

Proof. We set $\delta = \mu - d(f(x), y)$. Note that $\delta > 0$.



By the triangle inequality we have $B_\delta(f(x)) \subseteq B_\mu(y)$. Furthermore by our hypothesis on f there exists an $\epsilon > 0$ such that $f(B_\epsilon(x)) \subseteq B_\delta(f(x))$. Thus we see that $B_\epsilon(x) \subseteq f^{-1}(B_\mu(y))$. ■

The final lemma of this section shows that we can add and multiply continuous real-valued maps.

Lemma 4.11. (Sum-Product-Continuity Lemma) Let X be a topological space. If $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous maps, then the maps

$$\begin{array}{llll} X \to \mathbb{R} & X \to \mathbb{R} & X \to \mathbb{R} & X \to \mathbb{R} \\ x \mapsto f(x)+g(x) & x \mapsto f(x)\cdot g(x) & x \mapsto \max\{f(x),g(x)\} & x \mapsto \min\{f(x),g(x)\} \end{array}$$

are also continuous. The obvious analogs of the first two statements also hold if we replace $\mathbb{R}$ by $\mathbb{C}$.

Example. Let A be a real $(m \times n)$–matrix. It follows easily from the previous example, the Continuous Composition Lemma 4.2 and the Sum-Product-Continuity Lemma 4.11 that the following map is continuous:

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v. \end{array}$$

□

Proof. The proof of the lemma rests on two claims.

Claim 1. The maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x \cdot y \end{array}$$

are continuous.

Proof. The claim can for example be proved using the Metric Continuity Proposition 4.9. We will fill in the details in Exercise 4.38. ⊞

Claim 2. Let $f = (f_1, \ldots, f_n)\colon X \to \mathbb{R}^n$ be a map from a topological space X to $\mathbb{R}^n$. Then f is continuous if and only if each $f_i\colon X \to \mathbb{R}$ is continuous.

Proof. The claim can be proved, with some effort, using the Subbasis-Continuity Proposition 4.4 and the JH-Openness Criterion 3.3. We will fill in the missing details in Exercise 4.39. ⊞

Now we turn to the actual proof of the lemma. Thus let $f, g\colon X \to \mathbb{R}$ be two continuous maps. Note that the map $x \mapsto f(x) + g(x)$ equals the map

$$X \xrightarrow{x \mapsto (f(x),g(x))} \mathbb{R}^2 \xrightarrow{\alpha} \mathbb{R}.$$

By Claim 2 the first map is continuous and by Claim 1 we know that α is continuous. From the Continuous Composition Lemma 4.2 we deduce that the map $f+g$ is continuous. The same argument also takes care of the map $x \mapsto f(x) \cdot g(x)$ and of the setting where we replace $\mathbb{R}$ by $\mathbb{C}$. Finally note that for any $a, b \in \mathbb{R}$ we have

$$\max\{a,b\} = \tfrac{1}{2}(a+b+|a-b|) \quad \text{and} \quad \min\{a,b\} = \tfrac{1}{2}(a+b-|a-b|).$$

Using this observation one can now show easily, by a slight modification of the previous arguments, that the maps $x \mapsto \max\{f(x), g(x)\}$ and $x \mapsto \min\{f(x), g(x)\}$ are continuous. ■

4.2. **Open and closed maps.** We continue with the following straightforward definition.

Definition 4.12. Let $f\colon X \to Y$ be a continuous map between topological spaces.
(1) We say that f is open if the image of every open subset of X is open in Y.
(2) We say that f is closed if the image of every closed subset of X is closed in Y.

Examples.
(1) The projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is easily seen to be an open map, but it is not a closed map.[51]
(2) On the other hand the inclusion $i\colon \mathbb{R} \to \mathbb{R}^2$, $i(x) = (x, 0)$ is evidently not an open map, but it is a closed map. □

Lemma 4.13. (Open-Closed Inclusion Lemma) Let X be a topological space.
(1) If $U \subseteq X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.
(2) If $A \subseteq X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.

Proof.
(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subseteq X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1), we just need to appeal to the Subspace-Characterization Lemma 3.17 (2) to make the necessary adjustments. ■

We have the following analog to the Subbasis-Continuity Proposition 4.4.

Lemma 4.14. (Subbasis-Open Map Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of X (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds:

$$f \text{ is open} \iff \text{for each } C \in \mathcal{C} \text{ the image } f(C) \text{ is open in } Y.$$

Proof. The logic of the proof is very similar to the proof of the Subbasis-Continuity Proposition 4.4. Eventually the proof boils down to the Union-of-Basis Sets Lemma 3.53 and the Image-Preimage Lemma 1.9. ■

4.3. **The Compact Image Lemma.** Statement (1) of the following lemma is perhaps the most important feature of compact spaces.

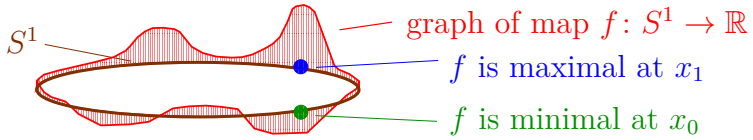
Lemma 4.15. (Compact Image Lemma)
(1) Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.

[51] Why is not a closed map?

(2) Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that

$$f(x_0) \;\leq\; f(x) \;\leq\; f(x_1) \quad \text{for all } x \in X.$$

(3) Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.



Remark. The statement of the Compact Image Lemma 4.15 (1) is really quite astounding. Let us elaborate: We already saw on page 179 that the image of an open (respectively closed) set under a continuous map is not necessarily open (respectively closed) and similarly for "closed" instead of "open". We will discuss such examples in greater detail in Exercise 4.36. □

Proof.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of the topological space $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have

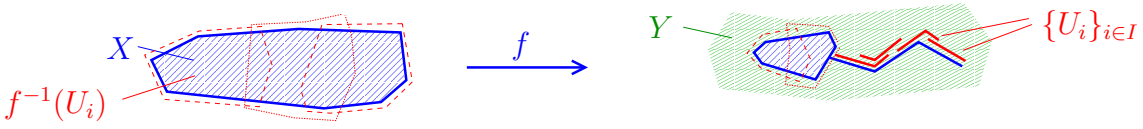
$$X \underset{\substack{\uparrow\\ \text{by the Image-Preimage}\\ \text{Lemma 1.9 (1)}}}{=} f^{-1}(f(X)) \subseteq f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \underset{\substack{\uparrow\\ \text{by the Image-Preimage}\\ \text{Lemma 1.9 (6)}}}{=} \bigcup_{i\in I} f^{-1}(U_i).$$

By hypothesis the map $f\colon X \to Y$ is continuous. It follows from the elementary Subspace Continuity Lemma 4.3 (3) that the map $f\colon X \to f(X)$ is also continuous, in particular each $f^{-1}(U_i)$ is an open subset of X. This observation together with the above discussion shows that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact there exist $i_1, \ldots, i_k \in I$ such that

$$X \;=\; f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k}).$$

Therefore

$$\begin{aligned} f(X) &= f\left(f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\substack{\uparrow\\ \text{by the Image-Preimage}\\ \text{Lemma 1.9 (3)}}}{=} f\left(f^{-1}(U_{i_1})\right) \cup \cdots \cup f\left(f^{-1}(U_{i_k})\right) \underset{\substack{\uparrow\\ \text{by the Image-Preimage}\\ \text{Lemma 1.9 (2)}}}{\subseteq} U_{i_1} \cup \cdots \cup U_{i_k}. \end{aligned}$$



(2) It follows from (1) and the Heine-Borel Theorem 3.44 that $f(X) \subseteq \mathbb{R}$ is bounded and closed. Since X is non-empty we thus see that $y_0 := \inf(f(X))$ and $y_1 := \sup(f(X))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$ have the desired properties.

(3) This statement is an immediate consequence of Statement (1) and of the Compact-Closed Lemma 3.39 (1). ■

The Compact Image Lemma 4.15 is one of the results that we will cite most often. In the remainder of this section we want to give a first application to the study of norms on finite-dimensional real vector spaces. This result is somewhat tangential to the main story of the lecture notes, but it falls into the category of "nice to know".

For the reader's convenience let us recall Definition 2.41:

Definition 4.16. Let V be a real vector space. A norm on V is a map $n\colon V \to \mathbb{R}_{\geq 0}$ that satisfies the following three properties:

(1) For every $v \in V$ we have $n(v) = 0 \iff v = 0$.
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $n(\lambda \cdot v) = |\lambda| \cdot n(v)$.
(3) For every $v, w \in V$ we have $n(v + w) \leq n(v) + n(w)$.

Recall that in the Norm–Metric Lemma 2.47 and the Metric-Topology Lemma 3.7 we saw that a norm gives rise to a metric and hence to a topology.

Definition 4.17. Let V be a real vector space. We say that two norms n and $\widetilde{n}$ on V are equivalent if there exist $c, C \in \mathbb{R}_{>0}$ such that for all $v \in V$ we have

$$c \cdot n(v) \leq \widetilde{n}(v) \leq C \cdot n(v).$$

Lemma 4.18. (Equivalent Norms–Same Topology Lemma) Let V be a real vector space.

(1) The notion of "equivalence" of norms on V is indeed an equivalence relation.
(2) Equivalent norms give rise to the same topology on V

Proof. Statement (1) is basically trivial. Statement (2) follows from the elementary Equivalent Metric Lemma 3.11. ∎

The following proposition is the promised result on norms on finite-dimensional real vector spaces.

Proposition 4.19. (Norm Equivalence Proposition) Any two norms on a finite-dimensional real vector space are equivalent.

Remark. In Exercise 4.7 we will see that the analogous statement for infinite-dimensional vector spaces does not necessarily hold. □

Proof. Evidently it suffices to consider the vector space $\mathbb{R}^k$. Furthermore note that by the Equivalent Norms–Same Topology Lemma 4.18 it suffices to show that every norm n on $\mathbb{R}^k$ is equivalent to the Euclidean norm $\| - \|$. As usual we denote by $\mathfrak{e}_1, \ldots, \mathfrak{e}_k$ the standard basis of $\mathbb{R}^k$. We set

$$C := \sqrt{n(\mathfrak{e}_1)^2 + \cdots + n(\mathfrak{e}_k)^2}.$$

Note that $C > 0$.

Claim 1. For every $v \in \mathbb{R}^k$ we have $n(v) \leq C \cdot \|v\|$.

Proof. Let $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$. We see that

$$\begin{aligned} n(v) = n(v_1 \cdot \mathfrak{e}_1 + \cdots + v_k \cdot \mathfrak{e}_k) &\overset{\text{since } n \text{ is a norm}}{\leq} |v_1| \cdot n(\mathfrak{e}_1) + \cdots + |v_k| \cdot n(\mathfrak{e}_k) \\ &\overset{\text{the Cauchy-Schwarz Inequality 2.28 applied to the standard inner product on } \mathbb{R}^k \text{ and the vectors } (|v_1|, \ldots, |v_k|) \in \mathbb{R}^k \text{ and } (n(\mathfrak{e}_1), \ldots, n(\mathfrak{e}_k)) \in \mathbb{R}^k, \text{ see also the example on page 123}}{\leq} \sqrt{(v_1^2 + \cdots + v_k^2) \cdot (n(\mathfrak{e}_1)^2 + \cdots + n(\mathfrak{e}_k)^2)} = \|v\| \cdot C. \end{aligned}$$

⊞

Since n is a norm it follows from Claim 1 together with the Metric Continuity Proposition 4.9 that $n\colon \mathbb{R}^k \to \mathbb{R}$ is continuous with respect to the standard topology on $\mathbb{R}^k$ given by the Euclidean norm. As we saw on page 158, it follows from the Heine-Borel Theorem 3.44 that $S^{n-1} = \{v \in \mathbb{R}^k \mid \|v\| = 1\}$ is compact. Therefore we know by the Compact Image Lemma 4.15 that there exists a $u \in S^{n-1}$ such that $n(u) \leq n(w)$ for all $w \in S^{n-1}$. Note that by Property (1) of the norm n we know that $n(u) > 0$. We set $c \coloneqq \frac{1}{n(u)}$. Evidently $c > 0$. At this stage it remains to prove the following claim:

Claim 2. For every $v \in \mathbb{R}^k$ we have $\|v\| \leq c \cdot n(v)$.

Proof. Let $v \in \mathbb{R}^k$. If $v = 0$, then there is nothing to show. So suppose that $v \neq 0$. We see that

$$\|v\| \underset{\substack{\uparrow\\ \text{definition of } c}}{=} \|v\| \cdot c \cdot n(u) \underset{\substack{\uparrow\\ \text{choice of } u}}{\leq} \|v\| \cdot c \cdot n\Big(\frac{v}{\|v\|}\Big) \underset{\substack{\uparrow\\ \text{since } n \text{ is a norm}}}{=} c \cdot n(v).$$

■

4.4. **Homeomorphisms.** We introduce the main objects of interest of this section.

Definition 4.20.

(1) We say that a map $f\colon X \to Y$ between two topological spaces X and Y is a homeomorphism if the following three properties are satisfied:
   (a) f is continuous,
   (b) f is bijective,
   (c) the inverse map $f^{-1}\colon Y \to X$ is also continuous.
   If there exists a homeomorphism between X and Y we say that X and Y are homeomorphic and sometimes we write $X \cong Y$.

(2) We say that a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$, where $f(X)$ is equipped with the subspace topology, is a homeomorphism. We say that the embedding is closed if $f(X)$ is a closed subset of Y and we say that the embedding is open if $f(X)$ is an open subset of Y.

(3) Given a topological space X we refer to the class of all topological spaces that are homeomorphic to X as the homeomorphism type of X.

Remark.

(1) If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology. For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.

(2) It follows from the Open-Closed Inclusion Lemma 4.13 that an open (respectively closed) embedding is also an open (respectively closed) map. □

Before we turn to our first examples we introduce some standard notation.

Notation 4.21. Given $n_0 \in \mathbb{N}$ we write

$$\begin{aligned} S^n_{\geq 0} &\coloneqq \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\} && \text{the upper hemisphere} \\ S^n_{\leq 0} &\coloneqq \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} \leq 0\} && \text{the lower hemisphere} \\ S^n_{=0} &\coloneqq \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} = 0\} && \text{the equator.} \end{aligned}$$

Similarly we define

$$\begin{aligned} B^n_{\geq 0} &:= \{(x_1, \ldots, x_n) \in B^n \mid x_n \geq 0\} \\ B^n_{\leq 0} &:= \{(x_1, \ldots, x_n) \in B^n \mid x_n \leq 0\} \end{aligned}$$

Finally, the same way we also define $\overline{B}^n_{\geq 0}$ and $\overline{B}^n_{\leq 0}$.



Examples.

(1) The maps

$$\begin{aligned} \varphi\colon \overline{B}^n &\to S^n_{\geq 0} \\ x &\mapsto \left(x, \sqrt{1 - \|x\|^2}\right) \end{aligned} \qquad \text{and} \qquad \begin{aligned} p\colon S^n_{\geq 0} &\to \overline{B}^n \\ (x_1, \ldots, x_n, x_{n+1}) &\mapsto (x_1, \ldots, x_n) \end{aligned}$$

are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.

(2) Let $A \in \mathrm{GL}(n, \mathbb{R})$ be an invertible real matrix. It follows from the discussion on page 178 that the map $\mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by A is a homeomorphism. This simple observation has a useful consequence: Let V be an n-dimensional real vector space. We pick an isomorphism $\varphi\colon \mathbb{R}^n \to V$ and we equip V with the unique topology that turns φ into a homeomorphism. By the above discussion this definition of a topology does not depend on the choice of φ. Unless we say something else any finite-dimensional real vector space will be equipped with this topology. Note that with these topologies any isomorphism between real vector spaces of the same finite-dimension is also a homeomorphism.

(3) Let $n \in \mathbb{N}$. It follows easily from the definitions of the standard metrics on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ that the map

$$\begin{aligned} \mathbb{R}^{2n} &\to \mathbb{C}^n \\ (x_1, y_1, \ldots, x_n, y_n) &\mapsto (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) \end{aligned}$$

is an isometry of metric spaces. Recall that on page 139 we defined the standard topology on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ to be the metric topology corresponding to the standard metrics. Thus we see that the above map is a homeomorphism. We will use this homeomorphism to make the identification $\mathbb{C}^m =_{\mathrm{i}} \mathbb{R}^{2m}$.[52]

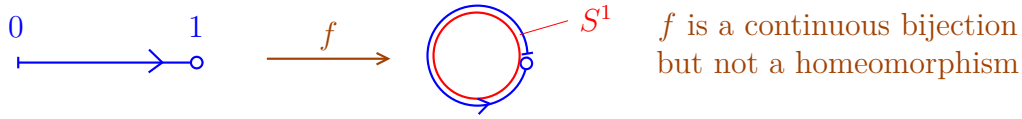
(4) We consider the map

$$\begin{aligned} f\colon [0,1) &\to S^1 \\ t &\mapsto \exp(2\pi \mathrm{i} t). \end{aligned}$$

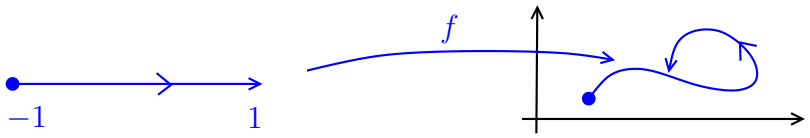
The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away

[52]By an "identification" we mean in these notes that we picked once and for all a homeomorphism. If in the following we write $\mathbb{C}^m =_{\mathrm{i}} \mathbb{R}^{2m}$, then this means that we use the given homeomorphism to go back and forth between these two topological spaces.

under the warm, but mistaken, impression that one now has found a homeomorphism.



(5) To drive home the point we made in the previous example we show in the figure below an injective continuous map $f\colon [-1,1) \to \mathbb{R}^2$ which is *not* an embedding.



Especially once we move to more abstract topological spaces it is easy to loose track of the fact that an injective continuous map is not necessarily an embedding. □

In many situations one is given an explicit continuous bijection $f\colon X \to Y$ between two topological spaces and one would like to show that f is in fact a homeomorphism. Unfortunately usually it is very painful to write down the explicit inverse map $f^{-1}\colon Y \to X$, let alone to verify that it is continuous. Fortunately we have several results which often help us in circumventing this difficulty.

Lemma 4.22. (Open-Closed–Injective Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that f is open or that f is closed.

(1) If f is an injection, then f is an open (respectively closed) embedding.
(2) If f is a bijection, then f is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces.

(2) We assume that f is a bijection. Furthermore we now assume that f is open. We want to show that f is a homeomorphism. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subseteq X$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by the Continuity-via-Closed Subsets Lemma 4.5.

(1) We assume that f is an injection and we assume that f is open. We write $Z := f(X)$. Since $f\colon X \to Y$ is open it follows basically immediately from the definition of the subspace topology that $f\colon X \to Z$ is also open. Thus we obtain from (2) that f is an embedding. Furthermore, since f is open we see that f is an open embedding. The proof in the case that f is closed is basically identical. ■

Example. Let U be an open subset of $\mathbb{R}$ and let $f\colon U \to \mathbb{R}$ be a strictly monotonously increasing map, i.e. for any $x < y$ in U we have $f(x) < f(y)$. Note that open intervals form a basis for the topology of U. By the Intermediate Value Theorem the image of an open interval under f is again an open interval. Thus it follows from the Subbasis-Open

Map Lemma 4.14 that $f\colon U \to \mathbb{R}$ is an open map. The Open-Closed–Injective Lemma 4.22 implies that $f\colon U \to f(U)$ is a homeomorphism. □

The next proposition is certainly one of the most frequently used results in topology.

Proposition 4.23. (Compact-Hausdorff Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. If X is compact and if Y is Hausdorff, then the following statements hold:

(1) The map $f\colon X \to Y$ is closed.
(2) If f is an injection, then f is a closed embedding.
(3) If f is a bijection, then f is a homeomorphism.

Remark.

(1) On page 183 we gave an example of a continuous bijective map $f\colon [0,1) \to S^1$ that is not a homeomorphism. This shows that we cannot drop the condition that X is compact from the proposition. But what about the "Hausdorff condition"? In Exercise 4.50 we will come up with an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.
(2) If X is non-compact, then it is often a major nuisance to try to show that a given map $f\colon X \to Y$ is in fact a closed embedding. Later on, in the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 and the Proper-Closed Proposition 4.64 we give two criteria which can often be used to show that an injective continuous map starting from a non-compact space to a Hausdorff space is a closed embedding after all. □

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff.

(1) Let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of Y. In fact we see that

$$A \subseteq X \text{ closed} \overset{\text{by the Compact-Closed Lemma 3.39 since } X \text{ is compact}}{\Longrightarrow} A \text{ is compact} \overset{\text{by the Compact Image Lemma 4.15 (1)}}{\Longrightarrow} f(A) \text{ is compact} \overset{\text{by the Compact-Closed Lemma 3.39 (2) since } Y \text{ is Hausdorff}}{\Longrightarrow} f(A) \subseteq Y \text{ is closed.}$$



(2) If f is an injection, then it follows immediately from (1) together with the Open-Closed–Injective Lemma 4.22 (1) that f is a closed embedding.
(3) This follows immediately from (2). ■

We formulate our first application of the Compact-Hausdorff Proposition 4.23 (3) as a lemma.

Lemma 4.24. (Stereographic Projection Lemma) Let $n \in \mathbb{N}$. We consider the map

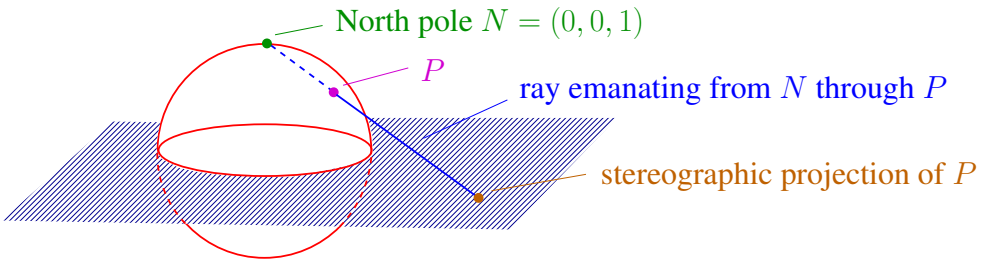
$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we introduced on page 152. We refer to Φ as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N = (0, \ldots, 0, 1) \in S^n$ to ∞.
(2) For any $P \in S^n$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^n$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^n \times \{0\}$ in $(\Phi(P), 0)$.
(3) For any $(v_1, \ldots, v_n, 0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ we have $\Phi(v_1, \ldots, v_n, 0) = (v_1, \ldots, v_n)$.
(4) The map Φ is a homeomorphism.
(5) (a) The restriction of Φ to a map $S^n \setminus \{N\} \to \mathbb{R}^n$ is also a homeomorphism, where the inverse is given by

$$\begin{array}{rcl} \mathbb{R}^n & \to & S^n \setminus \{N\} \\ x & \mapsto & \left(x \cdot \frac{2}{\|x\|^2+1}, \underbrace{\frac{\|x\|^2-1}{\|x\|^2+1}}_{=1-\frac{2}{\|x\|^2+1}}\right). \end{array}$$

(b) The restriction of Φ to a map $S^n_{\leq 0} \to \overline{B}^n$ is a homeomorphism and the restriction of Φ to a map $S^n_{\geq 0} \to (\mathbb{R}^n \cup \{\infty\}) \setminus B^n$ is a homeomorphism



Identification 4.25. Let $n \in \mathbb{N}$. On many occasions we will use the homeomorphism from the Stereographic Projection Lemma 4.24 to make the identification $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$. In particular we have the identification $S^2 =_{\mathrm{i}} \mathbb{C} \cup \{\infty\}$.

Proof.

(1) This statement follows immediately from the definition of Φ.
(2) This statement follows from an elementary calculation which we leave to the reader.
(3) This statement follows immediately from the definition of Φ.
(4) In Exercise 4.55 we will show that Φ is continuous. We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 3.44. Furthermore note that on page 152 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from the Compact-Hausdorff Proposition 4.23 (3) that Φ is a homeomorphism.
(5) It follows almost immediately from (4) that the two maps are homeomorphisms. One can easily verify that the inverse of Φ is indeed given by the map we provided. ■

The existence of non-compact spaces is one of those unfortunate facts of life. As we saw, in general the conclusion of the Compact-Hausdorff Proposition 4.23 does not hold if the domain is non-compact. The following proposition gives us a criterion for an injective continuous map to be a closed embedding which will deal with most cases that we will encounter.

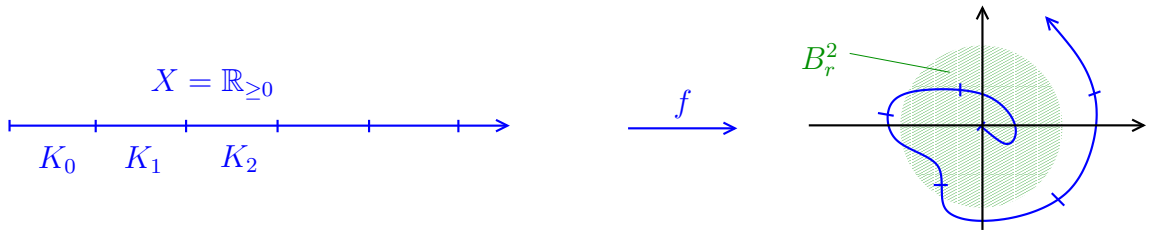
Proposition 4.26. (Closed Embedding-to-$\mathbb{R}^n$ Proposition) Let X be a topological space, let $n \in \mathbb{N}$ and let $f\colon X \to \mathbb{R}^n$ be an injective continuous map. Furthermore suppose that X is the union of compact subsets $\{K_i\}_{i\in I}$ such that the following condition is satisfied:

(∗) Given any $r > 0$ there exist only finitely many $i \in I$ with $f(K_i) \cap \overline{B}_r^n \neq \varnothing$.

Then f is a closed embedding.

Remark. Later we will prove the Proper-Closed Proposition 4.64 which gives a more general criterion for an injective continuous map to be a closed embedding. On page 213 we will use it to give a different, arguably more conceptual proof of the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26. □

Example. Let $X = \mathbb{R}_{\geq 0}$. We consider the "outward spiraling" map $f\colon X \to \mathbb{R}^2$ that is sketched in the figure below.



Given $i \in \mathbb{N}_0$ we set $K_i := [i, i+1]$. These compact subsets cover all of $X = \mathbb{R}_{\geq 0}$ and they satisfy the condition (∗) of the proposition. Thus we obtain from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 that the map f is a closed embedding. □

Proof. By the Open-Closed–Injective Lemma 4.22 (1) it remains to show that the map $f\colon X \to \mathbb{R}^n$ is closed. Thus let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of $\mathbb{R}^n$. Given $s \in \mathbb{N}_0$ we consider $C_s := \overline{B}_{s+1}^n \setminus B_s^n$. Clearly $\{C_s\}_{s\in\mathbb{N}_0}$ is a locally finite closed cover of $\mathbb{R}^n$. By the Cover-Open Subset Lemma 3.20 (2b) it suffices to show that each $C_s \cap f(A)$ is closed in $\mathbb{R}^n$. Thus let $s \in \mathbb{N}_0$. By hypothesis we know that $J := \{i \in I \mid f(K_i) \cap \overline{B}_s^n \neq \varnothing\}$ is a finite set. Now we see that

$$\begin{aligned} C_s \cap f(A) &= C_s \cap f\Big(\bigcup_{i\in I} K_i\Big) \overset{\text{by the definition of } J}{\downarrow =} C_s \cap f\Big(\bigcup_{j\in J} K_j\Big) \overset{\text{by the Compact–Union-Intersection Lemma 3.41 (1), since each } K_j \text{ is compact, and since } J \text{ is finite}}{\downarrow =} C_s \cap f(\text{compact subset}) \\ &\overset{\text{by the Compact Image Lemma 4.15}}{\uparrow =} C_s \cap \text{compact subset of } \mathbb{R}^n \overset{\text{by the Compact-Closed Lemma 3.39}}{\uparrow =} C_s \cap \text{closed subset of } \mathbb{R}^n \overset{\text{by the definition of the subspace topology}}{\uparrow =} \text{closed subset of } C_s. \end{aligned}$$

■

4.5. **Convex subsets.**

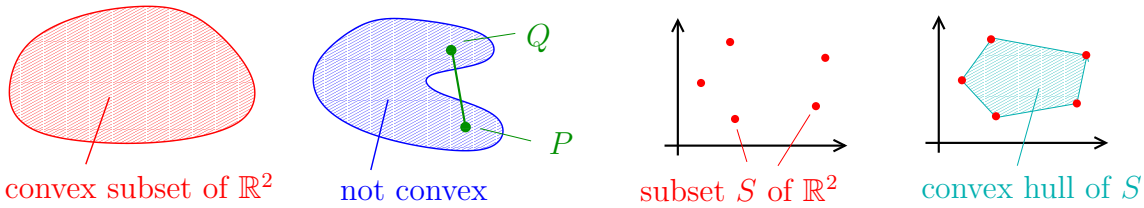
We start out with the definition of a convex subset:

Definition 4.27.

(1) Given $P, Q \in \mathbb{R}^n$ we refer to $\overline{PQ} := \{t \cdot P + (1-t) \cdot Q \,|\, t \in [0,1]\}$ as a Euclidean segment. Sometimes we also just say segment

(2) We say that a subset $A \subseteq \mathbb{R}^n$ is convex, if for every two points P and Q in A the Euclidean segment $\overline{PQ} = \{t \cdot P + (1-t) \cdot Q \,|\, t \in [0,1]\}$ also lies in A.

(3) Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.



Examples.

(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B^n_\epsilon(x)$ is convex. Indeed, let $P, Q \in B^n_\epsilon(x)$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} \|t\cdot P + (1-t)\cdot Q - x\| &= \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| \\ &\underset{\uparrow}{\leq} t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| \underset{\uparrow}{<} t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{aligned}$$

follows from the triangle inequality and the homogeneity of $\|-\|$

since $P, Q \in B^n_\epsilon(x)$

We have thus shown that $t \cdot P + (1-t) \cdot Q \in B^n_\epsilon(x)$. Basically the same way one shows that the closed ball $\overline{B}^n_\epsilon(x)$ is also convex.

(2) It is straightforward to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.

(3) In Exercise 4.46 we will show that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set $\left\{ \sum_{i=1}^n t_i \cdot P_i \,\middle|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1 \right\}$.

(4) One of the most useful properties of convex subsets is that the intersection of arbitrarily many convex subsets is, basically by definition, again convex. □

The following proposition gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Proposition 4.28. (Convex-to-Ball Proposition)

(1) Let A be a bounded *open* convex subset of $\mathbb{R}^n$ and let $Q \in A$.[53] We consider the map

$$\begin{aligned} \rho_A \colon A \setminus \{Q\} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sup\left\{ \|r \cdot (x-Q)\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot (x-Q) + Q \in A \right\}. \end{aligned}$$

This map is well-defined in the sense that the supremum exists for each $x \in A \setminus \{Q\}$. Furthermore the map ρ_A is continuous and the map

$$\begin{aligned} f_Q \colon A &\to B^n \\ x &\mapsto \begin{cases} (x-Q) \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{aligned}$$

is a homeomorphism with $f_Q(Q) = 0$.

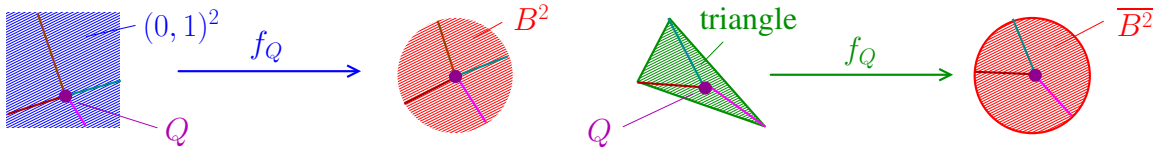
(2) Let A be a bounded *closed*[54] convex subset of $\mathbb{R}^n$. If Q is a point in the interior[55] of A, then the same expression as in (1) defines a homeomorphism[56]

$$f_Q \colon A \;\to\; \overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

with the following properties:

(a) the homeomorphism f_Q sends Q to the origin,

(b) the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.



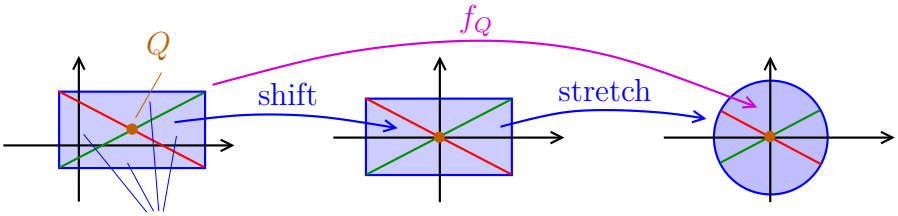
Example.

(1) Note that for $A = \overline{B}^n$ and $Q = 0$ the resulting homeomorphism $f_Q \colon \overline{B}^n \to \overline{B}^n$ is just the identity.

(2) For $i = 1, \ldots, n$ let $a_i < b_i$ be real numbers. We consider the corresponding "hyperrectangle" $[a_1, b_1] \times \cdots \times [a_n, b_n]$. It follows from the discussion on page 188 and from the Hyperrectangle-Compact Proposition 3.47 that the hyperrectangle is convex and compact. The Convex-to-Ball Proposition 4.28 (2) gives us an explicit homeomorphism

$$f_Q \colon [a_1, b_1] \times \cdots \times [a_n, b_n] \;\to\; \overline{B}^n$$

that sends the boundary of the hyperrectangle to the sphere S^{n-1} and that sends the "midpoint" $Q := (\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2})$ to the origin. Throughout this course we will use this homeomorphism to identify the left-hand side with the right-hand side. It follows easily from the explicit description of f_Q that the restriction of f_Q to each open cone on a side is smooth.[57]



the restriction of f_Q to each of the four open triangles is smooth

(3) Let A be a bounded convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.

---

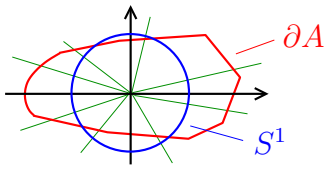
[53] The statement becomes slightly more readable if one sets $Q = 0$.

[54] Of course "bounded and closed" is in this instance, by the Heine-Borel Theorem 3.44 equivalent to "compact". In this instance we prefer to talk of "bounded closed" to stress the analogy to Statement (1) of the proposition.

[55] Note that the interior might well be empty, i.e. such a point Q might not exist.

[56] This says in particular that if A is a bounded closed convex subset of $\mathbb{R}^n$ with *non-empty interior*, then A is homeomorphic to $\overline{B}^n$.

[57] We leave it to the reader to make it precise what "open cone on a side" means. We will make use of this observation only once, and then it should be clear what we mean.



The Convex-to-Ball Proposition 4.28 (2), applied to $Q = 0$, implies in particular that the map

$$\begin{array}{rccc} f\colon \partial A & \to & S^{n-1} \\ x & \mapsto & \frac{x}{\|x\|} \end{array}$$

is a homeomorphism. □

Remark. The reader might be familiar with the concept of a diffeomorphism. In this context we will prove the following two variations on the Convex-to-Ball Proposition 4.28:

(1) In the Smooth Convex-to-Ball Proposition 30.22 we will show that every compact non-empty codimension-zero smooth submanifold of $\mathbb{R}^n$ is diffeomorphic to $\overline{B}^n$.
(2) In the Star Shaped-Diffeomorphism Type Theorem 82.2 we will show that every open convex subset of $\mathbb{R}^n$ is in fact diffeomorphic to $\mathbb{R}^n$, and thus diffeomorphic to B^n. The diffeomorphism that we will provide in that later instance is, in contrast to the above proposition, not explicit. □

The proof of the Convex-to-Ball Proposition 4.28 relies on the following lemma:

Lemma 4.29. (Distance-to Boundary Continuous Lemma) Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.

(1) The map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ defined in the Convex-to-Ball Proposition 4.28 is continuous.
(2) The map

$$\begin{array}{rccc} f_0\colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \end{array}$$

is continuous and it satisfies $f_0^{-1}(S^{n-1}) = \partial A$.

Proof of Lemma 4.29. We start out with the following claim:

Claim. The map $\rho_A\colon A\setminus\{0\} \to \mathbb{R}$ is continuous, it is bounded from above and it is bounded from below by 1.

Proof. First note that it follows from the hypothesis that A is bounded, that ρ_A is defined and that it is bounded from above. It follows from the definition of ρ_A that ρ_A is bounded from below by 1. Therefore it remains to show that ρ_A is continuous. Now let $x \in A \setminus \{0\}$ and let $\epsilon > 0$. It suffices to show the following two statements:

(1) there exists a neighborhood U of $x \in A \setminus \{0\}$ with $\rho_A(y) > \rho_A(x) - \epsilon$ for all $y \in U$,
(2) there exists a neighborhood V of $x \in A \setminus \{0\}$ with $\rho_A(y) < \rho_A(x) + \epsilon$ for all $y \in V$.

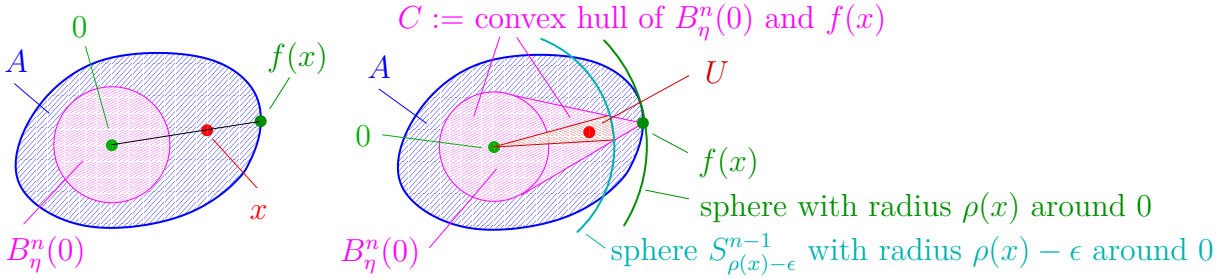
We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho_A(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho_A(x))$.

Since A is closed we have $f_0(x) \in A$. Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B^n_\eta(0) \subseteq A$. Since A is convex, the convex hull C of $B^n_\eta(0) \cup \{f_0(x)\}$ is

still contained[58] in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f_0(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho_A(x)-\epsilon}$ the sphere of radius $\rho_A(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho_A(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho_A(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho_A(x)-\epsilon}$ that is contained in C'. We set

$$U \;:=\; \{r \cdot z \,|\, z \in U' \text{ and } r \in (0,1)\}.$$

This is an open neighborhood of x and for any $y \in U$ we have $(\rho_A(x)-\epsilon)\cdot y \in A$, i.e. for any $y \in A$ we have $\rho_A(y) \geq \rho_A(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists.



The existence of V is proved in a very similar way. We refer to [**Berger2009**, Chapter 11.3] for full details. ⊞

From the continuity of ρ_A it follows that f is continuous on $A \setminus \{0\}$. From the fact that ρ_A is bounded it follows easily that f_Q is continuous in 0. Finally one can easily verify, using the convexity of A, that $f_0^{-1}(S^{n-1}) = \partial A$. We leave it to the reader to fill in the details. ■

Proof of the Convex-to-Ball Proposition 4.28.

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that $Q = 0$ is the origin, in particular we can assume that the origin 0 lies in the interior of A. By the Distance-to Boundary Continuous Lemma 4.29 we know that the map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ is defined and continuous. Next we consider the map

$$\begin{array}{rccl} f_0 \colon A & \to & \overline{B}^n & \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map is bijective[59] and by the Distance-to Boundary Continuous Lemma 4.29 we know that f_0 is continuous and that $f_0^{-1}(S^{n-1}) = \partial A$. Note that our hypothesis on A and the Heine-Borel Theorem 3.44 imply that A is compact. Thus we obtain from the Compact-Hausdorff Proposition 4.23 (3) that f_0 is in fact a homeomorphism. Finally note that it follows from $f_0^{-1}(S^{n-1}) = \partial A$ that f_0 restricts to a homeomorphism $\partial A \to S^{n-1}$.

[58] Why is that the case?

[59] It follows easily from the convexity of A, the hypothesis that $\mathring{A} \neq \varnothing$ and the definition of f_0 that for each $v \in S^{n-1}$ the map f_0 restricts to a bijection on the "ray defined by v", i.e. it restricts to a bijection

$$A \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\} \;\to\; \overline{B}^n \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\}.$$

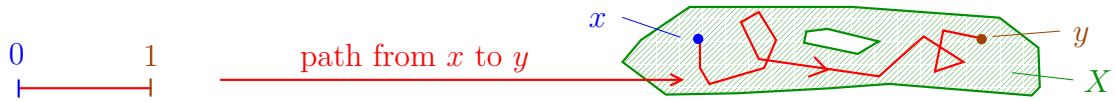
(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex we know by Exercise 4.10 that $A = \mathring{C}$. One verifies fairly easily that C is convex. It is clear that C is bounded and it follows from the Interior-Closure Lemma 3.22 that C is closed. One verifies easily that the map f_0 defined for $C = \overline{A}$ restricts to the map f_0 defined for A. Thus it follows from (2) that the map $f_0 \colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_0 \colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. ■

4.6. **The two notions of connected spaces.** In this section we will see that there are two notions of connectedness of a topological space. As we will see, for most "reasonable" topological spaces the two notions mercifully coincide. We start out with the more obvious definition.

Definition 4.30.

(1) Let X be a topological space. A path is a continuous map $p \colon [0,1] \to X$. Often we say that p is a path from $p(0)$ to $p(1)$.

(2) We say that a topological space X is path-connected if given any two points x and y there exists a path from x to y.



Before we discuss examples let us immediately provide the other, less obvious definition of connectedness.

Definition 4.31. We say that a topological space X is connected if the following holds[60][61]

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open subsets } U, V \implies X = U \text{ or } X = V.$$

A topological space which is not connected is called disconnected.

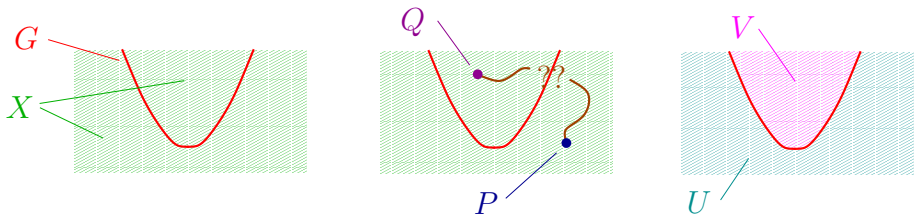
Example.

(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.

(2) Let $G \subseteq \mathbb{R}^2$ be the graph of the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. As in the figure below we consider the complement $X = \mathbb{R}^2 \setminus G$. It is pretty clear that X is not path-connected, surely there is no path from a point P below G to a point above G. But how can one show the non-existence of such a path?[62] But it is clear that X is not connected, since X is the disjoint union of the two open non-empty sets $U := \{(x,y) \in \mathbb{R}^2 \mid y < x^2\}$ and $V := \{(x,y) \in \mathbb{R}^2 \mid y > x^2\}$.

(3) It follows from the Intermediate Value Theorem that the rational numbers $\mathbb{Q}$ are not path-connected. But $\mathbb{Q}$ is also not connected, for example we can write the topological space $\mathbb{Q}$ as the disjoint union of the two open non-empty subsets $\{x \in \mathbb{Q} \mid x^2 < 2\}$ and $\{x \in \mathbb{Q} \mid x^2 > 2\}$.

[60] Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V, i.e. if $X = U \cup V$ and if $U \cap V = \varnothing$.
[61] This definition was first introduced by Felix Hausdorff [**Hausd1914**, p. 244].
[62] The fact that such a path does not exist can be shown, with some dexterity, using real analysis. How?



So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round. □

It is basically obvious that any interval is path-connected. But as we will see, it takes some effort to show that the interval $[0,1]$ is connected.

Proposition 4.32. (Interval-Connected Proposition) The interval $[0,1]$ is connected.

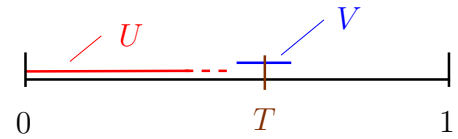
Proof. Let U and V be two disjoint open subsets of $[0,1]$ with $U \cup V = [0,1]$. Without loss of generality we can assume that $0 \in U$. We want to show that $U = [0,1]$. We consider

$$A \coloneqq \big\{ t \in [0,1] \,\big|\, [0,t] \subseteq U \big\}.^{63}$$

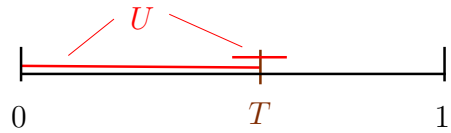
The set A contains 0, hence it is non-empty. Evidently the set is bounded thus we can consider $T \coloneqq \sup A$. We need to show that $T = 1$ and that $T \in U$.

First let us suppose that $T \notin U$. This implies that $T \in V$. Since V is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1]$ is also contained in V. In particular we have $(T-\epsilon, T+\epsilon) \cap A = \varnothing$ which implies that $\sup A \leq T - \epsilon$. Thus we have obtained a contradiction.

Now we know that $T \in U$. We still need to show that $T = 1$. Since U is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1] \subseteq U$.



if $T \in V$ then $(T-\epsilon, T+\epsilon) \cap [0,1]$ is also contained in V



since $T \in U$ we see that $[0, T+\epsilon) \cap [0,1]$ is contained in U

Thus we set that $[0, T+\epsilon) \cap [0,1] \subseteq U$. In light of the definition of T that is only possible if $T = 1$. ■

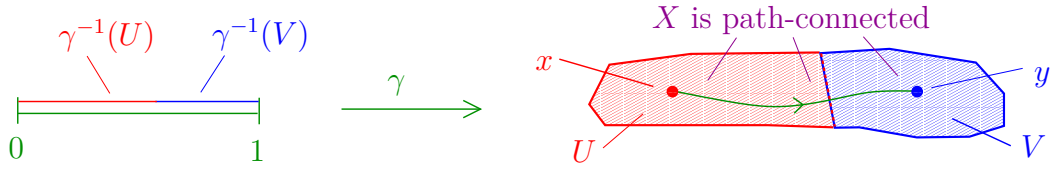
Corollary 4.33. (Path-Connected-Implies-Connected Corollary) Every path-connected topological space is also connected.

Proof. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since

[63] Put differently,

$$A \;=\; \text{the set of all } t \in [0,1] \text{ such that the interval } [0,t] \text{ is contained in } U.$$

X is path-connected there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0,1]$ into two disjoint open non-empty subsets.



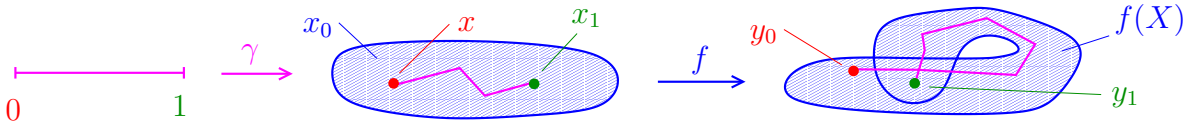
But that is a contradiction to the Interval-Connected Proposition 4.32. ■

Lemma 4.34. (Image-Connected Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces.

(1) If X is path-connected, then $f(X)$ is also path-connected.
(2) If X is connected, then $f(X)$ is also connected.

Proof.

(1) We suppose that X is path-connected. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$ be two points in $f(X)$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to x_1. But then $f \circ \gamma\colon [0,1] \to f(X)$ is a path from $f(x_0)$ to $f(x_1)$.



(2) We suppose that $f(Y)$ is disconnected. This means that there exist disjoint open non-empty subsets U and V of $f(Y)$ with $f(Y) = U \cup V$. By the Image-Preimage Lemma 1.9 we know that $f^{-1}(U)$ and $f^{-1}(V)$ are also disjoint and we know that $X = f^{-1}(U) \cup f^{-1}(V)$. By definition of a continuous map $f^{-1}(U)$ and $f^{-1}(V)$ are open. Finally, since $U, V \subseteq f(X)$ we see that $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Thus we have shown that X is also disconnected. ■

In the following lemma we show that most of the topological spaces that we had encountered so far are path-connected and connected. We will use this lemma on numerous occasions without explicitly referring to it.

Lemma 4.35. (Ball-Sphere-Connected Lemma)

(1) Let $n \geq 1$. Every open ball $B_r^n(y)$, every closed ball $\overline{B}_r^n(y)$ and $\mathbb{R}^n$ are path-connected and connected.
(2) Every sphere $S_r^n(y)$ with $n \geq 1$ is path-connected and connected.

Proof. By the Path-Connected-Implies-Connected Corollary 4.33 it suffices to show that all these topological spaces are path-connected.

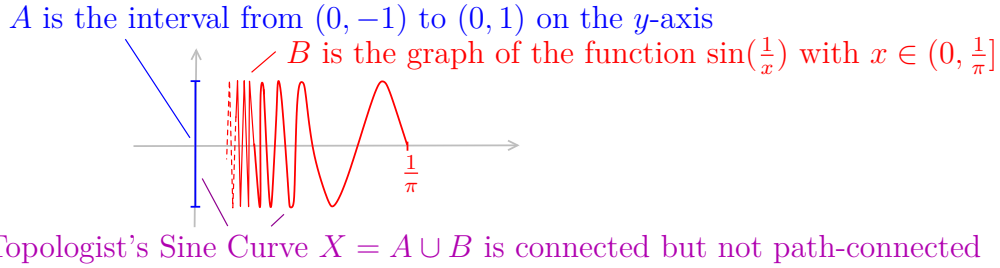
(1) It follows from the example on page 188 that all balls (open or closed) and $\mathbb{R}^n$ are convex. But as we already pointed out on page 192, convex subsets are basically by definition path-connected.
(2) We will show in Exercise 4.63 that spheres of dimension ≥ 1 are path-connected. ■

For better or worse, the converse to the Path-Connected-Implies-Connected Corollary 4.33 is not always correct.

Example. The *Topologist's Sine Curve* is defined as the following topological space

$$X \coloneqq \{(0,y)\,|\,y\in[-1,1]\}\cup\{(x,\sin(\tfrac{1}{x}))\,|\,x\in(0,\tfrac{1}{\pi}]\} \subseteq \mathbb{R}^2,$$

equipped with the subspace topology.



In other words, the Topologist's Sine Curve is the union of the graph of $x\mapsto\sin(\frac{1}{x})$ for $x\in(0,\frac{1}{\pi}]$ and an interval on the y-axis. It is a very instructive exercise, see Exercise 4.56, to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected. □

Remark. The fact that an example of a topological space that is connected but not path-connected is so "wild" indicates that perhaps for most "reasonable" spaces the two notions of connectedness and path-connectedness agree. In fact in basically all examples of topological spaces that we encounter throughout these notes the two notions of connectedness will agree. □

By a clever use of the notion of connectedness one can sometimes prove statements which at first glance have nothing to do with connectedness. For example we can now prove the following lemma:

Lemma 4.36. ($\mathbb{R}$-$\mathbb{R}^2$-Non Homeomorphism Lemma) Let $n\in\mathbb{N}_{\geq 2}$.
(1) The topological spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.
(2) The topological spaces $[0,1]$ and $[0,1]^n$ are not homeomorphic.

Remark.
(1) The statement of the $\mathbb{R}$-$\mathbb{R}^2$-Non Homeomorphism Lemma 4.36 is not overly surprising. But note that in the Space-Filling Curve Theorem 10.21 we will see that for any $n\in\mathbb{N}$ there exists a *surjective* map $[0,1]\to[0,1]^n$.
(2) Later on, in the $\mathbb{R}^k$-$\mathbb{R}^l$-Non Homeomorphism–Proposition 149.6, we will study the question whether for $k\neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic. □

Proof.
(1) Let us suppose that there exists a homeomorphism $f\colon\mathbb{R}\to\mathbb{R}^n$. Let $P\in\mathbb{R}$ be a point. Then f restricts to a homeomorphism from $\mathbb{R}\setminus\{P\}$ to $\mathbb{R}^2\setminus\{f(P)\}$. The topological space $\mathbb{R}\setminus\{P\}=(-\infty,P)\cup(P,\infty)$ is not path-connected[64], while one can easily verify that $\mathbb{R}^n\setminus\{f(P)\}$ is path-connected. But as we remarked on page 182, homeomorphic spaces have the same topological properties. In particular two homeomorphic spaces

[64]Why is this topological space not path-connected? How can one show this?

are either both path-connected or they are both not path-connected. Thus we have obtained a contradiction.

(2) The proof of this statement is fairly similar to the proof of (1). We leave it to the reader to fill in the details. ■

We continue with the following elementary but useful lemma.

Lemma 4.37. (Connected-to-Discrete Lemma) Every continuous map $f\colon X \to Y$ from a connected space X to a discrete space is constant.

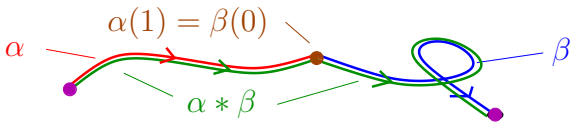
Remark. One could argue that the Connected-to-Discrete Lemma 4.37 is the lemma in topology that is most relevant to real life. More precisely, an immediate consequence of the Connected-to-Discrete Lemma 4.37 is that any *non-constant* map $(a, b) \to A$ from an interval to a finite set is not continuous. This is of huge significance since humanity likes to squeeze a real-parameter (e.g. time, intelligence, performance) into a finite set of values (e.g. historical stages, grades in school and exams, hired-or-fired). Any such function is non-continuous and hence very problematic. □

Proof. We need to show that for every $x \in X$ we have $f(\{x\}) = f(X)$. Since Y is a discrete space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. ■

Before we continue we introduce the concatenation of two paths.

Definition 4.38. Let X be a topological space and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by[65]

$$\begin{array}{rccl} \alpha * \beta\colon [0,1] & \to & X & \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta((2t-1)), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \end{array}$$



The concatenation of paths will play a major role throughout these notes, especially when we introduce the fundamental group of a pointed topological space. In a more humble fashion it also plays a minor role in the proof of the following lemma, which gives a criterion for the union of (path-) connected subsets to be (path-) connected.

Lemma 4.39. (Union Connected Lemma) Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap\limits_{i\in I} B_i$ is non-empty.

(1) If each B_i, $i \in I$ is path-connected, then $\bigcup\limits_{i\in I} B_i$ is path-connected.

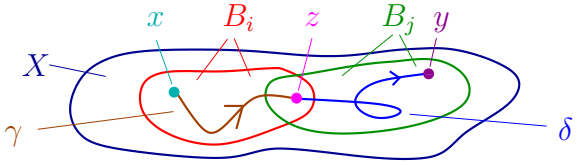
(2) If each B_i, $i \in I$ is connected, then $\bigcup\limits_{i\in I} B_i$ is connected.

[65] It follows from the Pasting Proposition 4.7 (2) and our hypothesis $\alpha(1) = \beta(0)$ that $\alpha * \beta\colon [0,1] \to X$ is a continuous map.

Remark. In Exercises 4.29, 10.12 and 23.9 we will provide a partial converse to the Union Connected Lemma 4.39. □

Proof. Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ contains a point z. We write $Z := \bigcup_{i\in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma\colon [0,1] \to B_i$ from x to z and we can pick a path $\delta\colon [0,1] \to B_j$ from z to y. Note that the concatenation of the paths γ and δ is a path from x to y.



(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \varnothing$ we see that $B_i = U \cap B_i$. Thus we have shown that

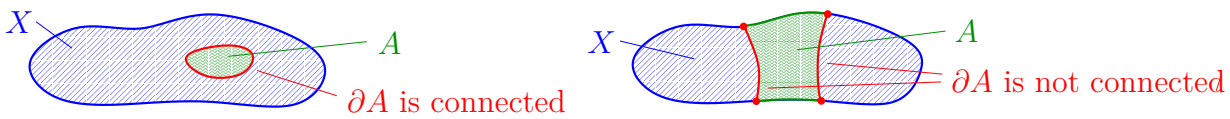
$$U \;=\; \bigcup_{i\in I} (B_i \cap U) \;=\; \bigcup_{i\in I} B_i \;=\; Z.$$

■

Sometimes it is also useful to have criteria which show that removing a subset does not change connectedness:

Lemma 4.40. (Complement Connected Lemma) Let X be a topological space and let $A \subseteq X$ be a subset.

(1) If X is path-connected and if ∂A is path-connected, then $X \setminus \mathring{A}$ is also path-connected.
(2) If X is connected and if ∂A is connected, then $X \setminus \mathring{A}$ is also connected.



Proof. We will prove the Complement Connected Lemma 4.40 in Exercise 4.58. ■

4.7. **The (path-) components of a topological space.** This section is a follow-up on the previous section. Let us start with the following simple-minded definition.

Definition 4.41. Let x and y be two points in a topological space X. We say that the points x and y are path-equivalent, if there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 4.42. (Path-Equivalence Lemma) For every topological space X the notion of path-equivalence is indeed an equivalence relation.

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive can be shown by concatenating paths. More precisely, if $\alpha\colon [0,1] \to X$ is a path from $P = \alpha(0)$ to $Q = \alpha(1)$ and if $\beta\colon [0,1] \to X$ is a

path from $Q = \beta(0)$ to $R = \beta(1)$, then the concatenation $\alpha * \beta \colon [0,1] \to X$ is a path from P to R. ■

The Path-Equivalence Lemma 4.42 allows us to introduce the following definition:

Definition 4.43. Let X be a topological space.

(1) We call the path-equivalence classes of points the path-components of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X.[66] Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.



Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ defines a path-component of $\mathbb{Q}$. □

Lemma 4.44. (Path-Component Lemma) Let X be a topological space.

(1) As a set, X is the disjoint union of its path-components.
(2) Each path-component of X is path-connected.
(3) Each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.
(4) Let A be a subset of X.
    (a) The following two statements are equivalent:
        (i) No point in A is path-equivalent to a point in $X \setminus A$.
        (ii) The subset A is the union of path-components of X.
    (b) If A is path-connected, then A is contained in a path-component of X.
    (c) If A is path-connected and if no point in A is path-equivalent to a point in $X \setminus A$, then A is a path-component of X.

Proof. The lemma follows immediately from the definitions and basic properties of equivalence relations, see the Equivalence Class Lemma 1.19. We leave it to the reader to fill in the few remaining details. ■

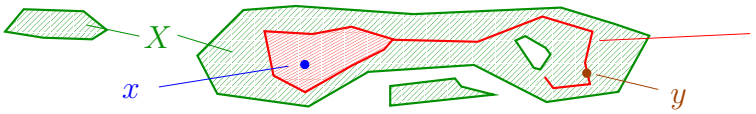
Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. It follows from the Intermediate Value Theorem that no point in $\mathbb{R}_{<0}$ is path-equivalent to a point in $\mathbb{R}_{>0}$. Thus it follows from the Path-Component Lemma 4.44 (4c) that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R} \setminus \{0\}$. □

Our next goal is to define the "components" of a topological space that should play the same role as the path-components, i.e. we want to introduce the notion of "components" such that the statements of the Path-Component Lemma 4.44 hold, but with "path-connected" replaced by "connected" and "path-component" replaced by "component".

We start out with the following definition:

[66] The notation "$\pi_0(X)$" might look rather odd at the moment, but it will become clearer later on, when we introduce the fundamental groups and the higher homotopy groups, why this is a perfectly sensible notation. At that point even the weird definition of "$\pi_0(X) = 0$" will start making more sense.

Definition 4.45. Let x and y be two points in a topological space X. We say that the points x and y are connectable, if there exists a connected subset of X that contains both x and y.



Lemma 4.46. (Connectable Lemma) For every topological space X the notion of being connectable is indeed an equivalence relation.

Proof. All properties of an equivalence relation are trivial except for transitivity. So suppose that x and y are connectable and that y and z are connectable. By definition there exists a connected subset A that contains x and y and there exists a connected subset B that contains y and z. By the Union Connected Lemma 4.39 (2) the union $A \cup B$ is connected and it thus follows that x and z are connectable. ■

The Connectable Lemma 4.46 allows us to introduce the following definition:

Definition 4.47. Let X be a topological space. We call the equivalence classes of connectable points on X the components of X.

Examples.

(1) In Exercise 4.56 we will show that the Topologist's Sine Curve, that we illustrated in the figure on page 195, has two path-components but only a single component.
(2) We consider $X = \mathbb{Q}$ with the subspace topology coming from $\mathbb{R}$. In Exercise 4.28 we show that the only connected subsets of $\mathbb{Q}$ are the subsets consisting of at most a point. It follows that the components of $\mathbb{Q}$ are precisely the one-point subsets. □

The following is in particular the promised analog of the Path-Component Lemma 4.44.

Lemma 4.48. (Component Lemma) Let X be a topological space.
(1) As a set X is the disjoint union of its components.
(2) Each component of X is connected.
(3) Each component C of X is a maximal connected subset of X, i.e. if $C \subsetneq D$, then D is not connected.
(4) Let A be a subset of X.
  (a) If C is a component of X, then $A \cap C$ is a union of components of A.
  (b) If A is closed and open in X, then A is the union of components of X.
  (c) If A is closed and open in X and if A is connected, then A is a component of X.
(5) Each component of X is the union of path-components of X.

Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. By the Path-Connected-Implies-Connected Corollary 4.33 we know that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are also connected. Since these two subsets are closed and open in $\mathbb{R} \setminus \{0\}$ we deduce from the Component Lemma 4.48 (4c) that the two subsets are the components of $\mathbb{R} \setminus \{0\}$. □

Proof. Let X be a topological space.

(1) This statement is clear since components are defined to be equivalence classes, see also the Equivalence Class Lemma 1.19.

(2) Now let C be a component of X. We pick $c \in C$. For each $d \in C$ there exists by definition a connected subset Y_d of X that contains c and d. We have

$$C = \bigcup_{c \in C} \{c\} \subseteq \bigcup_{c \in C} Y_c \subseteq C \quad \text{which implies that} \quad C = \bigcup_{c \in C} Y_c.$$

by definition all points in Y_d are connectable to c, thus we have $Y_d \subseteq C$

It follows from the Union Connected Lemma 4.39 (2) that C is connected.

(3) This statement follows immediately from the definitions.

(4) Let A be a subset of X.

(a) Let C be a component of X. Note that if two points of A are connectable in A, then they are also connectable in X. This implies that no point in $A \cap C$ is connectable to a point of $A \setminus (A \cap C)$. The statement now follows from (1) applied to A.

(b) Suppose that A is closed and open. Let C be a component of X. Note that $A \cap C$ is a subset of C that is closed and open. By (2) we know that C is connected, hence we see that either $A \cap C = \varnothing$ or $A \cap C = C$. This shows that A is precisely the union of all the components of X that A intersects.

(c) This statement follows immediately from (b).

(5) Let C be a component of X and let P be a path-component of X with $P \cap C \neq \varnothing$. We need to show that $P \subseteq C$. We do so as follows:

(a) Note that P is connected by the Path-Connected-Implies-Connected Corollary 4.33.

(b) It follows from (a) and the Union Connected Lemma 4.39 that $P \cup C$ is connected.

(c) It follows from (b) and (3) that $P \cup C \subseteq C$. This of course implies that $P \subseteq C$. ■

We continue our discussion of the components of a topological space with the following, possibly slightly counterintuitive, lemma.

Lemma 4.49. (Components-are-Closed Lemma) Let X be a topological space.

(1) Every component of X is a closed subset of X.

(2) If X has only finitely many components, then each component is also an open subset of X.[67]

Example. We consider $X = \mathbb{Q}$. As we mentioned on page 199, in this case every point is already a component. We see that every component is closed, but in this case no component is an open subset. Therefore we cannot drop the condition on the number of components in Statement (2) of the lemma. □

Proof. Let X be a topological space.

(1) Let C be a component of X. We need to show that $X \setminus C$ is open. By the JH-Openness Criterion 3.3 it suffices to show that every $P \in X \setminus C$ admits a neighborhood that does not intersect C. Thus let $P \in X \setminus C$. By the Component Lemma 4.48 (3) we know that $C \cup \{P\}$ is not connected. Therefore there exists a decomposition $C \cup \{P\} = U \sqcup V$

[67] In the Locally Path-Connected Lemma 4.57 we give a criterion which often can be used to show that components are in fact open.

with U and V open subsets such that both U and V are non-empty. Without loss of generality $C \cap U \neq \varnothing$. Since C is connected and since $C = (U \cap C) \sqcup (V \cap C)$ we see that $C = U \cap C$. Since V is also non-empty we see that $V = \{P\}$. By the definition of the subspace topology that means that there exists an open subset W of X such that $W \cap (C \cup \{P\}) = V = \{P\}$. But this shows that W is an open neighborhood of P that does not intersect C. We have thus shown that the complement of C is open.



(2) Now suppose that X has only finitely many components. Let C be a component of X. We already know from (1) that C is closed. By definition the complement of C is the union of all other components. Since there are only finitely many and since they are closed by (1) it follows that the complement of C is closed, i.e. C is open. ■

We conclude this section with the following lemma that can be viewed as a generalization of the Image-Connected Lemma 4.34 and the Connected-to-Discrete Lemma 4.37.

Lemma 4.50. (Image-Preimage-Component Lemma) Let $f\colon X \to Y$ be a continuous map between two topological spaces.

(1) Let C be a (path)-component of Y. The preimage $f^{-1}(C)$ is the union of (path)-components of X.
(2) Let B be a (path)-component of X. There exists a (path)-component C of Y with $f(B) \subseteq C$.

Proof.

(1) We only need to show that if a point $a \in f^{-1}(C)$ is path-equivalent (respectively connectable) to a point $b \in X$, then $b \in f^{-1}(C)$. If a and b are path-equivalent (connectable), then as in the Image-Connected Lemma 4.34 we see that $f(a)$ and $f(b)$ are path-equivalent (connectable), thus $f(a)$ and $f(b)$ both lie in C. Therefore a and b both lie in $f^{-1}(C)$.



(2) This statement is proved by the same elementary argument as in (1). ■

4.8. **The (path-) components of matrix groups.** In this section we will discuss the (path-) connectedness of certain matrix groups over $\mathbb{R}$ and over $\mathbb{C}$. First we introduce the main players.

Notation 4.51. Given a commutative ring R we write

$$\begin{aligned} \mathrm{GL}(n,R) &:= \{A \in \mathrm{M}(n \times n, R) \mid \det(A) \text{ is a multiplicative unit in } R\} \\ \mathrm{SL}(n,R) &:= \{A \in \mathrm{M}(n \times n, R) \mid \det(A) = 1\}. \end{aligned}$$

For $\mathbb{K} = \mathbb{R}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{GL}_+(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) > 0\} \\ \mathrm{GL}_-(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) < 0\} \\ \mathrm{O}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^{\mathrm{T}}\cdot A = \mathrm{id}\} \\ \mathrm{SO}(n) &:= \mathrm{O}(n)\cap \mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^{\mathrm{T}}\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

Furthermore for $\mathbb{K} = \mathbb{C}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{U}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^{\mathrm{T}}\cdot A = \mathrm{id}\} \\ \mathrm{SU}(n) &:= \mathrm{U}(n)\cap \mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^{\mathrm{T}}\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

The following convention explains in what sense these matrix groups are topological spaces.

Identification 4.52. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $m, n \in \mathbb{N}$ we make the identification

$$\mathrm{M}(m\times n,\mathbb{K}) =_{\mathrm{i}} \mathbb{K}^{m\cdot n}$$

by "juxtaposing" the rows of the matrix. We equip $\mathbb{K}^{m\cdot n}$ and thus also $\mathrm{M}(m\times n,\mathbb{K})$ with the standard topology that we introduced on page 139. We equip each subset of $\mathrm{M}(m\times n,\mathbb{K})$ with the corresponding subspace topology that we introduced in Definition 3.15.

Example. Using the Compact-Hausdorff Proposition 4.23 (3) one can easily verify that the following maps are homeomorphisms:

$$\begin{array}{rcl} S^1 & \to & \mathrm{SO}(2) \\ (x,y) & \mapsto & \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} S^1 & \to & \mathrm{SU}(1) \\ (x,y) & \mapsto & (x + y\mathrm{i}). \end{array}$$

□

Lemma 4.53. (Matrix Group Lemma) Let $n \in \mathbb{N}$.

(1) $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$, are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.
(4) $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathrm{M}(n\times n,\mathbb{C})$.
(5) $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{C})$.
(6) $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact.

Remark. We will generalize the Matrix Group Lemma 4.53 (1) in Exercise 4.49. □

Proof. We consider the following two maps

$$\begin{array}{rccl} \det\colon & \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R} \\ & A & \mapsto & \det(A) \end{array} \quad \text{and} \quad \begin{array}{rccl} \varphi\colon & \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathrm{M}(n\times n,\mathbb{R}) \\ & A & \mapsto & A^{\mathrm{T}}\cdot A. \end{array}$$

It follows from the Leibniz Formula 2.18, the fact that by the discussion on page 177 each projection from $\mathrm{M}(n\times n,\mathbb{R})$ onto an entry is continuous, and the rather elementary Sum-Product-Continuity Lemma 4.11 that both maps are continuous.

(1) It follows from the above that $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$, $\mathrm{GL}_+(n,\mathbb{R}) = \det^{-1}((0,\infty))$ and $\mathrm{GL}_-(n,\mathbb{R}) = \det^{-1}((-\infty,0))$ are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) Note that it follows from the above and the Continuity-via-Closed Subsets Lemma 4.5 that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(\{1\})$, $\mathrm{O}(n) = \varphi^{-1}(\{\mathrm{id}\})$ and $\mathrm{SO}(n) = \mathrm{SL}(n,\mathbb{R})\cap \mathrm{O}(n,\mathbb{R})$ are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.

(3) Note that each column of an orthogonal matrix has norm one. This implies that $\mathrm{O}(n) \subseteq \mathbb{R}^{n^2}$ is a bounded subset of $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$. It follows from this observation together with (2) and the Heine-Borel Theorem 3.44 that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.

The proofs of Statements (4), (5) and (6) are almost identical to the above proofs. ■

The following proposition will hopefully quench the reader's thirst for information on the (path-) connectedness of the matrix groups.

Proposition 4.54. (Matrix Group-π_0-Proposition) Let $n \in \mathbb{N}$.

	topological space	path-components
(1)	$\mathrm{GL}(n, \mathbb{R})$	path-components are $\mathrm{GL}_+(n, \mathbb{R})$ and $\mathrm{GL}_-(n, \mathbb{R})$
(2)	$\mathrm{SL}(n, \mathbb{R})$	path-connected
(3)	$\mathrm{O}(n)$	path-components are $\mathrm{SO}(n)$ and $\mathrm{O}(n) \cap \mathrm{GL}_-(n, \mathbb{R})$
(4)	$\mathrm{GL}(n, \mathbb{C})$	path-connected
(5)	$\mathrm{SL}(n, \mathbb{C})$	path-connected
(6)	$\mathrm{U}(n)$	path-connected
(7)	$\mathrm{SU}(n)$	path-connected.

Furthermore the path-components are also the components.

Remark.

(1) In Exercise 299.11 we will provide a somewhat different proof of some of the statements of the Matrix Group-π_0-Proposition 4.54. An alternative proof of the Matrix Group-π_0-Proposition 4.54 is also given in [**Bak2002**, Chapter 9.2].

(2) In the Matrix Groups (Smooth) Deformation Retract Propositions 19.4 and 41.15 we collect more information on the relationships between the topological spaces that are given by the various matrix groups. □

We break the proof of the Matrix Group-π_0-Proposition 4.54 into two parts. First we consider the matrix groups defined over $\mathbb{R}$, afterwards we consider the matrix groups defined over $\mathbb{C}$.

Proof of Proposition 4.54 (1), (2) and (3). The proof of the three statements rests mostly on the following three claims.

Claim 1. Given any matrix $A \in \mathrm{GL}_+(n, \mathbb{R})$ there exists a path $\alpha\colon [0,1] \to \mathrm{GL}_+(n, \mathbb{R})$ with $\alpha(0) = A$ and with $\alpha(1) \in \mathrm{SL}(n, \mathbb{R})$.

Proof. Let $A \in \mathrm{GL}_+(n, \mathbb{R})$. Since $\det(A) > 0$ we can consider the path that is given by $\alpha(t) = \det(A)^{-t/n} \cdot A$. Evidently $\alpha(0) = A$ and one can easily verify that $\det(\alpha(1)) = 1$ ⊞

Claim 2. Given any $A \in \mathrm{SL}(n, \mathbb{R})$ there exists a path $\beta\colon [0,1] \to \mathrm{SL}(n, \mathbb{R})$ with $\beta(0) = A$ and $\beta(1) \in \mathrm{SO}(n)$.

Proof. First we apply the Gram-Schmidt Orthonormalization Process 2.32 to the column vectors $v_1, \ldots, v_n$ of the given matrix $A \in \mathrm{SL}(n, \mathbb{R})$. For $i = 0, \ldots, n$ we obtain sets of vectors $u_{i,1}, \ldots, u_{i,n} \in \mathbb{R}^n$. For $i = 0, \ldots, n$ we write $A_i := (u_{i,1} \ \ldots \ u_{i,n}) \in \mathrm{M}(n \times n, \mathbb{R})$. By construction $A_0 = A$. Note that it follows from Properties (1), (2), (3) and (4) stated in the Gram-Schmidt Orthonormalization Process, together with the fact that the determinant of an orthogonal matrix is ± 1, that each matrix A_i lies in $\mathrm{SL}(n, \mathbb{R})$ and that $A_n \in \mathrm{SO}(n)$.

Let $i \in \{1, \dots, n\}$. We will now show that there exists path in $\mathrm{SL}(n, \mathbb{R})$ from A_{i-1} to A_i. To simplify the notation we write $u_1, \dots, u_n$ instead of $u_{i-1,1}, \dots, u_{i-1,n}$. First we consider the path

$$\begin{aligned} \varphi\colon [0,1] &\to \mathrm{SL}(n,\mathbb{R}) \\ t &\mapsto \left(u_1 \ \dots \ u_{i-1} \ \ \frac{u_i}{\lambda(t)} \ \ \lambda(t)\cdot u_{i+1} \ \ u_{i+2} \ \dots \ u_n\right) \text{ where } \lambda(t) = (1-t) + t\cdot\|u_i\|. \end{aligned}$$

We write $\widetilde{u}_i \coloneqq \frac{u_i}{\|u_i\|}$ and $\widetilde{u}_{i+1} \coloneqq \|u_i\| \cdot u_{i+1}$. Next we consider the path

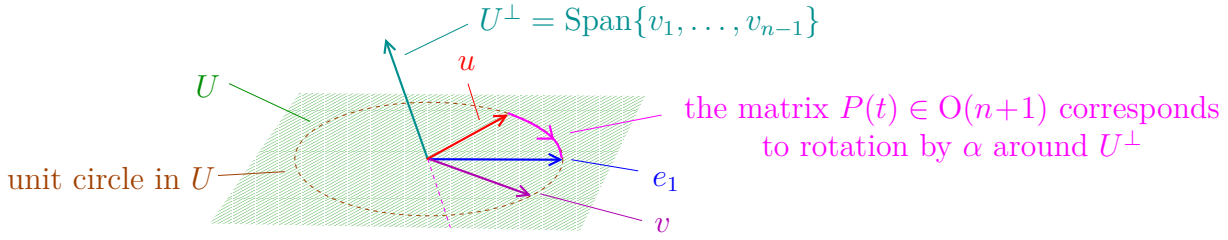
$$\begin{aligned} \psi\colon [0,1] &\to \mathrm{SL}(n,\mathbb{R}) \\ t &\mapsto \left(u_1 \ \dots \ u_{i-1} \ \ \frac{z(t)}{\|z(t)\|} \ \ \|z(t)\|\cdot\widetilde{u}_{i+1} \ \dots \ u_n\right) \text{ where } z(t) = \widetilde{u}_i - t\cdot\sum_{j=1}^{i-1}\langle\widetilde{u}_i, u_j\rangle\cdot u_j. \end{aligned}$$

Note that $\varphi(0) = A_{i-1}$ and note that $\varphi(1) = \psi(0)$. A short moment's thought shows that $\psi(1) = A_n$. Thus the concatenation of the paths φ and ψ is a path in $\mathrm{SL}(n, \mathbb{R})$ from A_{i-1} to A_i. The concatenation of these finitely many paths is the desired path $\beta\colon [0,1] \to \mathrm{SL}(n,\mathbb{R})$ from A to the orthogonal matrix $A_n \in \mathrm{SO}(n)$. ⊞

Claim 3. Given any $A \in \mathrm{SO}(n)$ there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = A$ and $\gamma(1) = \mathrm{id}_n$.

Proof. We prove this claim by induction on n. The case $n = 1$ is clear since $\mathrm{SO}(1)$ consists of precisely the matrix $(1) = \mathrm{id}_1$. Now suppose that the statement holds for some $n \geq 1$. Let $A \in \mathrm{SO}(n+1)$. We denote by $u \in \mathbb{R}^{n+1}$ the first column of the matrix A. We write $\mathfrak{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$. Since $n+1 \geq 2$ there exists a vector $v \in \mathbb{R}^{n+1}$ with $\|v\| = 1$, such that $u \perp v$ and such that $\mathfrak{e}_1 \in \mathrm{span}\{u, v\}$. We set $U \coloneqq \mathrm{span}\{u, v\}$. Since u and v are orthonormal there exists an $\alpha \in \mathbb{R}$ such that $\mathfrak{e}_1 = \cos(\alpha)\cdot u + \sin(\alpha)\cdot v$. We extend u, v to an orthonormal basis $u, v, w_1, \dots, w_{n-1}$. Given $t \in [0,1]$ we consider the matrix

$$P(t) \ \coloneqq \ Q \cdot \underbrace{\begin{pmatrix} \cos(\alpha\cdot t) & -\sin(\alpha\cdot t) & 0 \\ \sin(\alpha\cdot t) & \cos(\alpha\cdot t) & 0 \\ 0 & 0 & \mathrm{id}_{n-1} \end{pmatrix}}_{\in \mathrm{O}(n+1)} \cdot Q^{-1} \qquad \text{where } Q = \underbrace{(u \ v \ w_1 \ \dots \ w_{n-1})}_{\in \mathrm{O}(n+1)}.$$



Next we consider the path

$$\begin{aligned} \gamma\colon [0,1] &\to \mathrm{SO}(n+1) \\ t &\mapsto P(t)\cdot A. \end{aligned}$$

Evidently $\gamma(0) = A$. An elementary calculation shows that the first column of the matrix $\gamma(1) \in \mathrm{SO}(n+1)$ equals $\mathfrak{e}_1$. Since $\gamma(1)$ is orthogonal this implies that there exists an $\widetilde{A} \in \mathrm{SO}(n)$ with

$$\gamma(1) \ = \ \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{A} \end{pmatrix}.$$

By induction hypothesis there exists a path $\widetilde{\delta}\colon [0,1] \to \mathrm{SO}(n)$ from $\widetilde{A}$ to id_n. By "adding a 1 at the top left" this path $\widetilde{\delta}$ gives rise to a path δ in $\mathrm{SO}(n+)$ from $\gamma(1)$ to id_{n+1}. The concatenation of the two paths γ and δ is the promised path in $\mathrm{SO}(n+1)$ from A to the identity matrix id_{n+1}. ⊞

Now we turn to the actual proofs of the statements (1), (2) and (3) of the Matrix Group-π_0-Proposition 4.54.

(1) It follows from Claims 1, 2 and 3, together with the fact that path-equivalence is an equivalence relation, that $\mathrm{GL}_+(n,\mathbb{R})$ is path-connected. Next note that multiplication by a matrix in $\mathrm{GL}_-(n,\mathbb{R})$ gives rise to a homeomorphism $\mathrm{GL}_+(n,\mathbb{R}) \to \mathrm{GL}_-(n,\mathbb{R})$. Thus it follows that $\mathrm{GL}_-(n,\mathbb{R})$ is also path-connected.

Next we consider the continuous map $f\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}\setminus\{0\}$ that is given by the determinant. On page 198 we saw that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R}\setminus\{0\}$. It follows from the above together with the Image-Preimage-Component Lemma 4.50 that $\mathrm{GL}_-(n,\mathbb{R}) = f^{-1}(\mathbb{R}_{<0})$ and $\mathrm{GL}_+(n,\mathbb{R}) = f^{-1}(\mathbb{R}_{>0})$ are the path-components of $\mathrm{GL}(n,\mathbb{R})$.

This discussion, together with the Path-Connected-Implies-Connected Corollary 4.33 and the Component Lemma 4.48 shows that $\mathrm{GL}_-(n,\mathbb{R})$ and $\mathrm{GL}_+(n,\mathbb{R})$ are also the components of $\mathrm{GL}(n,\mathbb{R})$.

(2) It follows from Claims 2 and 3 that $\mathrm{SL}(n,\mathbb{R})$ is path-connected. By the Path-Connected-Implies-Connected Corollary 4.33 this implies that $\mathrm{SL}(n,\mathbb{R})$ is connected.

(3) Note that it follows from Claim 3 that $\mathrm{SO}(n,\mathbb{R})$ is path-connected. By basically the same argument as in (1) we see that $\mathrm{O}(n)$ has precisely two path-components, namely $\mathrm{SO}(n)$ and $\mathrm{O}(n)\cap\mathrm{GL}_-(n,\mathbb{R})$, and by the same argument as above one sees that these are also the components of $\mathrm{O}(n)$. ■

We turn to the proof of Proposition 4.54 for the matrix groups that are defined over $\mathbb{C}$:

Proof of Proposition 4.54 (4), (5), (6) and (7).

Claim 4.

(1) Given any $A \in \mathrm{GL}(n,\mathbb{C})$ there exists a path $\delta\colon [0,1] \to \mathrm{GL}(n,\mathbb{C})$ with $\delta(0) = A$ and $\delta(1) \in \mathrm{SL}(n,\mathbb{C})$.

(2) If $A \in \mathrm{U}(n)$, then we can find such a path that takes values in $\mathrm{U}(n)$.

Proof. Let $A \in \mathrm{GL}(n,\mathbb{C})$. We pick $r \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{R}$ with $\det(A) = r\cdot\exp(\mathrm{i}\varphi)$. The map

$$\begin{array}{rcl} \delta\colon [0,1] & \to & \mathrm{GL}(n,\mathbb{C}) \\ t & \mapsto & r^{-t/n}\cdot\exp(-\mathrm{i}\varphi\cdot t/n)\cdot A \end{array}$$

is a path from A to a matrix in $\mathrm{SL}(n,\mathbb{C})$. If $A \in \mathrm{U}(n)$, then $r = 1$ and we see that the path takes values in $\mathrm{U}(n)$. ⊞

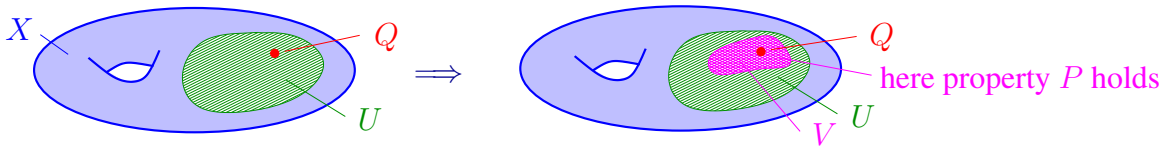
Using Claim 4 one sees that it suffices to show that $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$ are connected. This is achieved by making straightforward modifications to the Claims 2 and 3 that we considered in the proof of the real case. We leave it to the reader to fill in the details. ■

4.9. **Local and regional properties.** As so often, let us start out with a definition.

Definition 4.55. Let P be a property of topological spaces and let X be a topological space.

(1) We say that X is regionally P if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.
(2) We say that X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.

Note that by definition a topological space that is locally P is also regionally P.



Warning. Our usage of the expression "locally P" is non-standard. In fact most textbooks, see e.g. [**Hat2002, Bre1993, Rot1988**], define "locally P" as what we call "regionally P". (Note though that, as is apparent from [**Mun1975**, p. 161], in [**Mun1975**] our notion of "locally P" is being used.) The property that we now call "locally P" does not seem to have an established name. For our purpose it is useful to give names to both concepts since both have their uses and justifications. Later on we will also introduce notion of a local property of a map, in this context most books implicitly or explicitly seem to work with open neighborhoods. Thus it seems reasonable to us to use the word "local" whenever we deal with open neighborhoods and to rename the concept when any neighborhood is allowed. □

Examples.

(1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subseteq U$. The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.
(2) The topological space $X = (-2, 1) \cup [3, 6] \subseteq \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected.[68] On the other hand $X = \mathbb{Q}$ is not even locally path-connected.
(3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 152, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is locally Hausdorff. □

The following lemma can at times be useful.

Lemma 4.56. (Basis-with-Property Lemma) Let P be a property of topological spaces and let X be a topological space. If X is locally P, then the open subsets of X that have P form a basis for the topology of X.

Proof. This lemma follows easily from the Basis Characterization Lemma 3.57 (1). ■

The following lemma is the key to showing that for "reasonable" topological spaces the notions of being connected and being path-connected are actually the same.

Lemma 4.57. (Locally Path-Connected Lemma) Let X be a topological space.
(1) If X is locally path-connected, then the following statements hold:
  (a) X is connected if and only if X is path-connected.
  (b) The components of X are precisely the path-components of X.

[68] Why is that?

(2) If X is locally path-connected, then it is also locally connected.
(3) If X is locally connected, then every component of X is a closed and an open subset of X.[69]

Proof.

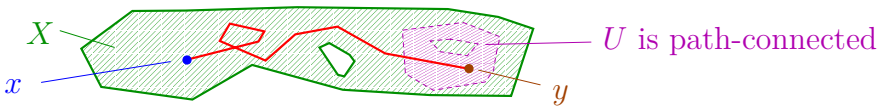
(1) First note that if X is path-connected, then we know by the Path-Connected-Implies-Connected Corollary 4.33 that X is connected. This gives us the cheap "if"-direction of (a).

We now assume that X is locally path-connected. Note that the "only if"-direction of (a) is a special case of (b). Thus it remains to prove (b). We start out with the following claim:

Claim 1. Each path-component of X is open.

Proof. Let P be a path-component. We pick $x \in X$. To show that P is open it suffices by the JH-Openness Criterion 3.3 to show that each $y \in P$ admits a neighborhood that is contained in P.

Thus let $y \in P$. We pick a path $p\colon [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. Since X is locally path-connected there exists an open neighborhood U of y that is path-connected.



It follows from the Image-Connected Lemma 4.34 (1) together with the Union Connected Lemma 4.39 that $p([0,1]) \cup U$ is path-connected. It follows from the Path-Component Lemma 4.44 (4b) that $p([0,1]) \cup U \subseteq P$. Finally note that since U is open we know that $p([0,1]) \cup U$ is a neighborhood of y. ⊞

Claim 2. Each path-component of X is closed.

Proof. By Claim 1 we now know that each path-component is open. By the Path-Component Lemma 4.44 (1) we know that the complement of a path-component is the union of path-components. Since the union of open subsets is open we see that each path-component is also closed. ⊞

Now let C be a component of X. By the Component Lemma 4.48 (2) and (5) we know that C is connected and that it is the union of a family of path-component $\{P_i\}_{i\in I}$. By Claim 1 and Claim 2 we know that each P_i is open and connected. But since C is connected this implies that I consists of a single element, i.e. C is a path-component of X.

(2) This statement follows from the Path-Connected-Implies-Connected Corollary 4.33.
(3) Let C be a component of X. By the Components-are-Closed Lemma 4.49 we know that C is a closed subset of X. We now assume that X is locally connected. We want to show that C is an open subset of X. By the Subspace-Characterization Lemma 3.17 it suffices to show that each $x \in C$ admits an open neighborhood that is contained in C. Thus let $x \in C$. Since X is locally connected we see that there exists an open neighborhood U of $x \in X$ such that U is connected. It follows immediately from the

[69] Recall that on page 199 we saw that for the topological space $X = \mathbb{Q}$ every point defines a component. Thus in general components are, very annoyingly, not necessarily open.

definition of a component, see Definition 4.47, that every point in U is also contained in C. ■

In the following we discuss the notion of being regionally compact.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is regionally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B_\epsilon^n(x) \subseteq U$. The closed ϵ-ball $\overline{B}_{\frac{\epsilon}{2}}^n(x)$ is a neighborhood of x and it is compact by the Heine-Borel Theorem 3.44.

(2) Similar to (1) one can easily show that every interval is regionally compact. In fact later on we will mostly need the fact that the interval $[0, 1]$ is regionally compact. □

In Exercise 4.66 we will see that there exists a topological space that is compact but not regionally compact. The following lemma shows that this slightly unintuitive phenomenon cannot occur if we restrict ourselves to topological spaces that are Hausdorff.

Lemma 4.58. (Compact Hausdorff-Regionally Compact Lemma) Let X be a Hausdorff space.

(1) If X is compact, then X is also regionally compact.

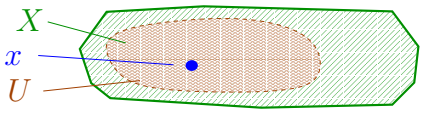
(2) If given any point $P \in X$ there exists a compact neighborhood of P, then X is regionally compact.

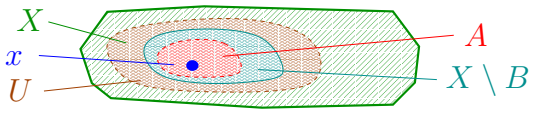
Proof.

(1) Let X be a compact Hausdorff space. We need to show that X is regionally compact. Thus let $x \in X$ and let U be a neighborhood of x.

Claim. There exists an open neighborhood A of x with $\overline{A} \subseteq U$.

Proof. Note that it follows from the hypothesis that X is Hausdorff and the Points-in-Hausdorff Lemma 3.32 that $\{x\}$ is a closed subset of X. Furthermore note that by the Compact Hausdorff-Normal Lemma 3.49 we know that the compact Hausdorff X is normal. Thus we can separate the closed subsets $\{x\}$ and $X \setminus \mathring{U}$ by open neighborhoods A and B. More precisely, there exist disjoint open neighborhoods A of $\{x\}$ and B of $X \setminus \mathring{U}$.

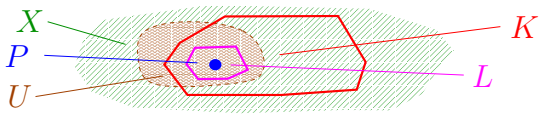




It remains to show that $\overline{A} \subseteq U$. Note that $A \subseteq X \setminus B \subseteq U$. Since B is open we know that $X \setminus B$ is closed. By Definition 3.21 of the closure $\overline{A}$ we have $\overline{A} \subseteq X \setminus B$. ⊞ Finally note, since $\overline{A}$ is a closed subset of the compact space X we know by the Compact-Closed Lemma 3.39 (1) that $\overline{A}$ is compact. We are done.

(2) Let X be a Hausdorff space such that every $P \in X$ admits a compact neighborhood. We need to show that X is regionally compact. Thus let $P \in X$ and let U be a neighborhood of P.

By hypothesis we know that P admits a compact neighborhood K. By (1) we know that K is regionally compact, which implies that there exists a compact neighborhood L of $P \in K$ with $P \in L \subseteq U \cap K$. ■



Our next definition is a closely related concept.

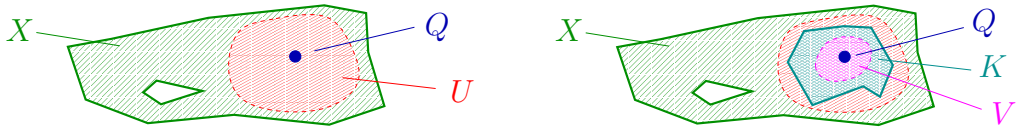
Definition 4.59. Let X be a topological space. A subset $A \subseteq X$ is called precompact if the closure $\overline{A}$ is compact.

Lemma 4.60. (Regionally Compact–Locally Precompact Lemma) Let X be a Hausdorff space that is regionally compact.

(1) Given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q such that $\overline{V}$ is compact and such that $Q \in V \subseteq \overline{V} \subseteq U$.
(2) X is locally precompact.

Proof. Let X be a Hausdorff space that is regionally compact.

(1) Let $Q \in X$ and let U be a neighborhood of Q. Since X is regionally compact we know that there exists a compact neighborhood K of Q with $K \subseteq U$. Since K is a neighborhood of Q we know that there exists an open neighborhood V of Q with $Q \in V \subseteq K$. Since X is Hausdorff we know by the Compact-Closed Lemma 3.39 (2) that K is a closed subset of X. Hence $\overline{V} \subseteq K$. Since $\overline{V}$ is a closed subset of the compact space K we obtain from the Compact-Closed Lemma 3.39 (1) that $\overline{V}$ is compact.



(2) It follows immediately from (1) that X is locally precompact. ■

We conclude the discussion of regionally compact spaces with another health warning.

Warning. Recall that what we call "regionally P" is usually called "locally P". Even more confusing, in the literature frequently "locally compact" is used with different definitions. For example in [**HY1988**, p. 71] and [**Dug1966**, p. 237] a topological space is called "locally compact" if every point admits an open neighborhood such that its closure is compact. Furthermore in [**SS1978**, p. 20], [**Bre1993**, p. 31] and [**Jos1983**, p. 266] a topological space is called "locally compact" if every point is contained in a compact neighborhood. The latter two notions of "locally compact" are in general different, see e.g. [**SS1978**, Example 52]. Furthermore note that for the above two concepts of "locally compact" every compact space is already locally compact. But as we mentioned above, in Exercise 4.66 we will see that there exists a compact space that is not regionally compact. Nonetheless, it is a "metatheorem" that all concepts of "locally compact" in the literature agree for Hausdorff spaces and they agree with our notion of "regionally compact". □

Next we turn from properties of topological spaces to properties of maps.

Definition 4.61. Let P be a property of maps between topological spaces. Furthermore let $f\colon X \to Y$ be a map between topological spaces.

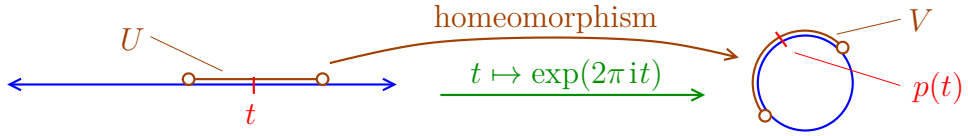
(1) We say that $f\colon X \to Y$ is locally P at $x \in X$ if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.
(2) We say that f is locally P if it is locally P at any $x \in X$.

Example.

(1) The map

$$\begin{array}{rccc} p\colon & \mathbb{R} & \to & S^1 \\ & t & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a local homeomorphism. Indeed, let $t \in \mathbb{R}$. We set $U = (t - \frac{1}{4}, t + \frac{1}{4})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore it is not difficult to show that p is an open map, see also Exercise 4.44. It follows from the Open-Closed–Injective Lemma 4.22 (2) that $p\colon U \to V$ is a homeomorphism.



(2) The statement of the Pasting Proposition 4.7 (1) can be reformulated as saying that a map between two topological spaces is continuous, if it is locally continuous. □

The following lemma gives a convenient criterion for showing that a bijective continuous map is actually a homeomorphism.

Lemma 4.62. (Bijective-Local Homeomorphism Lemma) If a map $f\colon X \to Y$ between topological spaces is bijective and a local homeomorphism, then it is actually a homeomorphism.

Proof. It follows immediately from the hypothesis that both f and f^{-1} are locally continuous. But then it follows from the Pasting Proposition 4.7 that both f and f^{-1} are actually continuous. ■

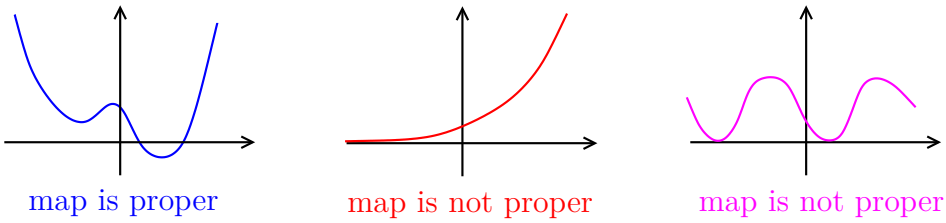
4.10. **Proper maps and closed embeddings.** Perhaps the best loved result in topology is the Compact-Hausdorff Proposition 4.23 (3) which says that a bijective continuous map $f\colon X \to Y$ is actually a homeomorphism if X is compact and Y is Hausdorff. In this section we will prove a result which at times can deal with the case that X is non-compact.

First we need to introduce the following definition which will appear time and again.

Definition 4.63. A map $f\colon X \to Y$ between topological spaces is called proper if the preimage of every compact subset of Y is a compact subset of X.

Example.

(1) In the figure below we show graphs of continuous maps $\mathbb{R} \to \mathbb{R}$, some of the maps are proper and some are not proper.
(2) A continuous map from a non-compact space to a compact space *cannot* be proper. In particular we see that inclusion maps are not necessarily proper, for example the inclusion $i\colon (0,1) \to [-2,2]$ is not proper.
(3) Evidently every homeomorphism is proper.



(4) It follows easily from the Compact-Closed Lemma 3.39 (1) and (2) that every continuous map from a compact space to a Hausdorff space is proper. □

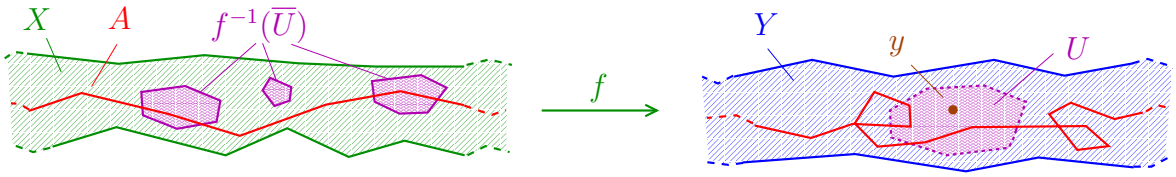
Now we can formulate and prove the following variation on the Compact-Hausdorff Proposition 4.23.

Proposition 4.64. (Proper-Closed Proposition) Let $f\colon X \to Y$ be a continuous map from a topological space X to a topological space Y. We assume that Y is Hausdorff and regionally compact.

(1) If f is proper, then $f\colon X \to Y$ is a closed map.
(2) The following two statements are equivalent:
  (a) The map $f\colon X \to Y$ is injective and proper.
  (b) The map $f\colon X \to Y$ is a closed embedding.

Proof.

(1) Let $A \subseteq X$ be a closed subset. We need to show that $f(A)$ is a closed subset of Y. By the JH-Openness Criterion 3.3 it suffices to show that every $y \in Y \setminus f(A)$ admits an open neighborhood V with $V \subseteq Y \setminus f(A)$ or equivalently with $V \cap f(A) = \varnothing$. Thus let $y \in Y$. The proof proceeds in the following steps, which are mostly forced on us:
- Since Y is regionally compact and Hausdorff we know by the Regionally Compact–Locally Precompact Lemma 4.60 that y admits an open neighborhood U such that the closure $\overline{U}$ is compact.
- Since f is proper we know that $f^{-1}(\overline{U})$ is compact.
- Since A is a closed subset we obtain from the Compact-Closed Lemma 3.39 (1) that $A \cap f^{-1}(\overline{U})$ is also compact.
- By the Compact Image Lemma 4.15 we know that $f(A \cap f^{-1}(\overline{U}))$ is compact.
- By the elementary Exercise 1.2 we know that $f(A \cap f^{-1}(\overline{U})) = f(A) \cap \overline{U}$.
- Since Y is Hausdorff we obtain immediately from the Compact-Closed Lemma 3.39 (2) that $f(A \cap f^{-1}(\overline{U})) = f(A) \cap \overline{U}$ is a closed subset of Y.
- Therefore $V := U \setminus (f(A) \cap \overline{U}) = U \setminus f(A)$ is an open neighborhood of $y \in Y$ that does not intersect $f(A)$.



(2) First we prove the "(a)⇒(b)"-direction. Thus we assume that $f\colon X \to Y$ is injective and proper. Since Y is Hausdorff and regionally compact we know by Statement (1)

that $f\colon X \to Y$ is a closed map. Since f is furthermore injective we obtain from the Compact-Hausdorff Proposition 4.23 that $f\colon X \to Y$ is indeed a closed embedding.

Now we prove the "(b)⇒(a)"-direction. We only need to show that $f\colon X \to Y$ is proper. Thus let $K \subseteq Y$ be a compact subset.

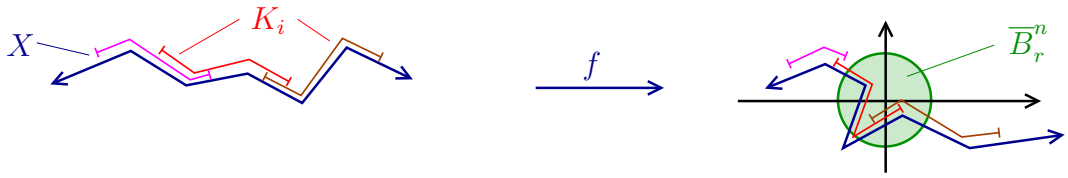
- Since $f\colon X \to Y$ is a closed map we know by the Compact-Closed Lemma 3.39 (1) that $K \cap f(X)$ is also compact.
- Since $f\colon X \to f(X)$ is a homeomorphism we know that $f^{-1}(K \cap f(X))$ is compact.
- Since $f^{-1}(K \cap f(X)) = f^{-1}(K)$ we have indeed shown that $f^{-1}(K)$ is compact. ■

We can use the Proper-Closed Proposition 4.64 to give a new proof of the following popular proposition.

Proposition 4.26. (Closed Embedding-to-$\mathbb{R}^n$ Proposition) Let X be a topological space, let $n \in \mathbb{N}$ and let $f\colon X \to \mathbb{R}^n$ be an injective continuous map. Furthermore suppose that X is the union of compact subsets $\{K_i\}_{i\in I}$ such that the following condition is satisfied:

(∗) Given any $r > 0$ there exist only finitely many $i \in I$ with $f(K_i) \cap \overline{B}^n_r \neq \varnothing$.

Then f is a closed embedding.



For the new proof of the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 we will need the following lemma, which often simplifies the task of showing that a certain map is proper.

Auxiliary Lemma 4.65. (Proper Criterion) Let $f\colon X \to Y$ be a continuous map between two topological spaces. We suppose that Y is Hausdorff.

(1) We suppose that we are given a family $\{L_j\}_{j\in J}$ of compact subsets of Y such that each compact subset of Y is contained in some L_j. If each $f^{-1}(L_j)$ is compact, then f is proper.

(2) Suppose that X is covered by compact subsets $\{K_i\}_{i\in I}$. Let $L \subseteq Y$ be compact. If there exist only finitely many $i \in I$ with $f(K_i) \cap L \neq \varnothing$, then $f^{-1}(L)$ is compact.

Proof of Lemma 4.65.

(1) Let L be a compact subset of Y. By hypothesis there exists a $j \in J$ with $L \subseteq L_j$. Note that

$$f^{-1}(L) \overset{\text{since } L\subseteq L_j}{\downarrow =} f^{-1}(L \cap L_j) \overset{\text{by the Image-Preimage Lemma 1.9 (7)}}{\downarrow =} f^{-1}(L) \cap f^{-1}(L_j)$$
$$= \underset{\uparrow}{\text{closed subset}} \cap \text{compact subset} = \underset{\uparrow}{\text{compact}}.$$

since Y is Hausdorff we know by the Compact-Closed Lemma 3.39 that L is closed, therefore $f^{-1}(L)$ is closed, furthermore by hypothesis we know that $f^{-1}(L_j)$ is compact

by the Compact-Closed Lemma 3.39

(2) Let $L \subseteq Y$ be a compact subspace with the property that there exist only finitely many $i \in I$ with $f(K_i) \subseteq L$. Note that

$$\begin{aligned} f^{-1}(L) &= f^{-1}(L) \cap \bigcup_{i \in I \text{ with } f(K_i) \cap L \neq \varnothing} K_i \\ &= \underset{\uparrow}{\text{closed subset}} \cap \text{compact subset} \underset{\uparrow}{=} \text{compact.} \end{aligned}$$

since Y is Hausdorff we know by the Compact-Closed Lemma 3.39 that L is closed, therefore $f^{-1}(L)$ is closed, furthermore by the Compact–Union-Intersection Lemma 3.41 we know that the union of finitely many compact subspaces is again compact

by the Compact-Closed Lemma 3.39

■

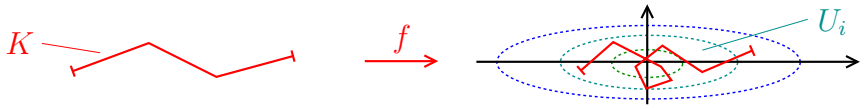
Proof of Proposition 4.26. Since $\mathbb{R}^n$ is Hausdorff it follows from the Proper Criterion Lemma 4.65 (2) that for each $r > 0$ the preimage $f^{-1}(\overline{B}^n_r)$ is compact. It follows from the Proper Criterion Lemma 4.65 (1) that f is actually proper. On page 208 we pointed out that $\mathbb{R}^n$ is regionally compact. We now obtain from the Proper-Closed Proposition 4.64 that f is in fact a closed embedding. ■

4.11. Exercises for Chapter 4.

Exercise 4.1. Let $f\colon K \to X$ be a continuous map from a compact space K to a topological space X. Suppose that there exists a sequence $U_1, U_2, \ldots$ of subsets of X with the following properties:

(1) each U_i is open,
(2) the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$,
(3) we have $X = \bigcup_{i \in \mathbb{N}} U_i$.

Show that there exists a $k \in \mathbb{N}$ with $f(K) \subseteq U_k$.



Exercise 4.2.

(a) Let X be a topological space that is locally connected. Show that the continuous maps $f\colon X \to \mathbb{R}$ that are locally constant are precisely the maps that are constant on all of the components of X.
(b) Show that the conclusion in (a) does not necessarily hold if we drop the hypothesis that X is locally connected.

Exercise 4.3. Let X be a topological space and let D be a *discrete* topological space. Show that a map $f\colon X \to D$ is continuous if and only if it is locally constant.

Exercise 4.4. We consider the following set of half-open intervals:

$$\mathcal{B} \coloneqq \{[a,b) \mid a,b \in \mathbb{R}\} \subseteq \mathcal{P}(\mathbb{R})$$

It is clear that $\mathcal{B}$ has the basis property introduced in Definition 3.52. Let $\mathcal{T} \coloneqq \mathcal{T}(\mathcal{B})$ be the topology generated by $\mathcal{B}$. This topology on $\mathbb{R}$ is called the *lower limit topology* and the topological space $(\mathbb{R}, \mathcal{T})$ is called the *Sorgenfrey line*.

(a) Let $\mathcal{S}$ be the standard topology on $\mathbb{R}$. Which of the two identity maps $(\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{S})$ is continuous?

(b) Determine the closure of the intervals $(1, \sqrt{2})$ and $(\sqrt{2}, 3)$ in the Sorgenfrey line.
(c) Is the Sorgenfrey line path-connected?
(d) Is the Sorgenfrey line regionally compact?

Exercise 4.5. Are the topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ (equipped with the subspace topology coming from $\mathbb{R}$) homeomorphic?

Exercise 4.6.

(a) Show that every self-homeomorphism of B^1 extends to a self-homeomorphism of $\overline{B}^1$.
(b) Give an example of a self-homeomorphism of B^2 that does not extend to a self-homeomorphism of $\overline{B}^2$.

Exercise 4.7. Let $C([0,1])$ be the real vector space of continuous maps from the interval $[0,1]$ to $\mathbb{R}$. We consider the

$$\text{supremum norm } s(f) := \sup\{|f(x)| \,|\, x \in [0,1]\} \quad \text{and the integral norm} \quad i(f) := \int_0^1 |f(x)|\, dx.$$

We denote the topologies corresponding to these norms by $\mathcal{S}$ and $\mathcal{I}$. Show that one topology is contained in the other and show that the two topologies are not the same.
Remark. This shows in particular that the identity map $\mathrm{id}\colon (C([0,1], \mathcal{S}) \to (C([0,1], \mathcal{I})$ is not a homeomorphism.

Exercise 4.8. Let $C([0,1])$ be the real vector space of continuous maps from the interval $[0,1]$ to $\mathbb{R}$ and let $C^1([0,1])$ be the real vector space of continuously differentiable maps from the interval $[0,1]$ to $\mathbb{R}$. We equip both vector spaces with the supremum norm that is given by $s(f) := \sup\{|f(x)| \,|\, x \in [0,1]\}$. We consider the two maps

$$\begin{array}{rcl} D\colon C^1([0,1]) & \to & C([0,1]) \\ f & \mapsto & f' \end{array} \qquad \text{and} \qquad \begin{array}{rcl} I\colon C([0,1]) & \to & C^1([0,1]) \\ f & \mapsto & \left(x \mapsto \int_0^x f(t)\, dt\right). \end{array}$$

Note that D and I are inverses of one another, in particular both are isomorphisms of real vector spaces.

(a) Show that I is continuous.
(b) Show that D is not continuous.

Remark 1. This exercise shows that integration is usually a well-behaved operation whereas differentiation can be problematic.
Remark 2. This exercise is a variation on Exercise 4.7.

Exercise 4.9. Let $f\colon [a,b] \to \mathbb{R}$ be an injective continuous map. Show that $f([a,b])$ is an interval with endpoints $f(a)$ and $f(b)$ and show that the map $f\colon [a,b] \to f([a,b])$ is a homeomorphism.
Hint. Use the Intermediate Value Theorem.

Exercise 4.10. Let U be an open subset of $\mathbb{R}^n$. We denote by $C = \overline{U}$ the closure of U.

(a) Suppose that U is convex. Show that $U = \mathring{C}$.
(b) Give an example of an open subset U of $\mathbb{R}^n$ where $U \neq \mathring{C}$.

Exercise 4.11. Let X be a topological space.

(1) A point $x_0 \in X$ is called *isolated* if $\{x_0\}$ itself is an open subset.

(2) Let $U \subseteq X$ be a subset. We say that $x \in X$ is an *accumulation point of* U, if every neighborhood of x contains a point in $U \setminus \{x\}$.

With these definitions we can formulate our exercise:

(a) Let $x \in U$. Show that x is an accumulation point of U if and only if x is not isolated in U.
(b) We denote by $A(U)$ the accumulation points of U and we denote by $I(U)$ the isolated points of U. Show that the closure $\overline{U}$ equals $A(U) \cup I(U)$.
(c) Let $f, g\colon X \to \mathbb{R}$ be continuous maps which agree on U. Show that for every accumulation point x of U we also have $f(x) = g(x)$.

Remark. In the literature accumulation points are sometimes also called limit points or cluster points.

Exercise 4.12. Let V be a finite-dimensional real vector space. Recall that given a norm $n\colon V \to \mathbb{R}_{\geq 0}$ its norm ball is defined as the subset $B(n) \coloneqq \{v \in V \mid n(v) \leq 1\}$.

(a) Show that for any norm $n\colon V \to \mathbb{R}_{\geq 0}$ the norm ball $B(n)$ is a convex subset with non-empty interior.
(b) Show that for any convex subset $C \subseteq \mathbb{R}^n$ such that the origin $0 \in \mathring{C}$ there exists a norm $n\colon V \to \mathbb{R}_{\geq 0}$ with $B(n) = C$.

Exercise 4.13. Let $n \in \mathbb{N}$. Show that any compact non-empty convex subset of $\mathbb{R}^n$ is homeomorphic to some $\overline{B}^k$ with $k \in \{0, \dots, n\}$.
Remark. Use the Convex-to-Ball Proposition 4.28 (2).

Exercise 4.14. Let $f\colon X \to Y$ be a map between topological spaces.

(a) Show that the following statements are equivalent:
  (i) The map is continuous.
  (ii) For every subset $B \subseteq Y$ we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
  (iii) For every subset $B \subseteq Y$ we have $f^{-1}(\mathring{B}) \subseteq$ interior of $f^{-1}(B)$.
  Remark. We showed (i)⇒(ii) in the Closure-Interior Mapping Lemma 4.6.
(b) Given an example of a continuous map $f\colon X \to Y$ that shows that in general we do not have an equality in (ii).
(c) Give an example of a continuous map $f\colon X \to Y$ that shows that in general we do not have an equality in (iii).

Exercise 4.15. Does there exist a continuous map $f\colon S^2 \to S^2$ with a single fixed point, i.e. with a unique point x such that $f(x) = x$?

Exercise 4.16. Let $f\colon X \to Y$ be a continuous injection such that $f(X)$ is a closed subset of Y. Is f an embedding?

Exercise 4.17. Let (X, d) and (Y, e) be two metric spaces and let $\varphi\colon X \to Y$ be a map.

(a) Suppose that there exists some $C > 0$ such that for every choice of $x, x' \in X$ we have $e(\varphi(x), \varphi(x')) < C \cdot d(x, x')$. Show that φ is continuous.
(b) Suppose that there exists some $c > 0$ such that for every choice of $x, x' \in X$ we have $e(\varphi(x), \varphi(x')) > c \cdot d(x, x')$. Furthermore suppose that φ is surjective. Does it follow that φ is open?

Exercise 4.18. Let (X, d) be a metric space and let A be a subset of X. Show that the map

$$\begin{array}{rcl} A & \to & [0,1] \\ a & \mapsto & \min\{1, \sup\{r \in \mathbb{R}_{\geq 0} \mid B_r(a) \subseteq A\}\} \end{array}$$

is continuous.

Exercise 4.19. Let X be a set and let d and e be two metrics on X that are equivalent, i.e. that induce the same topology on X. We use this common topology to view X as a topological space.

(a) Suppose that X is compact. Show that given any $\mu > 0$ there exists a $\nu > 0$ such that the following statement holds: If $x, y \in X$ satisfy $d(x, y) < \mu$, then they also satisfy $e(x, y) < \nu$.

(b) Let $\alpha \colon X \times X \to \mathbb{R}_{>0}$ be a continuous map. Show that there exists in fact a continuous map $\beta \colon X \times X \to \mathbb{R}_{>0}$ such that the following conclusion holds: If $x, y \in X$ satisfy $d(x, y) < \alpha(x, y)$, then they also satisfy $e(x, y) < \beta(x, y)$.

Hint. Use Exercise 4.18

Exercise 4.20. Let C be the Cantor set from Definition 3.45.

(a) Show that for every $x \in C$ there exists a unique sequence $a_1, a_2, \ldots$ of elements of $\{0, 1\}$ such that

$$x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}.$$

(b) We define the *Cantor function* as follows:

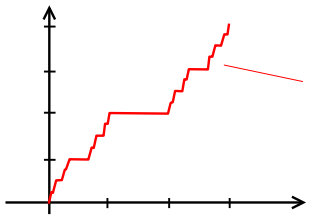
$$\begin{array}{rcl} c \colon [0,1] & \to & [0,1] \\ x & \mapsto & \begin{cases} \sum_{k=1}^{\infty} \frac{a_k}{2^k}, & \text{if } x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \in C, \\ \sup\{c(y) \mid y \in C \text{ and } y \leq x\}, & \text{if } x \notin C. \end{cases} \end{array}$$

Show that the Cantor function is continuous.

(c) Show that the Cantor function is surjective.

(d) Show that the Cantor set is uncountable.

Remark. The graph of the Cantor function is sometimes called the *Devil's Staircase*.



Exercise 4.21. Show that every connected non-empty subset of $\mathbb{R}$ is an interval.

Exercise 4.22. Let X be a topological space. Show that the following two statements are equivalent:

(i) The topological space X is disconnected.

(ii) There exists a continuous non-constant map $X \to \{0, 1\}$.

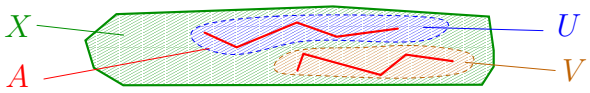
Remark. This shows that connectivity is characterized by maps out of the topological space. On the other hand, path-connectivity is by Definition 4.30 characterized by maps to the topological space.

Exercise 4.23. Let X be a topological space and let $A \subseteq X$ be a subset.

(a) Show that if A is connected, then so is its closure $\overline{A}$.
(b) If A is path-connected, does it follow, that its closure $\overline{A}$ is also path-connected?

Exercise 4.24. Let X be a topological space and let $A \subseteq X$ be a subspace. We say that A *can be disconnected in* X if there exist disjoint open subsets U and V of X such that $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$ and $A \subseteq U \cup V$.

(a) Show that if X is a metric space, then any disconnected subspace A can be disconnected in X.
(b) Give an example of a topological space and a disconnected subspace A that cannot be disconnected in X.



Exercise 4.25.

(a) Show that every self-homeomorphism of $\mathbb{R}$ is monotonously increasing or monotonously decreasing.
(b) Is every self-homeomorphism of $\mathbb{Q}$ monotonously increasing or monotonously decreasing?

Exercise 4.26. Let $X = \{0,1\}$ be the set with two elements which we equip with the topology $\mathcal{T} = \{\varnothing, X, \{0\}\}$. The topological space $X = (X, \mathcal{T})$ is called the *Sierpiński space*.

(a) Is every continuous map $X \to \mathbb{R}$ constant?
(b) Is every continuous map $\mathbb{R} \to X$ constant?
(c) Is X Hausdorff?
(d) Is X connected?
(e) Is X path-connected?
(f) Determine all self-homeomorphisms of X.

Exercise 4.27. Let $\{0,1\}$ be the set with two elements which we equip with the topology $\mathcal{T} = \{\varnothing, \{0,1\}, \{0\}\}$. As in Exercise 4.26 the topological space $(\{0,1\}, \mathcal{T})$ is called the *Sierpiński space*. Next let X be a topological space and let $U \subseteq X$. We consider the characteristic map

$$\begin{array}{rccl} \chi_U \colon X & \to & \{0,1\} & \\ x & \mapsto & \begin{cases} 1, & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

Show that $\chi_U \colon X \to \{0,1\}$ is continuous (with respect to the Sierpinski topology) if and only if $U \subseteq X$ is open.

Exercise 4.28. We consider $X = \mathbb{Q}$ with the subspace topology coming from $\mathbb{R}$. Show that the only connected subsets of $\mathbb{Q}$ are the subsets consisting of at most a point.

Exercise 4.29. Let X be a topological space and let $A, B \subseteq X$ be two *open* subsets such that $X = A \cup B$. Show that if X and $A \cap B$ are connected and non-empty, then A and B are also connected.

Remark. This statement can be viewed as a partial converse to the Union Connected Lemma 4.39 (2). We will deal with the "path-connected" setting in Exercise 10.12.

Exercise 4.30. Let $f\colon X \to Y$ be a continuous map between topological spaces, let $U \subseteq X$ be a subset and let $A \subseteq Y$ be a subset.

(a) Suppose that A is closed and suppose that $f(U) \subseteq A$. Show that $f(\overline{U}) \subseteq A$.
(b) Suppose that A is open and that $A \subseteq f(U)$. Show that $A \subseteq f(\mathring{U})$.

Exercise 4.31. Let X be a topological space. We say that a subset $A \subseteq X$ is a *retract of* X if there exists a *retraction* $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$. (Evidently we demand that r is continuous).

(a) Suppose that X is Hausdorff. Show that A is a closed subset of X.
(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that X is Hausdorff.

Exercise 4.32. We consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ as a subset of $Y = \mathbb{R}$. Show that there is no open neighborhood of X in Y that admits a retraction onto X.

Exercise 4.33. Let $f\colon (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a (possibly non-continuous) map between two topological spaces.

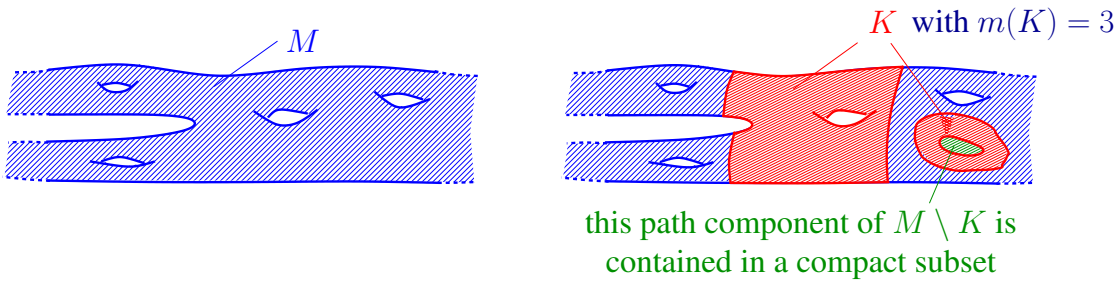
(a) Show that if f is continuous, then for every $(x_n)_{n\in\mathbb{N}}$ sequence in X we have
$$\lim_{n\to\infty} x_n = x \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = f(x).$$
(b) We suppose that for every sequence $(x_n)_{n\in\mathbb{N}}$ in X we have
$$\lim_{n\to\infty} x_n = x \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = f(x).$$
  (i) Give an example where this condition is satisfied but f is not continuous.
  (ii) A topological space is called *first-countable*, if for every $x \in X$ there exists a countable family of open neighborhoods $\{U_i\}_{i\in I}$ of x such that each open neighborhood of x contains at least one U_i. Show that if X is first-countable, then f is continuous.

Exercise 4.34. Let X be a topological space. Given a compact subset $K \subseteq X$ we define $m(K)$ to be the number of non-compact path-components of $X \setminus K$ that are *not* contained in a compact subset of X. We refer to
$$\mathrm{End}(X) \coloneqq \max\{m(K) \mid K \text{ is a compact subset of } X\} \in \mathbb{N}_0 \cup \{\infty\}$$
as the *number of ends of* X.

(a) Given $n \in \mathbb{N}$ determine $m(\mathbb{R}^n)$.
  Remark. This exercise shows that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.
(b) Can you find connected spaces with arbitrarily large number of ends?

Remark. We will introduce the "ends" of a topological space in Exercise 229.8. We will see that what we now call the "number of ends" equals the cardinality of the set of ends.

Exercise 4.35. Let $C \subseteq \mathbb{R}$ be the Cantor set. We equip C with the subspace topology. Determine the components of C.



Exercise 4.36. Let $f\colon X \to Y$ be a continuous map between topological spaces. By the Compact Image Lemma 4.15 we know that the image of a compact subset of X under f is a compact subset of Y.

(a) Give an example of a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ and an open subset $U \subseteq \mathbb{R}$ such that $f(U)$ is not an open subset of $\mathbb{R}$.
(b) Give an example of a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ and a closed subset $U \subseteq \mathbb{R}$ such that $f(U)$ is not a closed subset of $\mathbb{R}$.

Exercise 4.37. In this exercise we will see that for maps between subsets of $\mathbb{R}^2$ the notions of "open", "closed" and "continuous" are completely independent. More precisely for (a)–(f) provide an example of a map $\varphi\colon X \to Y$ between subsets of $\mathbb{R}^2$, equipped with the subspace topology coming from $\mathbb{R}^2$, that have the stated property.

(a) The map φ is open but neither closed nor continuous.
(b) The map φ is closed but neither open nor continuous.
(c) The map φ is continuous but neither open nor closed.
(d) The map φ is continuous and open but not closed.
(e) The map φ is continuous and closed but not open.
(f) The map φ is open and closed but not continuous.

Exercise 4.38. Show that the maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x \cdot y \end{array}$$

are continuous.
Hint. Use the Metric Continuity Proposition 4.9.

Exercise 4.39. Let $f = (f_1, \ldots, f_n)\colon X \to \mathbb{R}^n$ be a map from a topological space X to $\mathbb{R}^n$. Show that f is continuous if and only if each $f_i\colon X \to \mathbb{R}$ is continuous.
Remark. You could use the Subbasis-Continuity Proposition 4.4 and the JH-Openness Criterion 3.3.

Exercise 4.40. Let $f\colon X \to Y$ be a continuous map between topological spaces. Let $Q \subseteq Y$ be a subset and let U be an open neighborhood of $f^{-1}(Q) \subseteq X$.

(a) Show that if X is compact and if Y is Hausdorff, then there exists an open neighborhood V of $Q \subseteq Y$ such that $f^{-1}(V) \subseteq U$.
(b) Show that in (a) we cannot drop the hypothesis that X is compact.

Exercise 4.41. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces.

(a) Suppose that Y is Hausdorff. Show that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that Y is Hausdorff.
(c) Show that if f and g agree on a subset A and if Y is Hausdorff, then f and g also agree on the closure $\overline{A}$ of A in X.

Exercise 4.42. Let $\mathbb{R}^n \cup \{\infty\}$ be "$\mathbb{R}^n$ with a point at infinity". Let $v \in \mathbb{R}^n$. Show that the map

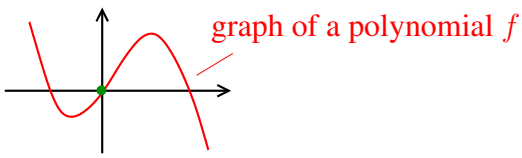
$$\begin{array}{rcl} \mathbb{R}^n \cup \{\infty\} & \to & \mathbb{R}^n \cup \{\infty\} \\ Q & \mapsto & \begin{cases} Q+v, & \text{if } Q \in \mathbb{R}^n, \\ \infty, & \text{if } Q = \infty \end{cases} \end{array}$$

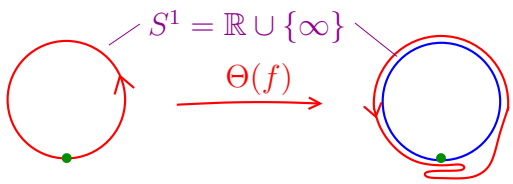
is continuous.

Exercise 4.43. Let f be a complex polynomial of degree ≥ 1. Show that the map

$$\begin{array}{rcl} \Theta(f)\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ t & \mapsto & \begin{cases} f(t), & \text{if } t \in \mathbb{C}, \\ \infty, & \text{if } t = \infty \end{cases} \end{array}$$

is continuous.
Remark. By the Stereographic Projection Lemma 4.24 we can make the identification $S^2 =_{\mathrm{i}} \mathbb{C} \cup \{\infty\}$. Thus we see that complex polynomials of degree ≥ 1 give rise to interesting self-maps of S^2.
Remark. The analogous statement also holds for a polynomial f with real coefficients, it gives rise to a self-map $\Theta(f)$ of $\mathbb{R} \cup \{\infty\} =_{\mathrm{i}} S^1$.



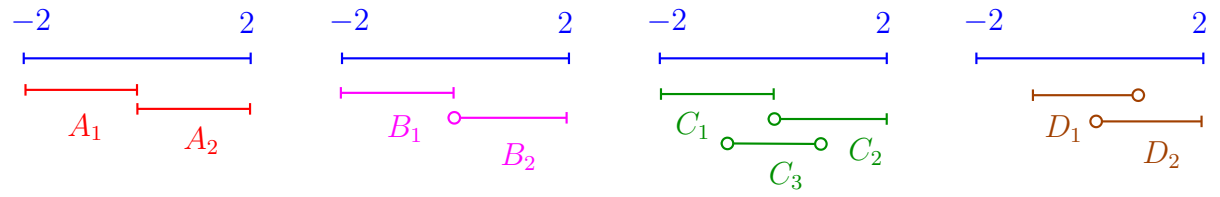


Exercise 4.44. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

(a) Show that f is an open map.
(b) Is f also a closed map?

Exercise 4.45. Let X be a set and let $\{A_i\}_{i \in I}$ be a family of subsets. Suppose that each A_i is equipped with a topology.
(a) We say that $U \subseteq X$ is open if for each $i \in I$ the intersection $U \cap A_i$ is an open subset of A_i. Show that this defines a topology on X.
(b) We consider the set $X = [-2, 2]$.
  (i) We consider the subsets $A_1 = [-2, 0]$ and $A_2 = [0, 2]$.
  (ii) We consider the subsets $B_1 = [-2, 0]$ and $B_2 = (0, 2]$.
  (iii) We consider the subsets $C_1 = [-2, 0]$, $C_2 = (0, 2]$ and $C_3 = (-1, 1)$.
  (iv) We consider the subsets $D_1 = [-1, 1)$ and $D_2 = (0, 2]$.
  For each of these four settings we equip the subsets with the obvious topology and we equip $X = [-2, 2]$ with the corresponding topology. Describe the resulting four topological spaces. Are these topological spaces that we are familiar with?



Exercise 4.46. Let $P_1, \ldots, P_n \in \mathbb{R}^k$.

(a) Show that the convex hull of the set $\{P_1, \ldots, P_n\}$ is given by the set
$$\left\{ \sum_{i=1}^{n} t_i \cdot P_i \,\middle|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}.$$
Note: We make no hypothesis on the points $P_1, \ldots, P_n$, in particular some of them could be identical and we do not assume say that the vectors $P_2 - P_1, \ldots, P_n - P_1$ are linearly independent.

(b) Show that the set in (a) is compact.

(c) Is the convex hull of a compact subset of $\mathbb{R}^k$ necessarily compact?

(d) Is the convex hull of an open subset of $\mathbb{R}^k$ necessarily an open subset?

Exercise 4.47. Let $f, g \colon X \to Y$ be two continuous maps between two topological spaces. Let $A \subseteq X$ be a subset such that f and g agree on A. Show that if A is dense, if Y is Hausdorff and if f and g are continuous, then f and g agree on all of X.

Exercise 4.48.

(a) Show that the half-open interval $[0, 1)$ and the open interval $(0, 1)$ are not homeomorphic.

(b) Is the topological space $[0, 1) \cap \mathbb{Q}$ homeomorphic to the topological space $(0, 1) \cap \mathbb{Q}$?

Exercise 4.49. Let $m, n \in \mathbb{N}$.

(a) Let $A = (a_1 \ \ldots \ a_n) \in \mathrm{M}(m \times n, \mathbb{R}) = (\mathbb{R}^m)^n$. Show that A has rank m if and only there exist $k_1, \ldots, k_m \in \{1, \ldots, n\}$ with $\det(a_{k_1} \ \ldots \ a_{k_m}) \neq 0$.

(b) Show that the set of matrices of rank m is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$.
Remark. In the Matrix Group Lemma 4.53 we dealt with the case $m = n$.

Exercise 4.50. Give an example of a bijective continuous map $f \colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.
Remark. By the Compact-Hausdorff Proposition 4.23 you know that Y cannot be Hausdorff.

Exercise 4.51. Let X and Y be topological spaces and let $i \colon X \to Y$ be a continuous map. We suppose that there exists a continuous map $r \colon Y \to X$ with $r \circ i = \mathrm{id}_X$. Show that i is an embedding.

Exercise 4.52. Let $f \colon [0, 1] \to \mathbb{R}$ be a continuous function. Let $a < b$ be two real numbers. Show that f can only go finitely many times from a to b. More precisely, show that it is not possible to find numbers $x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < \ldots$ in $[0, 1]$ such that for each $i \in \mathbb{N}$ we have $f(x_i) = a$ and we have $f(y_i) = b$.

Exercise 4.53. Let $A \in \mathrm{M}(m \times n, \mathbb{R})$. We define the *norm of* A as follows:

$$\|A\| \;:=\; \max\{\|Av\| \,|\, v \in S^{n-1}\} \;\in\; \mathbb{R}_{\geq 0}.$$

(Note that by the Compact Image Lemma 4.15 (2) it does indeed make sense to take the maximum.)

(a) Show that for all $v \in \mathbb{R}^n$ we have $\|Av\| \leq \|A\| \cdot \|v\|$.

(b) (i) Show that for all $A, B \in \mathrm{M}(m \times n, \mathbb{R})$ we have $\|A + B\| \leq \|A\| + \|B\|$.
(ii) Show that for any $A \in \mathrm{M}(m \times n, \mathbb{R})$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot A\| = |\lambda| \cdot \|A\|$.
Note that this shows that $\| - \|$ turns $\mathrm{M}(m \times n, \mathbb{R})$ into a normed vector space.

(c) Show that for $A \in \mathrm{O}(n)$ we have $\|A\| = 1$.

(d) Show that for $A \in \mathrm{M}(l \times m, \mathbb{R})$ and $B \in \mathrm{M}(m \times n, \mathbb{R})$ we have $\|A \cdot B\| \leq \|A\| \cdot \|B\|$.

(e) Give examples of matrices $A, B \in \mathrm{M}(m \times m, \mathbb{R})$ with $\|A \cdot B\| \neq \|A\| \cdot \|B\|$.

(f) Show that for $A \in \mathrm{GL}(n, \mathbb{R})$ we have $\|A^{-1}\| = \|A\|^{-1}$.

(g) Let $A \in \mathrm{GL}(n, \mathbb{R})$ be an invertible matrix. Show that there exists a $\lambda > 0$ such that $\|Av\| \geq \lambda \cdot \|v\|$ for all $v \in \mathbb{R}^n$.

(h) Show that the map

$$\begin{array}{rcl} \mathrm{M}(m \times n, \mathbb{R}) & \to & \mathbb{R}_{\geq 0} \\ A & \mapsto & \|A\| \end{array}$$

is continuous.

(i) Let $A \colon [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$ be a continuous map. Show that

$$\left\| \int_a^b A(t)\, dt \right\| \;\leq\; \int_a^b \|A(t)\|\, dt.$$

Hint. Make use of the Integral Inequality Lemma 2.45.

Exercise 4.54. Let A be an invertible real $(m \times m)$–matrix. Show that the map

$$\begin{array}{rcl} \mathbb{R}^m \cup \{\infty\} & \to & \mathbb{R}^m \cup \{\infty\} \\ x & \mapsto & \begin{cases} A \cdot x, & \text{if } x \in \mathbb{R}^m, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is continuous.
Hint. Use Exercise 4.53 (g).

Exercise 4.55. Let $n \in \mathbb{N}$. We consider the stereographic projection

$$\begin{array}{rcl} \Phi \colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1 \end{cases} \end{array}$$

that we introduced in the Stereographic Projection Lemma 4.24. Show that Φ is continuous.

Exercise 4.56. We consider the Topologist's Sine Curve

$$X \;:=\; \{(0, y) \,|\, y \in [-1, 1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2.$$

which we introduced on page 195.

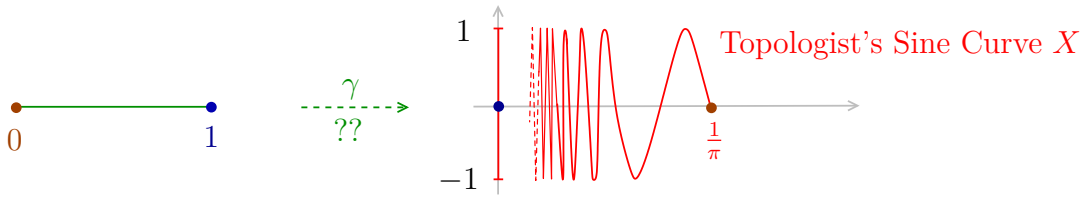
(a) Show that X is connected.

(b) Show that there does not exist a continuous map $\gamma \colon [0, 1] \to X$ with $\gamma(0) = (\frac{1}{\pi}, 0)$ and with $\gamma(1) = (0, 0)$.
Hint. Assume such a map exists. We denote by $q \colon X \to \mathbb{R}$ the projection onto the

second coordinate. Use the Intermediate Value Theorem to construct sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $[0,1]$ which both converge to the same number but such that for each $n \in \mathbb{N}$ we have $q(\gamma(a_n)) = -1$ and $q(\gamma(b_n)) = +1$.

Remark. This exercise implies of course that X is *not* path-connected.

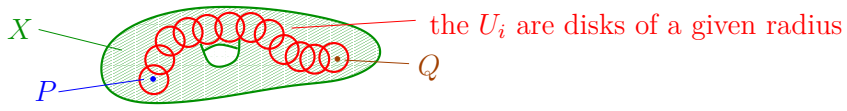
(c) Show that X has two path components.



Exercise 4.57. Let X be a topological space and let $\mathcal{U} \subseteq \mathcal{P}(X)$ be a cover of X such that each $U \subseteq \mathcal{U}$ is open and connected. Show that the following two statements are equivalent:

(1) The topological space X is connected.
(2) Given any two points $P, Q \in X$ there exist $U_0, \ldots, U_k \in \mathcal{U}$ with $P \in U_0$, $Q \in U_k$, with $U_i \cap U_{i+1} \neq \varnothing$ for $i = 0, \ldots, k-1$ and such that $U_i \cap U_j = \varnothing$ whenever $|i - j| \geq 2$.

Remark. Some people refer to this statement as the *Sausage Lemma*.



Exercise 4.58. Let X be a topological space and let $A \subseteq X$ be a subset.

(a) Suppose that X is path-connected and that ∂A is path-connected. Show that $X \setminus \mathring{A}$ is also path-connected.
(b) Suppose that X is connected and that ∂A is connected. Show that $X \setminus \mathring{A}$ is also connected.

Remark. This exercise is the content of the Complement Connected Lemma 4.40.

Exercise 4.59. We set $K := \{\frac{1}{n} \mid n \in \mathbb{N}\}$. The *K-topology* on $\mathbb{R}$ is defined as the topology $\mathcal{K}$ that is generated by the open intervals (a, b) and by all sets of the form $(a, b) \setminus K$ where $a, b \in \mathbb{R}$.

(a) Show that $(\mathbb{R}, \mathcal{K})$ is connected.
(b) Show that $(\mathbb{R}, \mathcal{K})$ is not path-connected, more precisely, show that it has precisely two path-components, namely $(-\infty, 0]$ and $(0, \infty)$.

Exercise 4.60. Let X be a topological space and let $A \subseteq X$ be a subset.

(a) Show that if A is connected, then its closure $\overline{A}$ is also connected.
(b) We suppose that A is path-connected. Is the closure $\overline{A}$ also path-connected?

Exercise 4.61. A topological space X is called *arc-connected* if given any two points $P, Q \in X$ there exists a continuous map $f\colon [0,1] \to X$ with $f(0) = P$, $f(1) = Q$ and which has the property that f is an embedding.[70]

(a) Show that the line with two zeros is not arc-connected.
(b) Let $n \geq 2$ and let X be the "$\mathbb{R}^n$ with two origins", which is defined in the hopefully obvious way. Is X arc-connected?

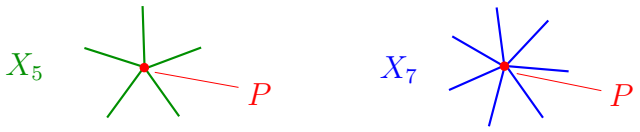
Remark. A rather amazing (non-trivial!) theorem says that any topological space that is path-connected *and which is Hausdorff* is in fact arc-connected. We refer to [**Will1970**, Corollary 31.6], [**Enge1989**, p. 381] and [**HY1988**, p. 118 and p. 129] for a proof of this statement.

Exercise 4.62.

(a) Let $n \in \mathbb{N}_{\geq 3}$. We consider

$$X_n \;:=\; \{r \cdot \exp(2\pi\,\mathrm{i} \cdot \tfrac{k}{n}) \,|\, r \in [0,1) \text{ and } k \in \{0, \ldots, n-1\}\}.$$

We write $P := 0$. Show that every self-homeomorphism $f\colon X_n \to X_n$ preserves P.
(b) Does the statement in (a) also hold for $n = 1$ and/or $n = 2$?



Exercise 4.63.

(a) Let $n \geq 1$. Show that the sphere S^n is path-connected.
Remark. This is the content of the Ball-Sphere-Connected Lemma 4.35 (2).
(b) Is the line with two zeros connected?

Exercise 4.64. A subset of a topological space is sometimes called *clopen* if it is open and closed. Let $n \in \mathbb{N}$. Determine all clopen subsets of $\mathbb{R}^n$.

Exercise 4.65. Let X be a path-connected subset of $\mathbb{R}^2$. Is it also necessarily regionally path-connected?

Exercise 4.66.

(a) Let $a < b \in \mathbb{Q}$. Show that $(a,b) \cap \mathbb{Q}$, equipped with the subspace topology coming from $\mathbb{R}$, is not compact.
(b) We consider $X := [0,1] \cap \mathbb{Q}$ where the open sets are the open sets of $[0,1) \cap \mathbb{Q}$ (which as in (a)) we equip with the subspace topology coming from $\mathbb{R}$, together with the set $X = [0,1]$. Show that X is compact and show that X is not regionally compact.

Exercise 4.67. Let X be a topological space. We consider the set $X^* := X \cup \{\infty\}$. We say that $U \subseteq X^*$ is open if one of the following conditions is satisfied:

(1) U is an open subset of X, or
(2) $\infty \in U$ and there exists a compact, closed subset K of X such that $U = X^* \setminus K$.

[70] Sometimes, see e.g. [**Will1970, Enge1989**], such topological spaces are also called *arcwise connected*. Rather confusingly some other books, e.g. [**Bre1993**], use the term *arcwise connected* for what we call path-connected. The lesson is, that in any given book one always should look up the definitions.

The inclusion $c\colon X \to X^*$ is called the *Alexandroff extension*. Show that the following statements hold:

(a) The open subsets do indeed define a topology on X^*.
(b) The map c is continuous and open.
(c) Show that the topological space X^* is compact.
(d) Show that if X is non-compact, then $c(X) \subseteq X^*$ is dense.
(e) Show that X^* is Hausdorff if and only if X is Hausdorff and regionally compact.

Remark. The topological space X^* is called the *one-point compactification of* X or *Alexandroff compactification of* X.[71]
Remark. One can show easily that the one-point compactification of $\mathbb{R}^n$ equals the "$\mathbb{R}^n$ with a point at infinity" that we introduced on page 152.

Exercise 4.68. We consider $\mathbb{Q} \subseteq \mathbb{R}$, equipped with the subspace topology. Is $[0,1] \cap \mathbb{Q}$ compact?

Exercise 4.69. Let $X = \mathbb{N}$ with the discrete topology. Show that the one-point compactification of X, as defined in Exercise 4.67, is homeomorphic to a subspace of $\mathbb{R}$.

Exercise 4.70. Give an example of a subspace of $\mathbb{R}^2$ that is path-connected but that is not locally path-connected.

Exercise 4.71. Using the natural map $\mathbb{C}^n \to \mathbb{R}^{2n}$ we obtain a map

$$\iota\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R}).$$

Show that $\iota(\mathrm{GL}(n,\mathbb{C})) \subseteq \mathrm{GL}_+(2n,\mathbb{R})$.

Exercise 4.72. Let X be a subspace of some $\mathbb{R}^n$. Show that X is homeomorphic to a closed subset of $\mathbb{R}^{n+1}$.

Exercise 4.73. Let X be a topological space.

(a) Let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be two continuous maps. We assume that f is proper and we assume that there exists an $\epsilon \in \mathbb{R}$ such that for every $x \in X$ we have $\|f(x) - g(x)\| < \epsilon$. Show that g is also proper.
(b) Does the conclusion of (a) hold if we replace the metric space $(\mathbb{R}^n, d(x,y) \coloneqq \|x-y\|)$ by any other metric space (Y,d)?

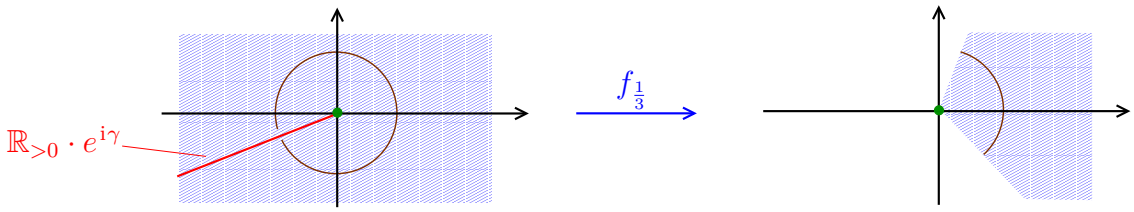
Exercise 4.74. Let X be a set, let Y be a topological space and let $f\colon X \to Y$ be a continuous map. Now suppose that we are given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X such that $f\colon (X,\mathcal{T}) \to Y$ and $f\colon (X,\mathcal{T}') \to Y$ are local homeomorphisms. Show that $\mathcal{T} = \mathcal{T}'$.
Hint. Use the Bijective-Local Homeomorphism Lemma 4.62.

Exercise 4.75. Let $\gamma \in \mathbb{R}$ and let $s \in \mathbb{R}$. We consider the map

$$\begin{array}{rcl} f_s\colon \mathbb{C} \setminus \{t \cdot \exp(\mathrm{i}\gamma) \mid t \in \mathbb{R}_{>0}\} & \to & \mathbb{C} \\ r \cdot \exp(\mathrm{i}\varphi) & \mapsto & r \cdot \exp(\mathrm{i}\varphi \cdot s). \end{array}$$

(a) Show that f_s is well-defined and continuous.
(b) Suppose that $s \neq 0$. Show that f_s is a local homeomorphism. (See the discussion in Definition 4.61 for the definition of a local homeomorphism.)

[71] Some authors use the expression "one-point compactification" only if X is non-compact, Hausdorff and regionally compact.


$\mathbb{R}_{>0} \cdot e^{\mathrm{i}\gamma}$
$f_{\frac{1}{3}}$

CHAPTER 5

Categories, functors and natural transformations

In the last two chapters we introduced the concept of topological spaces, we studied several natural examples and we discussed various basic concepts. We will now take a short break from discussing topological spaces.

In this chapter we will introduce the notion of a category, of a functor and of a natural transformation that many readers will already have encountered in different courses. In the subsequent chapter we will introduce the notions of (co-) products and (co-) limits in categories. These notions might initially appear to be rather abstract, but they play an essential role in topology and they are regularly used in many branches of pure mathematics. These concepts will also guide us for how to proceed with the discussion of topological spaces.

Nonetheless, readers who want to learn more about topological spaces as quickly as possible can also safely skip these two chapters. Having a background in categories is helpful for the later chapters on topological spaces, but not completely essential.

This chapter is organized as follows:

- In Section 5.1 we will introduce categories and we will provide some basic examples.
- In Section 5.2 we will study properties of isomorphisms in categories.
- In Section 5.3 we will monomorphisms and epimorphisms in categories. These notions generalize the concepts which are familiar from linear algebra.
- In Section 5.4 we will introduce the notion of initial and terminal objects in categories.
- In Section 5.5 we will introduce *covariant* functors between two categories.
- In Section 5.6 we will introduce *contravariant* functors between two categories.
- In Section 5.7 we will introduce natural transformations between two functors that go between the same categories.
- In Section 5.8 we will define the notion of an equivalence of categories.
- In Appendix 5.9 we will discuss the Yoneda Lemma.
- In Section 5.10 we will provide exercises for Chapter 5.

In the next Chapter 6 we will introduce products and coproducts in categories, and more generally, limits and colimits.

5.1. **Definition and examples of categories.** We start out with the definition of a category. As we will see, the notion of a category is extremely versatile and encompasses many situations.

Definition 5.1. A category $\mathcal{C}$ consists of the following data:
(1) A class[72] $\mathrm{Ob}(\mathcal{C})$, the elements of which are called the objects of the category.

(2) For each pair (X, Y) of objects there exists a set[73] [74] $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, the elements of which are called the morphisms from X to Y.
(3) For any three objects X, Y and Z we are provided a map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

called composition map, that satisfies the following axioms:
(A) (associativity): For every $f \in \mathrm{Mor}_{\mathcal{C}}(W,X), g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$(h \circ g) \circ f \;=\; h \circ (g \circ f) \;\in\; \mathrm{Mor}_{\mathcal{C}}(W,Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$, and
(ii) $f \circ \mathrm{id}_X = f$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Before we give a long list of examples let us first state a lemma which will look familiar from a basic algebra course.

Lemma 5.2. (Identity-Uniqueness Lemma) Let $\mathcal{C}$ be a category. For every $X \in \mathrm{Ob}(\mathcal{C})$ there exists a *unique* morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$,
(ii) $f \circ \mathrm{id}_X = f$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Proof. The existence of id_X is given by property (3B) of a category. Now suppose that $I \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ has the same property. Then we have

$$\mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (i) of } I \\ \text{applied to } f = \mathrm{id}_X}}{=} I \circ \mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (ii) of } \mathrm{id}_X \\ \text{applied to } f = I}}{=} I.$$

■

The Identity-Uniqueness Lemma 5.2 allows us to introduce the following definition:

Definition 5.3. Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}(\mathcal{C})$. We refer to the morphism that is uniquely defined in the Identity-Uniqueness Lemma 5.2 as the identity morphism id_X.

In the following we introduce all kinds of categories. Many of those will be faithful companions throughout these notes. These we bundle in the following definition:

Definition 5.4.

[72] As many "working mathematicians" the author has a slightly guilty conscience for not knowing what a "set" and what a "class" is. Therefore the reader is referred to [**Cie1997, FPr1985**] for details. Truth be told, most mathematicians learn from logicians at some point when to say "class" instead of "set", but many mathematicians do not know exactly what they are doing at that point.
[73] One could ask why $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is only required to be a class. Here's a long discussion which gives a satisfactory answer to people in category theory:

`http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set`

I myself do not understand the answer.
[74] If the category is clear from the context, then we just write $\mathrm{Mor}(X,Y)$ instead of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.

(a) We refer to the category $\mathcal{Set}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Set}) &:= \text{all sets,} \\ \mathrm{Mor}(X,Y) &:= \text{all maps from } X \text{ to } Y, \end{aligned}$$

and together with the usual composition of maps as the category of sets.

(b) We refer to the category $\mathcal{Top}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Top}) &:= \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) &:= \text{all continuous maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps[75] as the category of topological spaces.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{Vec}_{\mathbb{K}}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Vec}_{\mathbb{K}}) &:= \text{all } \mathbb{K}\text{-vector spaces,} \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the category of $\mathbb{K}$-vector spaces.

(d) We refer to the category $\mathcal{Group}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Group}) &:= \text{all groups,} \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the category $\mathcal{Group}$ of groups. Similarly we define the category $\mathcal{AbGroup}$ of abelian groups, the category $\mathcal{Ring}$ of rings[76] and the category $\mathcal{CRing}$ of commutative rings.

(e) A pair of sets is a pair (X, A), where X is a set and A is a subset of X. We refer to the category $\mathcal{PairSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairSet}) &:= \text{all pairs of sets,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subseteq B \end{aligned}$$

with the usual composition of maps as the category of pairs of sets. With the same approach we could also introduce the category of pairs of topological spaces and so on.

The name "morphisms" suggests that morphisms are necessarily maps. But as we will see in the following two examples, this is not necessarily always the case.

Example. Let G be an arbitrary group. We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{the set with the unique element } *, \\ \mathrm{Mor}(*,*) &:= G. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) &\to \mathrm{Mor}(*,*) \\ (f,g) &\mapsto g \circ f := \underbrace{g \cdot f}_{\text{multiplication in } G} \end{aligned}$$

via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be deduced easily from the group axioms of G. □

Our last example deserves being a proper definition again.

[75] By the Continuous Composition Lemma 4.2 we know that the composition of continuous maps is indeed continuous.

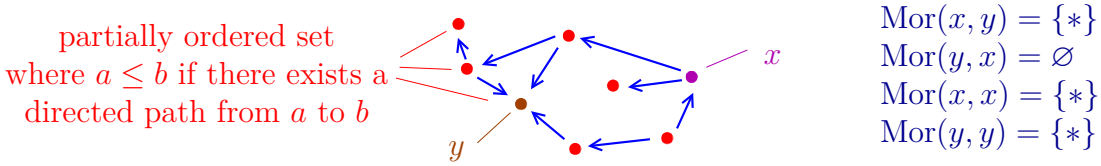
[76] We refer to Definition 2.6 for our definition of a ring and of a ring homomorphism.

Definition 5.5.

(f) Let $(P, \leq)$ be a partially ordered set, as introduced in Definition 1.25. We fix a set $\{*\}$ consisting of a single element. We define the order category $\mathit{Order}_{(P,\leq)}$ to be the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathit{Order}_{(P,\leq)}) &= P \\ \mathrm{Mor}(x,y) &= \begin{cases} \{*\}, & \text{if } x \leq y, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

together with the only possible way to compose morphisms.[77]



Given a category $\mathcal{C}$ we now introduce its doppelgänger, namely the corresponding opposite category $\mathcal{C}^{\mathrm{op}}$. The idea is very simple, we keep the objects, but we "invert the direction of the morphisms". This simple concept can be very convenient in future discussions and gives, almost for free, a rich source of new categories.

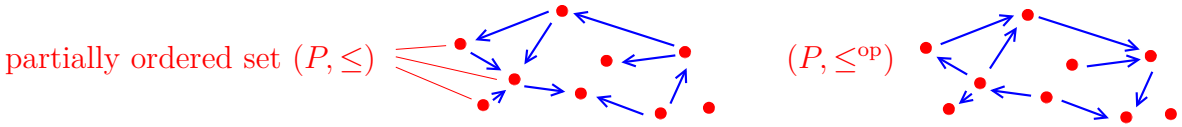
Definition 5.6. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined as the category where

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) &:= \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X,Y) &:= \mathrm{Mor}_{\mathcal{C}}(Y,X) \end{aligned}$$

with the composition given by $f \circ_{\mathcal{C}^{\mathrm{op}}} g := g \circ_{\mathcal{C}} f$.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathit{Order}_{(P,\leq)}$. The opposite of the order category $\mathit{Order}_{(P,\leq)}$ is the order category corresponding to the partially ordered set $(P, \leq^{\mathrm{op}})$ where $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$.



(2) For any category $\mathcal{C}$ we evidently have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. □

We continue with another general construction of categories:

Definition 5.7. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be categories.

(1) The product category $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ is defined as the category where

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}_1 \times \ldots \mathcal{C}_n) &:= \{C_1 \times \cdots \times C_n \,|\, C_i \in \mathrm{Ob}(\mathcal{C}_i)\} \\ \mathrm{Mor}_{\mathcal{C}_1 \times \cdots \times \mathcal{C}_n}(C_1 \times \cdots \times C_n, D_1 \times \cdots \times D_n) &:= \mathrm{Mor}_{\mathcal{C}_1}(C_1, D_1) \times \ldots \mathrm{Mor}_{\mathcal{D}}(C_n, D_n) \end{aligned}$$

with the composition given by $(f_1, \ldots, f_n) \circ (g_1, \ldots, g_n) := (f_1 \circ g_1, \ldots, f_n \circ g_n)$.

[77] For example, we have $\mathrm{Mor}(x,y) \neq \varnothing$ and $\mathrm{Mor}(y,z) \neq \varnothing$ precisely if $x \leq y$ and $y \leq z$. But since we are dealing with a partial order we have $x \leq z$, which implies that $\mathrm{Mor}(x,z) = \{*\}$. So the composition of the unique morphism in $\mathrm{Mor}(x,z)$ and the unique morphism in $\mathrm{Mor}(y,z)$ is the unique morphism in $\mathrm{Mor}(x,z)$.

(2) Given a category $\mathcal{C}$ and given $n \in \mathbb{N}$ we write $\mathcal{C}^n := \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n \text{ times}}$.

For completeness we provide following definition, even though it is almost self-explanatory.

Definition 5.8. Let $\mathcal{C}$ be a category. A subcategory is a category $\mathcal{D}$ such that $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ and such that for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Mor}_{\mathcal{D}}(X,Y) \subseteq \mathrm{Mor}_{\mathcal{C}}(X,Y)$. We say that $\mathcal{D}$ is a full subcategory if for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Mor}_{\mathcal{D}}(X,Y) = \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Examples.

(1) The category $\mathcal{AbGroup}$ of abelian groups is a full subcategory of the category $\mathcal{Group}$ of groups.
(2) Given a category $\mathcal{C}$ we can consider the subcategory $\mathcal{C}^{\cong}$ which has the same objects, but where we only consider isomorphisms. This is a subcategory of $\mathcal{C}$, but in general it is certainly not a full subcategory of $\mathcal{C}$. □

5.2. **Isomorphisms.** In this section we introduce the notion of an isomorphism, which generalizes the notion that is well-known from linear algebra.

Definition 5.9. Let $\mathcal{C}$ be a category and let X, Y be two objects in $\mathcal{C}$.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ be a morphism and let $g \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$.
  (a) We say that g is left inverse of f if $g \circ f = \mathrm{id}_X$.
  (b) We say that g is right inverse of f if $f \circ g = \mathrm{id}_Y$.
  (c) We say that g is an inverse of f if g is a left and a right inverse of f.
(2) A morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called invertible, or an isomorphism, if it admits an inverse.
(3) We denote by $\mathrm{Mor}_{\mathcal{C}}(X,Y)_{\mathrm{inv}}$ the set of all invertible morphisms.
(4) Two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ are called isomorphic, if there exists an isomorphism in $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.
(5) Given an object $X \in \mathrm{Ob}(\mathcal{C})$ we refer to the class of all objects that are isomorphic to X as the isomorphism type of X.

Example.

(1) In the category of vector spaces the invertible morphisms are precisely the isomorphisms which are familiar from linear algebra.
(2) In the category of sets the isomorphisms are precisely the bijections.
(3) In the category of topological spaces the isomorphisms are precisely the homeomorphisms.
(4) On page 229 we considered, given a group G, the category where the set of objects consists of a single element $\{*\}$ and where $\mathrm{Mor}(\{*\},\{*\}) := G$. Thus every morphism is an isomorphism. In fact categories in which every morphism is an isomorphism are called groupoids. □

We move on to an elementary but useful lemma:

Lemma 5.10. (Inverse Uniqueness Lemma) Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$. If f admits a left inverse $\widetilde{g}$ and a right inverse g, then they agree, i.e. $g = \widetilde{g}$. In particular f is invertible.

(2) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}(X, Y)$. If f admits an inverse, then the inverse is unique.
(3) Let $X \in \mathrm{Ob}(\mathcal{C})$. The set of invertible morphisms $\mathrm{Mor}_{\mathcal{C}}(X, X)_{\mathrm{inv}}$ forms a group under the composition of morphisms.

Definition 5.11. Let $\mathcal{C}$ be a category, let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ be an invertible morphism. By the Inverse Uniqueness Lemma 5.10 there exists a *unique* morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We refer to this morphism as the inverse of f and usually we denote it by f^{-1}.

Proof of Lemma 5.10.

(1) Let $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ be a morphism with $f \circ g = \mathrm{id}_Y$ and let $\widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ be a morphism with $\widetilde{g} \circ f = \mathrm{id}_X$. We have the following equalities of morphisms:

$$g \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \mathrm{id}_X}}{=} \mathrm{id}_X \circ g \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \widetilde{g}}}{=} (\widetilde{g} \circ f) \circ g \underset{\substack{\uparrow \\ \text{associativity of } \circ}}{=} \widetilde{g} \circ (f \circ g) \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } g}}{=} \widetilde{g} \circ \mathrm{id}_Y \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \mathrm{id}_Y}}{=} \widetilde{g}.$$

(2) Suppose we have $g, \widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ and with $f \circ \widetilde{g} = \mathrm{id}_Y$ and $\widetilde{g} \circ f = \mathrm{id}_X$. It follows immediately from (1) that $g = \widetilde{g}$.
(3) Let $X \in \mathrm{Ob}(\mathcal{C})$. First note that by definition $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is actually a set. Next note that by definition of a category we can compose any two morphisms that lie in $\mathrm{Mor}_{\mathcal{C}}(X, X)$. It follows easily from the associativity of "$\circ$" that the composition of two invertible morphisms is again invertible. It now follows from Properties (3A) and (3B) of a category that $\mathrm{Mor}_{\mathcal{C}}(X, X)_{\mathrm{inv}}$ forms a group. ∎

Lemma 5.12. (Isomorphism Lemma) Let $\mathcal{C}$ be a category.
(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ be morphisms. If two morphisms out of $f, g, g \circ f$ are invertible, then so is the third and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
(2) "Being isomorphic" is an equivalence relation in the class of objects of $\mathcal{C}$.[78]

Proof. The two statements are basically trivial. We leave it to the nervous reader to fill in the details. ∎

5.3. **Monomorphisms and epimorphisms.** In the context of categories the following definition is very natural.

Definition 5.13. Let $\mathcal{C}$ be a category.
(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called a monomorphism if for every two morphisms f_1, f_2 in $\mathrm{Mor}_{\mathcal{C}}(W, X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.

[78] The perspicacious reader might notice that here things get logically a little dicey. More precisely, in Definition 1.13 we had recalled the notion of an equivalence relation. But we had only defined the notion of an "equivalence relation" on a set and "all topological spaces" do not form a set they "only" form a class. One can also define the notion of an "equivalence relation" on a class, see e.g. [**FPr1985**, Kapitel 2]. But we do not want to get into the discussion "what is a class?". As a cheap way out we interpret Statement (2) of the lemma as saying that reflexivity, symmetry and transitivity hold for "homotopy equivalence" with the obvious interpretation.

(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called an epimorphism if for every two morphisms h_1, h_2 in $\mathrm{Mor}_{\mathcal{C}}(Y,Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

Of course the names "monomorphism" and "epimorphism" sound familiar. We will now see that in many, but perhaps not all settings, the new notions coincide with the familiar notions.

Proposition 5.14. (Monomorphism Proposition) Suppose we are in one of the following categories:

(1) The category *Group* of groups.
(2) The category *AbGroup* of abelian groups.
(3) The category $\mathcal{V}ec_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.
(4) The category *Ring* of rings.

If $\varphi\colon G \to H$ is a monomorphism in any of the above categories, then φ, viewed as a map between sets, is injective.

Proof. Let $\varphi\colon G \to H$ be a monomorphism in one of the first three categories. We set $K := \ker(\varphi) := \{g \in G \mid \varphi(g) = e\}$, where e is the trivial element of the group or the zero in the vector space. Note that in the first three categories we listed K is again an object of the category and φ is injective if and only if K is trivial.

We consider the trivial homomorphisms $\alpha\colon K \to G$ and we consider the inclusion homomorphism $\beta\colon K \to G$. Note that $\varphi \circ \alpha = \varphi \circ \beta$. Since φ is a monomorphism we see that $\alpha = \beta$. But this implies that K is trivial.

Finally we turn to the category *Ring* of rings. Note that the argument above does not apply since the multiplicatively neutral element never lies in the kernel, which implies that the kernel of a ring homomorphism is not a ring. Therefore we have to work a little harder. Let $\varphi\colon A \to B$ be a monomorphism in the category *Ring* of rings and let $a_1, a_2 \in A$ with $\varphi(a_1) = \varphi(a_2)$. We consider the polynomial ring $\mathbb{Z}[x]$ and we consider the two homomorphisms $f_1, f_2\colon \mathbb{Z}[x] \to A$ that are defined by $f_1(x) = a_1$ and $f_2(x) = a_2$. Then $(\varphi \circ f_1)(x) = (\varphi \circ f_2)(x)$ and hence $\varphi \circ f_1 = \varphi \circ f_2\colon \mathbb{Z}[x] \to B$. Since φ is a monomorphism we see that $f_1 = f_2$ and thus $a_1 = a_2$. ■

Proposition 5.15. (Epimorphism Proposition) Suppose we are in one of the following categories:

(1) The category *Group* of groups.
(2) The category *AbGroup* of abelian groups.
(3) The category $\mathcal{V}ec_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.

If $\varphi\colon G \to H$ is an epimorphism in any of the above categories, then φ, viewed as a map between sets, is surjective.

Example. The perspicacious reader will have noticed that in the Epimorphism Proposition 5.15 we do not say anything about the category of rings. In fact in Exercise 5.3 we will see that in the category *Ring* of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism. □

Proof. The case of the category *AbGroup* of abelian groups and the category of $\mathbb{K}$-vector spaces is straightforward, basically one just needs to consider the "naive cokernel". We will deal with these two cases in Exercise 5.3.

The case of the category *Group* of groups is trickier, since the cokernel of a group homomorphism is not necessarily a group. We will now deal with this case in detail, following the argument given in [**Lin1970**] and [**Pare1970**, p. 16]. (An alternative proof is given in [**Bur1965**, p. 767] and in Exercise 98.33.) Thus let $\varphi\colon G \to H$ be an epimorphism in the category *Group* of groups. We consider the set $H/\varphi(G)$ of left cosets of $\varphi(G)$ in H.[79] Let $\{*\}$ be a symbol that is not an element of $H/\varphi(G)$. We set $Y := \{*\} \cup H/\varphi(G)$. We denote by $\mathrm{Bij}(Y)$ the group of bijections $Y \to Y$. We consider the map

$$\begin{array}{rcl} \alpha\colon H & \to & \mathrm{Bij}(Y) \\ h & \mapsto & \left(\begin{array}{ccl} Y & \to & Y \\ y & \mapsto & \begin{cases} (h\cdot k)\cdot\varphi(G), & \text{if } y = k\cdot\varphi(G) \in H/\varphi(G), \\ *, & \text{if } y = *. \end{cases} \end{array} \right) \end{array}$$

One can easily verify that $\alpha\colon H \to \mathrm{Bij}(Y)$ is in fact a homomorphism.

Next let $\sigma \in \mathrm{Bij}(Y)$ be the bijection that swaps $\varphi(G) \in Y$ with $* \in Y$ and that keeps all other elements of Y fixed. Let $\beta\colon H \to \mathrm{Bij}(Y)$ be the homomorphism that is given by $h \mapsto \sigma \circ \alpha(h) \circ \sigma^{-1}$.

Claim. We have $\alpha \circ \varphi = \beta \circ \varphi\colon G \to \mathrm{Bij}(Y)$.

Proof. Let $g \in G$. It follows immediately from the definitions that $\alpha(\varphi(g)) \in \mathrm{Bij}(Y)$ fixes the coset $\varphi(G)$ and that it fixes $*$. It follows that the support[80] of $\alpha(\varphi(g)) \in \mathrm{Bij}(Y)$ and $\sigma \in \mathrm{Bij}(Y)$ are disjoint. This implies that $\alpha(\varphi(g))$ and σ commute. Now we see that

$$\alpha(\varphi(g)) \;=\; \sigma^{-1} \circ (\sigma \circ \alpha(\varphi(g))) \underset{\substack{\uparrow \\ \text{since } \alpha(\varphi(g)) \text{ and } \sigma \text{ commute}}}{=} \sigma^{-1} \circ \alpha(\varphi(g)) \circ \sigma \underset{\substack{\uparrow \\ \text{definition of } \beta}}{=} \beta(\varphi(g)).$$

Thus we have shown that $\alpha \circ \varphi = \beta \circ \varphi\colon G \to \mathrm{Bij}(Y)$. ⊞

Since φ is an epimorphism we obtain from the claim that $\alpha = \beta\colon H \to \mathrm{Bij}(Y)$. We need to show that $\varphi(G) = H$. Let $h \in H$. As we just noticed, we have $\alpha(h) = \beta(h) \in \mathrm{Bij}(Y)$. By definition of β this means that $\alpha(h) = \sigma \circ \alpha(h) \circ \sigma^{-1}$. In other words, we see that $\alpha(h)$ commutes with σ. This in turn implies that $\alpha(h) \in \mathrm{Bij}(Y)$ leaves $\varphi(G) \in Y$ fixed, i.e. we have $h \cdot \varphi(G) = \varphi(G)$. But that implies that $h \in \varphi(G)$. We have thus shown that $\varphi(G) = H$. ■

5.4. Initial and terminal objects.

In this very short section we study the following definition:

Definition 5.16. Let $\mathcal{C}$ be a category.

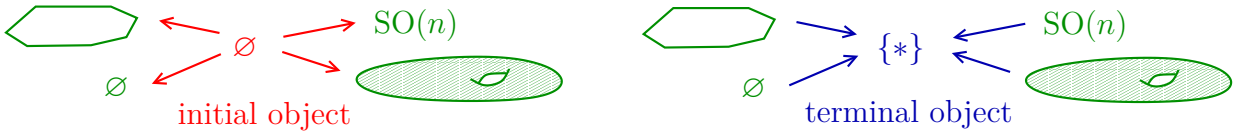
(1) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called initial if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from X to Y.
(2) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called terminal if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from Y to X.

Example.

[79] In general, if $A \subseteq H$ is a subgroup, then we obtain an equivalence relation on H by setting $x \sim y$ if there exists an $h \in H$ with $x \cdot h = y$. The set of equivalence classes is denoted by H/A. Note that each equivalence class is of the form $hA = \{h \cdot a \mid a \in A\}$.

[80] Given $\tau \in \mathrm{Bij}(Y)$ we define its support as the subset $\{y \in Y \mid \tau(y) \neq y\}$.

(1) In the category *Set* of sets an initial object is given by the empty set and a terminal object is given by any set consisting of a single element. Similarly in the category *Top* of topological spaces an initial object is given by the empty topological space and a terminal object is given by a topological space consisting of a single element.



(2) In the category *Group* of groups any group consisting of a single element is an initial and also a terminal object. The same applies to the category *AbGroup* of abelian groups and the analogous statement holds in the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$.
(3) In Exercise 5.11 we will see that the category *Field* of fields has neither an initial nor a terminal object.
(4) Let X be an initial object of a category $\mathcal{C}$. It follows easily from the definitions that it is also a terminal object of the opposite category $\mathcal{C}^{\mathrm{op}}$. □

Lemma 5.17. (Initial-Terminal Object Uniqueness Lemma) Let $\mathcal{C}$ be a category.
(1) If an initial object exists, then it is unique up to isomorphism.
(2) If a terminal object exists, then it is unique up to isomorphism.

Proof. The statements follow easily from the definition. We will fill in the details in Exercise 5.11. ■

5.5. **Covariant functors.** In this and the following section we introduce the all-important notions of covariant and contravariant functors.

Definition 5.18. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ of a map

$$F\colon \mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X),F(Y)),$$

such that the following axioms are satisfied:
(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ the following equality holds

$$F(\psi \circ \phi) = F(\psi) \circ F(\phi) \quad \in \mathrm{Mor}_{\mathcal{D}}(F(X),F(Z)).$$

Convention 5.19. If the covariant functor F is understood from the context, then given a morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ_* instead of $F(\phi)$. The first axiom (F1) then becomes $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and the second axiom (F2) becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Examples.

(1) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ (V,+,\cdot) & \mapsto & V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(V,W) & \to & \mathrm{Mor}_{\mathcal{Set}}(V,W) \\ (\varphi\colon V \to W) & \mapsto & (\varphi\colon V \to W) \end{array}$$

define a covariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to the category $\mathcal{Set}$ of sets. This example can be generalized to many contexts, for example there is a similar covariant functor from the category of topological spaces to the category of sets. Functors of this type are colloquially called *forgetful functor*.

(2) Given any category there is the identity functor with the obvious definition. Funnily enough this functor will soon, namely on page 241, play an interesting role.

(3) Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{Set}$ be the category of sets. It follows easily from the Image-Preimage-Component Lemma 4.50 (2) that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Top}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \pi_0(X) \coloneqq \text{set of path-components of } X, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\pi_0(X),\pi_0(Y)) \\ f & \mapsto & \left(\begin{array}{rcl} \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & \text{unique path-component} \\ & & \text{of } Y \text{ that contains } f(C) \end{array} \right) \end{array}$$

define a covariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{Set}$ of sets.

(4) The maps

$$\begin{array}{rcl} X & \mapsto & (X, \text{discrete topology}) \\ f & \mapsto & f \end{array}$$

define a covariant functor from the category $\mathcal{Set}$ of sets to the category $\mathcal{Top}$ of topological spaces. Evidently we can also replace the discrete topology by the trivial topology.

(5) Given partially ordered sets $(P,\leq)$ and $(Q,\leq)$ we introduced in Definition 5.5 the corresponding order categories $\mathcal{Order}_{(P,\leq)}$ and $\mathcal{Order}_{(Q,\leq)}$. It is an amusing exercise to verify that the covariant functors from $\mathcal{Order}_{(P,\leq)}$ to $\mathcal{Order}_{(Q,\leq)}$ correspond precisely to the order-preserving maps $(P,\leq)$ and $(Q,\leq)$ from Definition 1.27. □

We state our next example of a covariant functor as a lemma.

Lemma 5.20. (Fixed Domain-Covariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F = \mathrm{Mor}(W,-)\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(W,X) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\mathrm{Mor}_{\mathcal{D}}(W,X),\mathrm{Mor}_{\mathcal{D}}(W,Y)) \\ \phi & \mapsto & \left(\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(W,X) & \to & \mathrm{Mor}_{\mathcal{D}}(W,Y) \\ g & \mapsto & \phi \circ g \end{array} \right) \end{array}$$

is a covariant functor from the category $\mathcal{D}$ to the category $\mathcal{S}et$ of sets. This functor is sometimes called a covariant Hom-functor.

Proof. The statement follows easily from the axioms of a category. ∎

Example. The example of a covariant functor in the Fixed Domain-Covariant Functor Lemma 5.20 comes with many variations, in particular if $\mathrm{Mor}_{\mathcal{D}}(W, X)$ has more structure than that of a set. For instance let W be a real vector space. Then the above map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) \\ V & \mapsto & \underbrace{\mathrm{Hom}_{\mathbb{R}}(W, V)}_{\text{real vector space}}, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(U, V) & \to & \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(F(U), F(V)) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(W, U) & \to & \mathrm{Hom}_{\mathbb{R}}(W, V) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to itself. □

5.6. **Contravariant functors.** In this section we introduce the confusing evil cousin of the covariant functor, namely the contravariant functor.

Definition 5.21. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ of a map

$$\mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{D}}(F(Y), F(X))$$

such that the following axioms are satisfied:

(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ we have

$$F(\psi \circ \phi) = F(\phi) \circ F(\psi) \quad \in \mathrm{Mor}_{\mathcal{D}}(F(Z), F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor "reverses the direction of morphisms" and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom (F2) are swapped. In fact, it follows immediately from the definitions that a contravariant functor $\mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ to the opposite category.

Convention 5.22. If the contravariant functor F is understood from the context, then given a morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ we often write ϕ^* instead of $F(\phi)$. The first axiom then becomes $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and the second axiom becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

We state our first example as a lemma, namely we have the following analog of the Fixed Domain-Covariant Functor Lemma 5.20.

Lemma 5.23. (Fixed Target-Contravariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rccl} F = \mathrm{Mor}(-,W)\colon & \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ & X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(X,W) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\mathrm{Mor}_{\mathcal{D}}(Y,W),\mathrm{Mor}_{\mathcal{D}}(X,W)) \\ f & \mapsto & \left(\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(Y,W) & \to & \mathrm{Mor}_{\mathcal{D}}(X,W) \\ g & \mapsto & g\circ f \end{array}\right) \end{array}$$

is a contravariant functor from the category $\mathcal{D}$ to the category $\mathcal{Set}$ of sets. This functor is sometimes called a contravariant Hom-functor.

Proof. Again the statement follows easily from the axioms of a category. ■

Example. The Fixed Target-Contravariant Functor Lemma 5.23 can be viewed as the mother of all contravariant functors. As in the case of the Fixed Domain-Covariant Functor Lemma 5.20 it spawns many variations if $\mathrm{Mor}_{\mathcal{D}}(X,W)$ has more structure than just being a set.

(1) The map

$$\begin{array}{rccl} F\colon & \mathrm{Ob}(\mathcal{Set}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ & X & \mapsto & \mathrm{Map}(X,\mathbb{R}) := \underbrace{\text{set of maps } X\to\mathbb{R}}_{\text{vector space in the obvious way}}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Set}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(\mathrm{Map}(Y,\mathbb{R}),\mathrm{Map}(X,\mathbb{R})) \\ (\phi\colon X\to Y) & \mapsto & \left(\begin{array}{ccc} \mathrm{Map}(Y,\mathbb{R}) & \to & \mathrm{Map}(X,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{array}\right) \end{array}$$

is a contravariant functor from the category $\mathcal{Set}$ of sets to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

(2) The map

$$\begin{array}{rccl} F\colon & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ & V & \mapsto & V^* := \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U,V) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(V^*,U^*) \\ (\phi\colon U\to V) & \mapsto & \left(\begin{array}{ccc} \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) & \to & \mathrm{Hom}_{\mathbb{R}}(U,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{array}\right) \end{array}$$

is a contravariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to itself. □

The following lemma shows that contravariant functors are covariant functors in disguise.

Lemma 5.24. (Functor–Opposite Category Lemma) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote by $\mathcal{D}^{\mathrm{op}}$ the opposite category as introduced in Definition 5.6. If

$$\begin{array}{l} F\colon \mathcal{C} \to \mathcal{D} \\ \quad X \mapsto F(X) \\ (X \xrightarrow{\varphi} Y) \mapsto (F(Y) \xrightarrow{F(\varphi)} F(X)) \end{array} \quad \text{is a contravariant functor between two categories, then} \quad \begin{array}{l} \mathcal{C} \to \mathcal{D}^{\mathbf{op}} \\ \quad X \mapsto F(X) \\ (X \xrightarrow{\varphi} Y) \mapsto (F(Y) \xrightarrow{F(\varphi)} F(X)) \end{array}$$

is a covariant functor. The same statement also holds with the roles of "covariant" and "contravariant" swapped.

Proof. The statement follows immediately from the definitions. ■

The Functor–Opposite Category Lemma 5.24 can be useful for developing the general study of functors. When we work with "real life" categories, then personally we find that the Functor–Opposite Category Lemma 5.24 is not all that useful and therefore we prefer to work with both notions of functors.

The next lemma says that functors can be composed in a fairly obvious way.

Lemma 5.25. (Functor Composition Lemma) Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be three categories.

(1) Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be covariant functors. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & \to & \mathrm{Ob}(\mathcal{E}) \\ X & \mapsto & G(F(X)) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{E}}(G(F(X)), G(F(Y))) \\ (\varphi\colon X \to Y) & \mapsto & (G(F(\varphi))\colon G(F(X)) \to G(F(Y))) \end{array}$$

define a covariant functor from $\mathcal{C}$ to $\mathcal{E}$. We denote this functor by $G \circ F$.

(2) Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be *two contravariant functors*. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & \to & \mathrm{Ob}(\mathcal{E}) \\ X & \mapsto & G(F(X)) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{E}}(G(F(X)), G(F(Y))) \\ (\varphi\colon X \to Y) & \mapsto & (G(F(\varphi))\colon G(F(X)) \to G(F(Y))) \end{array}$$

define a *covariant functor* from the category $\mathcal{C}$ to the category $\mathcal{E}$.

(3) Similarly we can define the composition of a covariant functor with a contravariant functor and we obtain a contravariant functor.

Proof. The lemma again follows easily from the definitions. Nonetheless it is worth going through the argument to make sure that all maps actually make sense and go into the desired direction. ■

Example. We consider the covariant functors $F\colon \mathcal{Set} \to \mathcal{PairSet}$ and $G\colon \mathcal{PairSet} \to \mathcal{Set}$ that are given by

$$F\colon \begin{array}{rcl} \mathrm{Ob}(\mathcal{Set}) & \to & \mathrm{Ob}(\mathcal{PairSet}) \\ X & \mapsto & (X, \varnothing) \end{array} \quad \text{and} \quad G\colon \begin{array}{rcl} \mathrm{Ob}(\mathcal{PairSet}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ (X, A) & \mapsto & X \end{array}$$

together with the obvious maps on morphisms. The composition $G \circ F$ is just the identity functor on $\mathcal{Set}$ and the composition $F \circ G$ is the covariant functor that sends (X, A) to $(X, \varnothing)$. □

We conclude this section with the following basically obvious lemma, which often gets used subconsciously.

Lemma 5.26. (Isomorphism Preservation Lemma) Every functor (regardless of whether it is covariant or contravariant) sends invertible morphisms to invertible morphisms.

Example. Let $\mathcal{C}$ be a category and let $Z \in \mathrm{Ob}(\mathcal{C})$. It follows from the Isomorphism Preservation Lemma 5.26 applied to the functors coming from the Fixed Domain-Covariant Functor Lemma 5.20 and the Fixed Target-Contravariant Functor Lemma 5.23 that for every invertible morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ the maps

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(Y, Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X, Z) \\ g & \mapsto & g \circ f \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(Z, X) & \to & \mathrm{Mor}_{\mathcal{C}}(Z, Y) \\ g & \mapsto & f \circ g \end{array}$$

are isomorphisms in the category of sets, i.e. they are bijections. □

Proof of Lemma 5.26. The lemma follows immediately from the definitions. ■

5.7. **Natural transformations.** In this section we introduce the notion of a natural transformation between two functors. This topic might eventually look even more esoteric, but once one keeps one's eyes open one realizes that the mathematical world is stuffed with natural transformations and natural isomorphisms. They just suffer from the sad fate that often they do not recognized as such.

After this long preamble, let us give the definition of natural transformations:

Definition 5.27. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G \colon \mathcal{C} \to \mathcal{D}$ be two covariant functors.

(1) A natural transformation between the functors F and G assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X \colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $\mu \colon X \to Y$ in $\mathcal{C}$ the following diagram of morphisms in $\mathcal{D}$ commutes:

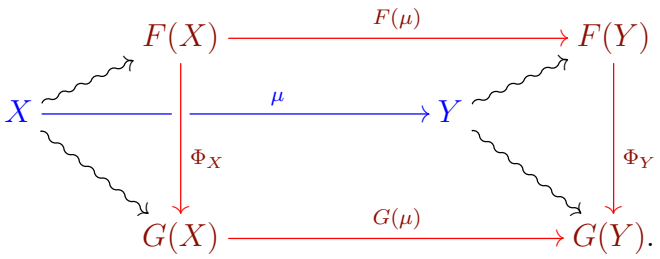
$$\begin{array}{ccc} F(X) & \xrightarrow{F(\mu)} & F(Y) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ G(X) & \xrightarrow{G(\mu)} & G(Y). \end{array}$$

(2) If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X \colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a natural isomorphism. In this case, given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are naturally isomorphic.[81]

The same way we can of course also define the notion of a natural transformation and a natural isomorphism between two contravariant functors.

Remark. The notion of a natural transformation can also be summarized in the following diagram:

[81] In the literature the words "natural isomorphism" and "canonical isomorphism" are often used indiscriminately, but whereas the expression "natural isomorphism" has a precise meaning, the expression "canonical isomorphism" is not a well-defined mathematical concept. However, in the lecture notes we will on occasions write that "some object exists canonically". This means that in the proof we show the existence of the object by giving an explicit construction that does not rely on any choices.



Here the wiggly arrows are not morphisms but they are just meant to remind us which objects a functor associates to the objects X and Y. □

Now we will see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Example.

(1) Let $\mathbb{K}$ be a field. We consider the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. Given a $\mathbb{K}$-vector space V we write $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and given $f\colon V \to W$ we denote by $f^*\colon W^* \to V^*$ the induced map. We consider the following two covariant functors from $\mathcal{V}ec_{\mathbb{K}}$ to $\mathcal{V}ec_{\mathbb{K}}$:

$$\begin{array}{rcl} \mathrm{id}\colon V & \mapsto & V \\ (f\colon V \to W) & \mapsto & (f\colon V \to W) \end{array} \quad \text{and} \quad \underbrace{\begin{array}{rcl} F\colon V & \mapsto & (V^*)^* \\ (f\colon V \to W) & \mapsto & ((f^*)^*\colon (V^*)^* \to (W^*)^*). \end{array}}_{\substack{\text{by the Functor Composition Lemma 5.25} \\ \text{this is a covariant functor}}}$$

For each $\mathbb{K}$-vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{K} \\ (\varphi\colon V \to \mathbb{K}) & \mapsto & \varphi(v) \end{array} \right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor $\mathrm{id}\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$ to the functor $F\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$. If we restrict ourselves to the category $\mathcal{V}ec_{\mathbb{K}}^{\dim<\infty}$ of finite-dimensional $\mathbb{K}$-vector spaces, then the above maps show that any finite-dimensional $\mathbb{K}$-vector space V is naturally isomorphic to its double dual $(V^*)^*$. We will discuss this example in even greater detail in Exercise 5.12.

(2) Let $\mathbb{K}$ be a field. We denote by $\mathcal{E}pi$ the following category:

- The objects are epimorphisms $\varphi\colon V \to W$ between vector spaces over $\mathbb{K}$.
- A morphism between objects $\varphi\colon V \to W$ and $\varphi'\colon V' \to W'$ consists of two $\mathbb{K}$-linear maps $\alpha\colon V \to V'$ and $\beta\colon W \to W'$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ V' & \xrightarrow{\varphi'} & W'. \end{array}$$

The composition of two morphisms (α, β) and $(\widetilde{\alpha}, \widetilde{\beta})$ is defined as $(\widetilde{\alpha} \circ \alpha, \widetilde{\beta} \circ \beta)$, provided $\widetilde{\alpha} \circ \alpha$ and $\widetilde{\beta} \circ \beta$ are defined.

The following maps define covariant functors from the category $\mathcal{Epi}$ to the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$:

$$\begin{array}{rcl} F\colon \mathcal{Epi} & \to & \mathcal{Vec}_{\mathbb{K}} \\ (\varphi\colon V \to W) & \mapsto & V/\ker(\varphi) \\ (\alpha,\beta) & \mapsto & \begin{pmatrix} V/\ker(\varphi) & \to & V'/\ker(\varphi') \\ [v] & \mapsto & [\alpha(v)] \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{Epi} & \to & \mathcal{Vec}_{\mathbb{K}} \\ (\varphi\colon V \to W) & \mapsto & W \\ (\alpha,\beta) & \mapsto & \beta. \end{array}$$

It is easy to verify that the maps

$$\begin{array}{rcl} V/\ker(\varphi) & \to & W \\ [v] = v + \ker(\varphi) & \mapsto & \varphi(v) \end{array}$$

form a natural isomorphism from the functor F to the functor G.

(3) We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{Ring} & \to & \mathcal{Ring} \\ R & \mapsto & \mathbb{Z} \\ (f\colon R \to S) & \mapsto & (\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{id}\colon \mathcal{Ring} & \to & \mathcal{Ring} \\ R & \mapsto & R \\ (f\colon R \to S) & \mapsto & (f\colon R \to S). \end{array}$$

For each commutative ring R we consider the ring homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathbb{Z} & \to & R \\ n & \mapsto & n \cdot 1_R. \end{array}$$

By our definition of a ring homomorphism in Definition 2.6 it follows that for each ring homomorphism $f\colon R \to S$ we have $f(1_R) = 1_S$. This implies that the following diagram commutes:

$$\begin{array}{ccc} F(R) = \mathbb{Z} & \xrightarrow{F_*(f)=\mathrm{id}} & \mathbb{Z} = F(S) \\ {\scriptstyle \Phi_R}\downarrow & & \downarrow{\scriptstyle \Phi_S} \\ \mathrm{id}(R) = R & \xrightarrow{\mathrm{id}_*(f)=f} & S = \mathrm{id}(S). \end{array}$$

But this shows that the ring homomorphisms Φ_R define a natural transformation from the covariant functor F to the identity functor id. Thus, given a commutative ring, we refer to this ring homomorphism $\mathbb{Z} \to R$ as the *natural ring homomorphism*.

(4) Let $\mathcal{CRing}$ be the category of commutative rings and let $\mathcal{Group}$ be the category of groups. Let $n \in \mathbb{N}$. We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{CRing} & \to & \mathcal{Group} \\ R & \mapsto & \mathrm{GL}(n,R) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{CRing} & \to & \mathcal{Group} \\ R & \mapsto & R^\times := \underset{\uparrow}{\text{units in } R.} \end{array}$$

see Definition 2.6

For each commutative ring R we consider the group homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathrm{GL}(n,R) & \to & R^\times \\ A & \mapsto & \det(A). \end{array}$$

We claim that these group homomorphisms define a natural transformation from F to G. Thus let $f\colon R \to S$ be a homomorphism between commutative rings and let $A \in F(R)$, i.e. $A = (a_{ij})$ is a matrix in $\mathrm{GL}(n,R)$. We have the following equalities in the abelian group $S^\times$:

$$\begin{aligned}\Phi_S(F_*(f)(A)) &= \det((f(a_{ij}))_{i,j=1,\dots,n}) \overset{\text{by the Leibniz Formula 2.18}}{=} \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\cdot\prod_{i=1}^{n} f(a_{i,\sigma(i)}) \\ &\underset{\text{since } f \text{ is a ring homomorphism}}{=} f\Big(\sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\cdot\prod_{i=1}^{n} a_{i,\sigma(i)}\Big) \underset{\text{by the Leibniz Formula 2.18}}{=} f(\det(A)) = G_*(f)(\Phi_R(A)).\end{aligned}$$

This means, by definition, that the Φ_R define a natural transformation from F to G.□

Perhaps in this instance it is also important to give a "non-example".

Non-Example. Let $\mathcal{FGAbGroup}$ be the category of finitely generated abelian groups. Given an abelian group A we denote by $\operatorname{Tor}(A) \coloneqq \{g \in A \,|\, \text{there exists an } n \in \mathbb{N} \text{ with } n \cdot g = 0\}$ its torsion subgroup and we denote by $\mathrm{F}A \coloneqq A/\operatorname{Tor}(A)$ the maximal torsion-free quotient. We consider the following two covariant functors that are given by[82]

$$\begin{array}{rcl} F\colon \mathcal{FGAbGroup} & \to & \mathcal{FGAbGroup} \\ A & \mapsto & A \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{FGAbGroup} & \to & \mathcal{FGAbGroup} \\ A & \mapsto & \operatorname{Tor}(A)\oplus \mathrm{F}A \end{array}$$

together with the obvious maps on morphisms. It follows from the Abelian Group-Classification Theorem 97.24 that for any finitely generated abelian group A the groups $F(A) = A$ and $G(A) = \operatorname{Tor}(A) \oplus \mathrm{F}A$ are isomorphic. Now we claim that nonetheless there is *no* natural isomorphism between F and G. Indeed, suppose that there was such a natural isomorphism Φ. Let us consider the group $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ and the homomorphism

$$\begin{array}{rcl} f\colon A = \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z} = A \\ ([a],b) & \mapsto & ([b],0). \end{array}$$

Since we assume that Φ is a natural isomorphism we obtain the following commutative diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow[\cong]{\Phi_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow[\cong]{\Phi_A} & G(A) \end{array} \quad \text{which specializes in our setting to the following commutative diagram} \quad \begin{array}{ccc} \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[\cong]{\Phi_A} & \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z} \\ {\scriptstyle F(f)=f}\downarrow & & \downarrow{\scriptstyle G(f)=0} \\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[\cong]{\Phi_A} & \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}. \end{array}$$

Note that the vertical map to the left is non-trivial whereas a close inspection shows that the vertical map to the right is trivial. But this collides with the statement that the horizontal maps are isomorphisms and that the diagram commutes. □

On many occasions we, and many authors, say that a certain map is "natural". This means that the map is in fact a natural transformation between two suitable functors between two suitable categories. Sometimes it takes some thought to figure out what the right context is.

[82]Let us explain why G is in fact a covariant functor. Let $f\colon A \to B$ be a homomorphism between two finitely generated abelian groups. Since a homomorphism sends torsion elements to torsion elements we see that f restricts to a homomorphism $\operatorname{Tor}(A) \to \operatorname{Tor}(B)$. This also implies that f descends to a homomorphism $\mathrm{F}\,A = A/\operatorname{Tor}(A) \to \mathrm{F}\,B = B/\operatorname{Tor}(B)$. Summarizing we see that $f\colon A \to B$ induces a homomorphism $G(f)\colon G(A) = \operatorname{Tor}(A)\oplus \mathrm{F}\,A \to G(B) = \operatorname{Tor}(B)\oplus \mathrm{F}\,B$.

Example. Let G be a group and let A and B be normal subgroups of G with $A \subseteq B$. The Third Isomorphism Theorem in group theory, see [**Hun1980**, Corollary 5.10], says that the map

$$\begin{array}{rccc} \Phi_{(G,A,B)}\colon & G/B & \to & (G/A)/(B/A) \\ & g \cdot B & \mapsto & (g \cdot A) \cdot B/A \end{array}$$

is an isomorphism. This isomorphism is a natural isomorphism in the following sense. We consider the category $\mathcal{TripGr}$ of triples of groups which is defined as follows:

- The objects are the triples (G, A, B) where G is a group and A and B are normal subgroups of G with $A \subseteq B$.
- The morphisms from (G, A, B) to (G', A', B') are the homomorphisms $\varphi\colon G \to G'$ with $\varphi(A) \subseteq A'$ and $\varphi(B) \subseteq B'$.

We consider the two functors

$$\begin{array}{rccc} F\colon & \mathcal{TripGr} & \to & \mathcal{Group} \\ & (G, A, B) & \mapsto & G/B \end{array} \quad \text{and} \quad \begin{array}{rccc} G\colon & \mathcal{TripGr} & \to & \mathcal{Group} \\ & (G, A, B) & \mapsto & (G/A)/(B/A). \end{array}$$

One can easily verify that the above isomorphisms $\Phi_{(G,A,B)}$ from the third Isomorphism Theorem define a natural isomorphism from F to G. □

All of the above examples of natural transformations are perhaps not particularly impressive. But as we will see, the concept of a natural transformation will eventually become one of the key notions in this course.

5.8. **Equivalence of categories.** We start out with the obvious notion of an isomorphism between categories:

Definition 5.28. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an isomorphism between the categories $\mathcal{C}$ and $\mathcal{D}$ if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$.

Example. As we will discuss on page 5961, there exists an "obvious" isomorphism from the category $\mathcal{AbGroup}$ of abelian groups to the category of $\mathbb{Z}$-modules. □

For many purposes the notion of an isomorphism between categories is a little too strict. In the remainder of this section we will thus discuss what it means for two categories to be equivalent:

Definition 5.29. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

(1) An equivalence of the categories $\mathcal{C}$ and $\mathcal{D}$ is a tuple (F, G, μ, ν) where $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are two covariant functors and $\mu\colon G \circ F \to \mathrm{id}_{\mathcal{C}}$ and $\nu\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural isomorphisms.
(2) A weak equivalence of the categories $\mathcal{C}$ and $\mathcal{D}$ is a covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ that admits a covariant functor $G\colon \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms $\mu\colon G \circ F \to \mathrm{id}_{\mathcal{C}}$ and $\nu\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ such that (F, G, μ, ν) is an equivalence of categories.[83]
(3) We say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories, if there exists an equivalence of categories.

[83] In other words, a weak equivalence of categories is the same as an equivalence of categories, except that G, μ and ν are not part of the data. In particular an equivalence of categories exists if and only if a weak equivalence of categories exists. This particular author finds this naming convention very confusing and refused to accept responsibility for this language.

Remark. Given a weak equivalence F of categories $\mathcal{C}$ and $\mathcal{D}$ the corresponding G, μ, ν are not uniquely defined. □

Proposition 5.30. (Weak Equivalence of Categories Proposition) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. The following two statements are equivalent:

(1) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a weak equivalence of categories.
(2) The following three statements hold:
   (a) The functor F is full, i.e. for any two objects C_1, C_2 of the category $\mathcal{C}$ the map $\mathrm{Mor}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Mor}_{\mathcal{D}}(F(C_1), F(C_2))$ is surjective.
   (b) The functor F is faithful, i.e. for any two objects C_1, C_2 of the category $\mathcal{C}$ the map $\mathrm{Mor}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Mor}_{\mathcal{D}}(F(C_1), F(C_2))$ is injective.
   (c) The functor F is essentially surjective (or dense), i.e. each object D in the category $\mathcal{D}$ is isomorphic to an object of the form $F(C)$ where C is an object of the category $\mathcal{C}$.

Proof. The "(1)⇒(2)"-direction is elementary. The "(2)⇒(1)" builds on the Axiom of Choice 1.1 and can be proved "by hand" with a little bit of an effort. The details are worked out in [**Rie2016**, Theorem 1.5.9]. ■

Example. Let $\mathcal{C}$ be the category given by the objects $\{\mathbb{R}^n \mid n \in \mathbb{N}_0\}$, where the morphisms $\mathrm{Mor}_{\mathcal{C}}(\mathbb{R}^m, \mathbb{R}^n)$ are all real $(n \times m)$-matrices and where composition of morphisms is given by multiplication of matrices. Furthermore let $\mathcal{D}$ be the category where the objects are all finite-dimensional real vector spaces and where the morphisms are all homomorphisms. Note that it follows quite easily from the Weak Equivalence of Categories Proposition 5.30 and very elementary linear algebra that the obvious functor $\mathcal{C} \to \mathcal{D}$ is an equivalence of categories. □

5.9. Appendix: The Yoneda Lemma.

In this short appendix we want to formulate the Yoneda Lemma which is named for the Japanese mathematician Nobuo Yoneda. We will not make use of the Yoneda Lemma, but it does get used in many other contexts, and should not be missing from an introduction to category theory. To formulate the Yoneda Lemma it is helpful to introduce the following piece of notation:

Notation 5.31. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors. We denote by $\mathrm{Nat}(F, G)$ the sets of all natural transformations from F to G.

Now we are ready to formulate the Yoneda Lemma:

Lemma 5.32. (Yoneda Lemma)[84] Let $\mathcal{C}$ be a category and let F be a covariant functor from the category $\mathcal{C}$ to the category $\mathcal{Set}$ of sets. Then for any $W \in \mathrm{Obj}(\mathcal{C})$ the map[85] [86]

$$\begin{array}{rcl} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(W, -), F) & \to & F(W) \\ \Phi & \mapsto & \Phi_W(\mathrm{id}_W) \end{array}$$

is a bijection.

Example. Let $\mathcal{C}$ be a category. It follows immediately from the Yoneda Lemma 5.32 that for any $W, X \in \mathrm{Obj}(\mathcal{C})$ the map

$$\begin{array}{rcl} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(W,-), \mathrm{Mor}_{\mathcal{C}}(X,-)) & \to & \mathrm{Mor}_{\mathcal{C}}(X,W) \\ \Phi & \mapsto & \Phi_W(\mathrm{id}_W) \end{array}$$

is a bijection. □

Proof. We outsource the proof of the Yoneda Lemma 5.32 to Exercise 5.20, since reading the proof is not very enlightening. Alternatively we refer to [**Rie2016**, Theorem 2.2.4] or [**Awo2010**, Chapter 8.3] for a detailed proof. ■

We conclude this appendix with the following corollary to the Yoneda Lemma 5.32:

Proposition 5.33. (Yoneda Isomorphism Characterization Proposition) Let $\mathcal{C}$ be a category and let $W, X \in \mathrm{Ob}(\mathcal{C})$ be two objects. The following three statements are equivalent:

(1) The objects W and X are isomorphic.
(2) The covariant Hom-functors $\mathrm{Mor}_{\mathcal{C}}(W,-)$ and $\mathrm{Mor}_{\mathcal{C}}(X,-)$ are naturally isomorphic.
(3) The contravariant Hom-functors $\mathrm{Mor}_{\mathcal{C}}(-,W)$ and $\mathrm{Mor}_{\mathcal{C}}(-,X)$ are naturally isomorphic.

Remark. Loosely speaking the Yoneda Isomorphism Characterization Proposition 5.33 says that two objects W and X in a category are isomorphic if they have "the same interactions with all objects of the category". □

Proof. The equivalence of (1) and (2) follows easily from the Yoneda Lemma 5.32 (see also the above example). The equivalence of (1) and (3) follows easily from the Yoneda Lemma 5.32 applied to the opposite category $\mathcal{C}^{\mathrm{op}}$. ■

5.10. Exercises for Chapter 5.

Exercise 5.1. Let $\mathcal{C}$ be a category and let $f\colon A \to B$ and $g\colon B \to A$ be morphisms. We suppose that $f \circ g$ and $g \circ f$ are both invertible.

(a) Show that f and g are invertible.
(b) Does it follow that f and g are inverses of one another?

Exercise 5.2. Show that there exists a one-to-one correspondence between partially ordered sets and categories where each set of morphisms has at most one element.

Exercise 5.3.

(a) Show that in the category $\mathcal{AbGroup}$ of abelian groups the epimorphisms are precisely the surjective homomorphisms.

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[84] The Yoneda Lemma is sometimes also called the "lemme du gare du Nord".

[85] Here $\mathrm{Mor}(W,-)\colon \mathcal{C} \to \mathcal{Set}$ is the covariant Hom-functor from the Fixed Domain-Covariant Functor Lemma 5.20.

[86] Recall that according to Definition 5.27 any natural transformation Φ between covariant functors $E, F\colon \mathcal{C} \to \mathcal{Set}$ assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a map $\Phi_X\colon E(X) \to F(X)$ of sets. Thus, in our context it assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a map $\Phi_X\colon \mathrm{Mor}_{\mathcal{C}}(X,W) \to F(X)$. We apply this logic to $X = W$ and $\mathrm{id}_W \in \mathrm{Mor}_{\mathcal{C}}(W,W)$.

Remark. Ideally you should give an argument that also applies to the category $\mathcal{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces over some field $\mathbb{K}$.

(b) Show that in the category $\mathcal{Ring}$ of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

(c) An abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. We define the category of divisible abelian groups in the obvious way. Show that in this category the obvious homomorphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Exercise 5.4.

(a) Is every monomorphism in the category $\mathcal{Top}$ of topological spaces injective?

(b) Is every epimorphism in the category $\mathcal{Top}$ of topological spaces surjective?

(c) Is every epimorphism in the category of Hausdorff spaces surjective?

Exercise 5.5. Make precise, and prove the following statement: Epimorphisms in a category $\mathcal{C}$ correspond to monomorphisms in the opposite category.

Exercise 5.6. Let $\mathcal{C}$ be a category and let $f\colon X \to Y$ be a morphism. Show that the following two statements are equivalent:

(1) The morphism $f\colon X \to Y$ is an isomorphism.

(2) For every object $A \in \mathrm{Obj}(\mathcal{C})$ the following two maps

$$\begin{array}{ccc} \mathrm{Mor}(A,X) & \to & \mathrm{Mor}(A,Y) \\ \varphi & \mapsto & f \circ \varphi \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Mor}(Y,A) & \to & \mathrm{Mor}(X,A) \\ \varphi & \mapsto & \varphi \circ f \end{array}$$

are surjective.

Exercise 5.7.

(a) Let φ be a morphism in a category $\mathcal{C}$ that is a monomorphism and an epimorphism. Is f necessarily an isomorphism?

(b) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor.

(i) If φ is a monomorphism in $\mathcal{C}$, is $F(\varphi)$ a monomorphism in $\mathcal{D}$?

(ii) If φ is an epimorphism in $\mathcal{C}$, is $F(\varphi)$ an epimorphism in $\mathcal{D}$?

Exercise 5.8. Let $\mathcal{C}$ be a category and let $\Phi\colon A \to B$ and $\Psi\colon B \to C$ be morphisms. We assume that $\Phi \circ \Psi = \mathrm{id}_B$. Show that if Ψ is an epimorphism, then $\Psi \circ \Phi = \mathrm{id}_A$.

Exercise 5.9. Let S be a set and let $R \subseteq S \times S$ be a subset. Given $s, t \in S$ we set

$$\mathrm{Mor}(s,t) \;:=\; \begin{cases} \{(s,t)\}, & \text{if } (s,t) \in R, \\ \varnothing, & \text{if } (s,t) \notin R. \end{cases}$$

Show that the set S with the above morphisms defines a category if and only if R is a relation on S that is reflexive and transitive in the sense of Definition 1.14.

Exercise 5.10. Let $\mathcal{PairTop}$ be the category of pairs of topological spaces. Recall that this means that the objects are pairs (X, A), where X is a topological space and $A \subseteq X$ is a subset and morphisms $(X, A) \to (Y, B)$ are continuous maps $f\colon X \to Y$ with $f(A) \subseteq B$. Are the pairs $([0,1], \{1\})$ and $([0,2], \{1\})$ of topological spaces isomorphic in the category $\mathcal{PairTop}$ of pairs of topological spaces?

Exercise 5.11.

(a) Let $\mathcal{C}$ be a category. Show that if an initial object exists, then it is unique up to isomorphism.

(b) Which of the following categories admits an initial object and which ones admit a terminal object?
  (i) The category $\mathcal{Ring}$ of rings.
  (ii) The full subcategory $\mathcal{Field}$ of the category $\mathcal{Ring}$ that consists only of fields.
  (iii) Let $(P, \leq)$ be a partially order set. We consider the corresponding order category $\mathcal{Order}_{(P,\leq)}$.
(c) Give an example of a category that contains an initial object but not a terminal object.
(d) Give an example of a category that contains a terminal object but that does not contain an initial object.

Exercise 5.12. Let $\mathbb{K}$ be a field. Given a $\mathbb{K}$-vector space V we write $V^* \coloneqq \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$. Given a homomorphism $f\colon V \to W$ we denote by $f^*\colon W^* \to V^*$ the dual map. As on page 241 we consider the covariant functor $F\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$ F that is given by the maps

$$\begin{array}{rcl} V & \mapsto & (V^*)^* \\ (f\colon V \to W) & \mapsto & ((f^*)^*\colon (V^*)^* \to (W^*)^*). \end{array}$$

Furthermore for each $\mathbb{K}$-vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} \Phi_V\colon V & \to & (V^*)^* \\ v & \mapsto & \begin{pmatrix} & V^* & \to & \mathbb{K} \\ (\varphi\colon V \to \mathbb{K}) & & \mapsto & \varphi(v) \end{pmatrix}. \end{array}$$

On page 241 we saw that the maps Φ_V define a natural transformation from the identity functor $\mathrm{id}\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$ to the functor $F\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$.

(a) Show that the maps Φ_V are in general not isomorphisms.
(b) We denote by $\mathcal{Vec}_{\mathbb{K}}^{\mathrm{fin}}$ the full subcategory of $\mathcal{Vec}_{\mathbb{K}}$ that is given by the finite-dimensional $\mathbb{K}$-vector spaces. Note that F restricts to a covariant functor $\mathcal{Vec}_{\mathbb{K}}^{\mathrm{fin}} \to \mathcal{Vec}_{\mathbb{K}}^{\mathrm{fin}}$. Show that this functor is naturally isomorphic to the identity functor.

Exercise 5.13. Let $f\colon X \to Y$ be a continuous map between topological spaces. On page 236 we saw that we get an induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$.

(a) Show that if g admits a right-inverse $g\colon Y \to X$, then $f_*\colon \pi_0(X) \to \pi_0(Y)$ is surjective.
(b) Show that if f is injective, if X is compact, if X has finitely many components and if Y is Hausdorff and normal, then $f_*\colon \pi_0(X) \to \pi_0(Y)$ is injective.

Exercise 5.14. Let $\mathbb{K}$ be a field. We denote by $\mathcal{Vec}_{\mathbb{K}}^n$ the category where the objects are all vector spaces over $\mathbb{K}$ of dimension n and where the morphisms are isomorphisms of vector spaces. We consider the following two covariant functors:

$$\begin{array}{rcl} F\colon \mathcal{Vec}_{\mathbb{K}}^n & \to & \mathcal{Vec}_{\mathbb{K}}^n \\ V & \mapsto & \mathbb{K}^n \\ (\varphi\colon V \to W) & \mapsto & (\mathrm{id}\colon \mathbb{K}^n \to \mathbb{K}^n) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{Vec}_{\mathbb{K}}^n & \to & \mathcal{Vec}_{\mathbb{K}}^n \\ V & \mapsto & V \\ (\varphi\colon V \to W) & \mapsto & (\varphi\colon V \to W). \end{array}$$

Are the two functors naturally isomorphic?

Exercise 5.15. Let $\mathbb{K}$ be a field, let W be a $\mathbb{K}$-vector space and let $U, V \subseteq W$ be vector subspaces. In what sense is the isomorphism

$$\begin{array}{rcl} V/(U \cap V) & \to & (U+V)/U \\ v + U \cap V & \mapsto & v + U \end{array}$$

a natural isomorphism?

Exercise 5.16. Let $G = (G, *)$ be a group. We define the *opposite group* $G^{\mathrm{op}} = (G, *^{\mathrm{op}})$ as the group

$$\begin{array}{ccc} G \times G & \to & G \\ (g, h) & \mapsto & g *^{\mathrm{op}} h \coloneqq h * g. \end{array}$$

(a) Show that the maps $G \mapsto G^{\mathrm{op}}$ and $f \mapsto f$ define a covariant functor from the category *Group* of groups to itself.
(b) Show that the functor in (a) is naturally isomorphic to the identity functor on the category *Group* of groups.

Exercise 5.17. We say that two covariant functors $F, G \colon \mathcal{C} \to \mathcal{D}$ are *pointwise isomorphic* if for each $X \in \mathrm{Ob}(\mathcal{C})$ the objects $F(X)$ and $G(X)$ in $\mathcal{D}$ are isomorphic. Let $\mathcal{C}$ be the category corresponding to the partially ordered set $(\{0, 1\}, 0 \leq 1)$. Give examples of two covariant functors $\mathcal{C} \to \mathcal{S}et$ that are pointwise isomorphic but that are not naturally isomorphic.

Exercise 5.18. Let $F \colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories between two categories and let $V \colon \mathcal{C} \to \mathcal{V}$ be a covariant functor.

(a) Show that there exists a covariant functor $W \colon \mathcal{D} \to \mathcal{V}$ such that $W \circ F$ is naturally isomorphic to V.
(b) Let $W \colon \mathcal{D} \to \mathcal{V}$ and $\widetilde{W} \colon \mathcal{D} \to \mathcal{V}$ be two covariant functors such that $W \circ F$ and $\widetilde{W} \circ F$ are naturally isomorphic to V. Show that W and $\widetilde{W}$ are naturally isomorphic.

Exercise 5.19.

(a) Let $\mathcal{C}$ be a category that admits products and that admits a terminal object T. Show that the product of any object C with T is naturally isomorphic to C.
(b) Let $\mathcal{C}$ be a category that admits coproducts and that admits an initial object I. Show that the coproduct of any object C with I is naturally isomorphic to C.

Exercise 5.20. Prove the Yoneda Lemma 5.32.

CHAPTER 6

Products and Coproducts – Limits and colimits

In this chapter we will continue with our general discussion of categories. First we will introduce the notion of products and coproducts. Afterwards we will discuss the even more general notions of limits and colimits. These notions will guide us in our exploration of topological spaces, which we will resume in the next chapter.

This chapter is organized as follows:

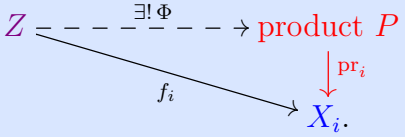
- In Section 6.1 we will introduce the notion of a *product* of a family of objects of a given category.
- In Section 6.2 we will introduce the notion of a *coproduct* of a family of objects of a given category.
- In Section 6.3 we will introduce the notion of a diagram in a category. We will then define what we mean by a limit of a diagram. This notion can be viewed as a generalization of the notion of a product. In particular we discuss two examples of limits in greater detail, namely the pullback and the equalizer.
- In Section 6.4 we will introduce the notion of a colimit of a family of morphisms between objects of a given category. This notion can be viewed as a generalization of the notion of a coproduct. In particular we discuss two examples of colimits in greater detail, namely the pushout and the coequalizer.
- In Section 6.5 we will introduce the concept of adjoint functors. We will prove that if a covariant functor admits a left-adjoint, then the functor commutes with limits and we will prove that if a covariant functor admits a right-adjoint, then the functor commutes with colimits.
- In Section 6.6 we will provide exercises for Chapter 6.

In the next Chapter 7 we will introduce products and coproducts (also called disjoint unions) of topological spaces.

6.1. Products.

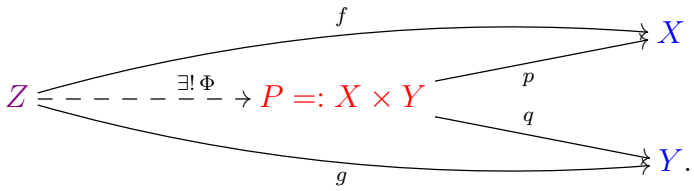
We start out with the following, initially technical looking, definition:

Definition 6.1. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $P \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{\mathrm{pr}_i \colon P \to X_i\}_{i\in I}$ such that the following universal property is satisfied: Whenever we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i \colon Z \to X_i\}_{i\in I}$ there exists a unique morphism $\Phi \colon Z \to P$ such that for each $i \in I$ we have $f_i = \mathrm{pr}_i \circ \Phi$, i.e. such that the following diagram commutes:[87]



Example. Let $\mathcal{C}$ be a category.

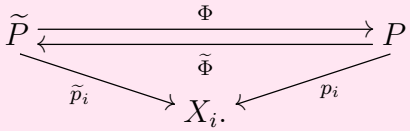
(1) First we consider the basic case of the product of two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$: A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in a category $\mathcal{C}$, then it follows easily from the definitions that a corresponding product is the same as a terminal object in the category $\mathcal{C}$. □

As usual objects with a universal property are unique in a suitable sense:

Lemma 6.2. (Product Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two products $(P, \{\mathrm{pr}_i\colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i\colon \widetilde{P} \to X_i\}_{i\in I})$ then there exist uniquely determined isomorphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Sketch of proof. The two morphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ come from the universal properties of P and $\widetilde{P}$. By comparing $\mathrm{id}_P\colon P \to P$ and $\Phi \circ \widetilde{\Phi}\colon P \to P$ one sees, by the uniqueness property, that $\mathrm{id}_P = \Phi \circ \widetilde{\Phi}$ and similarly one sees that $\mathrm{id}_{\widetilde{P}} = \widetilde{\Phi} \circ \Phi$, i.e. Φ and $\widetilde{\Phi}$ are isomorphisms. For more details we refer the reader to Exercise 6.1. ■

Next we introduce the standard notation for products:

Notation 6.3. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a product exists.

(1) We usually denote the product by $\prod_{i\in I} X_i$ and we refer to the morphisms $\prod_{i\in I} X_i \to X_i$ as natural projections. Furthermore, given morphisms $f_i\colon Z \to X_i$ we denote the corresponding morphism $Z \to \prod_{i\in I} X_i$ by $\prod_{i\in I} f_i$.

(2) If $I = \{1, \dots, k\}$, then often we denote the product by $X_1 \times \cdots \times X_k$. Furthermore given morphisms $f_i\colon Z \to X_i$, $i = 1, \dots, k$ we denote the corresponding morphism $Z \to X_1 \times \cdots \times X_k$ by $(f_1, \dots, f_k)$.

Example. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i=1,\dots,k}$, $\{Y_i\}_{i=1,\dots,k}$ be two families of objects in $\mathcal{C}$ such that a product exists. Furthermore let $\{g_i\colon X_i \to Y_i\}_{i=1,\dots,k}$ be a family of morphisms. For $j = 1, \dots, k$ we denote by $\mathrm{pr}_j\colon \prod_{i=1}^{k} X_i \to X_j$ the natural projection. We obtain a

morphism

$$\operatorname{diag}(f_1,\dots,f_k) := (f_1\circ \mathrm{pr}_1,\dots,f_k\circ \mathrm{pr}_k)\colon \prod_{i=1}^{k} X_i \to \prod_{i=1}^{k} Y_i.$$

Sometimes we refer to this as the *diagonal map*. □

Now we turn to the familiar concept of products in the category of sets and groups:

Lemma 6.4. (Product-of-Sets-and-Groups Lemma)

(1) Let $\mathcal{Set}$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The product in $\mathcal{Set}$ is precisely the product

$$\prod_{i\in I} X_i := \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i\in I \text{ we have } f(i)\in X_i \right\}$$

from Definition 1.10, together with the obvious maps to the X_i that are given by $f\mapsto f(i)$.

(2) Let $\mathcal{Group}$ be the category of groups and let $\{X_i\}_{i\in I}$ be a family of groups. The product in $\mathcal{Group}$ is the set

$$\prod_{i\in I} X_i = \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i\in I \text{ we have } f(i)\in X_i \right\}$$

as in (1), which we now view as a group by the obvious "entry wise" multiplication, together with the obvious homomorphisms to X_i.[88]

(3) The same approach as in (2) also applies to the category $\mathcal{AbGroup}$ of abelian groups and the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.[89]

Proof.

(1) So let $Z\in \mathrm{Ob}(\mathcal{Set})$ and let $\{f_i\colon Z\to X_i\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rccc} \Phi\colon & Z & \to & \prod_{i\in I} X_i \\ & z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

It follows immediately from the definition that for each $i\in I$ we have $f_i = \mathrm{pr}_i\circ\Phi$. It remains to show that Φ is unique. So let $\Psi\colon Z\to\prod_{i\in I} X_i$ be a map such that for each $i\in I$ we have $f_i=\mathrm{pr}_i\circ\Psi$. We need to show that $\Phi=\Psi$, i.e. we need to show that for each $z\in Z$ and $i\in I$ we have $\Phi(z)(i)=\Psi(z)(i)\in X_i$. But this follows from the observation that $\Phi(z)(i)=\mathrm{pr}_i(\Phi(z))=f_i(z)=\mathrm{pr}_i(\Psi(z))=\Psi(z)(i)\in X_i$.

(2) The same argument as in (1) also works for the category $\mathcal{Group}$ of groups.

(3) The argument of (2) also works in (3). ■

Example. Let $\{X_i\}_{i\in\varnothing}$ be an empty family of sets respectively groups. It follows from the discussion on page 98 that the product contains precisely one element, so the product is the set with a unique element (sometimes called *singleton*) respectively the trivial group.□

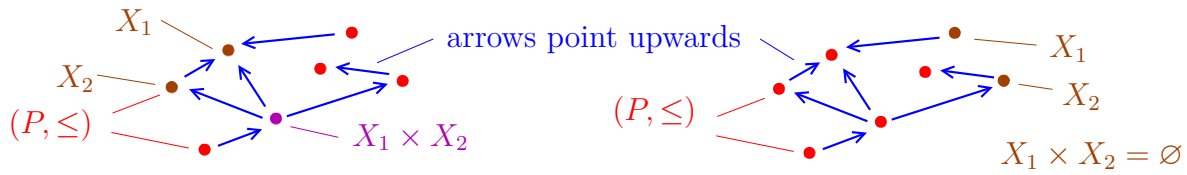
After these examples one might assume that one gets the hang of the product in categories. The following examples might rob the reader of this illusion.

Example.

(1) Let $(P,\leq)$ be a partially ordered set. We consider the corresponding order category $\mathcal{Order}_{(P,\leq)}$ from Definition 5.5. Given a family $\{X_i\}_{i\in I}$ of objects in $\mathcal{Order}_{(P,\leq)}$, i.e. given

a family $\{X_i\}_{i\in I}$ of elements of P, one can easily show that[90]

$$\prod_{i\in I} X_i := \begin{cases} \inf(\{X_i\}_{i\in I}), & \text{i.e. a maximal element among all lower bounds on } \{X_i\}_{i\in I} \\ \varnothing, & \text{if no such infimum exists.} \end{cases}$$



(2) We introduce a new category. A pointed set is a pair (X, x_0), where X is a set and x_0 is a point in X. We refer to the category $\mathcal{PSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PSet}) &:= \text{all pointed sets,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

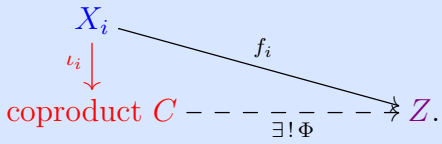
with the usual composition of maps as the category $\mathcal{PSet}$ of pointed sets. Now let (X, x_0) and (Y, y_0) be two pointed sets. In Exercise 6.6 we will see that the product of (X, x_0) and (Y, y_0) in the category $\mathcal{PSet}$ is given by the set

$$(X \times Y)/\underbrace{((X \times \{y_0\}) \cup (\{x_0\} \times Y))}_{\substack{\text{the points in this subset get} \\ \text{identified to a single point } *}} \quad \text{together with the point} \quad *$$

and the obvious morphisms to (X, x_0) and (Y, y_0). □

6.2. **Coproducts.** We now reverse the direction of the morphisms in the definition of a product and we obtain a new interesting definition:

Definition 6.5. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{\iota_i \colon X_i \to C\}_{i\in I}$ such that the following property is satisfied: If we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i \colon X_i \to Z\}_{i\in I}$ there exists a unique morphism $\Phi \colon C \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$, i.e. such that the following diagram commutes:[91]



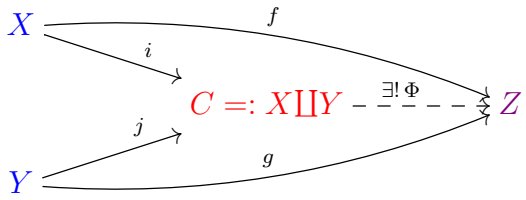
Remark. Note that by definition coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. So we did not introduce a new concept at all. □

Example. Let $\mathcal{C}$ be a category.

(1) First we consider the basic case of the product of two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$: A coproduct of X and Y in the category $\mathcal{C}$ is some object $C \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $i \colon X \to C$ and $j \colon Y \to C$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f \colon X \to Z$ and $g \colon Y \to Z$ there exists a unique morphism $\Phi \colon C \to Z$ such that $f = \Phi \circ i$ and

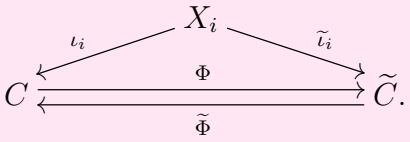
[90] In fact it is a very good exercise to provide a proof.

$g = \Phi \circ j$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in $\mathcal{C}$, then it follows easily from the definitions that the corresponding coproduct is an initial object in the category $\mathcal{C}$. □

Lemma 6.6. (Coproduct Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two coproducts $(C, \{\iota_i \colon X_i \to C\}_{i\in I})$ and $(\widetilde{C}, \{\widetilde{\iota}_i \colon X_i \to \widetilde{C}\}_{i\in I})$, then there exist unique isomorphisms $\Phi \colon C \to \widetilde{C}$ and $\widetilde{\Phi} \colon \widetilde{C} \to C$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Proof. The lemma is just a reformulation of the Product Uniqueness Lemma 6.2 since coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. Alternatively the statement also follows easily from the definitions. ■

Next we introduce a notation for coproducts, which is a little less standard than the notation for products:

Notation 6.7. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a coproduct exists. Often[92] we denote the coproduct by $\coprod_{i\in I} X_i$ and we refer to the morphisms $X_i \to \coprod_{i\in I} X_i$ as natural injections.[93]

We formulate our first example as a lemma:

Lemma 6.8. (Coproduct-of-Sets-and-Abelian Groups Lemma)

(1) Let $\mathcal{S}et$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The coproduct in $\mathcal{S}et$ is the disjoint union

$$\coprod_{i\in I} X_i \;:=\; \bigcup_{i\in I} (X_i \times \{i\}),$$

from Notation 1.12, together with the natural injections of the X_i.

[92] In fact, as we will see shortly, in many settings we will see that the coproduct already has an established different notation.

[92] The name *natural injection* is somewhat unfortunate since in general for a morphism it makes no sense to say that it is injective. Even if there is a reasonable notion of injective, the natural injection does not need to be an injection. On the other hand, in all examples we ever care about the natural injection will be an injective map.

(2) Let $\mathcal{AbGroup}$ be the category of *abelian* groups and let $\{X_i\}_{i\in I}$ be a family of abelian groups. The coproduct in $\mathcal{AbGroup}$ is given by

$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and the homomorphisms

$$\begin{array}{rcl} X_i & \to & \bigoplus_{i\in I} X_i \\ x & \mapsto & \left(\begin{array}{lcl} I & \to & \bigcup_{i\in I} X_i \\ j & \mapsto & \begin{cases} x_i, & \text{if } i = j, \\ 0_{X_j}, & \text{if } i \neq j. \end{cases} \end{array} \right) \end{array}$$

(3) The same approach as in (2) also applies to the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.[93]

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{Set})$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi \colon \bigsqcup_{i\in I} X_i & \to & Z \\ (z_i, i) & \mapsto & f_i(z_i). \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let $\Psi \colon \bigsqcup_{i\in I} X_i \to Z$ be a map such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $i \in I$ and each $x_i \in X_i$ we have $\Phi((x_i, i)) = \Psi((x_i, i)) \in Z$. But this follows from the observation that $\Phi((x_i, i)) = \Phi(\iota_i(x_i)) = f_i(x_i) = \Psi(\iota_i(x_i)) = \Psi((x_i, i)) \in Z$.

(2) Let a $Z \in \mathrm{Ob}(\mathcal{AbGroup})$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of homomorphisms. We set[94]

$$\begin{array}{rcl} \Phi \colon \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \text{all but finitely many } g_i \text{ are trivial} \right\} & \to & Z \\ (g_i)_{i\in I} & \mapsto & \sum_{\substack{i\in I \text{ with} \\ g_i \neq e}} f_i(g_i). \end{array}$$

Note that this map makes sense since we take the sum only of finitely elements of Z. Furthermore note that this map is a homomorphism since we work with the category $\mathcal{AbGroup}$ of abelian groups.

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let Ψ be another homomorphism such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$. So suppose that we are given $(g_i)_{i\in I} \in \prod_{i\in I} X_i$ such that there exist only finitely many $i_1, \ldots, i_k \in I$ such that the corresponding group element is non-trivial. Then

$$\Phi((g_i)_{i\in I}) \;=\; \Phi\Big(\sum_{r=1}^{k} \iota_{i_r}(g_{i_r}) \Big) \;=\; \sum_{r=1}^{k} \Phi(\iota_{i_r}(g_{i_r})) \;=\; \sum_{r=1}^{k} f_{i_r}(g_{i_r}) \;\underset{\text{same argument backwards}}{\uparrow}\; \Psi((g_i)_{i\in I}).$$

(3) This statement is proved the same way as we proved (2). ∎

The Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 leads us to the following notation:

[93] In the Coproduct-of-Modules Lemma 309.11 we will state the analogous statement for the category of R-modules.

[94] We use addition notation for the abelian groups.

Notation 6.9. Let $\{X_i\}_{i\in I}$ be objects in the category $\mathcal{AbGroup}$ of abelian groups.

(1) We write

$$\bigoplus_{i\in I} X_i \coloneqq \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and we refer to it as the direct sum of the X_i. For a finite family $\{X_i\}_{i=1,\dots,k}$ we often write $X_1 \oplus \cdots \oplus X_k$.

(2) Let Z be an abelian group and let $\{\varphi_i \colon X_i \to Z\}_{i\in I}$ be a family of morphisms. We write

$$\bigoplus_{i\in I} \varphi_i \colon \bigoplus_{i\in I} X_i \to Z$$

for the induced homomorphism. We use obvious variations of this notation for a finite family of abelian groups.

We proceed similarly with objects in the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Remark.

(1) Note that for a finite family $X_1, \dots, X_k$ of abelian groups the product and the coproduct have the same underlying group $X_1 \times \cdots \times X_k$. In a category, an object that is a product and also a coproduct is sometimes called a *biproduct*.

(2) In Exercise 6.9 we will see that the construction of the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 does *not* give the coproduct in the category $\mathcal{Group}$ of all groups. It turns out that for the construction of the coproduct in the category $\mathcal{Group}$ will require a more clever construction, which we will provide in Section 98.1. □

6.3. **Limits.** In this section and the following section we will introduce the very general notion of a limit and of a colimit, which in particular generalize the notion of a product and a coproduct.

This section is organized as follows:

- In Section 6.3.0 we will introduce diagrams.
- In Section 6.3.1 we will introduce limits.
- In Section 6.3.2 we will introduce pullbacks, which are an important special case of a limit.
- In Section 6.3.3 we will introduce equalizers.

6.3.0. **Diagrams.** Before we can introduce the new concepts of limit and colimit we need to provide the following definition:

Definition 6.10. Let $(P, \leq)$ be a partially ordered set and let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ over $(P, \leq)$ is a covariant functor from the order category $\mathcal{Order}_{(P,\leq)}$ to $\mathcal{C}$.[95]

Remark. In plain English a diagram F in a category $\mathcal{C}$ over a partially ordered set $(P, \leq)$ is the following data:

(1) for each $x \in P$ we are given an object $F(x) \in \mathrm{Ob}(\mathcal{C})$,

[95] If this was a proper course on category theory, then we would replace the partially ordered set $(P, \leq)$ by a small category I (where "small" means that the objects form a set). This greater generality has the advantage that for two objects we can now have multiple morphisms in a diagram. For most of our purposes this extra flexibility has no real advantages. In Definition 6.17 and Definition 6.27 we use a little hack to get around the limitations of our definition.

(2) for each $x < y$ we are given a morphism $F(x < y)\colon F(x) \to F(y)$,[96]

such that the following condition is satisfied:

($*$) if $x < y < z$, then the following diagram needs to commute:

$$\begin{array}{ccc} F(x) & \longrightarrow & F(z). \\ & \searrow \quad \nearrow & \\ & F(y) & \end{array}$$

□

Example.

(1) We consider the partially ordered set $(P, \leq_{\mathrm{tr}})$ that is given by a set P with the trivial partial order $\leq_{\mathrm{tr}}$, i.e. $x \leq_{\mathrm{tr}} y$ if and only if $x = y$. In this setting a diagram over $(P, \leq_{\mathrm{tr}})$ is the same data as a family of objects $\{X_i\}_{i \in P}$.

(2) We consider the partially ordered set $(\mathbb{N}, \leq)$. It is easy to see that in this setting a diagram over $(\mathbb{N}, \leq)$ is the same data as a sequence

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \dots$$

of morphisms $\varphi_i\colon X_i \to X_{i+1}$. For $i < j$ the morphisms $X_i \to X_j$ are understood to be given by $\varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i$. □

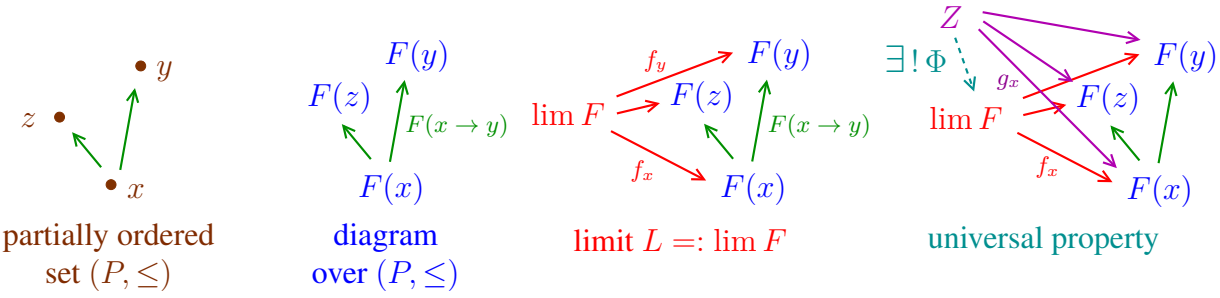
6.3.1. **Definition of limits.** The above Definition 6.10 allows us to introduce the following all important notion:

Definition 6.11. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A limit of F consists of an object $L \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x\colon L \to F(x)$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For every $x \leq y$ we have $f_y = F(x \to y) \circ f_x\colon L \to F(y)$.

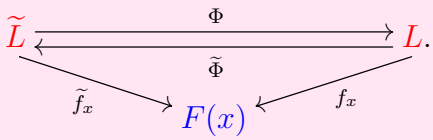
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x\colon Z \to F(x)$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi\colon Z \to L$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

If a limit exists, then often we denote it by $\lim F$. (We will see in the Limit Uniqueness Lemma 6.12 below that limits, if they exist, are essentially unique.)



[96]Note that for $x \in P$ we have $x \leq x$ and the morphism $* \in \mathrm{Mor}(x, x)$ is the identity morphism. Since F is a covariant functor we obtain from Definition 5.18 that $F(*) = \mathrm{id}_{F(x)}$. This is the reason why we only need to specify morphisms for $x < y$.

Lemma 6.12. (Limit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two limits $(L, \{f_x \colon L \to F(x)\}_{x \in P})$ and $(\widetilde{L}, \{\widetilde{f}_x \colon \widetilde{L} \to F(x)\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon \widetilde{L} \to L$ and $\widetilde{\Phi} \colon L \to \widetilde{L}$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



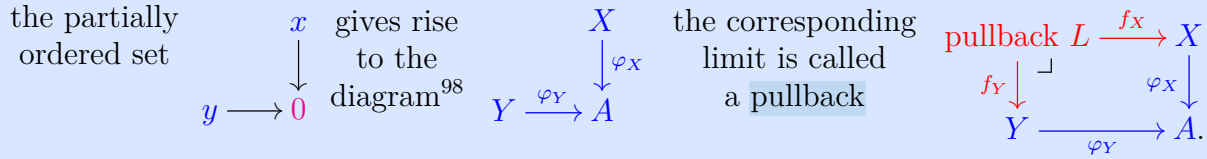
Proof. As in the case of the Product Uniqueness Lemma 6.2 the statement follows easily from the definitions. ■

Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the limit is by definition precisely the product in the sense of Definition 6.1.
(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{Set}$ of sets where each $\varphi_i \colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. In Exercise 6.11 we will see that in this case the limit is the first set X_1. □

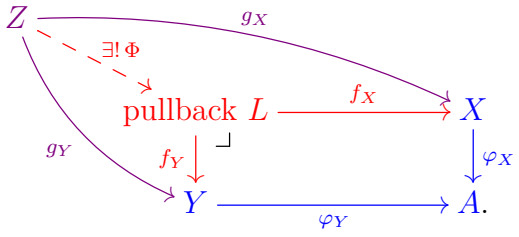
6.3.2. **Pullbacks.** We continue our discussion of limits with a particularly frequently used special case:

Definition 6.13. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $x \leq 0$ and $y \leq 0$. Furthermore let $\mathcal{C}$ a category and let $\varphi_A \colon X \to A$ and $\varphi_Y \colon Y \to B$ be two morphisms. In this setup[97]



Here the symbol "⌟" indicates that L is the pullback of the "⌟-shaped" diagram.

Remark. The following diagram summarizes the definition of a pullback:



(1) We start out with a pullback diagram.

[97] We do not need to specify the morphisms from the pullback L to A since it is agrees with $\varphi_X \circ f_X = \varphi_Y \circ f_Y$.
[98] In contrast to the notation above we now write φ_X, φ_Y instead of φ_x, φ_y, since we feel that in this setting this notation is more natural.

(2) The pullback L comes with morphisms $f_X\colon L \to X$ and $f_Y\colon L \to Y$ such that the rectangle commutes.
(3) If we are given an object Z of $\mathcal{C}$ with morphisms $g_X\colon Z \to X$ and $g_Y\colon Z \to Y$ such that the big outer quadrilateral commutes, then there exists a unique morphism $\Phi\colon Z \to L$ that makes the diagram commute. □

Example. We consider the following pullback diagram in the category *Group* of groups:

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \varphi} \\ \{e\} & \longrightarrow & H \end{array}$$

one can easily show that the corresponding pullback is given by

$$\begin{array}{ccc} \ker(\varphi) & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \varphi} \\ \{e\} & \longrightarrow & H \end{array}$$

□

We formulate our next example as a lemma:

Lemma 6.14. (Pullback–Product Lemma) Let $\mathcal{C}$ be a category. Suppose that $\mathcal{C}$ admits a terminal object $*$. Then the pullback of the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow[\varphi_Y]{} & * \end{array}$$

is given by the product $X \times Y$.

Proof. This statement follows easily from the definitions. ■

Next we see that one can "compose" pullbacks:

Lemma 6.15. (Pullback–Composition Lemma) Let $\mathcal{C}$ be a category. If we are given pullbacks

$$\begin{array}{ccc} L & \xrightarrow{f_X} & X \\ {\scriptstyle f_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow{\varphi_Y} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Z \\ {\scriptstyle \varphi_X}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi_Z} \\ A & \xrightarrow{\psi_A} & B \end{array}$$

then the "composition"

$$\begin{array}{ccc} L & \xrightarrow{\theta \circ f_X} & Z \\ {\scriptstyle f_y}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi_Z} \\ Y & \xrightarrow{\psi_A \circ \varphi_Y} & B \end{array}$$

is also a pullback.

Proof. This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

Finally we record a very visual property of pullbacks:

Lemma 6.16. (Pullback–Opposite Lemma) Let $\mathcal{C}$ be a category. Let

$$\begin{array}{ccc} L & \xrightarrow{f_X} & X \\ {\scriptstyle f_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow[\varphi_Y]{} & A. \end{array}$$

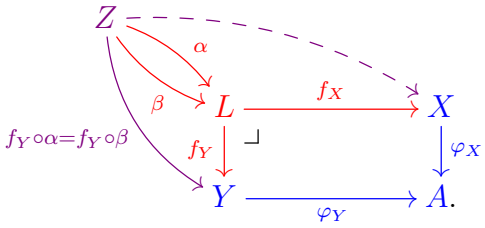
[98] It is a "meta theorem" that in many categories two morphisms in a pullback diagram, that are opposite of one another, have similar properties.

be a pullback.

(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.
(b) If φ_X is a monomorphism, then so is the "opposite" morphism f_Y.

The analogous statements of course also hold with the roles of X and Y swapped.

Proof. We assume that φ_X is a monomorphism in the sense of Definition 5.13. We need to show that f_Y is also a monomorphism. Thus let $\alpha\colon Z \to L$ and $\beta\colon Z \to L$ be two morphisms such that $f_Y \circ \alpha = f_Y \circ \beta$. We summarize all morphisms in the following figure:

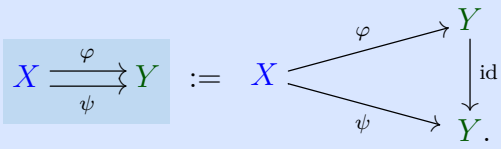


We need to show that $\alpha = \beta$. We proceed as follows:

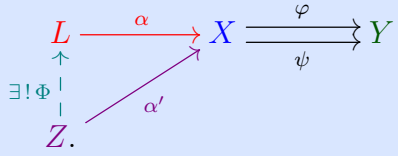
(i) We assume that $f_Y \circ \alpha = f_Y \circ \beta$.
(ii) By (i) we also have $\varphi_Y \circ f_Y \circ \alpha = \varphi_Y \circ f_Y \circ \beta$.
(iii) Since the diagram commutes we get $\varphi_X \circ f_X \circ \alpha = \varphi_X \circ f_X \circ \beta\colon Z \to A$.
(iv) Since φ_X is a monomorphism we see that $f_X \circ \alpha = f_X \circ \beta\colon Z \to X$.
(v) Since we dealing with a pullback there exists a *unique* morphism $\Phi\colon Z \to L$ such that $f_Y \circ \Phi = f_Y \circ \alpha = f_Y \circ \beta$ and such that $f_X \circ \Phi = f_X \circ \alpha = f_X \circ \beta$.
(vi) From (i), (iv) and (v) we obtain that $\alpha = \Phi = \beta$. ∎

6.3.3. **Equalizers.** We conclude this introduction to limits with another example:

Definition 6.17. Let $\mathcal{C}$ be a category, let X and Y be two objects and let $\varphi, \psi\colon X \to Y$ be two morphisms in $\mathcal{C}$. We write[99] [100]



Any corresponding limit is called the equalizer of φ and ψ. In plain English the equalizer is equivalent to a morphism $\alpha\colon L \to Y$ with $\varphi \circ \alpha = \psi \circ \alpha$ which has the universal property that for any other object Z and morphism $\alpha'\colon Z \to Y$ with $\varphi \circ \alpha' = \psi \circ \alpha'$ there exists a unique morphisms $\Phi\colon Z \to L$ that makes the following diagram commute:



Example.

[99] Note that the right hand side is a diagram in the sense of Definition 6.10, where the partially ordered set is given by $\{x, y', y''\}$ with $x < y' < y''$.

[100] The notation suggests that one could also define a diagram, where the left-hand side is a diagram, without having to go through slightly awkward formulation to the right. But this would need a more

(1) We consider the category *Group* of abelian groups. Let $\varphi\colon X \to Y$ be a group homomorphism. It is elementary to show that the equalizer of φ and the trivial homomorphism is naturally isomorphic to $\ker(\varphi)$.
(2) We now work in the category *Set* of sets. One can easily verify that the equalizer of two maps $\varphi, \psi\colon X \to Y$ is naturally in bijection to the subset $\{x \in X \,|\, \varphi(x) = \psi(x)\}$. Similarly it is straightforward to show that the analogous statement also holds in the topological category *Top*, if we equip $\{x \in X \,|\, \varphi(x) = \psi(x)\}$ with the subspace topology from Definition 3.15. □

We continue with the following lemma which lets us construct any limit out of products and equalizers (always assuming these exist). In fact, we give two constructions, one which gives us an elegant general construction, and a second construction, which in many situations leads to "fewer" restrictions to check:

Lemma 6.18. (Limit-via-Product+Equalizer Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$ in the sense of Definition 6.10.

(a) The limit of the diagram F over $(P, \leq)$ is given by the equalizer of

$$\prod_{x \in P} F(x) \underset{(v_x)_x \mapsto (v_x)_{x \leq y}}{\overset{(v_x) \mapsto (F(x \leq y)(v_x))_{x \leq y}}{\rightrightarrows}} \prod_{x \leq y} F(y),$$

always assuming these products and the equalizer exist.

(b) We can also give a more "efficient" construction: Let $I \subseteq \{(x, y) \in P^2 \,|\, x \leq y\}$ be a subset such that for any $x \leq y$ there exists a sequence $x = x_0 < x_1 < \cdots < x_n = y$ in P such that each (x_i, x_{i+1}) lies in I.[101] Then the limit of the diagram F over $(P, \leq)$ is given by the equalizer of

$$\prod_{x \in P} F(x) \underset{(v_x)_x \mapsto (v_x)_{(x,y) \in I}}{\overset{(v_x) \mapsto (F(x \leq y)(v_x))_{(x,y) \in I}}{\rightrightarrows}} \prod_{(x,y) \in I} F(x).$$

Proof. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$ in the sense of Definition 6.10.

(a) We suppose that the equalizer of the two morphisms in Statement (1) of the lemma exists. This leads us to the following notation:

$$\begin{array}{ccccc} L & \xrightarrow{\alpha} & \prod_{x \leq y} F(x) & \underset{(v_x)_x \mapsto (v_y)_{x \leq y}}{\overset{(v_x) \mapsto (F(x \leq y)(v_x))_{x \leq y}}{\rightrightarrows}} & \prod_{x \in P} F(x) \\ & \searrow^{f_x} & \downarrow \mathrm{pr}_x & & \\ & & F(x). & & \end{array}$$

general notion of a diagram, where instead of partially ordered sets $(P, \leq)$ we work with arbitrary small categories. But for our purposes the more restrictive notion of a diagram is usually general enough.

[101] For example, for $(P, \leq) = (\mathbb{N}, \leq)$ we could use $I = \{(i, i+1) \,|\, i \in \mathbb{N}\}$.

For each $x \in P$ let $\mathrm{pr}_x \colon \prod_{x \leq y} F(x) \to F(x)$ be the projection. For each $x \in P$ we consider the morphism

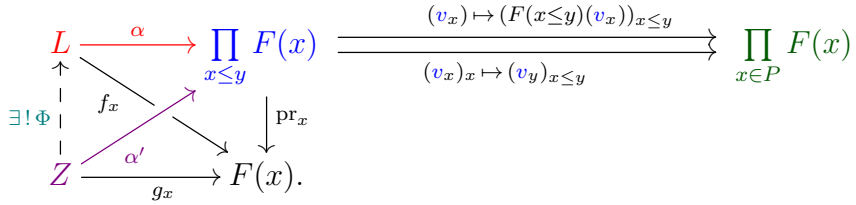
$$\begin{aligned} f_x \colon L &\to F(x) \\ l &\mapsto \mathrm{pr}_x(\alpha(l)). \end{aligned}$$

We claim that L together with these morphisms satisfies the conditions of Definition 6.11 of a limit. Thus we need to verify the following claim:

Claim.

(1) For every $x \leq y$ we have $f_x = F(x \to y) \circ f_y \colon L \to F(x)$.
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon Z \to F(x)$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon Z \to L$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

Proof. We summarize all morphisms in the following figure:



First note that one can easily verify that morphisms α' as in the above diagram, that make the diagram commute, are in a natural one-to-one correspondence to a family of morphisms $\{f'_x \colon Z \to F(x)\}_{x \in P}$ which satisfy $f'_x = F(x \to y) \circ f'_y \colon Z \to F(x)$. The claim now follows easily from these observations.

(b) This statement follows also easily from elementary logic and the argument of (1). Writing down details does not improve clarity. So we leave them to the reader. ∎

We conclude this section with a short discussion of the following basic concept:

Definition 6.19. A category is called complete if all limits exist.

Proposition 6.20. (Category–Completeness Criterion) Let $\mathcal{C}$ be a category. If all products and all equalizers exist, then $\mathcal{C}$ is complete.

Proof. This proposition is an immediate consequence of the Limit-via-Product+Equalizer Lemma 6.18. ∎

6.4. **Colimits.** In the last section we introduced limits. In this section we will reverse all arrows and we end up with the notion of a colimit, which we will discuss in great detail. More precisely, in this section we will do the following:

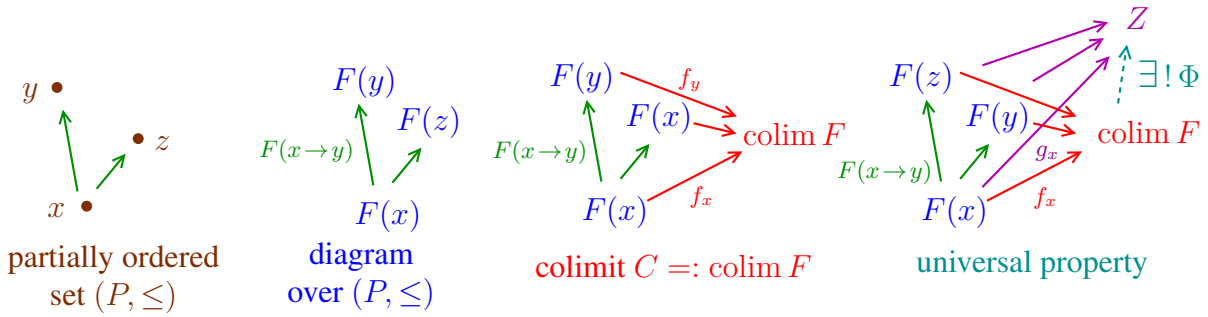
- In Section 6.4.1 we will introduce colimits.
- In Section 6.4.2 we will introduce pushouts, which are an important special case of a colimit.
- In Section 6.4.3 we will introduce coequalizers.

6.4.1. **Definition of colimits.** In this subsection we will introduce the notion of a colimit, which is very similar to the notion of a limit, except that the directions of the morphisms are reversed:

Definition 6.21. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A colimit of F consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x \colon F(x) \to C$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For every $x \leq y$ we have $f_x = f_y \circ F(x \to y) \colon F(x) \to C$.
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon F(x) \to Z$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon C \to Z$ such that $g_x = \Phi \circ f_x$ for all $x \in P$.

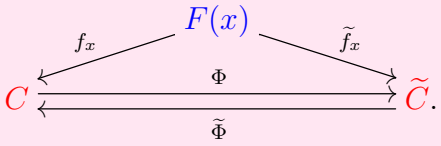
If a colimit exists, then often we denote it by $\operatorname{colim} F$.



Remark. Note that by definition colimits in $\mathcal{C}$ are limits in the opposite category $\mathcal{C}^{\mathrm{op}}$ with respect to the opposite diagram $(P, \leq)^{\mathrm{op}}$. So we did not introduce a new concept at all. □

Not surprisingly we have the following analog of the Limit Uniqueness Lemma 6.12:

Lemma 6.22. (Colimit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two colimits $(C, \{f_x \colon F(x) \to C\}_{x \in P})$ and $(\widetilde{C}, \{\widetilde{f}_x \colon F(x) \to \widetilde{C}\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon C \to \widetilde{C}$ and $\widetilde{\Phi} \colon \widetilde{C} \to C$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As always the proof follows easily from the definitions. ■

Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the colimit is by definition precisely the coproduct from Definition 6.5.
(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{Set}$ of sets where each $\varphi_i \colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. In Exercise 6.11 we will see that in this case the colimit is the union of the X_i. □

6.4.2. **Pushouts.** Next we introduce the analog of the pullback from Definition 6.13:

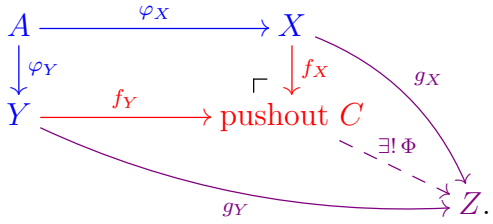
Definition 6.23. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$.

Furthermore let $\mathcal{C}$ a category and let $\varphi_A\colon A \to X$ and $\varphi_Y\colon A \to Y$ be two morphisms. In this setup

the partially ordered set $\begin{array}{ccc} 0 & \longrightarrow & x \\ \downarrow & & \\ y & & \end{array}$ gives rise to the diagram $\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & & \\ Y & & \end{array}$ the corresponding colimit is called a pushout $\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & \ulcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow[f_Y]{} & \text{pushout } C. \end{array}$

Here the symbol "$\ulcorner$" indicates that C is the pushout of the "$\ulcorner$-shaped" diagram.

Remark. The following diagram summarizes the definition of a pushout:



(1) We start out with a pushout diagram.
(2) The pushout C comes with morphisms $f_X\colon X \to C$ and $f_Y\colon Y \to C$ such that the rectangle commutes.
(3) If we are given an object Z with morphisms $g_X\colon X \to Z$ and $g_Y\colon Y \to Z$ such that the big outer quadrilateral commutes, then there exists a unique morphism $\Phi\colon C \to Z$ that makes the diagram commute. □

Example. Let G be a group and let A be a *normal* subgroup of G. We consider the following pushout diagram in the category *Group* of groups:

$\begin{array}{ccc} A & \longrightarrow & \{e\} \\ \downarrow & & \\ G & & \end{array}$ one can easily show that the corresponding pushout is given by $\begin{array}{ccc} A & \longrightarrow & \{e\} \\ \downarrow & \ulcorner & \downarrow \\ G & \longrightarrow & G/A. \end{array}$

In Exercise 6.13 we will consider a generalization of this situation. □

We continue with the following analog of the Pullback–Product Lemma 6.14.

Lemma 6.24. (Pushout–Coproduct Lemma) Let $\mathcal{C}$ be a category. If $\mathcal{C}$ has an initial object $*$, then the pushout of

$$\begin{array}{ccc} * & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & \ulcorner & \downarrow{\scriptstyle \iota_X} \\ Y & \xrightarrow{\iota_Y} & X \coprod Y \end{array}$$

is given by the coproduct $X \coprod Y$.

Proof. This statement follows easily from the definitions. ■

The following lemma is the analog of the Pullback–Composition Lemma 6.15:

Lemma 6.25. (Pushout–Composition Lemma) Let $\mathcal{C}$ be a category. If we are given pushouts

$$\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ {\scriptstyle \varphi_Y}\downarrow & \ulcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow{f_Y} & L \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Z \\ {\scriptstyle f_X}\downarrow & \ulcorner & \downarrow{\scriptstyle g_Z} \\ L & \xrightarrow{g_L} & W \end{array} \quad \text{then the "composition"} \quad \begin{array}{ccc} A & \xrightarrow{\theta\circ\varphi_X} & Z \\ {\scriptstyle \varphi_Y}\downarrow & \ulcorner & \downarrow{\scriptstyle g_Z} \\ Y & \xrightarrow{g_L\circ f_Y} & W \end{array}$$

is also a pushout.

Proof. This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

Finally the following is an analog of the Pullback–Opposite Lemma 6.16

Lemma 6.26. (Pushout–Opposite Lemma) Let $\mathcal{C}$ be a category and let

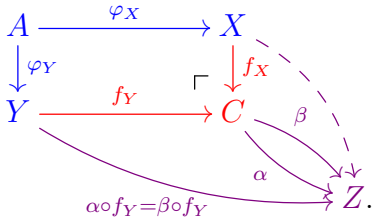
$$\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ {\scriptstyle \varphi_Y}\downarrow & \ulcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow{f_Y} & C. \end{array}$$

be a pushout.

(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.
(b) If φ_X is an epimorphism, then so is the "opposite" morphism f_Y.

The analogous statements of course also hold with the roles of X and Y swapped.

Proof. The lemma follows almost directly from the Pullback–Opposite Lemma 6.16 applied to the opposite category. But for clarity's sake, let us provide a direct proof. Thus we assume that φ_X is an epimorphism in the sense of Definition 5.13. We need to show that the morphism f_Y is also an epimorphism. Let $\alpha\colon C \to Z$ and $\beta\colon C \to Z$ be two morphisms such that $\alpha \circ f_Y = \beta \circ f_Y\colon Y \to Z$. We summarize all morphisms in the following figure:



We need to show that $\alpha = \beta$. We proceed as follows:

(i) We assume that $\alpha \circ f_Y = \beta \circ f_Y\colon Y \to Z$.
(ii) From (i) we obtain that $\alpha \circ f_Y \circ \varphi_Y = \beta \circ f_Y \circ \varphi_Y\colon A \to Z$.
(iii) Since the diagram commutes we get $\alpha \circ f_X \circ \varphi_X = \beta \circ f_X \circ \varphi_X\colon A \to Z$.
(iv) Since φ_X is an epimorphism we see that $\alpha \circ f_X = \beta \circ f_X\colon X \to Z$.
(v) Since we dealing with a pushout there exists a *unique* morphism $\Phi\colon C \to Z$ such that $\Phi \circ f_Y = \alpha \circ f_Y = \beta \circ f_Y$ and such that $\Phi \circ f_X = \alpha \circ f_X = \beta \circ f_X$.
(vi) It follows from (i), (iv) and (v) that $\alpha = \Phi = \beta$. ■

6.4.3. **Coequalizers.** We conclude this section with a discussion of the "co"-analog of the equalizer from Definition 6.17:

Definition 6.27. Let $\mathcal{C}$ be a category, let X and Y be two objects and let $\varphi, \psi\colon X \to Y$ be two morphisms in $\mathcal{C}$. As in Definition 6.17 we write

$$X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y \quad := \quad \begin{array}{ccc} & \overset{\varphi}{\nearrow} & Y \\ X & & \downarrow \mathrm{id} \\ & \underset{\psi}{\searrow} & Y. \end{array}$$

Any corresponding colimit is called the coequalizer of φ and ψ. In everyday math language, the coequalizer is equivalent to a morphism $\alpha\colon Y \to C$ with $\alpha \circ \varphi = \alpha \circ \psi$ which has the universal property that for any other object Z and morphism $\alpha'\colon Y \to Z$ with $\alpha' \circ \varphi = \alpha' \circ \psi$ there exists a unique morphisms $\Phi\colon C \to Z$ that makes the following diagram commute:

$$\begin{array}{ccccc} X & \underset{\psi}{\overset{\varphi}{\rightrightarrows}} & Y & \xrightarrow{\alpha} & C \\ & & & \searrow^{\alpha'} & \downarrow \exists!\,\Phi \\ & & & & Z. \end{array}$$

As a first example we consider coequalizers in the category $\mathcal{S}et$ of sets. Here we will see that coequalizers correspond to quotients by equivalence relations. More precisely, we have the following lemma:

Lemma 6.28. (Relation–Coequalizer Lemma) Let Y be a set.

(1) Let $\varphi, \psi\colon X \to Y$ be two maps. We consider the relation $R = \{(\varphi(x), \psi(x)) \,|\, x \in X\}$ and we denote by "$\sim$" the equivalence relation on Y that is generated by the relation R in the sense of Definition 1.22. Then the coequalizer of φ and ψ is naturally in bijection to the set $Y/\sim$ of equivalence classes. Put differently,

$$\operatorname*{colim}_{\longrightarrow} \left(X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y \right) \;=\; Y/\varphi(x) \sim \psi(x).$$

(2) Let $R \subseteq Y \times Y$ be an equivalence relation in the sense of Definition 1.13. We denote by $\mathrm{pr}_1\colon Y \times Y \to Y$ and $\mathrm{pr}_2\colon Y \times Y \to Y$ the natural projection onto the first, respectively the second factor. The coequalizer of $\mathrm{pr}_1\colon R \to Y$ and $\mathrm{pr}_2\colon R \to Y$ is naturally in bijection to the set Y/R of equivalence classes.

Proof.

(1) This statement follows easily from the definitions.

(2) Let $R \subseteq Y \times Y$ be an equivalence relation. We consider the maps $\mathrm{pr}_1\colon R \to Y$ and $\mathrm{pr}_2\colon R \to Y$. As in (1) we consider the set $\{(\mathrm{pr}_1(r), \mathrm{pr}_2(r)) \,|\, r \in R\}$. Note that by definition this set is already R, in particular the equivalence relation on Y that is generated by that set is just the original equivalence relation. Thus the promised statement is just a reinterpretation of Statement (1). ∎

Next we consider certain coequalizers in the category $\mathcal{G}roup$ of groups:

Lemma 6.29. (Cokernel-as-Coequalizer) We consider the category $\mathcal{A}b\mathcal{G}roup$ of *abelian* groups. Let $\varphi\colon X \to Y$ be a group homomorphism. The coequalizer of φ and the trivial homomorphism is naturally isomorphic to the cokernel $\mathrm{coker}(\varphi) = Y/\mathrm{im}(\varphi)$.

Proof. The lemma follows easily from elementary logic. We leave it to the reader to fill in the details. ∎

We continue with an analog of the Limit-via-Product+Equalizer Lemma 6.18 More precisely we have the following lemma which lets us construct any colimit out of coproducts and coequalizers (always assuming these exist). In fact, we give two constructions, one which gives us an elegant general construction, and a second construction, which in many situations leads to "smaller" models.

Lemma 6.30. (Colimit-via-Coproduct+Coequalizer Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$ in the sense of Definition 6.10.

(a) The colimit of the diagram F over $(P, \leq)$ is given by the coequalizer of

$$\coprod_{x \leq y} F(x) \underset{(v, x\leq y) \mapsto (v,x)}{\overset{(v, x\leq y) \mapsto (F(x\leq y)(v), y)}{\rightrightarrows}} \coprod_{x \in P} F(x),$$

always assuming these coproducts and the coequalizer exist.

(b) We can also give a more "efficient" construction: Let $I \subseteq \{(x, y) \in P^2 \,|\, x \leq y\}$ be a subset such that for any $x \leq y$ there exists a sequence $x = x_0 < x_1 < \cdots < x_n = y$ in P such that each (x_i, x_{i+1}) lies in I. Then the colimit of the diagram F over $(P, \leq)$ is given by the coequalizer of

$$\coprod_{(x,y) \in I} F(x) \underset{(v,(x,y)) \mapsto (v,x)}{\overset{(v,(x,y)) \mapsto (F(x\leq y)(v), y)}{\rightrightarrows}} \coprod_{x \in P} F(x).$$

Proof. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$ in the sense of Definition 6.10.

(a) We suppose that the coequalizer of the two morphisms in Statement (1) of the lemma exists. This leads us to the following notation:

$$\coprod_{x \leq y} F(x) \underset{(v, x\leq y) \mapsto (v,x)}{\overset{(v, x\leq y) \mapsto (F(x\leq y)(v), y)}{\rightrightarrows}} \coprod_{x \in P} F(x) \xrightarrow{\alpha} C$$

$$F(x) \uparrow \qquad \nearrow f_x$$

For each $x \in P$ we consider the morphism

$$\begin{aligned} f_x \colon F(x) &\to C \\ v &\mapsto \alpha(v, x). \end{aligned}$$

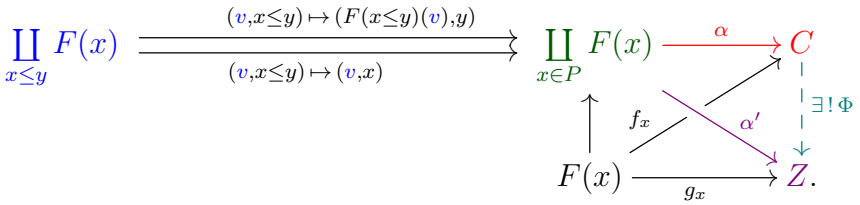
We claim that C together with these morphisms satisfies the conditions of Definition 6.21 of a colimit. Thus we need to verify the following claim:

Claim.

(1) For every $x \leq y$ we have $f_x = f_y \circ F(x \to y) \colon F(x) \to C$.

(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon F(x) \to Z$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon C \to Z$ such that $g_x = \Phi \circ f_x$ for all $x \in P$.

Proof. We summarize all morphisms involved in the following figure:



One can easily verify that morphisms α' as in the above diagram, that make the diagram commute, are in a natural one-to-one correspondence to a family of morphisms $\{f'_x\colon F(x) \to Z\}_{x\in P}$ which satisfy $f'_x = f'_y \circ F(x \to y)\colon F(x) \to Z$. The claim now follows easily from these observations.

(b) This statement follows also easily from elementary logic and the argument of (1). Writing down details does not improve clarity. So we leave them to the reader. ■

We conclude this section with the following analog of Definition 6.19 and of the Category–Completeness Criterion 6.20:

Definition 6.31. A category is called cocomplete if all colimits exist.

Proposition 6.32. (Category–Cocompleteness Criterion) Let $\mathcal{C}$ be a category. If all coproducts and all coequalizers exist, then $\mathcal{C}$ is cocomplete.

Proof. This proposition is an immediate consequence of the Colimit-via-Coproduct+Coequalizer Lemma 6.30. ■

6.5. **Adjoint functors.** We start out with the following definition which was first introduced by Daniel Kan [**Kan1958**]:

Definition 6.33. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

(1) We say that two covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ form an adjoint pair if given any $X \in \mathrm{Ob}(\mathcal{C})$ and given any $Y \in \mathrm{Ob}(\mathcal{D})$ there exists a bijection

$$\Xi_{X,Y}\colon \mathrm{Mor}_{\mathcal{D}}(F(X), Y) \to \mathrm{Mor}_{\mathcal{C}}(X, G(Y))$$

that is natural in X and that is natural in Y.[102] [103] We write $F \dashv G$.

(2) (a) In the setting of (1) we say that the functor F is a left adjoint of G. If F is left adjoint of some functor, then we say F is a left adjoint.

(b) In the setting of (1) we say that the functor G is a right adjoint of F. If G is right adjoint of some functor, then we say F is a right adjoint.

As of right now we have few examples of adjoint pairs, but throughout these notes we will find more and more such pairs.

Example. We now give several examples of adjoint pairs, even though we did not define all notions yet. But hopefully these show that adjoint pairs do indeed appear in "mathematical nature":

- Let $\mathcal{Set}$ be the category of sets and let $\mathcal{AbGroup}$ be the category of abelian groups. In the Free Abelian–Forgetful Functor Lemma 97.10 we will show that the two functors
$$\begin{array}{rcl} F\colon \mathcal{Set} & \to & \mathcal{AbGroup} \\ X & \mapsto & \underbrace{\mathbb{Z}^{(X)}}_{\text{free abelian group generated by } X} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{AbGroup} & \to & \mathcal{Set} \\ (Y, +) & \mapsto & Y \end{array}$$
form an adjoint pair, i.e. we will see that for any set X and any abelian group Y there exists a natural bijection
$$\Xi_{X,Y}\colon \operatorname{Hom}(\mathbb{Z}^{(X)}, Y) \to \operatorname{Map}(X, Y).$$
Furthermore, in the Free–Forgetful Functor Lemma 98.12 we will consider a non-commutative analog of that statement.
- Let $\mathcal{AbGroup}$ be the category of abelian groups and let A be an abelian group. The Hom–Tensor Adjunction Lemmas 159.13 and Hom–Tensor Adjunction Lemma 312.13 say that the two functors
$$\begin{array}{rcl} F\colon \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ X & \mapsto & \underbrace{A \otimes X}_{\text{tensor product}} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ Y & \mapsto & \operatorname{Hom}(A, Y) \end{array}$$
form an adjoint pair, i.e. we will see that for any two abelian groups X and Y there exists a natural bijection
$$\Xi_{X,Y}\colon \operatorname{Hom}(A \otimes X, Y) \to \operatorname{Hom}(X, \operatorname{Hom}(A, Y)).$$

---

[102] This means that for any morphism $\varphi\colon X \to X'$ in $\mathcal{C}$ and any morphism $\psi\colon Y \to Y'$ in $\mathcal{D}$ the following two diagrams commutes:
$$\begin{array}{ccc} \operatorname{Mor}_{\mathcal{D}}(F(X'), Y) & \xrightarrow{\Xi_{X,Y}} & \operatorname{Mor}_{\mathcal{C}}(X', G(Y)) \\ \downarrow{\scriptstyle F(\varphi)^*} & & \downarrow{\scriptstyle \varphi^*} \\ \operatorname{Mor}_{\mathcal{D}}(F(X), Y) & \xrightarrow{\Xi_{X',Y}} & \operatorname{Mor}_{\mathcal{C}}(X, G(Y)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{Mor}_{\mathcal{D}}(F(X), Y) & \xrightarrow{\Xi_{X,Y}} & \operatorname{Mor}_{\mathcal{C}}(X, G(Y)) \\ \downarrow{\scriptstyle F(\psi)_*} & & \downarrow{\scriptstyle \psi_*} \\ \operatorname{Mor}_{\mathcal{D}}(F(X), Y') & \xrightarrow{\Xi_{X',Y}} & \operatorname{Mor}_{\mathcal{C}}(X, G(Y'). \end{array}$$

[103] The naturality of Ξ can also be reformulated as follows: If
$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & G(Y) \\ {\scriptstyle\varphi}\uparrow & \nearrow_{\alpha} & \\ X & & \end{array} \quad \text{commutes in } \mathcal{C} \text{ then} \quad \begin{array}{ccc} F(X') & \xrightarrow{\Xi(\alpha')} & Y \\ {\scriptstyle F(\varphi)}\uparrow & \nearrow_{\Xi(\alpha)} & \\ F(X) & & \end{array} \quad \text{commutes in } \mathcal{D}$$
and, if
$$\begin{array}{ccc} F(X') & \xrightarrow{\beta'} & Y \\ {\scriptstyle F(\varphi)}\uparrow & \nearrow_{\beta} & \\ F(X) & & \end{array} \quad \text{commutes in } \mathcal{D} \text{ then} \quad \begin{array}{ccc} X' & \xrightarrow{\Xi(\beta')} & G(Y) \\ {\scriptstyle\varphi}\uparrow & \nearrow_{\Xi(\beta)} & \\ X & & \end{array} \quad \text{commutes in } \mathcal{C}.$$

- The Cylinder–Path Space Adjunction Lemma 16.11 says that for a fixed regionally compact space T the two covariant functors

$$\begin{array}{rccc} F\colon & \mathcal{T}op & \to & \mathcal{T}op \\ & X & \mapsto & X \times T \end{array} \qquad \text{and} \qquad \begin{array}{rccc} G\colon & \mathcal{T}op & \to & \mathcal{T}op \\ & Y & \mapsto & Y^T. \end{array}$$

  Y^T: set of continuous maps $T \to Y$ with the compact-open topology

  form an adjoint pair.

More examples of adjoint pairs are given in Exercise 6.20, Exercise 98.24 and the Suspension–Loop–Adjunction Proposition 308.25. □

For completeness sake we state the following proposition which we will not use (and hence we will not prove it), but which is nice-to-have:

Proposition 6.34. (Adjoint Uniqueness Proposition) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.
(1) Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor. If $F, F'\colon \mathcal{C} \to \mathcal{D}$ are two left adjoints of G, then they are naturally isomorphic.
(2) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If $G, G'\colon \mathcal{D} \to \mathcal{C}$ are two right adjoints of F, then they are naturally isomorphic.

Proof. A proof of the proposition, using the Yoneda Lemma 5.32, is given in [**Rie2016**, p. 133] or alternatively in [**Awo2010**, Proposition 9.9]/ ■

The following two propositions give a rather miraculous way to show that certain functors preserve (co-) limits:

Proposition 6.35. (Left Adjoint–Colimit Proposition) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. If F is a left adjoint functor, then F preserves all colimits.[104]

Proposition 6.36. (Right Adjoint–Limit Proposition) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $G\colon \mathcal{D} \to \mathcal{C}$ be a covariant functor. If G is a right adjoint functor, then G preserves all limits.[105]

Proof of Propositions 6.35 and Proposition 6.36. In the following we will prove the Left Adjoint–Colimit Proposition 6.35. The proof of the Right Adjoint–Limit Proposition 6.36 is of course almost the same: we just have to reverse all arrows.

Thus let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a left adjoint covariant functor. We pick a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ for F and we let

$$\Xi_{X,Y}\colon \operatorname{Mor}_{\mathcal{D}}(F(X), Y) \to \operatorname{Mor}_{\mathcal{C}}(X, G(Y))$$

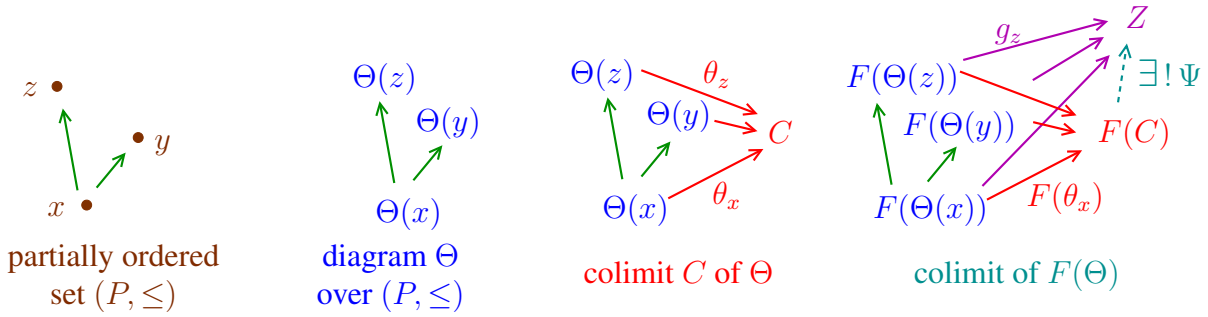
be a natural bijection.

Let $(P, \leq)$ be a partially ordered set and let Θ be a diagram in $\mathcal{C}$ over $(P, \leq)$, which admits a colimit in $\mathcal{C}$. We denote its colimit by C and we denote by $\theta_x\colon \Theta(x) \to C$ the corresponding morphisms for all $x \in P$ with the usual properties.

[104] The proof of the proposition explains in somewhat greater detail what the statement "preserves all colimits" is actually supposed to mean.

Now we consider the corresponding diagram $F(\Theta)$ in $\mathcal{D}$ over $(P, \leq)$. We claim that $F(C)$ together with the morphisms $F(\theta_x)\colon F(\Theta(x)) \to \Theta(C)$ is a colimit of the diagram $F(\Theta)$ in the category $\mathcal{D}$.

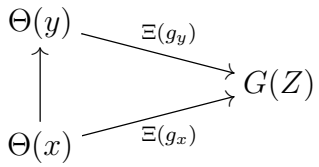
So suppose we are given an object Z of $\mathcal{D}$ and morphisms $g_x\colon F(\Theta(x)) \to Z$ such that for every $x \leq y$ we have $g_x = g_y \circ \Theta(F(x \to y))\colon F(\Theta(x)) \to Z$.



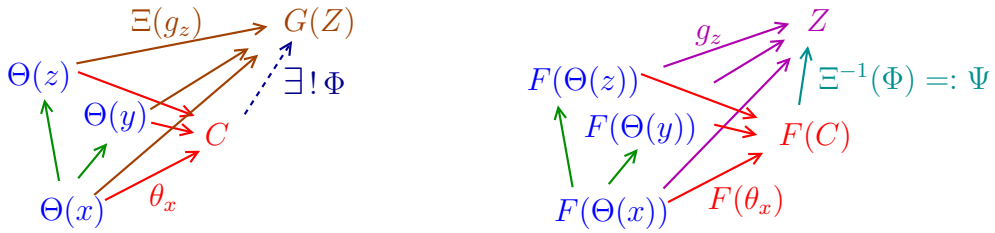
It remains to prove the following claim:

Claim. There exists a unique morphism $\Psi\colon F(C) \to Z$ in the category $\mathcal{D}$ such that for all $x \in P$ we have $g_x = \Psi \circ F(\theta_x)$

Proof. In the category $\mathcal{C}$ we consider the object $G(Z)$ together with the morphisms $\Xi(g_z)\colon \Theta(z) \to G(Z)$. As we discussed in the second diagram of Footnote 103, it follows from the naturality of the bijection Ξ that for any $x \leq y$ the following diagram commutes:



It thus follows from the universal property of the colimit C that there exists a unique morphism $\Phi\colon C \to G(Z)$ that makes the usual diagrams commute.

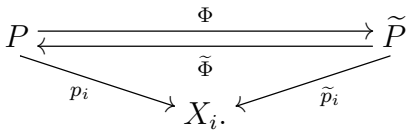


We obtain the morphism $\Psi := \Xi^{-1}(\Phi)\colon F(C) \to Z$. Again by the naturality of Ξ we see that this morphism makes all relevant diagrams commute. Running the argument backwards one also obtains the uniqueness of Ψ from the uniqueness of Φ. ■

6.6. Exercises for Chapter 6.

Exercise 6.1. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. Suppose we are given two products $(P, \{p_i\colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i\colon \widetilde{P} \to X_i\}_{i\in I})$. Show that there exist uniquely determined isomorphisms $\Phi\colon P \to \widetilde{P}$ and $\widetilde{\Phi}\colon \widetilde{P} \to P$ which are inverses of

one another and such that for each $i \in I$ the following diagram commutes:



Remark. This exercise is the content of the Product Uniqueness Lemma 6.2.

Exercise 6.2. Show that the category *Field* of fields does not admit products.

Exercise 6.3. Let *Metric* be the category of metric spaces where the objects are metric spaces and the morphisms are continuous maps.

(a) Do any two metric spaces X, Y admit a product in *Metric*?
(b) Do any two metric spaces X, Y admit a coproduct in *Metric*?

Exercise 6.4. Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathit{Order}_{(P,\leq)}$ from Definition 5.5. What is the coproduct of a family $\{X_i\}_{i\in I}$ of objects in $\mathit{Order}_{(P,\leq)}$, if it exists?

Exercise 6.5. Does the coproduct of the rings $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ in the category *Ring* of rings exist, and if yes, what is it?

Exercise 6.6. A *pointed set* is a pair (X, x_0), where X is a set and x_0 is a point in X. We refer to the category *PSet* with

$$\begin{aligned} \mathrm{Ob}(\mathit{PSet}) &:= \text{all pointed sets,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the *category of pointed sets*.

(a) Does the category of pointed sets admit an initial and a terminal object?
(b) Let (X, x_0) and (Y, y_0) be two pointed sets. Show that a product of (X, x_0) and (Y, y_0) in the category *PSet* is given by the set

$$(X \times Y)/\underbrace{((X \times \{y_0\}) \cup (\{x_0\} \times Y))}_{\substack{\text{the points in this subset get} \\ \text{identified to a single point } *}} \quad \text{together with the point} \quad *$$

and the obvious morphisms to (X, x_0) and (Y, y_0).
(c) Let (X, x_0) and (Y, y_0) be two pointed sets. What is the coproduct of (X, x_0) and (Y, y_0)?

Exercise 6.7. We consider the partially ordered set $(\mathbb{N}, \geq)$. For each $n \in \mathbb{N}$ we consider the topological space $X_n = \{1, \ldots, n\}$ together with the discrete topology and for $n \geq m$ we consider the map

$$\begin{aligned} \gamma_{mn} \colon X_n &\to X_m \\ i &\mapsto \begin{cases} i, & \text{if } i \in \{1, \ldots, m\}, \\ m, & \text{if } i \in \{m+1, \ldots, n\}. \end{cases} \end{aligned}$$

We equip $\mathbb{N} \cup \{\infty\}$ with the subspace topology coming from $\mathbb{R} \cup \{\infty\}$.

(a) We consider the limit $\lim_{i\in\mathbb{N}} X_i$ in the category of topological spaces. Show that there exists a continuous non-constant map $\mathbb{N} \cup \{\infty\} \to \lim_{i\in\mathbb{N}} X_i$.
(b) Is $\mathbb{N} \cup \{\infty\}$ homeomorphic to $\lim_{i\in\mathbb{N}} X_i$?

Exercise 6.8. Let $\mathcal{C}$ be a category that admits pullbacks. Given any two objects X and B and given any morphism $f\colon C \to B$ the pullback of the diagram

$$\begin{array}{ccc} X \times F & \xrightarrow{p_X} & B \times F \\ {\scriptstyle p_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle p_B} \\ C & \xrightarrow[f]{} & B \end{array}$$

is given by the product $X \times C$.

Exercise 6.9. We consider the category *Group* of all groups. Show that in general for two groups A, B the product $A \times B$ is not the coproduct in the category *Group*.
Remark. This exercise shows that the construction in the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 does not provide the coproduct in the category *Group*.

Exercise 6.10. Let $\mathcal{C}$ be a diagram.

(a) Describe the limit of the empty diagram in $\mathcal{C}$.
(b) Describe the colimit of the empty diagram in $\mathcal{C}$.

Exercise 6.11. We consider the partially ordered set $(\mathbb{N}, \leq)$. Let Y be a set. Furthermore let $X_1 \subseteq X_2 \subseteq \ldots$ be a nested sequence of subsets of Y. For each $i \in \mathbb{N}$ we denote by $\varphi_i\colon X_i \to X_{i+1}$ the inclusion. The X_i and φ_i define a diagram in the category *Set* of sets over the diagram $(\mathbb{N}, \leq)$.

(a) Show that the limit of this diagram equals the set X_1.
(b) Show that the colimit of this diagram equals the union of the X_i.

Exercise 6.12. Let $\mathcal{C}$ be a category. A covariant functor $F\colon \mathcal{C} \to \mathcal{S}$ is called *presentable* if there exists a $W \in \mathrm{Ob}(\mathcal{C})$ such that the functor F is naturally isomorphic to the covariant Hom-functor $\mathrm{Mor}(W, -)$ from the Fixed Domain-Covariant Functor Lemma 5.20. Show that any presentable covariant functor preserves all limits.

Exercise 6.13. Let $\varphi\colon A \to G$ be a group homomorphism. What is the pushout of the following pushout diagram in the category *Group* of groups:

$$\begin{array}{ccc} A & \longrightarrow & \{e\} \\ {\scriptstyle \varphi}\downarrow & & \\ G & & \end{array}$$

Exercise 6.14. We consider the category *Set* of sets.

(a) Show that pullbacks always exist.
(b) Show that pushouts always exist.

Exercise 6.15. We consider the category *AbGroup* of abelian groups.

(a) Show that pullbacks always exist.
(b) Show that pushouts always exist.

Exercise 6.16. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle r} \\ C & \xrightarrow{s} & X. \end{array}$$

be a pushout in the category $\mathcal{Set}$ of sets.

(a) We suppose that $f\colon A \to B$ is injective. Show that the opposite map $s\colon C \to X$ is also injective.

(b) If $f\colon A \to B$ is surjective, does it follow that the opposite map $s\colon C \to X$ is also surjective?

Exercise 6.17. Let $\mathcal{C}$ be some category. We consider the following commutative diagram of morphisms in $\mathcal{C}$:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C'. \end{array}$$

We suppose that the big rectangle forms a pushout square.

(a) We suppose that the square to the left is a pushout square. Show that the square to the right is also a pushout square.

(b) We suppose that the square to the right is a pushout square. Is the square to the left also a pushout square?

Exercise 6.18. Let $\mathcal{C}$ be a category. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be morphisms in $\mathcal{C}$ and let $\varphi\colon H \to G$ be an epimorphism in $\mathcal{C}$ in the sense of Definition 5.13. Let C be the pushout of α and β, i.e. we have the following setup:

$$\begin{array}{ccccc} H & \xrightarrow{\varphi} & G & \xrightarrow{\alpha} & A \\ & & {\scriptstyle\beta}\downarrow & & \downarrow \\ & & B & \longrightarrow & C. \end{array}$$

be a pushout in the category $\mathcal{Set}$ of sets. Finally let C' be the pushout of $\alpha \circ \varphi$ and $\beta \circ \varphi$. Show that there is a natural isomorphism $C \to C'$.

Exercise 6.19. We consider the category $\mathcal{Group}$ of groups. Let $\varphi\colon X \to Y$ be a group homomorphism. Describe the coequalizer of φ and the trivial homomorphism.

Exercise 6.20. We consider the functor[106]

$$\begin{array}{rcl} H\colon \mathcal{Top} & \to & \mathcal{Set} \\ (X, \mathcal{T}) & \mapsto & X \end{array}$$

from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{Set}$ of sets.

(a) Show that H is a left adjoint.

(b) Show that H is a right adjoint.

[106]Such a covariant functor, where we "forget" some of the given structure is often called a *forgetful functor*.

CHAPTER 7

Products and disjoint unions of topological spaces

In Chapter 6 we introduced the concepts of (co-) products and more generally (co-) limits in categories. In this and the following chapter we will see how these concepts play out in the category of topological spaces.

This chapter is organized as follows:

- In Section 7.1 we will introduce the product of a family of topological spaces in the category of topological spaces.
- In Section 7.2 we study properties of products of topological spaces. Our focus will be on the product of finitely many topological spaces.
- In Section 7.3 we will generalize some results from the Section 7.2 to products of arbitrary families of topological spaces.
- In Section 7.4 we will consider some applications of the product topology. For example we will see that we can use this topology to give a characterization of a topological space to be Hausdorff.
- In Section 7.5 we will use the product topology to show that the category of topological spaces has pullbacks.
- In Section 7.6 we will introduce the coproduct (or equivalently, the disjoint union) of any family of topological spaces.
- In Appendix 7.7 we will use the product topology to introduce the notion of topological groups and we give some basic examples.
- In Section 7.8 we will provide exercises for Chapter 7.

In the next Chapter 8 we will introduce quotients of topological spaces.

7.1. **The product of topological spaces.** In this section we will see that the category of topological spaces has products. To do so it is convenient to recall Definition 1.10:

Definition 7.1. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of these sets is defined as the set $$\prod_{i\in I} X_i := \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i\in I \text{ we have } f(i)\in X_i \right\}.$$
Sometimes we also denote an element of $\prod_{i\in I} X_i$ by $(x_i)_{i\in I}$.

Proposition 7.2. (Topological-Product Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The following statements hold:

(0) The set

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \subseteq \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

has the basis property that we introduced in Definition 3.52. Thus it follows from the Topology-from-Basis Lemma 3.51 that $\mathcal{B}$ defines a topology on $\prod_{i\in I} X_i$.

(1) (a) For any $i \in I$ the projection

$$\begin{array}{rccc} p_i\colon & \prod_{i\in I} X_i & \to & X_i \\ & f & \mapsto & f(i) \end{array}$$

is continuous with respect to the topology from (0).

(b) Given a topological space Z and given a family $\{f_i\colon Z \to X_i\}_{i\in I}$ of continuous maps the map

$$\begin{array}{rccc} F\colon Z & \to & \prod_{i\in I} X_i & \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

is continuous with respect to the topology from (0).

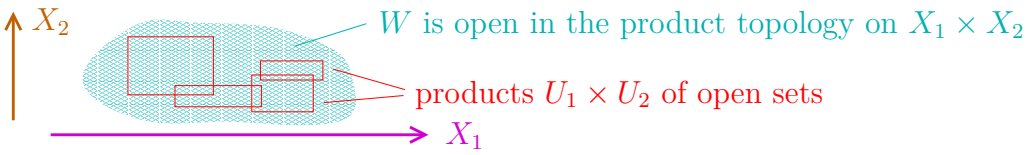
(∗) A map $F\colon Z \to \prod_{i\in I} X_i$ from some topological space Z is continuous if and only if each map $p_i \circ F\colon Z \to X_i$ is continuous.

(2) The topological space $\prod_{i\in I} X_i$ together with the natural projections $p_j\colon \prod_{i\in I} X_i \to X_j$ is a product (in the sense of Definition 6.1) of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces.

Example.

(1) It is worth considering the case that we will use most frequently, namely the case that we are given finitely many topological spaces $X_1, \dots, X_k$. In this case the product topology on $X_1 \times \cdots \times X_k$ is by definition the topology on $X_1 \times \cdots \times X_k$ that is generated by

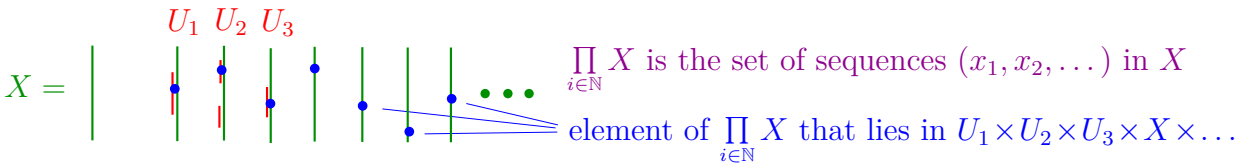
$$\mathcal{B} \;=\; \{U_1 \times \cdots \times U_k \,|\, \text{ each } U_i \text{ is open in } X_i\}.$$



(2) Let X be a topological space. We consider the product

$$X^{\mathbb{N}} \;=\; X \times X \times \ldots \;=\; \text{Maps}(\mathbb{N} \to X) \;=\; \text{set of sequences of points in } X.$$

A basis of the topology is given by the sets of the form $U_1 \times \cdots \times U_k \times X \times \ldots$ where $U_1, \dots, U_k \subseteq X$ are open. In the figure below we try to illustrate this definition.



Proof of Proposition 7.2. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(0) We need to verify that

$$\mathcal{B} \coloneqq \left\{ \prod_{i\in I} U_i \subseteq \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

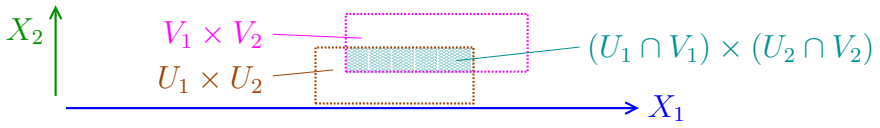
satisfies the conditions (B1) and (B2) formulated in Definition 3.52.

(B1) Evidently we have $\prod_{i\in I} X_i \in \mathcal{B}$. It follows immediately that $\mathcal{B}$ satisfies (B1).

(B2) Let $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$ be two sets in $\mathcal{B}$. Then

$$\prod_{i\in I} \underbrace{U_i}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}} \cap \prod_{i\in I} \underbrace{V_i}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}} = \prod_{i\in I} \underbrace{(U_i \cap V_i)}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}}$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2).



(1) (a) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.

(b) Suppose we are given a topological space Z and continuous maps $f_i \colon Z \to X_i$. We denote by $F \colon Z \to \prod_{i\in I} X_i$ the map that is given by $z \mapsto (i \mapsto f_i(z))$. We claim that F is continuous. By the Subbasis-Continuity Proposition 4.4 it suffices to verify that the preimage of any set in the basis $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. By definition there exists a finite set $J \subseteq I$ such that $U_i = X_i$ for every $i \notin J$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{j\in J} \underbrace{f_j^{-1}(U_j)}_{\text{open in } Z} = \begin{array}{c}\text{intersection of } \textit{finitely} \\ \text{many open subsets of } Z.\end{array}$$

This concludes the proof that F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i \colon Z \to X_i\}_{i\in I}$ be a family of continuous maps. By the definition of a product in Definition 6.1 we need to show that there exists a unique continuous map $\Phi \colon Z \to \prod_{i\in I} X_i$ such that $f_i = p_i \circ \Phi$ for all $i \in I$. To do so, first note that it follows from the Product-of-Sets-and-Groups Lemma 6.4 that the map

$$\begin{array}{rcl} \Phi \colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & (f_i(z))_{i\in I} \end{array}$$

is the unique map (of sets) $\Phi \colon Z \to \prod_{i\in I} X_i$ such that for each $i \in I$ we have the equality $f_i = p_i \circ \Phi$. In (1b) we just showed that this map is fortunately continuous. These two observations give us the desired result. ∎

Definition 7.3. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on the set $\prod_{i\in I} X_i$ that we defined in the Topological-Product Proposition 7.2 (0) as the product topology on $\prod_{i\in I} X_i$.

Example.

(1) It follows from the Topological-Product Proposition 7.2 (1a) and (1b) that given any topological space X the

diagonal map $\begin{array}{rcl} d\colon X & \to & X\times X \\ x & \mapsto & (x,x) \end{array}$ and the projection $\begin{array}{rcl} p\colon X\times X & \to & X \\ (x,y) & \mapsto & x \end{array}$

are continuous. Since $p\circ d = \mathrm{id}_X$ we see that $d\colon X\to X\times X$ is an embedding. This fact will come in handy on many occasions.

(2) Let $\{f_i\colon X_i\to Y_i\}_{i\in I}$ be a family of continuous maps between topological spaces. It follows from the Topological-Product Proposition 7.2 (1) together with the Continuous Composition Lemma 4.2 that the map

$$\begin{array}{rcl} \prod_{i\in I} f_i\colon \prod_{i\in I} X_i & \to & \prod_{i\in I} Y_i \\ (x_i)_{i\in I} & \mapsto & (f(x_i))_{i\in I} \end{array}$$

is continuous. □

The following lemma gives us a reinterpretation of the product topology in some of the most frequently studied cases.

Lemma 7.4. (Product Metric Lemma)

(1) Let $(X_1,d_1),\dots,(X_m,d_m)$ be metric spaces. We define the product metric d on the product $X_1\times\cdots\times X_m$ as follows:

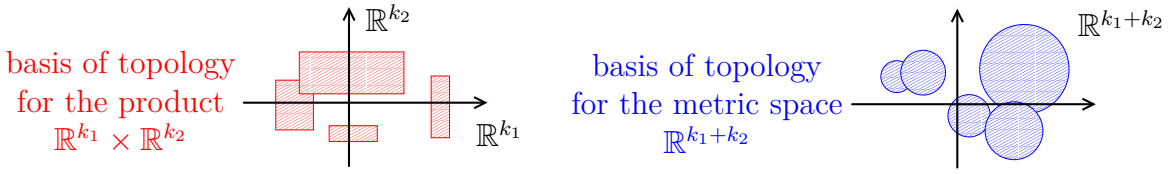
$$d((x_1,\dots,x_m),(y_1,\dots,y_m)) \;:=\; \sum_{i=1}^{m} d_i(x_i,y_i).$$

The product topology on $X_1\times\cdots\times X_m$ agrees with the metric topology coming from the product metric.

(2) Let $k_1,\dots,k_m\in\mathbb{N}_0$. The map

$$\begin{array}{rcl} \overbrace{\mathbb{R}^{k_1+\cdots+k_m}}^{\text{Euclidean topology}} & \to & \overbrace{\mathbb{R}^{k_1}\times\cdots\times\mathbb{R}^{k_m}}^{\text{product of Euclidean topologies}} \\ (x_{1,1},\dots,x_{1,k_1},\dots,x_{m,1},\dots,x_{k,k_m}) & \mapsto & ((x_{1,1},\dots,x_{1,k_1}),\dots,(x_{m,1},\dots,x_{m,k_m})) \end{array}$$

is a homeomorphism. The analogous statement also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$".



Remark. As we just pointed out, by design the product topology is set up in such a way that it is usually easy to show that a map into a product is continuous. On the other hand, except for the projections there is no obvious way to show that a map out of a product of topological spaces is continuous. This is where the Product Metric Lemma 7.4 (2) comes in.

If we want to show that a map out of the product topological space $\mathbb{R}^k \times \mathbb{R}^l$ is continuous, it suffices to show that the map from $\mathbb{R}^{k+l}$ is continuous, which is often easier, since we can use the tools that we acquired in analysis. □

Proof.

(1) To simplify the notation we deal with the case $k = 2$, the general case is proved in the same fashion. First note that one can easily verify that the given product metric on $X_1 \times X_2$ is indeed a metric. We need to show that the product topology $\mathcal{P}$ on $X_1 \times X_2$ agrees with the metric topology $\mathcal{M}$ coming from the product metric.

First we show that $\mathcal{P} \subseteq \mathcal{M}$. Thus let $U \in \mathcal{P}$ and let $(x_1, x_2) \in U$. By definition of $\mathcal{P}$ there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $B^{d_1}_{\epsilon_1}(x_1) \times B^{d_2}_{\epsilon_2}(x_2) \subseteq U$. We set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Note that $(x_1, x_2) \in B^d_\epsilon(x_1, x_2) \subseteq B^{d_1}_{\epsilon_1}(x_1) \times B^{d_2}_{\epsilon_2}(x_2) \subseteq U$. It follows from the definition of $\mathcal{M}$ that $U \in \mathcal{M}$.

Finally we show that $\mathcal{M} \subseteq \mathcal{P}$. Thus let $U \in \mathcal{M}$ and let $(x_1, x_2) \in U$. First note that by definition of $\mathcal{M}$ there exists an $\epsilon > 0$ such that $B^d_\epsilon(x_1, x_2) \subseteq U$. Next note that $(x_1, x_2) \in B^{d_1}_{\frac{\epsilon}{2}}(x_1) \times B^{d_2}_{\frac{\epsilon}{2}}(x_2) \subseteq B^d_\epsilon(x_1, x_2) \subseteq U$. It follows from the definition of $\mathcal{P}$ that $U \in \mathcal{P}$.

(2) It follows from (1) that it suffices to show that the product metric on $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ and the Euclidean metric on $\mathbb{R}^{k_1+\cdots+k_m}$ define the same topology. This in turn follows easily from the Equivalent Metric Lemma 3.11. We leave it to the reader to fill in the details. ■

Identification 7.5. Given $k_1, \ldots, k_m \in \mathbb{N}_0$ we will use the Product Metric Lemma 7.4 (2) to identify the product topological space $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ with the topological space $\mathbb{R}^{k_1+\cdots+k_m}$. We proceed the same way with "$\mathbb{R}$" replaced by "$\mathbb{C}$".

Example. It follows almost immediately from the definition of the topology on the vector space $\mathrm{M}(r \times s, \mathbb{R}) = \mathbb{R}^{r \cdot s}$ in Identifiation 4.52, the example on page 177, the Sum-Product-Continuity Lemma 4.11 and the Product Metric Lemma 7.4 (2) that for any $k, l, m \in \mathbb{N}_0$ the map

$$\begin{aligned} \mathrm{M}(k \times l, \mathbb{R}) \times \mathrm{M}(l \times m, \mathbb{R}) &\to \mathrm{M}(k \times m, \mathbb{R}) \\ ((a_{ij})_{i,j}, (b_{rs})_{r,s}) &\mapsto \Big(\sum_{j=1}^{l} a_{ij} \cdot b_{js} \Big)_{i,s} \end{aligned}$$

is continuous. The same also applies to matrices over $\mathbb{C}$. □

After these examples let us turn to the study of the more formal properties of the product topology. On several occasions we will use, consciously or unconsciously, the following lemma:

Lemma 7.6. (Product Topology-Basics Lemma)

(1) Let $\{X_i\}_{i \in I}$ be a family of topological spaces.
  (a) For each $i \in I$ let A_i be a subspace of X_i.
    (i) The subspace topology of $\prod_{i \in I} A_i \subseteq \prod_{i \in I} X_i$ agrees with product topology on $\prod_{i \in I} A_i$.
    (ii) The natural map $\prod_{i \in I} A_i \to \prod_{i \in I} X_i$

given by the inclusions $A_i \to X_i$ is an embedding, i.e. it is a homeomorphism onto its image.[107]

(b) If $J \subseteq I$ is a subset such that for each $j \in J$ the set X_j consists of a single element x_j, then the map

$$\begin{array}{ccc} \prod\limits_{i\in I\setminus J} X_i & \to & \prod\limits_{i\in I} X_i \\ (f\colon I\setminus J \to \bigcup\limits_{i\in I\setminus J} X_i) & \mapsto & \left(\begin{array}{ccc} I & \to & \bigcup\limits_{i\in I} X_i \\ i & \mapsto & \begin{cases} x_i, & \text{if } i \in J, \\ f(i), & \text{if } i \notin J \end{cases} \end{array}\right). \end{array}$$

is a homeomorphism.

(2) If we are given families of topological spaces $\{\{X_i\}_{i\in I_j}\}_{j\in J}$, then the obvious map

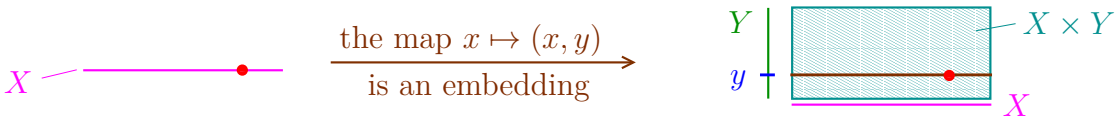
$$\prod_{j\in J}\left(\prod_{i\in I_j} X_i\right) \to \prod_{i\in \bigsqcup_{j\in J} I_j} X_i$$

is a homeomorphism.

(3) If $\{X_i\}_{i\in I}$ is a family of topological spaces and if for each $i \in I$ we are given a basis $\mathcal{B}_i$ of the topological space X_i, then

$$\mathcal{C} = \left\{ \prod_{i\in I} U_i \subseteq \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \neq X_i \text{ is contained in } \mathcal{B}_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

is a basis for $\prod\limits_{i\in I} X_i$.



Example.

(1) On page 165 we saw that the *rational* intervals form a basis for the usual topology on $\mathbb{R}$. It follows from the Product Topology-Basics Lemma 7.6 (3) that the set $\mathcal{B}$ of rational hyperrectangles, i.e. that the set

$$\mathcal{B} = \{\text{hyperrectangle } [a_1, b_1] \times \cdots \times [a_k, b_k] \,|\, a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Q}\} \subseteq \mathcal{P}(\mathbb{R}^k)$$

forms a basis for the standard topology on $\mathbb{R}^k$. It follows from this observation together with the Countability Lemma 1.37 (3) that $\mathbb{R}^k$ admits a basis for the topology that consists of only countably many sets.

(2) Let X be a topological space and let $Y = \{y\}$ be a topological space consisting of a single point. It follows from the Product Topology-Basics Lemma 7.6 (1) that the map $x \mapsto (x, y)$ defines a homeomorphism $X \to X \times \{y\}$. □

[107] If I is finite and if each $A_i \subseteq X_i$ is open (closed), then it follows immediately from the Definition 7.3 of the product topology and the Product-of-Subsets Lemma 7.8 (1) that the natural map $\prod\limits_{i\in I} A_i \to \prod\limits_{i\in I} X_i$ is an open (closed) embedding.

Convention 7.7. Given $X_i \subseteq \mathbb{R}^{k_i}$, $i = 1, \ldots, m$ we use the Product Metric Lemma 7.4 (2) and the Product Topology-Basics Lemma 7.6 (1a) as an excuse to view $X_1 \times \cdots \times X_m$ as a subspace of $\mathbb{R}^{k_1+\cdots+k_m}$.

Proof of Lemma 7.6.

(1) (a) (i) This statement follows immediately from the Subspace-Basis Lemma 3.58 together with the observation that if we take the basis of the topology of the product $\prod_{i\in I} X_i$ from the Topological-Product Proposition 7.2 (0) and if we intersect each basis element with $\prod_{i\in I} A_i$ we obtain precisely the basis of the product $\prod_{i\in I} A_i$ from the Topological-Product Proposition 7.2 (0).

(ii) The given map is clearly an injection. It follows from the Topological-Product Proposition 7.2 (0) that this map is continuous and it follows from (i) that this map is a homeomorphism onto its image.

(b) It follows from the Topological-Product Proposition 7.2 that the given map is continuous. It has an obvious inverse, which is also continuous by the Topological-Product Proposition 7.2.

(2) It is clear that the map is a bijection. It follows from the Topological-Product Proposition 7.2 (1) that the maps in both directions are continuous.

(3) The main idea is to prove the lemma using the Basis Characterization Lemma 3.57 (1). First note that by definition the sets in $\mathcal{C}$ are open. Next let $W \subseteq \prod_{i\in I} X_i$ be an open subset and let $(x_i)_{i\in I} \in W$. By Definition 7.3 of the product topology there exist open subsets $V_i \subseteq X_i$, $i \in I$ such that $X_i = V_i$ for all but finitely many $i \in I$ and such that $(x_i)_{i\in I} \int \prod_{i\in I} V_i \subseteq W$. Since $\mathcal{B}_i$ is a basis for X_i we see that for each i with $V_i \neq X_i$ there exists a $U_i \in \mathcal{B}_i$ with $x \in U_i \subseteq V_i$. For all other i we set $U_i := V_i = X_i$. It follows that

$$(x_i)_{i\in I} \in \prod_{i\in I} U_i, \subseteq \prod_{i\in I} V_i \subseteq W.$$

We deduce from the Basis Characterization Lemma 3.57 (1) that the given set $\mathcal{C}$ is indeed a basis for $\prod_{i\in I} X_i$. ∎

7.2. **Properties of products of topological spaces I.** Throughout these notes we will almost invariably work with the product of only finitely many topological spaces. In this section we will therefore discuss properties of the product of *finitely* many topological spaces. This simplifies some of the arguments and it makes the notation arguably more natural. In the following section we will, for completeness sake, also discuss some of the properties of products of arbitrarily many topological spaces.

Lemma 7.8. (Product-of-Subsets Lemma) Let $X_1, \ldots, X_k$ be topological spaces.

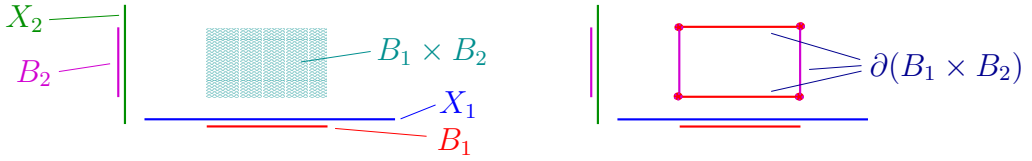
(1) If $A_i \subseteq X_i$, $i = 1, \ldots, k$ are closed subsets, then $A_1 \times \cdots \times A_k$ is a closed subset of $X_1 \times \cdots \times X_k$.

(2) Let $B_i \subseteq X_i$, $i = 1, \ldots, k$ be subsets. The following equalities hold:

(a) interior of $B_1 \times \ldots \times B_k$ in $X_1 \times \ldots \times X_k$ $= \overset{\circ}{B}_1 \times \ldots \times \overset{\circ}{B}_k$

(b) closure of $B_1 \times \ldots \times B_k$ in $X_1 \times \ldots \times X_k$ $= \overline{B}_1 \times \ldots \times \overline{B}_k$

(c) boundary of $B_1 \times \ldots \times B_k$ in $X_1 \times \ldots \times X_k$ $= \bigcup_{i=1}^{k} (\overline{B}_1 \times \ldots \times \partial B_i \times \ldots \times \overline{B}_k)$.

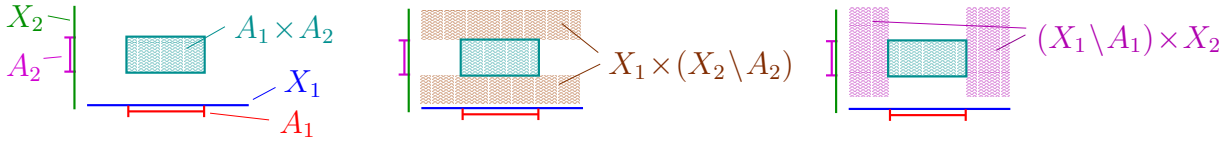


Proof.

(1) We have

$$(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k) \overset{\text{basic set theory}}{=} \bigcup_{i=1}^{k} \underbrace{X_1 \times \cdots \times (X_i \setminus A_i) \times \cdots \times X_k}_{\text{open in } X_1 \times \cdots \times X_k \text{ since } A_i \text{ is closed}}.$$

Thus we see that $(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k)$ is the union of open subsets, i.e. the set itself is open, which implies by definition that $A_1 \times \cdots \times A_k$ is closed.



(2) (b) By (1) and by definition we have $\overline{B_1 \times \cdots \times B_k} \subseteq \overline{B}_1 \times \cdots \times \overline{B}_k$. It remains to prove the reverse inclusion. Thus let $x = (x_1, \ldots, x_k) \in \overline{B}_1 \times \cdots \times \overline{B}_k$. We want to show that $x \in \overline{B_1 \times \cdots \times B_k}$. By the Interior-Closure Characterization Lemma 3.23 (2) we need to show that every neighborhood U of x has non-empty intersection with $B_1 \times \cdots \times B_k$. Given such a neighborhood U we can find, by Definition 7.3 of the product topology, open subsets $U_i \subseteq X_i$, $i = 1, \ldots, k$, such that

$$(x_1, \ldots, x_k) \in U_1 \times \cdots \times U_k \subseteq U.$$

In particular each U_i is an open neighborhood of $x_i \in X_i$. Thus it follows from the Interior-Closure Characterization Lemma 3.23 (2) that for every i we have $U_i \cap B_i \neq \varnothing$. But this implies that $U \cap (B_1 \times \cdots \times B_k) \neq \varnothing$.

(a) The proof of this statement is similar to the proof of (b). We will verify this statement in Exercise 7.3.

(a) This statement follows easily from (a) and (b) and the fact that by definition we have $\partial C = \overline{C} \setminus \overset{\circ}{C}$. ∎

The next proposition deals with some of our favorite properties of topological spaces.

Proposition 7.9. (Product Topology-Properties Proposition) Let $X_1, \ldots, X_k$ be non-empty[108] topological spaces.

(1) The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.

(2) The product $X_1 \times \cdots \times X_k$ is (path)-connected if and only if each X_i is (path)-connected.

(3) The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.

Remark. The Product Topology-Properties Proposition 7.9 (1) also shows how the case $n = 1$ in the Hyperrectangle-Compact Proposition 3.47 implies the general case of the Hyperrectangle-Compact Proposition 3.47. □

Before we head to the lengthy proof of the Product Topology-Properties Proposition 7.9 let us first discuss some examples in detail.

Definition 7.10.
(1) We refer to $(S^1)^n$ as the n-dimensional torus. For $n = 2$ we just say the torus.
(2) We refer to $S^1 \times \overline{B}^2$ as the solid torus and we refer to $S^1 \times B^2$ as the open solid torus.

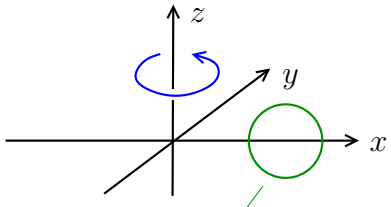
We deduce from the Product Topology-Properties Proposition 7.9 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

The following lemma allows us to view (solid) tori as subspaces of $\mathbb{R}^3$.

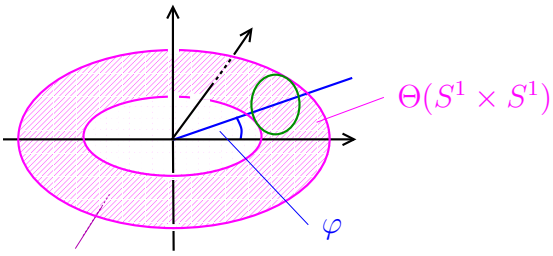
Lemma 7.11. (Torus Embedding Lemma) The map[109]

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \mathbb{R}^3 \\ ((x,y),\exp(\mathrm{i}\varphi)) & \mapsto & \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{array}$$

is an embedding. The restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are also embeddings.



circle in the xz-plane
is rotated around the z-axis



$\Theta(B^2 \times S^1)$ is in the "interior"

Proof. The map $\Theta\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$ can be written as the composition of the maps

$$\begin{array}{rcl} \overline{B}^2 \times S^1 & \to & \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \overset{\cong}{\leftarrow} \{(x,y,a,b) \in \mathbb{R}^4 \mid a^2+b^2 > 0\} \to \mathbb{R}^3 \\ & & (x,y,a,b) \mapsto \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y. \end{pmatrix} \end{array}$$

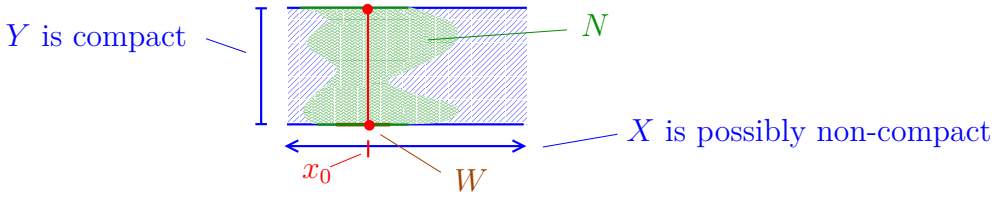
The first map is continuous by the Product Topology-Properties Proposition 7.9 (1), the second map is a homeomorphism by the Product Metric Lemma 7.4 and the last map is continuous by the Sum-Product-Continuity Lemma 4.11. This shows that Θ is continuous.

[108]The "if"-directions also hold if one the X_i is the empty space, since in this case the product is the empty space.
[109]The map is chosen in such a way that it is orientation-preserving (we will introduce the notions of orientation and orientation-preserving later in Definition 31.29) and such that $\Theta(\{(0,0)\} \times S^1) = S^1 \times \{0\}$.

An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. By the Product Topology-Properties Proposition 7.9 (1) we know that the solid torus $\overline{B}^2 \times S^1$ is compact. Thus we obtain from the Compact-Hausdorff Proposition 4.23 (3) that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is an embedding. But then it follows from the Iterated Subspace Lemma 3.18 that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are embeddings. ■

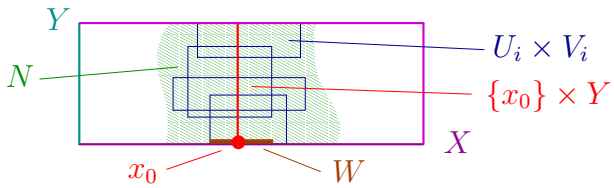
Now the time of procrastination is over, we need to provide the proof of the Product Topology-Properties Proposition 7.9. In preparation we will prove the following lemma:

Lemma 7.12. (Tube Lemma) Let X and Y be topological spaces, let $x_0 \in X$ and let N be an open subset of $X \times Y$ that contains $\{x_0\} \times Y$. If Y is compact, then there exists an open neighborhood W of x_0 such that $W \times Y \subseteq N$.



Proof of the Tube Lemma 7.12. By Definition 7.3 of the product topology there exist sets of the form $U_i \times V_i$, $i \in I$ with the following properties:

(1) each U_i is open in X and each V_i is open in Y,
(2) each U_i contains x_0, i.e. for each i we have $\{x_0\} \times V_i \subseteq U_i \times V_i$,
(3) each $U_i \times V_i$ is contained in N,
(4) $\{x_0\} \times Y$ is contained in the union of the $U_i \times V_i$.



It follows from the hypothesis that Y is compact and the Product Topology-Basics Lemma 7.6 (1) that $\{x_0\} \times Y$ is compact. Since the $U_i \times V_i$ form an open cover of $\{x_0\} \times Y$ we can arrange that I is a finite set, i.e. we can assume that $I = \{1, \dots, n\}$. We set $W = U_1 \cap \dots \cap U_n$. This is an open subset of X that contains x_0. It follows immediately from (3) and the definition of W that $W \times Y \subseteq N$. ■

Proof of the Product Topology-Properties Proposition 7.9 (1). By induction we only have to deal with the case $k = 2$. Thus let X and Y be two non-empty topological spaces. The"$\Rightarrow$"-direction of Statement (1) is fortunately elementary: If $X \times Y$ is compact, then it follows immediately from the Compact Image Lemma 4.15 (1) applied to the natural projections $X \times Y \to X$ and $X \times Y \to Y$ (which are continuous by the Topological-Product Proposition 7.2 (1a)), that X and Y are compact.[110]

[110]Note that the projections are surjective since we assume that X and Y are non-empty.

Next we deal with"$\Leftarrow$"-direction of Statement (1). This is by far the hardest part of the proposition. So we now assume that X and Y are both compact. We need to show that $X \times Y$ is compact.

Thus suppose that we are given an open cover $\{A_i\}_{i\in I}$ of $X \times Y$. Let $x \in X$. Since Y is compact there exists a finite set $I(x) \subseteq I$ such that $\{x\} \times Y \subseteq \bigcup_{i\in I(x)} A_i$. By the Tube Lemma 7.12 there exists an open neighborhood $W(x)$ of x such that $W(x) \times Y \subseteq \bigcup_{i\in I(x)} A_i$.

All these open subsets $\{W(x)\}_{x\in X}$ are an open cover of X. Since X is compact there exist $x_1, \ldots, x_k \in X$ such that $X = W(x_1) \cup \cdots \cup W(x_k)$. But then we see that

$$X \times Y \;=\; (W(x_1) \cup \cdots \cup W(x_k)) \times Y \;=\; \bigcup_{j=1}^{k} W(x_j) \times Y \;\subseteq\; \bigcup_{j=1}^{k} \bigcup_{i\in I(x_j)} A_i \;\subseteq\; X \times Y.$$

Thus we have successfully covered $X \times Y$ by finitely many sets in the given open cover. ■

Proof of the Product Topology-Properties Proposition 7.9 (2). As in the proof of Statement (1) we only need to deal with the case $k = 2$. Thus let X and Y be two non-empty topological spaces.

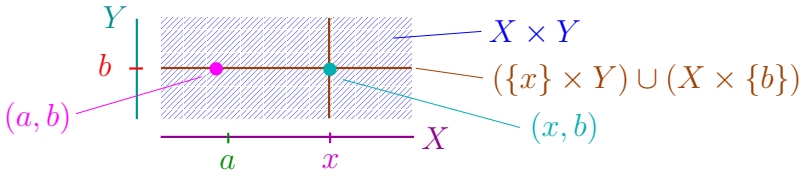
The"$\Rightarrow$"-direction of Statement (2) is once again basically elementary. Indeed, if $X \times Y$ is (path)-connected, then it follows immediately from the Image-Connected Lemma 4.34 applied to the natural projections $X \times Y \to X$ and $X \times Y \to Y$ (which are continuous by the Topological-Product Proposition 7.2 (1a)), that X and Y are (path)-connected.[111]

Next we deal with"$\Leftarrow$"-direction of Statement (2). First we suppose that X and Y are path-connected. We need to show that $X \times Y$ is also path-connected. Thus let (x, y) and (x', y') be two points in $X \times Y$. Since X is path-connected we can find a path $p\colon [0, 1] \to X$ from x to x' and similarly we can find a path $q\colon [0, 1] \to Y$ from y to y'. Then

$$\begin{array}{ccc} [0,1] & \to & X \times Y \\ t & \mapsto & (p(t), q(t)) \end{array}$$

is a path from (x, y) to (x', y').[112]

Finally we suppose that X and Y are connected. We want to show that $X \times Y$ is also connected. We pick $a \in X$ and $b \in Y$. Let $x \in X$. Note that it follows from our hypothesis on X and Y and the Product Topology-Basics Lemma 7.6 (1) that the subspaces $\{x\} \times Y$ and $X \times \{b\}$ are connected by our hypothesis. Since these have the point (x, b) in common we deduce from the Union Connected Lemma 4.39 (2) that $(\{x\} \times Y) \cup (X \times \{b\})$ is also connected.



Note that

$$X \times Y \;=\; \bigcup_{x\in X} (\{x\} \times Y) \cup (X \times \{b\}).$$

[111]Note that the projections are surjective since we assume that X and Y are non-empty.

[112]It follows from the Topological-Product Proposition 7.2 (1b) that this map is indeed continuous.

All the subspaces on the right are connected and they have the point (a, b) in common. Thus it follows again from the Union Connected Lemma 4.39 (2) that the union on the right-hand side is connected. ■

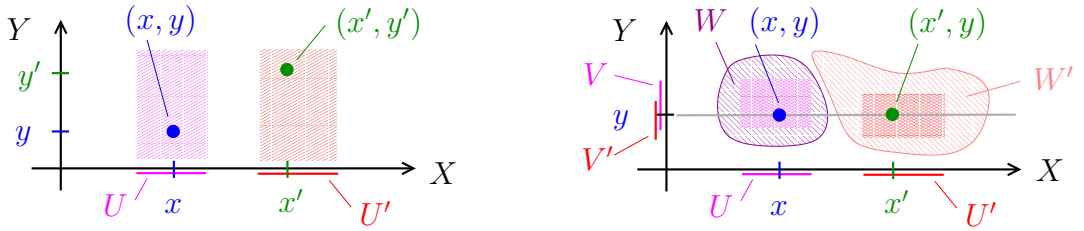
Proof of the Product Topology-Properties Proposition 7.9 (3). As in the proof of Statement (1) we only need to deal with the case $k = 2$. Thus let X and Y be two non-empty topological spaces.

First we deal with the"$\Leftarrow$"-direction of Statement (3). Thus we assume that X and Y are Hausdorff. Let (x, y) and (x', y') be two distinct points in $X \times Y$. Without loss of generality we can assume that $x \neq x'$. By our hypothesis there exist open neighborhoods U of x and U' of x' with $U \cap U' = \varnothing$. Note that

$$(U \times Y) \cap (U' \times Y) \;=\; (U \cap U') \times Y \;=\; \varnothing \times Y \;=\; \varnothing.$$

By Definition 7.3 of the product topology the sets $U \times Y$ and $U' \times Y$ are open in $X \times Y$. We have thus found disjoint open neighborhoods of (x, y) and (x', y'). Therefore $X \times Y$ is Hausdorff. (This argument is sketched in the figure below to the left.)

Now we deal with the"$\Rightarrow$"-direction of Statement (1). Therefore we suppose that $X \times Y$ is Hausdorff. We want to show that X is Hausdorff. Thus let x and x' be two different points in X. Since Y is non-empty there exists a $y \in Y$. Furthermore, since $X \times Y$ is Hausdorff there exist disjoint open neighborhoods W and W' of (x, y) and (x', y). By Definition 7.3 of the product topology there exist open neighborhoods U of x and U' of x' and open neighborhoods V of y and V' of y such that $(U \times V) \cap (U' \times V') = \varnothing$. In particular we see that $(U \times \{y\}) \cap (U' \times \{y\}) = \varnothing$. But this implies that $U \cap U' = \varnothing$. This implies that U and U' are disjoint open neighborhoods of x and x'. Thus we have shown that X is Hausdorff.



The proof that Y is also Hausdorff is evidently basically the same. ■

7.3. **Properties of products of topological spaces II.** In this section we now return to the study of the product of arbitrarily many topological spaces. We are not aware of an interesting generalization of the Product-of-Subsets Lemma 7.8. Thus we jump straight to the following generalization of the Product Topology-Properties Proposition 7.9.

Proposition 7.13. (Product Topology-Properties Proposition) Let $\{X_i\}_{i \in I}$ be a family of non-empty topological spaces.

(1) The product $\prod_{i \in I} X_i$ is compact if and only if each X_i is compact.

(2) The product $\prod_{i \in I} X_i$ is (path)-connected if and only if each X_i is (path)-connected.

(3) The product $\prod_{i \in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. The proof of all statements of the Product Topology-Properties Proposition 7.13 is basically the same as the proof of the corresponding statements of the Product Topology-Properties Proposition 7.9, *except*, for the "if"-direction of Proposition 7.13 (1) and the "if"-direction of Proposition 7.13 (2) in the connected case. We will deal with these two statements in the following discussion. ■

First we deal with the "if"-direction of the Product Topology-Properties Proposition 7.13 (1). This proof is now considerably harder and thus the proposition gets upgraded to a theorem:

Theorem 7.14. (Tychonoff's Theorem[113]) The product of arbitrarily many compact spaces is again a compact space.

Sketch of proof. We will not really make use of Tychonoff's Theorem, therefore we limit ourselves to a proof for $I = \mathbb{N}$. Thus let $\{X_i\}_{i\in\mathbb{N}}$ be a family of compact spaces. Furthermore let $\{U_j\}_{j\in J}$ be a family of open subsets of $\prod\limits_{i\in\mathbb{N}} X_i$ such that we cannot cover $\prod\limits_{i\in\mathbb{N}} X_i$ by finitely many sets in the given open cover $\{U_j\}_{j\in J}$. We need to show that $\{U_j\}_{j\in J}$ is not a cover of $\prod\limits_{i\in\mathbb{N}} X_i$. In other words, we need to show that $\bigcup\limits_{j\in J} U_j$ is a proper subset of $\prod\limits_{i\in\mathbb{N}} X_i$.

Claim 1. There exist $x_i \in X_i$, $i \in \mathbb{N}$ with the following property: Given any $k \in \mathbb{N}_0$ and given any open neighborhoods V_i of x_i, $i = 1, \dots, k$ the product

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots$$

cannot be covered by finitely many sets of $\{U_j\}_{j\in J}$.

Proof. The case $k = 0$ holds since we cannot cover $\prod\limits_{i\in\mathbb{N}} X_i$ by finitely many sets of $\{U_j\}_{j\in J}$. Suppose that for some $k \in \mathbb{N}_0$ we have already found $x_i \in X_i$, $i = 1, \dots, k$ with the desired properties. Suppose that there is no such $x_{k+1} \in X_{k+1}$. This means that for every $y \in X_{k+1}$ there exist open neighborhoods $V_1^y, \dots, V_k^y, W^y$ of $x_1, \dots, x_k, y$ such that the product $V_1^y \times \cdots \times V_k^y \times W^y \times X_{k+2} \times \dots$ can be covered by finitely many of the U_j, i.e. there exists a finite subset $J_y \subseteq J$ such that $V_1^y \times \cdots \times V_k^y \times W^y \times X_{k+2} \times \dots \subseteq \bigcup\limits_{j\in J_y} U_j$.

Since X_{k+1} is compact there exists a finite set $Y \subseteq X_{k+1}$ such that $\bigcup\limits_{y\in Y} W^y = X_{k+1}$. Give $i = 1, \dots, k$ we set $\widetilde{V}_i := \bigcap\limits_{y\in Y} V_i^y$. Since Y is finite we see that each $\widetilde{V}_i$ is actually an open neighborhood of x_i. Now note that

$$\widetilde{V}_1 \times \cdots \times \widetilde{V}_k \times \underbrace{X_{k+1}}_{=\bigcup\limits_{y\in Y} W^y} \times X_{k+2} \times \dots = \bigcup_{y\in Y} \widetilde{V}_1 \times \cdots \times \widetilde{V}_k \times W^y \times X_{k+2} \times \dots \subseteq \bigcup_{y\in Y} \bigcup_{j\in J_y} U_j.$$

In summary we have now found open neighborhoods $\widetilde{V}_1, \dots, \widetilde{V}_k$ of $x_1, \dots, x_k$ such that the product $\widetilde{V}_1 \times \cdots \times \widetilde{V}_k \times X_{k+1} \times X_{k+2} \times \dots$ can be covered by *finitely many* of the U_j. But this contradicts the defining property of $x_1, \dots, x_k$. ⊞

The following claim concludes the proof of Tychonoff's Theorem for $I = \mathbb{N}$.

[113]The theorem has its origin in the work of Andrey Tikhonov [**Tyc1935**]. The spelling for the theorem is due to the fact that Tikhonov initially published in German and his name got transcribed from the Cyrillic alphabet to the Latin alphabet as Tychonoff.

Claim 2. The point $(x_1, x_2, \dots) \in \prod_{i\in\mathbb{N}} X_i$ does not lie in any of the U_j.

Proof. Suppose that there exists a $j \in J$ with $(x_1, x_2, \dots) \in U_j$. By definition of the topology on $\prod_{i\in\mathbb{N}} X_i$ (see the Topological-Product Proposition 7.2 (0) together with the Topology-from-Basis Lemma 3.51) there exists a $k \in \mathbb{N}$ and open neighborhoods V_i of x_i, $i = 1, \dots, k$ such that

$$V_1 \times \dots \times V_k \times X_{k+1} \times X_{k+2} \times \dots \subseteq U_j.$$

But by construction of $(x_1, x_2, \dots)$ this is impossible. ⊞

The proof of the general case of the Tychonoff Theorem can be done in a very similar way by equipping the index set I with a total order, which we can do by the Well-Ordering Theorem 1.32. We refer to [**Wri1994**] for details. Alternatively proofs of Tychonoff's Theorem are provided in most textbooks on general topology, see e.g. [**Mun1975**, Chapter 5.1] or [**Jä2005**, Chapter X]. A particularly concise proof is also given in [**Chf1992**]. Note that all proofs of the Tychonoff Theorem rely on the Axiom of Choice 1.1, in fact, as is shown in [**Je1973**, p. 26] or [**BTT2020**, Theorem 3.14], the Tychonoff Theorem is equivalent to the Axiom of Choice 1.1. ■

Finally the following proposition gives us the "if"-direction of the Product Topology-Properties Proposition 7.13 (2) in the connected case.

Proposition 7.15. (Product Connected Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. If each X_i is connected, then the product $\prod_{i\in I} X_i$ is connected.

In the proof of the Product Connected Proposition 7.15 we will make use of the following lemma, which is of independent interest, and which we actually proved in Exercise 4.23.

Lemma 7.16. (Closure-Connected Lemma) Let X be a topological space and let $A \subseteq X$ be a subset. If A is connected, then so is its closure $\overline{A}$.

Proof of Lemma 7.16. Let $\overline{A} = C \cup D$ be a decomposition of $\overline{A}$ into two open subsets. We need to show that $\overline{A} = C$ or $\overline{A} = D$. It follows from the Iterated Subspace Lemma 3.18 that $C \cap A$ and $D \cap A$ are open subsets of A. Since A is connected we can assume $C \cap A = A$ and $D \cap A = \varnothing$. This means that $A \subseteq C$. Note that $C = \overline{A} \setminus D$ is the complement of an open subset, thus C is closed in $\overline{A}$. Since $\overline{A}$ is closed in X it follows from the Open-Closed Inclusion Lemma 4.13 that C is a closed subset of X. Since it contains A we know, by Definition 3.21 of the closure, that $\overline{A} \subseteq C$. Thus we see that $\overline{A} \subseteq C \subseteq \overline{A}$, i.e. $\overline{A} = C$. ■

Proof of Proposition 7.15. Let $\{X_i\}_{i\in I}$ be a family of connected topological spaces. We need to show that the product $\prod_{i\in I} X_i$ is also connected. If one of the X_i is empty, then there is nothing to show. So we can assume that every X_i is non-empty. For each $i \in I$ we pick a point $P_i \in X_i$. Furthermore, for each finite subset $F \subseteq I$ we set

$$Y_F \coloneqq \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \text{for each } i \notin F \text{ we have } x_i = P_i \right\}.$$

Claim 1. The subspace $Y \coloneqq \bigcup_{F \subseteq I \text{ finite}} Y_F$ is connected.

Proof. Let $F \subseteq I$ be a finite subset. There is an obvious bijection between Y_F and $\prod_{i\in F} X_i$. Using the Topological-Product Proposition 7.2 (1) and using the Subspace Continuity Lemma 4.3 (3) one can easily verify that the bijections in both directions are actually continuous. In other words, the bijections are homeomorphisms. By the initial Product Topology-Properties Proposition 7.13 (3) we know that $\prod_{i\in F} X_i$, and thus also Y_F, is connected. All the subspaces Y_F have the point $(P_i)_{i\in I}$ in common. Thus it follows from the Union Connectedness Lemma 4.39 that the union of the Y_F, namely Y, is connected. ⊞

By the Closure-Connected Lemma 7.16 together with the Density-Interior-Closure Lemma 3.26 it remains to prove the following claim:

Claim 2. The subset $Y = \bigcup_{F\subseteq I \text{ finite}} Y_F$ is dense in $\prod_{i\in I} X_i$, in the sense of Definition 3.25.

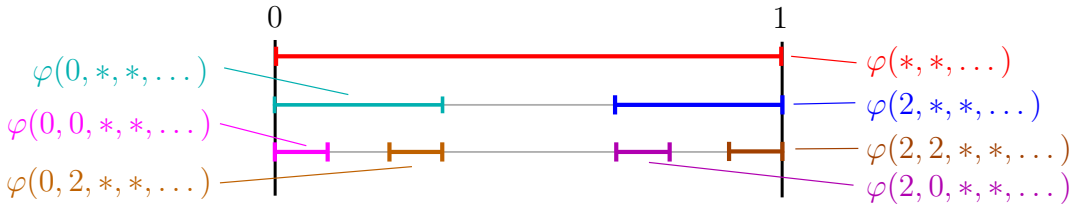
Proof. Let U be an open non-empty subset X. Since U is non-empty we know that U contains a subset of the form $\prod_{i\in I} V_i$ where each $V_i \subseteq X_i$ is open and non-empty and such that there exists a finite subset $F \subseteq I$ such that for each $i \notin F$ we have $V_i = X_i$. It is now clear that $Y_F \cap \prod_{i\in I} V_i \neq \varnothing$. But this implies that $Y \cap U \neq \varnothing$. ■

Now that we are finally done with the proof of the Product Topology-Properties Proposition 7.13 we can study a very instructive example.

Example. We equip $\{0,2\}$ with the discrete topology. We consider the map

$$\begin{aligned} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} &\to C = \text{Cantor set from Definition 3.45} \\ (f\colon \mathbb{N} \to \{0,2\}) &\mapsto \sum_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{aligned}$$

It is elementary to see that φ takes values in C.



It is a refreshing exercise, left to the enthusiastic reader, to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 7.14 and the right-hand is evidently Hausdorff. In Exercise 7.21 we will verify, using the Subbasis-Continuity Proposition 4.4 and the JH-Openness Criterion 3.3, that φ is continuous. Thus it follows from the Compact-Hausdorff Proposition 4.23 (3) that φ is a homeomorphism. □

Initially the definition of the Cantor set is nicely explicit, but the definition looks rather arbitrary. The above example gives a much more natural definition of the Cantor set, viewed as a topological space. Amazingly there exists a characterization of the Cantor set, up to homeomorphism, in terms of basic topological properties. To state the proposition we need two easy definitions.

Definition 7.17. Let X be a topological space. A point $x_0 \in X$ is called isolated if $\{x_0\}$ itself is an open subset.

Examples.

(1) We consider $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$. The only point in X that is not isolated is 0.
(2) One can easily verify that no point in the Cantor set is isolated.
(3) It follows immediately from the definitions that no point in $\prod_{i\in\mathbb{N}} \{0, 2\}$ is isolated.
(4) The topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ equipped with the subspace topology coming from $\mathbb{R}$ do not contain any isolated points.
(5) Let X be a topological space. If every point is isolated, then by definition X has the discrete topology. If X is furthermore finite, then it follows from the Compact-Discrete Lemma 3.40 (1) that X is finite. □

Definition 7.18. A topological space X is called totally disconnected if the only connected subsets are the empty set and the subsets consisting of a single point.

Example. The following topological spaces are totally disconnected:

(1) Discrete spaces.
(2) The Cantor set.
(3) The infinite product $\prod_{i\in\mathbb{N}} \{0, 2\}$.
(4) The topological spaces $\mathbb{Q}$ equipped with the subspace topology coming from $\mathbb{R}$.

Here (1) is obvious and (4) was dealt with in Exercise 4.28. We will deal with the remaining two examples in Exercise 7.23. □

We conclude this section with the following amusing proposition.

Proposition 7.19. (Cantor Set Characterization Proposition) Let X be a topological space that is compact and totally disconnected. If X contains no isolated points, then X is homeomorphic to the Cantor set.

Proof. We will not make use of the proposition, hence we will not prove it. A proof is for example given in [**Val2013**, Theorem 7.8] or [**Will1970**, Theorem 30.3]. ■

7.4. **Applications of the product topology.** In this short section we see in two instances that the product topology can be quite convenient and that it appears in unexpected settings. First of all we have the following lemma which gives us a surprising characterization of Hausdorff spaces:

Lemma 7.20. (Hausdorff-Diagonal Lemma) For every topological space the following two statements are equivalent:[114]

(i) X is Hausdorff.
(ii) The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$.

Proof. The lemma follows really quite easily from the definitions. Therefore we outsource the proof to Exercise 7.9 for the reader to get some practice with the definitions. ■

[114]Note that this lemma gives a solution to Exercise 4.41.

Using the Hausdorff-Diagonal Lemma 7.20 we can give an elegant proof of the following lemma:

Lemma 7.21. (Maps-agree-on-Closure Lemma) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. We assume that Y is Hausdorff.

(1) The set $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.
(2) If f and g agree on a subset A, then f and g also agree on the closure $\overline{A}$ of A in X.

Proof.

(1) We consider the diagonal $\Delta = \{(y, y) \in Y \times Y \mid y \in Y\}$. We also consider the map $(f, g)\colon X \to Y \times Y$ that is given by $x \mapsto (f(x), g(x))$. We now see that

$$\{x \in X \mid f(x) = g(x)\} \underset{\text{follows easily from the definitions}}{=} (f,g)^{-1}(\Delta) \underset{\text{since } Y \text{ is Hausdorff we know by the Hausdorff-Diagonal Lemma 7.20 that } \Delta \text{ is a closed subset of } Y \times Y}{=} (f,g)^{-1}(\text{closed subset of } Y \times Y)$$
$$\underset{\text{it follows from the Topological-Product Proposition 7.2 (1b) that } (f,g) \text{ is continuous}}{=} \text{closed subset of } X.$$

(2) This statement follows from (a) and Definition 3.21 of the closure as the intersection of all closed subsets of X that contain A. ∎

7.5. **The pullback of topological spaces.** In Definition 6.13 we introduced the general notion of pullbacks in a category. In this section we show that the category of topological spaces always has pullbacks. It is rather reassuring to hear that this is the case, even though the pullback will only really play a role much later when we study bundles. Thus this section can also be skipped without any major loss.

Proposition 7.22. (Topological-Pullback Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. We consider the topological space

$$f^*X := \{(c, x) \in C \times X \mid f(c) = p(x)\}$$

and the maps

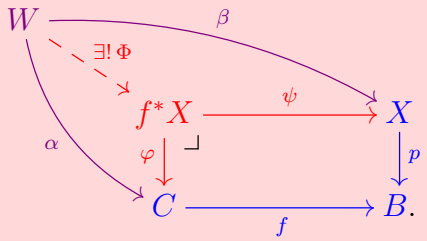
$$\begin{array}{rcl} \varphi\colon f^*X & \to & C \\ (c,x) & \mapsto & c \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon f^*X & \to & X \\ (c,x) & \mapsto & x. \end{array}$$

The following statements hold:

(1) (a) The maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ are continuous.
(b) Suppose we are given a topological space W and two continuous maps $\alpha\colon W \to C$ and $\beta\colon W \to X$ such that $f \circ \alpha = p \circ \beta\colon W \to B$. The map

$$\begin{array}{rcl} \Phi\colon W & \to & f^*X \\ w & \mapsto & (\alpha(w), \beta(w)) \end{array}$$

is continuous. In particular we obtain the following commutative diagram:



(∗) A map $\Phi\colon W \to f^*X$ is continuous if and only if the maps $\varphi \circ \Phi\colon W \to C$ and $\psi \circ \Phi\colon W \to X$ are continuous.

(2) The topological space f^*X with the natural maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ is the pullback, in the sense of Definition 6.13, in the category $\mathcal{T}op$ of topological spaces.

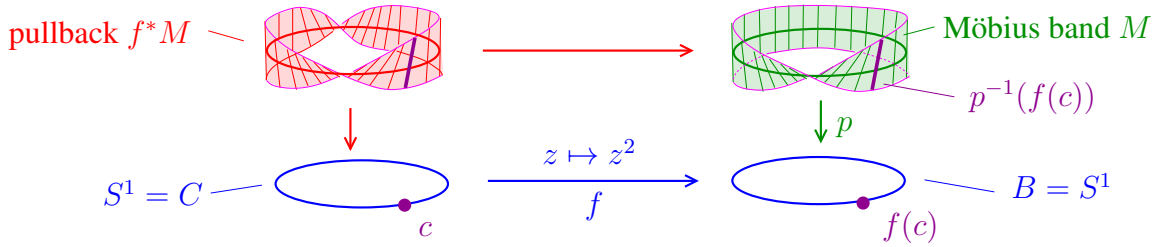
Proof.

(1) (1) This statement follows from the Topological-Product Proposition 7.2 (1a) and the Subspace Continuity Lemma 4.3 (2).
(2) This statement follows from the Topological-Product Proposition 7.2 (1b) and the Subspace Continuity Lemma 4.3 (3).
(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) It remains to show uniqueness of Φ. This follows immediately from the definitions and the fact that a point in $C \times X$ is uniquely determined by its projections. ■

Example.

(1) In the figure below we show the projection $p\colon M \to S^1$ from the Möbius band to S^1 and we see the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$.



We want to understand the pullback f^*M. The definition of the pullback says loosely that for each $c \in C$ the pullback f^*M contains a "copy" of $p^{-1}(f(c))$. With this intuition it becomes perhaps reasonable that the pullback f^*M is a "doubly twisted band". This "double twisted band" is of course homeomorphic to $S^1 \times [-1, 1]$.

(2) Let $p\colon X \to B$ be a continuous map of topological spaces and let $C \subseteq B$ be a subspace. We denote by $\iota\colon C \to B$ the inclusion.



It follows easily from the Topological-Product Proposition 7.2 (1) and the Subspace Continuity Lemma 4.3 (2) and (4) that the maps

$$\begin{array}{rcl} p^{-1}(C) & \to & \iota^*X \\ x & \mapsto & (p(x),x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \iota^*X & \to & p^{-1}(C) \\ (c,x) & \mapsto & x \end{array}$$

are continuous and inverses of one another. Thus we see that ι^*X is naturally homeomorphic to $p^{-1}(C)$. □

For completeness we also study our favorite three properties of topological spaces for pullbacks. We will not really make use of this result.

Proposition 7.23. (Pullback-Properties Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map.

(1) If X and C are compact and if X, B, C are Hausdorff, then the pullback f^*X is also compact.
(2) In Exercise 7.28 we will see that all naive guesses for a connectedness statement are wrong.
(3) If X and C Hausdorff, then the pullback f^*X is also Hausdorff.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \mu\colon C \times X & \to & B \times B \\ (c,x) & \mapsto & (f(c),p(x)). \end{array}$$

It follows from the Topological-Product Proposition 7.2 (1) that μ is continuous. We denote by $\Delta \coloneqq \{(b,b) \mid b \in B\} \subseteq B \times B$ the diagonal. Since B is Hausdorff it follows from the Hausdorff-Diagonal Lemma 7.20 that $\Delta \subseteq B \times B$ is a closed subset. By definition of the pullback we have $f^*X = \mu^{-1}(\Delta) \subseteq C \times X$. Since X and C are compact and Hausdorff we know by the Product Topology-Properties Proposition 7.9 (1) and (3) that $C \times X$ is compact and Hausdorff. It follows from this discussion and the Compact-Closed Lemma 3.39 (2) that f^*X is indeed compact.

(3) This statement follows immediately from the Product Topology-Properties Proposition 7.9 (3) and the Subspace-Hausdorff Lemma 3.31. ■

7.6. **The disjoint union of topological spaces.** In this section we will discuss the coproduct in the category of topological spaces. Before we get to topological spaces it is worth recalling the following definitions from Notation 1.12.

Definition 7.24. Let $\{A_i\}_{i\in I}$ be a family of sets.

(1) We define the disjoint union as the set

$$\bigsqcup_{i\in I} A_i \coloneqq \bigcup_{i\in I} (A_i \times \{i\}).$$

As usual we take some liberties with the notation if we have finitely many sets.

(2) Given $j \in I$ we refer to the map $A_j \to \bigsqcup_{i\in I} A_i$ given by $a \mapsto (a,j)$ as the natural injection.

In the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 we saw that the disjoint union, together with the obvious natural injections, gives us the coproduct in the category of sets.

In a second we will see that a very similar statement holds in the context of the category of topological spaces. But before we get to this point, let us introduce the following convention.

Convention 7.25. Let $\{A_i\}_{i\in I}$ be a family of sets. For any $i \in I$ we use the natural injection $A_i \to \bigsqcup_{i\in I} A_i$ to think of A_i as a subset of the disjoint union.

The following proposition is the analog of the Topological-Product Proposition 7.2.

Proposition 7.26. (Topological-Coproduct Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We consider the disjoint union $\bigsqcup_{i\in I} X_i$ from Notation 1.12.

(0) (a) We say that $U \subseteq \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $X_i \cap U$ is an open subset of X_i. This defines a topology on $\bigsqcup_{i\in I} X_i$. We will work with this topology throughout the proposition.

(b) A subset $A \subseteq \bigsqcup_{i\in I} X_i$ is closed if and only if for each $i \in I$ the intersection $X_i \cap A$ is a closed subset of X_i.

(1) (a) Each natural injection $\iota_j \colon X_j \to \bigsqcup_{i\in I} X_i$ is continuous. In fact it is an embedding that is open and closed.[115]

(b) If Z is a topological space and if $\{f_i \colon X_i \to Z\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{array}{rcl} \bigsqcup_{i\in I} X_i & \to & Z \\ (x,i) & \mapsto & f_i(x) \end{array}$$

is continuous.

(∗) A map $f \colon \bigsqcup_{i\in I} X_i \to Z$ to a topological space Z is continuous if and only if for each $i \in I$ the map $f \circ \iota_i \colon X_i \to Z$ is continuous.

(2) The topological space $\bigsqcup_{i\in I} X_i$ together with the natural injections ι_i is a coproduct of $\{X_i\}_{i\in I}$ in the category *Top* of topological spaces.

Proof.

(0) (a) This statement follows easily from the definitions.

(b) This statement follows from the elementary Topology-Defined-by-Family Lemma 4.8.

(1) (a) It follows immediately from writing down the definitions that each natural injection $X_j \to \bigsqcup_{i\in I} X_i$ is continuous, and that it is open and closed. It follows from the Open-Closed-Injective Lemma 4.22 that the natural injection is an embedding that is open and closed.

(b) Let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. We consider the corresponding map $F \colon \bigsqcup_{i\in I} X_i \to Z$. Let $U \subseteq Z$ be an open subset. For each $i \in I$ we have $F^{-1}(U) \cap X_i = f_i^{-1}(U)$. This shows that $F^{-1}(U)$ is open, i.e. F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. By the definition of the coproduct in Definition 6.5 we have to show that there exists

[115] In most cases this statement allows us to safely view each X_j as a closed and open subspace of the disjoint union $\bigsqcup_{i\in I} X_i$.

a *unique* continuous map $\Phi\colon \bigsqcup_{i\in I} X_i \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. In (1b) we defined such a map Φ and we saw that it is continuous. It remains to prove uniqueness. The proof of uniqueness is basically identical to the argument in the proof of the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 (1) where we dealt with the coproduct in the category of sets. ■

Definition 7.27. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on $\bigsqcup_{i\in I} X_i$ that we introduced in the Topological-Coproduct Proposition 7.26 as the disjoint union topology. We refer to this topological space as the disjoint union of the topological spaces X_i.[116]

The following lemma plays the role of the Product Topology-Properties Proposition 7.9.

Lemma 7.28. (Disjoint Union Topology-Properties Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) The following two statements are equivalent:
  (a) The disjoint union $\bigsqcup_{i\in I} X_i$ is compact.
  (b) Each X_i is compact and there are only finitely many $i \in I$ such that $X_i \neq \varnothing$.

(2) (a) Each X_i is an open and closed subset of $\bigsqcup_{i\in I} X_i$.
  (b) The (path-) components of the disjoint union $\bigsqcup_{i\in I} X_i$ are precisely the (path-) components of the X_i.

(3) The disjoint union $\bigsqcup_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. We prove the statements of the lemma in a different order than they were stated.

(2) (a) Let $i \in I$. Note that for $j \neq i$ we have $X_i \cap X_j = \varnothing$, which is open and closed. Thus it follows from the Topological-Coproduct Proposition 7.26 (0a) and that X_i is an open and closed subset.
  (b) This statement follows from (2a), the Component Lemma 4.48 (4a) and (5) and the Path-Component Lemma 4.44 (4).

(1) We first prove the "(b)⇒(a)"-direction. Thus we assume that each X_i is compact and that there are only finitely many $i_1, \ldots, i_k \in I$ such that the corresponding topological space is non-empty. Let $\{U_j\}_{j\in J}$ be an open cover of $\bigsqcup_{i\in I} X_i$. Since each X_i is compact we can find for each $i \in I$ a finite subset $J_i \subseteq J$ such that $\bigsqcup_{j\in J_i} (U_j \cap X_i) = X_i$. It is now clear that $X = \bigcup_{r\in\{1,\ldots,k\}} X_i = \bigcup_{r\in\{1,\ldots,k\}} \bigsqcup_{j\in J_{i_r}} U_j$.

We turn to the proof of the "(a)⇒(b)"-direction. Thus we assume that $\bigsqcup_{i\in I} X_i$ is compact. It follows from (2a) together with the Compact-Closed Lemma 3.39 (1) that each X_i is compact. Next note that by (2a) we also know that the $\{X_i\}_{i\in I}$ form an open cover of the compact topological space $\bigsqcup_{i\in I} X_i$. Thus we see that there exists a finite subset $J \subseteq I$ with $\bigsqcup_{j\in J} X_j = \bigsqcup_{i\in I} X_i$. In particular $X_i = \varnothing$ for $i \notin J$.

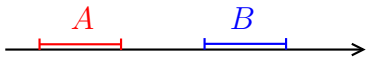
(3) This statement follows easily from the definitions and (2a). ■

[116] It would perhaps make more sense to refer to $\bigsqcup_{i\in I} X_i$ as the coproduct of the X_i. But we prefer to stick to the more classical name.

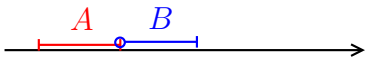
Example. Let X be a topological space and let $A, B \subseteq X$ be subsets of X. It follows from the Topological-Coproduct Proposition 7.26 (1b) that the obvious map

$$\underbrace{A \sqcup B}_{\substack{\text{disjoint union}\\ \text{topology}}} \quad \to \quad \underbrace{A \cup B}_{\text{subspace topology}}$$

is continuous. If A and B are disjoint, then this map is also a bijection. To be more specific, let $X = \mathbb{R}$. For $A = [0,1]$ and $B = [2,3]$ one can show by hand that the map is open, thus a homeomorphism.



$A \sqcup B$ is homeomorphic to $A \cup B \subseteq \mathbb{R}$



$A \sqcup B$ is *not* homeomorphic to $A \cup B \subseteq \mathbb{R}$

On the other hand for $A = [0,1]$ and $(1,2]$ the above map is certainly not a homeomorphism, for example $A \sqcup B$ is not compact whereas $A \cup B = [0,2]$ is compact. □

The following lemma now gives a criterion for showing that a union of subspaces of a given topological space is indeed homeomorphic to the disjoint union of the subspaces.

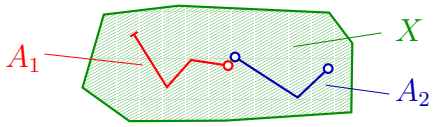
Lemma 7.29. (Disjoint Union Embedding Lemma) Let X be a topological space and let $\{A_i\}_{i\in I}$ be a family of disjoint subsets of X. The following two statements are equivalent:

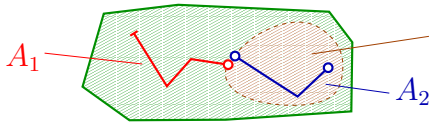
(1) For each $i \in I$ there exists an open neighborhood of $A_i \subseteq X$ that does not intersect any of the A_j with $j \neq i$.[117]

(2) The map

$$\begin{array}{ccc} \underbrace{\bigsqcup_{i\in I} A_i}_{\substack{\text{disjoint union}\\ \text{topology}}} & \to & \underbrace{\bigcup_{i\in I} A_i}_{\text{subspace of } X} \\ (a,i) & \mapsto & a \end{array}$$

is a homeomorphism.





Proof. First we point out that the given map

$$\begin{array}{rccc} \Phi\colon & \bigsqcup_{i\in I} A_i & \to & \bigcup_{i\in I} A_i \\ & (a,i) & \mapsto & a \end{array}$$

is a bijection since the A_i are disjoint subsets of X. By the Subspace Continuity Lemma 4.3 (1) we know that the inclusion maps $A_i \to \bigcup_{i\in I} A_i$ are continuous. It follows from the Topological-Coproduct Proposition 7.26 (1b) that the above map Φ is continuous.

We first prove (1)⇒(2). Thus we assume that for each $i \in I$ there exists an open neighborhood V_i of $A_i \subseteq X$ that does not intersect any of the A_j with $j \neq i$. We need to show that Φ is a homeomorphism. By the above discussion and the Open-Closed–Injective Lemma 4.22 it remains to show that the map Φ is open. Thus let U be an open subset of

[117] Of course this hypothesis is satisfied if the A_i are disjoint open subsets of X.

$\bigsqcup_{i\in I} A_i$. By definition of the disjoint union topology each $A_i \cap U$ is an open subset of A_i. By the definition of the subspace topology on A_i this means that there exists an open subset W_i of X with $A_i \cap U = A_i \cap W_i$. We now see that

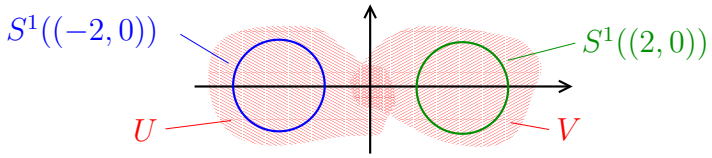
$$\begin{array}{rcccc} \Phi(U) & = & \Phi\Big(\bigsqcup_{i\in I}(A_i\cap U)\Big) & \overset{\text{definition of }\Phi}{=} & \bigcup_{i\in I}(A_i\cap U) \overset{\text{since } A_i\subseteq V_i \text{ and since } A_j\cap V_i=\varnothing \text{ for } i\neq j}{=} \bigcup_{i\in I}((A_i\cap U)\cap V_i) \\ & = & \bigcup_{i\in I}(A_i\cap(V_i\cap W_i)) & = & \Big(\bigcup_{i\in I}A_i\Big)\cap\Big(\bigcup_{j\in J}(V_j\cap W_j)\Big). \\ & \overset{\text{choice of } W_i}{=} & & \overset{\text{since } A_i\subseteq V_i\cap W_i \text{ and since } A_i\cap(V_j\cap W_j)=\varnothing \text{ for } i\neq j}{=} & \end{array}$$

Since $\bigcup_{j\in J}(V_j\cap W_j)$ is an open subset of X we obtain, by definition of the subspace topology, that $\Phi(U)$ is indeed an open subset of $\bigcup_{i\in I} A_i$.

Finally we prove (2)⇒(1). Thus we suppose that Φ is a homeomorphism. Let $i \in I$. By the Disjoint Union Topology-Properties Lemma 7.28 (2) we know that A_i is an open subset of $\bigsqcup_{j\in I} A_j$. Thus $\Phi(A_i) = A_i$ is an open subset of $\bigcup_{j\in I} A_j$. By definition of the subspace topology this means that there exists an open subset V_i such that $V_i \cap \bigcup_{j\in J} A_j = A_i$. But this means that V_i is an open neighborhood of $A_i \subseteq X$ that does not intersect any other A_j. ■

Examples.

(1) Using the Disjoint Union Embedding Lemma 7.29 one can show that the topological space $S^1 \sqcup S^1$ is homeomorphic to the topological space $S^1((-2,0)) \cup S^1((2,0)) \subseteq \mathbb{R}^2$.



(2) Using the Disjoint Union Embedding Lemma 7.29 one can also easily show that the topological space $[0,1] \sqcup [2,3]$ is homeomorphic to $[0,1] \cup [2,3] \subseteq \mathbb{R}$.

(3) Let X be a topological space and let $\{X_i\}_{i\in I}$ be its set of components. Note that the Component Lemma 4.48 (1) says that as a *set* we have $X = \bigsqcup_{i\in I} X_i$, thus the natural map $\bigsqcup_{i\in I} X_i \to X$ is a bijection. Furthermore we know by the Subspace Continuity Lemma 4.3 (1) and the Topological-Coproduct Proposition 7.26 that the map is continuous.

(a) If I is finite, then we know by the Components-are-Closed Lemma 4.49 that each X_i is closed and open, thus it follows from the Disjoint Union Embedding Lemma 7.29 that the natural map $\bigsqcup_{i\in I} X_i \to X$ is in fact a homeomorphism.

(b) If I is infinite, this logic does not necessarily apply. An easy counterexample is given by $X = \mathbb{Q}$, where, as we saw on page 199, the components are precisely the one-point subsets.
On the other hand, if X is locally connected, then it follows from the Locally Path-Connected Lemma 4.57 that each X_i closed and open, and thus it follows from the

Disjoint Union Embedding Lemma 7.29 that the natural map $\bigsqcup_{i\in I} X_i \to X$ is in fact a homeomorphism. □

7.7. **Appendix: Topological groups.** Now that we know the notion of the product topology we can introduce the notion of a topological group. This notion does not play an essential role in the lecture notes, but it will pop up every now and then.

Definition 7.30.

(1) A topological group is a topological space X together with a group structure such that the two maps[118]
$$\begin{array}{ccc} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$
are continuous.

(2) A morphism between topological groups X and Y is a group homomorphism $f\colon X \to Y$ that is furthermore continuous.

(3) We denote by $\mathcal{TopGr}$ the category of topological groups.

The most basic example of a topological group is given by taking any group and equipping it with the discrete topology. As we will see in the Topological Subgroup Lemma 7.33 (2), somewhat surprisingly even this banal example can turn out to be interesting.
The following lemma gives us much more interesting examples of topological groups.

Lemma 7.31. (Matrices–Topological Group Lemma) For any $n \in \mathbb{N}$ the matrix groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are topological groups.

Proof. As we pointed out on page 279, matrix multiplication is continuous. Furthermore by the Adjugate Matrix Lemma 2.21 we know that taking the inverse is given by the map
$$\begin{array}{ccc} \mathrm{GL}(n,\mathbb{R}) & \to & \mathrm{GL}(n,\mathbb{R}) \\ A & \mapsto & \frac{1}{\det(A)} \cdot \underbrace{\begin{array}{c}\text{matrix whose } (i,j)\text{-entry is given by taking the determinant} \\ \text{of the matrix that is obtained from } A \text{ by removing} \\ \text{the } i\text{-th column and the } j\text{-th row of } A.\end{array}}_{=\text{called the adjugate of } A} \end{array}$$
But this map is clearly continuous. This shows that $\mathrm{GL}(n,\mathbb{R})$ is a topological group. Basically the same argument also shows that $\mathrm{GL}(n,\mathbb{C})$ is a topological group. ■

After this first list of examples let us formalize the setting a little bit by introducing the category of topological groups.

Definition 7.32. We refer to the category $\mathcal{TopGr}$ with
$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{TopGr}) & := & \text{all topological groups,} \\ \mathrm{Mor}(X,Y) & := & \text{all homomorphisms from } X \text{ to } Y \text{ that are continuous,} \end{array}$$
with the usual composition of maps as the category of topological groups.

We obtain more examples of topological groups by taking subgroups and products:

Lemma 7.33. (Topological Subgroup Lemma)

[118] Here we evidently equip $X \times X$ with the product topology.

(1) Let X be a topological group and let Y be a subgroup. If we equip Y with the subspace topology, then Y is also a topological group.
(2) Let $\{X_i\}_{i\in I}$ be a family of topological groups. If we equip $\prod_{i\in I} X_i$ with the product topology and the obvious group structure, then it is again a topological group.

Proof.

(1) This statement follows immediately from the Subspace Continuity Lemma 4.3 together with the Product Topology-Basics Lemma 7.6 (1a).
(2) This statement follows easily from the Topological-Product Proposition 7.2 and the Product Topology-Basics Lemma 7.6 (2). ■

Example. It follows from the Matrices–Topological Group Lemma 7.31 and the Topological Subgroup Lemma 7.33 (1) that $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups. □

Lemma 7.34. (Components-of-Topological Group Lemma) Let G be a topological group.

(1) Given any $g \in G$ the two maps
$$\begin{array}{rcl} l_g\colon G & \to & G \\ h & \mapsto & g\cdot h \end{array} \quad \text{and} \quad \begin{array}{rcl} r_g\colon G & \to & G \\ h & \mapsto & h\cdot g \end{array}$$
are continuous.
(2) If G is a topological group, then any two path-components of G are homeomorphic.

Proof.

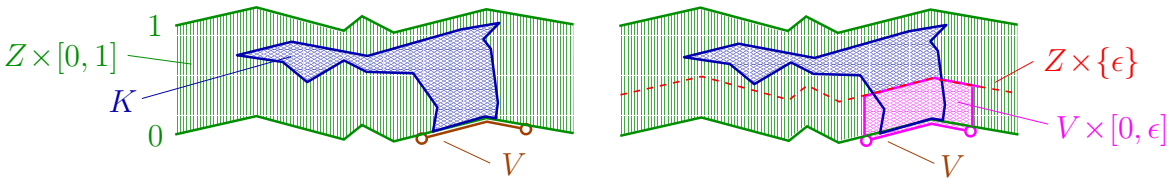
(1) The map l_g is the composition of the two maps
$$\begin{array}{rcl} G & \to & G\times G \\ h & \mapsto & (g,h) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\times G & \to & G \\ (x,y) & \mapsto & x\cdot y. \end{array}$$
The first map is continuous by the Topological-Product Proposition 7.2 (1∗) and the second map is continuous since G is a topological group. Thus l_g itself is continuous. The same argument shows that r_g is continuous.
(2) Let X, Y be two path-components of G. We pick $x \in X$ and $y \in Y$. We consider the map $\varphi\colon G \to G$ given by $g \mapsto yx^{-1}g$. By (1) we know that φ is continuous. Furthermore it follows from the Image-Preimage-Component Lemma 4.50 that $\varphi(X) \subseteq Y$.

We also consider the map $\psi\colon G \to G$ given by $g \mapsto xy^{-1}g$. The same argument as above shows that ψ is continuous and that $\psi(Y) \subseteq X$. Clearly φ and ψ are inverses of one another. But this implies that φ and ψ are homeomorphisms between X and Y. ■

7.8. Exercises for Chapter 7.

Exercise 7.1. Show that there are non-homeomorphic topological spaces A and B and a non-empty topological space X such that $A \times X$ is homeomorphic to $B \times X$.
Hint. you could take $A = \{0\}$ and $B = \{0, 1\}$.

Exercise 7.2. Let Z be a topological space and let $K \subseteq Z \times [0,1]$ be a compact subset. Furthermore let $V \subseteq Z$ be an open subset such that $(K \cap (Z \times \{0\}) \subseteq V \times \{0\}$. Show that there exists an $\epsilon > 0$ such that $K \cap (Z \times [0,\epsilon]) \subseteq K \cap (V \times [0,\epsilon])$.



Exercise 7.3. Let $X_1, \ldots, X_k$ be topological spaces and let $B_i \subseteq X_i$, $i = 1, \ldots, k$ be subsets. Show that

$$\text{interior of } B_1 \times \ldots \times B_k \text{ in } X_1 \times \ldots \times X_k \;=\; \mathring{B}_1 \times \ldots \times \mathring{B}_k.$$

Remark. This exercise provides the proof of the Product-of-Subsets Lemma 7.8 (2a).

Exercise 7.4. Let X and Y be topological spaces and let $W \subseteq X \times Y$ be an open subset. Let $y \in Y$. Show that $U_y := \{x \in X \,|\, (x, y) \in W\}$ is an open subset of X.

Exercise 7.5. Let X and Y be topological spaces and let $U \subseteq X$ and $V \subseteq Y$ be subspaces.

(a) Show that if $U \times V \subseteq X \times Y$ is open, then $U \subseteq X$ and $V \subseteq Y$ are open.
(b) Show that if $U \times V \subseteq X \times Y$ is closed then $U \subseteq X$ and $V \subseteq Y$ are closed.

Exercise 7.6. Let X and Y be topological spaces and let $K \subseteq X \times Y$ be a compact subset. Let $y \in Y$. We set $K_y := \{x \in X \,|\, (x, y) \in K\}$.

(a) Show that if Y is Hausdorff, then K_y is a compact subspace of X.
(b) Show that in general it is not true that K_y is a compact subspace of X.

Exercise 7.7. Let $k \in \mathbb{N}$ and let X be a topological space. We refer to

$$\mathrm{Conf}_k(X) \;:=\; \{(x_1, \ldots, x_k) \in X^k \,|\, \text{ for all } i \neq j \text{ we have } x_i \neq x_j\}$$

as the *configuration space of k points in X*.

(a) Determine the homeomorphism type of $\mathrm{Conf}_2(\mathbb{R})$.
(b) Determine the homeomorphism types of $\mathrm{Conf}_2(S^1)$ and $\mathrm{Conf}_3(S^1)$.

Remark. By "determine the homeomorphism type" we mean: show that these topological spaces are homeomorphic to more familiar topological spaces.

Exercise 7.8. Let X and Y be topological spaces.

(a) Show that if X is compact, then the natural projection $p\colon X \times Y \to Y$ is a closed map.
Remark. This exercise is related to the Tube Lemma 7.12.
(b) Give an example that shows that in general in (a) we cannot drop the hypothesis that X is compact.

Somewhat surprisingly, the converse to (a) holds. More precisely, if a topological space X has the property, that for any other topological space Y the projection $p\colon X \times Y \to Y$ is a closed map, then X is actually compact. This non-trivial statement is sometimes known as the *Kuratowski-Mrówka Theorem*, it was first proved by Stanislaw Mrówka [**Mró1959**, p. 21] in 1959. See also [**Herl1996**, p. 3] for a proof and see [**Enge1989**, Theorem 3.1.16] for a proof under the extra hypothesis that X is Hausdorff.

Exercise 7.9. Let X be a topological space. Show that the following two statements are equivalent:

(i) X is Hausdorff.

(ii) The diagonal $\Delta = \{(x,x) \,|\, x \in X\}$ is a closed subset of $X \times X$.

Exercise 7.10. Let $m, n \in \mathbb{N}_0$.
(a) Show that $\overline{B}^m \times \overline{B}^n$ is homeomorphic to $\overline{B}^{m+n}$.
(b) Show that $B^m \times B^n$ is homeomorphic to B^{m+n}.

Exercise 7.11. Let $k \in \mathbb{N}$ and let X be a topological space. We refer to

$$\mathrm{Conf}_k(X) \;:=\; \{(x_1, \dots, x_k) \in X^k \,|\, \text{ for all } i \neq j \text{ we have } x_i \neq x_j\}$$

as the *configuration space of k points in X*. We equip X^k with the product topology and $\mathrm{Conf}_k(X)$ with the subspace topology.
(a) We consider $X = [0,1]$. For which $k \in \mathbb{N}$ is $\mathrm{Conf}_k(\mathbb{R})$ path-connected?
(b) We consider the topological space Y which looks like a Y. For which $k \in \mathbb{N}$ is $\mathrm{Conf}_k(\mathbb{R})$ path-connected?
(c) We consider the topological space $X = \mathbb{R}^2$. For which $k \in \mathbb{N}$ is $\mathrm{Conf}_k(\mathbb{R}^2)$ path-connected?

Exercise 7.12. Let $f\colon X \to Y$ be a continuous map between topological spaces. We consider the set

$$S \;:=\; \{x \in X \,|\, \text{there exists } x' \in X \setminus \{x\} \text{ with } f(x) = f(x')\}.$$

(a) Show that if f is locally injective, if X is compact and Hausdorff and if Y is Hausdorff, then S is a closed subset of X.
(b) Show that in (a) we cannot drop the hypothesis that f is locally injective.
(c) Show that in (a) we cannot drop the hypothesis that X is compact.

Exercise 7.13. Let X, Y, Z be topological spaces and let $f\colon X \to Y$ and $g\colon X \to Z$ be continuous maps. Show that if f is proper and if Y and Z are Hausdorff, then the map

$$\begin{array}{rccc} (f,g)\colon & X & \to & Y \times Z \\ & x & \mapsto & (f(x), g(x)) \end{array}$$

is also proper.

Exercise 7.14. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced in Exercise 4.4. We refer to the product $S \times S$ as the *Sorgenfrey plane*.
(a) Show that the diagonal $\Delta := \{(x,x) \,|\, x \in \mathbb{R}\}$ is a closed subset of the Sorgenfrey plane.
(b) Show that the subspace topology on the anti-diagonal $A := \{(x,-x) \,|\, x \in \mathbb{R}\}$ is the discrete topology.

Exercise 7.15. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on the set $\prod_{i \in I} X_i$ is the topology that is generated by the set

$$\Big\{ \prod_{i \in I} U_i \,\Big|\, U_i \subseteq X_i \text{ open} \Big\}.$$

(a) Show that the product of compact topological spaces, equipped with the box topology, is not necessarily compact.
(b) Let X be a topological space. We equip $X^{\mathbb{N}} = \prod_{i \in \mathbb{N}} X$ with the box topology. Is the diagonal map

$$\begin{array}{ccc} X & \to & X^{\mathbb{N}} \\ x & \mapsto & (x, x, \dots) \end{array}$$

continuous?

Exercise 7.16. Let X and Y be topological spaces and let $M \subseteq X \times Y$ be a compact subset. Show that there exist compact subsets $K \subseteq X$ and $L \subseteq Y$ such that $M \subseteq K \times L$.

Exercise 7.17. Let X be a topological space that is compact and Hausdorff. Furthermore let $f\colon X \to X$ be a continuous map. We denote by $F := \{x \in X \mid f(x) = x\}$ the fixed point set of f. Show that F is a discrete subset of X if and only if F is finite.

Exercise 7.18. Let $C \subseteq \mathbb{R}$ be the Cantor set from Definition 3.45. Is the disjoint union $C \sqcup C$ homeomorphic to C itself?

Exercise 7.19.

(a) Let $\mathcal{C}$ be a category $\mathcal{C}$ that admits products and coproducts. Let X, Y, A be objects of $\mathcal{C}$. Show that there exists a natural map $(X \times A) \amalg (Y \times A) \to (X \amalg Y) \times A$.

(b) Show that the product topology is "distributive" with respect to the disjoint union topology. More precisely, let X and Y be two topological spaces and let A be another topological space. Show that the natural map $(X \times A) \sqcup (Y \times A) \to (X \sqcup Y) \times A$ from (a) is a homeomorphism.

(c) Show that the statement of (b) is not a consequence of the formal properties of products and coproducts. More precisely give an example of a category $\mathcal{C}$ that admits products and coproducts and objects X, Y, A of $\mathcal{C}$ such that the natural map

$$(X \times A) \amalg (Y \times A) \;\to\; (X \amalg Y) \times A$$

is not an isomorphism.

Exercise 7.20. Let X be a topological space.

(a) Let $\{Y_i\}_{i \in I}$ be a family of topological spaces. We consider the natural map

$$\begin{array}{rcl} \coprod_{i \in I} (X \times Y_i) & \to & X \times \coprod_{i \in I} Y_i \\ ((x,y),i) & \mapsto & (x,(y,i)). \end{array}$$

(i) Show that it follows from formal properties of the product and disjoint union topology that this natural map is continuous.

(ii) Show that the map is a homeomorphism.

(b) Let I be a set. We equip I with the discrete topology. Show that the natural map

$$\begin{array}{rcl} \overbrace{\coprod_{i \in I} X}^{\substack{\text{disjoint union} \\ \text{topology}}} & \to & \overbrace{X \times I}^{\substack{\text{product} \\ \text{topology}}} \\ (x,i) & \mapsto & (x,i) \end{array}$$

is a homeomorphism.

Exercise 7.21. Show "by hand" that the map

$$\begin{array}{rcl} \varphi\colon \prod_{i \in \mathbb{N}} \{0,2\} & \to & C = \text{Cantor set from Definition 3.45} \\ (f\colon \mathbb{N} \to \{0,2\}) & \mapsto & \sum_{i=1}^{\infty} \frac{f(i)}{3^i}, \end{array}$$

which we defined on page 289, is continuous.

Exercise 7.22. Let $\{X_i\}_{i\in\varnothing}$ be the empty family of topological spaces.

(a) Describe the product $\prod_{i\in\varnothing} X_i$.

(b) Describe the disjoint union $\bigsqcup_{i\in\varnothing} X_i$.

Exercise 7.23.

(a) Show that the Cantor set is totally disconnected.

(b) Show that $\prod_{i\in\mathbb{N}} \{0,2\}$ is totally disconnected.

Exercise 7.24. Let X be a topological space. Let $A \subseteq X$ be a subset that admits a retraction $r\colon X \to A$.

(a) We denote by $f\colon X \to X \times X$ the map that is given by $x \mapsto (x, r(x))$. Show that f is continuous and that $f^{-1}(\Delta) = A$.

(b) Show that if X is Hausdorff, then A is closed.

Remark. This gives a new solution to Exercise 4.31.

Exercise 7.25. Let C be the Cantor from Definition 3.45. We recall two facts:

(1) On page 289 we showed that C is homeomorphic to the product $\{0,1\}^{\mathbb{N}}$.

(2) In Exercise 4.20 we showed that there exists a surjective continuous map $\varphi\colon C \to [0,1]$.

Use the above to prove the following statements:

(a) The Cantor set C is homeomorphic to $C \times C$.

(b) Any continuous map $C \to \mathbb{R}^n$ extends to a continuous map $[0,1] \to \mathbb{R}^n$.
Remark. This statement follows from the Tietze Extension Theorem 12.6. But it also allows for an elementary proof.

(c) Use (a) and (b) to show that there exists a surjective continuous map $[0,1] \to [0,1] \times [0,1]$.
Remark. Such a map is called a *space-filling path*. We will provide alternative constructions of space-filling paths in the Space-Filling Curve Theorem 10.21 and Exercise 10.16.

Exercise 7.26. Given $v, w \in \mathbb{R}^2 \setminus \{(0,0)\}$ we define

$$\sphericalangle(v,w) := \text{unique } \varphi \in (-\pi,\pi] \text{ such that } w = r\cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \cdot v \text{ for some } r \in \mathbb{R}_{>0}.$$

(a) Show that the map
$$\begin{array}{rccc} \Phi\colon & (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & (-\pi,\pi] \\ & (v,w) & \mapsto & \sphericalangle(v,w) \end{array}$$
is not continuous.

(b) Show that the map
$$\begin{array}{rccc} \Phi\colon & (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & \mathbb{R}/2\pi\cdot\mathbb{Z} \\ & (v,w) & \mapsto & [\sphericalangle(v,w)] \end{array}$$
is continuous. (Here $\mathbb{R}/2\pi\cdot\mathbb{Z}$ denotes the quotient of the topological space $\mathbb{R}$ by the group $(2\pi\mathbb{Z},+)$.)

Exercise 7.27. A *pointed topological space* is a pair (X,x), where X is a topological space and x is a point in X. We refer to the category $\mathcal{PT}op$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PT}op) & := & \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x),(Y,y)) & := & \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x) = y \end{array}$$

with the usual composition of maps as the *category of pointed topological spaces*. Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces.

(a) Give an explicit construction of a product of $(X_i, x_i)_{i\in I}$ in the category $\mathcal{PTop}$.
(b) Give an explicit construction of a coproduct of $(X_i, x_i)_{i\in I}$ in the category $\mathcal{PTop}$.

Exercise 7.28. Let $p\colon X \to B$ be a continuous map of topological spaces let C be a topological space and let $f\colon C \to B$ be a continuous map.

(a) Show that even if X, B and C are connected, it does not necessarily follow that the pullback f^*X is connected.
(b) Suppose that X, B and C are connected. We also assume that for each $b \in B$ the preimage $p^{-1}(b) \subseteq X$ is connected. Does it follow that the pullback f^*X is connected?

Exercise 7.29. Let X be a topological group and let U be an open neighborhood of $e \in X$. Show that there exists an open neighborhood V of e with the following properties:

(a) V is symmetric, i.e. for every $x \in V$ we also have $x^{-1} \in V$.
(b) We have $V \cdot V \coloneqq \{x \cdot y \,|\, x, y \in V\} \subseteq U$.

Exercise 7.30. Let G be a topological group. We denote by X the set of subgroups of G that are closed subsets. We consider the following types of subsets of G:

$$\begin{aligned} \text{Miss} &\coloneqq \{H \in X \,|\, \text{there exists a compact subset } K \subseteq G \text{ with } H \cap K = \varnothing\}, \\ \text{Hit} &\coloneqq \{H \in X \,|\, \text{there exists an open subset } U \subseteq G \text{ with } H \cap U \neq \varnothing\}. \end{aligned}$$

The topology on X that is generated by Miss and Hit is called the *Chabauty topology on* X, see [**Chab1950**].

(a) Show that the Chabauty topology is Hausdorff.
(b) Let $G = \mathbb{R}$.
  (i) Determine the closed subgroups of $G = \mathbb{R}$.
  (ii) Show that in this case X, equipped with the Chabauty topology, is homeomorphic to $[0, 1]$.

CHAPTER 8

Quotients of topological spaces

In Chapter 7 we introduced the product and disjoint union (vulgo coproduct) of topological spaces. In this chapter we introduce the very flexible notion of a quotient of a topological space by an equivalence relation. This will allow us to introduce some of our favorite topological spaces, namely surfaces and projective spaces, and it will allow us to study the pushout in the category of topological spaces.

This chapter is organized as follows:

- In Section 8.1 we will introduce, given an equivalence relation "$\sim$" on a topological space X, a suitable topology on the set $X/\sim$ of all equivalence classes.
- In Section 8.2 we will use the quotient topology to introduce the Möbius band and the Klein bottle.
- In Section 8.3 we will introduce, given a topological space X and given a subset $A \subseteq X$, the quotient space X/A.
- In Section 8.4 we will use the disjoint union topology and the quotient topology to show that the category of topological spaces has pushouts.
- In Appendix 8.5 we will discuss the coequalizer in the topological category.
- In Section 8.6 we will provide exercises for Chapter 8.

In the next Chapter 9 we will discuss a special case of the quotient topology, namely the one induced by a group action, in greater detail.

8.1. **The quotient topology.** In Definition 1.13 we introduced the notion of an equivalence relation. We recall Notation 1.20.

Notation 8.1. Let $\sim$ be an equivalence relation on a set X.

(1) Given $x \in X$ we denote by $[x]$ the unique equivalence class that contains x.

(2) We refer to the map
$$\begin{array}{rccc} p\colon & X & \to & X/\sim \\ & x & \mapsto & [x] \end{array}$$
as the natural projection.

(3) Let $f\colon X \to Y$ be a map between sets such that $f(x) = f(y)$ whenever $x \sim y$. By the Quotient-Factorization Lemma 1.21 the map
$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$ such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow^{f} & \downarrow g \\ & & Y. \end{array}$$

We refer to this map g as the induced map.

In this chapter we will introduce and study the quotient of topological space by an equivalence relation. We start out with the following proposition which can be viewed as the analog of the Topological-Product Proposition 7.2 and of the Topological-Coproduct Proposition 7.26.

Proposition 8.2. (Topological-Quotient Proposition) Let $\sim$ be an equivalence relation on a topological space X. We denote by $p\colon X \to X/\sim$ the natural projection from X onto the set of equivalence classes $X/\sim$. The following statements hold:

(0) (a) The set
$$\mathcal{T} \coloneqq \{U \subseteq X/\sim \mid p^{-1}(U) \text{ is open in } X\}$$
is a topology on $X/\sim$. We use this topology throughout this proposition.[119]

(b) A subset $A \subseteq X/\sim$ is closed if and only if the preimage $p^{-1}(A)$ is closed in X.

(1) (a) The natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. The induced map
$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$
is also continuous.

(∗) A map $f\colon X/\sim \to Y$ is continuous if and only if the map $f \circ p\colon X \to Y$ is continuous.

(2) If $\mathcal{S}$ is a topology on $X/\sim$ such that $(X/\sim, \mathcal{S})$ has properties (1a) and (1b), then $\mathcal{S}$ equals the topology from (0).

Remark. In Appendix 8.5 we will reinterpret the construction from the Topological-Quotient Proposition 8.2 as a coequalizer in the category $\mathcal{T}op$ of topological spaces. This puts the Topological-Quotient Proposition 8.2 into a more categorical framework. □

Proof. Let $\sim$ be an equivalence relation on a topological space X.

(0) (a) This statement follows easily from the Image-Preimage Lemma 1.9 (6) and (7).

(b) This statement follows easily from the Image-Preimage Lemma 1.9 (8).

(1) (a) It follows immediately from the definition of the topology on $X/\sim$ that the natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let $f\colon X \to Y$ be a continuous map such that $f(x) = f(y)$ whenever $x \sim y$. We need to show that the induced map $g\colon X/\sim \to Y$ is continuous as well. Let $U \subseteq Y$ be open. We need to show that $g^{-1}(U) \subseteq X/\sim$ is open. By Definition 8.3 of the quotient topology on $X/\sim$ we need to show that $p^{-1}(g^{-1}(U))$ is open in X. We

[119] In other words, $U \subseteq X/\sim$ is open if and only if $p^{-1}(U) \subseteq X$ is open.

observe that

$$p^{-1}(g^{-1}(U)) \quad = \quad (g\circ p)^{-1}(U) \quad \underset{\substack{\uparrow\\ \text{by the commutative}\\ \text{diagram in Notation 8.1 (3)}}}{=} \quad f^{-1}(U) \quad \underset{\substack{\uparrow\\ \text{since } f \text{ continuous}}}{=} \quad \text{open subset}\,.$$

We have thus shown that g is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let $\mathcal{S}$ be a topology on $X/\!\sim$ such that the following conditions are satisfied:

(a′) The natural projection $p\colon X \to (X/\!\sim, \mathcal{S})$ is continuous.

(b′) Whenever we are given a topological space Y and a continuous map $f\colon X \to Y$ with the property that $f(x) = f(y)$ whenever $x \sim y$, then the corresponding induced map $(X/\!\sim, \mathcal{S}) \to Y$ is also continuous.

We apply (b) and (a′) to obtain that the identity map $\Phi\colon (X/\!\sim, \mathcal{T}) \to (X/\!\sim, \mathcal{S})$ is continuous. This implies that $\mathcal{S} \subseteq \mathcal{T}$. Similarly we apply (a) and (b′) to obtain that the identity map $\Psi\colon (X/\!\sim, \mathcal{S}) \to (X/\!\sim, \mathcal{T})$ is continuous, which tells us that $\mathcal{T} \subseteq \mathcal{S}$. In summary we see that $\mathcal{S} = \mathcal{T}$. ■

The following definition will not come as a surprise.

Definition 8.3. Let $\sim$ be an equivalence relation on a topological space X.

(1) We refer to the topology on $X/\!\sim$ from the Topological-Quotient Proposition 8.2 (0a) as the quotient topology and we refer to $X/\!\sim$ as a quotient space of X.

(2) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$.

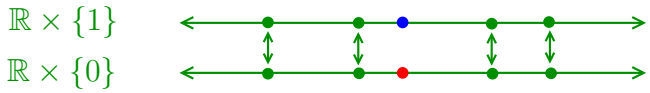
(a) We refer to the corresponding map $X/\!\sim \to Y$ as the induced map.

(b) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.

(c) Sometimes we also say that the map $f\colon X \to Y$ descends to a map $X/\!\sim \to Y$.

As before we care about compactness, connectedness and the Hausdorff property of topological spaces. We will see shortly that compactness and connectedness is preserved by going to quotients. The Hausdorff property is more tricky and we start out with the following cautionary example:

Example. We consider the topological space $X = (\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})$ and the equivalence relation that is generated by $(x,0) \sim (x,1)$ for all $x \neq 0$.



We denote by $p\colon X \to X/\!\sim$ the natural projection. In Exercise 9.6 we will see that $p((0,0))$ and $p((0,1))$ cannot be separated by open neighborhoods. Alternatively note that in Exercise 9.6 we will see that $X/\!\sim$ is homeomorphic to the line with two zeros, which is not Hausdorff. This shows that the quotient of a Hausdorff space is not necessarily Hausdorff. □

Definition 8.4. Let X be a topological space and let $\sim$ be an equivalence relation on X.

Recall that an equivalence relation is, by Definition 1.13, a subset of $X \times X$ with suitable properties. We say that the relation $\sim$ is closed if $\sim$ is a closed subset of $X \times X$.

The following lemma is the analog of the Product Topology-Properties Proposition 7.9 and the Disjoint Union Topology-Properties Lemma 7.28.

Proposition 8.5. (Quotient Topology-Properties Proposition) Let $\sim$ be an equivalence relation on a topological space X.

(1) If X is compact, then $X/\sim$ is also compact.
(2) If X is (path)-connected, then $X/\sim$ is also (path)-connected.
(3) If X is Hausdorff and compact and if $\sim$ is a closed equivalence relation on X, then $X/\sim$ is Hausdorff.

Remark. In Exercise 12.4 we will see that in the Quotient Topology-Properties Proposition 8.5 (3) we cannot drop the hypothesis that X is compact. □

Proof. By the Topological-Quotient Proposition 8.2 (1a) we know that the natural projection $X \to X/\sim$ is continuous. The first two statements of the lemma thus follow from the Compact Image Lemma 4.15 and the Image-Connected Lemma 4.34.

We turn to the proof of Statement (3). Let X be a topological space that is Hausdorff and compact and let $\sim$ be a closed equivalence relation on X. For psychological reasons we denote by $G \subseteq X \times X$ the subset given by the relation $\sim$. By hypothesis we know that G is a closed subset of $X \times X$. We denote by $p\colon X \to X/\sim$ the natural projection. We start out with the following claim:

Claim 1. Given any closed subset C of X the projection $p(C)$ is a closed subset of $X/\sim$.

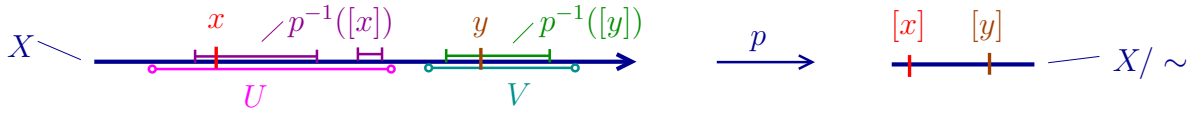
Proof. Let C be a closed subset of X. We need to show that $p(C)$ is a closed subset of $X/\sim$. By the Topological-Quotient Proposition 8.2 it suffices to show that $p^{-1}(p(C))$ is closed in X. We denote by $p_1, p_2\colon X \times X \to X$ the projection onto the first respectively the second factor. Then one easily sees that

$$p^{-1}(p(C)) \;=\; \{y \in X \,|\, \text{there exists an } x \in C \text{ with } y \sim x\} \;=\; p_2(p_1^{-1}(C) \cap G).$$

Note that $p_1^{-1}(C) = C \times X$ is the product of two closed subsets of X, hence it follows from the Product-of-Subsets Lemma 7.8 (1) that $p_1^{-1}(C) = C \times X$ is a closed subset of $X \times X$. Furthermore by our hypothesis G is a closed subset of $X \times X$. Thus we see that $p_1^{-1}(C) \cap G$ is closed. Furthermore, by the Product Topology-Properties Proposition 7.9 (1) and our hypothesis on X we know that $X \times X$ is compact. It follows from the above and the Compact-Closed Lemma 3.39 (1) that $p_1^{-1}(C) \cap G$ is a compact subset of $X \times X$. It is now a consequence of the Compact Image Lemma 4.15 (1) that $p_2(p_1^{-1}(C) \cap G)$ is a compact subset of X. But since X is Hausdorff we obtain from the Compact-Closed Lemma 3.39 (2) that $p_2(p_1^{-1}(C) \cap G)$ is a closed subset of X. ⊞

Now let $[x], [y] \in X/\sim$ be two distinct points in $X/\sim$. Since X is Hausdorff we know from the Compact-Closed Lemma 3.39 (2) that $\{x\}$ and $\{y\}$ are closed subsets of X. By Claim 1 we see that $p(\{x\}) = \{[x]\}$ and $p(\{y\}) = \{[y]\}$ are closed subsets of $X/\sim$. Since the projection $p\colon X \to X/\sim$ is continuous we see that $p^{-1}([x])$ and $p^{-1}([y])$ are two disjoint closed subsets of X. Since X is compact and Hausdorff we know by the Compact Hausdorff-Normal Lemma 3.49 that X is normal, i.e. that there exist disjoint open neighborhoods

U and V of the disjoint closed subsets $p^{-1}([x])$ and $p^{-1}([y])$ of X. Next we consider the complements $U^c := X \setminus U$ and $V^c := X \setminus V$.



It remains to prove the following claim:

Claim 2. The sets $(X/\sim) \setminus p(V^c)$ and $(X/\sim) \setminus p(U^c)$ are disjoint open neighborhoods of $[x]$ and $[y]$ in $X/\sim$.

Proof. Note that U^c and V^c are closed subsets of X. By Claim 1 the images of U^c and V^c are closed subsets of $X/\sim$. Thus we see that $(X/\sim) \setminus p(U^c)$ and $(X/\sim) \setminus p(V^c)$ are open subsets of $X/\sim$. Since $p^{-1}([x]) \cap V^c = \varnothing$ we see that $x \in (X/\sim) \setminus p(V^c)$. Similarly we have $y \in (X/\sim) \setminus p(U^c)$. It remains to show that these sets are disjoint. Indeed, we have

$$\begin{aligned}((X/\sim) \setminus p(U^c)) \cap ((X/\sim) \setminus p(V^c)) &= (X/\sim) \setminus (p(U^c) \cup p(V^c)) \overset{\text{by the Image-Preimage Lemma 1.9 (3)}}{=} (X/\sim) \setminus p(U^c \cup V^c) \\ &= (X/\sim) \setminus p((X \setminus U) \cup (X \setminus V)) \\ &= (X/\sim) \setminus p(\underbrace{X \setminus (U \cap V)}_{=X}) \quad \varnothing.\end{aligned}$$

■

A careful reading of these notes shows that the following two lemmas gets used on many occasions without giving them their due credit.

Lemma 8.6. (Quotient-to-Quotient Lemma) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $\sim_X$ and $\sim_Y$ be equivalence relations on X respectively Y. If $x_1 \sim_X x_2$ implies that $f(x_1) \sim_Y f(x_2)$, then the map

$$\begin{array}{ccc} X/\sim_X & \to & Y/\sim_Y \\ [x] & \mapsto & [f(x)] \end{array}$$

is well-defined and continuous.

Proof. First note that it follows from the Topological-Quotient Proposition 8.2 (1a) that the map $X \xrightarrow{f} Y \xrightarrow{y \mapsto [y]} Y/\sim_Y$ is continuous. Next note that it follows from our hypothesis on f and the Topological-Quotient Proposition 8.2 (1b) that this map descends to a continuous map $X/\sim_X \to Y/\sim_Y$. ■

Lemma 8.7. (Double-Quotient Lemma) Let X be a topological space and let $\sim_1$ be an equivalence relation on X. Furthermore let $\sim_2$ be an equivalence relation on $X/\sim_1$. Given $x, y \in X$ we set

$$x \sim y \quad :\Longleftrightarrow \quad [x]_{\sim_1} \sim_2 [y]_{\sim_1}.$$

Then the following two statements hold:

(a) "$\sim$" is an equivalence relation.

(b) The map

$$\begin{array}{ccc} X/\sim & \to & (X/\sim_1)/\sim_2 \\ [x] & \mapsto & [[x]_{\sim_1}]_{\sim_2} \end{array}$$

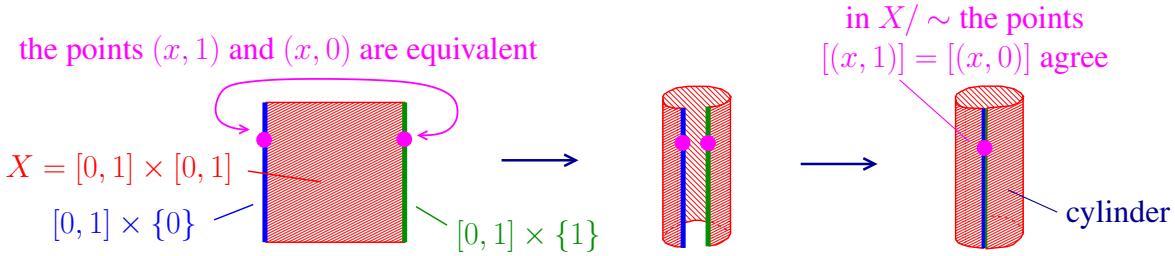
is a homeomorphism.

Proof.

(1) This statement follows from elementary logic.
(2) It follows again from elementary logic that the given map is a bijection. Next note that it follows from the Topological-Quotient Proposition 8.2 (1a) that the map $X \to X/\sim_1$ is continuous. It then follows from the Quotient-to-Quotient Lemma 8.6 that the given map $X/\sim \to (X/\sim_1)/\sim_2$ is continuous. Finally we consider the obvious inverse. Note that it follows from the Quotient-to-Quotient Lemma 8.6 that the map $X/\sim_1 \to X/\sim$ is continuous. But it then follows from the Topological-Quotient Proposition 8.2 (1b) that the map $(X/\sim_1)/\sim_2 \to (X/\sim_1)/\sim_2$ is continuous. ■

8.2. **Examples: Surfaces I.** In this section we will use the quotient topology construction to introduce some of the author's favorite topological spaces, namely surfaces. In the following we use the word "surface" in a colloquial manner. Later, in Chapter 23 we will introduce the notion of an n-dimensional topological manifold. We will then see that surfaces are 2-dimensional topological manifolds.

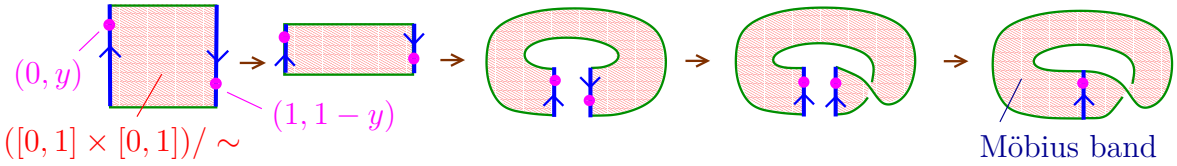
(1) We consider $X = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ and the equivalence relation that is generated by
$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1].$$
We will give two descriptions of this topological space, there first one is slightly informal, the second one is more rigorous.
(a) The quotient space $X/\sim$ is obtained from the square $X = [0,1]\times[0,1]$ by identifying each point on the upper edge with the corresponding point on the lower edge. In other words, "we glue the upper edge to the lower edge". This interpretation is illustrated in the figure below.



(b) We consider the map
$$\begin{array}{ccc} X/\sim & \to & S^1 \times [0,1] \\ [(s,t)] & \mapsto & (\exp(2\pi\,\mathrm{i}t), s). \end{array}$$
It follows from the Topological-Product Proposition 7.2 (1b) and the Topological-Quotient Proposition 8.2 (1b) that this map is continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 8.5 (1) we know that $X/\sim$ is compact and by the Product Topology Properties Proposition 7.9 (3) we know that $S^1 \times [0,1]$ is Hausdorff. Thus we obtain from the Compact-Hausdorff Proposition 4.23 (3) that the above map $X/\sim \to S^1 \times [0,1]$ is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the cylinder or the annulus.
(2) We consider $X = [0,1] \times [0,1] \subseteq \mathbb{R}^2$, this time with the equivalence relation that is generated by
$$(0,y) \sim (1,1-y) \quad \text{for all } y \in [0,1].$$

We refer to the quotient $X/\sim$ as the Möbius band. Again we give two descriptions:

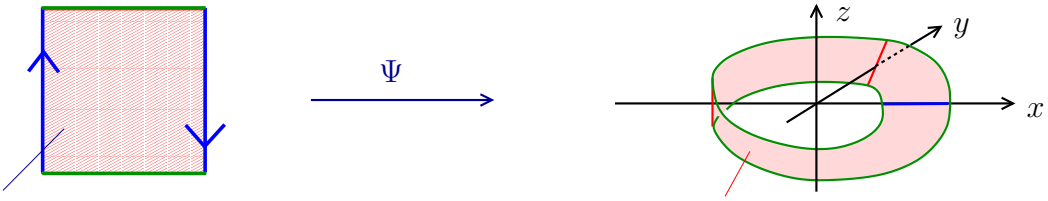
(a) The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on the right, but this time "we glue the edges with a twist".



(b) The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. We consider the map

$$\begin{aligned}\Psi\colon ([0,1]\times[0,1])/\sim &\to \mathbb{R}^3\\ [(\varphi,r)] &\mapsto \underbrace{\begin{pmatrix}\cos(2\pi\cdot\varphi) & -\sin(2\pi\cdot\varphi) & 0\\ \sin(2\pi\cdot\varphi) & \cos(2\pi\cdot\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix}2+(r-\frac{1}{2})\cdot\sin(\pi\cdot\varphi)\\ 0\\ (r-\frac{1}{2})\cdot\cos(\pi\cdot\varphi)\end{pmatrix}}_{\substack{\text{describes interval}\\ \text{that makes half a twist}\\ \text{as we go around the } z\text{-axis}}}\end{aligned}$$

Similar to (1b) one can see that Ψ is an embedding. Thus the Möbius band, defined abstractly as above, is indeed homeomorphic to a subspace of $\mathbb{R}^3$ that looks like what we had in mind.



$M = ([0,1]\times[0,1])/(0,y)\sim(1,1-y)$ Möbius band viewed as a subspace of $\mathbb{R}^3$

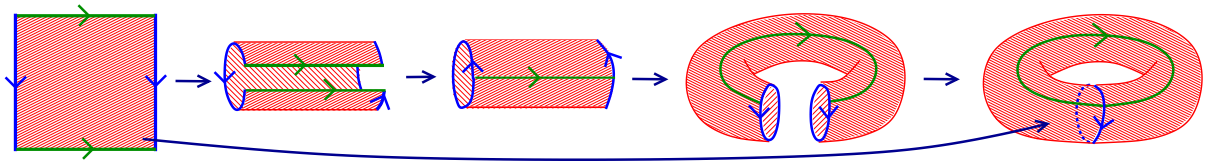
(3) We consider again $X = [0,1]\times[0,1]\subseteq\mathbb{R}^2$, but this time with the equivalence relation that is generated by

$$(x,0)\sim(x,1) \quad \text{for all } x\in[0,1]$$

and by

$$(0,y)\sim(1,y) \quad \text{for all } y\in[0,1].$$

(a) We obtain the quotient space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. The figure below illustrates that the quotient space $X/\sim$ is indeed a torus.



by gluing of the opposite edges of a square we obtain a torus

(b) As in (1b) one can write down an obvious homeomorphism $X/\sim\,\cong S^1\times S^1$. Thus we can make the identification $([0,1]\times[0,1])/\sim\,=_{\mathrm{i}} S^1\times S^1$.

(4) We consider $X = [0,1] \times [0,1] \subseteq \mathbb{R}^2$, this time with the equivalence relation that is generated by

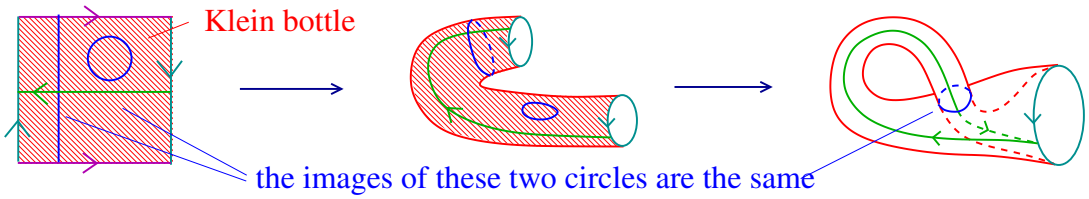
$$(x,0) \sim (x,1) \qquad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,1-y) \qquad \text{for all } y \in [0,1].$$

The resulting topological space is called the Klein bottle.[120]

(a) We obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. In the figure below we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.



(b) Let $0 < b < a$ be real numbers. It is elementary, but painful, to show that the map

$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & \mathbb{R}^4 \\ [(x,y)] & \mapsto & \begin{pmatrix} (a + b \cdot \cos(y \cdot 2\pi)) \cdot \cos(x \cdot 2\pi) \\ (a + b \cdot \cos(y \cdot 2\pi)) \cdot \sin(x \cdot 2\pi) \\ b \cdot \sin(y \cdot 2\pi) \cdot \cos(x \cdot \pi) \\ b \cdot \sin(y \cdot 2\pi) \cdot \sin(x \cdot \pi) \end{pmatrix} \end{array}$$

is an embedding. Thus we can view the Klein bottle as a subspace of $\mathbb{R}^4$.

In Exercise 8.2 we will show, using the Quotient Topology-Properties Proposition 8.5 (3), that the Klein bottle is Hausdorff.

We will introduce more surfaces a little later in Section 9.4.

8.3. **Quotients by subspaces.** In this section we will study another important source of quotient spaces.

Definition 8.8. Let X be a topological space and let $A \subseteq X$ be a subset.

(1) First we suppose that A is a non-empty subset. For $P, Q \in X$ we define

$$P \sim_A Q \quad :\Longleftrightarrow \quad P = Q \text{ or } P, Q \text{ both lie in } A.$$

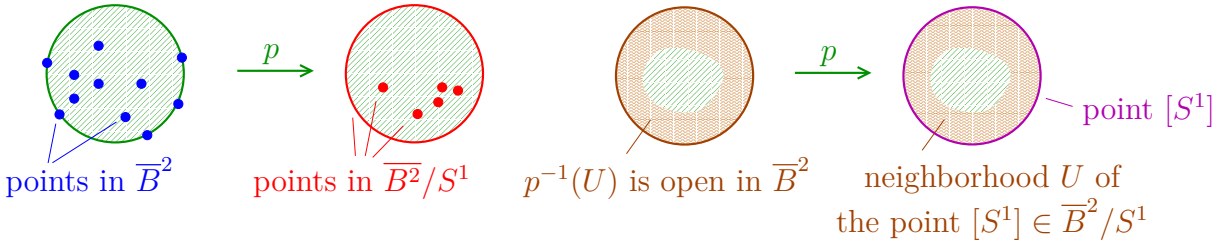
This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim_A$ and we always equip X/A with the quotient topology. We refer to X/A as the quotient of X by A. Note that $A \subseteq X$ is an equivalence class, thus it defines a point of X/A. In an attempt to avoid confusion we write $[A]$ for the point in X/A that is given by the equivalence class A.

(2) If $A = \varnothing$ is the empty set, then we define[121] $X/\varnothing := X \sqcup \{X\}$, i.e. $X/\varnothing$ equals the disjoint union of the topological space X with the topological space consisting of a single point $\{X\}$. For consistency we denote this point by $[\varnothing]$ as well.[122][123]

[120] The Klein bottle was first introduced by Felix Klein in 1881, see [**Kle1881, Kle1882b, Kle2018**].

Remark. Let us attempt to digest the definition of X/A. As a set X/A corresponds precisely to the points in $X \setminus A$ together with the point $[A]$. Furthermore, by Definition 8.3 of the quotient topology a set $U \subseteq X/A$ is open if and only if $p^{-1}(U)$ is an open subset of X. Note that given any subset $Z \subseteq X/A = (X \setminus A) \cup \{[A]\}$ we have[124]

$$p^{-1}(Z) = \begin{cases} Z, & \text{if } [A] \notin Z, \\ Z \cup A, & \text{if } [A] \in Z. \end{cases}$$



This shows that the open subsets in X/A correspond precisely to the open subsets of X that lie in $X \setminus A$ and the open subsets of X that contain A. □

Examples.

(1) We consider the quotient of $X = [0,3]$ by the compact interval $A = [1,2]$. This example has several interesting features:

- Using the Quotient Topology-Properties Proposition 8.5 (3) one can show directly that $[0,3]/[1,2]$ is Hausdorff.
- We want to show that the quotient $X/A = [0,3]/[1,2]$ is homeomorphic to an old acquaintance. To do so we consider the map

$$\begin{array}{rcl} f\colon [0,3] & \to & [0,2] \\ x & \mapsto & \begin{cases} x & \text{if } x \in [0,1], \\ 1, & \text{if } x \in [1,2], \\ x-1, & \text{if } x \in [2,3]. \end{cases} \end{array}$$

  It follows from the Pasting Proposition 4.7 (2) that this map is continuous. Furthermore it follows from the Topological-Quotient Proposition 8.2 (1b) that this map descends to a continuous map $\overline{f}\colon [0,3]/[1,2] \to [0,2]$, which is evidently a bijection. By the Quotient Topology-Properties Proposition 8.5 (1) we know that $[0,3]/[1,2]$ is compact. Thus it follows from the Compact-Hausdorff Proposition 4.23 that $\overline{f}\colon [0,3]/[1,2] \to [0,2]$ is a homeomorphism.
- Above we saw that projections $X \to X/\sim$ are basically by definition, continuous. One might think that projections should also be open. But in this example the projection $p\colon X = [0,3] \to X/A = [0,3]/[1,2]$ is *not* an open map, for example it

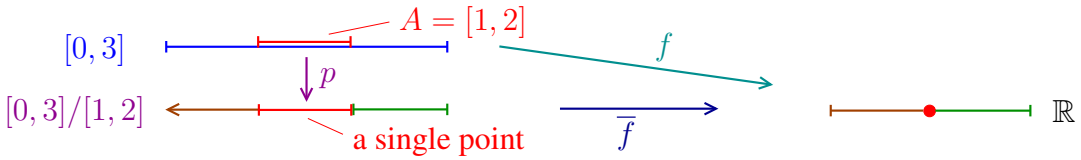
[121] Note that according to [**Cie1997**, Therem 4.1.1] $X \notin X$.

[122] In particular $\varnothing/\varnothing$ is the topological space consisting of the single point $\{\varnothing\}$.

[123] The reader will have spotted that the case $A = \varnothing$ is rather ad hoc. On page 318 we will give a less ad hoc interpretation. In general the case $A = \varnothing$ is a major nuisance. It is conceivable that later on in proofs we will subconsciously suppress this special case. Whenever this occurs we leave it to the reader to deal with the case $A = \varnothing$ separately.

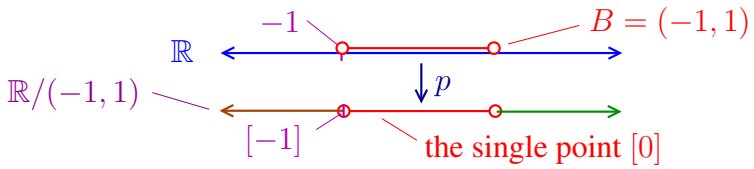
[124] In an attempt to lighten the notation, given $x \in X \setminus A$ we do not distinguish between the point $x \in X \setminus A$ and the equivalence class $\{x\} \subseteq X$.

follows immediately from the previous remark that the image of the open subset $(1,2) \subseteq [0,3]$ is not an open subset of $[0,3]/[1,2]$.



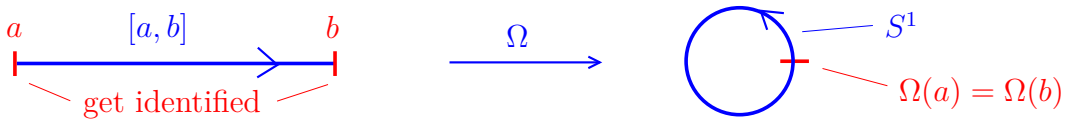
(2) We consider the quotient of $X = \mathbb{R}$ by the open interval $B = (-1,1)$. We denote by $p\colon \mathbb{R} \to \mathbb{R}/(-1,1)$ the projection. Even though this example looks very similar to the previous one, it shows very different behavior.

- The projection $X \to X/\sim$ is not closed. For example using the Topological-Quotient Proposition 8.2 one can easily see that the image of the closed subset $\{0\}$ is not closed.
- The quotient $X/B = \mathbb{R}/(-1,1)$ is not Hausdorff. More precisely we consider the two points $[-1]$ and $[0]$ in $\mathbb{R}/(-1,1)$. The previous remark shows that any open neighborhood U of $[-1]$ in $\mathbb{R}/(-1,1)$ has the property that $p^{-1}(U)$ is an open subset of $\mathbb{R}$ that contains -1. In particular $p^{-1}(U)$ has non-empty intersection with B. But this implies that $U = p(p^{-1}(U))$ contains the point $[0]$. Thus $[-1]$ and $[0]$ cannot be separated by open neighborhoods.



(3) Let $a < b$ be two real numbers. It follows easily from the Quotient Topology-Properties Proposition 8.5 (1), the Topological-Quotient Proposition 8.2 (1b) and the Compact-Hausdorff Proposition 4.23 (3) that the following map is a homeomorphism:

$$\begin{array}{rcl} \Omega\colon [a,b]/\{a,b\} & \to & S^1 \\ [t] & \mapsto & \exp\left(\mathrm{i} \cdot \frac{2\pi(t-a)}{b-a}\right) \end{array}$$



In particular in many settings we will use that $[0,1]/\{0,1\} = [0,1]/0 \sim 1$ is homeomorphic to S^1. □

We formulate our next examples as lemmas.

Lemma 8.9. (Cylinder Quotient-Ball Lemma)

(1) The map

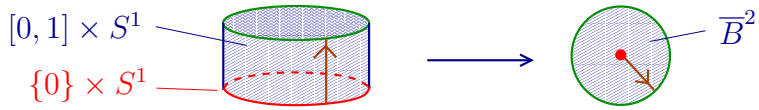
$$\begin{array}{rcl} ([0,1] \times S^n)/(\{0\} \times S^n) & \to & \overline{B}^{n+1} \\ [(r,v)] & \mapsto & r \cdot v \end{array}$$

is a homeomorphism.

(2) Let X be a topological space and let $f\colon [0,1]\times S^n \to X$ be a continuous map. If $f(0,v)=f(0,w)$ for all $v,w\in S^n$, then the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & X \\ r\cdot v & \mapsto & f(r,v) \text{ with } r\in[0,1] \text{ and } v\in S^n \end{array}$$

is well-defined and continuous.



Proof.

(1) As in the proof of the Torus Embedding Lemma 7.11 it follows from the Topological-Quotient Proposition 8.2 (1b) that the given map is continuous. It is basically clear that the map is a bijection. The left-hand side is compact by the Topological-Quotient Proposition 8.2 (1b). It follows from the Compact-Hausdorff Proposition 4.23 (3) that the map is a homeomorphism.

(2) It follows from the hypothesis that the map is well-defined. Thus it follows from (1) and the Topological-Quotient Proposition 8.2 (1b) that the map is continuous. ■

Lemma 8.10. (Ball Quotient-Sphere Lemma) Let $n\in\mathbb{N}$. The map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r\cdot v] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & 0 \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi\cdot r) \\ \sin(\pi\cdot r)\cdot v \end{pmatrix}}_{\in\mathbb{R}\times\mathbb{R}^n=\mathbb{R}^{n+1}} \end{array} \quad \text{with } r\in[0,1] \text{ and } v\in S^{n-1}$$

is a homeomorphism. This homeomorphism has the following properties:

(1) The map Ω sends the point $[S^{n-1}]\in\overline{B}^n/S^{n-1}$ to the North Pole $(0,\dots,0,1)$.

(2) The map Ω sends the origin to the South Pole $(0,\dots,0,-1)$.

(3) The map $B^n\to\overline{B}^n/S^{n-1}\xrightarrow{\Omega} S^n$ is a smooth embedding in the sense of Definition 30.9. Furthermore, if we equip B^n and S^n with the standard orientations from page 812 and page 817, then the smooth embedding is orientation-preserving in the sense of Definition 31.29.[125]

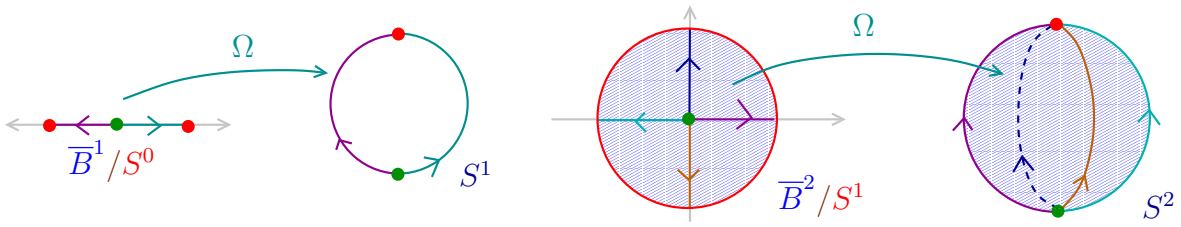
Furthermore, for $n=0$ the map

$$\begin{array}{rcl} \Omega\colon \overline{B}^0/S^{-1} & \to & S^0 \\ [x] & \mapsto & \begin{cases} -1, & \text{if } x=0\in\overline{B}^0, \\ 1, & \text{if } [x]=[S^{-1}] \end{cases} \end{array}$$

is a homeomorphism with the properties (1) and (2) above[126].

[125] We hope that the reader is not too fazed by the fact that we use some expressions that we will introduce only later. But we think it is useful to state all properties of Ω now.

[126] Unfortunately for $n=0$ we cannot arrange to have (2) and (3). This might lead later on to slight inconsistencies for $n=0$.



Identification 8.11. For each $n \in \mathbb{N}_0$ we use the homeomorphism from the Ball Quotient-Sphere Lemma 8.10 to make the identification $S^n =_{\mathrm{i}} \overline{B}^n/S^{n-1}$.

Proof of Lemma 8.10. Evidently we first deal with the case $n \neq 0$. It follows quite easily from the Cylinder Quotient-Ball Lemma 8.9 (2), the Topological-Quotient Proposition 8.2 (1b), the Quotient Topology-Properties Proposition 8.5 (1) and the Compact-Hausdorff Proposition 4.23 (3) that the map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ is a homeomorphism. It is clear that $\Omega([S^{n-1}]) = (0,\dots,0,1)$ and that $\Omega(0) = (0,\dots,0,-1)$.

Thus it remains to prove Statement (3). To show smoothness it is convenient to rewrite the definition of Ω. First note that by elementary real analysis we know that there exists a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \neq 0$ we have $f(x) = \frac{\sin(\pi\cdot x)}{x}$. One can now easily verify that the map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ has the following, in some sense cleaner, description:

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [w] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi\cdot\|w\|) \\ f(\|w\|)\cdot w \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}} \quad \text{with } w \in \overline{B}^n. \end{array}$$

It is now clear that the map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is smooth. One can easily verify that the differential at each $w \in B^n$ is invertible, i.e. the map $B^n \to S^n$ is an immersion in the sense of Definition 30.9. Using the above one can also easily verify that $B^n \to S^n$ is an embedding. In summary, we have shown that the map $B^n \to S^n$ is a smooth embedding.

It remains to show that the smooth embedding $B^n \to S^n$ is orientation-preserving. By the Manifold–Orientation Basics Lemma 31.31 it suffices to show that the map is orientation-preserving at the origin. At the origin the standard basis $\mathfrak{e}_1,\dots,\mathfrak{e}_n$ of $\widetilde{\mathrm{T}}_0 B^n = \mathbb{R}^n$ is a positive basis. A rather elementary calculation shows that it gets sent to the basis $(-1)^{n-1}\cdot\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_n$ of the tangent space of S^n at $\Phi(0) = -\mathfrak{e}_{n+1}$. Since we have the inequality $\det(-\mathfrak{e}_{n+1}\ \ (-1)^{n-1}\cdot\mathfrak{e}_1\ \ \mathfrak{e}_2\ \dots\ \mathfrak{e}_n) > 0$ we see, by the definition on page 817 of the standard orientation on S^n, that $(-1)^{n-1}\cdot\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_n$ is a positive basis of the tangent space of S^n at $\Phi(0) = -\mathfrak{e}_{n+1}$. We have thus shown that the smooth embedding $B^n \to S^n$ is orientation-preserving at the origin and thus everywhere.

Finally note that the case $n = 0$ follows easily from the observation that in Definition 8.8 we defined $X/\varnothing := X \sqcup \{\varnothing\}$. ∎

8.4. **The pushout in the topological category.** We conclude our long list of various types of quotients of topological spaces with a discussion of pushouts in the category of

topological spaces. The following proposition is formulated in the tradition of the Topological-Product Proposition 7.2,f the Topological-Coproduct Proposition 7.26 and the Topological-Quotient Proposition 8.2.

Proposition 8.12. (Topological-Pushout Proposition) Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the topological space[127]

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$

and the maps

$$\begin{array}{rcl} i\colon Y & \to & Y \cup_A Z \\ y & \mapsto & [y] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Z & \to & Y \cup_A Z \\ z & \mapsto & [z] \end{array}$$

The following statements hold:

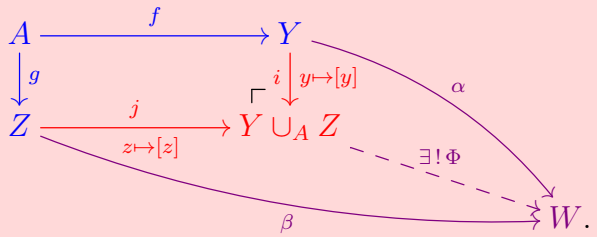
(0) A subset $U \subseteq Y \cup_A Z$ is open if and only if $i^{-1}(U) \subseteq Y$ and $j^{-1}(U) \subseteq Z$ are open. The same statement holds for "closed" instead of "open".

(1) (a) The maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ are continuous.

(b) Suppose we are given two continuous maps $\alpha\colon Y \to W$ and $\beta\colon Z \to W$ to some topological space W such that $\alpha \circ f = \beta \circ g\colon A \to W$. The map

$$\begin{array}{rcl} \Phi\colon Y \cup_A Z & \to & W \\ [P] & \mapsto & \begin{cases} \alpha(y) & \text{if } P = [y] \text{ for some } y \in Y, \\ \beta(z) & \text{if } P = [z] \text{ for some } z \in Z \end{cases} \end{array}$$

is well-defined and it is the unique map $\Phi\colon Y \cup_A Z \to W$ with $\Phi \circ i = \alpha$ and $\Phi \circ j = \beta$. Furthermore, if α and β are continuous, then Φ is also continuous. In particular we obtain the following commutative diagram:



(∗) A map $h\colon Y \cup_A Z \to W$ to some topological space W is continuous if and only if the maps $h \circ i\colon Y \to W$ and $h \circ j\colon Z \to W$ are continuous.

(2) The topological space $Y \cup_A Z$ with the natural maps $Y \to Y \cup_A Z$ and $Z \to Y \cup_A Z$ is the pushout, in the sense of Definition 6.23, in the category $\mathcal{T}op$ of topological spaces.

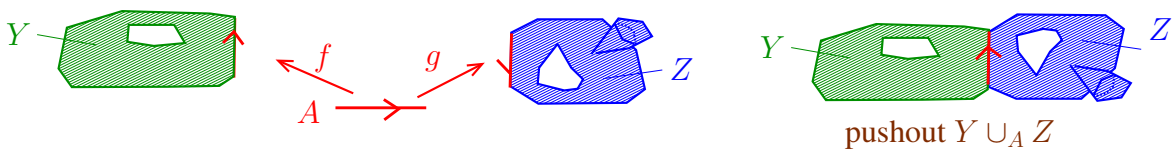
Definition 8.13. Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We refer to

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

as the pushout of f and g and we refer to the maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ as the two natural maps.

Proof.

[127] According to Notation 1.12 we have $Y \sqcup Z = (Y \times \{1\}) \cup (Z \times \{2\})$. Thus we should really write $(f(a), 1) \sim (g(a), 2)$. To keep the notation at a manageable level we ignore this issue.



(0) This statement follows from the Topological-Coproduct Proposition 7.26 (0) and the Topological-Quotient Proposition 8.2 (0).

(1) (a) This statement follows immediately from (0). Alternatively this statement follows from the Topological-Coproduct Proposition 7.26 (1a) and the Topological-Quotient Proposition 8.2 (1a).

(b) This existence of the continuous map follows immediately from the Topological-Coproduct Proposition 7.26 (1b) and the Topological-Quotient Proposition 8.2 (1b) (together with the Quotient-Factorization Lemma 1.23). We leave the proof of the uniqueness to the reader.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ■

Example. Let X be a topological space, let $A \subseteq X$ be a subspace, let $f\colon A \to \{*\}$ be the constant map to the topological space consisting of a single point and let $g\colon A \to X$ be the inclusion map. Using the uniqueness statement of the Quotient-Factorization Lemma 1.21 and using the Topological-Quotient Proposition 8.2 one can show that the quotient X/A, as introduced in Definition 8.8, is a pushout of the pushout diagram. In summary we have the following pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & \{*\} \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{x\mapsto[x]} & X/A. \end{array}$$

From the uniqueness statements for pushouts, see the Colimit Uniqueness Lemma 6.22, it follows that X/A is homeomorphic to the pushout $X \cup_A \{*\}$.[128] □

The following proposition is the analog of the Disjoint Union Topology-Properties Lemma 7.28 and the Quotient Topology-Properties Proposition 8.5.

Proposition 8.14. (Pushout-Properties Proposition) Let $f\colon A \to Y$, $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the corresponding pushout $Y \cup_A Z := (Y \sqcup Z)/\sim$.

(1) If Y and Z are compact, then so is the pushout $Y \cup_A Z$.

(2) If Y and Z are (path-) connected and if A is non-empty, then the pushout $Y \cup_A Z$ is also (path-) connected.

(3) If A, Y and Z are compact, if Y and Z are Hausdorff and if $f\colon A \to Y$ and $g\colon A \to Z$ are injective, then $Y \cup_A Z$ is also Hausdorff.

[128] Note that this even works if $A = \varnothing$.

Proof.

(1) If Y and Z are compact, then it follows from the Disjoint Union Topology-Properties Lemma 7.28 (1) and the Quotient Topology-Properties Proposition 8.5 (1) that $Y \cup_A Z$ is compact.

(2) If Y and Z are (path-) connected and if A is non-empty, then it follows easily from the Topological Pushout Proposition 8.12, the Image-Connected Lemma 4.34 and the Union Connected Lemma 4.39 that $Y \cup_A Z$ is also (path-) connected.

(3) Since f and g are injective we see that the relation on $Y \sqcup Z$ is given by

$$\{(f(a), g(a)) \,|\, a \in A\} \cup \{(w, w) \,|\, w \in Y \sqcup Z\} \subseteq (Y \sqcup Z) \times (Y \sqcup Z).$$

It follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2), together with the hypotheses that A, Y, Z are compact and Y, Z are Hausdorff, that this is a closed subset. It now follows from the Quotient Topology-Properties Proposition 8.5 (3) that $Y \cup_A Z$ is Hausdorff. ■

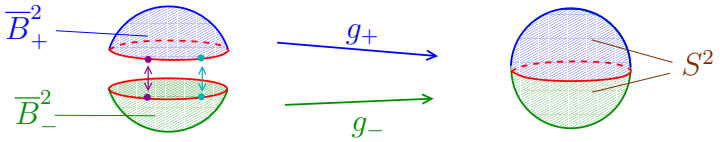
Example. Let $n \in \mathbb{N}_0$. Let $\overline{B}^n_\pm$ be two copies of $\overline{B}^n$. We consider the two obvious inclusion maps $S^{n-1} \to \overline{B}^n_\pm$ and the corresponding pushout $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$. As in the Stereographic Projection Lemma 4.24 we consider for $\epsilon = -1, 1$ the homeomorphisms[129]

$$\begin{array}{rcl} g_\epsilon \colon \overline{B}^n_\epsilon & \to & S^n \\ x = (x_1, \ldots, x_n) & \mapsto & \frac{1}{\|x\|^2+1} \cdot \Big((-1)^n \cdot 2x_1, 2x_2, \ldots, 2x_n, -\epsilon \cdot (\|x\|^2 - 1)\Big). \end{array}$$

Note that $g_+ = g_-$ on S^{n-1}. Thus it follows from the Topological Pushout Proposition 8.12 (1b) that the map

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$

is continuous. One can easily verify that it is a bijection.



It follows from the Pushout-Properties Proposition 8.14 (1) combined with the Compact-Hausdorff Proposition 4.23 (3) that the map is a homeomorphism.[130][131][132] □

[129] We put in the superfluous looking sign $(-1)^n$ for the following reason: once we have introduced orientations of smooth manifolds and once we have introduced the standard orientations of $\overline{B}^n$ and S^n this sign assures us that the map $g_+ \colon \overline{B}^n \to S^n$ is orientation-preserving and it means that $g_- \colon \overline{B}^n \to S^n$ is orientation-reversing.

[130] On page 1229 we will see that we can also identify $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$ and S^n as "smooth manifolds".

[131] In [**Lip2014**, Chapter 2] and [**Petn1979**] the argument is made that the universe in Dante's Divina Commedia consists of two 3-balls (corresponding to heaven and hell) that are glued along the boundary. In other words, Dante describes the 3-sphere.

[132] We could also have worked with the maps

$$\begin{array}{rcl} h_\pm \colon \overline{B}^n = \overline{B}^n_\pm & \to & S^n \\ (x_1, \ldots, x_n) & \mapsto & \big((-1)^n \cdot x_1, x_2, \ldots, x_n, \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}\big). \end{array}$$

Identification 8.15. We use the homeomorphism from the above example to make the identification $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- =_{\mathrm{i}} S^n$.

Lemma 8.16. (Topological Pushout-Maps Lemma) Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We obtain the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i}\ {\scriptstyle y\mapsto[y]} \\ Z & \xrightarrow[z\mapsto[z]]{\ j\ } & Y \cup_A Z. \end{array}$$

The following statements hold:

(1) If f is a homeomorphism, then the "opposite map" j is a homeomorphism.

(2) If f is injective, then the "opposite map" j is injective. The same statement holds for "surjective" and "bijective".[133]

(3) Suppose that at least one of the two maps f or g is injective.

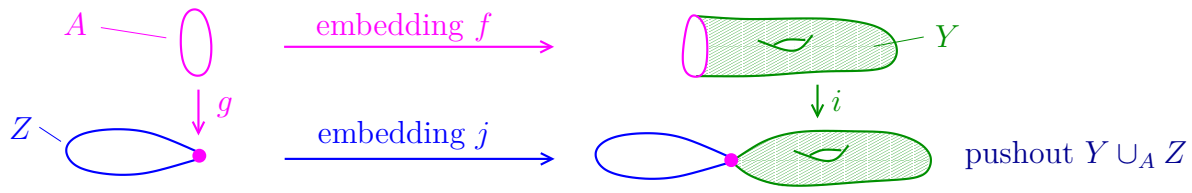
(α) If the image of f is a closed subset of Y, then the image of the "opposite map" j is a closed subset of $Y \cup_A Z$.

(β) If the image of g is a closed subset of Z, then the image of the "opposite map" i is a closed subset of $Y \cup_A Z$.

Both statements also hold with "closed" replaced by "open".

(4) If f is an embedding, then the "opposite map" j is an embedding. In fact if f is an open (closed) embedding, then the "opposite map" j is also an open (closed) embedding.

By symmetry the statements in (1), (2) and (4) hold also for g and its "opposite map" i.



Proof. Throughout the proof we denote by $p\colon Y \sqcup Z \to Y \cup_A Z$ the natural projection.

(1) This statement is a special case of the Pushout–Opposite Lemma 6.26.

The advantage of our maps $g_\pm$ is that they are not only homeomorphisms but they are actually "diffeomorphisms".

[133] This statement is very close to the content of Exercise 6.16. In fact it has nothing to do with topological spaces, the statement also holds for sets.

(2) First we assume that $f\colon A \to Y$ is injective. We need to show that $j\colon Z \to Y \cup_A Z$ is injective. Let $z \in Z$. Note that

$$\begin{aligned} j^{-1}(j(z)) \;=\; \{z' \in Z \mid j(z') = j(z)\} \;&=\; \begin{array}{c}\text{set of elements in } Z \text{ equivalent to } z \\ \text{with respect to } \sim \text{ on } Y \sqcup Z\end{array} \\ &= \bigcup_{k\in\mathbb{N}_0} \underbrace{g(f^{-1}(f(g^{-1}(\dots g(f^{-1}(f(g^{-1}}_{\text{applied } k \text{ times}}(\{z\}))))\dots)))) \\ &= \bigcup_{k\in\mathbb{N}_0} g(g^{-1}(\dots g(g^{-1}(\{z\}))\dots)) \;=\; \{z\}. \end{aligned}$$

by the Image-Preimage Lemma 1.9 (1b), since f is injective

by the Image-Preimage Lemma 1.9 (2a)

This shows that the map j is injective.

Now we assume that $f\colon A \to Y$ is surjective. We need to show that $j\colon Z \to Y \cup_A Z$ is surjective. Recall that by definition $Y \cup_A Z \coloneqq (Y \sqcup Z)/\!\sim$ where $f(a) \sim g(a)$ for all $a \in A$. Let $[w] \in Y \cup_A Z$. If $w \in Z$, then evidently $j(w) = [w]$. Now consider the case that $w \in Y$. Since $f\colon A \to Y$ is surjective there exists an $a \in A$ with $f(a) = w$. But now we see that $j(g(a)) = i(f(a)) = i(w) = [w]$.

(3) Without loss of generality we can assume that f is injective. The statements for "closed" follow almost immediately from the following observations:

- By the Topological-Quotient Proposition 8.2 a subset W of $Y \cup_A Z$ is closed if and only if $p^{-1}(W)$ is closed.
- Since f is injective it follows from the argument in (2) that
$$p^{-1}(i(Y)) = Y \sqcup g(A) \quad \text{and} \quad p^{-1}(j(Z)) = f(g^{-1}(Z)) \sqcup Z = f(A) \sqcup Z.$$
- By the Topological-Coproduct Proposition 7.26 a subset $U \sqcup V$ of $Y \sqcup Z$ is closed if and only if U and V are closed.

The proof for "open" is basically the same.

(4) We suppose that $f\colon A \to Y$ is an embedding. We need to show that $j\colon Z \to Y \cup_A Z$ is an embedding. By the Topological Pushout Proposition 8.12 (1a) we know that j is continuous and by (2) we know that $j\colon Z \to Y \cup_A Z$ is injective. By the Open-Closed–Injective Lemma 4.22 (2) and by definition of the subspace topology on $j(Z)$ it remains to prove the following claim:

Claim. If U is an open subset of Z, then there exists an open subset W of $Y \cup_A Z$ such that $j(U) = W \cap j(Z)$.

Proof. Let U be an open subset of Z. Since g is continuous we know that $g^{-1}(U)$ is an open subset of A. Since $f\colon X \to Y$ is an embedding we know that there exists an open subset $V \subseteq Y$ with $f(g^{-1}(U)) = f(A) \cap V$. We set $W \coloneqq p(U \sqcup V)$. An elementary argument as in (2), that uses that f is injective, shows that $p^{-1}(W) = U \sqcup V$. Thus it follows from the Definition 8.3 of the quotient topology and the disjoint union topology that W is indeed an open subset of $Y \cup_A Z$. Basically by construction we have $j(U) = W \cap j(Z)$. ⊞

The statement about open or closed embeddings follows immediately from (3) and the Open-Closed Inclusion Lemma 4.13. ■

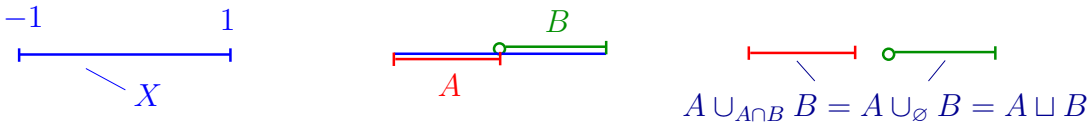
The following lemma is quite subtle and gets used, often inadvertently, on many occasions.

Lemma 8.17. (Regluing Lemma) Let X be a topological space and let A and B be two subsets of X with $X = A \cup B$. We suppose that the following two conditions holds:
(∗) Either both subsets A and B are closed or both subsets A and B are open.
Then the map

$$\begin{array}{rcl} A \cup_{A\cap B} B & \to & X \\ [x] & \mapsto & x \end{array}$$

is a homeomorphism.

Remark. In general the conclusion of the lemma does not hold without some assumptions on A and B. For example, consider $X = [-1, 1]$ together with the subsets $A = [-1, 0]$ and $B = (0, 1]$. In this case $A \cap B = \varnothing$, thus $A \cup_{A\cap B} B = A \cup_{\varnothing} B = A \sqcup B$ is the disjoint union of A and B. It follows from the Disjoint Union Topology-Properties Lemma 7.28 (1) and the Heine-Borel Theorem 3.44 Theorem that $A \sqcup B$ is not compact.

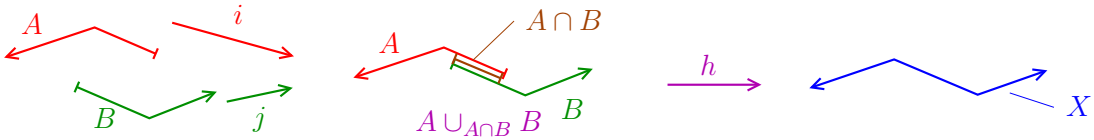


On the other hand $X = [-1, 1]$ is compact by the Heine-Borel Theorem 3.44. Thus the two topological spaces $A \cup_{A\cap B} B = A \sqcup B$ and X are not homeomorphic. □

Proof of Lemma 8.17. Let X be a topological space and let A and B be two subsets of X with $X = A \cup B$. In the following we consider the case that A and B are both closed subsets of X. The case that both A and B are open subsets is dealt with in almost entirely the same way.

It follows from the Topological Pushout Proposition 8.12 (1b) that the given map $h\colon A \cup_{A\cap B} B \to X$ is continuous. Furthermore, basically by definition, the map h is a bijection. By the Open-Closed–Injective Lemma 4.22 (2) it remains to show that the map h is closed.

Thus let $W \subseteq A \cup_{A\cap B} B$ be a closed subset. Since the natural maps $i\colon A \cap B \to A \cup_{A\cap B} B$ and $j\colon A \cap B \to A \cup_{A\cap B} B$ are both injective it follows easily from the Topological Pushout-Maps Lemma 8.16 (1) that we have $h(W) = i^{-1}(W) \cup j^{-1}(W) \subseteq X = A \cup B$.



Since i and j are continuous we know that $i^{-1}(W) \subseteq A$ and $j^{-1}(W) \subseteq B$ are closed subsets. Since A and B are closed subsets of X we obtain from the Open-Closed Inclusion Lemma 4.13 that $h(W) = i^{-1}(W) \cup j^{-1}(W)$ is a closed subset of X.[134] ■

We continue with an amusing description of spheres:

Example.

[134] One can also argue that $A \cup B$ is connected whereas X is not connected.

(1) We write $I = [-1, 1]$. Note that it follows from the Product-of-Subsets Lemma 7.8 (2c) that given $k \in \mathbb{N}_0$ we have

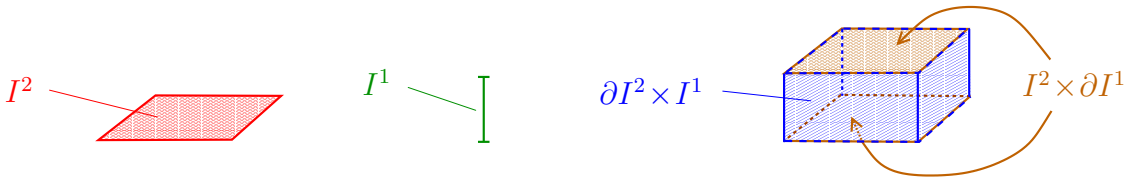
$$\partial I^k \;=\; \{(x_1, \ldots, x_k) \in I^k \mid \text{ at least one } x_i \text{ lies in } \{\pm 1\}\}.$$

Now let $m, n \in \mathbb{N}_0$. An elementary set-theoretic argument shows that

$$((\partial I^m) \times I^n) \;\cup\; (I^m \times \partial I^n) \;=\; \partial I^{m+n}$$

and the two subsets to the left intersect in $(\partial I^m) \times (\partial I^n)$. Since both subsets are closed it follows from the Regluing Lemma 8.17 that the inclusions induce a homeomorphism

$$((\partial I^m) \times I^n) \cup_{\partial I^m \times \partial I^n} (I^m \times \partial I^n) \;\xrightarrow{\cong}\; \partial I^{m+n}.$$



(2) Recall that by the Convex-to-Ball Proposition 4.28 there exists a canonical homeomorphism

$$\varphi_k \colon (\overline{B}^k, S^{k-1}) \;\xrightarrow{\cong}\; (I^k, \partial I^k)$$

with $\varphi_k(0) = 0$. It follows from this observation together with (1) that we obtain a canonical homeomorphism

$$(S^{m-1} \times \overline{B}^n) \cup_{S^{m-1} \times S^{n-1}} (\overline{B}^m \times S^{n-1}) \;\xrightarrow{\cong}\; S^{m+n-1}.$$

For example, if we take $m = n = 2$, then we see that we can write S^3 as the union of two solid tori. We will discuss this particular case in greater detail later in Section 111.2.□

We conclude this discussion with the following lemma, which inadvertently gets used frequently:

Lemma 8.18. (Pushout–Closed Embedding Lemma) Suppose that we are given two pushouts

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & Y \cup_A Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle \widetilde{g}}\downarrow & & \downarrow \\ \widetilde{Z} & \longrightarrow & \widetilde{Y} \cup_{\widetilde{A}} \widetilde{Z} \end{array}$$

of topological spaces and suppose that we are given continuous maps $a \colon A \to \widetilde{A}$, $y \colon Y \to \widetilde{Y}$ and $z \colon Z \to \widetilde{Z}$ such that $f \circ a = z \circ f$ and $\widetilde{g} \circ a = y \circ g$. Then we obtain an induced continuous map

$$Y \cup_A Z \;\to\; \widetilde{Y} \cup_{\widetilde{A}} \widetilde{Z}.$$

If $f \colon A \to Y$ and $\widetilde{f} \colon \widetilde{A} \to \widetilde{Y}$ are closed embeddings and if $g(A) \subseteq Z$ and $\widetilde{g}(\widetilde{A}) \subseteq \widetilde{Z}$ are closed, then the induced map $Y \cup_A Z \;\to\; \widetilde{Y} \cup_{\widetilde{A}} \widetilde{Z}$ is a closed embedding.

Sketch of proof.

(1) This statement follows immediately from the Definition 6.23 of a pushout.

(2) This statement follows fairly easily from the Open-Closed–Injective Lemma 4.22, Cover-Open Subset Lemma 3.20 (2b) together with the Topological Pushout-Maps Lemma 8.16. We leave it to the reader to fill in the details. ■

8.5. Appendix: The coequalizer in the topological category.

In the Relation–Coequalizer Lemma 6.28 we saw that in the category *Set* of sets the notion of a quotient by an equivalence relation is essentially the same as a coequalizer. We will now see that the analogous statement also in the category *Top* of topological spaces:

Lemma 8.19. (Topological Relation–Coequalizer Lemma) Let Y be a topological space.

(1) Let X be a topological space and let $\varphi, \psi\colon X \to Y$ be two continuous maps. We consider the relation $R = \{(\varphi(x), \psi(x)) \,|\, x \in X\}$ and we denote by "$\sim$" the equivalence relation on Y that is generated by the relation R in the sense of Definition 1.22. Then the coequalizer of φ and ψ in the category *Top* is naturally homeomorphic to the quotient space $Y/\sim$ from the Topological-Quotient Proposition 8.2. Put differently, the natural projection $Y \to Y/\sim$ induces a natural homeomorphism

$$\operatorname*{colim}\left(X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y \right) \xrightarrow{\cong} Y/\varphi(x) \sim \psi(x).$$

(2) Let $R \subseteq Y \times Y$ be an equivalence relation in the sense of Definition 1.13. We denote by $\mathrm{pr}_1\colon Y \times Y \to Y$ and $\mathrm{pr}_2\colon Y \times Y \to Y$ the natural projection onto the first, respectively the second factor. The coequalizer of $\mathrm{pr}_1\colon R \to Y$ and $\mathrm{pr}_2\colon R \to Y$ in *Top* is naturally homeomorphic to the quotient space Y/R. Put differently, the natural projection $Y \to Y/\sim$ induces a natural homeomorphism

$$\operatorname*{colim}\left(R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} Y \right) \xrightarrow{\cong} Y/\sim.$$

Proof. As in the setting of the Relation–Coequalizer Lemma 6.28 we see that the first statement follows easily from the definitions and that the second statement is just a reinterpretation of Statement (1). ∎

We conclude this chapter with the following satisfying proposition:

Proposition 8.20. (Topological Spaces–Complete+Cocomplete Proposition) The category *Top* of topological spaces is complete and cocomplete.

Proof. First note that it follows from the Category–Completeness Criterion 6.20 together with the Topological-Product Proposition 7.2 and the discussion on page 261 that *Top* is complete.

Next note that it follows from the Category–Cocompleteness Criterion 6.32 together with the Topological-Coproduct Proposition 7.26 and the Topological Relation–Coequalizer Lemma 8.19 that *Top* is cocomplete. ∎

8.6. Exercises for Chapter 8.

Exercise 8.1. Let X be a topological space and let $\sim$ be an equivalence relation on X. Furthermore let $\mathcal{B}$ be a basis of the topology of X.

(a) Show that if the projection $p\colon X \to X/\sim$ is an open map, then $p(\mathcal{B}) = \{p(B) \,|\, B \in \mathcal{B}\}$ is a basis for the topology of $X/\sim$.

(b) Show that in general the conclusion of (a) does not hold, if the projection map is not open.

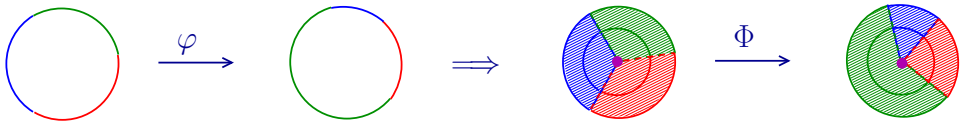
Exercise 8.2. As on page 312 we consider the Klein bottle $K = ([0,1] \times [0,1])/\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$. Show that this relation $\sim$ is a closed relation, i.e. show that $\sim$ is a closed subset of $([0,1] \times [0,1]) \times ([0,1] \times [0,1])$.
Remark. Together with the Topology-Properties Proposition 8.5 (3) this implies that the Klein bottle is Hausdorff.

Exercise 8.3. Let $n \in \mathbb{N}$ and let $\varphi\colon S^n \to S^n$ be a homeomorphism. We consider the map

$$\begin{array}{rccl} \Phi\colon \overline{B}^{n+1} & \to & \overline{B}^{n+1} & \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that Φ is continuous.
(b) Show that Φ is a homeomorphism.
(c) Give any example of a *diffeomorphism* $\varphi\colon S^n \to S^n$ such that the corresponding map $\Phi\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ is *not* a diffeomorphism.

Hint. You might want to use the Cylinder Quotient-Ball Lemma 8.9.
Remark 1. The fact that we can extend φ to a homeomorphism of $\overline{B}^{n+1}$ is sometimes referred to as the Alexander trick. We will get to know a more fancy Alexander trick later on page 512.
Remark 2. We have thus shown that every *homeomorphism* of S^n extends to a *homeomorphism* of $\overline{B}^{n+1}$. In contrast, in the Diffeomorphism–Non Extension Proposition 282.11 we will see that exists a *diffeomorphism* $\varphi\colon S^6 \to S^6$ that does not extend to a *diffeomorphism* of $\overline{B}^7$.



Exercise 8.4. Let

$$S \;:=\; \{(0,y) \,|\, y \in [-1,1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2$$

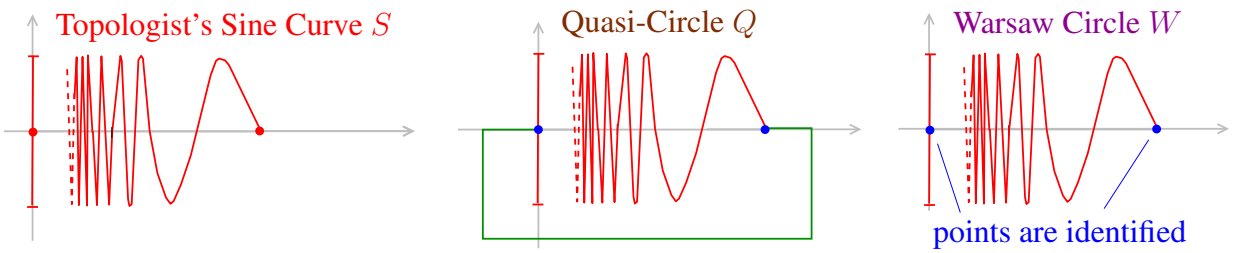
be the "Topologist's Sine Curve" from page 195.

- We turn the Topologist's Sine curve into the *Quasi-Circle* Q by connecting the point $(0,0)$ with $(\frac{1}{\pi},0)$ via the green curve shown in the figure below.
- We refer to quotient space $W := S/(0,0) \sim (\frac{1}{\pi},0)$ as the *Warsaw Circle*.

Now perform the following tasks:

(a) Show that the Topologist's Sine Curve, the Quasi-Circle and the Warsaw Circle are compact.
(b) Show that the Warsaw Circle is homeomorphic to the Quasi-Circle.
(c) Show that the Warsaw Circle and the Quasi-Circle are path-connected.

Remark. The Warsaw Circle is sometimes also called the *Polish Circle*.



Exercise 8.5. We consider the topological space $X = S^1$ together with the equivalence relation that is generated by the relation $\sim$ which says that we have $P \sim Q$ if both lie in $S^1_{>0} = \{(x, y) \in S^1 \mid y > 0\}$ or if both lie in $S^1_{<0} = \{(x, y) \in S^1 \mid y < 0\}$. Show that the quotient space $S^1/\sim$ is homeomorphic to the pseudocircle that we introduced on page 172.

Exercise 8.6. Let X be a topological space, let $A \subseteq X$ be a subspace and let $\sim$ be an equivalence relation on X. There are two topologies on $A/\sim$:

($\mathcal{A}$) We can consider the subspace topology on $A \subseteq X$ and afterwards the quotient topology on $A/\sim$.

($\mathcal{B}$) We can consider the quotient topology on $X/\sim$ and then the subspace space topology on $A/\sim \subseteq X/\sim$.

Now we can formulate the following exercises:

(a) Show that we always have $\mathcal{B} \subseteq \mathcal{A}$.

(b) Suppose that one of the following conditions holds:
  (i) $A \subseteq X$ is open.
  (ii) $A \subseteq X$ is closed.
  (iii) The projection $X \to X/\sim$ is an open map.
  (iv) The projection $X \to X/\sim$ is a closed map.
  Show that in all of these four cases we have $\mathcal{A} = \mathcal{B}$.
  Remark. These statements get used subconsciously on many occasions.

(c) We consider $X = \{1, 2, 3, 4\}$ with the topology given by $\{\varnothing, \{1, 3\}, \{2, 4\}, X\}$, the subspace $A = \{1, 2\}$ and the equivalence relation that is generated by $3 \sim 4$. Show that in this case $\mathcal{A} \neq \mathcal{B}$.

Exercise 8.7. Let $n \in \mathbb{N}$.

(a) Let $p\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ be the map that is given by $x \mapsto \frac{x}{\|x\|}$. Show that the preimage of a dense subset of S^{n-1} is a dense subset of $\mathbb{R}^n \setminus \{0\}$.

(b) Let $q\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ be the projection. Show that the preimage of a dense subset of $\mathbb{R}\mathrm{P}^{n-1}$ is a dense subset of S^{n-1}.

Exercise 8.8. Let $\sim$ be the equivalence relation on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ that is generated by

(1) $(0, \varphi) \sim (0, \varphi')$ for all $\varphi, \varphi' \in \mathbb{R}$ and by
(2) $(r, \varphi) \sim (r, \varphi + k \cdot 2\pi)$ for all $r \in \mathbb{R}_{\geq 0}$, $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$.

We equip $(\mathbb{R}_{\geq 0} \times \mathbb{R})/\sim$ with the quotient topology.

(a) Show that the map

$$\begin{array}{rccc} \Theta\colon & (\mathbb{R}_{\geq 0} \times \mathbb{R})/\sim & \to & \mathbb{C} \\ & [(r, \varphi)] & \mapsto & r \cdot \exp(\mathrm{i}\varphi) \end{array}$$

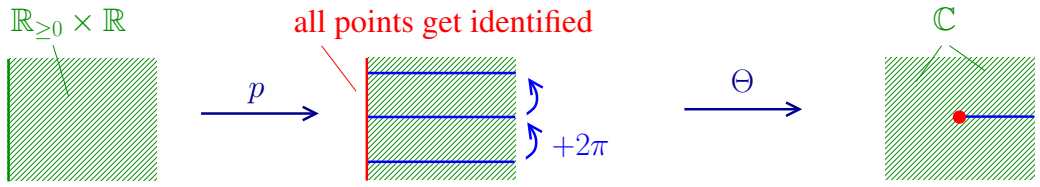
is a homeomorphism.
Hint. To show that the map is a homeomorphism you might want to use the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26.

(b) Let $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to X$ be a continuous map to some topological space X such that $f(P) = f(Q)$ whenever $P \sim Q$. Show that the map

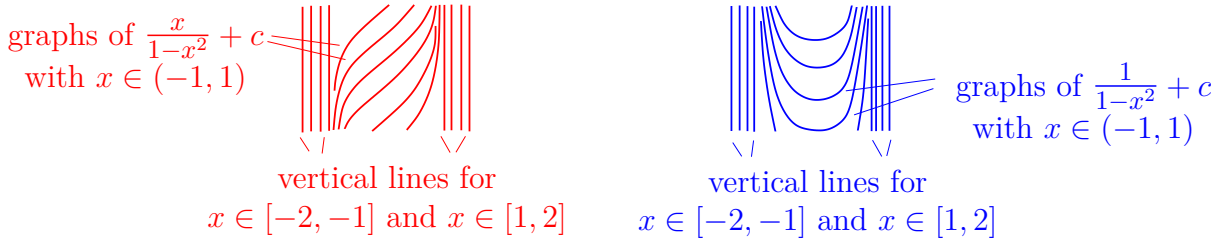
$$\begin{array}{ccc} \mathbb{C} & \to & X \\ r \cdot \exp(\mathrm{i}\varphi) & \mapsto & f(r, \varphi) \end{array}$$

is continuous.

Remark. This exercise is a variation on the Cylinder Quotient-Ball Lemma 8.9.



Exercise 8.9. In the figure below we see $[-2, 2] \times \mathbb{R}$ with two equivalence relations, more precisely we see $[-2, 2] \times \mathbb{R}$ with two decompositions into equivalence classes. One of the two quotient spaces is Hausdorff, the other one is not. Which one is Hausdorff?



Exercise 8.10.

(a) Let X be a compact Hausdorff space and let $A \subseteq X$ be a closed subset. Show that X/A is Hausdorff.

(b) Let $A \subseteq \mathbb{R}^n$ be a closed subset. Show that $\mathbb{R}^n/A$ is Hausdorff.

Exercise 8.11. Let $\sim$ be an equivalence relation on a topological space X. Furthermore let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. Show that if $f\colon X \to Y$ is an open map, then the induced map $X/\sim \to Y$ is also an open map.

Exercise 8.12. Let X be a topological space. We denote by $\sim$ the equivalence relation on the product $X \times [0, 1]$ that is generated by $(x, 0) \sim (x, 1)$ for all $x \in X$. Show that $(X \times [0, 1])/\sim$ is homeomorphic to $X \times S^1$.
Remark. It is straightforward to write down a continuous bijection $(X \times [0, 1])/\sim \to X \times S^1$. But you still need to show that the inverse is also continuous.

Exercise 8.13. Show that the topological space $\mathbb{R}/\{\pm 1\} = \mathbb{R}/x \sim -x$ is homeomorphic to the half-open interval $[0, \infty)$.

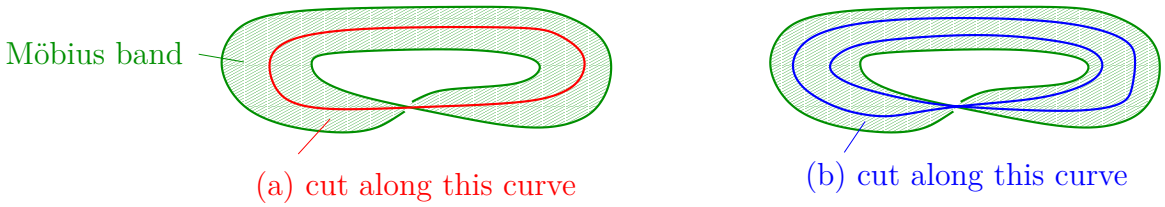
Exercise 8.14. We consider the following subsets of $[0,1]$:

$$A = \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}, \quad B = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \quad \text{and} \quad C = \{0\} \cup B.$$

(a) Show that the quotient $[0,1]/A$ is homeomorphic to a subspace of $\mathbb{R}^2$.
(b) Is the quotient $[0,1]/B$ homeomorphic to a subspace of $\mathbb{R}^2$?
(c) Is the quotient $[0,1]/C$ homeomorphic to a subspace of $\mathbb{R}^2$?

Exercise 8.15. Construct the Möbius band $M = ([0,1] \times [0,1])/(0,y) \sim (1,1-y)$ by gluing two of the opposite sides of a strip of paper with a twist.

(a) Cut M along the central curve, i.e. start cutting at the point $[(0, \frac{1}{2})]$ and continue along the Möbius band till you reach the starting point. What topological space do you obtain that way?
(b) Start cutting at the point $[(0, \frac{1}{3})]$ and continue along the Möbius band till you reach the starting point. What topological space do you obtain that way?
(c) Let $n \in \mathbb{N}$. Start cutting at $[(0, \frac{1}{n})]$ and continue along the Möbius band till you reach the starting point. What topological space do you obtain that way?



Exercise 8.16. Let $T = S^1 \times S^1$ be the torus. Let $\sim$ be the equivalence relation on T generated by $(z,w) \sim (w,z)$. Show that the quotient space $T/\sim$ is homeomorphic to the Möbius band.

Exercise 8.17.

(a) Show that S^3 is homeomorphic to $\mathrm{SU}(2)$.
(b) Show that the Möbius band is homeomorphic to $\mathrm{U}(2)/\sim$ where $A \sim B$ if A and B are conjugate in $\mathrm{U}(2)$. (Bonus question: Which matrices correspond to the boundary of the Möbius band?)
Remark. Exercise 8.16 might be useful.

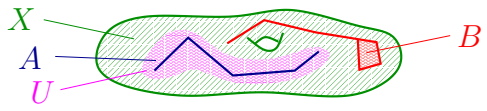
Exercise 8.18. Let A be the annulus and let M be the Möbius band. Are the product topological spaces $A \times [0,1]$ and $M \times [0,1]$ homeomorphic?

Exercise 8.19. Let X be a topological space and let A and B be two disjoint subsets. We equip A with the subspace topology and we equip X/B with the quotient topology. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/B$ the projection.

(a) Show that the map $p \circ i\colon A \to X/B$ is continuous and injective.
(b) Show that the map $p \circ i\colon A \to X/B$ is not necessarily an embedding.
(c) Suppose that there exists an open subset $U \subseteq X$ with $A \subseteq U$ and $U \cap B = \varnothing$. Show that the map $p \circ i\colon A \to X/B$ is an embedding.

Exercise 8.20. Let X be a topological space and let A be a closed subset of X.

(a) Show that for every open subset $U \subseteq X \setminus A$ the obvious map $U \to X/A$ is an open embedding.



(b) Show that for every open neighborhood W of A the map $W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism.
(c) Show that in (a) and (b) we cannot drop the hypothesis that A is closed.



Exercise 8.21. Similar to the description of the Möbius band in the Möbius Band-as-Quotient Lemma 9.6 we consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times \{-1, 1\}$ that is given by

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times \{-1,1\}) & \to & \mathbb{R} \times \{-1,1\} \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

Show that $X/G = X/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 8.22. Give an example of continuous maps $f\colon X \to Y$ and $X \to Z$ between infinite topological spaces such that the pushout $Y \cup_X Z$ consists of a single point.

Exercise 8.23. Let X be a topological space and let A and B be two subsets with $A \cup B = X$. We write $C \coloneqq A \cap B$. We consider the disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ and we consider $Y = (A \sqcup B)/\sim$ where for each $c \in C$ we identify the point $(c, 1) \in A \times \{1\}$ with the corresponding point $(c, 2) \in B \times \{2\}$. We consider the map

$$p\colon Y = (A \sqcup B)/\sim \quad \to \quad X$$

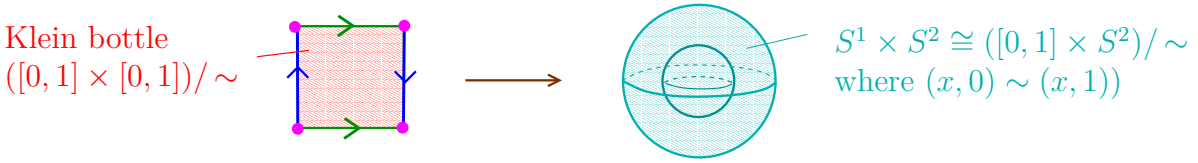
that is given by

$$\begin{array}{rcll} [(a,1)] & \mapsto & a & \text{for } a \in A, \\ [(b,2)] & \mapsto & b & \text{for } b \in B. \end{array}$$

(a) Is the map p necessarily continuous?
(b) Is the map p necessarily open?

Remark. In the Regluing Lemma 8.17 we saw that p is a homeomorphism if both A and B are open subsets or if both are closed subsets of X. The fact that p is not necessarily a homeomorphism can lead to nasty surprises.

Exercise 8.24. Let $K = ([0,1] \times [0,1])/\sim$ be the Klein bottle. Show that there exists an embedding $f\colon K \to S^1 \times S^2$.
Remark. You can use the fact that $S^1 \times S^2$ is homeomorphic to $([0,1] \times S^2)/(0,x) \sim (1,x)$ and that we can think of $[0,1] \times S^2$ as a subspace of $\mathbb{R}^3$.



Exercise 8.25. Let $f\colon A \to Y$ and $g\colon A \to Z$ be continuous maps between topological spaces and let $f'\colon A' \to Y'$ and $g'\colon A' \to Z'$ also be continuous maps between topological spaces. Suppose that we are given continuous maps $\alpha\colon A \to A'$, $\mu\colon Y \to Y'$ and $\nu\colon Z \to Z'$ such that the obvious diagrams commute.

(a) Show that there exists a continuous map $Y \cup_A Z \to Y' \cup_{A'} Z'$ which makes all the obvious diagrams commute.

(b) Show that if α, μ, ν are all embeddings, then so is the map in (a).

Remark. This part is related to Exercise 8.6.

Exercise 8.26. We consider the following diagram:

$$\begin{array}{ccc} \mathbb{R} \setminus \{0\} & \hookrightarrow & \mathbb{R} \\ \downarrow & & \\ \mathbb{R} & & \end{array}$$

Show that the corresponding pushout is homeomorphic to the line with two zeros.

CHAPTER 9

Group actions on topological spaces

In Chapter 8, given a topological space X and given an equivalence relation $\sim$ on X we introduced the quotient topology on the set $X/\sim$ of equivalence classes. In this chapter we will study equivalence relations that arise from group actions. This construction gets used frequently throughout these notes. In this chapter we will use this construction to introduce the real and complex projective spaces.

This chapter is organized as follows:

- In Section 9.1 we will study actions of groups on topological spaces. These give rise to equivalence relations and thus to quotient spaces.
- In Section 9.2 we will prove the Group Action-Hausdorff Quotient Proposition 9.5 which gives a criterion for a quotient space to be Hausdorff.
- In Section 9.3 we will introduce the real projective spaces $\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$.
- In Section 9.4 we will use the quotient topology to introduce the (non-) orientable surfaces of any genus.
- In Appendix 9.5 we will use the quotient topology to show that the zero sets of complex polynomials are, in a suitable sense, continuous in the coefficients.
- In Section 9.6 we will provide exercises for Chapter 9.

In the next Chapter 10 we will discuss the topology of metric spaces in greater detail.

9.1. Group actions on topological spaces.

We recall Definition 2.36 and Definition 2.39.

Definition 9.1. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties:

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (g \cdot_G h) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

($\cdot$: group action; $\cdot_G$: multiplication in the group G)

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

(3) If X is a topological space, then we say that the action is continuous, if for every $g \in G$ the map[135]

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g \cdot x \end{array}$$

is continuous.

(4) Let X be a topological space and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 2.38 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We set $X/G := X/\sim_G$ and we view X/G as a topological space equipped with the quotient topology.

We formulate our first example as a lemma:

Lemma 9.2. (Torus-as-Quotient Lemma)

(1) The map

$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$

defines a free and continuous action of the group $(\mathbb{Z}^n, +)$ on the topological space $X = \mathbb{R}^n$.

(2) The map

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1, \ldots, t_n)] & \mapsto & \big(\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)\big) \end{array}$$

is a homeomorphism.

Identification 9.3. We use the homeomorphism of the Torus-as-Quotient Lemma 9.2 to make the identification $\mathbb{R}^n/\mathbb{Z}^n =_{\mathrm{i}} (S^1)^n$.

Proof of Lemma 9.2. We start out with the following observations:

(1) This statement follows easily from the definitions.

(2) (a) The map is continuous by the Topological-Product Proposition 7.2 (1b) and the Topological-Quotient Proposition 8.2 (1b). Furthermore it is straightforward to see that it is a bijection.

(b) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By the Hyperrectangle-Compact Proposition 3.47 we know that $[0,1]^n$ is compact. It follows from the Compact Image Lemma 4.15 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

(c) From the Product Topology-Properties Proposition 7.9 (3) we obtain that $(S^1)^n$ is Hausdorff.

It follows from the above together with the Compact-Hausdorff Proposition 4.23 (3) that the given map $\mathbb{R}^n/\mathbb{Z}^n \to (S^1)^n$ is indeed a homeomorphism. ■

Example. If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that the quotient space X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$. It is a moderately insightful exercise, see Exercise 9.2, to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff. □

This example leads us to the following definition:

Definition 9.4. Let X be a topological space and let G be a group that acts on X. We say that G acts properly if for every two points[136] x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.[137]

[135] At times we will implicitly use Convention 2.40 which says that we always assume that an action preserves the given structure. In our context Convention 2.40 says that implicitly we always assume that actions on topological spaces are given by continuous maps.

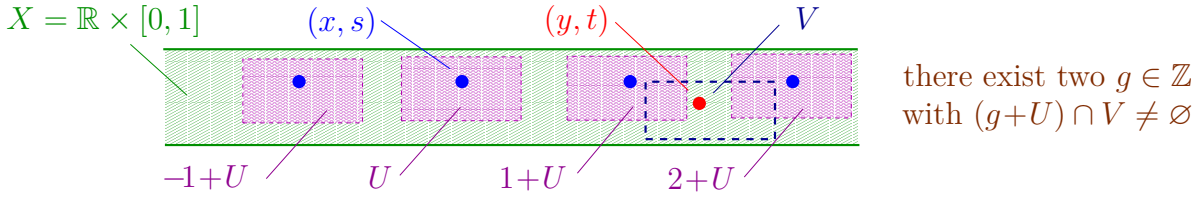
Examples.

(1) The action of any finite group is proper by definition.

(2) We consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times [0,1]$ that is given by

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [0,1]) & \to & \mathbb{R} \times [0,1] \\ (n,(x,y)) & \mapsto & (x+n,y). \end{array}$$

This action is proper. For example if we are given by $(x,s),(y,t) \in \mathbb{R} \times [0,1]$, then $U := \{(a,b) \in \mathbb{R} \times [0,1] \,|\, |a-x| < 1\}$ and $V := \{(a,b) \in \mathbb{R} \times [0,1] \,|\, |a-y| < 1\}$ have the desired properties. With some artistic freedom this situation is shown in the figure below.



(3) Similar to (2) one can easily verify that the action of $G = (\mathbb{Z}^n, +)$ on $X = \mathbb{R}^n$ is proper.

(4) In Exercise 9.2 we will see that the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$ is not proper. □

Proposition 9.5. (Group Action-Hausdorff Quotient Proposition) Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.

Proof. We already saw in the proof of the Quotient Topology-Properties Proposition 8.5 (3) that it can be quite painful, and unenlightening, to prove that a quotient space is Hausdorff. Therefore we outsource the proof of the Group Action-Hausdorff Quotient Proposition 9.5 to the next section. ■

Instead of dealing with the proof of the above proposition, let us look at an example:

Lemma 9.6. (Möbius Band-as-Quotient Lemma)

(1) The map

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n,(-1)^n \cdot y) \end{array}$$

defines an action of the group $(\mathbb{Z},+)$ on the topological space $X = \mathbb{R} \times [-1,1]$ that is free, continuous and proper.

(2) As on page 310 let $([0,1] \times [0,1])/\sim$ be the Möbius band, where $(0,y) \sim (1,1-y)$. The map

$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & (\mathbb{R} \times [-1,1])/\sim \\ [(x,y)] & \mapsto & [(x,2y-1)] \end{array}$$

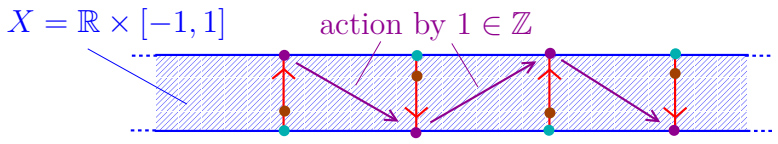
is a homeomorphism.

(3) The Möbius band $M = ([0,1] \times [0,1])/\sim$ is compact and Hausdorff.

Identification 9.7. We will use the homeomorphism from the Möbius Band-as-Quotient Lemma 9.6 to identify the Möbius band $([0,1] \times [0,1])/\sim$ with $(\mathbb{R} \times [-1,1])/\sim$.

[136] Note that we allow $x = y$.

[137] The notion of a "proper action" is defined in different ways by different authors.



Proof of Lemma 9.6.

(1) It is clear that the action is free and continuous. The fact that the action is proper is proved as in the above example.
(2) It follows from the Quotient-to-Quotient Lemma 8.6 that the map is well-defined and continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 8.5 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. Furthermore it follows from (1) and the Group Action-Hausdorff Quotient Proposition 9.5 that $(\mathbb{R} \times [0,1])/\sim$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map $([0,1] \times [0,1])/\sim \to (\mathbb{R} \times [0,1])/\sim$ is a homeomorphism.
(3) We showed in the proof of (2) that the Möbius band is compact and Hausdorff. ■

9.2. Proof of the Group Action-Hausdorff Quotient Proposition 9.5.

This section is totally and completely dedicated to the proof of the Group Action-Hausdorff Quotient Proposition 9.5. The proof relies on the following three lemmas.

Lemma 9.8. (Group Action-Projection Lemma) Let X be a topological space and let G be a group that acts continuously on X.

(1) The projection p is open.
(2) If $A \subseteq X$ is a closed subset with $G \cdot A = A$, then $p(A)$ is a closed subset of X/G.

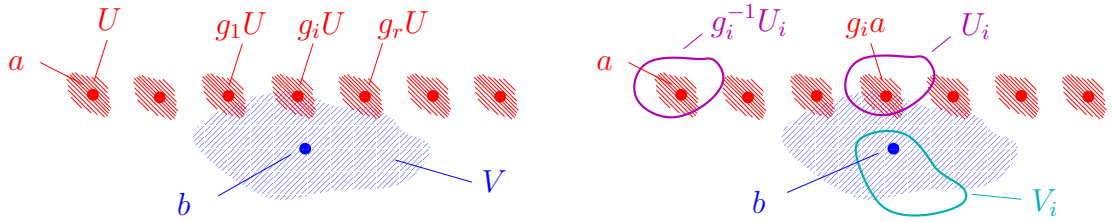
Remark. Note that even if G acts continuously on a topological space, it is not true in general that the projection $p\colon X \to X/G$ is closed. For example, let $X = \mathbb{R}$ and $G = \mathbb{Z}$ with the additive action on $\mathbb{R}$. The set $A \coloneqq \{n + \frac{1}{n} \,|\, n \in \mathbb{Z}\}$ is a closed subset of $X = \mathbb{R}$, but using the Topological-Quotient Proposition 8.2 one can easily show that $p(A)$ is not a closed subset of $X/G = \mathbb{R}/\mathbb{Z}$. □

Proof.

(1) Let $U \subseteq X$ be an open subset. We need to show that $p(U)$ is open in X/G. Thus we need to show that $p^{-1}(p(U))$ is an open subset of X. It follows easily from the definitions that
$$p^{-1}(p(U)) \;=\; \bigcup_{g \in G} g \cdot U.$$
Since G acts continuously the map $x \mapsto g \cdot x$ is a homeomorphism. Thus we see that $p^{-1}(p(U))$ is the union of open subsets, thus it is open itself.
(2) Let $A \subseteq X$ be a closed subset with $G \cdot A = A$. We see that
$$(X/G) \setminus p(A) \underset{\substack{\uparrow \\ \text{by the Image-Preimage Lemma 1.9 (5), since } G \cdot A = A \text{ implies that } p^{-1}(p(A)) = A}}{=} p(X \setminus A) \underset{\substack{\uparrow \\ \text{since } A \text{ closed}}}{=} p(\text{open subset of } X) \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \text{open subset of } X/G.$$
But this means that $p(A)$ is indeed a closed subset of X/G. ■

Lemma 9.9. (Disjoint Translated Neighborhoods Lemma) Let G be a group that acts continuously and properly on a topological space. If X is Hausdorff, then given any two points a and b in X there exist open neighborhoods A of a and B of b with the following property: For every $g \in G$ with $ga \neq b$ we also have $gA \cap B = \varnothing$.

Proof. Let G be a group that acts continuously and properly on a topological space X that is Hausdorff. Let a and b be two points in X. Since G acts properly on X there exist open neighborhoods U of a and V of b such that $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is a finite set. We denote the elements of this finite set by $g_1, \ldots, g_r$. We set



$$I \;:=\; \{i = 1, \ldots, r \,|\, g_i a \neq b\}.$$

Since X is Hausdorff we can find for every $i \in I$ an open neighborhood U_i of $g_i a$ and an open neighborhood V_i of b such that $U_i \cap V_i = \varnothing$. We set

$$A \;:=\; U \cap \bigcap_{i \in I} g_i^{-1} U_i \quad \text{and} \quad B \;:=\; V \cap \bigcap_{i \in I} V_i.$$

The sets A and B are the intersection of finitely many open subsets,[138] thus they are open. In particular A is an open neighborhood of a and B is an open neighborhood of b.

Now one can easily verify that A and B have the desired properties. Indeed, let $g \in G$ with $ga \neq b$. If $g \notin \{g_1, \ldots, g_r\}$, then $gU \cap V = \varnothing$, therefore $gA \cap B = \varnothing$. On the other hand, if $g \in \{g_1, \ldots, g_r\}$, then there exists an $i \in I$ with $g = g_i$. It follows that $g_i(g_i^{-1}U_i) \cap V_i = U_i \cap V_i = \varnothing$, hence $g_i A \cap B = \varnothing$. ∎

Lemma 9.10. (Disjoint Quotient Lemma) Let G be a group that acts on a set X. We denote by $p\colon X \to X/G$ the projection. Let A and B be subsets of X. Then

$$p(A) \cap p(B) \;=\; \varnothing \iff \text{for every } g \in G \text{ we have } gA \cap B = \varnothing,$$

or equivalently:

$$p(A) \cap p(B) \;\neq\; \varnothing \iff \text{there exists a } g \in G \text{ with } gA \cap B \neq \varnothing.$$

Proof. We prove Statement (2). Statement (1) is easily seen to be equivalent to Statement (2). Note that

$$\begin{aligned} p(A) \cap p(B) \;\neq\; \varnothing &\iff \text{there exist } a \in A \text{ and } b \in B \text{ with } p(a) = p(b) \in X/G \\ &\iff \text{there exist } a \in A,\ b \in B \text{ and } g \in G \text{ with } ga = b \\ &\iff \text{there exists a } g \in G \text{ with } gA \cap B \neq \varnothing. \end{aligned}$$

∎

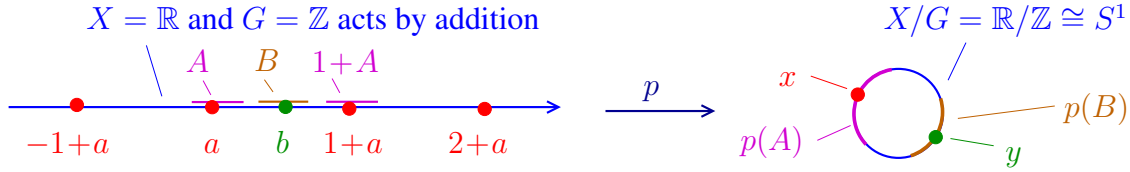
We turn to the actual proof of the Group Action-Hausdorff Quotient Proposition 9.5.

Proof of the Group Action-Hausdorff Quotient Proposition 9.5. Let G be a group that acts on a topological space X. We assume that X is Hausdorff and that the action is

[138] Here we use that G acts continuously, since this implies that the sets $h^{-1} \cdot U_i$ are also open.

proper and continuous. We want to show that the quotient space X/G is also Hausdorff. We denote by $p\colon X \to X/G$ the projection.

Let x and y be two distinct points in X/G. We pick a and b in X with $p(a) = x$ and $p(b) = y$. Since $p(a) = x \neq y = p(b)$ we see that for every $g \in G$ we have $ga \neq b$. By the Disjoint Translated Neighborhoods Lemma 9.9 there exist open neighborhoods A of a and B of b such that $gA \cap B = \varnothing$ for every $g \in G$. It follows from the Disjoint Quotient Lemma 9.10 that $p(A)$ and $p(B)$ are disjoint.



Since A and B are open it follows from the Group Action-Projection Lemma 9.8 (1) that $p(A)$ and $p(B)$ are open subsets of X/G, in particular they are open neighborhoods of $x = p(a)$ and $y = p(b)$. We have thus found the desired disjoint open neighborhoods of x and y in X/G. ■

9.3. **Projective spaces.** In this section we introduce two related families of topological spaces, namely the real and complex projective spaces, which initially might look artificial, but which will play an essential role throughout these notes.

Definition 9.11.

(1) The map
$$\begin{array}{rcl}(\mathbb{R}\setminus\{0\})\times(\mathbb{R}^{n+1}\setminus\{0\}) & \to & \mathbb{R}^{n+1}\setminus\{0\}\\ (z,P) & \mapsto & z\cdot P\end{array}$$
defines a continuous action of the multiplicative group $(\mathbb{R}\setminus\{0\},\cdot)$ on the topological space $\mathbb{R}^{n+1}\setminus\{0\}$. The quotient space $(\mathbb{R}^{n+1}\setminus\{0\})/(\mathbb{R}\setminus\{0\})$ is called the *n-dimensional real projective space* $\mathbb{R}\mathrm{P}^n$.[139] For $n = 2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the *real projective plane*.

(2) In (1) we can replace "$\mathbb{R}$" by "$\mathbb{C}$" and we obtain the *n-dimensional complex projective space* $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1}\setminus\{0\})/(\mathbb{C}\setminus\{0\})$.[140]

Remark.

(1) Informally speaking we can view $\mathbb{R}\mathrm{P}^n$ as the set of lines in $\mathbb{R}^{n+1}$ through the origin, or equivalently as the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$.

(2) For fans of projective spaces we will introduce in Definition 215.2 the quaternionic projective spaces and in Definition 215.6 we will introduce the octonionic projective plane. □

In our treatments of real and complex projective spaces we will often use the following notation:

[138] In the Projective Space–Manifold Proposition 25.13 we will see that $\mathbb{R}\mathrm{P}^n$ is an n-dimensional "smooth manifold".

[139] In the Projective Space–Manifold Proposition 25.13 we will see that $\mathbb{C}\mathrm{P}^n$ is a $2n$-dimensional "smooth manifold" and in the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we will see that $\mathbb{C}\mathrm{P}^n$ is an n-dimensional "complex manifold".

Notation 9.12. For $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \backslash \{0\}$ we denote by $[x_0 : \ldots : x_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$. We also use the same notation for points in complex projective spaces.

The following lemma gives us a convenient alternative description of projective spaces.

Lemma 9.13. (Projective Space-Sphere Quotient Lemma) Let $n \in \mathbb{N}_0$.

(1) The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (w, (x_1, \ldots, x_{n+1})) & \mapsto & (w \cdot x_1, \ldots, w \cdot x_{n+1}) \end{array}$$

defines a continuous action of the group $\{\pm 1\}$ on S^n. The maps

$$\begin{array}{rcl} S^n/\{\pm 1\} & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [x] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}\mathrm{P}^n & \to & S^n/\{\pm 1\} \\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right] \end{array}$$

are inverses of one another and they are homeomorphisms.

(2) The map

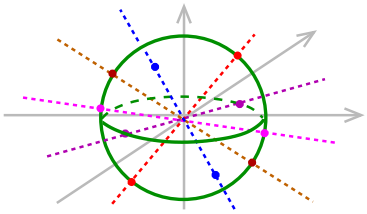
$$\begin{array}{rcl} S^1 \times \overbrace{S^{2n+1}}^{\subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}} & \to & S^{2n+1} \\ (w, (z_1, \ldots, z_{n+1})) & \mapsto & (w \cdot z_1, \ldots, w \cdot z_{n+1}) \end{array}$$

defines a continuous action of the group $(S^1, \cdot)$ on S^{2n+1}. The maps

$$\begin{array}{rcl} S^{2n+1}/S^1 & \to & \mathbb{C}\mathrm{P}^n \\ [z] & \mapsto & [z] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{C}\mathrm{P}^n & \to & S^{2n+1}/S^1 \\ [z_1 : \ldots : z_{n+1}] & \mapsto & \left[\frac{(z_1, \ldots, z_{n+1})}{\sqrt{|z_1|^2 + \ldots + |z_{n+1}|^2}}\right] \end{array}$$

are inverses of one another and they are homeomorphisms.

$S^2/\{\pm 1\}$ is the "set of pairs of antipodal points on S^2"



$\mathbb{R}\mathrm{P}^2 = (\mathbb{R}^3 \setminus \{0\}/(\mathbb{R} \setminus \{0\})$ is the "set of lines through 0"

Identification 9.14. We use the homeomorphisms from the Projective Space-Sphere Quotient Lemma 9.13 to make the identifications $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ and $\mathbb{C}\mathrm{P}^n =_{\mathrm{i}} S^{2n+1}/S^1$.

Proof of Lemma 9.13. It follows easily from the Quotient-to-Quotient Lemma 8.6 that all four maps are continuous. It is easy to verify that in (1) and (2) the maps are inverses of one another. ∎

The following lemma shows that real and complex projective spaces have some of our favorite properties.

Proposition 9.15. (Projective Spaces-Compact+Hausdorff Proposition) The real projective and the complex projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ are compact, path-connected and Hausdorff.

Example. Let $n \in \mathbb{N}_0$ and let $m \leq n$. We consider the maps

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 : \ldots : x_m] & \mapsto & [x_0 : \ldots : x_m : 0 : \ldots : 0] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \ldots : z_m] & \mapsto & [z_0 : \ldots : z_m : 0 : \ldots : 0]. \end{array}$$

It follows from the Quotient-to-Quotient Lemma 8.6, the Projective Spaces-Compact-+Hausdorff Proposition 9.15 and the Compact-Hausdorff Proposition 4.23 (2) that both

maps are embeddings. Sometimes we will use these maps to view $\mathbb{R}\mathrm{P}^m$ as a subspace of $\mathbb{R}\mathrm{P}^n$ and similarly we will view $\mathbb{C}\mathrm{P}^m$ as a subspace of $\mathbb{C}\mathrm{P}^n$. □

Example. For $n = 1$ we obtain homeomorphisms

$$\begin{array}{ccccc} & \overset{\text{homeomorphism from the Quotient Lemma 9.13}}{\downarrow \cong} & & \overset{\text{homeomorphism Compact-Hausdorff Proposition 4.23 (3)}}{\downarrow \cong} & \\ \mathbb{R}\mathrm{P}^1 & \longrightarrow & S^1/\{\pm 1\} & \longrightarrow & S^1 \\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right] & & \\ & & z & \mapsto & z^2. \end{array}$$

□

Proof. Let $n \in \mathbb{N}_0$. By the Projective Space-Sphere Quotient Lemma 9.13 it suffices to show that $S^n/\{\pm 1\}$ and S^{2n+1}/S^1 are compact, path-connected and Hausdorff. In fact it follows immediately from the fact that S^n and S^{2n+1} are compact and path-connected together with the Quotient Topology-Properties Proposition 8.5 (1) and (2) that both quotients are compact and path-connected. Furthermore note that it follows from the Group Action-Hausdorff Quotient Proposition 9.5 that $S^n/\{\pm 1\}$ is Hausdorff. It remains to show that $\mathbb{C}\mathrm{P}^n =_{\mathrm{i}} S^{2n+1}/S^1$ is Hausdorff. It follows from the Quotient Topology-Properties Proposition 8.5 (3) that S^{2n+1}/S^1 is Hausdorff once we have shown that the equivalence relation is closed. Thus we need to show that

$$G \;=\; \{(v_1, v_2) \in S^{2n+1} \times S^{2n+1} \,|\, \text{there exists } w \in S^1 \text{ with } w \cdot v_1 = v_2\}$$

is a closed subset of $S^{2n+1} \times S^{2n+1}$. Note that G is the image of the map

$$\begin{array}{rccc} \Phi\colon & S^1 \times S^{2n+1} & \to & S^{2n+1} \times S^{2n+1} \\ & (z, v) & \mapsto & (v, \underbrace{z \cdot v}_{\substack{\text{above action of} \\ S^1 \text{ on } S^{2n+1}}}). \end{array}$$

This map is continuous and the left-hand side is compact by the Product Topology-Properties Proposition 7.9 (1). Thus the image of Φ is compact by the Compact Image Lemma 4.15. By the Product Topology-Properties Proposition 7.9 (3) we know that $S^{2n+1} \times S^{2n+1}$ is Hausdorff. Thus we obtain from the Compact-Closed Lemma 3.39 that the image of Φ is a closed subset of $S^{2n+1} \times S^{2n+1}$.[140] ■

In the following lemma we give alternative descriptions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^1$ that will be used on many occasions. For technical reasons the lemma is split into three parts.

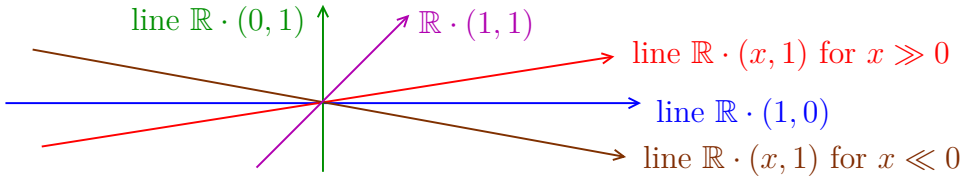
Lemma 9.16. (Projective Space-Avatar Lemma)

(1) The maps

$$\overbrace{\mathbb{C} \cup \{\infty\}}^{=_{\mathrm{i}}\, S^2, \text{ see Identification 4.25}} \to \mathbb{C}\mathrm{P}^1, \quad z \mapsto \begin{cases} [z:1], & \text{if } z \in \mathbb{C}, \\ [1:0], & \text{if } z = \infty \end{cases} \qquad \& \qquad \mathbb{C}\mathrm{P}^1 \to \overbrace{\mathbb{C} \cup \{\infty\}}^{=_{\mathrm{i}}\, S^2, \text{ see Identification 4.25}}, \quad [z_0 : z_1] \mapsto \begin{cases} z_1 \cdot z_0^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1:0] \end{cases}$$

[140] In Exercise 9.8 we provide an alternative proof that $\mathbb{C}\mathrm{P}^n$ is Hausdorff.

are inverses of one another and both are homeomorphisms.[141] Similarly we see that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to $\mathbb{R} \cup \{\infty\}$, which in turn is homeomorphic to S^1. [142]



Lemma 9.16. (Projective Space-Avatar Lemma)

(2) Let $n \in \mathbb{N}_0$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P \sim -P$ for $P \in S^{n-1} = \partial\overline{B}^n$, then the maps[143]

$$\begin{array}{rcl} \overline{B}^n/\!\sim & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [(x, \sqrt{1-\|x\|^2})] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\!\sim \\ [\underbrace{(v, x_{n+1})}_{v\in\mathbb{R}^n,\, x_{n+1}\in\mathbb{R}}] & \mapsto & \begin{cases} [v], & \text{if } x_{n+1} \geq 0, \\ [-v], & \text{if } x_{n+1} \leq 0 \end{cases} \end{array}$$

are homeomorphisms that are inverses of one another.

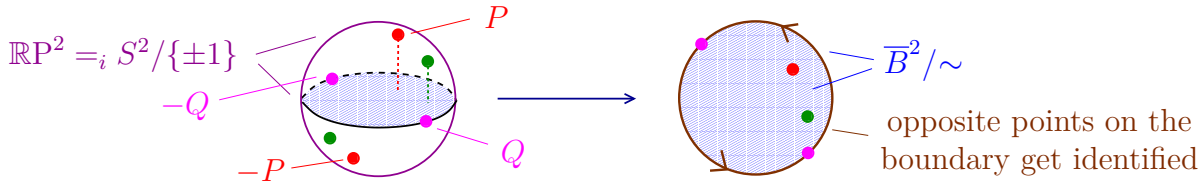


Illustration of the homeomorphism $\mathbb{R}\mathrm{P}^2 \to \overline{B}^2/\!\sim$

Lemma 9.16. (Projective Space-Avatar Lemma)

(3) As in (2) let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$. The map

$$\begin{array}{rcl} F\colon \overline{B}^3/\!\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r\in[0,1] \text{ and } v\in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi \cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0} \cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. In particular SO(3) is homeomorphic to $\mathbb{R}\mathrm{P}^3$.[144]

Identification 9.17.

(1) We will use the homeomorphism from the Projective Space-Avatar Lemma 9.16 (1) to make the identification $S^1 =_{\mathrm{i}} \mathbb{R}\mathrm{P}^1$.

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[141] In Exercise 39.10 we will give an alternative proof that S^2 is homeomorphic to $\mathbb{C}\mathrm{P}^1$.

[142] In the Projective Space–Manifold Proposition 25.13 and the Standard Smooth Structures Proposition 25.6 we will equip the projective spaces and the spheres with smooth structures. In Exercise 25.11 we will then show that $\mathbb{R}\mathrm{P}^1$ is actually diffeomorphic to S^1 and we will show that $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to S^2.

[143] Note that the map to the left is in fact well-defined even if $x_{n+1} = 0$.

[144] In the SU(2)-SO(3)-Covering Theorem 94.3 we will give an alternative, arguably more insightful prove that SO(3) is homeomorphic to $\mathbb{R}\mathrm{P}^3$.

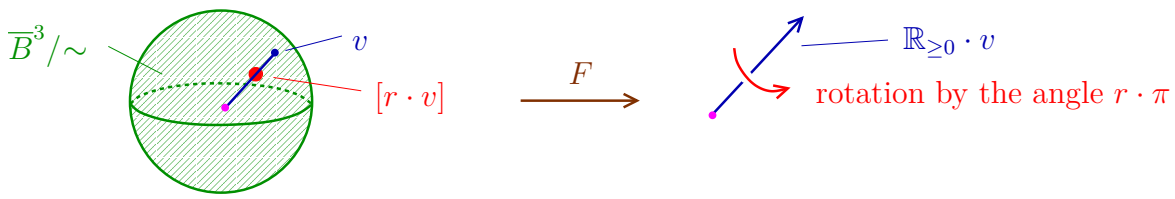


Illustration of the homeomorphism $\overline{B}^3/\sim \to \mathrm{SO}(3)$

(2) Recall that in the Stereographic Projection Lemma 4.24 (1) we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Combining this homeomorphism with the homeomorphism from the Projective Space-Avatar Lemma 9.16 (1) we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we use to make the identification $S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$. Note that under this identification the North Pole $(0,0,1) \in S^2$ corresponds to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.

(3) We will use the homeomorphism from the Projective Space-Avatar Lemma 9.16 (2) to make the identification $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} \overline{B}^n/\sim$.

Proof of Lemma 9.16.

(1) One can easily verify that the given maps are inverses of one another. Next we show that the given map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is continuous. First note that it follows from the Topological-Quotient Proposition 8.2 (1b) that it suffices to show that the corresponding map $\Theta\colon \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{C} \cup \{\infty\}$ is continuous. Next note that for any $C \in \mathbb{R}$ we have

$$\Theta^{-1}(\text{complement of } \overline{B}_C^n) = \{(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\} \mid |z_0| > |z_1|\}$$

which is an open subset of $\mathbb{C}^2 \setminus \{(0,0)\}$. It follows quite easily from this discussion together with the Subbasis-Continuity Proposition 4.4 and together with the discussion of the basis of $\mathbb{R}^2 \cup \{\infty\}$ on page 162 that the map Θ is continuous. By the Projective Spaces-Compact+Hausdorff Proposition 9.15 we know that each $\mathbb{C}\mathrm{P}^1$ is compact. Furthermore we know from Exercise 3.15 that $\mathbb{R}^2 \cup \{\infty\}$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is a homeomorphism.

(2) It is elementary to verify that the given continuous maps are inverses of one another, in particular they are bijections. We still need to show that the maps are continuous.

(a) We consider the map $S^n/\{\pm 1\} \to \overline{B}^n/\sim$. By the Topological-Quotient Proposition 8.2 (1b) it suffices to show that the corresponding map $S^n \to \overline{B}^n/\sim$ is continuous. One can easily verify that the restrictions of the map $S^n \to \overline{B}^n/\sim$ to the closed subsets $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are continuous. Thus it follows from the Pasting Proposition 4.7 (2) that the map $S^n \to \overline{B}^n/\sim$ is continuous.

(b) It follows almost immediately from the Quotient-to-Quotient Lemma 8.6 that the given map $\overline{B}^n/\sim \to \mathbb{R}\mathrm{P}^n$ is continuous.

(3) First we provide a more precise definition of the given map $F\colon \overline{B}^3/\sim \to \mathrm{SO}(3)$. Let $v \in S^2$ and let $\varphi \in \mathbb{R}$. We pick $a, b \in S^2$ such that $(v\ a\ b) \in \mathrm{SO}(3)$. We set

$$\rho(v,\varphi) \coloneqq \underbrace{(a\ b\ v)}_{\in \mathrm{SO}(3)} \cdot \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{(a\ b\ v)^{-1}}_{\in \mathrm{SO}(3)} \quad \in \mathrm{SO}(3).$$

In Exercise 9.11 we will show, using the fact that SO(2) is abelian, that this matrix $\rho(v,\varphi)$ does *not* depend on the choice of a and b.

Claim 1. The map $\rho\colon S^2 \times \mathbb{R} \to \mathrm{SO}(3)$ is continuous.

Proof. Given $i \in \{1,2,3\}$ we set $U_i := S^2 \backslash \{\mathfrak{e}_i\}$, where $\mathfrak{e}_i$ is the i-th standard basis vector of $\mathbb{R}^3$. Given $v \in U_i$ we set $\widetilde{a}(v) := \mathfrak{e}_i - \langle \mathfrak{e}_i, v\rangle \cdot \mathfrak{e}_i$. Since $v \neq \mathfrak{e}_i$ we see that $\widetilde{a}(v) \neq 0$, thus we can set $a(v) := \frac{\widetilde{a}(v)}{\|\widetilde{a}(v)\|}$. Finally we define $b(v)$ to be the cross product[145] of v and $a(v)$, i.e. we set $b(v) := v \times a(v)$. It follows from the Gram-Schmidt Orthogonalization Process 2.32 and the Cross Product Lemma 31.26 that $(v\ a(v)\ b(v)) \in \mathrm{SO}(3)$. It should now be clear that the restriction of ρ to $U_i \times \mathbb{R}$ is continuous. Since $U_1 \times \mathbb{R}$, $U_2 \times \mathbb{R}$ and $U_3 \times \mathbb{R}$ together form an open cover of $S^2 \times \mathbb{R}$ we deduce from the Pasting Proposition 4.7 (1) that the map $\rho\colon S^2 \times \mathbb{R} \to \mathrm{SO}(3)$ is continuous. ⊞

In Exercise 9.11 we will see that $\rho(v,\pi) = \rho(-v,\pi)$. It follows from this observation that the map

$$\begin{array}{ccc} \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \rho(v, r\cdot\pi) \end{array}$$

is well-defined. In fact this map is the formalization of the map that we gave in the statement in the lemma. Next note that it follows from the claim and the Cylinder Quotient-Ball Lemma 8.9 (2) that the above map $\overline{B}^3/\sim \to \mathrm{SO}(3)$ is continuous.

Claim 2. The map $\rho\colon S^2 \times \mathbb{R} \to \mathrm{SO}(3)$ is a bijection.

Proof. We provide a sketch of a proof. First recall the following facts from linear algebra:
(1) All eigenvalues of orthogonal matrices lie in S^1.
(2) If $z \in \mathbb{C}$ is an eigenvalue of a real matrix, then so is $\overline{z}$.
(3) The product of the eigenvalues of a matrix A equals the determinant of A.
It follows from these three observations that every matrix $A \in \mathrm{SO}(3)$ has an eigenvector $v \in \mathbb{R}^3$ to the eigenvalue $= 1$. Since multiplication by A preserves the inner product we see that multiplication by A preserves the orthogonal complement $(\mathbb{R}\cdot v)^\perp$ and it is an orientation-preserving isometry of the 2-dimensional vector space $(\mathbb{R}\cdot v)^\perp$. In other words, the restriction to $(\mathbb{R}\cdot v)^\perp$ is a rotation. This implies that multiplication by A is a rotation around the line $\mathbb{R}\cdot v$. We have thus shown that $\rho\colon S^2 \times \mathbb{R} \to \mathrm{SO}(3)$ is surjective.

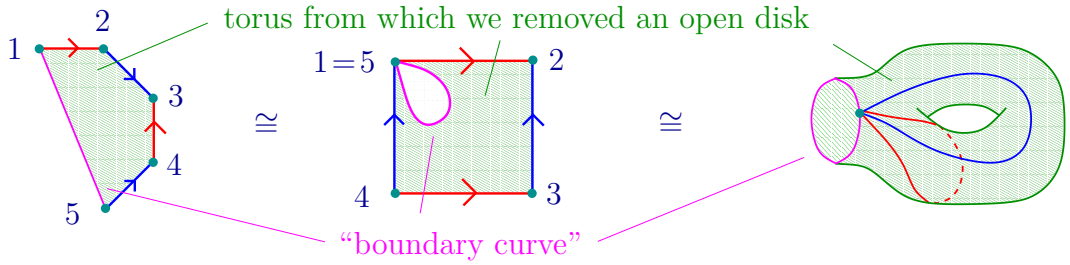
Using eigenvectors and basics about rotations in the plane one can also show that $\rho\colon S^2 \times \mathbb{R} \to \mathrm{SO}(3)$ is injective. We will fill in the details in Exercise 9.11. ⊞

Note that by the Projective Spaces-Compact+Hausdorff Proposition 9.15 we know that $\overline{B}^3/\sim \cong \mathbb{R}\mathrm{P}^3$ is compact. Evidently $\mathrm{SO}(3) \subseteq \mathrm{M}(3\times 3,\mathbb{R}) =_{\mathrm{i}} \mathbb{R}^9$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 4.23 (3) that the given map $\overline{B}^3/\sim \to \mathrm{SO}(3)$ is a homeomorphism. ■

9.4. **Examples: Surfaces II.** In Section 8.2 we already introduced some surfaces. It is time to introduce a few more.

[145] In Definition 31.25 we will recall the cross product from linear algebra.

Example. We start out with a slightly informal discussion of an example. In the figure below we see to the left a pentagon, where we identify two pairs of sides.



In the figure we illustrate that this topological space is homeomorphic to the torus minus an open disk. □

The above example motivates our next example:

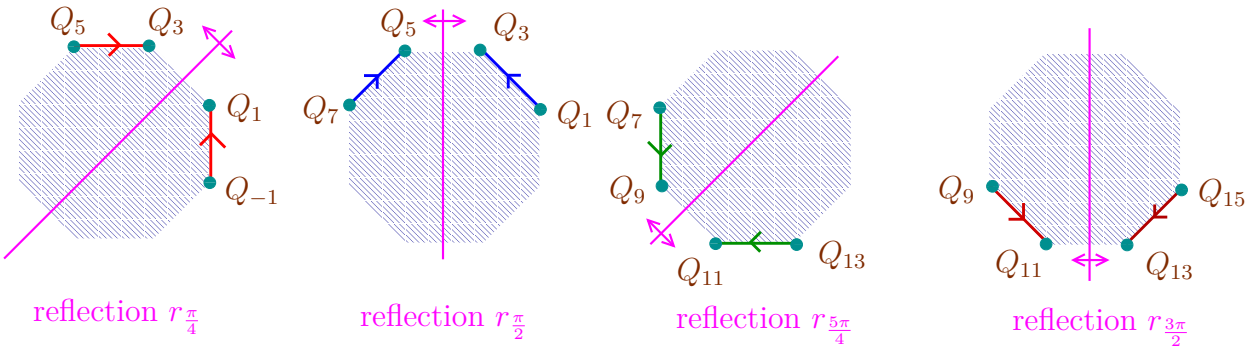
Example. We introduce the following notation:

(1) Let E_8 be the octagon in $\mathbb{C}$ with the vertices $Q_k = \exp(2\pi \mathrm{i} k/16)$, $k = 1, 3, 5, 7, \ldots, 15$, i.e. E_8 is the convex hull of these eight points in $\mathbb{R}^2$.
(2) Given two points $A, B \in \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ let $\overline{AB}$ be the Euclidean segment from A to B.
(3) Given $\varphi \in \mathbb{R}$ let $r_\varphi \colon \mathbb{C} \to \mathbb{C}$ be the reflection in the Euclidean line $\{t \cdot \exp(\mathrm{i}\varphi) \,|\, t \in \mathbb{R}\}$.

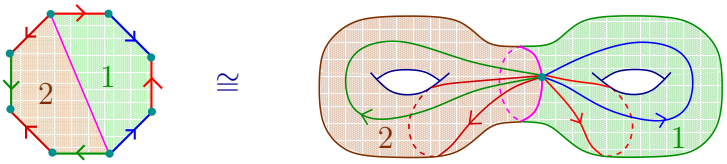
We denote by $\sim$ the equivalence relation on E_8 that is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \;\sim\; r_{2\pi(2k+2)/16}(P) \;\in\; \overline{Q_{2k+3}Q_{2k+5}}$$

for $k = 0, 1, 4, 5$. We consider the quotient space $E_8/\sim$.



Note that $E_8/\sim$ is given by two copies of the previous example glued together along the "boundary curves".



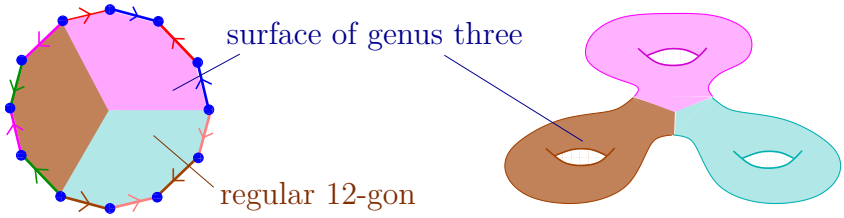
In other words, $E_8/\sim$ is homeomorphic to the "physical" surface of genus 2 in $\mathbb{R}^3$. □

The above example leads us to the following definition:

Definition 9.18.

(1) Let $g \in \mathbb{N}$. We take a regular $4g$-gon E_{4g} in $\mathbb{C}$ and for $j = 0, \ldots, g-1$ we use reflections to identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. We refer to the quotient space $E_{4g}/\sim$ as the surface of genus g.

(2) We refer to the sphere S^2 as the surface of genus 0.



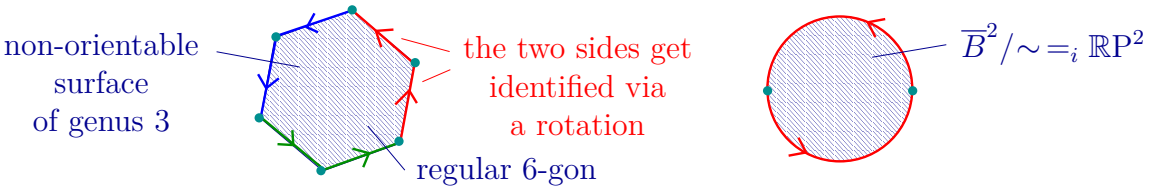
Example. On page 311 we saw that the surface of genus 1 is homeomorphic to the torus $S^1 \times S^1$. □

We move on to a closely related definition.

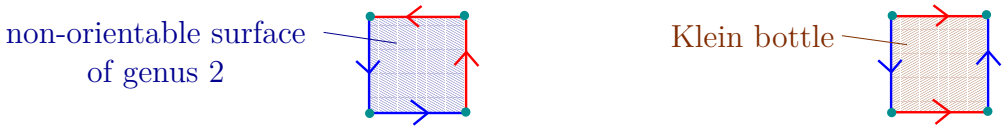
Definition 9.19.

(1) Given $g \in \mathbb{N}_{\geq 2}$ we consider the regular $2g$-gon E_{2g} with the identification of the boundary segments shown in the figure below. We refer to this topological space as the non-orientable surface N_g of genus g.[146] [147]

(2) We refer to the real projective space $\mathbb{R}\mathrm{P}^2$ as the non-orientable surface of genus 1.



Example. In the figure below we illustrate the non-orientable surface of genus 2 and the Klein bottle.



Note that the definitions are similar, but different. In Exercise 9.17 we will see that the non-orientable surface of genus 2 is indeed homeomorphic to the Klein bottle. □

Remark. Later in the Surface-is-Topological Manifold Proposition 23.7 we will see that every (non-) orientable surface of genus g is a closed "2-dimensional topological manifold". Furthermore in the Surface Classification Theorem 101.16 we will see that every closed connected non-empty "2-dimensional topological manifold" is homeomorphic to a (non-) orientable surface of some genus. In that sense the above list of examples is complete. □

[145] Later on in the Surface Non-Orientability Lemma 31.45 we will see why N_g deserves its epithet.

We have now introduced many examples of surface, some of them, for example the Möbius band, the torus and the surface of genus 2, can easily be pictured as subsets of "our universe", i.e. they can be viewed as subsets of $\mathbb{R}^3$. This raises the following question.

Question 9.20. Can the real projective space $\mathbb{R}\mathrm{P}^2$ and the Klein bottle, or more generally the non-orientable surfaces of some genus ≥ 3, be viewed[146] as subsets of $\mathbb{R}^3$?

We will give (partial) answers to this question in the Surface-No Smooth Embedding Corollary 51.18 and in the Surface-No Topological Embedding-Corollary Corollary 218.10.

9.5. **Appendix: Continuity of zeros.** We conclude this long chapter with a slightly peripheral discussion on the zeros of complex polynomials. This discussion shows in particular that group actions can appear in unexpected situations. We start out with the following slightly vague question.

Question 9.21. Let $n \in \mathbb{N}$. Do the zeros of a polynomial of degree n vary continuously with the coefficients of the polynomial?

The first question is, what does it mean that the "zeros of a polynomial" vary "continuously"? What are the topological spaces involved so that we can talk about continuity? The following proposition kills two birds at once: it makes the question precise and it gives a complete answer.

Proposition 9.22. (Zeros Continuity Proposition) Let $n \in \mathbb{N}$. We consider the action by the permutation group S_n on $\mathbb{C}^n$ that is given by permuting the coordinates. The map[147] [148]

$$\begin{array}{rcl} f\colon \mathbb{C}^n & \to & \mathbb{C}^n/S_n \\ (v_0,\dots,v_{n-1}) & \mapsto & \left[\begin{array}{c} \text{set of zeros of the polynomial} \\ v_0 + v_1\cdot z + \cdots + v_{n-1}\cdot z^{n-1} + z^n, \\ \text{counted with multiplicities} \end{array}\right] \end{array}$$

is continuous, in fact it is a homeomorphism.

Remark. For $n = 1, 2, 3, 4$ one can hope to prove the Zeros Continuity Proposition 9.22 by the formulas for determining zeros of polynomials of degree n. But since no such formula exist for general polynomials of higher degree this approach is doomed for $n \geq 5$. □

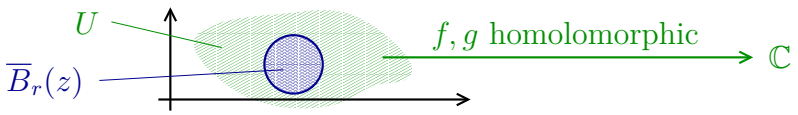
The proof of the Zeros Continuity Proposition 9.22 makes use of Rouché's Theorem from complex analysis.

Theorem 9.23. (Rouché's Theorem) Let $U \subseteq \mathbb{C}$ be an open subset. Furthermore let $f, g\colon U \to \mathbb{C}$ be two holomorphic[149] functions and let $z \in U$ and $r \in \mathbb{R}_{>0}$ be such that $\overline{B}_r(z) = \{w \in \mathbb{C} \,|\, |w - z| \leq r\}$ is contained in U. If for all $w \in \partial\overline{B}_r(z)$ we have the inequality $|g(w)| < |f(w)|$, then f and $f + g$ have the same number of zeros (counted with multiplicities) within the open disk $B_r(z)$.

[146] The word "view" is intentionally vague.

[147] Note that it follows from the Fundamental Theorem of Algebra that this map is actually defined. We will provide three proofs of the Fundamental Theorem of Algebra, namely on Section 32.2, 1751 and on page 2898.

[148] Slightly more rigorously the map f is defined as follows: given $(v_0,\dots,v_{n-1}) \in \mathbb{C}^n$ we define $f(v_0,\dots,v_{n-1}) \coloneqq [(\alpha_1,\dots,\alpha_n)]$ where $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ are determined uniquely (up to reordering) by the condition that $v_0 + v_1 z + \cdots + v_{n-1}z^{n-1} + z^n = (z - \alpha_1)\cdot\ldots\cdot(z - \alpha_n) \in \mathbb{C}[z]$.



Proof. This theorem is proved in every self-respecting book on complex analysis, see e.g. [**Lang1999**, Theorem VI.1.6]. ■

Example. We consider $f(z) = z^5$ and the disk $\overline{B}_1(0)$. The polynomial $f(z)$ has one zero of multiplicity five in $\overline{B}_1(0)$. For any $a, b \in \mathbb{C}$ with $|a| + |b| < 1$ we have $|az + b| < |z^5|$ for all points $z \in \partial\overline{B}_1(0) = S^1$. Thus by Rouché's Theorem 9.23 we see that for such a, b the polynomial $z^5 + az + b$ also has five zeros (counted with multiplicities) in $\overline{B}_1(0)$. □

We will also need the following definition:

Definition 9.24. Given $w = [(w_1, \dots, w_n)]$ and $z = [(z_1, \dots, z_n)]$ in $\mathbb{C}^n/S_n$ we set

$$d(w, z) \coloneqq \min_{\sigma \in S_n} \max_{1 \le j \le n} \left| w_j - z_{\sigma(j)} \right|.$$

The following lemma summarizes two key statements regarding Definition 9.24.

Lemma 9.25. (Metric-on-$\mathbb{C}^n$-Quotient Lemma)

(1) The map

$$\begin{array}{rcl} d\colon \mathbb{C}^n/S_n \times \mathbb{C}^n/S_n & \to & \mathbb{R}_{\ge 0} \\ (w, z) & \mapsto & d(w, z) \end{array}$$

is a metric on $\mathbb{C}^n/S_n$.

(2) The topology defined by the metric d on $\mathbb{C}^n/S_n$ agrees with the quotient topology on $\mathbb{C}^n/S_n$.

Proof. We hand over the task providing the proof of the lemma to the surely very meticulous reader. ■

Now we are in a position to prove the Zeros Continuity Proposition 9.22.

Proof of the Zeros Continuity Proposition 9.22. Recall that we need to show that the map

$$\begin{array}{rcl} f\colon \mathbb{C}^n & \to & \mathbb{C}^n/S_n \\ (v_0, \dots, v_{n-1}) & \mapsto & \left[\begin{array}{c} \text{set of zeros of the polynomial} \\ v_0 + v_1 \cdot z + \dots + v_{n-1} \cdot z^{n-1} + z^n, \\ \text{counted with multiplicities} \end{array} \right] \end{array}$$

is a homeomorphism. To do so we consider the map[150]

$$\begin{array}{rcl} g\colon \mathbb{C}^n/S_n & \to & \mathbb{C}^n \\ [(\alpha_1, \dots, \alpha_n)] & \mapsto & \text{coefficients of the polynomial } (z - \alpha_1) \cdot \ldots \cdot (z - \alpha_n). \end{array}$$

It is straightforward to check that g is continuous and that f and g are inverses of one another.

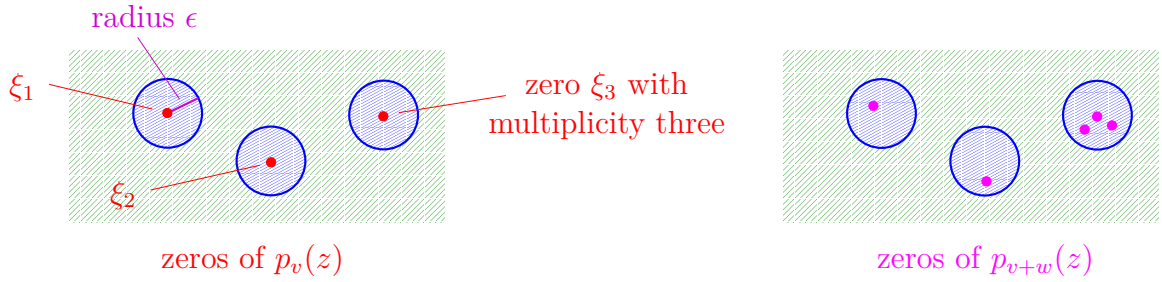
[149] We will introduce the notion of a holomorphic function in Definition 39.3. But at this point it suffices to know that polynomial functions are holomorphic.

[150] Slightly more precisely the map g is defined as follows: Given $[(\alpha_1, \dots, \alpha_n)] \in \mathbb{C}^n/S_n$ we have $f([(\alpha_1, \dots, \alpha_n)]) = (v_0, \dots, v_{n-1})$ where $v_0, \dots, v_{n-1} \in \mathbb{C}$ are uniquely determined by the condition that $v_0 + v_1 z + \dots + v_{n-1} z^{n-1} + z^n = (z - \alpha_1) \cdot \ldots \cdot (z - \alpha_n) \in \mathbb{C}[z]$.

It remains to show that f is continuous. We introduce the following definitions: We denote by d the above metric on $\mathbb{C}^n/S_n$. It follows from the Metric-on-$\mathbb{C}^n$-Quotient Lemma 9.25 that we can view f as a map between the two metric spaces $(\mathbb{C}^n, \|-\|)$ and $(\mathbb{C}^n/S_n, d)$. Therefore it follows from the Metric Continuity Proposition 4.9 that it suffices to prove the following claim:

Claim. $$\forall_{v\in\mathbb{C}^n}\ \forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{\substack{w\in\mathbb{C}^n \text{ with}\\ \|w\|<\delta}}\quad d(f(v), f(v+w)) < \epsilon.$$

Proof. Before we start with the proof of the claim we introduce the following definition: Given $v = (v_0, \dots, v_{n-1}) \in \mathbb{C}^n$ we denote by $p_v(z) \coloneqq v_0 + v_1 \cdot z + \cdots + v_{n-1} \cdot z^{n-1} + z^n$ the corresponding polynomial. Now we turn to the actual proof of the claim. Thus let $v = (v_0, \dots, v_{n-1}) \in \mathbb{C}^n$ and let $\epsilon > 0$. We denote by $\xi_1, \dots, \xi_k$ the *distinct* zeros of the corresponding polynomial $p_v(z)$. (Note that if $p_v(z)$ has zeros of higher multiplicity, then $k < n$.) After possibly replacing ϵ by a smaller non-negative number we can assume, without loss of generality, that $\epsilon < \frac{1}{2} \cdot |\xi_i - \xi_j|$ for all $i \neq j$.



We denote by Γ the union of the circles of the radius ϵ around $\xi_1, \dots, \xi_k$. Furthermore we set

$$\eta \coloneqq \inf_{z\in\Gamma} |p_v(z)| \in \mathbb{R}_{>0}.$$

It is elementary to show that there exists a $\delta > 0$ such that for all $w = (w_0, \dots, w_{n-1}) \in \mathbb{C}^n$ such that $\|w\| < \delta$ we have

$$|w_0 + w_1 \cdot z + \cdots + w_{n-1} \cdot z^{n-1}| < \eta \quad \text{for all } z \in \Gamma.$$

We claim that this δ has the desired property. Thus let $w \in \mathbb{C}^n$ with $|w| < \delta$. It follows from the definition of δ and Rouché's Theorem 9.23 that the polynomials $p_v(z)$ and $p_{v+w}(z)$ have the same number of zeros (counted with multiplicities) in the disks $B_\epsilon(\xi_1), \dots, B_\epsilon(\xi_k)$. This means that the zeros of $p_v(z)$ and can be matched up with the zeros of $p_{v+w}(z)$ such that each zero is less then ϵ removed from its match. This statement implies immediately that $d(f(v), f(v+w)) < \epsilon$. ∎

Remark. With some effort one can also prove the Zeros Continuity Proposition 9.22 without using Rouché's Theorem 9.23. We outline the key steps, using the notation that we introduced in the above proof of the Zeros Continuity Proposition 9.22.

(1) First note that it follows from the Open-Closed–Injective Lemma 4.22 (2) it suffices to show that the map $g\colon \mathbb{C}^n/S_n \to \mathbb{C}^n$ is open.
(2) By the Subbasis-Open Map Lemma 4.14 it suffices to show that for each $\epsilon > 0$ and each $z \in \mathbb{C}^n/S_n$ the image of the open ball $B_\epsilon(z) \subseteq (\mathbb{C}^n/S_n, d)$ under g is open.
(3) Thus let $\epsilon > 0$ and $z \in \mathbb{C}^n/S_n$.

(4) We pick $C \in \mathbb{R}_{\geq 0}$ such that $g(\overline{B}_\epsilon(z)) \subseteq B_C(0) \subseteq \mathbb{C}^n$.
(5) There exists a $D \in \mathbb{R}_{\geq 0}$ such that for every $v \in \mathbb{C}^n$ which satisfies $\|v\| \leq C$ we have $f(v) \in B_D(0) \subseteq \mathbb{C}^n/S_n$. (This statement requires some unraveling and a little bit of thought. It can be proved using Rouché's Theorem 9.23, but with a worse bound it can also be proved by hand.)
(6) Note that by the Compact-Hausdorff Proposition 4.23 (3) we know that g restricted to $\overline{B}_D(0) \to g(\overline{B}_D(0))$ is a homeomorphism. Thus $g(B_\epsilon(z))$ is an open subset of $g(\overline{B}_D(0))$.
(7) By (5) and (6), and since f and g are inverses of one another, we know that $g(B_\epsilon(z))$ is also open in $B_D(0)$. But $B_D(0)$ is an open subset of $\mathbb{C}^n$, hence $g(B_\epsilon(z))$ is also open in $\mathbb{C}^n$. □

9.6. Exercises for Chapter 9.

Exercise 9.1.

(a) Show that any matrix $A \in \mathrm{O}(2n)$ with $\det(A) = -1$ has 1 as an eigenvalue.
Hint. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then so is $\overline{\lambda}$.
(b) Let $G \subseteq \mathrm{O}(2n)$ be a subgroup that acts freely on S^{2n-1}. Show that G is contained in $\mathrm{SO}(2n)$.
(c) Show that the analog of (b) does not hold for subgroups of $G \subseteq \mathrm{O}(2n-1)$.

Exercise 9.2. Let the group $(\mathbb{Q}, +)$ act on the topological space $\mathbb{R}$ by addition.

(a) Show, using only the definitions, that the action of $\mathbb{Q}$ on $\mathbb{R}$ is not proper.
(b) Show that the quotient topology on $\mathbb{R}/\mathbb{Q}$ is actually the trivial topology from page 136.
Remark. This of course shows in particular that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Exercise 9.3. We consider the map

$$\begin{array}{rccc} f\colon & [0,1) & \to & \mathbb{R}/\mathbb{Z}. \\ & x & \mapsto & [x]. \end{array}$$

Give an example of a subset U of $[0,1)$ such that $f(U)$ not open in $\mathbb{R}/\mathbb{Z}$.

Exercise 9.4. Let X be a Hausdorff space which comes with a free and continuous action of $(\mathbb{Z}, +)$ on X. Is the quotient space $X/\sim$ necessarily Hausdorff?

Exercise 9.5. Let $k \in \mathbb{N}$ and let X be a topological space. We refer to

$$\mathrm{Conf}_k(X) \;:=\; \{(x_1, \ldots, x_k) \in X^k \mid \text{ for all } i \neq j \text{ we have } x_i \neq x_j\}$$

as the *configuration space of k points in X*. Note that the permutation group S_k acts in an obvious way on $\mathrm{Conf}_k(X)$. We refer to

$$\overline{\mathrm{Conf}}_k(X) \;:=\; \mathrm{Conf}_k(X)/S_k$$

as the *unordered configuration space of k points in X*.

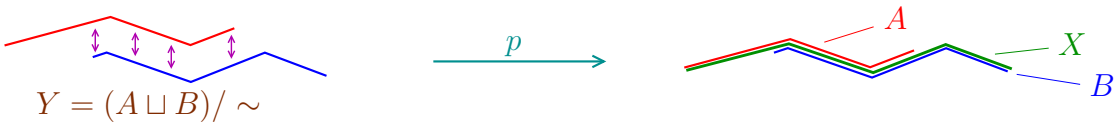
(a) Determine the homeomorphism type of $\overline{\mathrm{Conf}}_2(\mathbb{R})$.
(b) Determine the homeomorphism types of $\overline{\mathrm{Conf}}_2(S^1)$ and $\overline{\mathrm{Conf}}_3(S^1)$.

Remark. By "determine the homeomorphism type" we mean: show that these topological spaces are homeomorphic to more familiar topological spaces.

Exercise 9.6. We consider the topological space $X = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\})$ and the equivalence relation that is generated by $(x, 0) \sim (x, 1)$ for all $x \neq 0$.

(a) Show that the points $p((0, 0))$ and $p((0, 1))$ in $X/\sim$ cannot be separated by open neighborhoods.

(b) Show that $X/\sim$ is homeomorphic to the line with two zeros.
Remark. It is straightforward to write down a bijection, but why is it a homeomorphism?



Exercise 9.7. Let Σ be the surface of genus 4.

(a) Show that Σ admits a free and continuous action by $\mathbb{Z}/2\mathbb{Z}$.

(b) Show that Σ admits a free and continuous action by $\mathbb{Z}/3\mathbb{Z}$.

(c) Show that Σ admits a free and continuous action by $\mathbb{Z}/6\mathbb{Z}$.

Remark. In the Free Smooth Action-on-Manifold Proposition 156.20 together with the Manifold-χ-Proposition 190.24(5) we will see that if a finite group G admits a free and smooth action on Σ, then the order of G divides 6.

Exercise 9.8. We consider the map

$$\begin{array}{rcl} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) & \mapsto & z \cdot P. \end{array}$$

By definition we have $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$.

(1) Show that the obvious map $p\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ is open.

(2) In the Projective Spaces-Compact+Hausdorff Proposition 9.15 we showed that $\mathbb{C}\mathrm{P}^n$ is Hausdorff. In this exercise we want to give a more direct, down-to-earth proof that $\mathbb{C}\mathrm{P}^n$ is Hausdorff. Thus let $[x], [y] \in \mathbb{C}\mathrm{P}^n$ be two distinct points. Let $\langle\ ,\ \rangle$ be the standard Hermitian inner product on $\mathbb{C}^{n+1}$ that we introduced on page 124. We consider

$$U_x = \big\{z \in \mathbb{C}^{n+1} \setminus \{0\} \,\big|\, |\langle z, y\rangle| < |\langle z, x\rangle|\big\} \ \&\ U_y = \big\{z \in \mathbb{C}^{n+1} \setminus \{0\} \,\big|\, |\langle z, x\rangle| < |\langle z, y\rangle|\big\}.$$

(a) Show that U_x and U_y are open subsets of $\mathbb{C}^{n+1} \setminus \{0\}$.

(b) Show that $p(U_x)$ and $p(U_y)$ are disjoint. (Note that they are open by (1) and (2a).)

(c) Show that $[x] \in p(U_x)$ and $[y] \in p(U_y)$.
Hint. Use the Cauchy-Schwarz Inequality 2.31.

Exercise 9.9. Let $n \in \mathbb{N}_0$. Given $k \in \{0, \dots, n\}$ we consider the map

$$\begin{array}{rcl} f_k\colon \mathbb{C}^n & \to & \mathbb{C}\mathrm{P}^n \\ (z_1, \dots, z_n) & \mapsto & [z_1 : \cdots : z_k : 1 : z_{k+1} : \cdots : z_n]. \end{array}$$

(a) To what topological space is $\mathbb{C}\mathrm{P}^n \setminus f_k(\mathbb{C}^n)$ homeomorphic to?

(b) Show that $f_k(\mathbb{C}^n)$ is an open subset of $\mathbb{C}\mathrm{P}^n$.

(c) Show that each f_k is an open embedding, i.e. show that each f_k is a homeomorphism onto its image and that the image is an open subset of $\mathbb{C}\mathrm{P}^n$.

Exercise 9.10. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2 & \to & \mathbb{R}\mathrm{P}^2 \\ (x,y) & \mapsto & [x:y:1]. \end{array}$$

(a) Show that f is an embedding, i.e. show that f is a homeomorphism onto its image.
(b) What is the closure of $f(\mathbb{Z}^2) \subseteq \mathbb{R}\mathrm{P}^2$?

Exercise 9.11. Let $v \in S^2$, let $\varphi \in \mathbb{R}$ and let $a, b \in S^2$ be points such that $(v\ a\ b) \in \mathrm{SO}(3)$. We set

$$\rho(v,\varphi) \coloneqq \underbrace{(a\ b\ v)}_{\in \mathrm{SO}(3)} \cdot \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{(a\ b\ v)}_{\in \mathrm{SO}(3)}^{-1} \quad \in \mathrm{SO}\,(3).$$

(a) Show that $\mathrm{SO}(2)$ is abelian and show that 1 is an eigenvalue of every matrix in $\mathrm{SO}(3)$.
(b) Show that the matrix $\rho(v,\varphi)$ does *not* depend on the choice of a and b.
(c) Show that $\rho(v,\pi) = \rho(-v,\pi)$.
(d) Show that the map

$$\begin{array}{rcl} \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \rho(v, r\cdot\pi) \end{array}$$

is a bijection.

Exercise 9.12. We say that a group G *acts discretely* on a topological space X, if it acts continuously and if for each $x \in X$ there exists an open neighborhood U of x such that we have $U \cap gU = \varnothing$ for all $g \neq e$. Can the group $\mathbb{Z}$ act discretely on S^2?

Exercise 9.13. We consider $X = \mathbb{R}^2 \setminus \{(0,0)\}$ together with the action of $G = \mathbb{Z}$ that is given by $n \cdot (x,y) = (2^n \cdot x, 2^{-n} \cdot y)$.
(a) Show that this is action is not proper in the sense of Definition 9.4.
(b) Is the quotient space $X/\mathbb{Z}$ Hausdorff?
(c) Show that this action is discrete in the sense of Exercise 9.12.
(d) Give an example of a compact subspace $K \subseteq X$ such that $\{g \in G \,|\, gK \cap K \neq \varnothing\}$ is infinite.

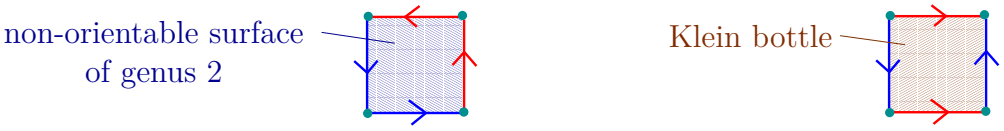
Exercise 9.14. Let X be a topological space and let G be a group that acts freely, discretely (in the sense of Exercise 9.12) and continuously on X. We suppose that the quotient X/G is Hausdorff and regionally compact. Show that the action is also proper.

Exercise 9.15. Let X be a topological space and let G be a group that acts continuously and properly on X.
(a) Show that for compact subspace $K \subseteq X$ the set $\{g \in G \,|\, g \cdot K \cap K \neq \varnothing\}$ is finite.
(b) Show that if G is infinite, then for any compact non-empty subspace $K \subseteq X$ and every $g \in G$ we have $g \cdot K \nsubseteq K$.

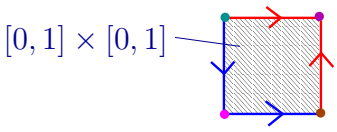
Exercise 9.16. Let $g \in \mathbb{N}_{\geq 0}$. We denote by Σ_g the surface of genus g from Definition 9.18.
(a) Show that each $x \in \Sigma_g$ admits an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^2$.
(b) Show that Σ_g is Hausdorff.

Remark. If you enjoyed this exercise you can continue with the non-orientable surface of genus g from Definition 9.19.

Exercise 9.17. Show that the Klein bottle, as defined on page 312, is homeomorphic to the non-orientable surface N_2 of genus 2 from Definition 9.19.



Exercise 9.18. We consider the square $[0,1] \times [0,1]$ with the equivalence relation $\sim$ that is illustrated in the figure below. The topological space $([0,1] \times [0,1])/\sim$ is homeomorphic to a topological space that we already introduced. Which one?



Exercise 9.19.

(a) Show that the surface of genus $g = 2$ is locally homeomorphic to an open subset of $\mathbb{R}^2$.

(b) Show that the non-orientable surface of genus $g = 3$ is locally homeomorphic to an open subset of $\mathbb{R}^2$.

Part II

General Topology II - Advanced Topics

In this chapter we continue our discussion of topological spaces. We prove several famous theorems, like the Urysohn's Lemma 12.5 and Urysohn Metrization Theorem 13.8, which we will use later on. The chapters are of course interesting in their own right. But the reader who is eager to learn about the topology of smooth manifolds or about algebraic topology is advised to skip this part of the lecture notes.

This part of the lecture notes is organized as follows:

- In Chapter 10 we will discuss in greater detail the most classical example of topological spaces, namely metric spaces. The high points include the construction of continuous surjective maps $[0,1] \to [0,1]^2$ in the Space-Filling Curve Theorem 10.21 and the proof of the Baire Category Theorem 10.26.
- In Chapter 11 we will introduce the notion of an oriented abstract graph and the notion of the topological realization of an oriented abstract graph. These form a fairly elementary but useful source of examples of topological spaces.
- In Chapter 12 we will state and prove Urysohn's Lemma 12.5 and the Tietze Extension Theorem 12.6. Both are important tools for constructing continuous maps.
- In Chapter 13 we will introduce the notion of a second-countable topological space. Furthermore we will state and prove the Urysohn Metrization Theorem 13.8 which gives a complete characterization for a second-countable topological space to be metrizable.
- In Chapter 14 we will introduce the notion of a paracompact topological space. Using Urysohn's Lemma 12.5 we will see in the Paracompact-Partition of Unity Theorem 14.15 that paracompact topological spaces can be characterized by the existence of suitable partitions of unity.
- In Chapter 15 we will introduce the covering dimension of a topological space. This is one of the many ways how one can associate a dimension to any topological space.
- In Chapter 16 we will introduce, given two topology spaces X and Y, the compact-open topology on the set $C(X,Y)$ of all continuous maps from X to Y. Experience shows that for many purposes this is the "right" topology on $C(X,Y)$.
- In Chapter 17 we will see that unfortunately taking products and quotients do not necessarily commute. But we will also see in the Whitehead Quotient Theorem 17.6 and the Product-Quotient Map Proposition 17.10 that this issue goes away, if we work with topological spaces that are "regionally compact".

CHAPTER 10

Topology of metric spaces

In this chapter we will discuss several topics that are germane to metric spaces. Some of these results, like the modest Lebesgue Lemma 10.8 and the more fancy Baire Category Theorem 10.26 will play important roles later on.

This chapter is organized as follows:

- In Section 10.1 we will, to get started, recall the definition of a metric space and we will make one quick observation.
- In Section 10.2 we will introduce the notion of the distance of two subsets of a given metric space.
- In Section 10.3 we will prove the Lebesgue Lemma 10.8, which despite its modest name, gets used frequently.
- In Section 10.4 we will study uniformly continuous maps between metric spaces.
- In Section 10.5 we will study complete metric spaces.
- In Section 10.6 we will show that there exist space-filling curves, i.e. for any $n \in \mathbb{N}$ there exist continuous *surjective* maps $[0,1] \to [0,1]^n$.
- In Section 10.7 we will state and prove the Baire Category Theorem 10.26 which, despite its historic name, has nothing to do with category theory. This theorem says if X is complete metric space, then the union of countably many, nowhere dense subsets, has empty interior.
- In Section 10.8 we will introduce, inspired by the Baire Category Theorem 10.26, Baire spaces and completely metrizable spaces.
- In Section 10.9 we will study applications of the Baire Category Theorem 10.26 to sequences of maps.
- In Section 10.10 we will provide exercises for Chapter 10.

In the next Chapter 11 we will introduce (topological) graphs.

10.1. **Basics of metric spaces.** For the reader's convenience we recall Definition 2.46 of a metric space:

Definition 10.1. A metric space is a pair (X,d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \coloneqq \{x \in \mathbb{R} \,|\, x \geq 0\} \\ (x,y) & \mapsto & d(x,y) \end{array}$$

with the following properties:

(1) $d(x,y) = 0 \iff x = y$

(2) for all $x,y \in X$ we have $d(x,y) = d(y,x)$ (symmetry)

(3) for all $x,y,z \in X$ we have $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

Example. The most familiar example of a metric space is given by $X = \mathbb{R}^n$ equipped with the Euclidean metric. □

As always we equip a metric space (X, d) with the topology that we introduced in the Metric-Topology Lemma 3.7 and we equip $X \times X$ with the product topology from Topological-Product Proposition 7.2. With these conventions it now makes sense to formulate the following lemma:

Lemma 10.2. (Metric-is-Continuous Lemma) Let (X, d) be a metric space. The metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ is continuous.

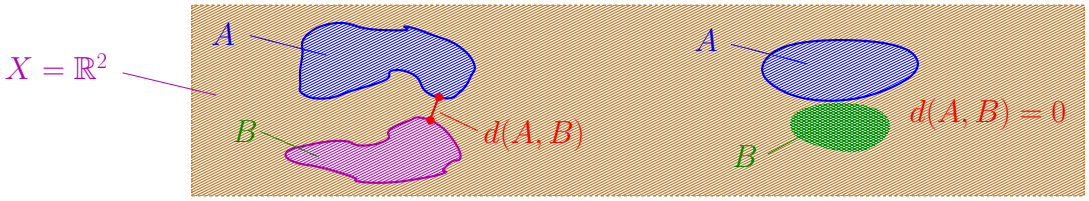
Proof. We prove the lemma in Exercise 10.2. ■

10.2. **Distance in metric spaces.** We will now generalize the notion of distance between points to distance between non-empty subsets:

Definition 10.3. Let X be a metric space. Given two non-empty subsets A and B we will refer to

$$d(A, B) \;:=\; \inf\{d(a, b) \,|\, a \in A \text{ and } b \in B\}$$

as the distance between A and B. If $A = \{a\}$ consists of a single point, then we write $d(a, B) := d(\{a\}, B)$.



Example. If two subsets A and B intersect, then we evidently have $d(A, B) = 0$. But in general the converse does not hold. For example if we consider $X = \mathbb{R}$, then

$$d\big(\underbrace{\text{open interval } (0,1)}_{=A}, \underbrace{\text{closed interval } [1,2]}_{=B}\big) \;=\; 0,$$

but evidently the intervals $A = (0, 1)$ and $B = [1, 2]$ do not intersect. □

The reader might justly suspect that the phenomenon of the last example is due to the fact that one of the subspaces is not compact. In fact the next two lemmas, which will play a useful role time and again, confirms this suspicion.

Lemma 10.4. (Distance-to-Subspace Lemma) Let X be a metric space and let $B \subseteq X$ be a non-empty subset.

(1) The map

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x, B) := \inf\{d(x, b) \,|\, b \in B\} \end{array}$$

is continuous.

(2) If B is closed, then given $x \in X$ we have

$$d(x, B) = 0 \quad \Longleftrightarrow \quad x \in B.$$

Proof. We will prove Statement (1) in Exercise 10.6. Thus we can immediately proceed to the proof of Statement (2). Let B be a closed subset of X. If $x \in B$, then evidently $d(x, B) = 0$. Now suppose that $x \notin B$. Since B is closed we know that $X \setminus B$ is open.

This implies that there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq X \setminus B$. But this implies that $d(x, B) \geq \epsilon$. ■

Lemma 10.5. (Disjoint Compact-Closed–Distance Lemma) Let X be a metric space and let K and L be two non-empty disjoint subsets. If one of the two subsets is compact and if the other subset is a closed subset (e.g. a compact subset) of X, then $d(K, L) > 0$.



Proof. We can assume that K is compact and that L is a closed subset of X. We consider the map

$$\begin{array}{ccc} K & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x, L). \end{array}$$

By the Distance-to-Subspace Lemma 10.4 we know that the map is continuous. Since K is compact we know by the Compact Image Lemma 4.15 (1) that f assumes a minimum, i.e. there exists an $x \in K$ such that for all $y \in K$ we have $d(x, L) \leq d(y, L)$. This implies that

$$d(K, L) \underset{\text{by Definition 10.3}}{=} \inf\{d(y, L) \,|\, y \in K\} \underset{\text{by choice of } x}{=} d(x, L) \overset{\text{by the Distance-to-Subspace Lemma 10.4 (2), since } L \text{ closed}}{>} 0.$$

■

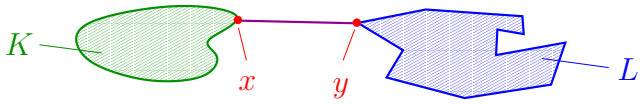
Sometimes it is nice to know that points of minimal distance exist. This leads us to the following lemma:

Lemma 10.6. (Distance-Minimizer Lemma) Let X be a metric space and let K and L be two non-empty subsets. We assume that one of the following holds:

(i) Both K and L are compact, or

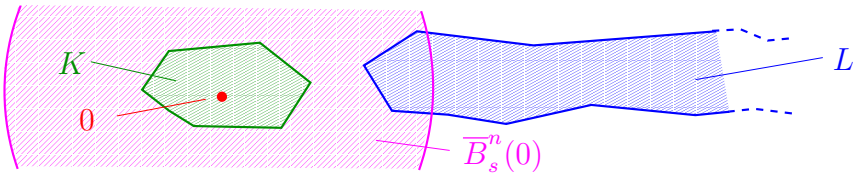
(ii) $X = \mathbb{R}^n$, K is compact and L is a closed subset of $X = \mathbb{R}^n$.

There exist $x \in K$ and $y \in L$ with $d(x, y) = d(K, L)$.



Proof. First we suppose that we are in the setting (i). Thus let K and L be two compact non-empty subsets of X. By the Product Topology-Properties Proposition 7.9 we know that $K \times L$ is compact. By the Metric-is-Continuous Lemma 10.2 we know that the metric $d\colon K \times L \to \mathbb{R}_{\geq 0}$ is continuous. It follows from the above, together with the fact that $K \times L$ is non-empty and the Compact Image Lemma 4.15 (2) that $d\colon K \times L \to \mathbb{R}_{\geq 0}$ assumes a minimum at some point $(x, y) \in K \times L$. But this means precisely that $d(x, y) = d(K, L)$.

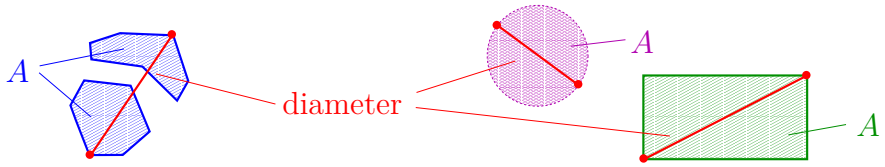
Now suppose that we are in the setting (ii). We will reduce this case to the setting (i). To do so we set $s := \max\{\|x\| \,|\, x \in K\} + 1 + d(K, L)$.
Next we set $L' := L \cap \overline{B}_s^n(0)$. Note that it follows from the hypothesis that L is closed and the Heine-Borel Theorem 3.44 together with the Compact-Closed Lemma 3.39 (2) that L' is compact. We leave it to the reader to verify that it follows easily from the choice of s and the triangle inequality that $L' \neq \varnothing$ and that $d(K, L') = d(K, L)$. We apply (i) to



$X = \mathbb{R}^n$ together with the compact subsets K and L'. We obtain $x \in K$ and $y \in L' \subseteq L$ with $d(x, y) = d(K, L') = d(K, L)$. ■

10.3. **The Lebesgue Lemma.** In this section we intend to formulate and prove the Lebesgue Lemma. Despite its humble name it does actually play an important role in topology. The formulation of the Lebesgue Lemma requires the following definition:

Definition 10.7. Let X be a metric space. The diameter of a non-empty subset A of X is defined as $\operatorname{diam}(A) := \sup\{d(a, b) \mid a, b \in A\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.
To avoid any issues later on we also define $\operatorname{diam}(\varnothing) := 0$.



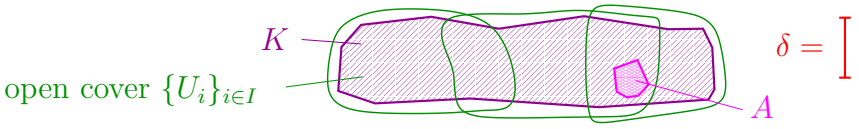
Example.

(1) Let (X, d) be a metric space, let $x \in X$ and $r \in \mathbb{R}_{\geq 0}$. Using the triangle inequality one can easily show that $\operatorname{diam}(\overline{B}_r^d(x)) = \operatorname{diam}(B_r^d(x)) \leq 2r$. Furthermore, if we consider balls in some $\mathbb{R}^n$, then it is straightforward to see that the above inequality is in fact an equality.

(2) We consider the metric space $(\mathbb{R}^n, d_{\text{Eucl}})$. We leave it to the reader to show that

$$\operatorname{diam}\big(\text{hyperrectangle } [a_1, b_1] \times \cdots \times [a_n, b_n]\big) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}. \quad \square$$

Lemma 10.8. (Lebesgue Lemma) Let K be a compact metric space and let $\{U_i\}_{i \in I}$ be an open cover of K. There exists a $\delta > 0$ such that for every subset A of K with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subseteq U_i$.



Remark. Any $\delta > 0$ that has the property stated in the Lebesgue Lemma 10.8 is usually called a *Lebesgue number*. □

Proof. Since K is compact we can cover K with finitely many of the U_i's. Put differently, without loss of generality we can assume that $I = \{1, \ldots, n\}$ is a finite set.

If there exists an $i \in \{1, \ldots, n\}$ with $K = U_i$, then any $\delta > 0$ has the desired property. Now suppose that this is not the case, i.e. we suppose that for every i we have $U_i \subsetneq K$.

Claim 1. The map

$$\begin{aligned} f\colon K &\to \mathbb{R} \\ x &\mapsto f(x) := \frac{1}{n}\cdot\sum_{i=1}^{n} d(x, \underbrace{K\setminus U_i}_{\neq\varnothing}) \end{aligned}$$

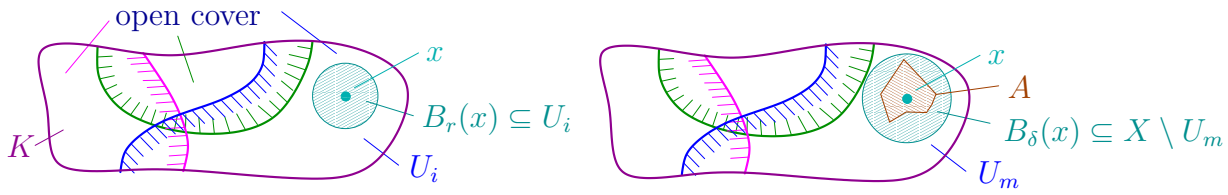
is continuous and positive for every $x \in K$.

Proof. By the Distance-to-Subspace Lemma 10.4 (1) we know that f is continuous. Now let $x \in K$. We want to show that $f(x) > 0$. Since $K = U_1 \cup \cdots \cup U_n$ there exists an i with $x \in U_i$. Since U_i is open and since $x \in U_i$ we obtain from the Distance-to-Subspace Lemma 10.4 (2) that $d(x, K \setminus U_i) > 0$. Evidently this implies that $f(x) > 0$. ⊞

Since f is continuous and since K is compact it follows from the Compact Image Lemma 4.15 that the map f has a global minimum δ on K. It follows from Claim 1 that this minimum δ is greater than 0. Now we want to show that this δ has the desired property. More precisely, we want to prove the following claim:

Claim 2. Let A be a subset of K. If $\operatorname{diam}(A) < \delta$, then there exists an $i \in \{1, \ldots, n\}$ with $A \subseteq U_i$.

Proof. If $A = \varnothing$, then there is nothing to show. If $A \neq \varnothing$, then we pick $x \in A$. We choose an $m \in \{1, \ldots, n\}$ such that $d(x, K \setminus U_m)$ is maximal. We want to show that $A \subseteq U_m$.



Since the diameter of A is less than δ we have $A \subseteq B_\delta(x)$. Thus it suffices to show that $B_\delta(x) \subseteq U_m$. Put differently, we need to show that $d(x, K \setminus U_m) \geq \delta$. Indeed we have

$$d(x, K\setminus U_m) \underset{\text{since } d(x,K\setminus U_m) \geq d(x,K\setminus U_i)}{\geq} \frac{1}{n}\cdot\sum_{i=1}^{n} d(x, K\setminus U_i) = f(x) \underset{\text{by the choice of } \delta}{\geq} \delta.$$

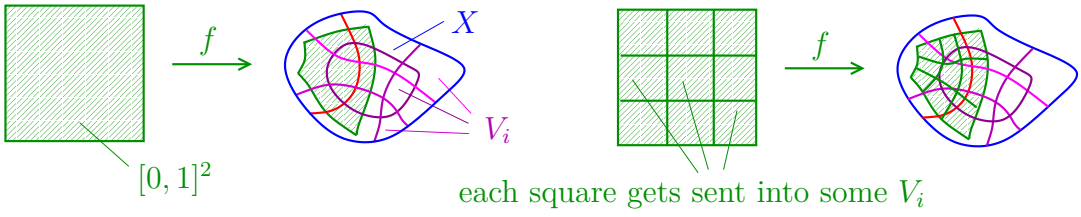
■

In many cases the following corollary is even more useful.

Lemma 10.9. (Lebesgue Lemma II) Let $f\colon [0,1]^n \to X$ be a continuous map from the cube $[0,1]^n$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. Then there exists an $N \in \mathbb{N}$ such that for every $a_1, \ldots, a_n \in \{0, \ldots, N-1\}$ there exists an $i \in I$ such that

$$f\big(\big[\tfrac{a_1}{N}, \tfrac{a_1+1}{N}\big] \times \cdots \times \big[\tfrac{a_n}{N}, \tfrac{a_n+1}{N}\big]\big) \subseteq V_i.$$

More pictorially speaking the corollary says that if $f\colon [0,1]^n \to X$ is a continuous map, and if we are given an open cover of X, then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open subsets cover X.



Proof. Let $f\colon [0,1]^n \to X$ be a continuous map from the cube $[0,1]^n$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. By the Hyperrectangle-Compact Proposition 3.47 we know that $[0,1]^n$ is compact. Therefore we can apply Lemma 10.8 to the open cover $U_i \coloneqq f^{-1}(V_i)$, $i \in I$ of $[0,1]^n$ and we obtain a $\delta > 0$ such that for each subset A of $[0,1]^n$ with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subseteq U_i = f^{-1}(V_i)$, which means that $f(A) \subseteq V_i$. It follows from the example on page 356 that the diameter of the cube of side length $\frac{1}{m}$ equals $\frac{\sqrt{n}}{m}$. Thus we see that for any $N \in \mathbb{N}_0$ with $N \geq \frac{\sqrt{n}}{\delta}$ any cube of side length $\frac{1}{N}$ has diameter less than δ. ■

10.4. **Uniformly continuous maps.** In this section we introduce the concept of uniformly continuous maps that the reader might be familiar with from a course on real analysis. We will show the following: If $f\colon X \to Y$ is a continuous map between metric spaces and if X is compact, then f is actually uniformly continuous. This is a rather technical result which we will make use of only once, namely in Section 141.3 when we discuss "CW-complexes". Most readers will be best served by passing on to the next section.

Definition 10.10. Let $f\colon X \to Y$ be a map between two metric spaces X and Y. By the Metric Continuity Proposition 4.9 we know that

$$f \text{ is continuous} \iff \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in B_\delta(x)}{\forall}\ d(f(x),f(y)) < \epsilon.$$

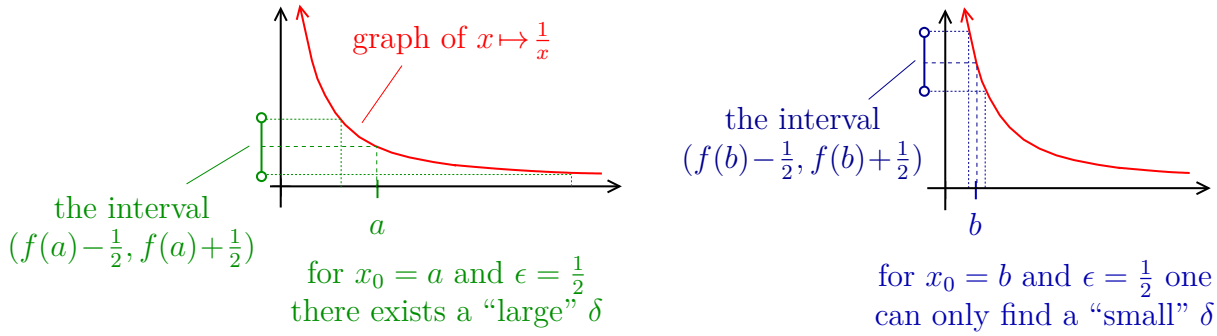
Now we define

$$f \text{ is uniformly continuous} :\iff \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in X}{\forall}\ \underset{y\in B_\delta(x)}{\forall}\ d(f(x),f(y)) < \epsilon.$$

Example. The map

$$\begin{array}{rcl} f\colon (0,\infty) & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array}$$

is not uniformly continuous.



For example for $\epsilon = \frac{1}{2}$ there is no "universal" δ that works for all $x \in (0,\infty)$. □

Proposition 10.11. (Compact–Uniformly Continuous Proposition) Let $f\colon X \to Y$ be a continuous map between two metric spaces. If X is compact, then f is uniformly continuous.

Proof. Let $f\colon X \to Y$ be a continuous map between two metric spaces and let $\epsilon > 0$. The"$\Leftarrow$"-direction of the Metric Continuity Proposition 4.9 tells us in particular that given any $P \in X$ there exists a $\delta(P) > 0$ such that for all $y \in B_{\delta(P)}(P)$ we have $d(f(P), f(y)) < \frac{\epsilon}{2}$. Since the open balls $B_{\delta(P)/2}(x)$ form an open cover of X and since X is compact there exist

$P_1, \dots, P_k \in X$ such that $B_{\delta(P_1)/2}(P_1) \cup \dots \cup B_{\delta(P_1)/2}(P_1) = X$. We claim that

$$\delta \;:=\; \min\left\{\frac{\delta(P_1)}{2}, \dots, \frac{\delta(P_k)}{2}\right\}$$

has the desired properties. Thus let $x, y \in X$ with $d(x,y) < \delta$. There exists a $j \in \{1, \dots, k\}$ with $x \in B_{\delta(P_j)/2}(P_j)$. From the triangle inequality and from $d(x,y) < \delta \leq \frac{\delta(P_j)}{2}$ we obtain that $y \in B_{\delta(P_j)}(P_j)$. It follows that

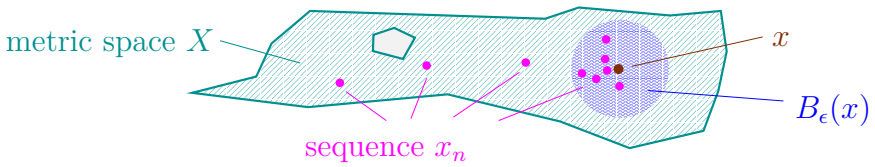
$$d(f(x), f(y)) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} d(f(P_j), f(x)) + d(f(P_j), f(y)) \underset{\substack{\uparrow \\ \text{since } x \in B_{\delta(P_j)}(P_j) \text{ and} \\ \text{since } y \in B_{\delta(P_j)}(P_j)}}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

We have thus shown that δ does indeed have the desired properties. ■

10.5. **Complete metric spaces.** In Definition 3.33 we defined for any topological space X what it means for a sequence of points in X to converge to some other point. In metric spaces the following lemma gives an alternative characterization of convergent sequences.

Lemma 10.12. (Definitions-of-Limit Lemma) Let (X,d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. For each $x \in X$ we have

$$\begin{array}{c} x \text{ is a limit of the sequence } (x_n)_{n\in\mathbb{N}} \\ \text{according to Definition 3.33} \end{array} \iff \underset{\epsilon>0}{\forall}\, \underset{N\in\mathbb{N}}{\exists}\, \underset{n\geq N}{\forall}\, \underbrace{d(x_n, x) < \epsilon.}_{\text{i.e. } x_n \in B_\epsilon(x)}$$



Proof. We need to show

$$\underset{\substack{\text{neighborhoods } U \text{ of } x}}{\forall}\, \underset{N\in\mathbb{N}}{\exists}\, \underset{n\geq N}{\forall}\, x_n \in U \iff \underset{\epsilon>0}{\forall}\, \underset{N\in\mathbb{N}}{\exists}\, \underset{n\geq N}{\forall}\, \underbrace{d(x_n, x) < \epsilon.}_{\text{i.e. } x_n \in B_\epsilon(x)}$$

Evidently that is just a dull routine exercise which is best left to the reader's office mate. ■

Next we introduce the standard notation for limits:

Definition 10.13. By the Metric Space-Hausdorff Proposition 3.28 we know that metric spaces are Hausdorff. It follows from the Unique Limit Lemma 3.34 that if a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space X converges, then its limit is unique. Thus it makes sense to denote its limit by $\lim\limits_{n\to\infty} x_n \in X$.

Definition 10.14. Let X be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called Cauchy sequence if

$$\underset{\epsilon>0}{\forall}\, \underset{N\in\mathbb{N}}{\exists}\, \underset{m,n\geq N}{\forall}\, d(x_n, x_m) < \epsilon.$$

One can easily show that every convergent sequence is a Cauchy sequence. In general the converse does of course not hold though. For example, if we now consider the metric space $(\mathbb{R}\setminus\{0\}, d(x,y) = |x-y|)$, then we see that the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence, but it does not converge in the metric space $\mathbb{R}\setminus\{0\}$.

Definition 10.15. We say that a metric space (X, d) is complete if every Cauchy sequence in (X, d) converges. Otherwise we say that (X, d) is incomplete.

Examples.

(1) It is one of the defining properties of the real numbers that the metric space given by $\mathbb{R}$ and the usual metric $d(x, y) = |x - y|$ is complete.
(2) It follows without too much sweat from (1) that the metric space given by $\mathbb{R}^n$ and the Euclidean metric is complete. □

Lemma 10.16. (Closed Subspace-Completeness Lemma) Let (X, d) be a complete metric space. If $A \subseteq X$ is a closed subset, then (A, d) is also complete.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (A, d). Since (X, d) is complete we know that $(a_n)_{n\in\mathbb{N}}$ converges to some point $x \in X$. But since A is closed we obtain from the Closed Subset Limit Lemma 3.35 that $x \in A$. ■

Proposition 10.17. (Compact-Complete Proposition) Every compact metric space is complete.

Proof. Let (X, d) be a compact metric space and let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d). We want to show that the sequence converges, i.e. we want to show that

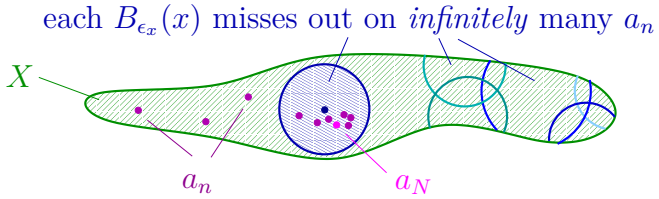
$$\underset{x\in X}{\exists}\ \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\geq N}{\forall}\ a_n \in B_\epsilon(x).$$

Let us assume that no such x exists. Thus we negate the above statement and we obtain that

$$\underset{x\in X}{\forall}\ \underset{\epsilon_x>0}{\exists}\ \underset{N\in\mathbb{N}}{\forall}\ \underset{n\geq N}{\exists}\ a_n \notin B_{\epsilon_x}(x).$$

Evidently the family $\{B_{\epsilon_x}(x)\}_{x\in X}$ of subsets is an open cover of X. Since X is compact it follows from the Lebesgue Lemma 10.8 that there exists a $\mu > 0$ such that any subset A of X of diameter less than μ is contained in some $B_{\epsilon_x}(x)$.

Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $d(a_m, a_n) < \frac{\mu}{4}$.



This means in particular that for all $m \geq N$ we have $d(a_N, a_m) < \frac{\mu}{4}$. It follows from the triangle inequality that the diameter of the set $A := \{a_n \,|\, n \geq N\}$ is $\leq \frac{\mu}{2} < \mu$. By the above there exists an $x \in X$ such that $A \subseteq B_{\epsilon_x}(x)$. But this means that for all $n \geq N$ we have $a_n \in B_{\epsilon_x}(x)$. But this contradicts the defining property of ϵ_x. ■

The following lemma shows that one can replace a metric by a bounded metric without affecting the topology.

Lemma 10.18. (Upper Bound-to-Metric Lemma) Let (M, d) be a metric space.
(1) The map

$$\begin{aligned} \widetilde{d}\colon M \times M &\to \mathbb{R}_{\geq 0} \\ (x,y) &\mapsto \widetilde{d}(x,y) \coloneqq \min\{1, d(x,y)\} \end{aligned}$$

also defines a metric on M.

(2) This new metric $\widetilde{d}$ induces the same topology as d on M.

(3) (M,d) is complete if and only if $(M,\widetilde{d})$ is complete.

Proof. All three statements are elementary and they follow easily from the definitions. We refrain from boring the reader with details. ■

Notation 10.19. Let (M,d) be a metric space and let X be a topological space. We denote by $C(X,M)$ the set of continuous maps from X to M.

Proposition 10.20. (Function Space–Metric Proposition) Let (M,d) be a metric space and let X be a non-empty topological space. The map

$$\begin{aligned} \widetilde{d}\colon C(X,M) \times C(X,M) &\to \mathbb{R}_{\geq 0} \\ (f,g) &\mapsto \widetilde{d}(f,g) \coloneqq \sup \underbrace{\{\widetilde{d}(f(x),g(x)) \,|\, x \in X\}}_{\substack{\text{non-empty and bounded, here} \\ \widetilde{d} \text{ is the metric on } M \text{ defined in the} \\ \text{Upper Bound-to-Metric Lemma 10.18}}} \end{aligned}$$

has the following properties:

(1) The map $\widetilde{d}$ defines a metric on $C(X,M)$.

(2) If X is compact, then the map

$$\begin{aligned} \widetilde{d}\colon C(X,M) \times C(X,M) &\to \mathbb{R}_{\geq 0} \\ (f,g) &\mapsto \widetilde{d}(f,g) \coloneqq \sup \underbrace{\{d(f(x),g(x)) \,|\, x \in X\}}_{\substack{\text{non-empty and bounded} \\ \text{by the Compact Image Lemma 4.15}}} \end{aligned}$$

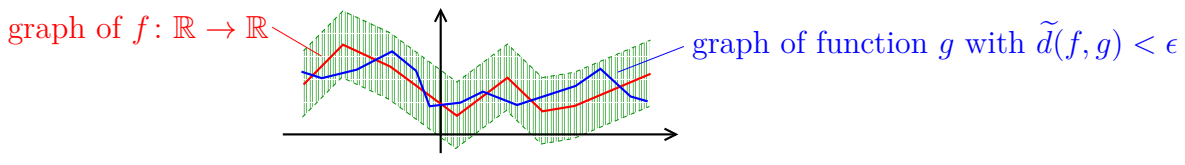
is also a metric on $C(X,M)$ which is equivalent, in the sense of Definition 3.10 to the above metric $\widetilde{d}$

(3) If (M,d) is complete, then given any Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $(C(X,M),\widetilde{d})$ the map

$$\begin{aligned} f\colon X &\to M \\ x &\mapsto \lim_{n\to\infty} f_n(x) \end{aligned}$$

exists (i.e. each limit exists), it is a continuous map (i.e. it defines an element of $C(X,M)$) and in the metric space $(C(X,M),\widetilde{d})$ it is the limit of the sequence $(f_n)_{n\in\mathbb{N}}$.

(4) If (M,d) is complete, then so is $(C(X,M),\widetilde{d})$.



Remark. The proposition might be familiar from a real analysis course, at least for the case that $X \subseteq \mathbb{R}^n$ and $M = \mathbb{R}$. The proof of the general case does not differ one iota from the special case. □

Proof. Statement (1) follows easily from the definitions and the properties of the supremum. We leave it to the reader to fill in the details. Statement (2) follows easily from the definitions and the Equivalent Metric Lemma 3.11. Furthermore the last statement is contained in Statement (3).

Thus it remains to prove Statement (3) of the proposition. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(C(X,M),\widetilde{d})$. The following three claims imply Statement (3) of the proposition.

Claim 1. For each $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges.

Proof. Let $x \in X$. Note that we have $\widetilde{d}(f_n(x), f_m(x)) \leq \widetilde{d}(f_n, f_m)$. Thus we see that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $(M,\widetilde{d})$. By our hypothesis (M,d) is complete. By the Upper Bound-to-Metric Lemma 10.18 we know that $(M,\widetilde{d})$ is also complete, thus the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in $(M,\widetilde{d})$ to a point which we call $f(x)$. ⊞

We can now consider the map

$$\begin{array}{rcl} f\colon X & \to & M \\ x & \mapsto & \lim\limits_{n\to\infty} f_n(x). \end{array}$$

Claim 2. $$\forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\geq N}\ \forall_{x\in X}\ \widetilde{d}(f_n(x), f(x)) < \epsilon.$$

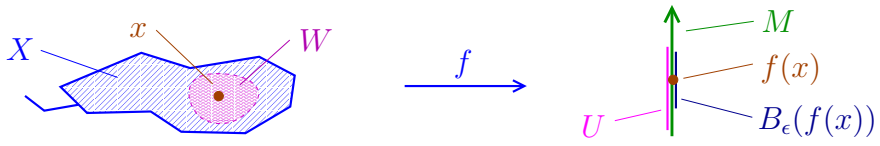
Proof. Let $\epsilon > 0$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C(X,M),\widetilde{d})$ we know that there exists an $N \in \mathbb{N}$ such that for each $m, n \geq N$ we have $\widetilde{d}(f_n, f_m) < \frac{\epsilon}{2}$. Now let $n \geq N$ and let $x \in X$. We see that

$$\widetilde{d}(f_n(x), f(x)) = \widetilde{d}\Big(f_n(x), \lim_{m\to\infty} f(x_m)\Big) \overset{\text{by continuity of metrics, see the Metric-is-Continuous Lemma 10.2, and Exercise 4.33}}{\downarrow} = \lim_{m\to\infty} \underbrace{\widetilde{d}(f_n(x), f_m(x))}_{\leq \widetilde{d}(f_n,f_m) < \frac{\epsilon}{2}} \leq \frac{\epsilon}{2} < \epsilon.$$

⊞

Claim 3. The map $f\colon X \to M$ is continuous.

Proof. Let $U \subseteq M$ be an open subset. We need to show that $f^{-1}(U)$ is open. By the JH-Openness Criterion 3.3 it suffices to show that for every $x \in f^{-1}(U)$ there exists an open neighborhood W of $x \in X$ with $f(W) \subseteq U$. Thus let $x \in f^{-1}(U)$. Since $U \subseteq M$ is open there exists an $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq U$.



Next note that by Claim 2 and by definition of $\widetilde{d}$ there exists an $n \in \mathbb{N}$ such that for every $x \in X$ we have $d(f_n(x), f(x)) < \frac{\epsilon}{3}$. Furthermore, by hypothesis $f_n\colon X \to M$ is continuous, which implies that there exists an open neighborhood W of x with $f_n(W) \subseteq B_{\frac{\epsilon}{3}}(f_n(x))$. Now we see that for each $y \in W$ we have

$$d(f(x), f(y)) \leq \underbrace{d(f(x), f_n(x))}_{\leq \frac{\epsilon}{3} \text{ by choice of } n} + \underbrace{d(f_n(x), f_n(y))}_{\leq \frac{\epsilon}{3} \text{ by choice of } W} + \underbrace{d(f_n(y), f(y))}_{\leq \frac{\epsilon}{3} \text{ by choice of } n} < \epsilon.$$

But this means precisely that $f(W) \subseteq B_\epsilon(f(x)) \subseteq U$. ■

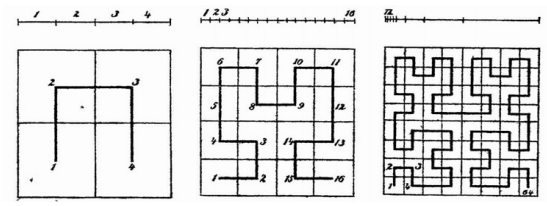
10.6. **Space-filling curves.** The goal of this section is to prove the following, possibly rather counterintuitive theorem.

Theorem 10.21. (Space-Filling Curve Theorem) For every $m \in \mathbb{N}$ there exists a continuous map $f\colon [0,1] \to [0,1]^m$ that is surjective.

Definition 10.22. Let $m \geq 2$. A surjective continuous map from $[0,1]$ to $[0,1]^m$ is called a space-filling path.[151]

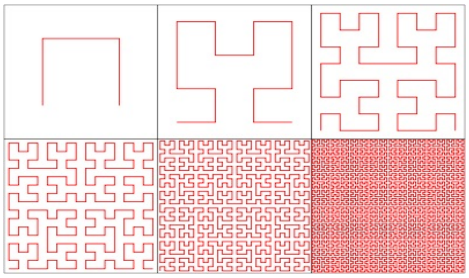
Remark.

(1) The first example of a space-filling path was provided by Giuseppe Peano [**Pean1890**] in 1890. Shortly afterwards David Hilbert [**Hilb1891**] gave the first "pictorial" description of a space-filling path, see the figure below. Nowadays there are many constructions of space-filling paths. In fact two such constructions are given in Exercises 7.25 and 10.16. Further constructions are considered in [**Sagan1994**, Theorem 2.1] and [**Bad2013**].



[**Hilb1891**]

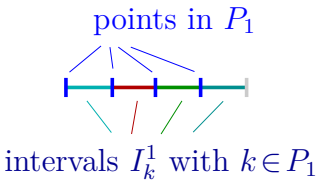
first stages in the construction of the Hilbert space-filling curve

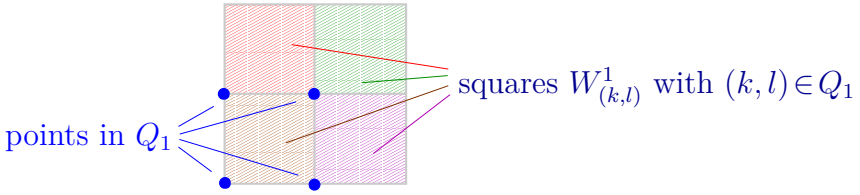


(2) In Exercise 10.14 we will use the Space-Filling Curve Theorem 10.21 to show that for every $n \in \mathbb{N}$ there exists a surjective continuous map $\mathbb{R} \to \mathbb{R}^n$. □

Proof of Theorem 10.21 for $m = 2$. For $n \in \mathbb{N}$ we set

$$P_n := \{0, \tfrac{1}{4^n}, \ldots, \tfrac{4^n-1}{4^n}\} \quad \text{and for } k \in P_n \text{ we set} \quad I_k^n := [k, k+\tfrac{1}{4^n}],$$
$$Q_n := \{0, \tfrac{1}{2^n}, .., \tfrac{2^n-1}{2^n}\} \times \{0, \tfrac{1}{2^n}, .., \tfrac{2^n-1}{2^n}\} \quad \text{and for } (k,l) \in Q_n \text{ we set } W_{(k,l)}^n := [k, k+\tfrac{1}{2^n}] \times [l, l+\tfrac{1}{2^n}].$$





Claim 1. For $n = 0, 1, 2, \ldots$ we can find a bijection $\alpha_n\colon P_n \to Q_n$ and a (continuous) map $f_n\colon [0,1] \to [0,1] \times [0,1]$ such that the following conditions are satisfied:

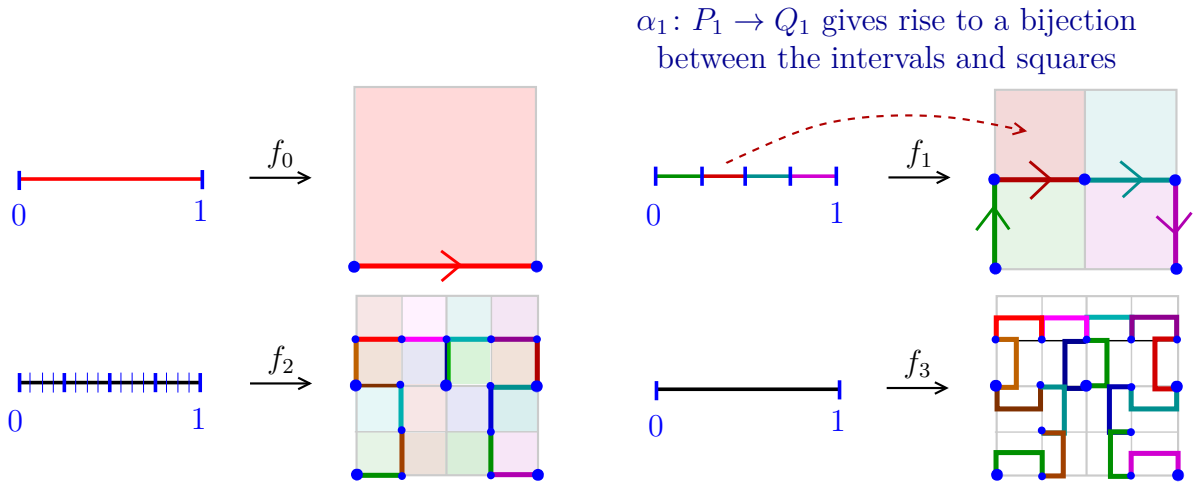
(a) For each $n \in \mathbb{N}_0$ and each $k \in P_n$ we have $f_n(I_k^n) \subseteq W_{\alpha_n(k)}^n$.

(b) For each $n \in \mathbb{N}_0$ and each $k \in P_n$ the two endpoints of the interval I_k^n get sent to adjacent vertices of the square $W_{\alpha_n(k)}^n$.

(c) For each $n \geq m \in \mathbb{N}$ and each $k \in P_m$ and $l \in P_l$ with $I_k^n \subseteq I_l^m$ we have the inclusion $f_n(I_k^n) \subseteq W_{\alpha_m(l)}^m$.[152]

[151] The same name is also used for obvious variations on this concept.

[152] Note that Condition (a) is the special case of (c) with $n = m$ and $l = k$. But It seems to us that it is easier to first digest (a) and then to try to digest (c).

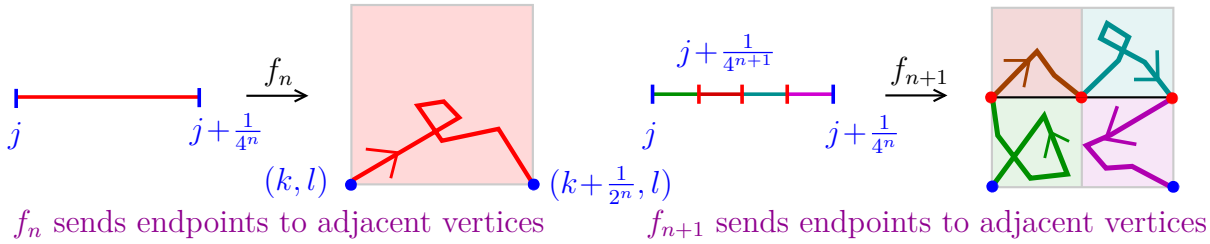


Proof. We define the maps iteratively.

- For $n = 0$ we have $P_1 = \{0\}$ and $Q_n = \{(0,0)\}$, thus we let $\alpha_0 \colon P_1 \to Q_0$ be the unique bijection. Furthermore we set
$$\begin{array}{rcl} f_1 \colon I_1^0 = [0,1] & \to & W_{(1,1)}^0 = [0,1] \times [0,1] \\ t & \mapsto & (t,0). \end{array}$$
It is clear that (a) and (b) are satisfied.
- Now suppose that we have constructed bijections $\alpha_0, \dots, \alpha_n$ and continuous maps $f_0, \dots, f_n$ with the desired properties. Next we need to construct the bijection $\alpha_{n+1} \colon P_{n+1} \to Q_{n+1}$ and the map $f_{n+1} \colon [0,1] \to [0,1] \times [0,1]$.

  Our first goal is, given $j \in P_n$, to define f_{n+1} on $I_j^n = [j, j + \frac{1}{4^n}]$ and to define α_{n+1} on $j, j + \frac{1}{4^{n+1}}, j + \frac{2}{4^{n+1}}, j + \frac{3}{4^{n+1}} \in P_{n+1}$. We write $\alpha_n(j) =: (k,l) \in Q_n$. It follows easily from (b) that we can find a bijection
$$\alpha_{n+1} \colon \{j, j+\tfrac{1}{4^{n+1}}, j+\tfrac{2}{4^{n+1}}, j+\tfrac{3}{4^{n+1}}\} \to \{(k,l), (k+\tfrac{1}{2^{n+1}}, l), (k, l+\tfrac{1}{2^{n+1}}), (k+\tfrac{1}{2^{n+1}}, l+\tfrac{1}{2^{n+1}})\}$$
and a map $f_{n+1} \colon I_j^n \to W_k^n$ with the following three properties:
(a) The map $f_{n+1} \colon I_j^n \to W_k^n$ agrees with the map f_n on the endpoints.
(b) Each interval $I^{n+1}_{j+\frac{i}{4^{n+1}}}$ gets sent to the square $W^{n+1}_{\alpha_{n+1}(j+\frac{i}{4^{n+1}})}$.
(c) The endpoints of each interval $I^{n+1}_{j+\frac{i}{4^{n+1}}}$ get sent to adjacent vertices of the square $W^{n+1}_{\alpha_{n+1}(j+\frac{i}{4^{n+1}})}$.
This can be done in a systematic way[153] (as is shown in the figure above) but does not have to be so (as is shown in the figure below). We will not clutter the pages with writing down explicit maps.
These maps $f_{n+1} \colon I_j^n \to [0,1] \times [0,1]$ agree on the endpoints. Therefore it follows from the Pasting Proposition 4.7 that they define a continuous map $f_{n+1} \colon [0,1] \to [0,1] \times [0,1]$. Furthermore the bijections α_{n+1} combined give a bijection $\alpha_{n+1} \colon P_{n+1} \to Q_{n+1}$. We leave it to the reader to verify that f_{n+1} and α_{n+1} have the desired properties. ⊞

[153] In fact one can easily see that α_{n+1} is uniquely determined. A canonical way of defining f_{n+1} is now given by connecting the endpoints via a Euclidean segment.



f_n sends endpoints to adjacent vertices f_{n+1} sends endpoints to adjacent vertices

As above we consider the set $C([0,1],[0,1]\times[0,1])$ of continuous maps from the interval $[0,1]$ to the square $[0,1]\times[0,1]$. We view this as a metric space with respect to the metric

$$d(f,g) \;:=\; \sup\{\|f(t)-g(t)\|_{\text{Eucl}} \,|\, t\in[0,1]\}$$

that we introduced in the Function Space–Metric Proposition 10.20 (2).

Claim 2. The sequence $(f_n)_{n\in\mathbb{N}_0}$ of maps $[0,1]\to[0,1]\times[0,1]$ is a Cauchy sequence in the metric space $C([0,1],[0,1]\times[0,1])$.

Proof. Let $\epsilon>0$. We pick $N\in\mathbb{N}_0$ with $\frac{1}{2^N}<\frac{\epsilon}{2}$. Let $m,n\geq N$. For any $t\in[0,1]$ we have

$$\|f_m(t)-f_n(t)\|_{\text{Eucl}} \;\underset{\uparrow}{\leq}\; \text{diameter of square of side length } \tfrac{1}{2^N} \;\underset{\uparrow}{=}\; \sqrt{2}\cdot\tfrac{1}{2^N} \;\underset{\uparrow}{<}\; \tfrac{\epsilon}{\sqrt{2}}.$$

it follows from Property (c) that for $m,n\geq N$ and $t\in[0,1]$ $f_m(t),f_n(t)$ are contained in a square of side length $\frac{1}{2^N}$ — see page 356 — choice of N

This implies that for $m,n\geq N$ we have $d(f_m,f_n)\leq\frac{\epsilon}{\sqrt{2}}<\epsilon$. ⊞

It follows from Claim 2 together with the Function Space–Metric Proposition 10.20 that the resulting map

$$\begin{array}{rcl} f\colon [0,1] & \to & [0,1]\times[0,1] \\ t & \mapsto & f(t)=\lim\limits_{n\to\infty} f_n(t) \end{array}$$

is continuous.[154]

Claim 3. The image of the map $f\colon[0,1]\to[0,1]\times[0,1]$ is dense.

Proof. Let U be an open subset of $[0,1]\times[0,1]$. We need to show that $f([0,1])\cap U\neq\varnothing$. It follows from an elementary argument, similar to the discussion on page 280, that there exists an $N\in\mathbb{N}_0$ and a $(k,l)\in Q_N$ such that $W^N_{(k,l)}\subseteq U$. By Property (a) there exists an $x\in[0,1]$ with $f_N(x)\in W^N_{(k,l)}$. By Property (c) we know that for each $n\geq N$ we have $f_n(x)\in W^N_{(k,l)}$. Since $W^N_{(k,l)}$ is closed we obtain from the Closed Subset Limit Lemma 3.35 that $f(x)=\lim\limits_{n\to\infty}f_n(x)\in W^N_{(k,l)}\subseteq U$. We have thus shown that $f([0,1])\cap U\neq\varnothing$. ⊞

Claim 4. The map $f\colon[0,1]\to[0,1]\times[0,1]$ is surjective.

Proof. Since $[0,1]$ is compact and since $[0,1]\times[0,1]$ is Hausdorff we obtain from the Compact Image Lemma 4.15 together with the Compact-Closed Lemma 3.39 (2) that $f([0,1])$ is a closed subset of $[0,1]\times[0,1]$. By Claim 3 we know that $f([0,1])$ is dense in $[0,1]\times[0,1]$. It follows immediately from the definition of a dense subset that the only closed subset that is dense is the total space. Thus we have shown that $f([0,1])=[0,1]\times[0,1]$. ■

Proof of Theorem 10.21 for $m\geq 2$. We need to show that for each $n\geq 2$ there exists a surjective continuous map $[0,1]\to[0,1]^m$. We prove this statement by induction on m. By

[154] Note though that f is nowhere differentiable.

the above we know that there exists a surjective continuous map $f\colon [0,1] \to [0,1]^2$. Now suppose we have found a surjective continuous map $\varphi\colon [0,1] \to [0,1]^m$ for some $m \geq 2$. The map[155]

$$[0,1] \xrightarrow{f} [0,1] \times [0,1] \xrightarrow{\operatorname{diag}(\mathrm{id},\varphi)} [0,1] \times [0,1]^m = [0,1]^{m+1}$$

is evidently surjective. ■

Let $m \geq 2$. Recall that in the $\mathbb{R}$-$\mathbb{R}^2$-Non Homeomorphism Lemma 4.36 we showed that $[0,1]$ and $[0,1]^m$ are not homeomorphic. By the Compact-Hausdorff Proposition 4.23 (3) this implies that there is no continuous map $[0,1] \to [0,1]^m$ that is surjective and injective. Nonetheless, the following variation on the Space-Filling Curve Theorem 10.21 holds:

Proposition 10.23. (Injective Curve-but-Positive Area Proposition) There exists an *injective* continuous map $f\colon [0,1] \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure.

Remark. An injective continuous map $f\colon S^1 \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure is called an *Osgood curve*. □

Proof. In 1903 William Osgood [**Osg1903**] and Henri Lebesgue [**Leb1903**] independently showed that there exist injective continuous maps $f\colon [0,1] \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure. Many more examples of such maps are given in [**Sagan1994**, Chapter 8.2]. In Exercise 10.15 we will see how this fact can be used to find injective continuous maps $g\colon S^1 \to [0,1]^2$ such that $g(S^1)$ has non-zero 2-dimensional Lebesgue measure. ■

10.7. **The Baire Category Theorem.** The formulation of the Baire Category Theorem requires the following basic definition.

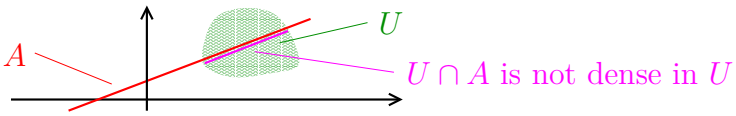
Definition 10.24. Let X be a topological space and let $A \subseteq X$.

(1) As in Definition 3.25 we say that A is dense in X if every open non-empty subset of X has a non-empty intersection with A.

(2) We say that A is nowhere dense in X if there is no open non-empty subset U of X such that $U \cap A$ is dense in U.

Examples.

(1) Let $A \subseteq \mathbb{R}^n$ be an affine hyperplane[156] of codimension ≥ 1. In Exercise 10.17 we will see that A is a nowhere dense subset of $\mathbb{R}^n$.

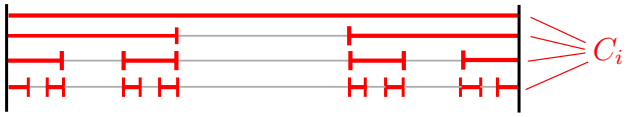


(2) In the following we recall the definition of the Cantor set from Definition 3.45. We set $C_0 := [0,1]$ and iteratively we define sets C_i, $i \in \mathbb{N}$ as follows:

$$C_i := \tfrac{1}{3} \cdot C_{i-1} \cup \left(\tfrac{2}{3} + \tfrac{1}{3} \cdot C_{i-1}\right).$$

[155]Note that it follows from the Topological-Product Proposition 7.2 (1) that the middle map is indeed continuous.

The Cantor set is defined as the intersection $C \coloneqq \bigcap_{i \in \mathbb{N}_0} C_i$. Note that for $i \in \mathbb{N}_0$ the set C_i does not contain an interval of length $> \frac{1}{3^i}$.



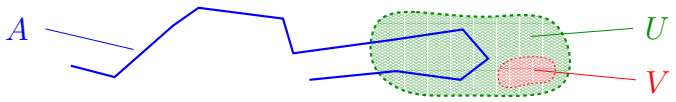
It follows that the Cantor set is nowhere dense. On the other hand we saw in Exercise 4.20 that the Cantor set is uncountable. This shows that $\mathbb{R}$ contains an uncountable nowhere dense subset. □

We have the following analog to the Density-Interior-Closure Lemma 3.26.

Lemma 10.25. (Nowhere Dense–Characterization Lemma) Let X be a topological space and let $A \subseteq X$ be a subset. The following two statements are equivalent:
(1) A is a nowhere dense in X.
(2) The interior of the closure $\overline{A}$ is the empty set, or more succinctly, if $\operatorname{int}(\overline{A}) = \varnothing$.

Proof. We have the following equivalences of statements:

$$
\begin{aligned}
& A \text{ is nowhere dense in } X \\
\iff & \forall_{\varnothing \neq U \subseteq X \text{ open}} \quad U \cap A \text{ is not dense in } U \\
\iff & \forall_{\varnothing \neq U \subseteq X \text{ open}} \ \exists_{\varnothing \neq V \subseteq U \text{ open}} \quad A \cap V = \varnothing \\
\iff & \forall_{\varnothing \neq U \subseteq X \text{ open}} \ \exists_{\varnothing \neq V \subseteq U \text{ open}} \quad A \subseteq (X \setminus V) \\
\iff & \forall_{\varnothing \neq U \subseteq X \text{ open}} \ \exists_{B \subsetneq X \text{ closed with } X \setminus U \subseteq B} \quad A \subseteq B \qquad \text{substitution } B = X \setminus V \text{ or } V = X \setminus B \\
\iff & \forall_{\varnothing \neq U \subseteq X \text{ open}} \ \exists_{C \subseteq X \text{ closed with } A \subseteq C} \quad U \nsubseteq C \qquad \text{substitution } C = (X \setminus U) \cup B \text{ or } B = C \\
\iff & \text{the interior of the closure } \overline{A} \text{ is the empty set.}
\end{aligned}
$$



This discussion gives us the desired result. ■

Theorem 10.26. (Baire Category Theorem) Let X be one of the following:
(1) X is a complete metric space, or
(2) X is a regionally compact Hausdorff space.[157]
Then the union of countably many nowhere dense subsets of X has empty interior.

Remark. The Baire Category Theorem 10.26 has its origins in the work of René-Louis Baire [**Bair1899**]. The name of the theorem predates the advent of category theory. In particular the theorem has absolutely nothing to do with category theory. □

[156] A *hyperplane* of a vector space is a codimension-one vector subspace. An *affine hyperplane* of a vector space V is any subset of the form $P + H$ where $P \in V$ and where H is a hyperplane of V.
[157] We will only make use of Statement (1). Nonetheless it is nice to have Statement (2). It is a fun little exercise to show that the two hypotheses given in Statements (1) and (2) are independent of one another.

Examples.

(1) It follows from the Baire Category Theorem 10.26 (1) and the example on page 366 that the union of countably many affine hyperplanes of $\mathbb{R}^n$ of codimension ≥ 1 has empty interior.
(2) The union of countable many nowhere dense sets might well be dense. For example take $X = \mathbb{R}$ and let $\varphi\colon \mathbb{N} \to \mathbb{Q}$ be a bijection. Then the sets $A_n \coloneqq \{\varphi(n)\}$ are nowhere dense, but the union of the A_n is $\mathbb{Q}$, which is of course dense. □

To clarify the exposition we prove the two cases of the Baire Category Theorem separately.

Proof of Theorem 10.26 for complete metric spaces. After possibly taking duplicates we only have to deal with the case that we have a countably *infinite* family $\{A_i\}_{i\in\mathbb{N}}$ of nowhere dense subsets of X. We need to show that no open non-empty subset U of X is contained in $\bigcup_{i\in\mathbb{N}} A_i$. Thus let U be an open non-empty subset of X. Our next goal is to find a point x in $U \setminus \bigcup_{i\in\mathbb{N}} A_i$.

Claim 1. There exists a sequence of open non-empty subsets $V_1, V_2, V_3, \dots$ of X with the following three properties:
(0) $V_1 = U$,
(1) for each $i \in \mathbb{N}$ we have $\overline{V}_{i+1} \subseteq V_i \setminus \overline{A}_i$, and
(2) for each $i \in \mathbb{N}_{\geq 2}$ we have $\operatorname{diam}(V_i) \leq 2^{-i}$.

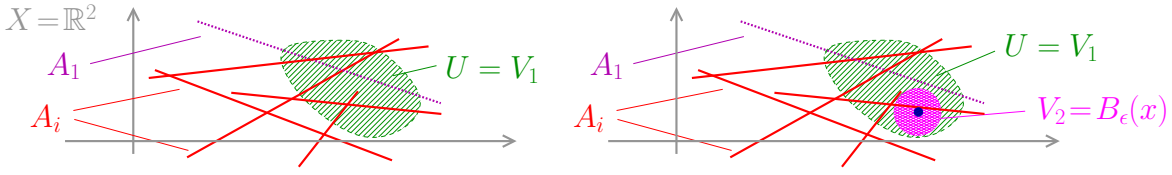
Proof. We construct the sequence iteratively. We set $U = V_1$ and we suppose that we have already constructed $V_1, \dots, V_i$. Since A_i is nowhere dense and since V_i is non-empty and open we know by the Nowhere Dense–Characterization Lemma 10.25 that $V_i \setminus \overline{A}_i$ is non-empty. By the Interior-Closure Lemma 3.22 we know that $\overline{A}_i$ is a closed subset. Thus we see that $V_i \setminus \overline{A}_i$ is in fact an open non-empty subset of X. Therefore there exists an open ball $B_\epsilon(x) \subseteq V_i \setminus \overline{A}_i$. We can arrange that $\epsilon \leq \frac{1}{2^i}$. We set $V_{i+1} = B_{\frac{\epsilon}{2}}(x)$. One can now easily verify that $V_1, \dots, V_{i+1}$ have the desired properties:

(1) Since $\overline{B}_{\frac{\epsilon}{2}}(x)$ is closed we see that $\overline{V}_{i+1} \subseteq \overline{B_{\frac{\epsilon}{2}}(x)} = \overline{B}_{\frac{\epsilon}{2}}(x) \subseteq B_\epsilon(x) \subseteq V_i \setminus \overline{A}_i$.
(2) By the discussion on page 356 we have $\operatorname{diam}(V_{i+1}) = \dim(B_{\frac{\epsilon}{2}}(x)) = 2 \cdot \frac{\epsilon}{2} \leq 2^{-i}$. ⊞

Let $i \in \mathbb{N}$. Since V_i is non-empty we can pick a point $x_i \in V_i$. Note that $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence since for $2 \leq i \leq j$ we have $d(x_i, x_j) \leq \operatorname{diam}(V_i) \leq 2^{-i}$. By hypothesis the metric space (X, d) is complete. This means that the Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ has a limit point $x \in X$.

Claim 2. The point x lies in each $\overline{V}_i$.

Proof. Let $i \in \mathbb{N}$. By construction we know that for all $n \geq i$ we have $x_n \in \overline{V}_i$. Since $\overline{V}_i$ is closed we obtain from the Closed Subset Limit Lemma 3.35 that $x \in \overline{V}_i$. ⊞



Finally note that

$$\begin{array}{ccccccccc} x & \in & \bigcap_{i\in\mathbb{N}} \overline{V}_{i+1} & \subseteq & \bigcap_{i\in\mathbb{N}} (V_i \setminus \overline{A}_i) & \subseteq & \bigcap_{i\in\mathbb{N}} (U \setminus A_i) & \subseteq & U \setminus \bigcup_{i\in\mathbb{N}} A_i. \\ & \uparrow & & & \uparrow & & \uparrow & & \uparrow \\ & \text{by the above} & & & \text{by construction of } V_{i+1} & & \text{since } V_i \subseteq U & & \text{de Morgan's Laws 1.7} \end{array}$$

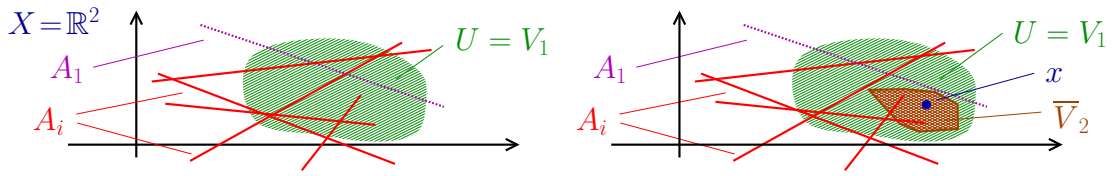
This shows that $U \not\subseteq \bigcup_{i\in\mathbb{N}} A_i$. ■

Proof of Theorem 10.26 for regionally compact Hausdorff spaces. As in the previous setting we need to show that if $\{A_i\}_{\in\mathbb{N}}$ is a family of nowhere dense subsets and if U is an open non-empty subset U of X, then $U \cap \bigcup_{i\in\mathbb{N}} A_i \neq \varnothing$. Thus let U be an open non-empty subset of X.

Claim. There exists a sequence of open non-empty subsets $V_1, V_2, V_3, \ldots$ with the following three properties:

(0) $V_1 = U$,
(1) for each $i \in \mathbb{N}$ we have $\overline{V}_{i+1} \subseteq V_i \setminus \overline{A}_i$, and
(2) for each $i \in \mathbb{N}_{\geq 2}$ the subspace $\overline{V}_i$ is compact.

Proof. We construct the sequence iteratively. We set $U = V_1$ and we suppose that we have already constructed $V_1, \ldots, V_i$. Since A_i is nowhere dense and since V_i is open we know by the Nowhere Dense–Characterization Lemma 10.25 that $V_i \setminus \overline{A}_i$ is non-empty. Since $\overline{A}_i$ is a closed subset we see that $V_i \setminus \overline{A}_i$ is in fact an open non-empty subset of X. We pick $x \in V_i \setminus \overline{A}_i$.



Since X is regionally compact and Hausdorff we know by the Regionally Compact–Locally Precompact Lemma 4.60 (1) that there exists an open neighborhood V_{i+1} of x such that $\overline{V}_{i+1}$ is compact and such that $\overline{V}_{i+1}$ is contained in $V_i \setminus \overline{A}_i$. Evidently $V_1, \ldots, V_{i+1}$ have the desired properties. ⊞

We make the following observations:

(1) The intersection of finitely many $\overline{V}_i$ is non-empty by construction.
(2) Since each $\overline{V}_i$ is closed and since X is Hausdorff we obtain from the Compact-Closed Lemma 3.39 that each $\overline{V}_i$ is a closed subset of $\overline{V}_1$.

Since $\overline{V}_1$ is compact it follows from the above, together with Exercise 3.27[158] that $\bigcap_{i\in\mathbb{N}} \overline{V}_i$ is non-empty. As in the previous case we note that we have $\bigcap_{i\in\mathbb{N}} \overline{V}_i \subseteq U \setminus \bigcup_{i\in\mathbb{N}} \overline{A}_i$. This shows that $U \subsetneq \bigcup_{i\in\mathbb{N}} \overline{A}_i$. ■

10.8. **Baire spaces and completely metrizable spaces.** The Baire Category Theorem 10.26 motivates several definitions.

[158]The content Exercise 3.27 is frankly just reformulation of the definition of a compact space using de Morgan's Laws 1.7.

Definition 10.27. A topological space that has the property that the union of countably many nowhere dense subsets of X has empty interior is called a Baire space.

Lemma 10.28. (Baire Space Characterization Lemma) Let X be a topological space. The following two statements are equivalent:

(1) X is a Baire space.
(2) For each countable collection of open dense subsets $\{U_n\}_{n\in\mathbb{N}}$ the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense.

Proof. We have the following equivalences of statements:

X is a Baire space
$\iff$ for every family $\{W_n\}_{n\in\mathbb{N}}$ of nowhere dense subsets the interior of $\bigcup_{n\in\mathbb{N}} W_n$ is $\varnothing$
$\overset{10.25}{\iff}$ for every family $\{W_n\}_{n\in\mathbb{N}}$ of subsets with $\operatorname{int}(\overline{W}_n) = \varnothing$ the interior of $\bigcup_{n\in\mathbb{N}} W_n$ is $\varnothing$
$\overset{3.22(2)}{\iff}$ for every family $\{A_n\}_{n\in\mathbb{N}}$ of closed subsets with $\operatorname{int}(A_n) = \varnothing$ the interior of $\bigcup_{n\in\mathbb{N}} A_n$ is $\varnothing$
$\underset{U_n=X\setminus A_n}{\overset{3.26}{\iff}}$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets the interior of $\bigcup_{n\in\mathbb{N}} (X \setminus U_n)$ is $\varnothing$
$\overset{3.26}{\iff}$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets the subset $X \setminus \bigcup_{n\in\mathbb{N}} (X \setminus U_n)$ is dense
$\overset{1.7}{\iff}$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense. ■

Lemma 10.29. If X is a Baire space, then every open subset V is also a Baire space.

Proof. The statement follows easily from the Open-Closed Inclusion Lemma 4.13 and the definitions. We leave it to the reader to fill in the details. ■

Definition 10.30. A topological space X is called completely metrizable if there exists a *complete* metric d on X such that the given topology agrees with the topology coming from d.

Example.

(1) By the Compact-Complete Proposition 10.17 we know that every compact metrizable space is already completely metrizable.
(2) The metric space $(\mathbb{R}, d^{\text{eucl}})$ is of course complete, hence $\mathbb{R}$ is completely metrizable.
(3) If a topological space X is homeomorphic to a completely metrizable topological space, then X is evidently also completely metrizable.
(4) We consider $X = (-1, 1)$. If we equip $(-1, 1)$ with the usual Euclidean metric d^{eucl}, then $((-1, 1), d^{\text{eucl}})$ is of course not complete. But the map $(-1, 1) \to \mathbb{R}$ given by $t \mapsto \tan(\frac{\pi}{2}t)$ is a homeomorphism. Thus it follows from (2) and (3) that $(-1, 1)$ is completely metrizable. □

With these definitions we can now reformulate the Baire Category Theorem 10.26 (1).

Theorem 10.26 (Baire Category Theorem)
(1) If X is a topological space that is completely metrizable, then X is a Baire space.

We conclude this section with the following amusing corollary. It is an amusing exercise to try to prove the corollary from scratch, i.e. without appealing to the Baire Category Theorem 10.26 (1).

Corollary 10.31. (ℚ–Not Completely Metrizable Corollary) The topological space ℚ is metrizable but not completely metrizable.

Proof. Evidently ℚ is metrizable. But ℚ is not a Baire space. This can be seen as follows. Since ℚ is countable there exists a bijection $\varphi\colon \mathbb{N} \to \mathbb{Q}$. For each $n \in \mathbb{N}$ we set $U_n := \{\varphi(n)\}$. Each U_n is nowhere dense, but $\bigcup_{n\in\mathbb{N}} U_n = \mathbb{Q}$. So we have found a countably many nowhere dense subsets such that the interior of the union is non-empty. Since ℚ is not a Baire space it follows from the Baire Category Theorem 10.26 (1) that ℚ is not completely metrizable. ■

10.9. Application of the Baire Category Theorem 10.26 to sequences of maps.

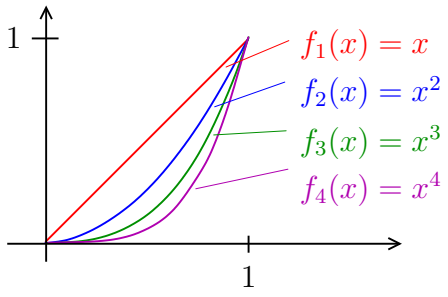
We conclude this chapter with an application of the Baire Category Theorem 10.26 to analysis. We will not make use of the concepts and results of this section. But hopefully the Limit Map–Discontinuities Proposition 10.34 gives the reader an idea of what the Baire Category Theorem 10.26 "is good for" in analysis.

Definition 10.32. Let X be a topological space and let Y be a Hausdorff space. Furthermore let $\{f_n\colon X \to Y\}_{n\in\mathbb{N}}$ be a sequence of continuous maps. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise, if for each $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in Y. We refer to the map[159]

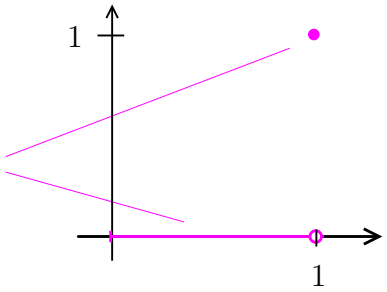
$$\begin{array}{ccc} X & \to & Y \\ x & \mapsto & \lim\limits_{n\to\infty} f_n(x) \end{array}$$

as the limit map.

Example. A popular way to shock students in a real analysis course is to show that the limit map of a pointwise convergent sequence of continuous maps is not necessarily continuous:



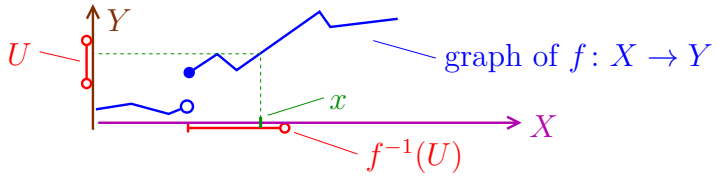
the sequence of maps $f_n(x) = x^n$ converges pointwise, but the limit map is not continuous at $x = 1$



The figure above shows one of standard examples for this phenomenon. □

[159] Since Y is Hausdorff we know by the Unique Limit Lemma 3.34 that we can unambiguously talk about the limit of the sequence $(f_n(x))_{n\in\mathbb{N}}$.

Definition 10.33. Let $f\colon X \to Y$ be a map between two topological spaces X and Y and let $x \in X$. We say that f is continuous at the point $x \in X$, if for each open neighborhood U of $f(x) \in Y$ the preimage $f^{-1}(U)$ is a neighborhood of $x \in X$.



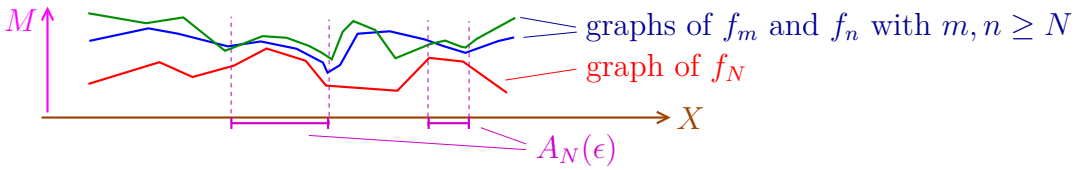
Remark. It follows easily from the JH-Openness Criterion 3.3 that a map $f\colon X \to Y$ is continuous at every point $x \in X$ if and only if the map f is continuous in the sense of Definition 4.1. □

The following proposition shows that the set of discontinuities of a limit map cannot get completely out of hand.

Proposition 10.34. (Limit Map–Discontinuities Proposition) Let X be a topological space, let (M, d) be a metric space and let $(f_n\colon X \to M)_{n\in\mathbb{N}}$ be a sequence of continuous maps which converges pointwise. If X is a Baire space, then the set of points at which the limit map $f\colon X \to M$ is continuous is dense in X.

Proof. Given $N \in \mathbb{N}$ and $\epsilon > 0$ we set

$$A_N(\epsilon) \;:=\; \{x \in X \,|\, d(f_m(x), f_n(x)) \le \epsilon \text{ for all } m, n \ge N\}$$



Claim 1.
(a) Each $A_N(\epsilon)$ is closed.
(b) $\bigcup_{N\in\mathbb{N}} A_N(\epsilon) = X$.

Proof.

(a) Note that $$A_N(\epsilon) \;=\; \bigcap_{m,n\ge N} (d \circ ((f_m, f_n)\colon X \to M \times M)\colon X \to \mathbb{R})^{-1}([-\epsilon, \epsilon]).$$
By the Metric-is-Continuous Lemma 10.2 we know that d is continuous. It follows from this description that $A_N(\epsilon)$ is closed.

(b) Let $x \in X$. By hypothesis $(f_n(x))_{n\in\mathbb{N}}$ is a convergent sequence. This implies that it is in particular a Cauchy sequence, which by definition implies that there exists an $N \in \mathbb{N}$ such that $x \in A_N(\epsilon)$. ⊞

Given $\epsilon > 0$ we set $$U(\epsilon) \;:=\; \bigcup_{N\in\mathbb{N}} \text{interior of } A_N(\epsilon).$$

By hypothesis X is a Baire space. Thus it follows from the Baire Space Characterization Lemma 10.28 that it suffices to prove the following claim:

Claim 2.
(a) For each $\epsilon > 0$ the set $U(\epsilon)$ is open and dense in X.
(b) The map f is continuous at every point in $\bigcap_{m\in\mathbb{N}} U(\frac{1}{m})$.

Proof.

(a) Since interiors of subsets are open we see that $U(\epsilon)$ is open. To show that $U(\epsilon)$ is dense it suffices, by definition of $U(\epsilon)$, to show that for each open subset $V \subseteq X$ there exists an $N \in \mathbb{N}$ such that $V \cap \operatorname{int}(A_N(\epsilon)) \neq \varnothing$. Thus let $V \subseteq X$ be an open subset.
  (i) It follows from Claim 1 (a) that each $V \cap A_N(\epsilon)$ is a closed subset of V.
  (ii) Since X is a Baire space and since $V \subseteq X$ is open we know by Lemma 10.29 that V itself is a Baire space.
  (iii) By Claim 1 (b) we know that the sets $V \cap A_N(\epsilon)$ form a cover of V. Since V is a Baire space we see that not all $V \cap A_N(\epsilon)$ can be nowhere dense in V. In other words there exists an $N \in \mathbb{N}$ such that $V \cap A_N(\epsilon)$ is not nowhere dense in V.
  (iv) By the Nowhere Dense–Characterization Lemma 10.25 we know that the interior of the closure of $V \cap A_N(\epsilon)$ in V is non-empty.
  (v) By (i) and (iv) we know that the interior of $V \cap A_N(\epsilon)$ is non-empty.
  In summary, we have shown that there exists an $N \in \mathbb{N}$ such that $V \cap A_N(\epsilon)$ contains a subset $W \subseteq V$ that is open in V. Since V is open in X it follows from the elementary Open-Closed Inclusion Lemma 4.13 that W is open in X. In particular we see that W is contained in the interior of $A_N(\epsilon)$.
(b) Let $x \in \bigcap_{m\in\mathbb{N}} U(\frac{1}{m})$. We need to show that f is continuous at the point x. It follows easily from the definition, see Definition 10.33, that it suffices to show that for every $\epsilon > 0$ there exists a neighborhood W of x such that for all $y \in W$ we have $d(f(x), f(y)) < \epsilon$. We make a few preparations:
  (i) We pick $k \in \mathbb{N}$ with $\frac{1}{k} < \frac{\epsilon}{3}$.
  (ii) Note that we know for free that $x \in U(\frac{1}{k})$. Therefore there exists an $n \in \mathbb{N}$ such that $x \in \operatorname{int}(A_n(\frac{1}{k}))$.
  (iii) Since f_n is continuous and since $\operatorname{int}(A_n(\frac{1}{k}))$ there exists a neighborhood W of $x \in X$ with $W \subseteq A_n(\frac{1}{k})$ such that $d(f_n(x), f_n(y)) < \frac{\epsilon}{3}$ for all $y \in W$.
  Now we see that for all $z \in W$ we have

$$d(f(z), f_n(z)) = \Big(\lim_{m\to\infty} f_m(z), f_n(z)\Big) \overset{\text{by continuity of metrics, see the Metric-is-Continuous Lemma 10.2, and Exercise 4.33}}{=} \lim_{m\to\infty} \underbrace{d(f_m(z), f_n(z))}_{< \frac{1}{k} \text{ for } m \geq n \text{ since } z \in W \subseteq A_n(\frac{1}{k})} \leq \frac{1}{k} < \frac{\epsilon}{3}.$$

  Finally note that for all $y \in W$ we have the following (in-) equalities:

$$d(f(x), f(y)) \overset{\text{triangle inequality}}{\leq} \underbrace{d(f(x), f_n(x))}_{< \frac{\epsilon}{3} \text{ by the above, since } x \in W} + \underbrace{d(f_n(x), f_n(y))}_{< \frac{\epsilon}{3}, \text{ since } y \in W, \text{ see (iii)}} + \underbrace{d(f_n(y), f(y))}_{< \frac{\epsilon}{3} \text{ by the above, since } y \in W} < \epsilon.$$

■

Remark. The Baire Category Theorem 10.26 has many other applications to analysis. For example one can use it to show that "most" continuous maps $\mathbb{R} \to \mathbb{R}$ are nowhere differentiable. We refer to [**JP2015**, Chapter 7] or alternatively to [**Ross2013**, Theorem 38.3] for details. □

10.10. Exercises for Chapter 10.

Exercise 10.1. Let (X, d) be a metric space, let $K \subseteq X$ be a compact subspace and let U be an open neighborhood of $K \subseteq X$. Show that there exists an $r > 0$ such that for each $x \in K$ we have $B_r(x) \subseteq U$.

Exercise 10.2. Let (X, d) be a metric space. Show that the map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

is continuous.

Exercise 10.3. Let (X, d) be a metric space. We suppose that X is compact. Furthermore let $f\colon X \to X$ be a continuous map with $f(x) \neq x$ for all $x \in X$. Show that there exists an $\epsilon > 0$ such that $d(f(x), x) \geq \epsilon$ for all $x \in X$.

Exercise 10.4. Let (X, d) be a metric space. Does there exist a metric e on X such that e induces same the topology on X as d, but such that (X, e) has finite diameter?

Exercise 10.5. Let (X, d) be a metric space and let $f\colon X \to \mathbb{R}$ be a continuous map. We now define

$$\begin{array}{rcl} \widetilde{d}\colon X \times X & \to & \mathbb{R}_0 \\ (x, y) & \mapsto & d(x, y) + |f(x) - f(y)|. \end{array}$$

Show that $\widetilde{d}$ is a metric and show that $\widetilde{d}$ induces the same topology as d.

Exercise 10.6. Let (X, d) be a metric space and let $B \subseteq X$ be a non-empty subset. Show that the map

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x, B) \coloneqq \inf\{d(x, b) \mid b \in B\} \end{array}$$

is continuous.



Exercise 10.7. Suppose that X is a metrizable topological space. Show that given any closed subset $A \subseteq X$ there exists a continuous map $f\colon X \to \mathbb{R}$ such that $f^{-1}(0) = A$.
Hint. Use the Distance-to-Subspace Lemma 10.4.

Exercise 10.8. Show that the statement of the Distance-Minimizer Lemma 10.6 does not generalize to general metric spaces. More precisely, give an example of a metric space X, a compact subset $K \subseteq X$ and a closed subset $L \subseteq X$ such that there do not exist $x \in K$ and $y \in L$ with $d(x, y) = d(K, L)$.

Exercise 10.9. Let (X, d) be a metric space. We denote by $\mathcal{P}(X)$ its power set. We consider the map

$$\begin{array}{rcl} d\colon \mathcal{P}(X)\setminus\{\varnothing\} \times \mathcal{P}(X)\setminus\{\varnothing\} & \to & \mathbb{R}_{\geq 0} \\ (A,B) & \mapsto & d(A,B). \end{array}$$

Which properties of a metric does this map satisfy?

Exercise 10.10. We consider the metric space $X = \mathbb{R}$. Let K and L be two closed non-empty disjoint subsets of $\mathbb{R}$. Is it necessarily true that $d(K,L) > 0$?

Exercise 10.11. Let X be a metric space and let A, B be two non-empty subsets of X. Suppose that $\operatorname{diam}(A) + \operatorname{diam}(B) < d(A,B)$. Show that $A \cap B = \varnothing$.

Exercise 10.12. Let X be a topological space and let $A, B \subseteq X$ be two *open* subsets such that $X = A \cup B$. Show that if X and $A \cap B$ are path-connected and non-empty, then A and B are also path-connected.
Hint. Use the Lebesgue Lemma 10.8.
Remark. This statement can be viewed as a partial converse to the Union Connected Lemma 4.39 (1). We proved the analog for "connected" in Exercise 4.29.

Exercise 10.13. Let $A \subseteq \mathbb{R}^n$ be a subset. We equip A with the usual Euclidean metric. Show that A is complete if and only if A is a closed subset of $\mathbb{R}^n$.

Exercise 10.14. Show that for any $n \in \mathbb{N}$ there exists a surjective continuous map $\mathbb{R} \to \mathbb{R}^n$.

Exercise 10.15. Recall that by the Injective Curve-but-Positive Area Proposition 10.23 we know that there exists an injective continuous map $f\colon [0,1] \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure. Show that there exists an injective continuous map $g\colon S^1 \to [0,1]^2$ such that $g(S^1)$ has non-zero 2-dimensional Lebesgue measure.

Exercise 10.16. Let $f\colon \mathbb{R} \to [0,1]$ be a continuous map that satisfies the following three conditions:
(1) for every $t \in \mathbb{R}$ we have $f(t+2) = f(t)$,
(2) for every $t \in [0, \frac{1}{3}]$ we have $f(t) = 0$,
(3) for every $t \in [\frac{2}{3}, 1]$ we have $f(t) = 1$.
Given $t \in \mathbb{R}$ we set

$$\varphi(t) \coloneqq \sum_{n=1}^{\infty} 2^{-n} \cdot f(3^{2n-2} \cdot t), \qquad \psi(t) \coloneqq \sum_{n=1}^{\infty} 2^{-n} \cdot f(3^{2n-1} \cdot t).$$

Prove the following three statements:
(a) The maps $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ are continuous.
(b) The map $t \mapsto (\varphi(t), \psi(t))$ is a surjection $[0,1] \to [0,1] \times [0,1]$.
(c) Let $n \in \mathbb{N}$. How can you modify the above construction of continuous maps to get a space-filling path $[0,1] \to [0,1]^n$?

Remark. This explicit description of a space-filling path was found by Isaac Schoenberg [**Schoe1938**]. It nicely complements the constructions given in Exercise 7.25 and the Space-Filling Curve Theorem 10.21.

Exercise 10.17. Show that every affine hyperplane in $\mathbb{R}^n$ of codimension ≥ 1 is nowhere dense.

Exercise 10.18. The Baire Category Theorem 10.26 (1), together with the Nowhere Dense–Characterization Lemma 10.25, implies that $\mathbb{R}$ cannot be written as a countable

union of *closed* subsets having empty interiors. Show that this conclusion does not hold, if the subsets are not required to be closed.

Exercise 10.19. Let X be a topological space.

(a) Show that the union of finitely many nowhere dense subsets is again nowhere dense.
(b) Is the union of countably many nowhere dense subsets necessarily nowhere dense?

Exercise 10.20. Show that the conclusion of the Baire Category Theorem 10.26 does not necessarily hold for incomplete metric spaces. More precisely, provide an example of a metric space X, a countable family $\{U_i\}_{i\in I}$ of nowhere dense subsets such that the union $\bigcup_{i\in I} U_i$ has non-empty interior.

Exercise 10.21. Let (M,d) be a complete metric space with no isolated points. Show that the set M is uncountable.
Remark. Recall that a point $x_0 \in X$ is called *isolated* if $\{x_0\}$ itself is an open subset.

Exercise 10.22. Let $f\colon (0,1) \to \mathbb{R}$ be an infinitely differentiable function such that for each $x \in (0,1)$ there exists an $n \in \mathbb{N}_0$ such that $f^{(n)}(x) = 0$. We want to show that f is a polynomial function. In the following we outline a proof by contradiction. Suppose that f is not a polynomial function.

(a) We consider the sets

$$\begin{aligned} S_n &:= \{x \in (0,1) \mid f^{(n)}(x) = 0\} \\ X &:= \left\{ x \in (0,1) \,\middle|\, \begin{array}{c}\text{there is no open neighborhood of } x \in (0,1) \text{ on} \\ \text{which } f \text{ is a polynomial function}\end{array} \right\} \end{aligned}$$

Show that S_n and X are closed and show that X has no isolated points.
(b) Apply the Baire Category Theorem 10.26 (1) to the covering $\{X \cap S_n\}_{n\in\mathbb{N}}$ of X to show that there exists an open interval (a,b) and an $n \in \mathbb{N}$ such that $(a,b) \cap X \neq \varnothing$ and such that $(a,b) \cap X \subseteq S_n$.
(c) Show that $(a,b) \cap X \subseteq S_m$ for all $m \geq n$.
Hint. Use that every point in X is an accumulation point and use the definition of derivatives.
(d) Let (s,t) be a maximal interval in $X \setminus (a,b)$. Note that f is described by a polynomial function of some degree d on (s,t). Use the fact that at least one of s,t lies in X to show that $d < n$.
Hint. Consider the d-th derivative.
(e) Use (d) to show that $f^{(n)} = 0$ on (a,b).
(f) Argue why (e) is a contradiction to $(a,b) \cap X$ being non-empty.

Remark. This statement was first proved in [**CSB1954**]. The details for this challenging exercise are worked out in [**Ross2013**, Theorem 38.5].

Exercise 10.23. We say that a map $f\colon \mathbb{R} \to \mathbb{R}$ is *nowhere monotonic* if there is no open non-empty subset $U \subseteq \mathbb{R}$ such that $f|_U\colon U \to \mathbb{R}$ is monotonic. Show that the set of nowhere monotonic maps is dense in the topological space $C(\mathbb{R},\mathbb{R})$.

Exercise 10.24. Give an example of a sequence $\{f_n\colon [0,1] \to \mathbb{R}\}_{n\in\mathbb{N}}$ of continuous maps that converges pointwise and such that the limit map $f\colon [0,1] \to \mathbb{R}$ is discontinuous at infinitely many points.

Exercise 10.25. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous map with the following property: For each $x \in \mathbb{R}_{>0}$ we have $\lim_{n\to\infty} f(nx) = 0$. Show that $\lim_{x\to\infty} f(x) = 0$.
Hint.

(a) Let $\epsilon > 0$. We set $D_n := \{x \in \mathbb{R}_{>0} \,|\, |f(mx)| \leq \epsilon \text{ for all } m \geq n\}$. Use the Baire Category Theorem 10.26 (1) to show that there exists an $n_0 \in \mathbb{N}$ such that D_{n_0} has non-empty interior.

(b) Use (a) to show that there exists an $r > 0$ such that $|f(x)| < \epsilon$ for all $x > r$.

Exercise 10.26. A topological space that has the property that every sequence admits a convergent subsequence is called *sequentially compact*.

(a) Show that every compact metric space is sequentially compact.
Remark. In fact a metric space is sequentially compact if and only if it is compact. The "only if"-direction is shown in [**Mun1975**, §28].
Remark. This exercise can be viewed as a generalization of the Bolzano-Weierstraß-theorem from real analysis.

(b) By Tychonoff's Theorem 7.14 we know that $[0,1]^{\mathbb{N}}$ is compact. Show that $[0,1]^{\mathbb{N}}$ is not sequentially compact.
Remark. Together with (a) this shows that $[0,1]^{\mathbb{N}}$ is not metrizable.

Exercise 10.27. Let X be a metric space. We say that a map $f\colon X \to X$ is an *isometry* if for every $x, y \in X$ we have $d(f(x), f(y)) = d(x, y)$. Show that if $f\colon X \to X$ is an isometry and if X is compact, then f is a homeomorphism.
Hint. It is clear that f is injective. By the Compact-Hausdorff Proposition 4.23 (3) it suffices to show that f is a surjection. Suppose that there exists an $a \in X \setminus f(X)$. There exists an $\epsilon > 0$ with $B_\epsilon(a) \subseteq X \setminus f(X)$. We consider the sequence given by $x_0 := a$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}_0$. Show that for every $m \neq n$ we have $d(x_m, x_n) > \epsilon$. Why is this not possible?

CHAPTER 11

Graphs

In this chapter we will first introduce the purely combinatorial concept of an oriented abstract graph. Afterwards we introduce the corresponding topological realizations. These notions are of interest for various reasons. Abstract graphs are ubiquitous in mathematics and at times they will also play a useful role in our discussions. The corresponding topological realizations form fairly simple, but nonetheless interesting examples of topological spaces.

This chapter is organized as follows:

- In Section 11.1 we will introduce the topology-free, purely combinatorial notion of an oriented abstract graph.
- In Section 11.2 we will introduce the topological realization of an oriented abstract graph. This concepts relates combinatorics to topology.
- In Section 11.3 we will introduce the notion of a topological graph. As we will see a topological graph is essentially just a topological space together with a homeomorphism to the topological realization of an abstract.
- In Section 11.4 we will introduce the notion of an *(unoriented)* abstract graphs and we will define their topological realizations. These notions are closely related to the earlier notion of an oriented abstract graph and their realizations.
- In Section 11.5 we will introduce the graph-theoretic notion of a tree.
- In Section 11.6 we will introduce the Euler characteristic of a finite graph.
- In Section 11.7 we will introduce the quotient of an oriented abstract graph by a subgraph.
- In Appendix 11.8 we will introduce the "topological valence" of a point in a topological space. We will see for topological graphs the topological valence agrees with the combinatorial valence.
- In Section 11.9 we will provide exercises for Chapter 11.

In the next Chapter 12 we will state and prove Urysohn's Lemma 12.5 and the Tietze Extension Theorem 12.6.

11.1. **Abstract graphs.** We start out with the following rather abstract definition.

Definition 11.1. An oriented abstract graph G is a quadruple (V, E, i, t) where V is a set, E is a set and $i, t\colon E \to V$ are maps from E to V.

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.

(2) Given $e \in E$ the point $i(e)$ is called the initial point of e and $t(e)$ is called the terminal point of e. We refer to $i(e)$ and $t(e)$ as the vertices of e.

(3) We say that an edge $e \in E$ is a loop if $i(e) = t(e)$.
(4) We say that G is finite (countable) if both V and E are finite (countable).

Remark. The origins of graph theory go back to Leonhard Euler [**Eul1736**] and his work in 1736 on the problem on the seven bridges of Königsberg, see Exercise 11.3. □

Example.

(A) We take $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$. Furthermore let $i\colon E \to V$ and $t\colon E \to V$ be the maps that are given by the following table:

	e	f	g	h
i	A	B	B	A
t	B	C	C	A

The quadruple (V, E, i, t) is an oriented abstract graph. We can try to visualize such an oriented abstract graph by drawing a point for each vertex and by drawing an arrow for each edge e that points from the initial point $i(e)$ to the endpoint $t(e)$.

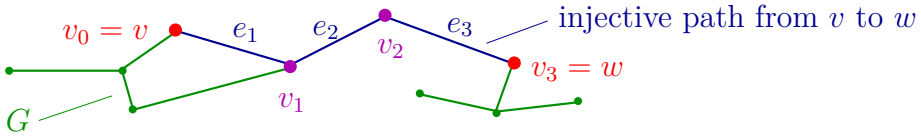


(B) Let $a, b \in \mathbb{Z}$. We denote by $G_{a,b} = (V, E, i, t)$ the oriented abstract graph with vertex set $V = \mathbb{Z}$, edge set $E = \mathbb{Z} \times \{0, 1\}$ and where $i(n, 0) = i(n, 1) = n$ and $t(n, 0) = n + a$ and $t(n, 1) = n + b$. We leave it to the reader to draw visualizations of $G_{a,b}$ for different values of a and b. □

The above figure already indicates that oriented abstract graphs give rise to topological spaces. The following definition is already somewhat topology-like.

Definition 11.2. Let $G = (V, E, i, t)$ be an oriented abstract graph.

(1) Let $v, w \in V$. A path of length $k \in \mathbb{N}$ from v to w is a sequence of edges $e_1, \dots, e_k$ such that there exist vertices $v = v_0, v_1, \dots, v_k = w$ such that for each $i \in \{1, \dots, k\}$ the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \dots, e_k$ is injective if the vertices $v_0, \dots, v_k$ are distinct. If $v = w$, then we refer to the empty path as an injective path of length zero from v to v.
(2) We say that G is connected if for every pair of vertices $v, w \in V$ there exists a path from v to w. One can easily show that if G is connected, then any two vertices can also be connected via an injective path.



Example. We want to determine whether or not the above examples (A) and (B) of oriented abstract graphs are connected.

(A) This oriented abstract graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property.

(B) Let $a, b \in \mathbb{Z}$. Basically by definition the oriented abstract graph $G_{a,b}$ is connected if and only if for every two integers $x, y \in \mathbb{Z}$ there exist $n, m \in \mathbb{Z}$ such that $x - y = m \cdot a + n \cdot b$. Elementary number theory says that this is the case if and only if a and b are coprime.

□

Definition 11.3.

(1) Let
$$\begin{aligned} G &= (V, E, i\colon E \to V, t\colon E \to V) \\ \text{and} \quad \widetilde{G} &= (\widetilde{V}, \widetilde{E}, \widetilde{i}\colon \widetilde{E} \to \widetilde{V}, \widetilde{t}\colon \widetilde{E} \to \widetilde{V}) \end{aligned}$$
be two oriented abstract graphs in the sense of Definition 11.1. We define a morphism $G \to \widetilde{G}$ of oriented abstract graphs to be a pair $(\alpha\colon V \to \widetilde{V}, \beta\colon E \to \widetilde{E})$ of maps such that for each $e \in E$ we have $\widetilde{i}(\beta(e)) = \alpha(i(e))$ and $\widetilde{t}(\beta(e)) = \alpha(t(e))$.

(2) We refer to the category $\mathcal{AbsGraph}$ with
$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsGraph}) &:= \text{all oriented abstract graphs,} \\ \mathrm{Mor}(G, \widetilde{G}) &:= \text{all maps from } G \text{ to } \widetilde{G} \text{ in the above sense} \end{aligned}$$
with the usual composition of maps as the category of oriented abstract graphs.

Example. The maps
$$\begin{aligned} (V, E, i\colon E \to V, t\colon E \to V) &\mapsto (V, E, t\colon E \to V, i\colon E \to V) \\ (f\colon G \to \widetilde{G}) &\mapsto (f\colon G \to \widetilde{G}) \end{aligned}$$
define a covariant functor from the category $\mathcal{AbsGraph}$ to itself. The effect on each oriented abstract graph is that it swaps the roles of the initial points and the endpoints. Pictorially speaking the functor "inverts the orientations of the arrows". □

The following definition is almost self explanatory.

Definition 11.4. Let $G = (V, E, i, t)$ be an oriented abstract graph. A subgraph of $G = (V, E, i, t)$ is an oriented abstract graph $G' = (V', E', i', t')$ such that $V' \subseteq V$ and $E' \subseteq E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$. Since i' and t' are determined by V' and E' we often suppress both of them from the notation.

Example.

(1) Every $v \in V$ gives rise to the subgraph with a single vertex v and no edge.

(2) Let $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be an oriented abstract graph and let $G = (V, E, i, t)$ be a subgraph. It follows immediately from the definitions that the inclusions $V \to \widetilde{V}$ and $E \to \widetilde{E}$ define a map $\iota\colon G \to \widetilde{G}$ of oriented abstract graphs in the above sense. □

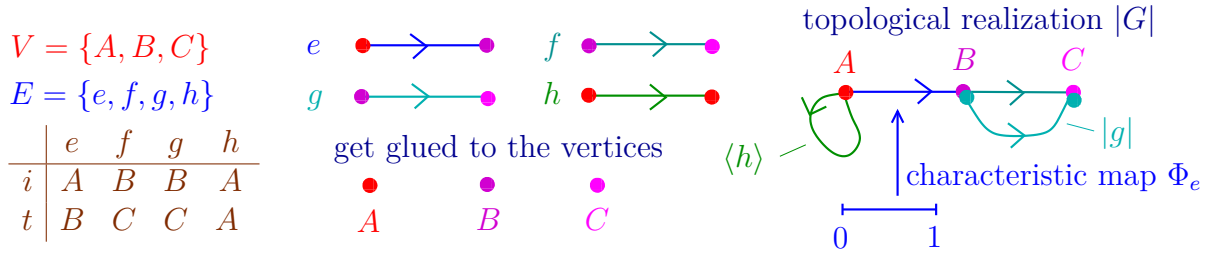
11.2. **The topological realization of an oriented abstract graph.** Surely the reader is bothered by the absence of topology. We will change this sorry state of affairs by introducing the topological realization of a graph. The idea is, basically, that the topological realization of an oriented abstract graph $G = (V, E, i, t)$ is given by one point for each vertex $v \in V$ and one interval for each edge $e \in E$ with endpoints given by $i(e)$ and $t(e)$. The definition below is the formalization of that idea.

Definition 11.5. Let $G = (V, E, i, t)$ be an oriented abstract graph.

(1) We define the topological realization of G as the topological space

$$|G| \;\coloneqq\; \big(\underset{\substack{\uparrow\\ \text{here } V \text{ is equipped}\\ \text{with the } \textit{discrete} \text{ topology}}}{V} \sqcup \bigsqcup_{e\in E} [0,1]\big)\big/\sim \qquad \begin{array}{l}\text{where for any } e \in E \text{ we have}\\ (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).\end{array}$$

(2) Let $e \in E$.
   (a) We refer to the map $\Phi_e \colon [0,1] \xrightarrow{x\mapsto(e,x)} \{e\} \times [0,1] \to |G|$ as the characteristic map of the edge e.
   (b) We write $|e| = \Phi_e([0,1])$ and we write $\langle e\rangle \coloneqq \Phi_e((0,1))$.

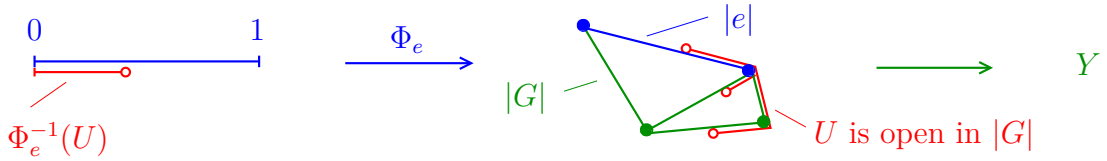


The following lemma gives us a practical way to understand the topology on the topological realization of an oriented abstract graph.

Lemma 11.6. (Abstract Graph-Topology Lemma) Let $G = (V, E, i, t)$ be an oriented abstract graph.

(1) For each $e \in E$ the characteristic map $\Phi_e \colon [0,1] \to |G|$ is continuous.
(2) Let $U \subseteq |G|$ be a subset.
   (a) $U \subseteq |G|$ is open $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is open in $[0,1]$.
   (b) $U \subseteq |G|$ is closed $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is closed in $[0,1]$.
(3) Let $f \colon |G| \to X$ be a map to some topological space X.

$$f \text{ is continuous} \iff \text{for each } e \in E \text{ the map } f \circ \Phi_e \colon [0,1] \to X \text{ is continuous.}$$



Example. Let $G = (V, E, i, t)$ be an oriented abstract graph. The following statements follow easily from the Abstract Graph-Topology Lemma 11.6:

(1) The obvious map $V \to |G|$ is injective and the image is a discrete subset of $|G|$.[160]
(2) Let $e \in E$. Let $\Phi_e \colon [0,1] \to |G|$ be the characteristic map.
   (a) $|e|$ is a closed subset of $|G|$ and $\langle e\rangle$ is an open subset of $|G|$.
   (b) If $i(e) \neq t(e)$, then Φ_e restricts to a homeomorphism $[0,1] \to |e|$.
   (c) If $i(e) = t(e)$, then Φ_e induces a homeomorphism $[0,1]/0 \sim 1 \to |e|$. □

[160]Here are the details why $V \subseteq |G|$ is discrete. Let $v \in V$. We consider $U \coloneqq |G| \setminus (V \setminus \{v\})$. Using the Abstract Graph-Topology Lemma 11.6 (2a) one can easily verify that U is an open neighborhood of $v \in |G|$. By construction U does not contain any other point of $V \subseteq |G|$. This shows that $V \subseteq |G|$ is discrete.

Proof of Lemma 11.6.

(1) This statement follows immediately from the Topological-Coproduct Proposition 7.26 (1a) and the Topological-Quotient Proposition 8.2 (1a).
(2) (a) This statement follows easily from the Topological-Coproduct Proposition 7.26 (0a) and the Topological-Quotient Proposition 8.2 (0a) and the observation that any subspace of a topological space consisting of a single point is open.
(b) This statement follows easily from the Topological-Coproduct Proposition 7.26 and the Topological-Quotient Proposition 8.2 and the observation that any subspace of a topological space consisting of a single point is closed.
(3) This statement follows immediately from (1) and (2a). ∎

The following lemma makes the unsurprising assertion that the above assignment of a topological space to an oriented abstract graph complex is functorial.

Lemma 11.7. (Topological Realization–Functor Lemma) The following two statements hold:

(1) Let $G = (V, E, i, t)$ and $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be two oriented abstract graphs. Furthermore let $f := (\alpha\colon V \to \widetilde{V}, \beta\colon E \to \widetilde{E})$ be a map between the two oriented abstract graphs. We consider the following map between the topological realizations of G and $\widetilde{G}$:

$$\begin{array}{rcl} & \overset{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)}{\downarrow} & \overset{\text{similarly } (e,0)\sim \widetilde{i}(e) \text{ and } (e,1)\sim \widetilde{t}(e)}{\downarrow} \\ |f|\colon |G| = \Big(V \sqcup \bigsqcup_{e\in E}[0,1]\Big)/\sim & \to & |\widetilde{G}| = \Big(\widetilde{V} \sqcup \bigsqcup_{e\in \widetilde{E}}[0,1]\Big)/\sim \\ [x] & \mapsto & \begin{cases} [\alpha(x)], & \text{if } x \in V, \\ [(\beta(e), t)], & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

This map $|f|\colon |G| \to |\widetilde{G}|$ is well-defined and continuous.

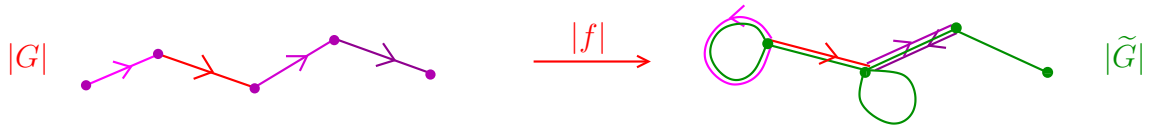
(2) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{AbsGraph}) & \to & \mathrm{Ob}(\mathcal{Top}) \\ G & \mapsto & |G|, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{AbsGraph}}(G, \widetilde{G}) & \to & \mathrm{Mor}_{\mathcal{Top}}(|G|, |\widetilde{G}|) \\ f & \mapsto & |f| \end{array}$$

form a covariant functor from the category $\mathcal{AbsGraph}$ of oriented abstract graphs to the category $\mathcal{Top}$ of topological spaces.



Example. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be an oriented abstract graph and let $G = (V, E, i, t)$ be a subgraph. The inclusions $V \to \widetilde{V}$ and $E \to \widetilde{E}$ define a map $\iota\colon G \to \widetilde{G}$ of oriented abstract graphs in the sense of the definition on Definition 11.3. Using the Topological Realization–Functor Lemma 11.7 (1), the Abstract Graph-Topology Lemma 11.6 (2a) and the Open-Closed–Injective Lemma 4.22 one can easily show that the induced map $|\iota|\colon |G| \to |\widetilde{G}|$ between the topological realizations is a closed embedding. In the following

we will sometimes use this observation as an excuse to view $|G|$ as a closed subset of $|\widetilde{G}|$. □

Proof of Lemma 11.7.

(1) It follows almost immediately from the definitions that the map is well-defined. Furthermore it follows easily from the Abstract Graph-Topology Lemma 11.6 (1) and (3) that the map is continuous.
(2) This statement follows easily from the definitions. ■

The following lemma can be viewed as the analog of the Product Topology-Properties Proposition 7.9 and the Disjoint Union Topology-Properties Lemma 7.28.

Lemma 11.8. (Topological Realization–Properties Lemma) Let $G = (V, E, i, t)$ be an oriented abstract graph.

(1) (a) Every compact subspace of $|G|$ is contained in the topological realization of a finite subgraph.
    (b) $|G|$ is compact if and only if G is finite.
(2) The following statements are equivalent:
    (a) G is connected.
    (b) Given any two distinct vertices v, w there exists an embedding $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = w$ and such that $\gamma([0,1]) \subseteq |G|$ is the union of edges.
    (c) The topological realization $|G|$ is path-connected.
    (d) The topological realization $|G|$ is connected.
(3) $|G|$ is Hausdorff.

Proof.

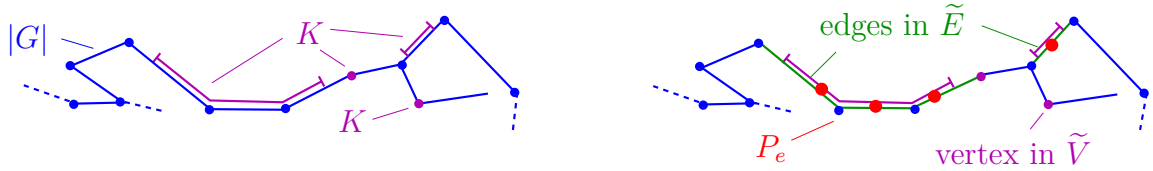
(1) (a) Let $K \subseteq |G|$ be a compact subspace. We set

$$\begin{aligned} \widetilde{E} &:= \{e \in E \,|\, K \cap \langle e\rangle \neq \varnothing\} \\ \widetilde{V} &:= \{v \in V \,|\, v \in K \text{ and there is no } e \in \widetilde{E} \text{ with } v \in |e|\}. \end{aligned}$$

We do the following:
- For each $e \in \widetilde{E}$ we pick a $P_e \in K \cap \langle e\rangle$ and we set $A_e := \langle e\rangle$.
- For each $v \in \widetilde{V}$ we set $B_v := \{v\} \cup \bigcup_e \langle e\rangle$, where we run over all $e \in E$ such that v is a vertex of e.
- We set $C := (|G| \setminus \bigcup_{e\in\widetilde{E}} \{P_e\}) \setminus \widetilde{V}$.

It follows easily from the Topological Realization–Properties Lemma 11.8 that each A_e, B_v and C is an open subset of $|G|$. Note that $\{A_e\}_{e\in\widetilde{E}}$, $\{B_v\}_{v\in\widetilde{V}}$ and C form an open cover of K. Since K is compact we see that we can cover K by finitely many of these open subsets. But each A_e contains a point of K, namely P_e, that does not lie in any of the other sets in the open cover. Similarly we see that each B_v contains a point in K, namely v, that does not lie in any of the other sets in the open cover. This shows that $\widetilde{E}$ and $\widetilde{V}$ are finite. Now we see that K is contained in the topological realization of the finite oriented abstract graph that is formed by $\widetilde{E}$, the vertices of the $e \in \widetilde{E}$ and by $\widetilde{V}$.

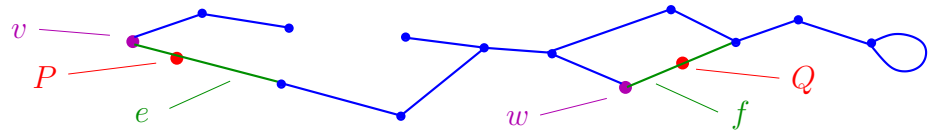
(b) If $|G|$ is compact, then it follows from (a) that G is finite. Next we assume that $G = (V, E, i, t)$ is finite. It follows from the Compact Image Lemma 4.15 that each



$|e|$ is compact. Since there are only finitely edges we obtain from the Compact–Union-Intersection Lemma 3.41 that $|G|$ is compact.

(2) (a)⇒(b). Let $v \neq w \in V$. Since we assume that G is connected there exists an injective path $e_1, \dots, e_k$ from v to w. Recall that this means that there exists distinct vertices $v = v_0, \dots, v_k = w$ such that the vertices of each e_i are v_{i-1}, v_i. We obtain a map $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$ such that on each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ the map γ is just a reparametrization of the characteristic map Φ_{e_i}, which now goes from v_{i-1} to v_i. It follows from the Abstract Graph-Topology Lemma 11.6 (1) and the Pasting Proposition 4.7 (2) that γ is continuous. Using the Abstract Graph-Topology Lemma 11.6 (2b) one can easily show that γ is a closed embedding.

(b)⇒(c). Let $P, Q \in |G|$. We pick edges e, f with $P \in |e|$ and $Q \in |f|$ and we pick vertices v of e and w of f. Since the characteristic maps of e and f are continuous one sees that there exists a path from P to v and a path from Q to w. By (b) there exists also a path from v to w. It follows from the Path-Equivalence Lemma 4.42 that there exists a path from P to Q.



(c)⇒(d). This follows from the Path-Connected-Implies-Connected Corollary 4.33.

(d)⇒(a). It follows easily from the definitions that G can be written as the disjoint union of subgraphs $\{G_i\}_{i\in I}$ such that each G_i is connected. It follows that $|G| = \bigsqcup_{i\in I} |G_i|$ and it follows easily from the Abstract Graph-Topology Lemma 11.6 (2a) that each $|G_i| \subseteq |G|$ is open. Since $|G|$ is connected we see that $\#I = 1$, i.e. G is connected.

(3) We will prove this statement in Exercise 11.2. ∎

11.3. **Topological graphs.** Even though Definition 11.5 of the topological realization of an oriented abstract graph is a perfectly fine, rigorous and precise definition, it is perhaps not the most intuitive definition, since the topological realizations are just abstract topological spaces. It would be much more fun to work with topological spaces that we can "see", i.e. for example we would like to work with subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$. This leads us to the following definition:

Definition 11.9. A topological graph is a triple $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ where the following holds:

(1) X is a topological space,
(2) $G = (V, E, i, t)$ is an oriented abstract graph,
(3) $\Theta\colon |G| \to X$ is a homeomorphism between the topological realization $|G|$ of G and the topological space X.

Given a topological graph $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ we introduce the following extra definitions:

(1) Given $v \in V$ we refer to $\Theta(v)$ as a vertex of the topological graph.
(2) Given $e \in E$ we refer to $\Theta(|e|)$ as an edge of the topological graph. Given a loop e we refer to $\Theta(|e|)$ as a loop of the topological graph.
(3) We refer to the oriented abstract graph $G = (V, E, i, t)$ as the underlying oriented abstract graph.
(4) Sometimes we refer, somewhat incorrectly, to X as a topological realization of G.

Sometimes we will just say that X is a topological graph. Furthermore on occasions we will suppress $G = (V, E, i, t)$ and $\Theta\colon |G| \to X$ from the notation.

The following lemma gives us a way to construct topological graphs with a given underlying oriented abstract graph.

Lemma 11.10. (Topological Realization–Embedding Lemma) Let $G = (V, E, i, t)$ be an oriented abstract graph and let $n \in \mathbb{N}_0$. Suppose we are given the following data:

(1) An injective map $\alpha\colon V \to \mathbb{R}^n$.
(2) For each $e \in E$ we are given a continuous map $\beta_e\colon [0,1] \to \mathbb{R}^n$ which is injective on $(0,1)$, which satisfies $\beta_e((0,1)) \subseteq \mathbb{R}^n \setminus \alpha(V)$ and which satisfies $\beta_e(0) = \alpha(i(e))$ and $\beta_e(1) = \alpha(t(e))$.
(3) For $e \neq f \in E$ we have $\beta_e((0,1)) \cap \beta_f((0,1)) = \varnothing$.

If G is infinite, then we also demand that the following condition is satisfied:

(4) For every $r \in \mathbb{R}_{\geq 0}$ there exist only finitely many $v \in V$ with $\alpha(v) \cap B_r^n(0) \neq \varnothing$ and there exist only finitely many $e \in E$ with $\beta_e([0,1]) \cap B_r^n(0) \neq \varnothing$.

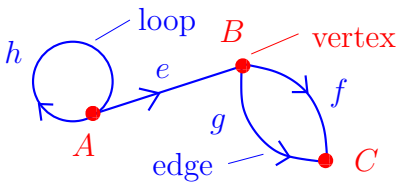
Then the following map is a closed embedding:

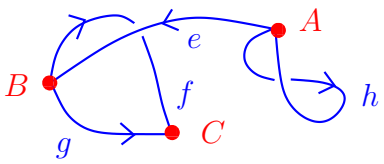
$$\begin{array}{rcl} |G| = \big(V \sqcup (E \times [0,1])\big)\big/\overset{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)}{\downarrow \sim} & \to & \mathbb{R}^n \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ \beta_e(t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

Proof. It follows easily from our hypotheses that the map is well-defined and that it is an injection. Furthermore we obtain from the Topological-Quotient Proposition 8.2 (1b) that the map is continuous. If G is finite, then it follows from the Topological Realization–Properties Lemma 11.8 (1) together with the Compact-Hausdorff Proposition 4.23 (3) that the given map $|G| \to \mathbb{R}^n$ is a closed embedding. In the general case the same conclusion follows from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26. ■

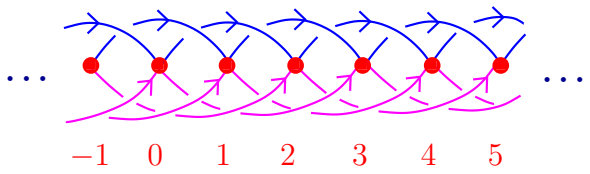
Example. We return to the two examples of oriented abstract graphs that we gave on page 379.

(A) As on page 379 we consider $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$ together with the maps $i, t\colon E \to V$ given by $i(e) = A$, $i(f) = B$, $i(g) = B$ and $i(h) = A$ and given by $t(e) = B$, $t(f) = C$, $t(g) = C$ and $t(h) = A$. In the figure below we show two topological graphs in $\mathbb{R}^3$ that are homeomorphic to $|G|$.

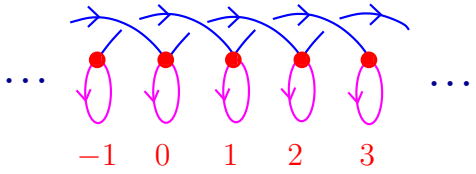




(B) Let $a, b \in \mathbb{Z}$. As on page 379 we consider the oriented abstract graph with vertex set $V = \mathbb{Z}$, edge set $\mathbb{Z} \times \{0, 1\}$ and where $i(n, 0) = i(n, 1) = n$ and $t(n, 0) = n + a$ and $t(n, 1) = b$.



topological realization of $G_{2,3}$ in $\mathbb{R}^3$



topological realization of $G_{0,2}$ in $\mathbb{R}^3$

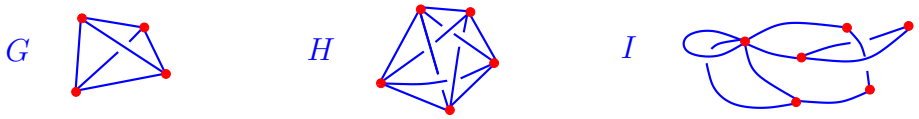
In the figure above we show topological realizations of the graphs $G_{2,3}$ and $G_{0,2}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$. □

Remark. In Exercise 11.6 we will see that given any finite oriented abstract graph G there exists an embedding of $|G|$ into $\mathbb{R}^3$. (Alternatively see [**CELR1994**, p. 2] or [**Mun1975**, p. 309] for a proof of this fact). □

We conclude this section with a question.

Question 11.11. Let G be a finite oriented abstract graph. Does there necessarily exist an embedding of $|G|$ into $\mathbb{R}^2$?

Example. We consider the finite oriented abstract graphs G, H and I that are (implicitly) illustrated in the figure below. Do their topological realizations admit embeddings into $\mathbb{R}^2$?



Surely the answer to Question 11.11 must be no, but it is not clear how one can prove this statement rigorously. We will return to this question in Section 157.4. □

11.4. **Undirected oriented abstract graphs and their topological realizations.** In this section we will introduce the notion of an (non-oriented) abstract graph and the corresponding topological realizations. In many settings abstract graphs are more suitable than the oriented abstract graphs from the previous section. But as we will see, the topological realization of an abstract graph is more unwieldy. Thus in these notes we will mostly work with oriented abstract graphs and we will leave it to the reader to generalize some of the later results to the setting of abstract graphs.

Without further ado, here is the definition.

Definition 11.12. An abstract graph G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

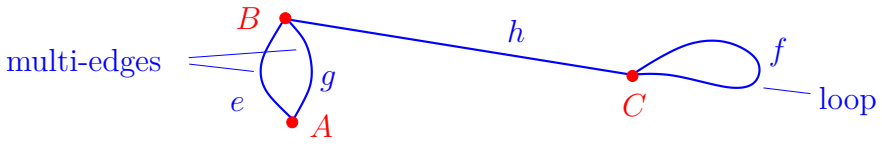
$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

Similar to the discussion in Definition 11.1 we will introduce the following extra definitions:

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.
(2) We say that an edge $e \in E$ is a loop if $\#\varphi(e) = 1$.
(3) Given $e \in E$ the points in $\varphi(e)$ are called the vertices of e.
(4) We say that G has multi-edges if there exist distinct edges e and e' with $\varphi(e) = \varphi(e')$.
(5) We say that G is finite (countable) if both V and E are finite (countable).

Example.

(1) Let $V = \{A, B, C\}$, $E = \{e, f, g, h\}$ and let $\varphi\colon E \to \mathcal{P}(V)$ be the map that is given by $\varphi(e) = \{A, B\}$, $\varphi(f) = \{C\}$, $\varphi(g) = \{A, B\}$ and $\varphi(h) = \{B, C\}$. This defines an abstract graph with loop f and multi-edges e and g. As before we can visualize such an abstract graph by drawing points for the vertices and by either drawing a Euclidean segment that either connects the two endpoints of an edge or by drawing a loop if the edge has a single endpoint.



(2) We take V to be the set of all subway stations in München and we take E to be the set of all segments of the subway system. The map φ then assigns to each $e \in E$ the two stations that are connected by the segment e. The triple (V, E, φ) is an abstract graph. □

So how is Definition 11.12 of an abstract graph related to Definition 11.1 of an oriented abstract graph? Clearly, if (V, E, i, t) is an oriented abstract graph, then $(V, E, \varphi(e) := \{i(e), t(e)\})$ is an abstract graph. For the inverse direction we introduce the following definition, even for the most part we will not really make use of it:

Definition 11.13. Let $G = (V, E, \varphi)$ be an abstract graph. An orientation for G is a pair of maps $i, t\colon E \to V$ such that for each $e \in E$ we have $\varphi(e) = \{i(e), t(e)\}$.[161]

If $G = (V, E, \varphi)$ is an abstract graph and if $i, t\colon E \to V$ is an orientation for G, then (V, E, i, t) is clearly an oriented abstract graph in the sense of Definition 11.1. Thus we see that an oriented graph is essentially the same as an abstract graph that is equipped with an orientation.

The following definition is the analog of Definition 11.3.

Definition 11.14.

(1) Let $G = (V, E, \varphi\colon E \to \mathcal{P}(V))$ and $G' = (V', E', \varphi'\colon E' \to \mathcal{P}(V'))$ be two abstract graphs. We define a morphism $G \to G'$ between the two abstract graphs to be a pair

$(\alpha\colon V \to V', \beta\colon E \to E')$ of maps such that for every $e \in E$ we have the equality $\varphi'(\beta(e)) = \alpha(\varphi(e))$.

(2) We refer to the category *UndirAbsGraph* with

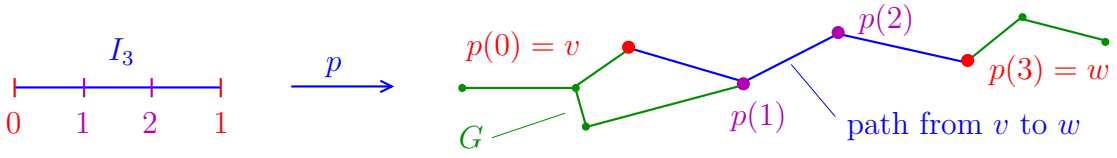
$$\begin{aligned} \mathrm{Ob}(\mathit{UndirAbsGraph}) &:= \text{all abstract graphs,} \\ \mathrm{Mor}(G, G') &:= \text{all maps from } G \text{ to } G' \text{ in the above sense} \end{aligned}$$

with the hopefully obvious composition of maps as the category of abstract graphs.

Definition 11.15.

(1) Let $n \in \mathbb{N}_0$. We denote by I_n the graph that is given by vertex set $\{0, \dots, n\}$, the edge set $0, \dots, n-1$ and the map $\varphi(i) = \{i, i+1\}$ for $i = 0, \dots, n-1$.

(2) We say that an abstract graph G is connected if given any vertices v, w there exists a map $p\colon I_n \to G$ of abstract graphs with $p(0) = v$ and $p(n) = w$.



The following totally elementary lemma can be viewed as a combinatorial analog of the Connected-to-Discrete Lemma 4.37. We will make use of it precisely once.

Lemma 11.16. (Combinatorial-Connected-to-Discrete Lemma) In the following let $G = (V, E, i, t)$ be an oriented abstract graph or let $G = (V, E, \varphi)$ be an abstract graph. Let $f\colon V \to X$ be a map to some set X such that for every $e \in E$ the values of f on the vertices of e agree. If G is connected, then f is constant.

Proof. This lemma follows almost immediately from the definitions. ■

Remark. Let *AbsGraph* be the category of oriented abstract graphs and let *UndirAbsGraph* be the category of abstract graphs. It follows easily from the definitions that the map

$$\begin{aligned} F\colon \mathrm{Ob}(\mathit{AbsGraph}) &\to \mathrm{Ob}(\mathit{UndirAbsGraph}) \\ (V, E, i\colon E \to V, t\colon E \to V) &\mapsto \begin{pmatrix} V, E, & E & \to & \mathcal{P}(V) \\ & e & \mapsto & \{i(e), t(e)\} \end{pmatrix} \end{aligned}$$

together with the hopefully obvious maps from the morphisms in *AbsGraph* to the morphisms in *UndirAbsGraph* is a covariant functor from the category *AbsGraph* of oriented abstract graphs to the category *UndirAbsGraph* of abstract graphs.[162] One can easily verify that if G is an oriented abstract graph, then G is connected if and only if $F(G)$ is connected. □

Now comes the slightly intimidating definition of the topological realization of an abstract graph.

[162]It follows from the Axiom of Choice 1.1 that given any abstract graph G there exists an oriented abstract graph G' with $F(G') = G$.

Definition 11.17. Let $G = (V, E, \varphi)$ be an abstract graph. We define the topological realization of G as the topological space[163]

$$|G| \coloneqq \left(V \sqcup \bigsqcup_{\substack{e \in E \\ \#\varphi(e) = 1}} [0,1] \sqcup \bigsqcup_{\substack{e \in E \\ \#\varphi(e) = 2}} [0,1] \times \mathrm{Bij}(\{0,1\}, \varphi(e)) \right) \Big/ \sim$$

where $\sim$ is generated by the following two relations:

(1) For each $e \in E$ with $\#\varphi(e) = 1$ we identify 0 and 1 in the corresponding copy of $[0,1]$ with the unique point $\varphi(e) \in V$.
(2) For each $e \in E$ with $\#\varphi(e) = 2$ we make the following identifications:
  (a) For each $\gamma \in \mathrm{Bij}(\{0,1\}, \varphi(e))$ we set $(0, \gamma) \sim \gamma(0)$ and $(1, \gamma) \sim \gamma(1)$.
  (b) For $\alpha \neq \beta \in \mathrm{Bij}(\{0,1\}, \varphi(e))$ and each $t \in [0,1]$ we set $(t, \alpha) \sim (1-t, \beta)$.

This definition spawns the following analog of the definition of a topological graph in Definition 11.9.

Definition 11.18. A topological graph is a triple

$$(X, G = (V, E, \varphi), \Theta \colon |G| \to X)$$

where the following holds:

(1) X is a topological space,
(2) $G = (V, E, \varphi)$ is an abstract graph,
(3) $\Theta \colon |G| \to X$ is a homeomorphism between the topological realization $|G|$ of G and the topological space X.

Example. Let us consider the following figure:



[163]Let $e \in E$ be a vertex with $\#\varphi(e) = 2$. Note that in this case $\mathrm{Bij}(\{0,1\}, \varphi(e))$ has precisely two elements. The problem is that there is no "preferred" bijection, so we need to consider both in our definition.

Using a variation on the Topological Realization–Embedding Lemma 11.10, or alternatively using the Topological Realization–Homeomorphism Lemma 11.19 below one can see that the subset of $\mathbb{R}^2$ shown in the figure above is homeomorphic to the topological realization of the subway system of München, in particular it is a topological graph. □

Lemma 11.19. (Topological Realization–Homeomorphism Lemma) Suppose we are given an oriented abstract graph $G = (V, E, i, t)$. As above we consider the abstract graph $\widetilde{G} = (V, E, \varphi)$ where $\varphi\colon E \to \mathcal{P}(V)$ is the map that is given by $\varphi(e) = \{i(e), t(e)\}$. The map

$$\begin{array}{rcl} \Theta_G\colon |G| & \to & |\widetilde{G}| \\ [x] & \mapsto & \begin{cases} [x], & \text{if } x \in V, \\ [t \in [0,1]_e], & \text{if } x = (e,t) \in E \times [0,1] \text{ with } i(e) = t(e), \\ [(t,\gamma)], & \text{if } x = (e,t) \in E \times [0,1] \text{ with } i(e) \neq t(e), \text{ where } \gamma \text{ is} \\ & \text{given by } \gamma(0) = i(e),\ \gamma(1) = t(e) \end{cases} \end{array}$$

is a homeomorphism.

Proof. It follows fairly easily from the definitions that the map is well-defined and a bijection. Furthermore one can deduce easily from the Quotient-to-Quotient Lemma 8.6 that the map is continuous. Working directly with the definitions one sees that the map is open. In summary it follows from the Open-Closed–Injective Lemma 4.22 that the map is indeed a homeomorphism. ■

Remark.

(1) Perhaps by now it has become clear why the definition of the topological realization of an *undirected* oriented abstract graph is more painful than the definition of the topological realization of an oriented abstract graph. In both cases we want to attach one interval for each edge. If we are given maps $i, t\colon E \to V$, then we have a preferred way of doing so. If we are only given a map $\varphi\colon E \to \mathcal{P}(V)$ none of the two ways is preferred, thus we need to attach an interval in both ways and then we need to "anti-identify" the two intervals.

(2) As above let $\mathcal{Top}$ be the category of topological spaces, let $\mathcal{AbsGraph}$ be the category of oriented abstract graphs and let $\mathcal{UndirAbsGraph}$ be the category of abstract graphs. In the Topological Realization–Functor Lemma 11.7 we defined a covariant functor $|-|$ from $\mathcal{AbsGraph}$ to $\mathcal{Top}$. Furthermore on page 388 we defined a covariant functor F from $\mathcal{AbsGraph}$ to $\mathcal{UndirAbsGraph}$ such that for every oriented abstract graph the homeomorphism $\Theta_G\colon |G| \to |F(G)|$ between the topological realizations is a homeomorphism. It is natural to ask whether one can define a covariant functor Ψ from $\mathcal{UndirAbsGraph}$ to $\mathcal{Top}$ such that the homeomorphisms Θ_G are natural? Somewhat depressingly there is no clean way of defining an induced map $|f|\colon |G| \to |\widetilde{G}|$ if an edge with two vertices gets sent to a loop. In this case there is no canonical way to get an induced map on topological realizations. □

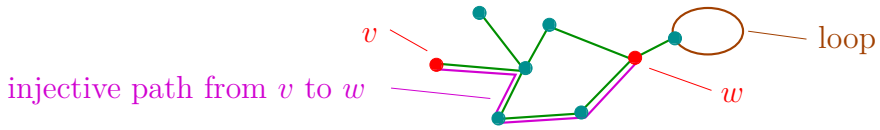
Remark. We now have two closely related notions, namely the notion of an oriented abstract graph (V, E, i, t) and of an abstract graph (V, E, φ). Both concepts have advantages and disadvantages:

(1) The definition of the topological realization in *AbsGraph* is much easier than the definition in *UndirAbsGraph*.
(2) The topological realization in *AbsGraph* is functorial, which is unfortunately not the case in *UndirAbsGraph*.
(3) In *AbsGraph* we have a well-defined notion of a characteristic map $\Phi_e\colon [0,1] \to |G|$. In *UndirAbsGraph* this notion is only well-defined up to precomposition by the homeomorphism $[0,1] \to [0,1]$ given by $t \mapsto 1-t$.
(4) The definition of connectedness in *AbsGraph* is less elegant than the corresponding definition in *UndirAbsGraph*. □

11.5. **Trees.** In this and the following section we again leave the realm of topology and we stay strictly in the world of oriented abstract graphs. For the reader's convenience we recall Definition 11.2:

Definition 11.20. Let G be an (undirected) oriented abstract graph.

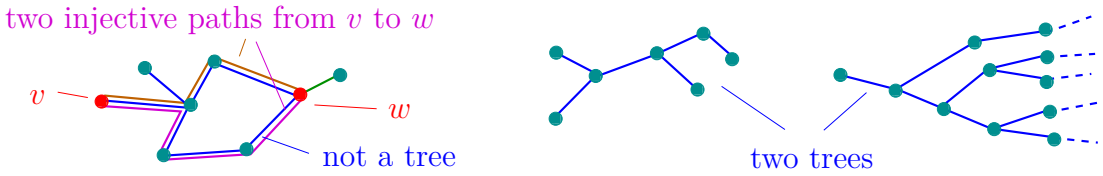
(1) An edge is called a loop if it has precisely one vertex.
(2) Let $v, w \in V$. A path of length k from v to w is a sequence of edges $e_1, \dots, e_k$ such that there exist vertices $v = v_0, v_1, \dots, v_k = w$ such that for each $i \in \{1, \dots, k\}$ the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \dots, e_k$ is injective if the vertices $v_0, \dots, v_k$ are distinct.



Now we can introduce the following basic but surprisingly useful class of oriented abstract graphs.

Definition 11.21. Let G be an (undirected) oriented abstract graph.

(1) We say that G is a tree if G is non-empty, if G does not contain a loop, and if given any $v, w \in V$ there exists a unique injective path from v to w.
(2) A subgraph that is a tree is called a subtree.



Lemma 11.22. (Tree–Basics Lemma)

(1) Any connected non-empty subgraph of a tree is again a tree.

(2) Let G be a graph. Let $T = (V, E)$ be a subtree of G and let $\{e_i\}_{i\in I}$ be a family of edges of G with the following properties:
   (a) The edge e_i has two vertices, one vertex t_i of e_i lies in T and the other vertex v_i does not lie in T.
   (b) The vertices v_i are distinct.
   Then $(V \cup \bigcup_{i\in I} \{v_i\}, E \cup \bigcup_{i\in I} \{e_i\})$ is also a subtree of G.

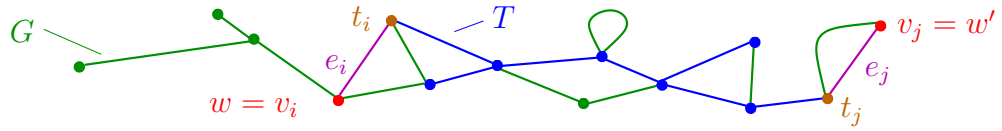
(3) Let G be a graph. If $T_1 \subseteq T_2 \subseteq \dots$ is a sequence of subtrees of G, then $\bigcup_{i\in\mathbb{N}} T_i$ is also a subtree of G.

Proof.

(1) This statement follows easily from the definitions.

(2) We set $W := V \cup \bigcup_{i\in I} \{v_i\}$ and $F := E \cup \bigcup_{i\in I} \{e_i\}$. We claim that (W, F) is a subtree. Since none of the e_i's is a loop we see that (W, F) has no loops. Next let $w, w' \in W$ be two vertices. We need to show that there exists a unique injective path from w to w'. There are several cases to distinguish.

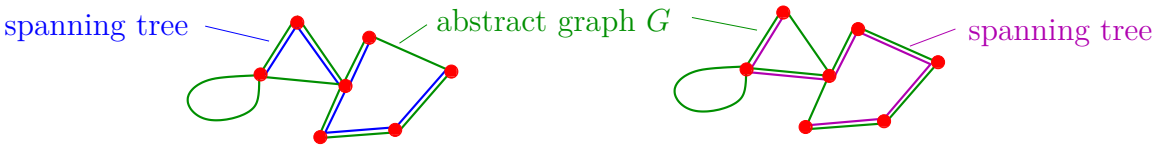
First assume that there exist $i, j \in I$ with $w = v_i$ and $w' = v_j$. Since (V, E) is a subtree we know that there exists an injective path in (V, E) from t_i to t_j. Together with e_i and e_j these form an injective path in (W, F) from $w = v_i$ to $w' = v_j$. We need to show that this is the unique injective path in (W, F) from $w = v_i$ to $w' = v_j$. Thus let $f_1, \dots, f_k$ be an injective path in (W, F) from w to w'. It follows from our hypothesis (a) and (b) that $f_1 = e_i$ and $f_k = e_j$ and that $f_2, \dots, f_{k-1}$ is an injective path which cannot involve any of the other e_l. Thus it is an injective path in (V, E), which is unique since T is a subtree. All other cases are treated in a similar fashion.



(3) We will prove this statement in Exercise 11.13. ■

Now we provide a few more definitions on (undirected) oriented abstract graphs.

Definition 11.23. Let G be an (undirected) oriented abstract graph. A spanning tree for G is a subtree of G that contains all vertices of G.

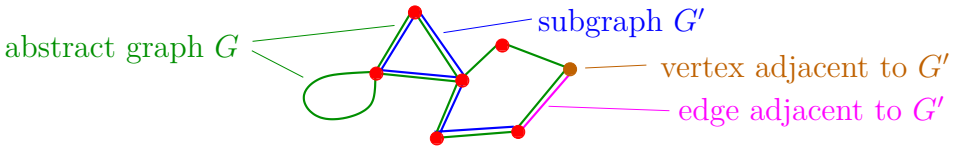


Remark. The example of the figure above shows that an oriented abstract graph usually has many different spanning trees. □

Definition 11.24. Let G be an (undirected) oriented abstract graph and let G' be a subgraph.

(1) We say that an edge e of G is adjacent to G' if e does not lie in G' and if at least one vertex of e lies in G'.

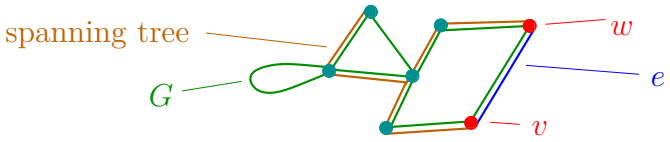
(2) We say that a vertex v of G is adjacent to G' if v does not lie in G' and if v is a vertex of an edge that is adjacent to G'.



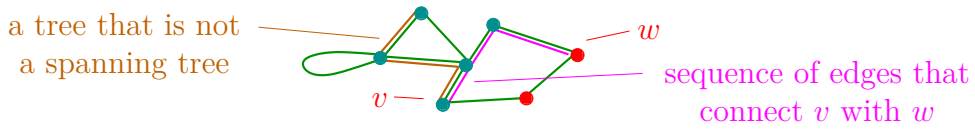
Proposition 11.25. (Spanning Tree–Characterization Proposition) Let G be a connected (undirected) oriented abstract graph and let $T \subseteq G$ be a tree. The following statements are equivalent:

(1) T is a spanning tree.
(2) T is maximal in the sense that if T is contained in a subtree T' of G, then $T = T'$.[164]
(3) T does not admit an adjacent vertex.

Proof.
(1)⇒(2). Let T be a spanning tree. We claim that T is maximal. Let H be a subgraph of G with $T \subsetneq H$. We need to show that H cannot be a tree. Since T contains all vertices and since $T \subsetneq H$ we see that there exists an edge e in H that does not lie in T. If e is a loop, then we know already that H is not a tree. Thus it remains to consider the case that e is not a loop. This means that e has two vertices v and w. Since T is connected there exists an injective path from v to w in T. On the other hand e is an injective path from v to w in H. Thus we see that there exist two injective paths in H from v to w. Thus H is not a tree.



(2)⇒(3). Let $T = (W, F)$ be a subtree of G that admits an adjacent vertex x. We need to show that T is not a maximal subtree. Let f be an edge such that one vertex is x and such that the other vertex lies in W. It follows from the Tree–Basics Lemma 11.22 (2) that $(W \cup \{x\}, F \cup \{f\})$ is a subtree of G that contains T.
(3)⇒(1). Let T be a subtree of G. We will show that if T is not a spanning tree, then T admits an adjacent vertex. Thus suppose that there exists a vertex w that does not lie in T. Let v be a vertex in T. Since G is connected there exists a path $e_1, \ldots, e_n$ from v to w.



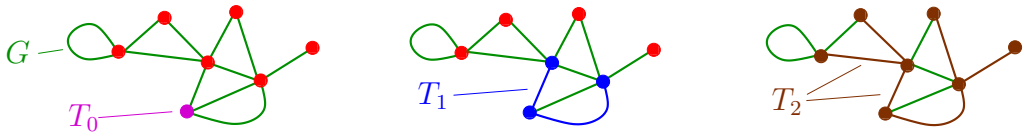
By definition this means that there exist vertices $v = v_0, \ldots, v_n = w$ such that the vertices of each e_i are v_{i-1}, v_i. The first vertex that does not lie in T is adjacent to T. ∎

We conclude this section with the following proposition:

[164] The equivalence of (1) and (2) explains why a spanning tree is often also called a *maximal tree*.

Proposition 11.26. (Spanning Tree–Existence Proposition) Every connected non-empty (undirected) oriented abstract graph admits a spanning tree.

Proof. Let $G = (V, E)$ be a connected non-empty (undirected) oriented abstract graph. Since G is non-empty we can pick a vertex $v \in V$. We set $T_0 := \{v\}$ and view T_0 as a subtree of G. Now we construct iteratively a sequence of subtrees $T_0 \subseteq T_1 \subseteq \dots$ as follows: Suppose we have a subtree $T_i = (V_i, E_i)$. We denote by W_i the set of all vertices of G that are adjacent to T_i. For each $w \in W_i$ we pick an edge e_w that is adjacent to T_i and such that w is a vertex of e_w.[165] We set $V_{i+1} := V_i \cup W_i$ and $E_{i+1} := E_i \cup \bigcup_{w \in W_i} \{e_w\}$. It follows from the Tree–Basics Lemma 11.22 (2) that T_{i+1} is also a tree.

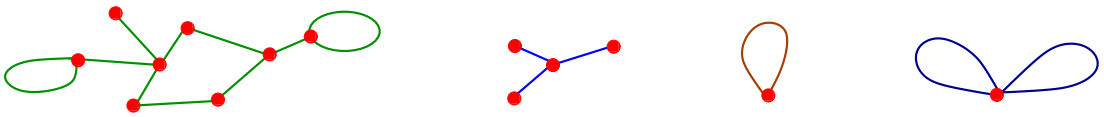


We set $T := \bigcup_{i \in \mathbb{N}_0} T_i$. It follows from the Tree–Basics Lemma 11.22 (3) that T is a tree. We claim that T is a spanning tree. By the (3)⇒(1)-Implication of the Spanning Tree–Characterization Proposition 11.25 it suffices to show that there is no vertex of G that is adjacent to T. In other words, we have to show that if there is an edge e of G with one vertex v in T, then the other vertex w also lies in T. If we are given such an edge e with one vertex v in T, then there exists an $i \in \mathbb{N}_0$ with $v \in T_i$, but then it follows from the definition of T_{i+1} that $w \in T_{i+1} \subseteq T$. ∎

11.6. **The Euler characteristic.** The following essential definition introduces a concept that will stay with us, in many different guises, throughout these lecture notes.

Definition 11.27. Let $G = (V, E)$ be a finite (undirected) oriented abstract graph. The Euler characteristic of G is defined as

$$\chi(G) := \#V - \#E = \text{number of vertices} - \text{number of edges}.$$



The following proposition gives a characterization of finite trees in terms of the Euler characteristic.

Proposition 11.28. (Tree Characterization Proposition) Let $G = (V, E)$ be a finite (undirected) oriented abstract graph. The following statements are equivalent:

(1) G is a tree.
(2) G is connected and $\chi(G) = 1$.

The proof of the Tree Characterization Proposition 11.28 rests on a lemma which is of independent interest. To formulate the lemma we need to introduce the following natural definition.

[165] Note that here we secretly use the Axiom of Choice 1.1.

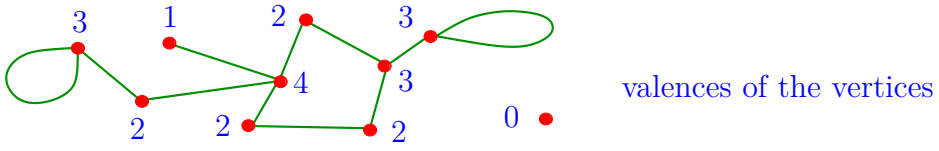
Definition 11.29.
(1) If $G = (V, E, i, t)$ is an oriented abstract graph, then the valence of a vertex v is defined as

$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

(2) If $G = (V, E, \varphi)$ is an abstract graph, then the valence of a vertex v is defined as

$$\begin{aligned}\text{valence}(v) \;:=\;& \#\{e \in E \,|\, v \in \varphi(e) \text{ and } \#\varphi(e) = 2\} \\ &+2 \cdot \#\{e \in E \,|\, v \in \varphi(e) \text{ and } \#\varphi(e) = 1\}.\end{aligned}$$

In a slightly less formal way, the valence of a vertex v is the number of edges at v, but hereby we have to count any loop at v twice, since it "goes twice into v".



Lemma 11.30. (Tree–Valence One Edge Lemma)[166] Let $G = (V, E)$ be a finite (undirected) oriented abstract graph. If $\chi(G) = 1$ and if G has at least one edge, then G admits at least two vertices of valence one.



Proof of Lemma 11.30. We prove the Tree–Valence One Edge Lemma 11.30 for oriented abstract graphs, the proof for abstract graphs is verbatim the same. Thus let $G = (V, E, i, t)$ be a finite oriented abstract graph with $\chi(G) = 1$ that has at least one edge.

(a) Since G has at least one edge and since it is by definition connected, we see that the oriented abstract graph G does not have a vertex of valence 0.
(b) We calculate that

$$\begin{aligned}\sum_{\text{vertex } v} \underbrace{\text{valence}(v)}_{\geq 1 \text{ by (a)}} &= \sum_{\text{vertex } v} \left(\#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}\right) \\ &\underset{\text{each edge contributes 2 to the sum}}{=} 2 \cdot \#\text{ edges} \underset{\text{by hypothesis we have } \chi(G) = 1}{=} 2 \cdot \#\text{vertices} - 2.\end{aligned}$$

By (a) the equality can only hold if there are at least two vertices with valence one. ■

Proof of the Tree Characterization Proposition 11.28. Let $G = (V, E)$ be a finite (undirected) oriented abstract graph.
(2)⇒(1). We now assume that G is connected and that $\chi(G) = 1$. We need to show that G is a tree. We will do so by inducting on the number of edges of G. If G has no edge, then it follows from $\chi(G) = 1$ that G has one vertex. Clearly this implies that G is a tree.

Now suppose that the statement holds for all graphs with n edges. Let $G = (V, E)$ be a connected graph with $n+1$ edges and with $\chi(G) = 1$. It follows from the Tree–Valence One Edge Lemma 11.30 that G admits a vertex v of valence one. Let e be the corresponding

edge. Note that $G' = (V \setminus \{v\}, E \setminus \{e\})$ is connected and that

$$\chi(G') \;=\; (\#V - 1) - (\#E - 1) \;=\; \#V - \#E \;=\; \chi(G) \;=\; 1.$$

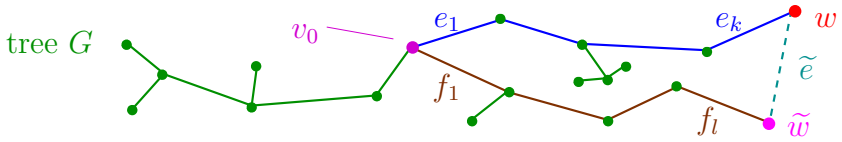
By induction G' is a tree. But then it follows from the Tree–Basics Lemma 11.22 (2) that G itself is also a tree.

(1)⇒(2). We suppose that G is a tree. If follows immediately from the definition of a tree that G is connected. We still need to show that $\chi(G) = 1$. We prove this statement by induction on the number of edges. If there is no edge then it follows from the fact that trees are non-empty and connected that there is precisely one vertex, thus $\chi(G) = 1$.

Next suppose that the statement holds if we have n edges. Let G be a tree with $n+1$ edges. We pick a vertex $v_0 \in V$. Let $w \in V$ be a vertex of maximal distance to v_0.

Claim. There exists precisely one edge e that has w as a vertex.

Proof. Let $e_1, \dots, e_k$ be a path of minimal length from v_0 to w. Since the path has minimal length we see that it is injective. Evidently w is a vertex of e_k. Suppose that there was a second edge $\widetilde{e}$ for which w is a vertex. As we mentioned before, a tree does not contain a loop. This implies that the edge $\widetilde{e}$ has a second vertex besides w. We denote this second vertex by $\widetilde{w}$. Since G is connected and since w has maximal distance from v_0 there exists an injective path $f_0, \dots, f_l$ from v_0 to $\widetilde{w}$ of length $\leq k$. Thus this path cannot be the path $e_1, \dots, e_k, \widetilde{e}$, which has length $k+1$.



We have thus found two different injective paths from v_0 to $\widetilde{w}$, namely $e_1, \dots, e_k, \widetilde{w}$ and $f_1, \dots, f_l$. But this contradicts the fact that we started out with a tree. ⊞

We can now consider the subgraph G' that is given by $V \setminus \{w\}$ and $E \setminus \{e\}$. One can easily show that this graph is again connected, thus it follows from the Tree–Basics Lemma 11.22 (1) that G' is a tree. We now see that

$$\chi(G) \;=\; \#V - \#E \;=\; (\#V - 1) - (\#E - 1) \;=\; \chi(G') \underset{\substack{\uparrow \\ \text{by our induction} \\ \text{hypothesis}}}{=} 1.$$

■

11.7. **Quotients of oriented abstract graphs.** In this section we will define and discuss quotients of oriented abstract graphs and of the corresponding topological realizations.

Definition 11.31. Let $G = (V, E, i, t)$ be an oriented abstract graph and let $H = (W, F)$ be a non-empty subgraph.

(1) We define $\sim$ to be the equivalence relation generated by $v \sim v'$ if v and v' are both vertices in H. We denote by $*$ the equivalence class defined by the vertices in H.
(2) We denote by $\pi \colon V \to V/\!\sim$ the map that is given by $v \mapsto [v]$.
(3) Finally we define the oriented abstract graph

$$G/H \;:=\; (V/\!\sim, E \setminus F, \pi \circ i \colon E \setminus F \to V/\!\sim, \pi \circ t \colon E \setminus F \to V/\!\sim).$$

We refer to G/H as the quotient of the oriented abstract graph G by the oriented abstract graph H.

The following lemma summarizes a few properties of the oriented abstract graph G/H.

Lemma 11.32. (Abstract Graph–Quotient Lemma) Let $G = (V, E, i, t)$ be an oriented abstract graph and let $H = (W, F)$ be a non-empty subgraph. We continue with the above notation.

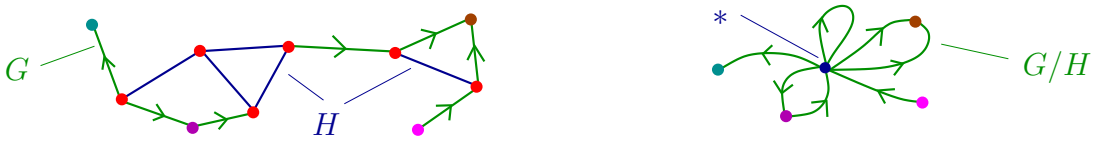
(1) If G is finite, then we have $\chi(G/H) = \chi(G) - \chi(H) + 1$.

(2) We consider the map

$$p\colon |G| = (V \sqcup (E \times [0,1]))/\overset{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)}{\sim} \to |G/H| = (V/\sim \sqcup ((E\setminus F)\times[0,1]))/\overset{\text{similarly } (e,0)\sim \pi(i(e)) \text{ and } (e,1)\sim \pi(t(e))}{\sim}$$

$$[x] \mapsto \begin{cases} [*], & \text{if } x \in |H|, \\ [\pi(x)], & \text{if } x \in V, \\ [(e,t)], & \text{if } x = (e,t) \text{ with } e \in E\setminus F \text{ and } t \in [0,1]. \end{cases}$$

The induced map $\overline{p}\colon |G|/|H| \to |G/H|$ is well-defined and it is a homeomorphism.



Proof.

(1) We calculate that

$$\begin{aligned} \chi(G/H) &= \#(V/\sim) - \#(E\setminus F) = (\#V - \#W + 1) - (\#E - \#F) \\ &= \chi(G) - \chi(H) + 1. \end{aligned}$$

(2) First of all note that it follows quite easily from the Abstract Graph-Topology Lemma 11.6 (3) together with the Topological-Quotient Proposition 8.2 (1b) that the map $\overline{p}\colon |G|/|H| \to |G/H|$ is continuous. We construct an inverse map $|G/H| \to |G|/|H|$ as follows.

(a) For each $e \in E \setminus F$ we consider the edge $X_e \subseteq |G/H|$ and we consider the edge $Y_e \subseteq |G|$. We have an obvious map $Y_e \to X_e$ which descends to a homeomorphism $Y_e/\sim \to X_e$ where $\sim$ is either the trivial relation or it is generated by identifying the two endpoints. The continuous map $Y_e \to |G|/|H|$ descends to a continuous map $Y_e/\sim \to |G|/|H|$. We now define $f_e\colon X_e \to |G|/|H|$ as the composition of the inverse of $Y_e/\sim \to X_e$ and the map $Y_e/\sim \to |G|/|H|$.

(b) One can easily verify that the maps f_e, together with the obvious maps on the vertices of G/H define a well-defined map $|G/H| \to |G|/|H|$.

(c) By construction the above map $|G/H| \to |G|/|H|$ is continuous on each X_e. Thus we obtain from the Abstract Graph-Topology Lemma 11.6 (3) that the map is continuous.

(d) We leave it to the reader to verify that the map in (c) is an inverse to the given map $\overline{p}\colon |G|/|H| \to |G/H|$.

We have thus shown that $\overline{p}$ is a homeomorphism. ∎

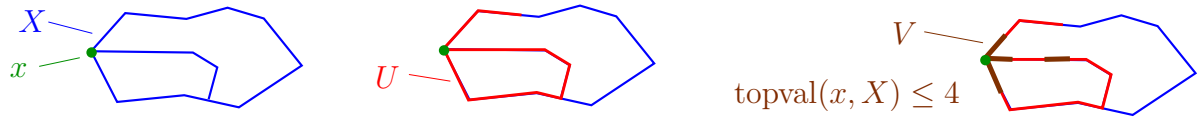
[166] It follows from the discussion on page 382 that it is legitimate to view $|H|$ as a subspace of $|G|$.

11.8. **Appendix: Valence of a vertex and topological valence.** In Definition 11.29 we introduced the notion of the valence of a vertex of an (undirected) oriented abstract graph. In this last section we will define the notion of the valence of a point in a topological space. We will see that for vertices of topological graphs the new definition agrees with the earlier definition. This is a fun and instructive topic, but the impatient reader can also skip the material without too much harm.

The definition of the valence of a vertex of an (undirected) oriented abstract graph was intuitive and natural definition. The following generalization of points of topological spaces does perhaps not qualify for these two adjectives.

Definition 11.33. Let X be a topological space and let $x \in X$. We define the topological valence of x as

$$\operatorname{topval}(x, X) := \min \left\{ n \in \mathbb{N} \cup \{\infty\} \;\middle|\; \begin{array}{c} \text{for every open neighborhood } U \text{ of } x \\ \text{there exists an open neighborhood } V \\ \text{of } x \text{ with } V \subseteq U \text{ such that } V \setminus \{x\} \text{ has} \\ \text{at least } n \text{ components} \end{array} \right\}.$$



The following lemma summarizes a few basic properties about the topological valence of points.

Lemma 11.34. (Topological Valence–Basics Lemma) Let X be a topological space and let $x \in X$.

(1) For every neighborhood W of x we have $\operatorname{topval}(x, W) = \operatorname{topval}(x, X)$.
(2) For every homeomorphism $f\colon X \to Y$ we have $\operatorname{topval}(x, X) = \operatorname{topval}(f(x), f(X))$.

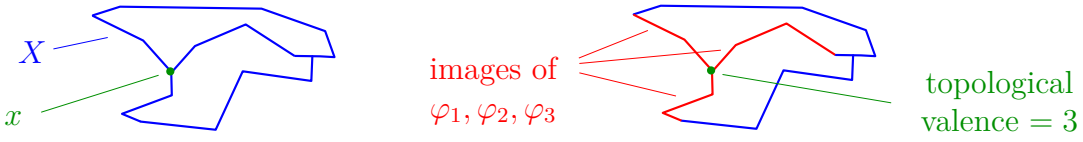
Proof. The lemma follows immediately from the definitions. ■

The following proposition allows for the calculation of the valence in many interesting settings.

Proposition 11.35. (Topological Valence Determination Proposition) Let X be a Hausdorff space and let $x \in X$ be a point. We now suppose that there exist embeddings $\varphi_1, \ldots, \varphi_n\colon [0,1] \to X$ with the following properties:

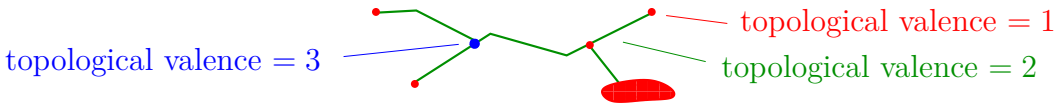
(a) $\varphi_1(0) = \cdots = \varphi_n(0) = x$,
(b) the images $\varphi_i((0,1])$, $i = 1, \ldots, n$ are pairwise disjoint,
(c) $\varphi_1([0,1]) \cup \cdots \cup \varphi_n([0,1])$ is a neighborhood of x.

Then $\operatorname{topval}(X, x) = n$.



Example.

(1) In the figure below we use the Topological Valence Determination Proposition 11.35 to determine some valences.



(2) Let I be an interval that consists of more than one point. It follows easily from the Topological Valence Determination Proposition 11.35 that the points in the interior of I have topological valence 2 and that the endpoints of I, if they are included, have topological valence 1. This observation implies for example that the half-open interval $[0,1)$ and the open interval $(0,1)$ are not homeomorphic. This gives in particular a solution to Exercise 4.48. □

Proof. We start out with the following claim:

Claim. There exists a $\lambda \in (0,1]$ such that for every $\epsilon \in (0,\lambda]$ the following holds:
(1) For each $i \in \{1,\dots,n\}$ the set $\varphi_i((0,\epsilon))$ is open in X.
(2) For each $i \in \{1,\dots,n\}$ the set $\varphi_i((0,\epsilon))$ is closed in $\varphi_1((0,\epsilon)) \cup \dots \cup \varphi_n((0,\epsilon))$.
(3) $\varphi_1([0,\epsilon)) \cup \dots \cup \varphi_n([0,\epsilon))$ is an open neighborhood of x.

Proof. Note that by Hypothesis (c) we know that there exists an open neighborhood W of x that is contained $\varphi_1([0,1]) \cup \dots \cup \varphi_n([0,1])$. By looking at the preimages of W under the maps $\varphi_1,\dots,\varphi_n$ we see that there exists a $\lambda \in (0,1]$ such that $\varphi_1([0,\lambda]),\dots,\varphi_n([0,\lambda])$ are contained in W. Now let $\epsilon \in (0,\lambda]$.

(1) Note that

$$\varphi_i((0,\epsilon)) \underset{\substack{\uparrow \\ \text{choice of } x \text{ and} \\ \text{Hypothesis (b)}}}{=} W \cap \Big(X \setminus \Big(\underbrace{\varphi_i([\epsilon,1]) \cup \bigcup_{j\neq i} \varphi_j([0,1])}_{\substack{\text{compact by the Compact Image Lemma 4.15,} \\ \text{since } X \text{ is Hausdorff we obtain from the} \\ \text{Compact-Closed Lemma 3.39 (2),} \\ \text{that the set is closed}}}\Big)\Big).$$

Thus we see that $\varphi_i((0,\epsilon))$ is an open subset of X.
(2) This statement follows immediately from (1).
(3) Almost the same argument as in (1) also implies (3). ⊞

We now turn to the actual proof that $\operatorname{topval}(X,x) = n$. First we want to show the inequality $\operatorname{topval}(X,x) \leq n$. Thus let U be a neighborhood of x. There exists a $\mu \in (0,\epsilon)$ such that for each $i \in \{1,\dots,n\}$ we have $\varphi_i((0,\mu)) \subseteq U$. By the above claim we know that $V := \varphi_1([0,\mu)) \cup \dots \cup \varphi_1([0,\mu))$ is an open neighborhood of X. Next note that $V \setminus \{x\}$ consists of the n open and closed subsets $\varphi_1((0,\mu)),\dots,\varphi_n((0,\mu))$. By the Path-Connected-Implies-Connected Corollary 4.33 we know that each of these sets is connected. Thus we obtain from the Component Lemma 4.48 (4c) that $V \setminus \{x\}$ has precisely n components.

Now let us show that $\operatorname{topval}(X,x) \geq n$. We set $U := \varphi_1((0,\epsilon)),\dots,\varphi_n((0,\epsilon))$. Let V be an open neighborhood of x that is contained in U. For each $i \in \{1,\dots,n\}$ we know that $\varphi_i^{-1}([0,\epsilon]) \cap V$ is open and that it contains 0. This implies that $\varphi_i^{-1}((0,\epsilon]) \cap (V \setminus \{x\})$ is also non-empty. By (2) this set is open and closed in $V \setminus \{x\}$. It follows from the Component Lemma 4.48 (4b) that $V \setminus \{x\}$ has at least n components. ■

The final proposition of this section shows that given an abstract graph the valence of a vertex agrees with the topological valence of the corresponding point on the topological realization.

Proposition 11.36. (Valence–Equality Proposition)

(1) Let $G = (V, E, \varphi)$ be an abstract graph. For every vertex $v \in V$ we have

$$\text{valence}(v) \;=\; \text{topval}(v, |G|) \;\in\; \mathbb{N}_0 \cup \{\infty\}.$$

(2) If $f\colon X = |G| \to Y = |H|$ is a homeomorphism between the topological realizations of abstract graphs G and H that sends vertices to vertices, then this homeomorphism preserves valences, viewed as numbers in $\mathbb{N}_0 \cup \{\infty\}$.

Remark. An alternative proof of the Valence–Equality Proposition 11.36 will be given in Exercise 150.16. □

Proof.

(1) If the valence(v) is finite, then the statement follows easily from the Topological Valence Determination Proposition 11.35. We leave it to the reader to modify the argument if valence(v) is infinite.

(2) This statement follows immediately from (1) and the Topological Valence–Basics Lemma 11.34 (2). ■

11.9. Exercises for Chapter 11.

Exercise 11.1. We consider the category *AbsGraph* of oriented abstract graphs. Let X, Y be oriented abstract graphs.

(a) Does the product of X and Y exist in the category *AbsGraph*?
(b) Does the coproduct of X and Y exist in the category *AbsGraph*?

Exercise 11.2. Let $G = (V, E, i, t)$ be an oriented abstract graph. Show that the topological realization $|G|$ is Hausdorff.

Exercise 11.3. In the below we see the seven bridges of Königsberg. Is it possible to cross all seven bridges in a walk without crossing a single bridge twice?

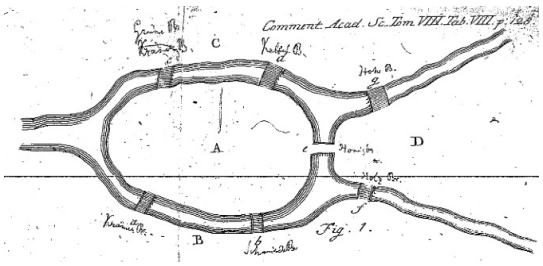
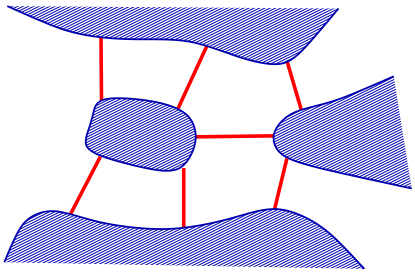


Illustration from Euler's paper [**Eul1736**]



Exercise 11.4. Let $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ be two abstract graphs. If the topological realizations $|G|$ and $|G'|$ are homeomorphic, does it follow that the abstract graphs G and G' are isomorphic?

Exercise 11.5. Are the two topological graphs G and H shown in the figure below homeomorphic?



Exercise 11.6.

(a) Let $A \subseteq \mathbb{R}^3$ be a finite set. We say that the points in A are in *general position* if no distinct three points in A are collinear and if no four points in A are coplanar. Show that given any $k \in \mathbb{N}$ there exists a subset $A \subseteq \mathbb{R}^3$ consisting of k points that is in general position.

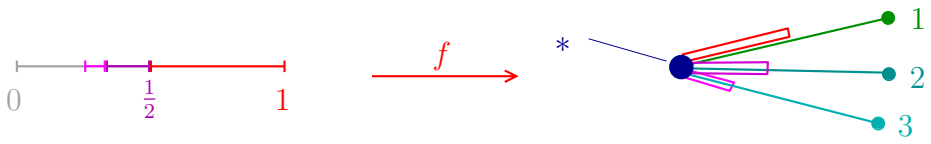
Remark. One approach is as follows. Show that if $t_1 < \cdots < t_k$ are real numbers, then the points $\{(t_i, t_i^2, t_i^3) \,|\, i = 1, \ldots, k\}$ are in general position.

(b) Show that given any finite oriented abstract graph G there exists an embedding of the topological realization $|G|$ into $\mathbb{R}^3$.

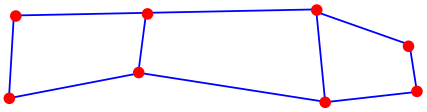
Exercise 11.7. Let $G = (V, E, i, t)$ be the oriented abstract graph with vertex set $V = \{*\} \cup \mathbb{N}$ and edge set $E = \mathbb{N}$ and with $i(n) = *$ and $t(n) = n$. For each $n \in \mathbb{N}$ let $\Phi_n \colon [0,1] \to |G|$ be the characteristic map of the edge $n \in \mathbb{N}$. We consider the following map

$$\begin{array}{rcl} f \colon [0,1] & \to & |G| \\ t & \mapsto & \begin{cases} *, & \text{if } t = 0, \\ \Phi_n\left(\frac{2n+1}{2n(n+1)} - \left|t - \frac{2n+1}{2n(n+1)}\right|\right), & \text{if } t \in \left[\frac{1}{n+1}, \frac{1}{n}\right]. \end{cases} \end{array}$$

In other words, on the interval $[\frac{1}{n+1}, \frac{1}{n}]$ the map f starts at $*$, goes into the n-th edge, and returns to $*$. Is this map continuous? If yes, why? If not, what is an open subset of $U \subseteq |G|$ such that preimage $f^{-1}(U)$ is not open in $[0,1]$?



Exercise 11.8. We consider the graph shown in the figure below. How many pairwise non-homeomorphic maximal trees can you find?



Exercise 11.9. We consider the oriented abstract graph $G = (V, E, \varphi)$ that is given by $V = \mathbb{Z} \cup \{*\}$, $E = \mathbb{Z}$ and where $\varphi \colon E \to \mathcal{P}(V)$ is given by $\varphi(n) = \{n, *\}$. We also consider the following subspace of $\mathbb{R}^2$:

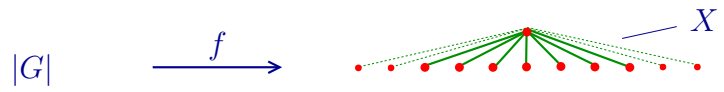
$$X \;:=\; \{(n, 0) \,|\, n \in \mathbb{Z}\} \cup \{(0, 1)\}.$$

There is an obvious bijection $f \colon |G| \to X$ that is given by $f(n) = (n, 0)$, $f(*) = (0, 1)$ and which sends each edge of $|G|$ linearly to the Euclidean segment connecting the corresponding endpoints in $\mathbb{R}^2$.

(a) Show that the map $f \colon |G| \to X$ is continuous.

(b) Show that the map $f \colon |G| \to X$ is *not* open.

Remark. This of course implies that f is not a homeomorphism.



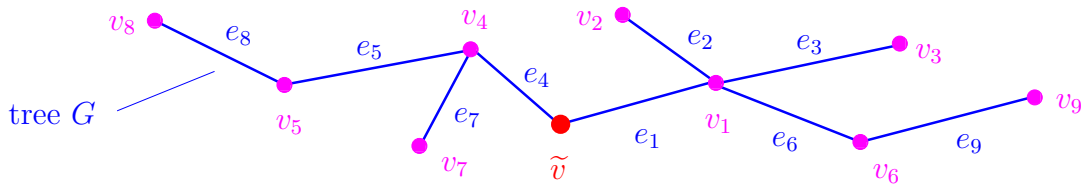
Exercise 11.10. Let $G = (V, E, i, t)$ be an oriented abstract graph.

(a) Show that if there exists a $v \in V$ of infinite valence, then there is no embedding of $|G|$ into any $\mathbb{R}^n$.
Hint. You could also first look at the more explicit Exercise 11.9.

(b) Suppose that every $v \in V$ has finite valence and that V is countable. Does $|G|$ necessarily admit an embedding into some $\mathbb{R}^n$?

Exercise 11.11. Let $G = (V, E, i, t)$ be an oriented abstract graph and let $\widetilde{v} \in V$ be a vertex. We suppose that G is a tree. Show that there exist distinct edges $e_1, \dots, e_k$ and distinct vertices $v_1, \dots, v_k$ with the following properties:

(1) We have $V = \{\widetilde{v}, v_1, \dots, v_k\}$ and $E = \{e_1, \dots, e_k\}$.
(2) For each $i \in \{1, \dots, k\}$ the vertex v_i is a vertex of the edge e_i.
(3) If some vertex v_i is a vertex of some e_j, then $i \leq j$.



Exercise 11.12. Let $G = (V, E, i, t)$ be a countable connected non-empty oriented abstract graph. We pick a vertex $v \in V$. Show that there exists a sequence $\{G_j = (V_j, E_j, i, t)\}_{j \in \mathbb{N}}$ of subgraphs with the following properties:

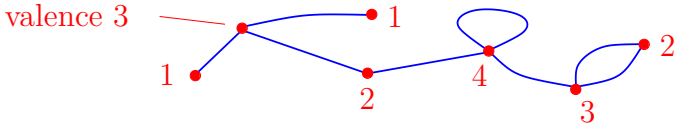
(1) Each G_j is finite and connected.
(2) Each G_j contains v.
(3) We have $G = \bigcup_{j \in \mathbb{N}} G_j$ in the sense that $V = \bigcup_{j \in \mathbb{N}} V_j$ and $E = \bigcup_{j \in \mathbb{N}} G_j$.

Exercise 11.13. Let G be a graph and let $T_1 \subseteq T_2 \subseteq \dots$ be a sequence of subtrees of G. Show that $\bigcup_{i \in \mathbb{N}} T_i$ is also a subtree.

Exercise 11.14. Let $G = (V, E, i, t)$ be a finite oriented abstract graph. Recall that the valence of a vertex v is defined as

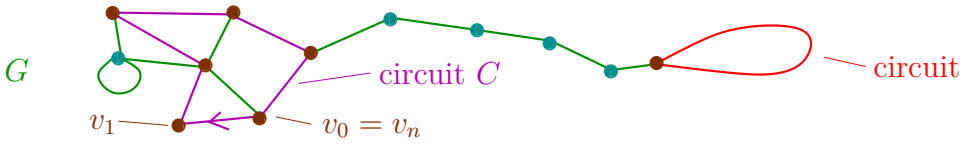
$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

Show that the number of vertices of odd valence is even.
Remark. This statement is known as the *Handshaking Lemma*, the name comes from the observation that the statement means that at an event where some people shake hands, an even number of people have shaken an odd number of other people's hands.



Exercise 11.15. We consider the topological space $X = \mathbb{Q}$. What is the topological valence $\mathrm{topval}(0, \mathbb{Q})$?

Exercise 11.16. Let $G = (V, E, i, t)$ be an oriented abstract graph. A *circuit of length n* in G is a sequence of edges $e_1, \dots, e_k$ such that there exist vertices $v_0, v_1, v_2, \dots, v_k = v_0$ such that the vertices $v_0, \dots, v_{n-1}$ are distinct and such that for each $i \in \{1, \dots, k\}$ the vertices of e_i are v_{i-1}, v_i. Show that G is a tree if and only if G is connected, non-empty and if G contains no circuits.



Exercise 11.17. An oriented abstract graph G is called *trivalent* if every vertex has valence three. We define a *circuit* as in Exercise 11.16. What is the number of isomorphism types of connected trivalent oriented abstract graphs with two disjoint circuits?

Exercise 11.18. Let $G = (V, E, \varphi)$ be an abstract graph and let π be a group. A *π-action on G* is a homomorphism $\Phi\colon \pi \to \mathrm{Aut}(G)$. We say that $v \in V$ is a *fixed vertex of the π-action* if for every $g \in \pi$ we have $\Phi(g)(v) = v$.

(a) Let T be an abstract graph that is a tree and let π be a finite group that acts on T. Show that there exists a finite subtree $\widetilde{T}$ such that the π-action restricts to a π-action on $\widetilde{T}$.

(b) Let $\widetilde{T}$ be a finite tree and let π be a finite group that acts on $\widetilde{T}$. Show that this action admits a fixed vertex or a fixed edge.
Remark. You can prove the statement by induction on $\#T$. You can perform the induction step by considering the vertices of valency ≥ 2.

CHAPTER 12

Urysohn's Lemma and the Tietze Extension Theorem

In this chapter we will state and prove Urysohn's Lemma 12.5 and the Tietze Extension Theorem 12.6. For the most part the proofs of the results we stated so far are fairly straightforward and could be assigned as (at times slightly challenging) homework exercises. The results of this chapter are the first really non-trivial statements.

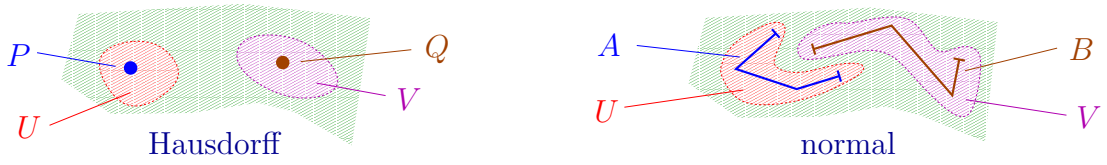
This chapter is organized as follows:

- In Section 12.1 we will (re-) introduce normal topological spaces.
- In Section 12.2 we will state and prove Urysohn's Lemma 12.5. This lemma says that if X is a normal topological space and $A, B \subseteq X$ are closed disjoint subsets of X, then there exists a continuous map $f\colon X \to \mathbb{R}$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.
- In Section 12.3 we will state and prove (using Urysohn's Lemma 12.5) the Tietze Extension Theorem 12.6. This theorem says, that under suitable hypothesis a continuous map from a subspace to $\mathbb{R}$ can be extended to a continuous map on the total space.
- In Section 12.4 we will provide exercises for Chapter 12.

In the next Chapter 13 we will introduce second-countable topological spaces and we will state and prove the Urysohn Metrization Theorem 13.8.

12.1. **Normal topological spaces.** Recall that according to Definition 3.27 a topological space X is called *Hausdorff* if any two distinct points $P \neq Q$ can be separated by disjoint open subsets, i.e. if there exist disjoint open neighborhoods U of P and V of Q. As we will see in the following two chapters, and especially in Section 13.4, there are many variations on this definition. In this chapter we will mostly care about the following concept, which we already introduced in Definition 3.48.

Definition 12.1. A topological space X is called normal if any two disjoint closed subsets A and B of X can be separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.



Remark.

(1) Note that the definitions of a normal topological space can differ slightly in the literature. For example [**Mun1975**, p. 195] imposes the extra condition that one-point sets are closed.

(2) Let X be a normal topological space. If every one-point subset is closed, then X is also Hausdorff.[167] But, as we will see in Exercise 12.1, without this condition there do in fact exist normal spaces that are not Hausdorff.
(3) Let X be the line with two zeros that we introduced in Definition 3.30. One can easily verify that every one-point subset is closed. We saw that X is not Hausdorff. Thus it follows from (2) that X is not normal either.
(4) In Exercise 12.3 we will give an example of a topological space that is Hausdorff but which is not normal.
(5) It follows easily from the Open-Closed Inclusion Lemma 4.13 (2) that any closed subset of a normal space is again normal. A topological space that has the property that every subspace is normal is called *hereditarily normal*. As we will see in the Separation Axioms Proposition 13.26 together with Exercise 13.18, there exist normal spaces that are not hereditarily normal. □

For the reader's convenience we recall the Compact Hausdorff-Normal Lemma 3.49. Together with the subsequent Metric Spaces-Normal Proposition 12.2 it provides us with a long list of examples of normal spaces.

Lemma 3.49. (Compact Hausdorff-Normal Lemma) Every compact Hausdorff space is normal.

The following proposition gives us an often convenient approach to showing that a topological space is normal. It implies, among many other things, that every subspace of some $\mathbb{R}^n$ is normal.

Proposition 12.2. (Metric Spaces-Normal Proposition) Let (X, d) be a metric space. If we equip X with the corresponding topology, then X is normal.

Proof. Let (X, d) be a metric space. We recall Definition 10.3: Given $x \in X$ and given a non-empty subset C of X we write

$$d(x, C) \coloneqq \inf\{d(x, c) \mid c \in C\}.$$

If C is closed, then we know by the Distance-to-Subspace Lemma 10.4 (2) that $d(x, C) = 0$ if and only if $x \in C$.

Now let A and B be two disjoint closed subsets of X. If one of A or B is empty, then the desired disjoint neighborhoods are given by the empty set and X. Thus it remains to consider the case that both A and B are non-empty.



In our context, since A and B are disjoint and closed, we conclude from the above that for every $x \in X$ we have $d(x, A) + d(x, B) > 0$. Thus it makes sense to consider the map

$$\begin{aligned} f\colon X &\to \mathbb{R} \\ x &\mapsto \frac{d(x, A)}{d(x, A) + d(x, B)}. \end{aligned}$$

[167] In Definition 13.25 we define a *T_0-space* to be a topological space that has the property that each one-point subset is closed.

By the Distance-to-Subspace Lemma 10.4 (1) we know that f is continuous. Evidently $f|_A \equiv 0$ and $f|_B \equiv 1$. It follows that $U := f^{-1}((-\infty, \frac{1}{4}))$ and $V := f^{-1}((\frac{3}{4}, \infty))$ have the desired properties. ■

The following lemma, which is an analog of the Points-in-Hausdorff Lemma 3.32, contains a convenient and frequently used property of normal spaces.

Lemma 12.3. (Normal-Open-Closed-Neighborhood Lemma) Let X be a topological space and let $A \subseteq X$ be a closed subset and $U \subseteq X$ be an open subset with $A \subseteq U$. If X is normal, then there exists an open subset $V \subseteq X$ with $A \subseteq V \subseteq \overline{V} \subseteq U$.



Proof. Let X be a topological space, let $A \subseteq X$ be a closed subset and let $U \subseteq X$ be an open subset with $A \subseteq U$. We set $B := X \setminus U$. This is a closed subset with $A \cap B = \varnothing$. Since X is normal there exist disjoint neighborhoods V of A and W of B. It remains to show that $\overline{V} \subseteq U$. In fact we have $V \subseteq X \setminus W \subseteq U$. But since W is open we know that $X \setminus W$ is closed. By Definition 3.21 of the closure$\overline{V}$ we have $\overline{V} \subseteq X \setminus W$. ■

Remark. The combination of the Paracompact-Criterion Lemma 14.5 and Dieudonné's Theorem 14.9 gives us many more examples of normal topological spaces. □

We conclude this short introduction to normal topological spaces with the following lemma which often gets used subconsciously.

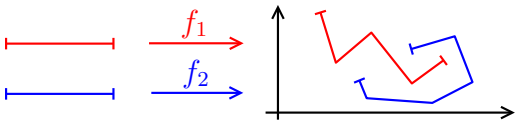
Lemma 12.4. (Closed Disjoint Embedding Lemma) Let Y be a normal space.

(1) Let $A_1, \dots, A_k \subseteq Y$ be disjoint subsets. If the A_i are all closed, then the natural map $A_1 \sqcup \dots \sqcup A_k \to A_1 \cup \dots \cup A_k$ is a homeomorphism.

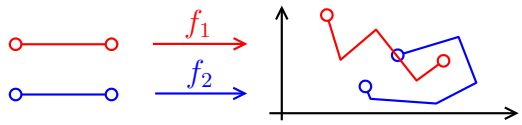
(2) Let $X_1, \dots, X_k$ be compact spaces and let $\varphi_i \colon X_i \to Y$ be embeddings with disjoint images. If Y is Hausdorff, then the map

$$\coprod_{i=1}^{k} f_i \colon \coprod_{i=1}^{k} X_i \;\to\; Y$$

is an embedding.



$f_1 \sqcup f_2 \colon X_1 \sqcup X_2 \to \mathbb{R}^2$ is an embedding



$f_1 \sqcup f_2 \colon X_1 \sqcup X_2 \to \mathbb{R}^2$ is *not* an embedding

Proof. Let Y be a normal space.

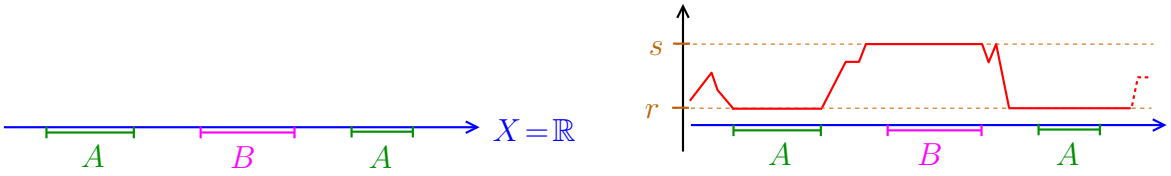
(1) Let $A_1, \dots, A_k \subseteq Y$ be disjoint closed subsets. Note that for any $i \in \{1, \dots, k\}$ the union of the A_j with $i \neq j$ is a finite union of closed subsets, thus it is a closed subset itself. Since Y is normal we see that there exists an open neighborhood of A_i that does not intersect any of the other A_j. It follows from this observation and the Disjoint Union Embedding Lemma 7.29 that the natural map $A_1 \sqcup \dots \sqcup A_k \to A_1 \cup \dots \cup A_k$ is a homeomorphism.

(2) Since $X_1, \dots, X_k$ are compact and since Y is in particular Hausdorff we know by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that the images $\varphi_1(X_1), \dots, \varphi_k(X_k)$ are closed subsets of Y. It follows from (1) that $\bigsqcup_{i=1}^{k} f_i \colon \bigsqcup_{i=1}^{k} X_i \to Y$ is indeed an embedding. ■

12.2. **Urysohn's Lemma.** In this section we will state and prove Urysohn's Lemma. Despite its modest name it is actually very much non-trivial.

Lemma 12.5. (Urysohn's Lemma) Let X be a topological space. The following two statements are equivalent:

(1) For any two disjoint closed subsets A and B and any $r < s \in \mathbb{R}$ there exists a continuous map $f \colon X \to [r, s]$ with $f|_A \equiv r$ and $f|_B \equiv s$.
(2) X is normal.



The statement of Urysohn's Lemma 12.5 is really quite amazing, just using the existence of suitable open subsets it builds up an interesting map.

Remark.

(1) Urysohn's Lemma 12.5 was first proved by Paul Urysohn [**Ury1925a**, p. 290] in 1924. The paper was finished three days before Paul Urysohn died in a swimming accident [**Gron2010**].
(2) A very nice motivation for the standard proof is given in [**Jä2005**, p. 109]. A somewhat different proof, using the Cantor function from page 216, is given in [**Cîr2021**].
(3) Note that Statement (1) in Urysohn's Lemma 12.5 does *not* say that there exists a continuous map $f \colon X \to [r, s]$ with $f^{-1}(r) = A$ and $f^{-1}(s) = B$. In fact, as we will see in Exercise 12.5, in general we cannot demand to find such a map on a normal space.
(4) The proof of the Metric Spaces-Normal Proposition 12.2 shows that the conclusion of Urysohn's Lemma 12.5 holds trivially for metric spaces.
(5) Urysohn's Lemma 12.5 motivates the interest in normal spaces. The combination of the Paracompact-Criterion Lemma 14.5 and the Dieudonné Theorem 14.9 shows that every topological space that is regionally compact, "second-countable" and Hausdorff is normal. In many settings this result will be strong enough to conclude that a given topological space is normal.
(6) In Lemma 27.10 we will prove a "smooth" version of Urysohn's Lemma 12.5. □

Proof. The "(1)⇒(2)"-implication is straightforward. Indeed, suppose that A and B are two disjoint closed subsets of X. If (1) holds, then there exists a continuous function $f \colon X \to [0, 1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$. Then $U := f^{-1}((-\infty, \frac{1}{2}))$ and $V := f^{-1}((\frac{1}{2}, \infty))$ have the desired properties.

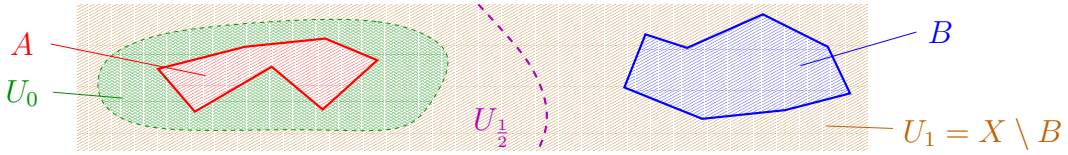
The "(2)⇒(1)"-implication is the non-trivial direction of Urysohn's Lemma and its proof will occupy the remainder of the proof. Let A and B be two disjoint closed subsets. An elementary rescaling argument shows that it suffices to deal with the case $r = 0$ and

$s = 1$, i.e. it suffices to show that there exists a continuous map $f\colon X \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.

Claim 1. We set $P = [0,1] \cap \mathbb{Q}$. There exists a family $\{U_s\}_{s\in P}$ of open subsets of X with the following three properties:

(i) $A \subseteq U_0$,
(ii) for each $p \in P$ with $p < 1$ we have $\overline{U_p} \subseteq X \setminus B$,
(iii) for each $s, t \in P$ with $s < t$ we have $\overline{U_s} \subseteq U_t$.

Proof. Since P is countable there exists a bijection $\varphi\colon \mathbb{N}_0 \to P$. We can arrange that $\varphi(0) = 0$ and $\varphi(1) = 1$. We set $U_1 := X \setminus B$. Since B is closed we know that U_1 is open. Since X is normal and since A is closed we obtain from the Normal-Open-Closed-Neighborhood Lemma 12.3 that there exists an open neighborhood U_0 of A with $\overline{U_0} \subseteq U_1$. Now suppose that for some $n \geq 1$ we have already defined $U_{\varphi(0)}, U_{\varphi(1)}, \ldots, U_{\varphi(n)}$ with the required properties. We set $P_n = \{\varphi(0), \ldots, \varphi(n)\}$ and we set $y := \varphi(n+1)$. Since $\varphi(0) = 0$ and $\varphi(1) = 1$ we see that $y \in (0,1)$. Since P_n is a finite subset of $[0,1]$ and since $\varphi(0) = 0$ and $\varphi(1) = 1$ we see that the set P_n contains a largest number x less than $y = \varphi(n+1)$ and it contains a smallest number z larger than $y = \varphi(n+1)$.



Note that our induction hypothesis implies that $\overline{U_x} \subseteq U_z$. Thus, since X is normal, we obtain from the Normal-Open-Closed-Neighborhood Lemma 12.3 that there exists an open subset U_y with $\overline{U_x} \subseteq U_y \subseteq \overline{U_y} \subseteq U_z$. One can now easily verify that $U_{\varphi(0)}, U_{\varphi(1)}, \ldots, U_{\varphi(n)}, U_{\varphi(n+1)}$ have the required properties. ⊞

For each $p \in (-\infty, 0) \cap \mathbb{Q}$ we set $U_p := \varnothing$ and for each $p \in (1, \infty) \cap \mathbb{Q}$ we set $U_p := X$. Next we consider

$$\begin{array}{rccl} f\colon & X & \to & \mathbb{R} \\ & x & \mapsto & \inf\{p \in \mathbb{Q} \,|\, x \in U_p\}. \end{array}$$

It follows easily from Properties (i) and (ii) that $f|_A \equiv 0$, that $f|_B \equiv 1$ and that f takes values in $[0,1]$. Thus it remains to prove that f is continuous. The proof of continuity requires some preparations.

Claim 2. Let $r \in \mathbb{Q}$.

(a) $x \in \overline{U_r} \implies f(x) \leq r$ (a′) $f(x) > r \Rightarrow x \notin \overline{U_r}$.
(b) $x \notin U_r \implies f(x) \geq r$ (b′) $f(x) < r \Rightarrow x \in U_r$.

Proof. Statements (a′) and (b′) are evidently just reformulations of Statements (a) and (b). Statements (a) and (b) are trivial if $r < 0$ or $r > 1$. Now assume that $r \in P = [0,1] \cap \mathbb{Q}$. In this case the statements follow from the following straightforward arguments:

$$\begin{array}{lll} & \text{by Property (iii)} & \text{by definition of the infimum and since } P \text{ is dense in } [0,1] \\ x \in \overline{U_r} & \overset{\downarrow}{\implies} x \in U_s \text{ for every } s \in P \text{ with } s > r & \overset{\downarrow}{\implies} f(x) \leq r \\ x \notin U_r & \implies x \notin U_s \text{ for every } s \in P \text{ with } s < r & \implies f(x) \geq r. \end{array}$$

⊞

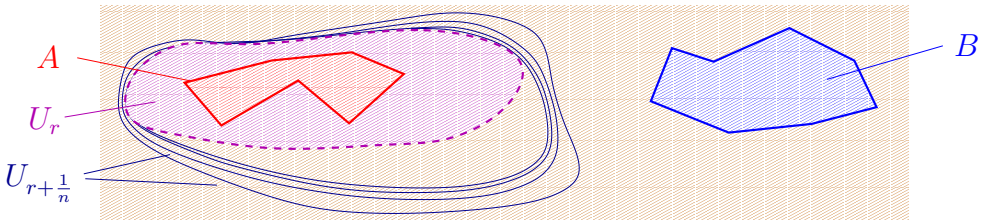
Claim 3. Given any $r \in \mathbb{Q}$ we have

$$f^{-1}([0,r]) = \bigcap_{n\in\mathbb{N}} \overline{U}_{r+\frac{1}{n}} \qquad \text{and} \qquad f^{-1}([0,r)) = \bigcup_{n\in\mathbb{N}} U_{r-\frac{1}{n}}.$$

Proof. Let $r \in \mathbb{Q}$. We see that

$$\begin{array}{rcccccl} f^{-1}([0,r]) & = & f^{-1}\Big(\bigcap_{n\in\mathbb{N}} [0, r+\frac{1}{n})\Big) & \overset{\text{by the Image-Preimage Lemma 1.9 (7)}}{=} & \bigcap_{n\in\mathbb{N}} f^{-1}([0,r+\frac{1}{n})) & \overset{\text{by Claim 2 (b')}}{\subseteq} & \bigcap_{n\in\mathbb{N}} U_{r+\frac{1}{n}} \subseteq \bigcap_{n\in\mathbb{N}} \overline{U}_{r+\frac{1}{n}} \\ & \subseteq & \bigcap_{n\in\mathbb{N}} f^{-1}([0,r+\frac{1}{n}]) & = & f^{-1}\Big(\bigcap_{n\in\mathbb{N}} [0,r+\frac{1}{n}]\Big) & = & f^{-1}([0,r]). \\ & \uparrow & & \uparrow & & & \\ & \text{by Claim 2 (a)} & & \text{by the Image-Preimage Lemma 1.9 (7)} & & & \end{array}$$

Note that at the beginning and at the end we have the same set. Thus all inclusions have to be equalities. Therefore we obtain the first equality of the claim.



The proof of the second equality is very similar to the proof of the first equality. For completeness' sake we carry out the argument:

$$\begin{array}{rcccccl} f^{-1}([0,r)) & = & f^{-1}\Big(\bigcup_{n\in\mathbb{N}} [0, r-\frac{1}{n})\Big) & \overset{\text{by the Image-Preimage Lemma 1.9 (6)}}{=} & \bigcup_{n\in\mathbb{N}} f^{-1}([0,r-\frac{1}{n})) & \overset{\text{by Claim 2 (b')}}{\subseteq} & \bigcup_{n\in\mathbb{N}} U_{r-\frac{1}{n}} \subseteq \bigcup_{n\in\mathbb{N}} \overline{U}_{r-\frac{1}{n}} \\ & \subseteq & \bigcup_{n\in\mathbb{N}} f^{-1}([0,r-\frac{1}{n}]) & = & f^{-1}\Big(\bigcup_{n\in\mathbb{N}} [0,r-\frac{1}{n}]\Big) & = & f^{-1}([0,r)). \\ & \uparrow & & \uparrow & & & \\ & \text{by Claim 2 (a)} & & \text{by the Image-Preimage Lemma 1.9 (6)} & & & \end{array}$$

As above we deduce the desired equality. ⊞

Claim 4. The map $f\colon X \to \mathbb{R}$ is continuous.

Proof. On page 165 we pointed out that the set of rational intervals forms a basis for the topology of $\mathbb{R}$. It follows from the Subbasis-Continuity Proposition 4.4 that it suffices to show that for every choice of $c, d \in \mathbb{Q}$ the preimage $f^{-1}((c,d))$ is open. Thus let $c, d \in \mathbb{Q}$. We see that

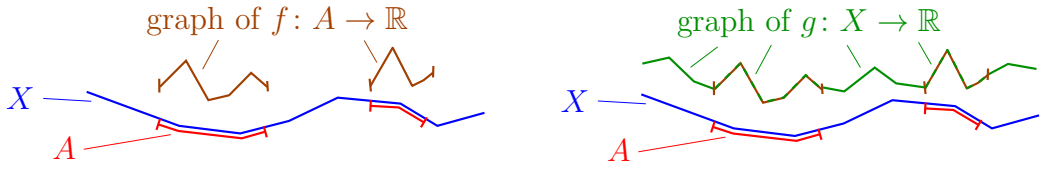
$$\begin{array}{rcccl} f^{-1}((c,d)) & \overset{\text{by the Image-Preimage Lemma 1.9 (8) and since } f(X)\subseteq[0,1]}{=} & f^{-1}([0,d)) \setminus f^{-1}([0,c]) & \overset{\text{by Claim 3}}{=} & \underbrace{\bigcup_{n\in\mathbb{N}} U_{d-\frac{1}{n}}}_{\substack{\text{open, since each} \\ U_p \text{ is open}}} \setminus \underbrace{\bigcap_{n\in\mathbb{N}} \overline{U}_{c+\frac{1}{n}}}_{\substack{\text{closed by the Interior-Closure} \\ \text{Lemma 3.22 (2) and the} \\ \text{Closed Subset-Lemma 3.5 (2)}}}. \end{array}$$

We have thus proved the desired result. ■

12.3. **The Tietze Extension Theorem.** Using Urysohn's Lemma 12.5 we can now prove the following major theorem.

Theorem 12.6. (Tietze Extension Theorem) Let X be a topological space, let $A \subseteq X$ be a closed subset and let $f\colon A \to \mathbb{R}$ be a continuous map. If X is normal, then there exists a continuous map $g\colon X \to \mathbb{R}$ with $g|_A = f$. If A is non-empty then one can find such g such that

$$\inf g(X) \;=\; \inf f(A) \;\in\; \mathbb{R} \cup \{-\infty\} \qquad \text{and} \qquad \sup g(X) \;=\; \sup f(A) \;\in\; \mathbb{R} \cup \{\infty\}.$$



Remark.

(1) The Tietze Extension Theorem was first proved for metric spaces by Heinrich Tietze [**Tie1915**] in 1915. The extension to normal topological spaces was proved by Paul Urysohn [**Ury1925a**, p. 293].

(2) Let X be a topological space and let A and B be two disjoint closed subsets. We consider the map on $A \sqcup B$ that is given by $f|_A \equiv 0$ and $f|_B \equiv 1$. By the Pasting Proposition 4.7 we know that f is continuous. If X is normal, then we can extend f by the Tietze Extension Theorem 12.6 to a continuous map on all of X. This shows that the Tietze Extension Theorem 12.6 is an extension of Urysohn's Lemma 12.5.

(3) In Exercise 12.11 we will see, borrowing some ideas from (2), that the Tietze Extension Theorem 12.6 is optimal in the sense, that if the conclusion holds for a topological space X, then X is normal.

(4) In the Smooth Extension Proposition 27.12 we will prove a "smooth" version of the Tietze Extension Theorem 12.6. □

For many applications the following formulation of the Tietze Extension Theorem 12.6 is more convenient.

Theorem 12.7. (Tietze Extension Theorem) Let X be a topological space and let A be a closed subset of X. Suppose that for $i = 1, \dots, n$ we are given a closed interval $I_i \subseteq \mathbb{R}$ and suppose that we are given a continuous map $f\colon A \to I_1 \times \cdots \times I_n \subseteq \mathbb{R}^n$. If X is normal, then there exists a continuous map $g\colon X \to I_1 \times \cdots \times I_n$ with $g|_A = f$.

Proof. We apply the earlier Tietze Extension Theorem 12.6 to the various coordinate functions $f_k\colon A \to I_k$, $k = 1, \dots, n$ and we find maps $g_1, \dots, g_k\colon X \to \mathbb{R}$ with the same infima and suprema as $f_1, \dots, f_k$. But since each I_k is closed we see that the infima and suprema lie in the corresponding I_k. This shows that each g_k takes again values in I_k. ■

The proof of the Tietze Extension Theorem 12.6 relies on the following proposition.

Proposition 12.8. (Series–Continuous Proposition) Let X be a non-empty topological space. First we introduce some notation: given a bounded map[168] $g\colon X \to \mathbb{R}$ we

set

$$\|g\| \;:=\; \sup\{|g(x)|\,|\,x\in X\} \;\in\; \mathbb{R}.$$

Let $(g_n\colon X\to\mathbb{R})_{n\in\mathbb{N}}$ be a sequence of bounded maps. If $\sum_{n=1}^{\infty}\|g_n\|<\infty$, then for each $x\in X$ the series $\sum_{n=1}^{\infty} g_n(x)$ converges in $\mathbb{R}$ and the map $x\mapsto\sum_{n=1}^{\infty} g_n(x)$ is continuous.

Proof of Proposition 12.8. Let X be a non-empty topological space. As in the Function Space–Metric Proposition 10.20 we consider the set $C(X,\mathbb{R})$ of continuous maps $X\to\mathbb{R}$ and given $f,g\in C(X,\mathbb{R})$ we write again

$$\widetilde{d}(f,g) \;=\; \sup\big\{\,\min\{|f(x)-g(x)|,1\}\,\big|\,x\in X\big\}.$$

Now let $(g_n\colon X\to\mathbb{R})_{n\in\mathbb{N}}$ be a sequence of bounded maps such that $\sum_{n=1}^{\infty}\|g_n\|<\infty$. Note that for each $k\geq l\in\mathbb{N}$ we have

$$\widetilde{d}\Big(\sum_{n=1}^{k} g_n \sum_{n=1}^{l} g_n\Big) \;\leq\; \Big\|\sum_{n=1}^{k} g_n \;-\; \sum_{n=1}^{l} g_n\Big\| \;=\; \Big\|\sum_{n=l+1}^{k} g_n\Big\| \;\underset{\text{triangle inequality}}{\leq}\; \sum_{n=l+1}^{k}\|g_n\|.$$

It follows easily from this observation and from the hypothesis that the series $\sum_{n=1}^{\infty}\|g_n\|$ converges in $\mathbb{R}$ that the partial sums $\sum_{n=1}^{k} g_k$ define a Cauchy sequence in $(C(X,\mathbb{R}),\widetilde{d})$. The desired statements now follow from the Function Space–Metric Proposition 10.20. ■

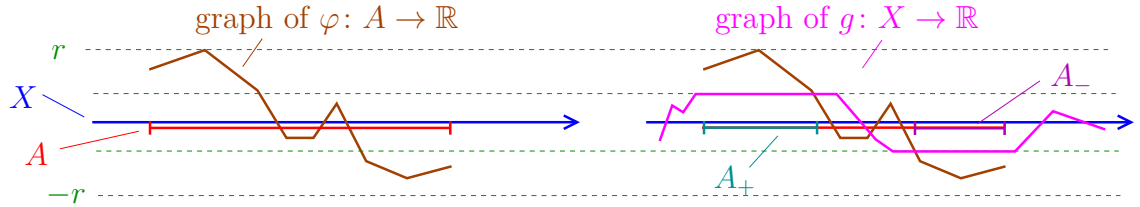
Proof of the Tietze Extension Theorem 12.6 if f is bounded. Let X be a topological space which is normal and let $A\subseteq X$ be a closed subset. If X is empty, then there is nothing to show. Thus we can assume that X is non-empty. Given a bounded map[168] $\psi\colon X\to\mathbb{R}$ we write again

$$\|\psi\| \;:=\; \sup\{|\psi(x)|\,|\,x\in X\} \;\in\; \mathbb{R}.$$

Claim. Let $r\in\mathbb{R}_{\geq 0}$. Given any continuous map $\varphi\colon A\to[-r,r]$ there exists a *continuous* map $g\colon X\to\mathbb{R}$ such that

$$\|g\|\leq\tfrac{1}{3}\cdot r \qquad\text{and such that}\qquad |\varphi(a)-g(a)| \;\leq\; \tfrac{2}{3}\cdot r \qquad\text{for all } a\in A.$$

Proof. We set $A_- = \varphi^{-1}([-r,-\frac{1}{3}r])$, $A_0=\varphi^{-1}([-\frac{1}{3}r,\frac{1}{3}r])$ and $A_+=\varphi^{-1}([\frac{1}{3}r,r])$. Since A_- and A_+ are disjoint closed subsets of X it follows from Urysohn's Lemma 12.5 that there exists a continuous map $g\colon X\to[-\frac{1}{3}r,\frac{1}{3}r]$ such that $g|_{A_-}\equiv-\frac{1}{3}r$ and $g|_{A_+}\equiv\frac{1}{3}r$. In particular we have $\|g\|\leq\frac{1}{3}r$.



Now let $a\in A$. In each of the three cases $a\in A_-$, $a\in A_0$ and $a\in A_+$ one can easily verify, using the triangle inequality, that $|\varphi(a)-g(a)|\leq\frac{2}{3}\cdot r$. ⊞

[168] We say that $g\colon X\to\mathbb{R}$ is bounded if there exists a $C\in\mathbb{R}$ such that $|g(x)|\leq C$ for all $x\in X$.

Now let $f\colon A \to \mathbb{R}$ be a bounded map. This means that there exists an $s \geq 0$ such that f takes values in $[-s, s]$. Thus we can apply the claim to $\varphi = f$ and $r = s$. We obtain a continuous map $g_1\colon X \to \mathbb{R}$ such that

$$\|g_1\| \leq \tfrac{1}{3} \cdot s \qquad \text{and} \qquad |f(a) - g_1(a)| \leq \tfrac{2}{3} \cdot s \qquad \text{for all } a \in A.$$

Next we apply the claim to the map $\varphi = f - g_1$ and $r = \frac{2}{3} \cdot s$. We obtain a continuous map $g_2\colon X \to \mathbb{R}$ such that

$$\|g_2\| \leq \tfrac{1}{3} \cdot \tfrac{2}{3} \cdot s \qquad \text{and} \qquad |\underbrace{(f(a) - g_1(a))}_{=\varphi(a)} - g_2(a)| \leq (\tfrac{2}{3})^2 \cdot s \text{ for all } a \in A.$$

Iterating this procedure we obtain a sequence of continuous maps $g_1, g_2, \ldots : X \to \mathbb{R}$ such that for each $n \in \mathbb{N}$ we have

$$\|g_n\| \leq \tfrac{1}{3} \cdot (\tfrac{2}{3})^{n-1} \cdot s \quad \text{and} \quad |f(a) - g_1(a) - \cdots - g_n(a)| \leq (\tfrac{2}{3})^n \cdot s \text{ for all } a \in A.$$

Note that $\sum_{n=1}^{\infty} \|g_n\| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot (\frac{2}{3})^{n-1} \cdot s = \frac{1}{3} \cdot s \cdot \sum_{n=0}^{\infty} (\frac{2}{3})^n = s < \infty$. Thus it follows from the Series–Continuous Proposition 12.8 that the map

$$\begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & g(x) \coloneqq \sum_{n=1}^{\infty} g_n(x) \end{array}$$

is defined and continuous. Next note that for any $a \in A$ we have

$$|f(a) - g(a)| = \left|f(a) - \sum_{n=1}^{\infty} g_n(a)\right| = \left|f(a) - \lim_{n\to\infty} \sum_{k=1}^{n} g_k(a)\right| = \lim_{n\to\infty} \underbrace{\left|f(a) - \sum_{k=1}^{n} g_k(a)\right|}_{\leq (\frac{2}{3})^n \cdot s \text{ by the second property}} = 0.$$

In other words, the maps g and f agree on A.

Finally assume that A is non-empty. Since $f\colon X \to \mathbb{R}$ is by hypothesis bounded it makes sense to set $a \coloneqq \inf f(A) \in \mathbb{R}$ and to set $b \coloneqq \sup f(A) \in \mathbb{R}$. By the Sum-Product-Continuity Lemma 4.11 we know that the map

$$x \mapsto \min(\max(g(x), a), b)$$

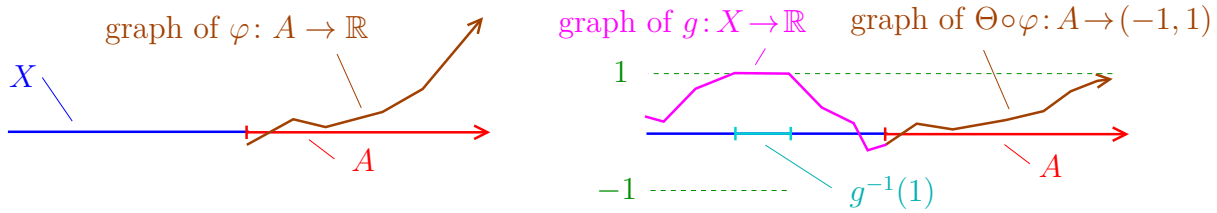
is continuous. Evidently this map has all the desired properties. ∎

Proof of the Tietze Extension Theorem 12.6 if f is unbounded. Let X be a topological space which is normal, let $A \subseteq X$ be a closed subset and let $f\colon A \to \mathbb{R}$ be a possibly unbounded continuous map.

We pick a homeomorphism $\Theta\colon \mathbb{R} \to (-1, 1)$, e.g. we could pick $x \mapsto \frac{2}{\pi} \cdot \arctan(x)$. The map $\Theta \circ f\colon A \to \mathbb{R}$ is bounded and continuous. By the previous case of the Tietze Extension Theorem we know that there exists a continuous map $g\colon X \to [-1, 1]$ which agrees with $\Theta \circ f$ on A. If $g(X) \subseteq (-1, 1)$, then $\Theta^{-1} \circ g\colon X \to \mathbb{R}$ is the desired extension of f.

It remains to consider the case that $g^{-1}(-1) \cup g^{-1}(1)$ is non-empty. Note that A and $g^{-1}(-1) \cup g^{-1}(1)$ are two disjoint closed subsets of X. Therefore we can apply Urysohn's Lemma 12.5 and we obtain a continuous map $\gamma\colon X \to [0, 1]$ with $\gamma|_A \equiv 1$ and which vanishes on $g^{-1}(-1) \cup g^{-1}(1)$. The map $x \mapsto \gamma(x) \cdot g(x)$ takes values in $(-1, 1)$. The map

$x \mapsto \Theta^{-1}(\gamma(x) \cdot g(x))$ is therefore well-defined and it is a continuous extension of $f\colon A \to \mathbb{R}$ to all of X.



By basically the same argument as in the bounded case we can arrange that f and g have the same infimum and the same supremum. ■

12.4. Exercises for Chapter 12.

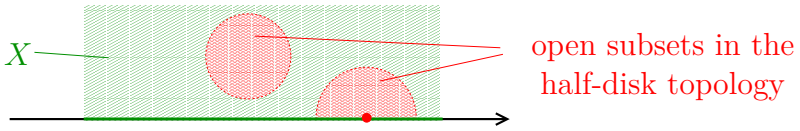
Exercise 12.1. Give an example of a topological space that is normal but which is not Hausdorff.

Exercise 12.2. Let Y be a topological space and let $X_1, \dots, X_k$ be disjoint closed subsets of Y. Show that if Y is normal, then the X_i are contained in disjoint open subsets.

Exercise 12.3. We consider the upper half-plane $X := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ together with the *half-disk topology* $\mathcal{H}$ that is generated by the usual open disks $B_r(x,y)$ with $r < y$ and the subsets of the form

$$\{(x,0)\} \;\cup\; (B_r(x,0) \cap \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}).$$

(a) Show that $(X, \mathcal{H})$ is Hausdorff.
(b) Show that $(X, \mathcal{H})$ is not normal.



Exercise 12.4. Let X be a topological space that is Hausdorff but not normal. (In Exercise 12.3 we showed that such a topological space exists.) Show that there exists a closed equivalence relation $\sim$ on X such that $X/\sim$ is not Hausdorff.
Remark. This shows that in the Quotient Topology-Properties Proposition 8.5 (3) we need to put some condition on the topological space beyond X being Hausdorff.

Exercise 12.5. Give an example of a normal topological space X and two closed subsets A and B such that there exists no continuous map $f\colon X \to \mathbb{R}$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

Exercise 12.6. Show that every topological graph, as introduced in Definition 11.9, is normal.

Exercise 12.7. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced in Exercise 4.4.
(a) Show that the Sorgenfrey line is normal.

As in Exercise 7.14 we refer to the product $S \times S$ as the *Sorgenfrey plane*.

(b) Show that $A \coloneqq \{(x, -x) \mid x \in \mathbb{Q}\}$ and $B \coloneqq \{(x, -x) \mid x \in \mathbb{R} \setminus \mathbb{Q}\}$ are closed subsets of the Sorgenfrey plane.
Remark. You could make use of Exercise 7.14.

(c) Show that the Sorgenfrey plane is not normal.

Exercise 12.8. Let $A \subseteq \mathbb{R}^n$ be a subset. Show that if A is not closed, then there exists a continuous map $f\colon A \to \mathbb{R}$ which cannot be extended to a continuous map on all of $\mathbb{R}^n$.

Exercise 12.9. Let X be a topological space that is normal. Let $U_1, \ldots, U_m$ be open subsets such that $X = \bigcup_{i=1}^{m} U_i$. Show that there exist open subsets $V_i \subseteq U_i$ such that $\overline{V}_i \subseteq U_i$ and such that $X = \bigcup_{i=1}^{m} V_i$.
Remark. A much stronger version of the statement of this exercise can be found in [**Dug1966**, Theorem VII.6.1] or alternatively in [**Grote1969**, Satz 43].

Exercise 12.10. Let X be a metric space.

(a) Let $A, B \subseteq X$ be two disjoint closed subsets of X. Does there exist a continuous map $f\colon X \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$?

(b) Let $A, B \subseteq X$ be two disjoint open subsets of X. Does there exist a continuous map $f\colon X \to \mathbb{R}$ and disjoint open intervals I and J such that $A = f^{-1}(I)$ and $B = f^{-1}(J)$?

(c) Let $A, B \subseteq X$ be two subsets of X such that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$. Do there exist disjoint open neighborhoods U of A and V of B?

Exercise 12.11. Let X be a topological space that satisfies the conclusion of the Tietze Extension Theorem 12.6, i.e. we assume that given any closed subset A of X and given any continuous map $f\colon A \to \mathbb{R}$ there exists a continuous map $g\colon X \to \mathbb{R}$ with $g|_A = f$. Show that X is normal.

CHAPTER 13

Second-countability and the Urysohn Metrization Theorem

In this chapter we will first introduce the notion of a second-countable topological space. This notion is used in the formulation of the Urysohn Metrization Theorem. Depending on the reader's taste, even more importantly, the notion of a second-countable topological space plays an essential role in the definition of topological manifolds in Definition 23.2.

This chapter is organized as follows:

- In Section 13.1 we will introduce second-countable topological spaces. These are topological spaces that admit a countable basis of the topology.
- In Section 13.2 we will state and prove the Urysohn Metrization Theorem 13.8 which says that
- In Section 13.3 we will introduce the notion of a regular topological spaces. For a topological space, where all singletons are closed, the notion of being regular is stronger than being Hausdorff and it is weaker than being normal.
- In Section 13.4 we will give a short survey on various other "separation axioms" for topological spaces. The only notions we will ever need though are Hausdorff, regular and normal.
- In Section 13.5 we will provide exercises for Chapter 13.

In the next Chapter 14 we will introduce paracompact topological spaces and partitions of unity.

13.1. **Second-countable topological spaces.** Recall that on Definition 3.52 we introduced the notion of a basis of a topological space. With this notion we can define what it means for a topological space to be second-countable.

Definition 13.1. A topological space is called second-countable if there exists a countable basis for the topology.[169]

Non-Example. Of course not every topological space is second-countable. For example consider the uncountable set $\mathbb{R}$, but for once equipped with the discrete topology $\mathcal{D}$. It follows easily from the definitions that $X = (\mathbb{R}, \mathcal{D})$ is *not* second-countable. There are also more subtle examples, for example we will see in Exercise 13.6 that there exist *compact* topological spaces that are not second-countable. Furthermore in the Long Line Theorem 28.20 we will see that there exists a path-connected space that is not second-countable. □

Using the next lemma we will see that many of the topological spaces that we are interested in are actually second-countable.

[169] This notion was first introduced in [**Hausd1914**, Kapitel XIII.3].

Lemma 13.2. (Second Countability Lemma)

(1) For every $n \in \mathbb{N}_0$ the topological space $\mathbb{R}^n$ is second-countable.
(2) If X is a topological space that is second-countable, then every subspace is also second-countable.
(3) (a) If $X_1, \dots, X_k$ are topological spaces that are second-countable, then $X_1 \times \cdots \times X_k$ is second-countable.
    (b) Let $\{X_i\}_{i \in I}$ be a countable family of topological spaces. If each X_i is second-countable, then $\prod_{i \in I} X_i$ is also second-countable.
(4) Let X be a topological space. If there exists a countable family of open subsets $\{U_i\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = X$ and such that each U_i is second-countable, then X itself is second-countable.
(5) Let X be a topological space and let $\sim$ be an equivalence relation on X. If X is second-countable and if the projection $X \to X/\sim$ is an open map, then $X/\sim$ is also second-countable.

The proof of the Second Countability Lemma 13.2 (5) rests on the following lemma, which is also the content of Exercise 8.1.

Auxiliary Lemma 13.3. (Quotient Topology-Basis Lemma) Let X be a topological space and let $\sim$ be an equivalence relation on X. Furthermore let $\mathcal{B}$ be a basis of the topology of X. If the projection $p\colon X \to X/\sim$ is an open map, then

$$p(\mathcal{B}) \;=\; \{p(B) \,|\, B \in \mathcal{B}\}$$

is a basis for the topology of $X/\sim$.

Proof of the Quotient Topology-Basis Lemma 13.3. First note that the sets in $p(\mathcal{B})$ are indeed open subsets of $X/\sim$ since we assume that the projection is open. We use the criterion provided by the Basis Characterization Lemma 3.57 (1) to show that $p(\mathcal{B})$ is a basis for the topology of $X/\sim$. Thus let $U \subseteq X/\sim$ be an open subset and let $y \in U$. We need to show that there exists a $B \in \mathcal{B}$ with $y \in p(B) \subseteq U$.

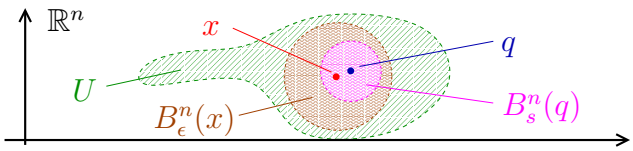
We choose an $x \in X$ with $p(x) = y$. Since $\mathcal{B}$ is a basis for the topology of X and since $p^{-1}(U)$ is open in X there exists by the Basis Characterization Lemma 3.57 (1) a $B \in \mathcal{B}$ with $x \in B \subseteq p^{-1}(U)$. But then we also have that $y = p(x) \in p(B) \subseteq p(p^{-1}(U)) = U$. ■

Proof of the Second Countability Lemma 13.2.

(1) This statement is basically the content of Exercise 3.8. For completeness' sake we provide the argument. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{B}$ the set of all open balls around points in $\mathbb{Q}^n$ with rational radii. By the Countability Lemma 1.37 we know that $\mathcal{B}$ is countable. Thus it remains to show that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}^n$. Note that each set in $\mathcal{B}$ is an open subset of $\mathbb{R}^n$. By the definition of the basis of a topology, see Definition 3.52, it now suffices to prove the following claim:

Claim. Given any open subset $U \subseteq \mathbb{R}^n$ and given any $x \in U$ there exists a $B \in \mathcal{B}$ with $x \in B \subseteq U$.

Proof. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $x \in U$. Since U is open there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subseteq U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists a $q \in \mathbb{Q}^n$ with $\|x - q\| < \frac{1}{3}\epsilon$. Next we pick an $s \in \mathbb{Q}$ with $s \in (\frac{1}{3}\epsilon, \frac{2}{3}\epsilon)$.



Note that a straightforward argument, using the triangle inequality, shows that $x \in B_s^n(q) \subseteq B_\epsilon^n(x) \subseteq U$. ⊞

(2) This statement follows from the Subspace-Basis Lemma 3.58 and the Countability Lemma 1.37 (4).

(3) (a) This statement follows immediately from the Product Topology-Basics Lemma 7.6 (3) and the Countability Lemma 1.37 (4).

(b) Let $\{X_i\}_{i\in I}$ be a countable family of second-countable topological spaces. For each $i \in I$ we pick a countable basis $\mathcal{B}_i$ of X_i. It follows from the Product Topology-Basics Lemma 7.6 (3) that a basis for the topological space $\prod_{i\in I} X_i$ is given by

$$\bigcup_{\{i_1,\ldots,i_r\}\subseteq I} \underbrace{\left\{ \prod_{i\in I} U_i \;\middle|\; \begin{array}{l} \text{for } i \notin \{i_1,\ldots,i_r\} \text{ we have } U_i = X_i, \\ \text{for } j = 1,\ldots,r \text{ we have } U_{i_j} \in \mathcal{B}_{i_j} \end{array} \right\}}_{\text{countable by the Countability Lemma 1.37}}.$$

Since I is countable it follows from the Countability Lemma 1.37 (1), (3) and (4) that there are only countably many finite subsets of I. It now follows that the above topology of $\prod_{i\in I} X_i$ is indeed countable.

(4) For each U_i we pick a countable basis $\mathcal{B}_i$ of the topology of U_i. Using the Basis Characterization Lemma 3.57 it is straightforward to verify that $\bigcup_{i\in I} \mathcal{B}_i$ is a basis for the topology of X. By the Countability Lemma 1.37 (2) this is a countable set.

(5) This statement follows from the Quotient Topology-Basis Lemma 13.3 and the Countability Lemma 1.37 (5). ■

The following lemma gives us a convenient property of second-countable topological spaces.

Lemma 13.4. (Countable Cover Lemma) Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. If X is second-countable, then there exists a countable subset $J \subseteq I$ with $X = \bigcup_{j\in J} U_j$.

Proof. Let $\mathcal{B}$ be a countable basis for the topology of X. We write

$$\mathcal{B}' \;:=\; \{V \in \mathcal{B} \,|\, V \subseteq U_i \text{ for some } i \in I\}.$$

Given $V \in \mathcal{B}'$ we pick $j \in I$ with $V \subseteq U_j$. It follows from the Countability Lemma 1.37 (1) that the set $J \subseteq I$ of all such j is countable. It remains to show that $X = \bigcup_{j\in J} U_j$.

Thus let $x \in X$. By hypothesis there exists an $i \in I$ with $x \in U_i$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $V \in \mathcal{B}$ with $x \in V \subseteq U_i$. Note that $V \in \mathcal{B}'$. But this implies that $x \in U_j$ for some $j \in J$. ■

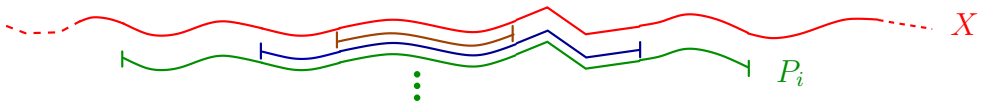
We conclude this introduction to second-countable topological spaces with the discussion of exhaustions of topological spaces, which play a role later on.

Definition 13.5. Let X be a topological space. We say that a sequence $\{P_i\}_{i\in\mathbb{N}}$ of subsets of X is an exhaustion if the following two conditions are satisfied:

(1) We have $X = \bigcup_{i\in\mathbb{N}} P_i$.

(2) For each $i \in \mathbb{N}$ we have $\overline{P}_i \subseteq \overset{\circ}{P}_{i+1}$.

We say that the exhaustion is compact, if each P_i is compact.



Example. A compact exhaustion of $\mathbb{R}^n$ is given by $P_i := \overline{B}_i^n(0)$ with $i \in \mathbb{N}$. □

Lemma 13.6. (Compact Exhaustion Lemma) Let X be a topological space which is second countable, Hausdorff and regionally compact.

(1) There exists an exhaustion of X by compact subsets.

(2) Given any compact subset $K \subseteq X$ there exists an open neighborhood P of K such that the closure $\overline{P}$ is compact.

Remark. In the Exhaustion Function Lemma 14.12 we will prove a refinement of the Compact Exhaustion Lemma 13.6 (1). □

Proof.

(1) First recall that in Definition 4.59 we said that a subset $A \subseteq X$ is *precompact in* X if the closure $\overline{A}$ of A in X is compact. By hypothesis we know that X is regionally compact and Hausdorff. It follows from the Regionally Compact–Locally Precompact Lemma 4.60 that X is locally precompact, which implies by the Basis-with-Property Lemma 4.56 that the topology of X admits a basis of precompact open subsets. Since X is second-countable we know by the Countable Cover Lemma 13.4 that X is in fact covered by a countable family $\{U_i\}_{i\in\mathbb{N}}$ of precompact open subsets. It remains to prove the following claim:

Claim. There exists a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of X such that for each $i \in \mathbb{N}$ we have $U_i \subseteq K_i$ and such that for each $i \geq 2$ we have $K_{i-1} = \overline{K}_{i-1} \subseteq \overset{\circ}{K}_i$.

Proof. We set $K_1 := \overline{U}_1$. Suppose we have already constructed $K_1, \ldots, K_n$ with the above properties. Since K_n is compact there exists an $r \in \mathbb{N}$ with $K_n \subseteq U_1 \cup \cdots \cup U_r$. We set

$$K_{n+1} := \overline{U}_1 \cup \cdots \cup \overline{U}_r \cup \overline{U}_{n+1}.$$

Since the $U_1, \ldots, U_r, U_{n+1}$ are precompact, their closures are by definition compact. Thus it follows from the Compact–Union-Intersection Lemma 3.41 that K_{n+1} is compact. Evidently $U_{n+1} \subseteq K_{n+1}$. Finally observe that

$$\overline{K}_n \underset{\uparrow}{=} K_n \subseteq U_1 \cup \cdots \cup U_r \subseteq \overset{\circ}{K}_{n+1} \subseteq K_{n+1}.$$

since X is Hausdorff we know by the Compact-Closed Lemma 3.39 (2) that compact subsets are closed

By iterating this procedure we get the desired sequence.

(2) Let K be a compact subset of X. We continue with the notation from (1). The sets $\mathring{K}_n$ form an open cover of K. Since K is compact and since we have $\mathring{K}_n \subseteq \mathring{K}_{n+1}$ we see that there exists an $n \in \mathbb{N}$ with $K \subseteq \mathring{K}_n$. But K_n is compact, thus we are done. ■

13.2. **The Urysohn Metrization Theorem.** In this section we will state and prove a major consequence of Urysohn's Lemma 12.5, namely the Urysohn Metrization Theorem. Before we can state the theorem, let us recall Definition 3.12.

Definition 13.7. A topological space X is called metrizable if there exists a metric d on X such that the given topology agrees with the topology coming from d.

For second-countable topological spaces the following theorem gives a complete characterization of metrizability.

Theorem 13.8. (Urysohn Metrization Theorem) Let X be a topological space. If X is second-countable, then the following two statements are equivalent:
(1) X is normal and every one-point subset is closed[170].
(2) X is metrizable.

Remark.
(1) The Urysohn Metrization Theorem 13.8 was proved by Paul Urysohn in 1924. Urysohn's proof was written up posthumously by Paul Alexandroff [**Ury1925b**].
(2) The later Completely Metrizable Criterion 14.13 can often be used to show that a given topological space is in fact completely metrizable.
(3) In Exercise 13.21 we will show that a compact metrizable space is second-countable. □

The proof of the Urysohn Metrization Theorem 13.8 relies on a proposition which is interesting in its own right. To formulate this proposition we need to introduce or recall the following notation:

Notation 13.9. We write

$$\mathbb{R}^{\mathbb{N}} \;:=\; \{\text{all maps } \mathbb{N} \to \mathbb{R}\} \;=\; \{(x_1, x_2, \dots) \,|\, \text{all } x_i \in \mathbb{R}\}.$$

We equip $\mathbb{R}^{\mathbb{N}}$ with the product topology introduced in Definition 7.3, namely this is the topology which is defined by the condition that a basis for the topology is given by the sets of the form

$$U_1 \times \cdots \times U_k \times \mathbb{R} \times \mathbb{R} \times \dots \quad \subseteq \; \mathbb{R}^{\mathbb{N}}$$

where $U_1, \dots, U_k$ are open subsets in $\mathbb{R}$.

Proposition 13.10. ($\mathbb{R}^{\mathbb{N}}$–Metrizable Proposition) The topological space $\mathbb{R}^{\mathbb{N}}$ is metrizable.

Proof. Throughout the proof we adopt the following notation:

- Given $a, b \in \mathbb{R}$ we write $d(a,b) := \min\{|b-a|, 1\}$.
- Given $x \in \mathbb{R}^{\mathbb{N}}$ we denote by $x_1, x_2, \dots$ the "coordinates" of x.
- Given $x, y \in \mathbb{R}^{\mathbb{N}}$ we set[171]

$$D(x,y) \;:=\; \sup_{k \in \mathbb{N}} \left\{ \tfrac{1}{k} \cdot d(x_k, y_k) \right\} \;\in\; \mathbb{R}.$$

[170] By the Points-in-Hausdorff Lemma 3.32 (1) this condition is satisfied if X is Hausdorff.
[171] By definition of d we see that the set $\{\frac{1}{k} \cdot d(x_k, y_k) \,|\, k \in \mathbb{N}\}$ is bounded by 1. It follows that the supremum does in fact exist.

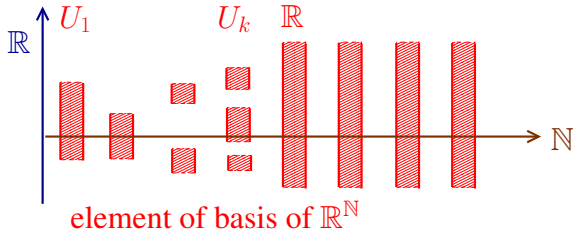
Claim 1. The map D defines a metric on $\mathbb{R}^{\mathbb{N}}$.

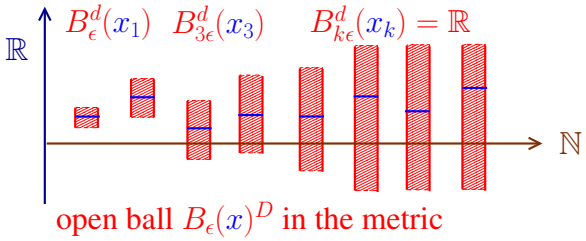
Proof. It is clear that D is symmetric and that $D(x,y)=0$ if and only if $x=y$. Thus it remains to prove the triangle inequality. Let $x,y,z\in\mathbb{R}^{\mathbb{N}}$. We have

$$D(x,z) \;=\; \sup_{k\in\mathbb{N}}\{\tfrac{1}{k}\cdot d(x_k,z_k)\} \;\underset{\text{follows from the straightforward observation that } d \text{ is a metric}}{\leq} \sup_{k\in\mathbb{N}}\{\tfrac{1}{k}\cdot d(x_k,y_k)+\tfrac{1}{k}\cdot d(y_k,z_k)\} \;\underset{\text{by definition of the supremum}}{\leq}\; D(x,y)+D(y,z).$$

It remains to prove the following claim:

Claim 2. The topology defined by the metric D equals the product topology on $\mathbb{R}^{\mathbb{N}}$.





Proof. First we show that the metric topology is contained in the product topology. Thus let $U\subseteq\mathbb{R}^{\mathbb{N}}$ be open in the metric topology. We want to show that U is open in the product topology. By the JH-Openness Criterion 3.3 it suffices to show that for each $x\in U$ there exists a set V, which is open in the product topology, with $x\in V\subseteq U$. Note that since U is open in the metric topology we know that there exists an $\epsilon>0$ with $B^D_\epsilon(x)\subseteq U$. We pick $n\in\mathbb{N}$ with $\frac{1}{n}<\epsilon$ and we set

$$V \;=\; (x_1-\epsilon,x_1+\epsilon)\times\cdots\times(x_n-\epsilon,x_n+\epsilon)\times\mathbb{R}\times\mathbb{R}\times\ldots \qquad\subseteq\; \mathbb{R}^{\mathbb{N}}.$$

It follows easily from the definitions that $V\subseteq B^D_\epsilon(x)$. By the above discussion we know that V is open in the product topology. Thus we have verified that U is open in the product topology.

Now we need to show that the product topology is contained in the metric topology. It follows easily from the JH-Openness Criterion 3.3 that it suffices to show that for each subset

$$U \;\coloneqq\; U_1\times\cdots\times U_k\times\mathbb{R}\times\mathbb{R}\times\ldots \qquad (\text{where } U_1,\ldots,U_k \text{ are open subsets in } \mathbb{R})$$

and each $x\in U$ there exists an open subset V of the metric topology with $x\in V\subseteq U$. Note that since the U_i are open we know that for $i=1,\ldots,k$ there exists a $\mu_i\in(0,1)$ with $(x_i-\mu_i,x_i+\mu_i)\subseteq U_i$. We set

$$\epsilon \;\coloneqq\; \min\{\tfrac{\mu_1}{1},\tfrac{\mu_2}{2},\ldots,\tfrac{\mu_k}{k}\}.$$

It follows easily from the definitions that $V\coloneqq B^D_\epsilon(x)\subseteq U$. ∎

Now we can finally provide the proof of the Urysohn Metrization Theorem 13.8.

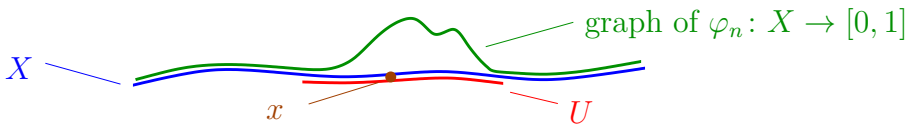
Proof of the Urysohn Metrization Theorem 13.8. Let X be a topological space which is second-countable. The "(2)⇒(1)"-direction is basically clear. Indeed, if X is metrizable, then it follows from the Metric Space-Hausdorff Proposition 3.28 that X is Hausdorff which in turn implies by the Points-in-Hausdorff Lemma 3.32 (1) that every one-point subset is

closed. Furthermore by the Metric Spaces-Normal Proposition 12.2 we know that X is normal.

We turn to the proof of the significantly more interesting "(1)⇒(2)"-direction. Thus we now assume that X is normal and that every one-point set is closed.

The idea, at this stage not particularly surprising, is to attempt to find an embedding $\Phi\colon X \to \mathbb{R}^{\mathbb{N}}$ and to use the fact that by the $\mathbb{R}^{\mathbb{N}}$–Metrizable Proposition 13.10 we know that $\mathbb{R}^{\mathbb{N}}$ is metrizable. A map $\Phi\colon X \to \mathbb{R}^{\mathbb{N}}$ is basically the same as a sequence of maps $\{\varphi_n\colon X \to \mathbb{R}\}_{n\in\mathbb{N}}$.

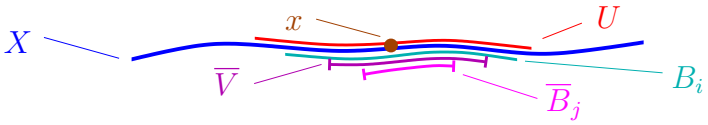
Claim 1. There exists a family $\{\varphi_n\colon X \to [0,1]\}_{n\in\mathbb{N}}$ of continuous maps with the following property: For each $x \in X$ and each open neighborhood U of x there exists an $n \in \mathbb{N}$ such that $\varphi_n(x) = 1$ and such that $\varphi_n|_{X\setminus U} \equiv 0$.



Proof. Since X is by hypothesis second-countable there exists a countable basis $\{B_j\}_{j\in J}$ for the topology of X. For every choice of $i, j \in J$ with $\overline{B}_i \subseteq B_j$ we can and will apply Urysohn's Lemma 12.5[172] to the disjoint closed subsets $\overline{B}_i$ and $X \setminus B_j$ and we obtain a map $g_{i,j}\colon X \to [0,1]$ with $g_{i,j}|_{\overline{B}_i} \equiv 1$ and $g_{i,j}|_{X\setminus B_j} \equiv 0$. Since J is countable we know by the Countability Lemma 1.37 that the set

$$K \;:=\; \{(i,j) \in J \times J \,|\, \overline{B}_i \subseteq B_j\} \;\subseteq J \times J$$

is again countable, thus in bijection with a subset of $\mathbb{N}$. Therefore it remains to show that these maps $\{g_{i,j}\}_{(i,j)\in K}$ have the desired properties. Now let $x \in X$ and let U be a neighborhood of x. By definition of a basis of the topology there exists a $j \in J$ with $x \in B_j \subseteq U$. By hypothesis X is normal and every one-point subset is closed. Therefore we obtain from the Normal-Open-Closed-Neighborhood Lemma 12.3 an open subset V of X with $x \in V \subseteq \overline{V} \subseteq B_j$. By definition of a basis there exists an $i \in J$ with $x \in B_i \subseteq V$. But then we also have $\overline{B}_i \subseteq \overline{V} \subseteq B_j$.



Note that the map $g_{i,j}$ satisfies $g_{i,j}(x) = 1$ and it vanishes on $X \setminus B_j$. From $B_j \subseteq U$ we obtain that $g_{i,j}$ vanishes in particular on $X \setminus U$. ⊞

We consider the product $\mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}$, equipped with the product topology. By the $\mathbb{R}^{\mathbb{N}}$–Metrizable Proposition 13.10 we know that $\mathbb{R}^{\mathbb{N}}$ is metrizable. Thus it suffices to prove the following claim:

Claim 2. The map

$$\begin{array}{rccc} \Phi\colon & X & \to & \mathbb{R}^{\mathbb{N}} \\ & x & \mapsto & (\varphi_1(x), \varphi_2(x), \dots) \end{array}$$

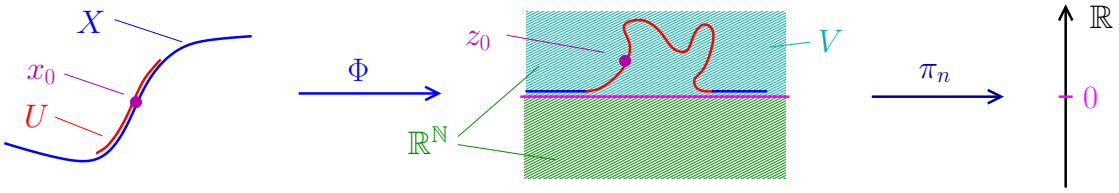
is an embedding, i.e. it is a homeomorphism onto its image.

[172] Here we use our hypothesis that X is normal

Proof. It follows immediately from the continuity of the φ_n and the Topological-Product Proposition 7.2 that Φ is continuous. Next we show that Φ is injective. Thus let $x \neq y$ be two points in X. Since X is normal and since each one-point set is closed there exists an open neighborhood U of x that does not contain y. By construction there exists an $n \in \mathbb{N}$ with $\varphi_n(x) = 1$ and with $\varphi_n(y) = 0$. This shows that Φ is injective.

We have now shown that Φ is an injective continuous map. We want to show that Φ is actually a homeomorphism onto its image. By the Open-Closed–Injective Lemma 4.22 it now suffices to show that Φ is an open map.

Thus let $U \subseteq X$ be an open subset. We need to show that $\Phi(U)$ is open in $\Phi(X)$. By the JH-Openness Criterion 3.3 it remains to show that for every $z_0 \in \Phi(U)$ there exists an open subset $W \subseteq \Phi(U)$ with $z_0 \in W \subseteq \Phi(U)$. We pick $x_0 \in U$ with $\Phi(x_0) = z_0$. By construction there exists an $n \in \mathbb{N}$ with $\varphi_n(x_0) = 1$ and with $\varphi_n|_{X\setminus U} \equiv 0$. We denote by $\pi_n \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ the projection onto the n-th coordinate. By the Topological-Product Proposition 7.2 we know that π_n is continuous.



We set $V \coloneqq \pi_n^{-1}((0,\infty))$ and we set $W \coloneqq V \cap \Phi(X)$. By definition of the subspace topology we know that W is open in $\Phi(X)$. It remains to show that $z_0 \in W$ and that $W \subseteq \Phi(U)$.

- It follows from $\pi_n(z_0) = (\pi_n \circ \Phi)(x_0) = \varphi_n(x_0) = 1 > 0$ that $z_0 \in W$.
- Let $z \in W$. We denote by $x \in X$ the unique element with $\Phi(x) = z$. Note that $\pi_n(z) = (\pi_n \circ \Phi)(x) = \varphi_n(x) > 0$. Since $\varphi_n|_{X\setminus U} \equiv 0$ we see that $x \in U$. In particular $z = \Phi(x) \in \Phi(U)$. ∎

13.3. **Regular topological spaces.** It turns out that one can formulate the Urysohn Metrization Theorem 13.8 with an a priori weaker hypothesis. This weaker hypothesis is used in most formulations of the Urysohn Metrization Theorem 13.8 in the literature. Therefore, for completeness' sake we will discuss this weaker hypothesis, even though we will not really make use of the results of this section.
We start out with the following notion.

Definition 13.11. A topological space X is called regular if given any $x \in X$ and given any closed subset $A \subseteq X$ with $x \notin A$ there exist disjoint open subsets U and V with $x \in U$ and $A \subseteq V$.

Remark.

(1) Regular topological spaces were introduced by Leopold Vietoris [**Vie1921**].
(2) Note that in the definition of a regular topological space we do not demand, in contrast say to [**Mun1975**, p. 195], that one-point sets are closed.
(3) In Section 13.4 we will discuss the relationship between the various "separation axioms", i.e. between the notions "Hausdorff", "regular" and "'normal". □

We have the following analog of the Points-in-Hausdorff Lemma 3.32 and the Normal-Open-Closed-Neighborhood Lemma 12.3.

Lemma 13.12. (Regular-Open-Closed-Neighborhood Lemma) Let X be a topological space, let $x \in X$ and let $U \subseteq X$ be an open neighborhood of x. If X is regular, then there exists an open subset V with $x \in V \subseteq \overline{V} \subseteq U$.



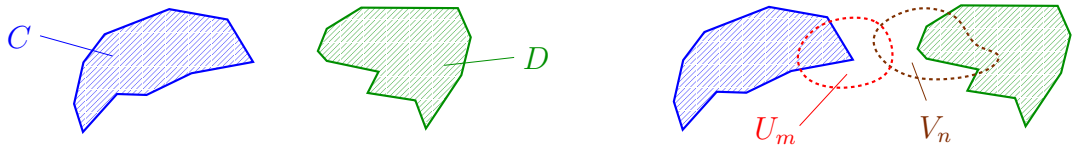
Proof. The proof of the lemma is almost identical to the proof of the Normal-Open-Closed-Neighborhood Lemma 12.3. For peace of mind we nonetheless provide the argument. Let U be an open neighborhood of x. We set $B := X \setminus U$. This is a closed subset with $x \notin B$. Since X is regular there exist disjoint neighborhoods V of x and W of B. It remains to show that $\overline{V} \subseteq U$. In fact we have $V \subseteq X \setminus W \subseteq U$. But since W is open we know that $X \setminus W$ is closed. By definition of $\overline{V}$ we have $\overline{V} \subseteq X \setminus W$. ■

The following proposition, together with the previous remark shows that for a large class of topological spaces the notions of being regular and normal coincide.

Proposition 13.13. (Regular+Second Countable Proposition) Let X be a topological space. If X is regular and if X is second-countable, then X is also normal.

Proof. Let X be a topological space that is regular and that is second-countable. The latter condition means that there exists a countable basis $\{B_j\}_{j\in J}$ for the topology of X. Now let C and D be two disjoint closed subsets of X.

Claim 1. There exist open subsets $\{U_m\}_{m\in\mathbb{N}}$ such that $C \subseteq \bigcup_{m\in\mathbb{N}} U_m \subseteq \bigcup_{m\in\mathbb{N}} \overline{U}_m \subseteq X \setminus D$.



Proof. Since J is countable we know that the set $\{j \in J \mid \overline{B}_j \subseteq X \setminus D\} \subseteq J$ is also countable. It remains to show that the union of the corresponding B_j covers C. Thus let $c \in C$. Since X is regular and since D is closed we know by the Regular-Open-Closed-Neighborhood Lemma 13.12 that there exists an open neighborhood U of c such that $\overline{U} \subseteq X \setminus D$. By definition of a basis of the topology there exists a $j \in J$ with $c \in B_j \subseteq U$. By construction $\overline{B}_j \subseteq \overline{U} \subseteq X \setminus D$, thus j lies in the given index set. ⊞

We swap the roles of C and D in the previous claim and we obtain open subsets $\{V_n\}_{n\in\mathbb{N}}$ such that $D \subseteq \bigcup_{n\in\mathbb{N}} V_n \subseteq \bigcup_{n\in\mathbb{N}} \overline{V}_n \subseteq X \setminus C$. Note that we do not know whether $\bigcup_{m\in\mathbb{N}} U_m$ and $\bigcup_{n\in\mathbb{N}} V_n$ are disjoint. Thus we have to go into extra time and do some more work. Given $m, n \in \mathbb{N}$ we now set

$$\widetilde{U}_m := U_m \setminus \underbrace{\bigcup_{i=1}^{m} \overline{V}_i}_{\substack{\textit{finite} \text{ union of} \\ \text{closed subsets}}} \quad \text{and} \quad \widetilde{V}_n := V_n \setminus \underbrace{\bigcup_{i=1}^{n} \overline{U}_i}_{\substack{\textit{finite} \text{ union of} \\ \text{closed subsets}}}.$$

Since the *finite* union of closed subsets is closed we see that the subsets $\widetilde{U}_m$ and $\widetilde{V}_n$ are open. Furthermore, since each $\overline{V}_i$ is disjoint from C we see that the $\widetilde{U}_m$ cover C. Similarly we see that the $\widetilde{V}_n$ cover D. It remains to prove the following claim:

Claim 2. The open subsets $\bigcup_{m\in\mathbb{N}} \widetilde{U}_m$ and $\bigcup_{n\in\mathbb{N}} \widetilde{V}_n$ are disjoint.

Proof. It suffices to show that for each $m, n \in \mathbb{N}$ we have $\widetilde{U}_m \cap \widetilde{V}_n = \varnothing$. If $m \leq n$, then we have by construction that even $\widetilde{U}_m \cap \overline{V}_n = \varnothing$. Similarly, if $n \leq m$, we have by construction that even $\widetilde{V}_n \cap \overline{U}_m = \varnothing$. ■

Now we can give the standard formulation of the Urysohn Metrization Theorem.

Theorem 13.14. (Urysohn Metrization Theorem) Let X be a topological space. If X is second-countable, the following two statements are equivalent:

(1) X is regular and every one-point subset is closed.
(2) X is metrizable.

Proof. Let X be a topological space which is second-countable.

First we consider again the basically trivial "(2)⇒(1)"-direction. If X is metrizable, then it follows from the Metric Space-Hausdorff Proposition 3.28 that X is Hausdorff which in turn implies by the Points-in-Hausdorff Lemma 3.32 (1) that every one-point subset is closed. Furthermore by the Metric Spaces-Normal Proposition 12.2 we know that X is normal. Since one-point subsets are closed this also implies that X is regular.

Now we consider the "(1)⇒(2)"-direction. Thus we assume that X is regular and that every one-point subset is closed. Since X is second-countable it follows from the Regular+Second Countable Proposition 13.13 that X is normal. Thus it follows from the "(1)⇒(2)"-direction of the original Urysohn Metrization Theorem 13.8 that X is metrizable. ■

The Urysohn Metrization Theorems 13.8 and 13.14 give a satisfactory characterization of metrizable topological spaces among all second-countable topological spaces.

For completeness' sake we state the following theorem which gives, rather astonishingly, a complete characterization of metrizable topological spaces among all topological spaces:

Theorem 13.15. (Nagata-Smirnov Metrization Theorem) Let X be a topological space. The following two statements are equivalent:

(1) X is metrizable.
(2) The following three conditions are satisfied:
  (a) X is regular.
  (b) X is Hausdorff.
  (c) The topology of X admits a basis that is the union of countably many locally finite families of subsets of X. In other words, there exists a countable set I and for each $i \in I$ there exists a locally finite family $\mathcal{B}_i$ of subsets of X such that $\bigcup_{i\in I} \mathcal{B}_i$ is a basis for the topology of X.

Remark. The theorem was proved independently by Junichi Nagata [**Naga1950**] and Yurii Smirnov [**Smi1951**]. A variation on the metrization theorem was proved by R. H. Bing [**Bin1951**].[173] □

[173]The first and middle name of Bing are "R." and "H.".

Example.

(1) Let X be a topological space that is second-countable, i.e. it admits a countable basis $\{U_i\}_{i\in I}$ of the topology. We set $\mathcal{B}_i := \{U_i\} \subseteq \mathcal{P}(X)$ and we see that condition (c) is satisfied. This shows that the Nagata-Smirnov Metrization Theorem 13.15 implies the Urysohn Metrization Theorem 13.14.

(2) Let X be a set that is equipped with the discrete topology. The family of all one-point subsets is locally finite. Thus, if we set $I := \{*\}$ and $\mathcal{B}_* := \{\{x\}\}_{x\in X}$, then we see that X satisfies Properties (a), (b) and (c), thus X is metrizable. In fact in this case it is easy to provide an explicit example of a metric, namely we just need to consider the discrete metric that is given by

$$d(x,y) := \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$ □

Proof. Despite its beauty we will not make use of the Nagata-Smirnov Metrization Theorem. We use this as an excuse for not providing the proof. The theorem is of course proved in the original sources [**Naga1950**] and Yurii Smirnov [**Smi1951**]. A textbook proof is given in [**Mun1975**, Chapters 6.2 and 6.3], [**Enge1989**, Theorem 4.4.7] and [**Pat2009**, Chapter 7.3]. ■

13.4. **Separation Axioms.** We have now introduced the properties of being Hausdorff, normal and regular. Each such property states to what degree points and closed subsets can be separated by open subsets. The limitless imagination of mathematicians has produced many variations on these separation axioms. In this section we will state, for reference, several other time-honored separation axioms, even though we will not really make use of the axioms that we introduce in this section.

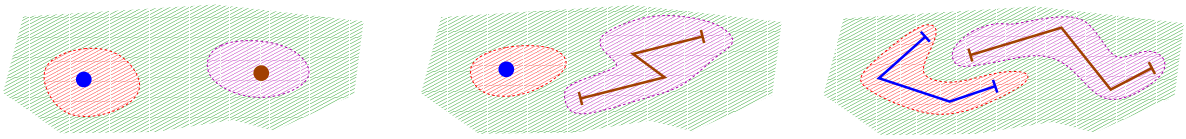
Definition 13.16. Let X be a topological space. Let A and B be two subsets of X. We say that A and B are separated by subsets U and V if $A \subseteq U$ and $B \subseteq V$ and if $U \cap V = \varnothing$. We use the obvious extensions of these definitions if we are dealing with points instead of subsets.

With these definitions we can reformulate the definitions of being "Hausdorff", "regular" and "normal":

(1) A topological space X is *Hausdorff* precisely if any two distinct points can be separated by open neighborhoods.

(2) A topological space X is *regular* precisely if every point x can be separated from every closed subset A with $x \notin A$ by open neighborhoods.

(3) A topological space X is *normal* if any two disjoint closed subsets of X can be separated by open neighborhoods.



Hausdorff regular normal

Definition 13.17. Let X be a topological space. Let A and B be two subsets of X.

(1) We say that the two subsets A and B are separated by a map if there exists a continuous map $f\colon X \to \mathbb{R}$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.
(2) We say that A and B are precisely separated by a map if there exists a continuous map $f\colon X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

We use the obvious extensions of these definitions if we are dealing with points instead of subsets.

By Urysohn's Lemma 12.5 we know that given a topological space X the following two statements are equivalent:

(1) Any two disjoint closed subsets can be separated by open neighborhoods.
(2) Any two disjoint closed subsets can be separated by maps.

It is less clear whether these statements are equivalent to the following statement:

(3) Any two disjoint closed subsets can be *precisely* separated by maps.

This leads us to the following refinement of the notion of being normal.

Definition 13.18. A topological space is called perfectly normal if any disjoint closed subsets can be precisely separated by a map.

Example. In the proof of the Metric Spaces-Normal Proposition 12.2 we implicitly showed that metric spaces are perfectly normal. □

For some purposes it is convenient to work with a reformulation of being "perfectly normal and Hausdorff". To do so we will need the following definition:

Definition 13.19. Let X be a topological space. We say that a subset is a G_δ-subset if it is the intersection of countably many open subsets.

Lemma 13.20. (Perfectly Normal Lemma) Let X be a Hausdorff space. The following three statements are equivalent:

(1) X is perfectly normal.
(2) For every closed subset $A \subseteq X$ there exists a continuous map $f\colon X \to [0,1]$ such that $A = f^{-1}(0)$.
(3) X is normal and every closed subset of X is a G_δ-subset.

Remark. In many, arguably most, textbooks, e.g. [**Kel1975**, p. 134], [**SS1978**, p. 16] and [**Jos1983**, p. 187], the characterization of the Perfectly Normal Lemma 13.20 (3) is used for the definition of a perfectly normal topological space. □

Proof. First we prove the "(1)⇒(2)"-direction. Let A be some closed subset of X. If $A = X$, then we can take f to be the constant map. Now suppose that $A \subsetneq X$. We pick $x \in X \setminus A$. Since X is Hausdorff we know by the Points-in-Hausdorff Lemma 3.32 that $\{x\}$ is a closed subset. Since X is perfectly normal there exists a continuous map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$ and with $\{x\} = f^{-1}(1)$.

We turn to the proof of the "(2)⇒(1)"-direction. Let A and B be two disjoint closed subsets. By hypothesis there exists a continuous map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$ and a continuous map $g\colon X \to [0,1]$ with $B = g^{-1}(0)$. Since A and B are disjoint we see that for every $x \in X$ we have $f(x) + g(x) > 0$. As in the proof of the Metric Spaces-Normal

Proposition 12.2 we can consider the map

$$\begin{array}{rcl} h\colon X & \to & [0,1] \\ x & \mapsto & \dfrac{f(x)}{f(x)+g(x)}. \end{array}$$

Evidently $A = h^{-1}(0)$ and $B = h^{-1}(1)$.

Next we prove the "(2)⇒(3)"-direction. It follows from the "(2)⇒(1)"-direction that X is perfectly normal. Thus it suffices to prove that every closed subset of X is a G_δ-subset. Now let A be a closed subset of X. By hypothesis there exists a continuous map $f\colon X \to [0,1]$ with $A = f^{-1}(0)$. We now see that

$$A = f^{-1}(0) = f^{-1}\Big(\bigcap_{n\in\mathbb{N}} [0,\tfrac{1}{n})\Big) = \bigcap_{n\in\mathbb{N}} \underbrace{f^{-1}([0,\tfrac{1}{n}))}_{\text{open subset of } X}.$$

by the Image-Preimage Lemma 1.9 (7)

Finally we provide the proof of the "(3)⇒(2)"-direction. Thus we assume that X is normal and we assume that every closed subset of X is a G_δ-subset. Let A be a closed subset of X. By hypothesis there exists a countable family $\{U_n\}_{n\in\mathbb{N}}$ of open subsets such that $A = \bigcap_{n\in\mathbb{N}} U_n$. Let $n \in \mathbb{N}$. Since X is normal and since A and $X\setminus U_n$ are disjoint we obtain from Urysohn's Lemma 12.5 a continuous map $f_n\colon X \to [0,1]$ with $f_n|_A \equiv 0$ and $f_n|_{X\setminus U_n} \equiv 1$. It follows from the Function Space–Metric Proposition 10.20 that the map

$$\begin{array}{rcl} f\colon X & \to & [0,1] \\ x & \mapsto & \displaystyle\sum_{n\in\mathbb{N}} \frac{1}{2^n}\cdot f_n(x) \end{array}$$

is continuous. Furthermore note that

$$X\setminus A = X\setminus \bigcap_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \underbrace{(X\setminus U_n)}_{\text{here } f_n\equiv 1}.$$

It follows almost immediately from this observation that $f^{-1}(0) = A$. ■

Since this was so much fun we want to introduce one more variation on being normal. For this we need the following definition:

Definition 13.21. Let X be a topological space. We say that two subsets A and B are separate if $A\cap\overline{B} = \varnothing$ and if $\overline{A}\cap B = \varnothing$.[174]

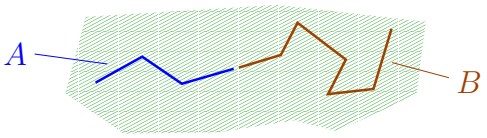
Example. Let $X = \mathbb{R}$. The subsets $A = (-1,0)$ and $B = (0,1)$ are separate. □

Definition 13.22. A topological space is called completely normal if separate subsets are separated by open neighborhoods.

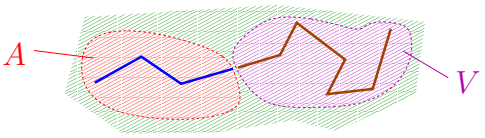
Lemma 13.23. (Perfectly Normal-implies-Completely Normal Lemma) Let X be a topological space. If X is perfectly normal, then it is also completely normal.

Proof. Let X be a perfectly normal Hausdorff space and let A and B be two separate subsets of X. By the Perfectly Normal Lemma 13.20 and the Interior-Closure Lemma 3.22

[174] Evidently disjoint closed subsets are separate.

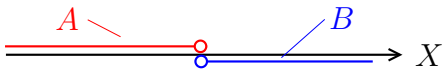


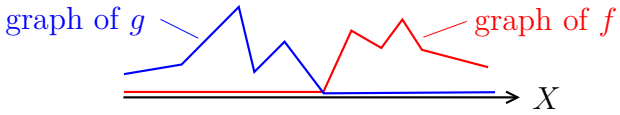
A and B are separate



completely normal

(2) there exist continuous maps $f\colon X \to [0,1]$ and $g\colon X \to [0,1]$ with $f^{-1}(0) = \overline{A}$ and $g^{-1}(0) = \overline{B}$.





We consider the map $h = f - g$. By construction $A \subseteq h^{-1}((-\infty, 0))$ and $B \subseteq h^{-1}((0,\infty))$. ■

Definition 13.24. Let X be a topological space. We say that two points x and y are topologically indistinguishable if x and y have the same neighborhoods, i.e. if a subset $U \subseteq X$ is a neighborhood of x if and only if it is a neighborhood of y. Otherwise we say that the points are topologically distinguishable.

Example. If X is a set that is equipped with the trivial topology $\{\varnothing, X\}$, then any two points are topologically indistinguishable. □

We can now formulate the following hierarchy of separations axioms, which has its origins in the work of Heinrich Tietze [**Tie1923**].

Definition 13.25. Given a topological space X we introduce the following separability axioms:[175]

T_0	Kolmogorov	distinct points are topologically distinguishable
T_1	Fréchet	every one-point subset is closed
T_2	Hausdorff	distinct points are separated by open neighborhoods
$T_{2\frac{1}{2}}$	Urysohn	distinct points are separated by closed neighborhoods
T_3	regular Hausdorff	X is Hausdorff & every point x is separated from any closed subset A with $x \notin A$ by open neighborhoods
$T_{3\frac{1}{2}}$	Tychonoff	X is Hausdorff & every point x is separated from every closed subset A with $x \notin A$ by a map
T_4	normal Hausdorff	X is Hausdorff & disjoint closed subsets are separated by open neighborhoods
T_5	completely normal Hausdorff	X is Hausdorff & separate subsets are separated by open neighborhoods
T_6	perfectly normal Hausdorff	X is Hausdorff & disjoint closed subsets are precisely separated by a map.

We say that X is a T_i-space if the corresponding condition is satisfied.

[175] The "T" in the names of the separation axioms goes back to the German word "Trennungsaxiom".

Remark.

(1) Fréchet spaces were introduced in [**Frec1928**].
(2) Note that the definitions can differ slightly in the literature, for example in [**vQuin2001**, Chapter 6] and in [**SS1978**, Section 2] the T_3 and the T_4-axiom is stated without the Hausdorff condition. The advantage of the above Definition 13.25 is that the separability axioms are increasingly stronger. More precisely, in the Separation Axioms Proposition 13.26 we will see that for T_i and T_j in the list with $i < j$ every T_j-space is also a T_i-space.
(3) In Exercise 13.18 we give an alternative characterization of completely normal Hausdorff spaces. □

Proposition 13.26. (Separation Axioms Proposition) For every two axioms T_i and T_j with $i < j$ in the above Definition 13.25 the following holds:
(1) If a topological space is a T_j-space, then it is also a T_i-space.
(2) There exists a topological space that is T_i but not T_j.

Remark. The Separation Axioms Proposition 13.26 says in particular that the obvious analog of Urysohn's Lemma 12.5 for regular spaces does *not* hold. More precisely, the Separation Axioms Proposition 13.26 says in particular that there exists a topological space X which satisfies T_3 but does not satisfy $T_{3\frac{1}{2}}$. In plain English this means that the topological space X has the following two properties:

(1) X is regular space X and every one-point subset is closed.
(2) There exists a point $x \in X$ and a closed subset $A \subseteq X$ with $x \notin A$ such that there is no continuous map $f\colon X \to \mathbb{R}$ with $f(x) = 0$ and $f|_A \equiv 1$. □

Proof.

(1) In the Points-in-Hausdorff Lemma 3.32 we showed that in a Hausdorff space every one-point subset is closed. In most cases this shows that for $i < j$ a T_j-space is also a T_i-space. The implications "$T_6 \Rightarrow T_5$" and "$T_{3\frac{1}{2}} \Rightarrow T_3$" rely on the simple observation, which we already used in the easy direction of Urysohn's Lemma 12.5, that if two subsets can be separated by a map, then they can also be separated by open subsets. Perhaps the only implication which is not entirely obvious is why a T_6-space is a T_5-space. Fortunately this implication follows immediately from the Perfectly Normal-implies-Completely Normal Lemma 13.23.
(2) The following list of examples, exercises and references provides us with the promised topological spaces:

T_0	but not	T_1	Sierpiński space, see Exercise 4.26
T_1	but not	T_2	line with two zeros, see Definition 3.30
T_2	but not	$T_{2\frac{1}{2}}$	see Exercise 13.15
$T_{2\frac{1}{2}}$	but not	T_3	see Exercise 13.16 or alternatively Exercise 12.3
T_3	but not	$T_{3\frac{1}{2}}$	see [**SS1978**, Example 91]
$T_{3\frac{1}{2}}$	but not	T_4	see Exercise 13.17
T_4	but not	T_5	see [**SS1978**, Examples 87 and 105]
T_5	but not	T_6	see [**SS1978**, Example 32].

Many more examples regarding these and other separation axioms are discussed in [**SS1978**], which is a never-ending source of amusing topological spaces. ■

13.5. Exercises for Chapter 13.

Exercise 13.1. Let X be a topological space. Is the following statement correct: If X is second-countable, then X has only countably many components?

Exercise 13.2. Let X be a topological space. Show that if X is second-countable, then every locally finite open cover $\{U_i\}_{i\in I}$ of X is countable.

Exercise 13.3. Let $n \in \mathbb{N}$. We equip $\mathbb{R}^n$ with the standard topology $\mathcal{T}$.

(a) Show that the open balls with rational centers and rational radii, i.e. the set

$$\mathcal{C} \;:=\; \{B_r^n(q) \,|\, r \in \mathbb{Q}_{\geq 0} \text{ and } q \in \mathbb{Q}^n\},$$

is a basis for the topological space $\mathbb{R}^n$.

(b) Use (a) to show that for the cardinality of $\mathcal{T}$ we have $\#\mathcal{T} \leq \#\mathcal{P}(\mathbb{Q})$.

(c) Use the Bernstein-Schröder Theorem 1.41 to show that $\#\mathcal{T} = \#\mathcal{P}(\mathbb{Q})$.

Remark. By Exercise 1.12 we know that $\#\mathcal{P}(\mathbb{Q}) = \#\mathbb{R}$ and that $\#\mathbb{R} < \#\mathcal{P}(\mathbb{R})$. Evidently $\#\mathcal{P}(\mathbb{R}) \leq \#\mathcal{P}(\mathbb{R}^n)$. In summary we have thus shown that $\#\mathcal{T} < \#\mathcal{P}(\mathbb{R}^n)$. Informally speaking this means that open subsets are "rare" among all subsets of $\mathbb{R}^n$.

Exercise 13.4. Given a non-empty subset $I \subseteq \mathbb{R}$ we denote by $C^b(I, \mathbb{R})$ the set of all bounded continuous maps $I \to \mathbb{R}$ and given $f \in C^b(I, \mathbb{R})$ we set

$$\|f\| \;:=\; \sup\{|f(x)| \,|\, x \in I\}.$$

Note that $(C^b(I, \mathbb{R}), \|-\|)$ is a metric space. We will equip $C^b(I, \mathbb{R})$ with the corresponding topology.

(a) Show that if I is a compact interval, then $C^b(I, \mathbb{R})$ is second-countable.

(b) Show that if I is a non-compact interval, then $C^b(I, \mathbb{R})$ is not second-countable.

Exercise 13.5. Let X be a topological space that admits a compact exhaustion $\{P_i\}_{i\in\mathbb{N}}$ in the sense of Definition 13.5. Suppose that X is connected and locally connected. Show that there exists also a compact exhaustion $\{Q_i\}_{i\in\mathbb{N}}$ such that each Q_i is connected.

Exercise 13.6. Let I be an uncountable set. Show that $[0,1]^I$ is not second-countable.

Remark. By the Tychonoff Theorem 7.14 we know that $[0,1]^I$ is compact. Thus the exercise shows that there exist compact spaces that are not second-countable.

Exercise 13.7. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced in Exercise 4.4. Show that the Sorgenfrey line is not second-countable.

Exercise 13.8. A topological space is called *separable* if it admits a subset that is countable and dense.

(a) Show that every second-countable topological space is separable.

(b) Let X be a set. We equip X with the cofinite topology that we defined on page 136.

(i) Show that X is separable.

(ii) Suppose that X is uncountable. Show that X is not second-countable.

Remark. This example is taken from [**SS1978**, Example 19].

Exercise 13.9. As in Exercise 13.8 we say that a topological space is *separable* if it admits a subset that is countable and dense.

(a) Show that the Sorgenfrey line S, which we introduced in Exercise 4.4, and the Sorgenfrey plane $S \times S$ are separable.
(b) Show that the diagonal $\Delta = \{(x,x) \in S \times S \,|\, x \in S\}$ is separable.
(c) Show that the anti-diagonal $\Delta = \{(x,-x) \in S \times S \,|\, x \in S\}$ is not separable.
Hint. Use Exercise 7.14.

Remark. This exercise shows that in general separability is not inherited by subsets.

Exercise 13.10. Let (X,d) be a metric space. Show that the corresponding topological space is second-countable if and only if X admits a countable dense subset.

Exercise 13.11. Show that every regionally compact Hausdorff space is regular.

Exercise 13.12. Let X be the *Sierpiński space* which we introduced in Exercise 4.26, i.e. let $X = \{0,1\}$ be the set with two elements which we equip with the topology $\mathcal{T} = \{\varnothing, X, \{0\}\}$. Show that X is normal and show that X is not regular.

Exercise 13.13. Show that the product of two regular topological spaces is again regular.
Remark. Together with Exercise 12.7 we see that the Sorgenfrey plane is regular but not normal. One can easily show that the Sorgenfrey plane is Hausdorff. Thus we see that the Sorgenfrey plane is an example of a topological space that is T_3 but not T_4.

Exercise 13.14. Provide an example of a topological space that is not T_0.

Exercise 13.15. Let $D := B_1^2(0) \subseteq \mathbb{R}^2$ be the open disk of radius 1 centered on the origin and let

$$X \;:=\; D \cup \{(-1,0)\} \cup \{(0,1)\}.$$

We equip X with the topology that is generated by the following subsets:

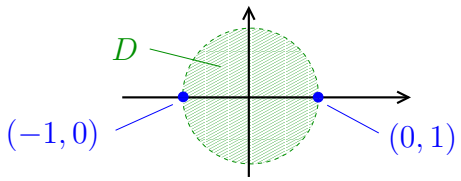
- For every $P \in D$ and every $r > 0$ we consider $B_r^2(P) \cap X$.
- For every $h \in (0,1)$ we consider the sets

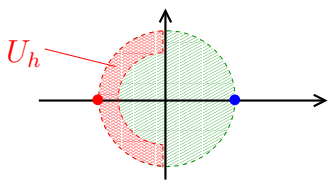
$$\begin{aligned} U_h &:= \{(x,y) \in D \,|\, x < 0 \text{ and } x^2 + y^2 > h\} \cup \{(-1,0)\} \\ V_h &:= \{(x,y) \in D \,|\, x > 0 \text{ and } x^2 + y^2 > h\} \cup \{(0,1)\}. \end{aligned}$$

The following two exercises show that $(X,\mathcal{T})$ is T_2 but not $T_{2\frac{1}{2}}$.

(a) Show that $(X,\mathcal{T})$ is Hausdorff.
(b) Show that the points $(-1,0)$ and $(0,1)$ cannot be separated by closed neighborhoods.

Remark. This exercise also solves Exercise 3.14.





Exercise 13.16. We set $K := \{\frac{1}{n} \mid n \in \mathbb{N}\}$. As in Exercise 4.59 we define the *K-topology* on $\mathbb{R}$ as the topology $\mathcal{K}$ that is generated by the open intervals (a, b) and by all sets of the form $(a, b) \setminus K$ where $a, b \in \mathbb{R}$.

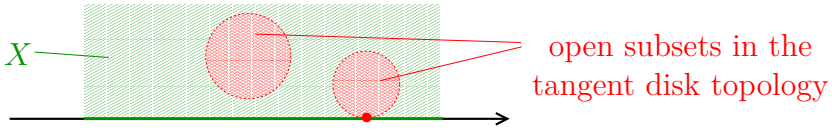
(a) Show that $(\mathbb{R}, \mathcal{K})$ is Hausdorff.
(b) Show that $(\mathbb{R}, \mathcal{K})$ is Urysohn.
(c) Show that $(\mathbb{R}, \mathcal{K})$ is not regular.
Hint. First show that K is closed in $(\mathbb{R}, \mathcal{K})$.

Remark. The exercises show that $(\mathbb{R}, \mathcal{K})$ is $T_{2\frac{1}{2}}$ but not T_3.

Exercise 13.17. We consider the upper half-plane $X := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ together with the *tangent disk topology* $\mathcal{D}$, i.e. the topology that is generated by the usual open disks $B_r(x, y)$ with $r < y$ and the subsets $B_y(x, y) \cup \{(0, x)\}$.

(a) Show that $(X, \mathcal{D})$ is Hausdorff.
(b) Show that $(X, \mathcal{D})$ is Tychonoff.
(c) Show that $(X, \mathcal{D})$ is not normal.
Hint. Show that $\{(x, 0) \mid x \in \mathbb{Q}\}$ and $\{(x, 0) \mid x \notin \mathbb{Q}\}$ are closed subsets.

Remark. The exercises show that $(X, \mathcal{D})$ is $T_{3\frac{1}{2}}$ but not T_4.



Exercise 13.18. Let X be a Hausdorff space. Show that the following two statements are equivalent:

(1) X is T_5, i.e. X is completely normal and Hausdorff.
(2) X is *hereditarily normal*, i.e. if every subspace of X is normal.

Exercise 13.19. Let X be a perfectly normal topological space and let $A \subseteq X$ be a closed subset. By the Perfectly Normal Lemma 13.20 we know that if X is Hausdorff, then there exists a continuous map $f\colon X \to [0, 1]$ with $A = f^{-1}(0)$. Do we need the hypothesis that X is Hausdorff?

Exercise 13.20. We consider the set

$$H := \prod_{n \in \mathbb{N}} [0, \tfrac{1}{n}] = \{(x_1, x_2, \dots) \mid \text{for every } n \in \mathbb{N} \text{ we have } x_n \in [0, \tfrac{1}{n}]\}$$

with the metric

$$d(x, y) := \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

This metric space, and the corresponding topological space, are called the *Hilbert Cube*.

(a) Show that d is defined, i.e. show that the series converges and show that d does indeed define a metric on H.
(b) Show that the Hilbert Cube, viewed as a topological space, is homeomorphic to the product space $[0, 1]^{\mathbb{N}}$.
Remark. By Tychonoff's Theorem 7.14 this implies that the Hilbert Cube is compact.

(c) Let X be a metric space that is second-countable.[176] Show that there exists an embedding $f\colon X \to H$ of topological spaces.
Hint. Reread the proof of the Urysohn Metrization Theorem 13.8.
Remark. Statements (b) and (c) imply that any second-countable metric space admits an embedding, as a topological space, into a *compact* metrizable topological space.

(d) Show that every Hausdorff compact second-countable topological space is homeomorphic to a closed subset of the Hilbert Cube.

Exercise 13.21. Let X be a compact metrizable space.

(a) Show that X is second-countable.
Hint. Use the fact that given any $n \in \mathbb{N}$ there exists a finite subset $A_n \subseteq X$ with $\bigcup_{a\in A_n} B_{\frac{1}{n}}(a) = X$.

(b) Show that X is regionally compact.

[176] In Exercise 13.8 we showed that a metric space is second-countable if and only if it is separable. With this reformulation Statement (c) is commonly used in the literature.

CHAPTER 14

Paracompact spaces and partitions of unity

In this chapter we will introduce the notion of a paracompact topological space. As the name suggests, paracompact spaces are a suitable generalization of the notion of a compact topological space.

This chapter is organized as follows:

- In Section 14.1 we will introduce paracompact spaces which, as the name suggests, generalizations of compact spaces.
- In Section 14.2 we will introduce partitions of unity. We will see in the Paracompact-Partition of Unity Theorem 14.15 that for Hausdorff spaces the existence of suitable partitions of unity is equivalent to the space being paracompact.
- In Section 14.3 we will state and prove the tricky Neighborhood Embedding Theorem 14.19 which at times allows us for paracompact spaces to extend an embedding on a closed subspace A to an embedding of an open neighborhood of A.
- In Section 14.4 we will provide exercises for Chapter 14.

In the next Chapter 15 we will introduce the covering dimension of topological spaces.

14.1. **Paracompact spaces.** We recall the following definitions from Definition 3.19.

Definition 14.1. Let X be a topological space.

(1) A cover of a subset $A \subseteq X$ is a family $\{U_i\}_{i\in I}$ of subsets of X with $A \subseteq \bigcup_{i\in I} U_i$.
(2) We say that a cover $\{U_i\}_{i\in I}$ is open if each U_i is open.
(3) We say that a family $\{U_i\}_{i\in I}$ of subsets is locally finite if each $x \in X$ admits an open neighborhood W such that $\{i \in I \,|\, U_i \cap W \neq \varnothing\}$ is finite.

We now add one and a half definitions in the context of covers:

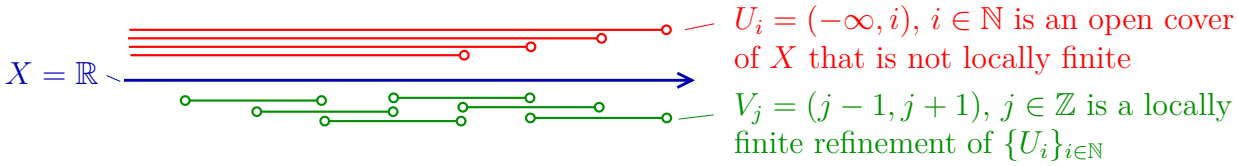
Definition 14.2. Let X be a topological space and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two covers of X.

(1) We say that $\{V_j\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$ if each V_j is contained in some U_i.
(2) If furthermore each V_j is open, then we say that $\{V_j\}_{j\in J}$ is an open refinement of $\{U_i\}_{i\in I}$.



Now we can give the key definition of this chapter, which was introduced by Jean Dieudonné in [**Die1944**, p. 66]:

Definition 14.3. A topological space X is called paracompact if every open cover of X admits a locally finite open refinement.



Remark. Many books, e.g. [**Bre1993**, Definition I.12.3] and [**Kel1975**, p. 156] demand that a paracompact space is also Hausdorff. □

As the name suggests, paracompact spaces are somewhat related to compact spaces. It is clear that a compact space is paracompact. The following lemma gives a useful example of a topological space that is not compact, but which is nonetheless paracompact.

Lemma 14.4. ($\mathbb{R}^n$-Paracompact Lemma) For every $n \in \mathbb{N}_0$ the topological space $\mathbb{R}^n$ is paracompact.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $\mathbb{R}^n$. Let $m \in \mathbb{N}$. We consider

$$\begin{array}{rrcl} \text{the closed annulus} & A_m & := & \{x \in \mathbb{R}^n \,|\, m \leq \|x\| \leq m+1\} \\ \text{the open annulus} & W_m & := & \{x \in \mathbb{R}^n \,|\, m-1 < \|x\| < m+2\}. \end{array}$$

By the Heine-Borel Theorem 3.44 we know that A_m is compact. Therefore there exists a finite subset $J_m \subseteq I$ such that the sets $\{U_j\}_{j\in J_m}$ form an open cover of A_m. One can now easily verify that

$$\{W_m \cap U_j\}_{m\in\mathbb{N}_0, j\in J_m}$$

is a locally finite open refinement of the open cover $\{U_i\}_{i\in I}$. ■

In fact, according to the principle "what properties of $\mathbb{R}^n$ did we really use?" we can generalize the above lemma to the following statement:

Lemma 14.5. (Paracompact-Criterion) Let X be a topological space. If X is second countable, Hausdorff and regionally compact, then X is also paracompact.

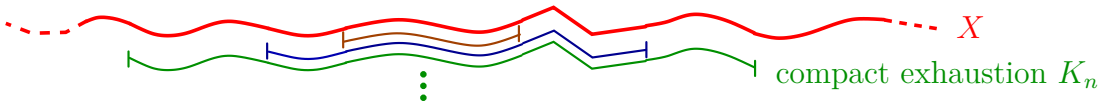
Remark. It follows from the discussion on page 208 and 280 that the $\mathbb{R}^n$-Paracompact Lemma 14.4 is actually a special case of the Paracompact-Criterion Lemma 14.5. □

Proof. Let X be a topological space that is second countable, Hausdorff and regionally compact. We want to show that X is paracompact. Thus let $\{U_i\}_{i\in I}$ be an open cover of X.

By the Compact Exhaustion Lemma 13.6 (1) there exists a sequence of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ of X such that the following two conditions are satisfied:

(1) for every $n \in \mathbb{N}$ we have $K_n \subseteq \mathring{K}_{n+1}$,
(2) we have $X = \bigcup_{n\in\mathbb{N}} K_n$.

We set $K_0 = \varnothing$. Let $n \in \mathbb{N}$. Since K_n is compact we obtain from the elementary Compact



Subspace Covering Lemma 3.38 that there exists a finite subset $J_n \subseteq I$ such that the sets

$\{U_j\}_{j\in J_n}$ form an open cover of K_n. One can now easily verify that[177]

$$\{(\overset{\circ}{K}_{n+1} \setminus K_{n-1}) \cap U_j\}_{n\in\mathbb{N}, j\in J_n}$$

is a locally finite open refinement of the open cover $\{U_i\}_{i\in I}$. ■

The following lemma can be viewed as an analog of the Compact-Closed Lemma 3.39 (1). In combination with the $\mathbb{R}^n$-Paracompact Lemma 14.4 and the Paracompact-Criterion 14.5 this new lemma gives us already a long list of paracompact spaces.

Lemma 14.6. (Closed Subset-Paracompact Lemma) Let X be a paracompact space. Every closed subset of X is also paracompact.

Remark. As is shown in [**Mun1975**, p. 254], in general a subspace of a paracompact space need not be paracompact. □

Proof. The proof of the Closed Subset-Paracompact Lemma 14.6 is almost identical to the proof of the Compact-Closed Lemma 3.39 (1). Indeed, let X be a paracompact space and let $A \subseteq X$ be a closed subset. Let $\{U_i\}_{i\in I}$ be an open cover of A. By definition of the subspace topology there exists for each $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is a closed subset of X we see that $\{V_i\}_{i\in I}$ together with $X \setminus A$ is an open cover of X. Since X is paracompact there exists a locally finite open refinement $\{W_j\}_{j\in J}$ of this cover of X. Evidently $\{W_j \cap A\}_{j\in J}$ is a locally finite open refinement of our original cover $\{U_i\}_{i\in I}$ of A. ■

We conclude our initial list of examples of paracompact spaces with the following difficult theorem. We will not make use of it and thus we will not provide the proof.

Theorem 14.7. (Stone's Theorem) Every metric space is paracompact.

Proof. The theorem was first proved by Arthur Stone [**Stone1948**] in 1948. A much shorter proof can be found in [**RudiM1969**]. Alternatively a proof is given in [**Mun1975**, Theorem 41.4]. ■

Remark. As we will see, most topological spaces that are of interest to us, are in fact paracompact. For example we just saw that Stone's Theorem 14.7 shows that metrizable topological spaces are paracompact. Furthermore in the Manifold Paracompact-Normal Proposition 23.9 (1) we will see, using the Paracompact-Criterion Lemma 14.5 that "topological manifolds" are paracompact. Furthermore in the CW-Complex–Paracompact Proposition 136.33 and in the Simplicial-Implies-CW Lemma 174.51 we will see that "CW-complexes" and "simplicial complexes" are paracompact. Nonetheless, elementary topological psychology tells us that surely not every topological space is paracompact. Indeed in the Long Ray Properties Proposition 28.21 we will see that the "long line", that we will define in Definition 28.19, is not paracompact. □

Next we recall Definition 3.48:

Definition 14.8. A topological space X is called normal if any two disjoint closed subsets A and B of X are separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.

[177]Since K_{n-1} is compact and since X is Hausdorff we obtain from the Compact-Closed Lemma 3.39 that K_{n-1} is a closed subset. Thus it follows that $\overset{\circ}{K}_{n+1} \setminus K_{n-1}$ is an open subset of X.

In the Compact Hausdorff-Normal Lemma 3.49 (2) we showed that every compact Hausdorff space is normal. The following theorem, which is originally due to Jean Dieudonné [**Die1944**, Théorème 1], generalizes this result from compact to paracompact spaces.

Theorem 14.9. (Dieudonné's Theorem) Every paracompact Hausdorff space is normal.

The proof of Dieudonné's Theorem 14.9 rests on the following technical but useful lemma.

Lemma 14.10. (Closure-of-Locally Finite Cover Lemma) Let X be a topological space. If $\{U_i\}_{i\in I}$ is a locally finite cover of X, then $\{\overline{U}_i\}_{i\in I}$ is also a locally finite cover of X.

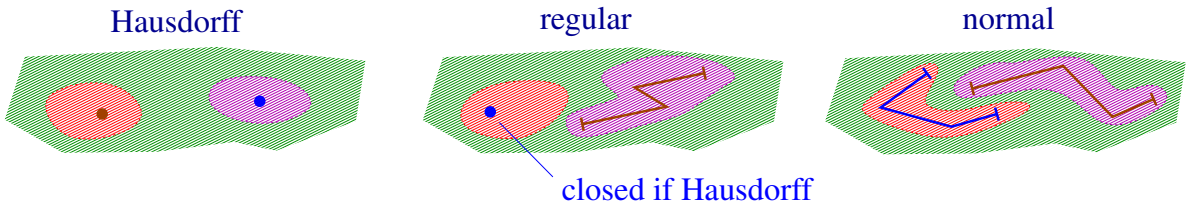
Proof of Lemma 14.10. Let X be a topological space and let $\{U_i\}_{i\in I}$ is a locally finite cover of X. The lemma follows immediately from the definitions and the following claim:

Claim. If W and U are two disjoint open subsets of X, then $\overline{U} \cap W = \varnothing$.

Proof. Let W and U be two disjoint open subsets of X. This means that U is contained in the closed subset $X \setminus W$. By Definition 3.21 of the closure this implies that $\overline{U} \subseteq X \setminus W$. In particular $\overline{U} \cap W = \varnothing$. ■

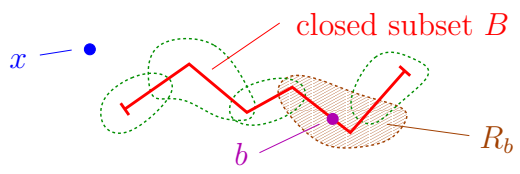
Proof of Dieudonné's Theorem 14.9. First recall that, according to Definition 13.11, a topological space X is called *regular* if given any $x \in X$ and given any closed subset $B \subseteq X$ with $x \notin A$ there exist disjoint open subsets U and V with $x \in U$ and $B \subseteq V$. Now let X be a paracompact Hausdorff space. In the following claim we first use paracompactness to upgrade from "Hausdorff" to "regular".

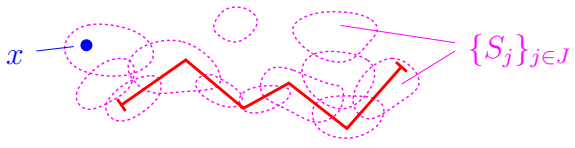
Claim. The topological space X is regular.



Proof. Let $x \in X$ and let $B \subseteq X$ be a closed subset. Since X is Hausdorff it follows from the Points-in-Hausdorff Lemma 3.32 (2) that we can find for each $b \in B$ an open neighborhood R_b of $b \in X$ such that $x \notin \overline{R}_b$. The open subsets $\{R_b\}_{b\in B}$ together with the open subset $X \setminus B$ form an open cover of X. Since X is paracompact there exists a locally finite open refinement $\{S_j\}_{j\in J}$ of this cover. We set

$$U \coloneqq X \setminus \bigcup_{j\in J \text{ with } S_j\cap B\neq\varnothing} \overline{S}_j \quad \text{and} \quad V \coloneqq \bigcup_{j\in J \text{ with } S_j\cap B\neq\varnothing} S_j.$$





We make the following observations:

- Evidently we have $U \cap V = \varnothing$.
- Since the S_j are an open cover of X we see that V is an open neighborhood of B.
- We claim that $x \in U$. If $j \in J$ with $S_j \cap B \neq \varnothing$, then we see that S_j is contained in one of the R_b's, since it cannot be contained in $X \setminus B$. Since $\overline{S}_j \subseteq \overline{R}_b$ we see that $x \notin \overline{S}_j$. This implies that $x \in U$.
- We need to show that U is open. Note that by construction the family $\{S_j\}_{j\in J}$ is locally finite. Next note that it follows from the Closure-of-Locally Finite Cover Lemma 14.10 that $\{\overline{S}_j\}_{j\in J}$ is also a locally finite family of subsets of X. By the Closed Subset-Lemma 3.5 (2b) we know that the union of a locally finite collection of closed subsets is again closed. This shows that U is open.

We have thus shown that U and V are disjoint open neighborhoods of x and B. ⊞
In the remainder we will use the argument of the claim to use paracompactness to upgrade from "regular" to "normal". Thus let A and B be two disjoint closed subsets of X. By the claim we know that X is regular. By the Regular-Open-Closed-Neighborhood Lemma 13.12 this implies that for each $a \in A$ there exists an open neighborhood R_a of a such that $\overline{R}_a \subseteq X \setminus B$.

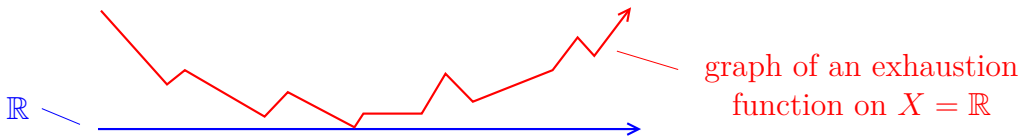
The open subsets $\{R_a\}_{a\in A}$ together with the open subset $X \setminus A$ form an open cover of X. Since X is paracompact there exists a locally finite open refinement $\{S_j\}_{j\in J}$ of this cover. We set

$$U \coloneqq \bigcup_{j\in J \text{ with } S_j\cap A\neq\varnothing} S_j \quad \text{and} \quad V \coloneqq X \setminus \bigcup_{j\in J \text{ with } S_j\cap A\neq\varnothing} \overline{S}_j.$$

By basically the same argument as above we see that U is an open neighborhood of A, that V is an open neighborhood of B and that $U \cap V = \varnothing$. ■

We conclude this section with a short discussion of exhaustion functions.

Definition 14.11. Let X be a topological space. An exhaustion function is a continuous map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X.[178]



Lemma 14.12. (Exhaustion Function Lemma) Let X be a topological space. If X is second countable, Hausdorff and regionally compact, then X admits an exhaustion function.

Remark. The Exhaustion Function Lemma 14.12 can be viewed as a refinement of the Compact Exhaustion Lemma 13.6 (1). □

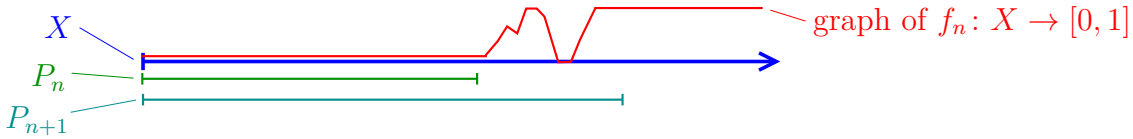
[178] One can easily show that a map $X \to \mathbb{R}_{\geq 0}$ is an exhaustion function if and only if it is proper in the sense of Definition 4.63.

Proof. Let X be a topological space which is second countable, Hausdorff and regionally compact. By the Compact Exhaustion Lemma 13.6 (1) we know that X admits an exhaustion by compact subsets. In other words we know that there exists a sequence $\{P_n\}_{n\in\mathbb{N}}$ of compact subsets of X such that the following two conditions are satisfied:

(1) We have $X = \bigcup_{n\in\mathbb{N}} P_n$.

(2) For each $n \in \mathbb{N}$ we have[179] $P_n \subseteq \mathring{P}_{n+1}$.

Furthermore, the combination of the Paracompact-Criterion Lemma 14.5 and Dieudonné's Theorem 14.9 shows that X is normal. Since X is Hausdorff we know by the Compact-Closed Lemma 3.39 that each P_n is closed. Furthermore, by (2) we know that for each $n \in \mathbb{N}$ the closed subsets P_n and $X \setminus \mathring{P}_{n+1}$ are disjoint. Since X is normal we obtain from Urysohn's Lemma 12.5 for each $n \in \mathbb{N}$ a map $f_n\colon X \to [0,1]$ such that $f_n|_{P_n} \equiv 0$ and $f_n\big|_{X\setminus\mathring{P}_{n+1}} \equiv 1$.



Claim 1. The sets $U_n = \{x \in X \mid f_n(x) \neq 0\}_{n\in\mathbb{N}}$ are a locally finite open cover of X.

Proof. Evidently each $U_n = f_n^{-1}((0,1])$ is open. Now let $x \in X$ be a point. According to Definition 3.19 we need to show that x admits an open neighborhood W such that W is contained in only finitely many U_n's. By (1) and (2) we know that x is contained in some open subset $\mathring{P}_k$. By (2) and by construction we know that $f_n(y) = 0$ for all $y \in P_k$ and all $n \geq k+1$. ⊞

By the claim it makes sense to consider the map

$$\begin{array}{rcl} f\colon X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sum\limits_{k\in\mathbb{N}} f_n(x). \end{array}$$

Furthermore it follows from the Pasting Proposition 4.7 that f is continuous. It remains to prove the following claim:

Claim 2. For every $c \in \mathbb{R}$ the preimage $f^{-1}([0,c])$ is compact.

Proof. Let $c \in \mathbb{R}$. We pick $N \in \mathbb{N}$ with $N \geq c$. Then

$$\begin{array}{rcl} f^{-1}([0,c]) & = & f^{-1}([0,c]) \cap f^{-1}([0,N]) \\ & \subseteq & f^{-1}([0,c]) \cap P_{N+1} \quad = \quad \text{closed subset} \cap \text{compact subset}. \end{array}$$

since $f(x) \geq N+1$ for all $x \in X \setminus \mathring{P}_{N+1}$

It follows from the Compact-Closed Lemma 3.39, see also the elementary Exercise 3.18, that $f^{-1}([0,c])$ is indeed compact. ■

Let X be a topological space that is second countable, Hausdorff and regionally compact. By the Paracompact-Criterion Lemma 14.5 we know that X is paracompact. It follows

[179]Here we use the Compact-Closed Lemma 3.39 (2) which tells us that every compact subspace of our Hausdorff space X is a closed subset.

from Dieudonné's Theorem 14.9 that X is normal. Since X is second-countable, normal and Hausdorff it follows from the elementary Points-in-Hausdorff Lemma 3.32 (1) and the much less elementary Urysohn Metrization Theorem 13.8 that X is metrizable. The concept of exhaustion functions allows us to prove the following proposition which upgrades the conclusion from "metrizable" to "completely metrizable".

Proposition 14.13. (Completely Metrizable Criterion) Let X be a topological space. If X is second countable, Hausdorff and regionally compact, then it is completely metrizable.

Proof. Let X be a topological space that is second countable, Hausdorff and regionally compact. As we just discussed, it follows from the Paracompact-Criterion Lemma 14.5, Dieudonné's Theorem 14.9, the Points-in-Hausdorff Lemma 3.32 (1) and the Urysohn Metrization Theorem 13.8 that X is metrizable.

Now let d be a metric on X which induces the given topology on X. Since X is second countable, Hausdorff and regionally compact we know by the Exhaustion Function Lemma 14.12 that X admits an exhaustion function, i.e there exists a continuous map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X.

We now set
$$\begin{array}{rccc} \widetilde{d}\colon & X \times X & \to & \mathbb{N}_0 \\ & (x,y) & \mapsto & d(x,y) + |f(x) - f(y)|. \end{array}$$
It follows from Exercise 10.5 that $\widetilde{d}$ is a metric and that $\widetilde{d}$ induces again the given topology.

Now let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X,\widetilde{d})$. It follows immediately from the definition of $\widetilde{d}$ that there exists a $c \in \mathbb{R}_{\geq 0}$ such that $|f(x_n)| \leq c$ for all $n \in \mathbb{N}$. In other words, the sequence $(x_n)_{n\in\mathbb{N}}$ takes place in the compact subspace $f^{-1}([0,c])$. But by the Compact-Complete Proposition 10.17 we know that any Cauchy sequence in a compact metric space converges. ∎

14.2. **Partitions of unity.** In this section we introduce the notion of a "partition of unity" and we will see that for Hausdorff spaces the existence of suitable partitions of unity is equivalent to being paracompact. On numerous occasions we will see that "partitions of unity" are very useful for building interesting maps "out of" a given topological space.

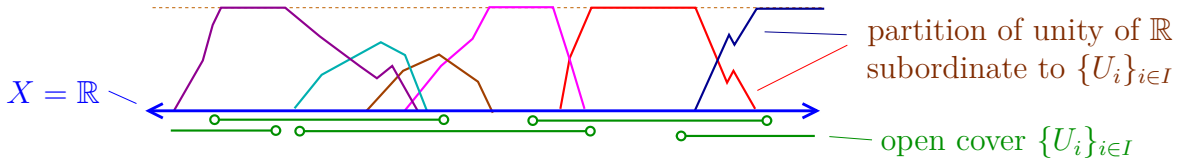
Definition 14.14. Let X be a topological space.

(1) The support of a map $f\colon X \to \mathbb{R}$ is defined as
$$\operatorname{supp}(f) \;:=\; \overline{\{x \in X \mid f(x) \neq 0\}}.$$

(2) A partition of unity on X is a family of continuous maps $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
   (a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i.[180]
   (b) For every $x \in X$ we have[181]
$$\sum_{i\in I} f_i(x) \;=\; 1.$$

(3) Let $\{U_i\}_{i\in I}$ be an open cover. A partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$ is a partition of unity $\{f_j\colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subseteq U_i$.

Example. In the figure below we consider the topological space $X = \mathbb{R}$ together with some open cover $\{U_i\}_{i\in I}$.



We also illustrate a partition of unity subordinate to this open cover. □

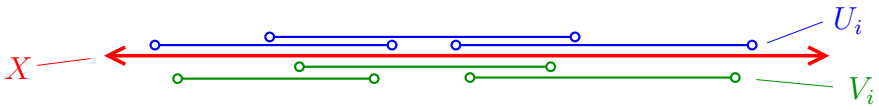
The following is the most interesting and useful theorem of this chapter.

Theorem 14.15. (Paracompact-Partition of Unity Theorem) Let X be a topological space that is *Hausdorff*. The following three statements are equivalent:

(1) The topological space X is paracompact.
(2) For every open cover $\{U_i\}_{i\in I}$ of X we can find a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$.
(3) For every open cover $\{U_i\}_{i\in I}$ of X we can find a partition of unity $\{f_i \colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\operatorname{supp}(f_i) \subseteq U_i$.

The proof of the "(1)⇒(3)"-direction of the Paracompact-Partition of Unity Theorem 14.15 rests on the shoulders of the following lemma which is interesting in its own right.

Lemma 14.16. (Shrinking Lemma) Let X be a paracompact Hausdorff space and let $\{U_i\}_{i\in I}$ be an open cover of X. There exists a locally finite open cover $\{V_i\}_{i\in I}$ of X such that for each $i \in I$ we have $\overline{V}_i \subseteq U_i$.



Proof of the Shrinking Lemma 14.16. Let X be a paracompact Hausdorff space and let $\{U_i\}_{i\in I}$ be an open cover of X. We define $\mathcal{A}$ to be the set of all open subsets A of X such that the closure $\overline{A}$ is contained in some subset U_i of the given open cover.

Claim 1. The sets in $\mathcal{A}$ form an open cover of X.

Proof. Since X is Hausdorff and paracompact we obtain from Dieudonné's Theorem 14.9 that X is normal. Furthermore since X is Hausdorff we know by the Points-in-Hausdorff Lemma 3.32 that every one-point subset of X is closed. It follows from these facts, together with the Normal-Open-Closed-Neighborhood Lemma 12.3, that given any U_i and given any $x \in U_i$ there exists an open subset A of X with $x \in A \subseteq \overline{A} \subseteq U_i$. Note that $A \in \mathcal{A}$. Since the U_i form an open cover of X we now see that the sets in $\mathcal{A}$ also form an open cover of X. ⊞

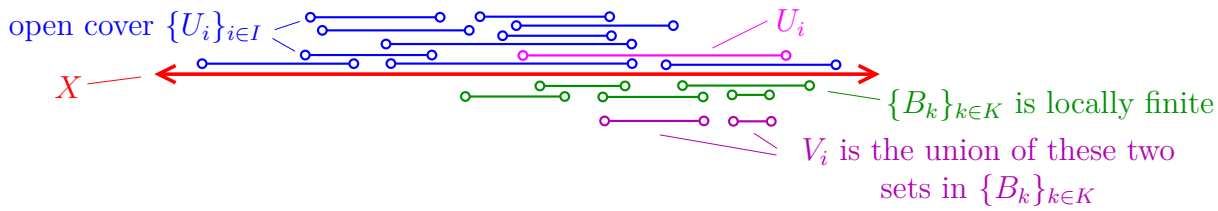
Since X is paracompact there exists a locally finite open cover $\{B_k\}_{k\in K}$ of X that is a refinement of $\mathcal{A}$. Note that every B_k is contained in some set in $\mathcal{A}$. Thus for every $k \in K$

[180] If U is a subset on which a map $f \colon X \to \mathbb{R}$ vanishes, and if U is open, then it follows from the Definition 3.21 of the closure that $\operatorname{supp}(f) \cap U = \varnothing$. This observation shows that (a) implies that for each $x \in X$ there exist only finitely many $i \in I$ such that $x \in \operatorname{supp}(f_i)$.

[181] Note that by (a) there are only finitely many non-zero summands, hence the expression on the left-hand side makes sense even if I is infinite.

we can and will pick an $f(k) \in I$ with $\overline{B}_k \subseteq U_{f(k)}$. Given $i \in I$ we set

$$V_i := \bigcup_{k \in K \text{ with } f(k)=i} B_k.$$



Claim 2. For each $i \in I$ we have $\overline{V}_i \subseteq U_i$.

Proof. Let $i \in I$. We see that

$$\overline{V}_i = \overline{\bigcup_{k \in K \text{ with } f(k)=i} B_k} = \bigcup_{k \in K \text{ with } f(k)=i} \overline{B}_k \subseteq U_i.$$

The first equality follows from the Interior-Closure-Cup-Cap Lemma 3.24 (2b'), since $\{B_k\}_{k\in K}$ is locally finite. The last inclusion is by definition of $f\colon K \to I$ we know that for each $k \in K$ with $f(k) = i$ we have $\overline{B}_k \subseteq U_i$. ⊞

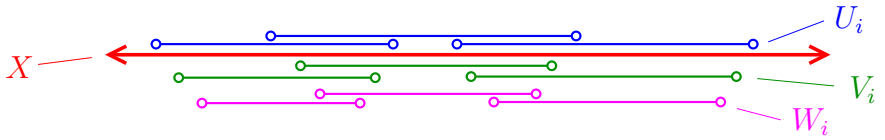
It remains to prove the following claim:

Claim 3. The family $\{V_i\}_{i\in I}$ is locally finite.

Proof. Let $x \in X$. Since $\{B_k\}_{k\in K}$ is locally finite there exists an open neighborhood W of $x \in X$ such that W has non-empty intersection with only finitely many B_k. Suppose that the corresponding indices are $k_1, \ldots, k_m \in K$. If $W \cap V_i \neq \varnothing$, then by definition of V_i there exists a $k \in K$ with $f(k) = i$ and with $W \cap B_k \neq \varnothing$, but this implies that $i \in \{f(k_1), \ldots, f(k_m)\}$. ■

Proof of the Paracompact-Partition of Unity Theorem 14.15. The "(2)⇒(1)"-direction is basically trivial. Indeed let $\{U_i\}_{i\in I}$ be an open cover of X. By hypothesis there exists a partition of unity $\{f_j\colon X \to [0,1]\}_{j\in J}$ subordinate to the open cover $\{U_i\}_{i\in I}$. It follows immediately from the definitions that $\{f_j^{-1}((0,\infty))\}_{j\in J}$ is a locally finite open cover that is a refinement of $\{U_i\}_{i\in I}$.

The "(3)⇒(2)"-direction is utterly trivial. Thus it remains to prove the "(1)⇒(3)"-direction. Let $\{U_i\}_{i\in I}$ be an open cover of X. By applying the Shrinking Lemma 14.16 twice we obtain locally finite open covers $\{V_i\}_{i\in I}$ and $\{W_i\}_{i\in I}$ such that for each $i \in I$ we have $\overline{W}_i \subseteq V_i \subseteq \overline{V}_i \subseteq U_i$.



Since X is Hausdorff and paracompact we know by Dieudonné's Theorem 14.9 that X is normal. Now let $i \in I$. Since X is normal we can apply Urysohn's Lemma 12.5 to the disjoint closed subsets $\overline{W}_i$ and $X \setminus V_i$ and we obtain a continuous map $g_i\colon X \to \mathbb{R}$ with $g|_{\overline{W}_i} \equiv 1$ and $g|_{X\setminus V_i} \equiv 0$.

Claim. For each $x \in X$ there exists an open neighborhood U of x such that g_i vanishes on U for all but finitely many i.

Proof. For every $i \in I$ we have $\{x \in X \mid g_i(x) \neq 0\} \subseteq V_i$. The claim now follows from the fact that the cover $\{V_i\}_{i\in I}$ is locally finite. ⊞

By the claim it makes sense to consider the map

$$\begin{array}{rcl} g\colon X & \to & \mathbb{R} \\ x & \mapsto & g(x) := \sum_{i\in I} g_i(x). \end{array}$$

It follows from the claim and the Pasting Proposition 4.7 that g is continuous. Furthermore note that it follows from the fact that the $\{W_i\}_{i\in I}$ form a cover of X and that $g_i|_{W_i} \equiv 1$ that for every $x \in X$ we have $g(x) > 0$. Therefore given $i \in I$ it makes sense to consider the map

$$\begin{array}{rcl} f_i\colon X & \to & \mathbb{R} \\ x & \mapsto & \frac{g_i(x)}{g(x)}. \end{array}$$

The maps f_i add up to the constant function 1. Together with the claim it follows that the maps $\{f_i\}_{i\in I}$ form a partition of unity.

Finally let $i \in I$. As we mentioned above, we have $\{x \in X \mid f_i(x) \neq 0\} \subseteq V_i$. It follows that $\operatorname{supp}(f_i) \subseteq \overline{V}_i \subseteq U_i$. ■

The following, slightly technical proposition, sometimes makes it possible to find a suitable *countable* open cover. We will only make use of it once, namely much later, when we study vector bundles. For the time being it is surely best to just ignore it.

Proposition 14.17. (Countable Cover Proposition) Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. If X is Hausdorff and paracompact, then there exists a *countable* locally finite open cover $\{V_j\}_{j\in J}$ with the following two properties:

(1) Each V_j is the disjoint union of open subsets of X, each of which is contained in some U_i.[182]

(2) Each component of any V_j is contained in some U_i.

Example. We consider the topological space $X = \bigsqcup_{i\in\mathbb{R}} (-2, 2)$ which is the disjoint union of $\mathbb{R}$-copies of the interval $(-2, 2)$. We consider the open cover that is given by the family $U_{i-} := (-2, 1) \times \{i\}$, $U_{i+} := (-1, 2) \times \{i\}$ with $i \in \mathbb{R}$. The finite family consisting of the two sets $V_- := \bigsqcup_{i\in I} U_{i-}$ and $V_+ := \bigsqcup_{i\in I} U_{i+}$ has the properties of the Countable Cover Proposition 14.17. □

Proof. Since X is Hausdorff and paracompact it follows from the Paracompact-Partition of Unity Theorem 14.15 (1)⇒(3) that there exists a partition of unity $\{f_i\colon X \to [0, 1]\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(f_i) \subseteq U_i$. Given a finite non-empty subset $S \subseteq I$ we define

$$V_S := \{x \in X \mid f_s(x) > f_i(x) \text{ for all } s \in S \text{ and all } i \notin s\}.$$

[182]Note that it follows from the Disjoint Union Embedding Lemma 7.29 that V_j is homeomorphic to the disjoint union of these open subsets.

Claim.

(a) Each V_S is open.
(b) Each $x \in X$ is contained in some V_S.
(c) Each V_S is contained in some U_i.
(d) For finite non-empty subsets S and T of I we have $V_S \subseteq V_T$ if and only if $S \subseteq T$.

Proof.

(a) Let $S \subseteq I$ be finite and non-empty. We need to show that V_S is open. By the JH-Openness Criterion 3.3 it suffices to show that given any $x \in V_S$ there exists an open neighborhood U of x such that $V_S \cap U$ is open. Thus let $x \in V_S$. By definition of a partition of unity there exists a neighborhood U on which all but finitely many of the functions f_i vanish. We set $T := \{i \in I \setminus S \mid f_i(x) \neq 0\}$. We see that

$$V_S \cap U \;=\; \bigcap_{t\in T}\bigcap_{s\in S} \underbrace{\{x \in X \mid f_s(x) > f_t(x)\}}_{=(f_t-f_s)^{-1}((-\infty,0))}.$$

This is the intersection of finitely many open subsets, thus open.
(b) Let $x \in X$. By definition of a partition of unity the set $S = \{i \in I \mid f_i(x) > 0\}$ is finite and non-empty. Evidently we have $x \in V_S$.
(c) Let $S \subseteq I$ be a finite non-empty subset. Given any $i \in S$ we have $V_S \subseteq \operatorname{supp}(f_i) \subseteq U_i$.
(d) This statement follows immediately from the definitions. ⊞

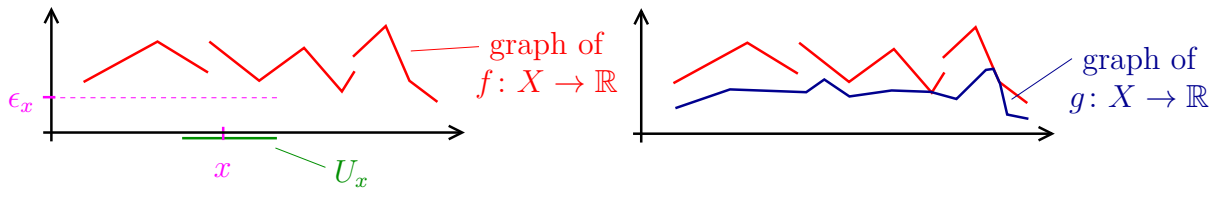
Now given $n \in \mathbb{N}_0$ we set $V_n := \bigcup_{\#S=n} V_S \subseteq X$. It follows from the above that the V_S with $\#S = n$ are disjoint and open. Therefore we deduce from the Component Lemma 4.48 (4b) that each component of V_n is contained in some V_S which in turn, as we saw above, is contained in some U_i. It is now clear that the family $\{V_n\}_{n\in\mathbb{N}_0}$ has all the desired properties. ■

14.3. The Neighborhood Embedding Theorem.

We conclude this chapter with the Neighborhood Embedding Theorem which shows that under favorable circumstances one can extend an embedding from a closed subset to an open neighborhood.

Before we discuss the Neighborhood Embedding Theorem let us first state and prove a lemma, which is of independent interest, but which often gets used in conjunction with the Neighborhood Embedding Theorem.

Lemma 14.18. (Locally Non-Zero Lemma) Let X be a topological space and let $f\colon X \to \mathbb{R}$ be a map (not necessarily continuous) with the property that for every $x \in X$ there exists an open neighborhood U_x of x and an $\epsilon_x > 0$ such that for each $y \in U_x$ we have $f(y) \geq \epsilon_x$. If X is paracompact, then there exists a continuous map $g\colon X \to (0,\infty)$, such that for all $x \in X$ we have $g(x) \leq f(x)$.



Proof. Since X is paracompact we know by the Paracompact-Partition of Unity Theorem 14.15 that we can pick a partition of unity $\{\varphi_x\}_{x\in X}$ such that for each $x \in X$ we have $\operatorname{supp}(\varphi_x) \subseteq U_x$. We set

$$g \coloneqq \sum_{x\in X} \varphi_x \cdot \epsilon_x.$$

As always it follows from the Pasting Proposition 4.7 that g is continuous. Since each $\epsilon_x > 0$ we see that $g(y) > 0$ for all $y \in X$. Finally, for every $y \in X$ we have

$$g(y) = \sum_{x\in X} \varphi_x(y)\cdot\epsilon_x \underset{\substack{\uparrow\\ \text{since } \operatorname{supp}(\varphi_x)\subseteq U_x}}{\leq} \sum_{x\in X \text{ with } y\in U_x} \varphi_x(y)\cdot\epsilon_x \underset{\substack{\uparrow\\ \text{property of } \epsilon_x}}{\leq} \sum_{x\in X \text{ with } y\in U_x} \varphi_x(y)\cdot f(y) \underset{\substack{\uparrow\\ \text{since } \{\varphi_x\}_{x\in X} \text{ is a partition of unity and since } \operatorname{supp}(\varphi_x)\subseteq U_x}}{=} f(y).$$

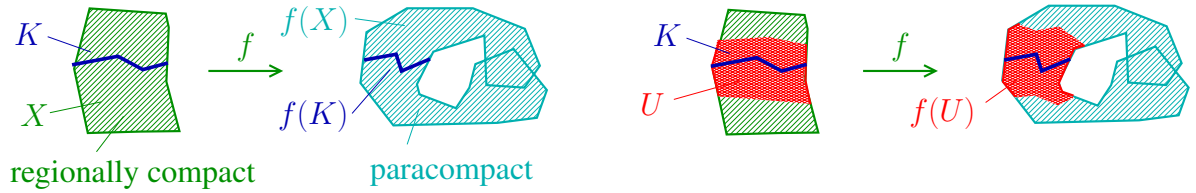
In summary, we have shown that $g(y) \leq f(y)$ for all $y \in X$. ■

Now we turn our attention to the actual hero of this section.

Theorem 14.19. (Neighborhood Embedding Theorem) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $K \subseteq X$ be a subset. We assume that the following conditions are satisfied.

(1) The restriction of f to $f\colon K \to Y$ is an embedding.
(2) K is a closed subset of X and $f(K)$ is a closed subset of Y.
(3) The map $f\colon X \to Y$ is a local homeomorphism at each $P \in K$.
(4) The topological space X is regionally compact.
(5) The topological space Y is Hausdorff and paracompact.

Then there exists an open neighborhood U of K such that the restriction of f to $f\colon U \to Y$ is an embedding and such that $f(U)$ is an open subset of Y.



Remark.

(1) This very useful theorem is based on [**Lang1985**, p. 97], [**Kos1993**, Lemma 9.2] and [**God1973**, p. 150]. Writing down the proof of the Neighborhood Embedding Theorem caused countless hours of pain for the author.
(2) The Neighborhood Smooth Embedding Theorem 27.16 gives the corresponding statement for smooth manifolds. □

Proof. We start out with the first of several claims.

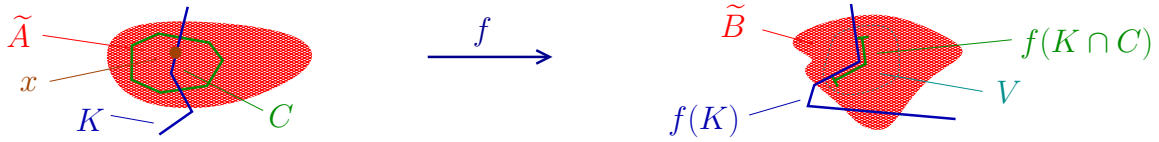
Claim 1. For each $x \in K$ there exists an open neighborhood A_x of $x \in X$ and an open neighborhood B_x of $f(x) \in Y$ such that $f|_{A_x}\colon A_x \to B_x$ is a homeomorphism and such that $f^{-1}(B_x \cap f(K)) = A_x \cap K$.

Proof. Let $x \in K$. Since f is a local homeomorphism at $x \in K$ we know that there exists an open neighborhood $\widetilde{A}$ of $x \in X$ and an open neighborhood $\widetilde{B}$ of $f(x) \in Y$ such that $f|_{\widetilde{A}}\colon \widetilde{A} \to \widetilde{B}$ is a homeomorphism.

Since X is regionally compact there exists a compact subset $C \subseteq X$ with $x \in C \subseteq \widetilde{A}$. Furthermore, since $K \subseteq X$ is closed we see that $K \cap C$ is a closed subset of C. It follows from the Compact-Closed Lemma 3.39 that $K \cap C$ is also compact. Next we consider $f(K \cap C)$. By the Compact Image Lemma 4.15 we know that $f(K \cap C)$ is compact. Since we assume that Y is Hausdorff we obtain from the Compact-Closed Lemma 3.39 that $f(K \cap C)$ is a closed subset of Y.

Next note that $K \cap (X \setminus \widetilde{A})$ is a closed subset of K. Since f is an embedding we see that $f(K \cap (X \setminus \widetilde{A}))$ is a closed subset of $f(K)$. Since $f(K)$ is a closed subset of Y we obtain from the Open-Closed Inclusion Lemma 4.13 (2) that $f(K \cap (X \setminus \widetilde{A}))$ is a closed subset of Y.

Since the map f is in particular injective we see that the two closed subsets $f(K \cap C)$ and $f(K \cap (X \setminus \widetilde{A}))$ are actually disjoint. Since Y is paracompact we know by Dieudonné's Theorem 14.9 that Y is normal. Thus there exist disjoint open neighborhoods V of $f(K \cap C)$ and W of $f(K \cap (X \setminus \widetilde{A}))$.



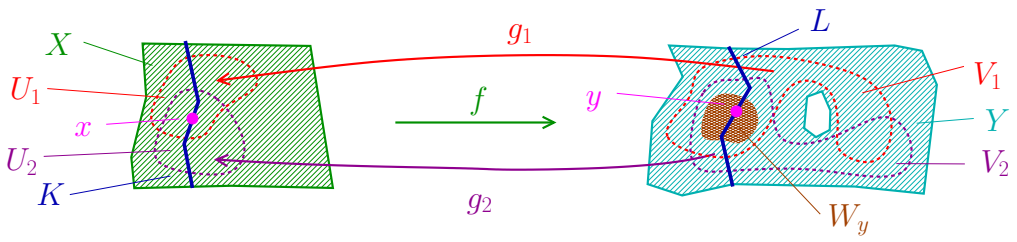
We set $B_x := \widetilde{B} \cap V$ and we set $A_x := \widetilde{A} \cap f^{-1}(B_x)$. Basically by construction these have the desired properties. ⊞

Claim 2. There exists a family of open subsets $\{U_i\}_{i \in I}$ of X with the following properties:
(a) The maps $f|_{U_i} \colon U_i \to V_i = f(U_i)$ are homeomorphisms.
(b) For each $i \in I$ we have $f_i^{-1}(V_i \cap f(K)) = U_i \cap K$.
(c) The sets $\{V_i\}_{i \in I}$ form a locally finite open cover of $f(K)$ in the sense that given any $y \in f(K)$ there exists an open neighborhood $\widetilde{W}$ of $y \in Y$ such that $\widetilde{W}$ has non-empty intersection with only finitely many V_i.

Proof. For each $x \in K$ we pick A_x and B_x as in Claim 1. Note that the sets $\{B_x\}_{x \in K}$ form an open cover of $f(K)$. By hypothesis $f(K)$ is a closed subset of Y. Thus the sets $\{B_x\}_{x \in K}$ and $Y \setminus f(K)$ form an open cover of Y. Since Y is paracompact we can find a *locally finite* open cover $\{V_i\}_{i \in I'}$ that is a refinement of the given open cover of Y. In particular each V_i is contained in some B_x or it is contained in $Y \setminus f(K)$. We set $I := \{i \in I' \mid V_i \text{ is contained in some } B_x\}$. For each $i \in I$ we set $U_i = f^{-1}(V_i)$. These sets have, almost by definition, the desired properties. ⊞

For each $i \in I$ we write $f_i := f|_{U_i} \colon U_i \to V_i$ and we write $g_i := f_i^{-1} \colon V_i \to U_i$. It follows from the hypothesis that Y is paracompact, the fact that $f(K)$ is a closed subset and the Shrinking Lemma 14.16 that there exists a locally finite open cover $\{D_i\}_{i \in I}$ of $f(K) \subseteq Y$ such that for each $i \in I$ we have $\overline{D}_i \subseteq V_i$.

Claim 3. Let $y \in f(K)$. There exists a subset $W_y \subseteq Y$ with the following properties:
(a) $\{i \in I \mid \overline{D}_i \cap W_y \neq \varnothing\}$ is finite.
(b) W_y is an open neighborhood of $y \in Y$.
(c) If $\overline{D}_i \cap W_y \neq \varnothing$, then $W_y \subseteq V_i$.
(d) For each $i, j \in I$ with $\overline{D}_i \cap W_y \neq \varnothing$ and $\overline{D}_j \cap W_y \neq \varnothing$ we have[183] $g_i|_{W_y} = g_j|_{W_y}$.



Proof. Let $y \in f(K)$. Since the open cover $\{V_i\}_{i\in I}$ is locally finite and since for each $i \in I$ we have $\overline{D}_i \subseteq V_i$ we know that there exists an open neighborhood $\widetilde{W}$ of $y \in Y$ such that $J \coloneqq \{i \in I \,|\, \overline{D}_i \cap \widetilde{W} \neq \varnothing\}$ is finite. We set $\widetilde{J} \coloneqq \{j \in J \,|\, y \in \overline{D}_j\}$ and we set

$$W_y \;\coloneqq\; \Big(\widetilde{W} \setminus \bigcup_{j \,\in\, J \text{ with } y \,\notin\, \overline{D}_j} \overline{D}_j\Big) \cap f\Big(\bigcap_{j\in\widetilde{J}} U_j\Big).$$

We claim that W_y has the desired properties:

(a) By construction $\{i \in I \,|\, \overline{D}_i \cap W_y \neq \varnothing\}$ is finite.

(b) First we make the following observations:

- We pick some $i \in \widetilde{J}$. Note that $\bigcap_{j\in\widetilde{J}} U_j$ is the intersection of finitely many open subsets of Y. It follows that it is an open subset of U_i. Since $f\colon U_i \to V_i$ is a homeomorphism and since V_i is an open subset of Y we obtain, using the elementary Open-Closed Inclusion Lemma 4.13, that $f\Big(\bigcap_{j\in\widetilde{J}} U_j\Big)$ is an open subset of Y.
- The set W_y is thus obtained by removing a finite number of closed subsets from $\widetilde{W}$ and by taking afterwards the intersection with an open subset of Y.

This discussion shows that W_y is an open subset of Y.

It remains to show that $y \in W_y$. Since $f\colon K \to Y$ is an embedding there exists a unique $x \in K$ with $f(x) = y$. It remains to show that for each $j \in \widetilde{J}$ we have $x \in U_j$. Thus let $j \in \widetilde{J}$. By definition we have $y \in \overline{D}_j \subseteq V_j$. By Property (b) of Claim 2 we have $f_j^{-1}(V_j \cap f(K)) = U_j \cap K$. It follows that $f_j^{-1}(y) \in U_j \cap K$, which means that $f_j^{-1}(y) = x$. Thus we have shown that $x \in U_j$.

(c) Suppose that we have an $i \in I$ with $\overline{D}_i \cap W_y \neq \varnothing$. By construction of W_y we have $y \in \overline{D}_i$. This in turn implies that $i \in \widetilde{J}$ which then implies, by construction, that $W_y \subseteq f(U_i) = V_i$.

(d) Now let $Q \in W_y$. Let $i \in I$ with $\overline{D}_i \cap W_y \neq \varnothing$.

- By definition $g_i(Q)$ is the unique element $P \in U_i$ with $f(P) = Q$.
- In (c) we just saw that $i \in \widetilde{J}$.
- By construction of W_y we have $Q \in f\big(\bigcap_{j\in\widetilde{J}} U_j\big)$.
- Since $i \in \widetilde{J}$ and since $f\colon U_i \to Y$ is injective we obtain from the above discussion that $P \in \bigcap_{j\in\widetilde{J}} U_j$.

In summary we have shown that $g_i(Q)$ is the unique $P \in \bigcap_{j\in\widetilde{J}} U_j$ with $f(P) = Q$. But this shows that $g_i(Q)$ is independent of the choice of $i \in J$. ⊞

Next we set

[183] Note that it follows from (c) that g_i and g_j are defined on W_y.

$$V := \bigcup_{y\in f(K)} W_y \qquad \text{and} \quad U := \bigcup_{y\in f(K)} \bigcup_{i\in I} g_i(W_y \cap D_i).$$

Since each W_y is open we see that V is an open neighborhood of $f(K)$. Furthermore, each $W_y \cap D_i$ is an open subset of V_i and thus each $g_i(W_y \cap D_i)$ is an open subset of U_i, in particular also of X. This shows that U is an open subset of X. Finally, since the D_i form an open cover of Y we see that U contains K. Thus we have shown that U is an open neighborhood of K.

The following claim now shows the desired statement, namely it shows that the restriction of f to $f\colon U \to Y$ is an embedding such that $f(U)$ is an open subset of Y.

Claim 4. The map

$$\begin{aligned} g\colon V &\to U \\ Q &\mapsto g_i(Q) \text{ where } i \in I \text{ is chosen such that } Q \in D_i \end{aligned}$$

is well-defined, continuous and it satisfies $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_U$.

Proof. It follows almost immediately from Claim 3 (d) that g is well-defined in the sense, that $g(Q)$ does not depend on the choice of $i \in I$. Furthermore, basically by definition of U we see that $g(V) \subseteq U$.

Next, note that on each subset $D_i \cap V$ the map g is given by the continuous map g_i. Since the $D_i \cap V$ are an open cover of V we obtain from the Pasting Proposition 4.7 that g is continuous.

Basically by definition we have $f \circ g = \mathrm{id}_V$. It remains to show that $g \circ f = \mathrm{id}_U$. Thus let $P \in U$. By definition P is of the form $g_i(Q) \subseteq U_i$ for some $Q \in W_y \cap D_i$. This implies that $f(P) = f(g_i(Q)) = Q \in W_y \cap D_i$. By definition we have $g(f(P)) = g_i(f(P))$. But $g_i(f(P))$ is the unique element of U_i which gets mapped to $f(P)$ under f. This element is P. We have thus shown that $g(f(P)) = P$. ∎

14.4. Exercises for Chapter 14.

Exercise 14.1. Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. Show that if there exists a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$, then there exists a partition of unity $\{f_i\colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\mathrm{supp}(f_i) \subseteq U_i$.

Remark. For Hausdorff spaces this is precisely the (2)⇒(3)-direction of the Paracompact-Partition of Unity Theorem 14.15. But even for Hausdorff spaces it is worth providing a direct proof.

Exercise 14.2. Let X be a paracompact space and let Y be a compact space. Show that the product $X \times Y$ is also paracompact.

Exercise 14.3. Let $S := (\mathbb{R}, \mathcal{T})$ be the Sorgenfrey line that we introduced in Exercise 4.4. Show that the Sorgenfrey line is Hausdorff and paracompact.

Remark. We consider the product $S \times S$, which is often called the *Sorgenfrey plane*. Since S is Hausdorff we know by the Product Topology-Properties Proposition 7.9 that the Sorgenfrey plane $S \times S$ is also Hausdorff. In Exercise 12.7 we saw that the Sorgenfrey plane $S \times S$ is not normal. It follows from these two observations, together with Dieudonné's

Theorem 14.9, that the Sorgenfrey plane $S \times S$ is not paracompact. This shows that the property of being paracompact is not preserved under taking products.

Exercise 14.4. Give an example of a paracompact space X together with an open cover $\{U_i\}_{i\in I}$ such that there is no subset $J \subseteq I$ such that $\{U_i\}_{i\in J}$ is a locally finite open cover of X.

Exercise 14.5.

(a) Show that every discrete space is paracompact.

(b) Provide an example of a continuous map $f\colon X \to Y$ such that X is paracompact, but such that the image $f(X)$ is not paracompact.

Remark. This exercise shows that the Compact Image Lemma 4.15 cannot be generalized from compact spaces to paracompact spaces.

CHAPTER 15

The covering dimension of topological spaces

In this chapter we introduce the covering dimension of a topological space. This topic is not essential for the main flow of the lecture notes, but it is interesting and amusing in its own right. The most difficult result of this chapter is the Topological Space–Euclidean Embedding Theorem 15.16 which shows that, under reasonably mild extra hypotheses, a topological space of covering dimension m admits a closed embedding into $\mathbb{R}^{2m+1}$. Later in Chapter 33 we will use some of the results of this chapter to show that every (possibly non-compact) "topological manifold" admits a closed embedding into some $\mathbb{R}^n$ and every "smooth manifold" admits a "smooth embedding" into some $\mathbb{R}^n$. The exposition of this chapter is based on [**Mun1975**, Chapter 50] and [**Flo2020**].

This chapter is organized as follows:

- In Section 15.1 we will define the covering dimension of a topological space.
- In Section 15.2 we will study basic properties of the covering dimension.
- In Section 15.3 we will state and prove the somewhat technical Covering Dimension Refinement Theorem 15.15.
- In Section 15.4 we will state and prove the Topological Space–Euclidean Embedding Theorem 15.16 which says that topological space X (with suitable adjective) of covering dimension d admits a closed embedding into $\mathbb{R}^{2d+1}$.
- In Section 15.5 we will provide exercises for Chapter 15.

In the next Chapter 16 we will introduce the compact-open topology on the set of continuous maps between two given topological spaces.

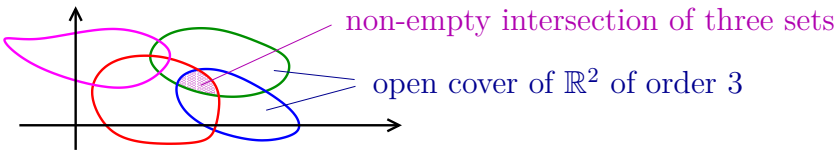
15.1. **The definition of the covering dimension of a topological space.** The key to introducing the covering dimension of a topological space is the following definition:

Definition 15.1. Let X be a set and let $m \in \mathbb{N}_0$. A family $\{U_i\}_{i\in I}$ of subsets of X has order m if the following two statements hold:

(1) There exists $J \subseteq I$ with $\#J = m$ such that $\bigcap_{j\in J} U_j \neq \varnothing$.

(2) For every $J \subseteq I$ with $\#J > m$ we have $\bigcap_{j\in J} U_j = \varnothing$.

If no such m exists, then we say that the family has infinite order.



Definition 15.2. Let X be a topological space. If X is non-empty, then we refer to

$$\operatorname{cdim}(X) \;:=\; \min\left\{ m \in \mathbb{N}_0 \cup \{\infty\} \;\middle|\; \begin{array}{c}\text{every open cover of } X \text{ admits an} \\ \text{open refinement of order} \leq m+1\end{array} \right\}$$

as the[184] covering dimension of X. If X is empty, then we set $\operatorname{cdim}(X) := -1$.



Remark. The above Definition 15.2 of the covering dimension of a topological space is equivalent to the definition given in [**Mun1975**, Chapter 50].[185] In many other books, e.g. [**Enge1995**, p. 42], [**Pear1975**, p. 111] and [**Chara2019**, p. 10], the covering dimension is defined by considering finite covers only. For many topological spaces these two notions agree, for example in [**Enge1995**, Proposition 3.2.2] it is shown that the two notions agree for paracompact spaces. □

The following trivial lemma, which is an immediate consequence of the definitions, says that the covering dimension is a homeomorphism invariant.

Lemma 15.3. (Covering Dimension–Homeomorphism Lemma) Let X and Y be two topological spaces. If X and Y are homeomorphic, then $\operatorname{cdim}(X) = \operatorname{cdim}(Y)$.

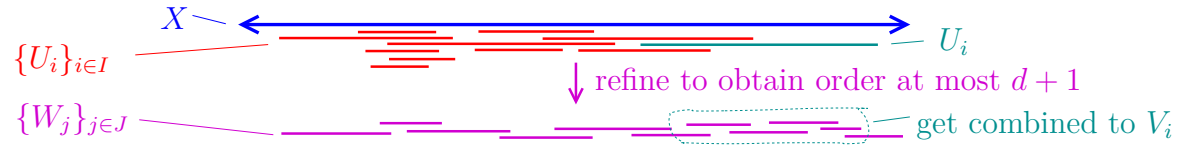
The next lemma simplifies life and the notation in applications significantly.

Lemma 15.4. (Covering Dimension–Refinement Lemma) Let X be a topological space with covering dimension $d := \operatorname{cdim}(X) \in \mathbb{N}_0$. Given any open cover $\{U_i\}_{i\in I}$ there exists an open cover $\{V_i\}_{i\in I}$ of order $\leq d+1$ such that for each $i \in I$ we have $V_i \subseteq U_i$.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. Since $d := \operatorname{cdim}(X) \in \mathbb{N}_0$ we know that there exists an open refinement $\{W_j\}_{j\in J}$ of order $\leq d+1$. Since $\{W_j\}_{j\in J}$ is a refinement for the original cover we see that for each $j \in J$ we can pick an $i(j) \in I$ such that $W_j \subseteq U_{i(j)}$. Given $i \in I$ we set

$$V_i \;=\; \bigcup_{j\,\in\, J \text{ with } i(j)\,=\,i} W_j \quad\subseteq\; U_i.$$

Evidently the family $\{V_i\}_{i\in I}$ is an open cover of X that is a refinement of the original cover $\{U_i\}_{i\in I}$. It remains to show that the cover $\{V_i\}_{i\in I}$ has order $\leq d+1$.



[184] In the mathematical literature the words "cover" and "covering" often get used indiscriminately. In these notes we usually reserve the word "cover" for the concept introduced in Definition 3.19 and we reserve the word "covering" for the concept that we will introduce in Definition 91.8. Thus it would make more sense to call the present concept "cover dimension". But since this term is not used at all in the literature we stick to the slightly illogical name "covering dimension".

[185] The definition is not entirely the same as the definition in [**Mun1975**, Chapter 50] since we consider families of subsets (where we allow repetitions), whereas [**Mun1975**, Chapter 50] works with subsets of the power set of X, in other words, families of subsets without repetition. Nonetheless, elementary logic shows that the definition of covering dimension that we give results in the same covering dimension as in [**Mun1975**, Chapter 50].

So let $x \in X$. Since $\{W_j\}_{j\in J}$ has order $\leq d+1$ we see that $\#\{j \in J \,|\, x \in W_j\} \leq d+1$. But then it follows that $\#\{i \in I \,|\, x \in V_i\} \leq d+1$. ■

The following theorem hopefully justifies somewhat the term covering *dimension*.

Theorem 15.5. (Compact Euclidean Subset-Covering Dimension Theorem) Let $n \in \mathbb{N}_0$. For every compact subspace $X \subseteq \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$.

Auxiliary Lemma 15.6. Let $n \in \mathbb{N}_0$. There exists an open cover $\{V_j\}_{j\in J}$ of $\mathbb{R}^n$ of order $n+1$ such that for each $j \in J$ we have $\operatorname{diam}(V_j) \leq \sqrt{n}$.

Remark. The proof we give of Lemma 15.6 is rather ad hoc. We will give an alternative proof using simplicial complexes in Exercise 175.8. □

Proof of Lemma 15.6. Let $x = (x_1, \ldots, x_n) \in (\frac{1}{2}\cdot\mathbb{Z})^n$.

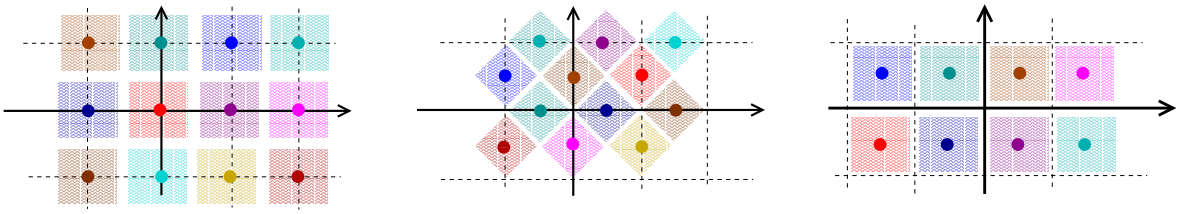
- We set $R_x := \{i \in \{1,\ldots,n\} \,|\, x_i \in \mathbb{Z} + \frac{1}{2}\}$ (the "rational coordinates" of x)
- We set

$$V(x) := x + \left\{ \sum_{i=1}^{n} \mathfrak{e}_i \cdot \lambda_i \;\middle|\; \begin{array}{l} \text{for } i \in R_x \text{ we have } \lambda_i \in (-\frac{1}{2}, \frac{1}{2}), \\ \text{for } i \notin R_x \text{ we have } |\lambda_i| < \max\{\frac{1}{2} - |\lambda_j| \,|\, j \in R_x\} \end{array} \right\}$$

Given $d \in \{0,\ldots,n\}$ we consider the following families of open subsets of $\mathbb{R}^n$:

$$\mathcal{V}_d := \{V(x)\}_{x\in X_d} \qquad \text{where} \qquad X_d := \{x \in (\tfrac{1}{2}\cdot\mathbb{Z})^n \,|\, \#R_x = d\}.$$

We claim that $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_d$ has the desired properties.



open sets in $\mathcal{V}_0$ open sets in $\mathcal{V}_1$ open sets in $\mathcal{V}_2$

It remains to prove the following claim:

Claim.

(1) Each $V(x)$ is an open subset of $\mathbb{R}^n$ with $\operatorname{diam}(V(x)) \leq \sqrt{n}$.
(2) $\mathcal{V}$ is a cover of $\mathbb{R}^n$.
(3) For every $d \in \{0,\ldots,n\}$ the sets in $\mathcal{V}_d$ are disjoint.

Proof.

(1) Let $x = (x_1,\ldots,x_n) \in (\frac{1}{2}\cdot\mathbb{Z})^n$. It is rather elementary to verify that $V(x)$ is open. We leave it to the reader to fill in the details. Furthermore note that for each point $y = (y_1,\ldots,y_n) \in V(x)$ and each $i \in \{1,\ldots,n\}$ we have $y_i \in (x_i - \frac{1}{2}, x_i + \frac{1}{2})$. This implies that $\operatorname{diam}(V(x)) \leq \sqrt{n}$.

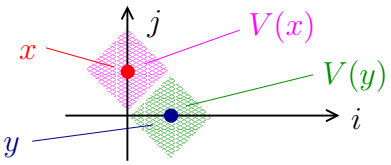
(2) Let $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$. For $i = 1,\ldots,n$ we set

$$x_i := \begin{cases} y_i, & \text{if } y_i \in \mathbb{Z}, \\ \lfloor y_i \rfloor + \frac{1}{2}, & \text{if } y_i \notin \mathbb{Z} \end{cases}$$

and we set $x = (x_1,\ldots,x_n)$ Note that for $i \in \{1,\ldots,n\}$ with $y_i \notin \mathbb{Z}$ we have $i \in R_x$ and $y_i \in (x_i - \frac{1}{2}, x_i + \frac{1}{2})$. It is now clear that $y \in V(x)$.

(3) Let $d \in \{0, \dots, n\}$ and let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X_d$ with $x \neq y$.

- If there exists an $i \in \{1, \dots, n\}$ with $|x_i - y_i| \geq 1$, then it follows from considering the i-th coordinate that $V(x) \cap V(y) = \varnothing$.
- Suppose that there exists no $i \in \{1, \dots, n\}$ with $|x_i - y_i| \geq 1$. Since $x \neq y$ we see that $R_x \neq R_y$. Since $\#R_x = \#R_y$ we see that there exists an $i \in \{1, \dots, n\}$ with $i \in R_x$ and $i \notin R_y$ and there exists a $j \in \{1, \dots, n\}$ with $j \notin R_x$ and $j \in R_y$.



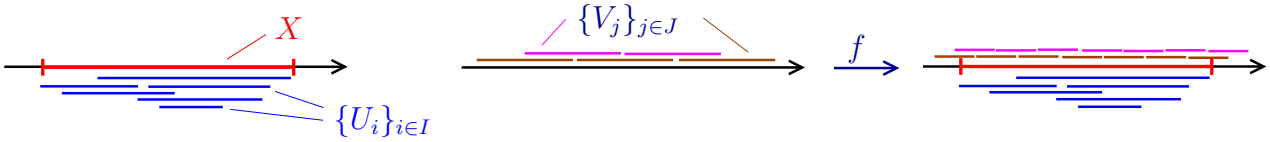
It is now elementary to see that the points in $V(x)$ and $V(y)$ differ in the i-th or the j-th coordinate. ■

Proof of Theorem 15.5. Let $X \subseteq \mathbb{R}^n$ be a compact subspace and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We equip X with the usual Euclidean metric coming from $\mathbb{R}^n$.[186] Since X is compact it follows from the Lebesgue Lemma 10.8 that there exists a $\delta > 0$ such that for every subset A of X with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subseteq U_i$.

By Lemma 15.6 there exists an open cover $\mathcal{V} = \{V_j\}_{j \in J}$ of order $\leq n + 1$ such that for each $j \in J$ we have $\operatorname{diam}(V_j) \leq \sqrt{n}$. We consider the homeomorphism

$$\begin{array}{rccc} f \colon & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & \frac{\delta}{2\sqrt{n}} \cdot x. \end{array}$$

It is clear that $\{f(V_j) \cap X\}_{j \in J}$ is an open cover of X of order $\leq n + 1$ such that for each $j \in J$ we have $\operatorname{diam}(f(V_j) \cap X) \leq \delta$.



But then it follows from the choice of δ that this is an open cover subordinate to the original open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X. ■

15.2. **Basic properties of the covering dimension.** The main goal of this section is to prove the Exhaustion-Covering Dimension Proposition 15.11 which allows us, in many cases, to obtain an upper bound on the covering dimension of a topological space in terms of covering dimensions of suitable subspaces. We start out with the following fairly straightforward lemma.

Lemma 15.7. (Closed Subspace Covering Dimension Lemma) Let X be a topological space and let $A \subseteq X$ be a subspace. If A is a closed subset, then $\operatorname{cdim}(A) \leq \operatorname{cdim}(X)$.

Proof. As we will see, the proof of the lemma is very similar to the proof of the Compact-Closed Lemma 3.39 (1) and of the Closed Subset-Paracompact Lemma 14.6. Indeed, let X be a topological space of covering dimension $d = \operatorname{cdim}(X)$ and let $A \subseteq X$ be a closed subset. If $d = \operatorname{cdim}(X) = \infty$, then the statement of the lemma is trivial. Thus we assume that $d = \operatorname{cdim}(X) < \infty$. We need to show that $\operatorname{cdim}(A) \leq d$. Let $\{U_i\}_{i \in I}$ be an open cover

[186] By Exercise 3.3 we know that this metric gives the topology of X.

of A. By definition of the subspace topology there exists for each $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$.



Since A is a closed subset of X see that $\{V_i\}_{i\in I}$ together with $X \setminus A$ is an open cover of X. Since $\operatorname{cdim}(X) = d$ there exists an open refinement $\{W_j\}_{j\in J}$ of this cover of X of order $\leq d+1$. Evidently $\{W_j \cap A\}_{j\in J}$ is an open refinement of our original cover of A of order $\leq d+1$. ■

Proposition 15.8. (Finite Union-Covering Dimension Proposition) Let X be a topological space and let $A_1, A_2, \ldots, A_k$ be finitely many closed subsets of X. Then

$$\operatorname{cdim}(A_1 \cup \cdots \cup A_k) \;=\; \max\{\operatorname{cdim}(A_1), \ldots, \operatorname{cdim}(A_k)\} \;\in\; \mathbb{N}_0 \cup \{\infty\}.$$

Example. Let $n \in \mathbb{N}_0$. We can write S^n as the union of the upper and lower hemispheres $S^n_{\geq 0}$ and $S^n_{\leq 0}$. We obtain that

$$\operatorname{cdim}(S^n) \underset{\uparrow}{=} \max\{\operatorname{cdim}(S^n_{\geq 0}), \operatorname{cdim}(S^n_{\leq 0})\} \overset{\downarrow}{=} \operatorname{cdim}(\overline{B}^n) \underset{\uparrow}{\leq} n.$$

by the Finite Union-Covering Dimension Proposition 15.8 since $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are closed subsets of S^n

by the Covering Dimension–Refinement Lemma 15.4 since on page 183 we saw that both hemispheres are homeomorphic to $\overline{B}^n$

by the Compact Euclidean Subset-Covering Dimension Theorem 15.5

□

The proof of the Finite Union-Covering Dimension Proposition 15.8 requires some preparations.

Definition 15.9. Let X be a topological space and let $A \subseteq X$ be a subset. We say that a family $\{U_i\}_{i\in I}$ of subsets of X has order $m \in \mathbb{N}_0 \cup \{\infty\}$ in A if the following two statements hold:[187]

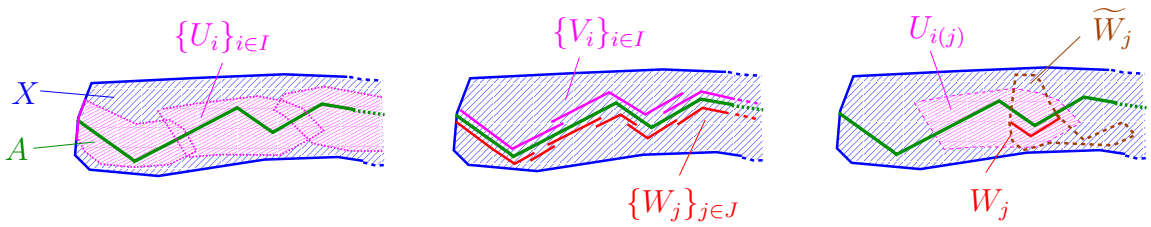
(1) There exists a $J \subseteq I$ with $\#J = m$ such that $A \cap \bigcap_{j\in J} U_j \neq \varnothing$.

(2) For each $J \subseteq I$ with $\#J > m$ we have $A \cap \bigcap_{j\in J} U_j = \varnothing$.

Auxiliary Lemma 15.10. (Cover–Order Reduction Lemma) Let X be a topological space, let $d \in \mathbb{N}_0$ and let $A \subseteq X$ be a closed subset with $\operatorname{cdim}(A) \leq d$. If $\{U_i\}_{i\in I}$ is an open cover of X, then there exists an open refinement of $\{U_i\}_{i\in I}$ that has order at most $d+1$ in A.

Proof of Lemma 15.10. For each $i \in I$ we set $V_i = U_i \cap A$. Since $\{V_i\}_{i\in I}$ is an open cover of A and since $\operatorname{cdim}(A) \leq d$ there exists an open refinement $\{W_j\}_{j\in J}$ of $\{V_i\}_{i\in I}$ of order $d+1$. For each $j \in J$ we can and will pick an $i(j) \in I$ with $W_j \subseteq V_{i(j)}$.
By definition of the subspace topology we can pick for each $j \in J$ an open subset $\widetilde{W}_j \subseteq X$ with $W_j = \widetilde{W}_j \cap A$. Since A is a closed subset we see that $\{\widetilde{W}_j \cap U_{i(j)}\}_{j\in J}$ together with

[187] In other words, the family $\{U_i\}_{i\in I}$ of subsets of X has order m in A if and only if the family $\{U_i \cap A\}_{i\in I}$ of subsets of A has order m.



the sets $\{U_i \setminus A\}_{i\in I}$ is an open cover of X that is a refinement of the original cover $\{U_i\}_{i\in I}$ and that has order at most $d+1$ in A. ■

Proof of the Finite Union-Covering Dimension Proposition 15.8. An elementary induction argument shows that it suffices to prove the proposition for $k = 2$. Furthermore we can assume that the two subspaces cover the total topological space. In other words, we can assume that we are in the following setting. We are given a topological space X and two closed subsets $A, B \subseteq X$ such that $X = A \cup B$. We set

$$d \;:=\; \max\{\operatorname{cdim}(A), \operatorname{cdim}(B)\} \quad \in\; \mathbb{N}_0 \cup \{\infty\}.$$

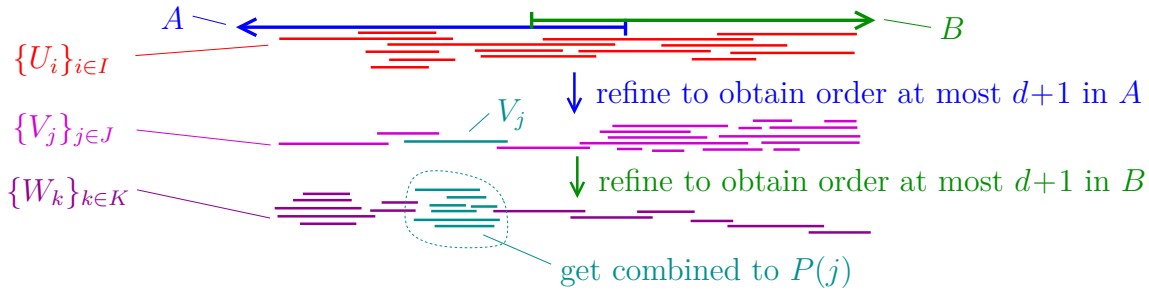
We need to show that $\operatorname{cdim}(X) = d$. Since A and B are closed subsets we obtain from the Closed Subspace Covering Dimension Lemma 15.7 that $\operatorname{cdim}(X) \geq d$. Thus it remains to show that $\operatorname{cdim}(X) \leq d$.

If $d = \infty$, then the statement is trivial. Thus we assume that $d \in \mathbb{N}_0$. Let $\{U_i\}_{i\in I}$ be an open cover of X. By the Cover–Order Reduction Lemma 15.10 there exists an open refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ that has order at most $d+1$ in A and there exists an open refinement $\{W_k\}_{k\in K}$ of $\{V_j\}_{j\in J}$ that has order at most $d+1$ in B.

For each $k \in K$ we pick $j(k) \in J$ with $W_k \subseteq V_{j(k)}$. Given $j \in J$ we set[188]

$$P(j) \;:=\; \bigcup_{k\,\in\, K \text{ with } j(k)\,=\,j} W_k \quad \subseteq\; V_j.$$

The following claim says that $\{P(j)\}_{j\in J}$ has all the desired properties.



Claim.
(1) $\{P(j)\}_{j\in J}$ is an open cover of X.
(2) $\{P(j)\}_{j\in J}$ is a refinement of the original cover $\{U_i\}_{i\in I}$ of X.
(3) The order of the open cover $\{P(j)\}_{j\in J}$ is at most $d+1$.

Proof.

[188]Here we use the same approach as in the proof of the Shrinking Lemma 14.16.

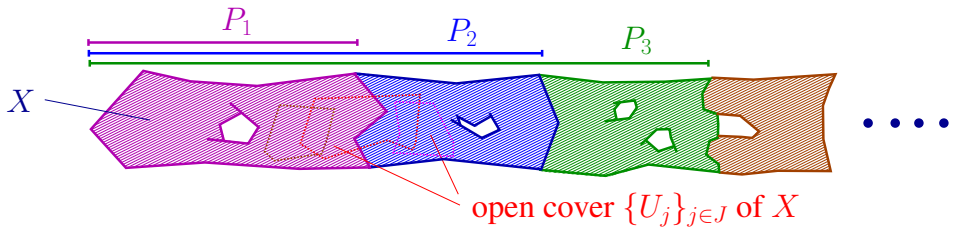
(1) It is clear that each $P(j)$ is open. Furthermore, given $x \in X$ there exists a $k \in K$ with $x \in W_k$ but this implies that $x \in P(j(k))$.
(2) For every $j \in J$ we have $P(j) \subseteq V_j$. Since $\{V_j\}_{j\in J}$ was a refinement of $\{U_i\}_{i\in I}$ we see that $\{P(j)\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$.
(3) Suppose we have distinct $j_1, \ldots, j_e \in J$ such that $P(j_1) \cap \cdots \cap P(j_e)$ contains a point x. We need to show that $e \leq d+1$. For $l = 1, \ldots, e$ we can and will pick a $k_l \in K$ with $j(k_l) = j_l$ and with $x \in W_{k_l}$. Note that $k_1, \ldots, k_l$ are also distinct. We see that

$$x \in W_{k_1} \cap \cdots \cap W_{k_e} \subseteq V_{j_1} \cap \cdots \cap V_{j_e}.$$

Since $X = A \cup B$ we know that $x \in A$ or $x \in B$.
- If $x \in A$, then $e \leq d+1$ since $\{V_j\}_{j\in J}$ has order at most $d+1$ in A.
- If $x \in B$, then $e \leq d+1$ since $\{W_k\}_{k\in K}$ has order at most $d+1$ in B. ■

The following proposition can be viewed as a generalization of the Finite Union-Covering Dimension Proposition 15.8.

Proposition 15.11. (Exhaustion-Covering Dimension Proposition) Let X be a topological space, let $d \in \mathbb{N}_0$ and let $\{P_i\}_{i\in\mathbb{N}}$ be a sequence of closed subsets of X such that the following three conditions are satisfied:[189]
(1) We have $X = \bigcup_{i\in\mathbb{N}} P_i$.
(2) For each $i \in \mathbb{N}$ we have $P_i \subseteq \mathring{P}_{i+1}$.
(3) For each $i \in \mathbb{N}$ we have $\operatorname{cdim}(P_{i+1} \setminus \mathring{P}_i) \leq d$.
Then $\operatorname{cdim}(X) \leq d$.



Remark. In many settings the combination of the Exhaustion-Covering Dimension Proposition 15.11, the Closed Subspace Covering Dimension Lemma 15.7 and the Finite Union-Covering Dimension Proposition 15.8 can be used to give an upper bound on the covering dimension of a non-compact space. For example in the Euclidean Subset–Covering Dimension Proposition 15.14 we will use this trinity to show that the covering dimension of any subspace of $\mathbb{R}^n$ is $\leq n$. Furthermore in the Covering Dimension-of-Manifolds Proposition 33.9 we will use these three results to show that the covering dimension of any n-dimensional "topological manifold" is $\leq n$. □

[189]The first two conditions just say that $\{P_i\}_{i\in\mathbb{N}}$ is an exhaustion of X in the sense of Definition 13.5.

Proof. [190] Let X be a topological space, let $d \in \mathbb{N}_0$ and let $\{P_i\}_{i\in\mathbb{N}}$ be a sequence of closed subsets of X such that the following two conditions are satisfied:

(1) We have $X = \bigcup_{i\in\mathbb{N}} P_i$.

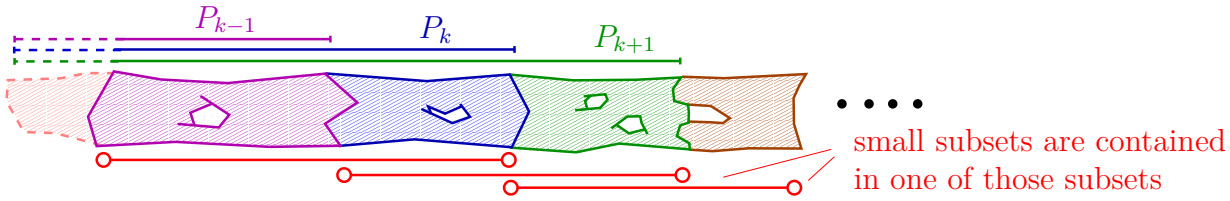
(2) For each $i \in \mathbb{N}$ we have $P_i \subseteq \mathring{P}_{i+1}$.

To be on the safe side we set $P_0 := \varnothing$. In the following we say that a subset $W \subseteq X$ is *small* if there exists a $j \in \mathbb{N}$ with $W \subseteq \mathring{P}_{j+1} \setminus P_{j-1}$.

Claim 1. Let $W \subseteq X$ be a small subset and let $k \in \mathbb{N}_0$.

(1) $W \cap P_k \neq \varnothing \implies W \subseteq \mathring{P}_{k+1}$

(2) $W \cap (X \setminus \mathring{P}_k) \neq \varnothing \implies W \cap \mathring{P}_{k-1} = \varnothing$.



Proof. Note that Statement (2) is just the contrapositive of Statement (1). Thus it suffices to prove Statement (1). We assume that $W \cap P_k \neq \varnothing$. Since W is small there exists a $j \in \mathbb{N}$ with $W \subseteq \mathring{P}_{j+1} \setminus P_{j-1}$. Note that $W \cap P_{j-1} = \varnothing$. Since $W \cap P_k \neq \varnothing$ we see that $k \geq j$. But this implies that $W \subseteq \mathring{P}_{j+1} \subseteq \mathring{P}_{k+1}$. ⊞

Now we turn to the actual proof of the Exhaustion-Covering Dimension Proposition 15.11. Thus we make the following extra assumption:

(3) For each $i \in \mathbb{N}$ we have $\operatorname{cdim}(P_{i+1} \setminus \mathring{P}_i) \leq d$.

Recall that we need to show that $\operatorname{cdim}(X) \leq d$. Thus let $\mathcal{U} = \{U_j\}_{j\in J}$ be an open cover of X. It follows easily from (1) and (2) that the sets $\{\mathring{P}_{i+1} \setminus P_{i-1}\}_{i\in\mathbb{N}}$ form an open cover of X. Thus, after possibly replacing $\mathcal{U}$ by the open cover $\{U_j \cap (\mathring{P}_{i+1} \setminus P_{i-1})\}_{j\in J, i\in\mathbb{N}}$ we can assume that each set in the given open cover $\mathcal{U}$ is small in the above sense.

Claim 2. There exists a sequence $\mathcal{V}_i$, $i \in \mathbb{N}_0$ of families of subsets of X with the following properties:

(0) $\mathcal{V}_0 = \mathcal{U}$.
(1) Each $\mathcal{V}_i$ is an open cover of X.
(2) Each $\mathcal{V}_i$ with $i \geq 1$ is a refinement of $\mathcal{V}_{i-1}$.
(3) If $i \geq 1$ and $A \in \mathcal{V}_{i-1}$ is a set that is contained in $\mathring{P}_{i-2}$, then $A \in \mathcal{V}_i$.
(4) Each set in $\mathcal{V}_i$ is small in the above sense.
(5) Each $\mathcal{V}_i$ has order at most $d+1$ in P_i.

Proof. We set $\mathcal{V}_0 = \mathcal{U}$. By the discussion preceding the claim we know that (4) is satisfied. Otherwise there is nothing to show.

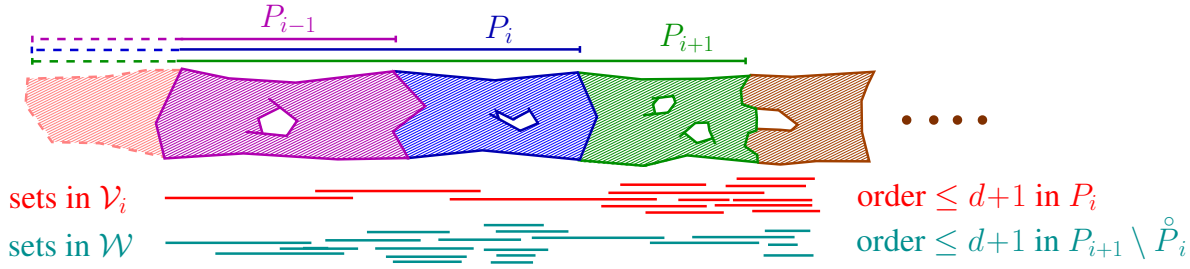
[190] As we mentioned above, the Exhaustion-Covering Dimension Proposition 15.11 can be viewed as a generalization of the Finite Union-Covering Dimension Proposition 15.8. In fact the proof of the Exhaustion-Covering Dimension Proposition 15.11 can be viewed as a refinement of the ideas of the proof of the Finite Union-Covering Dimension Proposition 15.8.

Now we suppose that we have already constructed $\mathcal{V}_0, \ldots, \mathcal{V}_i$ with the desired properties. By hypothesis P_{i+1} is a closed subset of X. It follows that $P_{i+1} \setminus \mathring{P}_i$ is also a closed subset of X. By hypothesis we know that $\operatorname{cdim}(P_{i+1} \setminus \mathring{P}_i) \leq d$. Thus it follows from the Cover–Order Reduction Lemma 15.10 that there exists an open refinement $\mathcal{W}$ of $\mathcal{V}_i$ that has order at most $d+1$ in $P_{i+1} \setminus \mathring{P}_i$. Since $\mathcal{W}$ is a refinement of $\mathcal{V}_i$ we can and will pick for each $W \in \mathcal{W}$ a set $\Psi(W) \in \mathcal{V}_i$ with $W \subseteq \Psi(W)$. For $A \in \mathcal{V}_i$ we set

$$\Phi(A) \;:=\; (A \cap \mathring{P}_i) \cup \bigcup_{\substack{W \in \mathcal{W} \text{ with } \Psi(W) = A \\ \text{such that } W \cap P_i \neq \varnothing \\ \text{and such that } W \cap \mathring{P}_{i-1} = \varnothing}} W \quad \subseteq A.$$

Note that for any $W \in \mathcal{W}$ with $W \cap P_i \neq \varnothing$ and with $W \cap \mathring{P}_{i-1} = \varnothing$ we have by definition $W \subseteq \Phi(\Psi(W))$. We define $\mathcal{V}_{i+1} \subseteq \mathcal{P}(X)$ to be the following two types of subsets of X:

(α) all $\Phi(A)$ for $A \in \mathcal{V}_i$,
(β) all $W \in \mathcal{W}$ such that $W \cap P_i = \varnothing$.



We need to verify that $\mathcal{V}_{i+1}$ has the desired properties:

(1) We want to show that $\mathcal{V}_{i+1}$ is an open cover of X. Evidently each set in $\mathcal{V}_{i+1}$ is open. Now let $x \in X$. We need to show that there exists a $U \in \mathcal{V}_{i+1}$ with $x \in U$.
   - First we suppose that $x \in \mathring{P}_i$. Since $\mathcal{V}_i$ is an open cover of X there exists an $A \in \mathcal{V}_i$ with $x \in A$. Since $x \in \mathring{P}_i$ we have $x \in A \cap \mathring{P}_i \subseteq \Phi(A)$. Thus we have shown that x is contained in a set of type (α).
   - Now we suppose that $x \notin \mathring{P}_i$. Since $\mathcal{W}$ is an open cover of X we know that there exists a $W \in \mathcal{W}$ with $x \in W$. If $W \cap P_i = \varnothing$, then W is a set of type (β). Thus it remains to consider the case that $W \cap P_i \neq \varnothing$. Since $x \in W$ and $x \notin \mathring{P}_i$ we have $W \cap (X \setminus \mathring{P}_i) \neq \varnothing$, thus we obtain from Claim 1 (2), that $W \cap \mathring{P}_{i-1} = \varnothing$. This implies that $x \in W \subseteq \Phi(\Psi(W))$. We have thus shown that x is contained in a set of type (α).

(2) It is clear from the construction that $\mathcal{V}_{i+1}$ is a refinement of $\mathcal{V}_i$.

(3) Let $A \in \mathcal{V}_i$ be a set that is contained in $\mathring{P}_{i-1}$. By definition $\Phi(A) = A$. Thus we see that $A = \Phi(A) \in \mathcal{V}_{i+1}$.

(4) Since each set $\mathcal{V}_i$ is small and since $\mathcal{V}_{i+1}$ is a refinement of $\mathcal{V}_i$ we see that each set in $\mathcal{V}_{i+1}$ is also small.

(5) We need to show that $\mathcal{V}_{i+1}$ has order at most $d+1$ in P_{i+1}. Let $U_1, \dots, U_k$ be distinct sets in $\mathcal{V}_{i+1}$ such that there exists an $x \in P_{i+1}$ with $x \in U_1 \cap \dots \cap U_k$. We need to show that $k \leq d+1$.
- First we suppose that $x \in P_i$. In this case $U_1, \dots, U_k$ are of type (α), i.e. there exist distinct $A_1, \dots, A_k \in \mathcal{V}_i$ such that each U_j is given by $U_j = \Phi(A_j)$. By construction we have $\Phi(A_j) \subseteq A_j$. Thus we see that x is contained in each A_j. Since $\mathcal{V}_i$ has order at most $d+1$ in P_i we see that $k \leq d+1$.
- We suppose that $x \in P_{i+1} \setminus P_i$. After possibly reordering we can assume that $U_1, \dots, U_l$ are of type (α) and $U_{l+1}, \dots, U_k$ are of type (β). By definition of the two types of sets this implies the following:
  (a) There exist distinct $A_1, \dots, A_l \in \mathcal{V}_i$ with $U_j = \Phi(A_j)$ for $j = 1, \dots, l$. Note that by definition of the sets of type (α) and the fact that $x \notin \mathring{P}_i$ there exist $W_1, \dots, W_l$ in $\mathcal{W}$ with $x \in W_j$ for $j = 1, \dots, l$ and such that $W_j \cap P_i \neq \varnothing$ for $j = 1, \dots, l$. Since $\Psi(W_j) = A_j$ for $j = 1, \dots, l$ are pairwise distinct we see that $W_1, \dots, W_l$ are pairwise distinct.
  (b) We have $U_{l+1}, \dots, U_k \in \mathcal{W}$ with $U_j \cap P_i = \varnothing$ for $j = l+1, \dots, k$.
  It follows that $W_1, \dots, W_l, U_{l+1}, \dots, U_k$ are pairwise distinct sets in $\mathcal{W}$ and each contains x. Since $\mathcal{W}$ has order at most $d+1$ in $P_{i+1} \setminus \mathring{P}_i$ we see that $k \leq d+1$. ⊞

The following claim completes the proof of the proposition.

Claim 3. We consider

$$\mathcal{V} \coloneqq \bigcup_{i \in \mathbb{N}} \{V \in \mathcal{V}_i \mid V \subseteq \mathring{P}_{i-1}\} \quad \subseteq \mathcal{P}(X).$$

We view $\mathcal{V}$ as a family of subsets. This family of subsets has the following properties:
(1) It is an open cover of X.
(2) It a refinement of $\mathcal{U}$.
(3) It has order at most $d+1$.

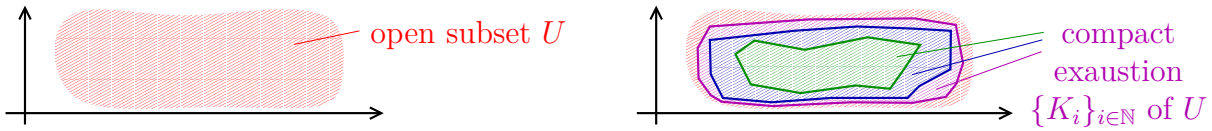
Proof.

(1) Let $x \in X$. Since $X = \bigcup_{i \in \mathbb{N}} P_i$ there exists an $i \in \mathbb{N}_{\geq 2}$ such that $x \in P_{i-2}$. Since $\mathcal{V}_i$ is an open cover of X there exists a $V \in \mathcal{V}_{i+2}$ with $x \in V$. By Claim 2 (4) we know that V is small. Therefore it follows from the fact that $V \cap P_{i-2} \neq \varnothing$ together with Claim 1 (1), that $V \subseteq \mathring{P}_{i-1}$, i.e. $V \in \mathcal{V}$. This shows that the sets in $\mathcal{V}$ form an open cover of X.
(2) By Claim 2 (2) each $\mathcal{V}_i$ is a refinement of $\mathcal{U}$, it follows immediately that $\mathcal{V}$ is a refinement of $\mathcal{U}$.
(3) Let $U_1, \dots, U_k \in \mathcal{V}$ be distinct sets in $\mathcal{V}$ with $U_1 \cap \dots \cap U_k \neq \varnothing$. We need to show that $k \leq d+1$.
We pick $x \in U_1 \cap \dots \cap U_k$ and we pick an $i_0 \in \mathbb{N}$ with $x \in P_{i_0}$. Next note that by definition of $\mathcal{V}$ there exists for each $j \in \{1, \dots, k\}$ an $i_j \in \mathbb{N}$ with $U_j \in \mathcal{V}_{i_j}$ and with $U_j \subseteq \mathring{P}_{i_j - 1}$. We set $i \coloneqq \max\{i_0, i_1, \dots, i_k\}$. It follows from Claim 2 (3), applied iteratively, that for each j we have $U_j \in \mathcal{V}_i$. By Claim 2 (5) we know that $\mathcal{V}_i$ has order at most $d+1$ in P_i. Since $x \in P_{i_0} \subseteq P_i$ we see that indeed $k \leq d+1$. ■

Corollary 15.12. (Open Euclidean Subset–Covering Dimension Corollary) For every open subset $U \subseteq \mathbb{R}^n$ we have $\operatorname{cdim}(U) \leq n$.

Proof. Let $U \subseteq \mathbb{R}^n$ be an open subset. By the Second Countability Lemma 13.2 we know that U is second-countable and by the discussion on page 208 we know that U is regionally compact. Evidently U is Hausdorff. It follows from the Compact Exhaustion Lemma 13.6 (1) that there exists a sequence of compact subsets $\{K_i\}_{i\in\mathbb{N}}$ of U such that the following two conditions are satisfied:

(1) for every $i \in \mathbb{N}$ we have $K_i \subseteq \overset{\circ}{K}_{i+1}$,
(2) we have $U = \bigcup_{i\in\mathbb{N}} K_i$.



Since $\mathbb{R}^n$ is Hausdorff we know by the Compact-Closed Lemma 3.39 that K_i is closed. Also note that each $K_{i+1} \setminus \overset{\circ}{K}_i$ is a closed subset of the compact subset K_{i+1}. In particular we obtain that $K_{i+1} \setminus \overset{\circ}{K}_i$ is compact by the Compact-Closed Lemma 3.39. By the Compact Euclidean Subset-Covering Dimension Theorem 15.5 we know that for each $i \in \mathbb{N}$ we have $\operatorname{cdim}(K_{i+1} \setminus \overset{\circ}{K}_i) \leq n$. It follows from the Exhaustion-Covering Dimension Proposition 15.11 that $\operatorname{cdim}(U) \leq n$. ■

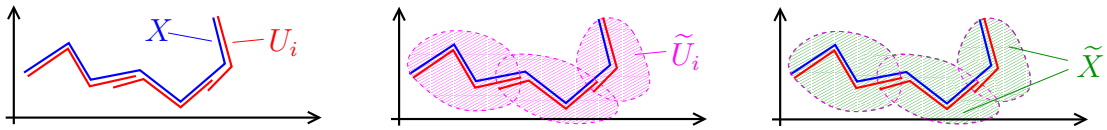
Question 15.13. Is the inequality from the Open Euclidean Subset–Covering Dimension Corollary 15.12 always an equality? In other words, is it true that for every open subset $U \subseteq \mathbb{R}^n$ we have $\operatorname{cdim}(U) = n$?

We will answer the above question in Section 33.3. We conclude this discussion of the Finite Union-Covering Dimension Proposition 15.8 and its consequences with the following proposition. The statement of the proposition is perhaps not a surprise, but the proof is exceedingly neat.

Proposition 15.14. (Euclidean Subset–Covering Dimension Proposition) For every subset $X \subseteq \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$.

Remark. Later, in the Covering Dimension–Interior Theorem 172.8 we will state a characterization of subsets X of $\mathbb{R}^n$ with $\operatorname{cdim}(X) = n$. □

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. By definition of the subspace topology we know that for each $i \in I$ there exists an open subset $\widetilde{U}_i$ of $\mathbb{R}^n$ with $U_i = X \cap \widetilde{U}_i$. We set $\widetilde{X} \coloneqq \bigcup_{i\in I} \widetilde{U}_i$.

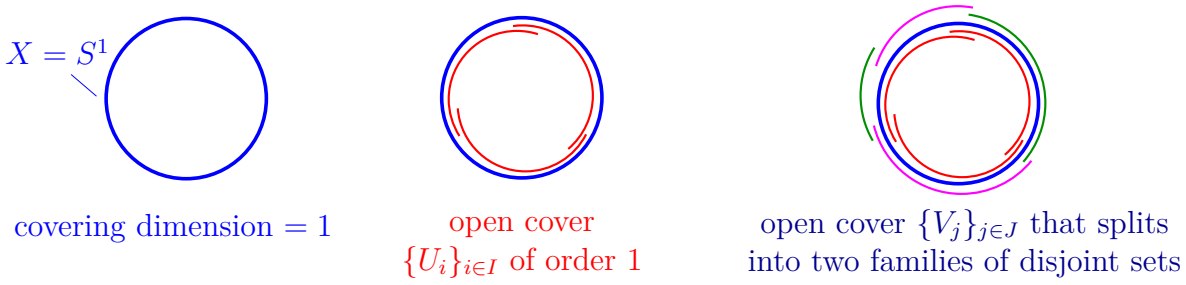


Evidently $\widetilde{X}$ is an open subset of $\mathbb{R}^n$ and $\{\widetilde{U}_i\}_{i\in I}$ is an open cover of $\widetilde{X}$. By the Open Euclidean Subset–Covering Dimension Corollary 15.12 we know that $\operatorname{cdim}(\widetilde{X}) \leq n$. This implies that there exists an open refinement $\{\widetilde{V}_j\}_{j\in J}$ of $\{\widetilde{U}_i\}_{i\in I}$ of order $\leq n+1$. But then $\{X \cap \widetilde{V}_j\}_{j\in J}$ is evidently an open refinement of $\{U_i\}_{i\in I}$ of order $\leq n+1$. ■

15.3. **The Covering Dimension Refinement Theorem.** The following theorem will play a crucial role in the proof of the Small Atlas Proposition 33.10 which says that every "smooth manifold" admits a "finite atlas". This result in turn is an essential ingredient in the proof of the Euclidean Embedding Theorem 33.2 that says that any "smooth manifold" admits a "smooth embedding" into some $\mathbb{R}^n$.

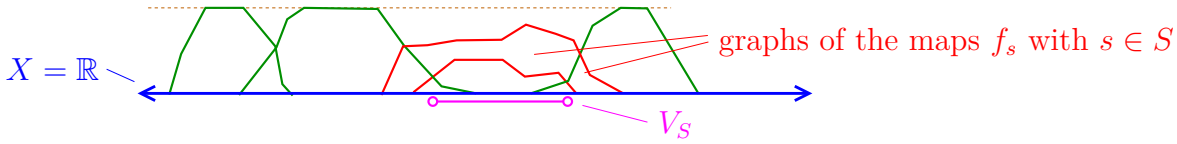
Theorem 15.15. (Covering Dimension Refinement Theorem) Let X be a Hausdorff space of covering dimension m and let $\{U_i\}_{i\in I}$ be an open cover of X.

(1) If X is paracompact, then there exists a locally finite refinement $\{V_j\}_{j\in J}$ of $\{U_i\}_{i\in I}$ and subsets $J_1, \ldots, J_{m+1}$ of J such that $J = J_1 \sqcup \cdots \sqcup J_{m+1}$ and such that for every $n \in \{1, \ldots, m+1\}$ and any $r, s \in J_n$ we have $V_r \cap V_s = \varnothing$.
(2) If X is second-countable, then we can arrange that J is countable.



Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. Since X has covering dimension m we might as well assume that the open cover $\{U_i\}_{i\in I}$ has order $m+1$. If X is second-countable we can assume, by the Countable Cover Lemma 13.4, that I is countable. Since X is Hausdorff and paracompact it follows from the Paracompact-Partition of Unity Theorem 14.15 that there exists a partition of unity $\{f_j \colon X \to [0,1]\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(f_i) \subseteq U_i$. Given a finite non-empty subset $S \subseteq I$ we set

$$V_S \;\coloneqq\; \{x \in X \mid f_s(x) > f_i(x) \text{ for all } s \in S \text{ and all } i \notin S\}.$$



Claim 1.

(a) Each V_S is open.
(b) Each $x \in X$ is contained in some V_S.
(c) If $i \in S$, then $V_S \subseteq U_i$. In particular if S is non-empty and finite, then V_S is contained in some U_i.
(d) Let S and T of I be two finite non-empty subsets of J. If $\#S = \#T$ and $S \neq T$, then $V_S \cap V_T = \varnothing$.

Proof. We basically proved this claim already in the proof of the Countable Cover Proposition 14.17. For the reader's convenience we recall the argument.

(a) Let $S \subseteq I$ be non-empty and finite. We need to show that V_S is open. By the JH-Openness Criterion 3.3 it suffices to show that given any $x \in V_S$ there exists an open

neighborhood U of $x \in X$ such that $V_S \cap U$ is open. Thus let $x \in V_S$. By definition of a partition of unity there exists a neighborhood U on which all but finitely many of the functions f_i vanish. We set $T \coloneqq \{i \in I \setminus S \,|\, f_i(x) \neq 0\}$. We see that

$$V_S \cap U \;=\; \bigcap_{t\in T}\bigcap_{s\in S} \underbrace{\{x \in X \,|\, f_s(x) > f_t(x)\}}_{=(f_s-f_t)^{-1}((0,\infty))}.$$

This is the intersection of finitely many open subsets, thus open.

(b) Let $x \in X$. By definition of a partition of unity the set $S = \{i \in I \,|\, f_i(x) > 0\}$ is finite and non-empty. Evidently we have $x \in V_S$.

(c) Let $S \subseteq I$ be a finite non-empty subset. Given any $i \in S$ we have $V_S \subseteq \operatorname{supp}(f_i) \subseteq U_i$.

(d) Let S and T of I be two different finite non-empty subsets of J with $\#S = \#T$. We can pick $s \in S \setminus T$ and we can pick $t \in T \setminus S$. If $x \in V_S$, then $f_s(x) > f_t(x)$ and if $x \in V_T$, then $f_t(x) > f_s(x)$. This implies that $V_S \cap V_T = \varnothing$. ⊞

Beyond this claim which is already familiar we also have the following new claim.

Claim 2. If $S \subseteq I$ is a finite subset with $\#S > m+1$, then $V_S = \varnothing$.

Proof. If $S \subseteq I$ is a finite subset with $\#S > m+1$, then

$$\begin{array}{ccccc} V_S & \subseteq & \bigcap_{j\in S} U_j & = & \varnothing. \\ & \uparrow & & \uparrow & \\ & \text{by Claim (1c)} & & \begin{array}{l}\text{since } \#S > m+1 \text{ and} \\ \text{since } \{U_i\}_{i\in I} \text{ has order } m+1\end{array} & \end{array}$$

⊞

Given $n \in \mathbb{N}$ we set

$$J_n \;\coloneqq\; \{\text{all subsets of } I \text{ with precisely } n \text{ elements}\}$$

and we set $J \coloneqq \bigsqcup_{n\in\mathbb{N}} J_n$. Now we can conclude the argument:

(1) It follows from Claim 1 (a)–(c) that $\{V_j\}_{j\in J}$ is an open refinement of the original cover $\{U_i\}_{i\in I}$. Furthermore it follows from Claim 2 that $J = J_1 \sqcup \cdots \sqcup J_{m+1}$. Finally it follows from Claim 1 (d) that for any $n \in \{1, \ldots, m+1\}$ and any $S, T \in J_n$ with $S \neq T$ the sets V_S and V_T are disjoint.

(2) Finally, if X is second-countable then we already arranged that I is countable. It follows from the Countability Lemma 1.37 (4) and (6) that J is also countable. ■

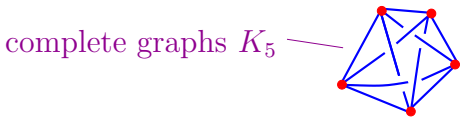
15.4. **The Euclidean Embedding Theorem for Topological Spaces.** We conclude this discussion of covering dimensions with the following Embedding Theorem.

Theorem 15.16. (Topological Space–Euclidean Embedding Theorem) Let X be a topological space such that the covering dimension $d \coloneqq \operatorname{cdim}(X)$ is finite. If X is second countable, Hausdorff and regionally compact, then there exists a closed embedding $X \to \mathbb{R}^{2d+1}$, in other words, X is homeomorphic to a closed subset of $\mathbb{R}^{2d+1}$.

Example. Let X be a compact topological graph. It is pretty clear that it is the union of finitely closed subsets each of which is homeomorphic to a compact interval. It thus follows from the Finite Union-Covering Dimension Proposition 15.8 and the Compact Euclidean

[190] It is perhaps worth recalling that the Urysohn Metrization Theorem 13.8 implies that a second-countable Hausdorff space is metrizable. This shows that (1) and (2) have a significant overlap. Nonetheless the two statements are independent.

Subset-Covering Dimension Theorem 15.5 that $\operatorname{cdim}(X) \leq 1$. It follows from the Topological Space–Euclidean Embedding Theorem 15.16, as in Exercise 11.10, that X admits an embedding into $\mathbb{R}^3$.



Now we consider the topological graph K_5 that is shown in the figure above (and which on page 3016 we will refer to as a "complete graph"). In the Non-Planar Graph Proposition 157.19 we will show that K_5 does not admit an embedding into $\mathbb{R}^2$. Thus we see that at least for $d = 1$ the dimension $2d + 1$ in the Topological Space–Euclidean Embedding Theorem 15.16 is optimal. □

Before we turn to the proof of the Topological Space–Euclidean Embedding Theorem 15.16 it is worth pointing out a little corollary.

Corollary 15.17. (Completely Metrizable Criterion Corollary) Let X be a topological space with $\operatorname{cdim}(X) < \infty$. If X is second countable, Hausdorff and regionally compact, then X is completely metrizable.[191] [192]

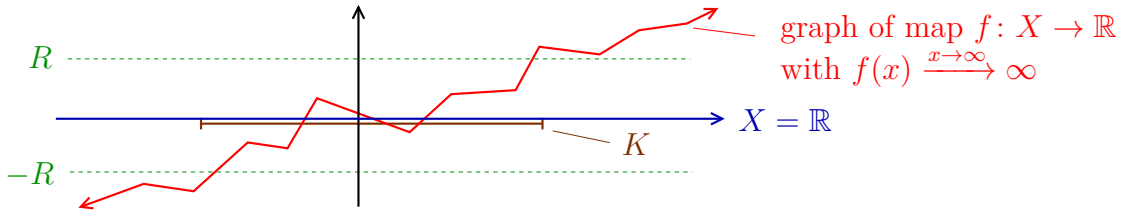
Proof. Let X be a topological space with $\operatorname{cdim}(X) < \infty$ that is second countable, Hausdorff and regionally compact. By the Topological Space–Euclidean Embedding Theorem 15.16 there exists an embedding $f\colon X \to \mathbb{R}^{2d+1}$ such that $f(X)$ is a closed subset. We equip $\mathbb{R}^{2d+1}$ with the usual Euclidean metric d. Since $f(X)$ is a closed subset we know by the Closed Subspace-Completeness Lemma 10.16 that $(f(X), d)$ is also complete and by Exercise 3.3 we know that the metric d on $f(X)$ gives the subspace topology.

Since $f\colon X \to f(X)$ is a homeomorphism we see that X admits a complete metric such that the metric topology agrees with the given topology. ■

The proof of the Euclidean Embedding Theorem 15.16 will require the remainder of this chapter. We start out with formulating a criterion for a continuous map $f\colon X \to \mathbb{R}^N$ to be a closed embedding.

Definition 15.18. Let X be a topological space and let $f \in C(X, \mathbb{R}^N)$. We write

$$f(x) \xrightarrow{x\to\infty} \infty \quad :\Longleftrightarrow \quad \begin{array}{c}\text{for every } R > 0 \text{ there exists a compact subset } K \subseteq X \\ \text{such that } \|f(x)\|_{\max} > R \text{ for all } x \in X \setminus K\end{array}$$



[191] Recall that in Definition 10.30 we said that a topological space X is *completely metrizable* if there exists a complete metric d on X such that the given topology agrees with the topology coming from d.

[192] Recall that by the Compact-Complete Proposition 10.17 we know that every *compact* metrizable space is already completely metrizable.

Lemma 15.19. (Closed Embedding Criterion) Let X be a topological space and let $f\colon X \to \mathbb{R}^N$ be an injective continuous map. If $f(x) \xrightarrow{x \to \infty} \infty$, then $f\colon X \to \mathbb{R}^N$ is a closed embedding.

Proof. Let $f\colon X \to \mathbb{R}^N$ be a continuous map which satisfies $f(x) \xrightarrow{x \to \infty} \infty$.

Claim. The map $f\colon X \to \mathbb{R}^N$ is proper.[193] [194]

Proof. Let $L \subseteq \mathbb{R}^N$ be a compact subset. By the Compact–Closed+Bounded Lemma 3.43 we know that L is closed and we know that it is bounded, i.e. there exists an $R \geq 0$, such that $L \subseteq \overline{B}_R^N(0)$. Since $f(x) \xrightarrow{x \to \infty} \infty$ there exists a compact subset $K \subseteq X$ with $f^{-1}(\overline{B}_R^N(0)) \subseteq K$. Now we see that

$$\begin{aligned} f^{-1}(L) &= f^{-1}\big(\overline{B}_R^N(0) \cap L\big) &&= f^{-1}\big(\overline{B}_R^N(0)\big) \cap f^{-1}(L) \\ &\subseteq K \cap f^{-1}(L) &&= \underset{\text{since } L \text{ is closed}}{\text{compact subset} \cap \text{closed subset of } X} &&= \underset{\text{by the Compact-Closed Lemma 3.39 (1)}}{\text{compact.}} \end{aligned}$$

We have thus shown that $f\colon X \to \mathbb{R}^N$ is proper. ⊞

Now we assume that $f\colon X \to \mathbb{R}^N$ is furthermore injective. By the discussion on page 208 we know that $\mathbb{R}^N$ is regionally compact. Evidently $\mathbb{R}^N$ is also Hausdorff. Thus it follows from the Proper-Closed Proposition 4.64[195] that $f\colon X \to \mathbb{R}^N$ is in fact a closed embedding. ■

We proceed with introducing the following notation which we will use throughout the remainder of this chapter.

Notation 15.20. Let $N \in \mathbb{N}_0$.

(1) Given $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we consider the maximum norm from page 129:

$$\|x\|_{\max} \coloneqq \max\{|x_1|, \ldots, |x_N|\}.$$

(2) As in the Upper Bound-to-Metric Lemma 10.18 we equip $\mathbb{R}^N$ with the metric that is given by

$$\rho(x, y) \coloneqq \min\{1, \|x - y\|_{\max}\}.$$

It follows from the Equivalent Metric Lemma 3.11 and the Norm Equivalence Proposition 4.19 together with the Upper Bound-to-Metric Lemma 10.18 that ρ induces the usual topology on $\mathbb{R}^N$. In particular $(\mathbb{R}^N, \rho)$ is also complete.

Let X be a topological space.

(3) We denote as usual by $C(X, \mathbb{R}^N)$ the set of continuous maps from X to $\mathbb{R}^N$.

(4) If X is non-empty, then we equip $C(X, \mathbb{R}^N)$ with the metric

$$\rho(f, g) \coloneqq \sup\{\rho(f(x), g(x)) \,|\, x \in X\}.$$

[193] Recall that in Definition 4.63 we called a map $f\colon X \to Y$ between two topological spaces *proper* if the preimage of every compact subset of Y is a compact subset of X.

[194] Let $f\colon X \to \mathbb{R}^N$ be a continuous map. The claim says that if $f(x) \xrightarrow{x \to \infty} \infty$, then f is proper. It is straightforward to show that the converse also holds. For our purpose though it is better work with the notion "$f(x) \xrightarrow{x \to \infty} \infty$" instead of the notion "proper".

[195] One can also deduce the statement from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26.

Since $(\mathbb{R}^N, \rho)$ is complete we know by the Function Space–Metric Proposition 10.20 that $(C(X, \mathbb{R}^N), \rho)$ is also complete.

Lemma 15.21. (Limit-to-Infinity Lemma) Let X be a topological space. Furthermore let $f, g \in C(X, \mathbb{R}^N)$ such that $\rho(f, g) < 1$. If $f(x) \xrightarrow{x \to \infty} \infty$, then also $g(x) \xrightarrow{x \to \infty} \infty$.

Proof. Let $R > 0$. Since $f(x) \xrightarrow{x \to \infty} \infty$ there exists a compact subset $K \subseteq X$ such that $\|f(x)\|_{\max} > R + 1$ for all $x \in X \setminus K$. For $x \in X \setminus K$ we now obtain that

$$\|g(x)\|_{\max} \underset{\uparrow}{\geq} \|f(x)\|_{\max} - \|f(x) - g(x)\|_{\max} \underset{\uparrow}{>} \|f(x)\|_{\max} - 1 \underset{\uparrow}{>} R + 1 - 1 = R.$$

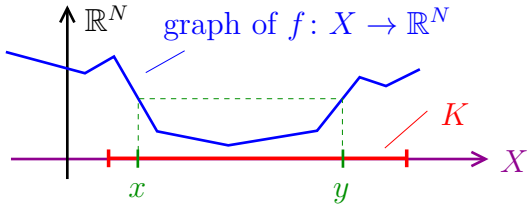
by the triangle inequality; since $\rho(f, g) < 1$ implies that $\rho(f(x), g(x)) < 1$ which implies that $\|f(x) - g(x)\|_{\max} < 1$; choice of K

■

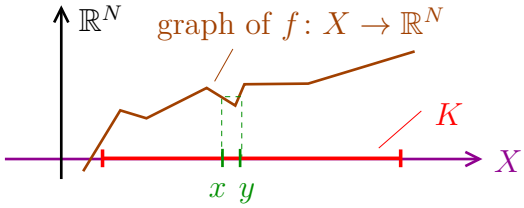
Definition 15.22. Let (X, d) be a metric space. Given $f \in C(X, \mathbb{R}^N)$ and given a compact non-empty subset $K \subseteq X$ we set

$$\Delta(f, K) \coloneqq \sup\{d(x, y) \,|\, x, y \in K \text{ with } f(x) = f(y)\}.$$

For safety, we extend this definition to $\Delta(f, \varnothing) \coloneqq 0$.



"large" $\Delta(f, K)$ means that $f\colon K \to \mathbb{R}^N$ is "far" from an injective map



"small" $\Delta(f, K)$ means that $f\colon K \to \mathbb{R}^N$ is "close" to an injective map

Lemma 15.23. (Delta Neighborhood Lemma) Let (X, d) be a metric space. We equip $C(X, \mathbb{R}^N)$ with the metric ρ from above. Let $\epsilon > 0$ and let $K \subseteq X$ be compact. Then

$$U_\epsilon(K) \coloneqq \{f \in C(X, \mathbb{R}^N) \,|\, \Delta(f, K) < \epsilon\}$$

is an open subset of $C(X, \mathbb{R}^N)$.

Proof. Note that by the definition of the topology in a metric space it suffices to show that for every $f \in U_\epsilon(K)$ there exists an $r > 0$ with $B_r^\rho(f) \subseteq U_\epsilon(K)$. Thus let $f \in U_\epsilon(K)$. By definition of $U_\epsilon(K)$ we can pick a $b \in \mathbb{R}$ with $\Delta(f, K) < b < \epsilon$. We set

$$A \coloneqq \{(x, y) \in K \times K \,|\, d(x, y) \geq b\}.$$

In the Metric-is-Continuous Lemma 10.2 we showed that $d\colon K \times K \to \mathbb{R}$ is continuous. Thus it follows that A is a closed subset of the compact space $K \times K$. It follows from the Compact-Closed Lemma 3.39 that A itself is compact.

Claim 1. The map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R} \\ (x, y) & \mapsto & \rho(f(x), f(y)) \end{array}$$

is strictly positive and continuous on A.

Proof. It follows from the Metric-is-Continuous Lemma 10.2, the Upper Bound-to-Metric Lemma 10.18 and the Topological-Product Proposition 7.2 (1) that the map is continuous. Now let $(x, y) \in A \subseteq K \times K$. By definition of A we have $d(x, y) \geq b > \Delta(f, K)$.

By definition of $\Delta(f,K)$ this implies that $f(x) \neq f(y)$. But this in turn implies that $d(f(x),f(y)) > 0$. ⊞

Since A is compact it follows from the claim and the Compact Image Lemma 4.15 that

$$r \;:=\; \tfrac{1}{2} \cdot \min\{\rho(f(x),f(y)) \,|\, (x,y) \in A\}$$

exists and that $r > 0$. It remains to prove the following claim:

Claim 2. We have $B_r^\rho(f) \subseteq U_\epsilon(K)$.

Proof. Let $g \in B_r^\rho(f)$, in other words, $\rho(f,g) < r$. We need to show that $g \in U_\epsilon(K)$. Since $b < \epsilon$ it suffices to show that for any $x,y \in K$ we have either $g(x) \neq g(y)$ or we have $d(x,y) \leq b$. Thus let $x,y \in K$. We distinguish two cases:

- Suppose that $(x,y) \in A$. In this case $\rho(f(x),f(y)) \geq 2r$. Note that we have the inequalities $\rho(f(x),g(x)) \leq \rho(f,g) < r$ and $\rho(f(y),g(y)) \leq \rho(f,g) < r$. In combination these show that $\rho(g(x),g(y)) > 0$, i.e. $g(x) \neq g(y)$.
- Suppose that $(x,y) \notin A$. In this case $d(x,y) < b$ by definition of A. ■

Definition 15.24.

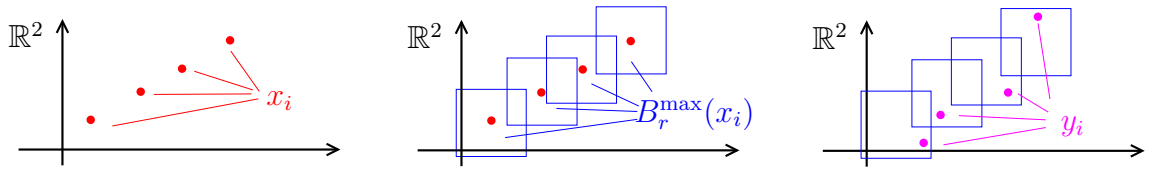
(1) We say that points $P_0,\dots,P_k \in \mathbb{R}^N$ are affinely independent if for every choice of $\lambda_0,\dots,\lambda_k \in \mathbb{R}$ we have[196]

$$\sum_{i=0}^{k} \lambda_i \cdot P_i \;=\; 0 \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i \;=\; 0 \quad \Longrightarrow \quad \lambda_0 = \dots = \lambda_k = 0.$$

(2) We say that a subset $S \subseteq \mathbb{R}^N$ is in general position if every collection of $\leq N+1$ distinct points in S is affinely independent.[197]

Lemma 15.25. (General Position–Existence Lemma) Let $x_1,\dots,x_n \in \mathbb{R}^N$ be distinct points and let $r > 0$. There exist distinct points $y_1,\dots,y_n \in \mathbb{R}^N$ with the following properties:

(1) For all $i = 1,\dots,n$ we have $\|x_i - y_i\|_{\max} < r$.
(2) The set $\{y_1,\dots,y_n\}$ is in general position.



Proof. Given $x \in \mathbb{R}^N$ and $r \geq 0$ we denote by $B_r^{\max}(x) = \{y \in \mathbb{R}^N \,|\, \|x-y\|_{\max} < r\}$ the r-ball with respect to the maximum norm. With this notation we can formulate the following claim:

Claim. Let $V_1,\dots,V_k \subseteq \mathbb{R}^N$ be affine subspaces each of which has codimension ≥ 1. For each $x \in \mathbb{R}^N$ and $r > 0$ the set $B_r^{\max}(x) \setminus (V_1 \cup \dots \cup V_k)$ is non-empty.

[196] In other words, $P_0,\dots,P_k$ are affinely independent if the affine vector subspace of $\mathbb{R}^N$ spanned by these points is k-dimensional. Or put differently, $P_0,\dots,P_k$ are affinely independent if $P_1 - P_0,\dots,P_k - P_0$ are linearly independent.

[197] We leave it to the reader to verify that for $N = 3$ this is precisely the notion of being in general position that we introduced on page 401.

Proof. Recall that by the Norm Equivalence Proposition 4.19 the maximum norm and the usual Euclidean norm define equivalent metrics on $\mathbb{R}^N$. Thus the desired statement follows immediately from the Baire Category Theorem 10.26 (1), the fact that the metric space $\mathbb{R}^N$ is complete and the fact that hyperplanes of codimension ≥ 1 are nowhere dense. ⊞

Let $x_1, \ldots, x_n \in \mathbb{R}^N$ be distinct points and let $r > 0$. We construct the points $y_1, \ldots, y_n$ iteratively. We set $y_1 := x_1$. Now suppose that points $y_1, \ldots, y_k$ have already been found with the desired properties.

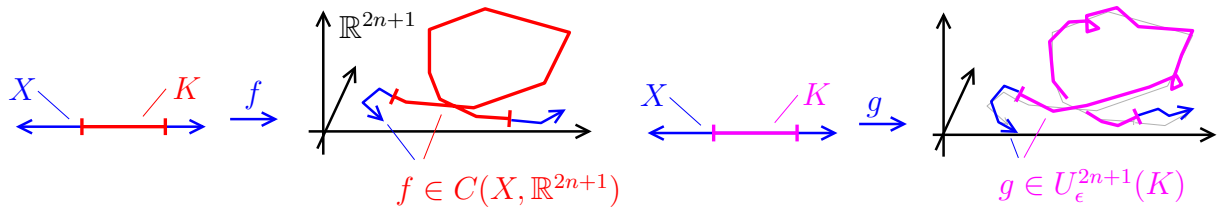
We denote by P the union of all affine subspaces of $\mathbb{R}^N$ that are generated by all subsets $A \subseteq \{y_1, \ldots, y_k\}$ with $\#A \leq N$. We Note that each such subspace in P has codimension $\geq N - \#A - 1 \geq 1$. By the above claim there exists a $y_{k+1} \in B_r^{\max}(x) \setminus P$. This point has the desired properties. ■

The following lemma is the key technical ingredient in the proof of the Topological Space–Euclidean Embedding Theorem 15.16.

Lemma 15.26. (Density Criterion) Let (X, d) be a metric space that is second-countable and regionally compact such that $\operatorname{cdim}(X) \leq n$. If $K \subseteq X$ is a compact subspace, then for every $\epsilon > 0$ the set

$$U_\epsilon^{2n+1}(K) \;:=\; \{g \in C(X, \mathbb{R}^{2n+1}) \,|\, \Delta(g, K) < \epsilon\}$$

is dense in $C(X, \mathbb{R}^{2n+1})$ with respect to the metric ρ that we introduced in Notation 15.20.



Proof. Let $f \in C(X, \mathbb{R}^{2n+1})$ and let $r \in (0, 1)$. Note that we need to show that there exists a $g \in U_\epsilon^{2n+1}(K)$ such that $\rho(f, g) \leq r$.

Claim 1. There exist finitely many open non-empty subsets $U_1, \ldots, U_m$ of K with the following properties:
(1) the sets $U_1, \ldots, U_m$ are an open cover of K,
(2) for all $i = 1, \ldots, m$ we have $\operatorname{diam}(U_i) < \frac{\epsilon}{2}$,
(3) for all $i = 1, \ldots, m$ we have $\operatorname{diam}(f(U_i)) < \frac{r}{2}$,
(4) the cover $\{U_1, \ldots, U_m\}$ has order at most $n + 1$.

Proof. For each $x \in K$ we consider

$$V_x \;:=\; \underbrace{B_{\frac{\epsilon}{2}}^X(x)}_{\substack{\text{open ball of radius } \frac{\epsilon}{2} \\ \text{around } x \in X}} \;\cap\; f^{-1}\Big(\underbrace{B_{\frac{r}{2}}^{\mathbb{R}^{2n+1}}(f(x))}_{\substack{\text{open ball of radius } \frac{r}{2} \\ \text{around } f(x) \in \mathbb{R}^{2n+1}}} \Big) \cap K.$$

These sets form an open cover of K. By hypothesis we have $\operatorname{cdim}(X) \leq n$. Since $K \subseteq X$ is compact and since X is Hausdorff we know by the Compact-Closed Lemma 3.39 (2) that $K \subseteq X$ is a closed subset. Thus it follows from the Closed Subspace Covering Dimension Lemma 15.7 that we also have $\operatorname{cdim}(K) \leq n$. Therefore there exists an open refinement

$\{U_i\}_{i\in I}$ of the open cover $\{V_x\}_{x\in K}$ that has order at most $n+1$. Since K is compact we know that we can cover K by finitely many of the U_i. Finally, since each U_i is contained in some V_x we see that each of non-empty U_i has the desired properties (2) and (3). ⊞

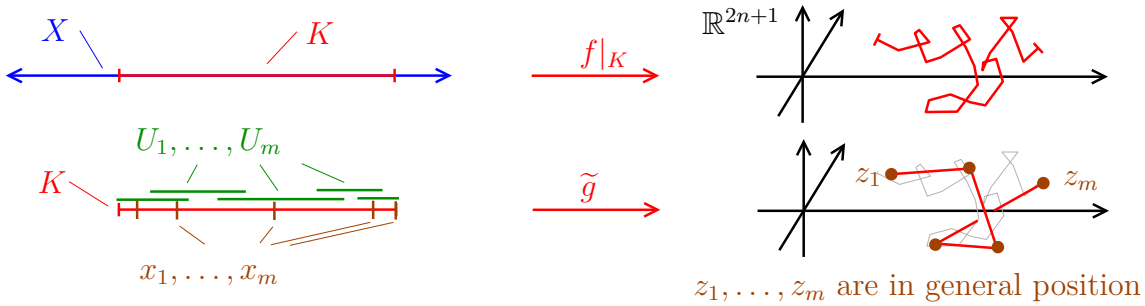
Since X is regionally compact and second-countable and since metric spaces are Hausdorff we know by the Paracompact-Criterion Lemma 14.5 that X is paracompact. As we just mentioned, $K \subseteq X$ is a closed subset. Thus we obtain from the Closed Subset-Paracompact Lemma 14.6 that K itself is also paracompact. Therefore it follows from the Paracompact-Partition of Unity Theorem 14.15 that there exists a partition of unity $\Phi_1, \dots, \Phi_m$ on K such that for each $i \in \{1, \dots, m\}$ we have $\operatorname{supp}(\Phi_i) \subseteq U_i$.

For each $i \in \{1, \dots, m\}$ we pick a point $x_i \in U_i$. By the General Position–Existence Lemma 15.25 there exist distinct points $z_1, \dots, z_m \in \mathbb{R}^{2n+1}$ with the following two properties:

(1) For each $i \in \{1, \dots, m\}$ we have $\|z_i - f(x_i)\|_{\max} < \frac{r}{2}$,
(2) The set $\{z_1, \dots, z_m\}$ is in general position.

Next we consider the map

$$\begin{aligned} \widetilde{g}\colon K &\to \mathbb{R}^{2n+1} \\ x &\mapsto \sum_{i=1}^{m} \Phi_i(x) \cdot z_i. \end{aligned}$$



Claim 2. For all $x \in K$ we have $\|\widetilde{g}(x) - f(x)\|_{\max} < r$.

Proof. Given $x \in K$ we see that

$$\begin{aligned} \|\widetilde{g}(x) - f(x)\|_{\max} &\overset{\text{since } \Phi_1(x)+\cdots+\Phi_m(x)=1}{=} \Big\| \sum_{i=1}^{m} \Phi_i(x) \cdot (z_i - f(x_i)) \; + \; \sum_{i=1}^{m} \Phi_i(x) \cdot (f(x_i) - f(x)) \Big\|_{\max} \\ &\le \sum_{i=1}^{m} \Phi_i(x) \cdot \underbrace{\|z_i - f(x_i)\|_{\max}}_{< \frac{r}{2} \text{ by choice of } z_i} + \sum_{i=1}^{m} \Phi_i(x) \cdot \underbrace{\|f(x_i) - f(x)\|_{\max}}_{\substack{\text{if } \Phi_i(x) \neq 0, \text{ then } x \in U_i, \\ \text{since } \operatorname{diam}(f(U_i)) < \frac{r}{2} \text{ this} \\ \text{implies that this term } < \frac{r}{2}}} \\ &< \sum_{i=1}^{m} \Phi_i(x) \cdot \tfrac{r}{2} + \sum_{i=1}^{m} \Phi_i(x) \cdot \tfrac{r}{2} \;=\; r \cdot \underbrace{\sum_{i=1}^{m} \Phi_i(x)}_{=1} \;=\; r. \end{aligned}$$

⊞

Claim 3. If $x, y \in K$ satisfy $\widetilde{g}(x) = \widetilde{g}(y)$, then $d(x, y) < \frac{\epsilon}{2}$.

Proof. Let $x, y \in K$ with $\widetilde{g}(x) = \widetilde{g}(y)$. Since each $\operatorname{diam}(U_i) < \frac{\epsilon}{2}$ it suffices to show that there exists an $i \in \{1, \dots, m\}$ with $x, y \in U_i$. We set

$$A \;:=\; \{i \in \{1, \dots, m\} \mid \Phi_i(x) \neq 0\} \qquad \text{and} \qquad B \;:=\; \{i \in \{1, \dots, m\} \mid \Phi_i(y) \neq 0\}.$$

Recall that the cover $\{U_1, \dots, U_m\}$ has order $n+1$. Furthermore recall that by choice we know that each $\operatorname{supp}(\Phi_i)$ is contained in the corresponding U_i. Thus we see that $\#A \leq n+1$ and we see that $\#B \leq n+1$. Next note that

$$\sum_{i \in A \cup B} (\Phi_i(x) - \Phi_i(y)) \cdot z_i \underset{\substack{\uparrow \\ \text{since the } i \notin A \cup B \\ \text{do not contribute}}}{=} \widetilde{g}(x) - \widetilde{g}(y) \underset{\substack{\uparrow \\ \text{since } \widetilde{g}(x) = \widetilde{g}(y)}}{=} 0. \qquad \text{and} \qquad \sum_{i \in A \cup B} (\Phi_i(x) - \Phi_i(y)) \underset{\substack{\uparrow \\ \text{since the } \Phi_i \text{ are a partition of unity and} \\ \text{since the } i \notin A \cup B \text{ do not contribute}}}{=} 0$$

Since the points $z_1, \dots, z_m$ are distinct, since the set $\{z_1, \dots, z_m\} \subseteq \mathbb{R}^{2n+1}$ is in general position and since

$$\#\{z_i \,|\, i \in A \cup B\} \;=\; \#(A \cup B) \;\leq\; \#A + \#B \;=\; 2n+2$$

we see that $\Phi_i(x) - \Phi_i(y) = 0$ for each $i \in \{1, \dots, m\}$. Since $\sum_{i=1}^{m} \Phi_i(x) = 1$ we see that there exists an $i \in \{1, \dots, m\}$ with $\Phi_i(x) > 0$. By the above we have $\Phi_i(x) = \Phi_i(y) > 0$. This implies that $x, y \in \operatorname{supp}(\Phi_i) \subseteq U_i$. ⊞

In Claim 1 and 2 we constructed an interesting map $\widetilde{g}\colon K \to \mathbb{R}^{2n+1}$. It remains to extend this map in a satisfactory way to a continuous map $g\colon X \to \mathbb{R}^{2n+1}$. To do so we consider the map

$$\begin{aligned} \Theta\colon K &\to \mathbb{R}^{2n+1} \\ x &\mapsto f(x) - \widetilde{g}(x). \end{aligned}$$

It follows from Claim 2 that this map takes values in $[-r, r]^{2n+1}$.

Since X is metric we know by the Metric Spaces-Normal Proposition 12.2 that X is normal. Furthermore, since K is compact and since X is Hausdorff we know by the Compact-Closed Lemma 3.39 that $K \subseteq X$ is a closed subset. This implies that we can apply the Tietze Extension Theorem 12.7 to obtain an extension of the above map $\Theta\colon K \to [-r, r]^{2n+1}$ to a continuous map $\widetilde{\Theta}\colon X \to [-r, r]^{2n+1}$. Finally we consider the map

$$\begin{aligned} g\colon X &\to \mathbb{R}^{2n+1} \\ x &\mapsto f(x) - \widetilde{\Theta}(x). \end{aligned}$$

By construction we have $g|_K = \widetilde{g}$. By Claim 3 we have $\Delta(g, K) \leq \frac{\epsilon}{2}$, which implies that $g \in U_\epsilon^{2n+1}(K)$. Furthermore, since Θ takes values in $[-r, r]^{2n+1}$ we see that $\rho(f, g) \leq r$. ■

Now we turn to the actual proof of the Topological Space–Euclidean Embedding Theorem 15.16.

Proof of the Euclidean Embedding Theorem 15.16. Let $n \in \mathbb{N}_0$ and let X be a topological space with $\operatorname{cdim}(X) \leq n$ that is second countable, Hausdorff and regionally compact. We need to show that X is homeomorphic to a closed subset of $\mathbb{R}^{2n+1}$. We make the following two preparations:

(1) Since X is second countable, Hausdorff and regionally compact we know by the Compact Exhaustion Lemma 13.6 (1) that there exists an exhaustion of X by compact subspaces. By Definition 13.5 this means that there exists a sequence $\{K_i\}_{i \in \mathbb{N}}$ of compact subspaces of X such that the following two conditions are satisfied:
   (a) We have $X = \bigcup_{i \in \mathbb{N}} K_i$.
   (b) For each $i \in \mathbb{N}$ we have $K_i \subseteq \mathring{K}_{i+1}$.

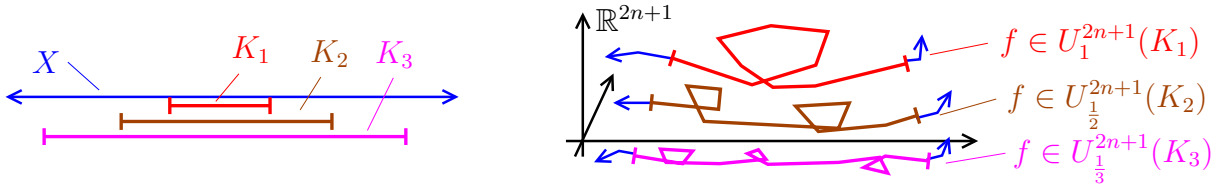
(2) Since X is second countable, Hausdorff and regionally compact we know by the Paracompact-Criterion Lemma 14.5 that X is paracompact, together with the Dieudonné

Theorem 14.9 this implies that X is normal. It follows from the Urysohn Metrization Theorem 13.8 that X is metrizable. We pick a metric d on X such that the corresponding metric topology equals the given topology.

Claim 1. The set[198] $\bigcap_{i=1}^{\infty} U_{\frac{1}{i}}^{2n+1}(K_i)$ is dense in $C(X, \mathbb{R}^{2n+1})$ with respect to the metric ρ that we introduced in Notation 15.20.

Proof. It follows from our hypothesis that $\operatorname{cdim}(X) \leq n$, together with the Delta Neighborhood Lemma 15.23 and the Density Criterion Lemma 15.26, that each $U_{\frac{1}{i}}^{2n+1}(K_i)$ is open and dense in $C(X, \mathbb{R}^{2n+1})$. By the Function Space–Metric Proposition 10.20 we know that the metric space $C(X, \mathbb{R}^{2n+1})$ is complete. The claim now follows from the Baire Category Theorem 10.26 (1) together with the Baire Space Characterization Lemma 10.28. ⊞

Claim 2. Every $f \in \bigcap_{i=1}^{\infty} U_{\frac{1}{i}}^{2n+1}(K_i)$ is injective.



Proof. Let $x, y \in X$ such that $f(x) = f(y)$. There exists an $i \in \mathbb{N}$ such that for all $j \geq i$ we have $x, y \in K_j$. Now let $j \geq i$. By hypothesis $f \in U_{\frac{1}{j}}^{2n+1}(K_j)$. Thus it follows from $f(x) = f(y)$ that $d(x, y) \leq \frac{1}{j}$. Since this holds for all $j \geq i$ we see that $d(x, y) = 0$. ⊞

Since X is second countable, Hausdorff and regionally compact we know by the Exhaustion Function Lemma 14.12 that X admits an exhaustion function $\widetilde{f}\colon X \to \mathbb{R}_{\geq 0}$. Recall that according to Definition 14.11 this is a continuous map $\widetilde{f}\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0, c])$ is a compact subset of X. We now consider the map

$$\begin{aligned} f\colon X &\to \mathbb{R}^{2n+1} \\ x &\mapsto (\widetilde{f}(x), 0, \ldots, 0). \end{aligned}$$

By comparing definitions one sees that $f(x) \xrightarrow{x \to \infty} \infty$.

By Claim 1 the set $\bigcap_{i=1}^{\infty} U_{\frac{1}{i}}^{2n+1}(K_i)$ is dense in $(C(X, \mathbb{R}^{2n+1}), \rho)$. In particular we see that there exists a $\iota \in \bigcap_{i=1}^{\infty} U_{\frac{1}{i}}^{2n+1}(K_i)$ with $\rho(f, \iota) < 1$. The theorem now follows from the following observations:

(1) By Claim 2 we know that $\iota\colon X \to \mathbb{R}^N$ is injective.
(2) It follows from $\rho(f, \iota) < 1$ and the Limit-to-Infinity Lemma 15.21 that $\iota(x) \xrightarrow{x \to \infty} \infty$.
(3) It follows from (1) and (2), together with the Closed Embedding Criterion Lemma 15.19, that $\iota\colon X \to \mathbb{R}^{2n+1}$ is a closed embedding. ■

Remark. There are many books that are completely dedicated to dimension theory, see e.g. [**HuW1948, Enge1978, Enge1995, Naga1983, Pear1975, Chara2019**]. We refer the reader, whose appetite got whetted, to the above books for more results. □

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[198] Recall that in the Delta Neighborhood Lemma 15.23 we set $U_{\epsilon}^{2n+1}(K) = \{f \in C(X, \mathbb{R}^N) \mid \Delta(f, K) < \epsilon\}$.

15.5. Exercises for Chapter 15.

Exercise 15.1. Give an example of a compact space X of covering dimension $\leq d$ and an open cover $\{U_i\}_{i\in I}$ such that there is no subset $J \subseteq I$ such that $\{U_j\}_{j\in J}$ is a cover of X of order $\leq d+1$.

Exercise 15.2. We consider the Topologist's Sine Curve

$$S \coloneqq \{(0,y) \,|\, y \in [-1,1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \subseteq \mathbb{R}^2.$$

Show that $\operatorname{cdim}(S) = 1$.

Exercise 15.3. Let X be a connected space with at least two points. Show that if X is T_1, in the sense of Definition 13.25, then $\operatorname{cdim}(X) \geq 1$.

Exercise 15.4. Let X be a topological space. Show that if $\operatorname{cdim}(X) = 0$, then X is normal.

Exercise 15.5. What is the covering dimension of $\mathbb{Q}$? Here we equip $\mathbb{Q} \subseteq \mathbb{R}$ as always with the subspace topology.

Exercise 15.6. Let C be the Cantor set from Definition 3.45. What is the covering dimension of the topological space $X \coloneqq (C \times [0,1]) \cup ([0,1] \times \{0\}) \subseteq \mathbb{R}^2$?

Exercise 15.7. Let $X = \{a,b,c,d\}$. We equip X with the topology that is generated by $\{a\}, \{d\}, \{b,d\}$ and $\{c,d\}$. What is $\operatorname{cdim}(X)$?

Exercise 15.8. Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of finite order. Is the open cover necessarily locally finite?

Exercise 15.9. Show that there exists a surjective continuous map $f\colon X \to Y$ between topological spaces such that $\operatorname{cdim}(X) < \operatorname{cdim}(Y)$.
Remark. You could use the Cantor function from page 216.

Exercise 15.10. Let X be a topological space and let A and B be disjoint subsets of X. We say that a subset $P \subseteq X$ is a *partition between A and B*, if there exist disjoint open subsets U and V of X with $A \subseteq U$, $B \subseteq V$ and $X \setminus P = U \cup V$.

Next we iteratively define classes $\mathcal{C}_{-1}, \mathcal{C}_0, \mathcal{C}_1, \ldots$ of topological spaces:

(1) We define $\mathcal{C}_{-1}$ to be the class consisting only of the empty topological space.
(2) Suppose that for some $n \in \mathbb{N}_0$ we have defined $\mathcal{C}_{-1}, \mathcal{C}_0, \ldots, \mathcal{C}_{n-1}$. Given a topological space X we say that $X \in \mathcal{C}_n$ if for every $x \in X$ and every closed subset $B \subseteq X$ with $x \notin B$ there exists a partition P between $\{x\}$ and B with $P \in \mathcal{C}_{n-1}$.

We define $\mathcal{C}_\infty$ to be the class of all topological spaces. Note that $\mathcal{C}_{-1} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \ldots$. Given a topological space X we define the *small inductive dimension* $\operatorname{ind}(X)$ as follows:

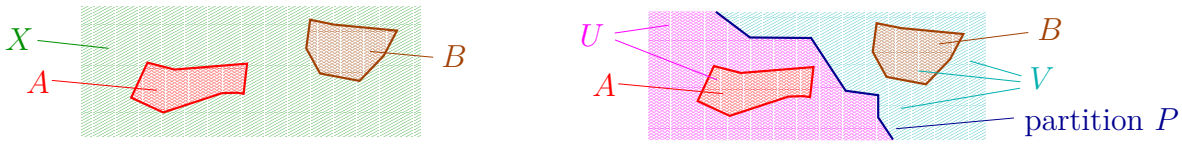
$$\operatorname{ind}(X) \coloneqq \min\{n \in \mathbb{Z}_{\geq -1} \cup \{\infty\} \,|\, X \in \mathcal{C}_n\}.$$

Prove the following statements:

(a) If X is a non-empty discrete space, then $\operatorname{ind}(X) = 0$.
(b) The small inductive dimension of $\mathbb{R}$ equals 1.
(c) For every subspace A of a topological space X we have $\operatorname{ind}(A) \leq \operatorname{ind}(X)$.

Exercise 15.11. We iteratively define classes $\mathcal{D}_{-1}, \mathcal{D}_0, \mathcal{D}_1, \ldots$ of topological spaces:

(1) We define $\mathcal{D}_{-1}$ to be the class consisting only of the empty topological space.



(2) Suppose that for some $n \in \mathbb{N}_0$ we have defined $\mathcal{D}_{-1}, \mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$. Given a topological space X we say that $X \in \mathcal{D}_n$ if for any two disjoint closed subsets A and B of X there exists a partition P between A and B with $P \in \mathcal{D}_{n-1}$. (We refer to Exercise 15.10 for the definition of a partition.)

We define $\mathcal{D}_\infty$ to be the class of all topological spaces. Note that $\mathcal{D}_{-1} \subseteq \mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \ldots$. Given a topological space X we define the *large inductive dimension* $\operatorname{ind}(X)$ as follows:

$$\operatorname{Ind}(X) \;:=\; \min\{n \in \mathbb{Z}_{\geq -1} \cup \{\infty\} \,|\, X \in \mathcal{D}_n\}.$$

Prove the following statements:

(a) The large inductive dimension of $[0,1]$ equals 1.
(b) For every closed subspace A of a topological space X we have $\operatorname{Ind}(A) \leq \operatorname{Ind}(X)$.

Exercise 15.12. Given a topological space X we denote by $\operatorname{ind}(X)$ the small inductive dimension that we defined in Exercise 15.10 and we denote by $\operatorname{Ind}(X)$ the large inductive dimension that we defined in Exercise 15.11.

(a) Let X be the Sierpiński space that we introduced in Exercise 4.26, i.e. let $X = \{0,1\}$ with the topology $\mathcal{T} = \{\varnothing, X, \{0\}\}$. Show that $\operatorname{cdim}(X) = \operatorname{ind}(X) = 0$ and show that $\operatorname{Ind}(X) = \infty$.
(b) Let $X = \{1,2,3\}$ with the topology that is generated by the subsets $\{1,2\}$ and $\{2,3\}$. Show that $\operatorname{cdim}(X) = 1$ and show that $\operatorname{ind}(X) = \operatorname{Ind}(X) = \infty$.

Remark. This exercise shows that the covering dimension, the small inductive and the large inductive dimension are different "dimension functions".

Exercise 15.13. Let X be the topological space that is obtained from the open disk B^2 by removing three points. By the Open Euclidean Subset–Covering Dimension Corollary 15.12 we know that $\operatorname{cdim}(X) \leq 2$. It follows from the Euclidean Embedding Theorem 15.16 that there exists an embedding $f\colon X \to \mathbb{R}^5$ such that $f(X)$ is a closed subset. Determine

$$\min\{n \in \mathbb{N}_0 \,|\, \text{there exists an embedding } f\colon X \to \mathbb{R}^n \text{ such that } f(X) \text{ is a closed subset}\}.$$

Exercise 15.14. Let $P_1, \ldots, P_5$ be five points in $\mathbb{R}^2$ that are in general position. Show that there exist four points among $P_1, \ldots, P_5$ that span a convex quadrilateral.
Remark. This result was proved by George Szekeres and Esther Klein in 1935. Paul Erdös called this statement the Happy Ending Theorem since it led to the marriage of George Szekeres and Esther Klein who were married for 68 years.

Exercise 15.15. Let $G = (V, E, \varphi)$ be an oriented abstract graph. Show that the following statements are equivalent:

(a) $|G|$ admits an embedding into some $\mathbb{R}^n$.
(b) $|G|$ admits an embedding into $\mathbb{R}^3$.
(c) V and E are countable and each vertex has finite valence.

Hint. Use the Topological Space–Euclidean Embedding Theorem 15.16 and make use of Exercise 11.10.

CHAPTER 16

The compact-open topology

Given two topological spaces X and Y we will introduce in this chapter the compact-open topology on the set of all continuous maps $X \to Y$. This topology generalizes several well-known concepts from analysis and it will play an interesting role on several occasions.

This chapter is organized as follows:

- In Section 16.1 we will introduce the compact-open topology on the X^Y of all continuous maps from a topological space X to another topological space Y.
- In Section 16.2 we will study the compact-open topology for metric spaces X and we will see that in some cases, e.g. matrices, it agrees with the standard topology.
- In Section 16.3 we will consider the evaluation $X^Y \times Y \to X$ and various exponential maps, and we will see under what hypotheses these groups are continuous.
- In Section 16.4 that for a topological space X, with suitable properties, the group of self-homeomorphisms, equipped with the compact-open topology, forms a topological group
- In Section 16.5 we will provide exercises for Chapter 16.

In the next Chapter 17 we will discuss the tricky relationship between the product topology and the quotient topology.

16.1. **Definition of the compact-open topology and basic properties.** First, recall that we saw in Definition 3.33 that a topology on a given set X makes it possible to talk of the convergence of a sequence in X. From real analysis we know that it is not only interesting to think about convergence of sequences of points but also to talk of convergence of functions. In fact, in real analysis one often encounters two different types of convergence that we now recall.

Definition 16.1. Let $U \subseteq \mathbb{R}^n$ be a subset, let $\{f_n \colon U \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of functions and let $f \colon U \to \mathbb{R}$ be a function. We define

$$(f_n)_{n\in\mathbb{N}} \text{ converges pointwise to } f \iff \underset{x\in U}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\geq N}{\forall}\ |f_n(x) - f(x)| < \epsilon$$

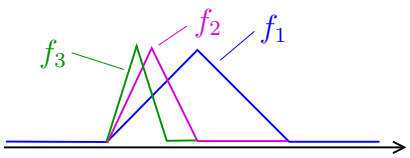
and

$$(f_n)_{n\in\mathbb{N}} \text{ converges uniformly to } f \iff \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{x\in U}{\forall}\ \underset{n\geq N}{\forall}\ |f_n(x) - f(x)| < \epsilon.$$

Example. We consider the following sequence of functions

$$\begin{array}{rcl} f_n \colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t < 0 \text{ or } t > \frac{2}{n}, \\ n \cdot t, & \text{if } t \in [0, \frac{1}{n}], \\ 2 - n \cdot t, & \text{if } t \in (\frac{1}{n}, \frac{2}{n}]. \end{cases} \end{array}$$

The graphs of these functions are shown in the figure below.



One easily verifies that this sequence of functions converges pointwise to the zero function, but it does not converge uniformly to the zero function. □

At times one also wants to study convergence of sequences of continuous maps between general topological spaces. This leads us to the following definition, which was first introduced by Ralph Fox [**Fox1945**]:

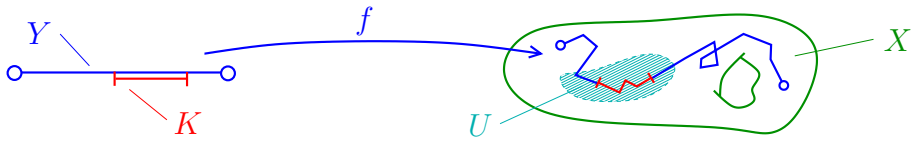
Definition 16.2. Let X and Y be topological spaces.

(1) We denote by $X^Y := C(Y,X)$ the set of all continuous maps from Y to X.

(2) The compact-open topology on X^Y is the topology generated, in the sense of Definition 3.56, by all the sets of the form

$$M(K,U) \;:=\; \{f \in X^Y \mid f(K) \subseteq U\}$$

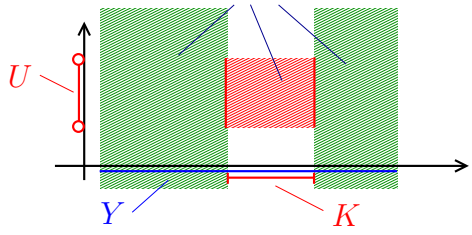
where $K \subseteq Y$ is compact and where $U \subseteq X$ is open.



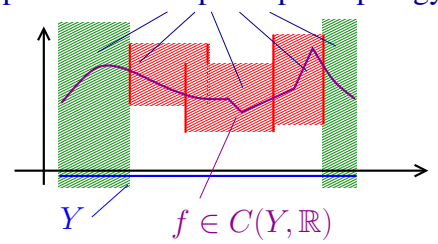
Examples.

(1) Let $Y \subseteq \mathbb{R}$ be an interval. Let us consider the compact-open topology on $C(Y,\mathbb{R}) = \mathbb{R}^Y$. As usual we picture elements of $C(Y,\mathbb{R})$ by their graphs. In the figure below we show on the left an open subset $M(K,U)$ as above and on the right we show the graph of some function $f \in C(Y,\mathbb{R})$ and an open neighborhood thereof.

$M(K,U)$ are the continuous functions whose graph is contained in this set



open neighborhood of $f \in C(Y,\mathbb{R})$ with respect to the compact-open topology



Now let $f_n \in C(Y,\mathbb{R})$, $n \in \mathbb{N}$ be a sequence of continuous functions and let $f \in C(Y,\mathbb{R})$. In Exercise 16.8 we will prove that the following holds:

$(f_n)_{n\in\mathbb{N}}$ converges[199] to f with respect to the compact-open topology $\iff$ for each compact subset $K \subseteq Y$ the sequence $(f_n|_K)_{n\in\mathbb{N}}$ converges uniformly to $f|_K$.

Alternatively a proof for this statement is given in [**Will1970**, Theorem 43.7] or in [**Jos1983**, Theorem 15.1.2].

(2) Let Y be a set, equipped with the discrete topology, and let X be a topological space. We consider the product $X^Y := \prod_{y\in Y} X$. In Exercise 16.7 we will see that the product topology[200] on $X^Y = \prod_{y\in Y} X$ agrees with the compact open topology on X^Y.

(3) Since this question is surely on everybody's lips, let us settle it now: In the Matrix Topology Lemma 16.8 we will show that given any $m, n \in \mathbb{N}_0$ the compact-open topology on the set $\mathrm{M}(m \times n, \mathbb{R}) \subseteq C(\mathbb{R}^n, \mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n}$ is just the usual topology on $\mathrm{M}(m \times n, \mathbb{R})$ that we introduced in Identifiation 4.52. □

To get some practice with the compact-open topology we will prove the following two lemmas in Exercises 16.2 and 16.3.

Lemma 16.3. (Compact Open–Hausdorff Lemma) Let X and Y be topological spaces. If X is Hausdorff, then X^Y is also Hausdorff.

Lemma 16.4. (Constant Map Lemma) If X and Y are non-empty topological spaces, then the map

$$\begin{array}{ccc} X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

is continuous and it is an embedding.

We continue with the following elementary but essential lemma:

Lemma 16.5. (Compact-Open Functoriality Lemma) Let X and Y be topological spaces.

(1) (a) If $\varphi\colon X \to \widetilde{X}$ is a continuous map, then the induced map

$$\begin{array}{ccc} \varphi_*\colon X^Y & \to & \widetilde{X}^Y \\ (f\colon Y \to X) & \mapsto & (\varphi \circ f\colon Y \to \widetilde{X}) \end{array}$$

is continuous.

(b) If $\varphi\colon X \to \widetilde{X}$ is an embedding, then so is $\varphi_*\colon X^Y \to \widetilde{X}^Y$.

(c) Let $\varphi, \psi\colon X \to \widetilde{X}$ be continuous maps. If φ and ψ are homotopic[201], then so are $\varphi_*, \psi_*\colon X^Y \to \widetilde{X}^Y$.

(d) If $\widetilde{X}$ is a topological space that is homotopy equivalent to X, then $\widetilde{X}^Y$ is also homotopy equivalent to X^Y.

(2) (a) If $\psi\colon Y \to \widetilde{Y}$ is a continuous map, then the induced map

$$\begin{array}{ccc} \psi^*\colon X^{\widetilde{Y}} & \to & X^Y \\ (f\colon \widetilde{Y} \to X) & \mapsto & (f \circ \psi\colon Y \to X) \end{array}$$

is continuous.

(b) Let X and Y be topological spaces and let $A \subseteq Y$ be a subset. If A is a closed subset of Y, then the restriction map $X^Y \to X^A$ is an open map.

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[199] We refer to Definition 3.33 for the definition of convergence of a sequence in a topological space.

[200] We refer to the Topological-Product Proposition 7.2 for the Definition 7.3 of the product topology.

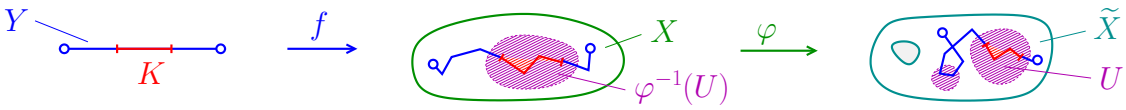
[201] We apologize to the reader for using the notion of being homotopic, even though it is only introduced in Definition 18.2. The reader who made it that far into this chapter will surely not be frightened by this inconvenience.

Remark. The Compact-Open Functoriality Lemma 16.5 (1a) says in particular, that given topological spaces X and Y and given any subspace A of X, we can view A^Y as a subspace of X^Y. □

Proof. To clarify the statements we replace in this proof the notation $M(K,U)$ by the notation $M_{X^Y}(K,U)$, with the obvious interpretation thereof.

(1) (a) By the Subbasis-Continuity Proposition 4.4 it suffices to prove the following claim:

Claim. Let $K \subseteq Y$ be a compact subset and let $U \subseteq \widetilde{X}$ be an open subset. The preimage $(\varphi_*)^{-1}(M_{\widetilde{X}^Y}(K,U))$ is an open subset of X^Y.



Proof. The claim follows from the following calculation:

$$\begin{array}{rcll}(\varphi_*)^{-1}(M_{\widetilde{X}^Y}(K,U)) &=& \{f\in X^Y \mid \varphi_*(f)\in M_{\widetilde{X}^Y}(K,U)\} &= \{f\colon Y\to X \mid \overbrace{(\varphi\circ f)(K)}^{=\varphi(f(K))}\subseteq U\}\\ &=& \{f\colon Y\to X \mid f(K)\subseteq \varphi^{-1}(U)\} &= M_{X^Y}(K,\underbrace{\varphi^{-1}(U)}_{\text{open in } X}).\\ &\uparrow& & \\ &\text{by Exercise 1.2}& & \end{array}$$

⊞

(b) Since φ is an embedding we can identify X with its image in $\widetilde{X}$. Put differently, it suffices to consider the case of an inclusion $i\colon X \to \widetilde{X}$. By the Open-Closed–Injective Lemma 4.22 (1) together with the Subbasis-Open Map Lemma 4.14 it suffices to prove the following claim:

Claim. For every compact subset $K \subseteq Y$ and every open subset $U \subseteq X$ the image $i_*(M_{X^Y}(K,U))$ is an open subset of $i_*(X^Y)$.

Proof. Since $U \subseteq X$ is open there exists, by definition of the subspace topology, an open subset $V \subseteq \widetilde{X}$ such that $U = V \cap X$. We have

$$\begin{array}{rcl} i_*(M_{X^Y}(K,U)) &=& \{i\circ f\colon Y\to \widetilde{X} \mid f\colon Y\to X \text{ and } f(K)\subseteq U\}\\ &=& \{f\colon Y\to X \mid f(K)\subseteq V\} \;=\; i_*(X^Y)\cap M_{\widetilde{X}^Y}(K,V).\\ &\uparrow& \\ &\text{since } f(Y)\subseteq X \text{ implies for any } x\in X& \\ &\text{that } f(x)\in U \text{ if and only if } f(x)\in V& \end{array}$$

By the definition of the subspace topology this is indeed an open subset of $i_*(X^Y) \subseteq \widetilde{X}^Y$. ⊞

(c) Let $F\colon X\times[0,1]\to\widetilde{X}$ be a homotopy between $F_0=\varphi$ and $F_1=\psi$. It remains to prove the following claim:

Claim. The map

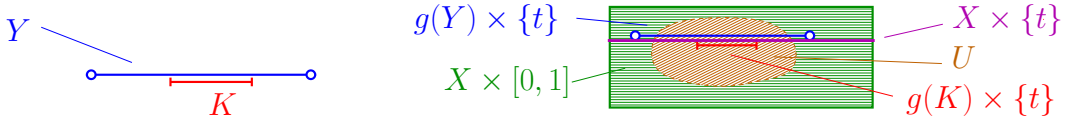
$$\begin{array}{rcl} X^Y\times[0,1] &\to& \widetilde{X}^Y\\ (g\colon Y\to X, t) &\mapsto& \begin{pmatrix} Y &\to& \widetilde{X}\\ y &\mapsto& F(g(x),t)\end{pmatrix}\end{array}$$

is continuous.

Proof. The map is the composition of the following two maps:

$$\begin{array}{rcccc} X^Y \times [0,1] & \xrightarrow{=:\Theta} & (X \times [0,1])^Y & \xrightarrow{F_*} & \widetilde{X}^Y \\ (g\colon Y \to X, t) & \mapsto & \begin{pmatrix} Y \to X \times [0,1] \\ y \mapsto (g(x), t) \end{pmatrix} & & \end{array}$$

By (1a) we know that the second map $F_*\colon (X \times [0,1])^Y \to \widetilde{X}^Y$ is continuous. Thus it remains to prove that the first map $\Theta\colon X^Y \times [0,1] \to (X \times [0,1])^Y$ is continuous. By the Subbasis-Continuity Proposition 4.4 it suffices to show that preimages of elements from the basis of the compact-open topology on $(X \times [0,1])^Y$ are open. Thus let $K \subseteq Y$ be a compact subset and let $U \subseteq X \times [0,1]$ be an open subset. Note that it follows from the JH-Openness Criterion 3.3 that it remains to show that for any $(g,t) \in \Theta^{-1}(M_{(X\times[0,1])^Y}(K,U)) \subseteq X^Y \times [0,1]$ there exists an open neighborhood N of $(g,t) \in X^Y \times [0,1]$ with $\Theta(N) \subseteq U$.



Since $g(K) \times \{t\} \subseteq U$ and since K is compact it follows, similar to the proof of the Tube Lemma 7.12, that there exists an open subset $V \subseteq X$ and an open neighborhood J of $t \in [0,1]$ with $g(K) \times \{t\} \subseteq V \times J \subseteq X \times [0,1]$. Finally we set $N := M_{X^Y}(V,K) \times J$. It follows easily from the definitions that we have indeed $(g,t) \in N \subseteq \Theta^{-1}(U)$.

(d) This statement follows almost immediately from (c).

(2) (a) Let $K \subseteq Y$ be a compact subset and let $U \subseteq X$ be an open subset. We have

$$\begin{aligned} (\psi^*)^{-1}(M_{X^Y}(K,U)) &= \{f \in X^{\widetilde{Y}} \mid \psi^*(f) \in M_{X^Y}(K,U)\} \\ &= \{f\colon \widetilde{Y} \to X \mid \underbrace{(f\circ\psi)(K)}_{=f(\psi(K))} \subseteq U\} = M_{X^{\widetilde{Y}}}(\psi(K),U). \end{aligned}$$

Since ψ is continuous we obtain from the Compact Image Lemma 4.15 that $\psi(K)$ is a compact subset of $\widetilde{Y}$. Thus we obtain again from the Subbasis-Continuity Proposition 4.4 that ψ^* is indeed continuous.

(b) By the Subbasis-Open Map Lemma 4.14 it suffices to prove the following claim:

Claim. For every open subset $U \subseteq X$ and every compact subset $K \subseteq Y$ the image of $M(K,U)$ under the restriction map $\rho\colon X^Y \to X^A$ is open.

Proof. First note that $\rho(M(K,U)) = M(K \cap A, U)$. Since A is closed it follows from the Compact-Closed Lemma 3.39 that $K \cap A$ is again compact. This implies that $M(K \cap A, U)$ is an open subset of X^A. ∎

16.2. **Compact-Open topology and metric spaces.** The following proposition is the main technical result of this section:

Proposition 16.6. (Compact Open-as-Metric Topology Proposition) Let (X,d) be a metric space and let Y be a topological space. If Y is non-empty and compact, then the compact-open topology on X^Y agrees with the topology coming from the supremum metric

$$\begin{array}{rcl} \sigma\colon X^Y \times X^Y & \to & \mathbb{R}_{\geq 0} \\ (f,g) & \mapsto & \sigma(f,g) := \underbrace{\sup\{d(f(y),g(y)) \,|\, y \in Y\}}_{\text{defined since } Y \text{ is compact and } \neq \varnothing}. \end{array}$$

Remark. A somewhat general statement, that does not need the requirement that Y is compact, is proved in [**Mun1975**, Theorem 46.8]. ■

Before we turn to the proof of the Compact Open-as-Metric Topology Proposition 16.6 we want to discuss the special case that is most useful for our purposes. To formulate this result we need to introduce the following convention:

Convention 16.7. Let $m, n \in \mathbb{N}$. We have the obvious injective map

$$\begin{array}{rcl} \mathrm{M}(m \times n, \mathbb{R}) & \to & C(\mathbb{R}^n, \mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n} \\ A & \mapsto & \left(\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v \end{array} \right). \end{array}$$

We will use this map to view $\mathrm{M}(m \times n, \mathbb{R})$ as a subset of $C(\mathbb{R}^n, \mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n}$.

The only application of the Compact Open-as-Metric Topology Proposition 16.6 that we really care about is now given by the following lemma:

Lemma 16.8. (Matrix Topology Lemma) Let $m, n \in \mathbb{N}$. The compact-open topology on $C(\mathbb{R}^n, \mathbb{R}^m)$ induces the usual topology on $\mathrm{M}(m \times n, \mathbb{R})$ that we introduced in Identification 4.52.

Proof of Lemma 16.8 assuming Proposition 16.6. We consider the following diagram:

$$\begin{array}{ccc} C(\mathbb{R}^n, \mathbb{R}^m) \cap \mathrm{M}(m \times n, \mathbb{R}) & \hookrightarrow & C(\mathbb{R}^n, \mathbb{R}^m) \\ \downarrow \text{restriction } f \mapsto f|S^{n-1} & & \downarrow \text{restriction } f \mapsto f|S^{n-1} \\ C(S^{n-1}, \mathbb{R}^m) \cap \mathrm{M}(m \times n, \mathbb{R}) & \hookrightarrow & C(S^{n-1}, \mathbb{R}^m). \end{array}$$

We make the following observations:

(1) First note that it follows easily from the Subspace-Basis Lemma 3.58 that the subspace topologies on $C(\mathbb{R}^n, \mathbb{R}^m) \cap \mathrm{M}(m \times n, \mathbb{R}) \subseteq C(\mathbb{R}^n, \mathbb{R}^m)$ and on $C(S^{n-1}, \mathbb{R}^m) \cap \mathrm{M}(m \times n, \mathbb{R}) \subseteq C(S^{n-1}, \mathbb{R}^m)$ are again compact-open topologies.
(2) We now consider the left vertical map in greater detail:
  (a) It follows from the Compact-Open Functoriality Lemma 16.5 (2a) that the left vertical map is continuous.
  (b) Since $S^{n-1} \subseteq \mathbb{R}^n$ is a closed subset it follows from the Compact-Open Functoriality Lemma 16.5 (2b) that the left vertical map is an open map.
  (c) Since linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are completely determined by the restriction to S^{n-1} we see that the left vertical map is a bijection.
  (d) It follows from the above together with the Open-Closed–Injective Lemma 4.22 (2) that the left vertical map is a homeomorphism.
(3) It follows from the Compact-Open–Metric Topology Proposition 16.6 that the compact-open topology on $C(S^{n-1}, \mathbb{R}^m)$ equals the topology arising from the supremum metric.
(4) The supremum metric on $(\mathbb{R}^m)^{S^{n-1}}$ arises from the supremum norm

$$\begin{array}{rcl} (\mathbb{R}^m)^{S^{n-1}} & \to & \mathbb{R}_{\geq 0} \\ (f\colon S^{n-1} \to \mathbb{R}^m) & \mapsto & \|f\| := \sup\{\|f(x)\| \,|\, x \in S^{n-1}\}. \end{array}$$

(5) By the Norm Equivalence Proposition 4.19 we know that any two norms on the finite-dimensional vector space $\mathrm{M}(m \times n, \mathbb{R})$ induce the same topology. In particular the supremum norm induces the same topology as the usual Euclidean norm. ■

The remainder of this section is occupied by the proof of the Compact-Open–Metric Topology Proposition 16.6:

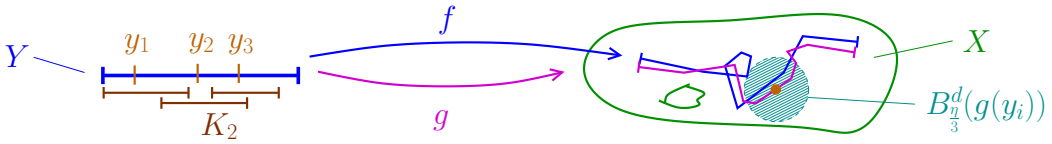
Proof of the Compact-Open–Metric Topology Proposition 16.6. Let (X, d) be a metric space and let Y be a compact space. Let $\mathcal{C}$ be the compact-open topology on X^Y and let $\mathcal{M}$ be the metric topology on X^Y coming from the above metric. We need to show that $\mathcal{C} = \mathcal{M}$. In the following two claims we will show that $\mathcal{M} \subseteq \mathcal{C}$ and that $\mathcal{C} \subseteq \mathcal{M}$.

Claim 1.

(1) For every $f \in X^Y$, every $\epsilon > 0$ and every $g \in B_\epsilon^\sigma(f)$ there exists a neighborhood N of g in the compact-open topology $\mathcal{C}$ with $N \subseteq B_\epsilon^\sigma(f)$.
(2) We have $\mathcal{M} \subseteq \mathcal{C}$.

Proof.

(1) Let $f \in X^Y$, let $\epsilon > 0$ and let $g \in B_\epsilon^\sigma(f)$. First we set $\eta := \epsilon - \sigma(f, g) > 0$. Next note that it follows from the Compact Image Lemma 4.15 that $g(Y) \subseteq X$ is compact. Thus there exist $y_1, \ldots, y_m \in Y$ such that $g(Y)$ is covered by the open balls $B^d_{\frac{\eta}{3}}(g(y_i))$. For $i = 1, \ldots, m$ we denote by $K_i \subseteq Y$ the closure of $g^{-1}(B^d_{\frac{\eta}{3}}(g(y_i)))$ in Y.



We make the following observations:
(a) Since Y is compact we obtain from the Interior-Closure Lemma 3.22 (2) together with the Compact-Closed Lemma 3.39 (1) that K_i is compact.
(b) It follows from the Image-Preimage Lemma 1.9 (6) that $Y = K_1 \cup \cdots \cup K_m$.
(c) For $i = 1, \ldots, m$ we have

$$g(K_i) \underset{\uparrow}{\subseteq} g(g^{-1}(\overline{B}^d_{\frac{\eta}{3}}(g(y_i)))) \underset{\uparrow}{\subseteq} \overline{B}^d_{\frac{\eta}{3}}(g(y_i)) \subseteq B_{\frac{\eta}{2}}(g(y_i)) =: U_i.$$

by Definition 3.21 of the closure have $K_i \subseteq g^{-1}(\overline{B}^d_{\frac{\eta}{3}}(g(y_i)))$

by the Image-Preimage Lemma 1.9 (2a)

First note that by (c) we have $g \in N := M(K_1, U_1) \cap \cdots \cap M(K_m, U_m)$. Next note that it follows from (a) that N is open with respect to the compact-open topology $\mathcal{C}$. Thus it now remains to show that $M(K_1, U_1) \cap \cdots \cap M(K_m, U_m) \subseteq B_\epsilon^\sigma(f)$. Let $h \in M(K_1, U_1) \cap \cdots \cap M(K_m, U_m)$. We need to show that $h \in B_\epsilon^\sigma(f)$. Thus let $y \in Y$. By (b) there exists an $i \in \{1, \ldots, m\}$ with $y \in K_i$. Now we see that

$$d(f(y), h(y)) \underset{\uparrow}{\leq} \underbrace{d(f(y), g(y))}_{\leq \sigma(f,g)} + \underbrace{d(g(y), g(y_i))}_{< \frac{\eta}{2}} + \underbrace{d(h(y), g(y_i))}_{< \frac{\eta}{2}} < \sigma(f, g) + \eta = \epsilon.$$

by the triangle inequality

$< \frac{\eta}{2}$, this follows from (c), since $y \in K_i$

$< \frac{\eta}{2}$, since $y \in K_i$ and since $h(K_i)$ is contained in $U_i := B_{\frac{\eta}{2}}(g(y_i))$

Thus we have shown that $h \in B_\epsilon^\sigma(f)$.

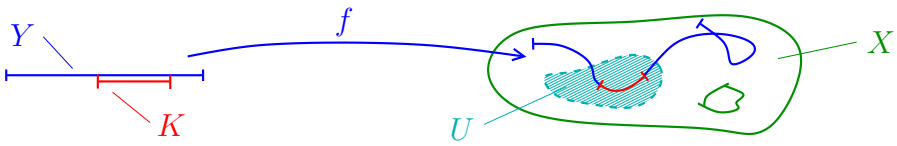
(2) It follows easily from (1) and the JH-Openness Criterion 3.3 that each $B_\epsilon^\sigma(f)$ lies in $\mathcal{C}$. It follows easily this observation and from the definition of a basis of a topology that have do indeed have an inclusion $\mathcal{M} \subseteq \mathcal{C}$. ⊞

Claim 2.

(1) For every open subset $M(K,U) \in \mathcal{C}$ and every $f \in M(K,U)$ there exists an $\epsilon > 0$ such that $B_\epsilon^\sigma(f) \subseteq M(K,U)$.
(2) We have $\mathcal{C} \subseteq \mathcal{M}$.

Proof.

(1) Suppose we are given an open subset $M(K,U) \in \mathcal{C}$. Let $f \in M(K,U)$. By the Compact Image Lemma 4.15 we know that $f(K) \subseteq X$ is compact.



Since $f(K) \subseteq U$ and since $U \subseteq X$ is open it follows from the Disjoint Compact-Closed–Distance Lemma 10.5 that $\eta := d(f(K), X \setminus U) > 0$. We claim that $B_{\frac{\eta}{2}}^\sigma(f) \subseteq M(K,U)$. Thus let $g \in B_{\frac{\eta}{2}}^\sigma(f)$. This means that $\sigma(f,g) < \frac{\epsilon}{2}$. By the triangle inequality this implies that $g(K) \subseteq U$, i.e. $g \in M(K,U)$.
(2) It follows again easily from (1) and the JH-Openness Criterion 3.3 that each $M(K,U)$ lies in $\mathcal{M}$. As before it follows from this observation and from the definition of a basis of a topology that have do indeed have an inclusion $\mathcal{C} \subseteq \mathcal{M}$. ■

16.3. **The evaluation and the exponential map.** We continue with a few more technical statements on the compact-open topology. In most applications later on we will consider the case that $Y = [0,1]$. In particular the extra hypothesis in the subsequent propositions on the topological space Y, namely that we require Y to be regionally compact, will not be an issue in our future applications.[202]

Proposition 16.9. (Evaluation Map Proposition) Let X and Y be topological spaces. If Y is regionally compact, then the following two statements hold:

(1) The evaluation map[203]
$$\begin{array}{rcl} \mathrm{ev}\colon X^Y \times Y & \to & X \\ ((f\colon Y \to X), y) & \mapsto & f(y) \end{array}$$
is continuous.
(2) For any $y \in Y$ the evaluation map
$$\begin{array}{rcl} \mathrm{ev}_y\colon X^Y & \to & X \\ (f\colon Y \to X) & \mapsto & f(y) \end{array}$$
is continuous.

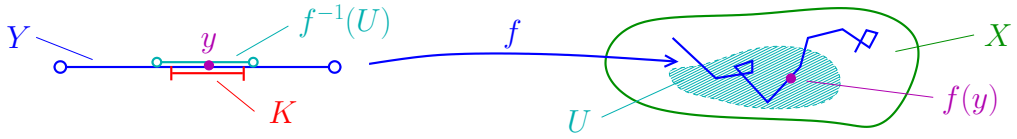
Proof.

(1) Let U be an open subset of X. We need to show that $\mathrm{ev}^{-1}(U)$ is open. By the JH-Openness Criterion 3.3 it suffices to prove the following claim:

[202]For many applications it is useful to know that if X is a topological space that is compact an Hausdorff, then it follows from the Compact Hausdorff-Regionally Compact Lemma 4.58 that X is regionally compact.
[203]Here we equip $X^Y \times Y$ with the product topology from the Topological-Product Proposition 7.2.

Claim. Given any $(f, y) \in \mathrm{ev}^{-1}(U)$ there exists a neighborhood W of $(f, y) \in X^Y \times Y$ such that $W \subseteq \mathrm{ev}^{-1}(U)$.

Proof. Let $f \in X^Y$ and $y \in Y$ with $\mathrm{ev}(f, y) = f(y) \in U$. Since f is continuous we know that $f^{-1}(U)$ is an open neighborhood of y. Since Y is regionally compact there exists a compact neighborhood K of $y \in Y$ with $y \in K \subseteq f^{-1}(U)$.



In other words, we have found a compact neighborhood K of y with $f(K) \subseteq U$. Therefore we have $f \in M(K, U)$ and the evaluation map ev sends $W := M(K, U) \times K$ into U. By the Definition 7.3 of the product topology the set $W = M(K, U) \times K$ is in fact a neighborhood of (f, y) in $X^Y \times Y$. ⊞

(2) This statement follows from (1) and the Topological-Product Proposition 7.2 (1∗). ■

On many occasions we will also need the following rather technical proposition.

Proposition 16.10. (Compact-Open Exponential Proposition) Let Y be a topological space that is regionally compact. Furthermore let X and T be two topological spaces. Let $H\colon Y \times T \to X$ be a map. Given $t \in T$ we denote by $H_t\colon Y \to X$ the map defined by $H_t(y) := H(y, t)$. Then

$$H\colon Y \times T \to X \text{ is continuous} \iff \begin{array}{l}(1)\ \text{each } H_t\colon Y \to X \text{ is continuous, and}\\ (2)\ \text{the map}^{204}\ T \to X^Y \text{ given by } t \mapsto H_t \text{ is continuous.}\end{array}$$

In other words, the map

$$\begin{array}{rcl} X^{Y\times T} & \to & (X^Y)^T \\ (H\colon Y \times T \to X) & \mapsto & \left(\begin{array}{rcl} T & \to & X^Y \\ t & \mapsto & (y \mapsto H(y,t)) \end{array}\right) \end{array}$$

is well-defined[205] and it is a bijection. (If Y and T are both regionally compact, then we will see in the Exponential Proposition 16.12 that the above map $X^{Y\times T} \to (X^Y)^T$ is actually a homeomorphism.)

Proof. [206] Note that the "in other words" part of the proposition is indeed just a reformulation of the first part of the proposition. Thus it remains to prove the equivalence of the two statements in the beginning.

First we prove the"$\Leftarrow$"-direction. We point out that $H\colon Y \times T \to X$ is the composition of the two maps

$$\begin{array}{rcl} Y \times T & \to & X^Y \times Y \\ (y, t) & \mapsto & (H_t, y) \end{array} \quad \text{and the evaluation map} \quad \begin{array}{rcl} X^Y \times Y & \to & X \\ (f, y) & \mapsto & f(y). \end{array}$$

It follows from the Evaluation Map Proposition 16.9 and the Topological-Product Proposition 7.2 (1) that the first map is continuous. Furthermore, since Y is regionally compact

[204] Note that by (1) we know that $H_t \in X^Y$.

[205] Here "well-defined" means that the map $T \to X^Y$ actually defines an element in $(X^Y)^T$. More precisely, the map $T \to X^Y$ given by $z \mapsto (y \mapsto H(y, z))$ actually lies in $(X^Y)^T$.

[206] I strongly advise against reading the proof.

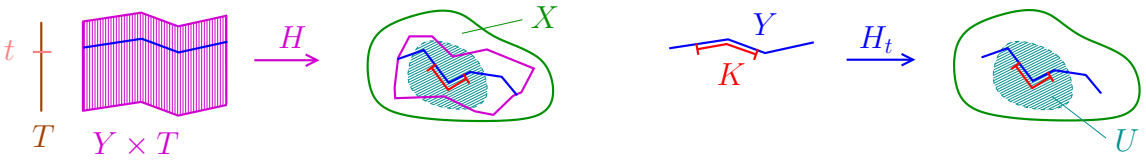
we obtain from the Evaluation Map Proposition 16.9 that the second map is continuous. Thus H itself is continuous.

Now we turn to the"$\Rightarrow$"-direction. Thus we suppose that $H\colon Y\times T\to X$ is continuous.

(1) Let $t\in T$. The map H_t is the composition of the inclusion $Y\to Y\times T$, $y\mapsto(y,t)$ with the map H. It follows from the Topological-Product Proposition 7.2 (1$*$) that the first map is continuous. The second map is continuous by our hypothesis. Hence H_t is also continuous. In particular we now know that H_t defines an element in X^Y.

(2) We need to show that the map $T\to X^Y$ given by $t\mapsto H_t$ is continuous. By the Subbasis-Continuity Proposition 4.4 and the JH-Openness Criterion 3.3 it suffices to show that for every set of the form $M(K,U)\subseteq X^Y$ and every $t\in T$ with $H_t\in M(K,U)$ there exists a neighborhood W of $t\in T$ such that for all $t'\in W$ we have $H_{t'}\in M(K,U)$.

Thus let K be a compact subset of Y, let U be an open subset of X and let $t\in T$ with $H_t\in M(K,U)$.



We start out with the following claim:

Claim. Given any $y\in K$ there exist open neighborhoods $V_y\subseteq Y$ of y and $W_y\subseteq T$ of t such that $H(V_y\times W_y)\subseteq U$.

Proof. By the continuity of $H\colon Y\times T\to X$ we know that $H^{-1}(U)$ is an open subset of $Y\times T$. Note that $H(y,t)=H_t(y)\in U$ since $H_t\in M(K,U)$ and $y\in K$. Thus $H^{-1}(U)$ is an open neighborhood of $(y,t)\in Y\times T$. The existence of V_y and W_y is now a consequence of the Definition 7.3 of the product topology of $Y\times T$. ⊞

Note that it follows from the compactness of K and the Compact Subspace Covering Lemma 3.38 that there exist points $y_1,\dots,y_n\in K$ with $K\subseteq V_{y_1}\cup\dots\cup V_{y_n}$. Next we set $W:=W_{y_1}\cap\dots\cap W_{y_n}\subseteq T$. Note that this is an open neighborhood of t. Then

$$H(K\times W)\underset{\substack{\uparrow\\ \text{since } K\subseteq\bigcup_{i=1}^n V_{y_i}}}{\subseteq} H\Big(\bigcup_{i=1}^n V_{y_i}\times W\Big)\underset{\substack{\uparrow\\ \text{since } W\subseteq W_{y_i}}}{\subseteq} H\Big(\bigcup_{i=1}^n V_{y_i}\times W_{y_i}\Big)=\bigcup_{i=1}^n\underbrace{H(V_{y_i}\times W_{y_i})}_{\subseteq U}\subseteq U.$$

It follows from this observation and the definition of $H_{t'}$ and $M(K,U)$ that for any $t'\in W$ we have $H_{t'}\in M(K,U)$. ■

The Compact-Open Exponential Proposition 16.10 has the following pretty consequence:

Lemma 16.11. (Cylinder–Path Space Adjunction Lemma) Let $\mathcal{T}op$ be the category of topological spaces. We fix a topological space Y that is regionally compact. The two covariant functors[207] [208]

$$\begin{array}{rcl} F\colon \mathcal{T}op & \to & \mathcal{T}op\\ T & \mapsto & Y\times T\end{array}\qquad\text{and}\qquad\begin{array}{rcl} G\colon \mathcal{T}op & \to & \mathcal{T}op\\ X & \mapsto & X^Y.\end{array}$$

form an adjoint pair in the sense of Definition 6.33.

Proof. Let $X, T \in \mathrm{Ob}(\mathcal{Top})$, i.e. let X and T be two topological spaces. We consider the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(\overbrace{Y \times T}^{=F(T)}, X) & \to & \mathrm{Mor}_{\mathcal{Top}}(T, \overbrace{X^Y}^{=G(X)}) \\ (H\colon Y \times T, X) & \mapsto & \left(\begin{array}{ccc} T & \to & X^Y \\ t & \mapsto & \left(\begin{array}{ccc} Y & \to & X \\ y & \mapsto & H(y,t) \end{array} \right) \end{array} \right). \end{array}$$

It follows from the Compact-Open Exponential Proposition 16.10, applied to the regionally compact space Y, that this map is well-defined and that it is a bijection. Furthermore note that it follows easily from the definitions that this bijection is natural in X and T. We have thus shown that F and G form an adjoint pair. ∎

On one later occasion we will also need the following equally technical proposition.

Proposition 16.12. (Exponential Proposition) Let X, Y and Z be topological spaces. If Y and Z are both regionally compact and Hausdorff, then the map[209]

$$\begin{array}{rcl} X^{Y \times Z} & \to & (X^Y)^Z \\ (f\colon Y \times Z \to X) & \mapsto & \left(\begin{array}{ccc} Z & \to & X^Y \\ z & \mapsto & (y \mapsto f(y,z)) \end{array} \right) \end{array}$$

is a homeomorphism.

Remark. In fact a slightly stronger version of the Exponential Proposition 16.12 holds. By [**Dug1966**, Theorem XII.5.3] one only needs to assume that Y is regionally compact, with no further restrictions on X, Y or Z. We will not need this stronger statement. □

Once again no sane person will ever voluntarily read the proof. For those readers who are still around we first provide the proof of the following lemma:

Lemma 16.13. (Compact-Open–Subbasis Lemma) Let X and Y be topological spaces. Furthermore let $\mathcal{V}$ be a subbasis for X. We consider

$$\mathcal{C} = \left\{ M(K,V) \in \mathcal{P}\left(X^Y\right) \,\middle|\, K \text{ a compact subset of } Y \text{ and } V \in \mathcal{V} \right\}.$$

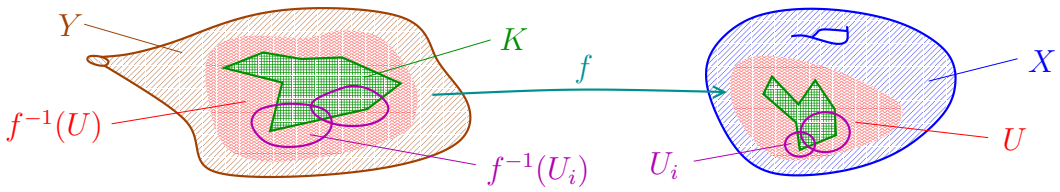
If Y is Hausdorff, then $\mathcal{C}$ is subbasis[210] for the topology of X^Y.

Proof. Let K be a compact subset of Y, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y \to X$ is a continuous map with $K \subseteq f^{-1}(U)$. By the Basis Characterization Lemma 3.57 (2) we need to show that there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $f \in \bigcap_{i=1}^{m} C_i \subseteq M(K,U)$.

Since $\mathcal{V}$ is a subbasis for the topology of X we can write $U = \bigcup_{i \in I} U_i$ where each U_i is of the form $U_i = \bigcap_{V \in P_i} V$ where P_i is a finite subset of $\mathcal{V}$. Since K is compact and since f is

[207] Here we equip $Y \times T$ with the product topology and we equip X^Y with the compact-open topology.
[208] Note that it follows from this lemma together with the Left Adjoint–Colimit Proposition 6.35 that taking products with a given regionally compact space Y preserves colimits.
[209] By the Compact-Open Exponential Proposition 16.10 we know that the map is well-defined and that it is a bijection.
[210] We refer to Definition 3.56 for the definition of a subbasis of a topological space.

continuous it follows from the Compact Subspace Covering Lemma 3.38 that there exists a finite subset $J \subseteq I$ such that $K \subseteq \bigcup_{j\in J} f^{-1}(U_j)$.



Since K is compact and Hausdorff we obtain from the Shrinking Lemma 14.16, together with the Compact-Closed Lemma 3.39 (1) and the hypothesis that Y is Hausdorff, that there exist compact subsets $K_j \subseteq f^{-1}(U_j)$, $j \in J$, with $K \subseteq \bigcup_{j\in J} K_j$.

Thus we see that

$$f \underset{\substack{\uparrow \\ \text{since } K_j \subseteq f^{-1}(U_j)}}{\in} \bigcap_{j\in J} M(K_j, U_j) \underset{\substack{\uparrow \\ \text{definition of } P_j}}{=} \bigcap_{j\in J} M\Big(K_j, \bigcap_{V\in P_j} V\Big) \overset{\substack{\text{since } M(K, A\cap B) = M(K,A)\cap M(K,B) \\ \downarrow}}{=} \bigcap_{j\in J}\bigcap_{V\in P_j} \underbrace{M(K_j, V)}_{\in \mathcal{C}} \underset{\substack{\uparrow \\ \text{since } K \subseteq \bigcup_{j\in J} K_j \text{ and since each } V \text{ lies in } U}}{\subseteq} M(K, U).$$

The finitely many sets $\{M(K_j, V)\}_{j\in J, V\in P_j}$ in $\mathcal{C}$ thus do the trick. ■

Now we are finally ready to provide the proof of the Exponential Proposition 16.12.

Proof of Proposition 16.12. Let X, Y and Z be topological spaces. We assume that Y and Z are both regionally compact and Hausdorff. We need to show that the map

$$\begin{array}{rcl} \Phi\colon X^{Y\times Z} & \to & (X^Y)^Z \\ (f\colon Y\times Z \to X) & \mapsto & \left(\begin{array}{ccc} Z & \to & X^Y \\ z & \mapsto & (y \mapsto f(y,z)) \end{array}\right) \end{array}$$

is a homeomorphism.

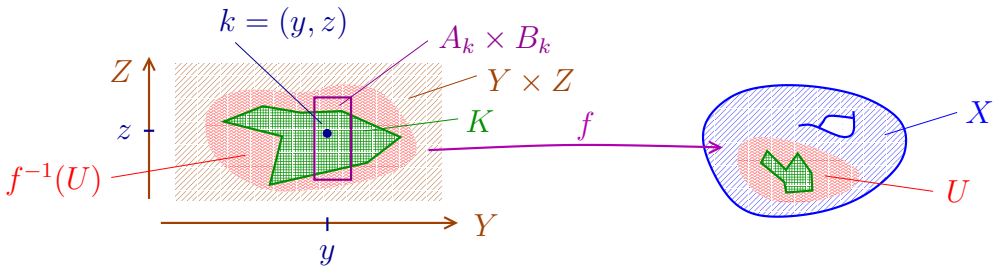
Claim. The set

$$\mathcal{C} = \left\{ M(A\times B, U) \in \mathcal{P}\big(X^{Y\times Z}\big) \;\middle|\; \begin{array}{l} A \text{ a compact subset of } Y,\ B \text{ a compact subset of } Z \\ \text{and } U \text{ an open subset of } X \end{array} \right\}$$

is a subbasis for the topological space $X^{Y\times Z}$.

Proof. Let $K \subseteq Y \times Z$ be a compact subset, let U be an open subset of X and let f be a point in $M(K, U)$, i.e. $f\colon Y \times Z \to X$ is a continuous map with $K \subseteq f^{-1}(U)$. By the Basis Characterization Lemma 3.57 (2) we need to show that there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $f \in \bigcap_{i=1}^{m} C_i \subseteq M(K, U)$.

Let $k \in K \subseteq Y \times Z$. We write $k = (y, z)$. Since $f^{-1}(U)$ is an open subset of $Y \times Z$ it follows from the Definition 7.3 of the product topology on $Y \times Z$ that there exist open neighborhoods $V_k \subseteq Y$ of y and $W_k \subseteq Z$ of z such that $k = (y, z) \in V_k \times W_k \subseteq f^{-1}(U)$. Since Y and Z are regionally compact we can find a compact neighborhood $A_k \subseteq Y$ of y and a compact neighborhood $B_k \subseteq Z$ of z such that $k \in A_k \times B_k \subseteq V_k \times W_k \subseteq f^{-1}(U)$.



Since K is compact it follows from the Compact Subspace Covering Lemma 3.38 that there exist finitely many points $k_1, \ldots, k_m \in K$ such that $K \subseteq \bigcup_{i=1}^{m} A_{k_i} \times B_{k_i}$. Now we see that

$$f \in \bigcap_{i=1}^{m} \overbrace{M(A_{k_i} \times B_{k_I}, U)}^{\in \mathcal{C}} \overset{\text{since } M(V \cup W, U) = M(V,U) \cap M(W,U)}{=} M\Big(\bigcup_{i=1}^{m} A_{k_i} \times B_{k_i}, U\Big) \subseteq M(K, U).$$

since $f(A_{k_i} \times B_{k_i}) \subseteq U$ ↑ ; since $K \subseteq \bigcup_{i=1}^{m} A_{k_i} \times B_{k_i}$ ↑ ; ⊞

As we already remarked in the Compact-Open Exponential Proposition 16.10, the given map $\Phi\colon X^{Y\times Z} \to (X^Y)^Z$ is clearly a bijection. Furthermore under this bijection the sets $M(A \times B, U)$ are precisely the sets $M(B, M(A, U))$. By the claim the former sets form a subbasis for $X^{Y\times Z}$. Thus it suffices to show that the latter sets form a subbasis for $(X^Y)^Z$. But that is an immediate consequence of the Compact-Open–Subbasis Lemma 16.13, applied twice, and our hypothesis that Y and Z are Hausdorff. ■

16.4. **Homeomorphisms form a topological group.** In this section we will see, that given a topological space X that is compact and Hausdorff, then the group of self-homeomorphisms of X, equipped with the compact-open topology, is a topological group.

To get to this point we first need to prove the following proposition:

Proposition 16.14. (Compact-Open Composition Proposition) Let X, Y and Z be topological spaces. If Y is regionally compact, then the map

$$\begin{array}{rcl} \Phi\colon X^Y \times Y^Z & \to & X^Z \\ (f, g) & \mapsto & f \circ g \end{array}$$

is continuous.

The proof of the Compact-Open Composition Proposition 16.14 rests on the following little lemma.

Lemma 16.15. (Compact Neighborhood Lemma) Let Y be a topological space that is regionally compact. If K is a compact subset that is contained in an open subset U of Y, then there exists a compact subset L with $K \subseteq \mathring{L} \subseteq L \subseteq U$.

Proof. Let $x \in K$. Since Y is regionally compact there exists a compact neighborhood $L_x \subseteq U$ of x. Since K is compact there exist $x_1, \ldots, x_m \in K$ with $K \subseteq \mathring{L}_{x_1} \cup \cdots \cup \mathring{L}_{x_m}$. We set $L := L_{x_1} \cup \cdots \cup L_{x_m}$. By the Compact–Union-Intersection Lemma 3.41 (1) we know that L is compact. We see that

$$K \subseteq \bigcup_{i=1}^{m} \mathring{L}_{x_i} \subseteq \text{interior of } \Big(\bigcup_{i=1}^{m} L_{x_i}\Big) = \mathring{L} \subseteq L \subseteq U.$$

↑ by the Interior-Closure-Cup-Cap Lemma 3.24 (1a)

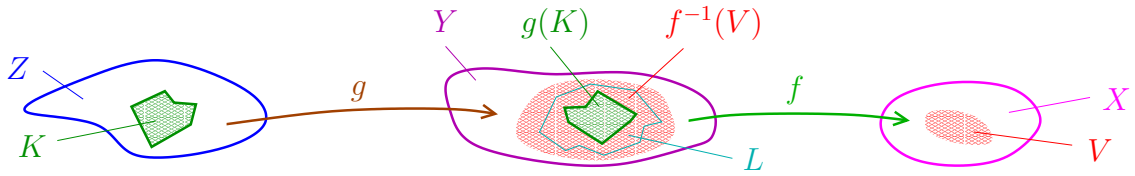
■

Proof of the Compact-Open Composition Proposition 16.14. By the Subbasis-Continuity Proposition 4.4 and the JH-Openness Criterion 3.3 and the Definition 7.3 of the product topology on $X^Y \times Y^Z$ it suffices to prove the following claim:

Claim. Let K be a compact subset of Z and let V be an open subset of X. If $f \in X^Y$ and $g \in Y^Z$ satisfy $f \circ g \in M(K,V)$, then there exists an open subset $U \subseteq Y$ and a compact subset $L \subseteq Y$ with

$$\Phi\big(\underbrace{M(L,V)}_{\subseteq X^Y} \times \underbrace{M(K,U)}_{\subseteq Y^Z}\big) \subseteq M(K,V).$$

Proof. First note that $g(K) \subseteq Y$ is compact by the Compact Image Lemma 4.15 and that $f^{-1}(V) \subseteq Y$ is open since f is evidently continuous.



Since Y is regionally compact we obtain from Lemma 16.15 that there exists a compact subset L with $g(K) \subseteq \mathring{L} \subseteq L \subseteq f^{-1}(U)$. It is clear that $U = \mathring{L}$ and L have the desired properties. ■

We continue with the following almost self-explanatory notation:

Notation 16.16. Given a topological space X we write

$$\mathrm{Homeo}(X) \;:=\; \text{the group of all self-homeomorphisms of } X.$$

We equip this set with the compact-open topology.

Given any topological space X we can now consider the group $\mathrm{Homeo}(X)$. It is natural to ask, under what conditions the compact-open topology on $\mathrm{Homeo}(X)$ turns $\mathrm{Homeo}(X)$ into a topological group. In light of the Compact-Open Composition Proposition 16.14 one might expect that this is the case if X is regionally compact and Hausdorff. Slightly disconcertingly it is shown in [**Dij2005**] that this is in general *not* the case. Fortunately it is shown in the following proposition that if X is furthermore locally connected, then $\mathrm{Homeo}(X)$ is in fact a topological group:

Proposition 16.17. (Homeomorphisms–Topological Group Proposition) Let X be a topological space. We equip the group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X with the subspace topology coming from X^X.

(1) If X is regionally compact then the map

$$\begin{array}{rcl} \mathrm{Homeo}(X) \times \mathrm{Homeo}(X) & \to & \mathrm{Homeo}(X) \\ (f,g) & \mapsto & f \circ g \end{array}$$

is continuous.

(2) If X is regionally compact, Hausdorff and locally connected, then $\mathrm{Homeo}(X)$ is a topological group.

Example. Let $n \in \mathbb{N}$ and let $X = \mathbb{R}^n$. Evidently $X = \mathbb{R}^n$ is regionally compact, Hausdorff and locally connected. It follows from the Homeomorphisms–Topological Group Proposition 16.17 that $\mathrm{Homeo}(\mathbb{R}^n)$ (equipped with the compact-open topology) is a topological group. In particular, $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$, equipped with the subspace topology, is a topological group. From this discussion and the Matrix Topology Lemma 16.8 we obtain a new proof of the Matrices–Topological Group Lemma 7.31, which says that the group $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{M}(n \times n, \mathbb{R})$ is in fact a topological group. □

In the proof of the Homeomorphisms–Topological Group Proposition 16.17 we will make use of the following lemma:

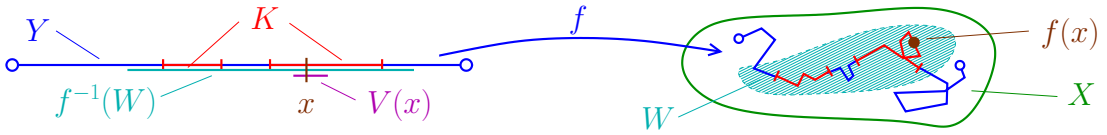
Lemma 16.18. (Compact-Open–Locally Connected–Subbasis Lemma) Let X and Y be topological spaces. If Y is locally connected and regionally compact, then a subbasis for the compact-open topology on X^Y is given by all sets of the form $M(K,U)$ where $U \subseteq X$ is open and where $K \subseteq Y$ is compact, *connected* and with non-empty interior.

Proof of the Compact-Open–Locally Connected–Subbasis Lemma 16.18. It follows from the Basis Characterization Lemma 3.57 that it suffices to prove the following claim:

Claim. Let $K \subseteq Y$ be compact and let $U \subseteq X$ be open. Given any $f \in M(K,U)$ there exist open subsets $U_1, \ldots, U_m \subseteq X$ and subsets $K_1, \ldots, K_m \subseteq Y$ which are compact, connected and with non-empty interior, such that

$$f \in M(K_1,U_1) \cap \cdots \cap M(K_m,U_m) \subseteq M(K,U).$$

Proof. Given any $x \in K$ we observe the following: Since $f(x) \in U$, since U is open and since f is continuous we see that $f^{-1}(U)$ is an open neighborhood of x. Since Y is regionally compact and locally connected we can thus find a connected open set $V(x) \subseteq Y$, whose closure is compact, and such that $x \in V(x) \subseteq f^{-1}(U)$.



Since K is compact there exist $x_1, \ldots, x_m \in K$ such that K is contained in the union of the interiors of the $V(x_i)$. For $i = 1, \ldots, m$ we set $K_i := \overline{V(x_i)}$. First note that by construction the subspace K_i is compact. Furthermore note that an elementary exercise, see Exercise 4.23, shows that the closure of a connected subset is again connected. This implies that $K_i = \overline{V(x_i)}$ is also connected. Finally note that by construction we have the inclusions $h \in M(K_1,U) \cap \cdots \cap M(K_m,U) \subseteq M(K,U)$. ■

Proof of the Homeomorphisms–Topological Group Proposition 16.17. [211] Let X be a topological space that is regionally compact. Note that it follows from the Compact-Open Composition Proposition 16.14 together with the Subspace Continuity Lemma 4.3 (2) and (3) that the multiplication map $\mathrm{Homeo}(X) \times \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ is continuous.

[211] We follow the argument provided in [**Are1946**, Theorem 4].

We now assume that X is furthermore Hausdorff and locally connected. By the above it remains to show that the inversion map

$$\begin{array}{rccc} I\colon & \mathrm{Homeo}(X) & \to & \mathrm{Homeo}(X) \\ & f & \mapsto & f^{-1} \end{array}$$

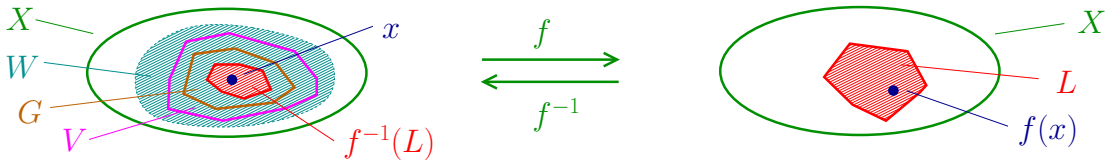
is continuous.

It follows from the Compact-Open–Locally Connected–Subbasis Lemma 16.18, the Subbasis-Continuity Proposition 4.4 and the JH-Openness Criterion 3.3 that it suffices to prove the following claim:

Claim. Let $L \subseteq X$ be a compact, connected subset with non-empty interior and let $W \subseteq X$ be an open subset. For any $f \in \mathrm{Homeo}(X)$ with $I(f) = f^{-1} \in M(L, W)$ there exists an open neighborhood U of $f \in \mathrm{Homeo}(X)$ such that $I(U) \subseteq M(L, W)$.

Proof. We start out with a few preparations:

- Since X is regionally compact we obtain from the Compact Neighborhood Lemma 16.15 an open subset G of X such that $f^{-1}(L) \subseteq G \subseteq \overline{G} \subseteq W$ and such that $\overline{G}$ is compact.
- By applying the Compact Neighborhood Lemma 16.15 again we see that there exists an open subset V of X such that $\overline{G} \subseteq V \subseteq \overline{V} \subseteq W$ and such that $\overline{V}$ is compact.
- Since the interior $\mathring{L}$ is non-empty we can pick $x \in X$ with $f(x) \in \mathring{L}$.



Note that

$$f((X \setminus G) \cap \overline{V}) \underset{\uparrow}{\subseteq} f(X \setminus f^{-1}(L)) \cap f(\overline{V}) \underset{\uparrow}{\subseteq} (X \setminus L) \cap f(W).$$

by the Image-Preimage Lemma 1.9 (4) and (8) and since $\overline{V} \subseteq W$

It follows from this observation and the choice of x that

$$U := M(\{x\}, \mathring{L}) \cap M(\underbrace{(X \setminus G) \cap \overline{V}}_{\substack{\text{compact by the} \\ \text{Compact-Closed} \\ \text{Lemma 3.39}}}, \underbrace{(X \setminus L) \cap f(W)}_{\substack{\text{open since} \\ X \text{ is Hausdorff}}})$$

is a neighborhood of $f \in X^X$ with respect to the compact-open topology.

It remains to show that for every $h \in U \cap \mathrm{Homeo}(f)$ we have $I(h) = h^{-1} \in M(L, W)$. Thus let $h \in U \cap \mathrm{Homeo}(f)$. By definition of U this implies that

$$h((X \setminus G) \cap \overline{V}) \subseteq (X \setminus L) \cap f(W).$$

By taking complements we obtain from De Morgan's Laws 1.7 and the Image-Preimage Lemma 1.9 (8) that

$$L \cup (X \setminus f(W)) \subseteq \underbrace{h(G \cup (X \setminus \overline{V}))}_{= h(G) \sqcup h(X \setminus \overline{V})}.$$

Since $h(G)$ and $h(X \setminus \overline{V})$ are disjoint open sets and since L is connected we obtain from the Image-Preimage-Component Lemma 4.50 (2) that L must be contained in one of the two sets. By definition of U we see that $h(x) \in \mathring{L}$. But this means that L is not contained

in $h(X \setminus \overline{V})$. Hence we have shown that $L \subseteq h(G)$. This implies that $h^{-1}(L) \subseteq G \subseteq W$. We have thus shown that $h^{-1} \in M(L, W)$. ■

We conclude this section with a different source of topological groups.

Lemma 16.19. (Function Space–Topological Group Lemma) Let X be a topological space.

(1) Let $\widetilde{X}$ and Y be topological spaces. If Y is regionally compact, then the map

$$\begin{array}{rcl} \Psi\colon X^Y \times \widetilde{X}^Y & \to & (X \times \widetilde{X})^Y \\ (f, \widetilde{f}) & \mapsto & \left(\begin{array}{rcl} f \times \widetilde{f}\colon Y & \to & X \times \widetilde{X} \\ y & \mapsto & (f(y), \widetilde{f}(y)) \end{array} \right) \end{array}$$

is continuous.

(2) Let G be a topological group. If X is regionally compact, then the topological space G^X together with the multiplication

$$\begin{array}{rcl} G^X \times G^X & \to & G^X \\ (f, g) & \mapsto & \left(\begin{array}{rcl} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{array} \right) \end{array}$$

is a topological group.

Proof.

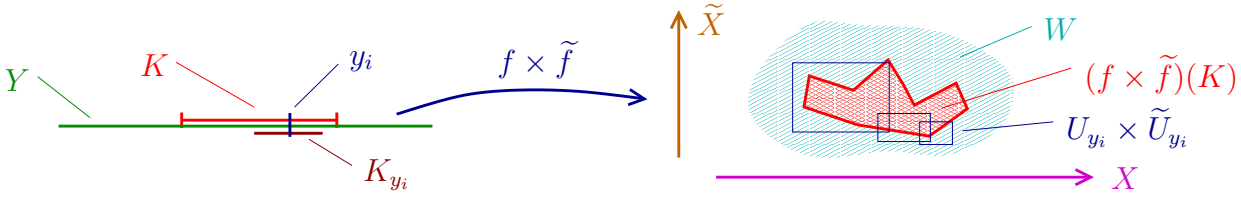
(1) By the Subbasis-Continuity Proposition 4.4 it suffices to show that for every compact subset $K \subseteq Y$ and every open subset $W \subseteq X \times \widetilde{X}$ the preimage $\Psi^{-1}\big(M_{(X\times\widetilde{X})^Y}(K, W)\big)$ is an open subset of $X^Y \times \widetilde{X}^Y$. We will do so by appealing to the JH-Openness Criterion 3.3.

Thus let $(f, \widetilde{f}) \in \Psi^{-1}(M_{(X\times\widetilde{X})^Y}(K, W))$. Note that this means that we are given $f\colon Y \to X$ and $\widetilde{f}\colon Y \to \widetilde{X}$ with $(f \times \widetilde{f})(K) \subseteq W$. For each $y \in K$ we can and will pick open subsets U_y of X and $\widetilde{U}_y$ of $\widetilde{X}$ such that $(f \times \widetilde{f})(y) \in U_y \times \widetilde{U}_y \subseteq W$. By the Topological-Product Proposition 7.2 (1) we know that $(f \times \widetilde{f})^{-1}(U_y \times \widetilde{U}_y)$ is an open subset of Y. Since Y is regionally compact there exists a compact neighborhood K_y of y such that $y \in K_y \subseteq (f \times \widetilde{f})^{-1}(U_y \times \widetilde{U}_y)$.

Since K is compact we obtain from the Compact Subspace Covering Lemma 3.38 that there exist $y_1, \ldots, y_m \in Y$ such that $K \subseteq K_{y_1} \cup \cdots \cup K_{y_m}$. Now we see that

$$(f, \widetilde{f}) \in \underset{\text{since } (f\times\widetilde{f})(K_{y_i}) \subseteq U_{y_i} \times \widetilde{U}_{y_i}}{\bigcap_{i=1}^{m}} M_{X^Y}(K_{y_i}, U_{y_i}) \times M_{\widetilde{X}^Y}(K_{y_i}, \widetilde{U}_{y_i}) \subseteq \underset{\text{since } U_{y_i}\times\widetilde{U}_{y_i} \subseteq W \text{ and since } K \subseteq \bigcup K_{y_i}}{\Psi^{-1}\big(M_{(X\times\widetilde{X})^Y}(K, W)\big)} \subseteq X^Y \times \widetilde{X}^Y.$$

It follows from the JH-Openness Criterion 3.3 that $\Psi^{-1}(M_{(X\times\widetilde{X})^Y}(K, U))$ is indeed an open subset of $X^Y \times \widetilde{X}^Y$.



(2) At this stage there is not much left that needs to be done, we outsource the argument to Exercise 16.5. ■

16.5. Exercises for Chapter 16.

Exercise 16.1. Let X and Y be two discrete topological spaces. Is the compact-open topology on X^Y the discrete topology on the set X^Y?

Exercise 16.2. Let X and Y be topological spaces. Suppose that X is Hausdorff. Show that X^Y is also Hausdorff.

Exercise 16.3. Let X and Y be non-empty topological spaces. We consider the following map:

$$\begin{array}{rcl} \varphi\colon X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

(a) Show that φ is continuous.
(b) Show that φ is an embedding.

Exercise 16.4. Let $X, \widetilde{X}$ and Y be topological spaces. We suppose that Y is regionally compact and Hausdorff. Show that the map

$$\begin{array}{rcl} X^Y \times \widetilde{X}^Y & \to & (X \times \widetilde{X})^Y \\ (f, \widetilde{f}) & \mapsto & \begin{pmatrix} Y & \to & X \times \widetilde{X} \\ y & \mapsto & (f(y), \widetilde{f}(y)) \end{pmatrix} \end{array}$$

is a homeomorphism.
Remark. We refer to [**Enge1989**, Exercise 3.4B, p. 165] for a similar, and for the most part, much more general statement.

Exercise 16.5. Let X be a topological space and let G be a topological group. We suppose that X is regionally compact. Show that G^X with the multiplication

$$\begin{array}{rcl} G^X \times G^X & \to & G^X \\ (f, g) & \mapsto & \begin{pmatrix} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{pmatrix} \end{array}$$

is a topological group.

Exercise 16.6. Let $n \in \mathbb{N}$. We consider the following map

$$\begin{array}{rcl} \Theta\colon \mathrm{SO}(n) & \to & \mathrm{Homeo}(S^{n-1}) \coloneqq \text{self-homeomorphisms of } S^{n-1} \\ A & \mapsto & (P \mapsto A \cdot P). \end{array}$$

(a) Show that the map Θ is continuous.
(b) Show that the map Θ is an embedding.

Hint. Use the Subspace Continuity Lemma 4.3 (3) and the Matrix Topology Lemma 16.8.

Exercise 16.7. Let Y be a set, equipped with the discrete topology, and let X be a topological space. We consider the product $X^Y \coloneqq \prod_{y \in Y} X$. Show that the product topology on $X^Y = \prod_{y \in Y} X$ agrees with the compact open topology on X^Y.

Exercise 16.8. Let $f_n \in C(Y, \mathbb{R})$, $n \in \mathbb{N}$ be a sequence of continuous functions and let $f \in C(Y, \mathbb{R})$. Show that the following holds:

$$\begin{array}{c}(f_n)_{n\in\mathbb{N}} \text{ converges to } f \text{ with respect} \\ \text{to the compact-open topology}\end{array} \iff \begin{array}{c}\text{for each compact subset } K \subseteq Y \\ \text{the sequence } (f_n|_K)_{n\in\mathbb{N}} \\ \text{converges uniformly to } f|_K.\end{array}$$

Hint. Use the Compact Open-as-Metric Topology Proposition 16.6.

Exercise 16.9. Let B and F be topological spaces. We denote by $p\colon B \times F \to B$ the natural projection. If F is regionally compact, Hausdorff and locally connected, then the map

$$\begin{array}{rcl} \mathrm{Homeo}(F)^B & \to & \{\Phi \in \mathrm{Homeo}(B \times F) \,|\, p \circ \Phi = p\} \\ (\mu\colon B \to \mathrm{Homeo}(F)) & \mapsto & \begin{pmatrix} B \times F & \to & B \times F \\ (b, f) & \mapsto & (b, \mu(b)(f)) \end{pmatrix} \end{array}$$

is well-defined and it is a bijection.

CHAPTER 17

Mixing product and quotient topologies

In this, rather technical chapter, we will see that mixing product and quotient topologies can be quite delicate. In particular we will give rather disturbing examples of natural continuous bijections that are not homeomorphisms. Fortunately we will also give some criteria which let us circumvent these issues.

This chapter is organized as follows:

- In Section 17.1 we will provide an example which shows that mixing product and quotient topologies carelessly can lead to serious problems.
- In Section 17.2 we will state and discuss the Whitehead Quotient Theorem 17.6 which, assuming suitable adjectives, allows us to mix product and quotient topologies.
- In Section 17.3 we will provide a proof of the Whitehead Quotient Theorem 17.6.
- In Section 17.4 we will prove the Product-with-Quotient Map Proposition 17.14 which is a variation on the Whitehead Quotient Theorem 17.6.

In the next Chapter 18 we will introduce the notion of a homotopy between two continuous maps.

17.1. **Mixing product and quotient topologies: a disturbing example.** We start out with the following elementary lemma:

Lemma 17.1. (Product-Quotient Continuity Lemma) Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$. The following map is a bijection and it is continuous:

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y). \end{array}$$

Proof. It follows basically immediately from the definition of $\sim_\times$ that the map φ is a bijection. We will walk cautiously through the continuity argument.

We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ and $q\colon X \to X/\sim$ the two natural projections. We have the following commutative diagram:

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q\times \mathrm{id}_Y} \\ (X \times Y)/\sim_\times & \xrightarrow{\quad\varphi\quad} & (X/\sim) \times Y. \end{array}$$

Let $W \subseteq (X/\sim) \times Y$ be an open subset. We need to show that $\varphi^{-1}(W)$ is an open subset of $(X \times Y)/\sim_\times$. By Definition 8.3 of the quotient topology we need to show that $p^{-1}(\varphi^{-1}(W))$ is an open subset of $X \times Y$.

By the JH-Openness Criterion 3.3 and the Definition 7.3 of the product topology it suffices to prove the following claim:

Claim. For each $(x,y) \in p^{-1}(\varphi^{-1}(W)) \subseteq X \times Y$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood V_y of $y \in Y$ such that $U_x \times V_y \subseteq p^{-1}(\varphi^{-1}(W))$.

Proof. Since $([x], y) \in W$ and since W is an open subset of $(X/\sim) \times Y$ we obtain, by Definition 7.3 of the product topology, an open neighborhood $\widetilde{U}_{[x]}$ of $[x] \in X/\sim$ and an open neighborhood V_y of $y \in Y$ with $\widetilde{U}_{[x]} \times V_y \subseteq W$. We set $U_x \coloneqq q^{-1}(\widetilde{U}_{[x]})$. By Definition 8.3 of the quotient topology this is an open subset of X. Therefore it remains to show that $U_x \times V_y \subseteq p^{-1}(\varphi^{-1}(W))$. In other words, we need to show that $\varphi(p(U_x \times V_y)) \subseteq W$. This inclusion follows from the following argument:

$$\varphi(p(U_x \times V_y)) \;=\; (\varphi \circ p)(U_x \times V_y) \;\overset{\text{by the above commutative triangle}}{=}\; (q \times \mathrm{id}_Y)(U_x \times V_y) \;=\; q(q^{-1}(\widetilde{U}_{[x]})) \times V_y \;=\; \widetilde{U}_{[x]} \times V_y \subseteq W.$$

(since $U_x = q^{-1}(\widetilde{U}_{[x]})$)

This concludes the proof of the claim and thus also of the lemma. ∎

One would expect that the map from the Product-Quotient Continuity Lemma 17.1 surely must be a homeomorphism. But perhaps rather shockingly, the next lemma shows that in general this is not the case:

Lemma 17.2. (Product-Quotient–Non-Homeomorphism Lemma) There exist topological spaces X and Y and an equivalence relation $\sim$ on X with the following property: If we denote by $\sim_\times$ the equivalence relation on $X \times Y$ generated by $(x,y) \sim_\times (x',y)$ whenever $x \sim x'$, then the map

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x,y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and it is continuous, but it is *not* a homeomorphism.

Remark. We refer to [**Fab2005**, Theorem 1], [**BrownR2006**, p. 111] and [**Bou1998**, p. 128] for other examples which also show that mixing product and quotient topologies is in general a very dangerous idea. □

Proof. We consider $X = \mathbb{R} \times \mathbb{N} = \{(x,n) \,|\, x \in \mathbb{R}, n \in \mathbb{N}\}$, equipped with the product topology. Let $\sim$ be the equivalence relation on X that is generated by $(0,m) \sim (0,n)$ for all $m,n \in \mathbb{N}$. We set $Y \coloneqq \prod_{i \in \mathbb{N}} \mathbb{R}$, equipped with the product topology introduced in Definition 7.3. By the Product-Quotient Continuity Lemma 17.1 we know that the map

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x,y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and that it is continuous. It remains to show that φ is not a homeomorphism. We will do so by exhibiting an open subset V of $(X \times Y)/\sim_\times$ such that the image $\varphi(V)$ is not an open subset of $(X/\sim) \times Y$.

Before we can proceed we need to introduce the following notation:

(1) We denote by $q\colon X \to X/\sim$ the natural projection.

(2) We write $f = q \times \mathrm{id}_Y \colon X \times Y \to (X/\sim) \times Y$.
(3) Given $n \in \mathbb{N}$ we set $U_n \coloneqq \{((t,n),(y_1,y_2,y_3,\dots)) \in X \times Y \mid |t \cdot y_n| < 1\}$.
and we set $U \coloneqq \bigcup_{n\in\mathbb{N}} U_n$.

The key to proving the lemma is the following claim:

Claim 1. The set U has the following properties:
(1) U is an open subset of $X \times Y$,
(2) $f^{-1}(f(U)) = U$,
(3) $f(U)$ is not open in $(X/\sim) \times Y$.

Proof.

(1) We want to show that U is an open subset of $X \times Y$. It suffices to show that each U_n is open. Note that the map

$$\begin{aligned} \gamma \colon X \times Y = \mathbb{R} \times \mathbb{N} \times \prod_{i\in\mathbb{N}} \mathbb{R} &\to \mathbb{R} \\ (t,n,y_1,y_2,\dots) &\mapsto t \cdot y_n \end{aligned}$$

is continuous by the Topological-Product Proposition 7.2. Furthermore note that $\{n\}$ is an open subset of $\mathbb{N}$. Now we see that

$$U_n = \gamma^{-1}((-1,1)) \cap \underbrace{\mathbb{R} \times \{n\} \times \prod_{i\in\mathbb{N}} \mathbb{R}}_{\substack{\text{open by Definition 7.3} \\ \text{of the product topology}}}$$

is the intersection of two open subsets, thus it is open itself.
(2) We need to show that $f^{-1}(f(U)) = U$. By the Image-Preimage Lemma 1.9 (1) it remains to show that $f^{-1}(f(U)) \subseteq U$. In other words, we need to show that if v is an element of $X \times Y$ with $f(v) \in f(U)$, then $v \in U$. But this statement follows from the following claim:

Subclaim. Let $v = ((s,m),(y_1,y_2,\dots))$ and $w = ((t,n),(z_1,z_2,\dots))$ be two elements in $X \times Y$. If $f(v) = f(w)$ and if $w \in U$, then we also have $v \in U$.

Proof of subclaim. If $v = w$, then there is nothing to show. On the other hand, if $v \neq w$, then it follows from the definition of the map f and the equivalence relation $\sim$ that $s = t = 0$. But this implies that v and w lie in every U_n, in particular they both lie in U. ⊞

(3) We prove this statement by contradiction. Thus suppose that $f(U)$ is, contrary to our hopes, an open subset of $(X/\sim) \times Y$. Note that $((0,0),(0,0,\dots)) \in U$. This implies that $f((0,0),(0,0,\dots)) = ([(0,0)],(0,0,\dots))$ is contained in $f(U)$. By Definition 7.3 of the product topology of $(X/\sim)\times Y$ and of the product topology of $Y = \prod_{i\in\mathbb{N}} \mathbb{R}$ there exists an open subset $V \subseteq X/\sim$ and open subsets $W_i \subseteq \mathbb{R}$ with

$$([(0,0)],(0,0,\dots)) \in V \times \prod_{i\in\mathbb{N}} W_i \subseteq f(U)$$

and such that all but finitely W_i are equal to $\mathbb{R}$. We pick an $N \in \mathbb{N}$ with $W_N = \mathbb{R}$. Since $[(0,0)] \in V$ we see that $q^{-1}(V)$ is an open subset of $X = \mathbb{R} \times \mathbb{N}$ that contains the subset $q^{-1}([(0,0)]) = \{0\} \times \mathbb{N}$. In particular $q^{-1}(V)$ contains some point (s,N) with

$s \neq 0$. We pick a real number $y_N > \frac{1}{|s|}$ and we consider the point

$$P \;:=\; ((s,N),(0,\dots,0,\underset{\substack{\uparrow\\ N\text{-th entry}}}{y_N},0,\dots)) \;\in X \times Y.$$

Note that

$$P \underset{\substack{\uparrow\\ \text{since } (s,N)\in q^{-1}(V)\\ \text{and since } W_N=\mathbb{R}}}{\in} q^{-1}(V) \times \prod_{i\in\mathbb{N}} W_i \underset{\substack{\uparrow\\ \text{since } f=q\times \mathrm{id}_Y}}{=} f^{-1}\Big(V \times \prod_{i\in\mathbb{N}} W_i\Big) \underset{\substack{\uparrow\\ \text{see above}}}{\subseteq} f^{-1}(f(U)) \underset{\substack{\uparrow\\ \text{by (2)}}}{=} U = \bigcup_{n\in\mathbb{N}} U_n.$$

By definition of the U_n's we see that the only possibility is that $P \in U_N$. But we have $|s \cdot y_N| > 1$, i.e. P cannot possibly lie in U_N. We have obtained a contradiction. □

It remains to prove the following claim:

Claim 2. We set $V := p(U)$.
(1) V is an open subset of $(X \times Y)/\sim_\times$,
(2) $\varphi(V)$ is not an open subset of $(X/\sim) \times Y$.

Proof. We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ the natural projection. We have the following commutative diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad f=q\times \mathrm{id}_Y \quad} & (X/\sim) \times Y \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle \varphi} & \\ & (X \times Y)/\sim_\times\ . & \end{array}$$

Note that $$p^{-1}(V) \;=\; p^{-1}(p(U)) \underset{\substack{\uparrow\\ \text{since } p=\varphi^{-1}\circ f \text{ and}\\ \text{since } \varphi \text{ is a bijection}}}{=} f^{-1}(f(U)) \underset{\substack{\uparrow\\ \text{by the previous claim}}}{=} U \underset{\uparrow}{=} \text{an open subset of } X \times Y.$$

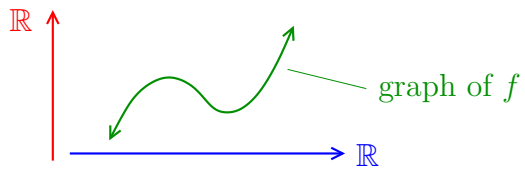
By the Definition 8.3 of the quotient topology on $(X \times Y)/\sim_\times$ this means that V is indeed an open subset of $(X \times Y)/\sim_\times$. Furthermore we have $\varphi(V) = \varphi(p(U)) = f(U)$. But by the previous claim we know that $f(U)$ is not an open subset of $(X/\sim) \times Y$. This concludes the proof of the claim and thus of the lemma. ■

17.2. **Whitehead's Quotient Theorem.** In this section we will state Whitehead's Quotient Theorem. It allows us to prove the Product-Quotient Mixing Lemma 17.8 which shows that the disturbing situation of the Product-Quotient Continuity Lemma 17.1 cannot arise if Y is regionally compact. In many applications later on we will consider the case $Y = [0,1]$ and the Product-Quotient Mixing Lemma 17.8 will ride to our rescue.

Definition 17.3. We say that a map $f\colon X \to Y$ between topological spaces is a quotient map if f is surjective and if a subset $U \subseteq Y$ is open in Y if and only if the preimage $f^{-1}(U)$ is open in X.

Examples.

(1) We consider the map $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is shown in the figure below.
This map is continuous, it is *not* an open map, but the reader is invited to verify that the map is in fact a quotient map.



(2) Let $X = \{0, 1\}$ be a set of two elements $0, 1$. Let $\mathcal{S} = \{\varnothing, X\}$ be the trivial topology on X and let $\mathcal{T} = \{\varnothing, X, \{0\}, \{1\}\}$ be the discrete topology on X. Clearly the identity map $(X, \mathcal{T}) \to (X, \mathcal{S})$ is a continuous bijection. But equally clearly it is not a quotient map. □

The following lemma gives us one of the main sources of quotient maps:

Lemma 17.4. (Projection–Quotient Map Lemma) Let X be a topological space and let $\sim$ be an equivalence relation on X. The projection $X \to X/\sim$ is a quotient map.

Proof. The lemma follows immediately from the Definition 8.3 of the quotient topology on the quotient $X/\sim$. ■

The following lemma summarizes a few elementary properties of quotient maps:

Lemma 17.5. (Quotient Map Lemma)

(1) Let $f\colon X \to Y$ be a quotient map, let Z be a topological space and let $g\colon Y \to Z$ be a map. The following two statements are equivalent:
- The map $g\colon Y \to Z$ is continuous.
- The map $g \circ f\colon X \to Z$ is continuous.

(2) If $f\colon X \to Y$ is a quotient map, then it is also continuous.
(3) If $f\colon X \to Y$ is a quotient map and a bijection, then f is a homeomorphism.

Proof.

(1) This statement follows easily from the definition of a quotient map.
(2) This statement follows immediately from (1) applied to $g = \mathrm{id}\colon Y \to Y$. Alternatively the statement follows immediately from the "only if" bit of the definition of a quotient map.
(3) By (2) we know that the quotient map $f\colon X \to Y$ is continuous. Now let $g\colon Y \to X$ be the inverse of $f\colon X \to Y$. Since $g \circ f = \mathrm{id}_X$ is continuous we now obtain from (1) that g is also continuous. ■

The following is the key technical theorem which in most reasonable situations will help us to navigate the choppy waters involving product and quotient maps.

Theorem 17.6. (Whitehead Quotient Theorem) Let $f\colon X \to Z$ be a continuous map between topological spaces that is a quotient map. Furthermore let Y be a topological space. If Y is regionally compact (e.g. if $Y = [0, 1]$), then the map

$$f \times \mathrm{id}_Y\colon X \times Y \;\to\; Z \times Y$$

is also a quotient map.

Remark. This theorem is sometimes just called the "Whitehead Theorem" since it was first proved by John Whitehead [**WhdJ1948**, Lemma 4] in 1948. For disambiguation

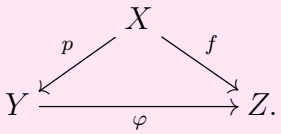
purposes we refer to it as the Whitehead Quotient Theorem and we reserve the name "Whitehead Theorem" for Theorem 305.1. □

Proof. We will provide the proof of the Whitehead Quotient Theorem 17.6 in the next section. ■

In the following we first state and prove some useful consequences of the Whitehead Quotient Theorem 17.6. In fairness to the reader one should perhaps say that it is arguably best to move on to the next chapter. It is only advisable to read the remainder of this section once one really faces acute technical problems involving the mix of product and quotient maps.

The following little lemma is the key to many applications of the Whitehead Quotient Theorem 17.6.

Lemma 17.7. (Quotient Map-Composition Lemma) Suppose we are given a commutative diagram of maps between topological spaces:



The following two statements hold:

(1) If p and f are quotient maps, then φ is also a quotient map.
(2) If p is a quotient map and if f is continuous, then φ is continuous.

Proof.

(1) Suppose that p and f are quotient maps. Let $V \subseteq Z$ be an open subset. We have the following implications:

$$V \text{ is open} \underset{\text{since } f \text{ is a quotient map}}{\overset{\uparrow}{\iff}} f^{-1}(V) \text{ is open} \underset{\text{since } f=\varphi\circ p \text{ we have } f^{-1}(V)=p^{-1}(\varphi^{-1}(V))}{\overset{\uparrow}{\iff}} p^{-1}(\varphi^{-1}(V)) \text{ is open} \underset{\text{since } p \text{ is a quotient map and since } p(p^{-1}(\varphi^{-1}(V)))=\varphi^{-1}(V)}{\overset{\uparrow}{\iff}} \varphi^{-1}(V) \text{ is open.}$$

Furthermore, it follows from the fact that the diagram commutes and the fact that f is surjective, that φ is surjective. We have thus verified that φ is a quotient map.

(2) Now we suppose that p is a quotient map and that f is continuous. We need to show that given an open subset $V \subseteq Z$ the preimage $\varphi^{-1}(V)$ is open. The argument is basically the same as in (1), except that the first arrow is only a right arrow, since we only assume that f is continuous. ■

We have the following addendum to the Product-Quotient Continuity Lemma 17.1.

Lemma 17.8. (Product-Quotient Mixing Lemma) Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$. If Y is regionally compact, then the following map is a homeomorphism:

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y). \end{array}$$

Old fashioned proof of the Product-Quotient Mixing Lemma 17.8. As we pointed out in the proof of the Product-Quotient Continuity Lemma 17.1, it is basically clear that

φ is a bijection. By the Quotient Map Lemma 17.5 (3) it remains to show that φ is a quotient map. We denote by $p\colon X\times Y\to (X\times Y)/\sim_\times$ and $q\colon X\to X/\sim$ the two natural projections. We have the following commutative diagram:

$$\begin{array}{ccc} & X\times Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle f=q\times \mathrm{id}_Y} \\ (X\times Y)/\sim_\times & \xrightarrow{\quad\varphi\quad} & (X/\sim)\times Y. \end{array}$$

By the Projection–Quotient Map Lemma 17.4 we know that p and q are quotient maps. Since Y is regionally compact we obtain from the Whitehead Quotient Theorem 17.6 that $q\times \mathrm{id}_Y$ is a quotient map. It follows from the Quotient Map-Composition Lemma 17.7 (1) that φ is also a quotient map. ■

Modern proof of the Product-Quotient Mixing Lemma 17.8. Let $R\subseteq X\times X$ be the given equivalence relation on X. We denote by $\mathrm{pr}_1\colon R\to X$ and $\mathrm{pr}_2\colon R\to X$ the two natural projections. By the Topological Relation–Coequalizer Lemma 8.19 (2) the natural projection $X\to X/\sim$ induces a homeomorphism

$$\operatorname*{colim}\left(R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X \right) \xrightarrow{\cong} X/\sim.$$

Next we consider the functor

$$\begin{array}{rcl} F\colon \mathcal{T}op & \to & \mathcal{T}op \\ Z & \mapsto & Z\times Y. \end{array}$$

Since Y is regionally compact we know by the Cylinder–Path Space Adjunction Lemma 16.11 that this functor is a left adjoint. It thus follows from the Left Adjoint–Colimit Proposition 6.35 that the above functor preserves colimits. Thus we see that the projection $X\times Y\to (X/\sim)\times Y$ induces a natural homeomorphism

$$\operatorname*{colim}\left(R\times Y \underset{\mathrm{diag}(\mathrm{pr}_2,\mathrm{id}_Y)}{\overset{\mathrm{diag}(\mathrm{pr}_1,\mathrm{id}_Y)}{\rightrightarrows}} X\times Y \right) \xrightarrow{\cong} (X/\sim)\times Y.$$

On the other hand it follows easily from the Topological Relation–Coequalizer Lemma 8.19 (1) and (2) that the projection $X\times Y\to (X\times Y)/\sim_\times$ induces a a natural homeomorphism

$$\operatorname*{colim}\left(R\times Y \underset{\mathrm{diag}(\mathrm{pr}_2,\mathrm{id}_Y)}{\overset{\mathrm{diag}(\mathrm{pr}_1,\mathrm{id}_Y)}{\rightrightarrows}} X\times Y \right) \xrightarrow{\cong} (X\times Y)/\sim_\times .$$

These two homeomorphisms put together give us the desired result. ■

Later on we will use the following special case of the Product-Quotient Mixing Lemma 17.8.

Lemma 17.9. (Gluing-Product-Homeomorphism Lemma) Let X and Y be two topological spaces, let $A\subseteq X$ be a subset and let $f\colon A\to Y$ be a continuous map. We consider the inclusion map $A\times[0,1]\to X\times[0,1]$ and the map $f\times\mathrm{id}\colon A\times[0,1]\to Y\times[0,1]$ and we form the pushout $(X\times[0,1])\cup_{A\times[0,1]}(Y\times[0,1])$. The map

$$\begin{array}{rcl} (X\times[0,1])\cup_{A\times[0,1]}(Y\times[0,1]) & \to & (X\cup_f Y)\times[0,1] \\ [(P,t)] & \mapsto & ([P],t) \end{array}$$

is a homeomorphism.

Proof. Once one unravels the definitions it is clear that this statement follows immediately from the Product-Quotient Mixing Lemma 17.8 together with the observation that $[0,1]$ is regionally compact. ■

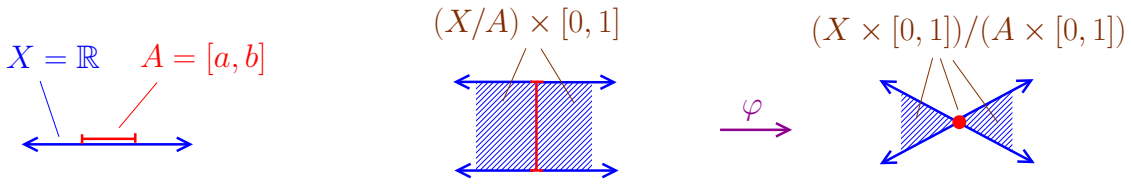
We conclude this list of applications of the Whitehead Quotient Theorem 17.6 with the following proposition that gives a fairly explicit illustration of how the Whitehead Quotient Theorem 17.6 can be used.

Proposition 17.10. (Product–Quotient Map Proposition) Let X be a topological space, let A be a subset of X and let Y be a topological space that is regionally compact. Then the map

$$\begin{array}{rcl} \varphi\colon (X/A)\times Y & \to & (X\times Y)/(A\times Y) \\ ([x],y) & \mapsto & [(x,y)] \end{array}$$

is a quotient map, in particular it is continuous.

Example. In the figure below we illustrate the map studied in the Product–Quotient Map Proposition 17.10 in the case that $X=\mathbb{R}$, $A=[a,b]$ and $Y=[0,1]$.



At least in the figure it seems clear that the map is continuous. □

Just for fun we provide two proofs of the Product–Quotient Map Proposition 17.10. The first one uses the Whitehead Quotient Theorem 17.6 and the second one makes use of the Left Adjoint–Colimit Proposition 6.35.

First proof of the Product–Quotient Map Proposition 17.10. In the following we denote by $p\colon X\times Y\to (X\times Y)/(A\times Y)$ and $q\colon X\to X/A$ the two natural projections. We have the following commutative diagram:

$$\begin{array}{ccc} & X\times Y & \\ {\scriptstyle q\times \mathrm{id}_Y}\swarrow & & \searrow{\scriptstyle p} \\ (X/A)\times Y & \xrightarrow{\quad\varphi\quad} & (X\times Y)/(A\times Y). \end{array}$$

By the Projection–Quotient Map Lemma 17.4 the maps p and q are quotient maps. Since Y is regionally compact we obtain from the Whitehead Quotient Theorem 17.6 that $q\times \mathrm{id}_Y$ is a quotient map. It follows from the Quotient Map-Composition Lemma 17.7 that φ is also a quotient map. By the Quotient Map Lemma 17.5 (2) this implies that φ is indeed continuous. ■

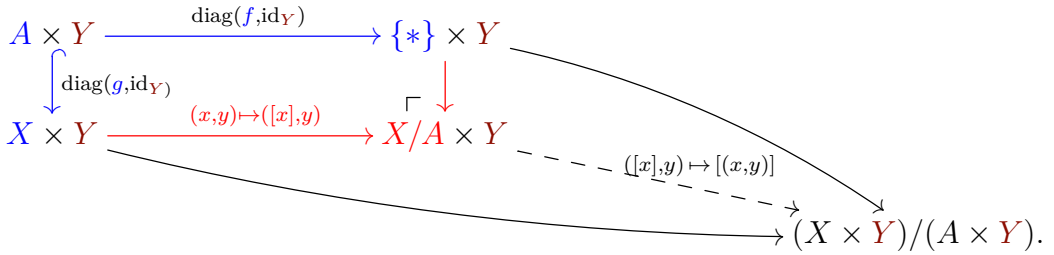
Second proof of the Product–Quotient Map Proposition 17.10. We start out with the observation that it follows from the discussion on page 318 that

$$\begin{array}{ccc} A & \xrightarrow{\quad f\quad} & \{*\} \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{\quad x\mapsto[x]\quad} & X/A. \end{array}$$

is a pushout diagram in the sense of Definition 6.23. Next we consider the functor

$$\begin{array}{rcl} F\colon \mathcal{T}op & \to & \mathcal{T}op \\ Z & \mapsto & Z \times Y. \end{array}$$

Since Y is regionally compact we know by the Cylinder–Path Space Adjunction Lemma 16.11 that this functor is a left adjoint. It thus follows from the Left Adjoint–Colimit Proposition 6.35 that the above functor preserves colimits. In particular it preserves pushouts. Thus we obtain the pushout rectangle in the following diagram:



The upper and lower diagonal map are the obvious maps and we know by the Topological-Quotient Proposition 8.2 (1a) that they are continuous. It now follows from the Topological-Pushout Proposition 8.12 (1b) that the middle diagonal map is, as promised, continuous. ∎

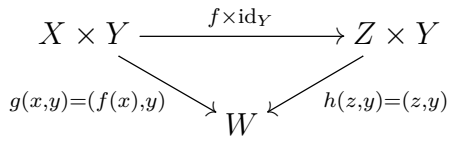
17.3. **Proof of the Whitehead Quotient Theorem 17.6.** Now that we procrastinated for as long as possible, we will finally run out of reasons not to turn to the proof of the Whitehead Quotient Theorem 17.6. In fact we provide two proofs.

The first proof of the Whitehead Quotient Theorem 17.6 cleverly reduces the proof to the Compact-Open Exponential Proposition 16.10. This approach pushes all the set-theoretic nastiness into the proof of the Compact-Open Exponential Proposition 16.10. The second proof of the Whitehead Quotient Theorem 17.6 is a bare-hands argument which can be read independently of the previous section on the compact-open topology.

Proof. First proof of Theorem 17.6, using the Compact-Open Exponential Proposition 16.10] We write $W = Z \times Y$ and we equip W with the topology that turns

$$\begin{array}{rcl} g\colon X \times Y & \to & W \\ (x,y) & \mapsto & (f(x),y) \end{array}$$

into a quotient map. In other words, as a set we have $W = Z \times Y$, but this time equipped with the topology that is defined by the condition that $U \subseteq W$ is open if and only if $g^{-1}(U)$ is open. Note that with this topology on W the map g is in particular continuous. Next we consider the following diagram:

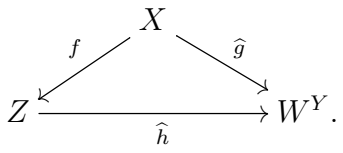


where $h\colon Z \times Y \to W$ is the identity. Since g is by definition a quotient map it remains to show that h is a homeomorphism. Evidently h is a bijection. Furthermore note that every open subset in $Z \times Y$ is open in W since $f \times \mathrm{id}_Y$ is continuous. Thus it remains to show that h is continuous.

We consider the maps

$$\begin{array}{rcl} \widehat{g}\colon X & \to & W^Y \\ x & \mapsto & \begin{pmatrix} Y & \to & W \\ y & \mapsto & g(x,y) \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widehat{h}\colon Z & \to & W^Y \\ z & \mapsto & \begin{pmatrix} Y & \to & W \\ y & \mapsto & h(z,y) \end{pmatrix} \end{array}$$

These maps fit into the following commutative diagram:

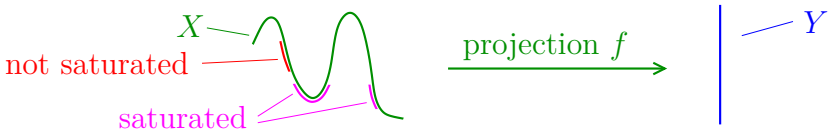


We make the following observations:

(1) Since $g\colon X \times Y \to W$ is continuous and since Y is by hypothesis regionally compact we obtain from the Compact-Open Exponential Proposition 16.10 that $\widehat{g}\colon X \to W^Y$ is continuous.
(2) By our hypothesis the map f is a quotient map.
(3) It follows from (1) and (2), together with the Quotient Map-Composition Lemma 17.7 (2), that the map $\widehat{h}\colon Z \to W^Y$ is continuous.
(4) It follows from the Compact-Open Exponential Proposition 16.10 and (3) that h is indeed continuous. ■

Our second proof of the Whitehead Quotient Theorem 17.6 is independent of earlier results. But it does mean that we will have to get our hands somewhat dirty.

Definition 17.11. Let $f\colon X \to Y$ be a continuous map between two sets. We say that $U \subseteq X$ is saturated with respect to f if $U = f^{-1}(f(U))$. If f is understood, then we just say that U is saturated.



Remark. Let $f\colon X \to Y$ be a continuous map between two sets.

(1) Let $U \subseteq Y$ be a subset. Note that by the Image-Preimage Lemma 1.9 (1) we know that $U \subseteq f^{-1}(f(U))$. Thus U is saturated if and only if $f^{-1}(f(U)) \subseteq U$.
(2) If f is an injection, then it follows from the Image-Preimage Lemma 1.9 (1) that every subset of X is saturated. □

The following is, besides the Image-Preimage Lemma 1.9, the only purely set-theoretic lemma of these notes.

Lemma 17.12. (Saturation Lemma) Let $f\colon X \to Z$ be a continuous map between two sets and let Y be another set. Let $\widetilde{W}$ be a subset of $X \times Y$ that is saturated with respect to the map $f \times \mathrm{id}_Y\colon X \times Y \to Z \times Y$. Let $K \subseteq Y$. We consider the set

$$U \;:=\; \{x' \in X \mid \{x'\} \times K \subseteq \widetilde{W}\}.$$

The set U is saturated with respect to f.

Proof. We write $h = f \times \mathrm{id}_Y$. By the above we need to show that $f^{-1}(f(U)) \subseteq U$. Thus let $x \in f^{-1}(f(U)) \subseteq X$. We have

$$h(\{x\}\times K) \underset{\uparrow}{=} (f\times \mathrm{id}_Y)(\{x\}\times K) = \{f(x)\}\times K \underset{\uparrow}{=} \{f(x')\}\times K \underset{\uparrow}{=} h(\{x'\}\times K) \underset{\uparrow}{\subseteq} h(\widetilde{W}).$$

definition of h — since $f(x) \in f(U)$ we know that $f(x) = f(x')$ for some $x' \in U$ — definition of $h = f \times \mathrm{id}_Y$ — since $x' \in U$

It follows that

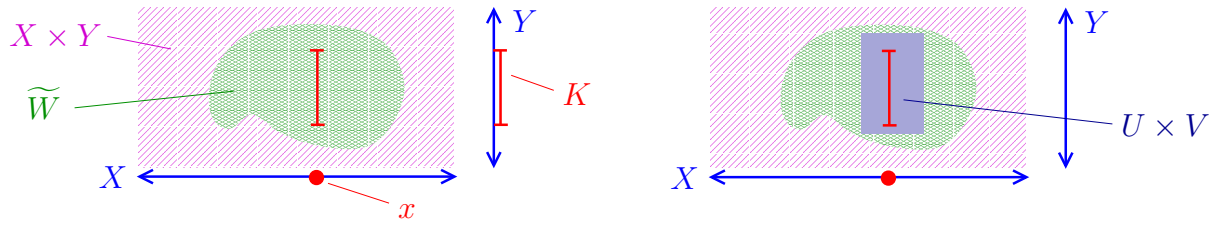
$$\{x\} \times K \underset{\uparrow}{\subseteq} h^{-1}(h(\{x\} \times K) \underset{\uparrow}{\subseteq} h^{-1}(h(\widetilde{W})) \underset{\uparrow}{=} \widetilde{W}.$$

by the Image-Preimage Lemma 1.9 (1) — by the above argument — since $\widetilde{W}$ is saturated

We have thus shown that $x \in U$. ∎

We will also need the following lemma which is again in the realm of topological spaces. It can be viewed as a minuscule generalization of the Tube Lemma 7.12.

Lemma 17.13. (Tube Lemma II) Let X and Y be topological spaces, let $x \in X$ and let $K \subseteq Y$ be a compact subset. Let $\widetilde{W} \subseteq X \times Y$ be an open subset that contains $\{x\} \times K$. There exist open neighborhoods $U \subseteq X$ of x and $V \subseteq Y$ of K such that $\{x\} \times K \subseteq U \times V \subseteq \widetilde{W}$.



Proof of the Tube Lemma 17.13. Since $\widetilde{W} \subseteq X \times Y$ is open and since $\{x\} \times K \subseteq \widetilde{W}$ we can find for any $k \in K$ open neighborhoods A_k of $x \in X$ and B_k of $k \in Y$ with $A_k \times B_k \subseteq \widetilde{W}$. Since K is compact there exist $k_1, \ldots, k_m \in K$ such that $K \subseteq V := B_{k_1} \cup \cdots \cup B_{k_m}$. We define $U := A_{k_1} \cap \cdots \cap A_{k_m}$. Since U is the intersection of finitely many open subsets each of which contain x we see that U is an open neighborhood of x. Clearly we have the inclusions $\{x\} \times K \subseteq U \times V \subseteq \widetilde{W}$. ∎

For convenience we remind the reader of the following lemma:

Lemma 3.3 (JH-Openness Criterion). Let X be a topological space and let $U \subseteq X$ be a subset. If given any $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.

Now we can provide our alternative proof of the Whitehead Quotient Theorem 17.6.

Second proof of Theorem 17.6. Let $f\colon X \to Z$ be a continuous map between topological spaces that is a quotient map. Furthermore let Y be a topological space that is regionally compact. We need to show that the map

$$f \times \mathrm{id}_Y \colon X \times Y \to Z \times Y$$

is also a quotient map.

Since f is a quotient map we know by the Quotient Map Lemma 17.5 (2) that f is continuous. Therefore it follows from the Topological-Product Proposition 7.2 (1) that the map $f \times \mathrm{id}_Y$ is continuous. Therefore it now remains to show that if $W \subseteq Z \times Y$ is a subset

such that the preimage $\widetilde{W} \coloneqq (f \times \mathrm{id}_Y)^{-1}(W)$ is an open subset of $X \times Y$, then W itself is open.

We intend to use the JH-Openness Criterion 3.3 to show that W is open. Therefore let $(z, y) \in W$. We pick $x_0 \in f^{-1}(z)$. We equip $\{x_0\} \times Y \subseteq X \times Y$ with the subspace topology. It follows from the Product Topology-Basics Lemma 7.6 (1a) and (1b) that the map

$$\begin{array}{rcl} i\colon Y & \to & \{x_0\} \times Y \\ y & \mapsto & (x_0, y) \end{array}$$

is an embedding. In particular $Y_{x_0} \coloneqq i^{-1}(\widetilde{W} \cap (\{x_0\} \times Y))$ is an open subset of Y. Note that $\{x_0\} \times Y_{x_0} \subseteq \widetilde{W}$. Since Y is regionally compact we can find a compact neighborhood K_{x_0} of x_0 that is contained in Y_{x_0}. By definition of a neighborhood there exists an open subset Y'_{x_0} of Y with $x_0 \in Y'_{x_0} \subseteq K_{x_0}$. We define

$$U_{x_0} \coloneqq \{x \in X \mid \{x\} \times K_{x_0} \subseteq \widetilde{W}\}.$$

Claim 1. The set $f(U_{x_0})$ is an open subset of Z.

Proof. First we intend to use the JH-Openness Criterion 3.3 to show that U_{x_0} is indeed an open subset of Y_{x_0}. Thus let $x \in U_{x_0}$. Since $\widetilde{W}$ is an open subset of $X \times Y$ and since K_{x_0} is compact it follows from the Tube Lemma 17.13 that there exist open subsets $A \subseteq X$ and $B \subseteq Y$ with $x \in A$ and $K_{x_0} \subseteq B$ such that $A \times B \subseteq \widetilde{W}$. Evidently this implies that $A \subseteq U_{x_0}$. Since $x \in A$ we have verified that the openness criterion of the JH-Openness Criterion 3.3 is satisfied. We have thus shown that U_{x_0} is an open subset of Y_{x_0}.

Now we want to show that $V \coloneqq f(U_{x_0})$ is an open subset of Z. By the Saturation Lemma 17.12 we know that U_{x_0} is saturated with respect to f. In other words, we know that $U_{x_0} = f^{-1}(V)$. Now we use that by hypothesis $f\colon X \to Z$ is a quotient map. Since $f^{-1}(V) = U_{x_0}$ is open by (1) we obtain that V itself is open. But that is what we had wanted to show. ⊞

By Definition 7.3 of the product topology we obtain that $f(U_{x_0}) \times Y'_{x_0}$ is an open subset of $X \times Y$. Evidently we have $(x, y) \in f(U_{x_0}) \times Y'_{x_0}$. It remains to prove the following claim:

Claim 2. We have $f(U_{x_0}) \times Y'_{x_0} \subseteq W$.

Proof. The claim follows from the following equalities and inclusions:

$$f(U_{x_0}) \times Y'_{x_0} = (f \times \mathrm{id}_Y)(U_{x_0} \times Y'_{x_0}) \underset{\text{since } Y'_{x_0} \subseteq K_{x_0}}{\subseteq} (f \times \mathrm{id}_Y)(U_{x_0} \times K_{x_0}) \overset{\text{definition of } U_{x_0}}{\subseteq} (f \times \mathrm{id}_Y)(\widetilde{W}) \underset{\text{since by definition } \widetilde{W} = (f \times \mathrm{id}_Y)^{-1}(W)}{=} W.$$

by the Image-Preimage Lemma 1.9 (2)

■

Remark. A different proof of the Whitehead Quotient Theorem 17.6, based on the Evaluation Map Proposition 16.9 and the Compact-Open Exponential Proposition 16.10, is given in [**Hat2002**, Lemma A.17]. □

17.4. **Mixing product and quotient topologies II.** In this section we prove a second statement regarding the interplay between product and quotient topologies. More precisely, we will prove the following proposition:

Proposition 17.14. (Product-with-Quotient Map Proposition) Let X and Y be topological spaces. Let B be a subset of Y. We denote by $q\colon Y \to Y/B$ the projection. If B is a compact subset of Y and if Y is Hausdorff, then the map

$$\begin{array}{rccc} \mathrm{id}_X \times q\colon & X \times Y & \to & X \times (Y/B) \\ & (x,y) & \mapsto & (x,[y]) \end{array}$$

is a quotient map.

Remark.

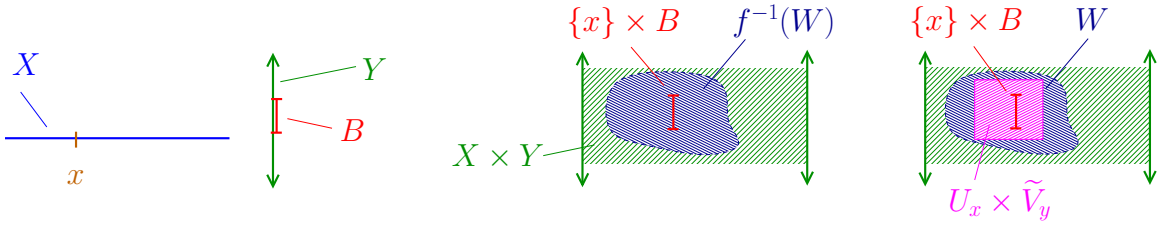
(1) At first glance the statement of the Product-with-Quotient Map Proposition 17.14 might sound quite similar to the statement of the Whitehead Quotient Theorem 17.6. But in a way it is quite different, this time there are no conditions at all on the "unaffected" factor X, but we do put a condition on the projection $Y \to Y/B$.
(2) According to [**tD2008**, Chapter 2.2, Exercise 16] one does not need to assume that Y is Hausdorff.
(3) Finally we refer to [**BrownR2006**, Chapter 4.3] for more information on the interplay between product and quotient topologies. □

In the proof of the Product-with-Quotient Map Proposition 17.14 we will need the Tube Lemma 17.13. For the reader's convenience we recall it with the notation that we will needs:

Proof. Let X and Y be topological spaces and let B be a compact subset of Y. We assume that Y is Hausdorff. We denote by $q\colon Y \to Y/B$ the natural projection. Note that $p = \mathrm{id}_X \times q$. First note that it follows from the Topological-Quotient Proposition 8.2 (1b) and the Topological-Product Proposition 7.2 (1) that p is continuous. This implies the "only if"-direction. It remains to show that "if"-direction. Thus let $W \subseteq X \times (Y/B)$ be a subset that has the property that $p^{-1}(W)$ is an open subset of $X \times Y$. We need to show that W is an open subset of $X \times (Y/B)$. By the JH-Openness Criterion 3.3 and the Definition 7.3 of the product topology it suffices to prove the following claim:

Claim. For each $(x,[y]) \in W \subseteq X \times (Y/B)$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood $V_{[y]}$ of $[y] \in Y/B$ such that $U_x \times V_{[y]} \subseteq W$.

Proof. As in the proof of the Product–Quotient Map Proposition 17.10 we consider two cases separately. First we consider the case that $[y] = [B] \in Y/B$. In this case we see that $p^{-1}((x,[y])) = \{x\} \times B$. By hypothesis $p^{-1}(W)$ is an open subset of $X \times Y$ that contains $p^{-1}((x,[y])) = \{x\} \times B$. Therefore it follows from the Tube Lemma 17.13 that there exists an open subset U_x of X and an open subset $\widetilde{V}_y \subseteq Y$ with $\{x\} \times B \subseteq U_x \times \widetilde{V}_y \subseteq p^{-1}(W)$.



We denote by $V_{[y]}$ the image of $\widetilde{V}_y$ under the projection $q\colon Y \to Y/B$. Note that $V_{[y]}$ is an open neighborhood of $[y] \in Y/B$.[212] Finally note that

$$U_x \times V_{[y]} \;=\; (\mathrm{id}_X \times q)(U_x \times \widetilde{V}_y) \;=\; p(U_x \times \widetilde{V}_y) \subseteq p(p^{-1}(W)) \;=\; W.$$

It remains to consider the case that $[y] \neq [B] \in Y/B$. We see that $(x, y) \in W$. Since B is compact and since Y is Hausdorff we know by the Compact-Closed Lemma 3.39 that B is a closed subset of Y, i.e. $Y \setminus B$ is open. It follows immediately from the Definition 7.3 of the product topology on $X \times B$, the fact that $Y \setminus B$ is open, the fact that $p^{-1}(W)$ is an open neighborhood of (x, y) and the fact that $y \notin B$ that there exists an open neighborhood U_x of x and an open neighborhood $\widetilde{V}_y$ of y with $\widetilde{V}_y \subseteq X \setminus B$ and with $U_x \times \widetilde{V}_y \subseteq p^{-1}(W)$. One easily verifies that U_x and $V_y = q(\widetilde{V}_y)$ have the desired properties. ■

17.5. Exercises for Chapter 17.

Exercise 17.1. Let $f\colon X \to Y$ be a map between topological spaces. We set $A := f(X)$. Show that the following two statements are equivalent:

(1) The map $f\colon X \to A$ is a quotient map.
(2) A subset $U \subseteq Y$ is open in Y (with respect to the subspace topology) if and only if the preimage $f^{-1}(U)$ is open in X.

[212] Indeed, since $B \subseteq \widetilde{V}_y$ we see that $q^{-1}(V_{[y]}) = q^{-1}(q(\widetilde{V}_y)) = \widetilde{V}_y \cup B = \widetilde{V}_y$, which is an open subset. By the Definition 8.3 of the quotient topology on Y/B we see that $q^{-1}(V_{[y]})$ is indeed open.

Part III

Basics of homotopy theory

Loosely speaking two continuous maps $f_0, f_1 \colon X \to Y$ between topological spaces are called homotopic if f_0 can be "continuously deformed" into f_1. This notion is quite natural and plays an absolutely essential role throughout these lecture notes. In this part of the lecture notes we define this notion properly and we discuss basic properties and examples. We also introduce the somewhat less intuitive notion of what it means for two topological spaces to be homotopy equivalent.

This part of the lecture notes is organized as follows:

- In Chapter 18 we will give a proper definition of what it means for two maps to be homotopic and we will discuss basic properties and many examples.
- In Chapter 19 we will use the notion of homotopies between maps to define what it means for two topological spaces to be homotopy equivalent. Again we will discuss basic properties of this notion and many examples.
- In Chapter 20 we will introduce the join $X * Y$ of two topological spaces and we will introduce the cone $\mathrm{Cone}(X)$ and the suspension $\Sigma(X)$ of a topological space. We will also study some basic (homotopy) properties.
- In Chapter 21 we will associate to a continuous map $f \colon X \to Y$ between two topological spaces three topological spaces, namely the mapping cylinder $\mathrm{Cyl}(f \colon X \to Y)$, the mapping cone $\mathrm{Cone}(f \colon X \to Y)$ and the mapping torus $\mathrm{Tor}(f \colon X \to Y)$. These constructions allow us to turn questions about continuous maps into questions about topological spaces. All these constructions play an important role throughout these lecture notes. But as a standalone chapter this chapter is not particularly interesting.
- In Chapter 22 we will introduce the notion of a cofibration which will appear here and there. Again, as for the previous chapter, as a topic on its own, it is not overly fun to read.

CHAPTER 18

Homotopies

Loosely speaking two continuous maps $f, g\colon X \to Y$ between topological spaces are called homotopic if one can be deformed "continuously" into the other. In the subsequent chapter we will use to this notion of homotopic maps to explain what it means for two topological spaces to be "homotopy equivalent".

This chapter is organized as follows:

- In Section 18.1 we will define what it means for two maps $f_0, f_1\colon X \to Y$ between topological spaces to be homotopic.
- In Section 18.2 we will use the Whitehead Quotient Theorem 17.6 to prove the Homotopy Quotient Theorem 18.11 which shows that homotopies descend to homotopies on quotients and pushouts.
- In Section 18.3 we will provide exercises for Chapter 18.

In the next Chapter 19 we will introduce the notion of a homotopy equivalence.

18.1. **Homotopic maps.** In its most basic form a homotopy is a continuous deformation of continuous maps between two topological spaces. To make this notion precise it is convenient to introduce the following notation:

Notation 18.1. Let X and Y be two topological spaces and let $I \subseteq \mathbb{R}$ be a subset. Given a continuous map $F\colon X \times I \to Y$ and given $t \in I$ we denote by F_t the continuous map[213]

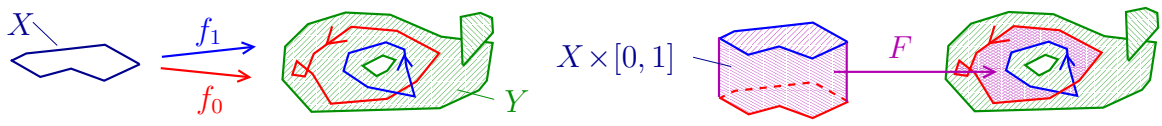
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$

Now we move on to the actual definition we are interested in:

Definition 18.2. Let X and Y be two topological spaces.

(1) Let $f_0, f_1\colon X \to Y$ be two continuous maps. A homotopy between the maps f_0 and f_1 is a continuous map $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are homotopic and sometimes we write $f_0 \simeq f_1$.

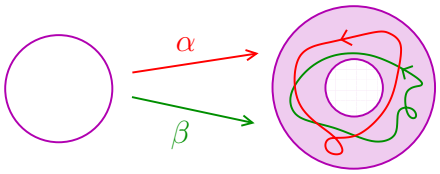
(2) A continuous map $f\colon X \to Y$ is called null-homotopic if it is homotopic to a constant map.



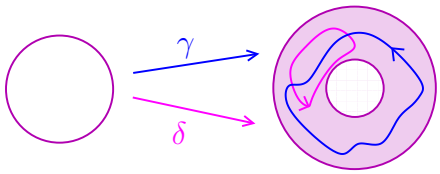
[213]Note that it follows from the Product Topology-Basics Lemma 7.6 (1∗) that the map F_t is indeed continuous.

Examples.

(1) In the figure below we show four maps α, β, γ and δ from the circle S^1 to the annulus $Y = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic whereas the maps γ and δ look as if they are not homotopic. But only much later, namely in Chapter 91, we will develop methods for showing that these maps are indeed not homotopic.



the maps α and β are homotopic



the maps γ and δ appear not to be homotopic

(2) Given an orthogonal matrix $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the map $S^{n-1} \to S^{n-1}$ that is given $x \mapsto A \cdot x$.

(a) Let $\gamma\colon [0,1] \to \mathrm{SO}(n)$ be a path from some matrix A to some matrix B. The map[214]

$$\begin{array}{rcl} S^{n-1} \times [0,1] & \to & S^{n-1} \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$

is a homotopy from $\rho(A)$ to $\rho(B)$.

(b) Let $A \in \mathrm{O}(n)$. If $\det(A) = 1$, i.e. if $A \in \mathrm{SO}(n)$, then the Matrix Group-π_0-Proposition 4.54 (3) provides us with a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ from $\gamma(0) = A$ to $\gamma(1) = \mathrm{id}$. It follows from (a) that $\rho(A)\colon S^{n-1} \to S^{n-1}$ is homotopic to the identity. Much later, on page 2878, we will see that if $\det(A) = -1$, then f is actually not homotopic to the identity.

(3) It follows immediately from the definitions that a topological space X is path-connected if and only if any two maps $\{*\} \to X$ are homotopic.

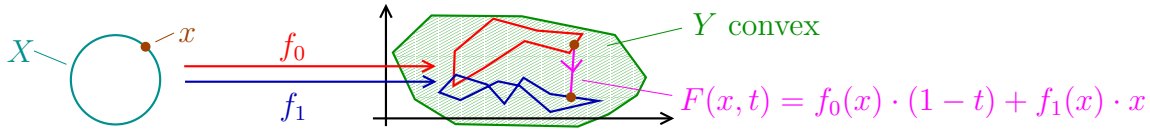
(4) Let Y be a topological space and let $X \subseteq Y$ be some subspace of Y. The homotopy $F\colon X \times [0,1] \to Y$ that is given by $F_t(x,t) = x$ is sometimes called a *stationary homotopy*. □

The next example of a homotopy is written down as a lemma.

Lemma 18.3. (Convex Homotopy Lemma) Let X be a topological space. Given any two continuous maps $f_0, f_1\colon X \to Y$ to a *convex* subset $Y \subseteq \mathbb{R}^n$ the map

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \underbrace{f_0(x) \cdot (1-t) + f_1(x) \cdot t}_{\in Y \text{ since } Y \text{ is convex}} \end{array}$$

is a homotopy between f_0 and f_1.



[214] The argument of the proof of the Torus Embedding Lemma 7.11 shows that the map $S^{n-1} \times [0,1] \to S^{n-1}$ is continuous.

Proof. Basically by definition we have $F_0 = f_0$ and $F_1 = f_1$. But we are not done yet, the conscientious reader will not have failed to notice that we need to show that the map F is actually continuous. The map F can be written as the composition of the following two maps:

$$X \times [0,1] \xrightarrow{(x,t)\mapsto(f_0(x),f_1(x),t)} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow{(x,y,t)\mapsto(x\cdot(1-t)+y\cdot t)} \mathbb{R}^n.$$

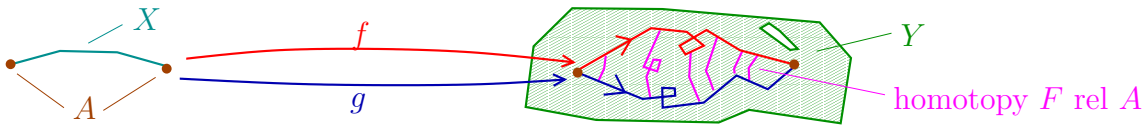
It follows from the Topological-Product Proposition 7.2 (1) that the map to the left is continuous. As in the proof of the Torus Embedding Lemma 7.11 we see that the map to the right is continuous. ■

Often it is more interesting to consider homotopies that fix a given subset. This leads us to the following definition:

Definition 18.4. Let X and Y be topological spaces and let $A \subseteq X$ be a subset.

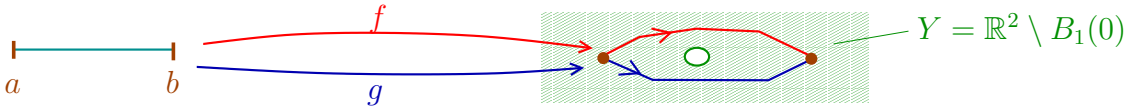
(1) We say that a continuous map $F\colon X \times [0,1] \to Y$ is a homotopy rel A if for all $a \in A$ and $s, t \in [0,1]$ we have $F(a,s) = F(a,t)$.

(2) We say that two continuous maps $f, g\colon X \to Y$ with $f|_A = g|_A$ are homotopic rel A if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

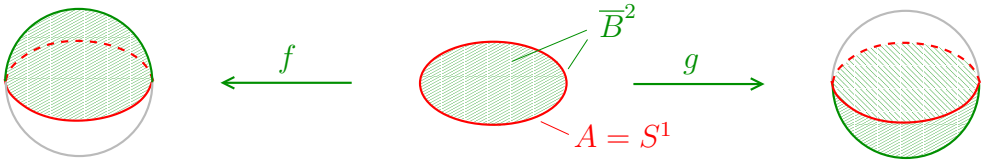


Examples.

(1) Let $f, g\colon [a,b] \to Y = \mathbb{R}^2 \setminus B_1(0)$ be the two continuous maps from an interval to the topological space Y that are shown in the figure below. Note that $f(a) = g(a)$ and $f(b) = g(b)$. As we will see in Exercise 18.1, any two continuous maps from an interval to a path-connected space are homotopic. On other hand it looks rather unlikely that the two maps are homotopic rel $\{a,b\}$, i.e. it does not look like we can deform f into g while keeping the endpoints of the interval fixed. Unfortunately we will have to wait till Chapter 91 to develop tools for showing that these two maps f and g are not homotopic rel $\{a,b\}$.



(2) We consider the two maps $f, g\colon \overline{B}^2 \to S^2$ that are shown in the figure below:



These maps agree on S^1. It is fairly easy to see that the two maps are homotopic but it seems rather unlikely that they are homotopic rel S^1. Unfortunately, as in (1), right now we lack the tools for showing the latter statement. Only much later, namely in Exercise 149.4, will we see that the two maps f and g are not homotopic rel S^1. □

We continue with the next variation on the notion of homotopy.

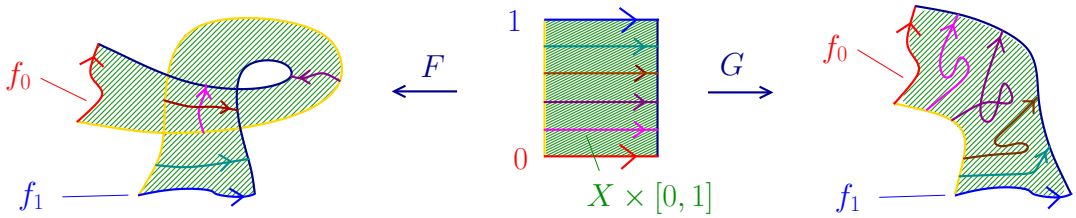
Definition 18.5. Let X and Y be two topological spaces.

(1) (a) An isotopy is a homotopy $F\colon X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t\colon X \to Y$ is an embedding.
(b) A homeotopy is a homotopy $F\colon X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t\colon X \to Y$ is a homeomorphism.

(2) Let $f_0, f_1\colon X \to Y$ be two continuous maps that are embeddings. We say that f_0 and f_1 are isotopic if there exists an isotopy $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. Similarly we define homeotopic.

Examples.

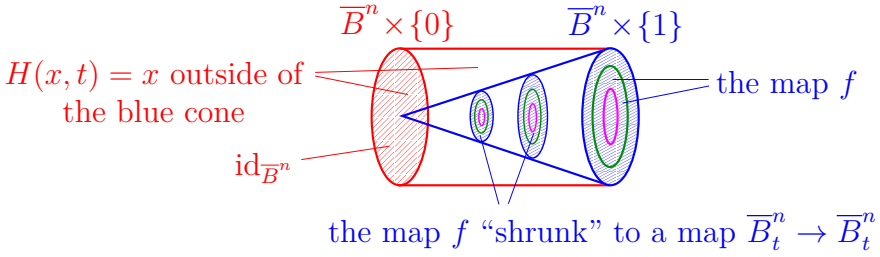
(1) In the figure below we show two maps $f_0, f_1\colon [0,1] \to \mathbb{R}^2$ that are embeddings. We also indicate two homotopies F and G between f_0 and f_1. The map F is an isotopy, the map G is not.



(2) Let $n \in \mathbb{N}$. Let $f\colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism. If $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, then f is isotopic to $\mathrm{id}_{\overline{B}^n}$. Indeed, in Exercise 18.6 we will see that the map

$$\begin{aligned} H\colon \overline{B}^n \times [0,1] &\to \overline{B}^n \\ (x,t) &\mapsto \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \le \|x\| < t, \\ x, & \text{otherwise} \end{cases} \end{aligned}$$

is continuous and that each H_t is a homeomorphism. It is clear that $H_0 = \mathrm{id}$ and $H_1 = f$. Thus we see that H is the desired isotopy.



The fact that f is isotopic to $\mathrm{id}_{\overline{B}^n}$ is often referred to as the *Alexander trick*.[215] □

In an ideal world we would be able to classify all self-homeomorphisms of a given topological space up to homeotopy. In general that might be a trifle too ambitious. But the following lemma gives us a classification in some simple but useful cases.

Lemma 18.6. (Homeomorphisms-of-S^1-Classification Lemma)

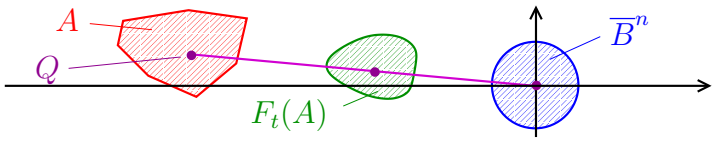
[215] The existence of such a homotopy was first observed by James Alexander in 1923, see [**Ale1924a**].

(1) Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.[216]
   (a) If f is monotonously increasing, then f is isotopic to the map $f(x) = x$.
   (b) If f is monotonously decreasing, then f is isotopic to the map $f(x) = -x$.
   An analogous statement also holds for $\mathbb{R}$ instead of $[0,1]$.
(2) Every self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.

Proof. We will prove the lemma in Exercise 18.9. (We will provide an alternative proof in Exercise 93.4.) ■

The next lemma is a refined version of the Convex-to-Ball Proposition 4.28 (2).

Lemma 18.7. (Convex–Isotopic-to-Ball Lemma) Let A be a bounded closed convex subset of $\mathbb{R}^n$ with non-empty interior. Let Q be a point in the interior of A. There exists a canonical isotopy $F\colon A \times [0,1] \to \mathbb{R}^n$ with the following three properties:
(1) We have $F_0 = \mathrm{id}_A$.
(2) The map F_1 is a homeomorphism from A to $\overline{B}^n$ and this map restricts to a homeomorphism $\partial A \to S^{n-1}$.
(3) For every $t \in [0,1]$ we have $F(Q,t) = (1-t)\cdot Q$.[217]



Remark. If A is "diffeomorphic" to $\overline{B}^n$, then the Ball-Isotopic-to-Diffeomorphism Lemma 57.27 and the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 give significant refinements of the conclusion of the Convex–Isotopic-to-Ball Lemma 18.7. □

Proof of Lemma 18.7. First consider the case that Q is the origin. As in the statement of the Convex-to-Ball Proposition 4.28 (2) we consider the map

$$\begin{aligned} \rho\colon A \setminus \{Q\} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in A\big\}. \end{aligned}$$

In the Distance-to Boundary Continuous Lemma 4.29 (1) we showed that ρ is continuous. Similar to the discussion on page 191 we consider the map

$$\begin{aligned} F\colon A \times [0,1] &\to \overline{B}^n \\ (x,t) &\mapsto \begin{cases} x \cdot \frac{1}{(1-t)+t\cdot\rho(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{aligned}$$

As in the Distance-to Boundary Continuous Lemma 4.29 (2) one can show that F is continuous. It is straightforward to verify that this map has all the desired properties. Finally we leave it to the reader to modify the above proof to deal with the case that Q is not the origin. ■

[216] It is an elementary real analysis exercise to show that every self-homeomorphism $f\colon [0,1] \to [0,1]$ is either monotonously increasing or monotonously decreasing.
[217] In particular, if Q is already the origin, then F is a homotopy rel the origin.

After these examples we return to developing the general theory of homotopies. On many occasions it will be useful to combine two homotopies or isotopies. This leads us to the following lemma.

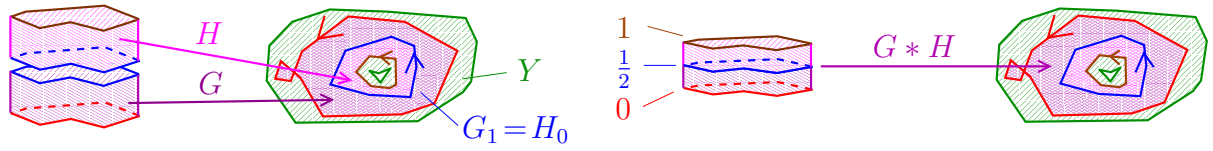
Lemma 18.8. (Homotopy Combination Lemma) Let X and Z be topological spaces.

(1) Let Y be a topological space and let $f\colon X\times Y\to Z$ be a continuous map. Suppose we are given finitely many closed subsets $A_1,\dots,A_m$ of Y with $\bigcup_{i=1}^{m} A_i = Y$. If each map $X\times A_i\to X\times Y\to Z$ is continuous, then f itself is continuous.

(2) Let $A\subseteq X$ be a (possibly empty) subset of X and let $G,H\colon X\times[0,1]\to Z$ be two homotopies rel A. If $G_1=H_0$, then the map

$$\begin{array}{rcl} G*H\colon X\times[0,1] & \to & Z \\ (x,t) & \mapsto & \begin{cases} G(x,2t), & \text{if } t\in[0,\frac{1}{2}], \\ H(x,2t-1), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{array}$$

is a homotopy rel A from G_0 to H_1. Furthermore, if G and H are isotopies (homeotopies), then $G*H$ is also an isotopy (homeotopy). We refer to $G*H$ as the combination of G and H.



Proof.

(1) Note that by the Product Topology-Basics Lemma 7.6 (1a) the following two topologies on the set $X\times A_i$ agree:
   (a) The product topology on $X\times A_i$ where we equip $X\subseteq A_i$ with the subspace topology.
   (b) The subspace topology on $X\times A_i\subseteq X\times Y$.
   It follows from this observation and our hypothesis that for each $i\in\{1,\dots,m\}$ the map $f|_{X\times A_i}\colon X\times A_i\to Z$ is continuous. By the Product-of-Subsets Lemma 7.8 (1) we know that $A_i\times Y$ is a closed subset of $X\times Y$. Now it follows from the Pasting Proposition 4.7 (2) that f is indeed continuous.

(2) We only need to verify that $G*H$ is indeed continuous, but that is a consequence of our hypothesis that $G_1=H_0$ and Statement (1). ∎

Lemma 18.9. (Homotopy-Equivalence Relation Lemma) Let X and Y be topological spaces and let A be a (possibly empty) subset of X. Then "homotopy rel A of maps" forms an equivalence relation on the set of continuous maps from X to Y. The same applies for "isotopy" and "homeotopy".

Proof. We have to show that "homotopy rel A" satisfies the three properties of an equivalence relation. To simplify the notation we only deal with the case $A=\varnothing$. In the following let $f,g,h\colon X\to Y$ be three continuous maps.

(1) The "stationary homotopy" given by $F(x,t):=f(x)$ is a homotopy between f and f.

(2) Let F be a homotopy from f to g. Then it is clear that the map F' that is defined by $F'(x,t):=F(x,1-t)$ is a homotopy from g to f.

(3) Let F be a homotopy from f to g and let G be a homotopy from g to h. We consider the map

$$\begin{array}{rcl} F * G \colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t \in \left[0, \frac{1}{2}\right], \\ G(x,2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

By the Homotopy Combination Lemma 18.8 (2) this is a homotopy from f to h.

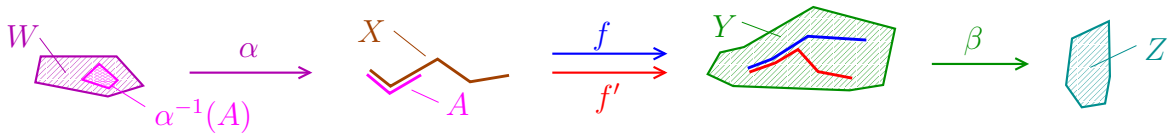
The statement for "isotopy" and "homeotopy" follows from the same argument. ■

Lemma 18.10. (Homotopy Composition Lemma)

(1) Let $f, f' \colon X \to Y$, let $\alpha \colon W \to X$ and $\beta \colon Y \to Z$ be continuous maps between topological spaces. Furthermore let $A \subseteq X$ be a (possibly empty) subset. The following conclusion holds:

$$\begin{array}{rcl} f \text{ is homotopic to } f' \text{ rel } A & \implies & f \circ \alpha \text{ is homotopic to } f' \circ \alpha \text{ rel } \alpha^{-1}(A) \\ f \text{ is homotopic to } f' \text{ rel } A & \implies & \beta \circ f \text{ is homotopic to } \beta \circ f' \text{ rel } A \end{array}$$

(2) Let X, Y and Z be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subsets. We suppose that $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ are continuous maps such that f and f' are homotopic rel A and such that g and g' are homotopic rel B. If $f(A) = f'(A) \subseteq B$, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.



Proof.

(1) This statement is basically obvious, we just need to precompose and postcompose a homotopy $H \colon X \times [0,1] \to Y$ between f and g by $\alpha \times \mathrm{id}_{[0,1]} \colon W \times [0,1] \to X \times [0,1]$ respectively by $\beta \colon Y \to Z$.

(2) Note that:

$$g \circ f \underset{\uparrow}{\simeq} g' \circ f \underset{\uparrow}{\simeq} g' \circ f'.$$

both follow from (1)



Furthermore one can easily verify that the "rel" business also works out. ■

18.2. **Homotopies on quotients and pushouts.** In this section we collect several subtle lemmas which show that, under the right circumstances, homotopies descend to quotient spaces and that given a pushout $X \cup_A Y$ we can also "pushout homotopies". Before we state these technical statements, we point out that the proof of these two statements eventually boils down to the rather subtle and delicate Whitehead Quotient Theorem 17.6.

After this preamble, let us move to the key result of this section which shows that homotopies descend to quotients.

Theorem 18.11. (Homotopy Quotient Theorem) Let X and Y be topological spaces. Furthermore let $\sim_X$ be an equivalence relation on X and let $\sim_Y$ be an equivalence relation

on Y. If $F\colon X\times[0,1]\to Y$ is a homotopy such that $F(x,t)\sim_Y F(x',t)$ whenever $x\sim_X x'$, then the map[218]

$$\begin{array}{rcl}\varphi\colon (X/\!\sim_X)\times[0,1] & \to & Y/\!\sim_Y\\ ([x],t) & \mapsto & [F(x,t)]\end{array}$$

is continuous.

Example. Let $n\in\mathbb{N}$ and let $\varphi\colon[0,1]\to \mathrm{GL}(n,\mathbb{R})$ be a continuous map. It follows from the argument of the proof of the Torus Embedding Lemma 7.11 that the map

$$\begin{array}{rcl}F\colon (\mathbb{R}^n\setminus\{0\})\times[0,1] & \to & \mathbb{R}^n\setminus\{0\}\\ (P,t) & \mapsto & \varphi(t)\cdot P\end{array}$$

is continuous. Next recall that $\mathbb{R}\mathrm{P}^{n-1}=(\mathbb{R}^n\setminus\{0\})/(\mathbb{R}\setminus\{0\})$. It follows from the Homotopy Quotient Theorem 18.11 that the map

$$\begin{array}{rcl}\mathbb{R}\mathrm{P}^{n-1}\times[0,1] & \to & \mathbb{R}\mathrm{P}^{n-1}\\ ([P],t) & \mapsto & [\varphi(t)\cdot P]\end{array}$$

is continuous. The same conclusion also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$". □

Proof of Theorem 18.11. We denote by $p\colon X\to X/\!\sim_X$ and $q\colon Y\to Y/\!\sim_Y$ the natural projections. We have the following commutative diagram:

$$\begin{array}{ccc} & X\times[0,1] & \\ {\scriptstyle p\times\mathrm{id}_{[0,1]}}\swarrow & & \searrow{\scriptstyle q\circ F} \\ (X/\!\sim_X)\times[0,1] & \xrightarrow[([x],t)\mapsto[F(x,t)]]{\varphi} & Y/\!\sim\,. \end{array}$$

By the Projection–Quotient Map Lemma 17.4 the map p is a quotient map. Since $[0,1]$ is regionally compact we obtain from the Whitehead Quotient Theorem 17.6 that $p\times\mathrm{id}_{[0,1]}$ is also a quotient map. Note that by the Topological-Quotient Proposition 8.2 (1a) and by hypothesis the map $q\circ F$ is continuous. Thus we obtain from the Quotient Map-Composition Lemma 17.7 (1) that φ is continuous. ■

For the reader who is wary of the Whitehead Quotient Theorem 17.6 we provide a second argument that works in many, but certainly not all cases.

Proof of Theorem 18.11 if X is compact and if $X/\!\sim_X$ is Hausdorff. We now assume that X is compact and that the quotient space $X/\!\sim_X$ is Hausdorff. On $X\times[0,1]$ we consider the equivalence relation $\sim$ given by

$$(x,t)\sim(x',t')\iff x\sim_X x' \text{ and } t=t'.$$

We denote by $h\colon X\times[0,1]\to(X\times[0,1])/\!\sim$ the natural projection. We consider the maps

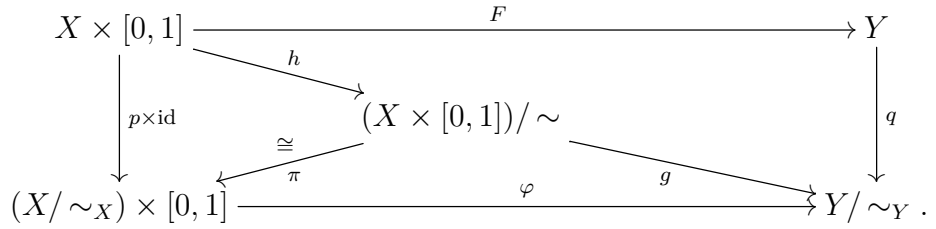
$$\begin{array}{rcl}p\times\mathrm{id}\colon X\times[0,1] & \to & X/\!\sim_X\times[0,1]\\ (x,t) & \mapsto & ([x],t)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}\pi\colon (X\times[0,1])/\!\sim & \to & X/\!\sim_X\times[0,1]\\ [(x,t)] & \mapsto & ([x],t).\end{array}$$

By the Topological-Quotient Proposition 8.2 (1a) the projection $p\colon X\to X/\!\sim_X$ is continuous. This shows that the map to the left is continuous. By the Topological-Quotient Proposition 8.2 (1b) this implies that the induced map π to the right is also continuous. One can easily verify that π is a bijection. By hypothesis X is compact. By the Product Topology-Properties Proposition 7.9 (1) this implies that $X\times[0,1]$ is compact. It follows from the Topological-Quotient Proposition 8.2 (1a) and the Compact Image Lemma 4.15

[218] It follows from our hypothesis on F that the map φ is well-defined.

that $(X \times [0,1])/\sim$ is compact. Next recall that by hypothesis $X/\sim_X$ is Hausdorff. By the Product Topology-Properties Proposition 7.9 (3) this implies that $X/\sim_X \times [0,1]$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 4.23 (3) that π is a homeomorphism.

Now we consider the following diagram:



We make the following clarifications and observations:

(1) We denote by $q\colon Y \to Y/\sim_Y$ the natural projection.

(2) First note that the map $q \circ F\colon X \times [0,1] \to Y/\sim_Y$ factors by our hypothesis through the map $h\colon X \times [0,1] \to (X \times [0,1])/\sim$. It follows again from the Topological-Quotient Proposition 8.2 (1b) that the induced map $g\colon (X \times [0,1])/\sim \to Y/\sim_Y$ is continuous.

(3) The map $\varphi\colon (X/\sim_X) \times [0,1] \to Y/\sim_Y$ equals the map $g \circ \pi^{-1}$. We just showed that the two maps π and g are continuous. Thus φ is continuous. ■

The following lemma is a hands-on special case of the Homotopy Quotient Theorem 18.11.

Lemma 18.12. (Homotopy Quotient-by-Subspace Lemma) Let (X, A) and (Y, B) be pairs of topological spaces. If $F\colon X \times [0,1] \to Y$ is a homotopy with $F(A \times [0,1]) \subseteq B$, then the induced map

$$\begin{array}{rcl} (X/A) \times [0,1] & \to & Y/B \\ ([x], t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Example. Let $\gamma\colon [0,1] \to \mathrm{O}(n)$ be a continuous map. We consider the maps

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{F}\colon (\overline{B}^n/S^{n-1}) \times [0,1] & \to & \overline{B}^n/S^{n-1} \\ ([x],t) & \mapsto & [\gamma(t) \cdot x]. \end{array}$$

As on page 510 we see that the map F to the left is continuous. We now obtain from the Homotopy Quotient-by-Subspace Lemma 18.12 that the map $\overline{F}$ to the right is also continuous. □

Proof. This statement follows immediately from the Homotopy Quotient Theorem 18.11 and the fact that quotients of topological spaces are defined via equivalence relations. ■

We conclude this section with the following lemma which shows how to define homotopies on pushouts.

Lemma 18.13. (Homotopy Pushout Lemma) Let $\alpha\colon A \to Y$ and $\beta\colon A \to Z$ be continuous maps between topological spaces. As in Definition 8.13 we define the pushout

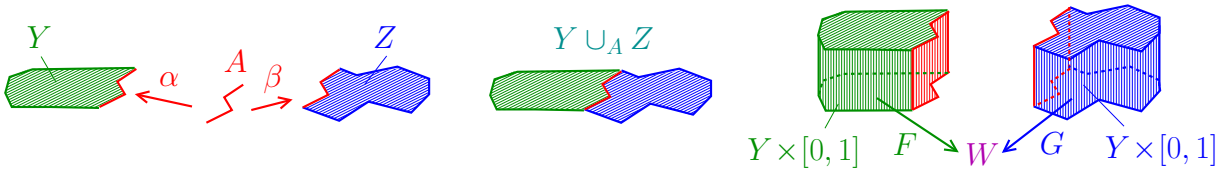
$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\sim \qquad \text{where } \alpha(a) \sim \beta(a) \text{ for all } a \in A.$$

Furthermore let W be another topological space. Suppose that we are given two homotopies $G\colon Y \times [0,1] \to W$ and $H\colon Z \times [0,1] \to W$ such that for every $a \in A$ and every

$t \in [0,1]$ we have $G(\alpha(a),t) = H(\beta(a),t)$. Then the map

$$\begin{array}{rcl} F\colon (Y \cup_A Z) \times [0,1] & \to & W \\ ([P],t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in Y, \\ H(P,t), & \text{if } P \in Z \end{cases} \end{array}$$

is well-defined and it is continuous.



Proof. It follows from Exercise 7.19 that the map

$$\begin{array}{rcl} (Y \sqcup Z) \times [0,1] & \to & W \\ (P,t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in Y, \\ H(P,t), & \text{if } P \in Z \end{cases} \end{array}$$

is continuous. It remains to show that the given map F is also continuous. We give two slightly different proofs:

- The statement follows from the Homotopy Quotient Theorem 18.11 and the definition of the pushout as a quotient of $Y \sqcup Z$.
- It follows from the Cylinder–Path Space Adjunction Lemma 16.11 together with the Left Adjoint–Colimit Proposition 6.35 that taking products with $[0,1]$ preserves all colimits, in particular it preserves pushouts. Using this observation one can easily verify that F is indeed continuous. ■

18.3. Exercises for Chapter 18.

Exercise 18.1. Let Y be a path-connected space and let $k \in \mathbb{N}_0$. Show that any two continuous maps $\overline{B}^k \to Y$ are homotopic.

Exercise 18.2. Let X be a topological space and let $f, g\colon X \to S^1$ be continuous maps. We assume that there exists a $\xi \in S^1$ such that for every $x \in X$ we have $f(x) \neq \xi \cdot g(x) \in S^1$. Show that f and g are homotopic.
Hint. Consider the map $X \to S^1$ that is given by $z \mapsto f(z) \cdot g(z)^{-1}$.

Exercise 18.3. Let X be a topological space and let $f, g\colon X \to S^1$ be continuous maps. We assume that for every $x \in X$ we have $f(x) \neq -g(x) \in S^1$. Show that f and g are homotopic.

Exercise 18.4. Let X be a topological space and let $f\colon X \to \mathbb{R}^n \setminus \{0\}$ be a continuous map. Show that f is homotopic to the map $X \to \mathbb{R}^n \setminus \{0\}$ given by $x \mapsto \frac{f(x)}{\|f(x)\|}$.

Exercise 18.5. Find as many dimensions $n \in \mathbb{N}$ as possible such that the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R}^n \\ P & \mapsto & -P \end{array}$$

is homotopic to the identity.

Exercise 18.6. Let $n \in \mathbb{N}$ and let $f\colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism with $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. We consider the map

$$\begin{array}{rccl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n & \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \leq \|x\| < t, \\ x, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that H is continuous.
(b) Show that for each $t \in [0,1]$ the map $H_t\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism.

Remark. The fact that f is isotopic to $\mathrm{id}_{\overline{B}^n}$ is often referred to as the *Alexander trick*.

Exercise 18.7. Let X and Y be topological spaces and let $f, g\colon X \to Y$ be homeomorphisms. Suppose that there exists a homeotopy $F\colon X \times [0,1] \to Y$ between f and g. Show that if X and Y are compact and Hausdorff, then $f^{-1}, g^{-1}\colon Y \to X$ are also homeotopic.
Remark 1. There is an obvious candidate map $Y \times [0,1] \to X$. But why is it continuous?
Remark 2. It seems like it follows from [**Are1946**, Theorem 4] that the conclusion also holds if X and Y are both regionally compact, locally connected and Hausdorff.

Exercise 18.8. Let X and Y be topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy. Show that there exists a homotopy $G\colon X \times [0,1] \to Y$ with the following properties:

(1) $G_0 = F_0$ and $G_1 = F_1$.
(2) There exists an $\epsilon \in (0,1)$ such that for each $t \in [0,\epsilon]$ we have $G_t = G_0$. and such that for each $t \in [1-\epsilon, 1]$ we have $G_t = G_1$
(3) For each $s \in [0,1]$ there exists a $t \in [0,1]$ with $G_s = F_t$.

Exercise 18.9.

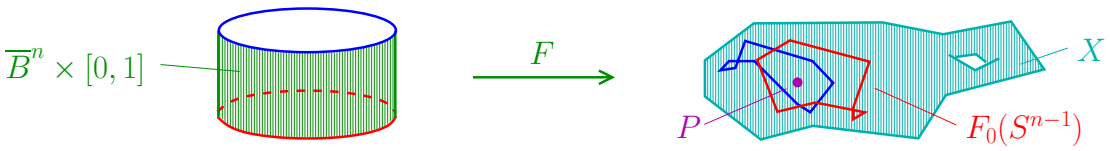
(a) Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.
  (i) Show that f is monotonously increasing or monotonously decreasing.
  (ii) Suppose that f is monotonously increasing. Show that f is isotopic to the map that is given by $g(x) = x$.
  (iii) Suppose that f is monotonously decreasing. Show that f is isotopic to the map that is given by $g(x) = -x$.
(b) Repeat (a) for $\mathbb{R}$ instead of $[0,1]$.
(c) Show that any self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.
  Hint. Let $f\colon S^1 \to S^1$ be a self-homeomorphism. First consider the case that $f(1) = 1$.

Remark. We will provide a different, and arguably more satisfying, proof of Statement (c) in Exercise 93.4.

Exercise 18.10. Let X be a Hausdorff space, let $P \in X$ and let $F\colon \overline{B}^n \times [0,1] \to X$ be a continuous map such that for every $t \in [0,1]$ we have $F_t(S^{n-1}) \subseteq X \setminus \{P\}$.

(a) Show that there exists a connected open neighborhood U of P such that for every $t \in [0,1]$ we have $F_t(S^{n-1}) \cap U = \varnothing$.
(b) Let U be as in (a). Show that for each $t \in [0,1]$ we have $U \subseteq F_t(B^n)$.

Exercise 18.11. Let h be a complex polynomial of degree ≥ 1. In Exercise 4.43 we showed that the map



$$\begin{array}{rcl} \Theta(h)\colon \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} \\ z & \mapsto & \begin{cases} h(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases} \end{array}$$

is continuous. Now let f and g be two polynomials with complex coefficients of degree ≥ 1. Suppose that $\deg(f) = \deg(g)$. Show that the maps $\Theta(f)$ and $\Theta(g)$ are homotopic.
Remark. It is quite easy to write down a candidate for a homotopy. But why is the map you wrote down actually continuous?

Exercise 18.12. Using the Stereographic Projection Lemma 4.24 we can make the identification $S^1 = \mathbb{R}\cup\{\infty\}$. Now let f be a polynomial with real coefficients of degree $n \geq 1$. In Exercise 4.43 we showed that the map

$$\begin{array}{rcl} F\colon S^1 = \mathbb{R}\cup\{\infty\} & \to & \mathbb{R}\cup\{\infty\} =_{\mathrm{i}} S^1 \\ t & \mapsto & \begin{cases} f(t), & \text{if } t\in\mathbb{R}, \\ \infty, & \text{if } t=\infty \end{cases} \end{array}$$

is continuous. (See the figure on page 220 for an illustration.) Show that F is homotopic to one of the following three types of self-maps of S^1:

(1) $a(z) = z$ (2) $b(z) = \overline{z}$ (3) $c(z) = 1$.

Exercise 18.13. Let X be a topological space and let Y be a topological space that is regionally compact. As in Definition 3.56 we consider the set X^Y of all continuous map $Y \to X$ and we equip X^Y with the compact-open topology. Let $f_0, f_1\colon Y \to X$ be two continuous maps. Show that the following holds:

f_0 and f_1 are homotopic $\iff$ f_0 and f_1 lie in the same path component of X^Y.

Remark. Use the Compact-Open Exponential Proposition 16.10.

Exercise 18.14. We consider the two maps $f, g\colon [0,1] \to \mathbb{R}^3$ shown in the figure below. Note that $f(0) = g(0)$ and $f(1) = g(1)$.
(a) Show that the maps f and g are homotopic rel $\{0\}$.
(b) Are the two maps f and g homotopic rel $\{0,1\}$?

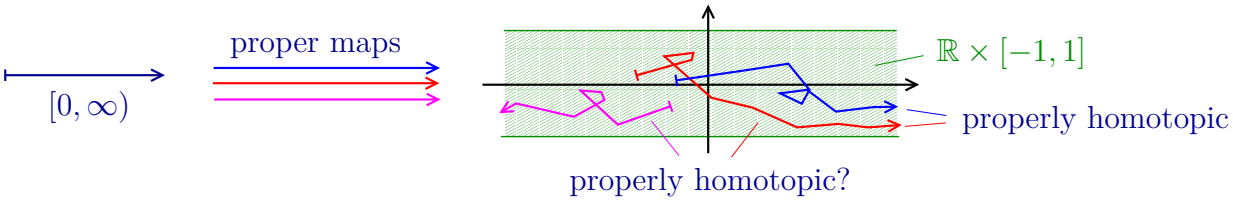


Exercise 18.15. Recall that according to Definition 4.63 a continuous map $f\colon A \to B$ between topological spaces is called *proper*, if the preimage of each compact subset of B is a compact subset of A. Let X and Y be topological spaces. Provide an example of a homotopy $F\colon X \times [0,1] \to Y$ such that each $F_t\colon X \to Y$ is proper but such that the map

$F\colon X \times [0,1] \to Y$ is not proper itself.
Hint. You can find examples with $X = Y = \mathbb{R}$ and $F_0 = F_1 = \mathrm{id}$.

Exercise 18.16. Let X and Y be topological spaces. We say that two continuous maps $f, g\colon X \to Y$ are *properly homotopic* if there exists a *proper homotopy* $F\colon X \times [0,1] \to Y$ between f and g, i.e. F is a homotopy between f and g and the map $F\colon X \times [0,1] \to Y$ is proper, in the sense of the above Exercise 18.15.

(a) Show that any two proper maps $[0,\infty) \to \mathbb{R}^2$ are properly homotopic.
(b) Show that there exist proper maps $f, g\colon [0,\infty) \to \mathbb{R} \times [-1,1]$ that are not properly homotopic.



CHAPTER 19

Homotopy equivalences and homotopy equivalent topological spaces

This chapter is a direct continuation of the previous chapter. In the last chapter we defined what it means for *maps* between topological spaces to be *homotopic*. Now in this chapter we will use this notion to define what it means for *topological spaces* to be *homotopy equivalent*. This notion is rather subtle and - at least for this author - not entirely intuitive.

This chapter is organized as follows:

- In Section 19.1 we will introduce the notion of a deformation retraction from a topological space to a subspace.
- In Section 19.2 we will introduce the slightly unintuitive notion what it means for two topological spaces to be homotopy equivalent. As we will see, a main example is given by deformation retracts. More precisely, if A is a deformation retract of X, then the inclusion $A \to X$ is a homotopy equivalence.
- In Section 19.3 we will introduce the *homotopy category*. In this category the objects are as usual topological spaces, but the morphisms between two topological spaces X and Y is now the set $[X, Y]$ of homotopy classes of continuous maps $X \to Y$. We will see that a map $f\colon X \to Y$ is a homotopy equivalence if and only if $[f]$ is invertible in the homotopy category.
- In Section 19.4 we will define that a topological space is *contractible*, if it is homotopy equivalent to a singleton. We will give some examples of contractible spaces.
- In Section 19.5 we will generalize homotopies and homotopy equivalences from topological spaces to pairs of topological spaces.
- In Section 19.6 we will provide exercises for Chapter 19.

In the next Chapter 20 we will introduce joins, cones and suspensions.

19.1. **Deformation retractions.** Before we introduce the notion of a homotopy equivalence we want to introduce the easier notions of a (deformation) retraction.

Definition 19.1. Let X be a topological space. We say that a subset $A \subseteq X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.



retraction r from X to A

Let us recall Definition 18.2:

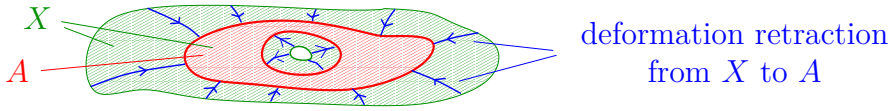
Definition 19.2. Let X and Y be two topological spaces.

(1) Given a continuous map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by F_t the continuous map
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$
(2) A continuous map $F\colon X \times [0,1] \to Y$ map is called a homotopy from F_0 to F_1.

Next we introduce an interesting special case of a homotopy.

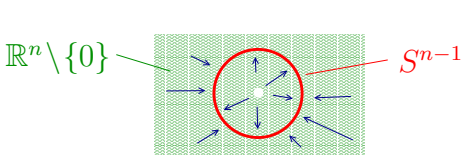
Definition 19.3. Let X be a topological space and let A be a subset.

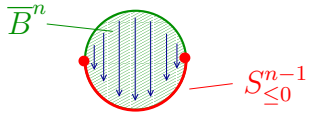
(1) A deformation retraction from X to A is a homotopy $F\colon X \times [0,1] \to X$ which has the following three properties:[219]
   (a) $F_0 = \mathrm{id}_X$,
   (b) F is a homotopy rel A, i.e. for all $a \in A$ and all $t \in [0,1]$ we have $F(a,t) = a$,
   (c) $F_1(X) = A$.
(2) We say that A is a deformation retract of X if there exists a deformation retraction from X to A. We say a point $a \in X$ is a deformation retract of X if the subset $\{a\}$ is a deformation retract of X.
(3) We say that a retraction $r\colon X \to A$ is realized by a deformation retraction if there exists a deformation retraction F from X to A with $F_1 = r$.

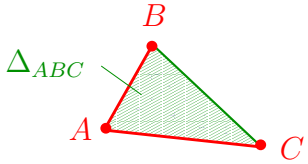


Examples. Let $n \in \mathbb{N}$.

(1) (a) The map[220]
$$\begin{array}{rccc} & (\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ & (v,t) & \mapsto & v \cdot (1 - t + t \cdot \frac{1}{\|v\|}) \end{array}$$
is a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to the sphere S^{n-1}.
In Exercise 19.24 we will prove the following two facts:
   (b) The hemisphere $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.
   (c) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.

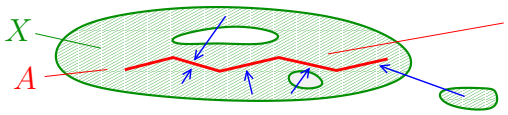






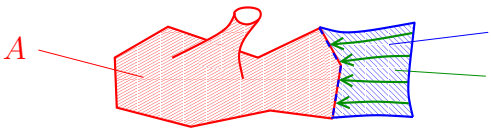
(2) If $F\colon X \times [0,1] \to X$ is a deformation retraction from X to A, then $F_1\colon X \to A$ is by definition a retraction from X to A. But, as we see in the figure below, not every retract is a deformation retract.

[219] Unfortunately in the literature the term "deformation retract" has different meanings. We will discuss the different possible definitions of "deformation retract" on page 533.
[220] The argument of the proof of the Torus Embedding Lemma 7.11 shows that this map is continuous.



A is a retract of X
if it was a deformation retract, any point $x \in X$ would be connected to A via the path $R(x,t)$

(3) Let X be a topological space. Suppose that we are given a decomposition $X = A \cup B$ into two *closed* subsets. If $A \cap B$ is a deformation retract of B, then A is a deformation retract of X.



Indeed, if $F\colon B \times [0,1] \to B$ is a deformation retraction from B to $A \cap B$, then it follows from the Homotopy Combination Lemma 18.8 (1) and our hypothesis that A and B are closed subsets of X, that the map

$$\begin{array}{rcl} G\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in A, \\ F(x,t), & \text{if } x \in B \end{cases} \end{array}$$

is continuous. It is now clear that G is the desired deformation retraction from X to A. We will use this remark on several occasions. □

We continue with an example of a deformation retraction which is so interesting, that it gets formulated as a proposition.

Proposition 19.4. (Matrix Groups Deformation Retract Proposition) Let $n \in \mathbb{N}_0$ and let $k \in \{1,\dots,n\}$. We write[221] [222]

$$\begin{array}{rcl} \mathrm{GL}(n,k) & = & \text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n \\ \mathrm{O}(n,k) & = & \text{set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^n. \end{array}$$

There exists a deformation retraction

$$R\colon\ \mathrm{GL}(n,k) \times [0,1] \ \to\ \mathrm{GL}(n,k)$$

from $\mathrm{GL}(n,k)$ to $\mathrm{O}(n,k)$ with the following properties:

(1) The map $R_1\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ is the map that is given by the Gram-Schmidt Orthonormalization Process 2.32.

(2) For every k-dimensional vector subspace V of $\mathbb{R}^n$ and every $t \in [0,1]$ we have

$$R_t(\text{basis } \underbrace{(v_1,\dots,v_k)}_{\text{matrix in } \mathrm{GL}(n,k)} \text{ of } V) \ = \ \text{basis of } V.$$

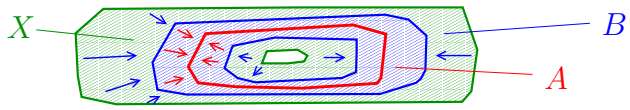
The analogous statements hold if we replace the pair $\mathrm{GL}(n,k)$ and $\mathrm{O}(n,k)$ by one of the following four pairs:

(a) $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{SO}(n)$
(b) $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n)$
(c) $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(n)$
(d) $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$.

Remark. Later, on page 4614 we will use the Polar Decomposition Theorem 239.18 to give an alternative proof of some of the main statements of the Matrix Groups Deformation Retract Proposition 19.4. □

The proof of the above Proposition 19.4 rests on the following lemma:

Auxiliary Lemma 19.5. (Iterated Deformation Retract Lemma) Let X be a topological space and let $A \subseteq B \subseteq X$ be subspaces. If A is a deformation retract of B and if B is a deformation retract of X, then A is a deformation retract of X.



Proof of Lemma 19.5. Let X be a topological space and let $A \subseteq B \subseteq X$ be subspaces. Furthermore let $F\colon B \times [0,1] \to B$ be a deformation retraction from B to A and finally let $G\colon X \times [0,1] \to X$ be a deformation retraction from X to B. We consider the map

$$\begin{array}{rcl} X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} G(x,2t), & \text{if } t \in [0,\frac{1}{2}], \\ F(\underbrace{G_1(x)}_{\in B}, 2t-1) = \underbrace{(F \circ (G_1 \times \mathrm{id}_{[0,1]}))}_{\text{continuous}}(x,2t-1), & \text{if } t \in [\frac{1}{2},t]. \end{cases} \end{array}$$

As in the Homotopy Combination Lemma 18.8 (1) we see that this map is continuous. Once continuity is verified it is clear that this map is indeed a deformation retraction from X to A. ■

Proof of Proposition 19.4. For $i = 1,\dots,k$ we consider the following subspaces of $\mathrm{GL}(n,k)$:

$$\begin{array}{rcl} X(i) & \coloneqq & \{(v_1 \dots v_k) \in \mathrm{GL}(n,k) \,|\, v_1,\dots,v_{i-1} \text{ are orthonormal and } v_i \perp v_1,\dots,v_{i-1}\} \\ Y(i) & \coloneqq & \{(v_1 \dots v_k) \in \mathrm{GL}(n,k) \,|\, v_1,\dots,v_i \text{ are orthonormal}\}. \end{array}$$

On page 204, in the proof of the Matrix Group Path-Component Proposition 4.54, we implicitly already showed that the $2k$ steps of the Gram-Schmidt Orthogonalization Process define deformation retractions from $X(i)$ to $Y(i)$, $i = 1,\dots,k$ and $Y(i)$ to $X(i+1)$, $i = 1,\dots,k-1$. Since $Y(1) = \mathrm{GL}(n,k)$ and since $Y(k) = \mathrm{O}(n,k)$ it follows from the Iterated Deformation Retract Lemma 19.5 that we can combine these $2k-1$ deformation retractions to obtain the desired deformation retraction

$$R\colon\ \mathrm{GL}(n,k) \times [0,1] \ \to\ \mathrm{GL}(n,k)$$

from $\mathrm{GL}(n,k)$ to $\mathrm{O}(n,k)$ such that R satisfies (2). If $n = k$, then it follows from the Gram-Schmidt Orthonormalization Process 2.32 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the complex cases are treated in a very similar fashion. ■

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[220] Note that for $k = n$ we have $\mathrm{GL}(n,k) = \mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n,k) = \mathrm{O}(n)$.

[221] In Definition 261.4 we will refer to $\mathrm{O}(n,k)$ as a Stiefel manifold.

[222] In the Matrix Groups–Smoth Deformation Retract Proposition 41.15 we will see that the deformation retraction can be chosen to be "smooth".

In the Matrix Groups Deformation Retract Proposition 19.4 we just showed that $\mathrm{SO}(n)$ is a deformation retract of $\mathrm{GL}_+(n,\mathbb{R})$. One can ask whether one can do better than that, i.e. whether one can find a deformation retraction to a subgroup or subspace that is a proper subspace of $\mathrm{SO}(n)$. Let us record this as a question, which we will answer in the $\mathrm{SO}(n)$-No Retraction Corollary 183.30.

Question 19.6. Does $\mathrm{SO}(n)$ admit a proper subspace that is a deformation retract of $\mathrm{SO}(n)$?

In the following we will see that deformation retractions descend to quotients under suitable hypotheses.

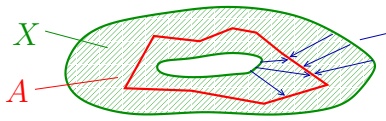
Theorem 19.7. (Deformation Retraction-Quotient Theorem) Let X be a topological space and let $A \subseteq X$ be a subset.

(1) Let "$\sim$" be an equivalence relation on X and let $R\colon X \times [0,1] \to X$ be a deformation retraction from X to A. If "$\sim$" is compatible with the deformation retraction R in the sense that $R(x,t) \sim R(x',t)$ whenever $x \sim x'$, then the map

$$\begin{array}{rcl} \overline{R}\colon (X/\sim) \times [0,1] & \to & X/\sim \\ ([x],t) & \mapsto & [R(x,t)] \end{array}$$

is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A.



deformation retraction $R\colon X \times [0,1] \to X$ from X to A
X/A admits a deformation retract to the point A/A
what does X/A "look like"?

Proof.

(1) It follows from the Homotopy Quotient Theorem 18.11 that $\overline{R}$ is continuous. It is now clear that $\overline{R}$ is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let $r\colon X \times [0,1] \to X$ be a deformation retraction from X to A. We consider the equivalence relation "$\sim_A$" on X that is generated by the condition that all points in A are equivalent. Note that by definition of a deformation retraction the homotopy r is compatible with the equivalence relation "$\sim_A$". Therefore it follows from (1) that the map

$$\begin{array}{rcl} (X/\sim_A) \times [0,1] & \to & X/\sim_A \\ ([x],t) & \mapsto & [r(x,t)]. \end{array}$$

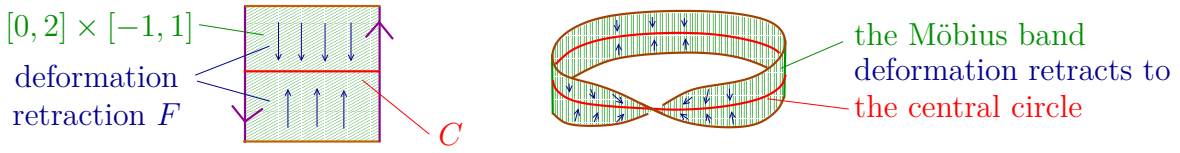
is a deformation retraction from $X/\sim_A = X/A$ to $A/\sim_A = A/A$. ∎

Example. We consider $X = [0,2] \times [-1,1]$ together with the equivalence relation "$\sim$" that is generated by

$$(0,y) \sim (2,-y) \quad \text{for all } y \in [-1,1].$$

It is straightforward to see that $X/\sim$ is homeomorphic to the Möbius band as defined on page 310. We consider the maps

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & X \\ ((x,y),t) & \mapsto & (x, y\cdot(1-t)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{F}\colon (X/\sim) \times [0,1] & \to & X/\sim \\ ([(x,y)],t) & \mapsto & [(x, y\cdot(1-t))]. \end{array}$$



Note that F is a deformation retraction from X to $C := [0,2]\times\{0\}$. One can easily verify that F is compatible with the equivalence relation "$\sim$". Thus we obtain from the Deformation Retraction-Quotient Theorem 19.7 (1) that $\overline{F}$ is a deformation retraction from the Möbius band to $C/\sim$. Note that by the discussion on page 314 we know that $C/\sim$ is homeomorphic to S^1. □

19.2. **Homotopy equivalent topological spaces.** For the reader's convenience we recall Definition 18.2:

Definition 19.8. Let X and Y be topological spaces. In the following we say that two maps $f_0, f_1\colon X \to Y$ are homotopic, written $f_0 \simeq f_1$, if there exists a continuous map $F\colon X\times[0,1]\to Y$ with $F_0 = f_0$ and with $F_1 = f_1$.

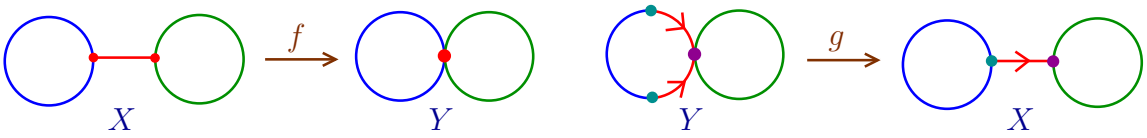
The following definition is one of the key definitions in topology. Unfortunately it usually takes some time to fully digest it.

Definition 19.9. Let X and Y be topological spaces.

(1) A continuous $f\colon X\to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y\to X$ with $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent.
(3) Given a topological space X we refer to the class of all topological spaces that are homotopy equivalent to X as the homotopy type of X.

Example.

(1) Basically by definition a homeomorphism is a homotopy equivalence.
(2) In the figure we see two subspaces X and Y of $\mathbb{R}^2$ and we indicate continuous maps $f\colon X\to Y$ and $g\colon Y\to X$.



With some effort one can show that $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$. Thus f and g are homotopy equivalences and X and Y are homotopy equivalent. We will return to this example in Exercise 19.11. □

The following lemma shows that in the previous section we already acquired a long list of homotopy equivalences.

Lemma 19.10. (Deformation Retract–Homotopy Equivalence Lemma) Let X be a topological space and let A be a subset. If $F\colon X\times[0,1]\to X$ is a deformation retraction

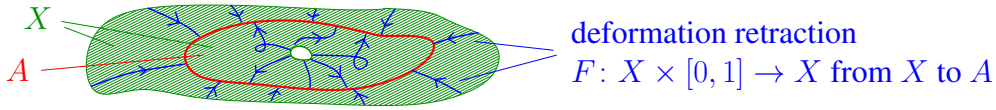
from X to A, then the inclusion map

$$\begin{array}{rcl} i\colon A & \to & X \\ a & \mapsto & a \end{array} \qquad \text{and the map} \qquad \begin{array}{rcl} r\colon X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are homotopy inverses of one another. In particular both maps are homotopy equivalences and in particular the deformation retract A is homotopy equivalent to the total space X.

Example. It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 and the discussions on page 523 and 527 that the sphere S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$ and that the Möbius band is homotopy equivalent to S^1. □

Proof. Let $F\colon X \times [0,1] \to X$ be a deformation retraction from X to A in the sense of Definition 19.3.



By definition of a deformation retraction the map $r \circ i = F_1 \circ i\colon A \to A$ is the identity id_A. Thus it remains to show that $i \circ r\colon X \to X$ is homotopic to the identity id_X. A homotopy between id_X and $i \circ r$ is given by $F\colon X \times [0,1] \to X$. ■

The name "homotopy equivalence" already suggests that this is an equivalence relation on the class of topological spaces. It is elementary, but slightly messy, to prove this by hand. Instead of proving this by hand we will first reformulate the notion of being homotopy equivalent in a more categorical language, and then the statement will become trivial.

19.3. The homotopy category.

In this section we introduce the homotopy category. Initially this might look like a detour, but in fact it simplifies and clarifies the discussion of homotopy equivalences.

Definition 19.11. Given two topological spaces X and Y we set

$$[X,Y] := \{\text{continuous maps } X \to Y\}/\sim$$

where $f \sim g$ if there exists a homotopy between f and g. We refer to $[X,Y]$ as the set of homotopy classes of continuous maps from X to Y. Sometimes, given a continuous map $f\colon X \to Y$ we denote by $[f]$ the equivalence class it represents in $[X,Y]$.

Example. It follows easily from the definitions that given a topological space the set $[\{*\}, X]$ is in bijection to $\pi_0(X)$, i.e. in bijection to the set of path-components of X. □

Definition 19.12. We define the homotopy category $\mathcal{HomTop}$ of topological spaces by[223]

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{HomTop}) & := & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

with the composition[224]

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

The following lemma changes subtly our point of view on homotopy equivalences.

Lemma 19.13. (Homotopy Equivalence-Reformulation Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomTop}}(X, Y)$ is an isomorphism in the homotopy category $\mathcal{HomTop}$.
(2) f is a homotopy equivalence.

In particular the following statements are equivalent:

(1′) X and Y are isomorphic in the homotopy category $\mathcal{HomTop}$.
(2′) X and Y are homotopy equivalent.

Proof. First we show the "(1)⇒(2)"-direction. Let $f\colon X \to Y$ be a continuous map. If $[f]$ is an isomorphism in the category $\mathcal{HomTop}$, then there exists a morphism $\varphi \in \mathrm{Mor}_{\mathcal{HomTop}}(Y, X)$ with $[f] \circ \varphi = \mathrm{id}_Y$ and $\varphi \circ [f] = \mathrm{id}_X$. We pick a continuous map $g\colon Y \to X$ with $\varphi = [g]$. Then we see that $[f \circ g] = [f] \circ [g] = [f] \circ \varphi = \mathrm{id}_Y = [\mathrm{id}_Y]$. But this means that $f \circ g \simeq \mathrm{id}_Y$. The same argument shows that $g \circ f \simeq \mathrm{id}_X$. Thus we have shown that f is a homotopy equivalence.

The argument for "(2)⇒(1)" follows almost immediately from the definitions. Finally note that the equivalence of (1′) and (2′) is an immediate consequence of the equivalence of (1) and (2). ■

Lemma 19.14. (Homotopy-Equivalence Basics Lemma)

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps between topological spaces. If two out of the three maps f, g and $g \circ f$ are homotopy equivalences, then so is the third.
(2) The relation "homotopy equivalence" is an equivalence relation on the class of all topological spaces.
(3) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. If f is a homotopy equivalence and if g is homotopic to f, then g is also a homotopy equivalence.
(4) Let X, Y and Z be topological spaces.
   (a) Let $f\colon X \to Y$ and let $g_0, g_1\colon Y \to Z$ be continuous maps such that $g_0 \circ f \simeq g_1 \circ f$. If f is a homotopy equivalence, then $g_0 \simeq g_1$.
   (b) Let $f_0, f_1\colon X \to Y$ and let $g\colon Y \to Z$ be continuous maps such that $g \circ f_0 \simeq g \circ f_1$. If g is a homotopy equivalence, then $f_0 \simeq f_1$.
(5) Let X, Y, X', Y' be a topological space. If $f\colon X \to X'$ and $g\colon Y \to Y'$ are homotopy equivalences, then so is the diagonal map $\mathrm{diag}(f, g)\colon X \times Y \to X' \times Y'$.

Proof. All the statements follow easily from the Homotopy Equivalence-Reformulation Lemma 19.13 and the Isomorphism Lemma 5.12. We leave it to the reader to fill in the details. ■

[223] Note that in this category the morphisms are not maps but they are equivalence classes of continuous maps.
[224] It follows from the Homotopy Composition Lemma 18.10 (2) that this composition map is well-defined. It follows immediately from the definitions that this composition satisfies the two conditions on a category, see Definition 5.1, namely associativity and existence of an identity morphism.

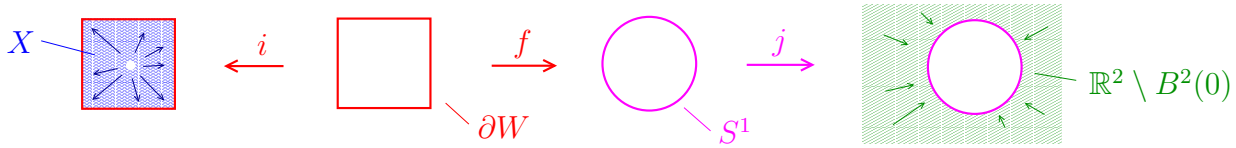
Example. We now have three methods for showing that topological spaces are homotopy equivalent:

(1) Homeomorphisms are almost by definition homotopy equivalences.
(2) The Homotopy-Equivalence Basics Lemma 19.14 (2) says that being homotopy equivalent is an equivalence relation.
(3) The Deformation Retract–Homotopy Equivalence Lemma 19.10 says that if A is a deformation retract of some topological space X, then the inclusion $i\colon A \to X$ is a homotopy equivalence.

The combination of these three methods can be very powerful: We set $W = [-1,1]\times[-1,1]$. We want to show that the topological spaces $X \coloneqq W\setminus\{0\}$ and $Y \coloneqq \mathbb{R}^2\setminus B^2(0)$ are homotopy equivalent. To do so we consider the maps

$$X = W\setminus\{0\} \xleftarrow{\ i\ } \partial W \xrightarrow[f]{x\mapsto \frac{x}{\|x\|}} S^1 \xhookrightarrow{\ j\ } \mathbb{R}^2\setminus B^2(0).$$

As on page 523 one can easily show that ∂W is a deformation retract of $X = W\setminus\{0\}$ and that S^1 is a deformation retract of $\mathbb{R}^2\setminus B^2(0)$. Furthermore f is a homeomorphism.



It follows from the combination of (1), (2) and (3) that $X = W\setminus\{0\}$ is homotopy equivalent to $Y = \mathbb{R}^2\setminus B^2(0)$.[225] □

Next we recall Definition 4.43 and a concept from page 236:

Definition 19.15. Let X be a topological space.

(1) We denote by $\pi_0(X)$ the set of path-components of X.
(2) Given a continuous map $f\colon X\to Y$ we write

$$\begin{aligned} f_*\colon \pi_0(X) &\to \pi_0(Y)\\ C &\mapsto \text{unique path-component of } Y \text{ that contains } f(C). \end{aligned}$$

Lemma 19.16. (Homotopy-π_0-Lemma) The maps

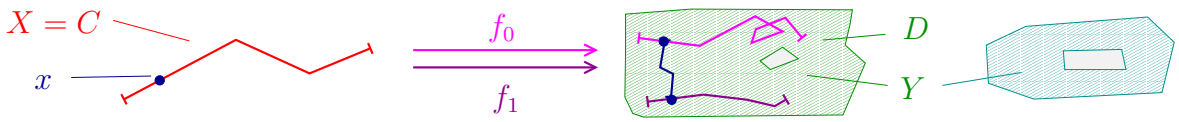
$$\begin{aligned} X &\mapsto \pi_0(X)\\ [f\colon X\to Y] &\mapsto [f_*\colon \pi_0(X)\to\pi_0(Y)] \end{aligned}$$

form a covariant functor from the homotopy category $\mathcal{HomTop}$ to the category $\mathcal{Set}$ of sets.

Proof. We only really need to show that the functor is well-defined. More precisely, we need to show that if $f_0, f_1\colon X\to Y$ are homotopic, then $f_{0*} = f_{1*}\colon \pi_0(X)\to\pi_0(Y)$. Thus

[225] In the Point Deformation Retract Lemma 19.19 and much later in Exercise **??** we will see that it is in general not true that homotopy equivalent topological spaces admit deformation retracts that are homeomorphic. Nonetheless in the Homotopy Equivalent–Characterization Corollary 22.21 we will show that (1), (2) and (3) are in fact enough to characterize homotopy equivalence. More precisely we will show that given two topological spaces X and Y the following two statements are equivalent:

- X and Y are homotopy equivalent.
- There exists a topological space W and closed embeddings $i\colon X\to W$ and $j\colon Y\to W$ such that $i(X)$ is a deformation retract of W and $j(Y)$ is a deformation retract of W.



let C be a path-component of X and let D be the unique path-component of Y that contains $f_0(C)$. Since f_0 and f_1 are homotopic we see that for every $x \in X$ the points $f_0(x)$ and $f_1(x)$ are path-equivalent in Y.
But that means, by Definition 4.43 of a path-component as a path-equivalence class, that we have an inclusion $f_1(C) \subseteq D$. This implies that $f_{1*}(C) = D = f_{0*}(C)$. ■

Lemma 19.17. (Homotopy-Equivalence π_0-Lemma)
(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a homotopy equivalence, then f induces a bijection $f_*\colon \pi_0(X) \to \pi_0(Y)$.
(2) If X and Y are homotopy equivalent topological spaces, then X is path-connected if and only if Y is path-connected.

Proof.
(1) This statement follows from the Homotopy-π_0-Lemma 19.16, the Homotopy Equivalence-Reformulation Lemma 19.13 and the Isomorphism Preservation Lemma 5.26.
(2) This statement is a special case of (1). ■

19.4. **Contractible topological spaces.** We turn to an important special case of homotopy equivalences:

Definition 19.18. Let X be a topological space. We say X is contractible, if X is homotopy equivalent to a topological space which consists of precisely one point.

Remark. Note that the empty topological space is *not* contractible. □

The following straightforward lemma gives an easy criterion for showing that a topological space is contractible.

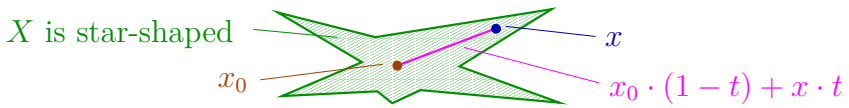
Lemma 19.19. (Point Deformation Retract Lemma) If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.

Proof. This lemma is just a special case of the Deformation Retract–Homotopy Equivalence Lemma 19.10. ■

Example.
(1) We say that a subset X of $\mathbb{R}^n$ is star-shaped if there exists a point $x_0 \in X$ such that for every $x \in X$ the Euclidean segment from x_0 to x lies in X. Basically by the same argument as in the Convex Homotopy Lemma 18.3 we see that the map

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \underbrace{x \cdot (1-t) + x_0 \cdot t}_{\in X, \text{ since } X \text{ is star-shaped}} \end{array}$$

is deformation retraction from X to the point x_0. It follows from the Point Deformation Retract Lemma 19.19 that X is contractible.



(2) Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. We just learned that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are contractible. It follows from the Homotopy-Equivalence Basics Lemma 19.14 (2) that $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent. □

The following elementary lemma gives a handy criterion for showing that a topological space is contractible.

Lemma 19.20. (Contractability Criterion) Let X be a topological space.

(1) X is contractible $\iff$ (2) there exists an $x_0 \in X$ such that the inclusion $\{x_0\} \to X$ is a homotopy equivalence
$\iff$ (3) there exists an $x_0 \in X$ such that the identity map id_X is homotopic to the constant map $x \mapsto x_0$.

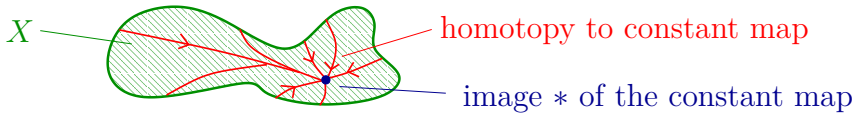
Furthermore, if Statement (2) or (3) of the Contractability Criterion Lemma 19.20 hold for a single point, then the statements also hold for all points in X.

Proof.

(1)⇒(2) First suppose that X is contractible. Let $Y = \{*\}$ be the topological space consisting of a single element. Since X is contractible we know that there exists a continuous map $g\colon \{*\} \to X$ that is a homotopy equivalence. We set $x_0 := g(*)$. Since $\{*\} \to \{x_0\}$ is a homeomorphism we see that the inclusion map $\{x_0\} \to X$ is also a homotopy equivalence.

(2)⇒(3) Let $x_0 \in X$ be a point such that the inclusion map $f\colon \{x_0\} \to X$ is a homotopy equivalence. We denote by $g\colon X \to \{x_0\}$ the unique map. Since there is only one such map we see that g is in particular the only homotopy inverse of f. It follows that $f \circ g\colon X \to X$ is homotopic to id_X. In other words, we have shown that id_X is homotopic to the map $f \circ g$, which is of course just the constant map $x \mapsto x_0$.

(3)⇒(1) Finally we suppose that the identity map of X is homotopic to a constant map $f\colon X \to X$. Let $*$ be the unique point in the image of f.

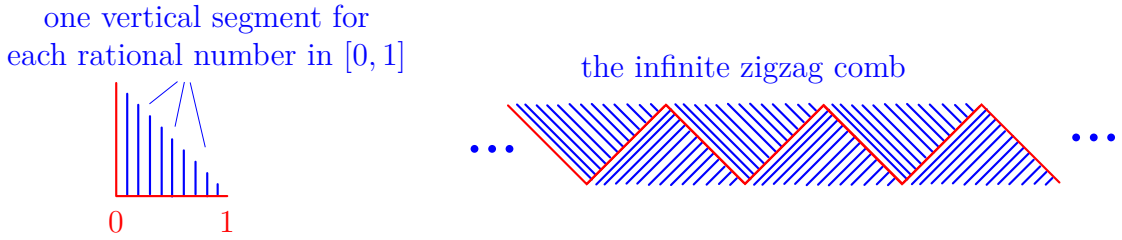


We set $Y := \{*\}$. Let $g\colon \{*\} \to X$ be the inclusion map. By hypothesis we have $g \circ f \simeq \mathrm{id}_X$. Furthermore we evidently have $f \circ g = \mathrm{id}_Y$. This shows that X is homotopy equivalent to $Y = \{*\}$.

We turn to the proof of the "furthermore statement". Note that it follows from the Homotopy-Equivalence π_0-Lemma 19.17 that every contractible space is path-connected, which means that any two maps $\{*\} \to X$ are homotopic. The desired statement now follows from the Homotopy-Equivalence Basics Lemma 19.14 (3). ■

In the Point Deformation Retract Lemma 19.19 we saw that if a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is contractible. The following lemma shows that in general the converse of this statement does *not* hold.

Lemma 19.21. (Zigzag Comb-Lemma) The "infinite zigzag comb" shown in the figure below is contractible, but it does not admit a deformation retraction to a single point.



Proof. This statement will be proved in Exercise 19.10. ■

Remark.

(1) The topological space in the Zigzag Comb-Lemma 19.21 is quite peculiar. In fact in the Whitehead Deformation Retract Proposition 305.2 we will see that if X is contractible and if X is a "CW-complex", then every point of X is in fact a deformation retract. We will also seen that most "reasonable" topological spaces are "CW-complexes".

(2) Unfortunately in the literature the term "deformation retract" is used differently by different authors. Our notion of a "deformation retract", as given in Definition 19.3, agrees with the definition in [**Hat2002**, p. 2] and [**Mun1984**, p. 208]. What we call a "deformation retract" is called a "strong deformation retract" in [**Bre1993**, p. 45] and [**Rot1988**, p. 209]. In these books this notion is used to distinguish it from the notion of a "weak deformation retract": A subspace A is called a weak deformation retract of X if there exists a continuous map $F\colon X \times [0,1] \to X$ such that
   (a) $F_0 = \mathrm{id}_X$,
   (b) $F_1|_A = \mathrm{id}_A$ (instead of the stronger condition that F is a homotopy rel A),
   (c) $F_1(X) \subseteq A$.
   It turns out that the two concepts of a weak and a strong deformation retract are indeed different. For example, let Z be the above infinite zigzag comb. It follows from the Zigzag Comb-Lemma 19.21 and the "furthermore statement" of the Contractability Criterion Lemma 19.20 that every point of Z is a weak deformation retract of Z but the Zigzag Comb-Lemma 19.21 says that no one-point subset of Z is a strong deformation retract of Z. □

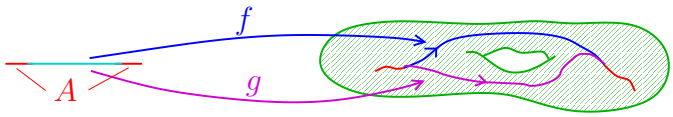
19.5. **Homotopies and homotopy equivalences for pairs of topological spaces.** In this section we provide a longish list of definitions that extend the definitions of the previous two chapters to the relative setting. This makes for slightly dull reading and in all likelihood the reader is best served by moving straight to the next chapter.

In Definition 18.2 we defined what it means for two maps $f, g\colon X \to Y$ between topological spaces to be homotopic and in Definition 19.9 we defined what it means for two topological spaces to be homotopy equivalent.

On many occasions one would like to restrict the freedom of a homotopy by fixing some subset pointwise or setwise. This idea leads us to two relative versions of "homotopy". The first version we already saw in Definition 18.4. For the reader's convenience we recall the definition.

Definition 19.22. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $A \subseteq X$ be a subset such that $f|_A = g|_A$.

(1) A homotopy between f and g rel A is a homotopy $H\colon X \times [0,1] \to Y$ between f and g such that $H(a,t) = f(a) = g(a)$ for all $t \in [0,1]$ and all $a \in A$.
(2) We say that f and g are homotopic rel A, if such a homotopy rel A exists.



Next we introduce the following harmless definition which is an analog of the category of pairs of sets that we had introduced in Definition 5.4.

Definition 19.23.

(1) A pair of topological spaces is a pair (X, A), where X is a topological space and A is a subset of X.
(2) A continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces is a continuous map $f\colon X \to Y$ of topological spaces with $f(A) \subseteq B$.
(3) We refer to the category $\mathcal{PairTop}$ with

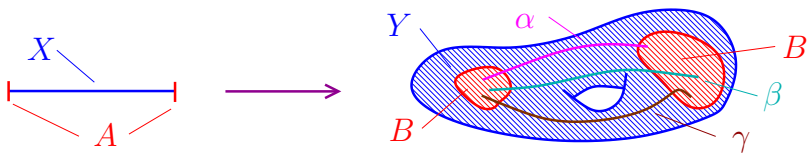
$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subseteq B \end{aligned}$$

with the usual composition of maps as the category of pairs of topological spaces.

We continue with another notion of a "relative homotopy".

Definition 19.24. Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subsets. Let $f, g\colon (X,A) \to (Y,B)$ be two continuous maps of pairs of topological spaces.

(1) A homotopy between f and g is a continuous map[226]
$$H\colon X \times [0,1] \to Y,$$
such that the following two conditions are satisfied:
   (a) $H_0 = f$ and $H_1 = g$, and
   (b) Each H_t is a map $(X,A) \to (Y,B)$. In other words, we have $H(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.
(2) If there exists a homotopy between f and g, then we say that f and g are homotopic and we write $f \simeq g$.
(3) We set
$$[(X,A),(Y,B)] := \{\text{continuous maps } (X,A) \to (Y,B)\}/\sim$$
where $f \sim g$ if there exists a homotopy between f and g.



$\alpha, \beta\colon (X,A) \to (Y,B)$ are homotopic but $\alpha, \gamma\colon (X,A) \to (Y,B)$ are not homotopic

[226] For $A = B = \varnothing$ we get the usual notion of a homotopy between maps $f, g\colon X \to Y$.

The above Definition 19.24 leads us straight to the following generalization of the notion of the homotopy category, which we introduced in Definition 19.12.

Definition 19.25. We define the homotopy category *HomPairTop* of pairs of topological spaces by

$$\begin{aligned} \mathrm{Ob}(\mathit{HomPairTop}) &:= \text{ all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= [(X,A),(Y,B)] \end{aligned}$$

with the composition[227]

$$\begin{aligned} \mathrm{Mor}((X,A),(Y,B)) \times \mathrm{Mor}((Y,B),(Z,C)) &\to \mathrm{Mor}((X,A),(Z,C)) \\ ([f],[g]) &\mapsto [g \circ f]. \end{aligned}$$

The Homotopy Equivalence-Reformulation Lemma 19.13 motivates the following relative analog of the concepts that we introduced in Definition 19.9.

Definition 19.26. Let (X,A) and (Y,B) be two pairs of topological spaces.

(1) A continuous map $f\colon (X,A) \to (Y,B)$ is a homotopy equivalence between (X,A) and (Y,B), if f represents an isomorphism in the homotopy category *HomPairTop* of pairs of topological spaces. In other words, if there exists a continuous map $g\colon (Y,B) \to (X,A)$ such that the following hold:
   (a) $g \circ f$ and id_X are homotopic as maps of (X,A) to itself,
   (b) $f \circ g$ and id_Y are homotopic as maps of (Y,B) to itself.

(2) If there exists a homotopy equivalence between (X,A) and (Y,B), then we say that (X,A) and (Y,B) are homotopy equivalent.

Examples.

(1) The two pairs of topological spaces (X,A) and (Y,B) shown in the figure below are homotopy equivalent.



(2) If (X,A) and (Y,B) are homotopy equivalent, then so are X and Y and also A and B.

(3) Let $A \subseteq Y$ be subsets of X. If Y is a deformation retract of X, then the inclusion map $(Y,A) \to (X,A)$ is easily seen to be a homotopy equivalence of pairs of topological spaces. □

The above notions can be generalized even further. For example, one could define triples of topological spaces and there is an obvious definition of what it means for continuous maps $f,g\colon (X,A,B) \to (Y,C,D)$ between triples of topological spaces to be homotopic. Similarly, given pairs of topological spaces (X,A) and (Y,B) and a subset Z of A there is an obvious definition for what it means for two maps $f,g\colon (X,A) \to (Y,B)$ to be homotopic rel Z. We will not spell out all the definitions. From the context it should always be clear what we mean.

Remark. In later chapters we will get to know many covariant functors from the homotopy category *HomTop* to an "algebraic category", e.g. the category *Group* of groups or the

[227] It follows again from the Homotopy Composition Lemma 18.10 (2) that this composition map is well-defined.

category *AbGroup* of abelian groups. The most important functors ("invariants") will be the fundamental groups, higher homotopy groups and homology groups.

By definition such functors turn homotopy equivalences into isomorphisms in the corresponding algebraic category. In particular these invariants can be used to show that maps between topological spaces are not homotopy equivalences and they can be used to show that certain pairs of topological spaces are not homotopy equivalent.

Much much later we will state and prove the Whitehead Theorem 305.1 which gives a partial converse, namely it gives a sufficient criterion for a continuous map between "CW-complexes" to be a homotopy equivalence. In many cases this gives a fairly practical criterion for showing that a given continuous map is a homotopy equivalence. □

19.6. Exercises for Chapter 19.

Exercise 19.1. We consider the interval $[0,1]$. Is $\{0,1\} \subseteq [0,1]$ a retract of $[0,1]$?

Exercise 19.2. Let X be a contractible topological space and let $A \subseteq X$ be a retract of X. Recall that this means that there exists a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. Is A necessarily contractible?

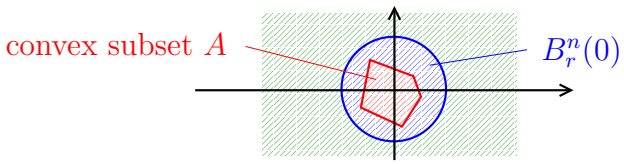
Exercise 19.3. Let X be a topological space and let $A \subseteq X$ be a subspace. We suppose that there exists a continuous map $H\colon X \times [0,1] \to X$ with the following properties:

(a) The map $H_0\colon X \to X$ is a homotopy equivalence.
(b) $H_1(X) \subseteq A$.
(c) We have $H(A \times [0,1]) \subseteq A$.
(d) The map $H_1\colon A \to A$ is a homotopy equivalence.

Show that the inclusion $i\colon A \to X$ and the map $H_1\colon X \to A$ are both homotopy equivalences.

Hint. Work in the homotopy category and make use of the Inverse Uniqueness Lemma 5.10 (1).

Exercise 19.4. Let A be a convex subset of $\mathbb{R}^n$ which contains the origin 0. Let $r > 0$ such that $A \subseteq B_r^n(0)$. Show that $\mathbb{R}^n \setminus B_r^n(0)$ is a deformation retract of $\mathbb{R}^n \setminus A$.

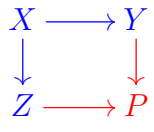


Exercise 19.5. Let X be a topological space and let Y be a contractible topological space.

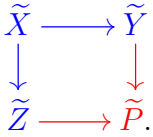
(a) Show that all continuous maps $X \to Y$ are homotopic.
(b) We suppose that X is path-connected. Show that all continuous maps $Y \to X$ are homotopic.

Exercise 19.6. Let $X \subseteq S^2$ be a contractible subspace. Is the complement $S^2 \setminus X$ also contractible?

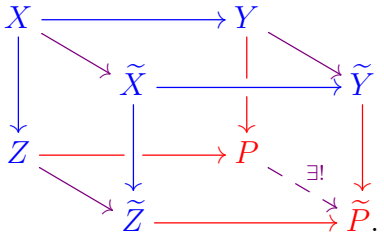
Exercise 19.7. Suppose we are given two pushouts



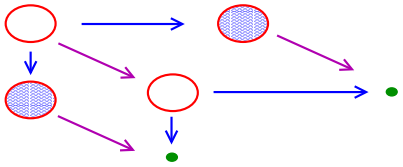
and



Furthermore suppose we are given a commutative diagram



An example for such a situation is sketched in the figure below.



(a) Show that there exists a unique map $P \to \widetilde{P}$ which makes the cube commute.
(b) Suppose that the maps $X \to \widetilde{X}$, $Y \to \widetilde{Y}$ and $Z \to \widetilde{Z}$ are homotopy equivalences. Does it follows that the map $P \to \widetilde{P}$ is a homotopy equivalence?

Exercise 19.8. Show that $\mathrm{GL}(2,\mathbb{R})$ is homotopy equivalent to a subspace of $\mathbb{R}^2$.

Exercise 19.9. Let X, Y, Z be topological spaces and let $g\colon Y \to Z$ be a continuous map. We get an induced map

$$\begin{array}{rcl} g_*\colon [X,Y] & \to & [X,Z] \\ [f] & \mapsto & [g \circ f]. \end{array}$$

Show that if g is a homotopy equivalence, then g_* is a bijection.
Hint. Use the Homotopy Equivalence-Reformulation Lemma 19.13 and work in the homotopy category and try to work as formally as possible.

Exercise 19.10.

(a) We consider the topological space

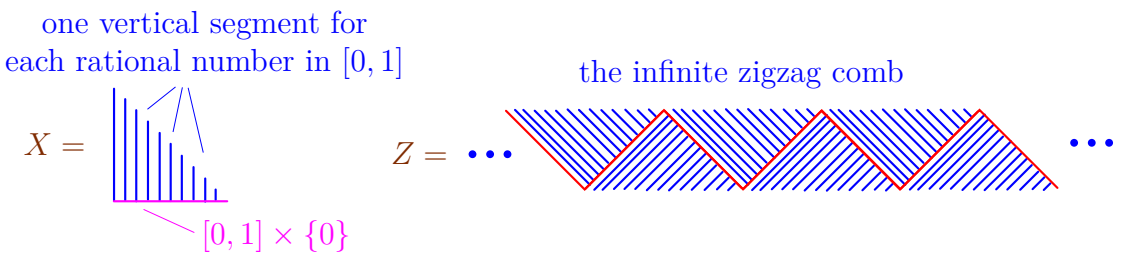
$$X \;=\; \{(x,y)\,|\,x \in [0,1] \cap \mathbb{Q}, y \in [0,1-x]\} \cup ([0,1] \times \{0\})$$

that is shown in the figure below to the left.
  (i) Show that given any point P in $[0,1] \times \{0\}$ there exists a deformation retraction from X to P.
  (ii) Show that for any point Q in X that does not lie on $[0,1] \times \{0\}$ there is no deformation retraction from X to Q.

(b) We consider the infinite zigzag comb Z that is shown in the figure below to the right.
  (i) Show that Z is contractible.
  (ii) Show that Z does not admit a deformation retraction to any point in Z.

Exercise 19.11. In the figure we see two subspaces X (called the *dumbbell graph*) and Y (called the wedge of two circles) of $\mathbb{R}^2$. Show that these two topological spaces are homotopy equivalent in two different ways:

(a) Find explicit maps $f\colon X \to Y$ and $g\colon Y \to X$ with homotopies from $g \circ f$ to id_X and from $f \circ g$ to id_Y.



(b) Find a topological space Z that contains (homeomorphic) copies of X' of X and Y' of Y such that X' and Y' are both deformation retracts of Z.



Exercise 19.12. Let $X = \{0, 1\}$ be the Sierpiński space, i.e. let X be the topological space with two elements 0 and 1 and where the open subsets are precisely $\varnothing, \{0\}, X$.

(a) Is $\{0\}$ a deformation retract of X?
(b) Is $\{1\}$ a deformation retract of X?

Exercise 19.13.

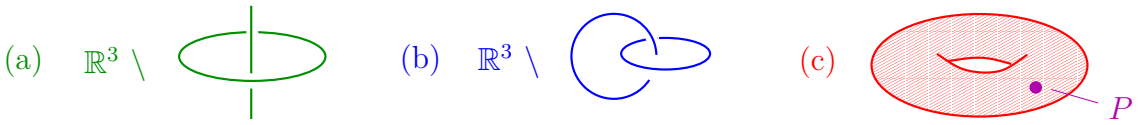
(a) Is $\mathrm{SL}(n, \mathbb{R})$ a deformation retract of $\mathrm{GL}(n, \mathbb{R})$?
(b) Is $\mathrm{SL}(n, \mathbb{C})$ a deformation retract of $\mathrm{GL}(n, \mathbb{C})$?

Remark. You might not give a complete answer, but you could try to make an educated guess.

Exercise 19.14. In the figure below we show the following three topological spaces:

(a) $\mathbb{R}^3$ with the z-axis and the unit circle in the xy-plane removed.
(b) $\mathbb{R}^3$ with two "linked" circles removed.
(c) The torus $S^1 \times S^1$ with one point P removed.

What are the "smallest" subspaces onto which you can find a deformation retraction?



Exercise 19.15.

(a) We consider the letters in the alphabet:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

How many different homotopy types of topological spaces can you find among the letters?

(b) How does the number of homotopy types change if you add the letters of your mother tongue?

Exercise 19.16. Let X and Y be topological spaces and let $x \in X$ and $y \in Y$.
(a) If X and Y are homotopy equivalent, are $X \setminus \{x\}$ and $Y \setminus \{y\}$ homotopy equivalent?
(b) If $X \setminus \{x\}$ and $Y \setminus \{y\}$ are homotopy equivalent, are X and Y homotopy equivalent?

Exercise 19.17. Let $\mathrm{Homeo}(S^1)$ be the group of self-homeomorphisms of S^1. We equip it with the compact-open topology. We view the group of rotations $\mathrm{SO}(2)$ as a subgroup of $\mathrm{Homeo}(S^1)$. Show that $\mathrm{O}(2)$ is a deformation retract of $\mathrm{Homeo}(S^1)$.

Exercise 19.18. Let $0 < r < s$ be two real numbers. Show that the obvious inclusion map $B_r^n(0) \setminus \{0\} \to B_s^n(0) \setminus \{0\}$ is a homotopy equivalence.

Exercise 19.19. Let $T = (V, E, i, t)$ be a finite tree in the sense of Definition 11.21. As in Definition 11.5 we consider the corresponding topological realization $|T|$. Show that the topological realization $|T|$ is contractible.
Remark. You could make use of the Tree–Valence One Edge Lemma 11.30. You should also explain carefully why the homotopies you write down are actually continuous.

Exercise 19.20. Let X be a topological space and let $H\colon X \times [0,1] \to X$ be a deformation retraction to a subset A. We consider the map

$$\begin{aligned} \phi\colon X &\to [0,1] \\ x &\mapsto \inf\{t \in [0,1] \mid H(x,t) \in A\}. \end{aligned}$$

Is the map ϕ necessarily continuous?

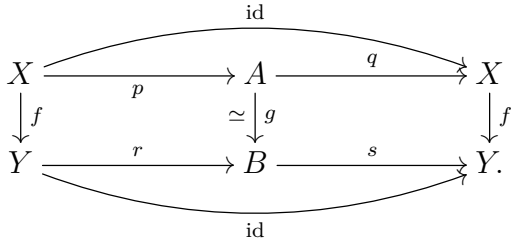
Exercise 19.21. Show that the topological spaces

$$X = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \quad \text{and} \quad Y = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{-1\}$$

are not homotopy equivalent.



Exercise 19.22. Let $f\colon X \to Y$ and $g\colon A \to B$ be continuous maps between topological spaces. We say *f is a retract of g* if there exists a commutative diagram of maps



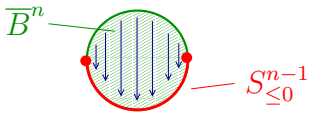
such that $q \circ p = \mathrm{id}_X$ and $s \circ r = \mathrm{id}_Y$. Show that if g is a homotopy equivalence, then f is also a homotopy equivalence.
Hint. Work with the homotopy category $\mathcal{HomTop}$.

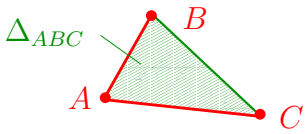
Exercise 19.23. Does there exist a topological space with three elements that is homotopy equivalent to the topological space $X = \{*\}$ given by a single point?

Exercise 19.24. Let $n \in \mathbb{N}$. Prove the following two statements:

(a) The hemispheres $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.

(b) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.
Hint. You could first consider the case that $A = (0,0)$, $B = (0,1)$ and $C = (1,0)$.





Exercise 19.25. Let

$$S \coloneqq \{(0,y) \,|\, y \in [-1,1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \subseteq \mathbb{R}^2.$$

be the "Topologist's Sine Curve" from page 195. As in Exercise 8.4 we refer to the quotient space $W \coloneqq S/(0,0) \sim (\frac{1}{\pi}, 0)$ as the *Warsaw Circle*. Our goal is to show that the circle is not contractible. To do so we introduce the following notation:

- We denote by $p\colon S \to W$ the projection.
- We set $Q = p(0,0)$ and given $n \in \mathbb{N}$ we set $Q_n = p((\frac{1}{\pi n}, 0))$.
- We set $L \coloneqq \{p(0,y) \,|\, y \in [-1,1]\} \subseteq W$ and $U \coloneqq \big\{p(x, \sin(\frac{1}{x})) \,\big|\, x \in (0, \frac{1}{2\pi}]\big\} \subseteq W$.

Now go through the following steps:

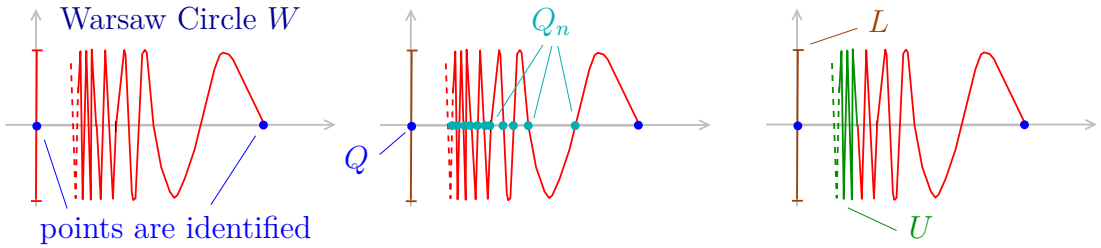
(a) Let $g\colon W \to W$ be a continuous map. We suppose that there exists an $n \in \mathbb{N}$ such that for all $m \geq n$ we have $g(Q_m) \in U$. Show that $g(Q) \in L \cup U$.

(b) Let $F\colon W \times [0,1] \to W$ be a homotopy. Show that

$$T \coloneqq \left\{ t \in [0,1] \,\middle|\, \begin{array}{c} F_t(Q) \in L \text{ and there exists an } n \in \mathbb{N} \text{ such} \\ \text{that for all } m \geq n \text{ we have } F_t(Q_m) \in U \end{array} \right\} \subseteq [0,1]$$

is open and closed.

(c) Show that W is not contractible.
Hint. Suppose that W is contractible. By the Contractability Criterion Lemma 19.20 this means that there exists a homotopy $F\colon W \times [0,1] \to W$ with $F_0 = \mathrm{id}$ and such that $F_1(w) = Q$ for all $w \in W$. Use (b) to show that such F cannot exist.



Exercise 19.26. We consider $X = [-1,1]/-1 \sim 1$ and we denote by $p\colon [-1,1] \to X$ the projection. We consider the two subspaces $A = p((-1,1)) \subseteq X$ and $B = p(\{0\}) \subseteq X$. Are the two pairs (X,A) and (X,B) of topological spaces homotopy equivalent?

Exercise 19.27. Let $U \subseteq \mathbb{R}^n$ be an open neighborhood of 0 and let $f\colon U \to \mathbb{R}^n$ be a smooth map with $f^{-1}(0) = \{0\}$ and such that the differential $\mathrm{D}\, f_0$ is invertible. Let $g\colon U \to \mathbb{R}^n$ be a continuous map such that $f(x) - g(x) = O(\|x\|^2)$, i.e. we assume that there exists a $C \geq 0$ and a $\delta > 0$ such that $\|f(x) - g(x)\| < C \cdot \|x\|^2$ for all $x \in \overline{B}^n_\delta$. Show that there exists an $s > 0$ such that the maps f and g are homotopic as maps of pairs $(\overline{B}^n_s, \overline{B}^n_s \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.

Exercise 19.28. Let X be a topological space and let $\varphi\colon S^n \to X$ be a continuous map. Show that the following two statements are equivalent:

(1) The map φ is null-homotopic, i.e. homotopic to a constant map.
(2) There exists a continuous map $\Phi\colon \overline{B}^{n+1} \to X$ with $\Phi|_{S^n} = \varphi$.

Exercise 19.29. Let X be Hausdorff space and let Y be a topological space that is homotopy equivalent to Y. Is Y necessarily also Hausdorff?

Exercise 19.30.

(a) Is $\mathrm{SL}(n,\mathbb{R})$ a retract of $\mathrm{GL}(n,\mathbb{R})$?
(b) Is $\mathrm{SL}(n,\mathbb{R})$ a deformation retract of $\mathrm{GL}(n,\mathbb{R})$?

Exercise 19.31. Let $K = \{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ be the trivial knot in the sphere $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. Let $X = S^3 \setminus K$ and let $Y = \mathbb{R}^3 \setminus K$. One of the two topological spaces X or Y is homotopy equivalent to S^1. Which one?

Exercise 19.32. Let $n \in \mathbb{N}$. We consider the topological space $\mathbb{R}^n \cup \{\infty\}$. Let $I \subseteq \mathbb{R}$ be an interval. We consider $B_I \coloneqq \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \in I\} \cup \{\infty\}$. We equip B_I with the subspace topology coming from $\mathbb{R}^n \cup \{\infty\}$.

(a) For which intervals I is the map

$$\begin{array}{rcl} F\colon B_I \times [0,1] & \to & \mathbb{R}^n \cup \{\infty\} \\ (P,t) & \mapsto & \begin{cases} \infty, & \text{if } P = \infty \text{ or } t = 1, \\ P + (\frac{1}{1-t} - 1) \cdot \mathfrak{e}_n, & \text{otherwise} \end{cases} \end{array}$$

continuous?
(b) For which intervals I does F define a deformation retraction from B_I to $\{\infty\}$?

Exercise 19.33. Show that topological graphs are locally contractible.
Remark. This exercise can be viewed as a special case of the CW-Complex–Local Properties Proposition 137.8 (2).

Exercise 19.34. Let X be a topological space and let $U \subseteq X$ be a subset with components $\{U_i\}_{i \in I}$. We suppose that for each $i \in I$ the inclusion $U_i \to X$ is null-homotopic and we assume that X is path-connected. Does it follow that the inclusion $U \to X$ is null-homotopic?

CHAPTER 20

Joins, cones and suspensions

In this chapter we will get to know three systematic ways to construct new topological spaces out of given topological space(s). More precisely, we will introduce the join $X * Y$ of two topological spaces and we will introduce the cone $\mathrm{Cone}(X)$ and the suspension $\Sigma(X)$ of a topological space X. As we will see, the latter two constructions are actually just special cases of the join construction.

These constructions in their own right are perhaps not overly interesting. But they will play a useful role throughout these notes. We could have introduced these topological spaces much earlier, the only reason that we introduce these notions at this moment is that we will also prove a few basic facts about their homotopy types.

This chapter is organized as follows:

- In Section 20.1 we will introduce the join $X * Y$ of two topological spaces X and Y.
- In Section 20.2 we will introduce the cone $\mathrm{Cone}(X)$ of a topological space X.
- In Section 20.3 we will introduce the suspension $\Sigma(X)$ of a topological space X.
- In Section 20.4 we will provide exercises for Chapter 20.

In the next Chapter 21 we will introduce mapping cylinders, mapping cones, and mapping tori.

20.1. **The join of two topological spaces.** In this section we introduce the join of two topological spaces X and Y. Loosely speaking the join $X * Y$ is obtained by connecting each point in X to each point in Y via an interval. The following definition makes this idea precise:

Definition 20.1. Let X and Y be two topological spaces.

(1) First we assume that both X and Y are non-empty. On the set $X \times [0,1] \times Y$ we consider the equivalence relation that is generated by the following relations:[228]

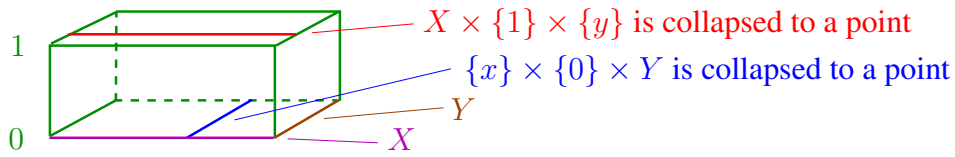
$$\begin{aligned} (x, 0, y_1) &\sim (x, 0, y_2), \quad \text{for all } x \in X \text{ and } y_1, y_2 \in Y \\ (x_1, 1, y) &\sim (x_2, 1, y), \quad \text{for all } x_1, x_2 \in X \text{ and } y \in Y. \end{aligned}$$

We refer to

$$X * Y \;:=\; (X \times [0,1] \times Y)/\sim$$

as the join of the topological spaces X and Y. Evidently we equip $X \times [0,1] \times Y$ with the product topology and we equip $(X \times [0,1] \times Y)/\sim$ with the corresponding quotient topology.

(2)
- For $Y = \varnothing$ we set $X * \varnothing := X \times \{0\}$.[229]
- For $X = \varnothing$ we set $\varnothing * Y := \{1\} \times Y$.[230]



We will do a few examples in a second, but before we can do so, let us get a few formalities out of the way:

Proposition 20.2. (Join Topology-Properties Proposition) Let X and Y be topological spaces.
(1) If X and Y are compact, then $X * Y$ is also compact.
(2) If X and Y are (path)-connected, then $X * Y$ is also (path-) connected.
(3) If X and Y are Hausdorff, then $X * Y$ is also Hausdorff.

Proof. If X or Y is the empty space, then the statements hold trivially. Now we suppose that X and Y are both non-empty. Note that in this case the first two statements follow immediately from the Product Topology-Properties Proposition 7.9 and the Quotient Topology-Properties Proposition 8.5 together with the observation that $[0,1]$ is compact and (path-) connected. Thus it remains to prove the following claim:

Claim. If X and Y are Hausdorff, then $X * Y$ is also Hausdorff.

Proof. Let $\pi\colon X \times [0,1] \times Y \to X * Y$ be the natural projection. Now let $P_1 = [(x_1, t_1, y_2)]$ and $P_2 = [(x_2, t_2, y_2)]$ be two distinct points in $X * Y$. We need to distinguish a few cases:

- If $t_1 = 0$ and $t_2 = 0$, then obtain easily from the definition of the join that $x_1 \neq x_2$. Since X is Hausdorff we can pick disjoint open neighborhoods $U_1, U_2 \subseteq X$ of $x_1, x_2 \in X$. An elementary argument shows that $\pi^{-1}(\pi(U_i \times [0,1) \times Y)) = U_i \times [0,1) \times Y$, which is an open subset of $X \times [0,1] \times Y$. It now follows from the Definition 8.3 of the quotient topology that $\pi(U_1 \times [0,1) \times Y)$ and $\pi(U_2 \times [0,1) \times Y)$ are the desired disjoint open neighborhoods of P_1 and P_2.
- The case $t_1 = t_2 = 1$ is dealt with like the previous case.
- If $t_1 \neq t_2 \in [0,1]$ then we pick disjoint open subsets $J_1, J_2 \subseteq [0,1]$ with $t_1 \in J_1$ and with $t_2 \in J_2$. One can easily verify that $\pi(X \times J_1 \times Y)$ and $\pi(X \times J_2 \times Y)$ are the desired disjoint open neighborhoods of P_1 and P_2.
- If $t_1 = t_2 \in (0,1)$, then we see that $(x_1, y_1) \neq (x_2, y_2)$. Note that it follows from the Product Topology-Properties Proposition 7.9 (3), together with the Definition 7.3 of the product topology, that there exist disjoint open neighborhoods $U_1 \times V_1, U_2 \times V_2 \subseteq X \times Y$ of $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then $\pi(U_1 \times (0,1) \times V_1)$ and $\pi(U_2 \times (0,1) \times V_2)$ are the desired disjoint open neighborhoods of P_1 and P_2. ∎

The following, otherwise broadly irrelevant lemma, justifies somewhat the idea that the join $X * Y$ is obtained by connecting each point in X to each point in Y via an interval.

[228] Loosely speaking we do the following:
- for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
- for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

[229] This slightly weird definition will be useful when we consider maps between joins.
[230] Note that for $X = Y = \varnothing$ we get $X * Y = \varnothing * \varnothing = \varnothing$.

Lemma 20.3. (Join Euclidean Embedding Lemma) Given $P, Q \in \mathbb{R}^n$ we denote by $\overline{PQ} := \{x \cdot (1-t) + y \cdot t \,|\, t \in [0,1]\}$ the Euclidean segment determined by P and Q. Let X and Y be two compact non-empty subsets of $\mathbb{R}^n$ which have the property that for any $x, x' \in X$ and $y, y' \in Y$ with $(x,y) \neq (x',y')$ the Euclidean segments $\overline{xy}$ and $\overline{x'y'}$ intersect at most in the endpoints. Then the map

$$\begin{array}{rcl} \Theta\colon X * Y = (X \times [0,1] \times Y)/\sim & \to & \overbrace{\bigcup_{x \in X, y \in Y} \overline{xy}}^{\text{equipped with the subspace topology coming from } \mathbb{R}^n} \\ [(x,t,y)] & \mapsto & x \cdot (1-t) + y \cdot t \end{array}$$

is a homeomorphism.

Proof. It follows easily from the Topological-Quotient Proposition 8.2 (1a) that the given map Θ is continuous. It is elementary to verify that the map is a bijection. It now follows from the Join Topology-Properties Proposition 20.2 (2) (here we use that X and Y are compact) together with the Compact-Hausdorff Proposition 4.23 (3) that the map Θ is a homeomorphism. ■

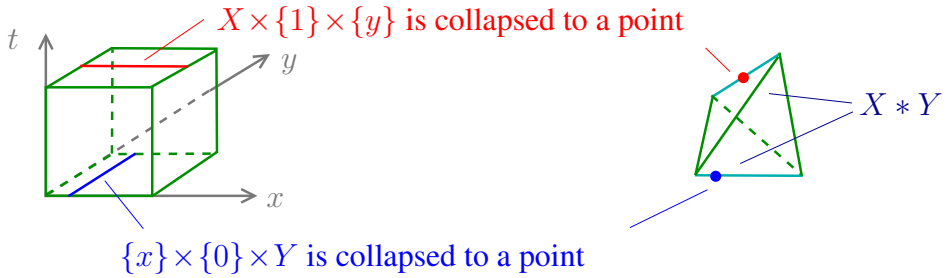
Example.

(1) In the figure below we show to the left a circle $X \subseteq \mathbb{R}^2 \times \{0\}$ and we also show the subspace Y that is given by the two "Poles" $\{(0,0,\pm 1)\}$ in $\mathbb{R}^3$.



It follows easily from the Join Euclidean Embedding Lemma 20.3 that the join $X * Y$ is homeomorphic to the "double cone" in $\mathbb{R}^3$ that is shown in the figure to the right.

(2) We consider the join of the two intervals $X = [0,1]$ and $Y = [0,1]$. The figure below suggests that the join $X * Y = [0,1] * [0,1]$ should be homeomorphic to a 3-dimensional pyramid.



In fact it follows from the Join Euclidean Embedding Lemma 20.3 that the map

$$\begin{array}{rcl} \overbrace{([0,1] \times [0,1] \times [0,1])/\sim}^{=[0,1]*[0,1]} & \to & \Delta = \{(t_1,t_2,t_3) \in [0,1]^3 \,|\, t_1 + t_2 + t_3 \leq 1\} \\ [(x,t,y)] & \mapsto & (x \cdot (1-t), y \cdot t, t) = x + (1 + y - x) \cdot t. \end{array}$$

is a homeomorphism.

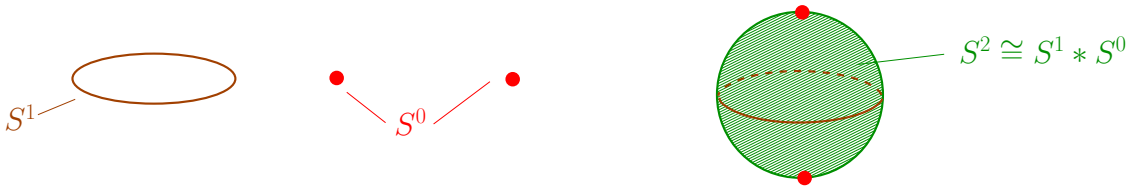
(3) In Exercise 20.4 we will see that in the Join Euclidean Embedding Lemma 20.3 we cannot drop the hypothesis that X and Y are compact. More precisely, we will see that the statement fails if X is an open interval and Y is a singleton. □

We record the next example of joins as a lemma:

Lemma 20.4. (Join-of-Spheres Lemma) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.



Proof. We just copy-paste the proof from the Join Euclidean Embedding Lemma 20.3: It follows easily from the Topological-Quotient Proposition 8.2 (1a) that the given map is continuous. It is elementary to verify that the map is a bijection. It now follows from the Join Topology-Properties Proposition 20.2 (2) together with the Compact-Hausdorff Proposition 4.23 (3) that the map is a homeomorphism. ■

Identification 20.5. Let $m, n \in \mathbb{N}_0$. We use the homeomorphism from the Join-of-Spheres Lemma 20.4 to add the identification $S^m * S^n =_{\mathrm{i}} S^{m+n+1}$ to our long list of avatars of the spheres.

Our next goal is to show that the join construction is functorial. This goal leads us the following definition:

Definition 20.6. Let X_1, Y_1, X_2, Y_2 be four non-empty topological spaces. Furthermore let $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ be continuous maps. We refer to the map

$$\begin{array}{rcl} f * g\colon X_1 * Y_1 & \to & X_2 * Y_2 \\ [(x,t,y)] & \mapsto & [(f(x),t,g(y))] \end{array}$$

as the join of f and g. We extend this definition in the obvious way to the case that one of the topological spaces is the empty topological space.[231]

Lemma 20.7. (Join–Functor Lemma)

(1) Let X_1, Y_1, X_2, Y_2 be topological spaces and let $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ be continuous maps. The join $f * g\colon X_1 * Y_1 \to X_2 * Y_2$ is continuous.

(2) The maps

[231] To be a little more precise: If $X_2 \neq \varnothing$ we define $\varnothing * Y_1 \to X_2 * Y_2$ as follows: We pick $x \in X_2$ and we consider the map $(1, y) \mapsto [(x, 1, y)]$. By definition of the join this map is independent of the choice of $x \in X_2$. It follows from the usual arguments that this map is continuous. One can deal with all other possible cases in a similar fashion.

$$\begin{array}{rcl}(X,Y) & \mapsto & X*Y \\ (f\colon X_1\to X_1, g\colon Y_1\to Y_2) & \mapsto & (f*g\colon X_1*Y_1\to X_2*Y_2)\end{array}$$

define a covariant functor from the product category $\mathcal{T}op^2$ of couples of topological spaces to the category of topological spaces. We refer to this functor as the join functor.

Proof.

(1) It follows easily from the Topological-Product Proposition 7.2 and the Quotient-to-Quotient Lemma 8.6 that the join $f*g$ is continuous.
(2) This statement follows easily from the definitions. ■

Lemma 20.8. (Join–Closed Embedding Lemma) Let $i\colon A\to X$ and $j\colon B\to Y$ be continuous maps of topological spaces. If both are closed embeddings, then the map

$$i*j\colon A*B \;\to\; X*Y$$

is also a closed embedding.

Example.

(1) The identity map $X\to X$ and the "empty map" $\varnothing\to Y$ are of course closed embeddings. It thus follows from the Join–Closed Embedding Lemma 20.8 that we have a natural closed embedding

$$i_X\colon X \xrightarrow[\cong]{x\mapsto(x,0)} X*\varnothing \;\to\; X*Y.$$

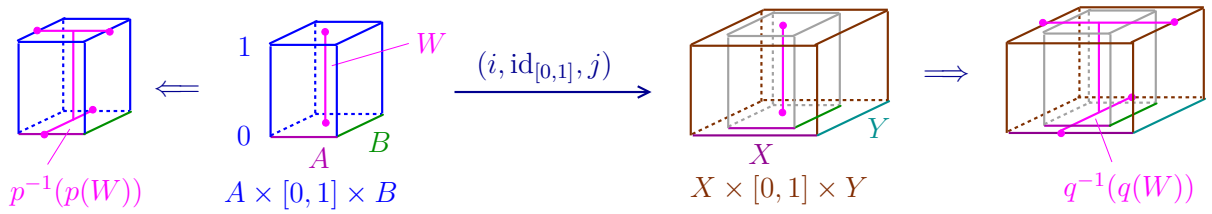
Similarly we of course also have a natural embedding $i_Y\colon Y\cong\varnothing*Y\to X*Y$.
(2) The combination of the Cone–Join Lemma 20.15 and Exercise 20.8 shows that in the above Join–Closed Embedding Lemma 20.8 we just can *not* drop the adjective "closed". More precisely, we will see that for the embedding $i\colon(-1,1)\to[-1,1]$ and the identity $j\colon\{y_0\}\to\{y_0\}$ the induced map $i*j\colon(-1,1)*\{y_0\}\to[-1,1]*\{y_0\}$ is in fact *not* an embedding. □

Proof. To simplify the notation we assume that we are given topological spaces X and Y and closed subsets $A\subseteq X$ and $B\subseteq Y$. We denote by $i\colon A\to X$ and $j\colon B\to Y$ the inclusion. We need to show that the map $i*j\colon A*B\to X*Y$ is a closed embedding. By the Join–Functor Lemma 20.7 we know that the map is continuous. Furthermore it is pretty clear that the map is an injection. By the Open-Closed–Injective Lemma 4.22 it remains to show that the map is closed.

We start out with a few preparations:

(1) Let $p\colon A\times[0,1]\times B\to A*B$ and $q\colon X\times[0,1]\times Y\to X*Y$ be the natural projections.
(2) We denote by $(i,\mathrm{id}_{[0,1]},j)\colon A\times[0,1]\times B\to X\times[0,1]\times Y$ the map that is induced by $i,\mathrm{id}_{[0,1]}$ and j.
(3) Given a topological space Z with an equivalence relation $\sim$ we know by the Topological-Quotient Proposition 8.2 (0b) that a subset of $Z/\sim$ is closed if and only if the preimage in Z is closed.
(4) We denote by $\pi_{A,0}\colon A\times\{0\}\times B\to A$ the natural projection. Similarly we define $\pi_{X,0}$, $\pi_{B,1}$ and $\pi_{Y,1}$.

(5) Let W be any subset of $X \times [0,1] \times Y$. It follows easily from the Definition 20.1 of a join that we have the following equalities:
  (a) $q^{-1}(q(W)) \cap (X \times \{0\} \times Y) = \pi_{X,0}^{-1}(\pi_{X,0}(W \cap (X \times \{0\} \times Y)))$.
  (b) $q^{-1}(q(W)) \cap (X \times (0,1) \times Y) = W \cap (X \times (0,1))$.
  (c) $q^{-1}(q(W)) \cap (X \times \{1\} \times Y) = \pi_{Y,1}^{-1}(\pi_{Y,1}(W \cap (X \times \{1\} \times Y)))$.



As we mentioned, we need to show that $i * j \colon A * B \to X * Y$ is a closed embedding. Thus let $Z \subseteq A * B$ be a closed subset. By the above we have the following commutative diagram:

$$\begin{array}{ccc} A \times [0,1] \times B & \xrightarrow{(i,\mathrm{id}_{[0,1]},j)} & X \times [0,1] \times Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \underset{Z}{A * B} & \xrightarrow{i*j} & \underset{(i*j)(Z)}{X * Y}. \end{array}$$

By (3) it suffices to show that $q^{-1}((i*j)(Z))$ is a closed subset of $X \times [0,1] \times Y$. Note that

$$\begin{aligned} q^{-1}((i*j)(Z)) &\overset{\text{by the Image-Preimage Lemma 1.9 (2b)}}{=} q^{-1}(((i*j)\circ p)(p^{-1}(Z))) \overset{\text{since the above diagram commutes}}{=} q^{-1}(q((i,\mathrm{id}_{[0,1]},j)(p^{-1}(Z)))) \\ &= \underbrace{p^{-1}(Z)}_{\text{(i)}} \cup \underbrace{\pi_{X,0}^{-1}(\pi_{X,0}(p^{-1}(Z) \cap (X \times \{0\} \times Y)))}_{\text{(ii)}} \cup \underbrace{\pi_{Y,1}^{-1}(\pi_{Y,1}(p^{-1}(Z) \cap (X \times \{1\} \times Y)))}_{\text{(iii)}}. \end{aligned}$$

by (5) and the Image-Preimage Lemma 1.9 (1a)

By the elementary Closed Subset-Lemma 3.5 (3) it now suffices to prove the following claim:

Claim.
(i) $p^{-1}(Z)$ is a closed subset of $X \times [0,1] \times Y$.
(ii) $\pi_{X,0}^{-1}(\pi_{X,0}(p^{-1}(Z) \cap (X \times \{0\} \times Y)))$ is a closed subset of $X \times [0,1] \times Y$.
(iii) $\pi_{Y,1}^{-1}(\pi_{Y,1}(p^{-1}(Z) \cap (X \times \{1\} \times Y)))$ is a closed subset of $X \times [0,1] \times Y$.

Proof.

(i) Since $A \subseteq X$ and $B \subseteq Y$ are closed subsets we obtain immediately from the Product-of-Subsets Lemma 7.8 and the Open-Closed Inclusion Lemma 4.13 that the map $(i, \mathrm{id}_{[0,1]}, j) \colon A \times [0,1] \times B \to X \times [0,1] \times Y$ is in fact a closed map. By (3) we know that $p^{-1}(Z)$ is a closed subset of $A \times [0,1] \times B$. Thus we are done.
(ii) We start out with two simple observations:
- First note that it follows easily from the Definition 20.1 of the equivalence relation in a join that $p^{-1}(Z) \cap (A \times \{0\} \times Y) = C \times \{0\} \times Y$ for some subset $C \subseteq A$.

- Since $p^{-1}(Z)$ is closed we obtain from an elementary argument, see Exercise 7.5, that $C \subseteq A$ is closed.

Now note that

$$\pi_{X,0}^{-1}(\pi_{X,0}(p^{-1}(Z) \cap (X \times \{0\} \times Y))) \overset{\text{since } p^{-1}(Z) \subseteq A \times [0,1] \times B}{\underset{\text{by the above observation}}{=}} \pi_{A,0}^{-1}(\pi_{X,0}(p^{-1}(Z) \cap (A \times \{0\} \times B))) = \pi_{X,0}^{-1}(\pi_{A,0}(C \times \{0\} \times Y)) = \pi_{X,0}^{-1}(C).$$

Since $C \subseteq A$ is closed and since $A \subseteq X$ is closed we see, by the Open-Closed Inclusion Lemma 4.13 (2), that $C \subseteq X$ is closed. Since $\pi_{X,0}$ is continuous we now see that $\pi_{X,0}^{-1}(C)$ is indeed closed.

(iii) The proof of this statement is of course almost the same as the proof of (ii). ■

Next the following lemma shows that a join is just a special case of a pushout. This lemma allows us to dip into the general theory of pushouts, which we developed in detail in Section 8.4.

Lemma 20.9. (Join–Pushout Lemma) Given any two topological spaces X and Y the following diagram is a pushout diagram:[232]

$$\begin{array}{ccc} X \times \{0,1\} \times Y & \hookrightarrow & X \times [0,1] \times Y \\ \downarrow & & \ulcorner \downarrow \\ X \sqcup Y & \xrightarrow[i_X \sqcup i_Y]{} & X * Y. \end{array}$$

Here the bottom horizontal map is given by the two natural closed embeddings i_X and i_Y that we introduced on page 546.

Proof. As always we ignore the trivial case that one of the topological spaces is the empty space. Now we consider the map[233]

$$\overbrace{(X \times [0,1] \times Y)/\sim}^{=X*Y} \to \overbrace{(X \times \{0,1\} \times Y) \cup_{X \sqcup Y} (X \times [0,1] \times Y)}^{\text{pushout by the Topological-Pushout Proposition 8.12}}$$
$$[P] \mapsto [(P,2)]$$

and we consider the map

$$(X \times \{0,1\} \times Y) \cup_{X \sqcup Y} (X \times [0,1] \times Y) \to (X \times [0,1] \times Y)/\sim$$
$$[((x,t,y),i)] \mapsto [(x,t,y)].$$

It is elementary to see that the maps are inverses of one another. Furthermore it follows easily from the Quotient-to-Quotient Lemma 8.6 that both maps are continuous. ■

Next we turn to the interaction between joins and homotopies:

[232] This lemma explains in particular the initially slightly odd convention in the definition of the join for the special case that at least one of X or Y is the empty topological space.

[233] Recall that by Notation 1.12 we have $A \sqcup B := (A \times \{1\}) \cup (B \times \{2\}$ and by the Topological-Pushout Proposition 8.12 we have $A \cup_C B = (A \sqcup B)/\sim$. With this definition we see that for $[P] \in (X \times [0,1] \times Y)/\sim$ we get a point $[(P,2)] \in (X \times \{0,1\} \times Y) \cup_{X \sqcup Y} (X \times [0,1] \times Y)$ on the right hand side.

Lemma 20.10. (Join–Homotopy Lemma)

(1) Let X_1, Y_1, X_2, Y_2 be topological spaces and let $f, f' \colon X_1 \to X_2$ and $g, g' \colon Y_1 \to Y_2$ be pairs of continuous maps. If f, f' are homotopic rel a (possible empty) subset $A_1 \subseteq X_1$ and if g, g' are homotopic rel a subset $B_1 \subseteq Y_1$, then $f * g \colon X_1 * Y_1 \to X_2 * Y_2$ is homotopic to $f' * g' \colon X_1 * Y_1 \to X_2 * Y_2$ rel the image of $A_1 * B_1$ in $X_1 \times Y_1$.

(2) If $f \colon X_1 \to X_2$ and $g \colon Y_1 \to Y_2$ are homotopy equivalences of topological spaces, then the join $f * g$ is also a homotopy equivalence.

Proof.

(1) We ignore the trivial case that one of the topological spaces is the empty space. Now we assume that all topological spaces involved are non-empty. Clearly it suffices to prove the following claim:

Claim. If $F \colon X_1 \times [0,1] \to X_2$ and $G \colon Y_1 \times [0,1] \to Y_2$ are continuous maps, then the map

$$\begin{array}{rcl} H \colon \overbrace{((X_1 \times [0,1] \times Y_1)/\sim)}^{=X_1 * Y_1} \times [0,1] & \to & \overbrace{(X_2 \times [0,1] \times Y_2)/\sim}^{=X_2 * Y_2} \\ ([(x,t,y)],s) & \mapsto & [(F(x,s),t,G(y,s))] \end{array}$$

is continuous.

Proof. First note that it follows easily from the Topological-Product Proposition 7.2 that the obvious map $X_1 \times [0,1] \times Y_1 \to X_2 \times [0,1] \times Y_2$ is continuous. It now follows from the Homotopy Quotient Theorem 18.11 that the given map under consideration is also continuous.

(2) It suffices to show that if $f \colon X_1 \to X_2$, $\widetilde{f} \colon X_2 \to X_1$ and $g \colon Y_1 \to Y_2$, $\widetilde{g} \colon Y_2 \to Y_1$ are continuous maps such that $\widetilde{f} \circ f \simeq \mathrm{id}_{X_1}$ and $\widetilde{g} \circ g \simeq \mathrm{id}_{Y_1}$, then $(\widetilde{f} * \widetilde{g}) \circ (f * g) \simeq \mathrm{id}_{X_1 * Y_1}$. This is indeed the case since we have

$$\begin{array}{ccccccc} (\widetilde{f} * \widetilde{g}) \circ (f * g) & \underset{\uparrow}{=} & (\widetilde{f} \circ f) * (\widetilde{g} \circ g) & \underset{\uparrow}{\simeq} & \mathrm{id}_{X_1} * \mathrm{id}_{Y_1} & \underset{\uparrow}{=} & \mathrm{id}_{X_1 * Y_1}\,. \\ & \text{by the Join–Functor Lemma 20.7} & & \text{by hypothesis and by (1)} & & \text{by the Join–Functor Lemma 20.7} & \end{array}$$

∎

Finally we consider the delicate question whether the join is associative.

Proposition 20.11. (Join Associativity Proposition) Let X, Y and Z be topological spaces. The map

$$\begin{array}{rcl} (X * Y) * Z & \to & X * (Y * Z) \\ [[(x,s,y)],t,z] & \mapsto & [(x,s,[(y,t,z)])] \end{array}$$

is a bijection. If X, Y and Z are compact and Hausdorff, then the map is in fact a homeomorphism.

Remark.

(1) When dealing with joins it seems that it is safest to stick to the case that X and Y are compact and Hausdorff. For example the Join Associativity Proposition 20.11 says that in this context the join is associative. But this statement is not true for all topological spaces. We refer to [**CoheD1957**], [**BrownR2006**, Chapter 5.7] and [**FF2016**, Chapter 2.4] for details.[234][235]

[234] As we saw in the Product-Quotient–Non-Homeomorphism Lemma 17.2, products and quotients of topological spaces do not mix well. It is easy to write down "continuous looking" maps that are not continuous.

[235] In light of the Product-Quotient–Non-Homeomorphism Lemma 17.2 it is perhaps not too shocking that the join is not associative.

(2) The join $X * Y$, viewed as a set, can be equipped with another reasonable topology, namely the "initial topology", see [**BrownR2006**, Chapter 5.7]. By [**BrownR2006**, Corollary 5.7.2] these two topologies agree if X and Y are compact and Hausdorff. It turns out that by [**BrownR2006**, Corollary 5.7.1] we know that the initial topology is in fact associative.

(3) Let $X_1, \ldots, X_k$ be finitely many topological spaces that are compact and Hausdorff. It follows from the Join Associativity Proposition 20.11 and the discussion on page 115 that we can unambiguously define the join $X_1 * \cdots * X_k$. □

Proof. The case that one of X, Y or Z is empty is trivial. Thus we now assume that all three topological spaces are non-empty. One can easily verify that the given map is a bijection. Now we assume that X, Y and Z are compact and Hausdorff. We need to show that the given map is now actually a homeomorphism.

To do we introduce an intermediate object, namely we now introduce the triple join $X * Y * Z$ as follows: we consider

$$\{(r, x, s, y, t, z) \in [0,1] \times X \times [0,1] \times Y \times [0,1] \times Z \mid r + s + t = 1\}/\sim$$

where "$\sim$" is generated by the following relations:

- $(0, x, s, y, t, z) \sim (0, x', s, y, t, z)$,
- $(r, x, 0, y, t, z) \sim (r, x, 0, y', t, z)$,
- $(r, x, s, y, 0, z) \sim (r, x, s, y, 0, z')$.

In the following we will use "$\sim$" with several meanings, we leave it to the reader to figure out at any given moment what is meant by this symbol.

Claim. The map

$$\begin{array}{rccc} \Theta\colon & \overbrace{(((X \times [0,1] \times Y)/\sim) \times [0,1] \times Z)/\sim}^{=(X*Y)*Z} & \to & \overbrace{(X \times [0,1] \times Y \times [0,1] \times Z)/\sim}^{=X*Y*Z} \\ & [([(x,s,y)],t,z)] & \mapsto & [((1-t)\cdot(1-s), x, (1-t)s, y, t, z)] \end{array}$$

is a homeomorphism.

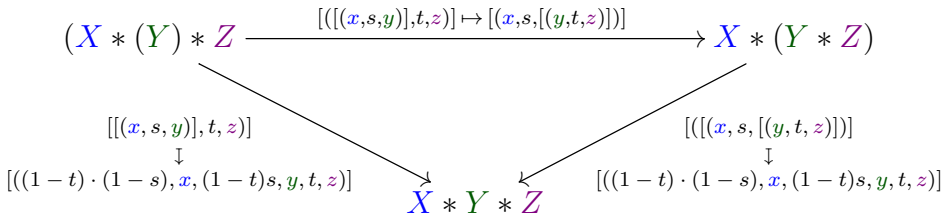
Proof. We consider the following diagram:

$$\begin{array}{ccc} (X \times [0,1] \times Y) \times ([0,1] \times Z) & \xrightarrow{\text{id}} & (X \times [0,1] \times Y) \times ([0,1] \times Z) \\ \downarrow & & \downarrow \\ ((X \times [0,1] \times Y) \times ([0,1] \times Z))/\sim & & \\ {\scriptstyle ([u],v)\mapsto[(u,v)]}\uparrow{\scriptstyle \varphi} & \overset{\Phi}{\searrow} & \\ ((X \times [0,1] \times Y)/\sim) \times ([0,1] \times Z) & \xrightarrow{\Psi} & X * Y * Z. \\ \downarrow & \overset{\Theta}{\nearrow} & \\ \underbrace{(((X \times [0,1] \times Y)/\sim) \times [0,1] \times Z)/\sim}_{=(X*Y)*Z} & & \end{array}$$

We make the following clarifications and observations:

(1) All the maps are the obvious (projection) maps, except for φ which is the map that is given by the Product-Quotient–Non-Homeomorphism Lemma 17.2. The diagram commutes basically by definition.
(2) It follows from the Quotient-to-Quotient Lemma 8.6 that Φ is continuous.
(3) Our hypothesis that Z is Hausdorff and compact implies by the Product Topology-Properties Proposition 7.9 together with the Compact Hausdorff-Regionally Compact Lemma 4.58 that $[0,1] \times Z$ is regionally compact. Thus we obtain from the Product-Quotient Mixing Lemma 17.8 that φ is actually a homeomorphism.
(4) It follows from (2) and (3) that Ψ is continuous.
(5) It follows from (4) together with the Topological-Quotient Proposition 8.2 (1b) that Θ is continuous.
(6) It follows easily from the definitions that Θ is a bijection.
(7) By our hypotheses on X, Y and Z we obtain from the Join Topology-Properties Proposition 20.2 (2) that the domain $(X * Y) * Z$ of the map Θ is compact. Furthermore, by an argument as in the Join Topology-Properties Proposition 20.2 (3) one sees that the target $X * Y * Z$ of the map Θ is Hausdorff.
(8) It follows from (5), (6) and (7) together with the Compact-Hausdorff Proposition 4.23 (3) that Θ is a homeomorphism. ⊞

We now consider the following diagram:



One can easily verify that the diagram commutes. In the claim we just showed that the left diagonal map is a homeomorphism. Essentially the same argument shows that the right diagonal map is a homeomorphism. It follows, as promised, that the top horizontal map is a homeomorphism. ■

20.2. **The cone of a topological space.** In this section we define and discuss the cone of a topological space. It turns out that the cone of a topological space is a particularly easy construction:

Definition 20.12. Given a topological space X the cone on X is defined as the topological space[236]

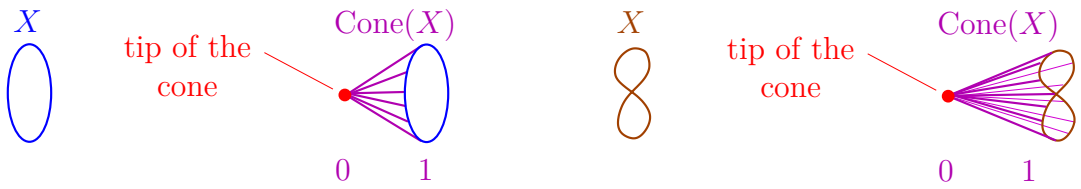
$$\mathrm{Cone}(X) \coloneqq (X \times [0,1])/(X \times \{0\}).$$

We refer to the point $[X \times \{0\}]$ as the tip of the cone.

[236]Note that if X is non-empty, then

$$\mathrm{Cone}(X) \coloneqq (X \times [0,1])/(X \times \{0\}) = (X \times [0,1])/\sim \text{ where } (x,0) \sim (x',0) \text{ for all } x, x' \in X.$$

If X is the empty topological space, then $X \times [0,1] = X \times \{0\} = \varnothing$ and we obtain from the definition of the quotient of a topological space Definition 8.8 that $\mathrm{Cone}(X) = \varnothing/\varnothing = \{\varnothing\}$ is the topological space consisting of the single point called $[\varnothing]$.



Examples.

(1) Let $n \in \mathbb{N}_0$. We claim that the map

$$\begin{array}{rcl} \mathrm{Cone}(S^n) & \to & \overline{B}^{n+1} \\ [(v,t)] & \mapsto & v \cdot t \end{array}$$

is a homeomorphism: First note that it follows from the Topological-Quotient Proposition 8.2 (1b) that the map is continuous. Next note that one can easily verify that the map is a bijection. It follows immediately from the Product Topology-Properties Proposition 7.9 (1) and the Quotient Topology-Properties Proposition 8.5 (1) that $\mathrm{Cone}(S^n)$ is compact. Since $\overline{B}^{n+1}$ is Hausdorff we obtain from the Compact-Hausdorff Proposition 4.23 (3) that the given map is a homeomorphism. On a few occasions we will use this specific homeomorphism to make the identification $\mathrm{Cone}(S^n) =_{\mathrm{i}} \overline{B}^{n+1}$.

(2) Let X be a subset of $\mathbb{R}^n$. We define its *physical cone* as

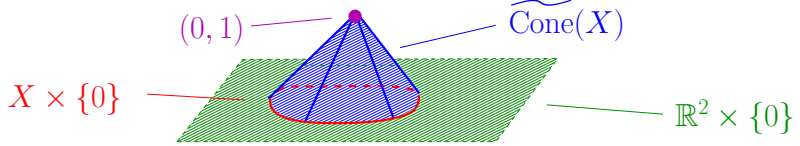
$$\widetilde{\mathrm{Cone}}(X) \;:=\; \text{union of all Euclidean segments from } \underbrace{(x,0)}_{\in\mathbb{R}^{n+1}} \text{ to } \underbrace{(0,1)}_{\in\mathbb{R}^{n+1}} \;\subseteq\; \mathbb{R}^{n+1},$$

which we equip with the subspace topology coming from $\mathbb{R}^{n+1}$.

(a) We consider the map

$$\begin{array}{rccl} f_X\colon & \overbrace{\mathrm{Cone}(X)}^{=(X\times[0,1])/(X\times\{0\})} & \to & \widetilde{\mathrm{Cone}}(X) \\ & [(x,t)] & \mapsto & t\cdot(x,0)+(1-t)\cdot(1,0). \end{array}$$

If X is a *compact* non-empty subset of $\mathbb{R}^n$, then one can show quite easily, using the Compact-Hausdorff Proposition 4.23, that the map is a homeomorphism. We will work out the details in Exercise 20.4.



(b) After looking at the figure above and considering Examples (1) and (2a) one might walk away with the feeling that one has a pretty good understanding of the cone construction. But this optimism is potentially ill-founded. For example in Exercise 20.4 we see will see that for $X = (0,1) \subseteq \mathbb{R}$ the above map $f_X\colon \mathrm{Cone}(X) \to \widetilde{\mathrm{Cone}}(X)$ from (2a) is *not* a homeomorphism. □

In the following we will aim for functoriality of the cone construction:

Definition 20.13. Let $f\colon X \to Y$ be a continuous map between topological space. We define

$$\begin{array}{rcl} \mathrm{Cone}(f)\colon\ \mathrm{Cone}(X) & \to & \mathrm{Cone}(Y) \\ [(x,t)] & \mapsto & [(f(x),t)]. \end{array}$$

The next lemma shows that the maps $\mathrm{Cone}(f)$ have several desirable properties:

Lemma 20.14. (Cone–Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. The map $\mathrm{Cone}(f)\colon \mathrm{Cone}(X) \to \mathrm{Cone}(Y)$ is continuous.

(2) The maps

$$\begin{array}{rcl} X & \mapsto & \mathrm{Cone}(X) \\ (f\colon X \to Y) & \mapsto & (\mathrm{Cone}(f)\colon \mathrm{Cone}(X) \to \mathrm{Cone}(Y)) \end{array}$$

define a covariant functor from the category $\mathcal{T}op$ of topological spaces to itself.

Proof. As in the Join–Functor Lemma 20.7 the first statement follows easily from the Topological-Product Proposition 7.2 and the Quotient-to-Quotient Lemma 8.6. Furthermore the second statement follows again easily from the definitions. ■

The reader will already have noticed that the cone of a topological space X looks awfully like the join $X * \{\star\}$ of X with a singleton $\{\star\}$. The following lemma confirms this impression:

Lemma 20.15. (Cone–Join Lemma) Let X be a topological space. The cone $\mathrm{Cone}(X)$ is naturally[237] homeomorphic to the join $X * \{\star\}$.

Proof. If X is the empty topological space, then both the cone $\mathrm{Cone}(X)$ and the join $X * \{\star\}$ consist of a single point, and the statement is trivial. Now assume that X is non-empty. Using the Quotient-to-Quotient Lemma 8.6 one can easily verify that the natural maps

$$\begin{array}{rcl} \mathrm{Cone}(X) & \to & X * \{\star\} \\ [(x,t)] & \mapsto & [(x,1-t,\star)] \end{array} \quad \text{and} \quad \begin{array}{rcl} X * \{\star\} & \to & \mathrm{Cone}(X) \\ [(x,t,\star)] & \mapsto & [(x,1-t)] \end{array}$$

are continuous and inverses of one another. ■

It is also convenient to express the cone as a pushout:

Lemma 20.16. (Cone–Pushout Lemma) Given any topological space X the following diagram is a pushout diagram:[238]

$$\begin{array}{ccc} X \times \{0\} & \hookrightarrow & X \times [0,1] \\ \downarrow & & \lrcorner\ \downarrow \\ \{\star\} & \longrightarrow & \mathrm{Cone}(X). \end{array}$$

Proof. This statement can be verified easily using the Topological-Pushout Proposition 8.12. We leave it to the reader to fill in the details. ■

The previous two lemmas allow us to appeal to some earlier results to obtain results on cones. First of all we can now easily deal with our standard properties of topological spaces:

[237] By the Join–Functor Lemma 20.7 and the Cone–Functor Lemma 20.14 it makes sense to talk of natural homeomorphisms.

[238] Note that this pushout lemma is "simpler" than the one we get from the combination of the Cone–Join Lemma 20.15 and the Join–Pushout Lemma 20.9.

Proposition 20.17. (Cone Topology-Properties Proposition) Let X be a topological space.

(1) If X is compact, then $\mathrm{Cone}(X)$ is also compact.
(2) If X is (path)-connected, then $\mathrm{Cone}(X)$ is also (path)-connected.
(3) If X is Hausdorff, then $\mathrm{Cone}(X)$ is Hausdorff.

Proof. The statements follow from the Cone–Join Lemma 20.15 together with the Join Topology-Properties Proposition 20.2. Alternatively it is not hard to prove the statements directly. ■

The Cone–Join Lemma 20.15 also comes in handy in the proofs of the following two lemmas:

Lemma 20.18. (Cone–Closed Embedding Lemma) Let $f\colon X \to Y$ be a closed embedding of topological spaces. Then the map

$$\mathrm{Cone}(f)\colon\ \mathrm{Cone}(X)\ \to\ \mathrm{Cone}(Y)$$

is also a closed embedding.

Remark. Note that we cannot just drop the adjective "closed" in the statement of the lemma. In fact in Exercise 20.8 we will see that the embedding $(-1,1) \to [-1,1]$ does *not* induce an embedding $\mathrm{Cone}((-1,1)) \to \mathrm{Cone}([-1,1])$. □

Proof. The lemma follows almost immediately from the Cone–Join Lemma 20.15 together with the Join–Closed Embedding Lemma 20.8. Alternatively it follows from the Cone–Pushout Lemma 20.16 together with the Pushout–Closed Embedding Lemma 8.18. Finally we will give a more direct proof in Exercise 20.8. ■

Lemma 20.19. (Space-Cone Embedding Lemma) Let X be a topological space. The natural map

$$\begin{array}{ccc} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array}$$

is a closed embedding. Sometimes we will use this map to view X as a closed subset of $\mathrm{Cone}(X)$.

Proof. We consider the following diagram:

$$\begin{array}{ccc} X * \{\varnothing\} & \xrightarrow{(x,0)\mapsto[(x,1,\star)]} & X * \{\star\} \\ {\scriptstyle (x,0)\mapsto x}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle [(x,t,\star)]\mapsto[(x,1-t)]} \\ X & \xrightarrow{x\mapsto[(x,1)]} & \mathrm{Cone}(X). \end{array}$$

First note that it follows from the explicit descriptions of the maps that the diagram commutes. Next note that it follows easily from the definitions and the Cone–Join Lemma 20.15 that the vertical maps are homeomorphisms. Since $\varnothing \to \{\star\}$ is a closed embedding we obtain from the Join–Closed Embedding Lemma 20.8 that the top horizontal map is a closed embedding. It follows from the above discussion that the bottom horizontal map is also a closed embedding. Of course it is also possible to prove the lemma directly from the definitions. ■

The definition of a cone does not seem to be particularly interesting, but we have one lemma whose proof is perhaps slightly more subtle than one might initially think.

Lemma 20.20. (Cone Contractability Lemma) Let X be a topological space. The tip of the cone $\mathrm{Cone}(X)$ is a deformation retract of $\mathrm{Cone}(X)$.[239]

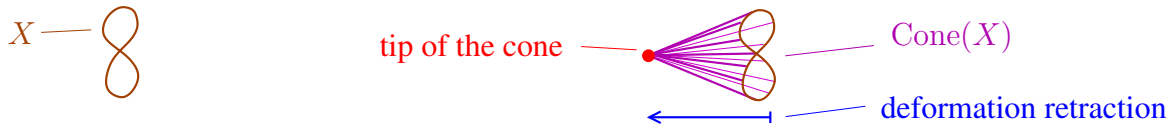
Proof. First note that the map

$$\begin{array}{rccc} F\colon & (X\times[0,1])\times[0,1] & \to & X\times[0,1] \\ & ((x,s),t) & \mapsto & (x,s\cdot(1-t)) \end{array}$$

is a deformation retraction from $X\times[0,1]$ to $X\times\{0\}$.[240] Next we consider the map

$$\begin{array}{rccc} \overline{F}\colon & ((X\times[0,1])/(X\times\{0\}))\times[0,1] & \to & (X\times[0,1])/(X\times\{0\}) \\ & ([(x,s)],t) & \mapsto & [(x,s\cdot(1-t))]. \end{array}$$

It follows easily from the definitions that the map is well-defined. Furthermore it follows from the Homotopy Quotient Theorem 18.11 that this map is continuous.



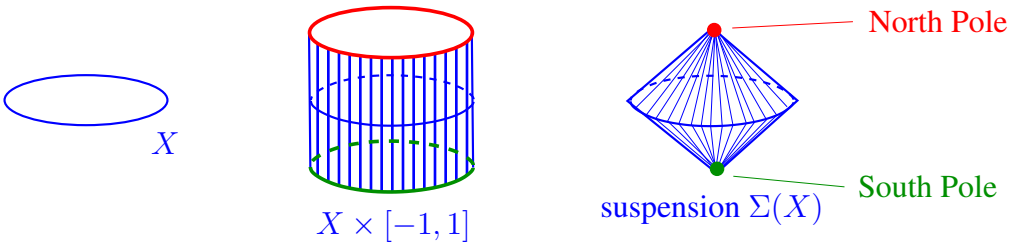
It is now clear that $\overline{F}$ is a deformation retraction from $\mathrm{Cone}(X)=(X\times[0,1])/(X\times\{0\})$ to the tip of the cone, namely $[X\times\{0\}]$. ■

20.3. **The suspension of a topological space.** In this section we introduce and discuss a close relative of the cone, namely the suspension of a topological space:

Definition 20.21. Let X be a topological space. We define the suspension of X to be the topological space[241]

$$\Sigma(X) \;:=\; ((X\times[-1,1])/(X\times\{-1\}))/(X\times\{1\}).$$

The point $[X\times\{1\}]$ is called the North Pole of $\Sigma(X)$ and the point $[X\times\{-1\}]$ is called the South Pole of $\Sigma(X)$.



Example. It is pretty clear from the above figure that the suspension of S^n is homeomorphic to S^{n+1}. In fact using the Compact-Hausdorff Proposition 4.23 (3) one can easily

[239] It follows in particular from the Point Deformation Retract Lemma 19.19 that cones are contractible.

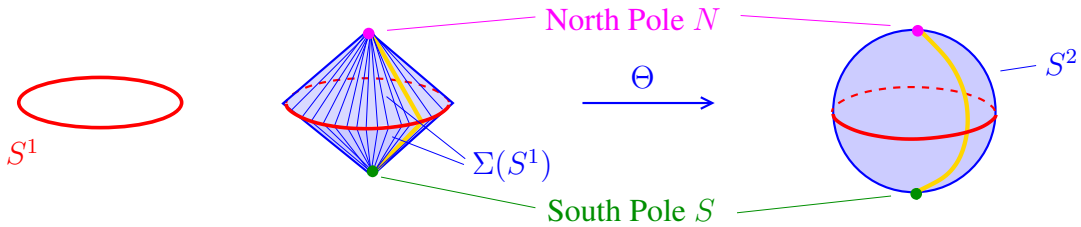
[240] It follows from the discussion on page 178 and the Topological-Product Proposition 7.2 (1) that this map is actually continuous.

[241] If X is non-empty, then it follows from the Double-Quotient Lemma 8.7 that $\Sigma(X)$ is naturally homeomorphic to the topological space $(X\times[-1,1])/\sim$ where we identify all points in $X\times\{-1\}$ to a single point and we identify all points in $X\times\{1\}$ to a single point. If X is empty, then it follows from the definition of the quotient space in Definition 8.8 that $\Sigma(X)$ is the discrete topological space given by the disjoint union $\{\{\varnothing\}\}\sqcup\{\{\{\varnothing\}\}\}$. In other words $\Sigma(\varnothing)$ is homeomorphic to the topological space $S^0=\{\pm1\}$.

verify that the following map is a homeomorphism:

$$\begin{array}{ccc} \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subseteq \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}\big). \end{array}$$

Clearly this homeomorphism sends the "North Pole" of $\Sigma(S^n)$ to the "North Pole" of S^{n+1}.



In fact throughout the course we will use the above homeomorphism to make the identification $\Sigma(S^n) =_{\mathrm{i}} S^{n+1}$. □

Next we provide the fairly obvious notion of the suspension of a map:

Definition 20.22. Given a continuous map $f\colon X \to Y$ between topological spaces we refer to

$$\begin{array}{ccc} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

as the suspension of f.

The following lemma will not come as a surprise:

Lemma 20.23. (Suspension Functor Lemma)

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then the map $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also continuous.

(2) The maps

$$\begin{array}{ccc} X & \mapsto & \Sigma(X) \\ (f\colon X \to Y) & \mapsto & (\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \end{array}$$

define a covariant functor from the category *Top* of topological spaces to itself.

Proof.

(1) This statement follows easily from the Quotient-to-Quotient Lemma 8.6.

(2) This statement follows easily from the definitions. ■

Again the reader will already have noticed that the suspension of a topological space looks like a join, this time, like the join with two points. The following lemma confirms the readers intuition:

Lemma 20.24. (Suspension–Join Lemma) Let X be a topological space. The suspension $\Sigma(X)$ is naturally[242] homeomorphic to the join $X * S^0$.

Proof. If X is the empty topological space, then both the suspension $\Sigma(X)$ and the join $X * S^0$ consist of two points with the discrete topology. Thus the statement is trivial. Now

[242]By the Suspension–Functor Lemma 20.23 and the Cone–Functor Lemma 20.14 it makes sense to talk of natural homeomorphisms.

assume that X is non-empty. We consider the following natural maps:

$$\begin{array}{ccc} \Sigma(X) & \to & X * S^0 \\ [(x,t)] & \mapsto & \begin{cases} [(x,t,1)], & \text{if } t \geq 0, \\ [(x,-t,-1)], & \text{if } t \leq 0 \end{cases} \end{array} \quad \text{and} \quad \begin{array}{ccc} X * S^0 & \to & \Sigma(X) \\ [(x,t,\epsilon)] & \mapsto & [(x,t\cdot\epsilon)]. \end{array}$$

Next note that, using the Topological-Quotient Proposition 8.2 (0b), one can easily verify that $\{[(x,t)] \,|\, x \in X, t \in [0,1]\}$ and $\{[(x,t)] \,|\, x \in X, t \in [-1,0]\}$ are closed subsets of $\Sigma(X)$. Using this observation together with the Pasting Proposition 4.7, the Topological-Product Proposition 7.2 (1) and the Topological-Quotient Proposition 8.2 (1) one can easily show that both maps are continuous. Finally note that it is elementary to verify that the maps are inverses of one another. ■

The following lemma tells us that we can express the suspension as a pushout:

Lemma 20.25. (Suspension–Pushout Lemma) For any topological space X the following diagram is a pushout diagram:[243]

$$\begin{array}{ccc} X \times \{-1,1\} & \hookrightarrow & X \times [-1,1] \\ \downarrow{\scriptstyle (x,\epsilon)\mapsto\epsilon} & & \downarrow \\ \{-1,1\} & \to & \Sigma(X). \end{array}$$

Proof. This statement can be verified easily using the Topological-Pushout Proposition 8.12. Again we leave it to the reader to fill in the details. ■

We will now use the Suspension–Join Lemma 20.24 to prove the next three statements:

Proposition 20.26. (Suspension Topology-Properties Proposition) Let X be a topological space.

(1) If X is compact, then $\Sigma(X)$ is also compact.
(2) If X is (path)-connected, then $\Sigma(X)$ is also (path)-connected.
(3) If X is Hausdorff, then $\Sigma(X)$ is Hausdorff.

Proof. One can either prove the statements directly (see e.g. Exercise 20.7), or one appeals to the Suspension–Join Lemma 20.24 and the Join Topology-Properties Proposition 20.2. ■

The following lemma elucidates the relationship between cones and suspensions:

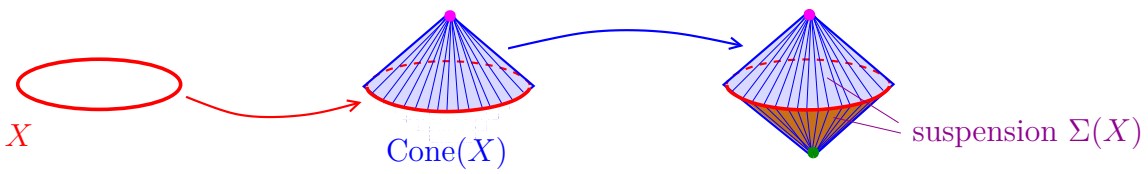
Lemma 20.27. (Cone-Suspension Embedding Lemma) Let X be a topological space. The natural maps

$$\begin{array}{ccc} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are closed embeddings. Sometimes we will use these embeddings to identify the domains with their images. For example we will use these maps to view X as a closed subset of $\mathrm{Cone}(X)$ and to view $\mathrm{Cone}(X)$ as a closed subset of $\Sigma(X)$. In particular we can and will at times view X as a closed subset of $\Sigma(X)$.

Proof. Note that we already know by the Space-Cone Embedding Lemma 20.19 that the first map $X \to \mathrm{Cone}(X)$ is a closed embedding. Next we turn to the second map

[243]Note that this pushout lemma is once again "simpler" than the one we get from the combination of the Suspension–Join Lemma 20.24 and the Join–Pushout Lemma 20.9.



$\mathrm{Cone}(X) \to \Sigma(X)$. We consider the following diagram:

$$\begin{array}{ccc} X * \{\star\} & \xrightarrow{[(x,t,\star)] \mapsto [(x,t,1)]} & X * S^0 \\ \cong \downarrow {\scriptstyle [(x,t,\star)] \mapsto [(x,1-t)]} & & \cong \downarrow {\scriptstyle [(x,t,\epsilon)] \mapsto [(x,t\cdot\epsilon)]} \\ \mathrm{Cone}(X) & \xrightarrow[[(x,t)] \mapsto [(x,1-t)]]{} & \Sigma(X) \end{array}$$

where the vertical maps are the homeomorphisms from the Cone-Join Lemma 20.15 and the Suspension–Join Lemma 20.24. It is clear that the diagram commutes. Since $\{\star\} \to S^0$ is a closed embedding we obtain from the Join–Closed Embedding Lemma 20.8 that the top horizontal map is a closed embedding. It follows from the above discussion that the bottom horizontal map is also a closed embedding. ■

We conclude this chapter with a little lemma on the homotopy type of suspensions:

Lemma 20.28. (Suspension–Homotopy Lemma)

(1) Let $f, g\colon X \to Y$ be continuous maps between two topological spaces. If f and g are homotopic rel a (possibly empty) subset A, then $\Sigma(f)$ and $\Sigma(g)$ are homotopic rel $\Sigma(A)$.

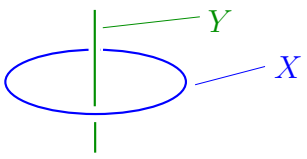
(2) If $f\colon X \to Y$ is homotopy equivalence of topological spaces, then the suspension $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also a homotopy equivalence.

Proof. This lemma follows easily from the Suspension–Join Lemma 20.24 together with the Join–Homotopy Lemma 20.10. ■

20.4. Exercises for Chapter 20.

Exercise 20.1. Show that for any choice of $m, n \in \mathbb{N}_0$ there exists a homeomorphism $\overline{B}^m * \overline{B}^n \xrightarrow{\cong} \overline{B}^{m+n+1}$.

Exercise 20.2. We let $X = S^1$ be the standard circle in the xy-plane of $\mathbb{R}^3$ and we let Y be the circle in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ that is given by the z-axis together with the point $\{\infty\}$. Show that we can "continuously" connect every point on X to every point Y such that the paths fill out S^3. In other words, try to visualize the fact that $S^1 * S^1 \cong X * Y$ is homeomorphic to S^3.

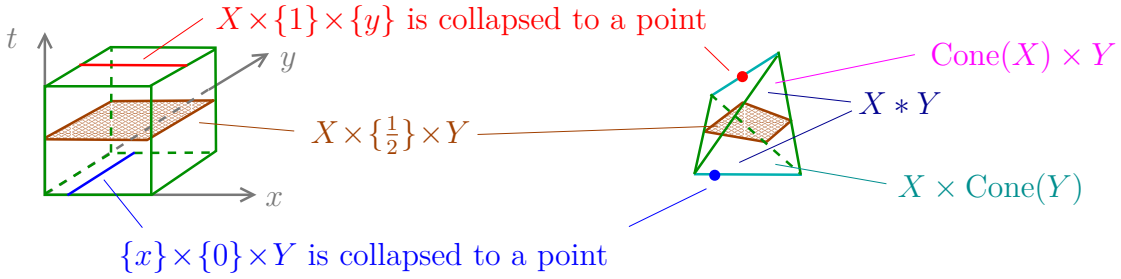


Exercise 20.3. Let X and Y be topological spaces. We consider the corresponding join $X * Y = (X \times [0,1] \times Y)/\sim$ from in Definition 20.1. Show that the map

$$\begin{array}{rcl} X * Y = (X \times [0,1] \times Y)/\sim & \cong & (\mathrm{Cone}(X) \times Y) \cup_{(X\times\{1\})\times Y = X\times(Y\times\{1\})} (X \times \mathrm{Cone}(Y)) \\ [(x,t,y)] & \mapsto & \begin{cases} (x,[(y,2t)]), & \text{if } t \in [0,\frac{1}{2}], \\ ([(x,2t-1)]),y), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homeomorphism.

Remark. Evidently you need to verify that all the maps are actually continuous.



Exercise 20.4. Given a subset X of $\mathbb{R}^n$ we define its *physical cone* as[244]

$$\widetilde{\mathrm{Cone}}(X) \;:=\; \text{union of all Euclidean segments from } \underbrace{(x,1)}_{\in\mathbb{R}^{n+1}} \text{ to } \underbrace{(0,0)}_{\in\mathbb{R}^{n+1}} \;\subseteq\; \mathbb{R}^{n+1}$$

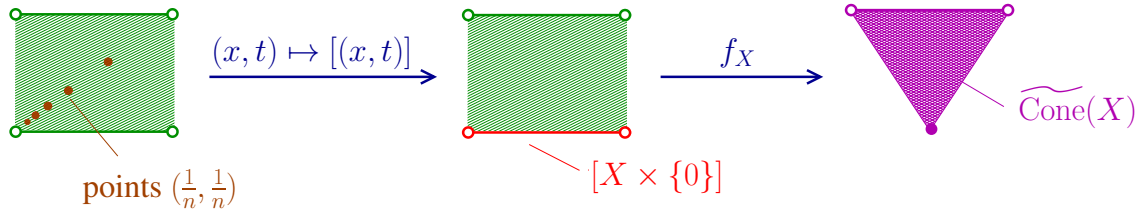
and we consider the map

$$\begin{array}{rcl} f_X\colon \mathrm{Cone}(X) & \to & \widetilde{\mathrm{Cone}}(X) \\ [(x,t)] & \mapsto & t\cdot(x,1). \end{array}$$

(a) Let X be a compact non-empty subset of $\mathbb{R}^n$. Show that the map f_X is a homeomorphism.

(b) Let $X = (0,1) \subseteq \mathbb{R}$. Show that the map f_X is not a homeomorphism.
Remark. It is actually true that f_X is continuous and a bijection. To show that f_X is not a homeomorphism we need to show that there exists an open subset U of $\mathrm{Cone}(X)$ such that the image under f_X is not open. You could try your hand at

$$U \;:=\; \mathrm{Cone}(X) \setminus \{[(\tfrac{1}{n},\tfrac{1}{n})] \mid n \in \mathbb{N}_{\geq 2}\}.$$



Exercise 20.5. Deduce the Cone Contractability Lemma 20.20 directly from the Join–Homotopy Lemma 20.10 and the Cone–Join Lemma 20.15.

Exercise 20.6. Let $X = (-1,1)$ be the open interval. We consider the cone $\mathrm{Cone}(X)$. Is the cone second-countable in the sense of Definition 13.1?

Exercise 20.7. Let X be a topological space. Give a direct proof for the following statement: If X is Hausdorff, then the suspension $\Sigma(X)$ is also Hausdorff.

[244] For completely random reasons we picked a slightly different definition from the one on page 552.

Exercise 20.8.

(a) Let $f\colon X \to Y$ be a closed embedding of topological spaces. Show directly that the induced map $\mathrm{Cone}(f)\colon \mathrm{Cone}(X) \to \mathrm{Cone}(Y)$ is also a closed embedding.

(b) Show that the obvious map $\mathrm{Cone}((-1,1)) \to \mathrm{Cone}([-1,1])$ is not an embedding.

Exercise 20.9. Let X be a topological space. Show directly that the injections

$$\begin{array}{ccc} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are closed embeddings.

Exercise 20.10. We consider the topological space $X = [-1,1]$, the subset $A = (-1,1)$ and the inclusion map $i\colon A \to X$. Show that the induced map $\Sigma(i)\colon \Sigma(A) \to \Sigma(X)$ is *not* an embedding.

Remark. It might help to have a look at the equally scary Exercise 20.4.

Exercise 20.11. Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion.

(a) Show that if A is a closed subset of X, then the induced map $\Sigma(i)\colon \Sigma(A) \to \Sigma(X)$ is a closed embedding.

(b) Show that in general the induced map $\Sigma(i)\colon \Sigma(A) \to \Sigma(X)$ is not an embedding.

Exercise 20.12. Is every topological space homotopy equivalent to the suspension of some other topological space?

CHAPTER 21

Mapping cylinders, mapping cones, and mapping tori

In the last chapter we associated to a (pair of) topological (s) new topological spaces, namely the join $X * Y$, the cone $\mathrm{Cone}(X)$ and the suspension $\Sigma(X)$. In this chapter we will present three systematic ways to construct new topological spaces out of a continuous map $f\colon X \to Y$ between given topological spaces, namely the mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$, the mapping cone $\mathrm{Cone}(f\colon X \to Y)$ and the mapping torus $\mathrm{Tor}(f\colon X \to Y)$. The main interest in these topological spaces is that these topological spaces encode some of the information contained in the actual map. On numerous occasions this will allow us to translate questions about maps between topological spaces into questions about topological spaces.

This chapter is organized as follows:

- In Section 21.1 we will introduce the mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$ of a continuous map $f\colon X \to Y$ between topological spaces.
- In Section 21.2 we will introduce the mapping cone $\mathrm{Cone}(f\colon X \to Y)$ of a continuous map $f\colon X \to Y$ between topological spaces.
- In Section 21.3 we will introduce the mapping torus $\mathrm{Tor}(f\colon X \to Y)$ of a continuous map $f\colon X \to Y$ between topological spaces.
- In Appendix 21.4 we will prove that under favorable circumstances taking the mapping cone and taking suspensions essentially commutes.
- In Section 21.5 we will provide exercises for Chapter 21.

In the next Chapter 22 we will introduce cofibrations.

21.1. **Mapping cylinder and mapping cone.** We start out with the definition of a mapping cylinder:

Definition 21.1. Let $f\colon X \to Y$ be a continuous map between topological spaces. We define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f) \coloneqq \mathrm{Cyl}(f\colon X \to Y) \coloneqq ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$



mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$

Remark. In many, arguably most, textbooks on algebraic topology the mapping cylinder is defined by identifying $(x,0)$ with $f(x)$ instead of identifying $(x,1)$ with $f(x)$. The definitions are evidently essentially equivalent, but as the author knows from experience, the usage of

two different definitions can cause some confusion. The totally non-mathematical advantage of our definition is that it leads to "horizontal" pictures which take up less space. □

As so often, let us first discuss the usual trinity of adjectives for topological spaces:

Proposition 21.2. (Mapping Cylinder Topology-Properties Proposition) In the following let $f\colon X \to Y$ be a continuous map between two topological spaces.

(1) (a) If X and Y are compact, then $\mathrm{Cyl}(f\colon X \to Y)$ is also compact.
(b) If X and Y are paracompact, then $\mathrm{Cyl}(f\colon X \to Y)$ is also paracompact.
(2) If X and Y are (path)-connected, then $\mathrm{Cyl}(f\colon X \to Y)$ is also (path)-connected.
(3) If X and Y are Hausdorff, then $\mathrm{Cyl}(f\colon X \to Y)$ is also Hausdorff.

Proof. Statements (1a) and (2) follow from the corresponding statements of the Product Topology-Properties Proposition 7.9, the Disjoint Union Topology-Properties Lemma 7.28 and the Topological-Quotient Proposition 8.2. We will prove Statements (1b) in Exercise 21.3 and we will prove Statement (3) in Exercise 21.2. ■

The next lemma shows that the mapping cylinder is just a special case of the more general pushout construction. This is very useful since it allows us to use all the results from the Topological Pushout-Maps Lemma 8.16.

Lemma 21.3. (Mapping Cylinder–Pushout Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The following is a pushout diagram:

$$\begin{array}{ccc} X & \xrightarrow{x\mapsto(x,1)} & X \times [0,1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (x,t)\mapsto[(x,t)]} \\ Y & \xrightarrow{y\mapsto[y]} & \mathrm{Cyl}(f\colon X \to Y). \end{array}$$

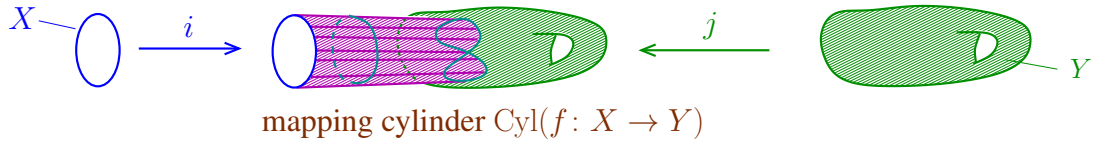
Example. If X is a topological space and if $f\colon X \to \{y\}$ is the constant map to the topological space consisting of a single element y, then it follows from comparing the Cone–Pushout Lemma 20.16 and the Mapping Cylinder–Pushout Lemma 21.3 that the mapping cylinder $\mathrm{Cyl}(f\colon X \to \{y\})$ is homeomorphic to $\mathrm{Cone}(X)$. □

Proof. This lemma follows immediately from the Topological-Pushout Proposition 8.12. ■

Lemma 21.4. (Mapping Cylinder–Embedding Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The natural maps

(1) $\begin{array}{rcl} i\colon X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & [(x,0)] \end{array}$ and (2) $\begin{array}{rcl} j\colon Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$

are closed embeddings. Sometimes we use these natural embeddings to identify X and Y with the corresponding subspaces of the mapping cylinder.



mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$

Proof.

(a) One can easily verify that the given map $i\colon X \to \mathrm{Cyl}(f\colon X \to Y)$ is injective and continuous. By the Open-Closed–Injective Lemma 4.22 it remains to show that the map $i\colon X \to \mathrm{Cyl}(f\colon X \to Y)$ is a closed map.

So let $A \subseteq X$ be a closed subset. The preimage of $i(A)$ under the projection $(X \times [0,1]) \sqcup Y \to \mathrm{Cyl}(f)$ is $A \times \{0\}$. It follows from the Topological-Coproduct Proposition 7.26 and the Product-of-Subsets Lemma 7.8 (1) that this is a closed subset. Therefore it follows from the Topological-Quotient Proposition 8.2 (0b) that $i(A)$ is indeed a closed subset of $\mathrm{Cyl}(f)$.

(b) We can prove this statement the same fashion as we proved (1). Alternatively we can argue more succinctly as follows: By the Mapping Cylinder–Pushout Lemma 21.3 we have the following pushout diagram:

$$\begin{array}{ccc} X & \xrightarrow{x\mapsto(x,1)} & X \times [0,1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (x,t)\mapsto[(x,t)]} \\ Y & \xrightarrow{y\mapsto[y]} & \mathrm{Cyl}(f\colon X \to Y). \end{array}$$

Clearly the top horizontal map $X \to X \times [0,1]$ is a closed embedding. It thus follows from the Mapping Cylinder–Pushout Lemma 21.3 that the "opposite map" $Y \to \mathrm{Cyl}(f\colon X \to Y)$ is also a closed embedding. ∎

The following lemma shows in particular that the homotopy type of a mapping cylinder is not very interesting:

Lemma 21.5. (Mapping Cylinder–Homotopy Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. We denote by $i\colon X \to \mathrm{Cyl}(f)$ and $j\colon Y \to \mathrm{Cyl}(f)$ the natural embeddings,

(1) The subspace $j(Y)$ is a deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$, in fact an explicit natural deformation retraction is given by the following map[245]

$$\begin{aligned} H\colon \mathrm{Cyl}(f\colon X \to Y) \times [0,1] &\to \mathrm{Cyl}(f\colon X \to Y) \\ (P,t) &\mapsto \begin{cases} P, & \text{if } P \in Y, \\ [(Q, s\cdot(1-t)+t)], & \text{if } P = [(Q,s)], \text{ where} \\ & \quad Q \in X \text{ and } s \in [0,1]. \end{cases} \end{aligned}$$

(2) We denote by $r = H_1$ the retraction $\mathrm{Cyl}(f\colon X \to Y) \to Y$ given by (1). The following statements hold:

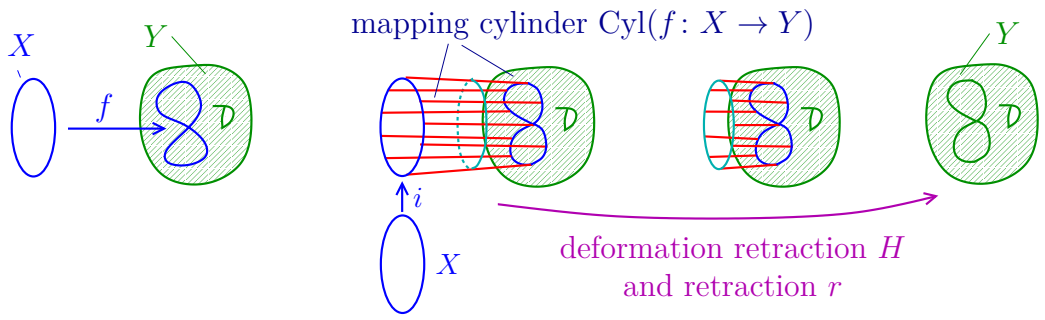
(a) The map $j\colon Y \to \mathrm{Cyl}(f\colon X \to Y)$ and $r\colon \mathrm{Cyl}(f\colon X \to Y) \to Y$ are homotopy equivalences.

(b) The map $i\colon X \to \mathrm{Cyl}(f\colon X \to Y)$ is a homotopy equivalence if and only if the map $f\colon X \to Y$ is a homotopy equivalence.[246]

Proof.

[245] On page 562 we saw that the mapping cylinder of a constant map $f\colon X \to \{y\}$ is homeomorphic to the cone on X. Thus we see that Statement (2) can be viewed as a generalization of the Cone Contractability Lemma 20.20.

[246] In the Homotopy Equivalence–Characterization Corollary 22.20 we will strengthen this statement: We will show that if $f\colon X \to Y$ is a homotopy equivalence, then $i(X)$ is in fact a deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$.



(1) We need to show that the map H is continuous. Using the Topological-Product Proposition 7.2 (1) and using the Topological-Quotient Proposition 8.2 (1a) one can easily verify that the maps

$$\begin{array}{rcl}(X\times[0,1])\times[0,1] & \to & \mathrm{Cyl}(f)\\ (Q,t) & \mapsto & [(Q,s\cdot(1-t)+t)]\end{array} \quad \text{and} \quad \begin{array}{rcl}Y\times[0,1] & \to & \mathrm{Cyl}(f)\\ (P,t) & \mapsto & P\end{array}$$

are continuous. But this implies, by the somewhat delicate Homotopy Pushout Lemma 18.13, that

$$H\colon ((X\times[0,1])\cup_{(x,1)\mapsto f(x)} Y)\times[0,1] \to \mathrm{Cyl}(f)$$

itself is continuous. Once continuity is out of the way it is basically clear that H is a deformation retraction.

(2) (a) It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the maps $j\colon Y\to\mathrm{Cyl}(f)$ and $r\colon \mathrm{Cyl}(f)\to Y$ are homotopy equivalences.

(b) We consider the map

$$\begin{array}{rcl}G\colon X\times[0,1] & \to & \mathrm{Cyl}(f\colon X\to Y)\\ (x,t) & \mapsto & [(x,t)].\end{array}$$

This map is continuous by the Topological-Quotient Proposition 8.2 (1b). We make the following observations:

$$\begin{array}{ccccc}i=G_0\colon X\to\mathrm{Cyl}(f) \text{ is a homotopy equivalence} & \Longleftrightarrow & i=G_1\colon X\to\mathrm{Cyl}(f) \text{ is a homotopy equivalence} & \Longleftrightarrow & \overbrace{r\circ G_1}^{=f}\colon X\to Y \text{ is a homotopy equivalence}\\ \uparrow & & \uparrow & & \\ \text{by the Homotopy-Equivalence Basics Lemma 19.14 (3), since } G_0 \text{ and } G_1 \text{ are homotopic} & & \text{by the Homotopy-Equivalence Basics Lemma 19.14 (1), since by (2a) we know that } r \text{ is a homotopy equivalence} & & \end{array}$$

We have thus shown that i is a homotopy equivalence if and only if $f=r\circ G_1$ is a homotopy equivalence. ■

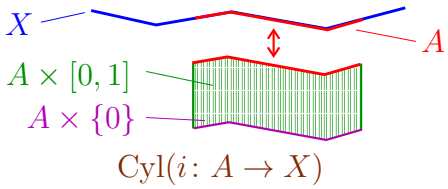
We conclude this discussion of mapping cylinders with a (non-) example. We will not really make use of the following lemma, but Statement (3) of the reader should warn the reader that the topology on a mapping cylinder is not necessarily what one might think it is.

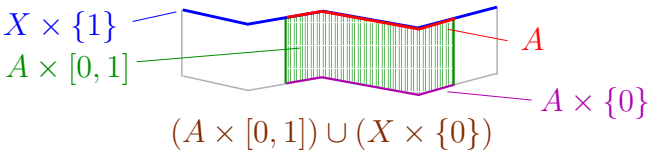
Lemma 21.6. (Mapping Cylinder-of-Embedding Lemma) Let X be a topological space and let $A\subseteq X$ be a subspace. We denote by $i\colon A\to X$ the inclusion map and we consider the map

$$\Theta\colon \operatorname{Cyl}(i) = ((A\times[0,1]) \sqcup X)/(a,1)\sim i(a) \;\to\; \overbrace{(A\times[0,1])\cup(X\times\{1\})}^{\text{equipped with the subspace topology coming from the product } X\times[0,1]}$$
$$[P] \;\mapsto\; \begin{cases} P, & \text{if } P\in A\times[0,1],\\ (P,1), & \text{if } P\in X. \end{cases}$$

The following three statements hold:

(1) The map Θ is continuous and it is a bijection.
(2) If A is a closed subset of X, then Θ is a homeomorphism.
(3) There exist situations, e.g. $X=[0,\infty)$ and $A=[0,1)$, in which Θ is *not* a homeomorphism.



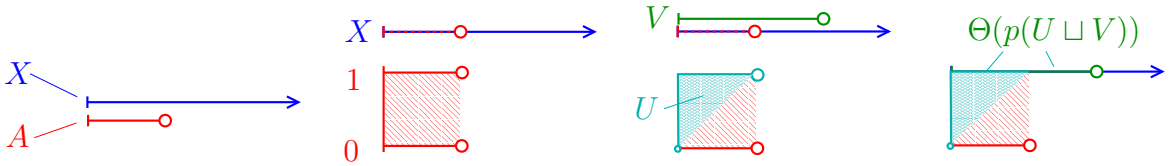


Proof.

(1) It follows easily from the Topological-Quotient Proposition 8.2 (1b) that the given map Θ is continuous. Furthermore it is basically clear that Θ is a bijection.
(2) Suppose that A is a closed subset of X. Note that $A\times\{1\}$ is a closed subset of $A\times[0,1]$. It thus follows basically from the Regluing Lemma 8.17 that Θ is a homeomorphism.
(3) We consider the case $X=[0,\infty)$ and $A=[0,1)$. We denote by

$$p\colon (A\times[0,1))\sqcup X \;\to\; \operatorname{Cyl}(i) = ((A\times[0,1))\sqcup X)/\!\sim$$

the projection. We consider the open subset $U := \{(x,t)\in[0,1)\times[0,1]\,|\,x>t\}$ of $A\times[0,1)$. Furthermore we consider the open subset $V:=[0,2)$ of $X=[0,\infty)$.



Note that $p^{-1}(p(U\sqcup V)) = U\sqcup V$. Thus it follows immediately from the Definition 8.3 of the quotient topology on $\operatorname{Cyl}(i) = ((A\times[0,1))\sqcup X)/\!\sim$ that $p(U\sqcup V)$ is an open subset of $\operatorname{Cyl}(i)$. On the other hand we have

$$\Theta(p(U\sqcup V)) \;=\; U\cup\{(x,1)\,|\,x\in V\},$$

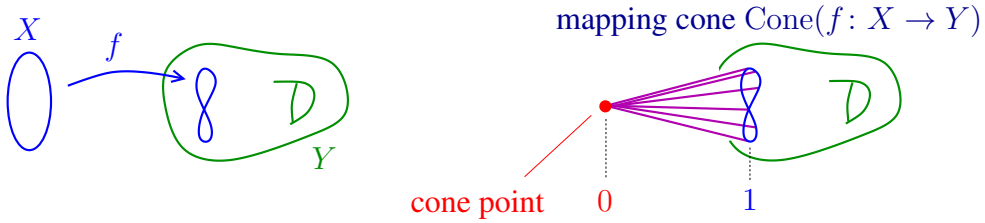
but considering the point $(1,1)$ one can easily verify that this is *not* an open subset of $(A\times[0,1])\cup(X\times\{1\})$. ■

21.2. **The mapping cone.** In the previous section we associated to a continuous map $f\colon X\to Y$ between topological spaces a new topological space, namely the mapping cylinder $\operatorname{Cyl}(f\colon X\to Y)$. But we saw in the Mapping Cylinder–Homotopy Lemma 21.5 (1) that the mapping cylinder is "homotopically indistinguishable" from Y. Therefore we move on to the definition of the mapping cone which is a much more interesting construction:

Definition 21.7. Given a continuous map $f\colon X \to Y$ between topological spaces the corresponding mapping cone is defined as

$$\mathrm{Cone}(f\colon X \to Y) \;:=\; (\mathrm{Cone}(X) \sqcup Y)/\!\sim \;\text{ where } [(x,1)] \sim f(x) \text{ for all } x \in X.$$

We refer to the point $[X \times \{0\}]$ as the tip of the cone of the mapping cone.



Remark. If $f\colon X \to Y$ is a continuous map between topological spaces, then it follows almost immediately from the definitions and the Double-Quotient Lemma 8.7 that we have a natural homeomorphism

$$\mathrm{Cone}(f\colon X \to Y) \;\xrightarrow{\cong}\; \mathrm{Cyl}(f\colon X \to Y)/(X \times \{0\}).$$ □

Example. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^2$. In Exercise 21.7 we will see that the corresponding mapping cone $\mathrm{Cone}(f\colon S^1 \to S^1)$ is homeomorphic to the real projective plane $\mathbb{R}\mathrm{P}^2$. □

The next proposition will not come as a surprise:

Proposition 21.8. (Mapping Cone Topology-Properties Proposition) In the following let $f\colon Y \to Y$ be a continuous map between two topological spaces.
(1) If Y and Y are compact, then $\mathrm{Cone}(f\colon Y \to Y)$ is also compact.
(2) If Y and Y are (path)-connected, then $\mathrm{Cone}(f\colon Y \to Y)$ is also (path)-connected.
(3) If Y and Y are Hausdorff, then $\mathrm{Cone}(f\colon Y \to Y)$ is also Hausdorff.

Proof. Statements (1) and (2) follow from the corresponding statements of the Cone Topology-Properties Proposition 20.17, the Disjoint Union Topology-Properties Lemma 7.28 and the Quotient Topology-Properties Proposition 8.5. The proof of Statement (3) is similar to the proofs of Statement (3) of the Quotient Topology-Properties Proposition 8.5 and the Mapping Cylinder Topology-Properties Proposition 21.2. We leave it to the reader to fill in the details. ■

Once again we will now see that our construction is a special case of the flexible pushout construction:

Lemma 21.9. (Mapping Cone–Pushout Lemma) If $f\colon X \to Y$ is a continuous map between topological spaces, then the following is a pushout diagram:

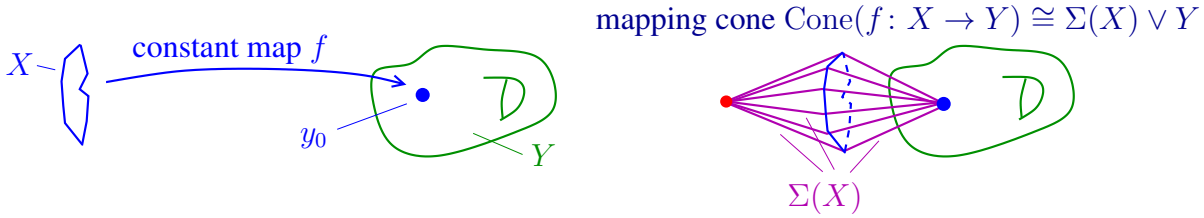
$$\begin{array}{ccc} X & \xrightarrow{x \mapsto [(x,1)]} & \mathrm{Cone}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P \mapsto [P]} \\ Y & \xrightarrow[y \mapsto [y]]{} & \mathrm{Cone}(f\colon X \to Y). \end{array}$$

Proof. This time the statement also follows immediately from the Topological-Pushout Proposition 8.12. ■

The following lemma doubles as an example:

Lemma 21.10. (Mapping Cone-of-Constant Map Lemma) Let X and Y be non-empty topological spaces and let $f\colon X \to Y$ be a constant map. (Recall that this means that there exists an $y_0 \in Y$ with $f(X) \subseteq \{y_0\}$.) We denote by $\Sigma(X) \vee Y$ the wedge that we obtain from the disjoint union $\Sigma(X) \sqcup Y$ by identifying the North Pole of $\Sigma(X)$ with the point $y_0 \in Y$.[247] There exists a natural homeomorphism

$$\mathrm{Cone}(f\colon X \to Y) \xrightarrow{\cong} \Sigma(X) \vee Y.$$



Proof. Let $p\colon \mathrm{Cone}(X) \to \Sigma(X)$ be the map that is given by $[(x,t)] \mapsto [(x, 2t-1)]$. Almost by definition of $\Sigma(X)$ this map induces a homeomorphism $\overline{p}\colon \mathrm{Cone}(X)/(X \times \{1\}) \to \Sigma(X)$. Next we consider the map

$$\varphi\colon \mathrm{Cone}(X) \sqcup Y \to \Sigma(X) \sqcup Y$$

that is induced by the map p on $\mathrm{Cone}(X)$ and by the identity on Y.

Claim. The above map descends to a homeomorphism

$$\overline{\varphi}\colon \underbrace{\mathrm{Cone}(f\colon X \to Y)}_{=(\mathrm{Cone}(X)\sqcup Y)/\sim} \to \Sigma(X) \vee Y.$$

Proof. First note that it follows from the Topological-Coproduct Proposition 7.26 that the original map φ is continuous. Next note that it follows from the Quotient-to-Quotient Lemma 8.6 that the induced map $\overline{\varphi}\colon \mathrm{Cone}(X) \sqcup Y \to \Sigma(X) \vee Y$ is also continuous. One can easily verify that this map is a bijection. It remains to show that the inverse of $\overline{\varphi}$ is continuous. It follows from the Topological-Quotient Proposition 8.2 (1b) and the Topological-Coproduct Proposition 7.26 (1b) (see also the later Wedge Proposition 96.2 (1b)) that it remains to show that the restriction of the map $\overline{\varphi}^{-1}$ to $\Sigma(X)$ is continuous. To do so we consider the following little diagram:

$$\begin{array}{ccccc}
\mathrm{Cone}(X) & \xrightarrow{x \mapsto [x]} & \mathrm{Cone}(X)/(X \times \{1\}) & \xrightarrow[\cong]{\overline{p}} & \Sigma(X) \\
\downarrow{\scriptstyle \iota} & \swarrow{\scriptstyle \overline{\iota}} & & & \downarrow \\
\mathrm{Cone}(f\colon X \to Y) & & \xrightarrow[\overline{\varphi}]{} & & \Sigma(X) \vee Y.
\end{array}$$

We make the following clarifications:

(1) It follows easily from the definitions that the diagram commutes.
(2) We denote by $\iota\colon \mathrm{Cone}(X) \to \mathrm{Cone}(f\colon X \to Y)$ the natural map.
(3) Since $f(X \times \{1\})$ is a single point we obtain from the Topological-Quotient Proposition 8.2 that ι descends to a continuous map

$$\overline{\iota}\colon \mathrm{Cone}(X)/(X \times \{1\}) \to \mathrm{Cone}(f\colon X \to Y).$$

[247] Later, in Definition 96.1, we will refer to $\Sigma(X) \vee Y$ as the *wedge* of $\Sigma(X)$ and Y.

It follows from the above discussion that the map $\Sigma(X) \to \mathrm{Cone}(f\colon X \to Y)$ we are interested in is given by the continuous map $\overline{i} \circ \overline{p}^{-1}$. ■

The following lemma is an analog of the Mapping Cylinder–Embedding Lemma 21.4:

Lemma 21.11. (Mapping Cone–Embedding Lemma) Let $f\colon Y \to Y$ be a continuous map between topological spaces.

(1) The natural map
$$\begin{array}{ccc} Y & \to & \mathrm{Cone}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$
is a closed embedding. Often we will use this natural embedding to identify Y with its image in $\mathrm{Cone}(f\colon X \to Y)$.

(2) If the map $f\colon X \to Y$ is a closed embedding, then the corresponding natural map $\mathrm{Cone}(X) \to \mathrm{Cone}(f\colon X \to Y)$ is also a closed embedding.

Proof. Recall that by the Mapping Cone–Pushout Lemma 21.9 we have the following pushout diagram:
$$\begin{array}{ccc} X & \xrightarrow{x \mapsto [(x,1)]} & \mathrm{Cone}(X) \\ {\scriptstyle f}\downarrow & & \ulcorner \downarrow{\scriptstyle P \mapsto [P]} \\ Y & \xrightarrow[y \mapsto [y]]{} & \mathrm{Cone}(f\colon X \to Y). \end{array}$$

(1) By the Space-Cone Embedding Lemma 20.19 we know that the top horizontal map $X \to \mathrm{Cone}(X)$ is a closed embedding. It follows from the Topological Pushout-Maps Lemma 8.16 (3) and (4) that the "opposite map" $Y \to \mathrm{Cone}(f\colon X \to Y)$ is also a closed embedding.

(2) If $f\colon X \to Y$ is a closed embedding, then we see as in (1) that the "opposite map" $\mathrm{Cone}(X) \to \mathrm{Cone}(f\colon X \to Y)$ is a closed embedding. ■

So what have we done so far? We have associated to a continuous map $f\colon X \to Y$ between topological spaces the topological space $\mathrm{Cone}(f\colon X \to Y)$. This construction will allow us to turn questions about maps between topological spaces into questions about topological spaces. As always in algebraic topology, not only do we want a map

$$\text{set of continuous maps between topological spaces} \;\to\; \text{set of topological spaces},$$

but we really want a covariant functor. It is clear what the category should be on the right. It is a little less clear what the category should be on the left. Here it is.

Definition 21.12. We refer to the category *MapTop* with

$$\begin{array}{rcl} \mathrm{Ob}(\mathit{MapTop}) & \coloneqq & \text{all continuous maps between topological spaces,} \\ \mathrm{Mor}(f\colon X \to Y, \widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) & \coloneqq & \begin{array}{c}\text{the set of pairs of}\\ \text{continuous maps}\\ (\varphi\colon X \to \widetilde{X}, \psi\colon Y \to \widetilde{Y})\\ \text{such that the diagram}\end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \end{array} \quad \text{commutes} \end{array}$$

with the (hopefully) obvious composition of morphisms as the category of maps between topological spaces.

The following lemma says that we can view "taking the mapping cone" as covariant functor:

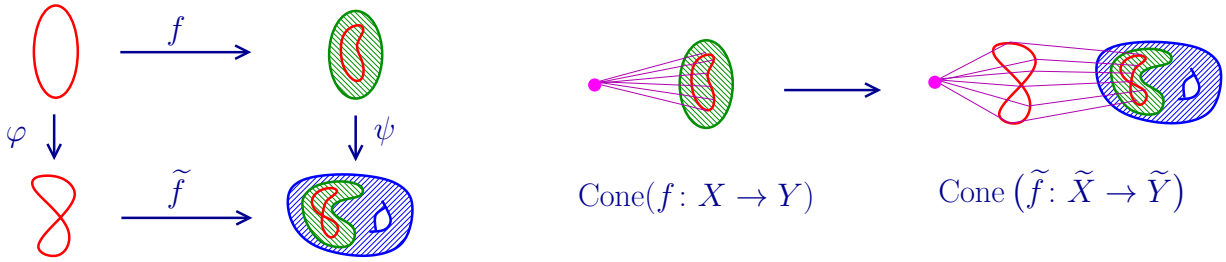
Lemma 21.13. (Mapping Cone–Functor Lemma) The maps[248]

$$(f\colon X \to Y) \;\mapsto\; \mathrm{Cone}(f\colon X \to Y)$$

and

$$(X \xrightarrow{\varphi} \widetilde{X}, Y \xrightarrow{\psi} \widetilde{Y}) \;\mapsto\; \left(\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y) & \to & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) \\ [P] & \mapsto & \begin{cases} [(\varphi(x),t)], & \text{if } P=[(x,t)] \text{ with,} \\ & \quad x \in X,\, t \in [0,1] \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array}\right).$$

define a covariant functor from the category *MapTop* of continuous maps between topological spaces to the category *Top* of topological spaces.



Proof. One needs to convince oneself that the maps between the mapping cones are in fact continuous. First note that by the Cone–Functor Lemma 20.14 (1) we know that the map

$$\begin{array}{rcl} \mathrm{Cone}(X) & \to & \mathrm{Cone}(\widetilde{X}) \\ [(x,t)] & \mapsto & [(\varphi(x),t)] \end{array}$$

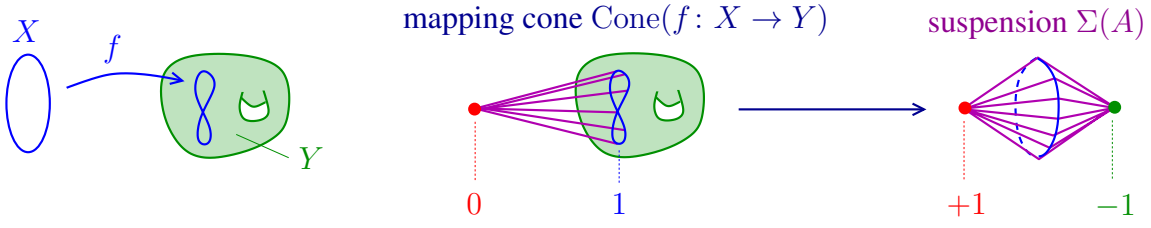
is continuous. The continuity of the map between the mapping cones now follows from the Topological Pushout Proposition 8.12 (1b). ■

On several occasions we will make use of the following lemma:

Lemma 21.14. (Mapping Cone–Quotient Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y)/Y & \to & \Sigma(X) \\ [P] & \mapsto & \begin{cases} [(x, 1-2t)], & \text{if } P=[(x,t)] \in \mathrm{Cone}(X), \\ [X \times \{-1\}], & \text{if } P \in Y \end{cases} \end{array}$$

is a natural homeomorphism.[249]



[248] An analogous statement also holds for mapping cylinders. Since mapping cylinders are a little dull there is no need to spell out this statement.

[249] Using the Mapping Cone–Functor Lemma 21.13 one could now make precise what we mean by "natural".

Proof. It is clear that the given map is a bijection. Using the Topological-Quotient Proposition 8.2 (1b) one can easily verify that the map is continuous. Next consider the map

$$\begin{array}{rcl}(X \times [-1,1])/(X \times \{1\}) & \to & \mathrm{Cone}(X) \\ [(x,t)] & \mapsto & [(x, \frac{1}{2} - \frac{1}{2}t)].\end{array}$$

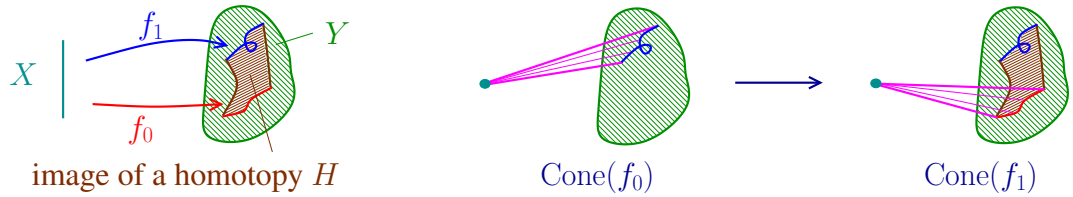
It follows from the Quotient-to-Quotient Lemma 8.6 that this map is continuous. Furthermore it follows once again from the Quotient-to-Quotient Lemma 8.6 that this map descends to a continuous map

$$\Sigma(X) \ \to \ \mathrm{Cone}(f\colon X \to Y)/Y$$

which is clearly the inverse to the original map. ■

The following proposition shows in particular that homotopic maps give rise to homotopy equivalent mapping cones. This proposition allows us to translate the question whether two maps are homotopic to the question whether two topological spaces are homotopy equivalent.

Proposition 21.15. (Homotopic Map–Mapping Cone Proposition) Suppose we are given continuous maps $f_0, f_1\colon X \to Y$ between topological spaces. If f_0, f_1 are homotopic, then there exists a homotopy equivalence $\mathrm{Cone}(f_0) \to \mathrm{Cone}(f_1)$ rel the union of Y and the tip of the cone.

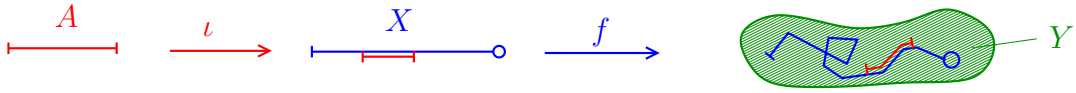


It is not that difficult to write down explicit maps between the cones and to write down explicit homotopies. (In fact we will take this approach in the closely related Homotopic Attaching Map Lemma 135.12). But we prefer to go take a more conceptual approach. Namely, instead of proving the Homotopic Map–Mapping Cone Proposition 21.15 directly we will prove it via two technical lemmas. We start out with the first one, which will come in handy at a few other occasions:

Lemma 21.16. (Mapping Cone–Closed Embedding Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $\iota\colon A \to X$ be a closed embedding. The induced map

$$\mathrm{Cone}(f \circ \iota\colon A \to Y) \ \to \ \mathrm{Cone}(f\colon X \to Y)$$

from the Mapping Cone–Functor Lemma 21.13 is a closed embedding.



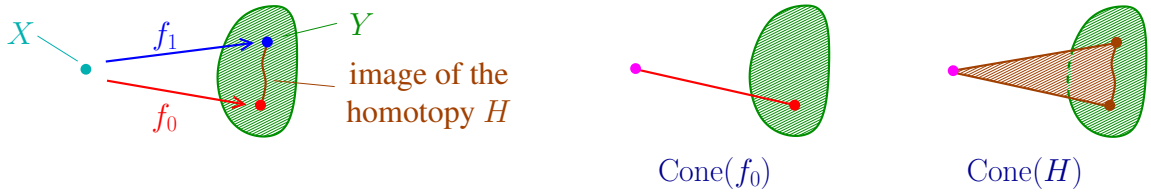
Proof. First note that it follows from the Cone–Closed Embedding Lemma 20.18 that the closed embedding ι induces a closed embedding $\mathrm{Cone}(A) \to \mathrm{Cone}(X)$. It now follows from the Pushout–Closed Embedding Lemma 8.18 and the Mapping Cone–Pushout Lemma 21.9 that the induced map $\mathrm{Cone}(f \circ \iota\colon A \to Y) \to \mathrm{Cone}(f\colon X \to Y)$ is indeed a closed embedding. ■

This lemma now allows us to prove the following lemma:

Lemma 21.17. (Mapping Cone-of-Homotopy Lemma) Let $f_0, f_1 \colon X \to Y$ be continuous maps between topological spaces. Furthermore let $H \colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. Let $j \in \{0,1\}$. We denote by $i_j \colon X \to X \times \{j\}$ the embedding that is given by $x \mapsto (x, j)$.

(1) By the Mapping Cone–Functor Lemma 21.13 we know that the map i_j induces a continuous map $\mathrm{Cone}(f_j) \to \mathrm{Cone}(H)$. This map is a closed embedding.

(2) If we use the map from (1) to view $\mathrm{Cone}(f_j)$ as a subspace of $\mathrm{Cone}(H)$, then it is a deformation retract of $\mathrm{Cone}(H)$.



Proof of Proposition 21.15 assuming Lemma 21.17. Let H be a homotopy between f_0 and f_1. By the Mapping Cone-of-Homotopy Lemma 21.17 we can view $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ as deformation retracts of $\mathrm{Cone}(H)$. It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 and the Homotopy-Equivalence Basics Lemma 19.14 that $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ are homotopy equivalent. A quick inspection of the maps involved shows that the two cones are homotopy equivalent rel the union of Y and the tip of the cone. ∎

Proof of the Mapping Cone-of-Homotopy Lemma 21.17. Let $f_0, f_1 \colon X \to Y$ be continuous maps between topological spaces. Furthermore let $H \colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. Let $j \in \{0,1\}$. We denote by $i_j \colon X \to X \times [0,1]$ the closed embedding that is given by $x \mapsto (x, j)$.

(1) First note that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_j} & Y \\ {\scriptstyle i_j}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ X \times [0,1] & \xrightarrow{H} & Y. \end{array}$$

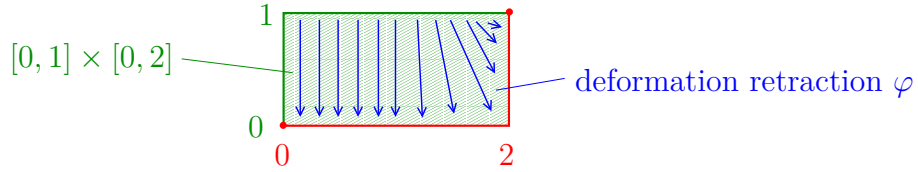
Thus we get from the Mapping Cone–Functor Lemma 21.13 an induced continuous map $\mathrm{Cone}(f_j) \to \mathrm{Cone}(H)$. Since i_j is a closed embedding we obtain from the Mapping Cone–Closed Embedding Lemma 21.16 that the map $\mathrm{Cone}(f_j) \to \mathrm{Cone}(H)$ is also a closed embedding.

(2) Since this proof suffers from an abundance of intervals playing different roles we rescale two of the intervals:
- We use the interval $[0,1]$ for the homotopy H.
- We use the interval $[0,2]$ for the definition of the mapping cone.
- We use the interval $[0,3]$ for the definition of the desired deformation retraction.

With this convention out of the way we pick a deformation retraction

$$\varphi \colon ([0,1] \times [0,2]) \times [0,3] \;\to\; [0,1] \times [0,2]$$

from $[0,1]\times[0,2]$ to $([0,2]\times\{0\})\cup(\{2\}\times[0,1])$ such that $\varphi(r,s,t)=(r,\frac{1}{3}st)$ whenever $r\in[0,\frac{1}{2}]$. It is straightforward to see that such a deformation retraction exists. We refer to the figure below for an illustration.



Now we consider the map

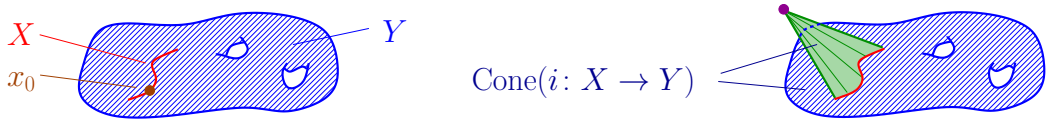
$$G\colon \overbrace{(((X\times[0,1]\times[0,2])\sqcup Y)/\sim)}^{=\mathrm{Cone}(H\colon X\times[0,1]\to Y)}\times[0,3] \to \mathrm{Cone}(H)$$
$$([P],z) \mapsto \begin{cases}[(x,\varphi(s,t,z))], & \text{if } P=(x,s,t),\\ P, & \text{if } P\in Y.\end{cases}$$

Using the Topological-Quotient Proposition 8.2 (1a) and the Homotopy Pushout Lemma 18.13 one can easily show that this map G is continuous. It is straightforward to verify that this map has the desired properties. ■

We conclude our discussion of mapping cones with the following proposition:

Proposition 21.18. (Mapping Cone-of-Contractible Proposition) Let Y be a topological space and let X be a closed subspace of Y. We denote by $i\colon X\to Y$ the inclusion map. We use the Mapping Cone–Embedding Lemma 21.11 (2) to view Y as a subspace of $\mathrm{Cone}(i\colon X\to Y)$.

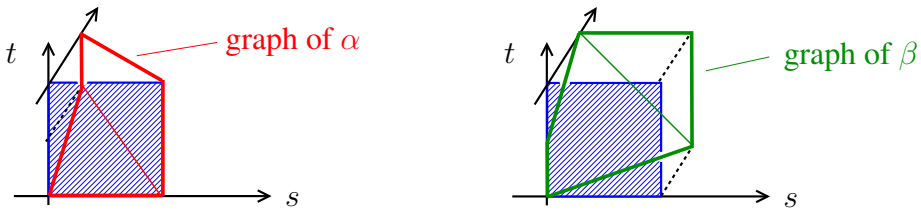
(1) If X is contractible, then given any $x_0\in X$ there exists a deformation retraction G from $\mathrm{Cone}(i\colon X\to Y)$ to Y such that $G_1(\text{tip of the cone } [X\times\{0\}])=\{x_0\}$.
(2) If $x_0\in X$ is a deformation retract of X, then there exists a deformation retraction G from $\mathrm{Cone}(i\colon X\to Y)$ to Y such that for any $t\in[0,1]$ we have $G_1([(x_0,t)])=\{x_0\}$.



Proof. We first prove Statement (1). Thus let $x_0\in X$. Since X is contractible it follows from the "furthermore statement" of the Contractability Criterion Lemma 19.20 that there exists a continuous map $r\colon X\times[0,1]\to X$ such that $r_0=\mathrm{id}$ and such that $r_1(x)=x_0$ for all $x\in X$. We pick continuous functions $\alpha,\beta\colon[0,1]\times[0,1]\to[0,1]$ with the following properties:

(a) For every $t\in[0,1]$ we have $\alpha(0,t)=1$ or $\beta(0,t)=0$.
(b) We have $\alpha(s,0)=0$ and $\beta(s,0)=s$ for all $s\in[0,1]$.
(c) We have $\beta(s,1)=1$ for all $s\in[0,1]$.
(d) We have $\alpha(1,t)=0$ and $\beta(1,t)=1$ for all $t\in[0,1]$.

We sketch graphs of such functions α and β in the figure below. We leave it to the reader to write down explicit descriptions of such maps.



Now we consider the map

$$\begin{array}{rcl} F\colon \overbrace{((X\times[0,1])/(X\times\{0\}))}^{=\mathrm{Cone}(X)}\times[0,1] & \to & \overbrace{(X\times[0,1])/(X\times\{0\})}^{=\mathrm{Cone}(X)} \\ ([(x,s)],t) & \mapsto & [(r(x,\alpha(s,t)),\beta(s,t))]. \end{array}$$

We make the following observations:

- Since α and β are continuous and since $[0,1]$ is regionally compact we obtain from the Topological-Quotient Proposition 8.2 (1b) and the Product-Quotient Mixing Lemma 17.8 that F is continuous.
- It follows from (a) that F is actually well-defined.
- It follows from (b) that $F_0 = \mathrm{id}$.
- It follows from (c) that F_1 takes values in $X\times\{1\}$.
- It follows from (d) that F is a homotopy rel $X\times\{1\}$.

In summary this shows that F is a deformation retraction from $\mathrm{Cone}(X)$ to $X\times\{1\}$. Now we consider the map

$$\begin{array}{rcl} G\colon \overbrace{(\mathrm{Cone}(X)\cup_{(x,1)=a} Y)}^{=\mathrm{Cone}(i\colon X\to Y)}\times[0,1] & \to & \mathrm{Cone}(i\colon X\to Y) \\ ([P],t) & \mapsto & \begin{cases} [F(P,t)] & \text{if } P\in\mathrm{Cone}(X), \\ [P], & \text{if } P\in Y. \end{cases} \end{array}$$

It follows from the Homotopy Pushout Lemma 18.13 that this map is continuous. Clearly G is a deformation retraction from $\mathrm{Cone}(i\colon X\to Y)$ to Y. By construction it sends the tip of the cone to x_0.

It remains to prove Statement (2). If we start out with a deformation retraction r from X to $\{x_0\}$, then the above homotopy F has the desired property. ∎

21.3. **The mapping torus.** In this last section we introduce the mapping torus of a continuous map between topological spaces:

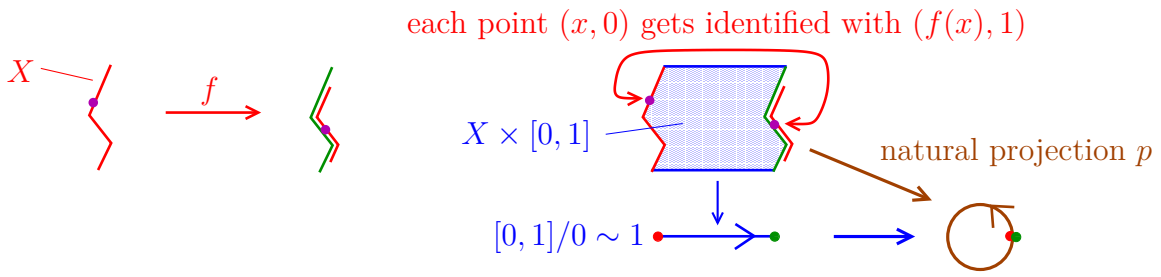
Definition 21.19. Let X be a topological space and let $f\colon X\to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])/\!\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X,f). We refer to the map

$$\begin{array}{rcl} \mathrm{Tor}(X,f)=(X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

as the natural projection onto S^1.



As always let us check the box at our favorite properties of topological spaces:

Proposition 21.20. (Mapping Torus Topology-Properties Proposition) In the following let $f\colon X \to Y$ be a continuous map between two topological spaces.

(1) If X and Y are compact, then $\mathrm{Tor}(f\colon X \to Y)$ is also compact.
(2) If X and Y are (path)-connected, then $\mathrm{Tor}(f\colon X \to Y)$ is also (path)-connected.
(3) If X and Y are Hausdorff, then $\mathrm{Tor}(f\colon X \to Y)$ is Hausdorff.

Proof. The first two statements follow immediately from the corresponding statements of the Product Topology-Properties Proposition 7.9 and the Quotient Topology-Properties Proposition 8.5. We will prove Statement (3) in Exercise 21.10. ■

The following lemma shows that the mapping torus is a pushout.

Lemma 21.21. (Mapping Torus–Pushout Lemma) Let X be a topological space and let $f\colon X \to X$ be a continuous map. The following is a pushout diagram:

$$\begin{array}{ccc} (X \times \{0\}) \sqcup (X \times \{1\}) & \hookrightarrow & X \times [0,1] \\ \downarrow{\scriptstyle \substack{(x,0)\mapsto x \\ (x,1)\mapsto f(x)}} & & \downarrow{\scriptstyle (x,t)\mapsto [(x,t)]} \\ X & \xrightarrow[x\mapsto [x]]{} & \mathrm{Tor}(X,f). \end{array}$$

Proof. First note that it follows from the Topological Pushout Proposition 8.12 (1b) that the given maps induce a continuous map

$$\Phi\colon \underbrace{(X \times [0,1]) \cup_{X\times\{0,1\}} X}_{\substack{\text{pushout by the Topological} \\ \text{Pushout Proposition 8.12 (1b)}}} \to \mathrm{Tor}(X,f).$$

Next we consider the natural map

$$\begin{array}{rccl} \Psi\colon & \overbrace{(X \times [0,1])/\sim}^{\mathrm{Tor}(X,f)} & \to & (X \times [0,1]) \cup_{X\times\{0,1\}} X \\ & [(x,t)] & \mapsto & [(x,t)]. \end{array}$$

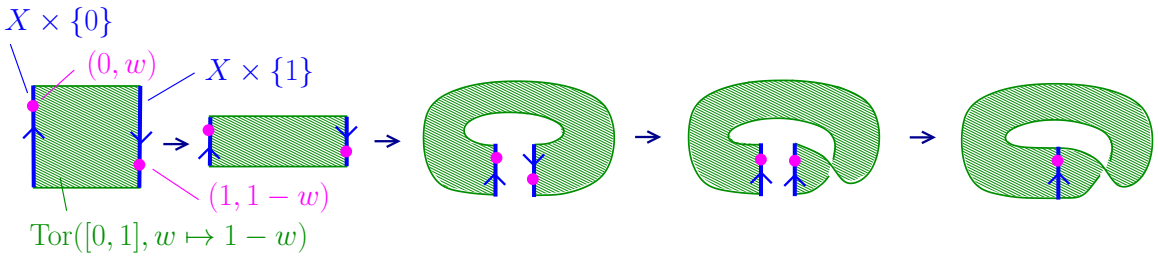
It follows from the Topological Pushout Proposition 8.12 (1a) and the Topological-Quotient Proposition 8.2 (1b) that Ψ is continuous. One can easily verify that Φ and Ψ are inverses of one another. Thus we have shown that both maps are homeomorphisms. ■

Next we will see that several well-established topological spaces are actually mapping tori.

[249] It follows from the Topological-Quotient Proposition 8.2 (1b) that the natural projection is indeed continuous.

Examples.

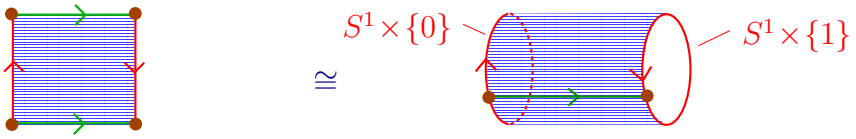
(1) We consider the topological space $[0,1]$ and the self-homeomorphism $f\colon [0,1] \to [0,1]$ that is given by $f(w) = 1 - w$. It follows immediately from the definitions that the corresponding mapping torus $\mathrm{Tor}([0,1], f)$ is precisely the Möbius band as defined on page 310.



(2) Let $X = S^1$ and let $f\colon S^1 \to S^1$ be the homeomorphism that is given by $f(z) = \overline{z}$. It follows easily from the Quotient-to-Quotient Lemma 8.6, the Compact-Hausdorff Proposition 4.23 (3) and the Mapping Torus Topology-Properties Proposition 21.20 (3) that the map

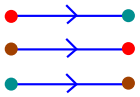
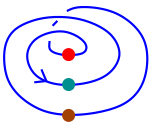
$$\begin{array}{rcl} \overbrace{([0,1]\times[0,1])\,/\,(x,0)\sim(x,1),(0,y)\sim(1,1-y)}^{\text{Klein bottle}} & \to & \mathrm{Tor}(S^1, f) \\ [(x,y)] & \mapsto & [\exp(2\pi\,\mathrm{i}y), x)] \end{array}$$

is a homeomorphism from the Klein bottle to the mapping torus $\mathrm{Tor}(S^1, f)$.



Klein bottle $([0,1]\times[0,1])/\sim$ mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$

(3) As a final example we consider $X = \mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$, equipped with the discrete topology. Let $f\colon X \to X$ be the homeomorphism that is given by $f(x) = x + 1 \bmod 3$.

the mapping torus $\mathrm{Tor}(\mathbb{Z}_3, f(x) = x+1)$  is homeomorphic to 

As we illustrate in the figure above, the corresponding mapping torus $\mathrm{Tor}(X, f)$ is homeomorphic to S^1. □

We slow down a little to consider the case $\varphi = \mathrm{id}_X$ in more detail. On page 310 we already saw that $\mathrm{Tor}(S^1, \mathrm{id}_{S^1})$ is homeomorphic to the torus $S^1 \times S^1$. Thus the following lemma does not come as a surprise:

Lemma 21.22. (Mapping Torus-of-id Lemma) Given any topological space X the map

$$\begin{array}{rcl} \Phi\colon \mathrm{Tor}(X, \mathrm{id}_X) & \to & X \times S^1 \\ [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a natural[250] homeomorphism.

[250] We leave it to the reader to figure out what "natural" means in this context.

Convention 21.23. Let X be a topological space. We use the homeomorphism from the Mapping Torus-of-id Lemma 21.22 to make the identification $\mathrm{Tor}(X, \mathrm{id}) =_{\mathrm{i}} X \times S^1$.

Proof. [251] It is easy to verify that the given map $\Phi\colon \mathrm{Tor}(X, \mathrm{id}_X) \to X \times S^1$ is natural and that it is a bijection. Using the Topological-Product Proposition 7.2 (1a) and the Topological-Quotient Proposition 8.2 (1b) one can easily show that Φ is continuous. It still remains to show that Φ is actually a homeomorphism. We provide two different proofs:

(1) In the following we provide an argument based on the Product-with-Quotient Map Proposition 17.14. First note that it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map $[0,1]/\{0,1\} \to S^1$ induced by $[t] \mapsto \exp(2\pi\,\mathrm{i}t)$ is a homeomorphism. We use this homeomorphism to replace S^1 by $[0,1]/\{0,1\}$. Next we consider the following diagram:

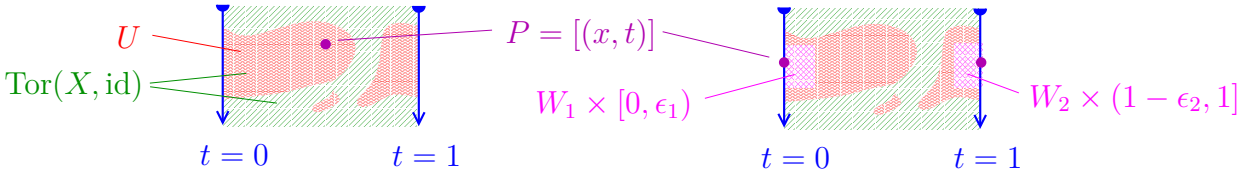
$$\begin{array}{ccc} & X \times [0,1] & \\ {\scriptstyle (x,t)\mapsto(x,[t])}\swarrow{\scriptstyle p} & & \searrow{\scriptstyle q}{\scriptstyle (x,t)\mapsto[(x,t)]} \\ X \times ([0,1]/\{0,1\}) & \xrightarrow[(x,[t])\mapsto[(x,t)]]{\Phi^{-1}} & (X \times [0,1])/(x,0) \sim (x,1). \end{array}$$

By the Product-with-Quotient Map Proposition 17.14, applied to $Y = [0,1]$ and the compact subset $B = \{0,1\}$ we know that p is a quotient map. By the Topological-Quotient Proposition 8.2 (1a) we know that q is continuous. Since the diagram commutes we obtain from the Quotient Map-Composition Lemma 17.7 that Φ^{-1} is continuous. We have thus shown that Φ is a homeomorphism.

(2) In the rather unlikely event that the reader skipped the Product-with-Quotient Map Proposition 17.14 we also give a more direct argument. By the Open-Closed–Injective Lemma 4.22 (2) it remains to show that Φ is an open map. So let U be an open subset of the mapping torus $\mathrm{Tor}(X, \mathrm{id}_X) = (X \times [0,1])/\sim$. Let $P \in U$ be a point. We can pick a representative $P = [(x,t)]$ with $t \in [0,1)$. We need to show that $Q := \Phi([(x,t)])$ admits an open neighborhood Y that is contained in $V := \Phi(U)$. In the following we denote by $p\colon X \times [0,1] \to \mathrm{Tor}(X, \mathrm{id}_X)$ the projection.

If $t \in (0,1)$, then by the definition of the topology on $\mathrm{Tor}(X, \mathrm{id}_X)$ there exists an open subset $W \subseteq X$ and an open interval $(a,b) \subseteq (0,1)$ with $P \in W \times (a,b) \subseteq U$. We set $Y := \Phi(W \times (a,b)) = W \times \{\exp(2\pi\,\mathrm{i}t) \mid t \in (a,b)\}$. Note that $p^{-1}(Y) = Y$ is an open subset of $X \times [0,1]$. Thus Y has the desired property

Now suppose that $t = 0$. In $X \times [0,1]$ we pick an open neighborhood $W_1 \times [0, \epsilon_1)$ of the point $(x,0)$ that is contained in $p^{-1}(U)$. Similarly we pick an open neighborhood $W_2 \times (1-\epsilon_2, 1]$ of the point $(x,1)$ that is also contained in $p^{-1}(U)$.



It is now straightforward to see that $Y := (W_1 \cap W_2) \times \{\exp(2\pi\,\mathrm{i}s) \mid s \in (-\epsilon_2, \epsilon_1)\}$ has the desired property. ∎

[251] The author wishes to thank UB for pointing out that this lemma is not "obviously true".

Next we want to explore the question, under what circumstances self-maps lead to homeomorphic mapping tori. This brings us to the next definition:

Definition 21.24. Let X and Y be topological spaces. We say that two homeomorphisms $f, g\colon X \to Y$ are homeotopic, if there exists a homeotopy from f to g in the sense of Definition 18.5. Recall that this means that there exists a continuous map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & H(x,t) \end{array}$$

such that the following hold:

(1) We have $H_0 = f$ and $H_1 = g$.
(2) For each $t \in [0,1]$ the map $H_t\colon X \to Y$ is a homeomorphism.

Example. We consider $X = [0,2]$ and the two self-homeomorphisms given by $f(x) = x$ and by $g(x) = \frac{1}{2} \cdot x^2$. Then[252]

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & f(x) \cdot (1-t) + g(x) \cdot t \end{array}$$

is a homeotopy from f to g. □

The above Definition 18.5 allows us to formulate the following lemma:

Lemma 21.25. (Mapping Tori–Homeomorphic Lemma) Let X be a topological space. If $f, g\colon X \to X$ are two homeomorphisms that are homeotopic, then the mapping tori $\mathrm{Tor}(X,f)$ and $\mathrm{Tor}(X,g)$ are homeomorphic.

Example. We consider $X = S^1$ and we consider the self-homeomorphism that is given by $g(z) = -z$. Note that the map $H(z,t) = \exp(\pi\,\mathrm{i}t) \cdot z$ defines a homeotopy from $f = \mathrm{id}$ to g. It follows from the Mapping Tori–Homeomorphic Lemma 21.25 that the mapping torus $\mathrm{Tor}(S^1, g(z) = -z)$ is homeomorphic to the (mapping) torus $\mathrm{Tor}(S^1, \mathrm{id}) \cong S^1 \times S^1$. □

Proof. Let $H\colon M \times [0,1] \to M$ be a homeotopy between $H_0 = f$ and $H_1 = g$. It follows from the Quotient-to-Quotient Lemma 8.6 that the following map is continuous:

$$\begin{array}{rcl} \mathrm{Tor}(M,f) & \to & \mathrm{Tor}(M,g) \\ [(x,t)] & \mapsto & [(H(f^{-1}(x),t),t]. \end{array}$$

An inverse is given by using the "opposite" homeotopy $H'(x,t) := H(x,1-t)$. ■

21.4. **Appendix: The mapping cone of a suspension.** In this appendix we show that under favorable circumstances the operation of taking a mapping cone (or mapping cylinder) commutes with the operation of taking the suspension of a map. This technical proposition will play a useful role much much later in Section 236.3, when we will study the relationship between Steenrod operations and suspensions. The proposition is also a good way to test whether we really understand the various topologies involved in the definition of mapping cones, mapping cylinders and suspensions.

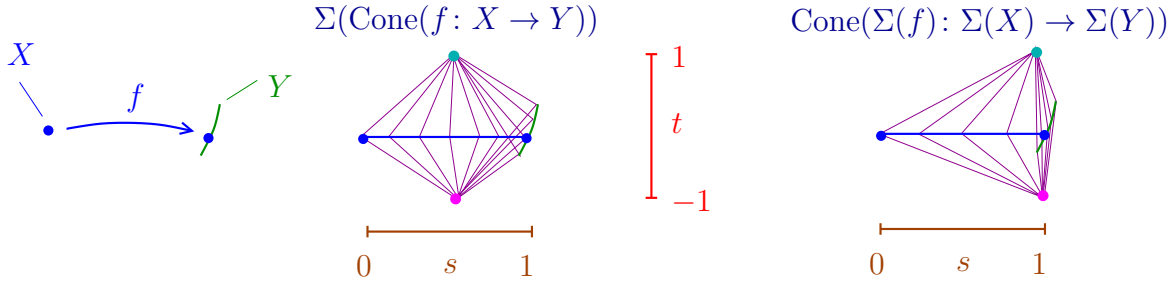
Proposition 21.26. (Mapping Cone-Suspension Proposition) If $f\colon X \to Y$ is a continuous map between topological spaces, then the

[252] Let $t \in [0,1]$. Why is the map $[0,2] \to [0,2]$ given by $x \mapsto H(x,t)$ a homeomorphism?

map[253]

$$\begin{array}{rcl}\Phi\colon \Sigma(\mathrm{Cone}(f\colon X\to Y)) & \to & \mathrm{Cone}(\Sigma(f)\colon \Sigma(X)\to\Sigma(Y)) \\ P & \mapsto & \begin{cases}[([(x,t)],s)], & \text{if } P=[([(x,s)],t)] \text{ with } x\in X,\ t\in[-1,1] \\ & \quad \text{and with } s\in[0,1], \\ [([y],t)], & \text{if } P=[([y],t)] \text{ with } y\in Y \text{ and } t\in[-1,1].\end{cases}\end{array}$$

is a homeomorphism.



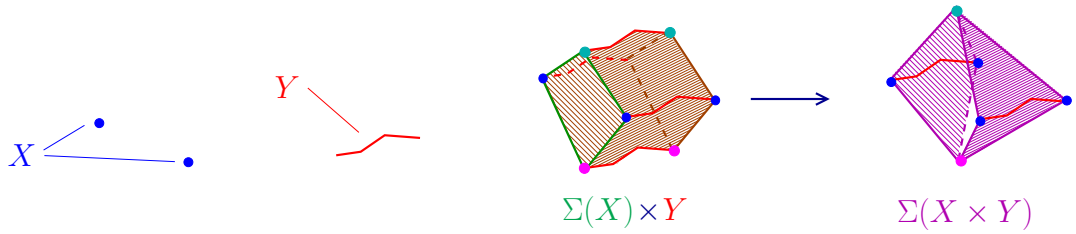
In the proof of the Mapping Cone-Suspension Proposition 21.26 we will need the following lemma which is quite similar to the Product–Quotient Map Proposition 17.10.

Lemma 21.27. (Suspension-Product Lemma) Let X be a topological space. If Y is a regionally compact space (e.g. if $Y=[0,1]$), then the map

$$\begin{array}{rcl}\varphi\colon \Sigma(X)\times Y & \to & \Sigma(X\times Y) \\ ([(x,t)],y) & \mapsto & [((x,y),t)]\end{array}$$

is continuous.

In the figure below we illustrate the Suspension-Product Lemma 21.27. This cute picture is not meant to hide the grim reality that for arbitrary topological spaces X it actually requires a non-trivial input to prove the Suspension-Product Lemma 21.27.



Proof of the Suspension-Product Lemma 21.27. In the following discussion we denote by $p\colon (X\times Y)\times[-1,1]\to\Sigma(X\times Y)$ and $q\colon X\times[-1,1]\to\Sigma(X)$ the two natural projections. We have the following commutative diagram:

$$\begin{array}{ccc}(X\times[-1,1])\times Y & \xrightarrow[\cong]{(x,t,y)\mapsto(x,y,t)} & (X\times Y)\times[-1,1] \\ {\scriptstyle q\times\mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle p} \\ \Sigma(X)\times Y & \xrightarrow{\qquad\varphi\qquad} & \Sigma(X\times Y).\end{array}$$

[253]It takes a major effort to disentangle the notation. We write $[-]$ to indicate the equivalence class corresponding to a suspension and we write $[-]$ to indicate the equivalence class corresponding to the equivalence relation used to define mapping cones.

By the Product Topology-Basics Lemma 7.6 (2) the top horizontal map is a homeomorphism. By the Projection–Quotient Map Lemma 17.4 the maps p and q are quotient maps. Since Y is regionally compact we obtain from the Whitehead Quotient Theorem 17.6 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from the Quotient Map-Composition Lemma 17.7 that φ is also a quotient map. By the Quotient Map Lemma 17.5 (2) this implies that φ is indeed continuous. ■

Proof of the Mapping Cone-Suspension Proposition 21.26. Let $f\colon X \to Y$ be a continuous map between topological spaces. One can easily verify that the given map Φ is a bijection. It remains to show that Φ and Φ^{-1} are continuous. We will do so in the following two claims. The domain and the target of Φ are both a scary mix of product and quotient topologies. Thus we need to tread super carefully.

Claim 1. The map Φ is continuous.

Proof. To prove the claim we introduce the following notation:

(a) We consider the map $\varphi\colon X \times \{1\} \to Y$ that is given by $\varphi(x,1) = f(x)$.
(b) We denote by $p\colon \mathrm{Cone}(f\colon X \to Y) \times [-1,1] \to \Sigma(\mathrm{Cone}(f\colon X \to Y))$ the projection.

We consider the following diagram:

$$\begin{array}{ccc}
(X\times[0,1])\times[-1,1] & & Y\times[-1,1] \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
((X\times[0,1])\times[-1,1]) \cup_{\varphi\times\mathrm{id}_{[-1,1]}} (Y\times[-1,1]) & & \\
\downarrow{\scriptstyle [(P,t)]\mapsto([P],t)} & \searrow^{=:\Psi} & \\
((X\times[0,1]) \cup_{\varphi} Y)\times[-1,1] & & \\
\downarrow{\scriptstyle \mathrm{id}} & & \\
\mathrm{Cone}(f\colon X \to Y)\times[-1,1] & & \\
\downarrow{\scriptstyle q\ ([P],t)\mapsto[(P,t)]} & & \\
\Sigma(\mathrm{Cone}(f\colon X \to Y)) & \xrightarrow{\quad\Phi\quad} & \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)).
\end{array}$$

We make the following observations:

(1) By the Gluing-Product-Homeomorphism Lemma 17.9 we know that the vertical map between the second and the third row is a homeomorphism.
(2) It follows from (1) together with the Topological-Quotient Proposition 8.2 (1b), applied to the projection q, that Φ is continuous if Ψ is continuous.
(3) It follows from the Topological Pushout Proposition 8.12 (1b) that Ψ is continuous if $\Psi \circ i$ and $\Psi \circ j$ are continuous.
(4) We have the following commutative diagram:

$$\begin{array}{ccc}
Y\times[-1,1] & \xrightarrow{\quad\Psi\circ j\quad} & \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)). \\
& {\scriptstyle (P,t)\mapsto[(P,t)]}\searrow \quad \nearrow{\scriptstyle Q\mapsto[Q]} & \\
& \Sigma(Y) &
\end{array}$$

The diagonal maps are continuous by the Topological-Quotient Proposition 8.2 (1a) and the Topological Pushout Proposition 8.12 (1a). Thus we see that $\Psi \circ j$ is continuous. Essentially the same argument shows that $\Psi \circ i$ is continuous.

The combination of (2), (3) and (4) shows that Φ is indeed continuous. ⊞

Claim 2. The map

$$\begin{array}{rcl} \overbrace{\mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y))}^{=((\Sigma(X)\times[0,1])\sqcup\Sigma(Y))/\sim} & \xrightarrow{\Phi^{-1}} & \Sigma(\mathrm{Cone}(f\colon X \to Y)) \\ P & \mapsto & \begin{cases} [([(x,s)],t)], & \text{if } P = [([(x,t)],s)] \text{ with } s \in [0,1] \text{ and} \\ & \text{with } [(x,t)] \in \Sigma(X), \text{ i.e. } x \in X,\ t \in [-1,1], \\ [([y],t)], & \text{if } P = [([y],t)] \text{ with } t \in Y,\ t \in [-1,1]. \end{cases} \end{array}$$

is continuous.

Proof. By the Topological Pushout Proposition 8.12 (1b) it suffices to show that the maps

$$\begin{array}{rcl} \Sigma(X) \times [0,1] & \to & \Sigma(\mathrm{Cone}(f\colon X \to Y)) \\ ([(x,t)],s) & \mapsto & [([(x,s)],t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Sigma(Y) & \to & \Sigma(\mathrm{Cone}(f\colon X \to Y)) \\ [(y,t)] & \mapsto & [([y],t)] \end{array}$$

are continuous. The second map is continuous by the Suspension Functor Lemma 20.23 (1). Furthermore it follows from the combination of the Suspension-Product Lemma 21.27 with the Suspension Functor Lemma 20.23 (1) that the first map is continuous. ■

21.5. Exercises for Chapter 21.

Exercise 21.1. Show that, up to homotopy, every continuous map is a closed embedding. More precisely prove the following: Given any continuous map $f\colon X \to Y$ between topological spaces there exists a topological space Z, a closed embeddding $i\colon X \to Z$ and a homotopy equivalence $g\colon Z \to Y$ such that $g \circ i = f$.

Exercise 21.2. Let $f\colon X \to Y$ be a continuous map between topological spaces. Show that if X and Y are Hausdorff, then the mapping cylinder $\mathrm{Cyl}(f)$ is also Hausdorff.
Remark. This exercise is the content of the Mapping Cylinder Topology-Properties Proposition 21.2 (3).

Exercise 21.3. Let $f\colon X \to Y$ be a continuous map between topological spaces. Show that if X and Y are paracompact, then the mapping cylinder $\mathrm{Cyl}(f)$ is also paracompact.
Remark. This exercise is the content of the Mapping Cylinder Topology-Properties Proposition 21.2 (1b).

Exercise 21.4. Let $f\colon X \to Y$ be a continuous map between topological spaces and let Z be a topological space.

(a) Show that the map

$$\begin{array}{rcl} \mathrm{Cyl}(f \times \mathrm{id}\colon X \times Z \to Y \times Z) & \to & \mathrm{Cyl}(f\colon X \to Y) \times Z \\ P & \mapsto & \begin{cases} ([(Q,t)],z), & \text{if } P = [((Q,z),t)] \text{ with } Q \in X, t \in [0,1], z \in Z, \\ ([Q],z), & \text{if } P = [(Q,z)] \text{ with } Q \in Y, z \in Z. \end{cases} \end{array}$$

is continuous.

(b) Suppose that Z is regionally compact. Show that the map Θ from (a) is a homeomorphism.
Hint. Use the Product-Quotient Mixing Lemma 17.8.

Exercise 21.5. Let X be a topological space and let A be a subset of X. We consider the natural injection

$$\begin{array}{rcl} \iota\colon A\times[0,1] & \to & \operatorname{Cyl}(i)=((A\times[0,1])\sqcup X)/\sim \\ (a,t) & \mapsto & [(a,t)]. \end{array}$$

(a) Show that if $A\subseteq X$ is closed, then $\iota\colon A\to X$ is a closed embedding.
(b) Show that if $A\subseteq X$ is not closed, then $\iota\colon A\to X$ is not necessarily an embedding.

Exercise 21.6. Let $f\colon X\to Y$ be a continuous map between two topological spaces. Show that if the topological spaces X and Y are both locally contractible, then so is the mapping cylinder $\operatorname{Cyl}(f\colon X\to Y)$.

Exercise 21.7. We consider the map $f\colon S^1\to S^1$ that is given by $z\mapsto z^2$. Show that the corresponding mapping cone $\operatorname{Cone}(f\colon S^1\to S^1)$ is homeomorphic to the real projective plane $\mathbb{R}\mathrm{P}^2$.
Hint. Make use of the fact that by the Projective Space-Avatar Lemma 9.16 we know that S^1 is homeomorphic to $\mathbb{R}\mathrm{P}^1$, which in turn is homeomorphic to $S^1/z\sim -z$.

Exercise 21.8. Let $n\in\mathbb{N}_0$. Show that there exists a continuous map $f\colon S^n\to\mathbb{R}\mathrm{P}^n$ such that the mapping cone $\operatorname{Cone}(f)$ is homeomorphic to $\mathbb{R}\mathrm{P}^{n+1}$.

Exercise 21.9. Let $f\colon X\to Y$ be a continuous map between topological spaces. Show that if $f\colon X\to Y$ is homotopic to a constant map $X\to Y$, then f extends to a continuous map $\operatorname{Cone}(f\colon X\to Y)\to Y$.

Exercise 21.10. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. We assume that X is Hausdorff. Show that the mapping torus $\operatorname{Tor}(X,f)$ is also Hausdorff.

Exercise 21.11. Let $\varphi\colon\overline{B}^2\to\overline{B}^2$ be the map that is given by $(x,y)\mapsto(x,-y)$. Is the corresponding mapping torus $\operatorname{Tor}(\overline{B}^2,\varphi)$ homeomorphic to the solid torus?

Exercise 21.12. We consider the map $f\colon S^1\to S^1$ that is given by $f(x,y)=(y,x)$. Is the mapping torus $\operatorname{Tor}(S^1,f)$ homeomorphic to the torus, to the Klein bottle or to neither?

Exercise 21.13. Let X be a topological space with finitely many path-components. Show that there exists a continuous map $f\colon X\to X$ such that the mapping torus $\operatorname{Tor}(f\colon X\to X)$ is path-connected.

Exercise 21.14. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. Furthermore let $\varphi\colon X\to X$ be another homeomorphism. Show that $\operatorname{Tor}(X,f)$ is homeomorphic to $\operatorname{Tor}(X,\varphi\circ f\circ\varphi^{-1})$.

Exercise 21.15. Let X be a non-empty path-connected topological space. Furthermore let $f\colon X\to X$ be a continuous map. Show that the natural projection

$$\begin{array}{rcl} p\colon \operatorname{Tor}(f\colon X\to X) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t), \end{array}$$

which we introduced in Definition 21.19, admits a right inverse. In other words, show that there exists a continuous map $g\colon S^1\to\operatorname{Tor}(f\colon X\to X)$ such that $p\circ g=\mathrm{id}_{S^1}$.

Exercise 21.16. Let Y be a topological space and let $f\colon Y \to Y$ be a homeomorphism. We consider the action of $\langle t \rangle$ on $Y \times \mathbb{R}$ that is given by

$$t^n \cdot (y, s) \ := \ (f^n(y), s + n).$$

Show that the quotient $(Y \times \mathbb{R})/\langle t \rangle$ is naturally homeomorphic to the mapping torus $\mathrm{Tor}(Y, f) := (Y \times [0, 1])/(y, 0) \sim (f(y), 1)$.

Exercise 21.17. Let $f, g\colon X \to X$ be two self-maps of a topological space X. Show that if f, g are homotopic, then the mapping tori $\mathrm{Tor}(X, f)$ and $\mathrm{Tor}(X, g)$ are homotopy equivalent.

CHAPTER 22

Cofibrations

Loosely speaking a map $i\colon A \to X$ between topological spaces is called a cofibration if it has the homotopy extension property, i.e. if a homotopy on A extends to a homotopy on X. As we will see many "reasonable" inclusion maps, e.g. the inclusion of a "submanifold" into a "smooth manifold", or of a "subcomplex" of a "CW-complex" into the CW-complex are cofibrations. The main technical achievement of this chapter is the daunting proof of the Homotopy Pushout Theorem 22.24 which will come in handy on several occasions throughout these notes.

Nonetheless, for the most part the results in this chapter are not essential for the understanding of the subsequent chapters, thus not much harm is done by initially skipping this chapter. It is best to return to it, when some of the results are actually needed.

This chapter is organized as follows:

- In Section 22.1 we will quickly recall the notion of a mapping cylinder and of a mapping cone.
- In Section 22.2 we will introduce the notion of a cofibrations. Loosely speaking an inclusion $A \to X$ is a cofibration if any homotopy $A \times [0,1] \to Y$ extends to a homotopy $X \times [0,1] \to Y$.
- In Section 22.3 we will study basic properties of cofibrations.
- In Section 22.4 we will give "real-life" examples of cofibrations.
- In Section 22.5 we will study the interplay between cofibrations and products.
- In Section 22.6 we will discuss the relationship between pushouts and cofibrations and we will state the Homotopy Pushout Theorem 22.24.
- In Section 22.7 we will provide the daunting proof of the Homotopy Pushout Theorem 22.24
- In Section 22.8 we will provide exercises for Chapter 22.

22.1. **The mapping cylinder.** Throughout this chapter we will on numerous occasions make use of the notion of a mapping cylinder and of a mapping cone. Thus we start out by recalling the definition of a mapping cylinder and a mapping cone that we had initially given in Definitions 21.1 and 21.7.

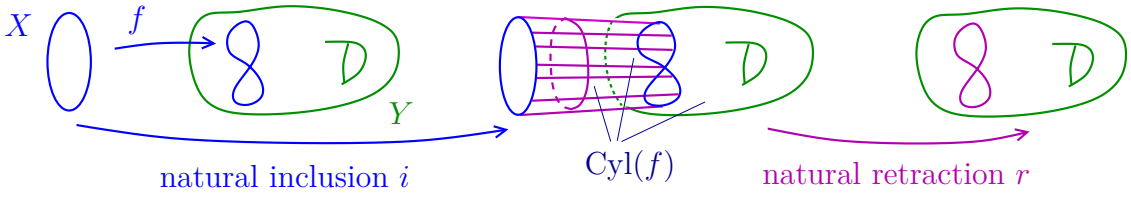
Definition 22.1. Let $f\colon X \to Y$ be a continuous map between topological spaces. We define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f\colon X \to Y) \;:=\; ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We refer to the maps

$$\begin{array}{ccc} X \xrightarrow{i} \mathrm{Cyl}(f) \\ x \mapsto (x,0) \end{array} \quad \text{respectively} \quad \begin{array}{ccc} \mathrm{Cyl}(f) \xrightarrow{r} Y \\ P \mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases} \end{array}$$

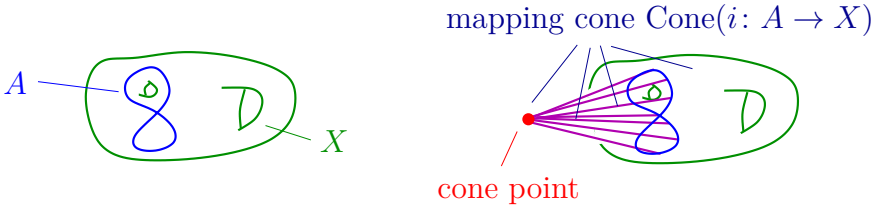
as the natural inclusion respectively the natural retraction.[254]



Definition 22.2. Given a continuous map $f\colon A \to X$ between topological spaces the corresponding mapping cone is defined as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \sqcup X)/\sim \quad \text{where } [(a,1)] \sim f(a) \text{ for all } a \in A.$$

We refer to the point $[A \times \{0\}]$ as the tip of the cone of the mapping cone.



22.2. **Cofibrations: Definition and basic examples.** We immediately introduce the main player of this section.

Definition 22.3.

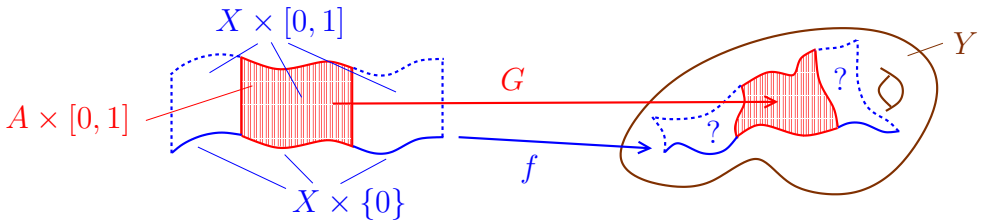
(1) A continuous map $i\colon A \to X$ between topological spaces is called a cofibration if it has the homotopy extension property, i.e. if given any topological space Y and any two continuous maps $f\colon X \times \{0\} \to Y$ and $G\colon A \times [0,1] \to Y$ with $f(i(a),0) = G(a,0)$ for all $a \in A$ there exists a continuous map $F\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times \{0\} & \hookrightarrow & A \times [0,1] \\ \downarrow{\scriptstyle i} & \overset{G}{\swarrow} & \downarrow{\scriptstyle i\times \mathrm{id}} \\ X \times \{0\} & \overset{f}{\nearrow} \; Y \; \overset{F}{\nwarrow} & X \times [0,1]. \end{array}$$

We illustrate the definition in the figure below.

[254]Note that by the Mapping Cylinder–Homotopy Lemma 21.5 we know that there exists in fact a natural deformation retraction $R\colon \mathrm{Cyl}(f) \times [0,1] \to \mathrm{Cyl}(f)$ from $\mathrm{Cyl}(f)$ to Y, viewed as a subspace of $\mathrm{Cyl}(f)$, such that $R_1 = r$.

(2) If $i\colon A \to X$ is a cofibration such that $i(A)$ is a closed subset of X, then we refer to i as a closed cofibration.

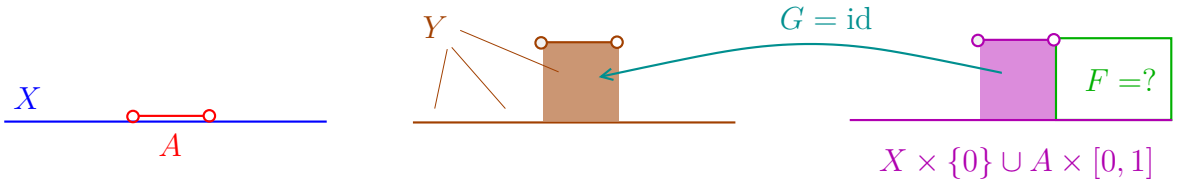


Remark.

(1) As we will see in the Cofibration-Closed Injective Lemma 22.9, in most cases of interest the map $i\colon A \to X$ is for all intents and purposes an inclusion. For such situations the definition of the homotopy extension property is illustrated in the figure above.

(2) In some books what we call a cofibration is called a "Hurewicz cofibration". Also, in some books, see e.g. [**Pic1992**], a cofibration is defined as what we call a closed cofibration. □

Non-Example. Let us start out with a non-example. Namely let us consider $X = \mathbb{R}$ and the subspace $A = (0,1)$. We write $Y = \mathbb{R} \times \{0\} \cup (0,1) \times [0,1]$ and let $f\colon X \to Y$ be the obvious inclusion map and let $G\colon (0,1) \times [0,1] \to Y$ be the obvious inclusion map.



It is an amusing exercise to show that G cannot be extended to a homotopy on all of $\mathbb{R}$.[255] It is an equally amusing exercise to show that the homotopy can be extended if throughout we replace the open interval $A = (0,1)$ by the closed interval $A = [0,1]$. □

We continue with the following fairly elementary lemma.

Lemma 22.4. (Cofibration Composition Lemma) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two (closed) cofibrations, then the composition $g \circ f\colon X \to Z$ is also a (closed) cofibration.

Proof. The lemma will be proved in Exercise 22.2. ■

Before we give examples of cofibrations it is convenient to add a characterization of cofibrations to our repertoire.

Lemma 22.5. (Cofibration Characterization Lemma) If $i\colon A \to X$ is a continuous map between two topological spaces, then the following two statements are equivalent:

(1) The map $i\colon A \to X$ is a cofibration.

[255] Here is a hint: If there was an extension of G to a homotopy $F\colon X \times [0,1] \to Y$, what would/should $F(1,1)$ be?

(2) The map[256]

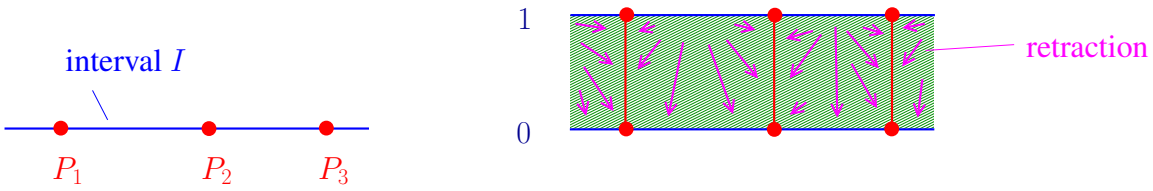
$$s\colon \begin{array}{rcl} \mathrm{Cyl}(i) & \to & X \times [0,1] \\ P & \mapsto & \begin{cases} (i(a), 1-t), & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1], \\ (P, 0), & \text{if } P \in X \end{cases} \end{array}$$

admits a continuous left inverse, i.e. there exists a continuous map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ such that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

If A is a subset of X and if $i\colon A \to X$ is the inclusion map, then (1) and (2) are equivalent to

(3) The set $(X \times \{0\}) \cup (A \times [0,1])$ is a retract of $X \times [0,1]$.

Example. Let I be an interval in $\mathbb{R}$ and let $P_1, \dots, P_k$ be finitely many distinct points in I. In the figure below we see a retraction from $I \times [0,1]$ to $(I \times \{0\}) \cup (\{P_1, \dots, P_k\} \times [0,1])$:

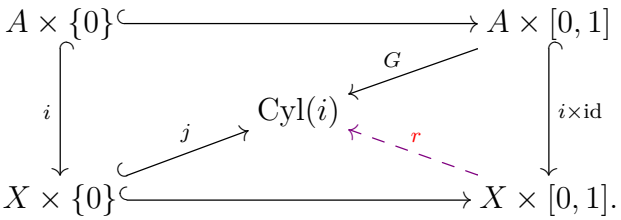


Together with the Cofibration Characterization Lemma 22.5 this shows that the inclusion $\{P_1, \dots, P_k\} \to I$ is a cofibration. □

Proof. First we prove the "(1)⇒(2)"-direction. This means that we now suppose that the map $i\colon A \to X$ is a cofibration. We consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x,0) = x$ and we consider the map

$$G\colon \begin{array}{rcl} A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a,s) & \mapsto & [(a, 1-s)]. \end{array}$$

We obtain the following diagram:



One easily verifies that the black diagram commutes. Since $i\colon A \to X$ is a cofibration we obtain a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. It is straightforward to verify that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

The proof of the converse, i.e. the proof of the "(2)⇒(1)"-direction is also quite straightforward. We will provide the proof in Exercise 22.4.

[256] It follows from the Topological Pushout Proposition 8.12 (1b) that this map is continuous.

Now we assume that A is a subset of X and that $i\colon A \to X$ is the inclusion map. We need to provide the proof for the equivalence of (2) and (3). This turns out to be quite subtle.

(a) First we assume that A is a closed subset of X.[257] Note that the image of $\mathrm{Cyl}(i)$ in $X \times [0,1]$ under the map s is precisely $(X \times \{0\}) \cup (A \times [0,1])$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Cyl}(i) & \xrightarrow{\quad s \quad} & (X \times \{0\}) \cup (A \times [0,1]) \\ & {\scriptstyle s}\searrow \quad \swarrow & \\ & X \times [0,1]. & \end{array}$$

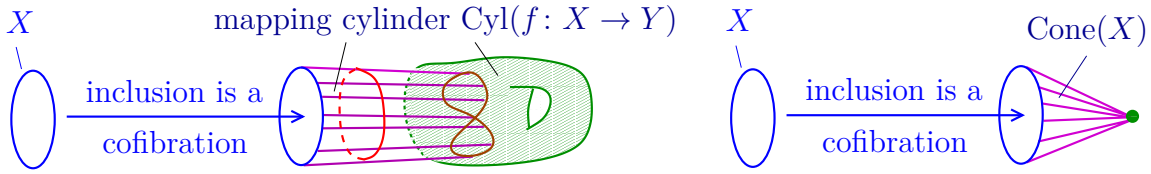
To show that (2) and (3) are equivalent it suffices to show that the top horizontal map is a homeomorphism. But this is a consequence of the fact that $A \subseteq X$ is a closed subset together with the Mapping Cylinder-of-Embedding Lemma 21.6.

(b) The logic of (a) breaks down if A is not a closed subset. More precisely, in the Mapping Cylinder-of-Embedding Lemma 21.6 (3) we saw that in general the top horizontal map is *not* a homeomorphism. Rather surprisingly it is nonetheless true that (2) and (3) are equivalent. This delicate argument is given in [**Strø1968**, Lemma 4] or alternatively in [**Hat2002**, Proposition A.18]. ■

The next proposition gives another rich source of examples of cofibrations.

Proposition 22.6. (Cone-Mapping Cylinder Inclusion Cofibration Proposition)

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then the natural inclusion $i\colon X \to \mathrm{Cyl}(f)$ is a closed cofibration.
(2) Let X be a topological space. The natural inclusion $X \to \mathrm{Cone}(X)$ is a closed cofibration.



The proof of the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 relies on the following elementary lemma:

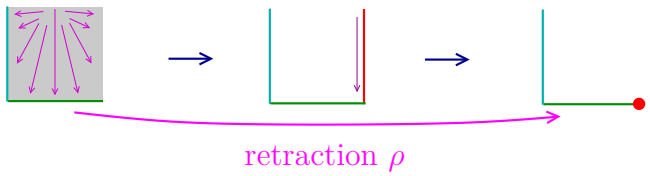
Lemma 22.7. (Square-Sides Retraction Lemma) There exists a retraction

$$\rho\colon [0,1] \times [0,1] \;\to\; (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$$

that has the property that $\rho(\{1\} \times [0,1]) = \{(1,0)\}$.

Proof of the Square-Sides Retraction Lemma 22.7. In the figure below we show how such a retraction ρ can be constructed:
We first retract to the $\sqcup$-shaped set $(\{0,1\} \times [0,1]) \cup ([0,1] \times \{0\})$ and afterwards we retract to $(\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ by collapsing $\{1\} \times [0,1]$ to the point $(0,1)$. ■

[257] This is also the only case we will ever need in practice.

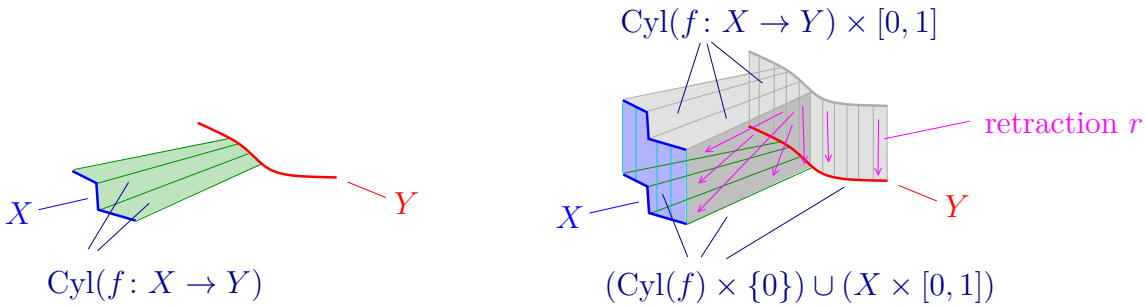


Proof of Proposition 22.6. First note that if X is a topological space and if $f\colon X \to \{*\}$ is the constant map, then, as we pointed out on page 562, the mapping cylinder $\mathrm{Cyl}(f)$ is basically the cone of X. Thus we see that Statement (2) is a special case of Statement (1).

It remains to prove Statement (1). Let $f\colon X \to Y$ be a continuous map between topological spaces. By the Mapping Cylinder–Embedding Lemma 21.4 we know that the image of X in $\mathrm{Cyl}(f)$ is a closed subset. Thus it remains to show that $i\colon X \to \mathrm{Cyl}(f)$ is a cofibration. By the Mapping Cylinder–Embedding Lemma 21.4 the map i is an embedding, thus we can identify X with its image in $\mathrm{Cyl}(f)$.

By the Cofibration Characterization Lemma 22.5 (3)⇒(1) it suffices to prove the following claim:

Claim. There is a retraction r from $\mathrm{Cyl}(f)\times[0,1]$ to its subset $(\mathrm{Cyl}(f)\times\{0\})\cup(X\times[0,1])$.



Proof. To construct the desired retraction r we recall that in the Square-Sides Retraction Lemma 22.7 we saw that there exists a retraction

$$s\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1]) \cup ([0,1]\times\{0\})$$

that has the property that $s(\{1\}\times[0,1]) = \{(1,0)\}$. Now the desired retraction is given by the map[258]

$$\begin{array}{rcl} r\colon\ \mathrm{Cyl}(f)\times[0,1] & \to & (\mathrm{Cyl}(f)\times\{0\}) \cup (X\times[0,1]) \\ P & \mapsto & \begin{cases} (x, s(t,u)), & \text{if } P=[(x,t,u)] \text{ with } x\in X \text{ and } t,u\in[0,1], \\ [(y,0)], & \text{if } P=[(y,u)] \text{ with } y\in Y \text{ and } u\in[0,1]. \end{cases} \end{array}$$

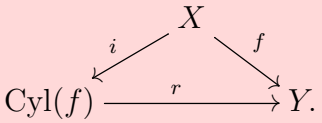
■

We conclude this section with a proposition which says that any continuous map $f\colon X \to Y$ between topological spaces can be "replaced" by a cofibration $X \to \mathrm{Cyl}(f)$.

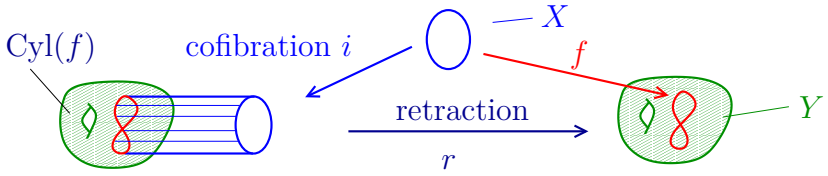
Proposition 22.8. (Replace-by-Cofibration Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. We denote by $i\colon X \to \mathrm{Cyl}(f)$ the natural inclusion. The natural retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence and the following

[258] This map is continuous by the Pasting Proposition 4.7 (2).

diagram commutes:



By the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 (1) we know that the left diagonal map $i\colon X \to \mathrm{Cyl}(f)$ is a cofibration.



Proof. We showed in the Mapping Cylinder–Homotopy Lemma 21.5 (2a) that the natural retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. It is clear that the given diagram commutes. Thus we are done by the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 (1). That was quick. ■

22.3. **Properties of cofibrations.** The following lemma summarizes a few key properties of cofibrations.

Lemma 22.9. (Cofibration-Closed Injective Lemma) If $i\colon A \to X$ is a cofibration, then the following three statements hold:

(1) the map $i\colon A \to X$ is injective,
(2) the map $i\colon A \to X$ is an embedding,
(3) if X is Hausdorff, then $i(A)$ is a closed subset of X, i.e. i is a closed cofibration.

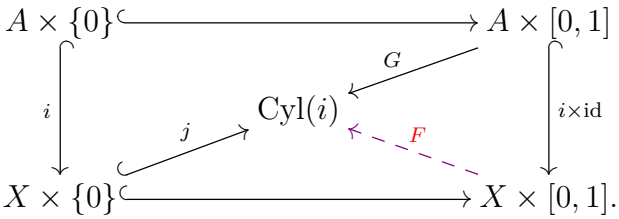
Remark. In Exercise 22.8 we will see that in the Cofibration-Closed Injective Lemma 22.9 (3) we cannot drop the hypothesis that X is Hausdorff. □

Proof. Let $i\colon A \to X$ be a cofibration.

(1) Similar to the proof of the Cofibration Characterization Lemma 22.5 we consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x,0) = x$ and we consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a,s) & \mapsto & [(a,1-s)]. \end{array}$$

We obtain the following diagram:



One easily verifies that the black diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. Note that by design for any $t \in (0,1]$ the map $G\colon A \times \{t\} \to \mathrm{Cyl}(i)$ is injective. Since the diagram commutes this implies that $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is also injective, i.e. $i\colon A \to X$ is injective.

(2) We continue with the argument and the notation in (1). Let $t \in (0,1]$. The map $G|_{A\times\{t\}}\colon A \times \{t\} \to A \times \{1-t\}$ is evidently a homeomorphism. Thus we see that

$$(G|_{A\times\{t\}})^{-1} \circ F|_{X\times\{t\}}\colon i(A) \times \{t\} \ \to \ A \times \{t\}$$

is an inverse to the map $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$. This implies that the map $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is an embedding, but this also implies that the map $i\colon A \to X$ is an embedding.

(3) Note that

$$\begin{array}{l} \text{for } x \in A \text{ we have } F(x,1) = G(x,1) = [(x,0)] \text{ by construction of } F, \text{ furthermore} \\ \text{for } x \notin A \text{ the point } [x] \text{ does not lie in } \mathrm{Cyl}(i), \text{ thus it does not lie in the image of } F \end{array}$$

$$\begin{aligned} A \ &\overset{\downarrow}{=} \ \{x \in X \mid F(x,0) = [x]\} \\ &= \left\{ \begin{array}{c} \text{the points in } X \text{ on} \\ \text{which the maps} \end{array} \begin{array}{ccc} X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & F(x,0) \end{array} \text{ and } \begin{array}{ccc} X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & [x] \end{array} \text{ agree} \right\}. \end{aligned}$$

It follows from this description of A, our hypothesis that X is Hausdorff (which implies by (2) and the Mapping Cylinder Topology-Properties Proposition 21.2 (3) that $\mathrm{Cyl}(i)$ is also Hausdorff) and the Maps-agree-on-Closure Lemma 7.21 that A is indeed a closed subset of X. ■

The following lemma shows that inclusion maps that are cofibrations are particularly nicely behaved.

Lemma 22.10. (Cofibration-Quotient Lemma) Let X be a topological space and let $A \subseteq X$ be a non-empty subset. We denote by $i\colon A \to X$ the inclusion map. If i is a closed cofibration, then the following three statements hold:

(1) The map[259]

$$\begin{array}{rccl} \varphi\colon \mathrm{Cone}(i\colon A \to X) & \to & X/A & \\ P & \mapsto & \left\{ \begin{array}{ll} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{array} \right. \end{array}$$

is a homotopy equivalence.

(2) If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.

(3) If $a_0 \in A$ is a point such that $\{a_0\}$ is a deformation retract of A, then the projection $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.

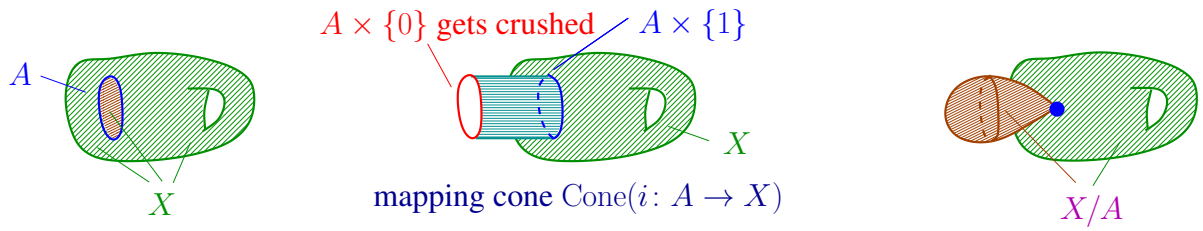
Remark. The statement of the Cofibration-Quotient Lemma 22.10 (2) is related in spirit to the statement of the Collapse-Cylinder Proposition 140.7. □

Proof. Let X be a topological space and let $A \subseteq X$ be a closed subset such that the inclusion $i\colon A \to X$ is a cofibration. We consider the corresponding mapping cone

$$\mathrm{Cone}(i\colon A \to X) \ := \ ((A \times [0,1])/(A \times \{0\}) \sqcup X)/(a,1) \sim f(a)$$

from Definition 21.7 and Definition 22.2.

[259] It follows from the ever popular the Pasting Proposition 4.7 (2) and the fact that A is a closed subset of X that the map φ is continuous.



We now turn to the actual proofs:

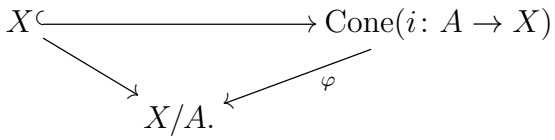
(1) Similar to the proof of the Cofibration-Closed Injective Lemma 22.9 (1) we consider the following two maps:
  (a) Let $j\colon X \times \{0\} \to \mathrm{Cone}(i)$ be the embedding which is given by $j(x,0) = x$.
  (b) We consider the map
$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cone}(i) \\ (a,s) & \mapsto & [(a, 1-s)]. \end{array}$$
  Note that $G_1\colon A \to \mathrm{Cone}(i)$ sends all points in A to the tip of the cone.
  We obtain the following diagram:
$$\begin{tikzcd} A \times \{0\} \arrow[rr, hook] \arrow[dd, hook, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[dd, hook, "i \times \mathrm{id}"] \\ & \mathrm{Cone}(i) & \\ X \times \{0\} \arrow[rr, hook] \arrow[ur, hook, "j"] & & X \times [0,1]. \arrow[ul, dashed, "F"'] \end{tikzcd}$$
  One easily verifies that the diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cone}(i)$ which makes the diagram commute. Note that the map $F_1\colon X \to \mathrm{Cone}(i)$ sends all points in A to the tip of the cone. Thus F_1 descends to a map $\psi\colon X/A \to \mathrm{Cone}(i)$. It remains to prove the following claim:

Claim. The maps $\varphi\colon \mathrm{Cone}(i) \to X/A$ and $\psi\colon X/A \to \mathrm{Cone}(i)$ are homotopy inverses of one another.

Proof. Note that $F(A \times [0,1]) = G(A \times [0,1]) \subseteq A$. Therefore it follows from the Homotopy Quotient-by-Subspace Lemma 18.12 that the map
$$\begin{tikzcd} H\colon (X/A) \times [0,1] \arrow[r, "{([x],t) \mapsto [(x,t)]}"] & (X \times [0,1])/(A \times [0,1]) \arrow[r, "{[(x,t)] \mapsto [F(x,t)]}"] & \mathrm{Cone}(i)/\mathrm{Cone}(A) \arrow[d, "{[x] \mapsto [\varphi(x)]}"] \\ & & X/A \end{tikzcd}$$
is continuous. It follows immediately from the definitions that H is a homotopy between $\mathrm{id}_{X/A}$ and $\varphi \circ \psi$.

Conversely we consider the map
$$\begin{array}{rcl} H\colon \mathrm{Cone}(i) \times [0,1] & \to & \mathrm{Cone}(i) \\ (P,s) & \mapsto & \begin{cases} F(P,s), & \text{if } P \in X, \\ [a, (1-s)\cdot t], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1]. \end{cases} \end{array}$$
One easily verifies that this is a homotopy from $\mathrm{id}_{\mathrm{Cone}(i)}$ to $\psi \circ \varphi$. ⊞

(2) We consider the following diagram:



It follows immediately from the definition of φ that this diagram commutes. Now suppose that A is contractible. Since A is hypothesis a closed subset we obtain from the Mapping Cone-of-Contractible Proposition 21.18 (1) that there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X. By Statement (1) we know that φ is a homotopy equivalence. Thus it follows from the Homotopy-Equivalence Basics Lemma 19.14 (1) that the map $X \to X/A$ is a homotopy equivalence.

(3) Now assume that $a_0 \in A$ is a point which has the property that $\{a_0\}$ is a deformation retract of A. By the Mapping Cone-of-Contractible Proposition 21.18 (2) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X such that for every $t \in [0,1]$ we have $G_1([(a_0,t)]) = a_0$. Next note that the homotopy equivalence H from (1) has the property that $H(a_0,t) = [(a_0, 1-t)]$. If one combines these observations with the argument in (2) it is not difficult to see that the projection is in fact a homotopy equivalence of the pointed topological spaces (X, a_0) and $(X/A, [a_0])$. ■

22.4. **"Real-life" Examples of cofibrations.** In future chapters we will discuss many interesting types of topological spaces, e.g. smooth and topological manifolds, CW-complexes and simplicial complexes. To make the search for results less confusing we now list several settings that give rise to cofibrations, even though we have not introduced the concepts yet.

In Definition 23.2 we will introduce the notion of a topological manifold M and its boundary ∂M and in Definition 25.5 we will introduce the notion of a smooth manifold M and its boundary ∂M. At this state we will not give the definitions, but in the figure below we show a cartoon version of a 2-dimensional topological manifold and of a 2-dimensional smooth manifold which hopefully gives the reader an idea of the concept.



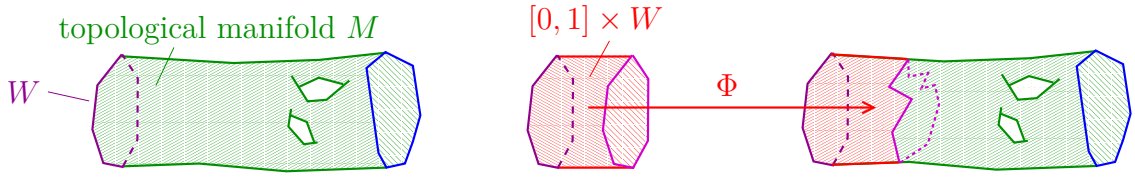
Proposition 22.11. (Manifold Boundary-Cofibration Proposition) Let M be a topological manifold. For any union W of components of ∂M the inclusion $i\colon W \to M$ is a closed cofibration.

Proof. Let M be a topological manifold and let W be a union of components of ∂M. It follows from the Topological Collar Neighborhood Theorem 35.14[260] that there exists a continuous map $\Phi\colon [0,1] \times W \to M$ with the following four properties:

(1) The map Φ is an embedding.

[260]For smooth manifolds we could also use the Smooth Collar Neighborhood Theorem 35.6.

(2) The image of $[0,1) \times W$ is an open subset of M.
(3) For every $P \in W$ we have $\Phi(0,P) = P$.
(4) The image of $[0,1] \times W$ is a closed subset of M.



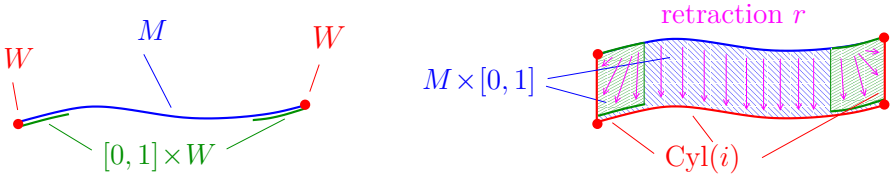
Recall that in the Square-Sides Retraction Lemma 22.7 we saw that there exists a retraction

$$\varphi = (\varphi_1, \varphi_2)\colon [0,1] \times [0,1] \;\to\; (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$$

that has the property that $\varphi(\{1\} \times [0,1]) = \{(1,0)\}$. We consider the map[261]

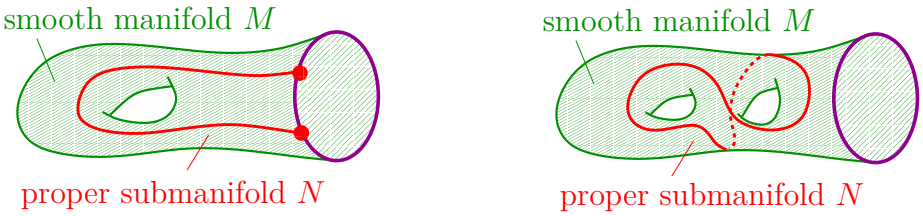
$$r\colon M \times [0,1] \;\to\; \mathrm{Cyl}(i) = ((W \times [0,1]) \sqcup M)/\!\sim \text{ where } (x,1) \sim x \text{ for } x \in W$$

$$(P,t) \;\mapsto\; \begin{cases} (\Phi(\underbrace{\varphi_1(s,t),Q}_{\in \Phi([0,1]\times W)}), \varphi_2(s,t)), & \text{if } P = \Phi(s,Q) \text{ with } s \in [0,1] \text{ and } Q \in W, \\ P, & \text{if } P \in M \setminus \Phi([0,1) \times W). \end{cases}$$



One can easily verify that r is a left inverse to the map $s\colon \mathrm{Cyl}(i) \to M \times [0,1]$ that we considered in the Cofibration Characterization Lemma 22.5 (2). Thus it follows from the Cofibration Characterization Lemma 22.5 that $i\colon W \to M$ is a cofibration. Finally it follows from the Topological Manifold–Boundary Proposition 23.15 (2a) that the union W of boundary components of the topological manifold M is a closed subset.[262] It follows that $i\colon W \to M$ is in fact a closed cofibration. ■

Next we turn to "proper submanifolds" of smooth manifolds. We will introduce this concept in Definition 25.22. In the figure below we again provide a cartoon to give the reader an idea what the concept of a "proper submanifold" is about.

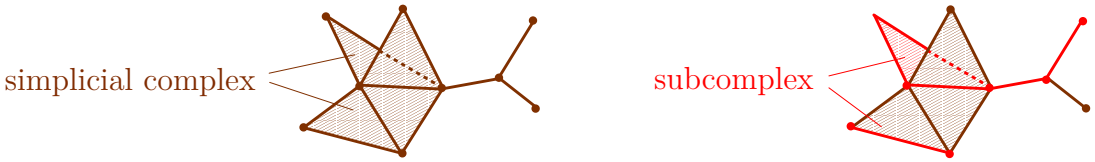


[261] Recall that by (2) and (4) the sets $\Phi([0,1] \times W)$ and $M \setminus \Phi([0,1) \times W)$ are actually closed subsets of M. It now follows from this observation together with the Basics-of-Collars Lemma 35.7 and the Pasting Proposition 4.7 (2) that the map r is indeed continuous.
[262] See also the Smooth Manifold–Boundary Proposition 25.27 (2a) for the corresponding statement for smooth manifolds which is much easier to prove.

Proposition 57.34. (Submanifold-Cofibration) Let M be a smooth manifold. For any compact proper submanifold N of M the inclusion $i\colon N \to M$ is a closed cofibration.

Proof. The proof of the Submanifold-Cofibration Proposition 57.34 relies on the Tubular Neighborhood–Existence Theorem 57.11 which can be viewed as a submanifold analog of the Topological Collar Neighborhood Theorem 35.14 and the Smooth Smooth Collar Neighborhood Theorem 35.6. Since the Tubular Neighborhood–Existence Theorem 57.11 is more delicate to formulate we refrain from doing so right now and we refer to the impatient reader to the proof of the Submanifold-Cofibration Proposition 57.34 on page 1313. ■

Finally we turn to CW-complexes and simplicial complexes. We will define the notion of a CW-complex in Definition 136.1 and we will define the notion of a simplicial complex in Definition 174.31. As we will see in Definition 136.4 and Definition 174.31 there are also natural notions of subcomplexes. We will not give any of the definitions. But as an illustration we show in the figure below a simplicial complex together with a subcomplex.



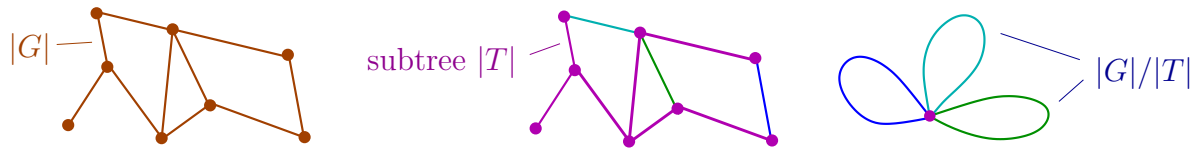
We also point out that a topological graph, as introduced in Definition 11.9, is essentially the same as a 1-dimensional CW-complex and that by the Simplicial-Implies-CW Lemma 174.51 every simplicial (sub-) complex is a CW- (sub-) complex.
The following theorem gives us one of the most important examples of a cofibration.

Theorem 22.12. (Subcomplex Cofibration Theorem) Let X be a CW-complex or a simplicial complex or a topological graph. Given any subcomplex A (or subgraph A) the corresponding inclusion map $i\colon A \to X$ is a closed cofibration.

Proof. By the Simplicial-Implies-CW Lemma 174.51 we only need to consider the case of a CW-complex. In this setting the desired statement is precisely the statement of the Homotopy Extension Theorem 140.2. The special case that X is the topological realization of a graph will also be dealt with in Exercise 22.5. ■

As a final example we turn to oriented abstract graphs and their topological realizations.

Proposition 22.13. (Graph-mod-Tree Proposition) Let G be an oriented abstract graph in the sense of Definition 11.1. For every finite subtree $T \subseteq G$ the natural projection $|G| \to |G|/|T|$ is a homotopy equivalence.

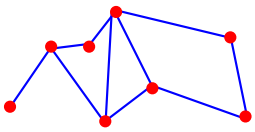


Remark. In the Graph-mod-Tree Proposition 96.11 we will give a hands-on proof of the Graph-mod-Tree Proposition 22.13 that, for better or worse, does not use the language of cofibrations. In the CW-Complex–Spanning Tree Proposition 140.5 we will generalize the Graph-mod-Tree Proposition 22.13 to infinite subtrees. □

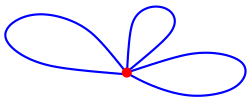
Proof. Let G be an oriented abstract graph. In Exercise 19.19 we showed that topological realizations of trees are contractible. Furthermore by the Subcomplex Cofibration Theorem 22.12, or alternatively by Exercise 22.5, we know that the inclusion map $i\colon |T| \to |G|$ is a closed cofibration. Thus it follows from the above together with the Cofibration-Quotient Lemma 22.10 that the projection $|G| \to |G|/|T|$ is a homotopy equivalence. ■

Corollary 22.14. (Graph-Homotopy Type Corollary) Let G be a finite oriented abstract graph.

(1) If G is connected and non-empty, then the topological realization $|G|$ is homotopy equivalent to the topological realization of an oriented abstract graph with a single vertex and $1-\chi(G)$ edges.
(2) Two finite connected topological graphs with the same Euler characteristic are homotopy equivalent.



is homotopy equivalent to



abstract graph with $\chi(G) = 8 - 10 = -2$ — one vertex and three edges

Proof. Let G be a finite oriented abstract graph.

(1) By the Spanning Tree–Existence Proposition 11.26 (1) we know that G admits a spanning tree T. In the Graph-mod-Tree Proposition 22.13 we just showed that the map $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. Thus it remains to show that $|G|/|T|$ is homeomorphic to the topological realization of an oriented abstract graph with a single vertex and $1-\chi(G)$ edges. We make the following observation:
   (a) By definition the spanning tree T contains all vertices of G. In other words, the oriented abstract graph G/T contains only a single vertex.
   (b) Note that
$$\chi(G/T) \underset{\uparrow}{=} \chi(G) - \chi(T) + 1 \underset{\uparrow}{=} \chi(G) - 1 + 1 = \chi(G).$$
   by the Abstract Graph–Quotient Lemma 11.32 (1); by the Tree Characterization Proposition 11.28
   (c) By (a) and (b) we know that G/T has $1-\chi(G)$ edges.
   (d) Note that by the Abstract Graph–Quotient Lemma 11.32 (2) we have a homeomorphism $|G|/|T| \cong |G/T|$.
   The desired statement now follows from (a), (c) and (d).
(2) The last statement is an immediate consequence of (1) and the Homotopy-Equivalence Basics Lemma 19.14. ■

22.5. **Products and cofibrations.** In this and the following two sections we collect several technical useful results on cofibrations. As stand-alone sections these are arguably not much fun to read. Thus it is better to move on and to come back to these three sections when the results are needed.

Proposition 22.15. (Product-Cofibration Proposition)

(1) If $f\colon A \to X$ and $g\colon B \to Y$ are two (closed) cofibrations, then the corresponding diagonal map $\operatorname{diag}(f,g)\colon A \times B \to X \times Y$ is also a (closed) cofibration.
(2) If $f\colon A \to X$ is a (closed) cofibration and if Z is a topological space, then the resulting map $f \times \mathrm{id}_Z\colon A \times Z \to X \times Z$ is also a (closed) cofibration.

Example. Let I be an interval and let $P \in I$. On page 586 we saw that the inclusion $\{P\} \to I$ is a cofibration. It follows from the Product-Cofibration Proposition 22.15 that given any topological space Z the inclusion $\{P\} \times Z \to I \times Z$ is a cofibration. □

Proof. It is clear that Statement (2) is an immediate consequence of Statement (1). Thus it remains to prove Statement (1). Now let $f\colon A \to X$ and $g\colon B \to Y$ be two cofibrations,

First note that by the Cofibration-Closed Injective Lemma 22.9 we can assume that the maps $f\colon A \to X$ and $g\colon B \to Y$ are inclusions. Next note that by the Cofibration Characterization Lemma 22.5 (1)⇒(3) there exist retractions $r\colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$ and $s\colon Y \times [0,1] \to (Y \times \{0\}) \cup (B \times [0,1])$. We denote by $r =: (r_X, r_{[0,1]})$ the coordinate functions of r viewed as a map with values in $X \times [0,1]$. Similarly we define $s =: (s_Y, s_{[0,1]})$. Next we consider the map

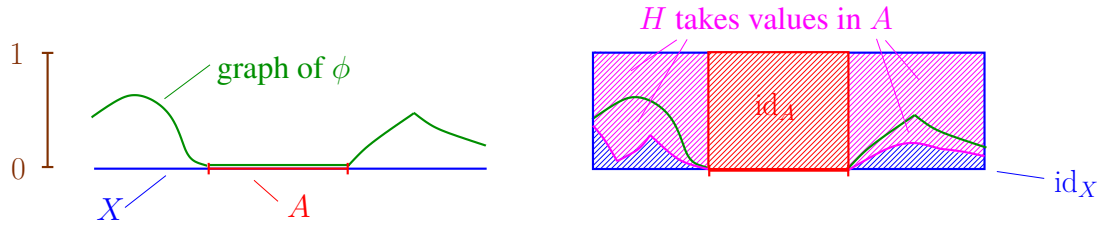
$$\begin{array}{rcl} (X \times Y) \times [0,1] & \to & ((X \times Y) \times \{0\}) \cup ((A \times B) \times [0,1]) \\ ((x,y),t) & \mapsto & \big(\underbrace{(r_X(x,t), s_Y(y,t))}_{\in X \times Y}, \underbrace{\tfrac{1}{2}(r_{[0,1]}(x,t) + s_{[0,1]}(y,t))}_{\in [0,1]}\big). \end{array}$$

One can easily verify that this map is a retraction. It follows from the Cofibration Characterization Lemma 22.5 (3)⇒(1) that $\operatorname{diag}(f,g)\colon A \times B \to X \times Y$ is indeed a cofibration.

If f and g are in fact closed cofibrations, then it follows from the Product-of-Subsets Lemma 7.8 (1) and the above that $\operatorname{diag}(f,g)$ is also a closed cofibration. ■

We continue with the following characterization of closed cofibrations that was initially proved in [**Strø1966, Strø1968**].

Theorem 22.16. (Strom's Cofibration Characterization Theorem) Let X be a topological space and let A be a closed subset. The following two statements are equivalent:
(1) The inclusion $i\colon A \to X$ is a cofibration.
(2) There exists a continuous map $\phi\colon X \to [0,1]$ such that $A = \phi^{-1}(0)$ and there exists a homotopy $H\colon X \times [0,1] \to X$ rel A such that $H_0 = \mathrm{id}_X$ and such that $H(x,t) \in A$ whenever $t > \phi(x)$.



Remark.

(1) We continue with the notation of the above Strom's Cofibration Characterization Theorem 22.16. We consider the open subset $U := \phi^{-1}([0,1))$. The homotopy H restricts to a homotopy $K\colon U \times [0,1] \to X$ rel A with $K_0 = \mathrm{id}$ and with $K_1(U) \subseteq A$. This almost

sounds like A is a deformation retract of U, but the given situation is somewhat looser, since K can take values outside of U.

(2) Sometimes a topological space X together with a closed subset A that admit H and ϕ as in Strom's Cofibration Characterization Theorem 22.16 (2) is called a *neighborhood deformation retract pair*, or short, *NDR-pair*. □

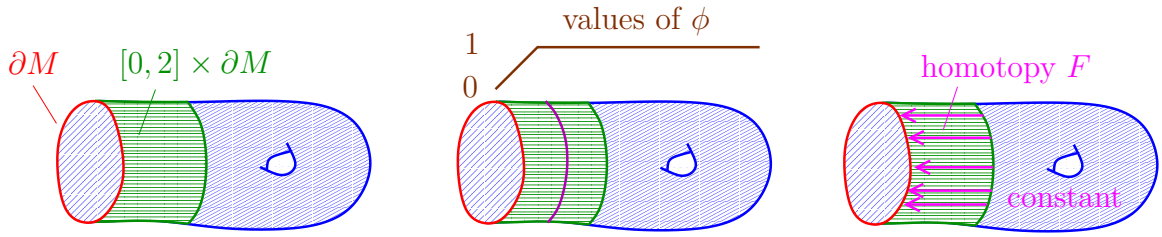
Example. Let M be a topological manifold. As in the proof of the Manifold Boundary-Cofibration Proposition 22.11 we appeal to the Topological Collar Neighborhood Theorem 35.14 to obtain a collar neighborhood $[0,2] \times \partial M$ of ∂M. Next we consider the maps

$$\begin{array}{rcl} \phi\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} s, & \text{if } P = (Q,s) \text{ with } Q \in \partial M \text{ and } s \in [0,1], \\ 1, & \text{otherwise} \end{cases} \end{array}$$

and

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (P,t) & \mapsto & \begin{cases} (Q, \frac{2(s-t)}{2-t}), & \text{if } P = (Q,s) \text{ with } Q \in \partial M \text{ and } s \in [t,2], \\ (Q,0), & \text{if } P = (Q,s) \text{ with } Q \in \partial M \text{ and } s \in [0,t], \\ P, & \text{otherwise.} \end{cases} \end{array}$$

One can easily verify that the conditions of Strom's Cofibration Characterization Theorem 22.16 are satisfied.



Note that this gives basically a new proof of the Manifold Boundary-Cofibration Proposition 22.11. □

Proof. First we deal with the "(1)⇒(2)"-direction. Thus we suppose that $i\colon A \to X$ is a cofibration. Note that by the Cofibration Characterization Lemma 22.5 (1)⇒(3) we know that there exists a retraction $r\colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$. We denote the two coordinate functions of r, viewed as a map with values in $X \times [0,1]$, by $r_X\colon X \times [0,1] \to X$ and $r_{[0,1]}\colon X \times [0,1] \to [0,1]$. We consider the following two maps:

$$\begin{array}{rcl} \phi\colon X & \to & [0,1] \\ x & \mapsto & \max\limits_{t \in [0,1]} |t - r_{[0,1]}(x,t)| \end{array} \quad \text{and} \quad \begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & r_X(x,t). \end{array}$$

Note that by definition of a retraction we have $H_0 = \mathrm{id}_X$.

Claim 1. The map ϕ is well-defined and continuous.

Proof. We consider the map $\theta\colon X \times [0,1] \to [0,1]$ that is given by $\theta(x,t) := |t - r_{[0,1]}(x,t)|$. Since $[0,1]$ is compact and since θ is continuous we obtain from the Compact Image Lemma 4.15 (2) that θ assumes a maximum, hence ϕ is well-defined.

It remains to show that $\phi\colon X \to [0,1]$ is continuous. Let V be an open subset of $[0,1]$. By the JH-Openness Criterion 3.3 it suffices to show that given any $x_0 \in \phi^{-1}(V)$ there exists an open neighborhood U of x_0 with $U \subseteq \phi^{-1}(V)$. First we consider the case that

$\phi(x_0) \in (0,1)$. This implies that we can pick $\epsilon > 0$ such that $(\phi(x_0) - \epsilon, \phi(x_0) + \epsilon) \subseteq V$. Note that by definition of ϕ and θ we have $\theta(\{x_0\} \times [0,1]) \subseteq [0, \phi(x_0)]$. Since $[0,1]$ is compact and since $[0, \phi(x_0) + \epsilon)$ is an open subset of $[0,1]$ we obtain from the Tube Lemma 7.12 that there exists an open neighborhood U_1 of $x_0 \in X$ such that

$$\{x_0\} \times [0,1] \ \subseteq \ U_1 \times [0,1] \ \subseteq \ \theta^{-1}([0, \phi(x_0) + \epsilon)).$$

Hence, for every $x \in U_1$ we have $\phi(x) \leq \phi(x_0) + \epsilon$. Next suppose that we are given $t_0 \in [0,1]$ such that $\theta(x_0, t_0) \in (\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)$. By Definition 7.3 of the product topology there exists an open subset $U_2 \subseteq X$ such that

$$(x_0, t_0) \ \in \ U_2 \times \{t_0\} \ \subseteq \ \theta^{-1}((\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)).$$

Note that this implies that for every $x \in U_2$ we have $\phi(x) \geq \phi(x_0) - \epsilon$. In particular we see that $U_1 \cap U_2 \subseteq \phi^{-1}((\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)) \subseteq V$. Basically the same argument also works if $\phi(x_0) = 0$ or $\phi(x_0) = 1$. ⊞

Claim 2. We have $A = \phi^{-1}(0)$.

Proof. First let $x \in A$. Since r is a retraction we know that $r(x,t) = (x,t)$ for all $t \in [0,1]$. This means that $r_{[0,1]}(x,t) = t$ for all $t \in [0,1]$. Thus we see that $\phi(x) = 0$. Conversely, suppose that we have $x \in X$ with $\phi(x) = 0$. One easily verifies that this implies that for every $t > 0$ we have $r(x,t) \in A \times [0,1]$, in other words, we have $\{x\} \times (0,1] \subseteq r^{-1}(A \times [0,1])$. Since $A \times [0,1]$ is a closed subset of $X \times [0,1]$ we see that the closure of $\{x\} \times (0,1]$ is contained in $r^{-1}(A \times [0,1])$. But the closure is precisely $\{x\} \times [0,1]$. Thus we have shown that $\{x\} \times [0,1]$ also belongs to $r^{-1}(A \times [0,1])$. But this implies that $x \in A$. ⊞

It remains to prove the following claim:

Claim 3. Given $(x,t) \in X \times [0,1]$ with $t > \phi(x)$ we have $H(x,t) \in A$.

Proof. So suppose that we are given $(x,t) \in X \times [0,1]$ with $t > \phi(x)$. Note that this implies that $t > \phi(x) \geq |t - r_{[0,1]}(x,t)|$. This is only possible if $r_{[0,1]}(x,t) > 0$. Thus we see that $r(x,t) \in A \times [0,1]$. This implies that $H(x,t) \in A$. ⊞

Finally we provide the proof for the "(1)⇒(2)"-direction. Given $\phi\colon X \to [0,1]$ and the homotopy $H\colon X \times [0,1] \to X$ we consider the map

$$\begin{array}{rcl} r\colon X \times [0,1] & \to & (X \times \{0\}) \cup (A \times [0,1]) \\ (x,t) & \mapsto & (H(x,t), \max\{t - \phi(x), 0\}). \end{array}$$

One can easily verify that this map is indeed a retraction. Thus it follows from the Cofibration Characterization Lemma 22.5 (3)⇒(1) that the inclusion $i\colon A \to X$ is a cofibration. ■

Proposition 22.17. (Mixed Product–Cofibration Proposition) Let $A \subseteq X$ and $B \subseteq Y$ be subsets of topological spaces. If the inclusion maps $i\colon A \to X$ and $j\colon B \to Y$ are closed cofibrations, then the inclusion map

$$(A \times Y) \cup (X \times B) \ \to \ X \times Y$$

is also a closed cofibration.

Remark. In Exercise 22.9 we will see that the conclusion of the Mixed Product–Cofibration Proposition 22.17 does not hold, if neither A nor B are closed subsets. □

Proof. Since i and j are closed cofibrations we can pick maps $\phi\colon X \to [0,1]$, $\psi\colon Y \to [0,1]$ and homotopies $G\colon X \times [0,1] \to X$ and $H\colon Y \times [0,1] \to Y$ as in Strom's Cofibration Characterization Theorem 22.16. We consider the maps

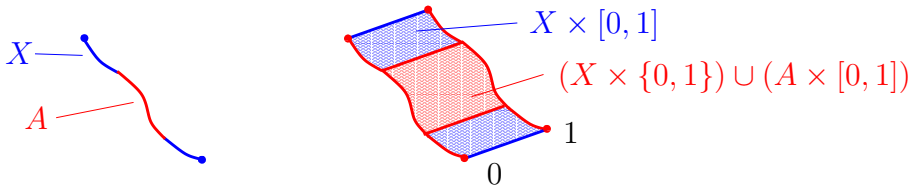
$$\begin{aligned} \gamma\colon X \times Y &\to [0,1] \\ (x,y) &\mapsto \min\{\phi(x), \psi(y)\} \end{aligned}$$

and

$$\begin{aligned} K\colon (X \times Y) \times [0,1] &\to X \times Y \\ ((x,y),t) &\mapsto (G(x, \min\{t, \psi(y)\}), H(y, \min\{t, \phi(x)\})). \end{aligned}$$

It is easy to see that $\gamma^{-1}(0) = (X \times B) \cup (A \times Y)$. Next note that it is basically clear that $K_0 = \mathrm{id}$. Finally note that by construction we have $K((x,y),t) \in (A \times Y) \cup (X \times B)$, whenever $t > \gamma(x,y)$. Therefore it follows from Strom's Cofibration Characterization Theorem 22.16 that the inclusion map $(A \times X) \cup (X \times B) \to X \times Y$ is a closed cofibration. ■

Corollary 22.18. (Interval Product–Cofibration Corollary) Let X be a topological space and let $A \subseteq X$ be a subset. If the inclusion $i\colon A \to X$ is a closed cofibration, then the inclusion $(X \times \{0,1\}) \cup (A \times [0,1]) \to X \times [0,1]$ is also a closed cofibration.



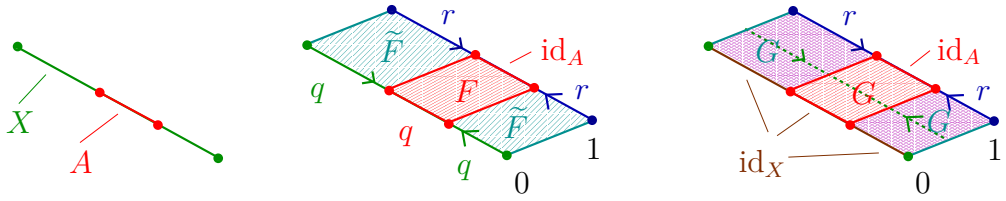
Proof. On page 586 we saw that the inclusion map $\{0,1\} \to [0,1]$ is a cofibration. It follows from this observation together with the Mixed Product–Cofibration Proposition 22.17 that the given inclusion map is a closed cofibration. ■

Proposition 22.19. (Homotopy Equivalence-Cofibration Proposition) Let $i\colon A \to X$ be a closed cofibration. If i is a homotopy equivalence, then $i(A)$ is a deformation retract of X.

Proof. Let $i\colon A \to X$ be a cofibration. By the Cofibration-Closed Injective Lemma 22.9 we can assume that A is a closed subset of X and that $i\colon A \to X$ is the inclusion. Let $q\colon X \to A$ be a homotopy inverse of i. Recall that this means that $q \circ i = q\colon A \to A$ is homotopic to id_A and $i \circ q$ is homotopic to id_X.

We pick a homotopy $F\colon A \times [0,1] \to A$ from $q \circ i = q\colon A \to A$ to id_A. Since $i\colon A \to X$ is a cofibration we can extend the map $q\colon X \to A$ on X and the homotopy $F\colon A \times [0,1] \to A$ to a homotopy $\widetilde{F}\colon X \times [0,1] \to A$. Note that $\widetilde{F}_1|_A = F_1|_A = \mathrm{id}_A$, in other words, $\widetilde{F}_1\colon X \to A$ is a retraction. We write $r \coloneqq \widetilde{F}_1\colon X \to A \subseteq X$.

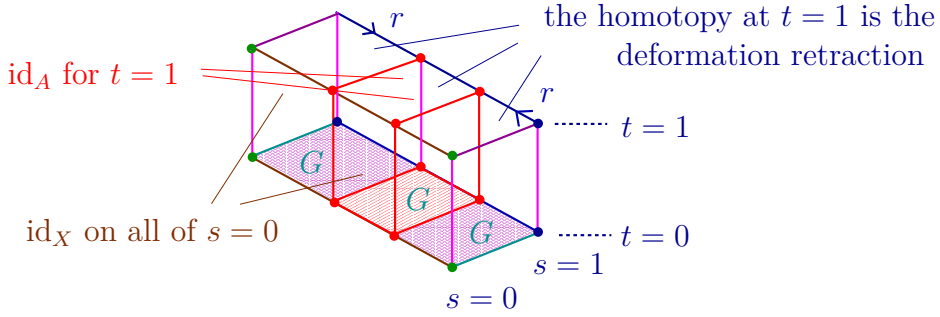
Note that $r\colon X \to X$ is homotopic to $q = i \circ q\colon X \to X$ which in turn is homotopic to id_X. Thus it follows from the Homotopy-Equivalence Relation Lemma 18.9 that there exists a homotopy G from id_X to r.



Now we consider the following map

$$K\colon ((X\times\{0,1\})\cup(A\times[0,1]))\times[0,1] \;\to\; X$$
$$(x,s,t) \;\mapsto\; \begin{cases} x, & \text{if } s=0,\\ G(r(x),1-t), & \text{if } s=1,\\ G(x,(1-t)s), & \text{if } x\in A,\\ G(x,s), & \text{if } t=0. \end{cases}$$

It follows easily from the hypothesis that A is a closed subset together with the Pasting Proposition 4.7 (2) that the map is continuous.



Next note that it follows from the Interval Product–Cofibration Corollary 22.18 we know that $(X\times\{0,1\})\cup(A\times[0,1])\to X\times[0,1]$ is a cofibration. Thus we can extend the homotopy K to a homotopy $\widetilde{K}\colon (X\times[0,1])\times[0,1]\to X$. The map $\widetilde{K}_1\colon X\times[0,1]\to X$ is now a homotopy rel A from id_X to the retraction r. We have thus shown that A is a deformation retract of X. ■

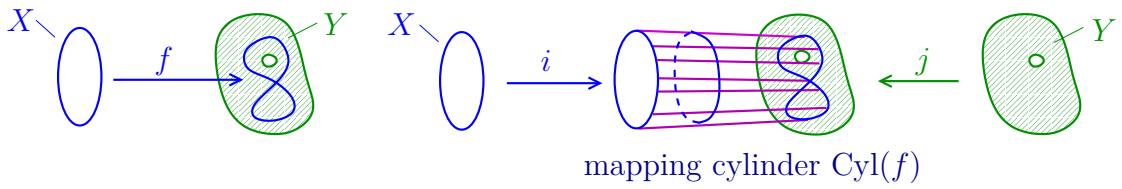
The Homotopy Equivalence-Cofibration Proposition 22.19 admits two interesting corollaries. Since both involve mapping cylinders, let us recall the definition. Given a continuous map $f\colon X\to Y$ between topological spaces we set

$$\mathrm{Cyl}(f) \;\coloneqq\; ((X\times[0,1])\sqcup Y)/\!\sim \quad \text{where } (x,1)\sim f(x) \text{ for all } x\in X.$$

Furthermore we consider the natural inclusions

$$\begin{array}{rcl} i\colon X & \to & \mathrm{Cyl}(f)\\ x & \mapsto & [(x,0)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Y & \to & \mathrm{Cyl}(f)\\ y & \mapsto & [y] \end{array}$$

The first corollary to the Homotopy Equivalence-Cofibration Proposition 22.19 is a strengthening of the Mapping Cylinder–Homotopy Lemma 21.5 (2b).



Corollary 22.20. (Homotopy Equivalence–Characterization Corollary) In the following let $f\colon X \to Y$ be a continuous map between two topological spaces. The following two statements are equivalent:

(1) f is a homotopy equivalence,
(2) $i(X)$ is a deformation retract of $\mathrm{Cyl}(f)$.

Proof. By the Mapping Cylinder–Homotopy Lemma 21.5 (2b) we know that the following two statements are equivalent:

(1) f is a homotopy equivalence,
(2′) $i\colon X \to \mathrm{Cyl}(f)$ is a homotopy equivalence.

Thus it remains to show that (2) and (2′) are equivalent. By the Deformation Retract–Homotopy Equivalence Lemma 19.10 we know that the "(2)⇒(2′)"-implication holds. It remains to prove the "(2')⇒(2)"-direction. Thus we assume that $i\colon X \to \mathrm{Cyl}(f)$ is a homotopy equivalence. By the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 we know that the natural inclusion $i\colon X \to \mathrm{Cyl}(f)$ is a closed cofibration. But now it follows from the Homotopy Equivalence-Cofibration Proposition 22.19 that $i(X)$ is indeed a deformation retract of X. ■

Corollary 22.21. (Homotopy Equivalent–Characterization Corollary) Let X and Y be two topological spaces. The following two statements are equivalent:

(1) X and Y are homotopy equivalent.
(2) There exists a topological space W and closed embeddings $i\colon X \to W$ and $j\colon Y \to W$ such that $i(X)$ is a deformation retract of W and $j(Y)$ is a deformation retract of W.

Proof. The "(2)⇒(1)"-direction is just a consequence of the Deformation Retract–Homotopy Equivalence Lemma 19.10 and the fact that being homotopy equivalent is an equivalence relation, see the Homotopy-Equivalence Basics Lemma 19.14 (2).

We turn to the proof of the "(1)⇒(2)"-direction. Thus we now assume that X and Y are homotopy equivalent. Let $f\colon X \to Y$ be a homotopy equivalence. We consider the corresponding mapping cylinder $\mathrm{Cyl}(f)$. By the Mapping Cylinder–Embedding Lemma 21.4 we know that the natural inclusions $i\colon X \to \mathrm{Cyl}(f)$ and $j\colon Y \to \mathrm{Cyl}(f)$ are closed embeddings. In the Mapping Cylinder–Homotopy Lemma 21.5 we showed that $j(Y)$ is always, regardless of the choice of f, a deformation retract of $\mathrm{Cyl}(f)$. Finally, since $f\colon X \to Y$ is by hypothesis a homotopy equivalence we now know by the Homotopy Equivalence–Characterization Corollary 22.20 that $i(X)$ is a deformation retract of $\mathrm{Cyl}(f)$. ■

22.6. **Pushouts and cofibrations.** In this section we will consider the relationship between cofibrations and pushouts. We recall the following notation from the Topological-Pushout Proposition 8.12.

Notation 22.22. Let $f\colon A \to X$ and $g\colon A \to Y$ be two continuous maps between topological spaces. We write

$$X \cup_A Y \;:=\; (X \sqcup Y)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$

In the Topological-Pushout Proposition 8.12 (2) we saw that $X \cup_A Y$ is the pushout of f and g.

The next proposition continues a theme that we already encountered in the Pushout–Opposite Lemma 6.26 and the Topological Pushout-Maps Lemma 8.16.

Proposition 22.23. (Opposite Map-Cofibration Proposition) Suppose we are given two continuous maps $f\colon A \to X$ and $g\colon A \to Y$ between topological spaces. We consider the corresponding pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & X \cup_A Y. \end{array}$$

The following two statements hold:

(1) If f is a (closed) cofibration, then the "opposite map" $j\colon Y \to X \cup_A Y$ is also a (closed) cofibration.

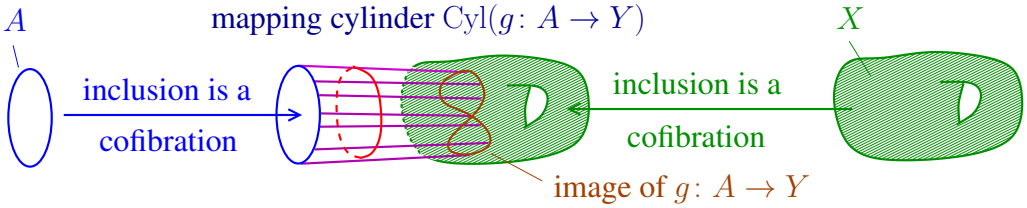
(2) If f is a closed cofibration and if f is a homotopy equivalence, then the "opposite map" $j\colon Y \to X \cup_A Y$ is also a homotopy equivalence.

The same statements hold, by symmetry, for g and the "opposite map" i.

Example. Let $g\colon A \to Y$ be a continuous map between topological spaces. We consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto (a,0)} & A \times [0,1] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & \underbrace{(A \times [0,1]) \cup_A Y}_{=\mathrm{Cyl}(g)}. \end{array}$$

Recall that by the discussion on page 596 we know that the top horizontal map is a cofibration. Therefore it follows from the Opposite Map-Cofibration Proposition 22.23 that the natural inclusion $Y \to \mathrm{Cyl}(g\colon A \to Y)$ is a cofibration.



Thus, together with the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 (1) we see that both the natural inclusions $A \to \mathrm{Cyl}(g\colon A \to Y)$ and $Y \to \mathrm{Cyl}(g\colon A \to Y)$ are cofibrations. □

Proof. Throughout the proof we assume that $f\colon A \to X$ is a cofibration. By the Cofibration-Closed Injective Lemma 22.9 we can assume that A is a subspace of X and that $f\colon A \to X$ is the inclusion.

First we want to show that $j\colon Y \to X \cup_A Y$ is also a cofibration. Since f is a cofibration we know by the Cofibration Characterization Lemma 22.5 (1)⇒(3) that there exists a retraction

$$r\colon X\times[0,1] \;\to\; (X\times\{0\}) \cup (A\times[0,1]).$$

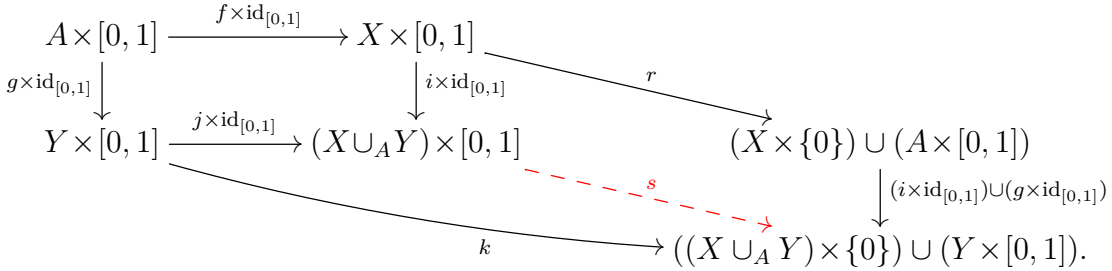
We introduce the following extra maps:

(1) We denote by $i\colon X \to X \cup_A Y$ and $j\colon Y \to X \cup_A Y$ the two natural maps.
(2) We consider the pushout diagram

$$\begin{array}{ccc} Y\times\{0\} & \longrightarrow & Y\times[0,1] \\ {\scriptstyle j\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle k} \\ ((X\cup_A Y)\times\{0\}) & \longrightarrow & ((X\cup_A Y)\times\{0\}) \cup (Y\times[0,1]). \end{array}$$

We denote by $k\colon Y\times[0,1] \to ((X\cup_A Y)\times\{0\}) \cup (Y\times[0,1])$ the natural map.

Now we consider the following diagram:



We make the following observations:

(1) It follows immediately from the Topological Pushout-Maps Lemma 8.16 (3) that we can view $Y\times[0,1]$ as a subspace of $(X\cup_A Y)\times[0,1]$.
(2) The diagonal dashed map s comes from the Homotopy Pushout Lemma 18.13 which shows that the pushout commutes with taking products with $[0,1]$.
(3) It follows from (1) and (2) that we can view the map s as a retraction. Thus it follows from the Cofibration Characterization Lemma 22.5 that the map $Y \to X\cup_A Y$ is indeed a cofibration.

Finally note that if f is a closed cofibration, then it follows from the above together with the Topological Pushout-Maps Lemma 8.16 (4) that the map $Y \to X\cup_A Y$ is also a closed cofibration.

Now we assume that f is a closed cofibration and that it is a homotopy equivalence. We need to show that $j\colon Y \to X\cup_A Y$ is a homotopy equivalence. By the Topological Pushout-Maps Lemma 8.16 (3) we know that j is an embedding. By the Deformation Retract–Homotopy Equivalence Lemma 19.10 it now suffices to show that $j(Y)$ is a deformation retract of $X\cup_A Y$. By the Homotopy Equivalence-Cofibration Proposition 22.19 we know that there exists a deformation retraction $G\colon X\times[0,1] \to X$ from X to A. We consider the map

$$\begin{array}{rcl} H\colon (X\cup_A Y)\times[0,1] & \to & X\cup_A Y \\ ([P],t) & \mapsto & \begin{cases} i(G(Q,t)), & \text{if } P = i(Q) \text{ with } Q\in X, \\ j(Q), & \text{if } P = j(Q) \text{ with } Q\in Y. \end{cases} \end{array}$$

It follows from the Homotopy Pushout Lemma 18.13 that this map is actually continuous. Now that we put our worries regarding continuity to rest it is clear that this map is a deformation retraction from $X \cup_A Y$ to $j(Y)$. ■

The following theorem is the main result of this chapter.

Theorem 22.24. (Homotopy Pushout Theorem) Suppose we are given the following commutative diagram of continuous maps between topological spaces:

$$\begin{array}{ccccc} X & \xleftarrow{\quad i \quad} & A & \xrightarrow{\quad f \quad} & Y \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_Y} \\ X' & \xleftarrow{\quad i' \quad} & A' & \xrightarrow{\quad f' \quad} & Y'. \end{array}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the left are closed cofibrations, then the induced map

$$\begin{array}{rcl} X \cup_A Y & \to & X' \cup_{A'} Y' \\ [P] & \mapsto & \begin{cases} \varphi_X(P), & \text{if } P \in X, \\ \varphi_Y(P), & \text{if } P \in Y \end{cases} \end{array}$$

between the pushouts is a homotopy equivalence.

Remark.

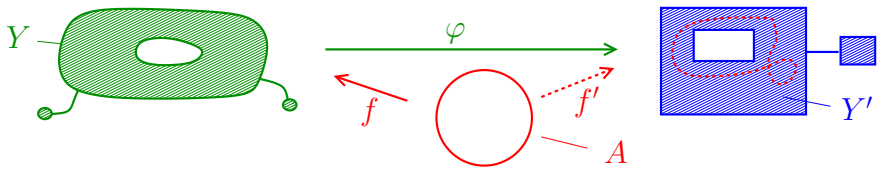
(1) We outsource the proof, which follows quite closely the argument in [**Rap2016**], to the next section. Alternative proofs are given in [**BrownR1968**, p. 249], [**BrownR2006**, Proposition 7.5.7] and [**tD2008**, Proposition 5.3.4].

(2) In Exercise 22.3 we will see that the naive analog of the Homotopy Pushout Theorem 22.24 for pullbacks does *not* hold. □

We will use the Homotopy Pushout Theorem 22.24 to prove the following convenient proposition:

Proposition 22.25. (Homotopy Equivalence-Mapping Cone Proposition) In the following let $\varphi\colon Y \to Y'$ be a homotopy equivalence between two topological spaces. Furthermore let A be a topological space and let $f\colon A \to Y$ and $f'\colon A \to Y'$ be two continuous maps such that $\varphi \circ f$ is homotopic to f'. Then there is a homotopy equivalence

$$\mathrm{Cone}(f\colon A \to Y) \;\to\; \mathrm{Cone}(f'\colon A \to Y')$$



Example. Let $f_0, f_1\colon A \to Y$ be two continuous maps between two topological spaces. If f_0 and f_1 are homotopic, then it follows immediately from the Homotopy Equivalence-Mapping Cone Proposition 22.25, applied to the map $\varphi = \mathrm{id}\colon Y \to Y' = Y$, that the mapping cones $\mathrm{Cone}(f\colon A \to Y)$ and $\mathrm{Cone}(f'\colon A \to Y)$ are homotopy equivalent. Note that this recovers for the most part the statement of the Homotopic Map–Mapping Cone Proposition 21.15. □

Remark. Later, in the Homotopy Equivalent Cell Attachment Proposition 135.13 we will give a self-contained proof of a special case of the Homotopy Equivalence-Mapping Cone Proposition 22.25. In fact the latter proof can be modified easily to give the general statement. □

Proof. Let $F\colon A \times [0,1] \to Y'$ be a homotopy between $\varphi \circ f$ and f'. We consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Cone}(A) & \hookleftarrow & A & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} \\
\mathrm{Cone}(A) & \hookleftarrow & A & \xrightarrow{\varphi\circ f} & Y' \\
\downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle a\mapsto(a,0)} & & \downarrow{\scriptstyle y\mapsto(y,0)} \\
\mathrm{Cone}(A)\times[0,1] & \hookleftarrow & A\times[0,1] & \xrightarrow{(a,t)\mapsto(F(a,t),t)} & Y'\times[0,1] \\
\uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle a\mapsto(a,1)} & & \uparrow{\scriptstyle y\mapsto(y,1)} \\
\mathrm{Cone}(A) & \hookleftarrow & A & \xrightarrow{f'} & Y'.
\end{array}$$

The vertical maps are evidently homotopy equivalences. Furthermore we know by the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 (2) and the Product-Cofibration Proposition 22.15 that the left horizontal maps are closed cofibrations. It follows from the Homotopy Pushout Theorem 22.24 together with the Homotopy-Equivalence Basics Lemma 19.14 (1) that the pushout on top is homotopy equivalent to the pushout at the bottom. But the pushout on top is precisely $\mathrm{Cone}(f\colon A \to Y)$ and the pushout at the bottom is precisely $\mathrm{Cone}(f'\colon A \to Y')$. ■

Remark. Another typical application of the Homotopy Pushout Theorem 22.24 is given in the proof of the Handle–Cell Attachment Proposition 254.2 where we show that "handle attachments" and "cell attachments" lead to homotopy equivalent topological spaces. □

22.7. **Proof of Theorem 22.24.** In this section we provide a proof of Theorem 22.24, following the argument of [**Rap2016**]. The least that can be said is that the proof is not for the faint of heart.

The following lemma is a key step towards the proof of Theorem 22.24.

Auxiliary Lemma 22.26. Let $f\colon A \to B$ and $g\colon A \to X$ be two continuous maps between topological spaces. We consider the corresponding pushout diagram

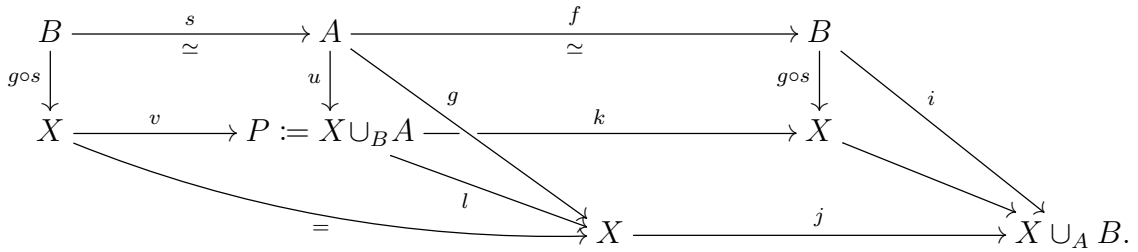
$$\begin{array}{ccc}
A & \underset{f}{\overset{s}{\rightleftarrows}} & B \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\
X & \xrightarrow{j} & X \cup_A B.
\end{array}$$

Let f be a homotopy equivalence. If g is a closed cofibration, and if there exists a continuous map $s\colon B \to A$ with $f \circ s = \mathrm{id}_B$ and which is a closed cofibration, then the map "opposite to f", i.e. the map j is a homotopy equivalence.

Remark. Sometimes it takes a while to spot the differences to earlier results. In the similar looking the Opposite Map-Cofibration Proposition 22.23 we had assumed that f

is not only a homotopy equivalence but also a closed cofibration. Now we assume g is a closed cofibration and that we have the extra map s to play with. □

Proof of Lemma 22.26. First note that it follows from the hypotheses that $f \circ s = \mathrm{id}_B$ and that f is a homotopy equivalence, together with the Homotopy-Equivalence Basics Lemma 19.14 (1), that s is also a homotopy equivalence. We consider the following diagram:



We make the following clarifications and observations:

(1) To the top left we have a pushout square. In particular the maps $u\colon A \to X \cup_B A$ and $v\colon X \to X \cup_B A$ are the corresponding natural maps.
(2) The map $k\colon X \cup_B A \to X$ is defined via the identity on X and the map $g \circ s \circ f$ on A.
(3) It follows easily from the Pushout–Opposite Lemma 6.26 and the Pushout–Composition Lemma 6.25[263] that the square to the top right is a pushout diagram.
(4) The map $l\colon X \cup_B A \to X$ is defined via the identity on X and the map g on A.
(5) It follows easily from the definitions that the diagram commutes.
(6) Since s is a closed cofibration and a homotopy equivalence it follows from the Opposite Map-Cofibration Proposition 22.23 (2) that the map $v\colon X \to X \cup_B A$ is also a closed cofibration and a homotopy equivalence.
(7) Note that $k \circ v = \mathrm{id}_X$ and $l \circ v = \mathrm{id}_X$. Since v and id_X are homotopy equivalences we obtain from the Homotopy-Equivalence Basics Lemma 19.14 (1) that k and l are also homotopy equivalences.
(8) Since $g \circ s$ is a closed cofibration we obtain from the Opposite Map-Cofibration Proposition 22.23 (1) that the map u is also a closed cofibration.

Next we consider the following two pushout diagrams:

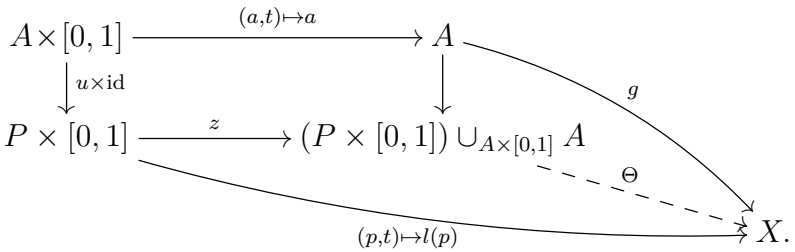
$$\begin{array}{ccc} P & \xrightarrow[\simeq]{l} & X \\ \downarrow{\scriptstyle p\mapsto(p,1)} & & \downarrow{\scriptstyle t} \\ (P\times[0,1]) \cup_{A\times[0,1]} A & \longrightarrow & \underbrace{((P\times[0,1])\cup_{A\times[0,1]} A)\cup_P X}_{=:\mathrm{Cyl}(l)_A} \end{array} \quad \text{and} \quad \begin{array}{ccc} P = X\cup_B A & \xrightarrow[\simeq]{k} & X \\ \downarrow{\scriptstyle \varphi(p)=(p,1)} & & \downarrow{\scriptstyle m} \\ \mathrm{Cyl}(l)_A & \xrightarrow{w} & \mathrm{Cyl}(l)_A \cup_P X. \end{array}$$

Providing the proof of the following claim is the main difficulty in the proof of the lemma.

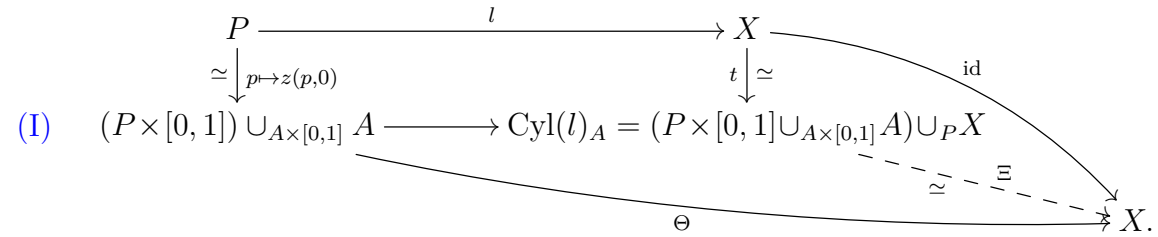
Claim. The map $w\colon \mathrm{Cyl}(l)_A \to \mathrm{Cyl}(l)_A \cup_P X$ is a homotopy equivalence.

[263] The Pushout Composition 6.25 implies that the natural map $X \cup_B B \to (X \cup_B A) \cup_A B$ is a homeomorphism and the Pushout Opposite Lemma 6.26 implies that the natural map $X \to X \cup_B B$ is a homeomorphism. We will apply this type of argument on several occasions.

Proof. To prove the claim we first consider the following pushout diagram:

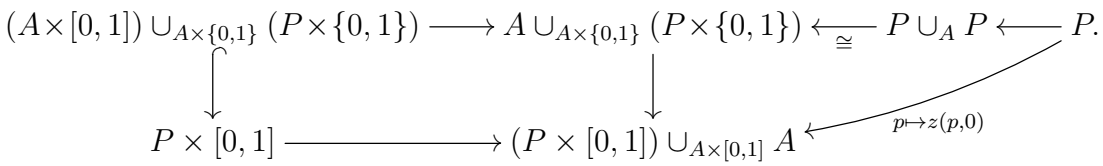


We take the resulting map Θ and use it to produce the following even more confusing pushout diagram



We continue with the following comments:

(9) We claim that the left-hand vertical map is a closed cofibration. To verify this statement we consider the following commutative diagram:



We make a few comments:

(a) The left top horizontal map is induced by the projection $A \times [0,1] \to A$ and the identity on $P \times \{0,1\}$.

(b) The left vertical map is the "obvious" map. It is a closed cofibration by the combination of the Interval Product–Cofibration Corollary 22.18 with the Regluing Lemma 8.17.

(c) The middle vertical map and the bottom horizontal map are the obvious maps.

(d) We leave it to the reader to verify that the square is in fact a pushout diagram.

(e) It follows from (b) and (d) together with the Opposite Map-Cofibration Proposition 22.23 that the middle vertical map is a closed cofibration.

(f) The map $P \cup_A P \to A \cup_{A \times \{0,1\}} (P \times \{0,1\})$ is given by sending the "left copy" of P to $P \times \{0\}$ and the "right copy" of P to $P \times \{1\}$. There is an obvious inverse which shows that this map is actually a homeomorphism.

(g) The map $P \to P \cup_A P$ is given by sending P to the "left copy" of P. Since $A \to P$ is a closed cofibration we obtain again from the Opposite Map-Cofibration Proposition 22.23 that the map $P \to P \cup_A P$ is a closed cofibration.

(h) It follows immediately from the definitions that the triangle to the right commutes. Thus we obtain from (e), (f) and (g) together with the Cofibration Composition

Lemma 22.4 that the diagonal map is a closed cofibration. But that is precisely what we wanted to show.

(10) We claim that the left-hand vertical map $P \to (P \times [0,1]) \cup_{A \times [0,1]} A$ of (I) is a homotopy equivalence. This can be seen as follows. By (9) and the Cofibration-Closed Injective Lemma 22.9 we can identify P with its image under the inclusion. It follows from the Homotopy Pushout Lemma 18.13 that the obvious deformation retraction on $P \times [0,1]$ and the constant map on A give rise to a deformation retraction from $(P \times [0,1]) \cup_{A \times [0,1]} A$ to the image of P. Thus it follows from the elementary Deformation Retract–Homotopy Equivalence Lemma 19.10 that the inclusion is indeed a homotopy equivalence.

(11) We consider the pushout square in (I). It follows from (9), (10) and the Opposite Map-Cofibration Proposition 22.23 (2) that the right-hand vertical map t in (I) is also closed cofibration and a homotopy equivalence.

(12) We deduce from (11) and the Homotopy-Equivalence Basics Lemma 19.14 (1) that Ξ is a homotopy equivalence.

Next we consider the following diagram:

(II)
$$\begin{array}{ccc} P = X \cup_B A & \xrightarrow[\simeq]{k} & X \\ \simeq \downarrow \varphi(p)=(p,0) & & \downarrow m \\ \mathrm{Cyl}(l)_A & \xrightarrow{w} & \mathrm{Cyl}(l)_A \cup_P X \\ \simeq \downarrow \Xi & & \\ X. & & \end{array} \qquad l$$

We make the following clarifications and observations:

(13) $\mathrm{Cyl}(l)_A \cup_P X$ is the pushout of the pushout diagram to the top.

(14) The composition $P \xrightarrow{\varphi} \mathrm{Cyl}(l)_A \xrightarrow{\Xi} X$ equals the map l.

(15) Since l is a homotopy equivalence and since we just saw that $\Xi \colon \mathrm{Cyl}(l)_A \to X$ is a homotopy equivalence we obtain again from the Homotopy-Equivalence Basics Lemma 19.14 (1) that $\varphi \colon P \to \mathrm{Cyl}(l)_A$ is a homotopy equivalence.

(16) We will now argue that the map $\varphi \colon P \to \mathrm{Cyl}(l)_A$ is a closed cofibration. To do so we consider the following diagram:

$$\begin{array}{ccc} & & P \\ & & \downarrow \\ (A \times [0,1]) \cup_{A \times \{0,1\}} (P \times \{0,1\}) & \longrightarrow & X \cup_A P \\ \downarrow & & \downarrow \\ P \times [0,1] & \longrightarrow & \mathrm{Cyl}(l)_A. \end{array} \qquad \varphi$$

We make the following observations:

(a) The top vertical map is a closed cofibration by the Opposite Map-Cofibration Proposition 22.23 and our hypothesis that $A \to X$ is a closed cofibration.

(b) The map $X \cup_A P \to \mathrm{Cyl}(l)_A$ is defined by t on X and by φ on P.

(c) The vertical map to the left is induced by the inclusions.

(d) The horizontal map on the top is induced by the projection $A \times [0,1] \to A$, the obvious identification $P \times \{0\} \to P$ and the map l on $P = P \times \{1\} \to X$.

(e) We consider the maps

$$(P \times [0,1]) \cup_{(A\times[0,1])\cup_{A\times\{0,1\}}(P\times\{0,1\})} (X \cup_A P) \;\leftrightarrow\; \underbrace{((P \times [0,1]) \cup_{A\times[0,1]} A) \cup_P X}_{=\mathrm{Cyl}(l)_A}$$

which are the obvious maps on $P \times [0,1]$, the obvious map on X and which going left-to-right is given by the obvious map $P \to P \times \{0\}$ and which right-to-left is given by the natural map $A \to X \cup_B A = P$. These maps are well-defined and inverses of one another, hence they are isomorphisms. These maps show that the square is a pushout diagram.

(f) It follows from the Interval Product–Cofibration Corollary 22.18 together with the Regluing Lemma 8.17 that the left vertical map in the square is a closed cofibration.

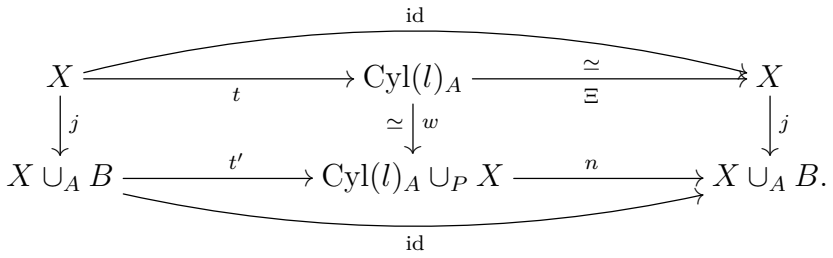
(g) It follows (f) together with the Opposite Map-Cofibration Proposition 22.23 (1) that the right vertical map in the square is also a closed cofibration.

(h) By (a) and (g) the map of our desire is a composition of two closed cofibrations, hence it is itself a closed cofibration by the Cofibration Composition Lemma 22.4.

(17) It follows from (15) and (16) together with the Opposite Map-Cofibration Proposition 22.23 (2), applied the diagram (II), that the map $m\colon X \to \mathrm{Cyl}(l)_A \cup_P X$ is a homotopy equivalence.

(18) By (7), (15) and (17) we know that m, k, φ are homotopy equivalences. By the diagram (II) we have $w \circ \varphi = m \circ k$. Thus we obtain from the Homotopy-Equivalence Basics Lemma 19.14 (1) that w is also a homotopy equivalence. ⊞

Finally we consider the following diagram:



Here are the customary clarifications and observations:

(19) Recall that the map $t\colon X \to \mathrm{Cyl}(l)_A$ is the natural inclusion.

(20) By definition of t and Ξ we have $\Xi \circ t = \mathrm{id}_X$.

(21) The map $t'\colon X \cup_A B \to \mathrm{Cyl}(l)_A \cup_P X$ is defined by $w \circ t\colon X \to \mathrm{Cyl}(l)_A$ and it is defined by $g \circ s\colon B \to X$. Furthermore the map $n\colon \mathrm{Cyl}(l)_A \cup_P X \to X \cup_A B$ is defined by Ξ on $\mathrm{Cyl}(l)_A$ and by the natural map $j\colon X \to X \cup_A B$ on X.

(22) One can easily verify that $n \circ t' = \mathrm{id}_{X\cup_A B}$.

(23) It follows immediately from the definitions that the diagram commutes.

It follows from (20), (22) and (23) and the above commutative diagram that j is a retract of the map w, in the sense of the definition in Exercise 19.22. Since w is a homotopy equivalence we obtain from Exercise 19.22 that j is also a homotopy equivalence. We are done! ■

The next lemma is the same statement as Lemma 22.26 except that we can drop the hypothesis on the existence of s.

Lemma 22.27. (Opposite Map-Homotopy Equivalence Lemma) Let $f\colon A \to B$ and $g\colon A \to X$ be two continuous maps between topological spaces. We consider the corresponding pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{j} & X \cup_A B. \end{array}$$

Let f be a homotopy equivalence. If g is a closed cofibration, then the map "opposite to f", i.e. the map j is a homotopy equivalence.

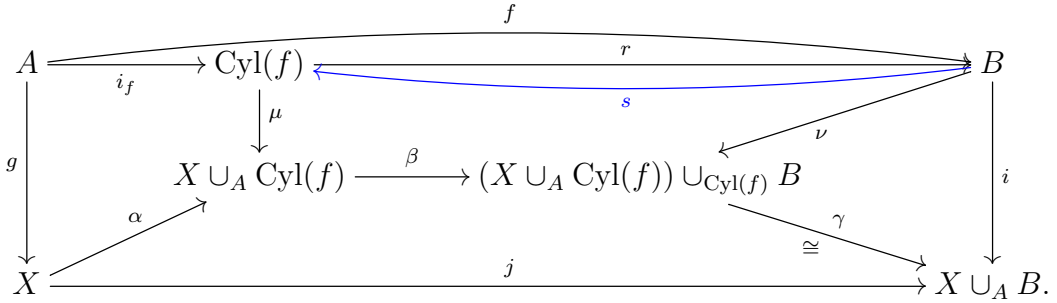
Proof of the Opposite Map-Homotopy Equivalence Lemma 22.27. We consider the maps

$$A \xrightarrow[a\mapsto(a,0)]{i_f} \operatorname{Cyl}(f) \underset{s}{\overset{r}{\rightleftarrows}} B, \qquad A \xrightarrow{f} B$$

where $i_f\colon A \to \operatorname{Cyl}(f)$ and $s\colon B \to \operatorname{Cyl}(f)$ are the natural inclusions and $r\colon \operatorname{Cyl}(f) \to B$ is the natural retraction. We make the following observations:

(1) It follows from the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 and the discussion on page 602 that i_f and s are closed cofibrations.
(2) It follows immediately from the definitions that $f = r \circ i_f$ and $r \circ s = \mathrm{id}_B$.
(3) By the Mapping Cylinder–Embedding Lemma 21.4 and the Mapping Cylinder–Homotopy Lemma 21.5 we know that r is a homotopy equivalence.
(4) It follows from the hypothesis that f is a homotopy equivalence, together with (2) and (3) and together with the Homotopy-Equivalence Basics Lemma 19.14 (1), that i_f is a homotopy equivalence.

Next we consider the following diagram:



We continue with the following clarifications and observations:

(5) The quadrilateral to the left is a pushout diagram. In particular α and μ are the natural maps of a pushout.
(6) The quadrilateral in the top center is also a pushout diagram. In particular β and ν are the natural maps of a pushout.
(7) The map γ is the natural homeomorphism given by the Pushout Composition Lemma 6.25. [264]

[264] More precisely, the Pushout Composition Lemma 6.25 implies that the natural map $X \cup_A C \to (X \cup_A C) \cup_C B$ is a homeomorphism.

(8) It follows immediately from the definitions that the diagram commutes.
(9) We consider the quadrilateral to the left.
  (a) By (1) and (4) we know that i_f is a homotopy equivalence and that it is a closed cofibration. Thus we obtain from (5) together with the Opposite Map-Cofibration Proposition 22.23 (2) that the opposite map $\alpha\colon X \to X \cup_A \mathrm{Cyl}(f)$ is a homotopy equivalence.
  (b) By hypothesis g is a closed cofibration, thus we obtain from (5) together with the Opposite Map-Cofibration Proposition 22.23 (1) that μ is also a closed cofibration.
(10) Next we obtain the quadrilateral in the top center. By (1), (3), (6) and (9b) we can apply Lemma 22.26 and we obtain that the map β is a homotopy equivalence.
(11) By (7), (9a) and (10) we know that α, β and γ are homotopy equivalences. Since the diagram commutes we see that the bottom map $j\colon X \to X \cup_A B$ is indeed a homotopy equivalence. ■

Proof of Theorem 22.24. We are given the following commutative diagram:

$$\begin{array}{ccccc} X & \xhookleftarrow{i} & A & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_Y} \\ X' & \xhookleftarrow{i'} & A' & \xrightarrow{f'} & Y' \end{array}$$

where all of the vertical maps are homotopy equivalences and where the maps i and i' to the left are closed cofibrations. Recall that we need to show that the induced map

$$\Phi\colon X \cup_A Y \;\to\; X' \cup_{A'} Y'$$

is a homotopy equivalence.

Case 1. First we assume that f and f' are homotopy equivalences. We consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & X \cup_A Y \\ {\scriptstyle\simeq}\downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \Phi} \\ X' & \xrightarrow{u'} & X' \cup_{A'} Y' \end{array} .$$

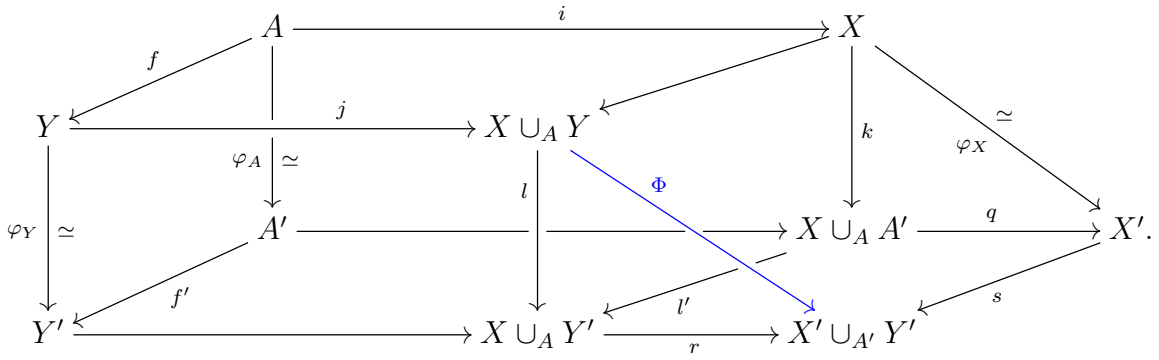
We make the following comments:

(1) The horizontal maps u and u' are the obvious natural maps coming from the following pushout diagrams

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{u} & X \cup_A Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A' & \xrightarrow[\simeq]{f'} & Y' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle v'} \\ X' & \xrightarrow{u'} & X' \cup_{A'} Y'. \end{array}$$

  Since f and f' are homotopy equivalences and since i and i' are closed cofibrations we obtain from the Opposite Map-Homotopy Equivalence Lemma 22.27, applied to these two pushout diagrams, that the horizontal maps u and u' are homotopy equivalences.
(2) The left vertical map of the diagram of the claim is a homotopy equivalence by hypothesis.
(3) It follows from (1) and (2), together with the Homotopy-Equivalence Basics Lemma 19.14 (1), that Φ is a homotopy equivalence.

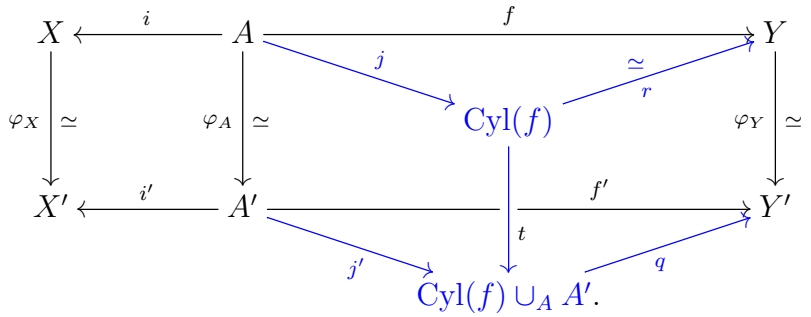
Case 2. Next we assume that f and f' are cofibrations. We consider the following diagram:



We make the following observations and clarifications:

(1) The top square and bottom left square are the obvious pushout diagrams.
(2) The vertical map $l\colon X \cup_A Y \to X \cup_A Y'$ is the map induced by φ_Y. The square in the front is a pushout diagram by the Pushout Composition Lemma 6.25.
(3) The square in the back is defined to be a pushout diagram.
(4) Since i is a closed cofibration and φ_A is a homotopy equivalence it follows from the Opposite Map-Homotopy Equivalence Lemma 22.27, applied to the pushout diagram in the back, that k is a homotopy equivalence.
(5) Since $i\colon A \to X$ is a closed cofibration we obtain, by applying the Opposite Map-Cofibration Proposition 22.23 to the pushout diagram on the top, that the corresponding "opposite map" $j\colon Y \to X \cup_A Y$ is a closed cofibration.
(6) Since j is a cofibration and since φ_Y is a homotopy equivalence we obtain again from the Opposite Map-Homotopy Equivalence Lemma 22.27, this time applied to the pushout diagram in front, that l is a homotopy equivalence.
(7) Since f' is a closed cofibration we obtain again from the Opposite Map-Cofibration Proposition 22.23 that the "opposite map" l' is a closed cofibration.
(8) We turn to the right part of the diagram. The map $q\colon X \cup_A A' \to X'$ is defined by φ_X on X and i' on A. Furthermore the map $r\colon X \cup_A Y' \to X' \cup_{A'} Y'$ is defined by φ_X on X' and id on Y'. Finally $s\colon X' \to X' \cup_{A'} Y'$ is the natural map. One can easily verify that the right hand part of the diagram commutes.
(9) Since φ_X and k are homotopy equivalences and since $q \circ k = \varphi_X$ we obtain from the Homotopy-Equivalence Basics Lemma 19.14 (1) that q is a homotopy equivalence.
(10) As we mentioned in (2), the bottom left square is a pushout square. Furthermore it follows immediately from the definitions that the big square formed by the two squares at the bottom is also a pushout square. It follows from a fairly elementary argument, see e.g. Exercise 6.17, that the bottom right square is also a pushout square.
(11) It follows from (7), (9) and (10) together with the hard-fought the Opposite Map-Homotopy Equivalence Lemma 22.27 that r is a homotopy equivalence.
(12) Since $\Phi = r \circ l$ we obtain from (6) and (11) that Φ is indeed a homotopy equivalence.

Case 3. Finally we deal with the case that f and f' are arbitrary maps. We consider the following diagram:



We make the following clarifications and observations:

(1) The black part of the diagram is precisely the commutative diagram of our setup.
(2) The top triangle is given by the usual factorization of the map $f\colon A \to Y$ into the natural inclusion $j\colon X \to \mathrm{Cyl}(f)$ and the natural retraction $r\colon \mathrm{Cyl}(f) \to Y$. By the Cone-Mapping Cylinder Inclusion Cofibration Proposition 22.6 we know that j is a closed cofibration. Furthermore, by the Mapping Cylinder–Homotopy Lemma 21.5 (2a) we know that r is a homotopy equivalence.
(3) The left parallelogram is defined to be the pushout of the maps $j\colon A \to \mathrm{Cyl}(f)$ and $\varphi_A\colon A \to A'$.
(4) Since j is a closed cofibration we obtain from (3) and the Opposite Map-Cofibration Proposition 22.23 that the map $j'\colon A' \to \mathrm{Cyl}(f) \cup_A A'$ is also a closed cofibration.
(5) Since φ_A is a homotopy equivalence and since j is a closed cofibration we obtain from the Opposite Map-Homotopy Equivalence Lemma 22.27 that t is a homotopy equivalence.
(6) The map $q\colon \mathrm{Cyl}(f) \cup_A A' \to Y'$ is defined via t on $\mathrm{Cyl}(f)$ and f' on A'.
(7) It follows easily from the definitions that the right parallelogram commutes, i.e. we have $q \circ t = \varphi_Y \circ r$. Since r, φ_Y and t are homotopy equivalences and since $q \circ t = \varphi_Y \circ r$ we obtain from the Homotopy-Equivalence Basics Lemma 19.14 (1) that q is a homotopy equivalence.

Next we consider the commutative diagram

$$\begin{array}{ccccc} X & \xhookleftarrow{i} & A & \xrightarrow{j} & \mathrm{Cyl}(f) \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle t} \\ X' & \xhookleftarrow{i'} & A' & \xrightarrow{j'} & \mathrm{Cyl}(f) \cup_A A' \end{array}$$

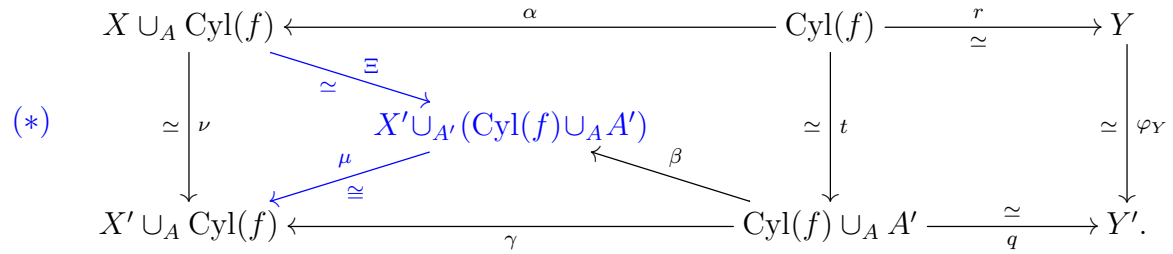
Here we just make a single observations:

(8) By (2) and (4) and by hypothesis we know that all horizontal maps are closed cofibrations. Furthermore by hypothesis and (5) we know that all vertical maps are homotopy equivalences. Thus we obtain from Case 2 that the induced map

$$\Xi\colon X \cup_A \mathrm{Cyl}(f) \;\to\; X' \cup_{A'} (\mathrm{Cyl}(f) \cup_A A')$$

is a homotopy equivalence.

We move on to consider the following diagram:



We make the following clarifications and observations:

(9) The lower left diagonal map μ is the natural homeomorphism that is provided by the Pushout Composition Lemma 6.25.[265]

(10) The horizontal map α at the top left and the diagonal map β are the natural maps coming from pushouts. The vertical map ν to the left is induced by φ_X and the horizontal map γ at the bottom left map is induced by i'. One can easily verify that the diagram commutes.

(11) Recall that by hypothesis $i\colon A \to X$ and $i'\colon A' \to X'$ are closed cofibrations. Thus we obtain from the Opposite Map-Cofibration Proposition 22.23, applied to the pushouts

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & X \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(f) & \xrightarrow{\ \alpha\ } & X \cup_A \mathrm{Cyl}(f) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A' & \xrightarrow{\ i'\ } & X' \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(f)\cup_A A' & \xrightarrow{\ \beta\ } & X'\cup_{A'} (\mathrm{Cyl}(f)\cup_A A') \end{array}$$

that the maps α and β are closed cofibrations. Since $\gamma = \mu \circ \beta$ and since μ is a homeomorphism we see that γ is also a closed cofibrations.

Finally we consider the following diagram:

$$\begin{array}{ccc} (X \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f)} Y & \longrightarrow & (X' \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f)\cup_A A'} Y' \\ \downarrow \cong & & \downarrow \cong \\ X \cup_A Y & \xrightarrow{\ \Phi\ } & X' \cup_{A'} Y'. \end{array}$$

As usual we cannot be refrained from making comments.

(12) The top horizontal map is induced by ν and φ_Y.

(13) By (5), (8), (9) and our hypothesis we know that all the vertical maps in the diagram (∗) are homotopy equivalences. Furthermore by (2) and (7) we know that the right horizontal maps of (∗) are homotopy equivalences. Thus we obtain from Case 1 that the top horizontal map in our present diagram is a homotopy equivalence.

(14) The vertical maps are the natural homeomorphism provided by the Pushout Composition Lemma 6.25.

(15) One can easily verify that the diagram commutes. Since the top horizontal map is a homotopy equivalence and since the vertical maps are homeomorphisms we see that

[265] More precisely we have the following: It follows immediately from the definitions that the pushout $C \cup_A B$ is homeomorphic to $B \cup_A C$. Furthermore the Pushout Composition Lemma 6.25 implies that the natural map $X \cup_A C \to X \cup_B (B \cup_A C)$ is a homeomorphism. These two homeomorphisms give us the desired result.

the bottom horizontal map Φ is also a homotopy equivalence. But that had been our goal since page 604. ■

This concludes our discussion of closed cofibrations. More information on closed cofibrations can be found in [**tD2008**, Chapter 5], [**BrownR2006**, Chapter 7], [**May1999a**, Chapter 6] and [**Pic1992**, Chapter 2.3].

22.8. Exercises for Chapter 22.

Exercise 22.1. Solve the following problems "by hand", i.e. without making use of any of above results. For example you should not use the Cofibration-Closed Injective Lemma 22.9.

(a) Let $A = [a, b]$ be a compact interval and let $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is a cofibration.
(b) Let $A = \mathbb{R} \setminus \{0\}$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.
(c) Let $A = (-1, 1)$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.

Exercise 22.2. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two (closed) cofibrations. Show that $g \circ f\colon X \to Z$ is a (closed) cofibration.

Exercise 22.3. We consider the two pullback diagrams

$$\begin{array}{ccc} & & \{0\} \\ & & \downarrow \\ \{1\} & \hookrightarrow & [0,1] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & \{*\} \\ & & \downarrow \\ \{*\} & \hookrightarrow & \{*\}. \end{array}$$

Note that we have obvious maps from the topological spaces to the left to the topological spaces on the left that are homotopy equivalences. Nonetheless show that the pullbacks are *not* homotopy equivalent. *Remark 1.* The solution to this conundrum is to consider the "homotopy pullback", which is the pullback in the homotopy category which we introduced in Definition 19.12.
Remark 2. More in the spirit of the Homotopy Pushout Theorem 22.24 one could summarize the pushout diagrams also as follows:

$$\begin{array}{ccccc} \{1\} & \hookrightarrow & [0,1] & \hookleftarrow & \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ \{*\} & \hookrightarrow & \{*\} & \hookleftarrow & \{*\}. \end{array}$$

Exercise 22.4. Prove the "(2)⇒(1)"-direction of the Cofibration Characterization Lemma 22.5.

Exercise 22.5. Let $X = (V, E, i, t)$ be an oriented abstract graph as introduced in Definition 11.1 and let A be a subgraph as in Definition 11.4. We denote by $|X|$ and $|A|$ the topological realizations of X and A as defined in Definition 11.5. Show that the corresponding inclusion map $i\colon |A| \to |X|$ is a closed cofibration.
Remark. This is a special case of the Subcomplex Cofibration Theorem 22.12.

Exercise 22.6. Give an example of a topological space X and a non-empty subset A such that the map $\mathrm{Cone}(i\colon A \to X) \to X/A$ given in the Cofibration-Quotient Lemma 22.10 (1) is not a homotopy equivalence.

Exercise 22.7. In the figure we see two subspaces X and Y of $\mathbb{R}^2$, namely the *dumbbell graph* X and the *wedge of two circles* Y. These are homotopy equivalent. By the Homotopy Equivalent–Characterization Corollary 22.21 this means that there exists a topological space Z that contains (homeomorphic) copies of X' and Y' of X and Y such that X' and Y' are both deformation retracts of Z. Provide an example of such Z.
Remark. This exercise is very similar to Exercise 19.11.



dumbbell graph wedge of two circles

Exercise 22.8. Let $X = \{x, y\}$ be a set with two elements and the trivial topology furthermore let $A = \{x\}$. Show that the inclusion $A \to X$ is a cofibration.
Remark. This is an example of a cofibration that is not closed.

Exercise 22.9. Let $X = \{a, b\}$ be the topological space with two elements a and b and where the open subsets are precisely $\varnothing, \{a\}, X$. We set $A \coloneqq \{a\}$.
(a) Show that the inclusion $A \to X$ is a cofibration.
(b) Show that the inclusion $(X \times A) \cup (A \times X) \to X \times X$ is not a cofibration.
Remark. This exercise shows that the conclusion of the Mixed Product–Cofibration Proposition 22.17 does not hold if neither A nor B are closed subsets.

Exercise 22.10. Let $i\colon A \to X$ be a closed cofibration. Show that the corresponding induced map $\mathrm{Cone}(A) \to \mathrm{Cone}(X)$ is also a cofibration.
Hint. Make sense of the following diagram and use it is as a guideline:

$$\begin{array}{ccccc}
A \times \{0\} & \longrightarrow & X \times \{0\} & \longrightarrow & \{*\} \\
\downarrow & & \downarrow & & \downarrow \\
A \times [0,1] & \longrightarrow & (X \times \{0\}) \cup_{A \times \{0\}} (A \times [0,1]) & \longrightarrow & \mathrm{Cone}(A) = \{*\} \cup_{A \times \{0\}} (A \times [0,1]) \\
 & & \uparrow & & \downarrow \\
 & & X \times [0,1] & \longrightarrow & \mathrm{Cone}(X).
\end{array}$$

Part IV

Topological and smooth manifolds

In this part of the lecture notes we introduce the concept of topological and smooth manifolds. We develop all the basic notions, like tangent spaces and orientations, and we will prove some of the foundational results like the Inverse Mapping Theorem 30.1, the Smooth Embedding Theorem 30.15, the Regular Value Theorem 32.5, the Euclidean Embedding Theorem 33.2 and Sard's Theorem 38.1. Throughout we will allow manifolds to have a boundary, which makes it technically more demanding to write down the correct statements and precise proofs.

This part of the lecture notes is organized as follows:

- In Chapter 23 we will introduce the notion of a topological manifold. An n-dimensional topological manifold is a Hausdorff space that locally is second countable and locally homeomorphic to an open subset of the upper half space of $\mathbb{R}^n$.
- In Chapter 24 we will recall concepts from real analysis which we will need in the study of smooth manifolds.
- In Chapter 25 we will a smooth manifold as a topological manifold together with a "smooth structure". This extra structure allows us to do analysis on smooth manifolds.
- In Chapter 26 we will introduce the rather technical, but unavoidable, topic of smooth manifolds with corner. As a topic on its own it is not very interesting, but they do naturally arise even in the study of the usual smooth manifolds.
- In Chapter 27 we will introduce (smooth) partitions of unity on topological (smooth) manifolds.
- In Chapter 28 we will classify topological and smooth manifolds of dimension one. We will also introduce the "long line", which satisfies all properties of a 1-dimensional topological manifold, except that it is not necessarily second countable.
- In Chapter 29 we first introduce the "physical" tangent space $\widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$ of a submanifold M of $\mathbb{R}^n$ at a given point $P \in M$. Afterwards we introduce the much more abstract notion of the tangent space $\mathrm{T}_P M$ at any point P on a smooth manifold M.
- In Chapter 30 we will prove the Inverse Mapping Theorem 30.1 and the Smooth Embedding Theorem 30.15. Both give us local behavior of smooth maps $f\colon M \to N$ just in terms of the behavior of the differentials.
- In Chapter 31 we will introduce the notion of the orientations of a smooth manifold. This notion extends the familiar notion for surfaces.
- In Chapter 32 we will prove the Regular Value Theorem 844 which says that the preimage of a regular value is a submanifold. We will use it in particular to show that various matrix groups are smooth manifolds.
- In Chapter 33 we will prove the Euclidean Embedding Theorem 33.2 which says that every smooth manifold is diffeomorphic to a smooth submanifold of some $\mathbb{R}^n$.
- In Chapter 34 we will introduce the notion of a Lie group. In a nutshell Lie groups are smooth manifolds that admit a compatible group structure.
- In Chapter 35 we first prove the Neighborhood Projection Theorem 35.2 which says that any submanifold $M \subseteq \mathbb{R}^n$ is a deformation retract of an open neighborhood of $M \subseteq \mathbb{R}^n$. Furthermore in the Smooth Collar Neighborhood Theorem 35.6 we will see that the boundary of every smooth manifold admits a "smooth collar". These play a technical but important role in the discussion of smooth manifolds.
- In Chapter 36 we will use the Neighborhood Projection Theorem 35.2 and the Smooth Collar Neighborhood Theorem 35.6 to prove the Whitney Approximation Theorem 36.1

which says that every continuous map between smooth manifolds is homotopic to a smooth map.

- In Chapter 37 we will introduce the "Subsets of measure zero" and we will show that the image of smooth map $f\colon M \to N$ with $\dim(M) < \dim(N)$ has measure zero. This result allows us to prove the Weak Whitney Embedding Theorem 37.9 which says that every m-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2m+1}$.
- In Chapter 38 we will use many of the results of Chapter 37 to prove Sard's Theorem 38.1 which says that given any smooth map $f\colon M \to N$ the set of singular values in N has measure zero. This theorem is one of the most fundamental results about smooth manifolds and will play an essential role in Part VI, where we discuss the topology of smooth manifolds in greater detail.

CHAPTER 23

Topological manifolds

In this chapter we introduce topological manifolds and study some basic properties. Afterwards, in the next chapter, we will introduce smooth manifolds, which for most applications are even more interesting.

This chapter is organized as follows:

- In Section 23.1 we will define topological manifolds and we will give a few basic examples.
- In Section 23.2 we will see that surfaces are 2-dimensional topological manifolds.
- In Section 23.3 will discuss basic properties of topological manifolds.
- In Section 23.4 we will discuss the boundary of topological manifolds.
- In Section 23.5 we will introduce the notion a submanifold of a topological manifold.
- In Section 23.6 we will provide exercises for Chapter 23.

In the next Chapter 24 we will recall some concepts from real analysis

23.1. **Topological manifolds: Definition and examples.** Before we can give the definition of topological manifolds we need to introduce the following notation:

Notation 23.1. Given $n \in \mathbb{N}_0$ we write

$$\begin{aligned} \mathbb{H}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\} \\ \partial\,\mathbb{H}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n = 0\}. \end{aligned}$$

We refer to $\mathbb{H}^n$ as the upper half-space in $\mathbb{R}^n$. We equip both sets with the subspace topology coming from $\mathbb{R}^n$. One can easily verify that $\partial\,\mathbb{H}^n$ is indeed the boundary of $\mathbb{H}^n$ viewed as a subspace of $\mathbb{R}^n$. Furthermore it is straightforward to show that the map $\mathbb{R}^{n-1} \to \partial\,\mathbb{H}^n$ given by $x \mapsto (x, 0)$ is a homeomorphism.

Now we can move on to the definition of a topological manifold.

Definition 23.2. Let X be a topological space.

(1) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and
   (i) V is an open subset of $\mathbb{R}^n$ or
   (ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\,\mathbb{H}^n$.
   In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii).

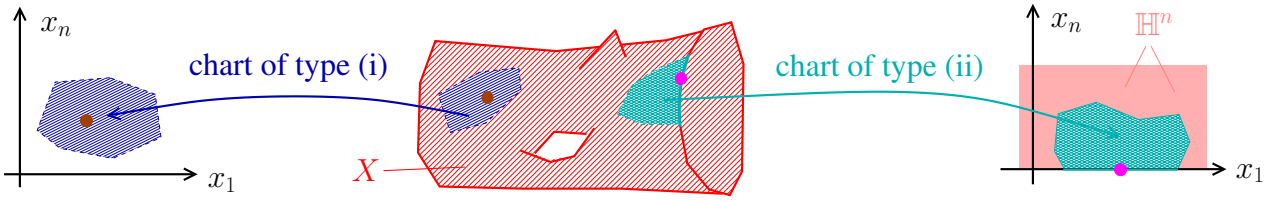
(2) We say X is an n-dimensional topological manifold, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.[266]

(3) An n-dimensional atlas for X is a collection of n-dimensional charts such that the domains of the charts cover all of X.

For topological manifolds we can furthermore define the notion of the boundary (points):

(4) We say that a point x on a topological manifold X is a boundary point if x does not admit a chart of type (i).[267] We denote by ∂X the set of all boundary points of X. We refer to ∂X as the boundary of X.

(5) We say that a topological manifold X is closed if X is compact with $\partial X = \varnothing$.[268]



Remark.

(1) In some references, see e.g. [**Hu1969, Aub2001**] and [**PoR2021**], a topological manifold is defined as above, except that the condition that X is second-countable is replaced by the condition that X is paracompact. We will see in the Manifold–Partitions-of-Unity Theorem 27.2 and the Paracompact–Second-Countable Proposition 27.8 that these two definitions of topological manifolds are equivalent. Furthermore in [**Wall2016**] the condition of being second-countable is replaced by the condition that X is the union of countably many compact subsets. We will see in the Topological Manifold Exhaustion Lemma 23.12 and the Second-Countable Criterion 27.9 that these two definitions of topological manifolds are also equivalent.

(2) Note that only in the Topological Manifold-Dimension Proposition 150.10 will we show that the dimension of a non-empty topological manifold is uniquely determined.

(3) Note that admitting a chart of type (i) is an open condition, which implies that the boundary of a topological manifold is a closed subset.

(4) It follows from the Countable Cover Lemma 13.4 that if a topological manifold admits an atlas $\{\Phi_i\}_{i\in I}$, then there exists a countable subset $J \subseteq I$ such that $\{\Phi_i\}_{i\in J}$ is also a cover. In particular we can always find a countable atlas. □

In the following we will give several examples of topological manifolds. When it comes to verifying that a topological space is a topological manifold the main focus is usually on establishing that any point admits a chart. It is easy to forget that one also needs to verify that the topological space is second-countable and Hausdorff. In many cases this task is taken care of by the following lemma:

Lemma 23.3. (Euclidean Subsets–Second Countable+Hausdorff Lemma) Let $n \in \mathbb{N}$. Every subset of $\mathbb{R}^n$ is second-countable and Hausdorff.

[265] The definition of a (topological) manifold goes back to the work of Hermann Weyl [**Wey1913, Wey1964**, Chapter I.4].

[266] It might sound more natural to define a boundary point to be a point that admits a chart of type (ii). In fact in the Unique Type-of-Chart Proposition 150.11 we will see that a point on a topological manifold admits *either* a chart of type (i) or a chart of type (ii). But the proof of this statement is not trivial and as of now we cannot prove this statement. As we will see in the subsequent examples our definition works fairly well, even without making use of the Unique Type-of-Chart Proposition 150.11.

[267] The latter condition just means that all points in X admit a chart of type (i).

Proof. Let A be a subset of $\mathbb{R}^n$. By the Second Countability Lemma 13.2 we know that A is second-countable. Furthermore, it follows from the Metric Space-Hausdorff Proposition 3.28 and the Subspace-Hausdorff Lemma 3.31 that A is also Hausdorff. ■

For the record we also state the following elementary lemma:

Lemma 23.4. (Countable Atlas-Second Countable Lemma) Let X be a topological space.

(1) If X admits a countable atlas, then X is second-countable.
(2) If X is compact and if it admits an atlas, then X is second-countable.

Proof.

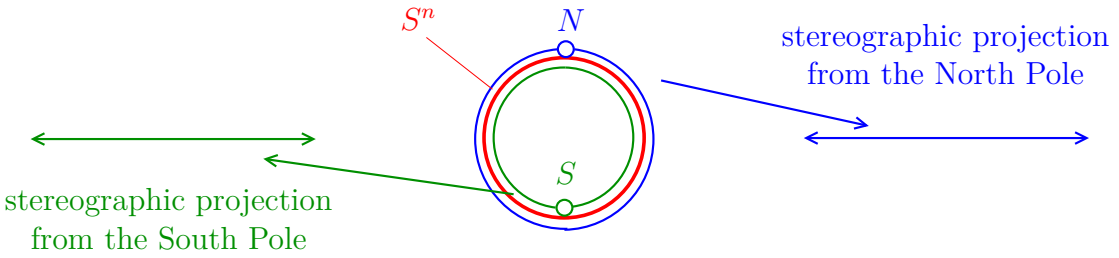
(1) Suppose that X admits a countable atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$. By the Euclidean Subsets–Second Countable+Hausdorff Lemma 23.3 we know that each V_i, and thus each U_i, is second-countable. It follows from the Second Countability Lemma 13.2 (4) that X itself is second-countable.
(2) Statement (2) in turn is an immediate consequence of Statement (1) since every compact space that admits an atlas also admits a finite atlas. ■

Now we turn to giving examples of topological manifolds.

Examples. Let $n \in \mathbb{N}$.

(1) We consider the sphere S^n. As usual we refer to $N := (0, \dots, 0, 1) \in S^n$ as the *North Pole* and similarly we refer to $S := (0, \dots, 0, -1) \in S^n$ as the *South Pole*. It follows quite easily from slight modifications of the arguments in the proof of the Stereographic Projection Lemma 4.24 that the stereographic projection $S^n \backslash \{N\} \to \mathbb{R}^n$ from the North Pole together with the stereographic projection $S^n \setminus \{S\} \to \mathbb{R}^n$ from the South Pole, as defined in the Stereographic Projection Lemma 4.24, are two n-dimensional charts of type (i) that cover all of S^n. By the Euclidean Subsets–Second Countable+Hausdorff Lemma 23.3 this shows that $S^n \subseteq \mathbb{R}^{n+1}$ is an n-dimensional topological manifold.

Note that both charts are of type (i), thus we see that the boundary of the topological manifold S^n is the empty set.[268] Finally note that S^n is compact, thus S^n is a closed n-dimensional topological manifold.

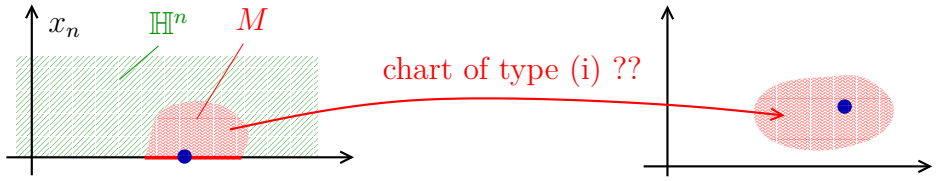


[268] The vigilant reader will have noticed that we have committed a major sin: we introduced an object called "boundary" twice, with quite different meanings:
(1) In Definition 3.21 we introduced the boundary ∂A of a subset A of a topological space X.
(2) In Definition 23.2 we introduced the boundary ∂M of a topological manifold M.
Usually it is clear from the context what we mean when we write ∂. This sorry state of affairs explain why some authors, e.g. [**Kel1975**], refer to the boundary of a subset of a topological space as the *frontier*.

(2) Every open subset U of $\mathbb{R}^n$ is an n-dimensional topological manifold. This follows again from the Euclidean Subsets–Second Countable+Hausdorff Lemma 23.3 and the observation that the identity map is a chart of type (i) for all points. In particular we see that the boundary of U is empty.

(3) Let M be an open subset of the upper half-space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$. Again, using the Euclidean Subsets–Second Countable+Hausdorff Lemma 23.3, it is straightforward to see that M is an n-dimensional topological manifold:
   (a) Given any point $x = (x_1, \ldots, x_n) \in M$ with $x_n > 0$ the identity map on the open subset $U \coloneqq \{(y_1, \ldots, y_n) \in M \,|\, y_n > 0\}$ is a chart of type (i) around x.
   (b) Given any point $(x_1, \ldots, x_n) \in M$ with $x_n = 0$ the identity map $M \to M$ is a chart of type (ii) around x.

   In this example we run into a major, and rather painful, subtlety. Namely one would think that the boundary of M as a topological manifold should be precisely the points in M with $x_n = 0$. But at the moment we cannot draw this conclusion. After all, in principle, such points also might admit charts of type (i). Only after having introduced and studied homology groups we will see in the Unique Type-of-Chart Proposition 150.11 that no such charts of type (i) exist.



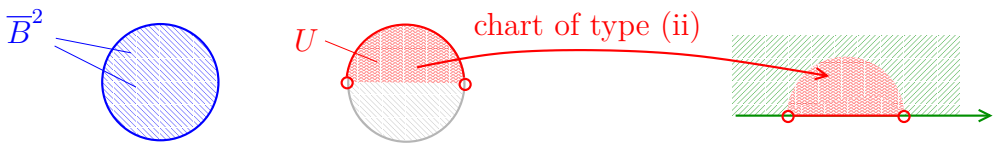
(4) Next we want to show that the closed n-ball $\overline{B}^n$ is an n-dimensional topological manifold. We provide the following charts:
   (a) We consider the chart of type (i) that is given by the identity on the open subset B^n of $\overline{B}^n$.
   (b) We set $U \coloneqq \{(x_1, \ldots, x_n) \in \overline{B}^n \,|\, x_n > 0\}$ and we consider the map

$$\begin{array}{rcl} U & \to & \mathbb{H}^n \\ (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_{n-1}, \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2} - x_n). \end{array}$$

   This is a chart of type (ii) for all points in $\{(x_1, \ldots, x_n) \in S^{n-1} \,|\, x_n > 0\}$. We refer to the figure below for an illustration of this chart.
   (c) We consider the obvious variations of the previous chart for $x_n < 0$ and for other coordinates.

   These charts form an atlas for $\overline{B}^n$, which together with the Euclidean Subsets–Second Countable+Hausdorff Lemma 23.3 shows that $\overline{B}^n$ is an n-dimensional topological manifold. We have found a chart of type (i) for each point in B^n and for the points on S^{n-1} we have found charts of type (ii). This shows that $\partial\overline{B}^n \subseteq S^{n-1}$. Furthermore we have the strong suspicion that $\partial\overline{B}^n = S^{n-1}$ but as in (3), as of right now we cannot confirm this. We will only be able to confirm our hunch by the time we have proved the Unique Type-of-Chart Proposition 150.11.

(5) Given $a < b$ in $\mathbb{R}$ the interval $[a, b]$ is easily seen to be a 1-dimensional topological manifold. Every point in (a, b) admits a chart of type (i). It follows from the Topological Valence–Basics Lemma 11.34 together with the Topological Valence Determination



Proposition 11.35 that the two points $a, b \in [a, b]$ do *not* admit charts of type (i).[269] Thus the boundary of $[a, b]$ consists, as expected, of the two points a and b. Similarly the boundary of the half-open interval $[a, b)$ equals $\{a\}$.

(6) In Exercise 23.1 we will see that a topological space X is a 0-dimensional topological manifold if and only if X is countable and X is equipped with the discrete topology. It follows from the Compact-Discrete Lemma 3.40 that a compact 0-dimensional topological manifold is precisely a finite set equipped with the discrete topology.

(7) We conclude this list of examples with the simplest of all examples: the empty set is a topological manifold of any dimension. □

It is also worth looking at a few non-examples.

Non-Examples.

(1) In Exercise 23.2 we will verify that the line with two zeros admits a chart of type (i) for every point. It is also easy to see that the line with two zeros is second-countable. But since the line with two zeros is not Hausdorff we see that it is not a topological manifold.

(2) We consider the topological space X that is given by the set $X = \mathbb{R}$ but for once we equip $X = \mathbb{R}$ with the discrete topology. Every point of X admits a 0-dimensional chart and X is obviously Hausdorff. But X is not second-countable, thus X is not a topological manifold.

(3) In Section 28.4 we will see that there exists a *connected* Hausdorff space which admits a 1-dimensional atlas, but which is not second-countable. In particular this topological space is not a topological manifold. □

The following lemma will get used, at least implicitly, on numerous occasions.

Lemma 23.5. (Chart-to-Balls Lemma) Let X be an n-dimensional topological manifold, let $P \in X$ and let N be a neighborhood of P.

(1) If $P \in X \setminus \partial X$, then there exists a chart $\Phi\colon U \to B^n$ with $\Phi(P) = 0$ and with $U \subseteq N$.

(2) If $P \in \partial X$, then there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ with $\Phi(P) = 0$ and with $U \subseteq N$.[270]

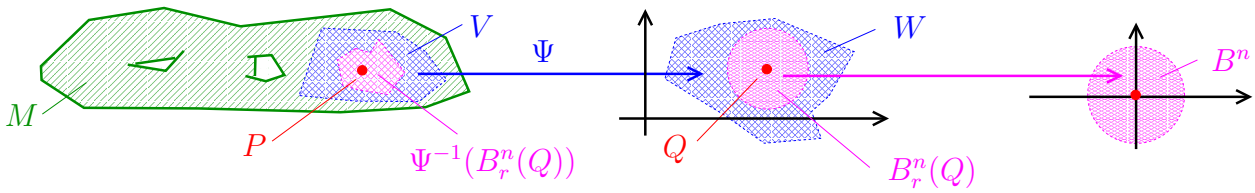
Proof.

(1) If $P \in X \setminus \partial X$, then by definition of ∂X there exists a chart $\Psi\colon V \to W$ for the point P of type (i). After possibly replacing Ψ by $V \cap \mathring{N} \to W \cap \Psi(\mathring{N})$ we can assume that $V \subseteq \mathring{N} \subseteq N$. We write $Q = \Psi(P)$. Note that W is an open subset of $\mathbb{R}^n$. This implies that there exists an $r > 0$ with $B^n_r(Q) \subseteq W$. The desired chart is given by composing $\Psi\colon \Psi^{-1}(B^n_r(Q)) \to B^n_r(Q)$ with the obvious homeomorphism $B^n_r(Q) \to B^n$.

(2) The proof of this statement is basically identical to the proof of (1). ■

[269] One could also prove this fact directly by hand, see Exercise 4.48.

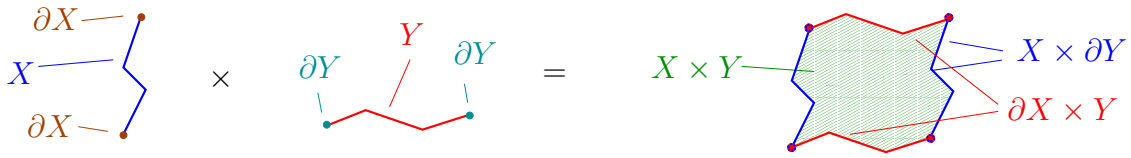
[270] Here we use the notation introduced in Notation 4.21, namely $B^n_{\geq 0} = \{(x_1, \dots, x_n) \in B^n \mid x_n \geq 0\}$.



The following proposition gives a useful method for constructing more topological manifolds.

Proposition 23.6. (Topological Manifold Product Proposition) Let X be an m-dimensional topological manifold and let Y be an n-dimensional topological manifold. Then the following hold:

(1) The product $X \times Y$ is an $(m+n)$-dimensional topological manifold.
(2) (a) Each point in $(X \times Y) \setminus ((\partial X \times Y) \cup (X \times \partial Y))$ admits a chart of type (i).
    (b) Each point in $(\partial X \times Y) \cup (X \times \partial Y)$ admits a chart of type (ii).
    (c) We have[271] $\partial(X \times Y) \subseteq (\partial X \times Y) \cup (X \times \partial Y)$.
(3) If X and Y are closed topological manifolds, then so is $X \times Y$.



Examples.

(1) It follows from the Topological Manifold Product Proposition 23.6 and the above discussion of S^1 that for any $m \in \mathbb{N}$ the m-dimensional torus $(S^1)^m$ is a closed m-dimensional topological manifold.
(2) It follows immediately from the Topological Manifold Product Proposition 23.6 and the above examples that the annulus $S^1 \times [0,1]$ is a 2-dimensional topological manifold with $\partial(S^1 \times [0,1]) \subseteq S^1 \times \{0,1\}$.
(3) Let $X = \overline{B}^2$ and $Y = [0,1]$, then $X \times Y = \overline{B}^2 \times [0,1]$ is the solid cylinder which is a 3-dimensional topological manifold and the boundary satisfies

$$\partial\big(\overline{B}^2 \times [0,1]\big) \underset{\substack{\uparrow \\ \text{in the Unique Type-of-Chart Proposition 150.11 we will see that equality holds}}}{\subseteq} (\partial\overline{B}^2 \times [0,1]) \cup (\overline{B}^2 \times \partial([0,1])) \underset{\substack{\uparrow \\ \text{since } \partial\overline{B}^2 \subseteq S^1}}{\subseteq} (S^1 \times [0,1]) \cup (\overline{B}^2 \times \{0,1\}).$$

□

Proof.

(1) Let X be an m-dimensional topological manifold and let Y be an n-dimensional topological manifold. We consider the topological space $X \times Y$. In the Product Topology-Properties Proposition 7.9 we remarked that the product of two Hausdorff spaces is

[271] Note that right now we can only prove the inclusion "$\subseteq$". Later on in the Unique Type-of-Chart Proposition 150.11 we will see that in fact we have an equality.

again Hausdorff. Thus we see that $X \times Y$ is Hausdorff. Furthermore note that it follows from the Second Countability Lemma 13.2 (3) that $X \times Y$ is second-countable.

It remains to find charts for $X \times Y$. So let $(x, y) \in X \times Y$. We distinguish three cases:

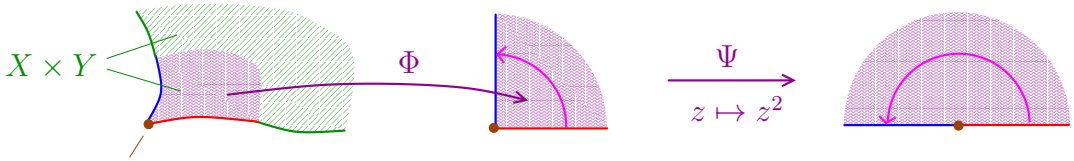
(a) If $x \in X \setminus \partial X$ and $y \in Y \setminus \partial Y$ then by taking "product charts" it is straightforward to find a chart of type (i) for $(x, y) \in X \times Y$.

(b) If $x \in X \setminus \partial X$ and $y \in \partial Y$ or if $x \in \partial X$ and $y \in Y \setminus \partial Y$ then it is equally straightforward to find a product chart of type (ii) for $(x, y) \in X \times Y$.

(c) Finally suppose that $x \in \partial X$ and $y \in \partial Y$. Our goal is to find a chart of type (ii) for $(x, y) \in X \times Y$. First note that using a product chart it is pretty straightforward to see that there exists an open neighborhood U of $(x, y) \in X \times Y$ and a homeomorphism $\Phi \colon U \to \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Next we consider the map

$$\begin{array}{rcl} \Psi \colon \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^{m+n-2} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \\ (x_1, \dots, x_{m+n-2}, \underbrace{r\cos(\varphi), r\sin(\varphi)}_{=:z \in \mathbb{C}}) & \mapsto & (x_1, \dots, x_{m+n-2}, \underbrace{r^2 \cdot \cos(2\varphi), r^2 \cdot \sin(2\varphi)}_{=z^2}). \end{array}$$

Note that the map $\mathbb{C} \to \mathbb{C}$ that is given by $z = x + \mathrm{i}y \mapsto z^2 = (x^2 - y^2) + 2xy\mathrm{i}$ is continuous. It follows that Ψ is continuous. It is straightforward to show, say using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26, that Ψ is indeed a homeomorphism. One can now easily verify that $\Psi \circ \Phi$ is a chart of type (ii) around (x, y).



(2) We proved Statements (a) and (b) in (1). Furthermore Statement (c) is an immediate consequence of (b) and the definition of the boundary of a topological manifold.

(3) Now suppose that X and Y are closed. By (2) we know that $\partial(X \times Y) = \varnothing$ and from the Product Topology-Properties Proposition 7.9 (1) we obtain that $X \times Y$ is compact. Thus we have shown that $X \times Y$ is a closed topological manifold. ■

23.2. **Surfaces are topological manifolds.** In 342 and Definition 9.19 we introduced the orientable surface of genus $g \in \mathbb{N}_0$ and the non-orientable surface of genus $g \in \mathbb{N}$. The names already suggests that these should be 2-dimensional topological manifolds, and the following proposition confirms our hunch.

Proposition 23.7. (Surface-is-Topological Manifold Proposition)

(1) For every $g \in \mathbb{N}_0$ the surface Σ_g of genus g is a closed 2-dimensional topological manifold.

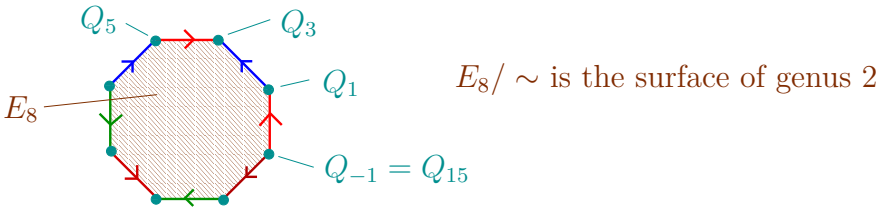
(2) For every $g \in \mathbb{N}$ the non-orientable surface N_g of genus g is a closed 2-dimensional topological manifold.

Proof. To simplify the discussion we will only prove Statement (1) of the proposition. We leave it to the reader to modify the argument to deal with the non-orientable case.



First recall that on page 622 we saw that any S^n is a closed n-dimensional topological manifold. The surface of genus 0 is by definition the sphere S^2, thus we are done by the above.

It remains to deal with the case $g \geq 1$. To simplify the notation we only deal with the case $g = 2$.[272] The proof of the general case is totally analogous. Recall that the surface of genus 2 is defined as $E_8/\sim$ where E_8 is the octagon with the vertices $Q_k = \exp(2\pi \mathrm{i} k/16)$, $k = 1, 3, \ldots, 15$ and where "$\sim$" is the equivalence relation that we introduced on page 342.



Note that E_8 is compact by the Heine-Borel Theorem 3.44. Thus we obtain from the Quotient Topology-Properties Proposition 8.5 (1) that $E_8/\sim$ is also compact. In the following we will see that $E_8/\sim$ admits a finite atlas, by the Countable Atlas-Second Countable Lemma 23.4 this implies in particular that $E_8/\sim$ is second-countable.[273] Note that E_8 is compact by the Heine-Borel Theorem 3.44. Thus we obtain from the Compact Image Lemma 4.15 that $E_8/\sim$ is also compact. We leave it to the reader to verify that $E_8/\sim$ is Hausdorff.[274]

Thus, to show that $E_8/\sim$ is a closed 2-dimensional topological manifold it remains to give a finite atlas for $E_8/\sim$ that consists only of charts of type (i).

First we consider the map

$$\begin{array}{rccc} \Phi\colon U := p\big(\mathring{E}_8\big) & \to & V = \mathring{E}_8 & \subseteq \mathbb{R}^2 \\ p(x) & \mapsto & x. & \end{array}$$

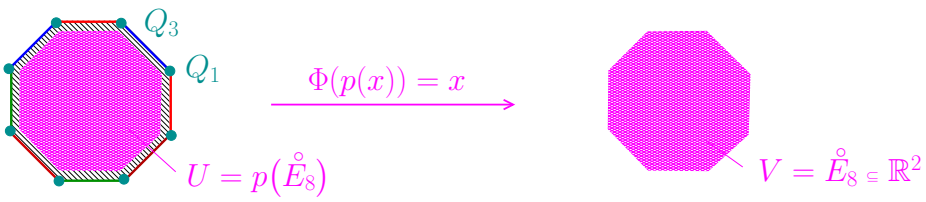
One can easily verify that this is a chart of type (i) for each point in the interior of E_8.

Next we introduce charts that cover the interiors of the edges. First we consider the edge given by Q_{-1} and Q_1. We denote by U and U' the half-disks that are illustrated in the figure below. (Note that it is understood that the Euclidean segment is included but that the circle is not included in U and U'.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line that

[272] The case $g = 1$ can also be treated as follows: On page 311 we saw that the surface of genus 1 is homeomorphic to the torus $S^1 \times S^1$. It follows from the Topological Manifold Product Proposition 23.6 that $S^1 \times S^1$ is a closed 2-dimensional topological manifold.

[273] It would also not be difficult to prove second-countability "by hand".

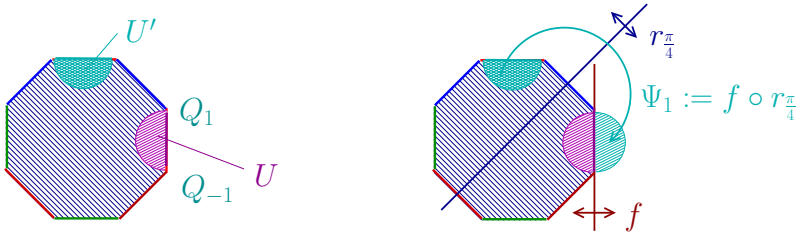
[274] This can either be done "by hand", or more elegantly but possibly with more work, by using the Quotient Topology-Properties Proposition 8.5 (3).



goes through Q_{-1} and Q_1. It is straightforward to verify that the map

$$\begin{array}{rcl} \Psi_1 \colon p(U \cup U') & \to & \text{the open disk } U \cup f(r_{\frac{\pi}{4}}(U)) \subseteq \mathbb{R}^2 \\ p(x) & \mapsto & \begin{cases} x, & \text{if } x \in U \\ f(r_{\frac{\pi}{4}}(x)), & \text{if } x \in U' \end{cases} \end{array}$$

is well-defined. Furthermore, using the Pasting Proposition 4.7 (2) one can show that Ψ_1 and its inverse are both continuous. Finally, it follows easily from the definitions that $p(U \cup U')$ is an open subset of $E_8/\sim$. Thus Ψ_1 is a chart of type (i) for all points in the interior of the edge from Q_{-1} to Q_1. Similarly we define charts Ψ_2, Ψ_3, Ψ_4 that take care of the other pairs of edges.



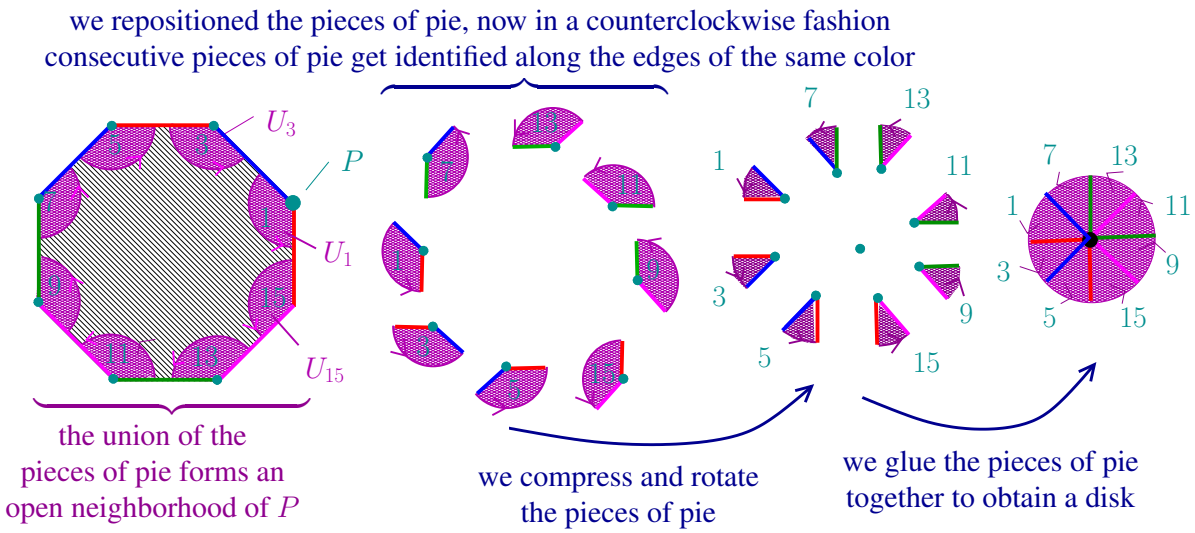
It remains to find a chart of type (i) that takes care of the point P in $E_8/\sim$ that is represented by the vertices. (It is a fun exercise to verify that all vertices get identified.) We pick an $\eta > 0$ such that the η-disks centered at $Q_1, \ldots, Q_{15}$ are disjoint. For $k = 1, 3, \ldots, 15$ we set $U_k := B_\eta(Q_k) \cap E_8$. The sets $U_1, U_3, \ldots, U_{15}$ are disjoint and one can easily verify that $p(U_1) \cup p(U_3) \cup \cdots \cup p(U_{15})$ is an open neighborhood of $p(Q_1) = p(Q_3) = \cdots = p(Q_{15})$.

Before we write down the chart rigorously it might help to visualize the situation. To the left of the figure below we see the sets $U_1, \ldots, U_{15}$. Next we arrange these "pieces of pie" as in the second part of the above. Thus we see the "pieces of pie" with an angle of $\frac{3\pi}{4}$ where we show the various identifications of the edges. The map from the second to the third part is given by compressing the angle of the "pieces of pie" to a third and by rotating the "pieces of pie". These eight "pieces of pie", each with an angle of $\frac{\pi}{4}$, can be combined to give a disk in $\mathbb{R}^2$.
Now we want to turn the above idea into formal mathematics. We consider the map

$$\Omega \colon p(U_1) \cup \cdots \cup p(U_{15}) \to B^2_\eta(0)$$

that is defined on each $p(U_k)$, $k \in \{1, 3, \ldots, 15\}$ to be the map

$$p\big(\underbrace{Q_k + r \cdot \exp(\mathrm{i}\alpha)}_{\substack{\text{where } r \in [0,\eta) \text{ and} \\ \alpha \in [\beta(k), \beta(k) + \frac{3}{4}\pi]}} \big) \mapsto r \cdot \exp(\mathrm{i}(\tfrac{1}{3}(\alpha - \beta(k)) + \gamma(k))).$$



Here $\beta(k)$ and $\gamma(k)$ are given by the following table:

k	1	3	5	7	9	11	13	15
$\beta(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$

It is straightforward, albeit slightly cumbersome, to verify that Ω is a bijection. Furthermore, using Exercise 4.75 together with the Pasting Proposition 4.7 (2) one can show fairly easily that Ω is continuous and that its inverse is also continuous. In other words, the map Ω is a homeomorphism. As pointed out above, it is straightforward to verify that $p(U_1) \cup \cdots \cup p(U_{15})$ is an open subset of $E_8/\sim$. In summary, we have shown that Ω is a chart of type (i) for P.

The six charts $\Phi, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ and Ω are now charts of type (i) that cover all of $E_8/\sim$. We have thus found the desired atlas. ■

23.3. **Properties of topological manifolds.** Topological manifolds are particularly important and also particularly nice topological spaces. For example the next lemma shows in particular that when it comes to topological manifolds, we do not have to worry about the two different definitions of connectedness.

Lemma 23.8. (Manifold–Local Properties Lemma)

(1) Every n-dimensional topological manifold is locally homeomorphic to an open ball in $\mathbb{R}^n$ or an open half-ball in $\mathbb{H}^n$, in particular it is locally homeomorphic to a convex subset of $\mathbb{R}^n$ or $\mathbb{H}^n$.

(2) Every topological manifold is locally contractible and in particular locally path-connected.

(3) (a) The components of a topological manifold M are precisely the path-components of M. In particular M is connected if and only if M is path-connected.[275]

(b) The components of a topological manifold M are open and closed subsets of M. In particular, if M is compact, then M has only finitely many components.

(c) If M is a topological manifold with components $\{M_i\}_{i\in I}$, then $\partial M = \bigcup_{i\in I} \partial M_i$.
(4) Every topological manifold is regionally compact.
(5) Every topological manifold is locally precompact.

Remark. Throughout the lecture notes we will on many occasions make use of the Manifold–Local Properties Lemma 23.8 (3) without referring to it explicitly. □

Proof. Let X be an n-dimensional topological manifold.

(1) We proved this statement already in the Chart-to-Balls Lemma 23.5.
(2) On page 531 we saw that convex sets are contractible. Thus it follows from (1) that topological manifolds are locally contractible. In particular we see that topological manifolds are locally path-connected.
(3) (a) This follows from (2) and the Locally Path-Connected Lemma 4.57 (1).
(b) The first part follows from (2) and the Locally Path-Connected Lemma 4.57 (3). The "in particular" statement follows now immediately from the definition of compactness.
(c) First note that it follows from (b) any chart of a component is a chart for the total topological manifold. The desired statement can easily be deduced from this observation.
(4) On page 208 we saw that open subsets of $\mathbb{R}^n$ are regionally compact. The same argument shows that open subsets of $\mathbb{H}^n$ are regionally compact. Using the Compact Image Lemma 4.15 it is now easy to see that topological manifolds are regionally compact.
(5) Since topological manifolds are by definition Hausdorff this statement follows immediately from (4) together with the Regionally Compact–Locally Precompact Lemma 4.60. ■

The humble Manifold–Local Properties Lemma 23.8, together with previous work, begets several interesting consequences.

Proposition 23.9. (Manifold Paracompact-Normal Proposition)
(1) Every topological manifold is paracompact.
(2) Every topological manifold X is normal, i.e. given any two disjoint closed subsets A and B of X there exist disjoint open neighborhoods U of A and V of B.

Proposition 23.10. (Manifold Metrization Proposition) Every topological manifold X is completely metrizable, i.e. there exists a complete metric d on X such that the given topology agrees with the topology coming from d.

Remark.
(1) In the Atlas Improvement Proposition 27.5 we will give an alternative proof that topological manifolds are paracompact.
(2) In the Manifold Metrization Proposition 33.5 we will give a different proof of the Manifold Metrization Proposition 23.10 which does not require the Urysohn Metrization Theorem 13.8. □

Proof of the Manifold Paracompact-Normal Proposition 23.9. Let X be a topological manifold.

[275] Very frequently we will use the fact that topological manifold is connected if and only if it is path-connected. At times we will not explicitly mention that secretly we are using this lemma.

(1) By the Manifold–Local Properties Lemma 23.8 we know that X is regionally compact. By definition X is Hausdorff and second-countable. Thus it follows from the Paracompact-Criterion Lemma 14.5 that X is paracompact.
(2) By definition topological manifolds are Hausdorff. We just saw in (1) that topological manifolds are paracompact. Therefore it follows from Dieudonné's Theorem 14.9 that topological manifolds are also normal. ■

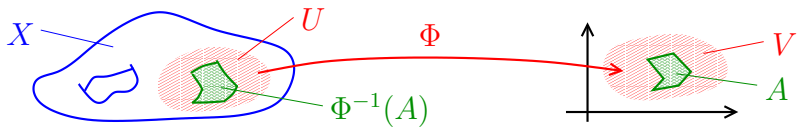
Proof of the Manifold Metrization Proposition 23.10. Let X be a topological manifold. By the Manifold Paracompact-Normal Proposition 23.9 (2) we know that X is normal. By definition X is second-countable and Hausdorff. The latter condition implies by the Points-in-Hausdorff Lemma 3.32 (1) that every one-point subset of X is closed. This shows that all the hypotheses of the Urysohn Metrization Theorem 13.8 are satisfied and we conclude that X is metrizable.

Furthermore, in the Manifold–Local Properties Lemma 23.8 we just saw that X is also regionally compact. Thus it follows from the Completely Metrizable Criterion 14.13 that we can upgrade "metrizable" to "completely metrizable". ■

On several occasions we will need the following, slightly technical lemma.

Lemma 23.11. (Compactness-Chart Lemma) Let X be a Hausdorff space (e.g. X could be a topological manifold), let U be an open subset of X and let $\Phi\colon U \to V$ be a homeomorphism (e.g. Φ could be a chart of type (i) or (ii).)

(1) If $A \subseteq V$ is a subset that is compact, then the corresponding subset $\Phi^{-1}(A)$ is a compact and in particular a closed subset of X.[276]
(2) If $B \subseteq V$ is a subset such that the closure of B is compact, then the closure of $\Phi^{-1}(B)$ is also compact.



Proof. Let X be a Hausdorff space, let U be an open subset of X and let $\Phi\colon U \to V$ be a homeomorphism. We denote by $\Psi = \Phi^{-1}\colon V \to U$ the inverse map.

(1) Let $A \subseteq V$ be a subset that is compact. It follows from the Compact Image Lemma 4.15 that $\Psi(A)$ is a compact subset of X. But by hypothesis X is Hausdorff. Hence it follows from the Compact-Closed Lemma 3.39 (2) that $\Psi(A) = \Phi^{-1}(A)$ is closed in X.
(2) Let $B \subseteq V$ be a subset such that the closure $A := \overline{B}$ is compact. By (1) we know that $\Phi^{-1}(A) = \Psi(A)$ is compact and closed. Since $\Phi^{-1}(B) \subseteq \Phi^{-1}(A)$ and the latter set is closed we see that the closure of $\Phi^{-1}(B)$ is contained in the compact set $\Phi^{-1}(A)$. But then it follows from the Compact-Closed Lemma 3.39 that the closure of $\Phi^{-1}(B)$ is also compact. ■

We continue with another technical lemma.

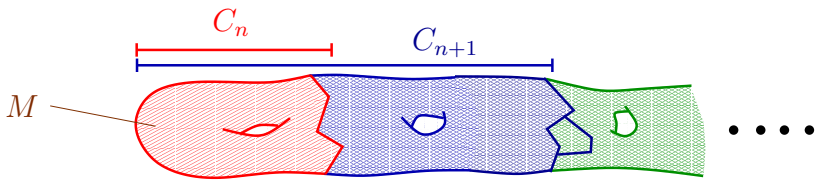
[276] It does not suffice that A is a closed subset of the topological space V. Indeed, $A = V$ is a closed subset of V, but $U = \Phi^{-1}(V)$ is in general not a closed subset of X.

Lemma 23.12. (Topological Manifold Exhaustion Lemma) Let M be a topological manifold. There exists a compact exhaustion of M in the sense of Definition 13.5. More precisely, there exists a family of compact subsets $\{C_n\}_{n\in\mathbb{N}}$ of M such that the following two conditions are satisfied:

(1) we have $M = \bigcup_{n\in\mathbb{N}} C_n$,

(2) for every $n \in \mathbb{N}$ we have $C_n \subseteq \mathring{C}_{n+1}$.[277]

Furthermore, if M is connected, then we can arrange that all the C_n are path-connected.



Proof. The first part of the lemma follows immediately from the Compact Exhaustion Lemma 13.6 (1) together with the Manifold–Local Properties Lemma 23.8 (4).

Finally we assume that M is connected. If one looks at the fairly explicit construction of the sequence C_n one sees that one can choose each C_n to be the union of finitely many path-connected subspaces. Since M is connected we know, by the Manifold–Local Properties Lemma 23.8 (3a), that M is also path-connected. So after after adding the image of finitely many paths we see (using the Union Connected Lemma 4.39) that each C_n is contained in a path-connected compact subspace C'_n of M. By (2) we know that the $\mathring{C}_n$ form an open cover of M. It now not hard to see that a subsequence of the C'_n has all the desired properties. ■

Remark. We will prove a refinement of the Topological Manifold Exhaustion Lemma 23.12 in the Manifold Exhaustion Lemma 27.4. For smooth manifolds we will prove a further refinement in the Compact Submanifold Exhaustion Proposition 38.2. □

23.4. **Boundary of topological manifolds.** In this section we consider boundaries of topological manifolds. As we will see, we are in a weird position. We can prove all kinds of properties of boundaries of topological manifolds, but we do lack the tools right now to show that the boundary is ever non-empty:

Proposition 23.13. (Unique Type-of-Chart Proposition) Let $n \in \mathbb{N}_0$. Precisely one of the following two statements holds:[278]

(α) Every point on every n-dimensional topological manifold admits either a chart of type (i) or it admits a chart of type (ii).

(β) The boundary of every n-dimensional topological manifold is empty.

Our proof of the Unique Type-of-Chart Proposition 23.13 relies on the following little lemma:

[277] Note that this implies that the interiors of the C_n form an open cover of M.

[278] Later, once we have introduced and studied homology groups we will show in the Unique Type-of-Chart Proposition 150.11 (i) that (α) holds.

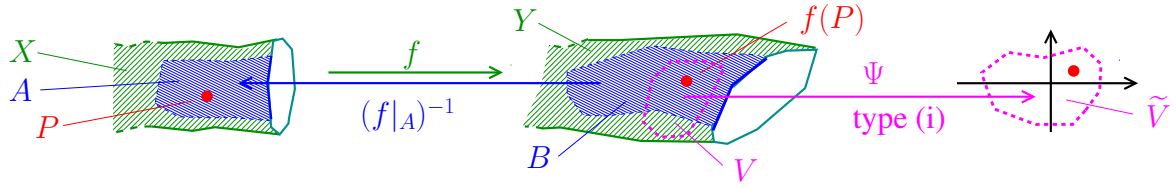
Lemma 23.14. (Boundary–Local Homeomorphism Lemma) Let X and Y be topological manifolds. If $f\colon X \to Y$ is a local homeomorphism, then $f(\partial X) = \partial Y \cap f(X)$.

Proof of Lemma 23.14. Let $f\colon X \to Y$ be a local homeomorphism. We need to prove the following claim:

Claim. Let $P \in X$. The following two statements hold:
(a) $\quad P \in \partial X \implies f(P) \in \partial Y$.
(b) $\quad f(P) \in \partial Y \implies P \in \partial X$.

Proof. Let $P \in X$. Since f is a local homeomorphism there exists an open neighborhood A of P such that $B := f(A)$ is an open neighborhood of $f(P)$ and such that $f\colon A \to B$ is a homeomorphism. We obtain in particular the inverse $(f|_A)^{-1}\colon B \to A$. In both of the above statements (a) and (b) we prove the contrapositive:

(a) We assume that $f(P) \in Y \setminus \partial Y$. This means that there exists a chart $\Psi\colon V \to \widetilde{V}$ of type (i) around $f(P)$. Note that $\Psi \circ f\colon A \cap f^{-1}(V) \to \Psi(B \cap V)$ is a chart of type (i) for P. Thus we have shown that $P \in X \setminus \partial X$.

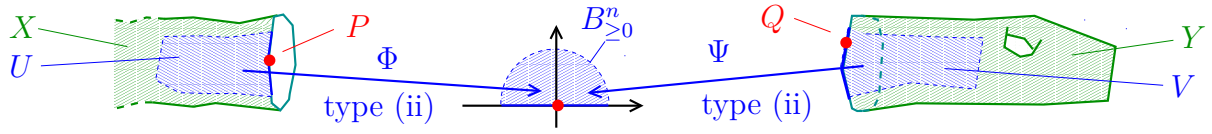


(b) We assume that $P \in X \setminus \partial X$. This means that there exists a chart $\Phi\colon U \to \widetilde{U}$ of type (i) around P. Note that $(\Phi \circ (f|_A)^{-1})\colon B \cap f(U) \to \Phi(A \cap U)$ is a chart of type (i) for $f(P)$. Thus we have shown that $f(P) \in Y \setminus \partial Y$. ∎

Proof of the Unique Type-of-Chart Proposition 23.13. Let $n \in \mathbb{N}_0$. We need to show that precisely one of the two statements (α) or (β) holds. First let us assume that (β) does not hold. We need to show that (α) holds.

Since we assume that (β) does not hold we know, by the definition of the boundary of a topological manifold in Definition 23.2, that there exists a topological manifold X and $P \in X$ such that P does not admit a chart of type (i). Since X is a topological manifold and P does not admit a chart of type (i), there does exist a chart $\Phi\colon U \to U'$ of type (ii). In fact after going to a suitable neighborhood of $\Phi(P)$ and using a translation we can arrange that $U' = B^n_{\geq 0} = \overline{B}^n \cap \mathbb{H}^n$ and that $\Phi(P) = 0$.

Now let Y be an n-dimensional topological manifold and let $Q \in Y$. We need to show that Q admits either a chart of type (i) or it admits a chart of type (ii). By definition of a topological manifold we know that Q admits a chart. Thus it suffices to show that if Q admits a chart of type (ii), then it does not admit a chart of type (i). If Q admits a chart of type (ii) then, as above, we see that there exists a chart $\Psi\colon V \to B^n_{\geq 0}$ with $\Psi(Q) = 0$.



We now see that

$$P \in \partial X \underset{\text{by the Boundary–Local Homeomorphism Lemma 23.14 applied to the homeomorphisms } \Phi \text{ and } \Psi \text{ and to the local homeomorphisms which are given by the inclusions } U \to X \text{ and } V \to Y}{\Longrightarrow} \Phi(P) = 0 \in \partial B^n_{\geq 0} \underset{\uparrow}{\Longrightarrow} Q \in \partial Y \underset{\text{by Definition 23.2}}{\Longrightarrow} Q \text{ does not admit a chart of type (i).}$$

Finally we assume assume that (β) holds. We need to show that (α) does not hold. So we now assume that the boundary of every n-dimensional topological manifold is empty. This implies in particular the boundary of $\mathbb{H}^n$ is empty. We now consider the point $0 \in \mathbb{H}^n$. Evidently the identity map gives us a chart of type (ii) for 0. But since we assume that the boundary of $\mathbb{H}^n$ is empty we see that $0 \in \mathbb{H}^n$ admits a chart of type (i). This violates (α). ■

The Unique Type-of-Chart Proposition 23.13 allows us to prove the following proposition:

Proposition 23.15. (Topological Manifold–Boundary Proposition) Let $n \in \mathbb{N}_0$ and let X be an n-dimensional topological manifold. The following statements hold:

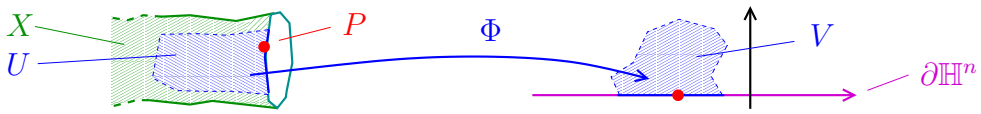
(1) The boundary ∂X is an $(n-1)$-dimensional topological manifold with empty boundary, i.e. we have $\partial(\partial X) = \varnothing$.

(2) (a) Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.

(b) If X is compact, then ∂X is also compact.

Proof. Let X be an n-dimensional topological manifold.

(1) If the Conclusion (α) of the Unique Type-of-Chart Proposition 23.13 holds, then there is clearly nothing to show. So we now assume that Conclusion (β) of the Unique Type-of-Chart Proposition 23.13 holds. First note that it follows from the Subspace-Hausdorff Lemma 3.31 and from the Second Countability Lemma 13.2 (2) that ∂X is Hausdorff and second-countable. Now it remains to show that ∂X admits an $(n-1)$-dimensional atlas. Let $P \in \partial X$. By definition of ∂X there exists a chart $\Phi\colon U \to V$ of type (ii) for X with $P \in U$. Note that Φ is a chart of type (ii) precisely for the points in $\Phi^{-1}(\partial\,\mathbb{H}^n)$. It follows from the assumption that (1β) holds together with the definition of ∂X that we have $\Phi^{-1}(\partial\,\mathbb{H}^n) = \partial X \cap U$.



In other words, under the obvious identification $\partial\,\mathbb{H}^n = \mathbb{R}^{n-1}$ we see that Φ restricts to an $(n-1)$-dimensional chart of type (i) for the topological space ∂X. This shows that ∂X is an $(n-1)$-dimensional topological manifold with empty boundary.

(2) (a) First note that, basically by definition, the set of points in X admitting a chart of type (i) is an open subset of X. It follows from the definition of ∂X that $X \setminus \partial X$ is an open subset. Thus the boundary ∂X is a closed subset of X. By (1) we already know that ∂X is a topological manifold. It thus follows from the Manifold–Local Properties Lemma 23.8 (3b) that every component of ∂X is an open subset of ∂X. This implies that any union of components of ∂X is an open subset of ∂X, which in

turn implies that every union of components is also a closed subset of ∂X. Since ∂X is a closed subset of X it now follows from the Open-Closed Inclusion Lemma 4.13 (2) that every union of components of ∂X is in fact a closed subset of X.

(b) Now suppose that X is compact. Thus ∂X is a closed subset of the compact space X. We obtain from the Compact-Closed Lemma 3.39 (1) that ∂X itself is compact. ■

The Topological Manifold–Boundary Proposition 23.15 allows us to prove the following lemma:

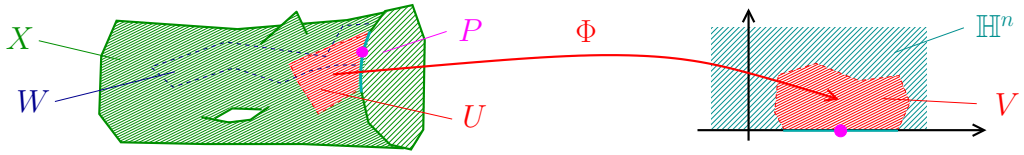
Lemma 23.16. (Interior-Non Compact Lemma) Let $n \in \mathbb{N}_0$ and let X be an n-dimensional topological manifold. The following statements hold:

(1) $X \setminus \partial X$ is an open and dense subset of X.
(2) If $\partial X \neq \varnothing$, then $X \setminus \partial X$ is non-compact, in particular $X \setminus \partial X \neq \varnothing$.

Proof. Let $n \in \mathbb{N}_0$ and let X be an n-dimensional topological manifold.

(1) It follows from the Topological Manifold–Boundary Proposition 23.15 that $X \setminus \partial X$ is an open subset of X.

Next we want to show that $X \setminus \partial X$ is dense in X. Thus let W be an non-empty open subset of X. Let $P \in W$. If $P \notin \partial X$, then we are already done. If $P \in \partial X$, then we pick a chart $\Phi\colon U \to V$ of type (ii) for P. After possibly replacing U by $U \cap W$ we can assume that $U \subseteq W$. Since $V \subseteq \mathbb{H}^n$ is an open subset we see that $V \cap (\mathbb{H}^n \setminus \partial \mathbb{H}^n) \neq \varnothing$. Finally note that Φ defines a chart of type (i) for every point in $\Phi^{-1}(V \cap (\mathbb{H}^n \setminus \partial\, \mathbb{H}^n))$. By definition this implies that $\Phi^{-1}(V \cap (\mathbb{H}^n \setminus \partial\, \mathbb{H}^n)) \subseteq X \setminus \partial X$. Thus we have shown that W has non-empty intersection with $X \setminus \partial X$.



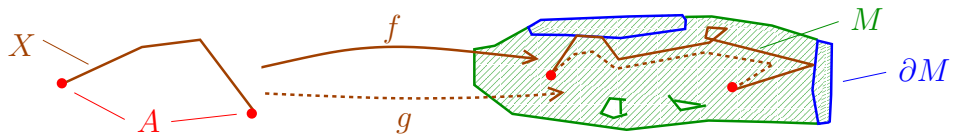
(2) It remains to show that if $X \setminus \partial X$ is compact, then $\partial X = \varnothing$. It follows from the elementary Disjoint Union Topology-Properties Lemma 7.28 (1) and the Manifold–Local Properties Lemma 23.8 (3b) that we only need to consider the case that X is actually connected. Thus assume that $Y := X \setminus \partial X$ is compact. Since X is Hausdorff we obtain from the Compact-Closed Lemma 3.39 that Y is closed. By the Topological Manifold–Boundary Proposition 23.15 we know that ∂X is also closed. Since X is connected, since $X = Y \sqcup \partial X$ and since $Y \neq \varnothing$ we see that $\partial X = \varnothing$. ■

We conclude this section with the following technical lemma.

Lemma 23.17. (Push Away-from-Boundary Lemma) Let M be a topological manifold.

(1) Let X be a compact space, let $A \subseteq X$ be closed and let $f\colon X \to M$ be a continuous map. If $f(A) \subseteq M \setminus \partial M$, then f is homotopic rel A to a map $g\colon X \to M \setminus \partial M$.
(2) If M is path-connected, then $M \setminus \partial M$ is also path-connected.

Remark. The Push Away-from-Boundary Lemma 23.17 foreshadows the Smooth Collar Neighborhood Theorem 35.6 and the Topological Collar Neighborhood Theorem 35.14 that



we will prove later. In Exercise 35.3 we will prove a "smooth" version of the Push Away-from-Boundary Lemma 23.17. □

Proof.

(2) Let us first show how to deduce (2) from (1). This also serves as a good example for (1). So let us assume that M is path-connected. Let $P, Q \in M \setminus \partial M$. Since M is path-connected there exists a continuous map $f\colon [0,1] \to M$ with $f(0) = P$ and $f(1) = Q$. We apply (1) to f and $A := \{0,1\} \subseteq X := [0,1]$ and we obtain a map $g\colon [0,1] \to M \setminus \partial M$ with $g(0) = f(0) = P$ and $g(1) = f(1) = Q$.

(1) Let X be a compact space, let $A \subseteq X$ be closed and let $f\colon X \to M$ be a continuous map with $f(A) \subseteq M \setminus \partial M$.

Claim. There exist charts $\Phi_i\colon U_i \to B_2^{n-1} \times [0,2)$, $i = 1, \dots, k$ of type (ii) such that

$$f(X) \cap \partial M \subseteq \bigcup_{i=1}^{k} \Phi_i^{-1}(B^{n-1} \times \{0\}) \quad \text{and such that} \quad f(A) \subseteq M \setminus \bigcup_{i=1}^{k} \underbrace{\Phi_i^{-1}(B_2^{n-1} \times [0,2))}_{=U_i}.$$

Proof. We make the following observations:

(a) Since $A \subseteq X$ is closed, since X is compact and since M is by hypothesis Hausdorff we obtain from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that the image $f(A) \subseteq M$ is closed. In other words, $M \setminus f(A)$ is open. Since $f(A) \subseteq M \setminus \partial M$ one can now easily see that for each $P \in f(X) \cap \partial M$ there exists a chart $\Phi_P\colon U_P \to B_2^{n-1} \times [0,2)$ of type (ii) such that $P \in U_P$, $\Phi_P(P) = 0$ and such that $U_P \cap f(A) = \varnothing$.

(b) Clearly the sets $\{\Phi_P^{-1}(B^{n-1} \times \{0\})\}_{P \in f(X) \cap \partial M}$ form an open cover of $f(X) \cap \partial M$.

(c) Since X is compact, we know by the Compact Image Lemma 4.15 that $f(X)$ is compact. By the Topological Manifold–Boundary Proposition 23.15 (2a) we know that ∂M is a closed subset of M. As we pointed out on page 155, this implies that $f(X) \cap \partial M$ is also compact.

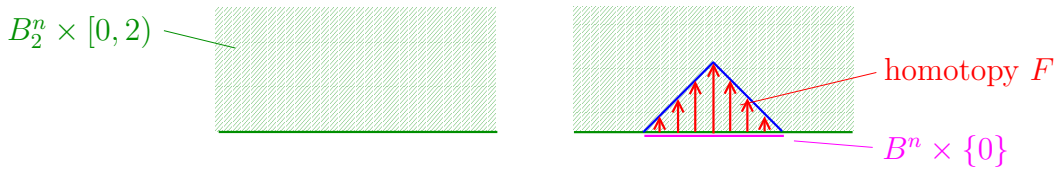
(d) It follows from (b), (c) and the Compact Subspace Covering Lemma 3.38 that there exist finitely many $P_1, \dots, P_k$ such that we can cover $f(X) \cap \partial M$ by the sets $\Phi_{P_i}^{-1}(B^{n-1} \times \{0\})$, $i = 1, \dots, k$. The charts $\Phi_i := \Phi_{P_i}$ are the charts we were looking for. ⊞

Next we consider the homotopy

$$\begin{aligned} F\colon B_2^{n-1} \times [0,2) \times [0,1] &\to B^{n-1} \times [0,2) \\ ((x,s),t) &\mapsto \begin{cases} (x,s), & \text{if } \|x\| > 1 \text{ or } s \geq 1 - \|x\|, \\ (x, s \cdot (1-t) + (1 - \|x\|) \cdot t), & \text{if } \|x\| \leq 1 \text{ and } s \leq 1 - \|x\| \end{cases} \end{aligned}$$

which is illustrated in the figure below:

Using the charts $\Phi_1, \dots, \Phi_k$ we can "transplant" the homotopy F and we obtain homotopies $G_1, \dots, G_k\colon M \times [0,1] \to M$. Now we apply these homotopies one after another



to obtain a homotopy $G\colon M \times [0,1] \to M$. More precisely, we consider the homotopy that is given by

$$\begin{array}{rcl} G\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & F_k(\dots F_2(F_1(x,t),t),\dots,t). \end{array}$$

One can easily verify that G has all the desired properties. ■

23.5. **Submanifolds.** In this section we want to specify what we mean by a submanifold of a topological manifold.

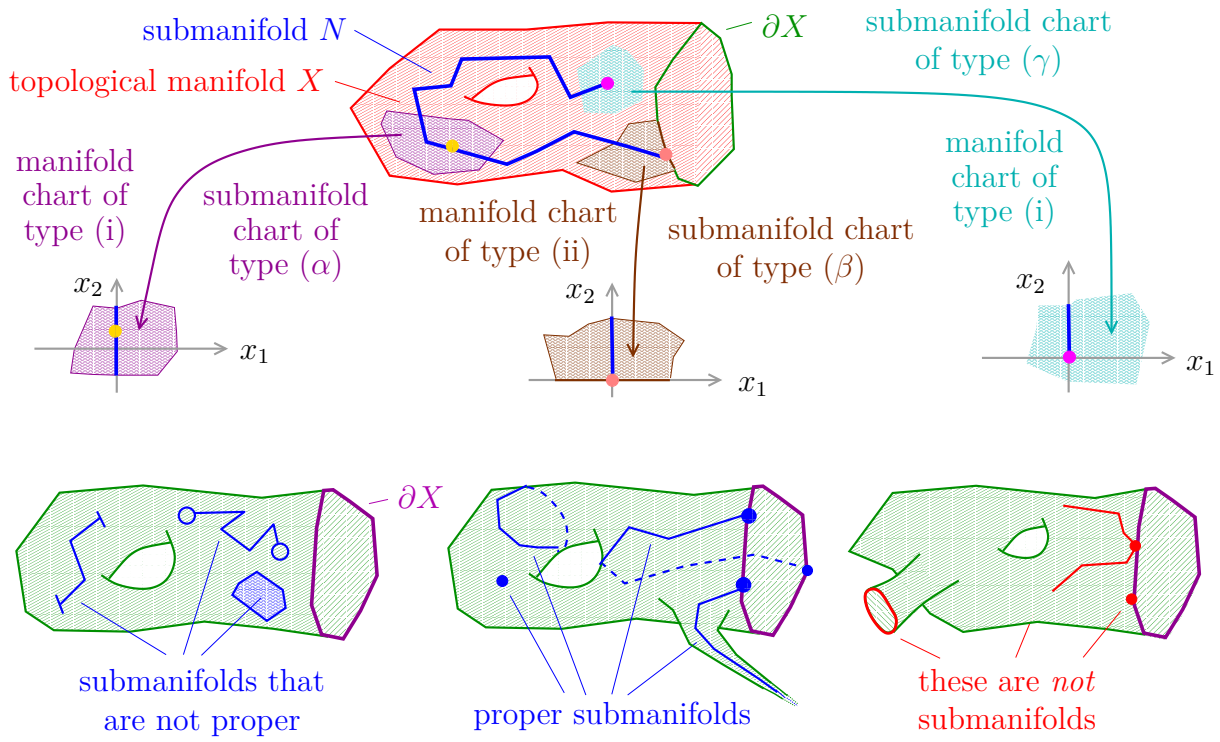
Definition 23.18. Let X be an n-dimensional topological manifold.

(1) We say that a subset $N \subseteq X$ is a k-dimensional submanifold if given any $P \in N$ one of the following holds:
   (α) there exists a chart $\Phi\colon U \to V$ of type (i) for X and P such that[279]
   $$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_i \in \mathbb{R}\},$$
   (β) or there exists a chart $\Phi\colon U \to V$ of type (ii) for X and P such that $\Phi(P)$ lies in $\partial\,\mathbb{H}^n$ and[280]
   $$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\},$$
   (γ) or there exists a chart $\Phi\colon U \to V$ of type (i) for X and P such that $\Phi(P)$ lies in $\partial\,\mathbb{H}^n$ and[281]
   $$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}.$$
   We refer to such charts as submanifold charts of N. We define ∂N as the set of all points in N that do not admit a submanifold chart of type (α).
(2) If $N \subseteq X$ is a k-dimensional submanifold, then we refer to $n-k$ as the codimension of N in X.
(3) We say that a submanifold N is proper if N is a closed subset of X and if for each $P \in N$ we can find submanifold charts of type (α) or (β).

Examples.

(1) In the figure below we give typical examples of submanifolds, some of which are proper, some are not. We also show some subsets that are not submanifolds. In particular it is worth pointing out that a non-empty submanifold of X is never a subset of ∂X.[282]

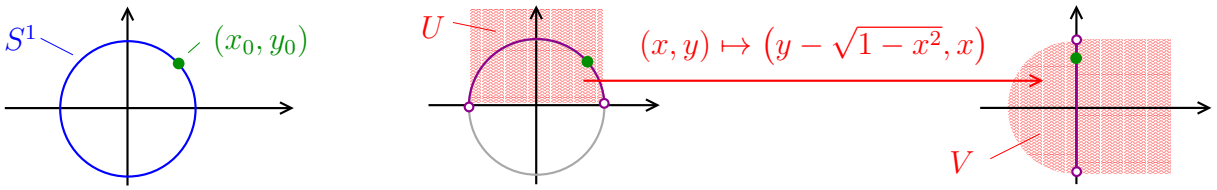
[279] Why are the zeros to the left and not to the right?
[280] For $k = 0$ we interpret $\{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}$ as the empty set.
[281] See the previous footnote.
[282] Indeed, suppose that N is a non-empty submanifold of a topological manifold X. It follows immediately from the definitions of a submanifold that N contains a point P that is contained in a chart of type (i) for X. But by the definition of ∂X in Definition 23.2 this means that $P \notin \partial X$.



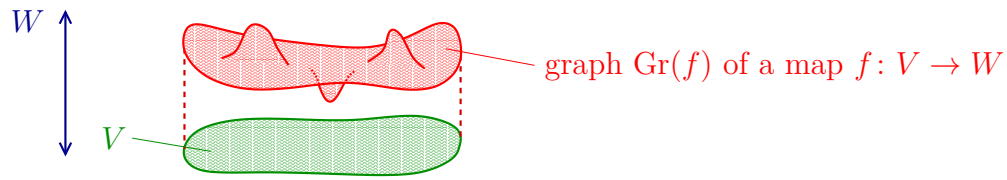
(2) Let us show that S^1 is a proper 1-dimensional submanifold of $\mathbb{R}^2$. First we consider the case of a point $P = (x_0, y_0) \in S^1$ with $x_0 \in (-1, 1)$ and with $y_0 > 0$. We consider the map

$$\begin{array}{rcl} \Phi\colon \overbrace{\{(x,y)\,|\,x \in (-1,1) \text{ and } y > 0\}}^{=:U} & \to & \overbrace{\left\{(x,y)\,\middle|\, x > -\sqrt{1-y^2} \text{ and } y \in (-1,1)\right\}}^{=:V} \\ (x,y) & \mapsto & \big(y - \sqrt{1-x^2}, x\big). \end{array}$$

One can easily verify that this is a chart of type (α) for $P = (x_0, y_0)$. By precomposing Φ by a rotation one obtains a chart of type (α) for any other point on S^1. Since S^1 is a closed subset of $\mathbb{R}^2$ we see that S^1 is a proper 1-dimensional submanifold of $\mathbb{R}^2$. A very mild variation on this argument shows that S^n is a proper n-dimensional submanifold of $\mathbb{R}^{n+1}$.



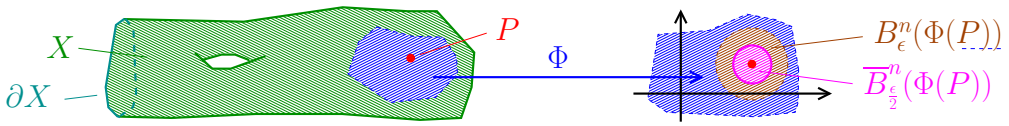
(3) Let $f\colon V \to W$ be a continuous map between two topological manifolds V and W with $f(V) \subseteq W \setminus \partial W$. In Exercise 23.11 we will see that the graph $\mathrm{Gr}(f) := \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the topological manifold $V \times W$.
An especially interesting example is given by the Cantor function $f\colon [0,1] \to \mathbb{R}$ that we introduced on page 216. The above discussion implies that the graph of the Cantor function is a proper 1-dimensional submanifold of $[0,1] \times \mathbb{R}$.



(4) Let X be a topological manifold. It follows easily from the definitions that a subset $N \subseteq X$ is a 0-dimensional submanifold of X if and only if N is a discrete subset of $X \setminus \partial X$.

(5) Let X be a topological manifold and let $U \subseteq X$ be an open subset. It follows almost immediately from the definitions that U is a codimension-zero submanifold of X. Furthermore, if we view U as a topological manifold in its own right, then it follows again basically immediately from the definitions, see also the Boundary–Local Homeomorphism Lemma 23.14, that $\partial U = U \cap \partial X$.

(6) Let X be an n-dimensional topological manifold and let $P \in X \setminus \partial X$. We claim that there exists a compact codimension-zero submanifold $W \subseteq X \setminus \partial X$ which is homeomorphic to $\overline{B}^n$, such that $P \in \mathring{W}$ and such that $X \setminus \mathring{W}$ is also a codimension-zero submanifold. This can be seen as follows: Since $P \in X \setminus \partial X$ there exists a chart $\Phi\colon U \to V$ of type (i) around P. Since $V \subseteq \mathbb{R}^n$ is open there exists an $\epsilon > 0$ with $B^n_\epsilon(\Phi(P)) \subseteq V$.



Using the homeomorphisms from page 623 one can easily show that $\overline{B}^n_{\frac{\epsilon}{2}}(\Phi(P))$ is a submanifold of V and that $V \setminus B^n_{\frac{\epsilon}{2}}(\Phi(P))$ is also a submanifold of V. It is now clear that $W := \Phi^{-1}(\overline{B}^n_{\frac{\epsilon}{2}}(\Phi(P)))$ has the desired properties. □

Remark. The notion of a "submanifold" is used differently by different authors. Therefore, reading the literature it is a good idea to check what definition of a submanifold the authors use. In particular our definition of a "proper submanifold" is often used as the definition of a submanifold. □

The following lemma shows that submanifolds are topological manifolds in their own right.

Lemma 23.19. (Submanifold-is-Manifold Lemma) Let X be a topological manifold. If $N \subseteq X$ is a k-dimensional submanifold, then N is a k-dimensional topological manifold.

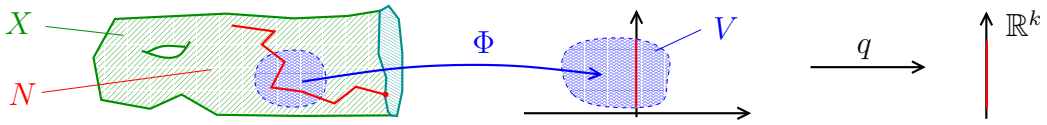
Remark. Let X be a topological manifold and let $N \subseteq X$ be a k-dimensional submanifold. In the Proper Submanifold Proposition 150.13 (1) we will see that N is a proper submanifold if and only if the following two conditions are satisfied:

(1) N is a closed subset of X.
(2) We have the equality $\partial N = (\partial X) \cap N$.[283] [284] □

[283] Here by ∂N we mean the boundary of N, viewed as a topological manifold in its own right.
[284] Note that if N is a proper submanifold, then as of right now we lack the means to show that we have the equality $\partial N = (\partial X) \cap N$, in fact we even lack the means to show either of the "$\subseteq$" and "$\supseteq$" inclusions.

Proof. Let X be a topological manifold and let $N \subseteq X$ be a k-dimensional submanifold. It follows from the Subspace-Hausdorff Lemma 3.31 and the Second Countability Lemma 13.2 that N is Hausdorff and second-countable. We denote by $q\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the last k coordinates. Let $P \in N$. We make the following slightly excessive observations:

(a) If P admits a submanifold chart $\Phi\colon U \to V$ of type (α), then this submanifold chart is a chart of type (i) for $P \in X$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (i) for $P \in N$.



(b) If P admits a submanifold chart $\Phi\colon U \to V$ of type (β), then this submanifold chart is a chart of type (ii) for $P \in X$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.

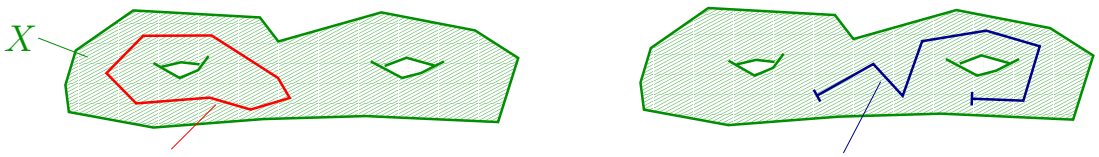
(c) If P admits a submanifold chart $\Phi\colon U \to V$ of type (γ), then this submanifold chart is a chart of type (i) for $P \in X$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.

It follows from these observations that N is indeed a k-dimensional topological manifold. ■

Remark. Unfortunately, when it comes to submanifolds the double usage of the word "closed" can lead to confusion. More precisely, let X be a topological manifold and let N be a submanifold.

(1) If N, viewed as a topological manifold in its own right, is closed, then it is in particular compact, and it follows from the Compact-Closed Lemma 3.39 (2) that N is actually a closed subset of X.

(2) If N is a closed subset of X, then it does not follow that N is a closed topological manifold. For example take $X = \mathbb{R}^2$ and $N = \{0\} \times \mathbb{R}$. In this case N is a closed subset and it is actually even a proper submanifold. But N is evidently not compact, hence it is not a closed topological manifold.



submanifold that is closed as a topological manifold in its own right

submanifold that is a closed subset of X

These two notions of "closed" are illustrated in the figure above. □

23.6. **Exercises for Chapter 23.**

Exercise 23.1. Let X be a topological space. Show that the following two statements are equivalent:

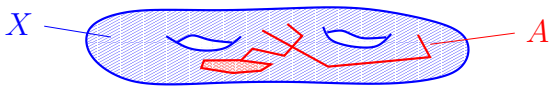
(1) X is a 0-dimensional topological manifold.
(2) The set X is countable and X is equipped with the discrete topology.

Exercise 23.2. Show that the line with two zeros admits a chart of type (i) for every point.

Exercise 23.3.

(a) Let $X = \mathbb{R} \cup \{*\}$ be the line with two zeros. Does there exist a continuous map $f\colon X \to \mathbb{R}$ with $f(0) = 0$ and $f(*) = 1$?
(b) Let X be a topological manifold and let $P, Q \in X$ be two distinct points. Show that there exists a continuous map $f\colon X \to \mathbb{R}$ with $f(P) = 0$ and $f(Q) = 1$.

Exercise 23.4. Let X be a connected topological manifold and let $A \subseteq X$ be a closed non-empty subset with $A \neq X$. Show that $X \setminus A$ is not compact.



Exercise 23.5. We consider the topological space $[0, 1]$. Show that $[0, 1]$ is a 1-dimensional topological manifold with boundary $\{0, 1\}$.

Exercise 23.6. Let M be a connected n-dimensional topological manifold with $n \geq 2$.

(a) Let $P \in M$. Show that $M \setminus \{P\}$ is connected.
(b) Let $D \subseteq M$ be a subset that is closed and discrete. Show that $M \setminus D$ is connected.

Remark. You could make use of the Union Connected Lemma 4.39.

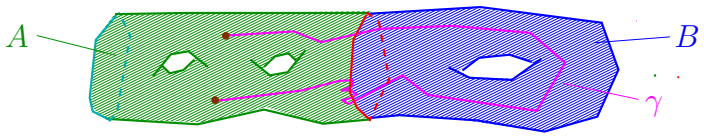
Exercise 23.7. Does there exist a compact space that satisfies all axioms of a closed 1-dimensional topological manifold except for the Hausdorff axiom?

Exercise 23.8. Let $f\colon X \to Y$ be a continuous map between two topological manifolds. Show that given any $x \in X$ there exist neighborhoods U of x and V of $f(x)$ with $f(U) \subseteq V$ such that both U and V are homeomorphic to closed balls.

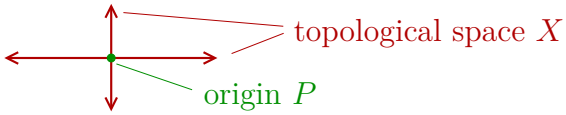
Exercise 23.9. Let M be a topological manifold and let $A, B \subseteq M$ be codimension-zero submanifolds with the following properties:

- A and B are closed subsets.
- $M = A \cup B$.
- The intersection $A \cap B$ is a component of ∂A and a component of ∂B.

Show that if M is path-connected and if $A \cap B$ is path-connected, then so are A and B.
Remark 1. This exercise is technically a little trickier than it might appear. The idea of course is, given points $a, a' \in A$ to modify a path γ in M from a to a' to obtain a path in A. But this path γ might hit $A \cap B$ infinitely many times.
Remark 2. This exercise can be viewed as a partial converse to the Union Connected Lemma 4.39. The exercise is also closely related to Exercises 4.29 and 10.12.



Exercise 23.10. We consider the topological space $X = \{(x, y) \in \mathbb{R}^2 \,|\, x \cdot y = 0\}$ shown in the figure below. Show that X is not a topological manifold.
Remark. It is clear that things go awry at the origin P. But it is somewhat tricky to nail down why there is no chart around that point.



Exercise 23.11. Let $f\colon V \to W$ be a continuous map between two topological manifolds such that $f(V) \subseteq W \setminus \partial W$. Show that the corresponding graph $\mathrm{Gr}(f) \coloneqq \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the topological manifold $V \times W$.

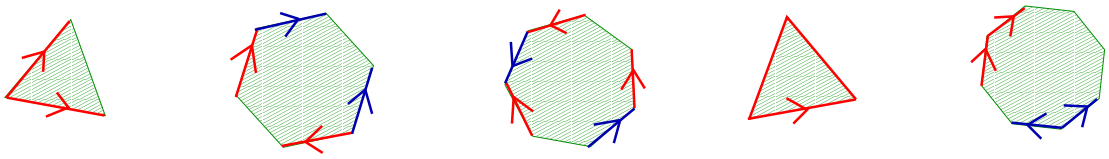
Exercise 23.12. Can you find an atlas for S^n that consists of a single chart?

Exercise 23.13. Let X be a topological space with countably many components such that each component is an n-dimensional topological manifold. Is X itself an n-dimensional topological manifold?

Exercise 23.14. Let W be an n-gon with sides $F_1, \ldots, F_n$. Suppose we identify some of the sides in an orientation-preserving or orientation-reversing way.

(a) Under what conditions on the identification scheme do we get a topological manifold?
(b) If we get a topological manifold, what is its boundary?
(c) If you have a sense of what an orientable topological manifold should be, under what conditions do we get an orientable topological manifold?

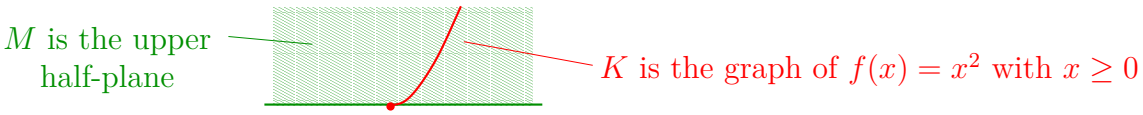
Remark. Strictly speaking we do not have the tools right now to rigorously prove the statements. So the exercise is to make an educated guess.



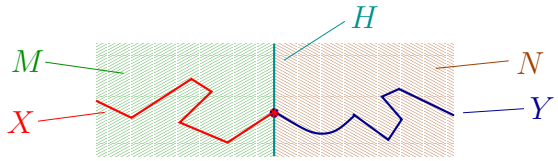
Exercise 23.15. In the figure below we show the topological manifold

$$X \;=\; \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$$

together with the subset $K \coloneqq \{(x, x^2) \,|\, x \geq 0\}$. Is K a submanifold of X?



Exercise 23.16. Let $M \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \leq 0\}$, $N \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \geq 0\}$ and $H \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 = 0\}$. Let $X \subseteq M$ be a k-dimensional topological submanifold with $\partial X = X \cap H$ and let $Y \subseteq N$ be a k-dimensional topological submanifold with $\partial Y = Y \cap H$. We suppose that $X \cap H = Y \cap H$. Show that $X \cup Y$ is a k-dimensional topological submanifold of $\mathbb{R}^n$.



Exercise 23.17. Let X be a topological manifold, let $P \in X$ and let $f\colon K \to X$ be a continuous map from a compact space K to X. Show that if $P \notin f(K)$, then there exists an open neighborhood U of P with $f(K) \cap U = \varnothing$.

Exercise 23.18. Let X be a topological manifold, let K be a compact subset and let U be an open neighborhood of K. Show that there exists an open subset V of X with $K \subseteq V \subseteq U$ such that $\overline{V} \subseteq U$ and such that $\overline{V}$ is compact.

Exercise 23.19. Let $n \in \mathbb{N}$. Let $\Phi_+\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from the North Pole $(0, \dots, 0, 1)$ and let $\Phi_-\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from the South Pole $(0, \dots, 0, -1)$. Show that the map $\Phi_- \circ \Phi_+^{-1}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by $x \mapsto \frac{1}{\|x\|^2} \cdot x$.

Exercise 23.20. Let $g \in \mathbb{N}_0$.

(a) Show that the surface of genus g admits an atlas with two charts.
(b) Show that the surface of genus g admits an atlas consisting of three charts whose domains are diffeomorphic to 2-balls.

Exercise 23.21. Let X be a topological manifold.

(a) Show that if X is non-empty, then there exists a surjective continuous map $X \to [0, 1]$.
(b) Show that if X is non-compact, then there exists a surjective continuous map $X \to \mathbb{R}$.

Exercise 23.22. Let X be a connected topological manifold.

(a) Suppose that X is compact and non-empty. Show that there exists a surjective continuous map $[0, 1] \to X$.
(b) Suppose that X is non-compact. Show that there exists a surjective continuous map $\mathbb{R} \to X$.

Hint. You might want to use the Space-Filling Curve Theorem 10.21.
Remark. Any surjective continuous map from a connected one-dimensional topological manifold to a non-empty topological manifold of dimension ≥ 2 is called a *space-filling path*.

Exercise 23.23. Let X be a topological manifold. Show that if X is connected, then $X \setminus \partial X$ is also connected.
Remark. Suppose that U and V are disjoint open subsets of $X \setminus \partial X$ with $X \setminus \partial X = U \cup V$. Show that there exists a decomposition $\partial X = A \sqcup B$ into open subsets such that $U \cup A$ and $V \cup B$ are open subsets of X.

Exercise 23.24. Let X be a topological manifold and let $A \subseteq X$ be a submanifold of codimension ≥ 2 that is also a closed subset. Show that if X is path-connected, then so is $X \setminus A$.

Remark. You could try to modify the argument of the Push Away-from-Boundary Lemma 23.17.

Exercise 23.25. Given a subset $I \subseteq \mathbb{R}$ we define

$$H(I) \coloneqq \{(x, y) \in \mathbb{R}^2 \,|\, y > 0 \text{ or } x \in I \text{ and } y = 0\}.$$

In other words, $H(I)$ is the open upper half-plane of $\mathbb{R}^2$ together with the points on the x-axis corresponding to I.

(a) Let $I \subseteq \mathbb{R}$ be a subset. Show that $H(I)$ is a submanifold of $\mathbb{R}^2$ if and only if I is an open subset of $\mathbb{R}$.

(b) We consider

$$\begin{aligned} X &\coloneqq H((-\infty, 0) \cup (0, \infty)) \\ Y &\coloneqq H((-\infty, -1) \cup (1, \infty)). \end{aligned}$$

Are the two topological manifolds X and Y homeomorphic?



CHAPTER 24

Interlude: Real Analysis

In the next chapter we will introduce the notion of a "smooth manifold". A "smooth manifold" will be defined as a topological manifold that is equipped with a "smooth atlas". This definition will require some notions from real analysis. In this chapter we collect some definitions and results from real analysis which we will need in the definition and the discussion of smooth manifolds. The reader might want to jump straight to the next Chapter 25 and return to the current chapter when the need arises.

This chapter is organized as follows:

- In Section 24.1 we will introduce and study smooth functions on arbitrary subsets of $\mathbb{R}^n$.
- In Section 24.2 we will recall Taylor's Theorem.
- In Section 24.3 we will recall the differential of a smooth map defined on an open subset of $\mathbb{R}^n$.
- In Section 24.4 we will consider a few special topics in the theory of smooth functions
- In Section 24.5 we will provide exercises for Chapter 24.

In Chapter 25 we will use the concepts from this chapter to introduce smooth manifolds.

24.1. Smooth functions on subsets of $\mathbb{R}^n$.

In the next section we want to define the notion of a "smooth structure" on a topological manifold. In this section we want to recall some basic definitions and facts regarding "classical" smooth maps.
Before we discuss properties of smooth maps we have to agree on what we mean by smooth maps on a subset of $\mathbb{R}^n$. The first definition is surely uncontroversial.

Definition 24.1. Let U be an open subset of $\mathbb{R}^n$.

(1) Let $f\colon U \to \mathbb{R}$ be a continuous map, let $i \in \{1, \dots, n\}$ and let $P \in U$. We denote by $\mathfrak{e}_i$ as usual the i-th vector of the standard basis of $\mathbb{R}^n$. We define the *i-th partial derivative of f at P* as follows:

$$\frac{\partial}{\partial x_i} f\big|_P \;:=\; \lim_{h\to 0} \frac{f(P+h\cdot\mathfrak{e}_i)-f(P)}{h},$$

if the limit exists.[285]

(2) We say that a map $f = (f_1, \dots, f_m)\colon U \to \mathbb{R}^m$ is *smooth*, if it is continuous and if for every coordinate function f_i all iterated partial derivatives $\frac{\partial}{\partial x_{k_1}} \dots \frac{\partial}{\partial x_{k_d}} f_i$, of arbitrarily large degree, exist.

[285] As usual the notation for partial derivatives can change slightly, depending on the circumstances, for example for $n = 1$ we might write $\frac{\partial}{\partial x}$ instead of $\frac{\partial}{\partial x_1}$.

Example. One can easily verify that given any $n \in \mathbb{N}_0$ the following maps are smooth:

$$\begin{array}{ccc} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (x,y) & \mapsto & x\cdot y \end{array} \qquad \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{R}^n\times\mathbb{R}^n & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y. \end{array}$$ □

Remark. Since partial derivatives are just classical derivatives in a slightly different context we see that all the classical rules for derivatives hold. For example on several occasions we will use the Product Rule for partial derivatives which says that for an open subset $U \subseteq \mathbb{R}^n$ and two smooth functions $f, g\colon U \to \mathbb{R}$ and $P \in U$ and $i \in \{1,\dots,n\}$ we have

$$\frac{\partial(f\cdot g)}{\partial x_i}(P) \;=\; \frac{\partial f}{\partial x_i}(P)\cdot g(P) + f(P)\cdot\frac{\partial g}{\partial x_i}(P).$$ □

Theorem 24.2. (Schwarz Theorem) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a smooth map. Furthermore let $i, j \in \{1,\dots,n\}$. For any $P \in U$ we have

$$\frac{\partial^2}{\partial x_i\,\partial x_j}(P) \;=\; \frac{\partial^2}{\partial x_j\,\partial x_i}(P).$$

Proof. This fundamental theorem is for example proved in [**RudiW1976**, Theorem 9.41]. ■

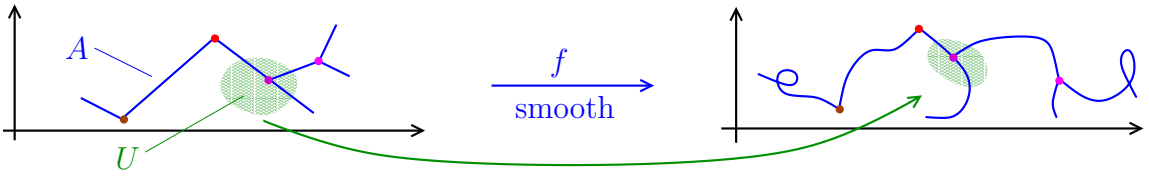
On many occasions it will also be interesting to consider maps which are not defined on open subsets of $\mathbb{R}^n$. This leads us to the following more risqué definition.

Definition 24.3. Let A be any subset of $\mathbb{R}^n$. We say that a map $f\colon A \to \mathbb{R}^m$ is smooth, if for every $a \in A$ there exists an open neighborhood U (in $\mathbb{R}^n$) of a, viewed as a point in $\mathbb{R}^n$, and a smooth map $\widetilde{f}\colon U \to \mathbb{R}^m$ (now "smooth" in the sense above) which coincides with f on $A \cap U$.[286]

Examples.

(1) Let $A = \{(x,y) \in \mathbb{R}^2 \mid x\cdot y = 0\} \subseteq \mathbb{R}$. The map $f\colon A \to \mathbb{R}$ given by $f(x,y) = x^2 - y^2$ is smooth, since it can be extended in an obvious way to a smooth map on all of $\mathbb{R}^2$.

(2) In the figure below we show a subset A of $\mathbb{R}^2$ and a smooth map $f\colon A \to \mathbb{R}^2$ which is more exotic than the lame example given in (1).



smooth map $\widetilde{f}$ on the open subset U that agrees with f on $A \cap U$

(3) The map $f\colon [0,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is not smooth, since it cannot be extended to a smooth function on any open interval (a,b) that contains 0. □

Definition 24.4. A subset of a topological space is called regular closed if $A = \overline{\mathring{A}}$, i.e. if it equals the closure of its interior.[287]

[286] Note that if A is an open subset of $\mathbb{R}^n$, then this definition of a smooth map $A \to \mathbb{R}^n$ actually agrees with the previous definition.

[287] The name "regular closed" is for example used in [**SS1978**, p. 6].

Example.

(1) By the discussion on page 147 every closed ball $\overline{B}_r^n(x)$ of radius > 0 is a regular closed subset of $\mathbb{R}^n$.
(2) One can easily show that every closed interval with non-empty interior is a regular closed subset of $\mathbb{R}$. □

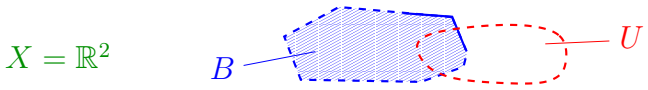
Lemma 24.5. (Regular Closed–∂_*-Lemma) Let $A \subseteq \mathbb{R}^n$ be a subset that is regular closed and let $f\colon A \to \mathbb{R}$ be a smooth map. Given any $i \in \{1, \dots, n\}$ there exists a unique smooth map $\varphi_i\colon A \to \mathbb{R}$ that agrees with the partial derivative $\frac{\partial f}{\partial x_i}$ on the open subset $\mathring{A}$.

Convention 24.6. Let $A \subseteq \mathbb{R}^n$ be a subset that is regular closed and let $f\colon A \to \mathbb{R}$ be a smooth map. Given $i \in \{1, \dots, n\}$ we refer to the map $\varphi_i\colon A \to \mathbb{R}$ from the Regular Closed–∂_*-Lemma 24.5 also as the partial derivative $\frac{\partial f}{\partial x_i}$.

Proof. We start out with the following set-theoretic claim.

Claim 1. For any topological space X, any regular closed subset $B \subseteq X$ and any subset open $U \subseteq X$ the closure of $U \cap \mathring{B}$ in $U \cap B$ equals $U \cap B$.

Proof. In this proof we denote by $\overline{U \cap \mathring{B}}$ the closure of $U \cap \mathring{B}$ in $U \cap B$. With this dangerous notation we need to show that $\overline{U \cap \mathring{B}} = U \cap B$. By definition we have the inclusion "$\subseteq$".



Next we show "$\supseteq$". In other words, we need to show that $U \cap B \subseteq \overline{U \cap \mathring{B}}$. Let C be a closed subset of $U \cap B$ with $U \cap \mathring{B} \subseteq C$. By the Subspace-Characterization Lemma 3.17 there exists a closed subset $\widetilde{C}$ of X with $(U \cap B) \cap \widetilde{C} = C$. Note that we have the following inclusions of (closures of) subsets of X:

$$B \underset{\substack{\uparrow \\ \text{since } B \text{ is regular closed}}}{=} \overline{\mathring{B}} = \overline{(\mathring{B} \setminus U) \cup (U \cap \mathring{B})} \subseteq \overline{\underbrace{(X \setminus U)}_{\substack{\text{closed in } X \\ \text{since } U \text{ open}}} \cup \widetilde{C}} \underset{\substack{\uparrow \\ \text{since } (X \setminus U) \cup \widetilde{C} \text{ is closed in } X}}{=} (X \setminus U) \cup \widetilde{C}.$$

We have thus shown that $B \subseteq (X \setminus U) \cup \widetilde{C}$. It follows that $U \cap B \subseteq (U \cap B) \cap \widetilde{C} = C$. Since $U \cap \mathring{B}$ is the intersection over such C we now see that $U \cap B \subseteq U \cap \mathring{B}$. ⊞

Now we turn to the actual proof of the lemma. Let $A \subseteq \mathbb{R}^n$ be a subset that is regular closed. Furthermore let $f\colon A \to \mathbb{R}$ be a smooth map and let $i \in \{1, \dots, n\}$.

Claim 2. Let $x \in A$.

(1) There exists an open neighborhood U_x of $x \in A$ and a smooth map $f_x\colon U_x \to \mathbb{R}$ such that $f_x = \frac{\partial f}{\partial x_i}$ on U_x.
(2) Suppose we are given two open neighborhoods U_x and U'_x of $x \in A$ and smooth maps $f_x\colon U_x \to \mathbb{R}$ and $f'_x\colon U_x \to \mathbb{R}$ such that $f_x = \frac{\partial f}{\partial x_i}$ on $U_x \cap \mathring{A}$ and such that $f'_x = \frac{\partial f}{\partial x_i}$ on $U'_x \cap \mathring{A}$. Then $f_x(x) = f'_x(x)$.

Proof.

(1) This statement is just the definition of the statement that $\frac{\partial f}{\partial x_i}$ exists and is a smooth map on A.

(2) Note that $f_x = f'_x$ on the open subset $U_x \cap U'_x \cap \mathring{A}$. It follows from Claim 1, the hypothesis that A is regular closed, the fact that $\mathbb{R}$ is Hausdorff and the Maps-agree-on-Closure Lemma 7.21 that $f_x = f'_x$ on $U_x \cap U'_x \cap A$. In particular they agree on the point x. ⊞

It follows immediately from Claim 2 (2) that we get a well-defined map $A \to \mathbb{R}$ that agrees with the partial derivative $\frac{\partial f}{\partial x_i}$ on the open subset $\mathring{A}$, and it follows from Claim 2 (1), that this map is smooth. ■

The following proposition can be viewed as the smooth analog of the Pasting Proposition 4.7 for continuous maps. It gives a convenient criterion for showing that a given map is smooth. This proposition will be used throughout the lecture notes, mostly without referring to it explicitly.

Proposition 24.7. (Smooth Pasting Proposition I) Let A be some subset of $\mathbb{R}^n$, let $f\colon A \to \mathbb{R}^m$ be a continuous map and let $\{V_i\}_{i\in I}$ be an open cover of A. The following holds:

$$f\colon A \to \mathbb{R}^m \text{ is smooth} \iff \text{every restriction } f|_{V_i}\colon V_i \to \mathbb{R}^m \text{ is smooth.}$$

Proof. The lemma follows immediately from the definitions. ■

Notation 24.8. Let $I \subseteq \mathbb{R}$ be an interval. Given a smooth map $f = (f_1, \dots, f_n)\colon I \to \mathbb{R}^n$ and given $x_0 \in \mathring{I}$ we set

$$f'(x_0) \coloneqq \left(\frac{\partial}{\partial x}\Big|_{x_0} f_1, \dots, \frac{\partial}{\partial x}\Big|_{x_0} f_n \right) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

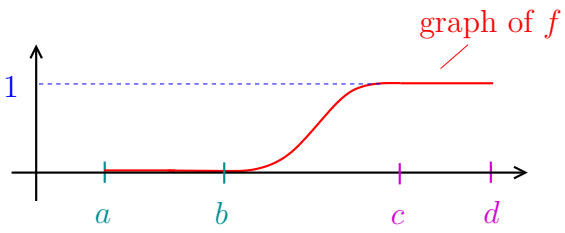
If I is closed and if I has non-empty interior, then we see that I is regular closed. Thus, by the above discussion, we can unambiguously extend $f'\colon \mathring{I} \to \mathbb{R}^n$ to a continuous map $f'\colon I \to \mathbb{R}^n$.

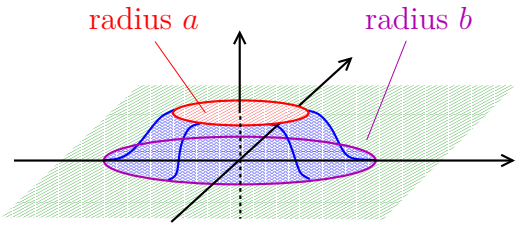
The following lemma is one of the key tools for constructing interesting smooth functions.

Lemma 24.9. (Smooth Transition Function Lemma)

(1) Suppose that we are given real numbers $a \le b < c \le d$ and $\epsilon > 0$. There exists a smooth function $f\colon [a,d] \to [0,1]$ with the following properties:
   (a) $f|_{[a,b]} \equiv 0$ and $f|_{[c,d]} \equiv 1$.
   (b) f is strictly monotonously increasing on $[b,c]$.
   (c) Given any $t \in (a,d)$ we have $f'(t) < (1+\epsilon)\cdot \frac{1}{c-b}$.

(2) Let $a < b$ be two positive real numbers and let $n \in \mathbb{N}$. There exists a smooth function $f\colon \mathbb{R}^n \to [0,1]$ such that $f(x) = 1$ whenever $\|x\| \le a$ and such that $f(x) = 0$ whenever $\|x\| > b$.





Sketch of proof.

(1) We consider the map

$$\begin{array}{rcl} \varphi\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \leq 0, \\ \exp(-\frac{1}{x^2}), & \text{if } x > 0. \end{cases} \end{array}$$

An induction argument, using l'Hopital's rule, shows that φ is in fact smooth, see [**Könr2004**, Chapter 9] for details. Next we consider the map

$$\begin{array}{rcl} \Phi\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \int_{t=0}^{t=x} \varphi(t) \cdot \varphi(1-t)\, dt. \end{array}$$

Using a little bit of massaging this map Φ can be turned into the desired map f. We leave it to the reader to fill in the remaining details.

(2) By (1) there exists a smooth function $\varphi\colon [0,\infty) \to [0,1]$ such that $\varphi(x) = 0$ for $x \in [0,a]$ and such that $\varphi(x) = 1$ for $x \in [b,\infty)$. Now consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & [0,1] \\ x & \mapsto & 1 - \varphi(\|x\|). \end{array}$$

Since $x \mapsto \|x\|$ is smooth on $\mathbb{R}^n \setminus \{0\}$ we see that f is smooth on $\mathbb{R}^n \setminus \{0\}$. Furthermore, since f is constant on a neighborhood of the origin it follows from the Smooth Pasting Proposition 24.7 that f is in fact smooth on all of $\mathbb{R}^n$. ■

In the following we give characterizations of smooth maps on intervals and certain subsets of $\mathbb{R}^n$. To the best of our knowledge we do not make use of the characterizations, neither explicitly nor implicitly. Nonetheless, it is good to know that they exist.

Proposition 24.10. (Smooth Functions-on-Compact Interval Proposition) Let $a < b$ be two real numbers and let $f\colon [a,b] \to \mathbb{R}$ be a function. The following two statements are equivalent:

(1) The function $f\colon [a,b] \to \mathbb{R}$ is smooth in the above sense.
(2) All derivatives of $f\colon (a,b) \to \mathbb{R}$ are defined on the open interval (a,b) and they extend to continuous maps on the closed interval $[a,b]$.

Proof. The "(1)⇒(2)"-implication is basically clear. The "(2)⇒(1)"-implication is a consequence of Borel's Theorem, see [**Strg1981**, Theorem 4.32]. ■

In fact an even more general statement holds:

Theorem 24.11. (Whitney Extension Theorem) Let $a < b$ be two real numbers, let U be an open subset of $\mathbb{R}^{n-1} \times [a,b]$ and let $f\colon U \to \mathbb{R}^m$ be a function. The following statements are equivalent:

(1) The map $f\colon U \to \mathbb{R}^m$ is smooth in the sense of Definition 24.1.
(2) The map f is smooth on the open subset $U \cap (\mathbb{R}^{n-1} \times (a,b))$ and all its partial derivatives extend to continuous maps on $U \cap (\mathbb{R}^{n-1} \times [a,b])$.

If $U = \mathbb{R}^{n-1} \times [a,b]$, the two statements are also equivalent to:

(3) The map f can be extended to a smooth map on all of $\mathbb{R}^n$.

The above statements also hold for half-spaces $\mathbb{R}^{n-1} \times (-\infty, a]$ and $\mathbb{R}^{n-1} \times [a,\infty)$.

Remark. The theorem was first proved by Hassler Whitney [**Why1934**] in 1934. □

Proof. The "(1)⇒(2)"-implication and the "(3)⇒(1)"-implication are basically clear. The "(2)⇒(3)"-implication is a consequence of a generalization of Borel's Theorem, see [**Hör1990a**, Theorem 2.3.6]. An outline of the argument is also given in [**KrP2002**, Theorem 2.3.2]. Finally note that the "(1)⇒(3)"-implication also follows from the Tietze Extension Theorem 12.7 combined with the Smooth Approximation Proposition 27.13. ■

24.2. **Taylor's Theorem.** In this short section we formulate Taylor's Theorem, so that later on we can refer to it. Since the reader is most likely aware of its existence it is arguably best to move on to the next section.

Definition 24.12. Let $n \in \mathbb{N}_0$.

(1) Let $m \in \mathbb{N}_0$ and let $I = (i_1, \ldots, i_m)$ be a sequence of length m with values in the set $\{1, \ldots, n\}$.
   (a) We write $|I| = m$.
   (b) We write $\partial_I = \frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}}$.
   (c) Given $x, a \in \mathbb{R}^n$ we write $(x-a)^I = (x_{i_1} - a_{i_1}) \cdot \ldots \cdot (x_{i_m} - a_{i_m})$ and similarly we define $|x-a|^I$.

(2) Let $U \subseteq \mathbb{R}^n$ be an open subset. Given a smooth map $f\colon U \to \mathbb{R}$, given $a \in U$ and given $k \in \mathbb{N}_0$ we define the k-th Taylor polynomial of f at a as follows:

$$p_k \;:=\; p(f,a)_k \;:=\; \sum_{m=0}^{k} \frac{1}{m!} \sum_{|I|=m} \partial_I f|_a \cdot (x-a)^I.$$

Furthermore, given any $x \in \mathbb{R}^m$ such that the Euclidean segment $\{(1-t)\cdot a + t \cdot x \,|\, t \in [0,1]\}$ lies in A we define the k-th remainder term of f at a as follows:

$$r_k \;:=\; r(f,a)_k \;:=\; \frac{1}{k!} \sum_{|I|=k+1} (x-a)^I \cdot \int_{t=0}^{t=1} (1-t)^k \cdot \partial_I f|_{(1-t)\cdot a + t\cdot x}\, dt.$$

Theorem 24.13. (Taylor's Theorem) Let $U \subseteq \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}$ be a smooth map, let $a \in U$ and let $k \in \mathbb{N}_0$. If W is a convex neighborhood of a in U, then for every $x \in W$ we have

$$f(x) \;=\; p(f,a)_k(x) + r(f,a)_k(x).$$

Proof. Taylor's Theorem gets proved in basically every single book on multivariable analysis. We refer to [**Lee2002**, p. 648] for a short proof. ■

Corollary 24.14. (Taylor Polynomial–Approximation Corollary) Let $U \subseteq \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}$ be a smooth map, let $a \in U$, let $k \in \mathbb{N}_0$ and let W be a convex neighborhood of a in U. Finally suppose that we are given $M \in \mathbb{R}$ such that

$$M \;\geq\; \text{absolute value of all } (k+1)\text{-st partial derivatives of } f \text{ on } W.$$

Then for every $x \in W$ we have

$$\big|f(x) - p(f,a)_k(x)\big| \;\leq\; \tfrac{n^{k+1}\cdot M}{(k+1)!} \cdot \|x-a\|^{k+1}.$$

Proof. The corollary follows easily from Taylor's Theorem 24.13, the definitions, the observation that for any I we have $|(x-a)^I| \leq \|x-a\|^{|I|}$ and some elementary estimates. ■

24.3. **The differential of a smooth map.** We start out with the following definition that should also be familiar from a course in real analysis.

Definition 24.15. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^m$ be a continuous map. We say that f is differentiable in P if there exists a matrix $\mathrm{D}f_P$ such that

$$\lim_{v\to 0} \frac{\|f(P+v) - f(P) - \mathrm{D}f_P \cdot v\|}{\|v\|} \;=\; 0$$

We refer to the $(m\times n)$-matrix $\mathrm{D}f_P$ (which is easily shown to be unique) as the differential of f at P.

Intuitively speaking a map $f\colon U \to \mathbb{R}^m$ is differentiable in P if in a neighborhood of P the map f is "close" to the affine linear map $w \mapsto \mathrm{D}f_P(w-P) + f(P)$.

Example. If $A \in \mathrm{M}(m\times n, \mathbb{R})$ and $v \in \mathbb{R}^m$, then it follows almost immediately from the definitions that for every $P \in \mathbb{R}^n$ the affine map $x \mapsto A\cdot x + v$ is differentiable with differential $\mathrm{D}f_P = A$. □

The following proposition is essential for showing differentiability and computing differentials. We will use it all the time without acknowledging it.

Proposition 24.16. (Differential-from-Partial Derivatives Proposition) Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f = (f_1, \ldots, f_m)\colon U \to \mathbb{R}^m$ be a continuous map. If f is smooth, then the map f is differentiable in every $P \in U$ and for every $P \in U$ we have

$$\mathrm{D}f_P \;:=\; \left(\frac{\partial f_i}{\partial x_j}\Big|_P\right)_{i=1,\ldots,m, j=1,\ldots,n}$$

Proof. This statement is contained in every book on multivariable real analysis, see e.g. [**Lee2002**, Proposition C.8] or [**Fors2008**, p. 70]. ■

Example. Let $I \subseteq \mathbb{R}$ be an open subset, let $f = (f_1, \ldots, f_m)\colon I \to \mathbb{R}^m$ be a smooth map and let $P \in I$. It follows immediately from the definitions that

$$f'(P) \;=\; \mathrm{D}\,f_P \cdot \underbrace{1}_{\in\mathbb{R}} \;=\; \text{the unique column vector of } \mathrm{D}f_P \;\in\; \mathbb{R}^m.$$

Put differently, the vector $f'(P) \in \mathbb{R}^m$, viewed as an $(m\times 1)$-matrix in the obvious way, equals $\mathrm{D}f_P$. □

We also recall the Chain Rule for differentials.

Proposition 24.17. (Chain Rule) Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$ be open subsets. Furthermore let $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ be two smooth maps such that $f(U) \subseteq V$. The composition $g\circ f\colon U \to \mathbb{R}^m$ is smooth and given any $P \in U$ we have

$$\underbrace{\mathrm{D}(g\circ f)_P}_{m\times k\text{-matrix}} \;=\; \underbrace{\mathrm{D}g_{f(P)}}_{m\times l\text{-matrix}} \cdot \underbrace{\mathrm{D}f_P}_{l\times k\text{-matrix}} \qquad \text{"Chain Rule".}$$

Proof. This proposition is proved in every book on multivariable real analysis, see e.g. [**Lee2002**, Proposition C.3]). ■

Example. Let $U \subseteq \mathbb{R}^k$ be an open subset and let $f, g\colon U \to \mathbb{R}$ be smooth maps. It follows from the Chain Rule 24.17 and the discussion on page 646 that the following maps are also smooth:[288]
$$\begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & f(x)\cdot g(x) \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & f(x)+g(x). \end{array}$$
□

We move on to the following definition:

Definition 24.18. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$.[289] We say that a map $f\colon U \to V$ is a diffeomorphism if f is smooth, if f is a bijection and if $f^{-1}\colon V \to U$ is also smooth. If such a diffeomorphism exists, then we say that U and V are diffeomorphic.

Examples.

(1) Let $n \in \mathbb{N}$ and $r, s > 0$ and $x, y \in \mathbb{R}^n$. It is straightforward to show that the open balls $B^n_r(x)$ and $B^n_s(y)$ are diffeomorphic.

(2) We consider the maps
$$\begin{array}{ccc} B^n & \to & \mathbb{R}^n \\ x & \mapsto & \dfrac{x}{\sqrt{1-\|x\|^2}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{R}^n & \to & B^n \\ x & \mapsto & \dfrac{x}{\sqrt{1+\|x\|^2}}. \end{array}$$
One can easily verify that they are smooth and that they are inverses of one another. This shows that both maps are diffeomorphisms. In particular we have shown that B^n is diffeomorphic to $\mathbb{R}^n$.[290]
□

Next we recall a proposition from the analysis courses. Since this proposition is one of the reasons why later on dealing with smooth manifolds is easier than dealing with topological manifolds we also provide the proof.

Proposition 24.19. (Differential-of-Diffeomorphism Proposition) Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets.

(1) If $f\colon U \to V$ is a diffeomorphism with inverse $f^{-1}\colon V \to U$, then for any $P \in U$ we have
$$\mathrm{D}(f^{-1})_{f(P)} \cdot \mathrm{D}f_P = \mathrm{id}_m \qquad \text{and} \qquad \mathrm{D}f_P \cdot \mathrm{D}(f^{-1})_{f(P)} = \mathrm{id}_n\,.$$
In particular the differential $\mathrm{D}f_P$ is an invertible matrix.

(2) If U and V are diffeomorphic and non-empty, then $m = n$.

Example. Suppose that U is an open subset, let's say connected, of some $\mathbb{R}^n$ and suppose that $f\colon U \to \mathbb{R}^n$ is a smooth map such that $\mathrm{D}f_P$ is invertible for every $P \in U$. One might be tempted to think that $f\colon U \to f(U)$ is already a diffeomorphism itself. But this is not necessarily the case, a counterexample is for example given by
$$\begin{array}{ccc} f\colon \mathbb{C}\setminus\{0\} & \to & \mathbb{C}\setminus\{0\} \\ z & \mapsto & z^2. \end{array}$$
One can easily verify that $\mathrm{D}f_z$ is invertible for every $z \in \mathbb{C}\setminus\{0\}$. But f is evidently not injective.
□

[288] This example is the smooth analog of the Sum-Product-Continuity Lemma 4.11.

[289] Note that we do *not* demand that U and V are open subsets.

[290] In the Star Shaped-Diffeomorphism Type Theorem 82.2 we will prove a much more general statement, namely we will see that every open star-shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Proof.

(1) Let $f\colon U \to V$ be a diffeomorphism between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. We denote by $f^{-1}\colon V \to U$ the inverse of f. Let $P \in U$. We see that

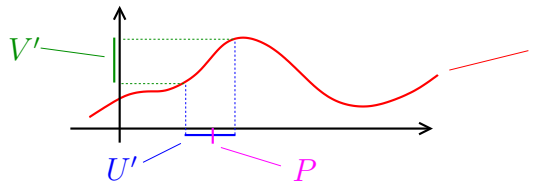
$$\mathrm{D}(f^{-1})_{f(P)} \cdot \mathrm{D}f_P \underset{\text{Chain Rule 24.17}}{=} \mathrm{D}(f^{-1} \circ f)_P = \mathrm{D}(\mathrm{id}_U)_P = \mathrm{id}_m \in \mathrm{M}(m \times m, \mathbb{R}).$$

The same way one shows that $\mathrm{D}f_P \cdot \mathrm{D}(f^{-1})_{f(P)} = \mathrm{id}_n \in \mathrm{M}(n \times n, \mathbb{R})$. In other words, the matrices $\mathrm{D}f_P$ and $\mathrm{D}(f^{-1})_{f(P)}$ are inverses of one another. This implies that $\mathrm{D}f_P$ is invertible.

(2) Since U is non-empty we can find a $P \in U$. By (1) the $(m \times n)$–matrix $\mathrm{D}f_P$ is invertible, this means that the matrix is a square matrix, i.e. $m = n$. ∎

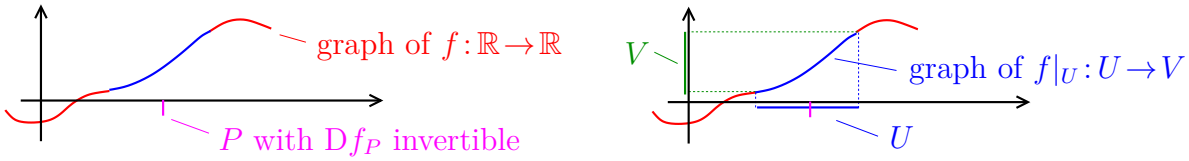
Before we state the next theorem let us introduce the following harmless definition, which is in line with our definition of a local property of a map that we gave in Definition 4.61.

Definition 24.20. Let $f\colon U \to V$ be a continuous map between two open subsets of $\mathbb{R}^n$ and let $P \in U$. We say that f is a local diffeomorphism at P if there exist open neighborhoods U' of P in U and V' of $f(P)$ in V such that $f|_{U'}\colon U' \to V'$ is a diffeomorphism.[291]



Now we can formulate the following theorem which can be viewed as a partial inverse to the Differential-of-Diffeomorphism Proposition 24.19 (1).

Theorem 24.21. (Inverse Function Theorem) Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^n$ be a smooth map. Furthermore let $P \in U$ be a point. If $\mathrm{D}f_P$ is invertible, then f is a local diffeomorphism at P.



Proof. This theorem is proved in most courses on multivariable real analysis. A proof is for example given in [**Lee2002**, Corollary C.34]. ∎

The Inverse Function Theorem 24.21 is one of the main results in real analysis. For example the proof of the following proposition rests heavily on the Inverse Function Theorem 24.21.

[291] In particular we demand that $f|_{U'}$ is smooth, whereas we do not demand that $f\colon U \to V$ is smooth everywhere.

Theorem 24.22. (Differential-of-Diffeomorphism Theorem) Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^n$ be a smooth map such that the differential $\mathrm{D}f_P$ is invertible at every point P in the open subset U. Then the following two statements hold:

(1) The map f is open, i.e. given any open subset $V \subseteq U$ the image $f(V)$ is an open subset of $\mathbb{R}^n$.
(2) If f is injective, then $f\colon U \to f(U)$ is a diffeomorphism.

Remark. In the Topological Invariance-of-Domain Theorem 170.9 we will see that the statement of the Differential-of-Diffeomorphism Theorem 24.22 (1) also holds if we only assume that f is continuous and injective. □

Proof. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^n$ be a smooth map such that the differential $\mathrm{D}f_P$ is invertible at every point P in the open subset U.

(1) Let $V \subseteq U$ be an open subset. We need to show that $f(V)$ is an open subset of $\mathbb{R}^n$. Let $Q \in f(V)$. We pick $P \in V$ with $f(P) = Q$. Since $\mathrm{D}f_P$ is invertible it follows from the Inverse Function Theorem 24.21, applied to the restriction $f\colon V \to \mathbb{R}^n$, that there exists an open neighborhood V' of P and an open neighborhood W' of $Q = f(P)$ such that $f\colon V' \to W'$ is a diffeomorphism. But this implies that $W' \subseteq f(V)$. It follows from this discussion, together with the elementary JH-Openness Criterion 3.3, that $f(V)$ is indeed an open subset of $\mathbb{R}^n$.
(2) We now suppose that $f\colon U \to \mathbb{R}^n$ is injective. By (1) we know that $f(U)$ is an open subset of $\mathbb{R}^n$. We consider the inverse $f^{-1}\colon f(U) \to U$. We need to show that $f^{-1}\colon f(U) \to U$ is smooth. Since f is a local diffeomorphism we see that f^{-1} is also a local diffeomorphism. This implies that f^{-1} is locally smooth which implies by the Smooth Pasting Proposition 24.7 that f^{-1} itself is smooth. ■

24.4. **Special topics in the theory of smooth functions.** In this short section we collect a ragtag of results on smooth functions, each of which will have its moment of glory later on. Nonetheless, in a first reading of this chapter, this section can be skipped.

The first proposition says that locally, up to translation, one can describe any smooth map as matrix multiplication, where the coefficients of the matrix are allowed to vary.

Proposition 24.23. (Map-as-Matrix Multiplication Proposition) Let $U \subseteq \mathbb{R}^n$ be some convex subset that contains the origin. Furthermore let $f\colon U \to \mathbb{R}^m$ be a smooth map with $f(0) = 0$.

(1) There exists a smooth map $A\colon U \to \mathrm{M}(m \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that
$$f(x) \;=\; A(x) \cdot x \qquad \text{for all } x \in U.$$
(2) If there exist $k \in \{0, \ldots, n\}$ and $l \in \{0, \ldots, m\}$ with $f((\mathbb{R}^k \times \{0\}) \cap U) \subseteq \mathbb{R}^l \times \{0\}$, then we can arrange that for each $x \in (\mathbb{R}^k \times \{0\}) \cap U$ the corresponding matrix $A(x)$ is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the top left corner is an $l \times k$-matrix.

Proof. First we take care of Statement (1). By considering the coordinate functions of a given map $f = (f_1, \ldots, f_m)\colon U \to \mathbb{R}^m$ separately we see that it suffices to deal with the case $m = 1$. In other words we need to prove the following claim:

Claim. [292]Let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0) = 0$. There exist smooth maps $a_1, \dots, a_n \colon U \to \mathbb{R}$ such that $a_i(0) = \frac{\partial f}{\partial x_i}(0)$ and with

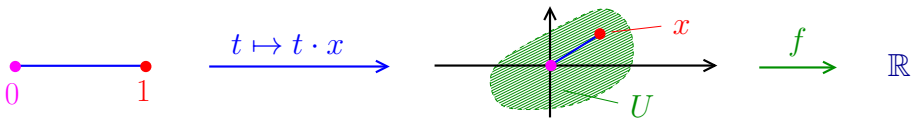
$$f(x) \;=\; \sum_{i=1}^{n} a_i(x_i) \cdot x_i \qquad \text{for all } x = (x_1, \dots, x_n) \in U.$$

Proof. Let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0) = 0$. We observe that given any $x = (x_1, \dots, x_n) \in U$ we have

$$f(x) \;=\; f(x) - \underbrace{f(0)}_{=0} \;=\; \int_{t=0}^{t=1} \frac{d}{dt} f(\underbrace{t \cdot x}_{\in U})\, dt \;=\; \int_{t=0}^{t=1} \sum_{i=1}^{n} \big(\tfrac{\partial f}{\partial x_i}\big)(t \cdot x) \cdot x_i \, dt$$

Fundamental Theorem of Calculus 73.27 — by the Chain Rule 24.17

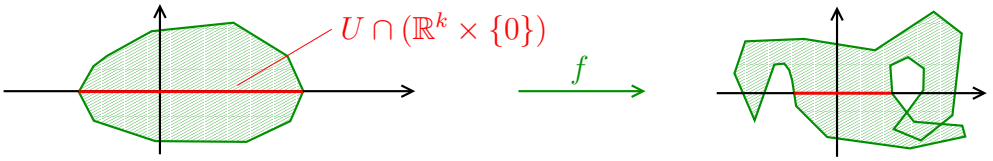
$$= \sum_{i=1}^{n} \underbrace{\int_{t=0}^{t=1} \big(\tfrac{\partial f}{\partial x_i}\big)(t \cdot x)\, dt}_{=:a_i(x)} \cdot x_i.$$



If U is open, then it follows from standard methods in analysis, e.g. the Leibniz Integral Rule 73.28, that each a_i is smooth. If U is not open, then this fact can be shown with more effort by hand. Alternatively one can use the Smooth Extension Proposition 27.12 to reduce the problem to the open case.

Finally note that it follows from the above equality and the Product Rule that we have the equality $a_i(0) = \frac{\partial f}{\partial x_i}(0)$. This shows that we have indeed $A(0) = \mathrm{D}f_0$. This concludes the proof of Statement (1).

Now we turn to the proof of Statement (2).



Note that it is straightforward to see that the map $A\colon U \to \mathrm{M}(m \times n, \mathbb{R})$ we constructed above has the desired property. Indeed, as above we only have to deal with the case $m = 1$ and $l = 1$. But if $f((\mathbb{R}^k \times \{0\}) \cap U) = \{0\}$, then it follows immediately from the construction of the matrix that for each $x \in (\mathbb{R}^k \times \{0\}) \cap U$ we obtain the zero matrix. ∎

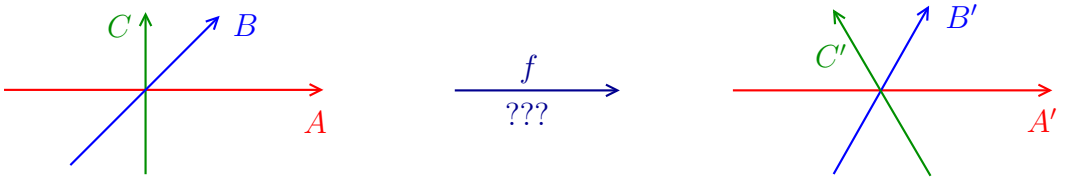
24.5. Exercises for Chapter 24.

Exercise 24.1. Show that there exists a diffeomorphism $[0, 1) \to [0, \infty)$ such that $f(x) = x$ for all $x \in [0, \frac{1}{2})$.

Hint. Use the Smooth Transition Function Lemma 24.9 and some creativity.

[292]The statement of this claim is often referred to as the *Hadamard Lemma*.

Exercise 24.2. Let $M \coloneqq A \sqcup B \sqcup C \subseteq \mathbb{R}^2$ be the subset shown in the figure below. Does there exist a smooth map $f\colon M \to \mathbb{R}^2$, as illustrated in the figure below, which restricts to diffeomorphisms $A \to A'$, $B \to B'$ and $C \to C'$?



Exercise 24.3. Let U be an open convex neighborhood of $0 \in \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0) = 0$. We assume that 0 is a critical point, i.e. we assume that the differential $\mathrm{D}f_0$ is zero. Show that there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$, $i, j = 1, \dots, n$ such that $h_{ij} = h_{ji}$ and such that

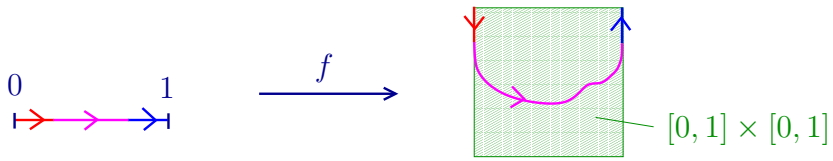
$$f(x) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j \qquad \text{for all } x = (x_1, \dots, x_n) \in U.$$

Hint. Make use of the Map-as-Matrix Multiplication Proposition 24.23.

Exercise 24.4. Show that there exists a smooth map $f = (f_1, f_2)\colon [0,1] \to [0,1] \times [0,1]$ with the following properties:

(a) For $t \in [0, \frac{1}{4}]$ we have $f(t) = (0, 1-t)$.
(b) For $t \in [\frac{3}{4}, 1]$ we have $f(t) = (1, t)$.
(c) The map $f_1\colon [0,1] \to [0,1]$ is monotonously increasing.
(d) For each $t \in [0,1]$ we have $f'(t) \neq 0 \in \mathbb{R}^2$.
(e) For $t \notin \{0,1\}$ we have $f_2(t) \in (0,1)$.

Hint. Use the Smooth Transition Function Lemma 24.9.



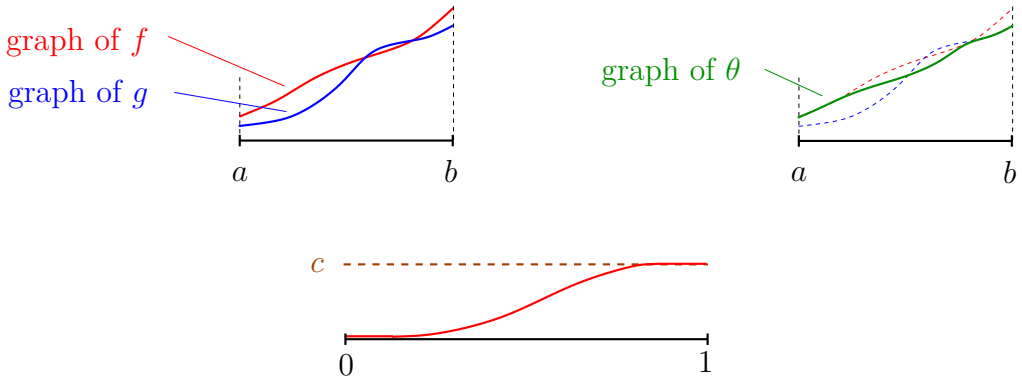
Exercise 24.5. Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be smooth maps such that $f'(x) > 0$ and $g'(x) > 0$ for all $x \in [a,b]$. We assume that $g(b) > f(a)$. Show that there exists an $\eta > 0$ and a smooth map $\theta\colon [a,b] \to \mathbb{R}$ with the following properties:

(1) $\theta'(x) > 0$ for all $x \in [a,b]$,
(2) θ agrees with f on $[a, a+\eta]$,
(3) θ agrees with g on $[b-\eta, b]$.

Hint. You could use the Smooth Transition Function Lemma 24.9 and consider the derivatives of f and g.

Exercise 24.6. Let $c > 0$ and let $d > \frac{4}{3}c$. Show that there exists a smooth function $\tau\colon [0,1] \to [0,c]$ with $\tau(t) = 0$ for $t \in [0, \frac{1}{8}]$, $\tau(t) = c$ for $t \in [\frac{7}{8}, 1]$ and $\tau'(t) \in [0,d)$ for all $t \in [0,1]$.

Hint. Use the Smooth Transition Function Lemma 24.9 to construct τ' and use the Fundamental Theorem 73.27 of Calculus.



Exercise 24.7. Let $v, w \in \mathbb{R}^2 \setminus \{0\}$. Show that there exists a smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ such that $f(P) > 0$ if and only $P \in \{\mu \cdot v + \lambda \cdot w \,|\, \mu, \lambda \in \mathbb{R}_{>0}\}$.

Exercise 24.8. Let $x, y \in B^n$. Show that there exists a diffeomorphism $f\colon B^n \to B^n$ with $f(x) = y$.

Exercise 24.9. Let $n \in \mathbb{N}$. Provide an explicit diffeomorphism between the open cube $(-1, 1)^n$ and the open ball B^n.

CHAPTER 25

Smooth manifolds

In this chapter we introduce smooth structures on topological manifolds. A topological manifold equipped with a smooth structure will just be called a smooth manifold. In the literature the term "smooth manifold" can be used with quite different conventions and meanings. So we try to state clearly the various definitions used throughout these notes.

This chapter is organized as follows:

- In Section 25.1 we will introduce the notion of a smooth manifold.
- In Section 25.2 we will give basic examples of smooth manifolds.
- In Section 25.3 we will see that we can view the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ as smooth manifolds.
- In Section 25.4 we will introduce smooth maps between smooth manifolds.
- In Section 25.5 we will introduce submanifolds of smooth manifolds.
- In Section 25.6 we will study the boundary of smooth manifolds.
- In Section 25.7 we will introduce the dimension of smooth manifolds and we will see that every non-empty smooth manifold has a well-defined dimension.
- In Section 25.8 we will show that for a connected smooth manifold we can connect any two points by a *smooth* path.
- In Section 25.9 we will introduce the product of smooth manifolds, where at least one factor has empty boundary.
- In Section 25.10 we will study group actions on smooth manifolds and we will see that the quotients often are naturally smooth manifolds.
- In Section 25.11 we will study proper codimension-zero submanifolds of smooth manifolds
- In Section 25.12 we will provide exercises for Chapter 25.

In the next Chapter 26 we will introduce smooth manifolds with corner.

25.1. **Definition of smooth manifolds.** In this section we will finally introduce one of the most important notions of these lecture notes, namely the concept of a "smooth manifold". For the reader's convenience we first recall the notion of a topological manifold that we introduced in Definition 23.2.

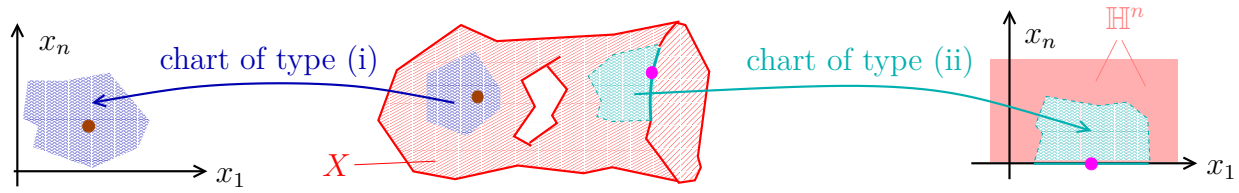
Definition 25.1. Let X be a topological space.

(1) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and
   (i) V is an open subset of $\mathbb{R}^n$ or
   (ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\,\mathbb{H}^n$.

In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii). We refer to the figure below for an illustration.

(2) We say X is an n-dimensional topological manifold, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.

(3) An n-dimensional atlas for X is a collection of n-dimensional charts such that the domains of the charts cover all of X.



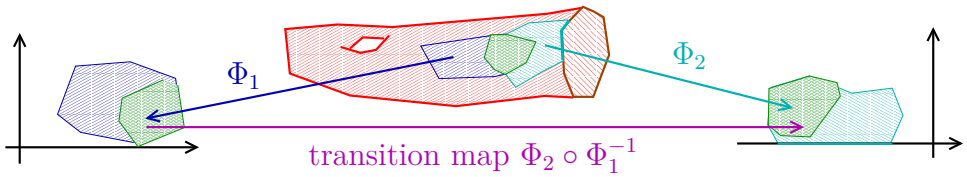
Definition 25.2. Let X be a topological manifold.

(1) We say that two n-dimensional charts $\Phi_1\colon U_1 \to V_1$ and $\Phi_2\colon U_2 \to V_2$ are smoothly compatible if the transition map[293]

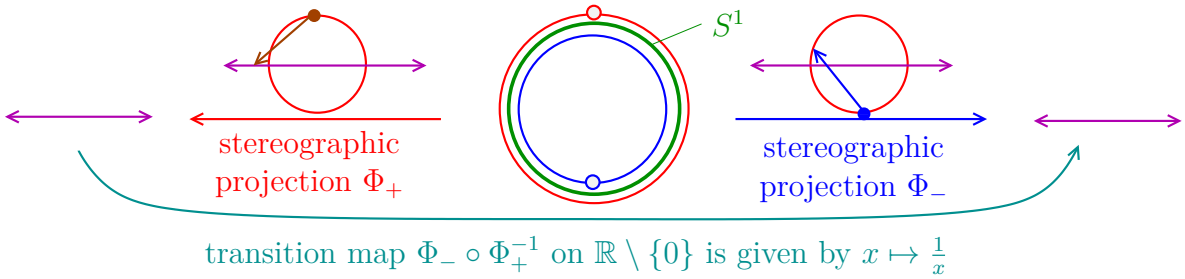
$$\Phi_2 \circ \Phi_1^{-1}\colon \underbrace{\Phi_1(U_1 \cap U_2)}_{\text{open subset in } \mathbb{R}^n \text{ or } \mathbb{H}^n} \to \underbrace{\Phi_2(U_1 \cap U_2)}_{\subseteq \mathbb{R}^n}$$

is a diffeomorphism.[294]

(2) An n-dimensional atlas of a topological manifold for X is called smooth if all charts in the atlas are smoothly compatible to one another.



Example. Let $n \in \mathbb{N}$. We denote by $N := (0, \dots, 0, 1)$ the North Pole of S^n and we denote by $S := (0, \dots, 0, -1)$ the South Pole of S^n. Let $\Phi_+\colon S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection from the North Pole N and let $\Phi_-\colon S^n \setminus \{S\} \to \mathbb{R}^n$ be the stereographic projection from the South Pole S, as defined in the Stereographic Projection Lemma 4.24.



[293] Once again we refer to Definition 24.3 for the definition of a smooth map $U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{H}^n$. We also refer to the Whitney Extension Theorem 24.11 for an alternative characterization of smooth maps on $\mathbb{H}^n$.

[294] Note that this condition is indeed symmetric in 1 and 2.

In Exercise 23.19 we saw that the map $\Phi_- \circ \Phi_+^{-1}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by the diffeomorphism $x \mapsto \frac{1}{\|x\|^2} \cdot x$. This shows that Φ_+ and Φ_- are smoothly compatible. Thus the atlas for S^n given by Φ_+ and Φ_- is a smooth atlas. □

Definition 25.3. Let M be a topological manifold and let $\mathcal{A}$ be a smooth atlas for M. We say that $\mathcal{A}$ is maximal if every chart Ψ that is smoothly compatible with all the charts in the smooth atlas $\mathcal{A}$ is already contained in $\mathcal{A}$.

Proposition 25.4. (Maximal Atlas Existence Proposition) Let M be a topological manifold and let $\mathcal{A}$ be a smooth atlas for M.

(1) The smooth atlas is contained in a unique maximal smooth atlas for M.
(2) The unique maximal smooth atlas from (1) is given by the set of all charts of type (i) and (ii) from Definition 23.2 that are smoothly compatible with all charts in the given smooth atlas $\mathcal{A}$.

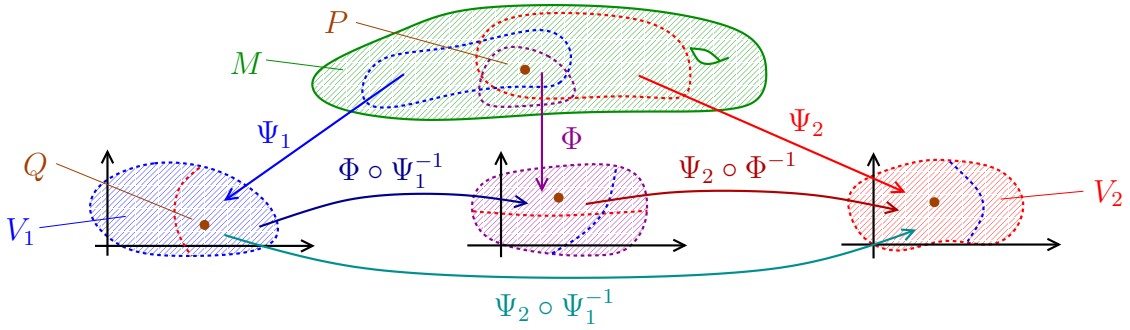
Proof. Let $\mathcal{A}$ be a smooth atlas for M. We denote by $\mathcal{B}$ the set of all charts of M that are smoothly compatible with respect to all charts in $\mathcal{A}$.

Claim 1. The charts in $\mathcal{B}$ form a smooth atlas.

Proof. Let $\Psi_1\colon U_1 \to V_1$ and $\Psi_2\colon U_2 \to V_2$ be two charts in $\mathcal{B}$. We need to show that Ψ_1 and Ψ_2 are smoothly compatible, i.e. we need to show that the transition map

$$\Psi_2 \circ \Psi_1^{-1}\colon \Psi_1(U_1 \cap U_2) \;\to\; \Psi_2(U_1 \cap U_2)$$

is smooth. By the Smooth Pasting Proposition 24.7 it suffices to show that the map is locally smooth. Thus let $Q \in \Psi_1(U_1 \cap U_j)$. We write $P := \Psi_1^{-1}(Q) \in M$. We pick a chart $\Phi\colon A \to B$ from our smooth atlas $\mathcal{A}$ with $P \in A$. Since Ψ_1 and Ψ_2 are smoothly compatible with all charts in $\mathcal{A}$ we know that the two transition maps $\Phi \circ \Psi_1^{-1}\colon \Psi_1(U_1 \cap A) \to \Phi(U_1 \cap A)$ and $\Psi_2 \circ \Phi^{-1}\colon \Phi(U_2 \cap A) \to \Psi_2(U_2 \cap A)$ are smooth.



Thus we see that the restriction of $\Psi_2 \circ \Psi_1^{-1}$ to the open neighborhood $\Psi_1(U_1 \cap U_2 \cap A)$ of P is smooth. ⊞

It remains to prove the following claim:

Claim 2. The smooth atlas $\mathcal{B}$ is maximal and it is the unique maximal smooth atlas that contains $\mathcal{A}$.

Proof. If Ψ is a chart that is smoothly compatible with all charts in $\mathcal{B}$, then it is also smoothly compatible with all charts in $\mathcal{A}$, thus Ψ is already contained in $\mathcal{B}$. This shows that $\mathcal{B}$ is maximal.

Now let $\mathcal{C}$ be a maximal smooth atlas that contains $\mathcal{A}$. We need to show that $\mathcal{B} = \mathcal{C}$. First let $\Phi \in \mathcal{C}$. By definition Φ is smoothly compatible with all charts $\mathcal{A}$, thus $\Phi \in \mathcal{B}$. Now let $\Phi \in \mathcal{B}$. The proof of Claim 1 shows that Φ is also smoothly compatible with any chart that is smoothly compatible with all charts in $\mathcal{A}$. Thus we see that $\Phi \in \mathcal{C}$. ■

Definition 25.5. Let X be a topological manifold.

(1) An n-dimensional smooth structure on X is a choice of a maximal n-dimensional smooth atlas for X.
(2) Let $\mathcal{A}$ be an n-dimensional smooth atlas for X. By the Maximal Atlas Existence Proposition 25.4 the smooth atlas $\mathcal{A}$ is contained in a unique maximal n-dimensional smooth atlas $\mathcal{M}$. We refer to $\mathcal{M}$ as the smooth structure defined by $\mathcal{A}$.[295]
(3) An n-dimensional smooth manifold is a pair $(X, \mathcal{M})$ consisting of a topological manifold X together with an n-dimensional smooth structure $\mathcal{M}$, i.e. together with a maximal smooth atlas $\mathcal{M}$ for X. (As with many other definitions we usually suppress the choice of a smooth atlas from the notation.) We refer to the topological space X as the underlying topological manifold of the smooth manifold $(X, \mathcal{M})$.

Remark.

(1) In the literature the term "manifold" gets used with wildly different meanings, in particular often it means "topological manifold" or "smooth manifold". To make matters worse, there are also other perfectly reasonable notions of a "manifold", namely in Definition 180.8 we will introduce "PL-manifolds" and in Definition 181.8 we will get to know "combinatorial manifolds". To prevent any misunderstanding we only rarely write "manifold" without any decorations. We only do so in informal discussions where the precise meaning is irrelevant.
(2) Since the underlying topological space of a smooth manifold is a topological manifold, many results on topological manifolds hold by fiat also for smooth manifolds. For example by the Manifold–Local Properties Lemma 23.8 we know that every topological manifold, and thus also every smooth manifold, is locally contractible.
(3) The question which topological manifolds admit smooth atlases is subtle. Perhaps rather shockingly in 1960 Michel Kervaire [**Kerv1960**] showed that there exists a 10-dimensional topological manifold that does not admit a smooth atlas. We will discuss this question in greater detail in **??**.
(4) A continuous map $f\colon U \to \mathbb{R}^m$ from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a *C^k-map*, if all iterated k-fold partial derivatives exist and if they are continuous. Throughout the above discussion one could replace "smooth" by "C^k". This leads to the notion of a C^k-atlas, a C^k-structure and a C^k-manifold. In [**Hirs1976**, Theorem 2.9] it is shown that every C^1-structure gives rise to an essentially unique smooth structure. □

25.2. **Examples of smooth manifolds.** The following proposition gives us interesting examples of smooth manifolds.

[295] In practice we might be a little sloppy every now and then and we might not distinguish in our notation between a smooth atlas and the corresponding maximal smooth atlas, i.e. the corresponding smooth structure.

Proposition 25.6. (Standard Smooth Structures Proposition) The following topological manifolds admit a smooth atlas:

(1) open subsets of $\mathbb{R}^n$ and open subsets of $\mathbb{H}^n$,
(2) all closed balls $\overline{B}^n$,
(3) all spheres $S^n \subseteq \mathbb{R}^{n+1}$,
(4) the n-dimensional torus $(S^1)^n$,
(5) the surfaces of genus $g \in \mathbb{N}_0$, as defined on page 342,
(6) the non-orientable surfaces of genus $g \in \mathbb{N}$ from Definition 9.19.

More precisely, the atlases that we provided in Section 23.1 and in the proof of the Surface-is-Topological Manifold Proposition 23.7 are smooth.

Proof. It is straightforward to verify that the atlases for examples (1) to (4) that we provided in Section 23.1 are smooth. For example on page 660, see also Exercise 23.19, we saw that the transition map for the atlas for S^n given by the stereographic projections from the Poles is smooth, which shows that the given atlas for S^n is smooth.

It remains to consider the (non-orientable) surfaces of genus $g \geq 2$. The atlas that we provided in the proof of the Surface-is-Topological Manifold Proposition 23.7 might initially look dubious. In fact all transition maps are compositions of (some of) the following types of maps:

(1) reflections in lines,
(2) translations,
(3) and for some $\eta > 0$ the map

$$\begin{array}{rcl} \{r \cdot \exp(\mathrm{i}\alpha) \,|\, r \in (0,\eta), \alpha \in (0, \frac{\pi}{4})\} & \to & \{r \cdot \exp(\mathrm{i}\alpha) \,|\, r \in (0,\eta), \alpha \in (0, \frac{3\pi}{4})\} \\ z = r \cdot \exp(\mathrm{i}\varphi) & \mapsto & r \cdot \exp(3\mathrm{i}\varphi) = \frac{1}{|z|^2} \cdot z^3, \end{array}$$

(4) the inverse of (3).

It is clear that the first three types of maps are smooth. It follows from the Differential-of-Diffeomorphism Theorem 24.22 (2) that the fourth map is also smooth. Thus the transition maps in the atlas are smooth. ■

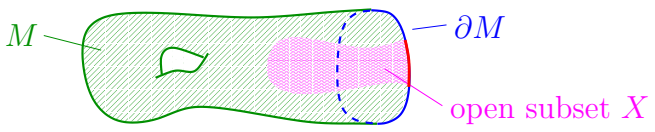
Convention 25.7. As we discussed in the Standard Smooth Structures Proposition 25.6, the atlases that we provided in Section 23.1 and in the proof of the Surface-is-Topological Manifold Proposition 23.7, give smooth atlases for the following topological manifolds:

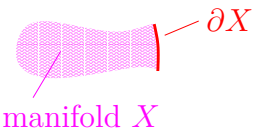
(1) open subsets of $\mathbb{R}^n$ and open subsets of $\mathbb{H}^n$,
(2) all closed balls $\overline{B}^n$,
(3) all spheres $S^n \subseteq \mathbb{R}^{n+1}$,
(4) the n-dimensional torus $(S^1)^n$,
(5) the surfaces of genus $g \in \mathbb{N}_0$, as defined page 342,
(6) the non-orientable surfaces of genus $g \in \mathbb{N}$ from Definition 9.19.

We will invariably work with the corresponding smooth structures, unless we say something different. We refer to these as the standard smooth structures on these topological manifolds.

For completeness we state the following lemma which is closely related to the discussion on page 639.

Lemma 25.8. (Open Subset-Submanifold Lemma) Let M be a smooth manifold and let X be an open subset of M. The restrictions of the charts of the smooth atlas of M to X define a smooth atlas for X.





Remark. In the Open Subset-Submanifold Boundary Lemma 25.11 we will prove an important addendum to the Open Subset-Submanifold Lemma 25.8. □

Proof. The lemma follows easily from the definitions. ■

Convention 25.9. We will constantly use the Open Subset-Submanifold Lemma 25.8 to view open subsets of a smooth manifold as smooth manifolds in their own right.

In Definition 23.2 we defined the boundary of a topological manifold. We can now similarly define the notion of the boundary of a smooth manifold.

Definition 25.10. Let M be a smooth manifold.

(1) We say that a point $x \in M$ is a boundary point if x does not admit a chart of type (i) in the smooth structure. We denote by ∂M the set of all boundary points of M. We refer to ∂M as the boundary of M.
(2) We say that a smooth manifold M is closed if M is compact with $\partial M = \varnothing$.

Remark. As we just mentioned, in Definition 23.2 we already defined the boundary of a topological manifolds. In the Unique Type-of-Chart Proposition 150.11 (ii) we will see that the boundary of a smooth manifold in the above sense agrees with the boundary of a topological manifold. □

The following lemma can be viewed as an addendum to the Open Subset-Submanifold Lemma 25.8.

Lemma 25.11. (Open Subset-Submanifold Boundary Lemma) Let M be a smooth manifold and let X be an open subset of M. We use the Open Subset-Submanifold Lemma 25.8 to view X as a smooth manifold in its own right. The boundary ∂X of X viewed as a smooth manifold is given by $\partial X = X \cap \partial M$.

Proof. The lemma follows easily from the definitions. ■

25.3. **Projective spaces as smooth manifolds.** We start out by recalling Definition 9.11:

Definition 25.12.

(1) The map
$$\begin{array}{rcl} (\mathbb{R}\setminus\{0\}) \times (\mathbb{R}^{n+1}\setminus\{0\}) & \to & \mathbb{R}^{n+1}\setminus\{0\} \\ (z, P) & \mapsto & z \cdot P \end{array}$$
defines a continuous action of the multiplicative group $(\mathbb{R}\setminus\{0\}, \cdot)$ on the topological space $\mathbb{R}^{n+1}\setminus\{0\}$. The quotient space $(\mathbb{R}^{n+1}\setminus\{0\})/(\mathbb{R}\setminus\{0\})$ is called the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$.
(2) In (1) we can replace "$\mathbb{R}$" by "$\mathbb{C}$" and we obtain the n-dimensional complex projective space $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1}\setminus\{0\})/(\mathbb{C}\setminus\{0\})$.

The next proposition shows that the real projective spaces $\mathbb{R}\mathrm{P}^n$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n$ can be viewed as smooth manifolds.

Proposition 25.13. (Projective Space–Manifold Proposition) Let $n \in \mathbb{N}$.

(1) For $i = 0, \dots, n$ we consider the map[296]

$$\begin{aligned} \Phi_i \colon \{[z_0 : \cdots : z_n] \in \mathbb{R}\mathrm{P}^n \mid z_i \neq 0\} &\to \mathbb{R}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\tfrac{z_0}{z_i}, \dots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \dots, \tfrac{z_n}{z_i}\right). \end{aligned}$$

These maps form a smooth atlas for $\mathbb{R}\mathrm{P}^n$ that turn the real projective space $\mathbb{R}\mathrm{P}^n$ into a closed n-dimensional real manifold.

(2) As in (1) we consider for $i = 0, \dots, n$ the map

$$\begin{aligned} \Phi_i \colon \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\} &\to \mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n} \\ [z_0 : \cdots : z_n] &\mapsto \left(\tfrac{z_0}{z_i}, \dots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \dots, \tfrac{z_n}{z_i}\right). \end{aligned}$$

These maps form a smooth atlas for $\mathbb{C}\mathrm{P}^n$ that turn the complex projective space $\mathbb{C}\mathrm{P}^n$ into a closed $2n$-dimensional smooth manifold.

Convention 25.14. We refer to the smooth structure on $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ that we just introduced as the standard smooth structure on $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

Remark. In the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we will see that $\mathbb{C}\mathrm{P}^n$ is in fact a "complex" manifold and in the Grassmannian–Manifold Proposition 270.3 we will generalize the Projective Space–Manifold Proposition 25.13 from projective spaces to "Grassmannians". □

Proof. In the following we provide the proof for the real projective spaces $\mathbb{R}\mathrm{P}^n$. The proof for the complex projective spaces $\mathbb{C}\mathrm{P}^n$ is basically identical. In the Projective Spaces-Compact+Hausdorff Proposition 9.15 (see also Exercise 9.8) we saw that each $\mathbb{R}\mathrm{P}^n$ is compact and Hausdorff. As we will show below, $\mathbb{R}\mathrm{P}^n$ admits a finite atlas, thus it follows from the Countable Atlas-Second Countable Lemma 23.4 that $\mathbb{R}\mathrm{P}^n$ is second-countable.

Claim 1. Let $i \in \{0, \dots, n\}$.

(a) $V_i \coloneqq \{[z_0 : \cdots : z_n] \in \mathbb{R}\mathrm{P}^n \mid z_i \neq 0\}$ is an open subset of $\mathbb{R}\mathrm{P}^n$.

(b) The map $\Phi_i \colon V_i \to \mathbb{R}^n$ is a homeomorphism.

Proof. We denote by $p \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^n$ the natural projection. We set $U_i \coloneqq p^{-1}(V_i)$. Note that $U_i = \{(z_0, \dots, z_n) \in \mathbb{R}^{n+1} \mid z_i \neq 0\}$.

(a) Note that $U_i = p^{-1}(V_i) = \{(z_0, \dots, z_n) \in \mathbb{R}^{n+1} \mid z_i \neq 0\}$ is clearly an open subset of $\mathbb{R}^{n+1} \setminus \{0\}$. It follows from the Definition 8.3 of the quotient topology that $V_i \subseteq \mathbb{R}\mathrm{P}^n$ is an open subset of $\mathbb{R}\mathrm{P}^n$.

(b) It follows from (a) together with Exercise 8.6 that the subspace topology on V_i agrees with the quotient topology on $U_i/(\mathbb{R} \setminus \{0\})$. It follows from this observation and the Topological-Quotient Proposition 8.2 (1b) that the map $\Phi_i \colon V_i \to \mathbb{R}^n$ is continuous. One can easily verify that $\Phi \colon V_i \to \mathbb{R}^n$ is a bijection. Next note that the map

$$\Phi_i^{-1} \colon \mathbb{R}^n \to V_i$$

[296] We use Notation 9.12, i.e. given a point $(z_0, \dots, z_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ we denote by $[z_0 : \cdots : z_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$.

is given by

$$(z_1, \dots, z_n) \;\mapsto\; [z_1 : \cdots : z_i : 1 : z_{i+1} : \cdots : z_n].$$

It follows from the Topological-Quotient Proposition 8.2 (1a) that this map is indeed continuous. ⊞

Evidently we have $\mathbb{R}\mathrm{P}^n = V_0 \cup \cdots \cup V_n$. Thus it follows from Claim 1 that the maps $\Phi_0, \dots, \Phi_n$ form an atlas for $\mathbb{R}\mathrm{P}^n$. It remains to prove the following claim:

Claim 2. The atlas is smooth.

Proof. Let $i, j \in \{0, \dots, n\}$. By symmetry it suffices to consider the case $i < j$. Using the calculation of Φ_i^{-1} in the proof of Claim 1 one can easily show that the corresponding transition map is given by

$$\begin{aligned} \Phi_j \circ \Phi_i^{-1} \colon \{(w_1, \dots, w_n) \in \mathbb{R}^n \,|\, w_j \neq 0\} &\;\to\; \{(w_1, \dots, w_n) \,|\, w_{i+1} \neq 0\} \\ (w_1, \dots, w_n) &\;\mapsto\; \left(\tfrac{w_1}{w_j}, \dots, \tfrac{w_i}{w_j}, \tfrac{1}{w_j}, \tfrac{w_{i+1}}{w_j}, \dots, \tfrac{w_{j-1}}{w_j}, \tfrac{w_{j+1}}{w_j}, \dots, \tfrac{w_n}{w_j}\right). \end{aligned}$$

This map is evidently a diffeomorphism. We have thus shown that the given continuous maps $\Phi_0, \dots, \Phi_n$ form a smooth atlas for $\mathbb{R}\mathrm{P}^n$. ■

25.4. **Smooth maps between smooth manifolds.** We give a few more standard definitions regarding smooth manifolds.

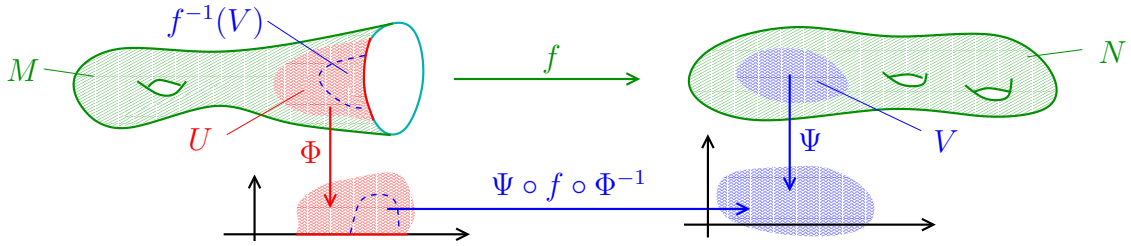
Definition 25.15. Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold.

(1) A map $f \colon M \to N$ is called smooth if it is continuous and if for every chart $\Phi \colon U \to \widetilde{U}$ of M and every chart $\Psi \colon V \to \widetilde{V}$ of N, in the respective smooth structures, the concatenation

$$\Psi \circ f \circ \Phi^{-1} \colon \underbrace{\Phi(U \cap f^{-1}(V))}_{\substack{\text{open subset of } \mathbb{R}^m \text{ or } \mathbb{H}^m, \\ \text{since } f \text{ is continuous}}} \;\to\; \widetilde{V} \;\subseteq\; \mathbb{R}^n$$

is smooth, in the sense of our Definition 24.3.

(2) We denote by $C^\infty(M, N)$ the set of all smooth maps from M to N.



Checking the condition in the definition of a smooth manifold for all charts in the smooth structure is evidently very very painful. Fortunately the following lemma simplifies the life of a working mathematician greatly.

Lemma 25.16. (Smooth-via-Atlas Lemma) Let $f \colon M \to N$ be a continuous map between an m-dimensional smooth manifold M and an n-dimensional smooth manifold N. Furthermore let $\{\Phi_i \colon U_i \to \widetilde{U}_i\}_{i \in I}$ be charts in the smooth structure with $M = \bigcup_{i \in I} U_i$ and let $\{\Psi_j \colon V_j \to \widetilde{V}_j\}_{j \in J}$ be charts in the smooth structure with $N = \bigcup_{j \in J} V_j$. If for all $i \in I$ and $j \in J$ the map

$$\Psi_j \circ f \circ \Phi_i^{-1} \colon \Phi(U_i \cap f^{-1}(V_j)) \ \to \ \widetilde{V}_j \ \subseteq \ \mathbb{R}^n$$

is smooth, then f is smooth.

Proof. The proof of the lemma is somewhat similar to the proof of Claim 1 in the proof of the Maximal Atlas Existence Proposition 25.4. Namely, the lemma follows easily from the Smooth Pasting Proposition 24.7 together with the fact, see the Chain Rule Proposition 24.17, that the composition of smooth maps between open subsets of some $\mathbb{R}^n$'s is again smooth. We leave it to the reader to fill in the details. ■

Example.

(1) We consider maps $f \colon S^n \to \mathbb{R}^m$. As on page 660 we equip S^n with the smooth atlas that is given by the stereographic projections and we equip $\mathbb{R}^m$ with the smooth atlas given by the identity chart. An entertaining calculation shows that the inverses of the stereographic projections $\Phi_\pm$ are given by

$$\begin{array}{rcl} \Phi_\pm^{-1} \colon \mathbb{R}^n & \to & \mathbb{R}^n \setminus \{(0,\dots,0,1)\} \\ (x_1,\dots,x_n) & \mapsto & \left(\frac{2x_1}{x_1^2+\dots+x_n^2+1}, \dots, \frac{2x_n}{x_1^2+\dots+x_n^2+1}, \pm 1 \mp \frac{2}{x_1^2+\dots+x_n^2+1} \right) \end{array}$$

the Smooth-via-Atlas Lemma 25.16 implies that $f \colon S^n \to \mathbb{R}^m$ is smooth precisely if $f \circ \Phi_\pm^{-1} \colon \mathbb{R}^n \to \mathbb{R}^m$ are smooth. For example it follows immediately that the inclusion map $S^n \to \mathbb{R}^{n+1}$ is smooth.

(2) Let U be an open subset of $\mathbb{R}^n$ and let $f \colon U \to \mathbb{R}^m$ be a continuous map. It follows easily from the definitions that the map f is smooth as a map between smooth manifolds if and only if it is smooth in the "classical" sense of Definition 24.1.

(3) Let $n \in \mathbb{Z}$. In Exercise 25.2 we will see that the following map is smooth:

$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

□

The following lemma can be viewed as an analog of the Continuous Composition Lemma 4.2:

Lemma 25.17. (Smooth Composition Lemma) The composition of two smooth maps between smooth manifolds is again smooth.

Proof. First recall that it follows from the Continuous Composition Lemma 4.2 that the composition of two continuos maps is again continuous. Next note that it follows from the Chain Rule 24.17 that the desired statement holds for smooth maps between open subsets of some $\mathbb{R}^n$'s. The statement for smooth maps between smooth manifolds follows fairly easily from this observation, together with the Smooth-via-Atlas Lemma 25.16 and the Smooth Pasting Proposition 24.7. We leave it to the reader to fill in the details. ■

Definition 25.18.

(1) We say that a map $f \colon M \to N$ between smooth manifolds is a diffeomorphism if f is smooth, if f is a bijection and if $f^{-1} \colon N \to M$ is also smooth. If such a diffeomorphism exists, then we say that M and N are diffeomorphic.

(2) Let $f\colon M \to N$ be a continuous map between two smooth manifolds and let $P \in M$. We say that f is a local diffeomorphism at P if there exist open neighborhoods U of P in M and V of $f(P)$ in N such that[297] $f|_U\colon U \to V$ is a diffeomorphism.

Example.

(1) Let $A \in \mathrm{GL}(n,\mathbb{R})$ be an invertible real matrix. It is clear that the map $\mathbb{R}^n \to \mathbb{R}^n$ that is given by multiplication by A is a diffeomorphism. This simple observation has a useful consequence: Let V be an n-dimensional real vector space. We pick an isomorphism $\varphi\colon \mathbb{R}^n \to V$ and as on page 183 we equip V with the unique topology that turns φ into a homeomorphism. Furthermore we now equip V with the unique smooth structure that turns φ into a diffeomorphism. Since any two such isomorphisms differ by multiplication by some $A \in \mathrm{GL}(n,\mathbb{R})$ we see that these constructions do not depend on the choice of φ. We refer to this smooth structure on V as the standard smooth structure on V.

(2) Let $A \in \mathrm{O}(n)$. It is true, but slightly nasty to verify directly from the definitions, that the map

$$\begin{array}{ccc} S^{n-1} & \to & S^{n-1} \\ v & \mapsto & A \cdot v \end{array}$$

is a diffeomorphism. We will give a satisfactory argument on page 673.

(3) Let $n \in \mathbb{N}$. In Exercise 25.3 we will see that the map

$$\begin{array}{ccc} \mathbb{R}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)) \end{array}$$

is a local diffeomorphism. □

Remark. The Inverse Function Theorem 24.21 gives a tremendously convenient way for showing that a smooth map between open subsets of $\mathbb{R}^n$ is a local diffeomorphism. Later on, in the Inverse Mapping Theorem 30.1 we will state and prove a generalization of the Inverse Function Theorem 24.21 to the more general context of smooth maps between smooth manifolds. □

The following harmless proposition contains two basic statements about smooth maps between smooth manifolds. Statement (1) can be viewed as a smooth analog of the the Pasting Proposition 4.7

Proposition 25.19. (Smooth Pasting Proposition II) Let $f\colon M \to N$ be a continuous map between two smooth manifolds.

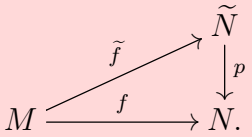
(1) (a) If there exists an open cover $\{U_i\}_{i\in I}$ of M such that for every $i \in I$ the restriction $f|_{U_i}\colon U_i \to N$ is smooth, then f itself is smooth.

(b) If there exists a smooth atlas $\{\Phi_i\colon U_i \to \widetilde{U}_i\}_{i\in I}$ for M and if there exists a smooth atlas $\{\Psi_j\colon V_j \to \widetilde{V}_j\}_{j\in J}$ for N such that $\bigcup_{i,j}(U_i \cap f^{-1}(V_j)) = M$ and such that each $\Psi_j \circ f \circ \Phi_i^{-1}\colon \Phi_i(U_i \cap f^{-1}(V_j)) \to \widetilde{V}_j$ is smooth, then f is smooth.

(2) If $f\colon M \to N$ is a bijection (e.g. if it is a homeomorphism) and if it is a local diffeomorphism, then it is in fact a diffeomorphism.[298]

[297] By the Open Subset-Submanifold Lemma 25.8 we can and will view U and V as smooth manifolds in their own right.

(3) Let $\widetilde{N}$ be a smooth manifold and let $p\colon \widetilde{N} \to N$ be a local diffeomorphism. Furthermore let $\widetilde{f}\colon M \to \widetilde{N}$ be a continuous map such that the following diagram commutes:



Then $f\colon M \to N$ is smooth if and only if $\widetilde{f}\colon M \to \widetilde{N}$ is smooth.

Remark.

(1) In contrast to the Pasting Proposition 4.7 it does not suffice to check smoothness on finitely many *closed* subsets. For example, consider the map $\mathbb{R} \to \mathbb{R}$ that is given by $x \mapsto |x|$. The restrictions to the closed subsets $(-\infty, 0]$ and $[0, \infty)$ are smooth, but the map is evidently not smooth on all of $\mathbb{R}$.

(2) In practice one can often control a map $f\colon M \to N$ on compact codimension-zero submanifolds instead of open subsets. As we just pointed out, one cannot conclude that f is smooth. But in the Smooth Manifold Pushout-Proposition 55.12 we will see that one can modify f slightly to turn f into a smooth map. □

Proof.

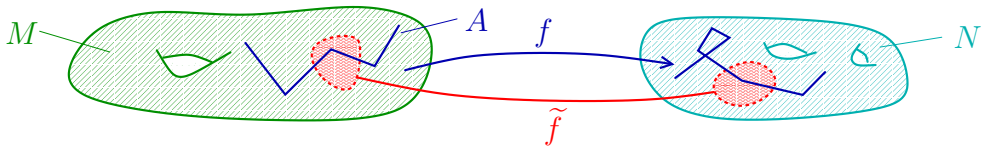
(1) Both statements follow immediately from the Smooth Pasting Proposition 24.7.

(2) Let $f\colon M \to N$ be a local diffeomorphism that is a bijection. We need to show that $f^{-1}\colon N \to M$ is also smooth. But that follows immediately from (1) and our hypothesis that f is a local diffeomorphism.

(3) This statement follows immediately from (1a). ■

For completeness' sake we provide the following definition which is a just a generalization of Definition 24.3.

Definition 25.20. Let M and N be smooth manifolds. Furthermore let $A \subseteq M$ be some subset and let $f\colon A \to N$ be a continuous map. We say that f is smooth if given any $a \in A$ there exists an open neighborhood U in M of a and a smooth map $\widetilde{f}\colon U \to N$[299] and which coincides with f on A.



Definition 25.21. Let M be a topological manifold. We say that two smooth structures $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exists a diffeomorphism between $(M, \mathcal{A})$ and $(M, \mathcal{B})$.

Remark.

[298] This statement is the analog of the Bijective-Local Homeomorphism Lemma 4.62.

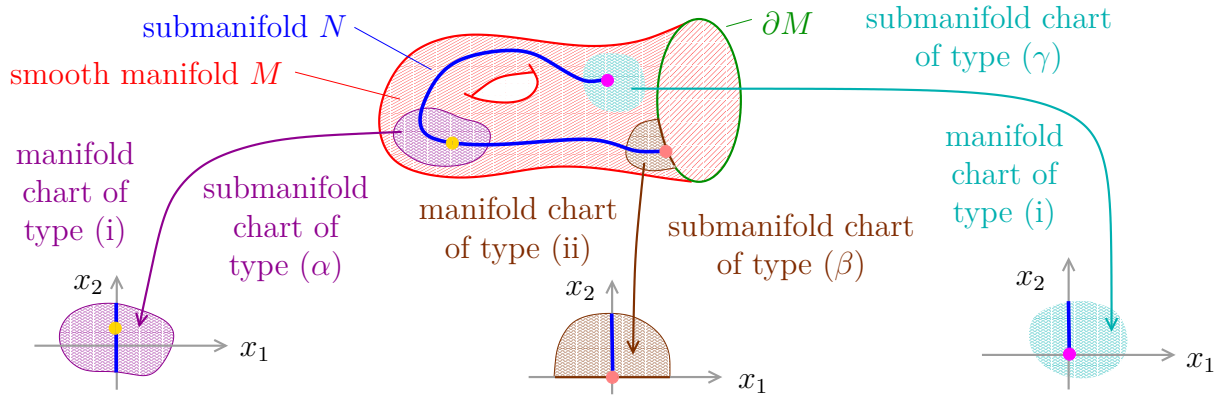
[299] Let $U \subseteq M$ be an open subset and let $\widetilde{f}\colon U \to N$ be a continuous map. We use the convention on page 663 to view U as a smooth manifold in its own right, and thus we have defined what it means for $\widetilde{f}$ to be smooth.

(1) There exist topological manifolds that admits inequivalent smooth structures. For example, as we will see in the 7-Sphere–Smooth Structures 282.6, John Milnor [**Miln1956a**] showed in 1956 that S^7 admits at least two inequivalent smooth structures.
(2) In the High-Dimensional Manifold–Smooth Structure Theorem **??** we will see that in dimensions $n \neq 4$ every closed topological manifold admits at most a finite number of inequivalent smooth structures. Amazingly, this statement is wrong in dimension 4. More precisely there is a plethora of closed 4-dimensional smooth manifolds that admit an infinite number of pairwise inequivalent smooth structures, see e.g. the Exotic K3-Theorem 228.24 for details. □

25.5. **Submanifolds of smooth manifolds.** In Section 23.5 we introduced the notion of a submanifold of a topological manifold. The definition of a submanifold of a smooth manifold is basically the same, the only difference is that now we demand that the submanifold charts are smooth. For the reader's convenience we write down the detailed definition, even though there is a serious overlap with our previous definition.

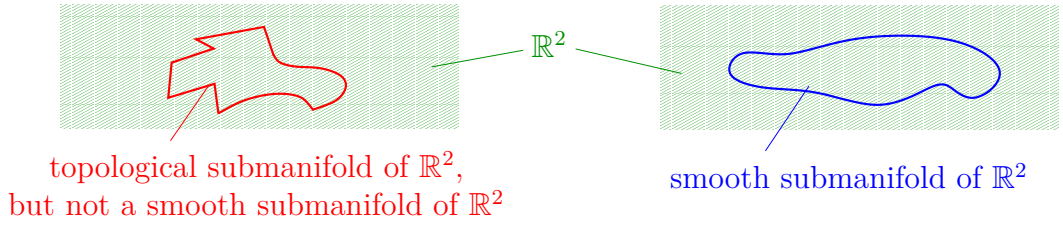
Definition 25.22. Let M be an n-dimensional smooth manifold.

(1) We say that a subset $N \subseteq M$ is a k-dimensional submanifold if given any $P \in N$ one of the following holds:
   (α) there exists a chart $\Phi\colon U \to V$ of type (i) in the smooth structure for M and P such that
   $$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_i \in \mathbb{R}\},$$
   (β) or there exists a chart $\Phi\colon U \to V$ of type (ii) in the smooth structure for M and P such that $\Phi(P)$ lies in $\partial\,\mathbb{H}^n$ and[300]
   $$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\},$$
   (γ) or there exists a chart $\Phi\colon U \to V$ of type (i) in the smooth structure for M and P such that $\Phi(P)$ lies in $\partial\,\mathbb{H}^n$ and
   $$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}.$$
   We refer to such charts as submanifold charts of N. Any collection of submanifold charts of N that covers N is called a submanifold atlas.
(2) We say that a submanifold N is proper if N is a closed subset of M and if for each $P \in N$ we can find submanifold charts of type (α) or (β).



[300]For $k = 0$ we interpret $\{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}$ as the empty set.

Remark. Sometimes the above language can be ambiguous. For example, if we consider $M = \mathbb{R}^2$ it is not clear whether we view M as a topological manifold or as a smooth manifold. If from the context it is not clear what we mean, then we distinguish in our language between a *topological submanifold* and a *smooth submanifold*.

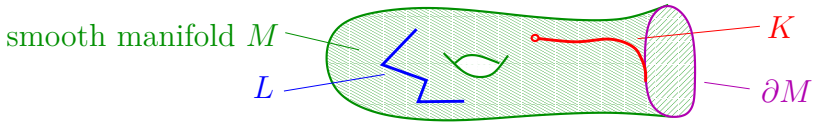


This distinction is illustrated in the figure above. □

Examples.

(1) We consider the topological manifold $M = \mathbb{R}^2$ with the smooth atlas $\mathcal{A}$ given by the identity $\Phi = \mathrm{id}_M$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. We also consider the subset $N = \{(x, f(x) \,|\, x \in \mathbb{R}\} \subseteq \mathbb{R}^2$. We cannot use the unique chart Φ in our atlas $\mathcal{A} = \{\Phi\}$ to show that N is indeed a submanifold of the smooth manifold $(\mathbb{R}^2, \mathcal{A})$. But the chart $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi(x, y) = (x, y - f(x))$ lies in the smooth structure, i.e. in the maximal smooth atlas, defined by $(\mathbb{R}^2, \mathcal{A})$ and we can use it to verify that N is indeed a submanifold of the smooth manifold $(\mathbb{R}^2, \mathcal{A})$.

(2) In the figure below we show a smooth manifold M with boundary together with two subsets K and L.



In Exercise 25.21 we will make the setting more precise and we will see that both K and L are submanifolds of the topological manifold M but that they are not submanifolds if we view M as a smooth manifold. □

Remark. As we remarked on page 639, there are various subtly different definitions of a "submanifold" out in the literature. This is even more pronounced when it comes to submanifolds of smooth manifolds, especially if M and N are allowed to have boundary.

(1) Surely there is no textbook that would consider L in the above figure as a submanifold of the smooth manifold M. As we remarked above, if we follow our convention or the convention of [**Wall2016**, p. 31], then K is not a submanifold of M. On the other hand the definition used in [**Lee2002**, p. 98] or [**Hirs1976**, p. 30] views K as a smooth submanifold of the smooth manifold M.

(2) Also note that what we call a proper submanifold is called a neat submanifold in [**Hirs1976**, p. 30]. □

The following lemma is an analog of the Submanifold-is-Manifold Lemma 23.19 and of the Subspace Continuity Lemma 4.3:

Lemma 25.23. (Submanifold Smooth Structure Lemma) Let M be a smooth manifold and let N be a smooth submanifold of M. The collection of all submanifold charts

of M gives rise naturally to a smooth atlas for N. The corresponding smooth structure has the following properties:

(1) (a) The inclusion map $N \to M$ is a smooth map.
    (b) If we start out with some other submanifold atlas for N, then this gives rise to the same smooth structure on N.
(2) Let $f\colon M \to Z$ be a smooth map to a smooth manifold Z. The restriction of f to $f\colon N \to Z$ is also smooth.
(3) Let $f\colon W \to M$ be a smooth map. If $f(W) \subseteq N$, then $f\colon W \to N$ is also smooth.
(4) We assume that N is a submanifold of codimension-zero. Let $f\colon N \to W$ be a continuous map to a smooth manifold. The following two statements are equivalent:
    (a) The map $f\colon N \to W$ is smooth as a map between smooth manifolds.
    (b) The map $f\colon N \to W$ is smooth in the sense of Definition 25.20, where we think of N as a subset of the smooth manifold M.

Convention 25.24. In the setting of the Submanifold Smooth Structure Lemma 25.23 we will refer to the smooth structure on the submanifold N, which we construct in the proof of the Submanifold Smooth Structure Lemma 25.23, as the standard smooth structure on N.[301]

Proof. We start out with the following claim:

Claim. Let U and V be open subsets of $\mathbb{R}^n$ and let $f\colon U \to V$ be a smooth map.
(a) If for every $y \in \mathbb{R}^k$ we have $f(0,y) \in \{0\} \times \mathbb{R}^k$, then the map

$$\underbrace{\{y \in \mathbb{R}^k \,|\, (0,y) \in U\}}_{\text{open subset of } \mathbb{R}^k} \xrightarrow{f} V \cap (\{0\} \times \mathbb{R}^k) \xrightarrow{(0,y)\mapsto y} \underbrace{\{y \in \mathbb{R}^k \,|\, (0,y) \in V\}}_{\text{open subset of } \mathbb{R}^k}$$

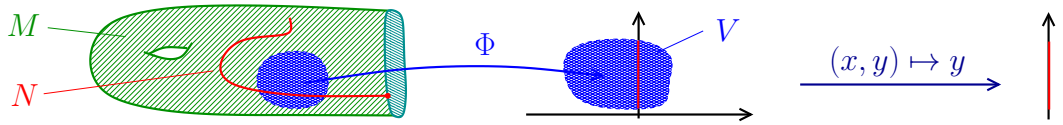
is smooth.
(b) The analog of (a) holds if we replace the $\mathbb{R}^k$ to the left by $\mathbb{H}^k$.

Proof. The claim follows from the observation that the map to the right is the restriction of the projection $\mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$, which evidently is smooth, to a suitable subset. ⊞

Let M be an n-dimensional smooth manifold and let $N \subseteq M$ be a k-dimensional submanifold. It follows from the Subspace-Hausdorff Lemma 3.31 and the Second Countability Lemma 13.2 that N is Hausdorff and second-countable. Now let $\Phi\colon U \to V$ be a submanifold chart of type (α). As we pointed out in the proof of the Submanifold-is-Manifold Lemma 23.19 it is pretty straightforward to verify that the map

$$U \cap N \xrightarrow{\Phi} V \cap \{(0,x) \in \mathbb{R}^{n-k} \times \mathbb{R}^k\} \xrightarrow{(0,y)\mapsto y} \{x \in \mathbb{R}^k \,|\, (0,x) \in V\}$$

is a chart of type (i) for N. In the same fashion submanifold charts of type (β) and (γ) give rise to charts of type (ii) for N.



[301] The reader might be worried that we equip certain topological manifolds with different "standard smooth structures". But, as we will see on page 672, for S^n this new standard smooth structure equals the standard smooth structure introduced earlier on.

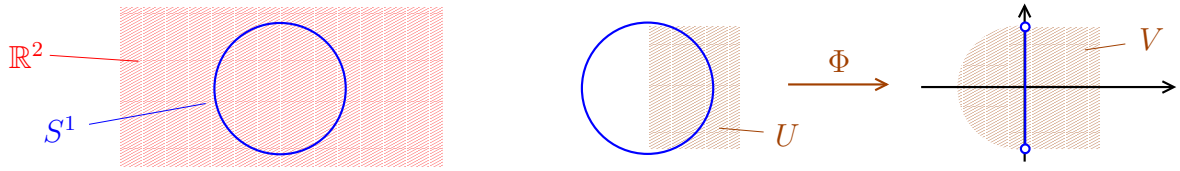
(1) (a) It follows easily from the claim that the above charts define a smooth atlas on N.
(b) Again using the claim one can easily show that any submanifold atlas for N gives rise to the same smooth structure on N.
(2) This statement follows immediately from (1) together with the Smooth Composition Lemma 25.17.
(3) Let $f\colon W \to M$ be a smooth map between smooth manifolds with $f(W) \subseteq N$. It follows easily from the fact that the projection $\mathbb{R}^n \xrightarrow{(x,y)\mapsto y} \mathbb{R}^k$ is smooth that the map $f\colon W \to N$ is also smooth.
(4) This statement follows easily, but not immediately, from the definitions. We leave it to the reader to fill in the details. ■

Examples.

(1) Let $n \in \mathbb{N}$. We consider the n-dimensional sphere $S^n \subseteq \mathbb{R}^{n+1}$. Similar to the discussion on page 623 we consider the map

$$\begin{array}{rcl} \Phi\colon U := (0,\infty) \times B^n & \to & V = \{(y,x) \in \mathbb{R} \mid y > -\|x\|^2\} \times B^n \\ (x,y) & \mapsto & (y - \|x\|^2, x). \end{array}$$

One can easily verify that this is a submanifold chart of type (α) from the smooth structure of $\mathbb{R}^{n+1}$. By changing the roles of the coordinates and changing signs we see that we can cover S^n by altogether $2(n+1)$ such submanifold charts which shows that S^n is an n-dimensional submanifold of the smooth manifold $\mathbb{R}^{n+1}$. In particular we obtain from the Submanifold Smooth Structure Lemma 25.23 a smooth atlas for S^n. We leave it to the reader to verify, using the Submanifold Smooth Structure Lemma 25.23 (1b), that this smooth atlas is smoothly compatible to the smooth atlas that we gave on page 622. In particular both define the same smooth structure.



(2) Let M be a smooth manifold and let $U \subseteq M$ be an open subset. As on page 639 one sees easily that U is a codimension-zero submanifold of M. Furthermore, if we use the Submanifold Smooth Structure Lemma 25.23 to view U as a smooth manifold in its own right, then we obtain, basically by definition, the same smooth structure as we obtained in the Open Subset-Submanifold Lemma 25.8. Finally note that it follows again basically immediately from the definitions that $\partial U = U \cap \partial M$.
(3) Let $k \leq n$. Using the charts in the Projective Space–Manifold Proposition 25.13 one can easily show that $\{[(x,0)] \in \mathbb{R}\mathrm{P}^b \mid x \in \mathbb{R}^{k+1} \setminus \{0\}\}$ is a k-dimensional smooth submanifold of $\mathbb{R}\mathrm{P}^n$ which is diffeomorphic to $\mathbb{R}\mathrm{P}^k$. The same applies for $\mathbb{C}\mathrm{P}^k$ and $\mathbb{C}\mathrm{P}^n$. □

Remark. If one is mathematically very honest, then one realizes that it is actually a pain to work directly with the definition of the standard smooth structure on a submanifold. For example, just using the definition, it is not that easy to show that a given map $f\colon N \to W$ is smooth. Fortunately life will improve once we have proved the Smooth Embedding Theorem 30.15. □

It is worth stressing the following special case of the Submanifold Smooth Structure Lemma 25.23.

Lemma 25.25. (Restrict-to-Submanifold-or-Boundary Lemma)
(1) Let M be a smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to N$ be a continuous map to some smooth manifold N. If there exists an open neighborhood U of $M \subseteq \mathbb{R}^n$ and a map $g\colon U \to N$ that is smooth in the sense from Definition 24.1 and which satisfies $g|_M = f$, then f itself is smooth.
(2) Let M be a smooth manifold and let N be a smooth submanifold of some $\mathbb{R}^n$. Let $f\colon M \to N$ be a continuous map such that the composition $f\colon M \to N \to \mathbb{R}^n$ is smooth. Then f itself is smooth.

Proof.
(1) We denote by $i\colon M \to U$ the inclusion map. It follows from the Submanifold Smooth Structure Lemma 25.23 (2) that i is smooth. By the discussion on page 666 we know that $g\colon U \to N$ is smooth as a map between smooth manifolds. Finally it follows from the Smooth Composition Lemma 25.17 that $f \circ i\colon M \to N$ is also smooth.
(2) This statement is just a special case of the Submanifold Smooth Structure Lemma 25.23 (3). ■

Example. Let $A \in \mathrm{O}(n)$. We consider again the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & S^{n-1} \\ v & \mapsto & A \cdot v. \end{array}$$

We denote by $g\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is also given by multiplication by A. By the discussion on page 651 we know that this defines a smooth map on $\mathbb{R}^n$. It follows from the Restrict-to-Submanifold-or-Boundary Lemma 25.25 (1) that the restriction $S^{n-1} \to \mathbb{R}^n$ is smooth. But this map takes values in S^{n-1}, so it follows from the Restrict-to-Submanifold-or-Boundary Lemma 25.25 (2) that the original map $f\colon S^{n-1} \to S^{n-1}$ is actually smooth. The same argument shows of course that f^{-1}, which is given by multiplication with A^{-1} is smooth. Thus we now have a convincing argument that for any $A \in \mathrm{O}(n)$ the map f is a diffeomorphism, solving the slight conundrum from page 667. □

25.6. **The boundary of smooth manifolds.** In this section and the following we will employ several results from Section 24.1 to draw conclusions on smooth manifolds and their boundaries. The fact that real analysis provides us with such powerful tools, like the Inverse Function Theorem 24.21 and the resulting Differential-of-Diffeomorphism Theorem 24.22, is one of the reasons why dealing with smooth manifolds is much easier than dealing with topological manifolds.

We start out with the following proposition which is the smooth analog (and improvement) of the Unique Type-of-Chart Proposition 23.13:

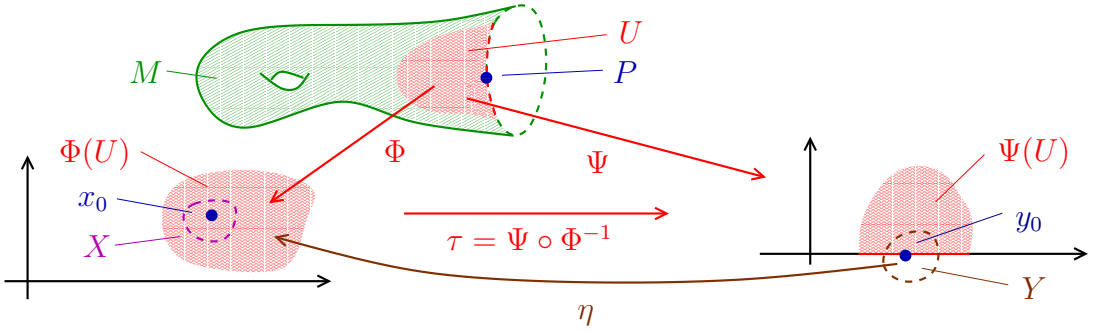
Proposition 25.26. (Unique Type-of-Chart Proposition) Let M be a smooth manifold and let $P \in M$. Precisely one of the following two statements holds:
(1) The point P admits a chart of type (i) from the smooth structure.
(2) The point P admits a chart of type (ii) from the smooth structure.

Remark.

(1) Let M be a smooth manifold. By the Unique Type-of-Chart Proposition 25.26 we now know that a point $P \in M$ is a boundary point if and only if Px admits a chart of type (ii).
(2) The proof of the Unique Type-of-Chart Proposition 25.26 makes use of smooth tools, more precisely it makes use of the Differential-of-Diffeomorphism Theorem 24.22. We have to postpone a proof of the analog of the Unique Type-of-Chart Proposition 25.26 for topological manifolds to the Unique Type-of-Chart Proposition 150.11 (i). □

Proof. Let M be an n-dimensional topological manifold that is equipped with a smooth atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$. Suppose that there exists a point P that admits a chart Φ of type (i) and a chart Ψ of type (ii) in the smooth structure. By intersecting the two domains, which are both open neighborhoods of P, we can assume that the charts have a common domain U. We set $x_0 = \Phi(P)$ and $y_0 = \Psi(P)$.

We set $\tau := \Psi \circ \Phi^{-1}$. Since Φ and Ψ belong to a smooth atlas we see that τ and τ^{-1} are smooth. Note that by Definition 24.3 of a smooth map on a (not necessarily open) subset of $\mathbb{R}^n$ there exists an open neighborhood Y of $y_0 \in \mathbb{R}^n$ and a smooth map $\eta \colon Y \to \mathbb{R}^n$ such that η agrees with τ^{-1} on $Y \cap \Psi(U)$.



Claim. There exists an open subset $X \subseteq \mathbb{R}^n$ with $x_0 \in X \subseteq \tau^{-1}(Y) \cap \Phi(U)$.

Proof. Note that $\tau \colon \Phi(U) \to \Psi(U)$ is in particular continuous and that, by definition of the subspace topology, $Y \cap \Psi(U)$ is an open neighborhood of $y_0 \in \Psi(U)$. Thus we see that $\tau^{-1}(Y)$ is an open neighborhood of x_0 in $\Phi(U)$. Since $\Phi(U)$ is open in $\mathbb{R}^n$ we know by the Open-Closed Inclusion Lemma 4.13 is also open in $\mathbb{R}^n$. Thus $X := \tau^{-1}(Y) \cap \Phi(U)$ has the desired property. ⊞

We make the following observations:

(a) Let $x \in X$. Note that $\eta \circ \tau = \mathrm{id}_X$. Since X is an open subset of $\mathbb{R}^n$ we can take differentials at x of both sides. Since X is open and since Y is open we can apply the Chain Rule 24.17 to the left-hand side and we see that $\mathrm{D}\tau_x$ is invertible.
(b) It follows from (a) and the Differential-of-Diffeomorphism Theorem 24.22 that $\tau(X)$ is an open subset of $\mathbb{R}^n$.
(c) Since $X \subseteq \Phi(U)$ we have $\tau(X) \subseteq \tau(\Phi(U)) = \Psi(U) \subseteq \mathbb{H}^n$. Furthermore we know that $\tau(x_0) = y_0 \in \Psi(U)$ and that $y_0 \in \partial\,\mathbb{H}^n$. In summary, we have that $\tau(X) \subseteq \mathbb{H}^n$ and that $\tau(X) \cap \partial\,\mathbb{H}^n \neq \varnothing$.
(d) Finally note that (b) and (c) are contradictory. ■

Next the following is the smooth analog of the Topological Manifold–Boundary Proposition 23.15.

Proposition 25.27. (Smooth Manifold–Boundary Proposition) Let M be an n-dimensional smooth manifold. The following statements hold:

(1) The restrictions of all of the charts of the smooth atlas to the boundary ∂M give rise to a natural smooth atlas on ∂M and turn ∂M into a smooth $(n-1)$-dimensional smooth manifold with $\partial(\partial M) = \varnothing$. The corresponding natural smooth structure has the following properties:[302]
   (a) The inclusion map $\partial M \to M$ is a smooth map.
   (b) Restrictions of any set of charts of M that cover ∂M define the same smooth structure on ∂M.
   (c) Let $f\colon W \to M$ be a smooth map between smooth manifolds. If $f(W) \subseteq \partial M$, then $f\colon W \to \partial M$ is also smooth.

(2) (a) Any union of components of ∂M is a closed subset of M. In particular ∂M itself is a closed subset of M.
   (b) If M is compact, then ∂M is also compact.

Convention 25.28. Let M be an n-dimensional smooth manifold. We refer to the smooth structure on ∂M that we obtain in the proof of the Smooth Manifold–Boundary Proposition 25.27 (1), by restricting charts from M, as the standard smooth structure on ∂M.

Remark. In practice the description of the smooth structure on ∂M can be a nuisance to work with. An alternative point of view is given in Exercise 30.16. □

Example. Let $n \in \mathbb{N}$. On page 623 we constructed an explicit smooth atlas for $\overline{B}^n$ which gives charts of type (ii) precisely for the points in S^{n-1}. Thus it follows from the Unique Type-of-Chart Proposition 25.26 that the boundary of the smooth manifold $\overline{B}^n$ is indeed given by S^{n-1}. We leave it to the reader to verify, the Smooth Manifold–Boundary Proposition 25.27 (1b), that the smooth structure on $\partial\overline{B}^n = S^{n-1}$ agrees with the standard smooth structure on S^{n-1} that we introduced on page 662. □

Proof. The remainder of the proof consists mostly of cleverly applying Unique Type-of-Chart Proposition 25.26. In the unlikely case that the reader does not feel like reading more technical proofs it is a good moment to move on.

(1) First note that it follows from the Subspace-Hausdorff Lemma 3.31 and from the Second Countability Lemma 13.2 (2) that ∂M is Hausdorff and second-countable.

Next we want to show that the smooth atlas on M gives rise to a natural smooth atlas on ∂M. We start out with the following claim:

Claim. Let U and V be open subsets of $\mathbb{H}^n$ and let $f\colon U \to V$ be a smooth map. If for every $y \in \mathbb{R}^{n-1}$ we have $f(y,0) \in \mathbb{R}^{n-1} \times \{0\}$, then the map

$$\underbrace{\{x \in \mathbb{R}^{n-1} \mid (x,0) \in U\}}_{\text{open subset of } \mathbb{R}^{n-1}} \xrightarrow{f} V \cap (\mathbb{R}^{n-1} \times \{0\}) \xrightarrow{(x,0)\mapsto x} \underbrace{\{x \in \mathbb{R}^{n-1} \mid (x,0) \in V\}}_{\text{open subset of } \mathbb{R}^{n-1}}$$

is smooth.

[301] This statement is similar in nature to the statement of the Submanifold Smooth Structure Lemma 25.23.
[302] Here ∂N denotes the boundary of N, where we use the Submanifold Smooth Structure Lemma 25.23 to view N as a smooth manifold in its own right.

Proof. This statement follows easily from the definition of a smooth map on open subsets of $\mathbb{H}^n$ and the fact that the composition of smooth maps is again smooth. ⊞

We continue with the following observation. Let $\Phi\colon U \to V$ be a chart of type (ii) for M. Note that it is a chart of type (ii) precisely for the points in $\Phi^{-1}(\partial\,\mathbb{H}^n)$. It follows from the Unique Type-of-Chart Proposition 25.26 and the definition of ∂M that $\Phi^{-1}(\partial\,\mathbb{H}^n) = \partial M \cap U$.

Given $i \in I$ we now set $\widetilde{V}_i = V_i \cap \partial\,\mathbb{H}^n$, we set $\widetilde{U}_i \coloneqq \Phi_i^{-1}(\widetilde{V}_i)$ and we set $\widetilde{\Phi}_i \coloneqq \Phi_i|_{\widetilde{U}_i}$. Furthermore we denote by $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the map that is given by the projection onto the first $n-1$ coordinates. It follows fairly easily from the above discussion and the claim that $\{p \circ \widetilde{\Phi}_i\colon \widetilde{U}_i \to p(\widetilde{V}_i)\}_{i\in I}$ is a smooth atlas for ∂M and that with respect to this atlas every point in ∂M admits a chart of type (i). In other words, $\partial(\partial M) = \varnothing$.

(a) It follows easily from the fact that the inclusion $\mathbb{R}^{n-1} \xrightarrow{x \mapsto (x,0)} \mathbb{R}^n$ is smooth that the inclusion $\partial M \to M$ is a smooth map.

(b) It follows easily from the claim that if we start out with some set of charts covering ∂M, that these define the same smooth structure on ∂M.

(c) Let $f\colon W \to M$ be a smooth map between smooth manifolds with $f(W) \subseteq \partial M$. It follows easily from the fact that the projection $\mathbb{H}^n \xrightarrow{(x,y) \mapsto x} \mathbb{R}^{n-1}$ is smooth that the map $f\colon W \to \partial M$ is also smooth.

(2) The proof of Statement (2) is basically identical to the proof of the Topological Manifold–Boundary Proposition 23.15 (2).[303] ■

For completeness we state the following lemma, which is the obvious analog of the Boundary–Local Homeomorphism Lemma 23.14.

Lemma 25.29. (Boundary–Diffeomorphism Lemma) Let M and N be two smooth manifolds. If $f\colon M \to N$ is a local diffeomorphism, then $f(\partial M) = \partial N \cap f(M)$.

Proof. The proof is basically identical to the proof of the sister Boundary–Local Homeomorphism Lemma 23.14. ■

Proposition 25.30. (Proper Submanifold Proposition) Let M be an n-dimensional smooth manifold. The following statements hold:

(1) Let N be a submanifold of M. The following two statements are equivalent:
  (a) N is a proper submanifold in the sense of Definition 25.22.
  (b) The following two conditions are satisfied:
    (i) N is a closed subset of M.
    (ii) We have the equality $\partial N = N \cap \partial M$.

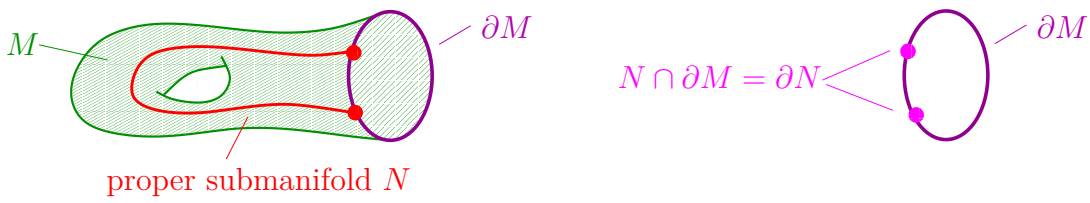
(2) If N is a proper submanifold of M, then[304] $\partial N = N \cap \partial M$ is a proper submanifold of ∂M.

Proof.

[303] Note that this statement does not follow directly from the Topological Manifold–Boundary Proposition 23.15 (2) since our definition of the boundary of a smooth manifold, given in Definition 25.10, is a priori different from the definition of the boundary of a topological manifold. Only later in the Unique Type-of-Chart Proposition 150.11 (ii) will we see that there is no difference.

[303] This statement is similar in nature to the statement of the Submanifold Smooth Structure Lemma 25.23.

[304] Here ∂N denotes the boundary of N, where we use the Submanifold Smooth Structure Lemma 25.23 to view N as a smooth manifold in its own right.



(1) Let $N \subseteq M$ be a k-dimensional submanifold. Recall that N is called *proper* if N is a closed subset of M and if every point in N admits a submanifold chart of type (α) or (β), as introduced in Definition 25.22. As in the proof of the Submanifold-is-Manifold Lemma 23.19 we denote by $q\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the last k coordinates and we make the following observations:
- If P admits a submanifold chart of type (α), then this submanifold chart is a chart of type (i) for $P \in M$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (i) for $P \in N$.
- If P admits a submanifold chart of type (β), then this submanifold chart is a chart of type (ii) for $P \in M$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.
- If P admits a submanifold chart of type (γ), then this submanifold chart is a chart of type (i) for $P \in M$ and the map $q \circ \Phi|_{U\cap N}$ is a k-dimensional chart of type (ii) for $P \in N$.

The equivalence of (a) and (b) now follows from these three observations together with the Unique Type-of-Chart Proposition 25.26.

(2) Let N be a proper submanifold of M. We want to show that ∂N is a proper submanifold of ∂M.
- As we saw in (1), a point $P \in \partial N$ admits a submanifold chart of type (β). But such a submanifold chart restricts to a submanifold chart of type (α) for ∂N viewed as a subset of ∂M.
- In (1) we just saw that $\partial N = N \cap \partial M$. Since N is a closed subset of M we obtain from the definition of the subspace topology that ∂N is a closed subset of ∂M.

These two observations put together show that ∂N is indeed a proper submanifold of ∂M. ■

25.7. **Dimension of smooth manifolds.** The following proposition says that the dimension of a non-empty smooth manifold is well-defined. Eventually, using homology groups, we will show in the Topological Manifold-Dimension Proposition 150.10 that the analogous statements also holds for topological manifolds.

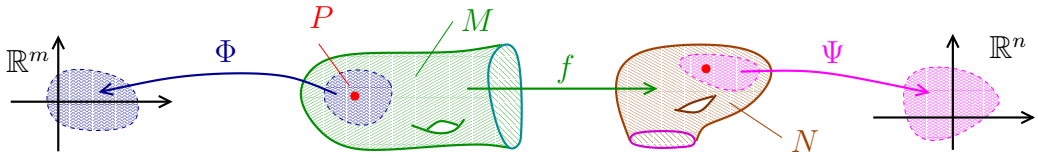
Proposition 25.31. (Smooth Manifold-Dimension Proposition) Let M be a non-empty m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. If M and N are diffeomorphic, then $m = n$.

Proof. Let M be a non-empty m-dimensional smooth manifold, let N be an n-dimensional smooth manifold and let $f\colon M \to N$ be a diffeomorphism.

Claim. There exists a $P \in M$ and a chart $\Phi\colon U \to \widetilde{U}$ of type (i) with $P \in U$ and where $\widetilde{U}$ is an open subset of $\mathbb{R}^m$.

Proof. Since M is non-empty we know that there exists a chart $\Phi\colon U \to \widetilde{U}$ with $U \neq \varnothing$. If Φ is a chart of type (i), then we are done. If Φ is a chart of type (ii), then we consider the chart $\Phi\colon \Phi^{-1}(\widetilde{U} \setminus \partial\mathbb{H}^n) \to \widetilde{U} \setminus \partial\mathbb{H}^n$. This is now a chart of type (i) with non-empty domain. ⊞

Since f is a diffeomorphism we see that $\Phi \circ f^{-1}\colon f(U) \to V$ is a chart of type (i) for N. In particular we see that $f(P) \in N$ admits a chart of type (i). Now let $\Psi\colon V \to \widetilde{V}$ be a chart of type (i) from the given smooth atlas of N for $f(P)$. Note that $\widetilde{V}$ is an open subset of $\mathbb{R}^n$.



By the Unique Type-of-Chart Proposition 25.26 we know that Ψ also has to be a chart of type (i). Furthermore note that $U \cap f^{-1}(V)$ contains P, thus it is non-empty. In summary we see that

$$\Psi \circ f \circ \Phi^{-1}\colon \Phi(U \cap f^{-1}(V)) \;\to\; \Psi(f(U) \cap V)$$

is a diffeomorphism between an open non-empty subset of $\mathbb{R}^m$ and an open non-empty subset of $\mathbb{R}^n$. It follows from the Differential-of-Diffeomorphism Proposition 24.19 that $m = n$. ■

25.8. **The Smooth Path-Connectivity Proposition.** We continue with a proposition which is somewhat technical, but which will have surprisingly many uses.

Proposition 25.32. (Smooth Path-Connectivity Proposition) Let M be a smooth manifold.

(1) Every continuous map $\gamma\colon [a,b] \to M$ is homotopic rel the endpoints $\{a,b\}$ to a smooth map.
(2) If M is path-connected, then given any $P, Q \in M$ there exists a smooth path from P to Q.[305]

The proof of the Smooth Path-Connectivity Proposition 25.32 requires the remainder of this section. The reader might be well advised to move on to the next section. For the reader who sticks around, we first formulate and prove the following useful lemma, which will be a key ingredient in the proof of the Smooth Path-Connectivity Proposition 25.32.

Lemma 25.33. (Smooth Path Composition Lemma) Let M be a smooth manifold. Let $\alpha\colon [a,b] \to M$ and $\beta\colon [b,c] \to M$ be two smooth paths with $\alpha(b) = \beta(b)$. There exists a smooth map $\gamma\colon [a,c] \to M$ which is homotopic to the map $[a,c] \to M$ that is defined by α and β.

Proof of Lemma 25.33. To simplify the notation we assume that $a = 0$, $b = 1$ and $c = 2$. By the Smooth Transition Function Lemma 24.9 there exists a smooth map $\tau\colon [0,1] \to [0,1]$

[305] In the Embedded Path-Corollaries 41.22 and **??** we will see that for $P \neq Q \in M \setminus \partial M$ there exists in fact a smooth embedding $f\colon [0,1] \to M$ with $f(0) = P$ and $f(Q) = Q$.

such that $\tau(t) = 0$ for $t \in [0, \frac{1}{3}]$ and such that $\tau(t) = 1$ for $t \in [\frac{2}{3}, 1]$. We consider the map

$$\begin{array}{rcl} \gamma\colon [0,2] & \to & M \\ t & \mapsto & \begin{cases} \alpha(\tau(t)), & \text{if } t \in [0,1], \\ \beta(\tau(t-1)), & \text{if } t \in [1,2]. \end{cases} \end{array}$$

Note that γ is constant on the interval $[\frac{2}{3}, \frac{4}{3}]$. Thus it follows easily from the Smooth Pasting Proposition 25.19 that γ is smooth. Finally the desired homotopy is given by the map

$$\begin{array}{rcl} [0,2] \times [0,1] & \to & M \\ (t,s) & \mapsto & \begin{cases} \alpha((1-s)\cdot t + s\cdot\tau(t)), & \text{if } t \in [0,1], \\ \beta((1-s)\cdot(t-1) + s\cdot\tau(t-1)), & \text{if } t \in [1,2]. \end{cases} \end{array}$$

■

Proof of the Smooth Path-Connectivity Proposition 25.32. Note that Statement (2) is an immediate consequence of Statement (1). Thus it suffices to prove Statement (1).

Let M be an n-dimensional smooth manifold. Let $\gamma\colon [a,b] \to M$ be a continuous map. We need to show that γ is homotopic rel $\{a,b\}$ to a smooth map $\widetilde{\gamma}\colon [a,b] \to M$.

Since $\gamma([a,b])$ is compact it follows easily from the Compact Subspace Covering Lemma 3.38 and the Chart-to-Balls 23.5 that we can find points $a = t_0 < t_1 < \cdots < t_k = b$ and charts $\{\Phi_i\colon U_i \to V_i\}_{0=1,\ldots,k-1}$ with the following properties:

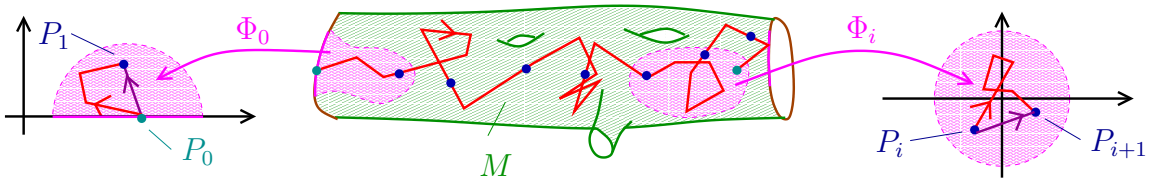
(a) For each $i \in \{0,\ldots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subseteq U_i$.

(b) Each V_i is an open convex subset either of $\mathbb{R}^n$ or of $\mathbb{H}^n$.

For $i = 0,\ldots,k-1$ we set $P_i = \Phi_i(\gamma(t_i))$ and for $i = 0,\ldots,k-1$ we consider the homotopy

$$\begin{array}{rcl} F_i\colon [t_i,t_{i+1}] \times [0,1] & \to & \mathbb{R}^n \\ (\lambda, s) & \mapsto & (1-s)\cdot\Phi_i(\gamma(\lambda)) + s\cdot \underbrace{\left(\frac{t_{i+1}-\lambda}{t_{i+1}-t_i}\cdot P_i + \frac{\lambda - t_i}{t_{i+1}-t_i}\cdot P_{i+1}\right)}_{\text{“straight path” from } P_i \text{ to } P_{i+1}}. \end{array}$$

Note that F_i is a homotopy rel $\{t_i, t_{i+1}\}$ from $\Phi_i \circ \gamma|_{[t_i,t_{i+1}]}$ to the “straight path” from P_i to P_{i+1}. Since V_i is a convex subset we see that this homotopy takes values in V_i. We now see that $\Phi_i^{-1} \circ F_i\colon [t_i,t_{i+1}] \times [0,1] \to M$ is a homotopy from $\gamma|_{[t_i,t_{i+1}]}$ rel $\{t_i,t_{i+1}\}$ to a smooth path $[t_i,t_{i+1}] \to M$.



Finally it follows from these results, together with the Smooth Path Composition Lemma 25.33, that there exists in fact a homotopy rel $\{a,b\}$ from $\gamma\colon [a,b] \to M$ to a smooth map $[a,b] \to M$. ■

25.9. **Products of smooth manifolds I.** In the Topological Manifold Product Proposition 23.6 we saw that the product of any two topological manifolds is again a topological manifold. The proof of the Topological Manifold Product Proposition 23.6 was mostly straightforward, except that if both topological manifolds have non-empty boundary, then we had to use a little ruse to find a chart. In the following proposition, where we will deal with smooth manifolds, we will sidestep this tricky business.

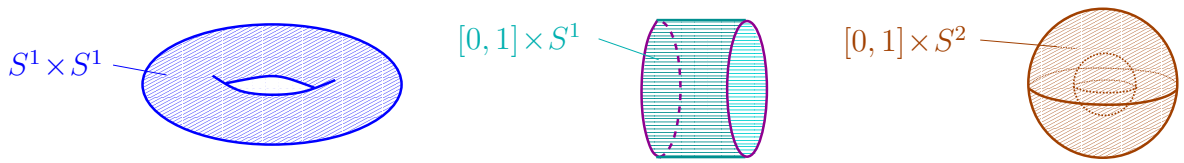
Proposition 25.34. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We assume that $\partial M = \varnothing$ or $\partial N = \varnothing$. We denote by

$$\begin{array}{rcl} p\colon M \times N & \to & M \\ (x,y) & \mapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon M \times N & \to & N \\ (x,y) & \mapsto & y \end{array}$$

the two projections. The following statements hold:

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold that admits a *unique* smooth manifold structure that satisfies the following two conditions:
   (a) The projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
   (b) For any smooth manifold W and any map $f\colon W \to M \times N$ the following holds: If the two maps $p \circ f\colon W \to M$ and $q \circ f\colon W \to N$ are smooth, then f itself is smooth.

(2) (a) If $\partial M = \varnothing$, then $\partial(M \times N) = M \times (\partial N)$.
   (b) If $\partial N = \varnothing$, then $\partial(M \times N) = (\partial M) \times N$.[306]



Remark. In the Smooth Manifold Product Proposition 26.12 and the Smooth Manifold Rounded Product Proposition 60.7 we will deal with the product of two smooth manifolds where both smooth manifolds are allowed to have non-empty boundary. □

Proof. We first consider the case that $\partial M = \varnothing$. In the Topological Manifold Product Proposition 23.6 we already saw that $M \times N$ is an $(m+n)$-dimensional topological manifold.

Now we turn to the construction of a smooth atlas. Let $\{\Phi_i\colon U_i \to \widetilde{U}_i\}_{i\in I}$ be the smooth atlas of M and let $\{\Psi_j\colon V_j \to \widetilde{V}_j\}_{j\in J}$ be the smooth atlas of N. One can easily verify that the maps

$$\begin{array}{rcl} \Phi_i \times \Psi_j\colon U_i \times V_j & \to & \widetilde{U}_i \times \widetilde{V}_j \\ (x,y) & \mapsto & (\Phi_i(x), \Psi_j(y)) \end{array}$$

form a *smooth* atlas for $M \times N$. We equip $M \times N$ with this smooth atlas and with the corresponding smooth structure. We make the following observations:

- It follows easily from the definitions that this smooth atlas has the properties (1a) and (1b).
- Since $\partial M = \varnothing$ it is also clear that $\Phi_i \times \Psi_j$ is of type (ii) if and only if Ψ_j is of type (ii). It follows from this observation, together with the Unique Type-of-Chart Proposition 25.26, that $\partial(M \times N) = M \times (\partial N)$.

The fact that the smooth structure is uniquely determined by (1a) and (1b) follows from a very general, formal argument. Suppose that $M \times N$ is equipped with two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ that both satisfy (1a) and (1b). Note that we need to show that the identity map

[306] Note that here, when we are dealing with smooth manifolds, we can determine the boundary. This is in contrast to the slightly unfortunate situation in the Topological Manifold Product Proposition 23.6.

$\mathrm{id}\colon (M\times N,\mathcal{A})\to(M\times N,\mathcal{B})$ is smooth. Since $\mathcal{A}$ satisfies (1a) we know that $p\circ\mathrm{id}$ and $q\circ\mathrm{id}$ are smooth. But since $\mathcal{B}$ satisfies (1b) this implies that id is smooth.

Finally we consider the case that $\partial M\neq\varnothing$ and $\partial N=\varnothing$. Recall that a chart of type (ii) takes values in the upper half-space which is defined by restricting the *last* coordinate to ≥ 0. Thus we proceed as above, except that we consider the product charts $\Psi_j\times\Phi_i$ to make sure that the product of a chart of type (i) and a chart of type (ii) is in fact a chart of type (ii). ■

Convention 25.35. Let M and N be two smooth manifolds, where at most one has non-empty boundary. Unless we say explicitly something else we will always equip $M\times N$ with the smooth structure introduced in the Smooth Manifold Product Proposition 25.34. We will refer to this as the standard smooth structure on $M\times N$.

Example.

(1) We equip each $\mathbb{R}^k$ with the standard smooth structure that we introduced on page 662. In Exercise 25.25 we will see that the map

$$\begin{array}{rcl}\mathbb{R}^{m+n} & \to & \mathbb{R}^m\times\mathbb{R}^n\\ (x_1,\dots,x_m,y_1,\dots,y_n) & \mapsto & ((x_1,\dots,x_m),(y_1,\dots,y_n))\end{array}$$

is a diffeomorphism. This observation can be viewed as a variation on the Product Metric Lemma 7.4.

(2) We now see that products of spheres are naturally smooth manifolds.

(3) Let M and N be two smooth manifolds, where at most one has non-empty boundary. Let $Q\in N$ and $P\in M$. It follows from the Smooth Manifold Product Proposition 25.34 (1b) that the two inclusion maps

$$\begin{array}{rcl}i_Q\colon M & \to & M\times N\\ x & \mapsto & (x,Q)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}j_P\colon N & \to & M\times N\\ y & \mapsto & (P,y)\end{array}$$

are smooth.

(4) Let $f\colon V\to W$ be a smooth map between two smooth manifolds with $\partial V=\varnothing$ and with $\partial W=\varnothing$. As on page 638 one sees that the graph $\mathrm{Gr}(f):=\{(x,f(x))\,|\,x\in V\}$ is a proper submanifold of the smooth manifold $V\times W$.[307] □

We conclude this section with the following variation on the Smooth Manifold Product Proposition 25.34:

Lemma 25.36. (Mapping Torus-of-Manifold Lemma) Let F be a k-dimensional smooth manifold F and let $\mu\colon F\to F$ be a self-diffeomorphism. The mapping torus

$$\mathrm{Tor}(F,\mu)\ :=\ (F\times[0,1])/(x,0)\sim(\mu(x),1)$$

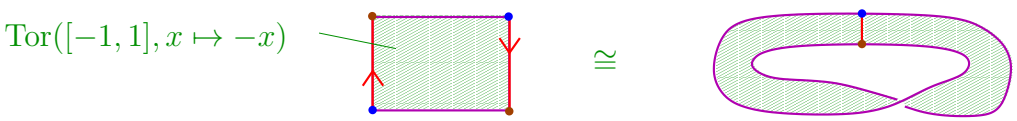
is naturally a $(k+1)$-dimensional smooth manifold with $\partial(\mathrm{Tor}(F,\mu))=\mathrm{Tor}(\partial F,\mu|_{\partial F})$.[308]

Example.

(1) Let $F=[-1,1]$ and let $\mu\colon[-1,1]\to[-1,1]$ be the diffeomorphism that is given by $x\mapsto -x$.

[307] In the Local Graph Lemma 30.5 we will prove a partial converse to this statement.

[308] Later, in the Manifold-Mapping Torus Lemma 59.11 we will prove a slight refinement of this lemma.



The corresponding mapping torus $\mathrm{Tor}(F,\mu)$ is, more or less by definition, diffeomorphic to the Möbius band.

(2) We can generalize the above example to higher dimensions: For example we can consider the disk $F=\overline{B}^2$ and the reflection $\mu\colon \overline{B}^2\to\overline{B}^2$ that is given by $(x,y)\mapsto(x,-y)$. The mapping torus $\mathrm{Tor}(\overline{B}^2,\mu)$ is sometimes called the *solid Klein bottle*. By the Mapping Torus-of-Manifold Lemma 25.36 we know that the boundary of the solid Klein bottle is given by the mapping torus $\mathrm{Tor}(S^1,\mu|_{S^1})$ which, by the discussion on page 575, is an incarnation of the Klein bottle. □

Proof. First note that it follows from the Mapping Torus Topology-Properties Proposition 21.20 (3) that $\mathrm{Tor}(F,\mu)$ is Hausdorff. Note that it follows from the Countable Cover Lemma 13.4 that there exists a *countable* smooth atlas $\{\Phi_i\colon U_i\to V_i\}_{i\in I}$ for F. As usual we denote by $p\colon F\times[0,1]\to\mathrm{Tor}(F,\mu)$ the projection map. One can easily verify that the maps

$$\begin{aligned} p(U_i\times(0,1)) &\to V_i\times(0,1)\\ p(x,t)&\mapsto(\Phi_i(x),t)\end{aligned}\qquad\&\qquad\begin{aligned} p(U_i\times[0,\tfrac12)\cup(\tfrac12\times1])&\to V_i\times(-\tfrac12,\tfrac12)\\ p(x,t)&\mapsto\begin{cases}(\Phi_i(x),t), & \text{if } t\in[0,\tfrac12)\\ (\mu(\Phi_i(x),t-1), & \text{if } t\in(\tfrac12,1]\end{cases}\end{aligned}$$

define a smooth atlas for $\mathrm{Tor}(F,\mu)$ and that $\partial(\mathrm{Tor}(F,\mu))=\mathrm{Tor}(\partial F,\mu|_{\partial F})$. One can easily verify that Since this atlas is again countable we obtain from the Countable Atlas-Second Countable Lemma 23.4 that $\mathrm{Tor}(F,\mu)$ is second-countable.[309] ■

25.10. **Manifolds and group actions.** In this section we will see that group actions can be used to construct interesting new examples of smooth manifolds.

For the reader's convenience we recall Definitions 2.36 and 9.4.

Definition 25.37. Let X be a set and let G be a group with trivial element e that acts on the given set X.

(1) An action is called free, if $g\cdot x=x$ for some $x\in X$ implies that $g=e$.

(2) Suppose that X is a topological space.

(a) We say that the action is continuous, if for every $g\in G$ the map $X\to X$ given by $x\mapsto g\cdot x$ is continuous.

(b) We say that the action is proper if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g\in G\,|\,gU\cap V\neq\varnothing\}$ is finite.

We add the following almost self-explanatory definition.

[309] One can also show easily, similar to the Second Countability Lemma 13.2, the following more general statement: If X is a second-countable topological space and if $f\colon X\to X$ is a homeomorphism, then the mapping torus $\mathrm{Tor}(X,f)$ is again second-countable.

Definition 25.38. Let M be a smooth manifold which is equipped with an action by a group G. We say that the action is smooth if for every $g \in G$ the map $M \to M$ given by $x \mapsto g \cdot x$ is smooth.

The following proposition is the main result of this section.

Proposition 25.39. (Manifold–Quotient Proposition) If a group G acts freely, properly and continuously on an n-dimensional topological manifold M, then the following hold:

(1) The quotient space M/G is also an n-dimensional topological manifold and the projection $p\colon M \to M/G$ is a local homeomorphism.
(2) If M is a smooth manifold and if G acts smoothly, then there exists a unique smooth structure on M/G such that the projection $p\colon M \to M/G$ is a local diffeomorphism.

Furthermore, in both cases the following four statements hold:

(a) The G-action restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.
(b) If M is compact, then M/G is compact.
(c) If M is closed, then M/G is closed.
(d) If $N \subseteq M$ is a (proper) submanifold such that the G-action restricts to an action on N, then N/G is a (proper) submanifold of M/G.

Convention 25.40. In the setting of the Manifold–Quotient Proposition 25.39 (2) we will refer to the smooth structure on M/G as the standard smooth structure on M/G.

Before we provide the proof of the Manifold–Quotient Proposition 25.39 let us first discuss some consequences and examples.

In the Topological-Quotient Proposition 8.2 (1b) we saw that the quotient topology has a universal property. The following proposition shows that the standard smooth structure on M/G also has a universal property:

Corollary 25.41. (Smooth Map-Descends-to-Quotient Corollary) Let G is a group that acts freely, properly and smoothly on an n-dimensional smooth manifold M. Furthermore let $f\colon M \to N$ be a smooth map such that for every $x \in M$ and every $g \in G$ we have $f(g \cdot x) = f(x)$. The induced map

$$\begin{array}{rccc} \overline{f}\colon & M/G & \to & N \\ & [x] & \mapsto & f(x) \end{array}$$

is also smooth. If f is in fact a local diffeomorphism, then $\overline{f}$ is also a local diffeomorphism.

Proof. Let $[x] \in M/G$. Since the projection $p\colon M \to M/G$ is a local diffeomorphism there exists an open neighborhood U of x and an open neighborhood V of $[x]$ such that $p\colon U \to V$ is a diffeomorphism. Note that $\overline{f}|_V = f \circ p^{-1}\colon V \to N$ is smooth. It follows from the Smooth Pasting Proposition 25.19 that $\overline{f}$ is smooth. If f is a local diffeomorphism then a very mild modification of the above argument shows that $\overline{f}$ is also a local diffeomorphism. ∎

We state our first example as a lemma.

Lemma 25.42. (Torus-as-Quotient Lemma)

(1) The action

$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$

from page 127 is free, proper and smooth.

(2) It follows from (1) and the Manifold–Quotient Proposition 25.39 that $\mathbb{R}^n/\mathbb{Z}^n$ has a standard smooth structure as an n-dimensional smooth manifold. We equip $(S^1)^n = S^1 \times \cdots \times S^1$ with the standard smooth structure coming from the Standard Smooth Structures Proposition 25.6 and the Smooth Manifold Product Proposition 25.34. The map

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ [(t_1, \ldots, t_n)] & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)). \end{array}$$

is a diffeomorphism.

Proof. The proof of Statement (1) is elementary. Thus let us turn to the proof of Statement (2). As we mentioned on page 667, the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots \exp(2\pi \mathrm{i} t_n)) \end{array}$$

is a local diffeomorphism. By the Torus-as-Quotient Lemma 9.2 the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ [(t_1, \ldots, t_n)] & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)) \end{array}$$

is a homeomorphism. By the Smooth Map-Descends-to-Quotient Corollary 25.41 (2) the map f is also a local diffeomorphism. Thus it follows from the Smooth Pasting Proposition 25.19 that the map f is in fact a diffeomorphism. ■

We continue with two more examples.

Examples.

(1) Using the action from the Projective Space-Sphere Quotient Lemma 9.13 of $G = \{\pm 1\}$ on $M = S^n$ we obtain that $S^n/\{\pm 1\}$ is naturally a closed n-dimensional smooth manifold. In Exercise 25.10 we will see that the obvious map $S^n/\{\pm 1\} \to \mathbb{R}\mathrm{P}^n$ is a diffeomorphism.

(2) We consider $M = \mathbb{R} \times [-1, 1]$ which we endow with the obvious smooth structure. Using the action of $G = \mathbb{Z}$ on $M = \mathbb{R} \times [-1, 1]$ from the Möbius Band-as-Quotient Lemma 9.6 we obtain that the Möbius band $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ has a canonical structure of a 2-dimensional smooth manifold. Furthermore, we have

$$\partial(\text{Möbius band}) \underset{\substack{\uparrow \\ \text{by the Manifold Quotient-by-Group} \\ \text{Action Proposition 25.39 (a)}}}{=} (\mathbb{R} \times \{-1, 1\})/\mathbb{Z} \underset{\substack{\uparrow \\ \text{see Exercise 8.21}}}{\cong} S^1.$$

□

Convention 25.43. In line with the above convention we refer to the above smooth structures on $\mathbb{R}^m/\mathbb{Z}^m$ and on the Möbius band as the standard smooth structures on $\mathbb{R}^m/\mathbb{Z}^m$ and on the Möbius band.

The proof of the Manifold–Quotient Proposition 25.39 relies on the following two somewhat technical lemmas.

Lemma 25.44. (Free-Proper Action Lemma) Let X be a topological space and let G be a group that acts freely, properly (this condition is automatically satisfied if G is

finite) and continuously on X. If X is Hausdorff, then for every $x \in X$ there exists an open neighborhood U such that $gU \cap U = \varnothing$ for all $g \neq e$.

Proof of the Free-Proper Action Lemma 25.44. Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts freely, properly and continuously on X. Finally let $x \in X$. Since G acts freely we know that for every non-trivial $g \in G$ we have $gx \neq x$.

Since the action is proper and continuous we can apply the Disjoint Translated Neighborhoods Lemma 9.9 to $a = b = x$ and we obtain open neighborhoods A and B of x such that $gA \cap B = \varnothing$ for every non-trivial $g \in G$. Thus $U := A \cap B$ has the desired property. ■

Lemma 25.45. (Projection Homeomorphism Lemma) Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts continuously on X. We denote by $p\colon X \to X/G$ the projection. If $U \subseteq X$ is an open subset such that the map $p\colon U \to X/G$ is injective, then the map $p\colon U \to p(U)$ is a homeomorphism.

Proof of the Projection Homeomorphism Lemma 25.45. By the Topological-Quotient Proposition 8.2 (1a) the projection $p\colon X \to X/G$ is continuous. Thus the restriction of p to U is also continuous. By our hypothesis the map $p\colon U \to p(U)$ is injective and evidently it is surjective. Thus the map is a bijection. It follows from the Open-Closed Inclusion Lemma 4.13 and the fact that U is open that $U \to X$ is an open map. Furthermore by the Group Action-Projection Lemma 9.8 (1) the map $p\colon X \to X/G$ is also open. In summary we see that $p\colon U \to p(U)$ is open. It follows from the Open-Closed–Injective Lemma 4.22 (2) that the map $p\colon U \to p(U)$ is in fact a homeomorphism. ■

Now we are in a position to prove the Manifold–Quotient Proposition 25.39. For clarity we deal with topological manifolds and smooth manifolds separately.

Proof of Proposition 25.39 for topological manifolds. Let M be an n-dimensional topological manifold and let G be a group that acts on M. We suppose that the action is continuous, free and proper. We want to show that M/G is an n-dimensional topological manifold. Thus we need to prove the following claim:

Claim.
(1) M/G is second-countable,
(2) M/G is Hausdorff, and
(3) every point $y \in M/G$ admits an n-dimensional chart.

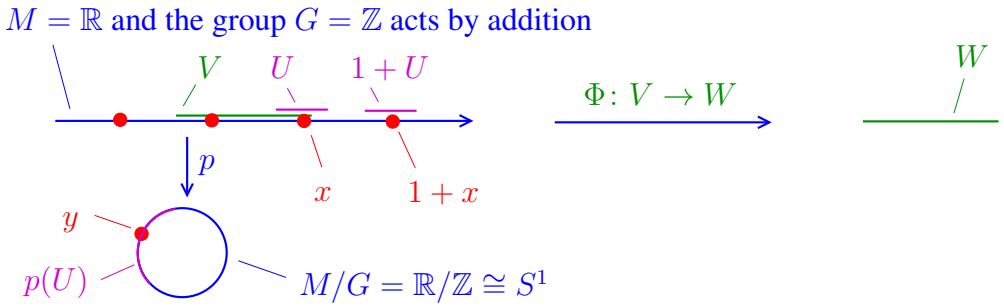
Proof. Statement (1) of the claim follows immediately from the Group Action-Projection Lemma 9.8 (1) and the Second Countability Lemma 13.2 (5). Statement (2) follows from the Group Action-Hausdorff Quotient Proposition 9.5. Thus it remains to deal with Statement (3). As usual we denote by $p\colon M \to M/G$ the projection. Let $y \in M/G$. We pick an $x \in M$ with $p(x) = y$. We make the following two observations:

- Since M is an n-dimensional topological manifold we know that there exists an n-dimensional chart $\Phi\colon V \to W$ with $x \in V$.
- By the Free-Proper Action Lemma 25.44 there exists an open neighborhood U of x such that $gU \cap U = \varnothing$ for every $g \neq e$. Note that this implies that the restriction of $p\colon M \to M/G$ to U is injective.

After possibly replacing U and V by $U \cap V$ we can assume that $U = V$. Note that by the Group Action-Projection Lemma 9.8 (1) the projection $p(U)$ is an open neighborhood of x. Finally we consider the maps

$$p(U) \xrightarrow{p^{-1}} U \xrightarrow{\Phi} W.$$

The map Φ is by definition a homeomorphism.



Furthermore it follows from the Projection Homeomorphism Lemma 25.45 that the map $p\colon U \to p(U) \subseteq M/G$ is a homeomorphism as well. Thus the above map $\Phi \circ p^{-1}\colon p(U) \to W$ is a homeomorphism, in particular it is an n-dimensional chart for M/G that contains y.⊞ We have now shown that M/G is an n-dimensional topological manifold. It remains to prove the following claim:

Claim.

(∗) The projection $p\colon M \to M/G$ is a local homeomorphism.
(a) The G-action on M restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.
(b) If M is compact, then M/G is compact.
(c) If M is closed, then M/G is closed.
(d) If $N \subseteq M$ is a (proper) submanifold such that the G-action restricts to an action on N, then N/G is a (proper) submanifold of M/G.

Proof.

(∗) Implicitly we had already proved Statement (1), indeed, given $x \in M$ we run through the above argument and we see, with the same notation, that $p(U)$ is an open subset of M/G and that $p\colon U \to p(U)$ is a homeomorphism.
(a) First note that it follows from the Boundary–Local Homeomorphism Lemma 23.14 that the G-action on M restricts to an action on ∂M. Furthermore note that it follows easily from the Boundary–Local Homeomorphism Lemma 23.14 and the above statement (∗) that $\partial(M/G) = (\partial M)/G$. We leave it to the reader to fill in the details.
(b) Statement (3) is of course just an immediate consequence of the Compact Image Lemma 4.15.
(c) This statement is an immediate consequence of statements (a) and (b).
(d) We now construct submanifold charts for $N/G \subseteq M/G$ from submanifold charts for $N \subseteq M$ the same way that we constructed manifold charts for M/G from manifold charts of M. Note that these submanifold charts for N/G have the same type as the submanifold charts for N.

Finally suppose that N is a proper submanifold. Recall that this means that N is a closed subset of M and that we require only submanifold charts of type (α) and (β).

As we just mentioned, the type of the submanifold charts involved does not change. Finally it follows from the hypothesis that the G-action on M restricts to a G-action on N together with the Group Action-Projection Lemma 9.8 (2) that N/G is a closed subset of M/G. ■

Proof of Proposition 25.39 for smooth manifolds. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold. Let G be a group that acts freely, properly and smoothly on M. We need to show that M/G admits a smooth atlas such that the projection $p\colon M \to M/G$ is a local diffeomorphism.

(1) We run through the proof of the Manifold–Quotient Proposition 25.39 (1), but this time only using charts for M that lie in $\mathcal{A}$, and we obtain an atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ for M/G. It is straightforward to see that any transition map $\Phi_j \circ \Phi_i^{-1}\colon \Phi_i(U_i \cap U_j) \to \Phi_j(U_i \cap U_j)$ can be written as the composition of (some of) the following maps:
- charts[310] and their inverses,
- restrictions to open subsets,
- the restriction of the action of $g \in G$ on M to some open subset.

All these maps are smooth maps between smooth manifolds.[311] Thus the transition maps are smooth.

(2) Note that in the above proof of the Manifold–Quotient Proposition 25.39 (1) we already saw that the projection $p\colon M \to M/G$ is a local homeomorphism. It is straightforward to verify that, with our choice of a smooth atlas for M/G, the projection is in fact a local diffeomorphism.

The proof of the remaining statements (a), (b), (c) and (d) is basically identical to the above proof for topological manifolds.

Finally, the statement that the smooth structure on M/G is uniquely determined by the smooth structure on M and the property that the projection $M \to M/G$ is a local diffeomorphism is a consequence of the Smooth Pasting Proposition 25.19 (1) and the fact that the action is proper. We leave it to the reader to fill in the details. ■

We conclude this section with the following amusing lemma. It gives an example of a diffeomorphism that will pop up on several occasions.

Lemma 25.46. (Maps-on-Torus Lemma) Let $n \in \mathbb{N}$. If $A \in \mathrm{GL}(n, \mathbb{Z})$ is a matrix, then the map

$$\begin{array}{rccc} f(A)\colon & \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ & [v] & \mapsto & [A \cdot v] \end{array}$$

is a diffeomorphism. (In Exercise 31.7 we will see that this diffeomorphism is "orientation-preserving" if and only if $A \in \mathrm{SL}(n, \mathbb{Z})$.)

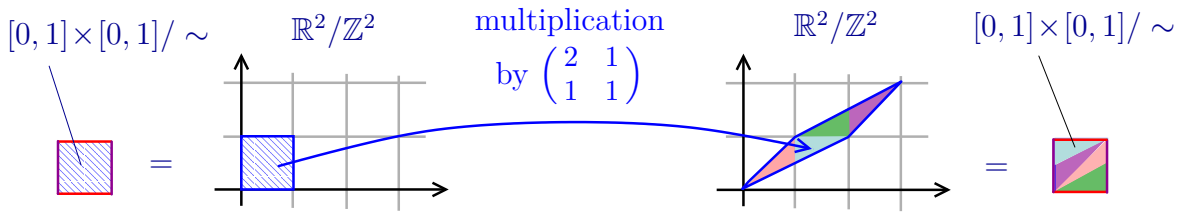
Proof. We will provide the proof in Exercise 25.13. ■

Example. We consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$.

The corresponding self-diffeomorphism $f(A)$ of $\mathbb{R}^2/\mathbb{Z}^2 = ([0,1] \times [0,1])/\sim$ is illustrated in the figure above. □

[310] Recall that we view open subsets of $\mathbb{R}^n$ as smooth manifolds. With this convention charts are in fact diffeomorphisms between smooth manifolds.

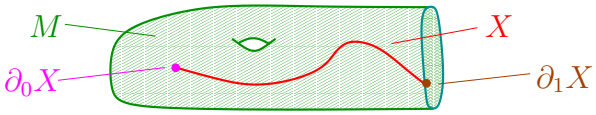
[311] Here the smooth manifolds are open subsets of $\mathbb{R}^n$ or of M.



25.11. **Codimension-zero submanifolds and decompositions of smooth manifolds.** Inspired by Definition 23.2 (4), Definition 25.10 and the Unique Type-of-Chart Proposition 25.26 we introduce the following notation:

Notation 25.47. Let M be a smooth manifold and let X be a submanifold. We set:

$\partial_0 X$:= the set of all $P \in X$ that admit a submanifold chart of type (γ),
$\partial_1 X$:= the set of all $P \in X$ that admit a submanifold chart of type (β).



Proposition 25.48. (Boundary-of-Submanifold Proposition) Let M be an n-dimensional smooth manifold and let X be a k-dimensional submanifold. The following statement holds:

(0) (a) $\partial_0 X = \partial X \setminus \partial M$,[312]
(b) $\partial_1 X = \partial X \cap \partial M$.
(1) $\partial_0 X$ and $\partial_1 X$ are unions of components of ∂X.
(2) $\partial_0 X$ is an $(k-1)$-dimensional submanifold of M. If X is a closed subset of M, then $\partial_0 X$ is a proper submanifold of M.

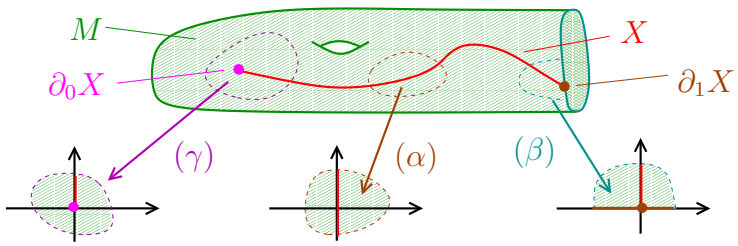
Proof. Let M be an n-dimensional smooth manifold and let X be a k-dimensional submanifold. Let $P \in X$. By the Submanifold Smooth Structure Lemma 25.23 we can and will view the submanifold X as a smooth manifold in its own right. We start out with recall the following observation which we already made in the proof of the Smooth Manifold–Boundary Proposition 25.27.

(α) A submanifold chart of type (α) for P is a manifold chart of type (i) for $P \in M$ and a manifold chart of type (i) for $P \in X$.
(β) A submanifold chart of type (β) for P is a manifold chart of type (ii) for $P \in M$ and a manifold of type (ii) for $P \in X$.
(γ) A submanifold chart of type (γ) for P is a manifold chart of type (i) for $P \in M$ and a manifold chart of type (ii) for $P \in X$.

Note that by the Unique Type-of-Chart Proposition 25.26 the chart type for a point is uniquely determined.

(0) This statement follows easily from the above discussion together with the Unique Type-of-Chart Proposition 25.26.

[312]Recall that by the Submanifold Smooth Structure Lemma 25.23 we can view the submanifold X as a smooth manifold in its own right. Here ∂X denotes the corresponding boundary of X viewed as a smooth manifold.



(1) From the above observations we see that the following hold:

$$\begin{aligned} \partial_0 X &= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\gamma), \\ \partial_1 X &= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\beta). \end{aligned}$$

Claim. The subsets $\partial_0 X$ and $\partial_1 X$ are both open in ∂X.

Proof. Let $P \in \partial_0 X$. By the above there exists a submanifold chart $\Phi\colon U \to V$ of type (γ) with $P \in U$. Note that $\partial X \cap U = \Phi^{-1}(V \cap (\{0\} \times \mathbb{R}^{k-1}))$ and note that Φ is also a submanifold chart of type (γ) for every point in $\Phi^{-1}(V \cap (\{0\} \times \mathbb{R}^{k-1} \times \{0\})$. Thus we have shown that an open neighborhood of P in ∂X is contained in $\partial_0 X$. It follows from the JH-Openness Criterion 3.3 that $\partial_0 X$ is an open subset of ∂X. The same argument also shows that $\partial_1 X$ is an open subset of ∂X. ⊞

It follows from the claim and the Component Lemma 4.48 (4) that $\partial_0 X$ and $\partial_1 X$ are unions of components of ∂X.

(2) Let $P \in \partial_0 X$. By the above there exists a submanifold chart of type (γ) for $P \in X$. It is easy to verify that this defines a submanifold chart of type (α) for $P \in \partial_0 X$. Thus we see that $\partial_0 X$ is an $(k-1)$-dimensional submanifold of M. If X is a closed subset of M, then so is $\partial_0 X$. Since we just saw that $\partial_0 X$ has only submanifold charts of type (α) we see that $\partial_0 X$ is a proper submanifold of M. ■

We now turn our gaze towards codimension-zero submanifolds. The following proposition is used frequently, often without noticing it.

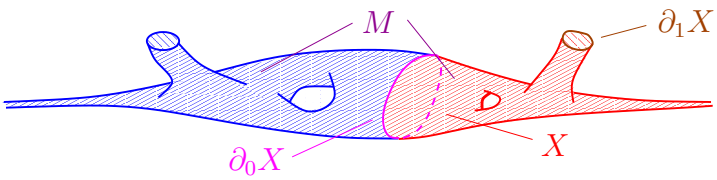
Proposition 25.49. (Codimension-Zero Submanifold Proposition) Let M be an n-dimensional smooth manifold and let X be a codimension-zero submanifold that is a closed subset of M (e.g. X could be a compact subset). Then the following statements hold:

(1) The boundary of X as a subset of M agrees with $\partial_0 X$.
(2) The interior $\mathring{X}$ of X as a subset of M agrees with $X \setminus \partial_0 X$.

We can furthermore make some statements about the complement $Y := M \setminus \mathring{X}$ of the interior:

(3) $Y := M \setminus \mathring{X}$ is a codimension-zero submanifold with $\partial_0 Y = \partial_0 X$ and $\partial_1 Y = \partial M \setminus \partial_1 X$.
(4) Y is a closed subset of M and if M is compact, then $Y = M \setminus \mathring{X}$ is also compact.
(5) If M is connected and if $\partial_0 X$ is connected, then $Y = M \setminus \mathring{X}$ is also connected.

Remark. In the Smooth Submanifold-with-Corner Lemma 26.16 we will prove a generalization of the the Codimension-Zero Submanifold Proposition 25.49 for codimension-zero submanifolds with corner. In the Codimension-Zero Submanifold Proposition 150.16 we



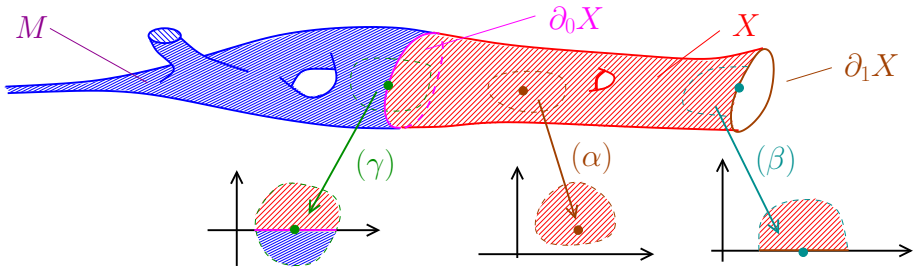
will prove an analog of the Codimension-Zero Submanifold Proposition 25.49 for topological manifolds. □

Proof. Let X be a codimension-zero submanifold of M that is a closed subset of M. By the Submanifold Smooth Structure Lemma 25.23 we can and will view the submanifold X as a smooth manifold in its own right. As in the proof of the Boundary-of-Submanifold Proposition 25.48 we start out with the following observation:

(α) A submanifold chart of type (α) for P is a manifold chart of type (i) for $P \in M$ and a manifold chart of type (i) for $P \in X$.
(β) A submanifold chart of type (β) for P is a manifold chart of type (ii) for $P \in M$ and a manifold of type (ii) for $P \in X$.
(γ) A submanifold chart of type (γ) for P is a manifold chart of type (i) for $P \in M$ and a manifold chart of type (ii) for $P \in X$.

Recall that by the Unique Type-of-Chart Proposition 25.26 the chart type for a point is uniquely determined. As in the proof of the Boundary-of-Submanifold Proposition 25.48 we point out that

$$\begin{aligned} \partial_0 X &= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\gamma), \\ \partial_1 X &= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\beta). \end{aligned}$$



(1) Let $P \in M$. The desired statement follows from the following observations:

this follows from the hypothesis that X is a closed subset and the Interior-Closure Characterization Lemma 3.23

P lies in the boundary of X as a subset of M $\Longleftrightarrow$ P lies in X and every open neighborhood of P contains a point in $M \setminus X$ $\Longleftrightarrow$ P admits a submanifold chart of type (γ) $\Longleftrightarrow$ $P \in \partial_0 X$.

since X is a submanifold of codimension zero

see above

(2) This statement follows immediately from (1) together with the Interior-Closure Lemma 3.22 (6).

Now we turn to the proofs of the statements about $Y = M \setminus \overset{\circ}{X}$.

(3) By the Interior-Closure Lemma 3.22 (6) and (7) together with the above Statements (1) and (2) we know that $\overset{\circ}{Y} = Y \setminus \partial_0 X$. Now let $P \in Y$.
   (a) If $P \in \overset{\circ}{Y}$, then the charts for M, intersected with $\overset{\circ}{Y}$, give rise to codimension-zero submanifold charts of type (α) or (β) for $P \in Y$.
   (b) If $P \in \partial_0 X$, then by the above we know that there exists a submanifold chart of type (γ) for $P \in X$. But flipping the last coordinate we see that this gives rise to a submanifold chart of type (γ) for $P \in Y = M \setminus \overset{\circ}{X}$.

   It follows from this discussion that Y is indeed a codimension-zero submanifold of M. Furthermore, using the discussion in the beginning of the proof we see easily that $\partial_0 Y = \partial_0 X$ and $\partial_1 Y = \partial M \setminus \partial_1 X$.

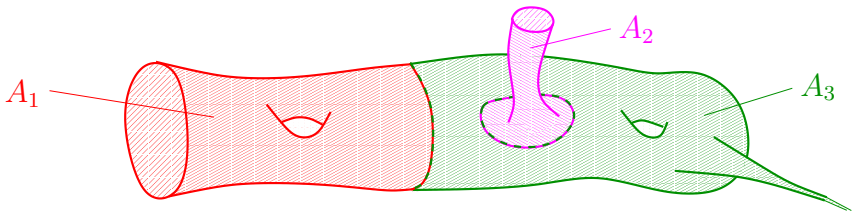
(4) By the Interior-Closure Lemma 3.22 (1) we know that $\overset{\circ}{X}$ is an open subset of M. Thus we see that $Y = M \setminus \overset{\circ}{X}$ is a closed subset of M. If M is compact, then we obtain from the Compact-Closed Lemma 3.39 that $Y = M \setminus \overset{\circ}{Y}$ is also compact.

(5) This statement follows immediately from Statement (1) together with the Complement Connected Lemma 4.40. ■

The Codimension-Zero Submanifold Proposition 25.49 partially motivates the following definition:

Definition 25.50. Let M be an n-dimensional smooth manifold. We say that M is decomposed into submanifolds $\{A_i\}_{i \in I}$ if the following conditions hold:

(1) We have $M = \bigcup_{i \in I} A_i$.
(2) Each A_i is a codimension-zero submanifold of M.
(3) Each A_i is a closed subset of M.
(4) For any $i, j \in I$ with $i \neq j$ the intersection $A_i \cap A_j$ is a union of boundary components of A_i and it is a union of boundary components of A_j.



On page 664 we equipped the real projective plane with a standard smooth structure. In Exercise 25.19 we will see that the Klein bottle $([0,1] \times [0,1])/\sim$ comes with a canonical smooth structure. The following cute lemma says that the Klein bottle and the real projective plane admit some interesting decompositions into submanifolds.

Lemma 25.51. (Möbius Band-in-Surface Lemma)

(1) We can decompose the Klein bottle into two submanifolds M and N where M and N are both diffeomorphic to the Möbius band.
(2) We can decompose the real projective plane into two submanifolds D and M where D is diffeomorphic to the disk $\overline{B}^2$ and M is diffeomorphic to the Möbius band.

Proof.

(1) The proof of Statement (1) is sketched in the figure below:

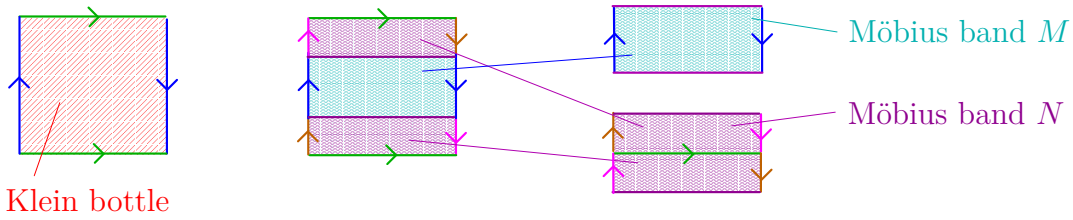


Figure A: Illustration of the proof of Lemma 25.51 (1)

In Figure B we attempt to sketch the equivalent statement that the Klein bottle is obtained from gluing two copies of the Möbius band along the common boundary.

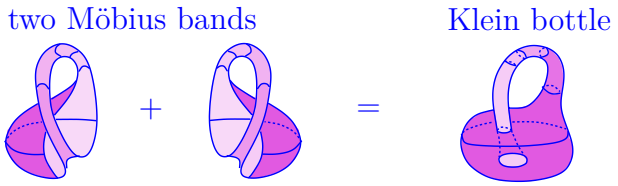


Figure B: Illustration of the proof of Lemma 25.51 (1).

(2) Let $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ be the projection. To simplify the notation we think in the following of $\overline{B}^2 \subseteq \mathbb{C}$ and $S^2 \subseteq \mathbb{C} \times \mathbb{R}$. We consider the following two subsets of S^2 which are illustrated in the figure below:

$$\widetilde{D} := \{(z,t) \in S^2 \,|\, t \geq \tfrac{1}{2}\} \qquad \text{and} \qquad \widetilde{M} := \{(z,t) \in S^2 \,|\, t \in [0, \tfrac{1}{2}]\}.$$



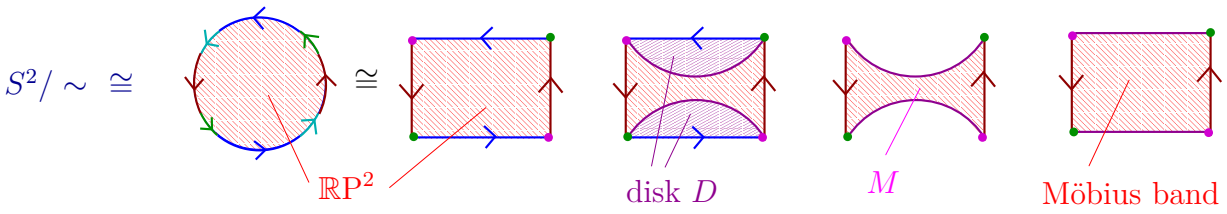
Furthermore we set $D := p(\widetilde{D})$ and $M := p(\widetilde{M})$. In the following we consider the Möbius band $\mathbb{R} \times [-1,1]/\sim$ as described in the Möbius Band-as-Quotient Lemma 9.6. We leave it to the reader to verify that the two maps

$$\begin{aligned} \overline{B}^2 &\to D \\ z &\mapsto p\Big(\tfrac{\sqrt{3}}{2}z, \sqrt{1 - \tfrac{3}{4}\cdot|z|^2}\Big) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{R} \times [-1,1]/\sim &\mapsto M \\ [(x,t)] &\mapsto p\Big(\sqrt{1 - \tfrac{1}{4}t^2}\cdot\exp(\pi\,\mathrm{i}\,x), \tfrac{1}{2}t\Big) \end{aligned}$$

are diffeomorphisms.[313] ■

Remark. If one does not care too much about smooth structures, then another decomposition of the real projective plane into a Möbius band and a real disk is sketched in the figure below.
Here, for $\mathbb{R}\mathrm{P}^2$ we use the homeomorphism $\mathbb{R}\mathrm{P}^2 \cong \overline{B}^2/\sim$ that we gave in the Projective Space-Avatar Lemma 9.16. □

[313] This can clearly be done, but is not particularly pleasant. The work becomes easier if one uses the Smooth Embedding Theorem 30.15.



25.12. Exercises for Chapter 25.

Exercise 25.1. We consider the topological space $X = \mathbb{R}$ and the following three charts:

$$\begin{array}{rcl} \Phi_1 \colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \qquad \begin{array}{rcl} \Phi_2 \colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array} \qquad \begin{array}{rcl} \Phi_3 \colon \mathbb{R} & \to & (-\frac{\pi}{2}, \frac{\pi}{2}) \\ x & \mapsto & \arctan(x). \end{array}$$

(a) Is the atlas given by $\{\Phi_1\}$ a smooth atlas for $X = \mathbb{R}$?
(b) Is the atlas given by $\{\Phi_2\}$ a smooth atlas for $X = \mathbb{R}$?
(c) Is the atlas given by $\{\Phi_1, \Phi_2\}$ a smooth atlas for $X = \mathbb{R}$?
(d) Is the atlas given by $\{\Phi_1, \Phi_3\}$ a smooth atlas for $X = \mathbb{R}$?

Exercise 25.2. Let $n \in \mathbb{Z}$. We consider the following map:

$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

(a) Show, just using the definitions and the Smooth-via-Atlas Lemma 25.16, that f is smooth.
(b) Show that if $n \neq 0$, then f is a local diffeomorphism.

Exercise 25.3.

(a) Let $r \in \mathbb{R}$. Show that the map

$$\begin{array}{rcl} (r - \frac{1}{2}, r + \frac{1}{2}) & \to & S^1 \setminus \{-\exp(2\pi \mathrm{i} \cdot r)\} \\ t & \mapsto & \exp(2\pi \mathrm{i} \cdot t) \end{array}$$

is a diffeomorphism.
(b) Let $n \in \mathbb{N}$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & (S^1)^n = S^1 \times \cdots \times S^1 \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} \cdot t_1) \ldots, \exp(2\pi \mathrm{i} \cdot t_n)) \end{array}$$

is a local diffeomorphism.

Remark. The point of this exercise is that one really needs to look up what are the standard smooth structures on the various smooth manifolds involved.

Exercise 25.4.

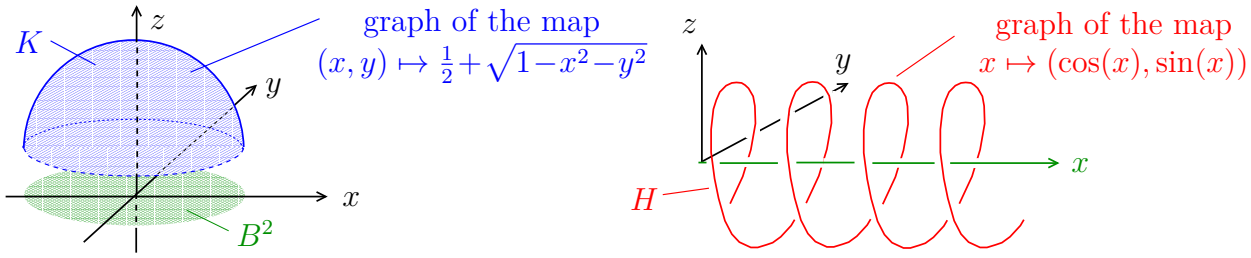
(a) Show that the topological space $X = \{(x, y) \in \mathbb{R}^2 \,|\, x \in [0, 1] \text{ and } y \in [0, x]\}$ is a topological manifold.
(b) Does the topological manifold X admit a smooth atlas?

Exercise 25.5. Give an example of two smooth atlases $\{\Phi_i\}_{i \in I}$ and $\{\Psi_j\}_{j \in J}$ for $X = \mathbb{R}$ such that for all $i \in I$ and $j \in J$ the map $\Psi_j \circ \Phi_i^{-1}$ is smooth, but such that the two atlases are not equivalent.

Exercise 25.6. Show that the following two statements are equivalent:

(1) S^n admits two inequivalent smooth structures.
(2) There exists a smooth manifold that is homeomorphic to S^n but not diffeomorphic to S^n, where we equip S^n with the standard smooth structure.

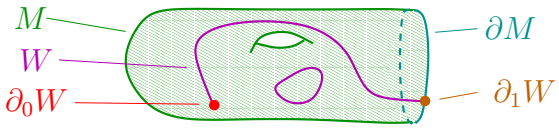
Exercise 25.7. Let V be some smooth manifold and let W be some smooth manifold with $\partial W = \varnothing$. Furthermore let $f\colon V \to W$ be a smooth map. Show that the corresponding graph $\mathrm{Gr}(f) \coloneqq \{(x, f(x)) \mid x \in V\}$ is a proper submanifold of the smooth manifold $V \times W$.
Remark. This is the smooth analog of Exercise 23.11.



Exercise 25.8. Let M be a smooth manifold and let W be a submanifold of M. We denote by ∂W the boundary of W viewed as a smooth manifold in its own right. Show that for every component C of ∂W we have either $C \subseteq \partial M$ or $C \subseteq M \setminus \partial M$.

Exercise 25.9. Let M be a smooth manifold and let W be a submanifold of M. We write $\partial_0 W \coloneqq \partial W \cap (M \setminus \partial M)$ and $\partial_1 W \coloneqq \partial W \cap \partial M$.

(a) (i) Show that if W is compact, then $\partial_0 W$ is a proper submanifold of M.
(ii) Show that if W is compact, then $\partial_1 W$ is a proper submanifold of ∂M.
(b) For which statements in (a) can we drop the hypothesis that W is compact?



Exercise 25.10. Let $n \in \mathbb{N}_0$. Recall that we introduced the following smooth structures:

- In the Projective Space–Manifold Proposition 25.13 we equipped the real projective space $\mathbb{R}\mathrm{P}^n$ with a smooth structure.
- On page 684 we equipped $S^n/\{\pm 1\}$ with a smooth structure.

Show that the map

$$\begin{array}{ccc} S^n/\{\pm 1\} & \to & \mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ [x] & \mapsto & [x] \end{array}$$

is a diffeomorphism.

Exercise 25.11. Recall that we introduced the following smooth structures:

- In the Standard Smooth Structures Proposition 25.6 we equipped S^1 and S^2 with a smooth structure.
- In the Projective Space–Manifold Proposition 25.13 we equipped the real projective space $\mathbb{R}\mathrm{P}^1$ and the complex projective space $\mathbb{C}\mathrm{P}^1$ with a smooth structure.

In the Projective Space-Avatar Lemma 9.16 we showed that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1 and we showed that $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2. Show that $\mathbb{R}\mathrm{P}^1$ is diffeomorphic to S^1 and show that $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to S^2.

Exercise 25.12. Let $n \in \mathbb{N}_0$. We consider the real projective space $\mathbb{R}\mathrm{P}^n$ and for $i = 0, \dots, n$ we consider the chart

$$\begin{array}{rccl} \Phi_i \colon & \overbrace{\{[x_0 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^n \mid x_i \neq 0\}}^{=:U_i} & \to & \mathbb{R}^n \\ & [x_0 : \cdots : x_n] & \mapsto & \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \end{array}$$

from the Projective Space–Manifold Proposition 25.13.

(a) Show that for each $k \in \{0, \dots, n\}$ the union $U_0 \cup \cdots \cup U_k$ is homotopy equivalent to the real projective space $\mathbb{R}\mathrm{P}^k$.
(b) Show that for each $k \in \{1, \dots, n\}$ the intersection $(U_0 \cup \cdots \cup U_{k-1}) \cap U_k$ is homotopy equivalent to S^{k-1}.

Exercise 25.13. Let $n \in \mathbb{N}$ and let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. Show that the map

$$\begin{array}{rccl} f(A) \colon & \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ & [v] & \mapsto & [A \cdot v] \end{array}$$

is a diffeomorphism.

Exercise 25.14. Let M be a smooth manifold and let $N \subseteq M$ be a smooth submanifold with $\dim(N) < \dim(M)$.

(a) Show that the interior of N in M is the empty set.
(b) Show that the closure of $M \setminus N$ in M equals M.

Exercise 25.15. We consider the stereographic projection

$$\begin{array}{rccl} \Phi \colon S^2 & \to & \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} \\ (x_1, x_2, x_3) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right), & \text{if } x_3 < 1, \\ \infty, & \text{if } x_3 = 1. \end{cases} \end{array}$$

Let $q(z)$ be a polynomial with complex coefficients. We denote by $N \coloneqq (0, 0, 1)$ the North Pole. Show that the map

$$\begin{array}{rccl} f \colon S^2 & \to & S^2 \\ P & \mapsto & \begin{cases} \Phi^{-1}(q(\Phi(P))), & \text{if } P \neq N, \\ N, & \text{if } P = N \end{cases} \end{array}$$

is smooth.
Hint. Use Exercise 23.19.

Exercise 25.16. Let M be a smooth manifold and let $U \subseteq M$ be an open subset. Furthermore let $f \colon U \to \mathbb{R}^k$ be a smooth map and let $P \in U$. Show that there exists an open neighborhood V of P and a smooth map $g \colon M \to \mathbb{R}^k$ such that f and g agree on V.
Hint. Use the Smooth Transition Function Lemma 24.9.

Exercise 25.17. Let M be a smooth manifold and let N be a proper codimension-zero submanifold of M. Show that if M is connected and if $N \neq \varnothing$, then $M = N$.

Exercise 25.18. Let $n \in \mathbb{N}_0$. We consider the smooth manifold S^n and we consider the upper hemisphere $S^n_{\geq 0} := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$.

(a) Show that $S^n_{\geq 0}$ is a submanifold of S^n.
(b) Show that the submanifold $S^n_{\geq 0}$ is diffeomorphic to $\overline{B}^n$.

Hint. Use a stereographic projection.

Exercise 25.19. We consider the square $X = [0,1] \times [0,1]$ with the equivalence relation that is generated by $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (x, 1)$. On page 312 we defined the Klein bottle to be the quotient $X/\sim$.

(a) Show that X admits a smooth structure such that the projection $X \to X/\sim$ is smooth.
(b) Show that $X/\sim$ is diffeomorphic to the non-orientable surface of genus two.

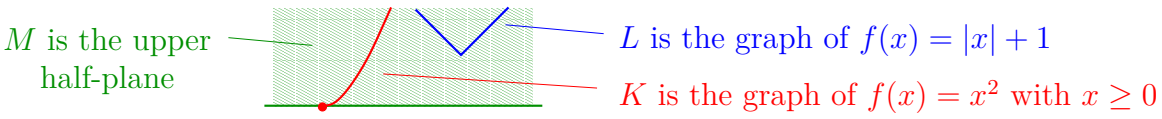
Remark. Note that part (b) is a refinement of Exercise 9.17. Also note that in Exercise 31.10 we will give an alternative, but equivalent, description of the smooth structure on the Klein bottle.

Exercise 25.20. Let $n \in \mathbb{N}_0$ and let $P, Q \in B^n = \overline{B}^n \setminus S^{n-1}$.

(a) Show that there exists a self-diffeomorphism φ of $\overline{B}^n$ with $\varphi(P) = Q$.
(b) Show that there exists a self-diffeomorphism φ of $\overline{B}^n$ with $\varphi(P) = Q$ and which is the identity in some neighborhood of S^{n-1}.

Exercise 25.21. In the figure below we show $M = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ together with two subsets $K := \{(x, x^2) \mid x \geq 0\}$ and $L := \{(x, |x| + 1) \mid x \in \mathbb{R}\}$.

(a) Show that both K and L are submanifolds of the topological manifold M.
Remark. You might have already shown in Exercise 23.15 that K is a submanifold.
(b) Show that K and L are not submanifolds if we view M as a smooth manifold.



Exercise 25.22. Let M be a smooth manifold and let $f \colon M \to N$ be a continuous map to some smooth manifold N. Furthermore let W be a smooth manifold and let $\varphi \colon W \to M$ be a smooth injective map. Show that if $f \circ \varphi \colon W \to N$ is smooth, then it is nonetheless in general not true that $f \colon M \to N$ is smooth[314] on the subset $\varphi(W) \subseteq M$.
Remark. In Exercise 25.23 we will see that in many important settings we can in fact deduce that f is smooth on the subset $\varphi(W) \subseteq M$.

Exercise 25.23. Let M be a smooth manifold and let $f \colon M \to N$ be a continuous map to some smooth manifold N. Recall that given a subset $A \subseteq M$ we explained in Definition 25.20 what it means for f to be smooth on A.

(a) We use the Smooth Manifold–Boundary Proposition 25.27 (1) to view the boundary ∂M as a smooth manifold in its own right. We denote by $\iota \colon \partial M \to M$ the inclusion map. Show that $f \circ \iota \colon \partial M \to N$ is smooth if and only if f is smooth on ∂M.

[314] Recall that given a subset $A \subseteq M$ we defined in Definition 25.20 what it means for f to be smooth on A.

(b) Let W be a submanifold of M. We use the Submanifold Smooth Structure Lemma 25.23 to view W as a smooth manifold in its own right. We denote by $\iota\colon W \to M$ the inclusion map. Show that $f \circ \iota\colon W \to N$ is smooth if and only if f is smooth on W.

Remark. Exercise 25.22 shows that the results of this exercise cannot be generalized ad infinitum.

Exercise 25.24. Show that the Klein bottle can be decomposed into two submanifolds that are diffeomorphic to the annulus $[0,1] \times S^1$.

Exercise 25.25. We equip each $\mathbb{R}^k$ with the standard smooth structure that we introduced on page 662 and we equip $\mathbb{R}^m \times \mathbb{R}^n$ with the product smooth structure introduced in the Smooth Manifold Product Proposition 25.34. Show that the map

$$\begin{array}{rcl} \mathbb{R}^{m+n} & \to & \mathbb{R}^m \times \mathbb{R}^n \\ (x_1, \dots, x_m, y_1, \dots, y_n) & \mapsto & ((x_1, \dots, x_m), (y_1, \dots, y_n)) \end{array}$$

is a diffeomorphism.

Exercise 25.26. Show that $N \coloneqq \{(x,y) \in \mathbb{R}^2 \,|\, y \geq x^2\}$ is not a smooth submanifold of the smooth manifold $M \coloneqq \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$.

Exercise 25.27. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Show that

$$\mathrm{T}M \;\coloneqq\; \{(P,w) \,|\, P \in M, w \in \mathrm{T}_P M\} \;\subseteq\; \mathbb{R}^n \times \mathbb{R}^n$$

is a $2m$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.
Remark. Later $\mathrm{T}M$ will play an important rule.

Exercise 25.28. Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^l$ be smooth submanifolds such that at most one has non-empty boundary. Show that $A \times B \subseteq \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ is a smooth submanifold of dimension $\dim(A) + \dim(B)$.

Exercise 25.29. Let M be a smooth manifold and let W be a (proper) submanifold.

(a) Show that $W \cap \partial M$ is a (proper) submanifold of ∂M.
(b) Let X be a (proper) submanifold of W. Show that X is also a (proper) submanifold of M.

Exercise 25.30. Let M be an n-dimensional smooth manifold with empty boundary. We consider the product $M \times M$.

(a) Show that the diagonal $\Delta \coloneqq \{(x,x) \,|\, x \in M\}$ is an n-dimensional submanifold of $M \times M$.
(b) Show that the diagonal Δ is *diffeomorphic* to M.
Remark. There is an obvious bijection, but why is this bijection a diffeomorphism?

Exercise 25.31. Let X be a topological space and let M be a smooth manifold. Furthermore let $f\colon X \to M$ be a continuous map. Now suppose that we are given two smooth structures $\mathcal{A}$ and $\mathcal{A}'$ on X such that $f\colon (X,\mathcal{A}) \to M$ and $f\colon (X,\mathcal{A}') \to M$ are local diffeomorphisms. Show that the two smooth structures on X are the same.
Hint. Use the Smooth Pasting Proposition 25.19.
Remark. This exercise can be viewed as the smooth manifold version of Exercise 4.74.

Exercise 25.32. Let $n \in \mathbb{N}$. A *lattice in* $\mathbb{R}^n$ is a subgroup of $(\mathbb{R}^n, +)$ that is a discrete subset of $\mathbb{R}^n$.

(a) Let $v_1, \dots, v_k \in \mathbb{R}^n$ be linearly independent vectors. Show that $\Lambda = \mathbb{Z} \cdot v_1 + \dots + \mathbb{Z} \cdot v_k$ is a lattice in $\mathbb{R}^n$. We refer to k as the rank of the lattice.

(b) Show that any lattice in $\mathbb{R}^n$ is of the form described in (a).

(c) Show that the obvious action of a lattice $\Lambda \subseteq \mathbb{R}^n$ on $\mathbb{R}^n$, i.e. the action that is given by addition, is proper.

If Λ is a lattice in $\mathbb{R}^n$, then it follows from (c) and the Manifold–Quotient Proposition 25.39 that $\mathbb{R}^n/\Lambda$ is naturally an n-dimensional smooth manifold.

(d) Show that if Λ and $\widetilde{\Lambda}$ are two lattices in $\mathbb{R}^n$ of the same rank, then $\mathbb{R}^n/\Lambda$ and $\mathbb{R}^n/\widetilde{\Lambda}$ are diffeomorphic.

Exercise 25.33. Let $g \in \mathbb{N}$. Let M be the smooth manifold that we obtain from removing a single point from the surface of genus g. Does there exist a smooth map $M \to S^2$ that is a local diffeomorphism?

Exercise 25.34. In the figure below we show to the left the usual way to depict the Möbius band M as submanifold in $\mathbb{R}^3$. Does there exist an embedding of the Möbius band in $\mathbb{R}^3$ such that the boundary is the closed curve K shown to the right?

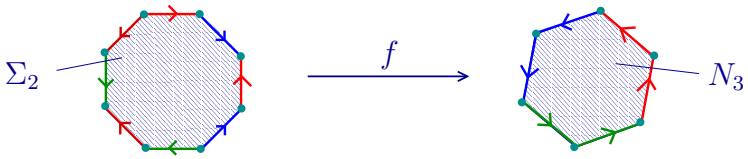


Exercise 25.35. Let W be an n-dimensional smooth manifold, let $M \subseteq W$ be an m-dimensional submanifold of W with $\partial M = \varnothing$ and let $K \subseteq M$ be a k-dimensional submanifold of M with $\partial K = \varnothing$. Show that for each $P \in K$ there exists an open neighborhood $U \subseteq W$, an open subset $V \subseteq \mathbb{R}^n$ and a diffeomorphism $\Phi \colon U \to V$ with the following properties:

(1) $\Phi(U \cap M) = V \cap (\mathbb{R}^m \times \{0\})$.

(2) $\Phi(U \cap K) = V \cap (\mathbb{R}^k \times \{0\})$.

Exercise 25.36. Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface g from Definition 9.18 and given $k \in \mathbb{N}$ we denote by N_k the non-orientable surface of genus k from Definition 9.19. Show that given any $k \in \mathbb{N}$ there exists a smooth map $f \colon \Sigma_{k-1} \to N_k$ that is a local diffeomorphism and that has the property that for each $x \in N_k$ we have $\#f^{-1}(x) = 2$.



CHAPTER 26

Smooth manifolds with corner

In this chapter we introduce smooth manifolds with corner. This might be viewed as a rather obscure topic, but it naturally arises in various situations, for example when one considers smooth homotopies of smooth manifolds with non-empty boundary.

This chapter is organized as follows:

- In Section 26.1 we will define smooth manifolds with corner.
- In Section 26.2 we will introduce the product of two smooth manifolds (with no restrictions on the boundary).
- In Appendix 26.3 we will introduce Submanifolds with corner.
- In Section 26.4 we will provide exercises for Chapter 26.

In the next Chapter 27 we will discuss partitions of unity on manifolds.

26.1. **Definition of smooth manifolds with corner.** In the Topological Manifold Product Proposition 23.6 we saw that the product of two topological manifolds X and Y is again a topological manifold. If both X and Y have non-empty boundary we had to make use of a delicate chart (see also the Corner Smooth Lemma 26.2 below.) It turns out that for smooth atlases this approach does not necessarily lead to a smooth atlas on $X \times Y$. This debacle requires us to introduce smooth manifolds with corner.

Notation 26.1. Given $n \in \mathbb{N}_{\geq 2}$ we refer to

$$\mathbb{R}^n_{++} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} \geq 0 \text{ and } x_n \geq 0\}$$

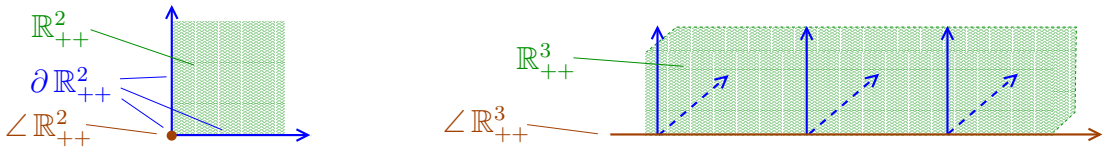
as the upper right quadrant of $\mathbb{R}^n$. We refer to

$$\partial\mathbb{R}^n_{++} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} = 0 \text{ or } x_n = 0\}$$

as the boundary of $\mathbb{R}^n_{++}$ and we refer to

$$\angle\mathbb{R}^n_{++} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_{n-1} = 0 \text{ and } x_n = 0\}$$

as the corner of $\mathbb{R}^n_{++}$.

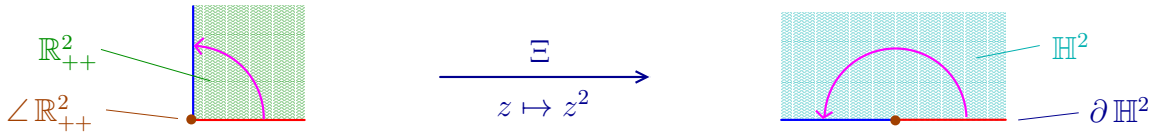


Lemma 26.2. (Corner Smooth Lemma) Let $n \in \mathbb{N}_{\geq 2}$.
(1) The map[315]

$$\begin{array}{rcl} \Xi\colon \mathbb{R}^n_{++} & \to & \mathbb{H}^n \\ (x_1,\dots,x_{n-2},\underbrace{r\cdot\cos(\varphi), r\cdot\sin(\varphi)}_{=:z\in\mathbb{C}}) & \mapsto & (x_1,\dots,x_{n-2},\underbrace{r^2\cdot\cos(2\varphi), r^2\cdot\sin(2\varphi)}_{=z^2}). \end{array}$$

is a homeomorphism which restricts to a homeomorphism $\partial\mathbb{R}^n_{++} \to \partial\mathbb{H}^n$.

(2) Let U be an open subset of $\mathbb{R}^n_{++}$ that contains a point of the corner $\angle\mathbb{R}^n_{++}$. There is no diffeomorphism[316] $f\colon U \to V$ from U to some open subset $V \subseteq \mathbb{H}^n$ such that $f(\partial\mathbb{R}^n_{++} \cap U) \subseteq \partial\mathbb{H}^n$.



Proof.

(1) We already showed this statement on page 626. But for the reader's convenience let us sketch the argument again. First of all note that the map $\mathbb{C} \to \mathbb{C}$ that is given by $z = x + \mathrm{i}y \mapsto z^2 = (x^2 - y^2) + 2xy\,\mathrm{i}$ is continuous. It follows that Ξ is continuous. It is straightforward to show, say using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26, that Ξ is indeed a homeomorphism.

(2) The proof of this statement is similar to the proof of the Unique Type-of-Chart Proposition 25.26. We outsource the proof to Exercise 26.1. ■

In the following we recall the definition of charts of type (i) and (ii) from Definition 23.2 and we introduce the notion of a corner chart.

Definition 26.3. Let X be a topological space and let $x \in X$. An n-dimensional chart at the point $x \in X$ is a homeomorphism $\Phi\colon U \to V$, where U is an open neighborhood of x and

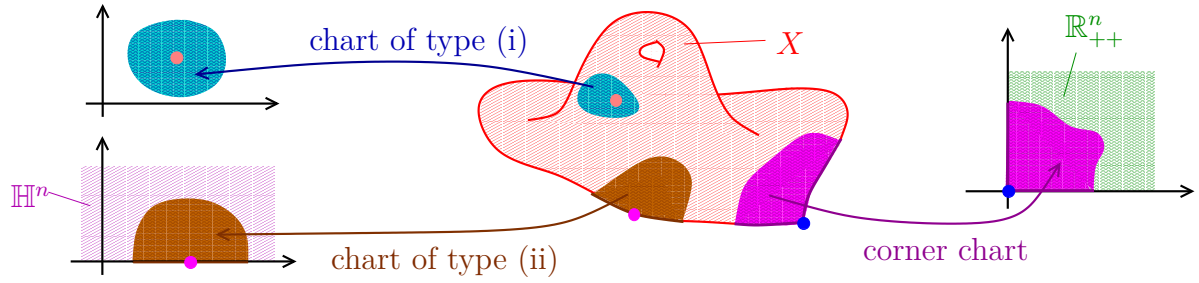
(i) V is an open subset of $\mathbb{R}^n$ or

(ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\mathbb{H}^n$.

An n-dimensional corner chart at the point $x \in X$ is a homeomorphism $\Phi\colon U \to V$, where U is an open neighborhood of x and

(c) V is an open subset of $\mathbb{R}^n_{++}$ and $\Phi(x)$ lies in $\angle\mathbb{R}^n_{++}$.

To simplify the discussion we refer to these as charts of type (i), (ii) and (c).



[315] We first encountered this nice map on page 626.

[316] Note that the definition of a diffeomorphism in Definition 24.18 also makes sense for subsets of $\mathbb{R}^n$ that are not open.

It follows from the Corner Smooth Lemma 26.2 (1) that if a point $x \in X$ admits a corner chart, then it also admits a chart of type (ii). Thus if we are only interested in topological manifolds such charts do not add anything new and they are basically irrelevant. As the Corner Smooth Lemma 26.2 (2) indicates, smoothly the situation is different. This leads us to the following definition:

Definition 26.4. Let X be a topological space.

(1) An n-dimensional corner atlas for the topological space X is a collection of n-dimensional charts of type (i), (ii) and (c) such that the domains cover all of X.

(2) We say that two n-dimensional charts $\Phi_1 \colon U_1 \to V_1$ and $\Phi_2 \colon U_2 \to V_2$ for X are smoothly compatible if the transition map

$$\Phi_2 \circ \Phi_1^{-1} : \underbrace{\Phi_1(U_1 \cap U_2)}_{\substack{\text{open subset in} \\ \mathbb{R}^n,\ \mathbb{H}^n \text{ or } \mathbb{R}^n_{++}}} \to \underbrace{\Phi_2(U_1 \cap U_2)}_{\subseteq \mathbb{R}^n}$$

is smooth, in the sense of our Definition 24.3.

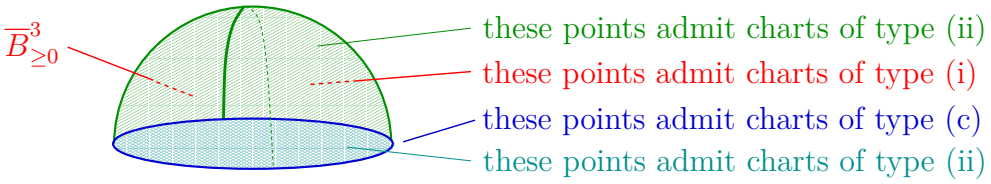
(3) A corner atlas $\mathcal{A}$ is called smooth if all charts in the corner atlas are smoothly compatible to one another.

(4) A smooth corner atlas $\mathcal{A}$ for M is called maximal if every chart Ψ that is smoothly compatible with all the charts in the smooth atlas $\mathcal{A}$ is already contained in $\mathcal{A}$. The proof of the Maximal Atlas Existence Proposition 25.4 shows that every smooth corner atlas is contained in a unique maximal smooth corner atlas.

(5) An n-dimensional smooth manifold with corner is a pair $(X, \mathcal{A})$ consisting of a second-countable Hausdorff space together with a maximal n-dimensional smooth corner atlas $\mathcal{A}$ for X.

Example. Let $n \geq 2$. We consider the upper half-ballex:upper-half-plane-corner

$$\overline{B}^n_{\geq 0} \;=\; \{(x_1, \ldots, x_n) \in \overline{B}^n \,|\, x_n \geq 0\}.$$



In Exercise 26.3 we will see that $\overline{B}^n_{\geq 0}$ is a smooth manifold with corner $\overline{B}^n_{\geq 0} = S^{n-1} \times \{0\}$, with charts as shown in the figure above. □

The following lemma says, as we already hinted at, that as topological manifolds we did not introduce anything new.

Lemma 26.5. (Corner–Topological Manifold Lemma) If $(X, \mathcal{A})$ is an n-dimensional smooth manifold with corner, then the underlying topological space X is an n-dimensional topological manifold.

Proof. This lemma follows, as will be clear to the reader, from the observation that the composition of a corner chart with the homeomorphism given in the Corner Smooth Lemma 26.2 is a chart of type (ii). ■

Definition 26.6. Many of the definitions for smooth manifolds extend in an obvious way to smooth manifolds with corner:

(1) As in the Open Subset-Submanifold Lemma 25.8 and the Open Subset-Submanifold Boundary Lemma 25.11, and as in the convention on page 663, we can view an open subset of a smooth manifold with corner as a smooth manifold with corner.
(2) A smooth map between smooth manifolds with corner is defined as in Definition 25.15.[317]

For completeness' sake we state the following generalizations of the Smooth Composition Lemma 25.17:

Lemma 26.7. (Smooth Composition Lemma) The composition of two smooth maps between smooth manifolds with corner is again smooth.

Proof. Again the proof is basically identical to the rather proof of the earlier Smooth Composition Lemma 25.17. ■

The following proposition is a generalization of the Unique Type-of-Chart Proposition 25.26:

Proposition 26.8. (Unique Type-of-Chart Proposition) Let M be a smooth manifold with corner and let $x \in M$. Precisely one of the following three statements holds:

- The point x admits a chart of type (i) from the smooth structure.
- The point x admits a chart of type (ii) from the smooth structure.
- The point x admits a chart of type (c) from the smooth structure.

Proof. The fact that a point that admits charts of type (i) in the smooth structure cannot also admit a chart of type (ii) in the smooth structure is the content of the Unique Type-of-Chart Proposition 25.26. The same argument also shows that such a point cannot admit a chart of type (c) in the smooth structure.

The fact that charts of type (ii) and (c) in the smooth structure are mutually exclusive follows easily from the Corner Smooth Lemma 26.2 (2). We outsource the proof to Exercise 26.2. Alternatively see also [**Lee2002**, Proposition 16.20] for details. ■

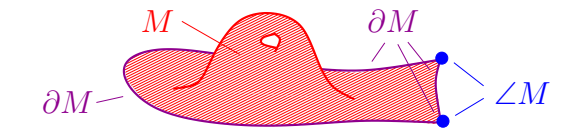
The Unique Type-of-Chart Proposition 26.8 leads us to the following analog and generalization of Definition 23.2 and Definition 25.10.

Definition 26.9. Let M be a smooth manifold with corner.

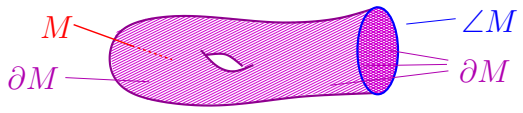
(1) We say that a point $x \in M$ is a boundary point if x admits a chart of type (ii) or (c) in the smooth structure. We denote by ∂M the set of all boundary points of M. We refer to ∂M as the boundary of M.
(2) We say that a point $x \in M$ is a corner point if x admits a chart of type (c) in the smooth structure. We denote by $\angle M$ the set of all corner points of M. We refer to $\angle M$ as the corner of M.

We conclude this section with the following proposition which is an analog of the Smooth Manifold–Boundary Proposition 25.27:

[317] Note that, as in Definition 25.15, a smooth map between smooth manifolds with corner is by definition continuous.



2-dimensional smooth manifold with corner



3-dimensional smooth manifold with corner

Proposition 26.10. (Smooth Manifold–Corner Proposition) Let M be an n-dimensional smooth manifold with corner. The following statements hold:

(1) (a) The subset $\partial M \angle M$ has a natural structure of an $(n-1)$-dimensional smooth manifold with $\partial(\partial M \setminus \angle M) = \varnothing$. Furthermore the inclusion map $\angle M \to M$ is a smooth map.

(b) The corner $\angle M$ has a natural structure of an $(n-2)$-dimensional smooth manifold with $\partial(\angle M) = \varnothing$. Furthermore the inclusion map $\angle M \to M$ is a smooth map.

(2) (a) Any union of components of ∂M is a closed subset of M.

(b) Any union of components of $\angle M$ is a closed subset of M.

Proof. The proof of these statements is very similar to the proof of the Smooth Manifold–Boundary Proposition 25.27. All we really have to do is to replace the Unique Type-of-Chart Proposition 25.26 by the Unique Type-of-Chart Proposition 26.8. We leave it to the reader to fill in the details. ■

Of course one could now develop the theory of smooth manifolds with corner in great detail. For example one could now introduce the notion of a submanifold of a smooth manifold with corner. We will spare the author and the reader to go through this discussion which would mostly be a case of "l'art pour l'art".

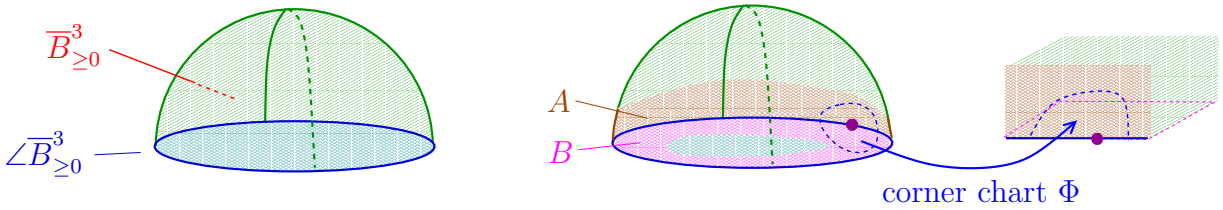
Instead, to pique the reader's interest we give a sneak preview of a later result. To formulate the result we will need the following definition:

Definition 26.11. Let M be an n-dimensional smooth manifold with corner. We say that the corner $\angle M$ is two-sided if there exist two disjoint open subsets A and B of ∂M such that the following conditions are satisfied:

(1) $A \cup \angle M \cup B$ is an open subset of ∂M,

(2) For every chart $\Phi\colon U \to V \subseteq \mathbb{R}^n_{++}$ of type (c) one of the following holds:

- $\Phi(A \cap U) \subseteq \{(x_1, \ldots, x_n) \,|\, x_{n-1} = 0\}$ and $\Phi(B \cap U) \subseteq \{(x_1, \ldots, x_n) \,|\, x_n = 0\}$, or
- $\Phi(B \cap U) \subseteq \{(x_1, \ldots, x_n) \,|\, x_{n-1} = 0\}$ and $\Phi(A \cap U) \subseteq \{(x_1, \ldots, x_n) \,|\, x_n = 0\}$.



The following proposition now says that smooth manifolds with corner can be turned into smooth manifolds without corner.

Proposition 60.5 (Rounding Corner Proposition–Simplified). Let M be a smooth manifold with corner $\angle M$. If $\angle M$ is two-sided, then any choice of a "corner-collar" gives rise to a smooth atlas on M which consists only of charts of type (i) and (ii).

26.2. **Products of smooth manifolds II.** In the Topological Manifold Product Proposition 23.6 we saw that the product of two topological manifolds is again a topological manifold. The case that both topological manifolds have non-empty boundary was somewhat tricky.

Thus, when in Section 25.9 we started to look at the product of two smooth manifolds we sidestepped the issue by only considering the product of two smooth manifolds where at least one has empty boundary. We now turn to the product of two smooth manifolds where both are allowed to have non-empty boundary.

Proposition 26.12. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We denote by

$$\begin{array}{rcl} p\colon M\times N & \to & M \\ (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon M\times N & \to & N \\ (x,y) & \mapsto & y \end{array}$$

the two projections. The following statements hold:

(1) The product $M\times N$ is an $(m+n)$-dimensional topological manifold that admits a *unique* structure as a smooth manifold with corner that satisfies the following two conditions:
   (a) The projections $p\colon M\times N\to M$ and $q\colon M\times N\to N$ are smooth.
   (b) For every smooth manifold with corner W and every map $f\colon W\to M\times N$ the following holds: If the two maps $p\circ f\colon W\to M$ and $q\circ f\colon W\to N$ are smooth, then f itself is smooth.

(2) (a) $\partial(M\times N)=(M\times\partial N)\cup(\partial M\times N)$.
   (b) $\angle(M\times N)=\partial M\times\partial N$ as $(n-2)$-dimensional smooth manifolds.

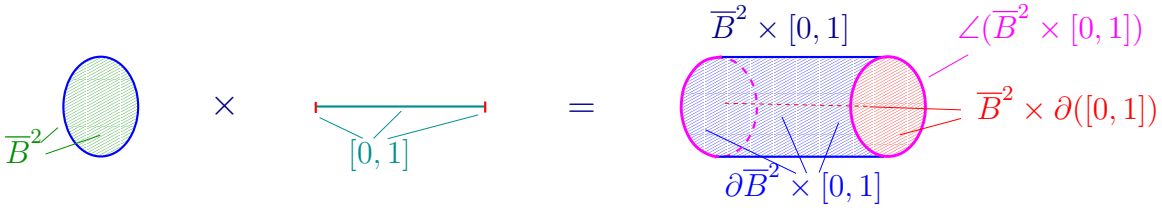
Proof. Similar to the proof of the Smooth Manifold Product Proposition 25.34 we obtain a corner atlas for $M\times N$ by taking products of charts of M and N. The proof of the proposition is thus just a straightforward modification of the already very straightforward proof of the Smooth Manifold Product Proposition 25.34. The only real difference is that for the proofs of Statement (2a) and (2b) we need to replace the Unique Type-of-Chart Proposition 25.26 by the Unique Type-of-Chart Proposition 26.8. ■

Example.

(1) If $\partial M=\varnothing$ or $\partial N=\varnothing$, then the Smooth Manifold Product Proposition 26.12 is of course just a special case of the Smooth Manifold Product Proposition 25.34.

(2) Let $M=\overline{B}^2$ and $N=[0,1]$, then $M\times N=\overline{B}^2\times[0,1]$ is the solid cylinder which is a 3-dimensional smooth manifold with corner such that

$$\begin{aligned} \text{boundary } \partial(\overline{B}^2\times[0,1]) &= (S^1\times[0,1])\cup(\overline{B}^2\times\{0,1\}) \\ \text{corner } \angle(\overline{B}^2\times[0,1]) &= S^1\times\{0,1\}. \end{aligned}$$



(3) Later, in Definition 41.2, given two smooth manifolds M and N and given smooth maps $f, g\colon M \to N$ we will introduce the notation of a smooth homotopy between f and g. This is just a smooth map $F\colon M \times [0,1] \to N$ with $F_0 = f$ and $F_1 = g$. At this point we will be very interested in products of the form $M \times [0,1]$.

(4) The courageous reader can show that the smooth manifold $\overline{B}^n \times [0,1)$ with corner $S^{n-1} \times \{0\}$ is diffeomorphic to the smooth manifold $\overline{B}^n_{\geq 0} \setminus \{(0,\ldots,0,1)\}$ with corner $S^{n-1} \times \{0\}$ which we introduced on page 1335. □

We continue with the following lemma:

Lemma 26.13. (Product-of-Smooth Maps Lemma) Let $f\colon M \to \widetilde{M}$ and $g\colon N \to \widetilde{N}$ be smooth maps between smooth manifolds. The map

$$\begin{array}{rccc} \operatorname{diag}(f,g)\colon & M \times N & \to & \widetilde{M} \times \widetilde{N} \\ & (x,y) & \mapsto & (f(x), g(y)) \end{array}$$

is a smooth map between smooth manifolds with corner.

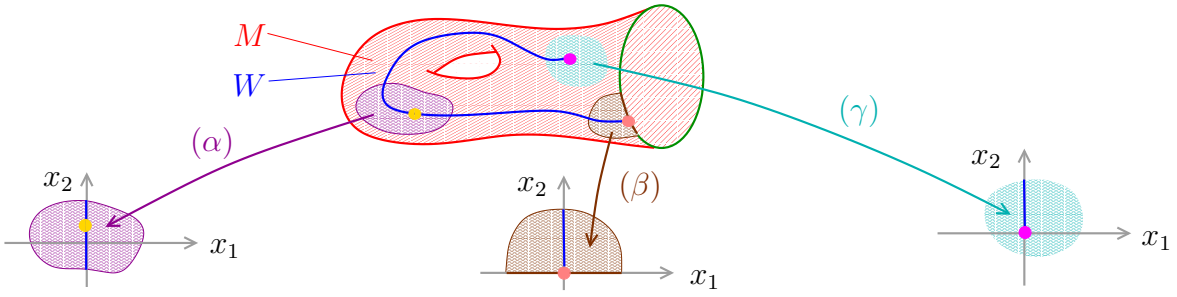
Proof. The proof is formally similar to the proof of the Topological-Product Proposition 7.2 (1), in the sense, that it follows easily from the Smooth Manifold Product Proposition 26.12 (1b). We encourage the reader to fill in the details. ■

By the Rounding Corner Proposition 60.5 we know that, upon the choice of a corner-collar, we can usually dispose of corners. For products of smooth manifolds this leads to the following proposition:

Proposition 60.7 (Rounding Corner Proposition–Simplified). Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. The corner $\angle(M \times N)$ is two-sided and any choice of a "corner-collar" gives rise to a smooth atlas on $M \times N$ which consists only of charts of type (i) and (ii).

If one keeps one's eyes open, smooth manifolds with corner occur naturally in many settings. Nonetheless, the theory of smooth manifolds with corner is an unloved (and some would say unlovable) topic which did not receive much treatment in the literature. We refer to [**Lee2002**, Chapter 16] and [**Joy2012**] for more information on smooth manifolds with corner and the obvious generalization where one allows charts that take values in open subsets of $\mathbb{R}^k \times (\mathbb{R}_{\geq 0})^{n-k}$.

26.3. **Appendix: Submanifolds with corner.** Recall that in Definition 23.18 we defined what it means for a subset W of a manifold M to be a submanifold, namely it means that given any $P \in W$ there is a chart for M of one of the three types (α), (β) and (γ) that are defined properly in Definition 23.18 and that are sketched in the figure below.



Now we introduce the more general notion of a submanifold with corner.

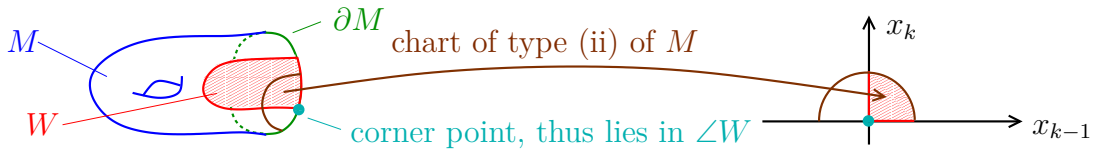
Definition 26.14. Let M be an n-dimensional topological manifold. We say that a subset $W \subseteq M$ is a k-dimensional submanifold with corner[318] if for every $P \in W$ there exists a chart of the type (α), (β) and (γ) from Definition 23.18 or if

(δ) there exists a chart $\Phi\colon U \to V$ of type (ii)[319] for M such that

$$\Phi(U \cap W) \subseteq \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_1,\dots,x_k \in \mathbb{R} \text{ with } x_{k-1} \geq 0 \text{ and } x_k \geq 0\}$$
$$\text{and with } \Phi(P) \in \{(0,\dots,0,x_1,\dots,x_{k-2},0,0) \,|\, x_1,\dots,x_{k-2} \in \mathbb{R}\}.$$

If M is an n-dimensional smooth manifold, then we demand that the charts come from the maximal smooth atlas. We introduce the following notation:

- We define ∂W as the set of all points in W that admit a submanifold chart of type (β), (γ) or (δ).
- We refer to the set of points in W that admit a chart of type (δ) as the corner set of W and we denote it by $\angle W$.

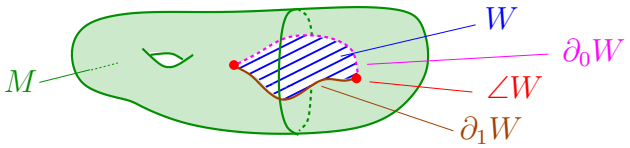


Remark. Throughout these lecture notes we will only consider submanifolds with corner of codimension *zero*. But the concept of a submanifold with corner can of course also play a role for other codimensions. □

The following definition generalizes some concepts from Notation 25.47:

Definition 26.15. Let M be an n-dimensional smooth manifold and let W be a submanifold with corner of M. We write

$$\begin{aligned} \partial_0 W &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\gamma) \text{ or } (\delta)\,, \\ \partial_1 W &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\beta) \text{ or } (\delta), \\ \angle W &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\delta). \end{aligned}$$



The following lemma is related to the Codimension-Zero Submanifold Proposition 25.49.

Lemma 26.16. (Smooth Submanifold-with-Corner Lemma) Let M be an n-dimensional smooth manifold and let W be a codimension-zero submanifold with corner that is a closed subset of M (e.g. X could be a compact subset). The following statements hold:

(0) (a) $\partial_0 W = \overline{\partial W \setminus \partial M}$,
(b) $\partial_1 W = \partial W \cap \partial M$,

[318] Often in the literature such objects are called "submanifold with corners". Following [**Wall2016**, p. 30] we write "submanifold with corner" instead.

[319] See Definition 23.2 for the definition of a chart of type (ii).

(c) $\angle W = \partial_0 W \cap \partial_1 W$.

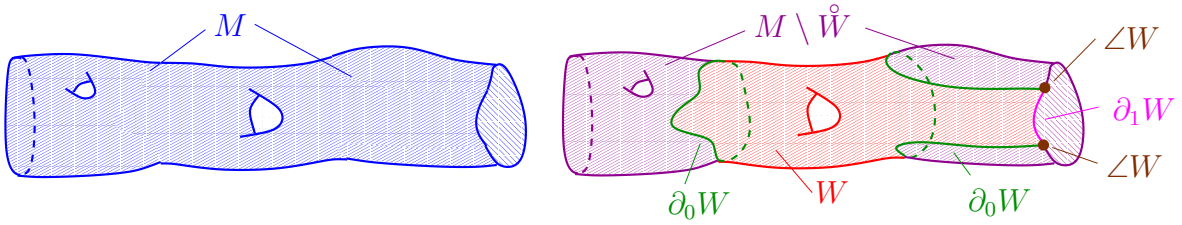
(1) W is itself an n-dimensional smooth manifold with corner $\angle W = \partial_0 W \cap \partial_1 W$.[320]

(2) (a) We have $\partial W = \partial_0 W \cup \partial_1 W$.
(b) $\partial_0 W$ and $\partial_1 W$ are codimension-zero submanifolds of the topological manifold ∂W and they are both closed subsets of ∂W.

(3) The set $\mathring{W} := W \setminus \partial_0 W$ is the interior of W in M, in particular $\mathring{W}$ is an open subset of M.

(4) The subset $M \setminus \mathring{W}$ is a codimension-zero submanifold of M with corner given by $\angle(M \setminus \mathring{W}) = \angle W$. Furthermore $\partial_0(M \setminus \mathring{W}) = \partial_0 W$ and $\partial_1(M \setminus \mathring{W}) = \overline{\partial M \setminus \partial_1 W}$.

(5) $\partial_0 W$ is a proper submanifold of M, in the sense of Definition 23.18.



Remark. We will formulate the topological analog of the Smooth Submanifold-with-Corner Lemma 26.16 in the Topological Submanifold-with-Corner Lemma 212.3. □

Sketch of proof. All the statements follow either easily from the definitions or by the arguments of the analogous statements in the Codimension-Zero Submanifold Proposition 25.49. We leave the slightly annoying task of filling in the details to the reader. ■

26.4. Exercises for Chapter 26.

Exercise 26.1. Let $n \in \mathbb{N}_{\geq 2}$. Let U be an open subset of $\mathbb{R}^n_{++}$ that contains a point of the corner $\angle \mathbb{R}^n_{++}$. Show that there is no diffeomorphism $f\colon U \to V$ from U to some open subset $V \subseteq \mathbb{H}^n$ such that $f(\partial \mathbb{R}^n_{++} \cap U) \subseteq \partial \mathbb{H}^n$.

Exercise 26.2. Let M be an n-dimensional smooth manifold with corner and let $x \in M$. Show that x cannot admit both a chart of type (ii) and a chart of type (c).

Exercise 26.3. Let $n \geq 2$. Show that the upper half-ball

$$\overline{B}^n_{\geq 0} \;=\; \{(x_1, \dots, x_n) \in \overline{B}^n \,|\, x_n \geq 0\}$$

admits the structure of a smooth manifold with corner such that

$$\angle \overline{B}^n_{\geq 0} \;=\; \{(x_1, \dots, x_{n-1}, 0) \in \overline{B}^n \,|\, (x_1, \dots, x_{n-1}) \in S^{n-2}\} \;=\; S^{n-2} \times \{0\}.$$

Exercise 26.4. Give an example of a smooth manifold M with corner such that the corner $\angle M$ is not two-sided.

Exercise 26.5. Let A be a codimension-zero submanifold of some smooth manifold M with $\partial M = \varnothing$. Furthermore let N be a smooth manifold.

(a) Show that $A \times N$ is a submanifold with corner of $M \times N$.
(b) Determine $\partial_0(A \times N)$, $\partial_1(A \times N)$ and $\angle(A \times N)$.

[320] In particular the notion of a corner of a submanifold and of a smooth manifold with corner agree.

CHAPTER 27

Partitions of unity on manifolds

In this short chapter we show in particular that (smooth) manifolds admit (smooth) partitions of unity and that they admit (smooth) exhaustion functions. For topological manifolds these results also follow from earlier results, like the Paracompact-Partition of Unity Theorem 14.15 which builds on Urysohn's Lemma 12.5. But for smooth manifolds the results of this chapter are new and for topological manifolds the arguments from this chapter are more elementary than the earlier ones.

The existence of (smooth) partitions of unity will be a key technical tool for constructing interesting (smooth) functions on (smooth) manifolds. In particular they will allow us to prove the Smooth Extension Proposition 27.12 and the Smooth Approximation Proposition 27.13.

This chapter is organized as follows:

- In Section 27.1 we will discuss partitions of unity on smooth manifolds.
- In Section 27.2 we will provide the proof of the Manifold–Partitions-of-Unity Theorem 27.2.
- In Section 27.3 we will discuss paracompactness and second-countability.
- In Section 27.4 we will discuss smooth extensions and approximations of smooth manifolds on (subspaces) of smooth manifolds.
- In Section 27.5 we will provide exercises for Chapter 27.

In Chapter 28 we will classify 1-dimensional topological and smooth manifolds.

27.1. **Existence on partitions of unity on manifolds.** First we recall the following definitions from Definition 14.14.

Definition 27.1. Let X be a topological space.

(1) The support of a function $f\colon X \to \mathbb{R}$ is defined as

$$\operatorname{supp}(f) \;:=\; \overline{\{x \in X \mid f(x) \neq 0\}}.$$

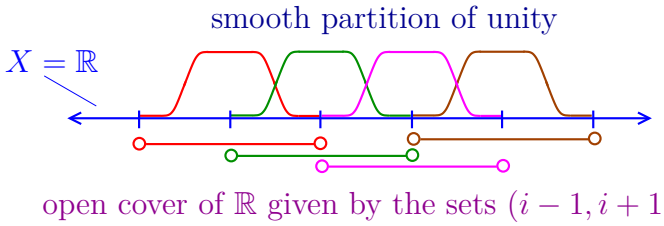
(2) A partition of unity on X is a family of continuous maps $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
   (a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i.[321]
   (b) For every $x \in X$ we have $\sum_{i\in I} f_i(x) = 1$.

[321] It follows from the elementary Closure-of-Locally Finite Cover Lemma 14.10 that Condition (1) is satisfied if and only if the supports $\{\operatorname{supp}(f_i)\}_{i\in I}$ form a locally finite family of subsets of X. In other words, Condition (1) is satisfied if and only if given any $x \in X$ there exists an open neighborhood U of x such that $U \cap \operatorname{supp}(f_i) = \varnothing$ for all but finitely many i.

The following theorem guarantees the existence of many interesting (smooth) partitions of unity.

Theorem 27.2. (Manifold–Partitions-of-Unity Theorem) Let X be a topological manifold.

(1) Given any open cover $\{U_i\}_{i\in I}$ there exists a partition of unity $\{f_i\}_{i\in I}$ such that for each $i \in I$ we have $\operatorname{supp}(f_i) \subseteq U_i$.
(2) If X is in fact a smooth manifold, then we can arrange that the maps f_i are smooth.



Remark.

(1) The Manifold–Partitions-of-Unity Theorem 27.2 is a great tool for constructing (smooth) maps *out* of a topological or smooth manifold. For constructing maps *into* a smooth manifold a particularly useful tool is given by the Neighborhood Projection Theorem 35.2.
(2) In the Manifold–Local Properties Lemma 23.8 we saw that topological manifolds are regionally compact. Since topological manifolds are by definition Hausdorff and second-countable we deduce from the Paracompact-Criterion Lemma 14.5 that topological manifolds are paracompact. In the Paracompact-Partition of Unity Theorem 14.15 we used Urysohn's Lemma 12.5 to show that any paracompact space satisfies the conclusion of the Manifold–Partitions-of-Unity Theorem 27.2 (1). Or, to cut a long story short, the above earlier results already provide a proof of the Manifold–Partitions-of-Unity Theorem 27.2. In this chapter we give a different proof, that does not make use of Urysohn's Lemma 12.5, and that has the distinct advantage that it also provides a proof of the Manifold–Partitions-of-Unity Theorem 27.2 (2).
(3) The combination of the Manifold–Partitions-of-Unity Theorem 27.2 and the Paracompact-Partition of Unity Theorem 14.15 shows that topological manifolds are paracompact. This provides a new proof of the Manifold Paracompact-Normal Proposition 23.9 (1). □

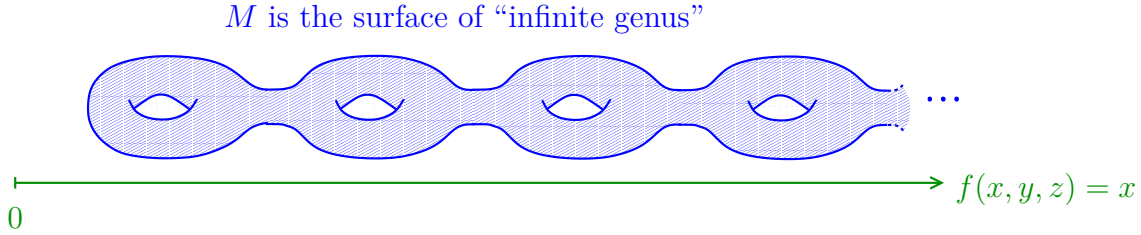
We postpone the proof of the Manifold–Partitions-of-Unity Theorem 27.2 to the next section. In the following we provide a typical application of the existence of partitions of unity. To do so we recall Definition 14.11:

Definition 27.3. Let X be a topological space. An exhaustion function is a continuous map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0, c])$ is a compact subset of X.[322]

Examples.

[322]One can easily show that a map $X \to \mathbb{R}_{\geq 0}$ is an exhaustion function if and only if it is proper in the sense of Definition 4.63.

(1) Let $M = [0, 1)$ be the half-open interval, viewed as a 1-dimensional smooth manifold. In this case the function defined by $f(x) = \frac{1}{1-x}$ is a smooth exhaustion function.
(2) We consider the surface of infinite genus illustrated in the figure below.



In this case the function $f(x, y, z) = x$ is a smooth exhaustion function. □

Lemma 27.4. (Manifold Exhaustion Lemma)
(1) Every topological manifold admits an exhaustion function.
(2) Every smooth manifold admits a smooth exhaustion function.

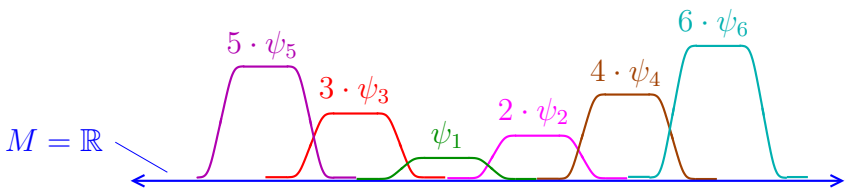
Remark.
(1) In the Exhaustion Function Lemma 14.12 we used Urysohn's Lemma 12.5 to show that every topological space that is regionally compact, second-countable and Hausdorff admits an exhaustion function. Thus, as in the previous remark, we see that we already gave a proof of the Manifold Exhaustion Lemma 27.4 (1). The interest of the present proof is that it does not make use of Urysohn's Lemma 12.5 and that it gives a stronger conclusion for smooth manifolds.
(2) The Manifold Exhaustion Lemma 27.4 can be viewed as a refinement of the Topological Manifold Exhaustion Lemma 23.12. □

Proof. In the following we provide the proof for smooth manifolds. The proof for topological manifolds is almost verbatim the same.[323] Thus let M be a smooth manifold. By the Manifold–Local Properties Lemma 23.8 (5) we know that M is locally precompact, i.e. given any $x \in M$ there exists a neighborhood U of x such that the closure of U in M is compact. It follows from the observation that M is by definition second-countable, together with the Countable Cover Lemma 13.4, that there exists a countable cover $\{U_k\}_{k\in\mathbb{N}}$ of M by precompact sets. Furthermore by the Manifold–Partitions-of-Unity Theorem 27.2 we now know that there exists a smooth partition of unity $\{\psi_k\}_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$ we have $\operatorname{supp}(\psi_k) \subseteq U_k$. We define

$$\begin{aligned} f\colon M &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sum_{k\in\mathbb{N}} k \cdot \psi_k(x). \end{aligned}$$

By definition of a partition of unity we know that locally this is a finite sum of smooth functions. It follows from the Smooth Pasting Proposition 25.19 that f is smooth.

[323] Also, as we just mentioned in the remark above, the statement for topological manifolds is a special case of the Exhaustion Function Lemma 14.12. In fact our proof for smooth manifolds is almost identical to the proof of the Exhaustion Function Lemma 14.12, the biggest difference is that we need to replace the Paracompact-Criterion Lemma 14.5 and the Paracompact-Partition of Unity Theorem 14.15 by the Manifold–Partitions-of-Unity Theorem 27.2.



It remains to prove the following claim:

Claim. For every $c \in \mathbb{R}$ the preimage $f^{-1}([0,c])$ is compact.

Proof. Let $c \in \mathbb{R}$. We pick $N \in \mathbb{N}$ with $N \geq c$. We start out with the following observation: Given any point $x \notin \overline{U}_1 \cup \cdots \cup \overline{U}_N$ we have

$$f(x) = \sum_{k=1}^{\infty} k \cdot \psi_k(x) \underset{\text{since } x \notin \overline{U}_1 \cup \cdots \cup \overline{U}_N \text{ and since } \operatorname{supp} \psi_k \subseteq U_k \subseteq \overline{U}_k}{=} \sum_{k=N+1}^{\infty} k \cdot \psi_k(x) \geq \sum_{k=N+1}^{\infty} N \cdot \psi_k(x) = N \cdot \underbrace{\sum_{k=1}^{\infty} \psi_k(x)}_{=1} = N > c.$$

It follows from this observation that

$$f^{-1}([0,c]) = \underbrace{f^{-1}([0,c])}_{\text{closed}} \cap \underbrace{\left(\overline{U}_1 \cup \cdots \cup \overline{U}_N\right)}_{\text{compact}}.$$

It follows from the Compact-Closed Lemma 3.39, see also Exercise 3.18, that $f^{-1}([0,c])$ is indeed compact. ∎

27.2. **Proof of the Manifold–Partitions-of-Unity Theorem 27.2.** In this section we now provide the long overdue proof of the Manifold–Partitions-of-Unity Theorem 27.2. The key ingredient is the following proposition which sounds technical, but which is interesting in its own right.

Proposition 27.5. (Atlas Improvement Proposition) Given an n-dimensional topological manifold M and given an open cover $\{U_i\}_{i \in I}$ there exists an atlas $\{\Phi_j \colon V_j \to W_j\}_{j \in J}$ with the following properties:

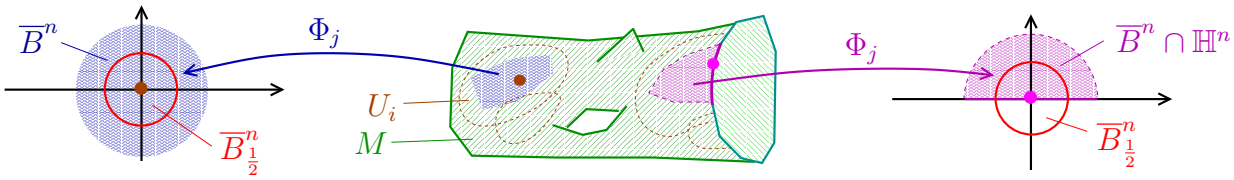
(1) The index set J is countable. If M is compact, then J can be chosen to be finite.
(2) The open cover $\{V_j\}_{j \in J}$ is a refinement of the open cover $\{U_i\}_{i \in I}$. In other words, given any $j \in J$ there exists an $i \in I$ with $V_j \subseteq U_i$.
(3) Each W_j is either the open n-ball B^n or the open half-ball $B^n \cap \mathbb{H}^n$.
(4) The open cover $\{V_j\}_{j \in J}$ is locally finite, i.e. given any point $x \in M$ there exists an open neighborhood X of x which has the property that $X \cap V_j = \varnothing$ for all but finitely many $j \in J$.
(5) We have $M = \bigcup_{j \in J} \Phi_j^{-1}(W_j \cap \overline{B}^n_{\frac{1}{2}})$.[324]

If M is in fact a smooth manifold, then we can also arrange that the following extra condition is satisfied:

(6) The charts are smooth, i.e. they lie in the maximal smooth atlas of M.

Remark.

[324]Note that it follows from the Compact Image Lemma 4.15 that the subsets $\Phi_j^{-1}(W_j \cap \overline{B}^n_{\frac{1}{2}})$ are compact. Thus we see in particular that we can cover M by a locally finite collection of compact subspaces.



(1) The Atlas Improvement Proposition 27.5 implies in particular that every topological manifold is paracompact. This provides a new proof of the Manifold Paracompact-Normal Proposition 23.9 (1).
(2) The Atlas Improvement Proposition 27.5 implies in particular that every topological manifold admits a locally finite cover by compact subsets.
(3) In the Atlas Improvement Proposition 27.5 we showed in particular that an n-dimensional topological manifold M admits a locally finite atlas. In the Covering Dimension-of-Manifolds Proposition 33.9 we will see that the covering dimension of M is $\leq n$. In the Small Atlas Proposition 33.10 we will use this fact to refine some of the conclusions of the Atlas Improvement Proposition 27.5. □

Now it is time to provide the proof of the Atlas Improvement Proposition 27.5.

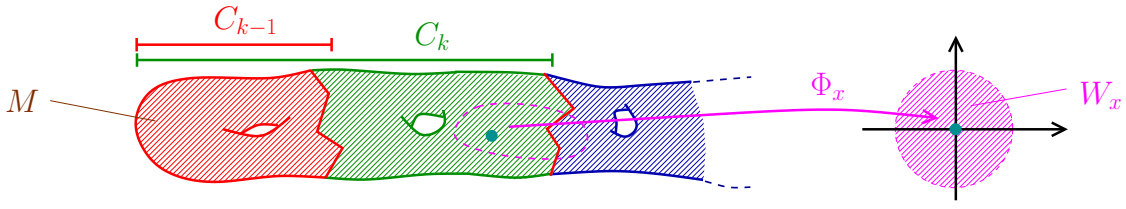
Proof. To simplify the discussion we only treat the case that M is a topological manifold with empty boundary. The proof of all other cases is basically the same and is left to the reader.

Thus let M be an n-dimensional topological manifold with empty boundary and let $\{U_i\}_{i\in I}$ be an open cover of M. By the Topological Manifold Exhaustion Lemma 23.12 there exists a family of compact subsets $\{C_k\}_{k\in\mathbb{N}}$ of M such that the following two conditions are satisfied:

(1) we have $M = \bigcup_{k\in\mathbb{N}} C_k$,
(2) for every $k \in \mathbb{N}$ we have $C_k \subseteq \mathring{C}_{k+1}$.

Note that it follows from the Compact-Closed Lemma 3.39 (2) that each $\mathring{C}_{k+1} \setminus C_{k-1}$ is an open subset of M. Furthermore, by design these open subsets cover all of M. Using this observation it is elementary to see that given any $x \in M$ there exists a chart $\Phi_x \colon V_x \to W_x$ with the following properties:

- V_x is an open neighborhood of x which is contained in some U_i,
- there exists a $k \in \mathbb{N}$ with $V_x \subseteq \mathring{C}_{k+1} \setminus C_{k-1}$,
- W_x is the open n-ball B^n of radius 1.



For each $x \in M$ we set $\widetilde{U}_x := \Phi_x^{-1}(B^n_{\frac{1}{2}})$. Evidently the $\widetilde{U}_x$ are open subsets which cover all of M. It follows from the Compact Subspace Covering Lemma 3.38 that for each $k \in \mathbb{N}$ there exists a finite set $J_k \subseteq M$ such that $C_k \subseteq \bigcup_{x\in J_k} \widetilde{U}_x$. We set $J := \bigcup_{k\in\mathbb{N}} J_k$. We now claim that the atlas $\{\Phi_j \colon V_j \to W_j\}_{j\in J}$ has all the desired properties. In fact all statements

basically hold by construction. Perhaps Property (4) is the least obvious one, so we provide the argument. Thus let $x \in M$. There exists a $k \in \mathbb{N}$ such that $x \in \mathring{C}_{k+1} \setminus C_{k-1}$. We set $X := \mathring{C}_{k+1} \setminus C_{k-1}$. By construction we have $X \cap V_j = \varnothing$ unless $j \in J_{k+1} \cup J_k \cup J_{k-1}$. ■

At this point it is convenient to recall Definition 14.14:

Definition 27.6. Let $\{U_i\}_{i\in I}$ be an open cover of a topological space X. A partition of unity subordinate to the open cover is a partition of unity $\{f_j \colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subseteq U_i$.

With this definition we can now formulate the following lemma:

Lemma 27.7. (Partition-of-Unity–Index Set Lemma) Let X be a topological space (smooth manifold) and let $\{U_i\}_{i\in I}$ be an open cover of X. Suppose that there exists a (smooth) partition of unity $\{\varphi_j \colon X \to [0,1]\}_{j\in J}$ subordinate to the given open cover, then there exists in fact a (smooth) partition of unity $\{\psi_i \colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\operatorname{supp}(\psi_i) \subseteq U_i$.

Remark. For topological spaces the statement of the Partition-of-Unity–Index Set Lemma 27.7 is precisely the (2)⇒(3)-direction of the Paracompact-Partition of Unity Theorem 14.15. But even for topological spaces our proof of the Partition-of-Unity–Index Set Lemma 27.7 is of interest, since it is a direct proof, whereas in the Paracompact-Partition of Unity Theorem 14.15 we took an indirect approach via Urysohn's Lemma 12.5. □

Proof. Let $\{\varphi_j \colon X \to [0,1]\}_{j\in J}$ be a (smooth) partition of unity subordinate to the given open cover $\{U_i\}_{i\in I}$ of the topological space X (smooth manifold X). For each $j \in J$ there exists therefore an $i(j) \in I$ with $\operatorname{supp}(\varphi_j) \subseteq U_{i(j)}$. Given $i \in I$ we set

$$\begin{array}{rcl} \psi_i \colon X & \to & [0,1] \\ x & \mapsto & \displaystyle\sum_{j\in J \text{ with } i(j)=i} \varphi_j(x). \end{array}$$

Note that ψ_i is locally a finite sum of (smooth) maps, hence the map ψ_i is locally continuous (locally smooth), which implies by the Pasting Proposition 4.7 (respectively the Smooth Pasting Proposition 25.19) that ψ_i is continuous (respectively smooth). Evidently the maps ψ_i add up to 1. Thus it remains to prove the following claim:

Claim. For each $i \in I$ we have $\operatorname{supp}(\psi_i) \subseteq U_i$.

Proof. Let $i \in I$. The claim follows from the following slightly delicate steps:

(1) By the definition of a partition of unity, see Definition 14.14, we know that the family $\{\varphi_j^{-1}((0,1])\}_{j\in J}$ is locally finite.
(2) It follows easily from (1) and the Closure-of-Locally Finite Cover Lemma 14.10, that the family $\{\operatorname{supp}(\varphi_j)\}_{j\in J}$ is also locally finite.
(3) We observe that it follows from (2) and the Interior-Closure-Cup-Cap Lemma 3.24 (2b') that $\bigcup_{j\in J \text{ with } i(j)=i} \operatorname{supp}(\varphi_j)$ is a closed subset of X.
(4) By definition of ψ_i we know that $\psi_i^{-1}((0,1])$ is contained in $\bigcup_{j\in J \text{ with } i(j)=i} \operatorname{supp}(\varphi_j)$.

Now we obtain the following inclusions:

$$\operatorname{supp}(\psi_i) \;=\; \overline{\psi_i^{-1}((0,1])} \;\subseteq\; \bigcup_{j\in J \text{ with } i(j)=i} \underbrace{\operatorname{supp}(\varphi_j)}_{\subseteq U_{i(j)}=U_i} \;\subseteq\; U_i.$$

$\uparrow$ follows from (3) and (4)

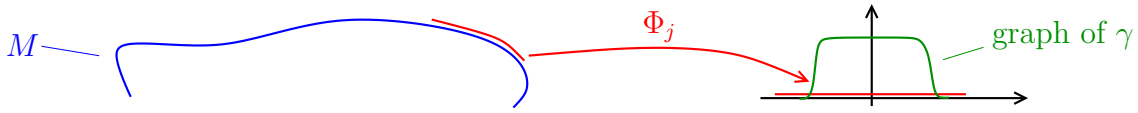
■

Now we can finally provide the proof of the Manifold–Partitions-of-Unity Theorem 27.2.

Proof of the Manifold–Partitions-of-Unity Theorem 27.2. Let M be a topological or a smooth manifold with a given open cover $\{U_i\}_{i\in I}$ of M. We apply the Atlas Improvement Proposition 27.5 and we obtain a (smooth) atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ with Properties (1)–(6). Note that according to the Smooth Transition Function Lemma 24.9 there exists a smooth function $\gamma\colon B^n \to [0,1]$ such that $\gamma(x) = 1$ whenever $\|x\| \leq \frac{1}{2}$ and such that $\gamma(x) = 0$ whenever $\|x\| > \frac{3}{4}$. For each $j \in J$ we consider the function

$$\begin{array}{rccl} g_j\colon M & \to & [0,1] & \\ x & \mapsto & \begin{cases} \gamma(\Phi_j(x)), & \text{if } x \in V_j, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We make the following observations:



(a) Let $j \in J$. Note that the restriction of g_j to the open subsets V_j and $M \setminus \Phi_j^{-1}(\overline{B}^n_{\frac{3}{4}})$ is continuous, hence g_j is continuous by the Pasting Proposition 4.7 (2). Using Property (6) of the atlas and using the Smooth Pasting Proposition 25.19 (1) we also see that the g_j are smooth, if we start out with a smooth manifold.

(b) It follows from Property (4) of the atlas that given any $x \in M$ there exists an open neighborhood W of x such that $g_j(x) = 0$ for all but finitely many $j \in J$.

(c) It follows from Property (2) of the atlas that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(g_j) \subseteq U_i$.

(d) It follows from Property (5) of the atlas that for each $x \in M$ there exists a $j \in J$ with $g_j(x) \neq 0$.

It follows from (b) that it makes sense to consider the function $G\colon M \to \mathbb{R}_{\geq 0}$ that is given by $G(x) \coloneqq \sum_{j\in J} g_j(x)$. By (d) we know that G is non-zero everywhere. Furthermore, in the setting of a smooth manifold this function is smooth. It is now clear that the functions $x \mapsto \varphi_j(x) \coloneqq \frac{g_j(x)}{G(x)}$ are a (smooth) partition of unity subordinate to the given open cover.

It follows from this discussion together with the Partition-of-Unity–Index Set Lemma 27.7 that there exists a (smooth) partition of unity $\{\psi_i\colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\operatorname{supp}(\psi_i) \subseteq U_i$. ■

27.3. **Paracompactness and second-countability.** In Definition 23.2 we defined an n-dimensional topological manifold as a topological space that satisfies the following three conditions:

(a) X is Hausdorff,

(b) X is locally homeomorphic to an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$,

(c) X is second-countable.

We now consider also the following two properties of topological spaces:

(d) X is paracompact,

(e) X is the union of countably many compact subsets.

In some references, see e.g. [**Hu1969, Aub2001**] and [**PoR2021**], a topological manifold is defined as a topological space that satisfies (a), (b) and (d). Furthermore in [**Wall2016**] a topological manifold is defined as a topological space that satisfies (a), (b) and (e). Our goal in this section is to show that if a topological space already satisfies (a) and (b), and if X is connected, then (c) is equivalent to (d) and also to (e).

In fact in the Manifold Paracompact-Normal Proposition 23.9 (1) we just saw that every topological manifold is paracompact. Furthermore in in the Topological Manifold Exhaustion Lemma 23.12 we showed in particular that every topological manifold satisfies (e). The following two results show that a connected space that satisfies (a), (b) and (d) respectively (e), also satisfies (a), (b) and (c).

We will not really make use of these results, except that it allows us to quote results from other resources without inhibition and with complete ease of mind. First we have the following proposition which can be viewed as a converse to the Manifold–Partitions-of-Unity Theorem 27.2.

Proposition 27.8. (Paracompact–Second-Countable Proposition) Let $n \in \mathbb{N}$ and let X be a topological space that is Hausdorff, connected and that is locally homeomorphic to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. If X is paracompact, then it is also second-countable.

The proof of the Paracompact–Second-Countable Proposition 27.8 rests on the following lemma, which can be viewed as the promised partial converse to the Topological Manifold Exhaustion Lemma 23.12.

Lemma 27.9. (Second-Countable Criterion) Let $n \in \mathbb{N}$ and let X be a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. If X can be written as a union of countably many compact subsets, then X is second-countable.

Proof of Lemma 27.9. Let $\{K_i\}_{i\in I}$ be a countable family of compact subsets of X that cover X.

Claim. Each K_i is contained in an open subset V_i that is second-countable.

Proof. Let $i \in I$. Since K_i is compact and since X is locally homeomorphic to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$ we see that K_i can be written as the union of finitely many sets $A_1, \ldots, A_n$ each of which is contained in an open subset U_i that is homeomorphic to a subspace of $\mathbb{R}^n$ or $\mathbb{H}^n$. By the Second Countability Lemma 13.2 (1) and (2) we know that each U_i is second-countable and by the Second Countability Lemma 13.2 (4) we know that $V_i := U_1 \cup \cdots \cup U_n$ is second-countable. ⊞

It follows from the claim that X is the union of countably many second-countable open subsets. Thus we obtain from the Second Countability Lemma 13.2 (4) that X itself is second-countable. ■

Proof of Proposition 27.8. Let X be a topological space that is Hausdorff and connected and that is furthermore paracompact. To simplify the notation we only deal with the case that X is locally homeomorphic to an open subset of $\mathbb{R}^n$. The proof of the general case is almost identical. The goal is to show, using the Second-Countable Criterion 27.9, that X is second-countable.

Claim 1. There exists a locally finite open cover $\mathcal{W}$ of X by subsets such that the closure of each $W \in \mathcal{W}$ is compact.

Proof. Note that by hypothesis for each $a \in X$ there exists an open neighborhood U_a and a homeomorphism $\Phi_a \colon U_a \to V_a$ where V_a is an open subset of $\mathbb{R}^n$. We pick an $r_a > 0$ such that $\overline{B}^n_{r_a}(\Phi_a(a)) \subseteq V_a$ and we set $W_a := \Phi_a^{-1}(B^n_{r_a}(a))$. These sets form an open cover of X. Note that it follows from our hypothesis that X is Hausdorff, from our choice of r_a, and from the Compactness-Chart Lemma 23.11 that the closure of W_a is compact. Since X is paracompact there exists a locally finite refinement $\mathcal{W}$ of the open cover $\{W_a\}_{a \in X}$. This refinement has all the desired properties. ⊞

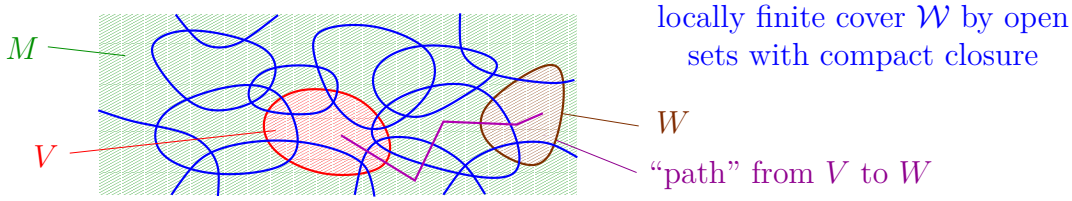
We pick a non-empty $V \in \mathcal{W}$.

Claim 2. Given any $W \in \mathcal{W}$ there exists a $k \in \mathbb{N}$ and sets $V = W_1, W_2, \ldots, W_k = W$ in $\mathcal{W}$ such that for each $i \in \{1, \ldots, k-1\}$ we have $W_i \cap W_{i+1} \neq \varnothing$.

Proof. We consider

$$U := \left\{ P \in M \;\middle|\; \begin{array}{c} \text{there exist sets } V = W_1, W_2, \ldots, W_k = W \text{ in } \mathcal{W} \text{ such} \\ \text{that each } W_i \cap W_{i+1} \neq \varnothing \text{ and such that } P \in W_k \end{array} \right\}.$$

Evidently $V \subseteq U$, thus U is non-empty. Since M is connected it remains to show that U is open and closed. First let $P \in U$. We pick a sequence $V = W_1, W_2, \ldots, W_k = W$ in $\mathcal{W}$ as in the definition of U.



Evidently $P \in W_k \subseteq U$. This shows that U is open. Conversely, let $P \notin U$. We pick $W \in \mathcal{W}$ with $P \in W$. If $U \cap W$ was non-empty, it would follow basically immediately from the definition of U that we have $W \subseteq U$ and thus also $P \in U$, which is impossible. This shows that U is closed. ⊞

Given $W \in \mathcal{W}$ we set

$$\varphi(W) := \text{minimal } k \in \mathbb{N} \text{ such that there exists a sequence as in Claim 2}$$

and given $k \in \mathbb{N}$ we set

$$Z_k := \overline{\bigcup_{W \in \mathcal{W} \text{ with } \varphi(W) \leq k} W}.$$

Claim 3.

(1) For each $k \in \mathbb{N}$ there exists only finitely many $W \in \mathcal{W}$ with $\varphi(W) \leq k$.
(2) For each $k \in \mathbb{N}$ the set Z_k is compact.

Proof. First we make the following observation. If for some $k \in \mathbb{N}$ there exist only finitely many $W \in \mathcal{W}$ with $\varphi(W) \leq k$, then we obtain from the Compact–Union-Intersection Lemma 3.41, the Interior-Closure-Cup-Cap Lemma 3.24 and the Compact-Closed Lemma 3.39 that Z_k is compact.

Thus it suffices to show Statement (1) of the claim. We prove this statement by induction. First let us deal with the case $k = 1$. Note that $\varphi(W) = 1$ if and only if $W = V$. Thus

we see that there exists a single $W \in \mathcal{W}$ with $\varphi(W) = 1$. Now assume that the statement holds for some $k \in \mathbb{N}$.

Since $\mathcal{W}$ is locally finite there exists for each $z \in Z_k$ an open neighborhood U_z which has non-empty intersection with at most finitely many sets in $\mathcal{W}$. By our induction hypothesis and the above observation we know that Z_k is compact. Hence we can cover Z_k by finitely many of such U_z. But this implies that Z_k has non-empty intersection only with finitely many $W \in \mathcal{W}$. By definition we know that each $W \in \mathcal{W}$ with $\varphi(W) = k + 1$ satisfies $W \cap Z_k \neq \varnothing$. Thus we have shown that there are only finitely many $W \in \mathcal{W}$ with $\varphi(W) \leq k + 1$. ⊞

It follows from Claim 2 and Claim 3 that X is the union of countably many compact subsets. It follows from the Second-Countable Criterion 27.9 that X is indeed second-countable. ■

27.4. **Smooth extensions and approximations.** In this section we will see that partitions of unity can be very helpful for constructing maps on manifolds. We start out with the following variation on Urysohn's Lemma 12.5 for topological and smooth manifolds.

Lemma 27.10. (Urysohn's Lemma for Manifolds) Let M be a topological manifold and let A and B be two disjoint closed subsets of M.

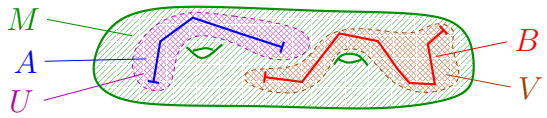
(1) There exists a continuous map $f\colon M \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.
(2) If M is in fact a smooth manifold, then we can arrange in (1) that f is smooth.

Proof. Let M be a topological manifold. By the Manifold Paracompact-Normal Proposition 23.9 (2) we know that M is normal. Thus Statement (1) is an immediate consequence of Urysohn's Lemma 12.5.

Note that this approach to proving Statement (1) does not generalize to Statement (2). In the following we will therefore provide a different proof of Statement (1), which does not appeal to Urysohn's Lemma 12.5, and which generalizes to smooth manifolds.

Thus let M be a topological manifold and let A and B be two disjoint closed subsets of M. As we mentioned above, we know by the Manifold Paracompact-Normal Proposition 23.9 (2) that M is normal. By definition this means that there exist open neighborhoods U of A and V of B such that $U \cap V = \varnothing$.

Note that $\{M \setminus B, V\}$ is an open cover of M. By the Manifold–Partitions-of-Unity Theorem 27.2 (1) there exists a corresponding partition of unity f, g with $\operatorname{supp}(f) \subseteq V$ and with $\operatorname{supp}(g) \subseteq M \setminus B$. It follows almost immediately from the various definitions that f has the desired properties.



Now assume that M is in fact a smooth manifold. We use the same argument as above. The only difference is that this time we use the Manifold–Partitions-of-Unity Theorem 27.2 (2). ■

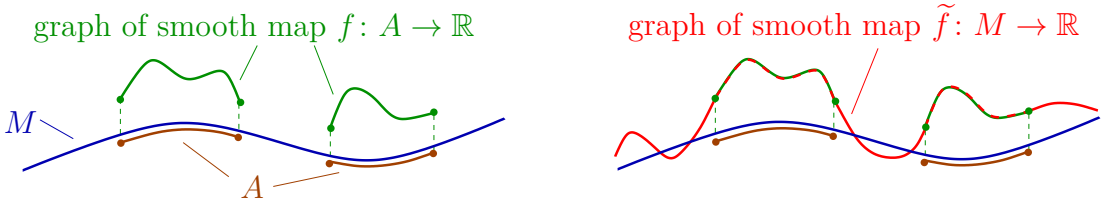
Proposition 27.11. (Continuous Extension Proposition) Let M be a topological manifold, let $A \subseteq M$ be a closed subset and let $f\colon A \to \mathbb{R}^n$ be a continuous map. Given

any open neighborhood U of A there exists a continuous map $\widetilde{f}\colon M \to \mathbb{R}^n$ such that $\widetilde{f}|_A = f$ and such that $\operatorname{supp}(\widetilde{f}) \subseteq U$.

Proof. The Continuous Extension Proposition 27.11 is an immediate consequence of the Manifold Paracompact-Normal Proposition 23.9 (2), Lemma 27.10 and the Tietze Extension Theorem 12.7. ■

In the remainder of this section we will use partitions of unity to construct useful smooth functions on smooth manifolds. First we have the smooth analog of the Continuous Extension Proposition 27.11.

Proposition 27.12. (Smooth Extension Proposition) Let M be a smooth manifold, let $A \subseteq M$ be a closed subset and let $f\colon A \to \mathbb{R}^n$ be a smooth map. Given any open neighborhood U of A there exists a smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$ such that $\widetilde{f}|_A = f$ and such that $\operatorname{supp}(\widetilde{f}) \subseteq U$.



Proof. It follows almost immediately from the definition of a smooth map $f\colon A \to \mathbb{R}^n$ that for each $P \in A$ there exists an open neighborhood $W_P \subseteq U$ and a smooth map $\widetilde{f}_P\colon W_P \to \mathbb{R}^n$ that agrees with f on $W_P \cap A$.

Since A is closed we see that the sets $\{W_P\}_{P\in A}$ and $M \setminus A$ form an *open* cover of M. By the Manifold–Partitions-of-Unity Theorem 27.2 there exists a smooth partition of unity $\{\varphi_P\}_{P\in M}$ and ψ such that for each $P \in M$ we have $\operatorname{supp}(\varphi_P) \subseteq W_P$ and such that $\operatorname{supp}(\psi) \subseteq M \setminus A$. In particular we have $\psi|_A \equiv 0$.



Given $P \in M$ we set

$$\begin{aligned} \Theta_P\colon M &\to \mathbb{R}^n \\ x &\mapsto \begin{cases} \varphi_P(x) \cdot \widetilde{f}_P(x), & \text{if } x \in W_P, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

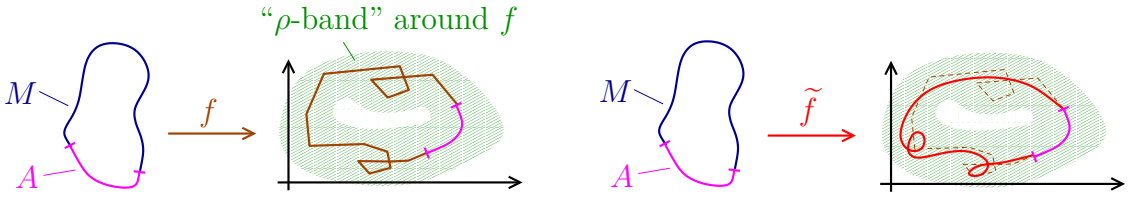
It follows almost immediately from the Smooth Pasting Proposition 25.19 that Θ_P is smooth. We consider the map

$$\begin{aligned} \widetilde{f}\colon M &\to \mathbb{R}^n \\ x &\mapsto \sum_{P\in A} \Theta_P(x). \end{aligned}$$

It follows easily from $\psi|_A \equiv 0$ and the fact that we are dealing with a partition of unity that $\widetilde{f}$ agrees with f on A. Furthermore it follows easily from the Interior-Closure-Cup-Cap Lemma 3.24 that $\operatorname{supp}(\widetilde{f}) \subseteq U$. ■

Proposition 27.13. (Smooth Approximation Proposition) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}^n$ be a continuous map that is smooth on a (possibly empty) closed subset A. Given any continuous map $\rho\colon M \to \mathbb{R}_{\geq 0}$ with $\rho(x) > 0$ for all $x \notin A$ there exists a *smooth* map $\widetilde{f}\colon M \to \mathbb{R}^n$ with the following two properties:

(1) The maps f and $\widetilde{f}$ agree on A.
(2) For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \rho(x)$.

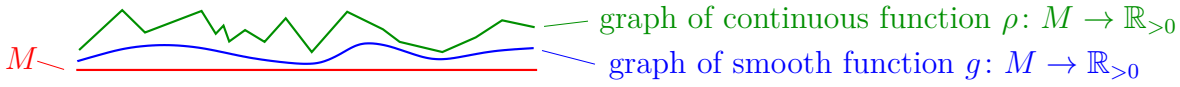


Remark. In the Whitney Approximation Theorem 36.1 we will generalize the above proposition from the target $\mathbb{R}^n$ to any smooth manifold N. □

Before we turn to the proof of the Smooth Approximation Proposition 27.13 let us formulate two useful corollaries.

Corollary 27.14. (Replace-by-Smooth Non-Negative Function Corollary) Let M be a smooth manifold.

(1) For every continuous map $\rho\colon M \to \mathbb{R}_{>0}$ there exists a smooth map $g\colon M \to \mathbb{R}_{>0}$ such that $g(x) \in (0, \rho(x)]$ for all $x \in M$.
(2) Let $f\colon M \to \mathbb{R}$ be a map (not necessarily continuous) with the property that for every $x \in M$ there exists an open neighborhood U_x of x and an $\epsilon_x > 0$ such that for each $y \in U_x$ we have $f(y) \geq \epsilon_x$. Then there exists a smooth map $g\colon M \to (0, \infty)$, such that for all $x \in M$ we have $g(x) \leq f(x)$.



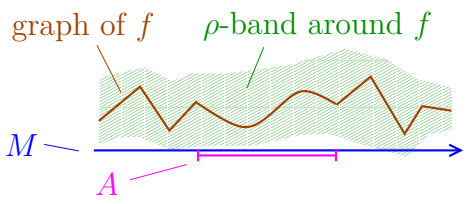
Proof of the Replace-by-Smooth Non-Negative Function Corollary 27.14.

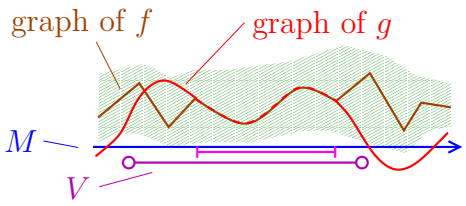
(1) This statement follows immediately from the Smooth Approximation Proposition 27.13 applied to the zero function $f \equiv 0$, to $A = \varnothing$ and to the given map $\rho\colon M \to \mathbb{R}_{>0}$.
(2) In the Manifold Paracompact-Normal Proposition 23.9 (1) we saw that topological manifolds are paracompact. Thus Statement (2) follows immediately from the Locally Non-Zero Lemma 14.18 together with Statement (1). ■

Proof of the Smooth Approximation Proposition 27.13. By the Smooth Extension Proposition 27.12 there exists a smooth map $g\colon M \to \mathbb{R}^n$ that agrees with f on A. We consider the open[325] subset

$$V \;:=\; \{y \in M \,|\, |g(y) - f(y)| < \rho(y)\}.$$

[325]Note that $|g(y) - f(y)| < \rho(y)$ if and only if $|g(y) - f(y)| - \rho(y) \in (-\infty, 0)$. Using this observation it is indeed clear that $V \subseteq M$ is open.





Claim. There exists a countable set $I \subseteq M \setminus A$ together with an open neighborhood $U_P \subseteq M \setminus A$ for each $P \in I$ such that $\bigcup_{P\in I} U_P = M \setminus A$ and such that for every $y \in U_P$ we have

$$|f(y) - f(P)| \;<\; \rho(y).$$

Proof. Let $x \in M \setminus A$. Since $\rho(x) > 0$ we obtain from the continuity of ρ and f that there exists an open neighborhood U_x of x in the open subset $M \setminus A$ such that for every $y \in U_x$ we have

$$|\rho(y) - \rho(x)| \;\leq\; \tfrac{1}{2} \cdot \rho(x) \qquad \text{and} \qquad |f(y) - f(x)| \;<\; \tfrac{1}{2} \cdot \rho(x).$$

It follows from the triangle inequality that given any $y \in U_x$ we have $|f(y) - f(x)| < \rho(y)$. It follows from the Second Countability Lemma 13.2 (2) and the Countable Cover Lemma 13.4 that there exists a countable $I \subseteq M \setminus A$ such that $M \setminus A$ is covered by the U_P. ⊞

By the Manifold–Partitions-of-Unity Theorem 27.2 there exists a smooth partition of unity $\{\varphi_P\}_{P\in I}, \psi$ such that for each $P \in I$ we have $\operatorname{supp}(\varphi_P) \subseteq U_P$ and with $\operatorname{supp}(\psi) \subseteq V$. We consider the map

$$\begin{array}{rccl} \widetilde{f}\colon & M & \to & \mathbb{R}^n \\ & x & \mapsto & \psi(x) \cdot g(x) + \sum\limits_{P\in I} \varphi_P(x) \cdot f(P). \end{array}$$

Note that $\widetilde{f}$ is smooth and that $\widetilde{f}$ agrees with f on A. It remains to prove the following claim:

Claim. For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \rho(x)$.

Proof. Let $x \in M$. We perform the following calculation:

$$\begin{aligned} \|f(x) - \widetilde{f}(x)\| &\overset{\text{since } \psi + \sum \varphi_P \equiv 1}{=} \left\| \Big(\psi(x) + \sum_{P\in I} \varphi_P(x)\Big) \cdot f(x) - \Big(\psi(x) \cdot g(x) + \sum_{P\in I} \varphi_P(x) \cdot f(P)\Big) \right\| \\ &\leq \psi(x) \cdot \underbrace{|f(x) - g(x)|}_{\leq \rho(x) \text{ if } \psi(x) \neq 0} + \sum_{P\in I} \varphi_P(x) \cdot \underbrace{|f(x) - f(P)|}_{\leq \rho(x) \text{ if } \varphi_P(x) \neq 0} \\ &\leq \psi(x) \cdot \rho(x) + \sum_{P\in I} \varphi_P(x) \cdot \rho(x) \overset{\text{since } \psi + \sum \varphi_P \equiv 1}{=} \rho(x). \end{aligned}$$

■

Let M be a smooth manifold. For any continuous function $f\colon M \to \mathbb{R}_{\geq 0}$ the preimage $f^{-1}(0)$ is of course a closed subset. Rather surprisingly the following proposition shows that every closed subset of M can be realized as the zero set of a smooth function.

Proposition 27.15. (Zero Set-Realization Proposition) Let M be a smooth manifold. Given any closed subset $C \subseteq M$ there exists a smooth function $f\colon M \to \mathbb{R}_{\geq 0}$ such that $C = f^{-1}(0)$.

Proof. Let M be a smooth manifold and let $C \subseteq M$ be a closed subset. Recall that by the Manifold Metrization Proposition 23.10 we know that M is metrizable, i.e. there exists a metric d such that the metric topology agrees with the given topology.

If C is the empty set, then we take f to be the constant function $= 1$. Now assume that C is non-empty. This allows us to consider the map

$$\begin{array}{rccl} g\colon & M & \to & \mathbb{R}_{\geq 0} \\ & x & \mapsto & d(x,C) \coloneqq \inf\{d(x,c) \,|\, c \in C\}. \end{array}$$

By the Distance-to-Subspace Lemma 10.4 we know that the map is continuous and, using that C is a closed subset, we know that we have

(∗) $g(x) = 0$ if and only if $x \in C$.

Note that g is constant $= 0$ on C. In particular g is a smooth function on C. By the Smooth Approximation Proposition 27.13 we know that there exists a smooth map $f\colon M \to \mathbb{R}$ such that for every $x \in M$ we have $|f(x) - g(x)| \leq \frac{g(x)}{2}$. In other words, for every $x \in M$ we have $f(x) \in [\frac{g(x)}{2}, \frac{3g(x)}{2}] \in \mathbb{R}_{\geq 0}$. It follows that

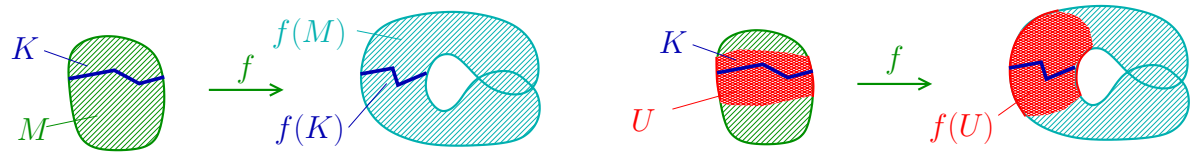
$$f(x) = 0 \quad \underset{\substack{\uparrow \\ \text{since } f(x) \in [\frac{g(x)}{2}, \frac{3g(x)}{2}]}}{\iff} \quad g(x) = 0 \quad \underset{\substack{\uparrow \\ \text{by } (*)}}{\iff} \quad x \in C.$$

Thus f has all the desired properties. ∎

We conclude this chapter with the following useful theorem.

Theorem 27.16. (Neighborhood Smooth Embedding Theorem) Let $f\colon M \to N$ be a continuous map between smooth manifolds. Furthermore let $K \subseteq M$ be a closed subset such that the map $f\colon K \to f(K)$ is a homeomorphism and such that $f(K)$ is a closed subset of N.

(1) If f is a local smooth embedding[326] at each $P \in K$, then there exists an open neighborhood U of K such that the restriction of f to U is a smooth embedding.
(2) If f is in fact a local diffeomorphism, then we can furthermore arrange that $f(U)$ is an open subset of N.



Proof. By the Manifold–Local Properties Lemma 23.8 we know that M is regionally compact. Furthermore, by the Manifold Paracompact-Normal Proposition 23.9 (1) we know that N is paracompact.

(2) The proof of Statement (2) of the Neighborhood Smooth Embedding Theorem 27.16 is now almost identical to the proof of the Neighborhood Embedding Theorem 14.19. All we have to do is to replace the Pasting Proposition 4.7 by the Smooth Pasting Proposition 25.19.
(1) The proof of Statement (1) is very similar. We leave the details to the reader. ∎

[326] We introduce the concept of a smooth embedding later in Definition 30.9. For the purpose of the proof this should not be an issue, since it is really a topological statement.

27.5. Exercises for Chapter 27.

Exercise 27.1. Provide an example of a connected space X that does not admit an exhaustion function $f\colon X \to \mathbb{R}_{\geq 0}$.

Exercise 27.2. Give an example of two topological manifolds M and N, a closed subset $A \subseteq M$ and a continuous map $f\colon A \to N$ such that f cannot be extended to a continuous map $\widetilde{f}\colon M \to N$.
Remark. This exercise shows that the Continuous Extension Proposition 27.11 cannot be extended from $N = \mathbb{R}^n$ to the completely general setting.

Exercise 27.3. Let $A = (a, b) \subseteq \mathbb{R}$ be an open interval where $-\infty < a < b < \infty$.

(a) Show that there exists a smooth map $f\colon A \to \mathbb{R}$ which cannot be extended to a smooth map on all of $\mathbb{R}$.
(b) Show that there exists a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ which is smooth on A but for which there does not exist a smooth map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ with the following two properties:
- We have $\widetilde{f}|_A = f|_A$.
- For all $x \in \mathbb{R}$ we have $|f(x) - \widetilde{f}(x)| \leq 1$.

Remark. This exercise shows that in the Smooth Extension Proposition 27.12 and the Smooth Approximation Proposition 27.13 the hypothesis that A is closed is absolutely essential.

Exercise 27.4. Let M be a smooth manifold and let $f\colon M \to S^n$ be a continuous map that is smooth on a (possibly empty) closed subset A and let $\rho\colon M \to \mathbb{R}_{\geq 0}$ be a continuous map such that $\rho(a) = 0$ for every $a \in A$. Show that there exists a *smooth* map $\widetilde{f}\colon M \to S^n$ such that for every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \rho(x)$.
Hint. Think of S^n as a subset of $\mathbb{R}^{n+1} \setminus \{0\}$.

CHAPTER 28

1-dimensional manifolds and the long line

This chapter can be seen as a short break from developing the general theory of smooth manifolds. In this chapter we will only consider "1-dimensional objects". More precisely, we will do the following:

(1) In the Classification Theorems 28.1 and 28.8 we will give the classification of connected 1-dimensional topological respectively smooth manifolds.
(2) We will construct the *long line*. As we will see in the Long Line Theorem 28.20, this is a path-connected topological space that satisfies all properties of a 1-dimensional topological manifold, except that it is not second-countable.

We will at times use these classification Theorems 28.1 and 28.8. But the proofs are somewhat technical and do not make for particularly leisurely reading. Thus the reader might be best served by having a quick glance at the, not very surprising, classification theorems and by moving on to the next chapter.

This chapter is organized as follows:

- In Section 28.1 we will classify 1-dimensional topological manifolds up to homeomorphism.
- In Section 28.2 we will classify 1-dimensional smooth manifolds up to diffeomorphism.
- In Section 28.3 we will take a short detour and we will introduce the order topology.
- In Section 28.4 we will introduce the "long line", which is a connected topological space that satisfies all conditions of a topological manifold except that it is not second-countable.
- In Section 28.5 we will provide the proof of the Long Line Theorem 28.20.
- In Section 28.6 we will provide exercises for Chapter 28.

In the next Chapter 29 we will introduce tangent spaces of smooth manifolds.

28.1. **The classification of 1-dimensional topological manifolds.** One of the goals of topology is to completely classify suitable classes of topological spaces up to homeomorphism. For example one could try to classify all connected topological and smooth manifolds of a given dimension. In this section we will give a complete classification in the 1-dimensional setting.

In the following we first deal with 1-dimensional *topological* manifolds.

Example. Some of the main examples of 1-dimensional topological manifolds are S^1, $\mathbb{R}$, the closed interval $[0,1]$ and the half-open interval $[0,1)$.

(1) Note that for S^1 and $\mathbb{R}$ we found atlases consisting only of charts of type (i), i.e. the boundary of S^1 and of $\mathbb{R}$ is empty. On the other hand, as we pointed out on page 624, it follows easily from the Topological Valence–Basics Lemma 11.34 together with the

Topological Valence Determination Proposition 11.35 that the boundary of $[0,1]$ equals $\{0,1\}$ and that the boundary of $[0,1)$ equals $\{0\}$.

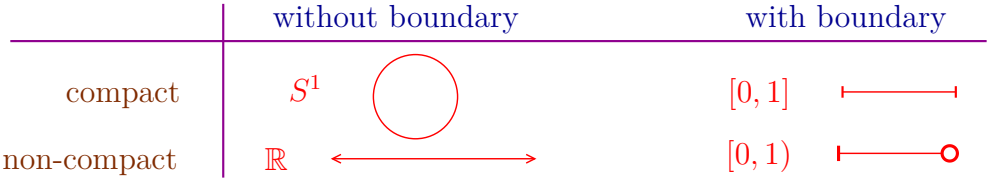
(2) By the Heine-Borel Theorem 3.44 we know that S^1 and $[0,1]$ are compact, whereas $\mathbb{R}$ and $[0,1)$ are non-compact.

This discussion shows in particular that S^1, $\mathbb{R}$, $[0,1]$ and $[0,1)$ are pairwise non-homeomorphic. □

The following theorem shows that the above list of 1-dimensional topological manifolds is in a suitable sense complete.

Theorem 28.1. (Topological 1-Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional topological manifold. The homeomorphism type of M is given by the following table:

	with empty boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0,1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0,1)$.



Remark. Note that the adjectives "Hausdorff" and "second-countable" in the definition of a topological manifold are both necessary:

(1) The line with two zeros, which we introduced in Definition 3.30, has all the properties of a 1-dimensional topological manifold except for the Hausdorff axiom.

(2) If one drops the condition that a topological manifold has to be second-countable, then there are two more examples of connected 1-dimensional topological manifolds, namely the "open long ray $\mathbb{L}^+$ and the "long line $\mathbb{L}$" which we will introduce later in Definition 28.19. □

Remark.

(1) Many proofs of the Topological 1-Manifold Classification Theorem 28.1 exist in the literature, see e.g. [**Gal1987**], [**Lee2000**, Theorem 6.1] and [**FuR1984**, Sections 3.1.1.16-19].

(2) We will classify 2-dimensional topological manifolds in Theorem 101.16. On the other hand we will see in the Manifold–Non-Classification Theorem 103.4 that closed smooth manifolds of dimension ≥ 4 cannot be classified. □

The proof of the Topological 1-Manifold Classification Theorem 28.1 requires several preparations. We start out with the following definition:

Definition 28.2. Let M be a topological space.

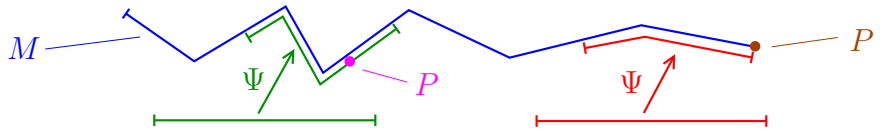
(1) A c-parametrization for M is a homeomorphism $\Psi\colon [r,s] \to C$ with the following properties:

(a) We have $r < s$.

(b) The image $C = \Psi([r,s])$ of the closed interval $[r,s]$ is a closed subset C of M.
(c) The image $\Psi((r,s))$ of the open interval (r,s) is an open subset of M.

(2) A c-atlas is a family of c-parametrizations $\{\Psi_i \colon [r_i,s_i] \to C_i\}_{i\in I}$ of M such that the union of the C_i's equals M.

Lemma 28.3. (C-Parametrization-for-Manifold Lemma) Let M be a 1-dimensional topological manifold. Given any $P \in M$ and given any neighborhood W of P there exists a c-parametrization $\Psi \colon [r,s] \to C$ such that C is a neighborhood of P and such that $C \subseteq W$.

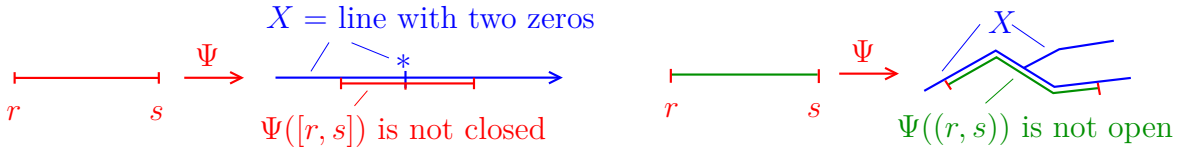


Proof. We distinguish the following cases:

- If $P \in M \setminus \partial M$, then we pick a chart $\Phi \colon U \to (-2,2)$ of type (i) with $\Phi(P) = 0$ and with $U \subseteq W$. We write $V = \Phi^{-1}((-1,1))$, $I = [-1,1]$ and $C = \Phi^{-1}([-1,1])$.
- If $P \in \partial M$, then we pick a chart $\Phi \colon U \to [0,2)$ of type (ii) with $\Phi(P) = 0$ and with $U \subseteq W$. We write $V = \Phi^{-1}([0,1))$, $I = [0,1]$ and $C = \Phi^{-1}([0,1])$.

It follows from the Compactness-Chart Lemma 23.11 that C is closed in M. (Note that here we use that topological manifolds are Hausdorff.) Also note that it follows from the Open-Closed Inclusion Lemma 4.13 that V is open in M and that V is a neighborhood of P. Thus the map $\Phi^{-1} \colon I \to C$ has all the desired properties. ■

Example. In the figure below we also show two "non-examples" of c-parametrizations for 1-dimensional non-Hausdorff-manifolds.



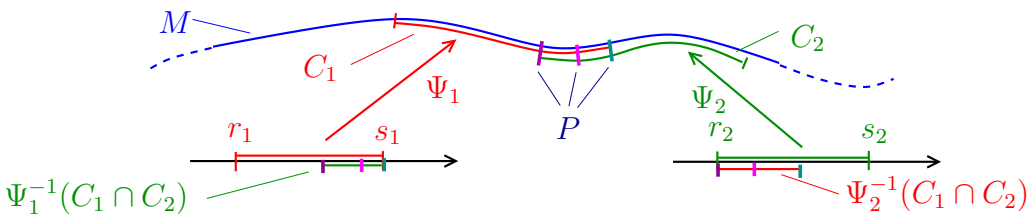
In the situation to the left the image of the map $\Psi \colon [r,s] \to X$ is not closed whereas in the situation to the right $\Psi((r,s))$ is not an open subset of X. □

In the following lemmas we study the "interaction" between two c-parametrizations whose images intersect.

Lemma 28.4. (C-Parametrization–Intersection Lemma) Let M be a topological space and let $\Psi_1 \colon [r_1,s_1] \to C_1$ and $\Psi_2 \colon [r_2,s_2] \to C_2$ be two c-parametrizations. Let $P \in C_1 \cap C_2$ and let $i \in \{1,2\}$. If $\Psi_i^{-1}(P)$ is not an endpoint of $[r_i,s_i]$, then one of the following happens:

(i) $\Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i},s_{3-i}]$, or
(ii) P lies in the interior of $C_1 \cap C_2$ in M.

Proof. Let $P \in C_i \cap C_{3-i}$ with $\Psi_i^{-1}(P) \in (r_i,s_i)$. If $\Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i},s_{3-i}]$, then we are done. Now suppose that $\Psi_{3-i}^{-1}(P)$ is not an endpoint of $[r_{3-i},s_{3-i}]$, i.e. we suppose that $\Psi_{3-i}^{-1}(P) \in (r_{3-i},s_{3-i})$. This means that $P \in \Psi_i((r_i,s_i)) \cap \Psi_{3-i}((r_{3-i},s_{3-i}))$.



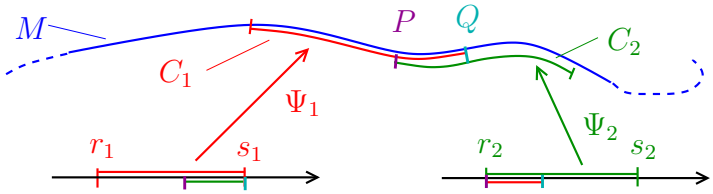
Since Ψ_1 and Ψ_2 are c-parametrizations we know that $\Psi_i((r_i, s_i)) \cap \Psi_{3-i}((r_{3-i}, s_{3-i}))$ is an open subset of M. It follows that P lies in the interior of $C_1 \cap C_2$ in M. ■

Lemma 28.5. (C-Parametrization–Intersection-Component Lemma) Let M be a topological space and let $\Psi_1 \colon [r_1, s_1] \to C_1$ and $\Psi_2 \colon [r_2, s_2] \to C_2$ be two c-parametrizations.

(1) For every component X of $C_1 \cap C_2$ there exist points $P, Q \in X$ (where we allow $P = Q$) such that for $i = 1, 2$ the following two statements hold:
   (a) The preimage $\Psi_i^{-1}(X)$ is the closed interval with endpoints $\Psi_i^{-1}(P)$ and $\Psi_i^{-1}(Q)$.
   (b) If $\Psi_i^{-1}(P)$ is not an endpoint of $[r_i, s_i]$, then $\Psi_{3-i}^{-1}(P)$ is an endpoint of $[r_{3-i}, s_{3-i}]$. The same statement also holds for Q.

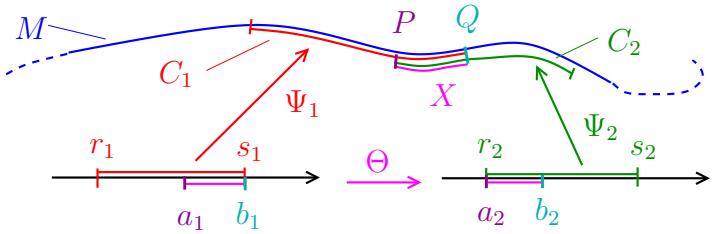
(2) If X is a component of $C_1 \cap C_2$ such that $\Psi_i^{-1}(X)$ is contained in (r_i, s_i), then C_{3-i} is contained in C_i.

(3) $C_1 \cap C_2$ has at most two components.



Proof. Let X be a component of $C_1 \cap C_2$. By the Component Lemma 4.48 we know that X is connected. Furthermore, since C_1 and C_2 are closed in M we know that $C_1 \cap C_2$ is closed in M and by the Components-are-Closed Lemma 4.49 this implies that the component X is also a closed subset of M.

(1) We consider $W_1 = \Psi_1^{-1}(X)$. Since Ψ_1 is a homeomorphism it follows from the above discussion that W_1 is a closed connected subset of the interval $[r_1, s_1]$. Since W_1 is a connected subset of the interval $[r_1, s_1]$ we obtain from Exercise 4.21 that W_1 is a closed interval contained in $[r_1, s_1]$. In other words $W_1 = [a_1, b_1]$ for some $r_1 \leq a_1 \leq b_1 \leq s_1$.



Claim. The points $P \coloneqq \Psi_1(a_1)$ and $Q \coloneqq \Psi_1(b_1)$ have the desired properties stated in (1).

Proof. We write $a_2 = \Psi_2^{-1}(P)$ and $b_2 = \Psi_2^{-1}(Q)$.

(a) We already know that the preimage $\Psi_1^{-1}(X)$ is the closed interval with endpoints $\Psi_1^{-1}(P)$ and $\Psi_1^{-1}(Q)$. It remains to prove the statement for $i = 2$. We consider the homeomorphism $\Theta := \Psi_2^{-1} \circ \Psi_1 \colon W_1 := \Psi_1^{-1}(X) \to \Psi_2^{-1}(X)$. Note that $\Theta(a_1) = a_2$ and $\Theta(b_1) = b_2$. It follows from an elementary argument, see Exercise 4.9, that Θ sends the interval $[a_1, b_1]$ homeomorphically to the interval with endpoints a_2 and b_2. It follows that the preimage $\Psi_2^{-1}(X)$ is the closed interval with endpoints $\Psi_2^{-1}(P)$ and $\Psi_2^{-1}(Q)$.

(b) We provide the proof for P, the proof for Q is evidently essentially the same. We argue by contradiction. Thus suppose that $a_1 = \Psi_1^{-1}(P)$ is neither an endpoint of $[r_1, s_1]$ nor that $a_2 = \Psi_2^{-1}(P)$ is an endpoint of $[r_2, s_2]$. If this should happen, then it would follow from the C-Parametrization–Intersection Lemma 28.4 that P is contained in the interior of $C_1 \cap C_2$. It follows that there exists an $\epsilon > 0$ such that $(a_1 - \epsilon, a_1 + \epsilon) \subseteq (r_1, s_1)$ and such that $(a_1 - \epsilon, a_1 + \epsilon) \subseteq \Psi_1^{-1}(C_1 \cap C_2)$. Of course this implies that $\Psi_1((a_1 - \epsilon, a_1 + \epsilon)) \subseteq C_1 \cap C_2$. We make three observations:
- We have $\Psi_1((a_1 - \epsilon, a_1 + \epsilon) \cup [a_1, b_1]) \subseteq C_1 \cap C_2$.
- We have $X = \Psi([a_1, b_1]) \subsetneq \Psi_1((a_1 - \epsilon, a_1 + \epsilon) \cup [a_1, b_1])$.
- The subset $\Psi_1((a_1 - \epsilon, a_1 + \epsilon) \cup [a_1, b_1])$ is connected.

By the Component Lemma 4.48 these three observations together provide a contradiction to X being a component of $C_1 \cap C_2$. ⊞

(2) We continue with the above notation. If neither a_i nor b_i is an endpoint of $[r_i, s_i]$, then it follows from the above that a_{3-i} and b_{3-i} are both endpoints of $[r_{3-i}, s_{3-i}]$. But this means that the interval given by a_{3-i} and b_{3-i} equals $[r_{3-i}, s_{3-i}]$, which implies that $C_{3-i} \subseteq C_i$.

(3) By (1) we know that each component of $C_1 \cap C_2$ contains at least two endpoints of the intervals $[r_1, s_1]$ and $[r_2, s_2]$. Since there are only four endpoints we see that there is only enough room for at most two components. ■

Lemma 28.6. (C-Parametrization–Combination Lemma) Let M be a 1-dimensional topological manifold.[327] Let $\Psi_1 \colon [r_1, s_1] \to C_1$ and $\Psi_2 \colon [r_2, s_2] \to C_2$ be two c-parametrizations. If $C_1 \cap C_2$ has one component, then there exist $\widetilde{r} \leq r_1 < s_1 \leq \widetilde{s}$ and a c-parametrization

$$\widetilde{\Psi} \colon [\widetilde{r}, \widetilde{s}] \ \to \ C_1 \cup C_2$$

such that $\widetilde{\Psi}|_{[r_1, s_1]} = \Psi_1$.

Proof of Lemma 28.6 if C_1 is not contained in C_2. First we deal with the case that C_1 is not contained in C_2. Since $C_1 \cap C_2$ consists of one component and since C_1 is not contained in C_2 we obtain from the C-Parametrization–Intersection-Component Lemma 28.5 that $\Psi_1^{-1}(C_1 \cap C_2)$ contains precisely one endpoint of $[r_1, s_1]$. We will now deal with the case that the endpoint is s_1. The other case is treated in essentially the same way.

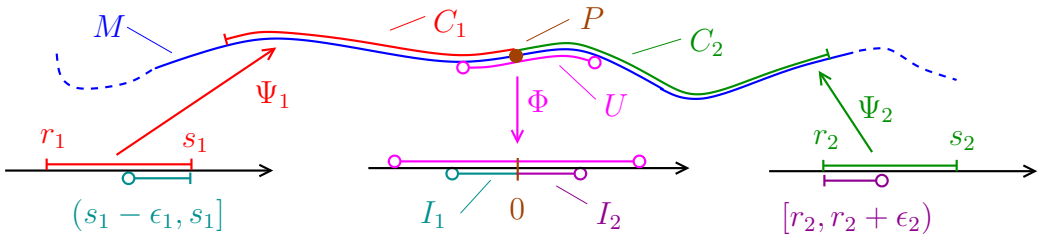
Next note that the C-Parametrization–Intersection-Component Lemma 28.5 now says that there exists a $b \in [r_1, s_1]$ with the following properties:

(1) $\Psi_1^{-1}(C_1 \cap C_2) = [b, s_1]$,

[327] Note that in contrast to the previous C-Parametrization–Intersection-Component Lemma 28.5 we now actually assume that we are dealing with a 1-dimensional topological manifold.

(2) $\Psi_2^{-1}(\Psi_1(b))$ is an endpoint of $[r_2, s_2]$.

Note that after possibly replacing $t \mapsto \Psi_2(t)$ by $t \mapsto \Psi_2(r_2 + s_2 - t)$ we can assume that $\Psi_2^{-1}(\Psi_1(b)) = r_2$. (Note that now we are conveniently enough in a setting which corresponds to the situation shown in the figure below.)



Next note that the C-Parametrization–Intersection-Component Lemma 28.5 now also says that $\Psi_2^{-1}(\Psi_1([b, s_1])) = [r_2, c]$ for some $c \in [r_2, s_2]$.

We consider the map

$$\begin{array}{rcl} \widetilde{\Psi}\colon [r_1, s_1 + (s_2 - c)] & \to & C_1 \cup C_2 \\ x & \mapsto & \begin{cases} \Psi_1(x), & \text{if } x \in [r_1, s_1], \\ \Psi_2(x - s_1 + c), & \text{if } x \in [s_1, s_1 + (s_2 - c)]. \end{cases} \end{array}$$

By construction we have $\widetilde{\Psi}|_{[r_1,s_1]} = \Psi_1$. Thus it suffices to prove the following claim:

Claim. The map $\widetilde{\Psi}$ is a c-parametrization.

Proof. We set $\widetilde{r} = r_1$ and $\widetilde{s} = s_1 + (s_2 - c)$. It follows from the Pasting Proposition 4.7 that $\widetilde{\Psi}\colon [\widetilde{r}, \widetilde{s}] \to C_1 \cup C_2$ is continuous and one can easily verify that $\widetilde{\Psi}\colon [\widetilde{r}, \widetilde{s}] \to C_1 \cup C_2$ is a bijection. Since M is Hausdorff we obtain from the Compact-Hausdorff Proposition 4.23 (3) that $\widetilde{\Psi}\colon [\widetilde{r}, \widetilde{s}] \to C_1 \cup C_2$ is in fact a homeomorphism and that $C_1 \cup C_2$ is a closed subset of M.

It remains to prove that $\widetilde{\Psi}((\widetilde{r}, \widetilde{s}))$ is an open subset of M. We set $P \coloneqq \Psi_1(s_1)$. Note that

$$\widetilde{\Psi}((\widetilde{r}, \widetilde{s})) \;=\; \underbrace{\Psi_1((r_1, s_1))}_{\substack{\text{open since } \Psi_1 \\ \text{c-parametrization}}} \;\cup\; \underbrace{\Psi_2((r_2, s_2))}_{\substack{\text{open since } \Psi_2 \\ \text{c-parametrization}}} \;\cup\; \{P\} \;\subseteq\; M.$$

If $P \in \Psi_2((r_2, s_2))$, then we are done. Therefore it remains to consider the slightly delicate case that $P \notin \Psi_2((r_2, s_2))$. Note that this means that $P = \Psi_2(r_2)$. In this case we still need to show that

$$\widetilde{\Psi}((\widetilde{r}, \widetilde{s})) \;=\; \Psi_1((r_1, s_1]) \cup \Psi_2([r_2, s_2))$$

is a neighborhood of P.

(i) First we assume that P admits a chart of type (i). This means that we can pick a chart $\Phi\colon U \to (-1, 1)$ of type (i) around P with $\Phi(P) = 0$.
- Since $\Psi_1^{-1}(U)$ is an open subset of $[r_1, s_1]$ that contains s_1 we see that there exists an $\epsilon_1 > 0$ such that $\Psi_1((s_1 - \epsilon_1, s_1]) \subseteq U$.
- Similarly we see that there exists an $\epsilon_2 > 0$ such that $\Psi_2([r_2, r_2 + \epsilon_2)) \subseteq U$.

Note that $I_1 = \Phi(\Psi_1((s_1 - \epsilon_1, s_1]))$ is a half-open interval in $(-1, 1)$ that contains 0 and that $I_2 = \Phi(\Psi_2([r_2, r_2 + \epsilon_2)))$ is a half-open interval in $(-1, 1)$ that contains 0. Furthermore note that $I_1 \cap I_2 = \{0\}$. It follows that $I_1 \cup I_2$ is an open neighborhood of

0. But that implies that $\Psi_1((s_1-\epsilon_1, s_1]) \cup \Psi_2([r_2, r_2+\epsilon_2))$ is a neighborhood of P. Note that this implies that $\widetilde{\Psi}((\widetilde{r}, \widetilde{s})) = \Psi_1((r_1, s_1]) \cup \Psi_2([r_2, s_2))$ is a neighborhood of P.

(ii) A slight modification of the above argument also shows that P actually cannot admit a chart of type (ii). ■

Proof of Lemma 28.6. We now deal with the case that C_1 is contained in C_2. We split C_2 into two c-parametrizations C_2' and C_2'' such that C_1 is contained in neither C_2' nor C_2'' and apply the previous case twice. ■

Lemma 28.7. (Homeomorphism-to-S^1-Lemma) Let M be a 1-dimensional topological manifold. Let $\Psi_1\colon [r_1, s_1] \to C_1$ and $\Psi_2\colon [r_2, s_2] \to C_2$ be two c-parametrizations. If $C_1 \cap C_2$ has two components, then there exists a homeomorphism

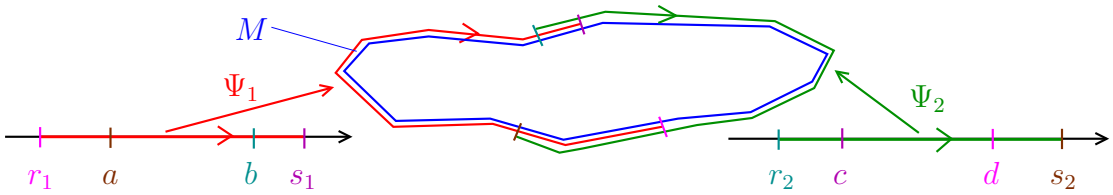
$$\Theta\colon S^1 \to C_1 \cup C_2$$

such that $\Theta(S^1) = C_1 \cup C_2$ is an open subset of M.

Proof. The proof is a variation on the proof of the C-Parametrization–Combination Lemma 28.6. Since $C_1 \cap C_2$ consists of two components we obtain from the C-Parametrization–Intersection-Component Lemma 28.5 that there exist $a, b \in \mathbb{R}$ with $r_1 \le a < b \le s_1$ such that the following hold:

(1) $\Psi_1^{-1}(C_1 \cap C_2) = [r_1, a] \cup [b, s_1]$,
(2) $\Psi_2^{-1}(\Psi_1(a))$ and $\Psi_2^{-1}(\Psi_1(b))$ are endpoints of $[r_2, s_2]$.

Note that after possibly replacing $t \mapsto \Psi_2(t)$ by $t \mapsto \Psi_2(r_2 + s_2 - t)$ we can assume that $\Psi_2(s_2) = \Psi_1(a)$ and $\Psi_2(r_2) = \Psi_1(b)$. Let $c \in [r_2, s_2]$ with $\Psi_2(c) = \Psi_1(s_1)$ and let $d \in [r_2, s_2]$ with $\Psi_2(d) = \Psi_1(r_1)$. Since C_2 is not contained in C_1 we see that $c < d$.



We consider the map

$$\begin{array}{rcl} \widetilde{\Psi}\colon [r_1, s_1 + (d-c)]/\sim & \to & C_1 \cup C_2 \\ [x] & \mapsto & \begin{cases} \Psi_1(x), & \text{if } x \in [r_1, s_1], \\ \Psi_2(x - s_1 + c), & \text{if } x \in [s_1, s_1 + (d-c)]. \end{cases} \end{array}$$

(we identify the two endpoints of the interval)

Note that $\widetilde{\Psi}$ is indeed well-defined.

As in the proof of the C-Parametrization–Combination Lemma 28.6 one can show fairly easily that $\widetilde{\Psi}$ is a homeomorphism, that the image is closed and that $C_1 \cup C_2$ is an open subset of M. Finally the desired homeomorphism $\Theta\colon S^1 \to C_1 \cup C_2$ is given by the observation, which we proved on page 314, that the quotient space $[r_1, s_1 + (d-c)]/\sim$ is homeomorphic to S^1. ■

Now we can provide the proof of the Topological 1-Manifold Classification Theorem 28.1. We break the proof into two parts. First we deal with the compact case, afterwards we will deal with the non-compact case.

Proof of Theorem 28.1 in the compact case. Let M be a compact connected 1-dimensional topological manifold. We start out with the following claim:

Claim 1. There exists a finite c-atlas for M.

Proof. Recall that by the C-Parametrization-for-Manifold Lemma 28.3 we know that for each $P \in M$ there exists a c-parametrization $\Psi_P \colon I_P \to C_P$ such that C_P is a neighborhood of P, i.e. such that there exists an open subset V_P of M with $P \in V_P \subseteq C_P$. In particular $\{V_P\}_{P\in M}$ is an open cover of M. Since M is compact, we see that there exist $P_1, \dots, P_k \in M$ such that the corresponding open subsets cover M. The corresponding c-parametrizations form a finite c-atlas for M. ⊞

Claim 2. Either M is homeomorphic to S^1 or M is homeomorphic to $[0, 1]$.

Proof. We prove the claim by induction on the number of elements of a finite c-atlas. If we have a c-atlas consisting of a single c-parametrization, then evidently M is homeomorphic to $[0, 1]$. Now assume that we have $k \geq 2$ and that the statement holds whenever we have a c-atlas with $< k$ c-parametrizations. Let $\{\Psi_i \colon I_i \to C_i\}_{i=1,\dots,k}$ be a c-atlas for M. Note that C_1 and $C_2 \cup \dots \cup C_k$ are both closed non-empty subsets of M. Since M is connected we see that there exists a $j \in \{2, \dots, k\}$ with $C_1 \cap C_j \neq \varnothing$. If $C_1 \subseteq C_j$ or $C_j \subseteq C_1$, then we can throw out of one of the c-parametrizations and we are done by induction.

Thus suppose that this is not the case. By the C-Parametrization–Intersection-Component Lemma 28.5 (3) we know that $C_1 \cap C_j$ consists of either one component or of two components. We deal with these two cases separately:

(1) If $C_1 \cap C_j$ consists of one component, then we know by the C-Parametrization–Combination Lemma 28.6 that there exists a c-parametrization $\widetilde{\Psi}$ with image $C_1 \cup C_j$. Thus $\{\Psi_i\}_{i\neq 2,j}$ together with $\widetilde{\Psi}$ is a c-parametrization of M with $k - 1$ charts. Thus we are done by induction.
(2) Finally suppose that $C_1 \cap C_j$ consists of two components. By the Homeomorphism-to-S^1-Lemma 28.7 we know that there exists a homeomorphism $\Theta \colon S^1 \to C_1 \cup C_j$ such that the image is a subset of M that is closed and open. Since M is connected we see that $C_1 \cup C_j = M$. ■

Now we turn to the classification of non-compact connected 1-dimensional topological manifolds.

Proof of Theorem 28.1 in the non-compact case. Let M be a non-compact connected 1-dimensional topological manifold. We start out with the following claim:[328]

Claim 1. There exists a countable c-atlas $\{\Psi_i \colon [r_i, s_i] \to C_i\}_{i\in\mathbb{N}}$ and a family $\{V_i\}_{i\in\mathbb{N}}$ of open subsets of M with the following properties:

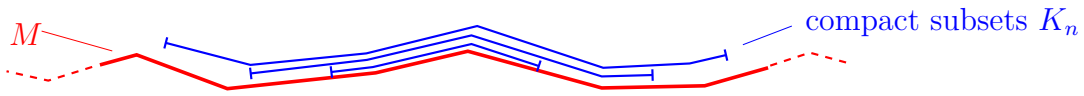
(1) The sets $\{C_i\}_{i\in\mathbb{N}}$ are a locally finite cover of M.[329]
(2) For each $i \in \mathbb{N}$ we have $V_i \subseteq C_i$ and the sets $\{V_i\}_{i\in\mathbb{N}}$ form an open cover of M.

Proof. By the Topological Manifold Exhaustion Lemma 23.12 there exists a family of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ of M such that the following two conditions are satisfied:

(a) for every $n \in \mathbb{N}$ we have $K_n \subseteq \overset{\circ}{K}_{n+1}$,

[328] Note that the claim is similar in spirit to some statements of the Atlas Improvement Proposition 27.5.
[329] Recall, that according to Definition 3.19 this means that each $x \in M$ admits an open neighborhood W such that $\{i \in \mathbb{N} \mid C_i \cap W \neq \varnothing\}$ is finite.

(b) we have $M = \bigcup_{n\in\mathbb{N}} K_n$.



Note that it follows from the Compact-Closed Lemma 3.39 (2) that each $\mathring{K}_{n+1} \setminus K_{n-1}$ is an open subset of M. Furthermore, by design these open subsets cover all of M.

It follows from the previous observation together with the C-Parametrization-for-Manifold Lemma 28.3 that for each $P \in M$ there exists a c-parametrization $\Psi_P \colon I_P \to C_P$ with the following properties:

(i) C_P is a neighborhood of P, i.e. this means that there exists an open subset V_P of M with $P \in V_P \subseteq C_P$,
(ii) there exists an $n \in \mathbb{N}$ with $V_P \subseteq \mathring{K}_{n+1} \setminus K_{n-1}$.

Let $n \in \mathbb{N}$. It follows from the compactness of K_n, the fact that the V_P are an open cover of M and the Compact Subspace Covering Lemma 3.38 that there exists a finite set $X_n \subseteq M$ such that $K_n \subseteq \bigcup_{P\in X_n} V_P$. We set $X \coloneqq \bigcup_{n\in\mathbb{N}} X_n$. One can now easily verify that the c-atlas $\{\Phi_P^{-1} \colon I_P \to C_P\}_{P\in X}$ together with the sets $\{V_P\}_{P\in X}$ has all the desired properties. ⊞

Claim 2. Given any $k, l \in \mathbb{N}$ there exist $r_0, \dots, r_m \in \mathbb{N}$ with $k = r_0$ and $l = r_m$ such that for every $i \in \{0, \dots, m-1\}$ we have $C_{r_i} \cap C_{r_{i+1}} \neq \varnothing$.

Proof. Let $k \in \mathbb{N}$. Recall that by construction the V_i introduced in the first claim form an open cover of M. We set

$$L \coloneqq \{l \in \mathbb{N} \,|\, \text{there exist } k = r_0, \dots, r_m = l \text{ with } V_{r_i} \cap V_{r_{i+1}} \neq \varnothing \text{ for } i = 0, \dots, m-1\}.$$

Note that $A \coloneqq \bigcup_{l\in L} V_l$ and $B \coloneqq \bigcup_{l\notin L} V_l$ are both open subsets of M with $A \cup B = M$. It follows easily from the definitions that $A \cap B = \varnothing$. Since $k \in L$ we know that A is non-empty. Since M is connected we see that $A = M$. It follows almost immediately from this observation that $L = \mathbb{N}$. ⊞

We define a map $\varphi \colon \mathbb{N} \to \mathbb{N}$ as follows. We set $\varphi(1) \coloneqq 1$. By the above claim it makes sense to iteratively set

$$\varphi(n) \coloneqq \min\left\{k \in \mathbb{N} \,\middle|\, k \notin \{\varphi(1), \dots, \varphi(n-1)\} \text{ and } C_k \cap (C_{\varphi(1)} \cup \dots \cup C_{\varphi(n-1)}) \neq \varnothing\right\}.$$

Given $n \in \mathbb{N}$ we also set $D_n \coloneqq C_{\varphi(1)} \cup \dots \cup C_{\varphi(n)}$. Note that it follows immediately from the claim that $\bigcup_{n\in\mathbb{N}} D_n = \bigcup_{n\in\mathbb{N}} C_{\varphi(n)} = M$.

Claim 3. There exists a monotonously decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, a monotonously increasing sequence $\{b_n\}_{n\in\mathbb{N}}$ of real numbers and a sequence of homeomorphisms $\Phi_n \colon D_n \to [a_n, b_n]$ such that for each $n \in \mathbb{N}$ we have $\Phi_n|_{D_{n-1}} = \Phi_{n-1}$.

Proof. We set $\Phi_1 = \Psi_1 \colon [r_1, s_1] \to C_1 = D_1$. We construct the sequences of real numbers and of homeomorphisms iteratively by applying the C-Parametrization–Combination Lemma 28.6. ⊞

Note that $I \coloneqq \bigcup_{n\in\mathbb{N}} [a_n, b_n]$ is an interval in $\mathbb{R}$ and that the maps Φ_n define a bijection

$$\begin{array}{rccl} \Phi \colon M & \to & I & \\ x & \mapsto & \Phi_n(x) & \text{if } x \in D_n. \end{array}$$

By construction the restriction of Φ_n to each D_n is continuous. This implies also that the restriction of Φ_n to each C_n is continuous. Since the $\{C_n\}_{n\in\mathbb{N}}$ form a locally finite cover of M of closed subsets we obtain from the Pasting Proposition 4.7 that Φ is actually continuous.

At this point it remains to show that the inverse $\Phi^{-1}\colon I \to M$ is continuous. Since each $\Phi_n\colon D_n \to [a_n, b_n]$ is homeomorphism we see that the restriction of Φ^{-1} to each $[a_n, b_n]$ is continuous. It follows from the Pasting Proposition 4.7 that $\Phi^{-1}\colon I \to M$ is continuous.

Since every non-compact interval is homeomorphic to either $\mathbb{R}$ or $[0, 1)$ we are finally done. ■

28.2. **The classification of 1-dimensional smooth manifolds.** If we consider smooth manifolds instead of topological manifolds and diffeomorphisms instead of homeomorphisms we get the same answer. More precisely the following proposition holds.

Theorem 28.8. (Smooth 1-Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional smooth manifold. Then the diffeomorphism type of M is given by the following table:

	with empty boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0, 1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0, 1)$.

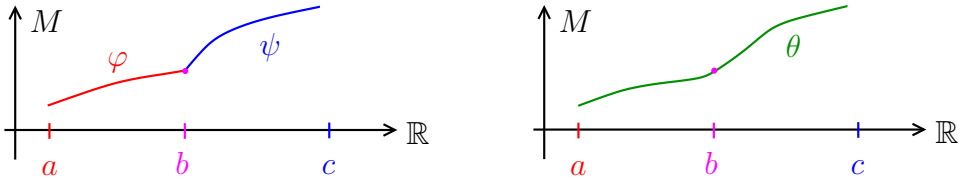
Remark.

(1) Other references for this theorem are [**GPol1974**, p. 212], [**Shas2011**, Theorem 5.4.1] and [**Miln1965a**, appendix].
(2) We will provide a different proof of the Smooth 1-Manifold Classification Theorem 28.8 in the Compact Smooth 1-Manifold Classification Proposition 255.1 and also in Exercise 255.3. □

The proof of the Smooth 1-Manifold Classification Theorem 28.8 relies on the following lemma:

Lemma 28.9. (Glue-Diffeomorphisms Lemma) Let M be a 1-dimensional smooth manifold. Let $\varphi\colon [a, b] \to M$ and $\psi\colon [b, c] \to M$ be smooth maps which are diffeomorphisms onto their images and which satisfy $\varphi([a, b]) \cap \psi([b, c]) = \{\varphi(b)\} = \{\psi(b)\}$. Then there exists a smooth map $\theta\colon [a, c] \to M$ which is a diffeomorphism onto its image with the following properties:

(1) $\theta([a, b]) = \varphi([a, b])$, with $\theta([b, c]) = \psi([b, c])$.
(2) There exists an $\epsilon > 0$ such that $\theta = \varphi$ on $[a, a + \epsilon]$ and $\theta = \psi$ on $[c - \epsilon, c]$.



Proof. If $a = b$ then we can take $\theta = \psi$ and if $b = c$, then we can take $\theta = \varphi$. So we only need to consider the case that $a < b < c$. By picking a chart from the smooth atlas around $\varphi(b) = \psi(b)$ we might as well assume that $M = \mathbb{R}$. After possibly applying the reflection

$t \mapsto -t$ to $M = \mathbb{R}$ we can furthermore assume that φ and ψ are strictly monotonously increasing.

We consider the map

$$\begin{array}{rccl} \chi\colon [a,c] & \to & \mathbb{R} \\ x & \mapsto & x - b + \varphi(b). \end{array}$$

It remains to show that the following hold:

(1) There exists an $\eta_a > 0$ and a smooth map $\theta_a\colon [a,b] \to \mathbb{R}$ which agrees with φ on $[a, a+\eta_a]$ and which agrees with χ on $[b-\eta_a, b]$.
(2) There exists an $\eta_c > 0$ and a smooth map $\theta_c\colon [b,c] \to \mathbb{R}$ which agrees with χ on $[b, b+\eta_c]$ and which agrees with ψ on $[c-\eta_c, c]$.

The existence of both was proved in Exercise 24.5, which in turn relies on the Smooth Transition Function Lemma 24.9. ■

Now we can easily provide a proof of the Smooth 1-Manifold Classification Theorem 28.8.

Proof of Theorem 28.8. Let M be a connected non-empty 1-dimensional smooth manifold. Evidently the idea is modify the proof of the Topological 1-Manifold Classification Theorem 28.1. In the following we discuss the case that M is non-compact with empty boundary. All other cases are treated in a similar fashion.

In the proof of the C-Parametrization-for-Manifold Lemma 28.3 we now use smooth charts to obtain smooth c-parametrizations for M. If we now go through the proof of the Topological 1-Manifold Classification Theorem 28.1 we see that there exists a homeomorphism $f\colon \mathbb{R} \to M$ which is smooth outside of a (possibly infinite) discrete subset $D \subseteq \mathbb{R}$. By applying the Glue-Diffeomorphisms Lemma 28.9 to each $x \in D$ we can turn f into a diffeomorphism. ■

28.3. **Interlude: The order topology.** In the next section we will introduce the "long line", which is a topological space that has all the properties of a connected 1-dimensional topological manifold, except that it is not second-countable. The construction of the "long line" makes use of the order topology which we will introduce in this section.

First we recall some definitions. At least this particular author invariably forgets the precise meanings of the different type of orders that exist.

Definition 28.10. A partial order on a set X is a relation $\leq$ that has the following properties:

(1) For every $x \in X$ we have $x \leq x$.
(2) Whenever $x \leq y$ and $y \leq x$, then we have $x = y$.
(3) Whenever $x \leq y$ and $y \leq z$, then we also have $x \leq z$.

A total order is a partial order that also satisfies the following condition:

(4) For every $x, y \in X$ we have $x \leq y$ or $y \leq x$.

Finally a well-order is a total order that satisfies the following condition:

(5) Every non-empty subset has a minimal element[330].

Such a pair $(X, \leq)$ is called a partially ordered set, totally ordered set respectively well-ordered set.

[330]Recall that in Definition 1.28 we said that an element q of a subset Q of a partial ordered set is *minimal*, if there is no $r \in Q$ with $r < q$.

Next we introduce some self-explanatory notation.

Notation 28.11. Let $(X, \leq)$ be a totally ordered set. As usual, given $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$. Let $x, y \in X$. As on page 103 we write

$$\begin{array}{rclcrcl} (x,y) & \coloneqq & \{a \in X \,|\, x < a < y\}, & & (x,y] & \coloneqq & \{a \in X \,|\, x < a \leq y\} \\ [x,y) & \coloneqq & \{a \in X \,|\, x \leq a < y\} & \text{and} & [x,y] & \coloneqq & \{a \in X \,|\, x \leq a \leq y\}. \end{array}$$

We refer to these subsets in the obvious way as open intervals, half-open intervals and closed intervals.

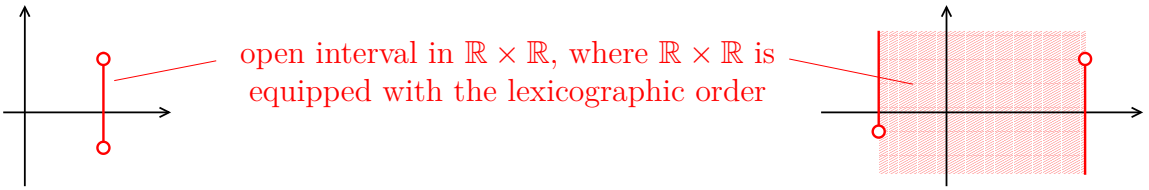
Definition 28.12. Let $(X, \leq)$ be a totally ordered set. Let $\mathcal{B}$ be the subset of the power set $\mathcal{P}(X)$ given by the subsets of X of the following four types:

(1) All intervals (a, b) in X.
(2) If x_0 is a smallest[331] element of $(X, \leq)$, then we also consider the intervals $[x_0, b)$.
(3) If x_1 is a largest[332] element of $(X, \leq)$, then we also consider the intervals $(a, x_1]$.
(4) The total set X.

We leave it to the reader to verify that $\mathcal{B}$ has the basis property, as introduced in Definition 3.52.[333] By the Topology-from-Basis Lemma 3.51 the basis $\mathcal{B}$ defines a topology on X. We refer to this topology as the order topology on X.

Examples.

(1) If we equip $X = \mathbb{R}$ with the usual total order $\leq$, then the resulting order topology equals, basically by definition, the usual topology on X.
(2) We equip $X = \mathbb{R} \times \mathbb{R}$ with the lexicographic order induced from the usual total order on the two $\mathbb{R}$-factors. Now we equip X with the corresponding order topology. In the figure below we show some open intervals in $\mathbb{R} \times \mathbb{R}$. We will discuss this fun example in greater detail in Exercise 28.7.



(3) We equip $\mathbb{N}_0$ and the half-open interval $[0, 1)$ with the usual total orders and we equip $\mathbb{N}_0 \times [0, 1)$ with the lexicographic order. In Exercise 28.6 we will show that if we equip $\mathbb{N}_0 \times [0, 1)$ with the corresponding order topology, then $\mathbb{N}_0 \times [0, 1)$ is homeomorphic to $[0, \infty)$. □

Convention 28.13. In this chapter, given a totally ordered set $(X, \leq)$ we equip X with the corresponding order topology. On numerous occasions we will use the obvious statement

[331] We say that $x_0 \in X$ is a *smallest element* if for all $b \in X$ we have $x_0 \leq b$.
[332] We say that $x_1 \in X$ is a *largest element* if for all $a \in X$ we have $a \leq x_1$.
[333] At first glance it might look unnecessary to explicitly throw in X into the set $\mathcal{B}$. But if X should happen to consist of a single element, then the sets of type (1), (2) and (3) actually do not satisfy property B1.

that if two totally ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are of the same order type[334], then the corresponding topological spaces are homeomorphic.

On several occasions we will use the following lemma, without explicitly referring to it.

Lemma 28.14. (Order–Subspace Topology Lemma) Let $(X, \leq)$ be a totally ordered set. Given any interval $I = (a, b)$, $[a, b)$, $(a, b]$ and $[a, b]$ in X the order topology on $(I, \leq)$ agrees with the subspace topology coming from X.

Proof. We will prove the lemma in Exercise 28.5. ■

Example. Later in this section the half-open interval $[0, 1) \subseteq \mathbb{R}$, equipped with the usual total order $\leq$, will play an important role. It follows from the Order–Subspace Topology Lemma 28.14 that the resulting order topology equals the usual topology on $X \subseteq \mathbb{R}$. □

We conclude this section with the following fairly straightforward lemma.

Lemma 28.15. (Order Topology–Hausdorff Lemma) Let $(X, \leq)$ be a totally ordered set. The topological space X that is given by the order topology is Hausdorff.

Proof. Let $(X, \leq)$ be a totally ordered set. We start out with the following claim:

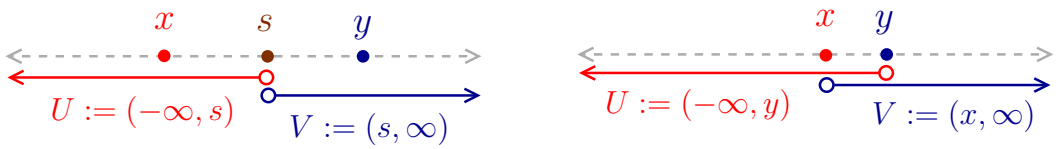
Claim. Given any $x \in X$ the sets

$$(-\infty, x) \coloneqq \{a \in X \mid a < x\} \qquad \text{and} \qquad (x, \infty) \coloneqq \{b \in X \mid x < b\}$$

are open subsets in the order topology.

Proof. If X has a minimal element x_0, then evidently we have $(-\infty, x) = [x_0, x)$. By definition this set is open. On the other hand, if X does not have a minimal element, then we have $(-\infty, x) = \bigcup_{a<x} (a, x)$ which is of course an open subset of X. Basically the same argument also shows that (x, ∞) is open. ⊞

Now let $x \neq y$ be two distinct points in X. Since $(X, \leq)$ is totally ordered we know that $x < y$ or $y < x$. Without loss of generality we can assume that $x < y$. If there exists an $s \in (x, y)$, then we set $U \coloneqq (-\infty, s)$ and $V \coloneqq (s, \infty)$.



Otherwise we set $U \coloneqq (-\infty, y)$ and $V \coloneqq (x, \infty)$. By the claim U and V are open. It is now clear that these are disjoint open neighborhoods of x and y. ■

28.4. **The long line.** Let us start with the following dull definition.

Definition 28.16. Let $(X, \leq)$ be a partially ordered set. Given $x \in X$ we refer to

$$S_x \coloneqq \{y \in X \mid y < x\}$$

as the x-section of $(X, \leq)$.

[334]Recall that in Definition 1.27 we introduced the notion that two partially ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ have the same *order type* if there exists a bijection $f\colon X_1 \to X_2$ which is order-preserving and such that the inverse $f^{-1}\colon X_2 \to X_1$ is also order-preserving.

We consider the well-ordered set $(\mathbb{N}, \leq)$. Evidently the set is *countable*, but every section is actually *finite*. The following, quite remarkable, theorem says that the same type of mismatch between the cardinalities of all sections and the cardinality of the actual well-ordered set can also happen for other cardinalities.

Theorem 28.17. (First Uncountable Ordinal Theorem)
(1) There exists an uncountable well-ordered set such that every section is countable.
(2) If X and Y are two well-ordered sets as in (1), then X and Y have the same order type.

We denote by ω_1 any well-ordered set as in (1).

Remark.
(1) The alert reader will have noticed that Statement (1) of the First Uncountable Ordinal Theorem 28.17 is precisely the content of Exercise 1.8.
(2) The order type of such a well-ordered set as in the First Uncountable Ordinal Theorem 28.17 is often called the *first uncountable ordinal*. We refer to [**Je2003**] and [**Dev1993**, Chapter 3] for an introduction into the concept of "ordinals".
(3) The construction of the well-ordered set ω_1 is a product of the ever-fertile mind of our friends the set-theorists. To the best of my knowledge there is no "down to earth" construction of ω_1.
(4) The well-ordered set ω_1 is studied, in "everyday language", in [**Mun1975**, p. 67] and [**Gaul2014**, Chapter B.2]. □

Proof.
(1) By the Well-Ordering Theorem 1.32 there exists a well-order on any set. In particular there exists a well-order $\leq$ on the uncountable set $X = \mathbb{R}$. If every section of $(X, \leq)$ is countable, then we are done since $\mathbb{R}$ is uncountable. If not, then we consider the following non-empty subset:

$$S := \{x \in X \mid \text{ the } x\text{-section is uncountable}\}.$$

Note that by the definition of a well-order we know that the non-empty subset S of X contains a minimal element z. Finally we consider the corresponding z-section $Y := S_z := \{x \in X \mid x < z\}$. Note that it follows from the construction of z that $Y = S_z$ is uncountable. Furthermore note that it follows immediately from the definition of z that every section of $Y = S_z$ is actually countable.
(2) We will not make use of this statement, thus we will not provide a proof. Instead we refer to [**Mun1975**, p. 73] or alternatively to [**Je2003**, Theorem I.2.12] for a proof. ■

Later on we will need the following simple lemma.

Lemma 28.18. (No Maximal Element-in-ω_1-Lemma) The well-ordered set ω_1 does not have a maximal element.

Proof. Let $x \in \omega_1$. We need to show that x is not a maximal element. By one of the two defining properties of ω_1 we know that the section $S_x = \{y \in X \mid y < x\}$ is countable. It follows from the Countability Lemma 1.37 (5) that $\{y \in X \mid y \leq x\} = S_x \cup \{x\}$ is also countable. Since ω_1 is uncountable we see that x cannot be a maximal element. ■

Now we are finally ready to give the definition of the legendary long line.

Definition 28.19.

(1) We equip $[0,1)$ with the usual total order. Next we equip $\mathbb{L}_{\geq 0} := \omega_1 \times [0,1)$ with the lexicographic order that we introduced in Definition 1.26 and we equip $\mathbb{L}_{\geq 0}$ with the corresponding order topology. We refer to this topological space $\mathbb{L}_{\geq 0}$ as the closed long ray.
(2) We denote by $* \in \omega_1$ the minimal element. We refer to $\mathbb{L}^+ := \mathbb{L}_{\geq 0} \setminus \{(*,0)\}$, equipped with the subspace topology coming from $\mathbb{L}_{\geq 0}$, as the open long ray.
(3) We write $\mathbb{L}_{\leq 0} := \mathbb{L}_{\geq 0}$. We refer to the topological space $\mathbb{L} := \mathbb{L}_{\leq 0} \sqcup_{(*,0)=(*,0)} \mathbb{L}_{\geq 0}$ as the long line. (In the literature the long line is sometimes also called the Alexandroff line.)

Remark. In many references, see e.g. [**Mun1966a**, p. 158], [**SS1978**, p. 71] and [**Kra2009**, p. 66], the open long ray is called the long line. □

The following proposition shows in particular that the open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ both satisfy all properties of a 1-dimensional topological manifold, except that neither is second-countable.

Theorem 28.20. (Long Line Theorem) The open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ both have the following properties:

(1) they are Hausdorff,
(2) they are locally homeomorphic to $\mathbb{R}$,
(3) they are path-connected and
(4) they are not second-countable.

Remark.

(1) The proof we give below is based on the sketch given in [**Mun1966a**, p. 158]. An alternative account of the proof is given in [**Gaul2014**, Chapter 1.2].
(2) The discovery of the long line $\mathbb{L}$ goes back to work of Georg Cantor [**Cant1883**] from 1883.
(3) A slightly less otherworldly example of a topological space that satisfies all properties of a path-connected topological manifold, except that it is not second-countable, is given in [**Bre1993**, Corollary 17.5] and [**Spr1957**, p. 56]. □

We will prove the Long Line Theorem 28.20 in the next section. In the remainder of this section we will discuss the open long ray and long line in greater detail. First of all the following proposition lists several weird and surprising properties of the open long ray and the long line.

Proposition 28.21. (Long Ray Properties Proposition)

(1) Every continuous map $f\colon \mathbb{L}^+ \to \mathbb{R}$ is eventually constant, i.e. there exists an $(x,s) \in \mathbb{L}^+$ such that $f((y,t)) = f(x,s)$ for every $(y,t) \geq (x,s)$.
(2) The topological space $\mathbb{L}^+$ is not paracompact.
(3) Every bounded continuous map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is homotopic to a constant map.
(4) Every unbounded continuous map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is homotopic to the identity map.
(5) An unbounded continuous map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is never homotopic to a bounded map.

(6) The open long line $\mathbb{L}^+$ is not contractible.

Proof.

(1) It is a great exercise to try to prove this statement. If, inexplicably, the reader is not inclined to provide the arguments, then we refer to [**Gaul2014**, Proposition 1.17] for a proof.
(2) We need to show that $\mathbb{L}^+$ admits an open cover that does not admit a locally finite open refinement. In fact one can show that the open subsets $U_{(x,s)} := ((*, 0), (x, s))$ of $\mathbb{L}^+$ is such an open cover. Since we will not make use of this statement we will not show that this cover does not admit a locally finite open refinement. The statement can be proved using a slight variation on the argument given on [**Jos1983**, Theorem 15.3.2(vi)].
(3)-(5) These statements are proved in [**Gaul2014**, Proposition 1.12].
(6) This statement follows immediately from (5) and the Contractability Criterion Lemma 19.20. ■

At this stage one might fear that now, when we no longer restrict ourselves to second-countable topological spaces, the genie is out of the bottle, and that thus any type of classification result is completely out of reach. Rather stunningly this is not the case.

The following theorem can now be viewed as a generalization of the Topological 1-Manifold Classification Theorem 28.1.

Theorem 28.22. (Non-Second Countable 1-Manifold Classification Theorem) Let X be a topological space that is Hausdorff and path-connected.

(1) If X is locally homeomorphic to an open subset of $\mathbb{R}$, then X is homeomorphic to one of the following: S^1, $\mathbb{R}$, $\mathbb{L}^+$ and $\mathbb{L}$.
(2) If X is locally homeomorphic to an open subset of $\mathbb{R}_{\geq 0}$, then either X is homeomorphic to one of the above, or one of the following: $[0, 1)$, $[0, 1]$ and $\mathbb{L}_{\geq 0}$.

Proof.

(1) This statement is proved in [**KnesH1958**].
(2) This statement can be deduced easily from (1) by first removing the "boundary points". We leave it to the reader to provide the details. ■

By the Long Line Theorem 28.20 we now know in particular that the long line $\mathbb{L}$ admits an atlas. In fact, as is shown in [**Gaul2014**, Proposition 7.9], there exists an atlas on $\mathbb{L}$ that is smooth, i.e. such that all transition maps are smooth. Much more impressively, it turns out that there are uncountable many mutually non-diffeomorphic smooth structures on $\mathbb{L}$. We refer to [**Gaul2014**, Theorem 7.14] and [**Nyi1992**] for a precise formulation and a proof of this assertion.

28.5. Proof of the Long Line Theorem 28.20.

In this section we finally provide the proof of the Long Line Theorem 28.20. We start out with some general definitions and statements regarding well-ordered sets.

Definition 28.23. Let $(X, \leq)$ be a well-ordered set. A subset A of X is called inductive if for every $a \in X$ with $S_a \subseteq A$ we also have $a \in A$.

Examples.

(1) Let $X = \mathbb{R}_{\geq 0}$. The interval $A = [0, 2)$ is not inductive whereas $A = [0, 2]$ is inductive.
(2) One can easily show that the only inductive subset of $\mathbb{N}_0$ is $\mathbb{N}_0$. □

The grandiose name of the following theorem is designed to impress friends and family.

Theorem 28.24. (Principle of Transfinite Induction) Let $(X, \leq)$ be a well-ordered set. If $A \subseteq X$ is an inductive subset of X, then $A = X$.

Proof. Let $A \subseteq X$ be an inductive subset. We suppose that $A \neq X$. Since X is well-ordered the set $X \setminus A$ has a minimal element y. Since y is the minimal element of $X \setminus A$ we see that for every $x < y$ we have $x \in A$. But this means that the y-section S_y is contained in A. But since A is inductive this implies that $y \in A$. We have thus obtained a contradiction. ■

In the proof of the Long Line Theorem 28.20 we will need the following fairly straightforward lemma.

Definition 28.25. Let $(X, \leq)$ be a partially ordered set.

(1) Let $x \in X$. We say that $a \in X$ is an immediate predecessor of x if $a < x$ and if there is no $b \in X$ with $a < b < x$.
(2) Let $x \in X$. We say that $a \in X$ is an immediate successor of x if $x < a$ and if there is no $b \in X$ with $x < b < a$.
(3) Let $A \subseteq X$ be a subset. We say that $x \in X$ is an upper bound of A if for every $a \in A$ we have $a \leq x$.
(4) Let $A \subseteq X$ be a subset. We say that $x \in X$ is a supremum of A if x is an upper bound of A and if there is no upper bound of A that is less than x.
(5) Let $x_1 < x_2 < \dots$ be an increasing sequence of elements in X. If $b = \sup\{x_1, x_2, \dots\}$ then we say that b is a limit of the sequence.

Examples.

(1) Every element in $\mathbb{Z}$ has an immediate predecessor and an immediate successor.
(2) No element in $\mathbb{R}$ has an immediate predecessor and no element in $\mathbb{R}$ has an immediate successor.
(3) If $(X, \leq)$ is a well-ordered set, then it follows easily from the definitions that every proper subset $A \subsetneq X$ admits a unique supremum. □

Lemma 28.26. (Countable Section Lemma) Let $(X, \leq)$ be a well-ordered set and let $x \in X$ such that the corresponding section S_x is countable. Precisely one of the following three statements holds:

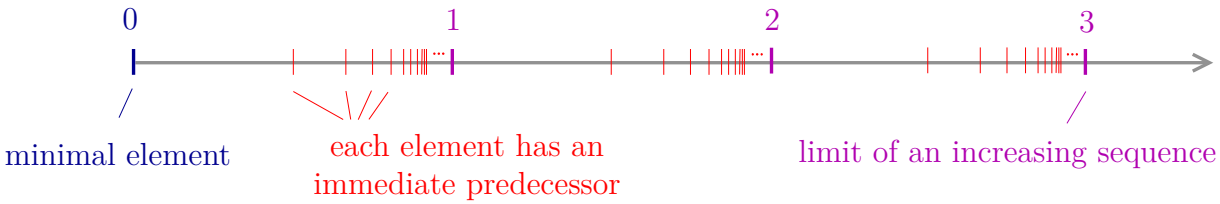
(1) x is the minimal element of X.
(2) There exists an immediate predecessor of x.
(3) x is the limit of an increasing sequence $x_0 < x_1 < x_2 < \dots$ of elements of X.

Proof. Let $(X, \leq)$ be a well-ordered set and let $x \in X$ such that the corresponding section $S_x = \{w \in X \mid w < x\}$ is countable. We only need to consider the case that $x \in X$ is not the minimal element. This means that the section S_x is non-empty. If the section S_x is finite, then it admits a maximal element, which is clearly an immediate predecessor of x. Next we suppose that the section S_x is infinite. By hypothesis on x we know that

S_x is also countable. Thus there exists a bijection $\varphi\colon \mathbb{N}_0 \to S_x$. Now given any $n \in \mathbb{N}_0$ we set $y_n := \max\{\varphi(0), \ldots, \varphi(n)\}$. By construction we have $y_1 \leq y_2 \leq y_3 \leq \ldots$. If the sequence $y_1, y_2, \ldots$ is eventually constant, i.e. if there exists an N such that $y_n = Y_N$ for every $n \geq N$, then Y_N is easily seen to be an immediate predecessor of x. Otherwise there exists a subsequence $y_{s_1} < y_{s_2} < y_{s_3} < \ldots$ which is strictly monotonously increasing. By construction x is the limit of this sequence.

Finally we leave it to the reader to verify that three possibilities are mutually exclusive. ■

Example. We consider the subset $X := \mathbb{N}_0 \cup \{n - \frac{1}{k} \mid n \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}_{\geq 2}\}$ of $\mathbb{R}$, equipped with the usual partial order. One can easily verify that $(X, \leq)$ is actually well-ordered. The number 0 is evidently the minimal element. Every x of the form $n - \frac{1}{k}$ with $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 2}$ has an immediate successor, namely $n - \frac{1}{k-1}$.



Finally every $n \in \mathbb{N}$ is the limit of the increasing sequence $x_i = n - \frac{1}{1+i}$. □

Lemma 28.27. (Glue Intervals Lemma) Let $(X, \leq)$ be a totally ordered set.
(1) Let $a < b < c$ be three elements of X. The interval $[a, c)$ has the order type of $[0, 1)$ if and only if each of the two intervals $[a, b)$ and $[b, c)$ has the order type of $[0, 1)$.
(2) Let $b \in X$ be the limit of an increasing sequence $x_0 < x_1 < \ldots$ of elements of X. If each $[x_n, x_{n+1})$ has the order type of $[0, 1)$, then so does $[x_0, b)$.

Proof. Throughout the proof we will make use of the following simple observation: If $(A, \leq)$ and $(B, \leq)$ are two totally ordered sets, then they have the same order type if and only if there exists an order-preserving bijection $f\colon A \to B$. In other words, the fact that A and B are totally ordered implies that the inverse $f^{-1}\colon B \to A$ is automatically order-preserving. Now we turn to the proofs of the two statements.

(1) Let $a < b < c$ be three elements of X. First suppose that there exist order-preserving bijections $f\colon [a, b) \to [0, 1)$ and $g\colon [b, c) \to [0, 1)$. It is clear that the map

$$\begin{array}{rcl} [a, c) & \to & [0, 1) \\ t & \mapsto & \begin{cases} \frac{1}{2} \cdot f(t), & \text{if } t \in [a, b), \\ \frac{1}{2} + \frac{1}{2} \cdot g(t), & \text{if } t \in [b, c) \end{cases} \end{array}$$

is an order-preserving bijection. Conversely suppose that there exists an order-preserving bijection $h\colon [a, c) \to [0, 1)$. We point out that we have $h([a, b)) = [0, h(b))$ and that we also have $h([b, c)) = [h(b), 1)$. Evidently the intervals $[0, h(b))$ and $[h(b), 1)$ have the order type of $[0, 1)$. Thus we are done.

(2) Let $b \in X$ be the limit of an increasing sequence $x_0 < x_1 < \ldots$ of elements of X. In the following we suppose that for each $n \in \mathbb{N}_0$ there exists an order-preserving bijection

$\varphi_n\colon [x_n, x_{n+1}) \to [0,1)$. One can now easily verify that the map

$$\begin{array}{ccccc} [x_0, b) & = & \bigsqcup_{n\in\mathbb{N}_0} [x_n, x_{n+1}) & \to & [0,1) \\ \uparrow & & t & \mapsto & \frac{1}{2^{n+1}} \cdot \varphi_n(t) + \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \quad \text{for } t \in [x_n, x_{n+1}) \end{array}$$

since b is the limit of the increasing sequence $x_0 < x_1 < \dots$

is an order-preserving bijection. ■

The next lemma contains the heart of the proof of the Long Line Theorem 28.20.

Lemma 28.28. (Interval–Order Type Lemma) As above we denote by $*$ the minimal element of ω_1 and let $x \in \omega_1 \setminus \{*\}$.

(1) The half-open interval $[(*,0),(x,0)) \subseteq \mathbb{L}_{\geq 0}$ has the order type of $[0,1)$.
(2) The open interval $((*,0),(x,0)) \subseteq \mathbb{L}_{\geq 0}$ has the order type of $(0,1)$.

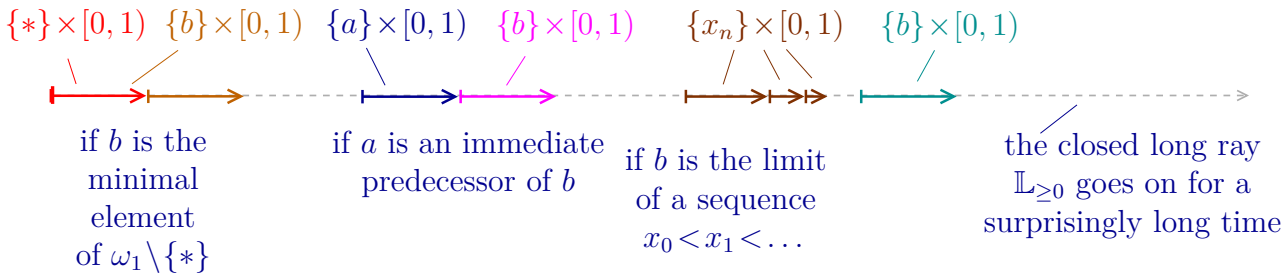
Proof. We consider the set

$$B := \big\{ b \in \omega_1 \setminus \{*\} \,\big|\, \text{the set } [(*,0),(b,0)) \text{ has order type of } [0,1) \big\}.$$

We want to show that $B = \omega_1 \setminus \{*\}$. Note that it follows from the fact that ω_1 is well-ordered that the complement of the minimal element $*$, i.e. the set $\omega_1 \setminus \{*\}$, is again well-ordered. Thus it follows from the Principle of Transfinite Induction Theorem 28.24 that it suffices to show that B is inductive.

Now let $b \in B$ such that $S_b \subseteq B$. By the Countable Section Lemma 28.26 we know that one of the following holds:

(1) b is the minimal element of B.
(2) b has an immediate predecessor a.
(3) b is the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X.



We will now treat these three cases separately.

(1) Suppose that b is the minimal element of $\omega_1 \setminus \{*\}$. In this case we see easily that $[(*,0),(b,0)) = \{(*,t) \,|\, t \in [0,1)\}$, which evidently has the order type of $[0,1)$.
(2) Suppose that b has an immediate predecessor a. Since $S_b \subseteq B$ and $a < b$ we see that $a \in B$, i.e. we see that $[(*,0),(a,0))$ has order type $[0,1)$. Next note that the fact that a is an immediate predecessor of b implies that $[(a,0),(b,0)) = \{(a,t) \,|\, t \in [0,1)\}$. Thus we see that this interval also has order type $[0,1)$. It follows from the Glue Intervals Lemma 28.27 (1) that $[(*,0),(b,0))$ has order type $[0,1)$.
(3) Now suppose that b is the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X. Since $S_b \subseteq B$ and since for each x_n we have $x_n < b$ we see as in (2) that each $[(*,0),(x_n,0))$ has order type $[0,1)$. It follows from the Glue Intervals Lemma 28.27 (1)

that each interval $[(x_n, 0), (x_{n+1}, 0))$ has order type $[0, 1)$. Finally it follows from the Glue Intervals Lemma 28.27 (2) that the interval $[(*, 0), (b, 0))$ has order type $[0, 1)$. ■

Now we can finally provide the proof of the Long Line Theorem 28.20.

Proof of the Long Line Theorem 28.20. We will prove the desired statements only for the open long ray $\mathbb{L}^+$. We leave it to the reader to make the minor modifications need to show that the analogous statements also hold for the long line $\mathbb{L}$. Now recall that we need to show the following four statements regarding the open long ray $\mathbb{L}^+$.

(1) $\mathbb{L}^+$ is Hausdorff,
(2) $\mathbb{L}^+$ is locally homeomorphic to an open subset of $\mathbb{R}$,
(3) $\mathbb{L}^+$ is path-connected,
(4) $\mathbb{L}^+$ is not second-countable.

We turn to the proofs of these four statements.

(1) This statement conveniently enough follows immediately from the Order Topology–Hausdorff Lemma 28.15.
(2) Let $(x, t) \in \mathbb{L}^+$. It follows from the No Maximal Element-in-ω_1-Lemma 28.18 that we can pick some $y > x$. Since $y > x$ we see that $U := ((*, 0), (y, 0))$ is an open neighborhood of (x, t). Note that by the Interval–Order Type Lemma 28.28 we know that the interval $[(*, 0), (y, 0))$ has the order type of $[0, 1)$. It follows that the open interval $U := ((*, 0), (y, 0))$ is homeomorphic to $(0, 1)$.
(3) Let (x, s) and (y, t) be two points in $\mathbb{L}^+$. By the No Maximal Element-in-ω_1-Lemma 28.18 we can pick $z \in X$ with $z > x$ and $z > y$. As in (2) we obtain from the Interval–Order Type Lemma 28.28 that the open interval $U := ((*, 0), (z, 0))$ is homeomorphic to $(0, 1)$. Since the open interval contains the points (x, s) and (y, t) and since $(0, 1)$ is path-connected we see that there exists a path $\gamma\colon [0, 1] \to \mathbb{L}^+$ that connects (x, s) and (y, t).
(4) Let $\mathcal{B}$ be a basis for the topological space $\mathbb{L}^+$. We need to show that $\mathcal{B}$ is uncountable. For each $x \in \omega_1$ we consider the open interval $U_x := ((x, \frac{1}{4}), (x, \frac{3}{4}))$. It follows from the definition of a basis of a topology that given any $x \in X$ there exists a set $B_x \in \mathcal{B}$ with $(x, \frac{1}{2}) \in B_x \subseteq U_x$. Since the U_x are disjoint we see that the B_x are distinct. Since X is uncountable we obtain from the Countability Lemma 1.37 (1) that $\mathcal{B}$ is uncountable. ■

28.6. Exercises for Chapter 28.

Exercise 28.1. An n-dimensional *non-Hausdorff-manifold* is a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$.

(a) In the figure we show the topological space M which is the "circle with two East Poles". Give a mathematical precise definition of M and show that it is a 1-dimensional non-Hausdorff-manifold.
(b) Are there infinitely many pairwise non-homeomorphic compact connected 1-dimensional non-Hausdorff-manifolds?
(c) Are there uncountable many pairwise non-homeomorphic compact connected 1-dimensional non-Hausdorff-manifolds?



Exercise 28.2. An n-dimensional *non-Hausdorff-manifold with boundary* is a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$ or of $\mathbb{H}^n$. Let M be a compact connected 1-dimensional non-Hausdorff manifold. What can you say about the number of boundary points of M?

Exercise 28.3. Let X be a non-empty 1-dimensional topological manifold. Use the $\mathbb{R}$-$\mathbb{R}^2$-Non Homeomorphism Lemma 4.36 to show that in this case the dimension is well-defined, i.e. there is no other $n \in \mathbb{N}_0$ such that X is an n-dimensional topological manifold.

Exercise 28.4. Let X be a 1-dimensional topological manifold. Show that X is homeomorphic to a proper submanifold of $\mathbb{H}^2$.
Remark. Since X is second-countable we know by Exercise 13.1 that X has only countably many components.

Exercise 28.5.

(a) Let $(X, \leq)$ be a totally ordered set. Show that given any interval $I = (a, b)$, $[a, b)$, $(a, b]$ and $[a, b]$ in X the order topology on $(I, \leq)$ agrees with the subspace topology coming from X.
(b) Let $(X, \leq)$ be a totally ordered set and let $A \subseteq X$ be a subset. Show that the order topology on $(A, \leq)$ does not necessarily agree with the subspace topology coming from X.

Exercise 28.6. We equip $\mathbb{N}_0$ and the half-open interval $[0, 1)$ with the usual total orders and we equip $\mathbb{N}_0 \times [0, 1)$ with the lexicographic order. Finally we equip $\mathbb{N}_0 \times [0, 1)$ with the corresponding order topology. Show that $\mathbb{N}_0 \times [0, 1)$ is homeomorphic to $[0, \infty)$.

Exercise 28.7. We consider $X \coloneqq \mathbb{R} \times \mathbb{R}$, for once equipped with the order topology $\mathcal{O}$.

(a) Let $\mathcal{T}$ be the usual topology on $X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Do we have an inclusion $\mathcal{O} \subseteq \mathcal{T}$? Do we have an inclusion $\mathcal{T} \subseteq \mathcal{O}$?
(b) Show that the topological space $(X, \mathcal{O})$ is Hausdorff.
(c) Is the topological space $(X, \mathcal{O})$ second-countable?
(d) Is $(X, \mathcal{O})$ locally homeomorphic to an open subset of $\mathbb{R}$?
(e) Is $(X, \mathcal{O})$ homeomorphic to the long line?

Exercise 28.8. Let ω_1 be the well-ordered set introduced in the First Uncountable Ordinal Theorem 28.17.

(a) Show that for every $x \in \omega_1$ the subset $\{y \in \omega_1 \,|\, x < y\}$ is uncountable.
(b) Let $X = \{x \in \omega_1 \,|\, x \text{ has no immediate predecessor}\}$. Show that X is uncountable.

Exercise 28.9.

(a) We consider the well-ordered set $\mathbb{L}_{\geq 0} \coloneqq \omega_1 \times [0, 1)$. Show that every increasing sequence in $\mathbb{L}_{\geq 0}$ converges.

(b) A topological space that has the property that every sequence admits a convergent subsequence is called sequentially compact. Show that the closed long ray $\mathbb{L}_{\geq 0}$ is sequentially compact.

Exercise 28.10. Show that the open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ are not homeomorphic.
Hint. What happens if you remove a point?

Exercise 28.11.

(a) Let $(X, \leq)$ be a well-ordered set. Let $x \in X$. Show that either x is the largest element of X or that x has an immediate successor.
(b) Give an example of a partially ordered set $(X, \leq)$ that is not well-ordered but that nonetheless has the property that every $x \in X$ has an immediate successor.

Exercise 28.12. Let $(X, \leq)$ be a partially ordered set and let $x \in X$.

(1) We say that x has *depth* -1 if x is a minimal element of X.
(2) We say that x has *depth* 0 if x has an immediate predecessor.

Now we iteratively define what it means for $x \in X$ to have depth $k \in \mathbb{N}$.

(3) We say that x has *depth* k if x is not of depth $k-1$ but if there exists an increasing sequence $x_1 < x_2 < \dots$ in X, such that each x_i has depth $< k$, and such that x is a limit of this sequence.

Show that there exists a countable well-ordered subset of $\mathbb{R}$ that admits elements of depth two.

Exercise 28.13. Does the long line $\mathbb{L}$ admit an embedding into $\mathbb{R}^2$?

Exercise 28.14. A topological space X is called *homogeneous* if given any $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$. Show that the long line $\mathbb{L}$ is homogeneous.

CHAPTER 29

Tangent spaces of smooth manifolds

In this chapter we introduce the notion of tangent spaces of smooth manifolds. As a warm-up we first consider tangent spaces of smooth submanifolds of some $\mathbb{R}^k$. In this setting the notion of a tangent space can be motivated easily. The notion of a tangent space for a general "abstract" manifold takes more time to get used to and to appreciate.

This chapter is organized as follows:

- In Section 29.1 will introduce the "classical" tangent spaces of smooth submanifolds of $\mathbb{R}^k$.
- In Section 29.2 we will see that tangent spaces give a functor from the category of pointed smooth submanifolds of $\mathbb{R}^k$ to the category of real vector spaces.
- In Section 29.3 we will introduce tangent spaces of (abstract) smooth manifolds.
- In Section 29.4 we will explain the relationship between the two definitions of tangent spaces for submanifolds of $\mathbb{R}^k$.
- In Section 29.5 will introduce the notion of a vector field on a smooth manifold.
- In Appendix 29.6 we will give an alternative construction of tangent spaces for smooth manifolds.
- In Section 29.7 we will provide exercises for Chapter 29.

In the next Chapter 30 we will state and prove the Inverse Mapping Theorem 24.21 and the Smooth Embedding Theorem 30.15.

29.1. **Tangent spaces of smooth submanifolds of $\mathbb{R}^k$.** In this section we study smooth submanifolds of some $\mathbb{R}^k$, where, as always, we equip $\mathbb{R}^k$ with the standard smooth structure. Note that in the Submanifold Smooth Structure Lemma 25.23 we saw that every smooth submanifold of $\mathbb{R}^k$ inherits a standard smooth structure.

Definition 29.1. Let M be an n-dimensional smooth submanifold of $\mathbb{R}^k$ and let $P \in M$.

(1) A curve through P is a continuous map $\gamma\colon I \to M$ where $I \subseteq \mathbb{R}$ is an open or half-open interval with $0 \in I$ and with $\gamma(0) = P$.

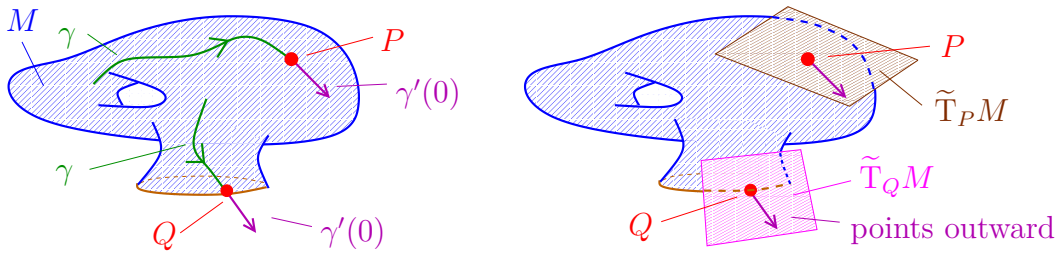
(2) Given a smooth curve $\gamma\colon I \to M$ through P we denote by $\gamma'(0) \in \mathbb{R}^k$ the derivative of γ viewed as a smooth map[335] $\gamma\colon I \to M \to \mathbb{R}^k$.

(3) We define the physical tangent space as the set of all derivatives of smooth curves on M through P, i.e. we define[336]

$$\widetilde{\mathrm{T}}_P M \;:=\; \{\gamma'(0) \,|\, \gamma\colon I \to M \text{ a smooth curve through } P\} \;\subseteq\; \mathbb{R}^k.$$

We refer to any vector in $\widetilde{\mathrm{T}}_P M$ as a tangent vector to M at P.

(4) Let $P \in \partial M$. We say that a tangent vector $v \in \widetilde{\mathrm{T}}_P M$ points outward if $v = \gamma'(0)$ for some smooth map $\gamma\colon (-\epsilon, 0] \to M$ and if $v \notin \widetilde{\mathrm{T}}_P \partial M$. Similarly we also define the notion that v points inward.

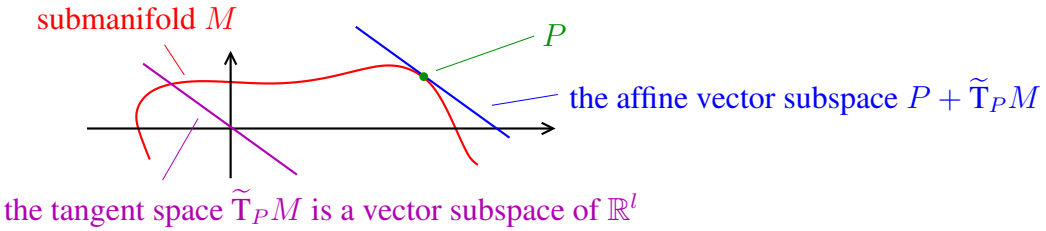


Remark.

(1) Let M be an n-dimensional smooth submanifold of $\mathbb{R}^k$ and let $P \in M$. If $P \notin \partial M$, then using charts and using the definition of a smooth map on a half-open interval, see Definition 25.20, one can easily show that

$$\widetilde{\mathrm{T}}_P M \;:=\; \{\text{all derivatives } \gamma'(0) \text{ of smooth maps } \gamma\colon (-\epsilon,\epsilon) \to M \text{ with } \gamma(0) = P\}.$$

(2) Let M be an n-dimensional smooth submanifold of $\mathbb{R}^k$ and let $P \in M$. In the Physical Tangent Space-Functor Proposition 29.3 (0) we will see that $\widetilde{\mathrm{T}}_P M$ is an n-dimensional vector subspace of $\mathbb{R}^k$. In pictures we usually show, for clarity, the affine vector subspace $P + \widetilde{\mathrm{T}}_P M$.



This affine vector space is the "best possible" approximation of M by an affine vector subspace. □

We formulate our first examples as a lemma.

Lemma 29.2. (Physical Tangent Spaces–Lemma) Let $n \in \mathbb{N}_0$.

(1) Let M be an open subset of $\mathbb{R}^n$ or of the half-plane $\mathbb{H}^n$. For every $P \in M$ we have the equality $\widetilde{\mathrm{T}}_P M = \mathbb{R}^n$.

(2) Let $W \subseteq \mathbb{R}^n$ be a k-dimensional vector space and let $Q \in \mathbb{R}^n$. The corresponding affine space $Q + W = \{Q + w \,|\, w \in W\}$ is a k-dimensional smooth submanifold of $\mathbb{R}^n$ and for any $P \in W$ we have $\widetilde{\mathrm{T}}_P W = W$.

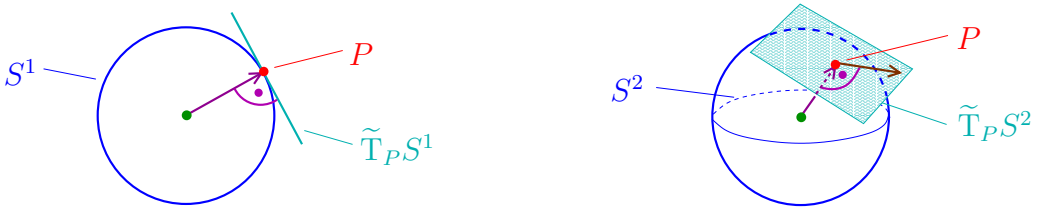
(3) Given any $P \in S^n \subseteq \mathbb{R}^{n+1}$ we have the equality $\widetilde{\mathrm{T}}_P S^n = \{v \in \mathbb{R}^{n+1} \,|\, \langle v, P\rangle = 0\}$.

Proof.

(1) First let M be an open subset of $\mathbb{R}^n$ and let $P \in M$. Note that by definition we have $\widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$. Conversely let $v \in \mathbb{R}^n$. Since M is open there exists an $\epsilon > 0$ such that $\{P + t \cdot v \,|\, t \in [0,\epsilon]\} \subseteq M$. Thus we can consider the smooth map $\gamma\colon [0,\epsilon) \to M$ given by $\gamma(t) = P + t \cdot v$. Evidently $\gamma'(0) = v$, which shows that $v \in \widetilde{\mathrm{T}}_P M$. The proof of

[335]Note that according to the Submanifold Smooth Structure Lemma 25.23 the inclusion $M \to \mathbb{R}^k$ is smooth, so the composition of the smooth map $\gamma\colon I \to M$ with the inclusion $M \to \mathbb{R}^k$ is also smooth.

[336]Note that if 0 is an endpoint of the interval I, then we use the convention from Notation 24.8 to define the derivative of γ at 0.



the case that M is an open subset of $\mathbb{H}^n$ is very similar, we just have to notice that for any $v \in \mathbb{R}^n$ we have $v \in \mathbb{H}^n$ or $-v \in \mathbb{H}^n$. We leave it to the reader to fill in the few remaining details.

(2) Note that it follows from elementary linear algebra that there exists an invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$ such that $A \cdot W = \{0\} \times \mathbb{R}^k$. The map

$$\begin{array}{rcl} Q + W & \to & \mathbb{R}^n \\ x & \mapsto & A \cdot (x - Q) \end{array}$$

is now a submanifold chart. Thus $Q + W$ is a k-dimensional smooth submanifold of $\mathbb{R}^n$. Using this chart one can also easily verify that for any $P \in W$ we indeed have $\widetilde{\mathrm{T}}_P W = W$. We leave it to the reader to fill in the details.

(3) Let $P \in S^n$. We need to show that $\widetilde{\mathrm{T}}_P M = \{v \in \mathbb{R}^{n+1} \mid \langle v, P \rangle = 0\}$. We prove the two inclusions separately.

(⊆) Let I be a (half-) open interval with $0 \in I$ and let $\gamma\colon I \to S^n$ be a smooth map with $\gamma(0) = P$. For any $t \in I$ we have

$$0 \underset{\substack{\uparrow \\ \text{since } \|\gamma(t)\| \equiv 1}}{=} \frac{d}{dt}\|\gamma(t)\|^2 = \frac{d}{dt}\langle \gamma(t), \gamma(t)\rangle \underset{\substack{\uparrow \\ \text{follows from the} \\ \text{Product Rule}}}{=} 2 \cdot \langle \gamma'(t), \gamma(t)\rangle.$$

In particular for $t = 0$ we see that $0 = \langle \gamma'(0), \gamma(0)\rangle = \langle \gamma'(0), P\rangle$.

(⊇) Suppose we are given $v \in \mathbb{R}^{n+1}$ with $\langle v, P\rangle = 0$. If $v = 0$, then v is the derivative of the constant curve through P. Now assume that $v \neq 0$. We consider the closed curve that is defined by $\gamma(t) \coloneqq P \cdot \cos(t \cdot \|v\|) + \frac{v}{\|v\|} \cdot \sin(t \cdot \|v\|)$. Evidently we have $\gamma(0) = P$ and $\gamma'(0) = v$. Finally note that it follows from $\langle v, P\rangle = 0$ that γ defines a curve on S^n. Thus we have shown that $v \in \widetilde{\mathrm{T}}_P S^n$. ■

Proposition 29.3. (Physical Tangent Space-Functor Proposition)

(0) If M is a d-dimensional submanifold of $\mathbb{R}^n$, then for each $P \in M$ the physical tangent space $\widetilde{\mathrm{T}}_P M$ is a d-dimensional vector subspace of $\mathbb{R}^n$.

(1) Let $f\colon M \to N$ be a smooth map[337] between smooth submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$. For any $P \in M$ the map[338]

$$\begin{array}{rcl} \mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M & \to & \widetilde{\mathrm{T}}_{f(P)} N \\ \gamma'(0) & \mapsto & (f \circ \gamma)'(0) \end{array}$$

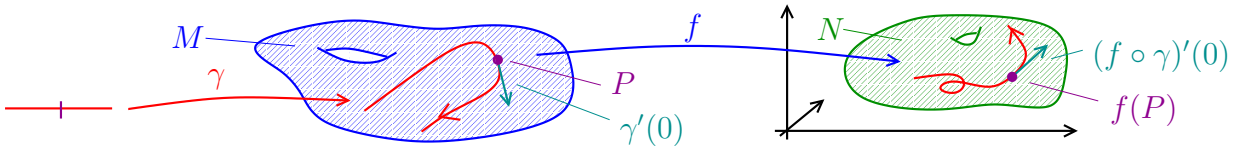
is well-defined and linear.

(2) The maps from (1) have the following two properties:

(a) If $\mathrm{id}\colon M \to M$ is the identity map, then for any $P \in M$ the corresponding map $\mathrm{D}\,\mathrm{id}_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_P M$ is also the identity.

(b) For any two smooth maps $f\colon L \to M$ and $g\colon M \to N$ between smooth submanifolds and any $P \in L$ we have

$$\mathrm{D}(g\circ f)_P \;=\; \mathrm{D}g_{f(P)} \circ \mathrm{D}f_P \colon \widetilde{\mathrm{T}}_P L \;\to\; \widetilde{\mathrm{T}}_{g(f(P))} N.$$



Definition 29.4. Let $f\colon M \to N$ be a smooth map between smooth submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$. Given any $P \in M$ we refer to the map

$$\begin{array}{rcl} \mathrm{D}f_P \colon \widetilde{\mathrm{T}}_P M & \to & \widetilde{\mathrm{T}}_{f(P)} N \\ \gamma'(0) & \mapsto & (f\circ\gamma)'(0), \end{array}$$

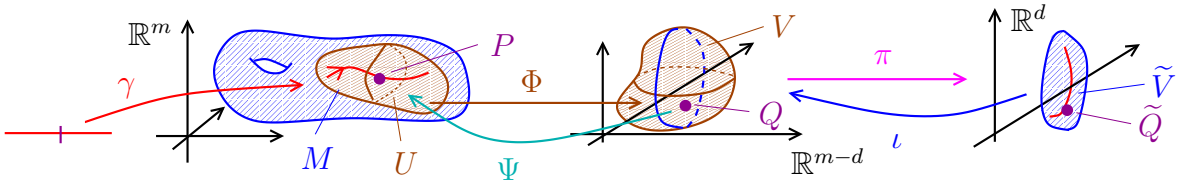
that we introduced in the Physical Tangent Space-Functor Proposition 29.3 (2), as the differential of f at P.

Proof. We start with a preamble. In the proofs of (0) and (1) we will consider smooth maps $h\colon W \to \mathbb{R}^t$ where W is an open subset of some $\mathbb{R}^s$. For each point $P \in W$ we denote by $\mathrm{D}h_P \in \mathrm{M}(t\times s, \mathbb{R})$ the differential as defined on Definition 24.15. Throughout the proofs of (0) and (1) we use this notation exclusively for such maps.

(0) Let M be a d-dimensional submanifold of $\mathbb{R}^m$ and let $P \in M$. First we consider the case that $P \in M \setminus \partial M$. Since M is a smooth submanifold of $\mathbb{R}^m$ we know that there exists a submanifold chart $\Phi\colon U \to V$ for $P \in M$. We write $Q := \Phi(P)$ and $\Psi := \Phi^{-1}\colon V \to U$. We write

$$\widetilde{V} \;:=\; \{y \in \mathbb{R}^d \,|\, (0,y) \in V\}.$$

This is an open subset of $\mathbb{R}^d$. We denote by $\iota\colon \widetilde{V} \to V$ the injective map that is given by $x \mapsto (0,x)$. Note that $Q = \iota(\widetilde{Q})$ for some $\widetilde{Q} \in \widetilde{V}$.



Claim.

$$\widetilde{\mathrm{T}}_P M \;=\; \underbrace{\mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}} \cdot \mathbb{R}^d}_{\uparrow} \;\subseteq\; \mathbb{R}^m.$$

note that $\Psi\circ\iota\colon \widetilde{V} \to \mathbb{R}^m$ is indeed defined on an open subset of $\mathbb{R}^d$, thus it makes sense to consider the "classical" differential

Proof. We show the two inclusions "$\supseteq$" and "$\subseteq$" separately.

(⊆) Let $\gamma\colon (-\epsilon,\epsilon) \to M$ be a smooth map with $\gamma(0) = P$. We need to show that $\gamma'(0) \in \mathrm{D}(\Psi\circ\iota)_{\widetilde{Q}}(\mathbb{R}^d)$. Note that after possibly reducing ϵ we can assume that

[337] At the danger of repeating ourselves, we use the Submanifold Smooth Structure Lemma 25.23 to view M and N as smooth manifolds in their own right. Thus by a smooth map we mean a smooth map between these two smooth manifolds.

[338] By the Submanifold Smooth Structure Lemma 25.23 the inclusion map $N \to \mathbb{R}^n$ is smooth, thus $f\circ\gamma$ is indeed a smooth curve that takes values in $\mathbb{R}^n$, thus it makes sense to take its derivative.

$\gamma((-\epsilon,\epsilon)) \subseteq U$. We denote by $\pi\colon \mathbb{R}^m = \mathbb{R}^{m-d} \times \mathbb{R}^d \to \mathbb{R}^d$ the projection that is given by $(x,y) \mapsto y$. We see that

$$\gamma'(0) \;=\; (\Psi \circ \iota \circ \pi \circ \Phi \circ \gamma)'(0) \;\underset{\substack{\uparrow\\ \text{by the Chain Rule 24.17}\\ \text{together with the remark}\\ \text{on page 651}}}{=}\; \mathrm{D}(\Psi \circ \iota)_{\widetilde{Q}} \cdot \underbrace{(\pi \circ \Phi \circ \gamma)'(0)}_{\text{vector in } \mathbb{R}^d}.$$

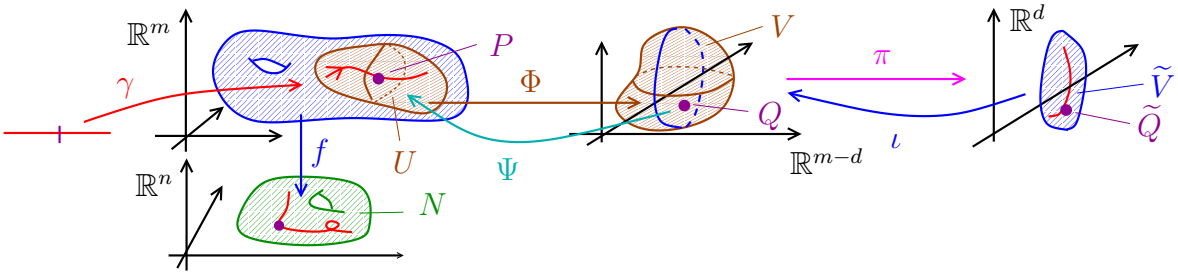
(⊇) Let $v \in \mathbb{R}^d$. Since $\widetilde{V}$ is open we see that there exists an $\epsilon > 0$ such that the corresponding curve $\gamma\colon (-\epsilon,\epsilon) \to \mathbb{R}^d$ given by $t \mapsto \widetilde{Q} + t \cdot v$ takes values in $\widetilde{V}$. Then

$$\mathrm{D}(\Psi \circ \iota)_{\widetilde{Q}} \cdot v \;=\; \mathrm{D}(\Psi \circ \iota)_{\widetilde{Q}} \cdot \gamma'(0) \;\underset{\substack{\uparrow\\ \text{by the Chain Rule 24.17}}}{=}\; (\Psi \circ \iota \circ \gamma)'(0).$$

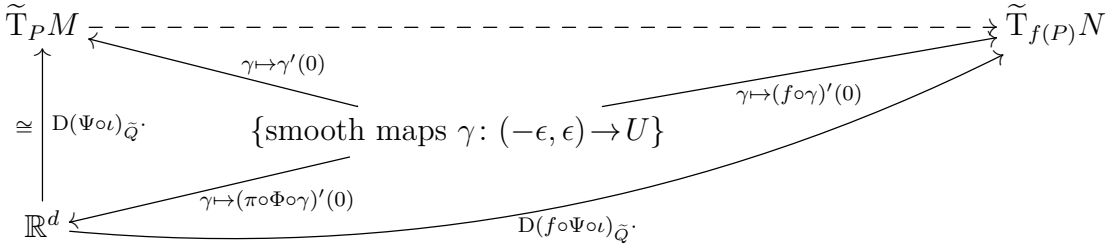
Since $\Psi \circ \iota \circ \gamma\colon (-\epsilon,\epsilon) \to M$ is a smooth map which sends 0 to P we have shown the "$\supseteq$"-inclusion. ⊞

This concludes the proof of Statement (0) if $P \in M \setminus \partial M$. The case that $P \in \partial M$ is treated in a very similar fashion. We leave it to the reader to fill in the details.

(1) Now let $f\colon M \to N$ be a smooth map and let $P \in M$. To simplify the notation a little bit we consider the case that $P \notin \partial M$. The case that $P \in \partial M$ is dealt with in essentially the same way. We continue with the notation and setup that we introduced in (0) for $M \subseteq \mathbb{R}^m$.



We consider the following diagram:



We make the following observations:

(a) In the proof of the "$\subseteq$"-statement in (0) we already showed that the triangle to the left commutes.

(b) In (0) we showed that the left vertical map is an epimorphism.

(c) By the Chain Rule 24.17 we have $\mathrm{D}(\Psi \circ \iota)_{\widetilde{Q}} = \mathrm{D}\Psi_Q \cdot \mathrm{D}\iota_{\widetilde{Q}}$.

(d) Recall that Ψ is a diffeomorphism. It follows from this fact together with the Differential-of-Diffeomorphism Proposition 24.19 that $\mathrm{D}\Psi_Q$ is an invertible matrix.

(e) Since the inclusion ι is a linear map we obtain from the discussion on page 651 that multiplication by the matrix $\mathrm{D}\iota_{\widetilde{Q}}$ equals the inclusion $\mathbb{R}^d \to \mathbb{R}^m$.

(f) It follows from (d) and (e) that the left vertical map is a monomorphism.
(g) The combination of (b) and (f) shows that the left vertical map is an isomorphism.
(h) It follows again from the Chain Rule 24.17 that the triangle to the bottom right commutes.
(i) Basically by definition of $\widetilde{\mathrm{T}}_P M$ the upper left diagonal map is a surjection.

The above observations imply that there is a unique horizontal map which makes the upper triangle commute. In other words, the map $\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_{f(P)} N$ given by $\gamma'(0) \to (f\circ\gamma)'(0)$ is well-defined. Furthermore, since the above diagram commutes and since the left vertical map is an isomorphism and since the right diagonal map is a homomorphism we see that the top horizontal map is indeed a homomorphism.

In the following proof of (2) we denote by D again the differential from Definition 29.4.

(2) (a) This statement is obvious.
(b) Let $f\colon L \to M$ and $g\colon M \to N$ be smooth maps between smooth submanifolds, let $P \in L$ and let $v \in \widetilde{\mathrm{T}}_P L$. Let $\gamma\colon I \to L$ be a smooth curve through P with $\gamma'(0) = v$. We see that

$$\mathrm{D}(g\circ f)_P(v) \;=\; ((g\circ f)\circ\gamma)'(0) \;=\; (g\circ(f\circ\gamma))'(0) \underset{\substack{\uparrow\\ \text{since } (f\circ\gamma)'(0) = \mathrm{D}f_P(v)}}{=} \mathrm{D}g_{f(P)}(\mathrm{D}f_P(v)).$$

■

Proposition 29.5. (Physical Tangent Space-Inclusion Proposition) Let M be a smooth submanifold of $\mathbb{R}^m$.

(1) Let W be a submanifold of M (e.g. W could be an open subset of M) and let $P \in W$. We denote by $i\colon W \to M$ the inclusion.
   (a) We have $\widetilde{\mathrm{T}}_P W \subseteq \widetilde{\mathrm{T}}_P M$ and the differential $\mathrm{D}i_P\colon \widetilde{\mathrm{T}}_P W \to \widetilde{\mathrm{T}}_P M$ is the inclusion.
   (b) If W is of codimension zero, then we have $\widetilde{\mathrm{T}}_P W = \widetilde{\mathrm{T}}_P M$.
(2) We denote by $i\colon \partial M \to M$ the inclusion. We have $\widetilde{\mathrm{T}}_P \partial M \subseteq \widetilde{\mathrm{T}}_P M$ and the differential $\mathrm{D}i_P\colon \widetilde{\mathrm{T}}_P \partial M \to \widetilde{\mathrm{T}}_P M$ is the inclusion.
(3) Let N be a smooth submanifold of some $\mathbb{R}^n$. If $f\colon M \to N$ is a local diffeomorphism, then for every $P \in M$ the differential $\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_{f(P)} N$ is an isomorphism.

Proof.

(1) (a) Recall that by the Submanifold Smooth Structure Lemma 25.23 we know that the inclusion $i\colon W \to M$ is smooth. Any smooth curve in W is thus a smooth curve in M. With this observation Statement (1a) is now obvious.
   (b) If W is of codimension zero, then we know by the Physical Tangent Space-Functor Proposition 29.3 (0) that the vector spaces $\widetilde{\mathrm{T}}_P W$ and $\widetilde{\mathrm{T}}_P M$ have the same dimension. Thus it follows from (1a) that the vector spaces are actually the same.
(2) As we already discussed in the Smooth Manifold–Boundary Proposition 25.27 (1), the inclusion map $\partial M \to M$ is smooth. Thus the statement is obvious.
(3) Let $f\colon M \to N$ be a local diffeomorphism and let $P \in M$. Since f is a local diffeomorphism there exists an open neighborhood U of P and an open neighborhood V of $Q = f(P)$ such that the restriction $\varphi = f|_U\colon U \to V$ is a diffeomorphism. We write $\psi = \varphi^{-1}\colon V \to U$. We denote by $i\colon U \to M$ and $j\colon V \to N$ the inclusions. We obtain the following diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{T}}_P M & \xrightarrow{\mathrm{D}f_P} & \widetilde{\mathrm{T}}_{f(P)} N \\ {\scriptstyle \mathrm{D}i_P}\uparrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \mathrm{id}}\uparrow{\scriptstyle \mathrm{D}j_{f(P)}} \\ \widetilde{\mathrm{T}}_P U & \underset{\mathrm{D}\psi_{f(P)}}{\overset{\mathrm{D}\varphi_P}{\rightleftarrows}} & \widetilde{\mathrm{T}}_{f(P)} V. \end{array}$$

We make the following three observations:

(a) We have $f \circ i = j \circ \varphi$. Thus it follows from the Physical Tangent Space-Functor Proposition 29.3 (2b) that the diagram commutes.

(b) It follows from $\psi \circ \varphi = \mathrm{id}_U$ and $\varphi \circ \psi = \mathrm{id}_V$, together with the Physical Tangent Space-Functor Proposition 29.3 (2) that $\mathrm{D}\varphi_P = \mathrm{D}\psi^{-1}_{f(P)}$. In particular we see that the bottom horizontal map is an isomorphism.

(c) By (1) we know that the vertical maps are the identity.

It follows from the above three observations that the top horizontal map is also an isomorphism. ■

The last lemma of this section relates the two notions of differentials that we have introduced so far.

Lemma 29.6. (Differential-equals-Differential Lemma) Let $f\colon M \to N$ be a smooth map between an *open subset of* $\mathbb{R}^m$ and a *smooth submanifold of* $\mathbb{R}^n$. For every $P \in M$ the following statement holds:

$$\underbrace{\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M = \mathbb{R}^m \to \widetilde{\mathrm{T}}_{f(P)} N \subseteq \mathbb{R}^n}_{\substack{\text{differential from} \\ \text{Definition 29.4}}} \text{ is given by multiplication by } \underbrace{\mathrm{D}f_P \in \mathrm{M}(n \times m, \mathbb{R}).}_{\substack{\text{the differential from} \\ \text{Definition 24.15}}}$$

Proof. Since $\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M = \mathbb{R}^m \to \widetilde{\mathrm{T}}_P N \subseteq \mathbb{R}^n$ is linear we only have to check this statement for the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_m$ of $\widetilde{\mathrm{T}}_P M = \mathbb{R}^m$. Let $i \in \{1, \ldots, m\}$. Since M is an open subset of $\mathbb{R}^m$ there exists an $\epsilon > 0$ such that the map $\gamma\colon (-\epsilon, \epsilon) \to \mathbb{R}^m$ given by $t \mapsto P + t \cdot \mathfrak{e}_i$ takes values in M. Now we see:

$$(\mathrm{D}f_P)(\mathfrak{e}_i) \;=\; (\mathrm{D}f_P)(\gamma'(0)) \underset{\substack{\uparrow \\ \text{see Definition 29.4}}}{=} (f \circ \gamma)'(0) \underset{\substack{\uparrow \\ \text{by the Chain Rule 24.17}}}{=} \mathrm{D}f_P \cdot (\gamma'(0)) \;=\; \mathrm{D}f_P \cdot \mathfrak{e}_i.$$

■

29.2. **Smooth manifolds and functors.** Before we move on to tangent spaces of general smooth manifolds it is convenient to formulate some of the above results in the language of category theory.

Definition 29.7.

(1) We refer to the category $\mathcal{SmMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfd}) &:= \text{all smooth manifolds,} \\ \mathrm{Mor}(X, Y) &:= \text{all smooth maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the category of smooth manifolds.

(2) We refer to the category $\mathcal{SmMfd}_{\mathbb{R}^n}$ with[340]

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfd}_{\mathbb{R}^n}) &:= \text{all smooth submanifolds of } \mathbb{R}^n \text{ for an arbitrary } n \in \mathbb{N}_0, \\ \mathrm{Mor}(X, Y) &:= \text{all smooth maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the category of smooth submanifolds of some $\mathbb{R}^n$.

In our setting we need a slight variation on the above categories.

Definition 29.8.

(1) A pointed smooth manifold is a pair (M, x_0) where M is a smooth manifold and x_0 is a point in M.
(2) We refer to the category $\mathcal{PSmMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PSmMfd}) &:= \text{all pointed smooth manifolds,} \\ \mathrm{Mor}((M,x_0),(N,y_0)) &:= \text{all smooth maps } f \text{ from } M \text{ to } N \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the category of pointed smooth manifolds.
(3) We define the category $\mathcal{PSmMfd}_{\mathbb{R}^n}$ of pointed smooth submanifolds of some $\mathbb{R}^n$ exactly the same way.

With these definitions we can rephrase (most of) the Physical Tangent Space-Functor Proposition 29.3 as follows.

Proposition 29.9. (Physical Tangent Space-Functor Proposition) The maps

$$\begin{aligned} (M,P) &\mapsto \widetilde{\mathrm{T}}_P M \\ (f\colon (M,P) \to (N,Q)) &\mapsto (\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_Q N) \end{aligned}$$

define a covariant functor from the category $\mathcal{PSmMfd}_{\mathbb{R}^n}$ of pointed smooth submanifolds of some $\mathbb{R}^n$ to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.

Note that $\mathcal{SmMfd}_{\mathbb{R}^n}$ is a full subcategory of $\mathcal{SmMfd}$ in the sense of the definition of a full subcategory in Definition 5.8. Similar $\mathcal{PSmMfd}_{\mathbb{R}^n}$ is a full subcategory of $\mathcal{PSmMfd}$. Thus the following question arises:

Question 29.10.

(1) Can we "extend" the covariant functor

$$\begin{aligned} \mathcal{PSmMfd}_{\mathbb{R}^n} &\to \mathcal{Vec}_{\mathbb{R}} \\ (M,P) &\mapsto \widetilde{\mathrm{T}}_P M \\ (f\colon (M,P) \to (N,Q)) &\mapsto (\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_Q N) \end{aligned}$$

to a covariant functor $\mathcal{PSmMfd} \to \mathcal{Vec}_{\mathbb{R}}$?

(2) If we can extend the covariant functor, is the extension "essentially unique"?

We will make Question 29.10 (1) more precise in the following section. There are many reasons why one should hope for a positive answer to Question 29.10 (1). For example, even if we only care about smooth submanifolds of some $\mathbb{R}^n$, very often it is convenient to work with "abstract" descriptions. For example in the Torus-as-Quotient Lemma 25.42 we showed that the torus $S^1 \times S^1$ is diffeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$. In many situations it is much more pleasant to work with the smooth manifold $\mathbb{R}^2/\mathbb{Z}^2$ instead of working directly with the torus $S^1 \times S^1$. The reader will thus be pleased to hear that we will give a positive answer in the Tangent Space-Isomorphism Proposition 29.24 (3).

[340] Here the dimension n is variable, thus the category contains as objects that are submanifolds of $\mathbb{R}^0$, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and so on.

Finally note that once we have proved the Euclidean Embedding Theorem 33.2 we will be able to provide an answer to Question 29.10 (2) in Exercise 33.1.

29.3. **Tangent spaces of smooth manifolds.** In this section we introduce the notion of tangent spaces to points on general smooth manifolds. We jump right into the fray by giving the definition. We will then spend some time trying to explain why this, initially invariably confusing, definition is actually meaningful.

Definition 29.11. Let M be a smooth manifold and let $P \in M$.[341]

(1) A derivation at P is an $\mathbb{R}$-linear map[342] $w\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M,\mathbb{R})$ we have

$$w(f \cdot g) \;=\; f(P) \cdot w(g) + w(f) \cdot g(P) \qquad \text{“Leibniz Rule”}.$$

Given a derivation w at P and given $f \in C^\infty(M,\mathbb{R})$ we sometimes call $w(f)$ the derivative of f at P in the w-direction.

(2) The set of all derivations at P is called the tangent space $\mathrm{T}_P M$ at P. We refer to elements of $\mathrm{T}_P M$, i.e. we refer to derivations at P, also as tangent vectors. Sometimes, especially when we also talk about the physical tangent space $\widetilde{\mathrm{T}}_P M$, we refer to $\mathrm{T}_P M$ also the abstract tangent space at P.

Lemma 29.12. (Tangent Space-is-Vector Space Lemma) Let M be a smooth manifold. For every $P \in M$ the tangent space $\mathrm{T}_P M$ is a vector subspace of the vector space of linear maps $C^\infty(M,\mathbb{R}) \to \mathbb{R}$.

Proof. Let v, w be two derivations at P. We need to show that $v + w$ is also a derivation at P. Thus let $f, g \in C^\infty(M,\mathbb{R})$. We perform the following calculation:

$$\begin{aligned}(v+w)(f\cdot g) \;=\; v(f\cdot g) + w(f\cdot g) \;&=\; f(P)\cdot v(g) + v(f)\cdot g(P) + f(P)\cdot w(g) + w(f)\cdot g(P)\\ &=\; f(P)\cdot(v(g)+w(g)) + (v(f)+w(f))\cdot g(P)\\ &=\; f(P)\cdot(v+w)(g) + (v+w)(f)\cdot g(P).\end{aligned}$$

An even easier argument shows that if $\lambda \in \mathbb{R}$, then $\lambda \cdot v$ is also a derivation at P. Finally note that evidently the zero homomorphism is a derivation. ∎

Lemma 29.13. (Derivation Lemma) Let M be a smooth manifold, let $P \in M$ and let $w \in \mathrm{T}_P M$.

(1) If $f\colon M \to \mathbb{R}$ is a constant map, then $w(f) = 0$.
(2) Let $f, g \in C^\infty(M,\mathbb{R})$. If $f(P) = g(P) = 0$, then $w(f \cdot g) = 0$.
(3) Let $f, g \in C^\infty(M,\mathbb{R})$. If there exists some open neighborhood U of P such that $f|_U = g|_U$, then $w(f) = w(g)$.

Proof.

[341] We should stress that we allow P to be in the boundary of M.

[342] Recall that according to Definition 25.15 we denote by $C^\infty(M,\mathbb{R})$ the vector space of all smooth maps $M \to \mathbb{R}$.

(1) Let $f\colon M \to \mathbb{R}$ be a constant map. We denote by c the map that is $= 1$ everywhere. We perform the following calculation:

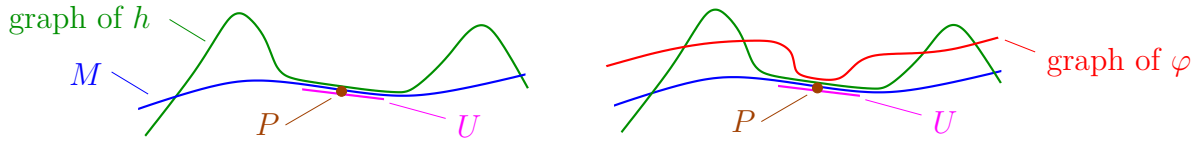
$$w(f) \underset{\substack{\uparrow\\ \text{follows from } f=f(P)\cdot c,\\ \text{since } w \text{ linear}}}{=} f(P)\cdot w(c) \underset{\substack{\uparrow\\ \text{since } c=c\cdot c}}{=} f(P)\cdot w(c\cdot c) \underset{\substack{\uparrow\\ \text{Leibniz Rule}\\ \text{see Definition 29.11}}}{=} f(P)\cdot(\underbrace{c(P)}_{=1}\cdot w(c) + w(c)\cdot\underbrace{c(P)}_{=1}) = f(P)\cdot 2\cdot w(c).$$

Comparing the second and the last term we see that those terms, and hence all terms, are zero.

(2) Let $f, g \in C^\infty(M,\mathbb{R})$. If $f(P) = g(P) = 0$, then

$$w(f\cdot g) \underset{\substack{\uparrow\\ \text{by Definition 29.11 of}\\ \text{a derivation}}}{=} f(P)\cdot w(g) + w(f)\cdot g(P) = 0\cdot w(g) + w(f)\cdot 0 = 0.$$

(3) Let $f, g \in C^\infty(M,\mathbb{R})$. We suppose that there exists some open neighborhood U of P such that $f|_U = g|_U$. We set $h \coloneqq f - g$. Note that h vanishes on U. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth map $\varphi\colon M \to \mathbb{R}$ which is $\equiv 1$ on $M \setminus U$ and which satisfies $\varphi(P) = 0$.



We see that $w(f) - w(g) \underset{\substack{\uparrow\\ \text{since } w \text{ is}\\ \text{linear}}}{=} w(h) \underset{\substack{\uparrow\\ \text{since } h=h\cdot\varphi}}{=} w(h\cdot\varphi) \underset{\substack{\uparrow\\ \text{by (2) since}\\ h(P)=\varphi(P)=0}}{=} 0.$ ∎

This definition of a tangent space might sound rather odd, but the following proposition should help in getting some intuition for the definition.

Proposition 29.14. (Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition) Let M be an open subset of $\mathbb{R}^n$. For $i = 1, \dots, n$ we denote by $\pi_i\colon M \to \mathbb{R}$ the projection onto the i-th coordinate. For any $P \in M$ the map

$$\begin{aligned} \Xi_P\colon \mathbb{R}^n &\to \mathrm{T}_P M \\ v &\mapsto \left(\begin{aligned} C^\infty(M,\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto \underbrace{\tfrac{d}{dt}\big|_{t=0} f(P+t\cdot v)}_{=\mathrm{D}f_P\cdot v \text{ by the Chain Rule 24.17}} \end{aligned} \right) \end{aligned}$$

is an isomorphism of vector spaces, where the inverse is given by

$$\begin{aligned} \Omega_P\colon \mathrm{T}_P M &\to \mathbb{R}^n \\ w &\mapsto (w(\pi_1), \dots, w(\pi_n)). \end{aligned}$$

Proof. Let M be an open subset of $\mathbb{R}^n$ and let $P = (P_1, \dots, P_n) \in M$. Our first claim implies that Ξ_P actually takes values in $\mathrm{T}_P M$.

Claim 1. For every $v \in \mathbb{R}^n$ the map

$$\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \mathrm{D}f_P \cdot v \end{array}$$

is a derivation.

Proof. It is clear that the map is linear in f. So it remains to show that it satisfies the Leibniz Rule from Definition 29.11. Let $f, g \in C^\infty(M,\mathbb{R})$. Note that the map $f \cdot g$ equals the following composition of maps:

$$M \xrightarrow[=:d]{x \mapsto (x,x)} M \times M \xrightarrow{\operatorname{diag}(f,g)} \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \xrightarrow[=:m]{(x,y)\mapsto x\cdot y} \mathbb{R}.$$

We see that

$$\begin{aligned} \mathrm{D}(f\cdot g)_P \cdot v &= \mathrm{D}(m \circ \operatorname{diag}(f,g) \circ d)_P \cdot v \overset{\text{by the Chain Rule 24.17}}{=} \mathrm{D}m_{(f(P),g(P))} \cdot (\mathrm{D}\operatorname{diag}(f,g)_{P\times P} \cdot \mathrm{D}d_P) \cdot v \\ &= (g(P)\ f(P)) \begin{pmatrix} \mathrm{D}f_P & 0 \\ 0 & \mathrm{D}g_P \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot v = g(P)\cdot \mathrm{D}f_P \cdot v + f(P) \cdot \mathrm{D}g_P \cdot v. \end{aligned}$$

We have thus successfully verified that the Leibniz Rule holds. ⊞

It is clear that the map $\Xi_P \colon \mathbb{R}^n \to \mathrm{T}_P M$ given by $v \mapsto (f \mapsto \mathrm{D}f_P \cdot v)$ is linear. Since $\mathbb{R}^n$ is finite-dimensional it remains to show that $\Omega_P \circ \Xi_P = \mathrm{id}_{\mathbb{R}^n}$ (which in particular implies that Ξ_P is injective) and that Ξ_P is surjective. We do so in the following two claims:

Claim 2. The map $\Omega_P \circ \Xi_P$ is the identity on $\mathbb{R}^n$.

Proof. Since Ω_P and Ξ_P are linear it suffices to show that $\Omega_P \circ \Xi_P$ is the identity on the standard basis vectors of $\mathbb{R}^n$. Now let $i, j \in \{1, \dots, n\}$. We see that

$$j\text{-th coordinate of } (\Omega_P \circ \Xi_P)(\mathfrak{e}_i) \underset{\text{definition of } \Omega_P \text{ and } \Xi_P}{=} \frac{d}{dt}\pi_j(P + t\cdot \mathfrak{e}_i)|_{t=0} \underset{\text{elementary real analysis}}{=} \delta_{ij}.$$

But this means that $(\Omega_P \circ \Xi_P)(\mathfrak{e}_i) = \mathfrak{e}_i$. ⊞

Claim 3. The map $\Xi_P \colon \mathbb{R}^n \to \mathrm{T}_P M$ is surjective.

Proof. Let $w \in \mathrm{T}_P M$. We set $v_i \coloneqq w(\pi_i) \in \mathbb{R}$ and we set $v \coloneqq (v_1, \dots, v_n) \in \mathbb{R}^n$. We claim that $\Xi_P(v) = w$. Let $f \in C^\infty(M,\mathbb{R})$. By Taylor's Theorem 24.13 we have

$$f(x) = f(P) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_P \cdot (\pi_i - P_i) + \underbrace{\sum_{i=1}^n \sum_{j=1}^n (\pi_i - P_i)(\pi_j - P_j) \cdot \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_{P+t(x-P)} dt}_{\text{it follows from the Derivation Lemma 29.13 (2) that } w(\dots) = 0}.$$

Finally we see that we have the following equalities:

$$w(f) \underset{\text{by the above observation}}{=} \overbrace{w(f(P))}^{=0 \text{ by the Derivation Lemma 29.13}} + w\Big(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_P \cdot (\pi_i - P_i)\Big) \underset{\text{linearity of } w}{=} \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_P \cdot (\underbrace{w(\pi_i)}_{=v_i} - \underbrace{w(P_i)}_{=0}) \underset{\text{by the Chain Rule 24.17}}{=} \Xi_P(v)(f).$$

Thus we have shown that $w = \Xi_P(v)$. ■

The following proposition summarizes some basic properties of the tangent space. It can be viewed as an analog of the Physical Tangent Space-Functor Proposition 29.3 (1) and (2) and of the Physical Tangent Space-Functor Proposition 29.9.

Proposition 29.15. (Tangent Space-Functor Proposition) Let M and N be two smooth manifolds.

(1) If $f\colon M \to N$ is a smooth map, then for every $P \in M$ the map

$$\begin{array}{rcl} \mathrm{D}f_P\colon \mathrm{T}_P M & \to & \mathrm{T}_{f(P)}N \\ v & \mapsto & \begin{pmatrix} C^\infty(N,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & v(g\circ f) \end{pmatrix} \end{array}$$

is well-defined and linear.

(2) The maps from (1) have the following two properties:

(a) If $\mathrm{id}\colon M \to M$ is the identity map, then for every point $P \in M$ the differential $\mathrm{D}\,\mathrm{id}_P\colon \mathrm{T}_P M \to \mathrm{T}_P M$ is also the identity.

(b) For any two smooth maps $f\colon L \to M$ and $g\colon M \to N$ between smooth manifolds and any $P \in L$ we have

$$\mathrm{D}(g\circ f)_P \;=\; \mathrm{D}g_{f(P)} \circ \mathrm{D}f_P\colon \mathrm{T}_P L \;\to\; \mathrm{T}_{g(f(P))}N \qquad \text{“Chain Rule”}.$$

(3) The maps

$$\begin{array}{rcl} (M,P) & \mapsto & \mathrm{T}_P M \\ (f\colon (M,P)\to(N,Q)) & \mapsto & (\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_Q N) \end{array}$$

define a covariant functor from the category $\mathcal{PSmMfd}$ of pointed smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.

Definition 29.16. Given a smooth map $f\colon M \to N$ between two smooth manifolds and given a point $P \in M$ we refer to the map

$$\begin{array}{rcl} \mathrm{D}f_P\colon \mathrm{T}_P M & \to & \mathrm{T}_{f(P)}N \\ v & \mapsto & ((g\colon N\to\mathbb{R}) \mapsto v(g\circ f\colon M\to\mathbb{R})) \end{array}$$

from the Tangent Space-Functor Proposition 29.15 (1) as the differential of f at P.

Proof.

(1) The only statement one needs to think about for a second is, why does $\mathrm{D}f_P$ take values in $\mathrm{T}_{f(P)}N$? Thus let $v \in \mathrm{T}_P M$. We need to show that $g \mapsto v(g\circ f)$ is indeed a derivation. Now let $g, h \in C^\infty(N,\mathbb{R})$. We perform the following little calculation:

$$v((g\cdot h)\circ f) \;=\; v((g\circ f)\cdot(h\circ f)) \;=\; (g\circ f)(P)\cdot v(h\circ f) + v(g\circ f)\cdot(h\circ f)(P).$$

We have thus verified that the Leibniz Rule from Definition 29.11 is satisfied.

(2) This statement follows almost immediately from the definitions.

(3) This statement just summarizes the contents of (1) and (2) and of the Tangent Space-is-Vector Space Lemma 29.12. ■

Example. Let M and N be two smooth manifolds. If $f\colon M \to N$ is a constant map, then it follows easily from the Derivation Lemma 29.13 that for each point $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is actually the zero homomorphism. □

The following proposition is the analog of the Physical Tangent Space-Functor Proposition 29.3 (0).

Proposition 29.17. (Tangent Space-Dimension Proposition) If M is an n-dimensional smooth manifold, then for every $P \in M$ the corresponding tangent space $\mathrm{T}_P M$ is an n-dimensional vector subspace of the vector space $\mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R})$.

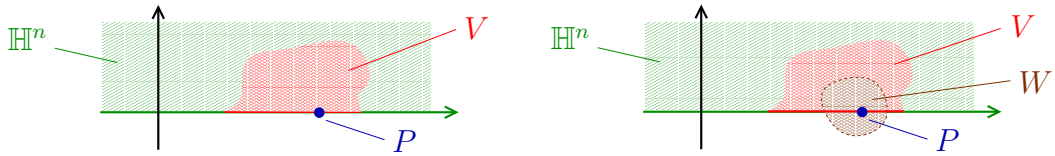
The proof of the Tangent Space-Dimension Proposition 29.17 relies on the following two lemmas.

Lemma 29.18. (Inclusion Differential Lemma) Let M be a smooth manifold and let $U \subseteq M$ be an open subset. We denote by $\iota\colon U \to M$ the inclusion. For every $P \in U$ the map $\mathrm{D}\iota_P\colon \mathrm{T}_P U \to \mathrm{T}_P M$ is an isomorphism.

Proof. The lemma follows, with a little bit of thought, from the Derivation Lemma 29.13 (3). We will fill in the details in Exercise 29.4. ∎

Lemma 29.19. (Inclusion Differential Lemma II) Let $n \in \mathbb{N}$ and let $V \subseteq \mathbb{H}^n$ be an open subset. We denote by $\iota\colon V \to \mathbb{R}^n$ the inclusion. For every $P \in \partial V$ the map $\mathrm{D}\iota_P\colon \mathrm{T}_P V \to \mathrm{T}_P\mathbb{R}^n$ is an isomorphism.

Proof. Let $P \in \partial V$. First we show that the map $\mathrm{D}\iota_P\colon \mathrm{T}_P V \to \mathrm{T}_P\mathbb{R}^n$ is injective. Thus let $w \in \mathrm{T}_P V$ with $\mathrm{D}\iota_P(w) = 0$. We need to show that $w(f) = 0$ for every smooth map $f\colon V \to \mathbb{R}$. Thus let $f\colon V \to \mathbb{R}$ be a smooth map. By definition of a smooth map on $\mathbb{H}^n$, see Definition 24.3, there exists an open neighborhood $W \subseteq \mathbb{R}^n$ of P and a smooth map $g\colon W \to \mathbb{R}$ with $f|_{V\cap W} = g|_{V\cap W}$.



By Exercise 25.16, applied to g, there exists furthermore an open neighborhood X of P and a smooth map $h\colon \mathbb{R}^n \to \mathbb{R}$ such that $h|_{W\cap X} = g|_{W\cap X}$. We calculate

$$w(f) \underset{\uparrow}{=} w(h|_W) = w(h\circ\iota) \underset{\uparrow}{=} (\mathrm{D}\iota_P(w))(h) = 0.$$

by the Derivation Lemma 29.13 (3), since f and $h|_W$ agree on $W \cap X$

by definition of $\mathrm{D}\iota_P(w)$

It remains to show that $\mathrm{D}\iota_P\colon \mathrm{T}_P W \to \mathrm{T}_P\mathbb{R}^n$ is surjective. For $i = 1,\dots,n$ we write

$$\begin{aligned} C^\infty(W,\mathbb{R}) &\xrightarrow{v_i} \mathbb{R} \\ f &\mapsto \lim_{h\to 0^+} \tfrac{1}{h}\cdot(f(\underbrace{P+h\cdot\mathfrak{e}_i}_{\in W \text{ for small } h>0}) - f(P)) \end{aligned} \quad \& \quad \begin{aligned} C^\infty(\mathbb{R}^n,\mathbb{R}) &\xrightarrow{w_i} \mathbb{R} \\ f &\mapsto \underbrace{\lim_{h\to 0} \tfrac{1}{h}\cdot(f(P+h\cdot\mathfrak{e}_i) - f(P))}_{=\frac{\partial f}{\partial x_i}|_P}. \end{aligned}$$

One can easily verify that $v_1,\dots,v_n$ are derivations. Evidently we have $(\mathrm{D}\iota_P)(v_i) = w_i$. By the Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition 29.14 we know that $w_1,\dots,w_n$ form a basis for $\mathrm{T}_P\mathbb{R}^n$. We have thus shown that $\mathrm{D}\iota_P$ is a surjection. ∎

Proof of the Tangent Space-Dimension Proposition 29.17. Let M be an n-dimensional smooth manifold. Furthermore let $P \in M$. First we consider the case that $P \notin \partial M$. Thus we can pick a chart $\Phi\colon U \to V$ of type (i) where V is an open subset of $\mathbb{R}^n$. We

denote by $i\colon U \to M$ the inclusion map. We have the following isomorphisms:

$$\underset{\text{isomorphism by the Inclusion Differential Lemma 29.18}}{\mathrm{T}_P M} \xleftarrow{\mathrm{D}i_P} \mathrm{T}_P U \xrightarrow[\text{by the Tangent Space-Functor Proposition 29.15 this is an isomorphism with inverse given by } \mathrm{D}(\Phi^{-1})_{\Phi(P)}]{\mathrm{D}\Phi_P} \mathrm{T}_{\Phi(P)} V \xrightarrow[\text{isomorphism given by the Open Subset-of-}\mathbb{R}^n\text{-Tangent Space Proposition 29.14}]{\cong} \mathbb{R}^n.$$

Now suppose that $P \in \partial M$. This means that there exists a chart $\Phi\colon U \to V$ of type (ii) where V is an open subset of $\mathbb{H}^n$ and $\Phi(P) \in \partial\,\mathbb{H}^n$. We denote by $j\colon V \to \mathbb{R}^n$ the inclusion map. In this setting we have the following isomorphisms:

$$\underset{\text{isomorphism by the Inclusion Differential Lemma 29.18}}{\mathrm{T}_P M} \xleftarrow{\mathrm{D}i_P} \mathrm{T}_P U \xrightarrow[\text{by the Tangent Space-Functor Proposition 29.15 this is an isomorphism with inverse given by } \mathrm{D}(\Phi^{-1})_{\Phi(P)}]{\mathrm{D}\Phi_P} \mathrm{T}_{\Phi(P)} V \xrightarrow[\text{isomorphism given by the Inclusion Differential Lemma 29.19}]{\mathrm{D}j_P} \mathrm{T}_{\Phi(P)} \mathbb{R}^n \xrightarrow[\text{isomorphism given by the Open Subset-of-}\mathbb{R}^n\text{-Tangent Space Proposition 29.14}]{\cong} \mathbb{R}^n.$$

■

The following proposition is the analog of the Physical Tangent Space-Inclusion Proposition 29.5.

Proposition 29.20. (Tangent Space-Inclusion Proposition) Let M be a smooth manifold.

(1) Let W be a submanifold of M (e.g. W could be an open subset of M) and let $P \in W$. We denote by $i\colon W \to M$ the inclusion.
   (a) The differential $\mathrm{D}i_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is a monomorphism.
   (b) If W is of codimension zero, then the differential $\mathrm{D}i_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is an isomorphism.

(2) We denote by $i\colon \partial M \to M$ the inclusion. The differential $\mathrm{D}i_P\colon \mathrm{T}_P \partial M \to \mathrm{T}_P M$ is a monomorphism.

(3) Let N be a smooth manifold. If $f\colon M \to N$ is a local diffeomorphism, then for every $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an isomorphism.

Convention 29.21. Let M be a smooth manifold. Since it is notationally dreadful to carry around inessential information we will usually use the following two conventions.

(1) Let W be a submanifold of M. We denote by $i\colon W \to M$ the inclusion map. By the Tangent Space-Inclusion Proposition 29.20 we know that the map $\mathrm{D}i_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is a monomorphism. On many occasions we will just identify $\mathrm{T}_P W$ with its image under $\mathrm{D}i_P$ in $\mathrm{T}_P M$. In particular we will view $\mathrm{T}_P W$ as a vector subspace of $\mathrm{T}_P M$.

(2) Let $P \in \partial M$. By the same logic as in (1) we will view $\mathrm{T}_P\,\partial M$ as a vector subspace of $\mathrm{T}_P M$.

This convention brings us in line with the corresponding situation for physical tangent spaces as described in the Physical Tangent Space-Inclusion Proposition 29.5 (1) and (2).

Proof. Let M be an n-dimensional smooth manifold.

(1) (a) Let W be a k-dimensional submanifold of M and let $P \in W$. In the following we deal with the case that P admits a submanifold chart of type (α) in the sense of Definition 25.22. Using the Inclusion Differential Lemma 29.19 the other cases

are treated in a similar fashion. Thus let $\Phi\colon U \to V$ be a submanifold chart of type (α). We write $H = \{0\} \times \mathbb{R}^k \subseteq \mathbb{R}^n$ and $Q = \Phi(P) \in H$. We denote by $\iota\colon H \to \mathbb{R}^n$ the inclusion and we denote by $\pi\colon \mathbb{R}^n \to H$ the projection that is given by $(x, y) \mapsto (0, y)$. We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{T}_P W & \longleftarrow & \mathrm{T}_P(W \cap U) & \xrightarrow{\mathrm{D}(\Phi|_{W\cap U})_P} & \mathrm{T}_Q(V \cap H) & \longrightarrow & \mathrm{T}_Q H \\
\downarrow{\scriptstyle \mathrm{D}i_P} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{D}\iota_Q} \quad \circlearrowleft {\scriptstyle \mathrm{D}\pi_Q} \\
\mathrm{T}_P M & \longleftarrow & \mathrm{T}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \mathrm{T}_Q V & \longrightarrow & \mathrm{T}_Q \mathbb{R}^n.
\end{array}$$

We make the following clarifications and observations:

- All maps that are not given explicitly are the differentials of the obvious inclusion maps.
- It follows from the Tangent Space-Functor Proposition 29.15 (2) that the diagram commutes.
- It follows from the Inclusion Differential Lemma 29.18 that the horizontal maps to the left and to the right are isomorphisms.
- Since Φ and also $\Phi|_{W\cap U}$ are diffeomorphisms we obtain from the Tangent Space-Functor Proposition 29.15 that the middle horizontal maps are isomorphisms.
- Note that $\pi \circ \iota\colon H \to H$ is the identity. It follows from the Tangent Space-Functor Proposition 29.15 (2) that $\mathrm{D}\pi_Q \circ \mathrm{D}\iota_Q = \mathrm{D}(\pi \circ \iota)_Q = \mathrm{D}\,\mathrm{id}_Q = \mathrm{id}$. Thus we see that $\mathrm{D}\iota_Q$ is a monomorphism.

The combination of the above shows that in fact all downward maps are monomorphisms.

(b) This statement follows immediately from (a) and the fact, which we just proved in the Tangent Space-Dimension Proposition 29.17, that $\mathrm{T}_P W$ and $\mathrm{T}_P M$ have the same dimension.

(2) The proof of this statement is very similar to the proof of Statement (1). We leave it to the reader to fill in the details.

(3) The proof of this statement is basically identical to the proof of the analogous statement in the Physical Tangent Space-Inclusion Proposition 29.5, all we have to do is to replace the Physical Tangent Space-Functor Proposition 29.3 (2) by the Tangent Space-Functor Proposition 29.15 (2). Again we leave it to the reader to fill in the details. ■

We conclude this section with a short discussion of outward and inward pointing tangent vectors. These notions are the analogs of the notions we introduced in Definition 29.1.

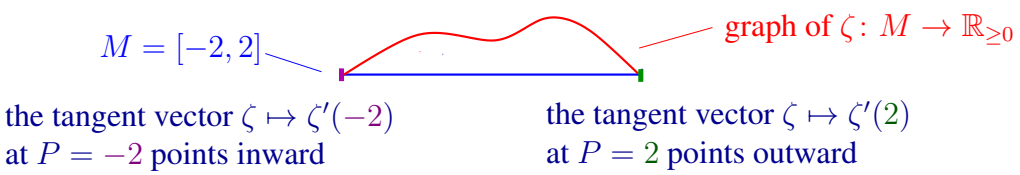
Definition 29.22. Let M be a smooth manifold and let $P \in \partial M$. We say that a tangent vector $w \in \mathrm{T}_P M$ points outward, if the following two conditions are satisfied:

(1) For every smooth map $\zeta\colon M \to \mathbb{R}_{\geq 0}$ with $\zeta|_{\partial M} \equiv 0$ we have $w(\zeta) \leq 0$.

(2) There exists a smooth map $\zeta\colon M \to \mathbb{R}_{\geq 0}$ with $\zeta|_{\partial M} \equiv 0$ such that $w(\zeta) < 0$.

By replacing "$\leq$" and "$<$" by "$\geq$" and "$>$" we obtain the notion of an inward pointing tangent vector.

The following lemma says that for a point $P \in \partial M$ there is a trichotomy for the tangent vectors in $\mathrm{T}_P M$.



Lemma 29.23. (Boundary Tangent Vector-Trichotomy Lemma) Let M be a smooth manifold, let $P \in \partial M$ and let $w \in \mathrm{T}_P M$. Precisely one of the following three statements holds:

(1) w points outward,
(2) w points inward,
(3) $w \in \mathrm{im}(\mathrm{D}\iota_P(\mathrm{T}_P\,\partial M \to \mathrm{T}_P M))$, where $\iota\colon \partial M \to M$ is the inclusion map.

Proof. We will prove the lemma in Exercise 29.2. ∎

29.4. **The relationship between the definitions of tangent spaces.** Given a k-dimensional smooth submanifold M of some $\mathbb{R}^m$ and given $P \in M$ we have now introduced two notions of tangent spaces. In Definition 29.1 we introduced the physical tangent space $\widetilde{\mathrm{T}}_P M$, which is a k-dimensional vector subspace of $\mathbb{R}^m$. Later, in Definition 29.11 we introduced the tangent space $\mathrm{T}_P M$ in a significantly more abstract way.

The next proposition elucidates the relationship between the physical tangent space $\widetilde{\mathrm{T}}_P M$ and the tangent space $\mathrm{T}_P M$.

Proposition 29.24. (Tangent Space-Isomorphism Proposition)

(1) Let M be a k-dimensional submanifold of $\mathbb{R}^m$ and let $P \in M$.
   (a) The map
$$\begin{array}{rccc} \Theta_M\colon \widetilde{\mathrm{T}}_P M & \to & \mathrm{T}_P M & \\ v & \mapsto & \left(\begin{array}{ccc} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & \mathrm{D}g_P(v) \end{array}\right) & \\ & & \uparrow & \\ & & \text{where } \mathrm{D}g_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_{g(P)}\mathbb{R} = \mathbb{R} & \end{array}$$
   is well-defined and it is an isomorphism of vector spaces.
   (b) For $P \in \partial M$ the isomorphism Θ_M restricts to a bijection between outward tangent vectors in $\widetilde{\mathrm{T}}_P M$ (as introduced in Definition 29.1) and outward tangent vectors in $\mathrm{T}_P M$ (as introduced in Definition 29.22). The same statement also holds with "outward" replaced by "inward".

(2) If $f\colon M \to N$ is a smooth map between submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$, then the following diagram commutes:
$$\begin{array}{ccc} \widetilde{\mathrm{T}}_P M & \xrightarrow{\Theta_M} & \mathrm{T}_P M \\ {\scriptstyle \mathrm{D}f_P}\downarrow & & \downarrow{\scriptstyle \mathrm{D}f_P} \\ \widetilde{\mathrm{T}}_{f(P)} N & \xrightarrow{\Theta_N} & \mathrm{T}_{f(P)} N. \end{array}$$
   Here to the left we consider the differential as defined on Definition 29.4 and to the right we consider the differential from Definition 29.16.

(3) The maps Θ_M defined in (1) give a natural isomorphism, in the sense of Definition 5.27, between the covariant functors

$$\begin{array}{rcl} \mathcal{PSmMfd}_{\mathbb{R}^n} & \to & \mathcal{Vec}_{\mathbb{R}} \\ (M,P) & \mapsto & \widetilde{\mathrm{T}}_P M \\ f & \mapsto & \mathrm{D}f_P \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathcal{PSmMfd}_{\mathbb{R}^n} & \to & \mathcal{Vec}_{\mathbb{R}} \\ (M,P) & \mapsto & \mathrm{T}_P M \\ f & \mapsto & \mathrm{D}f_P \end{array}$$

which we introduced in the Physical Tangent Space-Functor Proposition 29.9 and the Tangent Space-Functor Proposition 29.15.

Remark.

(1) The Tangent Space-Isomorphism Proposition 29.24 (3) gives an affirmative answer to Question 29.10 (1).

(2) What really matters in the end is that we managed to define a covariant functor $(M,P) \to \mathrm{T}_P M$ that satisfies the Tangent Space-Isomorphism Proposition 29.24. The actual definition of $\mathrm{T}_P M$ is in the following utterly irrelevant. In fact in Section 29.6 we will sketch a different construction of such a covariant functor, but at the end the actual definition of $\mathrm{T}_P M$ never gets used. □

Proof. We prove the statements in a different order than formulated.

(2) Let $f\colon M \to N$ be a smooth map between submanifolds of $\mathbb{R}^m$ respectively $\mathbb{R}^n$. Furthermore let $v \in \widetilde{\mathrm{T}}_P M$ and let $g \in C^\infty(N,\mathbb{R})$. We see that

$$(\mathrm{D}f_P \circ \Theta_M)(v)(g) \;=\; \mathrm{D}(g\circ f)_P(v) \underset{\uparrow}{=} (\mathrm{D}g_{f(P)} \circ \mathrm{D}f_P)(v) \;=\; (\Theta_N \circ \mathrm{D}f_P)(v)(g).$$

Chain Rule of the Physical Tangent Space-Functor Proposition 29.3 (2)

(1) (a) As in the proof of the Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition 29.14 we see that Θ_M is well-defined in the sense that for any $v \in \widetilde{\mathrm{T}}_P M$ the map $C^\infty(M,\mathbb{R}) \to \mathbb{R}$ given by $g \mapsto \mathrm{D}g_P(v)$ is indeed a derivation. It remains to show that the maps Θ_M are isomorphisms. Let M be a k-dimensional smooth submanifold of $\mathbb{R}^m$ and let $P \in M$. As always we use the Submanifold Smooth Structure Lemma 25.23 to view M as a smooth manifold in its own right. We pick a corresponding chart $\Phi\colon U \to V$ around P. We write $Q = \Phi(P)$. We denote by $i\colon U \to M$ and $j\colon V \to \mathbb{R}^k$ the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccccc} \widetilde{\mathrm{T}}_P M & \xleftarrow{\mathrm{D}i_P} & \widetilde{\mathrm{T}}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \widetilde{\mathrm{T}}_Q V & \xrightarrow{\mathrm{D}j_Q} & \widetilde{\mathrm{T}}_Q \mathbb{R}^k \\ \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_U} & & \downarrow{\scriptstyle \Theta_V} & & \downarrow{\scriptstyle \Theta_{\mathbb{R}^k}} \\ \mathrm{T}_P M & \xleftarrow{\mathrm{D}i_P} & \mathrm{T}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \mathrm{T}_Q V & \xrightarrow{\mathrm{D}j_Q} & \mathrm{T}_Q \mathbb{R}^k. \end{array}$$

We make the following observations:

- It follows from (2) that the diagram commutes.
- It follows from the Physical Tangent Space-Inclusion Proposition 29.5 and the Tangent Space-Inclusion Proposition 29.20 that the left and right horizontal maps are isomorphisms.
- Since Φ is a diffeomorphism we obtain from Physical Tangent Space-Functor Proposition 29.3 and the Tangent Space-Functor 29.15 that the middle horizontal maps are isomorphisms.

- By the Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition 29.14 we know that the right vertical map is an isomorphism.

The combination of the above shows that the left vertical map is indeed an isomorphism.

(b) Let M be a k-dimensional submanifold of $\mathbb{R}^m$ and let $P \in \partial M$. Let $v \in \widetilde{\mathrm{T}}_P M$ be a vector that points outward. Recall that this means that $v = \gamma'(0)$ where $\gamma\colon (-\epsilon, 0] \to M$ is a smooth map with $\gamma(0) = P$ but with $\gamma'(0) \notin \widetilde{\mathrm{T}}_P(\partial M)$. Let $g\colon M \to \mathbb{R}_{\geq 0}$ be a smooth map with $g|_{\partial M} \equiv 0$. We obtain that

$$(\Theta_M)(v)(g) \quad = \quad \mathrm{D}g_P(v) \quad = \quad \mathrm{D}g_P(\gamma'(0)) \quad \underset{\uparrow}{=} \quad (g \circ \gamma)'(0) \quad \underset{\uparrow}{\leq} \quad 0.$$

by the Chain Rule 29.3 (2); since $(g\circ\gamma)(t) \geq 0$ for $t \leq 0$ and since $(g \circ \gamma)(0) = 0$

Note that using charts it is not difficult to construct a smooth map $g\colon M \to \mathbb{R}_{\geq 0}$ with $g|_{\partial M} \equiv 0$ and with $(\Theta_M)(v)(g) < 0$. This shows that $\Theta_M(v)$ does indeed point outward. The converse can also be proved by considering charts. We leave it to the reader to fill in the details.

(3) This statement follows immediately from (1) and (3), and the definition of a natural isomorphism, see Definition 5.27. ■

The Tangent Space-Isomorphism Proposition 29.24 allows us to go back and forth between the two flavors of tangent spaces for a smooth submanifold of some $\mathbb{R}^k$. This leads us to the following convention.

Convention 29.25. Given a submanifold M of some $\mathbb{R}^k$ and $P \in M$ we will often, especially in the later chapters, use the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24 to make the identification $\widetilde{\mathrm{T}}_P M =_{\mathrm{i}} \mathrm{T}_P M$.

As a special case we have the following example:

Example. Let M be an open subset M of $\mathbb{R}^n$ or of $\mathbb{H}^n$ and let $P \in M$. The above convention says that at times we make the identification

$$\mathbb{R}^n \quad \underset{\uparrow}{=_{\mathrm{i}}} \quad \widetilde{\mathrm{T}}_P M \quad \underset{\uparrow}{=_{\mathrm{i}}} \quad \mathrm{T}_P M.$$

by the Physical Tangent Spaces–Lemma 29.2; identified via the natural isomorphism provided by the Tangent Space-Isomorphism Proposition 29.24

Going through the definitions one sees easily that this isomorphism $\mathbb{R}^n \to \mathrm{T}_P M$ is precisely the isomorphism provided by the Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition 29.14. □

Definition 29.26. Let $I \subseteq \mathbb{R}$ be an open subset, let M be a smooth manifold and let $f\colon I \to M$ be a smooth map. Given any $t \in I$ we write

$$f'(t) \ := \ \mathrm{D}f_t(\text{tangent vector } 1 \in \mathrm{T}_t\mathbb{R} =_{\mathrm{i}} \widetilde{\mathrm{T}}_t\mathbb{R} = \mathbb{R}) \ \in \ \mathrm{T}_{f(t)}M.$$

Remark. If M is a smooth submanifold of some $\mathbb{R}^n$, then under the above identification $\widetilde{\mathrm{T}}_P M = \mathrm{T}_P M$ one sees that the above definition of $f'(t) \in \mathrm{T}_{f(t)}M = \widetilde{\mathrm{T}}_{f(t)}M \subseteq \mathbb{R}^n$ agrees with Notation 24.8. □

29.5. **Vector fields.** In this section we will study vector fields on smooth manifolds and we will see that often they give rise to self-maps of smooth manifolds. These notions will play a role on many occasions throughout these notes.

Definition 29.27. Let $M \subseteq \mathbb{R}^n$ be a submanifold.

(1) A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for every $P \in M$ we have $v(P) \in \widetilde{\mathrm{T}}_P M$.
(2) A vector field on M is called smooth, if the map $v\colon M \to \mathbb{R}^n$ is smooth.[343]
(3) Given a vector field v on M we say that $x \in M$ is a zero of v if $v(x) = 0$.

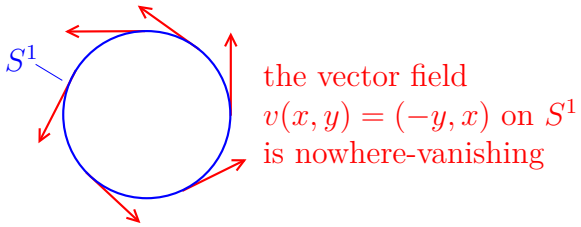
Example. Let $n \in \mathbb{N}$ and let P be a point on S^n. By the Physical Tangent Spaces–Lemma 29.2 we have

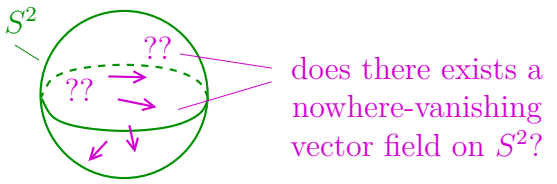
$$\widetilde{\mathrm{T}}_P S^n \;=\; \{v \in \mathbb{R}^{n+1} \mid \langle P, v\rangle = 0\}.$$

For an odd-dimensional sphere S^{2k-1} a smooth vector field is given by[344]

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{array}$$

This vector field has the property that it has no zeros.





In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction". □

The following question arises:

Question 29.28. Does there exist a nowhere-vanishing vector field on even-dimensional spheres?

We will answer this question with the Hairy Ball Theorem 46.18[345] or alternatively in the Euler Characteristic–Vector Field Theorem 182.22.

Now we define the notion of a vector field on any smooth manifold.

Definition 29.29. Let M be a k-dimensional smooth manifold.

(1) A vector field on M is a map

$$v\colon M \;\to\; \bigsqcup_{P \in M} \mathrm{T}_P M$$

such that for each $P \in M$ we have $v(P) \in \mathrm{T}_P M$ and such that the following continuity condition is satisfied: For every chart[346][347] $\Phi\colon U \to V$ from an open subset U of M to

[343] As always we view M as a smooth manifold in its own right and we use the definition of smoothness from Definition 25.15.
[344] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P)\rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\mathrm{T}_P S^{2k-1}$. It follows easily from the Restrict-to-Submanifold-or-Boundary Lemma 25.25 (1) that v is indeed smooth.
[345] We give alternative proofs of the Hairy Ball Theorem on page 1632, page 2882 and on page 3449.

an open subset V of $\mathbb{R}^k$ or of $\mathbb{H}^k$, the following map is continuous:

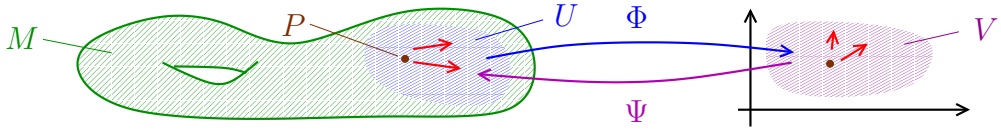
$$\begin{array}{ccl} V & \to & \mathbb{R}^k \\ Q & \mapsto & \big(\Theta_V^{-1} \circ \mathrm{D}\Phi_{\Psi(Q)} \circ \big(\underbrace{\mathrm{D}i_{\Psi(Q)}}_{\substack{\text{isomorphism by the} \\ \text{Tangent Space-Inclusion} \\ \text{Proposition 29.20}}} \big)^{-1} \big(\underbrace{v(\Psi(Q))}_{\in \mathrm{T}_{\Phi^{-1}(Q)}M} \big)\big) \in \widetilde{\mathrm{T}}_Q V = \mathbb{R}^k. \end{array}$$

Here we use the following notation:

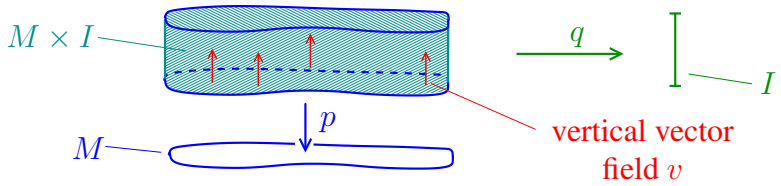
- We denote by $\Psi := \Phi^{-1} \colon V \to U$ the inverse of Φ.
- We denote by $i \colon U \to M$ the inclusion map.
- We denote by $\Theta_V \colon \widetilde{\mathrm{T}}_P V \to \mathrm{T}_P V$ the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24.

(2) We say that the vector field v is smooth if in the definition of (1) the resulting map $V \to \mathbb{R}^k$ is always smooth.

(3) Given a vector field v on M we say that $x \in M$ is a zero of v if $v(x) = 0$.



Example. Let M be a smooth manifold with $\partial M = \varnothing$ and let $I \subseteq \mathbb{R}$ be an interval of length > 0. We denote by $p \colon M \times I \to M$ and $q \colon M \times I \to I$ the two natural projections.



One can easily verify that for each $(x,t) \in M \times I$ there exists a unique tangent vector $v = v(x,t) \in \mathrm{T}_{(x,t)}(M \times I)$ with $\mathrm{D}p_{(x,t)}(v) = 0$ and with $\mathrm{D}q_{(x,t)}(v) = 1 \in \mathbb{R} = \widetilde{\mathrm{T}}_t\mathbb{R} =_{\mathrm{i}} \mathrm{T}_t\mathbb{R}$. Furthermore it is elementary to verify that this map $(x,t) \mapsto v(x,t)$ defines a smooth vector field on $M \times I$. Sometimes we refer to it as the *vertical vector field* on $M \times I$. □

On smooth submanifolds of $\mathbb{R}^n$ we now have a priori two different notions of (smooth) vector fields. The following lemma shows that these two notions are essentially the same.

Lemma 29.30. (Vector Field Definitions–Agree Lemma) Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. Given $P \in M$ we denote by $\Theta_P \colon \widetilde{\mathrm{T}}_P M \to \mathrm{T}_P M$ the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24. Let $v \colon M \to \mathbb{R}^n$ be a continuous map such that for each $P \in M$ we have $v(P) \in \widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$. The following statement holds:

(1) The map v is a vector field on M if and only if $P \mapsto \Theta_P(v(P))$ is a vector field on M.

[346] Recall that for a smooth manifold we mean by a "chart" a chart from the given smooth structure.

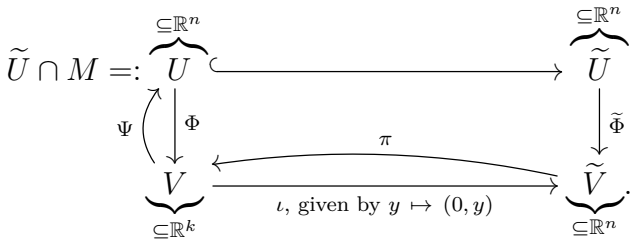
[347] As in the Smooth-via-Atlas Lemma 25.16 one can show fairly easily that it suffices to check the condition in part (1) of the definition only for a smooth atlas and not for all charts.

(2) The map v is a smooth vector field on M if and only if $P \mapsto \Theta_P(v(P))$ is a smooth vector field on M.

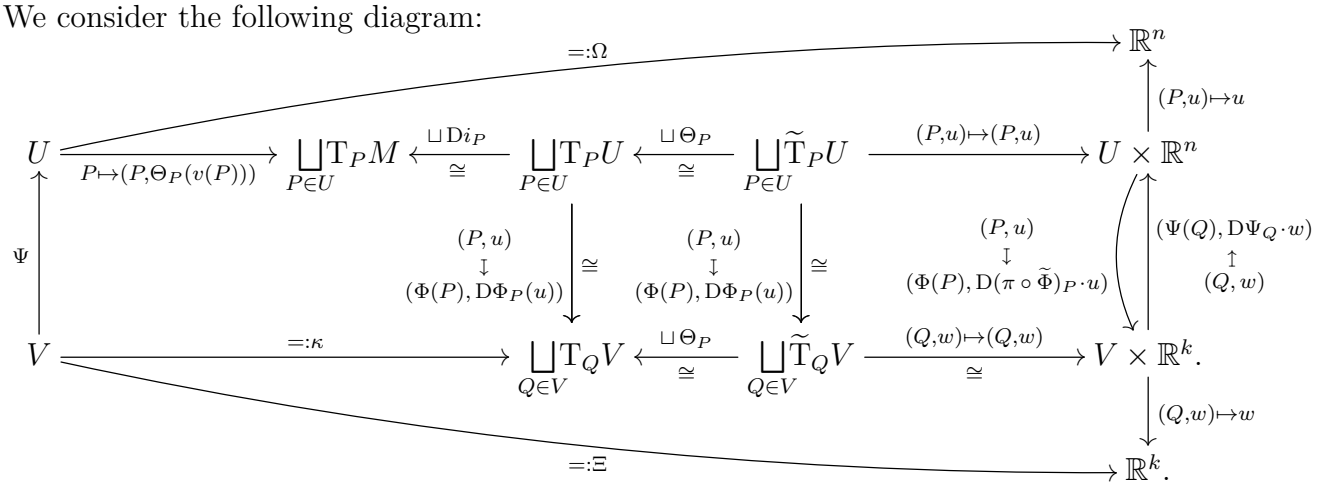
Proof. Let $M \subseteq \mathbb{R}^n$ be a k-dimensional smooth submanifold. In the following we deal with (2), the proof for (1) follows a similar (simplified) logic.

Let $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ be a submanifold chart for M. To simplify the notation we consider the case that $\widetilde{\Phi}$ is a submanifold chart of type (α) in the sense of Definition 25.22. The other cases are treated the same way. We write $U := \widetilde{U} \cap M$ and $V := \{y \in \mathbb{R}^k \mid (0, y) \in \widetilde{V}\}$. We denote by $\iota\colon V \to \widetilde{V}$ the map that is given by $y \mapsto (0, y)$ and we denote by $\pi\colon \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ the projection given by $(x, y) \mapsto y$. We denote by $\Phi\colon U \to V$ the diffeomorphism that is determined by the condition that $\iota \circ \Phi = \widetilde{\Phi}|_U$.

We denote by $\Psi\colon V \to U$ the inverse of Φ and we denote by $i\colon U \to M$ the inclusion. The following diagram summarizes all the maps that are involved so far:



We consider the following diagram:



We make the following clarifications and observations:

- The maps κ, Ω and Ξ are defined in such a way that the left square commutes and the top and bottom triangle commute.
- The middle square commutes by the naturality of the maps Θ.
- The right hand square commutes (in both possibilities) since $\Psi = \Phi^{-1}$ and by the Chain Rule.
- The two right vertical maps connecting $U \times \mathbb{R}^n$ and $V \times \mathbb{R}^k$ are smooth since the maps $P \mapsto \mathrm{D}(\pi \circ \widetilde{\Phi})_P$ and $Q \mapsto \mathrm{D}\Psi_Q$ are smooth.
- The question whether the vector field $P \mapsto \Theta_P(v)$ is smooth in the sense of Definition 29.29 boils down to the question whether $\Xi\colon V \to \mathbb{R}^n$ is smooth.

- Note that the naturality of the Θ maps implies that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{T}}_P U & \xrightarrow{\Theta} & \mathrm{T}_P U \\ {\scriptstyle \mathrm{D}i_P}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{D}i_P} \\ \widetilde{\mathrm{T}}_P M & \xrightarrow{\Theta} & \mathrm{T}_P M. \end{array}$$

  If one goes through the definitions, and one uses the above observation, then one sees that the map Ω is precisely the map $P \mapsto v(P)$. Thus the question whether v is a smooth vector field on M in the sense of Definition 29.27 boils down to the question whether Ξ is smooth.
- It follows easily from the above that Ξ is smooth if and only if Ω is smooth. ■

As a teaser we state the following question:

Question 29.31. Which closed connected 2-dimensional smooth manifolds admit a nowhere-vanishing vector field?

We will address the question in Exercise 29.10. For orientable 2-dimensional smooth manifolds we will prove an answer in the Sphere-Surface–Nowhere Vanishing Vector Field Corollary 53.30 (2). Furthermore we will give a complete answer on page 3450. While we are throwing out questions, let us also pose the following more general, but rather vague question.

Question 29.32. How can we tell whether a closed smooth manifold admits a nowhere-vanishing vector field?

Rather surprisingly we will be able to give a complete and satisfactory answer to this question in the Euler Characteristic–Vector Field Theorem 182.22 and the Nowhere Vanishing Vector Field Existence Theorem 223.7.

For completeness' sake we state the following basic lemma, which will get used on several occasions.

Lemma 29.33. (Vector Field-Pullback Lemma) Let $f\colon M \to N$ be a continuous map between two smooth manifolds and let v be a (smooth) vector field on N. If f is a local diffeomorphism, then for each $x \in M$ we can define $(f^*v)(x) = (\mathrm{D}f_x)^{-1}(v(f(x)) \in \mathrm{T}_x M$. These vectors define a (smooth) vector field f^*v on M. We refer to f^*v as the pullback of v under f.

Proof. The lemma follows easily from the definitions. We leave it to the reader to fill in the details. ■

Example. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ be the usual projection. Given any vector field v on $\mathbb{R}\mathrm{P}^n$ the pullback $p * v$ is a vector field on S^n. The number of zeros of p^*v equals twice the number of zeros of v. □

We will return to the study of vector fields on numerous occasions. They are obviously interesting in their own right. Furthermore, as we will see in the Flow Theorem 43.3, they can also be used to construct interesting self-maps of smooth manifolds.

29.6. **Appendix: Alternative construction of tangent spaces.** In this short appendix we present an alternative approach to answering Question 29.10 (1). To simplify the

discussion we will only work with smooth manifolds with empty boundary. Therefore we introduce the following notation:

Notation 29.34. We write:

$$\begin{aligned} \mathcal{SmMfd}_{\partial=\varnothing} &:= \text{category of smooth manifolds with empty boundary,} \\ \mathcal{PSmMfd}_{\partial=\varnothing} &:= \text{category of pointed smooth manifolds with empty boundary.} \end{aligned}$$

Similarly we define the analogs where we only consider smooth submanifolds with empty boundary of some $\mathbb{R}^n$.

Definition 29.35.

(1) Let M be a smooth manifold and let $P \in M$. Let us consider

$$\widehat{\mathrm{T}}_P M := \{\text{all smooth maps } \gamma\colon (-\epsilon,\epsilon) \to M \text{ with } \gamma(0) = P\}/\sim$$

where we define

$$\alpha \sim \beta \iff \begin{array}{c} \text{there exists a chart } \Phi\colon U \to V \text{ and an } \eta > 0 \text{ with} \\ (\Phi \circ \alpha|_{(-\eta,\eta)})'(0) = (\Phi \circ \beta|_{(-\eta,\eta)})'(0) \in \mathbb{R}^n. \end{array}$$

(2) If $f\colon M \to N$ is a smooth map between two smooth manifolds with empty boundary, then for any $P \in M$ we define

$$\begin{aligned} \mathrm{D}f_P\colon \widehat{\mathrm{T}}_P M &\to \widehat{\mathrm{T}}_{f(P)} N \\ [\alpha] &\mapsto [f \circ \alpha]. \end{aligned}$$

It is not that difficult to show that $\widehat{\mathrm{T}}_P M$ and $\mathrm{D}f_P$ have the same formal properties as the tangent spaces $\mathrm{T}_P M$ and the differentials $\mathrm{D}f_P$, in the sense that the fairly obvious analogs of the Tangent Space-Functor Proposition 29.15, the Tangent Space-Dimension Proposition 29.17 and the Tangent Space-Inclusion Proposition 29.20 hold. In particular we see that the maps

$$\begin{aligned} (M,P) &\mapsto \mathrm{T}_P M \\ (f\colon (M,P) \to (N,Q)) &\mapsto (\mathrm{D}f_P\colon \widehat{\mathrm{T}}_P M \to \widehat{\mathrm{T}}_Q N) \end{aligned}$$

define a covariant functor from the category $\mathcal{PSmMfd}_{\partial=\varnothing}$ of pointed smooth manifolds with empty boundary to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.

Furthermore, similar to the Tangent Space-Isomorphism Proposition 29.24 one can show that the maps

$$\begin{aligned} \Theta_M\colon \widehat{\mathrm{T}}_P M &\to \mathrm{T}_P M \\ [\gamma] &\mapsto \begin{pmatrix} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & (g \circ \gamma)'(0) \end{pmatrix} \end{aligned}$$

define a natural isomorphism.

Thus, at least in the absence of boundary, $\widehat{\mathrm{T}}_P M$ can be used as a replacement of $\mathrm{T}_P M$. One can also develop this theory for smooth manifolds with boundary, but as in the setting of the physical tangent space, this becomes rather messy. We refer to [**Jä93**, Chapter 3.3] and [**Nic2020**, Chapter 2] for details on $\widehat{\mathrm{T}}_P M$. We will not make use of this alternative point of view on tangent spaces of smooth manifolds. Another approach to defining the tangent space is given in [**Shah2016**, Chapter III.4.D].

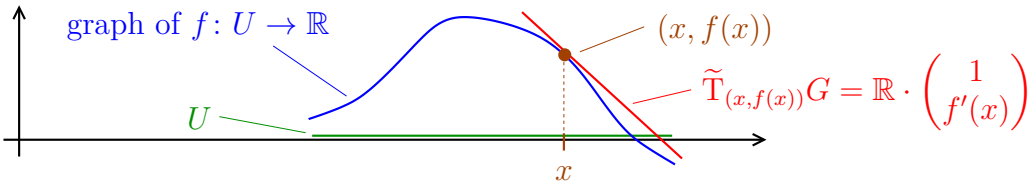
29.7. Exercises for Chapter 29.

Exercise 29.1. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^k$ be a smooth map. On page 681 we discussed that the graph

$$G \;:=\; \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k \,|\, x \in U \text{ and } y = f(x)\}$$

is an n-dimensional smooth submanifold of $\mathbb{R}^{n+k}$. Show that for every $x \in U$ we have

$$\widetilde{\mathrm{T}}_{(x,f(x))}G \;=\; \text{span of the columns of the } (n+k)\times n\text{-matrix } \begin{pmatrix} \mathrm{id}_n \\ \mathrm{D}f_x \end{pmatrix}.$$

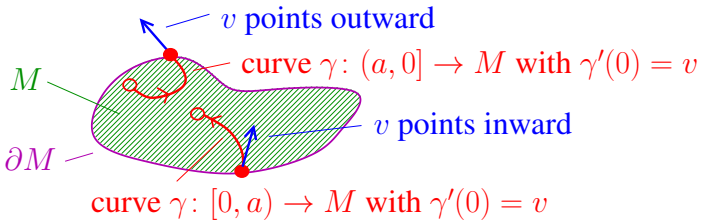


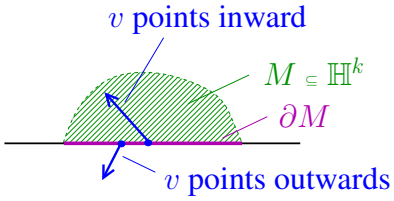
Exercise 29.2. Let M be a smooth manifold, let $P \in \partial M$ and let $v \in \mathrm{T}_P M$.

(a) Let N be a smooth manifold and let $f\colon M \to N$ be a local diffeomorphism with $f(P) \in \partial N$. Show that $v \in \mathrm{T}_P M$ points outward if and only if $\mathrm{D}f_P(v) \in \mathrm{T}_{f(P)}N$ points outward.

(b) Show that precisely one of the following three statements holds:
- (i) v points outward,
- (ii) v points inward,
- (iii) $v \in \mathrm{im}(\mathrm{D}\iota_P\colon \mathrm{T}_P\,\partial M \to \mathrm{T}_P M)$, where $\iota\colon \partial M \to M$ is the inclusion map.

Hint. First show that it suffices to consider the case that the smooth manifold M is an open subset of some $\mathbb{H}^k$.





Exercise 29.3. Let $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ be two closed smooth submanifolds. Show that $M \times N \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is a smooth submanifold and show that for any $x \in M$ and $y \in N$ we have

$$\widetilde{\mathrm{T}}_{(x,y)}(M \times N) \;=\; \widetilde{\mathrm{T}}_x M \;\times\; \widetilde{\mathrm{T}}_y N \;\subseteq\; \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}.$$

Exercise 29.4. Let M be a smooth manifold and let $U \subseteq M$ be an open subset. We denote by $\iota\colon U \to M$ the inclusion. Show that for every $P \in U$ the map $\mathrm{D}\iota_P\colon \mathrm{T}_P U \to \mathrm{T}_P M$ is an isomorphism.
Hint. Use the Derivation Lemma 29.13 (3).

Exercise 29.5. Let $M \subseteq \mathbb{R}^n$ be a k-dimensional smooth submanifold and let $P \in M \setminus \partial M$.

(a) Show that there exists an $A \in \mathrm{O}(n)$ with $A \cdot \widetilde{\mathrm{T}}_P M = \{(0,y) \in \mathbb{R}^n \,|\, y \in \mathbb{R}^k\}$.

(b) Show that there exists a neighborhood U of $P \in M$ such that the map

$$\begin{array}{ccccccc} U & \to & \widetilde{\mathrm{T}}_P M & \xrightarrow{x \mapsto A \cdot x} & \{(x,0) \,|\, x \in \mathbb{R}^k\} & \xrightarrow{(x,y) \mapsto x} & \mathbb{R}^k \\ x & \mapsto & \text{orthogonal projection} & & & & \\ & & \text{of } x - P \text{ onto } \widetilde{\mathrm{T}}_P M & & & & \end{array}$$

is a chart for M.

Exercise 29.6. Let M be a smooth manifold, let $X \subseteq M$ be a proper submanifold, let $P \in \partial X$ and let $w \in \mathrm{T}_P X$. We denote by $\iota \colon X \to M$ the inclusion map. Show that $w \in \mathrm{T}_P X$ points outward if and only if $\mathrm{D}\iota_P(w)$ points outward.

Exercise 29.7. We consider the following submanifold of $\mathbb{R}^3$:

$$M \;:=\; \{(x, y, x^2 + y^2) \,|\, x, y \in \mathbb{R}\}.$$

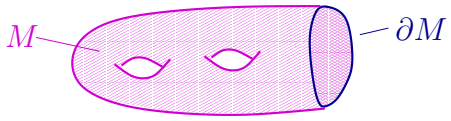
Determine $\widetilde{\mathrm{T}}_{(1,2,5)} M \subseteq \mathbb{R}^3$.

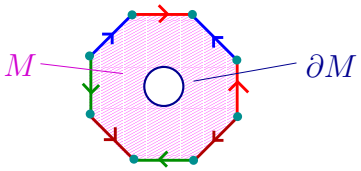
Exercise 29.8. A map $f \colon M \to N$ between smooth manifolds with empty boundary is called a *submersion* if f is smooth and if for every $P \in M$ the differential $\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an epimorphism. Show that for every $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} p \colon \mathbb{R}^{n+1} \setminus \{0\} & \to & \mathbb{R}\mathrm{P}^n \\ (x_0, \ldots, x_n) & \mapsto & [x_0 : \cdots : x_n] \end{array}$$

is a submersion.

Exercise 29.9. We consider the smooth manifold M shown in the figure below. Show that it admits a nowhere vanishing smooth vector field.





Exercise 29.10.

(a) For which closed connected 2-dimensional smooth manifolds can you find a nowhere-vanishing vector field?

(b) For which compact connected 2-dimensional smooth manifolds M can you find a nowhere-vanishing vector field v with the extra property that for each $P \in \partial M$ we have $v(P) \in \mathrm{T}_P(\partial M)$?

Remark. At this stage you cannot give a complete answer to these questions. The goal is to find the largest collection of examples where such vector fields exist.

Exercise 29.11.

(a) Can you find a vector field on S^2 with two zeros?

(b) Can you find a vector field on S^2 with one zero?

Exercise 29.12. Given an m-dimensional smooth submanifold $M \subseteq \mathbb{R}^n$ we consider

$$\mathrm{ST}M \;:=\; \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, x \in M \text{ and } v \in \widetilde{\mathrm{T}}_x M \text{ with } \|v\| = 1\}.$$

This topological space is sometimes called the *unit tangent bundle of* M.

(a) Show that $\mathrm{ST}M$ is a $(2m-1)$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.

(b) Show that $\mathrm{SO}(3)$ is diffeomorphic to $\mathrm{ST}S^2$.

Exercise 29.13. Let $W \subseteq \mathbb{R}^n$ be a smooth submanifold such that $0 \notin W$. We consider the map

$$\begin{aligned} \pi\colon W &\to S^{n-1} \\ P &\mapsto \tfrac{P}{\|P\|}. \end{aligned}$$

(a) Show that for each $P \in W$ the differential is given by

$$\begin{aligned} \mathrm{D}\pi_P(v)\colon \mathrm{T}_P W &\to \mathrm{T}_{\pi(P)} S^{n-1} = (\mathbb{R}\cdot P)^\perp \\ v &\mapsto \tfrac{1}{\|P\|} \cdot \underbrace{(v - \tfrac{P}{\|P\|} \cdot v)}_{\substack{\text{orthogonal projection} \\ \text{onto } (\mathbb{R}\cdot P)^\perp}}. \end{aligned}$$

(b) We now assume that W is of dimension $n-1$. Show that $\mathrm{D}\pi_P$ is an isomorphism if and only if $P \notin \mathrm{T}_P W$.

Exercise 29.14. Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension. Show that $\{P \in M \mid \mathrm{D}f_P \text{ is } \textit{not} \text{ an isomorphism}\}$ is a closed subset of M.
Hint. Use the Leibniz Formula 2.18.

Exercise 29.15. Let $f\colon M \to N$ be a smooth map between smooth manifolds and let $k \in \mathbb{N}_0$. Show that $\{P \in M \mid \mathrm{Rank}(\mathrm{D}f_P) \geq k\}$ is an open subset of M.
Hint. Use the Leibniz Formula 2.18.

Exercise 29.16. Let $n \in \mathbb{N}$. We consider $M = \mathbb{R}^n \cup \{\infty\}$. We denote by 0 the origin of $\mathbb{R}^n$.

(a) We consider the maps $\Phi = \mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ and

$$\begin{aligned} \Psi\colon M \setminus \{0\} &\to \mathbb{R}^n \\ x &\mapsto \begin{cases} \frac{1}{\|x\|^2} \cdot x, & \text{if } x \neq \infty, \\ 0, & \text{if } x = \infty. \end{cases} \end{aligned}$$

Show that Φ and Ψ form a smooth atlas for M.

(b) We consider the map

$$\begin{aligned} f\colon M &\to M \\ x &\mapsto \begin{cases} 3x, & \text{if } x \neq \infty, \\ \infty, & \text{if } x = \infty \end{cases} \end{aligned}$$

Note that $f(0) = 0$ and $f(\infty) = \infty$.

(i) Show that f is smooth.

(ii) Show that $\mathrm{D}f_0 = 3 \cdot \mathrm{id}_{\mathrm{T}_0 M}$ and show that $\mathrm{D}f_\infty = \frac{1}{3} \cdot \mathrm{id}_{\mathrm{T}_\infty M}$.

Exercise 29.17. We consider

$$B \coloneqq \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in S^2, v \in \widetilde{\mathrm{T}}_x(S^2) \text{ with } \|v\| = 1\}.$$

Show that B is diffeomorphic to $\mathbb{R}\mathrm{P}^3$.
Remark. The topological space B is often called the *tangent bundle of* S^2.

CHAPTER 30

The Inverse Mapping Theorem and the Smooth Embedding Theorem

The notion of a tangent space allows us to formulate the Inverse Mapping Theorem 30.1, which can be viewed as a smooth manifold version of the Inverse Function Theorem 24.21. We will also introduce immersions and smooth embeddings between smooth manifolds. This will lead us to the oft-cited Smooth Embedding Theorem 30.15 which says that the image of a smooth embedding is a smooth submanifold.

This chapter is organized as follows:

- In Section 30.1 we will state and prove the Inverse Mapping Theorem 30.1.
- In Section 30.2 we will state and prove the Local Graph Lemma 30.5 which says that every smooth submanifold of $\mathbb{R}^k$ "looks locally like a graph".
- In Section 30.3 we will study immersions and smooth embeddings of a smooth manifold into another smooth manifold.
- In Section 30.4 we will state and prove the Smooth Embedding Theorem
- In Section 30.5 we will study immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$.
- In Section 30.6 we will study immersions among smooth manifolds of the same dimension.
- In Appendix 30.7 we will prove the Smooth Convex-to-Ball Proposition 30.22 which says that every compact convex codimension-zero smooth submanifold of $\mathbb{R}^n$ is diffeomorphic to $\overline{B}^n$.
- In Appendix 30.8 we will study embeddings of smooth manifolds with corner.
- In Section 30.9 we will provide exercises for Chapter 30.

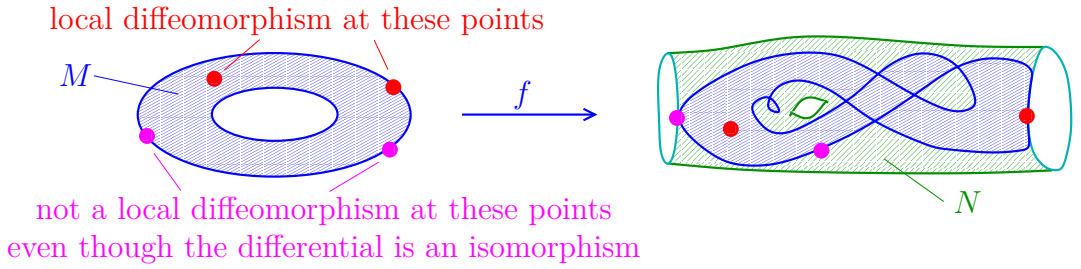
In the next Chapter 31 we will introduce the concept of orientations of smooth manifolds.

30.1. **The Inverse Mapping Theorem.** By the Tangent Space-Inclusion Proposition 29.20 we know that for a local diffeomorphism the differential at any point is an isomorphism. The following theorem gives the much more interesting converse. The theorem gives in particular a convenient criterion for showing that a map between smooth manifolds is a (local) diffeomorphism.

Theorem 30.1. (Inverse Mapping Theorem) Let $f\colon M \to N$ be a smooth map between two smooth manifolds.

(1) Let $P \in M$ such that $\mathrm{D}f_P$ is invertible.
   (a) If $P \notin \partial M$, then f is a local diffeomorphism at P.
   (b) If $P \in \partial M$ and if furthermore there exists an open neighborhood U of P such that $f(U \cap \partial M) \subseteq f(U) \cap \partial N$, then f is a local diffeomorphism at P.

(2) If $\mathrm{D}f_P$ is invertible for every $P \in M$, if f is a bijection and if $f(\partial M) = \partial N$, then f is a diffeomorphism.



Remark. In Exercise 30.1 we will consider the case $P \in \partial M$ and $f(P) \notin \partial N$. □

Examples.

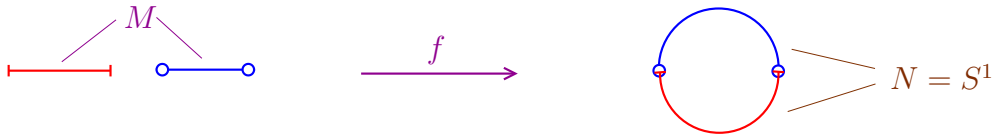
(1) The above figure indicates in particular a map $f\colon M \to N$ and two points $P \in \partial M$, such that $\mathrm{D}f_P$ is an isomorphism, but f is not a local diffeomorphism at P. This shows that in general in the formulation of the Inverse Mapping Theorem 30.1 we do indeed have to put an extra condition on f if $P \in \partial M$.

(2) We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & B^n \\ (x_1, \dots, x_n) & \mapsto & \frac{x_1^2+\dots+x_n^2}{1+x_1^2+\dots+x_n^2} \cdot (x_1, \dots, x_n). \end{array}$$

A slightly heroic calculation shows that f satisfies the conditions of the Inverse Mapping Theorem 30.1 (2), thus we see that f is a diffeomorphism. This is actually quite interesting since at first glance it is not totally apparent why $f^{-1}\colon B^n \to \mathbb{R}^n$ is smooth at the origin.

(3) We consider the smooth map $f\colon M \to N$ that is illustrated in the figure below.



This map is a bijection and for every $x \in M$ the differential $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_{f(x)}N$ is an isomorphism. But evidently f is not a diffeomorphism. This discussion shows that we cannot drop the condition $f(\partial M) = \partial N$ in the statement of the Inverse Mapping Theorem 30.1 (2). □

Evidently the proof of the Inverse Mapping Theorem 30.1 rests on the shoulders of the Inverse Function Theorem 24.21, which we now recall for the reader's convenience.

Theorem 24.21. (Inverse Function Theorem) Let $U \subseteq \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}^n$ be a smooth map and let $P \in U$ be a point. If $\mathrm{D}f_P \in \mathrm{M}(n \times n, \mathbb{R})$ is invertible, then f is a local diffeomorphism at P.

For the purpose of this section it is convenient to introduce the following notation that is modelled on the notation introduced in Notation 4.21.

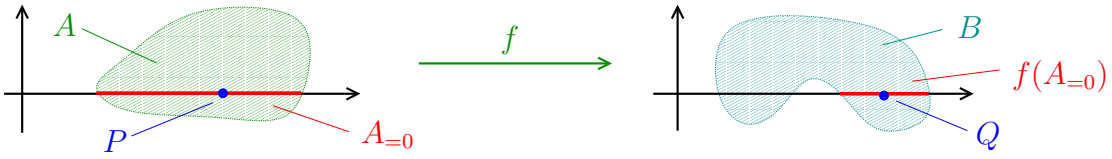
Notation 30.2. Given a subset A of $\mathbb{R}^n$ we write

$$A_{>0} \;:=\; A \cap \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

We define $A_{<0}$, $A_{=0}$ and $A_{\neq 0}$ in the analogous way.

The proof of the Inverse Mapping Theorem 30.1 for the case $P \in \partial M$ rests on the following lemma, which might actually be interesting in its own right.

Auxiliary Lemma 30.3. Let A and B be open subsets of $\mathbb{R}^n$ and let $f\colon A \to B$ be a diffeomorphism. If $f(A_{=0}) \subseteq B_{=0}$, then $f(A_{=0})$ is an open subset of $B_{=0}$.



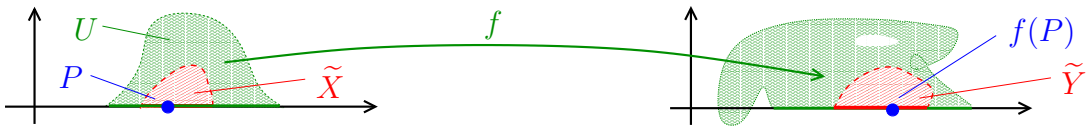
Proof. Let $Q \in f(A_{=0})$. We pick $P \in A_{=0}$ with $Q = f(P)$. We denote by $i\colon A_{=0} \to A$ the inclusion map and we denote by $j\colon B_{=0} \to B$ the inclusion map. Finally we denote by $f_{=0}\colon A_{=0} \to B_{=0}$ the restriction of f to $A_{=0}$. Note that $f \circ i = j \circ f_{=0}\colon A_{=0} \to B$. It follows from the Chain Rule that

$$\mathrm{D}f_P \circ \mathrm{D}i_P \;=\; \mathrm{D}j_Q \circ \mathrm{D}(f_{=0})_P\colon \widetilde{\mathrm{T}}_P A_{=0} \to \widetilde{\mathrm{T}}_Q B = \mathbb{R}^n.$$

Since f is a diffeomorphism we know that $\mathrm{D}f_P$ is an isomorphism. Clearly $\mathrm{D}i_P$ has rank $n-1$. It follows from the above that the map to the right also has rank $n-1$. But this is only possible if $\mathrm{D}(f_{=0})_P$ has rank $n-1$. This shows that $\mathrm{D}(f_{=0})_P\colon \widetilde{\mathrm{T}}_P A_{=0} \to \widetilde{\mathrm{T}}_Q B_{=0}$ is an isomorphism. It follows from the Inverse Function Theorem 24.21 that the image of $A_{=0}$ contains an open neighborhood of $Q \in B_{=0}$. It follows from the JH-Openness Criterion 3.3 that $f(A_{=0})$ is an open subset of $B_{=0}$. ∎

Using Lemma 30.3 we can now prove the following variation on the Inverse Function Theorem 24.21.

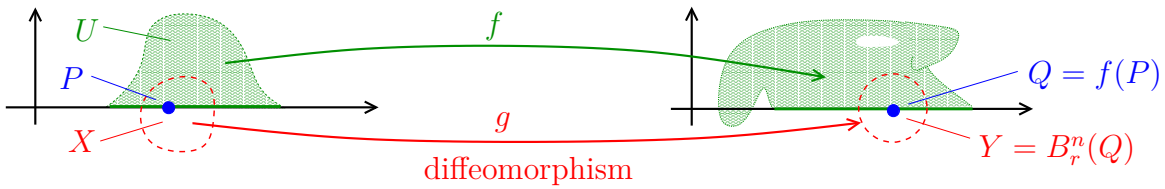
Lemma 30.4. (Inverse Function-with-Boundary Lemma) Let $U \subseteq \mathbb{H}^n$ be an open subset and let $f\colon U \to \mathbb{R}^n$ be a smooth map with $f(U \cap \partial\,\mathbb{H}^n) \subseteq \partial\,\mathbb{H}^n$. Furthermore let $P \in U \cap \partial\,\mathbb{H}^n$ be a point. If $\mathrm{D}f_P$ is an isomorphism and if $\mathrm{D}f_P(\mathbb{H}^n) \subseteq \mathbb{H}^n$, then f is a local diffeomorphism at P in the sense that there exists an open neighborhood $\widetilde{X}$ of $P \in U$ and an open neighborhood $\widetilde{Y}$ of $f(P) \in \mathbb{H}^n$ such that the map $f\colon \widetilde{X} \to \widetilde{Y}$ is a diffeomorphism between the two smooth manifolds $\widetilde{X}$ and $\widetilde{Y}$.



Proof. Note that by definition of a smooth map on an open subset of $\mathbb{H}^n$, see Definition 24.3, there exists an open neighborhood X of $P \in \mathbb{R}^n$ and a smooth map $g\colon X \to \mathbb{R}^n$ which agrees with f on $U \cap X$.

We set $Q := f(P)$. It follows easily from the Inverse Function Theorem 24.21 that, after possibly replacing X by a smaller open neighborhood of P, we can assume that there exists an open ball $Y := B^n_r(Q) \subseteq \mathbb{R}^n$ such that $g\colon X \to Y = B^n_r(Q)$ is a diffeomorphism. Next recall that given a subset A of $\mathbb{R}^n$ we write

$$A_{>0} \;:=\; A \cap \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\}.$$



and we define $A_{<0}$, $A_{=0}$ and $A_{\neq 0}$ in the analogous way. With this notation it remains to prove the following claim:

Claim. We have $f(X_{\geq 0}) = Y_{\geq 0}$ and the map $f = g\colon X_{\geq 0} \to Y_{\geq 0}$ is a diffeomorphism.

Proof. We already know that $g\colon X \to Y$ is a diffeomorphism. Thus it remains to prove that $g(X_{\geq 0}) = Y_{\geq 0}$.

First note that by hypothesis we have $g(X_{=0}) \subseteq Y_{=0}$. It follows from Lemma 30.3 that $g(X_{=0})$ is an open subset of $Y_{=0}$. Furthermore note that $X_{=0}$ is a closed subset of X, thus $g(X_{=0})$ is a closed subset of Y. But since $g(X_{=0})$ is contained in $Y_{=0}$ it follows from the elementary Subspace-Characterization Lemma 3.17 that $g(X_{=0})$ is a closed subset of $Y_{=0}$. In summary we see that $g(X_{=0})$ is a subset of $Y_{=0}$ that is open and closed. Since $Y_{=0} \cong B_r^{n-1}(Q)$ is connected and since $P \in X_{=0}$ we see that $g(X_{=0}) = Y_{=0}$.

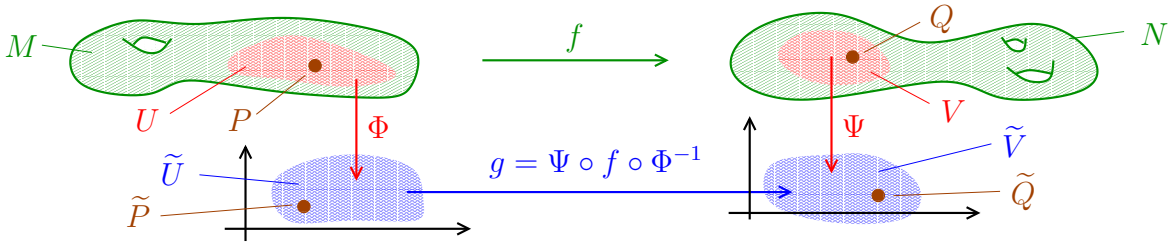
Since $g\colon X \to Y$ is a homeomorphism and since we have $g(X_{=0}) = Y_{=0}$ we see that $g\colon X_{\neq 0} \to Y_{\neq 0}$ is also a homeomorphism. Note that $Y_{\neq 0}$ has precisely two components, namely $Y_{>0}$ and $Y_{<0}$. Thus $X_{\neq 0}$ also has precisely two components. Since $X_{>0}$ and $X_{<0}$ are open, closed and non-empty it follows from the elementary Component Lemma 4.48 that the two components of $X_{\neq 0}$ are precisely $X_{>0}$ and $X_{<0}$. It follows from the fact that g gives a bijection of the components and from our hypothesis $\mathrm{D}f_P(\mathbb{H}^n) \subseteq \mathbb{H}^n$ that $g^{-1}(Y_{<0}) \subseteq X_{<0}$ and $g^{-1}(Y_{>0}) \subseteq X_{>0}$. ∎

Proof of the Inverse Mapping Theorem 30.1.

(1) Let $f\colon M \to N$ be a smooth map between smooth manifolds. We set $m := \dim(M)$ and $n := \dim(N)$.

(a) Let $P \in M \setminus \partial M$ be a point. We write $Q := f(P)$. In the following we assume that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_Q N$ is invertible. We pick a chart $\Phi\colon U \to \widetilde{U}$ of type (i) around P and we pick a chart $\Psi\colon V \to \widetilde{V}$ around $f(P)$.[348] We write $\widetilde{P} := \Phi(P)$ and $\widetilde{Q} := \Psi(Q)$.

After possibly replacing U by $U \cap f^{-1}(V)$ we can assume that $f(U) \subseteq V$. We consider the map $g := \Psi \circ f \circ \Phi^{-1}\colon \widetilde{U} \to \widetilde{V} \subseteq \mathbb{R}^n$.



[348] Note that a priori we do not know yet whether $Q = f(P) \in \partial N$, so Ψ could be a chart of type (i) or of type (ii).

Claim. The differential $\mathrm{D}g_{\widetilde{P}} \in \mathrm{M}(m \times n, \mathbb{R})$ is an invertible matrix.

Proof. Let $i\colon U \to M$ and $j\colon V \to N$ be the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{T}_P M & \xleftarrow{\mathrm{D}i_P} & \mathrm{T}_P U & \xrightarrow{\mathrm{D}\Phi_P} & \mathrm{T}_{\widetilde{P}}\widetilde{U} & \xrightarrow{\Theta_{\widetilde{U}}} & \widetilde{\mathrm{T}}_{\widetilde{P}}\widetilde{U} & \xrightarrow{\mathrm{id}} & \mathbb{R}^m \\
\downarrow{\scriptstyle \mathrm{D}f_P} & & \downarrow{\scriptstyle \mathrm{D}f_P} & & \downarrow{\scriptstyle \mathrm{D}g_{\widetilde{P}}} & & \downarrow{\scriptstyle \mathrm{D}g_{\widetilde{P}}} & & \downarrow{\scriptstyle \mathrm{D}g_{\widetilde{P}}} \\
\mathrm{T}_Q N & \xleftarrow{\mathrm{D}j_Q} & \mathrm{T}_Q V & \xrightarrow{\mathrm{D}\Psi_Q} & \mathrm{T}_{\widetilde{Q}}\widetilde{V} & \xrightarrow{\Theta_{\widetilde{V}}} & \widetilde{\mathrm{T}}_{\widetilde{Q}}\widetilde{V} & \xrightarrow{\mathrm{id}} & \mathbb{R}^n.
\end{array}$$

We make the following observations:

- It follows from the Inclusion Differential Lemma 29.18 that the left horizontal maps are isomorphisms.
- Since Φ and Ψ are diffeomorphisms we obtain from the Tangent Space-Functor Proposition 29.15 that the center-left horizontal maps are invertible.
- It follows from the Tangent Space-Functor Proposition 29.15 that the two rectangles to the left commute.
- The horizontal maps $\Theta_{\widetilde{U}}$ and $\Theta_{\widetilde{V}}$ are the natural isomorphisms from the Tangent Space-Isomorphism Proposition 29.24. In particular the third rectangle commutes.
- By the Differential-equals-Differential Lemma 29.6 we know that the rectangle to the right commutes.

The combination of all of the above, together with our hypothesis that $\mathrm{D}f_P$ is invertible implies that $\mathrm{D}g_{\widetilde{P}}$ is indeed an invertible matrix. ⊞

It follows from the claim and the Inverse Function Theorem 24.21 that there exists an open neighborhood Y of $\widetilde{P} \in \widetilde{U}$ and an open neighborhood Z of $\widetilde{Q} \in \widetilde{V}$ such that g restricts to a diffeomorphism $Y \to Z$. It follows that $f\colon \Phi^{-1}(Y) \to \Psi^{-1}(Z)$ is a diffeomorphism. We have thus shown that f is a local diffeomorphism at P.

(b) Now we assume that $P \in \partial M$ and we assume that there exists an open neighborhood U of P such that $f(U \cap \partial M) \subseteq f(U) \cap \partial N$, then f is a local diffeomorphism at P. The proof of this statement is almost identical to the proof of (a). We just need to replace the Inverse Function Theorem 24.21 by the Inverse Function-with-Boundary Lemma 30.4. (The condition that $\mathrm{D}g_P(\mathbb{H}^n) \subseteq \mathbb{H}^n$ is automatically satisfied since $g(P) \in \partial\mathbb{H}^n$ and $g(\widetilde{U}) \subseteq \mathbb{H}^n$.) We leave it to the reader to fill in the details.

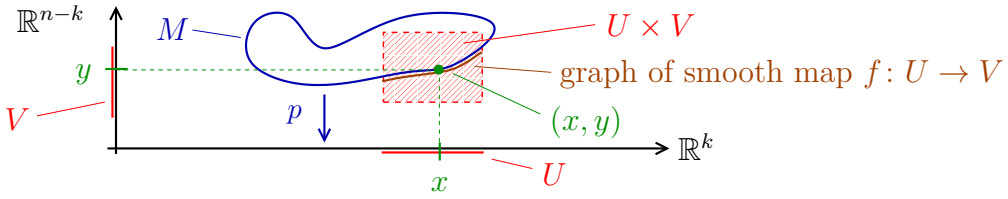
(2) Statement (2) follows immediately from (1) and the Smooth Pasting Proposition 25.19 (2). ■

30.2. **The Local Graph Lemma.** On page 681 we saw that graphs of smooth maps between smooth manifolds give rise to submanifolds. Using the Inverse Mapping Theorem 30.1 we can now show that locally, after possibly a rotation, any submanifold of $\mathbb{R}^n$ with empty boundary is a graph.

Lemma 30.5. (Local Graph Lemma) Let $M \subseteq \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ be a k-dimensional smooth submanifold. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first k coordinates. Let $(x, y) \in M \setminus \partial M$ such that $\mathrm{D}p_{(x,y)}\colon \widetilde{\mathrm{T}}_{(x,y)}M \to \mathbb{R}^k$ is an isomorphism. There exists an

open neighborhood U of $x \in \mathbb{R}^k$, an open neighborhood V of $y \in \mathbb{R}^{n-k}$ and a smooth map $f\colon U \to V$ such that

$$M \cap (U \times V) = \text{ Graph of } f\colon U \to V \ = \ \{(z, f(z)) \,|\, z \in U\}.$$



Remark.

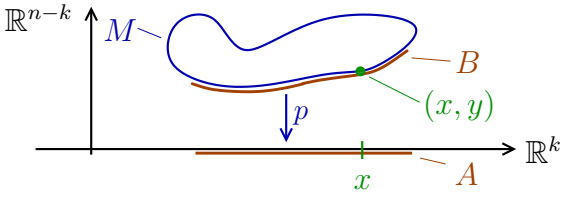
(1) Let $M \subseteq \mathbb{R}^n$ be a k-dimensional smooth submanifold and let $z \in M \setminus \partial M$. By the Orthogonal Action Lemma 2.37 (5) there exists an orthogonal matrix $A \in \mathrm{SO}(n)$ with $A \cdot \widetilde{\mathrm{T}}_z M = \mathbb{R}^k \times \{0\}$. It follows that, after a rotation, one can arrange that the hypotheses of the Local Graph Lemma 30.5 holds. For readers who are already familiar with the concept of a diffeotopy (which we will define in Definition 41.3) we point out that in the Diffeotopies-via-Matrices Lemma 41.12 we will see that a rotation can be realized by a diffeotopy of $\mathbb{R}^n$.

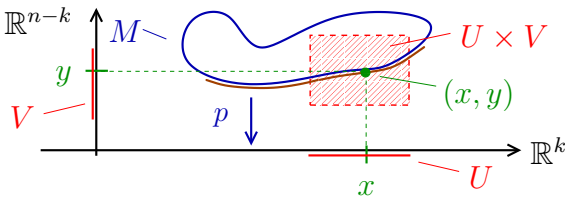
(2) Later in the Diffeomorphic-to-Euclidean Submanifold Corollary 33.3 we will see that every smooth manifold is diffeomorphic to a smooth submanifold of some $\mathbb{R}^d$. Thus we see that locally every smooth manifold with empty boundary is diffeomorphic to a graph. □

Proof. We view M as a smooth manifold in its own right. Note that by hypothesis the map $\mathrm{D}p_{(x,y)}\colon \widetilde{\mathrm{T}}_{(x,y)} M \to \widetilde{\mathrm{T}}_x \mathbb{R}^k = \mathbb{R}^k$ is an isomorphism. It follows from the Inverse Mapping Theorem 30.1 that there exists an open neighborhood A of $(x, y) \in M$ and an open neighborhood B of $x \in \mathbb{R}^k$ such that $p\colon A \to B$ is a diffeomorphism. Note that the inverse of $p\colon A \to B$ is necessarily of the form

$$\begin{array}{ccc} B & \to & A \\ z & \mapsto & (z, f(z)) \end{array}$$

where $f\colon B \to \mathbb{R}^{n-k}$ is a smooth map.





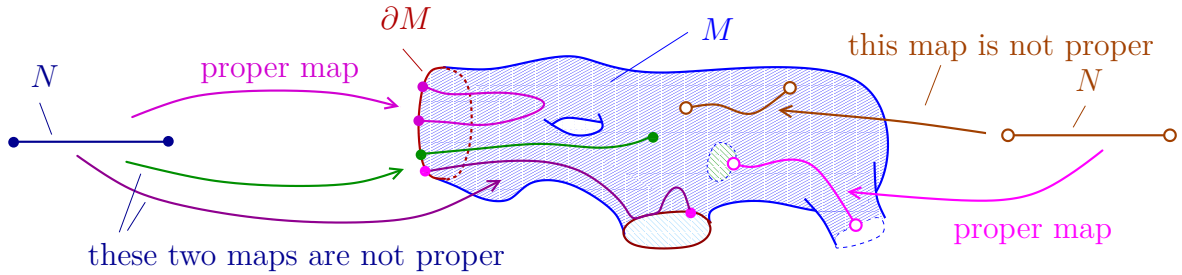
By the definition of the subspace topology on M and by the definition of the topology on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ there exists an open neighborhood U of x and an open neighborhood V of y such that $(U \times V) \cap M \subseteq A$. It is now clear that U, V and f have the desired properties. ■

30.3. **Immersions and smooth embeddings.** For the reader's convenience we recall Definition 4.20:

Definition 30.6. We say that a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$ is a homeomorphism, where $f(X)$ is equipped with the subspace topology.

Next we state a new definition.

Definition 30.7. Let N and M be topological manifolds. A continuous map $\varphi\colon N \to M$ is called proper if the following two conditions are satisfied:
(1) $\varphi^{-1}(\partial M) = \partial N$,
(2) $\varphi(N)$ is a closed subset of M.



For an *embedding* of topological manifolds the following, rather technical, lemma gives an alternative characterization of the second condition of a proper map between topological manifolds.

Lemma 30.8. (Manifold–Proper Map Lemma) Let M and N be topological manifolds and let $\varphi\colon N \to M$ be a continuous map. If φ is in fact an embedding, then the following two conditions are equivalent:[349]
(1) $\varphi(N)$ is a closed subset of M.
(2) For every compact subset K of M the preimage $\varphi^{-1}(K)$ is compact.

Remark. In Definition 4.63 we introduced the definition that a map $f\colon X \to Y$ between topological spaces is called *proper* if the preimage of every compact subset of Y is a compact subset of X. Unfortunately we now have two notions of the adjective "proper".

Let us start with the good news. Let M and N be two topological manifolds and let $\varphi\colon N \to M$ be a continuous map. If φ is an embedding, then the Manifold–Proper Map Lemma 30.8 says that the following two statements are equivalent:

(1) φ satisfies the second condition of a proper map between topological manifolds.
(2) $\varphi\colon N \to M$ is a proper map of topological spaces.

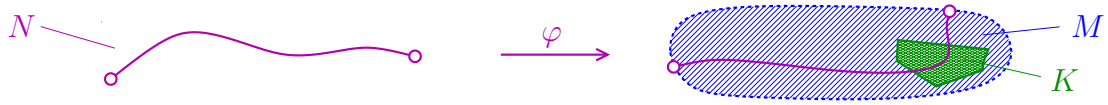
We continue with the bad news. In general the two notions of being "proper" diverge:

(1) The constant map $\mathbb{R} \to \{*\}$ is a proper map of topological manifolds but it is not a proper map of topological spaces.
(2) The inclusion map $f\colon N = [-1,1] \to \mathbb{R} = M$ is a proper map of topological spaces but it is not a proper map of topological manifolds.

[349]By the Manifold–Local Properties Lemma 23.8 we know that topological manifolds are regionally compact. Thus the content of the Manifold–Proper Map Lemma 30.8 is closely related to the Proper-Closed Proposition 4.64.

Since both definitions of "proper" are commonly used in the literature we will have to live with this very unfortunate ambiguity. We will strive to make sure that, given a continuous map between two topological manifolds, it is clear which of the two definitions of "proper" we are working with. □

Proof. First we prove the "(1)⇒(2)"-implication. Thus we assume that $\varphi(N)$ is a closed subset of M. Let K be a compact subset of M.



Since $\varphi(N)$ is a closed subset of M we know by the Subspace-Characterization Lemma 3.17 that $\varphi(N) \cap K$ is a closed subset of K. Thus we obtain from the Compact-Closed Lemma 3.39 that $\varphi(N) \cap K$ is compact. But $\varphi\colon N \to M$ is an embedding, i.e. $\varphi\colon N \to \varphi(N)$ is a homeomorphism. We denote by $\psi\colon \varphi(N) \to N$ the inverse map, which is a homeomorphism. Therefore we now see that $\varphi^{-1}(K) = \varphi^{-1}(\varphi(N) \cap K) = \psi(\varphi(N) \cap K)$ is compact.

Now we prove the "(2)⇒(1)"-implication. We need to show that $\varphi(N)$ is a closed subset of M. By the Atlas Improvement Proposition 27.5 (5) there exists a locally finite cover $\{K_j\}_{j\in J}$ of M by compact subsets. Since M is Hausdorff we know that each K_j is also a closed subset of M. Thus by the Cover-Open Subset Lemma 3.20 it suffices to show that each $\varphi(N) \cap K_j$ is a closed subset of M. Now let $j \in J$. By hypothesis we know that $\varphi^{-1}(K_j)$ is compact. Note that $\varphi(N) \cap K_j = \varphi(\varphi^{-1}(K_j))$. It follows from this observation, together with the Compact Image Lemma 4.15 (1) and the Compact-Closed Lemma 3.39 (2) that $\varphi(N) \cap K_j$ is indeed closed. ■

We continue with the following definition that is specific to smooth manifolds.

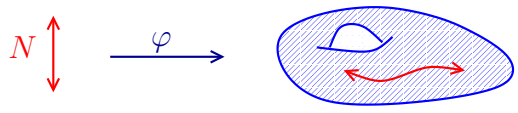
Definition 30.9. A map $\varphi\colon N \to M$ between two smooth manifolds is called an immersion if the following conditions are satisfied:

(1) The map φ is smooth.
(2) For each $P \in N$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism.
(3) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
  (a) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.[350]
  (b) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.[351 352]
  (c) For every component C of ∂N we have either $\varphi(C) \subseteq \partial M$ or $\varphi(C) \subseteq M \setminus \partial M$.[353]

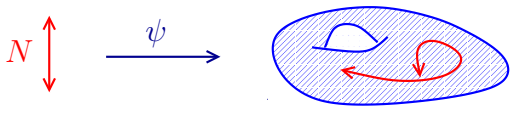
An immersion that also satisfies the following extra condition

(4) $\varphi\colon N \to M$ is an embedding,[354]

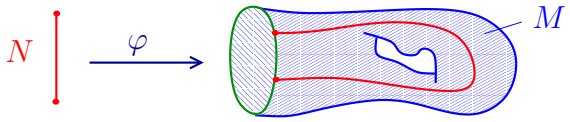
is called a smooth embedding.[355]



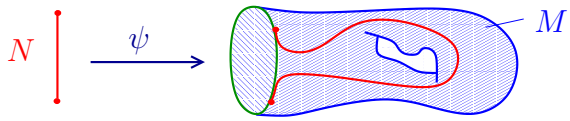
φ is a smooth embedding



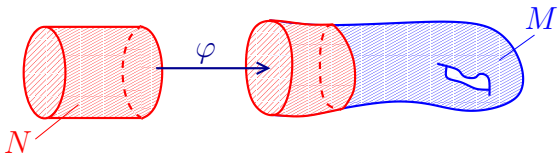
ψ is an immersion, but
ψ is not a smooth embedding since
$\varphi\colon N \to \varphi(N)$ is not a homeomorphism



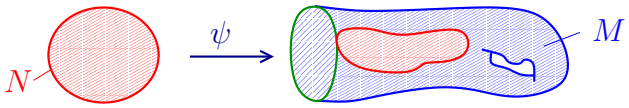
φ is a smooth embedding



ψ is not a smooth embedding since
condition (3b) is not satisfied



φ is a smooth embedding



ψ is not a smooth embedding since
condition (3c) is not satisfied

The following lemma gives an instructive example of immersions and smooth embeddings.

Lemma 30.10. (Torus–Curve Embedding Lemma) Given $\lambda \in \mathbb{R}$ we consider the following map

$$\begin{array}{rcl} \varphi_\lambda\colon \mathbb{R} & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ t & \mapsto & [(t, \lambda \cdot t)]. \end{array}$$

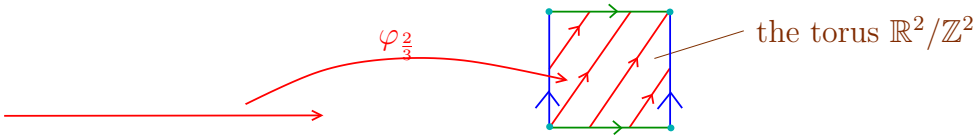
The following statements hold:

(1) The map φ_λ is an immersion.
(2) The map φ_λ is injective if and only if λ is irrational.
(3) If λ is irrational, then $\varphi_\lambda\colon \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ is *not* an embedding, in particular φ_λ is not a smooth embedding.
(4) Suppose that λ is rational. We write $\lambda = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q are coprime. The map

$$\begin{array}{rcl} f_\lambda\colon S^1 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ \exp(2\pi\,\mathrm{i}t) & \mapsto & [(qt, \lambda \cdot qt)] \end{array}$$

[350]Note that this condition is implied if $\varphi^{-1}(\partial M) = \partial N$, but the condition (3) is weaker than the condition that $\varphi^{-1}(\partial M) = \partial N$.
[351]Here we use the sanity preserving convention from page 758 which allows us to view $\mathrm{T}_{\varphi(P)}\partial M$ as a vector subspace of $\mathrm{T}_{\varphi(P)}M$.
[352]Another way of expressing this condition is that the map $\mathrm{T}_P N \to \mathrm{T}_{\varphi(P)}M/\mathrm{T}_{\varphi(P)}\partial M$ is an epimorphism.
[352]One can also state this condition as a local condition, namely using the fact that boundaries are closed subsets – see the Smooth Manifold–Boundary Proposition 25.27 (2) – one can easily show that this condition is equivalent to
(c′) For every $P \in \partial N$ with $\varphi(P) \in \partial M$, there exists a neighborhood V of P in ∂N such that $\varphi(V) \subseteq \partial M$.
[353]If N is compact, then it follows from the Compact-Hausdorff Proposition 4.23 (2) that it suffices to verify that φ is injective and continuous.
[354]The name "smooth embedding" is standard, but it is also rather unfortunate, since it sounds like an "embedding" that is a smooth map. But, as we will see in Exercise 30.5, our notion of a smooth embedding is much stricter.

is well-defined and it is a smooth embedding.



Proof. We write $\varphi = \varphi_\lambda$.

(1) We denote by $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ the projection and we denote by $\Phi\colon \mathbb{R} \to \mathbb{R}^2$ the map that is given by $t \mapsto (t, \lambda \cdot t)$. Evidently we have $\varphi = p \circ \Phi$. Let $t \in \mathbb{R}$. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{T}_t\mathbb{R} & & \xrightarrow{\mathrm{D}\Phi_t} & & \mathrm{T}_{\Phi(t)}\mathbb{R}^2 \xrightarrow{\mathrm{D}p_{\Phi(t)}} \mathrm{T}_{\varphi(t)}(\mathbb{R}^2/\mathbb{Z}^2) \\
\Theta \uparrow \cong & & & & \cong \uparrow \Theta \\
\mathbb{R} = \widetilde{\mathrm{T}}_t\mathbb{R} & & \xrightarrow{\mathrm{D}\Phi_t} & & \widetilde{\mathrm{T}}_{\Phi(t)}\,\mathbb{R}^2 = \mathbb{R}^2
\end{array}$$

(with $\mathrm{D}\varphi_t\colon \mathrm{T}_t\mathbb{R} \to \mathrm{T}_{\varphi(t)}(\mathbb{R}^2/\mathbb{Z}^2)$ the composite across the top)

We make the following observations:

(a) It follows from $\varphi = p \circ \Phi$ and the functoriality of differentials that the top of the diagram commutes.

(b) By the Manifold–Quotient Proposition 25.39 we know that the map $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism, hence the differential of p at any point is an isomorphism.

(c) The vertical maps are the natural isomorphisms provided by the Tangent Space-Isomorphism Proposition 29.24. In particular the square commutes.

(d) By the Differential-equals-Differential Lemma 29.6 the bottom horizontal map is given by multiplication by the matrix $\mathrm{D}\Phi_t = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$. In particular the bottom map is a monomorphism.

It follows from all of the above that the $\mathrm{D}\varphi_t\colon \mathrm{T}_t\mathbb{R} \to \mathrm{T}_{\varphi(t)}(\mathbb{R}^2/\mathbb{Z}^2)$ is a monomorphism.

(2) This statement is an amusing exercise that is left to the reader.

(3) The proof of this statement requires the simple number theoretic fact that given any $\alpha \in \mathbb{R}$ and any $N \in \mathbb{N}$ there exist $n, m \in \mathbb{Z}$ with $1 \leq n \leq N$ and with $|n\alpha - m| < \frac{1}{N}$.[355] We leave it to the reader to fill in the remaining details.

(4) Let λ be rational. We write $\lambda = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q are coprime. We consider the following diagram:

$$\begin{array}{ccc}
S^1 & \xrightarrow{f_\lambda} & \mathbb{R}^2/\mathbb{Z}^2 \\
\uparrow{\scriptstyle g([t])=\exp(2\pi\,\mathrm{i}\,t)} & & \uparrow{\scriptstyle \varphi_\lambda} \\
\mathbb{R}/\mathbb{Z} \xleftarrow[[t]\leftarrow t]{=:\pi} \mathbb{R} & \xrightarrow[t\mapsto q\cdot t]{=:h} & \mathbb{R}
\end{array}
\qquad \text{which gives rise to} \qquad
\begin{array}{ccc}
\mathrm{T}_{\exp(2\pi\,\mathrm{i}\,t)}S^1 & \xrightarrow{\mathrm{D}(f_\lambda)_{\exp(2\pi\,\mathrm{i}\,t)}} & \mathrm{T}_{\varphi(q\cdot t)}\mathbb{R}^2/\mathbb{Z}^2 \\
\cong \uparrow{\scriptstyle \mathrm{D}g_{[t]}} & & \uparrow{\scriptstyle \mathrm{D}(\varphi_\lambda)_t} \\
\mathrm{T}_{[t]}\mathbb{R}/\mathbb{Z} \xleftarrow[\cong]{\mathrm{D}\pi_t} \mathrm{T}_t\mathbb{R} & \xrightarrow[\cong]{\mathrm{D}h_t} & \mathrm{T}_{q\cdot t}\mathbb{R}.
\end{array}$$

Considering the commutative diagram on the right we see that f_λ is indeed an immersion. Furthermore it is straightforward to see that $f_\lambda\colon S^1 \to \mathbb{R}^2/\mathbb{Z}^2$ is injective. Thus it follows from the Compact-Hausdorff Proposition 4.23 (3) that f_λ is a smooth embedding. ∎

[355] This statement is known as Dirichlet's Approximation Theorem, but it is actually elementary to prove, see e.g. [**Lee2002**, Lemma 4.21].

The next lemma gives two interesting families of smooth embeddings.

Lemma 30.11. (Projective Space–Smooth Embedding Lemma) Let $m \leq n$. The maps

$$\begin{array}{rcl} \iota\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [(x,0)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \iota\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [x] & \mapsto & [(x,0)] \end{array}$$

are smooth embeddings.[356]

Proof. On page we showed these these maps are embeddings. It remains to show that these maps are immersions. Let us first consider the real projective spaces $\mathbb{R}\mathrm{P}^n$. In Exercise 25.10 we gave an explicit diffeomorphism $S^n/\{\pm 1\} \to \mathbb{R}\mathrm{P}^n$. It follows from this diffeomorphism together with the Manifold–Quotient Proposition 25.39 (2d) that $\iota(\mathbb{R}\mathrm{P}^m)$ is in fact a submanifold of the smooth manifold $\mathbb{R}\mathrm{P}^n$. The case of complex projective spaces is a little more complex, we will deal with it in Exercise 30.4. ■

We continue our discussion of smooth embeddings with the following modest lemma.

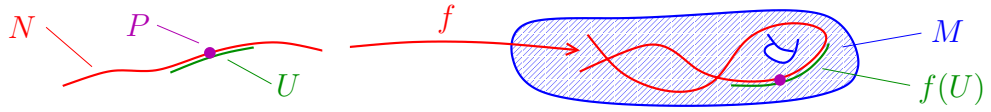
Lemma 30.12. (Restrict Smooth Embedding Lemma) Let $\varphi\colon N \to M$ be a continuous map between smooth manifolds.

(1) If φ is a smooth embedding, then $\varphi|_{\partial N}\colon \partial N \to M$ is also a smooth embedding.
(2) If φ is a proper smooth embedding, then $\varphi|_{\partial N}\colon \partial N \to \partial M$ is also a proper smooth embedding.

Proof. The statement follows almost immediately from the Tangent Space-Inclusion Proposition 29.20 (2) and the Smooth Manifold–Boundary Proposition 25.27 (2a). ■

We conclude this section with the Local Smooth Embedding Theorem which can be viewed as a generalization of the Inverse Mapping Theorem 30.1.

Theorem 30.13. (Local Smooth Embedding Theorem) Let $f\colon N \to M$ be a smooth map between two smooth manifolds. Let $P \in N$ be a point with $f(P) \notin \partial M$. We suppose that $\mathrm{D}f_P\colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$ is a monomorphism. Then there exists an open neighborhood U of P such that the restriction of f to a map $U \to M$ is a smooth embedding.



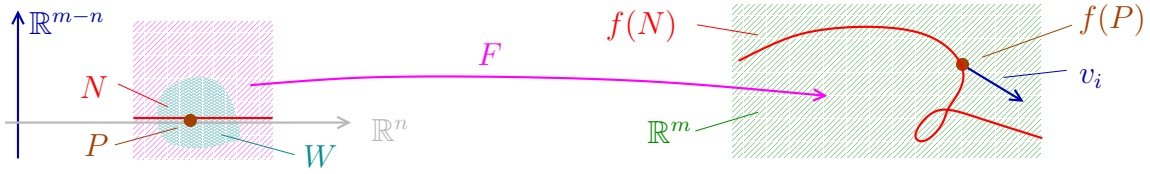
Proof. Since we were very meticulous in the proof of the Inverse Mapping Theorem 30.1, we feel that it is now OK to provide fewer details in the proof. Thus let $P \in N$ be a point such that $f(P) \notin \partial M$ and such that $\mathrm{D}f_P\colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$ is a monomorphism.

First we consider the case $P \in N \setminus \partial N$. Since $P \notin \partial N$ and $f(P) \notin \partial M$ we see that, after applying charts, we might as well assume that N is an open subset of $\mathbb{R}^n$ and that M is an open subset of $\mathbb{R}^m$. Since $\mathrm{D}f_P\colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$ is a monomorphism we can pick $v_1, \ldots, v_{m-n} \in \mathbb{R}^m$ such that the image of $\mathrm{D}f_P$ together with the vectors $v_1, \ldots, v_{m-n}$ spans $\mathbb{R}^m$. Now we consider the map

$$\begin{array}{rcl} F\colon N \times \mathbb{R}^{m-n} & \to & \mathbb{R}^m \\ (x, (\lambda_1, \ldots, \lambda_{m-n})) & \mapsto & f(x) + \sum_{i=1}^{m-n} \lambda_i \cdot v_i. \end{array}$$

[356]Recall that in the Projective Space–Manifold Proposition 25.13 we equipped the real and complex projective spaces with smooth structures.

Basically by design the differential $\mathrm{D}F_{(P,0)}$ is an epimorphism, and thus an isomorphism. Therefore it follows from the Inverse Function Theorem 24.21 that there exists an open neighborhood W of $(P,0)$ such that $F|_W\colon W \to F(W)$ is a diffeomorphism. It follows almost immediately that the restriction of f to $\{u \in N \,|\, (u,0) \in W\}$ has the desired property.



Finally we consider the case that $P \in \partial N$. After picking a chart we might as well assume that N is an open subset of $\mathbb{H}^n$. By definition of a smooth map on an open subset of $\mathbb{H}^n$, see Definition 24.3 and Definition 25.20, there exists an $r > 0$ such that $B^n_r(P) \cap \mathbb{H}^n \subseteq N$ and a smooth map $\widetilde{f}\colon B^n_r(P) \to M$ that agrees with f on $B^n_r(P) \cap \mathbb{H}^n$. Now that P lies in the interior of $B^n_r(P)$ it follows from the previous case that there exists an open neighborhood $\widetilde{U}$ of $P \in B^n_r(P)$ such that the restriction of $\widetilde{f}$ to a map $\widetilde{U} \to M$ is a smooth embedding. We now set $U := \widetilde{U} \cap \mathbb{H}^n$. Then the restriction of f to U is a smooth embedding. ■

30.4. **The Smooth Embedding Theorem.** We start out with a little lemma.

Lemma 30.14. (Submanifold-implies-Embedding Lemma) Let M be a smooth manifold and let W be a (proper) submanifold. The inclusion $\iota\colon W \to M$ is a (proper) smooth embedding.

Proof.

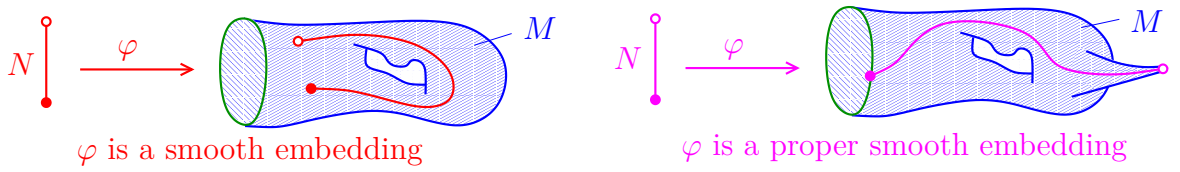
(1) By the Submanifold Smooth Structure Lemma 25.23 the inclusion ι is smooth.
(2) It follows from the Tangent Space-Inclusion Proposition 29.20 that each $\mathrm{D}\iota_P$ is a monomorphism.
(3) (a) Basically by definition we have $W \setminus \partial W \subseteq M \setminus \partial M$.
    (b) For $P \in \partial N$ with $\varphi(P) \in \partial M$ we have a submanifold chart of type (β) which implies that $\mathrm{T}_P N \nsubseteq \mathrm{T}_P \partial M$.
    (a) In Exercise 25.8 we showed that for every component C of ∂W we have either $C \subseteq \partial M$ or $C \subseteq M \setminus \partial M$.
(4) By definition of the subspace topology ι is an embedding.

If W is a proper submanifold, then it follows from an easy comparison of the definitions that ι is indeed a proper embedding. ■

The following theorem can be viewed as a converse to the simple-minded Submanifold-implies-Embedding Lemma 30.14.

Theorem 30.15. (Smooth Embedding Theorem) Let N be an n-dimensional smooth manifold and let M be a smooth manifold.

(1) If $\varphi\colon N \to M$ is a smooth embedding, then $\varphi(N)$ is an n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.[357]
(2) If $\varphi\colon N \to M$ is a proper smooth embedding, then $\varphi(N)$ is a proper n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.



Remark. Let M be a smooth manifold and let N be a smooth submanifold in the sense of Definition 25.22. The Submanifold-implies-Embedding Lemma 30.14 and the Smooth Embedding Theorem 30.15 imply that the standard smooth structure on N, which we introduced on page 671, is characterized by the property that it is the unique smooth structure that turns the inclusion $i\colon N \to M$ into a smooth embedding. □

Proof. Let N be an n-dimensional smooth manifold and let M be an m-dimensional smooth manifold. Let $\varphi\colon N \to M$ be a smooth embedding. By definition this means that φ satisfies the following:

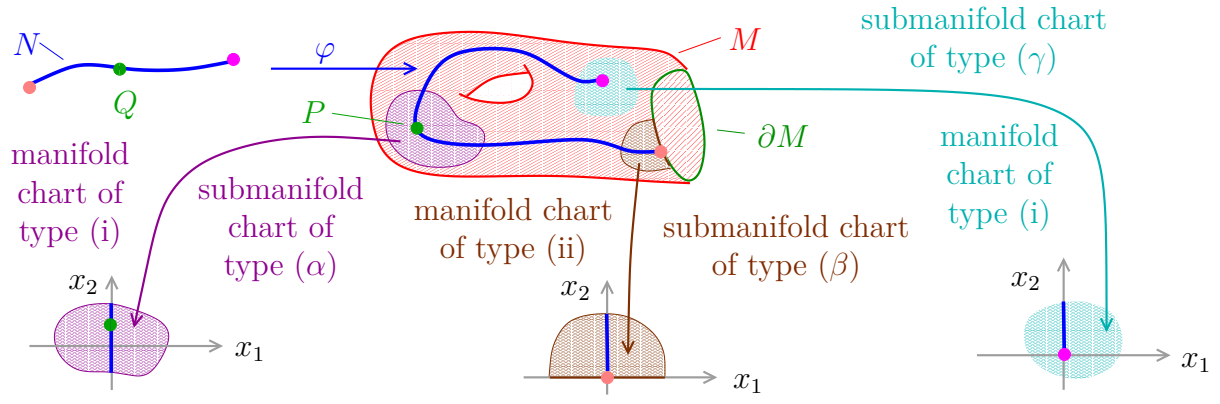
(1) The map φ is smooth.
(2) For each $P \in N$ the induced map $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism.
(3) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
  (a) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.
  (b) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.
  (c) For every component C of ∂N we have either $\varphi(C) \subseteq \partial M$ or $\varphi(C) \subseteq M \setminus \partial M$.
(4) $\varphi\colon N \to M$ is an embedding.

We need to show that $\varphi(N)$ is an n-dimensional submanifold of M and that the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.

Claim. Let $Q \in N$. The following statements hold:

(α) $Q \notin \partial N$ and $\varphi(Q) \notin \partial M$ $\Rightarrow$ there exists a submanifold chart of type (α) for $\varphi(Q)$,
(β) $Q \in \partial N$ and $\varphi(Q) \in \partial M$ $\Rightarrow$ there exists a submanifold chart of type (β) for $\varphi(Q)$,
(γ) $Q \in \partial N$ and $\varphi(Q) \notin \partial M$ $\Rightarrow$ there exists a submanifold chart of type (γ) for $\varphi(Q)$.

We refer to Definition 25.22 for the definitions of the various submanifold charts. A cartoon version of the definitions is also given in the figure below.



[357]Recall that a smooth submanifolds inherits by the Submanifold Smooth Structure Lemma 25.23 the structure of a smooth manifold. Thus it makes sense to say that $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.

Before we turn to the proof of the claim, let us show that the claim implies the statements of the theorem. First note that by hypothesis (3a) the above three cases are the only cases that can occur. Thus we see that the claim implies that $\varphi(N)$ is indeed a submanifold of M.

Next note that it follows from the Submanifold Smooth Structure Lemma 25.23 that $\varphi\colon N \to \varphi(N)$ is smooth. Furthermore, using the above charts and the Inverse Function Theorem 24.21 one can show that $\varphi\colon N \to \varphi(N)$ is a local diffeomorphism. It thus follows from the Smooth Pasting Proposition 25.19 (2) that $N \to \varphi(N)$ is in fact a diffeomorphism.

Finally, if φ is proper in the sense of the definition on Definition 30.7, then $\varphi(N)$ is a closed subset of M and $\varphi^{-1}(\partial M) = \partial N$. The latter fact implies, assuming the claim, that only submanifold charts of type (α) and (β) occur. Together with the fact that $\varphi(N)$ is closed we now know, by Definition 25.22, that $\varphi(N)$ is indeed proper submanifold. In other words, all that is left to do is to prove the claim.

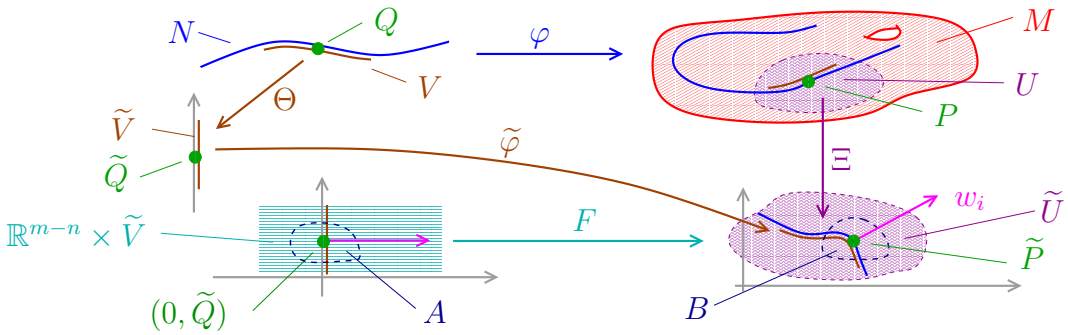
Proof of claim. Let $Q \in N$. We treat the three cases separately.

(α) We assume that $Q \notin \partial N$ and we assume that $P := \varphi(Q) \notin \partial M$.

- Since $Q \notin \partial N$ we can pick a chart $\Theta\colon V \to \widetilde{V}$ of type (i) around Q. We set $\widetilde{Q} := \Theta(Q)$.
- By hypothesis $\varphi\colon N \to M$ is an embedding which means that $\varphi\colon N \to \varphi(N)$ is a homeomorphism. This implies that $\varphi(V)$ is an open subset of $\varphi(N)$. By definition of the subspace topology this means that there exists an open subset $Z \subseteq M$ with $\varphi(V) = Z \cap \varphi(N)$.
- Since $P \notin \partial M$ we can pick a chart $\Xi\colon U \to \widetilde{U}$ of type (i) around P with $U \subseteq Z$. We set $\widetilde{P} := \Xi(P)$.
- After possibly replacing V by $V \cap \varphi^{-1}(U)$ we can and will assume that $\varphi(V) \subseteq U$.

Since $\varphi(V) \subseteq U$ we can consider $\widetilde{\varphi} := \Xi \circ \varphi \circ \Theta^{-1}\colon \widetilde{V} \to \widetilde{U}$. Since φ is an immersion we know that $\mathrm{D}\varphi_Q\colon \mathrm{T}_Q N \to \mathrm{T}_P M$ is a monomorphism. It follows from the Tangent Space-Isomorphism Proposition 29.24 that the corresponding map $\mathrm{D}\widetilde{\varphi}_{\widetilde{Q}}\colon \mathbb{R}^n \to \mathbb{R}^m$ is a monomorphism. Thus there exist vectors $w_1, \dots, w_{m-n} \in \mathbb{R}^m$ such that the image of $\mathrm{D}\widetilde{\varphi}_{\widetilde{Q}}$ together with these vectors spans $\mathbb{R}^m$. We consider the map

$$\begin{aligned} F\colon \mathbb{R}^{m-n} \times \widetilde{V} &\to \mathbb{R}^m \\ ((t_1, \dots, t_{m-n}), x) &\mapsto \widetilde{\varphi}(x) + \sum_{i=1}^{m-n} t_i \cdot w_i. \end{aligned}$$



Note that

$$\mathrm{D}F_{(0,\widetilde{Q})} \;=\; \begin{pmatrix} w_1 & \dots & w_{m-n} & \mathrm{D}\widetilde{\varphi}_{\widetilde{Q}} \end{pmatrix} \;\in\; \mathrm{M}(m \times m, \mathbb{R}).$$

Thus we see that $\mathrm{D}F_{(0,\widetilde{Q})}$ is invertible. It follows from the Inverse Function Theorem 24.21 that there exist open neighborhoods A of $(0,\widetilde{Q})$ and B of $\widetilde{P} \in \widetilde{U}$ such that $F\colon A \to B$ is a diffeomorphism. Finally note that it follows almost immediately from the construction that $(F|_A)^{-1} \circ \Xi\colon \Xi^{-1}(B) \to A$ is a submanifold chart of type (α) for P.

(β) The proof of this case is very similar to the proof of (α). We need to make only a few minor adjustments:

- This time we use charts of type (ii) for Q and for P. Our hypothesis (3c) implies that there exists an open neighborhood V of Q such that $\varphi(V \cap \partial N) \subseteq \partial M$.
- Our hypothesis (3b) implies that we can find $w_1, \dots, w_{m-n} \in \mathbb{R}^{m-n} \times \{0\}$ such that the image of $\mathrm{D}\varphi_Q$ together with these vectors span $\mathbb{R}^m$.

Using these two observations one can now show easily that the approach taken in (α) leads to a submanifold chart of type (β). The only difference is that we need to use the Inverse Function-with-Boundary Lemma 30.4 instead of the Inverse Function Theorem 24.21.

(γ) The proof of this statement is again very similar to the proof of (α). We start out with a chart of type (ii) for Q and a chart of type (i) for P. We obtain a smooth map $\widetilde{\varphi}\colon \widetilde{V} \to V$ where $\widetilde{V}$ is an open subset of $\mathbb{H}^n$. By the definition of a smooth map on $\mathbb{H}^n$, see Definition 24.3, we can (after possibly first shrinking $\widetilde{V}$) extend this map to an open subset of $\mathbb{R}^n$. The rest of the argument is now basically identical to the proof of (α). ∎

Examples.

(1) We consider the map

$$\begin{aligned} \Psi\colon \mathbb{R}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi)) &\mapsto \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1 + \frac{1}{2}x \\ 0 \\ -\frac{1}{2}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{aligned}$$

which we introduced in the Torus Embedding Lemma 7.11.

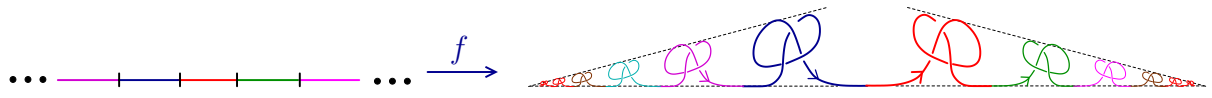
(a) Note that using the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24 one can show fairly easily that the map Θ restricts to a smooth embedding $\Psi\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$.

(b) It follows from (a) together with the Restrict Smooth Embedding Lemma 30.12 that Ψ also restricts to a smooth embedding $\Psi\colon S^1 \times S^1 \to \mathbb{R}^3$.

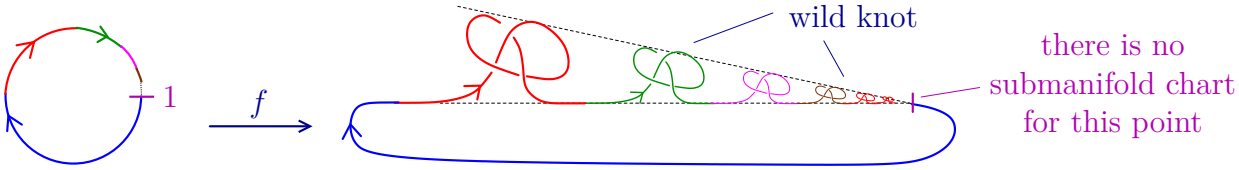
By the Smooth Embedding Theorem 30.15 this allows us to view $\overline{B}^2 \times S^1$ and $S^1 \times S^1$ as submanifolds of $\mathbb{R}^3$.

(2) Let $n \in \mathbb{N}_0$. In the Stereographic Projection Lemma 4.24 we used the stereographic projection to give an explicit homeomorphism $\Phi\colon S^n \to \mathbb{R}^n \cup \{\infty\}$. We leave it to the reader to verify that the map $\mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\} \xrightarrow{\Phi^{-1}} S^n$ is a smooth embedding. We will use this embedding to view $\mathbb{R}^n$ as a submanifold of the smooth manifold S^n.

(3) We consider the map $f\colon \mathbb{R} \to \mathbb{R}^3$ that is shown in the figure below. The map f is a smooth embedding, hence its image is a smooth submanifold of $\mathbb{R}^3$.



(4) Note that in the Smooth Embedding Theorem 30.15 it is essential that we are dealing with an immersion. For example consider the map $f\colon S^1 \to \mathbb{R}^3$ that is illustrated in the figure below.



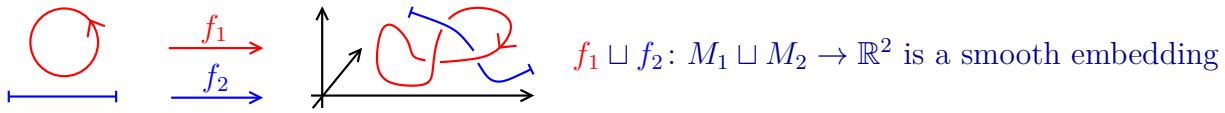
With a little bit of good will and effort one can arrange that f is smooth and that $\mathrm{D}f_P$ is a monomorphism at all points except for $P = 1$. Using the Compact-Hausdorff Proposition 4.23 (3) it is straightforward to show that this map is a homeomorphism onto its image, i.e. it is an embedding. But in the Wild Knot Existence Theorem 218.13 we will see that $f(S^1)$ is not even a topological submanifold of $\mathbb{R}^3$, more precisely we will see, that are no submanifold charts for the point $f(1) \in f(S^1)$.[358] □

The following lemma is often used implicitly when dealing with smooth embeddings of disconnected smooth manifolds.

Lemma 30.16. (Disjoint Union-Smooth Embedding Lemma) Let $M_1, \ldots, M_k$ be compact smooth manifolds, let N be a smooth manifold and let $f_i\colon M_i \to N$ be smooth embeddings. If the images are disjoint, then the map

$$\bigsqcup_{i=1}^{k} f_i\colon \bigsqcup_{i=1}^{k} M_i \;\to\; N$$

is also a smooth embedding.



Proof. The only statement one really needs to verify is that the map is an embedding. This follows from the Closed Disjoint Embedding Lemma 12.4 which we can apply by our hypothesis that $M_1, \ldots, M_k$ are compact, the fact that N is by definition Hausdorff and the observation that by the Manifold Paracompact-Normal Proposition 23.9 we know that N is normal. ■

We state our next example as a lemma.

Lemma 30.17. (Möbius Smooth Embedding Lemma) As we discussed in the Möbius Band-as-Quotient Lemma 9.6 and on page 684 the map

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$

[358] The image of an embedding $S^1 \to \mathbb{R}^3$ such that the image is not a topological submanifold is sometimes called a *wild knot*.

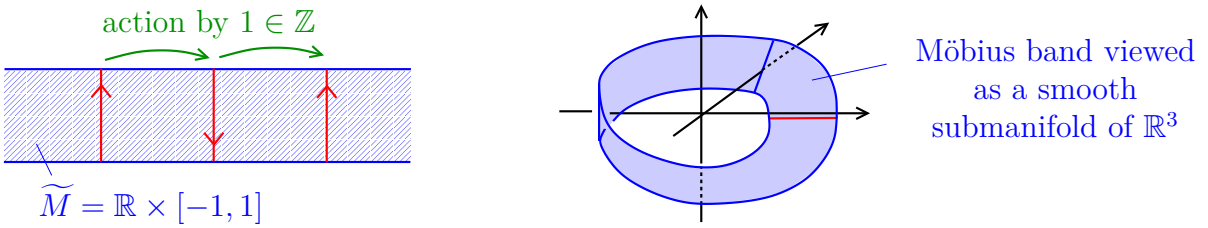
defines an action of $\mathbb{Z}$ on $\widetilde{M} := \mathbb{R} \times [-1,1]$ that is free and smooth. The quotient space $M := (\mathbb{R} \times [-1,1])/\mathbb{Z}$ is called the Möbius band. Given $\varphi, \psi \in \mathbb{R}$ and $r \in [-1,1]$ we set

$$A(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v(r,\psi) := \begin{pmatrix} 2 + r\sin\psi \\ 0 \\ r\cos\psi \end{pmatrix}.$$

The map

$$\begin{array}{rccc} \Psi\colon & (\mathbb{R} \times [-1,1])/\mathbb{Z} & \to & \mathbb{R}^3 \\ & [(\varphi, r)] & \mapsto & A(2\pi \cdot \varphi) \cdot v\big(r, \pi\varphi\big) \end{array}$$

is a smooth embedding. Thus the Smooth Embedding Theorem 30.15 allows us to view the Möbius band as a submanifold of $\mathbb{R}^3$.



Proof. On page 311 we already mentioned that Ψ is an embedding. It remains to show that it is an immersion. We denote by $\Phi\colon \widetilde{M} = \mathbb{R} \times [-1,1] \to \mathbb{R}^3$ the map that is given by $\Phi(\varphi, r) = A(2\pi \cdot \varphi) \cdot v(r, \pi\varphi)$ and we denote by $p\colon \widetilde{M} = \mathbb{R} \times [-1,1] \to (\mathbb{R} \times [-1,1])/\mathbb{Z}$ the projection. Given any $(\varphi, r) \in \widetilde{M} = \mathbb{R} \times [-1,1]$ we consider the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathrm{T}}_{(\varphi,r)}\widetilde{M} & & \xrightarrow{\Theta} & & \mathrm{T}_{(\varphi,r)}\widetilde{M} \xrightarrow{\mathrm{D}p_{(\varphi,r)}} \mathrm{T}_{(\varphi,r)}M \\ \big\downarrow{\scriptstyle \mathrm{D}\Phi_{(\varphi,r)}} & & & & \big\downarrow{\scriptstyle \mathrm{D}\Phi_{(\varphi,r)}} \quad \swarrow{\scriptstyle \mathrm{D}\Psi_{[\varphi,r]}} \\ \widetilde{\mathrm{T}}_{\Phi(\varphi,r)}\mathbb{R}^3 & & \xrightarrow{\Theta} & & \mathrm{T}_{\Phi(\varphi,r)}\mathbb{R}^3 \end{array}$$

We make the following observations and clarifications:

(1) The horizontal maps to the left are the natural isomorphisms provided by the Tangent Space-Isomorphism Proposition 29.24.
(2) A slightly heroic, but ultimately elementary calculation using the Differential-equals-Differential Lemma 29.6 shows that the vertical map to the left is a monomorphism.
(3) It follows from the Manifold–Quotient Proposition 25.39 that the horizontal map to the top right is an isomorphism.
(4) The triangle commutes by the functoriality of differentials.

The combination of the above shows that the diagonal map is a monomorphism. ∎

30.5. **Immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$ I.** In Question 9.20 we had asked whether the projective space $\mathbb{R}\mathrm{P}^2$ or the Klein bottle can be viewed as subsets of $\mathbb{R}^3$. Now that we have more mathematical language at our disposal we can turn this vague question into a precise question. In fact we can reinterpret this question in three different ways:

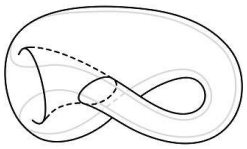
Question 30.18.

(1) Does $\mathbb{R}\mathrm{P}^2$ (respectively the Klein bottle) admit an immersion into $\mathbb{R}^3$?
(2) Does $\mathbb{R}\mathrm{P}^2$ (respectively the Klein bottle) admit a smooth embedding into $\mathbb{R}^3$?

(3) Does $\mathbb{R}P^2$ (respectively the Klein bottle) admit an embedding into $\mathbb{R}^3$?

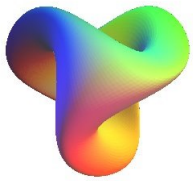
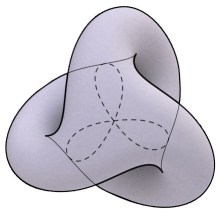
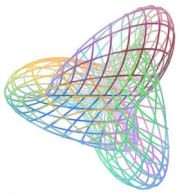
We start out with Question 30.18 (1).

(a) We saw in the figure on page 312 that there exists an immersion of the Klein bottle K into $\mathbb{R}^3$. In the figure below we show the images of two other immersions of the Klein bottle into $\mathbb{R}^3$. This gives a non-rigorous positive answer to Question 30.18 (1) for the Klein bottle. In Exercise 30.22 we will give a rigorous positive answer to Question 30.18 (1) for the Klein bottle. Finally we point out that explicit immersions of the Klein bottle into $\mathbb{R}^3$ are discussed in detail in [**Frai2012**].




(b) The case of the real projective plane is much more interesting. In 1901 David Hilbert, arguably the greatest German mathematician of his generation, gave his PhD student Werner Boy [**Boy1903**][359] the task to prove that there is no immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$. To everyone's surprise Boy did the opposite, he showed that it was possible to immerse $\mathbb{R}P^2$ into $\mathbb{R}^3$. In the figure below we give several illustrations of the image of the immersion. A detailed description of the immersion is given in [**Kir2007, Ap1987**]. But perhaps it is more helpful to just watch the following video:

`https://www.youtube.com/watch?v=uiq-EcQz_uU`


Boy's surface at Oberwolfach

As of right now we do not have the tools to answer Questions 30.18 (2) and (3). In the Surface-No Smooth Embedding Corollary 51.18 we will give an answer to Question 30.18 (2). Note that on Section 51.4 and 4154 we provide two different proofs of the Surface-No Smooth Embedding Corollary 51.18. Furthermore we will give an answer to Question 30.18 (3) in the Surface-No Topological Embedding-Corollary Corollary 218.10.

Since we cannot answer Question 30.18 (2) and (3), let us stick to statements we can actually prove. For example the next proposition says that any $\mathbb{R}P^n$ can be smoothly embedded into some $\mathbb{R}^N$.

Proposition 30.19. (Real Projective Space-Smooth Embedding Proposition) (1) The map

[359] Werner Boy (1879-1914) was a German mathematician who died in the first weeks of World War I.

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^2 = S^2/\pm 1 & \to & \mathbb{R}^4 \\ [(x,y,z)] & \mapsto & (yz, xz, xy, x^2+2y^2+3z^2) \end{array}$$

is a smooth embedding.

(2) Let $n \in \mathbb{N}$. The map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\pm 1 & \to & \mathbb{R}^{\frac{n\cdot(n+1)}{2}+1} \\ [(x_1,\ldots,x_{n+1})] & \mapsto & \left((x_i\cdot x_j)_{i=1,\ldots,n+1, j=1,\ldots,i-1}, \sum_{k=1}^{n+1} k\cdot x_k^2\right) \end{array}$$

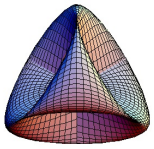
is a smooth embedding.

Proof. We will prove Statement (1) in Exercise 30.13. We leave the proof of Statement (2) to the intellectually intrepid reader. ■

Remark. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^2 & \to & \mathbb{R}^3 \\ [(x,y,z)] & \mapsto & (yz, xz, xy). \end{array}$$

In other words, we consider the composition of the smooth embedding $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$ from the Real Projective Space-Smooth Embedding Proposition 30.19 with the projection $\mathbb{R}^4 \to \mathbb{R}^3$ onto the first three coordinates. In Exercise 30.14 we will see that there are precisely 6 points on $\mathbb{R}\mathrm{P}^2$ such that $\mathrm{D}f_P$ is not a monomorphism.



Steiner surface in $\mathbb{R}^3$

By [**Ap1987**, p. 40] the image $f(\mathbb{R}\mathrm{P}^2) \subseteq \mathbb{R}^3$ is the so-called *Steiner surface* or *Roman surface*, which was first discovered by the Swiss mathematician Jacob Steiner in Rome in 1844. □

As always in mathematics, a result generates new questions. For example, now that by the Real Projective Space-Smooth Embedding Proposition 30.19 we know that we can smoothly embed each $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{\frac{n\cdot(n+1)}{2}+1}$, the following question naturally arises:

Question 30.20. Given $n \in \mathbb{N}$, what is the minimal dimension $d(n)$ such that $\mathbb{R}\mathrm{P}^n$ admits a smooth embedding into $\mathbb{R}^{d(n)}$?

Here are some partial answers:

- The Real Projective Space-Smooth Embedding Proposition 30.19 says that we have the inequality $d(n) \leq \frac{n\cdot(n+1)}{2} + 1$.
- In the Weak Whitney Embedding Theorem 37.9 we will show that $d(n) \leq 2n+1$.
- In the Surface-No Smooth Embedding Corollary 51.18 we will see that $d(2) \geq 4$.
- It follows from the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14 that $d(n) \geq n+2$.

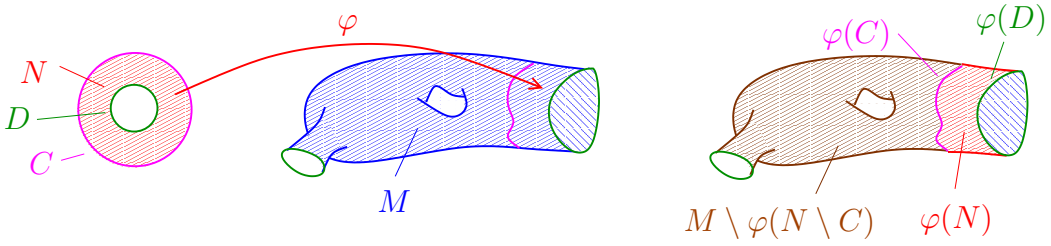
Over the years there have been lots of partial answers to Question 30.20, see e.g. [**Mah1962, MahM1968**]. Nonetheless it seems like Question 30.20 has not been fully answered yet. Some of the results are summarized in

`https://www.lehigh.edu/~dmd1/immtable`

30.6. **Immersions among smooth manifolds of the same dimension.** In the following proposition we consider immersions and smooth embeddings between smooth manifolds of the same dimension.

Proposition 30.21. (Codimension-Zero Smooth Embedding Proposition) Let M and N be smooth manifolds of the same dimension and let $\varphi\colon N \to M$ be an immersion. Recall that by the definition of an immersion in Definition 30.9 we know that for every component X of ∂N we have either $\varphi(X) \subseteq \partial M$ or $\varphi(X) \subseteq M \setminus \partial M$. We denote by C the union of all components of ∂N with $\varphi(C) \nsubseteq \partial M$ and we write $D \coloneqq \partial N \setminus C$.

(1) The image $\varphi(N \setminus C)$ is an open subset of M.
(2) If $\varphi\colon N \to M$ is a smooth embedding and if $\varphi(N)$ is a closed subset of M[360], then the following five statements hold:
   (a) $\varphi(N)$ is a submanifold of M with[361] $\partial_0(\varphi(N)) = \varphi(C)$ and $\partial_1(\varphi(N)) = \varphi(D)$.
   (b) M is decomposed, in the sense of Definition 25.50, into the submanifolds $\varphi(N)$ and $X \coloneqq M \setminus \varphi(N \setminus C)$.
   (c) The boundary of $M \setminus \varphi(N \setminus C)$ as a smooth manifold in its own right is given by $(\partial M \setminus \varphi(D)) \cup \varphi(C)$.
   (d) If ∂N is connected and if M is connected, then X is also connected.
   (e) If M is compact, then $M \setminus \varphi(N \setminus \partial N)$ is also compact.
   (We refer to the below for an illustration.)
(3) If $\varphi\colon N \to M$ is a smooth embedding, if N is closed and if M is connected, then φ is a diffeomorphism.



Example. It follows immediately from the Codimension-Zero Smooth Embedding Proposition 30.21 (3) that it is impossible to smoothly embed a closed n-dimensional smooth manifold into $\mathbb{R}^n$. □

Sketch of proof.

(1) This statement follows almost immediately from the Inverse Mapping Theorem 30.1.
(2) By the Smooth Embedding Theorem 30.15 we know that $\varphi(N)$ is a codimension-zero submanifold of N. The statement about $\partial_0(\varphi(N))$ and $\partial_1(\varphi(N))$ follows immediately from Notation 25.47.

   All the remaining statements follow from our hypothesis that $\varphi(N)$ is a closed subset of M together with the Codimension-Zero Submanifold Proposition 25.49.

[360]Since smooth manifolds are by definition Hausdorff we obtain from the Compact Image Lemma 4.15 together with the Compact-Closed Lemma 3.39 (2) that $\varphi(N)$ is automatically closed if N is compact.
[360]We refer to Notation 25.47 for the definition of ∂_0 and ∂_1 of a submanifold.

(3) We already know by the Smooth Embedding Theorem 30.15 that $\varphi\colon N \to \varphi(N)$ is a diffeomorphism. Thus it suffices to show that $\varphi(N) = M$. First note that it follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that $\varphi(N)$ is a closed subset of M. Furthermore, it follows easily from the Differential-of-Diffeomorphism Theorem 24.22 that $\varphi(N)$ is an open subset of M. Since M is connected we see that $M = \varphi(N)$. ■

30.7. **Appendix: Convex submanifolds of $\mathbb{R}^n$.** The goal of this appendix is to prove the following smooth analog of the Convex-to-Ball Proposition 4.28:

Proposition 30.22. (Smooth Convex-to-Ball Proposition) Let $M \subseteq \mathbb{R}^n$ be a compact codimension-zero smooth manifold. If M is convex and if $0 \in M \setminus \partial M$, then there exists a diffeomorphism

$$\Phi\colon M \to \overline{B}^n$$

with the following properties:

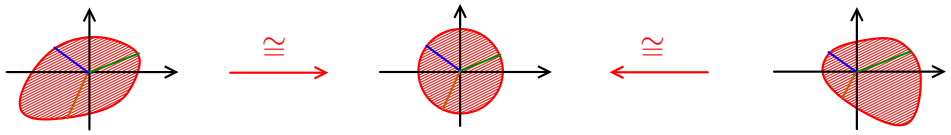
(1) On the boundary the map $\Phi\colon M \to \overline{B}^n$ is given by

$$\begin{array}{rcl} \partial M & \to & S^{n-1} \\ z & \mapsto & \frac{z}{\|z\|}. \end{array}$$

(2) For every $v \in S^{n-1}$ the map Φ sends $(\mathbb{R}_{\geq 0} \cdot v) \cap M$ to $\{t \cdot v \,|\, t \in [0,1]\}$.

(3) For every $v \in \partial M$ with $\|v\| = 1$ the map Φ is the identity on the Euclidean segment from 0 to v.

(4) There exists an $\epsilon \in (0,1)$ with $\overline{B}^n_\epsilon \subseteq M \setminus \partial M$ such that the restriction of Φ to $\overline{B}^n_\epsilon$ is the identity.



Remark. In the Star Shaped-Diffeomorphism Type Theorem 82.2 we will prove the related result that every open star-shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. □

Sketch of proof. Let $M \subseteq \mathbb{R}^n$ be a compact codimension-zero smooth manifold such that M is convex and such that $0 \in M \setminus \partial M$. Since $0 \in M \setminus \partial M$ we see that there exists an $\eta > 0$ with $\overline{B}^n_\eta \subseteq M \setminus \partial M$. After replacing M by $\frac{1}{\eta} \cdot M := \{\frac{1}{\eta} \cdot P \,|\, P \in M\}$ we can and will assume that $\overline{B}^n \subseteq M \setminus \partial M$.

We start out with the following claim, which does most of the hard work of the proof:

Claim 1. The map

$$\begin{array}{rcl} \pi\colon \partial M & \to & S^{n-1} \\ P & \mapsto & \frac{P}{\|P\|} \end{array}$$

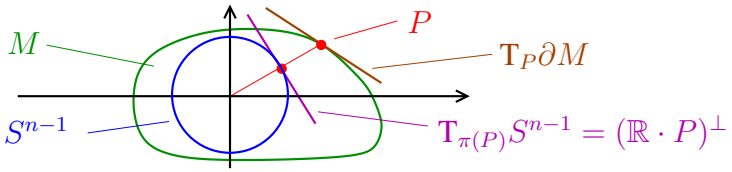
has the property that for each $P \in \partial M$ the differential $\mathrm{D}\pi_P\colon \mathrm{T}_P M \to \mathrm{T}_{\pi(P)} S^{n-1}$ is an isomorphism.

Proof. Let $P \in \partial M$. Note that by the Physical Tangent Spaces–Lemma 29.2 we know that $\mathrm{T}_{\pi(P)} S^{n-1} = (\mathbb{R} \cdot P)^\perp$. As we saw in Exercise 29.13 one can easily show, via a direct

calculation, that the differential is given by the following map:

$$\begin{array}{rcl} \mathrm{D}\pi_P(v)\colon \mathrm{T}_P\partial M & \to & \mathrm{T}_{\pi(P)}S^{n-1} = (\mathbb{R}\cdot P)^\perp \\ v & \mapsto & \frac{1}{\|P\|}\cdot \underbrace{(v - \langle \pi(P), v\rangle)}_{\substack{\text{orthogonal projection} \\ \text{onto } (\mathbb{R}\cdot P)^\perp}} . \end{array}$$

Elementary linear algebra shows that this map is an isomorphism if and only if $P \notin \mathrm{T}_P\partial M$.



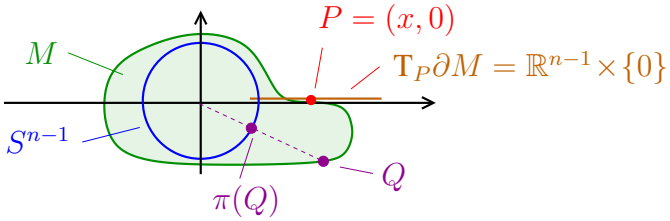
We now continue with a proof of the claim by contradiction. Suppose that the differential $\mathrm{D}\pi_P(v)\colon \mathrm{T}_P\partial M \to \mathrm{T}_{\pi(P)}S^{n-1}$ is not an isomorphism for all $P \in \partial M$. We consider the set $S := \{P \in \partial M \,|\, \mathrm{D}\pi_P \text{ is not an isomorphism}\}$. Recall that we assume that S is non-empty. Using charts and using the Leibniz Formula 2.18 one can easily verify that S is a closed subset of ∂M (see also Exercise 29.14). Since ∂M is compact we obtain from the Compact-Closed Lemma 3.39 (1) that S is compact. The distance function to the origin is continuous. Thus it follows from the Compact Image Lemma 4.15 (2) that there exists a $P \in S$ with minimal distance to the origin.

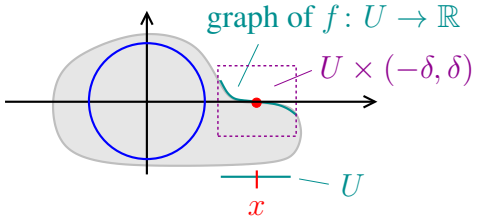
Next note that it follows easily from the Orthogonal Action Lemma 2.37 (3) that after applying an orthogonal matrix we can assume that $P \in \mathbb{R}^{n-1} \times \{0\}$. We can thus write $P = (x, 0)$ with $x \in \mathbb{R}^{n-1}$. Furthermore, after applying a rotation around $\mathbb{R}\cdot P$ we can assume that $\mathrm{T}_P\partial M = \mathbb{R}^{n-1} \times \{0\}$.

Next note that it follows from $\mathrm{T}_P\partial M = \mathbb{R}^{n-1} \times \{0\}$ and the Local Graph Lemma 30.5 that there exists a connected open neighborhood U of $x \in \mathbb{R}^{n-1}$, a $\delta > 0$ and a smooth function $f\colon U \to (-\delta, \delta)$ such that $(U \times (-\delta, \delta)) \cap M = \{(u, f(u)) \,|\, u \in U\}$. Since $P \in \partial M$ and since U is connected we see, via a pretty elementary argument, that one of the following holds:

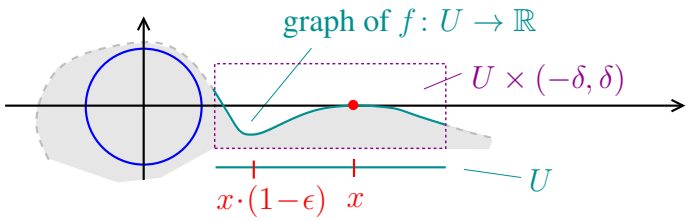
- $M \cap (U \times (-\delta, \delta)) = \{(u, y) \,|\, u \in U, y \geq f(y)\}$, or
- $M \cap (U \times (-\delta, \delta)) = \{(u, y) \,|\, u \in U, y \leq f(y)\}$.

After possibly flipping M in the $\mathbb{R}^{n-1} \times \{0\}$ we can assume that we are in the latter situation, i.e. "M is locally below the graph of f".





It follows from this discussion, from the convexity of M and from elementary real analysis that f has a local maximum at $x \in U \subseteq \mathbb{R}^{n-1}$. This implies in particular that there exists an $\epsilon > 0$ such that $\{x \cdot t \,|\, t \in (1-\epsilon, 1]\} \subseteq U$ and such that $f(x \cdot t) \leq f(x) = 0$ for all $t \in (1-\epsilon, 1]$.



Next note that it follows from the minimality of P that f cannot be locally constant to the left of x. This implies that there exists an $s \in (1-\epsilon, 1)$ such that $f(x \cdot s) < 0$, which means that $(x \cdot s, 0) \notin M$. But this contradicts the fact that M is convex, since the Euclidean segment between $(0,0)$ and $(x,0)$ is no longer contained in M. ⊞

Claim 2. The map
$$\begin{aligned} \pi\colon \partial M &\to S^{n-1} \\ z &\mapsto \tfrac{z}{\|z\|} \end{aligned}$$
is a diffeomorphism.

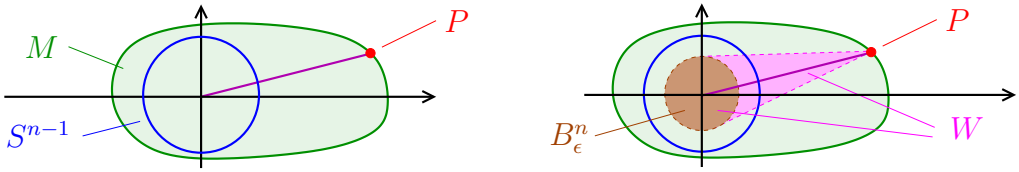
Proof. First note that we know by the Smooth Manifold–Boundary Proposition 25.27 (1) that ∂M has no boundary. It follows from this observation, the fact that S^{n-1} has no boundary, Claim 1 and the Inverse Mapping Theorem 30.1 that $\pi\colon \partial M \to S^{n-1}$ is a local diffeomorphism. By the Smooth Pasting Proposition 25.19 it just remains to show that $\pi\colon \partial M \to S^{n-1}$ is a bijection.

- We show that $\pi\colon \partial M \to S^{n-1}$ is surjective. First note that it follows from the Inverse Mapping Theorem 30.1 that $\pi\colon \partial M \to S^{n-1}$ is a local diffeomorphism that $\pi(\partial M)$ is an open subset of S^{n-1}.

  Next note that it follows from the hypothesis that M is compact together with the Smooth Manifold–Boundary Proposition 25.27 (2) that ∂M is compact. Thus we obtain from the Compact Image Lemma 4.15 together with the Compact-Closed Lemma 3.39 (2) that $\pi(\partial M)$ is also a closed subset of S^{n-1}.

  Next we point out that it follows from the Codimension-Zero Submanifold Proposition 25.49 (1) that $\partial M \neq \varnothing$.

  Finally, since S^{n-1} is connected and since $\partial M \neq \varnothing$ we obtain from the above and the Image-Preimage-Component Lemma 4.50 that $\pi\colon \partial M \to S^{n-1}$ is surjective.
- We show that $\pi\colon \partial M \to S^{n-1}$ is injective. So let $P, Q \in \partial M$ with $\pi(P) = \pi(Q)$, where $\|P\| \geq \|Q\|$. Since $0 \in \mathring{M}$ we see that there exists an $\epsilon > 0$ with $B^n_\epsilon \subseteq M$. It follows from the convexity of M that $W := \{r \cdot x + (1-r) \cdot P \,|\, x \in B^n_\epsilon, r \in (0,1)\}$ is contained in M.



  Note that it follows from the Inverse Mapping Theorem 30.1 (1), applied to the obvious map $B^n_\epsilon \times (0,1) \to W$, that $W \subseteq M$ is an open subset. Since $Q \in \partial M$ we see that $Q \notin W$, which implies that $Q = P$. ⊞

We need to introduce two more maps:

- We consider the map $\rho\colon S^{n-1} \to \mathbb{R}_{>0}$ that is given by $P \mapsto \|\pi^{-1}(P)\|$. One can easily verify that $\rho(P) = \sup\{\lambda \in \mathbb{R}_{\geq 0} \,|\, \lambda \cdot P \in M\}$.
- It follows from the Smooth Transition Function Lemma 24.9 that we can pick a monotonously increasing smooth map $\tau\colon [0,1] \to [0,1]$ such that $\tau(t) = 0$ for all $t \in [0, \frac{1}{4}]$ and such that $\tau(1) = 1$.

Now we finally turn to the denouement of the proof: Somewhat similar to the Convex-to-Ball Proposition 4.28, we consider the map

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & M \\ P & \mapsto & \begin{cases} 0, & \text{if } P = 0, \\ v \cdot \underbrace{r \cdot \big(1 - \tau(r) + \rho(v) \cdot \tau(r)\big)}_{\substack{= 0 \text{ for } P \in \overline{B}^n_{\frac{1}{4}} \text{ and } = \tau(P) \text{ for } P \in S^{n-1}, \\ \text{the derivative is } > 0 \text{ since } \rho(r) > 1}} & \text{if } P = r \cdot v \text{ with } r \in (0,1] \text{ and } v \in S^{n-1}. \end{cases} \end{array}$$

We make the following observations:

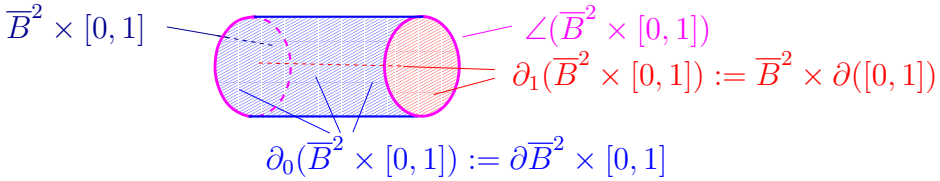
- First note that this map Φ is a "radial bijection" which sends 0 to 0. It follows from Claim 2 that Φ is a bijection which sends S^{n-1} to ∂M.
- Note that Φ is the identity on $\overline{B}^n_{\frac{1}{4}}$. It follows from this observation and the Smooth Pasting Proposition 25.19 that the map Φ is smooth.
- Note that the derivative of the radial stretching is > 0 everywhere. It follows from this observation that the differential of Φ is invertible for any $P \in \overline{B}^n$.

It follows from these observations and the Inverse Mapping Theorem 30.1 that Φ is a diffeomorphism. Clearly it has all the desired properties. ■

30.8. **Appendix: Embeddings of smooth manifolds with corner.** For some of the later purposes it is necessary to generalize some of the concepts and results of this chapter to smooth manifolds with corner.

Definition 30.23. Let N be a smooth manifold with corner $\angle N$. A boundary decomposition of N consists of two subspaces $\partial_0 N$ and $\partial_1 N$ of ∂N with $\partial_0 N \cap \partial_1 N = \angle N$ and with $\partial N = \partial_0 N \cup \partial_1 N$.

Example. Let A and B be two smooth manifolds. Recall that by the Smooth Manifold Product Proposition 26.12 we can consider the product $A \times B$ as a smooth manifold with corner $\angle(A \times B) = \partial A \times \partial B$.



If we set $\partial_0(A \times B) = \partial A \times B$ and $\partial_1(A \times B) = A \times \partial B$, then this clearly defines a boundary decomposition in the above sense. □

The following definition is a variation on the Definition 30.9.

Definition 30.24. A map $\varphi\colon N \to M$ between two smooth manifolds is called an immersion if the following conditions are satisfied:

(1) The map φ is smooth.
(2) For each $P \in N$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism.
(3) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
  (a) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.[361]
  (b) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.[362 363]
  (c) For every component C of ∂N we have either $\varphi(C) \subseteq \partial M$ or $\varphi(C) \subseteq M \setminus \partial M$.[364]

An immersion that also satisfies the following extra condition

(4) $\varphi\colon N \to M$ is an embedding,[365]

is called a smooth embedding.[366]

Definition 30.25. Let N be a smooth manifold with corner with a boundary decomposition $\partial N = \partial_0 N \cup \partial_1 N$. A map $\varphi\colon N \to M$ to a smooth manifold M is called a smooth embedding if the following conditions are satisfied:

(1) The map φ is smooth.
(2) For each $P \in N$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism.
(3) (a) $\varphi^{-1}(\partial M) = \partial_1 N$.
  (b) For each $P \in \partial_1 N$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.
  (c) For each $P \in \angle N$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P \partial_0 N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.
(4) $\varphi\colon N \to M$ is an embedding.

Remark. The above Definition 30.25 is related to Definition 30.9 as follows:

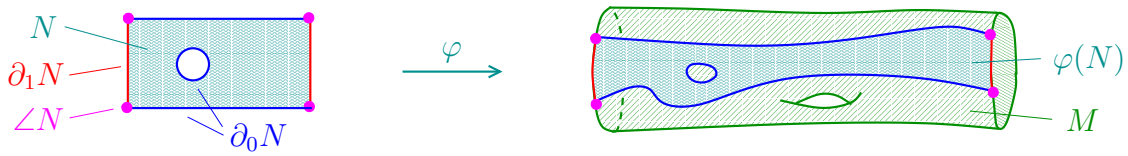
- If we are given a smooth manifold N with a boundary decomposition $\partial N = \partial_0 \cup \partial_1$ such that $\angle N = \partial_0 N \cap \partial_1 N = \varnothing$, then a smooth embedding in the sense of Definition 30.25 is a smooth embedding in the sense of Definition 30.9 and $\partial_i N$ equals $\partial_i N$ from Notation 25.47.
- If we are given a smooth manifold N (without corner) and a smooth embedding in the sense of Definition 30.9, then we get a boundary decomposition from Notation 25.47, and with this boundary decomposition we also have a smooth embedding in the sense of Definition 30.25. □

The following theorem is a variation on the Smooth Embedding Theorem 30.15 (1).

Theorem 30.26. (Smooth-with-Corner Embedding Theorem) Let N be a smooth manifold with corner with a boundary decomposition $\partial N = \partial_0 N \cup \partial_1 N$ and let M be a smooth manifold of the same dimension. If $\varphi\colon N \to M$ is a smooth embedding, then $\varphi(N)$ is a codimension-zero submanifold with corner of M, where[367] $\partial_0(\varphi(N)) = \varphi(\partial_0(N))$ and $\partial_1(\varphi(N)) = \varphi(\partial_1(N))$, and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.

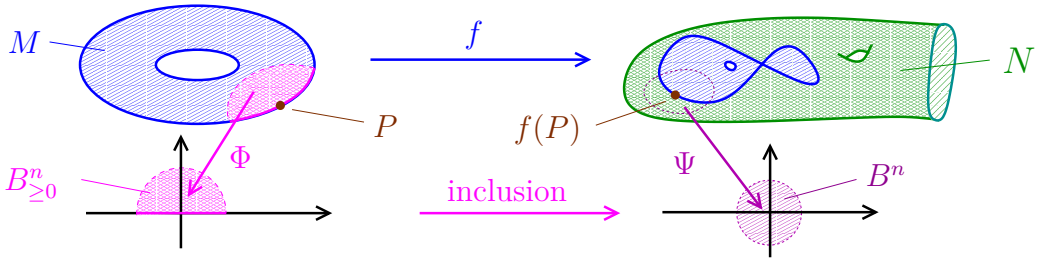
Proof. We leave it to the reader to modify the proof of the Smooth Embedding Theorem 30.15 to obtain the desired result. ■

[367] We refer to the Smooth Submanifold-with-Corner Lemma 26.16 for a definition of ∂_0 and ∂_1 of a codimension-zero submanifold with corner.



30.9. Exercises for Chapter 30.

Exercise 30.1. Let $f\colon M \to N$ be a smooth map between two n-dimensional smooth manifolds such that $f(\partial M) \subseteq N \setminus \partial N$. Let $P \in \partial M$ be a point such that $\mathrm{D}f_P$ is invertible. Show that there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ for M around P and a chart $\Psi\colon V \to B^n$ around $f(P)$ such that the map $\Psi \circ f \circ \Phi^{-1}\colon B^n_{\geq 0} \to B^n$ is the inclusion.



Exercise 30.2. At which points is the map

$$\begin{array}{rcl} f\colon S^2 & \to & S^2 \\ (x,y,z) & \mapsto & \frac{1}{x^4+y^4+z^4}(x^2,y^2,z^2) \end{array}$$

a local diffeomorphism?

Exercise 30.3. Let X and Y be two topological spaces and let $\varphi\colon X \to Y$ be an embedding. We consider the following two conditions on φ.

(I) $\varphi(X)$ is a closed subset of Y.
(II) For every compact subset K of Y the preimage $\varphi^{-1}(K)$ is compact.

In the Manifold–Proper Map Lemma 30.8 we saw that the two conditions (I) and (II) are equivalent if we are dealing with a map between topological manifolds.

(a) Give an example of a continuous map $\varphi\colon X \to Y$ between two topological spaces such that (I) is satisfied but (II) is not satisfied.
(b) Give an example of a continuous map $\varphi\colon X \to Y$ between two topological spaces such that (II) is satisfied but (I) is not satisfied.

Exercise 30.4. Let $m \leq n$. Show that the map

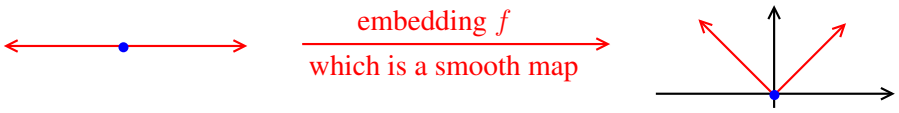
$$\begin{array}{rcl} \iota\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [x] & \mapsto & [(x,0)] \end{array}$$

is an immersion.

Exercise 30.5. Show that there exists an embedding $f\colon \mathbb{R} \to \mathbb{R}^2$ that is smooth in the sense of Definition 24.1 and such that $f(\mathbb{R}) = \{(x,|x|) \mid x \in \mathbb{R}\}$.

Exercise 30.6.

(a) Show that there is no immersion $S^1 \times S^1 \to \mathbb{R}^2$.



(b) Does there exist an immersion $(S^1 \times S^1) \setminus \{(1,1)\} \to \mathbb{R}^2$?

Exercise 30.7. Let M and N be smooth manifolds and let $f\colon M \to N$ be an immersion. We denote by $\Delta = \{(x,x) \in M \times M \,|\, x \in M\}$ the "diagonal" of $M \times M$.

(a) Show that there exists an open neighborhood U of the diagonal Δ such that for every $(x,y) \in U \setminus \Delta$ we have $f(x) \neq f(y)$.

(b) We suppose that M is compact. Show that every $Q \in N$ has only finitely many preimages.

(c) Is it true that f admits only finitely many double points, i.e. that there exist only finitely many distinct pairs $(x_1,y_1),\dots,(x_n,y_n)$ in $(M \times M) \setminus \Delta$ such that $f(x_i) = f(y_i)$ for $i = 1,\dots,n$?

Exercise 30.8.

(a) Show that a closed n-dimensional smooth manifold cannot admit an immersion into $\mathbb{R}^n$.

(b) Let $\varphi\colon \overline{B}^2 \to T := S^1 \times S^1$ be a smooth embedding.

(i) Does there exist an immersion of $M := T \setminus \varphi(B^2)$ to $\mathbb{R}^2$?

(ii) Does there exist a smooth embedding of $M := T \setminus \varphi(B^2)$ to $\mathbb{R}^2$?

Exercise 30.9. Let $m,n \in \mathbb{N}_0$. Show that the smooth manifold $S^m \times S^n$ is diffeomorphic to a smooth submanifold of $\mathbb{R}^{m+n+1}$.

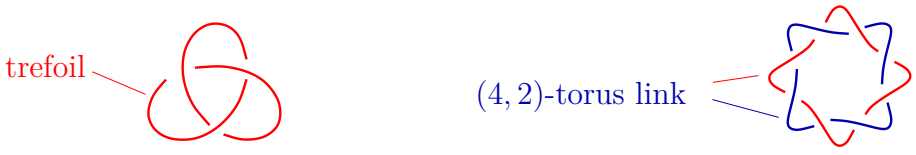
Exercise 30.10. Let $n \in \mathbb{N}$, let X be a compact topological space and let $f\colon X \to S^n$ be a continuous map that is not surjective. Show that there exists a smooth embedding $\Xi\colon \overline{B}^n \to S^n$ with $f(X) \subseteq \Xi(B^n)$.

Exercise 30.11. Let $A \in \mathrm{M}(m \times n, \mathbb{R})$ with rows $r_1,\dots,r_m$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v \end{array}$$

is a monomorphism if and only if there exist $i_1,\dots,i_n \in \{1,\dots,m\}$ such that the determinant of the $(n \times n)$-matrix with rows $r_{i_1},\dots,r_{i_n}$ has non-zero determinant.

Exercise 30.12. Consider the subsets of $\mathbb{R}^3$ that are shown in the figure below. Show that they are smooth submanifolds of $\mathbb{R}^3$ by giving explicit smooth embeddings of S^1 and $S^1 \sqcup S^1$ into $\mathbb{R}^3$ such that the images equals the shown subsets.



Exercise 30.13. Show that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^2 = S^2/\pm 1 & \to & \mathbb{R}^4 \\ [(x,y,z)] & \mapsto & (yz, xz, xy, x^2 + 2y^2 + 3z^2) \end{array}$$

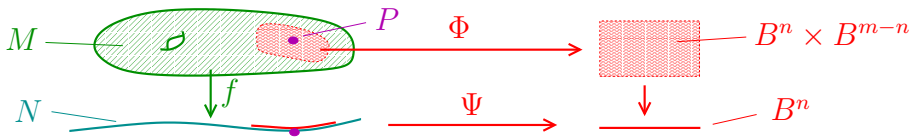
is a smooth embedding.

Exercise 30.14. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^2 & \to & \mathbb{R}^3 \\ [(x,y,z)] & \mapsto & (yz, xz, xy). \end{array}$$

Show that there are precisely 6 points on $\mathbb{R}\mathrm{P}^2$ such that $\mathrm{D}f_P$ is not a monomorphism.

Exercise 30.15. Let $f\colon M \to N$ be a smooth map between two smooth manifolds M and N of dimensions m respectively n. Let $P \in M \setminus \partial M$ such that $f(P) \in N \setminus \partial N$. We suppose that the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism.[368] Show that there exists a chart $\Phi\colon U \to B^n \times B^{m-n}$ around P and a chart $\Psi\colon V \to B^n$ around $f(P)$ with $f(U) \subseteq V$ such that the map $\Psi \circ f \circ \Phi^{-1}\colon B^n \times B^{m-n} \to B^n$ is just the projection onto the first factor.
Remark. This statement is sometimes called the *Local Submersion Theorem*.



Exercise 30.16. Let M be a smooth manifold. We equip ∂M with the smooth structure that we introduced on page 675. Let W be a smooth manifold and let $f\colon W \to M$ be a smooth map with $f(W) \subseteq \partial M$.

(a) Show that if $\mathrm{D}f_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is a monomorphism for each P, then $f\colon W \to \partial M$ is an immersion.

(b) Show that if f is a bijection and if $\mathrm{D}f_P\colon \mathrm{T}_P W \to \mathrm{T}_P \partial M \subseteq \mathrm{T}_P M$ is an isomorphism for each P, then $f\colon W \to \partial M$ is a diffeomorphism.

Remark. This exercise can be viewed as a variation on the Smooth Embedding Theorem 30.15.

Exercise 30.17. Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold and let $f\colon M \to N$ be a smooth map.

(a) Let $P \in M$ with $f(P) \notin \partial N$ such that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is a monomorphism. Show that there exists a chart $\Phi\colon U \to \widetilde{U}$ around P and a chart $\Psi\colon V \to \widetilde{V}$ around $f(P)$ such that the map

$$\Psi \circ f \circ \Phi^{-1}\colon \Phi(f^{-1}(V) \cap U) \to \mathbb{R}^n$$

is the map

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0).$$

(b) We now assume that there exists a $k \in \mathbb{N}$ such that for every $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ has rank k. Show that for each $P \in M$ with $f(P) \notin \partial N$ there exists a chart $\Phi\colon U \to \widetilde{U}$ around P and a chart $\Psi\colon V \to \widetilde{V}$ around $f(P)$ such that the map

$$\Psi \circ f \circ \Phi^{-1}\colon \Phi(f^{-1}(V) \cap U) \to \mathbb{R}^n$$

is the map

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

Remark. This statement is known as the *Constant Rank Theorem*.

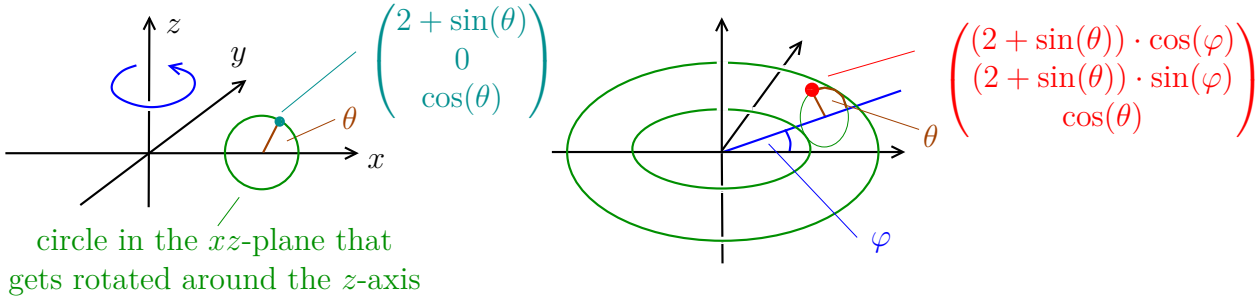
[368] Note that this hypothesis implies that $m \geq n$.

Exercise 30.18. Let M be a connected m-dimensional smooth manifold and let $N \subsetneq M$ be a submanifold. Show that if there exists a smooth retraction $f\colon M \to N$, then we have an inequality $\dim(N) < \dim(M)$.
Hint. Show that if N is a smooth retract of M with $\dim(N) = \dim(M)$, then N is closed and open in M.

Exercise 30.19. Show that the map

$$\begin{array}{rcl} \mathbb{R}^2/(2\pi\mathbb{Z})^2 & \to & \mathbb{R}^3 \\ [(\varphi,\theta)] & \mapsto & \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around the } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 2+\sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix}}_{\substack{\text{describes a circle} \\ \text{in the } xz\text{-plane}}} \end{array}$$

is a smooth embedding.
Remark. By the Torus-as-Quotient Lemma 25.42 we know that the torus $S^1 \times S^1$ is diffeomorphic to $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. This fact, together with the fact that f is a smooth embedding and the Smooth Embedding Theorem 30.15, proves the hardly surprising fact that the torus is diffeomorphic to a submanifold of $\mathbb{R}^3$.



Exercise 30.20. We consider the map

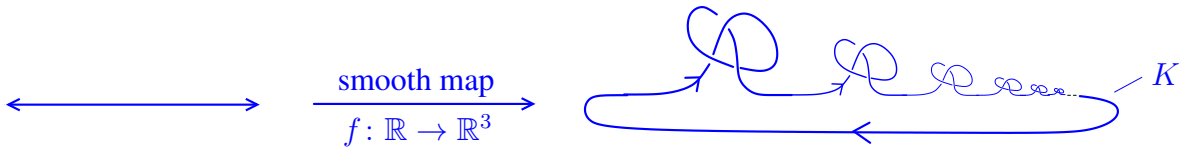
$$\begin{array}{rcl} f\colon \mathbb{R} \times \mathbb{R}_{>0} & \to & \mathbb{R}^3 \\ (\varphi, r) & \mapsto & \begin{pmatrix} r \cdot \cos(\varphi) \\ r \cdot \sin(\varphi) \\ \varphi \end{pmatrix} \end{array}$$

(a) Sketch the image of f.
(b) Show that f is a smooth embedding.

Exercise 30.21. We consider the subspace K of $\mathbb{R}^3$ that is shown in the figure below.
(a) Show that for every smooth map $f\colon \mathbb{R} \to \mathbb{R}^3$ with $f(\mathbb{R}) = K$ there exists a $t \in \mathbb{R}$ with $f'(t) = 0$.
(b) Show that $K \subseteq \mathbb{R}^3$ is not a smooth submanifold.
Remark. In the Wild Knot Existence Theorem 218.13 we will see that K is not even a topological submanifold of $\mathbb{R}^3$.

Exercise 30.22. On page 312 we considered the Klein bottle

$$K \coloneqq ([0,1] \times [0,1])/\sim \quad \text{where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y).$$



We now consider

$$\widetilde{K} \;:=\; \mathbb{R}^2/\!\sim \qquad \text{where } (u,v) \sim (u + 2k\cdot\pi, (-1)^k \cdot v + 2l\cdot\pi) \text{ for all } k,l \in \mathbb{Z}.$$

(a) In the description of $\widetilde{K}$ we implicitly gave an action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. This action is smooth. Show that this action is free. It follows from the Manifold-Quotient Proposition 25.39 that we can view $\widetilde{K} = \mathbb{R}^2/\mathbb{Z}^2$ as a smooth manifold.

(b) Show that K is diffeomorphic to $\widetilde{K}$.

(c) As in [**Frai2012**, p. 1079] we consider the map

$$\begin{array}{rcl} \widetilde{K} & \to & \mathbb{R}^3 \\ [(u,v)] & \mapsto & \begin{pmatrix} \frac{\cos(2u)\cdot\sin(v)}{1-\frac{1}{\sqrt{2}}(\sin(u)\cdot\cos(u)+\sin(2u)\cdot\sin(v))} \\ \frac{\sin(2u)\cdot\sin(v)-\sin(u)\cdot\cos(v)}{\sqrt{2}(1-\frac{1}{\sqrt{2}}(\sin(u)\cdot\cos(u)+\sin(2u)\cdot\sin(v))} \\ \frac{\cos(u)\cdot\cos(v)}{1-\frac{1}{\sqrt{2}}(\sin(u)\cdot\cos(u)+\sin(2u)\cdot\sin(v))} \end{pmatrix}. \end{array}$$

Show that this map is an immersion.

Remark. Note that a 3×2-matrix $(a \;\; b)$ has rank two precisely if the cross product $a \times b$ is non-zero, in particular if $\|a \times b\|^2 \neq 0$.

(d) Show that the map

$$\begin{array}{rcl} \widetilde{K} & \to & \mathbb{R}^4 \\ [(u,v)] & \mapsto & \begin{pmatrix} (2+\cos(2v))\cdot\cos(u) \\ (2+\cos(2v))\cdot\sin(u) \\ \sin(2v)\cdot\cos(\frac{u}{2}) \\ \sin(2v)\cdot\sin(\frac{u}{2}) \end{pmatrix} \end{array}$$

is a smooth embedding.

Remark. This embedding is essentially the same map that we considered on page 312.

Remark. Such maps are also discussed in `https://shorturl.at/QFiRb`.

Exercise 30.23. Does every smooth manifold N with corner $\angle N$ admit a boundary decomposition in the sense of Definition 30.23?

Exercise 30.24. Let M and N be smooth manifolds with $\partial N = \varnothing$. Furthermore let $\Theta\colon M\times N \to M\times N$ be a diffeomorphism such that there exists a diffeomorphism $g\colon N \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \xrightarrow{\;\Theta\;} & M \times N \\ & {\scriptstyle (x,y)\mapsto y} \searrow \quad \swarrow {\scriptstyle (x,y)\mapsto g(y)} & \\ & N. & \end{array}$$

We denote by $\pi\colon M \times N \to M$ the projection. Show that for each $y \in N$ the map $\Theta_y\colon M \to M$ that is given by $x \mapsto \pi(\Theta(x,y))$ is a diffeomorphism.

Hint. It is clear that the map Θ_y is a bijection. Use the Inverse Mapping Theorem 30.1 to

show that the map is a local diffeomorphism. Finally apply the Smooth Pasting Proposition 25.19 (2).

Exercise 30.25.

(a) Let $M = S^1 \times S^1$ be the torus. Does there exist an immersion $f\colon M \to \mathbb{R}^2$?

(b) Let $N = (S^1 \times S^1) \setminus \Phi(B^2)$ be the torus where we removed the interior of a smoothly embedded disk. Does there exist an immersion $g\colon N \to \mathbb{R}^2$?



CHAPTER 31

Orientations of smooth manifolds

In this chapter we will introduce the notion of an orientation of a smooth manifold. This notion captures something very intuitive, even though the technical definition takes a little while to get used to.

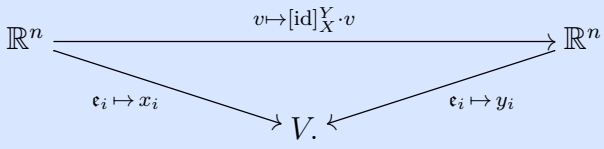
This chapter is organized as follows:

- In Section 31.1 we will introduce the purely algebraic notion of orientations of real vector spaces.
- In Section 31.2 will introduce orientations of smooth manifolds.
- In Section 31.3 will study orientations of codimension-one submanifolds of $\mathbb{R}^n$ and their relationship to unit normal fields.
- In Section 31.4 we will introduce orientation-preserving and reversing maps.
- In Section 31.5 we will see that the orientation of a smooth manifold induces an orientation on its boundary.
- In Section 31.6 we will study the interplay between orientations and group actions on a smooth manifold.
- In Section 31.7 we will study orientations on products of smooth manifolds.
- In Section 31.8 we will provide exercises for Chapter 31.

In the next Chapter 32 we will state and prove the Regular Value Theorem 32.5.

31.1. **Orientations of real vector spaces I.** Before we can introduce and study orientations of smooth manifolds we need to introduce the notion of an orientation of a finite-dimensional real vector space. This section is pure linear algebra.

Definition 31.1. Let V be some finite-dimensional real vector space. If $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ are bases[369] for V, then there exists a unique matrix $[\mathrm{id}]_X^Y = (p_{ij})$ such that given any $i \in \{1, \dots, n\}$ we have $x_i = \sum_{j=1}^{n} p_{ji} \cdot y_j$.[370] We refer to $[\mathrm{id}]_X^Y$ as the transition matrix from X to Y. Note that this matrix is the unique matrix that makes the following diagram commute:



[369] Here and throughout the notes a basis of a finite-dimensional vector space is always understood to be an ordered basis.

[370] In the language of Definition 2.11 the matrix $[\mathrm{id}]_X^Y$ is the matrix representing the identity map with respect to the bases X and Y.

Lemma 31.2. (Transition Matrix Lemma) Let V be a finite-dimensional real vector space.

(1) If X is a basis for V, then $[\mathrm{id}]_X^X = \mathrm{id}$.

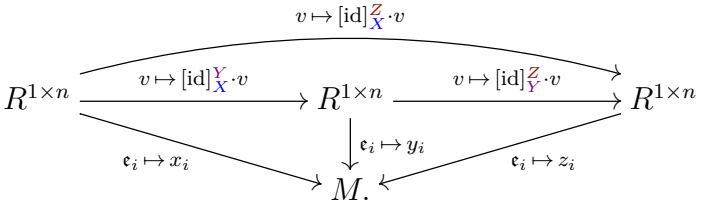
(2) If X, Y and Z are bases of V, then $[\mathrm{id}]_X^Z = [\mathrm{id}]_Y^Z \cdot [\mathrm{id}]_X^Y$.

(3) If X and Y are two bases of V, then $[\mathrm{id}]_Y^X = ([\mathrm{id}]_X^Y)^{-1}$.

Proof. [371]

(1) This statement follows immediately from the definitions.

(2) This statement follows easily from the observation that the following diagram commutes:



(3) This statement follows almost immediately from (1) and (2). ∎

Definition 31.3. Let V be a finite-dimensional real vector space of dimension > 0.

(1) We say that two bases of V are equivalent[372] if the determinant of the corresponding transition matrix is positive.[373]

(2) An equivalence class of bases is called an orientation for V. An oriented vector space is a vector space together with an orientation.

(3) Let $\mathcal{O}$ be an orientation for V. We say that a basis $(v_1, \ldots, v_k)$ of V is a positive basis, if $(v_1, \ldots, v_k)$ lies in $\mathcal{O}$, otherwise we say that it is a negative basis.

Examples. Let $n \in \mathbb{N}$.

(1) We refer to the orientation on the real vector space $\mathbb{R}^n$ that is given by the equivalence class of the canonical basis $(\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$ as the standard orientation of $\mathbb{R}^n$. It follows immediately from the definition that

$$(v_1, \ldots, v_n) \text{ is a positive basis for } \mathbb{R}^n \iff \det \underbrace{(v_1 \ \ldots \ v_n)}_{(n\times n)\text{-Matrix}} > 0.$$

Unless we say something else we always view the real vector space $\mathbb{R}^n$ as equipped with the standard orientation.

(2) Let $v \in \mathbb{R}^n$ be a non-zero vector. We consider the orthogonal complement of v, i.e. we consider the vector space $v^\perp = \{w \in \mathbb{R}^n \mid \langle v, w\rangle = 0\}$. In Exercise 31.1 we will show, using elementary linear algebra, that

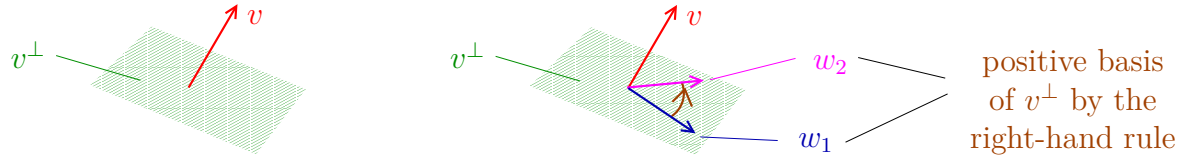
$$(w_1, \ldots, w_{n-1}) \text{ is a positive basis of } v^\perp \quad :\iff \quad \det(v \ w_1 \ \ldots \ w_{n-1}) > 0$$

[371] This lemma can be viewed as a special case of the Representing Matrix Lemma 2.12. For the reader's convenience we provide again a proof.

[372] It follows easily from the Transition Matrix Lemma 31.2 and the multiplicativity of the determinant that this notion of "equivalent" does indeed define an equivalence relation.

[373] If V is 0-dimensional, then it has only the empty basis and we use the convention that the determinant of the empty matrix is $+1$.

does indeed define an orientation on $v^\perp$. We say that this orientation is obtained by the *right-hand rule*.



(3) Let V be a real vector space of dimension ≥ 1. If $(v_1, v_2, \ldots, v_n)$ is a basis of V, then the basis $(-v_1, v_2, \ldots, v_n)$ is inequivalent to the original basis.

(4) In the Complex Vector Space-Orientation Lemma 40.16 we will see that every *complex* vector space of dimension $n \in \mathbb{N}$, viewed as a $2n$-dimensional real vector space, comes with a natural orientation. The algebraically inclined reader might want to figure out what this means and how this works. □

It follows easily from the previous example that any finite-dimensional real vector space of dimension ≥ 1 admits precisely two orientations. For a surprisingly large number of situations it is actually very convenient to extend the notion of an orientation to 0-dimensional real vector spaces in such a way that there are also precisely two orientations. This leads us to the following definition:

Definition 31.4. An orientation of a 0-dimensional real vector space is defined as a choice of a number in $\{-1, +1\}$.

Definition 31.5. Let V be a finite-dimensional real vector space. Given an orientation $\mathcal{O}$ for such a vector space V we denote by $-\mathcal{O}$ the other orientation. Sometimes we refer to it as the opposite orientation.

Definition 31.6. Let V and W be finite-dimensional real vector spaces.

(1) Let $f\colon V \to W$ be an isomorphism. If $\dim(V) = \dim(W) > 0$, then basic linear algebra shows that f induces a bijection

$$\begin{array}{rcl} f_*\colon \text{orientations of } V & \to & \text{orientations of } W \\ [(v_1, \ldots, v_n)] & \mapsto & [(f(v_1), \ldots, f(v_n))]. \end{array}$$

If $\dim(V) = \dim(W) = 0$, then given an orientation $\epsilon \in \{-1, 1\}$ for V we refer to ϵ also as the image of the orientation under the isomorphism $f\colon V \to W$.

(2) Now assume that V and W are oriented. We say that an isomorphism $f\colon V \to W$ is orientation-preserving if the image of the orientation of V under the map f_* described in (1) equals the orientation of W.[374] Otherwise we say that f is orientation-reversing.

For reference we state the following straightforward fact as a lemma.

Lemma 31.7. (Orientation Preserving-via-Determinants Lemma) In the following let $\Phi\colon V \to W$ be an isomorphism between two oriented finite-dimensional real vector

[374] In other words, if $\dim(V) = \dim(W) > 0$, then the map is orientation-preserving if the image of a positive basis of V is a positive basis of W.

spaces with $\dim(V) = \dim(W) > 0$. Furthermore let X be a basis for V and let Y be a basis for W. We make one of the following hypotheses:

(1) either both bases are positive or
(2) both bases are negative,

then

$$\Phi \text{ is orientation-preserving} \iff \det([\mathrm{id}]^Y_{\Phi(X)}) > 0.$$

The following lemma often gets used subconsciously:

Lemma 31.8. (Orientation-Composition Lemma) Let $V_0, \dots, V_k$ be oriented vector spaces. Furthermore let $f_i \colon V_i \to V_{i+1}$, $i = 0, \dots, k-1$ be isomorphisms. Then

$$\begin{array}{c}\text{the map } f_{k-1} \circ \cdots \circ f_0 \colon V_0 \to V_k \\ \text{is orientation-preserving}\end{array} \iff \begin{array}{c}\text{the number of } f_i \text{ which are} \\ \text{orientation-reversing is even.}\end{array}$$

In particular the composition of orientation-preserving isomorphisms is again orientation-preserving.

Proof. If all vector spaces are 0-dimensional, then the statement is completely elementary. Otherwise the lemma follows easily from the Transition Matrix Lemma 31.2 and the Orientation Preserving-via-Determinants Lemma 31.7 together with the multiplicativity of the determinant. ∎

We move on to a definition which will pop up on many occasions:

Definition 31.9. A short exact sequence of real vector spaces is a sequence of homomorphisms[375]

$$0 \to \mathrm{U} \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

between real vector spaces U, V and W such that the following conditions are satisfied:

(1) $\varphi \colon U \to V$ is a monomorphism,
(2) $\psi \colon V \to W$ is an epimorphism,
(3) $\mathrm{im}(\varphi) = \ker(\psi)$.

Example. Let V be a real vector space and let $U \subseteq V$ be a vector subspace. Then the maps $U \to V \to V/U$, given by the inclusion and the natural projection, form a short exact sequence of real vector spaces. In fact every short exact sequence is essentially of this form. More precisely, if $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$ is a short exact sequence, then we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xhookrightarrow{\varphi} & V & \xrightarrow{\psi} & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle w \mapsto [v] \text{ where } \psi(v) = w} & & \\ 0 & \longrightarrow & \varphi(U) & \xhookrightarrow{} & V & \longrightarrow & V/\varphi(U) & \longrightarrow & 0. \end{array}$$

where all vertical maps are isomorphisms. □

Lemma 31.10. (Short Exact Sequence–Orientation Lemma) Let

$$0 \longrightarrow U \xhookrightarrow{\varphi} V \xrightarrow{\psi} W \longrightarrow 0$$

[375]The 0 to the left and right is at this point just for decoration. But they make sense if one thinks in the more general context of exact sequences of arbitrary length.

be a short exact sequence of finite-dimensional real vector spaces. We suppose that U and W are oriented and that $\dim(U) > 0$ and $\dim(W) > 0$. We do the following:

- We pick a positive basis $(u_1, \dots, u_k)$ for U and a positive basis $(w_1, \dots, w_l)$ for W.
- We pick lifts $\widetilde{w}_1, \dots, \widetilde{w}_l \in V$ of $w_1, \dots, w_l \in W$, i.e. we pick $\widetilde{w}_1, \dots, \widetilde{w}_l \in V$ such that for $i = 1, \dots, l$ we have $\psi(\widetilde{w}_i) = w_i$.

Then $(\varphi(u_1), \dots, \varphi(u_k), \widetilde{w}_1, \dots, \widetilde{w}_l)$ is an ordered basis for V and the orientation it defines only depends on the orientations of U and W.

Sketch of proof. It is straightforward to verify that the given vectors define a basis for V. Furthermore note that any other choice of a positive basis for U, choice of positive basis for W and choice of lifts of $w_1, \dots, w_l \in W$ leads to a transition matrix of the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

with $\det(A) > 0$ and $\det(B) > 0$. Thus the two sets of choices of oriented bases lead to the same orientation for V. ■

The Short Exact Sequence–Orientation Lemma 31.10 allows us to introduce the following definition:

Definition 31.11. Let

$$0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

be a short exact sequence of finite-dimensional real vector spaces. We suppose that U, V and W are oriented. Next we do the following:

(1) If $\dim(U) > 0$ and $\dim(W) > 0$, then we say that the orientations are compatible if the orientations of U and W define, in the sense of the Short Exact Sequence–Orientation Lemma 31.10, the orientation of V.
(2) Suppose that $\dim(U) = 0$.
  (a) If the orientation of U is $+1$, then we say that the orientations are compatible if the isomorphism $\psi\colon V \to W$ is orientation-preserving.
  (b) If the orientation of U is -1, then we say that the orientations are compatible if the isomorphism $\psi\colon V \to W$ is orientation-reversing.
(3) The analog of (2) holds if $\dim(W) = 0$.

If the orientations are not compatible, then we say that they are incompatible.

Lemma 31.12. (Compatible Orientations Lemma) Let

$$0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

be a short exact sequence of real vector spaces.

(1) If two out of U, V and W are oriented, then there exists a unique orientation on the third vector space such that the three orientations are compatible.
(2) Suppose that U, V and W are oriented in a compatible way.
  (a) If we reverse an *even* number of the given three orientations, then the orientations are again *compatible*.
  (b) If we reverse an *odd* number of the given three orientations, then the orientations are *incompatible*.

Proof. The proof is straightforward and basically trivial. We will not bore the reader with details. ■

The Compatible Orientations Lemma 31.12 can be used to define various induced orientations.

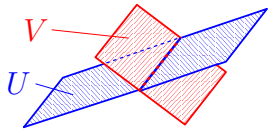
Definition 31.13. Let U and V be oriented finite-dimensional real vector spaces. We define the direct sum orientation on $U \oplus V$ to be the unique orientation[376] that makes the orientations in the following short exact sequence of real vector spaces compatible:

$$0 \;\to\; U \;\xrightarrow{u \mapsto (u,0)}\; U \oplus V \;\xrightarrow{(u,v) \mapsto v}\; V \;\to\; 0.$$

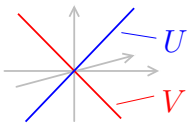
Remark. Note that if U and V are two 0-dimensional real vector spaces with orientations $\mu \in \{-1, 1\}$ and $\nu \in \{-1, 1\}$, then our conventions imply that $U \oplus V$ has the orientation $\mu \cdot \nu \in \{-1, 1\}$. □

Definition 31.14. Let W be a real vector space and let U be a vector subspace of W.

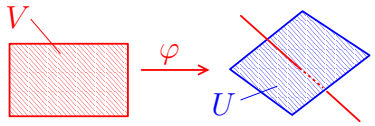
(1) Let V be another vector subspace of W. We say that U and V intersect transversally if $W = U + V$.
(2) Let V be a real vector space and let $\varphi\colon V \to W$ be a homomorphism. We say that U and φ intersect transversally if $W = U + \varphi(V)$.



U and V intersect transversally in $W = \mathbb{R}^3$



U and V do not intersect transversally in $W = \mathbb{R}^3$



U and φ intersect transversally in $W = \mathbb{R}^3$

Remark. Let W be a real vector space and let U, V be oriented vector subspaces. We denote by $\iota\colon V \to W$ the inclusion map. It follows immediately from the definitions that U and V are transverse if and only if U and ι are transverse. On many occasions this little observation allows us to treat (1) as a special case of (2). □

The following lemma is utterly elementary, but we use it so often that it is convenient to state it explicitly.

Auxiliary Lemma 31.15. Let W be a real vector space and let U be a vector subspace of W.

(1) Let V be another vector subspace of W. If U and V intersect transversally, then the map
$$\begin{array}{rcl} V/U \cap V & \to & W/U \\ [v] = v + U \cap V & \mapsto & [v] = v + U \end{array}$$
is an isomorphism.
(2) Let V be a real vector space and let $\varphi\colon V \to W$ be a homomorphism. If U and φ intersect transversally, then the map
$$\begin{array}{rcl} V/\varphi^{-1}(U) & \to & W/U \\ [v] = v + \varphi^{-1}(U) & \mapsto & [\varphi(v)] = \varphi(v) + U \end{array}$$
is an isomorphism.

[376] Here of course we use the Compatible Orientations Lemma 31.12.

Proof. As pointed out above, (1) is just a reader friendly special case of (2). Thus it remains to prove (2). It is clear that the map is a monomorphism. It follows almost immediately from the hypothesis that U and φ intersect transversally that the map is also an epimorphism. ■

Definition 31.16. Let W be an oriented real vector space and let U be an oriented vector subspace.

(1) Let V be another oriented vector subspace of W. We assume that U and V intersect transversally. We consider the following diagram:

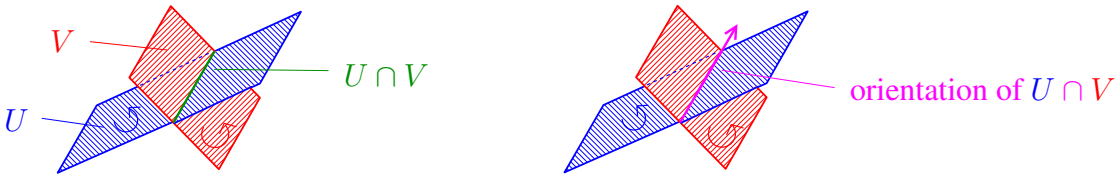
$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \hookrightarrow & W & \twoheadrightarrow & W/U & \longrightarrow & 0 \\ & & & & & & \cong\uparrow{\scriptstyle [v]\mapsto[v]} & & \\ 0 & \longrightarrow & U\cap V & \hookrightarrow & V & \twoheadrightarrow & V/(U\cap V) & \longrightarrow & 0. \end{array}$$

It follows from Lemma 31.15 that the right-hand vertical map is an isomorphism. Note that the horizontal sequences are short exact sequences. By the Compatible Orientations Lemma 31.12 the orientations on U and W induce an orientation on W/U and the isomorphism induces an orientation on $V/(U\cap V)$. Finally, again using the Compatible Orientations Lemma 31.12 we see that this orientation on $V/(U\cap V)$, together with the given orientation on V, induces an orientation on $U\cap V$. We refer to this as the intersection orientation on $U\cap V$.

(2) Let V be another real vector space and let $\varphi\colon V\to W$ be a homomorphism. We assume that φ intersects U transversally. We consider the following diagram:

$$\begin{array}{ccccc} U & \hookrightarrow & W & \twoheadrightarrow & W/U \\ & & & & \cong\uparrow{\scriptstyle [v]\mapsto[\varphi(v)]} \\ \varphi^{-1}(U) & \hookrightarrow & V & \twoheadrightarrow & V/\varphi^{-1}(U). \end{array}$$

By the same logic as in (1) we see that we obtain an orientation on $\varphi^{-1}(U)$. We refer to this as the preimage orientation on $\varphi^{-1}(U)$.



Remark.

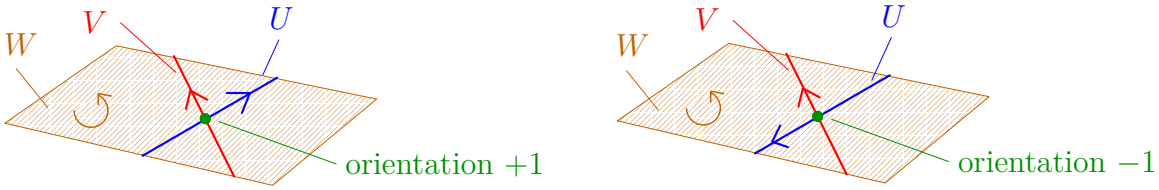
(1) A priori there are many reasonable ways to define the intersection orientation on $U\cap V$. Our convention agrees with the convention chosen in [**GPol1974**, p. 100].

(2) Our convention of the intersection orientation is designed as follows: Let A, B and C be oriented real vector spaces. We set $W := A\oplus B\oplus C$ and we consider the vector subspaces $U = A\oplus B\oplus\{0\}$ and $V = \{0\}\oplus B\oplus C$. If we equip U, V, W with the obvious orientations, then the intersection orientation on $U\cap V = \{0\}\oplus B\oplus\{0\}$ is the obvious orientation.

(3) Let W be an oriented real vector space and let U, V be oriented vector subspaces. We denote by $\iota\colon V\to W$ the inclusion map. As we remarked above, it follows immediately

from the definitions that U and V are transverse if and only if U and ι are transverse. Furthermore, if this is the case, then it follows again immediately from the definitions that the intersection orientation on $U \cap V$ agrees with the preimage orientation on $\iota^{-1}(U) = U \cap V$. □

Example. Let W be an oriented real vector space and let U and V be oriented vector subspaces with $W = U + V$ and with $U \cap V = 0$. We equip $U \oplus V$ with the direct sum orientation that we introduced in Definition 31.13.



It follows immediately from the definitions that the intersection-orientation of $U \cap V$ is given as follows:

$$\text{orientation of } U \cap V \;=\; \begin{cases} +1, & \text{if the direct sum orientation of } U \oplus V \text{ equals the} \\ & \quad \text{orientation of } W, \\ -1, & \text{otherwise.} \end{cases}$$ □

We conclude this discussion of orientations on real vector spaces and of orientation-preserving and orientation-reversing isomorphisms with a short discussion of *automorphisms* of finite-dimensional real vector spaces.

Definition 31.17. Let V be a finite-dimensional real vector space and let $f\colon V \to V$ be an automorphism.

(1) If $\dim(V) > 0$, then we say that f is orientation-preserving if for one choice of a basis X for V we have $\det([\mathrm{id}]^X_{f(X)}) > 0$. Otherwise we say that f is orientation-reversing.[377]
(2) If $\dim(V) = 0$, then we always say that f is orientation-preserving.

We will use this notion at several occasions. At this point there is not much to say except that we have the following very elementary lemma which we will use many times, often without noticing.

Lemma 31.18. (Orientation–Composition Lemma) Let V be a finite-dimensional real vector space.

(1) The composition of orientation-preserving automorphisms is again orientation-preserving.
(2) Let $\varphi\colon V \to W$ be an isomorphism and let $f\colon V \to V$ be an automorphism. Then the following two statements are equivalent:
   - $f\colon V \to V$ is orientation-preserving,
   - $\varphi \circ f \circ \varphi^{-1}\colon W \to W$ is orientation-preserving.

Proof. Both statements follow immediately from the Orientation-Composition Lemma 31.8. ■

[377] It follows immediately from the Transition Matrix Lemma 31.2 and the multiplicativity of the determinant that the sign of $\det(\mathrm{id}^X_{f(X)})$ does not depend on the choice of the basis X.

31.2. **Orientations of smooth manifolds.** After the preparations from the last section we can now introduce the notion of an orientation on a smooth manifold.
An orientation of a smooth manifold M is supposed to be an orientation for each tangent space $\mathrm{T}_P M$ which is "continuous". Turning the idea of "continuity" into a precise notion requires a little bit of thought.

Definition 31.19. Let M be a smooth manifold.

(1) An orientation for M is a map which assigns to every tangent space $\mathrm{T}_P M$, $P \in M$, an orientation such that the following continuity condition is satisfied:
   (∗) For every chart $\Phi\colon U \to V$ of the smooth structure of M and every component U' of U the map

$$\mathrm{T}_P M \xleftarrow[\uparrow]{\cong} \mathrm{T}_P U \xrightarrow[\cong]{\mathrm{D}\Phi_P} \mathrm{T}_{\Phi(P)} V \xrightarrow[\uparrow]{\Theta_V} \widetilde{\mathrm{T}}_{\Phi(P)} V \underset{\uparrow}{=} \mathbb{R}^k$$

   isomorphism by the Tangent Space-Inclusion Proposition 29.20 — natural isomorphism from Proposition 29.24 — the Physical Tangent Spaces–Lemma 29.2

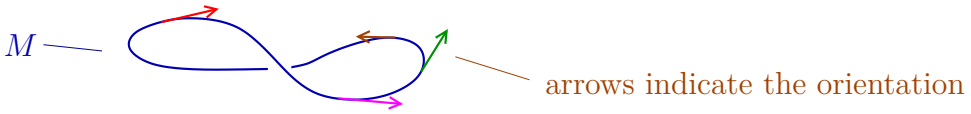
   is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

(2) A smooth manifold together with an orientation is called an oriented smooth manifold.

(3) A smooth manifold that admits an orientation is called orientable, otherwise we say that the smooth manifold is non-orientable.

Remark. Let M be a 0-dimensional smooth manifold. For each $P \in M$ the tangent space $\mathrm{T}_P M$ is a 0-dimensional real vector space. By Definition 31.4 an orientation of $\mathrm{T}_P M$ is a choice of an element of $\{-1, 1\}$. By the discussion on page 624 we know that the 0-dimensional smooth manifold has the discrete topology. Thus we see that an orientation of 0-dimensional smooth manifold is the same as a map $M \to \{-1, 1\}$. Given a 0-dimensional smooth manifold M we equip it, unless we say something else, with the orientation given by assigning $+1$ to each point in M. □

Convention 31.20. Given a picture of a 1-dimensional submanifold M of $\mathbb{R}^2$ or $\mathbb{R}^3$ we indicate an orientation by drawing a positive vector in $\mathrm{T}_P M = \widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^3$. (Note that in this case a non-zero vector is already a basis for the tangent space). On page 815 we will introduce two conventions for indicating orientations of 2-dimensional submanifolds of $\mathbb{R}^3$.



In practice the following lemma can be quite useful for verifying that orientations of tangent spaces actually define an orientation of a smooth manifold. In particular it says that instead of having to check the continuity condition for all charts, it suffices to consider charts that cover all of M.

Lemma 31.21. (Orientations-via-Atlas Lemma) Let M be a smooth manifold. Suppose that for each $P \in M$ we are given an orientation of $\mathrm{T}_P M$. The following statements are equivalent:

(1) The orientations of the tangent spaces define an orientation for M.
(2) Given any $x \in M$ there exists a chart $\Phi\colon U \to V$ in the given smooth structure of M which satisfies the above continuity condition, i.e. which has the property that for every component U' of U the map

$$\mathrm{T}_P M \xleftarrow[\uparrow]{\cong} \mathrm{T}_P U \xrightarrow[\cong]{\mathrm{D}\Phi_P} \mathrm{T}_{\Phi(P)} V \xrightarrow[\uparrow]{\Theta_V} \widetilde{\mathrm{T}}_{\Phi(P)} V \underset{\uparrow}{=} \mathbb{R}^k$$

isomorphism by the Tangent Space-Inclusion Proposition 29.20 (2) — natural isomorphism from Proposition 29.24 — by the Physical Tangent Spaces–Lemma 29.2

is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

Proof. The implication (1)⇒(2) is trivial. Now let us turn to the proof of the reverse implication (2)⇒(1).

Let M be a smooth manifold. The case $\dim(M) = 0$ is trivial. Therefore we now assume that $\dim(M) > 0$. Suppose that for each $P \in M$ we are given an orientation ω_P of $\mathrm{T}_P M$. We start out with introducing some notation:

Given a chart $\Phi\colon U \to V$ from the smooth structure we consider the map[378]

$$\begin{array}{rcl} \chi_\Phi\colon U & \to & \{\pm 1\} \\ P & \mapsto & \begin{cases} +1, & \text{if } (\Theta_V \circ \mathrm{D}\Phi_P)_*(\omega_P) \text{ equals the standard orientation of } \mathbb{R}^k, \\ -1, & \text{otherwise.} \end{cases} \end{array}$$

We equip $\{\pm 1\}$ with the discrete topology. We make the following observations:

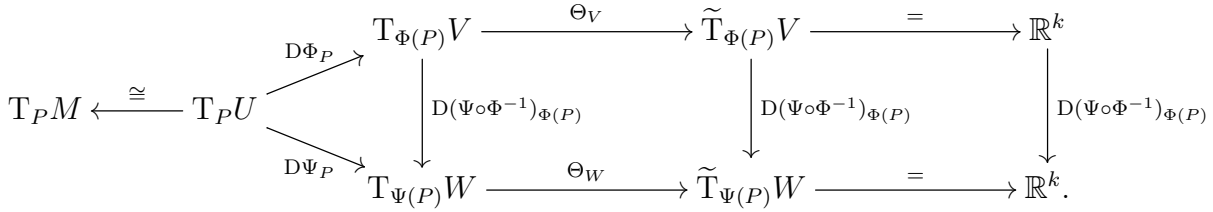
(a) The continuity condition for Φ is by definition satisfied if and only if χ_Φ is constant on each component of U.
(b) By the Connected-to-Discrete Lemma 4.37 a map $U \to \{\pm 1\}$ is continuous if and only if it is constant on each component of U.
(c) By the Pasting Proposition 4.7 a map is continuous if and only if it is locally continuous.
(d) Let $\widetilde{U} \subseteq U$ be an open subset. We write $\widetilde{\Phi} \coloneqq \Phi|_{\widetilde{U}}\colon \widetilde{U} \to \Phi(\widetilde{U})$. This is again a chart and one can easily verify that we have $\chi_\Phi|_{\widetilde{U}} = \chi_{\widetilde{\Phi}}$.

It is now straightforward to assemble a proof of the lemma from the above observations and the following claim:

Claim. Let $U \subseteq M$ be a connected subset and let $\Phi\colon U \to V$ and $\Psi\colon U \to W$ be two charts from the smooth atlas. Then $\chi_\Phi\colon U \to \{\pm 1\}$ is constant if and only if $\chi_\Psi\colon U \to \{\pm 1\}$ is constant.

[378] Given an isomorphism $f\colon V \to W$ we introduced in Definition 31.6 the induced map f_* from the set of orientations on V to the set of orientations on W.

Proof. Given $P \in U$ we consider the following diagram:



Here we also make a few observations:

(a) The diagram commutes by the functoriality of differentials and the naturality of the Θ maps.

(b) By the Differential-equals-Differential Lemma 29.6 the vertical map to the right is given by multiplication by the $(k \times k)$-matrix $\mathrm{D}(\Psi \circ \Phi^{-1})_{\Phi(P)}$.

(c) The map
$$\begin{array}{rcl} U & \to & \{\pm 1\} \\ P & \mapsto & \text{sign of } \det(\mathrm{D}(\Psi \circ \Phi^{-1})_{\Phi(P)}) \end{array}$$
is continuous. It follows from the connectedness of U and the Connected-to-Discrete Lemma 4.37 that this map is in fact constant.

The claim follows easily from combining all of the above. ■

As of right now there are not that many examples of oriented smooth manifolds that we can provide.

Example.

(1) Let $n \in \mathbb{N}$ and let U be an open subset of $\mathbb{R}^n$ or of the half-space $\mathbb{H}^n$. As always we equip U with the smooth structure given by the identity chart. We now use the Orientations-via-Atlas Lemma 31.21 in reverse to endow U with an orientation. More precisely, given any $P \in U$ we have the isomorphism
$$\mathbb{R}^n \underset{\substack{\uparrow \\ \text{by the Physical Tangent Spaces–Lemma 29.2}}}{=} \widetilde{\mathrm{T}}_P U \underset{\substack{\uparrow \\ \text{natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24}}}{\xleftarrow{\Theta_U}} \mathrm{T}_P U.$$
Now we use the standard orientation of the vector space $\mathbb{R}^n$ and the above isomorphism to equip $\mathrm{T}_P U$ with an orientation. It follows almost immediately from the Orientations-via-Atlas Lemma 31.21 that this defines an orientation on the smooth manifold U. We refer to this orientation on U as the standard orientation on U.

(2) We define the standard orientation on $\mathbb{R}^0 = \{0\}$ to be $+1$. □

We will construct more examples of oriented smooth manifolds in the subsequent sections. But first let us conclude this section with the following definition:

Definition 31.22. Let $M = (M, \mathcal{O})$ be an oriented smooth manifold. If for each tangent space we pick the opposite orientation then evidently we obtain again an orientation of M, which we refer to as the opposite orientation $-\mathcal{O}$. We denote the resulting oriented smooth manifold by $-M := (M, -\mathcal{O})$.

31.3. **Orientations of submanifolds of $\mathbb{R}^n$ and unit normal fields.** In this section we will consider "real-life" examples of smooth manifolds, namely we will consider submanifolds M of some $\mathbb{R}^n$. Throughout this section we will work with the physical tangent spaces $\widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$.

Observation 31.23. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. Under the natural isomorphism $\widetilde{\mathrm{T}}_P M \to \mathrm{T}_P M$ from the Tangent Space-Isomorphism Proposition 29.24 we see that an orientation of M is the same as a map which assigns to every physical tangent space $\widetilde{\mathrm{T}}_P M$, $P \in M$, an orientation such that the following continuity condition is satisfied:

(∗) For every chart $\Phi\colon U \to V$ of the smooth structure of the smooth manifold M (with the smooth structure defined in the Submanifold Smooth Structure Lemma 25.23) and every component U' of U the map

$$\begin{array}{ccccccc} \widetilde{\mathrm{T}}_P M & \xleftarrow{=} & \widetilde{\mathrm{T}}_P U & \xrightarrow[\cong]{\mathrm{D}\Phi_P} & \widetilde{\mathrm{T}}_{\Phi(P)} V & = & \mathbb{R}^k \\ \uparrow & & & & & & \uparrow \end{array}$$

Physical Tangent Space Inclusion Proposition 29.5 — Physical Tangent Spaces–Lemma 29.2

is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

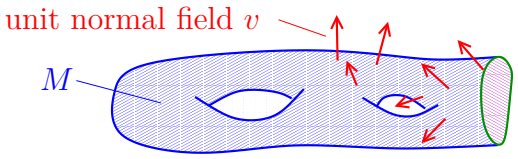
Furthermore, as in the Orientations-via-Atlas Lemma 31.21 the continuity condition only needs to be verified for charts in some smooth atlas of M.

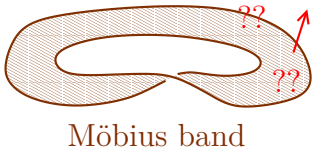
Definition 31.24. Let M be a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$. A unit normal field is a continuous map

$$v\colon M \to \mathbb{R}^{n+1}$$

such that the following conditions are satisfied:

(1) For each $P \in M$ we have $\|v(P)\| = 1$,
(2) for each $P \in M$ we have $v(P) \perp \widetilde{\mathrm{T}}_P M$.





Möbius band

Examples.

(1) Let $n \in \mathbb{N}$. It follows from the Physical Tangent Spaces–Lemma 29.2 that the map

$$\begin{array}{rcl} v\colon S^n & \to & \mathbb{R}^{n+1} \\ P & \mapsto & P \end{array}$$

is a unit normal field.

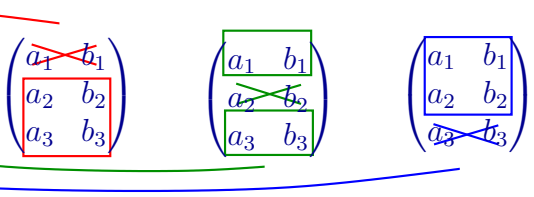
(2) In the Möbius Smooth Embedding Lemma 30.17 we saw that we can view the Möbius band as a smooth submanifold M of $\mathbb{R}^3$. Does M admit a unit normal field? This seems unlikely, but how do you show the non-existence of a unit normal field? □

Our next goal is to show that for a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$ there is a natural correspondence between orientations and unit normal fields. The key tool for doing so is the following definition from linear algebra:

Definition 31.25. Let $w_1, \dots, w_n \in \mathbb{R}^{n+1}$. As in [**Blo1979**, Chapter 5.14] we define the cross product[379]

$$w_1 \times \cdots \times w_n \coloneqq \sum_{k=1}^{n+1} (-1)^{k+1} \cdot \mathfrak{e}_k \cdot \det \begin{pmatrix} (n \times n)\text{-matrix obtained from} \\ \text{the } (n+1) \times n\text{-matrix } (w_1 \ \dots \ w_n) \\ \text{by crossing out the } k\text{-th row} \end{pmatrix}.$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$



In the following lemma we summarize a few properties of the cross product.

Lemma 31.26. (Cross Product Lemma)

(1) The map

$$\begin{array}{rcl} \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} & \to & \mathbb{R}^{n+1} \\ (v_1, \dots, v_n) & \mapsto & \underbrace{v_1 \times \cdots \times v_n}_{\text{cross product}} \end{array}$$

is smooth.

(2) Let $v_1, \dots, v_n \in \mathbb{R}^{n+1}$. The following statements hold:
   (a) $\det(v_1 \times \cdots \times v_n \ v_1 \ \dots \ v_n) \geq 0$.
   (b) $v_1 \times \cdots \times v_n$ is orthogonal to each $v_1, \dots, v_n$.
   (c) The cross product $v_1 \times \cdots \times v_n$ is non-zero if and only if the vectors $v_1, \dots, v_n$ are linearly independent.[380]

(3) Suppose that $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are both collections of linearly independent vectors in $\mathbb{R}^{n+1}$. The following statements hold:
   (a) $v_1 \times \cdots \times v_n$ and $w_1 \times \cdots \times w_n$ are linearly dependent if and only if we have $\operatorname{Span}(v_1, \dots, v_n) = \operatorname{Span}(w_1, \dots, w_n)$.
   (b) There exists a $\lambda > 0$ with $v_1 \times \cdots \times v_n = \lambda \cdot (w_1 \times \cdots \times w_n)$ if and only if the vectors span the same vector subspace of $\mathbb{R}^{n+1}$ and if $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are equivalent bases.

Sketch of proof.

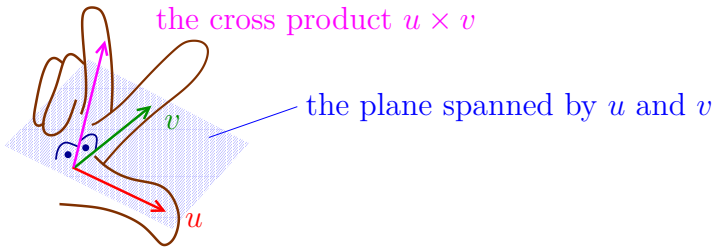
(1) This statement follows from the Leibniz Formula 2.18 for determinants.

(2) Let $v_1, \dots, v_n \in \mathbb{R}^{n+1}$. For any $u \in \mathbb{R}^{n+1}$ it follows from the Leibniz expansion by the first column that

$$\det(u \ v_1 \ \dots \ v_n) = \langle u, v_1 \times \cdots \times v_n \rangle.$$

[379] By the discussion in Footnote 29 the determinant of a 0×0-matrix is $+1$. It follows that for $n = 0$ the cross product is always the vector $\mathfrak{e}_1 = 1 \in \mathbb{R}$.

[380] In the Cross Product Characterization Proposition 76.8 we will see that the length of the cross product $v_1 \times \cdots \times v_n$ equals the "n-dimensional volume" of the parallelotope spanned by $v_1, \dots, v_n$.



(a) Using the above equality we see that

$$\det(v_1 \times \cdots \times v_n \; v_1 \; \ldots \; v_n) \;=\; \langle v_1 \times \cdots \times v_n, v_1 \times \cdots \times v_n \rangle \;\geq\; 0.$$

(b) Let $i \in \{1, \ldots, n\}$. We see that

$$\langle v_i, v_1 \times \cdots \times v_n \rangle \underset{\substack{\uparrow \\ \text{by the above equality}}}{=} \det(v_i \; v_1 \; \ldots \; v_n) \underset{\substack{\uparrow \\ \text{since two columns are the same}}}{=} 0.$$

(c) This statement follows from the fact, proved in Exercise 30.11, that an $(n+1) \times n$-matrix has rank n if and only if there exist n rows that are linearly independent.

(3) (a) Statement (3a) follows from Statement (2b).

(b) It follows from (3a) that we only have to consider the case that $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ span the same vector space. Let A be the transition matrix. It follows easily from the definitions that $v_1 \times \cdots \times v_n = \det(A) \cdot (w_1 \times \cdots \times w_n)$ which implies the statement. ■

In the following proposition we will see that for a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$ there is a natural bijection between orientations and unit normal fields.

Proposition 31.27. (Orientations-via-Normal Vector Fields Proposition) Let $n \in \mathbb{N}$ and let M be a codimension-one smooth submanifold of $\mathbb{R}^{n+1}$. The maps

$$\begin{array}{rcl} \{\text{unit normal fields on } M\} & \leftrightarrow & \{\text{orientations on } M\} \\ & & \text{given a point } P \in M \text{ we say that a} \\ v & \mapsto & \text{basis } w_1, \ldots, w_n \text{ of } \widetilde{\mathrm{T}}_P M \text{ is positive} \\ & & \text{if } \det\big(v(P) \; w_1 \; \ldots \; w_n\big) > 0 \\ \text{given a point } P \in M \text{ we pick a positive} & & \\ \text{basis } w_1, \ldots, w_n \text{ for } \widetilde{\mathrm{T}}_P M \text{ and normalize} & \leftarrow\!\shortmid & \text{orientation on } M \\ \text{the vector } w_1 \times \cdots \times w_n \text{ to length one} & & \end{array}$$

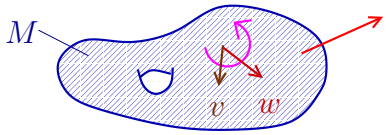
are well-defined and inverses of one-another.

Convention 31.28. Given a picture of a 2-dimensional submanifold M of $\mathbb{R}^3$ we have two conventions for indicating an orientation:

(1) We draw a part of a little circle together with a directional arrow. This indicates that a positive basis is given by (v, w) where w is obtained by rotating v into the indicated direction by an angle in $(0, \pi)$.

(2) Alternatively we draw a normal vector n, in this case a basis (v, w) of the tangent space is positive if (n, v, w) is a positive basis for $\mathbb{R}^3$.

both conventions indicate the same orientation of M



Proof. We start out with the following claim:

Claim 1. For every unit normal field v the construction defines an orientation on M.

Proof. It follows from the discussion on page 804 that for each $P \in M$ the construction does indeed define an orientation on $\widetilde{\mathrm{T}}_P M$. It remains to show that the continuity condition holds.

Let $\Phi\colon U \to V \subseteq \mathbb{R}^{n+1}$ be a submanifold chart with U connected. Recall that Φ gives rise to a chart $\Psi\colon U \cap M \to \widetilde{V} \subseteq \mathbb{R}^n$ of M viewed as an n-dimensional smooth manifold. For $Q \in U$ we now denote by $f(Q)$ the $(n+1)$-st coordinate of $\mathrm{D}\Phi_Q(v(Q)) \in \mathbb{R}^{n+1}$. Note that $f\colon U \to \mathbb{R}$ is continuous and note that f is non-zero everywhere. Now let $Q \in U \cap M$. For $i = 1, \dots, n$ we set $w_i \coloneqq \mathrm{D}\Psi^{-1}_{\Psi(Q)}(\mathfrak{e}_i)$. We have

$$\underbrace{\det(v(Q)\, w_1 \,\dots\, w_n)}_{\substack{\text{by definition} > 0 \\ \text{if } w_1, \dots, w_n \text{ is a} \\ \text{positive basis}}} = \det(\mathrm{D}\Phi_Q)^{-1} \cdot \begin{pmatrix} * & \mathfrak{e}_1 & \dots & \mathfrak{e}_n \\ f(Q) & 0 & \dots & 0 \end{pmatrix} = \det(\mathrm{D}\Phi_Q)^{-1} \cdot f(Q) \cdot (-1)^{n+1}.$$

$\uparrow$ follows from the by the Chain Rule 24.17

This shows that the isomorphism $\mathrm{D}\Psi^{-1}_{\Psi(Q)}\colon \mathbb{R}^n \to \widetilde{\mathrm{T}}_Q M$ is orientation-preserving if and only if $\det(\mathrm{D}\Phi_Q)^{-1} \cdot f(Q) \cdot (-1)^{n+1} > 0$. But this function in Q is non-zero everywhere. Hence, since U is connected, we see that the sign of this expression is independent of $Q \in U$. ⊞

Claim 2. For every orientation on M the construction defines a unit normal field on M.

Proof. It follows from the Cross Product Lemma 31.26 that for each $P \in M$ the vector $w_1 \times \dots \times w_n$ is indeed normal to $\widetilde{\mathrm{T}}_P M$ and that it is independent of the choice of the positive basis $w_1, \dots, w_n$ of $\widetilde{\mathrm{T}}_P M$. It only remains to show that the resulting map $v\colon M \to \mathbb{R}^{n+1}$ is continuous. Thus let $P \in M$. We pick a chart $\Phi\colon U \to V$ around P such that U is connected. After possibly composing Φ with a reflection we can assume that for each $Q \in V$ the map

$$\mathrm{D}\Phi^{-1}_Q\colon \mathbb{R}^n \to \widetilde{\mathrm{T}}_{\Phi^{-1}(Q)} M$$

is orientation-preserving. It now follows that on U the map v is given by

$$\begin{array}{ccl} U & \to & \mathbb{R}^{n+1} \\ x & \mapsto & \mathrm{D}\Phi^{-1}_{\Phi(x)}(\mathfrak{e}_1) \times \dots \times \mathrm{D}\Phi^{-1}_{\Phi(x)}(\mathfrak{e}_n) \text{ normalized to length } = 1. \end{array}$$

By the Cross Product Lemma 31.26 (1) we know that this map is continuous. This shows that v is locally continuous, hence it is continuous. ⊞

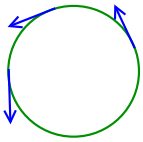
We have now shown that the maps given in the statement of the current proposition are well-defined. It follows easily from the Cross Product Lemma 31.26 (2) that the two maps are also inverses of one another. ■

Example.

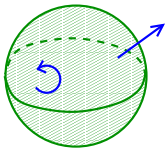
(1) Let $n \in \mathbb{N}$. As we mentioned above, it follows from the Physical Tangent Spaces–Lemma 29.2 that the map

$$\begin{array}{rcl} v\colon S^n & \to & \mathbb{R}^{n+1} \\ P & \mapsto & P \end{array}$$

is a unit normal field. Given $P \in S^n$ we say that a basis $(v_1, \dots, v_n)$ of the tangent space $\widetilde{\mathrm{T}}_P S^n \subseteq \mathbb{R}^{n+1}$ is positive if $\det(P\, v_1\, \dots\, v_n) > 0$.



orientation of S^1



orientation of S^2

It follows from the Orientations-via-Normal Vector Fields Proposition 31.27 and Observation 31.23 that these orientations define an orientation of the smooth manifold S^n. We refer to this as the standard orientation of S^n. Unless we explicitly say something else we will always view S^n as an oriented smooth manifold with the above orientation.

(2) We equip the 0-dimensional sphere $S^0 = \{\pm 1\}$ with the obvious orientation given by $\pm 1 \mapsto \pm 1$. □

31.4. **Orientation-preserving and reversing maps.** The next definition is basically self-explanatory.

Definition 31.29. Let $f\colon M \to N$ be a smooth map between two oriented smooth manifolds such that for every $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an isomorphism.[381]

(1) We say that the map f is orientation-preserving if for every $P \in M$ the induced isomorphism $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is orientation-preserving.

(2) We say that the map f is orientation-reversing if for every $P \in M$ the induced isomorphism $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is orientation-reversing.

Examples.

(1) The map

$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \subseteq \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \\ t & \mapsto & \exp(\mathrm{i}t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \end{array}$$

is a local diffeomorphism. If we equip $\mathbb{R}$ and S^1 with the standard orientation, then Ξ is orientation-preserving. Indeed, for any $t \in \mathbb{R}$ we have

$$\mathrm{D}(\Xi_t)\underbrace{(1)}_{\substack{\uparrow \\ \text{positive basis of } \widetilde{\mathrm{T}}_t\mathbb{R}}} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \in \widetilde{\mathrm{T}}_{\binom{\cos(t)}{\sin(t)}} S^1 \text{ which is positive since } \det \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} > 0.$$

(2) Let $A \in \mathrm{O}(n)$. In Exercise 31.5 we will see that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$.

(3) Let $f\colon M \to N$ be a local diffeomorphism between two oriented smooth manifolds. If M is non-empty and *connected*, then it follows from the Manifold–Orientation Basics

[381] In most applications f is a local diffeomorphism, but by the Tangent Space-Inclusion Proposition 29.20 the inclusion of a codimension-zero submanifold into its "surrounding" smooth manifold also has this property.

Lemma 31.31 below that f is either orientation-preserving or orientation-reversing. On the other hand for disconnected M one can easily come up with maps that are neither orientation-preserving nor orientation-reversing. □

The following lemma gives an easy criterion for showing that a diffeomorphism between two open subsets of $\mathbb{R}^n$ or of $\mathbb{H}^n$ is orientation-preserving.

Lemma 31.30. (Orientation Checking-via-Differentials Lemma) Let $n \in \mathbb{N}$ and let $f\colon X \to Y$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$. Recall that by the discussion on page 812 we equip X and Y with the standard orientations. Given $P \in X$ we have[382]

$$\underset{\substack{\uparrow\\ \text{differential from Definition 29.16}}}{\mathrm{D}f_P\colon \mathrm{T}_P X \to \mathrm{T}_{f(P)}Y \text{ is orientation-preserving}} \iff \det\Big(\underbrace{\text{differential } \mathrm{D}f_P}_{(n\times n)\text{-matrix}}\Big) > 0.$$

In particular we see

$$f\colon X \to Y \text{ is orientation-preserving} \iff \det(\mathrm{D}f_P) > 0 \text{ for all } P \in X.$$

Example. Let $U \subseteq \mathbb{R}^n$ be an open subset. Let $A \in \mathrm{GL}(n,\mathbb{R})$ with $\det(A) > 0$ and let $v \in \mathbb{R}^n$. It follows easily from the discussion on page 651 and the Orientation Checking-via-Differentials Lemma 31.30 that the map f given by $f(x) = x \mapsto A \cdot x + v$ defines an orientation-preserving diffeomorphism $U \to f(U)$. □

Proof. By the Tangent Space-Isomorphism Proposition 29.24 we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n = \widetilde{\mathrm{T}}_P X & \xrightarrow[\cong]{\Theta_X} & \mathrm{T}_P X \\ {\scriptstyle \mathrm{D}f_P}\downarrow & & \downarrow{\scriptstyle \mathrm{D}f_P} \\ \mathbb{R}^n = \widetilde{\mathrm{T}}_{f(P)} Y & \xrightarrow[\cong]{\Theta_Y} & \mathrm{T}_{f(P)} Y. \end{array}$$

The lemma follows from combining the following observations:

(1) As on page 803 we equip $\mathbb{R}^n$ with the standard orientation that is defined by the standard basis $(\mathfrak{e}_1, \dots, \mathfrak{e}_n)$ of $\mathbb{R}^n$.
(2) The horizontal maps are orientation-preserving by definition of the orientations of the oriented smooth manifolds X and Y.
(3) By the Differential-equals-Differential Lemma 29.6 the vertical map to the left is given by multiplication by the differential $\mathrm{D}f_P \in \mathrm{M}(n\times n, \mathbb{R})$. It follows from the Orientation Preserving-via-Determinants Lemma 31.7 that the vertical map to the left is orientation-preserving if and only if $\det(\mathrm{D}f_P) > 0$.
(4) It follows from the Orientation-Composition Lemma 31.8 that the left-hand vertical map is orientation-preserving if and only if the right-hand vertical map is orientation-preserving. ■

The following lemma summarizes a few elementary properties of orientations and of orientation-preserving maps.

[382] Unfortunately the symbol $\mathrm{D}f_P$ has several, subtly different meanings. To the left it stands for the induced map on tangent spaces. To the right it stands for the $(n\times n)$–matrix given by the partial derivatives of f.

Lemma 31.31. (Manifold–Orientation Basics Lemma) Let $n \in \mathbb{N}_0$ and let M be an n-dimensional smooth manifold.

(1) If two orientations on M agree at a point and if M is path-connected, then the two orientations agree everywhere.

(2) If M is path-connected and non-empty, then M is either non-orientable or it admits precisely two orientations, which are opposites of one another.

(3) Let $f\colon M \to N$ be a diffeomorphism. If M is path-connected and if there exists a point $P \in M$, such that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is orientation-preserving, then f is orientation-preserving.[383] The analogous statement also holds for "orientation-reversing" instead of "orientation-preserving".

(4) Suppose that M is equipped with an orientation $\mathcal{O}$. Let $f\colon M \to M$ be a self-diffeomorphism. Then

$$\begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } \mathcal{O} \end{array} \iff \begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } -\mathcal{O}. \end{array}$$

(5) Let $M_0, \dots, M_k$ be oriented smooth manifolds. Suppose that for $i = 0, \dots, k-1$ we are given a smooth map $f_i\colon M_i \to M_{i+1}$ such that for every $P \in M_i$ the differential $\mathrm{D}(f_i)_P\colon \mathrm{T}_P M_i \to \mathrm{T}_{f_i(P)}M_{i+1}$ is an isomorphism. Then

$$\begin{array}{c} \text{the map } f_{k-1} \circ \cdots \circ f_0\colon M_0 \to M_k \\ \text{is orientation-preserving} \end{array} \iff \begin{array}{c} \text{the number of } f_i \text{ which are} \\ \text{orientation-reversing is even.} \end{array}$$

In particular the composition of orientation-preserving diffeomorphisms is again orientation-preserving.

(6) Let N be a smooth manifold and let $f\colon N \to M$ be a smooth map such that all differentials $\mathrm{D}f_y\colon \mathrm{T}_y N \to \mathrm{T}_{f(y)}M$ are isomorphisms (e.g. f could be an immersion, or a local diffeomorphism or the inclusion of a codimension-zero submanifold), then there exists a unique orientation on N such that f is an orientation-preserving map.

Convention 31.32. Let M be an oriented smooth manifold and let N be a submanifold of codimension zero. We refer to the orientation on N that we introduced in the Manifold–Orientation Basics Lemma 31.31 (6b), i.e. the unique orientation on N such that the inclusion map $N \to M$ is orientation-preserving, as the standard orientation of a codimension-zero submanifold.

Example. It follows from the Manifold–Orientation Basics Lemma 31.31 that for each $n \in \mathbb{N}$ the closed ball $\overline{B}^n$ inherits an orientation from $\mathbb{R}^n$. We refer to this as the standard orientation on $\overline{B}^n$. Given $P \in \overline{B}^n$ it follows immediately from the definitions that the image of the standard basis $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ of $\mathbb{R}^n$ under the natural isomorphism $\Theta\colon \widetilde{\mathrm{T}}_P\overline{B}^n \to \mathrm{T}_P\overline{B}^n$ defines a positive basis for $\mathrm{T}_P\overline{B}^n$. □

[383] By definition $f\colon M \to N$ is orientation-preserving if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is orientation-preserving for every $P \in M$. If M is path-connected, then Statement (3) says that it suffices to verify this condition for a single point.

Proof. The case $n = 0$ follows from trivial considerations. Thus we now assume that $n \geq 1$.

(1) Suppose we are given two orientations on M. It follows immediately from the definitions that for any chart $\Phi\colon U \to V$ the two orientations either agree on all of U or they disagree on all of U.

But this implies that the set of points A where the orientations agree is open and that the set of points B where the orientations disagree is also open. If the orientations agree at a point we see that $A \neq \varnothing$. It follows from the Manifold–Local Properties Lemma 23.8 (3) that $A = M$.

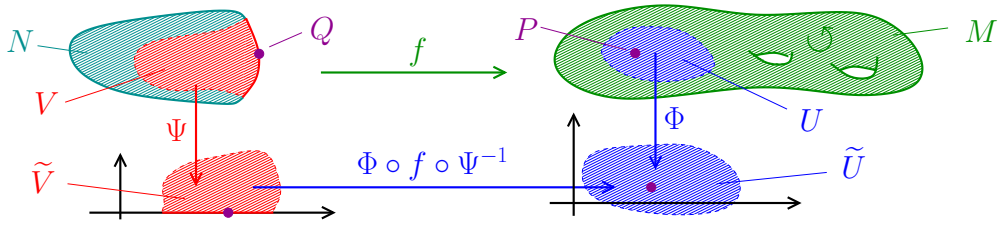
(2) This statement follows easily from (1).

(3) The proof of this statement is very similar to the proof of Statement (1). We leave it to the reader to fill in the details.

(4) This statement follows immediately from the definitions and the Orientation-Composition Lemma 31.8.

(5) Note only does the lemma smell like the Orientation-Composition Lemma 31.8, it is also an immediate consequence thereof.

(6) Given $Q \in N$ we equip the tangent vector space $\mathrm{T}_Q N$ with the orientation that turns the map $\mathrm{D}f_Q\colon \mathrm{T}_Q N \to \mathrm{T}_{f(Q)}M$ (which is by hypothesis an isomorphism) into an orientation-preserving isomorphism. It remains to show that these orientations define an orientation for N. So let $Q \in N$. We pick a chart $\Psi\colon V \to \widetilde{V}$ for $Q \in N$ and we pick an orientation-preserving chart $\Phi\colon U \to \widetilde{U}$ for $P := f(Q) \in M$. After possibly replacing V by $V \cap f^{-1}(U)$ we can assume that $V \subseteq f^{-1}(U)$. After further restrictions we can assume that V is path-connected.



For each $y \in V$ we now consider the following diagram:

$$\begin{array}{ccc} \mathrm{T}_y V & \xrightarrow{\quad \mathrm{D}f_y \quad} & \mathrm{T}_{f(y)} U \\ {\scriptstyle \mathrm{D}\Psi_y}\downarrow & & \downarrow{\scriptstyle \mathrm{D}\Phi_{f(y)}} \\ \mathbb{R}^k & \xrightarrow{\quad v \mapsto \mathrm{D}(\Phi \circ f \circ \Psi^{-1})_{\Psi(y)} \cdot v \quad} & \mathbb{R}^k. \end{array}$$

We make the following observations:

(a) It follows from the Chain rule that the diagram commutes.

(b) The right vertical map is orientation-preserving for all $x \in U$.

(c) The top horizontal map is, by definition, orientation-preserving for all $y \in V$.

(d) Since V is path-connected we see that the bottom horizontal map has the same orientation behavior for all points in $\widetilde{y} \in \widetilde{V}$.

It follows from the above that the left vertical map has the same orientation behavior for all $y \in V$. But that is what we needed to show. ∎

Definition 31.33. Let M be an oriented smooth manifold. An atlas $\{\Phi_i \colon U_i \to \widetilde{U}_i\}_{i\in I}$ is called orientation-preserving if each Φ_i is orientation-preserving. Hereby we work with the following orientations:

(1) Each open subset U_i is a codimension-zero submanifold of M, thus we can equip it with the above standard orientation.
(2) Each $\widetilde{U}_i$ is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Therefore it also comes with a standard orientation.

Remark. Note that if $\dim(M) > 0$, then any chart $\Phi \colon U \to V$ with V can be turned into an orientation-preserving respectively orientation-reversing chart by possibly composing Φ, on components of V, with the reflection in a hyperplane. In particular we see that any oriented smooth manifold of dimension ≥ 1 admits an orientation-preserving atlas. □

The following lemma gives a convenient criterion for showing that a given map is orientation-preserving.

Lemma 31.34. (Orientation-via-Atlas Lemma) Let $f \colon M \to N$ be a smooth map between two oriented n-dimensional smooth manifolds such that for each $P \in M$ the differential $\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an isomorphism. Let $\{\Phi_i \colon U_i \to \widetilde{U}_i\}_{i\in I}$ be an orientation-preserving atlas for M and let $\{\Psi_j \colon V_j \to \widetilde{V}_j\}_{j\in J}$ be an orientation-preserving atlas for N. The following two statements are equivalent:

(1) f is orientation-preserving.
(2) For each $i \in I$ and $j \in J$ and each $P \in U_i \cap f^{-1}(V_j)$ we have

$$\det\Big(\underbrace{\mathrm{D}(\Psi_j \circ f \circ \Phi_i^{-1})_{\Phi_i(P)}}_{\text{invertible } (n\times n)\text{-matrix}}\Big) \;>\; 0.$$

The analogous statement also holds for "orientation-reversing" and "< 0".

Proof. This lemma can be deduced easily from the Orientation Checking-via-Differentials Lemma 31.30 and the Manifold–Orientation Basics Lemma 31.31 (5) and the fact, shown in the Differential-equals-Differential Lemma 29.6 and the Tangent Space-Isomorphism Proposition 29.24 that the various definitions of differential are compatible. We leave it to the reader to fill in the few remaining details. ■

We continue with a lemma which gives another approach to defining an orientation on a given smooth manifold.

Lemma 31.35. (Orientations-via-Transition Maps Lemma) Let $(M, \mathcal{A})$ be a smooth manifold of dimension ≥ 1. Suppose that all transition maps between charts in $\mathcal{A}$ are orientation-preserving. Given $P \in M$ we define

$$\text{orientation of } \mathrm{T}_P M \;\coloneqq\; \overset{\text{see Definition 31.6}}{\overset{\downarrow}{(\mathrm{D}\Phi_P)}}{}_*^{-1}\Big(\underbrace{\text{standard orientation of } \mathrm{T}_{\Phi(P)} V}_{\text{as defined on page 812}}\Big)$$

where $\Phi \colon U \to V$ is a chart around P from the given atlas. The following statements hold:

(1) The definition of the orientation of $\mathrm{T}_P M$ does not depend on the choice of the chart.
(2) The orientations of the tangent space $\mathrm{T}_P M$ define an orientation for M.

(3) With respect to this orientation the given atlas is orientation-preserving.

Sketch of proof. Statement (1) follows easily from the hypothesis that the transition maps are orientation-preserving together with the Orientation-Composition Lemma 31.8. Statement (2) is a consequence of the Orientations-via-Atlas Lemma 31.21. Statement (3) holds basically by definition. ■

Example.

(1) (a) We consider the real projective space $\mathbb{R}\mathrm{P}^n$. One can easily verify that if n is odd, then the transition maps of the atlas provided by the Projective Space–Manifold Proposition 25.13 (1) are orientation-preserving. It follows from the Orientations-via-Transition Maps Lemma 31.35 that for odd n the real projective space $\mathbb{R}\mathrm{P}^n$ admits a unique orientation that makes the charts orientation-preserving. We refer to this as the standard orientation of $\mathbb{R}\mathrm{P}^n$.

(b) We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. One can easily verify that for any $n \in \mathbb{N}$ the transition maps of the atlas provided by the Projective Space–Manifold Proposition 25.13 (2) are orientation-preserving. It follows from the Orientations-via-Transition Maps Lemma 31.35 that the complex projective space $\mathbb{C}\mathrm{P}^n$ admits a unique orientation that makes the charts orientation-preserving. We refer to this as the standard orientation of $\mathbb{C}\mathrm{P}^n$.

It follows easily from Exercise 31.8 that under the identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ from Identifiation 4.25 and the Projective Space-Avatar Lemma 9.16 the above standard orientation on $\mathbb{C}\mathrm{P}^1$ is (unfortunately) the *opposite* of the standard orientation on S^2 which we introduced on page 817.

(2) We consider the surface Σ of genus 2. In the Surface-is-Topological Manifold Proposition 23.7 we gave an explicit atlas for Σ. In the Standard Smooth Structures Proposition 25.6 we saw that the transition maps are smooth. In Exercise 31.11 we will verify that all the transition maps are in fact orientation-preserving. Thus it follows from the Orientations-via-Transition Maps Lemma 31.35 that Σ is orientable. Basically the same argument shows that all surfaces of any genus are also orientable. Note that, as the name suggests, the same approach does not work for the non-orientable surfaces of some genus $g \in \mathbb{N}$. □

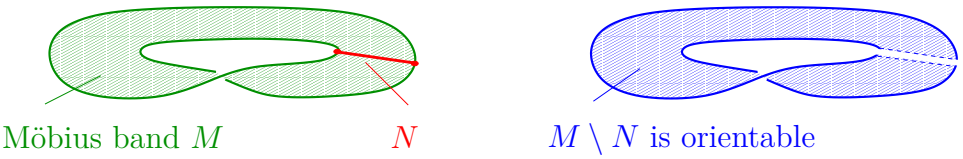
Lemma 31.36. (Complement-of-Boundary Orientation Lemma) Let M be a smooth manifold.

(1) Given any orientation on $M \setminus \partial M$ there exists a unique orientation on M such that the inclusion $M \setminus \partial M \to M$ is orientation-preserving.

(2) Let N be a submanifold of M of codimension ≥ 2 that is also a closed subset of M. Given any orientation on[384] $M \setminus N$ there exists a unique orientation on M such that the inclusion $M \setminus N \to M$ is orientation-preserving.

Non-Example. Let M be the Möbius band and let N be the proper 1-dimensional submanifold that is shown in the figure below. Then $M \setminus N$ is evidently orientable, whereas M is not orientable.

[384] Since $N \subseteq M$ is closed we know that $M \setminus N$ is an open subset of M, thus it is a smooth manifold.



Thus we see that in the Complement-of-Boundary Orientation Lemma 31.36 the condition on the codimension is necessary. □

Proof. Let M be an n-dimensional smooth manifold. In the following we assume that $n \geq 2$. The case $n = 1$ can be dealt with by a similar argument or one can directly appeal to the Smooth 1-Manifold Classification Theorem 28.8.

(1) We start out with a general claim.

Claim 1. Let $U \subseteq M$ be an open subset. We set $\widetilde{U} \coloneqq U \cap (M \setminus \partial M)$.
(a) If U is connected, then so is $\widetilde{U}$.
(b) The closure of $\widetilde{U}$ in M contains U.

Proof.

(a) Statement (1) follows from the Manifold–Local Properties Lemma 23.8 (3a), Open Subset-Submanifold Boundary Lemma 25.11 and the Push Away-from-Boundary Lemma 23.17 (2).

(b) This statement follows quite easily from the Interior-Closure Characterization Lemma 3.23. ⊞

Now let $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ be an atlas for M such that each U_i is connected. We equip the V_i, which are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, with the standard orientation. Given $i \in I$ we write $\widetilde{U}_i \coloneqq U_i \cap (M \setminus \partial M)$, $\widetilde{V}_i \coloneqq \Phi(\widetilde{U}_i)$ and we denote by $\widetilde{\Phi}_i$ the restriction of Φ_i to the diffeomorphism $\widetilde{\Phi}_i \colon \widetilde{U}_i \to \widetilde{V}_i$.

Now suppose that we are given an orientation for $M \setminus \partial M$. By Claim 1 (a) we know that each $\widetilde{U}_i$ is connected. Thus we know, say by the Manifold–Orientation Basics Lemma 31.31 (3), that $\widetilde{\Phi}_i \colon \widetilde{U}_i \to \widetilde{V}_i$ is orientation-preserving or orientation-reversing. We now do the following:

- In the first case we set $\rho_i \coloneqq \mathrm{id}_{\mathbb{R}^n}$.
- In the second case denote by $\rho_i \colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in the $(x_1 = 0)$-hyperplane.[385]

Note that in either case the map $\rho_i \circ \widetilde{\Phi}_i \colon \widetilde{U}_i \to \rho_i(\widetilde{V}_i)$ is orientation-preserving.

Claim 2. All transition maps of the atlas $\{\rho_i \circ \Phi_i \colon U_i \to \rho_i(V_i)\}_{i \in I}$ for M are orientation-preserving.

Proof. Let $i, j \in I$. By the Orientation Checking-via-Differentials Lemma 31.30 it suffices to show that the image of

$$\begin{aligned} f \colon U_i \cap U_j &\to \mathbb{R} \\ x &\mapsto \det(\underbrace{\mathrm{D}((\rho_i \circ \Phi_i) \circ (\rho_j \circ \Phi_j)^{-1})_{\rho_j(\Phi_j(x))}}_{(n \times n)\text{-matrix}}) \end{aligned}$$

[385]Here we use that $n \geq 2$, since for $n \geq 2$ the reflection in the $(x_1 = 0)$-hyperplane is an orientation-reversing diffeomorphism of $\mathbb{H}^n$.

consists only of positive real numbers. Note that this map f is smooth and non-zero everywhere. Furthermore, since $\rho_i \circ \widetilde{\Phi}_i$ and $\rho_j \circ \widetilde{\Phi}_j$ are orientation-preserving we see, say using the Orientation Checking-via-Differentials Lemma 31.30, that f is positive on $\widetilde{U}_i \cap \widetilde{U}_j$.

Now let $x \in U_i \cap U_j$. Since $U_i \cap U_j$ is an open neighborhood of x we obtain from Claim 1 that x is contained in the closure of $\widetilde{U}_i \cap \widetilde{U}_j = (U_1 \cap U_2) \cap (M \setminus \partial M)$ in M. Thus it follows from the Closure-Interior Mapping Lemma 4.6 that $f(x) \geq 0$. ⊞

It follows from the claim together with the Orientations-via-Transition Maps Lemma 31.35 that this atlas defines an orientation on M. It is clear that this orientation restricts to the given orientation on $M \setminus \partial M$.

(2) The argument is basically the same as in (1). In the proof of Claim 1 (a) we just need to replace the Push Away-from-Boundary Lemma 23.17 by Exercise 23.24. The remainder of the argument is identical. ■

31.5. **Boundary orientation.** In this section we will see that the boundary of an oriented smooth manifold inherits an orientation. Before we can formulate the lemma let us recall Definition 29.22.

Definition 31.37. Let M be a smooth manifold, let $P \in \partial M$ and let $w \in \mathrm{T}_P M$. We say that w points outward, if the following two conditions are satisfied:

(1) For every smooth map $\zeta\colon M \to \mathbb{R}_{\geq 0}$ with $\zeta|_{\partial M} \equiv 0$ we have $w(\zeta) \leq 0$.
(2) There exists a smooth map $\zeta\colon M \to \mathbb{R}_{\geq 0}$ with $\zeta|_{\partial M} \equiv 0$ such that $w(\zeta) < 0$.

By replacing "$\leq$" and "$<$" by "$\geq$" and "$>$" we obtain the notion of an inward pointing tangent vector.

Lemma 31.38. (Boundary Orientation Lemma) Let M be an oriented n-dimensional smooth manifold and let $P \in \partial M$.

(a) If $n = 1$, then we define the orientation of P as follows:

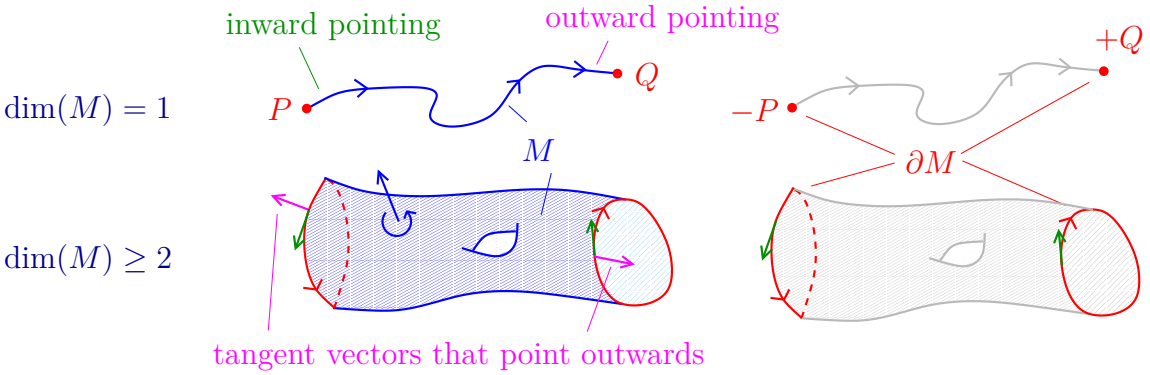
$$\text{orientation of } \mathrm{T}_P \partial M := \begin{cases} -1, & \text{if a positive tangent vector of } \mathrm{T}_P M \text{ points inward,} \\ +1, & \text{if a positive tangent vector of } \mathrm{T}_P M \text{ points outward.} \end{cases}$$

(b) Now we assume that $n \geq 2$. We pick a tangent vector $w \in \mathrm{T}_P M$ that points outward in the sense of Definition 29.22. We say that a basis $(v_1, \ldots, v_{n-1})$ for $\mathrm{T}_P(\partial M)$ is a positive basis of $\mathrm{T}_P(\partial M)$ if $(w, v_1, \ldots, v_{n-1})$ is a positive basis of $\mathrm{T}_P M$.

The following five statements hold:

(0) If M is a smooth submanifold of some $\mathbb{R}^k$, then the orientation convention in (a) and (b) also makes sense for the physical tangent spaces and it is compatible with the Tangent Space-Isomorphism Proposition 29.24.
(1) For $n \geq 2$ the definition of a positive basis for $\mathrm{T}_P M$ does not depend on the choice of w.
(2) These orientations of the tangent spaces of ∂M define an orientation on ∂M.
(3) We have $\partial(-M) = -\partial M$.
(4) Let $f\colon M \to N$ be an orientation-preserving (orientation-reversing) local diffeomorphism between oriented n-dimensional smooth manifolds. We equip ∂M and ∂N with

the orientations defined in (a) and (b). The restriction of f to a diffeomorphism $\partial M \to \partial N$ is also orientation-preserving (orientation-reversing).



Example. The boundary of the 1-dimensional smooth manifold $[a, b]$, which we equip with the standard orientation coming from $\mathbb{R}$, is easily seen to be given by $(-\{a\}) \cup \{b\}$. □

Proof. The proof in the case that $\dim(M) = 1$ is basically trivial. Thus in the remainder of the proof we only consider the case that $n = \dim(M) \geq 2$.

(0) This statement follows easily from the definitions together with Tangent Space-Isomorphism Proposition 29.24 (1b).

(1) Let $P \in \partial M$ and let $(v_1, \ldots, v_{n-1})$ be a basis for $\mathrm{T}_P(\partial M)$. We pick an orientation-preserving chart $\Phi\colon U \to V$ of type (ii) around P. We denote by $i\colon U \to M$ the inclusion map. We consider the isomorphisms

$$\mathrm{T}_P M \xleftarrow[\uparrow]{\mathrm{D}i_P} \mathrm{T}_P U \xrightarrow{\mathrm{D}\Phi_P} \mathrm{T}_P V \xrightarrow[\uparrow]{\Theta} \widetilde{\mathrm{T}}_P V \underset{\uparrow}{=} \mathbb{R}^n.$$

isomorphism by the Tangent Space-Inclusion Proposition 29.20

Tangent Space Isomorphism Proposition 29.24

by the Physical Tangent Spaces–Lemma 29.2

We denote by $\Omega\colon \mathrm{T}_P M \to \mathbb{R}^n$ the resulting isomorphism. Furthermore we denote by $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $(n-1)$ coordinates and we denote by $q\colon \mathbb{R}^n \to \mathbb{R}$ the projection onto the last coordinate. We make the following two observations:

(a) It follows fairly easily from the Tangent Space-Isomorphism Proposition 29.24 (1b) that a vector $w \in \mathrm{T}_P M$ points outward if and only if $q(\Omega(w))$ is negative.

(b) Given a vector $w \in \mathrm{T}_P M$ the following hold:

$$\begin{aligned} &(w, v_1, \ldots, v_{n_1}) \text{ is a positive basis for } \mathrm{T}_P M \\ &\quad\Longleftrightarrow (\Omega(w), \Omega(v_1), \ldots, \Omega(v_{n-1})) \text{ is a positive basis for } \mathbb{R}^n \\ &\quad\Longleftrightarrow \det(\Omega(w)\,\Omega(v_1)\,\ldots\,\Omega(v_{n-1})) > 0 \\ &\quad\Longleftrightarrow (-1)^n \cdot q(\Omega(w)) \cdot \det(p(\Omega(v_1))\,\ldots\,p(\Omega(v_{n-1}))) > 0. \end{aligned}$$

It follows that the definition of a positive basis is independent of the choice of the outward pointing vector w.

(2) Using the approach of (1) it is not difficult to verify this statement.

(3) This statement is basically obvious.

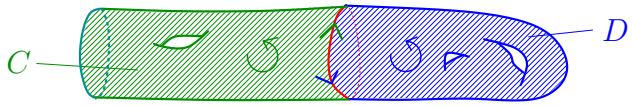
(4) This statement follows immediately from the observation that a local diffeomorphism sends outward pointing vectors to outward pointing vectors. ■

Convention 31.39. Let M be an oriented n-dimensional smooth manifold. Unless we say something else we always equip ∂M with the orientation given by the Boundary Orientation Lemma 31.38. We refer to this orientation of ∂M as the boundary orientation.

Example. Let $n \in \mathbb{N}$. As on page 819 we equip the smooth manifold $\overline{B}^{n+1}$ with the standard orientation coming from $\mathbb{R}^{n+1}$. It is not very hard to see that the boundary orientation of $\partial\overline{B}^{n+1} = S^n$ agrees with the standard orientation of S^n that we introduced on page 817. □

The boundary orientation can cause some confusion, since at times a smooth manifold is the boundary of two different smooth manifolds. For clarity's sake we state the following lemma:

Lemma 31.40. (Opposite Side–Opposite Orientation Lemma) Let M be a smooth manifold and let C and D be codimension-zero submanifolds such that $C \cap D = \partial C = \partial D$ and such that $M = C \cup D$. We suppose that M is oriented. We equip the codimension-zero submanifolds C and D with the standard orientations defined on page 819 and we equip ∂C and ∂D with the orientations from Convention 31.39. We then have the following equality of oriented smooth manifolds: $\partial C = -\partial D$.



Proof. The statement follows easily from the definitions. Working out the details is outsourced to Exercise 31.19. ■

At this point it makes sense to introduce the following definition:

Definition 31.41. Given $n \in \mathbb{N}_0$ we refer to the category $\mathcal{OrSmMfd}_n$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{OrSmMfd}_n) &:= \text{all oriented } n\text{-dimensional smooth manifolds,} \\ \mathrm{Mor}(X,Y) &:= \text{all orientation-preserving diffeomorphisms from } X \text{ to } Y, \end{aligned}$$

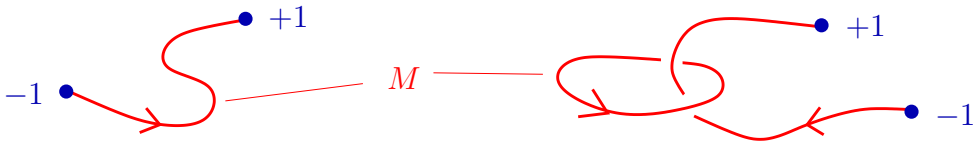
with the usual composition of maps as the category of oriented n-dimensional smooth manifolds.

Remark. Note that the Boundary Orientation Lemma 31.38 says in particular that the map $M \mapsto \partial M$ defines a covariant functor from the category $\mathcal{OrSmMfd}_n$ of oriented n-dimensional smooth manifolds to the category $\mathcal{OrSmMfd}_{n-1}$ of oriented $(n-1)$-dimensional smooth manifolds. □

We conclude this discussion of the orientations of the boundary of 1-dimensional smooth manifolds with the following subtle consequence of the Smooth 1-Manifold Classification Theorem 28.8.

Proposition 31.42. (Boundary-of-Compact Smooth 1-Manifold Proposition) Let M be a compact oriented 1-dimensional smooth manifold. We equip ∂M with the orientation defined on page 825. Then

$$\sum_{P\in\partial M} \underbrace{\text{orientation of } P}_{\in\{-1,1\}} = 0.$$



Proof. It suffices to prove the statement for a compact oriented connected 1-dimensional smooth manifold with non-empty boundary. By the Smooth 1-Manifold Classification Theorem 28.8 we know that there exists a diffeomorphism $f\colon M \to [0,1]$. After possibly composing this diffeomorphism with the orientation-reversing map $[0,1] \to [0,1]$ given by $t \mapsto 1-t$ we can assume that f is in fact orientation-preserving. Now we see that

$$\sum_{P\in\partial M} \epsilon(P) \underset{\substack{\uparrow\\ \text{by the Boundary Orientation Lemma 31.38}}}{=} \sum_{P\in\partial([0,1])} \epsilon(P) \underset{\substack{\uparrow\\ \text{see the example on page 825}}}{=} -1+1 = 0.$$

■

31.6. **Orientations and group actions.** We move on to considering the interplay between orientations and group actions.

Definition 31.43. Let M be an oriented smooth manifold that is equipped with a smooth action by a group G. We say that the action is orientation-preserving if for every $g \in G$ the diffeomorphism $M \to M$ given by $x \mapsto g \cdot x$ is orientation-preserving in the sense of Definition 31.29.

The first part of the following proposition can be viewed as a refinement of the Manifold–Quotient Proposition 25.39.

Proposition 31.44. (Orientation-Action Proposition) Let M be a smooth manifold. Furthermore let G be a group that acts freely, properly and smoothly on M. By the Manifold–Quotient Proposition 25.39 (2) the quotient M/G admits a standard smooth structure such that the projection $p\colon M \to M/G$ is a local diffeomorphism.

(1) If M is oriented and if the G-action is orientation-preserving, then the smooth manifold M/G admits a unique orientation such that the projection $p\colon M \to M/G$ is orientation-preserving.

(2) Conversely, if the smooth manifold M/G admits an orientation, then there exists a unique orientation on M such that the projection $p\colon M \to M/G$ is orientation-preserving. Furthermore this orientation has the property that the G-action on M is orientation-preserving.

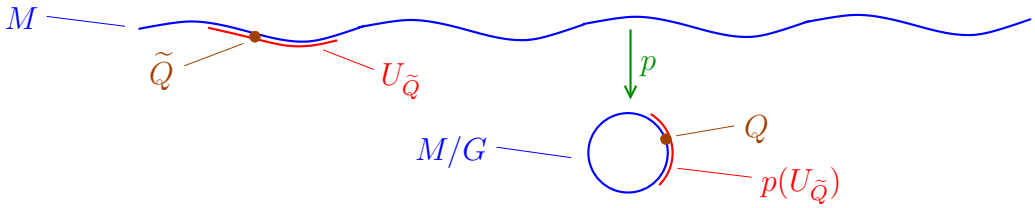
Remark. In Exercise 31.9 we will prove a partial converse to the Orientation-Action Proposition 31.44 (2). □

Proof. Both statements follow fairly easily from the definitions. But for completeness' sake let us sketch the argument.

(1) We assume that M is oriented and that for each $g \in G$ the map $M \to M$ given by $Q \mapsto g \cdot Q$ is orientation-preserving.

For each $Q \in M/G$ we pick some $\widetilde{Q} \in M$ with $p(\widetilde{Q}) = Q$ and we equip $\mathrm{T}_Q(M/G)$ with the orientation given by applying the isomorphism $\mathrm{D}p_{\widetilde{Q}} \colon \mathrm{T}_{\widetilde{Q}}M \to \mathrm{T}_Q(M/G)$ to the orientation of $\mathrm{T}_{p(\widetilde{Q})}M$. Since the G-action is orientation-preserving we see that this definition of the orientation is independent of the choice of $\widetilde{Q}$. We claim that these orientations of the tangent spaces of M/G form an orientation of the smooth manifold M/G.

For each $Q \in M/G$ we pick a chart $\Phi_{\widetilde{Q}} \colon U_{\widetilde{Q}} \to V_{\widetilde{Q}}$ around $\widetilde{Q}$ which has the property that the restriction of p to $U_{\widetilde{Q}}$ is a diffeomorphism onto its image. The maps $\Phi_{\widetilde{Q}} \circ (p|_{U_{\widetilde{Q}}})^{-1} \colon p(U_{\widetilde{Q}}) \to V_{\widetilde{Q}}$ form a smooth atlas for M/G. It follows almost immediately from the definitions that the orientations we just defined are continuous with respect to these charts. Since these charts form a smooth atlas for M/G we obtain from the Orientations-via-Atlas Lemma 31.21 that we have defined an orientation on M/G. By construction the map $p \colon M \to M/G$ is orientation-preserving.



(2) Now we assume that M/G is equipped with an orientation. For each $\widetilde{Q} \in M$ there exists an open neighborhood $U_{\widetilde{Q}}$ of $\widetilde{Q}$ such that the restriction of p to $p \colon U_{\widetilde{Q}} \to p(U_{\widetilde{Q}})$ is a diffeomorphism and such that there exists a smooth chart $\Phi_{\widetilde{Q}} \colon p(U_{\widetilde{Q}}) \to V_{\widetilde{Q}}$ for $p(\widetilde{Q}) \in M/G$.

We equip $\mathrm{T}_{\widetilde{Q}}M$ with the orientation that is obtained by applying the isomorphism $\mathrm{D}(p|^{-1}_{U_{\widetilde{Q}}})_{\widetilde{Q}} \colon \mathrm{T}_{p(\widetilde{Q})}(M/G) \to \mathrm{T}_{\widetilde{Q}}M$ to the given orientation of $\mathrm{T}_{p(\widetilde{Q})}(M/G)$.

Since the charts $\Phi_{\widetilde{Q}} \circ p \colon U_{\widetilde{Q}} \to V_{\widetilde{Q}}$ define a smooth atlas for M we see that these orientations do indeed define an orientation on M. Basically by construction these orientations are invariant under the G-action. ■

Example. Let $v \in \mathbb{R}^n$. We denote by $t \colon \mathbb{R}^n \to \mathbb{R}^n$ the translation by v. Note that given any $x \in \mathbb{R}^n$ the differential $\mathrm{D}t_x \in \mathrm{M}(n \times n, \mathbb{R})$, in the sense of Definition 24.15, is the identity matrix. Thus it follows from the Orientation Checking-via-Differentials Lemma 31.30 that the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ that is given by addition, is orientation-preserving. It follows from the Orientation-Action Proposition 31.44 that $\mathbb{R}^n/\mathbb{Z}^n$ is orientable. □

We formulate the next examples as a lemma.

Lemma 31.45. (Surface Non-Orientability Lemma) The following smooth manifolds are non-orientable:

(1) the Möbius band,

(2) the projective plane $\mathbb{R}\mathrm{P}^2$,
(3) the Klein bottle, and
(4) any non-orientable surface of genus ≥ 1.

Remark. In Exercises 31.6 and 31.10 we will give different, arguably more satisfactory proofs that the projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are non-orientable. □

Proof. Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. As in the Möbius Band-as-Quotient Lemma 9.6 we consider the action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1, 1]) & \to & \mathbb{R} \times [-1, 1] \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n \cdot y). \end{array}$$

As we remarked in the Möbius Band-as-Quotient Lemma 9.6 and on page 684, this action is free, proper and smooth. By definition the quotient $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ is the Möbius band.

Claim 1. The Möbius band is non-orientable.

Proof. Suppose that the Möbius band $X/\mathbb{Z}$ admits an orientation. It follows from the Orientation-Action Proposition 31.44 that $X = \mathbb{R} \times [-1, 1]$ admits an orientation such that the $\mathbb{Z}$-action is orientation-preserving. It follows from the Manifold–Orientation Basics Lemma 31.31 (3) and (4) that this $\mathbb{Z}$-action preserves the standard orientation on the smooth manifold $X = \mathbb{R} \times [-1, 1]$. Given $n \in \mathbb{Z}$ we consider the diffeomorphism of $X = \mathbb{R} \times [-1, 1]$ that is given by the action by n, i.e. we consider the diffeomorphism

$$\begin{array}{rcl} \Phi_n \colon \mathbb{R} \times [-1, 1] & \to & \mathbb{R} \times [-1, 1] \\ (x, y) & \mapsto & (x + n, (-1)^n \cdot y). \end{array}$$

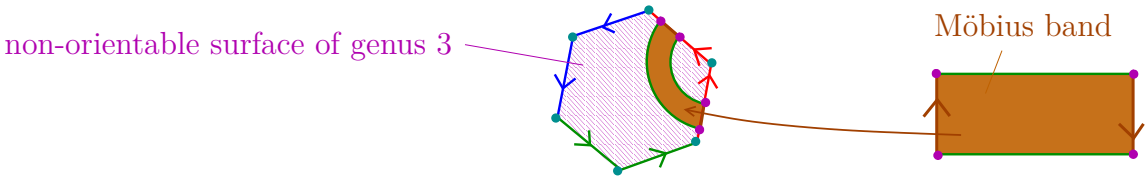
Given any $(x, y) \in X$ the differential of Φ_n is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$. It follows from the Orientation Checking-via-Differentials Lemma 31.30 that for n odd the diffeomorphism Φ_n is not orientation-preserving. We have thus obtained a contradiction. ⊞

We turn to the proof that the three given closed smooth manifolds are non-orientable. By the Manifold–Orientation Basics Lemma 31.31 (6) it suffices to prove the following claim:

Claim 2. The open Möbius band $([0, 1] \times (0, 1))/(0, y) \sim (1, 1 - y)$ is diffeomorphic to open subsets of the Möbius band, of the projective plane $\mathbb{R}\mathrm{P}^2$, of the Klein bottle and for any non-orientable surface of genus $g \geq 1$.

Proof. It is clear that the open Möbius band is an open subset of the Möbius band. The statements regarding the projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle follow immediately from the Möbius Band-in-Surface Lemma 25.51.

Recall that by Definition 9.19 the real projective plane is precisely the non-orientable surface of genus ≥ 1 and that by Exercise 9.17 the Klein bottle is diffeomorphic to the non-orientable surface of genus two. Thus it remains to deal with non-orientable surfaces of genus $g \geq 3$. In the figure below we indicate that for any $g \geq 3$ the non-orientable surface of genus g admits an open subset that is diffeomorphic to the open Möbius band.



We leave it to the reader to upgrade this proof to the desired level of rigor. ■

Remark. For most people starting out with mathematics the definition of a submanifold M of some $\mathbb{R}^n$ and the definition of a tangent space to M as a vector subspace of $\mathbb{R}^n$ are quite natural and reasonable. But initially it is perhaps not clear why one should introduce "abstract" smooth manifolds. Furthermore Definition 29.11 of the tangent space $\mathrm{T}_P M$ to such a smooth manifold traditionally leads to a lot of head scratching.

The above proof of the Surface Non-Orientability Lemma 31.45 is a good example why it is useful to introduce these abstract notions. In principle it is possible to show that the Möbius band, viewed as a submanifold of $\mathbb{R}^3$, is non-orientable purely by working in the explicit 3-dimensional setting, see e.g. [**Frie2016a**, Lemma 9.16]. But this proof is rather messy and not very illuminating. In contrast, the fact that we can work with abstract smooth manifolds, in this case with $(\mathbb{R} \times [-1,1])/\mathbb{Z}$, and their tangent spaces liberates us from the constraints of the 3-dimensional setting and using the Orientation Checking-via-Differentials Lemma 31.30 and the Manifold–Orientation Basics Lemma 31.31 we obtain a much clearer and much more elegant proof that the Möbius band is non-orientable. □

We conclude this section with the following lemma which generalizes the Surface Non-Orientability Lemma 31.45 (3).

Lemma 31.46. (Real Projective Space Orientability Lemma) Let $n \in \mathbb{N}$. The real projective space $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd.

Proof. On page 822 we already showed that if n is odd, then $\mathbb{R}\mathrm{P}^n$ is orientable. The converse, that if $\mathbb{R}\mathrm{P}^n$ is orientable, then n is odd will be shown in Exercise 31.6 ■

31.7. **Products of smooth manifolds and orientations.** In this section we will study products of oriented smooth manifolds. As discussed in Section 25.9 and Chapter 26, at this point it is natural to also study smooth manifolds with corner.
The following definition is copy-pasted from Definition 29.11.

Definition 31.47. Let M be a smooth manifold with *corner*. Given $P \in M$ we define the tangent space $\mathrm{T}_P M$ at P as the set of all derivations at P, i.e. $\mathrm{T}_P M$ is the vector space of all linear maps $w\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M,\mathbb{R})$ we have

$$w(f \cdot g) \;=\; f(P) \cdot w(g) + w(f) \cdot g(P) \qquad \text{"Leibniz Rule".}$$

Remark. The obvious generalizations of the results obtained in Section 29.3 hold with minor changes to the proofs. Furthermore one can also define, with almost the same definition as in Definition 31.19, the notion of an orientation on a smooth manifold with corner. Since every author's duty is to strive for brevity we will not go into detail. □

For the reader's convenience we recall the Smooth Manifold Product Proposition 26.12.

Proposition 26.12. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We denote by

$$\begin{array}{rcl} p\colon M \times N & \to & M \\ (x,y) & \mapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon M \times N & \to & N \\ (x,y) & \mapsto & y \end{array}$$

the two projections.

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold that admits a *unique* structure as a smooth manifold with corner which satisfies the following two conditions:
   (a) The projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
   (b) For any smooth manifold with corner W and any map $f\colon W \to M \times N$ the following holds: If the two maps $p \circ f\colon W \to M$ and $q \circ f\colon W \to N$ are smooth, then f itself is smooth.

(2) (a) $\angle(M \times N) = \partial M \times \partial N$,[386]
   (b) $\partial(M \times N) = (M \times \partial N) \cup (\partial M \times N)$.

We now have the following addendum to the Smooth Manifold Product Proposition 26.12.

Proposition 31.48. (Product Orientation Proposition) We continue with the notation of the Smooth Manifold Product Proposition 26.12. For the new statements we introduce the notation that given points $P \in M$ and $Q \in N$ we consider the two smooth embeddings

$$\begin{array}{rcl} i_Q\colon M & \to & M \times N \\ x & \mapsto & (x, Q) \end{array} \quad \text{and} \quad \begin{array}{rcl} j_P\colon N & \to & M \times N \\ y & \mapsto & (P, y). \end{array}$$

The following statements hold:

(3) (a) All the inclusion maps i_P and j_Q are smooth.
   (b) Let $P \in M$ and $Q \in N$. The map

$$(\mathrm{D}i_Q)_P \oplus (\mathrm{D}j_P)_Q\colon \mathrm{T}_P M \oplus \mathrm{T}_Q N \to \mathrm{T}_{(P,Q)}(M \times N)$$

   is a natural isomorphism, the inverse is given by $(\mathrm{D}p_{(P,Q)}, \mathrm{D}q_{(P,Q)})$. (Sometimes we use this isomorphism to identify the vector spaces to the left and right.)[387]
   (c) If M and N are oriented, then we equip the direct sum $\mathrm{T}_P M \oplus \mathrm{T}_Q N$ with the direct sum orientation from Definition 31.13. We use the isomorphism from (3b) to equip $\mathrm{T}_{(P,Q)}(M \times N)$ with the corresponding orientation. These orientations of tangent spaces define an orientation of $M \times N$.
   (d) If M and N are oriented, then

$$(-M) \times N \;=\; M \times (-N) \;=\; -(M \times N).$$

(4) We suppose that M and N are both oriented. If $\partial N = \varnothing$, then the above equality $\partial(M \times N) = (\partial M) \times N$ from (2b) is in fact an equality of oriented smooth manifolds.[388]

Convention 31.49. Let M and N be two oriented smooth manifolds. Unless we say something else we always equip the product $M \times N$ with the orientation given by the Product Orientation Proposition 31.48 (3). We refer to this orientation of $M \times N$ as the product orientation on $M \times N$.

Proof.

(3) (a) This statement follows from the Smooth Manifold Product Proposition 26.12 (1).
   (b) This statement follows easily from the functoriality of the differentials.

[386] Recall that in Definition 26.9 we introduced the corner $\angle W$ of a smooth manifold with corner.
[387] In the Tangent Bundle-of-Product Proposition 268.6 we will formulate and prove a refinement of this statement.
[388] In Exercise 31.13 we will determine the relationship between the orientations of $\partial(M \times N)$ and $M \times \partial N$ if N is allowed to have non-empty boundary.

(c) As so often, when it comes to verifying that orientations of tangent spaces define an orientation of the smooth manifold, we leave it to the reader to fill in the details.
(d) This statement follows from very elementary linear algebra.

(4) Let $P \in \partial M$ and $Q \in N$. Using Exercise 29.2 one can show that for an outward pointing vector $w \in \mathrm{T}_P M$ the image $\mathrm{D}(i_Q)_P(w) \in \mathrm{T}_{(P,Q)}(M \times N)$ is also outward pointing. We leave it to the reader to fill in the remaining details. ■

Remark. The definition of the product orientation is very slick, but it is perhaps worth spelling this out in more detail.

(1) If M and N are both of dimension ≥ 1, then it follows easily from the definitions that for any $(P, Q) \in M \times N$ the orientation on $\mathrm{T}_{(P,Q)}(M \times N)$ is determined by the condition that

$$((\mathrm{D}i_Q)_P(\text{positive basis of } \mathrm{T}_P M), (\mathrm{D}j_P)_Q(\text{positive basis of } \mathrm{T}_Q N))$$

is a positive basis of $\mathrm{T}_{(P,Q)}(M \times N)$.

(2) Let N be a smooth manifold and let $\{P\}$ be the 0-dimensional smooth manifold consisting of a single point P that is equipped with an orientation $\epsilon \in \{-1, 1\}$.
(a) If $\epsilon = 1$, then the product orientation on $\{P\} \times N$ is the orientation that turns the obvious map $N \to \{P\} \times N$ into an orientation-preserving diffeomorphism.
(b) If $\epsilon = -1$, then the production orientation $\{P\} \times N$ is the orientation that turns the obvious map $N \to \{P\} \times N$ into an orientation-reversing diffeomorphism.
For $N \times \{P\}$ we have the analogous statement.

(3) If $M = \{(P, \mu)\}$ and $N = \{(N, \nu)\}$ are two oriented 0-dimensional smooth manifolds, each consisting of a single point, then it follows from the above definition and the remark on page 807 that $M \times N = \{(P, Q)\}$ is equipped with the orientation $\mu \cdot \nu$. □

Example.

(1) Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. We equip both with the standard orientation that we defined on page 812. It follows fairly easily from the definitions that the product orientation on $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ equals the standard orientation of $U \times V$ viewed as an open subset of $\mathbb{R}^{m+n}$.

(2) Recall that on page 817 we equipped the 0-dimensional sphere $S^0 = \{\pm 1\}$ with the orientation given by $\pm 1 \mapsto \pm 1$. It follows immediately from the definitions that for any oriented smooth manifold M we have a natural orientation-preserving diffeomorphism $M \times S^0 \xrightarrow{\cong} (-M) \sqcup M$.

(3) Given any oriented smooth manifold N with empty boundary and given $a < b \in \mathbb{R}$ we have the equality

$$\partial([a,b] \times N) \underset{\uparrow}{=} \partial([a,b]) \times N \underset{\uparrow}{=} (-\{a\} \cup \{b\}) \times N = (-\{a\} \times N) \cup (\{b\} \times N).$$

by the Product Orientation Proposition 31.48 (4) — see page 825

□

We have one last, rather unsurprising lemma.

Lemma 31.50. (Product Orientation Preserving Lemma) Let $f\colon M \to \widetilde{M}$ and let $g\colon N \to \widetilde{N}$ be continuous maps between oriented smooth manifolds. We consider the map

$$\begin{array}{rccc} \operatorname{diag}(f,g)\colon & M \times N & \to & \widetilde{M} \times \widetilde{N} \\ & (x,y) & \mapsto & (f(x),g(y)). \end{array}$$

Let $x \in M$ and $y \in N$ be points such that $\mathrm{D}f_x\colon \mathrm{T}_x M \to \mathrm{T}_{f(x)}\widetilde{M}$ is an isomorphism and such that $\mathrm{D}g_y\colon \mathrm{T}_y N \to \mathrm{T}_{g(y)}\widetilde{N}$ is an isomorphism.

(1) The map $\mathrm{D}\operatorname{diag}(f,g)_{(x,y)}\colon \mathrm{T}_{(x,y)}(M \times N) \to \widetilde{\mathrm{T}}_{(\widetilde{x},\widetilde{y})}(\widetilde{M} \times \widetilde{N})$ is an isomorphism.

(2) The map $\operatorname{diag}(f,g)$ is orientation-preserving at (x,y) if and only if both or none of f and g are orientation-preserving at x respectively y.

Proof. The lemma can be proved easily using product charts. We leave it to the reader to fill in the details. ∎

This concludes our introduction to the notion of an orientation of smooth manifolds. Perhaps somewhat unintuitively there does also exist a perfectly fine notion of the orientation of a topological manifold. We will introduce this notion in Chapter 191. In the Smooth-Topological Orientation Proposition 191.25 we will see that this generalizes the present notion of an orientation. Furthermore in the Boundary Orientation Proposition 192.3 we will generalize the Boundary Orientation Lemma 31.38 to topological manifolds. Finally in the Product Orientation Proposition 192.7 (1) and the Product Orientation Proposition 192.7 (2) we will generalize the Product Orientation Proposition 31.48 to topological manifolds.

31.8. Exercises for Chapter 31.

Exercise 31.1. Let $v \in \mathbb{R}^n$ be a non-zero vector. We consider the orthogonal complement of v, i.e. we consider the vector space $v^\perp = \{w \in \mathbb{R}^n \mid \langle v, w\rangle = 0\}$. Show that

$$(w_1, \ldots, w_{n-1}) \text{ is a positive basis of } v^\perp \quad :\Longleftrightarrow \quad \det(v\ w_1\ \ldots\ w_{n-1}) > 0$$

defines an orientation on $v^\perp$.

Exercise 31.2. Let W be an oriented finite-dimensional vector space and let U, V be oriented vector subspaces of W with $W = U + V$. What is the relationship between the intersection orientation of $U \cap V$ and the intersection orientation of $V \cap U$?

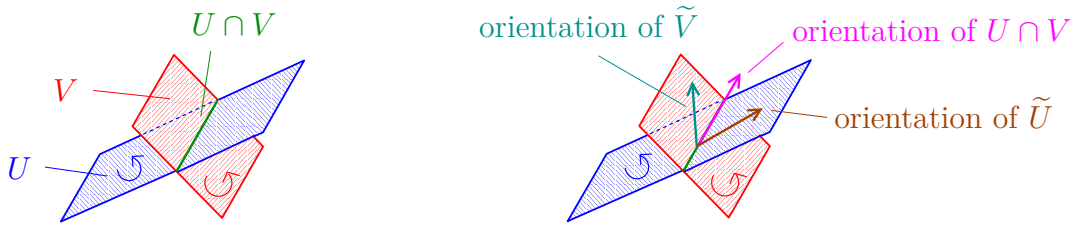
Exercise 31.3. Let W be an oriented real vector space and let U and V be oriented vector subspaces with $W = U + V$. We pick a vector subspace $\widetilde{U} \subseteq U$ such that $U = \widetilde{U} \oplus (U \cap V)$ and we pick a vector subspace $\widetilde{V} \subseteq V$ such that $V = (U \cap V) \oplus \widetilde{V}$.

(a) Show that there exist unique orientations on $U \cap V$, $\widetilde{U}$ and $\widetilde{V}$ such that the direct sum orientations of $U = \widetilde{U} \oplus (U \cap V)$, $V = (U \cap V) \oplus \widetilde{V}$ and $W = \widetilde{U} \oplus (U \cap V) \oplus \widetilde{V}$ are the given orientation.

(b) Show that the orientation of $U \cap V$ is precisely the intersection orientation.

Exercise 31.4. Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$ and let φ be the unique real number in $(-\pi, \pi)$ such that there exists a $\lambda > 0$ with

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} v_1 \ = \ \lambda \cdot v_2.$$



Show that the basis (v_1, v_2) is a positive basis of $\mathbb{R}^2$ if and only if $\varphi \in (0, \pi)$.

Exercise 31.5. Let $A \in \mathrm{O}(n)$. Show that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$.

Exercise 31.6. Let $n \in \mathbb{N}$.

(a) Show that the action of $G = \{\pm 1\}$ on S^n is orientation-preserving if and only if n is odd.
Hint. Use the Tangent Space-Isomorphism Proposition 29.24 (1b) and the Manifold–Orientation Basics Lemma 31.31 (3).

(b) Show that the real projective space $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd.
Hint. Use (a) together with the Manifold–Orientation Basics Lemma 31.31 and the Orientation-Action Proposition 31.44.

Exercise 31.7. Let $n \in \mathbb{N}$ and let $A \in \mathrm{GL}(n, \mathbb{Z})$. By the Maps-on-Torus Lemma 25.46 we know that the map

$$\begin{aligned} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n &\to \mathbb{R}^n/\mathbb{Z}^n \\ [v] &\mapsto [A \cdot v] \end{aligned}$$

is a diffeomorphism. Show that this diffeomorphism is orientation-preserving if and only if $A \in \mathrm{SL}(n, \mathbb{Z})$, i.e. if and only if $\det(A) = 1$.

Exercise 31.8.

(a) Let $p\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from the Stereographic Projection Lemma 4.24. It restricts to a diffeomorphism $\varphi\colon S^n \setminus \{(0, \dots, 0, 1)\} \to \mathbb{R}^n$. We equip both sides with the standard orientation. Show that φ is orientation-preserving if and only if n is odd.

(b) We consider the projection

$$\begin{aligned} \psi\colon S^n_{>0} = \{(x_1, \dots, x_{n+1}) \in S^n \,|\, x_{n+1} > 0\} &\to B^n \\ (x_1, \dots, x_{n+1}) &\mapsto (x_1, \dots, x_n). \end{aligned}$$

We equip both sides with the standard orientation. Show that ψ is orientation-preserving if and only if n is even.

Exercise 31.9. Let G be a group that acts freely, properly and smoothly on an oriented n-dimensional smooth manifold M. Suppose that M/G is orientable.

(a) Suppose that M is path-connected. Show that every $g \in G$ acts on M in an orientation-preserving way.
Hint. Use the Orientation-Action Proposition 31.44 (2).

(b) Show that the conclusion of (a) does not hold if M is not path-connected.

Remark. This exercise can be viewed as a converse to the Orientation-Action Proposition 31.44 (1).

Exercise 31.10. We consider the following two self-diffeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

We denote by G the subgroup of all diffeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means $G = \begin{array}{c} \text{all self-diffeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite composition of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$

(a) For $n \in \mathbb{Z}$, what is $A^{-n}BA^n$?

(b) We write $C = A^2$. We denote by H the subgroup of all diffeomorphisms of $\mathbb{R}^2$ that is generated by C and B. Does A lie in H?
Hint. Consider which self-diffeomorphisms of $\mathbb{R}^2$ are orientation-preserving.

(c) Show that any element X of G lies in H or it is of the form $G = AY$ with $Y \in H$.

(d) Show that the group G acts freely and properly on $\mathbb{R}^2$.

(e) We consider the square $X = [0,1] \times [0,1]$ with the equivalence relation that is generated by $(0,y) \sim (1, 1-y)$ and $(x,0) \sim (x,1)$. Show that the map

$$\begin{array}{rcl} X/\!\sim & \to & \mathbb{R}^2/G \\ [(x,y)] & \mapsto & [(x,y)] \end{array}$$

is a diffeomorphism. (Recall that in Exercise 25.19 we equipped $X/\!\sim$ with a smooth structure.) This implies that $\mathbb{R}^2/G$ is diffeomorphic to the Klein bottle that we defined on page 312.

Remark. This discussion, together with the Orientation-Action Proposition 31.44, gives a new proof of the fact, already shown in the Surface Non-Orientability Lemma 31.45, that the Klein bottle is non-orientable.

Exercise 31.11. We consider the surface Σ of genus 2 and we consider the explicit atlas that we constructed in the proof of the Surface-is-Topological Manifold Proposition 23.7. Show that the transition maps are all orientation-preserving.

Exercise 31.12. Can a closed non-orientable connected smooth manifold be the boundary of a compact smooth manifold?

Exercise 31.13. Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We equip the product smooth manifolds with corner $M \times N$ and $N \times M$ with the product orientations as defined in the Product Orientation Proposition 31.48.

(a) Show that the diffeomorphism

$$\begin{array}{rcl} M \times N & \to & N \times M \\ (x,y) & \mapsto & (y,x) \end{array}$$

is orientation-preserving if and only if $m \cdot n$ is even.

(b) We suppose that $\partial N = \varnothing$. We equip ∂M and $\partial(N \times M)$ with the boundary orientation and we equip $N \times \partial M$ with the product orientation. Show that we have the following equality of *oriented* smooth manifolds:

$$\partial(N \times M) = \begin{cases} N \times \partial M, & \text{if } n \text{ is even,} \\ -(N \times \partial M), & \text{if } n \text{ is odd.} \end{cases}$$

Exercise 31.14. Let M be an oriented smooth manifold with empty boundary. We denote by $-M$ the same smooth manifold with the opposite orientation. Show that the product orientations given by $M \times M$ and $(-M) \times (-M)$ are actually the same.

Exercise 31.15. Let F be a smooth manifold and let $\mu\colon F \to F$ be a self-diffeomorphism. In the Mapping Torus-of-Manifold Lemma 25.36 we saw that the mapping torus

$$\operatorname{Tor}(F,\mu) \;:=\; (F \times [0,1])/(x,0) \sim (\mu(x),1)$$

is naturally a smooth manifold. Show that $\operatorname{Tor}(F,\mu)$ is orientable if and only if F is orientable and if μ is orientation-preserving.

Exercise 31.16. Let M be a smooth codimension-one submanifold of $\mathbb{R}^n$, let $\gamma\colon [0,1] \to M$ be a curve and let $v \in (\widetilde{\mathrm{T}}_{\gamma(0)}M)^\perp$. Show that there exists a unique continuous map $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}^n$ such that $\widetilde{\gamma}(0) = \gamma(0) + v$ and such that for each $t \in [0,1]$ we have $\widetilde{\gamma}(t) - \gamma(t) \in (\widetilde{\mathrm{T}}_{\gamma(t)}M)^\perp$ with $\|\widetilde{\gamma}(t) - \gamma(t)\| = \|v\|$.
Remark. Evidently we want $\widetilde{\gamma}$ to be continuous. Note that we do not assume that M is orientable.



Exercise 31.17. We consider $\widetilde{M} := \mathbb{R} \times [-1,1]$ and the self-diffeomorphism

$$\begin{array}{rcl} g\colon \widetilde{M} = \mathbb{R} \times [-1,1] & \to & \widetilde{M} = \mathbb{R} \times [-1,1] \\ (r,s) & \mapsto & (r+1,-s). \end{array}$$

Recall that the Möbius band is defined as $M = \widetilde{M}/\!\sim$ where $P \sim g(P)$. We denote by $p\colon \mathbb{R} \times [-1,1] \to M$ the projection. We set $x := p(0,0)$. Show that there exists a smooth isotopy $F\colon M \times [0,1] \to M$ with the following properties:

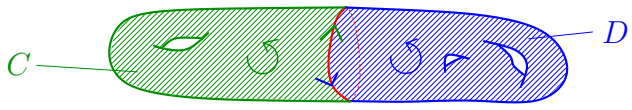
(1) Each F_t is a diffeomorphism. (Later we refer to such a map as a diffeotopy of M.)
(2) $F_0 = \mathrm{id}$.
(3) $F_1(x) = x$.
(4) $\mathrm{D}(F_1)_x\colon \mathrm{T}_xM \to \mathrm{T}_xM$ is orientation-reversing.
(5) Bonus exercise: Arrange that F is a homotopy rel ∂M.

Exercise 31.18. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an orientation-preserving local diffeomorphism. Show that f is strictly monotonously increasing, i.e. show that for any real numbers $s > t$ we have $f(s) > f(t)$.

Exercise 31.19. Let M be a smooth manifold and let C and D be codimension-zero submanifolds such that $C \cap D = \partial C = \partial D$ and such that $M = C \cup D$.

(a) We suppose that M is oriented. We equip the codimension-zero submanifolds C and D with the standard orientations defined on page 819 and we equip ∂C and ∂D with the orientations from Convention 31.39. Show that as oriented smooth manifolds we have $\partial C = -\partial D$.

(b) Suppose that the codimension-zero submanifolds C and D are oriented such that the oriented smooth manifolds ∂C and ∂D satisfy $\partial C = -\partial D$. Show that M admits an orientation that restricts to the orientations of C and D.
(c) Provide an example such that C and D are orientable, but M is not orientable.
(d) Show that if C is diffeomorphic to $\overline{B}^n$, then M is orientable if and only if D is orientable.



Exercise 31.20.

(a) Let M be a smooth manifold and let $k \in \mathbb{N}_0$. Show that if $M \times B^k$ is orientable, then so is M.
(b) Let M and N be non-empty smooth manifolds. Show that if $M \times N$ is orientable, then so are M and N.

Remark. This exercise can be viewed as a converse to the Product Orientation Proposition 31.48 (3c).

Exercise 31.21. Does there exist an oriented connected 1-dimensional smooth manifold that does not admit an orientation-reversing diffeomorphism?
Remark. In Exercise 51.16 we will study the 2-dimensional case.

Exercise 31.22. Let $f\colon M \to M'$ and $g\colon N \to N'$ be two diffeomorphisms of oriented connected smooth manifolds. Show that $\operatorname{diag}(f,g)\colon M \times N \to M' \times N'$ is orientation-preserving if and only if both f and g are orientation-preserving or if both are orientation-reversing.

Exercise 31.23. An n-dimensional *non-Hausdorff-manifold* is a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$. Note that the notion of a smooth structure and an orientation make sense for a non-Hausdorff-manifolds. Show that there exists a non-orientable 1-dimensional smooth non-Hausdorff-manifold.

Exercise 31.24. Given a ring $S \subseteq \mathbb{R}$ we consider the *Heisenberg group*

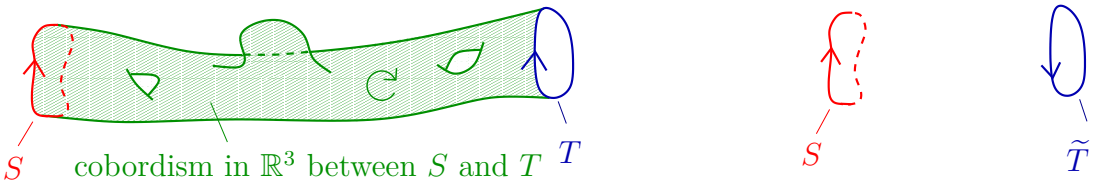
$$H_S := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{R}) \,\middle|\, x,y,z \in S \right\}.$$

(a) Show that the *Heisenberg manifold* $M := H_{\mathbb{R}}/H_{\mathbb{Z}}$ is a 3-dimensional smooth manifold.
(b) Is M compact?
(c) Is M orientable?

Exercise 31.25. Let $n \in \mathbb{N}$ be odd. Show that there exists an orientation-reversing diffeomorphism $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$.

Exercise 31.26. For which $n \in \mathbb{N}$ can you find an orientation-reversing diffeomorphism $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$?
Remark. The goal is to find as many n as possible. Currently we lack the tools to show that some oriented smooth manifolds do not admit an orientation-reversing diffeomorphism.

Exercise 31.27. Let S and T be two closed oriented submanifolds of $\mathbb{R}^n$. We say that S and T are *cobordant in* $\mathbb{R}^n$ if there exists a cobordism in $\mathbb{R}^n$, i.e. if there exists a compact oriented submanifold W of $\mathbb{R}^n$ with $\partial W = S \sqcup -T$. In the figure below to the left we give an example of two oriented curves S and T in $\mathbb{R}^3$ together with a cobordism in $\mathbb{R}^n$. Now we flip the orientation of T to obtain $\widetilde{T}$. Are S and $\widetilde{T}$ still cobordant in $\mathbb{R}^3$?



CHAPTER 32

The Regular Value Theorem

Let M be a smooth manifold, e.g. $M = \mathbb{R}^n$ or $M = \mathrm{M}(n \times n, \mathbb{R})$. In general it can be quite painful to show directly that a given subset, e.g. $\mathrm{O}(n) \subseteq \mathrm{M}(n \times n, \mathbb{R})$ is a smooth submanifold. Only the most enterprising reader will feel like writing down explicit submanifold charts for $\mathrm{O}(n) \subseteq \mathrm{M}(n \times n, \mathbb{R})$. In this chapter we will state and prove the Regular Value Theorem which can be used, among many other applications, to give an elegant proof that $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{R})$.

This chapter is organized as follows:

- In Section 32.1 we will introduce the notion of regular points and regular values
- In Section 32.2 we will use the notion of regular values to give another proof of the Fundamental Theorem of Algebra.
- In Section 32.3 we will state and prove the Regular Value Theorem 32.5 which says that the preimage of a regular value is a proper submanifold.
- In Section 32.4 we will see that matrix groups are smooth manifolds.
- In Section 32.5 we will provide exercises for Chapter 32.

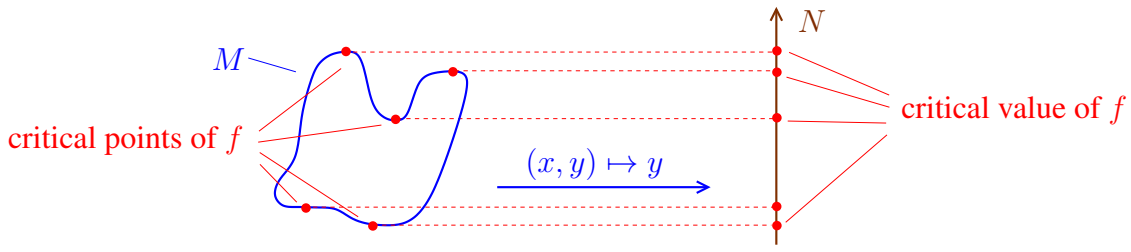
In the next Chapter 33 we will prove the Euclidean Embedding Theorem 33.2 which says that every smooth manifold admits a smooth embedding into some $\mathbb{R}^n$.

32.1. **Regular values.** Not surprisingly the formulation of the Regular Value Theorem is preceded by the definition of a regular value. This notion might be familiar to the reader from an earlier course on real analysis.

Definition 32.1. Let M and N be smooth manifolds (possibly with boundary). Furthermore let $f\colon M \to N$ be a smooth map.

(1) Let $P \in M$. We say that P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an epimorphism. Otherwise we say that P is a critical point.

(2) We say that $Q \in N$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a critical value. In other words, the critical values are precisely the images of the critical points.



The following simply observation will be used several times without explicitly mentioning it.

Observation. Let $f\colon M \to N$ be a smooth map between smooth manifolds.

(1) If $Q \in N$ is a point with $f^{-1}(Q) = \varnothing$, then Q is a regular value.
(2) If $\dim(M) < \dim(N)$, then the converse to (1) holds: it follows from the Tangent Space-Dimension Proposition 29.17 and elementary linear algebra that a point $Q \in N$ is a regular value if and only if we have $f^{-1}(Q) = \varnothing$.
(3) If $\varphi\colon X \to M$ is a local diffeomorphism, then it follows easily from the Chain Rule 24.17 that

$$\text{set of critical points of } f \circ \varphi\colon X \to N \;=\; \varphi^{-1}(\text{set of critical points of } f\colon M \to N).$$

From this observation together with the Image-Preimage Lemma 1.9 (2) it follows that

$$\text{set of critical values of } f \circ \varphi\colon X \to N \;\underset{\substack{\uparrow\\ \text{equality holds}\\ \text{if } \varphi \text{ is surjective}}}{\subseteq}\; \text{set of critical values of } f\colon M \to N.$$

The following lemma is slightly less trivial than the above observation.

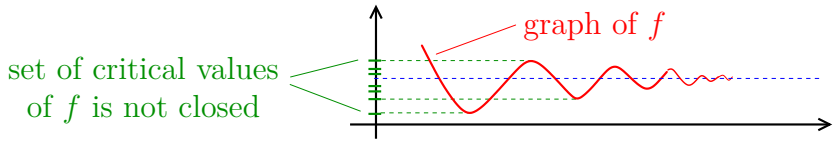
Lemma 32.2. (Regular Value-Open Subset Lemma) Let $f\colon M \to N$ be a smooth map between smooth manifolds.

(1) The set of regular points of f is an open subset of M.
(2) The set of critical points of f is a closed subset of M.

Example. We consider the smooth functions

$$\begin{array}{rcl} f\colon [1,\infty) & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x}\cdot\sin(x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} e^{-\frac{1}{x^2}}\cdot\sin(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

The first example shows that the set of critical values is not necessarily closed and the second example shows that the set of critical points is not necessarily discrete.



For purely artistic reasons we show in the figure above the graph of some other smooth function $f\colon (a,b) \to \mathbb{R}$ for which the set of critical values is not closed. □

Proof. Statement (2) is of course just a reader-friendly reformulation of Statement (1). We write $m := \dim(M)$ and $n := \dim(N)$. Let P be a regular point of f. Next we pick a chart $\Phi\colon U \to V$ around P and we pick a chart $\Psi\colon W \to X$ around $f(P)$. We consider the map

$$\begin{array}{rcl} \Xi\colon \Phi(U \cap f^{-1}(W)) & \to & \mathrm{M}(n \times m, \mathbb{R}) \\ Q & \mapsto & \mathrm{D}(\Psi \circ f \circ \Phi^{-1})_Q. \end{array}$$

The lemma follows easily from the hypothesis that P is regular point and the following four observations:

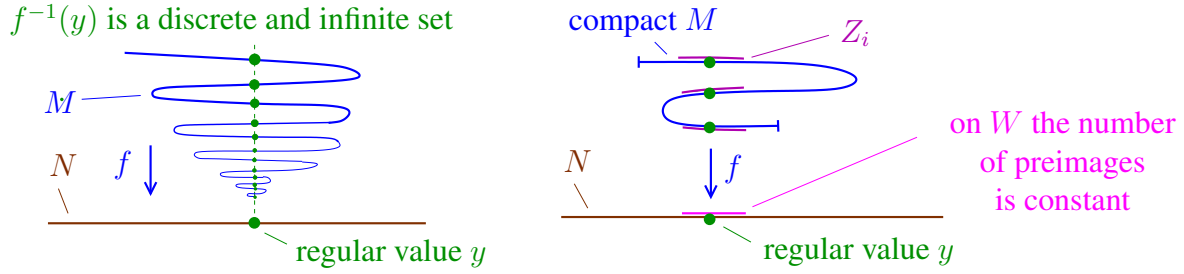
(a) Since f is smooth we see that Ξ is continuous.
(b) A point $Q \in U \cap f^{-1}(W)$ is a regular point of f if and only if $\Xi(\Phi(Q))$ is a matrix of rank m.
(c) $U \cap f^{-1}(W)$ is an open neighborhood of P.

(d) By Lemma 50.14, see also Exercise 4.49, we know that the set of matrices of rank m is an open subset of $\mathrm{M}(n \times m, \mathbb{R})$. ■

In the following lemma we summarize a few basic facts about regular points and values of smooth maps between smooth manifolds *of the same dimension*.

Lemma 32.3. (Stack-of-Records Lemma)[389] Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension.

(1) For every regular point $x \in M \setminus \partial M$ the map f is a local diffeomorphism at x.
(2) Let $y \in N$ be a regular value with $f^{-1}(y) \cap \partial M = \varnothing$.
  (a) The preimage $f^{-1}(y)$ is a discrete subset of M
  (b) If M is compact, then $f^{-1}(y)$ is a finite subset of M.
  (c) If M is compact, then there exists an open neighborhood W of $y \in N$ with the following properties:
    (i) $f^{-1}(W)$ consists of disjoint open subsets $Z_1, \dots, Z_k$ and the restriction of f to each Z_i defines a diffeomorphism $Z_i \to W$.
    (ii) Each $y' \in W$ is a regular value.
    (iii) For each $y' \in W$ we have the equality $\#f^{-1}(y) = \#f^{-1}(y')$.



Proof.

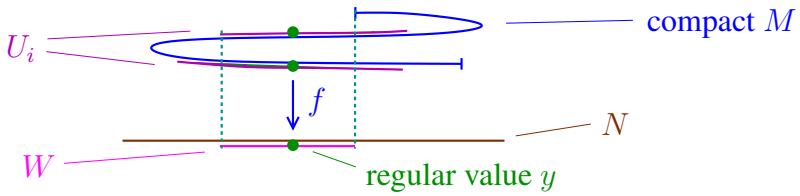
(1) Let $x \in M \setminus \partial M$ be a regular point of f. This means that $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_{f(x)}N$ is an epimorphism. Since M and N have the same dimension this implies that $\mathrm{D}f_x$ is an isomorphism. It follows from the hypothesis that $x \in M \setminus \partial M$ together with the Inverse Mapping Theorem 30.1 that f is a local diffeomorphism at x.
(2) Let y be a regular value of f with $f^{-1}(y) \cap \partial M = \varnothing$.
  (a) It follows immediately from (1) that for each $x \in f^{-1}(y)$ there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. This implies that $U \cap f^{-1}(y) = \{x\}$. We have thus shown that $f^{-1}(y)$ is a discrete subset of M.
  (b) By (a) we know that $f^{-1}(y)$ is a discrete subset of M. If M is compact, then it follows from the Compact-Discrete Lemma 3.40 that $f^{-1}(y)$ is actually a finite set.
  (c) We denote the points in $f^{-1}(y)$ by $x_1, \dots, x_k \in M$. Since $x_1, \dots, x_k$ are regular points we obtain from (a) open neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$ and open neighborhoods $V_1, \dots, V_k$ of $f(x_1), \dots, f(x_k)$ such that for each $i \in \{1, \dots, k\}$ the map $f_i\colon U_i \to V_i := f(U_i)$ is a diffeomorphism. Since M is Hausdorff we can and

[389] The idea behind the name is that W is a disk ("record") and "above it" lie k disks ("records"). The name might make more sense for the readers over 50.

will arrange that the U_i are pairwise disjoint. We set

$$W := \underbrace{\Big(\bigcap_{i=1}^{k} V_i \Big)}_{\text{open subset}} \setminus \underbrace{f\Big(M \setminus \bigcup_{i=1}^{k} U_i\Big)}_{\substack{\text{since } M \text{ is compact we obtain from the Compact Image Lemma 4.15 that this is a closed subset of } N}}.$$

Note that W is open and that it contains y.



Furthermore note that by construction we have $f^{-1}(W) \subseteq \bigcup_{i=1}^{k} U_i$. For $i = 1, \ldots, k$ we set $Z_i := f^{-1}(W) \cap U_i$. Evidently these sets are still disjoint.

(i) For each $i \in \{1, \ldots, k\}$ the map $f\colon Z_i \to W$ is a bijection and it is given by the restriction of a diffeomorphism to an open subset. Hence the map $f\colon Z_i \to W$ is actually a diffeomorphism.

(ii) It follows from (i) that each point in $f^{-1}(W)$ is a regular point, thus every point in W is a regular value.

(iii) By (i) the preimage of each $y' \in V$ consists of precisely k points, namely one x'_i in each U_i. In other words, we have shown that $\#f^{-1}(y) = k$ for all $y \in W$. ■

32.2. **The Fundamental Theorem of Algebra.** Perhaps rather surprisingly the straightforward Stack-of-Records Lemma 32.3 allows us to prove the Fundamental Theorem of Algebra:

Theorem 32.4. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Remark. We will provide two other proofs of the Fundamental Theorem of Algebra 32.4 on page 1751 and on page 2898. □

Proof. Let $q(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree $n \geq 1$ with complex coefficients. We will perform a proof by contradiction, so we now assume that $q(z)$ has no zeros. In our argument below we will use the following two inputs which we will not prove.

(i) Every complex polynomial of degree $d \geq 0$ has at most d zeros.

(ii) If p is a complex polynomial of degree ≥ 1, then the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto p(z)$ takes on infinitely many values.

We make a few preparations for the proof:

(1) We write $P_+ = (0,0,+1)$. As in the Stereographic Projection Lemma 4.24 we consider the stereographic projection from P_+ which is given as follows:

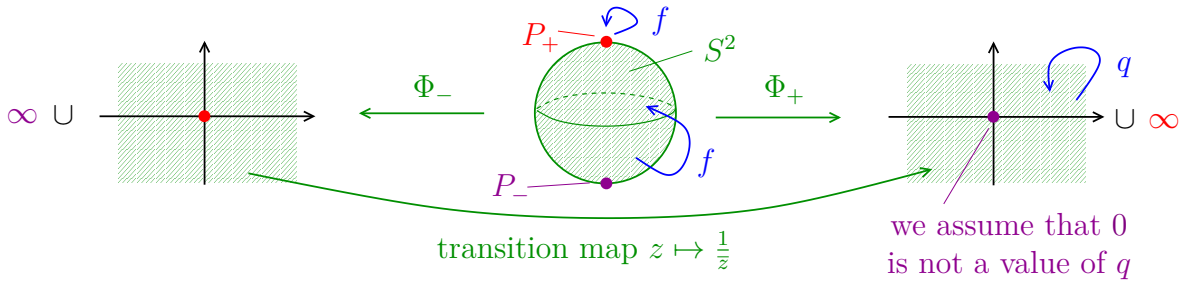
$$\begin{array}{rcl} \Phi_+ \colon S^2 & \to & \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} \\ (x_1,x_2,x_3) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right), & \text{if } x_3 \neq +1, \\ \infty, & \text{if } x_3 = +1. \end{cases} \end{array}$$

Similarly we write $P_- = (0,0,-1)$ and we denote by $\Phi_- \colon S^2 \to \mathbb{C} \cup \{\infty\}$ the stereographic projection from P_-. Note that $\Phi_\pm(P_\pm) = \infty$ and $\Phi_\pm(P_\mp) = 0$.

(2) By Exercise 23.19 we know that the transition map between the two charts is given by

$$\begin{array}{rcl} \Phi_+ \circ \Phi_-^{-1} \colon \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \frac{1}{|z|^2} \cdot z = \frac{1}{\bar{z}}. \end{array}$$

(3) As on page 622 and in the Standard Smooth Structures Proposition 25.6 we equip S^2 with the usual smooth atlas given by Φ_+ and Φ_-.



Claim. The map

$$\begin{array}{rcl} f \colon S^2 & \to & S^2 \\ P & \mapsto & \begin{cases} \Phi_+^{-1}(q(\Phi_+(P))), & \text{if } P \neq P_+, \\ P_+, & \text{if } P = P_+ \end{cases} \end{array}$$

has the following two properties:

(1) The map f is smooth.

(2) The map f has at most n critical points.

Proof.

(1)
- The map $\Phi_+ \circ f \circ \Phi_+^{-1} \colon \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto q(z)$, hence it is smooth.
- Note that our assumption that q has no zeros implies that P_- does not lie in the image of f. This in turn implies that ∞ does not lie in the image of $\Phi_- \circ f$. Thus we can consider the map

$$\Phi_- \circ f \circ \Phi_-^{-1} \colon \mathbb{C} \to \mathbb{C}.$$

It follows from the discussion of the transition map and the fact that $f(P_+) = P_+$ that this map is given by

$$z \mapsto \begin{cases} 0, & \text{if } z = 0, \\ \dfrac{1}{\overline{f(\frac{1}{\bar{z}})}} = \dfrac{z^n}{\bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n}, & \text{if } z \neq 0. \end{cases}$$

But since $a_n \neq 0$ we see that this map is smooth, even at zero.

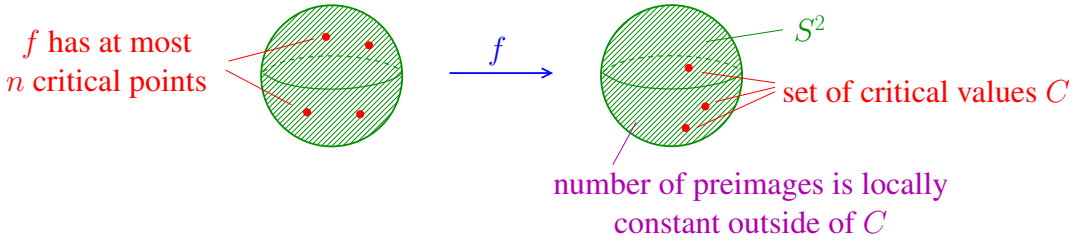
It follows from the above, together with the Smooth Pasting Proposition 25.19, that f is smooth.

(2) Note that by standard results in complex analysis, see e.g. the Smooth-Complex Differentiability Proposition 39.5, we know that for a holomorphic function the derivative corresponds to the differential. It follows almost immediately from this observation together with input (i) (applied to the derivative q') that the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto q(z)$ has at most $n-1$ critical points. This implies that f has at most n critical points on $\Phi_+(\mathbb{C}) = S^2 \setminus \{P_+\}$. This implies that f has at most n critical points. ⊞

Now we return to the main argument. First note that if $P_- = \Phi_+^{-1}(0)$ is a critical value of f, then this implies by Observation (3) on page 840 that 0 is a value of q, i.e. q has a zero. Now assume that $P_- = \Phi_+^{-1}(0)$ is not a critical value of f. We denote by C the set of critical values of f. It follows from the claim that C is finite. Next we consider the map

$$\begin{array}{rcl} d\colon S^2 \setminus C & \to & \mathbb{N}_0 \\ P & \mapsto & \#f^{-1}(P). \end{array}$$

It follows from the Stack-of-Records Lemma 32.3 that this map $d\colon S^2 \setminus C \to \mathbb{N}_0$ is locally constant, in particular it is continuous. Since C is finite we obtain from an elementary argument, see e.g. Exercise 23.6, that the left-hand side is connected. Since $\mathbb{N}_0$ is evidently discrete we obtain from the Connected-to-Discrete Lemma 4.37 that $d\colon S^2 \setminus C \to \mathbb{N}_0$ is actually a constant map.



Since q is not a constant polynomial it follows from input (ii) that the map $z \mapsto q(z)$, and thus also the map $f\colon S^2 \to S^2$, takes on infinitely many values. This implies that $d\colon S^2 \setminus C \to \mathbb{N}_0$ cannot be zero everywhere. But since this function is constant and since we assume that P_- is a regular value we see that $d(P_-) = \#f^{-1}(P_-) = \#q^{-1}(0)$ is non-zero. But this implies that q has at least one zero. ■

32.3. **The Regular Value Theorem.** The following theorem is often a convenient way for showing that a given subset of a smooth manifold, e.g. a given subset of some $\mathbb{R}^n$ or some $\mathrm{M}(m \times n, \mathbb{R})$ is in fact a smooth submanifold.

Theorem 32.5. (Regular Value Theorem) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold and let $f\colon M \to N$ be a smooth map. We denote by $f_\partial\colon \partial M \to N$ the restriction of f to ∂M.[390] Let $s \in N \setminus \partial N$ be a point which is a regular value of f and also of $f|_\partial$.

(1) The preimage $X := f^{-1}(s)$ is a proper $(m-n)$-dimensional submanifold of M with $\partial f^{-1}(s) = (f_\partial)^{-1}(s)$.

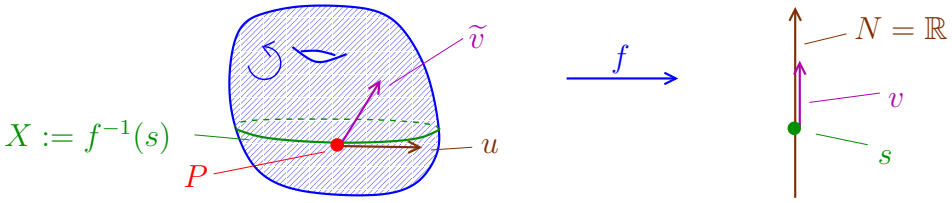
(2) We denote by $i\colon X \to M$ the inclusion. For every $P \in X$ we have

$$\mathrm{im}(\mathrm{D}i_P\colon \mathrm{T}_P X \to \mathrm{T}_P M) \;=\; \ker(\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_s N).$$

In particular, if M is a submanifold of some $\mathbb{R}^k$ and if N is a submanifold of some $\mathbb{R}^l$, then
$$\widetilde{\mathrm{T}}_P X \;=\; \ker(\mathrm{D}f_P\colon \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_s N).$$

(3) Suppose we have an orientation on M and an orientation on $\mathrm{T}_s N$.
   (a) Let $P \in X$. By (2) we have a short exact sequence[391]
$$0 \;\to\; \mathrm{T}_P X \;\xrightarrow{\mathrm{D}i_P}\; \mathrm{T}_P M \;\xrightarrow{\mathrm{D}f_P}\; \mathrm{T}_s N \;\to\; 0.$$
   By our hypothesis the vector spaces in the middle and to the right are oriented. It follows from the Compatible Orientations Lemma 31.12 that there exists a unique orientation on $\mathrm{T}_P X$ such that the orientations are compatible in the sense of Definition 31.11. These orientations on the tangent spaces $\mathrm{T}_P X$ define an orientation on the smooth manifold X.
   (b) If we equip $\partial f^{-1}(s)$ with the boundary orientation, then the orientation agrees with the orientation on $(f_\partial)^{-1}(s)$ defined by the definition of (3a) applied to the map $f_\partial\colon \partial M \to N$.

(4) The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.

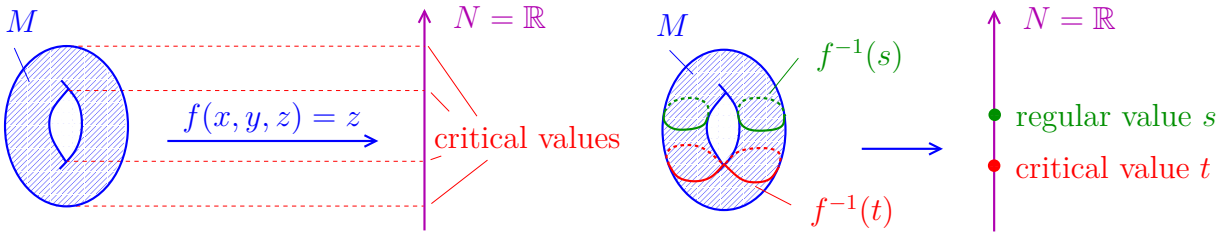


Remark. In the Neighborhood-Regular Value Theorem 56.9, the Transversal Preimage Theorem 50.8 and the Preimage-of-Interval Lemma 38.3 we will prove several extensions and refinements of the Regular Value Theorem 32.5. □

Convention 32.6. Recall that the Submanifold Smooth Structure Lemma 25.23 says that a submanifold of a smooth manifold has a natural smooth structure. We will use this structure to view preimages of regular values as smooth manifolds in their own right.

Example.

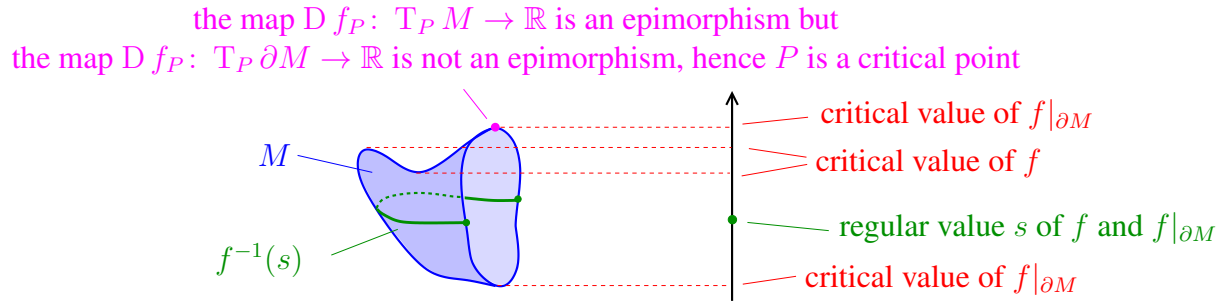
(1) In the figure below we show the torus M as a submanifold of $\mathbb{R}^3$ and we consider the "height function" $f\colon M \to \mathbb{R}$ given by $f(x, y, z) = z$.



[390]Note that it follows from the Smooth Manifold–Boundary Proposition 25.27 (1) that the restriction of a map on a smooth manifold to its boundary is also a smooth map.
[391]We refer to Definition 31.9 for the definition of a short exact sequence.

This function has precisely four critical points which are indicated in the figure on the left. On the right we show a regular value s and its preimage $f^{-1}(s)$ and we show a critical value t and its preimage $f^{-1}(t)$ which in this case is *not* a submanifold of M.

(2) In the figure below we consider again a smooth manifold $M \subseteq \mathbb{R}^3$ together with the "height function".



But this time M has non-empty boundary and $f^{-1}(s)$ is a proper submanifold of M with non-empty boundary.

(3) We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^{n+1} & \to & \mathbb{R} \\ (x_1, \dots, x_{n+1}) & \mapsto & x_1^2 + \dots + x_{n+1}^2. \end{array}$$

Since $\mathrm{D}f_{(x_1,\dots,x_{n+1})} = (2x_1 \ \dots \ 2x_{n+1})$ we see that each point in $\mathbb{R}^{n+1} \setminus \{0\}$ is a regular point and each non-zero $r \in \mathbb{R}$ is a regular value of f. In particular it follows from the Regular Value Theorem 32.5 that $S^n = f^{-1}(1)$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. This proof is significantly more pleasant than the original proof on page 672. Furthermore we see that for each $P \in S^n$ we have

$$\widetilde{\mathrm{T}}_P S^n \ = \ \ker(\mathrm{D}f_P\colon \mathbb{R}^{n+1} \to \mathbb{R}) \underset{\uparrow}{=} \{v \in \mathbb{R}^{n+1} \mid \langle v, P\rangle = 0\}.$$

by the above calculation of $\mathrm{D}f_P$

We have thus found a new proof of the Physical Tangent Spaces–Lemma 29.2 (3). In Exercise 32.14 we determine for which $\in \mathbb{N}$ the orientation defined by the Regular Value Theorem 32.5 (3a) on S^n agrees with the standard orientation that we introduced on page 817. □

Proof of the Regular Value Theorem 32.5. Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold, let $f\colon M \to N$ be a smooth map and let $s \in N \setminus \partial N$ be a regular value of f and of $f|_{\partial M}$. We write $X = f^{-1}(s)$.

(1) Let $P \in X$. Since s is a regular value we know that P is a regular point.

Claim.

(α) If $P \in M \setminus \partial M$, then there exists an $(m-n)$-dimensional submanifold chart of type (α) around P.

(β) If $P \in \partial M$, then there exists an $(m-n)$-dimensional submanifold chart of type (β) around P.

Proof. First we consider the case that $P \in M \setminus \partial M$. This means that we can pick a chart of type (i) for $P \in M$. Since by hypothesis $s \in N \setminus \partial N$ we can also pick a chart of type (i) for $s \in N$. In fact this shows that we might as well assume that M is an

open subset of $\mathbb{R}^m$ and that N is an open subset of $\mathbb{R}^n$. Furthermore after a translation we can assume that $s = 0$.

Next note that it follows from the fact that P is a regular point in $M \setminus \partial M$, together with the Tangent Space-Isomorphism Proposition 29.24, that the differential $\mathrm{D}f_P \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_s N = \mathbb{R}^n$ is an epimorphism. This implies that we can pick a linear map $\Theta \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \mathbb{R}^{m-n}$ such that the map

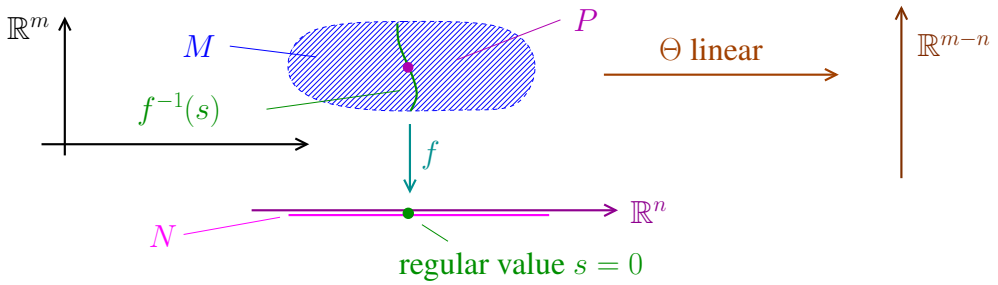
$$\mathrm{D}f_P \times \Theta \colon \underbrace{\widetilde{\mathrm{T}}_P M}_{=\mathbb{R}^m} \;\to\; \underbrace{\widetilde{\mathrm{T}}_s N}_{=\mathbb{R}^n} \times \mathbb{R}^{m-n} = \mathbb{R}^m$$

is an isomorphism. Next we consider the map

$$\Phi := f \times \Theta \colon M \;\to\; N \times \mathbb{R}^{m-n} \subseteq \mathbb{R}^m.$$

Note that $\mathrm{D}\Phi_P = \mathrm{D}f_P \times \Theta$ is an isomorphism. Thus it follows from the Inverse Mapping Theorem 30.1 that there exists an open neighborhood U of $P \in M$ and an open neighborhood V of $(0, \Theta(P)) \in N \times \mathbb{R}^{m-n}$ such that Φ restricts to a diffeomorphism $\Phi \colon U \to V$. We claim that Φ is a submanifold chart of type (α) for $P \in X = f^{-1}(0)$. In fact this follows immediately from the observation that given $Q \in U$ we have:

$$\Phi(Q) \in \{0\} \times \mathbb{R}^{m-n} \iff f(Q) = 0 \iff Q \in X = f^{-1}(0).$$



Now we consider the case that $P \in \partial M$. In this case we might as well assume that M is an open subset of $\mathbb{H}^m$ with $\partial M = M \cap \partial\, \mathbb{H}^m$. As above we can assume that N is an open subset of $\mathbb{R}^n$ and that $s = 0$. We denote by $\chi \colon \mathbb{R}^m \to \mathbb{R}$ the projection onto the last coordinate. Since P is also a regular point of $f|_{\partial M}$ we see that the map $\mathrm{D}f_P \colon \widetilde{\mathrm{T}}_P \partial M \to \widetilde{\mathrm{T}}_s N = \mathbb{R}^n$ is an epimorphism. Note that this implies that the map $\mathrm{D}f_P \times \chi \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \widetilde{\mathrm{T}}_s N \times \mathbb{R}$ is also an epimorphism. It follows that we can pick a linear map $\Theta \colon \mathbb{R}^m = \widetilde{\mathrm{T}}_P M \to \mathbb{R}^{m-n-1}$ such that the map

$$\mathrm{D}f_P \times \Theta \times \chi \colon \underbrace{\widetilde{\mathrm{T}}_P M}_{=\mathbb{R}^m} \;\to\; \underbrace{\widetilde{\mathrm{T}}_s N}_{=\mathbb{R}^n} \times \mathbb{R}^{m-n-1} \times \mathbb{R}$$

is an isomorphism. Next we consider the map

$$\Phi := f \times \Theta \times \chi \colon M \;\to\; N \times \mathbb{R}^{m-n-1} \times \mathbb{R}_{\geq 0} \;\subseteq\; \mathbb{H}^m.$$

The remainder of the argument is basically the same as above, except that this time we end up with a submanifold chart of type (β) for P. ⊞

We have thus found for each point in X a submanifold chart of type (α) or (β). This shows that X is an $(m-n)$-dimensional submanifold of M. Furthermore, note that $X = f^{-1}(s)$ is the preimage of a point, hence it is a closed subset of M. It follows

from this observation and the fact that we only need submanifold charts of type (α) and (β) that X is in fact a proper submanifold in the sense of Definition 23.18.

(2) We denote by $i\colon X \to M$ the inclusion. Since $f \circ i\colon X \to N$ is a constant map it follows from the discussion on page 756 that for every $P \in X = f^{-1}(s)$ we have the inclusion

$$\operatorname{im}(\mathrm{D}i_P\colon \mathrm{T}_P X \to \mathrm{T}_P M) \;\subseteq\; \ker(\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_s N).$$

Furthermore we know the following:

- By the Tangent Space-Inclusion Proposition 29.20 we know that $\mathrm{D}i_P$ is a monomorphism.
- By hypothesis we know that $\dim(\mathrm{T}_P M) = m$ and $\dim(\mathrm{T}_s N) = n$, furthermore by (1) we know that $\dim(\mathrm{T}_P X) = m - n$.
- By hypothesis $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_s N$ is an epimorphism.

It follows from the above observations that the above inclusion is in fact an equality.

Finally consider the case that M is a submanifold of some $\mathbb{R}^k$ and that N is a submanifold of some $\mathbb{R}^l$. By the Tangent Space-Isomorphism Proposition 29.24 we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{T}_P X & \xrightarrow{\mathrm{D}i_P} & \mathrm{T}_P M & \xrightarrow{\mathrm{D}f_P} & \mathrm{T}_s N \\
\cong\downarrow\Theta & & \cong\downarrow\Theta & & \cong\downarrow\Theta \\
\widetilde{\mathrm{T}}_P X & \xrightarrow{\mathrm{D}i_P} & \widetilde{\mathrm{T}}_P M & \xrightarrow{\mathrm{D}f_P} & \widetilde{\mathrm{T}}_s N
\end{array}$$

with exact top row. The statement now follows from the above discussion and the fact, shown in the Physical Tangent Space-Inclusion Proposition 29.5, that the map $\mathrm{D}i_P\colon \widetilde{\mathrm{T}}_P X \to \widetilde{\mathrm{T}}_P M$ is an inclusion.

(3) (a) We leave it to the reader to verify, using the chart from (1), that the given orientations on the tangent spaces $\mathrm{T}_P X$ do indeed define an orientation on the smooth manifold X.

(b) We need to prove that the two orientations on $\partial f^{-1}(s) = (f_\partial)^{-1}(s)$ agree. Thus let $P \in \partial f^{-1}(s)$. We work with the following orientations:

$\mathrm{T}_P(f^{-1}(s))$	with the orientation from (3a) applied to $f\colon M \to N$
$\mathrm{T}_P\,\partial(f^{-1}(s))$	with the boundary orientation from Convention 31.39
$\mathrm{T}_P(f_\partial)^{-1}(s))$	with the orientation from (3a) applied to $f_\partial\colon \partial M \to N$
$\mathrm{T}_P M$	with the given orientation
$\mathrm{T}_P\,\partial M$	with the boundary orientation from Convention 31.39.

To simplify the notation we view $\mathrm{T}_P f^{-1}(s)$ as a vector subspace of $\mathrm{T}_P M$. We pick an outward pointing vector $o \in \mathrm{T}_P f^{-1}(s)$. It follows from Exercise 29.6 that o, viewed as a vector in $\mathrm{T}_P M$, is also outward pointing. Next let $v_1, \dots, v_n$ be a positive basis for $\mathrm{T}_s N$. As before we denote by $\widetilde{v}_1, \dots, \widetilde{v}_n$ vectors in $\mathrm{T}_P M$ that project to $v_1, \dots, v_n$. Finally let $u_1, \dots, u_{m-n-1}$ be a basis of $\mathrm{T}_P\partial(f^{-1}(s)) = \mathrm{T}_P(f_\partial)^{-1}(s)$.

We see the following:

$$\begin{array}{rll} & u_1,\dots,u_{m-n-1} & \text{is a positive basis of } \mathrm{T}_P\,\partial(f^{-1}(s)) \\ \iff & o,u_1,\dots,u_{m-n-1} & \text{is a positive basis of } \mathrm{T}_P f^{-1}(s) \\ \iff & o,u_1,\dots,u_{m-n-1},\widetilde{v}_1,\dots,\widetilde{v}_n & \text{is a positive basis of } \mathrm{T}_P M \\ \iff & u_1,\dots,u_{m-n-1},\widetilde{v}_1,\dots,\widetilde{v}_n & \text{is a positive basis of } \mathrm{T}_P\,\partial M \\ \iff & u_1,\dots,u_{m-n-1} & \text{is a positive basis of } \mathrm{T}_P(f_\partial)^{-1}(s). \end{array}$$

(4) Let K be a compact subset of M. Since $X = f^{-1}(s)$ is a closed subset of M we know $X \cap K$ is a closed subset of K, hence $X \cap K$ is compact by the Compact-Closed Lemma 3.39. ■

32.4. **Matrix groups as smooth manifolds.** In this section we will see that several popular matrix groups are in fact smooth manifolds.

Convention 32.7.

(1) Given $m, n \in \mathbb{N}$ we will make the identification

$$\begin{array}{rcl} \mathrm{M}(m\times n,\mathbb{R}) & \to & \mathbb{R}^{m\cdot n} \\ A=(a_{ij}) & \mapsto & (a_{11},\dots,a_{1n},\dots,a_{m1},\dots,a_{mn}). \end{array}$$

In particular, given a smooth submanifold M of $\mathrm{M}(m\times n,\mathbb{R})$ we can consider the physical tangent space

$$\widetilde{\mathrm{T}}_P M \;\subseteq\; \mathrm{M}(m\times n,\mathbb{R}).$$

(2) We make the usual identification $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ and combining these identifications we will identify $\mathrm{M}(m\times n,\mathbb{C})$ with $\mathbb{C}^{m\cdot n} = \mathbb{R}^{2m\cdot n}$.

(3) Throughout this section we use the Tangent Space-Isomorphism Proposition 29.24, when appropriate, to identify the abstract tangent spaces $\mathrm{T}_P M$ with the physical tangent spaces $\widetilde{\mathrm{T}}_P M$.

We just made the identification $\mathrm{M}(m\times n,\mathbb{R}) = \mathbb{R}^{m\cdot n}$ which allows us to view $\mathrm{M}(m\times n,\mathbb{R})$ as an $m\cdot n$-dimensional smooth manifold. In particular we can thus view the open subset $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n\times n,R) \mid \det(A)\neq 0\}$ also as an n^2-dimensional smooth manifold. The next results show that other matrix groups are also smooth manifolds, albeit in a less trivial way.

Proposition 32.8. (Matrix Group-Smooth Manifold Proposition) Let $n \in \mathbb{N}$.

(1) The special linear group $\mathrm{SL}(n,\mathbb{R}) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A)=1\}$

is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. Furthermore for every $A \in \mathrm{SL}(n,\mathbb{R})$ we have

$$\widetilde{\mathrm{T}}_A\,\mathrm{SL}(n,\mathbb{R}) \;=\; A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{R}) \mid \mathrm{tr}(B)=0\}.$$

(2) The special linear group $\mathrm{SL}(n,\mathbb{C}) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \det(A)=1\}$

is an orientable proper $2(n^2-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C})$. Furthermore for every $A \in \mathrm{SL}(n,\mathbb{C})$ we have

$$\widetilde{\mathrm{T}}_A\,\mathrm{SL}(n,\mathbb{C}) \;=\; A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{C}) \mid \mathrm{tr}(B)=0\}.$$

In the proof of the Matrix Group-Smooth Manifold Proposition 32.8 we will make use of the following lemma:

Auxiliary Lemma 32.9. Given any smooth map $\gamma\colon (-\epsilon,\epsilon) \to \mathrm{M}(n\times n,\mathbb{R})$ with $\gamma(0) \in \mathrm{GL}(n,\mathbb{R})$ we have
$$\tfrac{d}{dt}\det(\gamma(t))\big|_{t=0} \;=\; \det(\gamma(0))\cdot \mathrm{tr}\left(\gamma(0)^{-1}\cdot\gamma'(0)\right).$$

Proof of Lemma 32.9. We start out with the following claim:

Claim. The map $\det\colon \mathrm{M}(n\times n,\mathbb{R}) \to \mathbb{R}$ is smooth and the differential at id is given by the map
$$\begin{array}{rcl}\mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R}\\ B & \mapsto & \mathrm{tr}(B).\end{array}$$

Proof. By the Leibniz Formula 2.18 the determinant of a matrix is a polynomial in the n^2 entries. It follows that the map $\det\colon \mathrm{M}(n\times n,\mathbb{R}) \to \mathbb{R}$ is smooth. Next let $B = (b_{ij}) \in \mathrm{M}(n\times n,\mathbb{R})$. Note that
$$\mathrm{D}\det_{\mathrm{id}}(B) \;=\; \tfrac{d}{dt}\det(\mathrm{id}+t\cdot B)\big|_{t=0}.$$
We now perform the following calculation:
$$\begin{aligned}\det(\mathrm{id}+t\cdot B) &= \sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\cdot\prod_{i=1}^{n}\big(\underbrace{\delta_{i\sigma(i)}}_{\substack{\text{Kronecker}\\ \text{delta}}}+t\cdot b_{i\sigma(i)}\big)\\ &\qquad\uparrow\ \text{Laplace expansion}\\ &= \underbrace{\prod_{i=1}^{n}(1+t\cdot b_{ii})}_{\text{corresponds to }\sigma=\mathrm{id}} + \sum_{\mathrm{id}\neq\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}\cdot\underbrace{\prod_{i=1}^{n}(\delta_{i\sigma(i)}+t\cdot b_{i\sigma(i)})}_{\substack{\text{since } \mathrm{id}\neq\sigma \text{ we have } i\neq\sigma(i)\\ \text{for at least two } i\in\{1,\dots,n\}\\ \text{which implies that we can factor out } t^2}}\\ &= \prod_{i=1}^{n}(1+t\cdot b_{ii}) + \text{terms of order at least } t^2\end{aligned}$$
It is now clear that if we take the derivative at $t=0$, then we obtain the trace of B. ⊞

Now let $\gamma\colon (-\epsilon,\epsilon)\to\mathrm{M}(n\times n,\mathbb{R})$ be a smooth map with $A \coloneqq \gamma(0)\in\mathrm{GL}(n,\mathbb{R})$. We see that
$$\tfrac{d}{dt}\det(\gamma(t))\big|_{t=0} \;=\; \det(A)\cdot\tfrac{d}{dt}\det(A^{-1}\cdot\gamma(t))\big|_{t=0} \;\underset{\substack{\uparrow\\ \text{by the Chain Rule 24.17 and}\\ \text{the above claim}}}{=}\; \det(A)\cdot\mathrm{tr}(A^{-1}\cdot\gamma'(0)).$$
■

Proof of the Matrix Group-Smooth Manifold Proposition 32.8.

(1) First we show that $\mathrm{SL}(n,\mathbb{R})$ is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. By the Regular Value Theorem 32.5 it suffices to show that 1 is a regular value of the smooth map $\det\colon \mathrm{M}(n\times n,\mathbb{R})\to\mathbb{R}$. In fact we will show that any matrix $A\in\mathrm{M}(n\times n,\mathbb{R})$ with $\det(A)\neq 0$ is a regular point. This in turn follows from the observation that
$$\tfrac{d}{dt}\det((1+t)\cdot A)\big|_{t=0} \;=\; \tfrac{d}{dt}(1+t)^n\cdot\det(A)\big|_{t=0} \;=\; (n\cdot(1+t)^{n-1}\cdot\det(A))\big|_{t=0} = \underbrace{n\cdot\det(A)}_{\neq 0}.$$
Next let $A\in\mathrm{SL}(n,\mathbb{R})$. If $\gamma\colon(-\epsilon,\epsilon)\to\mathrm{SL}(n,\mathbb{R})$ is a smooth path with $\gamma(0)=A$, then it follows immediately from Lemma 32.9 that $\mathrm{tr}(A^{-1}\cdot\gamma'(0))=0$. In other words, we see that
$$\widetilde{\mathrm{T}}_A\,\mathrm{SL}(n,\mathbb{R}) \;\subseteq\; \{B\in\mathrm{M}(n\times n,\mathbb{R})\mid \mathrm{tr}(A^{-1}\cdot B)=0\} \;=\; A\cdot\{B\in\mathrm{M}(n\times n,\mathbb{R})\mid \mathrm{tr}(B)=0\}.$$
But both are vector spaces of dimension n^2-1, so we see that equality holds.

(2) First we intend to show that $\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \det(A) = 1\}$ is an orientable proper $2(n^2-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C})$. Again we will go beyond the call of duty, namely we will show that in fact every matrix $A \in \mathrm{M}(n\times n,\mathbb{C})$ with $\det(A) \neq 0$ is in fact a regular point of $\det\colon \mathrm{M}(n\times n,\mathbb{C}) \to \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$. Similarly to (1) we calculate that

$$\frac{d}{dt}\det((1+\mathrm{i}t)\cdot A)\big|_{t=0} \;=\; \frac{d}{dt}(1+\mathrm{i}t)^n\cdot\det(A)\big|_{t=0} \;=\; \mathrm{i}\cdot n\cdot\det(A).$$

Together with the calculation in (1) this shows that $\mathrm{D}\det_A\colon \mathrm{M}(n\times n,\mathbb{C}) \to \mathbb{C}$ is indeed an epimorphism. The remainder of the argument is similarly an adaptation of the argument provided in (1). We leave it to the reader to fill in the details. ■

The following proposition deals with much more interesting matrix groups.

Proposition 32.10. (Matrix Group-Smooth Manifold Proposition II) Let $n \in \mathbb{N}$.

(1) (a) The set
$$\mathrm{O}(n) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^{\mathrm{T}}\cdot A = \mathrm{id}\}$$
of orthogonal matrices is an orientable proper[392] $\frac{1}{2}n(n-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$. Furthermore for every $A \in \mathrm{O}(n)$ we have
$$\widetilde{\mathrm{T}}_A\,\mathrm{O}(n) \;=\; A\cdot\underbrace{\{B \in \mathrm{M}(n\times n,\mathbb{R}) \mid B = -B^{\mathrm{T}}\}}_{\text{vector space of skew-symmetric matrices}}.$$

(b) The analogous statements also hold for $\mathrm{SO}(n)$.

(2) (a) The set
$$\mathrm{U}(n) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^{\mathrm{T}}\cdot A = \mathrm{id}\}$$
of unitary matrices is similarly an orientable proper n^2-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Furthermore for every $A \in \mathrm{U}(n)$ we have
$$\widetilde{\mathrm{T}}_A\,\mathrm{U}(n) \;=\; A\cdot\underbrace{\{B \in \mathrm{M}(n\times n,\mathbb{C}) \mid B = -\overline{B}^{\mathrm{T}}\}}_{\text{vector space of skew-Hermitian matrices}}.$$

(b) The set
$$\mathrm{SU}(n) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^{\mathrm{T}}\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}$$
of special unitary matrices is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Furthermore for every $A \in \mathrm{SU}(n)$ we have
$$\widetilde{\mathrm{T}}_A\,\mathrm{SU}(n) \;=\; A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{C}) \mid B = -\overline{B}^{\mathrm{T}} \text{ and } \mathrm{tr}(B) = 0\}.$$

All of the above examples, viewed as smooth manifolds in their own right, are closed.

Proof of the Matrix Group-Smooth Manifold Proposition 32.10 (1). First recall that by the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n)$ is a component of $\mathrm{O}(n)$. It is thus clear that Statement (1b) follows from Statement (1a).

We start out our proof of the Matrix Group-Smooth Manifold Proposition 32.10 (1a) with the following claim:

[392]The mnemonic for the dimension of the submanifold $\mathrm{O}(n)$ is as follows: Orthogonal matrices in $\mathrm{O}(n)$ are the real matrices of the form $(v_1\ v_2\ \ldots\ v_n)$ where each v_i has length one and where the v_i are orthogonal. In particular the first vector v_1 lies in S^{n-1}. The second vector v_2 lies in $S^{n-1}\cap v_1^{\perp}$ and so. So for the first vector the "possibilities" are $n-1$ dimensions, for the second $n-2$ dimensions, etc. In summary the dimension of $\mathrm{O}(n)$ is $(n-1)+(n-2)+\cdots+1 = \frac{1}{2}n(n-1)$.

Claim 1.

(i) The set

$$\mathrm{Sym}(n,\mathbb{R}) \;=\; \{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, B^{\mathrm{T}} = B\} \;=\; \{(b_{ij}) \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, b_{ij} = b_{ji}\}$$

is a submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n+1)$.

(ii) The map

$$\begin{array}{rccc} f\colon & \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathrm{Sym}(n,\mathbb{R}) \\ & A & \mapsto & A^{\mathrm{T}}\cdot A \end{array}$$

is smooth and id is a regular value.

Proof. Statement (i) follows immediately from the Physical Tangent Spaces–Lemma 29.2 (2). We turn to the proof of Statement (ii). It should be clear that f is smooth and that f does indeed take values in $\mathrm{Sym}(n,\mathbb{R})$. So it remains to show that id is a regular value.

In the following we again use the Physical Tangent Spaces–Lemma 29.2 (2) which implies that the physical tangent spaces of the submanifolds $\mathrm{Sym}(n,\mathbb{R})$ and $\mathrm{M}(n\times n,\mathbb{R})$ are just given by these vector subspaces. So let $A \in \mathrm{M}(n\times n,\mathbb{R})$ with $A^{\mathrm{T}}\cdot A = \mathrm{id}$ and let $C \in \mathrm{T}_{\mathrm{id}}\,\mathrm{Sym}(n,\mathbb{R}) = \mathrm{Sym}(n,\mathbb{R})$. Note that we need to show that there exists a B in $\mathrm{T}_A\,\mathrm{M}(n\times n,\mathbb{R}) = \mathrm{M}(n\times n,\mathbb{R})$ with $\mathrm{D}f_A(B) = C$.

We start out with the following little calculation: Given $B \in \mathrm{M}(n\times n,\mathbb{R})$ we have

$$\begin{aligned} \tfrac{d}{dt} f(A+tB)\big|_{t=0} &= \tfrac{d}{dt}(A^{\mathrm{T}}+tB^{\mathrm{T}})\cdot(A+tB)\big|_{t=0} \\ &= \tfrac{d}{dt}\big(A^{\mathrm{T}}\cdot A + tA^{\mathrm{T}}\cdot B + tB^{\mathrm{T}}\cdot A + t^2 B^{\mathrm{T}}\cdot B\big)\big|_{t=0} \;=\; A^{\mathrm{T}}\cdot B + B^{\mathrm{T}}\cdot A. \end{aligned}$$

Using that $A^{\mathrm{T}}\cdot A = \mathrm{id}$ and $C = C^{\mathrm{T}}$ it is now easy to see that $B \coloneqq \frac{1}{2}A\cdot C$ satisfies $\mathrm{D}f_A(B) = C$. ⊞

First note that by definition we have $\mathrm{O}(n) = f^{-1}(\mathrm{id})$. Thus it follows from the claim, together with the Regular Value Theorem 32.5, that $\mathrm{O}(n)$ is an orientable proper submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. Since $\mathbb{R}^{n^2}$ has no boundary and since $\mathrm{O}(n)$ is a proper submanifold we see that the boundary of $\mathrm{O}(n)$ is also empty. Finally note that in the Matrix Group Lemma 4.53 (3) we showed that $\mathrm{O}(n)$ is compact. These last two observations imply that $\mathrm{O}(n)$ is in fact a closed smooth manifold.

It remains to prove the following claim:

Claim 2. For every $A \in \mathrm{O}(n)$ we have the following equality:

$$\widetilde{\mathrm{T}}_A\,\mathrm{O}(n) \;=\; A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, B = -B^{\mathrm{T}}\}.$$

Proof. Note that both sides are vector subspaces of $\mathrm{M}(n\times n,\mathbb{R})$ of dimension $\frac{1}{2}n(n-1)$. Thus it suffices to prove the "$\subseteq$"-inclusion. Let $X \in \widetilde{\mathrm{T}}_A\,\mathrm{O}(n)$. We pick a smooth map $\gamma\colon(-\epsilon,\epsilon)\to\mathrm{O}(n)$ with $\gamma(0) = A$ and $\gamma'(0) = X$. We make the following observations:

$$\begin{aligned} \gamma(t)\in\mathrm{O}(n)\text{ for all }t &\implies \gamma(t)^{\mathrm{T}}\cdot\gamma(t) \;=\; \mathrm{id}\text{ for all }t \\ &\implies (\gamma(t)^{\mathrm{T}})'\cdot\gamma(t) + \gamma(t)^{\mathrm{T}}\cdot\gamma(t)' \;=\; 0\text{ for all }t \\ &\implies X^{\mathrm{T}}\cdot A + A^{\mathrm{T}}\cdot X \;=\; 0 \\ &\implies A^{\mathrm{T}}\cdot X = -(A^{\mathrm{T}}\cdot X)^{\mathrm{T}}. \end{aligned}$$

Next note that

$$X \overset{\text{since } A\in\mathrm{O}(n)}{=} A\cdot\underbrace{A^{\mathrm{T}}\cdot X}_{=:B}.$$

We are now done since $B = A^{\mathrm{T}} \cdot X = -(A^{\mathrm{T}} \cdot X)^{\mathrm{T}} = -B^{\mathrm{T}}$. ■

Proof of Proposition 32.10 (2). The proof for $\mathrm{U}(n)$ is almost verbatim the same as the proof for $\mathrm{O}(n)$. The proof for $\mathrm{SU}(n)$ follows from combining the ideas for the proof of $\mathrm{U}(n)$ with the ideas for the proof of the Matrix Group-Smooth Manifold Proposition 32.8 (2) where we showed that $\mathrm{SL}(n,\mathbb{C})$ is a $2(n^2-1)$-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. We will deal with $\mathrm{SU}(n)$ in Exercise 32.13. ■

We conclude this discussion with the following lemma which gives us an interesting diffeomorphism between various matrix groups:

Lemma 32.11. ($\mathbf{U}(n)$-Product Lemma) For every $n \in \mathbb{N}$ the maps

$$\begin{array}{rcl} \mathrm{U}(n) & \to & \mathrm{SU}(n)\times S^1 \\ A & \mapsto & \left(A\cdot\begin{pmatrix}\det(A)^{-1} & 0\\ 0 & \mathrm{id}_{n-1}\end{pmatrix}, \det(A)\right)\end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{SU}(n)\times S^1 & \to & \mathrm{U}(n) \\ (A,z) & \mapsto & A\cdot\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}\end{array}$$

are diffeomorphisms and they are inverses of one another.

Proof. It is clear that the two maps are inverses of one another. It remains to show that the two maps are smooth. We consider the following maps:

$$\begin{array}{rcl} \mathrm{GL}(n,\mathbb{C}) & \to & \mathrm{SL}(n,\mathbb{C})\times(\mathbb{C}\setminus\{0\}) \\ A & \mapsto & \left(A\cdot\begin{pmatrix}\det(A)^{-1} & 0\\ 0 & \mathrm{id}\end{pmatrix}, \det(A)\right)\end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{SL}(n,\mathbb{C})\times(\mathbb{C}\setminus\{0\}) & \to & \mathrm{GL}(n,\mathbb{C}) \\ (A,z) & \mapsto & A\cdot\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}.\end{array}$$

It follows easily from the Leibniz-Formula (see the Determinant–Characterization Proposition 2.18 that these two maps are smooth. But then it follows from the Submanifold Smooth Structure Lemma 25.23 that the restrictions we are interested in are also smooth. ■

32.5. Exercises for Chapter 32.

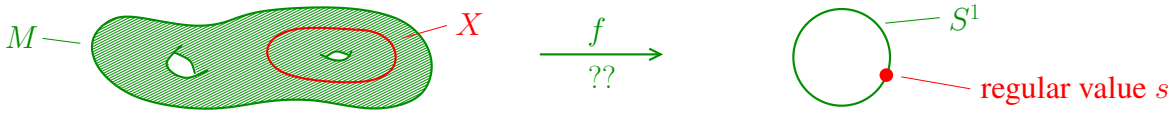
Exercise 32.1. Let M be a closed smooth manifold and let N be a connected smooth manifold. Let $f\colon M \to N$ be a smooth map. We assume that f has no critical values (in other words, we assume that all points of M are regular). Show that f is surjective.
Hint. First consider the case that $\dim(M) = \dim(N)$.

Exercise 32.2. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree n. We consider the map

$$\begin{array}{rcl} p\colon \mathbb{C} & \to & \mathbb{C} \\ w & \mapsto & p(w).\end{array}$$

Show that there exists a finite subset $Y \subseteq \mathbb{C}$ such that for every $w \notin Y$ we have the equality $\#p^{-1}(w) = n$.

Exercise 32.3. We consider the 2-dimensional smooth manifold M that is shown below in the figure together with the 1-dimensional submanifold X. Does there exist a smooth map $f\colon M \to S^1$ together with a regular value $s \in S^1$ such that $X = f^{-1}(s)$?



Exercise 32.4.

(a) We consider the smooth function

$$\begin{array}{rcl} f\colon \mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & (4x^2(1-x^2)-y^2)^2+z^2. \end{array}$$

Show that $\frac{1}{4}$ is a regular value of f. The picture shown in

`https://www.desmos.com/3d/856bbb08a2`

shows convincingly that $f^{-1}(\frac{1}{4})$ is a surface of genus two.

(b) Find polynomials $f(x,y,z)$ and regular values such that the preimages are surfaces of genus three and four.

Remark. By [**Hirs1976**, p. 28] there exists for any $g \in \mathbb{N}_0$ a polynomial function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that 0 is a regular value and such that $f^{-1}(0)$ is diffeomorphic to the surface of genus g.

Exercise 32.5. We consider

$$M \coloneqq \{(x,y,z) \in \mathbb{R}^3 \mid x^2-y^2-z^2=1\} \qquad \text{and} \qquad N \coloneqq \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2-z^2=1\}.$$

(a) Show that M and N are 2-dimensional smooth submanifolds of $\mathbb{R}^3$.

(b) Sketch M and N.

Exercise 32.6. Let A be a real symmetric $(n \times n)$-matrix.

(a) Show that $\{v \in \mathbb{R}^n \mid \langle v, A \cdot v\rangle = 1\}$ is an $(n-1)$-dimensional smooth submanifold of $\mathbb{R}^n$.

(b) Under what condition on the eigenvalues of A is the submanifold in (a) compact?

Exercise 32.7. We consider

$$T \coloneqq \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2}-2)^2+z^2=1\}.$$

(a) Show that T is rotationally symmetric around the z-axis.

(b) Provide a sketch of T.

(c) Show that T is a submanifold of $\mathbb{R}^3$.

(d) In Exercise 30.19 we showed that the map

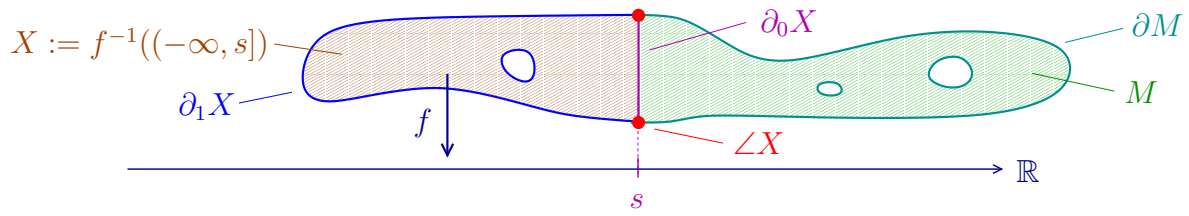
$$\begin{array}{rcl} f\colon \mathbb{R}^2/(2\pi\mathbb{Z})^2 & \to & \mathbb{R}^3 \\ (\varphi,\theta) & \mapsto & \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around the } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 2+\sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix}}_{\substack{\text{describes a circle} \\ \text{in the } xz\text{-plane}}} \end{array}$$

is a smooth embedding. Show that T is diffeomorphic to $f(\mathbb{R}^2/(2\pi\mathbb{Z})^2)$.

Exercise 32.8. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $s \in \mathbb{R}$ be a regular value of f. Show that $X \coloneqq f^{-1}((-\infty,s])$ is a smooth codimension-zero submanifold with corner of M with $\partial_0 X = f^{-1}(s)$, $\partial_1 X = f^{-1}((-\infty,s]) \cap \partial M$ and $\angle X = f^{-1}(s) \cap \partial M$.

Remark. We refer to Definition 26.14 for the definition of a submanifold with corner and

we refer to the Smooth Submanifold-with-Corner Lemma 26.16 for the definition of ∂_0 and ∂_1 of a submanifold with corner.



Exercise 32.9. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth map and let $z \in \mathbb{R}$ be a regular value. We set $M := f^{-1}(z)$. Given $P \in M$ we denote by

$$\operatorname{grad} f_P \;:=\; (\mathrm{D}f_P)^T \;=\; \big(\tfrac{\partial f}{\partial x_1}\big|_P, \dots, \tfrac{\partial f}{\partial x_n}\big|_P\big) \;\in\; \mathbb{R}^n$$

the gradient. Show that for every $P \in M$ we have

$$\widetilde{\mathrm{T}}_P M \;=\; (\operatorname{grad} f_P)^\perp \;=\; \{v \in \mathbb{R}^n \,|\, \langle v, \operatorname{grad} f_P\rangle = 0\}.$$

Exercise 32.10. Let $m, n \in \mathbb{N}$.

(a) Show that

$$\mathrm{O}(m,n) \;:=\; \text{set of } n\text{-tuples of orthonormal vectors of } \mathbb{R}^m$$

is a submanifold of $\mathrm{M}(m \times n, \mathbb{R})$.

(b) Given $A \in \mathrm{O}(m,n)$ determine $\widetilde{\mathrm{T}}_A\, \mathrm{O}(m,n) \subseteq \mathrm{M}(m \times n, \mathbb{R})$.

Exercise 32.11. Let $d \in \mathbb{N}_0$. Show that

$$M_d \;:=\; \Big\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \;\Big|\; z_0^d + \dots + z_n^d = 0 \text{ and } \sum_{k=0}^{n} |z_k|^2 = 2\Big\}$$

is a $(2n-1)$-dimensional submanifold of $\mathbb{C}^{n+1}$.

Remark. These smooth manifolds are sometimes called *Brieskorn manifolds*.

Exercise 32.12. Any matrix $A \in \mathrm{M}(n \times n, \mathbb{C})$ defines an endomorphism of $\mathbb{C}^n = \mathbb{R}^{2n}$. Thus we can view A also as a matrix in $\mathrm{M}(2n \times 2n, \mathbb{R})$.

(a) Show that with the above convention $\mathrm{U}(n)$ is a subgroup of $\mathrm{O}(2n)$.

(b) Show that $\mathrm{U}(n)$ is a submanifold of $\mathrm{O}(2n)$.

Exercise 32.13.

(a) Show that

$$\mathrm{SU}(n) \;=\; \big\{A \in \mathrm{M}(n \times n, \mathbb{C}) \,\big|\, \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id} \;\text{ and }\; \det(A) = 1\big\}$$

is a smooth manifold of dimension $n^2 - 1$.

(b) Show that for every $A \in \mathrm{SU}(n)$ we have

$$\widetilde{\mathrm{T}}_A\, \mathrm{SU}(n) \;=\; A \cdot \{B \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, B = -\overline{B}^{\mathrm{T}} \text{ and } \operatorname{tr}(B) = 0\}.$$

Exercise 32.14. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^{n+1} & \to & \mathbb{R} \\ (x_1, \dots, x_{n+1}) & \mapsto & x_1^2 + \dots + x_{n+1}^2. \end{array}$$

We equip $\mathbb{R}^{n+1}$ and $\mathbb{R}$ with the standard orientations. For which $n \in \mathbb{N}$ does the orientation on the sphere $S^n = f^{-1}(1)$ that is defined by the Regular Value Theorem 32.5 (3a) agree with the standard orientation on S^n that we introduced on page 817?

Exercise 32.15.

(a) Let $A \in \mathrm{M}(m \times n, \mathbb{R})$. Show that A has rank $\geq k$ if and only if there exist k columns and k rows such that the $(k \times k)$-matrix that is given by entries corresponding to these rows and columns has non-zero determinant.

(b) Let $m, n \in \mathbb{N}_0$ and let $k \in \{0, \dots, \min\{m, n\}\}$. Show that

$$X_k \coloneqq \{A \in \mathrm{M}(m \times n, \mathbb{R}) \mid A \text{ has rank } k\}$$

is a non-empty submanifold of $\mathrm{M}(m \times n, \mathbb{R})$ of dimension $m \cdot k + n \cdot k - k^2$.

Remark. This exercise is rather non-trivial. It is worth to first consider the case that $k = \min\{m, n\}$.

Exercise 32.16. Let $k \in \mathbb{N}$ and let M be a smooth manifold with empty boundary. We refer to

$$\mathrm{Conf}_k(M) \coloneqq \{(x_1, \dots, x_k) \in M^k \mid \text{ for all } i \neq j \text{ we have } x_i \neq x_j\}$$

as the *configuration space of k points in M*.

(a) Show that $\mathrm{Conf}_k(M)$ is an open subset of M^k. This shows that $\mathrm{Conf}_k(M)$ is a smooth manifold of dimension $\dim(M) \cdot k$.

(b) We now restrict ourselves to $k = 3$ and $M = \mathbb{R}^2$.

(i) We consider the map

$$\begin{array}{rccl} f\colon & \mathrm{Conf}_3(\mathbb{R}^2) & \to & \mathbb{R} \\ & (x_1, x_2, x_3) & \mapsto & \det \underbrace{(x_2 - x_1 \;\; x_3 - x_1)}_{2 \times 2\text{-matrix}}. \end{array}$$

Show that 0 is a regular value of f.

(ii) We say that $(x_1, x_2, x_3) \in \mathrm{Conf}_3(\mathbb{R}^2)$ are collinear if there exists a line that contains all three points. Show that the set of collinear (x_1, x_2, x_3) is a submanifold of $\mathrm{Conf}_3(\mathbb{R}^2)$ of dimension 5.

CHAPTER 33

The Euclidean Embedding Theorem

In this chapter we will show that every topological or smooth manifold (compact or non-compact) admits a (smooth) embedding into a suitable $\mathbb{R}^n$. This implies in particular that every smooth manifold is diffeomorphic to a "Euclidean submanifold", i.e. to a smooth submanifold of a suitable $\mathbb{R}^n$. This latter result will turn out to be incredibly useful for all kinds of purposes, especially when we want to construct maps into and out of smooth manifolds.

This chapter is organized as follows:

- In Section 33.1 we will state the Euclidean Embedding Theorem 33.2 which says that every smooth manifold admits a smooth embedding into some $\mathbb{R}^n$.
- In Section 33.2 we will prove the Euclidean Embedding Theorem 33.2 in the compact case.
- In Section 33.3 we will discuss the covering dimension of topological manifolds.
- In Section 33.4 we will prove the Euclidean Embedding Theorem 33.2 in the non-compact case.
- In Section 33.5 we will provide exercises for Chapter 33.

In the next Chapter 34 we will introduce the concept of a Lie group.

33.1. **The statement of the Euclidean Embedding Theorem.** For the reader's convenience we recall Definitions 4.20, 30.9 and 30.7:

Definition 33.1.

(1) A map $\varphi\colon Y \to X$ between topological spaces is called an embedding if $\varphi\colon Y \to \varphi(Y)$ is a homeomorphism. (Here we view $\varphi(Y)$ as a topological space equipped with the subspace topology coming from X.)

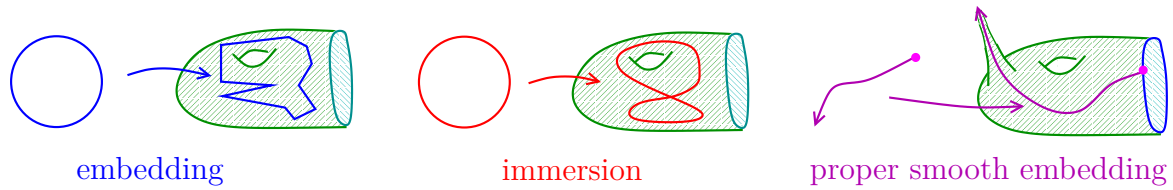
(2) A map $\varphi\colon N \to M$ between two smooth manifolds is called an immersion if the following conditions are satisfied:
   (a) φ is smooth,
   (b) for each $P \in N$ the differential $\mathrm{D}\varphi_P\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism,
   (c) If $\partial M \neq \varnothing$, then we require that also the following conditions are satisfied:
      - For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.
      - For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.
      - For every component C of ∂N we have $\varphi(C) \subseteq \partial M$ or $\varphi(C) \subseteq M \setminus \partial M$.

   An immersion that also satisfies the following extra condition
   (d) $\varphi\colon N \to M$ is an embedding,

is called a smooth embedding.

(3) Let M and N be two topological manifolds. A continuous map $\varphi\colon N \to M$ is called proper if the following two conditions are satisfied:[393]
   (a) $\varphi^{-1}(\partial M) = \partial N$,
   (b) $\varphi(N)$ is a closed subset of M.

(4) Given $n \in \mathbb{N}$ we write $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

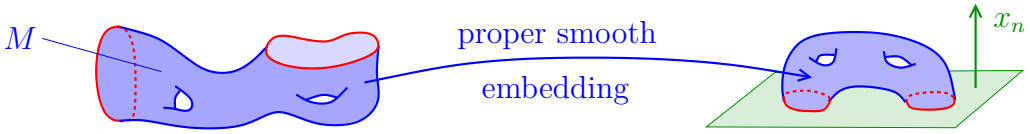


We can now formulate the main theorem of this chapter.

Theorem 33.2. (Euclidean Embedding Theorem)

(1) Given any topological manifold M there exists an $n \in \mathbb{N}$ and a proper embedding $M \to \mathbb{H}^n$.

(2) Given any smooth manifold M there exists an $n \in \mathbb{N}$ and a proper smooth embedding $M \to \mathbb{H}^n$.



We will provide the proof of the Euclidean Embedding Theorem 33.2 in the following sections.

It is already worth pointing out that the proof of the Euclidean Embedding Theorem 33.2 breaks into two cases:

(1) In the compact case the proof of the Euclidean Embedding Theorem 33.2 is fairly straightforward: Using a partition of unity subordinate to a finite atlas one can easily cobble together an argument. We refer to Section 33.2 for full details.

(2) The non-compact case is significantly harder. Our approach relies on the notion of the covering dimension that we had developed in Chapter 15. In Section 33.3 we will consider the covering dimension of topological manifolds. In particular in the Covering Dimension-of-Manifolds Proposition 33.9 we will show that the covering dimension of a k-dimensional topological manifold is $\leq k$. We will use this fact, together with the Covering Dimension Refinement Theorem 15.15, to show in the Small Atlas Proposition 33.10 that every k-dimensional topological manifold admits an atlas consisting of $\leq k+1$ charts. Using this result, together with the Covering Dimension Refinement Theorem 15.15, it is then straightforward to show that every non-compact topological manifolds admits a proper embedding into some $\mathbb{H}^n$. Furthermore the last two statements also work for smooth manifolds and provide us with a smooth atlas consisting of $\leq k+1$ charts and we obtain a proper smooth embedding $M \to \mathbb{H}^n$ for some n.

[393] On page 778 we gave a short discussion of the unfortunate fact that the adjective "proper" is oversubscribed.

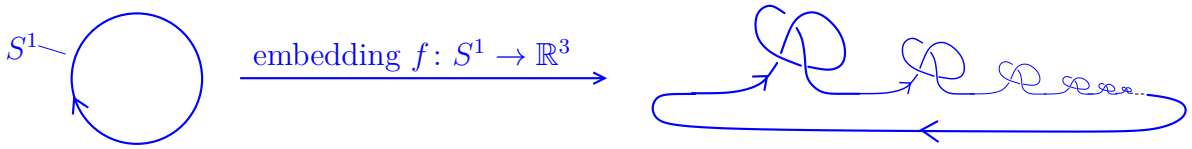
Remark. The fact that every non-compact smooth manifold (at least with empty boundary) admits a proper smooth embedding into some $\mathbb{H}^n$ is also proved, using a different approach, in [**Wall2016**, Corollary 4.7.8] and [**Lee2002**, Theorem 6.15]. We will sketch the argument provided in [**Lee2002**, Theorem 6.15] later on in Exercise **??**. □

In the remainder of this section we take the Euclidean Embedding Theorem 33.2 for granted and we discuss a few consequences.

Corollary 33.3. (Diffeomorphic-to-Euclidean Submanifold Corollary) Given any smooth manifold M there exists an $n \in \mathbb{N}$ such that M is diffeomorphic to a proper smooth submanifold of $\mathbb{H}^n$.

Proof. Let M be a smooth manifold. By the Euclidean Embedding Theorem 33.2 there exists an $n \in \mathbb{N}$ and a proper smooth embedding $M \to \mathbb{H}^n$. Furthermore by the Smooth Embedding Theorem 30.15 we know that $\varphi(N)$ is a proper n-dimensional smooth submanifold of $\mathbb{H}^n$ and that the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism. ■

Remark. Note that the logic of the Diffeomorphic-to-Euclidean Submanifold Corollary 33.3 does *not* apply to topological manifolds. For example let us consider the embedding $f\colon S^1 \to \mathbb{R}^3$ that is shown in the figure below and that we encountered already on page 786.



Using the Compact-Hausdorff Proposition 4.23 (3) one can easily verify that the map $f\colon S^1 \to \mathbb{R}^3$ is an embedding but, as we will see in the Wild Embedding Existence Proposition 218.16, the image $f(S^1)$ is not a topological submanifold of $\mathbb{R}^3$.[394] But it turns out that the situation can mostly be fixed:

(1) Let X and Y be topological manifolds with $\dim(Y) \geq \dim(X) + 3$. It was shown by Richard Miller [**MillR1972**, Theorem 5] (see also the work of Aleksei Chernavskii [**Cherna1969b**, Theorem 1] and M. A. Stan'ko [**Stank1973**, Theorem 1]) that given any embedding $f\colon X \to Y$, given any metric d on Y and given any $\epsilon > 0$ there exists an embedding $g\colon X \to Y$ such that $g(X)$ is a submanifold of Y and such that for all $x \in X$ we have $d(f(x), g(x)) < \epsilon$. It follows from this fact, together with the Euclidean Embedding Theorem 33.2, Exercise 4.73 and the Proper-Closed Proposition 4.64 that every topological manifold of dimension ≥ 2 is homeomorphic to a topological submanifold of $\mathbb{R}^{2n+1}$ that is a closed subset of $\mathbb{R}^{2n+1}$.[395]

(2) If X is a 1-dimensional topological manifold, then we know by the Topological 1-Manifold Classification Theorem 28.1 that each component of X is homeomorphic to one of S^1, $[0,1]$, $[0,1)$ or $(0,1)$. Since topological manifolds are second-countable they have, see Exercise 13.1, only countably many components. So it is straightforward to show that X is homeomorphic to a proper submanifold of $\mathbb{H}^2$. □

[394] In that proposition we will also provide a careful definition of f.

[395] The literature does not give control over the behavior of the boundary so we cannot arrange that we get a proper embedding into $\mathbb{R}^{2n+1}$. This burning issue could surely be resolved if somebody went back to the original literature and upgraded the results.

Corollary 33.4. (Smooth Manifold Category–Weak Equivalence Corollary) We denote by $\mathcal{SmMfd}$ the category of smooth manifolds and we denote by $\mathcal{SmMfd}_{\mathbb{R}^n}$ the category of smooth submanifolds of some $\mathbb{R}^n$. The functor $\mathcal{SmMfd}_{\mathbb{R}^n} \to \mathcal{SmMfd}$ that is given by

$$\begin{array}{rcl} M & \mapsto & (M, \text{standard smooth structure as defined on page 671}) \\ (f\colon M \to N) & \mapsto & (f\colon M \to N) \end{array}$$

is a weak equivalence of categories in the sense of Definition 5.29.

Proof. The Diffeomorphic-to-Euclidean Submanifold Corollary 33.3 says that the given functor is essentially surjective in the sense of the definition on page 245. The remaining conditions of the Weak Equivalence of Categories Proposition 5.30 are satisfied basically by definition. Thus we obtain from said proposition that the functor is a weak equivalence of categories. ■

In the Manifold Metrization Proposition 23.10 we showed, using the mighty Urysohn Metrization Theorem 13.8 and the less mighty Completely Metrizable Criterion 14.13 that every topological manifold is completely metrizable. In the following proposition we will use the Euclidean Embedding Theorem 33.2 to provide a new proof of this statement, that does not rely on the Urysohn Metrization Theorem 13.8. In fact for a smooth manifold M we can even strengthen the earlier result, namely we can now find a metric $d\colon M \times M \to \mathbb{R}_{\geq 0}$ that is smooth away from the "diagonal", i.e. away from $\{(x,x) \in M \times M \,|\, x \in M\}$.

Proposition 33.5. (Manifold Metrization Proposition)

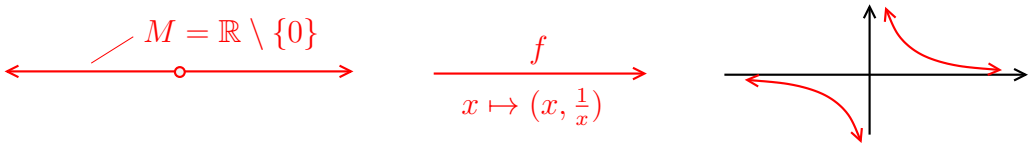
(1) Every topological manifold M is *completely* metrizable, i.e. there exists a *complete* metric d on M such that the given topology agrees with the topology coming from d.
(2) Every smooth manifold M admits a complete metric $d\colon M \times M \to \mathbb{R}_{\geq 0}$ such that the restriction of d to the open subset $\{(x,y) \in M \times M \,|\, x \neq y\}$ is smooth.

Remark. As we will see shortly, the Manifold Metrization Proposition 33.5 (1) is a straightforward consequence of the Euclidean Embedding Theorem 33.2 (1). As we mentioned above, for compact topological manifolds we provide in Section 33.2 an elementary proof of the Euclidean Embedding Theorem 33.2. On the other hand, for non-compact topological manifolds the proof of the Euclidean Embedding Theorem 33.2 relies on our detailed discussion of the covering dimension of topological manifolds in Chapter 15. For readers who are not familiar with the theory of covering dimensions we outline an alternative proof of the Manifold Metrization Proposition 33.5 (1) in Exercise 33.2. □

Example. Let $M = \mathbb{R} \setminus \{0\}$. The usual Euclidean metric on $\mathbb{R}$ shows that M is metrizable, but this metric is of course not complete. The map

$$\begin{array}{rcl} f\colon \mathbb{R} \setminus \{0\} & \to & \mathbb{R}^2 \\ x & \mapsto & (x, \frac{1}{x}) \end{array}$$

is easily seen to be a proper smooth embedding.



Thus $\mathbb{R} \setminus \{0\}$ is homeomorphic to a closed subset of $\mathbb{R}^2$. Now the Euclidean metric on $\mathbb{R}^2$ restricts to a *complete* metric on $f(\mathbb{R} \setminus \{0\})$ and thus it gives rise to a complete metric on $\mathbb{R} \setminus \{0\}$. □

Proof. For each $n \in \mathbb{N}_0$ we consider the usual Euclidean metric

$$\begin{aligned} e\colon \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ ((x_1, \dots, x_n), (y_1, \dots, y_n)) &\mapsto \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}. \end{aligned}$$

(1) Let M be a topological manifold. By the Euclidean Embedding Theorem 33.2 (1) there exists an $n \in \mathbb{N}$ and a closed embedding $\varphi\colon M \to \mathbb{R}^n$. In other words, M is homeomorphic to the closed subset $\varphi(M) \subseteq \mathbb{R}^n$. It follows from the elementary Exercise 3.3 that the Euclidean metric e on $\mathbb{R}^n$ restricts to a metric e on $\varphi(M)$ such that the topology coming from e agrees with the subspace topology on $\varphi(M) \subseteq \mathbb{R}^n$. Finally note that the fact that $\varphi(M)$ is closed in $\mathbb{R}^n$ implies, by the Closed Subspace-Completeness Lemma 10.16, that the metric space $\varphi(M)$ itself is complete.

(2) Let M be a smooth manifold. By the Euclidean Embedding Theorem 33.2 (2) there exists an $n \in \mathbb{N}$ and a closed smooth embedding $\varphi\colon M \to \mathbb{R}^n$. In (1) we showed that the map $e \circ (\varphi \times \varphi)\colon M \times M \to \mathbb{R}_{\geq 0}$ is a complete metric. It remains to prove the smoothness statement. We make two observations:
- A very elementary argument shows that $e\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a smooth map on the open subset $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \neq y\}$.
- Since φ is smooth it follows from the above discussion together with the Smooth Manifold Product Proposition 25.34 (and the Smooth Manifold Product Proposition 26.12 if M has non-empty boundary) that the map $\varphi \times \varphi\colon M \times M \to \mathbb{R}^n \times \mathbb{R}^n$ is smooth.

It follows from these two observations that the map $e \circ (\varphi \times \varphi)$ is smooth on the open subset $\{(x, y) \in M \times M \mid x \neq y\}$. ■

Remark.

(1) In the following chapters we will study several refinements of the Euclidean Embedding Theorem 33.2 for compact smooth manifolds:
   (a) In the Weak Whitney Embedding Theorem 37.9 we will show that any n-dimensional smooth manifold admits a proper smooth embedding into $\mathbb{H}^{2n+1}$.
   (b) In the Weak Unique Embedding Proposition **??** we will show that any two proper smooth embeddings $M \to \mathbb{H}^n$ are "properly smoothly isotopic" as long as we have the inequality $n > 2\dim(M) + 1$.
   (c) In Proposition **??** we will consider a refinement of Theorem 33.2 for smooth manifolds M that are equipped with a proper submanifold W.

(2) The Euclidean Embedding Theorem 33.2 is not only interesting in its own right, for us the fact that every smooth manifold can be viewed as a proper submanifold of $\mathbb{H}^n$ will come in handy on many occasions. One of the nice consequences of considering submanifolds of $\mathbb{H}^n$ is that it makes it relatively easy to construct many interesting and useful maps. In particular we will use the Euclidean Embedding Theorem 33.2, together with the Neighborhood Projection Theorem 35.2, to prove the following results:
   (a) The Smooth Collar Neighborhood Theorem 35.6.

(b) The Whitney Approximation Theorem 36.1.
(c) The Smooth Manifold-Simplicial Structure Theorem 179.5. □

As happens so often in mathematics, the proof of a result immediately invites new questions. For example, in light of the Euclidean Embedding Theorem 33.2 and the Weak Whitney Embedding Theorem 37.9 the following question arises:

Question 33.6. For a given (topological) smooth manifold M, e.g. $M = \mathbb{R}\mathrm{P}^n$, what is the minimal $k \in \mathbb{N}$ such that M admits a (topological) smooth embedding into $\mathbb{R}^k$?

We will give some partial answers for general smooth manifolds in the Weak Whitney Embedding Theorem 37.9 and the Strong Whitney Embedding Theorem 262.11. Furthermore we will consider some specific cases of topological and smooth manifolds in the Smooth Codimension-One Orientability Proposition 51.17, the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14 and the Topological Codimension-One Orientability Proposition 218.9.

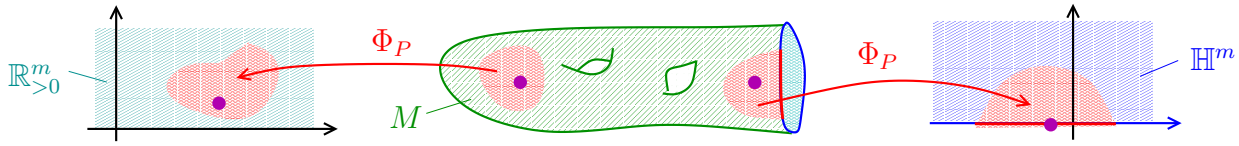
33.2. **Proof of the Euclidean Embedding Theorem 33.2 in the compact case.** We start out with the following elementary lemma:

Lemma 33.7. Let M be a compact m-dimensional topological manifold. There exists a finite atlas $\{\Phi_i \colon U_i \to V_i\}_{i=1,\dots,s}$ such that each V_i is an open subset of $\mathbb{H}^m$. If M is a smooth manifold, then we can choose the atlas to be smooth.

Remark. In the next section we will generalize Lemma 33.7 to the non-compact case. □

Proof. Let M be a compact m-dimensional topological manifold.

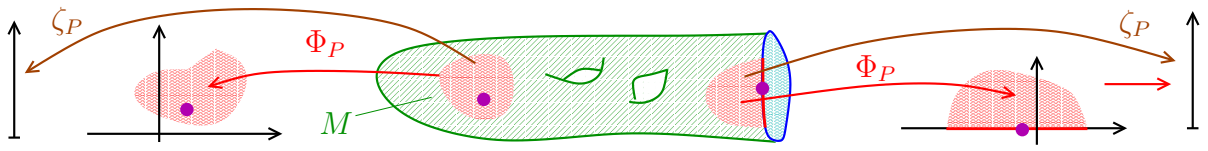
(1) For each $P \in M \setminus \partial M$ we can and will pick a chart $\Phi_P \colon U_P \to V_P$ of type (i) around P with $V_P \subseteq \mathbb{R}^m_{>0} = \{(x_1, \dots, x_m) \in \mathbb{R}^m \,|\, x_m > 0\} \subseteq \mathbb{H}^m$.
(2) For each $P \in \partial M$ we pick a chart $\Phi_P \colon U_P \to V_P$ of type (ii) around P, where V_P is an open subset of $\mathbb{H}^m$.



Since the sets $\{U_P\}_{P \in M}$ form an open cover, the desired statement follows immediately from the hypothesis that M is compact. If M is in fact smooth, then we pick charts from the smooth atlas and we are again done. ■

Proof of the Euclidean Embedding Theorem 33.2 in the compact case. In the following we go out of order, we first deal with the slightly more difficult smooth case and then we argue that the topological case follows by essentially the same argument as the smooth case - except for a technical nuisance.

(2) Let M be a compact m-dimensional smooth manifold. By Lemma 33.7 there exists a finite smooth atlas $\{\Phi_i \colon U_i \to V_i\}_{i=1,\dots,s}$ such that each V_i is an open subset of $\mathbb{H}^m$. For $i = 1, \dots, s$ we denote by $\zeta_i \colon U_i \to \mathbb{R}$ the composition of Φ_i with the projection $\mathbb{H}^m \to \mathbb{R}_{\geq 0}$ onto the last coordinate.
By the Manifold–Partitions-of-Unity Theorem 27.2 we can find a smooth partition of unity $\{\nu_i \colon M \to [0,1]\}_{i=1,\dots,s}$ such that for every $i \in \{1, \dots, s\}$ we have $\operatorname{supp}(\nu_i) \subseteq U_i$.



For $i = 1, \dots, s$ we define

$$\widetilde{\Phi}_i \colon M \to \mathbb{R}^m, \quad x \mapsto \begin{cases} \Phi_i(x) \cdot \nu_i(x), & \text{if } x \in U_i, \\ 0, & \text{else} \end{cases} \qquad \text{and} \qquad \widetilde{\zeta}_i \colon M \to \mathbb{R}, \quad x \mapsto \begin{cases} \zeta_i(x) \cdot \nu_i(x), & \text{if } x \in U_i, \\ 0, & \text{else.} \end{cases}$$

It follows easily from the Smooth Pasting Proposition 25.19 (1) that each of these maps $\widetilde{\Phi}_i$ and $\widetilde{\zeta}_i$ is smooth. It remains to prove the following claim:

Claim. The map

$$\begin{aligned} \varphi \colon M &\to \mathbb{H}^{s\cdot m+s+1} \\ x &\mapsto \Big(\widetilde{\Phi}_1(x), \dots, \widetilde{\Phi}_s(x), \nu_1(x), \dots, \nu_s(x), \sum_{i=1}^{s} \widetilde{\zeta}_i(x)\Big) \end{aligned}$$

is a proper smooth embedding.

Proof. For completeness' sake we provide the proof of the claim, even though the interest in the full details might be quite muted.

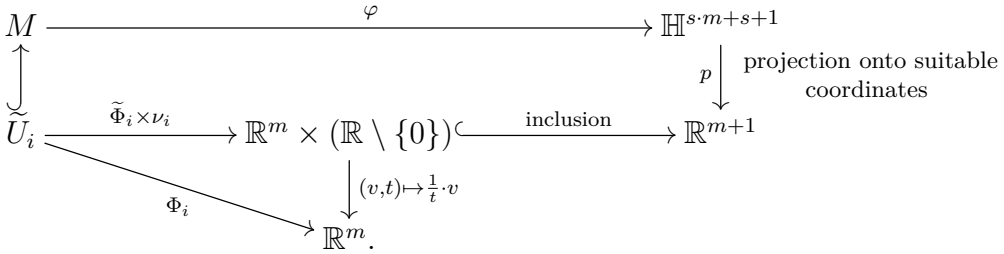
(i) First we show that φ is injective. Thus let $x, x' \in M$ be two distinct points. We pick i with $\nu_i(x) \neq 0$. If $\nu_i(x') \neq \nu_i(x)$, then we are done. Now suppose that $\nu_i(x') = \nu_i(x)$. In particular both are non-zero, thus $x, x' \in U_i$. Since $\Phi_i \colon U_i \to \mathbb{R}^m$ is injective we see that $\Phi_i(x) \neq \Phi_i(x')$. Since $\nu_i(x') = \nu_i(x) \neq 0$ we see that $\widetilde{\Phi}_i(x) \neq \widetilde{\Phi}_i(x')$. This implies that $\varphi(x) \neq \varphi(x')$.

(ii) It follows from (i), the hypothesis that M is compact and the Compact-Hausdorff Proposition 4.23 (2) that the map φ is an embedding.

(iii) We need to show that for each $x \in M$ the differential

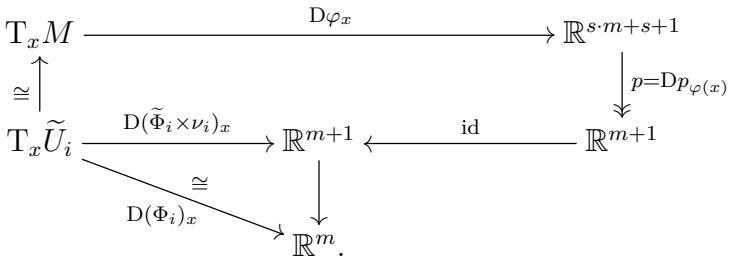
$$\mathrm{D}\varphi_x \colon \mathrm{T}_x M \to \mathrm{T}_{\varphi(x)} \mathbb{H}^{s\cdot m+s+1} = \mathbb{R}^{s\cdot m+s+1}$$

is a monomorphism. We pick $i \in \{1, \dots, s\}$ with $\nu_i(x) \neq 0$. We consider the open subset $\widetilde{U}_i := \{y \in U_i \,|\, \nu_i(y) \neq 0\}$. Next we consider the following diagram:



Here the right vertical map $p \colon \mathbb{H}^{s\cdot m+s+1} \to \mathbb{R}^{m+1}$ is the projection onto the coordinates that correspond to Φ_i and ν_i. It is straightforward to see that the diagram commutes. We now consider the corresponding commutative diagram of tangent

spaces and differentials:[396]



It follows from the fact that Φ_i is a local diffeomorphism and the Inclusion Differential Lemma 29.18 that the left vertical map and the diagonal map are isomorphisms. The combination of these observations now shows that the top horizontal map is a monomorphism.

(iv) Recall that by the Unique Type-of-Chart Proposition 25.26 a point on the smooth manifold M admits either a smooth chart of type (i) or of type (ii). Furthermore note that it follows easily from our various definitions that the last coordinate of $\varphi(P)$ is zero if and only if $P \in \partial M$. In other words, we have $\varphi^{-1}(\partial\,\mathbb{H}^{s\cdot m+s+1}) = \partial M$.

(v) Using our careful definition of the maps $\widetilde{\zeta}_i$ and using Footnote 352 one can verify easily that for every $P \in \partial M$ the image $\mathrm{D}\varphi_P(\mathrm{T}_PM)$ is not contained in $\mathrm{T}_{\varphi(P)}(\partial\,\mathbb{H}^{s\cdot m+s+1})$. ⊞

(1) The proof for compact topological manifolds is basically the same as in (2). There are only two differences:

(a) In this setting we obtain from the Manifold–Partitions-of-Unity Theorem 27.2 only a *continuous* partition of unity $\{\nu_i\colon M \to [0,1]\}_{i=1,\dots,s}$. In particular now the map $\varphi\colon M \to \mathbb{H}^{s\cdot m+s+1}$ that we constructed in (a) is an embedding, but evidently we cannot hope for any smoothness properties.

(b) In the proof of (iv) we used that by the Unique Type-of-Chart Proposition 25.26 we know that a point on a smooth manifold admits either a smooth chart of type (i) or of type (ii). For the proof that $\varphi^{-1}(\partial\mathbb{H}^{s\cdot m+s+1}) = \partial M$ we need the corresponding statement for topological manifolds which, somewhat awkwardly, we only prove in the Unique Type-of-Chart Proposition 150.11.[397] ■

33.3. **The covering dimension of topological manifolds.** In this section we provide the proof of the Euclidean Embedding Theorem 33.2 in the non-compact case. This case is much more intricate, and thus just so much more fun, than the compact case. The key trick for dealing with the non-compact case is to make use of the concept of the covering dimension of a topological space. For the reader's convenience we recall Definition 15.2.

[396] To preserve sanity we use the Convention 29.25, namely given a submanifold Y of some $\mathbb{R}^N$ we do not distinguish between the tangent spaces T_yY and $\widetilde{\mathrm{T}}_yY$.

[397] Note that we also use this statement to conclude that the boundary of the smooth manifold $\mathbb{H}^n$ is indeed $\mathbb{R}^{n-1}\times\{0\}$. If we view $\mathbb{H}^n$ as a topological manifold, then we also need the Unique Type-of-Chart Proposition 150.11 to obtain this result.

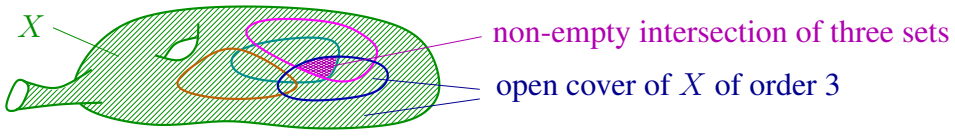
Definition 33.8. Let X be a topological space.

(1) A family $\{U_i\}_{i\in I}$ of subsets of X has order m if the following two statements hold:
   (a) There exists $J \subseteq I$ with $\#J = m$ such that $\bigcap_{j\in J} U_j \neq \varnothing$.
   (b) For every $J \subseteq I$ with $\#J > m$ we have $\bigcap_{j\in J} U_j = \varnothing$.
   If no such m exists, then we say that the family has infinite order.

(2) If X is non-empty, then we refer to

$$\operatorname{cdim}(X) \;:=\; \min\left\{ m \in \mathbb{N}_0 \cup \{\infty\} \;\middle|\; \begin{array}{c}\text{every open cover of } X \text{ admits an}\\ \text{open refinement of order} \leq m+1\end{array} \right\}$$

as the covering dimension of X. If X is empty, then we set $\operatorname{cdim}(X) := -1$.



Proposition 33.9. (Covering Dimension-of-Manifolds Proposition) For any n-dimensional topological manifold M we have $\operatorname{cdim}(M) \leq n$.

Remark.

(1) Note that as of right now we do not know yet whether a non-empty topological manifold has a uniquely defined dimension. In principle a non-empty topological manifold might have several dimensions. We will verify the uniqueness of dimensions only later in the Covering Dimension-of-Manifold Theorem 172.6 once we have introduced homology groups. But fortunately this issue does not have any impact on the Covering Dimension-of-Manifolds Proposition 33.9.

(2) Once we have introduced homology groups we can also prove the Covering Dimension-of-Manifold Theorem 172.6 which says that the dimension and the covering dimension of a non-empty topological manifold agree. □

Proof. Let M be an n-dimensional topological manifold. We start out with the following claim:

Claim. For every compact subspace $K \subseteq M$ we have $\operatorname{cdim}(K) \leq n$.

Proof. Since K is compact it follows from a straightforward argument that there exist finitely many charts $\Phi_i \colon U_i \to V_i$, $i = 1, \ldots, k$ and compact subsets $C_i \subseteq V_i$ such that $K \subseteq \Phi_1^{-1}(C_1) \cup \cdots \cup \Phi_k^{-1}(C_k)$.

(a) By the Compact Euclidean Subset-Covering Dimension Theorem 15.5 we know that for each i we have $\operatorname{cdim}(C_i) \leq n$.

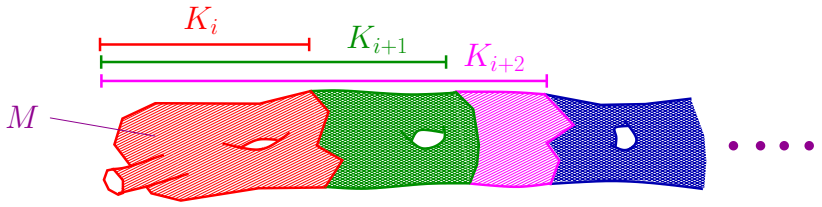
(b) By (a) and the Covering Dimension–Homeomorphism Lemma 15.3 we know that for each i we have $\operatorname{cdim}(\Phi_i^{-1}(C_i)) \leq n$.

(c) Note that from (b) and the Finite Union-Covering Dimension Proposition 15.8 we obtain that $\operatorname{cdim}(\Phi_1^{-1}(C_1) \cup \cdots \cup \Phi_k^{-1}(C_k)) \leq n$.

(d) Since K is compact and since M is by definition Hausdorff we know by the Compact-Closed Lemma 3.39 that K is a closed subset of $\Phi_1^{-1}(C_1) \cup \cdots \cup \Phi_k^{-1}(C_k)$. Therefore it follows from (c) together with the Closed Subspace Covering Dimension Lemma 15.7 that $\operatorname{cdim}(K) \leq n$. ⊞

We now proceed as in the proof of the Open Euclidean Subset–Covering Dimension Corollary 15.12. By the Topological Manifold Exhaustion Lemma 23.12, respectively the Compact Exhaustion Lemma 13.6 (1) we know that there exists a sequence of compact subsets $\{K_i\}_{i\in\mathbb{N}}$ of M such that the following two conditions are satisfied:

(1) for every $i \in \mathbb{N}$ we have $K_i \subseteq \overset{\circ}{K}_{i+1}$,
(2) we have $M = \bigcup_{i\in\mathbb{N}} K_i$.



Let $i \in \mathbb{N}$. Since $K_{i+1} \setminus \overset{\circ}{K}_i$ is a closed subset of the compact space K_{i+1} we know by the Compact-Closed Lemma 3.39 that $K_{i+1} \setminus \overset{\circ}{K}_i$ is also compact. Therefore, by the above claim, we know that for each i we have $\operatorname{cdim}(K_{i+1} \setminus \overset{\circ}{K}_i) \leq n$. It follows from the Exhaustion-Covering Dimension Proposition 15.11 that $\operatorname{cdim}(M) \leq n$. ■

Remark. For topological manifolds with empty boundary we now have an off-the-shelves proof of the Euclidean Embedding Theorem 33.2. Indeed, let X be an n-dimensional topological manifold. In the Covering Dimension-of-Manifolds Proposition 33.9 we just showed that $\operatorname{cdim}(X) \leq n$. By definition X is Hausdorff and second-countable. Furthermore, by the Manifold–Local Properties Lemma 23.8 we know that X is regionally compact. Thus it follows from the Topological Space–Euclidean Embedding Theorem 15.16 that there exists a closed embedding $X \to \mathbb{R}^{2n+1}$. Note though that this embedding does not give us any control over ∂X. Also this approach does not give us anything meaningful for smooth manifolds. □

33.4. Proof of the Euclidean Embedding Theorem 33.2 in the non-compact case.

We can now generalize Lemma 33.7 to the non-compact case. Since the proof is no longer trivial we upgrade the statement from a lemma to a proposition. This proposition might well be interesting in its own right.

Proposition 33.10. (Small Atlas Proposition) Let $m \geq 2$ and let M be an m-dimensional topological manifold.[398]

(1) There exists an atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ consisting of at most $m+1$ charts, such that each V_i is an open subset of $\mathbb{H}^m$.
(2) If M is a smooth manifold, then we can choose the atlas to be smooth.

Remark. Given an n-dimensional smooth manifold M we define, following [**Bers1968, Hopk1989**] the

$$\text{embedding covering number } N(M) \;:=\; \begin{array}{c}\text{minimal cardinality of an open cover } \{U_i\}_{i\in I} \\ \text{of } M \text{ such that each } U_i \text{ admits a} \\ \text{smooth embedding into } \mathbb{R}^n.\end{array}$$

[398]Note that the condition $m \geq 2$ is indeed necessary, namely the conclusion does *not* hold for the 1-dimensional smooth manifold $M = [0,1] \sqcup [0,1]$.

Note that the embedding covering number is precisely the minimal cardinality of a smooth atlas of M. By the Small Atlas Proposition 33.10 we know that $N(M) \leq \dim(M) + 1$. Evidently $N(M) \leq 1$ if and only if M is diffeomorphic to an open subset of some $\mathbb{R}^n$. It follows from this observation that for a closed non-empty smooth manifold we have the inequality $N(M) \geq 2$. We deduce from the Standard Smooth Structures Proposition 25.6 that $N(S^n) = 2$. In general it is very difficult to determine $N(M)$ for a given smooth manifold M. For example let us consider the real projective space $\mathbb{R}\mathrm{P}^n$. We write $n+1 = 2^k \cdot m$ with m odd. If $n \neq 31$ and $n \neq 47$, then it is shown in [**Hopk1989**, Theorem 1] (see also [**Bers1968**, Theorem 1.5]) that

$$N(\mathbb{R}\mathrm{P}^n) \;=\; \begin{cases} \max\{2, m\}, & \text{if } k \leq 3, \\ \lceil \frac{n+1}{2(k+1)} \rceil, & \text{if } k \geq 4. \end{cases}$$

This result is surely not what was on everybody's lips. Later on we will prove a related result, namely we will show in the LS Category–Projective Space Proposition 213.21 that any open cover of $\mathbb{R}\mathrm{P}^n$ by contractible open subsets (e.g. by open subsets that are diffeomorphic to $\mathbb{R}^n$) has cardinality at least $n+1$. □

Proof. In the following we only deal with the case of a smooth manifold. It is straightforward to modify the argument to deal with topological manifolds.

Let M be an m-dimensional smooth manifold. Similar to the compact case we make the following preparations:

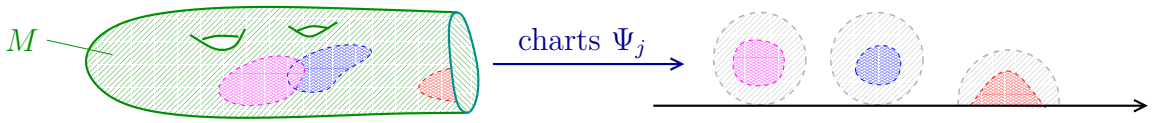
(a) For each $P \in M \setminus \partial M$ we pick a smooth chart $\Phi_P \colon U_P \to B^m((0, \ldots, 0, 1))$ of type (i) around P.
(b) For each $P \in \partial M$ we pick a smooth chart $\Phi_P \colon U_P \to B^m(0) \cap \mathbb{H}^m$ of type (ii) around P.

Note that $\{U_P\}_{P \in M}$ is an open cover of M. In the Covering Dimension-of-Manifolds Proposition 33.9 we just showed that $\operatorname{cdim}(M) \leq m$. By the Manifold Paracompact-Normal Proposition 23.9 (1) we know that M is paracompact and by definition we know that M is second-countable. Thus it follows from the Covering Dimension Refinement Theorem 15.15 that there exists a locally finite open refinement $\{V_j\}_{j \in J}$ of the open cover $\{U_P\}_{P \in M}$ and countable subsets $J_0, \ldots, J_m$ of J with $J = J_0 \sqcup \cdots \sqcup J_m$ such that for every $n \in \{0, \ldots, m\}$ and any $r, s \in J_n$ we have $V_r \cap V_s = \varnothing$.

Since $\{V_j\}_{j \in J}$ is a refinement of $\{U_P\}_{P \in M}$ we can pick for each $j \in J$ a $P_j \in M$ with $V_j \subseteq U_{P_j}$. Note that the images of the charts Φ_{P_j} have diameter 2. Thus, since J is countable and since $m \geq 2$ we can find a family of points $\{Q_j\}_{j \in J}$ in $\mathbb{R}^{m-1}$ such that the images of the maps

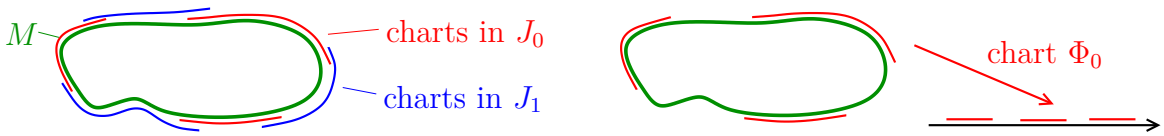
$$\begin{aligned} \Psi_j \colon V_j &\to \mathbb{R}^m \\ x &\mapsto \Phi_{P_j}(x) + (Q_j, 0) \end{aligned}$$

are disjoint.



Finally for $i \in \{0, \ldots, m\}$ we now consider the map

$$\Phi_i := \bigsqcup_{j \in J_i} \Psi_j \colon V_j \;\to\; \mathbb{R}^m.$$



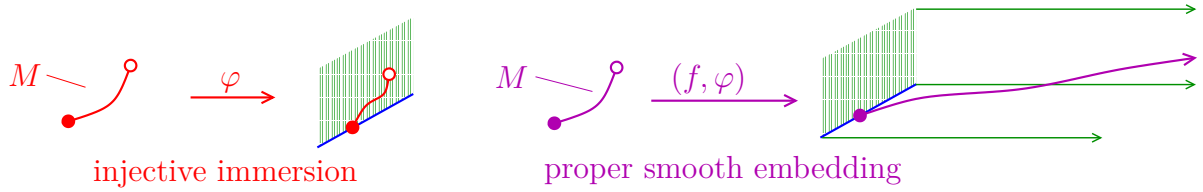
Note that, basically by construction, Φ_i is a chart and these charts $\Phi_0, \dots, \Phi_m$ form a smooth atlas for M. ■

Our proof of the Euclidean Embedding Theorem 33.2 in the non-compact case also rests on the following lemma:

Lemma 33.11. (Stretch-to-Embedding Lemma) Let M be a smooth manifold, let $\varphi\colon M \to \mathbb{H}^d$ be an injective immersion and let $f\colon M \to \mathbb{R}_{\geq 0}$ be a smooth exhaustion function, i.e. $f\colon M \to [0,\infty)$ is a smooth map such that the preimage of every compact interval is compact.[399] Then the map

$$\begin{array}{rcl}(f,\varphi)\colon M & \to & \mathbb{R}\times\mathbb{H}^d = \mathbb{H}^{d+1}\\ x & \mapsto & (f(x),\varphi(x))\end{array}$$

is a proper smooth embedding. The same statement holds if we replace $\mathbb{H}^d$ by any other smooth manifold.



Proof of the Stretch-to-Embedding Lemma 33.11. First note that it is clear that the map $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is still an injective immersion with $\partial M = (f,\varphi)^{-1}(\partial\mathbb{H}^{d+1})$. Using the fact that $f\colon M \to [0,\infty)$ is proper and using the elementary Exercise 7.13) one can easily show that $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is also proper.

It follows from this discussion together with the Proper-Closed Proposition 4.64, that the map $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is a closed embedding.[400] These observations show that the map $(f,\varphi)\colon M \to \mathbb{R}\times\mathbb{H}^d$ is a proper smooth embedding. ■

Proof of Theorem 33.2 in the non-compact case. In the following we deal with the smooth case. The case of topological manifolds is dealt with in a very similar fashion.

Let M be a (possibly non-compact) m-dimensional smooth manifold. If $m = 0$, then it follows easily from the fact that M is second-countable that we can find a smooth embedding $M \to \mathbb{R}$. If $m = 1$, then the statement can be proved easily "by hand" using the Topological 1-Manifold Classification Theorem 28.1 and using the fact that a smooth manifold, by virtue of being second-countable, has countably many components.

So now assume that $m \geq 2$. In the proof of the Euclidean Embedding Theorem 33.2 for compact smooth manifolds we used the compactness so that we could appeal to Lemma 33.7. Now, that M is possibly non-compact, we appeal to the Small Atlas Proposition 33.10, and

[399] Note that by the Manifold Exhaustion Lemma 27.4 we know that every smooth manifold M admits a smooth exhaustion function.

[400] Alternatively one could also argue using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26.

we obtain a smooth atlas $\{\Phi_i \colon U_i \to V_i\}_{i=1,\dots,m+1}$ such that each V_i is an open subset of $\mathbb{H}^m$. We apply the argument in the proof of the Euclidean Embedding Theorem 33.2 on page 862 and we see that there exists a $d \in \mathbb{N}$ and an injective immersion

$$\varphi \colon M \to \mathbb{H}^d$$

with $\partial M = \varphi^{-1}(\partial \mathbb{H}^d)$. If M is compact, then it follows from the Compact-Hausdorff Proposition 4.23 (2) that the map φ is an embedding.

If M is non-compact we have to put in some extra effort. First note that by the Manifold Exhaustion Lemma 27.4 we know that M admits a smooth exhaustion function, i.e. there exists a smooth map $f \colon M \to [0, \infty)$ such that the preimage of every compact interval is compact. By the Stretch-to-Embedding Lemma 33.11 we know that the map

$$\begin{array}{rcl} (f, \varphi) \colon M & \to & \mathbb{R} \times \mathbb{H}^d = \mathbb{H}^{d+1} \\ x & \mapsto & (f(x), \varphi(x)) \end{array}$$

is a proper smooth embedding. ∎

33.5. Exercises for Chapter 33.

Exercise 33.1. Recall that in Definition 29.8 we introduced the category $\mathcal{PSmMfd}$ of pointed smooth manifolds and we introduced the category $\mathcal{PSmMfd}_{\mathbb{R}^n}$ of pointed smooth submanifolds of some $\mathbb{R}^n$. In Question 29.10 (2) we asked whether the covariant functor

$$\begin{array}{rcl} \mathcal{PSmMfd}_{\mathbb{R}^n} & \to & \mathcal{Vec}_{\mathbb{R}} \\ (M, P) & \mapsto & \widetilde{\mathrm{T}}_P M \end{array}$$

can be extended in an "essentially unique" way to the category $\mathcal{PSmMfd}$. In the Tangent Space-Isomorphism Proposition 29.24 (3) we showed that such an extension exists. Turn the "essentially unique" into a precise statement and answer the question in the affirmative. *Hint.* Use the Euclidean Embedding Theorem 33.2 and reread Section 5.8.

Exercise 33.2. We consider

$$\ell^2(\mathbb{N}) := \left\{ (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \sum_{i=1}^{\infty} x_i^2 < \infty \right\}.$$

(a) We consider the map

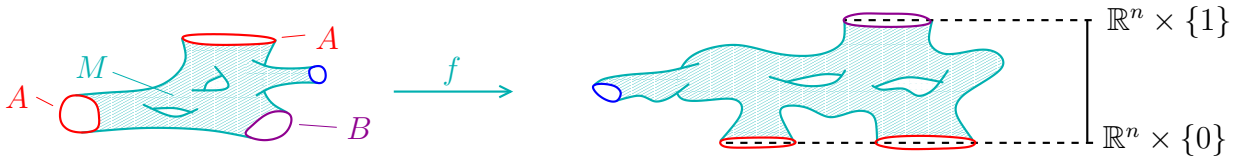
$$\begin{array}{rcl} d \colon \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) & \to & \mathbb{R}_{\geq 0} \\ ((x_1, x_2, \dots), (y_1, y_2, \dots)) & \mapsto & \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}. \end{array}$$

Show that this map does indeed take finite values and show that it defines a complete metric on $\ell^2(\mathbb{N})$.

(b) Let M be a topological manifold. Modify the argument of Section 33.2 to show that there exists a closed embedding $\varphi \colon M \to \ell^2(\mathbb{N})$.

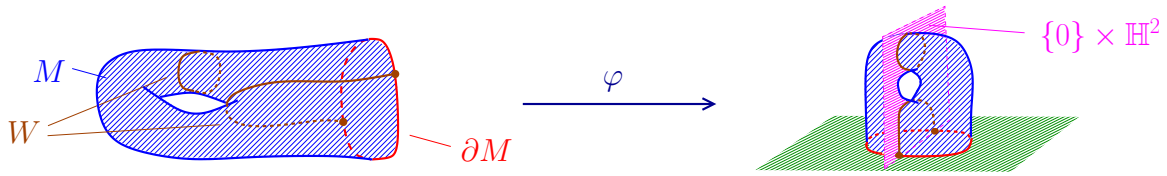
Remark. This exercise, together with the Closed Subspace-Completeness Lemma 10.16, provides a new proof that (possibly non-compact) topological manifolds are completely metrizable. In other words, it provides a proof of the Manifold Metrization Proposition 33.5 (1) that does not make use of the concept of covering dimension that we had developed in Chapter 15.

Exercise 33.3. Let M be a compact smooth manifold and let A and B be disjoint unions of components of ∂M such that $\partial M = A \sqcup B$. Show that there exists a proper smooth embedding $f\colon M \to \mathbb{R}^n \times [0,1]$ such that $A = f^{-1}(\mathbb{R}^n \times \{0\})$ and $B = f^{-1}(\mathbb{R}^n \times \{1\})$.

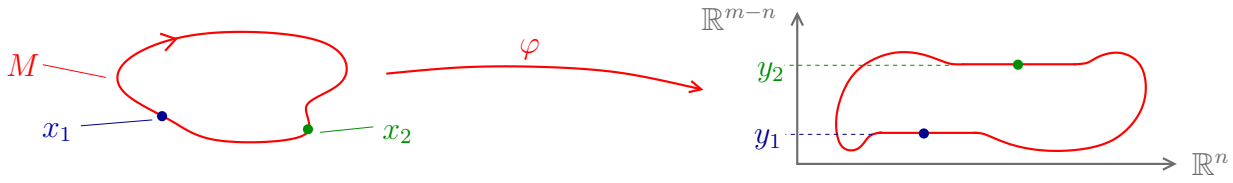


Exercise 33.4. Let M be a smooth manifold with empty boundary. Show that there is no metric $d\colon M \times M \to \mathbb{R}_{\geq 0}$ which is a smooth map from the smooth manifold $M \times M$ to the smooth manifold $\mathbb{R}_{\geq 0}$.

Exercise 33.5. Let M be a compact m-dimensional smooth manifold and let W be a (possibly empty) proper submanifold of M. Show that there exists a proper smooth embedding $\varphi\colon M \to \mathbb{H}^n$ and a $k \in \{0, \dots, n\}$ such that $\varphi(W) = \varphi(M) \cap (\{0\} \times \mathbb{H}^k)$.



Exercise 33.6. Let M be a closed n-dimensional smooth manifold and let $x_1, \dots, x_k \in M$ be pairwise distinct points. Show that there exists a smooth embedding $\varphi\colon M \to \mathbb{R}^m$ such that $\varphi(M)$ is "flat" near $\varphi(x_1), \dots, \varphi(x_k)$, in the sense that there exist disjoint neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$ and points $y_1, \dots, y_k \in \mathbb{R}^{m-n}$ such that for each $i \in \{1, \dots, k\}$ we have the inclusion $\varphi(U_i) \subseteq \mathbb{R}^n \times \{y_i\}$.



Exercise 33.7. Let M be the smooth manifold that is obtained from the open disk B^2 by removing the three points $(0, -\frac{1}{2}), (0,0)$ and $(0, \frac{1}{2})$. Give an explicit proper smooth embedding $f\colon M \to \mathbb{R}^3$.

Remark. This exercise is the "smooth version" of Exercise 15.13. In Exercise 51.12 we will see that there is no proper smooth embedding $f\colon M \to \mathbb{R}^2$.

Exercise 33.8. Given an n-dimensional smooth manifold M we define, following [**Bers1968, Hopk1989**] the

$$\text{embedding covering number } N(M) := \begin{array}{c}\text{minimal cardinality of an open cover } \{U_i\}_{i\in I} \\ \text{of } M \text{ such that each } U_i \text{ admits a} \\ \text{smooth embedding into } \mathbb{R}^n.\end{array}$$

In other words, the embedding covering number is precisely the minimal cardinality of a smooth atlas of M.

(a) Given $g \in \mathbb{N}$, what is the embedding covering number of the surface of genus g?
(b) Show that $N(\mathbb{R}\mathrm{P}^2) = 3$.
(c) What is the embedding covering number of the Klein bottle?

Remark. With the present knowledge one can give upper bounds to $N(M)$ but it is difficult to give very meaningful lower bounds on $N(M)$. So the task is to give the best possible upper bounds.

CHAPTER 34

Lie groups

In this very short chapter we will introduce the notion of a Lie group. Loosely speaking a Lie group is a smooth manifold with a compatible group structure. We will in particular see that the classical matrix groups form Lie groups and we will discuss a few few basic properties of Lie groups. As we will see, Lie groups will appear on various occasions throughout these notes, so it is worth introducing them separately in this chapter.

This chapter is organized as follows:

- In Section 34.1 we will define the notion of a Lie group and we will provide several examples.
- In Section 34.2 we will study quotients of Lie groups.
- In Section 34.3 we will provide exercises for Chapter 34.

In the next Chapter 35 we will will state and prove the Neighborhood Projection Theorem 35.2 which in turn we will use to prove the Smooth Collar Neighborhood Theorem 35.6.

34.1. **Definition of Lie groups and basic examples.** First we recall Definition 7.30:

Definition 34.1.

(1) A topological group is a topological space X together with a group structure such that the two maps

$$\begin{array}{ccc} X \times X & \to & X \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous.

(2) A morphism between topological groups X and Y is a group homomorphism $f\colon X \to Y$ that is furthermore continuous.

(3) We denote by $\mathcal{TopGr}$ the category of topological groups.

Example. On page 299 we already saw that for any $n \in \mathbb{N}$ the matrix groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups. □

The following definition of a Lie group is quite similar to the above definition of a topological group.

Definition 34.2.

(1) A Lie group is a smooth manifold X together with a group structure such that the two maps[401]

$$\begin{array}{ccc} X \times X & \to & X \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth.

(2) A morphism between Lie groups X and Y is a group homomorphism $f\colon X \to Y$ that is furthermore smooth.

(3) We denote by $\mathcal{LieGr}$ the category of Lie groups.

Remark.

(1) Recall that by Definition 25.15 a smooth map is in particular continuous. It follows that a Lie group is also a topological group.
(2) As we will see in Exercise 34.1, strictly speaking one does not need to demand that the map $x \mapsto x^{-1}$ is smooth, since it follows already from the other properties.
(3) A topological space X is called *homogeneous* if given any $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$. It is clear that topological groups (and thus Lie groups) are homogeneous. It follows easily that the boundary of Lie group is empty. □

Examples.

(1) It is straightforward to see that the smooth manifold $\mathbb{R}^n$ with the group structure given by addition and the smooth manifold S^1 with the group structure given by multiplication are Lie groups.
(2) One can easily verify that the Lie group structure on $\mathbb{R}^n$ descends to a Lie group structure on $\mathbb{R}^n/\mathbb{Z}^n$.
(3) The product $G_1 \times G_2$ of two Lie groups G_1 and G_2 admits a canonical smooth manifold structure by the Smooth Manifold Product Proposition 25.34 and the group structures on G_1 and G_2 define a group structure on $G_1 \times G_2$. One can easily verify that $G_1 \times G_2$ is again a Lie group. In particular the torus $S^1 \times S^1$ is a Lie group. □

Not surprisingly we will now see that various matrix groups are Lie groups:

Lemma 34.3. (Matrix–Lie Group Lemma) Let $n \in \mathbb{N}_0$.

(1) (a) Matrix multiplication turns $\mathrm{GL}(n, \mathbb{R})$ into a Lie group.
    (b) The submanifolds $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are also Lie groups.
(2) (a) Matrix multiplication turns $\mathrm{GL}(n, \mathbb{C})$ into a Lie group.
    (b) The submanifolds $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are also Lie groups.

Remark. In the later Matrix–Lie Group Lemma 299.5 we will use the quaternions to add one more type of matrix groups, namely the symplectic group $\mathrm{Sp}(n)$ to the above list. □

Proof. In the Matrices–Topological Group Lemma 7.31 we showed that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are topological groups. Almost the same argument given in that proof shows that matrix multiplication and taking inverses are smooth maps. This implies that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are Lie groups. In the Matrix Group-Smooth Manifold Proposition 32.8 and the Matrix Group-Smooth Manifold Proposition 32.10 we showed that the given subgroups are submanifolds of $\mathrm{GL}(n, \mathbb{R})$ respectively of $\mathrm{GL}(n, \mathbb{C})$. It follows that these are also Lie groups. ■

Above we showed that the torus $S^1 \times S^1$ admits a Lie group structure. It is a priori not clear what other 2-dimensional smooth manifolds admit a Lie group structure. But note that it follows from the following lemma, together with the Surface Non-Orientability Lemma 31.45, that the Klein bottle and the real projective plane are not Lie groups:

[401]Here we equip $X \times X$ with the product smooth manifold structure introduced in the Smooth Manifold Product Proposition 25.34 and in the Smooth Manifold Product Proposition 26.12.

Lemma 34.4. (Lie Group–Orientable Lemma) Every Lie group is orientable.

Remark.

(1) In the Lie Group–Parallelizability Lemmas 265.14 and 299.6 we will prove a stronger version of the Lie Group–Orientable Lemma 34.4 namely we will show that Lie groups are "parallelizable".
(2) In Exercise 53.7 we will see that the "Euler characteristic" of a Lie group is zero. It thus follows from the classification of compact 2-dimensional smooth manifolds and the calculations of Euler characteristics of surfaces and the Lie Group–Orientable Lemma 34.4 that, up to diffeomorphism, the torus $S^1 \times S^1$ is the only compact connected 2-dimensional smooth manifold that admits a Lie group structure. □

Sketch of proof. Let G be an n-dimensional Lie group with trivial element e. For each $g \in G$ we denote by $R_g \colon G \to G$ the map that is given by right-multiplication by g. Since G is a Lie group this map is in fact a diffeomorphism. We pick a basis $\mathcal{B} = (v_1, \dots, v_n)$ for $\mathrm{T}_e G$ and for each $g \in G$ we equip $\mathrm{T}_g G$ with the basis $\mathrm{D}(R_g)_e(\mathcal{B}) = (\mathrm{D}(R_g)_e(v_1), \dots, \mathrm{D}(R_g)_e(v_n))$. We need to prove the following statement:

Statement. These bases define an orientation on G.

We first show that these bases are continuous in an open neighborhood A of the trivial element e. We pick a chart $\Phi \colon U \to V$ around e with $\Phi(e) = 0$. Since the multiplication map $\mu \colon G \times G \to G$ is in particular continuous and since $(e, e) \in \mu^{-1}(e)$ we obtain from the definition of the product topology an open neighborhood A of e with $\mu(A \times A) \subseteq V$. After possibly replacing A by $A \cap U$ we can assume that $A \subseteq U$.

Claim. For each $v \in \mathrm{T}_e G = \mathrm{T}_e A$ the map

$$\begin{array}{rcl} A & \to & \mathbb{R}^n \\ a & \mapsto & \mathrm{D}\Phi_a((\mathrm{D}R_a)_e(v)) \end{array}$$

is smooth.

Proof. We set $B := \Phi(A)$ and we denote by $\Psi \colon V \to U$ the inverse of Φ. We consider the map

$$B \times B \xrightarrow{\mathrm{diag}(\Psi,\Psi)} A \times A \xrightarrow{\mu} U \xrightarrow{\Phi} V.$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=:\Theta}$$

Note that this map is the composition of smooth maps, hence it is smooth. Next note that it follows from this observation together with the Differential-equals-Differential Lemma 29.6 and the Differential-from-Partial Derivatives Proposition 24.16 that the map

$$\begin{array}{rcl} \Xi \colon B \times B \times \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ ((x, y), (r, s)) & \mapsto & \mathrm{D}\Theta_{(x,y)}(r, s) \end{array}$$

is also smooth. Next note that the map we are interested in the claim is precisely the map

$$\begin{array}{rcl} A & \to & \mathbb{R}^n \\ a & \mapsto & \Xi_{(e,\Phi(a))}(\mathrm{D}\Phi_e(v), 0). \end{array}$$

It follows from the above that the map is smooth. ⊞

Recall that we are given a basis $v_1, \dots, v_n \in \mathrm{T}_e G$. It follows from the claim that the map

$$\begin{array}{rcl} A & \to & \mathbb{R}\setminus\{0\} \\ a & \mapsto & \det((\mathrm{D}R_a)_e(v_1),\dots,(\mathrm{D}R_a)_e(v_n)) \end{array}$$

is continuous. It is now clear that the bases $(\mathrm{D}R_a)_e(v_1),\dots,(\mathrm{D}R_a)_e(v_n)$ form an orientation on A.

Now let $g \in G$ be arbitrary. We will now show that the orientations are continuous on the open neighborhood $R_g(A)$. In fact this follows from the fact that the orientations $(\mathrm{D}R_a)_e(v_1),\dots,(\mathrm{D}R_a)_e(v_n)$ are continuous on A and the fact that for $i = 1,\dots,n$ we have

$$(\mathrm{D}R_g)_a((\mathrm{D}R_a)_e(v_i)) \underset{\substack{\uparrow \\ \text{by the Chain rule}}}{=} (\mathrm{D}(R_g \circ R_a))_e(v_i) \underset{\substack{\uparrow \\ \text{since } R_g \circ R_a = R_{g\cdot a}}}{=} (\mathrm{D}R_{g\cdot a})_e(v_i).$$

■

Above we have seen that subgroups of matrix groups often give rise to Lie groups. Perhaps somewhat surprisingly, in a sense, all compact Lie groups arise that way. More precisely, the following theorem holds:[402]

Theorem 34.5. (Compact Lie Group Linearity Theorem) Every compact[403] Lie group is isomorphic, in the sense of Definition 34.2, to a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$.

Proof. The theorem is a consequence of the so-called Peter-Weyl Theorem. We only stated the theorem for "general culture". We will not make use of it and thus we refer to [**BtD1995**, Theorem III.4.1] or alternatively to [**Sep2007**, Theorem 3.28] for a proof. ■

34.2. **Quotients of Lie groups.** We start out with the following harmless lemma:

Lemma 34.6. (Topological Group–Subgroup Action Lemma) Let G be a topological group and let H be a subgroup of G. The map

$$\begin{array}{rcl} G \times H & \to & G \\ (g,h) & \mapsto & g \cdot h \end{array}$$

defines a right action of H on G that is free and continuous.

Proof. We leave the elementary proof of the lemma to the reader. ■

To avoid any misunderstanding we introduce the following convention.

Convention 34.7. Let G be a topological group and let H be a subgroup. We denote by G/H the topological space that is given by the equivalence relation[404]

$$g_1 \sim g_2 \iff \text{there exists an } h \in H \text{ with } g_1 \cdot h = g_2.$$

The following lemma gives us more examples of Lie groups.

Lemma 34.8. (Lie Group–Quotient Lemma) Let G be a Lie group and let K be a finite normal subgroup. The quotient G/K admits a unique smooth manifold structure such that the projection $G \to G/K$ is a local diffeomorphism. Furthermore the group structure on the quotient group G/K defines a Lie group structure on G/K.

[402] One could view the Compact Lie Group Linearity Theorem 34.5 as an analog of the Euclidean Embedding Theorem 33.2.

[403] On page 5758 we will point an example of a path-connected non-compact Lie group that is not isomorphic to a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$.

[404] The point of this convention is that in principle G/H could mean the quotient of the topological space G by the subset H.

Example. It follows from the Lie Group–Quotient Lemma 34.8 that for any $n \in \mathbb{N}$ the quotients $\mathrm{SU}(n)/\{\pm \mathrm{id}\}$ and $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\{\pm(\mathrm{id}, \mathrm{id})\}$ are Lie groups. □

Proof. In the Topological Group–Subgroup Action Lemma 34.6 we already saw that the action of K on G is free and continuous. Since G is a Lie group one also sees that the action is smooth. Finally, since G is finite we get for free that the action is proper. Thus we obtain from the Manifold–Quotient Proposition 25.39 that G/K admits a unique smooth manifold structure (with empty boundary) such that the projection $G \to G/K$ is a local diffeomorphism. Next note that clearly $G/K \times G/K$ is diffeomorphic to $(G \times G)/(K \times K)$ and that the projection $G \times G \to (G \times G)/(K \times K)$ is a local diffeomorphism. It follows easily from these observations that the induced multiplication map $G/K \times G/K \to G/K$ is again smooth. ■

34.3. Exercises for Chapter 34.

Exercise 34.1. Let M be a smooth manifold and let $g\colon M \times M \to M$ be a continuous map that is smooth and that defines a group structure on M. Show that (M, g) is a Lie group. In other words, show that the map $M \to M$ given by $x \mapsto x^{-1}$ is smooth.
Hint. Use the Implicit Function Theorem, see e.g. [**Lee2002**, p. 661].

Exercise 34.2. Let G be an n-dimensional Lie group. Show that there exist vector fields $v_1, \ldots, v_n$ on G, in the sense of Definition 29.29, such that for any $g \in G$ the vectors $v_1(g), \ldots, v_n(g)$ form a basis of $\mathrm{T}_g G$.

Exercise 34.3. Let $G \subseteq \mathrm{GL}(3, \mathbb{R})$ be the Lie group that is given by all upper diagonal matrices with 1's on the diagonal. Is this Lie group isomorphic to $\mathrm{SL}(2, \mathbb{R})$?

Exercise 34.4.

(1) We consider $(\mathbb{Q}, +)$ as a Lie group where we equip $\mathbb{Q}$ with the discrete topology. Is $(\mathbb{Q}, +)$ isomorphic to a Lie subgroup of some $\mathrm{GL}(n, \mathbb{R})$?
(2) We consider $(\mathbb{R}, +)$ as a Lie group where we equip $\mathbb{R}$ with the discrete topology. Is $(\mathbb{Q}, +)$ isomorphic to a Lie subgroup of some $\mathrm{GL}(n, \mathbb{R})$?

CHAPTER 35

Neighborhood Projections and Smooth Collar Neighborhoods

As the title of the chapter strongly suggests, in this chapter we will state, prove and discuss the Neighborhood Projection Theorem 35.2 and the Smooth Collar Neighborhood Theorem 35.6. Both are essential tools in the study of smooth manifolds and will be used frequently.

This chapter is organized as follows:

- In Section 35.1 will state and prove the Neighborhood Projection Theorem 35.2 which says that any submanifold of $\mathbb{R}^n$ is the deformation retraction of an open neighborhood in $\mathbb{R}^n$.
- In Section 35.2 we will state the Smooth Collar Neighborhood Theorem 35.6.
- In Section 35.3 we will use the Neighborhood Projection Theorem 35.2 to provide a proof of the Smooth Collar Neighborhood Theorem 35.6.
- In Appendix 35.4 we will use smooth collars to introduce the push-in of a submanifold of the boundary into the total smooth manifold.
- In Appendix 35.5 we will state and prove the Topological Collar Neighborhood Theorem 35.14 which is an analog of the Smooth Collar Neighborhood Theorem 35.6 for topological manifolds.
- In Section 35.6 we will provide exercises for Chapter 35.

In the next Chapter 36 we will state and prove the Whitney Approximation Theorem 36.1 which says that every continuous map between smooth manifolds is homotopic to a smooth map.

35.1. **The Neighborhood Projection Theorem.** In this section we state the Neighborhood Projection Theorem, which is a key tool in many of our future constructions. To state the theorem it is necessary to introduce the following definition:

Definition 35.1. A map $f\colon M \to N$ between two smooth manifolds with empty boundary is called a submersion if f is smooth and if given any point $P \in M$ the corresponding differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism.

With this definition in our bag we can state the Neighborhood Projection Theorem.

Theorem 35.2. (Neighborhood Projection Theorem) Let M be a proper smooth submanifold of $\mathbb{R}^n$.[405] Given $P \in M$ we write

$$(\widetilde{\mathrm{T}}_P M)^\perp \;=\; \{w \in \mathbb{R}^n \,|\, \langle v, w\rangle = 0 \text{ for all } v \in \widetilde{\mathrm{T}}_P M\}.$$

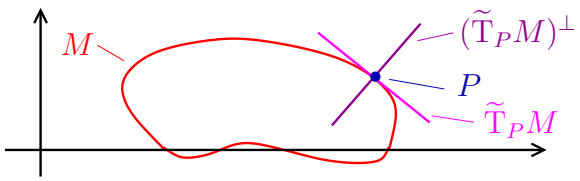
There exists a smooth map $\epsilon\colon M \to (0,\infty)$ such that the following conditions are satisfied:

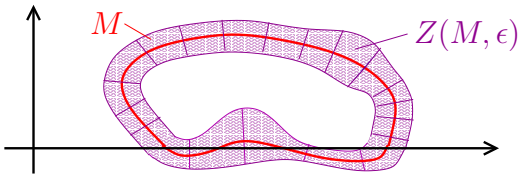
(1) The set $\quad Z(M,\epsilon) \coloneqq \big\{P + w \,|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp \text{ with } \|w\| < \epsilon(P)\big\} \;\subseteq\; \mathbb{R}^n$

is an open neighborhood of $M \subseteq \mathbb{R}^n$.

(2) The map
$$\begin{array}{rcl} \pi\colon Z(M,\epsilon) & \to & M \\ P+w & \mapsto & P \end{array} \quad \text{where } P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^{\perp} \text{ with } \|w\| < \epsilon(P)$$
is well-defined, it is smooth, it is a submersion, and it is a retraction $Z(M,\epsilon) \to M$. Furthermore there exists a smooth[406] deformation retraction
$$R\colon Z(M,\epsilon) \times [0,1] \ \to \ Z(M,\epsilon)$$
with $R_0 = \mathrm{id}$ and with $R_1 = \pi\colon Z(M,\epsilon) \to M$.

If M is compact, then we can choose ϵ to be a constant map.





Remark.

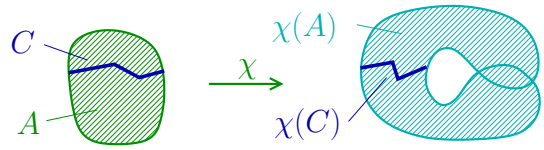
(1) In the (proof of the) Submanifold Neighborhood Retraction Proposition 56.5 we will prove a refinement of the Neighborhood Projection Theorem 35.2 if M has non-empty boundary.

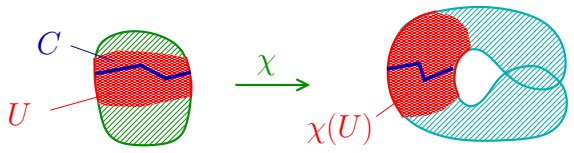
(2) In the Tubular Neighborhood–Existence Theorem 57.11 together with the Basics-of-Tubular Neighborhood Lemma 57.10 we prove an interesting variation on the above theorem.

(3) Later, in Borsuk's Theorem 189.7 we will prove a variation of the Neighborhood Projection Theorem 35.2 that also holds for closed subspaces of $\mathbb{R}^n$ that are topological manifolds. □

Before we can provide the proof of the Neighborhood Projection Theorem 35.2 we need to make a few preparations. First, for the reader's convenience we recall the following theorem which will play a role on three occasions in this chapter.

Theorem 27.16. (Neighborhood Smooth Embedding Theorem) Let $\chi\colon A \to B$ be a continuous map between smooth manifolds. Furthermore let $C \subseteq A$ be a closed subset such that the map $\chi\colon C \to \chi(C)$ is a homeomorphism and such that $\chi(C)$ is a closed subset of B. If χ is a local diffeomorphism at each $P \in C$, then there exists an open neighborhood U of C such that the restriction of χ to U is a smooth embedding and such that $\chi(U)$ is an open subset of B.





[405] Since $\mathbb{R}^n$ has no boundary we see that, by Definition 25.22, in our context "proper" just means that M is a closed subset of $\mathbb{R}^n$.

[406] Note that $Z(M,\epsilon)$ is a smooth manifold with empty boundary. Thus by the Smooth Manifold Product Proposition 25.34 we know that $Z(M,\epsilon) \times [0,1]$ is a smooth manifold. Therefore it makes sense for a map $Z(M,\epsilon) \times [0,1] \to Z(M,\epsilon)$ to be smooth.

Our proof of the Neighborhood Projection Theorem 35.2 relies on a lemma that will also come in handy when we prove the Smooth Collar Neighborhood Theorem 35.6. Therefore we formulate the lemma in a more general fashion than is strictly needed for just the proof of the Neighborhood Projection Theorem 35.2.

Lemma 35.3. (Normal Vectors Lemma) Let M be a proper k-dimensional smooth submanifold of $\mathbb{H}^n$. We set

$$\mathrm{N}M \;:=\; \{(P,w) \in \mathbb{H}^n \times \mathbb{R}^n \,|\, P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp\}.$$

The following three statements hold:

(1) The subset $\mathrm{N}M$ is an n-dimensional submanifold of $\mathbb{H}^n \times \mathbb{R}^n$ whose boundary is given by $\partial(\mathrm{N}M) = \mathrm{N}M \cap (\partial\mathbb{H}^n \times \mathbb{R}^n)$.

(2) Given any point $P \in M$ and given any open neighborhood U of $(P,0) \in \mathrm{N}M$ there exists an open neighborhood U_P of $P \in M$ and an $\epsilon_P > 0$ such that the subset $\{(x,w) \in \mathbb{H}^n \times \mathbb{R}^n \,|\, x \in U_P \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| < \epsilon_P\}$ is contained in U.

(3) The map

$$\begin{array}{rcl} \Pi\colon \mathrm{N}M & \to & M \\ (P,w) & \mapsto & P \end{array}$$

is a submersion.

An analogous statement also holds if we replace $\mathbb{H}^n$ by $\mathbb{R}^n$.

Proof of the Normal Vectors Lemma 35.3. In the following we deal with the case that M is a proper k-dimensional smooth submanifold of $\mathbb{H}^n$. The case $\mathbb{R}^n$ is dealt with in almost the same way. Let $\Phi\colon U \to V$ be a submanifold chart of $M \subseteq \mathbb{H}^n$. We write $\Psi := \Phi^{-1}\colon V \to U$.

> Note that $\mathrm{N}(\Phi(U \cap M))$ is given by setting some coordinates of $V \times \mathbb{R}^n \subseteq \mathbb{H}^n \times \mathbb{R}^n$ to zero. Unfortunately the map $\Psi = \Phi^{-1}\colon V \to U$ does not quite induce a map $\mathrm{N}(\Phi(U \cap M)) \to \mathrm{N}(U \cap M) \subseteq \mathrm{N}M$ since it does not preserve orthogonality. But this problem can be fixed by introducing a map that "makes vectors orthogonal".

As in the Matrix Groups Deformation Retract Proposition 19.4 we denote by $\mathrm{GL}(n,k)$ the set of k-tuples of linearly independent vectors in $\mathbb{R}^n$ and we denote by $\mathrm{O}(n,k)$ the set of k-tuples of orthonormal linearly independent vectors in $\mathbb{R}^n$. The Gram-Schmidt Orthogonalization Process 2.32 gives us a retraction $R\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$. Next we introduce the map $\Theta\colon (\mathbb{R}^n \setminus \{0\})^k \times \mathbb{R}^n \to \mathbb{R}^n$ which is defined as the composition of the following two maps:[407]

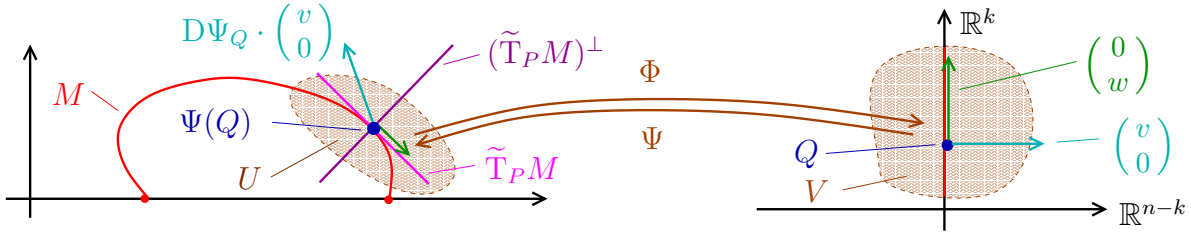
$$\begin{array}{rcccl} \mathrm{GL}(n,k) \times \mathbb{R}^n & \to & \mathrm{O}(n,k) \times \mathbb{R}^n & \to & \mathbb{R}^n \\ ((v_1,\dots,v_k),w) & \mapsto & (R(v_1,\dots,v_k),w) & & \\ & & (v_1,\dots,v_k,w) & \mapsto & \underbrace{w - \sum_{i=1}^{k} \langle w, v_i\rangle \cdot \frac{v_i}{\|v_i\|^2}}_{\text{orthogonal to } v_1,\dots,v_k} \end{array}$$

[407] Here $\langle\ ,\ \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the usual Euclidean bilinear symmetric form on $\mathbb{R}^n$.

Next we consider the map

$$\begin{aligned} G_\Psi \colon V \times \mathbb{R}^{n-k} \times \mathbb{R}^k &\to U \times \mathbb{R}^n \\ (Q, v, w) &\mapsto \Big(\Psi(Q), \underbrace{\Theta\Big(\underbrace{\big(\mathrm{D}\Psi_Q(\mathfrak{e}_{n-k+1}), \dots, \mathrm{D}\Psi_Q(\mathfrak{e}_n)\big)}_{\text{basis of } \widetilde{\mathrm{T}}_{\psi(Q)}M}\Big)}_{\in (\widetilde{\mathrm{T}}_{\psi(Q)}M)^\perp}, \mathrm{D}\Psi_Q \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}, \mathrm{D}\Psi_Q \cdot \begin{pmatrix} 0 \\ w \end{pmatrix}\Big). \end{aligned}$$

One can easily show that the map G_Ψ is a bijection and smooth. Furthermore using the Smooth Pasting Proposition 25.19 (2) one can show fairly easily that G_Ψ is in fact a diffeomorphism. Furthermore, once one has digested the various definitions, it is also quite clear that the $G_\Psi^{-1} \colon U \times \mathbb{R}^n \to V \times \mathbb{R}^n$ defines (after a suitable permutation of the coordinates) a submanifold chart for $\mathrm{N}M$. All the charts G_Ψ^{-1} show that $\mathrm{N}M$ is an n-dimensional submanifold of $\mathbb{H}^n \times \mathbb{R}^n$.



Furthermore note that it follows from the fact that M is a proper submanifold of $\mathbb{H}^n$ and the fact that the submanifold charts Ψ and G_Ψ have the same type, in the sense of Definition 25.22, that $\partial(\mathrm{N}M) = \mathrm{N}M \cap (\partial\mathbb{H}^n \times \mathbb{R}^n)$. In other words, we have completed the proof of Statement (1).

Next note that Statement (2) is a straightforward consequence of the Definition 7.3 of the product topology on $\mathbb{H}^n \times \mathbb{R}^n$ and the definition of the subspace topology on the subset $\mathrm{N}M \subseteq \mathbb{H}^n \times \mathbb{R}^n$.

Finally we turn to the proof of Statement (3). We need to show that $\Pi \colon \mathrm{N}M \to M$ is a submersion. Given $(P, v) \in \mathrm{N}M$ we pick a submanifold chart $\Phi \colon U \to V$ for $P \in M$ as above. We make use of the above notation and we obtain the following diagram:

$$\begin{array}{ccc} (V \cap (\{0\} \times \mathbb{R}^k)) \times \mathbb{R}^{n-k} \times \{0\} & \xrightarrow[\cong]{G_\Psi} & \mathrm{N}(U \cap M) \\ \downarrow{\scriptstyle (x,v)\mapsto x} & & \downarrow{\scriptstyle \Pi} \\ V \cap (\{0\} \times \mathbb{R}^k) & \xrightarrow[\cong]{\Psi} & U \cap M. \end{array}$$

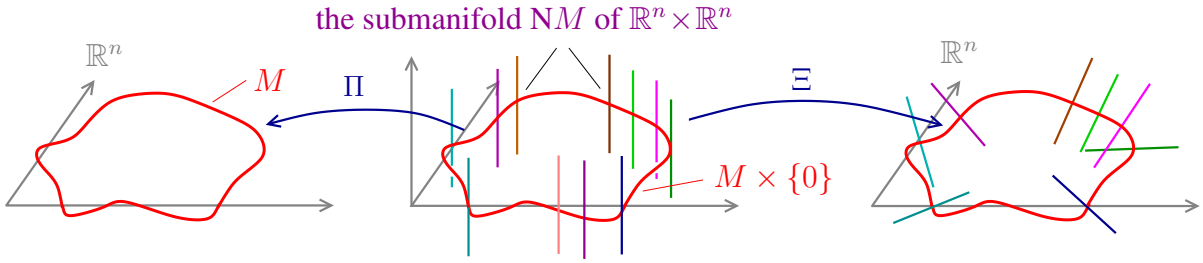
It follows easily from the earlier discussion that the horizontal maps are diffeomorphisms and that the diagram commutes. Evidently the map to the left is a submersion, thus the map to the right is also a submersion. This shows that Π is locally a submersion, but this implies evidently that Π is also a submersion as a whole. ∎

Proof of the Neighborhood Projection Theorem 35.2. Let M be a proper k-dimensional smooth submanifold of $\mathbb{R}^n$. As in the Normal Vectors Lemma 35.3 we consider again

$$\mathrm{N}M \;:=\; \big\{(P, w) \in \mathbb{R}^n \times \mathbb{R}^n \,\big|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp\big\}.$$

By the Normal Vectors Lemma 35.3 we know that $\mathrm{N}M$ is an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. We consider the maps

$$\begin{array}{rcl} \Pi\colon \mathrm{N}M & \to & M \\ (P,w) & \mapsto & P \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Xi\colon \mathrm{N}M & \to & \mathbb{R}^n \\ (P,w) & \mapsto & P+w. \end{array}$$



We make the following observations:

(a) The map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ that is given by $(P,w) \to P+w$ is evidently smooth. Since $\mathrm{N}M$ is a smooth submanifold we obtain from the Smooth Composition Lemma 25.17 and the Submanifold Smooth Structure Lemma 25.23 that the map $\Xi\colon \mathrm{N}M \to \mathbb{R}^n$ is smooth.

(b) Let $P \in M$. It follows easily from $\widetilde{\mathrm{T}}_P M \oplus (\widetilde{\mathrm{T}}_P M)^\perp = \mathbb{R}^n$, together with the discussion of the differential of a linear map on page 651 and the fact that $\mathrm{N}M$ is an n-dimensional smooth manifold that the differential of $\Xi\colon \mathrm{N}M \to \mathbb{R}^n$ at $(P,0) \in \mathrm{N}M$ is an isomorphism.

(c) It follows almost immediately from (b), our hypothesis that $\partial M = \varnothing$, the Local Smooth Embedding Theorem 30.13 and the Normal Vectors Lemma 35.3 (2) that for each point $P \in M$ there exists an open neighborhood W_P of $(P,0) \in \mathrm{N}M$ such that the restriction of $\Xi\colon \mathrm{N}M \to \mathbb{R}^n$ to W_P is a smooth embedding.

(d) Note that the restriction of $\Xi\colon \mathrm{N}M \to \mathbb{R}^n$ to $M \times \{0\}$ is given by $(P,0) \mapsto P$. This is evidently an embedding. By hypothesis M is a proper smooth submanifold of $\mathbb{R}^n$ which means in particular, by definition of "proper", that $M = \Xi(M \times \{0\})$ is a closed subset of $\mathbb{R}^n$.

(e) It follows from (b), (c) and (d), together with the Neighborhood Smooth Embedding Theorem 27.16, that there exists an open neighborhood U of $M \times \{0\} \subseteq \mathrm{N}M$ such that the restriction of Ξ to $\Xi|_U\colon U \to \mathbb{R}^n$ is a smooth embedding and such that $\Xi(U)$ is an open subset of $\mathbb{R}^n$.

(f) It follows from the Replace-by-Smooth Non-Negative Function Corollary 27.14 (2) together with the Normal Vectors Lemma 35.3 (2) that there exists a smooth map $\epsilon\colon M \to (0,\infty)$ such that for each $x \in M$ and each $w \in \big(\widetilde{\mathrm{T}}_x M\big)^\perp$ with $\|w\| < \epsilon(x)$ we have $(x,w) \in U$.

We set

$$\mathrm{N}(\epsilon) \;\coloneqq\; \big\{(P,w) \in \mathrm{N}M \,\big|\, P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^\perp \text{ with } \|w\| < \epsilon(P)\big\} \;\subseteq\; \mathrm{N}M.$$

We consider the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{N}M & \hookleftarrow & U & \hookleftarrow & \mathrm{N}(\epsilon) & & \\ \downarrow \Xi & & \cong \downarrow \Xi & & \cong \downarrow \Xi & \searrow^{\Pi} & \\ \mathbb{R}^n & \hookleftarrow & \Xi(U) & \hookleftarrow & Z(M,\epsilon) & \xrightarrow{\pi} & M. \end{array}$$

The theorem is now a consequence of the following observations and clarifications

(i) It follows easily from the definition of the subspace topology that $\mathrm{N}(\epsilon)$ is an open subset of $\mathrm{N}M$.

(ii) It follows from (i) together with (e) and (f) that the restriction of $\Xi\colon \mathrm{N}M \to \mathbb{R}^n$ to $\Xi\colon \mathrm{N}(\epsilon) \to \mathbb{R}^n$ is a smooth embedding such that the image is open.

(iii) The image of $\Xi\colon \mathrm{N}(\epsilon) \to \mathbb{R}^n$ is precisely $Z(M,\epsilon)$.

(iv) The map
$$\begin{array}{rcl} \pi\colon Z(M,\epsilon) & \to & M \\ P+w & \mapsto & P \end{array} \quad \text{where } P \in M \text{ and } w \in \big(\widetilde{\mathrm{T}}_P M\big)^{\perp} \text{ with } \|w\| < \epsilon(P)$$
is precisely the map $\Pi \circ \Xi^{-1}\colon Z(M,\epsilon) \to M$ and thus well-defined.

(v) It follows from the Smooth Embedding Theorem 30.15 that $\Xi\colon U \to \Xi(U)$ is a diffeomorphism and we know by the Normal Vectors Lemma 35.3 that $\Pi\colon \mathrm{N}M \to M$ is a submersion. It follows that $\pi = \Pi \circ \Xi^{-1}\colon \Xi(U) \to M$ is a submersion.

(vi) There exists an obvious smooth deformation retraction from $\mathrm{N}(\epsilon)$ to $M \times \{0\}$. Under the diffeomorphism $\Xi\colon \mathrm{N}(\epsilon) \to Z(M,\epsilon)$ it gives rise to the desired smooth deformation retraction $R\colon Z(M,\epsilon) \times [0,1] \to Z(M,\epsilon)$ with $R_0 = \mathrm{id}$ and $R_1 = \pi\colon Z(M,\epsilon) \to M$.

Finally if M is compact, then it follows from the Compact Image Lemma 4.15 that the map $\epsilon\colon M \to \mathbb{R}_{>0}$ assumes a minimal value > 0. Thus we can replace ϵ throughout by this minimal value > 0. ■

35.2. **Statement of the Smooth Collar Neighborhood Theorem.** In this section we will state the Smooth Collar Neighborhood Theorem which is another important tool in differential topology.

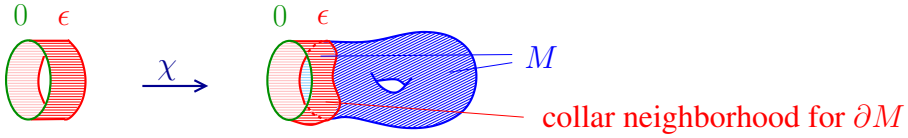
Definition 35.4. Let M be a smooth manifold. Furthermore let $B \subseteq \partial M$ be a union of components. A smooth collar (or just collar) of B is defined as a smooth embedding $\chi\colon [0,\epsilon] \times B \to M$ for some $\epsilon > 0$ that has the following three properties:

(a) The restriction of χ to $\{0\} \times B$ is the obvious map $(0,P) \mapsto P$.

(b) The image $\chi([0,\epsilon] \times B)$ is a closed subset of M.

(c) $\chi^{-1}(\partial M) = \{0\} \times B$.

We refer to the image of χ as a collar neighborhood of B.

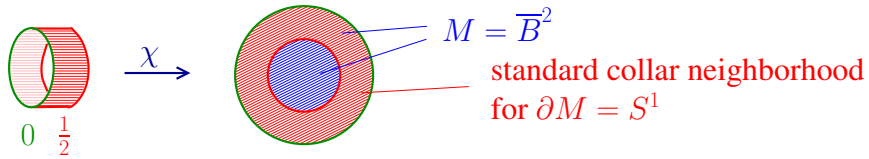


Example.

(1) Let $m \in \mathbb{N}$ and let W be a smooth manifold with $\partial W = \varnothing$. The map

$$\begin{array}{rcl} \chi\colon [0, \frac{1}{2}] \times (S^{m-1} \times W) & \to & \overline{B}^m \times W \\ (t, P, w) & \mapsto & ((1-t) \cdot P, w) \end{array}$$

is evidently a collar for the smooth manifold $\overline{B}^m \times W$.



We refer to this map as the standard collar of $\overline{B}^m \times W$ and we refer to the image of f as the standard collar neighborhood of $\overline{B}^m \times W$.

(2) Let N be a smooth manifold with empty boundary and let $a < b$ be two real numbers. We consider the product $N \times [a, b]$. We invariably equip the smooth boundaries $N \times \{a\}$ and $N \times \{b\}$ with obvious "vertical" collars for a suitable $\epsilon \in (0, b-a)$. □

Convention 35.5.

(1) We continue with the notation from the definition. By abuse of notation we sometimes denote the image of a collar $\chi\colon [0, \epsilon] \times B \to M$ by $[0, \epsilon] \times B$ as well. In particular sometimes we make the identification $\{0\} \times B = B$.

(2) Let $m \in \mathbb{N}$ and let W be a smooth manifold with $\partial W = \varnothing$. Unless we say something else we will equip $\overline{B}^m \times W$ with the above standard collar.

Warning. In the following discussion we suppose that the smooth manifold M is oriented. Recall that in Convention 31.39 we equipped B with a natural orientation. As usual the smooth manifold $[0, \epsilon]$ comes with its standard orientation. Let us now equip $[0, \epsilon] \times B$ with the product orientation that we introduced in Convention 31.49. Note that it follows from the Product Orientation Proposition 31.48 (3d) that, as an oriented smooth manifold, $\partial([0, \epsilon] \times B) = \partial([0, \epsilon]) \times B = -(\{0\} \times B) \cup (\{\epsilon\} \times B)$. Thus we see that every collar $[0, \epsilon] \times B \to M$ is *orientation-reversing*. In principle it would thus be better to define a collar as a smooth embedding $[-\epsilon, 0] \times B \to M$ which is the "identity" on $\{0\} \times M$. But such a choice would be notationally awkward at times. So we stick with our somewhat suboptimal convention. □

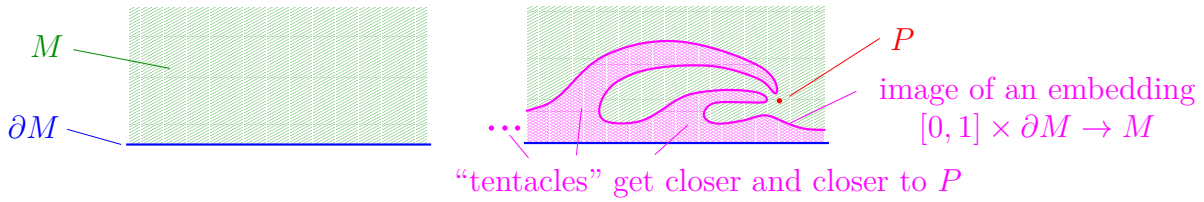
Remark. In the literature one can find several slightly different definitions of a collar and of a collar neighborhood.

(1) In [**Wall2016**, p. 32] the author works with $\partial M \times [0, 1]$ instead of $[0, 1] \times \partial M$. As we saw in the Product Orientation Proposition 31.48 (4), for products of two smooth manifolds it is usually preferable to have the smooth manifold with boundary as the first factor. As we just discussed, our orientation convention for collar neighborhoods is suboptimal, but working with $\partial M \times [0, 1]$ makes the situation even worse.

(2) In some books, e.g. [**Kos1993**, p. 21] and [**BJ1982**, Definition 13.5], a collar is defined as a smooth embedding $\chi\colon [0, 1) \times \partial M \to M$ such that $\chi(0, P) = P$ for all $P \in \partial M$. If we restrict a collar $\chi\colon [0, 1] \times \partial M \to M$ to $[0, 1) \times \partial M$ then this is evidently a smooth embedding. But as, we will see in Exercise 35.4 below, not every smooth embedding $f\colon [0, 1) \times \partial M \to M$ can be extended to a smooth embedding $[0, 1] \times \partial M \to M$. We

prefer to work with our definition since it makes it easier to formulate in what sense collars are unique, see the Smooth Collar-Uniqueness Proposition 54.2.

(3) Our condition (b), namely that a collar neighborhood needs to be closed, is non-standard. If ∂M is compact, which is anyway the case in most situations, the image of any smooth embedding $[0,1] \times \partial M \to M$ is a closed subset by the Product Topology-Properties Proposition 7.9, the Smooth Manifold–Boundary Proposition 25.27 (2b), the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39. Thus the condition only has content if ∂M is non-compact. It seems to the author that in the case of non-compact boundary it is often implicitly assumed that any subset that satisfies (a) is actually closed. But in general this is not the case. For example, in the figure below we consider $M = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ and the image of a smooth embedding $[0,1] \times \partial M$ that does not contain a certain point P, but such that P lies in the closure of the smooth embedding.



In other words, the image of $[0,1] \times \partial M$ is not a closed subset of M. In the Restrict-to-Closed Image Lemma 35.10 we will see that this issue can be resolved in a suitable sense. □

Now, after this lengthy discussion of the notion of a collar, we finally get a chance to state the Smooth Collar Neighborhood Theorem.

Theorem 35.6. (Smooth Collar Neighborhood Theorem) Let M be a smooth manifold and let B be a union of components of ∂M. For any neighborhood V of $B \subseteq M$ there exists a collar $\chi\colon [0,1] \times B \to M$ with $\chi([0,1] \times B) \subseteq V$.

We will provide a proof of the Smooth Collar Neighborhood Theorem 35.6 in the next section. In the remainder of this section we will discuss the Smooth Collar Neighborhood Theorem 35.6 and some of its consequences.

Remark.

(1) The Smooth Collar Neighborhood Theorem 35.6, sometimes in slightly different flavors, is proved in almost every book dealing with the topology of smooth manifolds. In particular proofs are provided in [**Wall2016**, Theorem 1.5.5], [**Lee2002**, Theorem 9.25], [**BJ1982**, Theorem 13.6], [**Muk2015**, Theorem 7.2.13], [**Miln1965b**, Corollary 3.5] and [**Kos1993**, Theorem I.7.3]. As we mentioned above, for non-compact smooth manifolds our definition of a collar is stricter than in the literature. In the Restrict-to-Closed Image Lemma 35.10 we will see that this is not really an issue.

(2) In Project 43.1 we will give an alternative proof of the Smooth Collar Neighborhood Theorem 35.6.

(3) In the Collar-for-Submanifold Proposition 56.3 we will state a refinement of the Smooth Collar Neighborhood Theorem 35.6 which can also take into account submanifolds.

Furthermore in the Smooth Collar Neighborhood Theorem 59.7 we will formulate a generalization for smooth manifolds with corner.

(4) In the Smooth Collar-Uniqueness Proposition 54.2 we will see that collars of compact smooth manifolds are unique in a suitable sense.

(5) The proof of the Smooth Collar Neighborhood Theorem 35.6 that we will provide does not generalize to the setting of topological manifolds. Nonetheless, in the Topological Collar Neighborhood Theorem 35.14 we will generalize the above theorem to the setting of topological manifolds, using a different argument. The approach taken in the proof for topological manifolds can, after some fine tuning, also be used to prove the above Smooth Collar Neighborhood Theorem 35.6. On the other hand, the argument which we use in this chapter for smooth manifolds has the advantage that it can also be used to prove several refinements, e.g. the Bicollar Neighborhood Theorem 55.2 and the Corner-Collar Neighborhood Theorem 60.3.

(6) At least for the case that B is compact one can also prove the Smooth Collar Neighborhood Theorem 35.6 using the Smooth Manifold Product Theorem 250.1. □

The following lemma summarizes a few more or less technical properties of collar neighborhood. The lemma in particular justifies the name "collar neighborhood".

Lemma 35.7. (Basics-of-Collars Lemma) Let M be an n-dimensional smooth manifold, let B be a union of components of ∂M and let $\chi\colon [0,\epsilon]\times B \to M$ be a collar on B for some $\epsilon > 0$.

(1) Let $t \in (0,\epsilon]$.
  (a) The image $\chi([0,t]\times B)$ is a codimension-zero submanifold of M with boundary given by $\chi(\{t\}\times B)\cup B$.
  (b) The image $\chi([0,t)\times B)$ is an open subset of M. In particular $\chi([0,t)\times B)$ and $\chi([0,t]\times B)$ are neighborhoods of B in the sense of Definition 3.2.

(2) Given any $t \in (0,\epsilon]$ the subset $W_t := M\setminus\chi([0,t)\times B)$ is a codimension-zero submanifold of M with boundary given by $(\chi(\{t\}\times B))\cup(\partial M\setminus B)$.

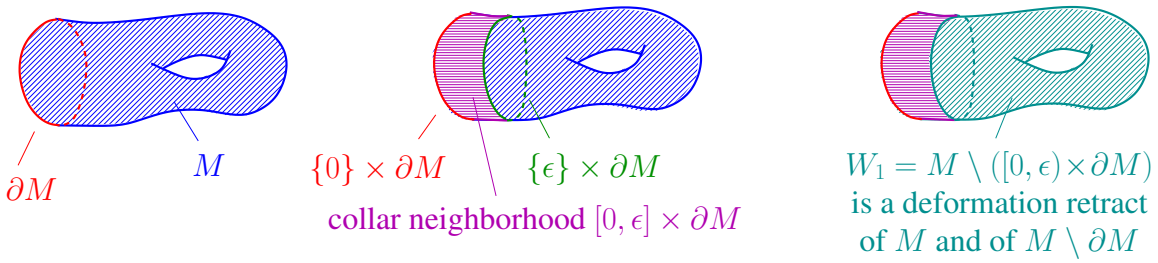
(3) (a) For every $t \in (0,\epsilon]$ the submanifold $W_t = M\setminus\chi([0,t)\times B)$ is a deformation retract of M and it is a deformation retract of $M\setminus B$.
  (b) For every $t \in [0,\epsilon)$ there exists a *smooth* homotopy[408] $F\colon M\times[0,1]\to M$ with the following properties:
    - F is a homotopy rel $W_1 = M\setminus\chi([0,\epsilon)\times B)$,
    - $F_0 = \mathrm{id}$,
    - for each $P\in B$ and $s\in[0,1]$ we have $F_s(P)\subseteq\chi([0,\epsilon]\times\{P\})$,
    - $F_1(M)\subseteq W_t$.

Proof.

(1) This statement is an immediate consequence of the Codimension-Zero Smooth Embedding Proposition 30.21 (1) and (2).

(2) Let $t\in[0,\epsilon]$. By hypothesis $\chi([0,\epsilon]\times B)$ is a closed subset of M. It follows from the Open-Closed Inclusion Lemma 4.13 that $\chi([0,t]\times B)$ is also a closed subset of M. The

[408]By the Smooth Manifold Product Proposition 26.12 we can view $M\times[0,\epsilon]$ as a smooth manifold (possibly with corner). Thus it makes sense to say that the map $F\colon M\times[0,\epsilon]\to M$ is smooth.



desired statement follows from this observation together with the Codimension-Zero Smooth Embedding Proposition 30.21 (2).[409]

(3) (a) Let $t \in (0, \epsilon]$. We consider the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (Q, s) & \mapsto & \begin{cases} \chi((1-s)\cdot u + s \cdot t, R), & \text{if } Q = \chi(u, R) \in \chi([0,t] \times B), \\ Q, & \text{if } Q \in W_t. \end{cases} \end{array}$$

Note that by definition the collar neighborhood $\chi([0, \epsilon] \times B)$ is a closed subset of M and that we know by (1) that $M \setminus \chi([0, \epsilon) \times B)$ is also a closed subset of M. Since the restriction of F to the products of these closed subsets with $[0, \epsilon]$ is continuous we obtain from the Pasting Proposition 4.7 (2) that the map $F\colon M \times [0, \epsilon] \to M$ is indeed continuous. This map defines a deformation retraction from M to W_t and also from $M \setminus B$ to W_t.

(b) Let $t \in [0, \epsilon)$. Using the Smooth Transition Function Lemma 24.9 one can show easily that there exists a smooth map $\eta\colon [0, \epsilon] \to [t, \epsilon]$ for which there exists an $\sigma > 0$ such that $\eta(s) = s$ for all $s \in [\epsilon - \sigma, \epsilon]$. We consider the homotopy

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & M \\ (Q, s) & \mapsto & \begin{cases} \chi((1-s)\cdot u + s \cdot \eta(u), R), & \text{if } Q = \chi(u, R) \in \chi([0,\epsilon] \times B), \\ Q, & \text{if } Q \in W_1. \end{cases} \end{array}$$

It follows from the Smooth Pasting Proposition 25.19 that this map is smooth.[410] It is now clear that this map has all the desired properties. ■

For technical purposes it is at times useful to arrange that maps take values "away from the boundary". In many cases this can be achieved by the following corollary, which can be viewed as a refinement of the Push Away-from-Boundary Lemma 23.17 in the setting of smooth manifolds.

Corollary 35.8. (Push-Away-from-Boundary Corollary) Let M be a smooth manifold.

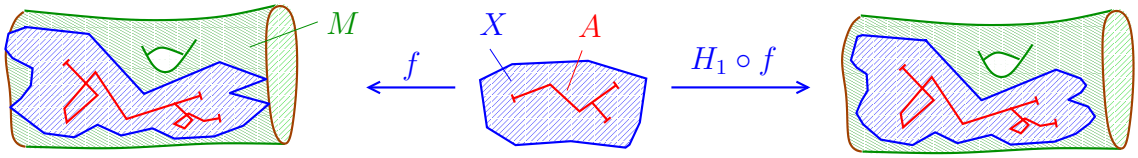
(1) The inclusion $M \setminus \partial M \to M$ is a homotopy equivalence[411]. In particular, if ∂M is non-empty, then M is homotopy equivalent to a non-compact n-dimensional smooth manifold with empty boundary.

[409] Note that the statement that the boundary equals $\chi(\{t\} \times B) \cup (\partial M \setminus B)$ rests on the fact that we have $\chi^{-1}(B) = \{0\} \times B$.

[410] The super vigilant reader will notice that we are really using the obvious extension of the Smooth Pasting Proposition 25.19 to smooth manifolds with corner.

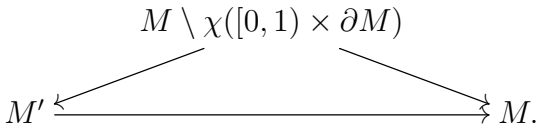
(2) (a) Let X be a topological space, let $f\colon X \to M$ be a continuous map and let $A \subseteq X$ be a (possibly empty) compact subset with $f(A) \subseteq M \setminus \partial M$. There exists a smooth homotopy $H\colon M \times [0,1] \to M$ rel $f(A)$ such that $H_0 = \mathrm{id}$ and such that $(H_1 \circ f)(X) \subseteq M \setminus \partial M$.

(b) If X is a smooth manifold and $f\colon X \to M$ is a smooth map, then f is smoothly homotopic to a map $X \to M \setminus \partial M$.



Proof. We set $M' \coloneqq M \setminus \partial M$. By the Smooth Collar Neighborhood Theorem 35.6 there exists a collar $\chi\colon [0,1) \times \partial M \to M$.

(1) We consider the following diagram of inclusions:



First note that it follows from the Basics-of-Collars Lemma 35.7 (3a) and the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the diagonal maps are homotopy equivalences. Since the diagram commutes we now obtain from the Homotopy-Equivalence Basics Lemma 19.14 that the inclusion $M' \to M$ is also a homotopy equivalence.

We now assume that $\partial M \neq \varnothing$. It follows from the argument of the Interior-Non Compact Lemma 23.16 that M' is non-compact, that it is an open subset of M (in particular it is an n-dimensional smooth manifold) and that the smooth manifold M' does not have any boundary.

(2) (a) We start out with the following claim:

Claim. There exists an $\epsilon \in (0,1]$ such that $\chi([0,\epsilon] \times \partial M) \cap f(A) = \varnothing$.

Proof. Since A is compact we obtain from the Compact Image Lemma 4.15 (1) that $f(A)$ is compact. Since $\chi([0,1] \times \partial M)$ is a closed subset we obtain from the elementary Lemma 3.39 that $f(A) \cap \chi([0,1] \times \partial M)$ is also compact. Finally note that it follows from this discussion, the Compact Image Lemma 4.15 (2) applied to the projection $[0,1] \times \partial M \to [0,1]$ and the hypothesis that $f(A) \subseteq M \setminus \partial M$ that there exists an $\epsilon \in (0,1]$ such that $\chi([0,\epsilon] \times \partial M) \cap f(A) = \varnothing$. ⊞

By the Basics-of-Collars Lemma 35.7 (3b), applied to the collar $(t,x) \mapsto \chi(t \cdot \epsilon, x)$, there exists a smooth homotopy $H\colon M \times [0,1] \to M$ rel $M \setminus \chi([0,\epsilon] \times \partial M)$ such that $H_0 = \mathrm{id}$ and such that $H_1(M) \subseteq M \setminus \partial M$.

[411] Later, on page 1608 we will show that the inclusion $M \setminus \partial M \to M$ is in fact a smooth homotopy equivalence.

(b) We pick a smooth homotopy $H\colon M \times [0,1] \to M$ as in (a). The desired smooth homotopy is now given given the map[412]

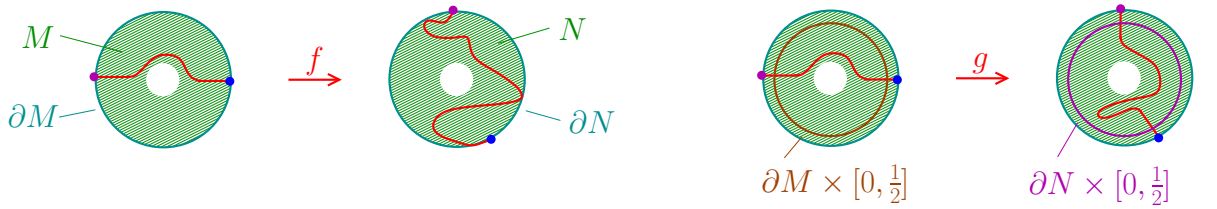
$$\begin{array}{rcl} X \times [0,1] & \to & M \\ (x,t) & \mapsto & H(f(x),t). \end{array}$$

■

We conclude this exhausting discussion of collar neighborhoods with the following technically quite useful lemma:

Lemma 35.9. (Homotopic-to-Product-Near-Boundary Lemma) Let M and N be smooth manifolds with collars $\mu\colon [0,1] \times \partial M \to M$ and $\nu\colon [0,1] \times \partial N \to N$. To simplify the notation we identify $[0,1] \times \partial M$ and $[0,1] \times \partial N$ with their images under μ and ν. Furthermore let $f\colon M \to N$ be a continuous (smooth) map with $f(\partial M) \subseteq \partial N$. There exists a (smooth) homotopy $F\colon M \times [0,1] \to N$ from f to a continuous (smooth) map g such that the following holds:

(1) For each $t \in [0,1]$ we have $F_t(\partial M) \subseteq \partial N$.
(2) The map $g\colon M \to N$ has the following two properties:
  (a) For every $(s,P) \in [0,\frac{1}{2}] \times \partial M$ we have $g(s,P) = (s, f(P)) \in [0,\frac{1}{2}] \times \partial N$.
  (b) We have $g^{-1}([0,\frac{1}{2}] \times \partial N) = [0,\frac{1}{2}] \times \partial M$.



Proof. First let $f\colon M \to N$ be a *continuous* map with $f(\partial M) \subseteq \partial N$. We consider the homotopy

$$\begin{array}{rcl} G\colon N \times [0,1] & \to & N \\ (Q,s) & \mapsto & \begin{cases} (\frac{s}{2}, \widetilde{Q}), & \text{if } Q = (t,\widetilde{Q}) \in [0,\frac{s}{2}] \times \partial N, \\ Q, & \text{if } Q \in N \setminus ([0,\frac{s}{2}) \times \partial N) \end{cases} \end{array}$$

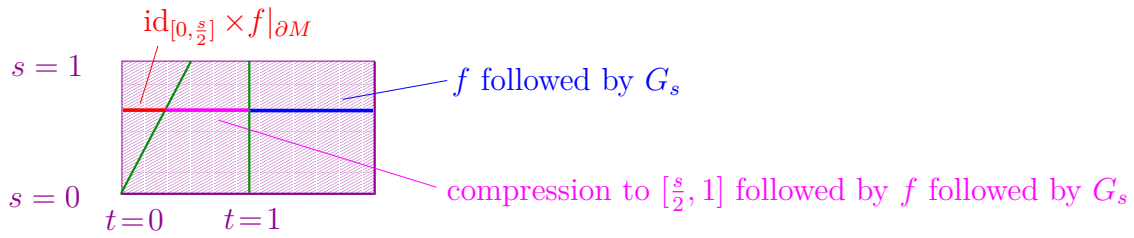
Note that each G_s "pushes everything in $[0,\frac{s}{2}] \times \partial N$ to $\{\frac{s}{2}\} \times \partial N$". Now we consider the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (P,s) & \mapsto & \left(\begin{array}{ll} (t, f(0,\widetilde{P})), & \text{if } P = (t,\widetilde{P}) \text{ with } t \in [0,\frac{s}{2}] \text{ and } \widetilde{P} \in \partial M, \\ G_s\big(f\big(\frac{t-\frac{1}{2}s}{1-\frac{1}{2}s}, \widetilde{P}\big)\big), & \text{if } P = (t,\widetilde{P}) \in [\frac{s}{2},1] \text{ and } \widetilde{P} \in \partial M, \\ G_s(f(P)), & \text{if } P \notin [0,1) \times \partial M. \end{array} \right) \end{array}$$

Note that $F_0 = f$.
One can now easily verify that this homotopy F has the desired properties. The smooth case is technically slightly more tricky and, as so often, rests on making good use of the Smooth Transition Function Lemma 24.9. We leave it to the reader to work out the details. ■

35.3. **Proof of the Smooth Collar Neighborhood Theorem 35.6.** This section is completely dedicated to the proof of the Smooth Collar Neighborhood Theorem 35.6. For

[412] It follows from the Smooth Manifold Product Proposition 26.12 that this map is indeed smooth.



non-compact smooth manifolds we will need the following technical lemma which is of interest only for diehard fans of noncompact smooth manifolds and their collars.

Lemma 35.10. (Restrict-to-Closed Image Lemma) Let M be a smooth manifold and let $f\colon [0,1]\times\partial M \to M$ be a smooth embedding such that for every $P\in\partial M$ we have $f(0,P)=P$. There exists a smooth map $\psi\colon \partial M\to(0,1]$ such that the map

$$\begin{array}{rcl} g\colon [0,1]\times\partial M & \to & M \\ (t,x) & \mapsto & f(\psi(x)\cdot t,x) \end{array}$$

is a collar, i.e. it is a smooth embedding such that $g(0,P)=P$ for all $P\in\partial M$ and such that $g([0,1]\times\partial M)$ is a closed subset of M.



Remark.

(1) We had alluded to the Restrict-to-Closed Image Lemma 35.10 earlier on page 884 as it allows us navigate between the different notions of a collar for a non-compact smooth manifolds.

(2) Our proof of the Restrict-to-Closed Image Lemma 35.10 is based on the proof provided in [**Lee2002b**]. □

Proof. By the Manifold Exhaustion Lemma 27.4 there exists a smooth map $\mu\colon M\to[0,\infty)$ with the property that given any $C\in\mathbb{R}$ the preimage $\mu^{-1}([0,C])$ is a compact subset of M. We consider the map

$$\begin{array}{rcl} \varphi\colon \partial M & \to & [0,1] \\ x & \mapsto & \sup\big\{t\in[0,1]\,|\,\mu(f(s,x))>\mu(\underbrace{f(0,x)}_{=x})-1 \text{ for all } s\in[0,t]\big\}. \end{array}$$

Using charts one can show fairly easily that the map φ is continuous and that $\varphi(x)>0$ for every $x\in\partial M$. By the Replace-by-Smooth Non-Negative Function Corollary 27.14 there exists a smooth map $\psi\colon\partial M\to(0,1]$ such that $\psi(x)\in(0,\frac{1}{2}\cdot\varphi(x)]$ for all $x\in\partial M$. It suffices to prove the following claim:

Claim. The map

$$\begin{array}{rcl} g\colon [0,1]\times\partial M & \to & M \\ (t,x) & \mapsto & f(\psi(x)\cdot t,x) \end{array}$$

is a smooth embedding such that $g(0,P) = P$ for all $P \in \partial M$ and such that $g([0,1] \times \partial M)$ is a closed subset of M.

Proof. The only statement which requires some thought is to show that $g([0,1] \times \partial M)$ is a closed subset of M. We start out with two little observations:

(1) For any $(t,x) \in [0,1] \times \partial M$ we have $0 \leq \psi(x) \cdot t \leq \frac{1}{2} \cdot \varphi(x) \cdot t < \varphi(x)$, which implies, by the definition of $\varphi(x)$, that we have $\mu(g(t,x)) = \mu(f(\psi(x) \cdot t, x)) > \mu(x,0) - 1$.

(2) It follows from (1) that for $(t,x) \in [0,1] \times \mu^{-1}((C+1,\infty))$ we have the inequality $\mu(g(t,x)) > C + 1 - 1 = C$. In other words, we have $g(t,x) \notin \mu^{-1}([0,C])$

Now we see that

$$M \setminus g([0,1]\times\partial M) = \underbrace{\Big(\bigcup_{C\in\mathbb{R}} \mu^{-1}([0,C))\Big)}_{=M} \setminus g([0,1]\times\partial M) = \bigcup_{C\in\mathbb{R}} \mu^{-1}([0,C)) \setminus g([0,1]\times\partial M)$$

$$= \bigcup_{C\in\mathbb{R}} \underbrace{\mu^{-1}([0,C))}_{\text{open}} \setminus \underbrace{g([0,1] \times (\partial M \cap \overbrace{\mu^{-1}([0,C+1])}^{\text{compact}}))}_{\substack{\text{closed by the Compact Image Lemma 4.15} \\ \text{and the Compact-Closed Lemma 3.39}}}.$$

compact by the Smooth Manifold Boundary Proposition 25.27 (2a) and Exercise 3.18

it follows from (2) that we have $\mu^{-1}([0,C) \cap g([0,1] \times \mu^{-1}((C+1,\infty))) = \varnothing$

Thus we have shown that $M \setminus g([0,1] \times \partial M)$ is the union of open subsets of M. It follows that $M \setminus g([0,1] \times \partial M)$ itself is open, i.e. $g([0,1] \times \partial M)$ is closed. ∎

Now we can finally turn to the actual proof of the Smooth Collar Neighborhood Theorem 35.6.

Proof of the Smooth Collar Neighborhood Theorem 35.6. Let M be a smooth manifold, let B be a union of boundary components and let V be an open neighborhood of ∂M. After replacing V by $V \setminus (\partial M \setminus B)$ we see that it suffices to deal with the case $B = \partial M$.

By the Euclidean Embedding Theorem 33.2 there exists an $n \in \mathbb{N}$ and a proper smooth embedding $\varphi\colon M \to \mathbb{H}^n$. Thus, by the Smooth Embedding Theorem 30.15, we might as well assume that M is a proper smooth submanifold of $\mathbb{H}^n$. By definition of "proper" this means in particular that the inclusion $M \to \mathbb{H}^n$ is a closed embedding. Note that it follows from the Smooth Manifold–Boundary Proposition 25.27 (1) and (2) that the inclusion $\partial M \to \mathbb{H}^n$ is also a closed embedding.

Our goal is to construct a collar $\chi\colon [0,1] \times \partial M \to M$. In the following three claims we first introduce three auxiliary maps. Afterwards we provide the actual construction of a collar.

Claim 1. The map

$$\begin{array}{rccc} \Omega\colon & [0,\infty) \times \partial M & \to & \mathbb{H}^n \\ & (t,P) & \mapsto & P + t \cdot \mathfrak{e}_n \end{array}$$

is smooth and it satisfies $\Omega^{-1}(\partial\mathbb{H}^n) = \{0\} \times \partial M$.

Proof. The two statements are basically obvious. ⊞

The idea now is to compose the map $\Omega\colon [0,\infty) \times \partial M \to \mathbb{H}^n$ with a projection onto M. This sounds like a case for the Neighborhood Projection Theorem 35.2. But we had formulated - for good reasons - the Neighborhood Projection Theorem 35.2 only for

smooth manifolds with *empty boundary*. Therefore we will have to work quite a bit to make the original idea work.

Recall that by the Normal Vectors Lemma 35.3 we know that

$$\mathrm{N}M \;:=\; \{(P,w)\in\mathbb{H}^n\times\mathbb{R}^n \mid P\in M \text{ and } w\in(\widetilde{\mathrm{T}}_PM)^\perp\}$$

is an n-dimensional submanifold of $\mathbb{H}^n\times\mathbb{R}^n$ with $\partial(\mathrm{N}M)=\mathrm{N}M\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$.

Claim 2. We consider the projections

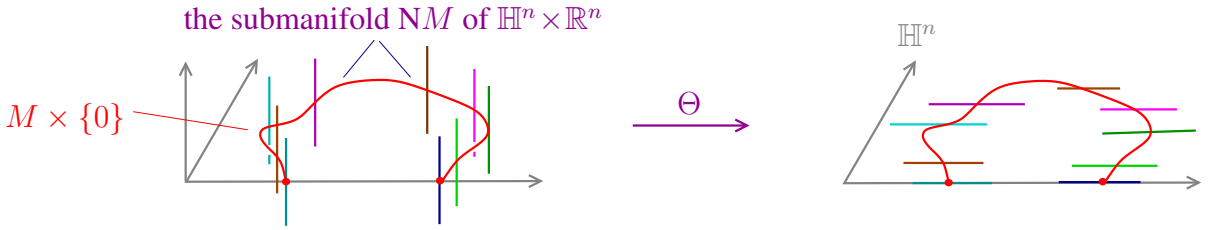
$$\begin{array}{rcl} p\colon \mathbb{H}^n & \to & \mathbb{R}^{n-1} \\ (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_{n-1}) \end{array} \quad\text{and}\quad \begin{array}{rcl} q\colon \mathbb{H}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\dots,x_n) & \mapsto & x_n. \end{array}$$

The map[413]

$$\begin{array}{rcl} \Theta\colon \mathrm{N}M & \to & \mathbb{R}^{n-1}\times\mathbb{R}_{\geq 0}=\mathbb{H}^n \\ (P,w) & \mapsto & (p(P+w),q(P)) \end{array}$$

has the following properties:

(1) For every $P\in\partial M$ the differential $\mathrm{D}\Theta_{(P,0)}\colon \widetilde{\mathrm{T}}_{(P,0)}\mathrm{N}M\to\mathbb{R}^n$ is an isomorphism.
(2) $\Theta^{-1}(\partial\mathbb{H}^n)=\partial(\mathrm{N}M)=\mathrm{N}M\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$.



Proof. Let $P\in\partial M$. Note that[414]

$$\widetilde{\mathrm{T}}_{(P,0)}\mathrm{N}M \;=\; \{(v,w)\mid v\in\widetilde{\mathrm{T}}_PM \text{ and } w\in\big(\widetilde{\mathrm{T}}_PM\big)^\perp\}.$$

Recall that by definition of a proper smooth submanifold we know that $\widetilde{\mathrm{T}}_PM\subsetneq\mathbb{R}^{n-1}\times\{0\}$. Using these two observations one can easily prove (1). Statement (2) follow easily from the definitions (see also the Normal Vectors Lemma 35.3 (1).) ⊞

Claim 3. The map

$$\begin{array}{rcl} \Pi\colon \mathrm{N}M & \to & M \\ (P,w) & \mapsto & P \end{array}$$

is smooth and it satisfies $\Pi^{-1}(\partial M)=\partial(\mathrm{N}M)=\mathrm{N}M\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$.

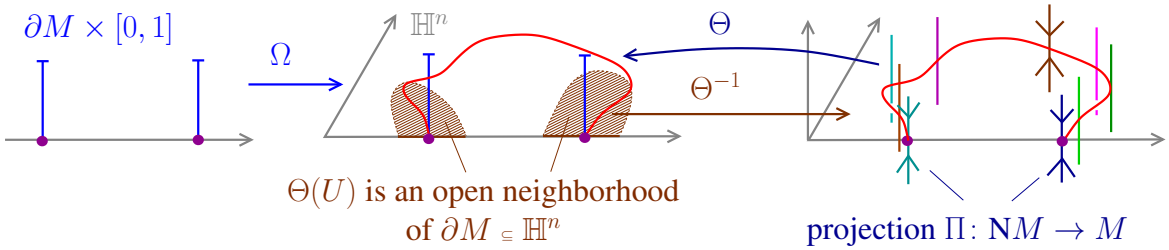
Proof. We proved already in the Normal Vectors Lemma 35.3 (3) that $\Pi\colon\mathrm{N}M\to M$ is smooth. (Alternatively this statement follows from the Smooth Manifold Product Proposition 25.34 together with the Submanifold Smooth Structure Lemma 25.23 (1).) The statement regarding $\Pi^{-1}(\partial\mathbb{H}^n)$ follows easily from the definitions. ⊞

As we mentioned above, the inclusion $\partial M\to\mathbb{H}^n$ is a closed embedding. Evidently for each $P\in\partial M$ we have $\Theta(P,0)=P$. It follows that the restriction of Θ to $\partial M\times\{0\}$ is a closed embedding. Furthermore note that it follows from Claim 2 (1) and (2), together with

[413]The map $\Theta\colon\mathrm{N}M\to\mathbb{H}^n$ is related to the map $\Xi\colon\mathrm{N}M\to\mathbb{R}^n$, $(P,w)\mapsto P+w$ that we introduced in the Normal Vectors Lemma 35.3. The advantage of the new map Θ is that it satisfies the equalities $\Theta^{-1}(\partial\mathbb{H}^n)=\partial(\mathrm{N}M)=\mathrm{N}M\cap(\partial\mathbb{H}^n\times\mathbb{R}^n)$, which is not true for Ξ.

[414]This equality can be seen as follows. One can easily verify that the right-hand side is contained in the left-hand side. Both are n-dimensional vector subspaces, thus they have to be the same.

the Inverse Mapping Theorem 30.1, that $\Theta\colon \mathrm{N}M \to \mathbb{H}^n$ is a local diffeomorphism at each $(P,0)$ with $P \in \partial M$. These arguments show that we can apply the Neighborhood Smooth Embedding Theorem 27.16 to $\Theta\colon \mathrm{N}M \to \mathbb{H}^n$ and the closed subset $\partial M \times \{0\}$. We obtain an open neighborhood U of $\partial M \times \{0\} \subseteq \mathrm{N}M$ such that $\Theta(U)$ is an open neighborhood of $\Theta(\partial M \times \{0\}) = \partial M \subseteq \mathbb{H}^n$ and such that $\Theta\colon U \to \Theta(U)$ is a diffeomorphism.



Claim 4. There exists a smooth map $\mu\colon \partial M \to (0,\infty)$ such that for each $P \in \partial M$ we have $\Omega([0,\mu(P)] \times \{P\}) \subseteq \Theta(U)$.

Proof. First note that it follows from the fact that $\Theta(U)$ is an open subset of $\mathbb{H}^n$ and the Definition 7.3 of the product topology on $[0,1] \times \partial M$ that for every $P \in \partial M$ there exists a $\mu > 0$ and an open neighborhood V of $P \in \partial M$ such that $\Omega([0,\mu] \times V) \subseteq \Theta(U)$. The claim follows from this observation together with the Replace-by-Smooth Non-Negative Function Corollary 27.14 (2). ⊞

For the reader's convenience we summarize all the maps involved so far:

$$\begin{array}{ccccccc}
\{(t,P)\,|\,t \in [0,\mu(P)] \text{ and } P \in \partial M\} & \xrightarrow{\Omega} & \Theta(U) & \underset{\cong}{\overset{\Theta}{\overset{\Theta^{-1}}{\rightleftarrows}}} & U & & \\
\downarrow & & \downarrow & & \downarrow & & \\
[0,\infty) \times \partial M & \xrightarrow{\Omega} & \mathbb{H}^n & \xleftarrow{\Theta} & \mathrm{N}M & \xrightarrow{\Pi} & M.
\end{array}$$

The following claim now finally proclaims that we can find a collar.

Claim 5. There exists a smooth map $\nu\colon \partial M \to (0,\infty)$ such that for each $P \in \partial M$ we have $\nu(P) \leq \mu(P)$ and such that the map[415]

$$\begin{array}{rcl} [0,1] \times \partial M & \to & M \\ (t,P) & \mapsto & (\Pi \circ \Theta^{-1} \circ \Omega)((\nu(P) \cdot t, P)) \end{array}$$

is a collar.

Proof. First we consider the map

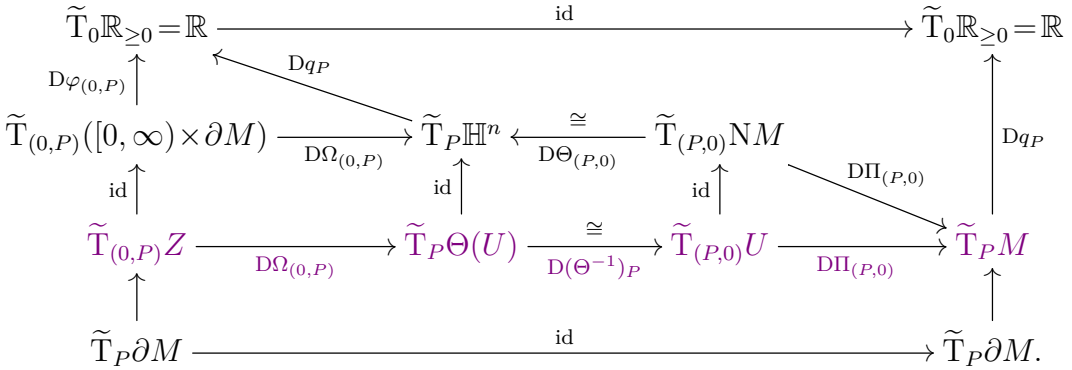
$$\begin{array}{rcl} g\colon \overbrace{\{(t,P) \in [0,\infty) \times \partial M\,|\,t \in [0,\mu(P)] \text{ and } P \in \partial M\}}^{=:Z} & \to & M \\ (t,P) & \mapsto & (\Pi \circ \Theta^{-1} \circ \Omega)(t,P). \end{array}$$

[415]Note that it follows from $\nu(P) \leq \mu(P)$ that $\Omega(\nu(P) \cdot t, P) \in \Theta(U)$. Thus $(\Pi \circ \Theta^{-1} \circ \Omega)((\nu(P) \cdot t, P))$ is indeed defined.

We make the following observations:

(1) It follows immediately from the definitions of the various maps that for every $P \in \partial M$ we have $g(0,P) = P$. Since ∂M is a closed subset of $\mathbb{H}^n$ we see that the restriction of g to $\partial M \times \{0\}$ is a closed embedding.

(2) We now want to show that for each $P \in \partial M$ the differential $\mathrm{D}g_{(0,P)} \colon \widetilde{\mathrm{T}}_{(0,P)}Z \to \widetilde{\mathrm{T}}_P M$ is an isomorphism. To do so we consider the following diagram:



We make the following clarifications and observations:

- As before we denote by $q \colon \mathbb{H}^n \to \mathbb{R}_{\geq 0}$ the projection onto the last coordinate.
- We denote by $\varphi \colon [0,\infty) \times \partial M \to \mathbb{R}_{\geq 0}$ the map that is given by the projection onto the first factor.
- The maps that have no names are the differentials of the obvious inclusions.
- It follows immediately from the definitions that $q \circ \Theta = q \circ \Pi \colon \mathrm{N}M \to \mathbb{R}_{\geq 0}$ and that $q \circ \Omega = \varphi \colon [0,\infty) \times \partial M \to \mathbb{R}_{\geq 0}$. Using these two observations and using the Chain Rule one can easily verify that the diagram commutes.
- The composition of the purple horizontal maps is precisely the map $\mathrm{D}g_{(0,P)}$.
- It is straightforward to verify that the vertical sequence to the left (ignoring the identity map) is exact.[416]
- It follows easily from the definition of a proper submanifold that the vertical sequence to the right is exact.

A gentle stroll through the diagram (or alternatively the later Five Lemma 145.11) now implies that $\mathrm{D}g_{(0,P)} \colon \widetilde{\mathrm{T}}_{(P,0)}Z \to \widetilde{\mathrm{T}}_P M$ is indeed an isomorphism.

(3) It follows immediately from the corresponding statements in Claim 1, 2 and 3 that $g^{-1}(\partial M) = \partial M \times \{0\}$.

It follows from (2) and (3), together with the Inverse Mapping Theorem 30.1, that for each $P \in \partial M$ the map g is a local diffeomorphism at $(0,P)$. Furthermore, using (1), we obtain from the Neighborhood Smooth Embedding Theorem 27.16 that there exists an open neighborhood Y of $\{0\} \times \partial M \subseteq Z = \{(t,P) \in [0,\infty) \times \partial M \mid t \in [0,\mu(P)]\}$ such that $g \colon Y \to M$ is a smooth embedding. After possibly replacing Y with $Y \cap V$ where V is the given neighborhood of ∂M - the reader surely remembers that we are given such a neighborhood - we can assume that $Y \subseteq V$.

[416]Recall that according to Definition 31.9 a sequence of homomorphisms $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$ between real vector spaces is exact if $\varphi \colon U \to V$ is a monomorphism, if $\psi \colon V \to W$ is an epimorphism and if $\mathrm{im}(\varphi) = \ker(\psi)$.

It follows again from the Replace-by-Smooth Non-Negative Function Corollary 27.14 (2) that there exists a smooth map $\omega\colon \partial M \to \mathbb{R}_{>0}$ such that for each $P \in \partial M$ we have $[0,\omega(P)] \times \{P\} \subseteq Y$. By definition of Y we have $\omega(P) \leq \mu(P)$. The choice of ω allows us to consider the smooth embedding

$$\begin{array}{rcl} [0,1] \times \partial M & \to & M \\ (t,P) & \mapsto & (\Pi \circ \Theta^{-1} \times \Omega)(\omega(P) \cdot t, P). \end{array}$$

Finally note that it follows from the Restrict-to-Closed Image Lemma 35.10 that there exists a smooth map $\rho\colon \partial M \to (0,\infty)$ such that the map

$$\begin{array}{rcl} [0,1] \times \partial M & \to & M \\ (t,P) & \mapsto & (\Pi \circ \Theta^{-1} \times \Omega)(\rho(P) \cdot \omega(P) \cdot t, P). \end{array}$$

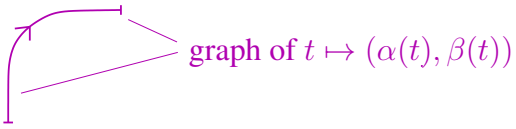
is a collar of M. It follows that the map $\nu\colon \partial M \to \mathbb{R}_{>0}$ that is given by $\nu(P) \coloneqq \rho(P) \cdot \omega(P)$ has the desired properties. ∎

35.4. **Appendix: The push-in of a submanifold.** In this appendix we introduce the notion of a push-in of a submanifold of the boundary ∂W into W. This notion gets used frequently in geometric topology. In these notes the notion does not get used very much.

But out of a sense of duty we still feel obliged to introduce this notion:

Definition 35.11. Let W be a smooth manifold with boundary and let X be a compact submanifold of the boundary ∂W[417]. By the Smooth Transition Function Lemma 24.9 we can pick smooth monotonously increasing function $\alpha\colon [0,1] \to [0,1]$, $\beta\colon [0,1] \to [0,1]$ with the following properties:

- $\alpha(t) = 0$ for $t \in [0, \frac{1}{3}]$ and $\alpha(t) = t$ for $t \in [\frac{2}{3}, 1]$.
- $\beta(t) = t$ for $t \in [0, \frac{1}{3}]$ and $\beta(t) = 1$ for $t \in [\frac{2}{3}, 1]$.
- For each $t \in [0,1]$ we have $\alpha'(t) > 0$ or $\beta'(t) > 0$.



Definition 35.11. (Continued) By applying the Smooth Collar Neighborhood Theorem 35.6 twice we can pick a collar $\mu\colon [0,1] \times \partial W \to W$ and we can also pick a collar $\nu\colon [0,1] \times \partial X \to X$. We consider the map

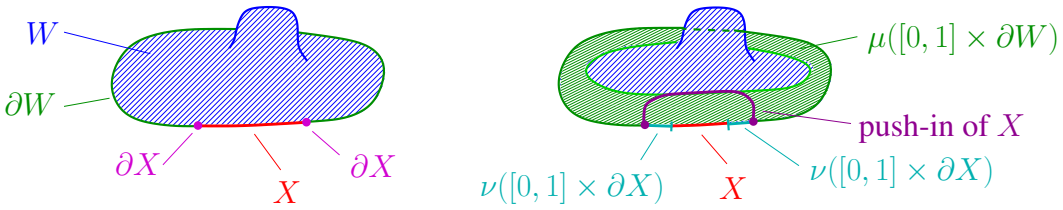
$$\begin{array}{rcl} f\colon X & \to & W \\ P & \mapsto & \begin{cases} \mu(\beta(t), \nu(\alpha(t), Q)), & \text{if } P = \nu(t,Q) \text{ for some } t \in [0,1] \text{ and some } Q \in \partial X, \\ \mu(1,P), & \text{otherwise.} \end{cases} \end{array}$$

We call $f(X)$ the push-in of X.[418]

The following lemma summarizes the key properties of push-ins.

[417] As we will see in **??** even the case $X = \partial W$ can be of interest.

[418] It follows from the Smooth Collar-Uniqueness Proposition 54.2 that the push-in of X is unique in a suitable sense.



Lemma 35.12. (Push-In Lemma) Let W be a smooth manifold with compact boundary and let X be a compact submanifold of ∂W.

(1) There exists a smooth homotopy $H\colon X \times [0,1] \to W$ rel ∂X with $H_0 = \mathrm{id}$ and such that $H_1(X)$ equals the push-in of X.
(2) The push-in Y of X is a compact proper submanifold of W which is diffeomorphic to X and whose boundary equals $\partial Y = \partial X$.

Proof. We continue with the notation from the definition of the push-in.

(1) We consider the map

$$\begin{aligned} X\times[0,1] &\xrightarrow{H} W \\ (P,s) &\mapsto \begin{cases} \mu(s\cdot\beta(t), \nu(s\cdot\alpha(t)+(1-s), Q)), & \text{if } P=\nu(t,Q) \text{ for } (t,Q)\in[0,1]\times\partial X, \\ \mu(s,P), & \text{otherwise.} \end{cases} \end{aligned}$$

First we claim that H is smooth. This can be seen as follows: It follows easily from the choice of α and β that the map H is smooth on $\nu([0,1)\times\partial X)\times[0,1]$ and also on $(X \setminus \nu([0,\frac{3}{4}]\times\partial X))\times[0,1]$. Since these sets are open and since they cover all of $X\times[0,1]$ we obtain from the Smooth Pasting Proposition 25.19 that H is indeed smooth. It is clear that H has all the desired properties.

(2) Using the properties of α and β one can easily verify that $H_1\colon X \to W$ is a proper smooth embedding. It follows from the Smooth Embedding Theorem 30.15 (2) that $H_1(X)$ is a proper submanifold of W which is diffeomorphic to X. It is now straightforward to verify that $Y \coloneqq H_1(X)$ has the desired properties. ∎

35.5. **Appendix: The Topological Collar Neighborhood Theorem.** We start out with the following topological analog of Definition 35.4:

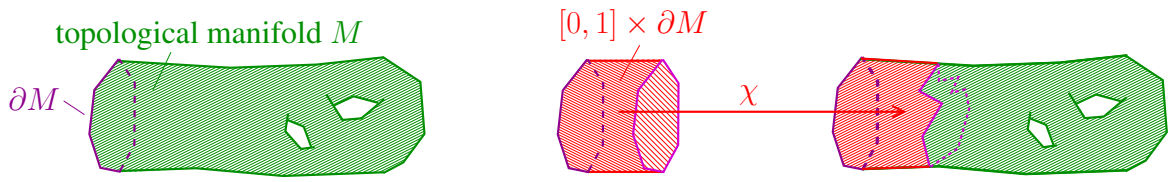
Definition 35.13. Let M be a topological manifold. Furthermore let $B \subseteq \partial M$ be a union of components. A topological collar (or just collar) of B is defined as an embedding $\chi\colon [0,\epsilon]\times B \to M$ for some $\epsilon > 0$ that has the following three properties:

(a) The restriction of χ to $\{0\}\times B$ is the obvious map $(0,P)\mapsto P$.
(b) The image $\chi([0,\epsilon]\times B)$ is a submanifold[419] of M and it is a closed subset of M.
(c) $\chi^{-1}(\partial M) = \{0\}\times B$.

We refer to the image of χ as a collar neighborhood of B.

The only goal of this appendix is to prove the analog of the Smooth Collar Neighborhood Theorem 35.6 for topological manifolds.

[419]For smooth collars it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 (1) and (2) that the image of a smooth collar is a smooth submanifold. There is no topological analog of the Codimension-Zero Smooth Embedding Proposition 30.21 (1) and (2), thus in the setting of topological manifolds we need to demand that the image is indeed a submanifold of M.



Theorem 35.14. (Topological Collar Neighborhood Theorem) Let M be a topological manifold and let B be a union of components of ∂M. For any neighborhood V of $B \subseteq M$ there exists a topological collar $\chi\colon [0,1] \times B \to M$ with $\chi([0,1] \times B) \subseteq V$.

Remark.

(1) In the compact case the Topological Collar Neighborhood Theorem 35.14 is also proved in [**Vic1994**, Theorem Appendix II.7] and [**Hat2002**, p. 253]. The general case, where M is also allowed to be non-compact, is also proved in [**BrownM1962a**, Theorem 3] and [**Arm1970**, Theorem 1]. Note that [**Arm1970**, Theorem 1] has a few refinements which could be useful in some applications. The proof we provide is based on [**Conny1971**].

(2) In Exercise 212.1 we will prove a variation on the Topological Collar Neighborhood Theorem 35.14 for the case that M is a codimension-zero submanifold *with corner* of some other topological manifold.

(3) In Exercise 35.13 we will use the Topological Collar Neighborhood Theorem 35.14 to give a new proof of the Push Away-from-Boundary Lemma 23.17. In particular we will show again that a topological manifold X is path-connected if and only if $X \setminus \partial X$ is path-connected.

(4) On page 1223 we will discuss the uniqueness of collars of topological manifolds. □

We will provide the proof of the Topological Collar Neighborhood Theorem 35.14 shortly. But while thinking about this theorem, a technical nuisance catches up with us: It is still a little unclear what the boundary of a given topological manifold actually is. To make this precise, it is worth recalling the definitions from Definition 23.2.

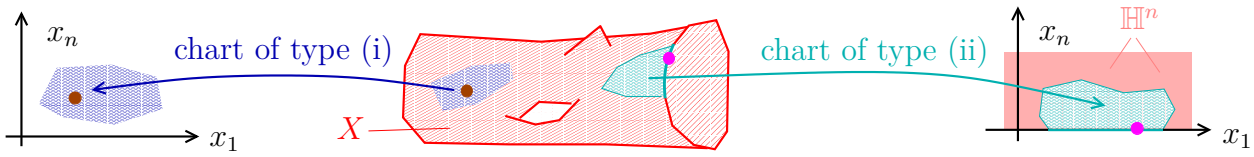
Definition 35.15. Let X be a topological space.

(1) An *n-dimensional chart for X at a point $x \in X$* is a homeomorphism $\chi\colon U \to V$ where U is an open neighborhood of x and
   (i) V is an open subset of $\mathbb{R}^n$ or
   (ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\chi(x)$ lies in $\partial\,\mathbb{H}^n$.
   In the former case we say that χ is a chart of type (i), in the latter case we say that χ is a chart of type (ii).

(2) We say X is an *n-dimensional topological manifold*, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\chi\colon U \to V$ at x.

(3) We say that a point x on a topological manifold X is a *boundary point* if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X. We refer to ∂X as the boundary of X.

We would expect that a point on a topological manifold will have *either* a chart of type (i) *or* a chart of type (ii). At this moment we cannot prove this statement. We can only prove the following weaker statement.



Auxiliary Lemma 35.16. Let $n \in \mathbb{N}_0$. We consider the corresponding upper half-ball $\overline{B}^n_{\geq 0} := \{(x_1, \ldots, x_n) \in \overline{B}^n \mid x_n \geq 0\}$.

(1) If there exists a homeomorphism between $\overline{B}^n_{\geq 0}$ and an open subset of $\mathbb{R}^n$,[420] then the boundary of every n-dimensional topological manifold is empty.

(2) If there does not exist a homeomorphism between $\overline{B}^n_{\geq 0}$ and an open subset of $\mathbb{R}^n$, then any point on a topological manifold admits either a chart of type (i) or of type (ii).

Sketch of proof of Lemma 35.16.

(1) If there exists a homeomorphism between $\overline{B}^n_{\geq 0}$ and an open subset of $\mathbb{R}^n$ then one can easily show that any point on a topological manifold that admits a chart of type (ii) also admits a chart of type (i). This implies that the boundary is empty.

(2) If there exists an n-dimensional topological manifold and a point that admits a chart of type (i) and a chart of type (ii), then one can easily show that there exists a homeomorphism between $\overline{B}^n_{\geq 0}$ and an open subset of $\mathbb{R}^n$. ∎

Now can turn to the actual proof of the Topological Collar Neighborhood Theorem 35.14.

Proof of Theorem 35.14 for compact M. We consider M with an "external collar" attached, i.e. we consider $W := M \cup_{\partial M = \{0\} \times \partial M} ([0,1] \times \partial M)$. Let us assume for a second that we have found a homeomorphism $F\colon M \to W$ with $F(\partial M) = \{1\} \times \partial M$. In this fortunate setting one can verify easily, say using the Topological Pushout-Maps Lemma 8.16 (3) and (4), that the map

$$\begin{aligned} \chi\colon [0,1] \times \partial M &\to M \\ (t,x) &\mapsto F^{-1}([(1-t,x)]) \end{aligned}$$

has all the desired properties. Thus it remains to show that such an F actually exists.

We start out our search for such F with the following claim:

Claim 1. There exist finitely many compact subsets $K_1, \ldots, K_m \subseteq \partial M$, embeddings $\chi_i\colon K_i \times [-1,0] \to M$ and continuous maps $f_i\colon \partial M \to [0,1]$ such that the following are satisfied:

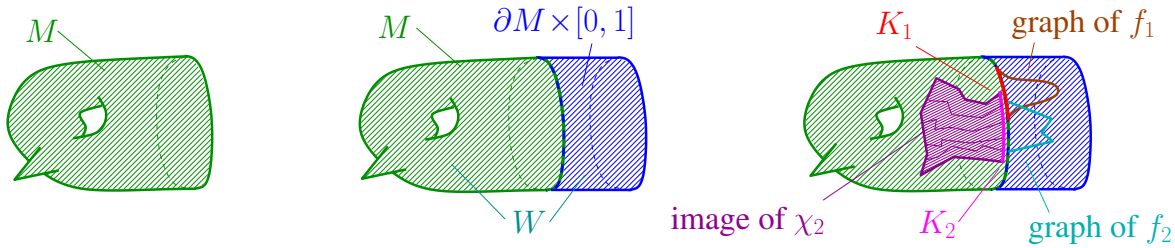
(1) For each $i \in \{1, \ldots, m\}$ we have $\chi_i^{-1}(\partial M) = K_i \times \{0\}$.

(2) For each $i \in \{1, \ldots, m\}$ we have $\operatorname{supp}(f_i) \subseteq K_i$.

(3) We have $f_1 + \cdots + f_m = 1$ on all of ∂M.

Proof. It follows easily from the definition of ∂M, together with the above Lemma 35.16 that for each $x \in \partial M$ there exists an open neighborhood U_x of x and an embedding $\chi_x\colon [-1,0] \times U_x \to M$ with $\chi_x^{-1}(\partial M) = \{0\} \times U_x$. For each x we can and will pick a compact neighborhood K_x with $K_x \subseteq U_x$. Since M is compact, we know by the Topological Manifold–Boundary Proposition 23.15 (2b) that ∂M is also compact. Evidently the interiors of the K_x cover all of ∂M. Since ∂M is compact we see that there exist

[420] In the Manifold-Local-H_*-Lemma 150.9 (1) we will show, using homology groups, that this case cannot happen.



$x_1, \dots, x_m \in \partial M$ such that ∂M is covered by the interiors of the K_{x_i}. Finally note that it follows from the Manifold–Partitions-of-Unity Theorem 27.2 that there exist continuous maps $f_1, \dots, f_m \colon \partial M \to [0,1]$ such that for each $i \in \{1, \dots, m\}$ we have $\operatorname{supp}(f_i) \subseteq K_{x_i}$ and such that $f_1 + \cdots + f_m = 1$. Renaming K_{x_i} and χ_{x_i} as K_i and χ_i we obtain the desired result. ⊞

Next, for $i = 0, \dots, m$ we set $g_i \coloneqq f_1 + \cdots + f_i$ and we set

$$W_i \coloneqq M \cup_{\partial M = \{0\} \times \partial M} \{(t, x) \,|\, x \in \partial M \text{ and } t \in [0, g_i(x)]\}.$$

By the Topological Pushout-Maps Lemma 8.16 (1) we know that W_0 is homeomorphic to M and evidently $W_m = W$.

Claim 2. For $i = 1, \dots, m$ there exists a homeomorphism $F_i \colon W_{i-1} \to W_i$.

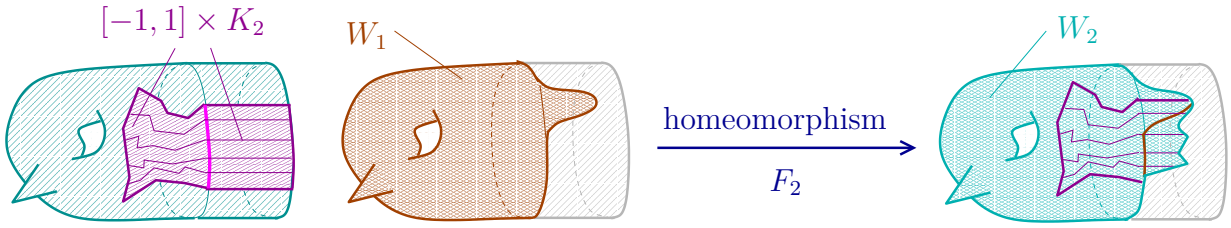
Proof. Let $i \in \{1, \dots, m\}$. We consider the map

$$\begin{array}{rcl} \Theta \colon [-1,1] \times K_i & \to & W \\ (t,x) & \mapsto & \begin{cases} [\chi_i((t,x))], & \text{if } t \in [-1,0], \\ [(t,x)], & \text{if } t \in [0,1]. \end{cases} \end{array}$$

It follows fairly easily from the Topological Pushout-Maps Lemma 8.16 (3) and (4), the Regluing Lemma 8.17 and the Compact-Hausdorff Proposition 4.23 (2) that this map is a closed embedding. To simplify the notation we will identify $[-1,1] \times K_i$ with its image in W. The desired homeomorphism $W_{i-1} \to W_i$ is now given by "stretching horizontally" along $[-1,1] \times K_i$. More precisely, we consider the map

$$\begin{array}{rcl} F_i \colon W_{i-1} & \to & W_i \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin [-1,1] \times K_i \\ \left(-1 + \frac{1+g_i(y)}{1+g_{i-1}(y)} \cdot (t+1), y\right), & \text{if } x = (t,y) \in [-1,1] \times K_i. \end{cases} \end{array}$$

It follows easily from the fact that $\operatorname{supp}(f_i) \subseteq K_i$, the Topological-Product Proposition 7.2 (1) and the Pasting Proposition 4.7 that this map is continuous.



Finally note that it follows from the Compact-Hausdorff Proposition 4.23 (2) that this map is in fact a homeomorphism. ⊞

It is now clear that $F \coloneqq F_m \circ \cdots \circ F_1 \colon M \to W$ is a homeomorphism which has the feature that $F(\partial M) = \{1\} \times \partial M$. ■

Proof of the Topological Collar Neighborhood Theorem 35.14. Now let M be any topological manifold. The proof of the Topological Collar Neighborhood Theorem 35.14 in this setting is not that different from the compact case. For space and time reasons we only sketch the argument.

First note that it follows from the fact that ∂M is in particular second-countable together with the Manifold–Partitions-of-Unity Theorem 27.2 that there exists an infinite sequence $K_1, K_2, \ldots$ of compact subsets, embeddings $\chi_i \colon K_i \times [-1, 0] \to M$ and continuous maps $f_i \colon \partial M \to [0, 1]$ such that

(0) For each $x \in \partial M$ there exists an open neighborhood U such that there exist only finitely many $i \in \mathbb{N}$ with $\operatorname{supp}(f_i) \cap U \neq \varnothing$.
(1) For each $i \in \mathbb{N}$ we have $\chi_i^{-1}(\partial M) = K_i \times \{0\}$.
(2) For each $i \in \mathbb{N}$ we have $\operatorname{supp}(f_i) \subseteq K_i$.
(3) For each $x \in \partial M$ the sum of the non-zero terms $f_i(x)$ equals one.[421]

As in the previous argument we now define functions $g_1, g_2, \ldots$, subsets $W_1, W_2, \ldots$ and homeomorphisms $F_i \colon W_{i-1} \to W_i$. It follows from (0) that the map

$$\begin{array}{rcll} F \colon M & \to & W & \\ x & \mapsto & (F_k \circ \cdots \circ F_1)(x) & \text{if } f_i(x) = 0 \text{ for } i \geq k \end{array}$$

is well-defined and continuous. In fact using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 it is straightforward to see that F is actually a homeomorphism. Furthermore, note that by construction we again have $F(\partial M) = \{1\} \times \partial M$. Thus, as in the proof of the compact case, we are done. ■

35.6. Exercises for Chapter 35.

Exercise 35.1. Let $n \in \mathbb{N}$. We consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$ and, as in the Normal Vectors Lemma 35.3, we consider the $(n+1)$-dimensional smooth manifold

$$\mathrm{N}S^n \;=\; \{(P, v) \,|\, P \in S^n \text{ and } v \in (\widetilde{\mathrm{T}}_P S^n)^\perp \underset{\substack{\uparrow \\ \text{follows from the Physical Tangent Spaces–Lemma 29.2 (3)}}}{=} \mathbb{R} \cdot P\} \;\subseteq\; \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$

(a) Given $(P, \lambda \cdot P) \in \mathrm{N}S^n$ show that the tangent space $\widetilde{\mathrm{T}}_{(P,\lambda\cdot P)}\mathrm{N}S^n$ is spanned by the vectors $\{(v, \lambda \cdot v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,|\, v \in \widetilde{\mathrm{T}}_P S^n\}$ and by $(0, P) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.
(b) We consider the endpoint map

$$\begin{array}{rcl} E \colon \mathrm{N}S^n & \to & \mathbb{R}^{n+1} \\ (P, v) & \mapsto & P + v. \end{array}$$

Show that $(P, \lambda \cdot P)$ is a critical point of E if and only if $\lambda = -1$ and show that $0 \in \mathbb{R}^{n+1}$ is the only critical value of E.

[421] Here we use that by (0) we already know that there are only finitely many non-zero terms.

Exercise 35.2. Let X and Y be two topological spaces and suppose that Y is equipped with a metric d. Let $\epsilon > 0$. We say that two maps $f, g\colon X \to Y$ are *ϵ-close* if for all $x \in X$ we have $d(f(x), g(x)) < \epsilon$.

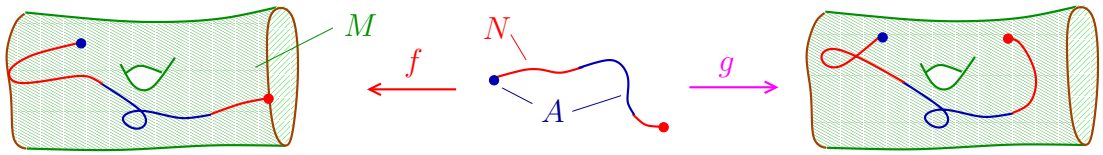
(a) Let M be a smooth submanifold of some $\mathbb{R}^n$ which is closed if viewed as a smooth manifold in its own right. Show that there exists an $\epsilon > 0$ with the following property: If two continuous maps $f, g\colon X \to M$ are ϵ-close, then they are actually homotopic.

(b) Extend the result in (a) to the case that M is a compact smooth submanifold of $\mathbb{R}^n$.

(c) Show that the conclusion of (a) and (b) does not hold if we do not assume that M is compact.

(d) Let M be a closed smooth manifold and let d be a metric on M that induces the topology on M. Show that there exists an $\epsilon > 0$ with the following property: If two continuous maps $f, g\colon X \to M$ are ϵ-close, then they are actually homotopic.
Hint. Try to reduce this statement to (b). In particular apply Exercise 4.19 to the metric d and the Euclidean metric coming from a smooth embedding into some $\mathbb{R}^n$.

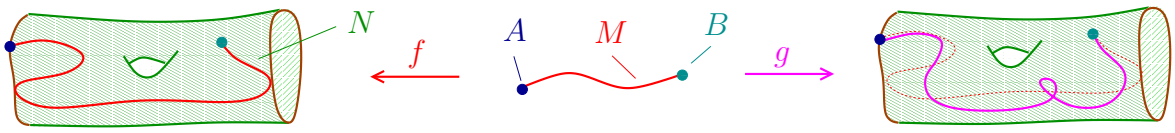
Exercise 35.3. Let M be a smooth manifold, let N be a compact smooth manifold, let $A \subseteq N$ be closed and let $f\colon N \to M$ be a (smooth) map with $f(A) \subseteq M \setminus \partial M$. Show that f is (smoothly) homotopic rel A to a map $g\colon N \to M \setminus \partial M$.
Remark. This exercise can be viewed as a smooth analog of the Push Away-from-Boundary Lemma 23.17. In particular one can use it to show again that if M is path-connected, then $M \setminus \partial M$ is also path-connected.



Exercise 35.4. Give an example of a smooth manifold M together with a smooth embedding $\chi\colon [0,1) \times \partial M \to M$ such that $\chi(0, P) = P$ for all $P \in \partial M$, which cannot be extended to a smooth embedding $f\colon [0,1] \times \partial M \to M$.

Exercise 35.5. Show that there is no smooth retraction from $[0,2]$ to $[0,1]$. More precisely, show that there is no smooth map $f\colon [0,2] \to [0,1]$ which satisfies $f(x) = x$ for all $x \in [0,1]$.

Exercise 35.6. Let M be a compact smooth manifold and let N be any smooth manifold. Furthermore let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Show that for every (smooth) map $f\colon M \to N$ with $f(A) \subseteq \partial N$ and with $f(B) \subseteq N \setminus \partial N$ there exists a (smooth) homotopy $F\colon M \times [0,1] \to N$ rel ∂M from $f\colon M \to N$ to a map $g\colon M \to N$ such that $g^{-1}(\partial N) = A$.
Hint. Apply the Smooth Collar Neighborhood Theorem 35.6 to A and ∂N.



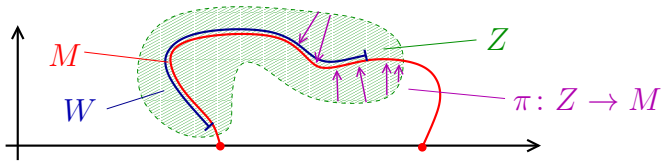
Exercise 35.7. Let M be a smooth manifold and let A and B be two distinct boundary components. Show that there exists a continuous map $f\colon M \to [0,1]$ such that $f|_A \equiv 0$

and such that $f|_B \equiv 1$.
Remark. One can prove this statement easily using Urysohn's Lemma 27.10 for Manifolds. But it is more fun to prove it using the Smooth Collar Neighborhood Theorem 35.6.

Exercise 35.8. Let M be a smooth manifold and let $f\colon \partial M \to S^n$ be a continuous map. Show that there exists a smooth map $g\colon M \to \overline{B}^{n+1}$ with $g|_{\partial M} = f$.

Exercise 35.9. Show that given any n-dimensional smooth manifold M there exists an n-dimensional smooth manifold N with $\partial N = \varnothing$ and a smooth embedding $\varphi\colon M \to N$ such that $\varphi(M)$ is a closed subset of N.
Hint. You could take $N = M \setminus \partial M$.

Exercise 35.10. Let M be a proper smooth submanifold of some $\mathbb{H}^n$ and let $W \subseteq M$ be a closed subset with $W \cap \partial M = \varnothing$. Show that there exists an open neighborhood Z of $W \subseteq \mathbb{R}^n$ with $Z \cap \partial M = \varnothing$ and a smooth map $\pi\colon Z \to M$ which has the following two properties:
(1) for each $P \in W$ we have $\pi(P) = P$,
(2) π is a submersion, i.e. for each $Q \in Z$ the differential $\mathrm{D}\pi_Q$ is an epimorphism.
Hint. You could try to modify the proof of the Neighborhood Projection Theorem 35.2.



Exercise 35.11. Let M be a compact smooth manifold and let $f\colon M \to M$ be a diffeomorphism such that $f|_{\partial M}\colon \partial M \to \partial M$ is diffeotopic to $\mathrm{id}_{\partial M}$. (A diffeotopy is a smooth homotopy $F\colon V \times [0,1] \to W$ such that each $F_t\colon V \to W$ is a diffeomorphism.) Show that f is diffeotopic to a map $g\colon M \to M$ such that $g|_{\partial M} = \mathrm{id}$.

Exercise 35.12. Let M be a compact smooth manifold and let A be a union of components of ∂M. Let $F\colon A \times [0,1] \to A$ be a diffeotopy, i.e. let F be a smooth homotopy such that each $F_t\colon A \to A$ is a diffeomorphism. Show that there exists a diffeotopy $G\colon M \times [0,1] \to M$ such that G restricts to F.

Exercise 35.13. Let M be a topological manifold, let X be a compact space, let $A \subseteq X$ be closed and let $f\colon X \to M$ be a continuous map with $f(A) \subseteq M \setminus \partial M$. Use the Topological Collar Neighborhood Theorem 35.14 to show that f is homotopic rel A to a map $g\colon X \to M \setminus \partial M$.
Remark. This exercise gives a new proof of the Push Away-from-Boundary Lemma 23.17. In particular one can use it to show again that if M is path-connected, then $M \setminus \partial M$ is also path-connected.
Remark. The exercise is the topological analog of Exercise 35.3.

CHAPTER 36

The Whitney Approximation Theorem

In Section 36.1 we will state and prove the Whitney Approximation Theorem 36.1 which says that every continuous map between smooth manifolds is homotopic to a smooth map. The surprisingly short proof makes use of the Neighborhood Projection Theorem 35.2 from the last chapter.

This chapter is organized as follows:

- In Section 36.1 we will state and prove the Whitney Approximation Theorem 36.1.
- In Section 36.2 we will discuss consequences of the Whitney Approximation Theorem 36.1.
- In Appendix 36.3 we will discuss a few more esoteric consequences of the Whitney Approximation Theorem 36.1.
- In Section 36.4 we will provide exercises for Chapter 36.

In the next Chapter 37 we will introduce the notion of a subset of measure zero on a smooth manifold.

36.1. **The statement of the Whitney Approximation Theorem.** As we will see later on, there are many ways to construct interesting continuous maps between two smooth manifolds. The following theorem often allows us to "replace" such a continuous map by a smooth map.

Theorem 36.1. (Whitney Approximation Theorem) Let M and N be two smooth manifolds. Furthermore let $f\colon M \to N$ be a continuous map.[422] There exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:

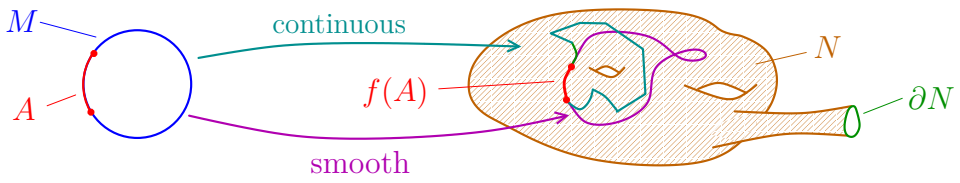
(1) We have $F_0 = f$.
(2) The map $F_1\colon M \to N$ is smooth.
(3) If there exists a union of components X of ∂M with $f(X) \subseteq \partial N$, then we can arrange that $F_t(X) \subseteq \partial N$ for all $t \in [0,1]$.

Suppose that f is smooth[423] on a (possibly empty) closed subset A and suppose that there exists an open neighborhood[424] of ∂N that does not intersect $f(A)$. Then we can find a homotopy as above which also has the following property:

(4) The homotopy is a homotopy rel A.

Finally suppose that we are given a continuous map[425] $\rho\colon M \to \mathbb{R}_{\geq 0}$ such that $\rho(x) > 0$ for every $x \notin A$ and suppose that we are given a metric d on N which induces the given topology on N.[426] Then we can find a homotopy F as above that also satisfies the following extra condition:

(5) For every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \rho(x)$.

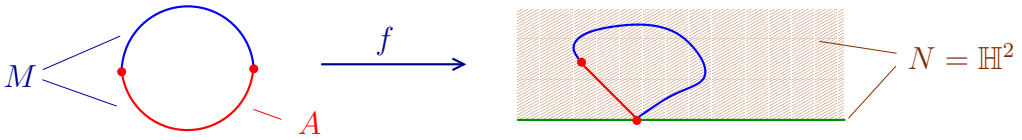


Remark. Suppose we are in the setting of the Whitney Approximation Theorem 36.1. Let $f\colon M \to N$ be a continuous map that is injective (respectively surjective, bijective). Now we know that we can find a homotopy from f to some smooth map $F_1 = g\colon M \to N$, but the theorem does not give us any control over whether the smooth map $F_1 = g$ is also injective (respectively surjective, bijective). For example, by the Space-Filling Curve Theorem 10.21 we know that there exists a surjective continuous map $[0,1] \to [0,1]^2$. It follows easily, see also Exercise 10.14, that there exists also a surjective continuous map $\mathbb{R} \to \mathbb{R}^2$. But, as we will see in the Measure-Zero Properties Proposition 37.3 and the Full Measure-Density Proposition 37.4 there exists no surjective smooth map $\mathbb{R} \to \mathbb{R}^2$. □

Example.

(1) Let M be a smooth manifold and let $f\colon [0,1] \to M$ be a path such that the endpoints $f(0)$ and $f(1)$ lie in $M \setminus \partial M$. By the Whitney Approximation Theorem 36.1, applied to f and the closed subset $\{0,1\} \subseteq [0,1]$, the map f is homotopic rel $\{0,1\}$ to a smooth path from $f(0)$ to $f(1)$. This gives a new proof of the Smooth Path-Connectivity Proposition 25.32 for smooth manifolds with empty boundary.

(2) In the setting of Whitney Approximation Theorem 36.1 (4) we demand that there exists an open neighborhood of ∂N that does not intersect $f(A)$.



In the figure above we show a map $f\colon M = S^1 \to \mathbb{H}^2 = N$ which is smooth on $A = S^1_{\leq 0}$ but which cannot be turned into a smooth map without touching the values on A. □

Remark. Proofs of various variations of the Whitney Approximation Theorem 36.1 are also provided in [**Lee2002**, Theorem 6.26], [**Lee2002**, Theorem 9.27], [**MTo1997**, Lemma A.9], [**Hirs1976**, Lemma 5.1.5] and [**DFN1985**, Theorem 10.1.2]. □

Before we turn to the proof of the Whitney Approximation Theorem 36.1 it is worth recalling the following proposition which basically takes care of the case $N = \mathbb{R}^n$.

---

[422]Note that M and N are both allowed to have non-empty boundary.

[423]Let M and N be smooth manifolds and let $A \subseteq M$ be an arbitrary subset. In Definition 25.20 we defined what it means for a map $f\colon A \to N$ to be smooth.

[424]This condition is satisfied if A is compact and $f(A) \cap \partial N = \varnothing$. Indeed, by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 we know that $f(A)$ is closed. Furthermore by the Smooth Manifold–Boundary Proposition 25.27 (2) we know that ∂N is closed. Finally note that by the Manifold Paracompact-Normal Proposition 23.9 (2) we know that N is normal. But this implies that there exists an open neighborhood of ∂N that does not intersect $f(A)$.

[425]Evidently we demand, as always, that ρ is continuous.

[426]Note that by the Manifold Metrization Proposition 23.10 we know that such a metric always exists.

Proposition 27.13. (Smooth Approximation Proposition) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}^n$ be a continuous map that is smooth on a (possibly empty) closed subset A. Given any continuous map $\eta\colon M \to \mathbb{R}_{\geq 0}$ such that $\eta(x) > 0$ for every $x \notin A$ there exists a smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$ with the following two properties:

(1) The maps f and $\widetilde{f}$ agree on A.
(2) For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \eta(x)$.

We break the proof of the Whitney Approximation Theorem 36.1 into three parts. First we consider the case that $\partial N = \varnothing$. Afterwards we deal with the case that $\partial N \neq \varnothing$ (where we distinguish between $X = \varnothing$ and $X \neq \varnothing$).

Proof of Theorem 36.1 if $\partial N = \varnothing$. Let M be a smooth manifold and let N be a smooth manifold with $\partial N = \varnothing$. Let $f\colon M \to N$ be a continuous map that is smooth on a (possibly empty) closed subset A.

By the Euclidean Embedding Theorem 33.2, together with the Smooth Embedding Theorem 30.15, we can view N as a smooth submanifold of some $\mathbb{R}^n$. Since $\partial N = \varnothing$ we know by the Neighborhood Projection Theorem 35.2 there exists an open neighborhood U of $N \subseteq \mathbb{R}^n$ together with a smooth retraction $\pi\colon U \to N$.

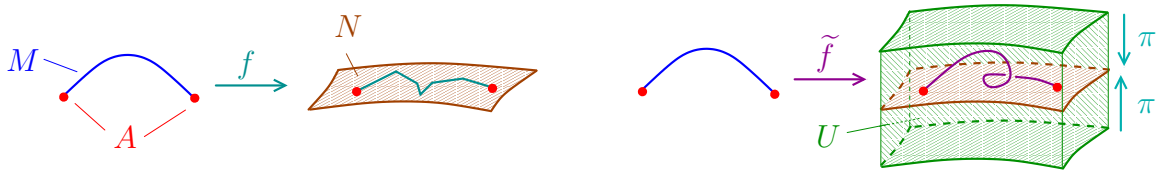
Since U is open it follows easily from the Locally Non-Zero Lemma 14.18 (see also Exercise 4.19) that there exists a continuous map $\mu\colon N \to \mathbb{R}_{>0}$ such that for each $y \in N$ we have $\overline{B}^n_{\mu(y)}(y) \subseteq U$. Next note that by the Smooth Approximation Proposition 27.13 we know that there exists a continuous map $\widetilde{f}\colon M \to \mathbb{R}^n$ with the following two properties:

(a) The maps f and $\widetilde{f}$ agree on A.
(b) For every $x \in M$ we have $\|f(x) - \widetilde{f}(x)\| \leq \mu(f(x))$.

It is straightforward to verify that the map

$$\begin{array}{rccl} F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \pi\big(\underbrace{f(x)\cdot(1-t) + \widetilde{f}(x)\cdot t}_{\in U}\big) \end{array}$$

has all the desired properties (1), (2) and (4). Since $\partial N = \varnothing$ we see that in our current setting Statement (3) is vacuous.



Finally we turn to the proof of (5). Thus suppose that we are given a continuous map $\rho\colon M \to \mathbb{R}_{\geq 0}$ such that $\rho(x) > 0$ for every $x \notin A$. Furthermore suppose that we are given a metric d on N that induces the given topology on N. We sketch how one needs to modify the above argument to obtain that Condition (5) is satisfied, i.e. we show that can arrange that for every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \rho(x)$. First we make the following two observations:

(a) Since $\|-\|$ and d give rise to the same topology we obtain from the Equivalent Metric Lemma 3.11 and the Locally Non-Zero Lemma 14.18 (see also Exercise 4.19) that there

exists a continuous map $\nu\colon M \to \mathbb{R}_{\geq 0}$ such that for all $x \in M$ and all $y \in N$ with $\|y - f(x)\| \leq \nu(x)$ we have $d(y, f(x)) \leq \rho(x)$.

(b) Since π is smooth one can show without problems that there exists a continuous function $\chi\colon N \to \mathbb{R}_{>0}$ such that for every $y \in N$ and every $z \in \overline{B}_{\mu(y)}(y)$ we have the inequality $\|\pi(z) - y\| < \chi(y) \cdot \|z - y\|$.

Now we apply the Smooth Approximation Proposition 27.13 to the map

$$\eta(x) \;:=\; \max\left\{\mu(f(x)), \tfrac{1}{\chi(f(x))} \cdot \nu(f(x))\right\}$$

and we obtain a suitable smooth map $\widetilde{f}\colon M \to \mathbb{R}^n$. As above one can now easily verify that the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \pi\big(\underbrace{f(x)\cdot(1-t) + \widetilde{f}(x)\cdot t}_{\in U}\big) \end{array}$$

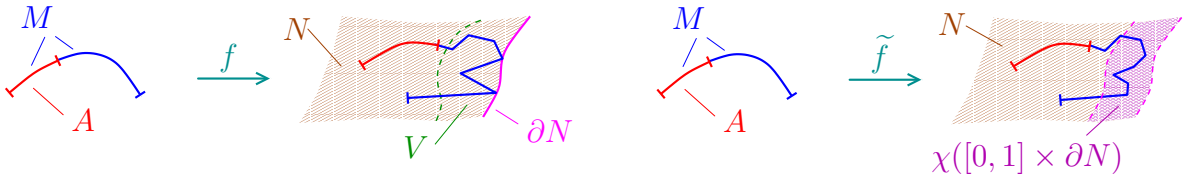
has all the desired properties. ∎

Proof of Theorem 36.1 if $\partial N \neq \varnothing$ and $X = \varnothing$. Let M be a smooth manifold and let N be a smooth manifold with $\partial N \neq \varnothing$. Let A be a (possibly empty) closed subset such that there exists an open neighborhood V of ∂N which satisfies $f(A) \cap V = \varnothing$. Finally let $f\colon M \to N$ be a continuous map that is smooth on A.

Note that by the Smooth Collar Neighborhood Theorem 35.6 there exists a collar $\chi\colon [0,1] \times \partial N \to N$ with $\chi([0,1] \times \partial N) \subseteq V$. As in the proof of the Basics-of-Collars Lemma 35.7 (3a) we consider the homotopy

$$\begin{array}{rcl} G\colon N \times [0,1] & \to & N \\ (Q,s) & \mapsto & \begin{cases} \chi((1-s)\cdot u + s, R), & \text{if } Q = \chi(u,R) \in \chi([0,1]\times\partial N), \\ Q, & \text{if } Q \in N \setminus \chi([0,1)\times\partial N) \end{cases} \end{array}$$

which describes the "obvious" deformation retraction from N to $N \setminus \chi([0,1) \times \partial N)$. Since $\chi([0,1] \times \partial N) \subseteq V$ and since $f(A) \cap V = \varnothing$ we see that G gives rise to a homotopy rel A from $f = G_0 \circ f$ to $\widetilde{f} := G_1 \circ f$. Thus we have shown that f is homotopic rel A to a map $\widetilde{f}$ that takes values in $\widetilde{N} := N \setminus \partial N$. By the Smooth Manifold–Boundary Proposition 25.27 (1) we know that $\widetilde{N}$ is a smooth manifold with empty boundary. Thus – by the above discussion of maps to smooth manifolds with empty boundary – we see that the map $\widetilde{f}\colon G_1 \circ f\colon M \to \widetilde{N}$ is homotopic rel A to a smooth map $M \to \widetilde{N}$.



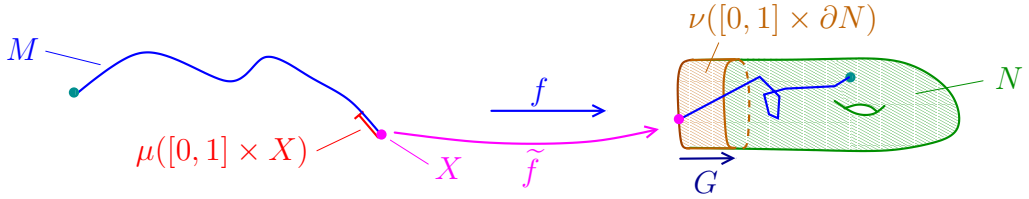
Finally suppose that we are given a continuous map $\rho\colon M \to \mathbb{R}_{>0}$ and suppose that we are given a metric d on N that induces the given topology on N. Dealing with these extra constraints requires a more careful choice of G. We leave it to the reader to fiddle around with our argument to produce a homotopy that also satisfies Condition (5). ∎

Proof of Theorem 36.1 if $\partial N \neq \varnothing$ and $X \neq \varnothing$. We now suppose that there exists a union of components X of ∂M with $f(X) \subseteq \partial N$. We need to show that we can arrange that $F_t(X) \subseteq \partial N$ for all $t \in [0,1]$.

By the Smooth Collar Neighborhood Theorem 35.6 there exist collars $\mu\colon [0,1]\times X \to M$ and $\nu\colon [0,1]\times \partial N \to N$. As above we consider the deformation retraction

$$\begin{array}{rcl} G\colon N\times[0,1] & \to & N \\ (Q,s) & \mapsto & \begin{cases} \nu((1-s)\cdot u + s, R), & \text{if } Q=\nu(u,R)\in\nu([0,1]\times\partial N), \\ Q, & \text{if } Q\in N\setminus\nu([0,1)\times\partial N) \end{cases} \end{array}$$

from N to $N\setminus\nu([0,1)\times\partial N)$. We write $\widetilde{f} := f|_X$. Since $\widetilde{f}\colon X\to\partial N$ is a map between smooth manifolds with empty boundary we know by the above that there exists a homotopy $\widetilde{F}\colon X\times[0,1]\to\partial N$ with $\widetilde{F}_0=\widetilde{f}$ such that $\widetilde{F}_1\colon X\to\partial N$ is smooth.



It follows from an elementary argument, see Exercise 18.8 or the Stationary-Near-The-Ends-Lemma 41.7, that we can modify the homotopy $\widetilde{F}$ such that there exists an $\epsilon>0$ such that $\widetilde{F}_t=\widetilde{F}_1$ for all $t\in[1-\epsilon,1]$. Next we consider the homotopy[427]

$$\begin{array}{rcl} H\colon M\times[0,1] & \to & N \\ (x,t) & \mapsto & \begin{cases} G(f(x),t), & \text{if } x\notin\mu([0,1]\times X), \\ G\big(f\big(\mu\big(\frac{s-\frac{t}{2}}{1-\frac{t}{2}},y\big)\big),t\big) & \text{if } x=\mu(s,y) \text{ with } (s,y)\in[\frac{t}{2},1]\times X, \\ \nu(\widetilde{F}(t-2s,y),2s) & \text{if } x=\mu(s,y) \text{ with } (s,y)\in[0,\frac{t}{2}]\times X. \end{cases} \end{array}$$

We set $h := H_1\colon M\to N$. Note that h is smooth on the closed subset $\mu([0,\frac{\epsilon}{2}]\times X)$. We set $\widetilde{M} := M\setminus\mu([0,\frac{\epsilon}{4})\times X)$ and consider the closed subset $\widetilde{A} := \mu([\frac{\epsilon}{4},\frac{\epsilon}{2}]\times X)\subseteq\widetilde{M}$.

Note that $h(\widetilde{M})\subseteq N\setminus\partial N$ and note that h is smooth on $\widetilde{A}$. Thus we can apply the first case to $h|_{\widetilde{M}}\colon \widetilde{M}\to N$ and we see that $h|_{\widetilde{M}}$ is homotopic rel $\widetilde{A}$ to a smooth map. We extend this smooth map via h to $X\times\mu([0,\frac{\epsilon}{4}])$ to obtain a map that is defined on all of M.

Note that the resulting map is smooth on the open subsets $M\setminus\mu([0,\frac{\epsilon}{4}]\times X)$ and $\mu([0,\frac{\epsilon}{2})\times X)$. Thus it follows from the Smooth Pasting Proposition 25.19 that this map is smooth.

Some adjustments of the above argument can also take into account the presence of a closed subset A and a continuous map $\rho\colon M\to\mathbb{R}_{>0}$ and a metric d on N. We leave this fun task to the reader who is surely tired of reading other people's proofs. ∎

36.2. **Consequences of the Whitney Approximation Theorem 36.1.** In this section we formulate and prove several straightforward but useful consequences of the Whitney Approximation Theorem 36.1.

Proposition 36.2. (Replace-by-Smooth-Fixed-Point-Free Map Proposition) Let M be a smooth manifold. Every continuous map $f\colon M\to M$ without fixed points is homotopic to a *smooth* map $g\colon M\to M$ without fixed points.

[427] This formula for H is clearly unreadable. Nonetheless, one can easily plug in the neuralgic values and see, using the Pasting Proposition 4.7, that this map is indeed continuous.

Proof. Let M be a smooth manifold and let $f\colon M \to M$ be a continuous map without fixed points. By the Manifold Metrization Proposition 23.10 there exists a metric d on M that induces the given topology on M. Given $x \in M$ we set $\rho(x) := d(x, f(x))$. Note that it follows from the Metric-is-Continuous Lemma 10.2 that ρ is continuous. By the Whitney Approximation Theorem 36.1 there exists a homotopy $F\colon M \times [0,1] \to M$ with $f = F_0$ such that $g := F_1$ is a smooth map $g\colon M \to M$ that satisfies the inequality $d(f(x), g(x)) < \rho(x)$ for every $x \in M$. From the triangle inequality we obtain that $d(x, g(x)) > 0$ for every $x \in M$. In other words, g has no fixed points. ■

For the reader's convenience we recall Definition 19.1:

Definition 36.3. Let X be a topological space. We say that a subset $A \subseteq X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

Proposition 36.4. (Replace-by-Smooth Retraction Proposition) Let M be a smooth manifold.

(1) Let A be a union of boundary components. If there exists a retraction $f\colon M \to A$, then there exists also a smooth retraction $g\colon M \to A$.
(2) Let B be a proper submanifold of M with $\partial B = \varnothing$. If there exists a retraction $f\colon M \to B$, then there exists also a smooth retraction $g\colon M \to B$.



Example. Note that in the Replace-by-Smooth Retraction Proposition 36.4 (2) it is essential that we work with *proper* submanifolds. For example in Exercise 36.5 we will see that for the smooth manifold $M = \mathbb{R}$ and the smooth submanifold $A = (-\infty, 0]$ there exists a retraction from M to A but there is no smooth retraction. □

The proof of the Replace-by-Smooth Retraction 36.4 makes use of the following lemma, which is interesting in its own right.

Lemma 36.5. (Smooth-on-Boundary-Submanifold Lemma) Let $f\colon M \to N$ be a continuous map between two smooth manifolds. We suppose that we are in one of the following two settings.

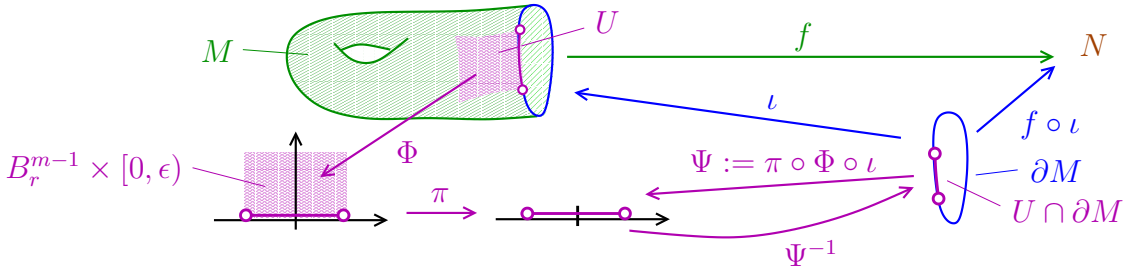
(1) We are given a union A of boundary components. We equip A with the standard smooth structure that we introduced on page 675
(2) We are given a submanifold A of M. We equip A with the standard smooth structure that we introduced on page 671.

We denote by $\iota\colon A \to M$ the inclusion map. If $f \circ \iota\colon A \to N$ is a smooth map between smooth manifolds, then $f\colon M \to N$ is smooth on the subset $A \subseteq M$ in the sense of Definition 25.20.

Remark. The Smooth-on-Boundary-Submanifold Lemma 36.5 is basically the content of Exercise 25.23. □

Proof of Lemma 36.5. We write $m = \dim(M)$. We provide the proof of Statement (1). It should be clear how to modify the proof to obtain a proof of Statement (2). Now let A

be a union of boundary components and let $P \in A$. Note that basically by definition of a boundary point there exists a chart $\Phi\colon U \to B_r^{m-1} \times [0,\epsilon)$ of type (ii) for $P \in M$. We denote by $\pi\colon B_r^{m-1} \times [0,\epsilon) \to B_r^{m-1}$ the natural projection.



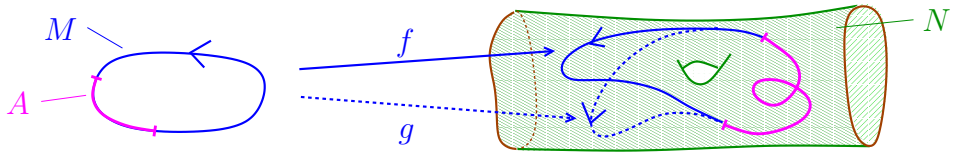
The map $\Psi := \pi \circ \Phi \circ \iota\colon U \cap \partial M \to B_r^{m-1}$ is, by definition of the smooth structure on ∂M, a chart for ∂M. We denote by $\Psi^{-1}\colon B_r^{m-1} \to U \cap \partial M$ the inverse.

Now we set $\widetilde{f} := (f \circ \iota) \circ \Psi^{-1} \circ \pi \circ \Phi$. This is a composition of smooth maps between smooth manifolds, thus it is smooth. Furthermore, by definition of Ψ we have $\widetilde{f}|_{U \cap \partial M} = f$. We have thus shown that the restriction of $f\colon M \to N$ to A is smooth in the sense of Definition 25.20. ■

Proof of the Replace-by-Smooth Retraction 36.4. We provide the proof of Statement (1). Again it should be clear how to modify the proof to obtain a proof of Statement (2). Let M be an n-dimensional smooth manifold and let A be a union of boundary components. Furthermore let $f\colon M \to A$ be a retraction.

Note that $f|_A = \mathrm{id}_A\colon A \to A$ is evidently smooth as a map from the smooth manifold A to itself. It follows from the Smooth-on-Boundary-Submanifold Lemma 36.5 that the map $f\colon M \to A$ is smooth on A in the sense of Definition 25.20. It follows that we can apply the Whitney Approximation Theorem 36.1 and we obtain a homotopy rel A from f to a smooth map $g\colon M \to A$. Since g is homotopic rel A to the retraction f we see that g is also a retraction. Thus g is the desired smooth retraction from M to A. ■

Proposition 36.6. (Replace-by-Smooth Homotopy Proposition) Let M be a smooth manifold with empty boundary and let N be a smooth manifold (with either non-empty or empty boundary). Let $f, g\colon M \to N$ be two smooth maps that agree on a (possibly empty) closed subset A. We suppose that there exists an open neighborhood V of ∂N such that $f(A) = g(A)$ is contained in $N \setminus V$.[428] If f and g are homotopic rel A, then there exists also a smooth homotopy rel A between f and g.

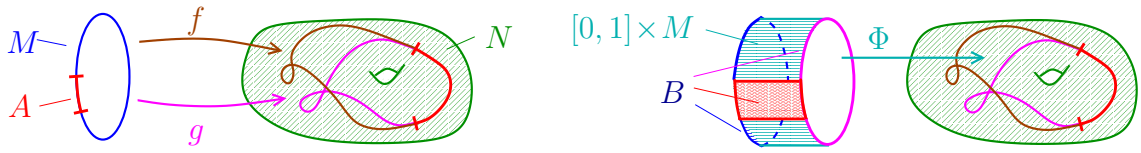


[428] As in Footnote 424 we see that this condition is satisfied if M is compact and if f and g take values in $N \setminus \partial N$. Indeed, in this case we know by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $f(M) \cup g(M)$ is a closed subset of N that is disjoint from the closed subset ∂N. Since topological manifolds are normal, see the Manifold Paracompact-Normal Proposition 23.9 (2), there exists an open neighborhood V of ∂N that is disjoint from $f(M) \cup g(M)$.

Proof. Let $f, g\colon M \to N \setminus \partial N$ be two smooth maps that agree on a closed subset A and let $\Phi\colon M \times [0,1] \to N$ be a homotopy rel A.

We first consider a special case, namely we suppose that there exists an open neighborhood V of ∂N such that actually $f(M)$ and $g(M)$ are contained in $N \setminus V$.

Note that $B := (M \times \{0\}) \cup (A \times [0,1]) \cup (M \times \{1\})$ is a closed subset of $M \times [0,1]$. Also note that $\Phi(B) \subseteq N \setminus V$. Since M has no boundary we see that $M \times [0,1]$ is a smooth manifold. Finally note that in Exercise 36.2, which is a slight generalization of the Smooth-on-Boundary-Submanifold Lemma 36.5 (2), we will see that Φ is smooth on B in the sense of Definition 25.20.



It follows from the above that we can apply the Whitney Approximation Theorem 36.1 to the map $\Phi\colon M \times [0,1] \to N$ and the closed subset B and we see that there exists a homotopy rel B from Φ to a smooth map $\Psi\colon M \times [0,1] \to N$. Since it is a homotopy rel B we see that $\Psi|_B = \Phi|_B$. But this implies that Ψ is now a smooth homotopy rel A from $\Psi_0 = \Phi_0 = f$ to $\Psi_1 = \Phi_1 = g$.

Now we consider the general case. Thus now we only suppose that there exists an open neighborhood V of ∂N such that $f(A) = g(A)$ is contained in $N \setminus V$. By a rather modest generalization of the Push-Away-from-Boundary Corollary 35.8 one sees that f and g are smoothly homotopic rel A to maps f' and g' such that there exists an open neighborhood W of ∂N such that $f'(M)$ and $g'(M)$ are contained in $N \setminus W$. By the above special case we know that f' and g' are smoothly homotopic rel A. Thus in summary we see that f and f', f' and g', g' and g are smoothly homotopic rel A. It is not hard to show that "smoothly homotopic rel A" is an equivalence relation[429], thus we see that f and g are indeed smoothly homotopic rel A. ∎

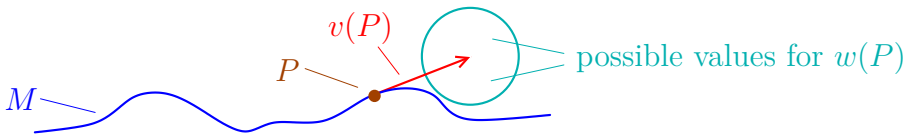
Proposition 36.7. (Replace-by-Smooth Vector Field Proposition) Let M be a smooth manifold. If M admits a nowhere-vanishing vector field, then it also admits a smooth nowhere-vanishing vector field.

Proof. Let M be a smooth manifold and let v be a nowhere vanishing vector field. By the Euclidean Embedding Theorem 33.2 together with the Smooth Embedding Theorem 30.15 we can view M as a smooth submanifold of some $\mathbb{R}^n$. Furthermore by the Tangent Space-Isomorphism Proposition 29.24 and the Vector Field Definitions–Agree Lemma 29.30 we can take the view of (smooth) vector fields that take values in the physical tangent spaces $\widetilde{\mathrm{T}}_P M$. Note that each $\widetilde{\mathrm{T}}_P M$ is a subspace of $\mathbb{R}^n$. Thus the vector field v is in particular a map $M \to \mathbb{R}^n$.

We apply the Whitney Approximation Theorem 36.1 to the map $v\colon M \to \mathbb{R}^n$ and the map $\rho\colon M \to \mathbb{R}_{>0}$ that is given by $\rho(x) := \frac{1}{2} \cdot \|v(x)\|$. We obtain a smooth map $w\colon M \to \mathbb{R}^n$ with the following property:

(∗) For every $x \in M$ we have $\|v(x) - w(x)\| \leq \frac{1}{2} \cdot \|v(x)\|$.

[429] We will provide the details in the Smooth Isotopy-Equivalence Relation Proposition 41.5



Finally we consider the map

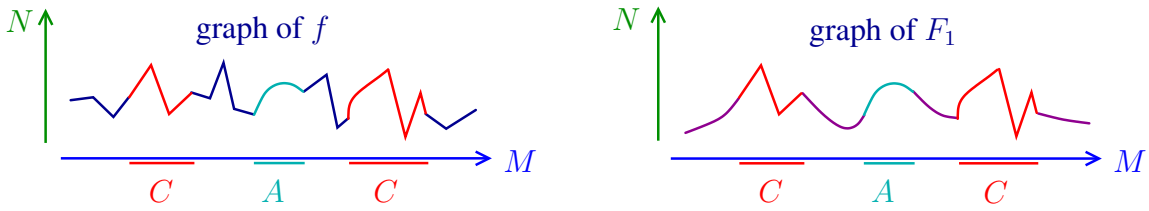
$$\begin{aligned} \widetilde{w}\colon M &\to \mathbb{R}^n \\ P &\mapsto \text{orthogonal projection of } w(P) \text{ onto } \widetilde{\mathrm{T}}_P M. \end{aligned}$$

Using an argument as in the proof of the Normal Vectors Lemma 35.3 one can show that this map is in fact smooth. It is elementary to see, using (∗), that for every $P \in M$ we have $\widetilde{w}(P) \neq 0$. Thus we have found a nowhere-vanishing smooth vector field on M. ■

36.3. **Appendix: Consequences of the Whitney Approximation Theorem 36.1.** We conclude this chapter with two and a half more technical refinements and consequences of the Whitney Approximation Theorem 36.1. These results are interesting in their own right, but the reader might want to skip them initially.

Proposition 36.8. (Whitney Approximation-outside-Closed Subset Proposition) Let M be a smooth manifold (with either non-empty or empty boundary) and let N be a smooth manifold with empty boundary. Furthermore let $f\colon M \to N$ be a continuous map. Finally let A and C be disjoint closed subsets of M such that the restriction of f to A is smooth and let $U \subseteq N$ be an open subset that contains $f(M \setminus C)$. There exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:

(1) The homotopy is a homotopy rel $C \cup A$.
(2) We have $F_0 = f$.
(3) The restriction of F_1 to the open subset $M \setminus C$ is smooth.[430]
(4) For each $t \in [0,1]$ we have $F_t(M \setminus C) \subseteq U$.



Remark. If $C = \varnothing$ and $U = N$, then the statement of the Whitney Approximation-outside-Closed Subset Proposition 36.8 is precisely the statement of the Whitney Approximation Theorem 36.1 (1)–(4). □

Proof. First we consider the case that $U = N$. By the Whitney Approximation Theorem 36.1 (1)–(4) it suffices to consider the case $C \neq \varnothing$. By the Manifold Metrization Proposition 23.10 we can pick a metric c on M such that the topology from the metric c agrees with the given topology on M. The same way we can pick a metric d on N. We consider the open subset $W \coloneqq M \setminus C$ and we view it as a smooth manifold. Since $C \neq \varnothing$

[430]Since C is closed we see that $M \setminus C$ is an open subset of M, thus we can view $M \setminus C$ as a smooth manifold in its own right.

we can consider the function

$$\begin{array}{rcl}\rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) \coloneqq c(w, C) = \inf\{c(w, x) \,|\, x \in C\}.\end{array}$$

Since C is closed and since the topology defined by c agrees with the topology on M we see that $\rho(w) > 0$ for every $w \in W$. By the Distance-to-Subspace Lemma 10.4 we know that $\rho\colon M \to \mathbb{R}_{\geq 0}$ is continuous. Since N is a smooth manifold with empty boundary we can apply the Whitney Approximation Theorem 36.1 to the map $f\colon W \to N$ and we obtain that there exists a homotopy $G\colon W \times [0,1] \to N$ with the following properties:

(a) The homotopy is a homotopy rel A.
(b) We have $G_0 = f|_W$.
(c) The map $G_1\colon W \to N$ is smooth.
(d) For every $(w,t) \in W \times [0,1]$ we have $d(G(w,t), f(w)) \leq \rho(w)$.

It follows easily from (d) and the definition of ρ that the map

$$\begin{array}{rcl}F\colon M \times [0,1] & \to & N \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x \in W, \\ f(x), & \text{if } x \in C \end{cases}\end{array}$$

is continuous. Now it is straightforward to verify that this map has all the desired properties.

Finally we consider the case that $U \neq N$. In this case we consider the map

$$\begin{array}{rcl}\rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) \coloneqq \min\{c(w, C), d(f(w), N \setminus U)\}.\end{array}$$

Since U is open and since $f(W) \subseteq U$ we see that ρ is again non-zero everywhere. The rest of the argument is basically the same and it follows from (d) that for each $t \in [0,1]$ we have $F_t(M \setminus C) \subseteq U$. ∎

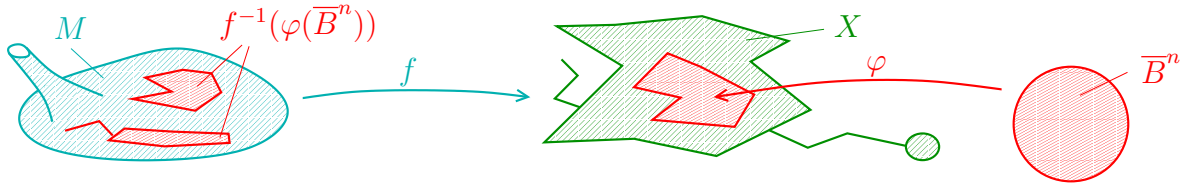
Loosely speaking the following proposition says that a map $M \to X$ from a smooth manifold to a topological space X can be made smooth with respect to open balls B^n contained in X.

Proposition 36.9. (Smooth-to-Cell Proposition) Let M be a smooth manifold and let $f\colon M \to X$ be a continuous map to a Hausdorff space X. Let $\varphi\colon \overline{B}^n \to X$ be an injective continuous map such that $\varphi(B^n)$ is an open subset of X. Given any $r \in (0,1)$ there exists a continuous map $g\colon M \to X$ with the following properties:

(1) The map g is homotopic to f rel the complement of $f^{-1}(\varphi(B^n_r))$ in M.
(2) The map

$$\begin{array}{rcl}f^{-1}(\varphi(B^n_r)) & \to & B^n_r \\ x & \mapsto & \varphi^{-1}(g(x))\end{array}$$

is smooth.



Proof. Since $\overline{B}^n$ is compact, since X is Hausdorff and since $\varphi\colon \overline{B}^n \to X$ is injective we know by the Compact-Hausdorff Proposition 4.23 (2) that φ is a closed embedding. Thus we might as well identify $\overline{B}^n$ with its image under the map $\varphi\colon \overline{B}^n \to X$. Now let $r \in (0,1)$. By the Open-Closed Inclusion Lemma 4.13 we know that B^n_r is also an open subset of X. Furthermore, note that it follows from the fact that X is Hausdorff, together with the Compact-Closed Lemma 3.39 (1), that $\overline{B}^n_r$ and $\overline{B}^n$ are closed subsets of X. We consider the map

$$\begin{array}{rcl} \alpha\colon M & \to & \overline{B}^n/S^{n-1} \\ x & \mapsto & \begin{cases} f(x), & \text{if } x \in f^{-1}(B^n), \\ [S^{n-1}], & \text{if } x \in M \setminus f^{-1}(B^n). \end{cases} \end{array}$$

Note that α is continuous.[431] Next note that by the Ball Quotient-Sphere Lemma 8.10 we can view $\overline{B}^n/S^{n-1}$ as a smooth manifold such that B^n is a submanifold. Therefore we can apply the Whitney Approximation-outside-Closed Subset Proposition 36.8 to the map $\alpha\colon M \to \overline{B}^n/S^{n-1}$ with $A = \varnothing$, with the closed subset $C \coloneqq M \setminus f^{-1}(B^n_r)$[432] and with the open subset $U \coloneqq B^n_r \subseteq \overline{B}^n \setminus S^{n-1} =: N$. We obtain a homotopy $F\colon M \times [0,1] \to \overline{B}^n/S^{n-1}$ with the following properties:

(1) The homotopy is a homotopy rel $C = M \setminus f^{-1}(B^n_r)$.
(2) We have $F_0 = \alpha$.
(3) The restriction of F_1 to $M \setminus C = f^{-1}(B^n_r)$ is smooth.
(4) For each $t \in [0,1]$ we have $F_t(f^{-1}(B^n_r)) \subseteq B^n_r$.

Finally we consider the map

$$\begin{array}{rcl} G\colon M \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x,t), & \text{if } x \in f^{-1}(B^n_r) \\ f(x), & \text{if } x \in M \setminus f^{-1}(B^n_r). \end{cases} \end{array}$$

Note that the restrictions of the map G to the two closed subsets $(M \setminus f^{-1}(B^n_r)) \times [0,1]$ and $f^{-1}(\overline{B}^n_r) \times [0,1]$ are continuous.[433] Thus we obtain from the Homotopy Combination Lemma 18.8 (1) that G is continuous. It is now straightforward to verify that $g \coloneqq G_1$ has all the desired properties. ∎

The following proposition is a slight variation on the Smooth-to-Cell Proposition 36.9.

Proposition 36.10. (Smooth-on-Sphere Proposition) Let M be a smooth manifold and let $f\colon M \to X$ be a continuous map to a Hausdorff space X. Let $n \in \mathbb{N}$ and let $* \in S^n$ be a point. Finally let $\varphi\colon S^n \to X$ be an injective continuous map such that $\varphi(S^n \setminus \{*\})$

[431] Indeed, the restrictions of α to the closed subset $M \setminus f^{-1}(B^n)$ is a constant map and the restriction of α to the closed subset $f^{-1}(\overline{B}^n)$ is given by the composition of the map f with the projection $\overline{B}^n \to \overline{B}^n/S^{n-1}$. Therefore we obtain from the Pasting Proposition 4.7 (2) that α is indeed continuous.

[432] Note the discreet subscript "r".

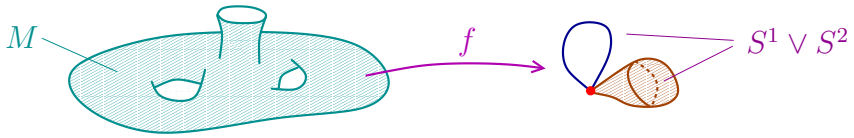
[433] For nervous people, like the author, let us verify in slightly greater detail that the restriction of G to $f^{-1}(\overline{B}^n_r) \times [0,1]$ is indeed continuous. First note that by (4) the restriction of the map F to $f^{-1}(B^n_r) \times [0,1]$ takes values in B^n_r. It follows from the Closure-Interior Mapping Lemma 4.6 together with the Product-of-Subsets Lemma 7.8 (2c) that the restriction of the map F to $f^{-1}(\overline{B}^n_r) \times [0,1]$ takes values in the closure of B^n_r in $\overline{B}^n/S^{n-1}$. Since $r \in (0,1)$ we see that the closure equals $\overline{B}^n_r \subseteq B^n$. Now we see that the restriction of the map G to $f^{-1}(\overline{B}^n_r) \times [0,1]$ equals the composition of the map $F\colon f^{-1}(\overline{B}^n_r) \times [0,1] \to \overline{B}^n_r$ with the inclusion $\overline{B}^n_r \to X$.

is an open subset of X. There exists a continuous map $g\colon M \to X$ with the following properties:

(1) The map g is homotopic to f rel the complement of $f^{-1}(\varphi(S^n \setminus \{*\}))$ in M.
(2) The map
$$\begin{array}{ccc} f^{-1}(\varphi(S^n \setminus \{*\})) & \to & S^n \setminus \{*\} \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$
is smooth.

Proof. We leave it to the reader to modify the proof of the Smooth-to-Cell Proposition 36.9 to obtain the desired result, see Exercise 36.1. As the reader will notice, in this case there is no need to work with some $r \in (0,1)$. In particular the conclusion we get is slightly more satisfactory. ■

Example. Let $\bigvee_{i=1}^{k} S^{n_i}$ be a wedge of spheres, i.e. we consider the topological space which is obtain from the disjoint union $\bigsqcup_{i=1}^{k} S^{n_i}$ by identifying the k North Poles to a single point $*$. Let M be a smooth manifold and let $f\colon M \to \bigvee_{i=1}^{k} S^{n_i}$ be a continuous map.



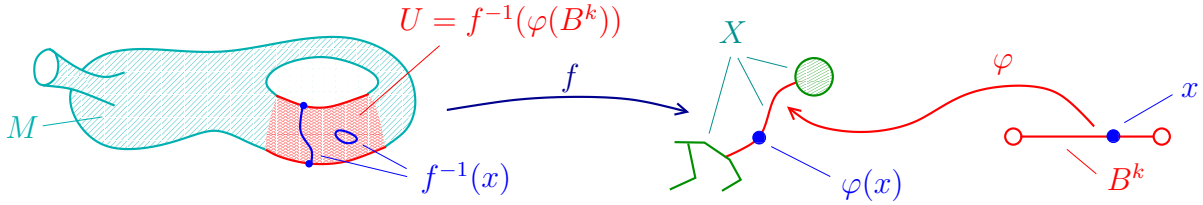
By iteratively applying the Smooth-on-Sphere Proposition 36.10 altogether k times we see that f is homotopic rel $f^{-1}(*)$ to a map that is smooth on the open subsets $f^{-1}(S^{n_i} \setminus \{*\})$, $i = 1, \ldots, k$. □

We conclude this appendix with a lemma which will be useful much much later:

Lemma 36.11. (Preimage-of-Point-in-Ball Lemma) Let M be an orientable n-dimensional smooth manifold, let $f\colon M \to X$ be a continuous map to some Hausdorff space X, let $\varphi\colon B^k \to X$ be an injective continuous map and let $x \in B^k$ be a point such that the following conditions are satisfied:

(1) The image $\varphi(B^k)$ is an open subset of X.
(2) The map $\varphi^{-1} \circ f\colon f^{-1}(\varphi(B^k)) \to B^k$ is smooth.
(3) The point $x \in B^k$ is a regular value of the map $\varphi^{-1} \circ f\colon f^{-1}(\varphi(B^k)) \to B^k$ and also of the map $\varphi^{-1} \circ f|_{\partial f^{-1}(\varphi(B^k)) \cap \partial M}$.

Then $f^{-1}(\varphi(x))$ is a proper $(n-k)$-dimensional submanifold of M.



Proof. Note that $U := f^{-1}(\varphi(B^k))$ is an open subset of M, in particular, as discussed in the Open Subset-Submanifold Lemma 25.8, it is a smooth manifold in its own right. It follows from this observation and (3) that we can apply the Regular Value Theorem 32.5 to the map $\varphi^{-1} \circ f\colon f^{-1}(\varphi(B^k)) \to \varphi(B^k)$ and the regular value x of $\varphi^{-1} \circ f\colon f^{-1}(\varphi(B^k)) \to B^k$

and of $\varphi^{-1} \circ f|_{\partial f^{-1}(\varphi(B^k))\cap\partial M}$. We obtain that $f^{-1}(\varphi(x))$ is a proper orientable $(n-k)$-dimensional submanifold of U. In particular $f^{-1}(\varphi(x))$ admits submanifold charts of type (α) and (β), as introduced in Definition 23.18, with respect to the smooth manifold U. Since U is an open subset of M we see that these submanifold charts are also submanifold charts of the same type for the set $f^{-1}(\varphi(x))$, viewed as a subset of M.

Since X is by hypothesis Hausdorff we know, by the Points-in-Hausdorff Lemma 3.32 (1) that $\{\varphi(x)\}$ is a closed subset of X. Finally, since f is continuous we see that $f^{-1}(\varphi(x))$ is a closed subset of M. In other words, we have shown that $f^{-1}(\varphi(x))$ satisfies all the conditions, as set out in Definition 23.18, of being a proper submanifold of M. ■

36.4. Exercises for Chapter 36.

Exercise 36.1. Provide a proof for the Smooth-on-Sphere Proposition 36.10.

Exercise 36.2. Let M and N be two smooth manifolds. We assume that $\partial M = \varnothing$. Furthermore let $f, g\colon M \to N$ be two smooth maps that agree on a subset A and let $\Phi\colon M \times [0,1] \to N$ be a homotopy rel A. Show that Φ is smooth, in the sense of Definition 25.20, on the subset $(M \times \{0\}) \cup (A \times [0,1]) \cup (M \times \{1\})$ of $M \times [0,1]$.

Exercise 36.3. In the figure below we show an explicit smooth manifold M together with a boundary component A and a submanifold B.
(a) Is A a retract of M?
(b) Is B a retract of M?



Exercise 36.4. Let M be a smooth manifold. The Smooth Path-Connectivity Proposition 25.32 says that every continuous map $\gamma\colon [a,b] \to M$ is homotopic rel the endpoints $\{a,b\}$ to a smooth map. Give a new proof of this statement, under the hypothesis that $f(a), f(b) \in M \setminus \partial M$, using the Whitney Approximation Theorem 36.1 and use the Smooth Collar Neighborhood Theorem 35.6 or the Push-Away-from-Boundary Corollary 35.8.

Exercise 36.5. We consider the smooth manifold $\mathbb{R}$ and the smooth submanifold $(-\infty, 0]$.
(a) Show that there exists a retraction from $\mathbb{R}$ to $(-\infty, 0]$.
(b) Show that there is no smooth retraction from $\mathbb{R}$ to $(-\infty, 0]$.

CHAPTER 37

Subsets of measure zero

Evidently a smooth manifold does not come with a natural notion of a "measure" or "volume". But perhaps somewhat surprisingly we will see that it makes perfect sense to talk of subsets of "measure zero" and of "full measure" of a smooth manifold. We will see that the image of a smooth map $f\colon M \to N$ with $\dim(M) < \dim(N)$ has "measure zero", and that this implies that $N \setminus f(M)$ is dense. This fact is surprisingly useful. In particular we will use it to show in the Weak Whitney Embedding Theorem 37.9 that every m-dimensional smooth manifold admits a smooth embedding $M \to \mathbb{R}^{2m+1}$.

The notion of subsets of measure zero also plays an essential role in the formulation of Sard's Theorem 38.1 that we will state and prove in the next chapter.

This chapter is organized as follows:

- In Section 37.1 will introduce the notion of subsets of measure zero of a smooth manifold and we will state several properties of subsets of measure zero.
- In Section 37.2 we will provide proofs for the statements in Section 37.1.
- In Section 37.3 we will state and prove the Weak Whitney Embedding Theorem 37.9 which says that every m-dimensional smooth manifold M admits a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$.
- In Section 37.4 we will discuss subsets of measure zero in products.
- In Section 37.5 we will provide exercises for Chapter 37.

In the next Chapter 38 we will prove Sard's Theorem 38.1 which says that for a smooth map $f\colon M \to N$ between smooth manifolds the set of critical values has measure zero in N.

37.1. **Sets of measure zero.** In our first definition of this chapter we introduce subsets of measure zero in $\mathbb{R}^n$. Fortunately this definition is actually quite harmless:

Definition 37.1.

(1) An open cube of side length $r \in \mathbb{R}_{\geq 0}$ is any set of the form $(c_1, c_1+r)\times\cdots\times(c_m, c_m+r)$. We define the volume of such an open cube as

$$\operatorname{vol}((c_1, c_1 + r) \times \cdots \times (c_m, c_m + r)) \;:=\; r^m.$$

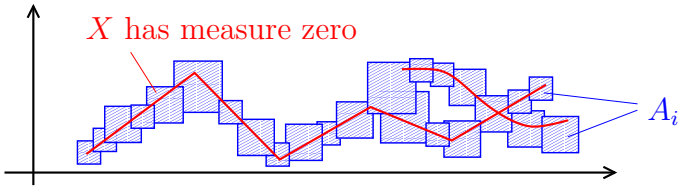
(2) Given a countable family $\{r_i\}_{i\in I}$ of non-negative real numbers we set

$$\sum_{i\in I} r_i \;:=\; \sup\left\{ \sum_{j\in J} r_j \,\middle|\, J \subseteq I \text{ is a finite subset}\right\} \;\in\; \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

(3) Let $X \subseteq \mathbb{R}^m$ be a subset. We say that X has measure zero if given any $\epsilon > 0$ there exists a countable family $\{A_i\}_{i\in I}$ of open cubes such that $X \subseteq \bigcup_{i\in I} A_i$ and such that $\sum_{i\in I} \operatorname{vol}(A_i) < \epsilon$. This definition is illustrated in the figure below.

(4) Let M be an m-dimensional smooth manifold.
   (a) We say that a subset $X \subseteq M$ is a subset of measure zero if for every chart $\Phi\colon U \to V$ from the smooth atlas the set $\Phi(X \cap U)$ is a subset of measure zero of $\mathbb{R}^m$.
   (b) We say that a subset $X \subseteq M$ is a subset of full measure if $M \setminus X$ is a subset of measure zero.

(5) Let P be a property of points in M. We say that almost every point in M has property P if the set of points which satisfy P has full measure.



Example.

(1) Let $n \geq 1$ and let $\{P = (p_1, \ldots, p_n)\}$ be a one-point subset of $\mathbb{R}^n$. Since $\{P\}$ is contained in any open cube $(p_1 - \epsilon, p_1 + \epsilon) \times (p_n - \epsilon, p_n + \epsilon)$ of volume $(2\epsilon)^n$ we see that $\{P\}$ is a subset of measure zero.

(2) Let M be a smooth manifold of dimension ≥ 1. Using (a) it is straightforward to show that any finite subset of M has measure zero. □

In the following we formulate three propositions which contain almost everything anybody ever wanted to know about subsets of measure zero. We will provide the proofs in the next section.

The definition of "measure zero" is not very practical since one needs to verify a certain condition for every chart. Our first proposition greatly simplifies life, namely it says, that it suffices to consider enough charts to cover all of M.

Proposition 37.2. (Measure Zero-via-Atlas Proposition) Let M be an m-dimensional smooth manifold and let $X \subseteq M$ be a subset. If there exists a family of smooth charts $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ with the following two properties:

(1) for each $i \in I$ the set $\Phi_i(X \cap U_i)$ is a subset of measure zero of $\mathbb{R}^m$,

(2) we have $X \subseteq \bigcup_{i\in I} U_i$,

then X has measure zero.

The second proposition summarizes a few basic properties of subsets of measure zero.

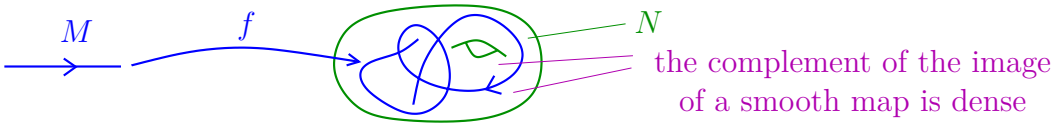
Proposition 37.3. (Measure-Zero Properties Proposition) Let M be an m-dimensional smooth manifold.

(1) If X is a subset of measure zero of M, then so is every subset of X.

(2) (a) The union of countably many subsets of measure zero is a subset of measure zero.
   (b) The intersection of countably many subsets of full measure is a subset of full measure.

(3) (a) Let $f\colon M \to N$ be a smooth map between smooth manifolds of the same dimension. If X is a subset of measure zero in M, then $f(X)$ has measure zero in N.
(b) If $f\colon M \to N$ is a diffeomorphism, then $X \subseteq M$ has measure zero (respectively full measure) if and only if $f(X)$ does.[434]
(4) If $f\colon M \to N$ is a smooth map between smooth manifolds and if $\dim(M) < \dim(N)$, then $f(M)$ is a subset of measure zero in N. In other words, $N \setminus f(M)$ is a subset of full measure.

Example. Let N be a smooth manifold and let M be a submanifold of N of codimension at least one.



By the Submanifold Smooth Structure Lemma 25.23 the inclusion map $\iota\colon M \to N$ is smooth. Therefore it follows from the Measure-Zero Properties Proposition 37.3 (4) that $M \subseteq N$ has measure zero. □

We conclude this section with one last proposition on subsets of measure zero.

Proposition 37.4. (Full Measure-Density Proposition) Let M be an m-dimensional smooth manifold.
(1) Given any subset X of measure zero of M the complement $M \setminus X$ is dense in M.[435]
(2) Every subset of M of full measure is dense in M.

Remark. The above propositions show that on smooth manifolds there is a sensible notion of subsets of measure zero. Denis Sullivan [**Sul1979**, p. 549] showed that for dimensions $n \neq 4$ any n-dimensional topological manifold has an essentially unique "Lipschitz structure", which makes it possible to again introduce a sensible notion of subsets of measure zero. We will not make use of this fact. □

37.2. **Proof of Propositions 37.2, 37.3 and 37.4.** We start out with the following self-explanatory definition.

Definition 37.5. Let $r \in \mathbb{R}_{\geq 0}$. A closed cube of side length r is any subset of $\mathbb{R}^m$ of the form $[c_1, c_1 + r] \times \cdots \times [c_m, c_m + r]$. We define the volume of such a closed cube as

$$\operatorname{vol}([c_1, c_1 + r] \times \cdots \times [c_m, c_m + r]) \;:=\; r^m.$$

The following lemma contains some very basic statements about subsets of measure zero.

Lemma 37.6. (Measure Zero-Basics Lemma)
(1) If C is a subset of measure zero of $\mathbb{R}^m$, then so is every subset of C.
(2) If $\{C_i\}_{i\in I}$ is countable family of subsets of $\mathbb{R}^m$ of measure zero, then $\bigcup_{i\in I} C_i$ also has measure zero.

[434] In Exercise 37.2 we will see that the conclusion does not hold if we replace "diffeomorphism" by "homeomorphism".
[435] By the Density-Interior-Closure Lemma 3.26 this is equivalent to saying that the interior of X is the empty set.

(3) Let C be a subset of measure zero of $\mathbb{R}^m$ and let $\sigma \in S_m$ be a permutation. Then
$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \,|\, (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in C\}$$
also has measure zero.

(4) Let C be a subset of $\mathbb{R}^m$. If given any $\epsilon > 0$ there exists a countable family $\{A_i\}_{i\in I}$ of closed cubes such that $X \subseteq \bigcup\limits_{i\in I} A_i$ and such that $\sum\limits_{i\in I} \text{vol}(A_i) < \epsilon$, then C has measure zero.

Proof.

(1) This statement is trivial.

(2) Let $\{X_i\}_{i\in I}$ be a countable family of subsets of measure zero of $\mathbb{R}^m$. Using the fact that $\sum\limits_{k\in\mathbb{N}} \frac{1}{2^k} = 1$ it is straightforward to show that $\bigcup\limits_{i\in I} X_i$ is also a subset of measure zero.

(3) This statement is clear since permuting coordinates turns cubes into cubes of the same volume.

(4) This statement follows from the straightforward observation that any closed cube is contained in an open cube of at most twice the volume. ■

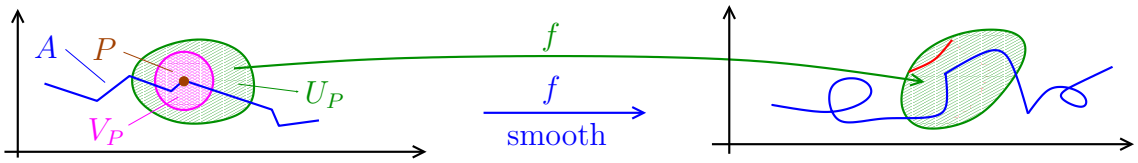
Lemma 37.7. (Smooth Image-of-Measure Zero Lemma) Let $A \subseteq \mathbb{R}^n$ be a measure zero subset and let $f\colon A \to \mathbb{R}^n$ be a smooth map. Then $f(A)$ also has measure zero.

Remark. The conclusion of the Smooth Image-of-Measure Zero Lemma 37.7 does not hold if we replace "smooth" by "continuous":

(1) It is clear that $I = \{(x, 0) \,|\, x \in [-1, 1]\} \subseteq \mathbb{R}^2$ has measure zero, but in the Space-Filling Curve Theorem 10.21 we saw that there exists a space-filling path, i.e. a continuous map $I \to [0, 1]^2$ that is surjective. But, as we will see in the Cubes-are-Not-Measure Zero Proposition 37.8, $[0, 1]^2 \subseteq \mathbb{R}^2$ does not have measure zero.

(2) In Exercise 37.2 we will see that there exists a homeomorphism $[0, 1] \to [0, 2]$ and a measure zero subset $C \subseteq [0, 1]$ such that $f(C)$ does not have measure zero. □

Proof. Let $A \subseteq \mathbb{R}^n$ be a measure zero subset and let $f\colon A \to \mathbb{R}^n$ be a smooth map. We need to show that $f(A)$ also has measure zero. First let $P \in A$. By definition of a smooth map there exists an open neighborhood U_P and a smooth function $f\colon U_P \to \mathbb{R}^n$ which agrees with the original f on $A \cap U_P$. Next we pick $r > 0$ such that $V_P \coloneqq \overline{B}^n_r(P) \subseteq U_P$. By the Second Countability Lemma 13.2 we know that A is second-countable. Thus it follows from the Countable Cover Lemma 13.4 that there exist countably many $P \in A$ such that A is covered by the V_P. Therefore, by the Measure Zero-Basics Lemma 37.6 it suffices to show that each $f(V_P \cap A)$ has measure zero.



So let $P \in A$, let $V_P = \overline{B}^n_r(P)$ and let $f\colon U_P \to \mathbb{R}^n$ be as above. Given a matrix $A \in \mathrm{M}(n \times n, \mathbb{R})$ we define its norm as
$$\|A\| \;\coloneqq\; \max\{\|Av\| \,|\, v \in S^{n-1}\} \;\in\; \mathbb{R}_{\geq 0}.$$

In Exercise 4.53 (h) we saw that the norm is a continuous map on $\mathrm{M}(n\times n,\mathbb{R})$. Thus, since V_P is compact, it follows that by the Compact Image Lemma 4.15 we can set

$$M \;:=\; \max\{\|\mathrm{D}f_P\| \,|\, P\in V_P\}.$$

Claim. For every $x,x'\in\overline{B}_r^n(P)$ we have

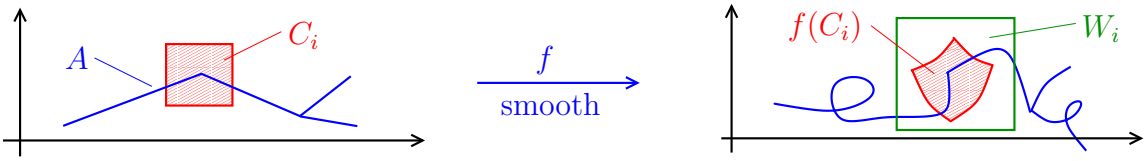
$$\|f(x)-f(x')\| \;\leq\; M\cdot\|x-x'\|.$$

Proof. Let $x,x'\in\overline{B}_r^n(P)$. We set $v:=x'-x$. We perform the following calculation:

$$\begin{array}{lllll}
 & \text{by the Fundamental Theorem of Calculus 73.27} & \text{by the Chain Rule 24.17} & & \text{by the Integral Inequality Lemma 2.45}\\
\|f(x)-f(x')\| & \downarrow & \downarrow & & \downarrow\\
 & = \left\|\int_{t=0}^{t=1}\frac{d}{dt}f(x+tv)\,dt\right\| & = \left\|\int_{t=0}^{t=1}\mathrm{D}f_{x+tv}\cdot v\,dt\right\| & & \leq \int_{t=0}^{t=1}\|\mathrm{D}f_{x+tv}\cdot v\|\,dt\\
 & \leq \int_{t=0}^{t=1}\|\mathrm{D}f_{x+tv}\|\cdot\|v\|\,dt & \leq \int_{t=0}^{t=1} M\cdot\|v\|\,dt & = M\cdot\|v\| & = M\cdot\|x-x'\|.\\
 & \uparrow & \uparrow & & \\
 & \text{Exercise 4.53 (a)} & \text{definition of } M & &
\end{array}$$

⊞

Let $\epsilon>0$. We pick $\delta>0$ with $\delta\cdot(2M)^n\leq\epsilon$. By the Measure Zero-Basics Lemma 37.6 (1) we know that $A\cap V_P$ is a subset of measure zero. Therefore there exists a countable set of closed cubes $\{C_i\}_{i\in\mathbb{N}}$ with $A\cap V_P\subseteq\bigcup_{i\in\mathbb{N}}C_i$ and with $\sum_{i\in\mathbb{N}}\mathrm{vol}(C_i)<\delta$. After possibly splitting each closed cube C_i into smaller closed cubes we can assume that each C_i is contained in U_P.[436] It follows from the claim, together with an elementary geometric argument regarding diameters and cubes, that the image $f(C_i)$ of the cube C_i is contained in a cube W_i whose side length is $2M$-times larger than the side length of C_i. In other words, $f(C_i)$ is contained in a closed cube with $\mathrm{vol}(W_i)=(2M)^n\cdot\mathrm{vol}(C_i)$.



We now see that

$$f(A\cap V_P)\;\subseteq\; f\Big(\bigcup_{i\in\mathbb{N}}C_i\Big)\;=\;\bigcup_{i\in\mathbb{N}}f(C_i)\;\subseteq\;\bigcup_{i\in\mathbb{N}}W_i.$$

The following calculation, together with the Measure Zero-Basics Lemma 37.6 (4), completes the proof:

$$\sum_{i=1}^{\infty}\mathrm{vol}(W_i)\;=\;\sum_{i=1}^{\infty}(2M)^n\cdot\mathrm{vol}(C_i)\;=\;(2M)^n\cdot\sum_{i=1}^{\infty}\mathrm{vol}(C_i)\;<\;(2M)^n\cdot\delta\;\leq\;\epsilon.$$

■

Now we can provide the proofs of Propositions 37.2 and 37.3:

[436] This can be seen as follows. Since U_P is open and since $V_P\subseteq U_P$ is compact we see that there exists an $\epsilon>0$ such that every closed cube that intersects V_P and which has side length $<\epsilon$ is actually contained in U_P. Now let C be a cube of side length d. We pick $m\in\mathbb{N}$ such that $\frac{d}{m}<\epsilon$ and we break C into m^n closed cubes of side length $\frac{d}{m}$. Now we only take those cubes which have non-empty intersection with V_P.

Proof of the Measure Zero-via-Atlas Proposition 37.2. Let M be an m-dimensional smooth manifold and let $X \subseteq M$ be a subset. Suppose that there exists a family of smooth charts $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ such that the following two conditions hold:

(1) for each $i \in I$ the set $\Phi_i(X \cap U_i)$ is a subset of measure zero of $\mathbb{R}^m$,
(2) we have $X \subseteq \bigcup_{i\in I} U_i$.

Note that it follows from the Second Countability Lemma 13.2 together with the Countable Cover Lemma 13.4 that X is already covered by countably many of the U_i. Thus we might as well assume that I is countable.

We need to show that X has measure zero. Thus let $\Psi \colon A \to B$ be a smooth chart. We need to show that $\Psi(A \cap X)$ has measure zero in $\mathbb{R}^m$. Note that

$$\Psi(A\cap X) = \Psi\Big(\bigcup_{i\in I}(U_i\cap X)\Big) = \Psi\Big(\bigcup_{i\in I}\Phi_i^{-1}(\Phi_i(U_i\cap X))\Big) = \bigcup_{i\in I} \underbrace{(\underbrace{\Psi\circ\Phi_i^{-1}}_{\text{smooth}})(\underbrace{\Phi_i(U_i\cap X)}_{\text{measure zero by (1)}})}_{\substack{\text{measure zero by the Smooth}\\ \text{Image-of-Measure Zero Lemma 37.7}}}$$

Since I is countable we now obtain from the Measure Zero-Basics Lemma 37.6 that $\Psi(A\cap X)$ does indeed have measure zero. ∎

Proof of the Measure-Zero Properties Proposition 37.3. Let M be an m-dimensional smooth manifold.

(1) Let X be a subset of measure zero of M. It follows almost immediately from the Measure Zero-Basics Lemma 37.6 (1) that any subset of X also has measure zero.
(2) (a) It follows easily from the Measure Zero-Basics Lemma 37.6 (2) and the definitions that the union of countably many subsets of measure zero is again a subset of measure zero.
    (b) It follows from (a) and de Morgan's Laws 1.7 that the intersection of countably many subsets of full measure is again a subset of full measure.
(3) (a) Let $f \colon M \to N$ be a smooth map between smooth manifolds of the same dimension and let X be a subset of measure zero in M. We need to show that $f(X)$ has measure zero in N. Let $\Psi \colon A \to B$ be a smooth chart for N. We need to show that $\Psi(A \cap f(X))$ has measure zero in $\mathbb{R}^n$. By the Countable Cover Lemma 13.4 we know that M admits a countable smooth atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$. Now we see that

$$\begin{aligned}\Psi(A\cap f(X)) &= \Psi\Big(\bigcup_{i\in I} f\big(f^{-1}(A)\cap X\cap U_i\big)\Big) = \Psi\Big(\bigcup_{i\in I} f\big(\Phi_i^{-1}(\Phi_i(f^{-1}(A)\cap X\cap U_i))\big)\Big)\\ &= \bigcup_{i\in I} \underbrace{(\underbrace{\Psi\circ f\circ\Phi_i^{-1}}_{\text{smooth}})(\underbrace{\Phi_i(f^{-1}(A)\cap X\cap U_i)}_{\text{measure zero by (1)}})}_{\substack{\text{measure zero by the Smooth}\\ \text{Image-of-Measure Zero Lemma 37.7}}}\end{aligned}$$

It follows from the Measure Zero-Basics Lemma 37.6 (2) that $\Psi(A \cap f(X))$ does indeed have measure zero.
    (b) This statement is an immediate consequence of (a).

(4) Let M and N be two smooth manifolds with $m := \dim(M) < \dim(N) =: n$. Furthermore let $f\colon M \to N$ be a smooth map. We need to show that $f(M)$ is a subset of measure zero in N. We consider the n-dimensional smooth manifold $W := M \times \mathbb{R}^{n-m}$.

It follows from $n-m > 0$ and the discussion on page 916 that $\{0\} \subseteq \mathbb{R}^{n-m}$ is a subset of measure zero. Using product charts and using the Measure Zero-via-Atlas Proposition 37.2 it follows easily that $M \times \{0\}$ is a subset of measure zero of $W = M \times \mathbb{R}^{n-m}$. Now consider the smooth map $g\colon W = M \times \mathbb{R}^{n-m} \to N$ that is given by $g(x,v) = f(x)$. Evidently $g(M \times \{0\}) = f(M)$. It follows from the above and (3a) that $g(M \times \{0\})$ has measure zero in N. ■

For the proof of the Full Measure-Density Proposition 37.4 we need the following subtle proposition.

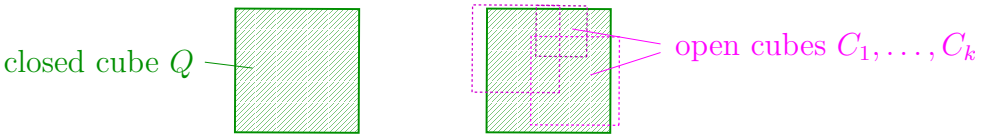
Proposition 37.8. (Cubes-are-Not-Measure Zero Proposition) If Q is a closed cube with side length > 0, then Q does not have measure zero.

Proof. The statement of the proposition follows of course from standard theory of Lebesgue measures, see e.g. Chapter 73. In the following we give a self-contained proof which in [**GPol1974**, p. 203] gets attributed to John von Neumann. Let Q be a closed cube with side length $q > 0$ in $\mathbb{R}^m$. Suppose that we have a countable covering $Q \subseteq \bigcup_{i\in I} C_i$ by open cubes. Given $i \in I$ we denote by c_i the side length of C_i. It suffices to prove the following claim:

Claim. We have the inequality $\sum_{i\in I} c_i^m \geq q^m$.

Proof. Since Q is compact we see that Q is covered by finitely many C_i. Thus we can assume that $I = \{1,\dots,k\}$ and that $Q \subseteq C_1 \cup \dots \cup C_k$. Given $n \in \mathbb{N}$ we set

$$X_n \;:=\; (\tfrac{1}{n}\cdot\mathbb{Z})^m \;=\; \{(r_1,\dots,r_m)\,|\,r_1,\dots,r_m \in \tfrac{1}{n}\mathbb{Z}\}.$$



We make the following crucial observation: Given any cube $D \subseteq \mathbb{R}^m$, closed or open, of side length d we have[437]

$$(*) \qquad \lceil d\cdot n\rceil^m \;\geq\; \#(D\cap X_n) \;\geq\; \lfloor d\cdot n\rfloor^m.$$

Now we see that

$$\sum_{i=1}^k \lceil c_i\cdot n\rceil^m \;\underset{\text{by }(*)}{\geq}\; \sum_{i=1}^k \#(X_n\cap C_i) \;\geq\; \#\Big(X_n\cap\bigcup_{i=1}^k C_i\Big) \;\underset{\text{since } Q\subseteq C_1\cup\dots\cup C_k}{\geq}\; \#(X_n\cap Q) \;\underset{\text{by }(*)}{\geq}\; \lfloor q\cdot n\rfloor^m.$$

[437]This observation follows from the fact that given any interval $I \subseteq \mathbb{R}$ of length d we have

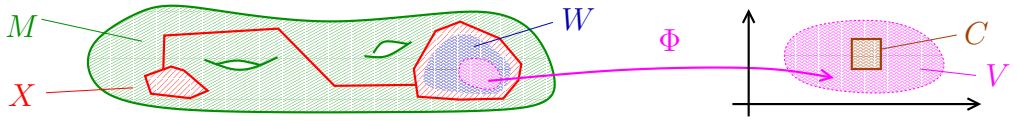
$$\lceil d\cdot n\rceil \;\geq\; \#\big(I\cap\{x\in\tfrac{1}{n}\cdot\mathbb{Z}\}\big) \;\geq\; \lfloor d\cdot n\rfloor.$$

Using the inequalities $x+1 \geq \lceil x \rceil$ and $\lfloor x \rfloor \geq x-1$ we see that

$$\sum_{i=1}^{k} (c_i \cdot n + 1)^m \;\geq\; (q \cdot n - 1)^m.$$

If we now divide both sides by n^m and if we take the limit $n \to \infty$, then we obtain the promised inequality $\sum_{i=1}^{k} c_i^m \geq q^m$. ■

Proof of the Full Measure-Density Proposition 37.4. Let M be an m-dimensional smooth manifold and let X be a subset. We will show that if $M \setminus X$ is not dense, then X cannot have measure zero. If $M \setminus X$ is not dense, then there exists an open non-empty subset W of M which has empty intersection with $M \setminus X$. In other words, $W \subseteq X$. Now let $\Phi\colon U \to V$ be a smooth chart with $U \subseteq W$. Since $V = \Phi(U)$ is an open non-empty subset of $\mathbb{R}^m$ it contains a closed cube C with side length > 0.



It follows from the Cubes-are-Not-Measure Zero Proposition 37.8 that $V = \Phi(U)$ is not a subset of measure zero. Since $V = \Phi(U) \subseteq W \subseteq X$ we obtain from the Measure Zero-Basics Lemma 37.6 that X is not a subset of measure zero. ■

37.3. **The Weak Whitney Embedding Theorem.** In the Euclidean Embedding Theorem 33.2 we saw that every smooth manifold M can be properly smoothly embedded into some $\mathbb{R}^n$. For compact smooth manifolds the following proposition nails down n in terms of the dimension of M.

Theorem 37.9. (Weak Whitney Embedding Theorem) Every m-dimensional smooth manifold M admits a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$ such that $f(M)$ is a closed subset of $\mathbb{R}^{2m+1}$.

Remark.

(1) In the Covering Dimension-of-Manifolds Proposition 33.9 we showed that the covering dimension of an m-dimensional topological manifold M is $\leq m$. As we already mentioned on page 866 it follows almost immediately from the Topological Space–Euclidean Embedding Theorem 15.16 that there exists a closed embedding $M \to \mathbb{R}^{2m+1}$. The Weak Whitney Embedding Theorem 37.9 betters the result in the sense that for smooth manifolds it provides a smooth embedding into $\mathbb{R}^{2m+1}$.

(2) The fact that the dimension $2m+1$ appears in the Topological Space–Euclidean Embedding Theorem 15.16 and the Weak Whitney Embedding Theorem 37.9 suggests that it is a dimension barrier that is hard to break. In fact the Strong Whitney Embedding Theorem 262.11, which is significantly harder to prove, shows that for smooth manifolds we can improve the dimension by 1, namely it says that any m-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2m}$.

(3) Later on we will prove some negative results, for example in the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14 and the Topological Codimension-One Orientability Proposition 218.9 we will see that the real projective space $\mathbb{R}\mathrm{P}^n$ does not admit a smooth embedding into $\mathbb{R}^{n+1}$. □

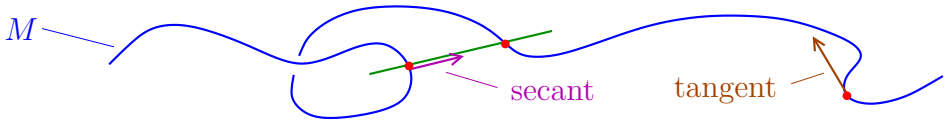
The proof of the Weak Whitney Embedding Theorem 37.9 requires the following lemma, which is perhaps of interest in its own right.

Lemma 37.10. (Generic Vector Lemma) Let M be an m-dimensional smooth submanifold of $\mathbb{R}^n$. We consider the vectors $v \in S^{n-1}$ with the following two properties:

(1) v does *not* lie in any tangent space $\widetilde{\mathrm{T}}_P M$.

(2) v is *not* secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

If $n > 2m + 1$ the set of such vectors is dense in S^{n-1}.



Proof of the Generic Vector Lemma 37.10. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. We start out with the following claim:

Claim. The subsets[438]

$$\begin{array}{lrcl} (1) & X & \coloneqq & \{(P,w) \,|\, P \in M, w \in \widetilde{\mathrm{T}}_P M \setminus \{0\}\} \subseteq \mathbb{R}^n \times \mathbb{R}^n \\ (2a) & Y_a & \coloneqq & \{(x,y) \in M \times (M \setminus \partial M) \,|\, x \neq y\} \subseteq \mathbb{R}^n \times \mathbb{R}^n \\ (2b) & Y_b & \coloneqq & \{(x,y) \in M \times \partial M \,|\, x \neq y\} \subseteq \mathbb{R}^n \times \mathbb{R}^n \end{array}$$

are smooth submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$ of dimension $\leq 2m$.

Proof. First we consider X. Thus let $(P, w) \in X$. We pick a submanifold chart $\Phi \colon U \to V$ around $P \in M$. The map

$$\begin{array}{rcl} \Psi \colon U \times (\mathbb{R}^n \setminus \{0\}) & \to & V \times (\mathbb{R}^n \setminus \{0\}) \\ (x, v) & \mapsto & (\Phi(x), \mathrm{D}\Phi_x(v)) \end{array}$$

is easily seen to be smooth and, using Φ^{-1}, one can easily write down a smooth inverse. Thus we see that Ψ is a diffeomorphism. Up to a permutation of coordinates the map Ψ is a $2m$-dimensional smooth submanifold chart for $(P, w) \in X$.

Now we turn to Y_a and Y_b. Using products of submanifold charts one can easily show that the product of two smooth submanifolds of $\mathbb{R}^n$, where at most one has non-empty boundary, is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ of the obvious dimension. The subsets Y_a and Y_b are open subsets of such products, thus they are smooth submanifolds with $\dim(Y_a) = 2m$ and $\dim(Y_b) = 2m - 1$. ⊞

In the following we suppose that $n > 2m + 1$. We consider the maps[439]

$$\begin{array}{rcl} f \colon X = \{(P,w) \,|\, P \in M, w \in \widetilde{\mathrm{T}}_P M \setminus \{0\}\} & \to & \mathbb{R}\mathrm{P}^{n-1} = (\mathbb{R}^n \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ (P, w) & \mapsto & [w] \end{array}$$

and

$$\begin{array}{rcl} g_a \colon Y_a = \{(x,y) \in M \times (M \setminus \partial M) \,|\, x \neq y\} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x, y) & \mapsto & [x - y] \\ g_b \colon Y_b = \{(x,y) \in M \times \partial M \,|\, x \neq y\} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x, y) & \mapsto & [x - y]. \end{array}$$

[438] This claim is closely related to Exercises 25.27 and 25.28.

[439] Here we use the identification $(\mathbb{R}^n \setminus \{0\})/(\mathbb{R} \setminus \{0\}) = \mathbb{R}\mathrm{P}^{n-1} \coloneqq S^{n-1}/\{\pm 1\}$ from the Projective Space-Sphere Quotient Lemma 9.13.

In the Projective Space–Manifold Proposition 25.13 we equipped $\mathbb{R}\mathrm{P}^{n-1}$ with a standard smooth structure and thus we can view $\mathbb{R}\mathrm{P}^{n-1}$ as an $(n-1)$-dimensional smooth manifold. As we just showed in the claim, the domains X, Y_a and Y_b of the above maps are smooth submanifolds of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ of dimension $\leq 2m$. It follows easily from the definitions of the smooth structures that the maps f, g_a and g_b are smooth. Since $2m < n-1$ we obtain from the Full Measure-Density Proposition 37.4 together with the Measure-Zero Properties Proposition 37.3 (2) that the complement of the union of the images of f, g_a and g_b is dense.

Let $p\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ be the projection. One can easily show, see Exercise 8.7, that the preimage of a dense subset of $\mathbb{R}\mathrm{P}^{n-1}$ is a dense subset of S^{n-1}. The lemma now follows from the above discussion and the observation that we have the following equality:

$$\begin{array}{c}\text{set of vectors in } \mathbb{R}^n \setminus \{0\} \\ \text{that satisfy (1) and (2)}\end{array} \quad = \quad p^{-1}(\text{complement of the images of } f,\ g_a \text{ and } g_b). \qquad \blacksquare$$

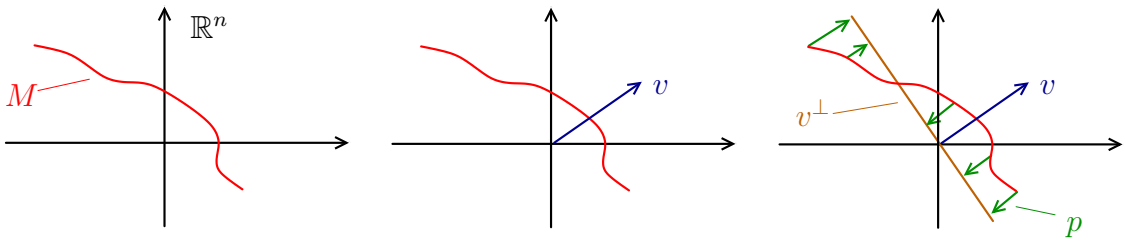
Now we turn to the proof of the Weak Whitney Embedding Theorem 37.9. As always, the case that M is compact is easier, so let us treat it first.

Proof of the Weak Whitney Embedding Theorem 37.9 if M is compact. Let M be an m-dimensional smooth manifold. We claim that we can smoothly embed M into $\mathbb{R}^{2m+1}$. By the Euclidean Embedding Theorem 33.2 together with the Smooth Embedding Theorem 30.15 we can assume that M is already a submanifold of some $\mathbb{R}^n$. If $n \leq 2m+1$ there is nothing to prove, so we may assume that $n > 2m+1$. By the Generic Vector Lemma 37.10 there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space $\widetilde{\mathrm{T}}_P M$ and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

We consider the projection of $\mathbb{R}^n$ onto the orthogonal complement of v, i.e. we consider the projection

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & v^\perp = \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\} \\ u & \mapsto & u - \langle u, v\rangle \cdot \frac{v}{\|v\|^2}. \end{array}$$

Since v does not lie in any tangent space $\widetilde{\mathrm{T}}_P M$ we see that $p\colon M \to v^\perp$ is an immersion. Furthermore, since v is not secant to M we see that $p\colon M \to v^\perp$ is injective. Since we assume that M is compact we obtain from the Compact-Hausdorff Proposition 4.23 (3) that $p\colon M \to v^\perp$ is actually an embedding.



In summary we have shown that the restriction of the projection p to M is a smooth embedding. Thus we have embedded M into a real vector space of dimension $n-1$. Iterating this process we get a smooth embedding of M into $\mathbb{R}^{2m+1}$. $\blacksquare$

Evidently we also want to prove the Weak Whitney Embedding Theorem 37.9 for possibly non-compact smooth manifolds. As an example, let us consider $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. This is a

non-compact 0-dimensional proper smooth submanifold of $\mathbb{R}^2$. the Generic Vector Lemma 37.10 shows that there exist (many) vectors $v \in S^1$ such that the projection of $\mathbb{Z}^2$ onto $v^\perp$ is injective. But, as we will see in Exercise 37.6, none of these projections will be an embedding, i.e. a homeomorphism onto its image. Thus, to deal with non-compact smooth manifolds we will have to come up with a new idea.

We will need the following elementary lemma, which is the content of Exercise 35.9.

Lemma 37.11. (Embed–Into No Boundary Lemma) Given any m-dimensional smooth manifold M there exists an m-dimensional smooth manifold N with $\partial N = \varnothing$ and a smooth embedding $\varphi\colon M \to N$ such that $\varphi(M)$ is a closed subset of N.

Proof. Let M be an m-dimensional smooth manifold. First note that it follows from the Smooth Collar Neighborhood Theorem 35.6 that we can pick a collar neighborhood $[0,1] \times \partial M$. Using the Smooth Transition Function Lemma 24.9 one can show easily that there exists a diffeomorphism $f\colon [0,1] \to [\frac{1}{4}, 1]$ such that $f(t) = t$ for $t \in [\frac{1}{2}, 1]$. It is now straightforward to verify that the map

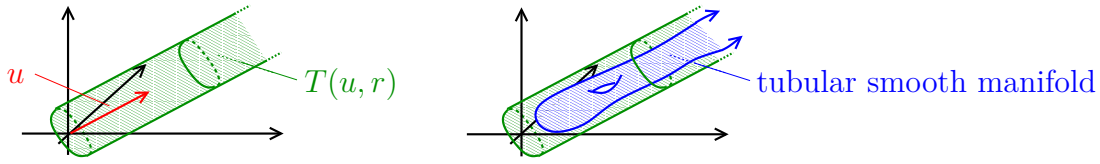
$$\begin{array}{rcl} \varphi\colon M & \to & N \coloneqq M \setminus \partial M \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin [0,1] \times \partial M, \\ (f(t), y), & \text{if } x = (t,y) \in [0,1] \times \partial M \end{cases} \end{array}$$

is a smooth embedding such that the image is a closed subset of N. ■

Proof of the Weak Whitney Embedding Theorem 37.9 if M is non-compact. Let M be an m-dimensional smooth manifold. We need to show that there exists a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$ such that $f(M)$ is a closed subset of $\mathbb{R}^{2m+1}$. It follows from the Embed–Into No Boundary Lemma 37.11 that we might as well assume that $\partial M = \varnothing$.

Given $N \in \mathbb{N}$ we say that M *is N-tubular* if there exists a $u \in \mathbb{R}^N$, an $r > 0$ and a proper smooth embedding of M into

$$T(u,r) \coloneqq \{\mu \cdot u + w \,|\, \mu \in \mathbb{R}_{\geq 0}, \text{ and } w \in u^\perp \text{ with } \|w\| \leq r\}.$$



Note that if M is N-tubular, then it follows from the Euclidean Embedding Theorem 33.2 that M is indeed diffeomorphic to a proper submanifold of $\mathbb{R}^N$. Thus it remains to show that M is $(2m+1)$-tubular. We start out with the following claim:

Claim 1. The smooth manifold M is tubular for some $N \in \mathbb{N}$.

Proof. First note that by the Euclidean Embedding Theorem 33.2 there exists a smooth embedding $f\colon M \to \mathbb{R}^k$ for some $k \in \mathbb{N}$. On page 652 we gave an explicit diffeomorphism $\alpha\colon \mathbb{R}^k \to B^k$. By the Manifold Exhaustion Lemma 27.4 we know that there exists a smooth exhaustion function $\varphi\colon M \to \mathbb{R}_{\geq 0}$, in other words, there exists a smooth map $\varphi\colon M \to \mathbb{R}_{\geq 0}$ such that the preimage of each compact interval $[0,c]$ is compact. Next we consider the map

$$\begin{array}{rcl} \Theta\colon M & \to & \mathbb{R} \times \mathbb{R}^k = \mathbb{R}^{k+1} \\ x & \mapsto & (\varphi(x), (\alpha \circ f)(x)). \end{array}$$

Note that $\Theta(M) \subseteq T(\mathfrak{e}_1, 1)$. Finally note that it follows from the Stretch-to-Embedding Lemma 33.11 that Θ is a proper smooth embedding. ⊞

We see that it remains to prove the following claim:

Claim 2. If M is N-tubular for some $N > 2m+1$, then M is also $(N-1)$-tubular.

Proof. We can assume that there exists a $u \in \mathbb{R}^N$ and an $r > 0$ such that M is a proper submanifold of $T(u,r) \subseteq \mathbb{R}^N$. By the Generic Vector Lemma 37.10 there exists a $v \in \mathbb{R}^N$ with $v \notin \mathbb{R} \cdot u$ with the following two properties:

(1) v does *not* lie in any tangent space $\widetilde{\mathrm{T}}_P M$.
(2) v is *not* secant to M.

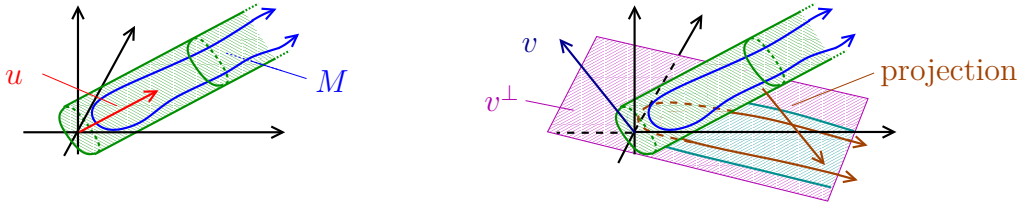
It follows from elementary linear algebra that there exists an isometry $\Theta\colon v^\perp \to \mathbb{R}^{N-1}$ of Euclidean real vector spaces. As on page 924 we consider the map

$$\begin{array}{rcl} p\colon \mathbb{R}^N & \to & v^\perp \\ x & \mapsto & x - \langle x, v\rangle \cdot \frac{v}{\|v\|^2} \end{array}$$

and now we also consider

$$\begin{array}{rcl} \Phi\colon \mathbb{R}^N & \to & \mathbb{R}^{N-1} \\ x & \mapsto & \Theta(p(v)). \end{array}$$

It follows easily from the fact that Θ is an isometry that $\Phi(T(v,r)) \subseteq T(\Phi(v), r)$. Thus it remains to show that $\Phi\colon M \to \mathbb{R}^{N-1}$ is a proper smooth embedding.



Since v does not lie in any tangent space $\widetilde{\mathrm{T}}_P M$ we see, similar to the discussion on page 924, that $\Phi\colon M \to v^\perp$ is an immersion and we see, using that v is not secant to M, that $\Phi\colon M \to v^\perp$ is injective. By the Proper-Closed Proposition 4.64 and the Proper Criterion Lemma 4.65 (1) it remains to show that the preimage of each compact ball $\overline{B}_s^{N-1}(0)$ under the map $\Phi\colon M \to \mathbb{R}^{N-1}$ is again compact. Thus let $s \geq 0$. Note that for each point $x = \mu \cdot u + w \in T(u,r)$ with $w \in u^\perp$ and $\|w\| \leq r$ we have

$$\|\Phi(x)\| \underset{\substack{\uparrow \\ \text{since } \Phi \text{ is} \\ \text{an isometry}}}{=} \|p(x)\| \underset{\substack{\uparrow \\ \text{since } p \text{ is linear and} \\ \text{by the triangle inequality}}}{\geq} \|p(\mu \cdot u)\| - \|p(w)\| \underset{\substack{\uparrow \\ \text{since } p \text{ is linear and} \\ \text{since } \|w\| \leq r}}{\geq} |\mu| \cdot \|p(u)\| - \max\left\{\|p(w)\| \,|\, w \in \overline{B}_r^N\right\}.$$

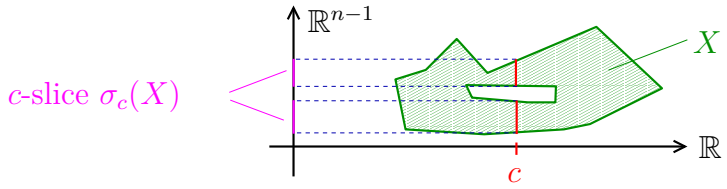
Recall that we cleverly picked $v \notin \mathbb{R} \cdot u$. Thus we see that $\|p(u)\| > 0$. It follows easily from this discussion that $\Phi^{-1}(\overline{B}_s^{N-1}(0)) \cap T(u,r)$ is bounded. Since $M \subseteq T(u,r)$ is closed we see that $\Phi^{-1}(\overline{B}_r^{N-1}(0))$ is a compact subset of M. ■

37.4. **The Mini-Fubini Theorem.** We conclude this chapter on subsets of measure zero with the "Mini-Fubini Theorem" which will play an important role in the proof of Sard's Theorem 38.1 in the next chapter. Our Mini-Fubini Theorem is a basic version of the Fubini Theorem [**StS2005**, Chapter 2.3.1] from measure and integration theory. To formulate the result we need to introduce the following harmless definition.

Notation 37.12. Let $X \subseteq \mathbb{R}^n$ be a subset. Given $c \in \mathbb{R}$ we refer to

$$\sigma_c(X) \;:=\; \{y \in \mathbb{R}^{n-1} \,|\, (c,y) \in X\}.$$

as the c-slice of X.



Now we can formulate the lemma which we will use in the proof of Sard's Theorem 38.1.

Lemma 37.13. (Mini-Fubini) Let $X \subseteq \mathbb{R}^n$ be a compact subset such that each slice $\sigma_c(X)$ is of measure zero in $\mathbb{R}^{n-1}$. Then X has measure zero in $\mathbb{R}^n$.

Remark. The Mini-Fubini Lemma 37.13 follows from the Fubini Theorem in measure theory. But as we will see, one can also give a completely elementary proof, thus there is no need to cite high-powered results. □

Proof. Given $U \subseteq \mathbb{R}^m$ we set

$$\mathrm{vol}_m(U) \;:=\; \inf\left\{ \sum_{i\in I} \mathrm{vol}(C_i) \;\middle|\; \begin{array}{c} \{C_i\}_{i\in I} \text{ is a countable} \\ \text{family of open cubes with } U \subseteq \bigcup_{i\in I} C_i \end{array} \right\} \;\in\; \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Claim 1. The following statements hold:

(i) A subset $U \subseteq \mathbb{R}^m$ has measure zero if and only if for every $\epsilon > 0$ we have $\mathrm{vol}_m(U) \leq \epsilon$.
(ii) Given $U \subseteq V \subseteq \mathbb{R}^m$ we have $\mathrm{vol}_m(U) \leq \mathrm{vol}_m(V)$.
(iii) For every interval (c,d) and every subset U of $\mathbb{R}^{m-1}$ we have

$$\mathrm{vol}_m((c,d) \times U) \;\leq\; (d-c) \cdot \mathrm{vol}_{m-1}(U).$$

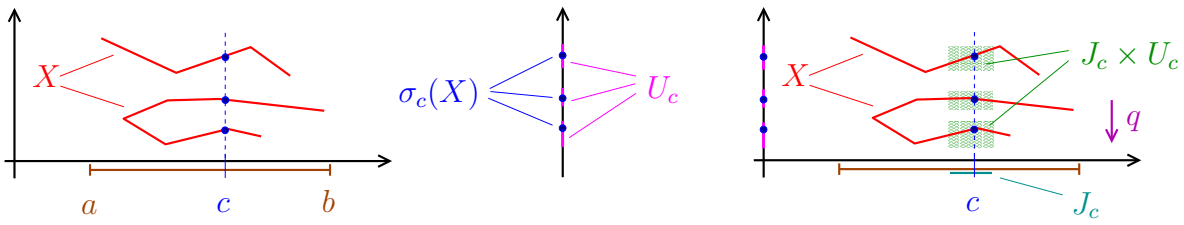
(iv) For $U_1, \ldots, U_l \subseteq \mathbb{R}^m$ we have $\mathrm{vol}_m(U_1 \cup \cdots \cup U_l) \leq \mathrm{vol}_m(U_1) + \cdots + \mathrm{vol}_m(U_l)$.

Proof. Statements (i), (ii) and (iv) are very elementary. The only proof which is not a one-liner is the proof of (iii). We leave it to the reader to provide the proofs. ⊞

Now we start with the actual proof of the lemma. Thus let $X \subseteq \mathbb{R}^n$ be a compact subset such that each slice $\sigma_c(X)$ is of measure zero in $\mathbb{R}^{n-1}$. We need to show that X has measure zero in $\mathbb{R}^n$. Since X is compact there exist $a, b \in \mathbb{R}$ with $X \subseteq [a,b] \times \mathbb{R}^{n-1}$. We fix some $\delta > 0$. Next let $c \in [a,b]$. By hypothesis we know that $\sigma_c(X) \subseteq \mathbb{R}^{n-1}$ has measure zero. By (i) there exists an open subset $U_c \subseteq \mathbb{R}^{n-1}$, given as the countable union of open cubes, with $\sigma_c(X) \subseteq U_c$ and $\mathrm{vol}_{n-1}(U_c) \leq \delta$.

Claim 2. There exists an open interval J_c with $c \in J_c$ and with $X \cap (J_c \times \mathbb{R}^{n-1}) \subseteq J_c \times U_c$.

Proof. We consider $K := X \setminus (\mathbb{R} \times U_c)$. Since U_c is open we see that K is a closed subset of X, hence it follows from the Compact-Closed Lemma 3.39 that K is also compact. Next consider the map $q\colon K \to \mathbb{R}$ that is given by projection onto the first coordinate. By construction $c \notin q(K)$. Since K is compact we know by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that $\mathbb{R} \setminus q(K)$ is open. This implies that there exists an open interval J_c that contains c and with $J_c \subseteq \mathbb{R} \setminus q(K)$. This open interval has the desired properties. ⊞



We have thus shown that for each $c \in [a,b]$ there exists an open subset $U_c \subseteq \mathbb{R}^{n-1}$ with $\mathrm{vol}_{n-1}(U_c) < \delta$ and an open interval J_c with $X \cap (J_c \times \mathbb{R}^{n-1}) \subseteq J_c \times U_c$. Since $[a,b]$ is compact we see that there exist $c_1, \ldots, c_l \in [a,b]$ such that $[a,b] \subseteq J_{c_1} \cup \cdots \cup J_{c_l}$. We write the intervals $J_{c_i} = (\alpha_i, \beta_i)$. An elementary argument[440] shows that after possibly shrinking the intervals we can arrange that $\sum_{i=1}^{l} (\beta_i - \alpha_i) \leq 2 \cdot (b-a)$. Altogether we see that

$$\mathrm{vol}_n(X) \underset{\text{by Claim 1 (ii)}}{\leq} \mathrm{vol}_n\left(\bigcup_{i=1}^{l} J_{c_i} \times U_{c_i}\right) \underset{\text{by Claim 1 (iv)}}{\leq} \sum_{i=1}^{l} \mathrm{vol}_n(J_{c_i} \times U_{c_i}) \underset{\text{by Claim 1 (iii)}}{\leq} \sum_{i=1}^{l} (\beta_i - \alpha_i) \cdot \underbrace{\mathrm{vol}_{n-1}(U_{c_i})}_{\leq \delta} \leq \delta \cdot 2 \cdot (b-a).$$

Since this inequality holds for every $\delta > 0$ we are done by Claim 1 (i). ■

On a later occasion we will also make use of the following converse to the Mini-Fubini Lemma 37.13.

Lemma 37.14. (Mini-Fubini II) Let $X \subseteq \mathbb{R}^n$ be a compact subset. If X has measure zero, then the set

$$\{c \in \mathbb{R}^{n-1} \mid \text{the } c\text{-slice } \sigma_c(X) \text{ has measure zero}\}$$

has full measure.

Proof. We will not provide a proof, we just point out that it follows from the usual Fubini Theorem in measure theory, see e.g. [**StS2005**, Chapter 2.3.1]. At least for $n = 2$ an elementary proof is given in [**Ox1971**, p. 51].[441] ■

37.5. Exercises for Chapter 37.

Exercise 37.1. Let $C \subseteq \mathbb{R}$ be the Cantor from Definition 3.45. Show that the Cantor set has measure zero.
Remark In Exercise 4.20 we saw that the Cantor set is uncountable. Thus we have shown that $\mathbb{R}$ contains uncountable subsets of measure zero.

Exercise 37.2. Let $c\colon [0,1] \to [0,1]$ be the Cantor function that we introduced on page 216. We consider the map

$$\begin{array}{rccc} f\colon & [0,1] & \to & [0,2] \\ & x & \mapsto & x + c(x). \end{array}$$

(a) Show that f is a homeomorphism.

[440] For example one can arrange that for $i \in \{1, \ldots, l-1\}$ we have $\beta_i > \alpha_{i+1}$. Now one can easily modify the intervals by going "from left to right".
[441] The reader who has a self-contained proof of this lemma should communicate this proof to the author.

(b) Let $C \subseteq [0,1]$ be the Cantor set. Show that $f(C)$ has Lebesgue measure one, in particular it is not a subset of measure zero.
Hint. Determine the Lebesgue measure of $f([0,1] \setminus C)$.

Remark. In principle this exercise requires familiarity with the Lebesgue measure. But one can also get joy out of the exercise by using a "naive" approach to length.

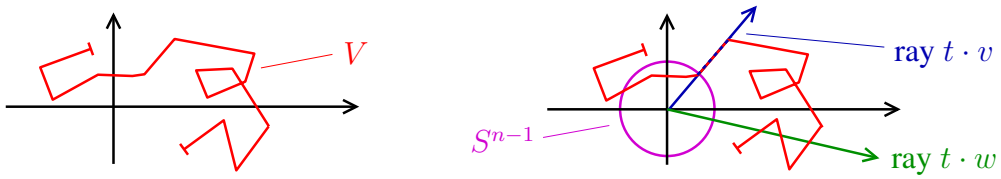
Exercise 37.3. Let $V \subseteq \mathbb{R}^n$ be a subset. Given $v \in S^{n-1}$ we write

$$\mu_v(V) \coloneqq \{t \in \mathbb{R}_{\geq 0} \,|\, t \cdot v \in V\}$$

Show that if V is compact with measure zero, then

$$\{v \in S^{n-1} \,|\, \mu_v(V) \text{ has measure zero}\} \subseteq S^{n-1}$$

has full measure in S^{n-1}.
Hint. Use the Mini-Fubini Lemma 37.14 and the Measure-Zero Properties Proposition 37.3 (3a).



Exercise 37.4. Let $A \subseteq \mathbb{R}^n$ be a subset of diameter d.
(a) Does there exist a closed ball of radius $\frac{d}{2}$ that contains A?
(b) Does there exist a closed cube of side length d that contains A?

Exercise 37.5. Let M be an m-dimensional smooth manifold. Show that there exists an immersion $M \to \mathbb{R}^{2m}$.
Hint. Modify the proof of the Weak Whitney Embedding Theorem 37.9.

Exercise 37.6. We consider the non-compact proper 0-dimensional smooth submanifold $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. Show that there is no $v \in S^1$ such that the projection of $\mathbb{Z}^2$ onto $v^\perp$ is a homeomorphism onto its image.

Exercise 37.7. Does there exist a closed 1-dimensional smooth submanifold $M \subseteq \mathbb{R}^3$ such that every vector in $\mathbb{R}^3$ is secant to M, i.e. such that for every $v \in \mathbb{R}^3$ there exist $x, y \in M$ with $v \in \mathbb{R} \cdot (x - y)$?

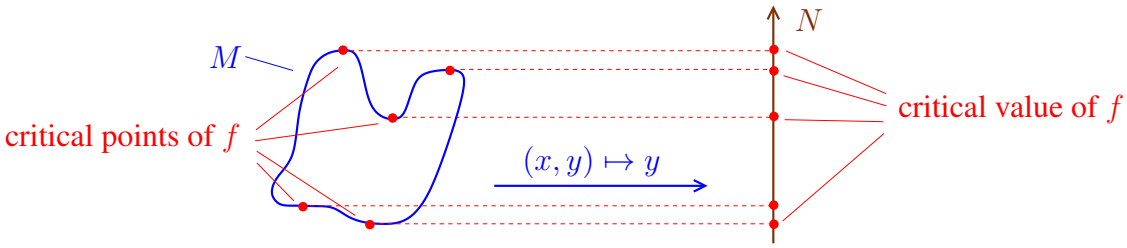
CHAPTER 38

Sard's Theorem

We recall the following definition:

Definition 32.1. Let M and N be smooth manifolds and let $f\colon M \to N$ be a smooth map.

(1) Let $P \in M$. We say that P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism. Otherwise we say that P is a critical point.
(2) We say that $Q \in N$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a critical value.



In this chapter we will prove Sard's Theorem 38.1 which says that for a smooth map $f\colon M \to N$ between smooth manifolds "almost every" point in N is a regular value. Sard's Theorem 38.1 is an absolutely essential tool in differential topology. As an amuse-bouche we will use it in this chapter to provide a proof of the Brouwer Fixed Point Theorem 38.7.

This chapter is organized as follows:

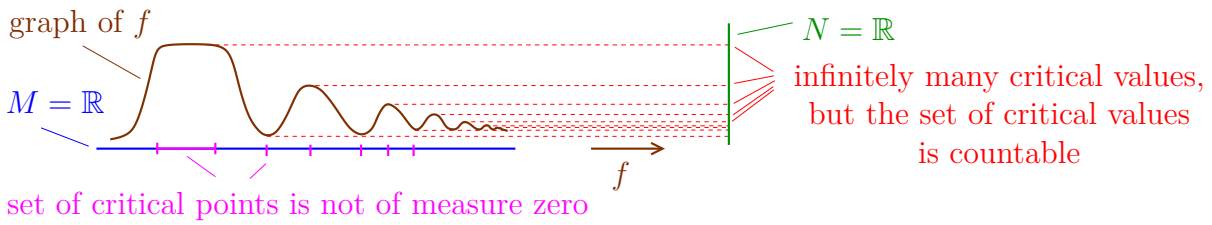
- In Section 38.1 we will state and prove Sard's Theorem 38.1 which says that the set of critical values of a smooth map between smooth manifolds always has measure zero.
- In Section 38.2 we will see that Sard's Theorem 38.1 can be used to show that every smooth manifolds can be exhausted by compact smooth manifolds.
- In Section 38.3 we will prove an another application of Sard's Theorem 38.1, namely we will see that boundaries of smooth manifolds are never retracts of the total manifold.
- In Section 38.4 we will state and prove the Brouwer Fixed Point Theorem 38.7 which says that every continuous self-map of a closed ball admits a fixed point.
- In Section 38.5 we will provide exercises for Chapter 38.

In the next Chapter 39 we will recall some concepts from complex analysis.

38.1. Sard's Theorem.

The following theorem is one of the key building blocks in the theory of smooth manifolds:

Theorem 38.1. (Sard's Theorem) If $f\colon M \to N$ is a smooth map between smooth manifolds (possibly with boundary)[442], then the set of critical values of f is of measure zero in N.



Remark.

(1) The theorem is named after Arthur Sard [**Sar1942**] who first proved a similar statement in 1942.
(2) As we pointed out on page 840, if $f\colon M \to N$ is a smooth map between smooth manifolds with $\dim(M) < \dim(N)$, then a point $Q \in N$ is a regular value if and only if $f^{-1}(Q) = \varnothing$. It follows from this observation that Sard's Theorem 38.1, together with the Measure-Zero Properties Proposition 37.3 (2), implies the Measure-Zero Properties Proposition 37.3 (4).
(3) Sometimes Sard's Theorem 38.1 is also called *Morse-Sard Theorem*, since the case that $N = \mathbb{R}$ was first proved by Anthony Morse [**MorsA1939**] in 1939. Furthermore the fact that the set of regular values is dense was first proved by Arthur Brown [**BrownA1935**] in 1935. □

The proof of Sard's Theorem 38.1 will be the content of the remainder of this section. Like many other textbooks we follow the account given in [**Miln1965a**, Chapter 3].

Proof. Let $f\colon M \to N$ be a smooth map between smooth manifolds. We need to show that the set of critical values of f is of measure zero in N. We set $m := \dim(M)$ and $n := \dim(N)$. We prove the theorem by induction on $m = \dim(M)$.

First let us consider the case $m = 0$. Note that this means that M is equipped with the discrete topology. Since smooth manifolds are in particular second-countable we see that M is a countable set. Thus the statement follows from the Measure-Zero Properties Proposition 37.3 (2).

Now assume that the conclusion of Sard's Theorem holds for smooth maps $M \to N$ with $\dim(M) < m$. Our goal is to prove Sard's Theorem for m-dimensional smooth manifolds M. From the observation that any smooth manifold is covered by a countable smooth atlas and from the definition of a smooth map on an open subset of $\mathbb{H}^m$ it follows easily, using the Measure-Zero Properties Proposition 37.3, that it suffices to deal with the case that M is an open subset of $\mathbb{R}^m$ and that $N = \mathbb{R}^n$. We denote by $f_1, \dots, f_n$ the coordinate functions of $f\colon M \to \mathbb{R}^n$.

We denote by $C \subseteq M \subseteq \mathbb{R}^m$ the set of critical points of $f\colon M \to N$. Given $k \in \mathbb{N}$ we set

$$C_k := \left\{ x \in C \;\middle|\; \begin{array}{c} \text{for all } j = 1, \dots, n \text{ and all } i = 1, \dots, k \text{ every} \\ \text{partial derivative of } f_j \text{ of order } i \text{ at } x \text{ vanishes} \end{array} \right\}.$$

Since f is smooth we know that all partial derivatives are continuous. Thus we see that C and all C_k's are closed in M. The proof of Sard's Theorem now proceeds in four steps.

[442] We stress that M and N are allowed to have boundary, since (a) we need these cases, and (b) since many proofs in the literature assume that M and N do not have boundary.

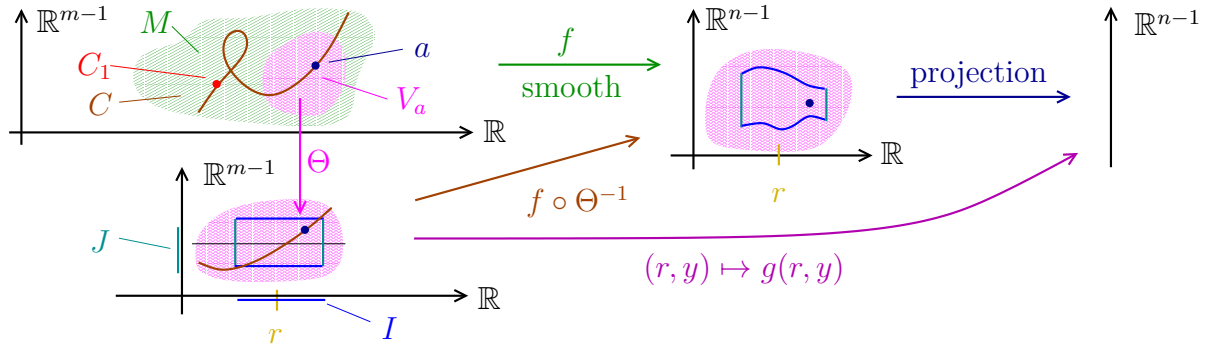
Step 1. *The set $f(C \setminus C_1) \subseteq \mathbb{R}^n$ has measure zero.*
Let $a \in C \setminus C_1$. By definition of C_1 we know that there exist a $j \in \{1, \dots, n\}$ and an $i \in \{1, \dots, k\}$ such that the partial derivative $\frac{\partial f_j}{\partial x_i}$ is non-zero at a. To simplify the notation we assume that $i = j = 1$. We write $\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$ and we consider the map

$$\begin{array}{rcl} \Theta\colon M & \to & \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1} \\ (\underbrace{x}_{\in\mathbb{R}}, \underbrace{y}_{\in\mathbb{R}^{m-1}}) & \mapsto & (f_1(x,y), y). \end{array}$$

Note that $\det(\mathrm{D}\Theta_a) = \frac{\partial f_1}{\partial x_1}(a) \neq 0$. Thus it follows from the Inverse Function Theorem 24.21 that Θ is a local diffeomorphism at a. This means that there exists an open neighborhood V_a of a in the open subset $M \setminus C_1$ such that $\Theta(V_a)$ is an open subset of $\mathbb{R}^m$ and such that $\Theta\colon V_a \to \Theta(V_a)$ is a diffeomorphism. By an elementary argument we see that $f \circ \Theta^{-1}$ is of the form

$$(*) \qquad (f \circ \Theta^{-1})(x,y) \;=\; (x, \underbrace{g(x,y)}_{\in\mathbb{R}^{n-1}}) \qquad \text{for every } (x,y) \in \Theta(V_a)$$

where $g\colon \Theta(V_a) \to \mathbb{R}^{n-1}$ is a smooth map. Next we pick a closed interval $I \subseteq \mathbb{R}$ and a closed ball $J \subseteq \mathbb{R}^{m-1}$, each with non-empty interior, such that $\Theta(a) \in I \times J \subseteq \Theta(V_a)$.



Claim. For each $r \in I$ we have

$$\begin{array}{l} r\text{-slice of the set of critical values} \\ \text{of the map } f \circ \Theta^{-1}\colon I \times J \to \mathbb{R} \times \mathbb{R}^{n-1} \end{array} \;=\; \text{set of critical values of } \begin{array}{rcl} \varphi_r\colon J & \to & \mathbb{R}^{n-1} \\ y & \mapsto & g(r,y). \end{array}$$

Proof. Let $r \in I$. Given $y \in J$ it follows from $(*)$ that we have the following equality:

$$\mathrm{D}(f \circ \Theta^{-1})_{(r,y)} \;=\; \begin{pmatrix} 1 & 0 \\ * & \mathrm{D}(y \mapsto g(r,y))_y \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ * & \mathrm{D}(\varphi_r)_y \end{pmatrix}.$$

This shows that (r,y) is a critical point of $f \circ \Theta^{-1}$ if and only if y is a critical point of the map $\varphi_r\colon J \to \mathbb{R}^{n-1}$ given by $y \mapsto g(r,y)$. The claim now follows from the above and the following two observations:

(a) For any s, z we have $(f \circ \Theta^{-1})(s,z) = (s, g(s,z)) = (s, \varphi_s(z))$.
(b) It follows from (a) that for $w \in \mathbb{R}^{n-1}$ we have

$$w \in r\text{-slice of } (f \times \Theta^{-1})(I \times J) \quad \Longleftrightarrow \quad w \in \varphi_r(J). \qquad \boxplus$$

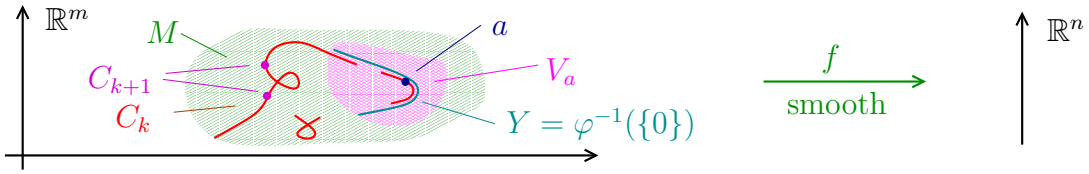
Now we can conclude the proof of Step 1:

(a) Since $\dim(J) = m - 1$ we see that for each $r \in I$ the right-hand side of the claim has measure zero by our induction hypothesis.

(b) As we mentioned above, C is closed. This implies that $C \cap (I \times J)$ is compact. Therefore it follows from (a) together with the Mini-Fubini Lemma 37.13 that the set of critical values of $f \circ \Theta^{-1}$ in $I \times J$ has measure zero.
(c) Since Θ is a diffeomorphism it follows from (b) and the trivial observation on page 840 that the set of critical values of f in $W_a := \Theta^{-1}(I \times J)$ has measure zero.
(d) We have now shown that each $a \in C \setminus C_1$ admits a neighborhood $W_a \subseteq M \setminus C_1$ such that $f(W_a \cap (C \setminus C_1))$ has measure zero.
(e) By the Countable Cover Lemma 13.4 we can cover $C \setminus C_1$ by countably many of the W_a. Therefore we can cover $f(C \setminus C_1)$ by countably many of the $f(W_a)$.
(f) It follows from (d), (e) and the Measure Zero-Basics Lemma 37.6 that $f(C \setminus C_1)$ has indeed measure zero.

Step 2. *For each $k \in \mathbb{N}$ the set $f(C_k \setminus C_{k+1}) \subseteq \mathbb{R}^n$ has measure zero.*
Let $a \in C_k \setminus C_{k+1}$. Since $a \notin C_{k+1}$ there exists a k-th partial derivative $\varphi\colon M \to \mathbb{R}$ of some f_j such that some partial derivative of φ does not vanish at a. Note that this means that a is a regular point of $\varphi\colon M \to \mathbb{R}$. By the Regular Value-Open Subset Lemma 32.2 we know that the set of regular *points* of φ is an open subset of M, thus there exists an open neighborhood $V_a \subseteq M \setminus C_{k+1}$ of a that consists only of regular points of $\varphi\colon M \to \mathbb{R}$.

We set $Y := \varphi^{-1}(0)$. Note that it follows from the Regular Value Theorem 32.5 that Y is a codimension-one submanifold of V_a. By definition of C_k all k-th partial derivatives of f vanish on C_k, in particular φ vanishes on C_k, which implies that $C_k \cap V_a \subseteq Y = \varphi^{-1}(0)$.



Claim. $f(C_k \cap V_a) \subseteq$ critical values of $f|_Y\colon Y \to \mathbb{R}^n$.

Proof. Let $P \in C_k \cap V_a$. Since $P \in C$ we see that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}^n$ is not an epimorphism. But this implies that $\mathrm{D}(f|_Y)_P = (\mathrm{D}f_P)|_{\mathrm{T}_P Y}\colon \mathrm{T}_P Y \to \mathbb{R}^n$ is also not an epimorphism. Thus we see that P is a critical point of $f|_Y\colon Y \to \mathbb{R}^n$ which implies that $f(P)$ is a critical value of $f|_Y\colon Y \to \mathbb{R}^n$. ⊞

Now we can conclude the proof of Step 2:

(a) Since Y is a smooth manifold of dimension $m-1$ we obtain from our induction hypothesis that the right-hand side of the claim is a set of measure zero.
(b) It follows from (a) together with the Measure-Zero Properties Proposition 37.3 (1) that each $a \in C_k \setminus C_{k+1}$ admits an open neighborhood $V_a \subseteq C_k \setminus C_{k+1}$ such that $f(V_a \cap (C_k \setminus C_{k+1}))$ has measure zero.
(c) By the Countable Cover Lemma 13.4 we can cover $C_k \setminus C_{k+1}$ by countably many of the V_a. Therefore we can cover $f(C_k \setminus C_{k+1})$ by countably many of the $f(V_a)$.
(d) It follows from (b), (c) and the Measure Zero-Basics Lemma 37.6 that $f(C_k \setminus C_{k+1})$ has indeed measure zero.

Step 3. *If $k > \frac{m}{n} - 1$, then the set $f(C_k) \subseteq \mathbb{R}^n$ has measure zero.*
Let $a \in C_k$ and let V_a be an open cube that is a neighborhood of a in M and which satisfies $\overline{V}_a \subseteq M$.

Claim. There exists some $D \in \mathbb{R}$ such that for each closed cube $W \subseteq \overline{V}_a$ of side length s and which contains a point $b \in C_k$ we have

$$\|f(x) - f(b)\| \;\leq\; D \cdot s^{k+1} \qquad \text{for each } x \in W.$$

Proof. Since f is smooth and since $\overline{V}_a$ is compact we can set

$$M \;\coloneqq\; \text{maximum of the absolute value of all } (k+1)\text{-st partial derivatives of } f \text{ on } \overline{V}_a.$$

We denote by $f_1, \ldots, f_n$ the coordinate functions of f.

Next let $W \subseteq \overline{V}_a$ be a closed cube of side length s which contains a point $b \in C_k$. Let $i \in \{1, \ldots, n\}$. Note that for $b \in C_k$ the k-th Taylor polynomial $p(f_i, b)_k$ of f_i at the point b, which we introduced in Definition 24.12, is actually the constant polynomial $f_i(b)$. Thus we see that for each $x \in W$ we have

$$|f_i(x) - f_i(b)| \;=\; |f_i(x) \underbrace{- p(f,b)_k(x)}_{= f_i(b)}| \;\underset{\text{by the Taylor Polynomial Approximation Corollary 24.14}}{\leq}\; \frac{m^{k+1} \cdot M}{(k+1)!} \cdot \|x - b\|^{k+1} \;\underset{\text{since } x, b \text{ are contained in an } m\text{-dimensional cube of side length } s}{\leq}\; \frac{m^{k+1} \cdot M}{(k+1)!} \cdot (\sqrt{m} \cdot s)^{k+1}.$$

Next recall that given any $(y_1, \ldots, y_n) \in \mathbb{R}^n$ we have the elementary inequality

$$\|(y_1, \ldots, y_n)\| \;=\; \sqrt{y_1^2 + \cdots + y_n^2} \;\leq\; n \cdot \max\{|y_1|, \ldots, |y_n|\}.$$

Combining these two inequalities we see that

$$\|f(x) - f(b)\| \;\leq\; \underbrace{n \cdot \frac{m^{k+1} \cdot M}{(k+1)!} \cdot (\sqrt{m})^{k+1}}_{=:D} \cdot s^{k+1}$$

⊞

Note that it follows from the claim for each cube $W \subseteq \overline{V}_a$ of side length s and which contains a point in C_k we have

$$f(\underbrace{W}_{\text{vol}=s^m}) \;\subseteq\; \begin{array}{c}\text{closed ball in } \mathbb{R}^n \text{ of} \\ \text{radius } D \cdot s^{k+1}\end{array} \;\subseteq\; \underbrace{\text{closed cube in } \mathbb{R}^n \text{ of side length } D \cdot s^{k+1}}_{\text{vol}=D^n \cdot s^{(k+1)n}}.$$

Here comes the key observation:

> If $k > \frac{m}{n} - 1$, then it follows that $(k+1)n > m$, which implies that the smaller s, the more the volume shrinks by applying f.

We now have to turn this observation into a proper argument.

Claim. The set $f(C_k \cap \overline{V}_a)$ has measure zero.

Proof. Let $\epsilon > 0$. We denote by r the side length of the cube $\overline{V}_a$. Since $m - (k+1) \cdot n < 0$ there exists an $L \in \mathbb{N}$ such that

$$D^n \cdot r^{(k+1) \cdot n} \cdot L^{m - (k+1) \cdot n} \;<\; \epsilon.$$

We choose L as above and we break $\overline{V}_a$ into L^m closed cubes $W_1, \ldots, W_{L^m}$ of side length $\frac{r}{L}$. We set

$$I \;\coloneqq\; \{i \in \{1, \ldots, L^m\} \,|\, C_k \cap W_i \neq \varnothing\}.$$

We see that

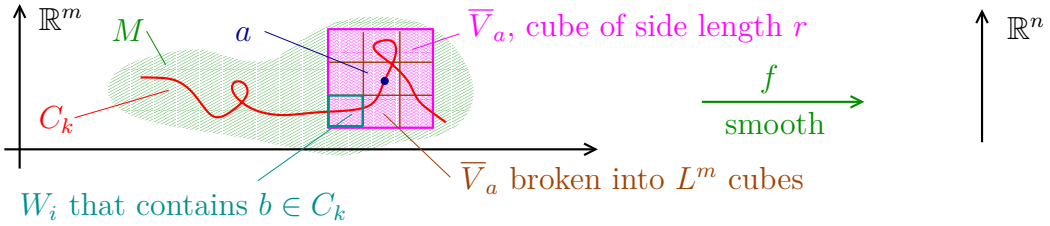
$$f(C_k \cap \overline{V}_a) \subseteq f\Big(\bigcup_{i\in I} W_i\Big) = \bigcup_{i\in I} f(W_i) \underset{\uparrow}{\subseteq} \bigcup_{i\in I} \text{closed cube of volume } D^n \cdot (\tfrac{r}{L})^{(k+1)n}.$$

by the above since each such W_i contains a point in C_k

It follows from the fact that $|I| \leq L^m$ that the sum of the volumes of the cubes to the right is at most $L^m \cdot D^n \cdot (\frac{r}{L})^{(k+1)n}$, which by choice of L is $< \epsilon$. We have now shown that we can cover $f(C_k \cap \overline{V}_a)$ by closed cubes whose volumes add up to less than ϵ. It follows from the Measure Zero-Basics Lemma 37.6 (4) that $f(C_k \cap \overline{V}_a)$ has indeed measure zero. ⊞

Now we can conclude the proof of Step 3:

(a) We have just shown that each $a \in C_k$ admits a neighborhood $\overline{V}_a \subseteq M$ such that $f(C_k \cap \overline{V}_a)$ has measure zero.
(b) By the Countable Cover Lemma 13.4 we can cover C_k by countably many of the $\overline{V}_a$.
(c) It follows from (a) and (b), together with the Measure Zero-Basics Lemma 37.6 (2), that $f(C_k)$ has indeed measure zero.



Step 4. *Conclusion of the argument.*
We pick some $k \in \mathbb{N}$ with $k > \frac{m}{n} - 1$. Then we have

$$f(C) = f\Big((C \setminus C_1) \cup \bigcup_{i=1}^{k-1} (C_i \setminus C_{i+1}) \cup C_k\Big) = \underbrace{f(C \setminus C_1)}_{\substack{\text{measure zero}\\ \text{by Step 1}}} \cup \bigcup_{i=1}^{k-1} \underbrace{f(C_i \setminus C_{i+1})}_{\substack{\text{measure zero}\\ \text{by Step 2}}} \cup \underbrace{f(C_k)}_{\substack{\text{measure zero}\\ \text{by Step 3}}}.$$

It follows from the Measure Zero-Basics Lemma 37.6 (2) that $f(C)$, i.e. the set of critical values of f, has measure zero. ■

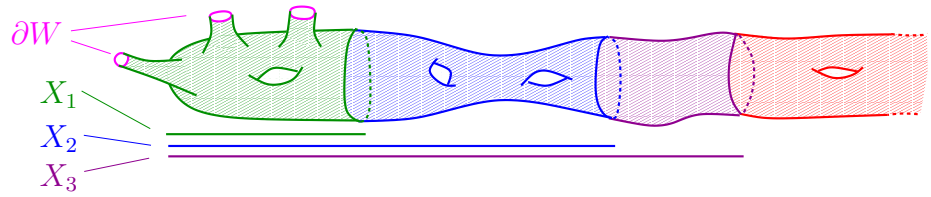
38.2. **Application of Sard's Theorem 38.1: Exhausting smooth manifolds.** In this section and the next section we give several applications of Sard's Theorem 38.1. Our first proposition of this section often allows us to reduce a statement about non-compact smooth manifolds to a statement about compact smooth manifolds (potentially with corner).

Proposition 38.2. (Compact Submanifold Exhaustion Proposition) Let W be a connected smooth manifold. There exists a sequence $X_1, X_2, \ldots$ of subspaces with the following four properties:[443]

(1) Each X_i is compact and connected.
(2) Each X_i is a codimension-zero smooth submanifold with corner[444] of W.
(3) For each $i \in \mathbb{N}$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$.
(4) We have $\bigcup_{i\in\mathbb{N}} \overset{\circ}{X}_i = W$.

Furthermore, if W has compact boundary, then we can arrange that all the X_i are smooth submanifolds (i.e. without corner).

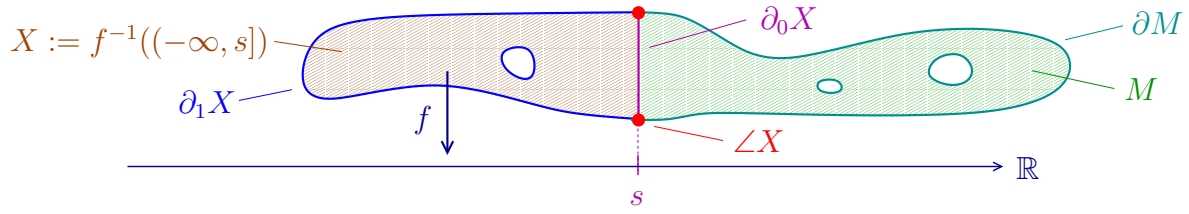
Example. In the figure below we show a non-compact smooth manifold W with compact boundary:



We also show a corresponding sequence $X_1, X_2, \ldots$ of compact submanifolds of W. □

The proof of the Compact Submanifold Exhaustion Proposition 38.2 requires the following variation on the Regular Value Theorem 32.5 (1), which will also be of use on a few other occasions.

Lemma 38.3. (Preimage-of-Interval Lemma) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. If $s \in \mathbb{R}$ is a regular value of $f\colon M \to \mathbb{R}$ and also of $f|_{\partial M}\colon \partial M \to \mathbb{R}$, then $X := f^{-1}((-\infty, s])$ is a smooth codimension-zero submanifold with corner of M such that $\partial_0 X = f^{-1}(s)$, $\partial_1 X = f^{-1}((-\infty, s]) \cap \partial M$ and $\angle X = f^{-1}(s) \cap \partial M$.[445] We have obvious variations on this statement for preimages of intervals of the form $[s, t]$ and $[s, \infty)$.



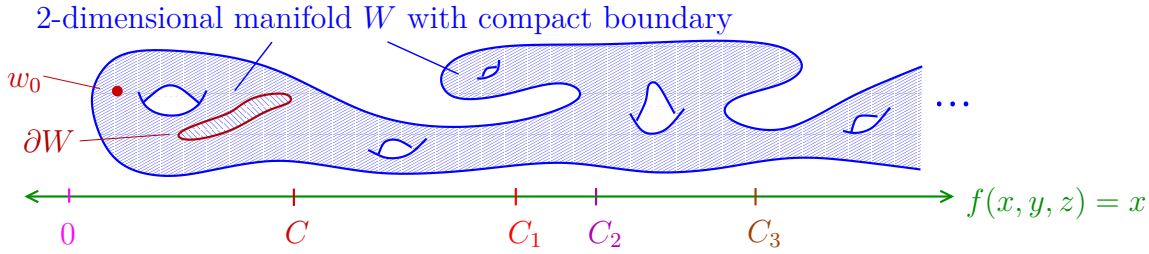
Proof of the Preimage-of-Interval Lemma 38.3. A careful reading of the proof of the Regular Value Theorem 32.5 (1) shows that this proof, with minor adaptations, also provides a proof for the present lemma. Alternatively see also Exercise 32.8. ■

Proof of the Compact Submanifold Exhaustion Proposition 38.2. Let W be a connected smooth manifold. If W is empty, then there is nothing to show, so we might as well assume that W is non-empty. We pick a point $w_0 \in W$. By the Manifold Exhaustion Lemma 27.4 there exists a smooth exhaustion function on W, i.e. there exists a smooth function $f\colon W \to \mathbb{R}_{\geq 0}$ with the property that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0, c])$ is a compact subset of W.

It follows from Sard's Theorem 38.1 and the Full Measure-Density Proposition 37.4 (2) that there exists a strictly monotonously increasing sequence $C_1 < C_2 < C_3 < \ldots$ of regular values of f and of $f|_{\partial W}$ such that $\lim_{i\to\infty} C_i = \infty$. For each $i \in \mathbb{N}$ we denote by X_i the path-component of $f^{-1}((-\infty, C_i])$ that contains w_0. By the Preimage-of-Interval Lemma 38.3 and by our choice of f we know that each X_i is a compact codimension-zero submanifold

[443] In particular we see that the X_i form a compact exhaustion of W in the sense of Definition 13.5.
[444] We refer to Definition 26.14 for the definition of a submanifold with corner.
[445] We refer to the Smooth Submanifold-with-Corner Lemma 26.16 for the definition of $\partial_0 X$ and $\partial_1 X$.

of W with $\partial_0 X_i = X_i \cap f^{-1}(C_i)$. It follows from the Codimension-Zero Submanifold Proposition 25.49 that $\mathring{X}_i = X_i \cap f^{-1}((-\infty, C_i))$. From this observation, and from the fact that the C_i are strictly monotonously increasing we obtain that for each $i \in \mathbb{N}$ we have $X_i \subseteq \mathring{X}_{i+1}$. We have now shown that the X_i satisfy Properties (1), (2) and (3).



Now let $P \in W$. Since W is path-connected there exists a path $\gamma\colon [0,1] \to W$ from w_0 to P. Since $[0,1]$ is compact it follows again from the Compact Image Lemma 4.15 and from $\lim_{i\to\infty} C_i = \infty$ that there exists an i with $\gamma([0,1]) \subseteq f^{-1}((-\infty, C_i))$. By the above this implies that $P \in \mathring{X}_i$. We have thus shown that (4) is also satisfied.

Finally we consider the setting that ∂W is compact. It follows from the Compact Image Lemma 4.15 that there exists a $C \in \mathbb{R}_{\geq 0}$ such that $f(\partial W) \subseteq (-\infty, C]$ and such that $C \geq f(w_0)$. In this case we can and will assume that all of the C_i are greater than C. We now obtain from the Preimage-of-Interval Lemma 38.3 that all the $f^{-1}((-\infty, C_i])$, and thus also all the X_i, are honest-to-G*d submanifolds. ■

We conclude this section with the following corollary which basically says that one can "wrap" any compact subset of a smooth manifold in a suitable codimension-zero compact smooth submanifold with corner.

Lemma 38.4. (Wrapping-in-Submanifold Lemma) Let M be a smooth manifold.

(1) Let K be a compact subset of $M \setminus \partial M$ and let U be a neighborhood of K in M. There exists a compact codimension-zero smooth submanifold X such that $K \subseteq \mathring{X} \subseteq X \subseteq U$.

(2) Let K be any compact subset of M, let U be a neighborhood of K in M and let V be some neighborhood of $K \cap \partial M$ (evidently one could take $V = \partial M$). Then there exists a compact codimension-zero smooth submanifold X with corner such that $K \subseteq \mathring{X} \subseteq X \subseteq U$ and such that $K \cap \partial M \subseteq \mathring{X} \cap \partial M \subseteq X \cap \partial M \subseteq V$.



Proof. Let M be a smooth manifold. Recall that by the Smooth Manifold–Boundary Proposition 25.27 (2) we know that ∂M is a closed subset of M.

(1) Let K be a compact subset of $M \setminus \partial M$ and let U be a neighborhood of K in M. We make a few initial observations:
- Basically by definition of a neighborhood we might as well assume that U is an open subset of M.
- Since ∂M is closed we can furthermore assume that U is an open subset of $M \setminus \partial M$.

- Since K is compact we see that we can assume that U has only finitely many components $U_1, \dots, U_k$.

Since U is an open subset of $M \setminus \partial M$ it follows from the Open Subset-Submanifold Lemma 25.8 and the Open Subset-Submanifold Boundary Lemma 25.11 that we can view it as a smooth manifold in its own right and that $\partial U = \varnothing$. Thus we can apply the Compact Submanifold Exhaustion Proposition 38.2 to the smooth manifolds $U_1, \dots, U_k$. It follows from the fact that K is compact, from the Compact Subspace Covering Lemma 3.38 and from the fact that the submanifolds provided by the Compact Submanifold Exhaustion Proposition 38.2 are nested that there exists a compact codimension-zero submanifold X of U with $K \subseteq \mathring{X} \subseteq X \subseteq U$. Since U is an open subset of M one obtains basically immediately that X is also a submanifold of M.

(2) We consider $W := U \setminus (\partial M \setminus V)$. It follows from the fact that ∂M is a closed subset of M, the fact that V is an open subset of ∂M and the Open-Closed Inclusion Lemma 4.13 that W is an open subset of M. By the Open Subset-Submanifold Lemma 25.8 and the Open Subset-Submanifold Boundary Lemma 25.11 we can view W as a smooth manifold in its own right with $\partial W = W \cap \partial M = \partial V \setminus (U \cup \partial M)$. The proof of (2) is now almost identical to the proof of (1). We leave it to the reader to make the minute modifications. ■

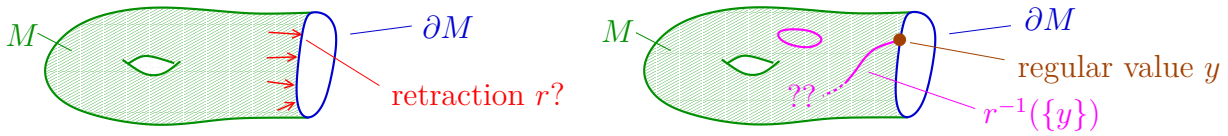
38.3. **Application of Sard's Theorem 38.1: Boundaries are not retracts.** To state our next application of Sard's Theorem 38.1 we need to recall Definition 19.1:

Definition 38.5. Let X be a topological space. We say that a subset $A \subseteq X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

Proposition 38.6. (Boundary Non-Retraction Proposition) Let M be a *compact* non-empty smooth manifold. There is no retraction from M to ∂M.[446]

Remark. In the Kernel-of-Boundary Inclusion Corollary 185.20 and the Boundary Non-Retraction Proposition 75.13 we will give alternative proofs of the Boundary Non-Retraction Proposition 38.6. Furthermore in the Topological Fundamental Class-of-Boundary Corollary 194.14 we will extend the conclusion of the Boundary Non-Retraction Proposition 38.6 to topological manifolds. □

Proof. Let M be a compact non-empty smooth manifold. We suppose that there exists a retraction $M \to \partial M$. It follows from the Replace-by-Smooth Retraction 36.4 (1) that there exists in fact a smooth retraction $r\colon M \to \partial M$. Since M is non-empty we obtain from Sard's Theorem 38.1 that the map $r\colon M \to \partial M$ admits a regular value y.



Since M is compact, and since evidently y is a regular value of $r|_{\partial M} = \mathrm{id}$, we obtain from the Regular Value Theorem 32.5 that $W := r^{-1}(y)$ is a compact proper 1-dimensional

[446]The adjective "compact" is of course necessary, for example the smooth manifold $M = [0, \infty)$ admits a retraction onto its boundary $\partial M = \{0\}$.

submanifold of M. Note that

$$\partial W \underset{\substack{\uparrow \\ \text{since } W \text{ is a proper} \\ \text{submanifold}}}{=} W \cap \partial M = \{x \in \partial M \,|\, r(x) = y\} \underset{\substack{\uparrow \\ \text{since } r = \mathrm{id} \text{ on } \partial M}}{=} \{y\}.$$

Thus we see that ∂W consists of a single point. But it follows from the Smooth 1-Manifold Classification Theorem 28.8, that the boundary of a compact 1-dimensional smooth manifold consists of an *even* number of points. ■

38.4. **The Brouwer Fixed Point Theorem I.** The Boundary Non-Retraction Proposition 38.6 from the last section allows us to prove the famous Brouwer Fixed Point Theorem which was first proved by Jacques Hadamard [**Had1910**] and Luitzen Brouwer [**Brou1912a**].

Theorem 38.7. (Brouwer Fixed Point Theorem) Every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.[447]

Remark. The Brouwer Fixed Point Theorem 38.7 admits many different proofs. We provide two more proofs on page 2843 and on page 3443. A fairly elementary proof is alternatively given in [**Miln1978**]. □

Proof. First let $f\colon \overline{B}^n \to \overline{B}^n$ be an arbitrary continuous map.

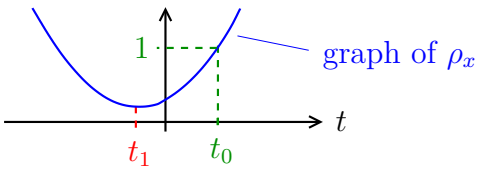
Claim 1. Given $x \in \overline{B}^n$ with $x \neq f(x)$ the open ray $r_x = \{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ contains a unique point on S^{n-1}.

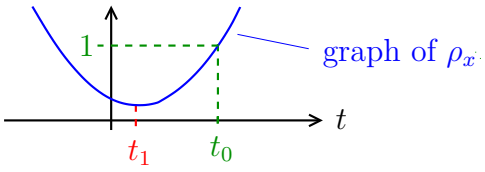
Proof. We consider the map

$$\begin{array}{rccl} \rho_x\colon & \mathbb{R} & \to & \mathbb{R}_{\geq 0} \\ & t & \mapsto & \|f(x) + t \cdot (x - f(x))\|^2 \underset{\substack{\uparrow \\ \text{follows from the equality} \\ \|v+w\|^2 = \langle v+w, v+w\rangle}}{=} \|f(x)\|^2 + 2t \cdot \langle f(x), x - f(x)\rangle + t^2 \cdot \|x - f(x)\|^2. \end{array}$$

Since $x \neq f(x)$ we see that ρ_x is a quadratic function with $\lim_{t \to \pm\infty} \rho_x(t) = \infty$. Also note that $\rho_x(0) = \|f(x)\|^2 \leq 1$. We obtain from the Intermediate Value Theorem that there exists a $t_0 \in \mathbb{R}_{\geq 0}$ with $\rho_x(t_0) = 1$.

Next note that there exists a $t_1 \in \mathbb{R}$ such that $\rho_x'(t) < 0$ for $t < t_1$ and $\rho_x'(t) > 0$ for $t > t_1$. Note that it follows from the Mean Value Theorem that on the interval $[t_1, \infty)$ the function ρ_x attains the value 1 at most once. If $t_1 \leq 0$ we are already done. If $t_1 > 0$, then it follows from the above data and the Mean Value Theorem that for all $t \in [0, t_1]$ we have $f(t) \leq f(0) < 1$.



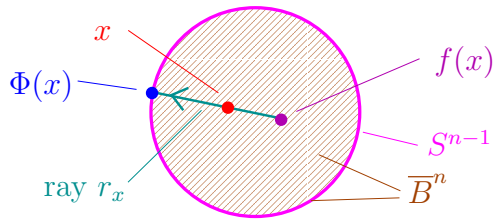


[447] How can one prove the statement for $n = 1$ using elementary methods from real analysis?

It follows from this discussion that there exists a unique $t_1 \in (0,\infty)$ with $\rho_x(t_1) = 1$. This shows that the open ray r_x contains a unique point on S^{n-1}. ⊞

Now we suppose that $f\colon \overline{B}^n \to \overline{B}^n$ has no fixed points. It follows from Claim 1 that for each $x \in \overline{B}^n$ the open ray $r_x = \{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ contains a unique point on S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \text{the unique intersection point of } S^{n-1} \text{ with the} \\ & & \text{uniquely determined open ray from } f(x) \text{ to } x. \end{array}$$



Claim 2. The map Φ is continuous.

Proof. As we discussed in Claim 1, for every $x \in \overline{B}^n$ there exists a unique $t_x > 0$ with $\rho_x(t_x) = \|f(x)+t_x\cdot(x-f(x))\|^2 = 1$, and by definition we have $\Phi(x) = f(x)+t_x\cdot(x-f(x))$. Thus it suffices to show that the map $x \mapsto t_x$ is continuous. One can determine t_x through the quadratic equation

$$1 \;=\; \|f(x) + t_x \cdot (x - f(x))\|^2 \;=\; \|f(x)\|^2 + t_x \cdot 2\langle f(x, x - f(x))\rangle + t_x^2 \cdot \|x - f(x)\|^2.$$

The quadratic formula for polynomials of degree two is continuous in the coefficients, thus we see that the map $x \mapsto t_x \in \mathbb{R}_{>0}$ is indeed continuous.[448] ⊞

It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in \overline{B}^n$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to the Boundary Non-Retraction Proposition 38.6 this is not possible. Thus we have obtained a contradiction. ■

The Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 38.8. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $(n \times n)$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Remark. There are more precise statements one can prove about matrices with positive entries we refer to [**Meyr2000**, Section 8.2] and also to

`https://en.wikipedia.org/wiki/Perron-Frobenius_theorem`

for details. Supposedly the Perron-Frobenius Theorem and some of the more refined results mentioned in [**Meyr2000**, Section 8.2] play a major role in Google's algorithm, see e.g.

`http://www.math.pku.edu.cn/teachers/yaoy/Fall2011/lecture07.pdf`

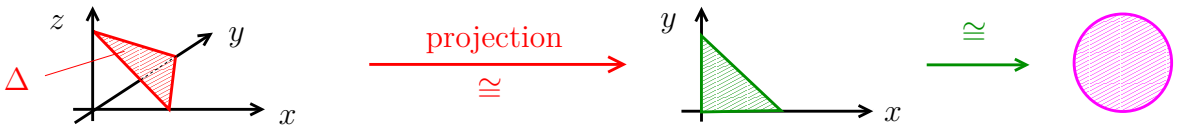
for more information. □

[448] One could also use the home grown Zeros Continuity Proposition 9.22, if one prefers more fancy arguments.

Proof. We introduce two players:

(1) Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum_{i=1}^{n} |v_i|$.

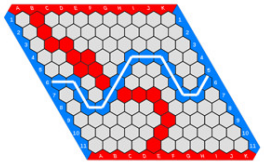
(2) We consider
$$\Delta \;:=\; \{(v_1, \ldots, v_n) \in [0,1]^n \,\big|\, |v| = 1\}.$$
By projecting Δ onto $\mathbb{R}^{n-1}$ and using the Convex-to-Ball Proposition 4.28 one sees that Δ is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 38.7 that any continuous self-map $f\colon \Delta \to \Delta$ has a fixed point.



Now let $A = (a_{ij})$ be an $(n \times n)$-real matrix such that all entries are positive. We consider the map
$$\begin{array}{rcl} f\colon \Delta & \to & \Delta \\ v = (v_1, \ldots, v_n) & \mapsto & \frac{A \cdot v}{|A \cdot v|}. \end{array}$$
Note that this map is defined since our hypothesis that all a_{ij} are positive and the fact that all entries of v are non-negative and that at least one entry of v is positive imply that $A \cdot v \neq 0$. As we had just discussed above, the map $f\colon \Delta \to \Delta$ has a fixed point $v \in \Delta$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

We conclude this discussion of the Brouwer Fixed Point Theorem 38.7 with two applications outside of mathematics:

(1) The board game "Hex" was invented in 1942 by Piet Hein[449] and it was re-invented in 1948 by John Nash.[450] The game is illustrated in the figure below and an online version can be found at
`http://www.lutanho.net/play/hex.html`
David Gale [**Gal1979**] showed in 1979 that the statement that the board game "Hex" cannot end in a draw, is equivalent to the Brouwer Fixed Point Theorem 38.7. We also refer to [**Gh2014**, p. 102] for a discussion of Hex and the Brouwer Fixed Point Theorem 38.7.



11 × 11 Hex gameboard showing
a winning configuration for Blue

(2) The Brouwer Fixed Point Theorem 38.7 can also be used to show the existence of a "Nash equilibrium" in game theory. We refer to the following website for more information:
`https://en.wikipedia.org/wiki/Nash_equilibrium`.

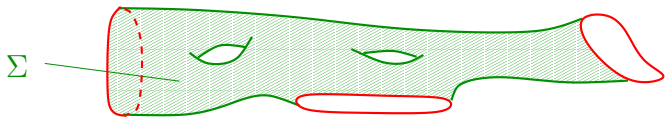
[449] Piet Hein (1905-1996) was a Danish mathematician.

[450] John Nash (1928-2015) was an American mathematician who was awarded the Nobel prize in economics in 1994. The movie "a beautiful mind" was based on his struggles with schizophrenia.

38.5. Exercises for Chapter 38.

Exercise 38.1. In the figure below we show Σ, the surface of genus two minus three open disks. By the Boundary Non-Retraction Proposition 38.6 we know that there is no retraction from Σ to $\partial\Sigma$.

(a) Does there exist a retraction to the union of two boundary components?
(b) Does there exist a retraction to a single boundary component?

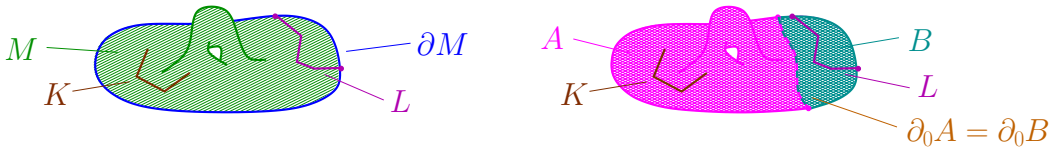


Exercise 38.2. We say that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has *bounded displacement* if there exists a $C \in \mathbb{R}$ such that for every $x \in \mathbb{R}^n$ we have $\|x - f(x)\| \leq C$. Show that every continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded displacement is onto.
Hint. Use the Brouwer Fixed Point Theorem 38.7.

Exercise 38.3. Let M be a smooth manifold and let $K, L \subseteq M$ be disjoint closed subsets of M with $K \cap \partial M = L \cap \partial M = \varnothing$. Show that there exist codimension-zero submanifolds A and B of M, which are closed subsets of M, such that $K \subseteq A$, $L \subseteq B$, $A \cup B = M$ and such that $A \cap B = \partial_0 A = \partial_0 B$.
Hint. Use the Preimage-of-Interval Lemma 38.3 and use Urysohn's Lemma 27.10 for Manifolds.



Exercise 38.4. Let M be a compact n-dimensional smooth manifold and let K and L be disjoint compact subsets of M. Show that there exist compact n-dimensional submanifolds with corner A and B of M such that $K \subseteq \mathring{A}$, $L \subseteq \mathring{B}$, $A \cup B = M$ and such that $A \cap B = \partial_0 A = \partial_0 B$.
Remark. This exercise is a generalization of Exercise 38.3.
Hint. Use Urysohn's Lemma 27.10 for Manifolds.



Exercise 38.5. Let M be a smooth manifold with $\partial M = \varnothing$, let U be a connected open subset of M and let K be a *compact* subset of M that is contained in U. Show that there exists a *compact* connected codimension-zero submanifold N of M with $K \subseteq N \setminus \partial N \subseteq N \subseteq U$.
Remark. The exercise is related to Exercise 38.3, but the compactness condition and requirement adds a welcome extra layer of difficulty.



Exercise 38.6. Let $n \in \mathbb{N}$. Show that the subset

$$\{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid \det(A) = 0\} \subseteq \mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$$

has measure zero.

Hint. There are many approaches to proving this statement. You could for example play with the map

$$\begin{aligned} \Phi\colon \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n\text{-1 copies}} \times \mathbb{R}^{n-1} &\to \mathrm{M}(n \times n, \mathbb{R}) \\ (v_1, \ldots, v_{n-1}, (\lambda_1, \ldots, \lambda_{n-1})) &\mapsto \Big(v_1 \ \ldots \ v_{n-1} \ \sum_{i=1}^{n-1} \lambda_i \cdot v_i\Big). \end{aligned}$$

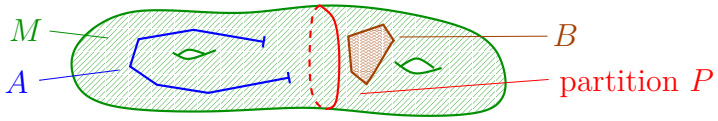
You could also make use of the fact that a matrix has determinant zero if and only if the column vectors are linearly dependent.

Exercise 38.7.

(a) Let X be a topological space and let A and B be disjoint subsets of X. As in Exercise 15.10 we say that a subset $P \subseteq X$ is a *partition between A and B*, if there exist disjoint open subsets U and V of X with $A \subseteq U$, $B \subseteq V$ and $X \setminus P = U \cup V$.

Show that if M is an n-dimensional smooth manifold, then given any two disjoint closed subsets A and B there exists a partition P between A and B that is an $(n-1)$-dimensional smooth manifold of M.

Remark. This part of the exercise is quite similar to Exercise 38.3.



(b) We iteratively define classes $\mathcal{D}_{-1}, \mathcal{D}_0, \mathcal{D}_1, \ldots$ of topological spaces:

- We define $\mathcal{D}_{-1}$ to be the class consisting only of the empty topological space.
- Suppose that for some $n \in \mathbb{N}_0$ we have defined $\mathcal{D}_{-1}, \mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$. Given a topological space X we say that $X \in \mathcal{D}_n$ if for any two disjoint closed subsets A and B of X there exists a partition P between A and B with $P \in \mathcal{D}_{n-1}$. (We refer to Exercise 15.10 for the definition of a partition.)

We define $\mathcal{D}_\infty$ to be the class of all topological spaces. Given a topological space X we define, as in Exercise 15.11, the *large inductive dimension* $\mathrm{Ind}(X)$ as follows:

$$\mathrm{Ind}(X) := \min\{n \in \mathbb{Z}_{\geq -1} \cup \{\infty\} \mid X \in \mathcal{D}_n\}.$$

Show that for every n-dimensional smooth manifold M we have $\mathrm{Ind}(M) \leq n$.

Part V

Complex manifolds

As a slight detour from the main story, namely the development of the theory of smooth manifolds, let us now give a short introduction to complex manifolds. Complex manifolds are topological manifolds that are equipped with a holomorphic atlas. As we will see, complex manifolds are smooth manifolds with particularly nice properties. Throughout these notes they will play a role every now and then.

This very short part of the lecture notes has a very simple organization:

- In Chapter 39 we will recall concepts from complex analysis which we will need in the study of complex manifolds.
- In Chapter 40 we will introduce complex manifolds, we will provide several basic examples and we will show a few results that are specific to complex manifolds. For example we will see that they have a natural orientation.

CHAPTER 39

Complex analysis

In Chapter 24 we recalled some basic concepts from real analysis. Similarly in this chapter we recall some basic concepts from complex analysis. These will come in helpful in the next Chapter 40, where we will introduce complex manifolds.

This chapter is organized as follows:

- In Section 39.1 we will recall the notion of a holomorphic function in the 1-dimensional setting.
- In Section 39.2 we will discuss various biholomorphisms in the one-dimensional setting.
- In Section 39.3 we will discuss holomorphic maps in the higher-dimensional setting.
- In Section 39.4 we will provide exercises for Chapter 39.

In the next Chapter 40 we will use the concepts from this chapter to introduce complex manifolds.

39.1. Holomorphic functions.

We start out with the following convention:

Convention 39.1. Let $n \in \mathbb{N}_0$. Throughout these notes we will use the isomorphism

$$\begin{array}{rcl} \mathbb{R}^{2n} = (\mathbb{R}^2)^n & \to & \mathbb{C}^n \\ (x_1, y_1, \ldots, x_n, y_n) & \mapsto & (x_1 + y_1\,\mathrm{i}, \ldots, x_n + y_n\,\mathrm{i}) \end{array}$$

of real vector spaces to make, as on page 183, the identification $\mathbb{R}^{2n} =_{\mathrm{i}} \mathbb{C}^n$.

The following elementary lemma is the essential tool for describing complex linear maps by real matrices.

Lemma 39.2. (Complex Multiplication-via-Matrix Lemma) Given any $a, b \in \mathbb{R}$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{v \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \cdot v} & \mathbb{R}^2 \\ {\scriptstyle (x,y)\mapsto x+y\mathrm{i}}\big\downarrow \cong & & \cong \big\downarrow {\scriptstyle (x,y)\mapsto x+y\mathrm{i}} \\ \mathbb{C} & \xrightarrow{z \mapsto (a+b\mathrm{i})\cdot z} & \mathbb{C}. \end{array}$$

Proof. The lemma follows immediately from an elementary calculation. ∎

Next we recall the definition of a holomorphic function.

Definition 39.3. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a continuous map.

(1) We say that f is holomorphic in $z_0 \in U$, if the limit

$$\frac{d}{dz} f(z_0) \;:=\; f'(z_0) \;:=\; \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \;\in \mathbb{C}$$

exists. We refer to $\frac{d}{dz} f(z_0) = f'(z_0)$ as the derivative of f at z_0.

(2) We say that f is holomorphic if f is holomorphic in each $z_0 \in U$.

Example. It follows immediately from the definition that the function $f(z) = z$ is holomorphic. Furthermore it is elementary to show products, fractions, sums and compositions of holomorphic functions are again holomorphic. This shows in particular that maps described by polynomials are holomorphic. □

Proposition 39.4. (Complex Power Series Convergence Proposition) Let $(c_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers and let $z_0 \in \mathbb{C}$. We consider the radius of convergence

$$R \coloneqq \sup\left\{|z - z_0| \,\middle|\, \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n \text{ converges}\right\}.$$

The following two statements hold:

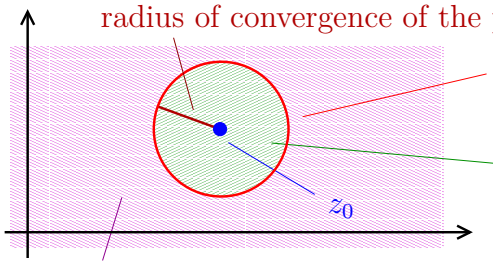
(1) Let $z \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n$ converges if $|z - z_0| < R$ and it diverges if $|z - z_0| > R$.

(2) The map

$$\begin{array}{rcl} B_R(z_0) & \to & \mathbb{C} \\ z & \mapsto & \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n \end{array}$$

is holomorphic with derivative

$$\begin{array}{rcl} B_R(z_0) & \to & \mathbb{C} \\ z & \mapsto & \sum_{n=0}^{\infty} c_n \cdot n \cdot (z - z_0)^{n-1}. \end{array}$$



radius of convergence of the power series $\sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n$

there is no general statement regarding the convergence on the circle $|z - z_0| = R$

the power series convergences on the open disk $B_R(z_0)$ and it defines a holomorphic function

the power series diverges outside of the disk $\overline{B_R(z_0)}$

Proof. This proposition is proved in [**Lang1999**, Theorem II.5.1] or alternatively [**Jä2011**, Chapter 1.2]. ■

Via the above identification $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ we can view a map $f\colon U \to \mathbb{C}$ as above also as a map from an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$. The following proposition clarifies the relationship between holomorphicity and smoothness in the sense of Definition 24.3.

Proposition 39.5. (Smooth-Complex Differentiability Proposition) Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a continuous map. Given $z \in U$ we write $u(z) \coloneqq \mathrm{Re}(f(z))$ and $v(z) = \mathrm{Im}(f(z))$.

(1) For $z_0 \in U$ and $a, b \in \mathbb{R}$ the following two statements are equivalent:

(a) The map $f\colon U \to \mathbb{C}$ is holomorphic in z_0 with derivative $f'(z_0) = a + b\mathrm{i}$.

(b) The map

$$\begin{array}{rcl} U & \to & \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \\ z & \mapsto & (u(z), v(z)) \end{array}$$

is differentiable in the point z_0, in the sense of Definition 24.15, with differential $\mathrm{D}f_{z_0} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.[451]

(2) The following two statements are equivalent:
  (a) The map $f\colon U \to \mathbb{C}$ is holomorphic.
  (b) The map $f\colon U \to \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ is smooth in the sense of Definition 24.3 and each differential is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Sketch of proof. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a continuous map.

(1) Given any $z_0 \in U$ and given any $a, b \in \mathbb{R}$ we see that:

$$
\begin{aligned}
&f \text{ is holomorphic in } z_0 \\
&\text{with derivative } a + b\mathrm{i}
\end{aligned}
\quad \Longleftrightarrow \quad \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = a + b\mathrm{i}
$$

$$
\Longleftrightarrow \quad \lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - (a+b\mathrm{i})\cdot h}{h} = 0
$$

$$
\Longleftrightarrow \quad \underbrace{\lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - (a+b\mathrm{i})\cdot h}{|h|}}_{\text{complex limit in } \mathbb{C}} = 0
$$

$$
\Longleftrightarrow \quad \underbrace{\lim_{w\to 0} \frac{f(z_0+w) - f(z_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\cdot w}{\|w\|}}_{\text{real limit in } \mathbb{R}^2} = 0
$$

(by the Complex Multiplication-via-Matrix Lemma 39.2)

$$
\Longleftrightarrow \quad f \text{ is differentiable in } z_0 \text{ with } \mathrm{D}f_{z_0} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.
$$

(see Definition 24.15)

(2) This statement follows from (1) and the fact, see the Power Series Expansion Theorem 39.9 below, that for every holomorphic map its derivative is again holomorphic. In other words, all higher derivatives exist. This corresponds to our convention, that by "smooth" we mean that all higher derivatives exist. ∎

The definition of a complex function being holomorphic is almost verbatim the same as the definition of a real function being differentiable, we just replace $\mathbb{R}$ by $\mathbb{C}$. But rather surprisingly being holomorphic has many consequences which one is not used to from the world of real valued differentiable functions. In the following we recall some of the main results from complex analysis. But before we can state the first theorem we need to introduce a few definitions.

Definition 39.6.
(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a continuous map $\gamma\colon [a,b] \to U$.

[451] By the Differential-from-Partial Derivatives Proposition 24.16 the condition that $\mathrm{D}f_{z_0}$ is of this form is equivalent to the statement that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These two equations are sometimes called the Cauchy-Riemann equations.

(2) Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a continuous map and let $\gamma\colon [a,b] \to U$ be a smooth path. We define the path integral of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} \underbrace{f(\gamma(t))}_{\in\mathbb{C}} \cdot \underbrace{\gamma'(t)}_{\in\mathbb{C}}\,dt \;\in \mathbb{C}.$$

product of complex numbers

Examples.

(1) Let $U = \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ be the path given by $\gamma(t) = \exp(2\pi\,\mathrm{i}t)$, i.e. γ is a path that "goes counterclockwise once around the origin". Then

$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \frac{1}{\exp(2\pi\,\mathrm{i}t)}\cdot\exp(2\pi\,\mathrm{i}t)\cdot 2\pi\,\mathrm{i}\,dt \;=\; 2\pi\,\mathrm{i}.$$

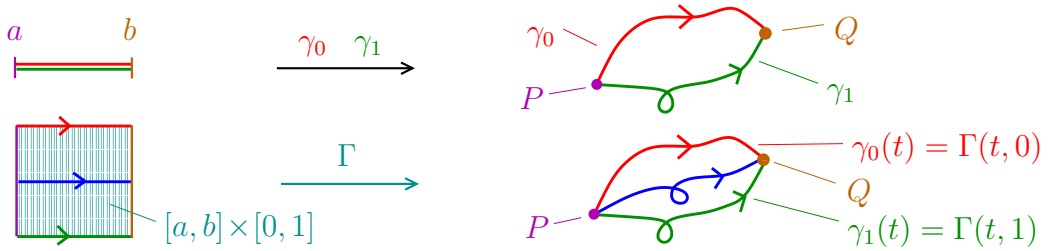
(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero. □

Definition 39.7. Let $U \subseteq \mathbb{C}$ be a subset. Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called path-homotopic in U, if there exists a homotopy rel $\{a,b\}$ from γ_0 to γ_1. In other words, if there exists a continuous map

$$\begin{aligned}\Gamma\colon [a,b]\times[0,1] &\to U\\ (s,t) &\mapsto \Gamma(s,t),\end{aligned}$$

with the following properties:

(a) for every $s \in [a,b]$ we have $\Gamma(s,0) = \gamma_0(s)$ and $\Gamma(s,1) = \gamma_1(s)$,
(b) for every $t \in [0,1]$ we have $\Gamma(a,t) = P$ and $\Gamma(b,t) = Q$.



The following theorem is the foundational result in complex analysis.

Theorem 39.8. (Cauchy Theorem) Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic in U, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

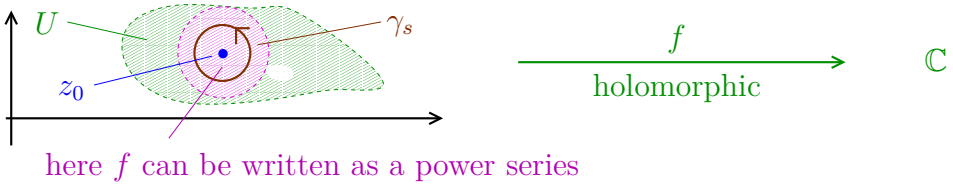
Proof. We will provide a proof of the theorem in Appendix 75.5. Alternatively a proof of Cauchy's Theorem is given in every book on complex analysis, see e.g. [**Lang1999**, Theorem III.5.1]. ■

The Cauchy Theorem 39.8 is the mother of all results in complex analysis. It is at the heart of many other results. We now give a hotchpotch list of theorems, which are more or less immediate consequences of the Cauchy Theorem 39.8, and which we will make use of later on.

Theorem 39.9. (Power Series Expansion Theorem) Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be holomorphic.

(1) If we are given a disk $B_r(z_0) \subseteq U$, then there exists a sequence $(c_n)_{n\in\mathbb{N}_0}$ of complex numbers such that
$$f(z) \;=\; \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n \qquad \text{for all } z \in B_r(z_0).$$
Furthermore for each $s \in (0, r)$ we have
$$c_n \;=\; \int_{\gamma_s} \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$
where $\gamma_s\colon [0, 1] \to U$ is the path given by $\gamma_s(t) = z_0 + s \cdot \exp(2\pi\, \mathrm{i} t)$.

(2) All derivatives of f are holomorphic.



Proof.

(1) This theorem is [**Lang1999**, Theorem III.7.2] or alternatively see [**Jä2011**, Chapter 3.2 Satz 3].

(2) This statement follows from Statement (1) together with the Complex Power Series Convergence Proposition 39.4. ■

Theorem 39.10. (Liouville's Theorem) Every bounded holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ is constant.

Remark. Later, in Picard's Theorem 132.27 we will prove a significant generalization of Liouville's Theorem 39.10, namely we will show that if $f\colon \mathbb{C} \to \mathbb{C}$ is a non-constant holomorphic function, then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f. □

Proof. As we will see in Exercise 39.1, one can deduce Liouville's Theorem 39.10 from the Power Series Expansion Theorem 39.9 (1). Alternatively we refer to [**Lang1999**, Theorem III.7.5] or [**Jä2011**, Chapter 3.2] for a proof. ■

Proposition 39.11. (Schwarz Reflection Principle) Let U be an open subset of the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ and let $f\colon U \to \mathbb{C}$ be a continuous function with the following properties:

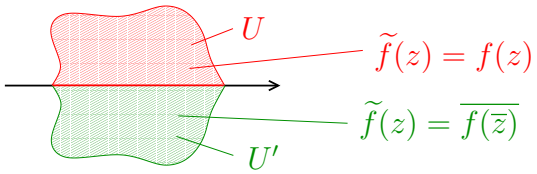
(1) f is holomorphic on $\mathring{U} = \{z \in U \mid \mathrm{Im}(z) > 0\}$,

(2) f only assumes real values on $U \cap \mathbb{R}$.

We set $U' \coloneqq \{\overline{z} \mid z \in U\}$, i.e. U' is the reflection of U in the real axis. Then the function[452]

$$\begin{array}{rcl} \widetilde{f}\colon U \cup U' & \to & \mathbb{C} \\ z & \mapsto & \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\overline{z})}, & \text{if } z \in U', \end{cases} \end{array}$$

is holomorphic.



Proof. This proposition is proved in [**Lang1999**, Chapter IX.1] or alternatively in [**Jä2011**, Chapter 3.3]. ■

Theorem 39.12. (Locally-a-Power Theorem) Let U be an open subset of $\mathbb{C}$. Furthermore let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $z_0 \in U$ be a zero of f.

(1) If all derivatives of f at z_0 vanish, then f is zero in an open neighborhood of z_0.
(2) If there exists a $k \in \mathbb{N}$ such that $f(z_0) = \cdots = f^{(k-1)}(z_0) = 0$ but $f^{(k)}(z_0) \neq 0$, then there exists an open neighborhood $V \subseteq U$ of z_0 and a holomorphic function $g\colon V \to \mathbb{C}$ with $g'(z_0) \neq 0$ such that $f(z) = (g(z))^k$ for all $z \in V$.

Proof. Statement (1) is an immediate consequence of the Power Series Expansion Theorem 39.9 together with the Complex Power Series Convergence Proposition 39.4 (2). Statement (2) follows from [**Jä2011**, Kapitel 3 Satz 10 and 12]. ■

Proposition 39.13. (Non-Constant+Holomorphic–Open Image Proposition) Let U be an open subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. If $V \subseteq U$ is a connected open subset such that $f|_V$ is not constant, then $f(V)$ is an open subset of $\mathbb{C}$.

Proof. We will prove the proposition in Exercise 39.5 using the Locally-a-Power Theorem 39.12 and using the Inverse Function Theorem 24.21. Alternatively see [**Jä2011**, Kapitel 3 Satz 13]. ■

Theorem 39.14. (Maximum Principle) Let U be an open connected subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. If the function $|f|\colon U \to \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.

Proof. We will prove the Maximum Principle in Exercise 39.7 using the Non-Constant+Holomorphic–Open Image Proposition 39.13. Alternatively see [**Lang1999**, p. 271] or [**Jä2011**, Kapitel 3 Satz 14]. ■

39.2. **Biholomorphisms in the one-dimensional setting.** The following definition gives us the holomorphic analog of a biholomorphism:

[452] It follows from the Cover-Open Subset Lemma 3.20 (2) that $U \cup U'$ is an open subset of $\mathbb{C}$.

Definition 39.15. Let $f\colon U \to V$ be a continuous map between two open subsets of $\mathbb{C}$. We say that f is a biholomorphism if f is holomorphic, f is a bijection and $f^{-1}\colon V \to U$ is also holomorphic.

The following proposition shows that the only biholomorphisms of $\mathbb{C}$ are the obvious ones.

Theorem 39.16. (Biholomorphism-of-$\mathbb{C}$-Theorem) Every biholomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is an affine linear map, i.e. there exist $a, b \in \mathbb{C}$ such that $f(z) = a \cdot z + b$ for all $z \in \mathbb{C}$.

Proof. The proposition is proved in [**Lang1999**, Theorem V.5.3]. ■

Example. In some cases one can explicitly write down a biholomorphism. For example, for the upper half-plane $\mathbb{H} = \{x + \mathrm{i}y \mid y > 0\}$ and the open disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ it is straightforward to verify that the maps

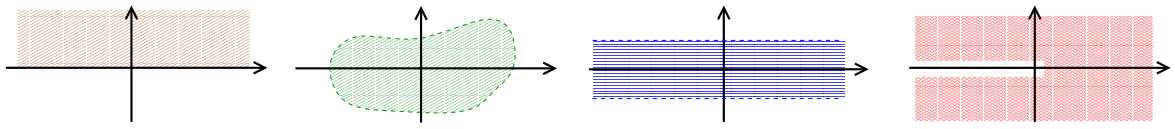
$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{\mathrm{i}+\mathrm{i}w}{1-w} \end{array}$$

are biholomorphisms that are inverse to one another. □

In general it is hard to write down an explicit biholomorphism between two open subsets of $\mathbb{C}$. One of the main sources of biholomorphisms is the Riemann Mapping Theorem. To formulate this fantastic theorem we need to introduce the following definition:

Definition 39.17. Let $U \subseteq \mathbb{C}$ be a subset. We say U is simply connected if U is non-empty, if U is path-connected and if each path $\gamma\colon [0,1] \to U$ with $\gamma(0) = \gamma(1)$ is path-homotopic to a constant map.

Theorem 39.18. (Riemann Mapping Theorem) Let $U \subseteq \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$. If $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.



these open subsets are biholomorphic to $\mathbb{D}$ by the Riemann Mapping Theorem 39.18

Remark. The Riemann Mapping Theorem was first stated by Bernhard Riemann in 1851 in his doctoral thesis. The first rigorous proof was given by William Osgood in 1900, see [**Wals1973**]. □

Proof. The Riemann mapping theorem is proved in most textbooks on complex analysis, see e.g. [**Jä2011**, Chapter 10.1] or alternatively [**Lang1999**, Chapter X]. ■

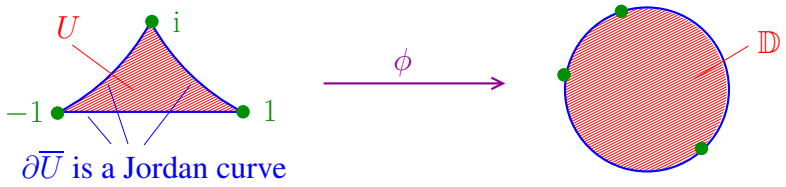
In many applications it is also useful to know that the biholomorphism from the Riemann Mapping Theorem extends to the boundary of U and $\mathbb{D}$. To state the corresponding theorem we need one more definition.

Definition 39.19.

(1) We write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, called "$\mathbb{C}$ with a point at infinity", where we equip the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the topology that we defined on page 152.
(2) Given $U \subseteq \overline{\mathbb{C}}$ we denote by $\overline{U}$ the closure of U in $\overline{\mathbb{C}}$.[453]
(3) A Jordan curve is the image of an embedding $S^1 \to \overline{\mathbb{C}}$.[454]

The following theorem was proved in 1913 by Constantin Carathéodory [**Cara1913**].[455]

Theorem 39.20. (Carathéodory Theorem) Let $U \subsetneq \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$ and let $\phi\colon U \to \mathbb{D}$ be a biholomorphism. If $\partial\overline{U}$ is a Jordan curve, then ϕ extends to a homeomorphism $\overline{U} \to \overline{\mathbb{D}}$.



Proof. A textbook proof of the theorem is provided in [**GM2005**, Chapter I.3]. ■

Example. We consider the open subset U of $\mathbb{C}$ that is shown in the figure above. It is a simply connected open proper subset of $\mathbb{C}$ and the boundary is clearly a Jordan curve. Thus it follows from the Riemann Mapping Theorem 39.18 and from Carathéodory's Theorem 39.20 that there exists a homeomorphism $\phi\colon \overline{U} \to \overline{\mathbb{D}}$ which restricts to a biholomorphism $\phi\colon U \to \mathbb{D}$. □

39.3. **Holomorphic maps in the higher-dimensional setting.** We now move from the one-dimensional setting to the higher-dimensional case.

Definition 39.21. Let $U \subseteq \mathbb{C}^m$ be an open subset and let $f = (f_1, \ldots, f_n)\colon U \to \mathbb{C}^n$ be a continuous map.

(1) We say that f is holomorphic if for every $i \in \{1, \ldots, m\}$, every $j \in \{1, \ldots, n\}$ and every point $(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_m) \in \mathbb{C}^{m-1}$ the map

$$\begin{array}{rcl} \overbrace{\{z \in \mathbb{C} \mid (w_1, \ldots, w_{i-1}, z, w_{i+1}, \ldots, w_m) \in U\}}^{\text{by the Tube Lemma 7.12 this is an open subset of } \mathbb{C}} & \to & \mathbb{C} \\ z & \mapsto & f_j(w_1, \ldots, w_{i-1}, z, w_{i+1}, \ldots, w_m) \end{array}$$

is holomorphic.

(2) We say that $f\colon U \to f(U)$ is a biholomorphism if f is holomorphic, if f is a bijection, if $f(U)$ is an open subset of $\mathbb{C}^n$ and if $f^{-1}\colon f(U) \to U$ is also holomorphic.

(3) Let $P \in U$. We say $f\colon U \to \mathbb{C}^m$ is a local biholomorphism at P if there exists an open neighborhood V of P and an open neighborhood W of $f(P)$ such that $f\colon V \to W$ is a biholomorphism.

Remark. Note that it follows immediately from the Smooth-Complex Differentiability Proposition 39.5 that every holomorphic map from an open subset $U \subseteq \mathbb{C}^m = \mathbb{R}^{2m}$ to $\mathbb{C}^n = \mathbb{R}^{2n}$ is in particular smooth. □

[453] Let $U \subseteq \mathbb{C}$. It follows easily from the definition of $\overline{\mathbb{C}}$ that $\infty \in \overline{U}$ if and only if U is unbounded.

[454] On page 152 we mentioned that $\overline{\mathbb{C}}$ is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 4.23 that a Jordan curve is the same as an injective map $S^1 \to \overline{\mathbb{C}}$.

[455] Constantin Carathéodory (1873-1950) was a Greek mathematician who was in particular a professor at the LMU in Munich.

Proposition 39.22. (Smooth-Complex Differentiability Proposition II) In the following let $U \subseteq \mathbb{C}^m$ be an open subset and let $f = U \to \mathbb{C}^n$ be a continuous map. The following two statements are equivalent:

(1) The map $f\colon U \to \mathbb{C}^n$ is holomorphic.
(2) The map $f\colon U \to \mathbb{C}^n$, viewed as a map from an open subset of $\mathbb{C}^m = \mathbb{R}^{2m}$ to $\mathbb{C}^n = \mathbb{R}^{2n}$ is smooth and for every $P \in U$ the differential $\mathrm{D}f_P\colon \mathbb{C}^m = \mathbb{R}^{2m} \to \mathbb{C}^n = \mathbb{R}^{2n}$ is a complex linear map.

Proof. The proposition can be deduced quite easily from the Smooth-Complex Differentiability Proposition 39.5. We leave it to the reader to fill in the details. ■

The following theorem is the holomorphic analog of the Inverse Function Theorem 24.21.

Theorem 39.23. (Holomorphic Inverse Function Theorem) Let $U \subseteq \mathbb{C}^n$ be an open subset and let $f\colon U \to \mathbb{C}^n$ be a holomorphic map. Furthermore let $P \in U$ be a point. If $\mathrm{D}f_P$ is invertible, then f is a local biholomorphism at P.

Example. The Holomorphic Inverse Function Theorem 39.23 is already interesting and non-trivial if $n = 1$. For example, let $k \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & z^k. \end{array}$$

Note that for every $z \in \mathbb{C} \setminus \{0\}$ we have $f'(z) = k \cdot z^{k-1} \neq 0$. Therefore it follows from the Smooth-Complex Differentiability Proposition 39.5 and the Holomorphic Inverse Function Theorem 39.23 that the above map $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a local biholomorphism at every point in $\mathbb{C} \setminus \{0\}$. □

Sketch of proof. First note that by the Inverse Function Theorem 24.21 we know that there exists an open neighborhood V of $P \in U \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$ and an open neighborhood W of $f(P) \in \mathbb{C}^n = \mathbb{R}^{2n}$ such that $f\colon V \to W$ is a bijection and such that $f^{-1}\colon W \to V$ is smooth. Next note that it follows from the Chain Rule 24.17 that for each $Q \in W$ we have $\mathrm{D}(f^{-1})_P = (\mathrm{D}f_{f^{-1}(Q)})^{-1}$. Since f is holomorphic we know by the Smooth-Complex Differentiability Proposition 39.22 that $\mathrm{D}f_{f^{-1}(Q)}\colon \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{C}^n = \mathbb{R}^{2n}$ is a complex linear map. Thus $\mathrm{D}(f^{-1})_P$ is also a complex linear map. It follows from the Smooth-Complex Differentiability Proposition 39.22 that $f^{-1}\colon W \to V$ is holomorphic. ■

39.4. Exercises for Chapter 39.

Exercise 39.1. Use the Power Series Expansion Theorem 39.9 (1) to show that every bounded holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is constant.

Exercise 39.2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic non-constant function. Use Liouville's Theorem 39.10 to show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Exercise 39.3. Use Liouville's Theorem 39.10 to show that every complex polynomial of degree ≥ 1 has a zero.

Exercise 39.4. Let $U \subseteq \mathbb{C}$ be an open connected subset and let $f, g\colon U \to \mathbb{C}$ be two holomorphic functions. Suppose that there exists a $z_0 \in U$ such that $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \in \mathbb{N}_0$. Use the Locally-a-Power Theorem 39.12 to show that $f = g$ on all of U.

Exercise 39.5. Let U be an open subset of $\mathbb{C}$, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $V \subseteq U$ be a connected open subset such that $f|_V$ is not constant. Show that $f(V)$ is an open subset of $\mathbb{C}$.
Hint. Use the Locally-a-Power Theorem 39.12.

Exercise 39.6. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Show that if f is injective, then $f\colon U \to f(U)$ is a biholomorphism.
Hint. Use the Locally-a-Power Theorem 39.12 and use the Inverse Function Theorem 24.21.

Exercise 39.7. Let U be an open connected subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Show that if the function $|f|\colon U \to \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.
Hint. Use the Non-Constant+Holomorphic–Open Image Proposition 39.13.

Exercise 39.8. Let $U \subseteq \mathbb{C}^n$ be an open subset and let $f\colon U \to \mathbb{C}^m$ be a smooth map. Show that f is holomorphic if and only if for every $P \in U$ the differential $\mathrm{D}f_P$ lies in $\mathrm{M}(m \times n, \mathbb{C}) \subseteq \mathrm{M}(2m \times 2n, \mathbb{R})$.

Exercise 39.9. Let $U \subseteq \mathbb{C}^k$ and $V \subseteq \mathbb{C}^l$ be open subsets. Furthermore let $f\colon U \to \mathbb{C}^l$ and $g\colon V \to \mathbb{C}^m$ be two holomorphic maps such that $f(U) \subseteq V$. Show that the composition $g \circ f\colon U \to \mathbb{C}^m$ is also holomorphic.
Hint. Make use of Exercise 39.8 and of the Chain Rule 24.17.

Exercise 39.10. Let $N = (0,0,1)$ be the North Pole of S^2 and let $S = (0,0,-1)$ be the South Pole of S^2. We consider the maps

$$\begin{array}{rcl} \alpha_N\colon S^2 \setminus \{N\} & \to & \mathbb{C} \\ (x,y,z) & \mapsto & \frac{x+\mathrm{i}y}{1-z} \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_S\colon S^2 \setminus \{S\} & \to & \mathbb{C} \\ (x,y,z) & \mapsto & \frac{x-\mathrm{i}y}{1+z}. \end{array}$$

We consider the map

$$\begin{array}{rcl} f\colon S^2 & \to & \mathbb{C}\mathrm{P}^1 \\ (x,y,z) & \mapsto & \begin{cases} [\alpha_N(x,y,z):1], & \text{if } z \neq 1, \\ [\alpha_S(x,y,z):1], & \text{if } z \neq -1. \end{cases} \end{array}$$

(a) Show that the map $f\colon S^2 \to \mathbb{C}\mathrm{P}^1$ is actually well-defined.
(b) Show that the map $f\colon S^2 \to \mathbb{C}\mathrm{P}^1$ is an orientation-reversing diffeomorphism.
Remark. This statement is just a special case of Exercise 31.8.

CHAPTER 40

Complex manifolds

In this chapter we will give the definition of complex manifolds and we discuss a few basic examples. In particular we will see that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ are complex manifolds. These will play a major role later on.

This chapter is organized as follows:

- In Section 40.1 we will introduce complex manifolds and we will consider basic examples, like the complex projective spaces.
- In Section 40.2 we will see that one can view tangent spaces of complex manifolds as complex vector spaces.
- In Section 40.3 we will use the results from Section 40.2 to show that complex manifolds come with a natural orientation.
- In Section 40.4 we will provide exercises for Chapter 40.

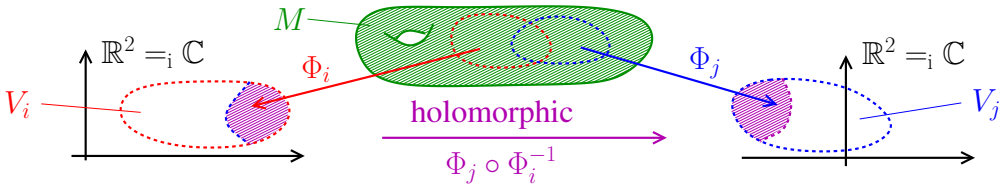
40.1. **Definition of complex manifolds and basic examples.** Given a topological manifold we now introduce the notion of a holomorphic atlas and of a complex manifold. The definition is almost the same as for a smooth atlas and a smooth manifold that we gave in Definition 25.2, except that we have to replace "smooth" by "holomorphic".

Definition 40.1.

(1) Let M be a $2n$-dimensional topological manifold M. An atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ of charts of type (i), i.e. of charts where each V_i is an open subset of $\mathbb{R}^{2n}$, for M is called holomorphic if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1} \colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open in } \mathbb{R}^{2n}=\mathbb{C}^n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subseteq \mathbb{R}^{2n}=\mathbb{C}^n}$$

is holomorphic.



Definition 40.2.

(2) We define the maximal holomorphic atlas of a complex manifold the same way as we defined the maximal smooth atlas of a smooth manifold, see Definition 25.3. Note that the argument of the Maximal Atlas Existence Proposition 25.4 shows that every holomorphic atlas is contained in a unique maximal holomorphic atlas.

(3) An n-dimensional complex manifold is a pair $(M, \mathcal{A})$ consisting of a $2n$-dimensional topological manifold M, together with a maximal holomorphic atlas for M.

Remark.

(1) As we mentioned above, by the Smooth-Complex Differentiability Proposition 39.5 we know that every holomorphic map from an open subset $U \subseteq \mathbb{C}^m$ to $\mathbb{C}^n$ is in particular smooth. It follows now easily that every holomorphic atlas is also a smooth atlas. In particular we see that an n-dimensional complex manifold is in particular a $2n$-dimensional smooth manifold.

(2) If M is a complex manifold, then every point admits a chart of type (i) from a smooth atlas, which according to Definitions 23.2 and Definition 25.10 means, that the boundary ∂M is empty. In particular a compact complex manifold is also closed in the sense of Definition 23.2 and Definition 25.10. □

Example. Let U be an open subset of $\mathbb{C}^n$. The identity map is an atlas for U and it is evidently a holomorphic atlas. This discussion shows that we can view U as an n-dimensional complex manifold. □

We move on straight to another set of closely related definitions, which are the holomorphic analogs of the smooth Definition 25.15 and Definition 25.5.

Definition 40.3.

(1) We say that a map $f\colon M \to N$ between two complex manifolds is holomorphic if f is continuous and if for every chart $\Phi\colon U \to V$ in the holomorphic atlas of M and every chart $\Psi\colon W \to Y$ in the holomorphic atlas of N the map
$$\Phi(f^{-1}(W) \cap U) \xrightarrow{\Psi \circ f \circ \Phi^{-1}} Y$$
is holomorphic.

(2) We say that a map $f\colon M \to N$ between two complex manifolds is a biholomorphism if f is a bijection and if both f and f^{-1} are holomorphic.

(3) We say that a map $f\colon M \to N$ between two complex manifolds is a local biholomorphism if given any $P \in M$ there exists an open neighborhood U of P and an open neighborhood V of $f(P)$ such that $f\colon U \to V$ is a biholomorphism.

(4) We say that two complex atlases for a given topological manifold are equivalent, if the identity map is a biholomorphism between the two complex manifolds corresponding to the complex atlases.

(5) A complex structure on a topological manifold is a choice of an equivalence class of holomorphic atlases.

The next proposition gives us an example of a complex manifold that will play a major role later on.

Proposition 40.4. ($\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition) Let $n \in \mathbb{N}$. In Definition 9.11 we introduced the complex projective space
$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

In the Projective Space–Manifold Proposition 25.13 we introduced for $i = 0, \dots, n$ the map[456]

$$\begin{aligned} \Phi_i \colon \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\} &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\tfrac{z_0}{z_i}, \dots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \dots, \tfrac{z_n}{z_i}\right). \end{aligned}$$

These maps form a holomorphic atlas for $\mathbb{C}\mathrm{P}^n$ that turn the complex projective space $\mathbb{C}\mathrm{P}^n$ into a compact n-dimensional complex manifold.

Convention 40.5. We refer to the complex structure on $\mathbb{C}\mathrm{P}^n$ that we just introduced as the standard complex structure on $\mathbb{C}\mathrm{P}^n$.

Remark. In the Grassmannian–Manifold Proposition 270.3 we will generalize the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 from projective spaces to "Grassmannians". □

Proof. In the proof of the Projective Space–Manifold Proposition 25.13 we already showed that $\mathbb{C}\mathrm{P}^n$ is a topological manifold and that the given charts form a smooth atlas. It remains to prove the following claim:

Claim. The atlas is holomorphic.

Proof. First note that for $i \in \{0, \dots, n\}$ the inverse of the chart Φ_i is given by the map

$$\begin{aligned} \Phi_i^{-1} \colon \mathbb{C}^n &\to \{[w_0 : \cdots : w_n] \in \mathbb{C}\mathrm{P}^n \mid w_i \neq 0\} \\ (w_1, \dots, w_n) &\mapsto [w_1 : \cdots : w_i : 1 : w_{i+1} : \cdots : w_n]. \end{aligned}$$

Now let $i, j \in \{0, \dots, n\}$. By symmetry it suffices to consider the case $i < j$. Using the above calculation of Φ_i^{-1} one can easily show that the corresponding transition map is given by

$$\begin{aligned} \Phi_j \circ \Phi_i^{-1} \colon \{(w_1, \dots, w_n) \in \mathbb{C}^n \mid w_j \neq 0\} &\to \{(w_1, \dots, w_n) \mid w_{i+1} \neq 0\} \\ (w_1, \dots, w_n) &\mapsto \left(\tfrac{w_1}{w_j}, \dots, \tfrac{w_i}{w_j}, \tfrac{1}{w_j}, \tfrac{w_{i+1}}{w_j}, \dots, \tfrac{w_{j-1}}{w_j}, \tfrac{w_{j+1}}{w_j}, \dots, \tfrac{w_n}{w_j}\right). \end{aligned}$$

This map is evidently a biholomorphism. We have thus shown that the given continuous maps $\Phi_0, \dots, \Phi_n$ form a holomorphic atlas for $\mathbb{C}\mathrm{P}^n$. ■

In Definition 25.38 we introduced the notion of a smooth action on a smooth manifold. Not surprisingly we have the following analog in our present context.

Definition 40.6. Let M be a complex manifold which is equipped with an action by a group G. We say that the action is holomorphic if for every $g \in G$ the map $M \to M$ given by $x \mapsto g \cdot x$ is holomorphic.

The following proposition is the complex analog of the Manifold–Quotient Proposition 25.39.

Proposition 40.7. (Complex Manifold–Quotient Proposition) If a group G acts freely, properly[457] and holomorphically on an n-dimensional complex manifold M, then M/G is a $2n$-dimensional topological manifold and there exists a unique complex structure on M/G such that the projection $p \colon M \to M/G$ is a local biholomorphism.

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[455] It follows in particular that $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ admits a complex structure. On page 966 we will discuss the other spheres.

[456] We use the Notation 9.12, i.e. given a point $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ we denote by $[z_0 : \cdots : z_n]$ the corresponding point in $\mathbb{C}\mathrm{P}^n$.

[457] We refer to Definition 25.37 for the definition of a free action and a proper action.

Proof. The proof of the proposition is verbatim the same as the proof of the corresponding statements in the Manifold–Quotient Proposition 25.39 for smooth manifolds. We leave it to the reader to fill in the details. ■

Convention 40.8. In the setting of the Complex Manifold–Quotient Proposition 40.7 we will refer to the complex structure on M/G as the standard complex structure on M/G.

Example. We consider the action

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{C}^n \setminus \{0\} & \to & \mathbb{C}^n \setminus \{0\} \\ (k, (z_1, \ldots, z_n)) & \mapsto & (2^k \cdot z_1, \ldots, 2^k \cdot z_n). \end{array}$$

One can easily show that this action is free, proper and holomorphic. Thus we see that the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ has a standard complex structure. Furthermore it is fairly elementary to verify that the map

$$\begin{array}{rcl} (\mathbb{C}^n \setminus \{0\})/\mathbb{Z} & \to & S^{2n-1} \times S^1 \\ [z] & \mapsto & \left(\frac{z}{\|z\|}, \exp(2\pi \mathrm{i} \cdot \ln_2(\|z\|))\right) \end{array}$$

is a diffeomorphism. Thus in summary we have shown that $S^{2n-1} \times S^1$ admits a complex structure. This complex manifold is called the *Hopf manifold* [**Hopf1948**]. □

We formulate our next example as a lemma.

Lemma 40.9. (Complex Torus Lemma) Let $v, w \in \mathbb{C}$ be two complex numbers that are linearly independent over $\mathbb{R}$.

(1) The action
$$\begin{array}{rcl} (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) \times \mathbb{C} & \to & \mathbb{C} \\ ((m \cdot v + n \cdot w), z) & \mapsto & z + m \cdot v + n \cdot w \end{array}$$
is free, proper and holomorphic.

(2) The quotient $T(v, w) := \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w)$ is a 2-dimensional topological manifold and it admits a unique complex structure which has the property that the natural projection $\mathbb{C} \to T(v, w) = \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w)$ is a local biholomorphism.

(3) The complex manifold $T(v, w) = \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w)$ is compact and, viewed as a smooth manifold, it is diffeomorphic to $S^1 \times S^1$.

(4) Let $\widetilde{v}, \widetilde{w}$ be two complex numbers that are linearly independent over $\mathbb{R}$. If there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) = \mathbb{Z} \cdot \widetilde{v} + \mathbb{Z} \cdot \widetilde{w}$, then the complex manifolds $T(v, w)$ and $T(\widetilde{v}, \widetilde{w})$ are biholomorphic.

Remark. In the Complex Tori–Classification Proposition 129.2 we will see that the converse to the Complex Torus Lemma 40.9 (4) holds. More precisely, we will see that if $T(v, w)$ and $T(\widetilde{v}, \widetilde{w})$ are biholomorphic, then there exists an $r \in \mathbb{C} \setminus \{0\}$ such that we have the equality $r \cdot (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) = \mathbb{Z} \cdot \widetilde{v} + \mathbb{Z} \cdot \widetilde{w}$. □

Proof.

(1) In Exercise 25.32 we showed that the action is proper. It is clear that the action is free and holomorphic.

(2) This statement follows from (1) together with the Complex Manifold–Quotient Proposition 40.7.

(3) We consider the map
$$\begin{array}{rcl} \mathbb{C}/(\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) & \to & S^1 \times S^1 \\ [r \cdot v + s \cdot w] & \mapsto & (\exp(2\pi \mathrm{i} r), \exp(2\pi \mathrm{i} s)). \end{array}$$

It follows from the Smooth Map-Descends-to-Quotient Corollary 25.41 that the map is a local diffeomorphism. Since v and w are linearly independent one sees that the map is a bijection. It follows from the Smooth Pasting Proposition 25.19 (2) that the map is in fact a diffeomorphism. Since $S^1 \times S^1$ is compact we see that $T(v, w)$ is also compact.

(4) Let $\widetilde{v}, \widetilde{w}$ be two complex numbers that are linearly independent over $\mathbb{R}$. We assume that there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot (\mathbb{Z} \cdot v + \mathbb{Z} \cdot w) = \mathbb{Z} \cdot \widetilde{v} + \mathbb{Z} \cdot \widetilde{w}$. In this case we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto r \cdot z} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ T(v, w) & \xrightarrow{[z] \mapsto [r \cdot z]} & T(\widetilde{v}, \widetilde{w}). \end{array}$$

Note that our hypothesis shows that bottom horizontal map is defined and a bijection. The top horizontal map is of course a biholomorphism. Since the vertical maps are local biholomorphisms we see that the bottom horizontal map is a bijection and a local biholomorphism. In other words, it is a biholomorphism. ■

We continue our discussion of complex manifolds with a pleasing proposition.

Proposition 40.10. (Holomorphic-on-Compact Proposition) Let M be a compact 1-dimensional complex manifold. Then any holomorphic function $f \colon M \to \mathbb{C}$ is locally constant, in particular it is constant if M is connected.

Proof. We provide the proof of the proposition in Exercise 40.1. ■

Remark. If $M = T(v, w) = \mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w)$, then we can also give another, arguably easier proof of the conclusion of the Holomorphic-on-Compact Proposition 40.10. Indeed, let $f \colon \mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w) \to \mathbb{C}$ be a holomorphic function. Since $\mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w) \cong S^1 \times S^1$ is compact we obtain from the Compact Image Lemma 4.15 that the function f is bounded. As usual we denote by $p \colon \mathbb{C} \to \mathbb{C}/(\mathbb{Z} \cdot v \oplus \mathbb{Z} \cdot w)$ the projection. Recall that p is holomorphic. Thus the composition $f \circ p \colon \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function. We deduce from the Liouville Theorem 39.10 that $f \circ p$ is constant. Since p is surjective it follows that f is also constant. □

Next we introduce the notion of a submanifold of a complex manifold. The definition is almost verbatim the same as the definition of a submanifold of a smooth manifold in Definition 25.22. The only difference is that we do not need to worry about boundary points.

Definition 40.11. Let M be an n-dimensional complex manifold.

(1) We say that a subset $N \subseteq M$ is a *k-dimensional complex submanifold* if given any $P \in N$ there exists a chart $\Phi \colon U \to V$ in the maximal complex atlas for M and P such that
$$\Phi(U \cap N) \;=\; V \cap \{(0, \ldots, 0, z_1, \ldots, z_k) \,|\, z_i \in \mathbb{C}\},$$
Any collection of charts covering N is called a *holomorphic submanifold atlas*.

(2) We say that a submanifold N is *proper* if N is a closed subset of M.

Example. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. Using a mild modification of the charts from the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 one can easily show that

$$\{[z_0 : \cdots : z_k : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^n \,|\, (z_0, \dots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is a submanifold of the complex manifold $\mathbb{C}\mathrm{P}^n$. □

The following lemma is an analog of the Submanifold Smooth Structure Lemma 25.23.

Lemma 40.12. (Submanifold Complex Structure Lemma) Let M be a complex manifold and let N be a complex submanifold of M. The collection of all submanifold charts of M gives rise naturally to a holomorphic atlas for N which we refer to as the standard complex structure on the submanifold. The standard complex structure has the following properties:

(1) The inclusion map $N \to M$ is a holomorphic map.
(2) If we start out with some submanifold atlas for M, then this gives rise to the same complex structure on N.
(3) Let $f\colon W \to M$ be a holomorphic map. If $f(W) \subseteq N$, then $f\colon W \to N$ is also holomorphic.

Example. Let $k \leq n$. Using the charts in the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 one can easily show that $\{[(x, 0)] \in \mathbb{C}\mathrm{P}^n \,|\, x \in \mathbb{C}^{k+1} \setminus \{0\}\}$ is a k-dimensional complex submanifold of $\mathbb{C}\mathrm{P}^n$ which is holomorphic to $\mathbb{C}\mathrm{P}^k$. □

Proof. The proof is basically identical to the proof of the Submanifold Smooth Structure Lemma 25.23. We leave it to the reader to make the few necessary modifications. ■

The following theorem is the complex analog of the Regular Value Theorem 32.5. As in the real case, it is a machine for providing examples of complex (sub-) manifolds.

Theorem 40.13. (Complex Regular Value Theorem) Let M be an m-dimensional complex manifold and let N be an n-dimensional complex manifold. Furthermore let $f\colon M \to N$ be a holomorphic map and let $s \in N$ be a regular value of f.

(1) The preimage $X \coloneqq f^{-1}(s)$ is an $(m-n)$-dimensional complex submanifold of M.
(2) The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.

Proof. The proof is almost identical to the proof of the "real" Regular Value Theorem 32.5. The main difference is that we need to replace the Inverse Function Theorem 24.21 by the Holomorphic Inverse Function Theorem 39.23. We refer to [**Huy2005**, p. 11 and p. 58] for some more details. ■

Examples.

(1) Let $n \in \mathbb{N}$. By basically the same argument as in the proof of the Matrix Group-Smooth Manifold Proposition 32.8 one can show that

$$\begin{aligned} \mathrm{M}(n \times n, \mathbb{C}) &\to \mathbb{C} \\ A &\mapsto \det(A) \end{aligned}$$

is holomorphic and that $1 \in \mathbb{C}$ is a regular value. It follows from the Complex Regular Value Theorem 40.13 that $\mathrm{SL}(n, \mathbb{C})$ is an $(n^2 - 1)$-dimensional complex submanifold of

$\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2}$. In particular by the Submanifold Complex Structure Lemma 40.12 we can view $\mathrm{SL}(n, \mathbb{C})$ as an $(n^2 - 1)$-dimensional complex submanifold.

(2) Let $n \in \mathbb{Z}$. In Exercise 40.8 we will use the Complex Regular Value Theorem 40.13 to show that $\{[x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$ is a compact 1-dimensional complex manifold. □

40.2. **Tangent spaces of complex manifolds.** Given a smooth manifold M and given $P \in M$ we introduced in Definition 29.11 the tangent space $\mathrm{T}_P M$. In the Tangent Space-is-Vector Space Lemma 29.12 we showed that $\mathrm{T}_P M$ is a real vector space. We will now see that for a complex manifold the tangent space $\mathrm{T}_P M$ can actually be viewed as a complex vector space:

Lemma 40.14. (Complex Manifold-Tangent Space Lemma)

(1) Let M be an n-dimensional complex manifold. Given any $P \in M$ we can use charts from the holomorphic atlas to equip $\mathrm{T}_P M$ naturally with the structure of a complex n-dimensional vector space.

(2) If $f \colon M \to N$ is a holomorphic map between two complex manifolds, then for each $P \in M$ the differential $\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is a complex linear map.

Sketch of a proof.

(1) Let M be an n-dimensional complex manifold and let $P \in M$. We pick a chart $\Phi \colon U \to V$ from the holomorphic atlas around P. We get an induced isomorphism $\mathrm{D}\Phi_P \colon \mathrm{T}_P M \to \mathrm{T}_P V = \mathbb{C}^n$. We use this isomorphism to equip $\mathrm{T}_P M$ with the structure of a complex vector space. Since Φ comes from a holomorphic atlas one obtains easily from the Smooth-Complex Differentiability Proposition 39.22 , see Exercise 39.8, that this complex vector space structure does not depend on the choice of the chart Φ.

(2) This statement follows fairly easily from the definitions and the Smooth-Complex Differentiability Proposition 39.22. We outsource the details to Exercise 40.7. ■

For completeness sake we provide the following analog of the Tangent Space-Functor Proposition 29.15:

Proposition 40.15. (Complex Tangent Space-Functor Proposition) We refer to the category $\mathcal{CplxMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{CplxMfd}) &:= \text{all complex manifolds,} \\ \mathrm{Mor}(X, Y) &:= \text{all holomorphic maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the category of complex manifolds. Furthermore we define the category $\mathcal{PCplxMfd}$ of pointed complex manifolds in the obvious way. The maps

$$\begin{aligned} (M, P) &\mapsto \mathrm{T}_P M \\ (f \colon (M, P) \to (N, Q)) &\mapsto (\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathrm{T}_Q N) \end{aligned}$$

define a covariant functor from the category $\mathcal{PCplxMfd}$ of pointed complex manifolds to the category $\mathcal{Vec}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces.

Proof. The proposition follows immediately from the Tangent Space-Functor Proposition 29.15 together with the Complex Manifold-Tangent Space Lemma 40.14. ■

40.3. **The natural orientation of a complex manifold.** In this section we will see that complex manifolds admit a natural orientation.

Lemma 40.16. (Complex Vector Space-Orientation Lemma)

(1) Let V be an n-dimensional complex vector space.
   (a) If $(v_1, \dots, v_n)$ is a basis of V viewed as a complex vector space, then the set of vectors $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ is a basis of V viewed as a real vector space.
   (b) If $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are two bases of V as a complex vector space, then the bases $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ and $(w_1, \mathrm{i}w_1, \dots, w_n, \mathrm{i}w_n)$ are equivalent in the sense from Definition 31.3.

(2) Let $f\colon V \to W$ be an isomorphism of n-dimensional complex vector spaces. If we equip V and W, viewed as real vector spaces, with the orientation from (1), then $f\colon V \to W$ is an orientation-preserving isomorphism.

Definition 40.17. Given a finite-dimensional complex vector space we refer to the orientation introduced in the Complex Vector Space-Orientation Lemma 40.16 (1) as the natural orientation.

Example. Let $n \in \mathbb{N}_0$. If we make the usual identification $\mathbb{C}^n = (\mathbb{R}^2)^n = \mathbb{R}^{2n}$, then the corresponding natural orientation on $\mathbb{C}^n = \mathbb{R}^{2n}$ agrees with the standard orientation. □

Proof.

(1) Let $(v_1, \dots, v_n)$ be a basis of V as a complex vector space. One can easily verify that $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ is a generating set over $\mathbb{R}$ and that it is linearly independent over $\mathbb{R}$. This concludes the proof of (a).

We turn to the proof of (b). Let $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ be two bases of V as a complex vector space. We consider the bases

$$X \coloneqq (v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n) \quad \text{and} \quad Y \coloneqq (w_1, \mathrm{i}w_1, \dots, w_n, \mathrm{i}w_n).$$

It remains to prove the following claim:

Claim. We have $\det([\mathrm{id}]_X^Y) > 0$.

Proof. There are two approaches: One can either make use of Exercise 4.71, or one performs a direct calculation. We now do the latter: We denote by $A \in \mathrm{GL}(n\times n, \mathbb{C})$ the transition matrix from Definition 31.1. We write $A = B + \mathrm{i}C$, where $B, C \in \mathrm{M}(n\times n, \mathbb{R})$. We consider the bases

$$\widetilde{X} \coloneqq (v_1, \dots, v_n, \mathrm{i}v_1, \dots, \mathrm{i}v_n) \quad \text{and} \quad \widetilde{Y} \coloneqq (w_1, \dots, w_n, \mathrm{i}w_1, \dots, \mathrm{i}w_n).$$

We now calculate the determinant of the transition matrix $[\mathrm{id}]_X^Y$ from X to Y:

$$\begin{aligned}
\det([\mathrm{id}]_X^Y) &\overset{\text{by the Transition Matrix Lemma 31.2 we have } [\mathrm{id}]_A^C = [\mathrm{id}]_B^C \cdot [\mathrm{id}]_{A,B}}{=} \det([\mathrm{id}]_{\widetilde{Y}}^{Y}) \cdot \det([\mathrm{id}]_{\widetilde{X}}^{\widetilde{Y}}) \cdot \det([\mathrm{id}]_X^{\widetilde{X}}) \\
&\overset{\text{since clearly } [\mathrm{id}]_{\widetilde{X}}^{X} = [\mathrm{id}]_{\widetilde{Y}}^{Y} \text{ and since } [\mathrm{id}]_{\widetilde{Y}}^{Y} = ([\mathrm{id}]_{\widetilde{Y}}^{\widetilde{Y}})^{-1}}{=} \det([\mathrm{id}]_{\widetilde{X}}^{\widetilde{Y}}) \\
&\overset{\text{by an elementary argument}}{=} \det\begin{pmatrix} B & -C \\ C & B \end{pmatrix} \overset{\text{add } i \text{ times bottom to the top}}{=} \det\begin{pmatrix} B+\mathrm{i}C & -C+\mathrm{i}B \\ C & B \end{pmatrix} \\
&\overset{\text{subtract } i \text{ times left from the right}}{=} \det\begin{pmatrix} B+\mathrm{i}C & 0 \\ C & B-\mathrm{i}C \end{pmatrix} \\
&= \det(B+\mathrm{i}C) \cdot \det(B-\mathrm{i}C) = \det(B+\mathrm{i}C) \cdot \overline{\det(B+\mathrm{i}C)} \\
&= |\det(B+\mathrm{i}C)|^2 > 0.
\end{aligned}$$

⊞

(2) This statement follows immediately from (1). ■

The following proposition says that complex manifolds admit a natural orientation, in particular they are orientable.

Proposition 40.18. (Complex Manifold-Orientation Proposition)

(1) Let M be an n-dimensional complex manifold. By the Complex Manifold-Tangent Space Lemma 40.14 we can view each tangent space $\mathrm{T}_P M$ as a complex n-dimensional vector space. If we equip each complex vector space $\mathrm{T}_P M$ with the corresponding natural orientation, then these orientations define an orientation on M, viewed as a $2n$-dimensional real smooth manifold. We refer to this orientation as the natural orientation of the complex manifold M.

(2) If $f\colon M \to N$ is a local biholomorphism between two complex manifolds, then f is an orientation-preserving diffeomorphism between M and N, viewed as oriented smooth manifolds.

Sketch of a proof.

(1) This statement follows easily from the Complex Manifold-Tangent Space Lemma 40.14 (1) and the Complex Vector Space-Orientation Lemma 40.16 (1). We leave it to the reader to fill in the details.

(2) This statement follows easily from the Complex Manifold-Tangent Space Lemma 40.14 (2) and the Complex Vector Space-Orientation Lemma 40.16 (2). ■

The Complex Manifold-Orientation Proposition 40.18 motivates the following definition:

Definition 40.19. Let M be an n-dimensional complex manifold. We refer to the orientation on the $2n$-dimensional smooth manifold M that is provided by the Complex Manifold-Orientation Proposition 40.18 as the standard orientation.

Example. In the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we equipped $\mathbb{C}\mathrm{P}^n$ with a complex structure. We leave it to the reader to show that the corresponding orientation on the $2n$-dimensional smooth manifold $\mathbb{C}\mathrm{P}^n$ equals the standard orientation on $\mathbb{C}\mathrm{P}^n$ which we introduced on page 822. □

We continue with the following corollary to the Complex Manifold-Orientation Proposition 40.18.

Corollary 40.20. (No Complex Structure Corollary) The Klein bottle and the real projective plane $\mathbb{R}\mathrm{P}^2$ do not admit the structure of a complex manifold.

Proof. In the Surface Non-Orientability Lemma 31.45 we showed that neither the Klein bottle nor the real projective plane $\mathbb{R}\mathrm{P}^2$ are orientable. This is in contrast to the fact, shown in the Complex Manifold-Orientation Proposition 40.18, that complex manifolds are orientable. ■

As we had already pointed out on page 340, the one-dimensional projective space $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2. This shows that S^2 can be viewed as a 1-dimensional complex manifold. We had just seen that the 2-sphere and the torus are complex manifolds and we saw that non-orientable smooth manifolds cannot be complex manifolds. The following question now arises:

Question 40.21. Let $g \geq 2$. Does the surface of genus g have the structure of a complex manifold?

We will discuss this question in the Surface–Möbius Structure Proposition 132.4.

Remark. It is often very hard to determine whether or not a given orientable even-dimensional smooth manifold admits that a complex structure. By the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we know that $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ admits a complex structure. It is natural to ask whether other even-dimensional spheres admit complex structure. Armand Borel and Jean-Pierre Serre [**BoS1953**, Proposition 15.1][458] showed in 1953 that for $n \neq 2, 6$ the n-dimensional sphere S^n cannot be a complex manifold. Furthermore it is shown in [**Stee1951**, Corollary 41.20] that the spheres S^{4n} with $n \geq 1$ do not admit a complex structure (see also [**Hsi1995**, p. 233]). But it is still an open question whether S^6 admits a complex structure. □

40.4. Exercises for Chapter 40.

Exercise 40.1. Let M be a compact connected 1-dimensional complex manifold. Show that any holomorphic function $f\colon M \to \mathbb{C}^n$ is constant.
Hint. Use the Maximum Principle, i.e. use Theorem 39.14.

Exercise 40.2. Let $q(z)$ be a non-constant complex polynomial. Show that the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^1 \\ [z:w] & \mapsto & \begin{cases} [q(a):1], & \text{if } [z:w]=[a:1] \text{ for some } a\in\mathbb{C}, \\ [1:0], & \text{if } [z:w]=[1:0] \end{cases} \end{array}$$

is holomorphic.
Remark. The discussion on page 843 shows, after a few translations, that the above map $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ is smooth.

Exercise 40.3. Let

$$0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

be a short exact sequence of finite-dimensional complex vector spaces. We equip each vector space with the natural orientation from Definition 40.17. Show that the orientations are compatible in the sense of Definition 31.11.

Exercise 40.4. For any oriented real vector space U we consider the isomorphism

$$\begin{array}{rccc} \Phi\colon & \underbrace{U\oplus U}_{\substack{\text{equipped with the}\\ \text{orientation from}\\ \text{Definition 31.13}}} & \to & \underbrace{U\otimes_{\mathbb{R}}\mathbb{C}}_{\substack{\text{equipped with the}\\ \text{orientation from}\\ \text{Definition 40.17}}} \\ & (u,u') & \mapsto & u+\mathrm{i}u'. \end{array}$$

Under which conditions is this isomorphism orientation-preserving?

Exercise 40.5.
(a) We view S^2 as a subspace of $\mathbb{C}\times\mathbb{R}=\mathbb{R}^3$. We consider the maps

$$\begin{array}{rcl} \Phi_+\colon S^2\setminus\{(0+0\mathrm{i},1)\} & \to & \mathbb{C} \\ (z,t) & \mapsto & \dfrac{z}{1-t} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Phi_-\colon S^2\setminus\{(0+0\mathrm{i},-1)\} & \to & \mathbb{C} \\ (z,t) & \mapsto & \dfrac{\overline{z}}{1+t}. \end{array}$$

[458] Armand Borel (1923–2003) was a Swiss mathematician, Jean-Pierre Serre (*1926) is a French mathematician who was awarded a Fields medal in 1954.

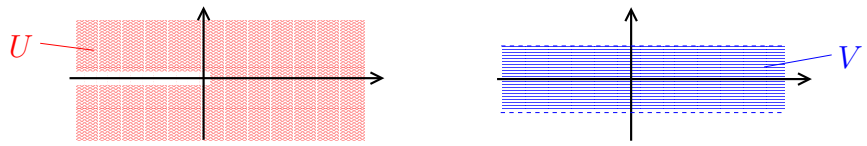
Note that Φ_+ is the stereographic projection from the North Pole $(0+0\mathrm{i}, 1)$ and that Φ_- is the stereographic projection the South Pole $(0+0\mathrm{i}, -1)$ composed with complex conjugation. In the Stereographic Projection Lemma 4.24 we showed that these two maps are homeomorphisms. Show that these two charts define a holomorphic atlas for S^2. We refer to this complex structure on S^2 as the *standard complex structure on* S^2.

(b) In the Stereographic Projection Lemma 4.24 and the Projective Space-Avatar Lemma 9.16 we gave an explicit homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$. Show that this map is a biholomorphism between the complex manifolds S^2 and $\mathbb{C}\mathrm{P}^1$.

Exercise 40.6. Show, without using the Riemann Mapping Theorem 39.18, that the complex manifolds

$$\begin{aligned} U &:= \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0 \text{ or } \mathrm{Im}(z) \neq 0\} \\ V &:= \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \in (-\pi, \pi)\} \end{aligned}$$

are biholomorphic.



Exercise 40.7. Let $f\colon M \to N$ be a holomorphic map between two complex manifolds. Show that for each $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is a complex linear map.

Exercise 40.8. Let $n \in \mathbb{Z}$. Show that $\{[x:y:z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$ is a compact 1-dimensional complex submanifold of $\mathbb{C}\mathrm{P}^2$.

Exercise 40.9. Let $n \in \mathbb{N}$. Let $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ be the complex projective space.

(a) Show that

$$M := \{([z], v) \in \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1} \,|\, v \in \mathbb{C} \cdot z\}$$

is a complex submanifold of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1}$.

(b) Show that the map

$$\begin{aligned} \pi\colon M &\to \mathbb{C}^{n+1} \\ ([z], v) &\mapsto v \end{aligned}$$

is holomorphic and show that $\pi\colon \pi^{-1}(\mathbb{C}^{n+1} \setminus \{0\}) \to \mathbb{C}^{n+1} \setminus \{0\}$ is a biholomorphism.

Remark. The complex manifold M is called the *blow up of* $\mathbb{C}^{n+1}$ *at the origin.*

Exercise 40.10. We refer to

$$\mathrm{K3} := \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,\middle|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\}$$

as the K3-*surface*. Use the Complex Regular Value Theorem 40.13 to show that the K3-surface is a closed 2-dimensional complex manifold.

Part VI

Differential topology

In the last part of the lecture notes we introduced topological and smooth manifolds and we introduced many of the essential notions and we formulated and proved many of the basic theorems. In this part of the lecture notes we intensify our study of smooth manifolds. As the name "differential topology" suggests, we are interested in the topological properties of smooth manifolds. We will start out with a long discussion of smooth homotopies on smooth manifolds. Afterwards we will introduce the degree of a smooth map between smooth manifolds. This notion has many applications, for example it allows us to prove the Hairy Ball Theorem 46.18 and the Borsuk-Ulam 47.12.

This part of the lecture notes is organized as follows:

- In Chapter 41 we will introduce the notion of smooth homotopies between smooth manifolds and we discuss several examples.
- In Chapter 42 we will classify the self-diffeomorphisms of $\mathbb{R}^n$ and S^1 up to diffeotopy.
- In Chapter 43 we will prove the Flow Theorem 43.3 which says that any smooth vector field gives rise to a diffeotopy. This theorem will play an essential role in the subsequent chapter.
- In Chapter 44 we will state and prove the Isotopy Extension Theorem 44.2 which is an essential tool in differential topology. Loosely speaking it says that any smooth isotopy of a proper smooth submanifold extends to a diffeotopy of the total smooth manifold.
- In Chapter 45 we will introduce the notion of orientation-preserving and orientation-reversing loops. We will use this concept to prove the Smooth Ball Embedding Theorem 45.10 which classifies smooth embeddings of codimension-zero balls into smooth manifolds, up to diffeotopy.
- In Chapter 46 we will introduce the notion of the degree $\deg(f)$ of a continuous map $f\colon M \to N$ between oriented smooth manifolds. We will use this concept to prove the Hairy Ball Theorem 46.18 and the Hopf Degree Theorem 46.22 which classifies continuous maps from a closed n-dimensional smooth manifold M to S^n, up to homotopy.
- In Chapter 47 we will use the concept of a degree to provide the first (of several) proofs of the famous Borsuk-Ulam Theorem 47.12 which says that given any continuous map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.
- In Chapter 48 we will discuss several applications of the Borsuk-Ulam Theorem 47.12. In particular we will prove the Ham-Sandwich Theorem 48.4, we will prove the combinatorial Kneser Conjecture 48.10 we will prove the purely algebraic Level Existence Theorem 48.16 and we will prove the fun Necklace Theorem 48.17.
- In Chapter 49 we will use the degree to define the rotation number of an immersion $S^1 \to \mathbb{R}^2$. Furthermore we will prove the Whitney-Graustein Theorem 49.11 which says that two immersions $S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if their rotation numbers agree. We will not make use of this result.
- In Chapter 50 we will prove the Submanifold Transversal Intersection Theorem 50.4 which says that the intersection of two transverse submanifolds is again a submanifold. More importantly we will also prove the Transversality Theorem 50.9 which says that we can always make two submanifolds transverse, via a smooth isotopy.
- In Chapter 51 we will introduce the algebraic intersection number of two oriented transverse submanifolds of complementary dimensions and we will study basic properties of the algebraic intersection number. We will use this concept to prove that there is no smooth embedding of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$.

- In Chapter 52 we will introduce the notion of a degree to introduce the linking number of two disjoint oriented knots in $\mathbb{R}^3$.
- In Chapter 53 we will use the algebraic intersection number to introduce the smooth Euler characteristic of a smooth manifold. In the Smooth Poincaré-Hopf Theorem 53.24 we will, given a smooth manifold, relate the smooth Euler characteristic to zeros of a vector field on M.
- Beginning with Chapter 54 we turn to more technical topics. In the Smooth Collar-Uniqueness Proposition 54.2 we will see that smooth collars are unique in a suitable sense. We will use this result to discuss what it means to glue smooth manifolds along a diffeomorphism of boundary components. In particular we will also consider the connected sum of two smooth manifolds and we will see that this operation is well-defined.
- In Chapter 55 we will study the reverse of the gluing operation on smooth manifolds, namely we will learn how to cut smooth manifolds along suitable codimension-one submanifolds.
- In the rather technical Chapter 56 we extend several earlier results to the context of proper submanifolds. Perhaps the most interesting result is that in the Weak Proper Whitney Embedding Theorem 56.7 we will see that every compact smooth manifold M admits a *proper* smooth embedding $M \to \mathbb{H}^{2\cdot\dim(M)+1}$.
- In Chapter 57 we will introduce the notion of a tubular neighborhood of a proper submanifold. In the Tubular Neighborhood–Existence Theorem 57.11 and the Tubular Neighborhood–Uniqueness Theorem 57.12 we will see that every compact proper submanifold admits an essentially unique tubular neighborhood.
- In Chapter 58 we will prove the Product Tubular Neighborhood 58.4 which says that in many settings a submanifold has actually a product tubular neighborhood.

CHAPTER 41

Smooth homotopies

In this chapter we will introduce the notion of smooth homotopies, smooth isotopies and diffeotopies. Furthermore, in this and the following chapter we will discuss some basic properties and many examples.

This chapter is organized as follows:

- In Section 41.1 we will introduce smooth homotopies.
- In Section 41.2 we will introduce smooth isotopies of (sub-) manifolds.
- In Section 41.3 we will give many examples of smooth homotopies.
- In Section 41.4 we will introduce homogeneous topological spaces and we will see that topological manifolds without boundary are homogeneous.
- In Appendix 41.5 we will state and prove the slightly technical Flattening Lemma 41.5.
- In Section 41.6 we will provide exercises for Chapter 41.

In the next Chapter 42 we will classify some diffeomorphisms up to diffeotopy.

41.1. **Smooth homotopies.** Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. Suppose we are given two smooth maps $f_0, f_1 \colon M \to N$. Recall that in Definition 18.5 we defined a homotopy between f_0 and f_1 to be a continuous map $F \colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$. On many occasions one feels much happier if one actually has a "smooth homotopy". To make sense of the notion of a smooth homotopy we need to recall some concepts we introduced earlier.

(1) If M is a smooth manifold with empty boundary and if $a < b$ are two real numbers, then by the Smooth Manifold Product Proposition 25.34 (1) we can equip $M \times [a,b]$ with a natural structure of an $(m+1)$-dimensional smooth manifold.
(2) More generally, if M is any smooth manifold, possibly with non-empty boundary, then by the Smooth Manifold Product Proposition 26.12 we can view $M \times [a,b]$ as an $(m+1)$-dimensional smooth manifold with corner $\angle(M \times [a,b]) = \partial M \times \{a,b\}$.

Either way, by Definitions 25.15 and 26.6 we now have a notion of a map $F \colon M \times [a,b] \to N$ to be smooth.

For people who happen to be allergic to smooth manifolds with corner, the following lemma often makes it possible to avoid dealing with the notion of smooth maps defined on smooth manifolds with corner.

Lemma 41.1. (Smooth Map Criterion) Let M and N be smooth manifolds and let $F \colon M \times [a,b] \to N$ be a continuous map.

(1) Let I be an interval with $[a,b] \subseteq I$. (We allow $I = [a,b]$.)
(2) Let $\widetilde{M}$ be a smooth manifold that contains M as a submanifold. (We allow $\widetilde{M} = M$.)

If there exists a smooth map[459] $G\colon \widetilde{M} \times I \to N$ such that the restriction of G to the subset $M \times [a,b]$ equals F, then $F\colon M \times [a,b] \to N$ is also smooth.

Proof. This lemma follows immediately from the Smooth Composition Lemma 26.7, the Product-of-Smooth Maps Lemma 26.13 and the fact that the inclusion of a submanifold with corner is smooth. ∎

After this preamble we now introduce the main protagonist of this chapter.

Definition 41.2. Let M and N be smooth manifolds.

(1) A smooth homotopy is a smooth map $F\colon M \times [0,1] \to N$.
(2) We say that two smooth maps $f_0, f_1\colon M \to N$ are smoothly homotopic if there exists a smooth homotopy $F\colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.[460]

There are various special types of smooth homotopies which we will now introduce.

Definition 41.3. Let M and N be smooth manifolds.

(1) A smooth isotopy of M in N is a smooth map
$$F\colon M \times [0,1] \;\to\; N$$
such that for each $t \in [0,1]$ the map $F_t\colon M \to N$ is a smooth embedding.[461] [462]
(2) We say that a smooth isotopy $F\colon M \times [0,1] \to N$ is proper if each F_t is a proper smooth embedding[463] *and* if the map $F\colon M \times [0,1] \to N$ is proper in the sense of Definition 4.63, i.e. the preimage of every compact subset of N is a compact subset of $M \times [0,1]$.
(3) We say that two (proper) smooth embeddings $f_0, f_1\colon M \to N$ are (properly) smoothly isotopic if there exists a (proper) smooth isotopy $F\colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.

If each F_t is a diffeomorphism, then we refer to a smooth isotopy as a diffeotopy and we say "diffeotopic" instead of "smoothly isotopic".

Example. We will soon give many examples of smooth isotopies. At this point it is perhaps better to give a non-example. More precisely, we consider the map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ that is sketched in the figure below.

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[459] If either I is an open interval, or if $\partial\widetilde{M} = \varnothing$, then it follows from the Smooth Manifold Product Proposition 25.34 (1) that we can treat $\widetilde{M} \times I$ as a smooth manifold without corner.

[460] Recall that given a homotopy $F\colon X \times [a,b] \to Y$ and $t \in [a,b]$ we denote by $F_t\colon X \to Y$ the map that is given by $x \mapsto F(x,t)$.

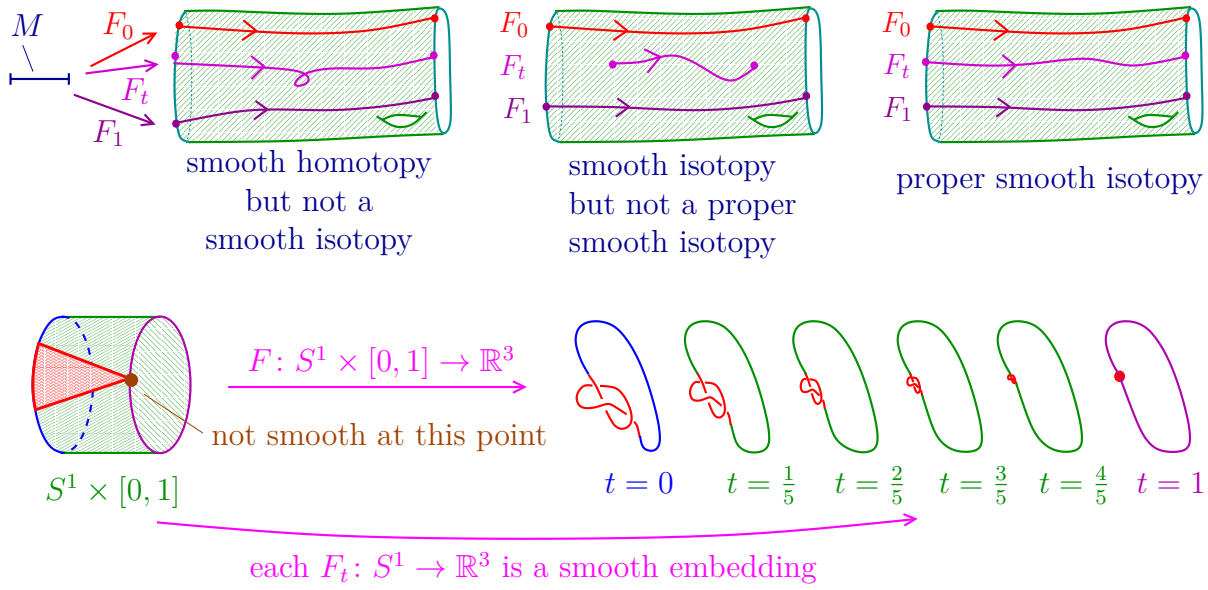
[461] Recall that by Definition 30.9 a map $f\colon M \to N$ between smooth manifolds is a *smooth embedding* if it is a homeomorphism onto its image, if it is smooth and if the following statements hold:

(1) For each $P \in M$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P M \to \mathrm{T}_{\varphi(P)} N$ is a monomorphism.
(2) For each $P \in M \setminus \partial M$ we have $\varphi(P) \in N \setminus \partial N$.
(3) For each $P \in \partial M$ with $\varphi(P) \in \partial N$ the image $\mathrm{D}\varphi_P(\mathrm{T}_P M)$ is not contained in $\mathrm{T}_{\varphi(P)}(\partial N)$.
(4) For every component C of ∂M we have either $\varphi(C) \subseteq \partial N$ or $\varphi(C) \subseteq N \setminus \partial N$.

[462] On a few occasions we also allow that M is a smooth manifold with corner, in this setting we need the definition of a smooth embedding that we gave in Definition 30.25.

[463] By Definition 30.7 a smooth embedding is called *proper* if furthermore the following hold:

(5) $f^{-1}(\partial N) = \partial M$,
(6) $f(M)$ is a closed subset of N.



This map has the property that for each $t \in [0,1]$ the map $F_t\colon S^1 \to \mathbb{R}^3$ is a smooth embedding. But the map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ is not smooth. Thus this map is not a smooth isotopy. □

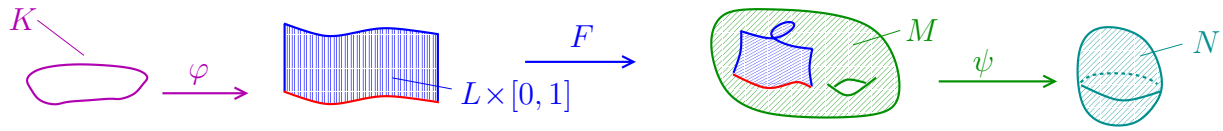
Remark. The term "diffeotopic" is (alas!) used differently by different authors. Our definition agrees with the definition given in most textbooks, see e.g. [**Wall2016**, p. 49] and [**BJ1982**, Definition 9.3]. But it differs from the definition provided in [**Hirs1976**, Chapter 8], there the map on $M \times [0,1]$ only needs to be continuous, but each F_t needs to be a diffeomorphism. As we just saw in the above example, this is a weaker condition. □

The following elementary lemma says in particular that one can precompose and postcompose smooth isotopies by smooth maps. It can be viewed as an analog of the Homotopy Composition Lemma 18.10 (1).

Lemma 41.4. (Smooth Homotopy Composition Lemma) Let K, L, M and N be smooth manifolds. If $F\colon L \times [0,1] \to M$ is a smooth homotopy and if $\varphi\colon K \to L$ and $\psi\colon M \to N$ are smooth maps, then the maps

$$\begin{array}{rcl} K \times [0,1] & \to & M \\ (x,t) & \mapsto & F(\varphi(x),t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} L \times [0,1] & \to & N \\ (x,t) & \mapsto & \psi(F(x,t)) \end{array}$$

are also smooth homotopies. If φ and ψ are diffeomorphisms, then a (proper) smooth isotopy (diffeotopy) $F\colon L \times [0,1] \to M$ becomes again a (proper) smooth isotopy (diffeotopy).



Proof. All of the statements are basically trivial, except that one needs to argue why the two maps $K \times [0,1] \to M$ and $L \times [0,1] \to N$ are smooth. Note that the map $K \times [0,1] \to M$

is the composition of the maps

$$K \times [0,1] \xrightarrow{\varphi \times \mathrm{id}_{[0,1]}} L \times [0,1] \xrightarrow{F} M.$$

It follows from the Smooth Manifold Product Proposition 26.12 that the map to the left is smooth. Thus the given map is the composition of two smooth maps between smooth manifolds with corner. Therefore it follows from the Smooth Composition Lemma 26.7 that it is smooth. Finally note that the map $L \times [0,1] \to N$ is precisely the map $\psi \circ F$, thus it is smooth by the Smooth Composition Lemma 26.7. ■

In the Homotopy-Equivalence Relation Lemma 18.9 we saw that being homotopic is an equivalence relation. The analogous statement also holds in the smooth context.

Proposition 41.5. (Smooth Isotopy-Equivalence Relation Proposition) Let M and N be smooth manifolds.

(1) For maps $M \to N$ being "smoothly homotopic", being "smoothly (properly) isotopic" and being "diffeotopic" is an equivalence relation.
(2) If $A \subseteq M$ is a subset, then the statements in (1) also hold "rel A".[464]

At first glance the Smooth Isotopy-Equivalence Relation Proposition 41.5 might appear trivial, but the proof of the transitivity statement is technically slightly involved. The reader who is happy to swallow the Smooth Isotopy-Equivalence Relation Proposition 41.5 as it is, can also press the fast-forward button and move on to the next section.

Before we provide the proof of the Smooth Isotopy-Equivalence Relation Proposition 41.5 we need to make a few preparations. We start out with the following definition:

Definition 41.6. Let X and Y be topological spaces. In the following we say that a homotopy $F\colon X \times [0,1] \to Y$ is trivial near the ends if there exists an $\epsilon > 0$ such that for all $P \in X$ and all $t \in [0,\epsilon]$ we have $F(P,t) = F(P,0)$ and $F(P,1-t) = F(P,1)$.

The proof of the Smooth Isotopy-Equivalence Relation Proposition 41.5 rests on the following two lemmas.

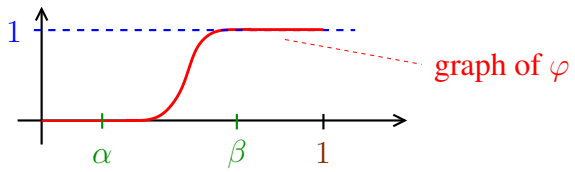
Lemma 41.7. (Stationary-Near-The-Ends-Lemma) [465] Let M and N be smooth manifolds, let $F\colon M \times [0,1] \to N$ be a smooth homotopy and let $0 \leq \alpha < \beta \leq 1$. There exists a smooth homotopy $G\colon M \times [0,1] \to N$ with the following properties:

(1) We have $G_0 = F_0$ and $G_1 = F_1$.
(2) For every $t \in [0,\alpha]$ we have $G_t = F_0$ and for every $t \in [\beta,1]$ we have $G_t = F_1$. In particular the smooth homotopy G is trivial near the ends.
(3) For each $s \in [0,1]$ there exists a $t \in [0,1]$ with $G_s = F_t$.

Furthermore, if F is a proper smooth homotopy or a diffeotopy, then so is G.

Proof of the Stationary-Near-The-Ends-Lemma 41.7. Let M and N be smooth manifolds and let $F\colon M \times [0,1] \to N$ be a smooth homotopy. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth map $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(t) = 0$ for all $t \in [0,\alpha]$ and $\varphi(t) = 1$ for all $t \in [\beta,1]$.

[464] Recall that according to Definition 18.4 we say that a map $F\colon X \times [0,1] \to Y$ that is a *homotopy rel A* if for all $x \in A$ and $s,t \in [0,1]$ we have $F(x,s) = F(x,t)$.
[465] This lemma is the smooth analog of Exercise 18.8.



It follows from the Smooth Composition Lemma 26.7 and the Product-of-Smooth Maps Lemma 26.13 that the map

$$\begin{array}{rcl} F \circ (\mathrm{id}_M \times \varphi)\colon M \times [0,1] & \to & N \\ (x,y) & \mapsto & F(x,\varphi(t)) \end{array}$$

is smooth. It is clear that this map has the desired Properties (1), (2) and (3).

Finally note that if F is a proper smooth homotopy (diffeotopy), then it follows from Property (3) that G is also a proper smooth homotopy (diffeotopy). ∎

We also need the following variation on the Homotopy Combination Lemma 18.8 (1).

Lemma 41.8. (Smooth Homotopy Combination Lemma) Let M be a smooth manifold and let $a < b < c$ be real numbers. Let $f\colon M \times [a,c] \to N$ be a continuous map to some smooth manifold N. We suppose that the following statements hold:

(1) The restrictions of f to $M \times [a,b]$ and to $M \times [b,c]$ (both viewed as smooth manifolds with corner) are smooth.

(2) There exists an $\epsilon > 0$ such that for all $t \in [a,c]$ with $t \in (b-\epsilon, b+\epsilon)$ and all $x \in M$ we have $f(x,t) = f(x,b)$.

Then f itself is smooth.

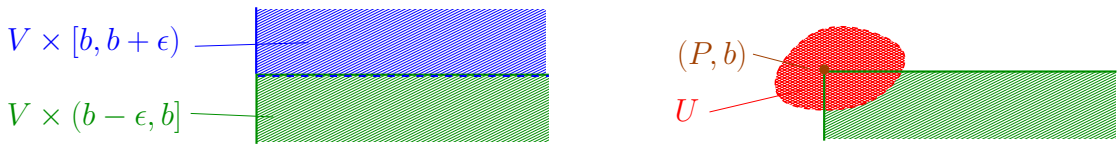
Proof of the Smooth Homotopy Combination Lemma 41.8. If $\partial M = \varnothing$, then the statement follows immediately from the Smooth Pasting Proposition 25.19. But as the meticulous reader will already have figured out, if $\partial M \neq \varnothing$, then there is one step which requires at least a little bit of thought. Namely, after taking charts for $P \in \partial M$ one sees that one still needs to prove the following claim:

Claim. Let V be an open subset of $\mathbb{H}^m$ and let $\varphi\colon V \times (b-\epsilon, b+\epsilon) \to \mathbb{R}^n$ be a continuous map with the following properties:

(1) The restrictions of φ to $V \times (b-\epsilon, b] \to \mathbb{R}^n$ and $V \times [b, b+\epsilon) \to \mathbb{R}^n$ are smooth in the sense of Definition 24.3.

(2) For every $P \in V$ and $t \in [a,c]$ with $t \in (b-\epsilon, b+\epsilon)$ we have $\varphi(P,t) = \varphi(P,b)$.

Then φ is smooth in the sense of Definition 24.3.



Proof. As the reader can verify, this claim does *not* follow immediately from the definition. So let $P \in V$. By hypothesis there exists an open neighborhood U of $(P,b) \in \mathbb{R}^{m+1}$ and a

smooth map $\widetilde{\varphi}\colon U \to \mathbb{R}^n$ with $\widetilde{\varphi} = \varphi$ on $U \cap (V \times (b-\epsilon, b])$. Now we consider the map

$$\begin{array}{rcl} \widetilde{\psi}\colon U & \to & \mathbb{R}^n \\ (P,t) & \mapsto & \widetilde{\varphi}(P,b). \end{array}$$

It follows easily from our hypothesis (2) that this map is smooth and that it agrees with φ on $U \cap (V \times (b-\epsilon, b+\epsilon))$. ■

Proof of the Smooth Isotopy-Equivalence Relation Proposition 41.5.

(1) The only non-trivial aspect is to show transitivity. We show how to prove this statement for "properly smoothly isotopic". All other cases are dealt with in the same way. Thus let $f, g, h\colon M \to N$ be proper smooth embeddings. We assume that f is properly smoothly isotopic to g and that g is properly smoothly isotopic to h. We need to show that f is properly smoothly isotopic to h.

Let $\Phi\colon M \times [0,1] \to N$ be a proper smooth isotopy from f to g. Furthermore let $\Psi\colon M \times [0,1] \to N$ be a proper smooth isotopy from g to h. After applying the Stationary-Near-The-Ends-Lemma 41.7 to Φ and Ψ we can arrange that there exists an $\epsilon \in (0,1)$ such that for each $P \in M$ and each $t \in [0,\epsilon]$ we have the equality $\Phi(P, 1-t) = g(P) = \Psi(P,t)$. Note that by the "furthermore" part of Stationary-Near-The-Ends-Lemma 41.7 we can guarantee that along the way we did not lose the property of Φ and Ψ being proper smooth isotopies. Now we consider the map

$$\begin{array}{rcl} M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \Phi(P, 2\cdot t), & \text{if } t \in [0, \frac{1}{2}], \\ \Psi(P, 2\cdot t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

By the Homotopy Combination Lemma 41.8 we know that this map is smooth. It is now clear that this map is a proper smooth isotopy between f and h.

(2) If we are given a subset $A \subseteq M$ and if Φ and Ψ are smooth isotopies rel A, then it follows from the construction and Part (3) of the Stationary-Near-The-Ends-Lemma 41.7 that the above smooth isotopy is again rel A. ■

We conclude this section with a proposition that says, loosely speaking, that a smooth homotopy starting at a smooth embedding is for "small parameters" actually a smooth isotopy.

Proposition 41.9. (Deform Smooth Embedding Proposition) Let Y and M be smooth manifolds, let S be a smooth manifold and let $F\colon Y \times S \to M$ be a smooth map which takes values in $M \setminus \partial M$. We assume that Y is compact.

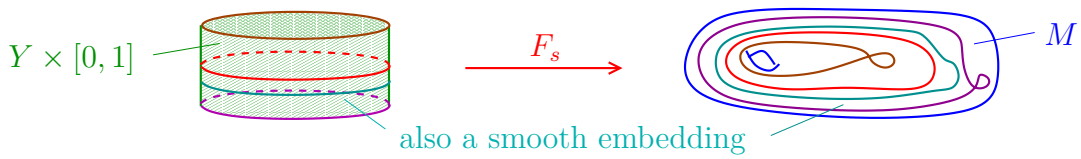
(1) Let $s \in S$ be a point such that the map $F_s\colon Y \to M$ is a smooth embedding. There exists an open neighborhood U of s such that for every $t \in U$ the map $F_t\colon Y \to M$ is a smooth embedding.

(2) If in the setting of (1) the map F_s is actually a diffeomorphism and if Y and M are closed connected smooth manifolds, then there exists an open neighborhood U of s such that for every $t \in U$ the map $F_t\colon Y \to M$ is a diffeomorphism.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \Phi\colon Y \times S & \to & M \times S \\ (y,t) & \mapsto & (F(y,t), t). \end{array}$$



Let $s \in S$ be a point such that the map $F_s \colon Y \to M$ is a smooth embedding. We observe that such $s \in S$ has the following two properties:

(a) The map $\Phi|_{Y \times \{s\}} \colon Y \times \{s\} \to M \times \{s\}$ is a smooth embedding.

(b) For each $y \in Y$ the differential $\mathrm{D}\Phi_{(y,s)}$ is a monomorphism. It follows from the Local Smooth Embedding Theorem 30.13 that for each $y \in Y$ the map Φ is locally a smooth embedding at (y, s).

We deduce from (a) and (b) together with the Neighborhood Smooth Embedding Theorem 14.19 that there exists an open neighborhood W of $Y \times \{s\} \subseteq Y \times S$ such that the restriction of Φ to $W \to M \times S$ is a smooth embedding. Since Y is compact we obtain from the Tube Lemma 7.12 that there exists an open neighborhood U of $s \in S$ such that $Y \times U \subseteq W$. It follows that for each $t \in U$ the map

$$\begin{array}{rcl} \Phi_t \colon Y & \to & M \times \{t\} \subseteq M \times S \\ y & \mapsto & (F_t(y), t) \end{array}$$

is a smooth embedding. Finally note that this in turn implies immediately that for each $t \in U$ the map $F_t \colon Y \to M$ is also a smooth embedding.

(2) Now we assume that $F_s \colon Y \to M$ is a diffeomorphism and that Y and M are closed connected smooth manifolds. In particular we have $\dim(Y) = \dim(M)$. By (1) we know there exists an open neighborhood U of s such that for every $t \in U$ the map $F_t \colon Y \to M$ is a smooth embedding. Since Y is closed, since M is connected and since $\dim(Y) = \dim(M)$ we obtain from the Codimension-Zero Smooth Embedding Proposition 30.21 (3) that each F_t is actually a diffeomorphism. ∎

41.2. **Smooth isotopies of (sub-) manifolds.** In this very short section we introduce a few basic definitions which are slight variations on some of the definitions from the last section. The need for these variations will arise every now and then.

Definition 41.10. Let M_0, M_1 be two (proper) submanifolds of a smooth manifold N.

(1) A (proper) smooth isotopy between M_0 and M_1 is defined as a (proper) smooth isotopy $F \colon M_0 \times [0,1] \to N$ with $F_0 = \mathrm{id}_{M_0}$ and with $F_1(M_0) = M_1$.[466]

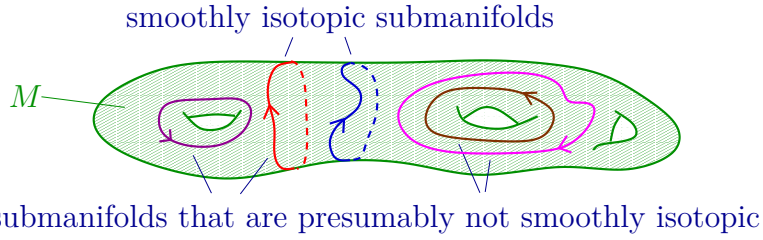
(2) We say that the two submanifolds M_0 and M_1 are (properly) smoothly isotopic if there exists a (proper) smooth isotopy between M_0 and M_1.

If M_0 and M_1 are oriented smooth manifolds, then we demand that the map $F_1 \colon M_0 \to M_1$ is orientation-preserving.

Remark. Let $F \colon M \times [0,1] \to N$ be a (proper) smooth isotopy. Recall that it follows from the Smooth Embedding Theorem 30.15 (1) that each $F_t(M)$ is a (proper) smooth submanifold of N. Thus, loosely speaking, two (proper) submanifolds are (properly) smoothly isotopic precisely if one submanifold can be smoothly deformed into the other through (proper) submanifolds. □

[466] Note that it follows from the Smooth Embedding Theorem 30.15 that $F_1 \colon M_0 \to M_1$ is a diffeomorphism.

Example. In the figure below we consider several oriented 1-dimensional submanifolds of a surface M.



In some cases it is clear that submanifolds are smoothly isotopic, in other cases this looks dubious. In Chapter 51 we will get to know techniques which allow us to show that the dubious pairs of oriented 1-dimensional submanifolds are indeed not smoothly isotopic. In particular in Exercise 51.14 we will return to the given submanifolds. □

For completeness' sake we state the following lemma which we will use implicitly on several occasions.

Lemma 41.11. (Smooth Isotopy–Equivalence Relation Lemma) Let N be a smooth manifold. The property of being (properly) smoothly isotopic is an equivalence relation on the set of (proper) submanifolds of (oriented) submanifolds of M.

Proof. It follows almost immediately from the Smooth Isotopy-Equivalence Relation Proposition 41.5 that being (properly) smoothly isotopic is an equivalence relation. ■

41.3. **Examples of smooth homotopies.** In this section we will give several examples of smooth homotopies which will come in handy on numerous occasions. We start out with an example of a diffeotopy.

Lemma 41.12. (Diffeotopies-via-Matrices Lemma) Let $n \in \mathbb{N}$.

(1) (a) If $\gamma\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ is a smooth path[467], then the map

$$\begin{array}{rcl} F\colon \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \gamma(t)\cdot x \end{array}$$

is a diffeotopy rel the origin.

(b) Given any $A \in \mathrm{GL}_+(n,\mathbb{R})$ there exists a diffeotopy of $\mathbb{R}^n$ from $\mathrm{id}_{\mathbb{R}^n}$ to the diffeomorphism of $\mathbb{R}^n$ that is given by multiplication by A.

(2) (a) If $\gamma\colon [0,1] \to \mathrm{O}(n)$ is a smooth path, then the map

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \gamma(t)\cdot x \end{array}$$

is a diffeotopy rel the origin.

(b) Given any $A \in \mathrm{SO}(n)$ there exists a diffeotopy of $\overline{B}^n$ from $\mathrm{id}_{\overline{B}^n}$ to the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.

The same statements hold with $\overline{B}^n$ replaced by B^n and S^{n-1}.

[467] Let $G = \mathrm{GL}(n,\mathbb{R})$ or $G = \mathrm{O}(n)$. By the discussion on page 849 and by the Matrix Group-Smooth Manifold Proposition 32.10 we know that G is a smooth submanifold of $\mathrm{M}(n\times n,\mathbb{R})$. Thus by the Submanifold Smooth Structure Lemma 25.23 we can and will view G as a smooth manifold in its own right. In this sense we demand that the map $\gamma\colon [0,1] \to G$ is smooth. In practice though we use the Submanifold Smooth

Proof. We first prove Statement (2) which is slightly more subtle.

(2) (a) By the Matrix Group-Smooth Manifold Proposition 32.10 we know that $\mathrm{O}(n)$ is a smooth submanifold of $\mathrm{M}(n \times n, \mathbb{R})$. By the Submanifold Smooth Structure Lemma 25.23 (1) the map γ, viewed as a map $\gamma\colon [0,1] \to \mathrm{M}(n \times n, \mathbb{R})$ is also smooth. It follows from this observation that the obvious extension of F to $\mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is smooth. We deduce from the Smooth Map Criterion Lemma 41.1, that the given map $F\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ is also smooth. It is now clear that it is in fact a diffeotopy.

(b) Let $A \in \mathrm{SO}(n)$. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n)$ is path-connected. It follows from this fact, together with the Smooth Path-Connectivity Proposition 25.32, that there exists a smooth map $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. It follows from (2a) that there exists a diffeotopy of $\overline{B}^n$ from $\mathrm{id}_{\overline{B}^n}$ to the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.

By the same argument the analogous statements also hold for B^n and S^{n-1}.

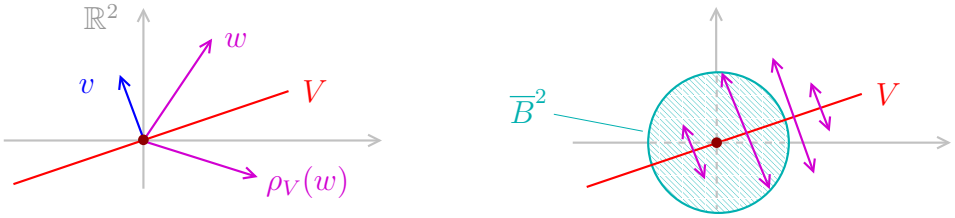
(1) The proof of Statement (1) is almost verbatim the same as the proof of (2). Note that by the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{GL}_+(n, \mathbb{R})$ is path-connected. ■

To formulate our next example we recall the following definition which should be familiar from a course in linear algebra.

Definition 41.13. Let $n \in \mathbb{N}$ and let $V \subseteq \mathbb{R}^n$ be a hyperplane, i.e. V is a vector subspace of $\mathbb{R}^n$ codimension one. We pick a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ that is orthogonal to V. We refer to the map[468] [469]

$$\begin{array}{rccc} \rho_V\colon & \mathbb{R}^n & \to & \mathbb{R}^n \\ & w & \mapsto & w - 2 \cdot \langle v, w \rangle \cdot v \end{array}$$

as the reflection in the hyperplane V.



Lemma 41.14. (Hyperplane–Reflection Lemma) Let $n \in \mathbb{N}$.

(1) Any reflection in a hyperplane of $\mathbb{R}^n$ has the following properties:
  (a) it is an orientation-reversing diffeomorphism,
  (b) it is an isometry (i.e. it preserves the Euclidean distance) and preserves the origin,
  (c) it restricts to orientation-reversing diffeomorphisms of $\overline{B}^n$ and S^{n-1}.

(2) Any two reflections in hyperplanes of $\mathbb{R}^n$ are related by a diffeotopy $F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that each F_t is also a reflection. In particular the diffeotopies are diffeotopies rel the origin. The diffeotopies restrict to diffeotopies of $\overline{B}^n$ and S^{n-1}.

Structure Lemma 25.23 which says that if $\gamma\colon [0,1] \to \mathrm{M}(n \times n, \mathbb{R})$ is smooth in the ordinary sense, and if γ takes values in G, then $\gamma\colon [0,1] \to G$ is smooth.

[468] Note that v is well-defined up to sign, it follows easily that ρ_V does not depend on the choice of v.

[469] Here $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Proof. Let $n \in \mathbb{N}$.

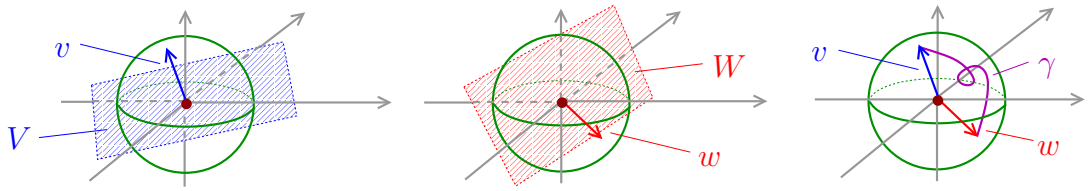
(1) Let ρ be a reflection in a hyperplane of $\mathbb{R}^n$. Clearly ρ is smooth and we have $\rho \circ \rho = \mathrm{id}_{\mathbb{R}^n}$. This implies immediately that ρ is a diffeomorphism. Next we want to show that the reflection $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation-reversing. By the Tangent Space-Isomorphism Proposition 29.24 and by the discussion on page 651 we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_0\mathbb{R}^n & \xrightarrow[\cong]{\Theta_{\mathbb{R}^n}} & \widetilde{\mathrm{T}}_0\mathbb{R}^n = \mathbb{R}^n \\ {\scriptstyle \mathrm{D}\rho_0}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \rho} \\ \mathrm{T}_0\mathbb{R}^n & \xrightarrow[\cong]{\Theta_{\mathbb{R}^n}} & \widetilde{\mathrm{T}}_0\mathbb{R}^n = \mathbb{R}^n. \end{array}$$

Elementary linear algebra shows that the right vertical map has negative determinant, which means that it is orientation-reversing. By the Orientation–Composition Lemma 31.18 (2) this implies that the vertical map $\mathrm{D}\rho_0\colon \mathrm{T}_0\mathbb{R}^n \to \mathrm{T}_0\mathbb{R}^n$ to the left is also orientation-reversing. It follows from the Manifold–Orientation Basics Lemma 31.31 that $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation-reversing.

A short amusing calculation shows that ρ preserves the standard inner product, hence length. Since $\rho(0) = 0$ and since ρ preserves the Euclidean metric we see that ρ restricts to self-maps of $\overline{B}^n$ and S^{n-1}. It follows from the Submanifold Smooth Structure Lemma 25.23 that these self-maps are in fact diffeomorphisms of $\overline{B}^n$ and S^{n-1}. The same argument as above shows that the self-diffeomorphism of $\overline{B}^n$ is orientation-reversing. Finally it follows from the Boundary Orientation Lemma 31.38 (4) that the self-diffeomorphism of $S^{n-1} = \partial\overline{B}^n$ is also orientation-reversing

(2) The case $n = 1$ is trivial since there is only one possible reflection. Now suppose that $n \in \mathbb{N}_{\geq 2}$ and suppose that we are given two hyperplanes V and W of $\mathbb{R}^n$. We pick vectors $v, w \in \mathbb{R}^n$ with $\|v\| = \|w\| = 1$ and with $v \perp V$ and $w \perp W$. It follows from $n \geq 2$ and the Ball-Sphere-Connected Lemma 4.35 that S^{n-1} is path-connected. It follows from the Smooth Path-Connectivity Proposition 25.32 that there exists in fact a smooth path $\gamma\colon [0,1] \to S^{n-1}$ with $\gamma(0) = v$ and $\gamma(1) = w$.



One can easily verify that the map

$$\begin{array}{rcl} F\colon \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (P,t) & \mapsto & P - 2 \cdot \langle \gamma(t), P \rangle \cdot \gamma(t) \end{array}$$

is smooth. By construction $F_0 = \rho_V$ and $F_1 = \rho_W$. Thus F is the desired diffeotopy of $\mathbb{R}^n$. As in (1) we see that this diffeotopy of $\mathbb{R}^n$ restricts to a diffeotopy of $\overline{B}^n$ and to a diffeotopy of S^{n-1}. ∎

Given a topological space X and a subset A we defined on Definition 19.3 the notion of a deformation retraction $R\colon X \times [0,1] \to X$ from X to A. If X is a smooth manifold, then it makes sense to talk of a smooth deformation retraction. The following proposition, which

is the smooth version of the Matrix Groups Deformation Retract Proposition 19.4, gives interesting examples of smooth deformation retractions.

Proposition 41.15. (Matrix Groups–Smooth Deformation Retract Proposition) Let $n \in \mathbb{N}_0$ and let $k \in \{1, \dots, n\}$. We write[470]

$$\begin{aligned} \mathrm{GL}(n,k) &= \text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n \\ \mathrm{O}(n,k) &= \text{set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^n. \end{aligned}$$

There exists a *smooth* deformation retraction

$$R\colon\ \mathrm{GL}(n,k) \times [0,1] \ \to\ \mathrm{GL}(n,k)$$

from $\mathrm{GL}(n,k)$ to $\mathrm{O}(n,k)$ with the following properties:

(1) The map $R_1\colon\ \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ is the map that is given by the Gram-Schmidt Orthonormalization Process 2.32.

(2) For every k-dimensional vector subspace V of $\mathbb{R}^n$ and every $t \in [0,1]$ we have

$$R_t(\text{basis of } V) \ =\ \text{basis of } V.$$

(3) For every $A \in \mathrm{O}(n)$, $P \in \mathrm{GL}(n,k)$ and $t \in [0,1]$ we have $R_t(A \cdot P) = A \cdot R_t(P)$.

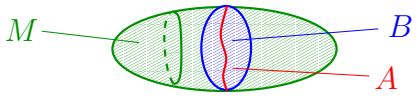
The analogous statements of (1) and (2) hold if we replace the pair $\mathrm{GL}(n,k)$ and $\mathrm{O}(n,k)$ by one of the following four pairs:

(a)	$\mathrm{GL}_+(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(b)	$\mathrm{SL}(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(c)	$\mathrm{GL}(n,\mathbb{C})$	and	$\mathrm{U}(n)$
(d)	$\mathrm{SL}(n,\mathbb{C})$	and	$\mathrm{SU}(n)$.

Remark. Later, on page 4614 we will use the Polar Decomposition Theorem 239.18 to give an alternative proof of some of the main statements of the Matrix Group–Homotopy Type Proposition 41.15. □

Our proof of the Matrix Groups–Smooth Deformation Retract Proposition 41.15 rests on the following elementary lemma:

Auxiliary Lemma 41.16. (Iterated Smooth Deformation Retract Lemma) Let M be a smooth manifold and let $A \subseteq B \subseteq X$ be submanifolds. If A is a smooth deformation retract of B and if B is a smooth deformation retract of M, then A is a smooth deformation retract of X.



Proof of the Iterated Smooth Deformation Retract Lemma 41.16. The proof of the lemma follows from combining the ideas of the proof of the Iterated Deformation Retract Lemma 19.5 and the proof of the Smooth Isotopy-Equivalence Relation Proposition 41.5. We leave it to the reader to fill in the details. ■

[469] Recall that by Definition 19.3 we say that a retraction $f\colon X \to A$ is realized by a deformation retraction $R\colon X \times [0,1] \to X$ if $R_1 = f$.

[470] In Exercise 41.6 we will see that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$. Therefore the word "smooth" makes sense in the most naive sense.

Sketch of a proof of Proposition 41.15. In the Matrix Groups Deformation Retract Proposition 19.4 we gave the same statement as above, except that we did not worry about smoothness. We will now see that demanding that the deformation retractions are smooth is not a big deal.

On page 204, in the proof of the Matrix Group-π_0-Proposition 4.54, we implicitly already showed that each of the $2k$ steps of the Gram-Schmidt Orthogonalization Process is realized by a smooth deformation retraction that satisfies (2). Furthermore, using the Gram-Schmidt Orthogonalization Process 2.32 (6) one can easily verify that (3) also holds in each case. It follows from the Iterated Smooth Deformation Retract Lemma 41.16 that these smooth deformation retractions give rise to the desired smooth deformation retraction

$$R\colon\ \mathrm{GL}(n,k)\times[0,1]\ \to\ \mathrm{GL}(n,k)$$

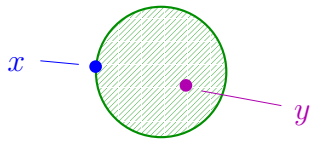
from $\mathrm{GL}(n,k)$ to $\mathrm{O}(n,k)$ such that R that satisfies (2) and (3). If $n=k$, then it follows from the Gram-Schmidt Orthonormalization Process 2.32 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the cases of the complex matrix groups are treated in a very similar fashion. ■

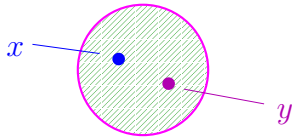
41.4. **Homogeneous topological spaces.** We start out with the following general definition.

Definition 41.17. Let X be a topological space.

(1) We say that X is homogeneous if given any two points $x,y\in X$ there exists a homeomorphism $f\colon X\to X$ with $f(x)=y$.
(2) Let $A\subseteq X$ be a subset. We say that X is homogeneous outside of A if given any two points $x,y\in X\setminus A$ there exists a homeomorphism $f\colon X\to X$ with $f(x)=y$.



$\overline{B}^2$ is *not* homogenous



$\overline{B}^2$ is homogenous outside of S^1

Examples.

(1) It is clear that the topological space $\mathbb{R}^n$ is homogeneous. Furthermore it follows from the Orthogonal Action Lemma 2.37 (3) that the sphere S^n is homogeneous.
(2) Every set that is equipped with the trivial topology or the discrete topology is homogeneous.
(3) In Exercise 28.14 we showed that the long line $\mathbb{L}$ is homogeneous.
(4) We consider the infinite product $X=\prod_{k\in\mathbb{N}}\{0,1\}=\{0,1\}^{\mathbb{N}}$ of the discrete space $\{0,1\}$. In Exercise 41.10 we will see that X is homogeneous. Note that on page 289 we saw that the Cantor set is homeomorphic to $\prod_{k\in\mathbb{N}}\{0,1\}=\{0,1\}^{\mathbb{N}}$. This gives us the not entirely intuitive conclusion, that the Cantor set is homogeneous.

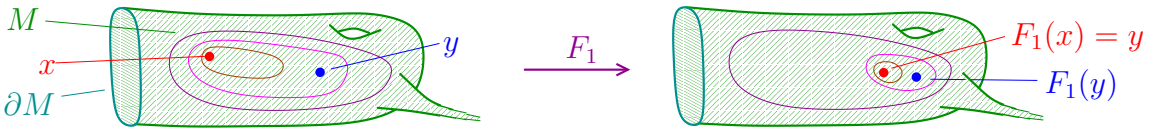
(5) It follows basically immediately from the definitions that a topological manifold with non-empty boundary and non-empty interior is not homogeneous. We are still a little unsure what the boundary of a given topological manifold is, but for example in Exercise 23.5 we showed that the boundary of $[0,1]$ is $\{0,1\}$. Furthermore, not surprisingly, it will follow from the Topological Manifold–Boundary Proposition 23.15 that the boundary of any cube $[0,1]^m$ is non-empty. This discussion shows that no cube $[0,1]^m$ is homogeneous. It is thus quite stunning that the Hilbert Cube[471], i.e. the topological space $\prod_{k\in\mathbb{N}}[0,1] = [0,1]^{\mathbb{N}}$, is actually homogeneous. A proof of this fact is given in [**Verb1969, HalW2012**]. □

For topological and smooth manifolds we are interested in finer notions of homogeneity, which leads us to the following definition:

Definition 41.18. Let M be a topological manifold and let $x, y \in M \setminus \partial M$. We say that x can be moved to y if there exists a homeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) $F_1(x) = y$,
(3) F is a homotopy rel some neighborhood of ∂M.
(4) F is a homotopy with compact support, i.e. there exists a compact subset $K \subseteq M$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

If M is a smooth manifold, then we demand that F is actually a diffeotopy.



The following proposition shows in particular that each compact interval $[a,b]$ is homogeneous outside of the set of endpoints $\{a,b\}$, that every sphere is homogeneous and that the closed ball $\overline{B}^n$ is homogeneous outside of the boundary sphere $\partial\overline{B}^n = S^{n-1}$.

Proposition 41.19. (Interval-Sphere-Ball Point Moving Proposition)

(1) Let $a < b \in \mathbb{R}$. Given any $x, y \in (a,b)$ we can move x to y in $[a,b]$.
(2) Let $n \in \mathbb{N}$. Given any $x, y \in S^n$ we can move x to y.
(3) Let $n \in \mathbb{N}$. Given any $x, y \in \overline{B}^n \setminus \partial\overline{B}^n = B^n$ we can move x to y in $\overline{B}^n$.

Remark.

(1) Note that the Interval-Sphere-Ball Point Moving Proposition 41.19 (3) gives in particular a solution to the rather challenging Exercise 25.20.
(2) In the Moving Points-in-Ball Proposition 43.22 we will use the Flow Theorem 43.3 to provide an alternative proof of the Interval-Sphere-Ball Point Moving Proposition 41.19 (3). □

The proof of the Interval-Sphere-Ball Point Moving Proposition 41.19 rests on the following little lemma.

[471] We encountered the Hilbert Cube in a slightly different avatar in Exercise 13.20.

Auxiliary Lemma 41.20. Let M be a smooth manifold and let $x, y, z \in M \setminus \partial M$. If we can move x to y and if we can move y to z, then we can move x to z.

Proof of Lemma 41.20. Let $F\colon M \times [0,1] \to M$ and $G\colon M \times [0,1] \to M$ be diffeotopies as in the above definition such that $F_1(x) = y$ and such that $G_1(y) = z$. Using the Stationary-Near-The-Ends-Lemma 41.7 we can arrange that F and G are trivial near the ends. One can easily verify that the map

$$\begin{array}{rccl} H\colon & \times [0,1] & \to & M \\ & (P,t) & \mapsto & \begin{cases} F(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ G(F_1(P),2t-1), & \text{if } t \in [\frac{1}{2},1] \end{cases} \end{array}$$

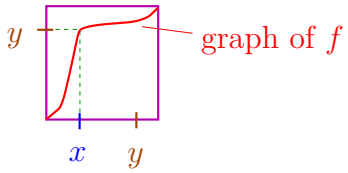
has the desired properties. ∎

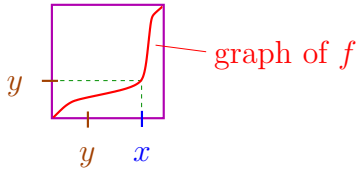
Proof of Proposition 41.19.

(1) Let $x, y \in (a,b)$. We leave it to the reader to show, say using the Smooth Transition Function Lemma 24.9, that there exists a diffeomorphism $f\colon [a,b] \to [a,b]$ with $f(x) = y$ and which is identity in a neighborhood of the two endpoints a and b. We consider the smooth homotopy

$$\begin{array}{rccl} F\colon [a,b] \times [0,1] & \to & [a,b] \\ (s,t) & \mapsto & s \cdot (1-t) + f(s) \cdot t \end{array}$$

which is evidently a homotopy rel an open neighborhood of $\{a,b\}$ with $F_0 = \mathrm{id}$ and $F_1 = f$. Since the derivative of f is strictly positive one sees easily, say using the Inverse Mapping Theorem 30.1 (2), that each F_t is a diffeomorphism of $[a,b]$, i.e. F is the promised diffeotopy.





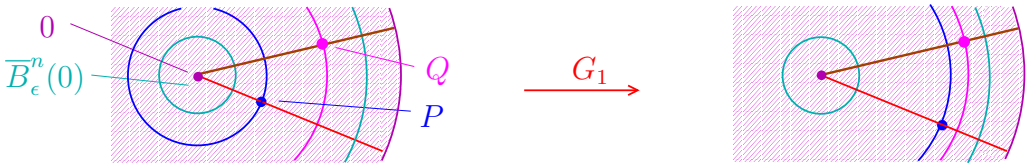
(2) Let $n \in \mathbb{N}$ and let $x, y \in S^n$. By the Orthogonal Action Lemma 2.37 (4) we know that there exists an $A \in \mathrm{SO}(n+1)$ with $A \cdot x = y$. The desired orientation-preserving diffeomorphism is thus given by multiplication by A. By the Diffeotopies-via-Matrices Lemma 41.12 we know that this map is diffeotopic to the identity.

(3) The case $n = 1$ is a special case of (1). Thus let us now suppose that $n \geq 2$. Let $P, Q \in \overline{B}^n \setminus \partial \overline{B}^n = B^n$. First we deal with the case that $P, Q \neq 0$. We start out with the following claim:

Claim. There exists a diffeotopy $G\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ with the following properties:
(a) $G_0 = \mathrm{id}$.
(b) For all $x \in \overline{B}^n$ with $\|x\| = \|P\|$ we have $\|G_1(x)\| = \|Q\|$.
(c) G is a homotopy rel a neighborhood of the boundary.

Proof. Since $P, Q \neq 0$ and since $P, Q \in B^n$ we see that $\|P\|, \|Q\| \in (0,1)$. Therefore it follows from (1) that there exists a diffeotopy $F\colon [0,1] \times [0,1] \to [0,1]$ with the following properties:
(a) We have $F_0 = \mathrm{id}$.
(b) The map $F_1\colon [0,1] \to [0,1]$ is a diffeomorphism with $F_1(\|P\|) = \|Q\|$.

(c) There exists an $\epsilon > 0$, such that $F(s,t) = s$ for all $s \in [0,\epsilon] \cup [1-\epsilon, 1]$ and all $t \in [0,1]$.



We now consider the following map:

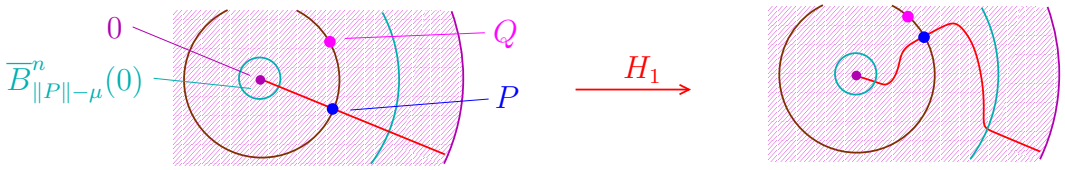
$$\begin{array}{rcl} G\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } \|x\| < \epsilon \\ \frac{x}{\|x\|} \cdot F(\|x\|, t), & \text{if } \|x\| > 0. \end{cases} \end{array}$$

Note that the map is indeed well-defined. Using the Smooth Pasting Proposition 24.7 one can easily now verify that the above map $G\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ is the desired diffeotopy. ⊞

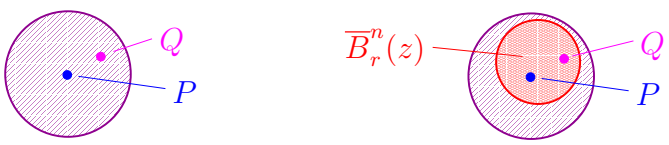
It follows from the claim and Lemma 41.20 that it remains to deal with the case that $\|P\| = \|Q\|$. We pick $\mu > 0$ with $0 < \|P\| - \mu < \|P\| + \mu < 1$. We write $v := \frac{P}{\|P\|}$ and $w := \frac{Q}{\|Q\|}$. Since $n \geq 2$ it follows from (2) and the Stationary-Near-The-Ends-Lemma 41.7 that there exists a diffeotopy $F\colon S^{n-1} \times [0,\mu] \to S^{n-1}$ with $F_0 = \mathrm{id}$, with $F_\mu(v) = w$ and which is trivial near the ends. Again one can easily verify that the desired diffeotopy is given by the map

$$\begin{array}{rcl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } \|x\| \in [0, \|P\| - \mu], \\ \|x\| \cdot F(\frac{x}{\|x\|}, t \cdot (\|x\| - \|P\| + \mu)), & \text{if } \|x\| \in [\|P\| - \mu, \|P\|], \\ \|x\| \cdot F(\frac{x}{\|x\|}, t \cdot (\|P\| + \mu - \|x\|)), & \text{if } \|x\| \in [\|P\|, \|P\| + \mu], \\ x, & \text{if } \|P\| \in [\|x\| + \mu, 1]. \end{cases} \end{array}$$

We have now settled the case that $P, Q \neq 0$.



Finally we need to deal with the case that one of two points $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n = B^n$ is the origin. In this case we pick a closed ball $\overline{B}^n_r(z) \subseteq \overline{B}^n$ with $P, Q \in B^n_r(P)$ but with $z \neq P, Q$. We apply the above scheme to find a diffeotopy of $\overline{B}^n_r(z)$ that sends P



to Q. Since the diffeotopy is a homotopy rel a neighborhood of the boundary of $\overline{B}^n_r(z)$ we see that we can extend this diffeotopy via the stationary homotopy to a diffeotopy of all of $\overline{B}^n$. This diffeotopy clearly has all the desired properties. ■

The following proposition implies in particular that connected topological manifolds with empty boundary are homogeneous.

Proposition 41.21. (Manifold–Homogeneous Proposition)
(1) Let M be a connected topological manifold. Given any $x, y \in M \setminus \partial M$ we can move x to y (in the sense of Definition 41.18) via a homeotopy.
(2) Let M be a connected smooth manifold. Given any $x, y \in M \setminus \partial M$ we can move x to y (in the sense of Definition 41.18) via a diffeotopy.

Remark. For smooth manifolds we will prove in the Homogeneous-via-Diffeotopies Proposition 44.6 a refinement of the Manifold–Homogeneous Proposition 41.21. □

Proof. We prove the Manifold–Homogeneous Proposition 41.21 for smooth manifolds. The case of topological manifolds follows by almost the same argument. Thus let M be a connected smooth manifold and let $x \in M \setminus \partial M$.

We say that $y \in M \setminus \partial M$ is *connected to x via balls* if there exist smooth embeddings $\Phi_i \colon \overline{B}^n \to M \setminus \partial M$, $i = 1, \dots, k$ such that $x \in \Phi_1(B^n)$, $y \in \Phi_k(B^n)$ and such that for each $i = 1, \dots, k-1$ we have $\Phi_i(B^n) \cap \Phi_{i+1}(B^n) \neq \varnothing$.

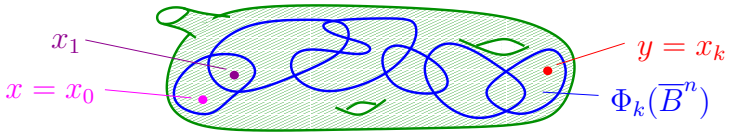
Claim 1. Every point in $M \setminus \partial M$ is connected to x via balls.

Proof. Recall that we assume that M is connected. It follows from the Manifold–Local Properties Lemma 23.8 and the Push Away-from-Boundary Lemma 23.17, see also Exercise 23.23, that $M \setminus \partial M$ is also connected. Now let U be the set of points in $M \setminus \partial M$ that are connected to x via balls.

- Since $x \in M \setminus \partial M$ we see that there exists a smooth embedding $\Phi \colon \overline{B}^n \to M \setminus \partial M$ with $x \in \Phi(B^n)$, thus $x \in U$, in particular U is non-empty.
- It is clear that U is open.
- Finally let $y \in \overline{U}$, where we take the closure of U in $M \setminus \partial M$. Again we see that there exists a smooth embedding $\Phi \colon \overline{B}^n \to M \setminus \partial M$ with $y \in \Phi(B^n)$. By the Codimension-Zero Smooth Embedding Proposition 30.21 we know that $\Phi(B^n)$ is an open neighborhood of y in $M \setminus \partial M$. Since $y \in \overline{U}$ we obtain from the Interior-Closure Characterization Lemma 3.23 (2) that there exists a point $\widetilde{y} \in \Phi(B^n) \cap U$. We can connect $\widetilde{y}$ to x via balls, but then we can also connect y to x via one extra ball. Thus we have shown that $y \in U$, i.e. we have shown that $U = \overline{U}$. By the Interior-Closure Lemma 3.22 (5) this implies that U is closed.

It follows from the above and the fact that $M \setminus \partial M$ is connected that $U = M \setminus \partial M$. ⊞

Now let $y \in M \setminus \partial M$. By the claim we know that y is connected to x via balls, i.e. there exist smooth embeddings $\Phi_i \colon \overline{B}^n \to M \setminus \partial M$, $i = 1, \dots, k$ such that $x \in \Phi_1(B^n)$, $y \in \Phi_k(B^n)$ and such that for each $i = 1, \dots, k-1$ there exists a point $x_i \in \Phi_i(B^n) \cap \Phi_{i+1}(B^n)$. We set $x_0 \coloneqq x$ and $x_k \coloneqq y$.



By Lemma 41.20 it remains to prove the following claim:

Claim 2. For each $i = 0, \dots, k-1$ we can move x_i to x_{i+1}.

Proof. Let $i \in \{0, \dots, k-1\}$. Note that by the Interval-Sphere-Ball Point Moving Proposition 41.19 (3) there exists an $r \in (0,1)$ and a diffeotopy $\Theta\colon \overline{B}^n \times [0,1] \to M$ rel $\{x \in \overline{B}^n \mid \|x\| \geq r\}$ with $\Theta_0 = \mathrm{id}$ and with $\Theta_1(\Phi_i^{-1}(x_i)) = \Phi_i^{-1}(x_{i+1})$. We set

$$\begin{array}{rccl} F_i\colon M \times [0,1] & \to & M & \\ (x,t) & \mapsto & \begin{cases} \Phi_i(\Theta(\Phi_i^{-1}(x),t)), & \text{if } x \in \Phi_i(B^n), \\ x, & \text{otherwise} \end{cases} \end{array}$$

Note that F_i is well-defined and that it is smooth on the open subsets[472] $\Phi_i(B^n) \times [0,1]$ and $(M \setminus \Phi_i(\overline{B}^n_r)) \times [0,1]$ of $M \times [0,1]$. Thus F_i is smooth by the Smooth Pasting Proposition 25.19. One can now easily verify that F_i has the desired properties. ∎

We conclude this section with a convenient corollary to the Manifold–Homogeneous Proposition 41.21, which in many settings can be viewed as a strengthening of the Smooth Path-Connectivity Proposition 25.32 (2):

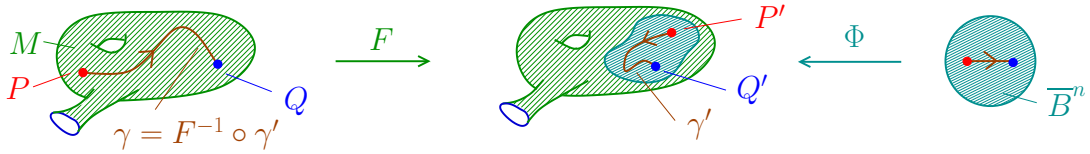
Corollary 41.22. (Embedded Path-Corollary) Let M be a connected smooth manifold and let $P \neq Q$ be two distinct point in $M \setminus \partial M$. There exists a smooth embedding $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.

Proof. Let M be a connected n-dimensional smooth manifold and let $P \neq Q \in M \setminus \partial M$. We need to show that there exists a smooth embedding $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = P$ and $\gamma(1) = Q$. The case $n = 1$ can be dealt with easily, e.g. using the Smooth 1-Manifold Classification Theorem 28.8. So we now assume that $n \geq 2$.

We pick a smooth embedding $\Phi\colon \overline{B}^n \to M \setminus \partial M$ and we set $P' := \Phi((-\frac{1}{2}, 0, \dots, 0))$ and $Q' := \Phi((\frac{1}{2}, 0, \dots, 0))$. We consider the smooth embedding

$$\begin{array}{rccl} \gamma'\colon [0,1] & \to & M \setminus \partial M & \\ t & \mapsto & \Phi((-\frac{1}{2} + t, 0, \dots, 0)). & \end{array}$$

Note that $\gamma'(0) = P'$ and $\gamma'(1) = Q'$.



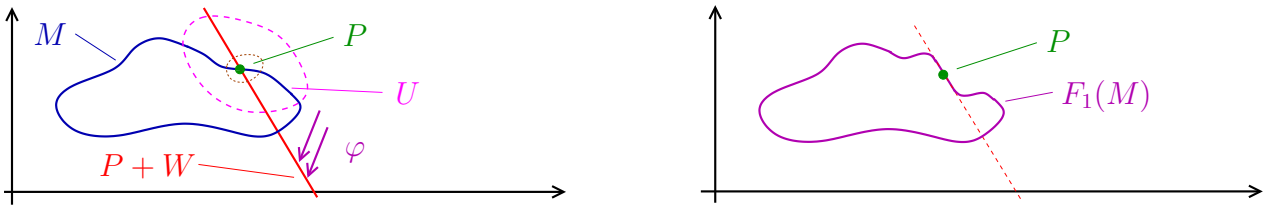
Next note that it follows from the hypothesis that $P \neq Q \in M \setminus \partial M$, the hypothesis that $n \geq 2$ and a slight extension of the Manifold–Homogeneous Proposition 41.21 (2) (see Exercise 41.11) that there exists a diffeomorphism $F\colon M \to M$ with $F(P) = P'$ and $F(Q) = Q'$. Evidently $\gamma := F^{-1} \circ \gamma'\colon [0,1] \to M \setminus \partial M$ is a smooth embedding with $\gamma(0) = P$ and $\gamma(1) = Q$. ∎

41.5. **Appendix: The Flattening Lemma.** We conclude this chapter with the following lemma, which is a little technical, but convenient at times.

[472] It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that $\Phi_i(B^n) \times [0,1]$ is an open subset of $M \times [0,1]$. Furthermore it follows from the Compact Image Lemma 4.15, the Compact-Closed Lemma 3.39 and the fact that smooth manifolds are Hausdorff that $(M \setminus \Phi_i(\overline{B}^n_r)) \times [0,1]$ is also an open subset of $M \times [0,1]$.

Lemma 41.23. (Flattening Lemma) Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. Furthermore let $P \in M \setminus \partial M$, let $W \subseteq \mathbb{R}^n$ be a vector subspace and let $\varphi\colon \mathbb{R}^n \to W$ be a homomorphism such that $\varphi = \mathrm{D}\varphi_P\colon \widetilde{\mathrm{T}}_P M \to W$ is a monomorphism. Given any open neighborhood U of $P \in M$ there exists a smooth isotopy $F\colon M \times [0,1] \to \mathbb{R}^n$ rel $(M \setminus U) \cup \{P\}$ and an open neighborhood $\widetilde{U}$ of $P \in M$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) $\widetilde{\mathrm{T}}_P(F_1(M)) = \varphi(\widetilde{\mathrm{T}}_P M)$,
(3) $F_1(\widetilde{U}) \subseteq P + W$,
(4) for each $x \in M$ and each $t \in [0,1]$ the point $F(x,t)$ lies on the Euclidean segment between x and $P + \varphi(x)$.



Proof. Let $M \subseteq \mathbb{R}^n$ be a k-dimensional smooth submanifold, let $P \in M \setminus \partial M$, let $W \subseteq \mathbb{R}^n$ be a vector subspace and let $\varphi\colon \mathbb{R}^n \to W$ be a homomorphism such that $\varphi\colon \widetilde{\mathrm{T}}_P M \to W$ is a monomorphism and let U be an open neighborhood of $P \in M$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first k coordinates. It follows from elementary linear algebra, see Exercise 41.15, that there exists a linear isomorphism $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and a monomorphism $\chi\colon \mathbb{R}^k \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{T}}_P M & \xrightarrow[\cong]{\Phi} & \Phi(\widetilde{\mathrm{T}}_P M) \\ \downarrow{\scriptstyle \varphi} & & {\scriptstyle p}\downarrow{\scriptstyle (x,y)\mapsto x} \\ W & \xleftarrow{\chi} & \mathbb{R}^k. \end{array}$$

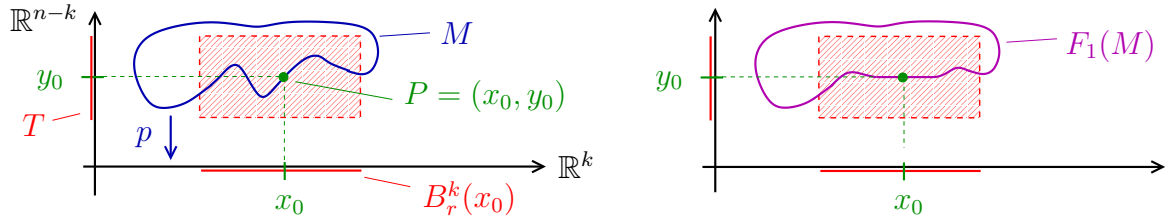
Since $\mathrm{D}\Phi_P = \Phi$ we can assume, after replacing M by $\Phi(M)$, that $\varphi = \chi \circ p$.

We write $P = (x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. It follows almost immediately from the Local Graph Lemma 30.5 that there exists an $r > 0$ and an open neighborhood T of $y_0 \in \mathbb{R}^{n-k}$ and a smooth map $f\colon B^k_r(x_0) \to T$ such that $B^k_r(x_0) \times T \subseteq U$ and such that

$$M \cap (B^k_r(x_0) \times T) = \text{ Graph of } f\colon B^k_r(x_0) \to T \ = \ \{(x, f(x)) \,|\, x \in B^k_r(x_0)\}.$$

Next note that by the Smooth Transition Function Lemma 24.9 there exists a smooth map $\gamma\colon B^k_r(x_0) \to [0,1]$ such that

(a) if $\|x - x_0\| \geq \frac{r}{2}$, then $\gamma(x) = 0$,
(b) if $\|x - x_0\| \leq \frac{r}{4}$, then $\gamma(x) = 1$.



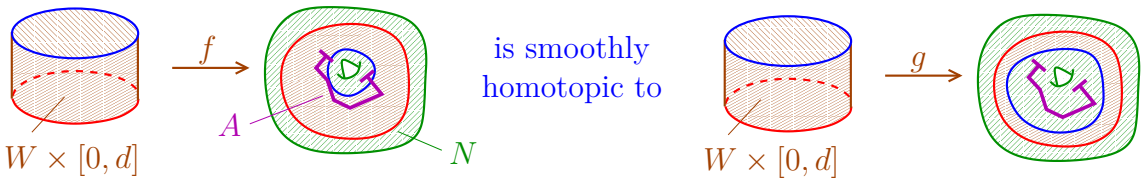
One can now easily verify that the map

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & M \\ (Q,t) & \mapsto & \begin{cases} Q, & \text{if } Q\notin (B_r^k(x_0)\times T \\ (x, f(x)+(y_0-f(x))\cdot\gamma(x)\cdot t), & \text{if } Q=(x,f(x)) \text{ with } x\in B_r^k(x_0) \end{cases} \end{array}$$

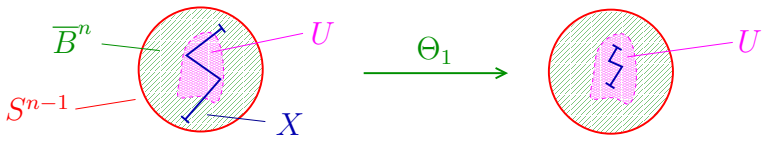
has the desired properties. ■

41.6. Exercises for Chapter 41.

Exercise 41.1. Let W be a compact smooth manifold with empty boundary and let N be a smooth manifold, possibly with boundary. Let $d>0$ and let $f\colon W\times[0,d]\to N$ be a smooth embedding. Let A be a closed subset of N with $f(W\times\{0\})\cap A=\varnothing$. Then the map f is smoothly isotopic rel $W\times\{0\}$ to a map $g\colon W\times[0,d]\to N$ with $g(W\times[0,d])\cap A=\varnothing$.



Exercise 41.2. Let $n\in\mathbb{N}$, let $X\subseteq B^n$ be compact and let U be an open neighborhood of the origin $0\in\overline{B}^n$. Show that there exists a diffeotopy $\Theta\colon\overline{B}^n\times[0,1]\to\overline{B}^n$ rel S^{n-1} such that $\Theta_0=\mathrm{id}$ and such that $\Theta_1(X)\subseteq U$.



Exercise 41.3. Let M be a smooth manifold and let $F\colon[0,1]\times\partial M\to\partial M$ be a diffeotopy. Show that we can extend F to a diffeotopy $[0,1]\times M\to M$ of all of M.
Hint. Use the Smooth Collar Neighborhood Theorem 35.6.

Exercise 41.4. Let M and N be smooth manifolds and let $F\colon M\times[0,1]\to N$ be a smooth isotopy.

(a) We suppose that there exists an open subset $V\subseteq N$ such that for each $t\in[0,1]$ we have $V\subseteq F_t(M\setminus\partial M)$. Show that the map

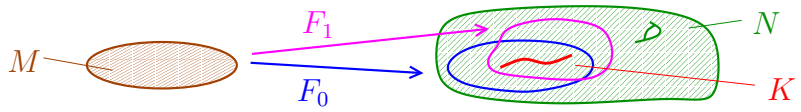
$$\begin{array}{rcl} \varphi\colon V\times[0,1] & \to & M \\ (P,t) & \mapsto & F_t^{-1}(P) \end{array}$$

is a smooth isotopy.
Hint. To show smoothness of φ you could use the Inverse Mapping Theorem 30.1 to show that the map $V\times[0,1]\to M\times[0,1]$ given by $(P,t)\mapsto(\varphi(P,t),t)$ is a local diffeomorphism.

(b) Let $K\subseteq N$ be a submanifold such that for each $t\in[0,1]$ we have $K\subseteq F_t(M\setminus\partial M)$. Show that the map

$$\begin{array}{rcl} K\times[0,1] & \to & M \\ (P,t) & \mapsto & F_t^{-1}(P) \end{array}$$

is a smooth isotopy.



Exercise 41.5. We consider the two compact 3-dimensional smooth submanifolds X and Y of $\mathbb{R}^3$ that are shown in the figure below. Show that there exists a smooth isotopy $F\colon X \times [0,1] \to \mathbb{R}^3$ with $F_0 = \mathrm{id}$ such that $F_1(X) = Y$.



Exercise 41.6. Let $n \in \mathbb{N}_0$ and let $k \in \{1, \dots, n\}$. We write

$$\mathrm{GL}(n,k) \;=\; \underbrace{\text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n}_{\mathbb{R}^n \times \dots \times \mathbb{R}^n = \mathbb{R}^{nk}}.$$

Show that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$.

Exercise 41.7. Let M be a connected smooth manifold.

(a) Let P, Q be points in $M \setminus \partial M$.
  (i) Show that if $\dim(M) \geq 2$, then there exists a diffeomorphism $f\colon M \to M$ that swaps P and Q and that is diffeotopic to the identity.
  (ii) For which 1-dimensional smooth manifolds does the conclusion of (i) hold?

(b) Let C be a component of ∂M and let P, Q be two points in C. We assume that $\dim(M) \geq 3$. Show that there exists a diffeomorphism $f\colon M \to M$ that swaps P and Q.
Hint. Use the Smooth Collar Neighborhood Theorem 35.6.

Exercise 41.8. Show that $\mathbb{R}/\mathbb{Q}$ is compact, that it is homogeneous and that it is not a topological manifold.

Exercise 41.9. Let M be a closed k-dimensional smooth manifold.

(a) Let $n \in \mathbb{N}_{\geq k+1}$ and let $f\colon M \to S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ be an embedding. Show that there exists a smooth isotopy $F\colon M \times [0,1] \to S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ with $F_0 = f$ and such that $F_1(M) \subseteq \mathbb{R}^n$.
Hint. Use the Measure-Zero Properties Proposition 37.3 and the Full Measure-Density Proposition 37.4, the Orthogonal Action Lemma 2.37 (4) and the Diffeotopies-via-Matrices Lemma 41.12 (2).

(b) Let $n \in \mathbb{N}_{\geq k+2}$ and let $f_0, f_1\colon M \to \mathbb{R}^n$ be two embeddings. Suppose that there exists a smooth isotopy $F\colon M \times [0,1] \to S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ with $F_0 = f_0$ and $F_1 = f_1$. Show that there exists also a smooth isotopy $G\colon M \times [0,1] \to \mathbb{R}^n$ with $G_0 = f_0$ and $G_1 = f_1$.
Remark. The basic idea of the proof is the same as in (a). But this time the argument requires a little more ingenuity.

(c) Show that the dimension restrictions on n in (a) and (b) are necessary.

Remark. The exercise, applied to $M = S^1$ and $n = 3$, shows in particular that the "theory of knots in $\mathbb{R}^3$" is the same as the "theory of knots in S^3".

Exercise 41.10. Show that the topological space $\prod_{k\in\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$ is homogeneous and show that it is not a topological manifold.

Remark. Note that by Tychonoff's Theorem 7.14 and the Product Topology-Properties Proposition 7.13 we know that $\prod_{k\in\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}}$ is compact and Hausdorff. Thus we see that there are compact Hausdorff spaces that are homogeneous but that are not topological manifolds.

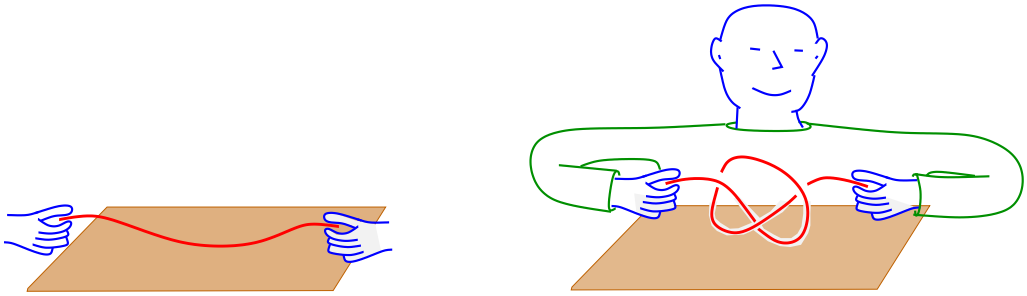
Exercise 41.11.

(a) Let $n \geq 2$ and let M be a connected n-dimensional smooth manifold. Suppose we are given distinct points $P_1, \ldots, P_m$ and distinct points $Q_1, \ldots, Q_m$ in $M \setminus \partial M$. Show that there exists a diffeotopy with compact support, rel a neighborhood of ∂M, from the identity to a diffeomorphism $\Phi\colon M \to M$ with $\Phi(P_i) = Q_i$ for $i = 1, \ldots, m$.
Hint. Apply the Manifold–Homogeneous Proposition 41.21 iteratively.
Remark. The same argument also works for topological manifolds where the conclusion is that we get an isotopy with compact support.

(b) Show that the conclusion of (a) is true for $M = S^1$ and $m \in \{1, 2\}$ and wrong for $M = S^1$ and $m \geq 3$.

(c) Show that the conclusion of (a) is wrong for $M = [0, 1]$ and $m \geq 2$.

Exercise 41.12. Can you pick up a loose unknotted string with both hands and turn it into a knotted string as shown in the figure below, while holding on to the ends of the string? Either way, which smooth isotopy types of knotted strings can you produce this way?[473]



Exercise 41.13. Let M and N be smooth manifolds and let $F\colon M \times \mathbb{R} \to N$ be a smooth map such that each $F_t\colon M \to N$ is a (proper) smooth embedding. Show that

$$\begin{array}{rcl} M \times \mathbb{R} & \to & N \times \mathbb{R} \\ (P, t) & \mapsto & (F(P, t), t) \end{array}$$

is a (proper) smooth embedding.

[473]Bonus problem: Where is the cheat in

`https://www.youtube.com/watch?v=idvHnV422_4`

or alternatively on 2:53 of

`https://web.archive.org/web/20170103163014/http://concise.britannica.com/new-multimedia/mp4/newdim025.mp4`

Exercise 41.14. Show that given any continuous map $\gamma\colon [0,1] \to S^n$ there exists a continuous map $A\colon [0,1] \to \mathrm{O}(n+1)$ such that for each $t \in [0,1]$ we have $A(t) \cdot \mathfrak{e}_1 = \gamma(t)$.
Hint. The Lebesgue Lemma 10.8 and the Gram-Schmidt Orthonormalization Process 2.32 might be helpful.

Exercise 41.15. Let $T \subseteq \mathbb{R}^n$ be a k-dimensional vector subspace, let $W \subseteq \mathbb{R}^n$ be a vector space and let $\varphi\colon \mathbb{R}^n \to W$ be a homomorphism such that $\varphi\colon T \to W$ is a monomorphism. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first k coordinates. Show that there exists a linear isomorphism $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and a monomorphism $\chi\colon \mathbb{R}^k \to W$ such that $\varphi = \chi \circ p \circ \Phi$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow[\cong]{\Phi} & \Phi(T) \\ \big\downarrow{\scriptstyle \varphi} & & {\scriptstyle p}\big\downarrow{\scriptstyle (x,y)\mapsto x} \\ W & \xleftarrow{\quad\chi\quad} & \mathbb{R}^k. \end{array}$$

CHAPTER 42

Classification of certain diffeomorphisms up to diffeotopy

In this chapter we will see that we can classify some diffeomorphisms and smooth embeddings up to diffeotopy, respectively smooth isotopy.

This chapter is organized as follows:

- In Section 42.1 we will settle some basic notation.
- In Section 42.2 we will see that diffeotopies preserve orientation.
- In Section 42.3 we will classify diffeotopies of $\mathbb{R}^n$ and we will classify smooth embeddings of balls into $\mathbb{R}^n$.
- In Section 42.4 we will classify diffeotopies of S^1
- In Section 42.5 we will provide exercises for Chapter 42.

In the next Chapter 43 we will state and prove the Flow Theorem 43.3. Later this will be essential for constructing smooth diffeotopies.

42.1. **Basic notation.** Given a smooth manifold one would like, ideally, to classify all diffeomorphisms up to diffeotopy and given two smooth manifolds one would like, ideally, to classify all smooth embeddings from one into the other up to smooth isotopy. Before we consider explicit examples let us introduce a few relevant definitions.

Definition 42.1. Let M and N be two smooth manifolds of the same dimension.

(1) (a) We denote by $\mathrm{Diff}(M,N)$ the set of diffeomorphisms from M to N.
(b) By the Smooth Isotopy-Equivalence Relation Proposition 41.5 we know that "diffeotopy" defines an equivalence relation on $\mathrm{Diff}(M,N)$. We write
$$\widetilde{\mathrm{Diff}}(M,N) := \mathrm{Diff}(M,N)/\sim \qquad \text{where } f \sim g \text{ if } f \text{ and } g \text{ are diffeotopic.}$$

(2) If M and N are oriented, then it makes sense to restrict one selves to orientation-preserving diffeomorphisms:
(a) We define $\mathrm{Diff}^+(M,N)$ to be the set of orientation-preserving diffeomorphisms from M to N.
(b) As in (1b) we define
$$\widetilde{\mathrm{Diff}}^+(M,N) := \mathrm{Diff}^+(M,N)/\sim \qquad \text{where } f \sim g \text{ if } f \text{ and } g \text{ are diffeotopic.}$$

Furthermore, given a single smooth manifold M we write $\mathrm{Diff}(M) := \mathrm{Diff}(M,M)$ and accordingly for all the related notions.

Remark. If M is a smooth manifold, then $\mathrm{Diff}(M)$ is evidently a group. In Exercise 42.1 we will show that this group structure descends to a group structure on $\widetilde{\mathrm{Diff}}(M)$. This group is also called the smooth mapping class group $\mathrm{MCG}^{\mathrm{sm}}(M)$. We will study it in greater detail in **??**. □

There is one elementary case that we can deal with immediately:

Lemma 42.2. (Diffeotopies-of-$[0,1]$-Lemma) Let $f\colon [0,1] \to [0,1]$ be a diffeomorphism.
(1) If f is orientation-preserving, then f is diffeotopic to the identity.
(2) If f is orientation-reversing, then f is diffeotopic to the map $x \mapsto 1-x$.
In particular $\widetilde{\mathrm{D}}^+([0,1]) = \{\mathrm{id}\}$ and $\widetilde{\mathrm{D}}([0,1]) = \{\mathrm{id}, x \mapsto 1-x\}$.

Proof. Let $f\colon [0,1] \to [0,1]$ be a diffeomorphism.

(1) If f is orientation-preserving, then $f'(x) > 0$ for all $x \in [0,1]$. It follows easily that the desired diffeotopy is given by

$$\begin{array}{rcl} [0,1] \times [0,1] & \to & [0,1] \\ (x,t) & \mapsto & \underbrace{f(x)\cdot(1-t) + x \cdot t}_{\substack{\text{for fixed } x \in [0,1] \\ \text{the derivative is } > 0}} \end{array}$$

(2) In the proof of (1) we just need to replace $f(x)\cdot(1-t)+t$ by $f(x)\cdot(1-t)+(1-x)\cdot t$. ■

Instead of diffeomorphisms one might also be interested in smooth embeddings. This leads us to the following definition:

Definition 42.3. Let M and N be two smooth manifolds.
(1) (a) We denote by $\mathrm{Emb}(M,N)$ the set of smooth embeddings $M \to N$.
(b) By the Smooth Isotopy-Equivalence Relation Proposition 41.5 we know that "smooth isotopy" defines an equivalence relation on $\mathrm{Emb}(M,N)$. We write
$\widetilde{\mathrm{Emb}}(M,N) := \mathrm{Emb}(M,N)/\sim$ where $f \sim g$ if f and g are smoothly isotopic.
(2) If M and N are oriented and of the same dimension, then it makes sense to restrict one selves to orientation-preserving smooth embeddings.
(a) We denote by $\mathrm{Emb}^+(M,N)$ the set of orientation-preserving smooth embeddings $M \to N$.
(b) We set
$\widetilde{\mathrm{Emb}}^+(M,N) := \mathrm{Emb}^+(M,N)/\sim$ where $f \sim g$ if f and g are smoothly isotopic.

Given two smooth manifolds M and N it is in general surely too difficult to describe $\mathrm{Diff}(M,N)$ and $\mathrm{Emb}(M,N)$ in a meaningful way. It is more realistic to try to get a handle on $\widetilde{\mathrm{Diff}}(M,N)$ and $\widetilde{\mathrm{Emb}}(M,N)$. But even this problem becomes very hard very quickly.

Example. In the figure below we see two smooth embeddings $f, g\colon S^1 \to \mathbb{R}^3$:



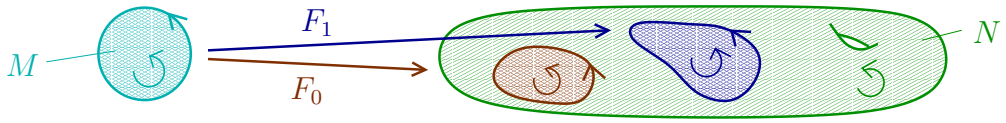
It is perhaps intuitively clear that the two smooth embeddings are not smoothly isotopic, i.e. that they represent different points in $\widetilde{\mathrm{Emb}}(S^1, \mathbb{R}^3)$. But a proper proof will have to wait till the Trefoil–Non-Trivial Proposition 111.19. □

The example shows that if at this stage we want to give a classification we will need to restrict ourselves to fairly basic situations. In fact in this chapter we will consider the following:

(1) diffeomorphisms of $\mathbb{R}^n$,
(2) smooth embeddings $\overline{B}^n \to \mathbb{R}^n$,
(3) smooth embedding $\overline{B}^k \to \mathbb{R}^n$ for $k < n$,
(4) diffeomorphisms of S^1.

42.2. **Diffeotopies and orientations.** We start out with the following proposition, which might be intuitively obvious, but which is somewhat subtle to prove.

Proposition 42.4. (Smooth Isotopy-Orientation Proposition) Let M and N be oriented connected smooth manifolds of the same dimension and let $F\colon M \times [0,1] \to N$ be a smooth isotopy.[474] Then either all smooth embeddings $F_t\colon M \to N$ are orientation-preserving or all F_t are orientation-reversing.



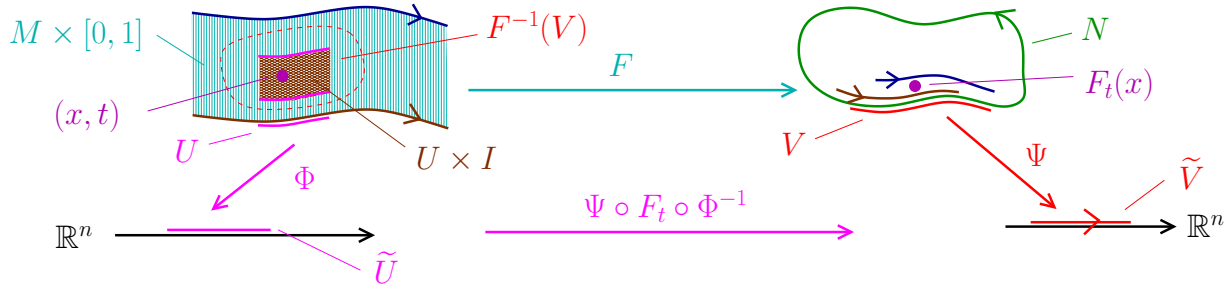
Proof. Let M and N be oriented connected n-dimensional smooth manifolds. Furthermore let $F\colon M \times [0,1] \to N$ be a smooth isotopy. We consider the sets

$$\begin{aligned} A &:= \{(x,t) \in M \times [0,1] \,|\, \mathrm{D}(F_t)_x\colon \mathrm{T}_x M \to \mathrm{T}_{F_t(x)} N \text{ is orientation-preserving}\} \\ B &:= \{(x,t) \in M \times [0,1] \,|\, \mathrm{D}(F_t)_x\colon \mathrm{T}_x M \to \mathrm{T}_{F_t(x)} N \text{ is orientation-reversing}\}. \end{aligned}$$

We need to show that $A = M \times [0,1]$ or $B = M \times [0,1]$. Since M is connected we know, by the Product Topology-Properties Proposition 7.9 (2), that $M \times [0,1]$ is also connected. Thus it suffices to prove the following statement:

Statement. The subsets A and B are both open subsets of $M \times [0,1]$.

Proof. We will show that A is open. The proof that B is open is basically the same. Now let $(x,t) \in A$. We pick an orientation-preserving chart[475] $\Psi\colon V \to \widetilde{V}$ for $F_t(x) \in N$. Since $F\colon M \times [0,1] \to N$ is in particular continuous we know that $F^{-1}(V)$ is open. Since $(x,t) \in F^{-1}(V)$ there exists an open subset of the form $U \times I$ of $M \times [0,1]$ such that $(x,t) \in U \times I \subseteq F^{-1}(V)$. After possibly making U even smaller we can assume that there exists an orientation-preserving chart $\Phi\colon U \to \widetilde{U}$ for M.



[474] The case of a diffeotopy is of course an important special case.

[475] Here we equip the open subset $\widetilde{V}$ of $\mathbb{R}^n$ with the standard orientation.

Claim. The map[476]
$$\begin{aligned}\theta\colon U\times I &\to \mathbb{R}\setminus\{0\}\\ (y,s) &\mapsto \det\big(\underbrace{\mathrm{D}(\Psi\circ F_s\circ\Phi^{-1})_{\Phi(y)}}_{(n\times n)\text{-matrix}}\big)\end{aligned}$$
has the following two properties:

(1) For $(y,s)\in U\times I$ we have $(y,s)\in A$ if and only if $\theta(y,s)>0$.
(2) The map θ is continuous.

Proof.

(1) Let $(y,s)\in U\times I$. By the Tangent Space-Functor Proposition 29.15 we have
$$\mathrm{D}(\Psi\circ F_s\circ\Phi^{-1})_{\Phi(y)} = \mathrm{D}\Psi_{F_s(y)}\circ\mathrm{D}(F_s)_y\circ\mathrm{D}(\Phi^{-1})_{\Phi(y)}.$$
Since the charts Ψ and Φ are by our choice orientation-preserving we know that the differentials $\mathrm{D}\Psi_{F_s(y)}\colon \mathrm{T}_{F_s(y)}M\to\mathrm{T}_{\Psi(F_s(y))}\widetilde{V}$ and $\mathrm{D}(\Phi^{-1})_{\Phi(y)}\colon \mathrm{T}_{\Phi(y)}\widetilde{U}\to\mathrm{T}_yU$ are orientation-preserving. It follows from the elementary Orientation-Composition Lemma 31.8 that the composition is orientation-preserving if and only if $\mathrm{D}(F_s)_y$ is orientation-preserving. But that is what we needed to show.

(2) Note that for each $\widetilde{y}\in\widetilde{U}$ and each $s\in I$ we have
$$(\Psi\circ F_s\circ\Phi^{-1})(\widetilde{y}) \;=\; (\Psi\circ F\circ \underbrace{(\Phi^{-1}\times \mathrm{id}_I)}_{\text{map }\widetilde{U}\times I\to U\times I})(\widetilde{y},s).$$
Since $\Psi\circ F\circ(\Phi^{-1}\times\mathrm{id}_{[0,1]})$ is a smooth map from $\widetilde{U}\times I\subseteq\mathbb{R}^n\times\mathbb{R}$ to $\mathbb{R}^n$ we see that the determinant of the differential is a continuous map on $\widetilde{U}\times I$. But this implies that θ is indeed continuous. ⊞

Since $(x,t)\in A$ we obtain from the claim that $\theta(x,t)>0$. By the claim we also know that θ is continuous. Since $U\times I$ is connected we see that θ is positive on all of $U\times I$. But again by the claim this means that $U\times I\subseteq A$. It follows from the JH-Openness Criterion 3.3 that A is indeed open. ■

42.3. **Diffeotopies of $\mathbb{R}^n$ and smooth embeddings of balls into $\mathbb{R}^n$.** The following proposition is the main result of this section:

Proposition 42.5. (Diffeotopies-of-$\mathbb{R}^n$-Proposition) Let $n\in\mathbb{N}_0$.

(1) Let $f\colon\mathbb{R}^n\to\mathbb{R}^n$ be a diffeomorphism.
  (a) If f is orientation-preserving, then f is diffeotopic to the identity map.
  (b) If f is orientation-reversing, then f is diffeotopic to the reflection $\rho\colon\mathbb{R}^n\to\mathbb{R}^n$ that is given by $(x_1,x_2,\dots,x_n)\mapsto(-x_1,x_2,\dots,x_n)$.
(2) Let $f\colon\overline{B}^n\to\mathbb{R}^n$ be a smooth embedding.
  (a) If f is orientation-preserving, then f is smoothly isotopic to the inclusion ι.
  (b) If f is orientation-reversing, then f is smoothly isotopic to the reflection $\rho|_{\overline{B}^n}$.

[476] As usual we use the Differential-equals-Differential Lemma 29.6 and the Tangent Space-Isomorphism Proposition 29.24 to freely go back and forth between the different notions of differentials.

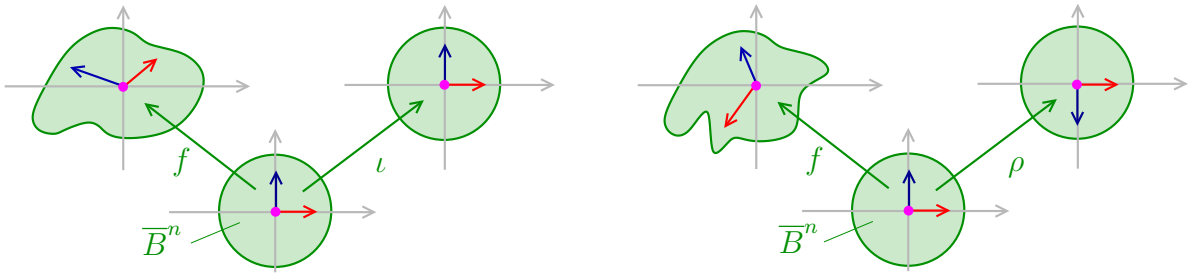
The same conclusion also holds for smooth embeddings $f\colon \overline{B}^n \to B^n_r(0)$ into a ball of radius $r > 1$.

(3) Let $k < n$. Every smooth embedding $f\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to the "standard" smooth embedding given by

$$\begin{array}{rcl} \iota\colon \overline{B}^k & \to & \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \\ x & \mapsto & (x, 0). \end{array}$$

The same conclusion also holds for smooth embeddings $f\colon \overline{B}^k \to B^n_r(0)$ into a ball of radius $r > 1$.

In any of the above cases we have the following refinement: If $f(0) = 0$, then we can find a diffeotopy rel $\{0\}$ respectively a smooth isotopy rel $\{0\}$.



f orientation-preserving $\qquad$ f orientation-reversing

Remark. In the Smooth Ball Embedding Theorem 45.10 and in the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 we will classify smooth embeddings of closed balls into smooth manifolds. This result can be viewed as a generalization of the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (2). □

The proof of the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 rests on the following lemma, which will play a role on a few other occasions.

Lemma 42.6. (Isotopic-to-Differential Lemma) Let $U \subseteq \mathbb{R}^n$ be a convex codimension-zero submanifold and let $f\colon U \to \mathbb{R}^n$ be a smooth embedding with $f(0) = 0$. The map

$$\begin{array}{rcl} F\colon U \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \begin{cases} \frac{1}{t} \cdot f(t \cdot x), & \text{if } t \neq 0, \\ \mathrm{D}f_0 \cdot x, & \text{if } t = 0 \end{cases} \end{array}$$

is a smooth isotopy from the map that is given by multiplication by $\mathrm{D}f_0$ to the original map $f\colon U \to \mathbb{R}^n$.

Proof of the Isotopic-to-Differential Lemma 42.6. We start out with the following observation: Since U is a *convex* codimension-zero submanifold of $\mathbb{R}^n$ that contains the origin and since $f(0) = 0$ we know by the Map-as-Matrix Multiplication Proposition 24.23 that there exists a smooth map $\Theta\colon U \to \mathrm{M}(n \times n, \mathbb{R})$ such that $\Theta(0) = \mathrm{D}f_0$ and such that for all $x \in U$ we have $f(x) = \Theta(x) \cdot x$. Note that the map $F\colon U \times [0,1] \to \mathbb{R}^n$ can be

written as follows:

$$\begin{array}{rcl} F\colon U\times[0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \underbrace{\Theta(t\cdot x)}_{\in \mathrm{M}(n\times n,\mathbb{R})}\cdot x. \end{array}$$

This map is evidently smooth. It remains to show that each F_t is a smooth embedding. This is clear for $t=0$. For $t\neq 0$ we see that

$$F_t(x) \;=\; \Theta(t\cdot x)\cdot x \;=\; \frac{1}{t}\cdot\Theta(t\cdot x)\cdot(t\cdot x) \;=\; \frac{1}{t}\cdot f(t\cdot x).$$

Thus we see that F_t is the composition of the three smooth embeddings $x\mapsto t\cdot x$, $y\mapsto f(y)$ and $z\mapsto \frac{1}{t}z$. Hence F_t is a smooth embedding itself. ∎

Proof of the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5. We start out with a few preparations:

- In the following we use on several occasions the fact, proved in the Smooth Isotopy-Equivalence Relation Proposition 41.5, that being diffeotopic and being smoothly isotopic are an equivalence relation.
- Given a matrix $A\in \mathrm{M}(m\times n,\mathbb{R})$ we denote by $\mu(A)\colon \mathbb{R}^n\to\mathbb{R}^m$ the map that is given by multiplication by A.
- We use the Differential-equals-Differential Lemma 29.6 as an excuse not to distinguish in our notation between the differential as a matrix and as a map.

Now we turn to the actual proof.

(1) First let $f\colon \mathbb{R}^n\to\mathbb{R}^n$ be an orientation-preserving diffeomorphism. Evidently there is a diffeotopy from f to a map that sends the origin to the origin. So by the Smooth Isotopy-Equivalence Relation Proposition 41.5 we might as well assume that $f(0)=0$. We write $A:=\mathrm{D}f_0$. First note that it follows from the Isotopic-to-Differential Lemma 42.6 that the map f is diffeotopic to the map $\mu(A)$.

Next note that our hypothesis that the diffeomorphism f is orientation-preserving implies by the Manifold–Orientation Basics Lemma 31.31 that $\det(A)>0$, in other words, $A\in \mathrm{GL}_+(n,\mathbb{R})$. By the Diffeotopies-via-Matrices Lemma 41.12 we know that multiplication by A is diffeotopic to the identity. Thus we are done by the Smooth Isotopy-Equivalence Relation Proposition 41.5.

Now we consider the case that f is orientation-reversing. There are two approaches to dealing with this situation. We can run exactly the same argument as above, except that we now use that $\det(\mathrm{D}f_0)<0$ and that therefore $\mathrm{D}f_0$ lies in the same path-component of $\mathrm{GL}(n,\mathbb{R})$ as the orthogonal matrix representing the reflection ρ.
Alternatively we cannot e that $\rho\circ f$ is orientation-preserving. We can apply the above result to $\rho\circ f$ and we obtain a diffeotopy $F\colon \mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ with $F_0=\rho\circ f$ and $F_1=\mathrm{id}$. Since $\rho\circ\rho=\mathrm{id}$ we see that $\rho\circ F$ is now the desired diffeotopy from f to ρ.

(2) Let $f\colon \overline{B}^n\to\mathbb{R}^n$ be a smooth embedding. It follows by basically the same argument as in (1) that the smooth embedding is either smoothly isotopic to the standard inclusion or to the reflection. We leave the few remaining details to the reader.

Now assume that we are given an $r>1$ and an orientation-preserving smooth embedding $f\colon \overline{B}^n\to B^n_r(0)$. As we saw in Exercise 24.1, using the Smooth Transition Function Lemma 24.9 one can easily find a diffeomorphism $\varphi\colon[0,r)\to[0,\infty)$ which is the identity on $[0,1]$. This diffeomorphism φ gives rise to the orientation-preserving

diffeomorphism
$$\begin{array}{rcl}\Phi\colon B_r^n(0) & \to & \mathbb{R}^n \\ v & \mapsto & \varphi(\|v\|)\cdot v\end{array}$$
which is the identity on $\overline{B}^n$. By the above discussion there exists a smooth isotopy $H\colon \overline{B}^n \times [0,1] \to \mathbb{R}^n$ with $H_0 = \Phi \circ f$ and such that H_1 is the inclusion. The desired smooth isotopy is now given by $(x,t) \mapsto \Phi^{-1}(H(x,t))$. The case that f is orientation-reversing is handled in the same way.

(3) Let $k < n$ and let $f\colon \overline{B}^k \to \mathbb{R}^n$ be a smooth embedding. As in (1) we might as well assume that $f(0) = 0$. Throughout the following discussion we make use of the following two projections:
$$\begin{array}{rcl}p\colon \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} & \to & \mathbb{R}^k \\ (x,y) & \mapsto & x\end{array} \quad \text{and} \quad \begin{array}{rcl}q\colon \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} & \to & \mathbb{R}^{n-k} \\ (x,y) & \mapsto & y.\end{array}$$

Claim 1. The smooth embedding $f\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to a smooth embedding $g\colon \overline{B}^k \to \mathbb{R}^n$ such that $\mathrm{D}(p\circ g)_0\colon \mathrm{T}_0\,\overline{B}^k \to \mathbb{R}^k$ is an isomorphism.

Proof. Since f is a smooth embedding we know that the matrix $\mathrm{D}f_0 \in \mathrm{M}(n\times k)$ has rank k. It follows from $k \leq n$ and some elementary linear algebra that there exists a matrix $A \in \mathrm{GL}_+(n,\mathbb{R})$ such that the first k rows of $A \cdot \mathrm{D}f_0$ have rank k. This is equivalent to the statement that
$$p \circ \underbrace{\mathrm{D}(\mu(A)\circ f)_0}_{\text{represented by } A\cdot \mathrm{D}f_0} = \mathrm{D}(p\circ\mu(A)\circ f)_0\colon \mathrm{T}_0\,\overline{B}^k \to \mathbb{R}^k$$
is an isomorphism. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{GL}_+(n)$ is path-connected. It follows from this fact, together with the Smooth Path-Connectivity Proposition 25.32, that there exists a smooth map $\gamma\colon [0,1] \to \mathrm{GL}_+(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. Similar to the Diffeotopies-via-Matrices Lemma 41.12 (1a) we now consider the smooth isotopy
$$\begin{array}{rcl}F\colon \overline{B}^k \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \gamma(t)\cdot f(x).\end{array}$$
⊞

Claim 2. The smooth embedding $g\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to a smooth embedding $h\colon \overline{B}^k \to \mathbb{R}^n$ such that $p\circ h\colon \overline{B}^k \to \mathbb{R}^k$ is a smooth embedding.

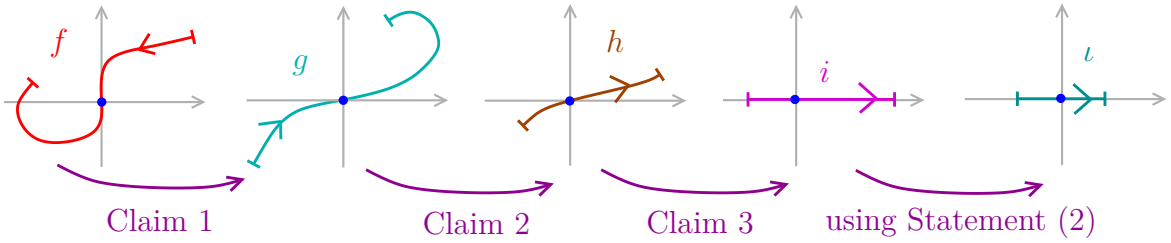
Proof. Recall that the map $\mathrm{D}(p\circ g)_0\colon \mathrm{T}_0\,\overline{B}^k \to \mathbb{R}^k$ is an isomorphism. It follows almost immediately from the Inverse Mapping Theorem 30.1 that there exists an $r > 0$ such that $p\circ g\colon \overline{B}^k_r(0) \to \mathbb{R}^k$ is a smooth embedding. The desired smooth isotopy is now given by
$$\begin{array}{rcl}G\colon \overline{B}^k \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & g(x\cdot(1-t+t\cdot r)).\end{array}$$
⊞

Claim 3. The smooth embedding $h\colon \overline{B}^k \to \mathbb{R}^n$ is smoothly isotopic to a smooth embedding $i\colon \overline{B}^k \to \mathbb{R}^n$ that takes values in $\mathbb{R}^k \times \{0\}$.

Proof. We consider the map
$$\begin{array}{rcl}H\colon \overline{B}^k \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & ((p\circ h)(x), (1-t)\cdot(q\circ h)(x)).\end{array}$$

Since $p \circ h \colon \overline{B}^k \to \mathbb{R}^k$ is a smooth embedding we see that each $H_t \colon \overline{B}^k \to \mathbb{R}^n$ is a smooth embedding. In particular H is a smooth isotopy. Note that $H_0 = h$. Finally note that H_1 takes values in $\mathbb{R}^k \times \{0\}$. ⊞

Note that it follows from (2) that there exists a smooth isotopy $I \colon \overline{B}^k \times [0,1] \to \mathbb{R}^k$ from $p \circ i$ to $\mathrm{id}_{\overline{B}^k}$. Finally note that this smooth isotopy I gives rise to a smooth isotopy $\overline{B}^k \times [0,1] \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ from i to the smooth embedding $\iota \colon \overline{B}^k \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ that is given by $x \mapsto (x, 0)$.



Now assume that we are given an $r > 1$ and an orientation-preserving smooth embedding $f \colon \overline{B}^k \to B^n_r(0)$. As in (2) we can reduce this case to the case of a smooth embedding $\overline{B}^k \to \mathbb{R}^n$, so we are done by the above discussion. ■

The following is our first classification result of this chapter.

Corollary 42.7. (Diffeomorphism Group-of-$\mathbb{R}^n$-Corollary) Given $n \in \mathbb{N}$ and given $k \in \{1, \dots, n\}$ we consider the maps

$$\begin{array}{rcl} \rho \colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, \dots, x_n) & \mapsto & (-x_1, x_2, \dots, x_n) \end{array} \quad \text{and} \quad \begin{array}{rcl} \iota_k \colon \overline{B}^k & \to & \mathbb{R}^n \\ x & \mapsto & (x, 0) \end{array}$$

With this notation the following statements hold:

(1) For $n \in \mathbb{N}$ we have $\widetilde{\mathrm{Diff}}^+(\mathbb{R}^n) = \{[\mathrm{id}]\}$ and $\widetilde{\mathrm{Diff}}(\mathbb{R}^n) = \{[\mathrm{id}], [\rho]\}$.

(2) For $n \in \mathbb{N}$ we have $\widetilde{\mathrm{Emb}}^+(\overline{B}^n, \mathbb{R}^n) = \{[\iota_n]\}$ and $\widetilde{\mathrm{Emb}}(\overline{B}^n, \mathbb{R}^n) = \{[\iota_n], [\rho|_{\overline{B}^n}]\}$.

(3) For $k < n$ we have $\widetilde{\mathrm{Emb}}(\overline{B}^k, \mathbb{R}^n) = \{[\iota_k]\}$.

Proof. The three statements follow immediately from the combination of the Smooth Isotopy-Orientation Proposition 42.4 and the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5. ■

42.4. **Diffeotopies of S^1.** In the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 we obtained a very good understanding of diffeotopies of $\mathbb{R}^n$. In this section we will use the statement for $n = 1$ to classify the diffeomorphisms of S^1 up to diffeotopy. More precisely we will prove the following proposition:

Proposition 42.8. (Diffeotopies-of-S^1-Proposition) Let $f \colon S^1 \to S^1$ be a diffeomorphism.

(1) If f is orientation-preserving, then f is diffeotopic to the identity.

(2) If f is orientation-reversing, then f is diffeotopic to the reflection $\rho \colon S^1 \to S^1$ that is given by $(x, y) \mapsto (x, -y)$.

Remark.

(1) In the Homeomorphisms-of-S^1-Classification Lemma 18.6 we proved an analogous statement for self-homeomorphisms of S^1.

(2) In Exercise 43.5 we will use the Flow Theorem 43.21 to give an alternative proof of the Diffeotopies-of-S^1-Proposition 42.8.
(3) The Munkres–Smale-Theorem **??** and the Cerf Theorem **??** say that the obvious analogs of the Diffeotopies-of-S^1-Proposition 42.8 hold for S^2 and S^3. But in the Exotic Diffeomorphism Proposition 282.10 and the S^n-Mapping Class Group–Homotopy Sphere Theorem **??** that in general the naive analogs in higher dimensions do *not* hold. More precisely, there exist infinitely many $n \in \mathbb{N}_{\geq 6}$ such that the sphere S^n admits an orientation-preserving self-diffeomorphism that is not diffeotopic to the identity. □

Before we turn to the proof of the Diffeotopies-of-S^1-Proposition 42.8 let us first state and prove the following almost immediate useful corollary.

Corollary 42.9. (Diffeomorphisms-of-S^1-Extension Corollary) Every diffeomorphism $f\colon S^1 \to S^1$ extends to a diffeomorphism of $\overline{B}^2$.

Remark.

(1) In Exercise 8.3 we encountered the Alexander trick, which shows that every homeomorphism of S^n extends to a homeomorphism of $\overline{B}^{n+1}$.
(2) In contrast to the setting of homeomorphisms the analog of the Diffeomorphisms-of-S^1-Extension Corollary 42.9 does *not* hold in all dimensions. More precisely, in the Diffeomorphism–Non Extension Proposition 282.11 we will see that there exists a diffeomorphism $f\colon S^6 \to S^6$ that does not extend to a diffeomorphism of $\overline{B}^7$. □

Proof of Corollary 42.9 assuming Proposition 42.8. Let $f\colon S^1 \to S^1$ be a diffeomorphism. By the Diffeotopies-of-S^1-Proposition 42.8 we know that there exists a diffeotopy $F\colon S^1 \times [0,1] \to S^1$ from $F_0 = f$ to the identity $F_1 = \mathrm{id}$ or to the reflection $F_1 = \rho$. By the Stationary-Near-The-Ends-Lemma 41.7 we can arrange that F is trivial near the ends. In particular we can assume that there exists an $\epsilon > 0$ such that $F_t = F_1$ for all $t \in [1-\epsilon, 1]$. We consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^2 & \to & \overline{B}^2 \\ P = (x,y) & \mapsto & \begin{cases} P = (x,y), & \text{if } \|P\| \leq \frac{1}{2} \text{ and if } f \text{ is orientation-preserving,} \\ (x,-y), & \text{if } \|P\| \leq \frac{1}{2} \text{ and if } f \text{ is orientation-reversing,} \\ \|P\| \cdot F\big(\frac{P}{\|P\|}, 2 - 2\cdot\|P\|\big), & \text{if } \|P\| \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Note that it follows easily from the Smooth Pasting Proposition 25.19 that the above map $\Phi\colon \overline{B}^2 \to \overline{B}^2$ is indeed smooth. Furthermore in Exercise 42.5 we will write down an explicit inverse to Φ and we will show that the inverse is smooth. This implies that Φ is indeed a diffeomorphism. ■

The proof of the Diffeotopies-of-S^1-Proposition 42.8 requires some preparations, which in the author's eyes are actually interesting in their own right. First we introduce the following notation:

Notation 42.10. Throughout this section we work with the map

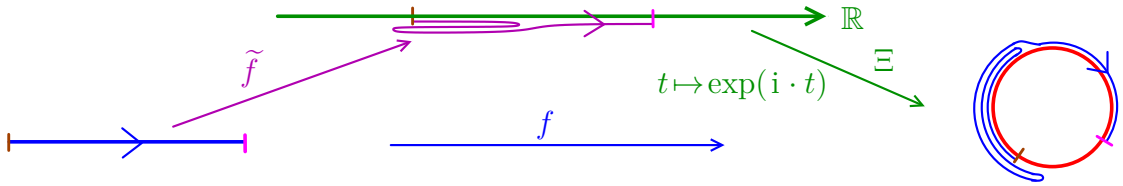
$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

Lemma 42.11. (Exponential-Local Diffeomorphism Lemma) Let $t \in \mathbb{R}$. The restriction of Ξ to $(t-\pi, t+\pi) \to S^1 \setminus \{-\Xi(t)\}$ is a diffeomorphism.

Proof. The lemma can be proved easily using the Smooth Embedding Theorem 30.15. Alternatively, as we saw in Exercise 25.3, it is not difficult to prove the lemma with one's bare hands. ■

Definition 42.12. Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \to S^1$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon I \to \mathbb{R}$ such that $\Xi \circ \widetilde{f} = f$, in other words, such that for all $t \in I$ we have

$$f(t) \;=\; \exp\big(\mathrm{i} \cdot \widetilde{f}(t)\big).$$



Remark. In Definition 92.1 and Definition 92.5 we will greatly generalize the notion of a lift. □

Lemma 42.13. (Smooth Lift Lemma) Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \to S^1$ be a continuous map. If f is smooth, then every lift is also smooth.

Proof. This statement follows from the Smooth Pasting Proposition 25.19 (3) together with the fact, implicitly shown in the Exponential-Local Diffeomorphism Lemma 42.11, that $\Xi\colon \mathbb{R} \to S^1$ is a local diffeomorphism. ■

In the proof of the Diffeotopies-of-S^1-Proposition 42.8 we will also make use of the following very interesting lemma.

Lemma 42.14. (Lift-from-S^1-to-$\mathbb{R}$ Lemma) Let $I \subseteq \mathbb{R}$ be an interval and let $x_0 \in I$. Given any continuous map $f\colon I \to S^1$ and given any $\widetilde{y}_0 \in \mathbb{R}$ with $f(x_0) = \exp(\mathrm{i} \cdot \widetilde{y}_0)$ there exists a *unique* lift $\widetilde{f}\colon I \to \mathbb{R}$ of f with $\widetilde{f}(x_0) = \widetilde{y}_0$.

Remark.

(1) The Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 is an immediate consequence of the general theory of coverings that we will study in great detail in Chapter 91. More precisely, it follows from the discussion on page 1776 and the Action–Covering Proposition 91.11 that the map $\Xi\colon \mathbb{R} \to S^1$ is a covering. The statement of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 follows now from a slight variation on the Path-Lifting Proposition 92.6 (1). Since at this stage we do not assume that the reader is familiar with covering theory we will now give a self-contained proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14.

(2) If $I = [a, b]$ is a compact interval, then we will provide an alternative proof in the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21. □

Proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14. Our first claim already deals with the uniqueness statement of the lemma.

Claim 1. Let J be an interval, let $f\colon J \to S^1$ be a continuous map and let $\widetilde{\varphi}, \widetilde{\psi}\colon J \to \mathbb{R}$ be two lifts of f. If $\widetilde{\varphi}$ and $\widetilde{\psi}$ agree at some point $t_0 \in J$, then they agree everywhere.

Proof. We consider the map

$$\begin{aligned} \theta\colon J &\to \mathbb{R} \\ s &\mapsto \theta(s) \coloneqq \widetilde{\varphi}(s) - \widetilde{\psi}(s). \end{aligned}$$

By hypothesis we have $\theta(t_0) = 0$. Furthermore note that for each $s \in J$ we have

$$\underbrace{\exp\big(\mathrm{i}\cdot(\widetilde{\varphi}(s) - \widetilde{\psi}(s))\big)}_{=\exp(\mathrm{i}\cdot\theta(s))} = \exp\big(\mathrm{i}\cdot\widetilde{\varphi}(s)\big)\cdot\exp\big(\mathrm{i}\cdot\widetilde{\psi}(s)\big)^{-1} \overset{\text{since } \widetilde{\varphi},\widetilde{\psi}\colon J\to\mathbb{R} \text{ are both lifts of } f}{=} f(s)\cdot f(s)^{-1} = 1.$$

This shows that the map $s \mapsto \theta(s) \coloneqq \widetilde{\varphi}(s) - \widetilde{\psi}(s)$ takes values in $2\pi\cdot\mathbb{Z} = \Xi^{-1}(1)$. Since J is connected it follows from the Connected-to-Discrete Lemma 4.37 that the map θ is constant. Since the map θ vanishes at t_0 we see that it is zero everywhere. ⊞

Claim 2. Let $b \in \mathbb{R}$ and let $s \in \mathbb{R}\cup\{\infty\}$ with $s > b$. If for every $c \in [b, s)$ there exists a lift of $f|_{[b,c]}$, then there exists also a lift of $f_{[b,s)}$.

Proof. It follows from the previous claim that the lifts on the various intervals agree on the overlaps. Thus they define a lift on all of $[b, s)$. ⊞

To simplify the notation we now only deal with the case that the interval is given by $I = [x_0, \infty)$. The proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 for all other types of intervals is done in a very similar fashion. We set

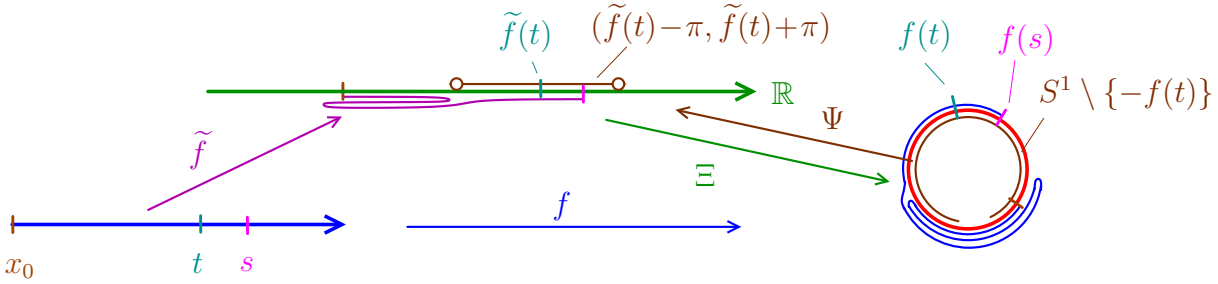
$$S \coloneqq \{s \in [x_0, \infty) \mid \text{there exists a lift of } \varphi|_{[x_0,s]}\}.$$

By Claim 2 it suffices to prove the following claim:

Claim 3. $S = [x_0, \infty)$.

Proof. Note that $x_0 \in S$. Since $[x_0, \infty)$ is connected it suffices to show that S is open and closed in $[x_0, \infty)$. It follows from elementary arguments, see the Interior-Closure Lemma 3.22 (5), that it suffices to show that for every $s \in [x_0, \infty)$ with $[x_0, s) \subseteq S$ there exists an $\epsilon > 0$, such that $[x_0, s+\epsilon) \subseteq S$.

Thus suppose that we are given an $s \in [x_0, \infty)$ with $[x_0, s) \subseteq S$. Note that by Claim 2 there exists a lift $\widetilde{f}\colon [x_0, s) \to \mathbb{R}$. We pick $t \in [x_0, s)$ such that $f([t, s]) \subseteq S^1 \setminus \{-f(s)\}$. Next note that by the Exponential-Local Diffeomorphism Lemma 42.11 we know that the map $\Xi\colon (\widetilde{f}(t) - \pi, \widetilde{f}(t) + \pi) \to S^1 \setminus \{-f(t)\}$ is a diffeomorphism. We denote its inverse by Ψ. Since f is continuous and since $f(s) \in S^1 \setminus \{-f(t)\}$ there exists an $\epsilon > 0$ such that $f([s, s+\epsilon)) \subseteq S^1 \setminus \{-f(t)\}$.



Finally we consider the map

$$\begin{array}{rcl} [x_0, s+\epsilon) & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \leq s, \\ \Psi(f(x)), & \text{if } x \in [s, s+\epsilon). \end{cases} \end{array}$$

Note that $\widetilde{f}$ and $\Psi \circ f$ are lifts of f on $[s,t]$ and note that they agree on t. Thus it follows from Claim 1 that they also agree on s. This shows that this map is in fact well-defined at s. It now follows from the Pasting Proposition 4.7 that this map is continuous. It is clear that it is a lift of $\varphi|_{[x_0,s+\epsilon)}$. Thus we see that $[x_0, s+\epsilon) \subseteq S$. ■

Now we can finally provide the long awaited proof of the Diffeotopies-of-S^1-Proposition 42.8.

Proof of the Diffeotopies-of-S^1-Proposition 42.8. Let $f\colon S^1 \to S^1$ be a diffeomorphism. First we consider the case that f is orientation-preserving. Note that by the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14, applied to the map $f \circ \Xi\colon \mathbb{R} \to S^1$, there exists a continuous map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle t \mapsto \exp(it)}\downarrow{\scriptstyle \Xi} & & {\scriptstyle \Xi}\downarrow{\scriptstyle t \mapsto \exp(it)} \\ S^1 & \xrightarrow{f} & S^1. \end{array}$$

We make the following observations regarding $\widetilde{f}$:

(a) As we observed in the Smooth Lift Lemma 42.13, it follows from the fact that Ξ is a local diffeomorphism, together with the hypothesis that f is a smooth map, that $\widetilde{f}$ is also smooth.
(b) Recall that on page 817 we showed that Ξ is an orientation-preserving local diffeomorphism. Since f is orientation-preserving we see that $\widetilde{f}$ is also orientation-preserving. As we saw in the elementary Exercise 31.18, the above implies that $\widetilde{f}$ is a strictly monotonously increasing function.
(c) Since f is a diffeomorphism and since the above diagram commutes we see that for any $x \in \mathbb{R}$ the restriction of $\widetilde{f}$ to the interval $[x, x+2\pi)$ defines a bijection with $[f(x), f(x)+2\pi)$.
(d) Given $x \in \mathbb{R}$ it follows immediately from the commutativity of the above diagram, that the equality $\widetilde{f}(x+2\pi) - \widetilde{f}(x) \in \mathbb{Z} \cdot 2\pi$ holds.
(e) It follows from (b), (c), (d) and the continuity of f that for any $x \in \mathbb{R}$ we have $\widetilde{f}(x+2\pi) = \widetilde{f}(x) + 2\pi$.

Now we consider the map

$$\begin{array}{rcl} \widetilde{F}\colon \mathbb{R} \times [0,1] & \to & \mathbb{R} \\ (x,t) & \mapsto & \widetilde{f}(x) \cdot (1-t) + x \cdot t. \end{array}$$

This map is evidently smooth. Since the map $x \mapsto x$ has the above Properties (a) to (e) one can see easily that for any $t \in [0,1]$ the map $\widetilde{F}_t\colon \mathbb{R} \to \mathbb{R}$ has the above Properties (a) to (e). This implies in particular that the map $\widetilde{F}$ descends to a map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^1 \\ (\Xi(x), t) & \mapsto & \Xi(\widetilde{f}(x) \cdot (1-t) + x \cdot t) \end{array}$$

where each $F_t\colon S^1 \to S^1$ is a homeomorphism. Since each $\widetilde{F}_t$ is a local diffeomorphism and since Ξ is a local diffeomorphism we obtain from the Smooth Pasting Proposition 25.19 (2) that each F_t is in fact a diffeomorphism. We have thus shown that F is indeed a diffeotopy from f to the identity.

Finally we consider the case that f is orientation-reversing. In this case $\rho \circ f$ is orientation-preserving. We apply the above result to $\rho \circ f$ and we obtain a diffeotopy $F\colon S^1 \times [0,1] \to S^1$ with $F_0 = \rho \circ f$ and $F_1 = \mathrm{id}$. Since $\rho \circ \rho = \mathrm{id}$ we see that $\rho \circ F$ is now the desired diffeotopy from f to ρ. ■

The following is our second classification result of this chapter.

Corollary 42.15. (Diffeomorphism Group-of-S^1-Corollary) Let $\rho\colon S^1 \to S^1$ be the reflection that is given by $(x,y) \mapsto (x,-y)$. With this notation the following statements hold: $\widetilde{\mathrm{Diff}}^+(S^1) = \{[\mathrm{id}]\}$ and $\widetilde{\mathrm{Diff}}(S^1) = \{[\mathrm{id}],[\rho]\}$.

Proof. The statements follow immediately from the combination of the Smooth Isotopy-Orientation Proposition 42.4 and the Diffeotopies-of-S^1-Proposition 42.8. ■

42.5. Exercises for Chapter 42.

Exercise 42.1. Let M be a smooth manifold.

(a) Let $f, \widetilde{f}, g, \widetilde{g}\colon M \to M$ be diffeomorphisms. We assume that f and $\widetilde{f}$ are diffeotopic and that g and $\widetilde{g}$ are diffeotopic. Show that $g \circ f$ and $\widetilde{g} \circ \widetilde{f}$ are diffeotopic.

(b) Let $f, \widetilde{f}\colon M \to M$ be two diffeomorphisms. We suppose that there exists a diffeotopy $F\colon M \times [0,1] \to M$ between f and $\widetilde{f}$. Show that f^{-1} and $\widetilde{f}^{-1}$ are diffeotopic.
Remark. There is an obvious candidate map $G\colon M \times [0,1] \to M$ with $G_0 = f^{-1}$ and $G_1 = \widetilde{f}^{-1}$. But why is it smooth? Note that this exercise is similar to Exercise 18.7.

(c) Show that the obvious group structure on $\mathrm{Diff}(M)$ descends to a group structure on the set $\widetilde{\mathrm{Diff}}(M)$ of diffeotopy classes of self-diffeomorphisms of M.

Exercise 42.2. Let $I \subseteq \mathbb{R}$ be an interval and let $x_0 \in I$. Let $f, g\colon I \to S^1$ be two continuous maps with $f(x_0) = g(x_0)$ and let $\widetilde{f}, \widetilde{g}\colon I \to \mathbb{R}$ be two lifts with $\widetilde{f}(x_0) = \widetilde{g}(x_0)$. Show that if $f, g\colon I \to S^1$ are homotopic rel $\{x_0\}$, then $\widetilde{f}$ and $\widetilde{g}$ are homotopic rel $\{x_0\}$.

Exercise 42.3. We consider the map

$$\begin{array}{rccc} \exp\colon & \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \\ & z & \mapsto & \exp(z). \end{array}$$

Show that given any continuous map $f\colon [a,b] \to \mathbb{C} \setminus \{0\}$ and given any $w \in \mathbb{C}$ with $\exp(z) = w$ there exists a unique continuous map $\widetilde{f}\colon [a,b] \to \mathbb{C} \setminus \{0\}$ with $\widetilde{f}(a) = z$ and such that $\exp \circ \widetilde{f} = f$, i.e. such that for all $t \in [a,b]$ we have $\exp(\widetilde{f}(t)) = f(t)$.

Exercise 42.4.

(a) Show that every diffeomorphism of the half-open interval $[0,1)$ is orientation-preserving.

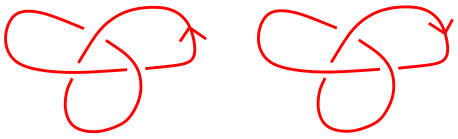
(b) Show that every diffeomorphism of the half-open interval $[0,1)$ is diffeotopic to the identity.

Exercise 42.5. Let $f\colon S^1 \to S^1$ be a diffeomorphism and let $F\colon S^1 \times [0,1] \to S^1$ be a diffeotopy with $F_0 = f$ and $F_1 = \mathrm{id}$. Furthermore let $\varphi\colon [0,1] \to [0,1]$ be a smooth map with $\varphi(t) = 0$ for $t \in [0, \frac{1}{4}]$ and with $\varphi(t) = 1$ for $t \in [\frac{3}{4}, 1]$. We consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^2 & \to & \overline{B}^2 \\ P = (x,y) & \mapsto & \begin{cases} P = (x,y), & \text{if } \|P\| \le \frac{1}{2}, \\ \|P\| \cdot F\big(\frac{P}{\|P\|}, \varphi(2 - 2\cdot\|P\|)\big), & \text{if } \|P\| \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Write down an explicit inverse of Φ and show that the inverse is smooth.

Exercise 42.6. For the purpose of this exercise an *oriented knot* is an oriented submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1. An oriented knot K is called *reversible* if there exists a smooth isotopy $F\colon K \times [0,1] \to \mathbb{R}^3$ such that $F_0\colon K \to \mathbb{R}^3$ is the inclusion and such that $F_1\colon K \to \mathbb{R}^3$ restricts to an orientation-reversing diffeomorphism $F_1\colon K \to K$. Show that the trefoil and the figure-8 knot are reversible. In other words, show that the two oriented knots shown in the figure to the left, respectively to the right, are smoothly isotopic.



trefoil with the two orientations

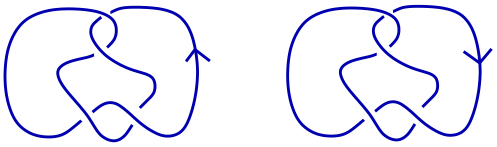


figure-8 knot with the two orientations

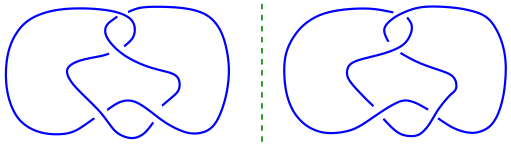
Exercise 42.7. For the purpose of this exercise a *knot* is a submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1. We say that two knots K and J are *smoothly isotopic* if there exists a smooth isotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ such that $F_0(S^1) = K$ and $F_1(S^1) = J$. We define the *mirror image* K^{mir} of a knot K to be the reflection of K in a hyperplane of $\mathbb{R}^3$.

(a) Show that any two mirror images of a given knot are smoothly isotopic.
(b) We say that a knot is *amphichiral* if it is smoothly isotopic to its mirror image. Show that the figure-8 knot is amphichiral.

Remark. In the Trefoil-is-Chiral Proposition 115.15 and also in the Trefoil-is-Chiral Proposition **??** we will see that the trefoil is not amphichiral.



smoothly isotopic knots



the figure-8 knot K and its mirror K^{mir}

Exercise 42.8. What is the cardinality of the set $\widetilde{\mathrm{Emb}}(\{1,\dots,n\}, S^1)$?

CHAPTER 43

The Flow Theorem

In Definition 29.29 we introduced the notion of a smooth vector field on a smooth manifold. In this section we state the Flow Theorem that shows in particular that smooth vector fields give rise to diffeotopies. The Flow Theorem cuts both ways. It will help us to construct interesting diffeotopies using smooth vector fields, this occurs for example in the proof of the Isotopy Extension Theorem 44.2. But later on, using information on diffeotopies, it will also help us to prove results on what type of smooth vector fields can occur on a smooth manifold. One such instance is given in the proof of the Smooth Poincaré-Hopf Theorem 53.24 and much later in the proof of the Euler Characteristic–Vector Field Theorem 182.22.

This chapter is organized as follows:

- In Section 43.1 we will state the Flow Theorem 43.3.
- In Section 43.2 we will prove the Flow Theorem 43.3.
- In Section 43.3 we will prove the Flow Fixed Point Theorem 43.9.
- In Appendix 43.4 we will study zeros of vector fields.
- In Appendix 43.5 we will discuss flows on non-compact smooth manifolds.
- In Section 43.6 we will Moving points on smooth manifolds
- In Section 43.7 we will provide exercises for Chapter 43.

In the next Chapter 44 we will use the Flow Theorem 43.3 to prove the Isotopy Extension Theorem 44.2.

43.1. Statement of the Flow Theorem.

The following definition might bring back fond memories from real analysis:

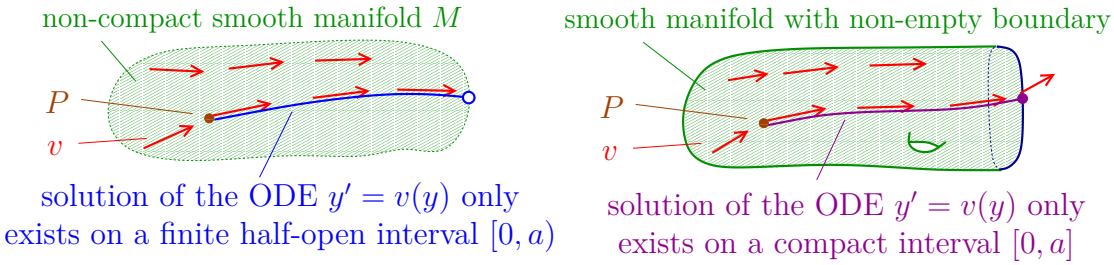
Definition 43.1. Let M be a smooth manifold and let v be a smooth vector field on M, as introduced in Definition 29.27.

(1) Let I be an interval with non-empty interior. We say that a smooth map $\varphi\colon I \to M$ is a[477] solution of the ODE $y' = v(y)$ if for every $t \in I$ we have the following equality:[478]

$$\varphi'(t) \;=\; v(\varphi(t)) \;\in\; \mathrm{T}_{\varphi(t)}M.$$

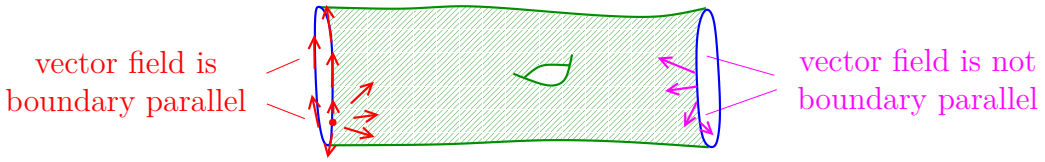
(2) A complete v-flow is a smooth map $F\colon M \times \mathbb{R} \to M$ that satisfies the following properties:
   (a) For each $P \in M$ we have $F(P, 0) = P$.
   (b) For each $P \in M$ the map $\mathbb{R} \to M$ given by $t \mapsto F(P, t)$ is a solution of the ODE $y' = v(y)$.

The reader might remember from an earlier course the Fundamental Theorem 43.15 for Autonomous ODEs which says, under mild hypotheses, that "short term" solutions of ODEs exist and that they are unique in a suitable sense. But note, as we can see in the figure below, these solutions might not be defined on all of $\mathbb{R}$. In particular there are hazards if the smooth manifold is non-compact and also if the smooth manifold has non-empty boundary.



We will get around these issues by considering only compact smooth manifolds and by restricting ourselves to the following type of vector fields.

Definition 43.2. Let M be a smooth manifold and let v be a vector field on M. We say that v is boundary parallel if for each $P \in \partial M$ we have[479] $v(P) \in \mathrm{T}_P \partial M$.



Now we are in a position to state the main theorem of this chapter.

Theorem 43.3. (Flow Theorem) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. There exists a unique complete v-flow $\Phi^v \colon M \times \mathbb{R} \to M$.

Remark. Statements that are very similar to our Flow Theorem 43.3 can also be found in [**Lee2002**, Theorem 9.12], [**Wall2016**, Theorem 1.4.2] and [**BJ1982**, Theorem 8.10]. □

The proof of the Flow Theorem 43.3 will consist of massaging the Fundamental Theorem for Autonomous ODEs 43.15, which, to most readers who made it that far, will be familiar from an earlier real analysis class. The proof of the Flow Theorem 43.3 is somewhat lengthy, thus we postpone the proof to the next section. In the following we prefer to discuss the Flow Theorem 43.3 and to consider an interesting addendum.

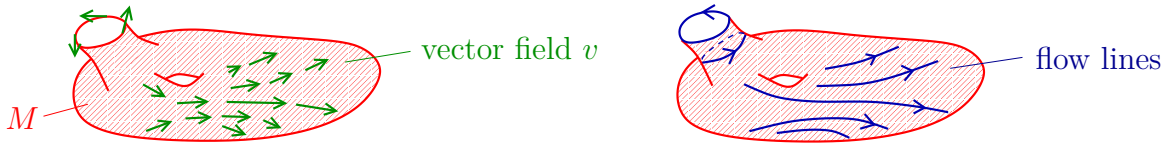
[477]Here ODE is short for "ordinary differential equation". Note that we do not define the notion of an ODE, we just define the expression "solution of an ODE $y' = v(y)$".

[478]We refer to Definition 29.26 for the definition of $\varphi'(t)$. As discussed in Notation 24.8, the fact that the interior of the interval is non-empty implies that we can talk sensibly of φ' at the endpoints of the interval.

[479]Here and throughout this chapter we will use the Convention 29.21 which says that given a smooth manifold M we can view the tangent spaces of the boundary ∂M and tangent spaces of submanifolds of M as vector subspaces of the tangent spaces $\mathrm{T}_P M$.

Definition 43.4. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel.

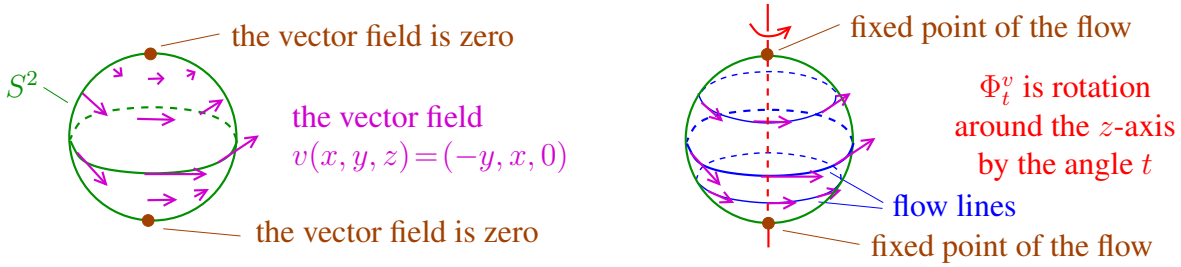
(1) For each $P \in M$ we refer to the subset $\Phi^v(\{P\} \times \mathbb{R}) \subseteq M$ as a flow line.[480]
(2) For $t \in \mathbb{R}$ the map $\Phi_t^v \colon M \to M$ that is given by $P \mapsto \Phi^v(P,t)$ is called the time-t map of the flow.



Example. We consider S^2 together with the smooth vector field that is given by

$$\begin{array}{rcl} v \colon S^2 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (-y,x,0). \end{array}$$

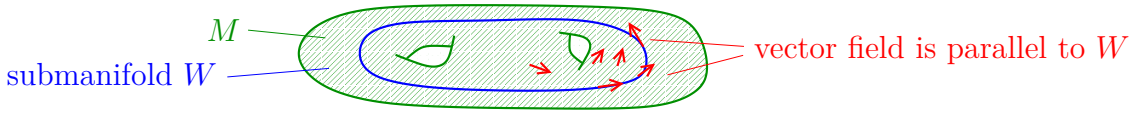
This vector field vanishes at the "North Pole" $(0,0,1)$ and at the "South Pole" $(0,0,-1)$. By the Flow Theorem 43.3 this vector field gives rise to a complex v-flow $\Phi^v \colon S^2 \times \mathbb{R} \to S^2$.



Note that it is pretty straightforward to see that for any $t \in \mathbb{R}$ the corresponding time-t map $\Phi_t^v \colon S^2 \to S^2$ is just given by rotation around the z-axis by the angle t. In particular we see that the flow lines are the circles of latitude. □

Our next definition is a self-explanatory variation on the notion of a boundary parallel vector field.

Definition 43.5. Let M be a smooth manifold and let v be a vector field on M. Let W be a submanifold of M. We say that v is parallel to the submanifold W if for each $P \in W$ we have $v(P) \in \mathrm{T}_P W$.



The following proposition summarizes a few properties of complete flows.

[480] We will not really make use of the concept of a flow line. But the concept will be useful in illustrations to draw pictures of flows.

Proposition 43.6. (Flow-Diffeotopy Proposition) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. We denote by $\Phi^v\colon M\times\mathbb{R}\to M$ the corresponding unique v-flow.

(1) For each $t\in\mathbb{R}$ the time-t map
$$\begin{array}{rccc}\Phi^v_t\colon & M & \to & M\\ & P & \mapsto & \Phi^v(P,t)\end{array}$$
is a diffeomorphism that is diffeotopic to the identity.

(2) Suppose that $W\subseteq M$ is a proper submanifold. If v is parallel to W, then the map Φ^v restricts to a map $W\times\mathbb{R}\to W$ and this map is the unique complete flow of the vector field $v|_W$ on the smooth manifold W.

Proof. We postpone the proof of the Flow-Diffeotopy Proposition 43.6 to the next section, more precisely, to page 1017. ■

We continue with the following definition that will pop up time and again throughout these notes.

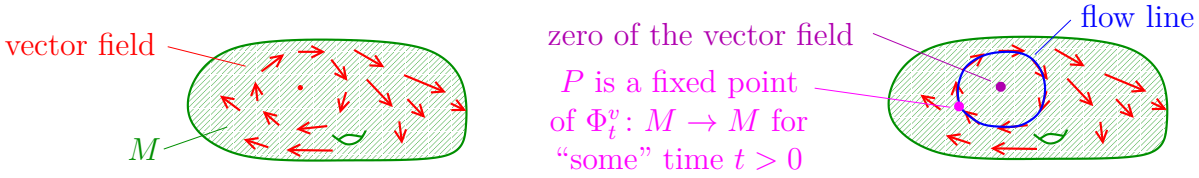
Definition 43.7. Let S be some set and let $f\colon S\to S$ be a continuous map. We say that $P\in S$ is a fixed point of f if $f(P)=P$.

The following lemma says that zeros of vector fields give rise to fixed points of the time-t maps of the complete v-flows.

Lemma 43.8. (Zero–Fixed Point Lemma) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M\times\mathbb{R}\to M$ be the corresponding unique complete v-flow from the Flow Theorem 43.3. If P is a zero of the vector field, i.e. if $v(P)=0$, then for every $t\in\mathbb{R}$ we have $\Phi^v(P,t)=P$. Put differently, for every $t\in\mathbb{R}$ the point P is a fixed point of $\Phi^v_t\colon M\to M$.

Proof. This statement follows immediately from the uniqueness statement in the Fundamental Theorem 43.15 for Autonomous ODEs, applied to the map $t\mapsto\Phi^v(P,t)$ and the constant map $t\mapsto P$. ■

Now let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M\times\mathbb{R}\to M$ be the corresponding unique complete v-flow from the Flow Theorem 43.3. If $P\in M$ is a point with $v(P)\neq 0$ it can of course still happen that there exists a $t\in\mathbb{R}$ such that $\Phi^v_t(P)=P$. For example let us consider the vector field v on S^2 that we studied on page 1010. In this case we see that $\Phi^v_{2\pi}=\mathrm{id}_{S^2}$.



The following theorem says that if a vector field has no zeros, then at least for a short time period each time-t map is fixed-point free.

Theorem 43.9. (Flow Fixed Point Theorem) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M\times\mathbb{R}\to M$ be the corresponding unique complete v-flow from the Flow Theorem 43.3. If v has no zeros,

then there exists an $\epsilon > 0$ such that for each $t \in (0, \epsilon]$ the map $\Phi_t^v \colon M \to M$ is fixed-point free. In other words, given any $t \in (0, \epsilon]$ and given any $P \in M$ we have $\Phi_t^v(P) \neq P$.

Proof. To keep the discussion of the main results as clean as possible we postpone the proof of the Flow Fixed Point Theorem 43.9 to page 1019. ■

Remark.

(1) In the Replace-by-Smooth Vector Field Proposition 36.7 we showed that if a smooth manifold admits a nowhere-vanishing vector field[481], then it also admits a smooth nowhere-vanishing vector field.

(2) Later, in the Euler Characteristic–Vector Field Theorem 182.22, we will use the combination of the Flow Fixed Point Theorem 43.9 and the Flow-Diffeotopy Proposition 43.6 to show that every smooth vector field on a closed smooth manifolds with non-zero "Euler characteristic" has at least one zero. □

Most of the remainder of this chapter is dedicated to the proofs of Flow Theorem 43.3 and the Flow Fixed Point Theorem 43.9. Only in the very last section we give one more application of the Flow Theorem 43.3. Readers who happen to be allergic to technical proofs are advised to move on either to Section 43.6 or straight to the next chapter.

43.2. Proof of the Flow Theorem 43.3.

In this section we will provide the proof of Flow Theorem 43.3. We will use one major ingredient from real analysis, namely the Fundamental Theorem 43.15 for Autonomous ODEs. Besides this input we will provide all details.

We start our discussion of the proof of the Flow Theorem 43.3 with the following sanity-preserving convention.

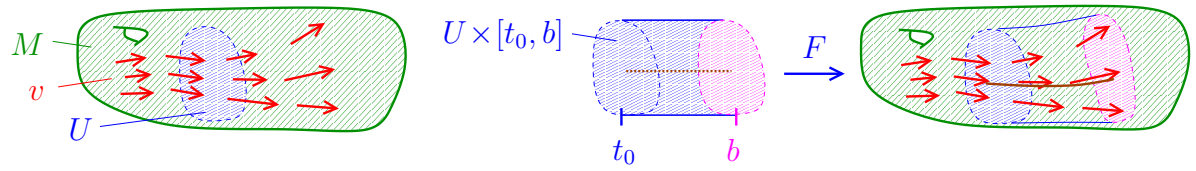
Convention 43.10. Given a smooth submanifold $U \subseteq \mathbb{R}^n$ we will use the Differential-equals-Differential Lemma 29.6 and the Tangent Space-Isomorphism Proposition 29.24 as an excuse for not distinguishing between the various notions of tangent spaces and differentials.

Throughout this section it is convenient to work with the following generalization of the notion of a complete flow.

Definition 43.11. Let M be a smooth manifold and let $t_0 \in \mathbb{R}$ and let v be a smooth vector field on M. A v-flow based at t_0 consists of an open subset $U \subseteq M$ and an interval neighborhood I of t_0 together with a smooth map $F \colon U \times I \to M$ that satisfies the following properties:

(1) For each $P \in U$ we have $F(P, t_0) = P$.

(2) For each $P \in U$ the map $I \to M$ that is given by $t \mapsto F(P, t)$ is a solution of the ODE $y' = v(y)$.



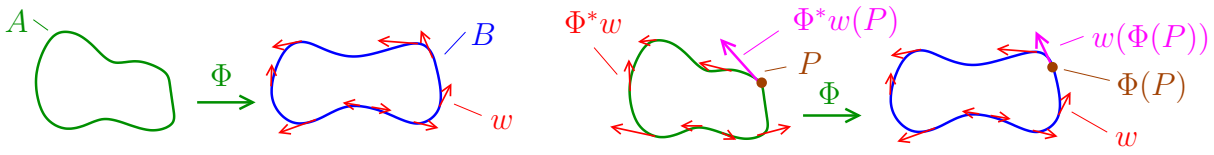
[481] Recall that according to our Definition 29.29 a vector field is by definition continuous, not more and not less.

Next we recall the following definition which we introduced in the Vector Field-Pullback Lemma 29.33.

Notation 43.12. Let $\Phi\colon A \to B$ be a local diffeomorphism between two smooth manifolds and let w be a vector field on B. We denote by Φ^*w the vector field on A which assigns to each $P \in A$ the tangent vector

$$\Phi^*w(P) \;:=\; (\mathrm{D}\Phi_P)^{-1}(\underbrace{w(\Phi(P))}_{\in \mathrm{T}_{\Phi(P)}B}) \;\in\; \mathrm{T}_P A.$$

By the Vector Field-Pullback Lemma 29.33 we know that if w is a smooth vector field on B, then Φ^*w is a smooth vector field on A.



Lemma 43.13. (Vector Field–Double Pullback Lemma) In the following let $\Phi\colon A \to B$ and $\Psi\colon B \to C$ be local diffeomorphisms between smooth manifolds and let w be a vector field on C. Then $\Phi^*(\Psi^*w) = (\Psi \circ \Phi)^*w$.

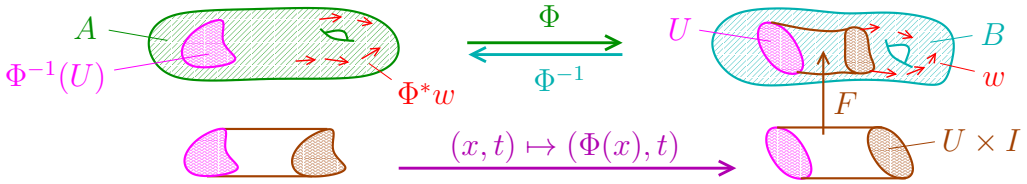
Proof. This statement follows immediately from the Chain Rule of differentials, see the Tangent Space-Functor Proposition 29.15 (2). ∎

The following lemma allows us to reduce questions on flows via diffeomorphisms to settings which are hopefully easier.

Lemma 43.14. (Flow-Diffeomorphism Lemma) Let $\Phi\colon A \to B$ be a diffeomorphism between two smooth manifolds and let w be a smooth vector field on B. Furthermore let $U \subseteq B$ be an open subset and let $t_0 \in \mathbb{R}$. If $F\colon U \times I \to B$ is a w-flow on B based at t_0, then the map

$$\begin{aligned} \Phi^{-1}(U) \times I &\to A \\ (x,t) &\mapsto \Phi^{-1}(F(\Phi(x),t)) \end{aligned}$$

is a Φ^*w-flow based at t_0.



Proof. Let $x \in \Phi^{-1}(U)$. First we calculate

$$\begin{aligned} \frac{d}{dt}\Phi^{-1}(F(\Phi(x),t)) &\overset{\text{by the Chain Rule, see the Tangent Space-Functor Proposition 29.15}}{=} \big(\mathrm{D}\Phi_{\Phi^{-1}(F(\Phi(x),t))}\big)^{-1} \cdot \frac{d}{dt}F(\Phi(x),t) \\ &\overset{\text{since } F \text{ is a } w\text{-flow}}{=} \big(\mathrm{D}\Phi_{\Phi^{-1}(F(\Phi(x),t))}\big)^{-1} \cdot w(F(\Phi(x),t)) \\ &\overset{\text{by Notation 43.12 of } \Phi^*w \text{ at the point } \Phi^{-1}(F(\Phi(x),t))}{=} \Phi^*(w)(\Phi^{-1}(F(\Phi(x),t))). \end{aligned}$$

Finally note that for each $x \in \Phi^{-1}(U)$ we have

$$\Phi^{-1}(F(\Phi(x), t_0)) \underset{\substack{\uparrow \\ \text{since the } w\text{-flow } F \text{ is} \\ \text{based at } t_0}}{=} \Phi^{-1}(\Phi(x)) = x.$$

We have thus verified that the given map is indeed a $\Phi^* w$-flow based at t_0. ■

The Flow-Diffeomorphism Lemma 43.14 gives us the tool to reduce many of our problems to the setting of open subsets of $\mathbb{R}^n$. This is the right moment for rolling out the following major theorem from real analysis.

Theorem 43.15. (Fundamental Theorem for Autonomous ODEs) Let $W \subseteq \mathbb{R}^n$ be an open subset and let $v\colon W \to \mathbb{R}^n$ be a smooth map.

(1) Given any $P \in W$ and any $t_0 \in \mathbb{R}$ there exists an open neighborhood U of P in W and an open neighborhood I of t_0 such that for every $x \in U$ there exists a solution $\varphi\colon I \to W$ of the ODE $y' = v(y)$ with $\varphi(t_0) = x$.

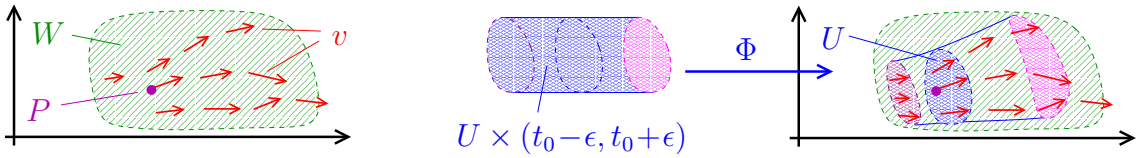
(2) Let $I \subseteq \mathbb{R}$ be an open interval. Furthermore let $f\colon I \to W$ and $g\colon I \to W$ be two solutions of the ODE $y' = v(y)$. If there exists a $t_0 \in I$ with $f(t_0) = g(t_0)$, then $f(t) = g(t)$ for all $t \in I$.

(3) Let $U \subseteq W$ be an open subset, let $t_0 \in \mathbb{R}$ and let $I \subseteq \mathbb{R}$ be an open neighborhood of t_0. Suppose that for each $x \in U$ we are given a smooth map $\varphi_x\colon I \to W$ that is a solution of the ODE $y' = v(y)$ and such that $\varphi_x(t_0) = x$. Then the map

$$\begin{array}{rcl} \Phi\colon U \times I & \to & W \\ (x,t) & \mapsto & \varphi_x(t) \end{array}$$

is smooth. In other words, Φ is a v-flow based at t_0.

(4) If $F\colon U \times I \to W$ and $G\colon V \times J \to W$ are two v-flows based at t_0, then F and G agree on the intersection of their domains.



Proof. The first three statements of the theorem get proved in every respectable course on ordinary differential equations. For example, a proof is provided in [**Lee2002**, Theorem D.1]. The last statement is an immediate consequence of (2) and the convenient fact that the intersection of two intervals is again an interval. ■

Lemma 43.16. (Flow–Diffeomorphism Lemma) Let M be a smooth manifold and let v be a smooth vector field on M. Let $F\colon M \times (a,b) \to M$ be a v-flow based at 0.

(1) For every μ, ν with $\mu \in (a,b)$ and $\mu + \nu \in (a,b)$ we have

$$F(F(P,\mu),\nu) = F(P, \mu+\nu).$$

(2) For every $\mu \in \mathbb{R}$ with $-\mu, \mu \in (a,b)$ the maps

$$\begin{array}{rcl} M & \to & M \\ P & \mapsto & F(P,-\mu) \end{array} \quad \text{and} \quad \begin{array}{rcl} M & \to & M \\ P & \mapsto & F(P,\mu) \end{array}$$

are diffeomorphisms and they are inverses of one another.

Proof.

(1) Let $P \in M$ and let $\mu \in (a,b)$. We consider the map

$$\begin{array}{rcl} \psi\colon (a-\mu, b-\mu) & \to & M \\ t & \mapsto & F(P, \mu + t). \end{array}$$

Note that for each $t \in (a-\mu, b-\mu)$ we have

$$\psi'(t) \underset{\substack{\uparrow \\ \text{Chain Rule}}}{=} F'(P,\mu+t)\cdot 1 = v(F(P,\mu+t)) = v(\psi(t)).$$

This shows that $t \mapsto \psi(t) = F(P,\mu+t)$ is a solution of the ODE $y' = v(y)$. Note that $\psi(0) = F(P,\mu)$. Another solution of the ODE $y' = v(y)$ with the same initial condition is given by $t \mapsto \varphi(t) \coloneqq F(F(P,\mu),t)$. It follows from the Fundamental Theorem 43.15 for Autonomous ODEs (3) that $F(P,t+\mu) = \psi(t) = \varphi(t) = F(F(P,\mu),t)$ for every $t \in (a-\mu, b-\mu)$. But that is what we needed to show.

(2) Let $\mu \in \mathbb{R}$ with $-\mu, \mu \in I$. We consider the maps

$$\begin{array}{rcl} \alpha\colon M & \to & M \\ P & \mapsto & F(P,-\mu) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon M & \to & M \\ P & \mapsto & F(P,\mu) \end{array}$$

The map $F\colon M \times (a,b) \to M$ is a flow, in particular F is smooth. It follows easily from the Smooth Manifold Product Proposition 25.34 and the Smooth Composition Lemma 25.17 that the two maps α and β are smooth. Thus it suffices to show that the maps α and β are inverses of one another. But this is an immediate consequence of (1). ∎

Now we can finally provide the proof of the Flow Theorem 43.3. To improve readability we first deal with the case of smooth manifolds with empty boundary. Afterwards we will explain how one needs to adapt the argument to deal with smooth manifolds with non-empty boundary.

Proof of the Flow Theorem 43.3 if $\partial M = \varnothing$. Let M be a closed n-dimensional smooth manifold and let v be a smooth vector field on M. We need to show that there exists a unique complete v-flow on M. In fact the uniqueness follows almost immediately from the uniqueness statement of the Fundamental Theorem 43.15 for Autonomous ODEs together with the Flow-Diffeomorphism Lemma 43.14. Thus it remains to show the existence of a complete v-flow $\Phi\colon M \times \mathbb{R} \to M$.

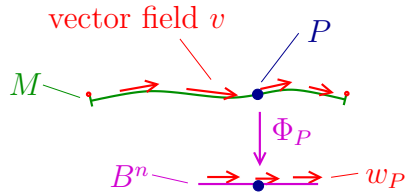
Claim 1. Given any $P \in M$ there exists an open neighborhood A_P of $P \in M$, an $\epsilon_P > 0$ and a v-flow $F_P\colon A_P \times (-\epsilon_P, \epsilon_P) \to M$ based at 0.

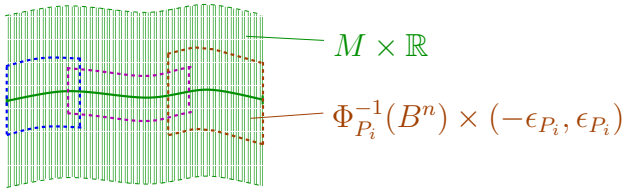
Proof. Let $P \in M$. Since we assume that $\partial M = \varnothing$ we know that there exists a chart $\Phi_P\colon U_P \to B^n$ with $\Phi_P(P) = 0$. We consider the smooth vector field $w \coloneqq (\Phi_P^{-1})^*(v)$ on B^n. Recall that we assume that v is a smooth vector field on M. By the definition of a smooth vector field on a smooth manifold, see Definition 29.29, this implies that w, viewed as a map $w\colon B^n \to \mathbb{R}^n$, is actually a smooth map. Therefore it follows from Statements (1) and (3) of the Fundamental Theorem 43.15 for Autonomous ODEs that there exists an open neighborhood N_P of $0 \in B^n$, an $\epsilon_P > 0$ and a w-flow $G\colon N_P \times (-\epsilon_P, \epsilon_P) \times B^n$

based at 0. We set $A_P \coloneqq \Phi^{-1}(N_P)$. The claim now follows from the Flow-Diffeomorphism Lemma 43.14 together with the Vector Field–Double Pullback Lemma 43.13. ⊞

Claim 2. There exists an $\epsilon > 0$ and a v-flow $F\colon M \times (-\epsilon, \epsilon) \to M$ based at 0.

Proof. By hypothesis the smooth manifold M is closed, in particular M is compact. It follows that there exist finitely many points $P_1, \dots, P_k \in M$ such that $M = A_{P_1} \cup \dots \cup A_{P_k}$. We set $\epsilon \coloneqq \min\{\epsilon_{P_1}, \dots, \epsilon_{P_k}\}$.





It follows from Statement (4) of the Fundamental Theorem 43.15 for Autonomous ODEs, together with the Flow-Diffeomorphism Lemma 43.14, that the flows $F_{P_1}, \dots, F_{P_k}$ agree on the overlaps of their domains. In particular these v-flows combine to give a v-flow $F\colon M \times (-\epsilon, \epsilon) \to M$ based at 0. ⊞

By the Flow–Diffeomorphism Lemma 43.16 we know that $\Theta \coloneqq F_{\frac{\epsilon}{2}}\colon M \to M$ is a diffeomorphism. For each $k \in \mathbb{Z}$ we consider the map

$$\begin{aligned} \widetilde{F}_k\colon M \times (\tfrac{k\cdot\epsilon}{2} - \epsilon, \tfrac{k\cdot\epsilon}{2} + \epsilon) &\to M \\ (P, t) &\mapsto F(\Theta^k(P), t - \tfrac{k\cdot\epsilon}{2}). \end{aligned}$$

It follows again from the Fundamental Theorem 43.15 for Autonomous ODEs (4) together with the Flow-Diffeomorphism Lemma 43.14 that these v-flows agree on the overlap of their domains. Thus these maps can be combined to define a v-flow $F\colon M \times \mathbb{R} \to \mathbb{R}$ that is based at 0. ■

The proof of the Flow Theorem 43.3 if $\partial M \neq \varnothing$ will build on the following lemma:

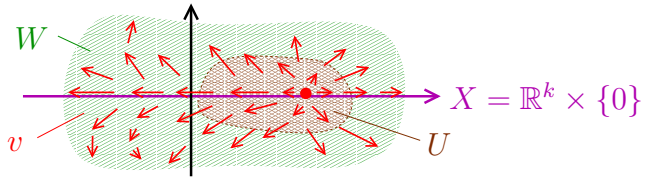
Lemma 43.17. (Flow-preserves-Subspace Lemma) Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. We consider the subspace $X \coloneqq \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$. Let W be an open subset of $\mathbb{R}^n$ and let $v\colon W \to \mathbb{R}^n$ be a smooth vector field on W that is parallel to $W \cap X$, i.e. we assume that for every $P \in X$ we have $v(P) \in \mathrm{T}_P X = \mathbb{R}^k \times \{0\}$. Furthermore let $U \subseteq W$ be an open subset and let $F\colon U \times I \to W$ be a v-flow on U based at some t_0.

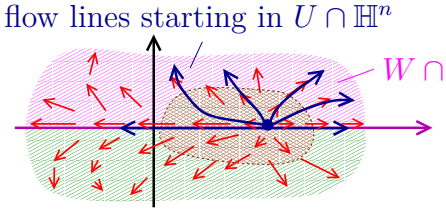
(1) The map $F\colon U \times I \to W$ restricts to a map

$$F\colon (U \cap X) \times I \to W \cap X.$$

(2) If $k = n - 1$, then the map $F\colon U \times I \to W$ restricts to a map

$$F\colon (U \cap \mathbb{H}^n) \times I \to W \cap \mathbb{H}^n.$$





Proof. Statement (1) follows immediately from the Mean Value Theorem. Statement (2) follows from Statement (1) and the Intermediate Value Theorem. We leave it to the reader to fill in the details. ■

Proof of the Flow Theorem 43.3 if $\partial M \neq \varnothing$. Let M be a compact n-dimensional smooth manifold with (possibly) non-empty boundary and let v be a smooth vector field on M that is boundary parallel. As in the proof for closed smooth manifolds it suffices to prove the following claim:

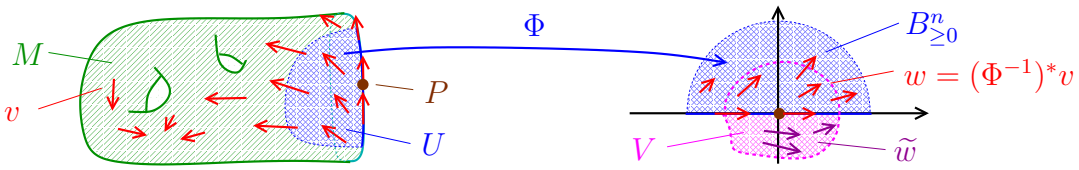
Claim 2. There exists an $\epsilon > 0$ and a v-flow $F\colon M \times (-\epsilon, \epsilon) \to M$ based at 0.

Furthermore, as in the proof for closed smooth manifolds, this Claim 2 follows easily from the following Claim 1.

Claim 1. Given any $P \in M$ there exists an open neighborhood A of $P \in M$, an $\epsilon > 0$ and a v-flow $F\colon A \times (-\epsilon, \epsilon) \to M$ based at 0.

Proof. Let $P \in M$. If $P \in M \setminus \partial M$, then we apply the same argument as on page 1015. Now let $P \in \partial M$. This implies that there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ with $\Phi(P) = 0$. We consider the vector field $w := (\Phi^{-1})^*(v)$ on $B^n_{\geq 0}$.

Since v is a smooth vector field on M we see, by the definition of a smooth vector field on a smooth manifold in Definition 29.29, that $(\Phi^{-1})^*v\colon B^n_{\geq 0} \to \mathbb{R}^n$ is a smooth map. By definition of a smooth map on $B^n_{\geq 0}$, see Definition 24.3, there exists an open neighborhood V of $0 \in \mathbb{R}^n$ and a smooth vector map $\widetilde{w}\colon V \to \mathbb{R}^n$ such that $\widetilde{w}$ and $(\Phi^{-1})^*v$ agree on the common subset $V \cap B^n_{\geq 0}$.



It follows again from Statements (1) and (3) of the Fundamental Theorem 43.15 for Autonomous ODEs that there exists an open neighborhood N of $0 \in V$, an $\epsilon > 0$ and a $\widetilde{w}$-flow $G\colon N \times (-\epsilon, \epsilon) \to V$ based at 0. Next note that it follows from our hypothesis that v is boundary parallel, together with the Flow-preserves-Subspace Lemma 43.17, that the map G restricts to a map $G\colon (N \cap B^n_{\geq 0}) \times (-\epsilon, \epsilon) \to V \cap B^n_{\geq 0}$. We set $A := \Phi^{-1}(N \cap B^n_{\geq 0})$. The desired statement now follows from the Flow-Diffeomorphism Lemma 43.14 together with the Vector Field–Double Pullback Lemma 43.13. ■

Finally, with all the tools that we have acquired by now the proof of the Flow-Diffeotopy Proposition 43.6 is only a formality.

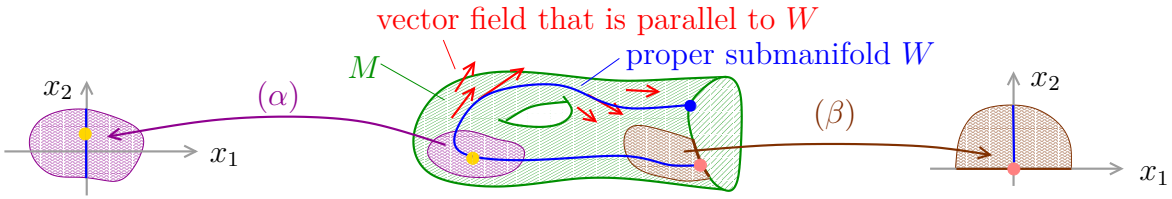
Proof of the Flow-Diffeotopy Proposition 43.6. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M \times \mathbb{R} \to M$ be the corresponding complete v-flow.

(1) Let $t \in \mathbb{R}$. We need to show that the map $\Phi^v_t\colon M \to M$ is a diffeomorphism that is diffeotopic to the identity. To do so we consider the map

$$\begin{array}{rcl} H\colon M \times [0,1] & \to & M \\ (P, s) & \mapsto & \Phi^v(P, t \cdot s). \end{array}$$

Since $\Phi^v\colon M \times \mathbb{R} \to M$ is a v-flow we know that $\Phi^v\colon M \times \mathbb{R} \to M$ is in particular smooth. It follows that $H\colon M \times [0,1] \to M$ is smooth. Furthermore by the Flow–Diffeomorphism Lemma 43.16 we know that each $H_s\colon M \to M$ is a diffeomorphism. This shows that H is a diffeotopy.

(2) Suppose that $W \subseteq M$ is a proper submanifold and suppose that v is parallel to W. We need to show that the map $\Phi^v\colon M \times \mathbb{R} \to M$ restricts to a map $W \times \mathbb{R} \to W$ and that this map is the unique complete flow of the vector field $v|_W$ on W. Recall that by the definition of a proper submanifold, see Definition 25.22, W is a closed subset of M and for each $P \in M$ we can find submanifold charts of type (α) or (β) as defined on Definition 25.22 and as illustrated in the figure below.



Given $P \in W$ we consider the map

$$\begin{aligned} \varphi\colon \mathbb{R} &\to M \\ t &\mapsto \Phi^v(P,t). \end{aligned}$$

It follows easily from the Flow-preserves-Subspace Lemma 43.17 together with the Flow-Diffeomorphism Lemma 43.14 (applied to charts), the compactness of M and the Tube Lemma 7.12 that $\varphi^{-1}(W)$ is an open subset of $\mathbb{R}$. Since W is a closed subset of M we know that $\varphi^{-1}(W)$ is a closed subset of $\mathbb{R}$. Since $\varphi(0) = P$ we see that $\varphi(W) \neq \varnothing$. By the Path-Connected-Implies-Connected Corollary 4.33 we know that $\mathbb{R}$ is connected. Thus we see that $\varphi^{-1}(W) = \mathbb{R}$. But this implies that $\Phi^v(W \times \mathbb{R}) \subseteq W$. Finally it follows from the uniqueness statement of the Flow Theorem 43.3 that $\Phi^v|_{W\times\mathbb{R}}$ is indeed the complete $v|_W$-flow. ■

43.3. **Proof of the Flow Fixed Point Theorem 43.9.** For the reader's convenience we recall the Flow Fixed Point Theorem 43.9.

Theorem 43.9. (Flow Fixed Point Theorem) Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel. Let $\Phi^v\colon M \times \mathbb{R} \to M$ be the unique complete v-flow from the Flow Theorem 43.3. If v has no zeros, then there exists an $\epsilon > 0$ such that for each $t \in (0, \epsilon]$ the time t-map $\Phi^v_t\colon M \to M$ is fixed-point free. In other words, given any $t \in (0, \epsilon]$ and given any $P \in M$ we have $\Phi^v_t(P) \neq P$.

Remark. We will give a bare hands proof of Flow Fixed Point Theorem 43.9. A more elegant proof can also be given using the Differential-of-Flow-at-Zero Proposition 43.19. □

Our proof rests on the following auxiliary lemma.

Auxiliary Lemma 43.18. (Injective Curve Lemma) Let I be an interval and let $\varphi\colon I \to U$ be a smooth map. If there exists a linear map $\pi\colon \mathbb{R}^n \to \mathbb{R}$ such that for every $t \in I$ we have $\pi(\varphi'(t)) > 0$, then φ is injective.

Proof of the Injective Curve Lemma 43.18. We consider the map $\pi\circ\varphi\colon I \to \mathbb{R}$. Note that for every $t \in I$ we have

$$\underset{\text{by the Chain Rule 24.17}}{(\pi \circ \varphi)'(t) \;=\; \mathrm{D}\pi_{\varphi(t)}(\varphi'(t))} \;\underset{\text{see page 651}}{=}\; \pi(\varphi'(t)) \;\underset{\text{by hypothesis}}{>}\; 0.$$

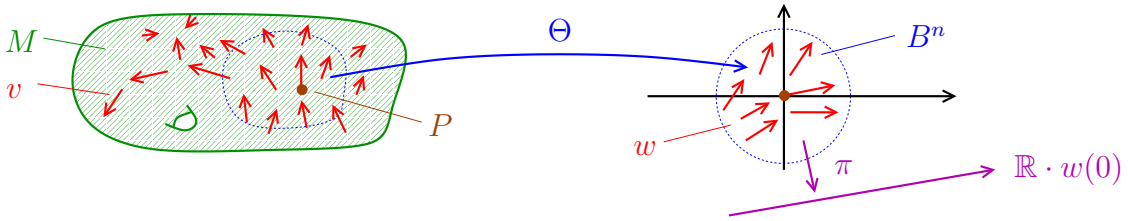
It follows from elementary real analysis that $\pi \circ \varphi$ is injective. But this implies of course that $\varphi\colon I \to U$ is injective. ■

Proof of the Flow Fixed Point Theorem 43.9. Let M be a compact smooth manifold and let v be a smooth vector field on M that is boundary parallel and which has no zeros. We denote by $\Phi^v\colon M \times \mathbb{R} \to M$ the corresponding unique complete v-flow.

Claim. For each $P \in M$ there exists an open neighborhood U_P of P and an $\epsilon_P > 0$ such that for every $x \in U_P$ the map $(-\epsilon_P, \epsilon_P) \to M$ given by $t \mapsto \Phi^v(x,t)$ is injective.

Proof. Let $P \in M$. In the following we deal with the case that $P \in M \setminus \partial M$. The case $P \in \partial M$ is deal with in a very similar fashion.

It follows easily from the hypothesis that $v(P) \neq 0$ that there exists a chart $\Theta\colon U \to B^n$ around P such that $\Phi(P) = 0$ and such that $w \coloneqq (\Theta^{-1})^* v$ has the property that for each $x \in B^n$ we have $\|w(x) - w(0)\| < \frac{1}{2} \cdot \|w(0)\|$.



Since U is an open subset of M we obtain from the continuity of Φ^v that $\Phi^v(U) \subseteq M \times \mathbb{R}$ is an open neighborhood of $(P, 0)$. Thus there exists an open neighborhood U_P of P and an $\epsilon_P > 0$ such that $\Phi^v(U_P \times (-\epsilon_P, \epsilon_P)) \subseteq U$.

Recall that $w(0) \neq 0$. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}$ be the linear map that corresponds to the orthogonal projection onto $\mathbb{R} \cdot w(0)$. Since $\|w(x) - w(0)\| < \frac{1}{2} \cdot \|w(0)\|$ for all $x \in B^n$ we obtain from an elementary calculation that $\pi(w(x)) > 0$ for all $x \in B^n$. It follows from the Injective Curve Lemma 43.18 that for each $x \in \Theta(U_P)$ the map

$$\begin{array}{rcl} (-\epsilon_P, \epsilon_P) & \to & B^n \\ x & \mapsto & \Theta(\Phi^v(\Theta^{-1}(x), t)) \end{array}$$

is injective. This shows that ϵ_P together with the open neighborhood U_P of P has the desired properties. ⊞

Since M is compact there exist $P_1, \ldots, P_k \in M$ with $U_{P_1} \cup \cdots \cup U_{P_k} = M$. It is clear that $\epsilon \coloneqq \min\{\epsilon_{P_1}, \ldots, \epsilon_{P_k}\}$ has the desired properties. ■

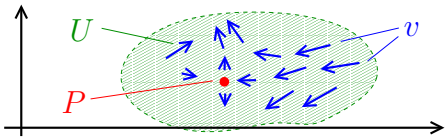
43.4. Appendix I: Zeros of vector fields.

In this first appendix to this chapter we state a proposition which gives us information about the flow of a vector field near a zero of the vector field. We will not make use of it. But perhaps readers, who want to extend some of the earlier results, will find it useful.

Proposition 43.19. (Differential-of-Flow-at-Zero Proposition) Let $U \subseteq \mathbb{R}^n$ be an open subset, let $v\colon U \to \mathbb{R}^n$ be a smooth vector field on U and let P be a zero of v. Furthermore let W be an open neighborhood of P and let $F\colon W \times (-\epsilon, \epsilon) \to U$ be a v-flow based at 0. Then for every $t \in (-\epsilon, \epsilon)$ we have[482]

$$\mathrm{D}(F_t)_P \;=\; \exp(t \cdot \mathrm{D}v_P).$$

Here, for a matrix $A \in \mathrm{M}(n \times n, \mathbb{R})$ we write $\exp(A) \coloneqq \sum_{k=0}^{\infty} \frac{A^k}{k!}$.



$$\mathrm{D}v_P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. The fact that P is a zero of v implies by the Zero–Fixed Point Lemma 43.8 that $F(P,t) = P$ for all $t \in (-\epsilon, \epsilon)$. Therefore it follows from [**PW2019**, Satz 4.3.1], after some slight unraveling of the definitions, that the map

$$\begin{array}{rcl} (-\epsilon, \epsilon) & \to & \mathrm{M}(n \times n, \mathbb{R}) \\ t & \mapsto & \mathrm{D}(F_t)_P \end{array}$$

satisfies the ODE

$$y' \;=\; \mathrm{D}v_P \cdot y.$$

But by [**PW2019**, p. 58] this means that $\mathrm{D}(F_t)_P = \exp(t \cdot \mathrm{D}v_P)$ for every $t \in (-\epsilon, \epsilon)$. ∎

43.5. **Appendix II: Dealing with non-compact smooth manifolds.** Let M be a smooth manifold and let v be a smooth vector field on M that is boundary parallel. As we saw on page 1009, if M is non-compact, then there does not necessarily exist a complete v-flow $\Phi^v\colon M \times \mathbb{R} \to M$. If one wants to show that such a complete v-flow exists after all one needs some restrictions on the vector fields under consideration. In this appendix we will sketch one approach. We will not make use of the results of this appendix.

The proof of the Flow Theorem 43.3 was based on Statement (1) of the Fundamental Theorem 43.15 for Autonomous ODEs which says that if $W \subseteq \mathbb{R}^n$ is an open subset and if $v\colon W \to \mathbb{R}^n$ is a smooth map, then given any $P \in W$ and any $t_0 \in \mathbb{R}$ there exists an open neighborhood U of P in W and an open neighborhood I of t_0 such that for every $x \in U$ there exists a solution $\varphi\colon I \to W$ of the ODE $y' = v(y)$ with $\varphi(t_0) = x$.

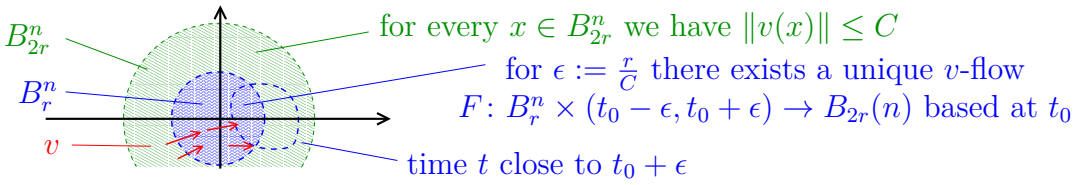
To deal with non-compact smooth manifolds we will need the following quantitative version of Statement (1) of the Fundamental Theorem 43.15 for Autonomous ODEs.

Proposition 43.20. (Time Interval-for-ODE Proposition) Let $n \in \mathbb{N}_0$, let $r > 0$ and let $v\colon B^n_{2r} \to \mathbb{R}^n$ be a smooth vector field on the open n-ball $B^n_{2r} \subseteq \mathbb{R}^n$ of radius $2r$. Suppose that we have a $C > 0$ such that for every $x \in B^n_{2r}$ we have $\|v(x)\| \leq C$. We set $\epsilon \coloneqq \frac{r}{C}$.

(1) For every $x_0 \in B^n_r$ and every $t_0 \in \mathbb{R}$ there exists a solution $\varphi\colon (t_0 - \epsilon, t_0 + \epsilon) \to B^n_{2r}$ of the ODE $y' = v(y)$ with $\varphi(t_0) = x_0$.

(2) For every $t_0 \in \mathbb{R}$ there exists a unique v-flow $F\colon B^n_r \times (t_0 - \epsilon, t_0 + \epsilon) \to B^n_{2r}$ based at t_0.

[482] Since $v(P) = 0$ we see in the Zero–Fixed Point Lemma 43.8 that P is a fixed point of every F_t. Hence $\mathrm{D}(F_t)_P$ is an automorphism of $\mathrm{T}_P U = \widetilde{\mathrm{T}}_P U = \mathbb{R}^n$. In other words, it is given by a matrix in $\mathrm{M}(n \times n, \mathbb{R})$.



Proof. Note that Statement (2) is an immediate consequence of Statement (1) together with Statements (3) and (4) of the Fundamental Theorem 43.15 for Autonomous ODEs. Thus it suffices to prove Statement (1).

We set $\epsilon := \frac{r}{C}$. Let $x_0 \in B^n_r$ and $t_0 \in \mathbb{R}$. In the following we will show that there exists a solution to the ODE $y' = v(y)$ with initial condition $\varphi(t_0) = y_0$ that is defined on $[t_0, t_0+\epsilon)$. Basically the same argument can then be used to get the extension "to the left", i.e. to the interval $(t_0-\epsilon, t_0]$. We consider

$$T := \sup\left\{ t \in (t_0,\infty) \;\middle|\; \begin{array}{c}\text{there exists a solution } \varphi\colon [t_0,t) \to B^n_{2r} \text{ of}\\ \text{the ODE } y' = v(y) \text{ with } \varphi(t_0) = x_0\end{array} \right\}.$$

If $T > t_0+\epsilon$ then we are evidently done. If $T = t_0+\epsilon$, then for every $a \in (t_0, T)$ there exists a solution φ_a. But by Statement (2) of the Fundamental Theorem 43.15 for Autonomous ODEs we know that for any $b > a$ we have $\varphi_b|_{[t_0,a)} = \varphi_a$. This shows that these maps φ_a combine to define a solution φ on all of $[t_0, T) = [t_0, t_0+\epsilon)$.

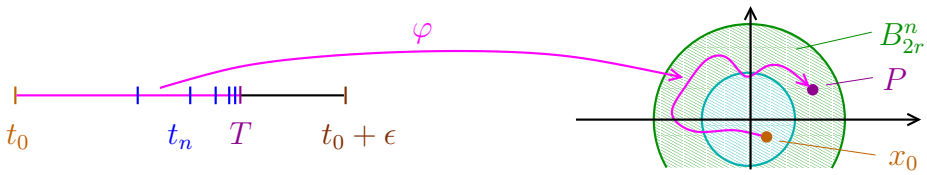
Finally suppose that $T < t_0+\epsilon$. As in the previous paragraph we see that there exists a solution $\varphi\colon [t_0, T) \to B^n_{2r}$. Note that by Statement (1) of the Fundamental Theorem 43.15 for Autonomous ODEs we know at least that $T > t_0$.

Claim. The limit $P := \lim_{t\to T} \varphi(t)$ exists and it lies in B^n_{2r}.

Proof. We start with a little inequality, namely given $x, y \in [t_0, T)$ we have

$$\|\varphi(y) - \varphi(x)\| \underset{\text{Fundamental Theorem of Calculus 73.27}}{=} \left\| \int_x^y \varphi'(s)\,ds \right\| \underset{\text{by the Integral Inequality Lemma 2.45}}{\le} \int_x^y \|\varphi'(s)\|\,ds \underset{\text{since } \varphi \text{ is a solution of the ODE } y' = v(y)}{=} \int_x^y \|v(\varphi(s))\|\,ds \underset{\text{since } \varphi \text{ takes values in } B^n_{2r} \text{ we have } \|v(\varphi(s))\| \le C}{\le} |y-x| \cdot C.$$

We pick a monotonously increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0, T)$ with $\lim_{n\to\infty} t_n = T$. It follows from the above inequality that $(\varphi(t_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in B^n_{2r}. We denote by $P \in \mathbb{R}^n$ its limit. It follows easily from the above inequality that in fact $\lim_{t\to T} \varphi(t) = P$. It remains to show that P actually lies in B^n_{2r}.



This follows from the following (in-) equalities:

$$\|P\| \leq \|x_0\| + \|P - x_0\| = r + \left\|\lim_{t\to T}(\varphi(t) - \varphi(t_0))\right\| \underset{\substack{\uparrow \\ \text{by the above inequality we} \\ \text{have } \|\varphi(t)-\varphi(t_0)\| \leq (t-t_0)\cdot C}}{\leq} r + (T - t_0)\cdot C \underset{\substack{\uparrow \\ \text{since } T < t_0+\epsilon \\ \text{and since } \epsilon = \frac{r}{C}}}{<} r + \epsilon \cdot \frac{r}{\epsilon} = 2r.$$

⊞

Since P is contained in the open subset B^n_{2r} it follows from Statement (1) of the Fundamental Theorem 43.15 for Autonomous ODEs that there exists an open neighborhood A of $P \in B^n_{2r}$, an open interval $(-\mu, \mu)$ and a v-flow

$$F\colon A \times (-\mu, \mu) \to B^n_{2r}$$

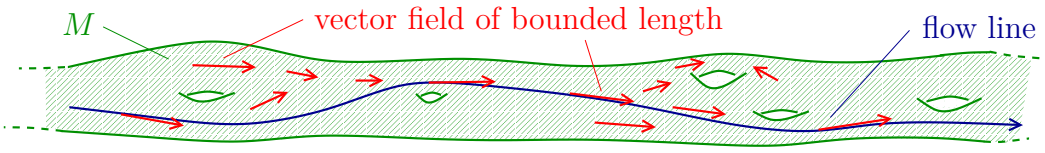
that is based at 0. Since $P = \lim_{t\to T}\varphi(t)$ and since A is an open neighborhood of U there exists a $t' \in (t_0, T)$ with $t' > T - \frac{\mu}{2}$ and with $\varphi(t') \in A$. We consider the map

$$\begin{array}{rcl} \psi\colon [t_0, t' + \mu) & \to & B^n_{2r} \\ s & \mapsto & \begin{cases} \varphi(s), & \text{if } s \in (t_0, T) \\ F(\varphi(t'), s - t'), & \text{if } s \in (t' - \mu, t' + \mu) \end{cases} \end{array}$$

Note that $s \mapsto \varphi(s)$ and $s \mapsto F(\varphi(t'), s - t')$ are both solutions to the ODE $y' = v(y)$ which agree at t'. Thus it follows from Statement (2) of the Fundamental Theorem 43.15 for Autonomous ODEs that these two maps agree on the common domain, in other words, the above map ψ is in fact unambiguously defined. By construction it is a solution to the ODE $y' = v(y)$ with $\psi(t_0) = x_0$. Since $t + \mu > T - \frac{\mu}{2} + \mu > T$ we have obtained a contradiction to the assumption that $T < t_0 + \epsilon$ and the definition of T a supremum. ■

Using the Time Interval-for-ODE Proposition 43.20 one can prove the following variation on the Flow Theorem 43.3.

Theorem 43.21. (Flow Theorem for Non-Compact Smooth Manifolds) Let M be a proper n-dimensional smooth submanifold of $\mathbb{R}^m$ and let v be a smooth vector field on M. If there exists a $C \in \mathbb{R}$, such that for each $x \in M$ we have $\|v(x)\| \leq C$, then there exists a unique complete v-flow $\Phi^v\colon M \times \mathbb{R} \to M$.



Proof. We leave the proof of the theorem as Project 43.2 to the reader. ■

43.6. **Moving points on smooth manifolds.** As a treat to the reader who made it that far we conclude this chapter with a neat application of the Flow Theorem 43.3, namely we can now give a new proof of the Interval-Sphere-Ball Point Moving Proposition 41.19 (2).

Proposition 43.22. (Moving Points-in-Ball Proposition) Given any two points $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n$ we can move P to Q in the sense of Definition 41.18.

Proof. Let $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n$.

Claim. There exists a smooth vector field v on $\overline{B}^n$ which has the following two properties: (1) The vector field vanishes on the open neighborhood $\overline{B}^n \setminus B^n_{1-\frac{\epsilon}{2}}$ of $\partial\overline{B}^n$.

(2) For every $t \in [0,1]$ we have $v(P \cdot (1-t) + t \cdot Q) = Q - P$.

Proof. We use Condition (2) as a definition of the vector field v on the Euclidean segment $\{P \cdot (1-t) + t \cdot Q \,|\, t \in [0,1]\}$. Since $P, Q \in \overline{B}^n \setminus \partial\overline{B}^n$ there exists an $\epsilon > 0$ such that this Euclidean segment is contained in $\overline{B}^n_{1-\epsilon}$. On $\overline{B}^n \setminus B^n_{1-\frac{\epsilon}{2}}$ we define v to be the zero vector field. We have thus defined v on a closed subset of $\overline{B}^n$.



It is straightforward to show that v is smooth in the sense of Definition 24.3. Thus it follows from the Smooth Extension Proposition 27.12 that we can extend v to a smooth map $\overline{B}^n \to \mathbb{R}^n$. ⊞

Note that the vector field is in particular parallel to the boundary. Thus it follows from the Flow Theorem 43.3 that there exists a complete v-flow $\Phi^v \colon \overline{B}^n \times \mathbb{R} \to \overline{B}^n$. It is clear that the map $[0,1] \to B^n$ that is given by $t \mapsto P + t \cdot (Q - P)$ is a solution to the ODE $y' = v(y)$. Since $\Phi_0(P) = P$ we obtain from the uniqueness statement of the Fundamental Theorem 43.15 for Autonomous ODEs that the solution $t \mapsto \Phi_t(P)$ agrees with the above solution. This implies that $\Phi_1(P) = Q$. Similarly we see that the restriction of each Φ^v_t to $\overline{B}^n \setminus B^n_{1-\frac{\epsilon}{2}}$ is the identity.

As in the proof of the Flow-Diffeotopy Proposition 43.6 we see that it follows from the Flow–Diffeomorphism Lemma 43.16 that the map

$$\begin{array}{rccc} H \colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,s) & \mapsto & \Phi^v(x,s) \end{array}$$

is a diffeotopy. From the above discussion it is clear that it has all the desired properties. ■

43.7. Exercises for Chapter 43.

Exercise 43.1. Let M be a smooth submanifold of $\mathbb{R}^n$ and let v be a "time dependent smooth vector field", i.e. let $v \colon M \times \mathbb{R} \to \mathbb{R}^n$ be a smooth map such that each $v_t \colon M \to \mathbb{R}^n$ is a smooth vector field on M.

- Let I be an interval with non-empty interior. We say that a smooth map $\varphi \colon I \to M$ is a *solution of the ODE* $y' = v(y,t)$ if for every $t \in I$ we have the following equality:
$$\varphi'(t) \;=\; v(\varphi(t), t) \;\in\; \mathrm{T}_{\varphi(t)} M.$$
- A *complete v-flow* is a smooth map $F \colon M \times \mathbb{R} \to M$ that satisfies the following properties:
  (a) For each $P \in M$ we have $F(P,0) = P$.
  (b) For each $P \in M$ the map $\mathbb{R} \to M$ given by $t \mapsto F(P,t)$ is a solution of the ODE $y' = v(y,t)$.

Now suppose that M, viewed as a smooth manifold, is closed. Show that there exists a complete v-flow.

Exercise 43.2. Give an example of a vector field v on a closed smooth manifold that has only finitely many zeros and such that there exists no $t > 0$ such that the map

$$\begin{array}{ccc} M & \to & M \\ P & \mapsto & \Phi^v(P,t) \end{array}$$

has infinitely many fixed-points.

Exercise 43.3. Let M be a compact smooth manifold.

(a) Let v be a smooth vector field on M that is boundary parallel. As usual we denote by $\Phi^v\colon M \times \mathbb{R} \to M$ the corresponding complete v-flow that exists by the Flow Theorem 43.3. Show that for every $s, t \in \mathbb{R}$ we have

$$\Phi^v_s \circ \Phi^v_t \;=\; \Phi^v_{s+t}\colon M \;\to\; M.$$

(b) We suppose that M is non-empty and that $\dim(M) > 0$. As in Definition 42.1 we denote by $\mathrm{Diff}(M)$ the group of self-diffeomorphism of M. Show that there exists a monomorphism $\mathbb{R} \to \mathrm{Diff}(M)$.

Exercise 43.4. We consider the smooth manifold $M = \mathbb{R}$. Does every smooth vector field v on M admit a complete v-flow?

Exercise 43.5. Use the Flow Theorem 43.21 to give a new proof of the Diffeotopies-of-S^1-Proposition 42.8. More precisely, let $f\colon S^1 \to S^1$ be a diffeomorphism, show that the following two statements hold:

(a) If f is orientation-preserving, then f is diffeotopic to the identity.
(b) If f is orientation-reversing, then f is diffeotopic to the reflection $\rho\colon S^1 \to S^1$ that is given by $(x, y) \mapsto (x, -y)$.

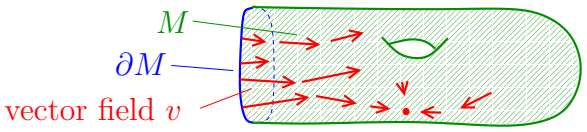
Project 43.1. Let M be a compact smooth manifold.

(a) Show that there exists a smooth vector field v on M such that for every $x \in \partial M$ the vector $v(x) \in \mathrm{T}_x M$ points inward in the sense of Definition 29.1.
Hint. Use partitions of unity.
(b) Show that there exists an $\epsilon > 0$ and a smooth map $F\colon \partial M \times [0,1] \to M$ such that for every $x \in M$ the following two statements hold:
(i) $F(x, 0) = x$,
(ii) for every $t_0 \in [0,1]$ we have $\frac{d}{dt}(t \mapsto F(x,t))|_{t=t_0} = v(F(x,t_0))$.
(c) Show that there exists a $\mu \in (0, \epsilon)$ such that $F\colon \partial M \times [0,\mu] \to M$ is a smooth embedding.
Hint. Use the Neighborhood Smooth Embedding Theorem 27.16.
Remark. This gives a new proof of the Smooth Collar Neighborhood Theorem 35.6 for smooth manifolds with compact boundary.

For readers with an insatiable appetite we have an extra exercise:

(d) Reprove the Smooth Collar Neighborhood Theorem 35.6 also for smooth manifolds with non-compact boundary.

Project 43.2. Prove the Flow Theorem 43.21 for Non-Compact Smooth Manifolds.


M
∂M
vector field v

CHAPTER 44

The Isotopy Extension Theorem

In this chapter we will state and prove the Isotopy Extension Theorem 44.2 which, loosely speaking, says that any smooth isotopy of a submanifold extends to a diffeotopy of the total manifold. This theorem is one of the unsung heroes of geometric topology.

This chapter is organized as follows:

- In Section 44.1 we will state the Isotopy Extension Theorem 44.2.
- In Section 44.2 we will discuss consequences of the Isotopy Extension Theorem 44.2.
- In Section 44.3 we will make preparations for the proof of the Isotopy Extension Theorem 44.2.
- In Section 44.4 we will prove the Isotopy Extension Theorem 44.2 for closed smooth manifolds.
- In Section 44.5 we will provide the proof of the Isotopy Extension Theorem 44.2 for the general case.
- In Appendix Section 44.6 we will formulate a variation on the Isotopy Extension Theorem 44.2 for pairs of smooth manifolds.
- In Section 44.7 we will provide exercises for Chapter 44.

In the next Chapter 45 we will introduce the notion of orientation-preserving and orientation-reversing loops in a smooth manifold.

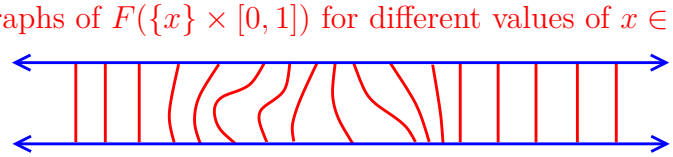
44.1. **The statement of the Isotopy Extension Theorem.** First let us recall Definition 18.4 and Definition 41.18:

Definition 44.1. Let X and Y be topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy.

(1) Let $A \subseteq X$ be a subset. We say that F is a homotopy rel A if for all $x \in A$ and $s,t \in [0,1]$ we have $F(x,s) = F(x,t)$.
(2) If $X = Y$, then we say that F has compact support if there exists a compact subset $K \subseteq X$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

Examples.

(1) Evidently every diffeotopy of a compact smooth manifold has compact support.
(2) In the figure below we sketch a non-trivial diffeotopy of $\mathbb{R}$ with compact support.



(3) The diffeotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ given by $F(x,t) = x + t$ does *not* have compact support. □

The following theorem is one of the key technical tools for dealing with smooth manifolds and their submanifolds.

Theorem 44.2. (Isotopy Extension Theorem) Let M be a compact smooth manifold and let N be a smooth manifold. Let A be a union of components of ∂M and let B be a union of components of ∂N. Let $F\colon M \times [0,1] \to N$ be a smooth isotopy that is trivial near the ends[483] and with the following property:

(∗) For each $t \in [0,1]$ we have $F_t(A) \subseteq B$ and we have $F_t(\partial M \setminus A) \subseteq N \setminus \partial N$.

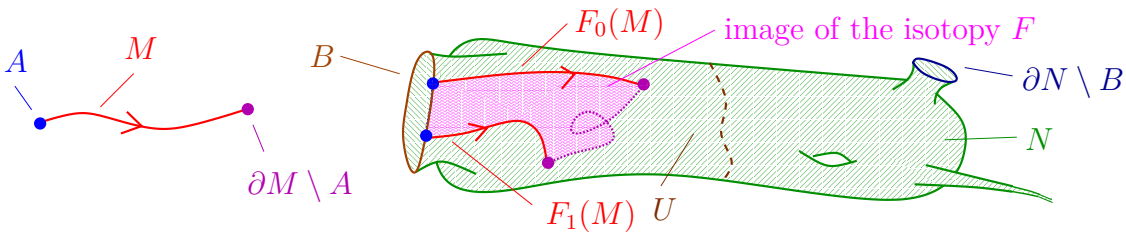
Furthermore suppose that we are given an open subset U of N with $F(M \times [0,1]) \subseteq U$. Then there exists a diffeotopy $G\colon N \times [0,1] \to N$ with the following properties:

(1) $G_0 = \mathrm{id}$.

(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon M \times [0,1] \to N$, in other words, we have

$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in M \text{ and } t \in [0,1].$$

(3) (a) G is a diffeotopy rel $N \setminus U$.
(b) G is a diffeotopy rel some neighborhood of $\partial N \setminus B$.
(c) G has compact support.
(d) G is trivial near the ends.



Proof. We procrastinate a bit and provide the proof in later sections. More precisely, in Sections 44.3 and 44.4 we will prove the Isotopy Extension Theorem 44.2 in the special case that M and N are closed smooth manifolds. The general case is dealt with in Section 44.5. ■

Remark.

(1) Our proof of the Isotopy Extension Theorem 44.2 is modelled on the proofs of Theorem 2.4.2 and Theorem 2.4.6 of [**Wall2016**]. Our statement is somewhat more general though than the corresponding statements in [**Wall2016**] since we allow maps that take values in ∂N. We also refer to [**Kos1993**, Theorem II.5.2], [**Hirs1961**, Theorem 8.1.3] and [**Pal1960b**, Chapter 6] for related results.

(2) In the Isotopy Extension Theorem 44.2 we can also replace the hypothesis that N is a smooth manifold by the slightly weaker hypothesis that N is the disjoint of smooth manifolds of varying dimension. For example we could consider embeddings of $S^1 \sqcup S^2$ into $\mathbb{R}^3$ such that the restrictions of the embeddings to the two components are smooth. The proof of this more general statement is identical.

(3) Note that there is a version of the Isotopy Extension Theorem 44.2 for topological manifolds and for "PL manifolds". We refer to [**EKi1971**, Corollary 1.4] and [**FNOP2025**,

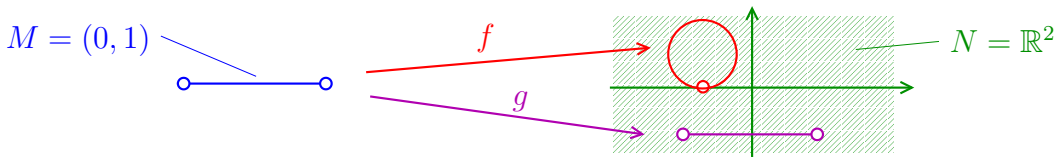
[483] By the Stationary-Near-The-Ends-Lemma 41.7 we know that any smooth isotopy $F\colon M \times [0,1] \to N$ can be modified easily to obtain a smooth isotopy $\widetilde{F}\colon M \times [0,1] \to N$ that is trivial near the ends.

Theorem 2.10] respectively to [**RS1972**, Theorem 4.24] for details. In both cases one should carefully read the statements before applying them. We have no intention of making use of these results. □

Example.

(1) Let N be a smooth manifold. The Isotopy Extension Theorem 44.2 often gets applied as follows: M is a (proper) submanifold in N and we start out with the inclusion map $i\colon M \to N$. Thus in this setting loosely speaking the Isotopy Extension Theorem 44.2 says, that if we deform M in N via (proper) submanifolds, then we can do so by "dragging N" along.

(2) In the statement of the Isotopy Extension Theorem 44.2 it is essential that M is compact. For example, let us consider the two smooth embeddings $f, g\colon (0,1) \to \mathbb{R}^2$ that are shown in the following figure:



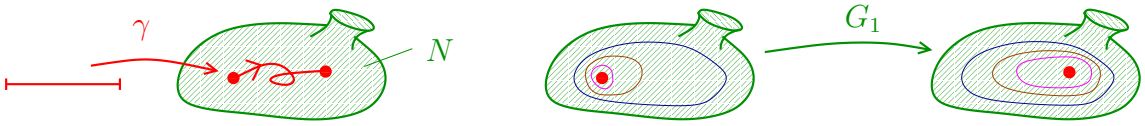
In Exercise 44.8 we will see that f and g are smoothly isotopic but we will also see that there is no diffeotopy $G\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ with $G_0 = \mathrm{id}$ and $G_1 \circ f = g$. Another, even more subtle example, will be studied in Exercise 44.13. □

To formulate our last example of this section we need to introduce the following definition:

Definition 44.3. Let N be a smooth manifold and let $\gamma\colon [0,1] \to N$ be a smooth path. We say that γ is trivial near the ends if there exists an $\epsilon > 0$ such that for each $t \in [0,\epsilon]$ we have $\gamma(t) = \gamma(0)$ and if for each $t \in [1-\epsilon, t]$ we have $\gamma(t) = \gamma(1)$.[484]

Lemma 44.4. (Path Extension-to-Diffeotopy Lemma) Let N be a smooth manifold and let $\gamma\colon [0,1] \to N \setminus \partial N$ be a smooth path that is trivial near the ends. There exists a diffeotopy $G\colon N \times [0,1] \to N$ with compact support, rel a neighborhood of ∂N, with $G_0 = \mathrm{id}$ and with $G(\gamma(0), t) = \gamma(t)$ for all $t \in [0,1]$.

Proof. Let $\gamma\colon [0,1] \to N \setminus \partial N$ be a smooth path that is trivial near the ends. Note that the map $\{0\} \times [0,1] \to N \setminus \partial N$ that is given by $(0,t) \mapsto \gamma(t)$ is a smooth isotopy that is trivial near the ends.



Finally note that it follows from the Isotopy Extension Theorem 44.2 that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ such that for each $t \in [0,1]$ we have $G(\gamma(0), t) = \gamma(t)$. ■

44.2. **Consequences of the Isotopy Extension Theorem 44.2.** The following corollary to the Isotopy Extension Theorem 44.2 is quite useful.

[484] As so often it follows from the Stationary-Near-The-Ends-Lemma 41.7 that we can turn any smooth path into a smooth path that is trivial near the ends.

Proposition 44.5. (Smooth Isotopies-Complements Proposition) Let M be a compact smooth manifold and let N be a smooth manifold. Let $f\colon M \to N$ and $g\colon M \to N$ be two proper smooth embeddings. We suppose that there exists a proper smooth isotopy $F\colon M \times [0,1] \to N$ from f to g. Then the following two statements hold:

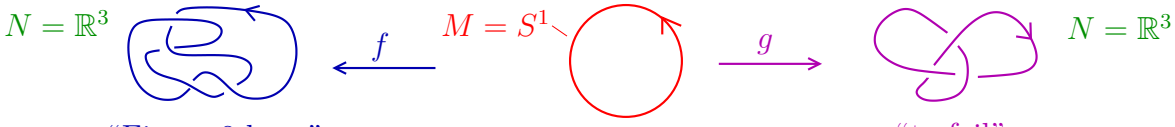
(1) There exists a diffeomorphism $\Phi\colon N \to N$ with $\Phi(f(M)) = g(M)$.
(2) Any diffeomorphism as in (1) restricts to a diffeomorphism[485] from the smooth manifold $N \setminus f(M)$ to the smooth manifold $N \setminus g(M)$.

If N is oriented, then we can arrange the diffeomorphisms in (1) and (2) to be orientation-preserving.

Proof. Let M be a compact smooth manifold and let N be a smooth manifold. Furthermore let $f\colon M \to N$ and $g\colon M \to N$ be two proper smooth embeddings. Note that here proper means that $f^{-1}(\partial N) = \partial M$ and $g^{-1}(\partial N) = \partial M$. We suppose that there exists a proper smooth isotopy $F\colon M \times [0,1] \to N$ from f to g. It follows from the Stationary-Near-The-Ends-Lemma 41.7 that we can assume that F is in fact trivial near the ends.

By the Isotopy Extension Theorem 44.2, applied with $A = \partial M$ and $B = \partial N$, we know that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ such that $G_1 \circ f = g$. But this implies that $G_1\colon N \to N$ is a diffeomorphism with $G_1(f(M)) = g(M)$. Evidently G_1 restricts to a diffeomorphism $N \setminus f(M) \to N \setminus g(M)$. Finally note that if N is oriented, then it follows from $G_0 = \mathrm{id}$ and the Smooth Isotopy-Orientation Proposition 42.4 that $\Phi := G_1$ is also orientation-preserving. ∎

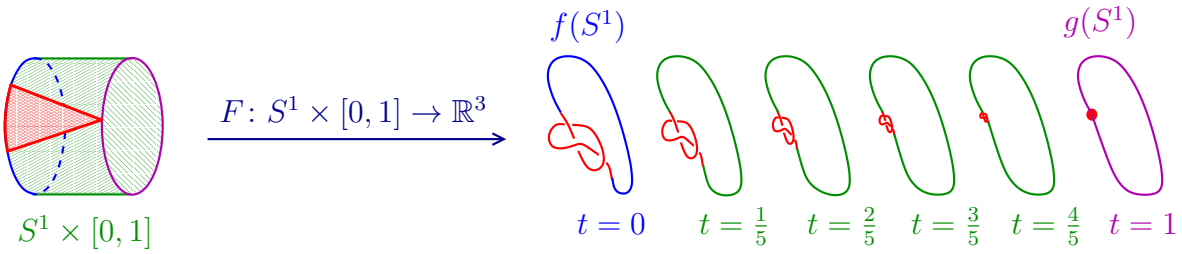
Example. A typical application of the Smooth Isotopies-Complements Proposition 44.5 is to consider smooth embeddings of S^1 into $\mathbb{R}^3$. In the following figure we see two such smooth embeddings $f, g\colon S^1 \to \mathbb{R}^3$:



The question arises, whether they are smoothly isotopic or not. In Section 88.1 we will introduce the "fundamental group $\pi_1(X)$ of a path-connected topological space X" whose isomorphism type is a homeomorphism invariant of X. In Chapter 111 we will see that the fundamental groups of the complements of $f(S^1)$ and $g(S^1)$ are not isomorphic. By the Smooth Isotopies-Complements Proposition 44.5 this implies that f and g are not smoothly isotopic. □

Example. The statement of the Smooth Isotopies-Complements Proposition 44.5 and the statement of the Isotopy Extension Theorem 44.2 are quite subtle. In the figure below, which is basically identical to the figure on page 974, we illustrate two smooth embeddings $f, g\colon S^1 \to \mathbb{R}^3$ and we illustrate a map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ with $F_0 = f$, $F_1 = g$ and with the property that given any $t \in [0,1]$ the corresponding map $F_t\colon S^1 \to \mathbb{R}^3$ is a smooth embedding.

[485] Since M is compact, we know by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $N \setminus f(M)$ and $N \setminus g(M)$ are open subsets of N, in particular they are smooth manifolds in their own right.



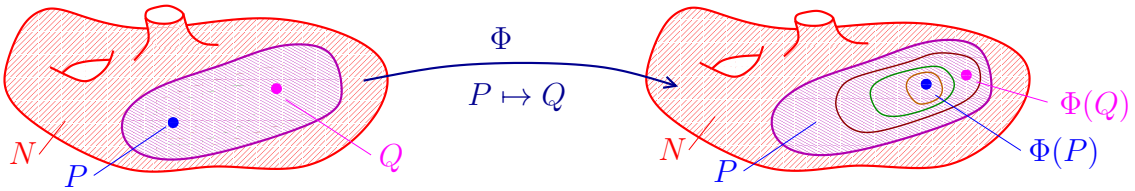
But similar to the previous example, we will see in Section 111.1 that the complements $\mathbb{R}^3 \setminus f(S^1)$ and $\mathbb{R}^3 \setminus g(S^1)$ are *not* diffeomorphic. This shows that in the formulations of the Smooth Isotopies-Complements Proposition 44.5 and the Isotopy Extension Theorem 44.2 it is essential that the map F is smooth on all of $M \times [0,1]$. □

We conclude this section with the following refinement of the Manifold–Homogeneous Proposition 41.21 for smooth manifolds.

Proposition 44.6. (Homogeneous-via-Diffeotopies Proposition) Let N be a connected n-dimensional smooth manifold.

(1) Given any two points P and Q in $N \setminus \partial N$ there exists a diffeotopy rel a neighborhood of ∂N from the identity to a diffeomorphism $\Phi\colon N \to N$ with $\Phi(P) = Q$.
(2) If $n \geq 2$, then given distinct points $P_1, \dots, P_m$ and distinct points $Q_1, \dots, Q_m$ in $N \setminus \partial N$ there exists a diffeotopy rel a neighborhood of ∂N from the identity to a diffeomorphism $\Phi\colon N \to N$ with $\Phi(P_i) = Q_i$ for $i = 1, \dots, m$.

Furthermore, in both cases we can furthermore arrange that the diffeotopy has compact support.[486]



Remark.

(1) In Exercise 41.11 we gave a down-to-earth proof of the Homogeneous-via-Diffeotopies Proposition 44.6. In fact the down-to-earth proof also works for topological manifolds.
(2) For *non-orientable* smooth manifolds we will prove in the Orientation-Flipping Proposition 45.6 a rather useful addendum to the above Homogeneous-via-Diffeotopies Proposition 44.6. □

Proof. Let N be a connected n-dimensional smooth manifold.

(1) Let P, Q be two points in $N \setminus \partial N$. By the Push Away-from-Boundary Lemma 23.17 we know that $N \setminus \partial N$ is also path-connected. It follows from the Smooth Path-Connectivity Proposition 25.32 that there exists a smooth path $F\colon [0,1] \to N \setminus \partial N$ from P to Q. As we saw in the proof of the Path Extension-to-Diffeotopy Lemma 44.4, a smooth path is

[486]Note that if N is oriented, then it follows from the Smooth Isotopy-Orientation Proposition 42.4 that in both cases Φ is in fact orientation-preserving.

basically the same as a smooth isotopy of a 0-dimensional manifold. By the Stationary-Near-The-Ends-Lemma 41.7 we can arrange that this smooth isotopy is trivial near the ends. The desired statement now follows from the Isotopy Extension Theorem 44.2.

(2) We prove this statement by induction on m. We dealt with the case $m = 1$ in (1). Now suppose that we know the statement for some $m \in \mathbb{N}$. Thus let $P_1, \dots, P_{m+1}$ and $Q_1, \dots, Q_{m+1}$ be two sets of distinct points in $N \setminus \partial N$. By induction we can without loss of generality already assume that $P_i = Q_i$ for $i = 1, \dots, m$.[487] Now pick disjoint closed n-balls around $P_1, \dots, P_m$ that are disjoint from $P := P_{m+1}$ and $Q := Q_{m+1}$. We denote by W the result of removing the corresponding open balls from M. By the Codimension-Zero Submanifold Proposition 25.49 (applied iteratively m times) we know, using the fact that $n \geq 2$, that W is a submanifold which is still connected. Thus we can apply (1) to P and Q in W to obtain a diffeotopy of W with all the desired properties. Since the diffeotopy is the identity near the boundary of W we can extend it via the identity to a diffeotopy of all of N. ■

The following corollary can be viewed as a strengthening of the result obtained in the Smooth Path-Connectivity Proposition 25.32.

Corollary 44.7. (Embedded Path Corollary) Let M be a connected smooth manifold. For any two distinct points $P, Q \in M \setminus \partial M$ there exists a smooth embedding $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.

Proof. Let $n := \dim(M)$. If $n = 1$, then the statement follows easily from the Smooth 1-Manifold Classification Theorem 28.8. Now assume that $n \geq 2$. Since M is non-empty there exists a non-empty chart and we can find a smooth embedding $\varphi\colon B^n \to M \setminus \partial M$. We pick $x, y \in B^n$. Since $n \geq 2$ we can apply the Homogeneous-via-Diffeotopies Proposition 44.6 and we see that there exists in particular a diffeomorphism $f\colon M \to M$ with $f(\varphi(x)) = P$ and $f(\varphi(y)) = Q$.



The map

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & M \\ t & \mapsto & f(\varphi(x \cdot (1-t) + y \cdot t)) \end{array}$$

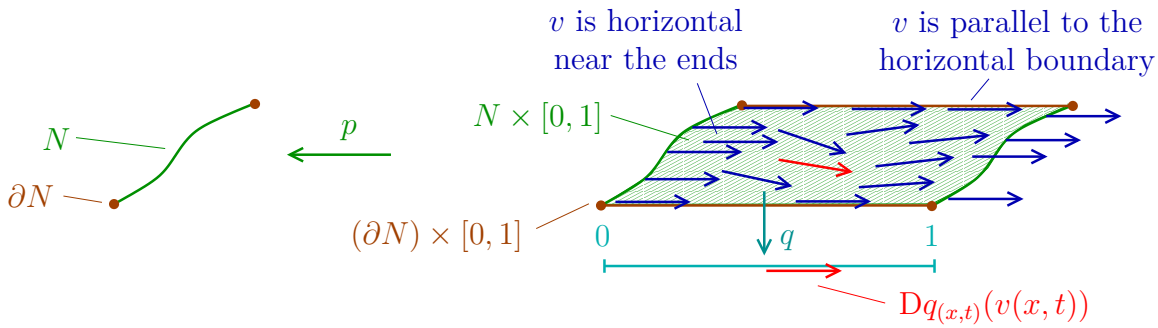
is a smooth embedding with $\gamma(0) = P$ and $\gamma(1) = Q$. ■

44.3. **Proof of the Isotopy Extension Theorem 44.2 I: Preparations.** In the following three sections we will provide a proof of the Isotopy Extension Theorem 44.2. First we introduce some notation.

[487] The super vigilant reader will have noticed that implicitly we are using the Smooth Isotopy-Equivalence Relation Proposition 41.5.

Definition 44.8. Let N be a smooth manifold. We consider the smooth manifold $N \times [0,1]$ with corner. We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two obvious projections. Let v be a vector field on[488] $N \times [0,1]$.

(1) We say that v is horizontal near the ends if there exists an $\epsilon > 0$ such that for all $x \in N$ and all $t \in [0,\epsilon] \cup [1-\epsilon, 1]$ we have $v(x,t) \in \ker(\mathrm{D}p_{(x,t)}\colon \mathrm{T}_{(x,t)}(\partial N \times [0,1]) \to \mathrm{T}_x N)$.
(2) We say that v is horizontally normal if for each $(x,t) \in N \times [0,1]$ we have
$$\mathrm{D}q_{(x,t)}(v(x,t)) \;=\; 1 \;\in\; \mathrm{T}_t\mathbb{R} \;=\; \widetilde{\mathrm{T}}_t\mathbb{R} \;=\; \mathbb{R}.$$
(3) We say that v is parallel to the horizontal boundary if for each $(x,t) \in \partial N \times [0,1]$ we have $v(x,t) \in \mathrm{T}_{(x,t)}(\partial N \times [0,1])$.
(4) (a) Given $(x,t) \in N \times [0,1]$ we denote by $\overline{h}$ the unique vector in $\mathrm{T}_{(x,t)}(N \times [0,1])$ with $\mathrm{D}q_{(x,t)}(h) = 1$ and $\mathrm{D}p_{(x,t)}(h) = 0$.
(b) We say that v is $\overline{h}$-trivial on a set $U \subseteq N \times [0,1]$ if $v(x,t) = \overline{h}$ for all $(x,t) \in U$.



The key to proving the Isotopy Extension Theorem 44.2 is the following consequence of the Flow Theorem 43.3.

Theorem 44.9. (Vector Fields-Diffeotopies Theorem) Let N be a smooth manifold. We denote by $p\colon N \times [0,1] \to N$ the projection. Suppose we are given a smooth vector field v on $N \times [0,1]$ with the following properties:

(1) The vector field is horizontal near the ends.
(2) The vector field is horizontally normal.
(3) The vector field is parallel to the horizontal boundary.
(4) The vector field is $\overline{h}$-trivial outside of a subset of the form $K \times [0,1]$ where K is a compact subset of N.

Then there exists a[489] diffeotopy $G\colon N \times [0,1] \to N$ of N with $G_0 = \mathrm{id}_N$ and with the property that for each $(Q, t_0) \in N \times [0,1]$ we have
$$\frac{d}{dt}(t \mapsto G(Q,t))|_{t=t_0} = \mathrm{D}p_{(G(Q,t_0),t_0)}(v(\underbrace{G(Q,t_0),t_0}_{\in N\times[0,1]})) \quad \in \mathrm{T}_{G(Q,t_0)}N.$$
Furthermore this diffeotopy is trivial near the ends and it is a diffeotopy rel $N \setminus K$.

[488] Strictly speaking we never introduced the tangent space of a smooth manifold with corner. But literally the same definition as in Definition 29.11 also works for smooth manifolds with corner and all the obvious statements hold again. As always we strive for brevity and we will thus not develop the concept of tangent spaces of smooth manifolds with corner in detail.

Example. Suppose we are already given a diffeotopy $G\colon W \times [0,1] \to W$ on some compact smooth manifold W. We assume that $G_0 = \mathrm{id}$ and that G is trivial near the ends. We consider the vector field w on $W \times [0,1]$ that is given by

$$w(P,t_0) \;=\; \big(\tfrac{d}{dt}(t \mapsto G(P,t))|_{t=t_0}, 1\big) \;\in\; \mathrm{T}_{(P,t_0)}(W \times [0,1]).$$

One can verify easily that this vector field satisfies the hypotheses of the Vector Fields-Diffeotopies Theorem 44.9. Thus the Vector Fields-Diffeotopies Theorem 44.9 says that we can recover G from the vector field w defined from G. □

The remainder of this section is dedicated to the proof of the Vector Fields-Diffeotopies Theorem 44.9. The proof consists mostly of massaging the setting till eventually one can apply the Flow Theorem 43.3. The proof does not contain any particularly interesting idea.

Proof. Let N be a smooth manifold. We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two projections. Suppose we are given a smooth vector field v on $N \times [0,1]$ with the following properties:

(1) The vector field is horizontal near the ends.
(2) The vector field is horizontally normal.
(3) The vector field is parallel to the horizontal boundary.
(4) The vector field is $\overline{h}$-trivial outside of a subset of the form $K \times [0,1]$ where K is a compact subset of N.

The proof idea is as follows: We can view v as a "time dependent" vector field on N. The Flow Theorem 43.3 does not apply to time dependent vector fields. We will get around this issue by applying the Flow Theorem 43.3 instead to the compact smooth manifold $N \times (\mathbb{R}/\mathbb{Z})$ for an appropriate vector field $\widetilde{v}$. We get a diffeotopy $\widetilde{G}\colon N \times (\mathbb{R}/\mathbb{Z}) \times [0,1] \to N \times (\mathbb{R}/\mathbb{Z})$ on $N \times (\mathbb{R}/\mathbb{Z})$ such that each corresponding time-t map is just given by addition of t in the second factor. The desired diffeotopy is then given by the map

$$G\colon N \times [0,1] \xrightarrow{(Q,s)\mapsto \widetilde{G}(Q,[0],s)} N \times (\mathbb{R}/\mathbb{Z}) \xrightarrow{\text{projection}} N.$$

We consider the compact smooth manifold $N \times (\mathbb{R}/\mathbb{Z})$ and we consider the following maps:

$$\begin{array}{rcl} \Psi\colon [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ t & \mapsto & [t] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{id}_N \times \Psi\colon N \times [0,1] & \to & N \times (\mathbb{R}/\mathbb{Z}) \\ (x,t) & \mapsto & (x,[t]). \end{array}$$

Claim 1.
(1) There exists a unique smooth vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ such that

$$\mathrm{D}(\mathrm{id}_N \times \Psi)_{(x,t)}(v(x,t)) \;=\; \widetilde{v}(x,[t]) \qquad \text{for each } (x,t) \in N \times [0,1].$$

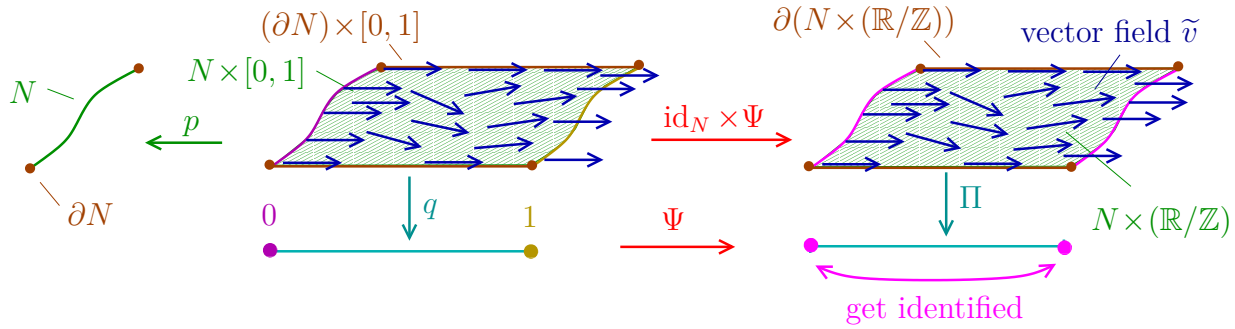
(2) The vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ is parallel to the boundary.

Proof.

[489] One can easily show, using the uniqueness statement of the Fundamental Theorem 43.15 for Autonomous ODEs, that the diffeotopy is uniquely determined by the stated properties. But we will not make use of the uniqueness statement and thus we do not claim it.

(1) Note that it follows from our hypothesis that v is horizontal near the ends and that v is horizontally normal that there exists an $\epsilon > 0$ such that for all $x \in N$ and $t \in [0, \epsilon] \cup [1-\epsilon, 1]$ we have $v(x,t) = \overline{h}$. The promised statement follows easily from this observation. We leave it to the reader to fill in the few remaining details.

(2) It follows easily from the hypothesis that the vector field v on $N \times [0,1]$ is parallel to the horizontal boundary that the vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ is also parallel to the boundary.[490] ⊞



Claim 2. There exists a unique complete $\widetilde{v}$-flow

$$\widetilde{G}\colon N \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \;\to\; N \times (\mathbb{R}/\mathbb{Z}).$$

Proof. As we saw in Claim 1, the vector field $\widetilde{v}$ on $N \times (\mathbb{R}/\mathbb{Z})$ is parallel to the boundary. Thus, if N is in fact compact, then the claim follows immediately from the above observation and the Flow Theorem 43.3 applied to the vector field $\widetilde{v}$ on the *compact* smooth manifold $N \times (\mathbb{R}/\mathbb{Z})$.

On the other hand, if N is non-compact, then unfortunately we cannot apply the Flow Theorem 43.3 as it is formulated, so we have to get our hands dirty a little bit. Recall that we assume that the vector field v is $\overline{h}$-trivial outside of a subset of the form $K \times [0,1]$ where K is a compact subset of N. This means that $v(x,t) = \overline{h}$ for all $(x,t) \in (N \times [0,1]) \setminus K$.

It follows from the Product Topology-Properties Proposition 7.9 that $K \times (\mathbb{R}/\mathbb{Z})$ is also compact. One can easily verify that the following map defines a $\widetilde{v}$-flow:

$$\begin{array}{rcl} \widetilde{G}\colon (N \setminus K) \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} & \to & N \times (\mathbb{R}/\mathbb{Z}) \\ ((x,[t]),s) & \mapsto & (x,[t+s]). \end{array}$$

Furthermore note that we can cover the compact set $K \times (\mathbb{R}/\mathbb{Z})$ by *finitely* many charts. It is now fairly straightforward to adapt the proof of the Flow Theorem 43.3 to give us the desired result.[491] We leave it to the reader, who is eager to show their technical prowess, to fill in the details. ⊞

[490] Initially it might appear strange to work with $N \times (\mathbb{R}/\mathbb{Z})$. This smooth manifold has several nice properties though. It has "less boundary" than $N \times [0,1]$, so the vector field $\widetilde{v}$ is now horizontal to the boundary. We could have achieved the same result by working with $N \times \mathbb{R}$. But in our setting we will make use of the fact that $K \times (\mathbb{R}/\mathbb{Z})$ is compact, whereas $K \times \mathbb{R}$ is not compact.

[491] If $\partial N = \varnothing$, then we could apply the Compact Submanifold Exhaustion Proposition 38.2 to arrange that K is actually a compact smooth manifold.

Now we know that there exists a unique complete $\widetilde{v}$-flow $\widetilde{G}$. By definition of a complete $\widetilde{v}$-flow this means in particular that the following two statements hold:

(a) For each $(Q,[s]) \in N \times (\mathbb{R}/\mathbb{Z})$ we have $\widetilde{G}(Q,[s],0) = (Q,[s])$.
(b) For each $(Q,[s]) \in N \times (\mathbb{R}/\mathbb{Z})$ the map $\mathbb{R} \to N \times (\mathbb{R}/\mathbb{Z})$ given by $t \mapsto \widetilde{G}(Q,[s],t)$ is a solution of the ODE $y' = \widetilde{v}(y)$, in other words, for every $t_0 \in \mathbb{R}$ we have the equality $\frac{d}{dt}(t \mapsto \widetilde{G}(Q,[s],t))|_{t=t_0} = \widetilde{v}(\widetilde{G}(Q,[s],t_0))$.

In the following we denote by $\Pi\colon N \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ the obvious projection.

Claim 3. For every $(Q,[s],t) \in N \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ we have $\Pi(\widetilde{G}(Q,[s],t)) = [s+t]$.

Proof. First note that we have the following commutative diagram:

$$\begin{array}{ccc} N \times [0,1] & \xrightarrow{\mathrm{id}_N \times \Psi} & N \times (\mathbb{R}/\mathbb{Z}) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \Pi} \\ [0,1] & \xrightarrow{\Psi} & \mathbb{R}/\mathbb{Z}. \end{array}$$

Now let $(Q,[s],t_0) \in N \times (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. We pick $(x,\tau) \in N \times [0,1]$ with $(x,[\tau]) = \widetilde{G}(Q,[s],t_0)$. We calculate

$$\begin{aligned} \tfrac{d}{dt}\big(t \mapsto \Pi(\widetilde{G}(Q,[s],t))\big)\big|_{t=t_0} &\overset{\text{Chain Rule 29.15}}{=} \mathrm{D}\Pi_{\widetilde{G}(Q,[s],t)}\big(\tfrac{d}{dt}(t \mapsto \widetilde{G}(Q,[s],t))\big|_{t=t_0}\big) \\ &\overset{\text{by the definition of } \widetilde{v} \text{ and the choice of } (x,\tau)}{=} (\mathrm{D}\Pi_{(x,[\tau])} \circ \mathrm{D}(\mathrm{id}_N \times \Psi)_{(x,\tau)})(v(x,\tau)) \\ &\overset{\text{by the above Statement (b) and the definition of } \tau}{=} \mathrm{D}\Pi_{(x,[\tau])}(\widetilde{v}(\widetilde{G}(Q,[s],t_0))) \\ &\overset{\text{by the above commutative diagram together with the Chain Rule 29.15}}{=} (\mathrm{D}\Psi_\tau \circ \mathrm{D}q_{(x,\tau)})(v(x,\tau)) \\ &\overset{\text{since } v \text{ is horizontally normal}}{=} \text{the image of } 1 \in \mathrm{T}_\tau[0,1] \text{ in } \mathrm{T}_{[\tau]}(\mathbb{R}/\mathbb{Z}). \end{aligned}$$

The claim follows almost immediately from this calculation and some elementary arguments from real analysis. ⊞

Finally we denote by $\Xi\colon N \times (\mathbb{R}/\mathbb{Z}) \to N$ the natural projection and we consider the map

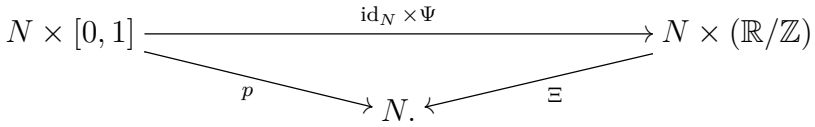
$$G\colon N \times [0,1] \xrightarrow{(Q,s) \mapsto \widetilde{G}(Q,[0],s)} N \times (\mathbb{R}/\mathbb{Z}) \xrightarrow{\Xi} N.$$

Claim 4. The map $G\colon N \times [0,1] \to N$ has the desired properties.

Proof. First note that it follows from (a) that $G_0 = \mathrm{id}_N$. Next note that it follows easily from Claim 3 together with the Submanifold Smooth Structure Lemma 25.23 (3) that G is smooth. Next we claim that for each $t \in [0,1]$ the map $G_t\colon N \to N$ is a diffeomorphism. This can be seen as follows: Using that $\widetilde{G}$ is a diffeotopy it is not hard to show that G_t is a bijection and that each differential of $G_t\colon N \to N$ is invertible. This implies by the Inverse Mapping Theorem 30.1 and the Smooth Pasting Proposition 25.19 (2) that each G_t is indeed a diffeomorphism.[492] In summary we have shown that G is a diffeotopy.

[492]The details are worked out in Exercise 30.24.

Next we want to show that the derivatives of $t \mapsto G(Q,t)$ are of the promised form. Note that we have the following simple minded commutative diagram:



Now let $(Q,t_0) \in N \times [0,1]$. By Claim 3 we know that $\widetilde{G}(Q,[0],t_0) = (x,[t_0])$ for some $x \in N$. We calculate

$$\begin{array}{lll}
\frac{d}{dt}(t \mapsto G(Q,t))|_{t=t_0} & \overset{\text{definition of } G}{=} & \frac{d}{dt}(t \mapsto \Xi(\widetilde{G}(Q,[0],t)))|_{t=t_0} \\
& = & \mathrm{D}\Xi_{(x,[t_0])}(\widetilde{v}(x,[t_0])) \\
& \overset{\text{by (b)}}{=} & \mathrm{D}p_{(x,t_0)}(v(x,t_0)) \\
& \overset{\text{by the above commutative diagram}}{=} &
\end{array}$$

$$\begin{array}{ll}
\overset{\text{Chain Rule 29.15}}{=} & \mathrm{D}\Xi_{(x,[t_0])}\big(\frac{d}{dt}(t \mapsto \widetilde{G}(Q,[0],t))\big|_{t=t_0}\big) \\
= & (\mathrm{D}\Xi_{(x,[t_0])} \circ \mathrm{D}(\mathrm{id}_N \times \Psi))(v(x,t_0)) \\
\overset{\text{by definition of } \widetilde{v}}{=} & \mathrm{D}p_{(G(Q,t_0),t_0)}(v(G(Q,t_0),t_0)). \\
\overset{\text{by the definition of } G \text{ and } x}{=} &
\end{array}$$

We conclude the proof of the claim with two observations:

(1) It follows from this calculation, the uniqueness statement of the Fundamental Theorem 43.15 for Autonomous ODEs and the fact that the vector field v is horizontal near the ends, that the diffeotopy $G\colon N \times [0,1] \to N$ is trivial near the ends.

(2) Recall that we assume that there exists a compact subset $K \subseteq N$ such that the vector field v is $\overline{h}$-trivial outside of $K \times [0,1]$. It follows from the above calculation and the uniqueness statement of the Fundamental Theorem 43.15 for Autonomous ODEs that the diffeotopy $G\colon N \times [0,1] \to N$ is a diffeotopy rel $N \setminus K$. ∎

44.4. Proof of the Isotopy Extension Theorem 44.2 II: Closed smooth manifolds.

We now turn to the actual proof of the Isotopy Extension Theorem 44.2. In this section we will prove the theorem if M and N are closed. This setting makes the discussion much easier and the key idea of the proof becomes clearer. The general case will be proved in the subsequent section.

Evidently, in the setting of the Isotopy Extension Theorem 44.2, we want to use the Vector Fields-Diffeotopies Theorem 44.9 to construct the desired diffeotopy. Thus we need to construct suitable vector fields. We will do so locally and patch them together. This leads us to the following variation on the above definitions from Definition 44.8.

Definition 44.10. Let N be a smooth manifold. Let $U \subseteq N \times [0,1]$ be a subset and let v be a vector field on U. We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two obvious projections.

(1) We say that the vector field v is horizontal near the ends if there exists an $\epsilon > 0$ such that for all points $(x,t) \in U$ with $t \in [0,\epsilon] \cup [1-\epsilon,1]$ we have $v(x,t) \in \ker(\mathrm{D}p_{(x,t)}\colon \mathrm{T}_{(x,t)}(\partial N \times [0,1]) \to \mathrm{T}_x N)$.

(2) We say that v is horizontally normal if for each $(x,t) \in U$ we have

$$\mathrm{D}q_{(x,t)}(v(x,t)) \;=\; 1 \;\in\; \mathrm{T}_t\mathbb{R} =_{\mathrm{i}} \widetilde{\mathrm{T}}_t\mathbb{R} \;=\; \mathbb{R}.$$

The following lemma is the key to patching together promising vector fields.

Lemma 44.11. (Horizontal Vector Fields Lemma) Let N be a smooth manifold, let $U \subseteq N \times [0,1]$ be an open subset, let v_1, v_2 be two vector fields on U and let $f_1, f_2 \colon U \to \mathbb{R}$ be two smooth maps.

(1) If v_1 and v_2 are horizontal near the ends, then so is $f_1 \cdot v_1 + f_2 \cdot v_2$.
(2) If $f_1 + f_2 \equiv 1$ and if v_1 and v_2 are horizontally normal, then so is $f_1 \cdot v_1 + f_2 \cdot v_2$.

Proof. We start out with the proof of Statement (1). We assume that the vector fields v_1 and v_2 are horizontal near the ends. With the hopefully obvious notation we pick suitable $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and we set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Now let $(x,t) \in U$ with $t \in [0,\epsilon] \cup [1-\epsilon, 1]$. We see that

$$\mathrm{D}p_{(x,t)}(f_1(x,t)\cdot v_1(x,t) + f_2(x,t)\cdot v_2(x,t)) \overset{\text{since differentials are linear}}{=} f_1(x,t)\cdot \underbrace{\mathrm{D}p_{(x,t)}(v_1(x,t))}_{=0} + f_2(x,t)\cdot \underbrace{\mathrm{D}p_{(x,t)}(v_2(x,t))}_{=0}.$$

Next we turn to the proof of Statement (2). Now we assume that $f_1 + f_2 \equiv 1$. We see that

$$\begin{aligned}\mathrm{D}q_{(x,t)}(f_1(x,t)\cdot v_1(x,t) + f_2(x,t)\cdot v_2(x,t)) &= f_1(x,t)\cdot \underbrace{\mathrm{D}q_{(x,t)}(v_1(x,t))}_{=1} + f_2(x,t)\cdot \underbrace{\mathrm{D}q_{(x,t)}(v_2(x,t))}_{=1} \\ &= f_1(x,t) + f_2(x,t) \;=\; 1. \qquad \blacksquare\end{aligned}$$

Now we turn to the proof of the Isotopy Extension Theorem 44.2 in the special case that M and N are both closed smooth manifolds.

Proof of Theorem 44.2 if M and N are closed smooth manifolds. Let M be a closed m-dimensional smooth manifold and let N be a closed n-dimensional smooth manifold. Let $F \colon M \times [0,1] \to N$ be a smooth isotopy which is trivial near the ends. We need to show that there exists a diffeotopy $G \colon N \times [0,1] \to N$ with the following properties:

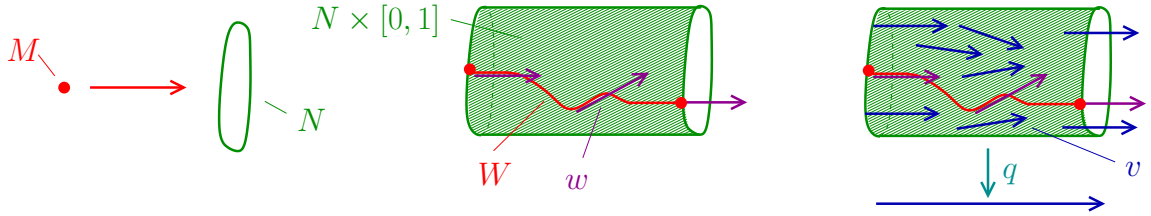
(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F \colon M \times [0,1] \to N$, in other words, we have
$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in M \text{ and } t \in [0,1].$$
(3) G is trivial near the ends.

The idea is to construct the diffeotopy using the Vector Fields-Diffeotopies Theorem 44.9. We introduce the following notation:

(1) Since $F \colon M \times [0,1] \to N$ is a smooth isotopy it follows as in the fairly straightforward Exercise 41.13 that
$$\begin{array}{rccc}\widetilde{F} \colon & M \times [0,1] & \to & N \times [0,1] \\ & (P,t) & \mapsto & (F(P,t),t)\end{array}$$
is a proper smooth embedding. In particular we obtain from the Smooth Embedding Theorem 30.15 that $W := \widetilde{F}(M \times [0,1])$ is a proper submanifold of $N \times [0,1]$ such that for every $(P,t) \in M \times (0,1)$ the point $Q = \widetilde{F}(P,t)$ admits a submanifold chart of type (α).
(2) We denote by $p \colon N \times [0,1] \to N$ and $q \colon N \times [0,1] \to [0,1]$ the two projections.
(3) As in the example on page 1033 we consider the vector field w on W that for $(F(P,t),t) \in W \subseteq N \times [0,1]$ is uniquely determined by
$$\begin{aligned}\mathrm{D}p_{(F(P,t),t)}(w(F(P,t),t)) &= \tfrac{d}{ds}\big(s \mapsto F(P,s)\big)|_{s=t} \;\in\; \mathrm{T}_{F(P,t)}N \\ \mathrm{D}q_{(F(P,t),t)}(w(F(P,t),t)) &= 1 \;\in\; \mathrm{T}_t\mathbb{R} = \mathbb{R}.\end{aligned}$$

One can easily verify that w is horizontal near the ends and horizontally normal.

Claim 1. There exists a smooth vector field v on $N \times [0,1]$ with the following properties:
(1) The vector field v is horizontal near the ends.
(2) The vector field v is horizontally normal.
(3) On $W \cap (N \times [0,1])$ the vector field v agrees with w.



Proof.

The idea is to construct such a vector field v by constructing suitable vector fields on open subsets of $N \times [0,1]$ which form an open cover of $N \times [0,1]$. We paste these vector fields together using a partition of unity and using the Horizontal Vector Fields Lemma 44.11.

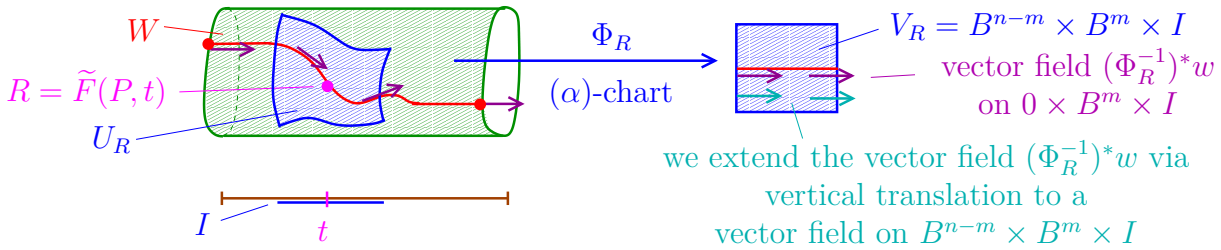
Since the smooth isotopy $F\colon M \times [0,1] \to N$ is by hypothesis trivial near the ends there exists an $\epsilon > 0$ such that $F(P,t) = F(P,0)$ for all $t \in [0,\epsilon]$ and all $P \in M$ and such that $F(P,t) = F(P,1)$ for all $t \in [1-\epsilon, 1]$ and all $P \in M$. Finally on the open subset $U_0 := N \times [0,1] \setminus \widetilde{F}(M \times [\epsilon, 1-\epsilon])$ we define the vector field v_0 to be given by the vector $\overline{h}$ at each point.

Now let $R = \widetilde{F}(P,t)$ with $P \in M$ and $t \in (\epsilon, 1-\epsilon)$. As we mentioned above, the point $R = \widetilde{F}(P,t) \in W \subseteq N \times [0,1]$ admits a submanifold chart of type (α). In fact, similar to the Smooth Embedding Theorem 30.15 one can show that there exists even an open neighborhood I of $t \in (0,1)$, an open neighborhood U_R of $R \in N \times [0,1]$ and a chart $\Phi_R\colon U_R \to V_R = B^{n-m} \times B^m \times I$ with the following properties:

- $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$,
- the map $U_R \xrightarrow{\Phi_R} B^{n-m} \times B^m \times I \xrightarrow{(x,y,t) \mapsto t} I$ equals q.

We now use Φ_R to extend w on $U_R \cap W$ to a vector field v_R on U_R as follows:
(x) We consider the vector field $(\Phi_R^{-1})^* w$ on $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$.
(y) We extend the vector field from (x) to a vector field on $B^{n-m} \times B^m \times I$ by "vertical translation".
(z) We use Φ_R to transport the vector field from (y) back to U_R.

One can can easily verify that the vector field v_0 and that the vector fields v_R are horizontal near the ends, that they are horizontally normal and that they agree with w on W.



Next note that by the Manifold–Partitions-of-Unity Theorem 27.2 we can find a smooth partition of unity $f_0 \colon N \times [0,1] \to \mathbb{R}$ and $\{f_R \colon N \times [0,1] \to \mathbb{R}\}_{R \in \widetilde{F}(M \times (\epsilon, 1-\epsilon))}$ such that $\operatorname{supp}(f_0) \subseteq U_0$ and such that we always have $\operatorname{supp}(f_R) \subseteq U_R$. We denote by $\widetilde{f_0 \cdot v_0}$ the extension of the vector field $f_0 \cdot v_0$ on U_0 to all of $N \times [0,1]$ by setting it zero outside of U_0. The same way we define $\widetilde{f_R \cdot v_R}$. The vector field v is now given by the sum of $\widetilde{f_0 \cdot v_0}$ and of all the vector fields $\widetilde{f_R \cdot v_R}$.

It now follows from a very mild generalization of the Horizontal Vector Fields Lemma 44.11 that v does indeed have all the desired properties. ⊞

We now take the smooth vector field v on $N \times [0,1]$ from Claim 1 and we plug it into the Vector Fields-Diffeotopies Theorem 44.9. We obtain a diffeotopy $G \colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ and such that for each $(Q,t) \in N \times [0,1]$ we have

$$\frac{d}{ds}(s \mapsto G(P,s))|_{s=t} \;=\; \mathrm{D}p_{(G(P,t),t)}(v(\underbrace{G(P,t),t}_{\in N \times [0,1]})) \;\in\; \mathrm{T}_{G(Q,t)}N.$$

Furthermore we saw in the Vector Fields-Diffeotopies Theorem 44.9 that G is trivial near the ends. Thus it remains to prove the following claim:

Claim 2. For each $x \in M$ and $t \in [0,1]$ we have $G(F_0(x),t) = F(x,t)$.

Proof. Let $x \in M$. We have $G(F_0(x),0) = F_0(x) = F(x,0)$. Furthermore, as we had just seen, for each $t \in [0,1]$ we have

$$\frac{d}{ds}\big(s \mapsto G(F_0(x),s)\big)\big|_{s=t} \underset{\text{see above}}{\overset{\uparrow}{=}} \mathrm{D}p_{(G(F_0(x),t),t)}(v(G(F_0(x),t),t)).$$

Next note that we have

$$\frac{d}{ds}(s \mapsto F(x,s))|_{s=t} \underset{\text{definition of } w, \text{ see page 1037}}{\overset{\uparrow}{=}} \mathrm{D}p_{(F(x,t),t)}(w(F(x,t),t)) \underset{\text{since } w = v \text{ on } W}{\overset{\uparrow}{=}} \mathrm{D}p_{(F(x,t),t)}(v(F(x,t),t)).$$

It follows from these observations that the maps

$$\begin{array}{rcl} [0,1] & \to & M \times [0,1] \\ t & \mapsto & (G(F_0(x),t),t) \end{array} \quad \text{and} \quad \begin{array}{rcl} [0,1] & \to & M \times [0,1] \\ t & \mapsto & (F(x,t),t) \end{array}$$

are both solutions of the ODE $(y,t)' = (\mathrm{D}p_{(y,t)}(v(y,t)),1)$ on $N \times [0,1]$ and both satisfy the same starting condition. Using charts we can apply the uniqueness statement of the Fundamental Theorem 43.15 for Autonomous ODEs and we see that for all $t \in [0,1]$ we have $G(F_0(x),t) = F(x,t)$. ■

44.5. **Proof of the Isotopy Extension Theorem 44.2 III: The general case.** In this section we finally provide the proof of the Isotopy Extension Theorem 44.2 in its fullest generality. The basic idea is identical to the idea of the proof in the special case considered in the previous section. But the presence of boundary and the fact that we allow N to be non-compact makes everything technically much more demanding and, let's face it, much less readable.

Definition 44.12. Let N be a smooth manifold. Let $U \subseteq N \times [0,1]$ be a subset and let v be a vector field on U.

(3) We say that v is parallel to the horizontal boundary if for each $(x,t) \in (\partial N \times [0,1]) \cap U$ we have $v(x,t) \in \mathrm{T}_{(x,t)}(\partial N \times [0,1])$.

We will need the following addendum to the Horizontal Vector Fields Lemma 44.11.

Lemma 44.11. (Horizontal Vector Fields Lemma) Let N be a smooth manifold, let $U \subseteq N \times [0,1]$ be an open subset, let v_1, v_2 be two vector fields on U and let $f_1, f_2 \colon U \to \mathbb{R}$ be two smooth maps.

(3) If v_1 and v_2 are parallel to the horizontal boundary, then so is $f_1 \cdot v_1 + f_2 \cdot v_2$.

Proof. If v_1 and v_2 are two vector fields that are parallel to the horizontal boundary, then for any $(x,t) \in (\partial N \times [0,1]) \cap U$ we have

$$\mathrm{D}p_{(x,t)}(f_1(x,t)\cdot v_1(x,t) + f_2(x,t)\cdot v_2(x,t)) = f_1(x,t)\cdot \underbrace{\mathrm{D}p_{(x,t)}(v_1(x,t))}_{\in \mathrm{T}_{(x,t)}(\partial N \times [0,1])} + f_2(x,t)\cdot \underbrace{\mathrm{D}p_{(x,t)}(v_2(x,t))}_{\in \mathrm{T}_{(x,t)}(\partial N \times [0,1])}.$$

Now we are done, since $\mathrm{T}_{(x,t)}(\partial N \times [0,1])$ is a vector subspace. ∎

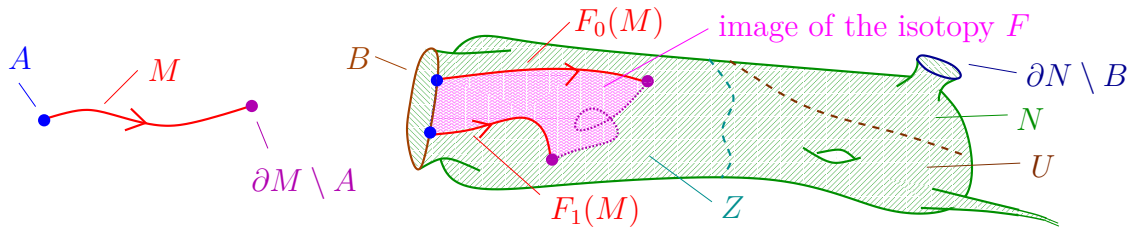
Proof of the Isotopy Extension Theorem 44.2. Let M be a compact n-dimensional smooth manifold and let N be an m-dimensional smooth manifold. Furthermore let A be a union of components of ∂M and let B be a union of components of ∂N. Finally let $F \colon M \times [0,1] \to N$ be a smooth isotopy which is trivial near the ends and which has the following property:

(∗) For each $t \in [0,1]$ we have $F_t(A) \subseteq B$ and we have $F_t(\partial M \setminus A) \subseteq N \setminus \partial N$.

Finally suppose that we are given an open subset U of N with $F(M \times [0,1]) \subseteq U$.

Claim 0. There exists an open subset Z of N with the following three properties:

(1) $F(M \times [0,1]) \subseteq Z \subseteq U$,
(2) $N \setminus Z$ is a neighborhood of $\partial N \setminus B$,
(3) the closure $\overline{Z}$ is compact.



Proof. By the Smooth Manifold–Boundary Proposition 25.27 we know that $\partial N \setminus B$ is a closed subset of N. Furthermore since M is compact we obtain from the Compact-Closed Lemma 3.39, the Product Topology-Properties Proposition 7.9 and the Compact Image Lemma 4.15 that $F(M \times [0,1])$ is a closed subset of N. Note that by hypothesis $F(M \times [0,1])$ and $\partial N \setminus B$ are disjoint. By the Manifold Paracompact-Normal Proposition 23.9 (2) we know that N is normal in the sense of Definition 3.48. Thus there exist disjoint open subsets S and T of N with $F(M \times [0,1]) \subseteq S$ and $\partial N \setminus B \subseteq T$.

The subset $U \cap S$ satisfies (1) and (2), but not necessarily (3). Thus we apply the Compact Exhaustion Lemma 13.6 (1) to the topological space $U \cap S$, which is second-countable, Hausdorff and regionally compact, and the compact subspace $F(M \times [0,1])$, and we obtain a subset $Z \subseteq U \cap S$ with all the desired properties. ⊞

It remains to show that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with the following properties:

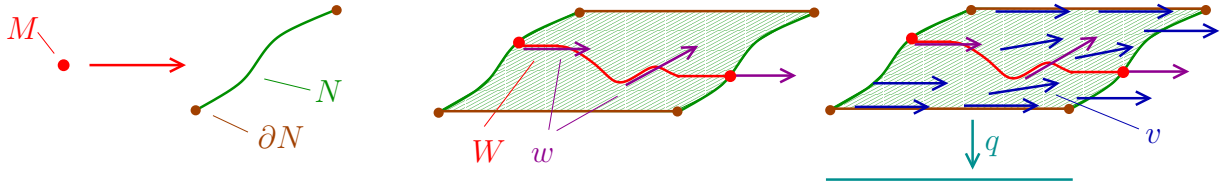
(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon M \times [0,1] \to N$, in other words, we have
$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in M \text{ and } t \in [0,1].$$
(3) G is trivial near the ends.
(4) G is a diffeotopy rel $N \setminus Z$.[493]

Again the idea is to construct the diffeotopy using the Vector Fields-Diffeotopies Theorem 44.9. We introduce the following notation:[494]

(1) We extend the smooth isotopy $F\colon M \times [0,1] \to N \times [0,1]$ to a map $F\colon M \times \mathbb{R} \to N \times \mathbb{R}$ by setting $F(P,t) = F(P,0)$ for $t \leq 0$ and $F(P,t) = F(P,1)$ for $t \geq 1$. Since F is a smooth isotopy that is trivial near the ends we obtain from the Smooth Pasting Proposition 25.19 that F is again smooth.
(2) Since F is a smooth isotopy it follows, as in the fairly elementary Exercise 41.13, that
$$\begin{array}{rcl} \widetilde{F}\colon M \times \mathbb{R} & \to & N \times \mathbb{R} \\ (P,t) & \mapsto & (F(P,t),t) \end{array}$$
is a smooth embedding such that $\widetilde{F}(M \times \mathbb{R})$ is a closed subset of $N \times \mathbb{R}$. In particular we obtain from the Smooth Embedding Theorem 30.15 and the proof thereof that $W := \widetilde{F}(M \times \mathbb{R})$ is a submanifold of $N \times \mathbb{R}$ such that for every $(P,t) \in M \times \mathbb{R}$ the following three statements hold:
   (a) If $P \in M \setminus \partial M$, then $Q = \widetilde{F}(P,t)$ admits a submanifold chart of type (α).
   (b) If $P \in A$, then $Q = \widetilde{F}(P,t)$ admits a submanifold chart of type (β).
   (c) If $P \in \partial M \setminus A$, then $Q = \widetilde{F}(P,t)$ admits a submanifold chart of type (γ).
(3) We denote by $p\colon N \times [0,1] \to N$ and $q\colon N \times [0,1] \to [0,1]$ the two projections.
(4) Basically as in the example on page 1033 we consider the vector field w on $W \cap (N \times [0,1])$ that for $(F(P,t),t) \in W \subseteq N \times [0,1]$ is uniquely determined by
$$\begin{array}{rcl} \mathrm{D}p_{(F(P,t),t)}(w(F(P,t),t)) & = & \frac{d}{ds}\big(s \mapsto F(P,s)\big)|_{s=t} \;\in\; \mathrm{T}_{F(P,t)}N \\ \mathrm{D}q_{(F(P,t),t)}(w(F(P,t),t)) & = & 1 \;\in\; \mathrm{T}_t\mathbb{R} = \mathbb{R}. \end{array}$$
One can easily verify that w is horizontal near the ends, parallel to the horizontal boundary and horizontally normal.

[493] Since $G_0 = \mathrm{id}$, since G is a diffeotopy rel $N \setminus Z$ and since $\overline{Z}$ is compact we see that G is a diffeotopy with compact support.
[494] In the following we partially work with $N \times \mathbb{R}$ which has the big advantage that this is a smooth manifold with boundary $\partial N \times \mathbb{R}$, whereas $N \times [0,1]$ is a smooth manifold with corner $\partial N \times \{0,1\}$, for which we have fewer ready-made results. For example we never developed the concept of a submanifold of a smooth manifold with corner.

Claim 1. There exists a smooth vector field v on $N \times [0,1]$ with the following properties:
(1) The vector field v is horizontal near the ends.
(2) The vector field v is horizontally normal.
(3) The vector field v is parallel to the horizontal boundary.
(4) The vector field v is $\overline{h}$-trivial outside of the compact set $\overline{Z}$.
(5) On $W \cap (N \times [0,1])$ the vector field v agrees with w.



Proof.

The idea is again to construct such a vector field v by constructing suitable vector fields around each point in $N \times [0,1]$ and by pasting them together using a partition of unity and using the Horizontal Vector Fields Lemma 44.11.

Thus let $R \in N \times [0,1]$.

(α) Suppose that $R = \widetilde{F}(P,t)$ with $P \in M \setminus \partial M$ and $t \in [0,1]$. As we mentioned above, the point $R = \widetilde{F}(P,t) \subseteq W \subseteq N \times \mathbb{R}$ admits a submanifold chart of type (α). We proceed as on page 1038: First of all, similar to the Smooth Embedding Theorem 30.15 one can show that there exists even an open neighborhood I of $t \in \mathbb{R}$, an open neighborhood U_R of $R \in N \times [0,1]$, with $U_R \subseteq (N \setminus \partial N) \times [0,1]$ and with $U_R \subseteq Z \times [0,1]$, and a chart $\Phi_R \colon U_R \to V_R = B^{n-m} \times B^m \times I$ with the following properties:
- $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$,
- the map $U_R \xrightarrow{\Phi_R} B^{n-m} \times B^m \times I \xrightarrow{(x,y,t) \mapsto t} I$ equals q,
- $U_R \cap U = \varnothing$.

We now use Φ_R to extend w on $U_R \cap W$ to a vector field v_R on U_R as follows:
(x) The combination of Φ_R and w defines a vector field on $\Phi_R(U_R \cap W) = \{0\} \times B^m \times I$.
(y) We extend the vector field from (x) to a vector field on $B^{n-m} \times B^m \times I$ by "vertical translation".
(z) We use Φ_R to transport the vector field from (y) back to U_R.

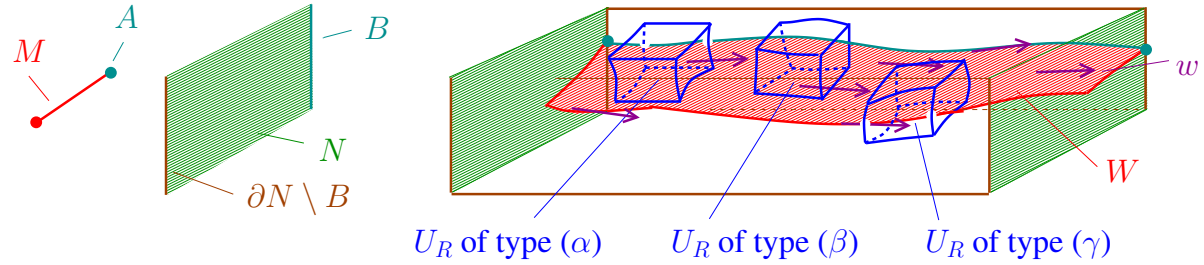
(β) Suppose that $R = \widetilde{F}(P,t)$ with $P \in A$ and $t \in [0,1]$. We pick a submanifold chart $\Phi_R \colon U_R \to V_R = B^{n-m} \times B^m \times I$ of type (β) around R with $U_R \subseteq Z \times [0,1]$ and we define a vector field v_R on U_R in a similar fashion to (α).

(γ) Suppose that $R = \widetilde{F}(P,t)$ with $P \in \partial M \setminus A$ and $t \in [0,1]$. We pick a submanifold chart $\Phi_R \colon U_R \to V_R = B^{n-m} \times B^m \times I$ of type (γ) around R with $U_R \subseteq Z \times [0,1]$. We need to modify the battle plan from (α).
(x) Since in our current setting we have a submanifold chart of type (γ) we have $\Phi_R(U_R \cap W) = \{0\} \times B^m_{\geq 0} \times I$. Therefore the combination of Φ_R and w now defines a smooth vector field on $\Phi_R(U_R \cap W) = \{0\} \times B^m_{\geq 0} \times I$.
(x̂) Using the Whitney Extension Theorem 24.11 we can extend the smooth vector field on $\{0\} \times B^m_{\geq 0}$ to a smooth vector field on $\{0\} \times B^m \times I$.
Now we perform (y) and (z) as above.

(δ) If $R \notin W = \widetilde{F}(M \times [0,1])$, then we set $U_R = (N \times [0,1]) \setminus W$ and we denote by v_R the unique $\overline{h}$-trivial vector field on U_R.

Note that one can easily verify that each of these four types of vector fields is horizontal near the ends, parallel to the horizontal boundary and horizontally normal on its domain.

By the Manifold–Partitions-of-Unity Theorem 27.2 we can find a partition of unity $\{f_R \colon N \times [0,1] \to \mathbb{R}\}_{R \in N \times [0,1]}$ such that for each $R \in N \times [0,1]$ we have $\operatorname{supp}(f_R) \subseteq U_R$. For each $R \in N \times [0,1]$ we denote by $f_R \cdot v_R$ the extension of the vector field $f_R \cdot v_R$ on $U_R \cap (N \times [0,1])$ to all of $N \times [0,1]$ by setting it zero outside of U_R. The vector field v is now given by the sum over all the vector fields $f_R \cdot v_R$.



It follows from a very mild generalization of the Horizontal Vector Fields Lemma 44.11 that v does indeed have all the desired properties. ⊞

We now take the smooth vector field v on $N \times [0,1]$ from Claim 1 and we plug it into the Vector Fields-Diffeotopies Theorem 44.9. We obtain a diffeotopy $G \colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ and such that for each $(Q,t) \in N \times [0,1]$ we have

$$\tfrac{d}{ds}(s \mapsto G(P,s))|_{s=t} \;=\; \mathrm{D}p_{(G(P,t),t)}(v(\underbrace{G(P,t),t}_{\in N \times [0,1]})) \;\in\; \mathrm{T}_{G(Q,t)}N.$$

Furthermore we saw in the Vector Fields-Diffeotopies Theorem 44.9 that G is trivial near the ends and that it is a diffeotopy rel $N \setminus Z$. In particular it is a diffeotopy rel $N \setminus U$, rel some neighborhood of $\partial N \setminus B$ and it has compact support.

It remains to prove the following claim:

Claim 2. For each $x \in M$ and $t \in [0,1]$ we have $G(F_0(x),t) = F(x,t)$.

Proof. Fortunately we had proved this claim on page 1039. ■

44.6. Appendix: The Isotopy Extension Theorem for pairs of smooth manifolds.

One a few occasions we will need the following variation on the Isotopy Extension Theorem 44.2:

Theorem 44.13. (Pairs–Isotopy Extension Theorem) Let M be a compact smooth manifold, let $V \subseteq M$ be a proper submanifold, let N be a smooth manifold and let $W \subseteq N$ be a proper submanifold. Let $F \colon M \times [0,1] \to N$ be a smooth isotopy that is trivial near the ends and with the following property:

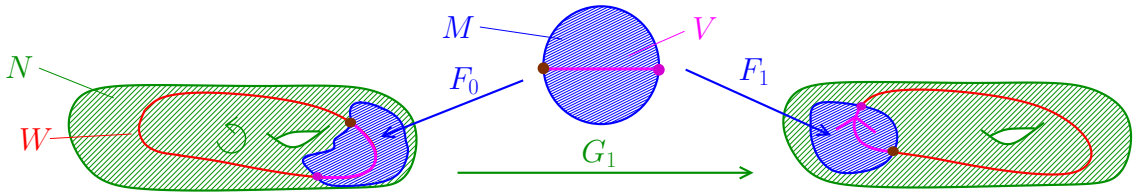
($*$) For each $t \in [0,1]$ we have $F_t(\partial M) \subseteq N \setminus \partial N$ and we have $F_t(V) \subseteq W$.

Then there exists a diffeotopy $G \colon N \times [0,1] \to N$ with the following properties:

(1) $G_0 = \mathrm{id}$.

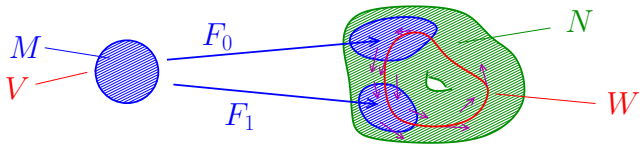
(2) We have $G \circ (F_0 \times \mathrm{id}) = F \colon M \times [0,1] \to N$, in other words, we have
$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in M \text{ and } t \in [0,1].$$
(3) (a) For each $t \in [0,1]$ we have $G_t(W) = W$.
(b) G has compact support.
(c) G is trivial near the ends.

Example. The most important setting for us will be that we are given an n-dimensional smooth manifold N and a k-dimensional manifold W together with smooth embeddings of $\overline{B}^n \to N$ which send $\overline{B}^k \times \{0\}$ into W.



This setting arises for example in the Respectful Embedding Proposition 56.16. □

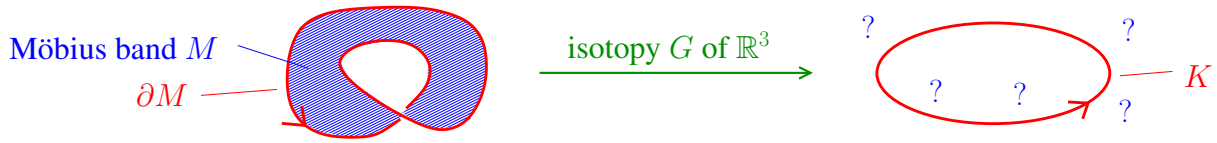
Proof. The proof of this theorem is of course essentially the same as the proof of the Isotopy Extension Theorem 44.2. One just needs to construct the vector field more carefully, namely such that it is tangential to the submanifold W.



We leave it to the reader to fill in all details. ■

44.7. Exercises for Chapter 44.

Exercise 44.1. We consider the Möbius band M in $\mathbb{R}^3$ as shown in the figure below. It should be pretty clear that there exists an isotopy F which turns ∂M into the trivial knot K. By the Isotopy Extension Theorem 44.2 this isotopy F can be extended to a diffeotopy G of $\mathbb{R}^3$. In particular we obtain a map $G_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ with $G_1(\partial M) = K$. What does $G_1(M)$ look like? Put differently, we just saw that there exists an embedded Möbius band in $\mathbb{R}^3$ with boundary given by K. What does it look like?

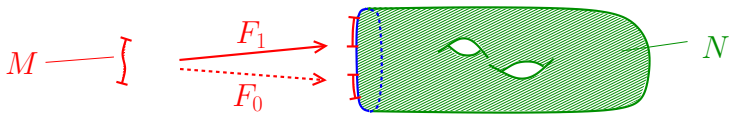


Exercise 44.2. Let M be a smooth manifold and let $F \colon [0,1] \times \partial M \to \partial M$ be a diffeotopy. Show that we can extend F to a diffeotopy $[0,1] \times M \to M$ of all of M.
Hint. Use the Smooth Collar Neighborhood Theorem 35.6.
Remark. This exercise is identical to Exercise 41.3.

Exercise 44.3.

(a) Give an example of a connected smooth manifold M with empty boundary such that for every $P \in M$ the two smooth manifolds $M \setminus \{P\}$ and M are diffeomorphic.
(b) Give an example of a connected smooth manifold M with non-empty boundary such that for every $P \in M$ the two smooth manifolds $M \setminus \{P\}$ and M are diffeomorphic.

Exercise 44.4. Let N be a compact smooth manifold, let M be a compact smooth manifold and let $F\colon M \times [0,1] \to \partial N$ be a smooth isotopy. Show that there exists a diffeotopy $G\colon N \times [0,1] \to N$ with $G_0 = \mathrm{id}$ and such that for each $(x,t) \in M \times [0,1]$ we have $F_t(G_0(x)) = G_t(x)$.
Hint. Use the Smooth Collar Neighborhood Theorem 35.6.
Remark. This exercise is an extension of Exercise 35.12 and it can be viewed as a variation on the Isotopy Extension Theorem 44.2.



Exercise 44.5. Let $n \in \mathbb{N}_{\geq 2}$.

(a) Let $P \subseteq \mathbb{R}^n$ be a finite set. Show that there exists a smooth embedding $f\colon S^{n-1} \to \mathbb{R}^n$ with $P \subseteq f(S^{n-1})$.
(b) Show that the conclusion of (a) in general does not hold if P is a countable bounded subset of $\mathbb{R}^n$.

Exercise 44.6. We consider the two compact 3-dimensional smooth submanifolds X and Y of $\mathbb{R}^3$ that are shown in the figure below. Are the complements $\mathbb{R}^3 \setminus X$ and $\mathbb{R}^3 \setminus Y$ diffeomorphic?



Exercise 44.7. Let M be a connected 2-dimensional smooth manifold with empty boundary. Furthermore let $P_1, \ldots, P_k \in M$ be pairwise distinct points. In the following we denote by $\mathrm{pr}_M\colon M \times [0,1] \to M$ and $\mathrm{pr}_{[0,1]}\colon M \times [0,1] \to [0,1]$ the natural projections. A *braid in* M is a smooth embedding $\varphi\colon \{1,\ldots,k\} \times \times [0,1] \to M \times [0,1]$ such that the following four conditions are satisfied:

(1)
- We have $\varphi(\{1,\ldots,k\} \times \{0\}) = \{P_1,\ldots,P_k\} \times \{0\}$.
- We have $\varphi(\{1,\ldots,k\} \times \{1\}) = \{P_1,\ldots,P_k\} \times \{1\}$.

(2) There exists an $\epsilon > 0$ such that the following conditions are satisfied:
- We have $\mathrm{pr}_M(\varphi(i,t)) = \mathrm{pr}_M(\varphi(i,0))$ for all $i = 1,\ldots,k$ and every $t \in [0,\epsilon]$.
- We have $\mathrm{pr}_M(\varphi(i,t)) = \mathrm{pr}_M(\varphi(i,1))$ for all $i = 1,\ldots,k$ and every $t \in [1-\epsilon,1]$.

(3) For each $i \in \{1,\ldots,k\}$ and $t \in [0,1]$ we have $\mathrm{pr}_{[0,1]}(\varphi(i,t)) = t$.

In the following we say that two braids φ_0, φ_1 as above are *equivalent* if there exists a smooth map $\Phi\colon \{1,\ldots,k\} \times \times [0,1] \times [0,1] \to M \times [0,1]$ such that $\Phi_0 = \varphi_0$, $\Phi_1 = \varphi_1$ and

such that each Φ_t is a braid. We denote by $\mathrm{B}_k(M) := \mathrm{B}_k^{P_1,\dots,P_k}(M)$ the set of equivalence classes of braids.

(a) Show that the map

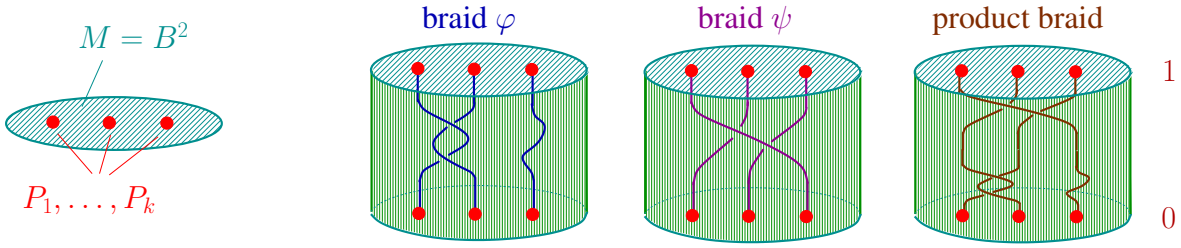
$$\begin{array}{rcl} \mathrm{B}_k(M) \times \mathrm{B}_k(M) & \to & \mathrm{B}_k(M) \\ ([\varphi],[\psi]) & \mapsto & \left[\begin{array}{rcl} \{1,\dots,k\}\times[0,1] & \to & M \\ (i,t) & \mapsto & \begin{cases} (\mathrm{pr}_M(\varphi(i,2t)),t), & \text{if } t\in[0,\frac{1}{2}], \\ (\mathrm{pr}_M(\psi(i,2t-1)),t), & \text{if } t\in[\frac{1}{2},1] \end{cases} \end{array}\right] \end{array}$$

defines a group structure on $\mathrm{B}_k(M)$.

(b) Show that the isomorphism type of $\mathrm{B}_k(M)$ does not depend on the choice of the distinct points $P_1,\dots,P_k$.

(c) Why do we only consider 2-dimensional smooth manifolds?

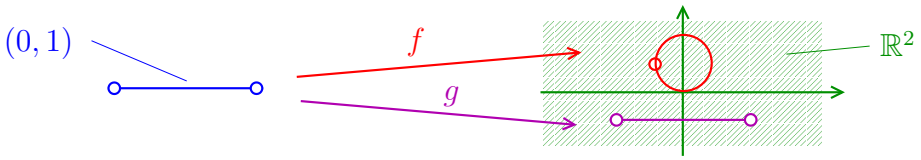
Remark. The group $\mathrm{B}_k(M)$ is called the *k-braid group of M*. The group $\mathrm{B}_k := \mathrm{B}_k(\mathbb{R}^2)$ is commonly called *the k-braid group*.



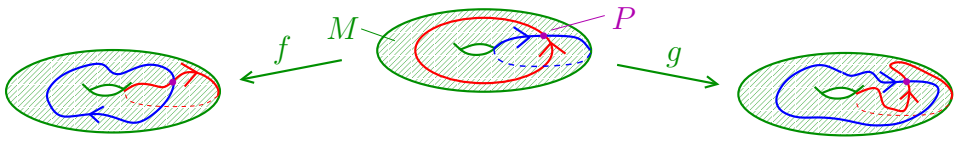
Exercise 44.8. We consider the two smooth embeddings

$$\begin{array}{rcl} f\colon (-1,1) & \to & \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C} \\ t & \mapsto & \exp(\pi\,\mathrm{i}\,t) + \mathrm{i} \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon (-1,1) & \to & \mathbb{R}^2 \\ t & \mapsto & (2t,-1) \end{array}$$

that are shown in the figure below. Show that f and g are smoothly isotopic and show that there is no diffeotopy $G\colon \mathbb{R}^2\times[0,1]\to\mathbb{R}^2$ with $G_0 = \mathrm{id}$ and $G_1 \circ f = g$.



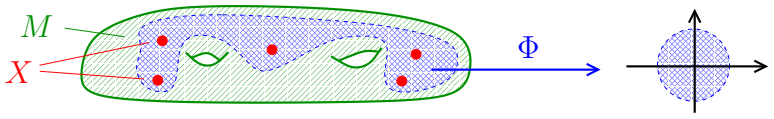
Exercise 44.9. Let M be a connected orientable smooth manifold and let $f\colon M\to M$ be an orientation-preserving diffeomorphism that fixes a point $P\in M\setminus\partial M$. Show that there exists a diffeotopy rel $\{P\}$ from f to a diffeomorphism $g\colon M\to M$ that is the identity on an open neighborhood of P.



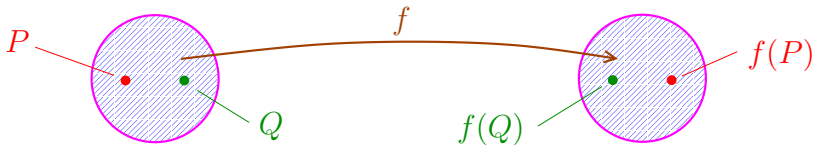
Exercise 44.10. Let M be a connected smooth manifold.

(a) Let $X\subseteq M\setminus\partial M$ be a finite set. Show that there exists a chart $\Phi\colon U\to B^n$ of type (i) such that $X\subseteq U$.

(b) We now suppose that ∂M is non-empty and connected. Let $X \subseteq \partial M$ be a finite set. Show that there exists a chart $\Phi\colon U \to B^n_{\geq 0}$ of type (ii) such that $X \subseteq U$.

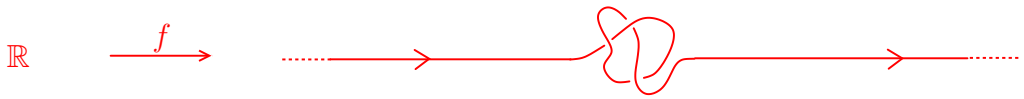


Exercise 44.11. We consider the disk $\overline{B}^2$ with two distinct points P and Q. By the Homogeneous-via-Diffeotopies Proposition 44.6 there exists an orientation-preserving diffeomorphism f of $\overline{B}^2$ that swaps P and Q and that is the identity on S^1. Try to visualize such a map.



Exercise 44.12. Let M be a connected n-dimensional smooth manifold. Show that for any two distinct points $P, Q \in M$ there exists a smooth embedding $\gamma\colon [0,1] \to M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.
Hint. If $P, Q \in M \setminus \partial M$, then this is just the content of Embedded Path Corollary 44.7. You could use the Smooth Collar Neighborhood Theorem 35.6 to reduce the general case to the Embedded Path Corollary 44.7.

Exercise 44.13. We consider the proper smooth embedding $f\colon \mathbb{R} \to \mathbb{R}^3$ that is shown in the figure below and we consider the standard smooth embedding $g\colon \mathbb{R} \to \mathbb{R}^3$ that is given by $x \mapsto (x, 0, 0)$. Show that there exists a *proper* smooth isotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ with $F_0 = f$ and $F_1 = g$.
Remark 1. Recall that according to Definition 41.3 a map $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ is a proper smooth isotopy if each F_t is a proper smooth embedding and if $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ is proper, i.e. if the preimage of each compact subspace of $\mathbb{R}^3$ is a compact subspace of $\mathbb{R} \times [0,1]$.
Remark 2. In Chapter 111 we will study the "fundamental groups" of the complements of $\mathbb{R}^3 \setminus f(\mathbb{R})$ and $\mathbb{R}^3 \setminus g(\mathbb{R})$ and we will see that they are not isomorphic. As in the Smooth Isotopies-Complements Proposition 44.5 this shows that there is no homeomorphism $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\Phi \circ f = g$. In particular there is no diffeotopy $G\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $G(f(x), t) = F(x, t)$ for all $(x, t) \in \mathbb{R} \times [0,1]$. This shows again that in the formulation of the Isotopy Extension Theorem 44.2 it is difficult to replace the hypothesis that M is compact by a weaker condition.



Exercise 44.14. Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, n-1\}$. Furthermore let $K \subseteq S^n$ be a k-dimensional submanifold that is diffeomorphic to S^k. We assume that K is smoothly isotopic to the "trivial knot" $U := \{(x, 0) \mid x \in S^k\} \subseteq \mathbb{R}^n \subseteq \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$. Show that there exists a smooth embedding $\tau\colon \overline{B}^{n-k} \times K \to S^n$ such that $\tau(0, P) = P$ for all $P \in K$.

(Later in Definition 58.2 we will refer to such a map as a *tubular map* for K.)
Hint. First deal with the case $K = U$.

CHAPTER 45

Orientation-preserving and orientation-reversing loops

In this chapter we will apply the Isotopy Extension Theorem 44.2 to introduce and study the concept of orientation-preserving and orientation-reversing loops. We will use this concept and various results from earlier chapters to classify smooth embeddings of balls into smooth manifolds.

This chapter is organized as follows:

- In Section 45.1 we will introduce orientation-preserving and orientation-reversing loops.
- In Section 45.2 we will see that non-orientable smooth manifolds admit orientation flipping diffeotopies.
- In Section 45.3 we will state and prove the Smooth Ball Embedding Theorem 45.10 which classifies smooth embeddings of n-dimensional closed balls into n-dimensional smooth manifolds.
- In Section 45.4 we will state and prove the Lower-Dimensional Smooth Ball Embedding Theorem 45.4.
- In Section 45.5 we will provide exercises for Chapter 45.

In the next Chapter 46 we will introduce the degree of a continuous map between oriented smooth manifolds.

45.1. **Orientation-preserving and orientation-reversing loops.** We start out with the following basic definitions:

Definition 45.1. Let X be a topological space and let $\gamma\colon [0,1] \to X$ be a path.

(1) We say that γ is a loop if $\gamma(0) = \gamma(1)$.

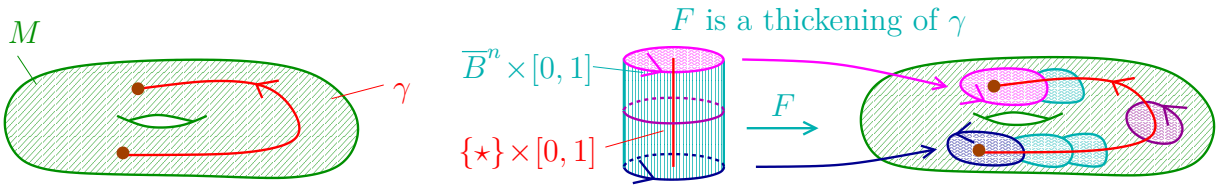
(2) As in Definition 44.3 we say that γ is trivial near the ends if there exists an $\epsilon > 0$ such that for each $t \in [0, \epsilon]$ we have $\gamma(t) = \gamma(0)$ and if for each $t \in [1-\epsilon, 1]$ we have $\gamma(t) = \gamma(1)$.[495]

Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a smooth path.

(3) Throughout this section we denote the origin of $\overline{B}^n$ by $\star$.

(4) A thickening of the smooth path γ is a smooth isotopy $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ such that for each $t \in [0,1]$ we have $F(\star, t) = \gamma(t)$.

[495] As so often it follows from the Stationary-Near-The-Ends-Lemma 41.7 that we can turn any smooth path into a smooth path that is trivial near the ends.



Proposition 45.2. (Path-Thickening Proposition) Let M be a smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a smooth path that is trivial near the ends.

(1) Given any smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ with $\varphi(\star) = \gamma(0)$ there exists a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ that is trivial near the ends with $F_0 = \varphi$.

(2) Given any smooth embedding $\psi\colon \overline{B}^n \to M \setminus \partial M$ with $\psi(\star) = \gamma(1)$ there exists a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ that is trivial near the ends with $F_1 = \psi$.

Proof. Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a smooth path that is trivial near the ends. We provide the proof of Statement (1). The proof of Statement (2) is very similar.

As in the proof of the Path Extension-to-Diffeotopy Lemma 44.4 we point out that the map $\{\star\} \times [0,1] \to M \setminus \partial M$ that is given by $(\star, t) \mapsto \gamma(t)$ is a smooth isotopy that is trivial near the ends. It follows from the Isotopy Extension Theorem 44.2 that there exists a diffeotopy $F\colon M \times [0,1] \to M$ that is trivial near the ends with $F_0 = \mathrm{id}$ and such that for each $t \in [0,1]$ we have $F(\star, t) = \gamma(t)$. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding with $\varphi(\star) = \gamma(0)$. The map

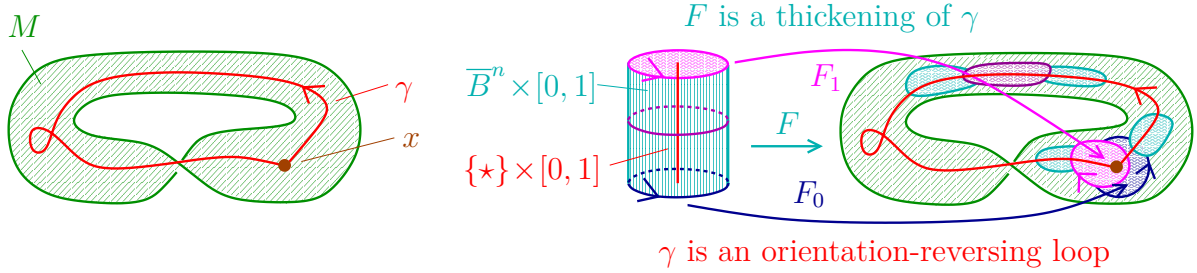
$$\begin{array}{rcl} F \circ (\varphi \times \mathrm{id}_{[0,1]})\colon \overline{B}^n & \to & M \\ (x,t) & \mapsto & F(\varphi(x), t) \end{array}$$

is a thickening of γ that is trivial near the ends with $F_0 = \varphi$. ■

Definition 45.3. Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a loop that is trivial near the ends. By the Path-Thickening Proposition 45.2 we can pick a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ. We say that γ is orientation-preserving if the automorphism

$$\mathrm{T}_\star \overline{B}^n \xrightarrow[\cong]{\mathrm{D}(F_0)_\star} \mathrm{T}_{\gamma(0)} M \xleftarrow[\cong]{\mathrm{D}(F_1)_\star} \mathrm{T}_\star \overline{B}^n$$

of the vector space $\mathrm{T}_\star \overline{B}^n$ is orientation-preserving. Otherwise we say that γ is orientation-reversing.



Remark. In Definition 95.18 we will give an alternative and slightly more general definition of orientation-preserving and orientation-reversing loops. In the Orientation–Smooth

Isotopy Proposition 95.20 we will see that the two definitions are equivalent for smooth paths that take values in $M \setminus \partial M$. □

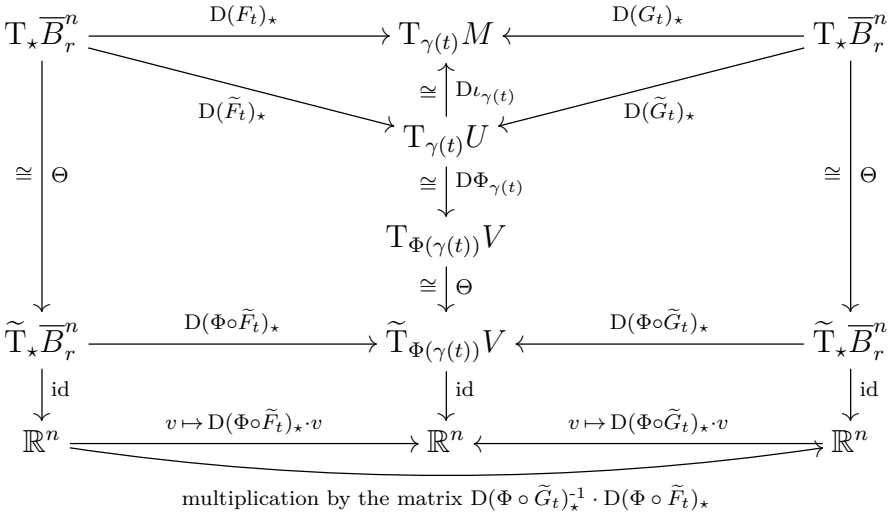
Proposition 45.4. (Orientation Preserving Loop–Definition Proposition) Let M be a smooth manifold. The above definition of a loop being orientation-preserving does not depend on the choice of the thickening.

Proof. Let M be an n-dimensional smooth manifold and let $\gamma\colon [0,1] \to M \setminus \partial M$ be a loop that is trivial near the ends. Let $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ and $G\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ be two thickenings of γ.

Claim. The map
$$\begin{array}{rcl} [0,1] & \to & \mathbb{R} \setminus \{0\} \\ t & \mapsto & \det\left((\mathrm{D}(G_t)_\star)^{-1} \circ \mathrm{D}(F_t)_\star\colon \mathrm{T}_\star \overline{B}^n \to \mathrm{T}_{\gamma(t)} M \to \mathrm{T}_\star \overline{B}^n\right) \end{array}$$
is continuous. In particular, either all values are positive or all values are negative.

Proof. We need to show that the given map is continuous. By the Pasting Proposition 4.7 it suffices to show that the given map is locally continuous.

Let $t \in [0,1]$. We pick a chart $\Phi\colon U \to V$ around $\gamma(t)$. We denote by $\iota\colon U \to M$ the inclusion map. Since $F, G\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ are in particular continuous and since U is open there exists an open neighborhood I of $t \in [0,1]$ and an $r > 0$ such that $F(\overline{B}^n_r \times I) \subseteq U$ and such that $G(\overline{B}^n_r \times I) \subseteq U$. We denote by $\widetilde{F}_t\colon \overline{B}^n \to U$ and $\widetilde{G}_t\colon \overline{B}^n \to U$ the unique maps with $F_t = \iota \circ \widetilde{F}_t$ and with $G_t = \iota \circ \widetilde{G}_t$. We consider the following diagram:



We make the following clarifications and observations:

(a) The maps Θ are the natural isomorphisms provided by the Tangent Space-Isomorphism Proposition 29.24.

(b) The pentagons commutes since the maps Θ are natural and since differentials are functorial.

(c) The triangles commute by the functoriality of differentials.

(d) By the Differential-equals-Differential Lemma 29.6 the differential of a smooth map between the physical tangent spaces of open subsets of $\mathbb{R}^n$ is just given by multiplication by the $(n \times n)$-matrix that is given by the various partial derivatives. In other words, the two bottom squares commute.

It follows from the above discussion and the fact that F and G are "smooth in t", that the map of the claim is continuous on I. We have thus shown that the given map is locally continuous which, as we mentioned above, implies by the Pasting Proposition 4.7, that the given map is continuous. ⊞

Finally we consider the following little commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_\star \overline{B}^n & \xrightarrow[\cong]{(\mathrm{D}(F_1)_\star)^{-1} \circ \mathrm{D}(F_0)_\star} & \mathrm{T}_\star \overline{B}^n \\ {\scriptstyle (\mathrm{D}(G_0)_\star)^{-1} \circ \mathrm{D}(F_0)_\star}\downarrow \cong & & \cong \downarrow{\scriptstyle (\mathrm{D}(G_1)_\star)^{-1} \circ \mathrm{D}(F_1)_\star} \\ \mathrm{T}_\star \overline{B}^n & \xrightarrow[\cong]{(\mathrm{D}(G_1)_\star)^{-1} \circ \mathrm{D}(G_0)_\star} & \mathrm{T}_\star \overline{B}^n \end{array}$$

It follows from the claim that the vertical isomorphism to the left is orientation-preserving if and only if the vertical isomorphism to the right is orientation-preserving. It follows from this observation and the Orientation-Composition Lemma 31.8 that the top horizontal map is orientation-preserving if and only if the bottom horizontal map is orientation-preserving. We have thus shown that it does not matter whether we work with the thickening F or the thickening G. ■

Proposition 45.5. (Orientability-via-Loops Proposition) Let M be a connected smooth manifold and let $P \in M \setminus \partial M$. The following four statements are equivalent:

(1) M is orientable.
(2) $M \setminus \partial M$ is orientable.
(3) Every loop $\gamma\colon [0,1] \to M \setminus \partial M$ that is trivial near the ends is orientation-preserving.
(4) Every loop $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = \gamma(1) = P$ that is trivial near the ends is orientation-preserving.

Proof. Let M be a connected n-dimensional smooth manifold and let $P \in M \setminus \partial M$. The implication (1)⇒(2) is basically obvious. The implication (2)⇒(1) is the content of the Complement-of-Boundary Orientation Lemma 31.36.

We turn to the proof of the implication (2)⇒(3). Thus we now assume that we can pick an orientation for $M \setminus \partial M$. Let $\gamma\colon [0,1] \to M \setminus \partial M$ be a loop that is trivial near the ends. We need to show that γ is orientation-preserving. To do so we pick a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ. By the Smooth Isotopy-Orientation Proposition 42.4 we know that each $\mathrm{D}F_t\colon \mathrm{T}_\star \overline{B}^n \to \mathrm{T}_{\gamma(t)} M$ is orientation-preserving or each such map is orientation-reversing. In particular the map is orientation-preserving for $t = 0$ if and only if it is for $t = 1$. It follows immediately that the loop γ is orientation-preserving.

The implication (3)⇒(4) is trivial. Therefore at this stage it remains to prove the implication (4)⇒(2).

Thus, in the following we assume that every smooth path $\gamma\colon [0,1] \to M \setminus \partial M$ with $\gamma(0) = \gamma(1) = P$ that is trivial near the ends is orientation-preserving. We need to show that $M \setminus \partial M$ is orientable.

We pick an orientation $\mathcal{O}_P$ of $\mathrm{T}_P M$. Let $Q \in M \setminus \partial M$. By the Push Away-from-Boundary Lemma 23.17 we know that $M \setminus \partial M$ is also path-connected. It follows from the Smooth Path-Connectivity Proposition 25.32 together with the Stationary-Near-The-Ends-Lemma 41.7 that there exists a smooth path $\gamma\colon [0,1] \to M \setminus \partial M$ from P to Q that is trivial near the ends. By the Path-Thickening Proposition 45.2 there exists a

thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ that is trivial near the ends. We equip $\mathrm{T}_Q M$ with the orientation $\mathcal{O}_Q$ that is given by the image of the orientation $\mathcal{O}_P$ of $\mathrm{T}_P M$ under the isomorphism

$$\mathrm{T}_P M \xrightarrow{(\mathrm{D}(F_0)_\star)^{-1}} \mathrm{T}_\star \overline{B}^n \xrightarrow{\mathrm{D}(F_1)_\star} \mathrm{T}_Q M.$$

Claim 1. This orientation of $\mathrm{T}_Q M$ does not depend on the choice of the smooth path γ.

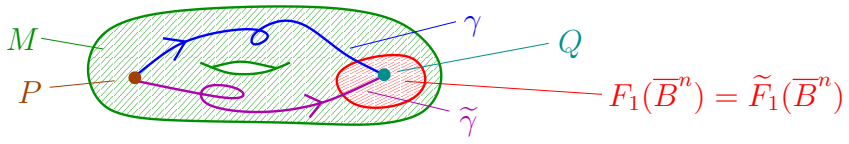
Proof. Let $\widetilde{\gamma}\colon [0,1] \to M \setminus \partial M$ be another smooth path from P to Q that is trivial near the ends. Recall that by the Path-Thickening Proposition 45.2 (2) there exists a thickening $\widetilde{F}\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of $\widetilde{\gamma}$ that is trivial near the ends *and* which has the property that $\widetilde{F}_1\colon \overline{B}^n \to M \setminus \partial M$ equals $F_1\colon \overline{B}^n \to M \setminus \partial M$. We consider the concatenations

$$\delta\colon [0,1] \to M \setminus \partial M, \quad t \mapsto \begin{cases} \gamma(2t), & \text{if } t \in [0, \tfrac{1}{2}], \\ \widetilde{\gamma}(2-2t), & \text{if } t \in [\tfrac{1}{2}, 1] \end{cases}$$

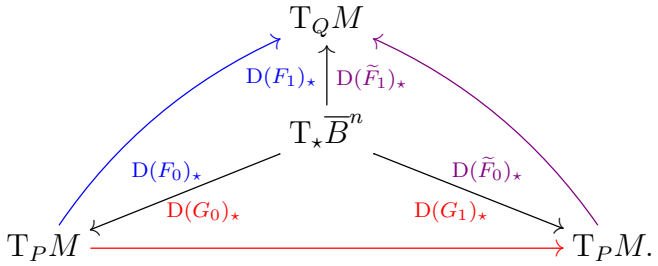
and

$$G\colon \overline{B}^n \times [0,1] \to M \setminus \partial M, \quad (x,t) \mapsto \begin{cases} F(x,2t), & \text{if } t \in [0, \tfrac{1}{2}], \\ \widetilde{F}(x,2-2t), & \text{if } t \in [\tfrac{1}{2}, 1]. \end{cases}$$

It follows from the fact that $F_1 = \widetilde{F}_1$ that the maps are well-defined and it follows from the fact that γ, $\widetilde{\gamma}$, F and $\widetilde{F}$ are trivial near the ends, together with the Smooth Pasting Proposition 25.19, that δ and G are actually smooth. Evidently δ is a loop and G is a thickening of δ.



Finally we consider the following diagram:



We make the following clarifications and observations:

- By construction we have $F_1 = \widetilde{F}_1$ and we have $G_0 = F_0$, $G_1 = \widetilde{F}_0$. Thus the two maps on the arrows are always the same map.
- The undecorated colored maps are defined to be the composition of the (inverse) of the two maps of the same color. In other words, the diagram commutes by definition.
- By hypothesis δ is orientation-preserving. This means that the horizontal red map at the bottom is orientation-preserving.
- It follows from the above that the blue map induces the same orientation on $\mathrm{T}_Q M$ as the violet map. ⊞

At this state it suffices to prove the following claim:

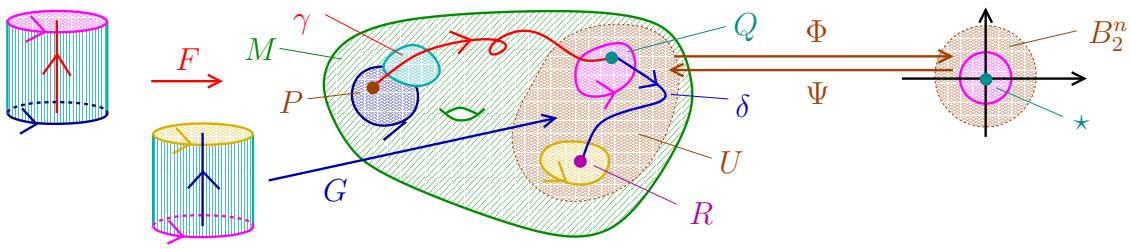
Claim 2. The orientations of the tangent spaces $\mathrm{T}_Q M$ define an orientation on $M \setminus \partial M$.

Proof. We will use the criterion from the Orientations-via-Atlas Lemma 31.21 to verify that these orientations define an orientation on $M \setminus \partial M$. Thus let $Q \in M \setminus \partial M$.

Before we do anything else we introduce the following little notation: For each $x \in B_2^n$ we denote by $\mathcal{S}_x \in \mathrm{T}_x\overline{B}^n$ the standard orientation.

Next we pick a chart $\Phi\colon U \to B_2^n$ around Q with $\Phi(Q) = \star$ and which has the property that the map $\mathrm{D}\Phi_Q\colon \mathrm{T}_Q U \to \mathrm{T}_\star \overline{B}_2^n$ sends the orientation $\mathcal{O}_Q$ to the standard orientation $\mathcal{S}_\star$ of $\mathrm{T}_\star\overline{B}_2^n$. As usual we denote the inverse of the chart Φ by $\Psi\colon B_2^n \to U$. By the Orientations-via-Atlas Lemma 31.21 it now suffices to show that Φ is orientation-preserving for every $R \in U$. Put differently, we need to show that for each $R \in U$ we have $\mathrm{D}\Phi_R(\mathcal{O}_R) = \mathcal{S}_{\Phi(R)}$.

We pick a smooth path $\gamma\colon [0,1] \to M \setminus \partial M$ from P to Q that is trivial near the ends. It follows immediately from the Path-Thickening Proposition 45.2 (2) that we can pick a thickening $F\colon \overline{B}^n \times [0,1] \to M \setminus \partial M$ of γ that is trivial near the ends and which has the property that $F_1\colon \overline{B}^n \to M$ equals $\Psi\colon \overline{B}^n \to U \subseteq M$.



Given $R \in U$ we pick a path $\delta\colon [0,1] \to U$ from Q to R that is trivial near the ends. We use the Path-Thickening Proposition 45.2 (1) to find a thickening $G\colon \overline{B}^n \times [0,1] \to U$ of δ with $G_0 = F_1$. We consider the concatenations

$$\gamma * \delta\colon [0,1] \to M, \quad t \mapsto \begin{cases} \gamma(2t), & \text{if } t \in [0,\tfrac{1}{2}], \\ \delta(2t-1), & \text{if } t \in [\tfrac{1}{2},1]. \end{cases}$$

$$F * G\colon \overline{B}^n \times [0,1] \to M, \quad (x,t) \mapsto \begin{cases} F(x,2t), & \text{if } t \in [0,\tfrac{1}{2}], \\ G(x,2t-1), & \text{if } t \in [\tfrac{1}{2},1]. \end{cases}$$

Note that $\gamma * \delta$ is a path from P to R that is trivial near the ends and that $F * G$ is a thickening of $\gamma * \delta$ that is trivial near the ends. We now do the following:

- We use the path γ and the thickening F to define the orientation $\mathcal{O}_Q$.
- We use the path $\gamma * \delta$ and the thickening $F * G$ to define the orientation $\mathcal{O}_R$.

Now we see that

$$\begin{aligned}
\mathrm{D}\Phi_R(\mathcal{O}_R) &\overset{\text{we use } F*G \text{ to define } \mathcal{O}_R}{=} \mathrm{D}\Phi_R((\mathrm{D}(\underbrace{(F*G)_1}_{=G_1})_\star \circ (\mathrm{D}(\underbrace{(F*G)_0}_{=F_0}))_\star^{-1})(\mathcal{O}_P)) \\
&= \mathrm{D}\Phi_R((\mathrm{D}(G_1)_\star \circ (\mathrm{D}(G_0)_\star)^{-1} \circ \mathrm{D}(F_1)_\star \circ (\mathrm{D}(F_0)_\star)^{-1})(\mathcal{O}_P)) \\
&\overset{\text{we use } F \text{ to define } \mathcal{O}_Q}{=} \mathrm{D}\Phi_R((\mathrm{D}(G_1)_\star \circ \mathrm{D}(G_0)_\star^{-1})(\mathcal{O}_Q)) \\
&\overset{\text{Chain Rule}}{=} (\mathrm{D}(\Phi \circ G_1)_\star \circ (\mathrm{D}(\Phi \circ G_0)_\star)^{-1})(\mathcal{S}_\star) \\
&\overset{\text{by choice of } \Phi}{=} \mathrm{D}\Phi_R((\mathrm{D}(G_1)_\star \circ \mathrm{D}(G_0)_\star^{-1})((\mathrm{D}\Phi_\star^{-1})(\mathcal{S}_\star))) \\
&\overset{\text{since } \Phi\circ G_0 \text{ and } \Phi \circ G_1 \text{ are smoothly isotopic maps from } \overline{B}^n \to B_2^n \text{ we obtain the equality from the Smooth Isotopy-Orientation Proposition 42.4}}{=} \mathcal{S}_{\Phi(R)}.
\end{aligned}$$

■

Remark. In Section 95.4 we will give an alternative and slightly more general definition of orientation-preserving and orientation-reversing loops. In the Orientation–Smooth Isotopy

Proposition 95.20 we will see that the two definitions are equivalent for smooth paths that take values in $M \setminus \partial M$. In fact in Section 95.4 we will study orientation-preserving and reversing loops in greater detail. For example we will see that if two loops are homotopic, then one is orientation-preserving if and only if the other is. In principle one could try to prove this statement with the techniques of the current section, but that promises to be a technical mess. □

45.2. **Orientation flipping diffeotopies I.** The main goal of this section is to prove the following proposition:

Proposition 45.6. (Orientation-Flipping Proposition) Let M be a *non-orientable* connected smooth manifold and let $X, Y \subseteq M \setminus \partial M$ be two disjoint finite subsets. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $X \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
   (a) For each $x \in X$ the map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
   (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Remark.

(1) It follows from the Smooth Isotopy-Orientation Proposition 42.4 that the conclusion of this proposition does *not* hold for *orientable* smooth manifolds.
(2) In Section 95.6 we will give a somewhat different proof of the Orientation-Flipping Proposition 45.6. □

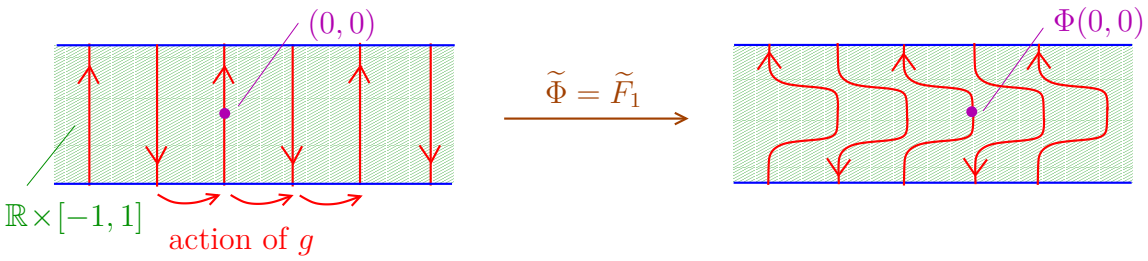
Example. We consider $\widetilde{M} := \mathbb{R} \times [-1,1]$ and the orientation-reversing self-diffeomorphism

$$\begin{array}{rcl} g\colon \widetilde{M} = \mathbb{R} \times [-1,1] & \to & \widetilde{M} = \mathbb{R} \times [-1,1] \\ (r,s) & \mapsto & (r+1,-s). \end{array}$$

Recall that the Möbius band is defined as $M = \widetilde{M}/\sim$ where $P \sim g(P)$. We denote by $p\colon \mathbb{R} \times [-1,1] \to M$ the natural projection. Let $x = p(0,0)$. We want to find a diffeotopy $F\colon M \times [0,1] \to M$ as in the Orientation-Flipping Proposition 45.6, with $X = \{x\}$ and $Y = \varnothing$. Using the Smooth Transition Function Lemma 24.9 we can find a smooth map $\tau\colon [-1,1] \to [0,1]$ such that $\tau(0) = 1$, such that $\tau(s) = 0$ for all $s \geq \frac{1}{2}$ and such that $\tau(-s) = \tau(s)$ for all $s \in [-1,1]$. We consider the map

$$\begin{array}{rcl} \widetilde{F}\colon \widetilde{M} \times [0,1] & \to & \widetilde{M} \\ ((r,s),t) & \mapsto & (r + t \cdot \tau(s), s) \end{array}$$

that is sketched in the figure below.
Note that $\widetilde{F}$ descends to a map $F\colon M \times [0,1] \to M$ which is a diffeotopy rel ∂M with $F_0 = \mathrm{id}$ and with $F_1(x) = x$. We set $\widetilde{\Phi} := \widetilde{F}_1$ and $\Phi := F_1$. We claim that $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is



orientation-reversing. To show this statement let us consider the following diagram:

$$\begin{array}{ccccc} \mathrm{T}_{(0,0)}(\mathbb{R}\times[-1,1]) & \xrightarrow{\mathrm{D}\widetilde{\Phi}_{(0,0)}} & \mathrm{T}_{(1,0)}(\mathbb{R}\times[-1,1]) & \xrightarrow{\mathrm{D}(g^{-1})_{(1,0)}} & \mathrm{T}_{(0,0)}(\mathbb{R}\times[-1,1]) \\ \downarrow{\scriptstyle \mathrm{D}p_{(0,0)}} & & & \searrow^{\mathrm{D}p_{(1,0)}} & \downarrow{\scriptstyle \mathrm{D}p_{(0,0)}} \\ \mathrm{T}_{p(0,0)}M & & \xrightarrow{\qquad\qquad\mathrm{D}\Phi_{p(0,0)}\qquad\qquad} & & \mathrm{T}_{p(0,0)}M. \end{array}$$

Note that $p\circ\widetilde{\Phi} = \Phi\circ p$ and $p\circ g = p$ (which implies that $p = p\circ g^{-1}$). Thus the diagram commutes by the functoriality of the differential.

Since $\widetilde{F}_0 = \mathrm{id}$ we know by the Smooth Isotopy-Orientation Proposition 42.4 that $\widetilde{F}_1 = \Phi$ is also orientation-preserving. In particular $\mathrm{D}\widetilde{\Phi}_{(0,0)}$ is orientation-preserving. On the other hand $\mathrm{D}(g^{-1})_{(1,0)} = (\mathrm{D}g_{(0,0)})^{-1}$ is orientation-reversing. In summary we see that the composition of the top horizontal maps is an orientation-reversing automorphism of $\mathrm{T}_{(0,0)}(\mathbb{R}\times[-1,1])$. The left and right vertical maps are honest-to-G*d the same isomorphism. Therefore it follows from the elementary Orientation–Composition Lemma 31.18 that the bottom horizontal map is also orientation-reversing.[496] □

For the proof of the Orientation-Flipping Proposition 45.6 we will need the following two little lemmas.

Auxiliary Lemma 45.7. Let X be a topological manifold, let $A\subseteq X$ be a closed subset and let $F\colon X\times[0,1]\to X$ be a homotopy rel A. If F has compact support, then there exists an open neighborhood U of A such that F is a homotopy rel U.

Proof of Lemma 45.7. Let $K\subseteq X$ be a compact subset such that $F(x,t) = x$ for all $x\notin K$ and all $t\in[0,1]$. Since X is in particular Hausdorff we know by the Compact-Closed Lemma 3.39 (2) that K is a closed subset of X. Since X is a topological manifold we know by the Manifold Paracompact-Normal Proposition 23.9 (2) that X is normal. This implies in particular that there exists an open neighborhood U of A that is disjoint from K. But then F is evidently a homotopy rel U. ■

Lemma 45.8. (Extend Diffeotopy Lemma) Let M be a smooth manifold and let U be an open subset of M. Furthermore let $G\colon U\times[0,1]\to U$ be a diffeotopy with compact support $K\subseteq U$. Then

$$\begin{array}{rcl} \widetilde{G}\colon M\times[0,1] & \to & M \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x\in K, \\ x, & \text{if } x\notin K \end{cases} \end{array}$$

[496] Note that this example gives a solution to Exercise 31.17.

is a diffeotopy of M.

Proof. Since K is a compact subset of U we know by the Compact Image Lemma 4.15 and the Product Topology-Properties Proposition 7.9 that $K \times [0,1]$ is compact. It follows from the Compact-Closed Lemma 3.39 that $K \times [0,1]$ is in fact a closed subset of $M \times [0,1]$. Thus we see that $\widetilde{G}$ is smooth on the open subsets $U \times [0,1]$ and $M \times [0,1] \setminus K \times [0,1]$. Since these open subsets cover $M \times [0,1]$, it follows from the Smooth Pasting Proposition 25.19, or to be precise its extension to smooth manifolds with corner, that $\widetilde{G}$ is smooth. Basically the same argument shows that each $\widetilde{G}_t$ is a diffeomorphism. ■

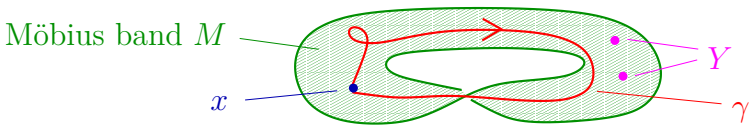
All the work for the proof of the Orientation-Flipping Proposition 45.6 is now done in the proof of the following auxiliary lemma, which deals with the case that $\#X = 1$.

Auxiliary Lemma 45.9. Let M be a connected *non-orientable* smooth manifold. Furthermore let $Y \subseteq M \setminus \partial M$ be a finite set and let $x \in M \setminus \partial M$ be a point with $x \notin Y$. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $\{x\} \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) The map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Proof of Lemma 45.9. Let M be a connected *non-orientable* smooth manifold. Note that by the Smooth 1-Manifold Classification Theorem 28.8 we know that $\dim(M) \geq 2$. Let $Y \subseteq M$ be a finite set and let $x \in M \setminus \partial M$ with $x \notin Y$.

(a) We pick an orientation $\mathcal{O}_x$ for $\mathrm{T}_x M$.
(b) We set $W := M \setminus Y$. Since Y is finite we see that Y is a closed subset. It follows from the Open Subset-Submanifold Lemma 25.8 and the Open Subset-Submanifold Boundary Lemma 25.11 that W is a smooth manifold with $\partial W = \partial M$. Furthermore it follows from the Complement-of-Boundary Orientation Lemma 31.36 (2) that W is still non-orientable.
(c) Since Y is a finite set and since $\dim(M) \geq 2$ we know by the elementary Exercise 23.6 that W is also path-connected.
(d) It follows from the above together with the Orientability-via-Loops Proposition 45.5 that there exists an orientation-reversing loop $\gamma\colon [0,1] \to W \setminus \partial W$ that is trivial near the ends with $\gamma(0) = \gamma(1) = x$.



By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $F\colon W \times [0,1] \to W$ with the following properties:

(0) For each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$.
(1) $F_0 = \mathrm{id}$.
(2) F has compact support.

(3) F is a diffeotopy rel ∂W.

Note that it follows from (2), together with the Extend Diffeotopy Lemma 45.8, that we can extend F to a diffeotopy $F\colon M \times [0,1] \to M$ rel Y on all of M by setting $F(x,t) = x$ for $x \notin W$. Finally we set $\Phi := F_1\colon M \to M$.

By design F has properties (1), (2), (3) and (4). Furthermore it follows almost immediately from (0), (1) and the fact that γ is orientation-reversing that (5a) is satisfied. Furthermore note that by Lemma 45.7 we know that F is actually a diffeotopy rel an open neighborhood of Y. This implies that for each $y \in Y$ we have $\mathrm{D}\Phi_y = \mathrm{id}_{\mathrm{T}_y M}$. Thus (5b) is also satisfied. ■

Proof of the Orientation-Flipping Proposition 45.6. Let $x_1, \dots, x_k$ be the elements of X. It follows from the ever useful Smooth Isotopy-Equivalence Relation Proposition 41.5 that it suffices to flip the signs of the points $x_1, \dots, x_k$ one-by-one. But this can be done by applying Lemma 45.9 altogether k times, namely to the points x_i and to the sets $Y_i := Y \cup \{x_1, \dots, \widehat{x}_i, \dots, x_k\}$. ■

45.3. **The Smooth Ball Embedding Theorem.** Our goal of this section is to prove the following theorem which was originally proved by Richard Palais [**Pal1960a**] in 1960.

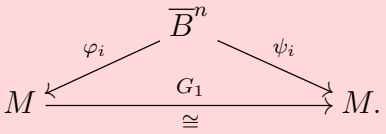
Theorem 45.10. (Smooth Ball Embedding Theorem) Let M be a connected n-dimensional smooth manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_k\colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k\colon \overline{B}^n \to M \setminus \partial M$ be two sets of k smooth embeddings of $\overline{B}^n$ with pairwise disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) We assume that we are in one of the following settings:

- $n \geq 2$, or
- $n = 1$ and $M \cong S^1$ and $k \in \{1, 2\}$, or
- $k = 1$.

Then there exists a diffeotopy[497]

$$G\colon M \times [0,1] \;\to\; M$$

with compact support, rel some neighborhood of ∂M, from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1\colon M \to M$ such that for each $i \in \{1, \dots, k\}$ the following diagram commutes:



Finally, if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$ and if M is orientable, then we can find a diffeotopy rel $\{\varphi_1(0), \dots, \varphi_k(0)\}$.

Remark.

(1) The Smooth Ball Embedding Theorem 45.10 has many applications, for example we will see in the Smooth Connected Sum-Proposition 54.13 that it can be used to show that the "connected sum" of connected orientable smooth manifolds is well-defined.

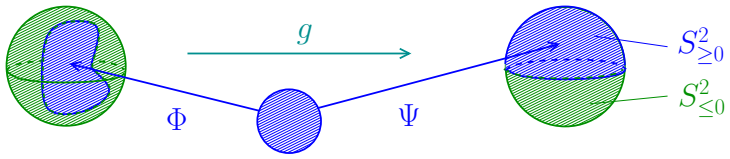
[497] Again note that if M is orientable, then the diffeomorphism G_1 is in fact orientation-preserving by the Smooth Isotopy-Orientation Proposition 42.4.

(2) In the Topological Ball Embedding Theorem 224.27 we will state an analog of the Smooth Ball Embedding Theorem 45.10 in the setting of topological manifolds. □

Before we turn to the proof of the Smooth Ball Embedding Theorem 45.10 let us state and prove a frequently used corollary:

Corollary 45.11. (Ball-in-Sphere Corollary) Let $n \in \mathbb{N}_0$ and let $\Phi\colon \overline{B}^n \to S^n$ be a smooth embedding. Then $S^n \setminus \Phi(B^n)$ is diffeomorphic to $\overline{B}^n$.

Proof. Let $\Phi\colon \overline{B}^n \to S^n$ be a smooth embedding. Furthermore let $\Psi\colon \overline{B}^n \to S^n$ be the smooth embedding from page 319 whose image is the upper hemisphere $S^n_{\geq 0}$. Finally let $\rho\colon \overline{B}^n \to \overline{B}^n$ be the reflection in a hyperplane. Note that Φ has the same orientation behavior as one of Ψ or $\Psi \circ \rho$. Since Ψ and $\Psi \circ \rho$ have the same image we now obtain from the Smooth Ball Embedding Theorem 45.10 that here exists a diffeomorphism $g\colon S^n \to S^n$ such that $g(\Phi(\overline{B}^n)) = \Psi(\overline{B}^n) = S^n_{\geq 0}$.



This implies that g restricts to a diffeomorphism $S^n \setminus \Phi(B^n) \to S^n \setminus \Psi(B^n) = S_{\leq 0}$. As on page 319 we see that the latter is diffeomorphic to $\overline{B}^n$. Thus we are done. ■

The key to proving the Smooth Ball Embedding Theorem 45.10 is the following proposition:

Proposition 45.12. (Smooth Ball Isotopy Proposition) Let M be a connected n-dimensional smooth manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_k\colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k\colon \overline{B}^n \to M \setminus \partial M$ be two sets of k smooth embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and all ψ_i are orientation-preserving or that all are orientation-reversing.) We assume that we are in one of the following settings:

- $n \geq 2$, or
- $n = 1$ and $M \cong S^1$ and $k \in \{1, 2\}$, or
- $k = 1$.

Then the smooth embeddings[498]

$$\bigsqcup_{i=1}^k \varphi_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M \setminus \partial M \qquad \text{and} \qquad \bigsqcup_{i=1}^k \psi_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M \setminus \partial M$$

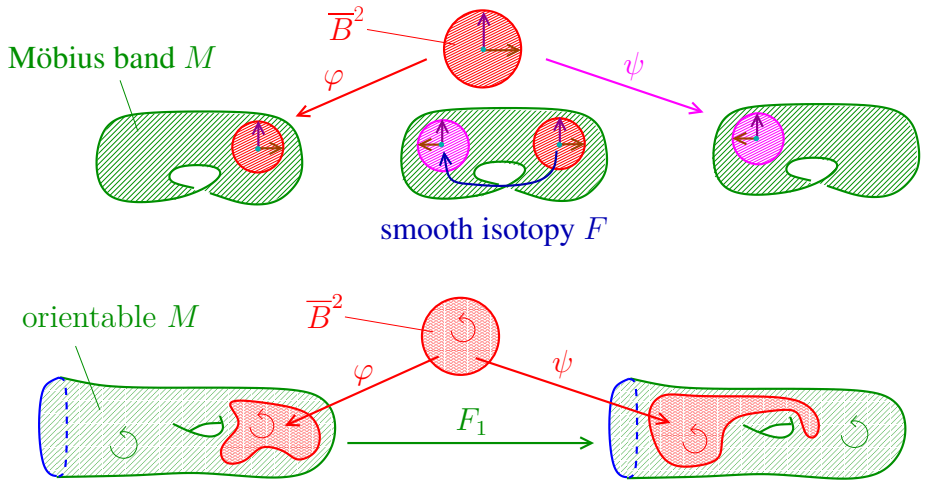
are smoothly isotopic.

Remark. In the language from Definition 42.3 we can say that the Smooth Ball Isotopy Proposition 45.12 calculates $\widetilde{\mathrm{Emb}}(\overline{B}^n \sqcup \dots \sqcup \overline{B}^n, M^n)$. □

Proof of Theorem 45.10 assuming Proposition 45.12. By the Smooth Ball Isotopy Proposition 45.12 there exists a smooth isotopy

$$F\colon \left(\bigsqcup_{i=1}^k \overline{B}^n_i \right) \times [0,1] \to M$$

[498]Since $\overline{B}^n$ is compact we obtain from the Disjoint Union-Smooth Embedding Lemma 30.16 that k smooth embeddings of $\overline{B}^n$ with disjoint images give rise to a smooth embedding of the disjoint union of k closed balls.



with $F_0 = \varphi_1 \sqcup \cdots \sqcup \varphi_k$ and $F_1 = \psi_1 \sqcup \cdots \sqcup \psi_k$. Furthermore a quick look at the proof of the Smooth Ball Isotopy Proposition 45.12 shows that if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$ and if M is orientable, then we can find a smooth isotopy rel the origins of the k balls. Finally, using the Stationary-Near-The-Ends-Lemma 41.7 we can arrange that F is actually trivial near the ends.

We now apply the Isotopy Extension Theorem 44.2 to the smooth isotopy F and we obtain a diffeotopy $G\colon M \times [0,1] \to M$ with all the desired properties. ■

Proof of the Smooth Ball Isotopy Proposition 45.12 if M is orientable. Note that after possibly precomposing all the smooth embeddings $\overline{B}^n \to M$ with the reflection in a hyperplane we only need to consider the case that all smooth embeddings are orientation-preserving.

We denote by $\iota\colon \overline{B}^n \to \overline{B}^n_2(0)$ the inclusion map. We pick orientation-preserving smooth embeddings $\{\Theta_i\colon \overline{B}^n_2(0) \to M \setminus \partial M\}_{i=1,\dots,k}$ with disjoint images. By the Smooth Isotopy-Equivalence Relation Proposition 41.5 it suffices to show that any orientation-preserving smooth embedding

$$\bigsqcup_{i=1}^k \alpha_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M \qquad \text{is smoothly isotopic to the smooth embedding} \qquad \bigsqcup_{i=1}^k \Theta_i \circ \iota\colon \bigsqcup_{i=1}^k \overline{B}^n \to M.$$

This statement follows from the subsequent three claims together with the Smooth Isotopy-Equivalence Relation Proposition 41.5.

Claim 1. The smooth embedding

$\bigsqcup_{i=1}^k \alpha_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M$ is smoothly isotopic to a smooth embedding $\bigsqcup_{i=1}^k \beta_i\colon \bigsqcup_{i=1}^k \overline{B}^n \to M$ such that for each $i \in \{1, \dots, k\}$ we have $\beta_i(0) = \Theta_i(0)$.

Proof. Recall that we assume that

- $n \geq 2$, or
- $n = 1$ and $M \cong S^1$ and $k \in \{1, 2\}$, or
- $k = 1$.

Also recall that the α_i and Θ_i take values in $M \setminus \partial M$, which implies in particular that the points $\alpha_i(0)$ and $\Theta_i(0)$ lie in $M \setminus \partial M$. Since M is by hypothesis connected it follows

from the Manifold–Homogeneous Proposition 41.21 and Exercise 41.11[499] that there exists a diffeotopy $F\colon M\times[0,1]\to M$ with $F_0=\mathrm{id}$ and such that $F_1(\alpha_i(0))=\Theta_i(0)$ for $i=1,\dots,k$. For $i=1,\dots,k$ we set $\beta_i:=F_1\circ\alpha_i$ and we have found the desired smooth embeddings. ⊞

Claim 2. The smooth embedding

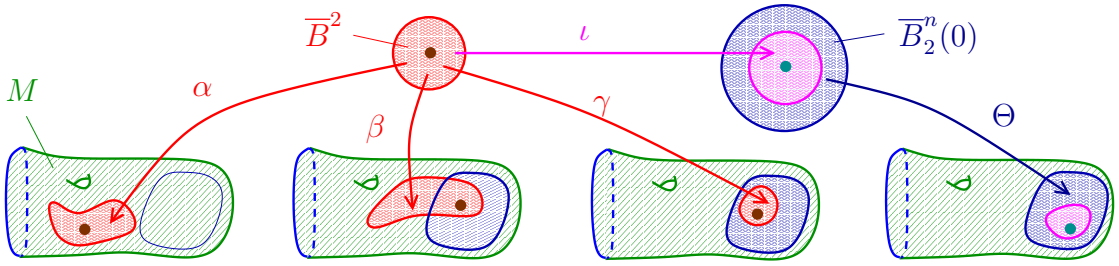
$$\bigsqcup_{i=1}^{k}\beta_i\colon\bigsqcup_{i=1}^{k}\overline{B}^n\to M \quad\text{is smoothly isotopic to a smooth embedding}\quad \bigsqcup_{i=1}^{k}\gamma_i\colon\bigsqcup_{i=1}^{k}\overline{B}^n\to M$$

such that for each $i\in\{1,\dots,k\}$ we have $\gamma_i(\overline{B}^n)\subseteq\Theta_i(B^n_2)$.

Proof. Let $i\in\{1,\dots,k\}$. Since Θ_i is a smooth embedding of an n-dimensional ball into an n-dimensional smooth manifold we obtain from the Codimension-Zero Smooth Embedding Proposition 30.21 that $\Theta_i(B^n_2(0))$ is an open subset of M. Since $\beta_i(0)=\Theta_i(0)$ we see that there exists an $\epsilon_i>0$ such that $\beta_i(\overline{B}^n_{\epsilon_i})\subseteq\Theta_i(B^n_2)$. We consider the smooth isotopy

$$\begin{aligned} B_i\colon\overline{B}^n\times[0,1] &\to M\\ (x,t) &\mapsto \beta_i(x\cdot((1-t)+t\cdot\epsilon_i)).\end{aligned}$$

This map is a smooth isotopy from β_i to a map $\gamma_i\colon\overline{B}^n\to M$ with $\gamma_i(\overline{B}^n)\subseteq\Theta_i(B^n_2(0))$.



Since these smooth isotopies have disjoint images it follows from the Disjoint Union-Smooth Embedding Lemma 30.16 that we have found the desired smooth isotopy. ⊞

Claim 3. The smooth embedding

$$\bigsqcup_{i=1}^{k}\gamma_i\colon\bigsqcup_{i=1}^{k}\overline{B}^n\to M \quad\text{is smoothly isotopic to the smooth embedding}\quad \bigsqcup_{i=1}^{k}\Theta_i\circ\iota\colon\bigsqcup_{i=1}^{k}\overline{B}^n\to M.$$

Proof. Let $i\in\{1,\dots,k\}$. Note that by the Smooth Embedding Theorem 30.15 together with the Codimension-Zero Smooth Embedding Proposition 30.21 we know that $\Theta_i(B^n_2(0))$ is an open subset of M and we know that $\Theta_i\colon B^n_2(0)\to\Theta_i(B^n_2(0))$ is a diffeomorphism. Therefore we can consider the inverse $\Theta_i^{-1}\colon\Theta_i(B^n_2(0))\to B^n_2(0)$. Since we arranged that $\gamma_i(\overline{B}^n)\subseteq\Theta_i(B^n_2(0))$ we can consider the corresponding orientation-preserving smooth embedding $\Theta_i^{-1}\circ\gamma_i\colon\overline{B}^n\to B^n_2(0)$. By the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (2) we know that $\Theta_i^{-1}\circ\gamma_i\colon\overline{B}^n\to B^n_2(0)$ is smoothly isotopic in $B^n_2(0)$ to the orientation-preserving smooth embedding $\iota\colon\overline{B}^n\to B^n_2(0)$. Therefore γ_i is smoothly isotopic in $\Theta_i(B^n_2(0))$ to $\Theta_i\circ\iota$. Since the $\Theta_i(\overline{B}^n_2(0))$ are disjoint we are done. ■

Proof of the Smooth Ball Isotopy Proposition 45.12 if M is non-orientable. Suppose that M is non-orientable. As before we denote by $\iota\colon\overline{B}^n\to\overline{B}^n_2(0)$ the inclusion map. We pick smooth embeddings $\{\Theta_i\colon\overline{B}^n_2(0)\to M\setminus\partial M\}_{i=1,\dots,k}$ with disjoint images. By the Smooth Isotopy-Equivalence Relation Proposition 41.5 it suffices to show that any smooth

[499] Alternatively one could also use the Homogeneous-via-Diffeotopies Proposition 44.6

embedding

$$\bigsqcup_{i=1}^{k} \alpha_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \qquad \text{is smoothly isotopic to the map} \qquad \bigsqcup_{i=1}^{k} \Theta_i \circ \iota \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M.$$

This statement follows again from three claims together with the Smooth Isotopy-Equivalence Relation Proposition 41.5.

Before we can state the first claim we need to introduce one new notion. We say that two smooth embeddings $\mu, \nu \colon \overline{B}^n \to M$ are *compatible* if $\mu(0) = \nu(0)$ and if the automorphism $(\mathrm{D}\nu_0)^{-1} \circ \mathrm{D}\mu_0$ of $\mathrm{T}_0\overline{B}^n$ is orientation-preserving.[500]

Now we can move to the modified three claims.

Claim 1. The smooth embedding

$$\bigsqcup_{i=1}^{k} \alpha_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \qquad \text{is smoothly isotopic to a smooth embedding} \qquad \bigsqcup_{i=1}^{k} \beta_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M$$

such that for $i = 1, \ldots, k$ the smooth embeddings β_i and $\Theta_i \circ \iota$ are compatible.

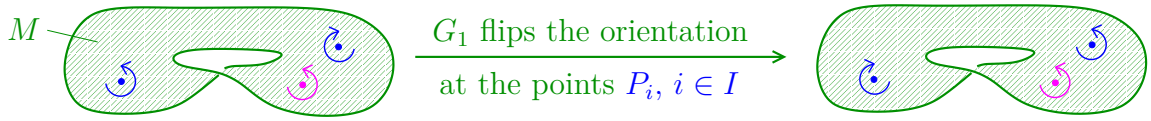
Proof. As in the oriented case we obtain from the Homogeneous-via-Diffeotopies Proposition 44.6 that there exists a diffeotopy $F \colon M \times [0,1] \to M$ with $F_0 = \mathrm{id}$ and such that $(F_1 \circ \alpha_i)(0) = (\Theta_i \circ \iota)(0)$ for $i = 1, \ldots, k$. For $i = 1, \ldots, k$ we set $P_i := (\Theta_i \circ \iota)(0)$. Next we define

$$I := \{i \in \{1, \ldots, k\} \,|\, \text{the smooth embeddings } F_1 \circ \alpha_i \text{ and } \Theta_i \circ \iota \text{ are not compatible}\}.$$

From the Orientation-Flipping Proposition 45.6 we obtain a diffeotopy $G \colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ with the following two properties:

(1) for all $i \in \{1, \ldots, k\}$ we have $G_1(P_i) = P_i$,
(2) the map $\mathrm{D}(G_1)_{P_i} \colon \mathrm{T}_{P_i}M \to \mathrm{T}_{P_i}M$ is orientation-reversing if and only if $i \in I$.

For $i = 1, \ldots, k$ we now set $\beta_i := G_1 \circ (F_1 \circ \alpha_i)$.



It follows easily from (1) and (2), together with the ever-useful Smooth Isotopy-Equivalence Relation Proposition 41.5, that these smooth embeddings $\beta_1, \ldots, \beta_k$ have the desired properties. ⊞

Claim 2. The smooth embedding

$$\bigsqcup_{i=1}^{k} \beta_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M \qquad \text{is smoothly isotopic to a smooth embedding} \qquad \bigsqcup_{i=1}^{k} \gamma_i \colon \bigsqcup_{i=1}^{k} \overline{B}^n \to M$$

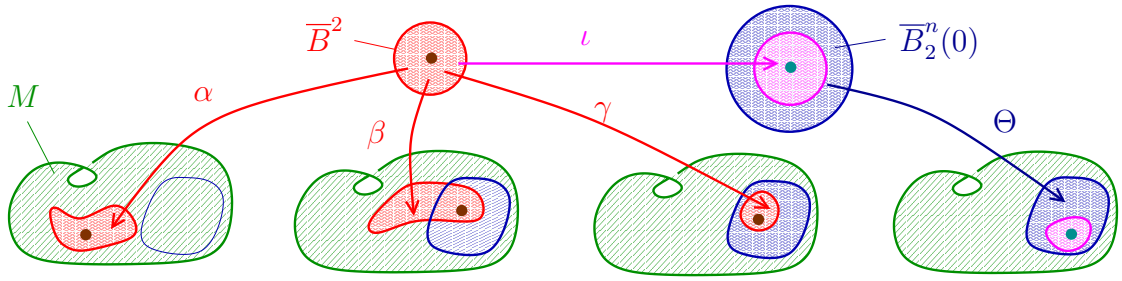
such that for each $i \in \{1, \ldots, k\}$ we have $\gamma_i(\overline{B}^n) \subseteq \Theta_i(B_2^n)$ and such that for $i = 1, \ldots, k$ the smooth embeddings γ_i and $\Theta_i \circ \iota$ are compatible.

Proof. The construction of the γ_i follows by the same recipe as in the oriented case. It is basically clear from the construction that for $i = 1, \ldots, k$ the smooth embeddings γ_i and $\Theta_i \circ \iota$ are indeed compatible. ⊞

[500] In Definition 31.17 we defined what it means for an *automorphism* of a finite-dimensional real vector space to be orientation-preserving.

Claim 3. The map

$\coprod_{i=1}^{k} \gamma_i \colon \coprod_{i=1}^{k} \overline{B}^n \to M$ is smoothly isotopic to the smooth embedding $\coprod_{i=1}^{k} \Theta_i \circ \iota \colon \coprod_{i=1}^{k} \overline{B}^n \to M$.



Proof. Let $i \in \{1, \dots, k\}$. Since $\gamma_i(\overline{B}^n) \subseteq \Theta_i(B^n_2(0))$ we can consider the corresponding smooth embedding $\iota \circ \Theta_i^{-1} \circ \gamma_i \colon \overline{B}^n \to B^n_2(0)$. We consider the following diagram:

$$\begin{array}{ccc} & \mathrm{T}_0 B^n_2(0) & \\ {\scriptstyle \mathrm{D}(\Theta^{-1}\circ\gamma_i)_0} \nearrow & & \nwarrow {\scriptstyle \mathrm{D}\iota_0} \\ \mathrm{T}_0\overline{B}^n & \xrightarrow[(\mathrm{D}(\Theta_i\circ\iota)_0)^{-1}\circ \mathrm{D}(\gamma_i)_0]{} & \mathrm{T}_0\overline{B}^n. \end{array}$$

Note that the diagram commutes by default. Since γ_i and $\Theta_i \circ \iota$ are compatible the bottom map is orientation-preserving. By definition of the orientations on $\overline{B}^n$ and $B^n_2(0)$ the right diagonal map is orientation-preserving. It follows that the left diagonal map is orientation-preserving. By the Manifold–Orientation Basics Lemma 31.31 (3) this implies that the smooth embedding $\Theta_i^{-1} \circ \gamma_i$ is orientation-preserving. At this point the argument proceeds as in the oriented case.

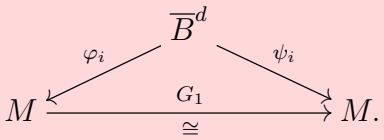
Indeed, recall that by the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (2) we now know that $\Theta_i^{-1} \circ \gamma_i \colon \overline{B}^n \to B^n_2(0)$ is smoothly isotopic in $B^n_2(0)$ to the orientation-preserving smooth embedding $\iota \colon \overline{B}^n \to B^n_2(0)$. Therefore γ_i is smoothly isotopic in $\Theta_i(B^n_2(0))$ to $\Theta_i \circ \iota$. Since the $\Theta_i(\overline{B}^n_2(0))$ are disjoint we are done. ∎

45.4. **The Lower-Dimensional Smooth Ball Embedding Theorem.** We also have a uniqueness theorem for smooth embeddings of lower dimensional balls. In fact it is slightly easier to state than the Smooth Ball Embedding Theorem 45.10 since we will not care about orientations.

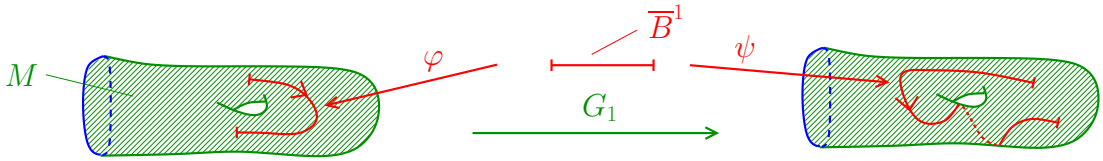
Theorem 45.13. (Lower-Dimensional Smooth Ball Embedding Theorem) Let M be an n-dimensional smooth manifold and let $d \in \{0, \dots, n-1\}$. In the following let $\varphi_1, \dots, \varphi_k \colon \overline{B}^d \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k \colon \overline{B}^d \to M \setminus \partial M$ be two sets of k smooth embeddings of $\overline{B}^d$ with disjoint images. If $n \geq 2$ or if $k = 1$, then there exists a diffeotopy

$$G \colon M \times [0,1] \to M$$

with compact support, rel some neighborhood of ∂M, from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1 \colon M \to M$ such that for each $i \in \{1, \dots, k\}$ the following diagram commutes:



Finally, if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$, then we can find a diffeotopy rel $\{\varphi_1(0), \dots, \varphi_k(0)\}$.



Remark.

(1) In the language from Definition 42.3 we can say that the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 calculates $\widetilde{\mathrm{Emb}}(\overline{B}^d \sqcup \dots \sqcup \overline{B}^d, M^n)$ for $d < n$.
(2) The case $d = 0$ is precisely the content of the Homogeneous-via-Diffeotopies Proposition 44.6. □

Proof. The same that we deduced the Smooth Ball Embedding Theorem 45.10 from the Smooth Ball Isotopy Proposition 45.12 we can now deduce the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 from the following claim:

Claim. If $n \geq 2$ or if $k = 1$, then the smooth embeddings

$$\bigsqcup_{i=1}^{k} \varphi_i \colon \bigsqcup_{i=1}^{k} \overline{B}^d \to M \setminus \partial M \qquad \text{and} \qquad \bigsqcup_{i=1}^{k} \psi_i \colon \bigsqcup_{i=1}^{k} \overline{B}^d \to M \setminus \partial M$$

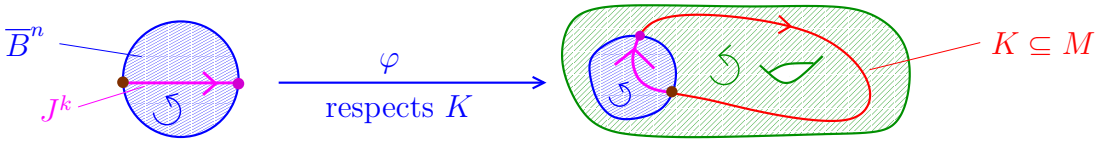
are smoothly isotopic.

Proof. The proof of the claim is basically identical to the proof of the Smooth Ball Isotopy Proposition 45.12 in the orientable case. All we have to do is to replace the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (2) by the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (3). We leave it to the reader to fill in the details. ■

Finally we want to consider a variation on the Smooth Ball Embedding Theorem 45.10 and the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 where we look at smooth embeddings of balls that "respect" a submanifold. This discussion is perhaps a little peripheral, but it plays a role when later on we introduce the connected sum of submanifolds and knots.

Definition 45.14. Let $n \in \mathbb{N}$ and let $k \in \{-1, 0, \dots, n\}$.

(1) We set $J^k := \{(x, 0) \in \overline{B}^n \mid x \in \overline{B}^k\} = \overline{B}^n \cap (\mathbb{R}^k \times \{0\})$ and we view equip J^k with the obvious orientation.
(2) Let M be an oriented n-dimensional smooth manifold and let $K \subseteq M$ be an oriented k-dimensional submanifold. Furthermore let $\varphi \colon \overline{B}^n \to M$ be a smooth embedding. We say that φ respects K if $\varphi(\overline{B}^n) \subseteq M \setminus \partial M$, if $\varphi(J^k) = \varphi(\overline{B}^n) \cap K$, if $\varphi \colon \overline{B}^n \to M$ is orientation-preserving and if $\varphi|_{J^k} \colon J^k \to \varphi(\overline{B}^n) \cap K$ is orientation-preserving.



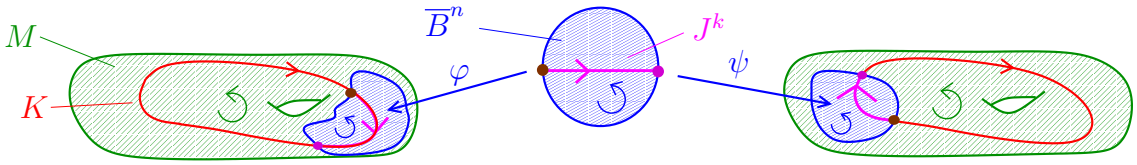
The following theorem gives a uniqueness statement for smooth embeddings of balls that respect a given submanifold.[501]

Theorem 45.15. (Ball-Submanifold Theorem) Let M be an oriented n-dimensional smooth manifold. Furthermore let $K \subseteq M$ be an oriented connected k-dimensional submanifold. Finally let $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to M$ be two smooth embeddings that respect K. There exists a diffeotopy

$$G\colon M \times [0,1] \to M$$

with compact support, rel some neighborhood of ∂M, from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1\colon M \to M$ such that $G_1 \circ \varphi = \psi$ with the following two properties:

- Each $G_t \circ \varphi\colon \overline{B}^n \to M$ respects K.
- For each $t \in [0,1]$ we have $G_t(K) = K$.



Sketch of proof. Similar to the proofs of the Smooth Ball Embedding Theorem 45.10 and of the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 we see, this time using the Pairs–Isotopy Extension Theorem 44.13, that it suffices to prove the following statement:

Statement. For any two smooth embeddings $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to M$ that respect K there exists a smooth isotopy $F\colon \overline{B}^n \times [0,1] \to M$ such that $F_0 = \varphi$, such that $F_1 = \psi$ and such that each $F_t \circ \varphi$ respects K.

The proof of the statement is a modification of the proof of the Smooth Ball Isotopy Proposition 45.12. Evidently now one has to go through the proof of the Smooth Ball Isotopy Proposition 45.12, and of all the sources it uses, to make the necessary changes. It seems to this author that most modifications are straightforward. In particular one can easily adapt the statements of Claim 1 and Claim 2. The one step which requires a little more thought is Claim 3 which built on the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (3).

Let us elaborate. Let $f\colon \overline{B}^n \to \mathbb{R}^n$ be a continuous map that respects $K := \mathbb{R}^k \times \{0\}$ and such that $f(0) = 0$. We write $A := \mathrm{D}f_0$ and we denote by $\mu(A)\colon \overline{B}^n \to \mathbb{R}^n$ the map that is given by multiplication by A. The fact that f respects K implies that A is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the top left corner is a $(k \times k)$-matrix. It follows fairly easily that $\mu(A)$ also respects $K = \mathbb{R}^k \times \{0\}$. We need to prove the following claim which is similar to the Isotopic-to-Differential Lemma 42.6:

Claim. There exists a smooth isotopy $F\colon \overline{B}^n \to \mathbb{R}^n$ with $F_0 = f$, $F_1 = \mu(A)$ and such that each F_t respects $K = \mathbb{R}^k \times \{0\}$

[501] The case $k = n$ and $K = M$ or also the case $k = -1$ and $K = \varnothing$ is in fact the statement of Smooth Ball Embedding Theorem 45.10 in the orientation-preserving case.

Proof. Since $\overline{B}^n$ is a convex subset of $\mathbb{R}^n$ that contains the origin, since $f(0) = 0$ and since f respects $K = \mathbb{R}^k \times \{0\}$ we know by the Map-as-Matrix Multiplication Proposition 24.23 (1) and (2) that there exists a smooth map $\Theta\colon \mathbb{R}^n \to \mathrm{M}(n \times n, \mathbb{R})$ with the following properties:

(1) we have $\Theta(0) = A = \mathrm{D}f_0$,
(2) for all $x \in \mathbb{R}^n$ we have $f(x) = \Theta(x) \cdot x$, and
(3) for all $x \in J^k$ we have $\Theta(x) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the top left square is a $(k \times k)$-matrix.

As in the proof of the Isotopic-to-Differential Lemma 42.6 we consider the map

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \underbrace{\Theta(t \cdot x)}_{\in \mathrm{M}(n\times n,\mathbb{R})} \cdot x. \end{array}$$

As in the proof of the Isotopic-to-Differential Lemma 42.6 we see that F is an isotopy with $F_0 = f$ and $F_1 = \mu(A)$. But now the above property (3) implies that each F_t respects $K = \mathbb{R}^k \times \{0\}$. ⊞

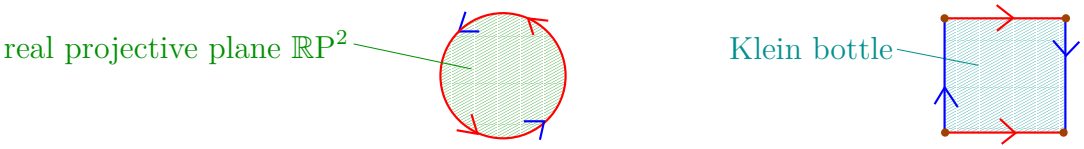
We leave all other details of the proof of the Ball-Submanifold Theorem 45.15 to the unquenchable thirst of the reader for technical details. ■

45.5. Exercises for Chapter 45.

Exercise 45.1. We consider the

$$\begin{array}{rcll} \text{real projective plane } \mathbb{R}\mathrm{P}^2 & = & \overline{B}^2/\!\sim \to & \text{where } x \sim -x \text{ for } x \in S^1 \\ \text{Klein bottle } K & = & ([0,1]\times[0,1])/\!\sim & \text{where } (x,0)\sim(x,1) \text{ and } (0,y)\sim(1,1-y). \end{array}$$

In both cases give examples of orientation-reversing loops.



Exercise 45.2. Let $n \in \mathbb{N}$ and let M be an orientable n-dimensional smooth manifold with empty boundary.

(a) Are any two orientation-preserving smooth embeddings of B^n into M related by a diffeotopy of M? More precisely, if we are given two orientation-preserving smooth embeddings $\varphi, \psi\colon B^n \to M$, does there exist a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ and $G_1 \circ \varphi = \psi$?
(b) Let $d \in \{1, \dots, n-1\}$ and let $\varphi, \psi\colon B^d \to M$ be smooth embeddings. Does there exist a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ and $G_1 \circ \varphi = \psi$?

Remark. So the question is, whether the analogs of the Smooth Ball Embedding Theorem 45.10 and the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 hold for open balls instead of closed balls.

Exercise 45.3. Let M be an oriented connected n-dimensional smooth manifold. Furthermore let $m \in \{0, \dots, n\}$ and let $f\colon \overline{B}^m \to M \setminus \partial M$ be a smooth embedding. Show that f extends to a smooth embedding $\overline{B}^m_2 \to M \setminus \partial M$.

Hint. Use the Smooth Ball Embedding Theorem 45.10 and the Lower-Dimensional Smooth Ball Embedding Theorem 45.13.

Exercise 45.4. Let M be an oriented connected smooth manifold, let $\Phi\colon M \to M$ be an orientation-preserving diffeomorphism and let $P \in M \setminus \partial M$. Show that there exists an open neighborhood U of P and a diffeotopy from Φ to a diffeomorphism $\Psi\colon M \to M$ such that $\Psi|_U = \mathrm{id}_U$.
Hint. Make good use of the Smooth Ball Embedding Theorem 45.10.

Exercise 45.5. Let $n \in \mathbb{N}$ and let $\varphi, \psi\colon \overline{B}^n \to S^n$ be smooth embeddings with disjoint images. Show that $S^n \setminus (\varphi(B^n) \cup \psi(B^n))$ is diffeomorphic to $S^{n-1} \times [0,1]$.

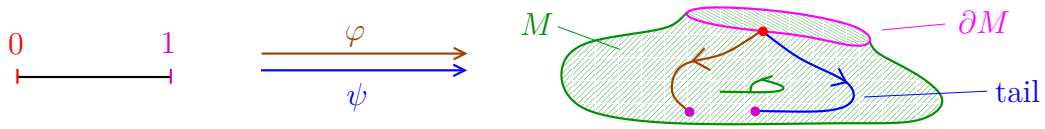
Exercise 45.6. Let M be a smooth manifold and let $\gamma, \delta\colon [0,1] \to M \setminus \partial M$ be smooth loops that are trivial near the ends with $\gamma(0) = \gamma(1) = \delta(0) = \delta(1)$. We consider the concatenation

$$\begin{array}{rccl} \gamma * \delta\colon [0,1] & \to & M \\ t & \mapsto & \begin{cases} \gamma(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \delta(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

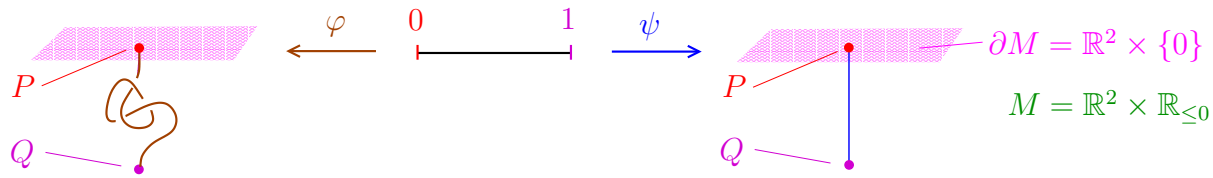
Show that $\gamma * \delta$ is orientation-preserving if and only if either both γ and δ are orientation-preserving or both are orientation-reversing.

Exercise 45.7. Let M be an n-dimensional smooth manifold and let $V \subseteq M \setminus \partial M$ be a submanifold that is diffeomorphic to $\overline{B}^n$. Furthermore let X be a compact smooth submanifold of dimension $\leq n-1$ and let $f\colon X \to M$ be a proper smooth embedding. Show that f is smoothly isotopic to a proper smooth embedding $f'\colon X \to M$ such that $f'(X) \cap V = \varnothing$.

Exercise 45.8. Let M be a smooth manifold and let $P \in \partial M$. A *tail at* P is a smooth embedding $\varphi\colon [0,1] \to M$ with the following properties:
(1) $\varphi(0) = P$ and $\varphi^{-1}(\partial M) = \{0\}$,
(2) $\varphi'(0) \notin \mathrm{T}_{\varphi(0)}(\partial M)$.
Show that if $\varphi, \psi\colon [0,1] \to M$ are two tails at P, then they are "smoothly isotopic as tails", i.e. show that there exists a smooth map $F\colon [0,1] \times [0,1] \to M$ with $F_0 = \varphi$, $F_1 = \psi$ and such that each F_t is a tail at P.



Exercise 45.9. Let M be a smooth manifold and let $P \in \partial M$. As in Exercise 45.8 we define a *tail at* P to be a smooth embedding $\varphi\colon [0,1] \to M$ with the following properties:
(1) $\varphi(0) = P$ and $\varphi^{-1}(\partial M) = \{0\}$,
(2) $\varphi'(0) \notin \mathrm{T}_{\varphi(0)}(\partial M)$.
We consider a "room" $M = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ with "ceiling" $\partial M = \mathbb{R}^2 \times \{0\}$. Furthermore we consider two cables φ and ψ from P to Q. Show that you can turn the "twisted cable φ" into the "untwisted cable ψ" (both of which are tails) while keeping P and Q fixed.


φ
0
1
ψ
P
Q
P
Q
$\partial M = \mathbb{R}^2 \times \{0\}$
$M = \mathbb{R}^2 \times \mathbb{R}_{\leq 0}$

CHAPTER 46

The degree of a map between oriented smooth manifolds I

In this chapter we will associate to any continuous map $f\colon M \to N$ between closed oriented smooth manifolds of the same dimension a degree $\deg(f) \in \mathbb{Z}$. This notion of a degree has many applications and will reappear in many different guises throughout these notes.

This chapter is organized as follows:

- In Section 46.1 we will define the degree of a map between closed smooth manifolds and we will state many properties of the degree.
- In Section 46.2 we will prove the statements of Section 46.1.
- In Section 46.3 we will provide several calculations of the degree.
- In Section 46.4 we will present applications of the degree.
- In Section 46.5 we will state and discuss the Hopf Degree Theorem 46.22 which classifies homotopy classes of maps to S^n in terms of the degree.
- In Section 46.6 we will prove the Hopf Degree Theorem 46.22.
- In Section 46.7 we will provide exercises for Chapter 46.

In the next Chapter 47 we will use the notion of a degree to prove the celebrated Borsuk-Ulam Theorem 47.12.

46.1. The definition of the degree of a map between closed smooth manifolds.

In preparation for the definition of a degree we first introduce the notion of a local degree:

Definition 46.1. Let $f\colon M \to N$ be a smooth map between two oriented n-dimensional smooth manifolds and let $x \in M \setminus \partial M$. If f is a local diffeomorphism at x, then we define the local degree of f at the point x as[502]

$$\deg(f,x) \coloneqq \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Remark. In Definitions 152.3 and 195.4 we will generalize the notion of local degree to maps that are not necessarily local diffeomorphisms and to suitable maps between topological manifolds. □

We continue with the following simplified version of the Stack-of-Records Lemma 32.3.

Lemma 32.3. (Stack-of-Records Lemma) Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds of the same dimension.
(1) For every regular value $y \in N$ the preimage $f^{-1}(y)$ is a finite subset of M.
(2) Given any regular point $x \in M$ the map $f\colon M \to N$ is a local diffeomorphism at x.

Proof.

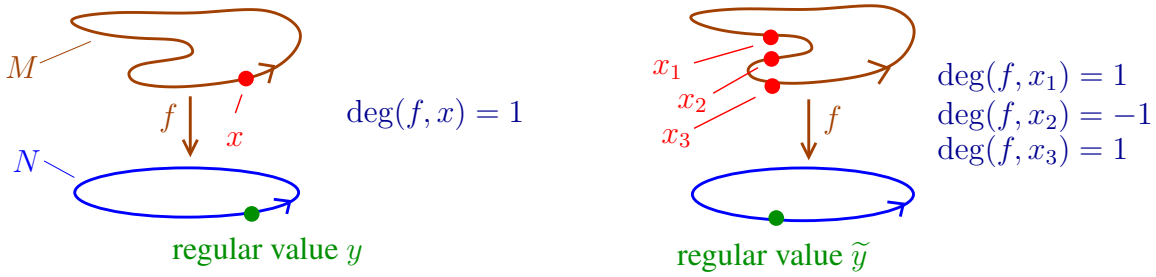
[502]Note that by the discussion on page 810 this definition also makes sense for 0-dimensional manifolds.

(1) By the Regular Value Theorem 32.5 we know that $f^{-1}(y)$ is a proper 0-dimensional submanifold of the closed smooth manifold M. It follows from the discussion on page 624 that $f^{-1}(y)$ is a discrete subset of M. Since M is in particular compact it follows from the Compact-Discrete Lemma 3.40 that $f^{-1}(y)$ is actually a finite set.
(2) Let $x \in M$ be a regular point of f. This means that $\mathrm{D}f_x \colon \mathrm{T}_{f(x)}M \to \mathrm{T}_{f(x)}N$ is an epimorphism. Since M and N have the same dimension this implies that $\mathrm{D}f_x$ is an isomorphism. It follows from the Inverse Mapping Theorem 30.1 that f is a local diffeomorphism at x. ■

Theorem 46.2. (Degree–Well-Defined Theorem) Let $f \colon M \to N$ be a smooth map between two closed oriented connected smooth manifolds of the same dimension. Let $y \in N$ be a regular value. The sum[503]

$$\sum_{x \in f^{-1}(y)} \deg(f, x) \ \in \ \mathbb{Z}$$

does not depend on the choice of y.



We postpone the proof of the Degree–Well-Defined Theorem 46.2 to the next section. The theorem allows us to introduce the following definition:

Definition 46.3. Let $f \colon M \to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension. By Sard's Theorem 38.1 there exists a regular value y. By the Degree–Well-Defined Theorem 46.2 we can unambiguously define the degree of the smooth map f as

$$\deg(f) \ := \ \sum_{x \in f^{-1}(y)} \deg(f, x) \ \in \ \mathbb{Z}.$$

Let $f \colon M \to N$ be a continuous map (not necessarily smooth) between two closed oriented connected non-empty smooth manifolds of the same dimension. By the Whitney Approximation Theorem 36.1 we know that f is homotopic to a smooth map $g \colon M \to N$. The idea now is to define the degree of f as the degree of the smooth map g. For this approach to make sense we need the following proposition:

Proposition 46.4. (Degree–Homotopy Proposition) Let $g \colon M \to N$ be a continuous map between two closed oriented connected non-empty smooth manifolds of the same dimension. If $f_0, f_1 \colon M \to N$ are smooth maps that are both homotopic to g, then $\deg(f_0) = \deg(f_1)$.

Proof. We postpone the proof of the Degree–Homotopy Proposition 46.4 to the next section. ■

[503] We know by the Stack-of-Records Lemma 32.3 that this is indeed a finite sum.

The Degree–Homotopy Proposition 46.4 allows us to make the following definition:

Definition 46.5. Let $f\colon M \to N$ be a continuous map between two closed oriented connected non-empty smooth manifolds of the same dimension. By the Whitney Approximation Theorem 36.1 we know that f is homotopic to a smooth map $g\colon M \to N$. By the Degree–Homotopy Proposition 46.4 it makes sense to define

$$\deg(f) \coloneqq \deg(g).$$

Remark.

(1) The notion of the degree was first introduced by Luitzen Brouwer [**Brou1912a**, p. 103], the degree is therefore sometimes also called the *Brouwer degree*.
(2) In Section 47.1 we will extend the definition of a degree to maps between smooth manifolds with non-empty boundary.
(3) In Definition 186.1 we will use homology groups to give a different definition of the degree of a smooth map between closed oriented smooth manifolds. In the Local-Global Degree Proposition 186.8 we will see that the two definitions agree.
(4) In Definition 195.1 we will again use homology groups to extend the notion of a degree from smooth manifolds to topological manifolds. □

In the following two theorems we state two interesting properties of the degree.

Theorem 46.6. (Degree Homotopy Invariance Theorem) Let $f_0, f_1\colon M \to N$ be continuous maps between two closed oriented connected non-empty smooth manifolds of the same dimension. If f_0 and f_1 are homotopic, then

$$\deg(f_0) = \deg(f_1).$$

Theorem 46.7. (Degree Multiplicativity Theorem) If $f\colon L \to M$ and $g\colon M \to N$ are continuous maps between closed oriented connected non-empty smooth manifolds of the same dimension, then

$$\deg(g \circ f) = \deg(g) \cdot \deg(f).$$

We provide the proofs of these two theorems in the next section.

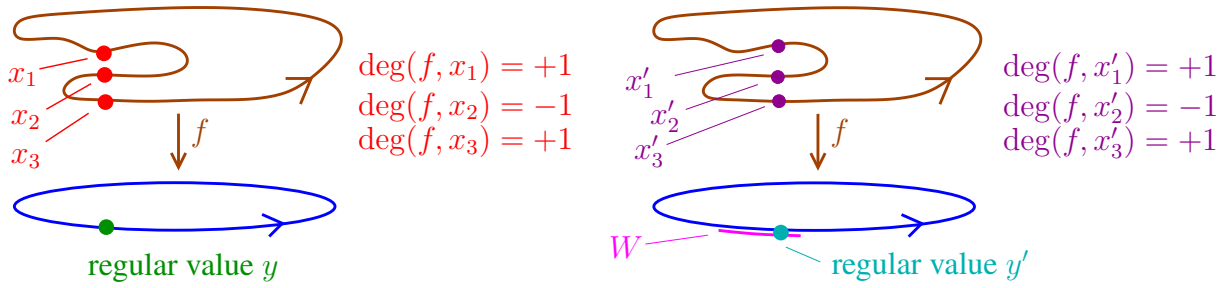
46.2. **Proof of the Theorems 46.2, 46.6 and 46.7 and of Proposition 46.4.** For the purpose of this section we introduce the following notation:

Notation 46.8. Let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty smooth manifolds of the same dimension. Let $y \in N$ be a regular value. We define

$$\deg^y(f) \coloneqq \sum_{x \in f^{-1}(y)} \deg(f, x) \in \mathbb{Z}.$$

The proofs of the Degree–Well-Defined Theorem 46.2 and 46.6 rely on three lemmas which we will prove in this section. The first lemma is a slight refinement of the Stack-of-Records Lemma 32.3.

Lemma 46.9. (Degree–Locally Constant Lemma) Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds of the same dimension. Every regular value $y \in N$ admits an open neighborhood W such that each $y' \in W$ is a regular value and such that for each $y' \in W$ we have $\deg^y(f) = \deg^{y'}(f)$.



Proof. Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds of the same dimension and let $y \in N$ be a regular value. We denote the points in the preimage $f^{-1}(y)$ by $x_1, \dots, x_k \in M$. Since $x_1, \dots, x_k$ are regular points we obtain from the Inverse Mapping Theorem 30.1 connected open neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$ such that each $f_k\colon U_k \to V_k := f(U_k)$ is a diffeomorphism. Since M is Hausdorff and since the $x_1, \dots, x_k$ are distinct we can arrange, see the elementary Exercise 3.12, that the open neighborhood $U_1, \dots, U_k$ are disjoint. We set

$$W := \left(\bigcap_{i=1}^{k} V_i\right) \setminus \underbrace{f\left(M \setminus \bigcup_{i=1}^{k} U_i\right)}_{\substack{\text{closed subset of } N \\ \text{since } M \text{ is compact}}}.$$

Note that W is open and that it contains y. Now let $y' \in W$. Note that the preimage of $y' \in W$ consists of precisely k points, namely precisely one x'_i in each U_i.[504] Since the U_i are connected it follows immediately from the Manifold–Orientation Basics Lemma 31.31 that for each $i \in \{1, \dots, k\}$ we have $\deg(f, x_i) = \deg(f, x'_i)$. This shows that $\deg^y(f) = \deg^{y'}(f)$. ∎

Lemma 46.10. (Same Regular Value Lemma) Let M and N be two closed oriented n-dimensional smooth manifolds. Furthermore let $f, g\colon M \to N$ be two smooth maps and let $y \in N$ be a regular value for f and for g. If f and g are smoothly homotopic, then $\deg^y(f) = \deg^y(g)$.

Proof. Let M and N be two closed oriented n-dimensional smooth manifolds. Furthermore let $f, g\colon M \to N$ be two smooth maps let $F\colon M \times [0,1] \to N$ be a smooth homotopy between f and g.

Next let $y \in N$ be a regular value for f and for g. It follows immediately from applying the Degree–Locally Constant Lemma 46.9 to f and g and taking the intersection of the resulting open neighborhoods of $y \in N$ that there exists an open neighborhood W of $y \in N$ such that for every $y' \in W$ we have $\deg^y(f) = \deg^{y'}(f)$ and $\deg^y(g) = \deg^{y'}(g)$.

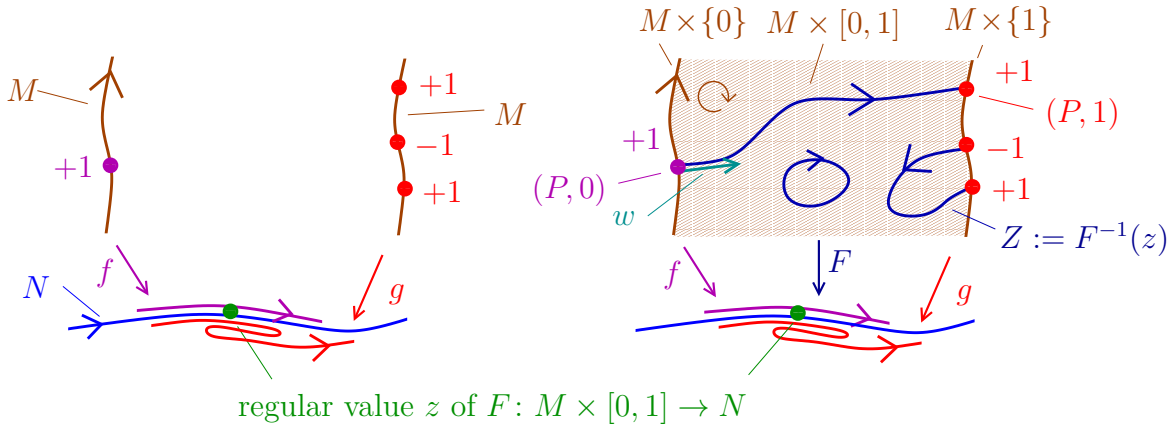
Since W is open and non-empty we obtain from Sard's Theorem 38.1 (together with the Measure-Zero Properties Proposition 37.3 (2b)) that there exists a $z \in W$ which is a regular value of $F\colon M \times [0,1] \to N$ and also of $F|_{\partial(M \times [0,1])}$. Note that by the Regular Value Theorem 32.5 we know that $F^{-1}(z)$ is a compact proper 1-dimensional submanifold

[504] This can be seen as follows. Let $y' \in W$. Let $i \in \{1, \dots, k\}$. By definition $y' \in V_i$, thus there exists a unique preimage of y' in U_i. Since $U_1, \dots, U_k$ are disjoint we see that y' has k preimages in $U_1 \cup \dots \cup U_k$. By choice of W the point y' has no preimage outside of $U_1 \cup \dots \cup U_k$.

of $M \times [0,1]$. Furthermore we saw in the Regular Value Theorem 32.5 (3) that we can equip the 1-dimensional smooth manifold $Z := F^{-1}(z) \subseteq M \times [0,1]$ with an orientation as follows:

(1) We equip $M \times [0,1]$ with the product orientation that we introduced in Convention 31.49.
(2) We pick a positive basis $v_1, \dots, v_n$ for $\mathrm{T}_z N$.
(3) We equip $Z = F^{-1}(z)$ with an orientation as follows: Given a point $Q \in Z = F^{-1}(z)$ we pick vectors[505] $\widetilde{v}_1, \dots, \widetilde{v}_n \in \mathrm{T}_Q(M \times [0,1])$ such that $\mathrm{D}F_Q(\widetilde{v}_i) = v_i$ for $i = 1, \dots, n$. Now we say that a non-zero vector $w \in \mathrm{T}_Q Z$ is positive if $\widetilde{v}_1, \dots, \widetilde{v}_n, w$ is a positive basis for $\mathrm{T}_Q(M \times [0,1])$.[506]

We equip ∂Z with the corresponding boundary orientation $\epsilon\colon \partial Z \to \{\pm 1\}$ given by the Boundary Orientation Lemma 31.38 (1).



Claim. Let $P \in M$.[507]
(1) If $(P,0) \in \partial Z$, then $\epsilon((P,0)) = -\deg(f, P)$.
(2) If $(P,1) \in \partial Z$, then $\epsilon((P,1)) = \deg(g, P)$.

Proof. First let $(P,0) \in \partial Z$. We make the following preparations:

(i) We denote by $q\colon M \times [0,1] \to [0,1]$ the projection. Note that it follows from the fact that Z is a proper submanifold of $M \times [0,1]$ and the fact that $\mathrm{T}_{(P,0)}\partial M = \ker(\mathrm{D}q_{(P,0)})$ that the image of $\mathrm{D}q_{(P,0)}\colon \mathrm{T}_{(P,0)} Z \to \mathrm{T}_{f(P)}\mathbb{R} = \mathbb{R}$ is non-trivial.
(ii) By (i) we can pick $w \in \mathrm{T}_{(P,0)} Z$ with $\mathrm{D}q_{(P,0)}(w) > 0$.
(iii) Since z is also a regular value of $F|_{\partial(M \times [0,1])}$ and since $(P,0) \in \partial Z$ we see that the differential $\mathrm{D}F_{(P,0)}\colon \mathrm{T}_{(P,0)}(M \times \{0\}) \to \mathrm{T}_{F(P,0)} N$ is also an epimorphism. Therefore we can actually pick vectors $\widetilde{v}_1, \dots, \widetilde{v}_n \in \mathrm{T}_{(P,0)}(M \times \{0\})$ such that $\mathrm{D}F_{(P,0)}(\widetilde{v}_i) = v_i$ for $i = 1, \dots, n$.
(iv) We denote by $\iota\colon M \to M \times [0,1]$ the inclusion given by $x \mapsto (x, 0)$.
(v) For $i = 1, \dots, n$ we pick $\widetilde{u}_i \in \mathrm{T}_P M$ such that $\mathrm{D}\iota_P(\widetilde{u}_i) = \widetilde{v}_i$.

[505] Since z is a regular value of F we know that $\mathrm{D}F_Q$ is an epimorphism.
[506] Note that this agrees with the orientation from the Regular Value Theorem 32.5 (3) only if n is even. For the purpose of this proof it is notationally easier to work with the orientation we just provided.
[507] Since we assume that z is also a regular value of $F|_{\partial(M \times [0,1])}$ we see that $z \in N$ is a regular value of $F_0 = f\colon M \to N$ and also of $F_1 = g\colon M \to N$. In particular $\deg(f, P)$ and $\deg(g, P)$ are actually defined.

Now we see that the following statements are equivalent:

we apply the definition of the boundary orientation from the Boundary Orientation Lemma 31.38 (1) to $\zeta := q\colon Z \to \mathbb{R}_{\geq 0}$ note that $\zeta((P,0)) = 0$ and $\mathrm{D}\zeta_{(P,0)}(w) = \mathrm{D}q_{(P,0)}(w) > 0$

$$\epsilon((P,0)) = -1 \overset{\downarrow}{\iff} w \in \mathrm{T}_{(P,0)}Z \text{ is a positive vector}$$
$$\iff \widetilde{v}_1, \ldots, \widetilde{v}_n, w \text{ is a positive basis for } \mathrm{T}_{(P,0)}(M \times [0,1])$$
$$\overset{\uparrow}{\iff} \widetilde{u}_1, \ldots, \widetilde{u}_n \text{ is a positive basis for } \mathrm{T}_P M \overset{\uparrow}{\iff} \mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N \text{ is orientation-preserving} \overset{\uparrow}{\iff} \deg(f,P) = +1.$$

follows from $\mathrm{D}q_{(P,0)}(w) > 0$ and the definition of the product orientation, see the Product Orientation Proposition 31.48

since $F_0 \circ \iota = f$ and since $v_1, \ldots, v_n$ is a positive basis for $\mathrm{T}_{f(P)}N$

by Definition 46.1 of $\deg(f,P)$

Since ϵ and deg take values in $\{\pm 1\}$ we are actually done with the proof of (1).

Now suppose that we are given a point $(P,1) \in \partial Z$. The argument is very similar to the above. Again we can and will pick $w \in \mathrm{T}_{(P,1)}W$ with $\mathrm{D}q_{(P,1)}(w) > 0$. Now we see that the following statements are equivalent:

we apply the definition of the boundary orientation from the Boundary Orientation Lemma 31.38 (1) to $\zeta(t) := q(1-t)\colon Z \to \mathbb{R}_{\geq 0}$ note that $\zeta((P,1)) = 0$ and $\mathrm{D}\zeta_{(P,1)}(w) = -\mathrm{D}q_{(P,0)}(w) < 0$

$$\epsilon((P,1)) = +1 \overset{\downarrow}{\iff} w \in \mathrm{T}_{(P,1)}W \text{ is a positive vector} \overset{\uparrow}{\iff} \ldots \overset{\uparrow}{\iff} \deg(g,P) = +1.$$

precisely as above

This concludes the proof of (2). ⊞

Now we can easily complete the argument, namely we perform the following calculation:

$$\deg^y(f) - \deg^y(g) \overset{\uparrow}{=} \deg^z(f) - \deg^z(g) \overset{\uparrow}{=} \sum_{P \in \partial Z \cap (M \times \{0\})} \epsilon(P) + \sum_{P \in \partial Z \cap (M \times \{1\})} \epsilon(P) \overset{\uparrow}{=} 0.$$

since $z \in W$

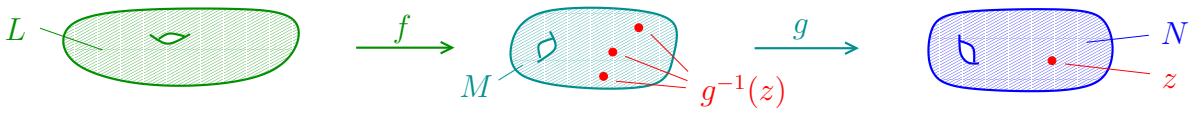
by the claim

by the Boundary-of-Compact 1-Dimensional Smooth Manifold Proposition 31.42 we know that the sum over the orientations of ∂Z is zero, the equality thus follows from the fact that $\partial Z \subseteq M \times \{0,1\}$

We have thus shown that $\deg^y(f) = \deg^y(g)$. ■

Lemma 46.11. (Degree Composition Lemma) Let $f\colon L \to M$ and $g\colon M \to N$ be two smooth maps between closed oriented connected non-empty smooth manifolds of the same dimension. If $z \in N$ is a regular value of $g\colon M \to N$ such that each point in $g^{-1}(z) \subseteq M$ is a regular value of $f\colon L \to M$, then

$$\deg^z(g \circ f) = \sum_{y \in g^{-1}(z)} \deg(g,y) \cdot \deg^y(f).$$



Proof. We perform the following calculation:

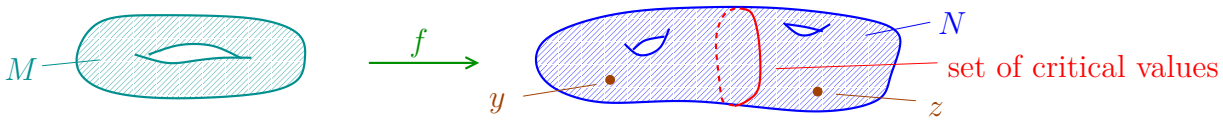
$$\deg^z(g\circ f) = \sum_{x\in(g\circ f)^{-1}(z)} \deg(g\circ f,x) \overset{\text{by the Orientation-Composition Lemma 31.8 we have } \deg(g\circ f,x)=\deg(g,f(x))\cdot\deg(f,x)}{\downarrow =} \sum_{y\in g^{-1}(z)}\sum_{x\in f^{-1}(y)} \deg(g,y)\cdot\deg(f,x)$$
$$= \sum_{y\in g^{-1}(z)} \deg(g,y)\cdot \underbrace{\sum_{x\in f^{-1}(y)} \deg(f,x)}_{=\deg^y(f)}.$$

■

After these preparations we can finally provide the proofs of the Degree–Well-Defined Theorem 46.2, 46.6 and 46.7.

Proof of the Degree–Well-Defined Theorem 46.2. Let $f\colon M\to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension. Let $y\in N$ and $z\in N$ be regular values of f. We need to show that $\deg^y(f)=\deg^z(f)$.

The first idea might be to try to connect y and z via a path and to use this path to show that the two degrees are the same. But this approach is doomed to fail since in general the set of regular values is *not* path-connected. Thus we need to use a different approach.



Recall that by the Manifold–Homogeneous Proposition 41.21 we know that there exists a diffeotopy $F\colon N\times[0,1]\to N$ with $F_0=\mathrm{id}$ and such that $F_1(y)=z$. We obtain the following equalities:

$$\deg^y(f) = \underbrace{\deg(F_1,y)}_{\substack{=1 \text{ since it follows from the Smooth Isotopy-Orientation Proposition 42.4 that } F_1 \text{ is an orientation-preserving diffeomorphism}}} \cdot \deg^y(f) \overset{\text{by the Degree Composition Lemma 46.11, since } F_1 \text{ is a diffeomorphism and since } y \text{ is a regular value of } f}{\downarrow =} \deg^{F_1(y)}(F_1\circ f) \overset{\text{since } F_1(y)=z}{\downarrow =} \deg^z(F_1\circ f) \underset{\substack{\uparrow \\ \text{follows from the Same Regular Value Lemma 46.10, since the two maps } F_1\circ f \text{ and } F_0\circ f = f \text{ are smoothly homotopic and since } z \text{ is a regular value of } F_1\circ f \text{ and of } f}}{=} \deg^z(f).$$

■

Proof of the Degree–Homotopy Proposition 46.4. Let $g\colon M\to N$ be a continuous map between two closed oriented connected non-empty smooth manifolds of the same dimension and let $f_0,f_1\colon M\to N$ be smooth maps that are both homotopic to g. We need to show that $\deg(f_0)=\deg(f_1)$.

It follows from the transitivity of being homotopic, see the Homotopy-Equivalence Basics Lemma 19.14, that f_0 and f_1 are homotopic. Furthermore it follows from the Replace-by-Smooth Homotopy Proposition 36.6 that f_0 and f_1 are actually smoothly homotopic.

Next note that it follows from Sard's Theorem 38.1 together with the Full Measure-Density Proposition 37.4 that we can find a $z\in N$ which is a regular value for f_0 and also for f_1. Thus it follows from the Same Regular Value Lemma 46.10 that $\deg(f_0)=\deg(f_1)$. ■

Proof of the Degree Homotopy Invariance Theorem 46.6. Let $f_0,f_1\colon M\to N$ be continuous maps between two closed oriented connected non-empty smooth manifolds of

the same dimension. We assume that f_0 and f_1 are homotopic. Recall that we need to show that $\deg(f_0) = \deg(f_1)$. By the Whitney Approximation Theorem 36.1 we know that for $i = 0, 1$ the map f_i is homotopic to a smooth map $\widetilde{f}_i \colon M \to N$. Now we see that

$$\deg(f_0) \overset{\text{by Definition 46.5}}{=} \deg(\widetilde{f}_0) \overset{\text{by the Degree–Homotopy Proposition 46.4 applied to } g = \widetilde{f}_0}{=} \deg(\widetilde{f}_1) \overset{\text{by Definition 46.5}}{=} \deg(f_1).$$

■

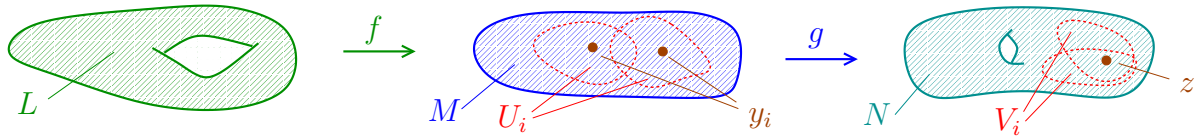
Proof of the Degree Multiplicativity Theorem 46.7 if f and g are smooth. In the following let $f \colon L \to M$ and $g \colon M \to N$ be smooth maps between closed oriented connected non-empty smooth manifolds of the same dimension. We need to show that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. Evidently the plan is to apply the Degree Composition Lemma 46.11, but to carry out this idea we need the following claim:

Claim. There exists a regular value z of $g \colon M \to N$ such that each point in $g^{-1}(z) \subseteq M$ is a regular value of $f \colon L \to M$.

Proof. We start out with some regular value z of $g \colon M \to N$. We denote the points in $g^{-1}(z)$ by $y_1, \ldots, y_k \in M$. As in the proof of the Degree–Locally Constant Lemma 46.9 we obtain from the Inverse Mapping Theorem 30.1 open neighborhoods $U_1, \ldots, U_k$ of $y_1, \ldots, y_k \in M$ such that each $g \colon U_k \to V_k \coloneqq g(U_k)$ is a diffeomorphism. We set

$$W \;\coloneqq\; \Big(\bigcap_{i=1}^{k} V_i\Big) \setminus g\Big(M \setminus \bigcup_{i=1}^{k} U_i\Big) \;\subseteq\; N.$$

As in the proof of the Degree–Locally Constant Lemma 46.9 we see that W is an open subset of N and that W contains z. For $i = 1, \ldots, k$ we set $\widetilde{U}_i = U_i \cap g^{-1}(W)$ and we denote by C_i the set of regular values of f in $\widetilde{U}_i$. By Sard's Theorem 38.1 we know that the set of critical values of f has measure zero in M. It follows almost immediately that each C_i is a set of full measure in the open subset $\widetilde{U}_i$ of M.



Since $g \colon \widetilde{U}_i \to W$ is a diffeomorphism we see that each $g(C_i)$ is also a set of full measure in W. It follows from the Full Measure-Density Proposition 37.4 that the set

$$\{\text{set of regular values of } g\} \cap W \cap \bigcap_{i=1}^{k} g(C_i)$$

is non-empty. Any point in this set has the desired property. ⊞

Now let $z \in N$ be a regular value of g such that each point in $g^{-1}(z)$ is a regular value of f. Then

$$\deg^z(g \circ f) \overset{\text{by the Degree Composition Lemma 46.11}}{=} \sum_{y \in g^{-1}(z)} \deg(g, y) \cdot \underbrace{\deg^y(f)}_{=\deg(f)} = \deg(f) \cdot \underbrace{\sum_{y \in g^{-1}(z)} \deg(g, y)}_{=\deg(g)} = \deg(f) \cdot \deg(g).$$

■

Proof of the Degree Multiplicativity Theorem 46.7. Let $f\colon L \to M$ and $g\colon M \to N$ be continuous maps between closed oriented connected non-empty smooth manifolds of the same dimension. By the Whitney Approximation Theorem 36.1 we know that there exists a homotopy $F\colon L \times [0,1] \to M$ with $F_0 = f$ and such that $\widetilde{f} \coloneqq F_1$ is smooth. Furthermore we know that there exists a homotopy $G\colon M \times [0,1] \to N$ with $G_0 = g$ and such that $\widetilde{g} \coloneqq G_1$ is smooth. Note that the map

$$\begin{array}{rccc} G \circ (F \times \mathrm{id}_{[0,1]})\colon & L \times [0,1] & \to & N \\ & (x,t) & \mapsto & G(F(x,t),t) \end{array}$$

is a homotopy from $g \circ f$ to the smooth map $\widetilde{g} \circ \widetilde{f}$. Thus we see that

$$\deg(g \circ f) \underset{\substack{\uparrow \\ \text{by the above and Definition 46.5 of the degree of } g \circ f}}{=} \deg(\widetilde{g} \circ \widetilde{f}) \underset{\substack{\uparrow \\ \text{by the previous discussion of the smooth case}}}{=} \deg(\widetilde{g}) \cdot \deg(\widetilde{f}) \underset{\substack{\uparrow \\ \text{by Definition 46.5 of the degrees of } g \text{ and } f}}{=} \deg(g) \cdot \deg(f).$$

■

46.3. **Calculations of the degree.** First we state a few basic properties of the degree.

Lemma 46.12. (Basics-of-Degree Lemma) Let $f\colon M \to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension.

(1) If f is not surjective, then $\deg(f) = 0$.
(2) If f is an orientation-preserving diffeomorphism, then $\deg(f) = 1$.
(3) If f is an orientation-reversing diffeomorphism, then $\deg(f) = -1$.

Proof.

(1) If $f\colon M \to N$ is not surjective, then we pick $z \in N \setminus f(M)$. Note that z is a regular value, and if we use this regular value for the calculation of $\deg(f)$, then we obtain immediately that $\deg(f) = 0$.
(2),(3) If $f\colon M \to N$ is a diffeomorphism, then every point in N is a regular value. The desired statements now follow from the Manifold–Orientation Basics Lemma 31.31.

■

Example. Let $A \in \mathrm{O}(n+1)$. In Exercise 31.5 we showed that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$. Combining this piece of information with the Basics-of-Degree Lemma 46.12 we see that

$$\deg\begin{pmatrix} S^n & \to & S^n \\ v & \mapsto & A \cdot v \end{pmatrix} = \det(A).$$

In particular, for $A = -\mathrm{id}_{n+1}$ we obtain that

$$\deg\left(\text{antipodal map } \begin{array}{ccc} S^n & \to & S^n \\ v & \mapsto & -v \end{array}\right) = \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$ □

The following lemma doubles as an interesting example of a calculation of the degree.

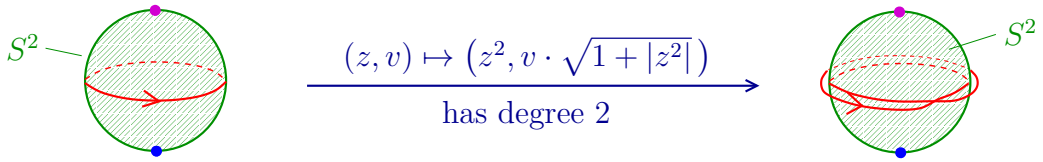
Lemma 46.13. (Degree Examples Lemma)

(1) Given any $d \in \mathbb{Z}$ we have $\deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^d \end{pmatrix} = d.$

(2) Let $n \in \mathbb{N}$. We think of S^n as a subspace of $\mathbb{C} \times \mathbb{R}^{n-1}$. Given any $d \in \mathbb{N}$ we have[508]

$$\deg \begin{pmatrix} S^n & \to & S^n \\ (z,v) & \mapsto & \left(z^d, v \cdot \sqrt{1+|z^2|+\cdots+|z^2|^{d-1}}\right) \end{pmatrix} = d.$$

(3) Given any $n \in \mathbb{N}$ and given any $d \in \mathbb{Z}$ there exists a continuous map $f\colon S^n \to S^n$ with degree $\deg(f) = d$.



Proof.

(1) The case $d = 0$ follows immediately from the Basics-of-Degree Lemma 46.12. Now assume that $d \neq 0$. In this case a straightforward calculation shows that each $x \in S^1$ is a regular point, thus every $y \in S^1$ is a regular value. We take $y = 1 \in S^1 \subseteq \mathbb{C}$ and we calculate

$$\deg(f) = \sum_{k=1}^{|d|} \deg(f, \exp(2\pi \mathrm{i} k/d)) = \sum_{k=1}^{|d|} \begin{cases} +1, & \text{if } d \geq 1, \\ -1, & \text{if } d \leq -1 \end{cases} = d.$$

using the Orientation Checking-via-Differentials Lemma 31.30 the orientation behavior can be deduced easily for the obvious extension of f to $\overline{B}^2$, the statement regarding S^1 then follows from the Boundary Orientation Lemma 31.38

(2) The proof of (2) is almost identical to the proof of (1). We leave it to the reader to fill in the details.

(3) Let $n \in \mathbb{N}$. The case $d > 0$ follows from (2). The case $d = -1$ follows from the example preceding the lemma which says in particular that any reflection in a hyperplane has degree -1. The case $d < 0$ follows from (2), the case $d = -1$ and the Degree Multiplicativity Theorem 46.7. Finally, by the Basics-of-Degree Lemma 46.12, the case $d = 0$ follows from taking a constant map $S^n \to S^n$. ∎

The following lemma often can be used to determine the degree of a continuous self-map of S^1. The lemma will come in handy in the proof of the Whitney-Graustein Theorem 49.11.

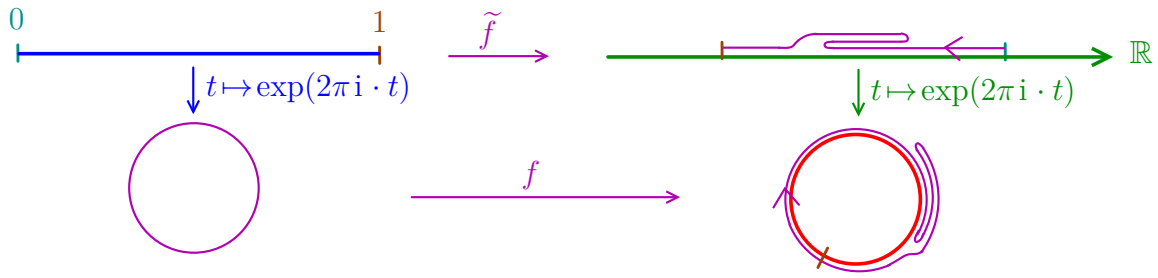
Lemma 46.14. (Degree-via-Lifts Lemma) Let $f\colon S^1 \to S^1$ be a smooth map. If $\widetilde{f}\colon [0,1] \to \mathbb{R}$ is a continuous map such that for every $t \in [0,1]$ we have the equality[509] $f(\exp(2\pi \mathrm{i} \cdot t)) = \exp(2\pi \mathrm{i} \cdot \widetilde{f}(t))$, then

$$\deg(f\colon S^1 \to S^1) = \widetilde{f}(1) - \widetilde{f}(0).$$

[508]Let $d \in \mathbb{N}$ and let $(z,v) \in S^n$. Note that $(z^d, v \cdot \sqrt{1+|z^2|+\cdots+|z^2|^{d-1}})$ does indeed lie in S^n: If $|z| = 1$, then $v = 0$, and in this case clearly $(z^d, 0) \in S^n$. Now assume that $|z| \neq 1$. From $(z,v) \in S^n$ it follows that $|z|^2 + \|v\|^2 = 1$, i.e. $1 - |z^2| = 1 - |z|^2 = \|v\|^2$. Thus we see that

$$|z^d|^2 + \left(\|v\| \cdot \sqrt{1+|z^2|+\cdots+|z^2|^{d-1}}\right)^2 = |z^2|^d + \|v\|^2 \cdot \frac{1-|z^2|^d}{1-|z^2|} = |z^2|^d + \|v\|^2 \cdot \frac{1-|z^2|^d}{\|v\|^2} = 1.$$

[509]In the Smooth Lift Lemma 42.13 we showed that such a map $\widetilde{f}$ always exists.



Example. Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. In this case $\widetilde{f}(t) = n \cdot t$ satisfies the conditions of the Degree-via-Lifts Lemma 46.14. Thus we see, once again, that $\deg(f) = \widetilde{f}(1) - \widetilde{f}(0) = n$. □

Proof. We consider the maps

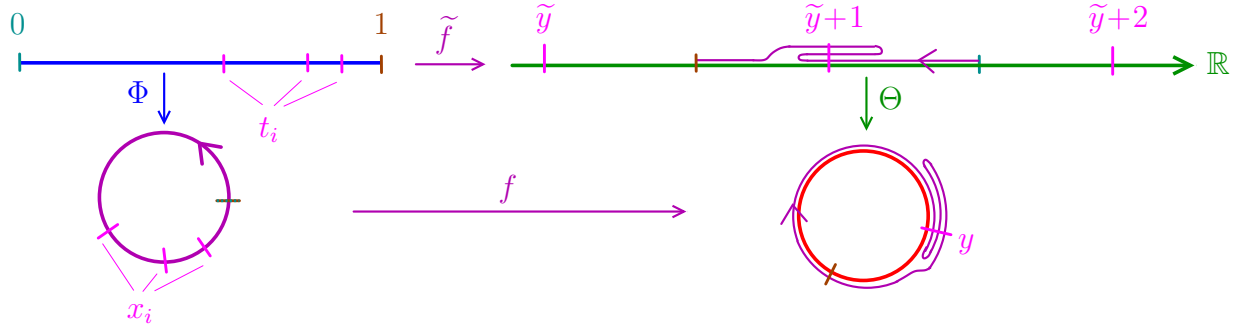
$$\begin{array}{rcl} \Phi\colon [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} \cdot t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Theta\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} \cdot t). \end{array}$$

Let $f\colon S^1 \to S^1$ be a smooth map and let $\widetilde{f}\colon [0,1] \to \mathbb{R}$ be a continuous map such that for every $t \in [0,1]$ we have $f(\exp(2\pi \mathrm{i} \cdot t)) = \exp(2\pi \mathrm{i} \cdot \widetilde{f}(t))$. In other words, we have $f \circ \Phi = \Theta \circ \widetilde{f}$, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} [0,1] & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Theta} \\ S^1 & \xrightarrow{f} & S^1. \end{array}$$

Recall that by the Exponential-Local Diffeomorphism Lemma 42.11 we know that Θ is a local diffeomorphism. It follows, as in the Smooth Lift Lemma 42.13, that $\widetilde{f}$ is also smooth.

By Sard's Theorem 38.1 there exists a regular value $y \in S^1 \setminus \{\Phi(0)\}$ of $f\colon S^1 \to S^1$. We denote the preimages of y by $x_1, \ldots, x_m \in S^1$. We denote by $t_1, \ldots, t_m$ the unique real numbers in $(0,1)$ with $x_i = \Phi(t_i)$. After reordering them we can assume that $t_1 < \cdots < t_m$. We pick $\widetilde{y} \in \mathbb{R}$ with $\Theta(\widetilde{y}) = y$.



We make the following observations:

(1) We equip $[0,1]$, S^1 and $\mathbb{R}$ with the standard orientations. Let $i \in \{1, \ldots, m\}$. Note that $\Phi\colon [0,1] \to S^1$ and $\Theta\colon \mathbb{R} \to S^1$ are orientation-preserving. It follows from $f \circ \Phi = \Theta \circ \widetilde{f}$ that $\widetilde{f}$ is orientation-preserving at t_i if and only if f is orientation-preserving at x_i. This implies that

$$\text{sign of } \underbrace{\widetilde{f}'(t_i)}_{\text{derivative of } \widetilde{f}} = \deg(f, x_i) \in \{-1, 1\}.$$

(2)
- As $\Theta(\widetilde{f}([0,t_1))) = f(\Phi([0,t_1))) \in S^1 \setminus \{y\}$ there is a unique $k_0 \in \mathbb{Z}$ such that we have an inclusion $\widetilde{f}([0,t_1)) \subseteq (\widetilde{y}+k_0, \widetilde{y}+k_0+1)$.
- Let $i \in \{1,\dots,m-1\}$. Since $\Theta(\widetilde{f}((t_i,t_{i+1}))) \in S^1 \setminus \{y\}$ there exists a unique $k_i \in \mathbb{Z}$ such that $\widetilde{f}((t_i,t_{i+1})) \subseteq (\widetilde{y}+k_i, \widetilde{y}+k_i+1)$.
- Finally, since $\Theta(\widetilde{f}((t_m,1))) \in S^1 \setminus \{y\}$ there exists a unique $k_m \in \mathbb{Z}$ such that we have an inclusion $\widetilde{f}((t_m,1]) \subseteq (\widetilde{y}+k_m, \widetilde{y}+k_m+1)$.

(3) It follows from (1) that for $i = 0,\dots,m-1$ we have $k_{i+1} = k_i + \deg(f, x_i)$.

Now we see that

$$\widetilde{f}(1) - \widetilde{f}(0) \underset{\uparrow}{=} k_m - k_0 = \sum_{i=0}^{m-1}(k_{i+1}-k_i) \underset{\uparrow}{=} \sum_{i=1}^{m}\deg(f,x_i) \underset{\uparrow}{=} \deg(f\colon S^1 \to S^1).$$

since $\widetilde{f}(1) \in (\widetilde{y}+k_m, \widetilde{y}+k_m+1)$ and since $\widetilde{f}(0) \in (\widetilde{y}+k_0, \widetilde{y}+k_0+1)$ we see that $\widetilde{f}(1)-\widetilde{f}(0) \in (k_m-k_0-1, k_m-k_0+1)$, the equality then follows from the fact that $\widetilde{f}(1)-\widetilde{f}(0) \in \mathbb{Z}$

by (3)

by Definition 46.3 of $\deg(f\colon S^1 \to S^1)$

We heave thus proved the desired equality. ■

46.4. **Applications of the degree.** In this section we discuss several hopefully interesting application of the concept of degree.

Theorem 46.15. (Sphere-Degree Fixed Point Theorem) Let $n \in \mathbb{N}$. Every continuous map $f\colon S^n \to S^n$ with $\deg(f) \neq (-1)^{n+1}$ has a fixed point.

Example. The Sphere-Degree Fixed Point Theorem Theorem 46.15 says in particular that every continuous map $f\colon S^2 \to S^2$ of degree one has a fixed point. Together with the Degree Homotopy Invariance Theorem 46.6 this implies that every continuous map $f\colon S^2 \to S^2$ that is homotopic to the identity has a fixed point. This means that we cannot "deform" the identity to a continuous map that is fixed point free. This is in contrast to the case $n = 1$ where we can "deform" the identity into the map that is given by rotation by a small angle, which is of course a fixed point free map. □

Proof. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a continuous map without fixed points. We need to show that $\deg(f) = (-1)^{n+1}$. The fact that f has no fixed points means by definition that for all $x \in S^n$ we have $f(x) \neq x$. It follows that for all $t \in [0,1]$ and all $x \in S^n$ we have[510] $(1-t)\cdot f(x) \neq t \cdot x$. Therefore we can consider the map

$$\begin{aligned} F\colon S^n \times [0,1] &\to S^n \\ (x,t) &\mapsto \frac{(1-t)\cdot f(x) - t\cdot x}{\|(1-t)\cdot f(x) - t\cdot x\|}. \end{aligned}$$

This map is a homotopy between the map $x \mapsto f(x)$ and the antipodal map $x \mapsto -x$. We obtain that $\deg(f) \underset{\uparrow}{=} \deg(\text{antipodal map } x \mapsto -x) \underset{\uparrow}{=} (-1)^{n+1}$.

by the Degree Homotopy Invariance Theorem 46.6 — see page 1077

[510] we know that the degree is preserved under homotopies

[510] This case be seen as follows: If $t \neq \frac{1}{2}$ we have $\|(1-t)\cdot f(x)\| = 1-t \neq t = \|tx\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)\cdot f(x) = \frac{1}{2}f(x) \neq \frac{1}{2}x = tx$.

■

Next we recall Definition 2.36:

Definition 46.16. Let G be a group that acts on a topological space X.

(1) We say that the action is continuous if for every $g \neq e$ the map $X \to X$ given by $x \mapsto g \cdot x$ is continuous.

(2) We say that the action is free if for every $g \neq e$ and every $x \in X$ we have $g \cdot x \neq x$.

Example. For each $k \in \mathbb{N}$ the action

$$\begin{array}{rcl} \mathbb{Z}/k\mathbb{Z} \times S^1 & \to & S^1 \\ (l, z) & \mapsto & \exp(2\pi \mathrm{i} l/k) \cdot z \end{array}$$

is free. We can easily generalize this example. More precisely, let S^{2n-1} be an odd-dimensional sphere. We view S^{2n-1} as a subspace of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then it is straightforward to show that

$$\begin{array}{rcl} \mathbb{Z}/k\mathbb{Z} \times S^{2n-1} & \to & S^{2n-1} \\ ([l], (z_1, \ldots, z_n)) & \mapsto & (\exp(2\pi \mathrm{i} l/k) \cdot z_1, \ldots, \exp(2\pi \mathrm{i} l/k) \cdot z_n) \end{array}$$

is a free and continuous action of $\mathbb{Z}/k\mathbb{Z}$ on S^{2n-1}. □

It is less clear which finite groups can act freely and continuously on even-dimensional spheres. Of course the group $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ acts freely and continuously on any sphere. The following proposition says that $\mathbb{Z}/2\mathbb{Z}$ is in fact the only finite non-trivial group that can act freely and continuously on an even-dimensional sphere.

Proposition 46.17. (Free Action-on-Sphere Proposition) If a finite group G acts freely and continuously on an even-dimensional sphere, then G is either trivial or $G \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Let G be a group that acts freely and continuously on a sphere S^n. We consider the map

$$\begin{array}{rcl} G & \to & \mathbb{Z} \\ g & \mapsto & \deg \begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & g \cdot x \end{pmatrix}. \end{array}$$

It follows from the Degree Multiplicativity Theorem 46.7 that this map takes values in $\{-1, 1\}$ and that the map $G \to \{-1, 1\}$ is actually a group homomorphism.

Now if n is even, then it follows from the Sphere-Degree Fixed Point Theorem 46.15 and our hypothesis that G acts freely that the above map $G \to \{-1, 1\}$ is actually a monomorphism. ■

Next we turn to the study of vector fields on spheres. On page 763 we saw that every odd-dimensional sphere admits a nowhere-vanishing vector field. In Question 29.28 we asked whether it is possible to find a nowhere-vanishing vector field on even-dimensional spheres. The following proposition says that this is not possible.

Theorem 46.18. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.[511]

[511] On S^{2k} we had so far found vector fields which vanish at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in Exercise 151.8.

Remark.

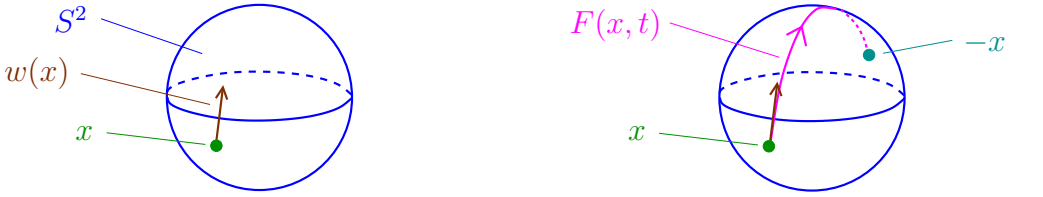
(1) We will discuss the Hairy Ball Theorem 46.18 again in somewhat greater detail on page 2882. At that point we will also give a different proof using homology groups.
(2) We will generalize the Hairy Ball Theorem 46.18 with the Smooth Poincaré–Hopf Theorem 53.24 and the Poincaré–Hopf Theorem 223.8.
(3) The literature contains many proofs of the Hairy Ball Theorem 46.18. Some, as our proofs, require some technology, some, see e.g. [**Miln1978**] get away with less technology. For the case of S^2 one can find especially short proofs, see e.g. [**EG1979, McG2016**]. □

Proof. Suppose that S^n admits a nowhere-vanishing vector field v. We have to show that n is odd. We consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

Next we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$



the vector field w gives rise to the homotopy F from id to $-$id

The map does indeed take values in S^n, since for any $x \in S^n$ and $t \in [0,1]$ we have

$$\begin{array}{rcl} \langle F(x,t), F(x,t)\rangle & = & \langle \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x), \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x)\rangle \\ & \underset{\substack{\uparrow \\ \text{since } \langle x, w(x)\rangle = 0}}{=} & \cos(\pi t)^2 \cdot \|x\|^2 + \sin(\pi t)^2 \cdot \|w(x)\|^2 \underset{\substack{\uparrow \\ \text{since } \|x\| = \|w(x)\| = 1}}{=} 1. \end{array}$$

Note that the map F is a homotopy between $F_0 = \mathrm{id}$ and the antipodal map $F_1 = -\mathrm{id}$. It follows that

$$1 \underset{\substack{\uparrow \\ \text{by the Basics-of-Degree} \\ \text{Lemma 46.12 (1)}}}{=} \deg(\mathrm{id}_{S^n}) \underset{\substack{\uparrow \\ \text{by the Degree Homotopy} \\ \text{Invariance Theorem 46.6} \\ \text{applied to the above homotopy}}}{=} \deg(-\mathrm{id}_{S^n}) \underset{\substack{\uparrow \\ \text{see page 1077}}}{=} (-1)^{n+1}.$$

But this means that n is odd. ■

46.5. **The Hopf Degree Theorem.** Before we turn to the Hopf Theorem let us first show that given any closed oriented smooth manifold any integer can be realized as the degree of a continuous map $M \to S^n$.

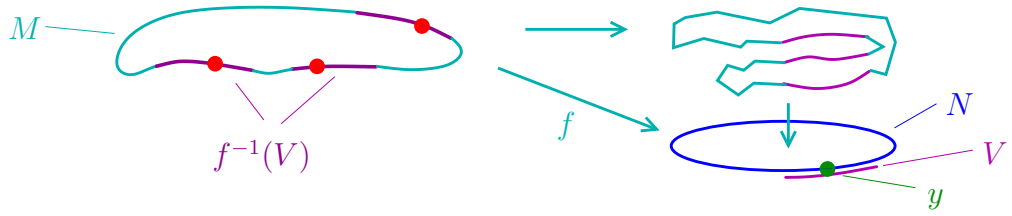
Proposition 46.19. (Degree Realization Proposition) Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Given any $k \in \mathbb{Z}$ there exists a continuous map $f\colon M \to S^n$ with $\deg(f) = k$.

Remark. We will reprove the Degree Realization Proposition 46.19 on page 3563. □

The proof of the Degree Realization Proposition 46.19 rests on the following lemma which can be useful for calculating degrees of non-smooth maps.

Lemma 46.20. (Partially Smooth–Degree Lemma) Let $f\colon M \to N$ be a continuous map between two closed oriented connected non-empty n-dimensional smooth manifolds. Let $y \in N$. Suppose that there exists an open neighborhood V of y such that f is smooth on $f^{-1}(V) \subseteq M$, such that $f^{-1}(y)$ consists of finitely many points and such that y is a regular value of the smooth map $f\colon f^{-1}(V) \to V$. Then

$$\deg(f) \;=\; \sum_{x \in f^{-1}(y)} \deg(f,x).$$



Proof of the Partially Smooth–Degree Lemma 46.20. The following claim is the key to proving the lemma.

Claim. The map f is homotopic to a smooth map g such that $f^{-1}(y) = g^{-1}(y)$ and such that for every $x \in f^{-1}(y)$ there exists an open neighborhood of x on which f and g agree.

Proof. We start out with a few preparations:

(a) We denote by $x_1, \ldots, x_m$ the points in $f^{-1}(y)$. For $i = 1, \ldots, m$ we pick a neighborhood U_i of x_i such that the closure $\overline{U}_i$ is contained in $f^{-1}(V)$.
(b) By the Manifold Metrization Proposition 23.10 or alternatively by the Manifold Metrization Proposition 33.5, we know that we can equip N with a metric d.
(c) Since M is compact and since $U_1, \ldots, U_m$ are open we know by the Compact-Closed Lemma 3.39 and the Compact Image Lemma 4.15 that $f(M \setminus (U_1 \cup \cdots \cup U_m))$ is a closed subset of N. Evidently $y \notin f(M \setminus (U_1 \cup \cdots \cup U_m))$.
(d) It follows from (b) and (c) together with the Distance-to-Subspace Lemma 10.4 that there exists an $\epsilon > 0$ such that $d(f(x), y) > \epsilon$ for all $x \in M \setminus (U_1 \cup \cdots \cup U_m)$.

It now follows from the Whitney Approximation Theorem 36.1 that there exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:

(1) We have $F_0 = f$.
(2) The map $F_1\colon M \to N$ is smooth.
(3) The homotopy is a homotopy rel $\overline{U}_1 \cup \cdots \cup \overline{U}_m$.
(4) For every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \frac{\epsilon}{2}$.

We set $g = F_1$. By (2) the map g is smooth and by construction the map g is homotopic to f. It follows easily from (3) and (4) that $f^{-1}(y) = g^{-1}(y)$. Furthermore it follows from (3) f and g agree on the open neighborhood $(U_1 \cup \cdots \cup U_m) \cap f^{-1}(B_{\frac{\epsilon}{2}}(y))$. Thus we see that for any $x \in f^{-1}(y)$ the maps f and g agree on an open neighborhood of x that is contained $U_1 \cup \cdots \cup U_m$. ⊞

Now we see that

$$\deg(f) \underset{\uparrow}{=} \deg(g) \underset{\uparrow}{=} \sum_{x\in g^{-1}(y)} \deg(g,x) \underset{\uparrow}{=} \sum_{x\in f^{-1}(y)} \deg(f,x).$$

by the Degree Homotopy Invariance Theorem 46.6

it follows from (4) and the choice of ϵ that $f^{-1}(y) = g^{-1}(y)$, we also know that maps f and g agree on neighborhoods around each x_i, in particular y is also a regular value for f and g and for each $x \in f^{-1}(y)$ we have $\deg(f,x) = \deg(g,x)$

■

Proof of the Degree Realization Proposition 46.19. Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $k \in \mathbb{Z}$. By the Degree Examples Lemma 46.13 there exists a continuous map $g\colon S^n \to S^n$ of degree k. It follows from this fact and the Degree Multiplicativity Theorem 46.7 that it suffices to prove the following claim:

Claim. There exists a continuous map $f\colon M \to S^n$ of degree $+1$.

Proof. Since M is non-empty we can pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n \to M$. Next, similar to the setting of the Ball Quotient-Sphere Lemma 8.10 we consider the map

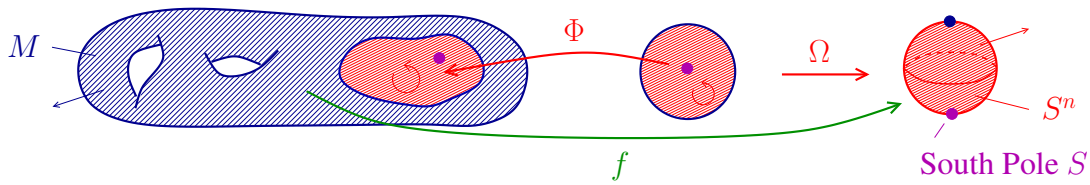
$$\begin{array}{rcl} \Omega\colon \overline{B}^n & \to & S^n \\ r\cdot v & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi\cdot r) \\ \sin(\pi\cdot r)\cdot v \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}} \quad \text{with } r\in[0,1] \text{ and } v \in S^{n-1}. \end{array}$$

By the Ball Quotient-Sphere Lemma 8.10 this map has the following properties:

(1) The map Ω sends every point in S^{n-1} to the North Pole $N \coloneqq (0,\dots,0,1)$.
(2) The map Ω sends the origin to the South Pole $S \coloneqq (0,\dots,0,-1)$.
(3) The map $B^n \to \overline{B}^n \xrightarrow{\Omega} S^n$ is a smooth embedding and, if we equip B^n and S^n with the standard orientations, the map is orientation-preserving.

Next we consider the map

$$\begin{array}{rcl} f\colon M & \to & S^n \\ x & \mapsto & \begin{cases} \Omega(\Phi^{-1}(x)), & \text{if } x \in \Phi(\overline{B}^n), \\ N, & \text{if } x \notin \Phi(\overline{B}^n). \end{cases} \end{array}$$



We make the following observations:

(1) It follows from the Compact Image Lemma 4.15, the Compact-Closed Lemma 3.39 (2) and the Pasting Proposition 4.7 (2) that f is continuous.
(2) The map f is smooth on the open subset $\Phi(B^n)$.
(3) Since Φ and $\Omega\colon B^n \to S^n$ are orientation-preserving smooth embeddings we see that the map $f|_{\Phi(B^n)}\colon \Phi(B^n) \to S^n$ is an orientation-preserving smooth embedding.

It follows from the Partially Smooth Map Lemma 46.20 applied to $y = S$ and $V = S^n \setminus \{N\}$ that $\deg(f) = 1$. ■

In the Degree Realization Proposition 46.19 we just showed that every closed orientable connected non-empty n-dimensional smooth manifold M admits a smooth map $f\colon M \to S^n$ of degree 1. In particular the torus $S^1 \times S^1$ admits a smooth map $f\colon S^1 \times S^1 \to S^2$ of degree 1. The question arises whether we can swap the roles of the torus and the sphere. In other words we have the following question:

Question 46.21. Does there exist a smooth map $f\colon S^2 \to S^1 \times S^1$ of degree one?

We will answer this question in the Sphere-to-Torus Proposition 86.8 or alternatively the Smaller-to-Larger Genus Proposition 186.11, once we have developed many more tools.

Now we can finally state the Hopf Degree Theorem, which is the main theorem of this section. It gives a very satisfying classification of continuous maps to S^n up to homotopy.

Theorem 46.22. (Hopf Degree Theorem) Let $n \in \mathbb{N}$, let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two continuous maps. Then

$$\deg(f) = \deg(g) \iff \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Remark.

(1) For $n = 2$ the theorem was first proved by Luitzen Brouwer [**Brou1921**, p. 281], the general case was first proved by Heinz Hopf [**Hopf1927a**, Satz IIb].
(2) We will give an alternative proof of the Hopf Degree Theorem 46.22 much later on page **??**, when we deduce it from the much more general Hopf Theorem **??**.
(3) The approach we use to prove the Hopf Degree Theorem 46.22 will also be used later in the proof of the $\Omega_0^{\text{th}}(\mathbb{R}^m)$-Proposition 300.18 where we calculate the cobordism group $\Omega_0^{\text{th}}(\mathbb{R}^n)$.
(4) The reader who already knows about higher homotopy groups will have noticed that the Degree Realization Proposition 46.19 and the Hopf Degree Theorem 46.22, both applied to $M = S^n$, imply that $\pi_n(S^n) \cong \mathbb{Z}$. □

Proof. We will prove the Hopf Degree Theorem 46.22 in the next and final section of this chapter. ■

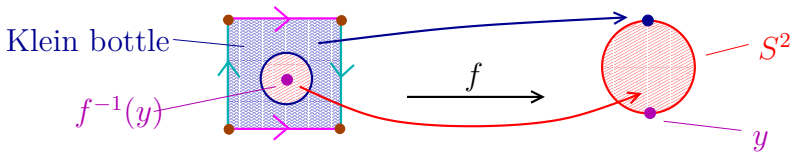
We conclude this section with the non-orientable analog of the Hopf Degree Theorem 46.22. To formulate this analog we need to introduce the $\mathbb{Z}/2\mathbb{Z}$-degree.

Definition 46.23. Let $f\colon M \to N$ be a continuous map between two closed (possibly non-orientable) connected non-empty n-dimensional smooth manifolds. By the Whitney Approximation Theorem 36.1 we know that f is homotopic to a smooth map $g\colon M \to N$. Let $y \in N$ be a regular value of g. We define the $\mathbb{Z}/2\mathbb{Z}$-degree of f as follows:

$$\deg_{\mathbb{Z}/2\mathbb{Z}}(f) \coloneqq \#g^{-1}(y) \in \mathbb{Z}/2\mathbb{Z}.$$

Using a variation on the Degree–Homotopy Proposition 46.4 (which holds by basically the same argument) one can show that this notion is well-defined.

One can easily show (see also Exercise 46.8) that the obvious analogs of the Degree–Homotopy Proposition 46.4, the Degree–Well-Defined Theorem 46.2, the Degree Homotopy Invariance Theorem 46.6, the Degree Multiplicativity Theorem 46.7 and the Degree Realization Proposition 46.19 hold in this context. As an example we show in the figure a map $f\colon K \to S^2$ from the Klein bottle K to the sphere S^2 with $\deg_{\mathbb{Z}/2\mathbb{Z}}(f) = 1 \in \mathbb{Z}/2\mathbb{Z}$.



Now we can formulate the analog of the Hopf Degree Theorem 46.22 for non-orientable manifolds.

Theorem 46.24. (Non-Orientable Hopf Degree Theorem) Let $n \in \mathbb{N}$, let M be a closed *non-orientable* connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two continuous maps. Then

$$\deg_{\mathbb{Z}/2\mathbb{Z}}(f) = \deg_{\mathbb{Z}/2\mathbb{Z}}(g) \iff \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Proof. At the very end of this chapter we will explain how to modify the proof of the Hopf Degree Theorem 46.22 to obtain a proof of the Non-Orientable Hopf Degree Theorem 46.24. ■

46.6. **Proof of the Hopf Degree Theorem 46.22.** This final section is totally dedicated to the proofs of the Hopf Degree Theorem 46.22 and of the Non-Orientable Hopf Degree Theorem 46.24. First we recall and introduce some notation.

Notation 46.25. As in the Ball Quotient-Sphere Lemma 8.10 we consider the map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r\cdot v] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi\cdot r) \\ \sin(\pi\cdot r)\cdot v \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}} \end{array} \quad \text{with } r \in [0,1] \text{ and } v \in S^{n-1}.$$

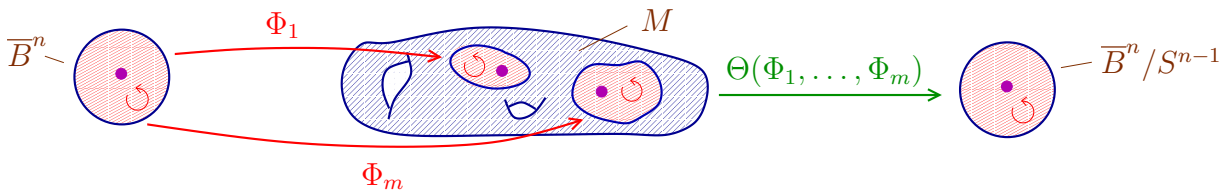
By the Ball Quotient-Sphere Lemma 8.10 this map has the following properties:

(1) The map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ is a homeomorphism. We use this homeomorphism and the standard smooth structure and orientation on S^n to equip $\overline{B}^n/S^{n-1}$ with a smooth structure and with an orientation.
(2) The map Ω sends the point $[S^{n-1}] \in \overline{B}^n/S^{n-1}$ to the North Pole $N := (0,\dots,0,1)$.
(3) The map Ω sends the origin to the South Pole $S := (0,\dots,0,-1)$.
(4) The map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is an orientation-preserving smooth embedding.

Remark. We use the diffeomorphism $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ to replace in the following dicussion the smooth manifold S^n by $\overline{B}^n/S^{n-1}$. This approach greatly simplifies the notation throughout the following arguments. □

Notation 46.26. Let M be a closed n-dimensional smooth manifold. Given smooth embeddings $\Phi_1,\dots,\Phi_m\colon \overline{B}^n \to M$ with disjoint images we define, similar to the construction in the proof of the Degree Realization Proposition 46.19, the map

$$\begin{array}{rcl} \Theta(\Phi_1,\dots,\Phi_m)\colon M & \to & \overline{B}^n/S^{n-1} \\ x & \mapsto & \begin{cases} [\Phi_i^{-1}(x)], & \text{if } x \in \Phi_i(\overline{B}^n), \\ [S^{n-1}], & \text{otherwise.} \end{cases} \end{array}$$



The proof of the Hopf Degree Theorem 46.22 rests on three lemmas.

Auxiliary Lemma 46.27. Let M be a closed connected n-dimensional smooth manifold and let $\Phi_1,\dots,\Phi_m\colon \overline{B}^n \to M$ be smooth embeddings with disjoint images.

(1) If M is oriented, then the degree of $\Theta(\Phi_1,\dots,\Phi_m)\colon M \to \overline{B}^n/S^{n-1}$ can be calculated as follows:

$$\deg(\Theta(\Phi_1,\dots,\Phi_m)) \;=\; \sum_{i=1}^{m} \begin{cases} +1, & \text{if } \Phi_i \text{ is orientation-preserving,} \\ -1, & \text{if } \Phi_i \text{ is orientation-reversing.} \end{cases}$$

(2) Let $\Psi_1,\dots,\Psi_m\colon \overline{B}^n \to M$ be another set of smooth embeddings with disjoint images. If the smooth embedding[512] $\Phi_1 \sqcup\dots\sqcup \Phi_m\colon \overline{B}^n \sqcup\dots\sqcup \overline{B}^n \to M$ is smoothly isotopic to the smooth embedding $\Psi_1 \sqcup\dots\sqcup \Psi_m\colon \overline{B}^n \sqcup\dots\sqcup \overline{B}^n \to M$, then the two maps $\Theta(\Phi_1,\dots,\Phi_m)\colon M \to \overline{B}^n/S^{n-1}$ and $\Theta(\Psi_1,\dots,\Psi_m)\colon M \to \overline{B}^n/S^{n-1}$ are homotopic.

Proof.

(1) The calculation of the degree follows from a slight variation on the argument of Proposition 46.19, using the general case of the Partially Smooth–Degree Lemma 46.20.

(2) We start out with a few preparations:

(a) By hypothesis we can pick a smooth isotopy $F\colon (\overline{B}^n \sqcup\dots\sqcup \overline{B}^n)\times[0,1] \to M$ from the smooth embedding $\Phi_1\sqcup\dots\sqcup\Phi_m\colon \overline{B}^n\sqcup\dots\sqcup\overline{B}^n \to M$ to the smooth embedding $\Psi_1 \sqcup\dots\sqcup \Psi_m\colon \overline{B}^n \sqcup\dots\sqcup \overline{B}^n \to M$.

(b) We denote by $p\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ and $q\colon \overline{B}^n \times [0,1] \to [0,1]$ the projections.

(c) Given $i = 1,\dots,m$ we consider the map

$$\begin{aligned} \widetilde{F}_i = (F|_{\overline{B}_i\times[0,1]} \times q)\colon \overline{B}^n \times [0,1] &\to M \times [0,1] \\ (x,t) &\mapsto (F(x,t),t) \end{aligned}$$

with x in the i-th copy of $\overline{B}^n$

This map is injective and continuous. Since $\overline{B}^n\times[0,1]$ is compact and since $M\times[0,1]$ is Hausdorff it follows from the Compact-Hausdorff Proposition 4.23 (3) that it is an embedding. We denote by $\widetilde{F}_i^{-1}\colon \widetilde{F}_i(\overline{B}^n \times [0,1]) \to \overline{B}^n \times [0,1]$ the inverse.

Finally we consider the map

$$\begin{aligned} G\colon M \times [0,1] &\to \overline{B}^n/S^{n-1} \\ z &\mapsto \begin{cases} [p(\widetilde{F}_i^{-1}(z))], & \text{if } z \in \widetilde{F}_i(\overline{B}^n \times [0,1]), \\ [S^{n-1}], & \text{otherwise.} \end{cases} \end{aligned}$$

[512]Since $\overline{B}^n$ is compact we obtain from the Disjoint Union-Smooth Embedding Lemma 30.16 that m smooth embeddings of $\overline{B}^n$ with disjoint images give rise to a smooth embedding of the disjoint union of m closed balls.

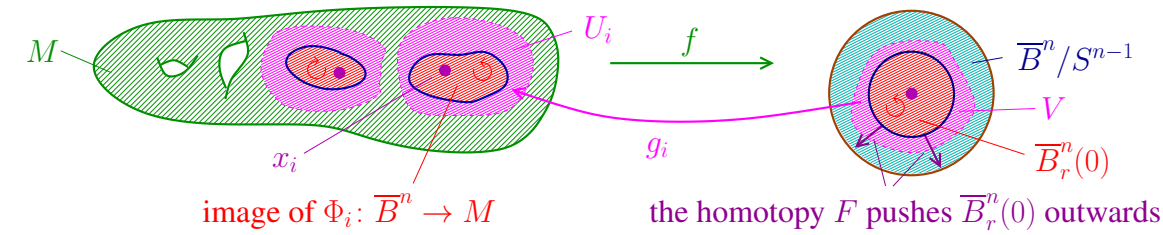
Basically by definition we have $G_0 = \Theta(\Phi_1, \dots, \Phi_m)$ and $G_1 = \Theta(\Psi_1, \dots, \Psi_m)$. Thus it remains to show that the map $G\colon M \times [0,1] \to \overline{B}^n/S^{n-1}$ is continuous. First note that it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that $\widetilde{F}_1(B^n \times [0,1]) \cup \dots \cup \widetilde{F}_m(B^n \times [0,1])$ is an open subset of $M \times [0,1]$. Furthermore it follows from the Product Topology-Properties Proposition 7.9, the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that $\widetilde{F}_1(\overline{B}^n \times [0,1]) \cup \dots \cup \widetilde{F}_1(\overline{B}^n \times [0,1])$ is a closed subset of $M \times [0,1]$. It follows from this lengthy discussion and the Pasting Proposition 4.7 (2) that this map $M \times [0,1] \to \overline{B}^n/S^{n-1}$ is indeed continuous. ■

Auxiliary Lemma 46.28. Let M be a closed n-dimensional smooth manifold. Given any continuous map $f\colon M \to \overline{B}^n/S^{n-1}$ there exist smooth embeddings $\Phi_1, \dots, \Phi_m\colon \overline{B}^n \to M$ with disjoint images such that f is homotopic to $\Theta(\Phi_1, \dots, \Phi_m)\colon M \to \overline{B}^n/S^{n-1}$.

Proof of Lemma 46.28. We start out with the following straightforward steps:

(1) By the Whitney Approximation Theorem 36.1 and the Degree Homotopy Invariance Theorem 46.6 we can assume that f is smooth.
(2) By Sard's Theorem 38.1 we know that the smooth map $f\colon M \to \overline{B}^n/S^{n-1}$ has a regular value. By applying the Manifold–Homogeneous Proposition 41.21 to the smooth manifold $\overline{B}^n/S^{n-1}$ we can assume that the "origin" $0 \in \overline{B}^n/S^{n-1}$ is a regular value of f.
(3) We denote the preimages of 0 by $x_1, \dots, x_m \in M$.
(4) Since each $\mathrm{D}f_{x_i}\colon \mathrm{T}_{x_i}M \to \mathrm{T}_0\overline{B}^n/S^{n-1}$ is an isomorphism we know by the Inverse Mapping Theorem 30.1 that there exist neighborhoods $U_1, \dots, U_m$ of $x_1, \dots, x_m$ and V of $0 \in \overline{B}^n/S^{n-1}$ such that for each $i \in \{1, \dots, m\}$ the map f restricts to a diffeomorphism $f\colon U_i \to V$.
(5) Since M is Hausdorff we can arrange that the neighborhoods $U_1, \dots, U_m$ of $x_1, \dots, x_m$ are disjoint.
(6) For $i = 1, \dots, m$ we denote by $g_i\colon V \to U_i$ the inverse of $f\colon U_i \to V$.
(7) We pick an $r \in (0,1)$ such that $\overline{B}^n_r(0) \subseteq V$.
(8) For $i = 1, \dots, m$ we consider the smooth embedding

$$\begin{array}{rccc} \Phi_i\colon & \overline{B}^n & \to & M \\ & x & \mapsto & g_i(x \cdot r). \end{array}$$



It now remains to prove the following claim:

Claim. The map $f\colon M \to \overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Phi_1, \dots, \Phi_m)\colon M \to \overline{B}^n/S^{n-1}$.

Proof. We consider the homotopy[513]

$$\begin{array}{rcl} F\colon \overline{B}^n/S^{n-1} \times [0,1] & \to & \overline{B}^n/S^{n-1} \\ ([x],t) & \mapsto & \begin{cases} [x \cdot ((1-t) + t \cdot \frac{1}{r})], & \text{if } \|x\| \leq r, \\ [x \cdot ((1-t) + t \cdot \frac{1}{\|x\|})], & \text{if } \|x\| \geq r \end{cases} \end{array}$$

from $F_0 = \mathrm{id}$ to a map $F_1\colon \overline{B}^n/S^{n-1} \to \overline{B}^n/S^{n-1}$ whose restriction to $\overline{B}^n_r(0)$ is induced by multiplication by $\frac{1}{r}$. Intuitively this homotopy F "expands" the r-ball till it covers all of the sphere $\overline{B}^n/S^{n-1}$ and everything outside of the r-ball gets pushed to the point $[S^{n-1}]$.

Finally note that it follows immediately from the definitions that the map

$$\begin{array}{rcl} G = F \circ (f \times \mathrm{id}_{[0,1]})\colon M \times [0,1] & \to & \overline{B}^n/S^{n-1} \\ (x,t) & \mapsto & F(f(x),t) \end{array}$$

is a homotopy from f to $\Theta(\Phi_1, \ldots, \Phi_m)$. ∎

Our next lemma is a refinement of the previous Lemma 46.28. The proof of the next lemma contains perhaps the most clever idea in the context of the proof of the Hopf Degree Theorem 46.22.

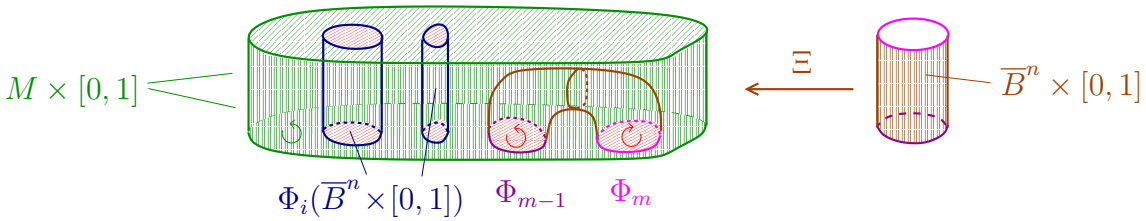
Auxiliary Lemma 46.29. Let M be a closed oriented n-dimensional smooth manifold. Furthermore let $f\colon M \to \overline{B}^n/S^{n-1}$ be a continuous map. We set $k := |\deg(f)|$. There exist k smooth embeddings $\Phi_1, \ldots, \Phi_k\colon \overline{B}^n \to M$ with disjoint images such that the map $f\colon M \to \overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Phi_1, \ldots, \Phi_k)\colon M \to \overline{B}^n/S^{n-1}$.

Proof of Lemma 46.29. Let M be a closed oriented n-dimensional smooth manifold and let $f\colon M \to \overline{B}^n/S^{n-1}$ be a continuous map of degree k. By Lemma 46.28 we know that there exist smooth embeddings $\Phi_1, \ldots, \Phi_m\colon \overline{B}^n \to M$ with disjoint images such that f is homotopic to $\Theta(\Phi_1, \ldots, \Phi_m)\colon M \to \overline{B}^n/S^{n-1}$.

It follows from Lemma 46.27 (1) that $m \geq k$. If $m = k$, then we are done. Now assume that $m > k$. It follows again from Lemma 46.27 (1) that, after possibly reordering the Φ_i, we can assume that the smooth embedding $\Phi_{m-1}\colon \overline{B}^n \to M$ is orientation-preserving and that the smooth embedding $\Phi_m\colon \overline{B}^n \to M$ is orientation-reversing. By induction on m it suffices to show that $\Theta(\Phi_1, \ldots, \Phi_m)$ is homotopic to $\Theta(\Phi_1, \ldots, \Phi_{m-2})$.

Claim. There exists a smooth embedding $\Xi\colon \overline{B}^n \times [0,1] \to M \times [0,1)$ with the following two properties:
(1) For every $x \in \overline{B}^n$ we have $\Xi(x,0) = (\Phi_{m-1}(x), 0)$ and $\Xi(x,1) = (\Phi_m(x), 0)$.
(2) The image of Ξ is disjoint from $\Phi_i(\overline{B}^n) \times [0,1]$, $i = 1, \ldots, m-2$.

[513]To show that the map is continuous actually requires a bit of footwork. First note that it follows from the Pasting Proposition 4.7 that the obvious corresponding map $\overline{B}^n \times [0,1] \to \overline{B}^n$ is continuous. By the Quotient-to-Quotient Lemma 8.6 this map descends to a continuous map $(\overline{B}^n \times [0,1])/\sim \to \overline{B}^n/S^{n-1}$. Finally note that it follows from the subtle Product-Quotient Mixing Lemma 17.8 that the given map $\overline{B}^n/S^{n-1} \times [0,1] \to \overline{B}^n/S^{n-1}$ is also continuous.



Proof. It follows fairly easily from the Smooth Ball Embedding Theorem 45.10 that there exists an open subset $U \subseteq M$ and an orientation-preserving chart $\Gamma\colon U \to B_6^n(0)$ with the following properties:[514]

(1) $\Phi_1(\overline{B}^n) \cup \cdots \cup \Phi_m(\overline{B}^n)$ are contained in U,
(2) $\Gamma(\Phi_1(\overline{B}^n)) \cup \cdots \cup \Gamma(\Phi_{m-2}(\overline{B}^n))$ is contained in $B_6^n(0) \setminus B_4^n(0)$,
(3) $\Gamma \circ \Phi_{m-1}\colon \overline{B}^n \to B_6^n(0)$ is given by $(x_1, \dots, x_n) \mapsto (-2 + x_1, x_2, \dots, x_n)$.
(4) $\Gamma \circ \Phi_m\colon \overline{B}^n \to B_6^n(0)$ is given by $(x_1, \dots, x_n) \mapsto (2 - x_1, x_2, \dots, x_n)$.



The desired smooth embedding is now given by

$$\begin{array}{rcl} \Xi\colon \overline{B}^n \times [0,1] & \to & M \times [0,1] \\ (x_1, x_2, \dots, x_n, t) & \mapsto & \big(\Gamma^{-1}(\underbrace{\cos(\pi \cdot t) \cdot (-2 + x_1), x_2, \dots, x_n)}_{\in B_6^n(0)}, \underbrace{\tfrac{1}{4} \sin(\pi \cdot t) \cdot (2 - x_1)}_{\in [0,1)}\big). \end{array}$$

⊞

In the following we will use the smooth embedding $\Xi\colon \overline{B}^n \times [0,1] \to M \times [0,1]$ to construct a homotopy from $\Theta(\Phi_1, \dots, \Phi_m)$ to $\Theta(\Phi_1, \dots, \Phi_{m-2})$. To do so we need to make a few preparations:

(1) For $i = 1, \dots, m-2$ we denote by $\widetilde{F}_i\colon \overline{B}^n \times [0,1] \to M \times [0,1]$ the embedding that is given by $\widetilde{F}_i(x,t) = (\Phi_i(x), t)$.
(2) We denote by $p\colon \overline{B}^n \times [0,1] \to \overline{B}^n$ the projection.

Similar to the proof of Lemma 46.27 we now consider the map

$$\begin{array}{rcl} G\colon M \times [0,1] & \to & \overline{B}^n / S^{n-1} \\ z & \mapsto & \begin{cases} p(\widetilde{F}_i^{-1}(z)), & \text{if } z \in \widetilde{F}_i(\overline{B}^n \times [0,1]), \\ p(\Xi^{-1}(z)), & \text{if } z \in \Xi(\overline{B}^n \times [0,1]), \\ [S^{n-1}], & \text{otherwise.} \end{cases} \end{array}$$

The argument in the proof of Lemma 46.27 shows that $G\colon M \times [0,1] \to \overline{B}^n / S^{n-1}$ is continuous, i.e. G is a homotopy. Finally note that it follows immediately from the definitions that we have the equalities $G_0 = \Theta(\Phi_1, \dots, \Phi_m)$ and $G_1 = \Theta(\Phi_1, \dots, \Phi_{m-2})$. ■

[514] This can be seen as follows: It is clear that we can pick an orientation-preserving chart $\widetilde{\Gamma}\colon \widetilde{U} \to B_6^n(0)$ and orientation-preserving smooth embeddings $\widetilde{\Phi}_i\colon \overline{B}^n \to \widetilde{U}$, $i = 1, \dots, m-1$ and an orientation-reversing smooth embedding $\widetilde{\Phi}_m\colon \overline{B}^n \to \widetilde{U}$ such that the desired statements hold. Let $\rho\colon \overline{B}^n \to \overline{B}^n$ be the reflection in a hyperplane. We now apply Smooth Ball Embedding Theorem 45.10 to the two sets of orientation-preserving smooth embeddings $\Phi_1, \dots, \Phi_m \circ \rho$ and $\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_m \circ \rho$. We obtain a diffeomorphism $G_1\colon M \to M$ such that $G_1^{-1}(\widetilde{U})$ and $\Gamma \circ G_1$ have the desired properties.

Proof of the Hopf Degree Theorem 46.22. Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. As we discussed in Notation 46.25, the smooth manifold S^n admits an orientation-preserving diffeomorphism to $\overline{B}^n/S^{n-1}$. Thus it suffices to prove the following claim:

Claim. Given any two continuous maps $f, g\colon M \to \overline{B}^n/S^{n-1}$ we have

$$\deg(f) = \deg(g) \iff \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Proof. The"$\Leftarrow$"-direction is just a special case of the Degree Homotopy Invariance Theorem 46.6. We turn to the proof of the"$\Rightarrow$"-direction. We set $k := |\deg(f)| = |\deg(g)|$.

First we point out that by Lemma 46.29 we know that there exist smooth embeddings $\Phi_1, \dots, \Phi_k\colon \overline{B}^n \to M$ with disjoint images such that $f\colon M \to \overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Phi_1, \dots, \Phi_k)\colon M \to \overline{B}^n/S^{n-1}$. The same way we also see that there exist smooth embeddings $\Psi_1, \dots, \Psi_k\colon \overline{B}^n \to M$ with disjoint images such that $g\colon M \to \overline{B}^n/S^{n-1}$ is homotopic to $\Theta(\Psi_1, \dots, \Psi_k)\colon M \to \overline{B}^n/S^{n-1}$.

We now assume that $\deg(f) = \deg(g) \geq 0$. Note that in this case it follows immediately from $k = \deg(f) = \deg(g)$ and Lemma 46.27 (1) that all of the smooth embeddings $\Phi_1, \dots, \Phi_k\colon \overline{B}^n \to M$ and $\Psi_1, \dots, \Psi_k\colon \overline{B}^n \to M$ are orientation-preserving. Thus it follows from the Smooth Ball Embedding Theorem 45.10 that there exists a smooth isotopy between $\Phi_1 \sqcup \dots \sqcup \Phi_k\colon \overline{B}^n \sqcup \dots \sqcup \overline{B}^n \to M$ and $\Psi_1 \sqcup \dots \sqcup \Psi_k\colon \overline{B}^n \sqcup \dots \sqcup \overline{B}^n \to M$. By Lemma 46.27 (2) this implies that the two maps $\Theta(\Phi_1, \dots, \Phi_k)\colon M \to \overline{B}^n/S^{n-1}$ and $\Theta(\Psi_1, \dots, \Psi_k)\colon M \to \overline{B}^n/S^{n-1}$ are homotopic.

Finally we assume that $\deg(f) = \deg(g) < 0$. Note that it follows from the equalities $k = -\deg(f) = -\deg(g)$ and Lemma 46.27 (1) that all of the smooth embeddings $\Phi_1, \dots, \Phi_k\colon \overline{B}^n \to M$ and $\Psi_1, \dots, \Psi_k\colon \overline{B}^n \to M$ are orientation-reversing. The remainder of the argument is identical to the above. ∎

Proof of the Non-Orientable Hopf Degree Theorem 46.24. The proof is almost entirely the same as the above proof of the Hopf Degree Theorem 46.24. The only real difference is that we have to reformulate Lemma 46.29 in the obvious way and in the proof of Lemma 46.29 we now apply the Smooth Ball Embedding Theorem 45.10 for non-orientable smooth manifolds. Note that in the non-orientable setting we do not have to distinguish between orientation-preserving and orientation-reversing smooth embeddings of balls. We leave it to the reader to fill in the details. ∎

46.7. Exercises for Chapter 46.

Exercise 46.1. Let M be a connected n-dimensional smooth manifold with *non-empty* boundary and let $X \subseteq M$ be a finite set. Show that there exists a collar $\chi\colon [0,1] \times \partial M \to M$ such that $X \subseteq \chi([0,1) \times \partial M)$.
Hint. Use the Homogeneous-via-Diffeotopies Proposition 44.6.

Exercise 46.2. Let M be a connected smooth manifold with *non-empty* compact boundary. We want to show that every continuous map $f\colon M \to S^n$ is null-homotopic. There are various approaches to proving this result. One way is as follows:

(a) Apply the Whitney Approximation Theorem 36.1.
(b) Apply Exercise 46.1 to the preimage of a regular value $y \in S^n$.

(c) Use (b) to show that f is homotopic to a map $M \to S^n$ that is not surjective.

Remark. A different approach is sketched in [**OR2009**, p. 193].

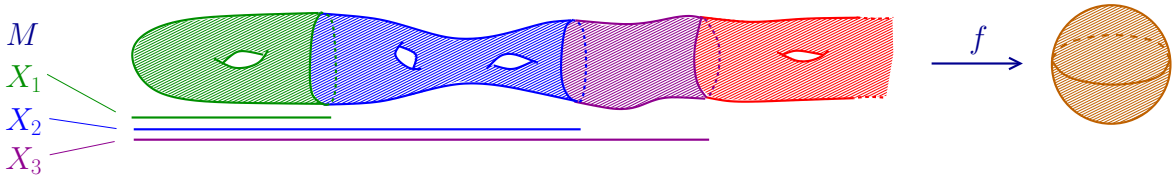
Exercise 46.3. Let M be a connected *non-compact* n-dimensional smooth manifold with empty boundary. We want to show that any continuous map $f\colon M \to S^n$ is null-homotopic. There are various approaches to proving this result. One way is as follows:

(a) By the Compact Submanifold Exhaustion Proposition 38.2 we can pick an exhaustion of M by a sequence of $\{X_k\}_{k\in\mathbb{N}}$ of compact submanifolds.

(b) Use Exercise 46.2 to show that for each $k \in \mathbb{N}$ there exists a homotopy F^k rel $M \setminus X_{k+1}$ such that $F^k_1(X_k) = \{S\}$, where $S = (0, \dots, 0, -1)$ is the "South Pole".

(c) Use the homotopies from (b) to define a homotopy $G\colon M \times [0,1] \to S^n$, where G is defined separately on the subsets $M \times [1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}]$.

Remark. A different approach is sketched in [**OR2009**, p. 193].



Exercise 46.4. Let $f, g\colon S^n \to S^n$ be two continuous maps. Show that $f \circ g$ and $g \circ f$ are homotopic.

Remark. In Exercise 46.6 we will see that the analogous statement is not true for the torus.

Exercise 46.5. Let $f, g\colon S^1 \to S^1$ be two continuous maps. We consider the map

$$\begin{aligned} f \cdot g\colon S^1 &\to S^1 \\ z &\mapsto f(z) \cdot g(z). \end{aligned}$$

Show that $\deg(f \cdot g) = \deg(f) + \deg(g)$.

Exercise 46.6. We consider the following maps:

$$\begin{array}{cccc} i_1\colon S^1 \to S^1 \times S^1 & i_2\colon S^1 \to S^1 \times S^1 & p_1\colon S^1 \times S^1 \to S^1 & p_2\colon S^1 \times S^1 \to S^1 \\ z \mapsto (z, 1) & z \mapsto (1, z) & (z_1, z_2) \mapsto z_1 & (z_1, z_2) \mapsto z_2. \end{array}$$

(a) Given a continuous map $f\colon S^1 \times S^1 \to S^1 \times S^1$ we define

$$\Theta(f) \coloneqq \begin{pmatrix} \deg(p_1 \circ f \circ i_1) & \deg(p_1 \circ f \circ i_2) \\ \deg(p_2 \circ f \circ i_1) & \deg(p_2 \circ f \circ i_2) \end{pmatrix} \in \mathrm{M}(2 \times 2, \mathbb{Z}).$$

Show that for any two continuous maps $f, g\colon S^1 \times S^1 \to S^1 \times S^1$ we have the equality $\Theta(f \circ g) = \Theta(f) \cdot \Theta(g)$.

Hint. Use Exercise 46.5 and use that $\mathrm{id} = (p_1 \circ i_1) \times (p_2 \circ i_2)\colon S^1 \times S^1 \to S^1 \times S^1$.

(b) Show that there exist continuous maps $f, g\colon S^1 \times S^1 \to S^1 \times S^1$ such that $f \circ g$ and $g \circ f$ are not homotopic.

Remark. This exercise shows that the conclusion of Exercise 46.4 does not hold for the torus.

Exercise 46.7. We defined the degree for a continuous map $f\colon M \to N$ between two closed oriented connected smooth manifolds of the same dimension. We used that M is compact to argue that the degree is actually defined, i.e. given by a finite sum of ± 1. We

did not really make use of the hypothesis that N is closed. In fact the same definition makes perfect sense if N is any smooth manifold with empty boundary. Show that if M is closed and N is non-compact, then for any smooth map $f\colon M \to N$ we have $\deg(f) = 0$.

Exercise 46.8. Let $f\colon M \to N$ be a continuous map between two closed connected non-empty n-dimensional smooth manifolds. We do not assume that M or N are orientable. By the Whitney Approximation Theorem 36.1 we know that f is homotopic to a smooth map $g\colon M \to N$. Let $y \in N$ be a regular value of g. We define the *$\mathbb{Z}/2\mathbb{Z}$-degree of f* as follows:

$$\deg_{\mathbb{Z}/2\mathbb{Z}}(f) \;:=\; [\#g^{-1}(y)] \;\in\; \mathbb{Z}/2\mathbb{Z}.$$

(a) Convince yourself that the obvious analogs of the Degree–Homotopy Proposition 46.4, the Degree–Well-Defined Theorem 46.2, the Degree Homotopy Invariance Theorem 46.6 and the Degree Multiplicativity Theorem 46.7 hold in this context. Put differently, reread the proofs and make mentally the necessary adjustments for the present case.

(b) Does there exist a continuous map f from the Klein bottle to the real projective plane $\mathbb{R}\mathrm{P}^2$ with $\deg_{\mathbb{Z}/2\mathbb{Z}}(f) = 1$?

Exercise 46.9. What is the degree of the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_n] & \mapsto & [z_0^2 : \cdots : z_n^2]\,? \end{array}$$

Exercise 46.10.

(a) Let $g\colon S^n \to S^n$ be a smooth map with $g(-x) = g(x)$ for all $x \in S^n$. Show that $\deg(g)$ is even.

(b) Let $f\colon S^n \to S^n$ be a smooth map of odd degree. Show that there exists antipodal points $x, -x$ in S^n which get sent to antipodal points, i.e. $f(x) = -f(-x)$.
Hint. Let $f\colon S^n \to S^n$ be a continuous map such that for each $x \in S^n$ we have $f(x) \neq -f(-x)$. Consider the homotopy

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-\frac{1}{2}t)\cdot f(x) + \frac{1}{2}t \cdot f(-x)}{\|(1-\frac{1}{2}t)\cdot f(x) + \frac{1}{2}t \cdot f(-x)\|}. \end{array}$$

Exercise 46.11. Let $n \in \mathbb{N}$ be odd. Recall that by the Real Projective Space Orientability Lemma 31.46 we know that $\mathbb{R}\mathrm{P}^n$ is a closed orientable smooth manifold. Determine the degree of the orientation-preserving map $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ that is given by $x \mapsto [x]$.

Exercise 46.12.

(a) Let $f\colon M \to N$ be a smooth map between two closed oriented connected non-empty n-dimensional smooth manifolds. Suppose that there exists a compact smooth manifold W with $\partial W = M$ and a smooth map $g\colon W \to N$ with $g|_M = f$. Show that $\deg(f) = 0$.
Hint. Pick up some ideas from the proof of the Degree Homotopy Invariance Theorem 46.6.

(b) Use (a) to give a new proof of the Degree Homotopy Invariance Theorem 46.6.

Exercise 46.13. As in the Stereographic Projection Lemma 4.24 we consider the stereographic projection from the North Pole $N := (0,0,1)$:

$$\begin{array}{rcl} \Phi\colon S^2 & \to & \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} \\ (x_1,x_2,x_3) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right), & \text{if } x_3 \neq 1, \\ \infty, & \text{if } x_3 = 1. \end{cases} \end{array}$$

Now let $q(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients. We consider the map

$$\begin{array}{rcl} \Theta(q)\colon S^2 & \to & S^2 \\ P & \mapsto & \begin{cases} \Phi^{-1}(q(\Phi(P))), & \text{if } P \neq N, \\ N, & \text{if } P = N. \end{cases} \end{array}$$

(For the record we mention that on page 843 we showed that this map is smooth.)

(a) Suppose that $a_n \neq 0$. Show that the maps $\Theta(q)$ and $\Theta(z^n)$ are homotopic.
Remark. This exercise is almost identical to Exercise 18.11.
(b) Show that $\deg(\Theta(z^n)) = n$.
(c) Show that if $n \geq 1$ and $a_n \neq 0$, then $q(z)$ has at least one zero.

Remark. We have just provided another proof of the Fundamental Theorem of Algebra 32.4.

Exercise 46.14. Let $p(z)$ be a complex polynomial.

(a) Let $z_0 \in \mathbb{C}$ be a zero and let $r > 0$ such that z_0 is the only zero of $p(z)$ on $\overline{B}_r(z_0) \subseteq \mathbb{C}$. We consider the map

$$\begin{array}{rcl} \varphi_{z_0,r}\colon S^1 & \to & S^1 \\ w & \mapsto & \dfrac{p(z_0 + r \cdot w)}{\|p(z_0 + r \cdot w)\|}. \end{array}$$

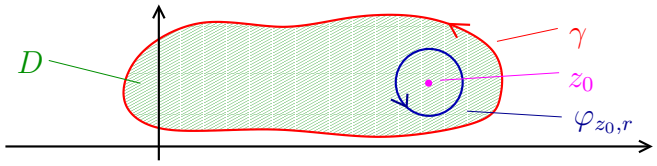
Show that

$$\deg(\varphi_{z_0,r}) \;=\; \text{order of the zero } z_0 \text{ of the polynomial } p(z).$$

(b) Let $D \subseteq \mathbb{C}$ be a compact codimension-zero submanifold such that there exists an orientation-preserving diffeomorphism $\gamma\colon S^1 \to \partial D$. We suppose that $p(z)$ does not have a zero on ∂D. Show that

$$\deg\left(\begin{array}{rcl} S^1 & \to & S^1 \\ w & \mapsto & \dfrac{p(\gamma(w))}{\|p(\gamma(w))\|} \end{array}\right) \;=\; \text{number of zeros of } p(z) \text{ in } D.$$

Hint. Pick disjoint disks around the zeros of $p(z)$ in D.
Remark. This statement is known as the *Argument Principle*.
(c) Use (b) to give one more proof of the Fundamental Theorem of Algebra.



Exercise 46.15. It is easy to come up with a self-homeomorphism of S^2 with two fixed points. Does there exist a self-homeomorphism of S^2 with precisely one fixed point?

Exercise 46.16. Given a smooth map $\gamma\colon S^1 \to \mathbb{C}\setminus\{0\}$ we define the *winding number*

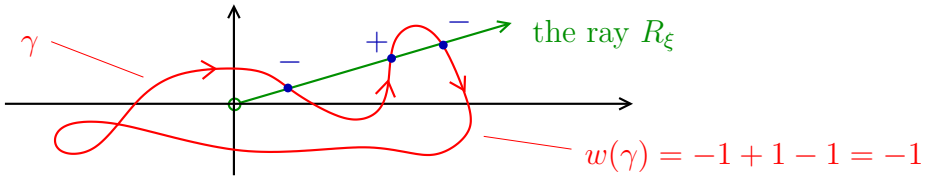
$$\mathrm{w}(\gamma) \;:=\; \deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & \dfrac{\gamma(z)}{\|\gamma(z)\|}\end{pmatrix}.$$

Let $\gamma\colon S^1 \to \mathbb{C}\setminus\{0\}$ be a smooth map. Given $\xi \in S^1$ we consider the corresponding ray $R_\xi := \{r\cdot\xi \,|\, r\in(0,\infty)\}$. We say that γ *intersects* R_ξ *transversally* if for every $z\in S^1$ with $\gamma(z)\in R_\xi$ the vectors $\gamma'(z)$ and ξ are linearly independent. Given such a $z\in S^1$ we define

$$\mathrm{sign}(\gamma,z) \;=\; \begin{cases} +1, & \text{if } \{\xi,\gamma'(z)\} \text{ is a positive basis of } \mathbb{C}=_{\mathrm{i}}\mathbb{R}^2,\\ -1, & \text{otherwise.}\end{cases}$$

Show that if γ intersects R_z transversally, then

$$\mathrm{w}(\gamma) \;=\; \sum_{z\in S^1 \text{ with } \gamma(z)\in R_\xi} \mathrm{sign}(\gamma,z).$$



Exercise 46.17. Given a smooth map $\gamma\colon S^1 \to \mathbb{C}\setminus\{0\}$ we denote by $\mathrm{w}(\gamma)$ the winding number that we introduced in Exercise 46.16. Suppose that at a time $t=0$ a person starts walking around a tree and the person keeps a dog on a leash. At $t=1$ both the person and the dog are precisely at the same starting position. We suppose that at any moment the distance of the person to the tree is longer than the leash. Show that the person and the dog have the same winding number around the tree.
Remark. This statement is colloquially known as the *Dog Walking Theorem*.

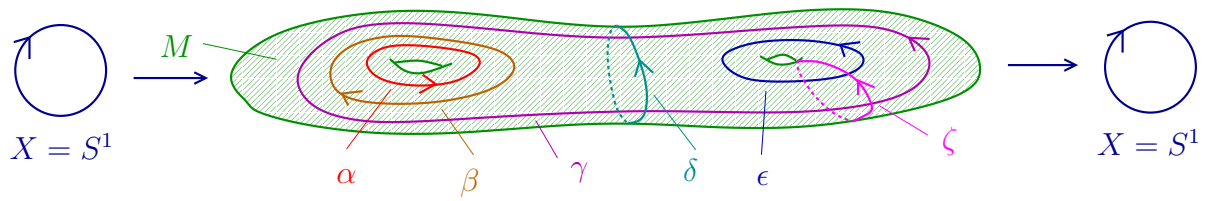
Exercise 46.18. Let $f\colon A\to M$ and $g\colon B\to N$ be smooth maps between closed oriented connected non-empty smooth manifolds with $\dim(A)=\dim(M)$ and $\dim(B)=\dim(N)$. Show that

$$\deg(\mathrm{diag}(f,g)\colon A\times B\to M\times N) \;=\; \deg(f\colon A\to M)\cdot\deg(g\colon B\to N).$$

Exercise 46.19.

(a) Let M be a smooth manifold and let X be a closed oriented smooth manifold. Furthermore let $f,g\colon X\to M$ be two smooth maps. Show that if there exists a smooth map $\pi\colon M\to X$ such that $\deg(\pi\circ f)\neq\deg(\pi\circ g)$, then f and g are not homotopic.

(b) Let M be the surface of genus 2. We consider the six maps $S^1\to M$ shown in the figure below. Show that these six maps are pairwise not homotopic.



Exercise 46.20. In the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 we saw that every orientation-preserving diffeomorphism of $\mathbb{R}^n$ is diffeotopic to the identity. Give examples of orientation-preserving diffeomorphisms of suitable closed connected smooth manifolds that are not diffeotopic to the identity.
Hint. Consider products of spheres.

Exercise 46.21. Let $k \in \mathbb{N}_0$. Show that for every $g \geq k$ there exists a degree-one map from the surface Σ_g of genus g to the surface Σ_k of genus k.

Exercise 46.22. Let $n \in \mathbb{N}_{\geq 2}$. In the Matrix Group-Smooth Manifold Proposition 32.10 we showed that $\mathrm{SU}(n)$ is a closed orientable smooth manifold of dimension $n^2 - 1$. For each $d \in \mathbb{Z}$ determine the degree of the map

$$\begin{array}{rcl} \mathrm{SU}(n) & \to & \mathrm{SU}(n) \\ A & \mapsto & A^d. \end{array}$$

Hint. Make use of the Spectral Theorem 2.35 (2b).
Remark. This exercise is a special case of a much more general calculation of the degree of a power map on a Lie group in terms of the dimension of a maximal torus. We refer to [**HoM2020**, Lemma 6.92] for details.

Exercise 46.23. Let $f, g \colon S^n \to S^n$ be two continuous maps. Show that if $\deg(f) \neq \deg(g)$, then there exists a point $x \in S^n$ with $\langle f(x), g(x) \rangle = 0$.
Hint. Consider $\| f(x) \cdot (1-t) + g(x) \cdot t \|$.

CHAPTER 47

The Borsuk-Ulam Theorem

In this chapter we will extend some of the definitions and the results from the previous chapter to maps between compact smooth manifolds, possibly with non-empty boundary. This extension is per se not particularly exciting. But we will use it to give our first of many proofs of the celebrated Borsuk-Ulam Theorem 47.12. In the next chapter we will then put the Borsuk-Ulam Theorem 47.12 to good use, namely we will use it to prove four theorems which at first and second glance are far removed from topology.

This chapter is organized as follows:

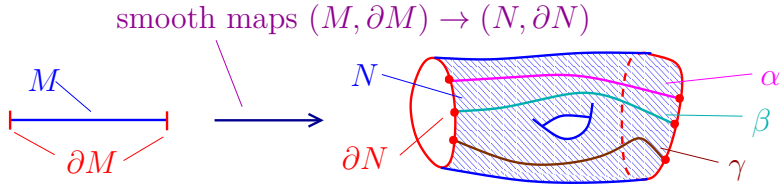
- In Section 47.1 we will extend the notion of the degree of a map between closed smooth manifolds to maps between smooth manifolds with non-empty boundary.
- In Section 47.2 we will relate the degree of a map between smooth manifolds to the degree of the restriction of the map to the boundary.
- In Section 47.3 we will state and prove the Borsuk-Ulam Theorem 47.12.
- In Section 47.4 we will provide exercises for Chapter 47.

In the next Chapter 48 we will discuss several applications of the Borsuk-Ulam Theorem 47.12.

47.1. **Extension of degrees to smooth manifolds with boundary.** In the following we also intend to study suitable maps between compact oriented smooth manifolds with *non-empty* boundary. At this point it is convenient to introduce the following definitions.

Definition 47.1.

(1) A pair of topological spaces is a pair (X, A) where X is a topological space and $A \subseteq X$.
(2) A map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces is a continuous map $f\colon X \to Y$ such that $f(A) \subseteq B$.
(3) We say that two maps $f_0, f_1\colon (X, A) \to (Y, B)$ between pairs of topological spaces are homotopic if there exists a homotopy $F\colon X \times [0, 1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$ and such that for each $t \in [0, 1]$ we have $F_t(A) \subseteq B$.



$\alpha, \beta\colon (M, \partial M) \to (N, \partial N)$ are homotopic as maps between pairs of topological spaces,
$\alpha, \gamma\colon (M, \partial M) \to (N, \partial N)$ are *not* homotopic as maps of pairs of topological spaces

Now we can formulate the following extension of the Degree–Well-Defined Theorem 46.2.

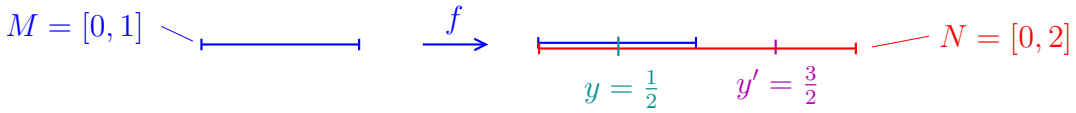
Theorem 47.2. (Degree–Well-Defined Theorem) Let M and N be two compact oriented connected non-empty smooth manifolds of the same dimension and let $f\colon (M,\partial M)\to(N,\partial N)$ be a smooth map. Let $y\in N\setminus\partial N$ be a regular value. The sum[515]

$$\sum_{x\in f^{-1}(y)}\deg(f,x)\ \in\ \mathbb{Z}$$

does not depend on the choice of y.

Proof. We postpone the proof of the Degree–Well-Defined Theorem 47.2 to the end of the section. ∎

Non-Example. We consider the inclusion $f\colon[0,1]\to[0,2]$. Both $y=\frac{1}{2}$ and $y'=\frac{3}{2}$ are regular values, but $f^{-1}(y)$ consists of a single point whereas $f^{-1}(\frac{3}{2})=\varnothing$.



This shows that in the Degree–Well-Defined Theorem 47.2 we cannot drop the hypothesis that $f(\partial M)\subseteq\partial N$. □

The Degree–Well-Defined Theorem 47.2 leads us to the following generalization of Definition 46.3.

Definition 47.3. Let M and N be two compact oriented connected non-empty smooth manifolds of the same dimension and let $f\colon(M,\partial M)\to(N,\partial N)$ be a smooth map. It follows from By Sard's Theorem 38.1 there exists a regular value $y\in N\setminus\partial N$. By the Degree–Well-Defined Theorem 47.2 we can unambiguously define the degree of the smooth map f as

$$\deg(f)\ :=\ \sum_{x\in f^{-1}(y)}\deg(f,x)\ \in\ \mathbb{Z}.$$

Example. Let $n\in\mathbb{Z}$. We view $\overline{B}^2$ as a subspace of $\mathbb{C}$ and we consider the map

$$\begin{array}{rcl} f\colon \overline{B}^2 & \to & \overline{B}^2 \\ z & \mapsto & z^n. \end{array}$$

One can easily show that each $y\in B^2\setminus\{0\}$ is a regular value. With $y=\frac{1}{2}$ we see that

$$\deg(f)\ =\ \sum_{k=1}^{|n|}\deg\Big(f,\underbrace{\exp\Big(\frac{-\ln(2)}{n}+2\pi\,\mathrm{i}\cdot\frac{k}{|n|}\Big)}_{n\text{-th root of }\frac{1}{2}}\Big)\ =\ \sum_{k=1}^{|n|}\begin{cases}+1, & \text{if } n\geq 1,\\ -1, & \text{if } n<1\end{cases}\ =\ n.$$

↑ can be deduced using the Orientation Checking-via-Differentials Lemma 31.30

□

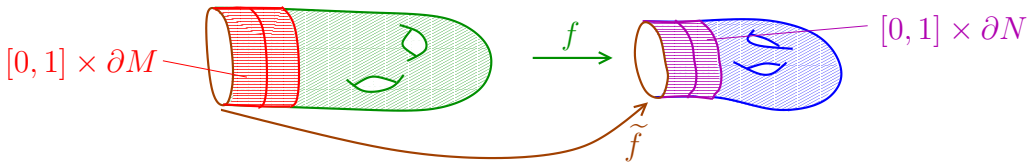
As before we would like to extend the definition of the degree to maps that are continuous but not necessarily smooth. To do so we need the following variation on the Whitney Approximation Theorem 36.1.

[515]It follows from the Regular Value Theorem 32.5 that $f^{-1}(y)$ is a finite set. Furthermore it follows from the hypothesis that $f(\partial M)\subseteq\partial N$ and $y\in N\setminus\partial N$ that $f^{-1}(y)\subseteq M\setminus\partial M$. Thus it follows from the Inverse Mapping Theorem 30.1 that for any $x\in f^{-1}(y)$ the map f is a local diffeomorphism at x. Therefore the local degree $\deg(f,x)$ is defined as in Definition 46.1.

Theorem 47.4. (Relative Whitney Approximation Theorem) Let M and N be compact smooth manifolds. Every continuous map $(M, \partial M) \to (N, \partial N)$ is homotopic, as a map of pairs of topological spaces, to a smooth map $(M, \partial M) \to (N, \partial N)$.

Proof. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a continuous map between compact smooth manifolds. By the Collar Neighborhood Theorem 35.6 there exist collars $[0,1] \times \partial M$ and $[0,1] \times \partial N$. We denote by $\widetilde{f}\colon \partial M \to \partial N$ the restriction of f. By the Whitney Approximation Theorem 36.1 we know that there exists a homotopy $\widetilde{F}\colon \partial M \times [0,1] \to \partial N$ with $\widetilde{F}_0 = \widetilde{f}$ and such that $\widetilde{F}_1$ is a smooth map. In fact, by the Stationary-Near-The-Ends-Lemma 41.7, we can assume that $\widetilde{F}_t = \widetilde{F}_1$ for all $t \in [\frac{1}{2}, 1]$. Now we consider the map

$$\begin{array}{rcl} F\colon M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \widetilde{F}(Q, t-2s), & \text{if } P = (s,Q) \in [0, \frac{1}{2}t] \times \partial M, \\ f((s - \frac{1}{2}t) \cdot \frac{1}{1-\frac{1}{2}t}, Q), & \text{if } P = (s,Q) \in [\frac{1}{2}t, 1] \times \partial M, \\ f(P), & \text{if } P \notin M \setminus ([0,1) \times M) \end{cases} \end{array}$$



It follows from the Pasting Proposition 4.7 together with the Basics-of-Collars Lemma 35.7 (1) that this map $F\colon M \times [0,1] \to N$ is continuous, thus it is indeed a homotopy. Note that for any $t \in [0,1]$ we have $F_t(\partial M) \subseteq \partial N$. Furthermore note that the restriction of $F_1\colon M \to N$ to $[0, \frac{1}{4}] \times \partial M$ is given by the composition of the projection $[0, \frac{1}{4}] \times \partial M$ to ∂M followed by the smooth map $\widetilde{F}_1$. Thus we see that the restriction of $F_1\colon M \to N$ to $[0, \frac{1}{4}] \times \partial M$ is smooth. In summary we have shown that the map $f\colon (M, \partial M) \to (N, \partial N)$ is homotopic to a map $F_1\colon (M, \partial M) \to (N, \partial N)$ that is smooth on $[0, \frac{1}{4}] \times \partial M$.

Now we can apply the Whitney Approximation Theorem 36.1 to the map $F_1\colon M \to N$ and the closed subset $A := [0, \frac{1}{4}] \times \partial M$. We see that there exists a homotopy rel A as maps $(M, \partial M) \to (N, \partial N)$ from $F_1\colon (M, \partial M) \to (N, \partial N)$ to a smooth map $(M, \partial M) \to (N, \partial N)$. We have now finally shown that the map $f\colon (M, \partial M) \to (N, \partial N)$ is indeed homotopic, as a map of pairs of topological spaces, to a smooth map $(M, \partial M) \to (N, \partial N)$. ∎

To extend the definition of the degree to non-smooth maps we will also need the following analog of the Degree–Homotopy Proposition 46.4.

Proposition 47.5. (Degree–Homotopy Proposition) Let $g\colon M \to N$ be a continuous map between two compact oriented connected non-empty smooth manifolds of the same dimension. If $f_0, f_1\colon (M, \partial M) \to (N, \partial N)$ are smooth maps that are both homotopic to g, then $\deg(f_0) = \deg(f_1)$.

Proof. We postpone the proof of the Degree–Homotopy Proposition 47.5 to the end of this section. ∎

The Relative Whitney Approximation Theorem 47.4 and the Degree–Homotopy Proposition 46.4 allow us to make the following definition:

Definition 47.6. Let $f\colon (M,\partial M)\to(N,\partial N)$ be a continuous map between two compact oriented connected non-empty smooth manifolds of the same dimension. By the Relative Whitney Approximation Theorem 47.4 we know that f is homotopic, as a map of pairs of topological spaces, to a smooth map $g\colon (M,\partial M)\to(N,\partial N)$. By the Degree–Homotopy Proposition 47.5 it makes sense to define

$$\deg(f) \;:=\; \deg(g).$$

Remark. In Definition 186.1 we will give an alternative definition of the degree using the "fundamental class". Fortunately we will see in the Local-Global Degree Proposition 186.8 that the two definitions of the degree agree. Furthermore in Definition 195.1 we will extend the notion of a degree to topological manifolds. □

The following two theorems are the obvious extensions of the Degree Homotopy Invariance Theorem 46.6 and the Degree Multiplicativity Theorem 46.7.

Theorem 47.7. (Degree Homotopy Invariance Theorem) Let M and N be two compact oriented connected non-empty smooth manifolds of the same dimension. Furthermore let $f_0, f_1\colon (M,\partial M)\to(N,\partial N)$ be continuous maps. If f_0 and f_1 are homotopic as maps of pairs of topological spaces, then

$$\deg(f_0) \;=\; \deg(f_1).$$

Theorem 47.8. (Degree Multiplicativity Theorem) Let L, M and N be three compact oriented connected non-empty smooth manifolds of the same dimension. Given any continuous maps $f\colon (L,\partial L)\to(M,\partial M)$ and $g\colon (M,\partial M)\to(N,\partial N)$ we have the following equality:

$$\deg(g\circ f) \;=\; \deg(g)\cdot\deg(f).$$

Proof of Theorems 47.2, 47.7 and 47.8 and the proof of Proposition 47.5. The proofs of Theorems 47.2, 47.7 and 47.8 and the proof of Proposition 47.5 for compact smooth manifolds with possibly non-empty boundary are, not surprisingly, very very similar to the proofs of Theorem 46.2, the Degree Homotopy Invariance Theorem 46.6 and 46.7 and of the Degree–Homotopy Proposition 46.4 of the corresponding statements for closed smooth manifolds. To generalize the earlier proofs we need to make three observations. So suppose that M and N are compact connected smooth manifolds.

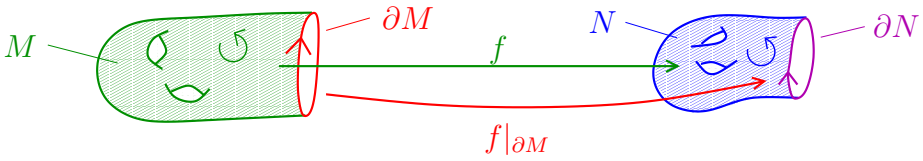
(1) If $y, z\in N\setminus\partial N$, then it follows from the Manifold–Homogeneous Proposition 41.21 that there exists a diffeotopy $F\colon N\times[0,1]\to N$ with $F_0=\mathrm{id}$ and $F_1(y)=z$.
(2) Let $f,g\colon (M,\partial M)\to(N,\partial N)$ be smooth maps. If f and g are homotopic as maps of pairs of topological spaces, then they are also smoothly homotopic as maps of pairs of smooth manifolds. We will prove this slight generalization of the Replace-by-Smooth Homotopy Proposition 36.6 in Exercise 47.1.
(3) Let $f,g\colon (M,\partial M)\to(N,\partial N)$ be smooth maps and let $F\colon M\times[0,1]\to N$ be a smooth homotopy between the maps f and g of pairs of topological spaces. Furthermore let $z\notin\partial N$ be a regular value of $F\colon M\times[0,1]\to N$. Since $F(\partial M\times[0,1])\subseteq\partial N$ we see that $F^{-1}(z)\subseteq(M\setminus\partial M)\times[0,1]$. In fact $Z:=F^{-1}(z)$ is a submanifold of $(M\setminus\partial M)\times[0,1]$ such that $\partial Z\subseteq(M\setminus\partial M)\times\{0,1\}$. Since $M\times[0,1]$ is compact we see that Z is compact. In summary we see that $Z=F^{-1}(z)$ is a compact proper submanifold of $(M\setminus\partial M)\times[0,1]$.

With these three observations it is now straightforward to modify the earlier proofs. We leave it to the reader to fill in the details. ■

47.2. **The Boundary Degree Proposition.** Now we turn to the key result on our new more general notion of degree.

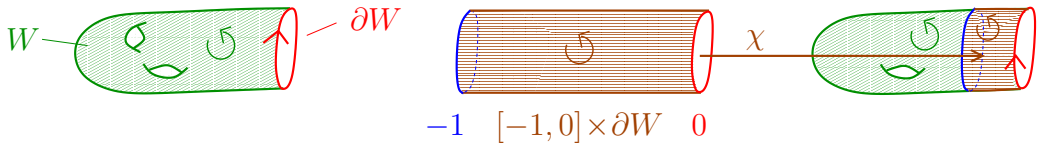
Proposition 47.9. (Boundary Degree Proposition) Let M and N be two compact oriented connected smooth manifolds of the same dimension, such that the boundaries ∂M and ∂N are both connected and non-empty. We equip ∂M and ∂N with the boundary orientation that we had introduced in Convention 31.39. If $f\colon (M, \partial M) \to (N, \partial N)$ is a continuous map, then

$$\deg(f\colon M \to N) \;=\; \deg(f|_{\partial M}\colon \partial M \to \partial N).$$



For the proof of the Boundary Degree Proposition 47.9 we will need the following slight variation on the Smooth Collar Neighborhood Theorem 35.6.

Auxiliary Lemma 47.10. Let W be a compact oriented smooth manifold. We equip ∂W with the boundary orientation, we equip $[-1, 0] \subseteq \mathbb{R}$ with the standard orientation and we equip $[-1, 0] \times \partial W$ with the product orientation. There exists an *orientation-preserving* smooth embedding $\chi\colon [-1, 0] \times \partial W \to W$ such that the restriction of χ to $\{0\} \times \partial W$ is the obvious map $(0, P) \mapsto P$.



Proof of Lemma 47.10. It follows immediately from the Smooth Collar Neighborhood Theorem 35.6 that we can find a smooth embedding $\chi\colon [-1, 0] \times \partial W \to W$ such that the restriction of χ to $\{0\} \times \partial W$ is the obvious map $(0, P) \mapsto P$. It follows from the Product Orientation Proposition 31.48 (3d) that, as an oriented smooth manifold, we have the equality

$$\partial([-1, 0] \times \partial W) \;=\; \partial([-1, 0]) \times \partial W \;=\; -(\{-1\} \times \partial W) \cup (\{0\} \times \partial W).$$

It follows from the Boundary Orientation Lemma 31.38 (3) and (4) that χ is orientation-preserving. ■

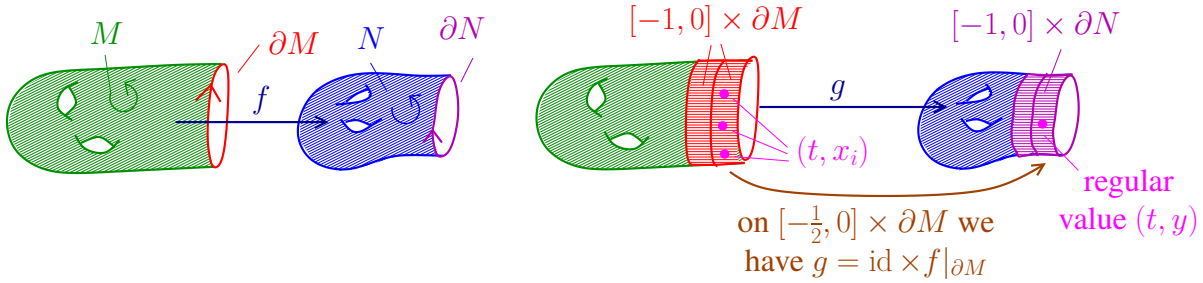
Proof of the Boundary Degree Proposition 47.9. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a continuous map. It follows from the Relative Whitney Approximation Theorem 47.4 and the Degree Homotopy Invariance Theorem 47.7 that it suffices to consider the case that f is smooth.

It follows from Lemma 47.10 that we can find *orientation-preserving* smooth embeddings $\Phi\colon [-1, 0] \times \partial M \to M$ and $\Psi\colon [-1, 0] \times \partial N \to N$ such that for all $P \in \partial M$ we have $\Phi(0, P) = P$ and such that for all $Q \in \partial N$ we have $\Psi(0, Q) = Q$. To simplify the notation

we identify $[-1,0]\times\partial M$ with its image under Φ and the same way we proceed with $[-1,0]\times\partial N$ and Ψ. Note that it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that $(-1,0]\times\partial M\subseteq M$ and $(-1,0]\times\partial N\subseteq N$ are open subsets.

Note that by the Homotopic-to-Product-Near-Boundary Lemma 35.9 we know that the map $f\colon (M,\partial M)\to(N,\partial N)$ is homotopic to a map $g\colon (M,\partial M)\to(N,\partial N)$ with the following two properties:

(1) For every $(t,P)\in[-\frac{1}{2},0]\times\partial M$ we have $g(t,P)=(t,f(P))\in[-\frac{1}{2},0]\times\partial N$.
(2) We have $g^{-1}([-\frac{1}{2},0]\times\partial N)=[-\frac{1}{2},0]\times\partial M$.



Note that a priori we do not know whether $g\colon M\to N$ is smooth. But note that the map $g\colon[-\frac{1}{2},0]\times\partial M\to[-\frac{1}{2},0]\times\partial N$ is smooth. Next: By Sard's Theorem 38.1, together with the Full Measure-Density Proposition 37.4, we know that there exists a regular value $(t,y)\in(-\frac{1}{2},0)\times\partial N$ of $g\colon[-\frac{1}{2},0]\times\partial M\to[-\frac{1}{2},0]\times\partial N$. It follows from (1) and (2) that the preimages of (t,y) are of the form $(t,x_1),\dots,(t,x_k)$ with $x_1,\dots,x_k\in\partial M$. We see that

$$\begin{aligned}
\deg(f\colon M\to N) &\overset{\text{by the Degree Homotopy Invariance Theorem 47.7, since } f \text{ and } g \text{ are homotopic}}{=} \deg(g\colon M\to N)\\
&\overset{\text{follows from (1) and (2) and the obvious generalization of the Partially Smooth Map Lemma 46.20 to compact smooth manifolds}}{=} \sum_{i=1}^{k}\deg(g,(t,x_i))\\
&= \sum_{i=1}^{k}\deg(\operatorname{diag}(\mathrm{id},f|_{\partial M})\colon[-\tfrac{1}{2},0]\times\partial M\to[-\tfrac{1}{2},0]\times\partial N,(t,x_i))\\
&\overset{\text{by (1) the map } g\colon[-\frac{1}{2},0]\times\partial M\to[-\frac{1}{2},0]\times\partial N \text{ is given by } \operatorname{diag}(\mathrm{id},f|_{\partial M}),\text{ here we implicitly use that } [-1,0]\times\partial M\to M \text{ and } [-1,0]\times\partial N\to N \text{ are orientation-preserving}}{=} \sum_{i=1}^{k}\deg(f|_{\partial M}\colon\partial M\to\partial N,x_i)\\
&\overset{\text{it follows from the Product Orientation Preserving Lemma 31.50 that the corresponding local degrees are the same}}{=} \deg(f|_{\partial M}\colon\partial M\to\partial N).
\end{aligned}$$

by Definition 46.3 of the degree

■

47.3. **The Borsuk-Ulam Theorem.** We start out with the following simple definition.

Definition 47.11. Let $A\subseteq S^n$ be a subset. We say that two points $x,y\in A$ are antipodal if $y=-x$.

The following theorem is arguably the most interesting result one can prove with the methods and tools of this chapter.

Theorem 47.12. (Borsuk-Ulam Theorem) Let $n\in\mathbb{N}_0$. Given any continuous map $f\colon S^n\to\mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x)=f(-x)$.

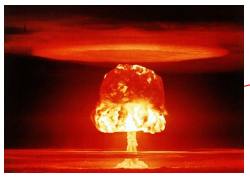
Example. If we think of the surface of earth as S^2, then it follows from the Borsuk-Ulam 47.12 that there exists a pair of antipodal points where the temperature and air pressure agree. □

Remark.

(1) The statement of the Borsuk-Ulam Theorem was conjectured by Stanislaw Ulam, see [**Bor1933**, p. 178]. The theorem was first proved by Karol Borsuk [**Bor1933**] in 1933.
   (a) Karol Borsuk (1905-1982) was a Polish mathematician. He lost his job at the University of Warsaw after the German invasion in 1939. To survive financially he developed a board game and produced and sold it during the war. Almost all copies of the game were lost during the Warsaw uprising in 1944, but in the 1990s one surviving copy was found and the game was republished. We refer to

   `https://en.wikipedia.org/wiki/Animal_Husbandry_(game)`

   for more details.
   (b) Stanislaw Ulam (1909-1984) was a Polish mathematician and nuclear scientist. He was one of the fathers of the hydrogen bomb.



Ulam's other contribution to mankind

(2) In Sections 165.3, 182.6 and 213.3 we will provide three other proofs of the Borsuk-Ulam Theorem 47.12. Furthermore in the 2D-Borsuk Ulam Theorem 89.1 we will provide a different proof for the special case $n = 2$.
(3) The Borsuk-Ulam Theorem 47.12 has many beautiful applications, in particular the Lusternik-Schnirelmann Theorem 48.2, the Ham-Sandwich Theorem 48.4, Pfister's Level Theorem 48.15 and, my favorite, the Necklace Theorem 48.17. Beyond these theorems the Borsuk-Ulam Theorem has an almost endless string of consequences, many of which lie in combinatorics and graph theory, we refer to [**Mato2008, Steinl1985, Steinl1993**] for an extensive discussion of the Borsuk-Ulam theorem and its many corollaries. □

The key ingredient in our proof of the Borsuk-Ulam Theorem 47.12 is the Smooth Odd Map–Odd Degree Proposition 47.14. To formulate this proposition we need the following straightforward definition.

Definition 47.13. A map $f\colon S^n \to S^n$ is called odd, if $f(-x) = -f(x)$ for all $x \in S^n$.

Proposition 47.14. (Smooth Odd Map–Odd Degree Proposition) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a smooth map. If f is odd, then $\deg(f)$ is odd.

Remark.

(1) In the Odd Map–Odd Degree Proposition 165.6 we will use homology theory to provide an alternative proof of the Smooth Odd Map–Odd Degree Proposition 47.14. In fact

at that point we will only need that f is continuous. Another proof will be given in the Odd Map–Even Lefschetz Number Proposition 182.24.

(2) In Exercise 47.11 we will also see that we can extend the Smooth Odd Map–Odd Degree Proposition 47.14 to continuous maps. □

Eventually we will prove the Smooth Odd Map–Odd Degree Proposition 47.14 by induction on n. For the starting point we will need the following elementary lemma:

Auxiliary Lemma 47.15. Let M and N be compact oriented connected 1-dimensional smooth manifolds with non-empty boundary and let $f\colon (M,\partial M) \to (N,\partial N)$ be a smooth map. If $f\colon \partial M \to \partial N$ is a diffeomorphism, then

$$\deg(f\colon M \to N) \;=\; \begin{cases} +1, & \text{if } f\colon \partial M \to \partial N \text{ is orientation-preserving,} \\ -1, & \text{if } f\colon \partial M \to \partial N \text{ is orientation-reversing.} \end{cases}$$



Proof of Lemma 47.15. By the Smooth 1-Manifold Classification Theorem 28.8 we can assume that $M = N = [0,1]$. Note that

$$\begin{array}{rccl} F\colon [0,1]\times[0,1] & \to & [0,1] \\ (x,t) & \mapsto & f(x)\cdot(1-t) + ((1-x)\cdot f(0) + x\cdot f(1))\cdot t \end{array}$$

is a homotopy of maps $([0,1],\{0,1\}) \to ([0,1],\{0,1\})$. If $f(0) = 0$ and $f(1) = 1$, then F is a homotopy from f to the map $x \mapsto x$ and if $f(0) = 1$ and $f(1) = 0$, then F is a homotopy from f to the map $x \mapsto 1 - x$. Either way the desired statement follows from the Degree Homotopy Invariance Theorem 46.6. ■

Proof of the Smooth Odd Map–Odd Degree Proposition 47.14. We want to prove the proposition by induction on n. In the following we will prove the case $n = 1$ and the induction step $n-1 \rightsquigarrow n$ by almost the same argument. Only at the very end will the two arguments diverge slightly.

In the following let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a smooth map that is odd. We introduce the following notation which we will use throughout this proof:

- We denote by $N \coloneqq (0,\dots,0,1)$ the North Pole of S^n.
- As in Notation 4.21 we denote by $S^n_{\geq 0}$ the upper hemisphere and we denote by $S^n_{=0}$ the equator of S^n. Given a map $g\colon S^n \to M$ to some set M we denote by $g_{\geq 0}$ and $g_{=0}$ the obvious restrictions.
- We consider the smooth embedding

$$\begin{array}{rccl} \Phi\colon (-1,1)\times S^{n-1} & \to & S^n \\ (t,x) & \mapsto & (x\cdot\sqrt{1-t^2},0) + t\cdot N. \end{array}$$

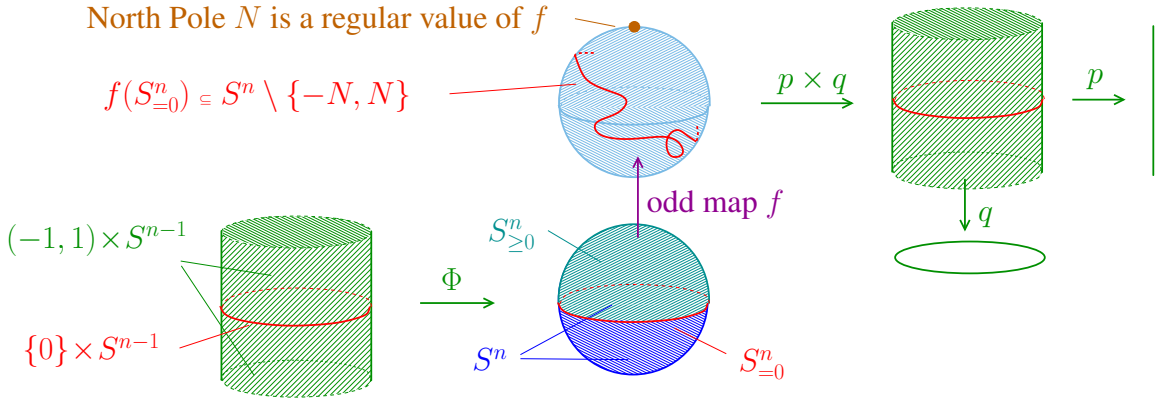
- We consider the maps

$$\begin{array}{rcl} p\colon S^n & \to & [-1,1] \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & x_{n+1} \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon S^n \setminus \{\pm N\} & \to & S^{n-1} \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & \dfrac{(x_1,\dots,x_n)}{\|(x_1,\dots,x_n)\|}. \end{array}$$

Evidently $(p \times q) \circ \Phi = \mathrm{id}_{(-1,1)\times S^{n-1}}$.

It follows from Sard's Theorem 38.1, together with the Measure-Zero Properties Proposition 37.3, that there exists a $v \in S^n$ which is a regular value of the smooth map $f\colon S^n \to S^n$ and also of the smooth map $f_{=0}\colon S^n_{=0} \to S^n$.

Note that by the Orthogonal Action Lemma 2.37 there exists an orthogonal matrix $A \in \mathrm{SO}(n+1)$ with $A \cdot v = N$. Next note that $x \mapsto A \cdot f(x)$ is still odd and note that by the Degree Multiplicativity Theorem 46.7 and the discussion on page 1077 we know that $\deg(x \mapsto A \cdot f(x)) = \deg(f)$. This shows that, after composing f with the map that is given by multiplication by A, we might as well assume that $v = N$. Note that since f is odd we see that in fact both N and $-N$ are regular values of $f\colon S^n \to S^n$ and of $f_{=0}\colon S^n_{=0} \to S^n$.



Claim 1. The map f is homotopic to an odd smooth map $g\colon S^n \to S^n$ such that $\pm N$ are again regular values of g and such that g satisfies $g(S^n_{=0}) \subseteq S^n_{=0}$.[516]

Proof. Since $S^n_{=0} = \Phi(\{0\} \times S^{n-1})$ is an $(n-1)$-dimensional smooth manifold we see that the fact that $\pm N$ is a regular value of $f\colon S^n_{=0} \to S^n$ implies that $\pm N \notin (f \circ \Phi)(\{0\} \times S^{n-1})$. It follows from the Tube Lemma 7.12 that there exists an $\epsilon \in (0,1)$ such that

$$(f \circ \Phi)([-\epsilon,\epsilon] \times S^{n-1}) \subseteq S^n \setminus \{\pm N\}.$$

It follows easily from the Smooth Transition Function Lemma 24.9 that we can pick a smooth map $\eta\colon [-\epsilon,\epsilon] \to [0,1]$ such that $\eta(x) = 1$ for $|x| \geq \frac{\epsilon}{2}$, $\eta(0) = 0$ and which is even in the sense that $\eta(-x) = \eta(x)$. Next we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (z,s) & \mapsto & \begin{cases} f(z), & \text{if } |p(z)| \geq \epsilon, \\ \Phi(\underbrace{p(f(z)) \cdot (1 - s + s \cdot \eta(p(z)))}_{\in[-\epsilon,\epsilon]}, q(f(z))), & \text{if } |p(z)| \leq \epsilon. \end{cases} \end{array}$$

Note that $F_0 = f$. Since η is smooth we see that each F_s is smooth. Furthermore, since η is symmetric one can easily verify that each F_s is odd. Furthermore note that it follows from

[516] Note that we do not claim that one can arrange that $g(S^n_{\geq 0}) \subseteq S^n_{\geq 0}$.

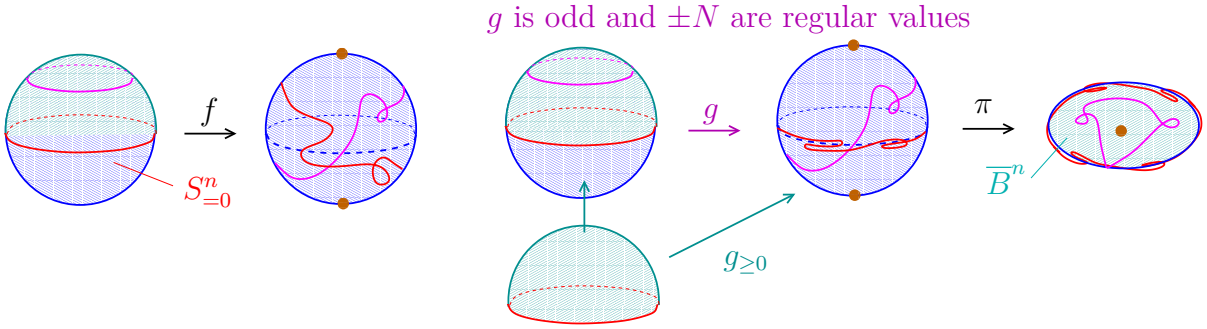
$\eta(0) = 0$ that $F_1(S^n_{=0}) \subseteq S^n_{=0}$. We set $g \coloneqq F_1$. It remains to show that $\pm N$ are also regular values for g. This can be seen as follows. First recall that $\pm N$ are regular values of f. Next note that by construction of g there exists an open neighborhood U of $\{-N, N\} \subseteq S^n$, namely $U \coloneqq S^n \setminus F(\Phi([-\epsilon, \epsilon] \times S^{n-1}) \times [0,1])$, such that $f^{-1}(U) = g^{-1}(U)$ and such that f and g agree on this subset. But this implies that $\pm N$ are also regular values for g. ⊞

Before we can state the next claim we need to introduce one more piece of notation:

- We denote by $\pi\colon S^n \to \overline{B}^n$ the projection onto the first n coordinates.

Claim 2. We equip $S^n_{\geq 0}$ with the submanifold orientation coming from S^n and we equip $\overline{B}^n$ with the standard orientation. With these orientations we have the following equality:

$$\deg(g\colon S^n \to S^n) \equiv \deg(\underbrace{\pi \circ g_{\geq 0}\colon S^n_{\geq 0} \to \overline{B}^n}_{\text{defined since } g_{=0}(S^n_{=0}) \subseteq S^n_{=0}}) \mod 2.$$



Proof. We calculate that in $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$, or $\mathbb{R}/2\mathbb{Z}$, we have following equalities:

$$\begin{array}{lll} \deg(g\colon S^n \to S^n) & \overset{\text{the equality follows from the fact that } \pm N \text{ are both regular values of } g \text{ and the observation that we calculate mod } \mathbb{Z}/2\mathbb{Z}, \text{ hence we can ignore signs}}{\equiv} & \frac{1}{2}\cdot\#g^{-1}(N) + \frac{1}{2}\cdot\#g^{-1}(-N) \\ & = & \#(\pi \circ g_{\geq 0})^{-1}(0) \\ & & \uparrow \text{ since } \pi^{-1}(0) = \{\pm N\} \end{array} \quad \begin{array}{ll} \overset{\text{since } g \text{ is odd and since } g(S^n_{=0}) \subseteq S^n \setminus \{\pm N\}}{=} & \#g^{-1}_{\geq 0}(N) + \#g^{-1}_{\geq 0}(-N) \\ \equiv & \deg(\pi \circ g\colon S^n_{\geq 0} \to \overline{B}^n). \\ \uparrow & \text{since } \mathrm{D}\pi_N \text{ is an isomorphism we see that 0 is a regular value of } \pi \circ g_{\geq 0} \end{array}$$

⊞

At this point we have to distinguish the cases $n = 1$ and $n \geq 2$. First we deal with the case $n = 1$. We perform the following calculation in $\mathbb{Z}/2\mathbb{Z}$:

$$\deg(f\colon S^1 \to S^1) \overset{\text{by the Degree Homotopy Invariance Theorem 46.6 since } f \text{ and } g \text{ are homotopic}}{=} \deg(g\colon S^1 \to S^1) \overset{\text{by Claim 2}}{=} \deg(\pi \circ g_{\geq 0}\colon S^1_{\geq 0} \to \overline{B}^1) = 1.$$

$=$ follows from Lemma 47.15, since the fact that $g_{=0}$ is odd implies that $\pi \circ g_{\geq 0}\colon S^1_{=0} \to S^0$ is a diffeomorphism

Finally let us do the induction step for some $n \in \mathbb{N}$. So now we assume that the conclusion holds for some $n - 1$ and we want to show that it holds for n. We perform a very similar

calculation in $\mathbb{Z}/2\mathbb{Z}$ as above:

$$\begin{array}{ccccccc}
\deg(f\colon S^n \to S^n) & \overset{\text{by the Degree Homotopy Invariance Theorem 46.6 since } f \text{ and } g \text{ are homotopic}}{\downarrow =} & \deg(g\colon S^n \to S^n) & & \overset{\text{by Claim 2}}{\downarrow =} & \deg(\pi \circ g_{\geq 0}\colon S^n_{\geq 0} \to \overline{B}^n) & \\
& \underset{\text{by the Boundary Degree Proposition 47.9}}{\uparrow =} & \deg(\pi \circ g_{=0}\colon S^n_{=0} \to S^{n-1}) & & \underset{\text{since } x \mapsto (x,0) \text{ is a diffeomorphism } S^n_{=0} \to S^{n-1}}{\uparrow =} & \deg\left(S^{n-1} \xrightarrow{x \mapsto (\pi \circ g)(x,0)} S^{n-1}\right) & \underset{\text{by our hypothesis on } n-1}{\uparrow =} \text{ odd.}
\end{array}$$

■

Now we can finally provide the proof of the Borsuk-Ulam Theorem 47.12. We split the proof into two parts. First we consider the case that f is smooth, afterwards we consider the general case.

Proof of the Borsuk-Ulam Theorem 47.12 for smooth maps. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to \mathbb{R}^n$ be a smooth map. We need to show that there exists an $x \in S^n$ with $f(-x) = f(x)$. The case $n = 0$ is trivial so we only need to deal with the case $n \in \mathbb{N}$.

Suppose that there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. Put differently, suppose that for every $x \in S^n$ we have $f(x) \neq f(-x)$. We consider the composition of the following two maps

$$\begin{array}{rccccc}
\Phi\colon S^{n-1} \times [0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\
(y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\
 & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}}_{\text{defined since } f(x) \neq f(-x)}
\end{array}$$

A quick look at the definitions shows that Φ_0 is odd and it shows that Φ_1 is a constant map. Thus we see that

$$\begin{array}{ccccccc}
\text{odd number} & \underset{\text{since } \Phi_0 \text{ is odd we obtain from the Smooth Odd Map–Odd Degree Proposition 47.14 that } \deg(\Phi_0) \text{ is odd}}{\uparrow =} & \deg(\Phi_0) & & \overset{\text{by the Degree Homotopy Invariance Theorem 46.6, since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{\downarrow =} & \deg(\Phi_1) & \underset{\text{by definition of } \Phi_1}{\uparrow =} \deg(\text{constant map}) \underset{\text{by the Basics-of-Degree Lemma 46.12 (1) since constant maps are not surjective}}{\uparrow =} 0.
\end{array}$$

We have thus obtained a contradiction. ■

Proof of the Borsuk-Ulam Theorem 47.12 for continuous maps. Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. We now suppose that there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. It follows that the map

$$\begin{array}{rcl}
\varphi\colon S^n & \to & \mathbb{R}_{\geq 0} \\
x & \mapsto & \|f(x) - f(-x)\|
\end{array}$$

is never zero. Since S^n is compact we see that there exists an $\epsilon > 0$ such for all $x \in S^n$ we have $\|f(x) - f(-x)\| \geq \epsilon$.

By the Smooth Approximation Proposition 27.13 there exists a *smooth* map $\widetilde{f}\colon S^n \to \mathbb{R}^n$ such that for every $x \in S^n$ we have $\|f(x) - \widetilde{f}(x)\| < \frac{\epsilon}{2}$. It follows from the triangle inequality

that for every $x \in S^n$ we have $\|\widetilde{f}(x) - \widetilde{f}(-x)\| > 0$. But this contradicts the Smooth Borsuk Ulam Theorem that we had just proved above. ■

47.4. Exercises for Chapter 47.

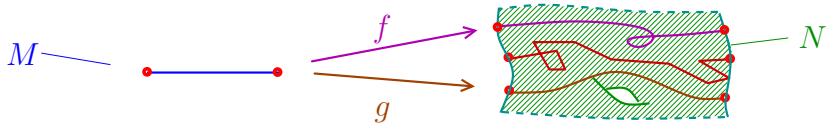
Exercise 47.1. Let M and N be compact smooth manifolds. Let $f, g\colon (M, \partial M) \to (N, \partial N)$ be two smooth maps. We suppose that there exists a homotopy $F\colon M \times [0,1] \to N$ of continuous maps of pairs of topological spaces from f to g.

(a) Show that there exists a homotopy $G\colon M \times [0,1] \to N$ of continuous maps of pairs (i.e. such that $G(\partial M \times [0,1]) \subseteq \partial N$) of topological spaces from f to g such that G is smooth on a neighborhood of $\partial M \times [0,1]$.

(b) Show that there exists a smooth homotopy $G\colon M \times [0,1] \to N$ of maps of pairs of topological spaces from f to g.

(c) Suppose that F is smooth on $\partial M \times [0,1]$. Show that in (a) and (b) we can arrange that $F|_{\partial M \times [0,1]} = G|_{\partial M \times [0,1]}$.

Hint. For (a) first apply the Replace-by-Smooth Homotopy Proposition 36.6 to the homotopy $F|_{\partial M \times [0,1]}\colon \partial M \times [0,1] \to \partial N$ and then use the Smooth Collar Neighborhood Theorem 35.6. Alternatively use the Homotopic-to-Product-Near-Boundary Lemma 35.9. For (b) modify the proof of the Replace-by-Smooth Homotopy Proposition 36.6.



Exercise 47.2. Let $n \in \mathbb{N}$ and let $f\colon \overline{B}^n \to \overline{B}^n$ be a smooth map such that the restriction $f\colon S^{n-1} \to S^{n-1}$ is a diffeomorphism. Show that $f\colon \overline{B}^n \to \overline{B}^n$ is surjective.
Remark. The case $n = 1$ is equivalent to the Intermediate Value Theorem from real analysis.

Exercise 47.3. For which $n \in \mathbb{N}$ does there exist a continuous map $f\colon S^n \to \mathbb{R}^n$ which has precisely one pair of antipodal points $\{x, -x\}$ with $f(x) = f(-x)$?

Exercise 47.4.

(a) Generalize the $\mathbb{Z}/2\mathbb{Z}$-degree from page 1085 to the more general setting of continuous maps $(M, \partial M) \to (N, \partial N)$ between compact connected smooth manifolds.

(b) Show that the analog of Boundary Degree Proposition 47.9 also holds for $\mathbb{Z}/2\mathbb{Z}$-degrees.

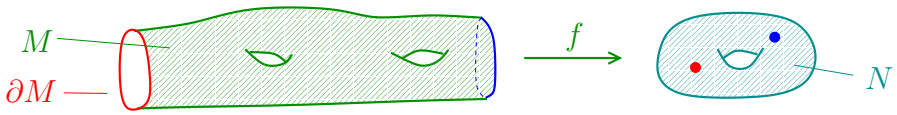
(c) Use (b) to give a new proof of the Boundary Non-Retraction Proposition 38.6.

Exercise 47.5. Let M be a compact connected smooth manifold with non-empty boundary and let N be a closed connected smooth manifold. We assume that M and N have the same dimension. Let $f\colon M \to N$ be a smooth map. Given a regular value $y \in N$ we define (as in Notation 46.8)

$$\deg^y(f) \;:=\; \sum_{x \in f^{-1}(y)} \deg(f, x) \;\in\; \mathbb{Z}.$$

(a) Show that in general the definition of $\deg^y(f)$ depends on the choice of the regular value y.

(b) We suppose that $f\colon M \to N$ is a smooth map which is locally constant on ∂M. Show that in this setting the definition of $\deg^y(f)$ does not depend on the choice of the regular value y.



Exercise 47.6. Let M be a compact oriented connected n-dimensional smooth manifold such that ∂M is connected and non-empty. Show that there exists a continuous map $f\colon M \to \overline{B}^n$ of degree $+1$.

Exercise 47.7. Prove the Borsuk-Ulam Theorem 47.12 for 1-dimensional spheres using methods from real analysis. In other words, show, just using methods from real analysis, that given any continuous map $f\colon S^1 \to \mathbb{R}$ there exists an $x \in S^1$ with $f(-x) = f(x)$.

Exercise 47.8. Let $n \in \mathbb{N}$. Show that the following are equivalent statements:

(a) For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = f(-x)$.
(b) For every odd continuous map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = 0$.
(c) There is no odd continuous map $f\colon S^n \to S^{n-1}$.
(d) There is no continuous map $f\colon \overline{B}^n \to S^{n-1}$ such that the restriction of f to $S^{n-1} \to S^{n-1}$ is odd.

Exercise 47.9. Let $n \in \mathbb{N}$. Show that the Borsuk-Ulam Theorem 47.12 implies the Brouwer Fixed Point Theorem 38.7.
Remark. From Exercise 47.8 you can pick your favorite formulation of the Borsuk-Ulam Theorem 47.12.

Exercise 47.10. Let $n \in \mathbb{N}$.

(a) Let $f\colon S^n \to S^n$ be a continuous map. Show that if the map f factors through the projection $p\colon S^n \to \mathbb{R}\mathrm{P}^n$, then $\deg(f)$ is even.
(b) Let $f\colon S^n \to S^n$ be a continuous map of odd degree. Show that there exists an $x \in S^n$ with $f(-x) = -f(x)$.
Hint. Prove the statement by contradiction and make use of (a).
Remark. This statement can be viewed as a partial converse to the Smooth Odd Map–Odd Degree Proposition 47.14.

Exercise 47.11. Let $n \in \mathbb{N}$.

(a) Show that any odd continuous map $S^n \to S^n$ is homotopic to an odd smooth map $S^n \to S^n$.
(b) Show that any odd continuous map $S^n \to S^n$ has odd degree.
Remark. This exercise generalizes the Smooth Odd Map–Odd Degree Proposition 47.14 to continuous maps.

Exercise 47.12. Let $k \in \mathbb{N}$ and let $l \in \{1, \dots, k-1\}$. Furthermore let $f\colon S^k \to \mathbb{R}^l$ be a continuous map with $f(-x) = -f(x)$ for all $x \in S^k$. Show that there exists an $x \in S^k$ with $f(x) = 0$.

CHAPTER 48

Applications of the Borsuk-Ulam Theorem

In this chapter we will state and prove several particularly interesting consequences of the Borsuk-Ulam Theorem 47.12.

This chapter is organized as follows:

- In Section 48.1 we will state and prove the Ham-Sandwich Theorem 48.4.
- In Section 48.2 we will determine the chromatic number of Kneser graphs.
- In Section 48.3 we will discuss the level of a commutative ring.
- In Section 48.4 we will state and prove the Necklace Theorem 48.17.
- In Section 48.5 we will provide exercises for Chapter 48.

In the next Chapter 49 we will formulate and prove Whitney-Graustein Theorem 49.11.

48.1. **The Ham-Sandwich Theorem.** Before we can state our first application of the Borsuk-Ulam Theorem 47.12 we need to recall the following harmless Definition 47.11.

Definition 48.1. Let $A \subseteq S^n$ be a subset. We say that two points $x, y \in A$ are antipodal if $y = -x$.

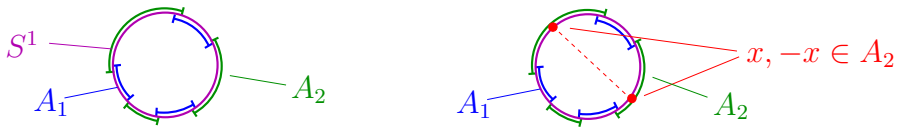
The following application of the Borsuk-Ulam Theorem 47.12 was first proved by the Soviet mathematicians Lazar Lusternik and Lev Schnirelmann [**LuS1930**, p. 26-31] in 1930 using different methods.

Theorem 48.2. (Lusternik-Schnirelmann Theorem) Let $A_1, \ldots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \cdots \cup A_{n+1} = S^n$. Then there exists an $i \in \{1, \ldots, n+1\}$ such that A_i contains a pair of antipodal points.

Remark. In the Generalized Lusternik-Schnirelmann Theorem 48.12 we will prove a slight generalization of the Lusternik-Schnirelmann Theorem 48.2. □

Examples.

(1) For $n = 1$ we obtain from the Lusternik-Schnirelmann Theorem 48.2 that if we have a decomposition $S^1 = A_1 \cup A_2$ where A_1 and A_2 are two closed subsets (which do not need to be connected), then the subset A_1 or the subset A_2 contains a pair of antipodal points x and $-x$. In Exercise 48.1 we will give a naive proof for the special case $n = 1$.



(2) We consider the surface of our earth, we take A_1 to be the closure of the land mass, we take A_2 to be the closure of the Pacific ocean and we take A_3 to be the closure of all seas except for the Pacific. Then the Lusternik-Schnirelmann Theorem 48.2 says that

at least one of A_1, A_2 or A_3 contains a pair of antipodal points. Which of A_1, A_2 and A_3 contains a pair of antipodal points?

(3) The Lusternik-Schnirelmann Theorem 48.2 is optimal in the following sense: in Exercise 48.2 we will see that given any $n \in \mathbb{N}_0$ we can find closed subsets $A_1, \ldots, A_{n+2}$ that cover all of S^n, but such that none of the A_i contains a pair of antipodal points. □

Proof. Let $A_1, \ldots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \cdots \cup A_{n+1} = S^n$. Without loss of generality we can assume that all $A_1, \ldots, A_{n+1}$ are non-empty.[517] Given a non-empty subset A of S^n and $x \in S^n$ we consider, as in Definition 10.3, the distance from x to A:

$$d(x, A) \coloneqq \inf\{\|x - a\| \mid a \in A\},$$

where $\|x - a\|$ denotes the usual Euclidean distance on $S^n \subseteq \mathbb{R}^{n+1}$. By the Distance-to-Subspace Lemma 10.4 together with Exercise 3.3 we know that the following two statements hold:

(a) If A is closed, then $d(x, A) = 0$ if and only if $x \in A$.
(b) The map $x \mapsto d(x, A)$ is continuous.

We consider the map

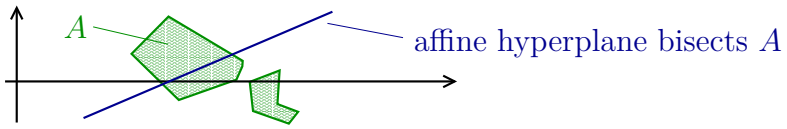
$$\begin{array}{rccl} f\colon & S^n & \to & \mathbb{R}^n \\ & x & \mapsto & (d(x, A_1), \ldots, d(x, A_n)). \end{array}$$

It follows from (b) that this map is indeed continuous. By the Borsuk-Ulam Theorem 47.12 there exists an $x \in S^n$ with $f(x) = f(-x)$. We distinguish three cases:

(1) If there exists an $i \in \{1, \ldots, n\}$ such that $x \in A_i$, then it follows from $f(x) = f(-x)$ that $d(-x, A_i) = d(x, A_i) = 0$. Since A_i is closed it follows from (a) that $-x \in A_i$.
(2) If there exists an $i \in \{1, \ldots, n\}$ such that $-x \in A_i$, then the argument of (1) shows that we also have $x \in A_i$.
(3) If neither x nor $-x$ lies in one of the A_i, then $x, -x$ both lie in A_{n+1}. ■

Before we can state the next consequence of the Borsuk-Ulam Theorem 47.12 we need to introduce one more definition.

Definition 48.3. Let $A \subseteq \mathbb{R}^n$ be a measurable subset[518] of finite volume. We say that an affine hyperplane $H \subseteq \mathbb{R}^n$ bisects A if for the two half-spaces V, V' determined by H we have the equality $\mathrm{vol}(A \cap V) = \mathrm{vol}(A \cap V')$.



We continue with the following pretty theorem.

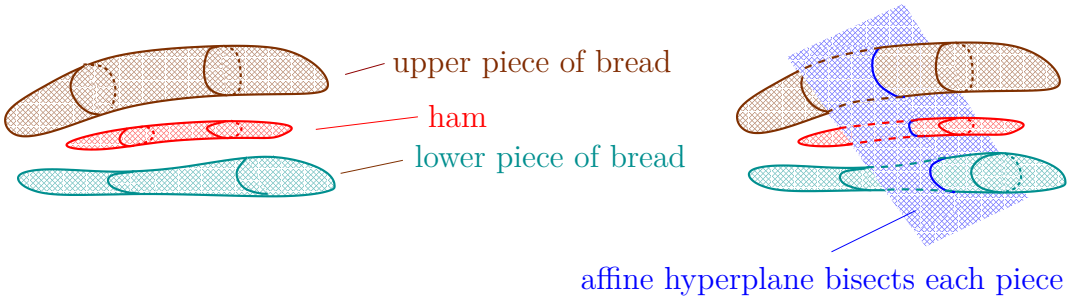
Theorem 48.4. (Ham-Sandwich Theorem) Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. Then there exists an $(n-1)$-dimensional affine hyperplane in $\mathbb{R}^n$ that bisects $A_1, \ldots, A_n$.

[517] Evidently at least one A_i is non-empty. So if one of the A_i is empty we just replace it with one of the non-empty ones.

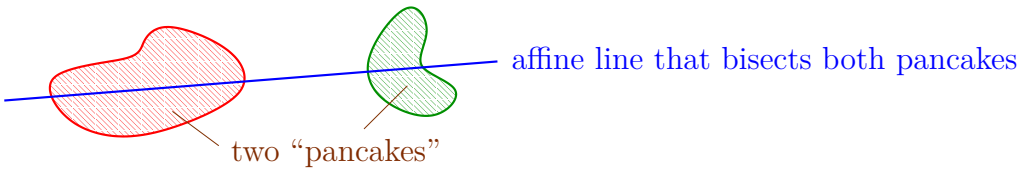
[518] Here "measurable" means Lebesgue-measurable in the sense of say [**Caro2000**] or [**RudiW1987**, Chapter 2].

Remark.

(1) The Ham-Sandwich Theorem 48.4 gets its name from the following "application". Suppose we are given a sandwich with an upper piece of bread A_1, some ham A_2 and a lower piece of bread A_3. Then using a knife one can cut the sandwich, along an affine hyperplane, into two pieces such that each piece of bread and the ham are bisected. This situation is, more or less convincingly, illustrated in the figure below.



(2) The 2-dimensional version of the Ham-Sandwich Theorem is sometimes referred to as the "Pancake Theorem", since one can bisect the two pancakes along a line into two equal sized halves.



(3) The Ham-Sandwich Theorem 48.4 is sometimes also called, rather more prosaically, the Stone-Tukey theorem after the two American mathematicians Arthur Stone and John Tukey [**StT1942**]. □

In the proof of the Ham-Sandwich Theorem 48.4 we will need the following result from measure theory.

Proposition 48.5. (Volume Continuous Proposition) Suppose that for each $t \in \mathbb{R}$ we have a measurable subset X_t of $\mathbb{R}^n$ such that for each $s \leq t \in \mathbb{R}$ we have $X_s \subseteq X_t$. Then $\bigcup_{t\in\mathbb{R}} X_t$ is also measurable and

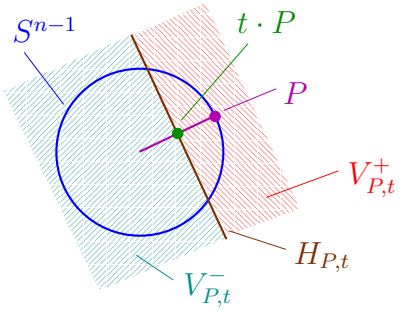
$$\lim_{t\to\infty} \operatorname{vol}(X_t) \;=\; \operatorname{vol}\Big(\bigcup_{t\in\mathbb{R}} X_t\Big).$$

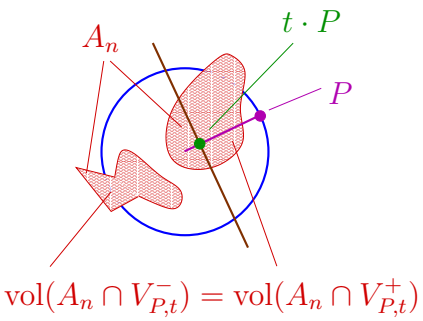
Proof of Proposition 48.5. The proposition can be deduced easily from basic properties of the Lebesgue measure, see [**Caro2000**, Chapter 16]. ■

Proof of the Ham-Sandwich Theorem 48.4. Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. If all A_i have volume zero, then there is nothing to prove. Thus, without loss of generality we can assume that the volume of A_n is positive. Given a point $P \in S^{n-1}$ and $t \in \mathbb{R}$ we write

$$\begin{aligned} H_{P,t} &:= \{t \cdot P + w \mid \langle P, w\rangle = 0\} \\ V^+_{P,t} &:= \{t \cdot P + w \mid \langle P, w\rangle \geq 0\} \\ V^-_{P,t} &:= \{t \cdot P + w \mid \langle P, w\rangle \leq 0\}. \end{aligned}$$

(We refer to the figure below on the left for an illustration.) Put differently, $H_{P,t}$ is the affine hyperplane through $t \cdot P$ orthogonal to P, and $V^{\pm}_{P,t}$ are the two half-spaces determined by $H_{P,t}$.





We make the following two observations:

(i) By definition we have $V^{\pm}_{-P,-t} = V^{\mp}_{P,t}$.
(ii) For a subset $A \subseteq \mathbb{R}^n$ of finite volume we have the equality

$$\mathrm{vol}(A \cap V^{\pm}_{P,t}) \underset{\uparrow}{=} \mathrm{vol}(A \cap (\mathbb{R}^n \setminus V^{\mp}_{P,t})) = \mathrm{vol}(A) - \mathrm{vol}(A \cap V^{\mp}_{P,t}).$$

since $H_{P,t} = V^+_{P,t} \cap V^-_{P,t}$ is a set of measure zero

We continue with the following claim:

Claim. There exists a function

$$t \colon S^{n-1} \to \mathbb{R}$$

with the following three properties:

(1) for every $P \in S^{n-1}$ we have $\mathrm{vol}(A_n \cap V^+_{P,t(P)}) = \frac{1}{2}\mathrm{vol}(A_n)$, i.e. the affine hyperplane $H_{P,t(P)}$ bisects A_n,
(2) for every $P \in S^{n-1}$ we have $t(-P) = -t(P)$,
(3) the function t is continuous.

Proof. We first fix $P \in S^{n-1}$. We consider the function

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}_{\geq 0} \\ t & \mapsto & \mathrm{vol}(A_n \cap V^+_{P,t}). \end{array}$$

This function is continuous[519] and we have

$$\lim_{t \to -\infty} \mathrm{vol}(A_n \cap V^+_{P,t}) \underset{\uparrow}{=} \mathrm{vol}\left(\bigcup_{t \in \mathbb{R}} (A_n \cap V^+_{P,t}) \right) = \mathrm{vol}(A_n).$$

by the Volume Continuous Proposition 48.5

and

$$\lim_{t \to \infty} \mathrm{vol}(A_n \cap V^+_{P,t}) \underset{\uparrow}{=} \mathrm{vol}(A_n) - \underbrace{\lim_{t \to -\infty} \mathrm{vol}(A_n \cap V^-_{P,t})}_{= \mathrm{vol}(A_n) \text{ by the Volume Continuous Proposition 48.5}} = 0.$$

by (ii)

[519] This requires at least a little bit of thought since we do not assume that A is bounded, we only assume it has finite volume. We leave it as a not entirely trivial exercise to prove this statement using basics facts of Lebesgue measures.

We deduce from the Intermediate Value Theorem that

$$T \coloneqq \{t \in \mathbb{R} \mid \operatorname{vol}(A_n \cap V^+_{P,t}) = \tfrac{1}{2}\operatorname{vol}(A_n)\}$$

is a non-empty set.[520] It is straightforward to see that it is a connected subset of $\mathbb{R}$, i.e. it is an interval. Since $\operatorname{vol}(A_n) > 0$ and since $\operatorname{vol}(A_n) < \infty$ we know that T is a finite interval. We define

$$t(P) \coloneqq \tfrac{1}{2}(\inf(T) + \sup(T)).$$

Note that it follows immediately from the definitions that $t(-P) = -t(P)$. Furthermore with some elementary measure theory one can show that the function $S^{n-1} \to \mathbb{R}$ given by $P \mapsto t(P)$ is continuous. We leave the verification of the details to the reader. ⊞

We consider the map

$$\begin{aligned} f\colon S^{n-1} &\to \mathbb{R}^{n-1} \\ P &\mapsto (\operatorname{vol}(A_1 \cap V^+_{P,t(P)}), \dots, \operatorname{vol}(A_{n-1} \cap V^+_{P,t(P)})). \end{aligned}$$

From the continuity of $t\colon S^{n-1} \to \mathbb{R}$ it follows that f itself is continuous. By the Borsuk-Ulam Theorem 47.12, there exists a point $P \in S^{n-1}$ with $f(P) = f(-P)$. We claim that the affine hyperplane $H_{P,t(P)}$ has the desired property.

By definition of $t(P)$ the affine hyperplane $H_{P,t(P)}$ bisects A_n. Furthermore note that for $i = 1, \dots, n-1$ we have

$$\operatorname{vol}(A_i \cap V^+_{P,t(P)}) \underset{\substack{\uparrow \\ \text{since } f(P)=f(-P)}}{=} \operatorname{vol}(A_i \cap V^+_{-P,t(-P)}) \underset{\substack{\uparrow \\ \text{since } t(-P)=-t(P)}}{=} \operatorname{vol}(A_i \cap V^+_{-P,-t(P)}) \underset{\substack{\uparrow \\ \text{since } V^+_{P,s}=V^-_{-P,-s}}}{=} \operatorname{vol}(A_i \cap V^-_{P,t(P)}).$$

This shows that the affine hyperplane $H_{P,t(P)}$ also bisects $A_1, \dots, A_{n-1}$. ■

Remark. The proof for the existence of the affine hyperplane that bisects the given sets $A_1, \dots, A_n$ is non-constructive, in the sense that even with some flexibility in mathematical rigor one cannot turn it into an algorithm that given $A_1, \dots, A_n$ finds the affine hyperplane that bisects all of the subsets. □

48.2. **The chromatic number of Kneser graphs.** In the following we give a definition of an abstract elementary graph.

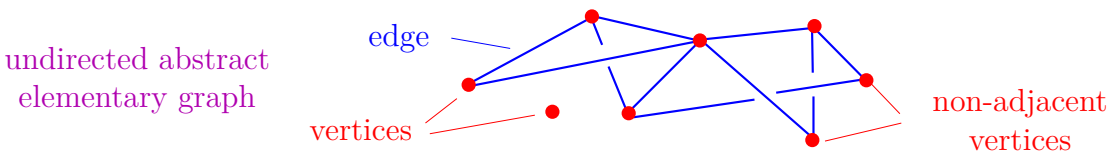
Definition 48.6.

(1) An abstract elementary graph is a pair (V, E) where V is a set, called the vertex set, and a set $E \subseteq \mathcal{P}(V)$, called the edge set, where each $e \in E$ contains precisely two elements of the vertex set.[521]
(2) Elements of the vertex set are called vertices and elements of the edge set are called edges.
(3) Two vertices v and w are called adjacent if $\{v, w\}$ is an edge.

In this section we only care about the following type of abstract elementary graph.

[520] One might think that T consists of just a point, but for example if A_n is disconnected, then it can happen that T is in fact an interval that is not just a point.

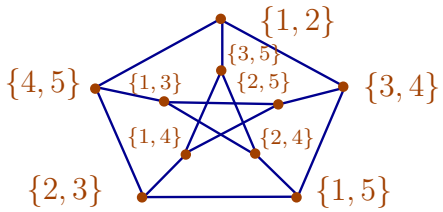
[521] In Definition 11.12 we introduced the notion of an abstract graph. The difference is that in an abstract elementary graph we no longer allow loops, i.e. edges where the two endpoints are the same, and we no longer allow multiple edges between two vertices.



Definition 48.7. Let $n \in \mathbb{N}_0$. The Kneser graph $K(n,k)$ is defined as the abstract elementary graph that is given by the following data:

(1) The vertex set $V(n,k)$ consists of all the k-sets in $\{1,\dots,n\}$. (A k-set is just a subset with precisely k elements.)
(2) We have an edge between $v \neq w \in V(n,k)$ precisely if $v \cap w = \varnothing$.

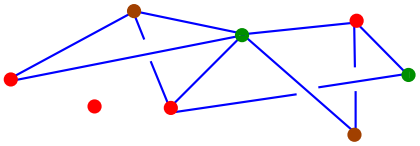
Kneser graph $K(5,2)$



Definition 48.8. Let $G = (V,E)$ be an abstract elementary graph in the above sense such that V is finite.

(1) Let $c \in \mathbb{N}_0$. A c-vertex coloring of $G = (V,E)$ is a map $\varphi\colon V \to \{1,\dots,c\}$ such that adjacent vertices are assigned different values (colloquially called "colors"). Note that $G = (V,E)$ admits a c-coloring if and only if there exists a decomposition $V = V_1 \sqcup \cdots \sqcup V_c$ such that no V_i contains an edge.
(2) The chromatic number $\chi(G)$ of $G = (V,E)$ is the minimal $c \in \mathbb{N}_0$ such that (V,E) admits a c-vertex coloring.

3-coloring

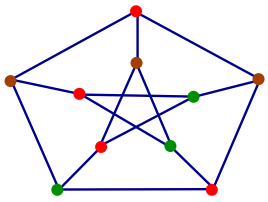


Lemma 48.9. (Kneser Graph–Chromatic Number–Lower Bound Lemma) Let $k,d \in \mathbb{N}_0$. The Kneser graph $K(2k+d,k)$ has chromatic number $\chi(K(2k+d,k)) \leq d+2$.

Proof. As we mentioned above, if we want to show that $\chi(K(n,k)) \leq d+2$ we need to find a decomposition $V(n,k) = V_1 \sqcup \cdots \sqcup V_{d+2}$ such that no V_i contains an edge. In other words, we need to find families $V_1,\dots,V_{d+2}$ of k-sets in $V(n,k)$ with $V(n,k) = V_1 \sqcup \cdots \sqcup V_{d+2}$ such that for every $i \in \{1,\dots,d+2\}$ any two distinct k-sets in V_i have at least one element in common. We do it as follows:

(1) Given $i = 1,\dots,d+1$ we denote by V_i the k-sets in $\{1,\dots,2k+d\}$ such that the minimal element is precisely i.
(2) We denote by V_{d+2} all the remaining k-sets. Note that V_{d+2} are precisely the k-sets that are contained in $\{d+2,\dots,2k+d\}$. Since $\{d+2,\dots,2k+d\}$ has $2k-1$ elements we see that there are no two disjoint k-sets in $\{d+2,\dots,2k+d\}$.

We have thus found the desired decomposition. ∎



3-vertex coloring of the Kneser graph $K(5,2)$:
$V_1 = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\}$
$V_2 = \{\{2,3\},\{2,4\},\{2,5\}\}$
$V_3 = \{\{3,4\},\{3,5\},\{4,5\}\}$

In 1955 Martin Kneser [**KnesM1955**] conjectured that the inequality in the Kneser Graph–Chromatic Number–Lower Bound Lemma 48.9 should actually be an equality:

Conjecture 48.10. (Kneser Conjecture 1955[522]**)** For any choice of $k, d \in \mathbb{N}_0$ the Kneser graph $K(2k+d,k)$ has chromatic number $\chi(K(2k+d,k)) = d+2$.

This purely combinatorial problem defeated all attempts to prove it using purely combinatorial methods. The conjecture was finally proved in 1978 by László Lovász[523] [**Lov1978**] using the Borsuk-Ulam Theorem 47.12.

Theorem 48.11. (Lovász Theorem 1978) For any $k, d \in \mathbb{N}_0$ the Kneser graph $K(2k+d,k)$ has chromatic number $\chi(K(2k+d,k)) = d+2$.

The proof of the Kneser Conjecture 48.10 we provide is a simplified version of the proof by László Lovász. This simplified proof was found by Josh Greene [**Greene2002**] in 2002. The proof is so charming that it made it in to "the book of proofs" [**AZ2014**].

The proof of the Lovász Theorem 48.11 rests on the following generalization of the Lusternik-Schnirelmann Theorem 48.2.

Theorem 48.12. (Generalized Lusternik-Schnirelmann Theorem) Let $A_1, \dots, A_{n+1}$ be subsets of S^n with $A_1 \cup \dots \cup A_{n+1} = S^n$. We assume that each of the first n sets $A_1, \dots, A_n$ is either an open or a closed subset of S^n. Then there exists an $i \in \{1, \dots, n+1\}$ such that A_i contains a pair of antipodal points.

Proof of the Generalized Lusternik-Schnirelmann Theorem 48.12. Let $A_1, \dots, A_{n+1}$ be subsets of S^n with $A_1 \cup \dots \cup A_{n+1} = S^n$. We assume that each of $A_1, \dots, A_n$ is closed or open in S^n. Without loss of generality we can assume that all $A_1, \dots, A_n$ are non-empty.[524] In the following we now assume that none of $A_1, \dots, A_{n+1}$ contains a pair of antipodal points.

Given a non-empty subset A of S^n and $x \in S^n$ we consider again the distance from x to A:

$$d(x,A) \;:=\; \inf\{\|x-a\| \mid a \in A\},$$

where $\|-\|$ denotes the usual Euclidean norm on $S^n \subseteq \mathbb{R}^{n+1}$. By the Distance-to-Subspace Lemma 10.4 together with Exercise 3.3 we know that the following two statements hold:

(a) If A is closed, then $d(x,A) = 0$ if and only if $x \in A$.
(b) The map $x \mapsto d(x,A)$ is continuous.

We consider the map

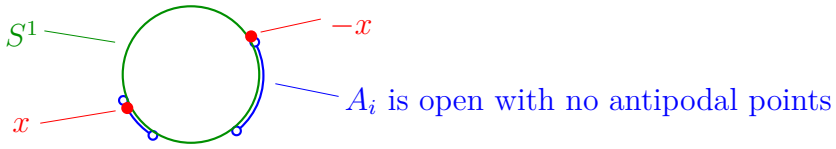
[522]Here the naming gets a bit confusing since in topology there is another Kneser Conjecture, see the Kneser Theorem **??**, which is named after Hellmuth Kneser, the father of Martin Kneser.
[523]László Lovász (*1948) is a Hungarian mathematician. He was awarded the Abels prize in 2021.
[524]If $A_{n+1} = S^n$ then we are done. Otherwise at least one of the $A_1, \dots, A_n$ is non-empty, and if one of A_i is empty we just replace it with one of the non-empty ones among $A_1, \dots, A_n$.

$$\begin{array}{rcl} f\colon S^n & \to & \mathbb{R}^n \\ x & \mapsto & (d(x,A_1),\dots,d(x,A_n)). \end{array}$$

It follows from (b) that this map is indeed continuous. By the Borsuk-Ulam Theorem 47.12 there exists an $x \in S^n$ with $f(x) = f(-x)$. Since A_{n+1} does not contain a pair of antipodal points we see that x or $-x$ is not contained in A_{n+1}. By symmetry we can assume that x is not contained in A_{n+1}. It follows that there exists an $i \in \{1,\dots,n\}$ such that x lies in A_i. We now distinguish the cases that A_i is closed or open.

(1) First we assume that A_i is closed. It follows immediately from $f(x) = f(-x)$ that $d(-x,A_i) = d(x,A_i) = 0$. Since A_i is closed it follows from (a) that $-x$ lies in A_i. Thus both x and $-x$ lie in A_i. We have obtained a contradiction.

(2) Now we assume that A_i is open.



The proof is given by a sequence of little arguments and observations:

(i) It follows from $f(-x) = f(x)$ and $x \in A_i$ that $d(-x,A_i) = d(x,A_i) = 0$.

(ii) It follows from (i) and (a) that $-x$ is contained in the closure $\overline{A}_i$ of A_i.

(iii) Since A_i is open we see that $S^n \setminus (-A_i)$ is a closed subset.

(iv) Recall that we assume that A_i has no pair of antipodal points. It follows that $A_i \cap -A_i = \varnothing$, i.e. A_i is contained in $S^n \setminus (-A_i)$.

(v) It follows immediately from (iii) and (iv), together with Definition 3.21 of the closure, that $\overline{A}_i \subseteq S^n \setminus (-A_i)$.

(vi) It follows from (ii) and (v) that $-x \in S^n \setminus (-A_i)$.

(vii) It follows from (vi) that $-x \notin -A_i$. This implies that $x \notin A_i$. We have obtained the desired contradiction. ■

We will also need the following elementary lemma:

Auxiliary Lemma 48.13. Given any $d \in \mathbb{N}_0$ and $n \in \mathbb{N}$ there exists a subset $X \subseteq S^{d+1}$ with n points that is in general position, in the sense that no $d+2$ points of X lie on a hyperplane through the origin.

Proof. [525] Let $d \in \mathbb{N}_0$. We prove the lemma by induction on $n \in \mathbb{N}_0$. The case $n = 0$ is clear. Now suppose that for some $n \in \mathbb{N}_0$ there exists an n-set in S^{d+1} that is in general position. Let V be the union of all hyperplanes in $\mathbb{R}^{d+2}$ that are formed by $(d+1)$-subsets in X. It follows from the Baire Category Theorem 10.26, or alternatively from the Measure-Zero Properties Proposition 37.3 and the Full Measure-Density Proposition 37.4, that $V \subsetneq S^{d+1}$. If we add any point in $S^{d+1} \setminus V$ to X we obtain an $(n+1)$-set in S^{d+1} that is in general position. ■

Proof of the Kneser Conjecture 48.10 aka Lovász Theorem 48.11. Let $k,d \in \mathbb{N}_0$. By the Kneser Graph–Chromatic Number–Lower Bound Lemma 48.9 it remains to show that $\chi(K(2k+d,k)) \geq d+2$. By Lemma 48.13 there exists a $(2k+d)$-set $X \subseteq S^{d+1}$ in general position. We now think of $V(2k+d,k)$ as the family of all k-sets in X.

[525] The proof is quite similar to the proof of the General Position–Existence Lemma 15.25.

Now let us assume that $\chi(K(2k+d,k)) \leq d+1$. This implies that $K(2k+d,k)$ admits a $(d+1)$-coloring. As we already mentioned in the proof of the Kneser Graph–Chromatic Number–Lower Bound Lemma 48.9, this means that there exists a decomposition

$$V(2k+d,k) \;=\; \{\text{all } k\text{-sets in } X \;=\; V_1 \sqcup \cdots \sqcup V_{d+1}\}$$

with the following property: For any V_i any two k-sets $A, B \in V_i$ have non-empty intersection. Our goal is to reach a contradiction. Thus we need to show that there exist disjoint k-sets A and B of X that belong to the same V_i.

For $x \in S^{d+1}$ we consider the corresponding open hemisphere

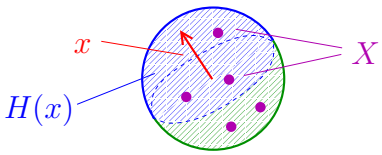
$$H(x) \;:=\; \{y \in S^{d+1} \mid \underbrace{\langle x, y\rangle}_{\text{standard inner product}} > 0\}.$$

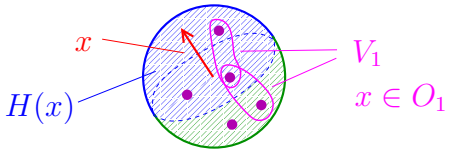
For $i = 1, \ldots, d+1$ we set

$$O_i \;:=\; \{x \in S^{d+1} \mid H(x) \text{ contains a } k\text{-set from } V_i\}.$$

Note that it follows easily from the Subspace-Characterization Lemma 3.17 that each O_i is an open subset of S^{d+1}. Evidently the open subsets $O_1, \ldots, O_{d+1}$ together with the closed subset $C := S^{d+1} \setminus (O_1 \cup \cdots \cup O_{d+1})$ form a cover of S^{d+1}.

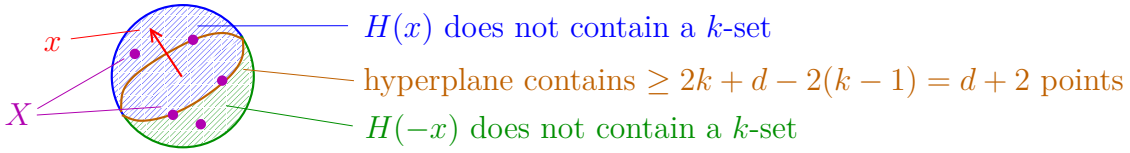
$d = 1$ and $k = 2$





Claim. The subset C does not contain a pair of antipodal points.

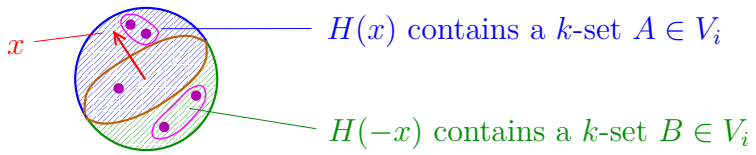
Proof. Suppose that $C = S^{d+1} \setminus (O_1 \cup \cdots \cup O_{d+1})$ contains a pair of antipodal points x and $-x$. Note that each k-set of X lies in one of the V_i. It follows from this observation and the definition of the O_i's that both open hemispheres $H(x)$ and $H(-x)$ contain less than k points. But this means that at least $d+2$ points lie on $S^{d+1} \setminus (H(x) \cup H(-x))$. But $S^{d+1} \setminus (H(x) \cup H(-x))$ is the intersection of S^{d+1} with a hyperplane. Thus we have obtained a contradiction to the fact that the points on X are in general position. ⊞



Since the O_i are open and since by the claim we know that $C = S^{d+1} \setminus (O_1 \cup \cdots \cup O_{d+1})$ does not contain a pair of antipodal points we deduce from the Generalized Lusternik-Schnirelmann Theorem 48.12 that there exists an $i \in \{1, \ldots, d+1\}$ such that O_i contains a pair of antipodal points x and $-x$. We make the following two simple observations:

(1) By definition of $x \in O_i$ there exists a k-set A from V_i with $A \subseteq H(x)$.
(2) By definition of $-x \in O_i$ there exists a k-set B from V_i with $B \subseteq H(-x)$.

In summary we see that there exist k-sets A and B, both of which lie in V_i, with $A \subseteq H(x)$ and $B \subseteq H(-x)$.



Since the open hemispheres $H(x)$ and $H(-x)$ are disjoint we see that A and B are disjoint. ■

48.3. **The level of a commutative ring.** In this section we provide an application of the Borsuk-Ulam Theorem 47.12 to algebra. First we have the following definition:

Definition 48.14. Let R be a commutative ring[526]. The level $s(R)$[527] of R is defined as the smallest number $n \in \mathbb{N}_0$ such that -1 can be written as the sum of n squares in R.[528] If -1 is not the sum of squares in R, then we define $s(R) = \infty$.

Examples.

(1) Evidently $s(\mathbb{C}) = 1$ and $s(\mathbb{R}) = \infty$. In fact a commutative ring R with $s(R) = \infty$ is sometimes called *formally real*.
(2) In Exercise 48.3 we will see that $s(\mathbb{Z}/4\mathbb{Z}) = 3$ and $s(\mathbb{Z}[\sqrt{5}\,\mathrm{i}]) = 2$. □

In 1966 Albrecht Pfister [**Pf1965**] proved the following theorem:

Theorem 48.15. (Pfister's Level Theorem) The level of a field is either ∞ or a power of 2.

Proof. The theorem is of course proved in [**Pf1965**]. There are many accounts of the proof, see e.g. also [**Pf1995**, Theorem 3.1.3] and [**Lam2005**, Theorem XI.2.2]. ■

Albrecht Pfister [**Pf1965**] also showed that any power of 2 appears as the level of a field. In 1977 Manfred Knebusch [**Kneb1977**] asked whether every $n \in \mathbb{N}_0$ is the level of some commutative ring. The following theorem, proved by Dai–Lang–Peng [**DLP1980**] gives an affirmative answer.

Theorem 48.16. (Level Existence Theorem) Given any $n \in \mathbb{N}$ there exists a commutative ring R with $s(R) = n$.

Proof. We consider the ring[529] $R = \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$. Evidently $s(R) \leq n$. We need to show that $s(R) \geq n$. So suppose that $s(R) < n$. This implies that there exist $f_0(x), f_1(x), \ldots, f_{n-1}(x) \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ such that

$$-1 \;=\; f_1(x)^2 + \cdots + f_{n-1}(x)^2 + f_0(x) \cdot (1 + x_1^2 + \cdots + x_n^2) \;\in\; \mathbb{R}[x_1, \ldots, x_n]. \tag{1}$$

Let $j \in \{0, \ldots, n-1\}$. Since $f_j(x)$ is a real polynomial we can write $f_j(\mathrm{i}x) = p_j(x) + \mathrm{i}q_j(x)$ where $p_j(x)$ is an even real polynomial and $q_j(x)$ is an odd real polynomial. Thus if we plug $\mathrm{i}x$ into Equality (1) and if we compare the real parts we obtain the following equality:

$$-1 \;=\; \sum_{j=1}^{n-1} (p_j(x)^2 - q_j(x)^2) + p_0(x) \cdot (1 - x_1^2 - \cdots - x_n^2). \tag{2}$$

[526] We refer to Definition 2.6 for the definition of a ring.
[527] The letter "s" stands for the German word "Stufe", which is the original name of "level".
[528] According to our definition of a ring we have $0 \neq 1$, thus we see that $s(R)$ can never be zero.
[529] Note that standard algebra, see e.g. [**Pf1995**, p. 45], shows that R is actually even a domain.

Next we consider the map $Q := (q_1, \dots, q_{n-1})\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$. Since the polynomials $q_1, \dots, q_{n-1}$ are odd we see that for every $a \in S^{n-1}$ we have $Q(-a) = -Q(a)$. Furthermore, by the Borsuk-Ulam Theorem 47.12 there exists an $a = (a_1, \dots, a_n) \in S^{n-1}$ with $Q(-a) = Q(a)$. Note that it follows from the above symmetry of Q that we have $Q(a) = 0$. Now we see that

$$-1 \underset{\uparrow}{=} \sum_{j=1}^{n-1} (p_j(a)^2 - q_j(a)^2) + p_0(a) \cdot \overbrace{(1 - a_1^2 - \cdots - a_n^2)}^{=0 \text{ since } a \in S^{n-1}} \underset{\uparrow}{=} \sum_{j=1}^{n-1} p_j(a)^2.$$

by Equality (2) — since $Q(a) = 0$ and $a \in S^{n-1}$

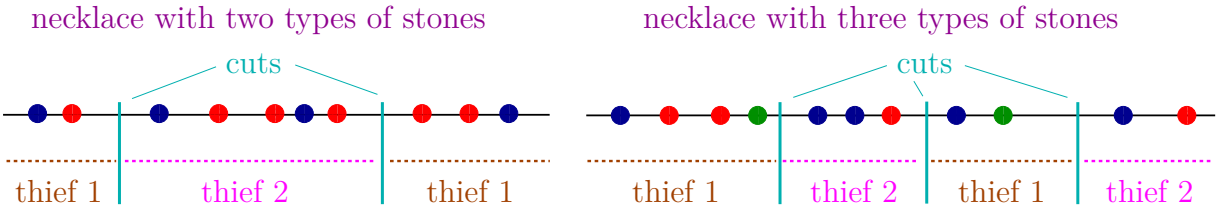
Since the right-hand side is non-negative we have thus obtained a contradiction. ■

Remark.

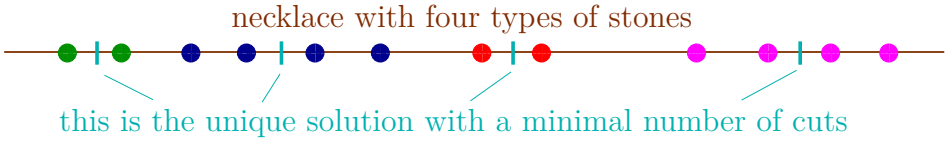
(1) The proof of the Level Existence Theorem 48.16 that we provide is rather ad hoc. One can also introduce a general notion of a level $s(X)$ of a topological space and use it to prove the Level Existence Theorem 48.16. We refer to [**Pf1995**, Chapter 3] for details.
(2) In the proof of the Level Existence Theorem 48.16 we only need to apply the Borsuk-Ulam Theorem 47.12 to maps that are given by polynomials. In this setting a purely algebraic proof of the Borsuk-Ulam Theorem 47.12 was given by Manfred Knebusch [**Kneb1982**]. □

48.4. **The Necklace Theorem.** We conclude this chapter with another, fun consequence of the Borsuk-Ulam Theorem. As in some of the earlier sections this application is again far-removed from what one would usually consider to be the realm of topology.

The setting is the following: Two thieves steal an open necklace with precious stones of k different types. Conveniently enough, of each type of stone there is an even number. The thieves want to cut the string of the open necklace such that they can distribute the resulting pieces in such a way that both thieves obtain the same number of stones of each type. The question is, what is the minimal number of cuts necessary? In the figure below we show two necklaces, one with two types of stones and one with three types of stones. Furthermore we show possible cuts to evenly distribute the stones.



Another open necklace is shown in the figure below. In this particular example the k different types of stones are grouped together. It is pretty clear that one will need at least k cuts to distribute the stones. It turns out that this is the worst case, more precisely, it



turns out that any string of k different types of stones can be distributed evenly using at most k cuts. The following theorem is a formalization of that statement.

Theorem 48.17. (Necklace Theorem) Let $S_1, \dots, S_k$ be disjoint finite subsets of the interval $[0,1)$ such that each subset S_i has an even number of elements. There exists a $d \in \{0, \dots, k\}$ and $0 = z_0 \leq z_1 \leq \cdots \leq z_d \leq z_{d+1} = 1$ such that for each "color" $c \in \{1, \dots, k\}$ we have

$$\underbrace{\sum_{i \text{ even}} \#([z_i, z_{i+1}) \cap S_c)}_{\substack{\text{number of } c\text{-colored pieces} \\ \text{in the even intervals}}} = \underbrace{\sum_{i \text{ odd}} \#([z_i, z_{i+1}) \cap S_c)}_{\substack{\text{number of } c\text{-colored pieces} \\ \text{in the odd intervals}}} .$$

Remark.

(1) The Necklace Theorem 48.17 was first formulated and proved by Charles Goldberg and Douglas West [**GW1985**] in 1985 using topological methods. The proof we give was found shortly afterwards by Noga Alon and Douglas West [**AW1986**].
(2) A topology-free proof of the Necklace Theorem 48.17 was found in 2008 by Frédéric Meunier [**Meu2008**].
(3) The Necklace Theorem 48.17 has been generalized by Noga Alon [**Nog1987**], again using topological methods, to the case of more than two thieves. It seems like there is no known proof for that more general statement that does not rely on topology.
(4) The proofs of the Necklace Theorem 48.17 and of the Continuous Necklace Theorem 48.18 are both nicely animated in the following video:

`https://www.youtube.com/watch?v=yuVqxCSsE7c&t=868s` □

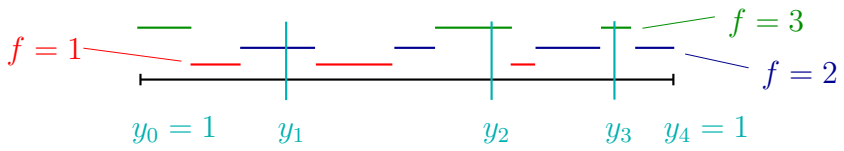
The Necklace Theorem 48.17 is a priori a discrete problem, far removed from the world of topological spaces. In fact the proof consists of two steps, first we will use the Borsuk-Ulam Theorem 47.12 to prove a "continuous analog" of the Necklace Theorem 48.17. Afterwards we will show how we can deduce the "discrete Necklace Theorem" from the "continuous Necklace Theorem". Since we will not make use of any of the results of this section we will feel free to use slightly more informal language than usual.

Without further ado, here is the promised continuous analog of the Necklace Theorem 48.17.

Theorem 48.18. (Continuous Necklace Theorem) Let $f\colon [0,1] \to \{1, \dots, k\}$ be a function such that each preimage $f^{-1}(c)$ is a measurable set (e.g. a union of finitely many intervals). There exists a $d \in \{0, \dots, k\}$ and $0 = y_0 < y_1 < \cdots < y_d < y_{d+1} = 1$ such that for each "color" $c \in \{1, \dots, k\}$ we have

$$\underbrace{\sum_{i \text{ even}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(c)\big)}_{\substack{\text{total length of } c\text{-colored} \\ \text{segments in the even intervals}}} = \underbrace{\sum_{i \text{ odd}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(c)\big)}_{\substack{\text{total length of } c\text{-colored} \\ \text{segments in the odd intervals}}}.$$

The same statement also holds for functions f which are defined on $[0,1)$.



Proof. We adopt the following notation: Given $(w_0, \dots, w_k) \in S^k$ we set $x_0 := 0$ and for $i = 1, \dots, k+1$ we write $x_i := w_0^2 + \cdots + w_{i-1}^2$. We make the following observation which is the key to the subsequent proof:

> Let $(w_0, \dots, w_k) \in S^k$. We note that $x_0 = 0 \le x_1 \le \cdots \le x_k \le x_{k+1} = 1$. Thus any point in S^k corresponds to a splitting of the interval $[0, 1)$ into $k+1$ (possibly empty) intervals. In fact, if we remember the signs of the w_i, then we obtain a way to assign each interval to one of the two thieves, say we award $[x_i, x_{i+1})$ to thief 1 if $w_i < 0$ and to thief 2 if $w_i > 0$. Thus the set of splittings of the interval corresponds to the topological space S^k.

For $c = 1, \dots, k$ we consider the following map:

$$\begin{aligned} g_c \colon S^k &\to \mathbb{R} \\ (w_0, \dots, w_k) &\mapsto \sum_{i \text{ with } w_i > 0} \operatorname{vol}\big(\big[\underbrace{x_i, x_{i+1}}_{\substack{\uparrow \\ \text{here we use the above} \\ \text{definition of } x_0, \dots, x_{k+1} \\ \text{as determined by } w_0, \dots, w_k}}\big) \cap f^{-1}(c)\big). \end{aligned}$$

We leave it to the meticulous reader to show, using standard properties of the Lebesgue-measure, that the map g_c is actually continuous.[530]

Next we combine all these maps and we obtain a map $g = (g_1, \dots, g_k) \colon S^k \to \mathbb{R}^k$. Note that it follows from the Borsuk-Ulam Theorem 47.12 that there exists a $(w_0, \dots, w_k) \in S^k$ with $g(-w_0, \dots, -w_k) = g(w_0, \dots, w_k)$. This just means that for each $c \in \{1, \dots, k\}$ we have

$$\underbrace{\sum_{i \text{ with } w_i < 0} \operatorname{vol}\big([x_i, x_{i+1}) \cap f^{-1}(c)\big)}_{= g_c(-w_0, \dots, -w_k)} = \underbrace{\sum_{i \text{ with } w_i > 0} \operatorname{vol}\big([x_i, x_{i+1}) \cap f^{-1}(c)\big)}_{= g_c(w_0, \dots, w_k)}.$$

Finally, we consider the numbers $x_0 \le x_1 \le \cdots \le x_{k+1} = 1$. Now we just need to combine adjacent intervals for which the w_i's have the same sign. More precisely, iteratively we remove each x_i for which $w_i \ge 0$ and $w_{i+1} \ge 0$ and we remove each x_i for which $w_i \le 0$ and $w_{i+1} \le 0$. It is now fairly elementary to verify that the remaining numbers, let us call them $0 = y_0 < y_1 < \cdots < y_d < y_{d+1} = 1$, have the desired properties. ∎

Proof of the Necklace Theorem 48.17. Let $S_1, \dots, S_k$ be disjoint finite subsets of the interval $[0, 1]$ such that each S_i has an even number of elements. Evidently we can assume that $S_1 \cup \cdots \cup S_k \neq \varnothing$. We set $s := \#(S_1 \cup \cdots \cup S_k)$. It is straightforward to see that without loss of generality we can assume that the points break $[0, 1]$ into intervals of the same lengths $\frac{1}{s}$. More precisely, we can assume that $\bigcup_{i=1}^{k} S_i = \{\frac{i}{s} \mid i = 0, \dots, s-1\}$. Now let $f \colon [0, 1) \to \{1, \dots, k\}$ be the function which is uniquely determined by the property

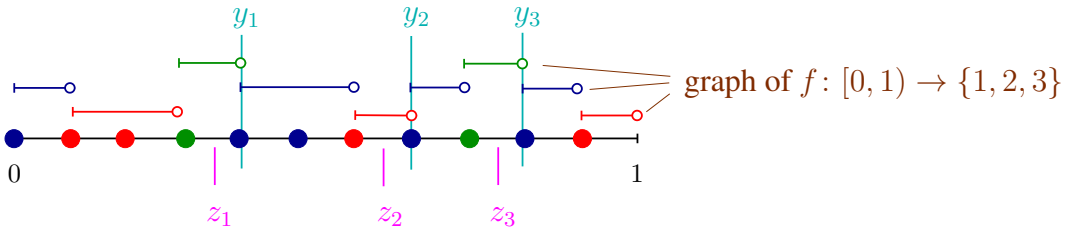
[530] The fact that we take the sum over all i with $w_i > 0$ might make the reader slightly nervous. But note that for $w_i = 0$ we have $x_{i+1} = x_i$, thus $\operatorname{vol}([x_i, x_{i+1}) \cap f^{-1}(c)) = 0$.

that for each $c \in S_i$ we have $f(x) = i$ for $x \in [c, c + \frac{1}{s})$. By the Continuous Necklace Theorem 48.18 there exists a $d \in \{0, \dots, k\}$ and $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ such that for each $c \in \{1, \dots, k\}$ we have

$$(*) \qquad \sum_{i \text{ even}} \operatorname{vol}([y_i, y_{i+1}) \cap f^{-1}(c)) \;=\; \sum_{i \text{ odd}} \operatorname{vol}([y_i, y_{i+1}) \cap f^{-1}(c)).$$

If each of the y_i lies in the "lattice" $L := \{\frac{i}{s} \mid i = 0, \dots, s\}$, then we can set $z_i := y_i - \frac{1}{2s}$, $i = 1, \dots, d$ and we are done.

necklace with three types of stones



Thus it remains to prove the following claim:

Claim. We can find $d \in \{0, \dots, k\}$ and $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ which satisfy $(*)$ and such that each y_i lies in $L = \{\frac{i}{s} \mid i = 0, \dots, s\}$.

Proof. Given a sequence $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ we do the following:

(1) We set $v(y_0 \leq y_1 \leq \cdots \leq y_{d+1}) := \#\{i \in \{0, \dots, d+1\} \mid y_i \notin L\}$.
(2) We consider

$$\text{complexity}(y_0 \leq y_1 \leq \cdots \leq y_{d+1}) \;:=\; (d, v(y_0 \leq y_1 \leq \cdots \leq y_{d+1})) \;\in \mathbb{N}_0 \times \mathbb{N}_0.$$

We equip $\mathbb{N}_0 \times \mathbb{N}_0$ with the lexicographic ordering.

We need to show that there exists a $d \in \{0, \dots, k\}$ and $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ which satisfy $(*)$ and such that $v = 0$. By the above discussion we know that there exists a $d \in \{0, \dots, k\}$ and a sequence $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ which satisfy $(*)$.

It suffices to show that whenever we have a sequence $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ which satisfies $(*)$ and with $v(y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1}) > 0$, then there exists a sequence which satisfies $(*)$ of lower complexity.

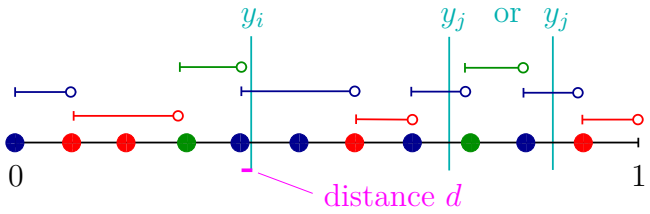
So now suppose that we are given a sequence $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ which satisfies $(*)$ and with $v > 0$. Since we assume that $v(y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1}) > 0$ we know that there exists a y_i that does not lie in L. We perform the following three steps:

(1) Among all y_i which do not lie in L we pick a y_i with minimal distance to a point in L on the left.
(2) We set $d := y_i - \lfloor y_i \rfloor$, where $\lfloor y_i \rfloor$ is the closed point in L to the left of y_i.
(3) It follows fairly easily from $(*)$ and the hypothesis that S_c has an even number of elements that there exists a $j \neq i$ such that $f(y_j) = f(y_i)$ and such that $y_j \notin L$. We distinguish two cases:
   (i) The parity of j is the opposite of the parity of i.
   - If $y_j > y_{j-1} + d$ or if $i = j - 1$, then we replace y_i by $\lfloor y_i \rfloor = y_i - d$ and we replace y_j by $y_j - d$.

- If $y_j \leq y_{j-1} + d$, then we replace y_i by $y_i - (y_j - y_{j-1})$ and we eliminate y_{j-1} and y_j from the sequence.

(ii) The parity of j is the same as the parity of i.

- If $y_j + d < y_{j+1}$, then we replace y_i by $\lfloor y_i \rfloor = y_i - d$ and we replace y_j by $y_j + d$.
- If $y_j + d \geq y_{j+1}$, then we replace y_i by $y_i - (y_{j+1} - y_j)$ and we eliminate y_j and y_{j+1} from the sequence.



One can easily verify that these new numbers still have all the required properties, except that now either y_i also lies in L or the number of y_i's was reduced by two. Thus the complexity went down. ■

48.5. Exercises for Chapter 48.

Exercise 48.1. Give a proof of the Lusternik-Schnirelmann Theorem 48.2 for $n = 1$ just using methods from real analysis. More precisely, let A, B be two closed subsets of S^1 with $A \cup B = S^1$. Show, just using methods from your real analysis course, that A or B contains a pair of antipodal points.

Exercise 48.2. Let $n \in \mathbb{N}$ and let $P_1, \ldots, P_{n+1}$ be $n+1$ points in $\mathbb{R}^n$ that are in general position, i.e. that are not contained in a hyperplane of $\mathbb{R}^n$. We consider the corresponding "simplex"

$$S(P_1, \ldots, P_{n+1}) \;:=\; \Big\{ \sum_{i=1}^{n+1} t_i \cdot P_i \;\Big|\; \sum_{i=1}^{n+1} t_i = 1 \text{ and } t_1, \ldots, t_{n+1} \in \mathbb{R}_{\geq 0} \Big\}.$$

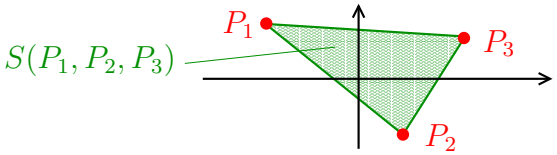
For $i = 1, \ldots, n+1$ we denote by F_i the "i-th face" of $S(P_1, \ldots, P_{n+1})$, i.e. the subspace that is given by setting $t_i = 0$.

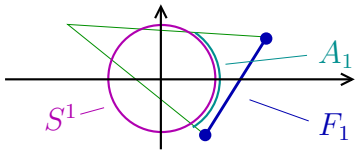
(1) Show that $\underbrace{\partial S(P_1, \ldots, P_{n+1})}_{\text{subspace of } \mathbb{R}^n} = \bigcup_{i=1}^{n+1} F_i$.

(2) Now assume that we have picked $P_1, \ldots, P_{n+1}$ such that the origin 0 lies in the interior of $S(P_1, \ldots, P_{n+1})$. For $i = 1, \ldots, n+1$ we set

$$A_i \;:=\; S^{n-1} \cap \{\lambda \cdot v \,|\, v \in F_i \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}.$$

Show that each A_i is closed, show that $S^{n-1} = A_1 \cup \cdots \cup A_{n+1}$ and show that no A_i contains a pair of antipodal points.





Exercise 48.3. Given a commutative ring R we denote by $s(R)$ the level of R as introduced in Definition 48.14.

(a) Show that $s(\mathbb{Z}/4\mathbb{Z}) = 3$.
(b) Show that $s(\mathbb{Z}[\sqrt{5}\mathrm{i}]) = 2$.

Exercise 48.4. We consider the string of jewels shown in the figure below. Use four cuts to distribute the jewels evenly.



Exercise 48.5. We consider again the setting of the Continuous Necklace Theorem 48.18. More precisely we are given a map $f\colon [0,1] \to \{1,\dots,k\}$ such that each preimage $f^{-1}(c)$ is a measurable set. For each "color" $c \in \{1,\dots,k\}$ let $y_c \in [0,1]$ be such that we have the equality $\operatorname{vol}\big(f^{-1}(c) \cap [0,y_c]\big) = \operatorname{vol}\big(f^{-1}(c) \cap [y_c,1]\big)$. Do these points $y_1,\dots,y_k$, after ordering them, necessarily satisfy the conclusion of the Continuous Necklace Theorem 48.18?

CHAPTER 49

Regular homotopies and the Whitney-Graustein Theorem

In this chapter we will take a short detour from explaining the key concepts of differential topology. Namely we will introduce and study the concept of regular homotopies. We will use the notion of the degree of a continuous self-map of S^1 to prove the Whitney-Graustein Theorem 49.11 which gives a complete classification of immersions of S^1 into $\mathbb{R}^2$ up to regular homotopies. We will not make use of the results of this chapter.

This chapter is organized as follows:

- In Section 49.1 we will introduce the notion of a regular homotopies between two immersions. Furthermore we will state the Unit Speed Proposition 49.6 and the Whitney-Graustein Theorem 49.11.
- In Section 49.2 we will provide a proof of the Unit Speed Proposition 49.6.
- In Section 49.3 we will conclude the proof of the Whitney-Graustein Theorem 49.11.
- In Section 49.4 we will provide exercises for Chapter 49.

In the next Chapter 50 we will state and prove the Transversality Theorem 50.9.

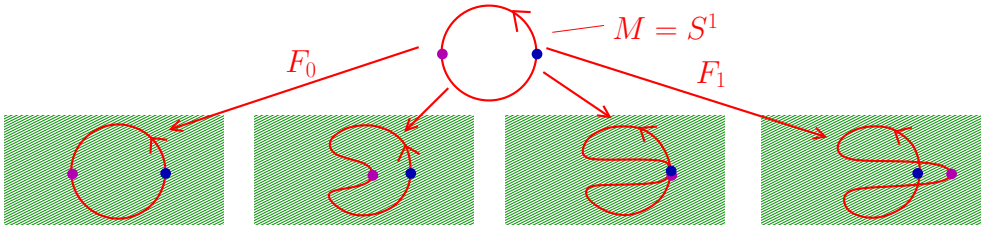
49.1. **Regular homotopies.** In Definition 30.9 we introduced the notion of an immersion. A smooth homotopy of immersions is called a regular homotopy. More precisely, we have the following definition:

Definition 49.1. Let M and N be smooth manifolds. We assume that $\partial N = \varnothing$.

(1) A map $f\colon M \to N$ is called an immersion if f is smooth and if for each $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is a monomorphism.

(2) A regular homotopy is a smooth map $F\colon M \times [0,1] \to N$ such that each $F_t\colon M \to N$ is an immersion.

(3) We say that two immersions $f_0, f_1\colon M \to N$ are regularly homotopic if there exists a regular homotopy $F\colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.



regular homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^2$

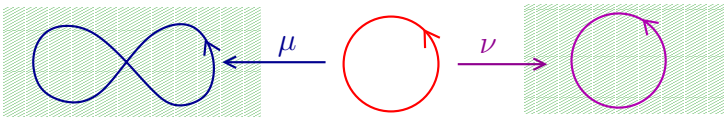
Lemma 49.2. (Regular Homotopy–Equivalence Lemma) Let M and N be smooth manifolds. We assume that $\partial N = \varnothing$. Being regularly homotopic is an equivalence relation on the set of immersions $M \to N$.

Proof. The proof of the lemma is basically identical to the proof of the Smooth Isotopy-Equivalence Relation Proposition 41.5. ∎

In the following we restrict ourselves to immersions $S^1 \to \mathbb{R}^n$. First note that any two smooth maps $\gamma, \delta\colon S^1 \to \mathbb{R}^n$ (in particular any two immersions) are smoothly homotopic. For example such a smooth homotopy is given by the map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & \mathbb{R}^n \\ (z,t) & \mapsto & \gamma(z)\cdot t + \delta(z)\cdot(1-t). \end{array}$$

In Exercise 49.2 we will see that if $n \geq 3$, then any two immersions $S^1 \to \mathbb{R}^n$ are regularly homotopic. This raises the natural question, whether any two immersions $S^1 \to \mathbb{R}^2$ are also regularly homotopic.



are these two immersions regularly homotopic?

In the following we will address this question. Throughout the following discussion we will adopt the following convention and notation.

Convention 49.3. We will use the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24 as an excuse not to distinguish in the notation between the physical tangent spaces and the abstract tangent spaces of S^1 and $\mathbb{R}^n$. Furthermore we will use the Differential-equals-Differential Lemma 29.6 to go back and forth freely between the different notions of differentials.

Definition 49.4.

(1) Given a smooth map $f\colon \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$ we write, following the discussion on page 651, $f'(t) := \mathrm{D}\, f_t \cdot 1 \in \mathbb{R}^n$.[531]

(2) Let $r \in \mathbb{R}$ and let X be a topological space. We say that a map $f\colon \mathbb{R} \to X$ is r-periodic if for every $x \in \mathbb{R}$ we have $f(x+r) = f(x)$.

(3) We consider the 2π-periodic map
$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ s & \mapsto & \exp(\mathrm{i}s). \end{array}$$

(4) Let $z \in S^1$. We write
$$v_z := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z = \mathrm{i}\cdot z.$$
($=$ under the usual identification $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$)

We make the following two observations:

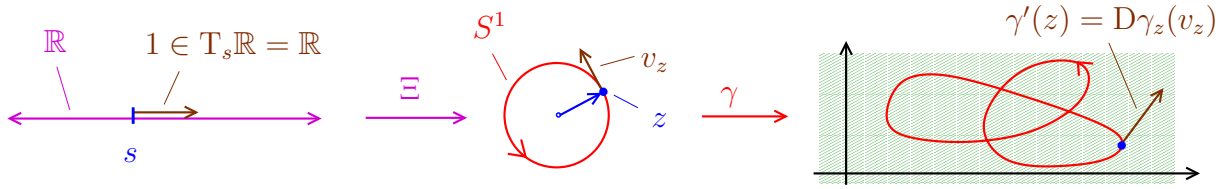
(a) The vector v_z is the unique vector of length one in $\mathrm{T}_z S^1$ that defines a positive basis of $\mathrm{T}_z S^1$ with respect to the standard orientation of S^1 that we introduced on page 817.

(b) For any $s \in \mathbb{R}$ we have $\mathrm{D}\Xi_s(1) = v_{\Xi(s)}$.

(5) Given a smooth map $\gamma\colon S^1 \to \mathbb{R}^n$ and given $z \in S^1$ we define
$$\gamma'(z) := \mathrm{D}\gamma_z(v_z) \in \mathrm{T}_z S^1,$$

where $D\gamma_z\colon T_z S^1_z \to T_{\gamma(z)}\mathbb{R}^n = \mathbb{R}^n$ denotes the differential of γ at z.

(6) We say that a smooth map $\gamma\colon S^1 \to \mathbb{R}^n$ has unit speed if $\|\gamma'(z)\| = 1$ for all $z \in S^1$.



When studying maps $\gamma\colon S^1 \to X$ it is often convenient to replace such a map by the map $\gamma \circ \Xi\colon \mathbb{R} \to X$, since $\mathbb{R}$ is a more familiar and much easier domain to work with. The following lemma summarizes in our context the relationship between the maps γ and $\gamma \circ \Xi$.

Lemma 49.5. ($\mathbb{R}$–S^1-Lemma)

(1) Let $\gamma\colon S^1 \to \mathbb{R}^n$ be a continuous map.
  (a) The map $\gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^n$ is 2π-periodic.
  (b) The map $\gamma\colon S^1 \to \mathbb{R}^n$ is smooth if and only if $\gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^n$ is smooth.

  We now assume that γ is smooth. Then the following statements hold:
  (c) For any $t \in \mathbb{R}$ we have $\gamma'(\Xi(t)) = (\gamma \circ \Xi)'(t)$.
  (d) The map $\gamma\colon S^1 \to \mathbb{R}^n$ is an immersion if and only if $(\gamma \circ \Xi)'(t) \neq 0$ for all $t \in \mathbb{R}$.

(2) (a) If $\widetilde{\rho}\colon \mathbb{R} \to \mathbb{R}^n$ is a (smooth) 2π-periodic map, then there exists a unique (smooth) map $\rho\colon S^1 \to \mathbb{R}^n$ with $\widetilde{\rho} = \rho \circ \Xi$.
  (b) If $\widetilde{F}\colon \mathbb{R} \times [0,1] \to \mathbb{R}^n$ is a smooth homotopy such that each $\widetilde{F}_t$ is a 2π-periodic map, then there exists a unique smooth homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^n$ with $\widetilde{F} = F \circ (\Xi \times \mathrm{id}_{[0,1]})$.

Proof.

(1) (a) This statement is clear.
  (b) This statement follows from the Smooth Pasting Proposition 25.19 (3) since Ξ is a local diffeomorphism.
  (c) Given $t \in \mathbb{R}$ we have

$$\gamma'(\Xi(t)) \underset{\uparrow}{=} (D\gamma_{\Xi(t)})(v_{\Xi(t)}) \underset{\uparrow}{=} \overbrace{D(\gamma \circ \Xi)_t}^{\text{differential}}(1) \underset{\uparrow}{=} \overbrace{D(\gamma \circ \Xi)_t}^{(2\times 1)\text{-matrix}} \cdot 1 \underset{\uparrow}{=} (\gamma \circ \Xi)'(t).$$

by definition and since $v_{\Xi(t)} = (D\Xi_t)(1)$ — Tangent Space-Functor Proposition 29.15 (2b) — Lemma 29.6 — see Definition 49.4

  (d) This statement is an immediate consequence of (1c).

(2) (a) By the Smooth Map-Descends-to-Quotient Corollary 25.41 we know that $\widetilde{\rho}$ descends to a smooth map $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^n$. The statement now follows from the fact, which we basically proved in the Torus-as-Quotient Lemma 25.42, that the following map is a diffeomorphism:

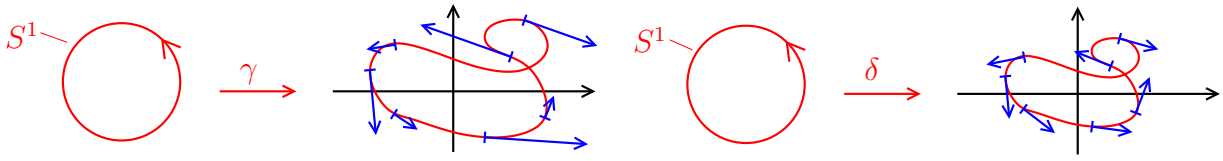
$$\begin{array}{ccc} \mathbb{R}/2\pi\mathbb{Z} & \to & S^1 \\ [t] & \mapsto & \Xi(t) \end{array}$$

  (b) The proof of Statement (b) is basically the same as the proof of Statement (a). ∎

We start out our discussions of immersions of S^1 into $\mathbb{R}^n$ with the following proposition:

[531] Here $1 \in T_t\mathbb{R} = \mathbb{R}$.

Proposition 49.6. (Unit Speed Proposition) Every immersion $\gamma\colon S^1 \to \mathbb{R}^n$ is regularly homotopic to an immersion $\delta\colon S^1 \to \mathbb{R}^n$ of unit speed.



Proof. The proposition is proved via a suitable reparametrization and rescaling of the immersion γ. We postpone the full proof of the Unit Speed Proposition 49.6, with all the technical details, to Section 49.2. ∎

Lemma 49.7. (Normalized Derivative Lemma) Given any immersion $\gamma\colon S^1 \to \mathbb{R}^2$ the following map is smooth:

$$\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & \dfrac{\gamma'(z)}{\|\gamma'(z)\|} \end{array}$$

Proof. Given $z \in S^1$ we set $g(z) \coloneqq \frac{\gamma'(z)}{\|\gamma'(z)\|}$. We need to show that the map $g\colon S^1 \to S^1$ is smooth. As usual let $\Xi\colon \mathbb{R} \to S^1$ be the map that is given by $\Xi(s) = \exp(\mathrm{i}s)$. By the $\mathbb{R}$–S^1-Lemma 49.5 (1b) it suffices to show that $g \circ \Xi\colon \mathbb{R} \to S^1$ is smooth. In fact it suffices to show that the map $s \mapsto \gamma'(\Xi(s))$ is smooth. Note that for any $s \in \mathbb{R}$ we have

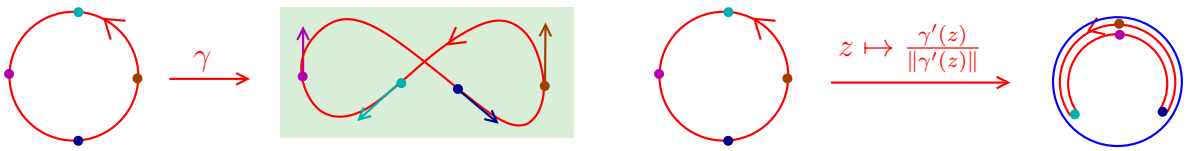
$$\gamma'(\Xi(s)) \overset{\text{by the } \mathbb{R}\text{–}S^1\text{-Lemma 49.5 (1c)}}{\overset{\downarrow}{=}} (\gamma \circ \Xi)'(s) \overset{\text{by Definition 49.4}}{\overset{\downarrow}{=}} \underbrace{\mathrm{D}(\gamma \circ \Xi)_s}_{(2\times 1)\text{-matrix}} \cdot 1.$$

Since γ and Ξ are smooth we know that $\gamma \circ \Xi\colon \mathbb{R} \to \mathbb{R}^2$ is smooth, which implies, by definition, that the entries of the differential $\mathrm{D}(\gamma \circ \Xi)_s \in \mathrm{M}(2 \times 1, \mathbb{R})$ are smooth in s. This implies the desired statement. ∎

The Normalized Derivative Lemma 49.7 allows us to introduce the following definition:

Definition 49.8. Given an immersion $\gamma\colon S^1 \to \mathbb{R}^2$ we define its rotation number as follows:[532] [533]

$$\rho(\gamma) \coloneqq \deg \begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & \dfrac{\gamma'(z)}{\|\gamma'(z)\|} \end{pmatrix}.$$



immersion γ

the corresponding map $z \mapsto \dfrac{\gamma'(z)}{\|\gamma'(z)\|}$ is null-homotopic, hence $\rho(\gamma) = 0$

[532] Since γ is an immersion we know that for every $z \in S^1$ we have $\gamma'(z) \neq 0$. This implies that it actually makes sense to normalize $\gamma'(z)$ to length one.

[533] The map $z \mapsto \frac{\gamma'(z)}{\|\gamma'(z)\|}$ is closely related to the Gauß map that we will introduce in Definition 53.31.

Remark. The rotation number of an immersion $\gamma\colon S^1 \to \mathbb{R}^2$ is sometimes also called the *turning number*, see [**BG1988**, Chapter 9.4] or *rotation index*, see [**doC2016**, p. 399] and [**MoR2009**, p. 329]. □

Lemma 49.9. (Rotation Number-of-Power Lemma) Let $n \in \mathbb{Z}\setminus\{0\}$. With the usual identification $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ we have

$$\rho\begin{pmatrix} S^1 & \to & \mathbb{C} \\ z & \mapsto & z^n \end{pmatrix} = n.$$

Proof. Let $n \in \mathbb{Z} \setminus \{0\}$. We denote by $\theta\colon S^1 \to \mathbb{C}$ the map that is given by $z \mapsto z^n$. Furthermore, very carefully, we denote by $\Theta\colon \mathbb{C} \to \mathbb{C}$ the map that is given by $z \mapsto z^n$. Note that

$$\theta'(z) \overset{\text{definition of } \theta'(z)}{\downarrow =} \mathrm{D}\theta_z(v_z) \overset{\text{since } \Theta|_{S^1} = \theta}{\downarrow =} \underbrace{\mathrm{D}\Theta_z}_{\text{Differential}}(v_z) \overset{\text{by the Differential-equals-Differential Lemma 29.6}}{\downarrow =} \underbrace{\mathrm{D}\Theta_z}_{2\times 2\text{-matrix}} \cdot v_z$$

$$\overset{\text{by viewing } \Theta \text{ as a holomorphic map, see the Smooth-Complex Differentiability Proposition 39.5, and using the definition of } v_z}{\uparrow =} \Theta'(z)\cdot \mathrm{i}z = n\cdot z^{n-1}\cdot \mathrm{i}z = n\cdot z^n\cdot \mathrm{i} = n\cdot z^n\cdot \mathrm{i}.$$

Thus we see that

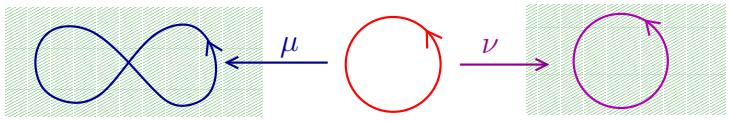
$$\rho(\theta) \overset{\text{by the above, since } |n\cdot z^n\cdot \mathrm{i}| = n}{\uparrow =} \deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & \mathrm{i}\cdot z^n \end{pmatrix} \overset{\text{same calculation as in the Degree Examples Lemma 46.13}}{\uparrow =} n.$$

■

The following proposition explains why we are interested in the rotation number.

Proposition 49.10. (Rotation Number–Regular Homotopy Proposition) Suppose we are given two immersions $\alpha, \beta\colon S^1 \to \mathbb{R}^2$. If α and β are regularly homotopic, then the rotation numbers agree.

Example. We return to the two immersions μ and ν that we encountered in the figure on page 1127:



the two immersions have different rotation numbers, thus they are not regularly homotopic

It follows from the figure on page 1129 and the Rotation Number-of-Power Lemma 49.9 that the rotation numbers are 0 and 1. It follows from the Rotation Number–Regular Homotopy Proposition 49.10 that these two immersions are not regularly homotopic. □

Proof. Let $F\colon S^1 \times [0,1] \to S^1$ be a regular homotopy with $F_0 = \alpha$ and $F_1 = \beta$. It follows immediately from the Degree Homotopy Invariance Theorem 46.6, together with the definition of the winding number, that it suffices to prove the following claim:

Claim. The map

$$\begin{array}{rcl} G\colon S^1 \times [0,1] & \to & S^1 \\ (z,t) & \mapsto & \dfrac{F_t'(z)}{\|F_t'(z)\|} \end{array}$$

is a smooth map, i.e. it is a smooth homotopy.

Proof. Not surprisingly the proof of the claim shares many features with the proof of the Normalized Derivative Lemma 49.7. As before we consider the map $\Xi\colon \mathbb{R} \to S^1$ that is given by $s \mapsto \exp(is)$. Note that $\Xi \times \mathrm{id}_{[0,1]}\colon \mathbb{R}\times[0,1] \to S^1 \times [0,1]$ is a local diffeomorphism. Thus it follows from the Smooth Pasting Proposition 25.19 (3) that it suffices to show that $G \circ \Xi\colon \mathbb{R} \times [0,1] \to S^1$ is smooth. In fact it suffices to show that the map

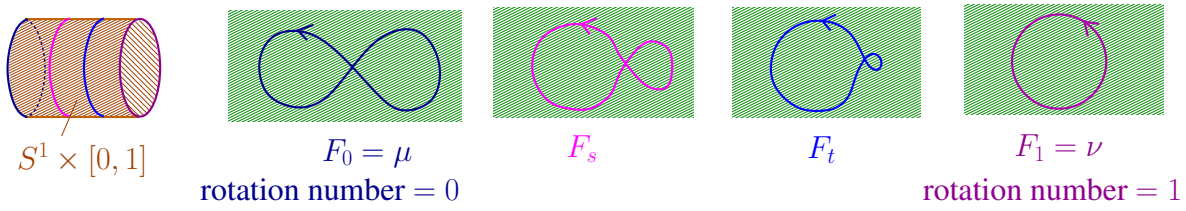
$$\begin{array}{rcl} \mathbb{R} \times [0,1] & \to & \mathbb{R}^2 \setminus \{(0,0)\} \\ (s,t) & \mapsto & F_t'(\Xi(s)) \end{array}$$

is smooth. Note that it follows from the $\mathbb{R}$–S^1-Lemma 49.5 that for any $s \in \mathbb{R}$ and $t \in [0,1]$ we have the following equalities:

$$\begin{array}{ccccc} & & \text{since } F_t \circ \Xi = (F \circ (\Xi \times \mathrm{id}_{[0,1]}))_t & & \\ & & \text{as maps } \mathbb{R} \to \mathbb{R}^2 & & \\ \text{by the } \mathbb{R}\text{–}S^1\text{-Lemma 49.5 (1c)} & & & & \\ F_t'(\Xi(s)) & \overset{\downarrow}{=} & \mathrm{D}(F_t \circ \Xi)_s \cdot 1 & \overset{\downarrow}{=} & \mathrm{D}((F \circ (\Xi \times \mathrm{id}_{[0,1]}))_t)_s \cdot 1 \\ & & & = & \text{vector in } \mathbb{R}^2 \text{ that is given by the two partial derivatives of} \\ & & & \uparrow & F \circ (\Xi \times \mathrm{id}_{[0,1]}) \text{ at } (s,t) \text{ with respect to the first coordinate.} \\ & & & \text{by definition of} & \\ & & & \text{the differential} & \end{array}$$

Since $F\colon S^1 \times [0,1] \to S^1$ is smooth and since $\Xi \times \mathrm{id}_{[0,1]}\colon \mathbb{R}\times[0,1] \to S^1 \times [0,1]$ are smooth we know that $F \circ (\Xi \times \mathrm{id}_{[0,1]})\colon \mathbb{R} \times [0,1] \to \mathbb{R}^2$ is smooth. This implies, by Definition 24.1, that the partial derivatives are smooth. By the above calculation this shows that the map $(s,t) \mapsto F_t'(\Xi(s))$ is smooth. ∎

Remark. Note that in the proof of the Rotation Number–Regular Homotopy Proposition 49.10 we made very much use of the fact that a regular homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ is by definition a smooth map. Note that in the figure below we show a continuous map $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ with $F_0 = \mu$ and $F_1 = \nu$, such that each $F_t\colon S^1 \to \mathbb{R}^2$ is an immersion.



This shows that we cannot easily replace the condition that the map $F\colon S^1 \times [0,1] \to \mathbb{R}^2$ is smooth by a weaker condition. □

There is nothing a topologist likes better than a complete characterization of a property. In this setting we are in this lucky situation, since the following theorem says that the converse to the Rotation Number–Regular Homotopy Proposition 49.10 also holds.

Theorem 49.11. (Whitney-Graustein Theorem) Two immersions $S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if their rotation numbers agree.

Proof. The "only if"-statement is of course the Rotation Number–Regular Homotopy Proposition 49.10. The "if"-statement was first proved by Hassler Whitney [**Why1937**, p. 279] in 1937.[534] We will give the "standard proof" for the Whitney-Graustein Theorem 49.11, which relies on the Unit Speed Proposition 49.6, whose proof we still owe to the reader. Thus the plan is as follows:

(1) In Section 49.2 we provide the overdue proof of the Unit Speed Proposition 49.6.
(2) In Section 49.3 we will then give the "standard proof" for the Whitney-Graustein Theorem 49.11.

Note that there are several other proofs in the literature, see for example [**Gei2008**, Theorem 6.3.11], [**Gei2009**] and [**Praso2006**, p. 68], finally see also see the discussion in [**Kau1987a**, p. 172]. ■

49.2. **Proof of the Unit Speed Proposition 49.6.** In this section we will finally prove the Unit Speed Proposition 49.6, i.e. we will show that any immersion is regularly homotopic to an immersion of unit speed. Before we get into the nitty-gritty of the proof let us start out with a little preparation.

Definition 49.12. Given a smooth map $\mu\colon S^1 \to \mathbb{R}^n$ we define its length as

$$\text{length of } \mu \;:=\; \int_{r=0}^{r=2\pi} \|(\mu \circ \Xi)'(r)\|\, dr.$$

Lemma 49.13. (Normalize-to-Length 2π Lemma) Every immersion $\mu\colon S^1 \to \mathbb{R}^n$ is regularly homotopic to an immersion of length 2π.

Proof. Let $\mu\colon S^1 \to \mathbb{R}^n$ be an immersion. We denote its length by L. By the $\mathbb{R}$–S^1-Lemma 49.5 (1c) we know that $(\mu \circ \Xi)'(r) = \mu'(\Xi(r))$. Since μ is an immersion we see that $L > 0$. One can easily verify that the map

$$\begin{array}{rcl} F\colon \mathbb{R} \times [0,1] & \to & \mathbb{R}^n \\ (s,t) & \mapsto & \dfrac{1}{(1-t) + \frac{L}{2\pi} \cdot t} \cdot \mu(s) \end{array}$$

is a regular homotopy from μ to an immersion of length 2π. ■

Now we are ready to provide the proof of the Unit Speed Proposition 49.6.

Proof of the Unit Speed Proposition 49.6. Let $\gamma\colon S^1 \to \mathbb{R}^n$ be an immersion. We need to show that γ is regularly homotopic to an immersion $\delta\colon S^1 \to \mathbb{R}^n$ of unit speed, i.e. such that $\|\delta'(z)\| = 1$ for all $z \in S^1$. It follows from the Regular Homotopy–Equivalence Lemma 49.2 and the Normalize-to-Length 2π Lemma 49.13 that it suffices to consider the case that the length of γ is equal to 2π.

[534] Whitney [**Why1937**, p. 279] writes "this theorem, together with a straightforward proof, was suggested to me by W. C. Graustein".

Claim 1. We set $\widetilde{\gamma} \coloneqq \gamma \circ \Xi \colon \mathbb{R} \to \mathbb{R}^n$ and we consider the map

$$\begin{array}{rccc} \lambda \colon & \mathbb{R} & \to & \mathbb{R} \\ & s & \mapsto & \int\limits_{r=0}^{r=s} \|\widetilde{\gamma}'(r)\| \, dr. \end{array}$$

(1) λ is a diffeomorphism.
(2) We have $\lambda(2\pi) = 2\pi$.
(3) For every $x \in \mathbb{R}$ we have $\lambda(x + 2\pi) = \lambda(x) + 2\pi$.

Proof.

(1) Recall that it follows from the $\mathbb{R}$–S^1-Lemma 49.5 (1c) that $r \mapsto \|\widetilde{\gamma}'(r)\|$ is non-zero for every $r \in \mathbb{R}$. Note that it follows from the Fundamental Theorem of Calculus 73.27 that $\lambda'(s) = \|\gamma'(s)\| \neq 0$ for all $s \in \mathbb{R}$. Furthermore note that it follows from the $\mathbb{R}$–S^1-Lemma 49.5 (1a) that $r \mapsto \|\widetilde{\gamma}'(r)\|$ is 2π-periodic. It follows easily from these two observations that $\lambda \colon \mathbb{R} \to \mathbb{R}$ is a bijection. Since the derivative is non-zero everywhere it follows from the Inverse Function Theorem 24.21 that λ is indeed a diffeomorphism.
(2) Since γ has length 2π we see that $\lambda(2\pi) = 2\pi$.
(3) Let $x \in \mathbb{R}$. We pick $n \in \mathbb{Z}$ with $n \cdot 2\pi \in [x, x + 2\pi]$. We calculate that

$$\begin{aligned} \lambda(x + 2\pi) &= \int_0^{x+2\pi} \|\widetilde{\gamma}'(r)\| \, dr = \int_0^{x} \|\widetilde{\gamma}'(r)\| \, dr + \int_x^{x+2\pi} \|\widetilde{\gamma}'(r)\| \, dr \\ &= \int_0^{x} \|\widetilde{\gamma}'(r)\| \, dr + \int_x^{n\cdot 2\pi} \|\widetilde{\gamma}'(r)\| \, dr + \int_{n\cdot 2\pi}^{x+2\pi} \|\widetilde{\gamma}'(r)\| \, dr \\ &= \int_0^{x} \|\widetilde{\gamma}'(r)\| \, dr + \underbrace{\int_{x-(n-1)\cdot 2\pi}^{2\pi} \|\widetilde{\gamma}'(r)\| \, dr + \int_0^{x-(n-1)\cdot 2\pi} \|\widetilde{\gamma}'(r)\| \, dr}_{=\lambda(2\pi)=2\pi} = \lambda(x) + 2\pi. \end{aligned}$$

$\uparrow$ since $\widetilde{\gamma}'$ is 2π-periodic

⊞

Next we consider the map

$$\begin{array}{rccc} \widetilde{\delta} \colon & \mathbb{R} & \to & \mathbb{R}^n \\ & s & \mapsto & \widetilde{\gamma}(\lambda^{-1}(s)). \end{array}$$

Claim 2. For every $s \in \mathbb{R}$ we have $\|\widetilde{\delta}'(s)\| = 1$.

Proof. Let $s \in \mathbb{R}$. We perform the following calculation:

$$\begin{aligned} \|\widetilde{\delta}'(s)\| &\overset{\text{by definition of } \widetilde{\delta}}{=} \|(\widetilde{\gamma} \circ \lambda^{-1})'(s)\| \overset{\text{Chain Rule 24.17}}{=} \|\widetilde{\gamma}'(\lambda^{-1}(s)) \cdot (\lambda^{-1})'(s)\| \\ &\overset{\text{by the Differential-of-Diffeomorphism Proposition 24.19 (1)}}{=} \left\| \widetilde{\gamma}'(\lambda^{-1}(s)) \cdot \frac{1}{(\lambda')(\lambda^{-1}(s))} \right\| \\ &= \left\| \widetilde{\gamma}'(\lambda^{-1}(s)) \cdot \frac{1}{\|\widetilde{\gamma}'(\lambda^{-1}(s))\|} \right\| = 1. \end{aligned}$$

$\uparrow$ by the Fundamental Theorem of Calculus 73.27

⊞

Claim 3. We consider the homotopy

$$\begin{array}{rccc} \widetilde{F} \colon & \mathbb{R} \times [0,1] & \to & \mathbb{R}^n \\ & (s,t) & \mapsto & \widetilde{\gamma}(s \cdot (1-t) + \lambda^{-1}(s) \cdot t) \end{array}$$

from $\widetilde{\gamma}$ to $\widetilde{\delta}$.
(1) Each $\widetilde{F}_t$ is 2π-periodic.
(2) Each $\widetilde{F}_t$ is an immersion.

Proof. Let $t \in [0,1]$ and let $s \in \mathbb{R}$.

(1) We see that

$$\widetilde{F}_t(s+2\pi) \overset{\text{by definition}}{=} \widetilde{\gamma}((s+2\pi)\cdot(1-t) + \lambda^{-1}(s+2\pi)\cdot t) \overset{\text{by Claim 1 (3), applied to } x=\lambda^{-1}(s+2\pi)=\lambda^{-1}(s)+2\pi}{=} \widetilde{\gamma}((s+2\pi)\cdot(1-t) + (\lambda^{-1}(s)+2\pi)\cdot t)$$
$$= \widetilde{\gamma}(s\cdot(1-t) + \lambda^{-1}(s)\cdot t + 2\pi) \overset{\text{since } \widetilde{\gamma}=\gamma\circ\Xi \text{ is clearly } 2\pi\text{-periodic}}{=} \widetilde{\gamma}(s\cdot(1-t) + \lambda^{-1}(s)\cdot t) = \widetilde{F}_t(s).$$

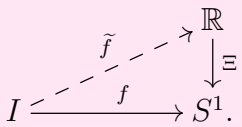
(2) Note that

$$\widetilde{F}_t'(s) \overset{\text{Chain Rule}}{=} \overbrace{\widetilde{\gamma}'(s\cdot(1-t)+\lambda^{-1}(s)\cdot t)}^{\neq 0} \cdot \overbrace{(1-t+\underbrace{(\lambda^{-1})'(s)}_{=\|\widetilde{\gamma}'(s)\| \text{ by FTC 73.27}}\cdot t)}^{>0} \neq 0.$$

Since $\widetilde{F}$ is evidently smooth and since by Claim 3 (1) we know that each $\widetilde{F}_t$ is 2π-periodic it follows immediately from the $\mathbb{R}$–S^1-Lemma 49.5 (2b) that there exists a smooth homotopy $F\colon S^1\times[0,1]\to\mathbb{R}^n$ such that $F\circ(\Xi\times \mathrm{id}_{[0,1]}) = \widetilde{F}$. By design we have $F_0=\gamma$. Note that it follows from Claim 2 and the $\mathbb{R}$–S^1-Lemma 49.5 (1c) that F_1 has unit speed. Furthermore it follows from Claim 3 (2) and the $\mathbb{R}$–S^1-Lemma 49.5 (1d) that each F_t is an immersion. In summary $F\colon S^1\times[0,1]\to S^1$ is the desired regular homotopy. ∎

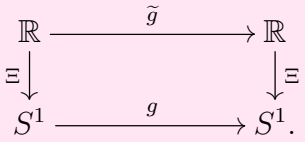
49.3. **Proof of the "if"-statement of the Whitney-Graustein Theorem 49.11.** For the reader's convenience we recall two earlier statements.

Lemma 42.14. (Lift-from-S^1-to-$\mathbb{R}$ Lemma). Let $I\subseteq\mathbb{R}$ be an interval and let $x_0\in I$. Given any continuous map $f\colon I\to S^1$ and given any $\widetilde{y}_0\in\mathbb{R}$ with $f(x_0)=\Xi(\widetilde{y}_0)$ there exists a unique continuous map $\widetilde{f}\colon I\to\mathbb{R}$ with $\widetilde{f}(x_0)=\widetilde{y}_0$ and such that $f=\Xi\circ\widetilde{f}$, i.e. such that the following diagram commutes:;



We refer to $\widetilde{f}$ as a *lift* of f.

Lemma 49.14. (Degree-via-Lifts Lemma) Let $g\colon S^1\to S^1$ be a smooth map and let $\widetilde{g}\colon\mathbb{R}\to\mathbb{R}$ be a continuous map such that $\Xi\circ\widetilde{g}=g\circ\Xi$, i.e. such that the following diagram commutes:



Then for any $s\in\mathbb{R}$ we have

$$\deg(g\colon S^1\to S^1) = \frac{1}{2\pi}\cdot\big(\widetilde{g}(s+2\pi)-\widetilde{g}(s)\big).$$

Proof. This lemma follows basically immediately from the Degree-via-Lifts Lemma 46.14 applied to the map $f\colon [0,1] \to \mathbb{R}$ that is given by $f(t) = g(s + 2\pi \cdot t)$. ■

We will also need the following lemma:

Lemma 49.15. (Integral Inequality Lemma) If $\Theta\colon [0,1] \to S^n$ is a non-constant map, then

$$\left\| \int_{r=0}^{r=1} \Theta(r)\, dr \right\| < \int_{r=0}^{r=1} \|\Theta(r)\|\, dr.$$

Proof. We view Θ as a map $[0,1] \to \mathbb{R}^{n+1}$. The Integral Inequality Lemma 2.45 says that the following inequality holds:

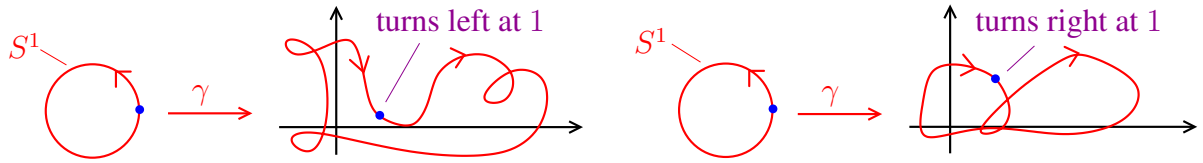
$$\left\| \int_{r=0}^{r=1} \Theta(r)\, dr \right\| \leq \int_{r=0}^{r=1} \|\Theta(r)\|\, dr \qquad \in \mathbb{R}.$$

Furthermore the Integral Inequality Lemma 2.45 says that equality holds if and only if there exists a $w \in \mathbb{R}^{n+1}$ and a continuous map $\lambda\colon [a,b] \to \mathbb{R}_{\geq 0}$ such that for every $t \in [a,b]$ we have $\Theta(t) = \lambda(t) \cdot w$. Since Θ takes values in S^n we see that equality only holds if Θ is a constant map. ■

In the remainder of this section we will need the following geometric definition.

Definition 49.16. Let $\gamma\colon S^1 \to \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ be an immersion of unit speed, i.e. such that $\gamma'(z) \in S^1$ for all $z \in S^1$. By the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 there exists a smooth map $\Lambda_\gamma\colon \mathbb{R} \to \mathbb{R}$ such that for all $s \in \mathbb{R}$ we have $\Xi(\Lambda_\gamma(s)) = (\gamma \circ \Xi)'(s) \in S^1$.

(1) We say that $\gamma\colon S^1 \to \mathbb{R}^2$ turns left at $1 \in S^1$ if $\Lambda'_\gamma(0) > 0$.[535]
(2) We say that $\gamma\colon S^1 \to \mathbb{R}^2$ turns right at $1 \in S^1$ if $\Lambda'_\gamma(0) < 0$.



Lemma 49.17. (Left-Right Turn Lemma) Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an immersion of unit speed.

(1) If the rotation number $\rho(\gamma) \geq 0$, then γ is regularly homotopic to an immersion of unit speed that turns left at $1 \in S^1$.
(2) If the rotation number $\rho(\gamma) \leq 0$, then γ is regularly homotopic to an immersion of unit speed that turns right at $1 \in S^1$.

Proof. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an immersion of unit speed. Evidently we only deal with one of the two cases of the lemma. For no particular reason we deal with the case that $\rho(\gamma) \geq 0$. Recall that by the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 there exists a continuous map $\Lambda_\gamma\colon \mathbb{R} \to \mathbb{R}$ such that for all $s \in \mathbb{R}$ we have $\Xi(\Lambda_\gamma(s)) = (\gamma \circ \Xi)'(s)$.

Claim. There exists a $w \in \mathbb{R}$ with $\Lambda'_\gamma(w) > 0$.

[535] The map Λ_γ is well-defined up to addition of a multiple of 2π. We see that the notion of turning left and right is independent of the choice of Λ_γ.

Proof. Note that

$$\text{(A)}\qquad \underset{\text{by the Fundamental Theorem of Calculus 73.27}}{\overset{t=2\pi}{\int_{t=0}} \Lambda'_\gamma(t)\,dt} = \Lambda_\gamma(2\pi) - \Lambda_\gamma(0) \underset{\text{by the Degree-via-Lifts Lemma 49.14}}{=} 2\pi \cdot \deg(\gamma') = \underset{\text{by Definition 49.8 of } \rho(\gamma)}{2\pi \cdot \rho(\gamma)}.$$

If $\rho(\gamma) > 0$ then it is clear from this calculation that there exists a $w \in [0, 2\pi]$ with $\Lambda'_\gamma(w) > 0$. It remains to consider the case $\rho(\gamma) = 0$. Note that we have

$$\text{(B)}\qquad 0 \underset{\text{since } \gamma\circ\Xi \text{ is } 2\pi\text{-periodic}}{=} (\gamma\circ\Xi)(2\pi) - (\gamma\circ\Xi)(0) \underset{\text{by the Fundamental Theorem of Calculus 73.27}}{=} \int_{t=0}^{t=2\pi} \overbrace{(\gamma\circ\Xi)'(t)}^{\in S^1}\,dt.$$

Since γ has unit speed we know that each $(\gamma\circ\Xi)'(t) \in S^1$. Thus the integral in (B) can only be zero if $(\gamma\circ\Xi)'(t)$ is not constant. This implies that Λ_γ is not constant. But this implies that Λ'_γ cannot be identically zero. But the only way how the integral in (A) can be zero is now that Λ'_γ assumes negative and positive values. ⊞

We have now shown that γ turns left at some point in S^1. It remains to shift this point to $1 \in S^1$. To do so we consider the following map

$$\begin{aligned} F\colon S^1 \times [0,1] &\to \mathbb{R}^2 \\ (z,t) &\mapsto \gamma(z \cdot \exp(\mathrm{i}\cdot w \cdot t)). \end{aligned}$$

One can easily verify that this map is smooth and that each F_t is an immersion of unit speed. In particular F is a regular homotopy from $F_0 = \gamma$ to $F_1 =: \delta$. Finally note that by construction we see that δ turns left at 1. ■

Now we can finally provide the proof of the "if"-statement of the Whitney-Graustein Theorem 49.11.

Proof of the "if"-statement of the Whitney-Graustein Theorem 49.11. In the following let $\alpha, \beta\colon S^1 \to \mathbb{R}^2$ be two immersions with $\rho(\alpha) = \rho(\beta)$. We need to show that α and β are regularly homotopic. In the following we will deal with the case that $\rho(\alpha) = \rho(\beta) \geq 0$. The case that $\rho(\alpha) = \rho(\beta) < 0$ is proved by an almost identical argument, namely, in the discussion below we just need to replace the Left-Right Turn Lemma 49.17 (1) by the Left-Right Turn Lemma 49.17 (2).

We start out with a few initial observations. In the Regular Homotopy–Equivalence Lemma 49.2 we showed that being regularly homotopic is an equivalence relation. It follows from this observation, together with the Unit Speed Proposition 49.6, the Left-Right Turn Lemma 49.17 (1) and the Rotation Number–Regular Homotopy Proposition 49.10 that we can assume that the two given immersions $\alpha, \beta\colon S^1 \to \mathbb{R}^2$ have the following properties:

(I) Both α and β are of unit speed.
(II) Both α and β turn left at $1 \in S^1$.

Furthermore, after a translation we can as well assume that the following condition is satisfied:

(III) We have $\alpha(1) = \beta(1) = 0 \in \mathbb{C}$.

Since $\alpha, \beta\colon S^1 \to \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ are immersions of unit speed we get maps $\alpha', \beta'\colon S^1 \to S^1$. Thus we obtain once again from the Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 that there exist smooth maps $\Lambda_{\alpha'}, \Lambda_{\beta'}\colon \mathbb{R} \to \mathbb{R}$ such that $\Xi \circ \Lambda_{\alpha'} = \alpha' \circ \Xi$ and such that $\Xi \circ \Lambda_{\beta'} = \beta' \circ \Xi$.

For orientation we summarize the various maps in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\Lambda_{\alpha'}} & \mathbb{R} \\ {\scriptstyle s\mapsto \exp(\mathrm{i}s)}\downarrow{\scriptstyle \Xi} & & {\scriptstyle \Xi}\downarrow{\scriptstyle s\mapsto \exp(\mathrm{i}s)} \\ S^1 & \xrightarrow[z\mapsto \alpha'(z)]{} & S^1. \end{array}$$

In Section 49.1 we saw that the standard homotopy $(z,t) \mapsto \alpha(z)\cdot(1-t) + \beta(z)\cdot t$ shows that α and β are homotopic. In general this homotopy is not a regular homotopy. To obtain a regular homotopy, under the hypothesis that $\rho(\alpha) = \rho(\beta)$ we will use a variation on the standard homotopy. More precisely, the idea is now as follows:

(a) We consider the lifts $\Lambda_{\alpha'}$ and $\Lambda_{\beta'}$ of the *derivatives* of $\Xi \circ \alpha$ and $\Xi \circ \beta$.
(b) We consider the standard homotopy of $\Lambda_{\alpha'}$ and $\Lambda_{\beta'}$.
(c) We apply Ξ to the homotopy from (b) and we integrate, to "undo" taking the derivative.
(d) We obtain a regular homotopy $\widetilde{F}\colon \mathbb{R}\times[0,1] \to \mathbb{R}^n$ between $\alpha\circ\Xi$ and $\beta\circ\Xi$,
(e) finally, and that is the most delicate step, we need to modify $\widetilde{F}$ to obtain a regular homotopy $\widetilde{G}$ that descends to a regular homotopy on S^1.

Next we consider the homotopy

$$\begin{array}{rcl} \Theta\colon \mathbb{R}\times[0,1] & \to & S^1 \\ (s,t) & \mapsto & \Xi\big(\Lambda_{\alpha'}(s)\cdot(1-t) + \Lambda_{\beta'}(s)\cdot t\big). \end{array}$$

Claim 1.
(1) Each Θ_t is 2π-periodic.
(2) Each $\Theta_t\colon \mathbb{R} \to \mathbb{R}$ is non-constant.

Proof. Let $t \in [0,1]$.

(1) Let $s \in \mathbb{R}$. We need to show that $\Theta_t(s+2\pi) = \Theta_t(s)$. The statement follows immediately from the fact that Ξ is 2π-periodic and the following calculation:

$$\begin{array}{rcl} \Lambda_{\alpha'}(s{+}2\pi)\cdot(1-t) + \Lambda_{\beta'}(s{+}2\pi)\cdot t & \overset{\text{by the Degree-via-Lifts Lemma 49.14 we have } \Lambda_{\alpha'}(s+2\pi)-\Lambda_{\alpha'}(s) = 2\pi\cdot\deg(\alpha') = 2\pi\cdot\rho(\alpha) \text{ and we have } \Lambda_{\beta'}(s+2\pi)-\Lambda_{\beta'}(s) = 2\pi\cdot\deg(\beta') = 2\pi\cdot\rho(\beta)}{=} & (\Lambda_{\alpha'}(s) + 2\pi\cdot\rho(\alpha))\cdot(1-t) + (\Lambda_{\beta'}(s) + 2\pi\cdot\rho(\beta))\cdot t \\ & = & \Lambda_{\alpha'}(s)\cdot(1-t) + \Lambda_{\beta'}(s)\cdot t + \underbrace{2\pi\cdot\rho(\alpha)}_{\in 2\pi\cdot\mathbb{Z}}. \\ & \overset{\text{follows from the hypothesis that } \rho(\alpha)=\rho(\beta)}{=} & \end{array}$$

(2) Let $t \in [0,1]$. Note that by (II) we know that $\Lambda_{\alpha'}(0) > 0$ and that $\Lambda_{\beta'}(0) > 0$. It follows that the derivative of $s \mapsto \Lambda_{\alpha'}(s)\cdot(1-t) + \Lambda_{\beta'}(s)\cdot t$ is positive at $s = 0$. It follows that Θ_t is non-constant. ⊞

We use this homotopy $\Theta\colon \mathbb{R}\times[0,1]\to S^1$ to define the following map:

$$\begin{array}{rcl}\widetilde{F}\colon \mathbb{R}\times[0,1] & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & \int\limits_{r=0}^{r=s}\Theta_t(r)\,dr.\end{array}$$

Claim 2. The map $\widetilde{F}$ has the following properties:
(1) $\widetilde{F}$ is smooth.
(2) $\widetilde{F}_0=\alpha\circ\Xi$ and $\widetilde{F}_1=\beta\circ\Xi$.

Proof.

(1) Note that α and β are smooth. As we mentioned in the Smooth Lift Lemma 42.13, it follows from the fact that $\Xi\colon\mathbb{R}\to S^1$ is a local diffeomorphism that $\Lambda_{\alpha'}$ and $\Lambda_{\beta'}$ are smooth. It now follows from standard facts in real analysis, e.g. the Leibniz Integral Rule 73.28, that $\widetilde{F}$ is smooth.
(2) Given any $s\in\mathbb{R}$ we perform the following calculation:

$$\widetilde{F}(s,0) \underset{\substack{\uparrow\\ \text{by definition}\\ \text{of } \widetilde{F}}}{=} \int_{r=0}^{r=s}\Theta(r,0)\,dr \underset{\substack{\uparrow\\ \text{by definition}\\ \text{of } \Theta(r,0)}}{=} \int_{r=0}^{r=s}(\Xi\circ\Lambda_{\alpha'})(r)\,dr \underset{\substack{\uparrow\\ \text{since we have}\\ \Xi\circ\Lambda_{\alpha'}=\alpha'\circ\Xi=(\alpha\circ\Xi)'}}{=} \int_{r=0}^{r=s}(\alpha\circ\Xi)'(r)\,dr \underset{\substack{\uparrow\\ \text{follows from the Fundamental}\\ \text{Theorem of Calculus 73.27}\\ \text{and the fact that } (\alpha\circ\Xi)(0)=0}}{=} (\alpha\circ\Xi)(s).$$

The same argument shows that $\widetilde{F}_1=\beta\circ\Xi$. ⊞

All this sounds very promising. But unfortunately there is a problem with the homotopy $\widetilde{F}\colon\mathbb{R}\times[0,1]\to\mathbb{R}^2$: there is no reason why the maps $\widetilde{F}_t$ should be 2π-periodic, i.e. why they should descend to a map $S^1\to\mathbb{R}^2$. To fix this issue we consider the following map:[536]

$$\begin{array}{rcl}\widetilde{G}\colon \mathbb{R}\times[0,1] & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & \widetilde{F}(s,t)-\frac{s}{2\pi}\cdot\widetilde{F}(2\pi,t) = \int\limits_{r=0}^{r=s}\Theta_t(r)\,dr - \frac{s}{2\pi}\cdot\int\limits_{r=0}^{r=2\pi}\Theta_t(r)\,dr.\end{array}$$

Claim 3. The map $\widetilde{G}$ has the following properties:
(1) Each $\widetilde{G}_t$ is 2π-periodic.
(2) $\widetilde{G}$ is smooth.
(3) $\widetilde{G}_0=\alpha\circ\Xi$ and $\widetilde{G}_1=\beta\circ\Xi$.
(4) For each $t\in[0,1]$ the map $\widetilde{G}_t\colon\mathbb{R}\to\mathbb{R}^2$ is an immersion.

Proof.

(1) Let $t\in[0,1]$. We need to show that for each $s\in\mathbb{R}$ we have $\widetilde{G}(s+2\pi,t)=\widetilde{G}(s,t)$. Evidently it suffices to show that for every $s\in[0,2\pi)$ and every $n\in\mathbb{Z}$ we have

[536] While writing this chapter the author regretted on numerous occasions the fact that 1 is not equal to 2π.

$\widetilde{G}(s+2\pi\cdot n,t)=\widetilde{G}(s,t)$. We perform the following elementary calculation:

$$\begin{aligned}
\widetilde{G}(s+2\pi n,t) &= \int_0^{2\pi n+s}\Theta_t(r)\,dr - \frac{2\pi n+s}{2\pi}\cdot\int_0^{2\pi}\Theta_t(r)\,dr \overset{\substack{\text{by definition and by the}\\ \text{additivity of the integral}}}{=}\\
&= \sum_{k=0}^{n-1}\int_{2\pi k}^{2\pi(k+1)}\Theta_t(r)\,dr - n\cdot\int_0^{2\pi}\Theta_t(r)\,dr + \int_{2\pi n}^{2\pi n+s}\Theta_t(r)\,dr - \frac{s}{2\pi}\cdot\int_0^{2\pi}\Theta_t(r)\,dr\\
&\underset{\substack{\text{since by Claim 1 (1) we}\\ \text{know that } \Theta_t \text{ is } 2\pi\text{-periodic}}}{=} \underbrace{\sum_{k=0}^{n-1}\int_0^{2\pi}\Theta_t(r)\,dr - n\cdot\int_0^{2\pi}\Theta_t(r)\,dr}_{=0} + \underbrace{\int_0^{s}\Theta_t(r)\,dr - \frac{s}{2\pi}\cdot\int_0^{2\pi}\Theta_t(r)\,dr}_{=\widetilde{G}(s,t)} = \widetilde{G}(s,t).
\end{aligned}$$

(2) It follows easily from Claim 2 (1) that $\widetilde{G}$ is smooth.

(3) Let $s\in\mathbb{R}$. We see that

$$\begin{aligned}
\widetilde{G}_0(s) &\overset{\text{by definition}}{=} \widetilde{F}_0(s)-\frac{s}{2\pi}\cdot\widetilde{F}_0(2\pi) \overset{\text{by Claim 2 (2)}}{=} (\alpha\circ\Xi)(s)-\frac{s}{2\pi}\cdot(\alpha\circ\Xi)(2\pi)\\
&\underset{\substack{\text{since } \alpha\circ\Xi \text{ is}\\ 2\pi\text{-periodic}}}{=} (\alpha\circ\Xi)(s)-\frac{s}{2\pi}\cdot\underbrace{(\alpha\circ\Xi)(0)}_{=0 \text{ by (III)}} = (\alpha\circ\Xi)(s).
\end{aligned}$$

The same argument shows that $\widetilde{G}_1=\beta\circ\Xi$.

(4) Let $t\in[0,1]$. We need to show that for every $s\in\mathbb{R}$ we have $\widetilde{G}_t{}'(s)\neq 0$. Thus let $s\in\mathbb{R}$. Note that

$$\begin{aligned}
\|\widetilde{G}_t{}'(s)\| &\underset{\substack{\text{by the Fundamental Theorem of Calculus 73.27}\\ \text{applied to the first term of } \widetilde{G}_t(s) \text{ and}\\ \text{by linearity of the second term of } \widetilde{G}_t(s)}}{=} \left\|\Theta_t(s)-\frac{1}{2\pi}\int_{r=0}^{r=2\pi}\Theta_t(r)\,dr\right\| \underset{\text{triangle inequality}}{\geq} \|\Theta_t(s)\|-\frac{1}{2\pi}\cdot\left\|\int_{r=0}^{r=2\pi}\Theta_t(r)\,dr\right\|\\
&\underset{\substack{\text{follows from the Integral Inequality Lemma 49.15, since we}\\ \text{know by Claim 1 (2) that the map } \Theta_t \text{ is not constant}}}{>} \underbrace{\|\Theta_t(s)\|}_{=1}-\frac{1}{2\pi}\cdot\int_{r=0}^{r=2\pi}\underbrace{\|\Theta_t(r)\|}_{=1}\,dr > 0.
\end{aligned}$$

⊞

It follows from Claim 3 (1) and (3) together with the $\mathbb{R}$–S^1-Lemma 49.5 (2b) that there exists a smooth map

$$G\colon S^1\times[0,1]\to\mathbb{R}^2$$

such that for all $(s,t)\in\mathbb{R}\times[0,1]$ we have $G(\Xi(s),t)=\widetilde{G}(s,t)$. It follows easily from the above properties of $\widetilde{G}$ that G is the desired regular homotopy between α and β. ■

49.4. Exercises for Chapter 49.

Exercise 49.1. In the figure below we see to the left the image of an immersion $S^1\to\mathbb{R}^2$. Provide an explicit regular smooth isotopy to the standard embedding $S^1\to\mathbb{R}^2$.



Exercise 49.2. Let $n \geq 3$. Show that any two immersions $S^1 \to \mathbb{R}^n$ are regularly homotopic.
Hint. Use the Measure-Zero Properties Proposition 37.3 and the Full Measure-Density Proposition 37.4.

Exercise 49.3. Let $n \in \mathbb{Z} \setminus \{0\}$. Use the Degree-via-Lifts Lemma 49.14 to give a new proof of the Rotation Number-of-Power Lemma 49.9, i.e. show that

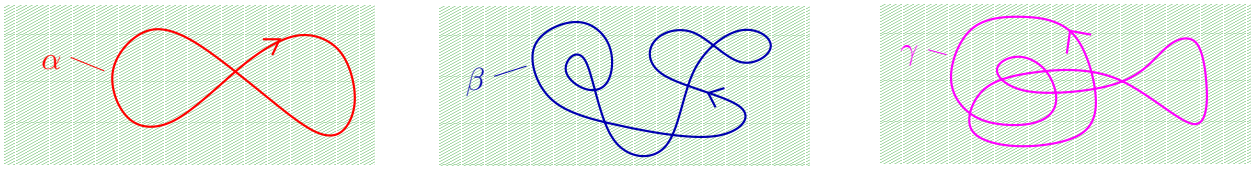
$$\rho \left(\begin{array}{ccc} S^1 & \to & \mathbb{C} \\ z & \mapsto & z^n \end{array} \right) = n.$$

Exercise 49.4. Let $\mu \colon S^1 \to \mathbb{R}^n$ be a smooth map. As in Definition 49.12 we define its length as

$$\text{length of } \mu \ := \ \int_{r=0}^{r=2\pi} \|(\mu \circ \Xi)'(r)\| \, dr.$$

(Here we use the Definition 49.4.) Show that if $\Phi \colon S^1 \to S^1$ is a diffeomorphism, then the lengths of μ and $\mu \circ \Phi$ agree.

Exercise 49.5. Determine the rotation numbers of the immersions indicated in the figure below.



Exercise 49.6. Let $\alpha, \beta \colon S^1 \to \mathbb{R}^2$ be immersions. We suppose that there exists a continuous map $F \colon S^1 \times [0,1] \to \mathbb{R}^2$ with the following properties:
(1) Each F_t is an immersion.
(2) $F_0 = \alpha$ and $F_1 = \beta$.
(3) For each $z \in S^1$ the map $[0,1] \to \mathbb{R}^2$ given by $t \mapsto F_t'(z)$ is continuous.
Does this imply that the rotation numbers of α and β agree?

CHAPTER 50

The transversality theorem

In this chapter we will do the following:

(1) We will define what it means for two submanifolds of a given smooth manifold to be transverse.
(2) We will see that if two submanifolds are transverse, then the intersection is again a submanifold.
(3) We will see that given two submanifolds we can always perform an isotopy of one of the two submanifolds to obtain a transverse intersection.

This chapter is organized as follows:

- In Section 50.1 we will point out a few conventions which we use throughout this chapter.
- In Section 50.2 we will state and prove the Submanifold Transversal Intersection Theorem 50.4
- In Section 50.3 we will state and prove the Transversal Preimage Theorem 50.8.
- In Section 50.4 we will state the Transversality Theorem 50.9 which says in a nutshell that smooth submanifolds can, after a smooth isotopy of one of them, be arranged to be transverse.
- In Section 50.5 we will provide the proof of the Transversality Theorem 50.9.
- In Section 50.6 we will provide the proof of the Transversality Theorem 50.10.
- In Section 50.7 we will provide exercises for Chapter 50.

In the next Chapter 51 we will use the Transversality Theorem 50.9 to introduce the algebraic intersection number.

50.1. **Conventions.** Throughout this chapter we will work with two already familiar conventions which will make our life much easier. First we recall the following Convention 29.21

Convention 50.1. Let M be a smooth manifold.

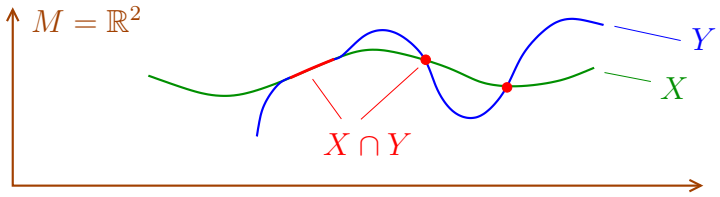
(1) Given $P \in \partial M$ we view $\mathrm{T}_P \partial M$ as a vector subspace of $\mathrm{T}_P M$.
(2) Given a submanifold W and given $P \in W$ we view $\mathrm{T}_P W$ as a vector subspace of $\mathrm{T}_P M$.

Secondly we adopt the following convention.

Convention 50.2. Given a smooth submanifold M of some $\mathbb{R}^n$ and given $P \in M$ we use the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24 as an excuse for not distinguishing in our notation between the abstract tangent space $\mathrm{T}_P M$ and the physical tangent space $\widetilde{\mathrm{T}}_P M$. We will always just write $\mathrm{T}_P M$.

50.2. **The Transversal Intersection Theorem.** In this section we want to study the following question: given a smooth manifold M and two submanifolds X and Y, is the intersection $X \cap Y$ again a submanifold of M? It is not difficult to show that in general the answer is negative. For example consider $M = \mathbb{R}^2$ with the two submanifolds X and Y

that are shown in the below. The intersection $X \cap Y$ has three components, one of which is a submanifold of dimension one, and the other two components consist of a single point. We see that $X \cap Y$ is not a smooth manifold (after all, what would its dimension be?), in particular it is not a submanifold of M.



If one spends some time pondering the question why in the previous example the intersection $X \cap Y$ is not a submanifold, then one is inexorably led towards the following definition:

Definition 50.3. Let M be a smooth manifold. Let X and Y be proper submanifolds of M.

(1) Let $P \in X \cap Y$. We say that P is a transverse intersection point if X and Y intersect transversally in P, i.e. if the equality[537]

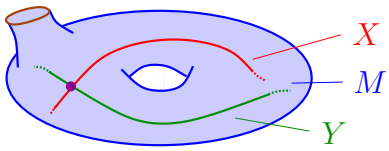
$$\mathrm{T}_P X + \mathrm{T}_P Y \;=\; \mathrm{T}_P M$$

holds. If $P \in \partial M$, then we also demand that[538]

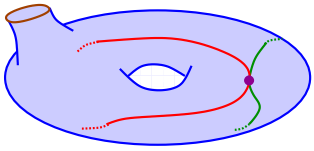
$$\mathrm{T}_P(X \cap \partial M) + \mathrm{T}_P(Y \cap \partial M) \;=\; \mathrm{T}_P\, \partial M.$$

(2) We say that X and Y intersect transversally if each intersection point is a transverse intersection point.

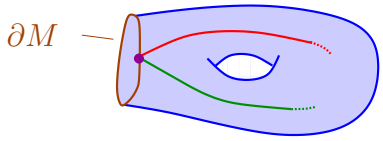
(3) We abbreviate the expression "X and Y intersect transversally" by $X \pitchfork Y$.



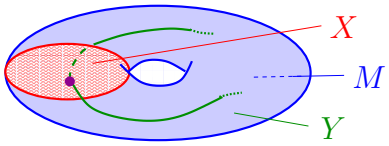
intersection point is transverse



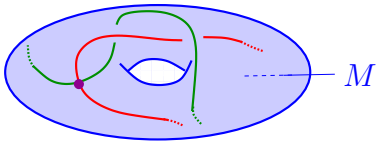
intersection point is not transverse



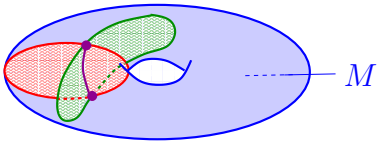
intersection point is not transverse



intersection point is transverse



intersection point is not transverse



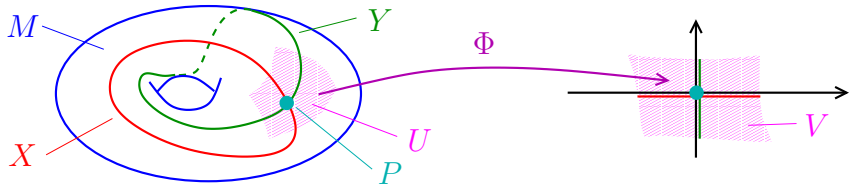
all intersection points are transverse

[537] Since X is a submanifold of M we can use the above convention to view the tangent space $\mathrm{T}_P X$ as a vector subspace of $\mathrm{T}_P M$.

[538] Note that in Exercise 25.29 (a) we showed that $X \cap \partial M$ and $Y \cap \partial M$ are submanifolds of ∂M, thus it makes sense to consider their tangent spaces.

Theorem 50.4. (Submanifold Transversal Intersection Theorem) Let M be an m-dimensional smooth manifold, let $X \subseteq M$ be a proper submanifold of codimension k and let Y be a proper submanifold of M of codimension l.

(1) Let $P \in M$ be a transverse intersection point of X and Y.
   (a) If $P \notin \partial M$, then there exists a chart $\Phi\colon U \to V$ of type (i) around P with
   $$\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k} \times \{0\}) \quad \& \quad \Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l}).$$
   (b) If $P \in \partial M$, then there exists a chart $\Phi\colon U \to V$ of type (ii) around P with
   $$\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k-1} \times \{0\} \times \mathbb{R}_{\geq 0}) \quad \& \quad \Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l-1} \times \mathbb{R}_{\geq 0}).$$
(2) If the submanifolds X and Y intersect transversally, i.e. if $X \pitchfork Y$, then the following statements hold:
   (a) The intersection $X \cap Y$ is a proper submanifold of M of codimension $k + l$. Furthermore $X \cap Y$ is a proper submanifold of X and of Y.
   (b) For every $P \in X \cap Y$ we have
   $$\mathrm{T}_P(X \cap Y) \;=\; \mathrm{T}_P X \cap \mathrm{T}_P Y \;\subseteq\; \mathrm{T}_P M.$$
   (c) Let M, X and Y be oriented.
      (i) By hypothesis for each $P \in X \cap Y$ the vector spaces $\mathrm{T}_P M$, $\mathrm{T}_P X$ and $\mathrm{T}_P Y$ are oriented. By (2b) we can equip $\mathrm{T}_P(X \cap Y) = \mathrm{T}_P X \cap \mathrm{T}_P Y$ with the intersection orientation that we introduced in Definition 31.16. These orientations define an orientation on $X \cap Y$.
      (ii) The orientations on the smooth manifolds $\partial X \cap \partial Y$ and $\partial(X \cap Y)$ agree.[539]
   (d) If one of X or Y is compact, then so is $X \cap Y$.

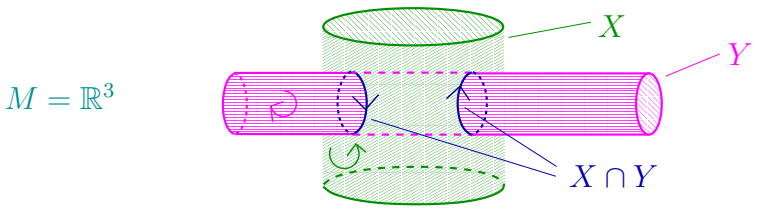


Remark. In the next section we will prove the Transversal Preimage Theorem 50.8 which is a generalization of the Submanifold Transversal Intersection Theorem 50.4 (2). □

Definition 50.5. Let M be an oriented smooth manifold and let X and Y be proper oriented submanifolds of M that intersect transversally. We refer to the orientation on $X \cap Y$ that we introduced in the Submanifold Transversal Intersection Theorem 50.4 (2) as the intersection orientation.

Example. In the following figure we see two oriented infinite cylinders X and Y in the oriented smooth manifold $\mathbb{R}^3$:

[539] Here we give $X \cap Y$ the orientation from (i) and we give $\partial(X \cap Y)$ the boundary orientation from Convention 31.39. Furthermore we equip ∂X, ∂Y and ∂M with the boundary orientations. By definition of transverse intersection we also have $\partial X \pitchfork \partial Y$ as submanifolds of ∂M. Thus we can use (i) to equip $\partial X \cap \partial Y$ with an orientation.

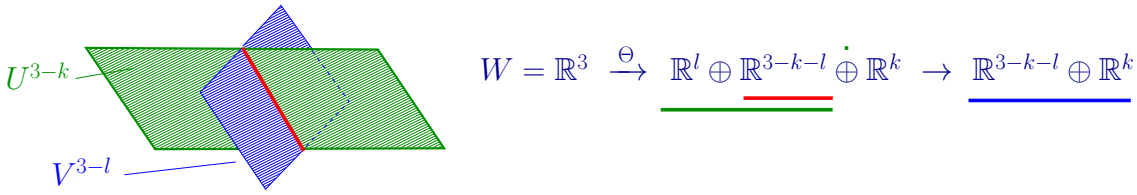


The cylinders are submanifolds of codimension one. They intersect transversally and the intersection consists of two circles, i.e. $X \cap Y$ is a one-dimensional submanifold of $\mathbb{R}^3$. Put differently, $X \cap Y$ is a submanifold of codimension two. □

Lemma 50.6. (Subspace Isomorphism Lemma) Let W be an m-dimensional real vector space, let U be an $(m-k)$-dimensional vector subspace of W and let V be an $(m-l)$-dimensional vector subspace of W. We assume that $U + V = W$. Furthermore let $\Theta\colon W \to \mathbb{R}^l \oplus \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$ be an isomorphism. If $\Theta(U) = \mathbb{R}^l \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$ and if $\Theta(U \cap V) = \{0\} \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$, then the map

$$V \xrightarrow{\Theta} \mathbb{R}^l \oplus \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k \xrightarrow{\text{projection}} \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$$

is an isomorphism.



Proof of the Subspace Isomorphism Lemma 50.6. First note that for dimension reasons we only have to show that the given map $V \to \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$ is an epimorphism. Thus let $(y, z) \in \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$. Since Θ is an isomorphism there exists a $w \in W$ with $\Theta(w) = (0, y, z)$. Since $U + V = W$ there exist $u \in U$ and $v \in V$ with $u + v = w$. By hypothesis $\Theta(u) = (x', y', 0)$ for some $x' \in \mathbb{R}^l$ and $y' \in \mathbb{R}^{m-k-l}$. Furthermore by hypothesis we have $\Theta(U \cap V) = \{0\} \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$. From $U + V = W$ it follows from elementary linear algebra that $U \cap V$ is also of dimension $m-k-l$. Thus the map $\Theta\colon U \cap V \to \{0\} \oplus \mathbb{R}^{m-k-l} \oplus \{0\}$ is actually an isomorphism. Thus there exists a $\widetilde{v} \in V$ with $\Theta(\widetilde{v}) = (0, y', 0)$. Now we see that

$$\Theta(v + \widetilde{v}) \underset{\substack{\uparrow \\ \text{since } u+v=w}}{=} \Theta(w) - \Theta(u) + \Theta(\widetilde{v}) = \underbrace{(0, y, z)}_{=\Theta(w)} - \underbrace{(x', y', 0)}_{=\Theta(u)} + \underbrace{(0, y', 0)}_{=\Theta(\widetilde{v})} = (-x', y, z).$$

Thus we have shown that $y \oplus z$ lies in the image of the given map $V \to \mathbb{R}^{m-k-l} \oplus \mathbb{R}^k$. ■

Proof of the Submanifold Transversal Intersection Theorem 50.4. Let M be an m-dimensional smooth manifold, let $X \subseteq M$ be a proper submanifold of codimension k and let Y be a proper submanifold of M of codimension l.

(1) Let $P \in M$ be a transverse intersection point of X and Y. In the following we will deal with the case that $P \notin \partial M$. The case $P \in \partial M$ is proved in a similar fashion, we leave it to the reader to fill in the details for the case $P \in \partial M$.

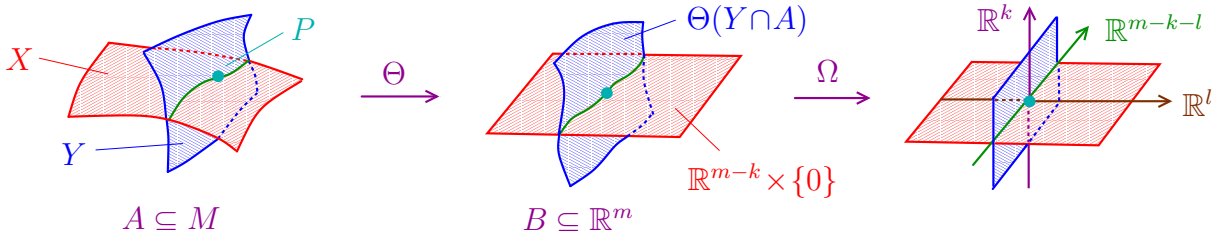
Since X is a proper submanifold of M and since $P \notin \partial M$ we can pick a submanifold chart $\Theta\colon A \to B$ of type (α) for X around P. We can arrange that $\Theta(P) = 0$. Note that $\mathrm{D}\Theta_P(\mathrm{T}_P X) = \mathbb{R}^{m-k} \times \{0\}$. Since P is a transverse intersection point we see that $\mathrm{D}\Theta_P(\mathrm{T}_P X \cap \mathrm{T}_P Y)$ is a vector subspace of $\mathbb{R}^{m-k} \times \{0\} \subseteq \mathbb{R}^m$ of dimension $m-k-l$.[540] Elementary linear algebra shows that there exists an $G \in \mathrm{GL}(m, \mathbb{R})$ such that $G \cdot (\mathbb{R}^{m-k} \times \{0\}) = \{0\} \times \mathbb{R}^{m-k}$ (this switch is for notational convenience) and such that $G \cdot \mathrm{D}\Theta_P(\mathrm{T}_P X \cap \mathrm{T}_P Y) = \{0\} \times \mathbb{R}^{m-k-l} \times \{0\} \subseteq \mathbb{R}^m$. In summary we see, that after composing Θ with multiplication by the matrix G we might as well assume that $\mathrm{D}\Theta_P(\mathrm{T}_P X) = \{0\} \times \mathbb{R}^{m-k}$ and that $\mathrm{D}\Theta_P(\mathrm{T}_P X \cap \mathrm{T}_P Y) = \{0\} \times \mathbb{R}^{m-k-l} \times \{0\} \subseteq \mathbb{R}^m$.

Next we consider the projections

$$\begin{array}{rcl} p_1\colon \mathbb{R}^m = \mathbb{R}^l \times \mathbb{R}^{m-l} & \to & \mathbb{R}^l \\ (x_1, x_2) & \mapsto & x_1 \end{array} \quad \text{and} \quad \begin{array}{rcl} p_2\colon \mathbb{R}^m = \mathbb{R}^l \times \mathbb{R}^{m-l} & \to & \mathbb{R}^{m-l} \\ (x_1, x_2) & \mapsto & x_2. \end{array}$$

Since P is a transverse intersection point and since $\mathrm{D}\Theta_P\colon \mathrm{T}_P(Y \cap A) \to \mathrm{T}_0(\Theta(Y \cap A))$ is an isomorphism we obtain from the elementary Subspace Isomorphism Lemma 50.6 that $(\mathrm{D}p_2)_0 = p_2\colon \mathrm{T}_0(\Theta(Y \cap A)) \to \mathbb{R}^{m-l}$ is an isomorphism. It follows from the Inverse Mapping Theorem 30.1 that $p_2\colon \Theta(Y \cap A) \to \mathbb{R}^{m-l}$ is a local diffeomorphism at 0. This implies that there exists an open neighborhood B_2 of $0 \in \mathbb{R}^{m-l}$ such that the restriction of p_2 to $\Theta(Y \cap A) \cap (\mathbb{R}^l \times B_2) \to B_2$ is a diffeomorphism. We denote by q_2 the inverse. We pick an open subset $B_1 \subseteq \mathbb{R}^l$ such that $B_1 \times B_2 \subseteq B$ and we consider the map

$$\begin{array}{rcl} \Omega\colon B_1 \times B_2 & \to & \mathbb{R}^l \times \mathbb{R}^{m-l} \\ (x_1, x_2) & \mapsto & (x_1 - (p_1 \circ q_2)(x_2), x_2). \end{array}$$



Claim. The map

$$\Phi \;:=\; \Omega \circ \Theta\colon \Theta^{-1}(B_1 \times B_2) \;\to\; \Omega(B_1 \times B_2)$$

has all the desired properties.

Proof. One can easily verify that an inverse to Φ is given by

$$\begin{array}{rcl} \Psi\colon \Omega(B_1 \times B_2) & \to & \Theta^{-1}(B_1 \times B_2) \\ (x_1, x_2) & \mapsto & \Theta^{-1}(x_1 + (p_1 \circ q_2)(x_2), x_2). \end{array}$$

This shows that Φ is a bijection. Furthermore one can easily verify that the differential of Φ at each point is invertible. Thus it follows from the Inverse Mapping Theorem 30.1 together with the Smooth Pasting Proposition 25.19 (2) that Φ is actually a

[540] This can be see as follows:

$$\mathbb{R}^l = \mathbb{R}^{\mathrm{codim}(Y)} \cong \frac{\mathrm{T}_P M}{\mathrm{T}_P Y} \underset{\text{since } X \text{ and } Y \text{ are transverse}}{=} \frac{\mathrm{T}_P X + \mathrm{T}_P Y}{\mathrm{T}_P Y} \underset{\text{linear algebra}}{\cong} \frac{\mathrm{T}_P X}{\mathrm{T}_P X \cap \mathrm{T}_P Y} \underset{\text{since } \Theta_P \text{ is an isomorphism}}{\cong} \frac{\mathbb{R}^{m-k} \times \{0\}}{\mathrm{D}\Theta_P(\mathrm{T}_P X \cap \mathrm{T}_P Y)}.$$

Thus $l = m - k - \dim(\mathrm{D}\Theta_P(\mathrm{T}_P X \cap \mathrm{T}_P Y))$.

diffeomorphism. This shows that Φ is a chart of type (i). Finally note that, setting $U := \Theta^{-1}(B_1 \times B_2)$ and $V := \Phi(U)$, we have, by construction, the two equalities $\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k} \times \{0\})$ and $\Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l})$. ⊞

(2) (a) The charts constructed in (1) define, after possibly reordering the coordinates, submanifold charts of type (α) and (β) for $X \cap Y \subseteq M$. Since X and Y are closed the intersection $X \cap Y$ is also closed. This shows that $X \cap Y$ is a proper submanifold of codimension $k + l$ of M.
A slight modification of the above argument shows that $X \cap Y$ is also a proper submanifold of X and of Y.

(b) Let $P \in X \cap Y$. By (2a) we know that $X \cap Y$ is a proper submanifold of M of codimension $k + l$. Thus $\mathrm{T}_P(X \cap Y)$ is a vector space of dimension $m - k - l$. Since P is a transverse intersection we obtain from an elementary linear algebra argument[541] that $\mathrm{T}_P X \cap \mathrm{T}_P Y$ is also of dimension $m - k - l$. As discussed in (2a), $X \cap Y$ is a proper submanifold of X and Y, thus $\mathrm{T}_P(X \cap Y) \subseteq \mathrm{T}_P X \cap \mathrm{T}_P Y$. Since the dimensions are equal we see that the vector spaces are the same.

(c) (i) We leave it to the reader to go through the dull task of verifying that the given recipe defines an orientation on $X \cap Y$.

(ii) In the Transversal Preimage Theorem 50.8 (3b) we will prove a generalization of this statement. The proof is a little delicate. To avoid lengthy repetitions we refer the reader to the later argument.

(d) By symmetry we can assume that Y is compact. Since X is a proper submanifold of M it is in particular a closed subset of M. By definition of the subspace topology on Y we see that $X \cap Y$ is a closed subset of the compact space Y. By the Compact-Closed Lemma 3.39 (1) this implies that $X \cap Y$ itself is compact. ■

50.3. **The Transversal Preimage Theorem.** The following definition can be viewed as generalization of Definition 50.3.

Definition 50.7. Let M be a smooth manifold and let X be a proper submanifold of M. Let Y be some other smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(\partial M)$ is a union of components of ∂Y.

(1) Let $Q \in Y$ with $f(Q) \in X$. We say that X intersects f transversally in Q if

$$\mathrm{D}f_Q(\mathrm{T}_Q Y) + \mathrm{T}_{f(Q)} X \;=\; \mathrm{T}_{f(Q)} M.$$

If $Q \in \partial Y$ (still with $f(Q) \in X$), then we also demand that $f(Q) \in \partial M$ and that [542]

$$\mathrm{D}f_Q(\mathrm{T}_Q(\partial Y)) + \mathrm{T}_{f(Q)}(\partial X) \;=\; \mathrm{T}_{f(Q)}(\partial M).$$

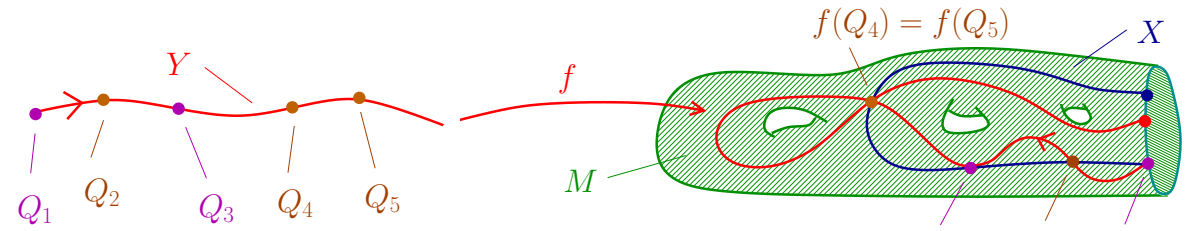
(2) Let C be a subset of Y. We say that X intersects f transversally on C if for every $Q \in C$ with $f(Q) \in X$ the submanifold X intersects the map f transversally in Q.

(3) We say that X intersects f transversally if for every $Q \in Y$ with $f(Q) \in X$ the submanifold X intersects the map f transversally in Q.

(4) We abbreviate the expression "X intersects f transversally" by $X \pitchfork f$.

[541] We performed this calculation in Footnote 540.

[542] If $Q \in \partial Y$ with $f(Q) \in \partial X$, then by hypothesis the component C of ∂Y that contains Q satisfies $f(C) \subseteq \partial M$. It follows from the hypothesis that f is smooth together with the Smooth Manifold–Boundary



the intersection of f with X is transverse at Q_2, Q_4, Q_5
the intersection of f with X is not transverse at Q_1 and Q_3

Examples.

(1) Let M be a smooth manifold and let X and Y be two proper submanifolds of M. We denote by $i\colon Y \to M$ the inclusion map. It follows immediately from the definitions that i intersects X transversally if and only if X intersects Y transversally in the sense of Definition 50.3.

(2) Let Y and M be smooth manifolds and let $f\colon Y \to M$ be a smooth map. Furthermore let $x \in M \setminus \partial M$. Note that this implies that $\{x\} \subseteq M$ is a proper smooth submanifold.



f is transverse to $\{x\} \subseteq M$ f is *not* transverse to $\{x\} \subseteq M$

It follows easily from comparing Definition 50.7 with Definition 32.1 that the map f is transverse to the 0-dimensional submanifold $\{x\}$ if and only if x is a regular value of $f\colon Y \to M$ and also of $f|_{\partial Y}\colon \partial Y \to M$. □

The following theorem might ring a bell. Considering the above examples we see that it is quite close to the statement of the Submanifold Transversal Intersection Theorem 50.4 (2) and to the statement of the Regular Value Theorem 32.5.

Theorem 50.8. (Transversal Preimage Theorem) Let M be a smooth manifold and let X be a proper submanifold of M of codimension l. Furthermore let Y be some other smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(X \setminus \partial X) \subseteq Y \setminus \partial Y$ and such that $f^{-1}(\partial M)$ is a union of components of ∂Y. We denote by $f_\partial\colon \partial Y \to \partial M$ the restriction of f to ∂Y. If X intersects f transversely, i.e. if $X \pitchfork f$, then the following statements hold:

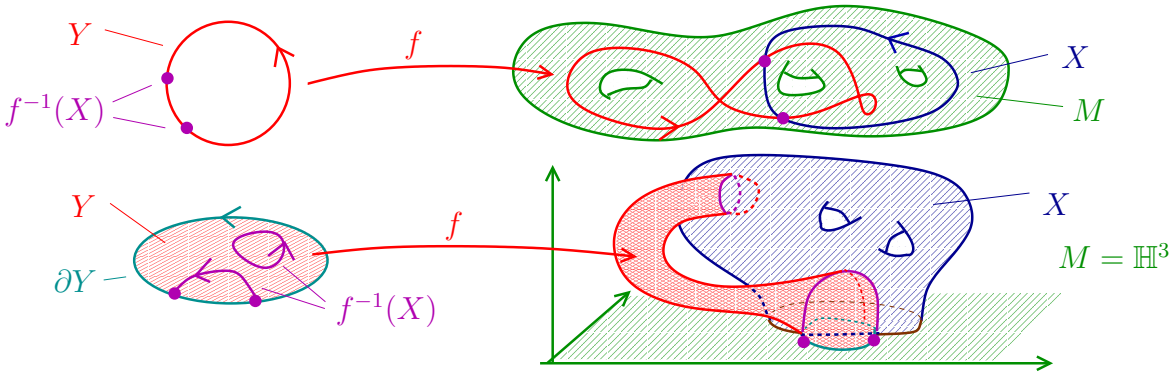
(1) The preimage $f^{-1}(X)$ is a proper submanifold of Y of codimension l where the boundary $\partial(f^{-1}(X))$ is given by $(f_\partial)^{-1}(X) = (f_\partial)^{-1}(\partial X)$.

(2) For every $P \in f^{-1}(X)$ we have

$$\mathrm{T}_P(f^{-1}(X)) \;=\; (\mathrm{D}f_P)^{-1}(\mathrm{T}_{f(P)}X).$$

Proposition 25.27 (1a) and (2c) that f restricts to a smooth map $f\colon C \to \partial M$. Thus we obtain that $\mathrm{D}f_Q(\mathrm{T}_Q(\partial Y)) = \mathrm{D}f_Q(\mathrm{T}_Q C)$ is a vector subspace of $\mathrm{T}_{f(Q)}(\partial M)$.

(3) Let M, X and Y be oriented. Furthermore let $P \in Y$.
   (a) By hypothesis the homomorphism $\mathrm{D}f_P \colon \mathrm{T}_P Y \to \mathrm{T}_{f(P)} M$ intersects $\mathrm{T}_P X$ transversally. Thus we can equip $\mathrm{T}_P(f^{-1}(X)) = (\mathrm{D}f_P)^{-1}(\mathrm{T}_{f(P)} X)$ with the preimage orientation that we introduced in Definition 31.16. These orientations define an orientation on the smooth manifold $f^{-1}(X)$.
   (b) The equality $\partial(f^{-1}(X)) = (f_\partial)^{-1}(\partial X)$ is an equality of oriented smooth manifolds.[543]
(4) If Y is compact, then so is $f^{-1}(X)$.



Remark.

(1) The preimage orientation that we introduced in Definition 31.16 is defined via the following diagram:

$$\begin{array}{ccccc} \mathrm{T}_{f(P)} X & \hookrightarrow & \mathrm{T}_{f(P)} M & \twoheadrightarrow & \mathrm{T}_{f(P)} M / \mathrm{T}_{f(P)} X \\ & & & & \cong \uparrow {\scriptstyle [v] \mapsto [\mathrm{D}f_P(v)]} \\ \mathrm{T}_P(f^{-1}(X)) & \hookrightarrow & \mathrm{T}_P Y & \twoheadrightarrow & \mathrm{T}_P Y / \mathrm{T}_P(f^{-1}(X)). \end{array}$$

The orientation on $\mathrm{T}_P(f^{-1}(X))$ is the unique orientation that makes the orientations in the short exact sequences compatible[544] and the right vertical map orientation-preserving.

(2) Let X and Y be proper submanifolds of a smooth submanifold M. If we apply the Transversal Preimage Theorem 50.8 to the inclusion $i \colon Y \to M$, then we see that $i^{-1}(X) = X \cap Y$ is a proper submanifold of Y. It follows from an elementary argument, see Exercise 25.29 (b), that $X \cap Y$ is also a proper submanifold of M. This discussion shows that we can recover the statement of the Submanifold Transversal Intersection Theorem 50.4 (2) from the Transversal Preimage Theorem 50.8.

(3) Suppose that in the setting of the Transversal Preimage Theorem 50.8 the submanifold X consists of a single point $\{x\}$. In this setting the Transversal Preimage Theorem 50.8

[543] It takes a while to figure out what orientations we are talking about. We give $f^{-1}(X)$ the orientation from (3a) and we give $\partial(f^{-1}(X))$ the boundary orientation from Convention 31.39. Furthermore we equip ∂X, ∂Y and ∂M with the boundary orientations. By definition of transverse intersection we also have $\partial X \pitchfork f_\partial$. Thus $(f_\partial)^{-1}(\partial(X))$ is a proper submanifold of ∂Y and we use (3a) to equip $(f_\partial)^{-1}(\partial X)$ with an orientation.

[544] We refer to Definition 31.11 for the definition of compatible orientations.

(1), (2) and (4) specializes to the Regular Value Theorem 32.5 (1), (2) and (4). Furthermore, if we equip this 0-dimensional manifold with the orientation given by $+1$, then an inspection of the definitions shows that the preimage orientation on $f^{-1}(x)$ that we just introduced agrees with the orientation on $f^{-1}(x)$ that we introduced in the Regular Value Theorem 32.5 (3). □

Proof. Let M be an m-dimensional smooth manifold and let X be a proper submanifold of M of codimension l. Let Y be some other smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(X \setminus \partial X) \subseteq Y \setminus \partial Y$, such that $f^{-1}(\partial M)$ is a union of components of ∂Y and such that $X \pitchfork f$.

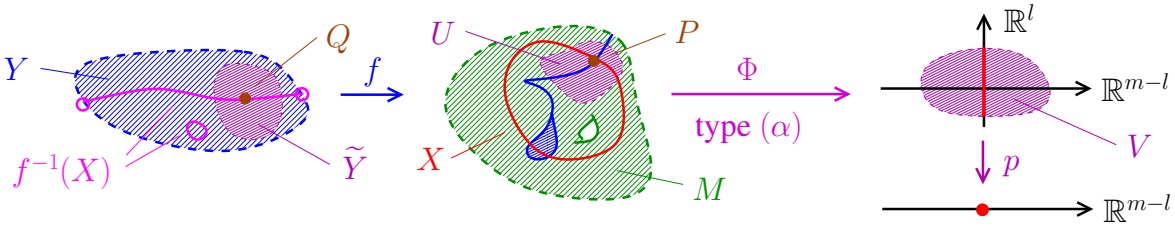
(1) We need to show that the preimage $f^{-1}(X)$ is a proper submanifold of Y of codimension l with boundary $\partial(f^{-1}(X)) = (f_\partial)^{-1}(\partial X)$. First note that X is a closed subset of M since X is a proper submanifold. Therefore $f^{-1}(X)$ is a closed subset of Y. Thus it remains to show that every point Q in $f^{-1}(X)$ admits a submanifold chart of type (α) or of type (β). To prove this statement it suffices to show that every point Q in $f^{-1}(X)$ admits an open neighborhood $\widetilde{Y} \subseteq Y$ such that $f^{-1}(X) \cap \widetilde{Y}$ is a proper submanifold of $\widetilde{Y}$.

Let $Q \in f^{-1}(X)$. We set $P := f(Q) \in X$. We need to distinguish the two cases that $P \in X \setminus \partial X$ and that $P \in \partial X$.

(a) First we consider the case that $P \in X \setminus \partial X$. Note that in this case it follows from the hypothesis that X is a proper submanifold of M that there exists a submanifold chart $\Phi\colon U \to V$ of type (α) around P such that $\Phi(P) = 0$.
Since we assume that $f^{-1}(X \setminus \partial X) \subseteq Y \setminus \partial Y$ we see that $\widetilde{Y} := f^{-1}(U) \cap (Y \setminus \partial Y)$ is an open neighborhood of Q. Now it suffices to prove the following claim:

Claim. The subset $f^{-1}(X) \cap \widetilde{Y}$ is a proper submanifold of $\widetilde{Y}$.

Proof. We write $\widetilde{f} := f|_{\widetilde{Y}}$. Since Φ is a submanifold chart of type (α) we know that we have $\Phi(X \cap U) \subseteq \{0\} \times \mathbb{R}^{m-l}$. We denote by $p\colon \mathbb{R}^m = \mathbb{R}^{m-l} \times \mathbb{R}^l \to \mathbb{R}^{m-l}$ the natural projection.



Note that

$$f^{-1}(X) \cap \widetilde{Y} \overset{\text{since } \widetilde{f} = f|_{\widetilde{Y}}}{=} \widetilde{f}^{-1}(X) \cap \widetilde{Y} \overset{\text{definition of } \widetilde{Y}}{=} f^{-1}(X) \cap f^{-1}(U) \cap (Y \setminus \partial Y)$$
$$\overset{\text{by the Image-Preimage Lemma 1.9 (7)}}{=} f^{-1}(X \cap U) \cap (Y \setminus \partial Y) \overset{\text{since } f = \widetilde{f} \text{ on } Y \setminus \partial Y}{=} \widetilde{f}^{-1}(X \cap U) = (p \circ \Phi \circ \widetilde{f})^{-1}(0).$$

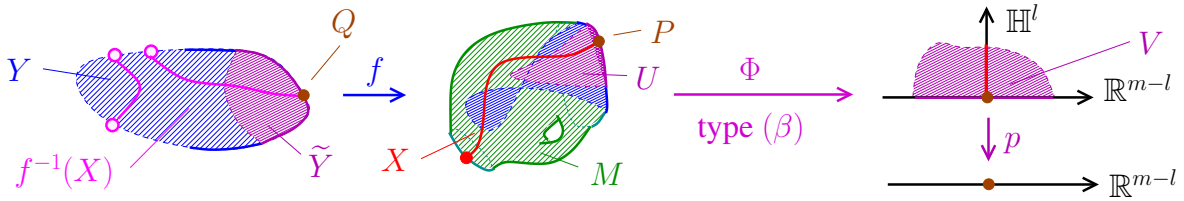
Note that the Regular Value Theorem 32.5 says that $(p \circ \Phi \circ \widetilde{f})^{-1}(0)$ is a proper submanifold of $\widetilde{Y}$ if 0 is a regular value of $p \circ \Phi \circ \widetilde{f}\colon \widetilde{Y} \to \mathbb{R}^{m-l}$ and if 0 is also a regular value of the restriction of $p \circ \Phi \circ \widetilde{f}$ to $\partial \widetilde{Y}$. First note that by our clever

choice of $\widetilde{Y}$ we know that the preimage of $\partial\widetilde{Y}$ is the empty set. So it remains to show that 0 is a regular value of $p \circ \Phi \circ \widetilde{f} \colon \widetilde{Y} \to \mathbb{R}^{m-l}$.
By definition of a regular value we need to show that each $Q' \in \widetilde{Y}$ is a regular point of $p \circ \Phi \circ \widetilde{f}$, i.e. we need to show that $\mathrm{D}(p \circ \Phi \circ \widetilde{f})_{Q'} \colon \mathrm{T}_{Q'}\widetilde{Y} \to \mathrm{T}_0\mathbb{R}^{m-l}$ is an epimorphism. The fact that this map is indeed an epimorphism follows easily from our hypothesis that f is transverse to X together with the Subspace Isomorphism Lemma 50.6 applied to $W = \mathrm{T}_P M$, $U = \mathrm{T}_P X$ and $V = \mathrm{D}\widetilde{f}_Q(\mathrm{T}_Q\widetilde{Y})$. ⊞

(b) Now we consider the case that $P \in \partial X$. Note that in this case it follows from the hypothesis that X is a proper submanifold of M that there exists a submanifold chart $\Phi \colon U \to V$ of type (β) around P such that $\Phi(P) = 0$. By our hypothesis on $f^{-1}(\partial M)$ we know that there exists a component C of ∂Y with $Q \in \partial Y$ and with $f(C) \subseteq \partial M$. By the Smooth Manifold–Boundary Proposition 25.27 (2) we know that $\partial M \setminus f(C)$ is a closed subset. Thus we see that $\widetilde{Y} \coloneqq (f^{-1}(U)) \cap (Y \setminus (\partial Y \setminus C))$ is an open neighborhood of Q. Now it suffices to prove the following claim:

Claim. The subset $f^{-1}(X) \cap \widetilde{Y}$ is a proper submanifold of the smooth manifold $\widetilde{Y}$ with boundary $(f_\partial)^{-1}(\partial X) \cap \widetilde{Y}$.

Proof. We write $\widetilde{f} \coloneqq f|_{\widetilde{Y}}$. Since Φ is a submanifold chart of type (β) we know that V is an open subset of $\mathbb{H}^m$ and that $\Phi(X \cap U) = \{0\} \times \mathbb{H}^l$. As in the previous claim we need to show that each $Q' \in \widetilde{Y}$ is a regular point of $p \circ \Phi \circ \widetilde{f}$ and that $Q' \in \widetilde{Y}$ is a regular point of $p \circ \Phi \circ \widetilde{f}|_{\partial\widetilde{Y}}$. Let $Q' \in \widetilde{Y} \setminus \partial\widetilde{Y}$. It follows as before that $\mathrm{D}(p \circ \Phi \circ \widetilde{f})_{Q'} \colon \mathrm{T}_{Q'}\widetilde{Y} \to \mathrm{T}_0\mathbb{R}^{m-l}$ is an epimorphism.



Now assume that $Q' \in \partial\widetilde{Y}$. From the definition of $\widetilde{Y}$ we see that $Q' \in C$, in particular $Q' \in \partial Y$ and $f(Q') \in \partial M$. It follows easily from Definition 50.7 that $\mathrm{D}(p \circ \Phi \circ \widetilde{f})_{Q'} \colon \mathrm{T}_{Q'}(\partial\widetilde{Y}) \to \mathrm{T}_0\mathbb{R}^{m-l}$ is also an epimorphism. ⊞

(2) Let $Q \in f^{-1}(X)$. Evidently we have $\mathrm{D}f_Q(\mathrm{T}_Q(f^{-1}(X))) \subseteq \mathrm{T}_{f(Q)}(X)$, which implies that $\mathrm{T}_Q(f^{-1}(X)) \subseteq (\mathrm{D}f_Q)^{-1}(\mathrm{T}_{f(Q)}(X))$. Both sides are vector spaces of dimension $\dim(Y) - l$, which implies that equality holds.

(3) (a) We leave the verification of this statement to the reader.

(b) We need to show that the two orientations on $\partial(f^{-1}(X)) = (f_\partial)^{-1}(\partial X)$ agree. Thus let $P \in \partial(f^{-1}(X))$ we write $Q = f(P)$. For orientation it might help to consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{T}_Q X & \longrightarrow & \mathrm{T}_Q M & \longrightarrow & \mathrm{T}_Q M/\mathrm{T}_Q X \\
 & & & & \uparrow{\scriptstyle \mathrm{D}f_P} \\
\mathrm{T}_P(f^{-1}(X)) & \longrightarrow & \mathrm{T}_P Y & \longrightarrow & \mathrm{T}_P Y/\mathrm{T}_P(f^{-1}(X)) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} \\
\mathrm{T}_P((f_\partial)^{-1}(\partial X)) & \longrightarrow & \mathrm{T}_P(\partial Y) & \longrightarrow & \mathrm{T}_P(\partial Y)/\mathrm{T}_P((f_\partial)^{-1}(\partial X))
\end{array}$$

In the following we make a few preparations.

- The orientations on $\mathrm{T}_Q X$ and $\mathrm{T}_Q M$ define an orientation on $\mathrm{T}_Q M/\mathrm{T}_Q X$. We pick vectors $v_1, \dots, v_l \in \mathrm{T}_Q M/\mathrm{T}_Q X$ which represent this orientation.
- It follows from the above diagram that we can pick $\widetilde{v}_1, \dots, \widetilde{v}_l \in \mathrm{T}_P(\partial Y)$ such that $\mathrm{D}f_P(\widetilde{v}_i) = v_i$ for $i = 1, \dots, l$.

We use the following orientations:

$\mathrm{T}_P(f^{-1}(X))$	with the orientation from (3a) applied to $f\colon Y \to M$
$\mathrm{T}_P\left(\partial(f^{-1}(X))\right)$	with the boundary orientation from Convention 31.39
$\mathrm{T}_P((f_\partial)^{-1}(\partial X))$	with the orientation from (3a) applied to $f_\partial\colon \partial Y \to M$
$\mathrm{T}_P Y$	with the given orientation
$\mathrm{T}_P\left(\partial Y\right)$	with the boundary orientation from Convention 31.39.

To simplify the notation we view $\mathrm{T}_P(f^{-1}(X))$ as a vector subspace of $\mathrm{T}_P Y$. We pick an outward pointing vector $o \in \mathrm{T}_P(f^{-1}(X))$. It follows from Exercise 29.6 that o, viewed as a vector in $\mathrm{T}_P Y$, is also outward pointing. Now let $u_1, \dots, u_d$ be a basis of $\mathrm{T}_P(\partial(f^{-1}(X))) = \mathrm{T}_P((f_\partial)^{-1}(\partial X))$. We see:

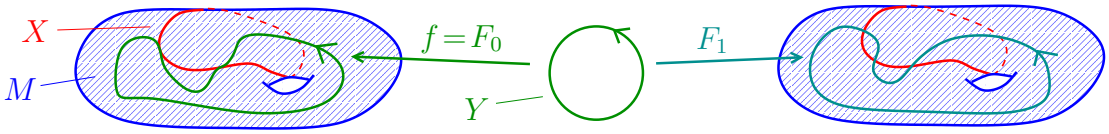
	$u_1, \dots, u_d$	is a positive basis of $\mathrm{T}_P\left(\partial(f^{-1}(X))\right)$
$\iff$	$o, u_1, \dots, u_d$	is a positive basis of $\mathrm{T}_P(f^{-1}(X))$
$\iff$	$o, u_1, \dots, u_d, \widetilde{v}_1, \dots, \widetilde{v}_l$	is a positive basis of $\mathrm{T}_P Y$
$\iff$	$u_1, \dots, u_d, \widetilde{v}_1, \dots, \widetilde{v}_l$	is a positive basis of $\mathrm{T}_P\left(\partial Y\right)$
$\iff$	$u_1, \dots, u_d$	is a positive basis of $\mathrm{T}_P((f_\partial)^{-1}(\partial X))$.

(4) Suppose that Y is compact. Since X is a proper submanifold of M it is in particular a closed subset of M. Therefore $f^{-1}(X)$ is a closed subset of the compact space Y. By the Compact-Closed Lemma 3.39 (1) this implies that $f^{-1}(X)$ itself is compact. ∎

50.4. **The Transversality Theorem.** Our goal of this section is to state the Transversality Theorem. For the reader's convenience we first state a simplified version before we state the bells-and-whistles version.

Theorem 50.9. (Transversality Theorem) Let M be a smooth manifold with $\partial M = \varnothing$ and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f\colon Y \to M$ be a smooth map.

(1) There exists a smooth homotopy F from $F_0 = f\colon Y \to M$ to a map $F_1\colon Y \to M$ that is transverse to X.
(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.

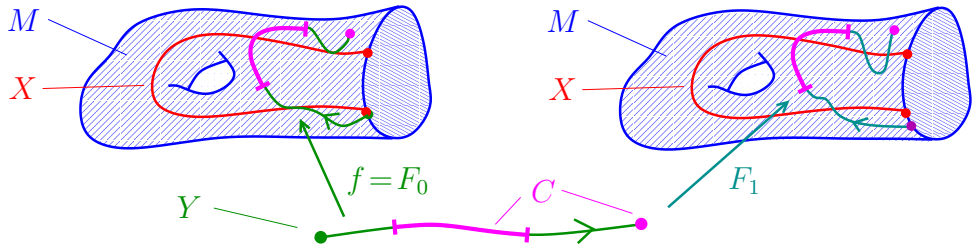


We postpone the proof of the Transversality Theorem 50.9 to Section 50.5. We move straight on to a more general statement.

Theorem 50.10. (Transversality Theorem II) Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f\colon Y \to M$ be a smooth map such that $f^{-1}(\partial M)$ is a union of components of ∂Y. We

assume that $X \pitchfork f$ on $\partial Y \setminus f^{-1}(\partial M)$. Finally let C be a closed subset of Y such that $X \pitchfork f$ on C.

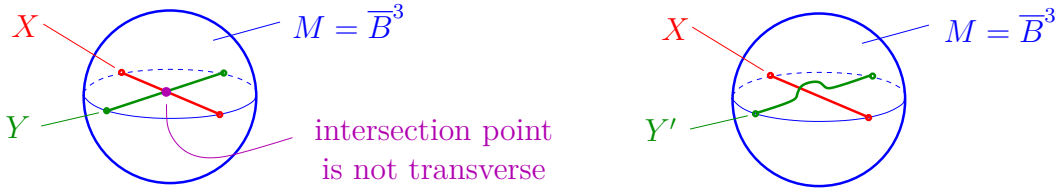
(1) There exists a smooth homotopy $F\colon Y \times [0,1] \to M$ with the following properties:
  (a) F is a smooth homotopy rel C from $F_0 = f$ to a map F_1 with $X \pitchfork F_1$.
  (b) For each $t \in [0,1]$ we have $F_t^{-1}(\partial M) = f^{-1}(\partial M)$.[545]
  (c) If[546] $\partial X \pitchfork f|_{\partial Y}$, then the set of $t \in [0,1]$ such that $X \pitchfork F_t$ has full measure.
(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.



Remark.

(1) Similar statements are also proved in [**Lee2002**, Theorem 6.36] and [**GPol1974**, p. 70]. Most statements of the theorem also follow from Proposition 4.4.4, Theorem 4.5.6 and Proposition 4.5.7 of [**Wall2016**].[547]
(2) In Exercise 50.11 we will prove a variation on the Transversality Theorem 50.10 which does not require X to be a proper submanifold of M.
(3) For many purposes the Avoid-Tubular Neighborhoods Proposition 263.16 can be a useful variation on the Transversality Theorem 50.9. □

Example. Let M be a smooth manifold. Furthermore let X and Y be two proper submanifolds of M. If $\dim(X) + \dim(Y) < \dim(M)$, then one deduces easily from the definitions that X and Y intersect transversally if and only if X and Y are disjoint. With this observation the Transversality Theorem 50.9 (2) says in particular that any two proper 1-dimensional submanifolds X and Y of the 3-dimensional ball $\overline{B}^3$ can be made disjoint after a smooth isotopy:



If the boundaries are already disjoint, then we can apply the Transversality Theorem 50.9 (2) with $C = \partial Y$ and we see that we can make X and Y disjoint while keeping X and the boundary of Y fixed. □

The following is a typical and often used consequence of the above transversality theorems.

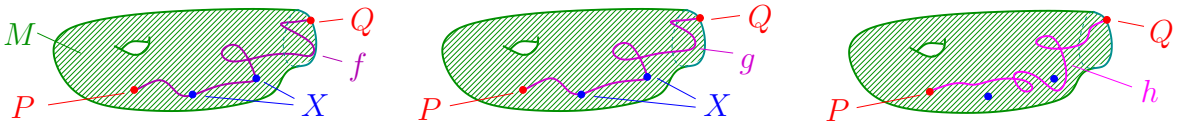
[545] In particular if f is proper, i.e. if $f^{-1}(\partial M) = \partial Y$, then for each t we have $F_t^{-1}(\partial M) = \partial Y$, i.e. each F_t is proper.
[546] This hypothesis is surely not necessary, but it makes our life much easier.
[547] A first look at [**Wall2016**, Theorem 4.5.6] might be slightly confusing since it demands that X is a "submanifold of $J^r(M,N)$" for some $r \in \mathbb{N}_0$. But as is explained on page 101 of [**Wall2016**], we have $J^0(M,Y) = M \times Y$, and $X \times \{*\}$ is clearly a submanifold of $M \times Y$ for any $* \in Y \setminus \partial Y$.

Corollary 50.11. (Complement-of-Submanifold Connected Corollary) Let M be a connected n-dimensional smooth manifold and let X be a proper k-dimensional smooth submanifold of M. If $k \leq n-2$, then $M \setminus X$ is path-connected.

Proof. Let $P, Q \in M \setminus X$. Since M is connected we know by the Manifold–Local Properties Lemma 23.8 (3a) together with the Smooth Path-Connectivity Proposition 25.32 that there exists a smooth map $f\colon [0,1] \to M$ with $f(0) = P$ and $f(1) = Q$.

Using the Smooth Collar Neighborhood Theorem 35.6 one can show, see Exercise 35.6, that there exists in fact a smooth path $g\colon [0,1] \to M$ with $g(0) = P$ and $g(1) = Q$ and with $g^{-1}(\partial M) \subseteq \{0,1\}$.

Note that the fact that $P, Q \in M \setminus X$ implies that $X \pitchfork f$ on $\partial([0,1]) = \{0,1\}$. Thus we can apply the Transversality Theorem 50.10 to the closed subset $C = \{0,1\}$ and we obtain a smooth map $h\colon [0,1] \to M$ with $h(0) = g(0) = f(0) = P$ and $h(1) = g(1) = f(1) = Q$ and such that $X \pitchfork h$.



But since $\dim(X)+\dim([0,1]) < \dim(M)$ the last condition just says that $X \cap h([0,1]) = \varnothing$. In other words, h is a path in $M \setminus X$ that connects P and Q. ∎

We now turn to the proofs of the Transversality Theorems. In Section 50.5 we provide the proof of the Transversality Theorem 50.9. This proof has a neat idea and is quite readable. Afterwards in Section 50.6 we provide the proof of the Transversality Theorem 50.10. This proof is much more technical and arguably much less fun for the author and the reader.

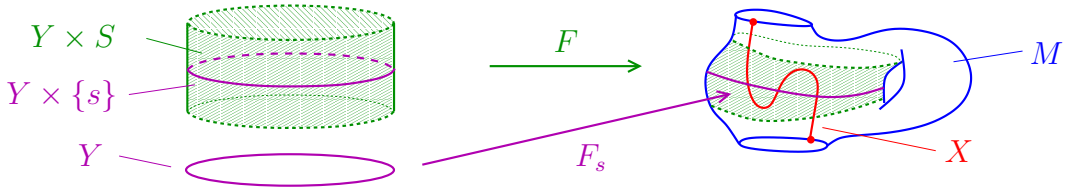
50.5. **Proof of the Transversality Theorem 50.9.** First we introduce or recall the following easy-to-digest notation.

Notation 50.12. Let $F\colon Y \times S \to M$ be a map. For each $s \in S$ we set

$$\begin{array}{rccc} F_s\colon & Y & \to & M \\ & y & \mapsto & F(y,s). \end{array}$$

The proofs of the Transversality Theorems 50.9 and 50.10 rest on the following proposition:

Proposition 50.13. (Parametric Transversality Proposition) Let Y and M be two smooth manifolds and let $X \subseteq M$ be a proper smooth submanifold. Furthermore let S be a smooth manifold with empty boundary. Finally let $F\colon Y \times S \to M$ be a smooth map. If $F\colon Y \times S \to M$ is transverse to X, then for almost every[548] $s \in S$ the map $F_s\colon Y \to M$ is also transverse to X.



[548]Recall that, according to Definition 37.1, the expression "almost every" means that the set of points $s \in S$ for which F_s is transverse to X has full measure.

Proof of the Parametric Transversality Proposition 50.13. Since $F\colon Y \times S \to M$ is transverse to the proper submanifold $X \subseteq M$ we obtain from the Transversal Preimage Theorem 50.8 that $W := F^{-1}(X)$ is a submanifold of $Y \times S$. Let $\pi_S\colon Y \times S \to S$ be the obvious projection. By Sard's Theorem 38.1 we know that the set of regular values of the smooth maps $\pi_S\colon W \to S$ and $\pi_S\colon \partial W \to S$ have full measure. Thus, according to Definition 50.7, it suffices to show the following claim:

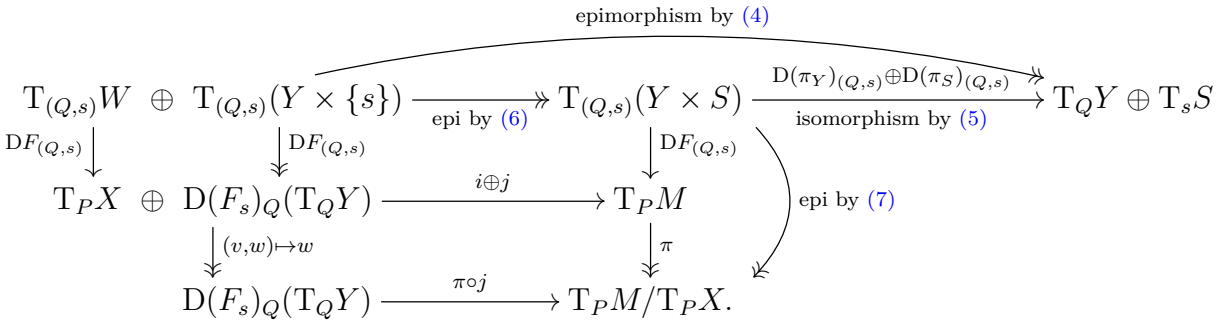
Claim.

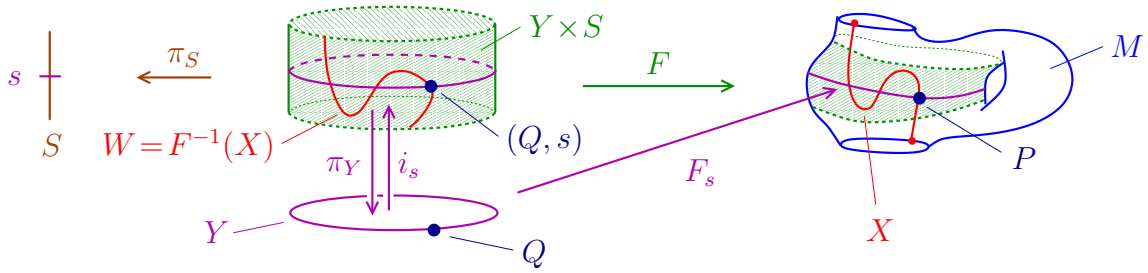
(1) If $s \in S$ is a regular value of $\pi_S|_W\colon W \to S$, then for any $Q \in Y$ with $F_s(Q) \in X$ we have $\mathrm{T}_{F_s(Q)}X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) = \mathrm{T}_{F_s(Q)}M$.

(2) If $s \in S$ is a regular value of $\pi_S|_{\partial W}\colon \partial W \to S$, then for any $Q \in \partial Y$ with $F_s(Q) \in \partial X$ we have $\mathrm{T}_{F_s(Q)}\partial X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q \partial Y) = \mathrm{T}_{F_s(Q)}\partial M$.

In other words, if $s \in S$ is a regular value of $\pi_S|_W\colon W \to S$ and if $s \in S$ is a regular value of $\pi_S|_{\partial W}\colon \partial W \to S$, then $F_s\colon Y \to M$ is transverse to X.

Proof. We prove Statement (1). The proof of Statement (2) is basically identical. Thus let $s \in S$ be a regular value of $\pi_S|_W\colon W \to S$ and let $Q \in F_s^{-1}(X)$. We set $P := F_s(Q) \in X$. We need to show that $\mathrm{T}_P X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) = \mathrm{T}_P M$. We consider the following diagram:





We make the following clarifications and observations:

(1) The top right horizontal map is induced by the projection $\pi_Y\colon Y \times S \to Y$ and the projection $\pi_S\colon Y \times S \to S$.

(2) The maps i, j are the inclusion maps and the map π is the natural projection. The remaining undecorated maps are given by inclusions.

(3) It is clear that the diagram commutes.

(4) Since s is a regular value of $\pi_S|_W\colon W \to S$ one sees easily that the composition of the top two horizontal maps is an epimorphism.

(5) Using the Product Orientation Proposition 31.48 (3b) one can easily show that the horizontal map to the top right is an isomorphism.

(6) It follows from (4) and (5) that the top left horizontal map is an epimorphism.
(7) Recall that by hypothesis the map $F\colon Y \times S \to M$ is transverse to X. Thus we know that $\mathrm{T}_P X + \mathrm{D}F_{(Q,s)}(\mathrm{T}_{(Q,s)}(Y \times S)) = \mathrm{T}_P M$. This implies that the composition of the middle two vertical maps is an epimorphism.
(8) By the Transversal Preimage Theorem 50.8 we have $\mathrm{T}_{(Q,s)}W = (\mathrm{D}F_{(Q,s)})^{-1}(\mathrm{T}_P X)$, which implies that $\mathrm{D}F_{(Q,s)}(\mathrm{T}_{(Q,s)}W) \subseteq \mathrm{T}_P X$.
(9) Clearly we have $\mathrm{D}F_{(Q,s)}(\mathrm{T}_{(Q,s)}(Y \times \{s\})) = \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y)$.
(10) It follows from the above that the map $\pi \circ j\colon \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) \to \mathrm{T}_P M/\mathrm{T}_P X$ is an epimorphism.
(11) It follows from (10) that $\mathrm{T}_P X + \mathrm{D}(F_s)_Q(\mathrm{T}_Q Y) = \mathrm{T}_P M$, which is exactly what we needed to show. ∎

Proof of the Transversality Theorem 50.9. Let M be a smooth manifold such that $\partial M = \varnothing$ and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f\colon Y \to M$ be a smooth map. First we need to show that there exists a smooth homotopy F from $F_0 = f\colon Y \to M$ to a map $F_1\colon Y \to M$ that is transverse to X. To do so, we make the following preparations:

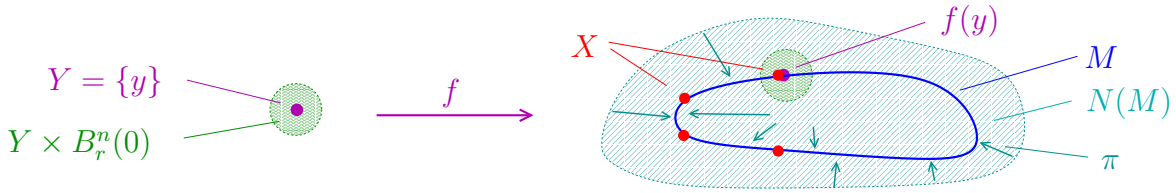
(1) By the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can assume that M is a proper smooth submanifold of some $\mathbb{R}^n$.
(2) By the Neighborhood Projection Theorem 35.2 there exists an open neighborhood $N(M)$ of $M \subseteq \mathbb{R}^n$ and a smooth map $\pi\colon N(M) \to M$ which has the following two properties:
   (a) π is a retraction,
   (b) π is a submersion, i.e. for each $Q \in N(M)$ the differential $\mathrm{D}\pi_Q$ is an epimorphism.

Since Y is compact we obtain from the Compact Image Lemma 4.15 together with the elementary Exercise 10.1 that there exists an $r > 0$ such that for each $y \in Y$ we have $B^n_r(f(y)) \subseteq N(M)$.

Therefore we can consider the map

$$\begin{array}{rcl} F\colon Y \times B^n_r(0) & \to & M \\ (y, v) & \mapsto & \pi(f(y) + v). \end{array}$$

Evidently the map $Y \times B^n_r(0) \to N(M)$ given by $(y, v) \mapsto f(y) + v$ is a submersion. Furthermore by (2b) we know that $\pi\colon N(M) \to M$ is a submersion. This discussion shows that for each $(y, v) \in Y \times B^n_r(0)$ the map $\mathrm{D}F_{(y,v)}$ is an epimorphism. This implies that $F\colon Y \times B^n_r(0) \to M$ is transverse to X.



By the Parametric Transversality Proposition 50.13 there exists a $v \in B^n_r(0)$ such that $F_v\colon Y \to M$ is transverse to X. The desired smooth homotopy is now given by the map

$$\begin{array}{rcl} H\colon Y \times [0,1] & \to & M \\ (y, t) & \mapsto & F_{t\cdot v}(y) = F(y, t \cdot v) = \pi(f(y) + t \cdot v). \end{array}$$

Now assume that f is a smooth embedding. Since $F_0 = f$ it follows from the Deform Smooth Embedding Proposition 41.9 (1) that, after possibly shrinking r, we can assume that for each $v \in B^n_r(0)$ the map $F_v \colon Y \to M$ is a smooth embedding. In this setting the above map H is now actually a smooth isotopy. ■

50.6. **Proof of the Transversality Theorem 50.10.** In hindsight the proof of the Transversality Theorems 50.9 was not all that bad. The proof of the considerably more elaborate Transversality Theorem 50.10 will be, well, considerably more elaborate.

We start out with the following lemma, which is of independent interest.

Lemma 50.14. Given any $m, n, r \in \mathbb{N}_0$ the set

$$\{A \in \mathrm{M}(m \times n, \mathbb{R}) \mid \operatorname{rank}(A) \geq r\}$$

is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$.

Remark. Lemma 50.14 is a slight generalization of Exercise 4.49. □

Proof. Given $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$ with $\#I = r$ and $\#J = r$ we consider the map

$$\begin{array}{rccl} \det_{I,J} \colon & \mathrm{M}(m \times n, \mathbb{R}) & \to & \mathbb{R} \\ & A = (a_{ij}) & \mapsto & \text{determinant of the } r \times r\text{-matrix } (a_{ij})_{i \in I, j \in J}. \end{array}$$

It follows from the Leibniz Formula 2.18 that $\det_{I,J}$ is continuous. Using this notation we can formulate the following claim:

Claim. Let $A = (a_{ij}) \in \mathrm{M}(m \times n, \mathbb{R})$. The following two statements are equivalent:

(1) $\operatorname{rank}(A) \geq r$.
(2) There exist subsets $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$ with $\#I = r$ and $\#J = r$ such that $\det_{I,J}(A) \neq 0$.

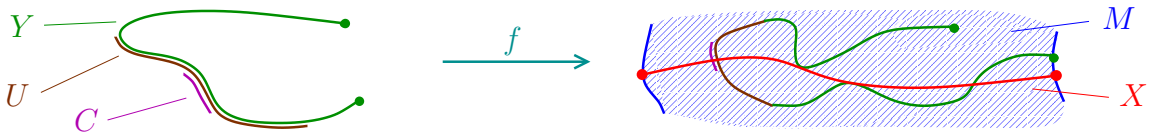
Proof. We leave it to the reader to pull out their favorite linear algebra book and to prove the claim. ⊞

It follows from the claim that

$$\{A \in \mathrm{M}(m \times n, \mathbb{R}) \mid \operatorname{rank}(A) \geq r\} = \bigcup_{\substack{I, J \text{ with} \\ \#I = r \text{ and } \#J = r}} (\det_{I,J})^{-1}(\mathbb{R} \setminus \{0\}).$$

Since each $\det_{I,J}$ is continuous we see that the set to the left is indeed open. ■

Lemma 50.15. (Transversality-Extends-to-Neighborhood Lemma) Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a smooth manifold and let $f \colon Y \to M$ be a smooth map. If C is a closed subset of Y such that $X \pitchfork f$ on C and such that $f^{-1}(\partial M) \cap C = (\partial Y) \cap C$, then there exists an open neighborhood U of C such that $X \pitchfork f$ on U.



Proof. It suffices to show that for each $P \in C \cap f^{-1}(X)$ there exists an open neighborhood V of $P \in Y$ such that $X \pitchfork f$ on V.

First we consider the case that $P \notin \partial Y$. After using charts of type (i) for $P \in Y$ and for $f(P) \in X$ we can assume that Y is an open subset of $\mathbb{R}^l$, that M is an open subset of $\mathbb{R}^m$ and that $X = \{0\} \times \mathbb{R}^k$. The fact that $X \pitchfork f$ on P implies that the $(m \times (l+k))$-matrix

$$(\mathrm{D}f_P \ \mathfrak{e}_{m-k+1} \ \dots \ \mathfrak{e}_m)$$

has rank m. It follows from Lemma 50.14 that this condition is satisfied in an open neighborhood of P.

The case that $P \in \partial Y$ is dealt with in almost the same way. In this case we use a chart of type (ii) for $P \in \partial Y$ and for $f(P) \in \partial X$. With the same argument as above one can find an open neighborhood V_1 of $P \in Y$ that satisfies the first condition for transversality and one can find an open neighborhood V_2 of $P \in \partial Y$ that satisfies the second condition for transversality of points on the boundary. On the open neighborhood $V = V_1 \cap (Y \setminus (\partial Y \setminus V_2))$ we now have $X \pitchfork f$. ∎

Now we turn to the actual proof of the Transversality Theorem 50.10. The following lemma implies in particular the Transversality Theorem 50.10 if $\partial M = \varnothing$.

Auxiliary Lemma 50.16. Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold, let $f\colon Y \to M$ be a smooth map and let C be a closed subset of Y such that $X \pitchfork f$ on C. We assume that $f^{-1}(\partial M) \subseteq C$.

(1) There exists a smooth homotopy $F\colon Y \times [0,1] \to M$ rel C with the following properties:
   (a) F is a smooth homotopy rel C from $F_0 = f$ to a map F_1 with $X \pitchfork F_1$.
   (b) For each $t \in [0,1]$ we have $F_t^{-1}(\partial M) = f^{-1}(\partial M)$.
   (c) The set of $t \in [0,1]$ such that $X \pitchfork F_t$ has full measure.
(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.

Proof. Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold, let $f\colon Y \to M$ be a smooth map and let C be a closed subset of Y such that $X \pitchfork f$ on C and such that $f^{-1}(\partial M) \subseteq C$.

By the Transversality-Extends-to-Neighborhood Lemma 50.15 there exists an open neighborhood U of $C \subseteq Y$ such that $X \pitchfork f$ on U. By the Manifold Paracompact-Normal Proposition 23.9 (2) we know that Y is normal. Using the Normal-Open-Closed-Neighborhood Lemma 12.3 twice we see that there exist open subsets V, W of Y with $C \subseteq W \subseteq \overline{W} \subseteq V \subseteq \overline{V} \subseteq U$.

Since $\overline{V}$ and $Y \setminus U$ are disjoint closed subsets of Y we obtain from Urysohn's Lemma 27.10 for Manifolds that there exists a smooth map $\tau\colon Y \to \mathbb{R}$ with $\tau|_{\overline{V}} \equiv 0$ and $\tau|_{Y \setminus U} \equiv 1$. In the following we proceed similar to the proof on page 1155.

(i) By the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can assume that M is a proper smooth submanifold of some $\mathbb{H}^n$.
(ii) By hypothesis we have $f^{-1}(\partial M) \subseteq C$. It follows that $f(Y \setminus W) \subseteq M \setminus \partial M$. Since Y is compact we know by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $f(Y \setminus W)$ is a closed subset of M that is disjoint from ∂M. We now know by Exercise 35.10, which is a mild generalization of the Neighborhood Projection Theorem 35.2, that there exists an open neighborhood N of $f(Y \setminus W) \subseteq \mathbb{R}^n$ with $N \cap \partial M = \varnothing$ and a smooth map $\pi\colon N \to M$ which has the following two properties:
   (a) For each $P \in f(Y \setminus W)$ we have $\pi(P) = P$.

(b) π is a submersion, i.e. for each $Q \in N$ the differential $\mathrm{D}\pi_Q$ is an epimorphism.

Since $Y \setminus W$ is compact we obtain from the Compact Image Lemma 4.15 together with the elementary Exercise 10.1 that there exists an $r > 0$ such that for each $y \in f(Y \setminus W)$ we have $B^n_r(f(y)) \subseteq N$. Therefore we can consider the map

$$\begin{array}{rcl} F\colon Y \times B^n_r(0) & \to & M \setminus \partial M \\ (y, v) & \mapsto & \begin{cases} \pi(f(y) + \tau(y) \cdot v), & \text{if } y \in Y \setminus \overline{W}, \\ y, & \text{if } y \in \overline{W}. \end{cases} \end{array}$$

Since $\tau|_{\overline{V}} \equiv 0$ we see that on the open subset $V \times B^n_r(0)$ the map F is just given by the smooth map $(y, v) \mapsto y$. Furthermore by construction F is smooth on $(Y \setminus \overline{W}) \times B^n_r(0)$. Since F is smooth on open subsets that cover $Y \times B^n_r(0)$ we obtain from the Smooth Pasting Proposition 25.19 that F is indeed smooth.

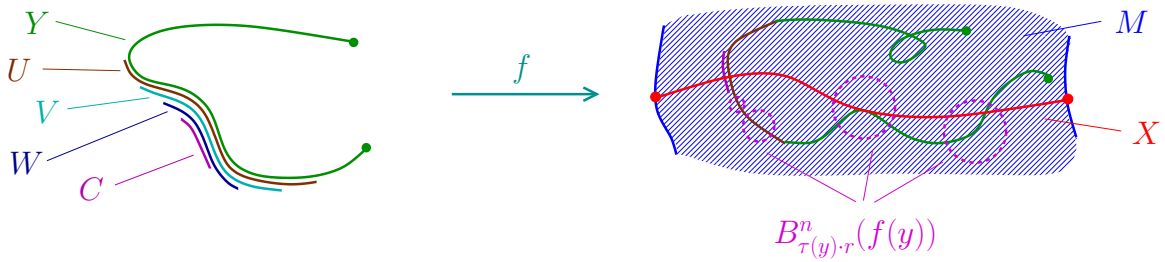
Claim. We have $X \pitchfork F$.

Proof. Let $y \in Y$.

- If $\tau(y) = 0$, then by construction of τ we have $y \in U$. By the choice of U this implies that $X \pitchfork f$ at y. But since $\tau(y) = 0$ this implies that $X \pitchfork F$ at every (y, v).
- If $\tau(y) \neq 0$, then it follows from the fact that $f(y) + \tau(y) \cdot B^n_r(0) = B^n_{r \cdot \tau(y)}(f(y))$ is an open neighborhood of $f(y)$ contained in $N \subseteq \mathbb{R}^n$ and the fact that π is a submersion that for each $(y, v) \in Y \times B^n_{r \cdot \tau(y)}(0)$ the map $\mathrm{D}F_{(y,v)}$ is an epimorphism, in particular we see that $X \pitchfork F$ at (y, v). ⊞

The remainder of the argument is similar to the proof on page 1155. We consider

$$V \coloneqq \{v \in B^n_r(0) \mid X \text{ is transverse to } F_v\}.$$



Note that it follows from the claim and the Parametric Transversality Proposition 50.13 that V has full measure in $B^n_r(0)$. It follows from Exercise 37.3 (which is a consequence of the Mini-Fubini Lemma 37.14), that there exists a $v \in B^n_r(0)$ such that $X \pitchfork F_v$, i.e. such that $F_v\colon Y \to M$ is transverse to X, and such that the following condition is satisfied:

(∗) The set of $t \in [0, 1]$ such that $X \pitchfork F_{t \cdot v}$ has full measure in $[0, 1]$.

Next we consider the map

$$\begin{array}{rcl} H\colon Y \times [0, 1] & \to & M \\ (y, t) & \mapsto & F_{t \cdot v}(y) = F(y, t \cdot v). \end{array}$$

It remains to verify that the map H has all the desired properties:

(a) Since F is smooth we see that H is smooth. By construction H is a homotopy from $H_0 = f$ to a map H_1 with $X \pitchfork H_1$. Note that for $y \in C$ we have, by the choice of τ, that $H(y, t) = y$ for all $t \in [0, 1]$. In other words, H is indeed a homotopy rel C.

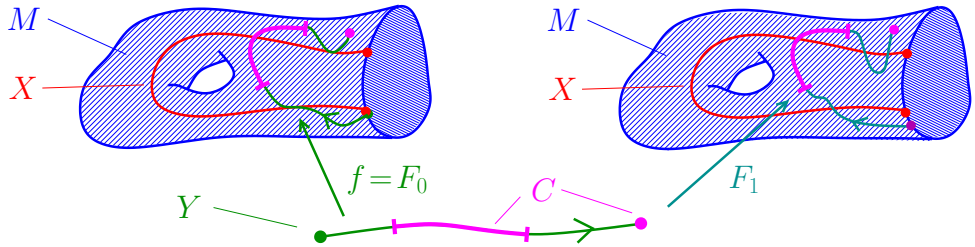
(b) Since $N \cap \partial M = \varnothing$ and since $\partial M \subseteq C$ we see that $H_t^{-1}(\partial M) = f^{-1}(\partial M)$ for any $t \in [0, 1]$.

(c) It follows from (∗) that the set of $t \in [0,1]$ such that $X \pitchfork H_t$ has full measure.

Finally assume that f is a smooth embedding. Since $F_0 = f$ it follows from the Deform Smooth Embedding Proposition 41.9 (1) that, after possibly shrinking r, we can assume that for each $s \in B^n_r(0)$ the map $F_s \colon Y \to M$ is a smooth embedding. In this setting the above map H is now actually a smooth isotopy. ■

Proof of the Transversality Theorem 50.10. Let M be a smooth manifold and let X be a proper submanifold of M. Furthermore let Y be a compact smooth manifold and let $f \colon Y \to M$ be a smooth map such that $f^{-1}(\partial M)$ is a union of components of ∂Y. To simplify the language and notation we assume that $f^{-1}(\partial M) = \partial Y$. Finally let C be a closed subset of Y such that $X \pitchfork f$ on C. Recall that we need to show the following:

(1) There exists a smooth homotopy $F \colon Y \times [0,1] \to M$ with the following properties:
  (a) F is a smooth homotopy rel C from $F_0 = f$ to a map F_1 with $X \pitchfork F_1$.
  (b) For each $t \in [0,1]$ we have $F_t^{-1}(\partial M) = \partial M$, i.e. F is a proper smooth homotopy.
  (c) If $\partial X \pitchfork f|_{\partial Y}$, then the set of $t \in [0,1]$ such that $X \pitchfork F_t$ has full measure.
(2) If f is a smooth embedding, then we can choose F to be a smooth isotopy.



We start out with the following claim:

Claim.

(1) There exists a proper smooth homotopy G rel C from f to a map $g \colon Y \to M$ such that $X \pitchfork g$ on $C \cup \partial Y$.
(2) If f is a smooth embedding, then we can choose G to be a smooth isotopy.

Proof. [549] We make the following preparations:

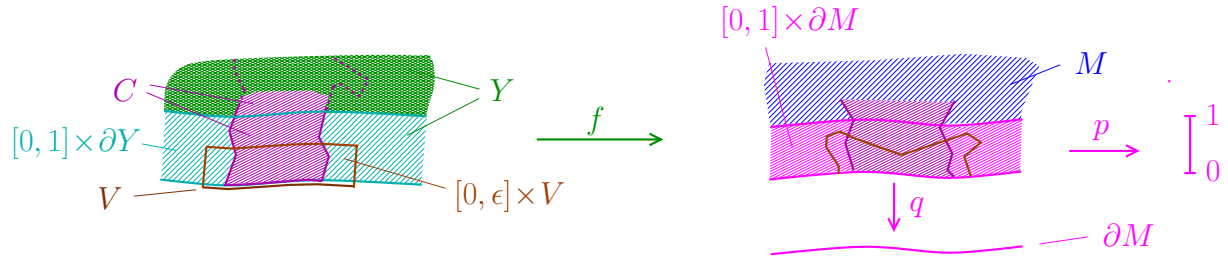
(a) We write $f_\partial := f|_{\partial Y} \colon \partial Y \to \partial M$.
(b) By the definition of transversality we know that the submanifold $\partial X \subseteq \partial M$ is transversal to the map $f_\partial \colon \partial Y \to \partial M$ on $\partial Y \cap C$.
(c) By the Transversality-Extends-to-Neighborhood Lemma 50.15 there exists an open neighborhood U of $\partial Y \cap C \subseteq \partial Y$ such that $\partial X \pitchfork f_\partial$ on U.
(d) By the Manifold Paracompact-Normal Proposition 23.9 (2) we know that ∂Y is normal. By the Normal-Open-Closed-Neighborhood Lemma 12.3 this implies that there exists an open subset $V \subseteq \partial Y$ with $\partial Y \cap C \subseteq V \subseteq \overline{V} \subseteq U$.
(e) By the Smooth Collar Neighborhood Theorem 35.6 we can pick a collar $[0,1] \times \partial Y$ for Y and we can pick a collar $[0,1] \times \partial M$ for M.
(f) Since Y is compact, since $C \subseteq Y$ is closed and since $[0,1] \times \partial Y$ is closed we see that $C \cap ([0,1] \times \partial Y)$ is compact. Furthermore since $[0,1] \times \partial Y$ is a neighborhood of ∂Y

[549] In many situations this claim can also be proved fairly easily using the Homotopic-to-Product-Near-Boundary Lemma 35.9 and Lemma 50.16 applied to ∂Y. But for the most general statements we have to get our hands dirty again.

and since V is a neighborhood of $C \cap \partial Y \subseteq \partial Y$ it follows from Exercise 7.2 that we can pick an $\epsilon > 0$ such that $C \cap ([0,\epsilon] \times \partial Y) = C \cap ([0,\epsilon] \times V)$.

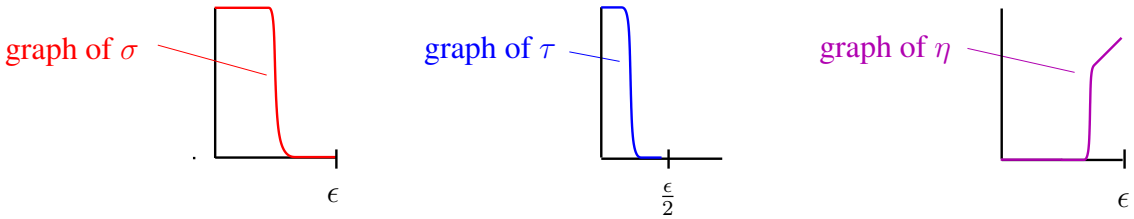
(g) It follows from the Basics-of-Collars Lemma 35.7, the compactness of the boundary ∂Y and the Tube Lemma 7.12, that after possibly shrinking ϵ, we can also assume that $f([0,\epsilon] \times \partial Y) \subseteq [0,1) \times \partial M$.

(h) We denote by $p\colon [0,1] \times \partial M \to [0,1]$ respectively $q\colon [0,1] \times \partial M \to \partial M$ the projection onto the first factor respectively second factor.



(i) It follows from the Smooth Transition Function Lemma 24.9 that we can pick the following types of smooth functions:

- $\sigma\colon [0,\epsilon] \to [0,1]$ such that $\sigma(t) = 1$ for $t \in [0, \frac{\epsilon}{4}]$ and $\sigma(t) = 0$ for $t \in [\frac{3\epsilon}{4}, \epsilon]$.
- $\tau\colon [0, \frac{\epsilon}{2}] \to [0,1]$ such that $\tau(t) = 1$ for $t \in [0, \frac{\epsilon}{8}]$ and $\tau(t) = 0$ for $t \in [\frac{3\epsilon}{8}, \frac{\epsilon}{2}]$.
- $\eta\colon [0,\epsilon] \to [0,\epsilon]$ such that $\eta(t) = 0$ for $t \in [0, \frac{5\epsilon}{8}]$ and $\eta(t) = t$ for $t \in [\frac{7\epsilon}{8}, \epsilon]$.



Now we can provide the actual proof of the claim. Since $\partial\partial M = \varnothing$ we know by Lemma 50.16 that there exists a smooth homotopy $F\colon \partial Y \times [0,1] \to \partial M$ rel $\partial Y \cap \overline{V}$ such that $F_0 = f_\partial$, $\partial X \pitchfork F_1$ and such that the following statement holds:

(∗) The set of $t \in [0,1]$ with $\partial X \pitchfork F_t$ has full measure in $[0,1]$.

Next we consider the maps

$$G'\colon Y \times [0,1] \to M$$
$$(y,t) \mapsto \begin{cases} f(y), & \text{if } y \notin [0,\epsilon] \times \partial Y, \\ ((1-t)\cdot p(f(s,\widetilde{y})) + t\cdot(s\cdot\sigma(s) + p(f(s,\widetilde{y}))\cdot(1-\sigma(s))), & \text{if } y = (s,\widetilde{y}) \text{ in } [0,\epsilon]\times\partial Y. \\ \quad q(f((1-t)\cdot s + t\cdot\eta(s),\widetilde{y}))), & \end{cases}$$

$$G''\colon Y \times [0,1] \to M$$
$$(y,t) \mapsto \begin{cases} f(y), & \text{if } y \notin [0,\epsilon] \times \partial Y, \\ (s\cdot\sigma(s) + p(f(s,\widetilde{y}))\cdot(1-\sigma(s)), F(\widetilde{y},\tau(s)\cdot t)), & \text{if } y = (s,\widetilde{y}) \in [0,\frac{\epsilon}{2}]\times\partial Y, \\ (\underbrace{s\cdot\sigma(s) + p(f(s,\widetilde{y}))\cdot(1-\sigma(s))}_{\in[0,1]}, \underbrace{q(f(\eta(s),\widetilde{y}))}_{\in\partial M}), & \text{if } y = (s,\widetilde{y}) \in [\frac{\epsilon}{2},\epsilon]\times\partial Y. \end{cases}$$

We make the following observations:

- (a) We have $G'_0 = f$.

(b) By design the above map $G' \colon Y \times [0,1] \to M$ is smooth on the two open subsets $([0,\epsilon) \times \partial Y) \times [0,1]$ and $(Y \setminus ([0, \frac{3\epsilon}{4}] \times \partial Y)) \times [0,1]$. It follows from the Smooth Pasting Proposition 25.19 that G' is smooth.
(c) For each $t \in [0,1]$ we have $(G'_t)^{-1}(\partial M) = \partial Y$.
(d) The homotopy G' is constant outside of $[0,\epsilon] \times \partial Y$ and it is constant on $[0,\epsilon] \times \overline{V}$. By the choice of ϵ this implies that G' is constant on C.

- (a) The maps G'_1 and G''_0 are the same.[550]
(b) By design the above map $G'' \colon Y \times [0,1] \to M$ is smooth on the two open subsets $([0,\epsilon) \times \partial Y) \times [0,1]$ and $(Y \setminus ([0, \frac{3\epsilon}{4}] \times \partial Y)) \times [0,1]$. It follows again from the Smooth Pasting Proposition 25.19 that G'' is smooth.
(c) For each $t \in [0,1]$ we have $(G''_t)^{-1}(\partial M) = \partial Y$.
(d) The homotopy G'' is constant outside of $[0,\epsilon] \times Y$ and it is constant on $[0,\epsilon] \times \overline{V}$. By the choice of ϵ this implies that G'' is constant on C.
(e) By choice of F_1 we have $\partial X \pitchfork G''_1|_{\partial Y}$. In fact, it follows easily from the definition of G'' that we have $X \pitchfork G''_1$ on ∂Y.

As in the proof of the Smooth Isotopy-Equivalence Relation Proposition 41.5 we can use the Stationary-Near-The-Ends-Lemma 41.7 to combine the two smooth homotopies G' and G'' to obtain a smooth homotopy $G \colon Y \times [0,1] \to M$ with $G_0 = G'_0$, such that $G_1 = G''_1$ and such that each G_t is either of the form G'_s or G''_s. It follows easily from the above discussion of G' and G'' that G is the desired smooth homotopy.

If f is a smooth embedding then we can arrange, using $(*)$ and the Deform Smooth Embedding Proposition 41.9 (2), by rescaling the original map G that G is actually a smooth isotopy. ⊞

It follows from the claim and the Smooth Isotopy-Equivalence Relation Proposition 41.5 that it remains to consider the case that $f \colon Y \to M$ is a smooth map with $f^{-1}(\partial M) = \partial Y$ such that $X \pitchfork f$ on $C \cup \partial Y$, where C is a closed subset of Y. But this case we have dealt with in Lemma 50.16. (Note that if $\partial X \pitchfork f|_{\partial Y}$, then we can ignore the claim and we can head straight to Lemma 50.16, which gives us a smooth isotopy such the set of $t \in [0,1]$ with $X \pitchfork F_t$ has full measure.) ■

50.7. Exercises for Chapter 50.

Exercise 50.1. Let M be a closed m-dimensional complex manifold, let $X \subseteq M$ be a proper complex submanifold (in the sense of Definition 40.11) of codimension k and let Y be a proper complex submanifold of M of codimension l. Show that if the submanifolds X and Y intersect transversally, then the intersection $X \cap Y$ is a complex submanifold of M.

Exercise 50.2. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. Show that the set

$$\{P \in \mathbb{R}^k \mid M \text{ is not transverse to the submanifold } \mathbb{R}^{n-k} \times \{P\}\}$$

has measure zero in $\mathbb{R}^k$.

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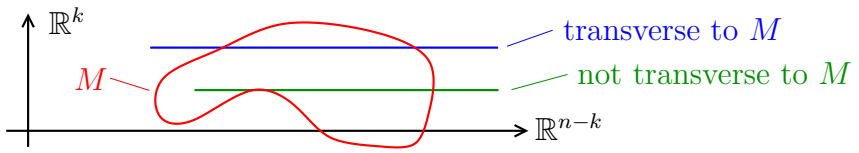
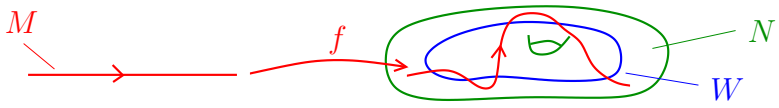
[550] By plugging in one sees that

$$\begin{array}{rccl} G'_1 = G''_0 \colon Y & \to & M & \\ y & \mapsto & \begin{cases} f(y), & \text{if } y \notin [0,\epsilon] \times \partial Y, \\ (s \cdot \sigma(s) + p(f(s,\widetilde{y})) \cdot (1-\sigma(s)), q(f(\eta(s),\widetilde{y}))), & \text{if } y = (s,\widetilde{y}) \in [0,\epsilon] \times \partial Y. \end{cases} \end{array}$$

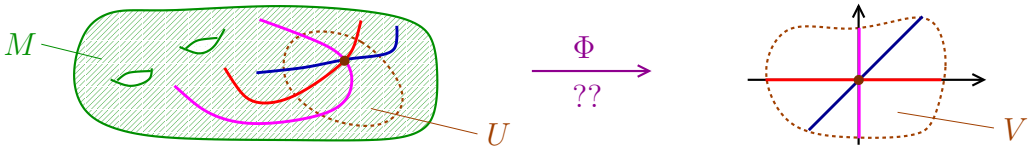


Exercise 50.3. Let M and N be two smooth manifolds with empty boundary. We suppose that $\dim(M) < \dim(N)$. Furthermore let $f\colon M \to N$ be a smooth map and let W be a submanifold of N with empty boundary. We suppose that for every point $P \in M$ with $f(P) \in W$ we have the equality $\mathrm{D}f_P(\mathrm{T}_P M) + \mathrm{T}_{f(P)}W = \mathrm{T}_{f(P)}N$. Show that $W \setminus f(M)$ has full measure in W.
Hint. Use the Local Smooth Embedding Theorem 30.13, the Submanifold Transversal Intersection Theorem 50.4 (2) and the Measure-Zero Properties Proposition 37.3.



Exercise 50.4. Let M be an oriented smooth manifold and let X and Y be two proper smooth submanifolds that intersect transversally. By the Submanifold Transversal Intersection Theorem 50.4 we obtain oriented submanifolds $X \cap Y$ and $Y \cap X$. What is the relationship between the orientations of $X \cap Y$ and $Y \cap X$?
Remark. This exercise is just a reminder for the reader to do Exercise 31.2.

Exercise 50.5. Let M be a 2-dimensional smooth manifold with $\partial M = \varnothing$, let A, B, C be three 1-dimensional smooth submanifolds of M and let $P \in A \cap B \cap C$ be a point such that all pairs among A, B and C intersect transversally. Does there exist a chart $\Phi\colon U \to V$ for $P \in M$ with $\Phi(P) = (0,0)$, $\Phi(A) \subseteq \mathbb{R} \times \{0\}$, $\Phi(B) \subseteq \{0\} \times \mathbb{R}$ and with $\Phi(C) \subseteq \{(x,x) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\}$?

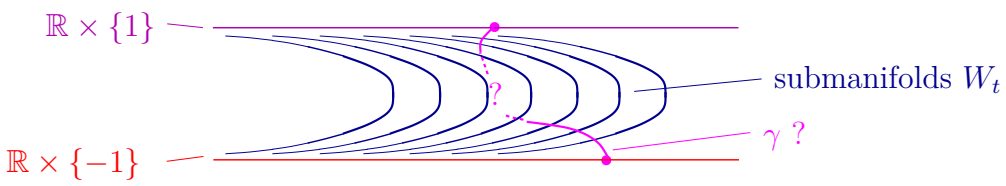


Exercise 50.6. Let $L \subseteq S^1 \times S^2$ be a 1-dimensional smooth submanifold. Show that L is smoothly isotopic to a 1-dimensional smooth submanifold that is contained in $S^1 \times S^2_{\geq 0}$.

Exercise 50.7. We consider the smooth manifold $M = \mathbb{R} \times [-1,1]$ together with the submanifolds

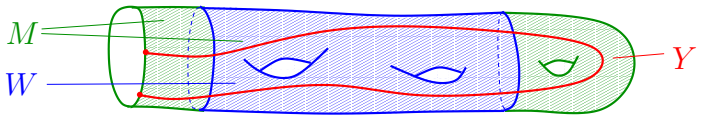
$$W_t := \{(t + \tfrac{1}{y^2-1}, y) \,|\, y \in (-1,1)\} \quad \text{where } t \in \mathbb{R}.$$

Show that there is no smooth path $\gamma\colon [-1,1] \to M$ that connects the bottom $\mathbb{R} \times \{-1\}$ with the top $\mathbb{R} \times \{1\}$ such that γ is transverse to every W_t, i.e. such that for every $s \in (-1,1)$ with $\gamma(s) \in W_t$ we have $\gamma'(s) \notin \mathrm{T}_{\gamma(s)}W_t$.
Remark. This decomposition of the interior of $M = \mathbb{R} \times [-1,1]$ into submanifolds is sometimes called a *Reeb foliation*.

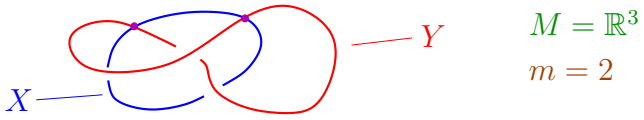


Exercise 50.8. Let M be a closed smooth manifold and let $\alpha\colon S^k \to M$ be a continuous map. Show that there exists an $r \in \mathbb{N}_0$ and a smooth embedding $\varphi\colon S^k \to \partial(M \times \overline{B}^r)$ such that the maps $S^k \xrightarrow{\alpha} M \xrightarrow{x \mapsto (x,0)} M \times \overline{B}^r$ and $\varphi\colon S^k \to M \times \overline{B}^r$ are homotopic.

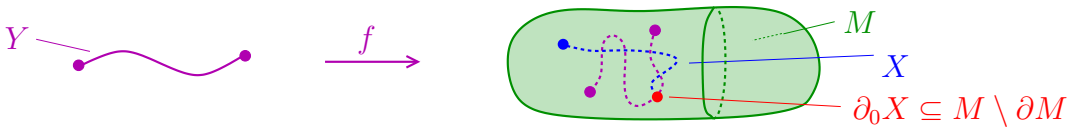
Exercise 50.9. Let M be an m-dimensional smooth manifold, let W be compact codimension-zero submanifold of M and let Y be a proper submanifold of M that intersects ∂W transversally. Show that $Y \cap W$ is a proper submanifold of W.



Exercise 50.10. Let M be an n-dimensional smooth manifold with $\partial M = \varnothing$ and let X and Y be closed submanifolds of M of dimensions k and l. We assume that there exists an $m \in \{0, \dots, n\}$ such that for every $P \in X \cap Y$ we have $\dim(\mathrm{T}_P X + \mathrm{T}_P Y) = m$. Show that $X \cap Y$ is a proper submanifold of M of dimension $k + l - m$.
Hint. Given any $P \in M$ you can assume, after applying a chart, that you are working in an open subset of $\mathbb{R}^n$. Apply the Submanifold Transversal Intersection Theorem 50.4 to a suitable setting.



Exercise 50.11. Let M be a smooth manifold and let X be a submanifold of M that is a closed subset of M. Furthermore let Y be a compact smooth manifold and let $f\colon Y \to M$ be a smooth embedding such that $f^{-1}(\partial M)$ is a union of components of ∂Y. We assume that $f^{-1}(\partial X) = \varnothing$. Show that exists a smooth isotopy $F\colon Y \times [0,1] \to M$ with $F_0 = f$ and such that $F_1(Y) \cap X = \varnothing$.
Hint. First apply the Transversality Theorem 50.10 to $\partial_0 X = \partial X \setminus \partial M$ and then apply the Transversality Theorem 50.10 again to a suitable choice of smooth manifold and submanifold.



Project 50.1. Formulate and prove a version of the Submanifold Transversal Intersection Theorem 50.4 for three, or more generally $n \in \mathbb{N}$, submanifolds.

CHAPTER 51

The algebraic intersection number

In this section we will use the results from the last chapter to introduce the algebraic intersection number between transverse submanifolds of complementary dimension. The algebraic intersection number is one of the most popular notions of geometric topology. In this chapter we will use this concept to prove the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15 which implies, as we will see in the Surface-No Smooth Embedding Corollary 51.18, that there is no smooth embedding of $\mathbb{R}\mathrm{P}^2$ or the Klein bottle into $\mathbb{R}^3$.

This chapter is organized as follows:

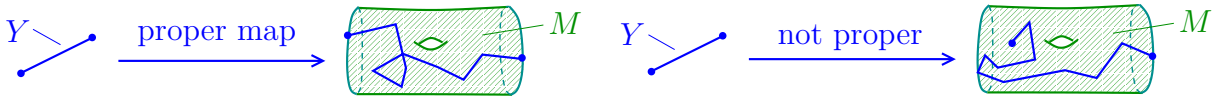
- In Section 51.1 we will introduce the algebraic intersection number between two oriented submanifolds of complementary dimensions.
- In Section 51.2 we will state the Intersection Number–Homotopy Invariance Theorem 51.9 which says that algebraic intersection numbers are unchanged under smooth homotopies.
- In Section 51.3 we will introduce the $\mathbb{Z}/2\mathbb{Z}$-algebraic intersection number for submanifolds of complementary dimension, without any orientability assumptions.
- In Section 51.4 we will state the Smooth Jordan-Brouwer Separation Theorem 51.15 and we will prove it using the $\mathbb{Z}/2\mathbb{Z}$-algebraic intersection number.
- In Section 51.5 we will provide exercises for Chapter 51.

In the next Chapter 52 we will introduce the linking number of say two disjoint oriented knots in S^3.

51.1. The algebraic intersection number between a submanifold and a map.

We start out with the following definition, which is a special case of Definition 30.7.

Definition 51.1. Let M be a topological manifold and let Y be a compact topological manifold. We say that a map $f\colon Y \to M$ is proper if $f^{-1}(\partial M) = \partial Y$.[551]



Definition 51.2. Let M be an m-dimensional smooth manifold and let $X \subseteq M$ be a compact proper submanifold of dimension k.

(1) Let Y be a compact smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a smooth map that intersects X transversally in the sense of Definition 50.7, i.e. such that $X \pitchfork f$.[552] By the Transversal Preimage Theorem 50.8 we know that $f^{-1}(X)$

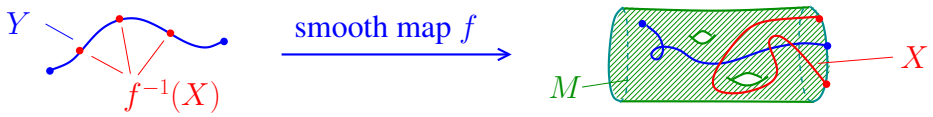
[551] It follows from the Compact Image Lemma 4.15 that for a compact topological manifold this definition is equivalent to Definition 30.7.

is a compact 0-dimensional manifold. As we remarked on page 624 this means that $f^{-1}(X)$ is a finite set. Thus we can define the

$$\text{geometric intersection number of } X \text{ and } f \;:=\; \#f^{-1}(X).$$

(2) If Y is a compact $(m-k)$-dimensional submanifold of M with $X \pitchfork Y$, then we define the

$$\text{geometric intersection number of } X \text{ and } Y \;:=\; \#(X \cap Y).$$



The geometric intersection number (except perhaps for its parity) is frankly not a very interesting number. In the presence of orientations we can introduce a significantly more interesting number.

Definition 51.3. Let M be an oriented m-dimensional smooth manifold and let $X \subseteq M$ be a compact oriented proper submanifold of dimension k.

(1) Let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a smooth map with $X \pitchfork f$. By the Transversal Preimage Theorem 50.8 we know that $f^{-1}(X)$ is a compact *oriented* 0-dimensional submanifold. Thus we can define

$$\text{algebraic intersection number of } X \text{ and } f \;:=\; X \cdot f \;:=\; \sum_{P \in f^{-1}(X)} \underbrace{\text{orientation of } P}_{\in\{-1,1\}}.$$

(2) If Y is a compact $(m-k)$-dimensional submanifold of M with $X \pitchfork Y$, then we define[554]

$$\text{algebraic intersection number of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \underbrace{\text{orientation of } P}_{\in\{-1,1\}}.$$

Example. Let M and Y be compact oriented smooth manifolds of the same dimension and let $f\colon Y \to M$ be a smooth map with $f^{-1}(\partial M) \subseteq \partial Y$. By Sard's Theorem 38.1 there exists a regular value $P \in M \setminus \partial M$. Basically by definition the fact that P is regular means that the map $f\colon Y \to M$ is transverse to the 0-dimensional submanifold $\{P\}$. As usual we give $\{P\}$ the orientation given by $+1$. If one goes through the definitions one sees that

$$\{P\} \cdot f \;=\; \deg(f\colon Y \to M),$$

where $\deg(f\colon Y \to M)$ denotes the degree that we introduced in Definition 46.3 and Definition 47.3. □

The above definition of the algebraic intersection number is very elegant, but it is arguably not very intuitive. Therefore we now give an alternative description of the algebraic intersection number.

[552] If one looks carefully at Definition 50.7 one sees that our dimension conditions imply that in our setting $X \pitchfork f$ implies that $X \cap f(\partial Y) = \varnothing$. Subconsciously this piece of information gets used frequently.

[554] If $\iota\colon Y \to M$ denotes the inclusion map, then it follows immediately from the remark on page 809 that $X \cdot Y = X \cdot \iota$.

Definition 51.4. Let M be an oriented m-dimensional smooth manifold and let $X \subseteq M$ be a compact oriented proper submanifold of dimension $k \in \{1, \dots, m-1\}$.

(1) Let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a smooth map with $X \pitchfork f$. Let $P \in f^{-1}(X)$. We set $\operatorname{sign}(P) := 1$, if

$$(\text{positive basis of } \mathrm{T}_{f(P)}X, \mathrm{D}f_P(\text{positive basis of } \mathrm{T}_P Y)) \;=\; \text{positive basis of } \mathrm{T}_{f(P)}M.$$

Otherwise we set $\operatorname{sign}(P) := -1$.[555]

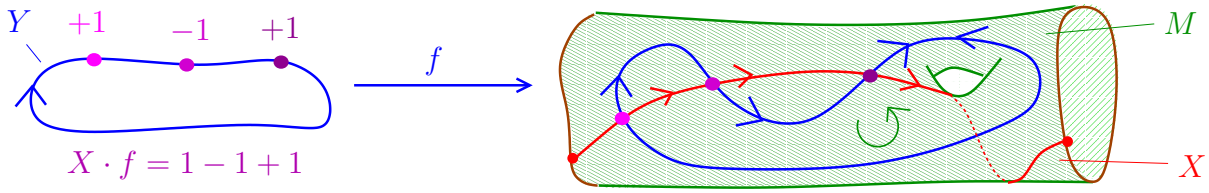
(2) Let Y be a compact oriented proper submanifold of M of dimension $m-k$. We denote by $i\colon Y \to M$ the inclusion. For any $P \in X \cap Y = i^{-1}(X)$ we define $\operatorname{sign}(P)$ as in (1).

Lemma 51.5. (Intersection Numbers-via-Signs Lemma) If we are in the setting of the above definition, then

$$X \cdot f \;=\; \sum_{P \in f^{-1}(X)} \operatorname{sign}(P).$$

In particular, if Y is a compact proper submanifold of M with $X \pitchfork Y$, then

$$X \cdot Y \;=\; \sum_{P \in X \cap Y} \operatorname{sign}(P).$$



geometric intersection number of X and f $= 3$
algebraic intersection number of X and f $= 1$

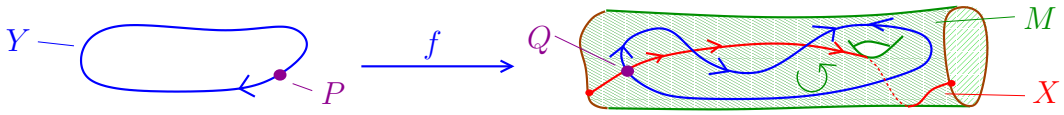
Proof. The lemma follows almost immediately from the definitions. But, for peace of mind, let us provide at least some details, to make sure that the sign conventions really do work out. It suffices to show the following claim:

Claim. For each $P \in f^{-1}(X)$ the preimage orientation on P equals $\operatorname{sign}(P)$.

Proof. Let $P \in f^{-1}(X)$. We write $Q = f(P)$. Note that by the Transversal Preimage Theorem 50.8 (2) we have $(\mathrm{D}f_P)^{-1}(\mathrm{T}_Q X) = \mathrm{T}_P\{P\}$. Using this observation, we consider, as in the definition of the preimage orientation in Definition 31.16, the following diagram:

$$\begin{array}{ccccc}
\mathrm{T}_Q X & \hookrightarrow & \mathrm{T}_Q M & \twoheadrightarrow & \mathrm{T}_Q M/\mathrm{T}_Q X \\
 & & \overset{\mathrm{D}f_P}{\cong}\uparrow & & \cong\uparrow [v]\mapsto[\mathrm{D}f_P(v)] \\
\mathrm{T}_P\{P\} & \hookrightarrow & \mathrm{T}_P Y & \overset{\cong}{\twoheadrightarrow} & \mathrm{T}_P Y/\mathrm{T}_P\{P\}.
\end{array}$$

The diagonal map exists since $\mathrm{T}_P\{P\}$ is 0-dimensional. We equip $\mathrm{T}_Q M/\mathrm{T}_Q X$ with the orientation given by the orientation on $\mathrm{T}_P Y$ and the isomorphism $\mathrm{D}f_P$.



[555] It is a straightforward exercise in linear algebra to show that the definition of $\operatorname{sign}(P)$ does not depend on the choice of the positive bases of $\mathrm{T}_P X$ and $\mathrm{T}_P Y$.

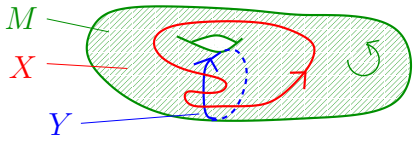
It follows immediately from the above diagram that the following three statements are equivalent:

(1) The preimage orientation of $P \in f^{-1}(X)$ equals $+1$.
(2) The orientations on $\mathrm{T}_Q X, \mathrm{T}_Q M$ and $\mathrm{T}_Q M/\mathrm{T}_Q X$ are compatible in the sense of Definition 31.11.
(3) $\mathrm{sign}(P) = 1$. ■

If we are given two oriented submanifolds X and Y of M, then of course it makes sense to consider $X \cdot Y$ as well as $Y \cdot X$. The following lemma elucidates the relationship between these algebraic intersection numbers.[556]

Lemma 51.6. (Intersection Number–Symmetry Lemma) Let M be an oriented smooth manifold and let $X, Y \subseteq M$ be compact oriented proper submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. If $X \pitchfork Y$, then also $Y \pitchfork X$ and

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \;\cdot\; X \cdot Y.$$



$$\begin{aligned} X \cdot Y &= +1 - 1 + 1 = +1 \\ Y \cdot X &= -1 + 1 - 1 = -1 \end{aligned}$$

Proof. We set $k = \dim(X)$, $l = \dim(Y)$ and $m = \dim(M)$. The cases $k = 0$ and $l = 0$ follow easily from the definitions. Thus let us now suppose that $k, l \in \{1, \dots, m-1\}$. In this case it follows from the claim in the proof of the Intersection Numbers-via-Signs Lemma 51.5 that at each $P \in X \cap Y = Y \cap X$ the orientations differ by the sign of the determinant of the following matrix:

$$\begin{pmatrix} 0_{l\times k} & \mathrm{id}_l \\ \mathrm{id}_k & 0_{k\times l} \end{pmatrix}.$$

We leave it to the reader to bring out their favorite tricks for computing determinants to show that the determinant of this matrix equals $(-1)^{k\cdot l}$. ■

The following elementary lemma shows how the algebraic intersection number is affected by several basic operations.

Lemma 51.7. (Intersection Number–Basics Lemma) Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold of dimension k, let Y be a compact oriented smooth manifold of dimension $m - k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$.

(1) As usual we denote by $-X$ the submanifold with the opposite orientation. We have

$$(-X) \cdot f \;=\; -X \cdot f.$$

(2) Let $\widetilde{M}$ be an oriented smooth manifold with $\dim(\widetilde{M}) = \dim(M)$. If $\Phi\colon M \to \widetilde{M}$ is a smooth embedding, then

$$\Phi(X) \cdot (\Phi \circ f) \;=\; \begin{cases} X \cdot f, & \text{if } \Phi \text{ is orientation-preserving} \\ -X \cdot f, & \text{if } \Phi \text{ is orientation-reversing.} \end{cases}$$

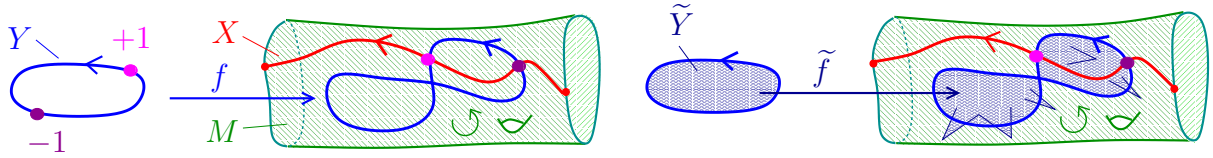
[556] The next lemma deals with a special case of Exercise 50.4.

(3) Let $\widetilde{Y}$ be an oriented smooth manifold. If $\Psi\colon \widetilde{Y} \to Y$ is a diffeomorphism, then

$$X \cdot (f \circ \Psi) \;=\; \begin{cases} X \cdot f, & \text{if } \Psi \text{ is orientation-preserving} \\ -X \cdot f, & \text{if } \Psi \text{ is orientation-reversing.} \end{cases}$$

Proof. All statements follow easily from the definitions together with the Compatible Orientations Lemma 31.12 (2). ■

Proposition 51.8. (Intersection-Number Zero-if-Extends Proposition) Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold of dimension k, let Y be a *closed* oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper[557] smooth map with $X \pitchfork f$. If there exists a compact oriented smooth manifold $\widetilde{Y}$ with[558] $\partial\widetilde{Y} = Y$ and if there exists a continuous map[559] $\widetilde{f}\colon \widetilde{Y} \to M$ with $\widetilde{f}|_{\partial\widetilde{Y}} = f$, then $X \cdot f = 0$.



Remark. The Intersection-Number Zero-if-Extends Proposition 51.8 gives in particular a solution to Exercise 46.12. □

Proof. Let $\widetilde{Y}$ be a compact oriented smooth manifold of dimension $m-k+1$ with $\partial\widetilde{Y} = Y$ and let $\widetilde{f}\colon \widetilde{Y} \to M$ be a continuous map with $\widetilde{f}|_{\partial\widetilde{Y}} = f$. We need to show that $X \cdot f = 0$.

Claim. There exists a smooth map $\gamma\colon \widetilde{Y} \to M \setminus \partial M$ with $\gamma|_{\partial\widetilde{Y}} = f$ such that $X \pitchfork \gamma$.

Proof. We will perform three upgrades to $\widetilde{f}$ to get the desired map.

(α) Since $\widetilde{f}|_{\partial\widetilde{Y}} = f$ is smooth we obtain from the Smooth-on-Boundary-Submanifold Lemma 36.5 together with the Whitney Approximation Theorem 36.1[560] that we can replace $\widetilde{f}$ by a *smooth* map $\alpha\colon \widetilde{Y} \to M$ with $\alpha|_{\partial\widetilde{Y}} = f$.

(β) By hypothesis the map $f\colon Y \to M$ is proper, which in the present setting just means that $f(Y) = \alpha(Y) \subseteq M \setminus \partial M$. It follows from the Push-Away-from-Boundary Corollary 35.8 that we can replace α by a smooth map $\beta\colon \widetilde{Y} \to M \setminus \partial M$ with $\beta|_{\partial\widetilde{Y}} = \alpha|_{\partial\widetilde{Y}} = f$.

(γ) It follows from $X \pitchfork f$ that $X \pitchfork \beta$ on $Y = \partial\widetilde{Y}$. Since β takes values in $M \setminus \partial M$ we can apply the Transversality Theorem 50.10[561] to the map $\beta\colon \widetilde{Y} \to M \setminus \partial M$ and the closed subset $C := Y = \partial\widetilde{Y}$ of $\widetilde{Y}$. We obtain a smooth map $\gamma\colon \widetilde{Y} \to M \setminus \partial M$ with $\gamma|_{\partial\widetilde{Y}} = \widetilde{f}$ and with $X \pitchfork \gamma$. ⊞

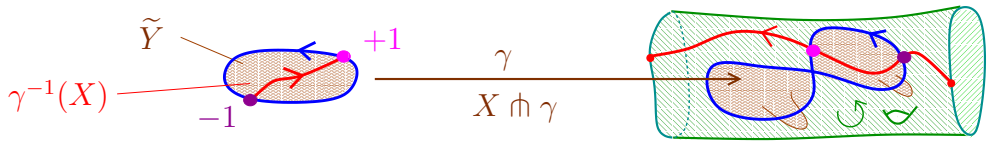
[557] Since Y is closed this just means that $f(Y) \subseteq M \setminus \partial M$.

[558] In Convention 31.39 we introduced the natural orientation of the boundary of an oriented smooth manifold. With this definition the equality $\partial\widetilde{Y} = Y$ is supposed to be an equality of oriented smooth manifolds.

[559] Note that we do not assume that $\widetilde{f}$ is smooth.

[560] Note that the hypothesis that $f\colon Y \to M$ is proper implies that $f(\partial\widetilde{Y}) \subseteq M \setminus \partial M$, so we can indeed apply the Whitney Approximation Theorem 36.1.

[561] There is a slight issue: We have $X \pitchfork \beta$ on Y which is not the same as the condition $X \pitchfork \alpha$ which is required in the Transversality Theorem 50.10.



Given a compact oriented 0-dimensional smooth manifold A we denote by $\epsilon(A) \in \mathbb{Z}$ the sum of the orientations of the points in M. We now see that

$$X \cdot f \underset{\uparrow}{=} \epsilon(f^{-1}(X)) \underset{\uparrow}{=} \epsilon(\partial\gamma^{-1}(X)) \underset{\uparrow}{=} \epsilon\left(\begin{array}{c}\text{boundary of compact oriented}\\ \text{1-dimensional smooth manifold}\end{array}\right) \underset{\uparrow}{=} 0.$$

by definition

by the Transversal Preimage Theorem 50.8 (3) and (4), here we use that $X \pitchfork \gamma$, that $\gamma|_{\partial\widetilde{Y}} = f$ and that $\widetilde{Y}$ is compact

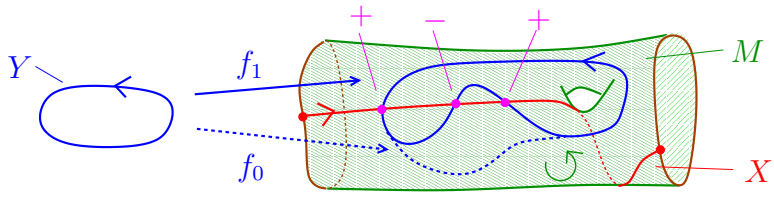
Boundary-of-Compact 1-Dimensional Smooth Manifold Proposition 31.42

We have thus proved that $X \cdot f = 0$. ■

51.2. **The Intersection Number–Homotopy Invariance Theorem.** The following theorem can be viewed as an analog of the Degree Homotopy Invariance Theorem 46.6.

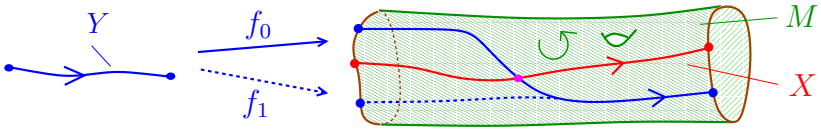
Theorem 51.9. (Intersection Number–Homotopy Invariance Theorem) Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold of dimension k, let Y be a *closed* oriented smooth manifold of dimension $m-k$ and let $f_0, f_1 \colon Y \to M$ be proper smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$. If there exists a homotopy[562] $F \colon Y \times [0,1] \to M$ between f_0 and f_1, then

$$X \cdot f_0 \;=\; X \cdot f_1.$$



$$X \cdot f_1 = -1+1-1 = +1$$
$$X \cdot f_0 = +1$$

Example. Note that in the statement of the Intersection Number–Homotopy Invariance Theorem 51.9 we cannot just drop the hypothesis that Y is closed.



$$X \cdot f_0 = -1$$
$$X \cdot f_1 = 0$$

More precisely, in the figure above we show a smooth manifold M, a proper submanifold X with $\partial X \neq \varnothing$ and two properly smoothly homotopic maps $Y \to M$ with $\partial Y \neq \varnothing$, which are both transverse to X, such that $X \cdot f_0 \neq X \cdot f_1$. □

As always in mathematics, under extra hypotheses one can in fact allow Y to have non-empty boundary. We will see one such instance with the Intersection Number–Homotopy Invariance Theorem II 51.10 and we will see more examples in Chapter 220.

Proof. Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold of dimension k, let Y be a closed oriented smooth manifold

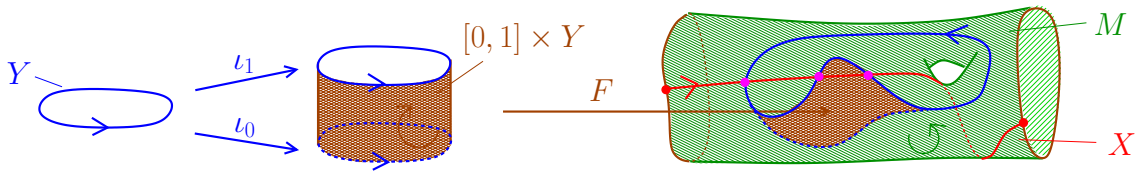
[562] Note that we do not demand that the homotopy is proper or smooth.

of dimension $m-k$ and let $f_0, f_1 \colon Y \to M$ be smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$ such that there exists a homotopy between f_0 and f_1. We need to show that $X \cdot f_1 - X \cdot f_0 = 0$.

Given $t \in [0,1]$ we equip $\{t\} \times Y$ with the product orientation that we introduced in Convention 31.49. We denote by

$$\begin{array}{rccc} \iota_t \colon & Y & \to & \{t\} \times Y \\ & y & \mapsto & (t,y) \end{array}$$

the obvious diffeomorphism. The hypothesis that f_0 and f_1 are homotopic implies that there exists a continuous map $F \colon [0,1] \times Y \to M$ with $F \circ \iota_0 = f_0$ and $F \circ \iota_1 = f_1$.



Now we perform the following calculation:

$$\begin{aligned} X \cdot f_1 - X \cdot f_0 &= X \cdot (F \circ \iota_1) - X \cdot (F \circ \iota_0) \\ &\overset{\text{by the Intersection Number–Basics Lemma 51.7 (1) since by the discussion on page 832 we know that } \iota_0 \text{ and } \iota_1 \text{ are orientation-preserving diffeomorphisms}}{=} X \cdot F|_{\{1\}\times Y} - X \cdot F|_{\{0\}\times Y} \\ &\overset{\text{by the Intersection Number Basics Lemma 51.7 (2)}}{=} X \cdot F|_{\{1\}\times Y} + X \cdot F|_{-(\{0\}\times Y)} \\ &\overset{\text{since the algebraic intersection number is additive under disjoint unions}}{=} X \cdot F|_{\{1\}\times Y \cup -(\{0\}\times Y)} \\ &\overset{\text{see page 832}}{=} X \cdot F|_{\partial([0,1]\times Y)} \\ &\overset{\text{by the Intersection-Number Zero-if-Extends Proposition 51.8}}{=} 0. \end{aligned}$$

■

Under suitable restrictions one can generalize the Intersection Number–Homotopy Invariance Theorem 51.9 from the case of a closed smooth manifold Y to the case of non-empty boundary. In the next section we will make use of the following theorem:

Theorem 51.10. (Intersection Number–Homotopy Invariance Theorem II) Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold of dimension k, let Y be a compact oriented smooth manifold of dimension $m-k$ and let $f_0, f_1 \colon Y \to M \setminus \partial M$ be smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$. We make the following hypothesis on the behavior of the maps on the boundary:

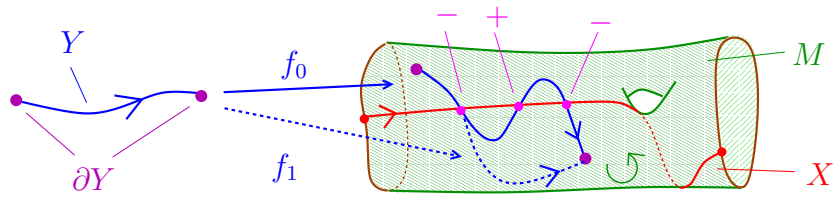
(1) $f_0|_{\partial Y} = f_1|_{\partial Y}$.
(2) $f_0(\partial Y) \cap X = f_1(\partial Y) \cap X = \varnothing$.

If f_0 and f_1 are homotopic rel ∂Y, then

$$X \cdot f_0 = X \cdot f_1.$$

Proof. Once again we need to show that $X \cdot f_1 - X \cdot f_0 = 0$. Since $[0,1] \times Y$ is a smooth manifold *with corner* $\partial Y \times \{0,1\}$ we cannot use the off-the-shelf approach that we took in the proof of the Intersection Number–Homotopy Invariance Theorem 51.9. Therefore in the following we also need to look under the hood of the proof of the Intersection-Number Zero-if-Extends Proposition 51.8 to make the argument work.

[562] Again we do not demand that the homotopy is smooth.



We will outline the changes that need to be made. As before, given $t \in [0,1]$ we equip $\{t\} \times Y$ with the product orientation and we denote by

$$\begin{array}{rccc} \iota_t \colon & Y & \to & \{t\} \times Y \\ & y & \mapsto & (t,y) \end{array}$$

the obvious diffeomorphism.

Claim 1. There exists a continuous map $\delta \colon [0,1] \times Y \to M$ with the following properties:
(1) We have $\delta \circ \iota_0 = f_0$ and $\delta \circ \iota_1 = f_1$.
(2) For each $t \in [0,1]$ and $P \in \partial Y$ we have $\delta(t,P) = f_0(P)$.
(3) We have $\delta([0,1] \times \partial Y) \cap \partial M = \varnothing$.
(4) The map δ is smooth.

Proof. The proof of the claim uses an approach similar to the one taken in the proof of the Intersection-Number Zero-if-Extends Proposition 51.8. Again we proceed in several steps:
(1) Since f_0 and f_1 are homotopic rel ∂Y there exists a continuous map $\alpha \colon [0,1] \times Y \to M$ which satisfies (1) and (2).
(2) The Push-Away-from-Boundary Corollary 35.8 implies that we can replace α by a map β which now satisfies (1), (2) and (3).
(3) Next note that by the Stationary-Near-The-Ends-Lemma 41.7 we can replace β by a homotopy $\gamma \colon [0,1] \times Y \to M$ which has the following properties:
   (a) For each $t \in [0, \frac{1}{4}]$ we have $\gamma \circ \iota_t = f_0$.
   (b) For each $t \in [\frac{3}{4}, 1]$ we have $\gamma \circ \iota_t = f_1$.
(4) We apply the Whitney Approximation Theorem 36.1 to[563] $\gamma \colon (0,1) \times Y \to M \setminus \partial M$ and the closed subset $A = ((0, \frac{1}{8}] \times Y) \cup ([\frac{7}{8}, 1) \times Y) \cup ((0,1) \times \partial Y)$ to obtain a smooth map $\delta \colon (0,1) \times Y \to M \setminus \partial M$. We extend δ to $[0,1] \times M \setminus \partial M$ via f_0 and f_1. It follows from the obvious extension of the Smooth Pasting Proposition 25.19 to smooth manifolds with corner that this map $\delta \colon [0,1] \times Y \to M \setminus \partial M$ is smooth. By construction it has all the desired properties. ⊞

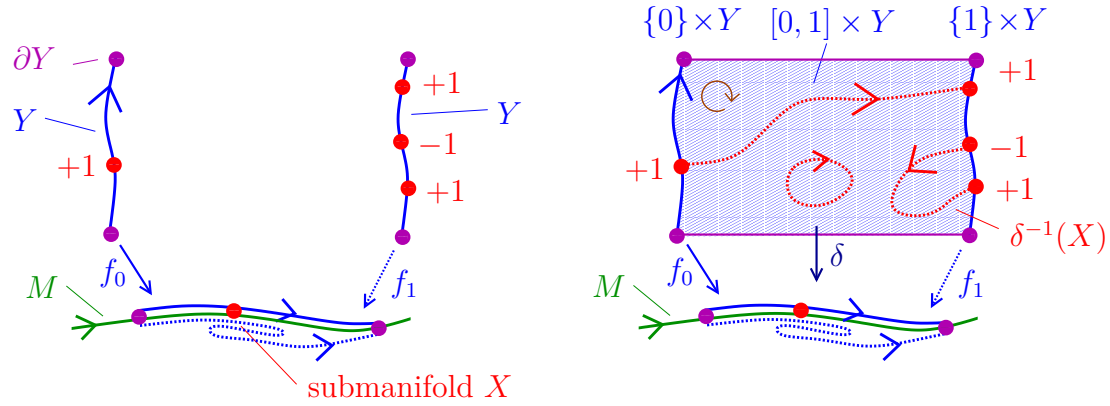
Claim 2. We have $X \cdot \delta_{\partial([0,1]) \times Y} = 0$.

Proof. If $\partial Y = \varnothing$, then this equality follows immediately from the Intersection-Number Zero-if-Extends Proposition 51.8 applied to $\widetilde{Y} = [0,1] \times Y$. We still need to consider the case that $\partial Y \neq \varnothing$. In the following we will adapt the argument of the Intersection-Number Zero-if-Extends Proposition 51.8 to obtain the desired result. We start out with the following two observations:
(i) By construction we have $\delta^{-1}(X) \subseteq [0,1] \times (Y \setminus \partial Y)$.

[563] Note that $[0,1] \times Y$ could be a smooth manifold with non-empty corner, but $(0,1) \times Y$ is fortunately a smooth manifold without corner. This is the reason why we want to work with $(0,1) \times Y$.

(ii) By hypothesis we have $X \pitchfork f_0$ and $X \pitchfork f_1$. This means that $X \pitchfork \delta|_{\{0,1\}\times Y}$.

It follows from these two observations, as in the proof of the Intersection-Number Zero-if-Extends Proposition 51.8, that we can apply the Push-Away-from-Boundary Corollary 35.8 and the Transversality Theorem 50.10 to find a smooth map $\widetilde{\delta}\colon [0,1]\times Y \to M \setminus \partial M$ such that $\delta = \widetilde{\delta}$ on $\partial([0,1])\times Y$, such that $\delta = \widetilde{\delta}$ on $[0,1]\times \partial Y$, and such that $X \pitchfork \widetilde{\delta}$.



Next we recall the following notation: Given a compact oriented 0-dimensional smooth manifold A we denote by $\epsilon(A) \in \mathbb{Z}$ the sum of the orientations of the points in M. We now see that

$$
\begin{array}{rcccl}
X \cdot \delta_{\partial([0,1])\times Y} & \underset{\downarrow}{=} & \epsilon((\delta_{\partial([0,1])\times Y})^{-1}(X)) & = & \epsilon((\widetilde{\delta}_{\partial([0,1])\times Y})^{-1}(X)) \\
& = & \epsilon(\partial(\widetilde{\delta}|_{(0,1)\times Y})^{-1}(X)) & \underset{\uparrow}{=} & \epsilon\left(\begin{array}{c}\text{boundary of compact oriented}\\ \text{1-dimensional smooth manifold}\end{array}\right) \\
& \underset{\uparrow}{=} & & &
\end{array}
$$

by Definition 51.3

by the Transversal Preimage Theorem 50.8 (3) and (4), note that it follows from the fact that X is closed and the fact that Y is compact that $\widetilde{\delta}_{\partial([0,1])\times Y})^{-1}(X)$ is compact

$$
= 0.
\underset{\uparrow}{}
$$

by the Boundary-of-Compact 1-Dimensional Smooth Manifold Proposition 31.42

We have thus proved that $X \cdot \delta_{\partial([0,1])\times Y} = 0$. ⊞

Now we perform the following calculation:

$$
X\cdot f_1 - X\cdot f_0 \;=\; X\cdot(\delta\circ\iota_1) - X\cdot(\delta\circ\iota_0) \quad \underset{\uparrow}{=} \quad X\cdot\delta|_{\{1\}\times Y} - X\cdot\delta|_{\{0\}\times Y}
$$

by the Intersection Number–Basics Lemma 51.7 (1) since by the discussion on page 832 we know that ι_0 and ι_1 are orientation-preserving diffeomorphisms

$$
\underset{\uparrow}{=} X\cdot\delta|_{\{1\}\times Y} + X\cdot\delta|_{-(\{0\}\times Y)} \underset{\uparrow}{=} X\cdot\delta|_{\{1\}\times Y\cup -(\{0\}\times Y)} \underset{\uparrow}{=} X\cdot\delta|_{\partial([0,1])\times Y} \underset{\uparrow}{=} 0.
$$

by the Intersection Number Basics Lemma 51.7 (2)

since the algebraic intersection number is additive under disjoint unions

see page 832

Claim 2

■

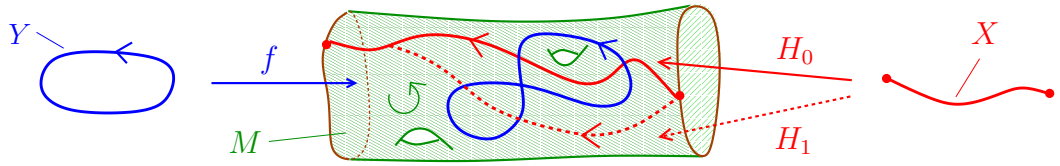
In the Intersection Number–Homotopy Invariance Theorem 51.9 we showed that under suitable hypotheses the algebraic intersection number $X\cdot f$ stays constant if we "deform" f. The question arises, what happens if we "deform" the submanifold X by a smooth isotopy. If we are dealing with the algebraic intersection number between two submanifolds, then

we can use the (anti-) symmetry from the Intersection Number–Symmetry Lemma 51.6 to swap the roles of the two inputs and we can reduce the problem to the Intersection Number–Homotopy Invariance Theorem 51.9.

In the general setting we have the following theorem:

Theorem 51.11. (Intersection Number–Homotopy Invariance Theorem III) Let M be an oriented m-dimensional smooth manifold. Furthermore let Y be a closed oriented smooth manifold of dimension $m-k$ and let $f\colon Y \to M \setminus \partial M$ be a smooth map. Finally let X be a compact oriented smooth manifold of dimension k and let $H\colon X \times [0,1] \to M$ be a proper smooth isotopy such that[564] $H_0(X) \pitchfork f$ and $H_1(X) \pitchfork f$. Then

$$H_0(X) \cdot f \;=\; H_1(X) \cdot f.$$



Proof. The natural idea would be to try to modify the argument of the Intersection Number–Homotopy Invariance Theorem 51.9. We will do it differently though, namely we will employ the Isotopy Extension Theorem 44.2, to sidestep all difficulties.

By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $\Phi\colon M \times [0,1] \to M$ with the following properties:

(1) $\Phi_0 = \mathrm{id}$.
(2) $\Phi_1 \circ H_0 = H_1\colon X \to M$.

We now see that

$$H_0(X) \cdot f \underset{\uparrow}{=} \Phi_1(H_0(X)) \cdot (\Phi_1 \circ f) = H_1(X) \cdot (\Phi_1 \circ f) \underset{\uparrow}{=} H_1(X) \cdot f.$$

since $\Phi_0 = \mathrm{id}$ we know by the Smooth Isotopy-Orientation Proposition 42.4 that Φ_1 is an orientation-preserving diffeomorphism, the statement now follows from the Intersection Number–Basics Lemma 51.7

by the Intersection Number–Homotopy Invariance Theorem 51.9, since $(x,t) \mapsto \Phi_t \circ f$ is a homotopy between $f = \mathrm{id} \circ f = \Phi_0 \circ f$ and $\Phi_1 \circ f$

■

51.3. **The $\mathbb{Z}/2\mathbb{Z}$-algebraic intersection number.** If we do not have orientations at hand, then we cannot define the $\mathbb{Z}$-valued algebraic intersection number. We can of course still consider the geometric intersection number, but the geometric intersection number is clearly not invariant under homotopies. But, playing around with pictures shows that the parity of the geometric intersection number has a chance of being invariant under homotopies. This hope leads us to the following definition:

[564] Note that it follows from the Smooth Embedding Theorem 30.15 that each $H_t(X)$ is indeed a proper smooth submanifold of M.

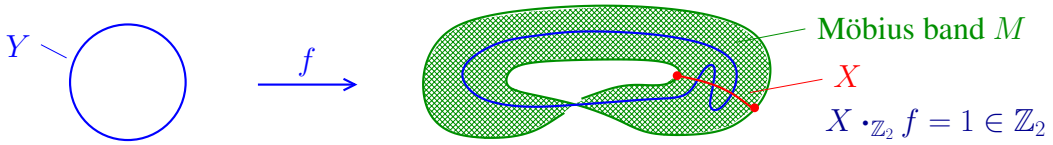
Definition 51.12. Let M be an m-dimensional smooth manifold and let $X \subseteq M$ be a compact proper submanifold of dimension k.

(1) Let Y be a compact smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$. We define the

$$\mathbb{Z}/2\mathbb{Z}\text{-valued intersection number of } X \text{ and } f \;:=\; X \cdot_{\mathbb{Z}/2\mathbb{Z}} f \;:=\; [\#f^{-1}(X)] \in \mathbb{Z}/2\mathbb{Z}.$$

(2) If Y is a compact proper $(m-k)$-dimensional submanifold of M with $X \pitchfork Y$, then we define

$$\mathbb{Z}/2\mathbb{Z}\text{-valued intersection number of } X \text{ and } Y \;:=\; X \cdot_{\mathbb{Z}/2\mathbb{Z}} Y \;:=\; [\#(X \cap Y)] \in \mathbb{Z}/2\mathbb{Z}.$$



Remark. In the setting of the above definition, if X, Y and M are actually oriented, then we evidently have $X \cdot_{\mathbb{Z}/2\mathbb{Z}} f = X \cdot f \mod 2$. □

Our previous results on the algebraic intersection numbers in the oriented setting also have obvious analogs in the unoriented "mod 2"-setting. We just need to delete the adjective "oriented" and replace $\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$. For example we have the following analog the Intersection-Number Zero-if-Extends Proposition 51.8.

Proposition 51.13. (Intersection-Number Zero-if-Extends Proposition) Let M be an m-dimensional smooth manifold, let $X \subseteq M$ be a compact proper submanifold of dimension k, let Y be a closed smooth manifold of dimension $m-k$ and let $f\colon Y \to M$ be a proper smooth map with $X \pitchfork f$. If there exists a compact smooth manifold $\widetilde{Y}$ with $\partial\widetilde{Y} = Y$ and a continuous map $\widetilde{f}\colon \widetilde{Y} \to M$ with $\widetilde{f}|_{\partial\widetilde{Y}} = f$, then $X \cdot_{\mathbb{Z}/2\mathbb{Z}} f = 0$.

Proof. The proof is almost identical to the proof of the Intersection-Number Zero-if-Extends Proposition 51.8. ■

At the end of the day, the only result about $\mathbb{Z}/2\mathbb{Z}$-value intersection numbers we really care about is the following analog of the Intersection Number–Homotopy Invariance Theorem 51.10. In fact we will put it to good use in the next section in our proof of the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15.

Theorem 51.14. (Intersection Number–Homotopy Invariance Theorem) Let M be an m-dimensional smooth manifold, let $X \subseteq M$ be a compact proper submanifold of dimension k and let Y be compact oriented smooth manifold of dimension $m-k$. Furthermore let $f_0, f_1\colon Y \to M \setminus \partial M$ be smooth maps with $X \pitchfork f_0$ and $X \pitchfork f_1$. We make the following hypotheses on the behavior of the maps on the boundary:

(1) $f_0|_{\partial Y} = f_1|_{\partial Y}$.
(2) $f_0(\partial Y) \cap X = f_1(\partial Y) \cap X = \varnothing$.

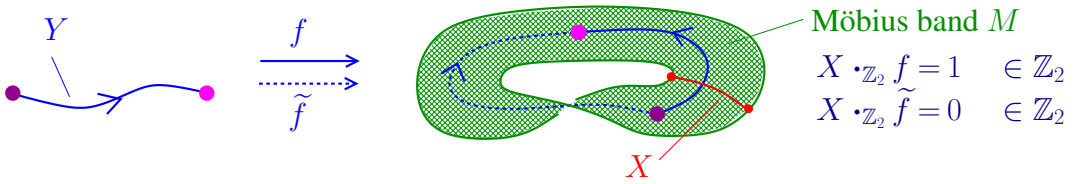
If f_0 and f_1 are homotopic rel ∂Y, then

$$X \cdot_{\mathbb{Z}/2\mathbb{Z}} f_0 \;=\; X \cdot_{\mathbb{Z}/2\mathbb{Z}} f_1.$$

Proof. Again the proof is a worry-free analog of the proof of the original Intersection Number–Homotopy Invariance Theorem 51.10. Along the way we just need to replace the

Intersection-Number Zero-if-Extends Proposition 51.8 by the above Intersection-Number Zero-if-Extends Proposition 51.13. ■

Example. Let M be the Möbius band. We consider the proper 1-dimensional submanifold X and the two maps $f, \widetilde{f}\colon [0,1] \to M$ shown in the following figure:



Since $X \cdot_{\mathbb{Z}/2\mathbb{Z}} f = 1 \in \mathbb{Z}/2\mathbb{Z}$ and $X \cdot_{\mathbb{Z}/2\mathbb{Z}} \widetilde{f} = 0 \in \mathbb{Z}/2\mathbb{Z}$ we obtain from the Intersection Number–Homotopy Invariance Theorem 51.14 that the two maps are not homotopic rel $\{0,1\}$. □

51.4. **The Smooth Jordan-Brouwer Separation Theorem.** The following theorem is the main result of this final section of the chapter:

Theorem 51.15. (Generalized Smooth Jordan-Brouwer Separation Theorem) Let $n \geq 2$. Let M be a connected non-empty smooth codimension-one submanifold of $\mathbb{R}^n$ that is closed if viewed as a smooth manifold in its own right.

(1) The complement $\mathbb{R}^n \setminus M$ has precisely two path-components[565], one of which is bounded and one which is unbounded.[566]
(2) Let A and B be the two path-components of $\mathbb{R}^n \setminus M$. The following statements hold:
  (a) The closures $\overline{A}$ and $\overline{B}$ are codimension-zero submanifolds of $\mathbb{R}^n$.
  (b) $\partial A = \partial \overline{A} = \partial B = \partial \overline{B} = \overline{A} \cap \overline{B} = M$.[567]



As an almost immediate corollary we obtain the following result:

Corollary 51.16. (Smooth Jordan-Brouwer Separation Theorem) Let $n \in \mathbb{N}$.

(1) Let M be a closed connected non-empty n-dimensional smooth manifold. Given any smooth embedding $f\colon M \to \mathbb{R}^{n+1}$ the complement $\mathbb{R}^{n+1} \setminus f(M)$ has precisely two path-components.
(2) If $f\colon S^n \to \mathbb{R}^{n+1}$ is a smooth embedding, then $\mathbb{R}^{n+1} \setminus f(S^n)$ has precisely two path-components.

[565]It follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that $\mathbb{R}^n \setminus M$ is an open subset of $\mathbb{R}^n$. Thus it follows from the Manifold–Local Properties Lemma 23.8 that the path-components of $\mathbb{R}^n \setminus M$ are also the components of $\mathbb{R}^n \setminus M$.
[566]Note that this statement is incorrect for $n = 1$.
[567]Since $\overline{A}$ and $\overline{B}$ are closed subsets and since they are codimension-zero submanifolds it follows from the Codimension-Zero Submanifold Proposition 25.49 that the boundaries of $\overline{A}$ and $\overline{B}$ viewed as smooth manifolds agree with the boundaries of $\overline{A}$ and $\overline{B}$ viewed as subsets of $\mathbb{R}^n$.



Proof of Corollary 51.16. Statement (1) follows from the Smooth Embedding Theorem 30.15 together with the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15. Statement (2) is evidently an immediate consequence of Statement (1). ■

Remark.

(1) The case $n = 1$ is sometimes referred to as the *Smooth Jordan Curve Theorem*.

(2) (a) In the Jordan-Brouwer Separation Theorem 170.3 we will generalize the Smooth Jordan-Brouwer Separation Theorem 51.16 from smooth embeddings $S^n \to \mathbb{R}^{n+1}$ to injective continuous maps $S^n \to \mathbb{R}^{n+1}$.

(b) In the Generalized Jordan-Brouwer Separation Theorem 218.12 we will generalize the Smooth Jordan-Brouwer Separation Theorem 51.16 (1) from smooth embeddings $M \to \mathbb{R}^{n+1}$ of closed connected non-empty smooth manifolds to embeddings $M \to \mathbb{R}^{n+1}$ of closed connected non-empty topological manifolds. □

Proof of the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15. Let $n \geq 2$ and let M be a connected non-empty smooth codimension-one submanifold of $\mathbb{R}^n$ that is closed if viewed as a smooth manifold in its own right. On several occasions we will use that it follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that M is a closed subset of $\mathbb{R}^n$.

Since M is closed we obtain from the Neighborhood Projection Theorem 35.2 that there exists an $\epsilon > 0$ such that

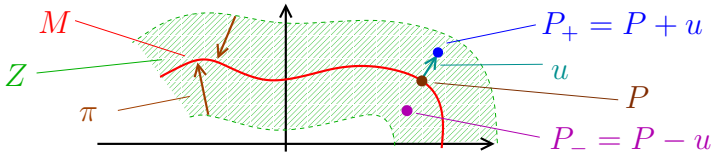
$$Z \coloneqq \left\{P + w \,\middle|\, P \in M \text{ and } w \in \left(\widetilde{\mathrm{T}}_P M\right)^\perp \text{ with } \|w\| < \epsilon\right\}$$

is an open neighborhood of $M \subseteq \mathbb{R}^n$ and such that the map

$$\begin{array}{rcll} \pi\colon Z & \to & M & \\ P + w & \mapsto & P & \text{where } P \in M \text{ and } w \in \left(\widetilde{\mathrm{T}}_P M\right)^\perp \end{array}$$

is well-defined. We also saw that the map π is smooth and that π is a retraction $Z \to M$.

Since M is non-empty we can pick a point $P \in M$. Since M has codimension 1 in $\mathbb{R}^n$ there exist precisely two vectors $u \in (\widetilde{\mathrm{T}}_P M)^\perp$ with $\|u\| = \frac{\epsilon}{2}$. Let u be one of the two vectors. We set $P_\pm = P \pm u$.



Claim 0. The topological space $Z \setminus M$ contains at most two path-components.

Proof. Let $Q \in Z \setminus M$. By the Path-Equivalence Lemma 4.42 we know that being connected via a path is an equivalence relation. Thus it suffices to show that there exists a path from Q to P_- or P_+. We pick $z \in M$ and $v \in (\widetilde{\mathrm{T}}_P M)^\perp$ with $\|v\| < \epsilon$ and with $z + v = Q$. Since M is connected there exists a path $\gamma\colon [0,1] \to M$ with $\gamma(0) = z$ and $\gamma(1) = P$. Since

M is a codimension-one submanifold one can show easily, either by hand[568] or using the Path-Thickening Proposition 45.2 together with the cross product from Definition 31.25, that there is a smooth map $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}^n$ with the following properties:

(a) $\widetilde{\gamma}(0) = \gamma(0) + v$,
(b) for each $t \in [0,1]$ we have $\widetilde{\gamma}(t) - \gamma(t) \in (\widetilde{\mathrm{T}}_{\gamma(t)}M)^{\perp}$,
(c) for each $t \in [0,1]$ we have $\|\widetilde{\gamma}(t) - \gamma(t)\| = \|v\|$.

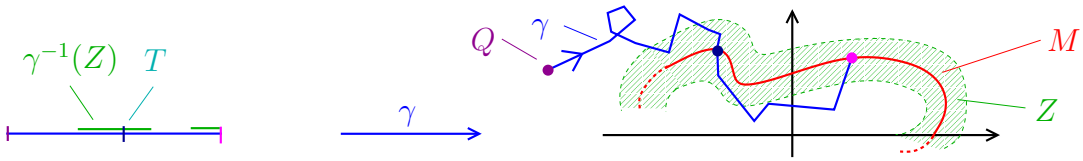
Note that $\widetilde{\gamma}(1) = P + \lambda \cdot u$ for some $\lambda \in (-2,0) \cup (0,2)$.



It follows that $\widetilde{\gamma}(1)$ is connected via a path in $Z \setminus M$ to $P + \lambda \cdot u$, thus $\widetilde{\gamma}(1)$ is also connected via a path in $Z \setminus M$ to P_- or to P_+. ⊞

Claim 1. The topological space $\mathbb{R}^n \setminus M$ contains at most two path-components.

Proof. Let $Q \in \mathbb{R}^n \setminus M$. Since $\mathbb{R}^n$ is path-connected we know that there exists a path $\gamma\colon [0,1] \to \mathbb{R}^n$ with $\gamma(0) = Q$ and $\gamma(1) \in M$. Since M is a closed subset of $\mathbb{R}^n$ we know that $\gamma^{-1}(M) \subseteq [0,1]$ is a closed subset of the compact space $[0,1]$. It follows from the Compact-Closed Lemma 3.39 that $\gamma^{-1}(M)$ is compact. Since $0 \notin \gamma^{-1}(M)$ we deduce from the Compact Image Lemma 4.15 (2) that $\gamma^{-1}(M)$ has a minimal element, i.e. there exists a $T \in \gamma^{-1}(M)$ such that $[0,T) \cap \gamma^{-1}(M) = \varnothing$.



Since Z is an open neighborhood of $M \subseteq \mathbb{R}^n$ there exists an $\epsilon > 0$ such that we have an inclusion $(T-\epsilon, T) \subseteq \gamma^{-1}(Z)$. Thus the restriction of γ to the interval $[0, T - \frac{\epsilon}{2}]$ gives us a path within $\mathbb{R}^n \setminus M$ from P to a point in $Z \setminus M$. It follows from Claim 0 that $\mathbb{R}^n \setminus M$ has indeed at most two path-components. ⊞

Claim 2.

(1) Let $P \in M$ and let $u \in (\widetilde{\mathrm{T}}_P M)^{\perp}$ with $\|u\| < \epsilon$. The points $P_{\pm} = P \pm u$ lie in two different path-components of $\mathbb{R}^n \setminus M$.
(2) The topological space $\mathbb{R}^n \setminus M$ contains at least two path-components.

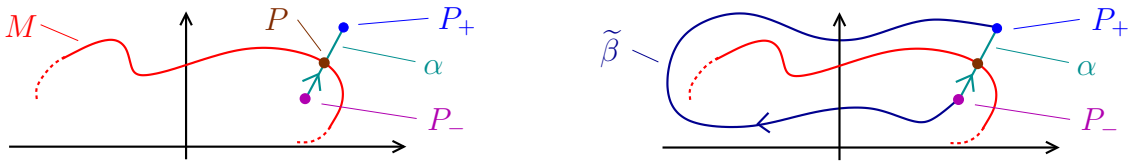
Proof. Since M is non-empty we see that (2) is an immediate consequence of (1). Therefore it suffices to prove (1). Thus let $P \in M$ and let $u \in (\widetilde{\mathrm{T}}_P M)^{\perp}$ with $\|u\| < \epsilon$. We set $P_{\pm} = P \pm u$. We need to show that the points $P_-, P_+ \in \mathbb{R}^n \setminus M$ lie in different path-components of $\mathbb{R}^n \setminus M$. Let $\alpha\colon [0,1] \to \mathbb{R}^n$ be the "straight path" from P_- to P_+ that is given by $\alpha(t) = (1-t) \cdot P_- + t \cdot P$. It follows from the properties of Z and the choice of u that the path α from $P_- = P - u$ to $P_+ = P + u$ intersects M precisely at $t = \frac{1}{2}$ and that this

[568] The "by hand" approach is the content of Exercise 31.16.

intersection point is transverse. Suppose that there exists also a path $\beta\colon [0,1] \to \mathbb{R}^n \setminus M$ from P_- to P_+. Since M is a closed subset of $\mathbb{R}^n$ we know that $\mathbb{R}^n \setminus M$ is an open subset, in particular it is a smooth submanifold of $\mathbb{R}^n$. Thus it follows from the Smooth Path-Connectivity Proposition 25.32 that there exists in fact a smooth path $\widetilde{\beta}\colon [0,1] \to \mathbb{R}^n \setminus M$ from P_- to P_+. Note that

$$\begin{aligned} F\colon [0,1] \times [0,1] &\to \mathbb{R}^n \\ (t,s) &\mapsto \alpha(t) \cdot (1-s) + \widetilde{\beta}(t) \cdot s \end{aligned}$$

is a homotopy rel $\{0,1\}$ from α to $\widetilde{\beta}$.



We obtain the following equality of intersection numbers of submanifolds of $\mathbb{R}^n$:

$$\begin{array}{ccccccc} 0 & = & M \cdot_{\mathbb{Z}/2\mathbb{Z}} \widetilde{\beta} & = & M \cdot_{\mathbb{Z}/2\mathbb{Z}} \alpha & = & 1 \in \mathbb{Z}/2\mathbb{Z}. \end{array}$$

by the Intersection Number–Homotopy Invariance Theorem 51.14

since $\widetilde{\beta}$ takes values in $\mathbb{R}^n \setminus M$

since α intersects M transversally precisely in the point $t = 0$

This contradiction proves the claim. ⊞

Claims 1 and 2 show that $\mathbb{R}^n \setminus M$ contains precisely two path-components. Since M is compact there exists an $r > 0$ such that $M \subseteq B_r(0)$. It follows from our hypothesis that $n \geq 2$ and the Complement Connected Lemma 4.40 that $\mathbb{R}^n \setminus B_r^n(0)$ is path-connected. Therefore $\mathbb{R}^n \setminus B_r^n(0)$ is contained in a path-component B of $\mathbb{R}^n \setminus M$. We denote the other path-component of $\mathbb{R}^n \setminus M$ by A. We set $\widetilde{A} := A \cup M$ and we set $\widetilde{B} := B \cup M$. By the Codimension-Zero Submanifold Proposition 25.49 it remains to prove the following claim:

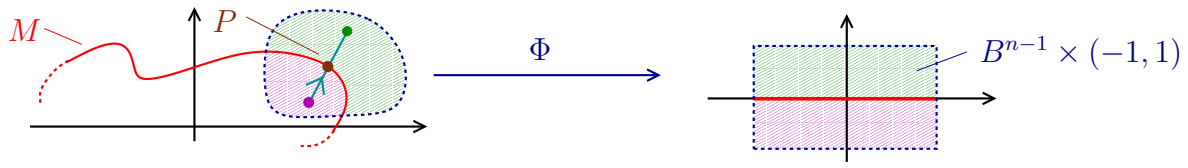
Claim 3.
(1) $\widetilde{A}$ and $\widetilde{B}$ are closed subsets of $\mathbb{R}^n$.
(2) $\widetilde{A}$ is compact.
(3) $\widetilde{A}$ and $\widetilde{B}$ are codimension-zero submanifolds of $\mathbb{R}^n$ with $\partial\widetilde{A} = \partial\widetilde{B} = \widetilde{A} \cap \widetilde{B} = M$.

Proof.

(1) As we remarked above, $\mathbb{R}^n \setminus M$ is an open subset of $\mathbb{R}^n$. In particular, as we remarked on page 206, $\mathbb{R}^n \setminus M$ is locally path-connected. It follows from the Locally Path-Connected Lemma 4.57 and the Components-are-Closed Lemma 4.49 that A and B are open subsets of $\mathbb{R}^n \setminus M$. Since M is a closed subset of $\mathbb{R}^n$ we see that A and B are both open subsets of $\mathbb{R}^n$. It follows that $\widetilde{A} = \mathbb{R}^n \setminus B$ and $\widetilde{B} = \mathbb{R}^n \setminus A$ are closed.

(2) By design we have $\widetilde{A} \subseteq B_r^n$. Since $\widetilde{A}$ is closed it follows from the Heine-Borel Theorem 3.44 that $\widetilde{A}$ is in fact compact.

(3) First we show that $\widetilde{A}$ is a codimension-zero submanifold of $\mathbb{R}^n$ with $\partial\widetilde{A} = M$. For the points in A the identity defines a submanifold chart. Now let $P \in M$. Since M is a codimension-one submanifold we know that there exists a chart $\Phi\colon U \to B^{n-1} \times (-1,1)$ around P such that $\Phi(U \cap M) = B^{n-1} \times \{0\}$.



Note that $B^{n-1} \times (0,1)$ and $B^{n-1} \times (-1,0)$ are path-connected. Next note that it follows almost immediately from Claim 2 (1) that $\Phi^{-1}(B^{n-1} \times (0,1))$ and $\Phi^{-1}(B^{n-1} \times (-1,0))$ lie in two different path-components of $\mathbb{R}^n \setminus M$. These two observations imply that either we have $\Phi(U \cap \widetilde{A}) = B^{n-1} \times [0,1)$ or we have $\Phi(U \cap \widetilde{A}) = B^{n-1} \times (-1,0]$. Either way we have found (possibly after composing Φ with a reflection) a submanifold chart of type (γ) for P.

The same argument works for $\widetilde{B}$. We have thus shown the desired statements. ■

Using the Smooth Jordan-Brouwer Separation Theorem 51.16 we can easily prove the following pretty proposition.

Proposition 51.17. (Smooth Codimension-One Orientability Proposition) Let M be some closed n-dimensional smooth manifold. If there exists a smooth embedding $M \to \mathbb{R}^{n+1}$, then M is orientable.

Remark.

(1) On page 4154 we will give an alternative proof of the Smooth Codimension-One Orientability Proposition 51.17.
(2) In the Topological Codimension-One Orientability Proposition 218.9 we will strengthen the result. □

Proof. Let M be a closed n-dimensional smooth manifold that admits a smooth embedding $M \to \mathbb{R}^{n+1}$. It is clear that we only need to deal with the case that M is non-empty and connected. Now let us show that M is orientable. The proof proceeds in three easy steps:

(1) By the Smooth Embedding Theorem 30.15 we know that $f(M)$ is a closed codimension-one smooth submanifold of $\mathbb{R}^{n+1}$ such that $f\colon M \to f(M)$ is a diffeomorphism.
(2) It follows from the Smooth Jordan-Brouwer Separation Theorem 51.16 that there exists a codimension-zero submanifold A of $\mathbb{R}^{n+1}$ such that $\partial A = f(M)$.
(3) By the Manifold–Orientation Basics Lemma 31.31 we know that A is orientable. It follows from the Boundary Orientation Lemma 31.38 that $\partial A = f(M)$ is orientable. Since f is a diffeomorphism we obtain that M itself is orientable. ■

In Question 30.18 (2) we asked whether $\mathbb{R}\mathrm{P}^2$ or the Klein bottle admit a smooth embedding into $\mathbb{R}^3$. We can now give a negative answer.

Corollary 51.18. (Surface-No Smooth Embedding Corollary) Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$.

Remark. In the Surface-No Topological Embedding-Corollary Corollary 218.10 we will see that $\mathbb{R}\mathrm{P}^2$ and the Klein bottle do not even admit a (possibly non-smooth) embedding into $\mathbb{R}^3$. □

Proof. By the Surface Non-Orientability Lemma 31.45 we know that $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are both non-orientable. Since both are closed 2-dimensional smooth manifolds it

follows from the Smooth Codimension-One Orientability Proposition 51.17 that neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$. ■

Proposition 51.19. (Projective Space–Non Embedding Proposition) If $n \in \mathbb{N}_{\geq 2}$ is even, then there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{n+1}$.

Proof. Let $n \in \mathbb{N}_{\geq 2}$ be even. It follows from the Real Projective Space Orientability Lemma 31.46 that $\mathbb{R}\mathrm{P}^n$ is non-orientable. Since $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional smooth manifold it follows from the Smooth Codimension-One Orientability Proposition 51.17 that there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^{n+1}$. ■

Remark. In the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14 we will see that the conclusion of the Projective Space–Non Embedding Proposition 51.19 holds for every natural number $n \in \mathbb{N}_{\geq 2}$, i.e. regardless of whether or not $\mathbb{R}\mathrm{P}^n$ is orientable, it cannot be smoothly embedded into $\mathbb{R}^{n+1}$. □

51.5. Exercises for Chapter 51.

Exercise 51.1. Let $f\colon A \to M$ and $g\colon M \to N$ be smooth maps between compact oriented smooth manifolds. Let $Y \subseteq N$ be a compact oriented proper submanifold such that $Y \pitchfork g$. By the Transversal Preimage Theorem 50.8 we know that $g^{-1}(Y)$ is a compact oriented proper submanifold of M. We suppose that $\dim(A) + \dim(Y) = \dim(M)$ and we suppose that $g^{-1}(Y) \pitchfork f$. Show that

$$Y \cdot (g \circ f) \;=\; g^{-1}(Y) \cdot f.$$

Exercise 51.2. Let Y be a closed oriented smooth manifold, let $f\colon Y \to M$ be a smooth map to a compact oriented smooth manifold and let $X \subseteq M$ be a compact oriented proper submanifold with $\dim(X) + \dim(Y) = \dim(M)$. Finally let $\widetilde{Y}$ be a closed oriented smooth manifold with $\dim(\widetilde{Y}) = \dim(Y)$ and let $\varphi\colon \widetilde{Y} \to Y$ be a smooth map. We assume that $X \pitchfork f$ and $X \pitchfork (f \circ \varphi)$. Show that

$$X \cdot (f \circ \varphi\colon \widetilde{Y} \to M) \;=\; (X \cdot (f\colon Y \to M)) \;\cdot\; \deg(\varphi\colon \widetilde{Y} \to Y).$$

Exercise 51.3. In the example in Section 51.2 we saw that the conclusion of Theorem 51.9 does not hold if we do not assume that Y is closed. Go through the proof of Theorem 51.9 and find out where the proof fails if Y is not closed.

Exercise 51.4. Let M be an oriented connected smooth manifold and let X, Y be two closed oriented submanifolds with $\dim(X)+\dim(Y) = \dim(M)$. We suppose that $X \cdot Y = 0$. Does it follow that there exists a smooth homotopy $F\colon Y \times [0,1] \to M$ such that $F_0 = \mathrm{id}$ and $X \cap F_1(Y) = \varnothing$?

Exercise 51.5. Is the closed curve that goes twice around the Möbius band null-homotopic, i.e. is it homotopic to the constant curve?

Exercise 51.6.

(a) Show that every smooth map $f\colon S^1 \to S^2$ is null-homotopic, i.e. homotopic to a constant map.

(b) Show that S^2 not diffeomorphic to $S^1 \times S^1$.

(c) Can you upgrade the argument to show that S^2 is not homeomorphic to $S^1 \times S^1$?

Exercise 51.7. In the Intersection Number–Homotopy Invariance Theorem 51.10 we studied the homotopy invariance of the algebraic intersection number under the following fairly strong hypotheses:

(1) $f_0|_{\partial Y} = f_1|_{\partial Y}$.
(2) $f_0(\partial Y) \cap X = f_1(\partial Y) \cap X = \varnothing$.
(3) There exists a homotopy F rel ∂Y.

Formulate the weakest hypothesis on the behavior of f_0, f_1 and F on ∂Y and prove the corresponding theorem.

Exercise 51.8. Let M be a codimension-one smooth submanifold of $\mathbb{R}^n$ which is a closed smooth manifold in its own right. Determine the number of components of $\mathbb{R}^n \setminus M$ in terms of the number of components of M.

Exercise 51.9. Let M be a smooth manifold. For $i = 1, \dots, k$ let Y_i be a compact smooth manifold and let $f_i \colon Y_i \to M$ be a proper smooth embedding. Show that there exist proper smooth embeddings $\widetilde{f}_i \colon Y_i \to M$, $i = 1, \dots, k$ with the following properties:

(1) Each $\widetilde{f}_i$ is smoothly isotopic to f_i.
(2) For any $i \neq j \in \{1, \dots, k\}$ we have $\widetilde{f}_i(Y_i) \pitchfork \widetilde{f}_j(Y_j)$.

Exercise 51.10. Let $\varphi \colon S^1 \to \mathbb{R}^2$ be a smooth embedding. Show that there exist distinct three points $x, y, z \in S^1$ such that $\varphi(x), \varphi(y), \varphi(z)$ from an equilateral triangle.
Hint. Use the Smooth Jordan-Brouwer Separation Theorem 51.16 for $n = 2$.

Exercise 51.11. Let

$$S \;\coloneqq\; \{(0, y) \,|\, y \in [-1, 1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2$$

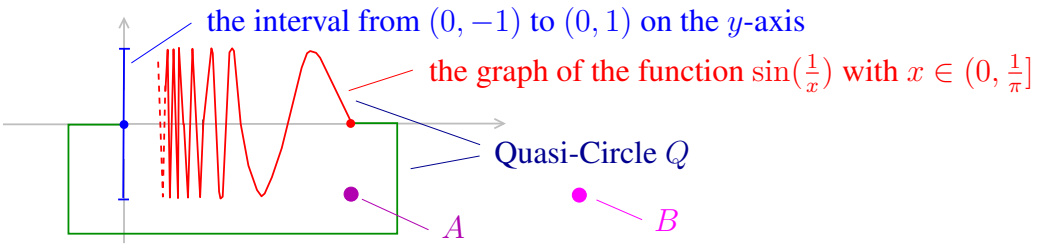
be the "Topologist's Sine Curve" from page 195. As in Exercise 8.4 we turn the Topologist's Sine curve into the *Quasi-Circle* Q by connecting the point $(0, 0)$ with $(\frac{1}{\pi}, 0)$ via the green curve shown in the figure below. We consider the points $A = (\frac{1}{\pi}, -1)$ and $B = (2, -1)$.

(a) Given $\epsilon > 0$ we consider the thickened up topological space

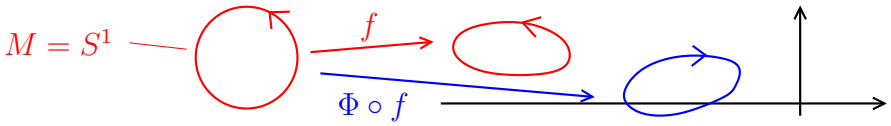
$$Q_\epsilon \;\coloneqq\; \{P \in \mathbb{R}^2 \,|\, \text{there exists } x \in Q \text{ with } d(x, P) < \epsilon\}.$$

Show that for every $\epsilon > 0$ there exists a smooth curve $\varphi \colon S^1 \to Q_\epsilon$ such that A and B lie in different path-components of $\mathbb{R}^2 \setminus \varphi(S^1)$.
Hint. Use the Smooth Jordan-Brouwer Separation Theorem 51.16.
(b) Show that there is no path $f \colon [0, 1] \to \mathbb{R}^2 \setminus Q$ from A to B.
Hint. By Exercise 8.4 we know that Q is compact. The Disjoint Compact-Closed–Distance Lemma 10.5 might come in handy.
(c) Show that $\mathbb{R}^2 \setminus Q$ has precisely two path components.

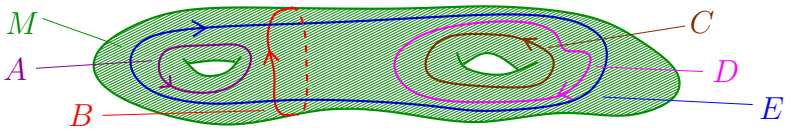
Exercise 51.12. Let M be the smooth manifold that is obtained from the open disk B^2 by removing the three points $(0, -\frac{1}{2}), (0, 0)$ and $(0, \frac{1}{2})$. Show that there is no *proper* smooth embedding $f \colon M \to \mathbb{R}^2$.
Remark. This exercise can be viewed as the complement of Exercise 15.13.



Exercise 51.13. Let M be a closed connected non-empty n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}^{n+1}$ be a smooth embedding. Furthermore let $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orientation-reversing diffeomorphism. Show that the smooth embeddings $f\colon M \to \mathbb{R}^{n+1}$ and $\Phi \circ f\colon M \to \mathbb{R}^{n+1}$ are not smoothly isotopic.
Hint. You could try to get the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15 into the game.



Exercise 51.14. We consider the smooth manifold M together with the oriented submanifolds A, B, C, D and E are that are shown in the figure below. Show that the smooth submanifolds are pairwise non isotopic.



Exercise 51.15. Let $g \in \mathbb{N}_0$ and let Σ_g be the orientable surface of genus g. We equip Σ_g with some orientation. For the purpose of this exercise, given a closed oriented 1-dimensional submanifold $C \subseteq \Sigma_g$ we define the self-intersection number $C \cdot C := 0$. What is the maximal $k \in \mathbb{N}_0$ such that there exist closed oriented 1-dimensional submanifolds $A_1, \dots, A_k \subseteq \Sigma_g$ such that any two A_i, A_j with $i \neq j$ are transverse and such that the matrix $(A_i \cdot A_j)_{i,j=1,\dots,k}$ has non-zero determinant?
Remark. With the current techniques it is perhaps difficult to show that you have achieved the maximal k, but at least you can try to aim for the highest k possible.

Exercise 51.16. We set

$$\begin{aligned} H &:= \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} \qquad \text{the open half-plane} \\ A_{-1} &:= \{(x,0) \,|\, x < 0\} \\ \text{and for } k \in \mathbb{N}_0 \text{ we set } A_k &:= \{(x,0) \,|\, x \in (2k+1, 2k+2)\}, \\ \text{finally we set} M &:= H \cup (A_{-1} \cup A_0 \cup A_1 \cup \dots). \end{aligned}$$

We need to introduce some more language:

- Let $k \neq l$. A *track* from A_k to A_l is an oriented connected proper 1-dimensional smooth manifold T of M such that $\partial T = \{-P\} \cup \{Q\}$ where $P \in A_k$ and $Q \in A_l$.

- We say that two components A_k and A_l of ∂M are *adjacent* if there exists a track T from A_k to A_l such that all other components of ∂M are contained in one component of $M \setminus T$.

Now we turn to the actual problems:

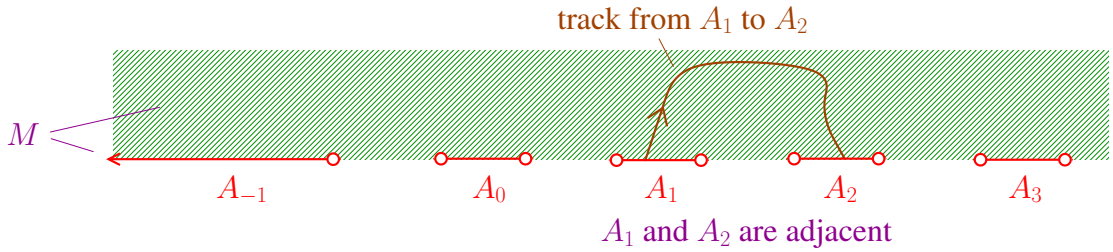
(a) Show that $M \coloneqq H \cup (A_{-1} \cup A_0 \cup A_1 \cup \dots)$ is a 2-dimensional smooth submanifold with $\partial M = A_{-1} \cup A_0 \cup A_1 \cup \dots$.

(b) Show that for any $k \neq l$ in $\mathbb{Z}_{\geq -1}$ the corresponding components A_k and A_l are adjacent if and only if $k - l \in \{-1, 1\}$.

(c) Let $f\colon M \to M$ be a self-diffeomorphism. Show that for any $k \in \mathbb{Z}_{\geq -1}$ we have $f(A_k) = A_k$.
Hint. Note that adjacency is preserved by diffeomorphisms.

(d) Show that every diffeomorphism of M is orientation-preserving.
Hint. Look at intersection numbers of tracks between components.

Remark. You can use the fact, which we will show later in Exercise 101.24, that any two tracks from A_k to A_l are properly isotopic.



CHAPTER 52

Linking numbers

In this short chapter we will use the techniques that we have developed over the last chapters to introduce and discuss linking numbers of suitable disjoint submanifolds in $\mathbb{R}^n$.

This chapter is organized as follows:

- In Section 52.1 we will introduce linking numbers and we will learn how to calculate them in practice for two knots in $\mathbb{R}^3$.
- In Section 52.2 we will state and prove several properties of linking numbers.
- In Section 52.3 we will provide exercises for Chapter 52.

In the next Chapter 53 we will formulate and prove the Smooth Poincaré–Hopf Theorem 53.24.

52.1. **Definition of linking numbers.** To define the linking numbers it is convenient to generalize slightly the notion of the degree of a continuous map between closed oriented connected smooth manifolds.

Definition 52.1. In the following let M be a closed oriented smooth manifold with components $M_1, \dots, M_k$ and let N be a closed oriented connected non-empty smooth manifold of the same dimension. Given any continuous map $f\colon M \to N$ we define

$$\deg(f) \coloneqq \sum_{i=1}^{k} \underbrace{\deg(f|_{M_i}\colon M_i \to N)}_{\text{see Definition 46.5}}.$$

Now we can turn to the key definition of this section.

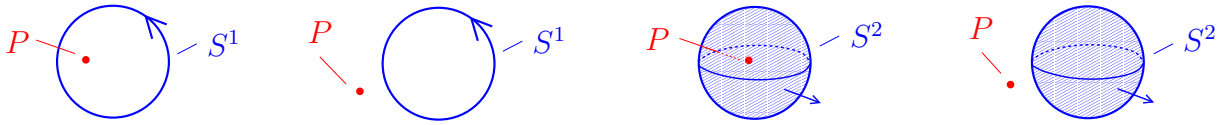
Definition 52.2. Let $n \in \mathbb{N}$. Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n - 1$. We define the linking number $\operatorname{lk}(K, L)$ as[569]

$$\operatorname{lk}(K,L) \coloneqq (-1)^{\dim(L)} \cdot \deg\begin{pmatrix} K \times L & \to & S^{n-1} \\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|} \end{pmatrix}$$

Remark. There are many ways think about the linking number. In particular in Definition 284.1 we will also give an alternative definition of the linking number which we will discuss in detail in Section 284.1. In the Linking Number Reformulation Proposition 285.8 we will see that our two definitions of the linking number agree. Our sign convention is chosen in a way such that the latter definition becomes pretty, which unfortunately forces us to add the slightly ugly "$(-1)^{\dim(L)}$"-factor to the above definition. □

[569]Here we endow $K \times L$ with the product orientation from Convention 31.49 and we equip S^{n-1} with the standard orientation that we introduced on page 817. Thus we see that $K \times L$ is a closed oriented smooth manifold of dimension $n - 1$. Therefore it makes sense to talk of the degree of a continuous map to S^{n-1}.

Example. Let $n \in \mathbb{N}$, let $L = S^{n-1} \subseteq \mathbb{R}^n$ be the sphere equipped with the standard orientation from page 817 and let $K = \{P\}$ be the 0-dimensional submanifold that consists of a single point $P \in \mathbb{R}^n \setminus S^{n-1}$ and that is equipped with the orientation $+1$.



$\mathrm{lk}(\{P\}, S^1) = 1$ $\mathrm{lk}(\{P\}, S^1) = 0$ $\mathrm{lk}(\{P\}, S^2) = 1$ $\mathrm{lk}(\{P\}, S^2) = 0$

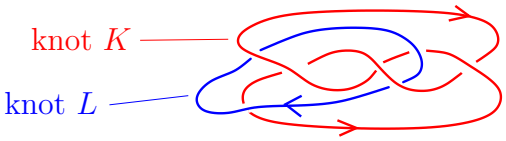
In Exercise 52.3 we will see that

$$\mathrm{lk}(\{P\}, S^{n-1}) \;=\; \begin{cases} 1, & \text{if } P \in B^n, \\ 0, & \text{otherwise.} \end{cases}$$

□

Our main interest will be in the case $n = 3$ and $\dim(K) = \dim(L) = 1$. It is thus convenient to introduce the following definition:

Definition 52.3. An oriented knot in $\mathbb{R}^3$ is an oriented submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1.



what is $\mathrm{lk}(K, L)$?

To formulate our next proposition we will need the following notation and definition.

Definition 52.4.

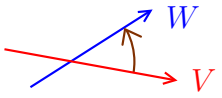
(1) Let $n \in \mathbb{N}$. We denote by

$$\begin{array}{rcl} \pi\colon \mathbb{R}^n & \to & \mathbb{R}^{n-1} \\ (x_1, \dots, x_n) & \mapsto & (x_1, \dots, x_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} z\colon \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \dots, x_n) & \mapsto & x_n \end{array}$$

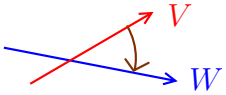
the projections onto the first $n-1$ coordinates and onto the last coordinate.

(2) Let $m \in \mathbb{N}_0$. Given oriented vector subspaces V, W of $\mathbb{R}^m$ with $\dim(V) + \dim(W) = m$ and with $V + W = \mathbb{R}^m$ we set

$$\mathrm{sign}(V, W) \;:=\; \begin{cases} +1, & \text{if } V \times W \to \mathbb{R}^m,\ (v, w) \mapsto v + w \text{ is orientation-preserving,} \\ -1, & \text{otherwise.} \end{cases}$$



$\mathrm{sign}(V, W) = +1$



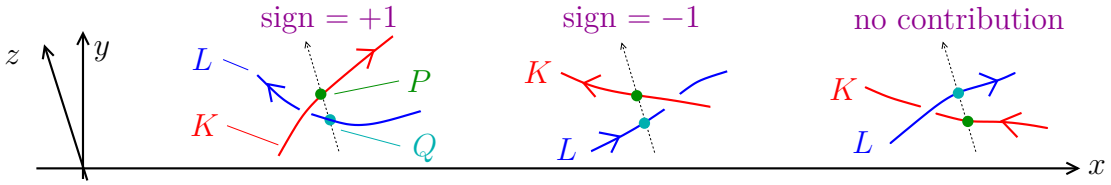
$\mathrm{sign}(V, W) = -1$

The next proposition presents one way to calculate linking numbers. In "real life" it gives a handy way to calculate the linking number of two disjoint oriented knots in $\mathbb{R}^3$. We will also make use of it for a more theoretical purpose.

Proposition 52.5. (Linking Number–Crossings Proposition) Let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n - 1$. We assume that for each $P \in K$ and $Q \in L$ with $\pi(P) = \pi(Q)$ we have an equality $\pi(\widetilde{\mathrm{T}}_P K) + \pi(\widetilde{\mathrm{T}}_Q L) = \mathbb{R}^{n-1}$. Then there are only finitely many $(P, Q) \in K \times L$ with $\pi(P) = \pi(Q)$

and with $z(P) > z(Q)$. Furthermore[570]

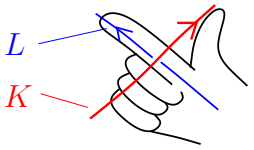
$$\underbrace{\mathrm{lk}(K,L)}_{\text{see Definition 52.2}} = \sum_{\substack{P\in K \text{ and } Q\in L \\ \text{with } \pi(P)=\pi(Q) \text{ and} \\ z(P)>z(Q)}} \mathrm{sign}(\pi(\widetilde{\mathrm{T}}_P K), \pi(\widetilde{\mathrm{T}}_Q L)).$$



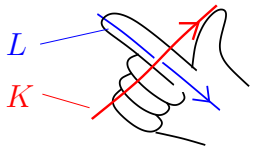
Example. We want to explain in slightly more detail how the Linking Number–Crossings Proposition 52.5 can be used to calculate the linking number between two disjoint oriented knots K and L in $\mathbb{R}^3$. In practice the knots K and L are given by a diagram, i.e. the images of the knots in the xy-plane and one remembers which z-coordinate is larger. In this setting the Linking Number–Crossings Proposition 52.5 says the following:

(1) To each point where K crosses over L we assign $+1$ if the following "right hand rule" is satisfied: If the thumb of the right hand points into the direction of the knot K, then the fingers point in the direction of the knot L. Otherwise we assign -1 to such a crossing point.

(2) We sum up over all signs defined in (1) and we obtain the linking number $\mathrm{lk}(K,L)$ of K and L.

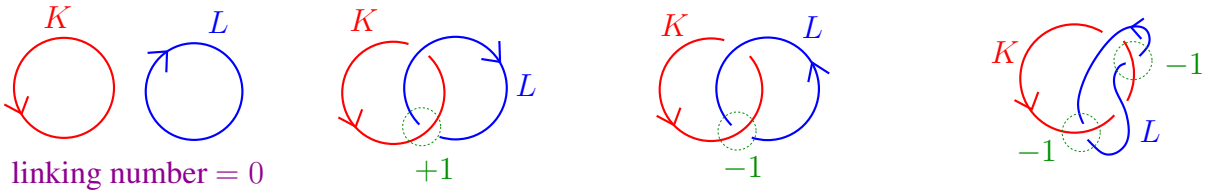


contributes $+1$ to $\mathrm{lk}(K,L)$



contributes -1 to $\mathrm{lk}(K,L)$

In practice it is now straightforward to calculate linking numbers of disjoint oriented knots in $\mathbb{R}^3$.



Some calculations are shown in the above figure. □

Proof of the Linking Number–Crossings Proposition 52.5. We consider the map

$$\begin{aligned} \Theta\colon K\times L &\to S^{n-1} \\ (P,Q) &\mapsto \tfrac{P-Q}{\|P-Q\|}. \end{aligned}$$

[570]Note that $\dim(K)+\dim(L) = n-1$ and $\pi(\widetilde{\mathrm{T}}_P K)+\pi(\widetilde{\mathrm{T}}_Q L) = \mathbb{R}^{n-1}$ imply that the restriction of π to $\widetilde{\mathrm{T}}_P K$ and to $\widetilde{\mathrm{T}}_Q L$ are both monomorphisms. Thus we can view $\pi(\widetilde{\mathrm{T}}_P K)$ and $\pi(\widetilde{\mathrm{T}}_Q L)$ as oriented vector subspaces of $\mathbb{R}^{n-1}$.

Claim.

(1) For any $(P,Q) \in K \times L$ we have $\widetilde{\mathrm{T}}_{(P,Q)}(K \times L) = \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \subseteq \mathbb{R}^n \times \mathbb{R}^n$.

(2) For any $(P,Q) \in K \times L$ with $P - Q = \lambda \cdot \mathfrak{e}_n$ for some $\lambda > 0$ the map

$$\mathrm{D}(\pi \circ \Theta)_{(P,Q)} \colon \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \;\to\; \widetilde{\mathrm{T}}_{\pi(\Theta(P,Q))}\mathbb{R}^{n-1} = \mathbb{R}^{n-1}$$

is given by

$$(v,w) \;\mapsto\; \lambda^{-1} \cdot (\pi(v) - \pi(w)).$$

(3) Given $P \in K$ and $Q \in L$ the following two statements are equivalent:
   (a) $\pi(P) = \pi(Q)$ and $z(P) > z(Q)$,
   (b) $\Theta(P,Q) = \mathfrak{e}_n$.

(4) $\mathfrak{e}_n \in S^{n-1}$ is a regular value of Θ.

(5) There are only finitely many $(P,Q) \in K \times L$ with $\pi(P) = \pi(Q)$ and with $z(P) > z(Q)$.

Proof.

(1) We know by Exercise 29.3 that $\widetilde{\mathrm{T}}_{(P,Q)}(K \times L) = \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L$.

(2) Let $(P,Q) \in K \times L$ with $P - Q = \lambda \cdot \mathfrak{e}_n$ where $\lambda > 0$. We introduce the following two maps:

$$\begin{array}{rcl} \tau \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (v,w) & \mapsto & v - w \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu \colon \mathbb{R}^n \setminus \{0\} & \to & S^{n-1} \\ x & \mapsto & \frac{x}{\|x\|}. \end{array}$$

Let $v \in \widetilde{\mathrm{T}}_P K$ and $w \in \widetilde{\mathrm{T}}_Q L$. We perform the following calculation:

$$\begin{array}{rcccl} \mathrm{D}(\pi \circ \Theta)_{(P,Q)}(v,w) & \overset{\text{by definition of } \Theta}{\underset{\downarrow}{=}} & \mathrm{D}(\pi \circ \nu \circ \tau)_{(P,Q)}(v,w) & \overset{\text{by the Chain Rule and since } \pi \text{ is linear}}{\underset{\downarrow}{=}} & \pi \circ (\mathrm{D}\nu_{\lambda \cdot \mathfrak{e}_n} \circ \mathrm{D}\tau_{(P,Q)})(v,w) \\ & = & \lambda^{-1} \cdot (\pi \circ \tau)(v,w) & = & \lambda^{-1} \cdot (\pi(v) - \pi(w)). \\ & & \underset{\uparrow}{} & & \\ & & \text{by an explicit calculation of } \mathrm{D}\nu_{\lambda \cdot \mathfrak{e}_n} \text{ and since } \tau \text{ is linear} & & \end{array}$$

(3) This statement follows easily from the definitions.

(4) Note that $\mathrm{D}(\pi|_{S^{n-1}})_{\mathfrak{e}_n} : \widetilde{\mathrm{T}}_{\mathfrak{e}_n} S^{n-1} \to \mathbb{R}^{n-1}$ is an isomorphism. It follows from this observation, together with (2) and (3) and our hypothesis in the proposition, that $\mathfrak{e}_n$ is a regular value of Θ.

(5) It follows from (3) and (4) and the Stack-of-Records Lemma 32.3, that there are only finitely $(P,Q) \in K \times L$ with $\pi(P) = \pi(Q)$ and with $z(P) > z(Q)$. ⊞

Before we move on to the actual calculation of the linking number let us introduce one more piece of notation: given any isomorphism $\varphi \colon V \to W$ between oriented vector spaces we write

$$\deg(\varphi) \;:=\; \begin{cases} +1, & \text{if } \varphi \colon V \to W \text{ is orientation-preserving,} \\ -1, & \text{if } \varphi \colon V \to W \text{ is orientation-reversing.} \end{cases}$$

Using this notation we calculate

$$\begin{array}{rcl}
\mathrm{lk}(K,L) & \overset{\text{by Definition 52.2}}{\downarrow} & \\
& = & (-1)^l \cdot \deg\begin{pmatrix} K\times L & \xrightarrow{\Theta} & S^{n-1} \\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|} \end{pmatrix} \downarrow \\
& = & (-1)^l \cdot \sum_{(P,Q)\in\Theta^{-1}(\mathfrak{e}_n)} \deg(\widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \xrightarrow{\mathrm{D}\Theta_{(P,Q)}} \widetilde{\mathrm{T}}_{\mathfrak{e}_n} S^{n-1}) \\
& \uparrow & \text{by the Definition 46.3 of deg, and since} \\
& & \text{by Claim 1 we know that } \mathfrak{e}_n \text{ is a regular value} \\
& = & (-1)^l \cdot \sum_{(P,Q)\in\Theta^{-1}(\mathfrak{e}_n)} \deg(\mathrm{D}(\pi\circ\Theta)_{(P,Q)} \colon \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L \to \mathbb{R}^{n-1}) \\
& \uparrow & \text{by the Chain Rule, and since by the definition of the} \\
& & \text{standard orientation on } S^{n-1}\text{, see page 817, we see that the} \\
& & \text{map } \mathrm{D}(\pi|_{S^{n-1}})_{\mathfrak{e}_n} \colon \widetilde{\mathrm{T}}_{\mathfrak{e}_n} S^{n-1} \to \mathbb{R}^{n-1} \text{ is an orientation-preserving isomorphism} \\
& = & (-1)^l \cdot \sum_{(P,Q)\in\Theta^{-1}(\mathfrak{e}_n)} \deg\begin{pmatrix} \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L & \to & \mathbb{R}^{n-1} \\ (v,w) & \mapsto & \pi(v)+\pi(-w) \end{pmatrix} \\
& \uparrow & \text{follows from Statement (2) of the claim,} \\
& & \text{the fact that } z(P-Q)>0 \text{ and} \\
& & \text{the fact that } -\pi(w)=\pi(-w) \\
& = & \sum_{(P,Q)\in\Theta^{-1}(\mathfrak{e}_n)} \deg\begin{pmatrix} \widetilde{\mathrm{T}}_P K \times \widetilde{\mathrm{T}}_Q L & \to & \mathbb{R}^{n-1} \\ (v,w) & \mapsto & \pi(v)+\pi(w) \end{pmatrix} \\
& \uparrow & \text{for the } l\text{-dimensional vector space } W = \widetilde{\mathrm{T}}_Q L \text{ the} \\
& & \text{degree of the map } w\mapsto -w \text{ equals } (-1)^l \\
& = & \sum_{(P,Q)\in\Theta^{-1}(\mathfrak{e}_n)} \mathrm{sign}(\pi(\widetilde{\mathrm{T}}_P K), \pi(\widetilde{\mathrm{T}}_Q L)). \\
& \uparrow & \text{by the definition of the sign}
\end{array}$$

It follows from Statement (3) of the claim that the sum on the right-hand side is precisely the sum on the right-hand side of the proposition. ■

52.2. **Properties of linking numbers.** We continue with the development of the theory of linking numbers.

Proposition 52.6. (Linking Number–Isotopy Proposition) Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L)=n-1$. If $F\colon (K\sqcup L)\times[0,1]\to\mathbb{R}^n$ is a smooth isotopy, then,

$$\mathrm{lk}(F_0(K),F_0(L)) \;=\; \mathrm{lk}(F_1(K),F_1(L)).$$

Proof. We write $k=\dim(K)$ and $l=\dim(L)$. We consider the map

$$\begin{array}{rcl} \Xi\colon K\times L\times[0,1] & \to & S^2 \\ (x,y,t) & \mapsto & \frac{F_t(x)-F_t(y)}{\|F_t(x)-F_t(y)\|}. \end{array}$$

Now we see that

$$\begin{array}{rcl}
\mathrm{lk}(F_0(K),F_0(L)) & = & (-1)^l\cdot\deg(K\times L\xrightarrow{\Xi_0} S^2) \\
& & \overset{\text{by the Degree Homotopy Invariance Theorem 46.6}}{\downarrow} \\
& = & (-1)^l\cdot\deg(K\times L\xrightarrow{\Xi_1} S^2) \\
& = & \mathrm{lk}(F_1(K),F_1(L)).
\end{array}$$

■

The next propositions is perhaps a little more surprising:

Proposition 52.7. (Linking Number–Symmetry Proposition) Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L)=n-1$. We have the following equality

$$\mathrm{lk}(L,K) \;=\; (-1)^{\dim(K)\cdot\dim(L)+1}\cdot \mathrm{lk}(K,L).$$

Proof. We consider the three maps

$$\begin{array}{rcl}\Theta\colon K\times L &\to& S^{n-1}\\ (x,y) &\mapsto& \frac{x-y}{\|x-y\|}\end{array} \qquad \begin{array}{rcl}\tau\colon K\times L &\to& L\times K\\ (x,y) &\mapsto& (y,x)\end{array} \qquad \text{and} \qquad \begin{array}{rcl}\sigma\colon S^{n-1} &\to& S^{n-1}\\ z &\mapsto& -z.\end{array}$$

Then

$$\begin{array}{rcl}\mathrm{lk}(L,K) &\overset{\text{by Definition 52.2}}{=}& (-1)^k\cdot\deg(\sigma\circ\Theta\circ\tau)\\ &=& (-1)^k\cdot(-1)^{k+l+1}\cdot\deg(\Theta)\cdot(-1)^{k\cdot l}\\ &\overset{\text{by the Basics-of-Degree Lemma 46.12 and since}}{\underset{\sigma \text{ is orientation-preserving if and only if } n=k+l+1 \text{ is even (see Exercise 31.5), and}}{\underset{\tau \text{ is orientation-preserving if and only if } k\cdot l \text{ is even (see Exercise 31.13)}}{=}}} & (-1)^{k\cdot l+1}\cdot\mathrm{lk}(K,L).\end{array} \qquad \begin{array}{rcl} &\overset{\text{by the Degree Multiplicativity Theorem 46.7}}{=}& (-1)^k\cdot\deg(\sigma)\cdot\deg(\Theta)\cdot\deg(\tau)\\ &=& (-1)^{k\cdot l+1}\cdot\mathrm{lk}(K,L).\end{array}$$

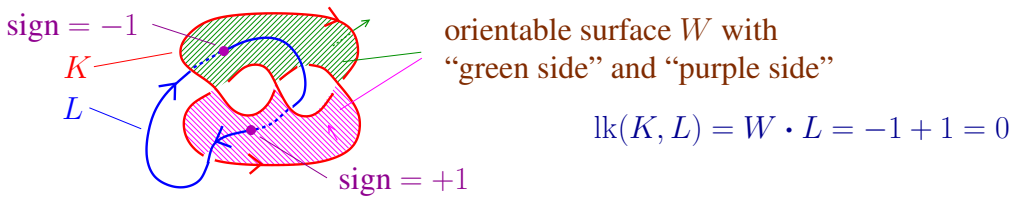
■

We conclude this chapter with the following proposition which can be very useful for calculating linking numbers of knots in $\mathbb{R}^3$.

Proposition 52.8. (Linking #-via-Intersection #-Proposition) Let $n\in\mathbb{N}$ and let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ such that we have the equality $\dim(K)+\dim(L)=n-1$. Suppose that there exists a compact orientable smooth submanifold W of $\mathbb{R}^n$ with $\partial W = K$ (as oriented smooth manifolds). If W intersects L transversally, then

$$\underbrace{\mathrm{lk}(K,L)}_{\text{see Definition 52.2}} \;=\; (-1)^{n+1}\cdot(W\cdot L)$$

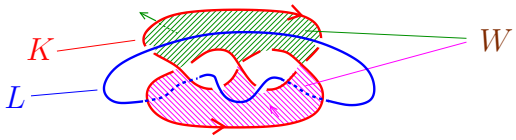
where $W\cdot L$ denotes the algebraic intersection number that we introduced in Definition 51.3.



Remark. In the Linking #-via-Intersection #-Proposition 285.5 we will give an alternative proof of the Linking #-via-Intersection #-Proposition 52.8 in the special case that $n=3$. □

We will prove the Linking #-via-Intersection #-Proposition 52.8 by first considering two special cases.

Lemma 52.9. (Linking Number Zero-Criterion) Let $n\in\mathbb{N}$ and let K and L be closed oriented smooth manifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L)=n-1$. If there exists a compact orientable smooth submanifold W of $\mathbb{R}^n$ such that $\partial W = K$ (as oriented smooth manifolds) and which is disjoint from L, then $\mathrm{lk}(K,L)=0$.



Proof of the Linking Number Zero-Criterion 52.9. Since W is disjoint from L we can consider the map

$$\begin{array}{rcl}\Theta\colon W\times L & \to & S^{n-1}\\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|}.\end{array}$$

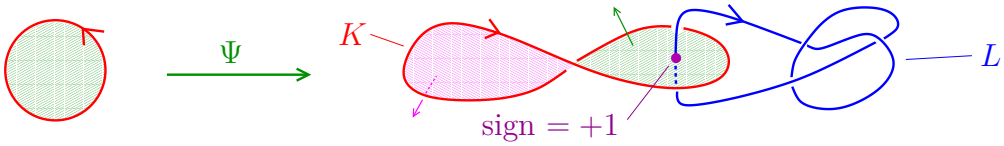
By Sard's Theorem 38.1 we know that $\Theta|_{K\times L}\colon K\times L\to S^{n-1}$ has a regular value $P\in S^{n-1}$. Now we see that[571]

$$\mathrm{lk}(K,L) \underset{\text{by Definition 52.2}}{=} (-1)^{\dim(L)}\cdot \deg(\Theta|_{K\times L}\to S^{n-1}) \overset{\text{see page 1165}}{=} (-1)^{\dim(L)}\cdot(\{P\}\cdot\Theta|_{K\times L}) \underset{\text{by the Intersection Number Zero-if-Extends Proposition 51.8}}{=} 0.$$

■

Lemma 52.10. (Linking Number–Disk Intersection Lemma) Let $n\in\mathbb{N}$. Furthermore let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ such that $\dim(K)+\dim(L)=n-1$. We set $k:=\dim(K)$. We suppose that there exists a smooth embedding $\Psi\colon \overline{B}^{k+1}\to\mathbb{R}^n$ such that $\Psi\colon S^k\to K$ is a diffeomorphism of oriented smooth manifolds. If $\Psi(\overline{B}^{k+1})$ intersects L transversally, then

$$\mathrm{lk}(K,L) \;=\; (-1)^{n+1}\cdot(\Psi(\overline{B}^{k+1})\cdot L).$$



Proof of Lemma 52.10. First we consider a special case. Afterwards we will reduce the general case to the special case.

Claim 1. Let $\iota\colon \overline{B}^{k+1}\to\mathbb{R}^n$ be the standard smooth embedding that is given by $x\mapsto(x,0)$. Let $\widetilde{L}$ be a closed oriented smooth submanifold of $\mathbb{R}^n$ that is disjoint from $\widetilde{K}:=\iota(S^k)$ and with $\dim(\widetilde{L})=n-k-1$. If $\widetilde{L}$ is transverse to $\overline{B}^{k+1}\times\{0\}$ and to $S^k\times\{0\}\times\mathbb{R}$, then

$$\mathrm{lk}(\widetilde{K},\widetilde{L}) \;=\; (-1)^{n+1}\cdot(\iota(\overline{B}^{k+1})\cdot\widetilde{L}).$$

Proof. Since $\widetilde{L}$ is compact there exists an $r<0$ such that $\widetilde{L}\subseteq\mathbb{R}^{n-1}\times(r,\infty)$. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth embedding

$$\begin{array}{rcl}\gamma\colon[0,1] & \to & \mathbb{R}^2\\ t & \mapsto & \gamma(t)=(\alpha(t),\beta(t))\end{array}$$

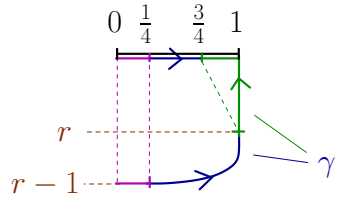
with the following properties:

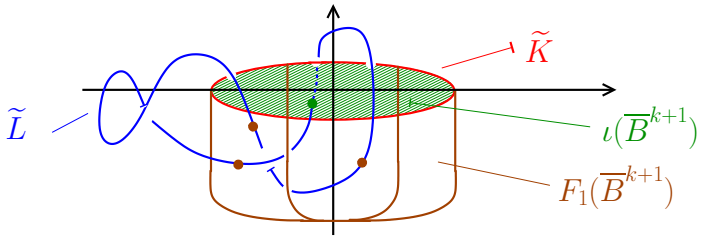
(1) For $t\in[0,\frac{1}{4}]$ we have $\gamma(t)=(t,r-1)$.
(2) For $t\in[\frac{1}{4},\frac{3}{4}]$ we have $\gamma(t)\in[\frac{1}{4},1]\times[r-1,r]$.
(3) For $t\in[\frac{3}{4},1]$ we have $\gamma(t)=(1,r\cdot(1-4(t-\frac{3}{4})))$.

[571] The argument becomes even more direct if one has solved Exercise 46.12.

Next we consider the following map:[572]

$$\begin{array}{rcl} F\colon \overline{B}^{k+1} \times [0,1] & \to & \mathbb{R}^n \\ (v,t) & \mapsto & \begin{cases} (v,0,\dots,0,t\cdot(r-1)), & \text{if } \|v\| < \frac{1}{4}, \\ (\frac{v}{\|v\|}\cdot(\|v\|\cdot(1-t)+t\cdot\alpha(\|v\|)),0,\dots,0,t\cdot\beta(\|v\|)), & \text{if } \|v\| >. \end{cases} \end{array}$$





We make the following observations and preparations:

(1) Using the Smooth Pasting Proposition 25.19 one can easily verify that F is smooth.
(2) Note that $F_1(\overline{B}^{k+1}) \cap (\mathbb{R}^{n-1} \times (r,0]) = S^k \times \{0\} \times (r,0]$.
(3) As above we denote by $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $n-1$ coordinates.
(4) Since $\widetilde{L}$ is transverse to $S^k \times \{0\} \times \mathbb{R}$ we obtain from the Submanifold Transversal Intersection Theorem 50.4 that $(S^k \times \{0\} \times \mathbb{R}) \cap \widetilde{L}$ is a compact 0-dimensional submanifold, i.e. it is a finite set. We denote the intersection points of $F_1(\overline{B}^{k+1})$ with $\widetilde{L}$ by $(x_i,0,t_i) \in S^k \times \{0\} \times (r,0)$, $i = 1,\dots,m$.
(5) Note that the points $(x_i,0,t_i) \in \widetilde{L}$, $i = 1,\dots,m$ are precisely the points on $\widetilde{L}$ such that the projection under π is contained in $\pi(\widetilde{K})$ and such that the last coordinate of the point on $\widetilde{K}$ is greater than the last coordinate of the point on $\widetilde{L}$.
(6) Note that it follows from the Boundary Orientation Lemma 31.38 (4) together with the Product Orientation Proposition 31.48 (4) that the map $\Theta\colon (\frac{3}{4},1] \times S^k \to \overline{B}^{k+1}$ given by $(t,v) \mapsto t \cdot v$ is orientation-preserving.
(7) It follows easily from (6) that the map

$$\begin{array}{rcl} F_1\colon \{v \in \overline{B}^{k+1} \mid \|v\| > \frac{3}{4}\} & \to & S^k \times \{0\} \times (r,0] \\ v & \mapsto & (\frac{v}{\|v\|},0,\dots,0,r\cdot(1-4\cdot(\|v\|-\frac{3}{4}))) \end{array}$$

is *orientation-reversing*.
(8) We have $\det \begin{pmatrix} 0 & \mathrm{id}_{n-1} \\ 1 & 0 \end{pmatrix} = (-1)^{n+1}$.
(9) Let $i \in \{1,\dots,m\}$. It follows easily from (7) and (8) and the various definitions that

$$\underbrace{\begin{array}{c}\text{sign of the intersection point} \\ (x_i,0,t_i) \text{ of } F_1(\overline{B}^{k+1}) \text{ and } \widetilde{L}\end{array}}_{\text{see Definition 51.4}} = (-1)^{n+1} \cdot \underbrace{\operatorname{sign}(\pi(\widetilde{\mathrm{T}}_{(x_i,0,t_i)}F_1(\overline{B}^{k+1})), \pi(\widetilde{\mathrm{T}}_{(x_i,0,t_i)}\widetilde{L}))}_{\text{see Definition 52.4}}$$

Then we see that

$$\iota(\overline{B}^{k+1}) \cdot \widetilde{L} = F_0(\overline{B}^{k+1}) \cdot \widetilde{L} \underset{\uparrow}{=} F_1(\overline{B}^{k+1}) \cdot \widetilde{L} \underset{\text{Lemma 51.5}}{\overset{\text{by the Intersection Numbers-via-Signs}}{\downarrow}} \sum_{i=1}^{m} \operatorname{sign}(x_i,0,t_i) \underset{\uparrow}{=} (-1)^{n+1} \cdot \operatorname{lk}(\widetilde{K},\widetilde{L}).$$

by the Intersection Number Homotopy Invariance Theorem 51.10

follows from (8) and the Linking Number–Crossings Proposition 52.5 ⊞

[572] Note that by our choice of γ the definition below for $\|v\| \in (0,\frac{1}{4})$ is consistent.

The statement of the following Claim 2 is identical to the statement of Claim 1, except that we will manage to drop the hypothesis that $\widetilde{L}$ is transverse to $S^k \times \{0\} \times \mathbb{R}$.

Claim 2. Let $\widehat{L}$ be a closed oriented smooth submanifold of $\mathbb{R}^n$ that is disjoint from the submanifold $\widetilde{K} \coloneqq \iota(S^k)$ and with $\dim(\widehat{L}) = n-k-1$. If $\widehat{L}$ is transverse to $\overline{B}^{k+1} \times \{0\}$, then
$$\operatorname{lk}(\widetilde{K}, \widehat{L}) \;=\; (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widehat{L}).$$

Proof. First note that it follows easily from the Transversality-Extends-to-Neighborhood Lemma 50.15 together with the Transversality Theorem 50.10, applied to the smooth manifold $M = \mathbb{R}^n \setminus (S^k \times \{0\})$, that there exists a smooth isotopy $G\colon \widehat{L} \times [0,1] \to \mathbb{R}^n$ rel a closed neighborhood of $\widehat{L} \cap (\overline{B}^{k+1} \times \{0\})$, such that $G_0 = \mathrm{id}_{\widehat{L}}$ and such that we have $(S^k \times \{0\} \times \mathbb{R}) \pitchfork G_1(\widehat{L})$. We set $\widetilde{L} \coloneqq G_1(\widehat{L})$. Now we see that

$$\operatorname{lk}(\widetilde{K}, \widehat{L}) \underset{\uparrow}{=} \operatorname{lk}(\widetilde{K}, \widetilde{L}) \overset{\text{Claim 1}}{\underset{\downarrow}{=}} (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widetilde{L}) \underset{\uparrow}{=} (-1)^{n+1} \cdot (\iota(\overline{B}^{k+1}) \cdot \widehat{L}).$$

by the Linking Number–Isotopy Proposition 52.6, since G and the constant isotopy on $\widetilde{K} = S^k \times \{0\}$ define a smooth isotopy from $\widetilde{K} \sqcup \widehat{L}$ to $\widetilde{K} \sqcup \widetilde{L}$

since $\widetilde{L} = \widehat{L}$ in a neighborhood of $\iota(\overline{B}^{k+1})$

⊞

Now we finally turn to the general setting of the proposition. Thus let K and L be two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K) + \dim(L) = n-1$. We set $k \coloneqq \dim(K)$. We suppose that there exists a smooth embedding $\Psi\colon \overline{B}^{k+1} \to \mathbb{R}^n$ such that $\Psi\colon S^k \to K$ is a diffeomorphism of oriented smooth manifolds and such that $\Psi(\overline{B}^{k+1})$ intersects L transversally. It follows from the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 that there exists a diffeotopy $F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that $F_0 = \mathrm{id}$ and such that $F_1 \circ \Psi\colon \overline{B}^{k+1} \to \mathbb{R}^n$ is the standard embedding $\iota\colon \overline{B}^{k+1} \to \mathbb{R}^n$ given by $x \mapsto (x, 0)$. Note that it follows from $F_0 = \mathrm{id}$ and the Smooth Isotopy-Orientation Proposition 42.4 that F_1 is orientation-preserving. Now we see that

$$\operatorname{lk}(K, L) \underset{\uparrow}{=} \operatorname{lk}(F_1(K), F_1(L)) \overset{\text{by Claim 2, since } F_1(K) = \widetilde{K}, \text{ we also use that by hypothesis } \Psi(\overline{B}^{k+1}) \text{ intersects } L \text{ transversally and hence } F_1(\Psi(\overline{B}^{k+1})) = \iota(\overline{B}^{k+1}) \text{ and } F_1(L) \text{ also intersect transversally}}{\underset{\downarrow}{=}} \iota(\overline{B}^{k+1}) \cdot F_1(L) \underset{\uparrow}{=} \Psi(\overline{B}^{k+1}) \cdot L.$$

by the Linking Number–Isotopy Proposition 52.6

by the Intersection Number–Basics Lemma 51.7 applied to the orientation-preserving diffeomorphism F_1^{-1}

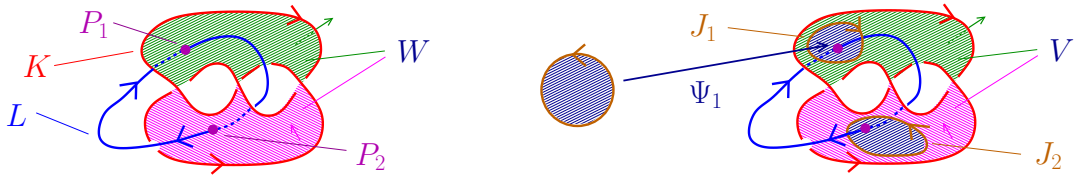
■

Proof of the Linking #-via-Intersection #-Proposition 52.8. Let K and L be disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L) = n-1$. We suppose that there exists a compact orientable smooth submanifold W of $\mathbb{R}^n$ with $\partial W = K$ and with $W \pitchfork L$.

It follows from the Submanifold Transversal Intersection Theorem 50.4 that $W \cap L$ is a compact 0-dimensional submanifold of W. This means that $W \cap L$ consists of finitely many points $P_1, \dots, P_m \in W \setminus \partial W$.

We set $k \coloneqq \dim(K)$. Evidently we can pick orientation-preserving smooth embeddings $\Psi_i\colon \overline{B}^{k+1} \to W \setminus \partial W$, $i = 1, \dots, m$, with disjoint images, and such that $\Psi_i(0) = P_i$ for

$i = 1, \dots, m$. We set $V \coloneqq W \setminus \bigcup_{i=1}^{m} \Psi_i(B^{k+1})$ and for $i = 1, \dots, m$ we set $J_i \coloneqq \Psi_i(S^k)$. We equip J_i with the orientation coming from the standard orientation of S^k.



By the Codimension-Zero Smooth Embedding Proposition 30.21 we know that V is a compact codimension-zero submanifold of W. If we equip V with the orientation coming from W, then it follows from the Opposite Side–Opposite Orientation Lemma 31.40 that $\partial V = K \cup \bigcup_{i=1}^{m} (-J_i)$. Finally we see that

$$\operatorname{lk}(K,L) \underset{\uparrow}{=} \operatorname{lk}\left(K \cup \bigcup_{i=1}^{m} (-J_i), L\right) + \sum_{i=1}^{m} \operatorname{lk}(J_i, L) \underset{\uparrow}{=} 0 + \sum_{i=1}^{m} (-1)^{n+1} \cdot \Psi_i(\overline{B}^{k+1}) \cdot L$$

follows easily from the Definition 52.2 of the linking number and basic properties of the degree

this equalities consists of two equalities that come from:
(1) the Linking Number Zero-Criterion 52.9, $\partial V = K \cup \bigcup_{i=1}^{m} (-J_i)$ and $V \cap L = \varnothing$
(2) the Linking Number–Disk Intersection Lemma 52.10 and $J_i = \partial \Psi_i(\overline{B}^{k+1})$

$$\underset{\uparrow}{=} (-1)^{n+1} \cdot W \cdot L.$$

since we have the same intersection points

■

52.3. Exercises for Chapter 52.

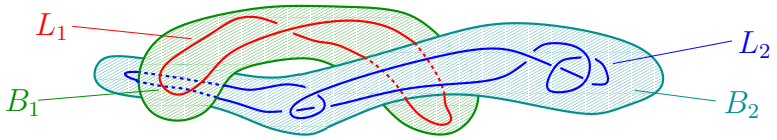
Exercise 52.1. An *oriented link in* $\mathbb{R}^3$ is a closed oriented 1-dimensional smooth submanifold of $\mathbb{R}^3$. We say an m-component link $L \subseteq \mathbb{R}^3$ is a *trivial link* if it bounds m disjoint smoothly embedded disks in $\mathbb{R}^3$. It follows immediately from the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 that all trivial m-component links are in fact smoothly isotopic.

(a) Give an example of two disjoint knots K, L in $\mathbb{R}^3$ such that K and L are smoothly isotopic to the trivial knot, such that $\operatorname{lk}(K,L) = 0$ but such that $K \sqcup L$ is not smoothly isotopic to a trivial link.
(b) Give an example of a 3-component link $L = L_1 \sqcup L_2 \sqcup L_3$ in $\mathbb{R}^3$ such that any two components are isotopic to a trivial link but such that L itself is not smoothly isotopic to a trivial link.
(c) Generalize (b) from $m = 3$ to any $m \in \mathbb{N}_{\geq 3}$.

Remark. We lack the tools right now to show that a link with zero linking numbers is not smoothly isotopic to a trivial link. Thus with the methods we have developed so far we cannot really give a complete answer to the challenges. But this should not prevent the reader from trying to find examples.

Exercise 52.2. An *oriented link in* $\mathbb{R}^3$ is a closed oriented 1-dimensional smooth submanifold of $\mathbb{R}^3$. An m-component link $L = L_1 \sqcup \cdots \sqcup L_m \subseteq \mathbb{R}^3$ is called *split link* if there exist disjoint smoothly embedded closed 3-dimensional balls $B_1, \dots, B_m \subseteq \mathbb{R}^3$ such that each L_i

is contained in B_i. Show that if $L = L_1 \sqcup \cdots \sqcup L_m \subseteq \mathbb{R}^3$ is a split link, then for any $i \neq j$ we have $\mathrm{lk}(L_i, L_j) = 0$.



Exercise 52.3. Let $n \in \mathbb{N}$, let $L = S^{n-1} \subseteq \mathbb{R}^n$ be the sphere equipped with the standard orientation from page 817 and let $K = \{P\}$ be the 0-dimensional submanifold that consists of a single point $P \in \mathbb{R}^n \setminus S^{n-1}$ and that is equipped with the orientation $+1$. Show that

$$\mathrm{lk}(\{P\}, S^{n-1}) \;=\; \begin{cases} (-1)^n, & \text{if } P \in B^n, \\ 0, & \text{otherwise.} \end{cases}$$

CHAPTER 53

The Smooth Poincaré-Hopf Theorem

The reader might already have heard of the Euler characteristic of a "simplicial complex" or a "CW-complex". In that setting the Euler characteristic is usually defined as the alternating sum of the number of simplices (or cells) in the various dimensions. In this chapter we use a totally different approach to define the Euler characteristic of a smooth manifold. We will later see that these notions of Euler characteristic are compatible.

This chapter is organized as follows:

- In Section 53.1 we will introduce the smooth Euler characteristic of a closed oriented smooth manifold.
- In Section 53.2 we will study zeros of vector fields. In particular we will introduce the index of a non-degenerate zero of a vector field.
- In Section 53.3 we will introduce the total tangent space of a smooth submanifold of $\mathbb{R}^n$ and we will reinterpret the index of a zero of a vector field as an algebraic intersection number.
- In Section 53.4 we will show that the total tangent space is diffeomorphic to a neighborhood of the diagonal of M in the product $M \times M$.
- In Section 53.5 we will state the Smooth Poincaré–Hopf Theorem 53.24 and we will provide the surprisingly short proof.
- In Section 53.6 we will discuss applications of the Smooth Poincaré–Hopf Theorem 53.24.
- In Section 53.7 we will state the Gauß Map-Degree Theorem 53.32 and we will prove it using the Smooth Poincaré–Hopf Theorem 53.24.
- In Section 53.8 we will provide exercises for Chapter 53.

In the next Chapter 54 we will learn how to glue two smooth manifolds together.

53.1. **The smooth Euler characteristic.** The following definition generalizes Definition 51.3 of the algebraic intersection number of two transverse submanifolds to the non-transverse case.

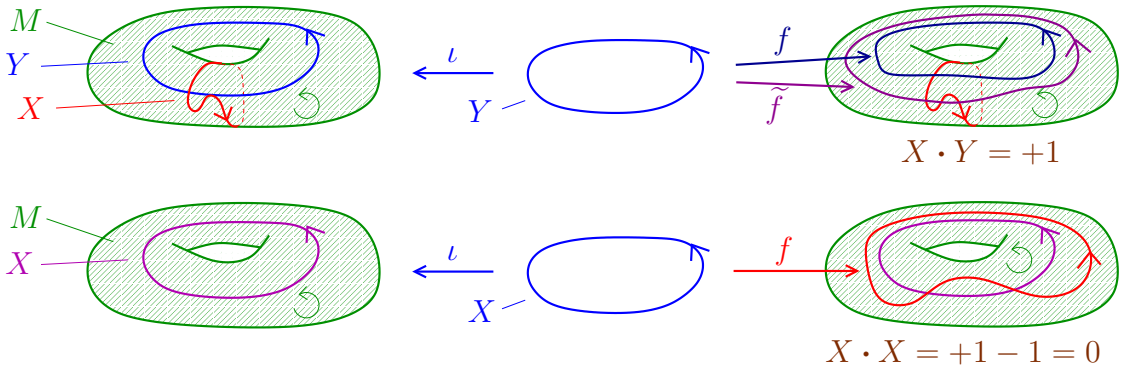
Definition 53.1. Let M be an oriented smooth manifold.

(1) Let X and Y be two closed oriented submanifolds of M of complementary dimension, i.e. we have $\dim(X)+\dim(Y) = \dim(M)$. Let $\iota\colon Y \to M$ be the inclusion map. By the Transversality Theorem 50.10 the map ι is smoothly isotopic to a smooth embedding $f\colon Y \to M$ with $X \pitchfork f$. It follows from the Intersection Number–Homotopy Invariance Theorem 51.9 that

$$X \cdot Y \coloneqq X \cdot f \in \mathbb{Z}$$

does not depend on the choice of f. We refer to $X \cdot Y$ as the algebraic intersection number of X and Y.

(2) If M is even-dimensional and if X is a closed oriented submanifold of M such that $2 \cdot \dim(X) = \dim(M)$, then in (1) we can take $Y = X$ and we refer to $X \cdot X \in \mathbb{Z}$ as the algebraic self-intersection number of X



We continue with an analog of the Intersection Number–Symmetry Lemma 51.6 and the Intersection Number–Basics Lemma 51.7

Lemma 53.2. (Intersection Number–Symmetry Lemma) Let M be an oriented m-dimensional smooth manifold and let $X, Y \subseteq M$ be two closed oriented submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$.

(1) The following equality holds:

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \;\cdot\; X \cdot Y.$$

(2) If we reverse the orientation of precisely one of X, Y or M, then the sign of the algebraic intersection number flips.

Proof.

(1) One might think that this statement is just a special case of the Intersection Number–Symmetry Lemma 51.6. But that is not the case: in the definition of the algebraic intersection number in the non-transverse case we apply a smooth isotopy only to the second submanifold. Thus the definition of the algebraic intersection number is not symmetric in X and Y.

We write $k = \dim(X)$ and $l = \dim(Y)$. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. By the Transversality Theorem 50.9 there exists a smooth isotopy $G\colon X \times [0,1] \to M$ with $G_0 = i\colon X \to M$ and such that $Y \pitchfork G_1$. By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $H\colon M \times [0,1] \to M$ with $H_0 = \mathrm{id}$

such that for each $(x,t) \in X \times [0,1]$ we have $H(x,t) = G(x,t)$. Now we see that

$$\begin{array}{ccccccc} & \text{by Definition 53.1} & & & \text{since } H_1^{-1} \circ G_1 = i & & \\ Y \cdot X & \overset{\downarrow}{=} & Y \cdot G_1 & = & H_1^{-1}(Y) \cdot (H_1^{-1} \circ G_1) & \overset{\downarrow}{=} & H_1^{-1}(Y) \cdot X \end{array}$$

follows from the Intersection Number–Basics Lemma 51.7 and the Smooth Isotopy-Orientation Proposition 42.4

$$\begin{array}{ccccccc} & = & (-1)^{k\cdot l} \cdot X \cdot H_1^{-1}(Y) & = & (-1)^{k\cdot l} \cdot X \cdot H_1^{-1} & = & (-1)^{k\cdot l} \cdot X \cdot Y. \\ & \text{by the Intersection Number–Symmetry Lemma 51.6} & & \text{by Definition 53.1} & & \text{by Definition 53.1 of } X \cdot Y \text{ since the map } \begin{array}{ccc} Y \times [0,1] & \to & M \\ (y,t) & \mapsto & H_t^{-1}(y) \end{array} \text{ is a smooth isotopy} & \end{array}$$

(2) This statement follows from the Compatible Orientations Lemma 31.12 (2). ■

Before we can introduce the key definition of this chapter we need to state and prove the following elementary lemma:

Lemma 53.3. (Diagonal–Submanifold Lemma) Let M be a smooth manifold with empty boundary.

(1) The diagonal $\Delta_M \coloneqq \{(x,x) \,|\, x \in M\}$ is a proper submanifold of the smooth manifold $M \times M$.

(2) The diagonal Δ_M is diffeomorphic to M.

Proof. [573] Let M be an n-dimensional smooth manifold with $\partial M = \varnothing$.

(1) Let $(x,x) \in \Delta_M$. We pick a chart $\Phi \colon U \to V$ for $x \in M$. It is elementary to verify that the map

$$\begin{array}{ccccc} U \times U & \xrightarrow{\Phi \times \Phi} & V \times V & \to & \{(c,d) \in \mathbb{R}^{2n} \,|\, c+d \in V \text{ and } d \in V\} \\ & & (a,b) & \mapsto & (a-b,b) \end{array}$$

is a submanifold chart for Δ_M. Since M is by definition Hausdorff we know by the Hausdorff-Diagonal Lemma 7.20 that the diagonal Δ_M is a closed subset of $M \times M$. This shows that Δ_M is a proper submanifold of $M \times M$.

(2) We consider the following two maps:

$$\begin{array}{ccccc} p \colon \Delta_M & \to & M & \text{and} & \begin{array}{ccccc} d \colon M & \to & \Delta_M \\ x & \mapsto & (x,x). \end{array} \\ (x,x) & \mapsto & x & & \end{array}$$

Evidently these two maps are inverses of one another. Furthermore it follows from the Smooth Manifold Product Proposition 25.34 and the Submanifold Smooth Structure Lemma 25.23 that p and d are smooth. This shows that both maps are diffeomorphisms. ■

The Diagonal–Submanifold Lemma 53.3 allows us to introduce the following surprising definition:

Definition 53.4. Let M be a closed orientable smooth manifold. We pick an orientation and we consider the product $M \times M$, equipped with the product orientation introduced in Convention 31.49, and we consider the submanifold $\Delta_M \coloneqq \{(x,x) \,|\, x \in M\}$, called the diagonal of M. By the Diagonal–Submanifold Lemma 53.3 (2) we know that Δ_M is

[573] Note that this proof solves Exercise 25.30.

orientable. We equip Δ_M with an orientation. We define the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of M as follows: $\chi^{\mathrm{sm}}(M) := \Delta_M \cdot \Delta_M$.

Remark. In Definitions 156.1 and 156.6 we will define the Euler characteristic $\chi(X)$ of suitable topological spaces X. Furthermore we will see in the Smooth Manifolds-Invariants Proposition 179.10 and alternatively on page 4851 that this more general Euler characteristic is defined for every compact smooth manifold. We will see in the Cup Product–Intersection Number Theorem 220.10 and the Lefschetz-Intersection Number Proposition 222.4, or alternatively in the Euler Characteristics–Equality Corollary 223.6 (which we prove twice, once on page 4309 and once on page **??**) that the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of a closed orientable smooth manifold M agrees with the more general Euler characteristic $\chi(M)$. □

Lemma 53.5. (Smooth-χ-Lemma)

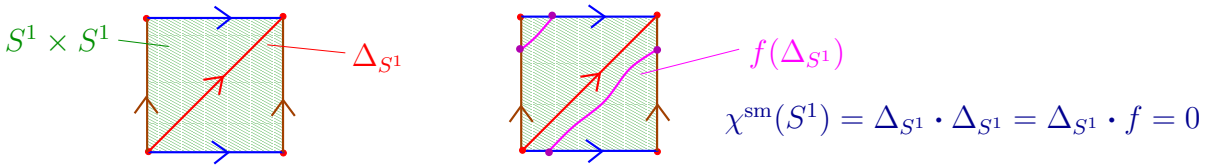
(1) Let M be a closed orientable smooth manifold. The smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ is independent of the choice of the orientation of M and independent of the choice of the orientation of Δ_M.
(2) Let M and N be two closed orientable smooth manifolds. If M and N are diffeomorphic, then $\chi^{\mathrm{sm}}(M) = \chi^{\mathrm{sm}}(N)$.

Proof.

(1) Let M be a closed orientable smooth manifold. The desired statement follows immediately from the following two facts:
- By the Product Orientation Proposition 31.48 (3) we have $(-M)\times(-M) = M\times M$.
- It follows from the Intersection Number–Symmetry Lemma 53.2 (2) that we have $(-\Delta_M)\cdot(-\Delta_M) = -\Delta_M\cdot(-\Delta_M) = \Delta_M\cdot\Delta_M$.

(2) This statement follows easily from the definitions. ■

Example. In the figure below we consider $M = S^1$. The diagonal $\Delta_{S^1} \subseteq S^1 \times S^1$ is the closed curve shown in the figure:



We denote by $\iota\colon \Delta_{S^1} \to S^1 \times S^1$ the inclusion and as in the figure we pick a continuous map $f\colon \Delta_{S^1} \to S^1 \times S^1$ that is smoothly isotopic to ι and which satisfies $\Delta_{S^1} \pitchfork f$. We see that $\Delta_{S^1} \cdot \Delta_{S^1} = \Delta_{S^1} \cdot f = 0$. In fact the next lemma shows that this is not a coincidence at all. □

Proposition 53.6. (Odd-Dimensional Manifold-χ-Proposition) The smooth Euler characteristic of any closed orientable *odd-dimensional* smooth manifold is zero.

Remark. In the Odd-Dimensional Manifold-χ-Proposition 210.9 and the Odd-Dimensional Manifold-χ-Proposition 250.9 we will reprove and generalize the Odd-Dimensional Manifold-χ-Proposition 53.6. □

Proof. Let M be a closed orientable $(2n+1)$-dimensional smooth manifold. By the Diagonal–Submanifold Lemma 53.3 we know that the diagonal Δ_M is also an $(2n+1)$-dimensional smooth manifold. Now we see that

$$\chi^{\mathrm{sm}}(M) \underset{\text{by Definition 53.4}}{\uparrow} = \Delta_M \cdot \Delta_M \underset{\substack{\text{by the Intersection Number} \\ \text{Symmetry Lemma 53.2 (1)}}}{\uparrow} = (-1)^{(2n+1)\cdot(2n+1)} \cdot \Delta_M \cdot \Delta_M = -\Delta_M \cdot \Delta_M \underset{\text{by Definition 53.4}}{\uparrow} = -\chi^{\mathrm{sm}}(M).$$

It follows that $\chi^{\mathrm{sm}}(M) = 0$. ∎

It is now natural to try to compute the smooth Euler characteristic of our favorite orientable smooth manifolds. We pose the following question:

Question 53.7.
(1) Let $n \in \mathbb{N}$ be even. What is the smooth Euler characteristic of S^n?
(2) Let $g \in \mathbb{N}$. What is the Euler characteristic of the surface Σ_g of genus g?

To answer Question 53.7 we will have to develop tools for calculating smooth Euler characteristics. In the following sections we will formulate and prove the Smooth Poincaré–Hopf Theorem 53.24 which allows us to compute the smooth Euler characteristic of closed orientable smooth manifolds in terms of "indices" of "non-degenerate" zeros of vector fields. We will then be able to answer Question 53.7 (1) in the Sphere-Euler Characteristic-Proposition 53.26 and we will be able to answer Question 53.7 (2) in the Surface-χ-Proposition 53.27.

53.2. **Zeros of vector fields.** As on several occasions we adopt for the remainder of this chapter the following popular convention.

Convention 53.8. We will use the natural isomorphism from the Tangent Space-Isomorphism Proposition 29.24 to go back and forth between the physical tangent space $\widetilde{\mathrm{T}}_P M$ and the abstract tangent space $\mathrm{T}_P M$ of a smooth submanifold M of some $\mathbb{R}^n$.

Recall that in Definition 29.29 we introduced the notion of a smooth vector field on a smooth manifold and recall the following definition from the Vector Field-Pullback Lemma 29.33:

Notation 53.9. Let $f\colon M \to N$ be a local diffeomorphism between two smooth manifolds M and N and let w be a vector field on N. We denote by f^*w the vector field on M which assigns to each $P \in M$ the tangent vector $(\mathrm{D}f_P)^{-1}(w(f(P)))$.[574]

We continue with a new aspect of vector fields:

Definition 53.10. Let M be a smooth manifold and let v be a smooth vector field on M. Let $P \in M \setminus \partial M$ be a zero of v. We pick a chart $\Phi\colon U \to V$. We set $Q := \Phi(P)$ and we write $\Psi := \Phi^{-1}$. We consider the map

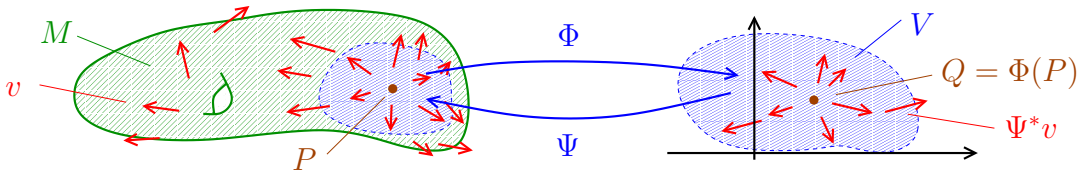
$$\begin{array}{rcl} \Psi^* v\colon V & \to & \mathbb{R}^n \\ x & \mapsto & (\Psi^* v)(x) = \underbrace{\mathrm{D}\Phi_{\Psi(x)}(v(\Psi(x)))}_{\in \mathrm{T}_x\mathbb{R}^n = \widetilde{\mathrm{T}}_x\mathbb{R}^n = \mathbb{R}^n}. \end{array}$$

[574] We will often use that by the Vector Field-Pullback Lemma 29.33 we know that if w is a smooth vector field, then f^*w is also smooth.

(1) If $\mathrm{D}(\Psi^* v)_Q$ is an invertible matrix, then we say that the zero is non-degenerate. Otherwise we call the zero degenerate.
(2) If the zero P is non-degenerate, then we define

$$\operatorname{index}(v, P) \coloneqq \begin{cases} +1, & \text{if } \det(\mathrm{D}(\Psi^* v)_Q) > 0, \\ -1, & \text{if } \det(\mathrm{D}(\Psi^* v)_Q) < 0. \end{cases}$$

In the Index-of-Zero Lemma 53.11 we will see that these notions do not depend on the choice of the chart Φ.



Lemma 53.11. (Index-of-Zero Lemma) The notions introduced in the previous definition do not depend on the choice of the chart Φ.

In the proof of the Index-of-Zero Lemma 53.11 we will need the following, potentially slightly confusing, little lemma:

Auxiliary Lemma 53.12. Let $A \subseteq \mathbb{R}^m$ be an open subset and let $\Xi\colon A \to \mathrm{M}(n \times k, \mathbb{R})$ and $u\colon A \to \mathbb{R}^k$ be smooth maps. Let $S \in A$. If $u(S) = 0$, then

$$\mathrm{D}(\underbrace{x \mapsto \Xi(x) \cdot u(x)}_{\text{map } A \to \mathbb{R}^n})_S = \Xi(S) \cdot \mathrm{D}u_S \quad \in \mathrm{M}(n \times m, \mathbb{R}).$$

Proof of Lemma 53.12. The lemma follows immediately from the Product Rule applied to each entry and the fact that $u(S) = 0$. ∎

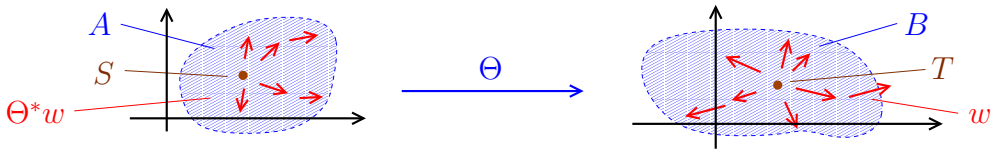
Proof of Lemma 53.11. Let M be an n-dimensional smooth manifold, let v be a smooth vector field on M and let $P \in M \setminus \partial M$ be a zero of v. We need to show that if we have two different charts $\Phi\colon U \to V$ and $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ around P, then we end up with the same notion of degeneracy and index.

We set $A \coloneqq \Phi(U \cap \widetilde{U})$, $B = \widetilde{\Phi}(U \cap \widetilde{U})$ and we denote by $\Theta \coloneqq \widetilde{\Phi} \circ \Phi^{-1}\colon A \to B$ the transition map between the two charts. It follows easily from the Vector Field–Double Pullback Lemma 43.13 that we only have to prove the following claim:

Claim. Let $\Theta\colon A \to B$ be a diffeomorphism between two open subsets of some $\mathbb{R}^n$. Let $w\colon B \to \mathbb{R}^n$ be a smooth vector field on B with a zero at some point $T \in B$. We set $S \coloneqq \Theta^{-1}(T) \in A$. We consider the maps

$$\begin{array}{rcl} \Theta^* w\colon A & \to & \mathbb{R}^n \\ x & \mapsto & (\Theta^* w)(x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} w\colon B & \to & \mathbb{R}^n \\ y & \mapsto & w(y). \end{array}$$

Then the differentiable $\mathrm{D}(\Theta^* w)_S$ is invertible if and only if $\mathrm{D}w_T$ is invertible. Furthermore we have $\det(\mathrm{D}(\Theta^* w)_S) > 0$ if and only if $\det(\mathrm{D}w_T) > 0$.



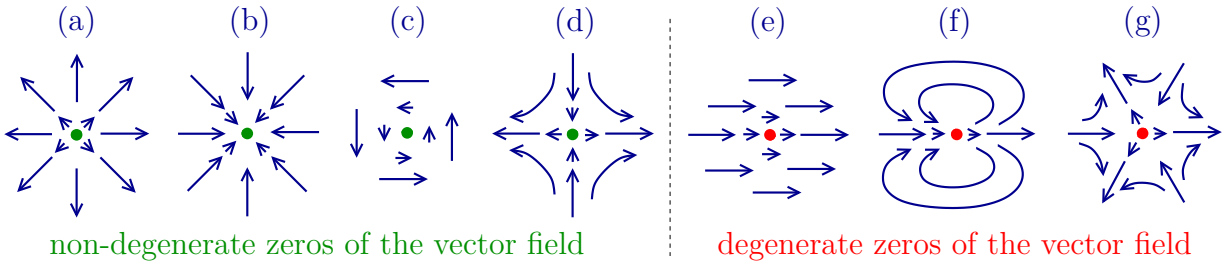
Proof. Note that

$$\begin{array}{lll} \mathrm{D}(x \mapsto (\Theta^* w(x)))_S & \overset{\text{by definition and by the Differential-equals-Differential Lemma 29.6}}{=} & \mathrm{D}\big(x \mapsto (\mathrm{D}\Theta_x)^{-1} \cdot (w \circ \Theta)(x)\big)_S \\ & = (\mathrm{D}\Theta_S)^{-1} \cdot \mathrm{D}(w \circ \Theta)_S & = (\mathrm{D}\Theta_S)^{-1} \cdot \mathrm{D}w_T \cdot \mathrm{D}\Theta_S. \\ & \underset{\text{since } (w \circ \Theta)(S) = 0}{\overset{\text{by Lemma 53.12 and}}{=}} & \underset{\text{since } \Theta(S) = T}{\overset{\text{by the Chain Rule and}}{=}} \end{array}$$

This shows that $\mathrm{D}(\Theta^* w)_S$ is invertible if and only if $\mathrm{D}w_T$ is invertible. Furthermore we see that $\det(\mathrm{D}(\Theta^* w)_S) > 0$ if and only if $\det(\mathrm{D}w_T) > 0$. ■

Remark. In Definition 223.1 we will introduce a more general notion of an index of a zero of a vector field which is defined also for degenerate zeros of vector fields. It follows from the Linearization–Homotopy Lemma 152.7 that the two definitions of the index agree, if both are defined. □

Example. We consider the vector fields on $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ with isolated zero at the origin that are shown in the following figure:



In the following we use the identity chart $\Phi = \mathrm{id}$ to determine the various indices. We obtain the following table:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
vector field v at $(x,y)=z$	(x,y)	$(-x,-y)$	$(-y,x)$	$(x,-y)$	$(x^2+y^2,0)$	z^2	z^{-2}
$\mathrm{D}v_{(0,0)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
	non-degenerate zero				degenerate zero		
index	1	1	1	-1	not defined		

□

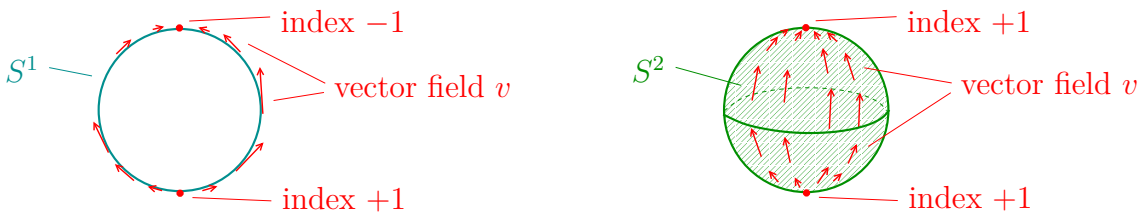
We state our last example of indices of zeros of vector fields as a lemma.

Lemma 53.13. (Vector Field-on-Sphere Lemma) Given any $n \in \mathbb{N}_0$ there exists a smooth vector field v on S^n with the following properties:

(1) The vector field has precisely two zeros, namely at the South Pole $S = (0, \ldots, 0, -1)$ and at the North Pole $N = (0, \ldots, 0, 1)$.

(2) The two zeros are non-degenerate and the indices are given by $\mathrm{index}(v, S) = 1$ and $\mathrm{index}(v, N) = (-1)^n$.

Proof. Statement (1) is taken care of by the following claim:



Claim. We think of S^n as a subspace of $\mathbb{R}^n \times \mathbb{R}$. The map

$$\begin{array}{rcl} v\colon S^n & \to & \mathbb{R}^{n+1} \\ \underbrace{(x,y)}_{x\in\mathbb{R}^n,\ y\in\mathbb{R}} & \mapsto & (x\cdot(-y), \|x\|^2) \end{array}$$

defines a smooth vector field on S^n with precisely two zeros, namely at S and N.

Proof. It follows from $\|x\|^2 = x_1^2 + \cdots + x_n^2$ that v is a smooth map. It is clear that for $(x,y) \in S^n \subseteq \mathbb{R}^n \times \mathbb{R}$ we have $v(x,y) = 0$ if and only if $x = 0$. In particular N and S are the only two zeros. Next note that for each $(x,y) \in S^n$ we have

$$\langle (x,y), v(x,y)\rangle \;=\; \langle (x,y), (x\cdot(-y), \|x\|^2)\rangle \;=\; -\langle x,x\rangle \cdot y + y\cdot\|x\|^2 \;=\; 0.$$

Thus it follows from the Physical Tangent Spaces–Lemma 29.2 that $v(x,y) \in \widetilde{\mathrm{T}}_{(x,y)}S^n$. This shows that v defines indeed a smooth vector field on the smooth manifold S^n. ⊞

It remains to prove Statement (2). For $\epsilon = \pm 1$ we consider the zero $P_\epsilon = (0,\ldots,0,\epsilon)$ of our vector field v. We consider the chart

$$\begin{array}{rcl} \Phi_\epsilon\colon U := S^n_{>0} = \{(x,y) \in S^n \subseteq \mathbb{R}^n \times \mathbb{R} \mid y\cdot\epsilon > 0\} & \to & B^n \\ (x,y) & \mapsto & x. \end{array}$$

The inverse of the chart is given by $\Psi_\epsilon(x) = (x, \epsilon\cdot\sqrt{1-\|x\|^2})$. A straightforward calculation shows that $\Psi_\epsilon^* v$ is given by

$$\begin{array}{rcl} \Psi_\epsilon^* v\colon B^n & \to & \mathbb{R}^n \\ x & \mapsto & -\epsilon\cdot x\cdot\sqrt{1-\|x\|^2}. \end{array}$$

An elementary calculation shows that $\mathrm{D}(\Psi_\epsilon^* v)_0 = (-\epsilon)\cdot\mathrm{id}$. This shows that the zero of v at P_ϵ is non-degenerate and that $\mathrm{index}(v, P_\epsilon) = (-\epsilon)^n$. ■

Lemma 53.14. (Non-Degenerate Zero Lemma) Let M be a smooth manifold and let v be a smooth vector field on M.

(1) Every non-degenerate zero of v is an isolated point in the set of zeros of v.
(2) If M is compact and if all zeros of v are non-degenerate, then v has only finitely many zeros.

Proof.

(1) Using a chart it is clear that we only need to deal with the case that M is an open subset of some $\mathbb{R}^n$ and that $v\colon M \to \mathbb{R}^n$ is a smooth vector field with a zero at $x = 0$. In this setting the statement follows easily from the Inverse Function Theorem 24.21.
(2) This statement follows from (1) together with the Compact-Closed Lemma 3.39, the Compact-Discrete Lemma 3.40 and the fact that smooth manifolds are by definition Hausdorff. ■

The above discussion motivates the following question.

Question 53.15. Does every closed smooth manifold admit a smooth vector field such that all zeros are non-degenerate?

To answer Question 53.15 we will develop some more general theory in the following section. We will then be able to answer the question in the Non-Degenerate Vector Field Existence Proposition 53.22.

53.3. **The total tangent space and vector fields.** In this section we will introduce the total tangent space and we will prove several statements that relate the total tangent space with vector fields. This discussion might initially appear tangential to the topics so far. But in fact we will be so successful in this section, that afterwards the proofs of the main result of this chapter, namely the Smooth Poincaré–Hopf Theorem 53.24, will be a breeze.

We start out with the following new notation.

Notation 53.16.

(1) Let M be a smooth submanifold of $\mathbb{R}^n$. We define the total tangent space

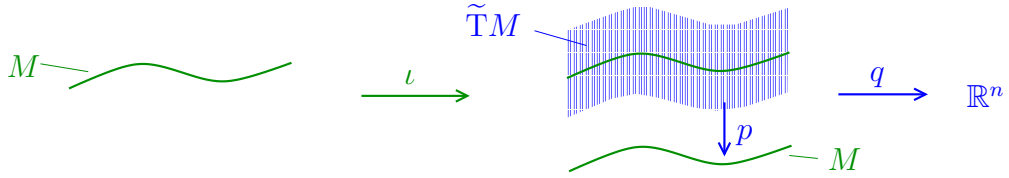
$$\widetilde{\mathrm{T}}M := \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, x \in M \text{ and } v \in \widetilde{\mathrm{T}}_x M\}.$$

(2) Let M be a smooth submanifold of $\mathbb{R}^m$ and let N be a smooth submanifold of $\mathbb{R}^n$. Given any smooth map $f\colon M \to N$ we obtain an induced map

$$\begin{array}{rcl} f_*\colon \widetilde{\mathrm{T}}M & \to & \widetilde{\mathrm{T}}N \\ (x,v) & \mapsto & (f(x), \mathrm{D}f_x(v)). \end{array}$$

(3) We consider the maps

$$\begin{array}{rcl} p\colon \widetilde{\mathrm{T}}M & \to & M \\ (x,y) & \mapsto & x \end{array} \qquad \begin{array}{rcl} q\colon \widetilde{\mathrm{T}}M & \to & \mathbb{R}^n \\ (x,y) & \mapsto & y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \iota\colon M & \to & \widetilde{\mathrm{T}}M \\ x & \mapsto & (x,0). \end{array}$$



Remark. Let M be a smooth submanifold of $\mathbb{R}^n$. The total tangent space $\widetilde{\mathrm{T}}M$ *together* with the map $p\colon \widetilde{\mathrm{T}}M \to M$ is called the *tangent bundle* of M. The tangent bundle will play a significant role later on. □

Lemma 53.17. (Physical Tangent Bundle Lemma) Let $k \in \mathbb{N}$ and let $n \in \mathbb{N}$.

(1) Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$.
   (a) The subset $\widetilde{\mathrm{T}}M$ is a $2k$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.
   (b) For each $(x,v) \in \widetilde{\mathrm{T}}M$ we have $\widetilde{\mathrm{T}}_{(x,v)}\widetilde{\mathrm{T}}M = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M \subseteq \mathbb{R}^n \times \mathbb{R}^n$.
   (c) If M is oriented, then there exists a unique orientation on $\widetilde{\mathrm{T}}M$ such that for each $(x,v) \in \widetilde{\mathrm{T}}M$ the equality $\widetilde{\mathrm{T}}_{(x,v)}\widetilde{\mathrm{T}}M = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is an equality of oriented vector spaces.

(2) Let M be a smooth submanifold of $\mathbb{R}^m$ and let N be a smooth submanifold of $\mathbb{R}^n$. Furthermore let $f\colon M \to N$ be a smooth map.
   (a) The map $f_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}N$ is smooth.
   (b) If f is an orientation-preserving local diffeomorphism, then so is $f_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}N$.
(3) The construction in (1a) and the maps in (2a) define a covariant functor from the category $\mathcal{SmMfd}_{\mathbb{R}^n}$ of smooth submanifolds of some $\mathbb{R}^n$, which we introduced in Definition 29.7, to itself.

Proof.

(1) (a) Let $(y,w) \in \widetilde{\mathrm{T}}M$. Let $\Phi\colon U \to V$ be a submanifold chart for $y \in M$. One can easily verify that

$$\begin{array}{rcl} U \times \mathbb{R}^n & \to & V \times \mathbb{R}^n \\ (x,v) & \mapsto & (\Phi(x), \mathrm{D}\Phi_x(v)) \end{array}$$

is, after a permutation of coordinates, a submanifold chart for $(y,w) \in \widetilde{\mathrm{T}}M$. Thus we have shown that $\widetilde{\mathrm{T}}M$ is a $2k$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.
   (b) Let $(x,v) \in \widetilde{\mathrm{T}}M$. One can easily verify that $\widetilde{\mathrm{T}}_xM \times \widetilde{\mathrm{T}}_xM \subseteq \widetilde{\mathrm{T}}_{(x,v)}\widetilde{\mathrm{T}}M$. But since both are $2k$-dimensional vector spaces we actually have equality.
   (c) We leave it to the reader to verify that this way we get an orientation for $\widetilde{\mathrm{T}}M$.
(2) We leave the verification of this statement to the reader.
(3) This statement follows easily from the definitions. ∎

Notation 53.18. Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$. Given a smooth vector field $v\colon M \to \mathbb{R}^n$ on M we consider the graph

$$\mathrm{graph}(v) \;:=\; \{(x, v(x)) \,|\, x \in M\} \;\subseteq \widetilde{\mathrm{T}}M.$$

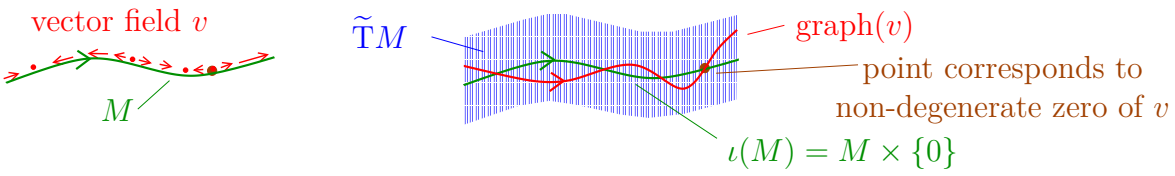
As on page 638 one can easily verify that $\mathrm{graph}(v)$ is a k-dimensional smooth submanifold of $\widetilde{\mathrm{T}}M$ such that the map $p\colon \mathrm{graph}(v) \to M$, $(x,y) \mapsto x$, is a diffeomorphism. If M is oriented, then we equip $\mathrm{graph}(v)$ with the corresponding orientation.[575]

Proposition 53.19. (Vector Field–Graph Proposition) Let M be a smooth submanifold of $\mathbb{R}^n$ with empty boundary and let v be a smooth vector field on M. The following statements hold:
(0) We have $(x,y) \in \iota(M) \cap \mathrm{graph}(v)$ if and only if $x \in M$, $y = 0$ and $v(x) = 0$.
(1) All zeros of the vector field v are non-degenerate if and only if the two submanifolds $\iota(M) = M \times \{0\}$ and $\mathrm{graph}(v)$ intersect transversally in $\widetilde{\mathrm{T}}M$, i.e. if $\iota(M) \pitchfork \mathrm{graph}(v)$.
(2) If x is a non-degenerate zero of v and if M is oriented[576], then

$$\mathrm{index}(v,x) \;=\; \text{sign of the intersection point } (x,0) \text{ of } \iota(M) \text{ and } \mathrm{graph}(v).$$

We first prove the Vector Field–Graph Proposition 53.19 for open subsets of $\mathbb{R}^k$. This is almost precisely the content of the following lemma:

[575] If v is the zero vector field, then $\mathrm{graph}(v) = M \times \{0\} = \iota(M)$, and we see that $\iota(M)$ is also an (oriented) k-dimensional smooth submanifold of $\widetilde{\mathrm{T}}M$.
[576] The orientation of M is strictly speaking not necessary. For the zero x of v it suffices to pick an orientation of M in a neighborhood of x. One can easily verify that the sign of the intersection point does not depend on the choice of the orientation.



Auxiliary Lemma 53.20. Let W be an open subset of $\mathbb{R}^k$ and let $w\colon W \to \mathbb{R}^k$ be a smooth vector field on W. The following statements hold:

(1) Let $y \in W$ be a zero of the vector field w. The zero y is non-degenerate if and only if the submanifolds $\iota(W) = W \times \{0\}$ and graph(w) of in $\widetilde{\mathrm{T}}W = W \times \mathbb{R}^k$ intersect transversally in $(y, 0)$.

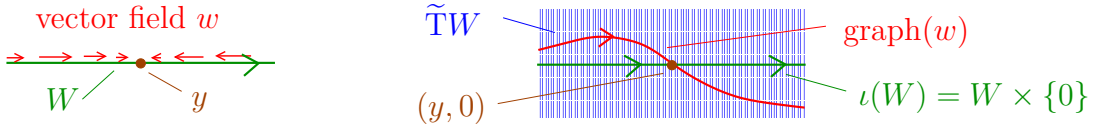
(2) If $y \in W$ is a non-degenerate zero of w, then

$$\mathrm{index}(w, y) \;=\; \text{sign of the intersection point } (y,0) \text{ of } \iota(W) \text{ and graph}(w).$$

Proof of Lemma 53.20. Let W be an open subset of $\mathbb{R}^k$, let $w\colon W \to \mathbb{R}^k$ be a smooth vector field on W and let $y \in W$ be a point with $w(y) = 0$. Evidently $(y, 0)$ is an intersection point of $\iota(W) = W \times \{0\}$ and graph(w). For the definition of a non-degenerate zero and the index of a non-degenerate zero we need to pick a chart. But since W is already an open subset of $\mathbb{R}^k$ we might as well use the identity chart. Next note that

$$\begin{aligned}\widetilde{\mathrm{T}}_{(y,0)}\iota(W) + \widetilde{\mathrm{T}}_{(y,0)}\,\mathrm{graph}(w) \;&=\; \mathrm{im}\begin{pmatrix} \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \\ a \mapsto \begin{pmatrix} a \\ 0 \end{pmatrix} \end{pmatrix} + \mathrm{im}\begin{pmatrix} \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \\ a \mapsto \begin{pmatrix} a \\ \mathrm{D}w_{(y,0)} \cdot a \end{pmatrix} \end{pmatrix} \\ &=\; \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{D}w_{(y,0)} \end{pmatrix} \begin{pmatrix} \mathbb{R}^k \\ \mathbb{R}^k \end{pmatrix}.\end{aligned}$$

We see that $\widetilde{\mathrm{T}}_{(y,0)}\iota(W) + \widetilde{\mathrm{T}}_{(y,0)}\,\mathrm{graph}(w) = \mathbb{R}^{2k} = \widetilde{\mathrm{T}}_{(y,0)}\widetilde{\mathrm{T}}W$ if and only if $\mathrm{D}w_{(y,0)}\colon \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism. This observation, together with Definition 50.3 and Definition 53.10 concludes the proof of Statement (1).



Finally Statement (2) follows from the following little calculation:

$$\begin{aligned}\text{sign of the intersection point } (y,0) \text{ of } \iota(W) \text{ and graph}(w) \;&\underset{\substack{\uparrow \\ \text{by the above discussion} \\ \text{and Definition 51.4}}}{=}\; \mathrm{sign}\left(\det\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & \mathrm{D}w_{(y,0)} \end{pmatrix}\right) \;=\; \mathrm{sign}(\det(\mathrm{D}w_{(y,0)})) \\ &\underset{\substack{\uparrow \\ \text{see Definition 53.10}}}{=}\; \mathrm{index}(w, y).\end{aligned}$$

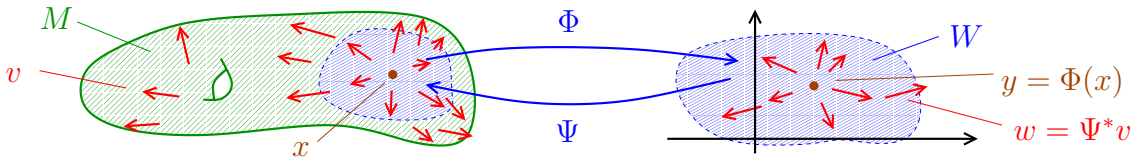
■

Proof of the Vector Field–Graph Proposition 53.19. Let M be an (oriented) k-dimensional smooth submanifold of $\mathbb{R}^n$ with empty boundary and let v be a smooth vector field on M. Statement (0) follows immediately from the definitions.

Let $x \in M$ be a zero of the vector field. We pick an (orientation-preserving) chart $\Phi\colon V \to W$ around x. We set $y := \Phi(x)$ and we set $\Psi := \Phi^{-1}$. We consider the vector field $w := \Psi^* v$. The desired statements follow from Lemma 53.20 and the following claim:

Claim.

(a) The point $y \in W$ is a zero of the vector field w.
(b) $x \in M$ is non-degenerate if and only if $y \in W$ is non-degenerate.
(c) $\operatorname{index}(v, x) = \operatorname{index}(w, y)$.
(d) $(x, 0)$ is a transverse intersection point of $\iota(M)$ and $\operatorname{graph}(v)$ if and only if $(y, 0)$ is a transverse intersection point of $\iota(W)$ and $\operatorname{graph}(w)$.
(e) The sign of the intersection point $(x, 0)$ equals the sign of the intersection point $(y, 0)$.



Proof.

(a) This statement is trivial.
(b),(c) These two statements hold by the definitions of non-degeneracy and of the index in Definition 53.10.
(d),(e) We denote by $i\colon V \to M$ the inclusion. We consider the following diagram:

$$\begin{array}{ccccc}
M & \xleftarrow{\ i\ } & V & \xrightarrow{\ \Phi\ } & W \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\
\widetilde{\mathrm{T}}M & \xleftarrow{\ i_*\ } & \widetilde{\mathrm{T}}V & \xrightarrow{\ \Phi_*\ } & \widetilde{\mathrm{T}}W \\
\uparrow & & \uparrow & & \uparrow \\
\operatorname{graph}(v\colon M \to \mathbb{R}^n) & \xleftarrow{\ i_*\ } & \operatorname{graph}(v\colon V \to \mathbb{R}^n) & \xrightarrow{\ \Phi_*\ } & \operatorname{graph}(w\colon W \to \mathbb{R}^k)
\end{array}$$

We make the following observations:
(a) It follows easily from the definitions that $i_*\colon \widetilde{\mathrm{T}}V \to \widetilde{\mathrm{T}}M$ and $\Phi_*\colon \widetilde{\mathrm{T}}V \to \widetilde{\mathrm{T}}W$ restrict to local diffeomorphisms $\operatorname{graph}(v\colon V \to \mathbb{R}^n) \to \operatorname{graph}(v\colon M \to \mathbb{R}^n)$ and $\operatorname{graph}(v\colon V \to \mathbb{R}^n) \to \operatorname{graph}(w\colon W \to \mathbb{R}^k)$.
(b) It follows immediately from the definitions that the diagram commutes and that the horizontal maps are orientation-preserving local diffeomorphisms.

The desired Statements (d) and (e) follow immediately from this discussion. ∎

Lemma 53.21. (Vector Field–Graph Transversality Lemma) Let M be a closed smooth submanifold of $\mathbb{R}^n$. There exists a smooth vector field v on M such that $\iota(M) \pitchfork \operatorname{graph}(v)$.

Proof. Let M be a closed smooth submanifold of $\mathbb{R}^n$. As before we consider the natural projection $p\colon \widetilde{\mathrm{T}}M \to M$.

Claim. There exists a smooth embedding $g\colon M \to \widetilde{\mathrm{T}}M$ such that $\iota(M) \pitchfork g$ and such that $p \circ g = \mathrm{id}_M$.

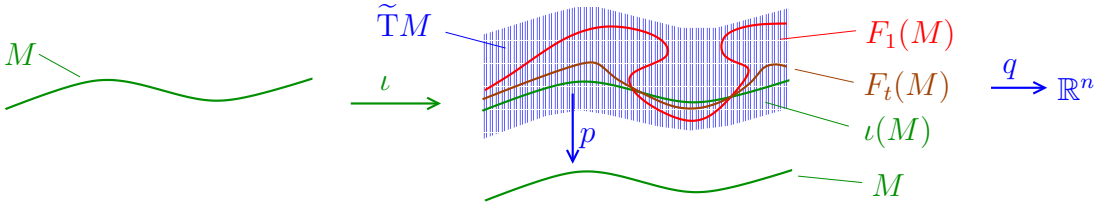
Proof. First note that by the Transversality Theorem 50.9 there exists a smooth homotopy $F\colon M \times [0,1] \to \widetilde{\mathrm{T}}M$ with $F_0 = \iota$ such that

$$T \;:=\; \{t \in [0,1] \,|\, \text{the map } F_t\colon M \to \widetilde{\mathrm{T}}M \text{ is transverse to } \iota(M) \text{ in } \widetilde{\mathrm{T}}M\}$$

has full measure in $[0,1]$. By the Measure-Zero Properties Proposition 37.3 this means that T is dense in $[0,1]$. Next we consider the smooth homotopy

$$\begin{array}{rcl} H\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & p(F(x,t)). \end{array}$$

It follows from $H_0 = \mathrm{id}$, the fact that M is closed, the Deform Smooth Embedding Proposition 41.9 (2) and the fact that T is dense in $[0,1]$ that there exists a $t \in T$ such that H_t is a diffeomorphism. By the Physical Tangent Bundle Lemma 53.17 we know that the diffeomorphism $H_t^{-1}\colon M \to M$ induces a diffeomorphism $(H_t^{-1})_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}M$. Finally we set $g := (H_t^{-1})_* \circ F_t\colon M \to \widetilde{\mathrm{T}}M$. We claim that g has the desired properties.



This claim follows from the following two observations:

- It follows immediately from the definitions that $(H_t^{-1})_*(\iota(M)) = \iota(M)$. Thus the fact that $\iota(M) \pitchfork F_t$ implies that $\iota(M) \pitchfork (H_t^{-1})_* \circ F_t$.
- We have the following equalities of maps $M \to M$:

$$p \circ g \;=\; p \circ ((H_t^{-1})_* \circ F_t) \;=\; (p \circ (H_t^{-1})_*) \circ F_t \;\overset{\text{by definition of the the map } (H_t^{-1})_*\colon \widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}M}{=}\; (H_t^{-1} \circ p) \circ F_t$$
$$= H_t^{-1} \circ \underbrace{(p \circ F_t)}_{=H_t} \;=\; \mathrm{id}\,.$$

⊞

Finally we study the vector field

$$\begin{array}{rcl} v\colon M & \to & \mathbb{R}^n \\ x & \mapsto & \underbrace{q(g(x))}_{\in \widetilde{\mathrm{T}}_x M}. \end{array}$$

Since $p \circ g = \mathrm{id}_M$ we see that $\mathrm{graph}(v) = g(M)$. Since $\iota(M) \pitchfork g$ and since $g\colon M \to \widetilde{\mathrm{T}}M$ is a smooth embedding it follows from the remark in Definition 50.3 that $\iota(M) \pitchfork \mathrm{graph}(v)$. ■

Now we can prove the following proposition, which gives an affirmative answer to Question 53.15.

Proposition 53.22. (Non-Degenerate Vector Field Existence Proposition) Every closed smooth manifold admits a vector field v such that each zero of v is non-degenerate.

Remark. In the Non-Degenerate Vector Field Existence Proposition 223.5, which we will prove twice on page 4310 and **??**, we will give an alternative proof of the Non-Degenerate Vector Field Existence Proposition 53.22. □

Proof. Let M be a closed smooth manifold. By the Euclidean Embedding Theorem 33.2 it suffices to deal with the case that M is a k-dimensional smooth submanifold of some $\mathbb{R}^n$

and vector fields that are defined in terms of the physical tangent spaces. It follows from the Vector Field–Graph Transversality Lemma 53.21 that there exists a smooth vector field v on M such that $\iota(M) \pitchfork \operatorname{graph}(v)$ in $\widetilde{\mathrm{T}}M$. By the Vector Field–Graph Proposition 53.19 (1) this implies that v is a vector field such that each zero of v is non-degenerate. ■

53.4. **The total tangent space and the diagonal.** The only goal of this section is to prove the following proposition:

Proposition 53.23. (Total Tangent Space-Diagonal Embedding Proposition) Let M be a closed smooth submanifold of $\mathbb{R}^n$. There exists a smooth embedding

$$f\colon \widetilde{\mathrm{T}}M \;\to\; M \times M$$

such that for each $x \in M$ we have $f(x,0) = (x,x)$. If M is oriented, then this smooth embedding can be chosen to be orientation-preserving.

The proof of Proposition 53.23 rests on the following tricky theorem from an earlier chapter.

Theorem 27.16. (Neighborhood Smooth Embedding Theorem) Let $b\colon X \to Y$ be a continuous map between smooth manifolds. Furthermore let $C \subseteq X$ be a compact subset such that the map $b\colon C \to Y$ is an embedding. If b is a local diffeomorphism at each $P \in C$, then there exists an open neighborhood U of C such that the restriction of b to U is a smooth embedding.



Proof of Proposition 53.23. Let M be a closed k-dimensional smooth submanifold of $\mathbb{R}^n$. Given $\sigma > 0$ we write

$$\widetilde{\mathrm{T}}M(\sigma) \;:=\; \{(x,v) \in \widetilde{\mathrm{T}}M \,|\, \|v\| < \sigma\}.$$

We prove a few basic facts about $\widetilde{\mathrm{T}}M(\sigma)$.

Claim 1.

(1) For every $\sigma > 0$ the above set $\widetilde{\mathrm{T}}M(\sigma)$ is an open subset of $\widetilde{\mathrm{T}}M$. Thus we can view $\widetilde{\mathrm{T}}M(\sigma)$ as an (oriented) smooth manifold.
(2) Given any open neighborhood W of $M \times \{0\} \subseteq \widetilde{\mathrm{T}}M$ there exists a $\sigma > 0$ such that $\widetilde{\mathrm{T}}M(\sigma) \subseteq W$.
(3) For every $\sigma > 0$ there exists a diffeomorphism $\widetilde{\mathrm{T}}M(\sigma) \to \widetilde{\mathrm{T}}M$ which is the identity on the common subset $M \times \{0\}$. If M is oriented, then the diffeomorphism can be chosen to be orientation-preserving.

Proof.

(1) This statement follows easily from the Definition 7.3 of the product topology on $\mathbb{R}^n \times \mathbb{R}^n$ and the definition of the subspace topology on $\widetilde{\mathrm{T}}M$.
(2) Since W is an open neighborhood of $M \times \{0\} \subseteq \widetilde{\mathrm{T}}M(\sigma)$ we know, by the Definition 7.3 of the product topology on $\mathbb{R}^n \times \mathbb{R}^n$ and by definition of the subspace topology on $\widetilde{\mathrm{T}}M$, that for each $P \in M$ there exists an open neighborhood U_P of $P \in M$ and a $\sigma_P > 0$ such that $\{(x,v) \in \widetilde{\mathrm{T}}M \,|\, x \in U_P \text{ and } \|v\| < \sigma_P\} \subseteq W$. Since M is compact there exist $P_1, \ldots, P_m \in M$ with $M = U_{P_1} \cup \cdots \cup U_{P_m}$. It is clear that $\sigma := \min\{\sigma_{P_1}, \ldots, \sigma_{P_m}\}$ has the desired properties.

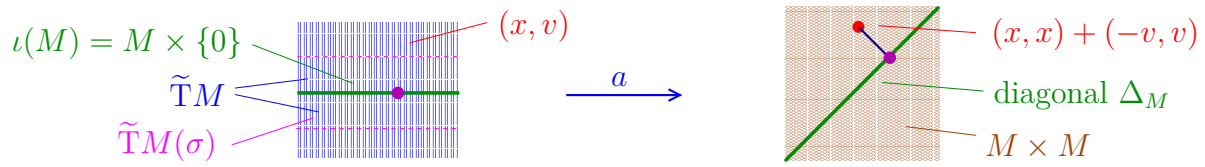
(3) Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists an orientation-preserving diffeomorphism $\chi\colon [0,\sigma) \to \mathbb{R}_{\geq 0}$ that is the identity on a neighborhood of 0. We consider the map

$$\begin{array}{rcl} g\colon \widetilde{\mathrm{T}}M(\sigma) & \to & \widetilde{\mathrm{T}}M \\ (x,v) & \mapsto & \begin{cases} (x,0), & \text{if } v = 0, \\ (x, \frac{v}{\|v\|} \cdot \chi(\|v\|)), & \text{if } v \neq 0. \end{cases} \end{array}$$

Note that g is the identity on a neighborhood of $M \times \{0\}$. Using the Smooth Pasting Proposition 24.7 one can now show that g is smooth. Using the inverse of the diffeomorphism χ one can easily write down a smooth inverse to the map g, thus we see that g is a diffeomorphism. If M is oriented, then g is evidently orientation-preserving. ⊞

We consider the map[577]

$$\begin{array}{rcl} a\colon \widetilde{\mathrm{T}}M & \to & \mathbb{R}^n \times \mathbb{R}^n \\ (x,v) & \mapsto & (x,x) + (-v,v). \end{array}$$



Claim 2. For each $x \in M$ the map

$$\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \mathbb{R}^n \times \mathbb{R}^n$$

restricts to an isomorphism

$$\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M.$$

If M is oriented, then this isomorphism is orientation-preserving.

Proof. First of all note that it follows from the Physical Tangent Bundle Lemma 53.17 that we have the equality $\widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$. Since a is a linear map it follows almost immediately that the map $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \mathbb{R}^n \times \mathbb{R}^n$ is given by the map

$$\begin{array}{rcl} \mathrm{D}a_{(x,0)}\colon \overbrace{\widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M}^{\subseteq \mathbb{R}^n \times \mathbb{R}^n} & \to & \mathbb{R}^n \times \mathbb{R}^n \\ \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & \begin{pmatrix} u \\ u \end{pmatrix} + \begin{pmatrix} -v \\ v \end{pmatrix}. \end{array}$$

Thus we see that the image of $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \mathbb{R}^n \times \mathbb{R}^n$ lies in $\widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$. Next we pick a basis for $\widetilde{\mathrm{T}}_x M$ and we equip $\widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ with the obvious corresponding basis. It follows immediately from the above discussion, that with this choice of a basis, the map $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is represented by the matrix $\begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ \mathrm{id} & \mathrm{id} \end{pmatrix}$. Therefore we see that the map $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is a monomorphism. Since both vector spaces are $2k$-dimensional we see that $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is in fact an isomorphism. Furthermore, if M is oriented, then we see that, with the appropriate orientations, the map $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is also orientation-preserving. ⊞

[577]There are many possible choices for a, for example $a(x,v) = (x,x) + (0,v)$ would also work. We picked the given map for subjective aesthetic reasons.

By the Neighborhood Projection Theorem 35.2 there exists an open neighborhood Z of $M \times M \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and a smooth retraction $\pi\colon Z \to M \times M$. Note that $a^{-1}(Z)$ is an open neighborhood of $M \times \{0\} \subseteq \widetilde{\mathrm{T}}M$. It follows from Claim 1 (2) that there exists a $\mu > 0$ such that $\widetilde{\mathrm{T}}M(\mu) \subseteq a^{-1}(Z)$. By Claim 1 (3) it now suffices to prove the following claim:

Claim 3. There exists a $\nu \in (0, \mu)$ such that the following map is a smooth embedding:

$$\begin{array}{rcl} b\colon \widetilde{\mathrm{T}}M(\nu) & \to & M \times M \\ (x, v) & \mapsto & \pi(a(x, v)). \end{array}$$

Proof. By the choice of μ it makes sense to consider the map

$$\begin{array}{rcl} b\colon \widetilde{\mathrm{T}}M(\mu) & \to & M \times M \\ (x, v) & \mapsto & \pi(a(x, v)). \end{array}$$

We make the following two observations:

(1) Since $\pi\colon Z \to M$ is a retraction we see that the restriction of b to the closed subset $M \times \{0\}$ of $\widetilde{\mathrm{T}}M(\mu)$ is given by the map $x \mapsto (x, x)$. It follows from the Compact-Hausdorff Proposition 4.23 that the map $b\colon M \times \{0\} \to \widetilde{\mathrm{T}}M(\mu)$ is an embedding.

(2) Let $(x, 0) \in M \times \{0\}$ be arbitrary. By Claim 2 we know that we get an isomorphism $\mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$. Since $\pi\colon Z \to M \times M$ is a retraction we know that the restriction of $\mathrm{D}\pi_{(x,x)}$ to $\widetilde{\mathrm{T}}_{(x,x)}(M \times M) = \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is the identity. In summary we see that the map $\mathrm{D}b_{(x,0)} = \mathrm{D}\pi_{(x,x)} \circ \mathrm{D}a_{(x,0)}\colon \widetilde{\mathrm{T}}_{(x,0)}\widetilde{\mathrm{T}}M \to \widetilde{\mathrm{T}}_x M \times \widetilde{\mathrm{T}}_x M$ is an isomorphism. It follows from the Inverse Mapping Theorem 30.1 that b is a local diffeomorphism at $(x, 0) \in M \times \{0\}$.

It follows from the above and the Neighborhood Smooth Embedding Theorem 27.16 that there exists an open neighborhood U of $M \times \{0\} \subseteq \widetilde{\mathrm{T}}M(\mu)$ such that the restriction of b to U is a smooth embedding. We obtain the desired $\nu \in (0, \mu)$ from Claim 1 (2). ∎

53.5. **The Smooth Poincaré–Hopf Theorem.** We move on to the most interesting result of this chapter.

Theorem 53.24. (Smooth Poincaré–Hopf Theorem) Let M be a closed orientable smooth manifold and let v be a smooth vector field on M such that all zeros of v are non-degenerate. Then we have the equality[578]

$$\sum_{x \text{ a zero of } v} \mathrm{index}(v, x) \;=\; \chi^{\mathrm{sm}}(M).$$

Remark.

(1) In case the reader is wondering, as we mentioned on page 1198, the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ equals the "usual" Euler characteristic $\chi(M)$, that we will introduce in Definition 156.1 and in Definition 156.6. In fact in the Non-Degenerate Vector Field Existence Proposition 223.5, which we will prove twice on page 4310 and on page **??**, we will construct a smooth vector field on M such that all zeros of v are non-degenerate and such that the sum of the indices equals $\chi(M)$. As we point out in the Euler Characteristics–Equality Corollary 223.6, this construction together with the Smooth Poincaré–Hopf Theorem 53.24 implies that $\chi(M) = \chi^{\mathrm{sm}}(M)$.

(2) In the Poincaré–Hopf Theorem 223.8 we will reprove the Smooth Poincaré–Hopf Theorem 53.24 and generalize it in two directions:
   (a) We will have a more general notion of the index of a zero of a vector field. Thus we no longer demand that the zeros are non-degenerate, we only demand, that there are only finitely many.
   (b) We will allow compact smooth manifolds with non-empty boundary. If M has non-empty boundary, then we will demand that along the boundary ∂M the vector field v is pointing outward. □

Our proof of the Smooth Poincaré–Hopf Theorem 53.24 rests on the following proposition, which is interesting in its own right.

Proposition 53.25. (Tangent Bundle–Self-Intersection Proposition) Let M be a closed orientable k-dimensional smooth submanifold of $\mathbb{R}^n$. As in Notation 53.16 we consider the oriented $2k$-dimensional smooth manifold $\widetilde{\mathrm{T}}M$ and the smooth embedding $\iota\colon M \to \widetilde{\mathrm{T}}M$ that is given by $x \mapsto (x,0)$. Let v be a smooth vector field on M such that all zeros of v are non-degenerate. Then[579]

$$\sum_{x \text{ a zero of } v} \operatorname{index}(v,x) \;=\; \underbrace{\iota(M)\cdot\iota(M)}_{\substack{\text{self-intersection}\\ \text{number in } \widetilde{\mathrm{T}}M}}.$$

Proof. We denote by $x_1,\dots,x_k$ the zeros of v. We consider the smooth homotopy

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & \widetilde{\mathrm{T}}M \\ (x,t) & \mapsto & (x,t\cdot v(x)). \end{array}$$

Note that $F_0 = \iota\colon M \to \widetilde{\mathrm{T}}M$ and that $F_1(M) = \operatorname{graph}(v)$. Note that by our hypothesis on v and by the Vector Field–Graph Proposition 53.19 (1) we know that $\iota(M) \pitchfork F_1$. The proposition now follows from the following calculation:

$$\begin{array}{ccccc} \displaystyle\sum_{x \text{ a zero of } v} \operatorname{index}(v,x) & = & \displaystyle\sum_{i=1}^{k} \operatorname{index}(v,x_i) & \overset{\text{by the Vector Field–Graph Proposition 53.19 (2)}}{=} & \displaystyle\sum_{i=1}^{k} \begin{array}{l}\text{sign of the intersection point}\\ (x_i,0) \text{ of } \iota(M) \text{ and } F_1(M)\end{array} \\ & = & \iota(M) \underset{\widetilde{\mathrm{T}}M}{\cdot} F_1(M) & = & \iota(M) \underset{\widetilde{\mathrm{T}}M}{\cdot} \iota(M). \\ & \overset{\text{by Definition 51.3}}{=} & & \overset{\text{by Definition 53.1 of the self-intersection number, here we use that } F_1 \text{ is smoothly homotopic to } \iota}{=} & \end{array}$$

■

Proof of the Smooth Poincaré–Hopf Theorem 53.24. Let M be a closed orientable smooth manifold and let v be a smooth vector field on M such that all zeros of v are non-degenerate. By the Euclidean Embedding Theorem 33.2 we can assume that M is a smooth submanifold of some $\mathbb{R}^n$.

[578] Note that by the Non-Degenerate Zero Lemma 53.14 we know that v has only finitely many zeros, so the sum on the left-hand side is actually defined.

[579] To make sense of the individual terms on the right-hand side we have to pick an orientation of M. This orientation then endows $\iota(M)$ and $\widetilde{\mathrm{T}}M$ with an orientation and the self-intersection number $\iota(M)\cdot\iota(M)$ is defined. One can easily verify that $\iota(M)\cdot\iota(M)$ does not depend on the choice of the orientation.

By the Total Tangent Space-Diagonal Embedding Proposition 53.23 there exists an orientation-preserving smooth embedding

$$f\colon \widetilde{\mathrm{T}}M \;\to\; M \times M$$

such that for each $x \in M$ we have $f(x,0) = (x,x)$. Now we see that

$$\sum_{x \text{ a zero of } v} \mathrm{index}(v,x) \;\underset{\uparrow}{=}\; \iota(M) \underset{\widetilde{\mathrm{T}}M}{\cdot} \iota(M) \;\underset{\uparrow}{=}\; \Delta_M \underset{M\times M}{\cdot} \Delta_M \;\underset{\uparrow}{=}\; \chi^{\mathrm{sm}}(M).$$

by the Tangent Bundle Self-Intersection Proposition 53.25 — follows from $f(\iota(M)) = \Delta_M$ and the Intersection Number Basics Lemma 51.7 (2) — by Definition 53.4

■

Remark. Let M be a closed orientable smooth manifold. In the proof of the Smooth Poincaré–Hopf Theorem 53.24 we showed that $\iota(M) \cdot_{\widetilde{\mathrm{T}}M} \iota(M) = \Delta_M \cdot_{M\times M} \Delta_M = \chi^{\mathrm{sm}}(M)$. In the literature this statement sometimes gets shortened to the catchy and mysterious looking "$M \cdot M = \chi^{\mathrm{sm}}(M)$". □

53.6. **Applications of the Smooth Poincaré–Hopf Theorem 53.24.** The beauty of the Smooth Poincaré–Hopf Theorem 53.24 is that it can be used for two different purposes:

(1) By constructing suitable vector fields we can use the Smooth Poincaré–Hopf Theorem 53.24 to calculate smooth Euler characteristics.
(2) Once we have calculated the smooth Euler characteristic of a smooth manifold M the Smooth Poincaré–Hopf Theorem 53.24 gives us information on the existence of certain vector fields on M.

In the following we will see both principles in action. First, recall that in Question 53.7 we were challenged to calculate the smooth Euler characteristics of spheres and surfaces of genus g. Fortunately we can now rise to the challenge.

Proposition 53.26. (Sphere-χ-Proposition) Given $n \in \mathbb{N}_0$ we have

$$\chi^{\mathrm{sm}}(S^n) \;=\; 1 + (-1)^n.$$

Proof. Let $n \in \mathbb{N}_0$. By the Vector Field-on-Sphere Lemma 53.13 we know that there exists a smooth vector field v on S^n with the following properties:

(1) The vector field has precisely two zeros, namely at the South Pole $S = (0,\dots,0,-1)$ and at the North Pole $N = (0,\dots,0,1)$.
(2) The two zeros are non-degenerate and the indices are given by $\mathrm{index}(v,S) = 1$ and $\mathrm{index}(v,N) = (-1)^n$.

It now follows that

$$\chi^{\mathrm{sm}}(S^n) \;\underset{\uparrow}{=}\; \sum_{x \text{ a zero of } v} \mathrm{index}(v,x) \;\underset{\uparrow}{=}\; \mathrm{index}(v,S) + \mathrm{index}(v,N) \;\underset{\uparrow}{=}\; 1 + (-1)^n.$$

by the Smooth Poincaré–Hopf Theorem 53.24 — by the above — by the above

■

Proposition 53.27. (Surface-χ-Proposition) We have

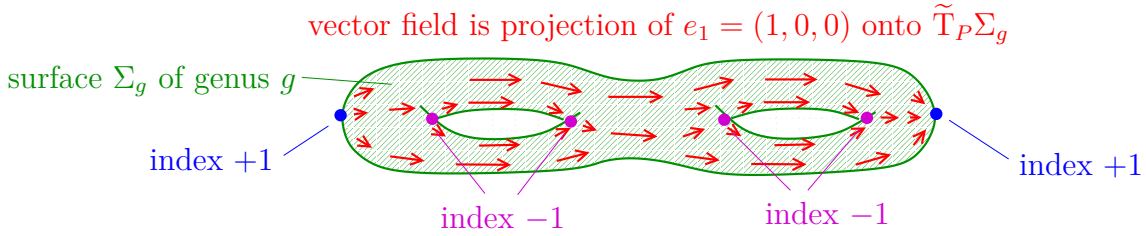
$$\chi^{\mathrm{sm}}(\text{surface of genus } g) \;=\; 2 - 2g.$$

Proof. Let Σ_g be the surface of genus g. We will prove the proposition using the Smooth Poincaré–Hopf Theorem 53.24 and an explicit smooth vector field. In fact we will provide

two constructions of smooth vector fields on Σ_g. The first one is slightly informal, the second one is more formal.

(1) First we view Σ_g as a smooth submanifold of $\mathbb{R}^3$ in the "usual way", as shown in the figure below. We denote by v the smooth vector field on Σ_g that is given at each $x \in \Sigma_g$ by projecting the vector $\mathfrak{e}_1 = (1,0,0)$ onto the tangent plane $\widetilde{\mathrm{T}}_x\Sigma_g$. This vector field has non-degenerate zeros of index $+1$ to the "very left" and it has a non-degenerate zero of index $+1$ to the "very right". Furthermore, for each "hole" there exist two non-degenerate zeros of index -1. The are no other zeros. It follows from this discussion that

$$\chi^{\mathrm{sm}}(\Sigma_g) \underset{\substack{\uparrow\\ \text{by the Smooth Poincaré–Hopf Theorem 53.24}}}{=} \sum_{x \text{ a zero of } v} \mathrm{index}(v,x) \underset{\substack{\uparrow\\ \text{by the above}}}{=} +1 + 2g \cdot (-1) + 1 = 2 - 2g.$$



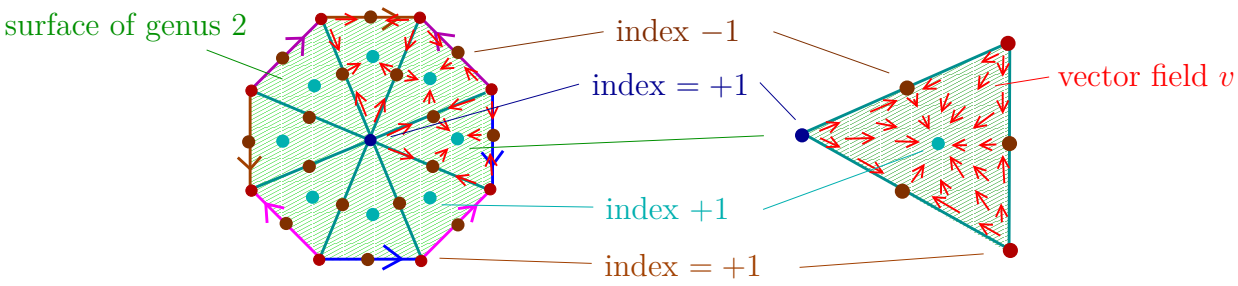
(2) We consider the surface $\Sigma_g = E_{4g}/\sim$ from Definition 9.18. We construct a smooth vector field v on $E_{4g}/\sim$ as follows:
   (a) We break $E_{4g}/\sim$ into $4g$ triangles, as shown in the figure below.
   (b) The vector field v is zero on each vertex, on each midpoint of an edge and on the midpoint of each triangle.
   (c) On the edges and on the triangles the vector field v points towards the midpoint.

   We make the following observations regarding the number of zeros and the indices of the zeros:[580]
   (a) at the two vertices the vectors point outward in all directions, thus the index equals $+1$,
   (b) at the midpoints of the $2g+4g$ edges the vectors point inward in one direction and outward in another direction, thus the index equals -1,
   (c) at the midpoints of the $4g$ triangles the vectors point inward from all directions, thus the index equals $+1$.

   The vector field is non-zero at all other points.
   In summary we see that

$$\chi^{\mathrm{sm}}(\Sigma_g) \underset{\substack{\downarrow\\ \text{by the Smooth Poincaré–Hopf Theorem 53.24}}}{=} \sum_{x \text{ a zero of } v} \mathrm{index}(v,x) = (+1) \cdot \underbrace{\#\text{vertices}}_{=2} + (-1) \cdot \underbrace{\#\text{edges}}_{=2g+4g} + (+1) \cdot \underbrace{\#\text{triangles}}_{=4g}$$
$$= 2 - 2g - 4g + 4g = 2 - 2g. \qquad \blacksquare$$

[580] On page 4310 we will generalize this construction and calculation to smooth manifolds that are equipped with a "smooth triangulation".



Corollary 53.28. (Surfaces-Non Diffeomorphic Corollary) If $g \neq h$ then the surface of genus g and the surface of genus h are not diffeomorphic.

Remark.

(1) In the Diffeomorphism-to-Standard Surfaces Theorem 255.14 we will show that any closed orientable connected non-empty 2-dimensional smooth manifold is diffeomorphic to some surface of genus g. This theorem together with the above Surfaces-Non Diffeomorphic Corollary 53.28 gives us a diffeomorphism classification of closed orientable connected non-empty 2-dimensional smooth manifolds.

(2) In the Surfaces-Non Homeomorphic Theorem 101.18 we will see that for $g \neq h$ the surface of genus g and the surface of genus h are in fact not even homeomorphic.

(3) We refer to Chapters 101 and 255 for more information on the classification of 2-dimensional smooth and topological manifolds. □

Proof. This corollary follows immediately from the fact, see the Smooth Euler Characteristic Lemma 53.5 (2), that the smooth Euler characteristic is a diffeomorphism invariant together with the calculation of the smooth Euler characteristic of surfaces in the Surface-χ-Proposition 53.27. ■

The following theorem is an almost immediate consequence of the Smooth Poincaré–Hopf Theorem 53.24

Theorem 53.29. (Smooth-χ-Vector Field Theorem) Let M be a closed orientable smooth manifold. If M admits a nowhere-vanishing vector field, then $\chi^{\mathrm{sm}}(M) = 0$.

Remark. In the Euler Characteristic–Vector Field Theorem 182.22 we will reprove the Smooth-χ-Vector Field Theorem 53.29 and we will extend the statement to non-orientable smooth manifolds. □

Proof. Let M be a closed smooth manifold that admits a nowhere-vanishing vector field. By the Replace-by-Smooth Vector Field Proposition 36.7 we know that M also admits a *smooth* nowhere-vanishing vector field. It follows immediately from the Smooth Poincaré–Hopf Theorem 53.24 that $\chi^{\mathrm{sm}}(M) = 0$. ■

The Smooth-χ-Vector Field Theorem 53.29 together with the above calculations of smooth Euler characteristics gives us the following pleasant corollary.

Corollary 53.30. (Sphere-Surface–Nowhere Vanishing Vector Field Corollary)

(1) If $n \in \mathbb{N}_0$ is even, then there is no nowhere vanishing vector field on S^n.

(2) If $g \neq 1$, then there is no nowhere vanishing vector field on the surface Σ_g of genus g.

Remark.

(1) The statement of the Sphere-Surface–Nowhere Vanishing Vector Field Corollary 53.30 (1) is of course precisely the content of the Hairy Ball Theorem 46.18.
(2) The Sphere-Surface–Nowhere Vanishing Vector Field Corollary 53.30 (2) answers Question 29.31 in the orientable case. □

Proof.

(1) Let $n \in \mathbb{N}_0$ be an even natural number. By the Sphere-χ-Proposition 53.26 we know that $\chi^{\mathrm{sm}}(S^n) = 1 + (-1)^n = 2 \neq 0$. It follows from the Smooth-χ-Vector Field Theorem 53.29 that S^n cannot admit a nowhere-vanishing vector field.
(2) Let $g \neq 1$. By the Surface-χ-Proposition 53.27 we know that $\chi^{\mathrm{sm}}(\Sigma_g) = 2 - 2g \neq 0$. Thus we see, as in (1), that Σ_g does not admit a nowhere-vanishing vector field. ■

We conclude this section with a sneak preview of a theorem which we will prove later on. This proposition is basically the converse to the Smooth-χ-Vector Field Theorem 53.29.

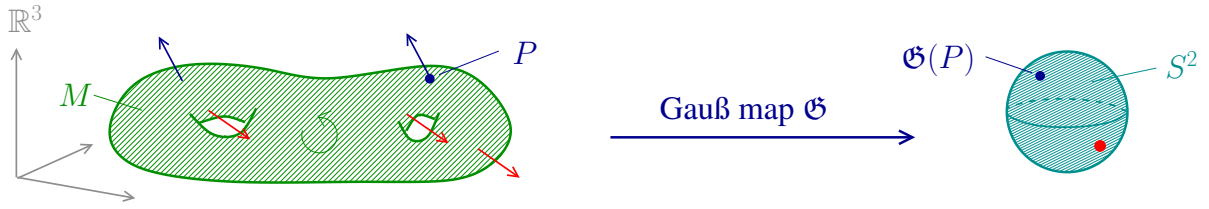
Theorem 223.7. (Nowhere Vanishing Vector Field Existence Theorem) Let M be a closed connected smooth manifold. If $\chi^{\mathrm{sm}}(M) = 0$, then M admits a nowhere-vanishing smooth vector field.

53.7. **The Gauß map I.** In this section we will introduce the Gauß map and we will determine its degree. We will not make use of the results of this section.
The following definition introduces the protagonist of this section.

Definition 53.31. Let $n \in \mathbb{N}_0$ and let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented smooth submanifold of codimension one. We refer to the map

$$\begin{aligned} \mathfrak{G}\colon M &\to S^n \\ P &\mapsto \text{the unique } \mathfrak{G}(P) \in \mathbb{R}^{n+1} \text{ that has the following properties:} \\ &\quad (1)\ \mathfrak{G}(P) \text{ is normal to the physical tangent space } \widetilde{\mathrm{T}}_P M, \\ &\quad (2)\ \mathfrak{G}(P) \text{ has length one, i.e. } \mathfrak{G}(P) \in S^n, \\ &\quad (3)\ \mathfrak{G}(P) \text{ followed by a positive basis for } \widetilde{\mathrm{T}}_P M \text{ is a positive basis for } \mathbb{R}^{n+1} \end{aligned}$$

as the Gauß map of M.[581] [582]



Example. Let $n \in \mathbb{N}$. We consider the submanifold $S^n \subseteq \mathbb{R}^{n+1}$ with the standard orientation introduced on page 817. It follows almost immediately from the definitions and conventions that the corresponding Gauß map $\mathfrak{G}\colon S^n \to S^n$ is the identity. □

[581] A variation on the Gauß map appeared also in the definition of the rotation number of an immersion $S^1 \to \mathbb{R}^2$ in Definition 49.8.
[582] As discussed in the Orientations-via-Normal Vector Fields Proposition 31.27, this map is smooth.

The following theorem was first proved by Heinz Hopf [**Hopf1925**] in 1925.

Theorem 53.32. (Gauß Map-Degree Theorem) Let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. If n is even, then

$$\deg(\text{Gauß map } \mathfrak{G}\colon M \to \underbrace{S^n}_{\substack{\uparrow \\ \text{equipped with the standard} \\ \text{orientation from page 817}}}) \;=\; \tfrac{1}{2} \cdot \chi^{\mathrm{sm}}(M).$$

Remark.

(1) Alternative proofs of the Gauß Map-Degree Theorem 53.32 are given in [**Bre1993**, Theorem VI.12.11], [**Cib2016**], [**Sakk1990**], [**Tu2011**, p. 174] and [**GPol1974**, p. 198]. Furthermore we will give a somewhat different proof in Section 223.3.

(2) Note that in general the conclusion of the Gauß Map-Degree Theorem 53.32 does not hold if n is odd. For example if we consider $M = S^1 \subseteq \mathbb{R}^2$, then we know by the above that the Gauß map $\mathfrak{G}$ is the identity, hence its degree equals one, whereas we saw in the Sphere-χ-Proposition 53.26 that $\chi^{\mathrm{sm}}(S^1) = 0$. □

Before we provide a proof of the Gauß Map-Degree Theorem 53.32, let us formulate and prove a neat corollary.

Corollary 53.33. If M is a closed orientable even-dimensional smooth manifold that admits a smooth embedding into $\mathbb{R}^{\dim(M)+1}$, then $\chi^{\mathrm{sm}}(M)$ is even.

Example. Let $n \in \mathbb{N}$. In Exercise 53.5 we will see that $\chi^{\mathrm{sm}}(\mathbb{C}\mathrm{P}^n) = n + 1$. Thus, if n is in fact even, then it follows from Corollary 53.33 that the $2n$-dimensional smooth manifold $\mathbb{C}\mathrm{P}^n$ does not admit a smooth embedding into $\mathbb{R}^{2n+1}$. No wonder you never saw a copy of $\mathbb{C}\mathrm{P}^2$ in $\mathbb{R}^5$. □

Proof of Corollary 53.33. We write $n = \dim(M)$. By the Smooth Embedding Theorem 30.15 we know that M is diffeomorphic to a smooth submanifold $N \subseteq \mathbb{R}^{n+1}$. It follows that

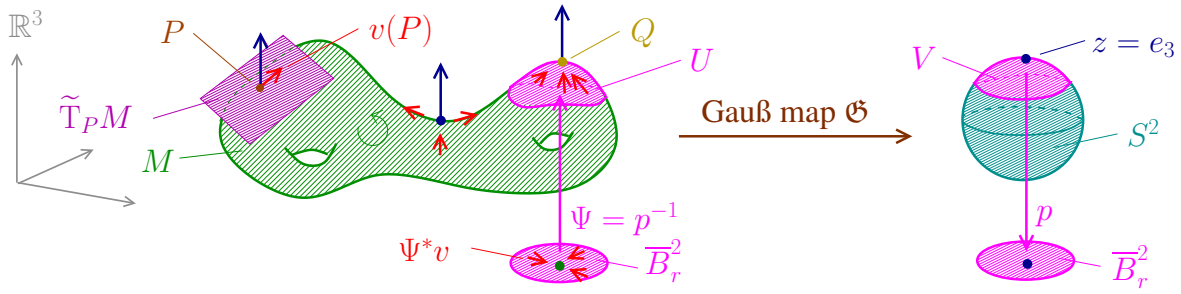
$$\chi^{\mathrm{sm}}(M) \;\underbrace{=}_{\substack{\uparrow \\ \text{by the Smooth-}\chi \\ \text{Lemma 53.5}}}\; \chi^{\mathrm{sm}}(N) \;\underbrace{=}_{\substack{\uparrow \\ \text{by the Gauß Map-Degree Theorem 53.32,} \\ \text{here we use that } n \text{ is even}}}\; 2 \cdot \deg(\text{Gauß map } \mathfrak{G}\colon N \to S^n) \;\in\; 2 \cdot \mathbb{Z}.$$

■

Proof of the Gauß Map-Degree Theorem 53.32. Let $n \in \mathbb{N}_0$ and let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. Recall that above we observed that the Gauß map $\mathfrak{G}\colon M \to S^n$ is smooth. Thus we can appeal to Sard's Theorem 38.1 which, together with the Measure-Zero Properties Proposition 37.3, implies that there exists some $z \in S^n$ such that z and $-z$ are both regular values of the Gauß map $\mathfrak{G}$. By the Orthogonal Action Lemma 2.37 we know that $\mathrm{SO}(n+1)$ acts transitively on S^n. It follows easily from this observation, together with the Diffeotopies-via-Matrices Lemma 41.12 and the Degree Homotopy Invariance Theorem 46.6 that, after a rotation, we can assume that $z = \mathfrak{e}_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Next we consider the vector field

$$\begin{aligned} v\colon M &\to \mathbb{R}^{n+1} \\ Q &\mapsto \underbrace{z - \langle z, \mathfrak{G}(Q)\rangle \cdot \mathfrak{G}(Q)}_{\substack{\text{orthogonal projection} \\ \text{of } z \text{ to } \widetilde{\mathrm{T}}_Q M}}. \end{aligned}$$

Clearly we have $v(Q) = 0$ if and only if $\mathfrak{G}(Q) = z$ or $\mathfrak{G}(Q) = -z$.



Let $Q \in M$ with $\mathfrak{G}(Q) = \pm z$. By design we know that Q is a regular value of $\mathfrak{G}$. By the Stack-of-Records Lemma 32.3 (1) we know that $\mathfrak{G} \colon M \to S^n$ is a local diffeomorphism at Q, thus, as in Definition 46.1, it makes sense to introduce the following definition:

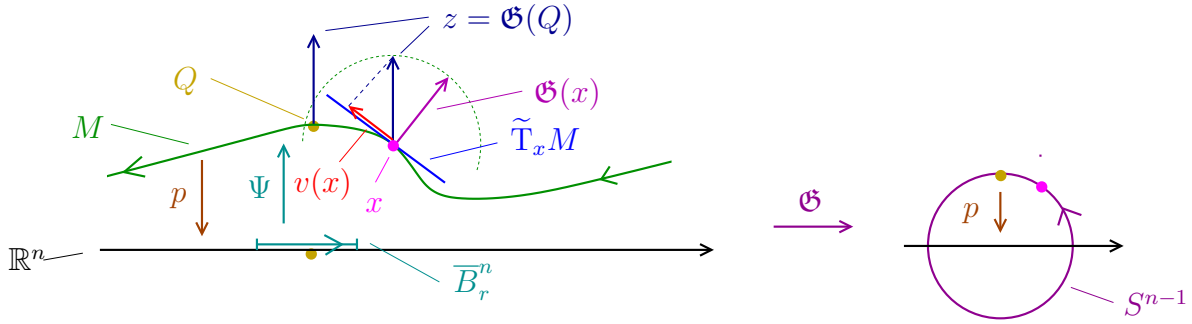
$$\deg(\mathfrak{G}, Q) \;=\; \begin{cases} +1, & \text{if } \mathfrak{G} \text{ is orientation-preserving at } Q, \\ -1, & \text{if } \mathfrak{G} \text{ is orientation-reversing at } Q. \end{cases}$$

Claim. Let $Q \in M$.
(1) If $\mathfrak{G}(Q) = z$, then $\operatorname{index}(v, Q) = (-1)^n \cdot \deg(\mathfrak{G}, Q)$.
(2) If $\mathfrak{G}(Q) = -z$, then $\operatorname{index}(v, Q) = \deg(\mathfrak{G}, Q)$.

Proof. We start out with the following observations and preparations.
(a) We denote by $p \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection onto the first n coordinates.
(b) It follows fairly easily from the Inverse Mapping Theorem 30.1, see also Exercise 29.5, that there exists a neighborhood U of $Q \in M$, a neighborhood V of $z \in S^n$ and an $r \in \mathbb{R}_{>0}$ with the following properties:
  (i) We have $\mathfrak{G}(U) \subseteq V$.
  (ii) The restriction of the projection $p \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ to $U \subseteq M$ and $V \subseteq S^n$ defines diffeomorphisms $p|_U \colon U \to \overline{B}^n_r$ and $p|_V \colon V \to \overline{B}^n_r$.
(c) We set $\Psi := (p|_U)^{-1} \colon \overline{B}^n_r \to U$.
(d) It follows immediately from the orientation conventions and some elementary linear algebra that the orientation behavior of $p|_U$ and $p|_V$ is the same. In particular the orientation behavior of Ψ and $p|_V$ is the same.

We now discuss the two cases separately:
(1) First we assume that $\mathfrak{G}(Q) = \mathfrak{e}_{n+1}$.

Subclaim. $$\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)} \;=\; \mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}.$$



Proof of subclaim. We start out with the following observation:

(∗) If W is an open subset of $\mathbb{R}^n$ and if $\lambda\colon W \to \mathbb{R}$ and $g\colon W \to \mathbb{R}^n$ are smooth maps, then it follows immediately from the Product Rule for partial derivatives for any $w \in W$ we have

$$\mathrm{D}(\lambda \cdot g)_w = (\tfrac{\partial \lambda}{\partial x_1}(w) \cdot g(w) \ldots \tfrac{\partial \lambda}{\partial x_n}(w) \cdot g(w)) + \lambda(w) \cdot \mathrm{D}g_w.$$

We now see that

$$\begin{aligned}
\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)} &= \mathrm{D}(p \circ (x \mapsto \overbrace{-\mathfrak{e}_{n+1} + \langle \mathfrak{e}_{n+1}, \mathfrak{G}(x)\rangle \cdot \mathfrak{G}(x)}^{=-v(x)}) \circ \Psi)_{p(Q)} \\
&= \mathrm{D}(\underbrace{(x \mapsto (\langle \mathfrak{e}_{n+1}, (\mathfrak{G} \circ \Psi)(x)\rangle))}_{=:\lambda} \cdot \underbrace{(p \circ \mathfrak{G} \circ \Psi)}_{=:g})_{p(Q)} \\
&\uparrow \text{ since } p \text{ is linear and since } x \mapsto -\mathfrak{e}_{n+1} \text{ is constant} \\
&= \mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}.
\end{aligned}$$

↑ follows from (∗) and the fact that $g(p(Q)) = (p \circ \mathfrak{G} \circ \Psi)(p(Q)) = p(\mathfrak{G}(Q)) = p(\mathfrak{e}_{n+1}) = 0$ and the observation that $\lambda(p(Q)) = \langle \mathfrak{e}_{n+1}, (\mathfrak{G} \circ \Psi)(p(Q))\rangle = \langle \mathfrak{e}_{n+1}, \underbrace{\mathfrak{G}(Q)}_{=\mathfrak{e}_{n+1}}\rangle = \langle \mathfrak{e}_{n+1}, \mathfrak{e}_{n+1}\rangle = 1$

⊞

We now see that

we know that p and Ψ are either both orientation-preserving or they are both orientation-reversing ↓

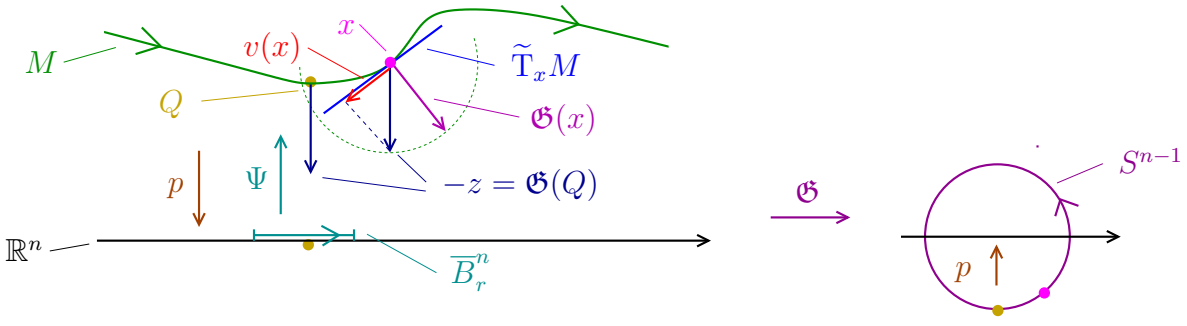
$$\begin{aligned}
\deg(\mathfrak{G}, Q) = +1 &\iff \det(\mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}) > 0 \\
&\iff \det(\mathrm{D}(\underbrace{\Psi^*(-v)}_{\text{pullback of } -v})_{p(Q)} > 0
\end{aligned}$$

by the above subclaim ↓

$$\begin{aligned}
&\iff \det(\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)}) > 0 \\
&\iff (-1)^n \cdot \det(\mathrm{D}(\Psi^* v)_{p(Q)} > 0 \\
&\iff \mathrm{index}(v, Q) = (-1)^n.
\end{aligned}$$

(2) Now we consider the case that $\mathfrak{G}(Q) = -\mathfrak{e}_{n+1}$.

Subclaim. $\mathrm{D}(p \circ (x \mapsto v(x)) \circ \Psi)_{p(Q)} = \mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}$.



Proof of subclaim. Similar to the previous subclaim we see that

$$\begin{aligned}
\mathrm{D}(p\circ(x\mapsto v(x))\circ\Psi)_{p(Q)} &= \mathrm{D}(p\circ(x\mapsto \overbrace{\mathfrak{e}_{n+1}-\langle\mathfrak{e}_{n+1},\mathfrak{G}(x)\rangle\cdot\mathfrak{G}(x)}^{=v(x)})\circ\Psi)_{p(Q)}\\
&= \mathrm{D}(\underbrace{(x\mapsto(-\langle\mathfrak{e}_{n+1},(\mathfrak{G}\circ\Psi)(x)\rangle))}_{=:\lambda}\cdot\underbrace{(p\circ\mathfrak{G}\circ\Psi)}_{=:g})_{p(Q)}\\
&\uparrow \text{ since } p \text{ is linear and since } x\mapsto\mathfrak{e}_{n+1} \text{ is constant}\\
&= \mathrm{D}(p\circ\mathfrak{G}\circ\Psi)_{p(Q)}.\\
&\uparrow \text{ follows from } (*) \text{ and the fact that } g(p(Q))=(p\circ\mathfrak{G}\circ\Psi)(p(Q))=p(\mathfrak{G}(Q))=p(\mathfrak{e}_{n+1})=0\\
&\text{and the observation that } \lambda(p(Q))=-\langle\mathfrak{e}_{n+1},(\mathfrak{G}\circ\Psi)(p(Q))\rangle=\langle\mathfrak{e}_{n+1},\underbrace{\mathfrak{G}(Q)}_{=-\mathfrak{e}_{n+1}}\rangle=\langle-\mathfrak{e}_{n+1},-\mathfrak{e}_{n+1}\rangle=1
\end{aligned}$$

⊞

We now see that

$$\begin{aligned}
\deg(\mathfrak{G},Q)=+1 &\overset{\text{we know that } p \text{ and } \Psi \text{ are either both orientation-preserving or they are both orientation-reversing}}{\iff} \det(\mathrm{D}(p\circ\mathfrak{G}\circ\Psi)_{p(Q)})>0 \overset{\text{by the above subclaim}}{\iff} \det(\mathrm{D}(p\circ(x\mapsto v(x))\circ\Psi)_{p(Q)})>0\\
&\iff \det(\mathrm{D}(\underbrace{\Psi^*(v)}_{\text{pullback of } -v})_{p(Q)}>0 \iff \det(\mathrm{D}(\Psi^*v)_{p(Q)}>0\\
&\iff \operatorname{index}(v,Q)=1.
\end{aligned}$$

⊞

We turn to the actual proof of the promised equality. Thus we now assume that n is even. We perform the following calculation:

$$\begin{aligned}
2\cdot\deg(\text{Gauß map } \mathfrak{G}\colon M\to S^n) &\overset{\text{since } z \text{ and } -z \text{ are regular values we can apply the Local-Global Degree Proposition 186.8 to } z \text{ and to } -z}{=} \sum_{\substack{Q\in M \text{ with}\\ \mathfrak{G}(Q)=z}}\deg(\mathfrak{G},Q)+\sum_{\substack{Q\in M \text{ with}\\ \mathfrak{G}(Q)=-z}}\deg(\mathfrak{G},Q)\\
&= \sum_{\substack{Q\in M \text{ with}\\ v(Q)=0}}\deg(\mathfrak{G},Q) = \sum_{\substack{Q\in M \text{ with}\\ v(Q)=0}}\operatorname{index}(v,Q) = \chi^{\mathrm{sm}}(M).\\
&\uparrow \text{since } v(Q)=0 \text{ if and only if } \mathfrak{G}(Q)=z \text{ or } \mathfrak{G}(Q)=-z \qquad \uparrow \text{by the claim and since } n \text{ is even} \qquad \uparrow \text{by the Smooth Poincaré-Hopf Theorem 53.24}
\end{aligned}$$

■

Remark. This concludes our discussion of the algebraic (self-) intersection number for the time being. We will return to the study of algebraic (self-) intersection numbers much later in Chapter 220 when we will relate it to the cup product on cohomology. □

53.8. Exercises for Chapter 53.

Exercise 53.1.

(a) Let M and N be two closed oriented smooth manifolds. Show that

$$\chi^{\mathrm{sm}}(M\times N) = \chi^{\mathrm{sm}}(M)\cdot\chi^{\mathrm{sm}}(N).$$

(b) Show that given any $k\in\mathbb{Z}_{\geq 2}$ there exist closed orientable connected $2k$-dimensional smooth manifolds of arbitrarily large and arbitrarily low values for the smooth Euler characteristic.

Exercise 53.2.

(a) Let M be a closed $2n$-dimensional smooth manifold and let $X \subseteq M$ be a closed n-dimensional submanifold. We do not make any assumptions on orientability. Show that one can still introduce a $\mathbb{Z}/2\mathbb{Z}$-algebraic self-intersection number $X \cdot_{\mathbb{Z}/2\mathbb{Z}} X \in \mathbb{Z}/2\mathbb{Z}$ in a reasonable way.
(b) Let $\iota\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^2$ be the smooth embedding given by $[x_0 : x_1] \mapsto [x_0 : x_1 : x_2]$. Determine the $\mathbb{Z}/2\mathbb{Z}$-algebraic self-intersection number of $\iota(\mathbb{R}\mathrm{P}^1) \subseteq \mathbb{R}\mathrm{P}^2$.

Exercise 53.3. We consider the smooth embedding

$$\begin{array}{rcl} \iota\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^2 \\ [z_0 : z_1] & \mapsto & [z_0 : z_1 : z_2]. \end{array}$$

Determine the algebraic self-intersection number of $\iota(\mathbb{C}\mathrm{P}^1) \subseteq \mathbb{C}\mathrm{P}^2$.
Remark. We know by the Projective Space–Manifold Proposition 25.13 and the discussion on page 965 that each $\mathbb{C}\mathrm{P}^n$ is a closed oriented $2n$-dimensional smooth manifold.

Exercise 53.4. Let M be an m-dimensional smooth submanifold of some $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $P \in M \setminus \partial M$ be a critical point. We pick a chart $\Phi\colon U \to V$ for P. We write $Q \coloneqq \Phi(P)$. In the Critical Point-via-Matrix Lemma 249.10 we will see that the congruence class of the Hessian matrix

$$\mathrm{Hess}(f \circ \Phi^{-1})_Q \;\coloneqq\; \left(\frac{\partial^2}{\partial x_i\, \partial x_j}\bigg|_Q f \circ \Phi^{-1} \right)_{i,j=1,\dots,m} \;\in\; \mathrm{M}(m \times m, \mathbb{R})$$

is independent of the choice of the chart. We say that the critical point P is *non-degenerate* if $\det(\mathrm{Hess}(f \circ \Phi^{-1})_Q) \neq 0$. We refer to

$$\mathrm{index}(f, P) \;\coloneqq\; \text{number of negative eigenvalues of } \mathrm{Hess}(f \circ \Phi^{-1})_Q$$

as the *index of the critical point* P. Next we consider the vector field that is given by

$$\begin{array}{rcl} \nabla f\colon M & \to & \mathbb{R}^n \\ x & \mapsto & \text{the unique } v(x) \in \widetilde{\mathrm{T}}_x M \text{ such that for all} \\ & & w \in \widetilde{\mathrm{T}}_x M \text{ we have } \langle v(x), w \rangle = \mathrm{D}f_x(w). \end{array}$$

We refer to it as the *gradient vector field of* f. Show that for every $x \in M$ the following statements hold:

(a) x is a critical point of f if and only if x is a zero of ∇f.
(b) If x is a critical point of f, then x is a non-degenerate critical point of f if and only if x is a non-degenerate zero of ∇f.
(c) If x is a non-degenerate critical point of f, then $\mathrm{index}(\nabla f, x) = (-1)^{\mathrm{index}(f,x)}$.

Exercise 53.5. Let $n \in \mathbb{N}_0$. We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the following functions on $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm \mathrm{id}\}$ and $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$:

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R} \\ [x_0 : \cdots : x_n] & \mapsto & \sum_{j=0}^{n} c_j \cdot x_j^2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \sum_{j=0}^{n} c_j \cdot |z_j|^2. \end{array}$$

By the Smooth Embedding Theorem 30.15 we can think of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ as smooth submanifolds of some $\mathbb{R}^N$. This allows us to consider the corresponding gradient vector fields ∇f and ∇g that we introduced in Exercise 53.4.

(a) Show that ∇f has precisely $n+1$ zeros, that these are non-degenerate and that the indices are $1, -1, \dots, (-1)^n$.

(b) Show that ∇g has precisely $n+1$ zeros, that these are non-degenerate and that the index is always $+1$.

Remark. You might want to use the explicit charts given in the Projective Space–Manifold Proposition 25.13

Remark. Note that it follows from this exercise together with the Smooth Poincaré–Hopf Theorem 53.24 that $\chi^{\mathrm{sm}}(\mathbb{R}\mathrm{P}^n) = \frac{1}{2}(1+(-1)^n)$ and that $\chi^{\mathrm{sm}}(\mathbb{C}\mathrm{P}^n) = n+1$.

Exercise 53.6. Let $p\colon \widetilde{M} \to M$ be a local diffeomorphism between two closed oriented connected non-empty smooth manifolds.

(a) Show that for any $x, y \in M$ we have $\#p^{-1}(x) = \#p^{-1}(y)$ and show that the cardinality of this set is finite.
(b) Show that the cardinality of the set in (a) equals $|\deg(p)|$.
(c) Show that $\chi^{\mathrm{sm}}(\widetilde{M}) = \deg(p) \cdot \chi^{\mathrm{sm}}(M)$.
(d) Given $k \in \mathbb{N}$ we denote by N_k the non-orientable surface of genus k from Definition 9.19. Show that N_k admits a nowhere vanishing smooth vector field if and only if $k = 2$.
Hint. Make use of Exercise 25.36.

Exercise 53.7. A *Lie group* is a smooth manifold M with empty boundary together with a group structure such that the two maps

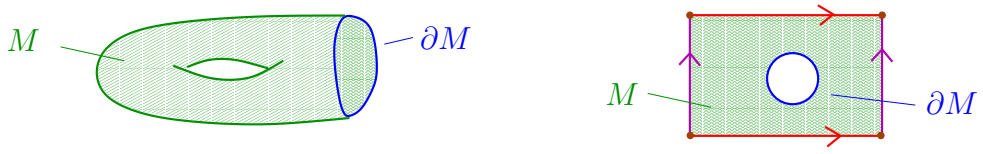
$$\begin{array}{rcl} M \times M & \to & M \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} M & \to & M \\ x & \mapsto & x^{-1} \end{array}$$

are smooth. For example it is not difficult to show that the matrix groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ with the usual multiplication are Lie groups.

(a) Show that every Lie group is orientable.
(b) Show that every Lie group admits a nowhere vanishing smooth vector field.

Remark. It follows from this exercise together with the Smooth-χ-Vector Field Theorem 53.29 that the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of any closed Lie group M of dimension ≥ 1 is zero.

Exercise 53.8. We consider the smooth manifold M shown in the figure below. Does M admit a nowhere vanishing smooth vector field that is nowhere tangential on the boundary?



CHAPTER 54

Gluing smooth manifolds

In this and the following chapter we will learn how to glue smooth manifolds along boundary components and we will get to know the complementary operation of cutting a smooth manifold along a codimension-one submanifold.

This chapter is organized as follows:

- In Section 54.1 we will finally show that collars of boundaries of smooth manifolds are in a suitable sense unique.
- In Section 54.2 we will learn how to glue topological manifolds along boundary components.
- In Section 54.3 we will learn how to glue smooth manifolds along boundary components. The construction of the new smooth structures uses the existence and uniqueness of collars.
- In Section 54.4 we will introduce the double of topological and smooth manifolds.
- In Section 54.5 we will introduce the connected sum operation for smooth manifolds.
- In Appendix 54.6 we will state and prove the Embedding Extension Theorem 54.6.
- In Section 54.7 we will provide exercises for Chapter 54.

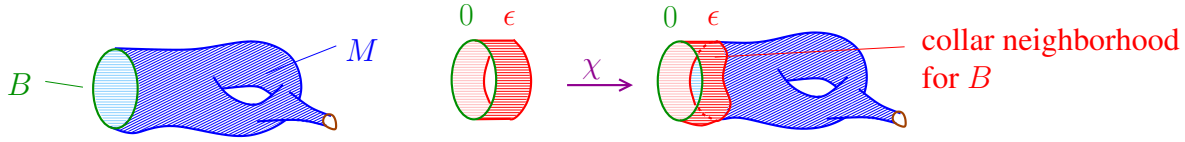
In the next Chapter 55 we will learn how to cut smooth manifolds along codimension-one submanifolds.

54.1. **Uniqueness of collars.** The gluing operation that we will introduce in this chapter relies on the notion of a collar. We recall the definition of a collar from Definition 35.4:

Definition 54.1. Let M be a smooth manifold and let $B \subseteq \partial M$ be a union of components of ∂M. A collar of B is defined as a smooth embedding $\chi\colon [0,\epsilon] \times B \to M$ for some $\epsilon > 0$ that has the following three properties:

(a) The restriction of χ to $\{0\} \times B$ is the obvious map $(0, P) \mapsto P$,
(b) The image $\chi([0,\epsilon] \times B)$ is a closed subset of M.
(c) $\chi^{-1}(\partial M) = \{0\} \times B$.

We refer to the image of χ as a collar neighborhood.

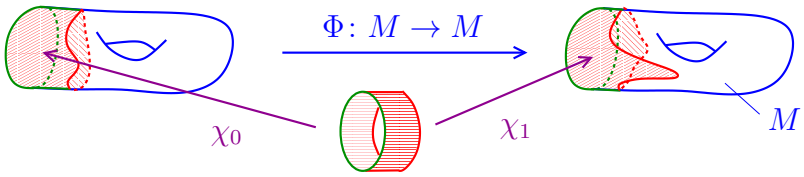


In the Smooth Collar Neighborhood Theorem 35.6 we showed that every smooth manifold admits a collar. For the gluing operation to be suitably well-defined we need not only that collars exist but we also need that they are essentially unique. This is precisely the content of the following proposition:

Proposition 54.2. (Smooth Collar-Uniqueness Proposition) Let M be a smooth manifold. Furthermore let $B \subseteq \partial M$ be a compact union of components of ∂M. Finally let $\chi_0 \colon [0,\epsilon_0] \times B \to M$ and $\chi_1 \colon [0,\epsilon_1] \times B \to M$ be two collars for B.

(1) There exists a diffeotopy
$$H \colon M \times [0,1] \ \to \ M$$
rel ∂M[583] from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1 \colon M \to M$ which has the property that $(\Phi \circ \chi_0)(t,b) = \chi_1(\frac{\epsilon_1}{\epsilon_0} \cdot t, b)$ for all $(t,b) \in [0,\epsilon_0] \times B$.

(2) Given any open subset U of M that contains the collar neighborhoods $\chi_0([0,\epsilon_0] \times B)$ and $\chi_1([0,\epsilon_1] \times B)$ one can find H as above such that H is furthermore a homotopy rel $M \setminus U$.



Remark.

(1) The Smooth Collar-Uniqueness Proposition 54.2 is also proved in [**Wall2016**, Proposition 2.5.7]. A very similar statement is also proved in [**BJ1982**, Theorem 13.7]. The statement about U is not contained in either of the two references, but the proofs can be modified accordingly without problems.

(2) The uniqueness of collars surely also holds if ∂M is non-compact, but we leave the task of proving this generalization to other authors. From what I can see nobody ever had the energy to carry out this task. Note though that in the non-compact setting the example on page 884 shows that it is essential to demand that the image of a collar $\chi \colon [0,\epsilon] \times \partial M \to M$ is a closed subset of M.

(3) The uniqueness statement for collar neighborhoods can be souped up, namely one can show that the "space of collars is contractible". We refer to [**Cer1961**, Chapitre II.5.2], [**Bud2007**, p. 8] and [**Hirs1976**, Section 4.5.3] for details.

(4) In the Topological Collar Neighborhood Theorem 35.14 we saw that topological manifolds also have collars. The obvious analog of the Smooth Collar-Uniqueness Proposition 54.2 holds in the context of topological manifolds. We have no use for this statement, thus we will not formulate it explicitly, let alone prove it. A proof of this uniqueness statement of collars of topological manifolds is sketched in [**KSi1969**, Essay I, Theorem A.2]. A proof (at least for compact topological manifolds) is also given in [**Arm1970**, Theorem 2]. □

Proof of the Smooth Collar-Uniqueness Proposition 54.2. In the following we first assume that M is compact and that $U = M$. Towards the end of the proof we will explain how to prove the general result.

Thus let M be a compact smooth manifold and let B be a union of components of ∂M. (It follows from the Smooth Manifold–Boundary Proposition 25.27 (2b) that B is compact.) Let $\chi_0 \colon [0,\epsilon_0] \times B \to M$ and $\chi_1 \colon [0,\epsilon_1] \times B \to M$ be two collars. Note that after

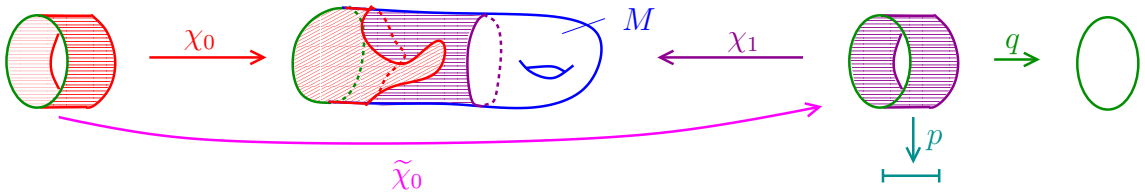
[583] Recall that this means that $H(P,t) = P$ for all $P \in \partial M$ and $t \in [0,1]$.

rescaling we can and will assume that $\epsilon_0 = \epsilon_1 = 1$. It follows from the Stationary-Near-The-Ends-Lemma 41.7 together with the Isotopy Extension Theorem 44.2[584][585], applied with $A = B \times \{0\}$ and $B = B$, that it suffices to prove the following claim:

Claim. The two collars $\chi_0, \chi_1 \colon [0,1] \times B \to M \setminus (\partial M \setminus B)$ are smoothly isotopic rel $\{0\} \times B$.

Proof. It follows from the Basics-of-Collars Lemma 35.7 (1) that $\chi_1([0,1) \times B)$ is an open neighborhood of B. Thus we see that $\chi_0^{-1}(\chi_1([0,1) \times B))$ is an open neighborhood of $\{0\} \times B \subseteq [0,1] \times B$. Since B is compact we obtain from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that $[0,\epsilon] \times B \subseteq \chi_0^{-1}(\chi_1([0,1) \times B))$. It is now clear that χ_0 is smoothly isotopic rel $\{0\} \times B$ to a collar whose image is contained in $\chi_1([0,1) \times B)$. It follows from the Smooth Isotopy-Equivalence Relation Proposition 41.5 that we might as well assume that $\chi_0([0,1] \times B) \subseteq \chi_1([0,1) \times B)$.

Next we set $\widetilde{\chi}_0 = \chi_1^{-1} \circ \chi_0 \colon [0,1] \times B \to [0,1] \times B$. Note that the restriction of $\widetilde{\chi}_0$ to $\{0\} \times B$ is the identity. In the following we also consider the two natural projections $p \colon [0,1] \times B \to [0,1]$ and $q \colon [0,1] \times B \to B$.



Next we consider the map

$$\begin{array}{rcl} \Theta \colon ([0,1] \times B) \times [0,1] & \to & ([0,1] \times B) \times [0,1] \\ ((s,x),t) & \mapsto & (\underbrace{(p \circ \widetilde{\chi}_0)(s,x) \cdot (1-t) + s \cdot t, (q \circ \widetilde{\chi}_0)(s \cdot (1-t))}_{\substack{= \widetilde{\chi}_0(s,x), \quad \text{if } t = 0, \qquad = (0,x), \quad \text{if } s = 0 \\ = (s,x), \quad \text{if } t = 1}}, t). \end{array}$$

Note that $\Theta_0 = \widetilde{\chi}_0$ and $\Theta_1 = \mathrm{id}$. We make two observations:

(1) Since χ_0 and χ_1 are both smooth embeddings[586] which agree on $\{0\} \times B$ we see, using a straightforward calculation, that for each $x \in B$ and each $t \in [0,1]$ the differential $\mathrm{D}\Theta_{(0,x,t)}$ is an isomorphism.

(2) The restriction of Θ to $\{0\} \times \partial M \times [0,1]$ is the identity.

It follows from (1) and (2) and the Inverse Mapping Theorem 30.1 together with the subtle Neighborhood Smooth Embedding Theorem 27.16 that there exists a $\mu > 0$ such that the restriction of Θ to $[0,\mu] \times B \times [0,1]$ is a smooth embedding. This implies that Θ defines a smooth isotopy of $[0,\mu] \times B$ rel $\{0\} \times \partial M$.

It follows from the above that the map

$$\begin{array}{rcl} ([0,1] \times \partial M) \times [0,1] & \to & M \\ ((s,x),t) & \mapsto & \chi_1(\Theta(\mu \cdot s, x, t)) \end{array}$$

[584]Note that we proved the Isotopy Extension Theorem 44.2 only for *compact* smooth manifolds.

[585]To apply the Isotopy Extension Theorem 44.2 we need that $\chi_i^{-1}(\partial M) = B$.

[586]Here we work with smooth manifolds with corner, we developed the theory of smooth manifolds with corner only partially. But it should be pretty clear that no serious issues arise.

is a smooth isotopy between $(t,x) \mapsto \chi_0(\mu \cdot t, x)$ and $(t,x) \mapsto \chi_1(\mu \cdot t, x)$. Evidently the maps $(x,t) \to \chi_i(x,t)$, $i = 0,1$ and $(x,t) \mapsto \chi_i(\mu \cdot t, x)$, $i = 0,1$ are also smoothly isotopic. It follows from the Smooth Isotopy-Equivalence Relation Proposition 41.5 that the three isotopies give rise to a smooth isotopy between χ_0 and χ_1. ⊞

We are now done with the proof of the proposition if the smooth manifold M is compact and if $U = M$. Now let M be any smooth manifold, and furthermore let $\chi_0, \chi_1 \colon [0,1] \times B \to M$ be two collars. Furthermore let $U \subseteq M$ be an open subset that contains the collar neighborhoods $\chi_0([0,1] \times B)$ and $\chi_1([0,1] \times B)$. Since B is compact it follows from the Compact Submanifold Exhaustion Proposition 38.2 (applied to the open subset $M \setminus (\partial M \setminus B)$ of M) that we can find a compact codimension-zero submanifold W of M, which contains ∂M, which contains the two collar neighborhoods and which is contained in U. Almost the same argument as above, now applied to W instead of M, gives the desired diffeotopy of W. In fact, using Statement (3a) of the Isotopy Extension Theorem 44.2 we can find a diffeotopy rel an open neighborhood of $\partial W \setminus B$, which means that we can extend the diffeotopy to a diffeotopy of M. ■

54.2. **Gluing topological manifolds.** The following proposition says that if, given a (possibly) disconnected topological manifold and if we glue boundary components via a homeomorphism, then the quotient is a topological manifold.

Proposition 54.3. (Gluing-Topological Manifolds-Proposition) Let M be an n-dimensional topological manifold[587]. Let A and B be disjoint unions of boundary components of M. Furthermore let $f \colon A \to B$ be a homeomorphism. Then the following statements hold:

(0) (a) If M is compact, then $M/a \sim f(a)$ is also compact.
(b) If A and B are connected and if each component of M contains A or B, then $M/a \sim f(a)$ is connected.

(1) The topological space
$$M/a \sim f(a) \quad \text{for } a \in A$$
is an n-dimensional topological manifold.

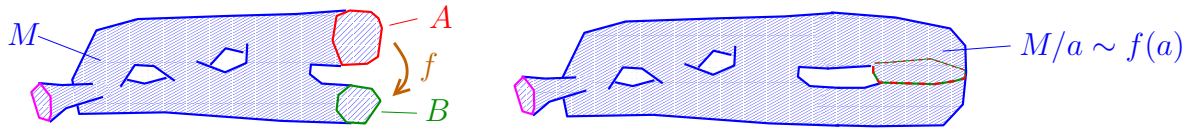
(2) The boundary[588] of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.

(3) (a) The image of A in $M/a \sim f(a)$ is a codimension-one submanifold of $M/a \sim f(a)$.
(b) If M consists of two components X and Y with $A \subseteq X$ and $B \subseteq Y$, then the following statements hold:
(i) The obvious maps $i \colon X \to M/a \sim f(a)$ and $j \colon Y \to M/a \sim f(a)$ are embeddings.
(ii) The images $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$ and they are codimension-zero submanifolds of $M/a \sim f(a)$.

In Chapter 191 we will introduce the notion of an orientation of a topological manifold. To preserve the symmetry with the Gluing-Smooth Manifolds-Proposition 54.5 we add the following statement, even though with the present knowledge it does not have any meaning yet.

(4) Assume that M is oriented. We equip A and B with the boundary orientations, as defined in the Boundary Orientation Proposition 192.3.
(a) If f is orientation-reversing, then $M/a \sim f(a)$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.

(b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.



Proof.

(0) (a) This statement follows from the Pushout-Properties Proposition 8.14 (1).
(b) This statement follows from the Pushout-Properties Proposition 8.14 (2).

(1) We write $W := M/a \sim f(a)$. We need to show that W admits an atlas. We set $M' := M \setminus (A \cup B)$. As we remarked on page 621, A and B are closed subsets of M. Thus we see that M' is an open subset of M and of W. We denote by $p\colon M \to W$ the projection. Now let $x \in W$. We need to distinguish two cases.

(α) First assume that $x = p(y)$ for some $y \in M'$. In this case we pick a chart $\Phi\colon U \to V$ for $y \in M$, we set $U' := U \cap M'$ and we obtain a chart $\Phi \circ (p|_{U'})^{-1}$ for $y \in W$.

(β) If (α) does not hold, then there exist $a \in A$ and $b \in B$ with $x = p(a) = p(b)$. We pick a chart $\Phi\colon U \to \widetilde{U}$ of type (ii) for $a \in M$ and we pick a chart $\Psi\colon V \to \widetilde{V}$ of type (ii) for $b \in M$. After possibly shrinking[589] U and V we can assume that $f(U \cap A) = V \cap B$ and we can assume that $U \cap V = \varnothing$. We introduce the following notation:

- We denote by $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the $(x_n = 0)$-hyperplane.
- We write $\widetilde{U}' := \{x \in \mathbb{R}^{n-1} \mid (x, 0) \in \widetilde{U}\}$ and we denote by $\iota\colon \widetilde{U}' \to \widetilde{U}$ the inclusion.
- We denote by $\pi\colon \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^{n-1}$ the obvious diffeomorphism.
- We write $\widetilde{f} := \pi \circ \Psi|_{V \cap B} \circ f \circ (\Phi_{U \cap A})^{-1} \circ \iota\colon \widetilde{U}' \to \mathbb{R}^{n-1}$.

One can now easily verify that

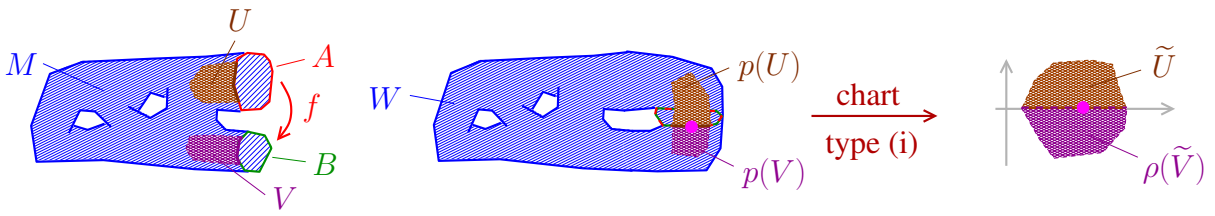
$$\begin{array}{rcl} p(U) \cup p(V) & \to & \widetilde{U} \cup \rho(\widetilde{V}) \\ x & \mapsto & \begin{cases} \Phi(u), & \text{if } x = p(u) \text{ for some } u \in U, \\ ((f|_{U \cap A} \times \mathrm{id}_{\mathbb{R}}) \circ \rho)(\Psi(v)), & \text{if } x = p(v) \text{ for some } v \in V \end{cases} \end{array}$$

is a chart of type (i) for x.

[588] We do not assume that M is connected.

[588] Recall that according to Definition 23.2 a point on a topological manifold is a boundary point if it does not admit a chart of type (i). For the record we mention that later in the Unique Type-of-Chart Proposition 150.11 we will see that a point admits *either* a chart of type (i) or a chart of type (ii). At this point we do not make use of this statement.

[589] To be more precise, we do the following: By the discussion on page 621 we know that A and B are closed subsets of M. By the Manifold Paracompact-Normal Proposition 23.9 we can find disjoint open neighborhoods $\widetilde{A}$ of A and $\widetilde{B}$ of B. After replacing U by $U \cap \widetilde{A}$ and V by $\widetilde{B}$ we can assume that U and V are disjoint. Furthermore, after replacing U by $U \setminus (A \setminus f^{-1}(V))$ and V by $V \setminus (B \setminus f(U))$ we can assume that $f(U \cap A) = V \cap B$.



As usual we still have to deal with the slightly annoying task of showing that W is Hausdorff and second-countable:

- Using the fact that A and B are closed subsets of M it is not hard to show that W is Hausdorff, one just needs to deal with several different cases. We leave it to the reader to fill in the details.
- It is straightforward to see that one can construct a countable atlas for W. Thus it follows from the Countable Atlas-Second Countable Lemma 23.4 that W is second-countable.

(2) Recall that in (1) for points in the image of $A \cup B$ we found charts of type (i), thus they are not part of the boundary. We also saw that $M \setminus (A \cup B)$ can be viewed as an open subset of $M/a \sim f(x)$. This implies that a point $x \in M \setminus (A \cup B)$ has a chart of type (i) in M if and only if x has a chart of type (i) in $M/a \sim f(x)$. The promised statement follows immediately from this observation.

(3) (a) The atlas that we constructed in (1) gives us submanifold charts for the image of A in $M/a \sim f(a)$.

(b) The statement follows from the Topological Pushout-Maps Lemma 8.16 (4) together with the observation that the atlas we constructed in (1) gives, with minor modifications, rise to the desired submanifold charts.

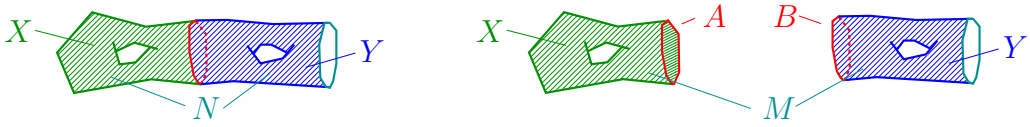
(4) We will prove this statement much later on page 3711, once we have understood the concept of an orientation of a topological manifold. ■

The following lemma can be viewed as a converse to the Gluing-Topological Manifolds-Proposition 54.3 (3b):

Lemma 54.4. (Regluing Topological Manifold Lemma) Let N be a topological manifold and let $X, Y \subseteq N$ be two closed codimension-zero submanifolds such that $X \cap Y$ is a union of boundary components $A \subseteq \partial X$ and a union of boundary components $B \subseteq \partial Y$. We set $M := X \sqcup Y = (X \times \{1\}) \cup (Y \times \{2\})$. Let $f\colon A \times \{1\} \to B \times \{2\}$ be the homeomorphism that is given by $(a, 1) \mapsto (a, 2)$. Then the map

$$\begin{array}{rcl} M/a \sim f(a) & \to & N \\ [(x, i)] & \mapsto & x \end{array}$$

is a homeomorphism.



Proof. This lemma is really just a special case of the Regluing Topological Lemma 8.17. ■

54.3. **Gluing smooth manifolds.** The following proposition is the smooth analog of the Gluing-Topological Manifolds-Proposition 54.3.

Proposition 54.5. (Gluing-Smooth Manifolds-Proposition) Let M be an n-dimensional smooth manifold, let A and B be disjoint unions of boundary components[590] of M and let $f\colon A \to B$ be a diffeomorphism.[591]

(0) (a) If M is compact, then $M/a \sim f(a)$ is also compact.
(b) If A and B are connected and if each component of M contains A or B, then $M/a \sim f(a)$ is connected.

(1) (a) Let $\chi\colon [0,\epsilon] \times (A \sqcup B) \to M$ be a collar[592] for $A \sqcup B \subseteq \partial M$. The topological manifold $M/a \sim f(a)$ admits the structure of an n-dimensional smooth manifold where the smooth structure is uniquely determined by the following two properties:
- The natural map $M \to M/a \sim f(a)$ is smooth and the differential at every point is an isomorphism.
- The map
$$\begin{array}{rcl} [-\epsilon,\epsilon] \times A & \to & M/a \sim f(a) \\ (t,a) & \mapsto & \begin{cases} \chi(t,a), & \text{if } t \in [0,\epsilon], \\ \chi(-t,f(a)), & \text{if } t \in [-\epsilon,0] \end{cases} \end{array}$$
is smooth and the differential at every point is an isomorphism.

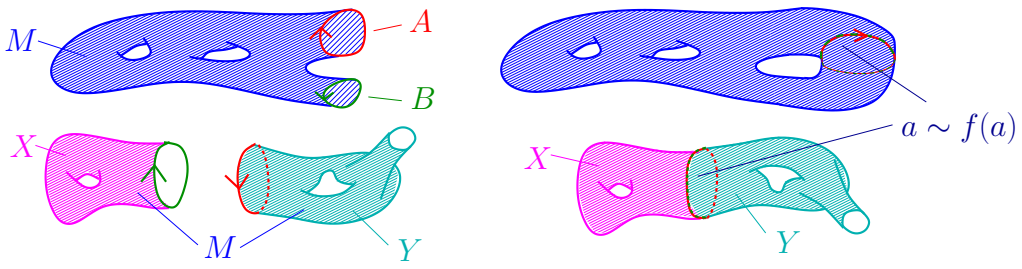
(b) Let χ and λ be two collars for $A \sqcup B$. By (a) these two collars give rise to two smooth structures $\mathcal{A}_\chi$ and $\mathcal{A}_\lambda$ on the topological manifold $M/a \sim f(a)$. If ∂M is compact[593], then given any neighborhood U of the image of A in $M/a \sim f(a)$ there exists a homeotopy $H\colon (M/a \sim f(a)) \times [0,1] \to M/a \sim f(a)$ with the following three properties:
(i) $H_0 = \mathrm{id}$.
(ii) $H_1\colon (M/a \sim f(a), \mathcal{A}_\chi) \to (M/a \sim f(a), \mathcal{A}_\lambda)$ is a diffeomorphism.
(iii) $H(x,t) = x$ for all $x \notin U$.

(c) If ∂M is compact, then the diffeomorphism type of $M/a \sim f(a)$ does not depend on the choice of the collar.

(2) The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.

(3) (a) The image of A in $M/a \sim f(a)$ under the natural map $A \to M \to M/a \sim f(a)$ is a codimension-one submanifold of $M/a \sim f(a)$.
(b) If M consists of two components X and Y with $A \subseteq X$ and $B \subseteq Y$, then the following statements hold:
(i) The obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are smooth embeddings.
(ii) The images $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$ and they are codimension-zero submanifolds of $M/a \sim f(a)$.

(4) Assume that M is oriented. We equip A and B with the boundary orientations.
(a) If f is orientation-reversing, then $M/a \sim f(a)$ admits a unique orientation such that the natural projection $M \to M/a \sim f(a)$ is orientation-preserving.
(b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.



Remark.

(1) The fact that the differential structure on $M/a \sim f(a)$ depends on the choice of a collar, and that only the diffeomorphism type is well-defined, is an awkward nuisance. Like many other sources we usually gloss over this issue.
(2) In Exercise 54.12 we will prove more statements about the tricky question under which circumstances $M/a \sim f(a)$ is orientable.
(3) In the next chapter we will prove the Bicollar Neighborhood Theorem 55.2 which will then allow us to prove the Regluing Smooth Manifold Lemma 55.5, which is the smooth analog of the Regluing Topological Manifold Lemma 54.4. □

Convention 54.6. In the setting of the Gluing-Smooth Manifolds-Proposition 54.5 (4a) we always equip $M/a \sim f(a)$ with the induced orientation.

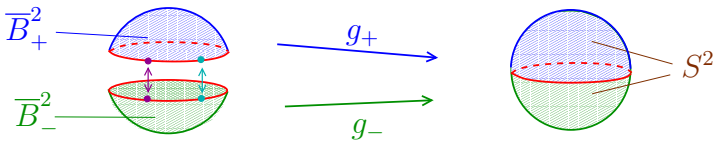
Example.

(1) Let $n \in \mathbb{N}_0$. As on page 319 we consider for $\epsilon = -1, 1$ the maps

$$\begin{array}{rcl} g_\epsilon \colon \overline{B}^n_\epsilon & \to & S^n \\ x = (x_1, \dots, x_n) & \mapsto & \frac{1}{\|x\|^2+1} \cdot \Big((-1)^n \cdot 2x_1, 2x_2, \dots, 2x_n, -\epsilon \cdot (\|x\|^2 - 1) \Big) \end{array}$$

which are given by stereographic projection from $(0, \dots, 0, \mp 1)$. We also consider the map

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$



[590] In the Gluing-Smooth Manifolds-Proposition 59.9 we will consider more general gluing.
[591] Recall that by the Smooth Manifold–Boundary Proposition 25.27 (1) the boundary components A and B are naturally $(n-1)$-dimensional smooth manifolds.
[592] By the Smooth Collar Neighborhood Theorem 35.6 we know that a collar always exists.
[593] The slightly annoying restriction to the case that the boundary ∂M is compact is due to the fact that we need to apply the Smooth Collar-Uniqueness Proposition 54.2. It might well be that the statement holds without any restrictions on A and B. Also note that in practice one deals almost always with the case of compact boundary components.

Recall that on page 319 we already discussed that this map is a homeomorphism. It is not difficult to show, using the standard collar for $\overline{B}^n_\pm$, that the above map is actually a diffeomorphism.[594]

(2) Let M be a smooth manifold with empty boundary and let $a < b < c$ be real numbers. We equip the products $M \times [a,b]$ and $M \times [b,c]$ with the (almost) obvious collars for $M \times \{b\} \subseteq M \times [a,b]$ and $M \times \{b\} \subseteq M \times [b,c]$. We glue $M \times [a,b]$ and $M \times [b,c]$ along $M \times \{b\}$ in the obvious way. It follows easily from the definitions that the obvious map to $M \times [a,c]$ is a diffeomorphism of smooth manifolds. □

Proof.

(0) These two statements follow immediately from the Gluing-Topological Manifolds-Proposition 54.3 (0).

(1) In the following we write $W := M/a \sim f(a)$ and $M' := M \setminus (A \cup B)$.

(a) On page 1226 we already showed that W is second-countable and Hausdorff. We also showed that W admits an n-dimensional atlas. But even if we start out with a smooth manifold M and a diffeomorphism the approach from page 1226 will not produce a smooth atlas. So we have to use a more sophisticated approach to get a smooth atlas.

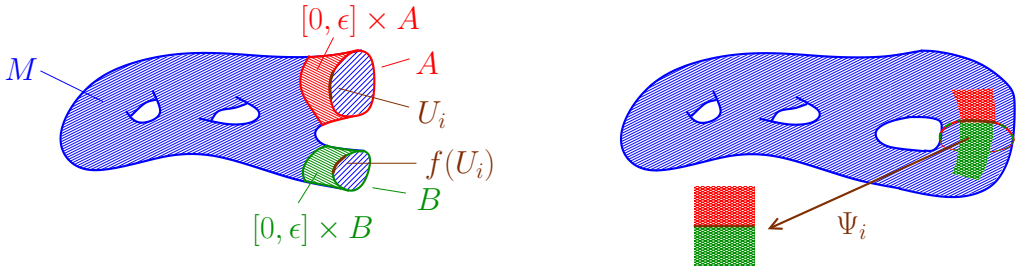
It follows from the Smooth Manifold–Boundary Proposition 25.27 (2a) that A and B are closed subsets of M. Thus we see that M' is an open subset of M and of W. We denote by $p\colon M \to W$ the projection. We will now see how a collar $\chi\colon [0,\epsilon] \times (A \sqcup B) \to M$ gives rise to smooth atlas for W. We do so as follows:

(α) We restrict all the charts in the given smooth atlas of M to the open subset M' of M.

(β) We equip A with the smooth $(n-1)$-dimensional atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ that is provided by the Smooth Manifold–Boundary Proposition 25.27 (1). Given $i \in I$ we define

$$\begin{array}{rcl} \overbrace{p(\chi([0,\epsilon)\times U_i)) \cup p(\chi([0,\epsilon)\times f(U_i))}^{\text{open subset of } M/a\sim f(a)} & \xrightarrow{\Psi_i} & \overbrace{(-\epsilon,\epsilon)\times V_i}^{\text{open in } \mathbb{R}^n} \\ p(\chi(t,y)) & \mapsto & \begin{cases} (t,\Phi_i(y)), & \text{if } t\in[0,\epsilon) \text{ and } y\in U_i, \\ (-t,\Phi_i(f^{-1}(y))), & \text{if } t\in[0,\epsilon) \text{ and } y\in f(U_i). \end{cases} \end{array}$$

One can verify easily that the charts from (α) and (β) define an n-dimensional smooth atlas for W. The corresponding maximal smooth atlas for W is called $\mathcal{A}_\chi$.



[594] The extra sign $(-1)^{n+1}$ assures us that the map $g_+\colon \overline{B}^n \to S^n$ is orientation-preserving and it means that $g_-\colon \overline{B}^n \to S^n$ is orientation-reversing.

We leave it to the reader to verify that this smooth structure has the following two desired properties:

- The natural map $M \to M/a \sim f(a)$ is smooth and the differential at every point is an isomorphism.
- The map
$$\begin{array}{rcl} [-\epsilon,\epsilon] \times A & \to & M/a \sim f(a) \\ (t,a) & \mapsto & \begin{cases} \chi(t,a), & \text{if } t \in [0,\epsilon], \\ \chi(-t,f(a)), & \text{if } t \in [-\epsilon,0] \end{cases} \end{array}$$
is smooth and the differential at every point is an isomorphism.

Using the Inverse Mapping Theorem 30.1 (2) one can verify without too much of an effort that this is the unique smooth structure on M with the above two properties.

(b) We now assume that the boundary ∂M is compact. Suppose we are given two collars $\chi\colon [0,\mu] \times (A \sqcup B) \to M$ and $\lambda\colon [0,\nu] \times (A \sqcup B) \to M$. By the Smooth Collar-Uniqueness Proposition 54.2 there exists a diffeotopy $H\colon M \times [0,1] \to M$ rel ∂M from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1\colon M \to M$ such that we have $(\Phi \circ \chi)(t,b) = (\lambda)(\frac{\nu}{\mu} \cdot t, b)$ for all $(t,b) \in [0,\mu] \times (A \sqcup B) \to M$. We consider the map
$$\begin{array}{rcl} F\colon (M/a \sim f(a)) \times [0,1] & \to & M/a \sim f(a) \\ (p(Q),t) & \mapsto & [p(H(Q,t))]. \end{array}$$
Note that it follows from the Homotopy Quotient Theorem 18.11 that this map is actually continuous. It is basically clear that each F_t is a homeomorphism. Finally note that F_1 defines a diffeomorphism from $(M/a \sim f(a), \mathcal{A}_\chi)$ to $(M/a \sim f(a), \mathcal{A}_\lambda)$. We leave it to the reader to refine the argument to ensure that the map F is a stationary homotopy outside of a given open neighborhood U of the image of A in $M/a \sim f(a)$.

(c) Note that this statement is an immediate consequence of (c).

(2) The atlas that we gave above for $W = M/a \sim f(a)$ has a chart of type (i) for every point in $M \setminus (\partial M \setminus (A \cup B))$ and a chart of type (ii) for every point in $\partial M \setminus (A \cup B)$. Therefore it follows from the Unique Type-of-Chart Proposition 25.26 that the boundary of $W = M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.

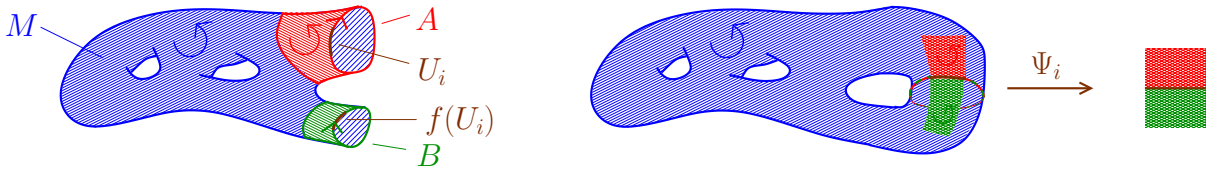
(3) (a) The atlas that we constructed in (1a) gives us submanifold charts for the image of A in $M/a \sim f(a)$.

(b) We already saw in the Gluing-Topological Manifolds-Proposition 54.3 (2) that the obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are embeddings with closed images. Furthermore it follows almost immediately from inspecting the atlas we had just constructed that the images $i(X)$ and $j(Y)$ are codimension-zero submanifolds of $M/a \sim f(a)$.

(4) We now assume that M is oriented. We equip A and B with the boundary orientation. We denote by $\pi\colon M \to M/a \sim f(a)$ the natural projection. Note that one can easily verify, using the explicit atlas for $M/a \sim f(a)$, that for each $Q \in M$ the differential $\mathrm{D}\pi_Q\colon \mathrm{T}_Q M \to \mathrm{T}_P(M/a \sim f(a))$ is an isomorphism.

(a) We now assume that the map $f\colon A \to B$ is orientation-reversing. We need to show that the smooth manifold $M/a \sim f(a)$ admits an orientation such that the natural projection $M \to M/a \sim f(a)$ is orientation-preserving.

First, for each $Q \in M \setminus B$ we equip $\mathrm{T}_{\pi(Q)}(M/a \sim f(a))$ with the orientation that we obtain from the given orientation of $\mathrm{T}_Q M$ by applying the isomorphism $\mathrm{D}\pi_Q \colon \mathrm{T}_Q M \to \mathrm{T}_P(M/a \sim f(a))$.



Claim. These orientations define an orientation on $M/a \sim f(a)$.

Proof. We will prove the claim via the Orientations-via-Atlas Lemma 31.21. We work with the atlas from (1). The charts of type (α) clearly are not a problem. We now turn to charts of type (β). Thus let $i \in I$. After restricting to the components of U_i we can assume that U_i is connected and after possibly composing the chart $\Phi_i \colon U_i \to V_i$ with a reflection we can assume that the corresponding chart $\Phi_i \colon U_i \to V_i$ is orientation-preserving. We consider the corresponding chart

$$\begin{aligned} p(\chi([0,\epsilon)\times U_i)) \cup p(\chi([0,\epsilon)\times f(U_i)) &\xrightarrow{\Psi_i} (-\epsilon,\epsilon)\times V_i \\ p(\chi(t,y)) &\mapsto \begin{cases} (t,\Phi_i(y)), & \text{if } t\in[0,\epsilon) \text{ and } y\in U_i, \\ (-t,\Phi_i(f^{-1}(y))), & \text{if } t\in[0,\epsilon) \text{ and } y\in f(U_i). \end{cases} \end{aligned}$$

that we constructed in (1a). We make the following observation:

- It follows easily from the definition of the boundary orientation that the restriction of Ψ_i to $p(\chi([0,\epsilon)\times U_i))$ is orientation-preserving.
- Note that it follows from the orientation-reversing "$-t$" in the definition of Ψ_i and the Product Orientation Preserving Lemma 31.50 that the restriction of Ψ_i to $p(\chi((-\epsilon,0])\times U_i))$ is orientation-preserving if and only if $f \colon U_i \to f(U_i)$ is orientation-reversing.

This discussion shows that the chart Ψ_i satisfies the continuity condition of the Orientations-via-Atlas Lemma 31.21. ⊞

(b) We now assume that M is connected. We need to show that if $M/a \sim f(a)$ is orientable, then f is orientation-reversing. First note that if $M/a \sim f(a)$ is orientable, then we can assume that (after possibly flipping the orientation of $M/a \sim f(a)$) the map $M \to M/a \sim f(a)$ is orientation-preserving at a point. But since M is path-connected it follows from the Manifold–Orientation Basics Lemma 31.31 (3) that the map is orientation-preserving everywhere.

At this point the same type of analysis as in the proof of the claim in (a) shows that f is orientation-reversing. ■

For many purposes and constructions it does not really make a difference if we modify a map by a diffeotopy. For example we have the following lemma which often gets used implicitly.

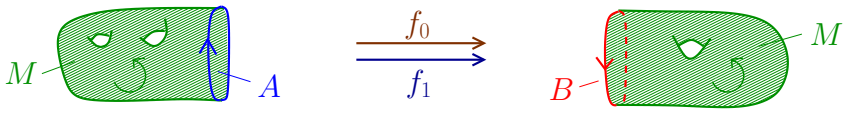
Lemma 54.7. (Diffeotopic-Gluings Lemma) Let M be a smooth manifold with compact boundary, let A and B be two distinct compact boundary components of M and let f_0 and f_1 be two diffeomorphisms from A to B. If f_0 and f_1 are diffeotopic, then the

results of gluing via f_0 respectively via f_1 are diffeomorphic, i.e. we have a diffeomorphism

$$M/x \sim f_0(x) \xrightarrow{\cong} M/x \sim f_1(x).$$

Furthermore, given a neighborhood U of $B \subseteq M$ one can find a diffeomorphism φ such that for any $x \in M \setminus U$ we have $\varphi([x]) = [x]$. Finally, if M is oriented and if f_0 and f_1 are orientation-reversing, then the above diffeomorphism is in fact orientation-preserving.

Sketch of proof. Since f_0 and f_1 are diffeotopic and since they are diffeomorphisms there exists a diffeotopy $F\colon B \times [0,1] \to B$ from $f_1 \circ f_0^{-1}$ to id_B. By the Stationary-Near-The-Ends-Lemma 41.7 we can arrange that $F_t = \mathrm{id}_B$ for $t \in [\frac{1}{2}, 1]$.

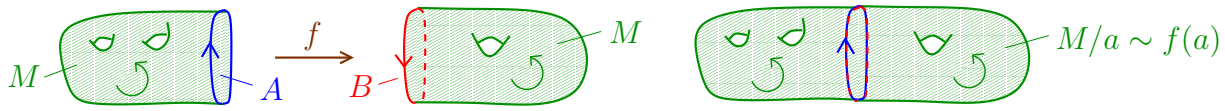


Next note that by the Smooth Collar Neighborhood Theorem 35.6 we can pick a collar neighborhood $[0,1]\times B$ for B. (Since B is compact it follows from the Tube Lemma 7.12 that we can, after possibly rescaling, arrange that $[0,1] \times B$ is contained in a given neighborhood of B.) One can now easily verify that

$$\begin{array}{rcl} M/x \sim f_0(x) & \to & M/x \sim f_1(x) \\ y & \mapsto & \begin{cases} y, & \text{if } y \in M \setminus ([0,1) \times B), \\ F(b,t) & \text{if } y = (t,b) \in [0,1] \times B \end{cases} \end{array}$$

is a diffeomorphism. Furthermore it is clear that this diffeomorphism has all the required properties. ∎

Example. Let M be a 2-dimensional smooth manifold and let A and B be two distinct boundary components of M. Suppose that A and B are equipped with orientations and let $f, g\colon A \to B$ be two orientation-reversing diffeomorphisms. It follows from the Smooth Manifold–Boundary Proposition 25.27 (1), the Smooth 1-Manifold Classification Theorem 28.8 and the Diffeotopies-of-S^1-Proposition 42.8 that f and g are diffeotopic. Thus it follows from the Diffeotopic-Gluings Lemma 54.7 that the resulting smooth manifolds $M/f(a) \sim a$ and $M/g(a) \sim a$ are diffeomorphic.



Loosely speaking this observation says that if we glue two oriented boundary components of a surface together in an orientation-reversing way, then the diffeomorphism type of the resulting smooth manifold does not depend on the specific choice of the orientation-reversing diffeomorphism. □

54.4. **The double of topological and smooth manifolds.** In this section we introduce the double of a manifold. This construction associates to a compact manifold with non-empty boundary a closed manifold. This procedure comes in handy on several occasions. In particular sometimes this procedure allows us to reduce the proof of a certain statement for compact manifolds to the closed case.

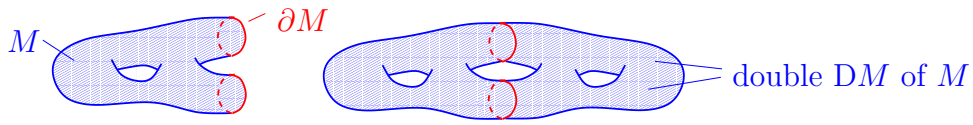
Definition 54.8. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to

$$\mathrm{D}_W M \;\coloneqq\; ((M \times \{1\}) \sqcup (M \times \{2\}))/\sim \qquad \text{where } (w,1) \sim (w,2) \text{ for all } w \in W$$

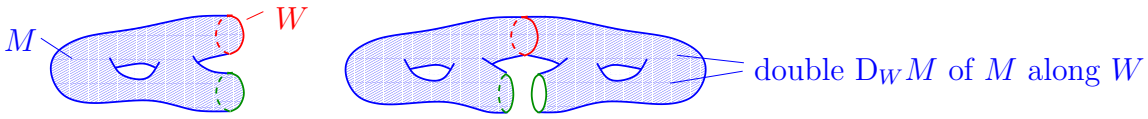
as the double of M along W. We refer to $\mathrm{D}M \coloneqq \mathrm{D}_{\partial M} M$ as the double of M.

Example.

(1) The double of M is given by taking two copies of M and identifying the points in W "left and right" in the obvious way. The figure below makes a convincing case that the double of the torus minus two open disks is homeomorphic to the surface of genus 3.



The next figure shows the effect of doubling along just one component of ∂M.



(2) For $W = \varnothing$, we have $\mathrm{D}_\varnothing M = M \sqcup M$. □

The following definition allows us to relate a topological manifold to its double.

Definition 54.9. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to the map

$$i\colon M \xrightarrow{P \mapsto (P,1)} M \times \{1\} \;\to\; (M \times \{1\}) \sqcup (M \times \{2\}) \;\to\; \mathrm{D}_W M$$

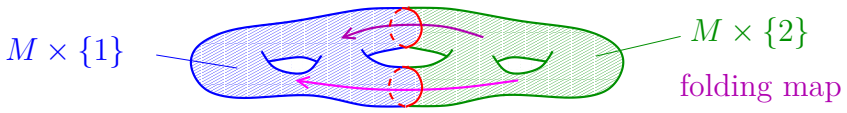
as the natural inclusion. Furthermore we refer to the map

$$\begin{aligned} r\colon \mathrm{D}_W M = ((M \times \{1\}) \cup (M \times \{2\}))/\sim \;&\to\; M \\ [(P,i)] \;&\mapsto\; P \end{aligned}$$

as the folding map. Finally we refer to the map

$$\begin{aligned} \rho\colon \mathrm{D}_W M \;&\to\; \mathrm{D}_W M \\ [(P,i] \;&\mapsto\; [(P,3-i)] \end{aligned}$$

as the natural involution on $\mathrm{D}_W M$.



The following lemma summarizes a few key properties of the doubling operation.

Lemma 54.10. (Doubling-Manifold Lemma) Let M be an n-dimensional topological (smooth) manifold and let W be the union of some components of ∂M. The following statements hold:

(0) (a) M is compact if and only if the double $\mathrm{D}_W M$ is compact.
(b) If M is connected and if $W \neq \varnothing$, then the double $\mathrm{D}_W M$ is also connected.

(1) The double $\mathrm{D}_W M$ along W is an n-dimensional topological manifold. If M is actually a smooth manifold, then the double $\mathrm{D}_W M$ is naturally an n-dimensional smooth manifold.
(2) The boundary of the double $\mathrm{D}_W M$ is given by two copies of $\partial M \setminus W$. In particular the boundary of the double $\mathrm{D}M$ is empty.
(3) (a) The natural inclusion $M \to \mathrm{D}_W M$ is a (smooth) embedding with closed image and its image is topological (smooth) submanifold of $\mathrm{D}_W M$.
(b) The folding map $\mathrm{D}_W M \to M$ is continuous and it satisfies $r \circ i = \mathrm{id}_M$.
(c) The natural involution $\mathrm{D}_W M \to \mathrm{D}_W M$ is continuous and it is an involution, i.e. $\rho \circ \rho = \mathrm{id}$.
(4) If M is a smooth manifold, then we have the following equivalence of statements:

$$M \text{ is orientable} \iff \text{the double } \mathrm{D}_W M \text{ is orientable.}$$

Furthermore, if M is equipped with an orientation, then $\mathrm{D}_W M$ admits a unique orientation such that the "first" embedding $M \to \mathrm{D}_W M$ is orientation-preserving and such that the "second" embedding $M \to \mathrm{D}_W M$ is orientation-reversing.[595]

Convention 54.11. Sometimes we will use the fact, established in the Doubling-Manifold Lemma 54.10, that the natural inclusion $M \to \mathrm{D}_W M$ is an embedding as an excuse to view M as a subspace of $\mathrm{D}_W M$ and to view $r\colon \mathrm{D}_W M \to M$ as a retraction.

Example.
(1) The discussion on page 1229 shows that the double of $\overline{B}^n$ is diffeomorphic to S^n.
(2) In the Non-Degenerate Vector Field Existence Proposition 53.22 we showed that every *closed* smooth manifold admits a smooth vector field with only finitely many zeros, all of which are non-degenerate. In Exercise 55.1 we will see that we can use the doubling construction to generalize the above result to compact smooth manifolds. □

Proof. Let M be an n-dimensional topological (smooth) manifold and let W be the union of some components of ∂M. Most of the statements follow immediately from the Compact–Union-Intersection Lemma 3.41, the Gluing-Topological Manifolds-Proposition 54.3 and the Gluing-Smooth Manifolds-Proposition 54.5. There are only a few statements that we still need to prove:

(3b) It follows from the Topological-Coproduct Proposition 7.26 and the Topological-Quotient Proposition 8.2 (1b) that the folding map $r\colon \mathrm{D}_W M \to M$ is continuous. It is clear that $r \circ i = \mathrm{id}_M$.
(3c) It follows from the Topological-Coproduct Proposition 7.26 and the Topological-Quotient Proposition 8.2 (1a) that the involution ρ is continuous. It is clear that $\rho \circ \rho = \mathrm{id}$.
(0a) We need to prove the "only if" statement of (0a). But this follows immediately from (3b) together with the Compact Image Lemma 4.15.
(4) First we suppose that M is an oriented smooth manifold. We equip $M \times \{1\}$ with the obvious orientation and we equip $M \times \{2\}$ with the opposite orientation. With these orientations it follows from the Boundary Orientation Lemma 31.38 (3) that the map $W \times \{1\} \xrightarrow{(P,1) \mapsto (P,2)} W \times \{2\}$ is orientation-reversing. Thus it follows from the Gluing-Smooth Manifolds-Proposition 54.5 (4) that the double $\mathrm{D}_W M$ admits an

[595] In the Double-Orientability Lemma 192.6 we will prove an analogous statement for topological manifolds.

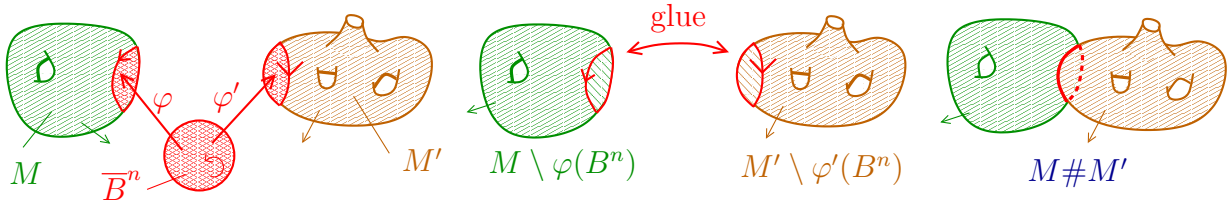
orientation such that the "first" embedding map $M \to \mathrm{D}_W M$ is orientation-preserving and such that the "second" embedding map $M \to \mathrm{D}_W M$ is orientation-reversing. It is clear that this orientation is uniquely determined.

Conversely, if $\mathrm{D}_W M$ is oriented, then it follows from (3a) and the Manifold–Orientation Basics Lemma 31.31 (4) that M inherits an orientation such that the inclusion $M \to \mathrm{D}_W M$ is orientation-preserving. ■

54.5. **The connected sum operation for smooth manifolds.** In this section we introduce the connected sum of two smooth manifolds. As we will see this construction combines features from the last two sections.

Definition 54.12. Let $n \in \mathbb{N}_0$ and let M and M' be two connected non-empty n-dimensional smooth manifolds. (If both M and M' are orientable, then we assume that M and M' are oriented.) We pick two smooth embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$. (If both M and M' are orientable, then we assume that precisely one of φ and φ' is orientation-preserving.) We define the connected sum of M and M' as

$$M \# M' := (M \setminus \varphi(B^n)) \sqcup (M' \setminus \varphi'(B^n))/\sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$



Remark.

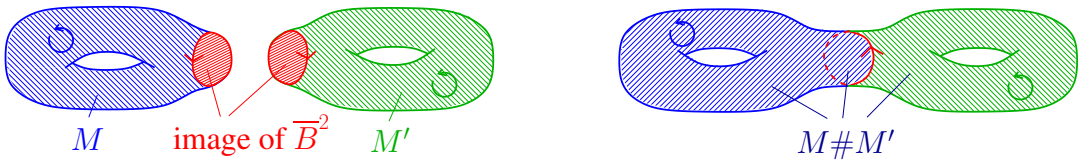
(1) A slightly different approach to defining the connected sum of two smooth manifolds is presented in [**KeM1963**, p. 505] and [**Kos1993**, p. 90]. In that reference the gluing is performed along open subsets which reduces some of the issues that arise while gluing along boundary components, i.e. while gluing along subsets that are not open. But at the end there is not real difference between the two approaches.

(2) In Section 224.7 we will introduce the connected sum of topological manifolds. □

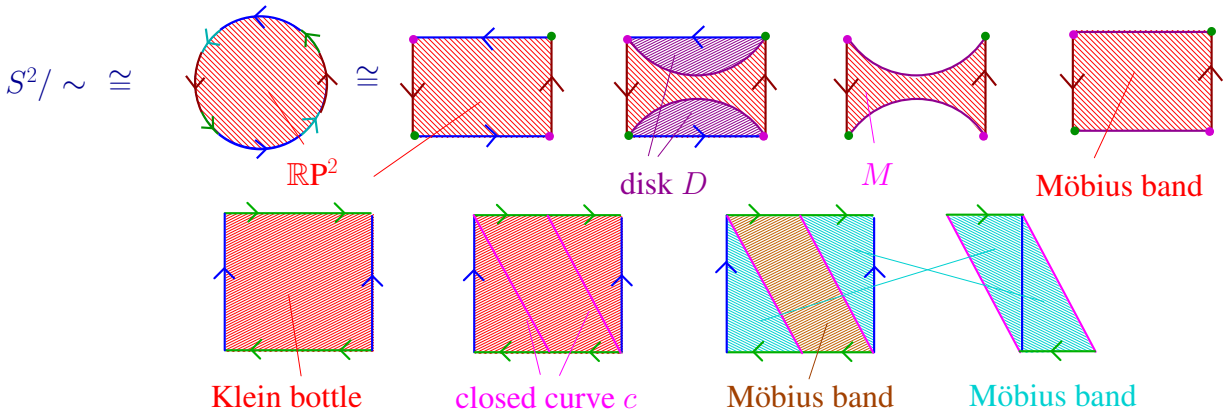
In the Smooth Connected Sum-Proposition 54.13 we will discuss the properties of the connected sum. In particular we will see that in a suitable sense the connected sum $M \# M'$ is well-defined, i.e. independent of any of the choices. So we will talk of "*the* connected sum" instead of "*a* connected sum". Before we go to the technicalities let us consider the connected sum in the 2-dimensional setting.

Examples.

(1) In the figure below we give a pictorial argument why the connected sum of two tori is homeomorphic to the surface of genus two. We will provide a somewhat more formal argument in the proof of the Surface–Connected Sum Lemma 54.15 (1).



(2) In the figure below we recall the argument why $\mathbb{R}\mathrm{P}^2$ minus an open disk is diffeomorphic to the Möbius band. This shows that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is given by gluing two Möbius bands along their common boundary. In the figure on the bottom we see a closed curve in the Klein bottle that decomposes the Klein bottle into two Möbius bands. Put differently, we see that both $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ and the Klein bottle are the result of gluing two Möbius bands along their boundary.



It is now straightforward to see that the Klein bottle is homeomorphic to the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ of two real projective planes. □

Now we turn to the technical proposition that contains all the statements anybody ever wanted to know about the topology of the connected sum of two smooth manifolds.

Proposition 54.13. (Smooth Connected Sum-Proposition)[596] Let $n \in \mathbb{N}_0$ and let M and M' be two connected n-dimensional smooth manifolds. (If both M and M' are orientable, then we demand that M and M' are oriented.) Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings. (If M and M' are orientable, then we demand that precisely one of φ, φ' is orientation-preserving and precisely one is orientation-reversing.) The following statements hold:

(0) (a) If M and M' are compact, then $M\#M'$ is also compact.
(b) If $n \geq 2$, then $M\#M'$ is also connected.[597]

(1) The resulting connected sum $M\#M'$ is an n-dimensional topological manifold that admits a smooth structure. The diffeomorphism type of $M\#M'$ is natural.

(2) (a) The boundary of $M\#M'$ is given by $\partial M \sqcup \partial M'$.
(b) If M and M' are closed, then $M\#M'$ is also closed.

(3) $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$ are closed subsets of $M\#M'$ and they are both codimension-zero submanifolds of $M\#M'$.

(4) If M and M' are oriented, then the smooth manifold $M\#M'$ admits a unique orientation that restricts to the given orientations of $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$.[598]

(5) The diffeomorphism type of $M\#M'$ does not depend on the choice of φ and φ'.

(6) (a) The connected sum operation is commutative, in the sense that there exists an (orientation-preserving) diffeomorphism from $M\#M'$ to $M'\#M$.
(b) If $n \geq 2$,[599] then the connected sum operation is associative, in the sense that if we are furthermore given a connected (oriented) n-dimensional smooth manifold M'',

then there exists an (orientation-preserving) diffeomorphism from $(M\#M')\#M''$ to $M\#(M'\#M'')$.

Remark. The Smooth Connected Sum-Proposition 54.13 (6), together with the discussion on page 115, allows us to unambiguously define the connected sum of any finite non-empty set of smooth manifolds of dimension $n \geq 2$. For safety we define the connected sum of the empty set of n-dimensional smooth manifolds as S^n. □

Convention 54.14. In the setting of the Smooth Connected Sum-Proposition 54.13 (4) we always equip $M\#M'$ with the induced orientation.

Proof. We write $W := M \setminus \varphi(B^n)$ and $W' := M' \setminus \varphi'(B^n)$. By the Codimension-Zero Smooth Embedding Proposition 30.21 (2) these are both n-dimensional smooth manifolds with boundary components $\varphi(S^{n-1})$ and $\varphi'(S^{n-1})$. By definition we have

$$M\#M' \;=\; (W \sqcup W')/a \sim \varphi'(\varphi^{-1}(a)) \quad \text{where } a \in \varphi(S^{n-1}).$$

(0) (a) If M and M' are compact, then it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 (2e) that W and W' are also compact. Thus it follows from the Gluing-Smooth Manifolds-Proposition 54.5 (1b) that $M\#M'$ is compact.

(b) We now assume $n \in \mathbb{N}_{\geq 2}$. In this case S^{n-1} is connected. It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 (2d) that W and W' are connected. We obtain from the Gluing-Topological Manifolds-Proposition 54.3 (0b) that $M\#M'$ is indeed connected.

(1) It follows from the Gluing-Smooth Manifolds-Proposition 54.5 that $M\#M'$ admits the structure of an n-dimensional smooth manifold and that the diffeomorphism type is well-defined.

(2) (a) It follows almost immediately from the Codimension-Zero Smooth Embedding Proposition 30.21 (2c) and the Gluing-Smooth Manifolds-Proposition 54.5 (2) that we have $\partial(M\#M') = \partial M \cup \partial M'$.

(b) This statement is an immediate consequence of (0a) and (2a).

(3) It follows from the Gluing-Smooth Manifolds-Proposition 54.5 (3) that W and W' are closed subsets of $M\#M'$ and that they are codimension-zero submanifolds of $M\#M'$.

(4) Now we suppose that M and M' are oriented. First note that it follows from the Manifold–Orientation Basics Lemma 31.31 that W and W' inherit an orientation from M and M'.

Next recall that in this case we suppose that precisely one smooth embedding out of $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ is orientation-preserving. This fact, together with the Boundary Orientation Lemma 31.38 (4), implies that the gluing map

[596] In the Topological Connected Sum-Proposition 224.24 we will prove an analog for topological manifolds.
[597] Note that the connected sum $\mathbb{R}\#\mathbb{R}$ of $\mathbb{R}$ with itself is the disjoint union of two copies of $\mathbb{R}$, thus it is disconnected. This shows that we cannot drop the hypothesis that $n \geq 2$.
[598] In Exercise 54.8 we will show the converse statement, that if the connected sum of two smooth manifolds is orientable, then each summand is orientable.
[599] We need to impose the condition $n \geq 2$ to ensure (using (1c)) that $M\#M'$ and $M'\#M''$ are connected, so that the connected sums $(M\#M')\#M''$ and $M\#(M'\#M'')$ are well-defined. It is a fun exercise to see that for $n = 1$ the expressions $(M\#M')\#M''$ and $M\#(M'\#M'')$ can be ambiguous.

$\varphi' \circ \varphi^{-1} \colon \varphi(S^{n-1}) \to \varphi'(S^{n-1})$ is orientation-reversing. Thus the desired statement follows from the Gluing-Smooth Manifolds-Proposition 54.5 (4).

(5) We need to show that the diffeomorphism type of $M\#M'$ is independent of the choices of the smooth embeddings φ and φ'. Given smooth embeddings $\varphi \colon \overline{B}^n \to M \setminus \partial M$ and $\varphi' \colon \overline{B}^n \to M' \setminus \partial M'$ we introduce the following cumbersome notation:

$$(M\#M')(\varphi,\varphi') \coloneqq \big(M \setminus \varphi\big(B^n\big)\big) \sqcup \big(M' \setminus \varphi'\big(B^n\big)\big) \,/\, \varphi(P) \sim \varphi'(P) \text{ for } P \in S^{n-1}.$$

Below we show independence of φ and φ'.

(α) We consider the case that M and M' are both orientable. We said that precisely one of φ and φ' is orientation-preserving. Let us show that it does not depend on which one we picked. Thus let $\varphi \colon \overline{B}^n \to M \setminus \partial M$ be an orientation-preserving smooth embedding and let $\varphi' \colon \overline{B}^n \to M' \setminus \partial M'$ be an orientation-reversing smooth embedding. Let $\rho \colon \overline{B}^n \to \overline{B}^n$ be a reflection in some hyperplane of $\mathbb{R}^n$. Note that the map $\varphi' \circ \rho \colon \overline{B}^n \to M' \setminus \partial M'$ is now an orientation-preserving embedding and that the map $\varphi \circ \rho \colon \overline{B}^n \to M \setminus \partial M$ is now an orientation-reversing embedding. If we use these two maps to define $M'\#M$ we see almost immediately that the "obvious" map $M\#M' \to M'\#M$ is an orientation-preserving diffeomorphism.

Now we can restrict ourselves to the case that the embeddings to M are orientation-preserving. Therefore let $\varphi, \psi \colon \overline{B}^n \to M \setminus \partial M$ be two orientation-preserving smooth embeddings and furthermore let $\varphi', \psi' \colon \overline{B}^n \to M' \setminus \partial M'$ be two orientation-reversing smooth embeddings. Next note that it follows from the Smooth Ball Embedding Theorem 45.10 there exist orientation-preserving diffeomorphisms $F \colon M \to M$ with $F \circ \varphi = \psi$ and $F' \colon M' \to M'$ with $F' \circ \varphi' = \psi'$. One can now easily verify that the map

$$\begin{array}{rcl} (M\#M')(\varphi,\varphi') & \to & (M\#M')(\psi,\psi') \\ x & \mapsto & \begin{cases} F(x) & \text{if } x \in M \setminus \varphi(B^n), \\ F'(x) & \text{if } x \in M' \setminus \varphi'(B^n) \end{cases} \end{array}$$

is an orientation-preserving diffeomorphism.

(β) We consider the case that M and M' are both non-orientable. Basically the same argument as in (α) applies. The only difference is that we now apply the Smooth Ball Embedding Theorem 45.10 to the case of non-orientable smooth manifolds, so we no longer have to make any assumptions on whether or not φ, ψ and φ', ψ' are orientation-preserving.

(γ) We consider the case that M is orientable and that M' is non-orientable. Thus let $\varphi, \psi \colon \overline{B}^n \to M \setminus \partial M$ and $\varphi', \psi' \colon \overline{B}^n \to M' \setminus \partial M'$ be two smooth embeddings. If φ and ψ are both orientation-preserving or if both are orientation-reversing, then the same argument as in (α) and (β) applies. Now suppose that this is not the case. Let $\rho \colon \overline{B}^n \to \overline{B}^n$ again be a reflection in a hyperplane. Then as before we see that $(M\#M')(\varphi,\varphi')$ is diffeomorphic to $(M\#M')(\psi \circ \rho, \psi' \circ \rho)$. But $(M\#M')(\psi \circ \rho, \psi' \circ \rho)$ is, basically by definition, the same as $(M\#M')(\psi,\psi')$.

(δ) The case that M is non-orientable and that M' is orientable is of course dealt with in the same way as in (γ).

(6) (a) We pick smooth embeddings $\varphi \colon \overline{B}^n \to M \setminus \partial M$ and $\varphi' \colon \overline{B}^n \to M' \setminus \partial M'$ (with appropriate orientation conditions). Let $\tau \colon \overline{B}^n \to \overline{B}^n$ be a reflection. We now use $\varphi' \circ \tau \colon \overline{B}^n \to M'$ and $\varphi \circ \tau \colon \overline{B}^n \to M$ (which by design have the appropriate

orientation conditions) to define the connected sum $M'\#M$. One can now easily verify that the obvious map $M\#M' \to M'\#M$ is an (orientation-preserving) diffeomorphism.

(b) We now pick four smooth embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$, $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$, $\psi'\colon \overline{B}^n \to M' \setminus \partial M'$ and $\psi''\colon \overline{B}^n \to M'' \setminus \partial M''$ (with appropriate orientation conditions) such that φ' and ψ' have disjoint images. Note that φ' also defines a smooth embedding $\overline{B}^n \to M\#M'$ and that ψ' also defines a smooth embedding $\overline{B}^n \to M'\#M''$. If we use all these smooth embeddings it is clear that there exists an (orientation-preserving) diffeomorphism from $(M\#M')\#M''$ to $M\#(M'\#M'')$. ■

Remark. The fact that the diffeomorphism type of the connected sum is well-defined is rather subtle.

(1) If M and M' are two *orientable* smooth manifolds, then the diffeomorphism type of the connected sum is defined only if both smooth manifolds are in fact *oriented*, i.e. they are equipped with an orientation. In general the definition does depend on the choice of orientation. More precisely, there exist oriented smooth manifolds M and M' such that $M\#M'$ is not diffeomorphic (in fact not even homeomorphic) to $M\#(-M')$, where $-M'$ denotes as usual the smooth manifold M' with the opposite orientation. We will see such examples later, namely in the Connected Sum–Orientation Dependence Proposition 226.19, once we have introduced the cup product on cohomology groups.

(2) Often the connected sum operation gets abbreviated as follows: *Given two smooth manifolds we remove an open ball from each smooth manifold and then glue M and M' together, along the boundary spheres that we had just created, in such a way that the orientations of the boundaries are opposite.* This description is rather dangerous since the diffeomorphism type of the connected sum is only well-defined if we glue the boundary components via the diffeomorphism given by the smooth embeddings of the closed balls. In Exercise 282.5 we will see that the above shortened description of the connected sum operation is incorrect. □

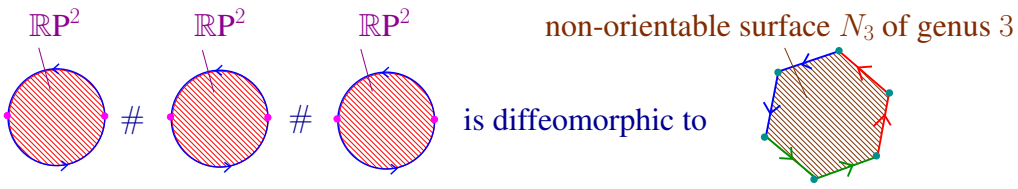
We conclude this section with several types of explicit examples of connected sums. First of all we consider the connected sums of (non-) orientable surfaces, as defined in Definitions 9.18 and 9.19.

Lemma 54.15. (Surface–Connected Sum Lemma)[600]

(1) Let $g, h \in \mathbb{N}$.
   (a) The connected sum of the surface of genus g and the surface of genus h is diffeomorphic to the surface of genus $g + h$.
   (b) The iterated connected sum of g tori is diffeomorphic to the surface of genus g.

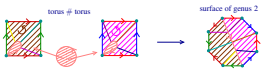
(2) Let $k, l \in \mathbb{N}$.
   (a) The connected sum of the non-orientable surface of genus k and the non-orientable surface of genus l is diffeomorphic to the non-orientable surface of genus $k + l$.
   (b) The iterated connected sum of k copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface of genus k.

Sketch of proof. In the following we sketch an argument. To simplify the discussion we just show that we have homeomorphisms. A proof that provides diffeomorphisms can be



given by fine tuning the approach below or by using the Surface Connected Sum–Handles Lemma 255.10, see page 4921.

(1) It follows from the associativity of the connected sum, see the Smooth Connected Sum-Proposition 54.13 (6b), that it suffices to show that the connected sum of the torus with the surface of genus g is diffeomorphic to the surface of genus $g+1$. In the following we provide a sketch of a proof of the statement for $g=1$. More precisely, in the figure below we show to the left two copies of the torus and two embeddings of $\overline{B}^2$, one is orientation-preserving and the other one is orientation-reversing. Furthermore we indicate a map from the corresponding connected sum to the surface of genus 2. With some good will one can see that this gives rise to the desired homeomorphism. We leave it to the reader to deal with the case $g \geq 2$.



(2) We outsource the proof of this statement to Exercise 54.9. ■

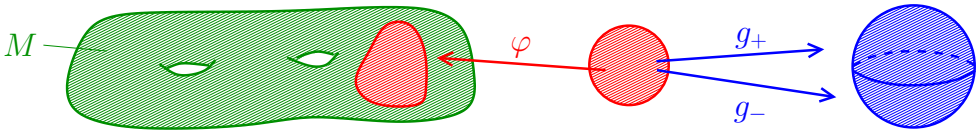
We move onto an elementary but essential special case:

Lemma 54.16. (Connected Sum-with-Sphere Lemma) Any connected sum of an oriented connected n-dimensional smooth manifold M with S^n is diffeomorphic to M.

Proof. If $M = \varnothing$ then there is nothing to show. Now assume that $M \neq \varnothing$. First we pick an (orientation-preserving) smooth embedding $\varphi\colon \overline{B}^n_2 \to M \setminus \partial M$. We consider, as on page 319, the two smooth embeddings

$$\begin{array}{rcl} g_+\colon \overline{B}^n & \to & S^n_{\geq 0} \subseteq S^n \\ x=(x_1,\ldots,x_n) & \mapsto & \frac{1}{\|x\|^2+1} \cdot \Big((-1)^n \cdot 2x_1, 2x_2, \ldots, 2x_n, -(\|x\|^2-1)\Big), \\ g_-\colon \overline{B}^n & \to & S^n_{\leq 0} \subseteq S^n \\ x=(x_1,\ldots,x_n) & \mapsto & \frac{1}{\|x\|^2+1} \cdot \Big((-1)^n \cdot 2x_1, 2x_2, \ldots, 2x_n, +(\|x\|^2-1)\Big). \end{array}$$

One can easily verify that g_+ is orientation-preserving and that g_- is orientation-reversing. It follows from the Smooth Connected Sum-Proposition 54.13 (5) that we can assume that we consider the connected sum that is obtain from φ and g_-.



To construct the smooth manifold $M \# S^n$ we need to provide collars. In the following we work with the collars

$$\begin{array}{rcl} \varphi(S^{n-1}) \times [0,1] & \to & M \setminus \varphi(B^n) \\ (\varphi(x),t) & \mapsto & \varphi((1+\frac{1}{2}t)\cdot x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \partial S^n_{\geq 0} \times [0,1] & \to & S^n_{\geq 0} \\ (g_+(x),t) & \mapsto & g_+(x\cdot(1-\frac{t}{2})). \end{array}$$

We use these collars to construct the smooth manifold $M\#S^n$. We leave it to the reader to verify that the map

$$\begin{array}{rcl} M\#S^n = (M\setminus\varphi(B^n)) \sqcup \underbrace{(S^n\setminus g_-(B^n))}_{=S^n_{\geq 0}}/\sim & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M\setminus\varphi(B^n), \\ \varphi(g_+^{-1}(P)), & \text{if } P \in S^n\setminus g_-(B^n) \end{cases} \end{array}$$

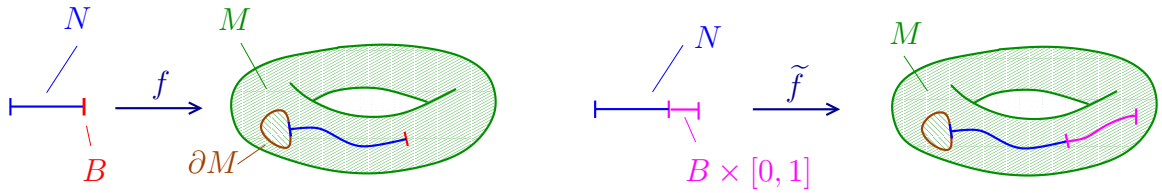
is a diffeomorphism. ■

54.6. **Appendix: The Embedding Extension Theorem.** In this appendix we will prove the following theorem, which we will only use once, but which nonetheless belongs to the toolbox of a well-equipped geometric topologist.

Theorem 54.17. (Embedding Extension Theorem) Let M be a smooth manifold, let N be a compact smooth manifold and let B be a union of components of ∂N. Furthermore let $f\colon N \to M$ be a smooth embedding such that $f(B) \subseteq M\setminus\partial M$ and let $\chi\colon [0,1]\times B \to N$ be a collar. There exists a smooth embedding[601]

$$\widetilde{f}\colon \underbrace{(N \sqcup (B \times [0,1]))/b \sim (b,0)}_{\substack{\text{using the collar } \chi \text{ this becomes} \\ \text{canonically a smooth manifold by the} \\ \text{Gluing-Smooth Manifolds-Proposition 54.5;} \\ \text{by Exercise 54.2 it is diffeomorphic to } N}} \to M$$

which restricts to f on N and such that the image of $B \times [0,1]$ lies in $M \setminus \partial M$.



Proof of the Embedding Extension Theorem 54.17. Let M be a smooth manifold, let N be a compact smooth manifold and let B be a union of components of ∂N. Furthermore let $f\colon N \to M$ be a smooth embedding such that $f(B) \subseteq M \setminus \partial M$ and let $\chi\colon [0,1] \times B \to N$ be a collar.

Using the Smooth Transition Function Lemma 24.9 it is straightforward to show that there exists a smooth map $\eta\colon [0,1] \to [0,1]$ with the following properties:

(1) $\eta'(t) > 0$ for all $t \in [0,1]$,
(2) $\eta(t) = \frac{1}{2} + t$ for $t \in [0, \frac{1}{4}]$,
(3) $\eta(t) = t$ for $t \in [\frac{7}{8}, 1]$,
(4) $\eta(0) = \frac{1}{2}$.



[601]Here the topological space $\big(N \sqcup (B\times[0,1])\big)/\sim$ is defined by the equivalence relation $P \sim (P,0)$ for any $P \in B$.

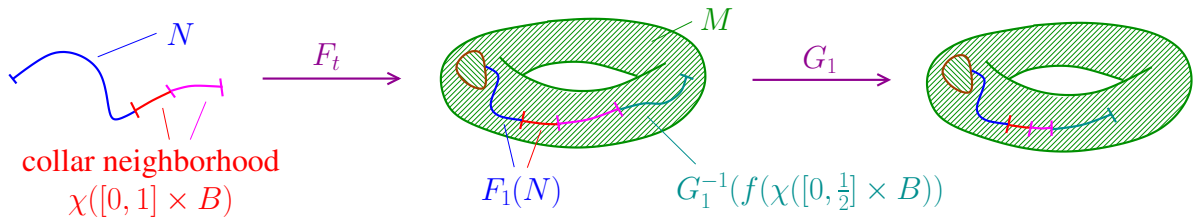
Next we consider the map

$$\begin{aligned} F\colon N\times[0,1] &\to M \\ (P,t) &\mapsto \begin{cases} f(P), & \text{if } P\notin \chi([0,1]\times B), \\ f(\chi(s\cdot(1-t)+\eta(s)\cdot t, Q)), & \text{if } P=\chi(s,Q) \text{ with } (s,Q)\in[0,1]\times B. \end{cases} \end{aligned}$$

One can easily verify that this map is a smooth isotopy with $F_0 = f$. Now note that it follows from the Isotopy Extension Theorem 44.2 (which we can apply since N is compact) that there exists a diffeotopy $G\colon M\times[0,1]$ with $G_0 = \mathrm{id}$ such that $G_t(f(P)) = F_t(P)$ for all $P\in N$. We pick a smooth map $\epsilon\colon[0,1]\to[0,1]$ with the following properties:

(1) $\epsilon'(t) < 0$ for all $t\in[0,1]$,
(2) $\epsilon(t) = \frac{1}{2} - t$ for all $t\in[0,\frac{1}{4}]$,
(3) $\epsilon(1) = 0$.

We could for example take $\epsilon(t) = \eta(1-t) - \frac{1}{2}$, but any other ϵ will do.



Finally we consider the map

$$\begin{aligned} \widetilde{f}\colon (N\sqcup(B\times[0,1]))/\!\sim &\to M \\ [P] &\mapsto \begin{cases} f(P), & \text{if } P\in N, \\ G_1^{-1}(f(\chi(\epsilon(t),Q))), & \text{if } P=(Q,t)\in B\times[0,1]. \end{cases} \end{aligned}$$

Using the Smooth Pasting Proposition 25.19 one can easily show that $\widetilde{f}$ is a smooth embedding.[602] ∎

54.7. Exercises for Chapter 54.

Exercise 54.1. Let M be a smooth manifold, let $B\subseteq\partial M$ be a compact union of components of ∂M and let $\chi\colon[0,1]\times B\to M$ be a collar. Show that we can extend χ to a collar $\chi\colon[0,2]\times B\to M$.

Hint. You could make use of the Smooth Collar-Uniqueness Proposition 54.2 by applying it to two collars built out of χ.

Exercise 54.2. Let M be a compact smooth manifold and let C be a union of components of ∂M. Let $\varphi\colon C\to C$ be a diffeomorphism.

(a) Show that the smooth manifold

$$(M\sqcup([0,1]\times C))/\!\sim \qquad \text{where } x\sim\varphi(0,x) \text{ for } x\in C$$

is diffeomorphic to M.

Remark. Recall that the definition of the smooth structure depends on the choice of a collar $\chi\colon[0,1]\times C\to M$.

[602] It suffices to show that $G_1\circ\widetilde{f}$ is a smooth embedding. Smoothness in a neighborhood of B follows from the definitions, the fact that $\eta(t) = \frac{1}{2} + t$ for small t and the fact that $\epsilon(t) = \frac{1}{2} - t$ for small t.

(b) In fact show that given any neighborhood U of $C \subseteq M$ there exists a diffeomorphism that is the identity on the common subset $M \setminus U$.

Remark. $[0,1] \times C \subseteq (M \sqcup ([0,1] \times C))/\sim$ is sometimes called an *external collar*.

Exercise 54.3. For $i = 1, 2$ let M_i be an n-dimensional smooth manifold, let A_i and B_i be two distinct boundary components of M_i and let $f_i \colon A_i \to B_i$ be a diffeomorphism. Furthermore suppose that we are given a diffeomorphism $\Phi \colon M_1 \to M_2$ such that the following diagram commutes:

$$\begin{array}{ccc} A_1 & \xrightarrow{\Phi|_{A_1}} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\Phi|_{B_1}} & B_2. \end{array}$$

We define the manifolds $N_i \coloneqq M_i/\sim$ as in the Gluing-Smooth Manifolds-Proposition 54.5.

(a) Show that the "obvious" map $N_1 \to N_2$ given by $[P] \mapsto [\Phi(P)]$ is a homeomorphism.

(b) Give an example where the map from (a) is not a diffeomorphism.

(c) Show that N_1 and N_2 are diffeomorphic.

Hint. Use the Smooth Collar Neighborhood Theorem 35.6.

Exercise 54.4. What is a familiar topological space that is homeomorphic to the double of the Möbius band? Furthermore, which one is homeomorphic to the double of the Klein bottle minus an open disk?

Exercise 54.5. Is it possible that the connected sum of two non-empty smooth manifolds is the empty manifold?

Exercise 54.6. Let M be a connected n-dimensional smooth manifold and let $P \in M \setminus \partial M$. Show that $M \setminus \{P\}$ is diffeomorphic to the connected sum $M \# \mathbb{R}^n$.

Exercise 54.7. Let $n \in \mathbb{N}$. What is the diffeomorphism type of $\mathbb{R}^n \# \mathbb{R}^n$?

Exercise 54.8. Let M and N be connected non-empty n-dimensional smooth manifolds. Suppose that their connected sum $M \# N$ is orientable. Show that M and N are also orientable.

Exercise 54.9. Let $g \in \mathbb{N}_{\geq 2}$. Show that the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface of genus g that we introduced in Definition 9.19.

Remark. We refer to the figure on page 1241 for an illustration.

Exercise 54.10. Let M be a compact smooth manifold. Show that there exists a compact smooth manifold W with $\partial W = \mathrm{D}M$.

Exercise 54.11. Let M and N be two oriented connected smooth n-dimensional manifolds and let W be a k-dimensional manifold. Is $(M \times W) \# (N \times W)$ diffeomorphic to $(M \# N) \times W$?

Exercise 54.12. Let M and N be two oriented connected n-dimensional smooth manifolds, let A be a union of boundary components of M and let B be a union of boundary components of N and let $\varphi \colon A \to B$ be a diffeomorphism.

(a) Show that if φ is orientation-preserving or orientation-reversing, then $M \cup_\varphi N$ admits an orientation such that the map $M \to M \cup_\varphi N$ is an orientation-preserving smooth embedding.

(b) Show that if $M \cup_\varphi N$ admits an orientation, then the map $A \to B$ is orientation-preserving or orientation-reversing.

Exercise 54.13. Give examples of oriented connected non-empty 1-dimensional smooth manifolds M, M' and M'' such that $M\#M'$ is disconnected and such that the expression $(M\#M')\#M''$ is ambiguous.

Exercise 54.14. Let M and N be compact connected k-dimensional smooth manifolds. Suppose that there exist smooth embeddings $M \to S^n$ and $N \to S^n$. Show that there exists a smooth embedding of the connected sum $M\#N$ into S^n.
Hint. Think of S^n as $S^n\#S^n$.

Exercise 54.15. Let M be a connected oriented k-dimensional smooth manifold and let $\Phi\colon \overline{B}^k \to M$ be a smooth embedding. Show that $\mathrm{D}(M \setminus \Phi(B^k))$ is diffeomorphic to $M\#(-M)$.

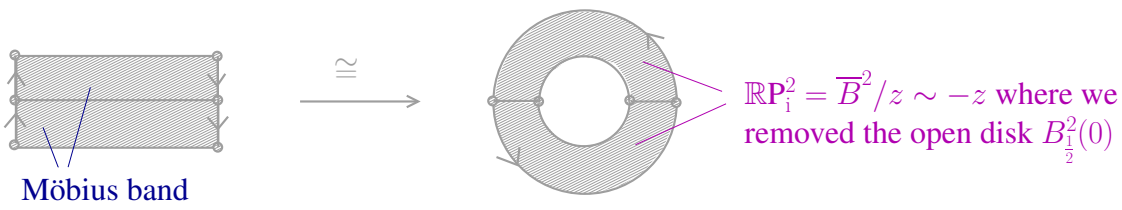
Exercise 54.16. Let W be a closed smooth manifold. We say that two diffeomorphisms $\Phi_0, \Phi_1\colon W \to W$ are *smoothly pseudo-isotopic* if there exists a *smooth pseudo-isotopy* between Φ_0 and Φ_1, i.e. a diffeomorphism $F\colon W \times [0,1] \to W \times [0,1]$ such that for all $w \in W$ we have $F(w,0) = (\Phi_0(w), 0)$ and $F(w,1) = (\Phi_1(w), 1)$.

(a) Show that if two diffeomorphisms are smoothly isotopic, then they are also smoothly pseudo-isotopic.

(b) Let M and N be two closed n-dimensional smooth manifolds and let $A \subseteq \partial M$ and $B \subseteq \partial N$ be boundary components. Finally let $f\colon A \to B$ be a diffeomorphism. Show that if $\Phi\colon A \to A$ is smoothly pseudo-isotopic to id_A, then $M \cup_f N$ is diffeomorphic to $M \cup_{f\circ\Phi} N$.

Remark. This exercise can be viewed as a generalization of the Diffeotopic-Gluings Lemma 54.7.

Exercise 54.17. On page 1237 we described a diffeomorphism from the Möbius band to the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z$ minus one open disk. Fill in the colors in the figure below to describe an alternative diffeomorphism.



Exercise 54.18. Let $K, L \subsetneq S^n$ be compact subspaces. The complements $S^n \setminus K$ and $S^n \setminus L$ are open subsets of S^n, thus we can view them as oriented smooth manifolds. Describe the connected sum $(S^n \setminus K)\#(S^n \setminus L)$.

CHAPTER 55

Cutting smooth manifolds

In Chapter 54 we learned how to glue two boundary components of a smooth manifold together. In this chapter we will learn how to cut a smooth manifold along a codimension-one submanifold and we will see that the operations of gluing and cutting are in a sense inverse operations.

In Chapter 54 we needed to further develop the theory of collars to make complete sense of the process of gluing boundary components. In a similar fashion, to make sense of the process of cutting along a codimension one submanifold, we now need to develop the theory of bicollars.

This chapter is organized as follows:

- In Section 55.1 we will see in the Bicollar Neighborhood Theorem 55.2 and the Bicollar-Uniqueness Proposition 55.4 that codimension-one submanifolds admit bicollars and that these are unique in a suitable sense.
- In Section 55.2 we will use bicollars to prove the smooth analog of the Regluing Topological Manifold Lemma 54.4.
- In Section 55.3 we will use bicollars to define what it means to cut smooth manifolds along codimension-one submanifolds.
- In Section 55.4 we will consider a frequently used special case of cutting along a codimension-one submanifold. The following lemma gives us an important example of cutting along a codimension-one submanifold.
- In Section 55.5 we will see that gluing along boundary components and cutting along codimension-one submanifolds are essentially inverse operations.
- In Appendix 55.6 we will state and prove the Smooth Manifold Pushout-Proposition 55.12.
- In Section 55.7 we will provide exercises for Chapter 55.

In the next Chapter 56 we will study the interaction between submanifolds and collars.

55.1. The Bicollar Neighborhood Theorem.

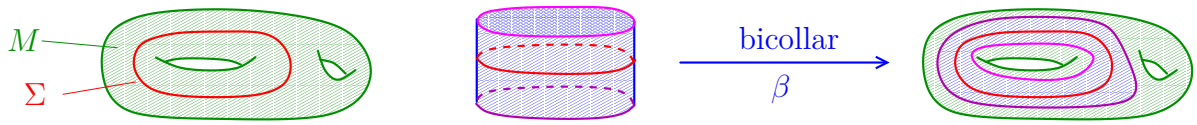
We start out with the following variation on the concept of a collar which we introduced in Definition 35.4.

Definition 55.1. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold of M with empty boundary[603] [604]. A bicollar for Σ is a smooth embedding $\beta\colon [-\epsilon,\epsilon] \times \Sigma \to M \setminus \partial M$ that has the following two properties:

(a) The restriction of β to $\{0\} \times \Sigma$ is the obvious map $(0, P) \mapsto P$.

(b) The image $\beta([-\epsilon,\epsilon] \times \Sigma)$ is a closed subset of M.[605]

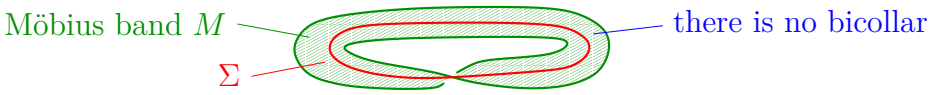
We refer to the image of β as a bicollar neighborhood of Σ.[606]



The following theorem is the analog of the Smooth Collar Neighborhood Theorem 35.6:

Theorem 55.2. (Bicollar Neighborhood Theorem) Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a proper oriented codimension-one submanifold of M with empty boundary[607]. There exists an orientation-preserving bicollar $[-1,1] \times \Sigma \to M$.

Remark. Let M be the Möbius band M and the submanifold Σ that is given by the central curve. It is pretty obvious that in this setting there is no bicollar $[-\epsilon,\epsilon] \times \Sigma \to M$.



This shows that in the Bicollar Neighborhood Theorem 55.2 we can certainly not drop the hypothesis that M is orientable. □

The proof of the Bicollar Neighborhood Theorem 55.2 rests on the following lemma.

Auxiliary Lemma 55.3. (Euclidean Submanifold Embedding Lemma)[608] Let M be an oriented smooth manifold with $\partial M = \varnothing$ and let $\Sigma \subseteq M$ be a proper oriented codimension-one submanifold of M with $\partial\Sigma = \varnothing$. There exists an $N \in \mathbb{N}_0$ and a proper smooth embedding $\Theta\colon M \to \mathbb{R}^N$ such that $\Theta(\Sigma) \subseteq \{0\} \times \mathbb{R}^{N-1}$ but such that for each $x \in \Sigma$ we have $\mathrm{D}\Theta_x(\mathrm{T}_x M) \not\subseteq \{0\} \times \mathbb{R}^{N-1}$.

Proof of Lemma 55.3. Let M be an oriented n-dimensional smooth manifold such that $\partial M = \varnothing$ and let $\Sigma \subseteq M$ be a proper oriented codimension-one submanifold of M such that $\partial\Sigma = \varnothing$. In the following we provide the proof if $n \geq 2$. The case $n = 1$ follows from a minor variation which we leave to the reader.

---

[603]Recall that according to Definition 23.18 a submanifold N is *proper* if N is a closed subset of X and if for each $P \in N$ we can find submanifold charts of type (α) or (β). Since in this definition we only consider submanifolds with empty boundary we see that we only have submanifold charts of type (α), as introduced in Definition 23.18.

[604]We impose the condition that $\partial\Sigma = \varnothing$ to avoid talking about smooth (sub-) manifolds with corner. In principle many of the results of this section can be extended in a suitable way to the case that Σ is a proper submanifold with non-empty boundary. In fact, if Σ is compact, then that is exactly what we do in the Product Tubular Neighborhood–Existence Theorem 58.4 (3). Furthermore we will discuss the general case later, in Definition 59.15.

[605]If Σ is compact, then it follows easily from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that this condition is automatically satisfied.

[606]We have a pretty obvious analog of the Basics-of-Collars Lemma 35.7 which we will use implicitly on occasion. In particular, as in the Basics-of-Collars Lemma 35.7 (1b) we note that it follows easily from the Codimension-Zero Smooth Embedding Proposition 30.21 (1) that $\beta((-\epsilon,\epsilon) \times \Sigma)$ is an open subset of M, in particular $\beta([-\epsilon,\epsilon] \times \Sigma)$ is indeed a neighborhood of Σ.

[607]We will discuss the case of non-empty boundary later in the Bicollar Neighborhood Theorem 59.16.

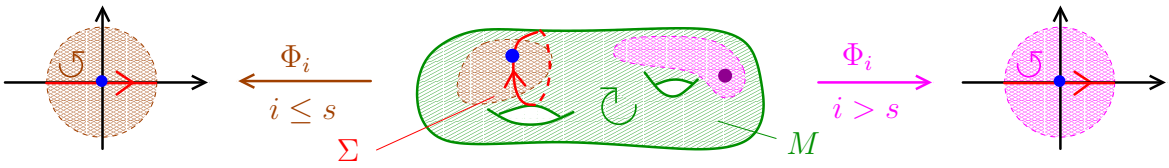
[608]This lemma can be viewed as a variation on the Euclidean Embedding Theorem 33.2.



Claim 1. There exists a smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,t}$ for M and an $s \in \{0,\dots,t\}$ such that the following conditions are satisfied:

(1) For each $i \in \{1,\dots,s\}$ the following statements hold:
   (a) We have $\Phi_i(U_i \cap \Sigma) = V_i \cap (\{0\} \times \mathbb{R}^{n-1})$.
   (b) The chart Φ_i is orientation-preserving and $\Phi_i\colon U_i \cap \Sigma \to V_i \cap (\{0\} \times \mathbb{R}^{n-1})$ is orientation-preserving.

(2) For each $i \in \{s+1,\dots,t\}$ we have $U_i \cap \Sigma = \varnothing$.



Proof. Since Σ is a proper submanifold we see that for each $x \in \Sigma$ we can find a chart $\Phi_x\colon U_x \to B^n$ that contains x and that satisfies (1a). It follows from $n \geq 2$ that, after possibly applying reflections, we can also arrange that (1b) holds. Furthermore, since Σ is a closed subset and since $\partial M = \varnothing$ we see that for each $x \in M \setminus \Sigma$ we can find a chart $\Phi_x\colon U_x \to B^n$ such that $U_x \cap \Sigma = \varnothing$.

If M is compact, then we can cover M by finitely many of these charts and we are done. In the non-compact case we need to work a little bit harder. More precisely, we need to apply the argument of the Small Atlas Proposition 33.10 which in turn relied on the Covering Dimension Refinement Theorem 15.15. We leave it to the reader to make the (few) modifications of the Small Atlas Proposition 33.10 to provide us with the desired proof of the claim. ⊞

Next let us apply the above Manifold–Partitions-of-Unity Theorem 27.2 to the open cover $U_1,\dots,U_t$ of M. We obtain a partition of unity $\{\nu_i\colon M \to [0,1]\}_{i=1,\dots,t}$ such that for each i we have $\operatorname{supp}(\nu_i) \subseteq U_i$. For $i = 1,\dots,t$ we define

$$\widetilde{\Phi}_i\colon M \to \mathbb{R}^n,\quad x \mapsto \begin{cases} \Phi_i(x)\cdot\nu_i(x), & \text{if } x \in U_i,\\ 0, & \text{else.}\end{cases}$$

For $i = 1,\dots,s$ we denote by $\zeta_i\colon U_i \to \mathbb{R}$ the map that is given by composing the chart $\Phi_i\colon U_i \to V_i$ with the projection $V_i \to \mathbb{R}$ onto the first coordinate and we set

$$\widetilde{\zeta}_i\colon M \to \mathbb{R},\quad x \mapsto \begin{cases} \zeta_i(x)\cdot\nu_i(x), & \text{if } x \in U_i,\\ 0, & \text{else.}\end{cases}$$

Claim 2. The map
$$\begin{aligned}\Theta\colon M &\to \mathbb{R}^{1+t\cdot n+t}\\ x &\mapsto \Big(\sum_{i=1}^{s}\widetilde{\zeta}_i(x), \widetilde{\Phi}_1(x),\dots,\widetilde{\Phi}_t(x),\nu_1(x),\dots,\nu_t(x)\Big)\end{aligned}$$
is an injective immersion with $\Theta(\Sigma)\subseteq\{0\}\times\mathbb{R}^{t\cdot n+n}$ such that for each $x\in\Sigma$ we have $\mathrm{D}\Theta_x(\mathrm{T}_xM)\not\subseteq\{0\}\times\mathbb{R}^{t\cdot n+n}$.

Proof. The proof of the Euclidean Embedding Theorem 33.2 shows that Θ is injective and an immersion. By construction we have $\Theta(\Sigma)\subseteq\{0\}\times\mathbb{R}^{t\cdot n+t}$. Finally let $x\in\Sigma$. Since Σ is a submanifold of codimension one we can pick $v\in\mathrm{T}_xM$ such that $\mathrm{T}_xM=\mathbb{R}\cdot v\oplus\mathrm{T}_x\Sigma$. After possibly flipping v we can arrange that if $w_1,\dots,w_{n-1}$ is a positive basis of $\mathrm{T}_x\Sigma$, then $v,w_1,\dots,w_{n-1}$ is a positive basis of T_xM. It follows easily from our choice of Φ_i, $i=1,\dots,s$ (which we chose to preserve orientations of M and of Σ) that for $i=1,\dots,s$ we have $\mathrm{D}\zeta_i(v)\in\mathbb{R}_{>0}$. It follows that
$$\mathrm{D}\Big(\sum_{i=1}^{s}\zeta_i\cdot\nu_i\Big)_x(v) \overset{\text{follows from the Product Rule}}{=} \sum_{i=1}^{s}\underbrace{\mathrm{D}(\zeta_i)_x(v)}_{>0}\cdot\underbrace{\nu_i(x)}_{\geq 0}+\sum_{i=1}^{s}\underbrace{\zeta_i(x)}_{=0 \text{ since } x\in\Sigma}\cdot\mathrm{D}(\nu_i)_x(v) \overset{\text{since at least one } \nu_i(x)>0}{>} 0.$$
It follows that $\mathrm{D}\Theta_x(v)\not\subseteq\{0\}\times\mathbb{R}^{s\cdot n+s}$, which implies that $\mathrm{D}\Theta_x(\mathrm{T}_xM)\not\subseteq\{0\}\times\mathbb{R}^{t\cdot n+n}$. ⊞

If M is compact, then we obtain from the Compact-Hausdorff Proposition 4.23 (3) that $\Theta\colon M\to\mathbb{R}^{1+t\cdot n+t}$ is an embedding. By Definition 30.9 a smooth embedding is a map that is an immersion and an embedding. This shows that we are done if M is compact.

If M is non-compact, then again we have to go into extra time. First recall that by the Manifold Exhaustion Lemma 27.4 we know that there exists a smooth exhaustion function $f\colon M\to\mathbb{R}_{\geq 0}$, i.e. there exists a smooth map $f\colon M\to[0,\infty)$ such that the preimage of every compact interval is compact. Next we consider the map
$$\begin{aligned}(\Theta,f)\colon M &\to \mathbb{R}\times\mathbb{R}^n=\mathbb{R}^{n+1}\\ x &\mapsto (\Theta(x),f(x)).\end{aligned}$$
In the Stretch-to-Embedding Lemma 33.11 we showed that (Θ,f) is a proper smooth embedding. One can now verify easily that this map has all the desired properties. ■

Proof of the Bicollar Neighborhood Theorem 55.2. [609] Let M be an oriented n-dimensional smooth manifold and let $\Sigma\subseteq M$ be a proper oriented codimension-one submanifold of M with $\partial\Sigma=\varnothing$. We make two initial observations:

(1) It follows from the Doubling-Manifold Lemma 54.10 that, after possibly replacing M by the double $\mathrm{D}M$, it suffices to deal with the case that the boundary of M is empty.
(2) It follows from (1), the Euclidean Submanifold Embedding Lemma 55.3 and the Smooth Embedding Theorem 30.15 that we can assume that M is actually a proper smooth submanifold of some $\mathbb{R}^N$ such that $\Sigma\subseteq\{0\}\times\mathbb{R}^{N-1}$ and such that for each $x\in\Sigma$ we have $\widetilde{\mathrm{T}}_xM\not\subseteq\{0\}\times\mathbb{R}^{N-1}$.

[609] As the name "bicollar" already suggests there are many similarities between collars and bicollars. In particular the proof of the Bicollar Neighborhood Theorem 55.2 is quite similar to the proof of the Smooth Collar Neighborhood Theorem 35.6.

Since M is a proper smooth submanifold of $\mathbb{R}^N$ we obtain from the Neighborhood Projection Theorem 35.2 that there exists a smooth map[610] $\epsilon\colon M \to (0,\infty)$ such that the following conditions are satisfied:

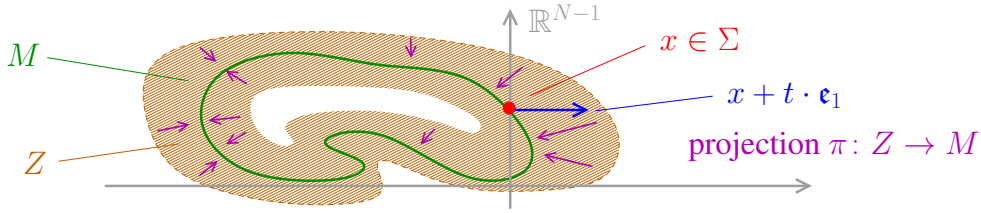
(1) The set
$$Z \coloneqq \{x + w \,|\, x \in M \text{ and } w \in (\widetilde{\mathrm{T}}_x M)^\perp \text{ with } \|w\| < \epsilon(x)\} \subseteq \mathbb{R}^N$$
is an open neighborhood of $M \subseteq \mathbb{R}^N$.

(2) The map
$$\begin{array}{rcl} \pi\colon Z & \to & M \\ x + w & \mapsto & x \end{array} \quad \text{where } x \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| < \epsilon(x)$$
is well-defined and it is smooth. Note that it follows fairly easily from the definitions that for any $y \in M$ the differential $\mathrm{D}\pi_y\colon \widetilde{\mathrm{T}}_y Z \to \widetilde{\mathrm{T}}_y M$ is just given by the orthogonal projection to $\widetilde{\mathrm{T}}_y M$.

Claim 1. There exists a smooth function $\mu\colon \Sigma \to \mathbb{R}_{>0}$ such that for every $x \in \Sigma$ and every $t \in (-\mu(x), \mu(x))$ we have $x + t \cdot \mathfrak{e}_1 \in Z$.

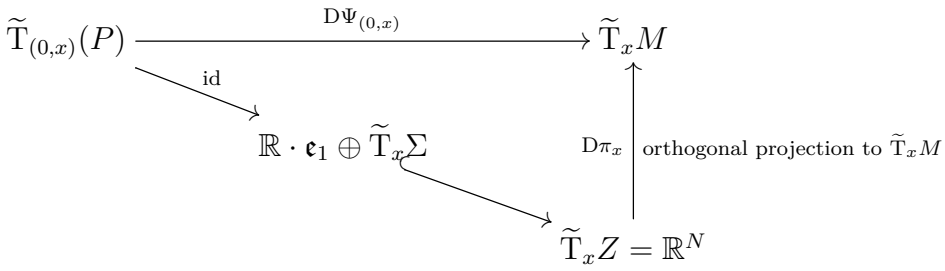
Proof. Let $x \in Z$. We consider the continuous map $\mathbb{R} \times M \to \mathbb{R}^N$ that given by $x \mapsto x + t \cdot \mathfrak{e}_1$. Since Z is an open neighborhood of Σ we see that the preimage of Z under the above map is an open neighborhood of $\{0\} \times \Sigma \subseteq \mathbb{R} \times \Sigma$. Thus there exists a $\nu(x) > 0$ such that for all $t \in (-\nu(x), \nu(x))$ we have $x + t \cdot \mathfrak{e}_1 \in Z$. The existence of the smooth map μ is now a consequence of the Replace-by-Smooth Non-Negative Function Corollary 27.14 (2) applied to the map $\nu\colon \Sigma \to \mathbb{R}_{>0}$. ⊞

We set
$$P \coloneqq \{(t, x) \in \mathbb{R} \times \Sigma \,|\, |x| < \mu(x)\}$$
and we consider the map
$$\begin{array}{rcl} \Psi\colon P & \to & M \\ (t, x) & \mapsto & \pi(x + t \cdot \mathfrak{e}_1). \end{array}$$



Claim 2. Given any $x \in \Sigma$ the differential $\mathrm{D}\Psi_{(0,x)}\colon \widetilde{\mathrm{T}}_{(0,x)}(P) \to \widetilde{\mathrm{T}}_x \mathbb{R}^N$ is a monomorphism.

Proof. First we point out that the restriction of Ψ to $\{0\} \times \Sigma$ defines a smooth embedding $\{0\} \times \Sigma \to \{0\} \times \mathbb{R}^{N-1}$. Thus it suffices to show that the image of $\mathrm{D}\Psi_{(0,x)}$ is not contained in $\{0\} \times \mathbb{R}^{N-1}$. We consider the following commutative diagram:



[610] If M is compact, then we can choose ϵ to be a constant map.

We consider $v := \mathrm{D}\pi_x(\mathfrak{e}_1)$. As we discussed above, v is the orthogonal projection of $\mathfrak{e}_1$ to $\widetilde{\mathrm{T}}_y M$. By definition of the orthogonal projection to $\widetilde{\mathrm{T}}_x M$ this means that v has the following two properties:

(a) $v \in \widetilde{\mathrm{T}}_x M$.
(b) $v - \mathfrak{e}_1$ is orthogonal to $\widetilde{\mathrm{T}}_x M$.

It follows from the fact that $\widetilde{\mathrm{T}}_x M \not\subseteq \{0\} \times \mathbb{R}^{N-1}$ and (b) that $v \neq 0$. From (b) it also follows that $\langle v - \mathfrak{e}_1, v\rangle = 0$. But this implies that $\langle v, \mathfrak{e}_1\rangle = \langle v, v\rangle$. It follows from $v \neq 0$ that $\langle v, \mathfrak{e}_1\rangle \neq 0$. This implies that $v = \mathrm{D}\pi_x(\mathfrak{e}_1) \notin \{0\} \times \mathbb{R}^{N-1}$. By (a) we have thus shown that $\widetilde{\mathrm{T}}_x M \not\subseteq \{0\} \times \mathbb{R}^{N-1}$.

It follows from this observation and the above commutative diagram that the image of $\mathrm{D}\Psi_{(0,x)}$ is not contained in $\{0\} \times \mathbb{R}^{N-1}$. ⊞

It follows easily from Claim 2, the Local Smooth Embedding Theorem 30.13, the Neighborhood Smooth Embedding Theorem 27.16 and the Replace-by-Smooth Non-Negative Function Corollary 27.14 (2) that there exists a smooth map $\nu\colon \Sigma \to \mathbb{R}_{>0}$ such that for each $x \in \Sigma$ we have $\nu(x) \leq \mu(x)$ and such that

$$\Psi\colon \{(t,x) \in \mathbb{R} \times \Sigma \,|\, |t| < \nu(x)\} \;\to\; M$$

is a smooth embedding. After possibly flipping in the first coordinate we can assume that the map is orientation-preserving. Finally, using the argument of the Restrict-to-Closed Image Lemma 35.10 one can show that there exists a smooth function $\omega\colon \Sigma \to \mathbb{R}_{>0}$ such that for each $x \in \Sigma$ we have $\omega(x) \leq \nu(x)$ and such that the map

$$\begin{array}{rcl} [0,1] \times \Sigma & \to & M \\ (t,x) & \mapsto & \Psi(x, \omega(x) \cdot \nu(x) \cdot t) \end{array}$$

is an orientation-preserving smooth embedding *with closed image*. It it clear that this map is now our desired bicollar. ■

We also have the following analog of the Smooth Collar-Uniqueness Proposition 54.2.

Proposition 55.4. (Bicollar-Uniqueness Proposition) Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a closed[611] oriented codimension-one submanifold. Furthermore let $\beta_0\colon [-\epsilon_0, \epsilon_0] \times \Sigma \to M$ and $\beta_1\colon [-\epsilon_1, \epsilon_1] \times \Sigma \to M$ be two orientation-preserving bicollars for Σ. There exists a diffeotopy

$$H\colon M \times [0,1] \;\to\; M$$

rel Σ from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1\colon M \to M$ which has the property that $(\Phi \circ \beta_0)(t,b) = \beta_1(\frac{\epsilon_1}{\epsilon_0} \cdot t, b)$ for all $(t,b) \in [-\epsilon_0, \epsilon_0] \times \Sigma$.

Sketch of proof. First note that after rescaling we can assume that $\epsilon_0 = \epsilon_1 = 1$. We denote by $\overline{\beta}_1$ the bicollar that is defined by $\overline{\beta}_1(t,x) = \beta_1(-t,x)$. Basically the same argument as in the proof of the Smooth Collar-Uniqueness Proposition 54.2 shows that

[611] In this proposition we now restrict ourselves to closed submanifolds, i.e. proper submanifolds, which have empty boundary and are compact. The case of a compact submanifold with non-empty boundary is discussed in the Bicollar-Uniqueness Proposition 59.17. But since later on we only need existence and not the uniqueness statement for non-compact submanifolds we refrain from carrying out the proof for the more general statement.

there exists a diffeotopy

$$H\colon M \times [0,1] \ \to \ M$$

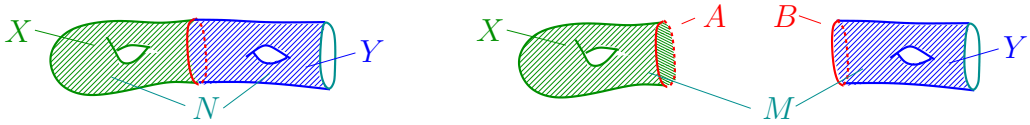
rel Σ from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1\colon M \to M$ such that for each component $\widetilde{\Sigma}$ of Σ we have $\Phi \circ \beta_0|_{[-1,1]\times\widetilde{\Sigma}} = \beta_1|_{[-1,1]\times\widetilde{\Sigma}}$ or $\Phi \circ \beta_0|_{[-1,1]\times\widetilde{\Sigma}} = \overline{\beta}_1|_{[-1,1]\times\widetilde{\Sigma}}$. Since we know that $H_0 = \mathrm{id}_M$ we obtain from the Smooth Isotopy-Orientation Proposition 42.4 that $\Phi = H_1$ is also orientation-preserving. Since β_0 and β_1 are both orientation-preserving we see, fortunately, that the second $\Phi \circ \beta_0|_{[-1,1]\times\widetilde{\Sigma}} = \overline{\beta}_1|_{[-1,1]\times\widetilde{\Sigma}}$ can never occur. ■

55.2. **Gluing codimension-zero submanifolds.** The following lemma is the obvious analog of the Regluing Topological Manifold Lemma 54.4. It can be viewed as a converse to the Gluing-Smooth Manifolds-Proposition 54.3 (3b):

Lemma 55.5. (Regluing Smooth Manifold Lemma) Let N be a smooth manifold and let $X, Y \subseteq N$ be two closed codimension-zero submanifolds such that $X \cap Y$ is a union of boundary components $A \subseteq \partial X$ and a union of boundary components $B \subseteq \partial Y$. We set $M := X \sqcup Y = (X \times \{1\}) \cup (Y \times \{2\})$. Let $f\colon A \times \{1\} \to B \times \{2\}$ be the diffeomorphism that is given by $(a,1) \mapsto (a,2)$. For a suitable choice of a collar for ∂M we see that the map

$$\begin{array}{rcl} M/a \sim f(a) & \to & N \\ [(x,i)] & \mapsto & x \end{array}$$

is a diffeomorphism.



Sketch of proof. We set $\Sigma := X \cap Y$. Using collars for $\Sigma \subseteq \partial X$ and for $\Sigma \subseteq \partial Y$ (which exist by the Collar Neighborhood Theorem 35.6) one can show that there exists an orientable open neighborhood U of $\Sigma \subseteq M$. It thus follows from the Bicollar Neighborhood Theorem 55.2 that $\Sigma \subseteq U$ admits a bicollar $\beta\colon [-1,1] \times \Sigma \to U \subseteq M$.

Note that β restricts to a collar $\chi\colon [0,1] \times \Sigma \to X$ for $A \subseteq \partial X$ and that it restricts to a collar $\lambda\colon [0,1] \times \Sigma \to Y$ for $B \subseteq \partial Y$. We use these collars and the Gluing-Smooth Manifolds-Proposition 54.5 to define the smooth manifold structure on $M/a \sim f(a)$. It now follows easily from the characterization of the Gluing-Smooth Manifolds-Proposition 54.5 (1a) together with the Inverse Mapping Theorem 30.1 (2) that the map

$$\begin{array}{rcl} M/a \sim f(a) & \to & N \\ [(x,i)] & \mapsto & x \end{array}$$

is indeed a diffeomorphism. ■

55.3. **Cutting along submanifolds.** In this section we will introduce the operation of cutting along a codimension-one submanifold. This construction can be seen as the inverse of the gluing operation that we introduced in Section 54.2.

Definition 55.6. Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a closed[612] oriented submanifold of codimension one. By the Bicollar Neighborhood Theorem 55.2 we

know that there exists an orientation-preserving bicollar $\beta\colon [-\epsilon,\epsilon]\times\Sigma \to M$. We define[613]

$$M \setminus\!\setminus \Sigma \;:=\; (M \setminus \Sigma) \sqcup (\Sigma \times \{\pm 0\})$$

with the topology that is generated by the following sets:

- The usual open subsets of $M \setminus \Sigma$.
- Sets of the following form:

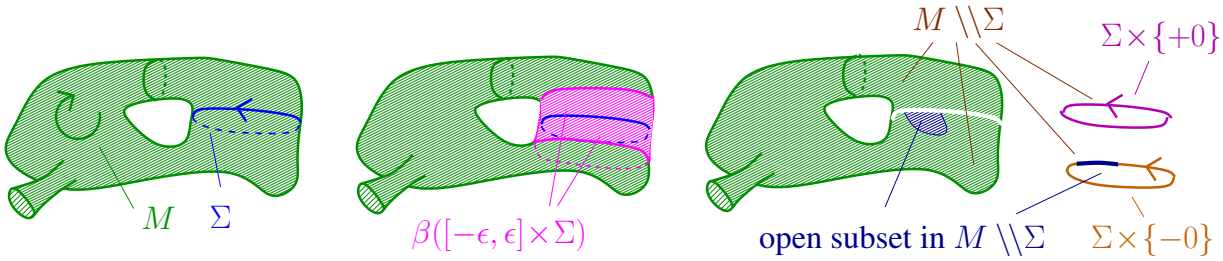
$$\begin{array}{rl} (U \cap \beta((0,\epsilon)\times\Sigma)) \cup ((U\cap\Sigma)\times\{+0\}) & \text{with } U \text{ an open subset of } \beta([0,\epsilon)\times\Sigma) \\ (U \cap \beta((-\epsilon,0)\times\Sigma)) \cup ((U\cap\Sigma)\times\{-0\}) & \text{with } U \text{ an open subset of } \beta((-\epsilon,0])\times\Sigma). \end{array}$$

We introduce the following objects:

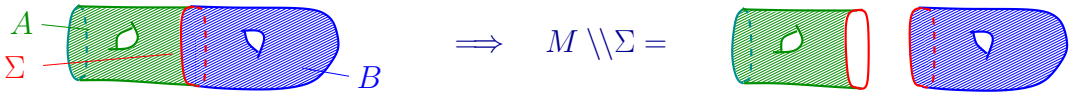
(1) We refer to $M \setminus\!\setminus \Sigma$ as M cut along Σ.
(2) We denote by $\Sigma_\pm$ the images of $\Sigma \times \{\pm 0\}$ in $M \setminus\!\setminus \Sigma$. .
(3) We denote by $i_\pm\colon \Sigma_\pm \to M \setminus\!\setminus \Sigma$ the obvious inclusion maps.
(4) We refer to the map

$$\begin{array}{rccl} \pi\colon M \setminus\!\setminus \Sigma & \to & M & \\ x & \mapsto & \begin{cases} x, & \text{if } x \in M\setminus\Sigma, \\ y, & \text{if } x = (y,\pm 0) \text{ for some } x \in \Sigma \end{cases} \end{array}$$

as the natural map from $M \setminus\!\setminus$ to M.



Example. Let M be a smooth manifold and let $A, B \subseteq M$ be codimension-zero smooth submanifolds such that $\Sigma := A \cap B$ is a union of components of ∂A and also a union of components of ∂B.



It follows easily from the definitions that $M \setminus\!\setminus \Sigma$ is homeomorphic to $A \sqcup B$. Once we have talked about smooth structures on $M \setminus\!\setminus \Sigma$ we will see in the Cutting Decomposed Smooth Manifold Lemma 55.9 that we can upgrade "homeomorphic" to "diffeomorphic". □

The following proposition summarizes some of the key properties of this construction.

[612] In Definition 59.18 we will generalize this notion to proper submanifolds.
[613] Here ± 0 are just understood to be the two symbols $+0$ and -0 with no further meaning.

Proposition 55.7. (Cutting Manifold Proposition) Let M be an oriented n-dimensional smooth manifold and let $\Sigma \subseteq M$ be a closed[614] oriented submanifold of codimension one.

(0) (a) The definition of the topological space $M \setminus\setminus \Sigma$ does not depend on the choice of the bicollar.[615]
(b) The maps $\iota_\pm \colon \Sigma_\pm \to M \setminus\setminus \Sigma$ and $\pi \colon M \setminus\setminus \Sigma \to M$ are continuous.
(1) The topological space $M \setminus\setminus \Sigma$ is a topological manifold. It admits a unique structure as an n-dimensional smooth manifold such that the natural map $\pi \colon M \setminus\setminus \Sigma \to M$ is smooth and such that at every point the differential of π is an isomorphism.
(2) With this smooth structure we have $\partial(M \setminus\setminus \Sigma) = \partial M \sqcup \Sigma_- \sqcup \Sigma_+$.
(3) Let $\beta \colon [-\epsilon, \epsilon] \times \Sigma \to M$ be an orientation-preserving bicollar. The following two statements hold:
(a) There exists a diffeomorphism[616] $M \setminus \beta((-\epsilon, \epsilon) \times \Sigma) \to M \setminus\setminus \Sigma$ which sends $\Sigma_\pm$ to $\beta(\{\pm\epsilon\} \times \Sigma)$.
(b) $M \setminus \beta((-\epsilon, \epsilon) \times \Sigma)$ is a deformation retract of $M \setminus\setminus \Sigma$.

Remark.

(1) In the figures we will use the Cutting Manifold Proposition 55.7 (3a) as an excuse to draw $M \setminus\setminus \Sigma$ the same way that we would draw $M \setminus \beta((-1, 1) \times \Sigma)$.
(2) In the literature the smooth manifold $M \setminus \beta((-1, 1) \times \Sigma)$ sometimes is also called "M cut along Σ". This name is partly justified by the Cutting Manifold Proposition 55.7 (3). But one should note that our notion of "M cut along Σ" is much more elegant since the definition does not depend on the choice of a bicollar.
(3) A slightly different approach to defining the operation of "cutting along a codimension one manifold" is given in [**Wall2016**, p. 65]. As we will see in Exercise 55.2, the two approaches essentially agree in our setting. □

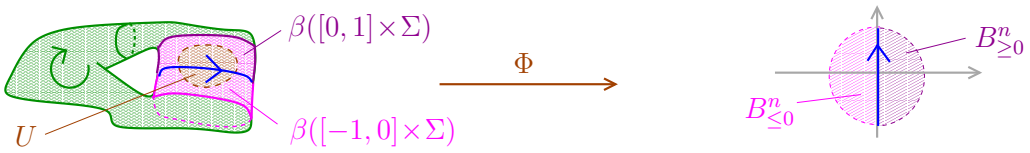
The proof of the Cutting Manifold Proposition 55.7 will require the remainder of this section. In the next Section 55.4 we will discuss a frequently used special case of cutting along a codimension-one submanifold.

The proof of the Cutting Manifold Proposition 55.7 relies on the following lemma:

Lemma 55.8. (Bicollar–Sides Lemma) Let M be an oriented n-dimensional smooth manifold and let $\Sigma \subseteq M$ be a closed oriented codimension-one submanifold.

(1) Let $\beta \colon [-1, 1] \times \Sigma \to M$ be an orientation-preserving bicollar for Σ. Furthermore let $\Phi \colon U \to B^n$ be an orientation-preserving chart, such that $U \subseteq \beta((-1, 1) \times \Sigma)$, such that $\Phi^{-1}(\{0\} \times B^{n-1}) = \Sigma$[617] and such that $\Phi \colon \Sigma \to \{0\} \times B^{n-1}$ is also orientation-preserving. Then $\Phi(\beta([0, 1) \times \Sigma) \cap U) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\Phi(\beta((-1, 0] \times \Sigma) \cap U) \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$.
(2) Let $\beta_0, \beta_1 \colon [-1, 1] \times \Sigma \to M$ be two orientation-preserving bicollars for Σ. There exists an $\epsilon \in (0, 1]$ such that $\beta_1([0, \epsilon) \times \Sigma) \subseteq \beta_0([0, 1] \times \Sigma)$.

[614] In the Cutting Manifold Proposition 59.19 we will discuss the case that Σ has non-empty boundary.
[615] Note that here "does not depend" is meant in the strong sense, that different choices of an orientation-preserving bicollar give precisely the same topological space.
[616] Note that we know by the Codimension-Zero Smooth Embedding Proposition 30.21 that the left-hand side is a smooth manifold.



Proof of the Bicollar–Sides Lemma 55.8. We set $n \coloneqq \dim(M)$.

(1) We write $U_{\geq 0} = \beta([0,1)\times\Sigma) \cap U$ and $B^n_{\geq 0} = \{(x_1,\dots,x_n) \in B^n \mid x_1 \geq 0\}$, and we use obvious variations on these two definitions.

Claim. We have $B^n_{>0} = \Phi(U_{>0})$ or $B^n_{>0} = \Phi(U_{<0})$.

Proof. From $U \subseteq \beta((-1,1)\times\Sigma)$ it follows that[618] $U = U_{>0} \sqcup U_{=0} \sqcup U_{<0}$. Note that $U_{>0}$ and $U_{<0}$ are disjoint open subsets of U. From $\Phi^{-1}(B^n_{=0}) = U_{=0}$ it follows immediately that $B^n_{>0} \subseteq \Phi(U_{>0}) \sqcup \Phi(U_{<0})$. Since $B^n_{>0}$ is connected we see that $B^n_{>0} \subseteq \Phi(U_{>0})$ or $B^n_{>0} \subseteq \Phi(U_{<0})$. Similarly we have $B^n_{<0} \subseteq \Phi(U_{>0})$ or $B^n_{<0} \subseteq \Phi(U_{<0})$. The claim follows from this discussion and the observation that $\Phi\colon U_{\neq 0} \to B_{\neq 0}$ is a homeomorphism (in fact we only need that it is a bijection). ⊞

First note that by considering closures we obtain easily from the above claim that $B^n_{\geq 0} = \Phi(U_{\geq 0})$ or that $B^n_{\geq 0} = \Phi(U_{\leq 0})$. Next note that it follows fairly easily from our extra information on orientations that only the first case of the claim can occur. But this implies that $\Phi(\beta([0,1)\times\Sigma)\cap U) = \Phi(U_{\geq 0}) = B^n_{\geq 0} \subseteq \mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}$.

(2) It follows from the compactness of Σ and from the Tube Lemma 7.12 that it suffices to prove the following claim:

Claim. For each $x \in \Sigma$ there exists an open neighborhood V of $(0,x) \in [0,1]\times\Sigma$ such that $\beta_1(V) \subseteq \beta_0([0,1]\times\Sigma)$.

Proof. Let $x \in \Sigma$. Clearly we can pick submanifold chart $\Phi\colon U \to B^n$ with the properties described in (1). After a possibly composing Φ with the reflection in the $(x_1 = 0)$-hyperplane we can arrange that Φ is furthermore orientation-preserving. Since $\beta_0, \beta_1\colon [-1,1]\times\Sigma \to M$ are both orientation-preserving we obtain from (1) that for $i = 1,2$ we have $\beta_i([0,1]\times\Sigma) \subseteq \mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}$ and that $\beta_i([-1,0]\times\Sigma) \subseteq \mathbb{R}_{\leq 0}\times\mathbb{R}^{n-1}$. It follows in particular that $\beta_0([0,1)\times\Sigma) \subseteq \mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}$ is an open neighborhood of $\Phi(x) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1}$. We now see that $V \coloneqq \beta_1^{-1}(\beta_0([0,1)\times\Sigma))$ has the desired property. ■

Proof of the Cutting Manifold Proposition 55.7. Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a closed oriented codimension-one submanifold.

(0) (a) Recall that we need to prove the following claim:

Claim. The definition of the topological space $M \setminus\setminus \Sigma$ does not depend on the choice of the bicollar.

Proof. Let $\beta_0\colon [0,1]\times[-\epsilon_0,\epsilon_0]\times\Sigma \to M$ and let $\beta_1\colon [0,1]\times[-\epsilon_1,\epsilon_1]\times\Sigma \to M$ be two orientation-preserving bicollars for Σ. After rescaling we can and will assume that $\epsilon_0 = \epsilon_1$. For the purpose of this proof we consider the topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ on the set $M \setminus\setminus \Sigma$ that are defined by β_0 and β_1. We need to show that the topologies

[617] This just means that Φ is a submanifold chart for $\Sigma \subseteq M$.

[618] Here we mean disjoint union of sets, not of topological spaces.

agree. By symmetry it suffices to show that $\mathcal{T}_0 \subseteq \mathcal{T}_1$. Thus let $U \in \mathcal{T}_0$. We need to show that $U \in \mathcal{T}_1$. By the JH-Openness Criterion 3.3 it suffices to show that for each $x \in U$ there exists an open subset $V \in \mathcal{T}_1$ with $x \in V \subseteq U$. If $x \in M \setminus \Sigma$ then this is almost obvious. Thus let us consider the slightly more interesting case that say $x \in \Sigma_+$. This means that $x = (y, +0)$ for some $y \in \Sigma$. The fact that $U \in \mathcal{T}_0$ implies that there exists an open subset W_0 of $\beta_0([0,1) \times \Sigma)$ with $y \in W_0$.
It follows from the Bicollar–Sides Lemma 55.8 (2) that there exists an $\epsilon \in (0,1]$ such that $\beta_1([0,\epsilon) \times \Sigma) \subseteq \beta_0([0,1] \times \Sigma)$. We set $W_1 := \beta_1([0,\epsilon) \times \Sigma) \cap W_0$. Note that $y \in W_1$. Since W_0 is open in $\beta_0([0,1) \times \Sigma)$ and since $\beta_1([0,\epsilon) \times \Sigma) \subseteq \beta_0([0,1) \times \Sigma)$ we see that W_1 is open in $\beta_1([0,\epsilon) \times \Sigma)$, and thus also in $\beta_1([0,1) \times \Sigma)$. It follows that W_1 gives rise to an open subset $V \in \mathcal{T}_1$ with $x \in V \subseteq U$.
The same way we of course deal with points in Σ_-. ⊞

(b) It follows immediately from the definitions that the maps $\iota_\pm \colon \Sigma_\pm \to M \setminus\!\setminus \Sigma$ are continuous. Furthermore note that it is straightforward to verify that the map $\pi \colon M \setminus\!\setminus \Sigma \to M$ is continuous on the open subsets $M \setminus \Sigma$ and $\beta((-\epsilon,\epsilon) \times \Sigma) \cup \Sigma_- \cup \Sigma_+$. It follows from the Pasting Proposition 4.7 (1) that π is continuous.

(1) We pick an orientation-preserving bicollar $\beta \colon [-\epsilon,\epsilon] \times \Sigma \to M$. Using this bicollar, and using the M, Σ and $[0,\epsilon]$ are Hausdorff and second-countable it is not hard to show that $M \setminus\!\setminus \Sigma$ is Hausdorff and second-countable. So let us concentrate on the more interesting task of producing a smooth atlas:

Claim. The topological manifold $M \setminus\!\setminus \Sigma$ admits a unique structure as an n-dimensional smooth manifold such that the natural map $\pi \colon M \setminus\!\setminus \Sigma \to M$ is smooth and such that the differential is an isomorphism at every point.

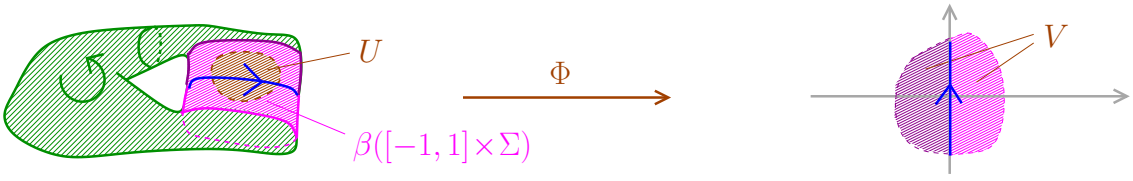
Proof. First of all note that it follows almost immediately from the fact that $\Sigma \subseteq M$ is a closed subset that $\Sigma_\pm$ are closed subsets of $M \setminus\!\setminus \Sigma$. Thus we can take all the smooth charts of M and restrict them to the open subset $(M \setminus\!\setminus \Sigma) \setminus (\Sigma_- \cup \Sigma_+)$. We still need to find charts for points in Σ_- and Σ_+. Let $x \in \Sigma_+$. This means that $x = (y, +0)$ for some $y \in \Sigma$. We pick a smooth orientation-preserving chart $\Phi \colon U \to V$ with $y \in \Sigma$ such that $\Phi(\Sigma) \subseteq \{0\} \times \mathbb{R}^{n-1}$, such that $\Phi|_\Sigma \colon \Sigma \to \{0\} \times \mathbb{R}^{n-1}$ is orientation-preserving and such that $U \subseteq \beta((-\epsilon,\epsilon) \times \Sigma)$. We set

$$U_+ \;:=\; (U \cap \beta((0,\epsilon) \times \Sigma)) \cup \{(x, 0_+) \mid x \in U \cap \Sigma\}.$$

One can now easily verify that

$$\begin{array}{rcl} \Phi_+ \colon U_+ & \to & V \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}) \\ P & \mapsto & \begin{cases} \Phi(P), & \text{if } P \in (U \cap \beta((0,\epsilon) \times \Sigma)) \\ \Phi(x), & \text{if } P = (x, 0+) \text{ for some } x \in U \cap \Sigma \end{cases} \end{array}$$

is a chart of type (ii) for $x \in M \setminus\!\setminus \Sigma$.



Almost the same way we also find charts of type (ii) for points in Σ_-. It follows almost immediately from the definitions that this atlas is smooth. We now constructed a smooth atlas with charts of type (i) for points in $(M \setminus \partial M) \cup \Sigma$ and charts of type (i) for points in ∂M and $\Sigma_- \cup \Sigma_+$. It follows from this observation together with the Unique Type-of-Chart Proposition 25.26 that $\partial(M \setminus\!\setminus \Sigma) = \partial M \cup \Sigma_- \cup \Sigma_+$. We now take the maximal smooth atlas that is generated by the above smooth atlas. One can easily verify that it has the desired properties. Finally, using the Inverse Mapping Theorem 30.1 one can show that the smooth structure is uniquely determined. ⊞

(2) it follows immediately from the explicit atlas that we constructed in (1) that $\partial(M \setminus\!\setminus \Sigma) = \partial M \sqcup \Sigma_- \sqcup \Sigma_+$.

(3) We pick an orientation-preserving bicollar $\beta\colon [-\epsilon,\epsilon]\times\Sigma \to M$. Evidently the map that is given by $\Phi(t,P) := \beta(\frac{1}{2}t, P)$ also defines an orientation-preserving bicollar. It follows from the Bicollar-Uniqueness Proposition 55.4 that it suffices to prove the statement for the bicollar Φ. We consider the map

$$\Theta\colon M \setminus\!\setminus \Sigma \to M \setminus \Phi((-\epsilon,\epsilon)\times\Sigma)$$
$$P \mapsto \begin{cases} P, & \text{if } P \in M \setminus \beta((-\epsilon,\epsilon)\times\Sigma), \\ \beta(\frac{1}{2}t + \frac{\epsilon}{2}, Q), & \text{if } P = \beta(t,Q) \text{ with } t \in (0,\epsilon), \\ \beta(\frac{1}{2}t - \frac{\epsilon}{2}, Q), & \text{if } P = \beta(t,Q) \text{ with } t \in (-\epsilon,0), \\ \beta(\pm\frac{\epsilon}{2}, Q), & \text{if } P = (Q,\pm 0) \text{ for some } Q \in \Sigma. \end{cases}$$

Note that $\Theta(\Sigma_\pm) = \Phi(\{\pm\epsilon\}\times\Sigma)$. Using the Pasting Proposition 4.7 (2) one can easily show that Θ is continuous. We leave it to the reader to write down an explicit inverse to Θ which then shows that Θ is a homeomorphism. As always, using the Smooth Transition Function Lemma 24.9 one can slightly adjust the map Θ to turn it into a diffeomorphism. We leave the details to the reader.

Finally we consider the map

$$R\colon (M \setminus\!\setminus \Sigma)\times[0,1] \to M \setminus\!\setminus \Sigma$$
$$(x,t) \mapsto \begin{cases} x, & \text{if } x \notin \Sigma_+ \cup \Sigma_- \cup \beta((-\epsilon,\epsilon)\times\Sigma), \\ x, & \text{if } x \in \Sigma_+ \cup \Sigma_- \text{ and } t = 0, \\ \beta(s\cdot(1-t) + \frac{s}{|s|}\cdot t, y), & \text{if } x = \beta(s,y) \text{ with } s \in (-\epsilon,0)\cup(0,\epsilon) \\ & \quad \text{and } x \in \Sigma, \\ \beta(\pm t, x), & \text{if } x = (x,\pm 0) \in \Sigma_- \cup \Sigma_+ \text{ and } t > 0. \end{cases}$$

We leave it to the reader to show that R is continuous.[619] Once one is satisfied with continuity it is clear that R is a deformation retraction from $M \setminus\!\setminus \Sigma$ to $M \setminus \Phi((-\epsilon,\epsilon)\times\Sigma)$. Since we will not make use of this statement we cannot be bothered with providing all the details. ■

55.4. **Cutting a manifold with a given decomposition.** The following lemma gives us an important example of cutting along a codimension-one submanifold.

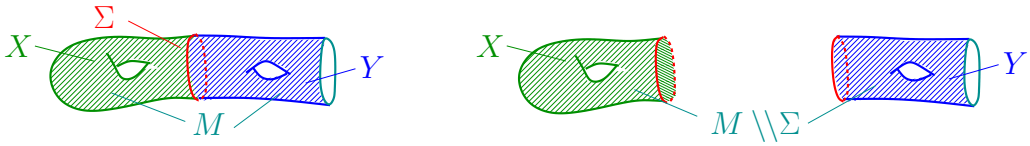
Lemma 55.9. (Cutting Decomposed Smooth Manifold Lemma) Let M be a smooth manifold and let $X, Y \subseteq N$ be two closed codimension-zero submanifolds such that

[619] Since the definition of $M \setminus\!\setminus \Sigma$ does not come from any of our usual constructions there is no real shortcut for showing that this map is continuous.

$\Sigma = X \cap Y$ is a union of boundary components of X and a union of boundary components of Y. We consider the disjoint union $X \sqcup Y := (X \times \{1\}) \cup (Y \times \{2\})$ from Notation 1.12. The map

$$\begin{array}{rcl} f\colon X \sqcup Y & \to & M \setminus\!\!\setminus \Sigma \\ (Q,i) & \mapsto & \begin{cases} Q, & \text{if } Q \notin \Sigma, \\ (Q,-), & \text{if } i = 1, \\ (Q,+), & \text{if } i = 2 \end{cases} \end{array}$$

is a diffeomorphism.



Sketch of proof. One can easily verify that the given map is a bijection. Furthermore it follows easily from the Boundary-of-Submanifold Proposition 25.48 (1), the Codimension-Zero Submanifold Proposition 25.49 (3) and the Cutting Manifold Proposition 55.7 (2) that $f(\partial(X \cup Y)) = \partial(M \setminus\!\!\setminus \Sigma)$. Finally, by the Inverse Mapping Theorem 30.1 (2) it remains to prove the following claim:

Claim. For each $P \in X \sqcup Y$ the differential $\mathrm{D}f_P$ is an isomorphism.

Proof. We consider the following simple commutative diagram

$$\begin{array}{ccc} X \sqcup Y & \xrightarrow{f} & M \setminus\!\!\setminus \Sigma \\ & {\scriptstyle \iota} \searrow & \downarrow {\scriptstyle \pi} \\ & & M \end{array}$$

and the induced commutative diagram of differentials

$$\begin{array}{ccc} \mathrm{T}_P(X \sqcup Y) & \xrightarrow{\mathrm{D}f_P} & \mathrm{T}_P(M \setminus\!\!\setminus \Sigma) \\ & {\scriptstyle \mathrm{D}\iota_P} \searrow & \downarrow {\scriptstyle \mathrm{D}\pi_P} \\ & & \mathrm{T}_P M \end{array}$$

where the diagonal map ι is induced by the inclusions $X \to M$ and $Y \to M$. Note that it follows from the characterization of the smooth structure in the Cutting Manifold Proposition 55.7 (1) that $\mathrm{D}\pi_P$ is an isomorphism and that it follows from the Tangent Space-Inclusion Proposition 29.20 (1a) that $\mathrm{D}\iota_P$ is an isomorphism. It follows that $\mathrm{D}f_P$ is indeed an isomorphism. ∎

55.5. **Gluing and cutting.** In this section we will see that gluing and cutting of smooth manifolds are essentially inverse operations.

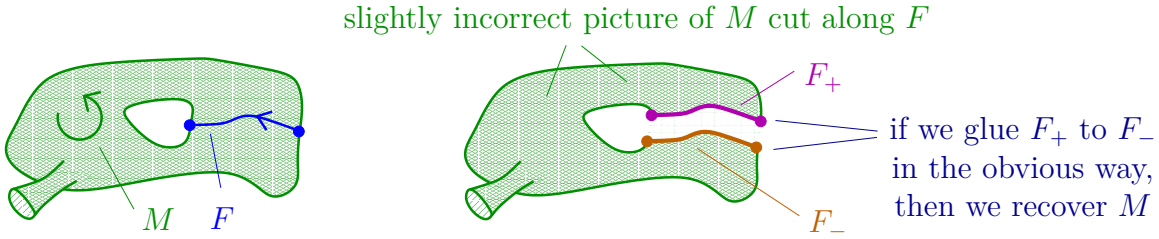
Proposition 55.10. (Regluing Proposition) Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a closed oriented submanifold of codimension one. We consider the natural map

$$\begin{array}{rcl} \chi_\Sigma\colon M \setminus\!\!\setminus \Sigma & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M \setminus \Sigma, \\ Q, & \text{if } P = [(Q, \pm 0)] \text{ for some } Q \in \Sigma \end{cases} \end{array}$$

from Definition 55.6. If we pick a bicollar $\beta\colon [-\epsilon, \epsilon] \times \Sigma \to M$ and if we equip the boundary components $\Sigma_- \cup \Sigma_+ \subseteq M \setminus\!\!\setminus \Sigma$ with the corresponding obvious collar, then the above map

χ_Σ descends to a diffeomorphism

$$\overline{\chi_\Sigma}\colon (M \setminus\!\setminus \Sigma)/\underset{\substack{\uparrow\\ \text{with } x \in \Sigma}}{(x, 0_-)} \sim (x, 0_+) \;\to\; M.$$



Proof. We pick a bicollar $\beta\colon [-\epsilon, \epsilon] \times \Sigma \to M$. We then equip $\Sigma_+ \cup \Sigma_- \subseteq \partial(M \setminus\!\setminus \Sigma)$ with the collar given by

$$\begin{array}{rcl} (\Sigma_+ \cup \Sigma_-) \times [0,1] & \to & M \setminus\!\setminus \Sigma \\ ((x,\epsilon),t) & \mapsto & \begin{cases} (x,\epsilon), & \text{if } t = 0, \\ \beta(\mu \cdot t, x), & \text{if } t > 0 \text{ and } \epsilon = 0_\mu \text{ with } \mu \in \{-,+\}. \end{cases} \end{array}$$

We equip $(M \setminus\!\setminus \Sigma)/\sim$ with the corresponding smooth structure coming from the Gluing-Smooth Manifolds-Proposition 54.5. By the Inverse Mapping Theorem 30.1 (2) it suffices to prove the following claim:

Claim. The induced map

$$\overline{\chi_\Sigma}\colon (M \setminus\!\setminus \Sigma)/(x, 0_-) \sim (x, 0_+) \;\to\; M$$

has the following properties:

(0) It is a bijection.
(1) (a) The map is smooth.
    (b) For every $P \in (M \setminus\!\setminus \Sigma)/(x, 0_-) \sim (x, 0_+)$ the differential $\mathrm{D}(\overline{\chi_\Sigma})_P$ is an isomorphism.
(2) We have $\overline{\chi_\Sigma}(\partial(M \setminus\!\setminus \Sigma)/(x, 0_-) \sim (x, 0_+)) = \partial M$.

Proof.

(0) This statement follows easily from the definitions.
(1) Both statements follow fairly easily from our choice of collar, the discussion of the smooth structures in the Cutting Manifold Proposition 55.7 (1) and the Gluing-Smooth Manifolds-Proposition 54.5 (1) and the Smooth Pasting Proposition 25.19.
(2) This follows easily from the Cutting Manifold Proposition 55.7 (2) together with the Gluing-Smooth Manifolds-Proposition 54.5 (2).

We leave it to the reader to fill in the details. ∎

The Regluing Manifold Proposition 55.10 together with the following proposition shows that cutting and gluing are indeed inverse operations.

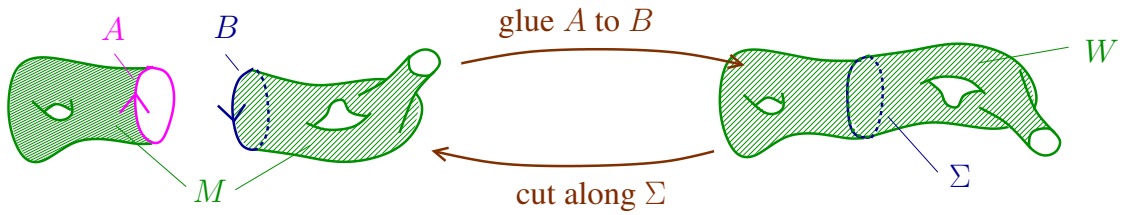
Proposition 55.11. (Recutting Proposition) Let M be an oriented smooth manifold and let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be an orientation-preserving diffeomorphism. We consider the smooth manifold $M/a \sim f(a)$ that is obtained from gluing A to B. We denote by Σ the image of A under

the map $\pi\colon A \to M \to M/a \sim f(a)$. By the Gluing-Smooth Manifolds-Proposition 54.5 we know that Σ is a closed codimension-one submanifold of the oriented smooth manifold $M/\sim$. Thus we can cut $M/\sim$ along Σ. The map

$$\begin{array}{rcl} f\colon (M/\sim)\setminus\!\setminus\Sigma & \to & M \\ x & \mapsto & \begin{cases} x, & \text{if } x \in M \setminus \Sigma, \\ a, & \text{if } x = (\pi(a), -0) \text{ for some } a \in A, \\ f(a), & \text{if } x = (\pi(a), +0) \text{ for some } a \in A \end{cases} \end{array}$$

is a diffeomorphism.

Proof. We pick a collar $\chi\colon [0,1] \times \partial M \to M$ and we use it to define the smooth structure on $M/a \sim f(a)$. This collar gives rise to a bicollar for the submanifold $\Sigma \subseteq M$ that is given by identifying A with B.



By the Inverse Mapping Theorem 30.1 (2) it suffices to prove the following claim:

Claim. The given map $f\colon (M/\sim)\setminus\!\setminus\Sigma \to M$ has the following properties:

(0) It is a bijection.
(1) (a) The map is smooth.
    (b) For every $P \in (M/\sim)\setminus\!\setminus\Sigma$ the differential $\mathrm{D}f_P$ is an isomorphism.
(2) We have $f((M/\sim)\setminus\!\setminus\Sigma) = \partial M$.

Proof.

(0) This statement follows easily from the definitions.
(1) Both statements follow fairly easily from our choice of collar, the discussion of the smooth structures in the Cutting Manifold Proposition 55.7 (1) and the Gluing-Smooth Manifolds-Proposition 54.5 (1).
(2) This follows easily from the Cutting Manifold Proposition 55.7 (2) and the Gluing-Smooth Manifolds-Proposition 54.5 (2).

We leave it to the reader to fill in the details. ∎

55.6. **Appendix: Diffeomorphisms and decompositions of smooth manifolds.** The gluing of smooth manifolds is not quite a pushout (at least not to the author's knowledge), but the following proposition is formulated in a way to make it as close as possible to the formulation of the Topological-Pushout Proposition 8.12:

Proposition 55.12. (Smooth Manifold Pushout-Proposition) Let M and N be two n-dimensional smooth manifolds, let A be an $(n-1)$-dimensional smooth manifold and let $f\colon A \to M$ and $g\colon A \to N$ be smooth maps which define a diffeomorphism between A and a union of components of ∂M respectively between A and a union of components of

∂N. Let $\chi\colon [0,\epsilon]\times f(A)\to M$ and $\lambda\colon [0,\epsilon]\times g(A)\to N$ be smooth collars. We equip

$$M\cup_A N \;:=\; (M\sqcup N)/\sim$$

with the n-dimensional smooth structure from the Gluing-Smooth Manifolds-Proposition 54.5.

(a) The natural maps $i\colon M\to M\cup_A N$ and $j\colon N\to M\cup_A N$ are closed smooth embeddings.[620]

(b) Let $\widetilde{Z}$ be a smooth manifold, let $\alpha\colon M\to\widetilde{Z}$ and $\beta\colon N\to\widetilde{Z}$ be closed *smooth* embeddings (in the sense of Definition 30.9) with $\alpha\circ f=\beta\circ g\colon A\to\widetilde{Z}$ such that the induced map $\Phi\colon M\cup_A N\to\widetilde{Z}$ is an embedding (in the sense of Definition 4.20). Then there exists an isotopy $\Theta\colon (M\cup_A N)\times[0,1]\to\widetilde{Z}$ with the following properties:

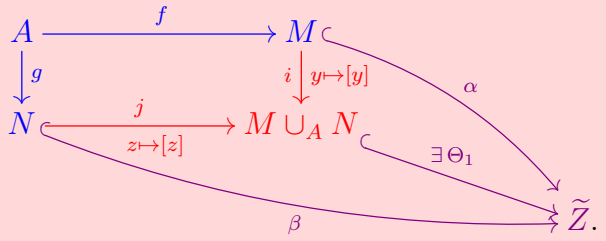
(i) $\Theta_0=\Phi$.

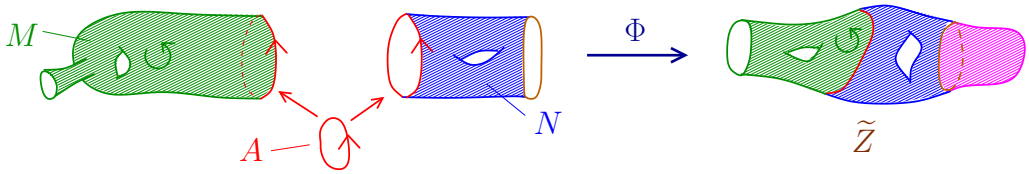
(ii) The homotopy $\Theta\circ\operatorname{diag}(i,\mathrm{id}_{[0,1]})\colon M\times[0,1]\to\widetilde{Z}$ is stationary.

(iii) The homotopy $\Theta\circ\operatorname{diag}(j,\mathrm{id}_{[0,1]})\colon N\times[0,1]\to\widetilde{Z}$ is a smooth isotopy and for each $t\in[0,1]$ we have $\Theta_t(j(N))=\Theta_0(j(N))$.

(iv) The map $\Theta_1\colon M\cup_A N\to\widetilde{Z}$ is a smooth embedding.[621]

For the reader's convenience we summarize the maps in the following "pushout-like" diagram:





The proof of the Smooth Manifold Pushout-Proposition 55.12 makes use of the following technical lemma:

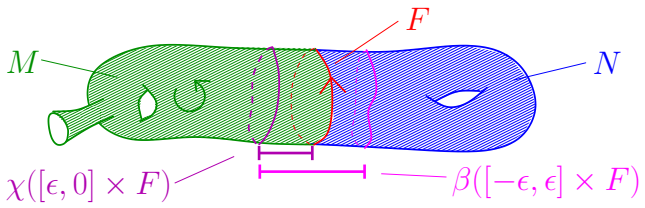
Lemma 55.13. (Collar-to-Bicollar Extension Lemma) Let Z be a compact smooth manifold and let M and N be two compact submanifolds such that $M\cup N=Z$ and such that $F:=M\cap N$ is a union of components of ∂M and a union of components of ∂N. Let $\chi\colon[0,\epsilon]\times F\to M$ be a smooth collar. Then there exists a bicollar $\beta\colon[-\epsilon,\epsilon]\times F\to Z$ such that for all $(t,P)\in[-\epsilon,0]\times F$ we have

$$\beta(t,P) \;=\; \chi(-t,P).$$

Proof of the Collar-to-Bicollar Extension Lemma 55.13. First note that it follows easily from the proof of the Bicollar Neighborhood Theorem 55.2 that there exists a bicollar

[620] It thus follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that the images of M and N are submanifolds of $M\cup_A N$.

[621] In particular, if $\Theta_0(M\cup_A N)=\widetilde{Z}$, then it follows easily from the conditions that $\Theta_1\colon M\cup_A N\to\widetilde{Z}$ is actually a diffeomorphism.



$\widetilde{\beta}\colon [-\epsilon,\epsilon]\times F \to Z$ such that the map

$$\begin{array}{rcl}\widetilde{\chi}\colon [0,\epsilon]\times F & \to & M\\ (t,x) & \mapsto & \widetilde{\chi}(t,x) := \widetilde{\beta}(-t,x)\end{array}$$

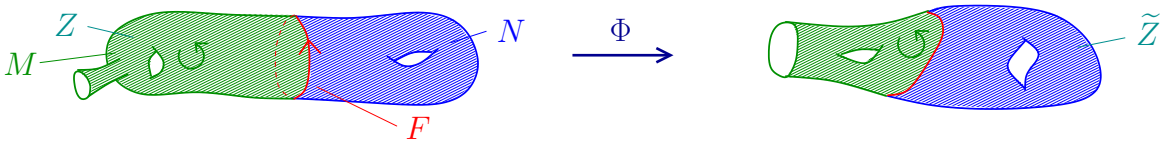
is a smooth collar for M. Next note that it follows from the Smooth Collar-Uniqueness Proposition 54.2 that there exists a diffeotopy $G\colon M\times[0,1]\to M$ with $G_0 = \mathrm{id}$ and such that $G_1\circ\widetilde{\chi} = \chi$. Finally note that it follows from the Isotopy Extension Theorem 44.2 that we can extend G to a diffeotopy $G\colon Z\times[0,1]\to Z$ with $G_0 = \mathrm{id}$. It is now clear that $\beta := G_1\circ\widetilde{\beta}\colon [-\epsilon,\epsilon]\times F\to Z$ has the desired properties. ■

Proof of the Smooth Manifold Pushout-Proposition 55.12. Let M and N be two smooth manifolds, let A be a smooth manifold and let $f\colon A\to M$ and $g\colon A\to N$ be smooth maps as above. Finally Let $\chi\colon [0,\epsilon]\times f(A)\to M$ and $\lambda\colon [0,\epsilon]\times g(A)\to N$ be smooth collars.

(a) This statement is an immediate consequence of the Gluing-Smooth Manifolds-Proposition 54.5.

(b) We start out with a few simplifications:

- We will use (a) to identify M and N with their images in $Z = M\cup_A N$. In particular note that now we have $\alpha := \Phi|_M$ and $\beta := \Phi|_N$.
- It follows easily from our hypothesis and the Codimension-Zero Smooth Embedding Proposition 30.21 that $\Phi(Z)$ is a submanifold of $\widetilde{Z}$. So might as well assume that $\Phi(Z) = \widetilde{Z}$.



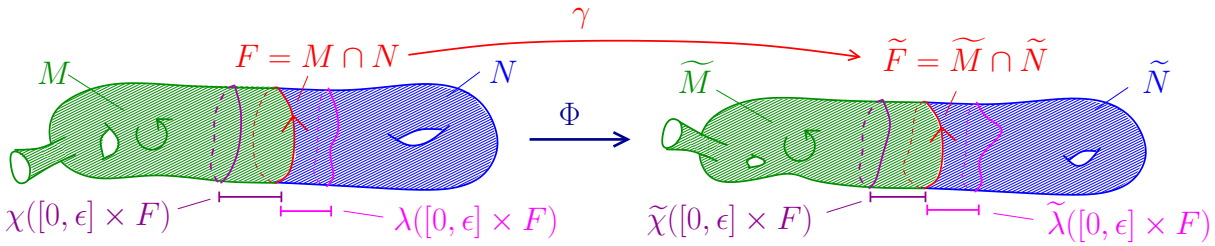
Next we introduce the following objects:

- We write $F := M\cap N$. Note that the maps χ and λ can be combined to give a map

$$\begin{array}{rcl}\beta\colon [-\epsilon,\epsilon]\times F & \to & Z\\ (t,a) & \mapsto & \begin{cases}\chi(-t,a), & \text{if } t\in[-\epsilon,0],\\ \lambda(t,a), & \text{if } t\in[0,\epsilon].\end{cases}\end{array}$$

  By construction of the smooth structure in the Gluing-Smooth Manifolds-Proposition 54.5 this map is a smooth embedding.
- We set $\widetilde{M} := \Phi(M)$ and $\widetilde{N} := \Phi(N)$. Note that it follows from our hypotheses and the Codimension-Zero Smooth Embedding Proposition 30.21 that $\widetilde{M}$ and $\widetilde{N}$ are smooth submanifolds of $\widetilde{Z}$ such that $\Phi\colon M\to\widetilde{M}$ and $\Phi\colon N\to\widetilde{N}$ are diffeomorphisms.

- We write $\widetilde{F} := \widetilde{M} \cap \widetilde{N}$ and we write $\gamma := \Phi|_F \colon F \to \widetilde{F}$.
- We consider the collars
$$\begin{array}{rcl} \widetilde{\chi} \colon [0,\epsilon] \times \widetilde{F} & \to & \widetilde{M} \\ (t, \widetilde{x}) & \mapsto & \Phi(\chi(t, \gamma^{-1}(\widetilde{x}))) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{\lambda} \colon [0,\epsilon] \times \widetilde{F} & \to & \widetilde{N} \\ (t, \widetilde{x}) & \mapsto & \Phi(\lambda(t, \gamma^{-1}(\widetilde{x}))) \end{array}$$
- Note that it follows from the Collar-to-Bicollar Extension Lemma 55.13 that there exists a bicollar $\widetilde{\beta} \colon [-\epsilon,\epsilon] \times \widetilde{F} \to \widetilde{Z}$ such that for all $(t, P) \in [-\epsilon, 0] \times F$ we have
$$\widetilde{\beta}(t, P) \;=\; \widetilde{\chi}(-t, P).$$
- As in the proof of Collar-to-Bicollar Extension Lemma 55.13 we see that it follows immediately from the Smooth Collar-Uniqueness Proposition 54.2 that there exists a diffeotopy $\widetilde{G} \colon \widetilde{N} \times [0,1] \to \widetilde{N}$ rel $\widetilde{F}$ such that $\widetilde{G}_0 = \mathrm{id}$ and $\widetilde{G}_1 \circ \widetilde{\lambda} = \widetilde{\beta}|_{[0,\epsilon] \times \widetilde{F}}$.
- We set $G := \widetilde{G} \circ \mathrm{diag}(\Phi, \mathrm{id}_{[0,1]}) \colon N \times [0,1] \to \widetilde{N}$.



Claim. The map
$$\begin{array}{rcl} \Theta \colon Z \times [0,1] & \to & \widetilde{Z} \\ (P, t) & \mapsto & \begin{cases} \alpha(P), & \text{if } P \in M, \\ \widetilde{G}(\beta(P), t), & \text{if } P \in N \end{cases} \end{array}$$
has all the desired properties.

Proof. First note that it follows from the Homotopy Pushout Lemma 18.13 that the map Θ is in fact continuous. Next note that it follows easily from the construction that Θ has the desired properties (i), (ii) and (iii). It remains to prove (iv), i.e. it remains to prove that $\Theta_1 \colon Z \to \widetilde{Z}$ is a diffeomorphism. To do so, first note that by construction the restriction of Θ_1 to the open subsets $M \setminus F$ and $N \setminus F$ is a diffeomorphism onto its image. Finally note that the restriction of Θ_1 to $\beta((-\epsilon,\epsilon) \times F)$ is given by the map
$$\beta((-\epsilon,\epsilon) \times F) \xrightarrow{\beta^{-1}} (-\epsilon,\epsilon) \times F \xrightarrow[\mathrm{diag}(\mathrm{id},\gamma)]{(t,P) \mapsto (t,\gamma(P))} (-\epsilon,\epsilon) \times \widetilde{F} \xrightarrow{\widetilde{\beta}} \widetilde{\beta}((-\epsilon,\epsilon) \times \widetilde{F}).$$
Thus we see that the restriction of Θ_1 to the open subset $\beta((-\epsilon,\epsilon) \times F)$ is also a diffeomorphism onto its image. Since we already know that Θ_1 is a homeomorphism it now follows from the Smooth Pasting Proposition 25.19 (1) and (2) that Θ_1 itself is a diffeomorphism. ∎

55.7. Exercises for Chapter 55.

Exercise 55.1.

(a) Let M be a compact smooth manifold. Show that M admits a smooth vector field with only finitely many zeros, all of which are non-degenerate.

Hint. Make use the Non-Degenerate Vector Field Existence Proposition 53.22 and use a doubling argument.

(b) Let M be a connected smooth manifold, let $F \subseteq M \setminus \partial M$ be a finite subset and let $U \subseteq M$ be an open subset. Show that there exists a diffeomorphism $\Phi\colon M \to M$ such that $\Phi(F) \subseteq U$.

(c) By (a) we know that every compact smooth manifold M admits a smooth vector field with finitely many zeros. Now let M be a compact connected smooth manifold with non-empty boundary. Show that M admits a smooth vector field that is non-zero everywhere.
Hint. Use the Smooth Collar Neighborhood Theorem 35.6.

(d) Let M be a closed connected non-empty smooth manifold. Show that M admits a smooth vector field with precisely one zero.
Hint. Use (c).

Exercise 55.2. Let M be an n-dimensional smooth manifold and let $\Sigma \subseteq M$ be a closed submanifold of codimension one. We consider the set

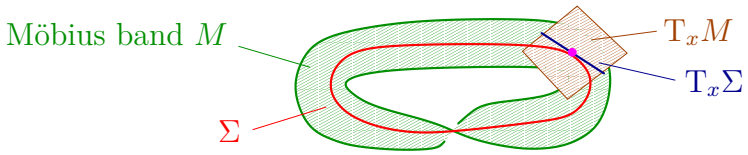
$$W \;:=\; (M \setminus \Sigma) \cup \{(x, v) \,|\, x \in \Sigma, v \text{ a component of } \mathrm{T}_x M \setminus \mathrm{T}_x \Sigma\}.$$

(a) Equip W with a suitable topology and show that W is an n-dimensional topological manifold.

(b) What is W if M is the Möbius band and Σ is the central curve?

(c) Equip W with a suitable smooth atlas and show that if M is oriented and if Σ is oriented, then W is diffeomorphic to $M \setminus\setminus \Sigma$.

Remark. This is basically the approach taken in [**Wall2016**, p. 65].



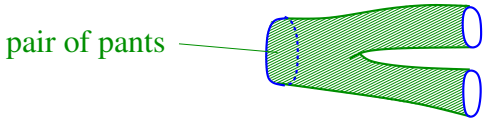
Exercise 55.3. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold of M with empty boundary. By definition a bicollar for Σ is a smooth embedding $\beta\colon [-1, 1] \times \Sigma \to M \setminus \partial M$ that has the following two properties:

(a) The restriction of β to $\{0\} \times \Sigma$ is the obvious map $(0, P) \mapsto P$.

(b) The image $\beta([-1, 1] \times \Sigma)$ is a closed subset of M.[622]

Show that (b) is necessary to get a suitable uniqueness statement for bicollars.

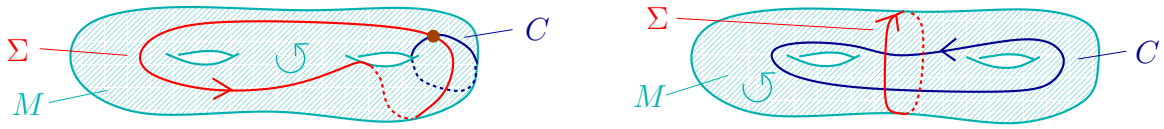
Exercise 55.4. The smooth manifold M that is obtain from S^2 by removing the interiors of three disjointly embedded disks is called a *pair of pants*. Which compact 2-dimensional smooth manifolds Σ contain a closed 1-dimensional submanifold C such that $\Sigma \setminus\setminus C$ is diffeomorphic to a disjoint union of pairs of pants?



Exercise 55.5. Let M be a compact oriented connected n-dimensional smooth manifold and let $\Sigma \subseteq M$ be a closed oriented proper codimension-one submanifold of M.

(a) Show that if $M \setminus\setminus \Sigma$ is connected, then there exists a closed oriented connected 1-dimensional smooth submanifold $C \subseteq M \setminus \partial M$ such that Σ intersects C transversally in a single point.

(b) Show that if $M \setminus\setminus \Sigma$ is not connected, then for any 1-dimensional smooth submanifold $C \subseteq M \setminus \partial M$ the algebraic intersection number $\Sigma \cdot C$ is zero.
Hint. Use the Intersection-Number Zero-if-Extends Proposition 51.8.



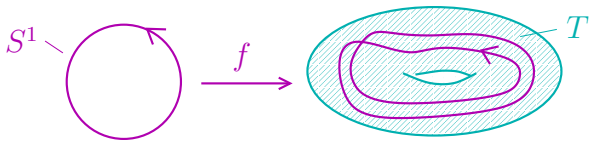
Exercise 55.6. We consider the map $f\colon S^1 \to T = S^1 \times S^1$ that is shown in the figure below. Show that f is not homotopic to a smooth embedding.
Hint.

(1) To make the exercise more precise, note that f is homotopic to the map

$$\begin{array}{ccc} S^1 & \to & T = S^1 \times S^1 \\ z & \mapsto & (z^2, 1). \end{array}$$

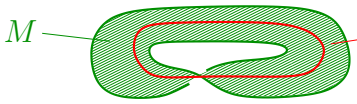
Thus it suffices to show that $\widetilde{f}$ is not homotopic to a smooth embedding.

(2) Use the algebraic intersection number of continuous maps $S^1 \to T$ and closed oriented connected 1-dimensional smooth submanifolds of T to show that $\widetilde{f}$ is not homotopic to a smooth embedding. You could make use of Exercises 51.2 and 55.5.



f ist not homotopic to a smooth embedding

Exercise 55.7. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a compact proper codimension-one submanifold. We do not assume that Σ is two-sided. Is there a sensible notion of cutting M along Σ?



does it make sense to cut M along this curve?

Exercise 55.8. Let M, N and W be compact smooth manifolds and let $f\colon \partial M \to \partial N$ be a diffeomorphism. We glue M to N via f and we obtain $M \cup_f N$. Furthermore $\varphi\colon M \cup_f N \to W$ be a homeomorphism such that $\varphi|_M\colon M \to W$ and $\varphi|_N\colon N \to W$ are smooth embeddings. Show that there exist collars for $\partial M \subseteq M$ and $\partial N \subseteq N$ such that if we use these to form the smooth manifold $M \cup_f N$, then $\varphi\colon M \cup_f N \to W$ is actually a diffeomorphism.

CHAPTER 56

Submanifolds and collars

Let W be a smooth manifold with a given collar $\chi\colon [0,1] \times \partial W \to W$ and let $M \subseteq W$ be a compact proper smooth submanifold. In this chapter we will prove the Collar-for-Submanifold Proposition 56.3 which says that we can always arrange M to be "a product" with respect to the given collar. This somewhat technical result is an essential tool in the proof of the following moderately exciting results:

(1) In the Submanifold Neighborhood Retraction Proposition 56.5 we will see that given a compact smooth manifold M and a proper smooth submanifold K of M there exists an open neighborhood Z of $K \subseteq M$ and a smooth retraction $\rho\colon Z \to K$.
(2) In the Weak Proper Whitney Embedding Theorem 56.7 we will see that every compact smooth manifold M admits a *proper* smooth embedding $M \to \mathbb{H}^{2\cdot\dim(M)+1}$.
(3) In the Neighborhood-Regular Value Theorem 56.9 we will prove a strengthening of the Regular Value Theorem 32.5.
(4) In the Gluing Submanifolds Proposition 56.12 we will see how, while gluing two smooth manifolds along the boundary, one can also glue submanifolds.

The Submanifold Neighborhood Retraction Proposition 56.5 will also play a major role in the proof of the Tubular Neighborhood–Existence Theorem 57.11 in the next chapter.

This chapter is organized as follows:

- In Section 56.1 we will see that for each proper submanifold we can find a collar of the total manifold such that the submanifold is a product with respect to a collar.
- In Section 56.2 we will prove the Submanifold Neighborhood Retraction Theorem which can be viewed as a version of the Neighborhood Projection Theorem 35.2 for smooth submanifolds with boundary.
- In Section 56.3 we will state and prove the Weak Whitney Embedding Theorem 56.7 for smooth manifolds with boundary.
- In Section 56.4 we will state and prove the Neighborhood-Regular Value Theorem 56.9.
- In Section 56.5 we will explain what it means to glue submanifolds.
- In Section 56.6 we will associate to two pairs (M_1, K_1) and (M_2, K_2) which consist of a smooth manifold M_i and submanifold K_i, with $\dim(M_1) = \dim(M_2)$ and also with $\dim(K_1) = \dim(K_2)$ a connected sum $(M_1 \# M_2, K_1 \# K_2)$.
- In Section 56.7 we will provide exercises for Chapter 56.

In the next Chapter 57 we will introduce the concept of a tubular neighborhood of a submanifold. We will see that these always exist and that they are essentially unique.

56.1. **Submanifolds that are products with respect to a collar.** For the reader's convenience we recall Definition 35.4

Definition 56.1. Let W be a smooth manifold and let $c > 0$. A collar of ∂W is a smooth embedding $\chi\colon [0,c] \times \partial W \to W$ that has the following three properties:
(a) The restriction of χ to $\{0\} \times \partial W$ is the obvious map $(0,x) \mapsto x$.
(b) The image $\chi([0,c] \times \partial W)$ is a closed subset of W.
(c) $\chi^{-1}(\partial W) = \{0\} \times \partial W$.
We refer to the image of χ as a collar neighborhood of ∂W.[623]

Example. We consider the smooth manifold $\mathbb{H}^n$. The map

$$\begin{array}{rcl} [0,1] \times \partial \mathbb{H}^n & \to & \mathbb{H}^n \\ (t,x) & \mapsto & x + t \cdot \mathfrak{e}_n \end{array}$$

is clearly a collar. We refer to it as the standard collar of $\partial \mathbb{H}^n$. □

For submanifolds of a given manifold W it is sometimes convenient if the submanifold is compatible with a given collar $\chi\colon [0,c] \times \partial W \to W$. This leads us to the following definition:

Definition 56.2. Let W be a smooth manifold and let $\chi\colon [0,c] \times \partial W \to W$ be a collar. Let M be a proper smooth submanifold of W. We say that M is a product with respect to the collar χ if there exists an $\epsilon \in (0,c]$ such that the following two conditions are satisfied:
(1) The map

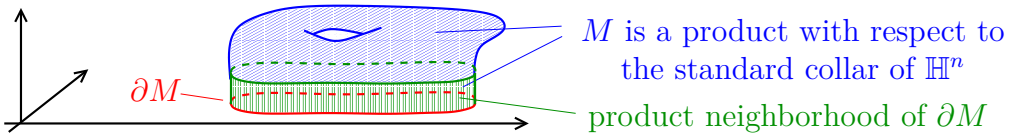
$$\begin{array}{rcl} [0,\epsilon] \times \partial M & \to & W \\ (t,P) & \mapsto & \chi(t,P) \end{array}$$

takes values in M and it defines a collar for M.
(2) We have $M \cap \chi([0,\epsilon] \times \partial W) = \chi([0,\epsilon] \times \partial M)$.
We refer to $\chi([0,\epsilon] \times \partial M) \subseteq M$ as a product neighborhood of ∂M.

Example. In many of our applications we care about the case $W = \mathbb{H}^n$ with the above standard collar. Note that a proper smooth submanifold M of $\mathbb{H}^n$ is a product with respect to the standard collar of $\mathbb{H}^n$ if there exists an $\epsilon \in (0,1]$ such that following two conditions are satisfied:
(1) The map

$$\begin{array}{rcl} [0,\epsilon] \times \partial M & \to & \mathbb{H}^n \\ (t,x) & \mapsto & x + t \cdot \mathfrak{e}_n \end{array}$$

takes values in M and it defines a collar for M.
(2) We have $M \cap (\mathbb{R}^{n-1} \times [0,\epsilon]) = \{x + t \cdot \mathfrak{e}_n \,|\, t \in [0,\epsilon] \text{ and } x \in \partial M\}$.



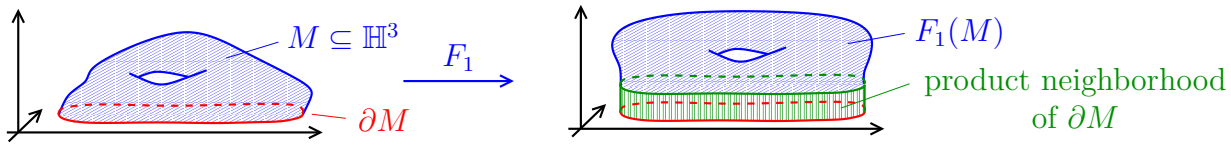
This observation is sketched in the figure above. □

The following proposition says that any compact proper submanifold can always be isotoped to be a product with respect to a given collar.

Proposition 56.3. (Collar-for-Submanifold Proposition) Let W be a smooth manifold and let $\chi\colon [0,c] \times \partial W \to W$ be a collar. Let M be a compact proper smooth

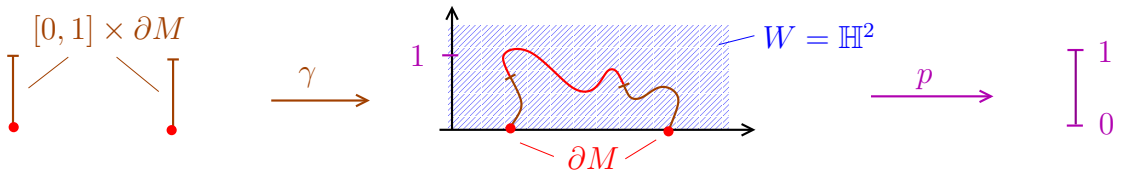
[623] By the Basics-of-Collars Lemma 35.7 (1) we know that the image of χ is indeed a neighborhood of ∂W.

submanifold of W. There exists a proper smooth isotopy $F\colon M \times [0,1] \to W$ rel ∂M with $F_0 = \mathrm{id}$ such that $F_1(M)$ is a product with respect to χ.



Sketch of proof. To simplify the notation a little bit we assume that $c = 1$. Furthermore we use χ to view $[0,1] \times \partial W$ as a codimension-zero submanifold of W. We denote by $p\colon [0,1] \times \partial W \to [0,1]$ and $q\colon [0,1] \times \partial W \to \partial W$ the natural projections.

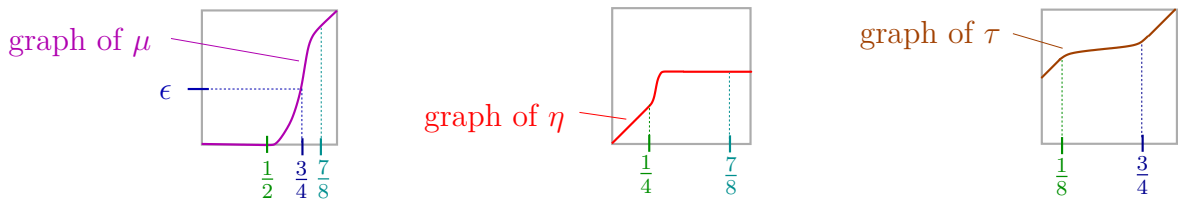
First note that by the Smooth Collar Neighborhood Theorem 35.6 there exists a collar $\gamma\colon [0,1] \times \partial M \to M$. Since M is compact it follows from the Smooth Manifold–Boundary Proposition 25.27 (2b) that ∂M is also compact. Also note that it follows from the Basics-of-Collars Lemma 35.7 (1) that $[0, \frac{1}{8}) \times \partial W$ is an open neighborhood of ∂W, therefore $\gamma^{-1}([0, \frac{1}{8}) \times \partial W)$ is an open neighborhood of $\{0\} \times \partial M$. It follows from the Tube Lemma 7.12 that there exists a $c > 0$ such that $\gamma([0,c] \times \partial M) \subseteq [0, \frac{1}{8}) \times \partial W$. After reparametrizing γ we can in fact assume that $\gamma([0,1] \times \partial M) \subseteq [0, \frac{1}{8}) \times \partial W$.



Since M is a proper submanifold we know that $\partial M \subseteq \partial W$. It follows immediately from the definition of a submanifold that for all $P \in \partial M$ we have $\mathrm{T}_P M \not\subseteq \mathrm{T}_P \partial W$. This implies that for all $P \in \partial M$ the differential $\mathrm{D}p_P$ is non-zero. Note that by Exercise 29.15 we know that the set $\{P \in M \mid \mathrm{D}\,p_P \neq 0\}$ is an open subset. Since ∂M is compact it follows from this discussion, again by the Tube Lemma 7.12, that there exists an $\epsilon \in (0, \frac{1}{2})$ such that the differential $\mathrm{D}p_{\gamma(t,P)}$ is non-zero for all $t \in [0, \epsilon]$ and all $P \in \partial M$.

Next note that using the Smooth Transition Function Lemma 24.9 one can easily construct the following three types of functions:

(1) A smooth non-decreasing function $\mu\colon [0,1] \to [0,1]$ with $\mu(t) = 0$ for all $t \in [0, \frac{1}{2}]$, $\mu(\frac{3}{4}) = \epsilon$, with $\mu'(t) > 0$ for all $t \in [\frac{3}{4}, 1]$ and such that $\mu(t) = t$ for all $t \in [\frac{7}{8}, 1]$.
(2) A smooth non-decreasing function $\eta\colon [0,1] \to [0,1]$ with $\eta(t) = t$ for all $t \in [0, \frac{1}{4}]$ and with $\eta(t) = \frac{1}{2}$ for all $t \in [\frac{7}{8}, 1]$.
(3) A smooth function $\tau\colon [0,1] \to [\frac{1}{2}, 1]$ such that $\tau'(t) > 0$ for all $t \in [0,1]$ and such that $\tau(t) = t + \frac{1}{2}$ for all $t \in [0, \frac{1}{8}]$ and $\tau(t) = t$ for all $t \in [\frac{3}{4}, 1]$.



Let $\Xi\colon W \to W$ be the map that is given by $(t,w) \mapsto (\tau(t), w)$ on $[0,1] \times \partial W$ and id outside of $[0,1) \times \partial W$. Using the Smooth Pasting Proposition 25.19, using the Basics-of-Collars

Lemma 35.7 (1) and using the fact that $\tau = \mathrm{id}$ on $[\frac{3}{4}, 1]$ one can easily verify that Ξ is smooth.

Next we consider the map

$$f\colon M \to W$$

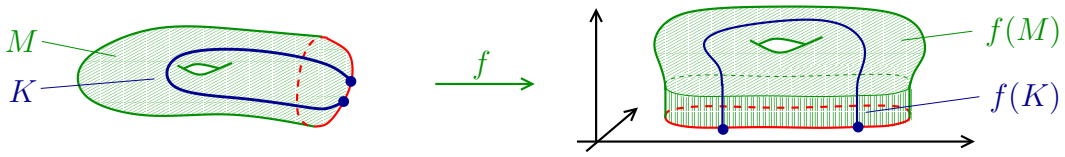
$$P \mapsto \begin{cases} \overbrace{(p(\gamma(\mu(t), Q)) + \eta(t), q(\gamma(\mu(t), Q)))}^{\in [0,1]\times \partial W}, & \text{if } P = \gamma(t, Q) \text{ with } t \in [0,1] \ \&\ Q \in \partial M, \\ \Xi(P), & \text{if } P \notin \gamma([0,1] \times \partial M). \end{cases}$$

Note that in an open neighborhood of $\chi(\{1\} \times \partial M)$ both maps in the above definition are given by "adding $\frac{1}{2}$ to the $[0,1]$ coordinate". This shows that f is in fact well-defined and it follows easily from the Smooth Pasting Proposition 25.19 that f is in fact smooth.

We leave it to the reader to verify that f is a proper smooth embedding and that $f(M)$ is a product with respect to the collar of W. Furthermore we leave it to the reader to write down a proper smooth isotopy $F\colon M \times [0,1] \to W$ with $F_0 = \mathrm{id}$ and $F_1 = f$. ■

The Collar-for-Submanifold Proposition 56.3 allows us to prove the following proposition, which which will be useful on several occasions.

Proposition 56.4. (Euclidean Collar-for-Submanifold Proposition) Let M be a compact smooth manifold and let K be a (possibly empty) proper smooth submanifold of M. There exists a proper smooth embedding $f\colon M \to \mathbb{H}^n$ such that $f(M) \subseteq \mathbb{H}^n$ and $f(K) \subseteq \mathbb{H}^n$ are both products with respect to the standard collar of $\partial\mathbb{H}^n$.



Proof. It follows from the Diffeomorphic-to-Euclidean Submanifold Corollary 33.3 that we can assume that M is a compact proper smooth submanifold of $\mathbb{H}^n$. Let K be a proper smooth submanifold of M. Recall that "proper" means in particular that $K \subseteq M$ is closed. Since M is compact, we know by the Compact-Closed Lemma 3.39 (1) that K is also compact.

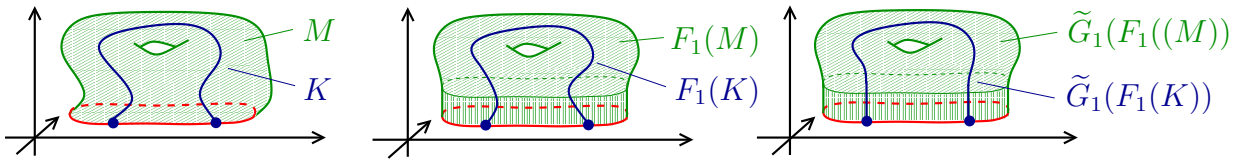
Next note that it follows from the Collar-for-Submanifold Proposition 56.3, applied to $M \subseteq \mathbb{H}^n$ with the standard collar $[0,1] \times \partial\mathbb{H}^n \to \mathbb{H}^n$, that there exists a proper smooth isotopy $F\colon M \times [0,1] \to \mathbb{H}^n$ with $F_0 = \mathrm{id}$ such that $F_1(M)$ is a product with respect to the standard collar $[0,1] \times \partial\mathbb{H}^n \to \mathbb{H}^n$. Recall that this means in particular that there exists a $c \in (0,1]$ such that the map

$$\begin{aligned} \sigma\colon [0,c] \times \partial F_1(M) &\to F_1(M) \\ (t,x) &\mapsto \sigma(x,t) = x + t \cdot \mathfrak{e}_n \end{aligned}$$

takes values in $F_1(M)$ and that it is a collar for $\partial F_1(M)$.

Next we apply the Collar-for-Submanifold Proposition 56.3 to the compact proper smooth submanifold $F_1(K)$ of $F_1(M)$ and the above collar $\sigma\colon [0,c] \times \partial F_1(M) \to F_1(M)$. We obtain a proper smooth isotopy $G\colon F_1(K) \times [0,1] \to F_1(M)$ with $G_0 = \mathrm{id}$ such that $G_1(F_1(K))$ is a product with respect to σ. By the Isotopy Extension Theorem 44.2 there

exists a diffeotopy $\widetilde{G}\colon F_1(M) \times [0,1] \to F_1(M)$ such that for all $(k,t) \in F_1(K) \times [0,1]$ we have $\widetilde{G}(k,t) = G(k,t)$.



Finally note that $\widetilde{G}_1(F_1(M)) = F_1(M)$ is a product with respect to σ and that $\widetilde{G}_1(F_1(K))$ is also a product with respect to σ. In summary we have thus shown that the smooth embedding $f \coloneqq \widetilde{G}_1 \circ F_1 \colon M \to \mathbb{H}^n$ has the desired properties. ■

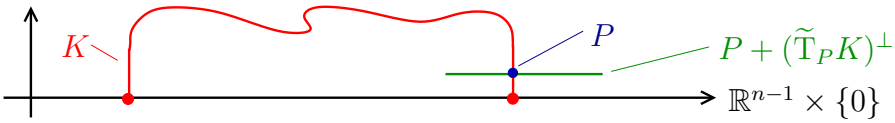
56.2. **The Submanifold Neighborhood Retraction Theorem.** The Collar-for-Submanifold Proposition 56.3 allows us to prove the following variation on the Neighborhood Projection Theorem 35.2:

Proposition 56.5. (Submanifold Neighborhood Retraction Proposition) Let M be a compact smooth manifold and let K be a proper smooth submanifold of M. There exists an open neighborhood Z of $K \subseteq M$ and a smooth retraction $\rho\colon Z \to K$ with $\rho^{-1}(\partial K) = \partial Z$.



In the proof of the Submanifold Neighborhood Retraction Proposition 56.5 we will make use of the following little lemma.

Lemma 56.6. (Vertical Boundary–Tangent Space Lemma) Let K be a proper smooth submanifold of $\mathbb{H}^n$. If K is a product with respect to the standard collar of $\partial\mathbb{H}^n$, then there exists an $\epsilon > 0$ such that for every $P \in K \cap (\mathbb{R}^{n-1} \times [0,\epsilon])$ we have $(\widetilde{\mathrm{T}}_P K)^\perp \subseteq \mathbb{R}^{n-1} \times \{0\}$.



Proof of Lemma 56.6. Let K be a proper smooth submanifold of $\mathbb{H}^n$ such that K is a product with respect to the standard collar of $\partial\mathbb{H}^n$. Recall that this means that there exists an $\epsilon > 0$ with the following properties:

(1) The map
$$\begin{array}{rcl} [0,\epsilon] \times \partial K & \to & \mathbb{H}^n \\ (t,x) & \mapsto & x + t \cdot \mathfrak{e}_n \end{array}$$
takes values in K and it defines a collar for K.

(2) We have $K \cap (\mathbb{R}^{n-1} \times [0,\epsilon]) = \{x + t \cdot \mathfrak{e}_n \,|\, t \in [0,\epsilon] \text{ and } x \in \partial K\}$.

Now let $P \in K \cap (\mathbb{R}^{n-1} \times [0,\epsilon])$. By (2) we know that $P = x + t \cdot \mathfrak{e}_n$ for some $t \in [0,\epsilon]$ and $x \in \partial K$. It follows pretty easily from (1) that $\widetilde{\mathrm{T}}_P K = \widetilde{\mathrm{T}}_x \partial K \oplus \mathbb{R} \cdot \mathfrak{e}_n$. This implies that $(\widetilde{\mathrm{T}}_P K)^\perp \subseteq \mathbb{R}^{n-1} \times \{0\}$. ■

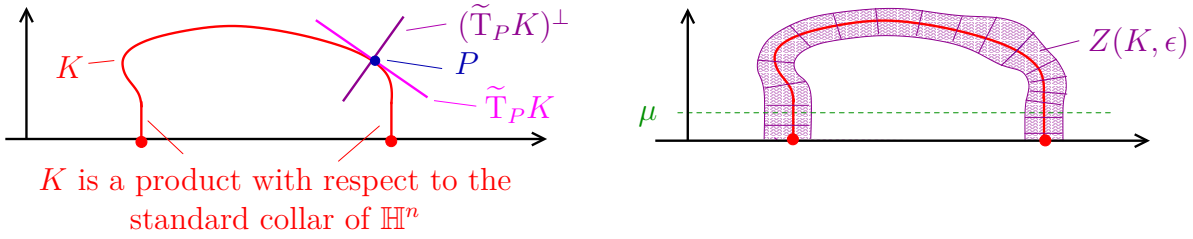
Proof of the Submanifold Neighborhood Retraction Proposition 56.5. It follows from the Euclidean Collar-for-Submanifold Proposition 56.4 and the Smooth Embedding Theorem 30.15 that we can assume that M is a proper smooth submanifold of some $\mathbb{H}^n$ such that $K \subseteq \mathbb{H}^n$ is a product with respect to the standard collar of $\partial\mathbb{H}^n$.

Claim. There exists an $\epsilon > 0$ such that the following conditions are satisfied:

(1) The set
$$Z(K,\epsilon) \coloneqq \big\{P + w \,\big|\, P \in K \text{ and } w \in \big(\widetilde{\mathrm{T}}_P K\big)^\perp \text{ with } \|w\| < \epsilon\big\} \subseteq \mathbb{R}^n$$
is actually contained in $\mathbb{H}^n$ and it is an open neighborhood of $K \subseteq \mathbb{H}^n$.

(2) The map
$$\begin{aligned} \rho\colon Z(K,\epsilon) &\to K \\ P + w &\mapsto P \quad \text{where } P \in K \text{ and } w \in \big(\widetilde{\mathrm{T}}_P K\big)^\perp \text{ with } \|w\| < \epsilon \end{aligned}$$
is well-defined, it is smooth and it is a retraction $Z(K,\epsilon) \to K$.

(3) We have[624] $\rho^{-1}(\partial K) = \partial Z(K,\epsilon)$.



Proof. The reader will have noticed that the statement of the claim is very similar to the statement of the Neighborhood Projection Theorem 35.2. The only sense in which the above claim is stronger is that we now claim that $Z(K,\epsilon)$ can be arranged to be an open subset of $\mathbb{H}^n$ and that we can arrange that $\rho^{-1}(\partial K) = \partial Z(K,\epsilon)$. As we will now see, both these extra features can be arranged using the Vertical Boundary–Tangent Space Lemma 56.6.

Indeed, by the Vertical Boundary–Tangent Space Lemma 56.6 we know that there exists a $\mu > 0$ such that for every $P \in K \cap (\mathbb{R}^{n-1} \times [0,\mu])$ we have $(\widetilde{\mathrm{T}}_P K)^\perp \subseteq \mathbb{R}^{n-1} \times \{0\}$. It follows easily from the definitions that $Z(K,\mu) \subseteq \mathbb{H}^n$. With this observation the argument of the Neighborhood Projection Theorem 35.2 shows that there exists an $\epsilon \in (0,\mu)$ with the desired properties. We leave it to the reader to fill in the details. ⊞

Now we are done: We just set $Z \coloneqq Z(K,\epsilon) \cap M$. It is clear that $\rho\colon Z \to K$ has the desired properties. ■

56.3. **Weak Whitney Embedding Theorem for manifolds with boundary.** In the Weak Whitney Embedding Theorem 37.9 we saw that given any smooth manifold M there exists a smooth embedding $M \to \mathbb{R}^{2\cdot\dim(M)+1}$ such that the image is a closed subset of $\mathbb{R}^{2\cdot\dim(M)+1}$. Note that that proposition did not make any claims about the embedding of the boundary. The following proposition rectifies this issue and it provides, for compact smooth manifolds, a useful refinement of the above statement.

[624] By (1) we know that $Z(K,\epsilon)$ is an open subset of $\mathbb{H}^n$. Thus $Z(K,\epsilon)$ is a smooth manifold with boundary $\partial Z(K,\epsilon) = (\mathbb{R}^{n-1} \times \{0\}) \cap Z(K,\epsilon)$.

Theorem 56.7. (Weak Proper Whitney Embedding Theorem) Given any smooth manifold M, such that the boundary ∂M is compact, there exists a *proper* smooth embedding $M \to \mathbb{H}^{2\cdot\dim(M)+1}$.

Remark. Surely the conclusion of the Weak Proper Whitney Embedding Theorem 56.7 also holds for non-compact smooth manifolds, but we leave it to other authors to work out the details. □

Sketch of proof. First note that it follows from the Euclidean Embedding Theorem 33.2 that every c smooth manifold admits a proper smooth embedding into some $\mathbb{H}^n$. Thus it suffices to prove the following statement:

Statement. If M is a proper m-dimensional smooth submanifold of some $\mathbb{H}^n$ such that $n > 2m+1$ and with compact boundary ∂M, then M also admits a proper smooth embedding into $\mathbb{H}^{n-1}$.

Thus let M be a proper m-dimensional smooth submanifold of $\mathbb{H}^n$ with $n > 2m+1$ and with compact boundary. It follows from the Collar-for-Submanifold Proposition 56.3 that we can arrange that there exists an $\epsilon > 0$ such that

$$(*) \qquad M \cap (\mathbb{R}^n \times [0,\epsilon]) \;=\; \{P + t\cdot \mathfrak{e}_n \,|\, P \in \partial M \text{ and } t \in [0,\epsilon]\}.$$

We consider

$$\widetilde{M} \;:=\; M \cup \{(P + t\cdot \mathfrak{e}_n \,|\, P \in \partial M \text{ and } t \in (-\infty, 0]\} \;\subseteq\; \mathbb{R}^n.$$

It is straightforward to verify that $\widetilde{M}$ is an m-dimensional submanifold of $\mathbb{R}^n$. We consider the vectors $v \in S^{n-1}$ with the following two properties:

(1) v does not lie in any tangent space $\widetilde{\mathrm{T}}_P\widetilde{M}$.
(2) v is not secant to $\widetilde{M}$, in other words, for any two distinct points $x, y \in \widetilde{M}$ the difference $x - y$ is not parallel to v.

Since $n > 2m+1$ it follows from the Generic Vector Lemma 37.10 that the set of such vectors is dense in S^{n-1}. In particular there exists a $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ which has the above properties and which satisfies

$$|v_1| \geq \tfrac{1}{2} \quad \text{and} \quad |v_n| \leq \tfrac{\epsilon}{4(d+1)} \quad \text{where} \quad d \;:=\; \underbrace{\max\{|x_1| \,|\, (x_1, \ldots, x_{n-1}, 0) \in \partial M\}}_{<\infty \text{ since } \partial M \text{ is compact}}.$$

We consider the matrix

$$A \;:=\; \begin{pmatrix} v_1 & 0 & \ldots & 0 \\ v_2 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ v_n & 0 & \ldots & 1 \end{pmatrix} \quad \text{and its inverse} \quad A^{-1} \;:=\; \begin{pmatrix} v_1^{-1} & 0 & \ldots & 0 \\ -v_2\cdot v_1^{-1} & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ -v_n\cdot v_1^{-1} & 0 & \ldots & 1 \end{pmatrix}$$

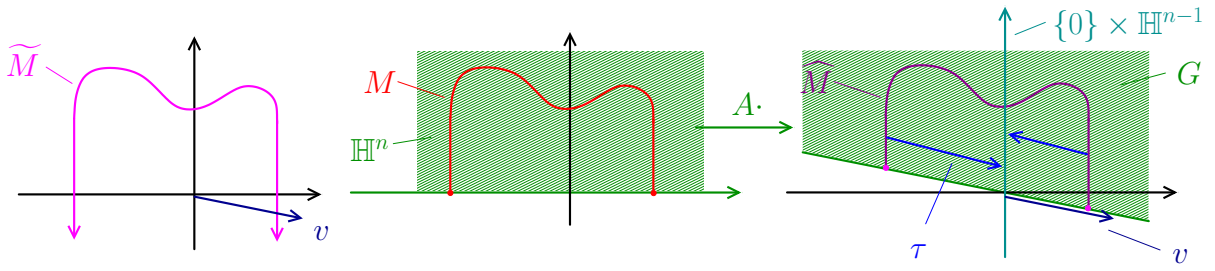
and we consider the half-space

$$G \;:=\; \{\lambda_1\cdot v + \lambda_2\cdot \mathfrak{e}_2 + \ldots \lambda_n \cdot \mathfrak{e}_n \,|\, \lambda_1, \ldots, \lambda_n \in \mathbb{R} \text{ and } \lambda_n \geq 0\} \;\subseteq\; \mathbb{R}^n.$$

Note that $A \cdot \mathbb{H}^n = G$, or equivalently, $A^{-1}\cdot G = \mathbb{H}^n$. We set $\widehat{M} := \widetilde{M} \cap G$. Next we consider the maps

$$\tau\colon \mathbb{R}^n \xrightarrow{w \mapsto A^{-1}\cdot w} \mathbb{R}^n \xrightarrow{(x_1,\ldots,x_n)\mapsto(x_2,\ldots,x_n)} \mathbb{R}^{n-1} \quad \text{and} \quad q\colon \mathbb{R}^{n-1} \xrightarrow{(x_2,\ldots,x_n)\mapsto x_n} \mathbb{R}.$$

It remains to prove the following claim:



Claim.

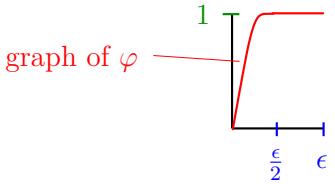
(1) There exists a proper smooth embedding $\Theta\colon M \to G$ with image $\widehat{M}$ (which by the Smooth Embedding Theorem 30.15 is diffeomorphic to M).
(2) The restriction of $\tau\colon G \to \mathbb{H}^{n-1}$ to $\widehat{M} \to \mathbb{H}^{n-1}$ is a proper smooth embedding.

Proof.

(1) We start out with an observation, namely we have $(q\circ\tau)(\partial M) \subseteq [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$. This can be seen as follows: given $(x_1,\dots,x_{n-1},0) \in \partial M$ we calculate that

$$|(q\circ\tau)(x_1,\dots,x_{n-1},0)| \;=\; |-v_n\cdot v_1^{-1}\cdot x_1| \;\underset{\substack{\uparrow\\ \text{by choice of } v=(v_1,\dots,v_n)}}{\leq}\; \tfrac{\epsilon}{4(d+1)}\cdot 2\cdot d \;<\; \tfrac{\epsilon}{2}.$$

Next using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth map $\varphi\colon [0,\epsilon] \to [0,1]$ with $\varphi(0)=0$ and $\varphi(t)=1$ for $t \in [\frac{\epsilon}{2},\epsilon]$.



The desired smooth embedding $\Theta\colon M \to G$ is given by the identity on $M\cap(\mathbb{R}^n\times[\epsilon,\infty))$ and by "stretching each interval $P\times[0,\epsilon]$". More precisely on $M\cap(\mathbb{R}^n\times[0,\epsilon])$ the map Θ is given by the map

$$\begin{array}{rcl} M\cap(\mathbb{R}^n\times[0,\epsilon]) & \to & \mathbb{R}^n \\ \underset{\substack{\uparrow\\ \text{with } P\in\partial M}}{P+t\cdot\mathfrak{e}_n} & \mapsto & P-(\varphi(t)\cdot t+(1-\varphi(t))\cdot\underbrace{(q\circ\tau)(P)}_{\geq-\frac{\epsilon}{2}})\cdot\mathfrak{e}_n. \end{array}$$

It follows easily from the definitions that $\Theta(M)=\widehat{M}$ and that $\Theta(M)=\widehat{M}$ is a proper submanifold of G.[625]

(2) First note that it follows from $A^{-1}\cdot G=\mathbb{H}^n$ that the map $\tau\colon \mathbb{R}^n\to\mathbb{R}^{n-1}$ restricts to a map $\tau\colon G\to\mathbb{H}^{n-1}$. Finally note that it follows easily from the choice of v and the fact that $\widehat{M}\subseteq\widetilde{M}$ that $\tau\colon \widehat{M}\to\mathbb{H}^{n-1}$ is indeed a proper smooth embedding. ∎

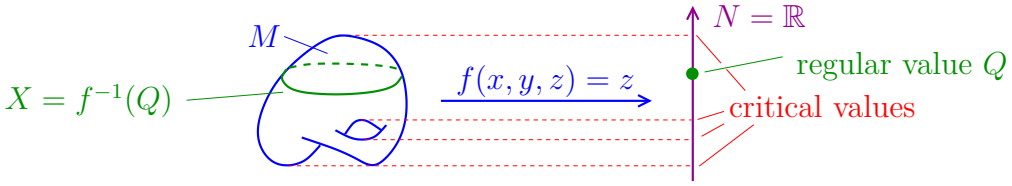
[625]As a reality check let us verify that $M\cap(\mathbb{R}^n\times\{0\})$ gets sent to the boundary ∂G. First note that $\partial G=\{(\lambda_1\cdot v+\lambda_2\cdot\mathfrak{e}_2+\dots+\lambda_{n-1}\cdot\mathfrak{e}_{n-1}\,|\,\lambda_1,\dots,\lambda_{n-1}\in\mathbb{R}\}$. Next we consider $P+0\cdot\mathfrak{e}_n$ with $P\in\partial M$. It follows from $\varphi(0)=0$ that $\Theta(P+0\cdot\mathfrak{e}_n)=P-(q\circ\tau)(P)\cdot\mathfrak{e}_n$. By definition $(q\circ\tau)(P)$ is the $\mathfrak{e}_n$-coefficient of P with respect to the basis $v_1,\mathfrak{e}_2,\dots,\mathfrak{e}_n$. This shows that the $\mathfrak{e}_n$-coefficient of $P-(q\circ\tau)(P)\cdot\mathfrak{e}_n$ is zero, which implies that $P-(q\circ\tau)(P)\cdot\mathfrak{e}_n\in\partial G$.

56.4. **The Neighborhood-Regular Value Theorem.** In this section we formulate a refinement of the Regular Value Theorem 32.5. Let us first recall the relevant definitions from Definition 32.1:

Definition 56.8. Let $f\colon M \to N$ be a smooth map between smooth manifolds.

(1) Let $P \in M$. If $P \in M \setminus \partial M$, then we say that the point P is a regular point if the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ is an epimorphism. Otherwise we say that P is a critical point.

(2) We say that $Q \in N$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a critical value.



We can now formulate the following refinement of the Regular Value Theorem 32.5.

Theorem 56.9. (Neighborhood-Regular Value Theorem) Let $f\colon M \to N$ be a smooth map between smooth manifolds M and N and let $Q \in N \setminus \partial N$ be a regular value of $f\colon M \to N$ and also of $f|_{\partial M}\colon \partial M \to N$.[626]

(1) $K := f^{-1}(Q)$ is a proper submanifold of M of dimension $\dim(M) - \dim(N)$.

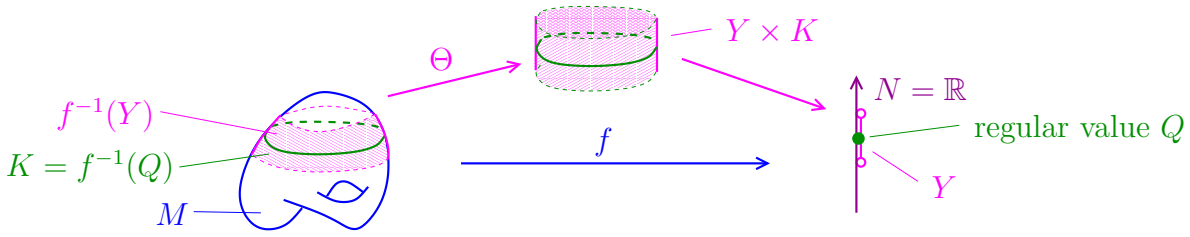
(2) If M is orientable, then K is also orientable.

(3) If M is compact, then K is also compact.

For the last statement we assume that M is compact and that $\partial N = \varnothing$.

(4) There exists an open neighborhood Y of $Q \in N$ together with a diffeomorphism $\Theta\colon f^{-1}(Y) \to Y \times K$ such that the following diagram commutes:

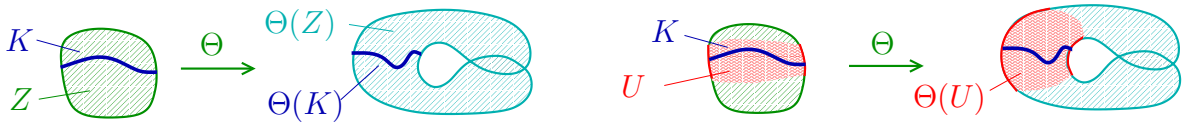
$$\begin{array}{ccc} f^{-1}(Y) & \xrightarrow{\ \Theta\ } & Y \times K \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle (y,k)\mapsto y} & \\ & Y. & \end{array}$$



Our proof of the Neighborhood-Regular Value Theorem 56.9 relies on the following theorem, which we have used before on several occasions to construct diffeomorphisms on neighborhoods of suitable subspaces:

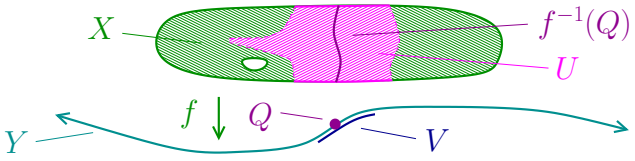
[626] The first three statements are precisely the content of the original Regular Value Theorem 32.5.

Theorem 27.16. (Neighborhood Smooth Embedding Theorem) Let $\Theta\colon Z \to B$ be a continuous map between smooth manifolds. Furthermore let $K \subseteq Z$ be a closed subset such that the map $\Theta\colon K \to \Theta(K)$ is a homeomorphism and such that $\Theta(K)$ is a closed subset of B. If Θ is a local diffeomorphism at each $P \in K$, then there exists an open neighborhood U of K such that the restriction of Θ to U is a smooth embedding and such that $\Theta(U)$ is an open subset of B.

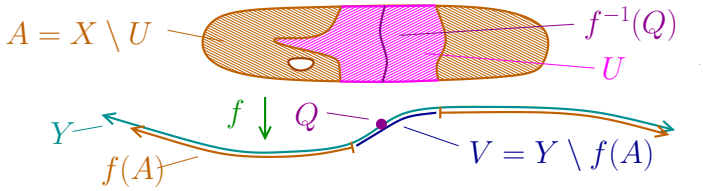


We will also need the following slightly subtle lemma, which is also the content of Exercise 4.40.

Lemma 56.10. (Preimage–Neighborhood Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. Let $Q \subseteq Y$ be a subset and let U be an open neighborhood of $f^{-1}(Q) \subseteq X$. If X is compact and if Y is Hausdorff, then there exists an open neighborhood V of $Q \subseteq Y$ such that $f^{-1}(V) \subseteq U$.



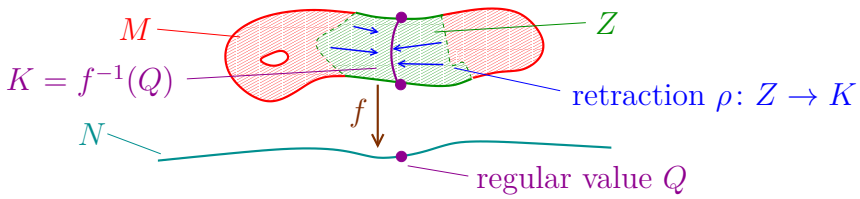
Proof. We consider $A \coloneqq X \setminus U$. Since $U \subseteq X$ is open we know that $A \subseteq X$ is closed. Since X is compact, we know by the Compact-Closed Lemma 3.39 (1) that A is compact. Since f is continuous it follows from the Compact Image Lemma 4.15 that $f(A)$ is compact.



Since Y is Hausdorff it follows from the Compact-Closed Lemma 3.39 (2) that $f(A)$ is a closed subset of Y. It follows from the Image-Preimage Lemma 1.9 (8) that $V \coloneqq Y \setminus f(A)$ has the desired properties. ∎

Proof of the Neighborhood-Regular Value Theorem 56.9. Let $f\colon M \to N$ be a smooth map between smooth manifolds M and N and let $Q \in N \setminus \partial N$ be a regular value of $f\colon M \to N$ and also of $f|_{\partial M}\colon \partial M \to N$. As mentioned above, the first three statements of the theorem form precisely the content of the original Regular Value Theorem 32.5. In particular we now know that $K \coloneqq f^{-1}(Q)$ is naturally a proper submanifold of M of dimension $\dim(M) - \dim(N)$.

So it remains to deal with the final statement. Thus we now assume that M is compact and that $\partial N = \varnothing$. Since M is compact it follows from the Submanifold Neighborhood Retraction Proposition 56.5 that there exists an open neighborhood Z of $K \subseteq M$ and a smooth retraction $\rho\colon Z \to K$ such that $\rho^{-1}(\partial K) = \partial Z$.
Next we consider the map



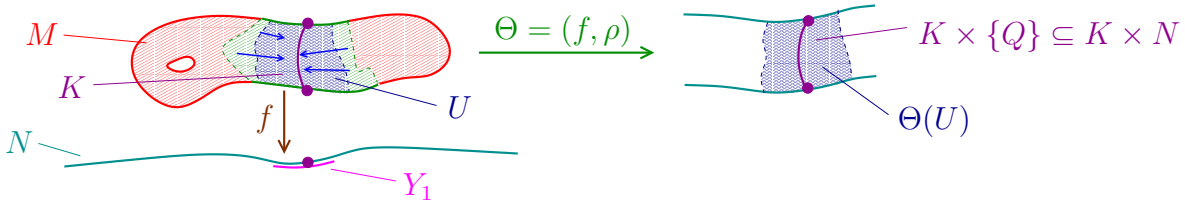
$$\begin{aligned}\Theta := (f,\rho)\colon Z &\to N \times K\\ P &\mapsto (f(P),\rho(P)).\end{aligned}$$

We start out with two observations:

(a) It follows easily from the fact that K is compact, that N and K are Hausdorff and the Compact-Hausdorff Proposition 4.23 that the restriction of Θ to K defines a homeomorphism $\Theta\colon K \to \{Q\} \times K$ and that $\Theta(K)$ is a closed subset of $N \times K$.

(b) Using the fact that $Q \in N \setminus \partial N$ is a regular value of $f\colon M \to N$ and using the fact that $\rho|_K = \mathrm{id}_K$ one can easily show that for each point $P \in K$ the corresponding differential $\mathrm{D}\Theta_P$ is an isomorphism. Furthermore note that it follows from $\partial N = \varnothing$ that we have $\Theta^{-1}(\partial(N \times K)) = \Theta^{-1}(N \times \partial K) = \rho^{-1}(\partial K) = \partial Z$. It thus follows from the Inverse Mapping Theorem 30.1 that for each $P \in K$ the map Θ is a local diffeomorphism at P.

It follows from the Observations (a) and (b) that we can apply the Neighborhood Smooth Embedding Theorem 27.16 to $\Theta\colon Z \to N \times K$ and $K \subseteq Z$. Thus we see that there exists an open neighborhood U of $K \subseteq Z$ such that $\Theta(U)$ is an open neighborhood of $\{Q\} \times K$ in $N \times K$ and such that $\Theta\colon U \to \Theta(U)$ is a diffeomorphism.



(a) Since M is compact and since the smooth manifold N is by definition Hausdorff we obtain from the Preimage–Neighborhood Lemma 56.10 that there exists an open neighborhood Y_1 of $Q \in N$ with $f^{-1}(Y_1) \subseteq U$.

(b) Since M is compact, we know, by (2), that K is also compact. We obtain from the Tube Lemma 7.12 that there exists an open neighborhood Y_2 of Q with $Y_2 \times K \subseteq \Theta(f^{-1}(Y_1))$.

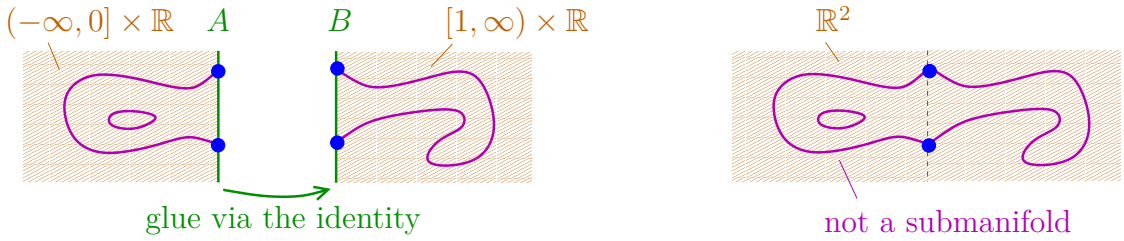
We set $Y := Y_1 \cap Y_2$. It is now straightforward to verify that Θ defines a diffeomorphism $\Theta\colon f^{-1}(Y) \to Y \times K$. Finally note that the diagram

$$\begin{array}{ccc} f^{-1}(Y) & \xrightarrow{\Theta=(f,\rho)} & Y \times K \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle (y,k)\mapsto y} & \\ & Y & \end{array}$$

commutes by definition of Θ. ∎

56.5. **Gluing submanifolds.** In the Gluing-Smooth Manifolds-Proposition 54.5 we saw that if we are given a smooth manifold M (possibly disconnected) and some diffeomorphism

$f\colon A \to B$ between disjoint components A and B of ∂M, then any choice of a collar neighborhood of ∂M lets us view $M/a \sim f(a)$ again as a smooth manifold. Now suppose that we are given a proper submanifold N of M with $f(\partial N \cap A) = \partial N \cap B$. We would like to view $N/a \sim f(a)$ as a submanifold of $M/a \sim f(a)$. But it is easy to find examples (see the figure below) where this is not the case, since in all likelihood $N/a \sim f(a)$ will not be smooth at the gluing points.



The following proposition gives an approach to circumventing this issue. For better readability it is formatted as if it was statement (5) of the Gluing-Smooth Manifolds-Proposition 54.5.

Proposition 56.11. (Gluing-Smooth Manifolds-Proposition) Let M be a smooth manifold, let A and B be two distinct boundary components of M and let $f\colon A \to B$ be a diffeomorphism.

(5) Let N be a proper submanifold of M such that the map f restricts to a diffeomorphism $A \cap N \to B \cap N$. If $\chi\colon [0, c] \times \partial M \to M$ is a collar such that N is a product with respect to χ and if we use this collar χ to define the smooth manifold structure on $M/a \sim f(a)$, then $N/a \sim f(a)$ is a submanifold of $M/a \sim f(a)$.

Proof. The proof follows easily from the definition of the smooth structure on $M/a \sim f(a)$ as provided in the proof of the Gluing-Smooth Manifolds-Proposition 54.5 (1). In contrast to many other proofs we will not bore the reader with unnecessary details. ∎

The above results now allow us to prove the following proposition which will be used on several occasions.

Proposition 56.12. (Gluing Submanifolds Proposition) Let M and N be two compact smooth manifolds with boundary components $A \subseteq M$ and $B \subseteq N$ and let $f\colon A \to B$ be a diffeomorphism. We set

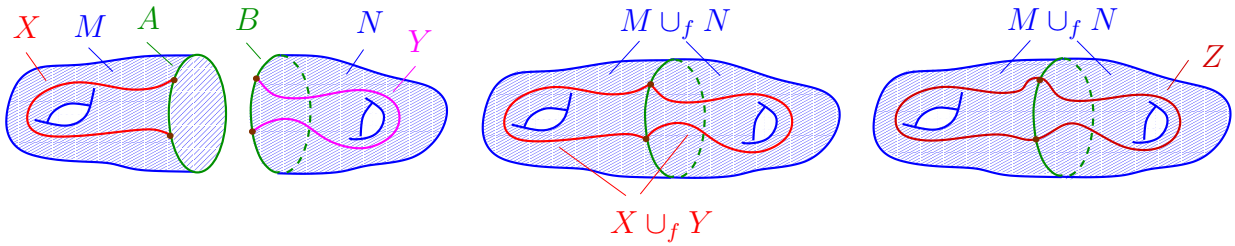
$$M \cup_f N \;:=\; (M \sqcup N)/a \sim f(a) \qquad \text{for all } a \in A.$$

Suppose we have chosen collar neighborhoods for ∂M and ∂N such that $M \cup_f N$ comes with a smooth manifold structure. Furthermore let $X \subseteq M$ and $Y \subseteq N$ be two proper m-dimensional submanifolds such that $f(X \cap A) = Y \cap B$. We set

$$X \cup_f Y \;:=\; (X \sqcup Y)/a \sim f(a) \qquad \text{for all } a \in X \cap A.$$

Then there exists a proper submanifold $Z \subseteq M \cup_f N$ with the following three properties:

(1) $Z \cap A = X \cap A$ (and thus also $Z \cap B = Y \cap B$).
(2) Outside of any previously chosen neighborhood of $A = B \subseteq M \cup_f N$ we have the equality $Z = X \cup_f Y := (X \sqcup Y)/a \sim f(a)$.
(3) The submanifold Z is isotopic to $X \cup_f Y$.

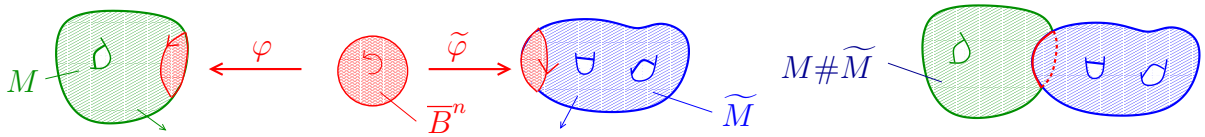


Proof. The proposition can be deduced quite easily from the Gluing-Smooth Manifolds-Proposition 56.11 (5) and the Collar-for-Submanifold Proposition 56.3. We leave it to the reader to fill in the details. ■

56.6. **The connected sum of oriented submanifolds.** In this section we will introduce the connected sum $(M,K)\#(\widetilde{M},\widetilde{K})$ of two suitable pairs consisting of a smooth manifold and a submanifold. To get to this point let us first recall Definition 54.12 of the connected sum of smooth manifolds:

Definition 56.13. Let $n \in \mathbb{N}_0$ and let M and $\widetilde{M}$ be two oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$. Furthermore we pick an orientation-reversing smooth embedding $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M} \setminus \partial\widetilde{M}$. We define the connected sum of M and $\widetilde{M}$ as

$$M\#\widetilde{M} := \big(M \setminus \varphi(B^n)\big) \sqcup \big(\widetilde{M} \setminus \widetilde{\varphi}(B^n)\big)/\!\sim \quad \text{where } \varphi(P) \sim \widetilde{\varphi}(P) \text{ for all } P \in S^{n-1}.$$



Example. The above figure shows (hopefully convincingly) that the connected sum of a surface of genus g with a surface of genus h is a surface of genus $g+h$.[627] □

The following proposition, which is a slightly simplified version of the Smooth Connected Sum-Proposition 54.13, summarizes some key facts about the connected sum of smooth manifolds:

Proposition 56.14. (Smooth Connected Sum-Proposition) Let $n \in \mathbb{N}_0$ and let M and M' be two oriented connected n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M\setminus\partial M$ and $\varphi'\colon \overline{B}^n \to M'\setminus\partial M'$ be smooth embeddings. We demand that precisely one of φ,φ' is orientation-preserving and precisely one is orientation-reversing. The following statements hold:

(0) (a) If M and M' are compact, then $M\#M'$ is also compact.
(b) If $n \geq 2$, then $M\#M'$ is also connected.
(1) The resulting connected sum $M\#M'$ is an n-dimensional topological manifold that admits a smooth structure. The diffeomorphism type of $M\#M'$ is natural.
(2) (a) The boundary of $M\#M'$ is given by $\partial M \sqcup \partial M'$.

[627] In the Surface–Connected Sum Lemma 54.15 we gave a fairly formal proof of this statement.

(b) If M and M' are closed, then $M\#M'$ is also closed.
(3) $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$ are closed subsets of $M\#M'$ and they are both codimension-zero submanifolds of $M\#M'$.
(4) The smooth manifold $M\#M'$ admits a unique orientation that restricts to the given orientations of $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$.
(5) The diffeomorphism type of $M\#M'$ does not depend on the choice of φ and φ'.
(6) (a) The connected sum operation is commutative, in the sense that there exists an (orientation-preserving) diffeomorphism from $M\#M'$ to $M'\#M$.
(b) If $n \geq 2$, then the connected sum operation is associative, in the sense that if we are furthermore given a connected (oriented) n-dimensional smooth manifold M'', then there exists an (orientation-preserving) diffeomorphism from $(M\#M')\#M''$ to $M\#(M'\#M'')$.

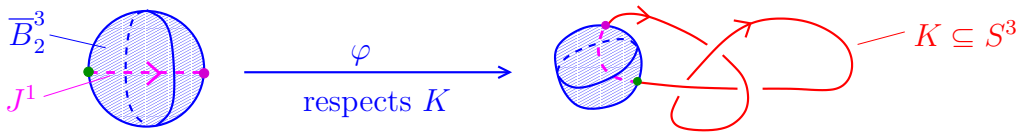
We also recall the following lemma which, which gives us a key example of a connected sum:

Lemma 54.16. (Connected Sum-with-Sphere Lemma) Any connected sum of an oriented connected n-dimensional smooth manifold M with S^n is naturally diffeomorphic to M.

Next we introduce the connected sum of submanifolds. To do so we recall and expand on some concepts from Definition 45.14.

Definition 56.15. Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$.
(1) We denote by $\overline{B}_2^n$ and $\overline{B}_2^k$ the closed balls of radius 2 in $\mathbb{R}^n$ and $\mathbb{R}^k$.
(2) We set $J^k := \{(x, 0) \in \overline{B}_2^n \,|\, x \in \overline{B}_2^k\}$ and we view equip J^k with the obvious orientation.
(3) Let M be an oriented n-dimensional smooth manifold and let $K \subseteq M$ be an oriented k-dimensional submanifold. Furthermore let $\varphi\colon \overline{B}_2^n \to M \setminus \partial M$ be a smooth embedding.
(a) We say that φ respects K if $\varphi^{-1}(\overline{B}_2^n \cap K) = J^k$, if $\varphi\colon \overline{B}_2^n \to M$ is orientation-preserving and if $\varphi|_{J^k}\colon J^k \to \varphi(\overline{B}_2^n) \cap K$ is orientation-preserving.
(b) We say that φ anti-respects K if the same conditions as in (a) hold, except that we now demand that both maps are orientation-reversing.

In the figure below we illustrate the definition of a map that respects K in the special case that $M = \mathbb{R}^3 \cup \{\infty\}_i = S^3$ and that K is in fact a knot.



The following proposition says that maps that respect a submanifold K always exist and that in a sense they are unique.

Proposition 56.16. (Respectful Embedding Proposition) Let M be an oriented n-dimensional smooth manifold. Furthermore let $k \in \{1, \dots, n-1\}$ and let $K \subseteq M$ be an oriented k-dimensional submanifold.
(1) If K is non-empty, then there exists a smooth embedding $\overline{B}_2^n \to M$ that respects K.

(2) If the submanifold K is connected, then for any two smooth embeddings $\varphi\colon \overline{B}^n_2 \to M$ and $\psi\colon \overline{B}^n_2 \to M$ that respect K there exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:
- We have $F_0 = \mathrm{id}$.
- We have $F_1 \circ \varphi = \psi$.
- For each $t \in [0,1]$ the map $F_t \circ \varphi$ respects K and we have $F_t(K) = K$.

The obvious analogs of (1) and (2) also hold for "anti-respect" instead of "respect".

Remark. An analog of the Respectful Embedding Proposition 56.16 for topological manifolds is proved in [**Liv2021**]. □

Sketch of proof. Statement (1) follows easily from the definition of a submanifold. (Note though that we need the hypothesis $k \in \{1, \ldots, n-1\}$ to arrange the orientation behavior by possibly flipping along hyperplanes.) Statement (2) is just a reformulation of the Ball-Submanifold Theorem 45.15. ■

Now we can define the connected sum of smooth submanifolds:

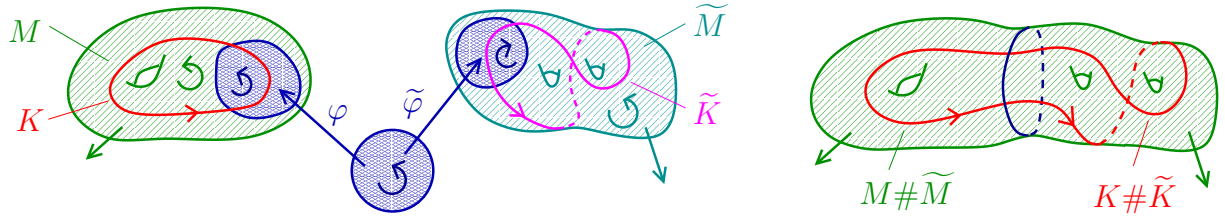
Definition 56.17. We start out with the following data:
- Let M be an oriented n-dimensional smooth manifold and let $K \subseteq M$ be an oriented non-empty k-dimensional submanifold.
- Let $\widetilde{M}$ be an oriented n-dimensional smooth manifold and let $\widetilde{K} \subseteq \widetilde{M}$ be an oriented non-empty k-dimensional submanifold.

We pick a smooth embedding $\varphi\colon \overline{B}^n_2 \to M$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^n_2 \to \widetilde{M}$ that anti-respects $\widetilde{K}$. Now we do the following:

(1) We use $\varphi\colon \overline{B}^n \to M$ and $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M}$ to define $M \# \widetilde{M}$.

(2) We use[628] $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\varphi} K$ and $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\widetilde{\varphi}} \widetilde{K}$ to define the connected sum $K \# \widetilde{K}$.

We refer to the resulting pair $(M \# \widetilde{M}, K \# \widetilde{K})$ as the connected sum of (M, K) and $(\widetilde{M}, \widetilde{K})$.



The next proposition can be viewed as an analog of the Smooth Connected Sum-Proposition 54.13:

Proposition 56.18. (Submanifold–Connected Sum Proposition) Let M be an oriented n-dimensional smooth manifold and let K be a closed oriented non-empty k-dimensional submanifold of M. Furthermore let $\widetilde{M}$ be an oriented n-dimensional smooth manifold and let $\widetilde{K}$ be a closed oriented non-empty k-dimensional submanifold of $\widetilde{M}$.

(1) The subset $K \# \widetilde{K}$ is a closed k-dimensional smooth submanifold of $M \# \widetilde{M}$.

[628] Note that here we deliberately restrict ourselves from balls of radius 2 to balls of radius 1.

(2) If M and $\widetilde{M}$ are both connected, then the oriented diffeomorphism type of the pair $(M\#\widetilde{M}, K\#\widetilde{K})$ does not depend on the choice of the map that respects K and the map that anti-respects $\widetilde{K}$.

Sketch of proof.

(1) We leave it to the reader to verify this statement. Note that here one needs to make use of the fact that the maps that (anti-) respect the submanifold give us control on $\overline{B}^n_2$ and not just on $\overline{B}^n$.

(2) This statement follows quite easily from the Respectful Embedding Proposition 56.16. ■

The following lemma can be viewed as a relative version of the Connected Sum-with-Sphere Lemma 54.16:

Lemma 56.19. (Connected Sum-with-Trivial Sphere Lemma) Any connected sum of a pair (M^n, K^k) and $(S^n, S^k \times \{0\})$ naturally diffeomorphic to (M, K).

Proof. We leave the proof to the reader. ■

As a special case we consider knots in spheres. Now we can finally specialize to our beloved setting of knots in S^3. Following most books in knot theory, see e.g. [**Rolf1990**, p. 40] and [**Kaw1996**, p. 30] we introduce the connected sum of two oriented knots as follows:

Definition 56.20. Let K and $\widetilde{K}$ be two oriented knots in S^3. We pick a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3_2 \to S^3$ that anti-respects $\widetilde{K}$. We perform the connected sum $(S^3\#S^3, K\#\widetilde{K})$. By the Connected Sum-with-Sphere Lemma 54.16 we have an orientation-preserving diffeomorphism $f\colon S^3\#S^3 \to S^3$. It follows from the Smooth Isotopy–Diffeomorphism Proposition 108.11 that the smooth isotopy of $f(K\#\widetilde{K})$ in S^3 is well-defined. We denote this smooth isotopy type by $K\#\widetilde{K}$. It follows from the Submanifold–Connected Sum Proposition 56.18 together with the Smooth Isotopy–Diffeomorphism Proposition 108.11 (2)⇒(1) that the smooth isotopy type of $K\#K'$ is well-defined.

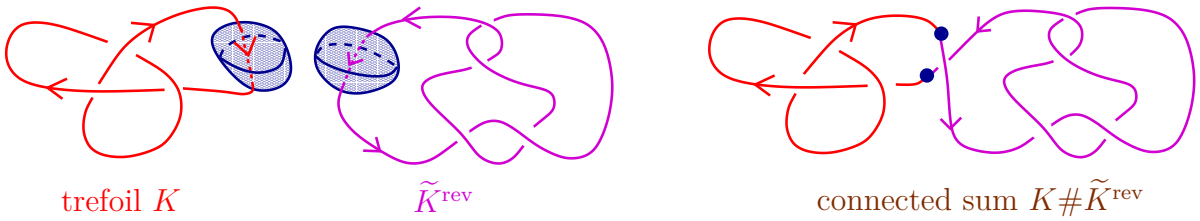
Example. In the figure below we show the connected sum of the trefoil K and the figure-8 knot $\widetilde{K}$. Admittedly it can take a minute to connect the picture to the actual definition.



trefoil K figure-8 knot $\widetilde{K}$ connected sum $K\#\widetilde{K}$

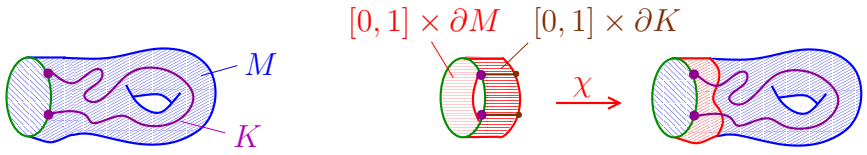
In the next figure we show the connected sum of the trefoil K and of the reverse of the figure-8 knot $\widetilde{K}^{\mathrm{rev}}$. The figure shows that the orientation of the knots plays a role.

Remark. In Section 187.7 we will introduce the "internal connected sum" of two oriented submanifolds. In particular we will introduce the "internal connected sum" of two oriented knots which, for most intents and purposes agrees with the connected sum of two oriented knots. □



56.7. Exercises for Chapter 56.

Exercise 56.1. Let M be a compact smooth manifold and let $K \subseteq M$ be a proper submanifold of M. Show that there exists a collar $\chi\colon [0,1] \times \partial M \to M$ such that χ restricts to a collar $\chi\colon [0,1] \times \partial K \to K$.



Exercise 56.2. A map $\varphi\colon M \to N$ between two smooth manifolds is called a *submersion* if the following conditions are satisfied:

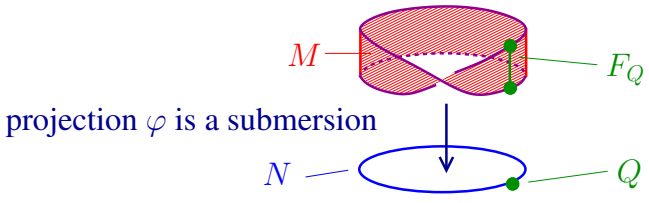
- The map φ is smooth.
- For each $P \in M \setminus \partial M$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P M \to \mathrm{T}_{\varphi(P)} N$ is an epimorphism.
- For each $P \in \partial M$ the map $\mathrm{D}\varphi_P\colon \mathrm{T}_P \partial M \to \mathrm{T}_{\varphi(P)} N$ is an epimorphism.

Now let $\varphi\colon M \to N$ be a submersion. Given $Q \in N$ we set $F_Q \coloneqq \varphi^{-1}(Q)$.

(a) We assume that M is compact, that N is connected and that $\partial N = \varnothing$. Show that all the F_Q are diffeomorphic.

(b) We assume that M is compact, that N is connected and that $\partial N = \varnothing$. Show that all the F_Q are properly smoothly isotopic submanifolds of M.

(c) Show that in (a) we cannot drop the hypothesis that M is compact.



Exercise 56.3. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} f\colon \mathrm{O}(n+1) & \to & S^n \\ (v_1 \ \ldots \ v_{n+1}) & \mapsto & v_{n+1}. \end{array}$$

(a) Show that every $Q \in S^n$ is a regular value of f.

(b) What is the homeomorphism type of each $f^{-1}(Q)$?

Remark. You could make use of Exercise 56.2.

Exercise 56.4. We consider the Hopf map

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

(a) Show that every $Q \in \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ is a regular value of H.
(b) What is the homeomorphism type of each $H^{-1}(Q)$?

Remark. You could make use of Exercise 56.2.

Exercise 56.5. We consider the oriented torus M and the two oriented submanifolds K_1 and K_2 of M that are shown in the figure below.

(a) Draw a figure for the connected sum $(M\#M, K_1\#K_1)$ from Definition 56.17.
(b) Draw a figure for the connected sum $(M\#M, K_1\#K_2)$ from Definition 56.17.



CHAPTER 57

Tubular neighborhoods

In Definition 35.4 we introduced the notion of a collar of the boundary of a smooth manifold and in the Smooth Collar Neighborhood Theorem 35.6 and the Smooth Collar-Uniqueness Proposition 54.2 we showed that collars always exist and they are in a suitable sense unique. These results are an essential tool for the working geometric topologist.

In this chapter we will introduce the analogous notion of a tubular neighborhood of a proper submanifold. Furthermore we will prove the existence and uniqueness of tubular neighborhoods of submanifolds. Again these results are fundamental to the study of differential manifolds.

Unfortunately these statements, let alone the proofs, are more technical than for collars. For many purposes it is enough to look at Chapter 58, where we deal with important special cases, which are much easier to formulate.

This chapter is organized as follows:

- In Section 57.1 we will introduce smooth fiber bundles.
- In Section 57.2 we will introduce linear $\overline{B}^d$-bundles.
- In Section 57.3 we will introduce tubular neighborhoods of smooth submanifolds.
- In Section 57.4 we will state the Tubular Neighborhood–Existence Theorem 57.11 and the Tubular Neighborhood–Uniqueness Theorem 57.12.
- In Section 57.5 we will provide the proof of the Tubular Neighborhood–Existence Theorem 57.11.
- In Section 57.6 we will provide the proof of the Tubular Neighborhood Uniqueness Theorem 57.12.
- In Appendix 57.7 we will see that one can extend tubular neighborhoods from the boundary and we will discuss tubular neighborhoods of two transverse smooth submanifolds.
- In Appendix 57.8 we will discuss tubular neighborhoods of compact non-proper submanifolds.
- In Appendix 57.9 we will use tubular neighborhoods to show that the inclusion of a submanifold into the total smooth manifold is a cofibration.
- In Section 57.10 we will provide exercises for Chapter 57.

In the next Chapter 58 we will study special cases of tubular neighborhoods, namely product tubular neighborhoods and tubular maps.

57.1. **Smooth fiber bundles.** The precise definition of a "tubular neighborhood" requires some preparations, which we will carry out in the next two sections.

Definition 57.1. Let F be a smooth manifold.

(1) Let B be a smooth manifold and let W be a smooth manifold (possibly with corner). A smooth F-trivialization of a given smooth map $p\colon W \to B$ is a diffeomorphism[629] $\Phi\colon W \to B \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow[\cong]{\Phi} & B \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (b,v)\mapsto b} & \\ & B. & \end{array}$$

(2) (a) A smooth F-bundle over a smooth manifold K is a smooth manifold N (possibly with corner) together with a smooth map $p\colon N \to K$ with the following property: Given any point $x \in K$ there exists a smooth F-trivialization of p around x, i.e. there exists an open neighborhood U of $x \in K$ and a smooth F-trivialization of $p\colon p^{-1}(U) \to U$. In other words, there exists a diffeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\Phi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (b,v)\mapsto b} & \\ & U. & \end{array}$$

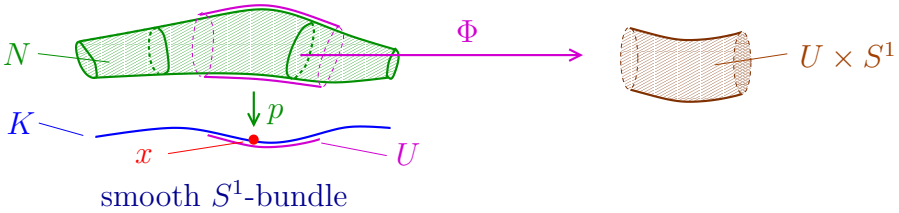
(b) For $x \in K$ we refer to $p^{-1}(x)$ as a fiber of p.

(c) An atlas for p is a family of smooth F-trivializations $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}$ such that $K = \bigcup_{i\in I} U_i$.

(3) An isomorphism between two smooth F-bundles $p\colon M \to K$ and $q\colon N \to K$ is defined to be a diffeomorphism $\Phi\colon M \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & N \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & K. & \end{array}$$

If such an isomorphism exists we say that the two smooth F-bundles are isomorphic.



Example. Let F and K be smooth manifolds. By the Smooth Manifold Product Proposition 26.12 the product $K \times F$ is naturally a smooth manifold with corner. Let $p\colon K \times F \to K$ be the natural projection. This data is evidently a smooth F-bundle over K and we refer to it as the *trivial smooth F-bundle over K*. □

Lemma 57.2. (Fiber Bundle Compactness Lemma) Let F be a smooth manifold and let $(N, p\colon N \to K)$ be a smooth F-bundle. If F and K are compact, then N is also compact.

[629] By the Smooth Manifold Product Proposition 26.12 the product $U \times F$ is naturally a smooth manifold with corner. Thus it makes sense to say that Φ is a diffeomorphism.

Sketch of proof. This lemma follows easily from the fact that the smooth manifold K is regionally compact, the Product Topology-Properties Proposition 7.9 (2) and the Compact–Union-Intersection Lemma 3.41. We leave it to the reader to fill in the details. ■

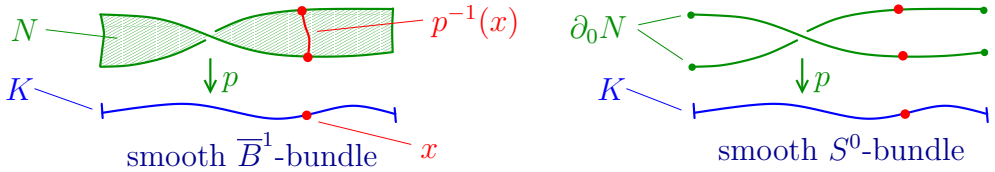
Lemma 57.3. (Boundary Fiber Bundle Lemma) Let F be a smooth manifold and let $(N, p\colon N \to K)$ be a smooth F-bundle.

(1) For each $x \in K$ the preimage $p^{-1}(x)$ is naturally a smooth manifold.

(2) We set

$$\partial_0 N \;:=\; \bigcup_{x\in K} \underset{\substack{\uparrow\\ \text{boundary of the}\\ \text{smooth manifold } p^{-1}(x)}}{\partial p^{-1}(x)}.$$

This topological space is naturally a smooth manifold and the pair $(\partial_0 N, p\colon \partial_0 N \to K)$ is a smooth ∂F-bundle.



Proof. The lemma follows easily from the definitions together with the elementary Boundary–Diffeomorphism Lemma 25.29. We leave the details to the reader. ■

57.2. **Linear $\overline{B}^d$-bundles.** In the pictures tubular neighborhoods look like smooth $\overline{B}^d$-bundles. But this does not quite capture the truth, to define tubular neighborhoods we in fact need the notion of a linear smooth $\overline{B}^d$-bundle, which we are introducing with the following definition:

Definition 57.4.

(1) Let W be a smooth manifold and let $p\colon W \times \overline{B}^d \to W$ and $q\colon W \times \overline{B}^d \to W$ be two smooth $\overline{B}^d$-bundles. We say that an isomorphism $\Phi\colon W \times \overline{B}^d \to W \times \overline{B}^d$ of smooth $\overline{B}^d$-bundles is linear if there exists a continuous map $g\colon W \to \mathrm{O}(d)$ such that for every $(w, v) \in W \times \overline{B}^d$ we have[630]

$$\Phi(w, v) \;=\; (w, g(w) \cdot v).$$

(2) Let $p\colon N \to K$ be a smooth $\overline{B}^d$-bundle. An atlas $\{(U_i, \Phi_i\colon p^{-1}(U_i) \to U_i \times \overline{B}^d)\}_{i\in I}$ for $p\colon N \to K$ is called linear if for every choice of $i, j \in I$ the corresponding isomorphism $\Phi_j \circ \Phi_i^{-1}\colon (U_i \cap U_j) \times \overline{B}^d \to (U_i \cap U_j) \times \overline{B}^d$ of smooth $\overline{B}^d$-bundles over $U_i \cap U_j$ is linear.

(3) A linear smooth $\overline{B}^d$-bundle is a smooth $\overline{B}^d$-bundle $p\colon N \to K$ together with a linear atlas of smooth trivializations.[631] Henceforth, by a trivialization of a linear smooth $\overline{B}^d$-bundle we always mean a trivialization from the given atlas.

Examples.

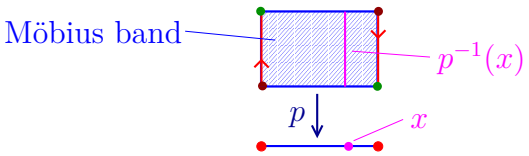
(1) Any trivial $\overline{B}^d$-bundle $p\colon K \times \overline{B}^d \to K$ over a smooth manifold defines a linear smooth $\overline{B}^d$-bundle in the obvious way.

[630] Note that the map $g\colon W \to \mathrm{O}(d)$ can be read off from the differential of Φ. It follows that the map $g\colon W \to \mathrm{O}(d)$ is necessarily smooth.

[631] Similar to many other occasions we will suppress this choice of atlas from the notation.

(2) We consider the Möbius band $M = ([0,1]\times[-1,1])/(0,y)\sim(1,-y)$ together with the projection

$$\begin{array}{rcl} p\colon M = ([0,1]\times[-1,1])/\sim & \to & [0,1]/0\sim 1 \\ [(x,y)] & \mapsto & [x]. \end{array}$$



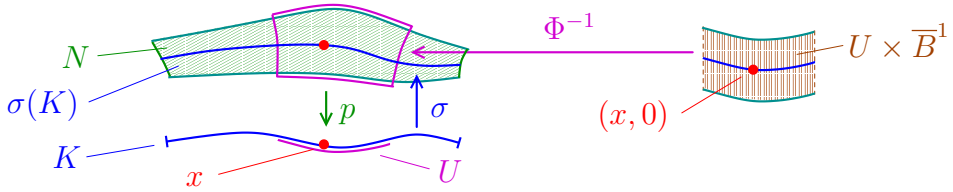
In Exercise 57.2 we will verify that, with suitable trivializations, this map is a linear smooth interval bundle. □

We conclude this section with two fairly elementary lemmas on linear smooth $\overline{B}^d$-bundles.

Lemma 57.5. (0-Section Lemma) Let $(N, p\colon N\to K)$ be a linear smooth $\overline{B}^d$-bundle. We consider the map

$$\begin{array}{rcl} \sigma\colon K & \to & N \\ x & \mapsto & \Phi^{-1}(x,0) \end{array}$$

where $\Phi\colon p^{-1}(U)\to U\times\overline{B}^d$ is any trivialization with $x\in U$ from the given linear atlas. The above map is well-defined and a proper smooth embedding. We refer to this map as the 0-section of $p\colon N\to K$. Sometimes we also call $\sigma(K)\subseteq N$ the 0-section of p.



Proof. First let us show that $\sigma\colon K\to N$ is well-defined. So let $\widetilde{\Phi}\colon p^{-1}(\widetilde{U})\to\widetilde{U}\times\overline{B}^d$ be another trivialization with $x\in\widetilde{U}$ from the given linear atlas. Note that by definition of a linear atlas there exists a map $g\colon U\cap\widetilde{U}\to \mathrm{O}(d)$ such that for every $(x,v)\in(U\cap\widetilde{U})\times\overline{B}^d$ we have $(\widetilde{\Phi}\circ\Phi^{-1})(x,v) = (x, g(x)\cdot v)$. It follows that

$$\Phi^{-1}(x,0) \;=\; \widetilde{\Phi}^{-1}((\widetilde{\Phi}\circ\Phi^{-1})(x,0)) \;=\; \widetilde{\Phi}^{-1}(x,\underbrace{g(x)\cdot 0}_{=0}) \;=\; \widetilde{\Phi}^{-1}(x,0).$$

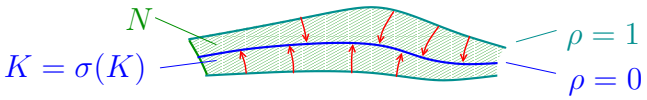
We have thus shown that σ is well-defined. Now that we know that σ is well-defined it follows easily from the observation that the map $U\to U\times\overline{B}^d$ given by $x\mapsto(x,0)$ is a proper smooth embedding that also $\sigma\colon K\to N$ is a proper smooth embedding. ■

Lemma 57.6. (Radial Distance Lemma) Let $p\colon N\to K$ be a smooth linear $\overline{B}^d$-bundle. We use the 0-Section Lemma 57.5 to identify K with its 0-section in N.

(1) Given $x\in N$ we pick a trivialization $\Phi\colon p^{-1}(U)\to U\times\overline{B}^d$ from the given linear atlas with $x\in p^{-1}(U)$, we write $\rho(x) =: (x,v)$ and we set $\rho(x) := \|v\|$. The map $\rho\colon N\to[0,1]$ is well-defined. We refer to $\rho\colon N\to[0,1]$ as the radial distance function.

(2) The radial distance function $\rho\colon N\to[0,1]$ is continuous and it satisfies $\rho|_{\partial_0 N}\equiv 1$ and $\rho^{-1}(0) = K$.

Proof.



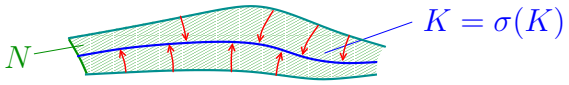
(1) As in the proof of the 0-Section Lemma 57.5 one can easily verify that any two trivializations from the linear atlas give the same result.
(2) This statement follows easily from the definitions. ■

Definition 57.7. Let $d \in \mathbb{N}_0$ and let $p\colon N \to K$ be a smooth linear $\overline{B}^d$-bundle. We denote by $\rho\colon N \to [0,1]$ the radial distance function. Given $\epsilon > 0$ we set

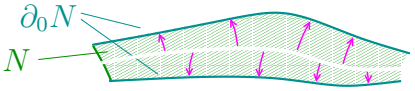
$$N(\epsilon) \coloneqq \{x \in N \mid \rho(x) \leq \epsilon\}.$$

Lemma 57.8. (Radial Shrinking Lemma) Let $(N, p\colon N \to K)$ be a linear smooth $\overline{B}^d$-bundle. We use the 0-Section Lemma 57.5 to identify K with its 0-section in N.

(1) For each $\epsilon \in (0,1]$ the restriction of p to $p\colon N(\epsilon) \to N$ is naturally a smooth linear $\overline{B}^d$-bundle.
(2) There exists a smooth map $R\colon N \times [0,1] \to N$ with the following properties:
  (a) • R is a deformation retraction from N to K.
  • For every $(x,t) \in N \times [0,1]$ we have $p(R(x,t)) = p(x)$.
  (b) For every choice of $s \in [0,1)$ the map $R_s\colon N \to N$ restricts to an isomorphism $R_s\colon N \to N(1-s)$ of linear smooth $\overline{B}^d$-bundles.
  (c) We now suppose that K is compact. For every open neighborhood U of K there exists an $\epsilon > 0$ such that $R(N \times [1-\epsilon, 1]) \subseteq U$.
(3) There exists a smooth map $S\colon (N \setminus K) \times [0,1] \to N \setminus K$ with the following two properties:
  • S is a deformation retraction from $N \setminus K$ to $\partial_0 N$.
  • For every $(x,t) \in N \times [0,1]$ we have $p(S(x,t)) = p(x)$.



deformation retraction R from N to K



deformation retraction S from $N \setminus K$ to $\partial_0 N$

Proof.

(1) This statement follows immediately from the observation that the map $\overline{B}^d \to \overline{B}^d_\epsilon$ given by $x \mapsto \epsilon \cdot x$ is a diffeomorphism that commutes with multiplication by orthogonal matrices.
(2) (a) For any smooth manifold U the map

$$\begin{array}{rcl} (U \times \overline{B}^d) \times [0,1] & \to & U \times \overline{B}^d \\ ((x,v),t) & \mapsto & (x, (1-t)\cdot v) \end{array}$$

[631] As in the proof of the 0-Section Lemma 57.5 one can easily verify that any two trivializations from the linear atlas give the same result.

is a smooth deformation retraction from $U \times \overline{B}^d$ to $U \times \{0\}$. Since this map commutes with fiberwise multiplication by an orthogonal matrix we obtain, using our linear atlas, the desired smooth deformation retraction from N to K. More precisely, let $\Phi\colon p^{-1}(U) \to U \times \overline{B}^d$ be any trivialization from the linear atlas. The map

$$p^{-1}(U)\times[0,1] \xrightarrow{\operatorname{diag}(\Phi,\mathrm{id}_{[0,1]})} (U\times\overline{B}^d)\times[0,1] \xrightarrow{((x,v),t)\mapsto(x,(1-t)\cdot v)} U\times\overline{B}^d \xrightarrow{\Phi^{-1}} p^{-1}(U)$$

is clearly a smooth deformation retraction from $p^{-1}(U)$ to $K \cap p^{-1}(U)$. As in the proof of the 0-Section Lemma 57.5 one can easily verify that for two trivializations from the linear atlas the corresponding maps agree on the overlap. It is now clear that $R\colon N \times [0,1] \to N$ is smooth. Finally note that it follows easily from the definition of p that for every $(x,t) \in N \times [0,1]$ we have $p(R(x,t)) = p(x)$.

(b) Let $s \in [0,1)$. It is straightforward to verify that $R_s(N) = N(1-s)$ and that $R_s\colon N \to N(1-s)$ is an isomorphism of linear smooth $\overline{B}^d$-bundles.

(c) We now assume that K is compact. It follows from the Fiber Bundle Compactness Lemma 57.2 that N is compact. Next let U be an open neighborhood of K. Note that $N \times \{1\} \subseteq R^{-1}(U)$. It follows from these two observations and the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that $R(N \times [1-\epsilon, 1]) \subseteq U$.

(3) For any smooth manifold U the map

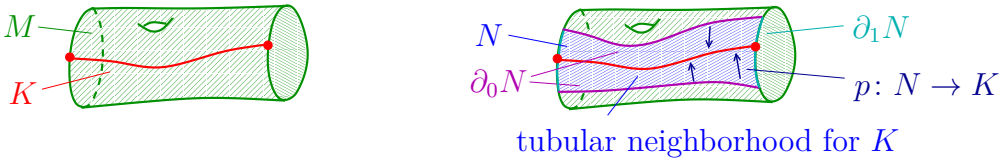
$$\begin{array}{rcl}(U \times (\overline{B}^d \setminus \{0\})) \times [0,1] & \to & U \times (\overline{B}^d \setminus \{0\}) \\ ((x,v),t) & \mapsto & (x, (1-t)\cdot v + t \cdot \frac{v}{\|v\|})\end{array}$$

is a smooth deformation retraction from $U \times (\overline{B}^d \setminus \{0\})$ to $U \times S^{d-1}$. As in (2) we see that these maps give rise to a smooth deformation retraction S from $N \setminus K$ to $\partial_0 N$ such that for every $(x,t) \in N \times [0,1]$ we have $p(S(x,t)) = p(x)$. ■

57.3. **Tubular neighborhoods.** Now we can formulate the definition that we had been aiming for.

Definition 57.9. Let M be an m-dimensional smooth manifold and let K be a proper k-dimensional submanifold of M. A tubular neighborhood for K is a pair $(N, p\colon N \to K)$ with the following properties:

(1) N is a codimension-zero submanifold of M with corner with $\partial_1 N = p^{-1}(\partial K)$.
(2) The pair $(N, p\colon N \to K)$ is a linear smooth $\overline{B}^{m-k}$-bundle such that K is the 0-section of p.
(3) The ∂_0-boundary of the submanifold N with corner, i.e. $\partial_0 N = (N \cap (\overline{M \setminus N})$, equals the ∂_0-boundary of the smooth $\overline{B}^{m-k}$-bundle $N \to K$.
(4) N is a closed subset of M.[632]



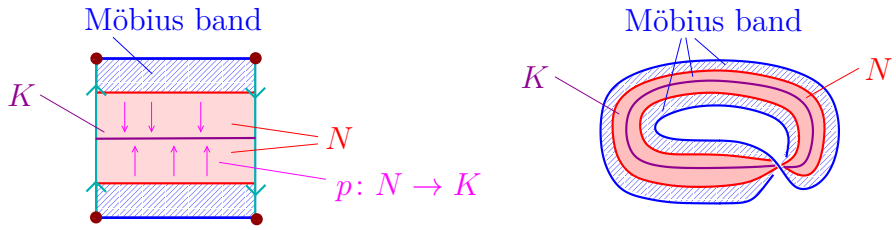
tubular neighborhood for K

[632]It follows from the Fiber Bundle Compactness Lemma 57.2 and the Compact-Closed Lemma 3.39 that this condition is automatically satisfied if K is compact.

Remark. There are many books that introduce the notion of a tubular neighborhood, see e.g. [**Kos1993, BJ1982, Bre1993, Lang2002a**] and [**Wall2016**]. The definitions and results tend to be subtly different and it requires some effort to go back and forth between these books. Our treatment is, by accident, very similar to the one given in [**Wall2016**, Chapter 2.5]. □

Example.

(1) As above we consider the Möbius band $M = ([0,1] \times [-1,1])/\sim$ and the submanifold $K = [0,1] \times \{0\} \subseteq M$ that is given by the central circle. In the figure below we illustrate a tubular neighborhood for K.



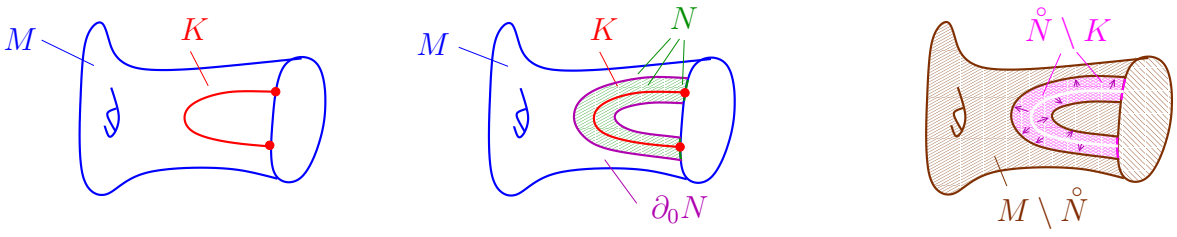
(2) Let M be a smooth manifold and let $K \subseteq M$ be a closed codimension-one submanifold of M. Recall that in Definition 55.1 we defined a bicollar for K to be a smooth embedding $\beta\colon [-1,1] \times K \to M \setminus \partial M$ such that $\beta(0,x) = x$ for all $x \in K$. Using the Smooth Embedding Theorem 30.15 and the Codimension-Zero Submanifold Proposition 25.49 one can easily verify that in this setting a bicollar is essentially the same as a tubular neighborhood.

(3) In the Projective Space-minus-Ball Lemma 238.11 we will give explicit tubular neighborhoods for $\mathbb{R}\mathrm{P}^1 \subseteq \mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$ and $\mathbb{H}\mathrm{P}^1 \subseteq \mathbb{H}\mathrm{P}^2$. □

The following lemma, which is loosely modeled on the Basics-of-Collars Lemma 35.7, justifies in particular the name tubular *neighborhood*.

Lemma 57.10. (Basics-of-Tubular Neighborhood Lemma) Let M be a smooth manifold, let K be a compact proper submanifold of M and let $(N, p\colon N \to K)$ be a tubular neighborhood of K.

(1) (a) $\mathring{N} := N \setminus \partial_0 N$ is the interior of $N \subseteq M$.
    (b) N is a neighborhood of $K \subseteq M$.
(2) $M \setminus \mathring{N}$ is a codimension-zero submanifold with corner of M.
(3) (a) $M \setminus \mathring{N}$ is a deformation retract of $M \setminus K$.
    (b) The inclusion $M \setminus K \to M \setminus \mathring{N}$ is a homotopy equivalence.



Proof.

(1) By definition N is a codimension-zero submanifold of M with corner. Statement (a) thus follows from the Smooth Submanifold-with-Corner Lemma 26.16 (3). Statement (b) follows immediately from Statement (a).
(2) This statement follows from the Smooth Submanifold-with-Corner Lemma 26.16 (4).
(3) (a) Recall that by the Radial Shrinking Lemma 57.8 (3) there exists a deformation retraction $R\colon (N \setminus K) \times [0,1] \to N \setminus K$ from $N \setminus K$ to $\partial_0 N$. We extend this deformation retraction to all of $M \setminus K$ in the obvious way. More precisely, we consider the map

$$\begin{aligned} F\colon (M \setminus K) \times [0,1] &\to M \setminus K \\ (P,t) &\mapsto \begin{cases} R(P,t), & \text{if } P \in N \setminus K, \\ P, & \text{if } P \notin \mathring{N}. \end{cases} \end{aligned}$$

Using the Homotopy Combination Lemma 18.8 (1) and using (1a) one can easily show that this map is indeed continuous. Once continuity is verified it is straightforward to show that F is indeed a deformation retraction from $M \setminus K$ to $M \setminus \mathring{N}$.
(b) It follows from (a) and the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the inclusion $M \setminus K \to M \setminus \mathring{N}$ is indeed a homotopy equivalence. ■

57.4. **Tubular neighborhoods: Existence and uniqueness.** Now is the time to state the first of the two main theorems of this chapter, namely the theorem which guarantees us the existence of tubular neighborhoods of compact proper submanifolds:

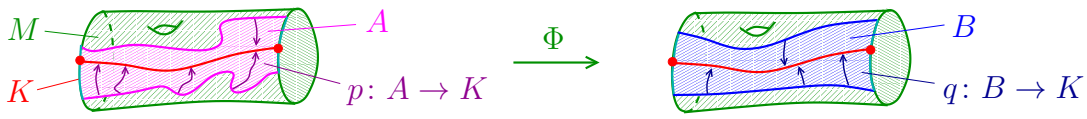
Theorem 57.11. (Tubular Neighborhood–Existence Theorem) Every compact[633] proper submanifold of a smooth manifold admits a tubular neighborhood.

Proof. The proof of the theorem is carried out in Section 57.5. ■

The following theorem shows that tubular neighborhoods are in a suitable sense unique.

Theorem 57.12. (Tubular Neighborhood–Uniqueness Theorem) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $p\colon A \to K$ and $q\colon B \to K$ be two tubular neighborhoods of K. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:
(1) $F_0 = \mathrm{id}$.
(2) F is a homotopy rel K.
(3) The diffeomorphism $\Phi = F_1\colon M \to M$ restricts to an isomorphism[634] $\Phi\colon A \to B$ of smooth $\overline{B}^{m-k}$-bundles.[635] [636]
(4) The diffeotopy F has compact support.

[633] The statement of the Tubular Neighborhood–Existence Theorem 57.11 surely also holds if K non-compact. We leave it to the reader to modify our argument to deal with the non-compact setting.
[634] See Definition 57.1 for the definition of an isomorphism of smooth F-bundles.
[635] Thus the diffeomorphism $\Phi\colon M \to M$ restricts to diffeomorphisms $\Phi\colon A \to B$ and $\Phi\colon M \setminus \mathring{A} \to M \setminus \mathring{B}$.
[636] An *isomorphism of linear smooth $\overline{B}^n$-bundles* $p\colon C \to K$ and $q\colon D \to K$ is defined as an isomorphism $\Theta\colon C \to D$ of smooth $\overline{B}^n$-bundles such that for every choice of trivializations $(U, \Phi\colon p^{-1}(U) \to U \times \overline{B}^n)$ of p and $(V, \Psi\colon q^{-1}(V) \to V \times \overline{B}^n)$ of q the isomorphism $\Phi \circ \Theta \circ \Psi^{-1}\colon (U \cap V) \times \overline{B}^n \to (U \cap V) \times \overline{B}^n$ of smooth $\overline{B}^n$-bundles over $U \cap V$ is linear. One would think that one can find F such that $F_1\colon A \to B$ is an isomorphism of smooth linear $\overline{B}^{m-k}$-bundles. It is not clear to me how one can arrange this.



Proof. We postpone the proof to Section 57.6. In fact, the proof of the Tubular Neighborhood–Uniqueness Theorem 57.12 was a pain to write and it is a pain to read. It is thus recommended to first read the proof of the Tubular Map–Uniqueness Theorem 58.5, which we provide in Section 58.4. This proof already contains many of key ideas. If there is still some appetite left for uniqueness proofs, then one should venture into Section 57.6. ■

Convention 57.13. By the Tubular Neighborhood–Uniqueness Theorem 57.12 we usually do not have to worry about the precise choice of a tubular neighborhood. Given a compact proper submanifold K of a smooth manifold M we usually denote by $N(K)$ some choice of a tubular neighborhood.

Remark.

(1) A proof of the Tubular Neighborhood–Existence Theorem 57.11 is also provided in [**Wall2016**, Theorem 2.3.3]. Closely related and similar results are also proved in [**Kos1993**, Section III.2], [**Bre1993**, Theorem II.11.14], [**Lee2002**, Theorem 6.24], [**BJ1982**, Theorem 12.11] and [**Lang2002a**, Chapter IV]. Furthermore, by [**Bre1972**, Theorem VI.2.2] there exists a strengthened version in case a Lie group G acts on M, preserving K.

(2) A proof of the Tubular Neighborhood–Uniqueness Theorem 57.12 is also provided in [**Wall2016**, Theorem 2.5.5 and Theorem 2.5.8]. We refer to [**BJ1982**, Theorem 12.13], [**Kos1993**, Corollary III.3.2] and [**Lang2001**, Theorem IV.6.2] for other approaches.

(3) With slight modifications the definition of a tubular neighborhood also makes sense for a proper submanifold K of a *topological* manifold M.
   (a) By [**FNOP2025**, Theorem 4.6] (which builds on [**FQ1990**, Chapter 9]) tubular neighborhoods exist for any compact proper submanifold of a 4-dimensional topological manifold. If the submanifold is one-dimensional and the topological manifold is orientable, then there exists a tubular neighborhood, which is homeomorphic to a product.
   (b) If K is a codimension 2 submanifold, then some type of tubular neighborhood seems to exist, see e.g. [**DV2009**, Theorem 6.8.1] and [**FQ1990**, p. 137] for details. We also refer to [**PedE1977**] and [**JohF1972**] for some positive results.
   (c) If K is a codimension 3 submanifold, then, according to [**DV2009**, p. 347] and [**Edw2009**, p. 9], a tubular neighborhood does not exist in general. □

In the remainder of this chapter (except for the appendix) we will only be concerned with the proofs of the Tubular Neighborhood–Existence Theorem 57.11 and the Tubular Neighborhood–Uniqueness Theorem 57.12. In the next chapter we will discuss several special cases.

57.5. **Proof of the Tubular Neighborhood–Existence Theorem 57.11.** In many ways the Tubular Neighborhood–Existence Theorem 57.11 can be viewed as a refined version of the Neighborhood Projection Theorem 35.2. It is thus no surprise that we will

borrow some ideas from the proof of the Neighborhood Projection Theorem 35.2. In fact we will see that the proof of the Tubular Neighborhood–Existence Theorem 57.11 consists of three steps:

(1) We construct an "external vector bundle" over K.
(2) The "disk bundle" corresponding to the "external vector bundle" gives us an "external" linear smooth $\overline{B}^{m-k}$-bundle over K.
(3) We embed this external linear smooth $\overline{B}^{m-k}$-bundle into M to obtain the desired tubular neighborhood.

We will perform these steps in the following three subsections.

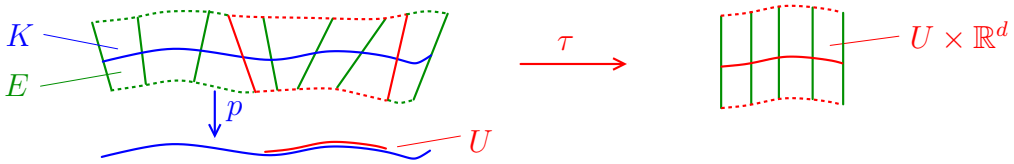
57.5.1. **Proof of the Tubular Neighborhood–Existence Theorem 57.11: Vector bundles.** As a first step towards showing the existence of linear smooth $\overline{B}^d$-bundles we will discuss the related concept of Euclidean $\mathbb{R}^d$-vector bundles:

Definition 57.14. A Euclidean $\mathbb{R}^d$-vector bundle is a triple $(K, E, p\colon E \to K)$ with the following properties:

(1) K is a smooth manifold.
(2) There exists an $n \geq 0$ such that E is a submanifold of $K \times \mathbb{R}^n$.
(3) The map $p\colon E \to K$ is the restriction of the natural projection $K \times \mathbb{R}^n \to K$.
(4) For each $x \in K$ there exists a vector subspace $V_x \subseteq \mathbb{R}^n$ with $p^{-1}(x) = \{x\} \times V_x$.
(5) For each $x \in K$ there exists an open neighborhood U of $x \subseteq K$ and a diffeomorphism $\tau\colon p^{-1}(U) \to U \times \mathbb{R}^d$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\tau} & U \times \mathbb{R}^d \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,v)\mapsto x} & \\ & U & \end{array}$$

and such that for each $x \in U$ the map $V_x \xrightarrow{v\mapsto(x,v)} \{x\} \times V_x \xrightarrow{\tau} \{x\} \times \mathbb{R}^d \xrightarrow{(x,v)\mapsto v} \mathbb{R}^d$ is an isomorphism of vector spaces. We call such a map τ a trivialization of $p\colon E \to K$.



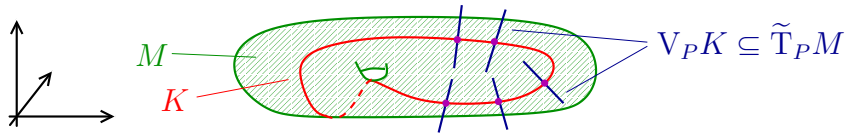
Lemma 57.15. (Submanifold Normal Vectors Lemma) Let M be an m-dimensional smooth submanifold of some $\mathbb{R}^n$ and let $K \subseteq M$ be a compact proper k-dimensional submanifold of M. For each $P \in K$ we denote by $\mathrm{V}_P K \subseteq \widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$ the orthogonal complement of $\widetilde{\mathrm{T}}_P K \subseteq \widetilde{\mathrm{T}}_P M$. We consider the topological space

$$\mathrm{V}K \;:=\; \{(P, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid P \in K \text{ and } v \in \mathrm{V}_P K\}$$

and the map

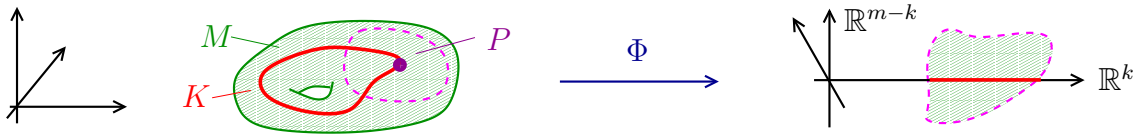
$$\begin{array}{rcl} \Pi\colon \mathrm{V}K & \to & K \\ (P, v) & \mapsto & P \end{array}$$

The triple $(\mathrm{V}K, K, \Pi\colon \mathrm{V}K \to K)$ is a Euclidean $\mathbb{R}^{m-k}$-bundle.

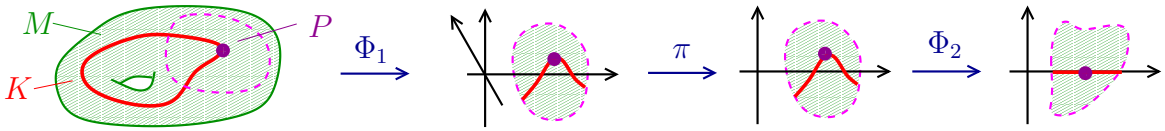


Proof. To simplify the notation a little bit we assume that $\partial M = \varnothing$ and that $\partial K = \varnothing$. The general case is proved basically the same way. We start out with the following very general claim, which is basically the content of Exercise 25.35.

Claim 1. For each $P \in K$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$, an open subset $V \subseteq \mathbb{R}^n$ and a diffeomorphism $\Phi\colon U \to V$ with the following properties:
(1) $\Phi(U \cap M) = V \cap (\mathbb{R}^m \times \{0\})$.
(2) $\Phi(U \cap K) = V \cap (\mathbb{R}^k \times \{0\})$.



Proof. Since M is a submanifold of $\mathbb{R}^n$ with $\partial M = \varnothing$ we know that there exists an open neighborhood $U_1 \subseteq \mathbb{R}^n$ of $P \in K \subseteq \mathbb{R}^n$, an open subset $V_1 \subseteq \mathbb{R}^n$ and a diffeomorphism $\Phi_1\colon U_1 \to V_1$ such that $\Phi_1(U_1 \cap M) = V_1 \cap (\mathbb{R}^m \times \{0\})$. We denote by $\pi\colon \mathbb{R}^m \times \{0\} \to \mathbb{R}^m$ the obvious diffeomorphism. Since K is a submanifold of M with $\partial K = \varnothing$ we see that $\pi(\Phi_1(U_1 \cap K))$ is a submanifold of $\mathbb{R}^m$ with empty boundary. This means that there exists an open neighborhood U_2 of $\pi(\Phi_1(P)) \in \pi(V_1) \subseteq \mathbb{R}^m$, an open subset V_2 of $\mathbb{R}^m$ and a diffeomorphism $\Phi_2\colon U_2 \to V_2$ such that $\Phi_2(\pi(\Phi_1(U_1 \cap K))) = V_2 \cap (\mathbb{R}^k \times \{0\})$.

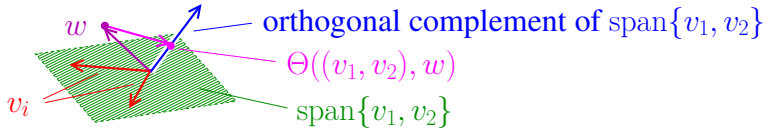


One can now easily verify that $U := \Phi_1^{-1}(\pi^{-1}(U_2) \cap V_1)$, the map $\Phi := (\Phi_2, \mathrm{id}_{\mathbb{R}^{n-m}}) \circ \Phi_1$ and $V := \Phi(U)$ have the desired properties. ⊞

Next we consider the following map:[637]

$$\begin{aligned} \Theta\colon (\mathbb{R}^n \setminus \{0\})^k \times \mathbb{R}^n &\to \mathbb{R}^n \\ ((v_1, \dots, v_k), w) &\mapsto \underbrace{w - \sum_{i=1}^{k} \langle w, v_i \rangle \cdot \frac{v_i}{\|v_i\|^2}}_{\text{orthogonal to } v_1, \dots, v_k}. \end{aligned}$$

Note that Θ is the map that projects the vector w orthogonally onto the orthogonal complement of the vector subspace spanned by $v_1, \dots, v_k$.



[637] Here $\langle\ ,\ \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the usual Euclidean bilinear symmetric form on $\mathbb{R}^n$.

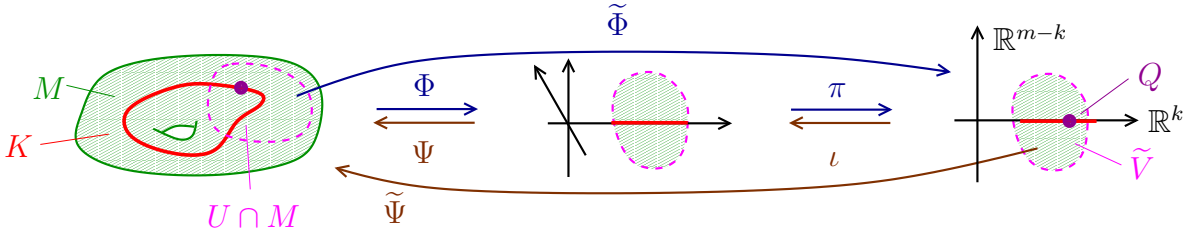
Claim 2. Let $\Phi\colon U \to V$ be a diffeomorphism as in Claim 1 for $K \subseteq M \subseteq \mathbb{R}^n$. We introduce the following notation:

- We write $\Psi := \Phi^{-1}\colon V \to U$.
- We denote by $\pi\colon \mathbb{R}^n \to \mathbb{R}^m$ the projection that is given by $(v,w) \mapsto v$.
- We write $\widetilde{\Phi} := \pi \circ \Phi\colon U \to \mathbb{R}^m$.
- We write $\widetilde{V} := \{v \in \mathbb{R}^m \mid (v,0) \in V\}$.
- We denote by $\iota\colon \widetilde{V} \to V$ the injection that is given by $v \mapsto (v,0)$.
- We write $\widetilde{\Psi} := \Psi \circ \iota\colon \widetilde{V} \to M$.

The map

$$\begin{aligned} \widetilde{\Phi}(K) \times \mathbb{R}^{m-k} &\to \Pi^{-1}(U \cap K) \\ (Q,v) &\mapsto \big(\widetilde{\Psi}(Q), \Theta\big(\underbrace{(\mathrm{D}\widetilde{\Psi}_Q(\mathfrak{e}_1), \dots, \mathrm{D}\widetilde{\Psi}_Q(\mathfrak{e}_k))}_{\text{basis of } \mathrm{T}_{\widetilde{\Psi}(Q)}K}, \underbrace{\mathrm{D}\widetilde{\Psi}_Q(w)}_{\in \mathrm{T}_{\widetilde{\Psi}(Q)}M}\big)\big)_{\in \mathrm{V}_{\widetilde{\Psi}(Q)}K}. \end{aligned}$$

is a diffeomorphism and its inverse is a trivialization of $\Pi\colon \mathrm{V}K \to K$.



Proof. The proof of Claim 2 is pretty similar to the proof of the Normal Vectors Lemma 35.3. We leave it to the reader to adapt the earlier argument. ⊞

Using a mild modification of the map given in Claim 2 one can also easily show that $\mathrm{V}K$ is a submanifold of $K \times \mathbb{R}^n$. We have thus verified that $\Pi\colon \mathrm{V}K \to K$ is a Euclidean $\mathbb{R}^{m-k}$-bundle. ■

57.5.2. **Proof of the Tubular Neighborhood–Existence Theorem 57.11: Disk bundles.** In this section we will build linear smooth $\overline{B}^d$-bundles out of the Euclidean $\mathbb{R}^d$-bundles from the previous section.

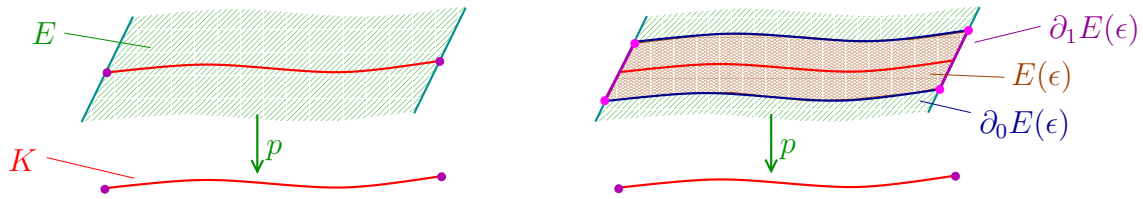
Definition 57.16. Let $(K, E, p\colon E \to K)$ be a Euclidean vector bundle where $E \subseteq K \times \mathbb{R}^n$. Given $\epsilon > 0$ we set

$$\begin{aligned} E(\epsilon) &:= \{(x,v) \in K \times \mathbb{R}^n \mid \|v\| \leq \epsilon\} \\ \mathrm{S}E(\epsilon) &:= \{(x,v) \in K \times \mathbb{R}^n \mid \|v\| = \epsilon\}. \end{aligned}$$

Lemma 57.17. ($\mathbb{R}^d$-Bundle-to-$\overline{B}^d$-Bundle Lemma) Let $(K, E, p\colon E \to K)$ be a Euclidean $\mathbb{R}^n$-vector bundle and let $\epsilon > 0$.

(1) $E(\epsilon)$ is a codimension-zero submanifold with corner of E where $\partial_0 E(\epsilon) = \mathrm{S}E(\epsilon)$ and $\partial_1 E(\epsilon) = p^{-1}(\partial K) \cap E(\epsilon)$.
(2) The restriction of p to $p\colon E(\epsilon) \to K$ defines a linear smooth $\overline{B}^d$-bundle.
(3) For the smooth $\overline{B}^d$-bundle $p\colon E(\epsilon) \to K$ we consider $\partial_0 E(\epsilon)$ as defined in the Boundary Fiber Bundle Lemma 57.3. With this definition we have $\partial_0 E(\epsilon) = \mathrm{S}E(\epsilon)$.

Proof. Let $(K, E, p\colon E \to K)$ be a Euclidean $\mathbb{R}^d$-vector bundle where $E \subseteq K \times \mathbb{R}^n$. Recall that this means in particular that for each $x \in K$ we have $p^{-1}(x) = \{x\} \times V_x$ where $V_x \subseteq \mathbb{R}^n$



is a vector subspace. Let $U \subseteq K$ be an open subset and let $\tau\colon p^{-1}(U) \to U \times \mathbb{R}^d$ be a trivialization. Recall that this means that $\tau\colon p^{-1}(U) \to U \times \mathbb{R}^d$ is a diffeomorphism such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\tau} & U \times \mathbb{R}^d \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,v)\mapsto x} & \\ & U & \end{array}$$

and such that for each $x \in U$ the map $V_x \xrightarrow{v\mapsto(x,v)} \{x\} \times V_x \xrightarrow{\tau} \{x\} \times \mathbb{R}^d \xrightarrow{(x,v)\mapsto v} \mathbb{R}^d$ is an isomorphism of vector spaces.

The problem we encounter is that even though each $V_x \to \mathbb{R}^d$ is linear, this map is in general not length preserving, so it will not send unit balls to unit balls. To be able rectify this problem we first need to introduce some notation:

- We denote by $\pi\colon K \times \mathbb{R}^n \to \mathbb{R}^n$ the projection.
- For $i = 1, \dots, d$ we consider the map

$$\begin{array}{rcl} \Xi_i\colon (\mathbb{R}^n)^d & \to & \mathbb{R}^n \\ & & \text{the } i\text{-th vector which we obtain from the} \\ (v_1, \dots, v_d) & \mapsto & \text{Gram-Schmidt Orthogonalization Process 2.32} \\ & & \text{applied to the } d \text{ vectors } v_1, \dots, v_d \in \mathbb{R}^n \end{array}$$

- Given $Q \in U$ we set $\alpha_i(Q) \coloneqq \Xi_i(\underbrace{\pi(\tau^{-1}(Q, \mathfrak{e}_1)), \dots, \pi(\tau^{-1}(Q, \mathfrak{e}_d))}_{\text{basis of } \pi(p^{-1}(Q)) \subseteq \mathbb{R}^n}) \in \pi(p^{-1}(Q))$.

Note that $\alpha_1(Q), \dots, \alpha_d(Q)$ form an orthonormal basis for $\pi(p^{-1}(Q)) \subseteq \mathbb{R}^n$.

Claim. The map

$$\begin{array}{rcl} \Psi_\tau\colon U \times \overline{B}^d & \to & E(\epsilon) \cap p^{-1}(U) \\ \left(Q, \sum_{i=1}^d \lambda_i \cdot \mathfrak{e}_i\right) & \mapsto & \left(Q, \epsilon \cdot \sum_{i=1}^d \lambda_i \cdot \alpha_i(Q)\right) \end{array}$$

is a diffeomorphism and its inverse is a trivialization of $p\colon E(\epsilon) \to K$ as a smooth $\overline{B}^d$-bundle.

Proof. Using the fact that $\alpha_1(G), \dots, \alpha_d(Q)$ form an orthonormal basis for $\pi(p^{-1}(Q)) \subseteq \mathbb{R}^n$ one can easily verify that $\Psi_\tau\colon U \times \overline{B}^d \to U \times \mathbb{R}^d$ takes values in $E(\epsilon)$. Furthermore it follows easily from the definitions that $\Psi_\tau^{-1}\colon E(\epsilon) \cap p^{-1}(U) \to U \times \overline{B}^d$ is smooth and a bijection. Using the Inverse Mapping Theorem 30.1 (3) one can now verify that Ψ_τ is a diffeomorphism. Finally it follows easily from the various definitions that the inverse $\Psi_\tau^{-1}\colon E(\epsilon) \cap p^{-1}(U) \to U \times \overline{B}^d$ is indeed a trivialization of $p\colon E(\epsilon) \to K$ as a smooth $\overline{B}^d$-bundle. We leave it to the reader to fill in the details. ⊞

Now we turn to the proof of the three statements of the lemma:

(1) It is pretty clear that $U \times \overline{B}^d$ is a codimension-zero submanifold with corner of $U \times \mathbb{R}^d$ where $\partial_0(U \times \overline{B}^d) = U \times S^{m-1}$ and $\partial_1(U \times \overline{B}^d) = \partial U \times \overline{B}^d$. It now follows easily from the claim that $E(\epsilon)$ is a codimension-zero submanifold with corner of E where $\partial_0 E(\epsilon) = \mathrm{S}E(\epsilon)$ and $\partial_1 E(\epsilon) = p^{-1}(\partial K)(\epsilon)$.

(2) Let $\{\tau_i \colon p^{-1}(U_i) \to U_i \times \mathbb{R}^d\}_{i\in I}$ be a family of trivializations of the Euclidean $\mathbb{R}^d$-vector bundle $p \colon E \to K$ with $K = \bigcup_{i\in I} U_i$. Note that from the above claim we obtain a corresponding family $\{\Psi_{\tau_i}^{-1} \colon E(\epsilon) \cap p^{-1}(U_i) \to U_i \times \overline{B}^d\}_{i\in I}$ of trivializations which show that $p \colon E(\epsilon) \to K$ is a smooth $\overline{B}^d$-bundle. Finally for $i, j \in I$ and $x \in U_i \cap U_j$ one sees easily that the map

$$\overline{B}^d \xrightarrow{v \mapsto (x,v)} \{x\} \times \overline{B}^d \xrightarrow{\Psi_{\tau_i}} p^{-1}(x) \xrightarrow{(\Psi_{\tau_j})^{-1}} \{x\} \times \overline{B}^d \xrightarrow{(x,v)\mapsto v} \overline{B}^d$$

is a diffeomorphism and that it is given by multiplication by a matrix. But these two statements together imply that the map is given by multiplication by an orthogonal matrix. This discussion shows that the above trivializations turn $p \colon E(\epsilon) \to K$ into a linear smooth $\overline{B}^d$-bundle.

(3) This statement follows immediately from the definitions. ∎

57.5.3. **Proof of the Tubular Neighborhood–Existence Theorem 57.11: Conclusion.** Let K be a compact proper k-dimensional submanifold of some m-dimensional smooth manifold M. In the previous two sections we constructed an "external linear smooth $\overline{B}^{m-k}$"-bundle. The idea is now to embed this external bundle to obtain the desired tubular neighborhood. Let's do it:

Proof of the Tubular Neighborhood–Existence Theorem 57.11. Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. First note that it follows easily from the Compact Submanifold Exhaustion Proposition 38.2 that we can assume that M is compact. Since M is compact, we obtain from the Euclidean Collar-for-Submanifold Proposition 56.4 that we can view M as a proper smooth submanifold of some $\mathbb{H}^n$ such that $M \subseteq \mathbb{H}^n$ and $K \subseteq \mathbb{H}^n$ are both products with respect to the standard collar of $\partial\mathbb{H}^n$.

As in the Submanifold Normal Vectors Lemma 57.15 we introduce the following notation:

- For each $P \in K$ we denote by $\mathrm{V}_P K \subseteq \widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$ the orthogonal complement of $\widetilde{\mathrm{T}}_P K \subseteq \widetilde{\mathrm{T}}_P M$.
- We consider the Euclidean $\mathbb{R}^{m-k}$-bundle which is given by the topological space

$$\mathrm{V}K := \{(P, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid P \in K \text{ and } v \in \mathrm{V}_P K\}$$

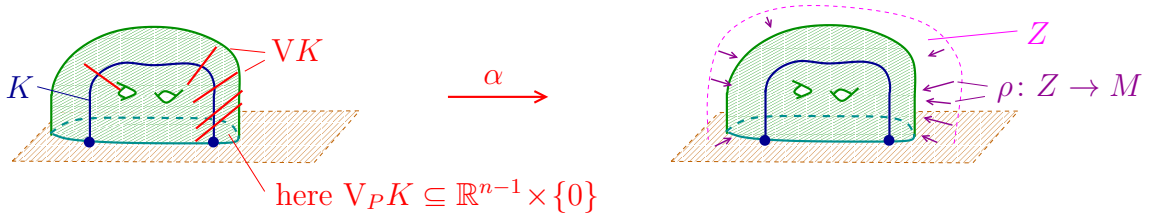
and the map

$$\begin{aligned} \Pi \colon \mathrm{V}K &\to K \\ (P, v) &\mapsto P. \end{aligned}$$

We make the following two preparations:

(1) By the Submanifold Neighborhood Retraction Proposition 56.5 there exists an open neighborhood Z of $M \subseteq \mathbb{H}^n$ and a smooth retraction $\rho \colon Z \to M$ with $\rho^{-1}(\partial M) = \partial V$.

(2) We denote by $\alpha \colon \mathrm{V}K \to \mathbb{R}^n$ the addition map that is given by $(P, v) \mapsto P + v$.



Claim 1.

(1) There exists an open neighborhood $\widetilde{U}$ of $K \subseteq \mathrm{V}K$ such that $\alpha(\widetilde{U}) \subseteq Z$ and such that $\alpha^{-1}(\partial Z) = \partial \widetilde{U}$.[638]

(2) There exists an open neighborhood U of $K \subseteq \mathrm{V}K$ such that the map

$$\begin{array}{rcl} \Theta\colon \widetilde{U} & \to & M \\ (P,v) & \mapsto & \rho(\underbrace{P+v}_{=\alpha(P,v)}) \end{array}$$

restricts to a smooth embedding on U with $\Theta^{-1}(\partial M) = \partial \widetilde{U}$ and such that $\Theta(U)$ is an open subset of M.

Proof.

(1) Since K is a product with respect to the standard collar of $\partial \mathbb{H}^n$ we obtain from the Vertical Boundary–Tangent Space Lemma 56.6 that there exists an $\epsilon > 0$ such that for every $P \in K \cap (\mathbb{R}^{n-1} \times [0,\epsilon])$ we have $(\widetilde{\mathrm{T}}_P K)^\perp \subseteq \mathbb{R}^{n-1} \times \{0\}$.

As in the proof of the Submanifold Neighborhood Retraction Proposition 56.5 one sees easily that $\widetilde{U} \coloneqq \alpha^{-1}(Z) \cap \{(P,v) \in \mathrm{V}K \mid \|v\| < \epsilon\}$ has all the desired properties.

(2) (a) Since K is compact it follows easily from the Compact-Hausdorff Proposition 4.23 (3) and the Compact-Closed Lemma 3.39 that the restriction of Θ to $K \times \{0\} \to K$ is a homeomorphism and that the image is a closed subset of M.

(b) One can verify quite easily that for each $P \in K$ the map $\mathrm{D}\Theta_{(P,0)}\colon \widetilde{\mathrm{T}}_{(P,0)}\widetilde{U} \to \widetilde{\mathrm{T}}_P M$ is an isomorphism.

(c) It follows from (1) and (b) and the Inverse Mapping Theorem 30.1 that Θ is a local diffeomorphism at each $(P,0) \in K \times \{0\}$.

(d) It follows from (a), (c) and the Neighborhood Smooth Embedding Theorem 27.16 that there exists an open neighborhood U of $K \times \{0\} \subseteq \widetilde{U}$ such that the restriction of the map Θ to U is a smooth embedding and such that the image is an open subset of M. ⊞

Given $\epsilon > 0$ we consider, as in Definition 57.16, the topological space

$$\mathrm{V}K(\epsilon) \;\coloneqq\; \{(P,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid P \in M \text{ and } v \in \mathrm{V}_P K \text{ with } \|v\| \leq \epsilon\}.$$

It follows easily from the $\mathbb{R}^d$-Bundle-to-$\overline{B}^d$-Bundle Lemma 57.17 together with the Radial Shrinking Lemma 57.8 that there exists an $\epsilon > 0$ with $\mathrm{V}K(\epsilon) \subseteq U$.

[638] Since $\widetilde{U}$ is an open subset of $\mathrm{V}K$ it is in particular a codimension-zero submanifold of $\mathrm{V}K$ and we can view it as a smooth manifold in its own right. By the elementary Open Subset-Submanifold Boundary Lemma 25.11 we have $\partial \widetilde{U} = \widetilde{U} \cap \partial \mathrm{V}K$.

We set $N := \Theta(\mathrm{V}K(\epsilon)) \subseteq M$ and we define $p := \Pi \circ \Theta^{-1} \colon N \to K$. We claim that $(N, p \colon N \to K)$ is a tubular neighborhood. Let us verify the four conditions from Definition 57.9:

(1) It follows easily from the $\mathbb{R}^d$-Bundle-to-$\overline{B}^d$-Bundle Lemma 57.17 (1), the fact that U is an open neighborhood of $K \times \{0\} \subseteq \mathrm{V}K$ and the fact that $\Theta(U)$ is an open subset of M that N is a submanifold of corner with $\partial_0 N = \Theta(\mathrm{SV}K(\epsilon))$ and with $\partial_1 N = p^{-1}(\partial K)$.
(2) It follows easily from the $\mathbb{R}^d$-Bundle-to-$\overline{B}^d$-Bundle Lemma 57.17 (2) and the fact that $\Theta \colon \mathrm{V}K(\epsilon) \to N$ is a diffeomorphism that $(N, p \colon N \to K)$ is a linear smooth $\overline{B}^{m-k}$-bundle. By construction the 0-section of p is precisely K.
(3) This statement follows from (1) and the observation that $\mathrm{SV}K(\epsilon)$ equals ∂_0 of the smooth $\overline{B}^d$-bundle $\mathrm{V}K(\epsilon)$.
(4) It follows from the Fiber Bundle Compactness Lemma 57.2, the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that N is a closed subset of M. ■

57.6. **Proof of the Tubular Neighborhood Uniqueness Theorem 57.12.** In this section we will provide the lengthy proof of the Tubular Neighborhood Uniqueness Theorem 57.12. Thus let M be a smooth manifold, let K be a compact proper submanifold of M and let $p \colon A \to K$ and $q \colon B \to K$ be two tubular neighborhoods of K. We have to "turn" A into B. To make this notion precise we need the following definition:

Definition 57.18. Let M be a smooth manifold and let $p \colon A \to K$ be a tubular neighborhood of some compact proper submanifold K of M. We equip A with the boundary decomposition, in the sense of Definition 30.23, given by $\partial_0 A$ and $\partial_1 A$. A smooth isotopy of the tubular neighborhood $p \colon A \to K$ is a smooth isotopy $F \colon A \times [0,1] \to M$.[639] in the sense of Definition 41.3.

In the above setting, as a first step, we want to show that we can isotope A to become a subspace of B. To do this it is helpful to recall the following notation:

Notation 57.19. Let $p \colon N \to K$ be a smooth linear $\overline{B}^n$-bundle. We denote by $\rho \colon N \to [0,1]$ the radial distance function that we introduced in the Radial Distance Lemma 57.6. Given $\epsilon > 0$ we set

$$N(\epsilon) := \{x \in N \mid \rho(x) \leq \epsilon\}.$$

Note that by the Radial Shrinking Lemma 57.8 (1) we can view $N(\epsilon)$ naturally as a smooth linear $\overline{B}^n$-bundle.

Lemma 57.20. (Neighborhood-into-Neighborhood Lemma) Let M be a smooth manifold and let K be a compact proper submanifold of M. Furthermore let $p \colon A \to K$ and $q \colon B \to K$ be two tubular neighborhoods of K. There exists a smooth isotopy $F \colon A \times [0,1] \to A$ rel K with the following two properties:
(1) $F_0 = \mathrm{id}$.
(2) $F_1(A) \subseteq \mathring{B}$.

Proof. We start out with two observations:

(1) By the Basics-of-Tubular Neighborhood Lemma 57.10 (1) we know that $\mathring{B}$ is an open neighborhood of $K \subseteq M$.

(2) By the Radial Shrinking Lemma 57.8 (2) we know that there exists a smooth map $H\colon A \times [0,1] \to A$ with the following properties:
   (a) For any $\epsilon \in [0,1)$ the map H_ϵ is an isomorphism $A \to A(1-\epsilon)$ of linear smooth $\overline{B}^n$-bundles.
   (b) For any open neighborhood U of K there exists an $\epsilon > 0$ with $H([1-\epsilon,1] \times A) \subseteq U$.

The desired isotopy is now given by $F(a,t) := H(a, t \cdot (1-\epsilon))$ where ϵ is chosen for $U = \mathring{B}$. ∎

To formulate the next steps it is convenient to introduce the following definition:

Definition 57.21. Let $p\colon A \to K$ and $q\colon B \to K$ be two linear smooth $\overline{B}^n$-bundles over some smooth manifold K.

(1) We say that a smooth map $f\colon A \to B$ is fiber preserving if for each $k \in K$ we have $f(p^{-1}(k)) \subseteq q^{-1}(k)$.
(2) We say that a fiber preserving smooth map $f\colon A \to B$ is fiberwise linear if for every $k \in K$, every choice of trivializations $\sigma\colon p^{-1}(U) \to U \times \overline{B}^n$ with $k \in U$ and every trivialization $\tau\colon q^{-1}(V) \to V \times \overline{B}^n$ with $k \in V$ the map
$$\overline{B}^n \xrightarrow{v \mapsto (k,v)} \{k\} \times \overline{B}^n \xrightarrow{\sigma^{-1}} p^{-1}(k) \xrightarrow{f} q^{-1}(k) \xrightarrow{\tau} \{k\} \times \overline{B}^n \xrightarrow{(k,v) \mapsto v} \overline{B}^n$$
is given by multiplication by a matrix.[640]
(3) We say that a fiber preserving smooth map $f\colon A \to B$ is an isomorphism of smooth $\overline{B}^n$-bundles if for each $k \in K$ the map $f\colon p^{-1}(k) \to q^{-1}(k)$ is a diffeomorphism.[641]

Now we can outline the three main steps of the proof of the Tubular Neighborhood Uniqueness Theorem 57.12. Let M be an m-dimensional smooth manifold, let K be a compact proper k-dimensional submanifold of M and let $p\colon A \to K$ and $q\colon B \to K$ be two tubular neighborhoods of K. By the Neighborhood-into-Neighborhood Lemma 57.20 it remains to consider the case that $A \subseteq B$. In the following three subsections we will prove the following three statements:

(1) Using an isotopy of A we can arrange that the inclusion $A \to B$ is fiber preserving.
(2) Using an isotopy of A we can arrange that the inclusion $A \to B$ is fiberwise linear.
(3) Using an isotopy of A we can arrange that the inclusion $A \to B$ is an isomorphism of smooth $\overline{B}^{m-k}$-bundles.

The desired conclusion of the Tubular Neighborhood Uniqueness Theorem 57.12 then follows from the Isotopy Extension Theorem 44.2.

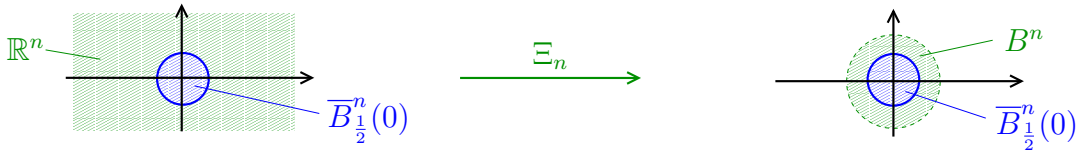
57.6.1. **Proof of the Uniqueness Theorem 57.12: Fibers to fibers.** Throughout the following sections we will make use of the following notation:

Notation 57.22. Given $P \in \mathrm{GL}(n,\mathbb{R})$ we denote by $\rho_P\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by multiplication by P. We use the same notation for the restriction of ρ_P to a map $\overline{B}^n \to \mathbb{R}^n$.

In the following discussion we will on several occasions deform a map $\overline{B}^n \to \overline{B}^n$ to a map $\overline{B}^n \to \mathbb{R}^n$. The following little lemma helps us in "reining" this map.

[640] It follows easily from the definitions of a linear $\overline{B}^n$-bundle that for each $k \in K$ we just have to verify the given condition for only one set of trivializations σ and τ.
[641] One can verify, say using the Inverse Mapping Theorem 30.1, that this definition is equivalent to the definition given in Definition 57.1.

Lemma 57.23. (Reining-In Lemma) Given any $n \in \mathbb{N}_0$ there exists a diffeomorphism $\Xi_n \colon \mathbb{R}^n \to B^n$ which is the identity on $\overline{B}^n_{\frac{1}{2}}(0)$ and which has the property that for every $P \in \mathrm{O}(n)$ we have $\Xi_n \circ \rho_P = \rho_P \circ \Xi_n$.



Proof. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a diffeomorphism $\xi \colon [0,\infty) \to [0,1)$ such that $\xi(t) = t$ for all $t \in [0, \frac{1}{2}]$. We consider the map

$$\begin{array}{rcl} \Xi_n \colon \mathbb{R}^n & \to & B^n \\ x & \mapsto & \begin{cases} \xi(\|x\|) \cdot \frac{x}{\|x\|}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

Note that Ξ_n is the identity on $\overline{B}^n_{\frac{1}{2}}(0)$. It now follows easily from the Smooth Pasting Proposition 24.7 that Ξ_n is smooth. Since ξ is a diffeomorphism one can write down an explicit smooth inverse of Ξ_n, which then shows that Ξ_n is a diffeomorphism. Finally note that it is clear that for any $P \in \mathrm{O}(n)$ and any $x \in \mathbb{R}^n$ we have $\Xi_n(P \cdot x) = P \cdot \Xi_n(x)$, i.e. we have $\Xi_n \circ \rho_P = \rho_P \circ \Xi_n$. ∎

Lemma 57.24. (Fiber Preserving Lemma) Let M be a smooth manifold and let K be a compact proper submanifold of M. Furthermore let $p \colon A \to K$ and $q \colon B \to K$ be two tubular neighborhoods of K. We assume that $A \subseteq B$. There exists a smooth isotopy $F \colon A \times [0,1] \to M$ rel K with the following two properties:

(1) $F_0 = \mathrm{id}$.
(2) The map $F_1 \colon A \to B$ is fiber preserving.

Proof. Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $p \colon A \to K$ and $q \colon B \to K$ be two tubular neighborhoods of K. We assume that $A \subseteq B$.

First we fix a notation: Given a topological space X we denote by $\pi \colon X \times \overline{B}^{m-k} \to \overline{B}^{m-k}$ the natural projection. We continue with a few basic considerations:

(i) If B is a product $K \times \overline{B}^{m-k}$, then one can easily verify that the map $f \colon A \to B$ given by $f(a) = (p(a), \pi(a))$ has the property that $f(p^{-1}(x)) \subseteq q^{-1}(x)$, i.e. f is fiber preserving.
(ii) If $\Phi \colon B \to K \times \overline{B}^{m-k}$ is trivialization, then the map $f_\Phi \colon A \to B$ that is given by $f_\Phi(a) = \Phi^{-1}(p(a), \pi(\Phi(a)))$ clearly has the property that $f_\Phi(p^{-1}(x)) \subseteq q^{-1}(x)$, i.e. f_Φ is fiber preserving.
(iii) Suppose we have two trivializations $\Phi \colon B \to K \times \overline{B}^{m-k}$ and $\widetilde{\Phi} \colon B \to K \times \overline{B}^{m-k}$ such that $\Phi \circ \widetilde{\Phi}^{-1}$ is linear. The corresponding maps $f_\Phi \colon A \to B$ and $f_{\widetilde{\Phi}} \colon A \to B$ (as defined in (ii)) are not necessarily the same.

But the fact that $\Phi \circ \widetilde{\Phi}^{-1}$ is linear implies that there exists a map $g\colon K \to \mathrm{O}(m-k)$ such that the following diagram commutes:

$$\begin{array}{ccc} & B & \\ {\scriptstyle\widetilde{\Phi}}\swarrow & & \searrow{\scriptstyle\Phi} \\ K \times \overline{B}^{m-k} & \underset{(k,v)\mapsto(k,g(k)^{-1}\cdot v)}{\overset{(k,v)\mapsto(k,g(k)\cdot v)}{\rightleftarrows}} & K \times \overline{B}^{m-k}. \end{array}$$

We consider the map

$$\begin{array}{rccl} \Theta\colon & \overbrace{\Phi(f_\Phi(A))}^{\subseteq K\times\overline{B}^{m-k}} & \to & K \times \overline{B}^{m-k} \\ & (k,v) & \mapsto & (k, \underbrace{g(k)\cdot g(q(f_\Phi^{-1}(\Phi^{-1}(k,v))))^{-1}}_{\in \mathrm{O}(m-k)} \cdot v). \end{array}$$

Claim. We have $(\Phi^{-1}\circ\Theta\circ\Phi)\circ f_\Phi = f_{\widetilde{\Phi}}\colon A \to B$, i.e. the following diagram commutes:

$$\begin{array}{ccc} & A & \\ {\scriptstyle f_\Phi}\swarrow & & \searrow{\scriptstyle f_{\widetilde{\Phi}}} \\ f_\Phi(A) & \xrightarrow{\Phi^{-1}\circ\Theta\circ\Phi} & B \\ {\scriptstyle\cong}\downarrow{\scriptstyle\Phi} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Phi} \\ \Phi(f_\Phi(A)) & \xrightarrow{\Theta} & K \times \overline{B}^{m-k}. \end{array}$$

Proof. For any $a \in A$ we have

$$\begin{array}{rcl} f_{\widetilde{\Phi}}(a) & \overset{\text{definition of } f_{\widetilde{\Phi}}}{=} & \widetilde{\Phi}^{-1}(p(a), \pi(\widetilde{\Phi}(a))) \\ & = & \Phi^{-1}(p(a), g(p(a))\cdot g(q(a))^{-1}\cdot \pi(\Phi(a))) \\ & \overset{\text{it follows easily from the definition of } g \text{ that } \pi(\widetilde{\Phi}(a)) = g(q(a))^{-1}\cdot\pi(\Phi(a))}{=} & (\Phi^{-1}\circ\Theta\circ\Phi)(f_\Phi(a)). \\ & \overset{\text{by definition of } f_\Phi \text{ we have } \Phi(f_\Phi(a)) = (p(a),\pi(\Phi(a)))}{=} & \end{array}$$

$$\begin{array}{rcl} & \overset{\text{by definition of } g}{=} & \Phi^{-1}(p(a), g(p(a))\cdot\pi(\widetilde{\Phi}(a))) \\ & = & \Phi^{-1}(\underbrace{\Theta(p(a),\pi(\Phi(a)))}_{=(\Phi\circ f_\Phi)(a)}) \\ & \overset{\text{definition of } \Theta}{=} & \end{array}$$

⊞

Here is the crucial observation which we will need:

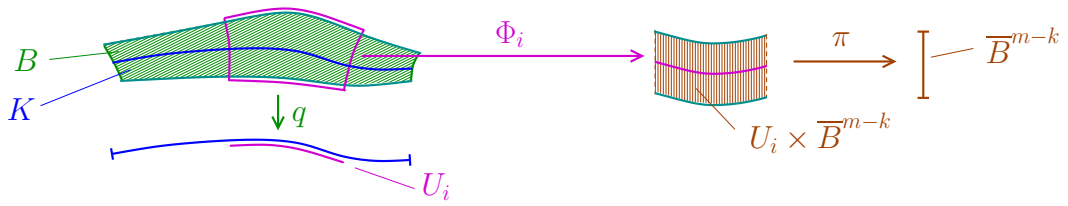
(∗) Let $(k,0) \in \Phi(f_\Phi(A)) \subseteq K \times \overline{B}^{m-k}$. Note that $\Phi(f_\Phi(k)) = (k,0)$. It follows that $q(f_\Phi^{-1}(\Phi^{-1}(k,0))) = q(k) = k$. This implies that for any $(k,0) \in \Phi(f_\Phi(A))$ we have $\mathrm{D}\Theta_{(k,0)} = \mathrm{id}$.

After these initial thoughts we turn to the actual proof of the lemma. In the following let[642] $\{\Phi_i\colon q^{-1}(U_i) \to U_i \times \overline{B}^{m-k}\}_{i=1,\dots,s}$ be a linear atlas for the linear smooth $\overline{B}^{m-k}$-bundle B. For each $i \in \{1,\dots,s\}$ we denote, as above, by $\pi\colon U_i \times \overline{B}^{m-k} \to \overline{B}^{m-k}$ the natural projection.

We introduce the following objects and notation:

(1) By the Manifold–Partitions-of-Unity Theorem 27.2 there exists a smooth partition of unity $\{\sigma_i\colon B \to [0,1]\}_{i=1,\dots,s}$ such that for each $i \in \{1,\dots,s\}$ we have the inclusion

[642] Since K is compact we can find a finite atlas.



$\operatorname{supp}(\sigma_i) \subseteq q^{-1}(U_i)$. For $i = 1, \ldots, s$ we consider the map

$$\begin{array}{rcl} \Psi_i \colon B & \to & \overline{B}^{m-k} \\ x & \mapsto & \begin{cases} \sigma_i(x) \cdot (\pi \circ \Phi_i)(x), & \text{if } x \in q^{-1}(U_i), \\ 0, & \text{if } x \notin q^{-1}(U_i). \end{cases} \end{array}$$

It follows easily from the Smooth Pasting Proposition 25.19 that Ψ_i is smooth.

(2) Let $x \in K$. Suppose that we are given $v_1, \ldots, v_t \in q^{-1}(x)$. We pick an $i \in \{1, \ldots, s\}$ with $x \in q^{-1}(U_i)$. We set

$$\alpha(v_1, \ldots, v_t) \;:=\; \Phi_i^{-1}\Big(x, \underset{\text{from the Reining-In Lemma 57.23}}{\Xi_{m-k}}\big(\underbrace{\pi(\Phi_i(v_1)) + \cdots + \pi(\Phi_i(v_t))}_{\substack{\text{sum of points in } \overline{B}^{m-k}, \\ \text{thus a point in } \mathbb{R}^{m-k}}}\big)\Big) \;\in\; q^{-1}(x).$$

(Loosely speaking α is the addition in the fiber $q^{-1}(x)$, except that we have to throw in the map Ξ_{m-k} to make sure that we do not "overshoot" $\overline{B}^{m-k}$.) Recall that Ξ_{m-k} has the property that for any $A \in \mathrm{O}(m-k)$ and any $v \in \mathbb{R}^{m-k}$ we have $\Xi_{m-k}(A \cdot v) = A \cdot \Xi_{m-k}(v)$. It follows easily from this observation and the fact that we are dealing with a linear atlas that the definition of α does not depend on the choice of $i \in \{1, \ldots, s\}$.

(3) Recall that by the Radial Shrinking Lemma 57.8 there exists a deformation retraction $R \colon A \times [0,1] \to A$ from A to K with $R_0 = \mathrm{id}$ and $R_1 = p \colon A \to K$.

Claim. Let $(a,t) \in A \times [0,1]$. Let $1 \le i_1 < \cdots < i_m \le s$ be the indices such that $q(R(a,t)) \in U_i$. We set

$$F(a,t) \;:=\; \underbrace{\alpha\big(\underbrace{\Phi_{i_1}^{-1}(q(R(a,t)), \Psi_{i_1}(a))}_{\in q^{-1}(R(a,t))}, \ldots, \underbrace{\Phi_{i_s}^{-1}(q(R(a,t)), \Psi_{i_s}(a))}_{\in q^{-1}(R(a,t))}\big)}_{\in q^{-1}(R(a,t))}.$$

The resulting map

$$F \colon A \times [0,1] \;\to\; B$$

has the following five properties:

(1) We have $F_0 = \mathrm{id}$ on $A(\frac{1}{2})$.
(2) The map $F_1 \colon A \to B$ is fiber preserving.
(3) For each $k \in K$ and each $t \in [0,1]$ we have $F(k,t) = k$.
(4) For each $t \in [0,1]$ and each $k \in K$ the differential $\mathrm{D}(F_t)_k$ is an isomorphism.
(5) There exists an open neighborhood U of $K \subseteq A$ such that $F \colon U \times [0,1] \to B$ is a smooth isotopy.

Proof.

(1) Let $a \in A(\frac{1}{2})$. We need to show that $F(a,0) = a$. For all $i \in \{1,\dots,s\}$ with $a \in q^{-1}(U_i)$ we write $\Phi_i(a) =: (q(a), v_i) \in U_i \times \overline{B}^{m-k}$. Next note that

$$\Phi_i^{-1}(q(R(a,0)), \Psi_i(a)) \underset{\substack{\uparrow \\ \text{since } R_0 = \mathrm{id} \text{ and by} \\ \text{definition of } \Psi_i}}{=} \Phi_i^{-1}(q(a), \sigma_i(a) \cdot (\pi \circ \Phi_i)(a)) \underset{\substack{\uparrow \\ \text{by the definition of } v_i}}{=} \Phi_i^{-1}(q(a), \sigma_i(a) \cdot v_i).$$

Using the fact that the σ_i form a partition of unity, the definition of α and the fact that $\Xi_{m-k} = \mathrm{id}$ on $\overline{B}^{m-k}_{\frac{1}{2}}$ one can now easily verify that $F(a,0) = a$.

(2) Let $a \in A$. Since $R_1 = p$ we see that $F_1(a) \in q^{-1}(R(a,1)) = q^{-1}(p(a))$. This means that F_1 is fiber preserving.

(3) This statement follows easily from the observation that for each $(k,t) \in K \times [0,1]$ we have $\Psi_i(k) = 0$ and $R(k,t) = k$.

(4) This statement follows, with a little bit of effort, from the observation $(*)$ and the observation that for any $(k,t) \in K \times [0,1]$ we have $q(R(k,t)) = q(k) = p(k)$.

(5) We consider the map

$$\begin{array}{rcl} \widetilde{F}\colon A \times [0,1] & \to & B \times [0,1] \\ (a,t) & \mapsto & (F(a,t),t). \end{array}$$

It follows from (3) that the restriction of $\widetilde{F}$ to $K \times [0,1] \to B \times [0,1]$ is just the inclusion map. Since A is compact we obtain from the Compact-Hausdorff Proposition 4.23 and the Compact-Closed Lemma 3.39 that this map $K \times [0,1] \to B \times [0,1]$ is an embedding with closed image. Furthermore note that it follows easily from (4) that for each $(k,t) \in K \times [0,1]$ the differential $\mathrm{D}\widetilde{F}_{(k,t)}$ is an isomorphism. It follows from the Neighborhood Smooth Embedding Theorem 14.19 that there exists an open neighborhood U of $K \times [0,1] \subseteq A \times [0,1]$ such that $\widetilde{F}$ is a smooth embedding. But this implies that $F\colon U \times [0,1] \to B$ is a smooth isotopy. ⊞

As in the proof of the Neighborhood-into-Neighborhood Lemma 57.20 we see that there exists an $\epsilon > 0$ such that $A(\epsilon) \subseteq U$. By the Radial Shrinking Lemma 57.8 we know that A is smoothly isotopic to $A(\epsilon)$. Combing this isotopy with the isotopy from the claim gives us the desired smooth isotopy. ■

57.6.2. **Proof of the Uniqueness Theorem 57.12: Linearity.** In the Fiber Preserving Lemma 57.24 we saw that we can arrange that the inclusion $A \to B$ is fiber preserving. In this section we will see that we can further arrange that the inclusion $A \to B$ is a fiber preserving fiberwise linear map.

We will need the following slight refinement of the Isotopic-to-Differential Lemma 42.6.

Lemma 57.25. (Isotopic-to-Differential Lemma) Let $f\colon \overline{B}^n \to \mathbb{R}^n$ be a smooth embedding with $f(0) = 0$.

(1) The map

$$\begin{array}{rcl} H(f)\colon \overline{B}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \begin{cases} \frac{1}{1-t} \cdot f((1-t)\cdot x), & \text{if } t \neq 1, \\ \mathrm{D}f_0 \cdot x, & \text{if } t = 1 \end{cases} \end{array}$$

is a smooth isotopy rel 0 from the original map $f\colon \overline{B}^n \to \mathbb{R}^n$ to the map that is given by multiplication by the matrix $\mathrm{D}f_0$.

(2) For any $P, Q \in \mathrm{O}(n)$ and any $t \in [0,1]$ we have $H(\rho_P \circ f \circ \rho_Q)_t = \rho_P \circ H(f)_t \circ \rho_Q$ as maps $\overline{B}^n \to \mathbb{R}^n$.

Proof.

(1) This statement is just the content of the original Isotopic-to-Differential Lemma 42.6.
(2) We need to show that the equality holds for any $(x,t) \in \overline{B}^n \times [0,1]$.
   (a) For $t \neq 1$ this follows immediately from the definitions and the fact that ρ_P and ρ_Q commute with scalar multiplication.
   (b) For $t = 1$ this follows from the Chain Rule 24.17 and the fact, pointed out on page 651, that the differential of a linear map is the linear map itself. ∎

Applying the Isotopic-to-Differential Lemma 57.25 "fiberwise" we obtain the following lemma:

Lemma 57.26. (Isotopic-to-Linear Lemma) Let M be a smooth manifold, let K be a compact proper submanifold of M and let $p\colon A \to K$ and $q\colon B \to K$ be two tubular neighborhoods of K. We assume that $A \subseteq B$ and that the inclusion $A \to B$ is fiber preserving. There exists a smooth isotopy $F\colon A \times [0,1] \to M$ rel K with the following properties:
(1) We have $F_0 = \mathrm{id}$.
(2) The map $F_1\colon A \to B$ is fiberwise linear.

Proof. It follows easily from the Radial Shrinking Lemma 57.8 (2) and the Smooth Isotopy-Equivalence Relation Proposition 41.5 that it suffices to prove the following claim:

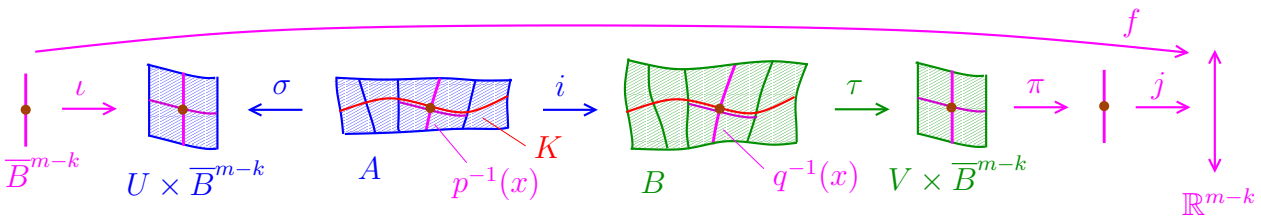
Claim. There exists a smooth isotopy $F\colon A \times [0,1] \to M$ rel K with the following properties:
(1) We have $F_0 = \mathrm{id}$.
(2) There exists an $\epsilon > 0$ such that map $F_1\colon A(\epsilon) \to B$ is fiberwise linear.

Proof. We write $m \coloneqq \dim(M)$ and we write $k \coloneqq \dim(K)$.

(1) By the Reining-In Lemma 57.23 there exists a diffeomorphism $\Xi_{m-k}\colon \mathbb{R}^{m-k} \to B^{m-k}$ with the following two properties:
   (a) Ξ_{m-k} is the identity on $\overline{B}^{m-k}_{\frac{1}{2}}(0)$.
   (b) For every $P \in \mathrm{O}(m-k)$ we have $\Xi_{m-k} \circ \rho_P = \rho_P \circ \Xi_{m-k}$.
(2) Given a smooth embedding $f\colon \overline{B}^n \to \mathbb{R}^n$ we denote by $H(f)\colon \overline{B}^n \times [0,1] \to \mathbb{R}^n$ the map that we introduced in the Isotopic-to-Differential Lemma 57.25.

Now let $x \in K$. We make the following further preparations:

(3) We pick a trivialization $\sigma\colon p^{-1}(U) \to U \times \overline{B}^{m-k}$ from the given linear atlas for A and a trivialization $\tau\colon q^{-1}(V) \to V \times \overline{B}^{m-k}$ from the given linear atlas for B such that $x \in U \cap V$.
(4) We denote by $\iota\colon \overline{B}^{m-k} \to \{x\} \times \overline{B}^{m-k}$ and $\pi\colon \{x\} \times \overline{B}^{m-k} \to \overline{B}^{m-k}$ the obvious diffeomorphisms and we denote by $j\colon \overline{B}^{m-k} \to \mathbb{R}^{m-k}$ the inclusion.
(5) We denote by $i\colon A \to B$ the inclusion.
(6) We write $f \coloneqq j \circ \pi \circ \tau \circ i \circ (\sigma|_{p^{-1}(x)})^{-1} \circ \iota\colon \overline{B}^{m-k} \to \mathbb{R}^{m-k}$.



We consider the map

$$\begin{array}{rcccccl} p^{-1}(x)\times[0,1] & \xrightarrow{(\pi\circ\sigma|_{p^{-1}(x)})\times\mathrm{id}} & \overline{B}^{m-k}\times[0,1] & \to & \mathbb{R}^{m-k} & \xrightarrow{\Xi_{m-k}} & \overline{B}^{m-k} \xrightarrow{\left(\tau|_{q^{-1}(x)}\right)^{-1}\circ\iota} q^{-1}(x). \\ & & (v,t) & \mapsto & H(f)_t(v). & & \end{array}$$

Using the Isotopic-to-Differential Lemma 57.25 (2) and using the Property (1b) of Ξ_{m-k} one can easily verify that this map is independent of the choice of σ and τ. It follows that these maps give rise to a well-defined map $F\colon A\times[0,1]\to B$. We leave it to the reader to verify that F has all the desired properties. ■

57.6.3. **Proof of the Uniqueness Theorem 57.12: Diffeomorphisms.** The following lemma is a variation on the Convex–Isotopic-to-Ball Lemma 18.7.

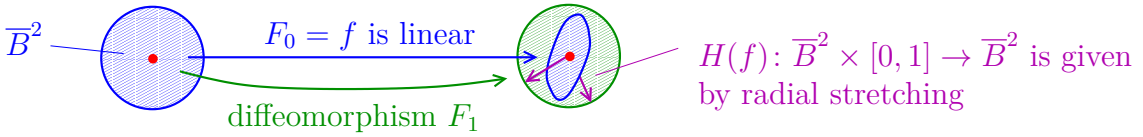
Lemma 57.27. (Ball-Isotopic-to-Diffeomorphism Lemma) Let $f\colon\overline{B}^n\to\overline{B}^n$ be a smooth embedding that is given by multiplication by some matrix. By the Smooth Transition Function Lemma 24.9 there exists a smooth function $\varphi\colon[0,1]\to[0,1]$ such that $\varphi(t)=0$ for $t\in[0,\frac{1}{2}]$, such that $\varphi'(t)\geq 0$ for all $t\in[0,1]$ and such that $\varphi(1)=1$.

(1) The map

$$\begin{array}{rcl} H(f)\colon \overline{B}^n\times[0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} 0, & \text{if } x=0, \\ f(x)\cdot\left(1+t\cdot\frac{\varphi(\|x\|)}{\|x\|}\cdot\left(\frac{1}{\|f(\frac{x}{\|x\|})\|}-1\right)\right), & \text{if } x\neq 0 \end{cases} \end{array}$$

is a smooth isotopy such that $H(f)_0=f\colon\overline{B}^n\to\overline{B}^n$ and such that $H(f)_1\colon\overline{B}^n\to\overline{B}^n$ is a diffeomorphism.[643]

(2) For any $P,Q\in\mathrm{O}(n)$ and any $t\in[0,1]$ we have $H(\rho_P\circ f\circ\rho_Q)_t=\rho_P\circ H(f)_t\circ\rho_Q$ as maps $\overline{B}^n\to\overline{B}^n$.



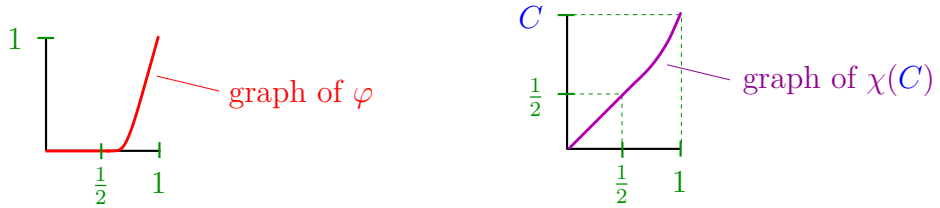
Remark. The Gram-Schmidt Orthogonalization Process 2.32 gives a way to turn a matrix in $\mathrm{GL}(n,\mathbb{R})$ into an orthogonal matrix, in particular it can be used to turn a linear embedding $\overline{B}^n\to\overline{B}^n$ into a diffeomorphism $\overline{B}^n\to\overline{B}^n$. But, as we saw in Exercise 2.7, the Gram-Schmidt Orthogonalization Process 2.32 does not commute with right-multiplication by an orthogonal matrix. Thus the Gram-Schmidt Orthogonalization Process 2.32 cannot be easily used to give an isotopy with the properties of the Ball-Isotopic-to-Diffeomorphism Lemma 57.27 (1) and (2). □

[643] Note that in general $H(f)_1$ will not be linear.

Proof.

(1) It is clear that $H(f)_0 = f$. Furthermore note that for each $t \in [0,1]$ we have $H(f)_t = f$ on $\overline{B}^n_{\frac{1}{2}}$. It follows from the Smooth Pasting Proposition 25.19 that $H(f)$ is smooth.

We still need to show that $H(f)$ is a smooth isotopy and that $H(f)_1 \colon \overline{B}^n \to \overline{B}^n$ is a diffeomorphism. For fixed $C \in \mathbb{R}$ we consider the function $\chi(C) \colon [0,1] \to [0,C]$ that is given by $\chi(C)(s) = s + \varphi(s) \cdot (C-1)$. If $C \geq 1$, then evidently we have $\chi'(C)(s) \geq 0$ for all $s \in [0,1]$ and in this case $\chi(C) \colon [0,1] \to [0,C]$ is a diffeomorphism.



Since f is given by multiplication by some matrix we see that for any $x \in \overline{B}^n \setminus \{0\}$ and $t \in [0,1]$ we have

$$H(f)(x,t) \;=\; f\big(\tfrac{x}{\|x\|}\big) \cdot \underbrace{\chi\Big(1 - t + t \cdot \tfrac{1}{\|f(\frac{x}{\|x\|})\|}\Big)(\|x\|)}_{= \|x\| \text{ if } \|x\| \leq \frac{1}{2}}.$$

Using the observation that $f \colon S^{n-1} \to f(S^{n-1})$ is a diffeomorphism and that for each $C \geq 1$ the map $\chi(C) \colon [0,1] \to [0,C]$ is a diffeomorphism which equals the identity on $[0, \frac{1}{2}]$ one can now easily verify that each $H(f)_t \colon \overline{B}^n \to \overline{B}^n$ is a smooth embedding. Since H is smooth we have now shown that H is a smooth isotopy. Finally note that it is now clear that $H(f)_1 \colon \overline{B}^n \to \overline{B}^n$ is a diffeomorphism.

(2) This statement follows easily from the definitions and the fact that multiplication by orthogonal matrices preserves lengths. ∎

Applying the Ball-Isotopic-to-Diffeomorphism Lemma 57.27 "fiberwise" we obtain the following lemma:

Lemma 57.28. (Isotopic-to-Diffeomorphism Lemma) Let M be an m-dimensional smooth manifold, let K be a k-dimensional compact proper submanifold of M and let $p \colon A \to K$ and $q \colon B \to K$ be two tubular neighborhoods of K. We now assume that $A \subseteq B$ and that the inclusion $A \to B$ is fiberwise linear. There exists a smooth isotopy $F \colon A \times [0,1] \to M$ rel K with the following properties:

(1) We have $F_0 = \mathrm{id}$.

(2) The map $F_1 \colon A \to B$ is an isomorphism of smooth $\overline{B}^{m-k}$-bundles.[644]

Proof. [645] It follows easily from the Radial Shrinking Lemma 57.8 (2) and the Smooth Isotopy-Equivalence Relation Proposition 41.5 that it suffices to prove the following claim:

[644] It is not clear to me whether one can arrange this to be an isomorphism of smooth linear $\overline{B}^{m-k}$-bundles.

[645] The proof of the Isotopic-to-Diffeomorphism Lemma 57.28 is basically identical to the proof of the Isotopic-to-Linear Lemma 57.26. For the most part we just need to replace the Isotopic-to-Differential Lemma 57.25 by the Ball-Isotopic-to-Diffeomorphism Lemma 57.27.

Claim. There exists a smooth isotopy $F\colon A \times [0,1] \to M$ rel K with the following properties:

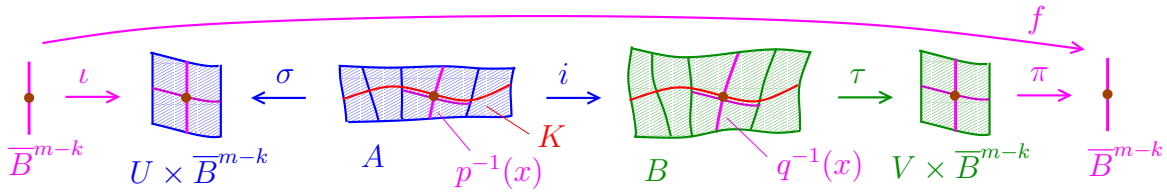
(1) We have $F_0 = \mathrm{id}$.
(2) There exists an $\epsilon > 0$ such that map $F_1\colon A(\epsilon) \to B$ is an isomorphism of smooth $\overline{B}^{m-k}$-bundles.

Proof. We start with two overall preparations:

(1) We use the Smooth Transition Function Lemma 24.9 to pick, once and for all, a smooth function $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(t) = 0$ for $t \in [0, \frac{1}{2}]$, such that $\varphi'(t) \geq 0$ for all $t \in [0,1]$ and such that $\varphi(1) = 1$.
(2) Given a smooth embedding $f\colon \overline{B}^n \to \overline{B}^n$ we use the map φ which we picked in (1), to define, as in the Ball-Isotopic-to-Diffeomorphism Lemma 57.27 the smooth isotopy $H(f)\colon \overline{B}^n \times [0,1] \to \overline{B}^n$.

Let $x \in K$. We continue with our preparations:

(3) We pick a trivialization $\sigma\colon p^{-1}(U) \to U \times \overline{B}^{m-k}$ from the given linear atlas for A and a trivialization $\tau\colon q^{-1}(V) \to V \times \overline{B}^{m-k}$ from the given linear atlas for B such that $x \in U \cap V$.
(4) We denote by $\iota\colon \overline{B}^{m-k} \to \{x\} \times \overline{B}^{m-k}$ and $\pi\colon \{x\} \times \overline{B}^{m-k} \to \overline{B}^{m-k}$ the obvious diffeomorphisms and we denote by $j\colon \overline{B}^{m-k} \to \mathbb{R}^{m-k}$ the inclusion.
(4) We denote by $i\colon A \to B$ the inclusion.
(5) We write $f \coloneqq \pi \circ \tau \circ i \circ (\sigma|_{p^{-1}(x)})^{-1} \circ \iota\colon \overline{B}^{m-k} \to \overline{B}^{m-k}$.



Using the Ball-Isotopic-to-Diffeomorphism 57.27 (2) one can easily verify that the map

$$\begin{array}{ccccccc} p^{-1}(x) \times [0,1] & \xrightarrow{(\pi \circ \sigma|_{p^{-1}(x)}) \times \mathrm{id}} & \overline{B}^{m-k} \times [0,1] & \to & \overline{B}^{m-k} & \xrightarrow{\left(\tau|_{q^{-1}(x)}\right)^{-1} \circ \iota} & q^{-1}(x). \\ & & (v,t) & \mapsto & H(f)_t(v) & & \end{array}$$

does not depend on the choice of σ and τ. We leave it to the reader to verify that these maps define a smooth isotopy $F\colon A \times [0,1] \to B$ which has all the desired properties with $\epsilon = \frac{1}{2}$. ∎

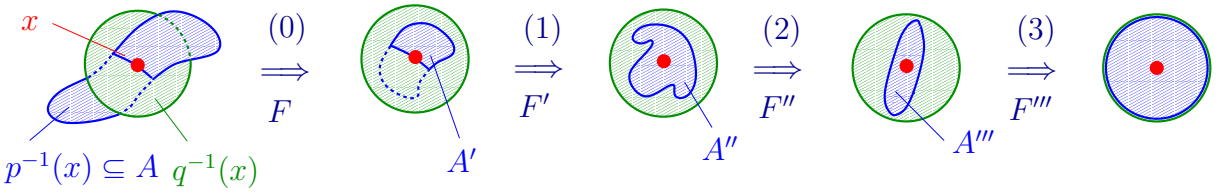
57.6.4. **Proof of the Uniqueness Theorem 57.12: Conclusion.** Using all of the above results it is now straightforward to provide a proof of the Tubular Neighborhood–Uniqueness Theorem 57.12.

Proof of the Tubular Neighborhood–Uniqueness Theorem 57.12. Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $p\colon A \to K$ and $q\colon B \to K$ be two tubular neighborhoods of K.

(0) It follows from the Neighborhood-into-Neighborhood Lemma 57.20 that there exists a smooth isotopy $F\colon A \times [0,1] \to M$ rel K such that
  (a) $F_0 = \mathrm{id}$, and
  (b) $F_1(A) \subseteq B$.

(1) We set $A' \coloneqq F_1(A)$. It follows from the Fiber Preserving Lemma 57.24 that there exists a smooth isotopy $F'\colon A' \times [0,1] \to B$ rel K such that
   (a) $F'_0 = \mathrm{id}$, and
   (b) $F'_1\colon A' \to B$ is fiber preserving.
(2) We set $A'' \coloneqq F'_1(A')$. It follows from the Isotopic-to-Linear Lemma 57.26 that there exists a smooth isotopy $F''\colon A \times [0,1] \to B$ rel K such that
   (a) $F''_0 = \mathrm{id}$, and
   (b) $F''_1\colon A'' \to B$ is fiberwise linear.
(3) We set $A''' \coloneqq F''_1(A'')$. It follows from the Isotopic-to-Diffeomorphism Lemma 57.28 that there exists a smooth isotopy $F'''\colon A''' \times [0,1] \to B$ rel K such that
   (a) $F'''_0 = \mathrm{id}$,
   (b) $F'''_1\colon A''' \to B$ is an isomorphism of smooth $\overline{B}^{m-k}$-bundles.

As in the Smooth Isotopy-Equivalence Relation Proposition 41.5 we can combine these isotopies to obtain a smooth isotopy $G\colon A \times [0,1] \to M$ such that $G_0 = \mathrm{id}$ and such that $G_1\colon A \to B$ is an isomorphism of smooth $\overline{B}^{m-k}$-bundles.



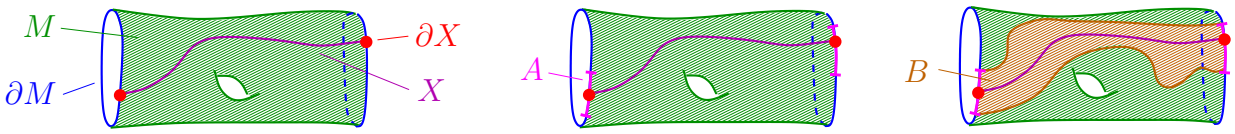
Finally note that we obtain the promised diffeotopy $\widetilde{G}\colon M \times [0,1] \to M$ from the (relatively straightforward) extension of the Isotopy Extension Theorem 44.2 to submanifolds with corner. ∎

57.7. **Appendix: Tubular Neighborhoods – Extensions and transversality.** In this appendix we state two theorems on tubular neighborhoods which we will not really make use of, but which which often get used implicitly in the literature, so we figured, it would be good to at least formulate them and hint at how one might prove them.

- In Subsection 57.7.1 we will see that we can extend tubular neighborhoods from the boundary and from codimension-zero submanifolds.
- In Subsection 57.7.2 we will see that for transverse submanifolds one can find "transverse" tubular neighborhoods.

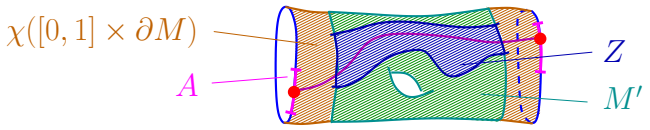
57.7.1. **Extending tubular neighborhoods.** Our first theorem of the appendix says that one can extend tubular neighborhoods from the boundary to the total manifold:

Theorem 57.29. (Tubular Neighborhood–Boundary Extension Theorem) Let M be a smooth manifold and let X be a compact proper submanifold of M. If $p\colon A \to \partial M$ is a tubular neighborhood for $\partial X \subseteq \partial M$, then there exists a tubular neighborhood $q\colon B \to M$ for $X \subseteq M$ such that $q|_A = p$.



Sketch of proof. It follows easily from the Smooth Collar Neighborhood Theorem 35.6 together with the Collar-for-Submanifold Proposition 56.3 that we can pick a smooth collar $\chi\colon [0,1] \times \partial M \to M$ such that X is a product with respect to the collar, in the sense of Definition 56.2. We set $M' := M \setminus \chi([0,1) \times \partial M)$.

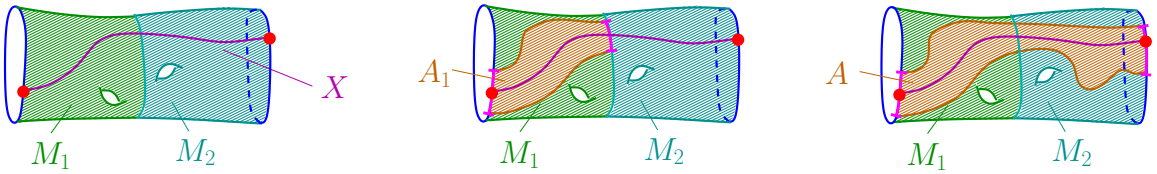
By the Tubular Neighborhood–Existence Theorem 57.11 we can find a tubular neighborhood $\gamma\colon Z \to X \cap M'$ for the compact proper submanifold $X \cap M' \subseteq M'$.



Next note that it follows from the Tubular Neighborhood–Uniqueness Theorem 57.12 that the two collars $\gamma|_{\partial M'}\colon Z \cap \partial M' \to \partial M'$ and $p\colon A \to \partial M$ are, under the obvious diffeomorphism $\partial M' \to \partial M$, smoothly isotopic, This smooth isotopy can now be used to "bridge the gap" between $\gamma|_{\partial M'}\colon Z \cap \partial M' \to \partial M'$ and $p\colon A \to \partial M$ on $\chi([0,1] \times \partial M)$. We leave to it to the reader to fill in the details. ■

The following theorem says that we can extend tubular neighborhoods that exist on codimension-zero submanifolds:

Theorem 57.30. (Tubular Neighborhood–Submanifold Extension Theorem) Let M be a smooth manifold and let X be a compact proper submanifold of M. Suppose that we have a decomposition $M = M_1 \cup M_2$ where M_1 and M_2 are codimension-zero submanifolds of M such that $M_1 \cap M_2$ is the union of boundary components of M_1 and the union of boundary components of M_2. If X intersects $M_1 \cap M_2$ transversally and if $p\colon A_1 \to M_1$ is a tubular neighborhood for $X \cap M_1$ then there exists a tubular neighborhood $q\colon A \to M$ for $X \subseteq M$ such that $q|_{A_1} = p$.



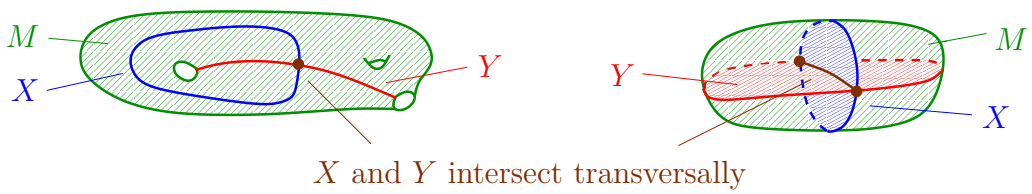
Proof. The proof of this theorem is very similar to the proof of the Tubular Neighborhood–Boundary Extension Theorem 57.29. We leave it to the reader to fill in the details. ■

57.7.2. **Transverse tubular neighborhoods.** In this subsection we state a theorem which says that for transverse submanifolds one can find "transverse" tubular neighborhoods.

First let us recall Definition 50.3:

Definition 57.31. Let M be a smooth manifold and let X and Y be two compact proper submanifolds of M. We say that X and Y intersect transversally (written $X \pitchfork Y$) if for each $P \in X \cap Y$ we have $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ and if for each $P \in X \cap Y \cap \partial M$ we furthermore have $\mathrm{T}_P \partial X + \mathrm{T}_P \partial Y = \mathrm{T}_P \partial M$. By the Submanifold Transversal Intersection Theorem 50.4 we know that if $X \pitchfork Y$, then $X \cap Y$ is a compact proper submanifold of M with $\operatorname{codim}(X \cap Y) = \operatorname{codim}(X) + \operatorname{codim}(Y)$.

We can now formulate the promised result.



Theorem 57.32. (Transverse Tubular Neighborhood Theorem) Let M be a smooth manifold and let X and Y be two compact proper submanifolds of M. If the submanifolds X and Y intersect transversally, then there exist tubular neighborhoods $p\colon A \to X$ and $q\colon B \to Y$ such that for every $x \in X \cap Y$ there exists a neighborhood U of $x \in M$, such that $U \cap X \cap Y$ is an open subset of $X \cap Y$, and a diffeomorphism

$$\Phi = (\Phi_U, \Phi_x, \Phi_y)\colon U \;\to\; ((X \cap Y) \cap U) \times \overline{B}^{\operatorname{codim}(X)} \times \overline{B}^{\operatorname{codim}(Y)}$$

of smooth manifolds with corner such that the following three conditions are satisfied:

(1) (a) $\Phi(X \cap U) = ((X \cap Y) \cap U) \times \{0\} \times \overline{B}^{\operatorname{codim}(Y)}$.

(b) $\Phi(Y \cap U) = ((X \cap Y) \cap U) \times \overline{B}^{\operatorname{codim}(X)} \times \{0\}$.

(2) (a) The map

$$\begin{aligned} U &\to (X \cap U) \times \overline{B}^{\operatorname{codim}(X)} \\ P &\mapsto (\underbrace{\Phi^{-1}(\Phi_U(P), 0, \Phi_y(P))}_{\in X \cap U \text{ by (1a)}}, \underbrace{\Phi_x(P)}_{\in \overline{B}^{\operatorname{codim}(X)}}) \end{aligned}$$

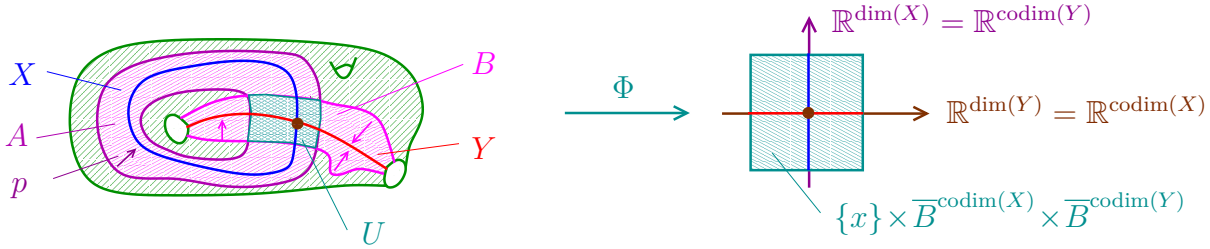
is a trivialization of $p\colon A \to X$.

(b) The map

$$\begin{aligned} U &\to (Y \cap U) \times \overline{B}^{\operatorname{codim}(Y)} \\ P &\mapsto (\underbrace{\Phi^{-1}(\Phi_U(P), \Phi_x(P), 0)}_{\in Y \cap U \text{ by (1b)}}, \underbrace{\Phi_y(P)}_{\in \overline{B}^{\operatorname{codim}(Y)}}) \end{aligned}$$

is a trivialization of $q\colon B \to Y$.

Furthermore in (a) and (b) the trivializations are compatible with the given linear smooth structures.



Sketch of proof. Implicitly this theorem gets used (subconsciously) frequently in geometric topology. But to the best of my knowledge no one ever had the energy to write up a proof. In the following we will also just provide a sketch of a proof:

(1) By the Submanifold Transversal Intersection Theorem 50.4 we know that $X \cap Y$ is a compact proper submanifold of M. A generalization of the argument in the proof of the Tubular Neighborhood–Existence Theorem 57.11 shows one can find a closed

neighborhood U of $X \cap Y$ and suitable tubular neighborhoods A' for $X \cap U$ and B' for $Y \cap U$.

(2) We use the Tubular Neighborhood–Submanifold Extension Theorem 57.30 to extend the tubular neighborhoods from (1) to tubular neighborhoods for all of X and Y.

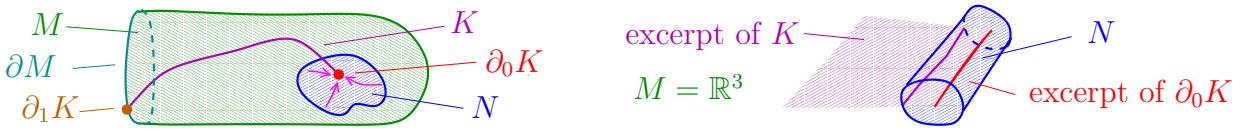


As the reader surely dreaded: We leave it to the reader to fill in the many missing details. ■

57.8. **Appendix: Tubular neighborhoods and compact non-proper submanifolds.** In this appendix we will state another variation on the Tubular Neighborhood–Existence Theorem 57.11:

Theorem 57.33. (Tubular Neighborhood–Boundary-of-Submanifold Theorem) Let M be a smooth manifold and let K be a compact (not necessarily proper) submanifold of M. As in Notation 25.47 we write $\partial_0 K := \partial K \cap (M \setminus \partial M)$ and $\partial_1 K := \partial K \cap \partial M$. There exists a tubular neighborhood $(N, p\colon N \to \partial_0 K)$ for the compact proper submanifold[646] $\partial_0 K$ and a diffeomorphism[647] $\Theta\colon N \cap K \to \partial_0 K \times [0,1]$ such that the following diagram commutes:

$$\begin{array}{ccc} N \cap K & \xrightarrow{\ \Theta\ } & \partial_0 K \times [0,1] \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,t)\mapsto x} & \\ & \partial_0 K. & \end{array}$$



Sketch of proof. [648] Evidently the proof of the theorem under discussion is a variation on the proof of the Tubular Neighborhood–Existence Theorem 57.11 on page 1297. First note that it follows easily from the Compact Submanifold Exhaustion Proposition 38.2 that we can assume that M is compact. The key to the proof is the following variation on the Euclidean Collar-for-Submanifold Proposition 56.4:

Claim. There exists an $n \in \mathbb{N}$ and a smooth embedding $\varphi\colon M \to \mathbb{R}^n$ with the following two properties:

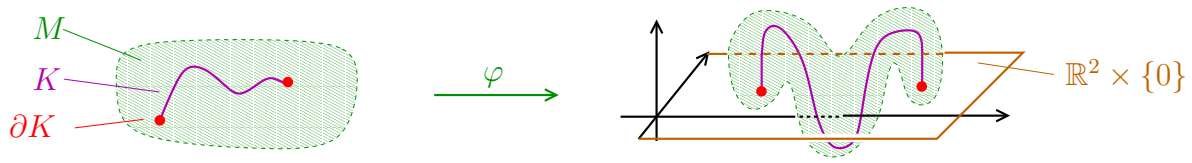
(1) $\varphi(\partial_0 K) \subseteq \mathbb{R}^{n-1} \times \{0\}$.

(2) There exists an open neighborhood U of $\partial_0 K$ in K such that $U \subseteq \mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ and such that $\varphi(U \cap K)$ is a product with respect to the standard collar of $\partial_0 \mathbb{H}^n$.

[646] Note that in Exercise 25.9 we showed that $\partial_0 K$ is a compact proper submanifold of M.

[647] Note that this implies that $\Theta^{-1}\colon \partial_0 K \times [0,1] \to N \cap K$ is a collar for $\partial_0 K \subseteq K$.

[648] In Exercise 57.10 we will give an alternative proof of the theorem.



Proof. This claim can be proved by a variation on the Euclidean Collar-for-Submanifold Proposition 56.4 (note that here it is convenient that M is compact). We leave, not surprisingly, the details to the reader. ⊞

Using the above claim the fairly constructive proof of the Tubular Neighborhood–Existence Theorem 57.11 on page 1297 will give, without too many problems, the desired statement. Once again all the details are left to the reader. ■

57.9. **Appendix: Submanifolds and cofibrations.** Using the Tubular Neighborhood–Existence Theorem 57.11 we can now prove the following proposition that we had already stated in Section 22.4.

Proposition 57.34. (Submanifold-Cofibration Proposition) Let M be a smooth manifold. For any compact proper submanifold K of M the inclusion $i\colon K \to M$ is a closed cofibration.

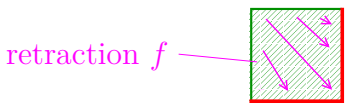
Since the author is slightly out of shape when it comes to cofibrations let us recall the following simplified version of the Cofibration Characterization Lemma 22.5.

Lemma 22.5 (Cofibration Characterization Lemma) Let M be a topological space and let $K \subseteq M$ be a subspace. If the set $(M \times \{0\}) \cup (K \times [0,1])$ is a retract of $M \times [0,1]$, then the inclusion $i\colon K \to M$ is a cofibration.

Proof of the Submanifold-Cofibration Proposition 57.34. Let M be a smooth manifold and let K be a compact proper submanifold of M. We want to show that the inclusion $i\colon K \to M$ is a cofibration. By definition of a proper submanifold we know that K is a closed subset of M. So it remains to show that the map $i\colon K \to M$ is a cofibration.

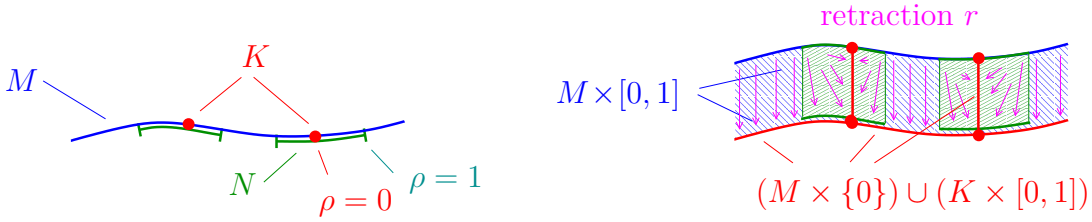
We make four preparations:

(1) By the Tubular Neighborhood–Existence Theorem 57.11 the submanifold K admits a tubular neighborhood $(N, p\colon N \to K)$.
(2) By the Radial Distance Lemma 57.6 there exists a continuous map $\rho\colon N \to [0,1]$ such that $\rho|_{\partial_0 N} \equiv 1$ and $\rho^{-1}(0) = K$.
(3) By the Radial Shrinking Lemma 57.8 there exists a continuous map $R\colon N \times [0,1] \to N$ such that $R_0 = \mathrm{id}$ and R_1 is a retraction $N \to K$.
(4) We pick a retraction $f = (f_x, f_y)\colon [0,1] \times [0,1] \to ([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$.



We consider the map

$$\begin{array}{rcl} r\colon M \times [0,1] & \to & (M \times \{0\}) \cup (K \times [0,1]) \\ (P,t) & \mapsto & \begin{cases} (P,0), & \text{if } P \notin \overset{\circ}{N}, \\ (R(P, f_x(1-\rho(P),t)), f_y(1-\rho(P),t)), & \text{if } P \in N. \end{cases} \end{array}$$



It follows from the definition of a tubular neighborhood, see Definition 57.9, together with the Pasting Proposition 4.7 (2) and the Smooth Submanifold-with-Corner Lemma 26.16 (3) that the map r is indeed continuous. It is now clear that this map is a retraction. We conclude from the Cofibration Characterization Lemma 22.5 that $i\colon K \to M$ is a cofibration. ∎

57.10. Exercises for Chapter 57.

Exercise 57.1. Let F be a smooth manifold. Show that every smooth F-bundle $p\colon N \to I$ over a compact interval I is isomorphic to the trivial smooth F-bundle $F \times I \to I$.

Exercise 57.2. We consider the Möbius band $M = ([0,1] \times [-1,1])/(0,y) \sim (1,-y)$ together with the projection

$$\begin{array}{rcl} p\colon M = ([0,1] \times [-1,1])/\sim & \to & [0,1]/0 \sim 1 \\ [(x,y)] & \mapsto & [x]. \end{array}$$

Show that there exist trivializations of this bundle which turn this map into a linear smooth interval bundle.

Exercise 57.3. Let F be a smooth manifold and let $p\colon N \to K$ be a smooth F-bundle. We consider the corresponding ∂F-bundle $p\colon \partial_0 N \to K$.

(1) If N is orientable, is $\partial_0 N$ necessarily also orientable?
(2) If $\partial_0 N$ is orientable, is N necessarily also orientable?

Exercise 57.4. Let F be a smooth manifold.

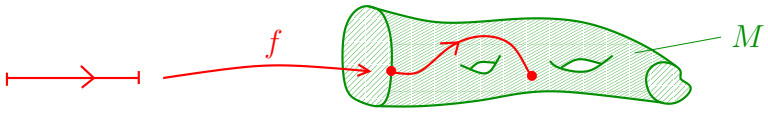
(a) Let $p\colon N \to K$ be a smooth F-bundle. Show that if F is connected and if K is connected, then so is N.
(b) Let M be a smooth manifold and let K be a proper submanifold of M. Show that if M is connected, if K is connected and if $\dim(M) - \dim(K) \geq 2$, then $M \setminus K$ is also connected.
Hint. Make use of Exercise 4.29 or of Exercise 23.9.

Exercise 57.5. Let M be a smooth manifold and let K be a proper submanifold of M. In the definition of a tubular neighborhood $(N, p\colon N \to K)$ in Definition 57.9 we demanded that N is a closed subset of K. If K is compact this condition is automatically satisfied. Show that for non-compact K the uniqueness statement of the Tubular Neighborhood–Uniqueness Theorem 57.12 would fail without the closedness condition.

Exercise 57.6. For which $k \in \mathbb{N}_0$ does there exist a 3-dimensional smooth manifold M and a proper k-dimensional smooth submanifold K such that the tubular neighborhood N of K is not isomorphic to the product bundle $K \times \overline{B}^{3-k}$?

Exercise 57.7. Let K be the Klein bottle. Show that there exists a closed *orientable* 3-dimensional smooth manifold M which contains K as a submanifold.
Hint. You could first try to find a compact 3-dimensional smooth manifold with toroidal boundary that contains M as a submanifold.

Exercise 57.8. Let M be a 2-dimensional smooth manifold. Furthermore let $f\colon [0,1] \to M$ be a smooth embedding such that $f(0) \in \partial M$ and such that $f((0,1]) \subseteq M \setminus \partial M$. Show that the inclusion $M \setminus f([0,1]) \to M$ is a homotopy equivalence.
Hint. You could construct a submanifold Σ of M such that $f([0,1])$ is a proper submanifold of Σ and apply the Tubular Neighborhood–Existence Theorem 57.11.



Exercise 57.9. Let M be a closed smooth manifold and let $X, Y \subseteq M$ be closed smooth submanifolds of M. Let $p\colon A \to X$ and $q\colon B \to Y$ be tubular neighborhoods. Show that if the compact smooth manifolds $M \setminus \mathring{A}$ and $M \setminus \mathring{B}$ are diffeomorphic, then so are the complements $M \setminus X$ and $M \setminus Y$.

Exercise 57.10. Let M be a smooth manifold, let K be a compact submanifold of M and let $(N, p\colon N \to \partial_0 K)$ be a tubular neighborhood for $\partial_0 K \coloneqq \partial K \cap (M \setminus \partial M)$. We consider the radial distance function $\rho\colon N \to [0,1]$ from the Radial Distance Lemma 57.6. Given $\epsilon \in [0,1]$ we write, as in Definition 57.7, $N(\epsilon) = r^{-1}([0,\epsilon])$.

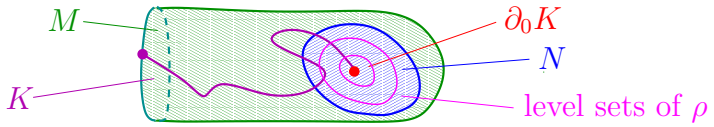
(a) Show that there exists an open neighborhood U of $\partial_0 K \subseteq K$ that is contained in N such that any $t \in [0,1]$ is a regular value of the map $r\colon U \to [0,1]$.

(b) Show that there exists an $\epsilon > 0$ and a diffeomorphism[649] $\Theta\colon N(\epsilon) \cap K \to \partial_0 K \times [0,\epsilon]$ such that the following diagram commutes:

$$\begin{array}{ccc} N(\epsilon) \cap K & \xrightarrow{\ \Theta\ } & \partial_0 K \times [0,\epsilon] \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,t)\mapsto x} & \\ & \partial_0 K. & \end{array}$$

Hint. Modify the proof of the Neighborhood-Regular Value Theorem 56.9 (4).

Remark. This exercise gives a new proof for the Tubular Neighborhood-Boundary-of-Submanifold Theorem 57.33.



CHAPTER 58

Product tubular neighborhoods and tubular maps

In this chapter we will introduce the reader friendly concept of a *product* tubular neighborhood. We will see that many (but not all) submanifolds admit a product tubular neighborhood.

This chapter is organized as follows:

- In Section 58.1 we will introduce the popular notion of a product tubular neighborhoods and of tubular maps.
- In Section 58.2 we will state and discuss the Product Tubular Neighborhood–Existence Theorem 58.4 and the Tubular Map–Uniqueness Theorem 58.5.
- In Section 58.3 we will prove the Product Tubular Neighborhood–Existence Theorem 58.4.
- In Section 58.4 we will prove the Tubular Map–Uniqueness Theorem 58.5.
- In Appendix 58.5 we will prove the Tubular-Regular Value Theorem 58.13.
- In Section 58.6 we will provide exercises for Chapter 58.

58.1. **Product tubular neighborhoods and tubular maps.** The following definition is almost self-explanatory:

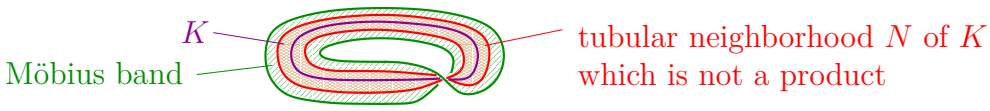
Definition 58.1.

(1) A linear smooth $\overline{B}^d$-bundle $p\colon N \to K$ is called a product if it is isomorphic, as a linear smooth $\overline{B}^d$-bundle, to the product $\overline{B}^d$-bundle $K \times \overline{B}^d \to K$.[650]

(2) Let M be an m-dimensional smooth manifold and let K be a k-dimensional submanifold of M. A tubular neighborhood $(N, p\colon N \to K)$ of K is called a product if the linear smooth $\overline{B}^{m-k}$-bundle $p\colon N \to K$ is a product.[651]

Example. We will see that in many important settings tubular neighborhoods are products. But, for better or worse, not all tubular neighborhoods are products:

(1) As on page 1290 we consider the Möbius band M and the submanifold K that is given by the central circle.



At least from the picture it is pretty clear that K does *not* admit a product tubular neighborhood.

[650] Such bundles are often also called trivial bundles.

[651] By the Tubular Neighborhood–Uniqueness Theorem 57.12 all tubular neighborhoods are isomorphic, thus if one is a product, all are.

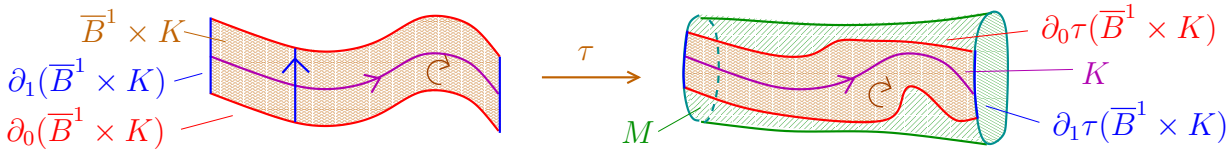
(2) In the Non-Product Tubular Neighborhood Proposition 220.22 (and also in Exercise 240.8) we will see that the tubular neighborhood of $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$ is not a product. □

The concept of a product tubular neighborhood leads us naturally to the following definition:

Definition 58.2. Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M.

(1) By the Smooth Manifold Product Proposition 26.12 we can view $\overline{B}^{m-k} \times K$ as a smooth manifold with corner $\angle(\overline{B}^{m-k} \times K) = \partial\overline{B}^{m-k} \times \partial K$. As on page 794 we equip the product $\overline{B}^{m-k} \times K$ with the boundary decomposition $\partial_0(\overline{B}^{m-k} \times K) := \partial\overline{B}^{m-k} \times K$ and $\partial_1(\overline{B}^{m-k} \times K) := \overline{B}^{m-k} \times \partial K$.

(2) A tubular map for K is a smooth embedding (in the sense of Definition 30.25) $\tau\colon \overline{B}^{m-k} \times K \to M$ such that for every $P \in K$ we have $\tau(0, P) = P$.



Lemma 58.3. (Tubular Map Lemma) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M.

(1) (a) If a tubular neighborhood N of K is a product, then there exists a tubular map $\tau\colon \overline{B}^{m-k} \times K \to M$ such that $\tau\colon \overline{B}^{m-k} \times K \to N$ is a diffeomorphism.

(b) The image of a tubular map $\tau\colon \overline{B}^{m-k} \times K \to M$ defines a product tubular neighborhood of K.

(2) (a) If[652 653] $\tau\colon \overline{B}^{m-k} \times K \to M$ is an orientation-preserving (respectively orientation-reversing) tubular map for (M, K), then it is an orientation-reversing (respectively orientation-preserving) tubular map for $(-M, K)$ and for $(M, -K)$.

(b) Let $\rho\colon \overline{B}^{m-k} \to \overline{B}^{m-k}$ be a reflection in a hyperplane. If $\tau\colon \overline{B}^{m-k} \times K \to M$ is an orientation-preserving (respectively orientation-reversing) tubular map for (M, K), then $\tau \circ (\rho \times \mathrm{id}_K)\colon \overline{B}^{m-k} \times K \to M$ is an orientation-reversing (respectively preserving) tubular map for (M, K).

(c) If M and K are oriented, if K admits a tubular map and if $m - k \geq 1$, then we can find an orientation-preserving tubular map $\overline{B}^{m-k} \times K \to M$.

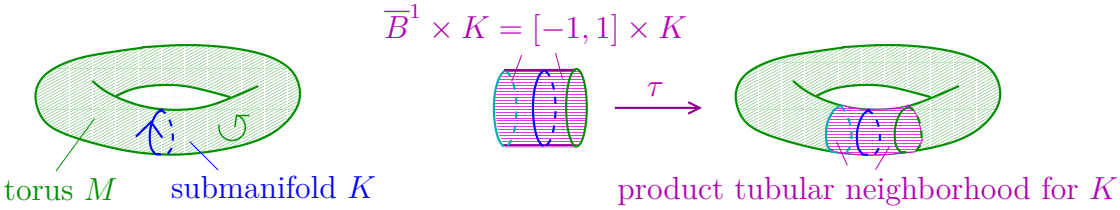
Example. We consider $M = S^1 \times S^1$ together with the submanifold $K = S^1 \times \{1\}$. We equip M and K with the standard orientations. Then

$$\begin{array}{rcl} \tau\colon [-1,1] \times (S^1 \times \{1\}) & \to & S^1 \times S^1 \\ (t, (z, 1)) & \mapsto & (\exp(\mathrm{i}t\tfrac{\pi}{2}), z) \end{array}$$

is an orientation-preserving tubular map for K.

[652] We equip $\overline{B}^{m-k}$ with the standard orientation introduced on page 819 and we equip $\overline{B}^{m-k} \times K$ with the product orientation introduced in Convention 31.49.

[653] Note that in the Product Orientation Proposition 31.48 (4) we showed that if C and D are oriented smooth manifolds with $\partial D = \varnothing$, then $\partial(C \times D) = \partial C \times D$ as *oriented* smooth manifolds. This is the reason why for tubular maps we use the ball $\overline{B}^{m-k}$ as the first factor.



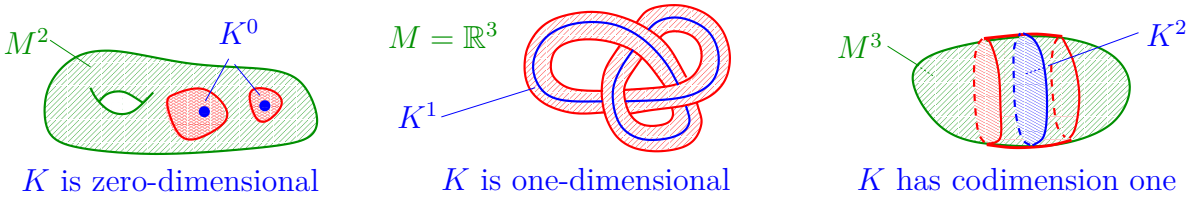
Proof.

(1) (a) This statement follows easily from the definitions.
(b) This statement follows from the Smooth-with-Corner Embedding Theorem 30.26.
(2) (a) This statement follows easily from the Product Orientation Proposition 31.48 (3d).
(b) The statement follows from the following two observations:
- It follows easily from the Orientation Checking-via-Differentials Lemma 31.30 that the map $\rho\colon \overline{B}^{m-k} \to \overline{B}^{m-k}$ is orientation-reversing.
- If $f\colon M \to M'$ and $g\colon N \to N'$ are two diffeomorphisms of oriented connected smooth manifolds and if f is orientation-reversing and g is orientation-preserving, then it follows from the pretty elementary Exercise 31.22 that the map $\operatorname{diag}(f,g)\colon M \times N \to M' \times N'$ is orientation-reversing.

(c) Since $m-k \geq 1$ there exists a hyperplane of $\mathbb{R}^{m-k}$ and we can consider the corresponding reflection ρ in the hyperplane.
Now let $\tau\colon \overline{B}^{m-k} \times K \to M$ be a tubular map. For each component K' of K for which the map $\tau\colon \overline{B}^{m-k} \times K' \to M$ is not orientation-preserving we precompose the map by the map $\rho \times \mathrm{id}_K$. It follows from (b) that this new tubular map is orientation-preserving. ∎

58.2. **Product tubular neighborhoods: existence and uniqueness.** The following theorem shows that in many interesting cases we do indeed have product tubular neighborhoods.

Theorem 58.4. (Product Tubular Neighborhood–Existence Theorem) Let M be a smooth manifold and let K be a compact proper submanifold of M. If we are in one of the following situations:
(1) K is zero-dimensional,
(2) (a) K is one-dimensional and each component of K has non-empty boundary,
(b) K is one-dimensional and M is orientable,
(3) K is of codimension one and M and K are both orientable,
then K admits a tubular map.[654]



[654]It follows from the Tubular Map Lemma 58.3 and the Tubular Neighborhood–Uniqueness Theorem 57.12 that in these settings every tubular neighborhood is a product.

Proof. We postpone the somewhat lengthy, not particularly inspiring proof of the Product Tubular Neighborhood–Existence Theorem 58.4 to Section 58.3. ■

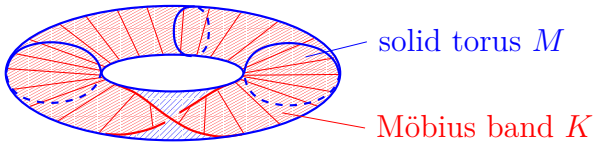
Remark.

(1) It follows from the Tubular Map Lemma 58.3 and (see also the discussion on page 1290) that the Product Tubular Neighborhood–Existence Theorem 58.4 (3) is a generalization of the Bicollar Neighborhood Theorem 55.2 from closed codimension-one submanifolds to compact proper codimension-zero submanifolds.

(2) In Exercise 58.4 (b), the Tubular-Regular Value Theorem 58.13, the Thicken–High Codimension Sphere Lemma 256.13 and the Product Tubular Neighborhood–Existence Theorem 243.12 we will give a few more conditions which guarantee that a tubular neighborhood is a product. □

Non-Example.

(1) Let $M = \overline{B}^2 \times S^1$ be the solid torus. We can view the Möbius band as a proper submanifold K of M. In Exercise 58.3 we will see that the tubular neighborhood of $K \subseteq M$ is not a product. This shows that in the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we cannot drop the hypothesis that K is orientable.



(2) In the above example and also in the example on page 1316 we saw that orientability issues imply that some tubular neighborhoods are not products. In the Non-Product Tubular Neighborhood Proposition 220.22 we will see that the *orientable* submanifold $\mathbb{C}\mathrm{P}^1$ of the *orientable* smooth manifold $\mathbb{C}\mathrm{P}^2$ also does not admit a product tubular neighborhood. □

Before we provide the proof of the Product Tubular Neighborhood–Existence Theorem 58.4 let us first formulate the corresponding uniqueness statement.

Theorem 58.5. (Tubular Map–Uniqueness Theorem) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $\sigma, \tau\colon \overline{B}^{m-k} \times K \to M$ be two tubular maps for K. There exists a diffeotopy $F\colon M \times [0,1] \to M$ and a smooth map $g\colon K \to \mathrm{O}(m-k)$ with the following properties:

(1) $F_0 = \mathrm{id}$.

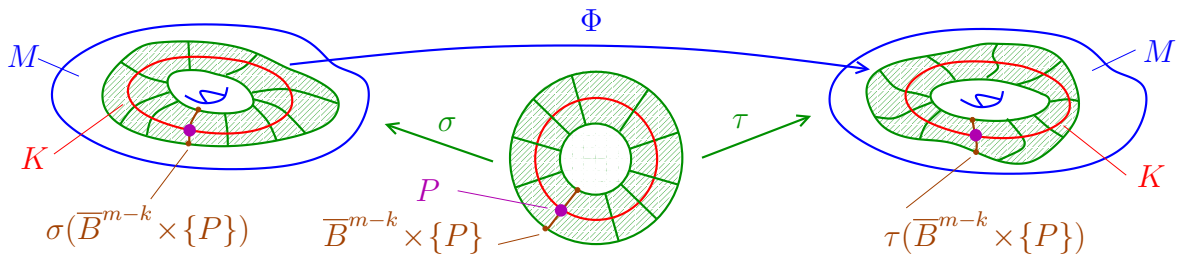
(2) F is a homotopy rel K.

(3) The diffeomorphism $\Phi = F_1\colon M \to M$ restricts to the map[655]

$$\begin{array}{rcl} \sigma(\overline{B}^{m-k} \times K) & \to & \tau(\overline{B}^{m-k} \times K) \\ \sigma(v,x) & \mapsto & \tau(g(x)\cdot v, x), \end{array}$$

(4) The diffeotopy F has compact support.

If M and K are oriented and if σ and τ are orientation-preserving, then all the above diffeomorphisms are orientation-preserving, in particular the map g from (2) takes values in $\mathrm{SO}(m-k)$.



Proof. We postpone the proof of the Tubular Map–Uniqueness Theorem 58.5 to Section 58.4. ■

Example. If M is a smooth manifold and $K = \{P\}$ is a zero-dimensional submanifold consisting of a single point, then the conclusion of the Tubular Map–Uniqueness Theorem 58.5 is very close to the conclusion of the Smooth Ball Embedding Theorem 45.10. In fact, as we will see, our proof of the Tubular Neighborhood–Uniqueness Theorem 57.12 borrows some ideas from the proof of the Smooth Ball Embedding Theorem 45.10. □

58.3. **Proof of the Product Tubular Neighborhood–Existence Theorem 58.4.** The proof of the Product Tubular Neighborhood–Existence Theorem 58.4 and also the proof of the Tubular Map–Uniqueness Theorem 58.5 make use of the following harmless lemma.

Lemma 58.6. (Linear $\overline{B}^k$-Bundle-Isomorphism Lemma) Let K be an oriented smooth manifold and let $n \in \mathbb{N}$.

(1) Let $\gamma\colon K \to \mathrm{SO}(n)$ be a continuous (smooth) map. The map

$$\begin{array}{rcl} K\times\overline{B}^n & \to & K\times\overline{B}^n \\ (x,v) & \mapsto & (x,\gamma(x)\cdot v) \end{array}$$

is an orientation-preserving isomorphism of linear (smooth) $\overline{B}^n$-bundles.

(2) Every orientation-preserving self-isomorphism of the linear (smooth) $\overline{B}^n$-bundle $K\times\overline{B}^n$ is of the form described in (1).

In the absence of orientations the analogous statement holds, except that we now consider maps $K \to \mathrm{O}(n)$ instead of maps $K \to \mathrm{SO}(n)$.

Proof of the Linear $\overline{B}^k$-Bundle-Isomorphism Lemma 58.6. This statement follows fairly easily from the definitions. We leave the details to the reader. ■

Proof of the Product Tubular Neighborhood–Existence Theorem 58.4. Let M be an m-dimensional smooth manifold and let K be a compact proper submanifold of M. It follows from the Tubular Neighborhood–Existence Theorem 57.11 and the Tubular Map Lemma 58.3 that it remains to show that in each of the given settings each tubular neighborhood $p\colon N \to K$ is already trivial. It is basically clear that in the following we can restrict ourselves to the case that K is connected and non-empty.

(1) We now assume that K is 0-dimensional. In this case the statement follows from the observation that evidently every linear $\overline{B}^{m-n}$-bundle over a point is trivial.

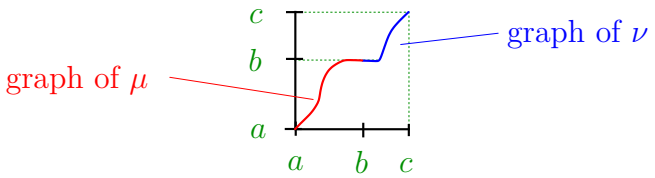
---

[655] Thus the diffeomorphism $\Phi\colon M \to M$ restricts to diffeomorphisms $\Phi\colon \sigma(\overline{B}^{m-k}\times K) \to \tau(\overline{B}^{m-k}\times K)$ and $\Phi\colon M\setminus\sigma(B^{m-k}\times K) \to M\setminus\tau(B^{m-k}\times K)$.

(2) We now turn to the case that K is 1-dimensional. Before we get into the specific situations let us state and prove the following technical claim which will on several occasions allows us to find isomorphisms of linear *smooth* $\overline{B}^k$-bundles.

Claim 1. Let $a < b < c$ and let $\sigma\colon [a,c] \times \overline{B}^d \to [a,c] \times \overline{B}^d$ be an isomorphism of linear $\overline{B}^d$-bundles such that the restrictions of σ to $[a,b] \times \overline{B}^d$ and to $[b,c] \times \overline{B}^d$ are smooth. Then there exists an isomorphism $\widetilde{\sigma}\colon [a,c] \times \overline{B}^d \to [a,c] \times \overline{B}^d$ of linear smooth $\overline{B}^d$-bundles which agrees with σ in a neighborhood of a and in a neighborhood of c.

Proof. By the Linear $\overline{B}^d$-Bundle-Isomorphism Lemma 58.6 we know that there exists a continuous map $\gamma\colon [a,c] \to \mathrm{O}(d)$ such that for any $(t,v) \in [a,c] \times \overline{B}^d$ we have $\sigma(t,v) = (t, \gamma(t) \cdot v)$ and such that $\gamma|_{[a,b]}$ and $\gamma|_{[b,c]}$ are smooth. Using the Smooth Transition Function Lemma 24.9 one can easily find smooth maps $\mu\colon [a,b] \to [a,b]$ and $\nu\colon [b,c] \to [b,c]$ such that $\mu = \mathrm{id}$ in a neighborhood of a, $\mu(t) = b$ in a neighborhood of b, $\nu(t) = b$ in a neighborhood of b and $\nu = \mathrm{id}$ in a neighborhood of c.



We consider

$$\begin{array}{rcl} \widetilde{\gamma}\colon [a,c] & \to & \mathrm{O}(d) \\ t & \mapsto & \begin{cases} \gamma(\mu(t)), & \text{if } t \in [a,b], \\ \gamma(\nu(t)), & \text{if } t \in [b,c]. \end{cases} \end{array}$$

Using the Smooth Pasting Proposition 24.7 one can easily verify that $\widetilde{\gamma}$ is smooth. It is now straightforward to verify that the map

$$\begin{array}{rcl} \widetilde{\sigma}\colon [a,c] \times \overline{B}^d & \to & [a,c] \times \overline{B}^d \\ (t,v) & \mapsto & (t, \widetilde{\gamma}(t) \cdot v) \end{array}$$

has the desired properties. ⊞

Now we turn to the actual proof of the statement. Thus let K be a compact connected proper non-empty 1-dimensional submanifold of M.

(a) We assume that $\partial K \neq \varnothing$. We need to show that every tubular neighborhood of K is a product $\overline{B}^{m-1}$-bundle. In fact we will prove the following a priori slightly stronger statement:

Claim 2. Every smooth linear $\overline{B}^n$-bundle $p\colon N \to K$ over K is a product.

Proof. First note that it follows from the Smooth 1-Manifold Classification Theorem 28.8 that there exists a diffeomorphism $\varphi\colon [0,1] \to K$. Next note that it follows easily from the Lebesgue Lemma 10.8 that there exists an $L > 0$ such that for $i = 0, \ldots, L-1$ there exists a trivialization of the form

$$\begin{array}{rcl} p^{-1}(\varphi([\frac{i}{L}, \frac{i+1}{L}])) & \to & \varphi([\frac{i}{L}, \frac{i+1}{L}]) \times \overline{B}^n \\ x & \mapsto & (p(x), \sigma_i(x)) \end{array}$$

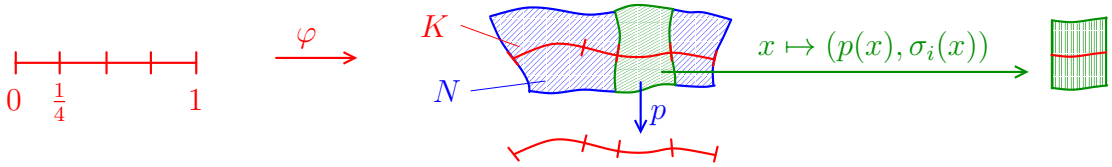
that comes from the restriction of a trivialization from the linear atlas. Note that by definition of a linear atlas there exist matrices $A_1, \ldots, A_{L-1} \in \mathrm{O}(n)$ such that for

every $i \in \{1, \dots, L-1\}$ and any $x \in p^{-1}(\varphi(\frac{i}{L}))$ we have $A_i \cdot \sigma_i(x) = \sigma_{i-1}(x) \in \overline{B}^n$. We consider the map

$$\begin{array}{rcl} N & \to & K \times \overline{B}^n \\ x & \mapsto & (p(x), A_1 \cdot \ldots \cdot A_i \cdot \sigma_i(x)). \end{array}$$

where i satisfies $x \in p^{-1}(\varphi([\frac{i}{L}, \frac{i+1}{L}]))$

Note that this map is well-defined and that it is an isomorphism of linear $\overline{B}^n$-bundles which is smooth on the subsets $p^{-1}([\frac{i}{L}, \frac{i+1}{L}])$.



Using Claim 1 we now get the desired isomorphism of linear smooth $\overline{B}^n$-bundles $N \to K \times \overline{B}^n$. ⊞

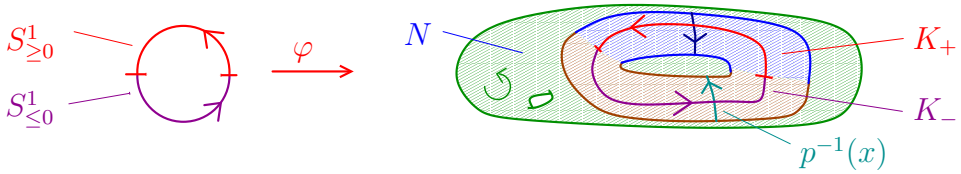
(b) We assume that $\partial K = \varnothing$ and that M is orientable. Let $p\colon N \to K$ be a tubular neighborhood for K. If $m = 1$, then there is nothing to show, so we assume that $m > 1$.

First note that it follows from the Smooth 1-Manifold Classification Theorem 28.8 that there exists a diffeomorphism $\varphi\colon S^1 \to K$. We set $K_+ := \varphi(S^1_{\geq 0})$ and we set $K_- := \varphi(S^1_{\leq 0})$. Note that K_+ and K_- are diffeomorphic to $[0,1]$. Thus we obtain from Claim 2 that there exist maps $\sigma_\pm\colon p^{-1}(K_\pm) \to \overline{B}^{m-1}$ such that the maps

$$\begin{array}{rcl} \tau_\pm\colon p^{-1}(K_\pm) & \to & K_\pm \times \overline{B}^{m-1} \\ x & \mapsto & (p(x), \sigma_\pm(x)) \end{array}$$

are isomorphisms of linear smooth $\overline{B}^{m-1}$-bundles.

Since M is orientable we can pick an orientation for M. Since K is diffeomorphic to S^1 we can pick an orientation for K. Given $x \in K$ we equip the submanifold $p^{-1}(x)$ with the orientation which is given by the condition that a positive basis for $\mathrm{T}_x K$ followed by a positive basis for $\mathrm{T}_x p^{-1}(x)$ gives a positive basis for $\mathrm{T}_x M$.



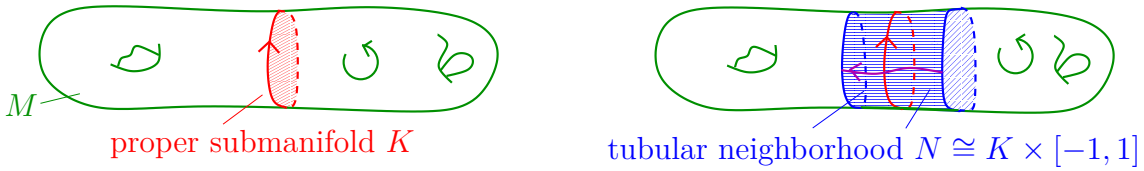
We equip the closed ball $\overline{B}^{m-1}$ with the standard orientation. Since K_+ is connected one can easily show that the maps $\{\sigma_+|_{p^{-1}(x)}\colon p^{-1}(x) \to \overline{B}^{m-1}\}_{x \in K_+}$ all have the same orientation behavior, i.e. they are either all orientation-preserving or they are all orientation-reversing. Since $m \geq 2$ we can assume that, after possibly composing σ_+ with a reflection $\overline{B}^{m-1} \to \overline{B}^{m-1}$ in a hyperplane, all these maps are actually *orientation-preserving* diffeomorphisms. Similarly we can also arrange that all the maps $\{\sigma_-|_{p^{-1}(x)}\colon p^{-1}(x) \to \overline{B}^{m-1}\}_{x \in K_-}$ are *orientation-preserving* diffeomorphisms. Next note that it follows from this discussion that for every $x \in K_+ \cap K_-$ the map $(\sigma_+|_{p^{-1}(x)}) \circ (\sigma_-|_{p^{-1}(x)})^{-1}\colon \overline{B}^{m-1} \to \overline{B}^{m-1}$ is given by multiplication by a matrix $\gamma(x) \in \mathrm{SO}(m-1)$. By the Matrix Group-$\pi_0$-Proposition 4.54 we know that $\mathrm{SO}(m-1)$ is path-connected. It follows from this observation, the fact that K_- is

diffeomorphic to $[0,1]$ and the Smooth Path-Connectivity Proposition 25.32 that there exists a smooth map $\gamma\colon K_- \to \mathrm{SO}(m-1)$ which agrees with the above values for γ on $K_- \cap K_+$. We now consider the map

$$\begin{array}{rcl} p^{-1}(K) & \to & K \times \overline{B}^{m-1} \\ x & \mapsto & \begin{cases} (p(x), \sigma_+(x)), & \text{if } x \in p^{-1}(K_+), \\ (p(x), \gamma(p(x)) \cdot \sigma_-(x)), & \text{if } x \in p^{-1}(K_-). \end{cases} \end{array}$$

One can easily verify that this map is an isomorphism of linear $\overline{B}^{m-1}$-bundles such that the restrictions to $p^{-1}(K_-)$ and $p^{-1}(K_+)$ are smooth. Using Claim 1 we can modify this map to obtain an isomorphism of linear smooth $\overline{B}^{m-1}$-bundles.

(3) We now suppose that K is of codimension one and that M and K are both orientable. Let $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times [-1,1]\}_{i\in I}$ be a linear atlas for $p\colon N \to K$. Since M and K are orientable we can equip M and K with an orientation. We equip $[-1,1]$ with the standard orientation, we equip each $p^{-1}(U_i) \subseteq M$ with the orientation coming from M and we equip each $U_i \times [-1,1]$ with the product orientation. By possibly using the reflection $x \mapsto -x$ on $[-1,1]$ we can arrange that all the maps $\Phi_i\colon p^{-1}(U_i) \to U_i \times [-1,1]$ are orientation-preserving. It follows from this observation and the fact that $\mathrm{SO}(1)$ consists of a single point, that for any $i, j \in I$ the maps Φ_i and Φ_j agree on the overlap.



But this means that the map

$$\begin{array}{rcl} N & \to & K \times [-1,1] \\ x & \mapsto & \Phi_i(x) \end{array}$$

where $i \in I$ satisfies $x \in p^{-1}(U_i)$

is well-defined and thus clearly an isomorphism of linear smooth $[-1,1]$-bundles. ■

58.4. **Proof of the Tubular Map–Uniqueness Theorem 58.5.** One might initially think that the Tubular Map–Uniqueness Theorem 58.5 is an immediate consequence of the Tubular Neighborhood–Uniqueness Theorem 57.12 together with the Linear $\overline{B}^k$-Bundle-Isomorphism Lemma 58.6. But this is not the case since the Tubular Neighborhood–Uniqueness Theorem 57.12 provides an isomorphism of smooth $\overline{B}^{m-k}$-bundles but it does not provide an isomorphism of smooth *linear* $\overline{B}^{m-k}$-bundles.

The bad news is that this means that we actually have to do some work to provide a proof of the Tubular Map–Uniqueness Theorem 58.5. The good news is that the proof shares some features of the Tubular Map–Uniqueness Theorem 58.5, and that the proof we will now provide is in fact much easier than the previous proof.

We start out with the following analog of Definition 57.18.

Definition 58.7. Let M be an m-dimensional smooth manifold and let K be a compact k-dimensional smooth manifold. As in Definition 58.2 we write $\partial_0(\overline{B}^{m-k} \times K) := S^{m-k-1} \times K$ and $\partial_1(\overline{B}^{m-k} \times K) := \overline{B}^{m-k} \times \partial K$. A smooth isotopy of $\overline{B}^{m-k} \times K$ is a smooth isotopy $\overline{B}^{m-k} \times K \times [0,1] \to M$ in the sense of Definition 30.25.

We continue with the following lemma, which is a variation on the Neighborhood-into-Neighborhood Lemma 57.20.

Lemma 58.8. (Tubular Map-into-Tubular Map Lemma) Let M be an m-dimensional smooth manifold, let K be a compact proper k-dimensional submanifold of M and let $\sigma, \tau \colon \overline{B}^{m-k} \times K \to M$ be two tubular maps for K. There exists a smooth isotopy $F \colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ with $F_0 = \sigma$ and with $F_1(\overline{B}^{m-k} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$.

Proof. By the Smooth-with-Corner Embedding Theorem 30.26 we know that $\tau(B^{m-k} \times K)$ is an open neighborhood of K. It follows that $\sigma^{-1}(\tau(B^{m-k} \times K))$ is a neighborhood of $\{0\} \times K \subseteq \overline{B}^{m-k} \times K$. We thus obtain from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that $\sigma(\overline{B}^{m-k}_{\epsilon} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$. The desired smooth isotopy is now given by the map

$$\begin{array}{rcl} H \colon \overline{B}^{m-k} \times K \times [0,1] & \to & M \\ ((v,k),t) & \mapsto & \sigma(v \cdot (1 - t + t \cdot \epsilon), k). \end{array}$$

■

The following definition is modelled Definition 57.21.

Definition 58.9. Let K be a smooth manifold.

(1) We say that a smooth map $f \colon \overline{B}^n \times K \to \overline{B}^n \times K$ is fiber preserving if for each $x \in K$ we have $f(\overline{B}^n \times \{x\}) \subseteq \overline{B}^n \times \{x\}$.

(2) We say that a fiber preserving smooth map $f \colon \overline{B}^n \times K \to \overline{B}^n \times K$ is fiberwise linear if for every $x \in K$ the map

$$\overline{B}^n \xrightarrow{v \mapsto (v,x)} \overline{B}^n \times \{x\} \xrightarrow{f} \overline{B}^n \times \{x\} \xrightarrow{(v,x) \mapsto v} \overline{B}^n$$

is given by multiplication by a matrix.

(3) We say that a fiber preserving smooth map $f \colon \overline{B}^n \times K \to \overline{B}^n \times K$ is a fiberwise linear isomorphism if for each $x \in K$ the map in (2) is given by multiplication by an orthogonal matrix.

The following lemma plays the role of the Fiber Preserving Lemma 57.24.

Lemma 58.10. (Fiber Preserving Lemma) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $\sigma, \tau \colon \overline{B}^{m-k} \times K \to M$ be two tubular maps for K with $\sigma(\overline{B}^{m-k} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$. There exists a smooth isotopy $G \colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ with the following three properties:

(1) $G_0 = \sigma$.

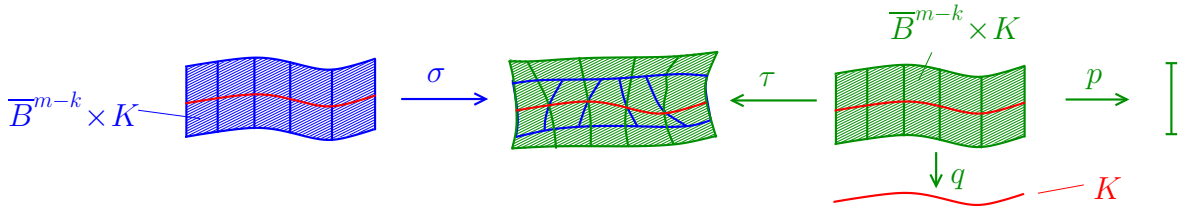
(2) We have $G_1(\overline{B}^{m-k} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$.

(3) The map $\tau^{-1} \circ G_1 \colon \overline{B}^{m-k} \times K \to \overline{B}^{m-k} \times K$ is fiber preserving.

Proof. Throughout the proof we denote $p \colon \overline{B}^{m-k} \times K \to \overline{B}^{m-k}$ and $q \colon \overline{B}^{m-k} \times K \to K$ the natural projections. Next we consider the map

$$\begin{array}{rcl} F \colon (\overline{B}^{m-k} \times K) \times [0,1] & \to & (\overline{B}^{m-k} \times K) \times [0,1] \\ ((v,k),t) & \mapsto & (((p \circ \tau^{-1} \circ \sigma)(v,k), (q \circ \tau^{-1} \circ \sigma)(t \cdot v, k)), t). \end{array}$$

One can easily verify that the restriction of F to $(\{0\} \times K) \times [0,1]$ is an embedding with closed image and that for each $((0,k),t) \in (\{0\} \times K) \times [0,1]$ the differential $\mathrm{D}F_{((0,k),t)}$ is an isomorphism. It follows from the Neighborhood Smooth Embedding Theorem 14.19 that



there exists an open neighborhood U of $(K\times\{0\})\times[0,1]\subseteq(\overline{B}^{m-k}\times K)\times[0,1]$ such that the restriction of F to U is a smooth embedding. By the Tube Lemma 7.12 we can pick $\epsilon>0$ with $K\times\overline{B}^{m-k}_{\epsilon}\times[0,1]\subseteq U$. We now see that the map

$$\begin{aligned} G\colon(\overline{B}^{m-k}_{\epsilon}\times K)\times[0,1] &\to M\\ ((v,k),t) &\mapsto \tau(\underbrace{(p\circ\tau^{-1}\circ\sigma)(v,k),(q\circ\tau^{-1}\circ\sigma)(t\cdot v,k)}_{\in\overline{B}^{m-k}\times K}) \end{aligned}$$

is a smooth isotopy with $G_0=\sigma$ and such that $\tau^{-1}\circ G_1$ is fiber preserving. By precomposing this isotopy with the obvious radial shrinking isotopy we get the desired smooth isotopy. ■

The following lemma plays the role of the Isotopic-to-Linear Lemma 57.26.

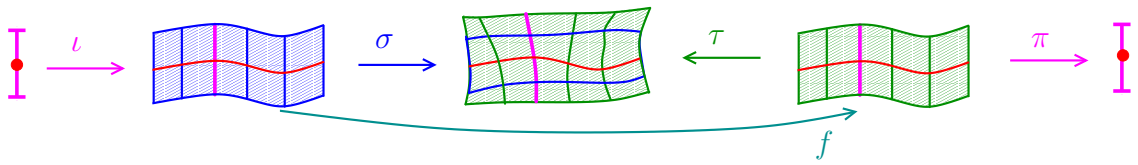
Lemma 58.11. (Isotopic-to-Linear Lemma) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $\sigma,\tau\colon\overline{B}^{m-k}\times K\to M$ be two tubular maps for K with $\sigma(\overline{B}^{m-k}\times K)\subseteq\tau(\overline{B}^{m-k}\times K)$ and such that $\tau^{-1}\circ\sigma\colon\overline{B}^{m-k}\times K\to\overline{B}^{m-k}\times K$ is fiber preserving. There exists a smooth isotopy $F\colon(\overline{B}^{m-k}\times K)\times[0,1]\to M$ rel $\{0\}\times K$ with the following three properties:

(1) $F_0=\sigma$.
(2) We have $F_1(\overline{B}^{m-k}\times K)\subseteq\tau(\overline{B}^{m-k}\times K)$.
(3) The map $\tau^{-1}\circ F_1\colon\overline{B}^{m-k}\times K\to\overline{B}^{m-k}\times K$ is fiberwise linear.

Proof. We write $f:=\tau^{-1}\circ\sigma$. Given $x\in K$ we introduce the following maps:

(1) $\iota\colon\overline{B}^{m-k}\to\overline{B}^{m-k}\times\{x\}$ and $\pi\colon\overline{B}^{m-k}\times\{x\}\to\overline{B}^{m-k}$ are the obvious diffeomorphisms.
(2) We write $f_x:=\pi\circ f|_{\overline{B}^{m-k}\times\{x\}}\circ\iota\colon\overline{B}^{m-k}\to\overline{B}^{m-k}$. Since f is fiber preserving and a diffeomorphism we see that f_x is a diffeomorphism.
(3) By the Isotopic-to-Differential Lemma 42.6 we know that the following map is a smooth isotopy:

$$\begin{aligned} H(f_x)\colon\overline{B}^{n}\times[0,1] &\to \mathbb{R}^n\\ (v,t) &\mapsto \begin{cases} \frac{1}{1-t}\cdot f_x((1-t)\cdot v), & \text{if } t\neq 1,\\ \mathrm{D}(f_x)_0\cdot v, & \text{if } t=1 \end{cases} \end{aligned}$$



Next we consider the map

$$\begin{aligned} G\colon(\overline{B}^{m-k}\times K)\times[0,1] &\to \mathbb{R}^{m-k}\times K\\ ((v,x),t) &\mapsto (H(f_x)_t(v),x). \end{aligned}$$

Note that $G_0 = \tau^{-1} \circ \sigma$ and that G_1 is fiberwise linear. Note though that this map G takes values in $\mathbb{R}^{m-k} \times K$. We still have to modify the map G such that we only take values in $\overline{B}^{m-k} \times K$. This requires just a little bit of an effort: First note that it follows quite easily from the fact that $\overline{B}^{m-k} \times K \times [0,1]$ is compact that there exists an $R \geq 1$ such that $G((\overline{B}^{m-k} \times K) \times [0,1]) \subseteq \overline{B}_R^{m-k} \times K$. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth map $r\colon [0,1] \times [1, \frac{1}{R}]$ such that $r(0) = 1$ and such that $r(t) = \frac{1}{R}$ for $t \in [\frac{1}{2}, 1]$. Similarly one can show that there exists a smooth map $\mu\colon [0,1] \to [0,1]$ such that $\mu(t) = 0$ for $t \in [0, \frac{1}{2}]$ and $\mu(1) = 1$. It follows now easily from the definitions that the map

$$\begin{array}{rcl} \widetilde{G}\colon (\overline{B}^{m-k} \times K) \times [0,1] & \to & \mathbb{R}^{m-k} \times K \\ ((v,x),t) & \mapsto & (r(t) \cdot H(f_x)_{\mu(t)}(v), x) \end{array}$$

is a smooth isotopy that takes values in $\overline{B}^{m-k} \times K$ with $\widetilde{G}_0 = \tau^{-1} \circ \sigma$ and such that $\widetilde{G}_1$ is fiberwise linear. Finally the desired smooth isotopy is given by $F(v,k) = \tau(\widetilde{G}(v,k))$. ∎

The following lemma plays the role of the Isotopic-to-Diffeomorphism Lemma 57.28. Note though that this time we can arrange that the map is fiberwise not only a diffeomorphism but in fact given by multiplication by an orthogonal matrix.

Lemma 58.12. (Isotopic-to-Isomorphism Lemma) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $\sigma, \tau\colon \overline{B}^{m-k} \times K \to M$ be two tubular maps for K with $\sigma(\overline{B}^{m-k} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$ and such that $\tau^{-1} \circ \sigma\colon \overline{B}^{m-k} \times K \to \overline{B}^{m-k} \times K$ is fiberwise linear. There exists a smooth isotopy $F\colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ with the following three properties:

(1) $F_0 = \sigma$
(2) We have $F_1(\overline{B}^{m-k} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$.
(3) The map $\tau^{-1} \circ F_1\colon \overline{B}^{m-k} \times K \to \overline{B}^{m-k} \times K$ is fiberwise an isomorphism.

Proof. By the Matrix Groups–Smooth Deformation Retract Proposition 41.15 there exists a smooth deformation retraction

$$R\colon \mathrm{GL}(m-k,\mathbb{R}) \times [0,1] \to \mathrm{GL}(m-k,\mathbb{R})$$

from $\mathrm{GL}(m-k,\mathbb{R})$ to $\mathrm{O}(m-k)$. One can verify fairly easily that the map R from the proof of the above proposition has the following extra property: If $A \subseteq \mathrm{GL}(m-k,\mathbb{R})$ is a matrix with $A \cdot \overline{B}^{m-k} \subseteq \overline{B}^{m-k}$, then for every $t \in [0,1]$ we have $R_t(A) \cdot \overline{B}^{m-k} \subseteq \overline{B}^{m-k}$.

The remainder of the proof is almost identical to the proof of the Isotopic-to-Linear Lemma 58.11. We write $f \coloneqq \tau^{-1} \circ \sigma$. Given $x \in K$ we introduce the following maps:

(1) $\iota\colon \overline{B}^{m-k} \to \overline{B}^{m-k} \times \{x\}$ and $\pi\colon \overline{B}^{m-k} \times \{x\} \to \overline{B}^{m-k}$ are the obvious diffeomorphisms.
(2) We write $f_x \coloneqq \pi \circ f|_{\overline{B}^{m-k} \times \{x\}} \circ \iota\colon \overline{B}^{m-k} \to \overline{B}^{m-k}$. Since f is fiberwise linear we see that f_x is given by multiplication by some matrix $A_x \in \mathrm{GL}(m-k,\mathbb{R})$.

The desired smooth isotopy is now given by the map

$$\begin{array}{rcl} F\colon \overline{B}^{m-k} \times K \times [0,1] & \to & M \\ ((v,x),t) & \mapsto & \tau((R(A_x)_t) \cdot v, x). \end{array}$$

∎

Proof of the Tubular Map–Uniqueness Theorem 58.5. Let M be an m-dimensional smooth manifold let K be a compact proper k-dimensional submanifold of M and let $\sigma, \tau\colon \overline{B}^{m-k} \times K \to M$ be two tubular maps for K.

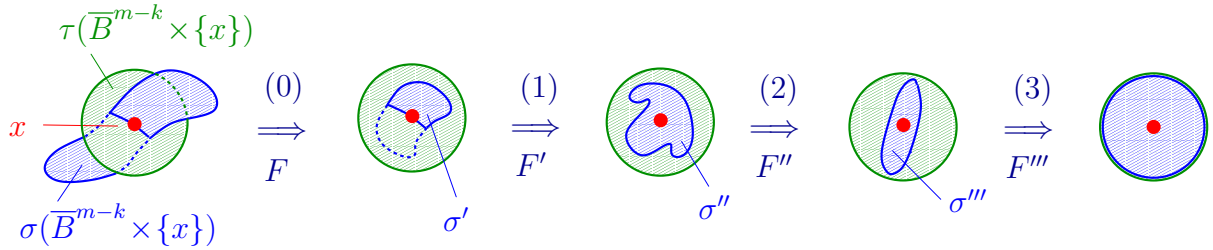
Similar to the proof of the Tubular Neighborhood–Uniqueness Theorem 57.12 on page 1308 we make the following observations:

(0) It follows from the Tubular Map-into-Tubular Map Theorem 58.8 that there exists a smooth isotopy $F\colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ such that
  (a) $F_0 = \sigma$, and
  (b) the inclusion $F_1(\overline{B}^{m-k} \times K) \subseteq \tau(\overline{B}^{m-k} \times K)$.

(1) We set $\sigma' \coloneqq F_1$. It follows from the Fiber Preserving Lemma 58.10 that there exists a smooth isotopy $F'\colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ such that
  (a) $F'_0 = \sigma'$, and
  (b) the map $\tau^{-1} \circ F'_1\colon \overline{B}^{m-k} \times K \to \tau(\overline{B}^{m-k} \times K)$ is fiber preserving.

(2) We set $\sigma'' \coloneqq F'_1$. It follows from the Isotopic-to-Linear Lemma 58.11 that there exists a smooth isotopy $F''\colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ such that
  (a) $F''_0 = \sigma''$, and
  (b) the map $\tau^{-1} \circ F''_1\colon \overline{B}^{m-k} \times K \to \tau(\overline{B}^{m-k} \times K)$ is fiberwise linear.

(3) We set $\sigma''' \coloneqq F''_1$. It follows from the Isotopic-to-Isomorphism Lemma 58.12 that there exists a smooth isotopy $F'''\colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ rel $\{0\} \times K$ such that
  (a) $F'''_0 = \sigma'''$, and
  (b) the map $\tau^{-1} \circ F'''_1\colon \overline{B}^{m-k} \times K \to \tau(\overline{B}^{m-k} \times K)$ is fiberwise an isomorphism.



As in the Smooth Isotopy-Equivalence Relation Proposition 41.5 we can combine these isotopies to obtain an isotopy $G\colon (\overline{B}^{m-k} \times K) \times [0,1] \to M$ such that $G_0 = \mathrm{id}$ and such that $G_1\colon \sigma(\overline{B}^{m-k} \times K) \to \tau(\overline{B}^{m-k} \times K)$ is an isomorphism of smooth $\overline{B}^{m-k}$-bundles.

Finally note that we obtain the promised diffeotopy $\widetilde{G}\colon M \times [0,1] \to M$ from the (relatively straightforward) extension of the Isotopy Extension Theorem 44.2 to submanifolds with corner. ■

58.5. **Appendix: The Tubular-Regular Value Theorem.** We conclude this chapter with the following addendum to the Regular Value Theorem 32.5:

Theorem 58.13. (Tubular-Regular Value Theorem) Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold, let $f\colon M \to N$ be a smooth map and let $Q \in N \setminus \partial N$ be a regular value of $f\colon M \to N$ and also of $f|_{\partial M}\colon \partial M \to N$.

(1) The preimage $K \coloneqq f^{-1}(Q)$ is a proper $(m-n)$-dimensional submanifold of M.
(2) If M is orientable, then K is also orientable.
(3) If M is compact, then K is also compact.

For the last statement we assume that M is compact and that $\partial N = \varnothing$.

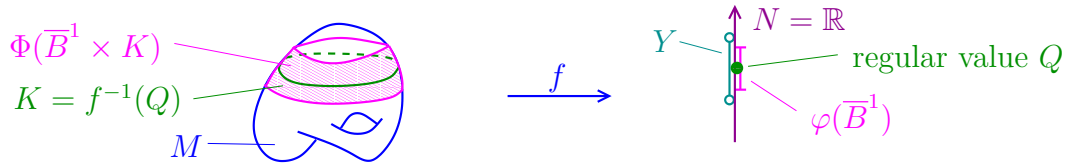
(4) There exists an open neighborhood Y of $Q \in N$ such that for every smooth embedding $\varphi\colon \overline{B}^n \to Y$ with $\varphi(0) = Q$ there exists a tubular map $\Phi\colon \overline{B}^n \times K \to M$ such that the following two statements hold:

(a) We have $f^{-1}(\varphi(\overline{B}^n)) = \Phi(\overline{B}^n \times K)$.

(b) The following diagram commutes:

$$\begin{array}{ccc} \overline{B}^n \times K & \xrightarrow{\Phi} & M \\ \downarrow{\scriptstyle (v,x)\mapsto v} & & \downarrow{\scriptstyle f} \\ \overline{B}^n & \xrightarrow{\varphi} & N. \end{array}$$

If M is oriented, then the orientation on K can be chosen such that Φ is orientation-preserving.

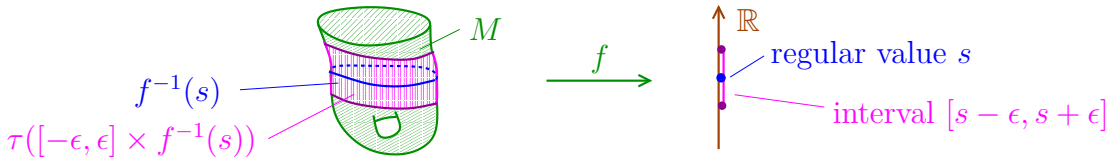


Proof. The first three statements are precisely the content of the original Regular Value Theorem 32.5. Statement (4) is an almost immediate consequence of the Neighborhood-Regular Value Theorem 56.9. ∎

Example. Let M be a compact smooth manifold.

(1) Let $f\colon M \to \mathbb{R}$ be a smooth map and let $s \in \mathbb{R}$ be a regular value of f and also of $f|_{\partial M}\colon \partial M \to \mathbb{R}$. We set $X := f^{-1}(s)$. Note that it follows easily from the Tubular-Regular Value Theorem 58.13 (4) that there exists an $\epsilon > 0$ and a smooth embedding $\tau\colon [-\epsilon, \epsilon] \times X \to M$ such that the following diagram commutes:

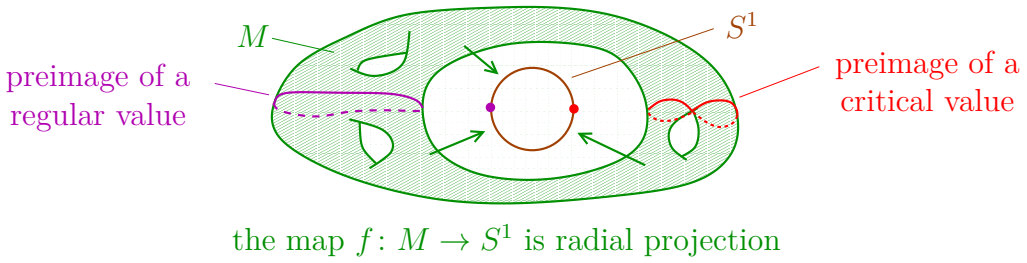
$$\begin{array}{ccc} [-\epsilon, \epsilon] \times X & \xrightarrow{\tau} & M \\ \downarrow{\scriptstyle (t,x)\mapsto t} & & \downarrow{\scriptstyle f} \\ [-\epsilon, \epsilon] & \xrightarrow{t\mapsto t+s} & \mathbb{R}. \end{array}$$



(2) Let $f\colon M \to S^1$ be a smooth map and let $z = \exp(2\pi\,\mathrm{i}t) \in S^1$ be a regular value. We set $X := f^{-1}(z)$.

It follows easily from the Tubular-Regular Value Theorem 58.13 (4) that there exists an $\epsilon > 0$ and a smooth embedding $\tau\colon [-\epsilon, \epsilon] \times X \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} [-\epsilon, \epsilon] \times X & \xrightarrow{\tau} & M \\ \downarrow{\scriptstyle (t,x)\mapsto t} & & \downarrow{\scriptstyle f} \\ [-\epsilon, \epsilon] & \xrightarrow{s\mapsto \exp(2\pi\,\mathrm{i}(t+s))} & S^1. \end{array}$$



□

58.6. Exercises for Chapter 58.

Exercise 58.1.

(a) Show that given any $n \geq 2$ there exists an n-dimensional smooth manifold M and a smooth submanifold K, diffeomorphic to S^1, such that the tubular neighborhood of $K \subseteq M$ is not a product.

(b) Let $g \in \mathbb{N}$. Show that given any $n \geq 3$ there exists an n-dimensional smooth manifold M and a smooth submanifold K, diffeomorphic to the surface of genus g, such that the tubular neighborhood of $K \subseteq M$ is not a product.

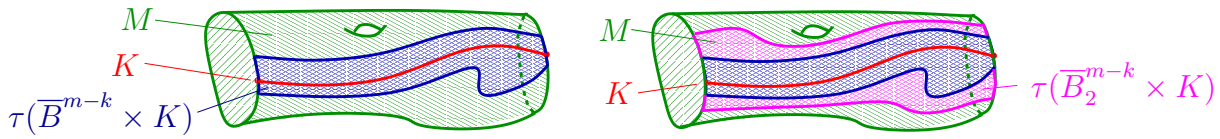
Exercise 58.2. Let M be an orientable m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M.

(a) Suppose that K admits a tubular map $\tau\colon K \times \overline{B}^{m-k} \to M$.

(i) Show that the tubular map can be extended to a tubular map $K \times \overline{B}_2^{m-k} \to M$.
Hint. You could make use of the Tubular Map–Uniqueness Theorem 58.5 in a clever fashion.[656]

(ii) Show that there exists a smooth isotopy $F\colon (M \setminus K) \times [0,1] \to M \setminus K$ with $F_0 = \mathrm{id}$ and $F_1(M \setminus K) = M \setminus \tau(K \times \overline{B}^{m-k})$.
Note that this implies in particular that $M \setminus \tau(K \times \overline{B}^{m-k})$ is diffeomorphic to $M \setminus K$.

(b) Now suppose that K does not admit a product tubular neighborhood. Formulate the appropriate generalization of (1) and prove it.



Exercise 58.3. Let M be a smooth manifold and let K be a proper submanifold of M.

(a) Show that if M is non-orientable and if K is orientable, then the tubular neighborhood is not a product.

(b) Show that if M is orientable and if K is non-orientable, then the tubular neighborhood is not a product.

---

[656] This exercise is similar in spirit to Exercise 45.3 and Exercise 54.1.

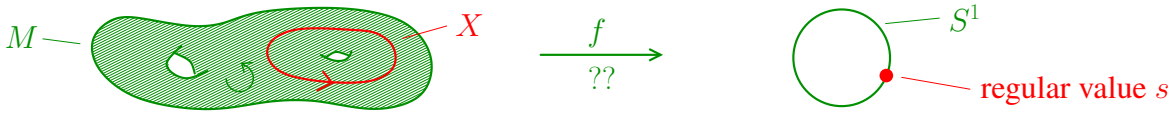
Exercise 58.4. Let M be an m-dimensional smooth manifold, let $\varphi\colon S^k \to M$ be a smooth embedding and let $p\colon N \to M$ be a tubular neighborhood of $\varphi(S^k) \subseteq M$.

(a) Show that the restrictions of p to $\varphi(S^k_{\geq 0})$ and $\varphi(S^k_{\leq 0})$ are products.
Hint. You could use the Smooth Ball Embedding Theorem 45.10.

(b) We assume that M is orientable and that $k = m - 2$. Show that p is a product.

(c) Show that the conclusion in (b) does not necessarily hold if M is non-orientable.

Exercise 58.5. Let M be a closed orientable n-dimensional smooth manifold M and let $X \subseteq M$ be a closed orientable submanifold of codimension one. Show that there exists a smooth map $f\colon M \to S^1$ together with a regular value $s \in S^1$ such that $X = f^{-1}(s)$.



Exercise 58.6. Let M be a connected m-dimensional smooth manifold. Suppose that M is oriented. Let $\varphi, \psi\colon \overline{B}^m \to M$ be two orientation-preserving smooth embeddings. Use the Homogeneous-via-Diffeotopies Proposition 44.6 and the Tubular Neighborhood–Uniqueness Theorem 57.12 to show that there exists a smooth isotopy $F\colon \overline{B}^m \times [0,1] \to M$ with $F_0 = \varphi$ and $F_1 = \psi$.

Remark. This exercise shows that the Tubular Map–Uniqueness Theorem 58.5 can be viewed as a generalization of a big chunk of the Smooth Ball Embedding Theorem 45.10.

Part VII

Differential topology for smooth manifolds with corner

In Definition 26.4 we introduced the notion of a smooth manifold with corner. This notion pops up naturally in various settings, but unfortunately it is also a technical headache. In this part of the lecture we will prove all kinds of (moderately) interesting and useful results on smooths manifolds with corner. In particular we will do the following:

(1) We will learn about collars of smooth boundaries, we will learn about gluing smooth manifolds with corner and we will learn how to cut smooth manifolds along compact proper two-sided submanifolds.
(2) In the Rounding Corner Proposition 60.5. we will see that many smooth manifolds with corner admit naturally the structure of a smooth manifold (without corner).[657]
(3) In the Introducing Corner Proposition 61.4 we will see that, if we feel like it, we can also introduce corners along suitable codimension-one submanifolds of the boundary.
(4) We will use the Introducing Corner Proposition 61.4 and the Gluing-Smooth Manifolds-Proposition 59.9 to define in Definition 61.7 what it means to glue a (possibly disconnected) smooth manifold along parts of the boundary. This notion will in particular allow us to define the "boundary connected sum" of two smooth manifolds.

Many of the results in this part of the lecture notes are extensions and variations on earlier results. For example the Smooth Collar Neighborhood Theorem 59.7 is a generalization of the Smooth Collar Neighborhood Theorem 59.7 and the Smooth Collar-Uniqueness Proposition 59.8 are generalization of the Smooth Collar-Uniqueness Proposition 54.2. For many of these new results we will leave it to the reader to generalize the earlier proofs.

Overall the goal of this part of the lecture notes is to get the statements right, which is hard enough. In most cases we will not attempt to provide full proofs, which, after all, in most cases are just adaptions of arguments which we are by now very comfortable with.

Arguably there is little point in reading this part for its own sake, unless one really really cares about rounding and introducing corners or about gluing smooth manifolds along parts of the boundary. It is better to continue with other parts of the lecture notes, and to return to this part, when the results are actually applied. This will most prominently be the case in Part XXVIII where we study handle decompositions of smooth manifolds.

This part of the lecture notes is organized as follows:

- In Chapter 59 we will extend some of our earlier results and concepts about smooth manifolds (without corner), which we learned about in Part VI, to smooth manifolds with corner.
- In Chapter 60 we will learn how to round corners, and we will also learn how to introduce corners.
- In Chapter 61 we will learn how to glue smooth manifolds along embeddings of a smooth manifold into their boundaries. In particular we will introduce the boundary connected sum of two smooth manifolds.

[657] On page 703 we already stated a simplified version of the Rounding Corner Proposition 60.5.

CHAPTER 59

Collars, gluing and cutting for smooth manifolds with corner

In Part VI we discussed in great detail many concepts about smooth manifolds (without corner). In this chapter we will first recall the notion of a smooth manifold with corner and we will then see how various earlier concepts generalize to smooth manifolds with corner.

This chapter is organized as follows:

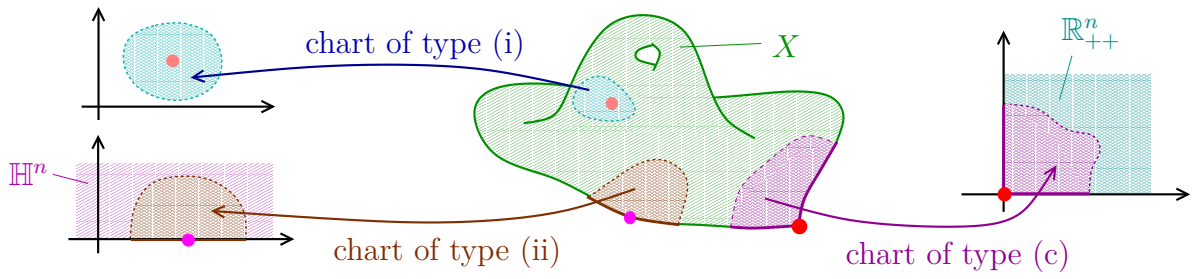
- In Section 59.1 we will recall the notion of a smooth manifold with corner.
- In Section 59.2 we will introduce the notion of a smooth boundary of a smooth manifold with corner.
- In Section 59.3 we will introduce collars of smooth boundaries of smooth manifolds with corner.
- In Section 59.4 we will learn how to glue smooth manifolds with corner along smooth boundaries.
- In Section 59.5 we will use the gluing operation from Section 59.4 to introduce the mapping torus of smooth manifolds.
- In Section 59.6 we will learn how to cut a smooth manifold along a proper codimension-one submanifold into a smooth manifold with corner.
- In Section 59.7 we will provide exercises for Chapter 59.

In the next Chapter 60 we will introduce the concept of rounding corners.

59.1. **Smooth manifolds with corner.** In the following we first want to remind ourselves of the notion of a smooth manifold with corner, so that we know what we are talking about. To do so we recall Definitions 23.2 and 26.3:

Definition 59.1. Let X be a topological space and let $x \in X$. An n-dimensional (corner) chart at the point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of $x \in X$ and

(i) V is an open subset of $\mathbb{R}^n$ or
(ii) V is an open subset of the upper half-space $\mathbb{H}^n$ and $\Phi(x)$ lies in $\partial\,\mathbb{H}^n$, or
(c) V is an open subset of $\mathbb{R}^n_{++} := \mathbb{R}^{n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and the point $\Phi(x)$ lies in the subspace $\angle\mathbb{R}^n_{++} := \mathbb{R}^{n-2} \times \{(0,0)\}$.



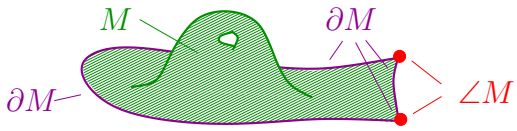
Definition 59.2. Let X be a topological space.

(1) An n-dimensional corner atlas for the topological space X is a collection of n-dimensional charts of type (i), (ii) and (c) such that the domains cover all of X.
(2) A corner atlas $\mathcal{A}$ is called smooth if all charts in the corner atlas are smoothly compatible to one another, in the sense of Definition 26.4.
(3) An n-dimensional smooth manifold with corner is a pair $(X, \mathcal{A})$ consisting of a second-countable Hausdorff space together with an n-dimensional smooth corner atlas $\mathcal{A}$ for X that is maximal in the sense of Definition 26.4.

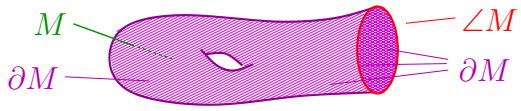
In the Unique Type-of-Chart Proposition 26.8 we showed that each point on a smooth manifold with corner admits charts of a *unique* type. This fact led us in Definition 26.9 to the following definition:

Definition 59.3. Let M be a smooth manifold with corner.

(1) We say that a point $x \in M$ is a boundary point if x admits a chart of type (ii) or (c) in the smooth structure. As always we denote by ∂M the set of all boundary points of M. We refer to ∂M as the boundary of M.
(2) We say that a point $x \in M$ is a corner point if x admits a chart of type (c) in the smooth structure. We denote by $\angle M$ the set of all corner points of M. We refer to $\angle M$ as the corner of M.[658]



2-dimensional smooth manifold with corner



3-dimensional smooth manifold with corner

The following proposition gives us the main source of smooth manifolds with corner:

Proposition 26.12. (Smooth Manifold Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. We denote by

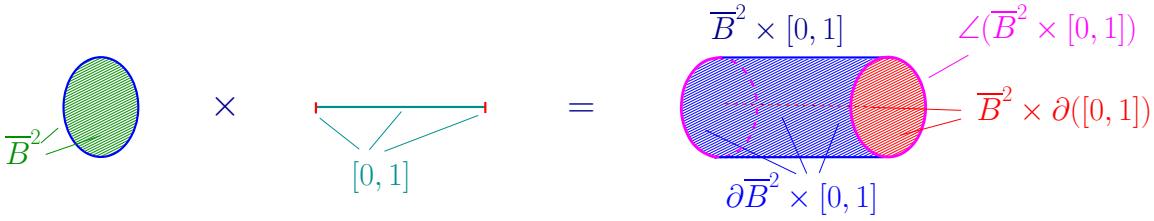
$$\begin{array}{rcl} p\colon M\times N & \to & M \\ (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon M\times N & \to & N \\ (x,y) & \mapsto & y \end{array}$$

the two projections. The following statements hold:

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold that admits a *unique* structure as a smooth manifold with corner that satisfies the following two conditions:
   (a) The projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
   (b) For every smooth manifold with corner W and every map $f\colon W \to M \times N$ the following holds: If the two maps $p \circ f\colon W \to M$ and $q \circ f\colon W \to N$ are smooth, then f itself is smooth.
(2) (a) $\partial(M \times N) = (M \times \partial N) \cup (\partial M \times N)$.
   (b) $\angle(M \times N) = \partial M \times \partial N$ as $(m+n-2)$-dimensional smooth manifolds.

Example. We consider the case that $M = \overline{B}^2$ and $N = [0,1]$.

[658] By the Smooth Manifold–Corner Proposition 26.10 we know that $\angle M$ is naturally a smooth manifold of dimension $\dim(M) - 2$ with empty boundary.



It follows from the Smooth Manifold Product Proposition 26.12 that the "solid cylinder" $M \times N = \overline{B}^2 \times [0,1]$ is a 3-dimensional smooth manifold with corner such that

$$\begin{aligned} \text{boundary } \partial(\overline{B}^2 \times [0,1]) &= (S^1 \times [0,1]) \cup (\overline{B}^2 \times \{0,1\}) \\ \text{corner } \angle(\overline{B}^2 \times [0,1]) &= S^1 \times \{0,1\}. \end{aligned}$$

□

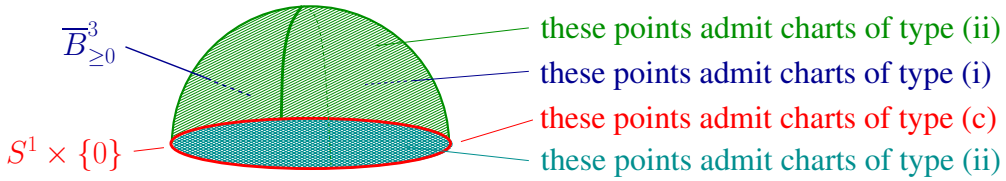
There are also many other examples of smooth manifolds with corner:

Example.

(1) Let $n \geq 2$. We consider the upper half-ball

$$\overline{B}^n_{\geq 0} = \{(x_1, \dots, x_n) \in \overline{B}^n \mid x_n \geq 0\}.$$

We will now equip $\overline{B}^n_{\geq 0}$ with the structure of an n-dimensional smooth manifold with corner $\angle\overline{B}^n_{\geq 0} = S^{n-1} \times \{0\}$.



We consider the map

$$\begin{aligned} \Theta\colon \overline{B}^{n-1} \times [0, \tfrac{1}{2}) &\to \overline{B}^n_{\geq 0} \\ (x,t) &\mapsto (x \cdot \sqrt{1-t^2}, t). \end{aligned}$$

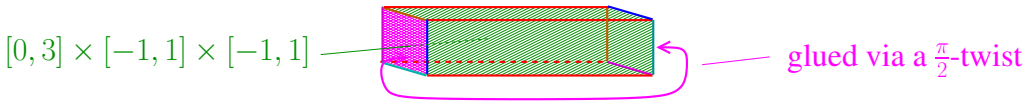
Note that it follows immediately from the Smooth Manifold Product Proposition 26.12 that $\overline{B}^{n-1} \times [0, \frac{1}{2})$ is naturally a smooth manifold with corner $S^{n-2} \times \{0\}$. One can easily verify that this map is an open embedding and that it is a smooth embedding outside of the corner $S^{n-2} \times \{0\}$. We equip $\overline{B}^n_{\geq 0}$ with the following charts:

- For all points outside of $S^{n-2} \times \{0\}$ we pick the hopefully rather obvious charts of type (i) and (ii).
- For each chart $\Phi\colon U \to V$ of $\overline{B}^{n-1} \times [0, \frac{1}{2})$ from the Smooth Manifold Product Proposition 26.12 we consider the chart $\Phi \circ \Theta^{-1}\colon \Theta(U) \to V$.

It follows easily from the fact that Θ is a smooth embedding outside of $S^{n-2} \times \{0\}$ that the above charts define a smooth corner atlas for $\overline{B}^n_{\geq 0}$.

(2) We consider the topological space

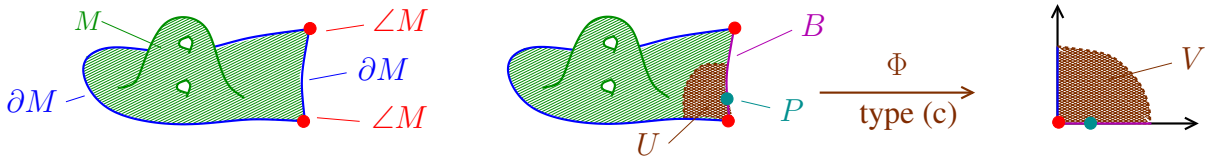
$$M := ([0,3] \times [-1,1] \times [-1,1])/(0,(x,y)) \sim (1,(-y,x)).$$



In Exercise 59.2 we will see that M has the structure of a smooth manifold with corner where $\angle M$ is diffeomorphic to S^1. □

59.2. **Smooth boundaries.** The "total" boundary of a smooth manifold with corner does not have a natural smooth structure and one should not try to view it as a smooth manifold. But as we will see, substantial "chunks" of the boundary have natural smooth structures. To make sense of this we need to introduce the following definition:

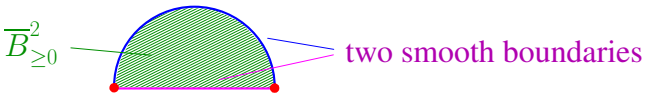
Definition 59.4. Let M be an n-dimensional smooth manifold with corner $\angle M$. We say that a subspace $B \subseteq \partial M$ is a[665] smooth boundary of M if B is a closed subset and if for every $P \in B$ there exists a chart $\Phi\colon U \to V$ of type (ii) or type (c) with $P \in U$ and such that $\Phi(U \cap B) = V \cap (\mathbb{R}^{n-1} \times \{0\})$.



Example.

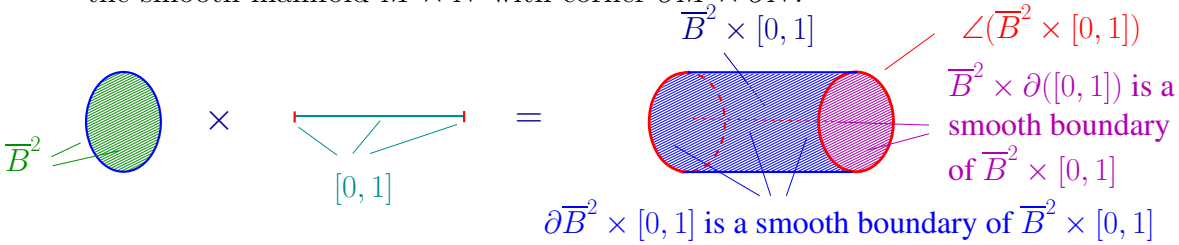
(1) Let M be an n-dimensional smooth manifold with empty corner $\angle M = \varnothing$. Furthermore let $B \subseteq \partial M$ be a subset. It follows easily from the definitions and the Component Lemma 4.48 (4b) that B is a smooth boundary in the above sense if and only if B is the union of components of ∂M.

(2) We consider the smooth manifold $\overline{B}^n_{\geq 0}$. One can show quite easily that $\overline{B}^n_{\geq 0}$ has precisely two smooth boundaries, namely the "bottom" $\overline{B}^{n-1} \times \{0\}$ and the "top" $S^{n-1}_{\geq 0}$.



(3) Let M and N be smooth manifolds.

(a) We consider the smooth manifold $M \times N$ with corner $\partial M \times \partial N$. If A is a union of components of ∂M, then one can easily verify that $A \times N$ is a smooth boundary of the smooth manifold $M \times N$ with corner $\partial M \times \partial N$.



Of course the same statement holds with the roles of M and N reversed.

(b) One can generalize the above example: Let $p\colon W \to M$ be a smooth N-bundle over M. In this case one can easily see that $p^{-1}(\partial M)$ and the corresponding ∂N-bundle are smooth boundaries where the corner is given by the restriction of the ∂N-bundle to ∂M. □

[665] Note that we define "a smooth boundary", not "the smooth boundary".

The following proposition justifies the name *smooth boundary*. Also note that by the above example the following proposition is a generalization of most statements of the Smooth Manifold–Boundary Proposition 25.27:

Proposition 59.5. (Smooth Manifold–Boundary Proposition) Let M be an n-dimensional smooth manifold with corner and let B be a smooth boundary of M in the sense of Definition 59.4. The following statements hold:

(1) The restrictions of all of the charts of the smooth atlas to the smooth boundary B give rise to a natural smooth atlas on B and turn B into a smooth $(n-1)$-dimensional smooth manifold with $\partial B = B \cap \angle M$. The corresponding natural smooth structure has the following properties:
   (a) The inclusion map $B \to M$ is a smooth map.
   (b) Restrictions of any set of charts of M that cover B define the same smooth structure on B.
   (c) Let $f\colon W \to M$ be a smooth map between smooth manifolds. If $f(W) \subseteq B$, then $f\colon W \to B$ is also smooth.

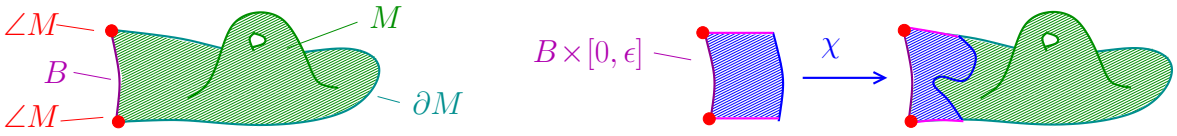
(2) If M is compact, then B is also compact.

Proof. The proof is very similar to the proof of the original Smooth Manifold–Boundary Proposition 25.27. ∎

59.3. **Collars of smooth boundaries.** Recall that on Definition 35.4, given a smooth manifold M, we introduced the notion of a collar for unions of components of the boundary of M. In this section we will generalize this concept to smooth boundaries of smooth manifolds with corner.

Definition 59.6. Let M be a smooth manifold with corner $\angle M$ and let $B \subseteq \partial M$ be a smooth boundary of M. A collar of B is defined as a smooth embedding $\chi\colon [0,\epsilon] \times B \to M$ for some $\epsilon > 0$ that has the following three properties:

(a) The restriction of χ to $\{0\} \times B$ is the obvious map $(0, P) \mapsto P$.
(b) The image $\chi([0,\epsilon] \times B)$ is a closed subset of M.
(c)
- $\chi^{-1}(\partial M) = (\{0\} \times B) \cup ([0,\epsilon] \times \partial B)$.
- $\chi^{-1}(\angle M) = \{0\} \times \partial B$.



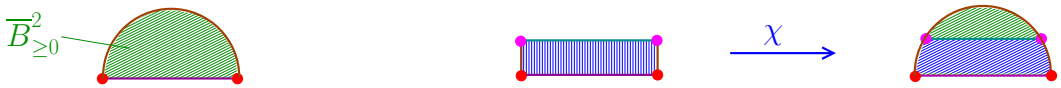
Example.

(1) Let N be a smooth manifold and let $a < b$ be two real numbers. We consider the product $N \times [a,b]$. We invariably equip the smooth boundaries $N \times \{a\}$ and $N \times \{b\}$ with obvious "vertical" collars for a suitable $\epsilon \in (0, b-a)$.

(2) As on page 883 we almost invariably equip $\overline{B}^n$ with the standard collar that is given by

$$\begin{array}{rcl} S^{n-1} \times [0, \frac{1}{2}] & \to & \overline{B}^n \\ (x,t) & \mapsto & x \cdot (1-t). \end{array}$$

(3) We consider $\overline{B}^n_{\geq 0}$ and the smooth boundary $\overline{B}^{n-1} \times \{0\}$.



Similar to the discussion on page 1335 we consider the map

$$\begin{array}{rccc} \chi\colon & \overline{B}^{n-1} \times [0, \frac{1}{2}] & \to & \overline{B}^n_{\geq 0} \\ & (x,t) & \mapsto & (x \cdot \sqrt{1-t^2}, t). \end{array}$$

It follows easily from the definitions that this map defines a collar for the bottom smooth boundary. □

Not surprisingly we claim that the following generalization of the Smooth Collar Neighborhood Theorem 35.6 holds:

Theorem 59.7. (Smooth Collar Neighborhood Theorem) Let M be a smooth manifold with corner $\angle M$ and let $B \subseteq \partial M$ be a smooth boundary of M. For any neighborhood V of the boundary $B \subseteq M$ there exists a collar $\chi\colon [0,1] \times B \to M$ with $\chi([0,1] \times B) \subseteq V$.

Proof. The proof is quite similar to the proof of the original Smooth Collar Neighborhood Theorem 35.6. We leave it to the reader to make the necessary changes. ■

Similarly we claim that the following generalization of the Smooth Collar-Uniqueness Proposition 54.2 holds:

Proposition 59.8. (Smooth Collar-Uniqueness Proposition) Let M be a smooth manifold with corner $\angle M$ and let $B \subseteq \partial M$ be a compact smooth boundary of M. For any two collars $\chi_0\colon [0,\epsilon_0] \times B \to M$ and $\chi_1\colon [0,\epsilon_1] \times B \to M$ there exists a diffeotopy

$$H\colon M \times [0,1] \to M$$

rel B from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1\colon M \to M$ which has the property that $(\Phi \circ \chi_0)(t,b) = \chi_1(\frac{\epsilon_1}{\epsilon_0} \cdot t, b)$ for all $(t,b) \in [0,\epsilon_0] \times B$.

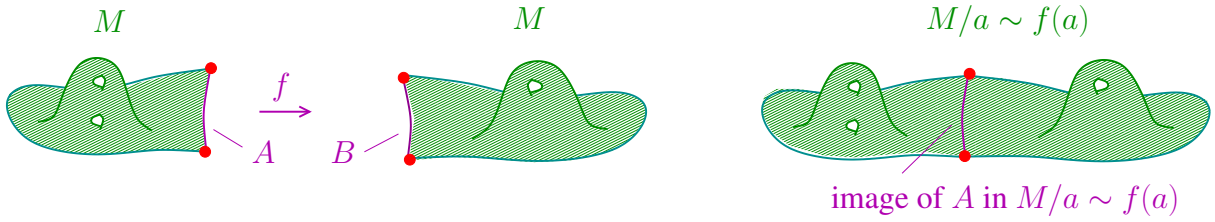
Proof. It will not come as a surprise to the disillusioned reader that we just claim that the proof is very similar to the proof of the original Smooth Collar-Uniqueness Proposition 54.2. ■

59.4. **Gluing smooth manifolds with corner.** Let M be a smooth manifold. In the Gluing-Smooth Manifolds-Proposition 54.5 we learned that if we glue distinct boundary components of M via a diffeomorphism, then we obtain again a smooth manifold. The following proposition generalizes this concept:

Proposition 59.9. (Gluing-Smooth Manifolds-Proposition) Let M be an n-dimensional smooth manifold with corner, let A and B be disjoint compact smooth boundaries of M and let $f\colon A \to B$ be a diffeomorphism.

(0) (a) If M is compact, then $M/a \sim f(a)$ is also compact.
(b) If A and B are connected and if each component of M contains A or B, then $M/a \sim f(a)$ is connected.

(1) The topological space $M/a \sim f(a)$ is an n-dimensional topological manifold. Furthermore it also admits a smooth structure:
  (a) Any choice of a collar $\chi\colon [0,\epsilon] \times (A \sqcup B) \to M$ for the smooth boundary $A \sqcup B$ equips the topological manifold $M/a \sim f(a)$ canonically with the structure of an n-dimensional smooth manifold with corner.
  (b) Let κ and λ be two collars. By (a) these two collars give rise to two smooth structures $\mathcal{A}_\kappa$ and $\mathcal{A}_\lambda$ on the topological manifold $M/a \sim f(a)$. Given any neighborhood U of the image of A in $M/a \sim f(a)$ there exists a homeotopy $H\colon (M/a \sim f(a)) \times [0,1] \to M/a \sim f(a)$ with the following three properties:
    (i) $H_0 = \mathrm{id}$.
    (ii) $H_1\colon (M/a \sim f(a), \mathcal{A}_\kappa) \to (M/a \sim f(a), \mathcal{A}_\lambda)$ is a diffeomorphism of smooth manifolds with corner.
    (iii) $H(x,t) = x$ for all $x \notin U$.
  (c) The diffeomorphism type of $M/a \sim f(a)$ does not depend on the choice of the collar.
(2) (a) The boundary of $M/a \sim f(a)$ is given by $(\partial M \setminus (\mathring{A} \cup \mathring{B}))/a \sim f(a)$ for $a \in \partial A$.
  (b) The corner of $M/a \sim f(a)$ is given by $\angle M \setminus (\partial A \cup \partial B)$.
(3) (a) The image of A in $M/a \sim f(a)$ under the natural map $A \to M \to M/a \sim f(a)$ is a proper codimension-one submanifold of $M/a \sim f(a)$.
  (b) If M consists of two components X and Y with $A \subseteq X$ and $B \subseteq Y$, then the following statements hold:
    (i) The obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are smooth embeddings.
    (ii) The images $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$ and they are codimension-zero submanifolds (with corner) of $M/a \sim f(a)$.
(4) Assume that M is oriented. We equip A and B with the boundary orientations.
  (a) If f is orientation-reversing, then $M/a \sim f(a)$ admits a unique orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.
  (b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.



Convention 59.10. In the setting of the Gluing-Smooth Manifolds-Proposition 59.9 (4a) we always equip $M/a \sim f(a)$ with the induced orientation.

Example.

(1) Let M be a smooth manifold and let $a < b < c$ be real numbers. As above we equip $M \times [a,b]$ and $M \times [b,c]$ with the (almost) obvious collars for $M \times \{b\} \subseteq M \times [a,b]$ and $M \times \{b\} \subseteq M \times [b,c]$. We glue $M \times [a,b]$ and $M \times [b,c]$ along $M \times \{b\}$ in the

obvious way. It follows easily from the definitions that the obvious map to $M \times [a,c]$ is a diffeomorphism of smooth manifolds.

(2) We consider $M = [0,1] \times [0,1]$ as smooth manifold with corner in the obvious way. We consider the smooth boundaries $A \coloneqq \{0\} \times [0,1]$ and $B \coloneqq \{1\} \times [0,1]$. We equip $A \sqcup B$ with the (almost) obvious collar for $\epsilon = \frac{1}{4}$.

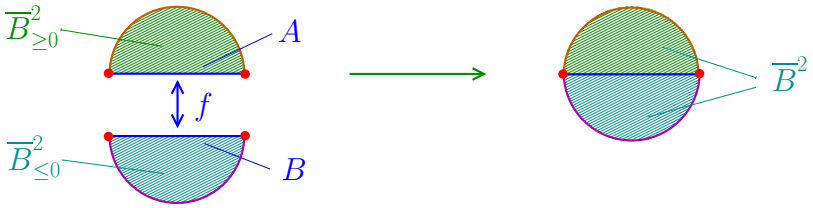
(a) We consider the diffeomorphism $f\colon A \to B$ that is given by $f(0,t) = (1,t)$. It follows easily from the definitions that the map

$$\begin{array}{rcl} ([0,1] \times [0,1])/(0,t) \sim (1,t) & \mapsto & S^1 \times [0,1] \\ [(s,t)] & \mapsto & (\exp(2\pi \operatorname{i} s), t) \end{array}$$

is, with the natural smooth structures, a diffeomorphism.

(b) Similarly to the above, if we use the diffeomorphism $f\colon A \to B$ that is given by $f(0,t) = (1,1-t)$, then one can write down an explicit diffeomorphism to the Möbius band.

(3) Let $M = \overline{B}^n_{\geq 0} \sqcup \overline{B}^n_{\leq 0}$. We view both as smooth manifolds with corner. We glue the bottom of the top to the top of the bottom.

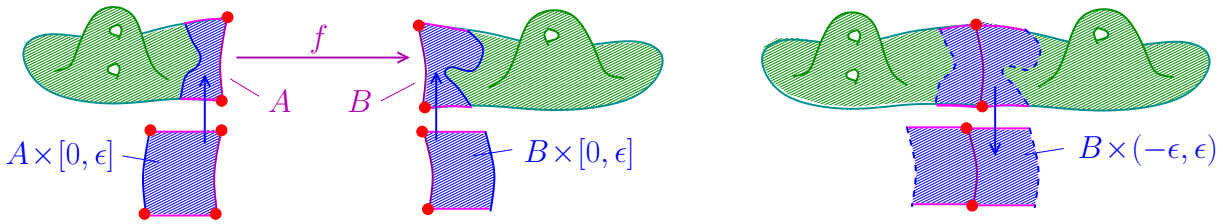


If we use the smooth collars from page 1338 with $\epsilon = \frac{1}{2}$, then it follows easily from the definitions that the obvious map to $\overline{B}^n$ is a diffeomorphism. □

Sketch of proof. The proof of the Gluing-Smooth Manifolds-Proposition 59.9 is basically identical to the proof of the Gluing-Topological Manifolds-Proposition 54.3 and the Gluing-Smooth Manifolds-Proposition 54.5. In the following we therefore just outline the key observation. We denote by $p\colon M \to M/a \sim f(a)$ the natural projection. Furthermore let $\chi\colon [0,\epsilon] \times (A \sqcup B) \to M$ be a collar. One can now easily verify that $p(\chi([0,\epsilon) \times (A \sqcup B)))$ is an open neighborhood of $p(A)$ and that the following map is a well-defined homeomorphism

$$\begin{array}{rcl} p(\chi([0,\epsilon) \times (A \sqcup B))) & \to & B \times (-\epsilon, \epsilon) \\ p(\chi(t,x)) & \mapsto & \begin{cases} (f(x), -t), & \text{if } x \in A, \\ (x,x), & \text{if } x \in B. \end{cases} \end{array}$$

Using this observation one can easily modify the aforementioned proofs to obtain the desired results, except for (1b).



Finally note that for the uniqueness statement in (1b) we need to replace the original Smooth Collar-Uniqueness Proposition 54.2 by the more advanced Smooth Collar-Uniqueness Proposition 59.8. We leave it to the zealous reader to fill in all the details. ■

59.5. **Mapping tori of smooth manifolds.** As a prime example for the gluing operation from the last section we now consider mapping tori. First we recall the following definition:

Definition 21.19. Let X be a topological space and let $f\colon X \to X$ be a continuous map. We refer to

$$\operatorname{Tor}(X,f) \;:=\; (X \times [0,1])\,/\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

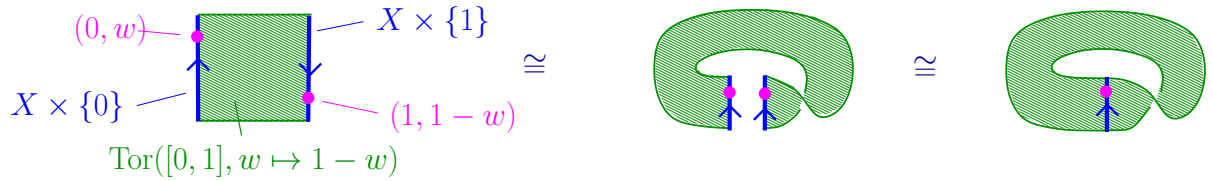
as the mapping torus of (X,f). We refer to the map

$$\begin{array}{rcl} \operatorname{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

as the natural projection onto S^1.

Example.

(1) In the figure below we illustrate that the mapping torus of the map $f\colon [0,1] \to [0,1]$ that is given by $f(w) = 1-w$ is homeomorphic to the Möbius band.



(2) On page 575 we saw that the Klein bottle can be viewed as the mapping torus of a self-map of S^1.

(3) By the Mapping Torus-of-id Lemma 21.22 we know that any mapping torus $\operatorname{Tor}(X,\mathrm{id})$ is naturally homeomorphic to $X \times S^1$. □

We now consider mapping tori in the context of smooth manifolds. This leads us to the following lemma, which can be viewed as a refinement of the Mapping Torus-of-Manifold Lemma 25.36:

Lemma 59.11. (Manifold-Mapping Torus Lemma) Let M be an n-dimensional smooth manifold and let $f\colon M \to M$ be a diffeomorphism. The following statements hold:

(0) (a) If M is compact, then $\operatorname{Tor}(M,f)$ is also compact.
(b) If M is connected, then $\operatorname{Tor}(M,f)$ is connected.

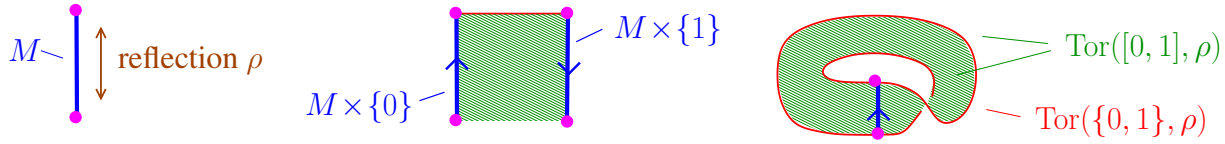
(1) The mapping torus $\operatorname{Tor}(M,f)$ is naturally an $(n+1)$-dimensional smooth manifold.

(2) The boundary $\partial \operatorname{Tor}(M,f)$ is given by the mapping torus $\operatorname{Tor}(\partial M, f|_{\partial M})$, in particular, if M is closed, then $\operatorname{Tor}(M,f)$ is also closed.

(3) The image of M in $\operatorname{Tor}(M,f)$ under the natural map $x \mapsto [(x,0)]$ is a proper n-dimensional smooth submanifold of $\operatorname{Tor}(M,f)$.

(4) We now assume that M is orientable.
(a) If f is orientation-preserving, then $\operatorname{Tor}(M,f)$ is also an orientable smooth manifold.

(b) If M is connected and if f is orientation-reversing, then $\mathrm{Tor}(M,f)$ is also an orientable smooth manifold.



Proof. By the Smooth Manifold Product Proposition 26.12 we know that $M \times [0,1]$ is naturally an $(n+1)$-dimensional smooth manifold with corner $\angle(M \times [0,1]) = \partial M \times \{0,1\}$. On page 1336 we pointed out that a smooth boundary of $M \times [0,1]$ is given by $M \times \{0,1\}$.[666] We equip this smooth boundary with the obvious collar $(M \times \{0,1\}) \times [0, \frac{1}{4}] \to M \times [0,1]$.

Note that we obtain the mapping torus $\mathrm{Tor}(M,f)$ from the smooth manifold $M \times [0,1]$ by identifying the boundary component $M \times \{0\}$ with $M \times \{1\}$ via the diffeomorphism f. Thus the statements are an immediate consequence of the Gluing-Smooth Manifolds-Proposition 59.9.[667] ∎

The following lemma is a smooth analog of the Mapping Tori–Homeomorphic Lemma 21.25.

Lemma 59.12. (Mapping Tori–Diffeomorphic Lemma) Let M be a smooth manifold. If $f, g\colon M \to M$ are two diffeomorphisms that are smoothly diffeotopic, then the mapping tori $\mathrm{Tor}(M,f)$ and $\mathrm{Tor}(M,g)$ are diffeomorphic.

Proof. The lemma is just a special case of the Diffeotopic-Gluings Lemma 54.7. For the reader's convenience we give a self-contained argument. Let $H\colon M \times [0,1] \to M$ be a diffeotopy between $H_0 = f$ and $H_1 = g$. By the Stationary-Near-The-Ends-Lemma 41.7 we can assume that $H_t = f$ for all $t \in [0, \frac{1}{4}]$ and that $H_t = g$ for all $t \in [\frac{3}{4}, 1]$. Using the Smooth Pasting Proposition 25.19 one can now easily verify that the map

$$\begin{array}{rcl} \mathrm{Tor}(M,f) & \to & \mathrm{Tor}(M,g) \\ [(x,t)] & \mapsto & [(H(f^{-1}(x),t),t] \end{array}$$

is a diffeomorphism. ∎

Now we can determine which surfaces are mapping tori:

Proposition 59.13. (Surfaces-as-Mapping Tori Proposition) The annulus, the Möbius band, the torus and the Klein bottle are, up to diffeomorphism, the only connected, compact 2-dimensional topological manifolds that are diffeomorphic to the mapping torus $\mathrm{Tor}(M,f)$ of a connected non-empty 1-dimensional smooth manifold M via a self-diffeomorphism.[668]

Proof. By the examples (1), (2) on page 575 and by the Mapping Torus-of-id Lemma 21.22 we know that there exist the following types of homeomorphisms:

$$\begin{array}{rcl} \text{annulus } S^1 \times [0,1] & \cong & \mathrm{Tor}([0,1],\mathrm{id}) \\ \text{Möbius band} & \cong & \mathrm{Tor}([0,1], x \mapsto 1-x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{torus } S^1 \times S^1 & \cong & \mathrm{Tor}(S^1,\mathrm{id}) \\ \text{Klein bottle} & \cong & \mathrm{Tor}(S^1, z \mapsto \overline{z}). \end{array}$$

[666] If $\partial M = \varnothing$, then this statement also follows from the Smooth Manifold Product Proposition 25.34 (4).

[667] Readers who only care about smooth manifolds without boundary can also make use of the Gluing-Smooth Manifolds-Proposition 54.5.

[668] Of course the analog in the topological setting also holds. In the proof of the proposition we just have to make the obvious adjustments. For example ewe need to replace the Smooth 1-Manifold Classification Theorem 28.8 by the Topological 1-Manifold Classification Theorem 28.1.

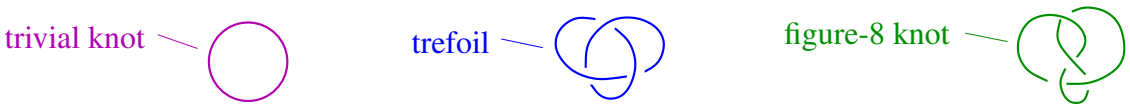
In fact, looking slightly more carefully at the smooth structures one can easily verify that the maps we gave are in fact diffeomorphisms. In particular all these four smooth manifolds are diffeomorphic to mapping tori. It remains to show that these are the only connected, compact 2-dimensional smooth manifolds that can be written as a mapping torus.

To do so, we make the following observation: It follows from the Smooth 1-Manifold Classification Theorem 28.8 that any compact connected non-empty 1-dimensional smooth manifold is diffeomorphic to $[0,1]$ or to S^1. We know from the Diffeotopies-of-$[0,1]$-Lemma 42.2 and the Diffeotopies-of-S^1-Proposition 42.8 that any self-diffeomorphism is diffeotopic to one of the self-diffeomorphisms that we had written down above. But then it follows from the Mapping Tori–Diffeomorphic Lemma 59.12 that any mapping torus is diffeomorphic to one from the above list. ■

Now that we understand which compact 2-dimensional smooth manifolds it is reasonable to ask, how about smooth manifolds of higher dimensions. We will not be able to give a complete answer, but we want to provide a few more examples:

Examples.

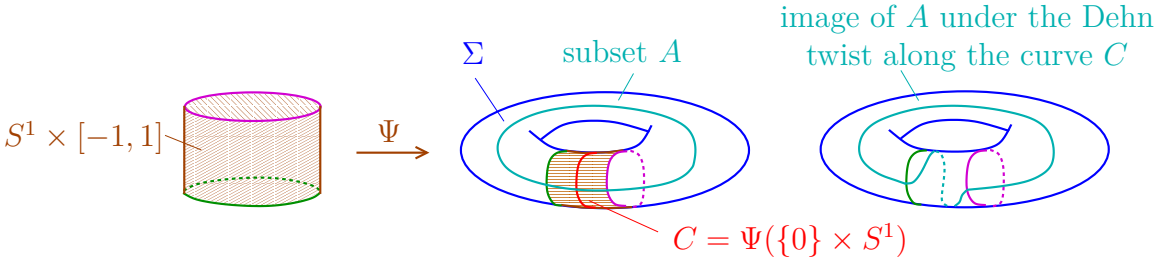
(1) In the Fibered Genus-One Knots Theorem 122.15 we will see that the complement of the trivial knot the complement of the trefoil and the complement of the figure-8 knot in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ are diffeomorphic to a mapping torus, where the fiber is given by removing a point from the torus.



(2) If one wants to construct new interesting smooth manifolds spaces out of a given smooth manifold we need to find "non-trivial" self-diffeomorphisms. For example let Σ be a surface, let $\Psi\colon [-1,1]\times S^1 \to \Sigma$ be an embedding and let $\tau\colon [-1,1]\to[0,2\pi]$ be a smooth map with $\tau(t)=0$ for $t\in[-1,-\frac{1}{2}]$ and with $\tau(t)=2\pi$ for $t\in[\frac{1}{2},1]$. Using the Smooth Pasting Proposition 25.19 one can easily verify that the map

$$\begin{aligned}\Sigma &\to \Sigma\\ P &\mapsto \begin{cases} P, & \text{if } P\notin \Psi([-1,1]\times S^1),\\ \Psi(t, z\cdot\exp(\mathrm{i}\cdot((t+1)\cdot\pi))) & \text{if } P=\Psi(t,z) \text{ for some } t\in[-1,1] \text{ and } z\in S^1\end{cases}\end{aligned}$$

is a diffeomorphism. This map is called a *Dehn twist*[669] *twist along the closed curve* $C := \Psi(\{0\}\times S^1)$.



[669]Max Dehn (1878-1952) was a German–Jewish mathematician. In 1939 he fled to Norway. After the German occupation of Norway he fled to America via Siberia and Japan. He died in 1952 in Black Mountain, North Carolina.

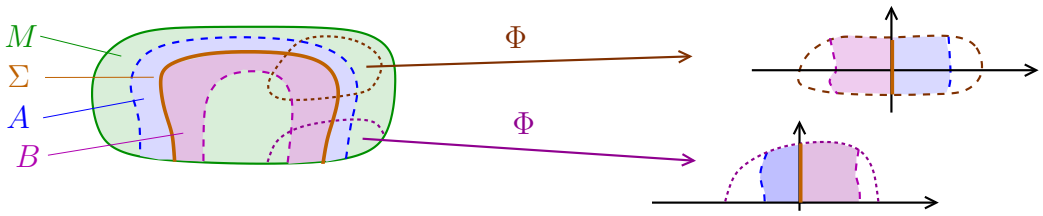
We will discuss Dehn twists in greater detail in **??**. □

59.6. **Cutting smooth manifolds.** In this section we want to get to know a construction that is the inverse to the gluing operation from the last section. To make sense of this we need the notion of a "two-sided" submanifold:

Definition 59.14. Let M be a k-dimensional smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold.

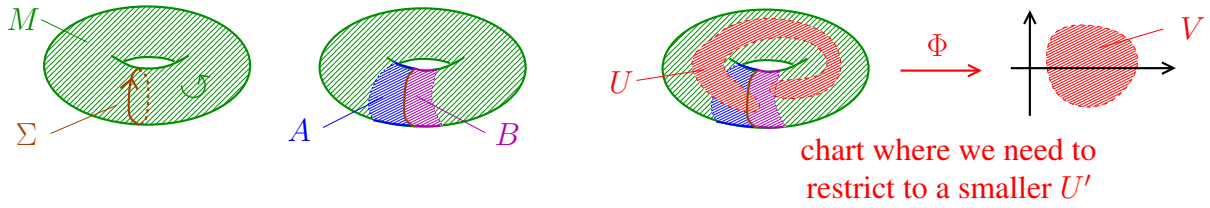
(1) A submanifold–two-coloring for Σ is a pair (A, B) of disjoint open subsets A and B of M such that the following conditions are satisfied:
   (a) $A \cup \Sigma \cup B$ is an open subset of M.
   (b) For every *submanifold chart* $\Phi\colon U \to V$ of type (α) or of type (β), in the sense of Definition 23.18[670], and every $P \in U \cap \Sigma$ there exists an open neighborhood U' of $P \in U$ such that one of the following holds:[671]
      - $\Phi(A \cap U') \subseteq \{(x_1, \dots, x_k) \,|\, x_1 < 0\}$ and $\Phi(B \cap U') \subseteq \{(x_1, \dots, x_k) \,|\, x_1 > 0\}$, or
      - $\Phi(B \cap U') \subseteq \{(x_1, \dots, x_k) \,|\, x_1 > 0\}$ and $\Phi(A \cap U') \subseteq \{(x_1, \dots, x_k) \,|\, x_1 < 0\}$.

(2) We say that the submanifold Σ is two-sided if Σ admits a submanifold–two-coloring.



Examples.

(1) Let M be an orientable smooth manifold and let $\Sigma \subseteq M$ be a proper orientable codimension-one submanifold. It follows from (a slight generalization of) the Bicollar Neighborhood Theorem 55.2 that that Σ admits a bicollar $\beta\colon [-1, 1] \times \Sigma \to M$. Then $A := \beta((-1, 0) \times \Sigma)$ and $B := \beta((0, 1) \times \Sigma)$ are easily seen to be a submanifold-two-coloring for Σ.



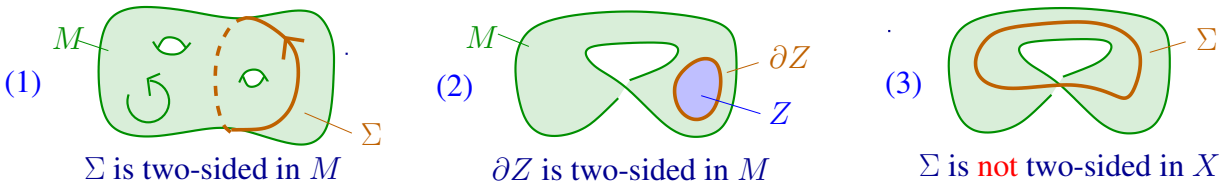
(2) Let M be a smooth manifold and let $Y \subseteq M$ be a compact codimension-zero submanifold of M. Using the Codimension-Zero Submanifold Proposition 25.49 one can easily show that $\partial Y \subseteq M$ is two-sided.

[670] Recall that a proper submanifold is by definition a submanifold that is a closed subset and that only requires submanifold charts of type (α) and (β).

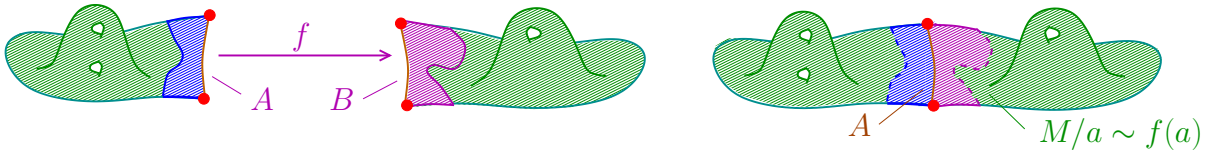
[671] Since Σ is submanifold of codimension we know, by the Definition 23.18 of a submanifold chart, that $\Phi(\Sigma \cap U) \subseteq \{(x_1, \dots, x_k) \,|\, x_1 = 0\}$.

[671] Indeed, the Codimension-Zero Submanifold Proposition 25.49 tells us that $Z := M \setminus \mathring{Y}$ is also a codimension-zero submanifold with $\partial Y = \partial Z = Y \cap Z$. It is now straightforward to verify that $U := \mathring{Y}$ and $V := \mathring{Z}$ have the desired properties.

(3) Let M be the Möbius band. It is pretty clear that the central curve Σ is not two-sided.



(4) Let M be a smooth manifold with corner, let A and B be disjoint compact smooth boundaries of M and let $f\colon A \to B$ be a diffeomorphism. In the Gluing-Smooth Manifolds-Proposition 59.9 (3a) we showed that A defines a compact proper submanifold of $M/a \sim f(a)$. One can easily verify that the collars of A and B define a submanifold-



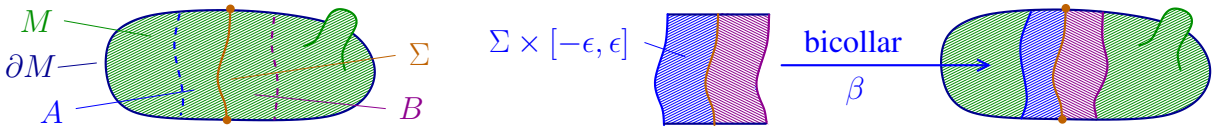
two-coloring of $A \subseteq M/a \sim f(a)$. This suggests that the opposite to gluing should be cutting along a *two-colored* submanifold. We will see shortly that this is indeed the case. □

The following definition is a generalization of the earlier definition of a bicollar in Definition 55.1:

Definition 59.15. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold of M. A bicollar for Σ is a smooth embedding $\beta\colon [-\epsilon,\epsilon] \times \Sigma \to M$ that has the following three properties:

(a) The restriction of β to $\{0\} \times \Sigma$ is the obvious map $(0, P) \mapsto P$.
(b) The image $\beta([-\epsilon,\epsilon] \times \Sigma)$ is a closed subset of M.
(c) We have $\beta^{-1}(\partial M) = [-\epsilon,\epsilon] \times \partial\Sigma$.

We refer to the image of β as a bicollar neighborhood of Σ. If (A, B) is a two-coloring for Σ, then we say that β respects the two-coloring, if $\beta([-\epsilon, 0) \times \Sigma) \subseteq A$ and if $\beta(0,\epsilon] \times \Sigma) \subseteq B$.



We can now formulate the following generalizations of the Bicollar Neighborhood Theorem 55.2 and the Bicollar-Uniqueness Proposition 55.4:

Theorem 59.16. (Bicollar Neighborhood Theorem) Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold of M. If (A, B) is a two-coloring for Σ, then there exists a bicollar $[-\epsilon,\epsilon] \times \Sigma \to M$ that respects the two-coloring.

Proof. If ∂M is orientable and if Σ is also orientable, then we are precisely in the setting of the earlier Bicollar Neighborhood Theorem 55.2. The general case is proved by almost exactly the same way as we proved the Bicollar Neighborhood Theorem 55.2. The only difference is that the submanifold–two-coloring allows us (after possibly reflecting in the

last coordinate) to work only with submanifold charts $\Phi\colon U \to V$ of type (α) for Σ such that we have $\Phi(A \cap U) \subseteq \{(x_1,\dots,x_n) \,|\, x_1 < 0\}$ and $\Phi(B \cap U) \subseteq \{(x_1,\dots,x_n) \,|\, x_1 > 0\}$. It follows that on page 1249 we can again conclude that "we have $D\zeta_i(v) \in \mathbb{R}_{>0}$". The rest of the proof of the Bicollar Neighborhood Theorem 55.2 is unchanged. We leave it to the reader to verify that our claim is correct. ■

Proposition 59.17. (Bicollar-Uniqueness Proposition) Let M be a smooth manifold and let $\Sigma \subseteq M$ be a compact proper codimension-one submanifold with two-coloring (A, B). Furthermore let $\beta_0\colon [-\epsilon_0,\epsilon_0]\times\Sigma \to M$ and $\beta_1\colon [-\epsilon_1,\epsilon_1]\times\Sigma \to M$ be two bicollars for Σ that respect the two-coloring. There exists a diffeotopy

$$H\colon M \times [0,1] \;\to\; M$$

rel Σ from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1\colon M \to M$ which has the property that $(\Phi \circ \beta_0)(t,b) = \beta_1(\frac{\epsilon_1}{\epsilon_0}\cdot t, b)$ for all $(t,b) \in [-\epsilon_0,\epsilon_0]\times\Sigma$.

Proof. The proof is again very similar to the proof of the earlier Bicollar-Uniqueness Proposition 55.4. We leave it to the reader to make the necessary changes. ■

After these preparations we can now define what it means to cut a smooth manifold along a two-sided proper codimension-one submanifold. More precisely, we have the following generalization of Definition 55.6.

Definition 59.18. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper submanifold of codimension one with two-coloring. By the Bicollar Neighborhood Theorem 59.16 we know that there exists a two-color preserving bicollar $\beta\colon [-\epsilon,\epsilon]\times\Sigma \to M$. We define

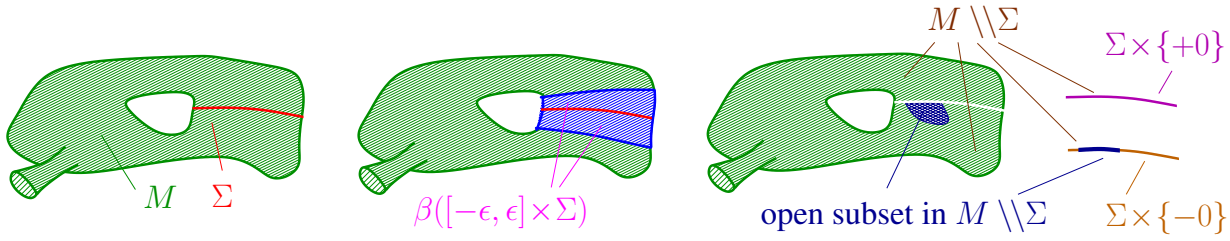
$$M \setminus\!\!\setminus \Sigma \;:=\; (M \setminus \Sigma) \sqcup (\Sigma \times \{\pm 0\})$$

with the topology that is generated by the following sets:

(1) The usual open subsets of $M \setminus \Sigma$.
(2) Sets of the following form:

$$\begin{array}{rl} (U \cap \beta((0,\epsilon)\times\Sigma)) \cup ((U\cap\Sigma)\times\{+0\}) & \text{with } U \text{ an open subset of } \beta([0,\epsilon)\times\Sigma) \\ (U \cap \beta((-\epsilon,0)\times\Sigma)) \cup ((U\cap\Sigma)\times\{-0\}) & \text{with } U \text{ an open subset of } \beta((-\epsilon,0])\times\Sigma). \end{array}$$

We refer to $M \setminus\!\!\setminus \Sigma$ as M cut along Σ. We denote by $\Sigma_\pm$ the images of $\Sigma\times\{\pm 0\}$ in $M \setminus\!\!\setminus \Sigma$. and we denote by $i_\pm\colon \Sigma_\pm \to M \setminus\!\!\setminus \Sigma$ the obvious inclusion maps. Sometimes we also write $\partial_\star(M \setminus\!\!\setminus \Sigma) := \partial(M \setminus \Sigma) \cup (\partial\Sigma \times \{\pm 0\})$.



The following proposition is a generalization of the earlier Cutting Manifold Proposition 55.7:

Proposition 59.19. (Cutting Manifold Proposition) Let M be an n-dimensional smooth manifold and let $\Sigma \subseteq M$ be a compact two-colored proper submanifold of codimension one.

(1) The definition of the topological space $M \setminus\setminus \Sigma$ does not depend on the choice of the bicollar.

(2) The topological space $M \setminus\setminus \Sigma$ admits a natural structure of an n-dimensional smooth manifold with corner such that

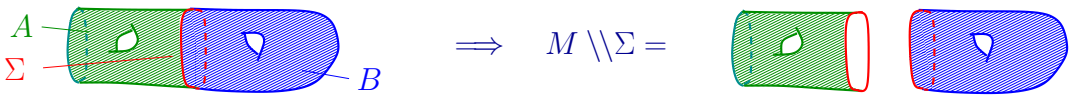
$$\begin{aligned} \partial(M \setminus\setminus \Sigma) &= \partial M \sqcup \Sigma_- \sqcup \Sigma_+, \\ \angle(M \setminus\setminus \Sigma) &= \partial\Sigma_- \sqcup \partial\Sigma_+ \end{aligned}$$

and such that Σ_-, Σ_+ and $\Sigma_\star$ are smooth boundaries of $M \setminus\setminus \Sigma$.



Proof. The proof of the proposition is very similar to the proof of the earlier Cutting Manifold Proposition 55.7. Again we leave it to the reader to make the necessary changes. ■

Example. Let M be an n-dimensional smooth manifold and let $A, B \subseteq M$ be codimension-zero smooth submanifolds such that $\Sigma := A \cap B$ is a union of components of ∂A and also a union of components of ∂B.



It follows easily from the definitions that $M \setminus\setminus \Sigma$ is diffeomorphic to $A \sqcup B$. □

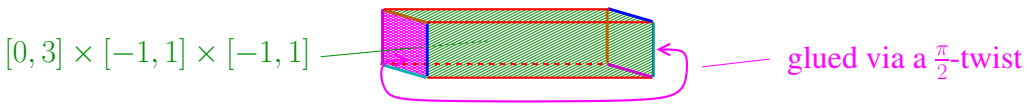
We have the obvious analogs of the Regluing Proposition 55.10 and of the Recutting Proposition 55.11. To save trees and memory we will not spell out these modified propositions.

59.7. Exercises for Chapter 59.

Exercise 59.1. Does every smooth manifold M with corner with non-empty boundary admit a smooth boundary?

Exercise 59.2. We consider the topological space

$$M := ([0,3] \times [-1,1] \times [-1,1])/(0,(x,y)) \sim (1,(-y,x)).$$



(a) Show that M has the structure of a smooth manifold with corner where $\angle M$ is diffeomorphic to S^1.

(b) Does M also admit a smooth atlas without corner charts?

Exercise 59.3. We consider the half-ball $\overline{B}^n_{\geq 0}$. On page 1338 we gave an explicit collar for the bottom $\overline{B}^{n-1} \times \{0\}$. Give an explicit collar of the smooth boundary $S^{n-1}_{\geq 0}$.

Exercise 59.4. Let M be a compact smooth manifold with corner and let C be a smooth boundary of M in the sense of Definition 59.4. Let $\varphi\colon C \to C$ be a diffeomorphism.

(a) Show that the smooth manifold

$$(M \sqcup ([0,1] \times C))/\sim \qquad \text{where } x \sim \varphi(0,x) \text{ for } x \in C$$

is diffeomorphic to M.

Remark. Recall that the definition of the smooth structure depends on the choice of a collar $\chi\colon [0,1] \times C \to M$.

(b) In fact show that given any neighborhood U of $C \subseteq M$ there exists a diffeomorphism that is the identity on the common subset $M \setminus U$.

Remark. $[0,1] \times C \subseteq (M \sqcup ([0,1] \times C))/\sim$ is sometimes called an *external collar*.

Remark. This exercise is essentially the same as Exercise 54.2.

Exercise 59.5. In the Gluing-Smooth Manifolds-Proposition 59.9 we learned how to glue a smooth manifold with corner along a diffeomorphism $f\colon A \to B$ of smooth boundaries. In the proof we saw that the resulting smooth structure on $M/a \sim f(a)$ depends on the choice of a collar. What is the minimal dimension for which different collars lead to different smooth structures on $M/a \sim f(a)$?

Exercise 59.6. Let M be a smooth manifold with corner. Furthermore let B be a smooth boundary of M. In Definition 59.6 we introduced a collar as suitable smooth embedding $\chi\colon [0,\epsilon] \times B \to M$ for some $\epsilon > 0$ such that the image is closed. Is the closed condition really necessary?

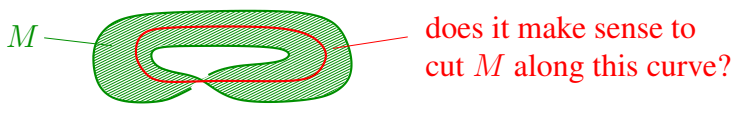
Exercise 59.7. In the Surfaces-as-Mapping Tori Proposition 59.13 we wrote the following: " The annulus, the Möbius band, the torus and the Klein bottle are, up to diffeomorphism, the only connected, compact 2-dimensional topological manifolds that are diffeomorphic to the mapping torus $\mathrm{Tor}(M,f)$ of a connected non-empty 1-dimensional smooth manifold M via a self-diffeomorphism of M."
What happens if we drop the condition connected?

Exercise 59.8.

(a) Let M be an orientable smooth manifold. Show that a closed non-orientable codimension-one submanifold is never two-sided.

(b) Provide an example of an orientable smooth manifold together with a closed non-orientable codimension-one submanifold.

Exercise 59.9. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a compact proper codimension-one submanifold. We do not assume that Σ is two-sided. Is there a sensible notion of cutting M along Σ?

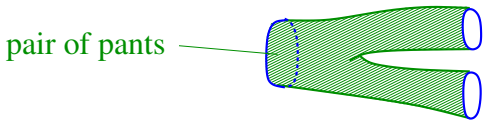


Exercise 59.10. Let M be an orientable smooth manifold with corner.

(a) Is every smooth boundary of M orientable?

(b) Is the corner of M orientable?

Exercise 59.11. The smooth manifold M that is obtain from S^2 by removing the interiors of three disjointly embedded disks is called a *pair of pants*. Which compact 2-dimensional smooth manifolds Σ contain a closed 1-dimensional submanifold C such that $\Sigma \setminus\setminus C$ is diffeomorphic to a disjoint union of pairs of pants?



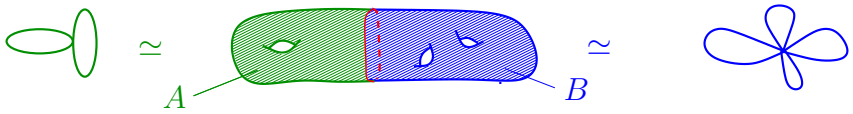
Exercise 59.12. Which compact 2-dimensional smooth manifolds can we cut iteratively into simply connected smooth manifolds (potentially with corner)?

Exercise 59.13. Let M be a closed n-dimensional smooth manifold. We say M admits a *decomposition into small pieces* if there exist compact codimension zero submanifolds A, B of M with the following properties:

(1) $M = A \cup B$.
(2) $A \cap B = \partial A = \partial B$.
(3) A and B are both homotopy equivalent to CW-complexes of dimension $\leq n-2$.

Which of the following smooth manifolds admit a decomposition into small pieces:

(a) The spheres S^n.
(b) The surface of genug g.
(c) The complex projective space $\mathbb{C}\mathrm{P}^m$.
(d) The matrix group $\mathrm{SO}(m)$.



Exercise 59.14. Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a proper oriented codimension-one submanifold of M. Suppose that $\alpha\colon [-1,1] \times \partial\Sigma \to \partial M$ is an orientation-preserving bicollar. Show that there exists an orientation-preserving bicollar $[-1,1] \times \Sigma \to M$ that restricts to α.

Hint. Use the Bicollar-Uniqueness Proposition 59.17. Smooth Collar Neighborhood Theorem 35.6

Exercise 59.15. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper submanifold of codimension one with two-coloring. Furthermore let $\beta\colon [-\epsilon,\epsilon] \times \Sigma \to M$ be a two-color preserving bicollar. Show that there exists a diffeomorphism

$$M \setminus\setminus \Sigma \xrightarrow{\cong} M \setminus \beta((-\epsilon,\epsilon) \times \Sigma)$$

such that $f(\Sigma_\pm) = \beta(\{\pm\epsilon\} \times \Sigma)$.

Hint. Use the Bicollar-Uniqueness Proposition 59.17.

Remark. In the literature often the right-hand side gets used as the definition of "M cut along Σ". But this is a rather awkward definition since it depends on the choice of the bicollar.

CHAPTER 60

Rounding corners

In this chapter we will see that, given a smooth manifold with corner, we can in many cases round corners. More precisely, in the Rounding Corner Proposition 60.5. we will see that many smooth manifolds with corner admit the structure of a smooth manifold (without corner). We will also study the "rounding corner" procedure in some explicit examples, in particular we will see that rounding the corner of the product $\overline{B}^k \times \overline{B}^l$ leads to a smooth manifold that is diffeomorphic to $\overline{B}^{k+l}$.

This chapter is organized as follows:

- In Section 60.1 we will introduce corner-collars of a smooth manifold with corner.
- In Section 60.2 we will use corner-collars to round corners.
- In Section 60.3 we will learn how to round the corners of the product of two smooth manifolds with non-empty boundary.
- In Section 60.4 we deal with rounding the single corner of $(0,1] \times (0,1]$.
- In Section 60.5 we will use the results from Section 60.4 to round the corners of a product of balls and of a half-ball.
- In Section 60.6 we will provide exercises for Chapter 60.

In the next Chapter 61 we will learn what it means to introduce corners.

60.1. **Corner-collars.** The goal of this chapter is to show that under mild hypotheses one can dispose of those pesky corners. To state our main result we need to recall and refine Definition 26.11:

Definition 60.1. Let M be an n-dimensional smooth manifold with corner $\angle M$.

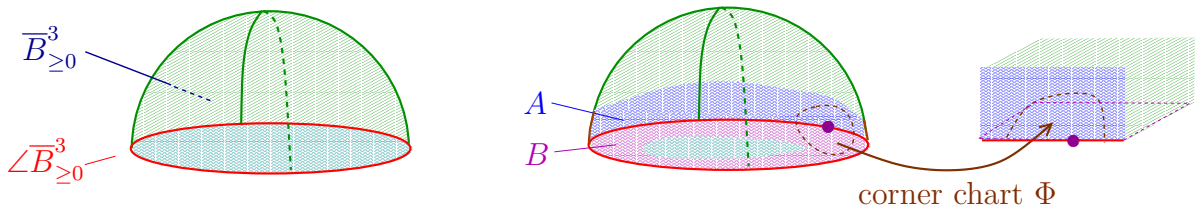
(1) A corner–two-coloring for $\angle M$ is a pair (A, B) of disjoint open subsets A and B of ∂M such that the following conditions are satisfied:
   (a) $A \cup \angle M \cup B$ is an open subset of ∂M,
   (b) For every chart $\Phi\colon U \to V$ of type (c) and every $P \in U \cap \angle M$ there exists an open neighborhood U' of $P \in U$ such that one of the following holds:
      - $\Phi(A \cap U') \subseteq \{(x_1, \dots, x_n) \,|\, x_{n-1} = 0\}$ & $\Phi(B \cap U') \subseteq \{(x_1, \dots, x_n) \,|\, x_n = 0\}$, or
      - $\Phi(B \cap U') \subseteq \{(x_1, \dots, x_n) \,|\, x_{n-1} = 0\}$ & $\Phi(A \cap U') \subseteq \{(x_1, \dots, x_n) \,|\, x_n = 0\}$.

(2) We say that the corner $\angle M$ is two-sided[672] if $\angle M$ admits a corner–two-coloring.

Example.

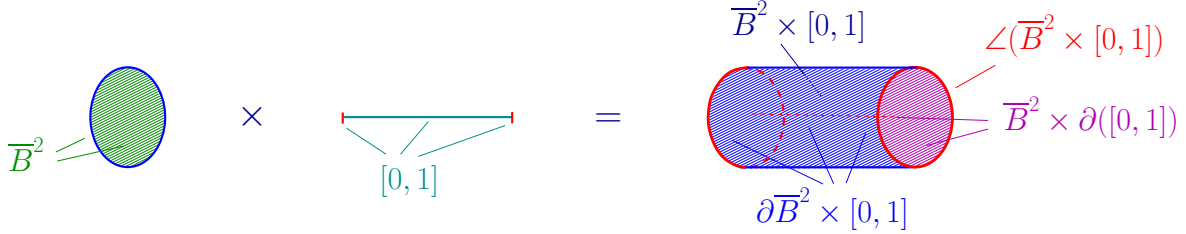
(1) Let M be a smooth manifold with empty boundary. We consider the smooth manifold $M \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with corner $M \times \{(0,0)\}$. One can easily show that a corner–two-coloring for $\angle M$ is given by $M \times \mathbb{R}_{>0} \times \{0\}$ and $M \times \{0\} \times \mathbb{R}_{>0}$.

[672] Our definition of "two-sided" is formulated differently from the definition in [**Wall2016**, p. 59], but it is not that difficult to show that the two definitions are equivalent.



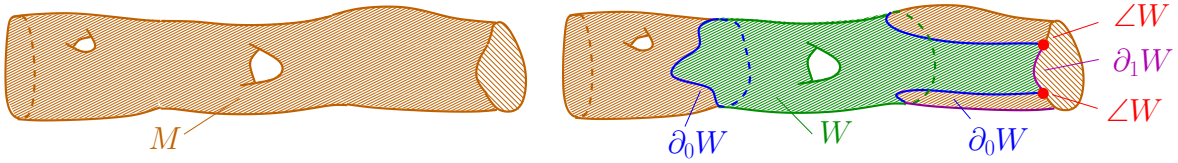
(2) Let M and N be two smooth manifolds.

(a) By the Smooth Manifold Product Proposition 26.12 we know that $M \times N$ is naturally a smooth manifold with corner $\partial M \times \partial N$. It follows easily from the definitions that $A := \partial M \times (N \setminus \partial N)$ and $B := (M \setminus \partial M) \times \partial N$ form a corner–two-coloring of $\angle(M \times N) = \partial M \times \partial N$.



(b) One can generalize the above example: As on page 1351 let $p\colon W \to M$ be a smooth N-bundle over M. In this case one can easily see that W is a smooth manifold where the corner is given by the restriction of the ∂N-bundle to ∂M. This corner is easily seen to be two-sided.

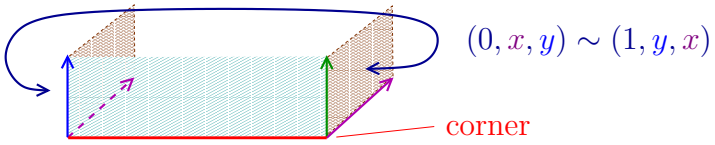
(3) Let M be a smooth manifold. In Definition 26.14 we introduced the notion of a codimension-zero submanifold W with corner. Given such a submanifold W we introduced in Notation 25.47 and the Boundary-of-Submanifold Proposition 25.48 (0) the subspaces $\partial_0 W := W \cap (\overline{M \setminus W})$ and $\partial_1 W := W \cap \partial M$. It follows from the Smooth Submanifold-with-Corner Lemma 26.16 that W is naturally a smooth manifold with corner $\angle W = \partial_0 W \cap \partial_1 W$ and that a corner–two-coloring of $\angle W$ is given by $\partial_0 W \setminus \angle W$ and $\partial_1 W \setminus \angle W$.



(4) The topological space

$$M := ([0,1] \times \mathbb{R}^2_{++})/(0,x,y) \sim (1,y,x)$$

admits the structure of a smooth manifold with corner, where $\angle M$ is represented precisely by the points $(t,0,0)$.



One can easily see that in this case the corner is *not* two-sided.[673] □

The following definition is again inspired by the definition of a collar on Definition 35.4:

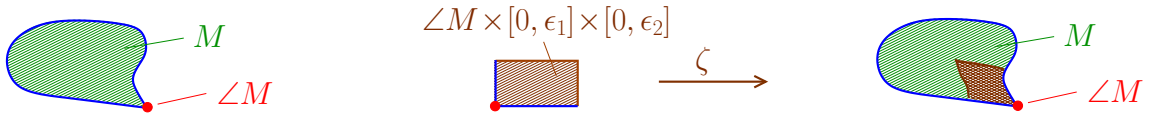
Definition 60.2. Let M be a smooth manifold with corner. A corner-collar[674] is a smooth embedding[675]

$$\zeta\colon \angle M \times [0,\epsilon_1] \times [0,\epsilon_2] \;\to\; M$$

for some $\epsilon_1, \epsilon_2 > 0$ with the following properties:

(a) The restriction of ζ to $\angle M \times \{(0,0)\}$ is the obvious map $(P,0,0) \mapsto P$.
(b) The image $\zeta(\angle M \times [0,\epsilon_1] \times [0,\epsilon_2])$ is a closed subset of M.
(c) $\zeta^{-1}(\partial M) = \angle M \times (([0,\epsilon_1] \times \{0\}) \cup (\{0\} \times [0,\epsilon_2]))$.

If we are given a corner–two-coloring (A,B) for $\angle M$, then we say that the corner-collar respects the corner–two-coloring if $\zeta(\angle M \times (0,\epsilon_1] \times \{0\}) \subseteq A$ and $\zeta(\angle M \times \{0\} \times (0,\epsilon_2]) \subseteq B$.



Example. Let M and N be smooth manifolds. Furthermore let $\chi\colon \partial M \times [0,\mu] \to M$ and $\lambda\colon \partial N \times [0,\nu] \to N$ be two collars. It follows easily from the definitions that the map

$$\begin{array}{rcl} \angle(M \times N) \times [0,\mu] \times [0,\nu] & \to & M \times N \\ ((x,y),s,t) & \mapsto & (\chi(x,s),\lambda(y,t)) \end{array}$$

is a corner-collar for the smooth manifold $M \times N$ with corner $\angle(M \times N) = \partial M \times \partial N$. □

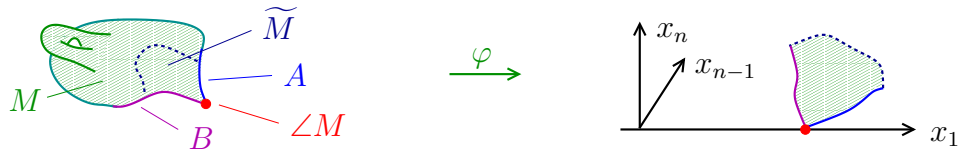
It is clear that if the corner admits a corner-collar, then it is two-sided. The following theorem, which can be viewed as an analog of the Smooth Collar Neighborhood Theorem 35.6, shows that the converse holds:

Theorem 60.3. (Corner-Collar Neighborhood Theorem) Let M be a smooth manifold with corner and let (A,B) be a corner–two-coloring for $\angle M$. There exists a corner-collar $\zeta\colon \angle M \times [0,1]^2 \to M$ that respects the corner–two-coloring.

Sketch of proof. The following proof follows the same logic as the proof of the Smooth Collar Neighborhood Theorem 35.6 which we gave on page 890.

(1) By the same logic as in the Euclidean Embedding Theorem 33.2 we see that there exists an open neighborhood $\widetilde{M}$ of $\angle M \subseteq M$ with $\partial\widetilde{M} \subseteq A \cup B$, an $n \in \mathbb{N}$ and a proper smooth embedding $\varphi\colon \widetilde{M} \to \mathbb{R}^n_{++}$ with the following two properties:
   (a) φ restricts to a proper embedding $A \to \{(x_1,\dots,x_{n-2},x_{n-1},0) \in \mathbb{R}^n \mid x_{n-1} \geq 0\}$.
   (b) φ restricts to a proper embedding $B \to \{(x_1,\dots,x_{n-2},0,x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

[673] This example gives in particular a solution to Exercise 26.4.
[674] Recall that by the Smooth Manifold–Corner Proposition 26.10 we know that $\angle M$ is a smooth manifold of dimension $\dim(M)-2$ with no boundary. It follows that $\angle M \times [0,\epsilon_1] \times [0,\epsilon_2]$ is a smooth manifold with corner.
[675] What we call a "corner-collar" is called a "bicollar" in [**Wall2016**, p. 60]. Recall that in Definition 55.1 we already used the term "bicollar" for a different concept.



To simplify the notation we now assume that $\widetilde{M} = M$ and we view M already as a submanifold of $\mathbb{R}^n_{++}$ where A and B have the above properties.

(2) As in Claim 1 on page 890 we consider the map

$$\begin{array}{rcl} \Omega\colon \angle M \times [0,1] \times [0,1] & \to & \mathbb{R}^n_{++} \\ (P, t_1, t_2) & \mapsto & P + t_1 \cdot e_{n-1} + t_2 \cdot e_n. \end{array}$$

(3) Similar to the Normal Vectors Lemma 35.3 one can show that

$$WM \coloneqq \{(P, w) \in \mathbb{R}^n_{++} \times \mathbb{R}^n \mid P \in M \text{ and } w \in (\mathrm{T}_P M)^\perp\}$$

is an n-dimensional submanifold of $\mathbb{R}^n_{++} \times \mathbb{R}^n$ with $\partial(WM) = WM \cap (\partial\mathbb{R}^n_{++} \times \mathbb{R}^n)$ and with $\angle(WM) = WM \cap (\angle\mathbb{R}^n_{++} \times \mathbb{R}^n)$.

(4) Similar to Claim 2 on page 891 we consider the projection

$$\begin{array}{rcl} p\colon \mathbb{R}^n_{++} & \to & \mathbb{R}^{n-2} \\ (x_1, \dots, x_n) & \mapsto & (x_1, \dots, x_{n-2}) \end{array} \qquad \text{and for } i = n-1, n \text{ we consider} \qquad \begin{array}{rcl} q_i\colon \mathbb{R}^n_{++} & \to & \mathbb{R}_{\geq 0} \\ (x_1, \dots, x_n) & \mapsto & x_i. \end{array}$$

Furthermore we consider the corresponding map

$$\begin{array}{rcl} \Theta\colon WM & \to & \mathbb{R}^{n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} = \mathbb{R}^n_{++} \\ (P, w) & \mapsto & (p(P + w), q_{n-1}(P), q_n(P)). \end{array}$$

(5) As in Claim 3 on page 891 we consider the projection

$$\begin{array}{rcl} \Pi\colon WM & \to & M \\ (P, w) & \mapsto & P. \end{array}$$



(6) Similar to the discussion on page 891 we see that there exists an open neighborhood U of $\angle M \times \{(0,0)\} \subseteq WM$ such that $\Theta(U)$ is an open neighborhood of $\angle M \subseteq \mathbb{R}^n_{++}$ and such that $\Theta\colon U \to \Theta(U)$ is a diffeomorphism.

(7) Finally, similar to Claim 5 on page 892 we see that there exist two smooth maps $\nu_1, \nu_2\colon \angle M \to (0, \infty)$ such that the map

$$\begin{array}{rcl} \angle M \times [0,1] \times [0,1] & \to & M \\ (P, (t_1, t_2)) & \mapsto & (\Pi \circ \Theta^{-1} \circ \Omega)((\nu_1(P) \cdot t_1, \nu_2(P) \cdot t_2, P)) \end{array}$$

is defined and such that it is a corner-collar.

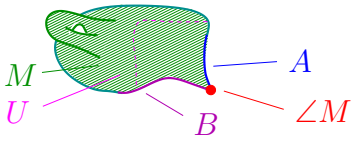
We will leave it to the reader to modify the detailed proof of the Smooth Collar Neighborhood Theorem 35.6 to fill in the details for the current proof. ■

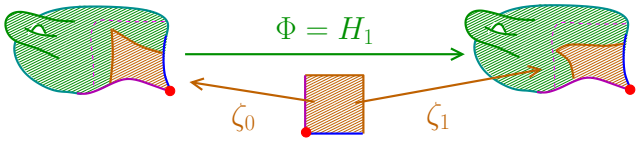
We conclude this discussion of corner-collars with the following analog of the Smooth Collar-Uniqueness Proposition 54.2.

Theorem 60.4. (Corner-Collar–Uniqueness Theorem) Let M be a smooth manifold with corner such that the corner $\angle M$ is compact and let (A, B) be a corner–two-coloring. Furthermore let $\zeta_i \colon \angle M \times [0, \epsilon_1^i] \times [0, \epsilon_2^i] \to M$, $i = 0, 1$ be two corner-collars that respect the corner–two-coloring.

(1) There exists a diffeotopy
$$H \colon M \times [0,1] \;\to\; M$$
rel $\angle M$ from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi := H_1 \colon M \to M$ such that we have $(\Phi \circ \zeta_0)(c, s_1, s_2) = \zeta_1(c, \frac{\epsilon_1^1}{\epsilon_1^0} \cdot s_1, \frac{\epsilon_2^1}{\epsilon_2^0} \cdot s_2)$ for all $(c, s_1, s_2) \in \angle M \times [0, \epsilon_1^0] \times [0, \epsilon_2^0]$.

(2) For any open subset U of the smooth manifold M that contains the two corner-collar neighborhoods $\zeta_i(\angle M \times [0, \epsilon_1^i] \times [0, \epsilon_2^i])$, $i = 0, 1$ one can find H as above such that H is furthermore a homotopy rel $M \setminus U$.





Outline of proof. Evidently the proof is related to the proof of the Smooth Collar-Uniqueness Proposition 54.2. But in fact it is even closer in terms of techniques to the proof of the Tubular Neighborhood–Uniqueness Theorem 57.12. Note that in the proof of the Smooth Collar-Uniqueness Proposition 54.2 and of the Tubular Neighborhood–Uniqueness Theorem 57.12 we made use of the Isotopy Extension Theorem 44.2, which we formulated and proved only for compact submanifolds. This is the main reason why we restrict to the case of a compact corner. We leave the unpleasant task of assembling a complete proof to the reader. ■

60.2. **Rounding corners.** After all these preparations we are finally ready to round corners:

Proposition 60.5. (Rounding Corner Proposition) Let $(M, \mathcal{C})$ be a smooth manifold with compact corner $\angle M := \angle(M, \mathcal{C})$. If $\angle M$ is two-sided, then any choice of a corner-collar for $\angle M$ (by the Corner-Collar Neighborhood Theorem 60.3 we know that such a corner-collar always exists) gives rise to a maximal smooth atlas $\mathcal{A}$ on M with the following properties:

(1) (a) $\mathcal{A}$ consists only of charts of type (i) and (ii).
(b) Each point in $\angle M$ admits a chart of type (ii) in $\mathcal{A}$.

(2) The identity map $(M \setminus \angle M, \mathcal{C}) \to (M \setminus \angle M, \mathcal{A})$ is a diffeomorphism.

Any two choices of corner-collars give rise to maximal atlases $\mathcal{A}$ and $\mathcal{A}'$ such that $(M, \mathcal{A})$ and $(M, \mathcal{A}')$ are diffeomorphic.

Notation 60.6. We continue with the notation from the Rounding Corner Proposition 60.5. We denote the resulting smooth manifold $(M, \mathcal{A})$ by $M^{\curvearrowleft}$.

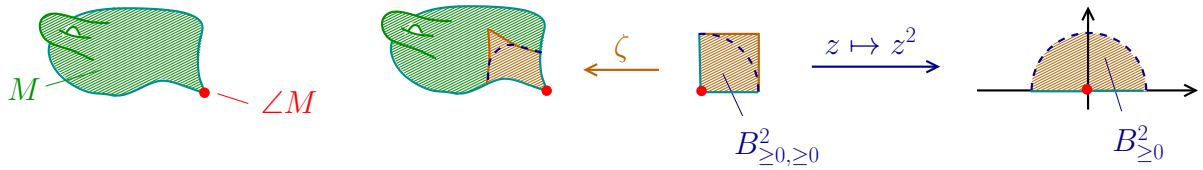


Proof. Let $(M, \mathcal{C})$ be an n-dimensional smooth manifold with two-sided corner $\angle M$ and let $\zeta\colon \angle M \times [0,\epsilon_1] \times [0,\epsilon_2] \to M$ be a corner-collar. After rescaling we can and will assume that $\epsilon_1 = \epsilon_2 = 1$. We now construct an atlas $\mathcal{A}$ for the topological manifold M as follows:

(α) We take all charts from $\mathcal{C}$ and restrict them to $M \setminus \angle M$.[676]

(β) Recall that by the Smooth Manifold–Corner Proposition 26.10 we know that the corner $\angle M$ is naturally an $(n-2)$-dimensional smooth manifold with empty boundary. Let $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ be a smooth atlas for $\angle M$. We consider the "open quarter-disk" $B^2_{\geq 0,\geq 0} := \{(x,y) \in B^2 \,|\, x,y \geq 0\}$ and the "open half-disk" $B^2_{\geq 0} := \{(x,y) \in B^2 \,|\, y \geq 0\}$. For each $i \in I$ we consider the map that is given by the composition of the following three maps:

$$\underbrace{\zeta(U_i \times B^2_{\geq 0,\geq 0})}_{\text{open in } M} \xrightarrow{\zeta^{-1}} U_i \times B^2_{\geq 0,\geq 0} \xrightarrow{\operatorname{diag}(\Phi_i,\mathrm{id})} V_i \times \underbrace{B^2_{\geq 0,\geq 0}}_{\subseteq \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}} \xrightarrow{\operatorname{diag}(\mathrm{id}, z\mapsto z^2)} \underbrace{V_i \times B^2_{\geq 0}}_{\text{open in } \mathbb{H}^n}.$$



The following claim says that we have found the desired smooth atlas:

Claim. The charts from (α) and (β) form a smooth atlas $\mathcal{A}$ for M consisting of charts of type (i) and (ii) and this atlas (and the corresponding maximal atlas) has the desired properties (1) and (2).

Proof. First of all note that the map $B^2_{\geq 0,\geq 0} \to B^2_{\geq 0}$ that is given by $z \mapsto z^2$ is a homeomorphism. It is now clear that all of the above maps are in fact charts. Next note that it is clear that any two charts of type (α) are smoothly compatible. Furthermore note that it follows easily from the Inverse Mapping Theorem 30.1 (2) that the map

$$\begin{array}{ccc} B^2_{\geq 0} \setminus \{0\} & \to & B^2_{\geq 0,\geq 0} \setminus \{0\} \\ z & \mapsto & z^2 \end{array}$$

is a diffeomorphism. Using this piece of information one can show fairly easily that every chart of type (α) is compatible with every chart of type (β). Finally if we are given two charts of type (β), then it follows easily from the definitions that the transition map is of the form $\operatorname{diag}(\Phi_j \circ \Phi_i^{-1}, \mathrm{id}_{B^2_{\geq 0}})$, which is of course smooth. ⊞

We pick a corner–two-coloring (A, B) for the corner $\angle M$. Suppose that we are given two corner-collars ζ_0, ζ_1. Note that after possibly swapping the two coordinates of the corner collars we can arrange that both corner-collars respect the corner–two-coloring. It now follows easily from the Corner-Collar–Uniqueness Theorem 60.4 that any two choices of

[676] Note that by the Smooth Manifold–Corner Proposition 26.10 (2b) we know that $\angle M$ is a closed subset of M, thus $M \setminus \angle M$ is an open subset of M.

corner-collars give rise to maximal atlases $\mathcal{A}$ and $\mathcal{A}'$ such that $(M, \mathcal{A})$ and $(M, \mathcal{A}')$ are diffeomorphic. Again we leave it to the reader to fill in the details. ■

60.3. **Rounding products.** In the Topological Manifold Product Proposition 23.6 we saw that the product of two topological manifolds is again a topological manifold. Furthermore, in the Smooth Manifold Product Proposition 25.34 we saw that if we have two smooth manifolds M and N, where at least one has empty boundary, then the product $M \times N$ comes with a natural smooth structure. On the other hand, if both smooth manifolds have non-empty boundary, then we saw in the Smooth Manifold Product Proposition 26.12 that the product $M \times N$ should really be viewed as a smooth manifold with corner $\partial M \times \partial N$.

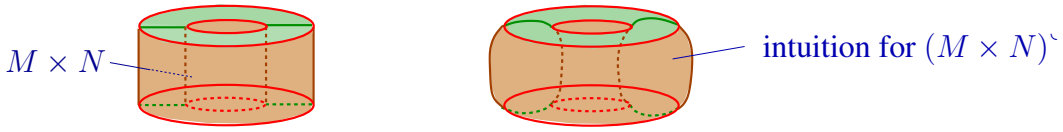
But since we just learned that we can round corners it is pleasant to record the following proposition:

Proposition 60.7. (Smooth Manifold Rounded Product Proposition)[677] Let M be a compact m-dimensional smooth manifold and let N be a compact n-dimensional smooth manifold.

(1) (a) By the Smooth Manifold Product Proposition 26.12 the product $M \times N$ is a smooth manifold with corner $\partial M \times \partial N$. This corner admits a corner-collar.
   (b) Given a choice of corner-collar we can equip $M \times N$ with the structure of an $(m+n)$-dimensional smooth manifold $(M \times N)^{\llcorner}$ with empty corner such that the inclusion maps $M \times (N \setminus \partial N) \to (M \times N)^{\llcorner}$ and $(M \setminus \partial M) \times N \to (M \times N)^{\llcorner}$ are both smooth embeddings.[678]
   (c) Any two corner-collars lead to diffeomorphic smooth structures on the topological manifold $M \times N$.

(2) The boundary $\partial(M \times N)^{\llcorner}$ of the smooth manifold $(M \times N)^{\llcorner}$ is naturally diffeomorphic to $(\partial M \times N) \cup (M \times \partial N)$.[679]

(3) If M and N are oriented, then the smooth manifold $(M \times N)^{\llcorner}$ admits a unique orientation such that the smooth embeddings[680] $M \times (N \setminus \partial N) \to (M \times N)^{\llcorner}$ and $(M \setminus \partial M) \times N \to (M \times N)^{\llcorner}$ are both orientation-preserving.



Sketch of proof.

(1) (a) As we discussed on page 1352, collars for $\partial M \subseteq M$ and $\partial N \subseteq N$ give rise to a corner-collar for $\angle(M \times N) = \partial M \times \partial N$.
   (b) This statement follows from the Smooth Manifold Product Proposition 26.12 together with the Rounding Corner Proposition 60.5.

[677] On page 705 we formulated a simplified version of this proposition.

[678] Note that $M \setminus \partial M$ and $N \setminus \partial N$ do not have boundary. Hence it follows from the Smooth Manifold Product Proposition 25.34 that we can view $M \times (N \setminus \partial N)$ and $(M \setminus \partial M) \times N$ as smooth manifolds.

[679] More precisely, a choice of a corner-collar for $\angle(M \times N)$ also defines collars for $\partial M \subseteq M$ and $\partial N \subseteq N$ which we then use, together with the Gluing-Smooth Manifolds-Proposition 59.9, to define the smooth structure on $(\partial M \times N) \cup (M \times \partial N)$.

[680] We equip $M \times (N \setminus \partial N)$ and $(M \setminus \partial M) \times N$ with the orientations coming from the Product Orientation Proposition 31.48.

(c) This statement follows quite easily from the uniqueness statement of the Rounding Corner Proposition 60.5.

(2) Let $\zeta\colon (\partial M \times \partial N) \times [0,\epsilon_1] \times [0,\epsilon_2] \to M$ be a corner-collar. One can easily verify that the maps

$$\begin{array}{rcl} \partial M \times \partial N \times [0,\epsilon_2] & \to & \partial M \times N \\ (z,t) & \mapsto & \zeta(z,0,t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \partial M \times \partial N \times [0,\epsilon_1] & \to & M \times \partial N \\ (z,t) & \mapsto & \zeta(z,t,0) \end{array}$$

define collars for $\partial M \times \partial N \subseteq \partial M \times N$ and also for $\partial M \times \partial N \subseteq M \times \partial N$. If we use these two collars (and the Gluing-Smooth Manifolds-Proposition 54.5) to define the smooth manifold $(\partial M \times N) \cup (M \times \partial N)$, then one can easily verify that the identity map $\partial(M \times N)^\backprime \to (\partial M \times N) \cup (M \times \partial N)$ is a diffeomorphism.

(3) We now assume that the smooth manifolds M and N are oriented. Recall that by the Product Orientation Proposition 31.48 we can equip $M \times (N \setminus \partial N)$ and $(M \setminus \partial M) \times N$ with natural orientations. Basically by definition these orientations agree on the overlap $(M \setminus \partial M) \times (N \setminus \partial N)$, thus we have an orientation on $(M \times N) \setminus (\partial M \times \partial N)$. Finally note that it follows from an argument as in the Complement-of-Boundary Orientation Lemma 31.36 (1) that the smooth manifold $(M \times N)^\backprime$ admits a unique orientation such that the inclusion $(M \times N) \setminus (\partial M \times \partial N) \to (M \times N)^\backprime$ is orientation-preserving. We will fill in the details in Exercise 60.3. ■

Remark. The Smooth Manifold Rounded Product Proposition 60.7 is less helpful than it might appear. In most situations it is the correct point of view to treat the product of two smooth manifolds with non-empty boundary as a smooth manifold with corner. Rounding corners depends on choices, and it is not really that clear, what good rounding corners of products actually does. □

60.4. **Explicit rounding of a single corner.** In this section we want to round corners in several explicit "real world" cases. The key to doing so is the following "baby case":

Proposition 60.8. (Single Corner-to-Disk Proposition) We consider the smooth manifold $(0,1] \times (0,1]$ with corner $\{(1,1)\}$ and the obvious corner-collar with $\epsilon = \frac{1}{4}$. There exists a diffeomorphism

$$\Xi\colon \overline{B}^2_{>0,>0} = \{(x,y) \in \overline{B}^2 \,|\, x,y > 0\} \;\to\; ((0,1] \times (0,1])^\backprime$$

which has the following properties:

(a) We have $\Xi(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (1,1)$.

(b) Ξ is the identity for all z with $|z| \leq \frac{1}{4}$.

(c) There exists a smooth function $\tau\colon [0,1] \to [0,1]$ with $\tau(t) = 0$ for $t \in [0, \frac{1}{4}]$ and with $\tau(t) = 1$ for $t \in [\frac{3}{4}, 1]$ such that for all $z = r \cdot \exp(\mathrm{i}\varphi)$ with $\varphi \in [0, \frac{\pi}{8}]$ we have[681]

$$\Xi(r \cdot \exp(\mathrm{i}\varphi)) \;=\; r \cdot \big((1 - \tau(r)) + \frac{1}{\sin(\varphi)} \cdot \tau(r)\big) \cdot \exp(\mathrm{i}\varphi)$$

and such that for all $z = r \cdot \exp(\mathrm{i}\varphi)$ with $\varphi \in [\frac{3\pi}{8}, \frac{\pi}{2}]$ we have

$$\Xi(r \cdot \exp(\mathrm{i}\varphi)) \;=\; r \cdot \big((1 - \tau(r)) + \frac{1}{\cos(\varphi)} \cdot \tau(r)\big) \cdot \exp(\mathrm{i}\varphi).$$

(d) For every $e^{\mathrm{i}\varphi} \in S^1$ with $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$ we have $\Xi(e^{\mathrm{i}\varphi}) = \frac{1}{\cos(\varphi)} \cdot e^{\mathrm{i}\varphi} = (\tan(\varphi), 1)$.



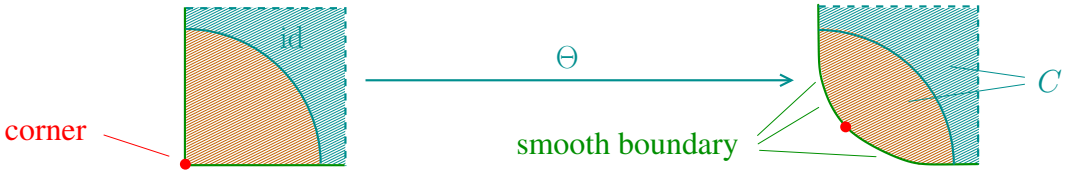
The proof of the Single Corner-to-Disk Proposition 60.8 will occupy the remainder of this section. In the next Section 60.5 we will use the Single Corner-to-Disk Proposition 60.8 to consider several other explicit examples of rounding corners.

The key to proving the Single Corner-to-Disk Proposition 60.8 is the following more technical lemma:

Lemma 60.9. (Single Corner-to-Convex Lemma) We consider the smooth manifold $[0,1) \times [0,1)$ with the corner $\{(0,0)\}$ and the obvious corner-collar with $\epsilon = \frac{1}{4}$. There exists a pair $(C, \Theta\colon [0,1) \times [0,1) \to C$ of the following form:

(1) C is a subspace of $[0,1) \times [0,1)$ with the following properties:
- C contains all $(x,y) \in [0,1) \times [0,1)$ with $x \geq \frac{3}{4}$ or $y \geq \frac{3}{4}$.
- C is a convex 2-dimensional submanifold of $\mathbb{R}^2$.

(2) $\Theta\colon [0,1) \times [0,1) \to C$ is a map with the following properties:
- Θ is the identity on $\{z \in [0,1) \times [0,1) \,|\, \|z\| \geq \frac{3}{4}\}$.
- Θ is a diffeomorphism $([0,1) \times [0,1))^\backprime \to C$.



Sketch of proof of the Single Corner-to-Convex Lemma 60.9. Using the Smooth Transition Function Lemma 24.9 one can show that there exists a smooth map $\tau\colon [\frac{1}{4}, 1] \to [0, \frac{1}{4}]$ with $\tau(t) = \frac{1}{2} - t$ for all $t \in [\frac{1}{4}, \frac{3}{8}]$, $\tau(t) = 0$ for all $t \in [\frac{3}{4}, 1]$ and such that $\tau'(t) \leq 0$ for all t and such that $\tau''(t) \geq 0$ for all t. We set

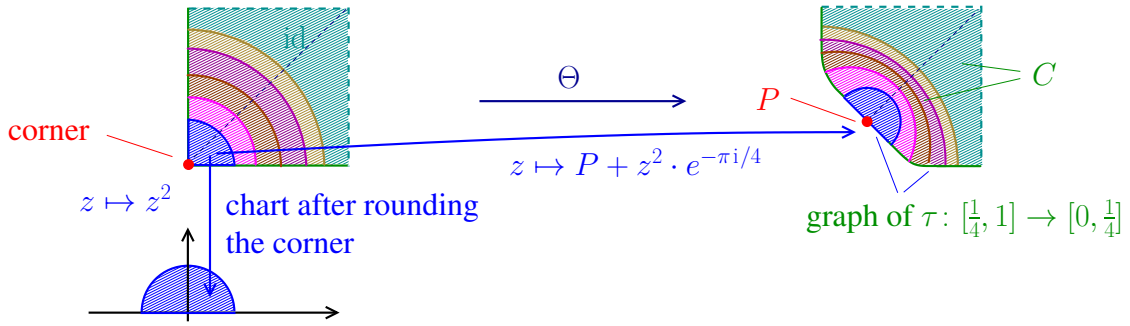
$$C := \{(x,y) \in [0,1) \times [0,1) \,|\, \text{ if } x \geq \tfrac{1}{4}, \text{ then } y \geq \tau(x) \text{ and if } y \geq \tfrac{1}{4}, \text{ then } x \geq \tau(y)\}.$$

It is pretty elementary to verify that C has the desired properties.



[681] This condition says that for the angles $\varphi \in [0, \frac{\pi}{8}]$ and $\varphi \in [\frac{3\pi}{8}, \frac{\pi}{2}]$ the map Ξ is given by "radial stretching". The fact that $\tau(t) = 0$ for $t \in [0, \frac{1}{4}]$ implies that the radial stretching factor is $= 1$ whenever $|z| \leq \frac{1}{4}$. This is consistent with Property (b).

Claim. There exists a bijective map $\Theta\colon [0,1)\times[0,1)\to C$ with the following properties:
(1) Θ sends a circle segment (in the sense of Definition 2.51) around $(0,0)$ to a circle segment.
(2) Θ is the identity on $\{z\in[0,1)\times[0,1)\mid \|z\|\geq\frac{3}{4}\}$.
(3) Θ is smooth.
(4) The differential of Θ is invertible for all $(x,y)\neq(0,0)$.
(5) We have $\Theta(0,0)=(\frac{1}{4},\frac{1}{4})=:P$.
(6) For $z=x+y\mathrm{i}$ with $|z|<\frac{1}{4}$ we have $\Theta(z)=P+z^2\cdot e^{-\pi\mathrm{i}/4}$.

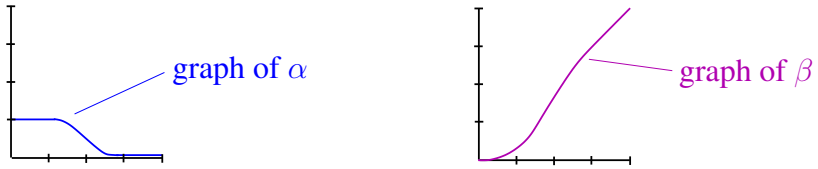


Proof. For $z=x+y\mathrm{i}\in[0,1)\times[0,1)$ with $|z|\leq 1$, i.e. for $z=r\cdot\exp(\mathrm{i}\varphi)$ with $r\in[0,1)$ and $\varphi\in[0,\frac{\pi}{2}]$ the desired function is of the form

$$r\cdot\exp(\mathrm{i}\varphi)\quad\to\quad \underbrace{\alpha(r)\cdot(1+\mathrm{i})}_{\text{center of the circle}}+\underbrace{\beta(r)}_{\text{radius}}\cdot\exp\Big(\mathrm{i}\big(\underbrace{\gamma(r)\cdot(\varphi-\tfrac{\pi}{4})+\tfrac{\pi}{4}}_{\text{angle}}\big)\Big)$$

where α, β and γ are the following types of functions:

- $\alpha\colon[0,1)\to[0,\frac{1}{4}]$ is a smooth function with $\alpha(r)=\frac{1}{4}$ for $r\in[0,\frac{1}{4}]$, $\alpha(r)=0$ for $r\in[\frac{3}{4},1]$ and $\alpha'(r)\in[-\frac{3}{4},0]$ for all t.
- $\beta\colon[0,1)\to[0,1)$ is a smooth function with $\beta(r)=t^2$ for $r\in[0,\frac{1}{4}]$, $\beta(r)=t$ for $r\in[\frac{3}{4},1]$ and $\alpha'(r)+\beta'(r)>0$ for all $r\in(0,1]$.
- $\gamma\colon[0,1)\to[0,2]$ is defined as the ratio of the length of the "new circle segment" to the length $r\cdot\frac{\pi}{2}$ of the "old circle segment". Note that $\gamma(r)=2$ for $r\in[0,\frac{1}{4}]$ and that $\gamma(r)=1$ for $r\in[\frac{3}{4},1]$. One can write down γ explicitly in terms of α, β and τ to show that γ is smooth. We leave it to the reader to work out the details.



We leave it to the technical real analysis II skills of the reader to show, using the Smooth Transition Function Lemma 24.9, that we can find such α and β and to verify that with these choices of α, β and γ all the properties are satisfied. ⊞

We now claim that $\Theta\colon[0,1)\times[0,1)\to C$ has all the desired properties. Indeed it follows from (4), (5) and the Inverse Mapping Theorem 30.1 that the map Θ restricts to a diffeomorphism outside of $\{(0,0)\}$ and P. Furthermore, the explicit description of the smooth chart of $([0,1)\times[0,1))^{\llcorner}$, corresponding to the obvious corner-collar with $\epsilon_1=\epsilon_2=\frac{1}{4}$,

shows that the map Θ with respect to this chart is given by the map $w \mapsto P + w \cdot e^{-\pi \mathrm{i}/4}$. This map is clearly a diffeomorphism. Thus we see that $\Theta\colon ([0,1) \times [0,1))^\backslash \to C$ is a diffeomorphism. ∎

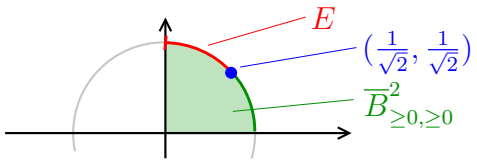
We will also need the following technical lemma:

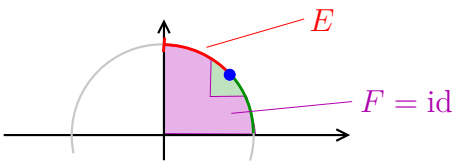
Lemma 60.10. (Eighth-of-Circle Lemma) We consider the "eighth of a circle", namely

$$E := \{(x,y) \in S^1 \,|\, x \in [0, \tfrac{1}{\sqrt{2}}], y > 0\}.$$

Let $f\colon E \to E$ be a diffeomorphism with $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ such that $f(x,y) = (x,y)$ for all $(x,y) \in E$ with $x \leq \frac{1}{2}$. Then there exists a diffeomorphism $F\colon \overline{B}^2_{\geq 0,\geq 0} \to \overline{B}^2_{\geq 0,\geq 0}$ with the following properties:

(1) The restriction of F to E equals f.
(2) F is the identity for all $(x,y) \in \overline{B}^2_{\geq 0,\geq 0}$ with $\|(x,y)\| \leq \frac{1}{2}$ and also whenever $y \leq \frac{1}{2}$ or $x \leq \frac{1}{2}$.





Proof. The proof is essentially a slightly painful exercise in real analysis which we leave to the reader. ∎

Finally we recall the following proposition:

Proposition 30.22. (Smooth Convex-to-Ball Proposition) Let $M \subseteq \mathbb{R}^n$ be a compact codimension-zero smooth manifold. If M is convex and if $0 \in M \setminus \partial M$, then there exists a diffeomorphism

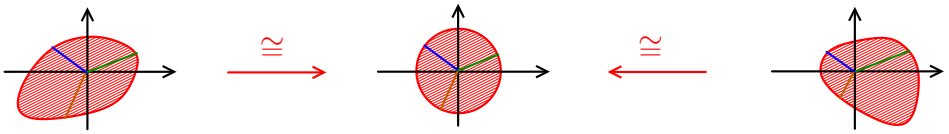
$$\Phi\colon M \to \overline{B}^n$$

with the following properties:

(1) On the boundary the map $\Phi\colon M \to \overline{B}^n$ is given by

$$\begin{array}{rcl} \partial M & \to & S^{n-1} \\ z & \mapsto & \frac{z}{\|z\|}. \end{array}$$

(2) For every $v \in S^{n-1}$ the map Φ sends $(\mathbb{R}_{\geq 0} \cdot v) \cap M$ to $\{t \cdot v \,|\, t \in [0,1]\}$.
(3) For every $v \in \partial M$ with $\|v\| = 1$ the map Φ is the identity on the Euclidean segment from 0 to v.

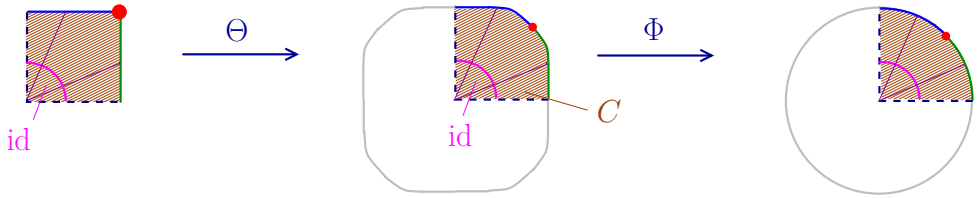


We are now in position to prove the Single Corner-to-Disk Proposition 60.8:

Proof of the Single Corner-to-Disk Proposition 60.8. We consider the smooth manifold $(0,1] \times (0,1]$ with corner $\{(1,1)\}$ and the obvious corner-collar with $\epsilon = \frac{1}{4}$.

(1) It follows from the Single Corner-to-Convex Lemma 60.9 (which we apply after reflecting along $(\frac{1}{2}, \frac{1}{2})$ to flip the corner at $(1,1)$ into a corner at $(0,0)$) that we obtain a pair $(C, \Theta\colon [0,1) \times [0,1) \to C$ where C is a convex smooth submanifold of $(0,1] \times (0,1]$.

(2) It follows easily from gently massaging the Smooth Convex-to-Ball Proposition 30.22 that there exists a diffeomorphism $\Phi\colon C \to \overline{B}^2_{>0,>0}$.



It follows easily from the definitions, and the explicit descriptions of Θ and Φ, that the composition $\Phi \circ \Theta\colon (0,1] \times (0,1] \to \overline{B}^2_{>0,>0}$ has the desired properties (a), (b) and (c). Finally note that using the Eighth-of-Circle Lemma 60.10 one can arrange that (d) also holds. ■

60.5. **Explicit rounding of corners.** In this last section of this chapter we will use the Single Corner-Proposition 60.8 and the Single Corner-to-Convex Lemma 60.9 to prove two more general statements about rounding corners:

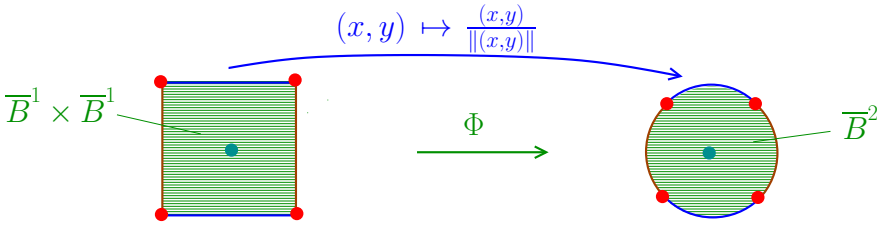
Proposition 60.11. (Product-of-Balls-Proposition) Let $k, l \in \mathbb{N}$. There exists a diffeomorphism

$$(\overline{B}^k \times \overline{B}^l)^{\backsim} \to \overline{B}^{k+l}$$

with the following two properties:

(1) The map sends the corner $S^{k-1} \times S^{l-1}$ to $\{(x,y) \in \mathbb{R}^k \times \mathbb{R}^l \,|\, \|x\| = \|y\| = \frac{1}{\sqrt{2}}\}$.

(2) For $(x,y) \in \overline{B}^k \times S^{l-1}$ the map is given by $(x,y) \mapsto \frac{(x,y)}{\|(x,y)\|}$.



Sketch of proof. First of all, we equip $\overline{B}^k$ and $\overline{B}^l$ with the standard collar from page 1337 and we equip $\overline{B}^k \times \overline{B}^l$ with the corresponding corner-collar, as defined on page 1352. We will use this corner-collar to define $(\overline{B}^k \times \overline{B}^l)^{\backsim}$. Next we introduce the following notation:

(1) Let

$$\Theta\colon (0,1]^2 \to \overline{B}^2_{>0,>0}$$

be the diffeomorphism from the Single Corner-to-Disk Proposition 60.8. It follows easily from the extra properties of this diffeomorphism that this map extends to a diffeomorphism

$$\Theta\colon [0,1]^2 \to \overline{B}^2_{\geq 0,\geq 0}$$

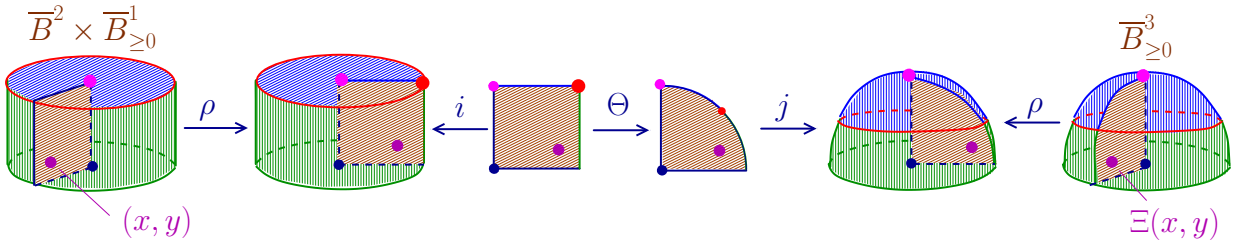
(2) Let $i\colon [0,1] \times [0,1] \to \overline{B}^k \times \overline{B}^l$ be the embedding that is given by $(s,t) \mapsto ((s \cdot e_1), (t \cdot e_1))$.

(3) Let $j\colon \overline{B}^2_{\geq 0,\geq 0} \to \overline{B}^{k+l}_{\geq 0}$ be the embedding that is given by $(s,t) \mapsto ((s \cdot e_1), (t \cdot e_1))$.

We define a map $\Xi\colon \overline{B}^k \times \overline{B}^l \to \overline{B}^{k+l}$ as follows: Let $(x,y) \in \overline{B}^k \times \overline{B}^l$.

(4) By the Orthogonal Action Lemma 2.37 (3) we can find $A \in \mathrm{O}(k)$ and $B \in \mathrm{O}(l)$ such that $A \cdot x \in \mathbb{R}_{\geq 0} \cdot e_1 \in \mathbb{R}^k$ and such that $B \cdot y \in \mathbb{R}_{\geq 0} \cdot e_1 \in \mathbb{R}^l$. Let $\rho = \mathrm{diag}(A, B)$ be the self-diffeomorphism of $\overline{B}^k \times \overline{B}^l$ and also of $\overline{B}^{k+l} \subseteq \mathbb{R}^k \times \mathbb{R}^l$ that is given by multiplication by A on the first entry and by B on the second entry.

We now set $\Xi(x,y) := (\rho^{-1} \circ j \circ \Theta \circ i^{-1} \circ \rho)(x,y)$.

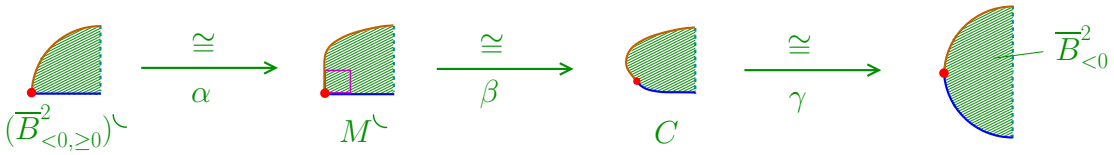


We leave it to the linear algebra skills of the reader to verify that this map does not depend on the choice of A and B. Once this is verified it is not that hard to show that the map is a diffeomorphism $(\overline{B}^k \times \overline{B}^l)^\backprime \to \overline{B}^{k+l}$. ■

Proposition 60.12. (Half Ball-Proposition) Let $n \in \mathbb{N}_{\geq 2}$. We consider the upper half-ball $\overline{B}^n_{\geq 0}$. As on page 1335 we view it as a smooth manifold with corner $\angle \overline{B}^n_{\geq 0} = S^{n-2} \times \{0\}$. There exists a diffeomorphism $(\overline{B}^n_{\geq 0})^\backprime \to \overline{B}^n$.



Sketch of proof. In the following diagram we sketch the case $n = 2$ for a single corner:



(α) First we set $\overline{B}^2_{<0,\geq 0} := \{(x,y) \in \overline{B}^2 \,|\, x < 0 \text{ and } y \geq 0\}$. In Exercise 60.8 we will see that there exists a diffeomorphism $\alpha\colon \overline{B}^2_{<0,\geq 0} \to M$ of smooth manifolds with corner where $M \subseteq [-1,0) \times [0,1)$ is a smooth manifold with corner $\{(-1,0)\} \subseteq M \subseteq \mathbb{R}^2$ and where a corner-collar is given by

$$\begin{array}{rcl} \{(-1,0)\} \times [0,\frac{1}{4}] \times [0,\frac{1}{4}] & \to & M \\ ((-1,0),(s,t)) & \mapsto & (-1+s,t). \end{array}$$

This diffeomorphism also induces a diffeomorphism $\alpha\colon (\overline{B}^2_{<0,\geq 0})^\backprime \to M^\backprime$.

(β) Next we apply the Single Corner-to-Disk Proposition 60.8 to obtain a diffeomorphism $\beta\colon M^\backprime \to C$ where $C \subseteq \mathbb{R}^2$ is a submanifold that is convex.

(γ) Finally note that from the Smooth Convex-to-Ball Proposition 30.22 we obtain easily a diffeomorphism $\gamma\colon C \to \overline{B}^2_{\leq 0}$.

As in the Product-of-Balls-Proposition 60.11 we can reduce the higher-dimensional case to the above case. We leave it to the reader to fill in the details. ■

60.6. Exercises for Chapter 60.

Exercise 60.1. We consider the "open quarter-disk" $B^2_{\geq 0,\geq 0} := \{(x,y) \in B^2 \mid x,y \geq 0\}$ and the "open half-disk" $B^2_{\geq 0} := \{(x,y) \in B^2 \mid y \geq 0\}$. We also consider the map

$$\begin{array}{rccc} S\colon & B^2_{\geq 0} & \to & B^2_{\geq 0,\geq 0} \\ & z & \mapsto & z^2. \end{array}$$

(a) Show that the map S is a homeomorphism and show that it is not a diffeomorphism.
(b) Show that the restriction of S to $B^2_{\geq 0} \setminus \{0\} \to B^2_{\geq 0,\geq 0} \setminus \{0\}$ is a diffeomorphism.

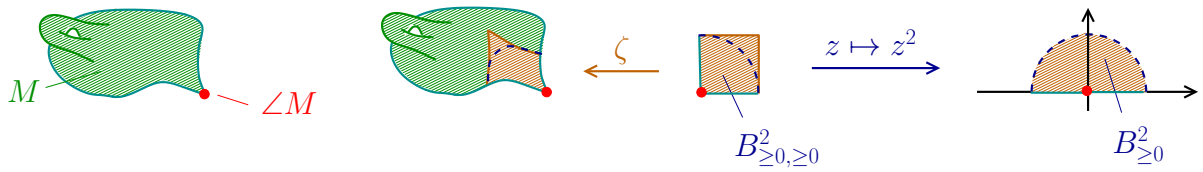
Now let M be a smooth atlas with corner. We say a corner chart is *standard*, if it is of the form $\Phi\colon U \to V \times B^2_{\geq 0,\geq 0}$ where $V \subseteq \angle M$ is an open subset. By restricting charts one can easily show that M admits a smooth atlas where all corner charts are standard.

(c) Given any corner chart of the form $\Phi\colon U \to V \times B^2_{\geq 0,\geq 0}$ we define the following chart

$$\underbrace{\zeta(U_i \times B^2_{\geq 0,\geq 0})}_{\text{open in } M} \xrightarrow{\zeta^{-1}} U_i \times \underbrace{B^2_{\geq 0,\geq 0}}_{\subseteq \mathbb{R}^2 = {}_i\mathbb{C}} \xrightarrow{\operatorname{diag}(\Phi_i,\mathrm{id})} V_i \times \underbrace{B^2_{\geq 0,\geq 0}}_{\subseteq \mathbb{R}^2 = {}_i\mathbb{C}} \xrightarrow{\operatorname{diag}(\mathrm{id},z\mapsto z^2)} \underbrace{V_i \times B^2_{\geq 0}}_{\text{open in } \mathbb{H}^n}.$$

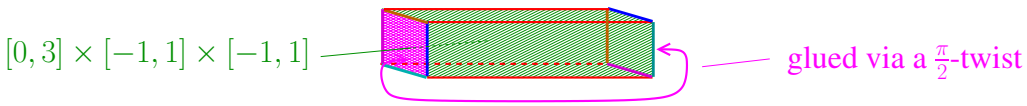
Show that these charts are of type (ii).
(d) Explain why the new atlas of M is in general *not* smooth.

This exercise shows that the naive approach to replacing corner charts by charts of type (ii) will not work.



Exercise 60.2. As in Exercise 59.2 we consider the topological space

$$M := ([0,3] \times [-1,1] \times [-1,1])/(0,(x,y)) \sim (1,(-y,x)).$$



We saw that M has the structure of a smooth manifold with the "obvious" red corner. Is this corner two-colorable?

Exercise 60.3. Let M be a smooth manifold with corner $\angle M$ and let $M^{\llcorner}$ be a smooth manifold that we obtain from M via rounding the corner. Show that if $M \setminus \angle M$ is oriented, then $M^{\llcorner}$ admits a unique orientation such that the inclusion $M \setminus \angle M \to M^{\llcorner}$ is orientation-preserving.
Remark. This statement is related to the Complement-of-Boundary Orientation Lemma 31.36 (1).

Exercise 60.4. Let M be a compact (orientable) n-dimensional smooth manifold. Show that for any $m \geq n$ there exists a compact (orientable) m-dimensional smooth manifold that is homotopy equivalent to M.

Exercise 60.5. Let M be a smooth manifold with two-sided corner. We consider the smooth manifold $M^{\llcorner\lrcorner}$.

(a) Is the identity map $M \to M^{\llcorner\lrcorner}$ smooth?
(b) Is the identity map $M^{\llcorner\lrcorner} \to M$ smooth?

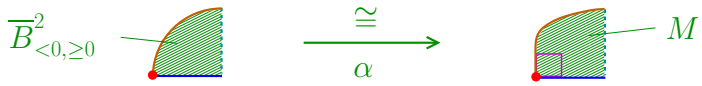
Exercise 60.6. What does the image of $[0,1] \times [0,1] \subseteq \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ under the map $z \mapsto z^2$ look like?

Exercise 60.7. Let M be a smooth manifold. Show that the double $\mathrm{D}M$ is diffeomorphic to $\partial((M \times [-1,1])^{\llcorner\lrcorner})$.

Exercise 60.8. We set $\overline{B}^2_{<0,\geq 0} := \{(x,y) \in \overline{B}^2 \,|\, x < 0 \text{ and } y \geq 0\}$. As on page 1335 we will view $\overline{B}^2_{<0,\geq 0}$ as a smooth manifold with corner $\{(-1,0)\}$. Let $M \subseteq [-1,0) \times [0,1)$ be a convex smooth manifold where the boundary is given by the union of $[-1,0) \times \{0\}$ and of $\{(f(y),y) \,|\, y \in [0,1)\}$ for a smooth function $f\colon [0,1) \to [-1,0)$ and with corner $\{(-1,0)\} \subseteq M \subseteq \mathbb{R}^2$ and where a corner-collar is given by

$$\begin{array}{rcl} \{(-1,0)\} \times [0,\frac{1}{4}] \times [0,\frac{1}{4}] & \to & M \\ ((-1,0),(s,t)) & \mapsto & (-1+s,t). \end{array}$$

Show that there exists a diffeomorphism $\alpha\colon \overline{B}^2_{<0,\geq 0} \to M$ between the two smooth manifolds with corner which has the property that for any point $(x,y) \in M$ with $y \in [0,\frac{1}{4}]$ it is given by $(x,y) \mapsto (x \cdot \sqrt{1-y^2}, y)$.



CHAPTER 61

Introducing corners and gluing via smooth embeddings

In the last chapter we used a lot of brain power to dispose of corners. In this chapter we will do the opposite: We will learn how to (re-) introduce corners.

Introducing corners sounds initially like an undesirable operation. But we will see in this chapter that introducing corners can have a purpose. More precisely, we will use the Introducing Corner Proposition 61.4 together with the Gluing-Smooth Manifolds-Proposition 59.9 to define what it means to glue a (possibly disconnected) smooth manifold along parts of the boundary. This notion will in particular allow us to define the "boundary connected sum" of two smooth manifolds. But the main use of this construction will be in Part XXVIII, when we discuss handle attachments and handle decompositions of smooth manifolds.

This chapter is organized as follows:

- In Section 61.1 we will learn how to introduce corners.
- In Section 61.2 we will learn how to glue two smooth manifolds along codimension-zero submanifolds of the boundary.
- In Section 61.3 we will introduce the boundary connected sum operation.
- In Section 61.4 we will provide exercises for Chapter 61.

61.1. **Introducing corners.** In the Rounding Corner Proposition 60.5 we showed that we can round a corner if it is two-sided. Similarly we will now see that we can introduce corners along a closed codimension-one submanifold W of ∂M, as long as W is two-sided in the sense of Definition 59.14.

To introduce corners we need the following variation on the definitions of a collar, a bicollar and a corner-collar from Definition 35.4, Definition 55.1 and Definition 60.2:

Definition 61.1. Let M be a smooth manifold and let $W \subseteq \partial M$ be a proper codimension-one submanifold of ∂M.[682] A double-collar for W is a smooth embedding

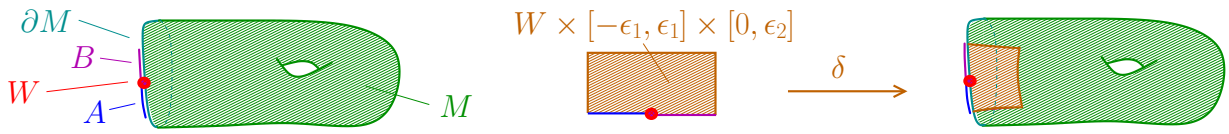
$$\delta\colon W \times [-\epsilon_1, \epsilon_1] \times [0, \epsilon_2] \ \to \ M$$

for some $\epsilon_1 > 0$, $\epsilon_2 > 0$ with the following properties:

(a) The restriction of δ to $W \times \{(0,0)\}$ is the obvious map $(w, 0, 0) \mapsto w$.
(b) The image of δ is a closed subset of M.
(c) $\delta^{-1}(\partial M) = W \times [-\epsilon_1, \epsilon_1] \times \{0\}$.

If we are given a two-coloring (A, B) for $W \subseteq \partial M$, then we say that the double-collar respects the two-coloring if $\delta(W \times [-\epsilon_1, 0) \times \{0\}) \subseteq A$ and $\delta(W \times (0, \epsilon_1] \times \{0\}) \subseteq B$.

[682] In other words, W is an $(n-2)$-dimensional smooth submanifold of ∂M with $\partial W = \varnothing$ and which is a closed subset of ∂M.



The following theorem and proposition are the analogs of the Smooth Collar Neighborhood Theorem 35.6, the Bicollar Neighborhood Theorem 59.16, the Corner-Collar Neighborhood Theorem 60.3 and the corresponding uniqueness statements.

Theorem 61.2. (Double Collar Theorem) Let M be a smooth manifold. Every proper two-colored codimension-one submanifold $W \subseteq \partial M$ of the boundary ∂M admits a double-collar $\delta\colon W \times [-1,1] \times [0,1] \to M$ that respects the two-coloring.

Proposition 61.3. (Double Collar Uniqueness Proposition) Let M be a smooth manifold and let $W \subseteq \partial M$ be a closed two-colored codimension-one submanifold of ∂M. Let $\delta_i\colon W \times [-\epsilon_1^i, \epsilon_1^i] \times [0, \epsilon_2^i] \to M$, $i = 0, 1$ be two double-collars of W that respect the two-coloring of $W \subseteq \partial M$. There exists a diffeotopy $H\colon M \times [0,1] \to M$ rel W from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi := H_1\colon M \to M$ such that we have

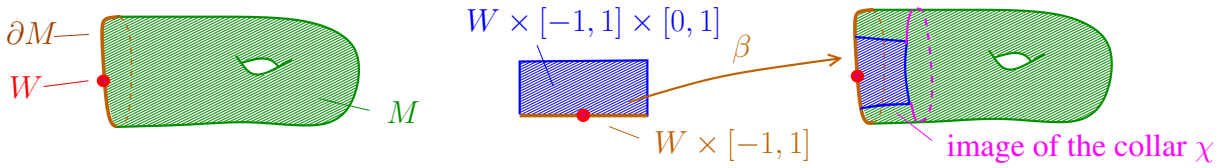
$$(\Phi \circ \delta_0)(w, s_1, s_2) \;=\; \delta_1(w, \tfrac{\epsilon_1^1}{\epsilon_1^0} \cdot s_1, \tfrac{\epsilon_2^1}{\epsilon_2^0} \cdot s_2)$$

for all $(w, s_1, s_2) \in W \times [-\epsilon_1^0, \epsilon_1^0] \times [0, \epsilon_2^0]$.

Sketch of a proof of the Double Collar Theorem 61.2. Let M be a smooth manifold and let $W \subseteq \partial M$ be a closed two-sided codimension-one submanifold of ∂M. We make the following two observations:

(1) It follows immediately from our hypothesis on W and the Bicollar Neighborhood Theorem 59.16 applied to $W \subseteq \partial M$ that there exists a bicollar $\beta\colon W \times [-1,1] \to \partial M$ that respects the two-coloring, i.e. there exists a smooth embedding $\beta\colon W \times [-1,1] \to \partial M$ such that for each $w \in W$ we have $\beta(w, 0) = w$ and such that $\beta(W \times [-1, 0)) \subseteq A$ and such that $\beta(W \times (0, 1]) \subseteq B$.

(2) It follows from the Smooth Collar Neighborhood Theorem 35.6 that there exists a collar $\chi\colon [0,1] \times \partial M \to BM$.



It follows easily from the various definitions that the map

$$\begin{array}{rcl} W \times [-1,1] \times [0,1] & \to & M \\ (w, s, t) & \mapsto & \chi(\beta(w, s), t) \end{array}$$

is a double-collar that respects the two-coloring. ∎

Sketch of a proof of the Double Collar Uniqueness Proposition 61.3. The uniqueness statement follows quite easily from the Bicollar-Uniqueness Proposition 59.17 together with a suitable generalization of the Smooth Collar-Uniqueness Proposition 54.2 for collars of codimension-zero submanifolds of ∂M. We leave it to the reader to turn these ideas into a formal proof. ∎

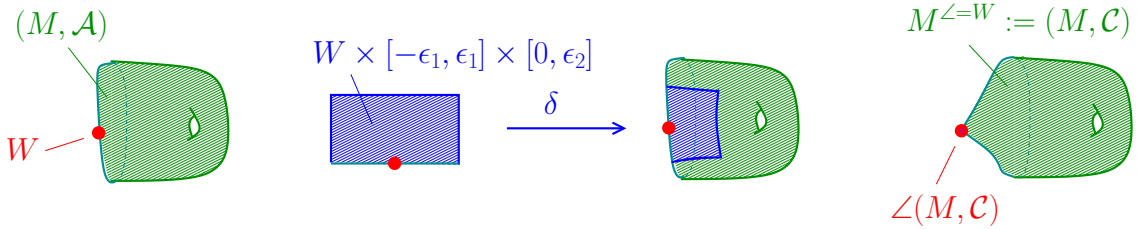
The following proposition is the "reverse" of the Rounding Corner Proposition 60.5.

Proposition 61.4. (Introducing Corner Proposition) Let $(M,\mathcal{A})$ be a smooth manifold (without corner) and let $W \subseteq \partial M$ be a closed two-sided codimension-one submanifold of ∂M. Any choice of a double-collar for W (by the Double Collar Theorem 61.2 we know that such a double-collar always exists) gives rise to a maximal smooth corner atlas $\mathcal{C}$ on M with the following properties:

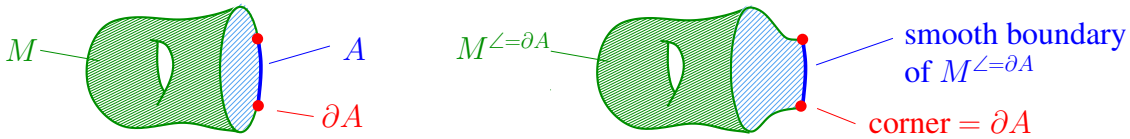
(1) $\angle(M,\mathcal{C}) = W$.
(2) The identity map $(M \setminus W, \mathcal{A}) \to (M \setminus W, \mathcal{C})$ is a diffeomorphism.

Any two choices of double-collars give rise to maximal smooth atlases $\mathcal{C}$ and $\mathcal{C}'$ on M which are equivalent in the sense that for any open neighborhood U of $W \subseteq M$ there exists a diffeomorphism between $(M,\mathcal{C})$ and $(M,\mathcal{C}')$ that is the identity outside of U.

Notation 61.5. Using the notation from the Introducing Corner Proposition 61.4 we denote by $M^{\angle=W}$ the smooth manifold with corner that we just constructed.



Example. Let M be a smooth manifold and let $A \subseteq \partial M$ be a compact *codimension-zero* submanifold. By the discussion on page 1345 we know that $\partial A \subseteq \partial M$ is two-sided. Thus the Introducing Corner Proposition 61.4 allows us to introduce a corner along ∂A.

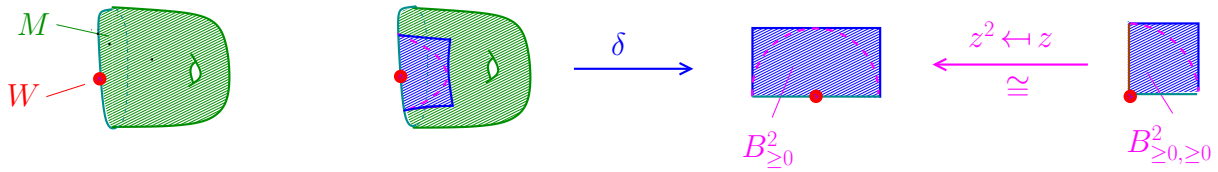


One can easily verify that A is a smooth boundary of the smooth manifold $M^{\angle=\partial A}$ with corner ∂A. □

Proof. Let $(M,\mathcal{A})$ be a smooth manifold (without corner), let $W \subseteq \partial M$ be a closed two-sided codimension-one submanifold of ∂M and let $\delta\colon W \times [-\epsilon_1,\epsilon_1] \times [0,\epsilon_2] \to M$ be a double-collar for W. After rescaling we can assume that $\epsilon_1 = \epsilon_2 = 1$. We now construct an atlas $\mathcal{C}$ for the topological manifold M as follows:

(α) We take all charts from $\mathcal{A}$ and restrict them to $M \setminus W$.
(β) Since W is a closed $(n-2)$-dimensional smooth manifold we can pick an $(n-2)$-dimensional smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ of charts of type (i) for W. We consider again the "open quarter-disk" $B^2_{\geq 0,\geq 0} := \{(x,y) \in B^2 \,|\, x,y \geq 0\}$ and the "open half-disk" $B^2_{\geq 0} := \{(x,y) \in B^2 \,|\, y \geq 0\}$. For each $i \in I$ we consider the map that is given by the composition of the following three maps:

$$\underbrace{\delta(U_i \times B^2_{\geq 0})}_{\text{open in } M} \xrightarrow{\delta^{-1}} U_i \times B^2_{\geq 0} \xrightarrow{\operatorname{diag}(\Phi_i,\mathrm{id})} V_i \times \underbrace{B^2_{\geq 0}}_{\subseteq \mathbb{R}^2 =_i \mathbb{C}} \xrightarrow{\operatorname{diag}(\mathrm{id},(z\mapsto z^2)^{-1})} \underbrace{V_i \times B^2_{\geq 0,\geq 0}}_{\text{open in } \mathbb{R}^n_{\geq 0,\geq 0}}.$$



As in the proof of the Rounding Corner Proposition 60.5 one can now easily verify that the charts from (α) and (β) define a smooth atlas for M and that the corresponding maximal smooth corner atlas $\mathcal{C}$ has the desired properties (1) and (2).

Finally note that it follows easily from the Double Collar Uniqueness Proposition 61.3 that any two choices of double-collars give rise to maximal smooth corner atlases $\mathcal{C}$ and $\mathcal{C}'$ such that $(M,\mathcal{C})$ and $(M,\mathcal{C}')$ are diffeomorphic as smooth manifolds with corner. Again we leave it to the reader to fill in the details. ■

The creative use of the tools of the English language leads us to the following proposition, which shows that the operations in the Rounding Corner Proposition 60.5 and the Introducing Corner Proposition 61.4 are inverses of one another:

Proposition 61.6. (Recornering–Rerounding Proposition)

(1) Let M be a smooth manifold with two-sided compact corner $\angle M$. If we round the corner, and then introduce a corner along $W \coloneqq \angle M$ in $M^{\llcorner}$, then the result is diffeomorphic to the original smooth manifold with corner. In other words, $(M^{\llcorner})^{\angle = \angle M} \cong M$.

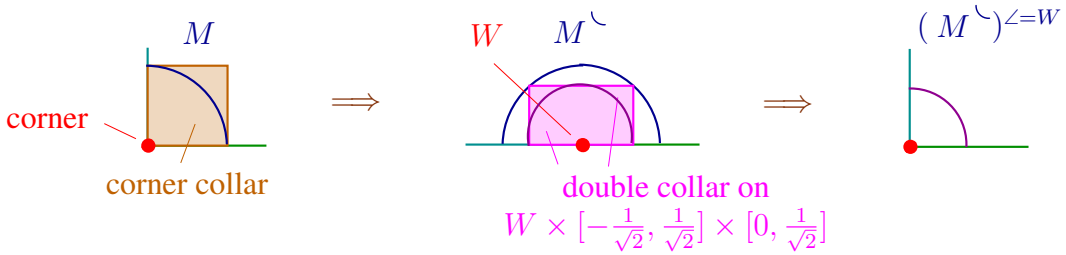
(2) Let M be a smooth manifold and let $W \subseteq \partial M$ be a compact closed two-sided codimension-one submanifold of ∂M. If we introduce a corner along W and then round the resulting corner, then the result is diffeomorphic to the original smooth manifold. In other words, $(M^{\angle = W})^{\llcorner} \cong M$.

In fact, in both cases, if we start out with a corner-collar (respectively double-collar), then we obtain naturally a double-collar (respectively corner-collar), and if we use that collar, we return to the previous structure.

Proof.

(1) Let $(M,\mathcal{C})$ be a smooth manifold with corner $\angle M$ and let $\zeta\colon \angle M \times [0,\epsilon_1] \times [0,\epsilon_2] \to M$ be a corner-collar for $\angle M$. After rescaling we can assume that $\epsilon_1 = \epsilon_2 = 1$. We use this corner-collar to round the corner and we obtain the smooth manifold $M^{\llcorner} = (M,\mathcal{A})$ with submanifold $W \coloneqq \angle M \subseteq \partial M^{\llcorner}$.

One can easily see that the corner-collar naturally gives rise to a double-collar $\delta\colon W \times [-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}] \times [0,\frac{1}{\sqrt{2}}] \to M$ for $W \subseteq M^{\llcorner}$ (which implies in particular that W is two-sided).



If we use this double-collar to introduce a corner[683], then it follows easily from the definitions that the new maximal smooth atlas $\mathcal{C}'$ on M agrees with the original maximal smooth atlas $\mathcal{C}$.

(2) The logic here is basically the same as in (1). We leave it to the reader to fill in the details. ■

Example. We consider the closed ball $\overline{B}^{k+l}$ with boundary S^{k+l-1} and we consider the submanifold $W := \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^l \,|\, \|x\| = \|y\| = \frac{1}{\sqrt{2}}\}$ of S^{k+l-1}.



It follows easily from the Recornering–Rerounding Proposition 61.6 together with the Product-of-Balls-Proposition 60.11 there exists a diffeomorphism

$$\Theta\colon \overline{B}^k \times \overline{B}^l \;\to\; (\overline{B}^{k+l})^{\angle = W}$$

between smooth manifolds with corner $S^{k-1} \times S^{l-1}$ respectively W which has the property that for $(x,y) \in \overline{B}^k \times S^l$ the map Θ is given by

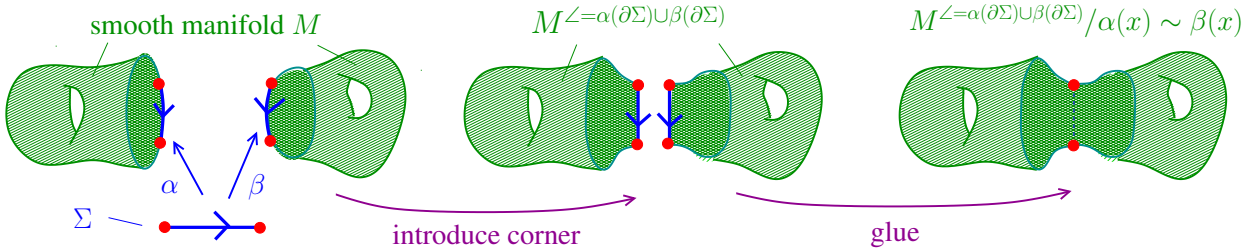
$$\Theta(x,y) \;=\; \frac{(x,y)}{\|(x,y)\|}.$$ □

61.2. **Gluing via smooth Embeddings.** The results of the previous section allow us to introduce the following definition:

Definition 61.7. Let M be a (possibly disconnected) n-dimensional smooth manifold, let Σ be a compact $(n-1)$-dimensional smooth manifold and let $\alpha, \beta\colon \Sigma \to \partial M$ be two smooth embeddings with disjoint images. We refer to[684]

$$M^{\angle = \alpha(\partial\Sigma) \cup \beta(\partial\Sigma)}/\!\sim \qquad \text{where } \alpha(x) \sim \beta(x) \text{ for all } x \in \Sigma$$

as the result of gluing along α and β.



The following proposition is almost a carbon copy of the Gluing-Smooth Manifolds-Proposition 54.5 and the Gluing-Smooth Manifolds-Proposition 59.9:

Proposition 61.8. (Gluing-via Embedding-Smooth Manifolds-Proposition) Let M be an n-dimensional smooth manifold, let Σ be a compact $(n-1)$-dimensional smooth

[683] Note that here we implicitly make use of the Double Collar Uniqueness Proposition 61.3.

[684] First note that it follows from the Smooth Embedding Theorem 30.15 and the discussion on page 1344 that $\alpha(\partial\Sigma) \sqcup \beta(\partial\Sigma) \subseteq \partial M$ is two-sided. Thus the Introducing Corner Proposition 61.4 allows us to introduce corners at $\alpha(\partial\Sigma) \cup \beta(\partial\Sigma)$. As we mentioned on page 1367, $\alpha(\Sigma)$ and $\beta(\Sigma)$ are smooth boundaries of $M^{\angle = \alpha(\partial\Sigma) \cup \beta(\partial\Sigma)}$. Thus the Gluing-Smooth Manifolds-Proposition 59.9 allows us to consider the smooth manifold $M^{\angle = \alpha(\partial\Sigma) \cup \beta(\partial\Sigma)}/\!\sim$.

manifold and let $\alpha, \beta\colon \Sigma \to \partial M$ be two smooth embeddings such that $A := \alpha(\Sigma)$ and $B := \beta(\Sigma)$ are disjoint. As above we consider

$$M/\sim \qquad \text{where } \alpha(x) \sim \beta(x) \text{ for all } x \in \Sigma.$$

(0) (a) If M is compact, then $M/\alpha(x) \sim \beta(x)$ is also compact.
(b) If Σ is connected and non-empty and if each component of M contains $\alpha(\Sigma)$ or $\beta(\Sigma)$, then $M/\alpha(x) \sim \beta(x)$ is connected.

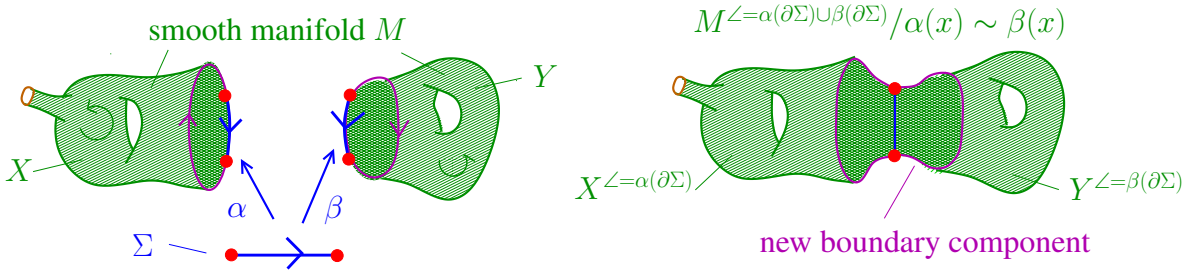
(1) The topological space $M/\alpha(x) \sim \beta(x)$ is an n-dimensional topological manifold. Furthermore it also admits a smooth structure:
(a) Any choice of a double collar for $\alpha(\partial\Sigma) \sqcup \beta(\partial\Sigma) \subseteq \partial M$ and any choice of a collar $\chi\colon [0,\epsilon] \times (A \sqcup B) \to M$ for the smooth boundary $A \sqcup B$ equips the topological manifold $M^{\angle=\alpha(\partial\Sigma)\cup\beta(\partial\Sigma)}/\alpha(x) \sim \beta(x)$ canonically with the structure of an n-dimensional smooth manifold.
(b) Given any two choices as in (a) and given any neighborhood U of $\alpha(\Sigma) \cup \beta(\Sigma)$ there exists a diffeomorphism between the two resulting smooth manifolds that is the identity outside of U.
(c) The diffeomorphism type of $M/\alpha(x) \sim \beta(x)$ does not depend on the choice of the collar.

(2) The boundary of $M/\alpha(x) \sim \beta(x)$ is given by $(\partial M \setminus (\mathring{A} \cup \mathring{B}))/\alpha(x) \sim \beta(x)$ for $x \in \partial\Sigma$.

(3) (a) The image of the smooth manifold Σ in $M/\alpha(x) \sim \beta(x)$ under the natural map $\Sigma \to M/\alpha(x) \sim \beta(x)$ is a proper codimension-one submanifold of $M/\alpha(x) \sim \beta(x)$.
(b) If M consists of two components X and Y with $A \subseteq X$ and $B \subseteq Y$, then the following statements hold:
(i) The natural maps $X^{\angle=\partial A} \xrightarrow{i} M/\alpha(x) \sim \beta(x)$ and $Y^{\angle=\partial B} \xrightarrow{j} M/\alpha(x) \sim \beta(x)$ are smooth embeddings.
(ii) The images $i(X)$ and $j(Y)$ are closed subsets of $M/\alpha(x) \sim \beta(x)$ and they are codimension-zero submanifolds (with corner) of $M/\alpha(x) \sim \beta(x)$.

(4) Assume that M is oriented. We equip A and B with the boundary orientations.
(a) If $\beta \circ \alpha^{-1}\colon A \to B$ is orientation-reversing, then $M/\alpha(x) \sim \beta(x)$ admits a unique orientation such that the inclusion $M \setminus (A \cup B) \to M/\alpha(x) \sim \beta(x)$ is orientation-preserving.
(b) If M is connected and if $\beta \circ \alpha^{-1}\colon \alpha(\Sigma) \to \beta(\Sigma)$ is not orientation-reversing, then $M/\alpha(x) \sim \beta(x)$ is non-orientable.

(5) The diffeomorphism type of $M/\alpha(x) \sim \beta(x)$ only depends on the smooth isotopy type of the smooth embeddings $\alpha, \beta\colon \Sigma \to M$ with disjoint images. Furthermore, given different gluings and given any neighborhood U of ∂M, one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.



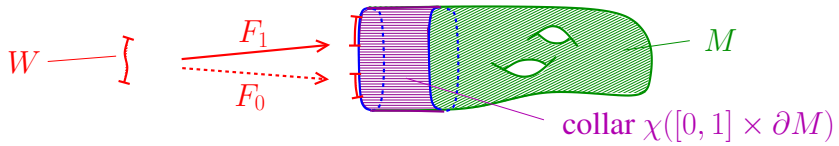
Proof. All the statements, except for (5), follow quite easily from the definitions and the Gluing-Smooth Manifolds-Proposition 59.9. We leave it to the reader to fill in the details.

It remains to prove Statement (5). Since this statement will be essential later on, we will provide some details. The key to proving this statement is the following general claim.[685]

Claim. Let W be a compact smooth manifold and let $f_0, f_1 \colon W \to \partial M$ be two embeddings. If the maps f_0, f_1 are smoothly isotopic, then there exists a diffeomorphism $\Theta \colon M \to M$ with $\Theta \circ f_0 = f_1 \colon W \to M$.

Proof. We make a few preparations:

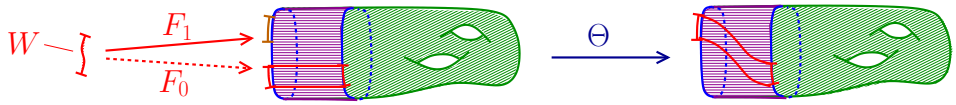
- By hypothesis there exists a smooth isotopy $F \colon W \times [0,1] \to \partial M$ with $F_0 = f_0$ and $F_1 = f_1$.
- Next note that it follows from the compactness of W, the fact that $\partial(\partial M) = \varnothing$, and the Isotopy Extension Theorem 44.2 that there exists a diffeotopy $G \colon \partial M \times [0,1] \to \partial M$ with $G_0 = \mathrm{id}$ and such that for each $(x,t) \in \partial M \times [0,1]$ we have $G_t(F_0(x)) = F_t(x)$.
- It follows from the Smooth Collar Neighborhood Theorem 35.6 that there exists a collar $\chi \colon [0,1] \times \partial M \to M$.



- Note that by the Smooth Transition Function Lemma 24.9 there exists a smooth function $\tau \colon [0,1] \to [0,1]$ with $\tau(0) = 1$ and $\tau(t) = 0$ for $t \in [\frac{3}{4}, 1]$.

We now consider the following map:

$$\begin{array}{rcl} \Theta \colon M & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \chi([0,1] \times \partial M), \\ \chi(t, G(Q, \tau(t))), & \text{if } P = \chi(t,Q) \text{ for some } (t,Q) \in [0,1] \times \partial M. \end{cases} \end{array}$$



One can easily verify, say using the Smooth Pasting Proposition 25.19, that Θ is a diffeomorphism with $\Theta \circ f_0 = f_1$. ⊞

Now let $\alpha_i \sqcup \beta_i \colon \Sigma \sqcup \Sigma \to \partial M$, $i = 0, 1$ be two smooth embeddings which are smoothly isotopic. It follows from the claim that there exists a diffeomorphism $\Theta \colon M \to M$ with $\Theta \circ (\alpha_0 \sqcup \beta_0) = \alpha_1 \sqcup \beta_1$. One can now easily verify that the map

$$\begin{array}{rcl} M^{\angle = \alpha_0(\partial\Sigma) \cup \beta_0(\partial\Sigma)} / \alpha_0(x) \sim \beta_0(x) & \to & M^{\angle = \alpha_1(\partial\Sigma) \cup \beta_1(\partial\Sigma)} / \alpha_1(x) \sim \beta_1(x) \\ [P] & \mapsto & [\Theta(P)] \end{array}$$

is a diffeomorphism.[686] ■

[685] Incidentally this claim also provides a solution to Exercise 44.4.

[686] More precisely, if we are given a double-collar and a collar on the left-hand side, then Θ induces a double-collar and a collar for the setting on the right-hand side. If we use consider the corresponding smooth structure, then Θ is almost by definition a diffeomorphism.

61.3. **The boundary connected sum operation.** In this section we will discuss one particularly interesting special case of the construction from the last section. Namely we will introduce the boundary connected sum of two smooth manifolds with non-empty boundary, which is a variation on the definition of the connected sum of smooth manifolds. For the reader's convenience we recall the Definition 54.12 of the connected sum of smooth manifolds:

Definition 61.9. Let $n \in \mathbb{N}_0$ and let M and M' be oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and an orientation-reversing smooth embedding $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$. We define the connected sum of M and M' as

$$M \# M' \coloneqq (M \setminus \varphi(B^n)) \sqcup (M' \setminus \varphi'(B^n))/\sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$

Examples.

(1) In the Surface–Connected Sum Lemma 54.15 we showed that the connected sum of the surface of genus g and the surface of genus k is the surface of genus $g+k$.

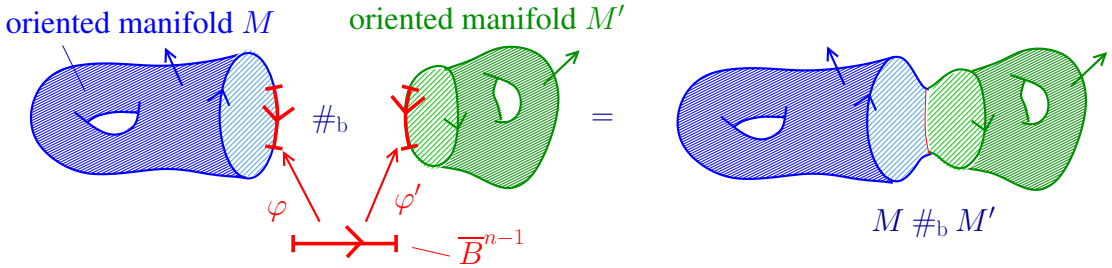


(2) In the Connected Sum-with-Sphere Lemma 54.16 we showed that for any $n \in \mathbb{N}_0$ and any oriented connected n-dimensional smooth manifold M we have a diffeomorphism $M \# S^n \cong M$. □

Definition 61.10. Let $n \in \mathbb{N}$, let M and M' be two n-dimensional smooth manifolds and let $\varphi\colon \overline{B}^{n-1} \to \partial M$ and $\varphi'\colon \overline{B}^{n-1} \to \partial M'$ be two smooth embeddings. If ∂M and $\partial M'$ are both orientable, then we assume that they are equipped with an orientation and that φ is orientation-preserving and φ' is orientation-reversing.[687] We consider the smooth manifold $M \sqcup M'$ and we glue along φ and φ'. In other words, we consider[688]

$$M \#_{\mathrm{b}} M' \coloneqq (M^{\angle = \varphi(S^{n-2})} \sqcup (M')^{\angle = \varphi'(S^{n-2})})/\sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in \overline{B}^{n-1}.$$

We refer to $M \#_{\mathrm{b}} M'$ as a boundary connected sum of M and M'.



[687] As always we use the Boundary Orientation Lemma 31.38 to equip the boundaries of oriented smooth manifolds with induced orientations.

[688] Sometimes in the literature one also writes $M \sharp M'$ or $M \#_\partial M'$ instead of $M \#_{\mathrm{b}} M'$.

In the following proposition, which is the analog of the Smooth Connected Sum-Proposition 54.13, we collect some basic facts on the boundary connected sum operation:

Proposition 61.11. (Boundary Connected Sum Proposition) Let $n \in \mathbb{N}$, let M and M' be two n-dimensional smooth manifolds and let $\varphi\colon \overline{B}^{n-1} \to \partial M$ and $\varphi'\colon \overline{B}^{n-1} \to \partial M'$ be two smooth embeddings. If ∂M and $\partial M'$ are both orientable, then we assume that they are equipped with an orientation and that φ is orientation-preserving and φ' is orientation-reversing.

(0) (a) If M and M' are compact, then $M \#_{\mathrm{b}} M'$ is also compact.
(b) If M and M' are connected and if $n \geq 1$, then $M \#_{\mathrm{b}} M'$ is also connected.

(1) The resulting boundary connected sum $M \#_{\mathrm{b}} M'$ is an n-dimensional topological manifold that admits the structure of a smooth manifold and the corresponding diffeomorphism type is natural.[689]

(2) We write $\partial M = A \sqcup B$ and $\partial M' = A' \sqcup B'$ where $\varphi(\overline{B}^{n-1}) \subseteq A$ and $\varphi'(\overline{B}^{n-1}) \subseteq A'$. Then

$$\partial(M \#_{\mathrm{b}} M') \;=\; \underbrace{A \# A'}_{\substack{\text{connected sum} \\ \text{of } A \text{ and } A'}} \sqcup B \sqcup B'.$$

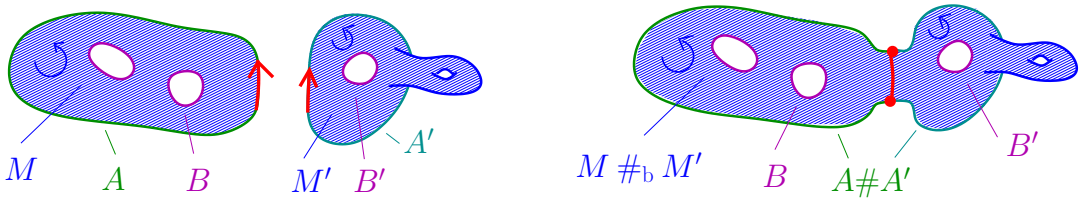
In particular, if the boundaries of M and M' are connected, then the boundary of $M \#_{\mathrm{b}} M'$ is the connected sum of the boundaries of M and M'.

(3) (a) The image of $\varphi(\overline{B}^{n-1})$ in $M \#_{\mathrm{b}} M'$ is a proper submanifold of $M \#_{\mathrm{b}} M'$.
(b) The images of M and M' in $M \#_{\mathrm{b}} M'$ are closed subsets and they are codimension-zero submanifolds with corner that is given by $\varphi(S^{n-2}) = \varphi'(S^{n-2})$.

(4) If M and M' are oriented and if we equipped ∂M and $\partial M'$ with the corresponding boundary orientations, then the boundary connected sum $M \#_{\mathrm{b}} M'$ admits a unique orientation such that the natural maps $M \to M \#_{\mathrm{b}} M'$ and $M' \to M \#_{\mathrm{b}} M'$ are orientation-preserving.

(5) Given boundary components A and A' the diffeomorphism type of the boundary connected sum $M \#_{\mathrm{b}} M'$ does not depend on the choice of the two smooth embeddings $\varphi\colon \overline{B}^{n-1} \to A$ and $\varphi'\colon \overline{B}^{n-1} \to A'$. In particular, if the boundary of M and M' is connected, then the diffeomorphism type of the boundary connected sum $M \#_{\mathrm{b}} M'$ is well-defined.

Convention 61.12. We use the Boundary Connected Sum Proposition 61.11 (4) to equip the boundary connected sum with an orientation.



[689] Here by "natural" we mean that it is natural with respect to diffeomorphisms. In plain English this means, that the diffeomorphism type does not depend on any of the choices.

Sketch of proof. Most of the statements follow easily from applying the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 to our current setting. We make just a few extra observations:

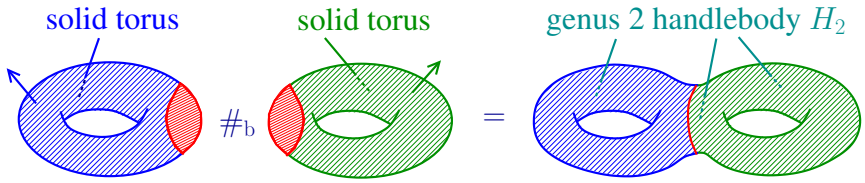
(2) We have the following diffeomorphisms:

$$\partial(M \#_{\mathrm{b}} M') \underset{\substack{\uparrow \\ \text{by the Gluing-via Embedding-Smooth} \\ \text{Manifolds-Proposition 61.8 (5)}}}{=} ((B \sqcup (A \setminus \varphi(B^{n-1})) \sqcup (B' \sqcup (A' \setminus \varphi'(B^{n-1})))/\sim \underset{\substack{\uparrow \\ \text{by the definition of the} \\ \text{connected sum given above}}}{=} B \sqcup B' \sqcup A \# A'.$$

(5) Recall that the Smooth Ball Embedding Theorem 45.10 says the following: Let W be a connected m-dimensional smooth manifold and let $\varphi, \psi \colon \overline{B}^m \to W$ be smooth embeddings. If W is orientable, then we assume (for a choice of orientation), that both φ, ψ are orientation-preserving. Then φ, ψ are smoothly isotopic.

The desired statement now follows easily from this fact, applied to the smooth embeddings of $\overline{B}^{n-1}$ into the oriented connected $(n-1)$-dimensional smooth manifolds A and A', together with the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (5). We leave it to the reader to fill in the details. ■

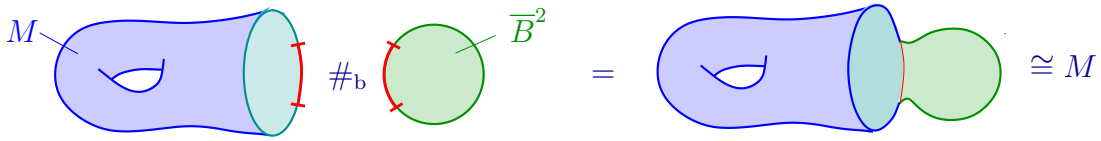
Example. Any 3-dimensional smooth manifold that is diffeomorphic to the boundary connected sum of g copies of $\overline{B}^2 \times S^1$ is sometimes called a *genus k-handlebody*. We denote it by H_g.[690]



It follows from the Boundary Connected Sum Proposition 61.11 (2) that the boundary of H_g is the connected sum of g copies of the torus $S^1 \times S^1$. It follows from this observation and the Surface–Connected Sum Lemma 54.15 that the boundary of the genus g-handlebody H_g is diffeomorphic to the surface of genus g. □

In the Connected Sum-with-Sphere Lemma 54.16 we showed that the connected sum of an oriented connected n-dimensional smooth manifold M with S^n is diffeomorphic to M. The following proposition is a similar statement in the context of boundary connected sums:

Proposition 61.13. (Boundary Connected Sum-with-Ball Proposition) Any boundary connected sum of an n-dimensional smooth manifold M with $\overline{B}^n$ is diffeomorphic to M. More succinctly: $M \#_{\mathrm{b}} \overline{B}^n \cong M$.



In the proof of the Boundary Connected Sum-with-Ball Proposition 61.13 we will need the following lemma:

[690] In the Boundary Connected Sum-with-Solid Torus Proposition 247.16 we will see where the name "handlebody" comes from.

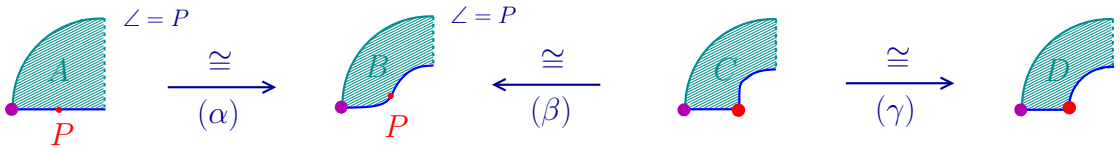
Lemma 61.14. (Corner-at-Bottom Lemma) Given any $n \in \mathbb{N}_0$ there exists a diffeomorphism
$$\Xi\colon (\overline{B}^n_{2,\geq 0})^{\angle = S^{n-2}\times\{0\}} \to S^{n-1}_{\geq 0} \times [1,2]$$
of smooth manifolds with corner such that for every $P \in S^{n-1}_{2,\geq 0}$ and every $t \in [\frac{3}{2}, 2]$ we have $\Xi(P \cdot t) = (x,t)$.



Sketch of proof. As in the proof of the Product-of-Balls-Proposition 60.11 it suffices to deal with the case $n = 2$ and a single corner. In other words, it suffices to show the following claim:

Claim. There exists a diffeomorphism
$$\Xi\colon \underbrace{\{(x,y) \in \overline{B}^2_2 \,|\, x < 0, y \geq 0\}}_{=:A}{}^{\angle\{(-1,0)\}} \to \underbrace{\{(x,y) \in S^1 \,|\, x < 0, y \geq 0\} \times [1,2]}_{=:D}$$
such that for every $P = (x,y) \in S^1$ with $x < 0, y \geq 0$ and every $t \in [\frac{3}{2}, 2]$ we have $\Xi(P \cdot t) = (P,t)$.

Proof. To prove this claim we consider the sequence of diffeomorphisms that is shown in the figure below.

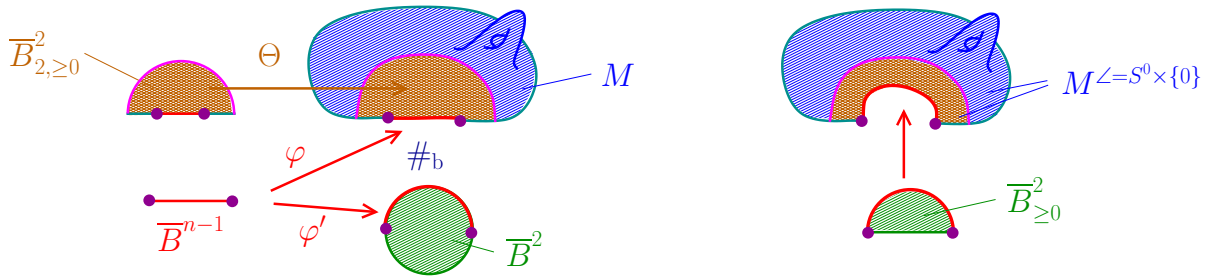


(α) This diffeomorphism follows from a variation on the Smooth Convex-to-Ball Proposition 30.22.
(β) This diffeomorphism is provided by the Single Corner-to-Convex Lemma 60.9.
(α) This diffeomorphism is given by a variation on the Exercise 60.8, which we also made use of in the proof of the Half Ball-Proposition 60.12.

We leave it to the reader to fill in the details. ∎

Sketch of proof of the Boundary Connected Sum-with-Ball Proposition 61.13. Let $n \in \mathbb{N}$ and let M be a connected n-dimensional smooth manifold. By the Boundary Connected Sum Proposition 61.11 (5) we have freedom how we choose our smooth embeddings of the ball into the boundaries. Thus we now carefully choose our smooth embeddings:

- Let $\overline{B}^n_{2,\geq 0} := \{(x_1, \dots, x_n) \in \overline{B}^n_2 \,|\, x_n \geq 0\}$ be the upper half-ball of radius two. Using a chart we can easily show that there exists an (orientation-preserving) smooth embedding $\Theta\colon \overline{B}^n_{2,\geq 0} \to M$ such that $\Theta^{-1}(\partial M) = \overline{B}^{n-1}_2 \times \{0\}$. We denote by $\varphi\colon \overline{B}^{n-1} \to \partial M$ the smooth embedding that is given by $\varphi(x) = \Theta(x,0)$.
- Let $\varphi'\colon \overline{B}^{n-1} \to S^{n-1} = \partial \overline{B}^n$ be the obvious smooth embedding into the upper hemisphere.



To simplify the notation we now use Θ to view $\overline{B}^n_{2,\geq 0}$ as a submanifold with corner of M. We set $W := M \setminus B^n_{2,\geq 0}$. We now see that

$$M\#_{\mathrm{b}}\overline{B}^n \overset{\text{by Definition 61.10 of } \#_{\mathrm{b}}}{\underset{\text{by the Corner-at-Bottom Lemma 61.14}}{\downarrow \; = \; \uparrow}} (M^{\angle=\varphi(S^{n-1})} \sqcup (\overline{B}^n)^{\angle=S^{n-1}\times\{0\}})/\sim \; = \; ((W \cup (S^{n-1}_{\geq 0}\times[1,2])) \sqcup \overline{B}^n_{\geq 0})/\sim \overset{\text{by the Half Ball Proposition 60.12}}{\underset{\text{since } (S^{n-1}_{\geq 0}\times[1,2])\cup \overline{B}^n_{\geq 0} \cong \overline{B}^n_{2,\geq 0}}{\downarrow \; = \; \uparrow}} (W \cup \overline{B}^n_{2,\geq 0})^{\angle=S^{n-1}\times\{0\}} \sqcup \overline{B}^n_{\geq 0})/\sim \; = \; (W \sqcup \overline{B}^n_{2,\geq 0})/\sim \; \cong \; M.$$

■

61.4. Exercises for Chapter 61.

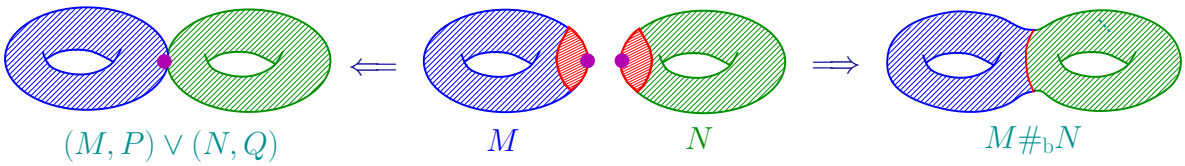
Exercise 61.1. We consider $M := S^1 \times S^1 \times [0,1]$. Let $\alpha\colon \overline{B}^2 \to S^1 \times S^1 \times \{0\} \subseteq \partial M$ and $\beta\colon \overline{B}^2 \to S^1 \times S^1 \times \{1\} \subseteq \partial M$ be orientation-preserving smooth embeddings. (Here we equip $S^1 \times S^1 \times \{i\}$ with the product orientation.) We consider the smooth manifold $W := M/\alpha(x) \sim \beta(x)$.

(a) What is the diffeomorphism type of ∂W?
(b) What is $\pi_1(W)$?
(c) What are the homology groups of W?
(d) Is W diffeomorphic to a genus g-handlebody?

Exercise 61.2. Let $k \in \mathbb{N}$ be *even*. Show that the non-orientable surface N_k of genus k is diffeomorphic to the boundary of a compact (necessarily non-orientable) 3-dimensional smooth manifold.
Hint. First deal with the case $k = 2$, i.e. with the Klein bottle.
Remark. It follows from the Boundary-χ-Corollary 210.10 that for $k \in \mathbb{N}$ *odd*, the non-orientable surface N_k is *not* diffeomorphic to the boundary of a compact 3-dimensional smooth manifold.

Exercise 61.3. Let M and N be oriented n-dimensional smooth manifolds with non-empty connected boundary. Show that the boundary connected sum $M \#_{\mathrm{b}} N$ is homotopy equivalent to the wedge $(M,P) \vee (N,Q)$ where $P \in \partial M$ and $Q \in \partial N$.
Remark. The statement is "clear" from the pictures. The challenge is to turn this into a formal proof.
Hint. Show that one can extend the smooth embeddings of $\overline{B}^n$ in the definition of the connected sum to smooth embeddings of $\overline{B}^n_2$. Then use a smooth collar to show that there

exists an embedding $(M,P) \vee (N,Q) \to M \#_{\mathrm{b}} N$ such that the image of the embedding is a deformation retraction of $M \#_{\mathrm{b}} N$.



Exercise 61.4. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we denote by $\Sigma_{g,n}$ the result of removing the interiors of n disjoint smoothly embedded disks of the surface Σ_g of genus g. Let $g, g' \in \mathbb{N}_0$ and $n, n' \in \mathbb{N}$.

(a) Show that the diffeomorphism type of the boundary connected sum $\Sigma_{g,n} \#_{\mathrm{b}} \Sigma_{g',n'}$ is independent of all choices.
(b) What is the genus and the number of boundary components of $\Sigma_{g,n} \#_{\mathrm{b}} \Sigma_{g',n'}$?

Exercise 61.5. Given $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we denote by $N_{k,n}$ the result of removing the interiors of n disjoint smoothly embedded disks of the non-orientable surface N_k of genus k. Let $k, k' \in \mathbb{N}_0$ and $n, n' \in \mathbb{N}$.

(a) Show that the diffeomorphism type of the boundary connected sum $N_{k,n} \#_{\mathrm{b}} N_{k',n'}$ is independent of all choices.
(b) Determine l, m such that $N_{k,n} \#_{\mathrm{b}} N_{k',n'}$ is diffeomorphic to $N_{l,m}$.

Exercise 61.6. Let $M = \{P\}$ and $N = \{Q\}$ be two 0-dimensional manifolds which consists of a single point each.

(a) Does it make sense to consider the connected sum $M \# N$? If yes, what is it?
(b) Does it make sense to consider the boundary connected sum $M \#_{\mathrm{b}} N$? If yes, what is it?

Exercise 61.7. Let M be a smooth manifold and let $W \subseteq \partial M$ be a two-sided closed codimension-one submanifold. We consider the smooth manifold $M^{\angle = W}$.

(a) Is the identity map $M \to M^{\angle = W}$ smooth?
(b) Is the identity map $M^{\angle = W} \to M$ smooth?

Exercise 61.8. Which diffeomorphism types of compact 2-dimensional smooth manifolds can you obtain if you start out with finitely many disks and you iteratively glue along smooth embeddings of compact 1-dimensional smooth manifolds into the boundary?

Exercise 61.9. Let M and M' be two compact oriented connected manifolds with non-empty boundary. Show that in general the diffeomorphism type of the boundary connected sum depends on the choice of the boundary component.

Exercise 61.10. Let $n \in \mathbb{N}$ and let $\alpha, \beta \colon \overline{B}^{n-1} \to S^{n-1}$ be two smooth embeddings. What are the possible diffeomorphism types for gluing $\overline{B}^n$ along α and β?

Part VIII

Conway, Jones and HOMFLY-PT polynomial

A *knot* is a submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1 and a *link* is a submanifold of $\mathbb{R}^3$ such that each component is diffeomorphic to S^1. Knots and links are a fun and much-loved topic in topology. In this part of the lecture notes we will introduce these concepts in detail and we will introduce various invariants of knots and links. More precisely, we will use the Burau representation to introduce the Alexander polynomial and the Conway polynomial of an oriented link. Furthermore, we will use diagrams to define the Kauffman bracket of a diagram, which we then use to introduce the Jones polynomial of an oriented link. Finally we will introduce the HOMFLY-PT polynomial which gives a common generalization of the Conway polynomial and the Jones polynomial.

This part of the lecture notes contains the following chapters:

- In Chapter 62 we will give a short introduction to knots and links in $\mathbb{R}^3$.
- In Chapter 63 we will introduce several constructions of knots and links.
- In Chapter 64 we will introduce the braid group and the Burau representation.
- In Chapter 66 we will study the relationship between braids and links. More precisely, we will introduce the closure of a braid and we will state Markov's Theorem which tells us when two braids lead to smoothly isotopic closures.
- In Chapter 67 we will use the fact that links are closures of braids, together with the reduced Burau representation, to introduce an invariant of oriented links, namely the Alexander polynomial.
- In Chapter 68 we will use the Burau representation to introduce the Conway polynomial of any oriented link.
- In Chapter 69 we will study properties of the Conway polynomial.
- In Chapter 70 we will introduce the Jones polynomial of any oriented link.
- In **??** we will introduce the HOMFLY-PT polynomial.

CHAPTER 62

Introduction to knots and links in $\mathbb{R}^3$

In this chapter we will define knots and links and we will discuss several examples, basic results and notions.

This chapter is organized as follows:

- In Section 62.1 we will introduce knots and link in $\mathbb{R}^3$s.
- In Section 62.2 we will introduce the notion of a smooth isotopy between two links.
- In Section 62.3 we will discuss orientations of knots and links.
- In Section 62.4 we will introduce the mirror of a link.
- In Section 62.6 we will provide exercises for Chapter 62.

In the next Chapter 63 we will introduce several constructions of knots and links.

62.1. Knots and links in $\mathbb{R}^3$.

We out with the definition of a knot:

Definition 62.1. A knot in $\mathbb{R}^3$ is a submanifold of that is diffeomorphic to S^1.[691]

We start out with the most boring but also the most important knot.

Definition 62.2. We define

$$\text{trivial knot} \;:=\; \{(x,y,0)\in\mathbb{R}^3 \,|\, x^2+y^2=1\}.$$

In the literature the trivial knot is often also called the unknot.

Example. In the figure below we show the trivial knot together with three other examples of knots, namely the trefoil, the figure-8 knot[692] and stevedore knot.



trivial knot (left-handed) trefoil figure-8 knot stevedore knot

In principle it is possible to give a precise description of all these four knots. For example we just gave a definition of the trivial knot as $\{(x,y,0)\in\mathbb{R}^3 \,|\, x^2+y^2=1\}$. Precise definitions, and 3-dimensional models of the trefoil and the figure-8 knot can also be found here:

trefoil:	`https://www.desmos.com/3d/3c1777edd9`	(thanks to Johannes Zander)
figure-8 knot:	`https://www.desmos.com/3d/aab54cb3df`	(thanks to Filip Misev)

Later, in Definition 62.16, we will give an alternative precise definition of the trefoil. It is also clear that one can give a precise description of the stevedore knot, but this description would be painful to write down and it would not add to our understanding. We therefore

[691] Sometimes, to distinguish this knot from the higher dimensional siblings that we will introduce on Definition 111.27 and Definition 258.1, we also refer to such a knot as a *classical knot*.

[692] The figure-8 knot is also known as Listing's knot [**Lis1847**].

stick with the picture, with the understanding, that if somebody was challenging us, we could write down a precise description in coordinates. But it is considered very impolite to challenge a topologist to give a rigorous description. □

The notion of a knot can be generalized to the notion of a link:

Definition 62.3. Let $m \in \mathbb{N}_0$.

(1) An m-component link in $\mathbb{R}^3$ is a submanifold of $\mathbb{R}^3$ that is diffeomorphic to the disjoint union of m copies of S^1 together with an ordering of the components of L, i.e. together with a bijection from $\{1, \dots, m\}$ to the set of components of L.

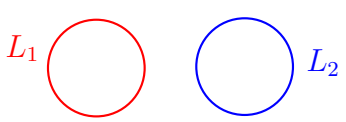
(2) Given an m-component link $L \subseteq \mathbb{R}^3$ and given $i \in \{1, \dots, m\}$ we denote by L_i the i-th component of L.

The next definition gives us an essential class of links:

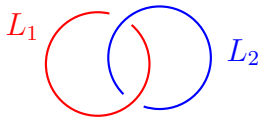
Definition 62.4. Let $m \in \mathbb{N}_0$. We define

$$\text{trivial } m\text{-component link} := \bigsqcup_{i=1}^{m} \{(x, y + 3(i-1), 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^3$$

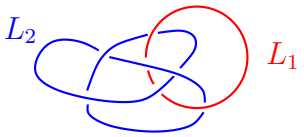
with the obvious numbering of the components of L.



trivial 2-component link



Hopf link



link where one component is a trefoil and one is the trivial knot

62.2. **Links up to smooth isotopy.** The main concept in knot theory is that we do not want to study knots and links as totally rigid submanifolds of $\mathbb{R}^3$, but we would like to say that two links are "the same" if one can be "deformed" into the other. Fortunately we already have the language to turn this idea into formal mathematical language, namely the notion of a smooth isotopy between (oriented) submanifolds that we introduced in Definition 41.10 is exactly what we need. For the reader's convenience we repeat the definition of "smoothly isotopic" in our context.

Definition 62.5. We say that two m-component links $L = L_1 \sqcup \dots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \dots \sqcup \widetilde{L}_m$ in $\mathbb{R}^3$ are smoothly isotopic if there exists a smooth isotopy from L to $\widetilde{L}$, i.e. if there exists a smooth map

$$\begin{array}{rccc} F\colon & L \times [0,1] & \to & \mathbb{R}^3 \\ & (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:[693]

(1) For each $t \in [0,1]$ the map $F_t\colon L \to \mathbb{R}^3$ is a smooth embedding.

(2) We have $F_0 = \mathrm{id}$ and for each $i \in \{1, \dots, m\}$ we have $F_1(L_i) = \widetilde{L}_i$.

Example. One can easily convince oneself that the three knots in $\mathbb{R}^3$ shown in the figure below are smoothly isotopic.

[693] As usual, given a map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by $F_t\colon X \to Y$ the map that is given by $x \mapsto F(x,t)$.



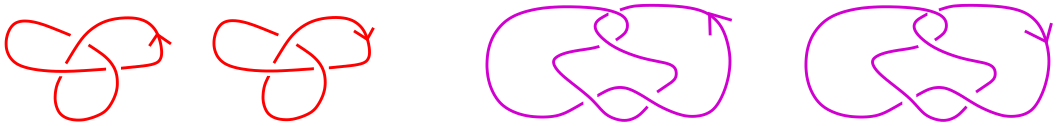
Usually we do not distinguish two links if they are smoothly isotopic. For example, any knot that is smoothly isotopic to a trivial knot is called *trivial knot*. Sometimes we also say that a knot that is smoothly isotopic to a trivial knot is *trivial*. □

We conclude this section with the following piece of notation:

Notation 62.6. Note that it follows from the Smooth Isotopy-Equivalence Relation Proposition 41.5 that the property of being smoothly isotopic defines an equivalence relation on the set of links. Thus it makes sense to define $\texttt{Link}(\mathbb{R}^3)$ the set of smooth isotopy classes of links.

62.3. **Orientations.** In many settings it is also natural to equip knots and links with orientations. This leads us to the following definition:

Definition 62.7. Let $m \in \mathbb{N}_0$. A link in $\mathbb{R}^3$ is called oriented if it is oriented as a 1-dimensional smooth manifold.



Definition 62.8. We say that two *oriented* links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_m$ in $\mathbb{R}^3$ are smoothly isotopic, if there exists a smooth isotopy $F\colon L \times [0,1] \to \mathbb{R}^3$ from L to $\widetilde{L}$ such that for each $i \in \{1, \ldots, m\}$ the map $F_1\colon L_i \to \widetilde{L}_i$ is an orientation-preserving diffeomorphism.

For completeness' sake we immediately get the following variation on Notation 62.6 out of the way:

Notation 62.9. We define $\texttt{Link}^{\text{or}}(\mathbb{R}^3)$ the set of smooth isotopy classes of oriented links.

Example. We consider the two Hopf links H^+ and H^- in $\mathbb{R}^3$ that are shown in the figure below.



The difference is that we flipped the orientation of the blue component. We will see later in Exercise 67.13 that these two links are not smoothly isotopic. □

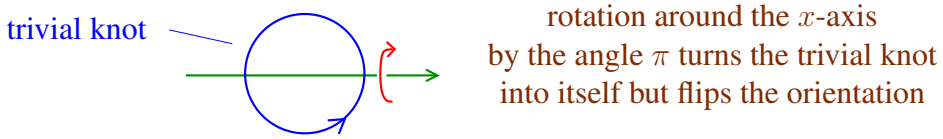
In light of the above example we now restrict ourselves for a minute to knots. For knots one can wonder whether the orientation of a knot really makes a difference. This suggests the following definition:

Definition 62.10.

(1) Let $K \subseteq \mathbb{R}^3$ be an oriented knot. We define the reverse K^{rev} of K to be the knot K with the opposite orientation.[694]

(2) A knot K is called reversible if the two orientations on K give smoothly isotopic oriented knots, otherwise we call the knot irreversible.[695]

Example. As we see in the figure below it is basically clear that the trivial knot is reversible.



In Exercise 62.2 we will show that the trefoil and the figure-8 knot are also reversible. □

62.4. **The mirror of a link.** We continue with the following nicely geometric definition:

Definition 62.11. Given a link $L \subseteq \mathbb{R}^3$ the reflection of L in any hyperplane of $\mathbb{R}^3$ is called mirror of L and it is denoted by L^{mir}.

Lemma 62.12. ($\mathbb{R}^3$-Link Mirror Lemma) Let $L \subseteq \mathbb{R}^3$ be a link. The reflections in any two hyperplanes of $\mathbb{R}^3$ give rise to smoothly isotopic links.

Proof. Let $L \subseteq \mathbb{R}^3$ be a link and let H_0 and H_1 be two hyperplanes of $\mathbb{R}^3$. By the Orthogonal Action Lemma 2.37 there exists an $X \in \mathrm{SO}(3)$ with $X \cdot H_0 = H_1$. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(3)$ is path-connected. It follows from the Smooth Path-Connectivity Proposition 25.32 that there exists a smooth path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = X$. We now consider the diffeotopy

$$\begin{array}{rcl} F\colon \mathbb{R}^3 \times [0,1] & \to & \mathbb{R}^3 \\ (x,t) & \mapsto & \text{reflection in the hyperplane } \gamma(t) \cdot H_0 \text{ applied to } x. \end{array}$$

The restriction of $F\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ to $L \times [0,1] \to \mathbb{R}^3$ gives us the desired smooth isotopy between the reflections of L in H_0 and H_1. ■

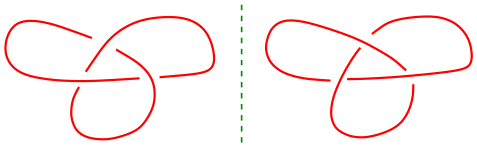
Convention 62.13. Let $L \subseteq \mathbb{R}^3$. Since usually we only care about links up to smooth isotopy we use the $\mathbb{R}^3$-Link Mirror Lemma 62.12 to talk of "the mirror" of L instead of the slightly more correct "a mirror of L".

Example. In the figure below we show the trefoil and its mirror and we also show the figure-8 knot and its mirror.
Note that the "double mirror" of a knot K is just the original knot. □

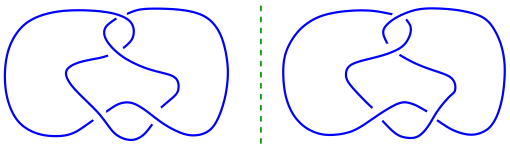
As for orientations the question arises, whether the mirror of a link is "truly different" from the original link. We are thus naturally led to the following definition:

[694] In Definition 31.22 we introduced the following notation: Given an oriented smooth manifold M we denote by $-M$ the smooth manifold with the opposite orientation. For better or worse in knot theory the expression "$-K$" is usually reserved for $(K^{\text{rev}})^{\text{mir}}$. (Here for a knot $J \subseteq \mathbb{R}^3$ the expression J^{mir} stands for the mirror of J, which we will introduce in Definition 108.23). This knot theoretic tradition explains our slightly incoherent notation.

[695] In the literature what we call a reversible knot is often called an invertible knot. We reserve the adjective invertible for a related, but different notion, see Definition 115.8.



the trefoil K and its mirror K^{mir}



the figure-8 knot J and its mirror J^{mir}

Definition 62.14. We say that a link L in $\mathbb{R}^3$ is amphichiral if it is smoothly isotopic to its mirror, otherwise we call the link L chiral.

Example.

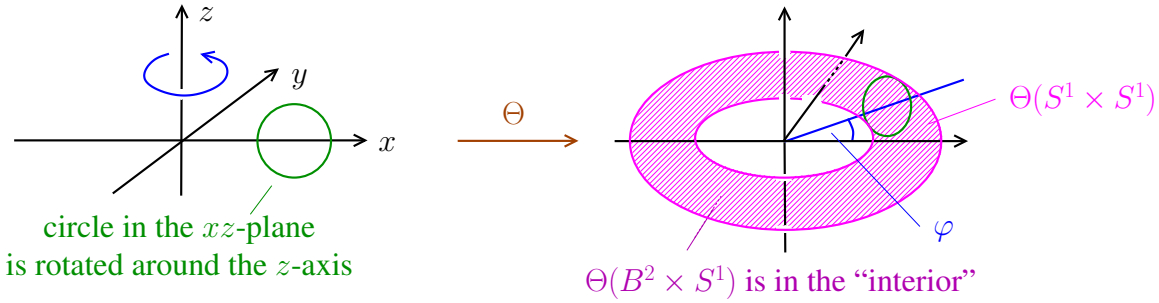
(1) The trivial knot equals of course its mirror, in particular the trivial knot is amphichiral.
(2) In Exercise 62.3 we will discuss the question whether the trefoil and or the figure-8 knot is amphichiral. □

62.5. **Torus knots.** In this section we will introduce an interesting infinite family of knots. To do so it is convenient to first introduce the following notation:

Notation 62.15. Recall that in the Torus Embedding Lemma 7.11 and on page 785 we saw that the map

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \mathbb{R}^3 \\ ((x,y),\exp(\mathrm{i}\varphi)) & \mapsto & \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in the } xz\text{-plane}}} \end{array}$$

is a smooth embedding.



circle in the xz-plane is rotated around the z-axis

$\Theta(B^2 \times S^1)$ is in the "interior"

Definition 62.16. We denote by $\Theta\colon S^1 \times S^1 \to \mathbb{R}^3$ the smooth embedding from Notation 62.15.

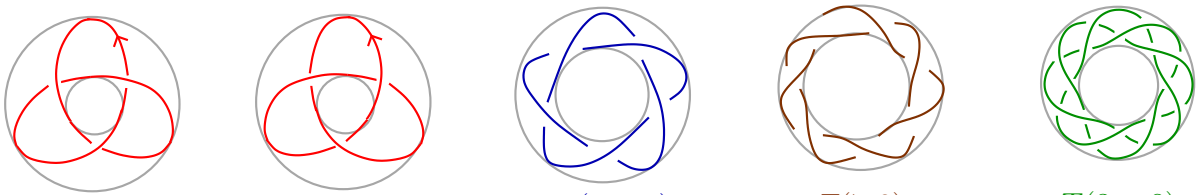
(1) Let $p, q \in \mathbb{Z}$ be coprime. We refer to the image of the map

$$\begin{array}{rcl} \varphi_{p,q}\colon \mathbb{R} & \to & \mathbb{R}^3 \\ t & \mapsto & \Theta(\exp(\mathrm{i}\cdot p\cdot t), \exp(\mathrm{i}\cdot q\cdot t)) \end{array}$$

as the (p,q)-torus knot $T(p,q)$. We equip the torus knot with the orientation which turns the map $\varphi_{p,q}\colon \mathbb{R} \to T(p,q)$ into an orientation-preserving local diffeomorphism.

(2) We define the left-handed trefoil to be the $(2,3)$-torus knot and we define the right-handed trefoil to be the $(-2,3)$-torus knot.

Visualization. In the following figure we illustrate several examples of torus knots:



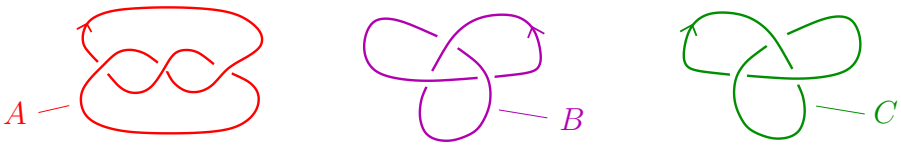
left-handed trefoil $T(2,3)$ | right-handed trefoil $T(-2,3)$ | $T(5,-2)$ | $T(7,2)$ | $T(3,-8)$

In each case we show in gray the underlying 2-dimensional torus. □

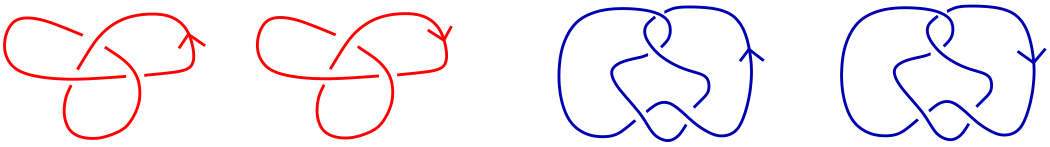
For the time being we do not want to go any further into the theory of torus knot. We will discuss torus knots in greater detail in Section 111.4.

62.6. Exercises for Chapter 62.

Exercise 62.1. In the figure below we show to three oriented knots A, B and C in $\mathbb{R}^3$. Precisely two of these are smoothly isotopic (you do not have to show this). Which two knots are smoothly isotopic?



Exercise 62.2. Show that the trefoil and the figure-8 knot are reversible. In other words, show that the two oriented knots shown in the figure to the left, respectively to the right, are smoothly isotopic.



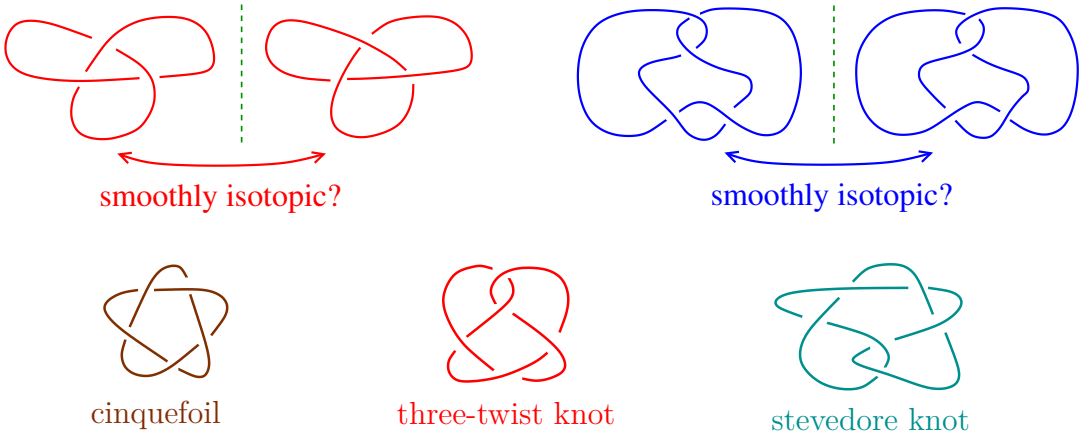
trefoil with the two orientations | figure-8 knot with the two orientations

Exercise 62.3. Precisely one knot out of the trefoil and the figure-8 knot is amphichiral.[696] Which one?

Exercise 62.4. For some knots one can give explicit descriptions of the unknot:

trefoil:	`https://www.desmos.com/3d/3c1777edd9`	(thanks to Johannes Zander)
figure-8 knot:	`https://www.desmos.com/3d/aab54cb3df`	(thanks to Filip Misev)

Give an explicit description for the cinquefoil, the three-twist knot and the stevedore knot that are shown below.



Exercise 62.5.

(a) Find a 3-component link that "seems to be linked" but such that each 2-component sublink is unlinked.

(b) For any $n \in \mathbb{N}$ find an n-component link that "seems to be linked" but such that each $(n-1)$-component sublink is unlinked.

Remark. Such links were first studied by Hermann Brunn [**Brun1892**] and are often called *Brunnian links*.

Exercise 62.6. For the purpose of this exercise we say that two knots K and $\widetilde{K}$ in $\mathbb{R}^3$ are *weakly equivalent* if there exists a continuous map

$$\begin{array}{rcl} F\colon K \times [0,1] & \to & \mathbb{R}^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following two conditions hold:

(1) For each $t \in [0,1]$ the map $F_t\colon K \to \mathbb{R}^3$ is a smooth embedding.

(2) We have $F_0 = \mathrm{id}$ and we have $F_1(K) = \widetilde{K}$.

If two knots are weakly equivalent, are they also smoothly isotopic in the sense of Definition 62.5?

Remark. The two definitions of "weakly equivalent" and "smoothly isotopic" are almost the same, except that in Definition 62.5 we demanded that F is smooth.

Exercise 62.7. Let $n \in \mathbb{N}$. Recall that in Definition 9.11 we introduced the complex projective space

$$\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

where $(z_1, \dots, z_{n+1}) \sim (\lambda \cdot z_1, \dots, \lambda \cdot z_{n+1})$ for all $(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$ and all $\lambda \in \mathbb{C} \setminus \{0\}$. Given $(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$ we denote the corresponding equivalence class by $[z_1 : \cdots : z_{n+1}] \in \mathbb{C}\mathrm{P}^n$. Also recall that in the Projective Space-Avatar Lemma 9.16 we gave an explicit homeomorphism $S^2 \xrightarrow{\cong} \mathbb{C}\mathrm{P}^1$. We consider the Hopf map

$$\begin{array}{rcl} H\colon \mathbb{R}^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

(a) Let $P \in S^2$. What does $H^{-1}(\{P\}) \subseteq \mathbb{R}^3$ look like?

[696] We will only see much later, why one of these two knots is not amphichiral.

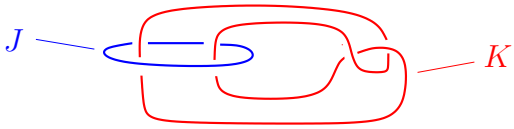
(b) Let $P, Q \in S^2$ be two distinct points. What does $H^{-1}(\{P, Q\}) \subseteq \mathbb{R}^3$ look like?
(c) Let $P, Q, R \in S^2$ be three distinct points. What does $H^{-1}(\{P, Q, R\}) \subseteq \mathbb{R}^3$ look like?

Remark. You can make use of the fact (shown in Exercise 56.2) that the answers do not depend on the choice of the points.

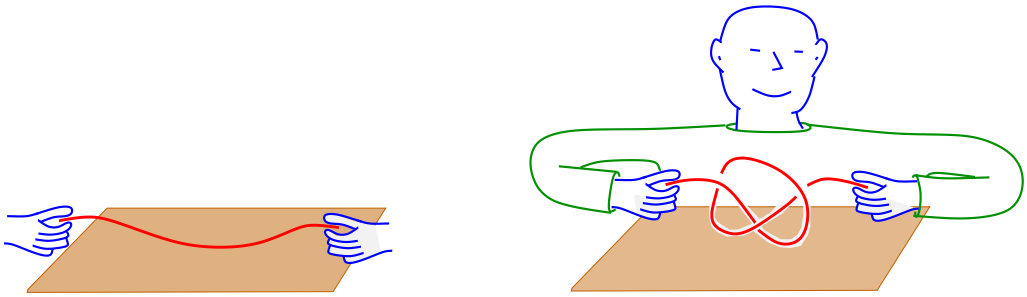
Exercise 62.8. We consider the two links shown in the figure below. In precisely one of the two cases the red component bounds a smoothly embedded disk that is disjoint from the other components. Which case is it?



Exercise 62.9. We consider the two disjoint knots J and K shown below. Is the link $L = J \cup K$ smoothly isotopic to the link $L = K \cup J$ (with the opposite ordering of the two components)?



Exercise 62.10. Can you pick up a loose unknotted string with both hands and turn it into a knotted string as shown in the figure below, while holding on to the ends of the string? Either way, which smooth isotopy types of knotted strings can you produce this way?



CHAPTER 63

Constructions of knots and links

In this chapter we will learn about several constructions of new knots and links out of given links. In particular we will introduce disjoint union of links and the connected sum of two oriented knots. These two constructions are explained fairly easily, but it is a little tricky to show that they are well-defined. Some of the more technical aspects of the constructions will be delayed to the later Chapter 109.

This chapter is organized as follows:

- In Section 63.1 we will introduce the split union of two links.
- In Section 63.2 we will introduce the connected sum of knots.
- In Section 63.3 we will provide exercises for Chapter 109.

In the next Chapter 64 we will introduce the braid group and the Burau representation.

63.1. **The split union of links.** In this section we will introduce the "split union" of two links. The idea is quite simple: We want to define the "split union" of two links L and $\widetilde{L}$ by putting each link into a ball, and then embed the two balls disjointly into $\mathbb{R}^3$. We now turn this idea into a precise mathematical concept:

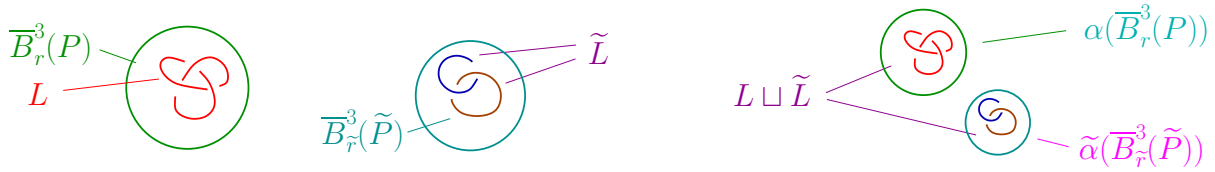
Definition 63.1. Let L and $\widetilde{L}$ be two oriented links in $\mathbb{R}^3$.

- We pick a compact ball $\overline{B}^3_r(P)$ with $L \subseteq B^3_r(P)$ and we pick a compact ball $\overline{B}^3_{\widetilde{r}}(\widetilde{P})$ with $L \subseteq B^3_{\widetilde{r}}(\widetilde{P})$.
- We pick two orientation-preserving affine linear embeddings $\alpha\colon \overline{B}^3_r(P) \to \mathbb{R}^3$ and $\widetilde{\alpha}\colon \overline{B}^3_{\widetilde{r}}(\widetilde{P}) \to \mathbb{R}^3$ with disjoint images.

We refer to

$$L \sqcup \widetilde{L} \;:=\; \alpha(L') \cup \widetilde{\alpha}(\widetilde{L}')$$

as the split union $L \sqcup \widetilde{L}$ of the links L and $\widetilde{L}$. One can easily show that the smooth isotopy type of $\alpha(L') \cup \widetilde{\alpha}(\widetilde{L}')$ is well-defined and that it only depends on the smooth isotopy type of L and $\widetilde{L}$.

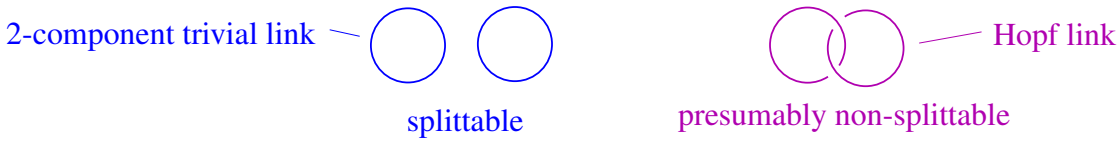


Definition 63.2. We say that a link is splittable if it is the split union of two non-empty links. Otherwise we call the link unsplittable.[697]

Example.

[697] It follows almost immediately from the definitions that a link is splittable if and only if there exist two disjoint smoothly embedded 3-balls in $\mathbb{R}^3$ each of which contains at least one component of L.

(1) It follows quite easily from the definitions that the split union of two copies of the trivial knot is (smoothly isotopic) to the trivial 2-component link as defined in Definition 62.4. This shows in particular that the trivial 2-component link is splittable.



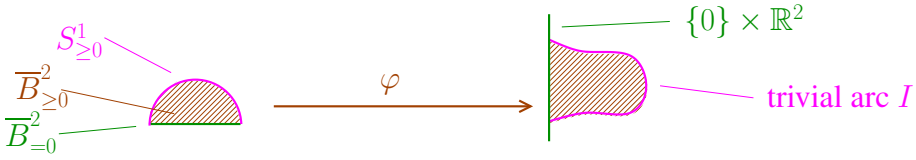
(2) It seems pretty obvious that the Hopf link is not splittable. But right now we lack the tools for formally proving that the Hopf link is indeed not splittable. □

63.2. **The connected sum of oriented knots.** In this section we will introduce a binary operation on the set of smooth isotopy classes of oriented knots, namely the connected sum operation. This definition requires some preparations.

Definition 63.3. Let $I \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ be a compact proper 1-dimensional smooth manifold that is diffeomorphic to $[-1,1]$. We say that I is a trivial arc if there exists a smooth embedding $\varphi\colon \overline{B}^2_{\geq 0} \to \mathbb{R}_{\geq 0} \times \mathbb{R} \times \{0\}$ such that the following conditions hold:

- $\varphi(\overline{B}^2_{=0}) \subseteq \{0\} \times \mathbb{R}^2$.
- $\varphi(S^1_{\geq 0}) = I$.

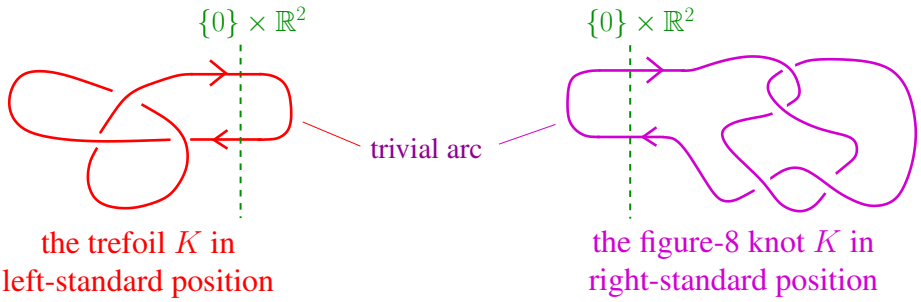
An argument as in the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 shows that any two trivial arcs are properly smoothly isotopic.



Definition 63.4. Let $K \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3$ be an oriented knot. We say that K is in left-standard position if there exists an $\epsilon > 0$ such that the following conditions hold:

(1) $K \cap ([-\epsilon, \epsilon] \times \mathbb{R}^2) = [-\epsilon, \epsilon] \times \{(\pm 1, 0)\}$.
(2) The orientation of K is given as follows:
  - The orientation of K points from $(-\epsilon, 1, 0)$ to $(\epsilon, 1, 0)$.
  - The orientation of K points from $(\epsilon, -1, 0)$ to $(-\epsilon, -1, 0)$.
(3) The intersection $K \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$ is a trivial arc.

Similarly we define what it means for K to be in right-standard position.



Next we will see that we can arrange any knot to be in standard position:

Lemma 63.5. (Knot-Standard Position Lemma) Every oriented knot K in $\mathbb{R}^3$ is smoothly isotopic to a knot in left-standard position and it is smoothly isotopic to a knot in right-standard position.

Proof. We leave the slightly technical proof of the lemma to the later Knot-Standard Position Lemma 109.16. ■

We are now ready to give the following definition:

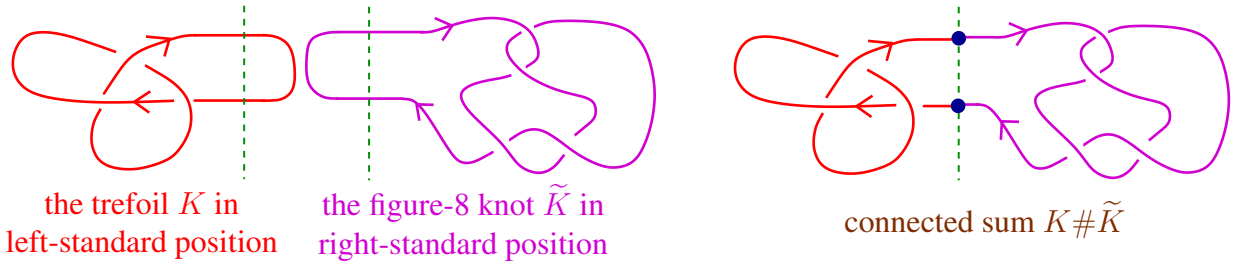
Definition 63.6. Let K and $\widetilde{K}$ be two oriented knots in $\mathbb{R}^3$. We pick a knot K^{st} be a knot in left-standard position that is smoothly isotopic to K and we pick a knot $\widetilde{K}^{\text{st}}$ in right-standard position that is smoothly isotopic to $\widetilde{K}$. We refer to

$$K\#\widetilde{K} \ := \ \underbrace{(K^{\text{st}} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2))}_{\subseteq \mathbb{R}^3_{\leq 0}} \cup \underbrace{(\widetilde{K}^{\text{st}} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2))}_{\subseteq \mathbb{R}^3_{\geq 0}}$$

as the connected sum of K and $\widetilde{K}$. It follows from the discussion in Definition 109.12 and the Standard Connected Sum Lemma 109.17 that the smooth isotopy class of $K\#\widetilde{K}$ does not depend on the choice of the left-standard and right-standard positions of K and $\widetilde{K}$.

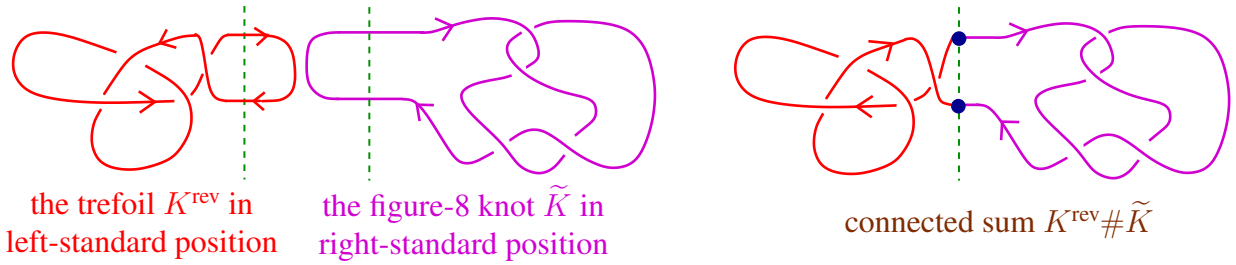
Example. The connected sum of any oriented knot K with the trivial knot is easily seen to be smoothly isotopic to K. □

Example. In the figure below we show the trefoil K in left-standard position and we show the figure-8 knot $\widetilde{K}$ in right-standard position:



By Definition 63.6 the knot to the right-hand side is the connected sum $K\#\widetilde{K}$ of the trefoil K and the figure-8 knot $\widetilde{K}$.

Note though that the connected sum operation depends on the orientation of the knots. In the following figure we show the trefoil K^{rev} "with the opposite orientation" and again the figure-8 knot.



At least a priori there is no reason why $K\#\widetilde{K}$ and $K^{\text{rev}}\#\widetilde{K}$ should be of the same smooth isotopy type. □

63.3. Exercises for Chapter 63.

Exercise 63.1.

(a) Show that the trivial knot is a neutral element with respect to the connected sum operation.
(b) Show that the connected sum operation is (up to smooth isotopy) abelian.
(c) Show that the connected sum operation is (up to smooth isotopy) associative.

Hint. Use that the connected sum operation is well-defined, i.e. independent of any of the choices.

Exercise 63.2. We consider the oriented trefoil K and the oriented figure-8 knot $\widetilde{K}$ as shown below. Is the connected sum $K\#\widetilde{K}$ smoothly isotopic to the connected sum $K^{\mathrm{rev}}\#\widetilde{K}$?



CHAPTER 64

The braid group

In this chapter we will first use topology to introduce for every $m \in \mathbb{N}_0$ the braid group B_m. Loosely speaking this group is given by considering m "downward pointing" strings in $\mathbb{R}^2 \times [0,1]$ with a fixed set of m starting points and m endpoints, where the group structure is given by "stacking one on top of the other". We will then see that each braid group B_m admits an explicit purely algebraic finite presentation.

It will be clear from the figures in this chapter that braids are closely related to knots and links. We will make this connection precise a little later, in Chapter 66.

This chapter is organized as follows:

- In Section 64.1 we will introduce the braid groups B_m.
- In Section 64.2 we will introduce the *pure* braid groups, which form finite-index subgroups of the usual braid groups.
- In Section 64.3 we will give a quick introduction to finite presentations of groups.
- In Section 64.4 we will give an explicit finite presentation for the braid group B_m.
- In Section 64.5 we will learn how to add and delete strands.
- In Section 64.6 we will provide exercises for Chapter 64.

In the next Chapter 65 we will explore the relationship between braids and links.

64.1. **The braid group.** We start out by providing the following definition:[698]

Definition 64.1. Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m_0 \in \mathbb{N}$. Furthermore let $P_1, \ldots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points. An *m-strand braid in the pair $(\Sigma, \{P_1, \ldots, P_m\})$* is defined as a proper 1-dimensional submanifold $X \subseteq (\Sigma \setminus \partial\Sigma) \times [0,1]$ with the following properties:

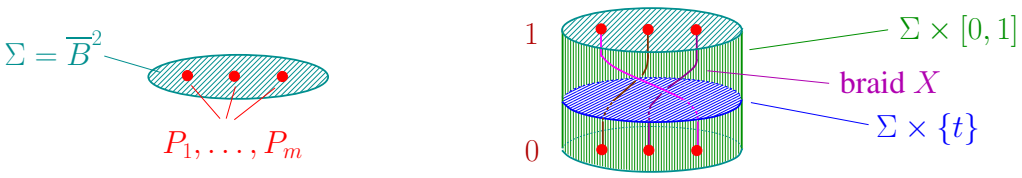
(1) X has precisely m components, each of which is diffeomorphic to an interval.[699]
(2) For each $t \in (0,1)$ the submanifold X is transverse to the submanifold $\Sigma \times \{t\}$.
(3) There exists an $\epsilon > 0$ such that the following conditions are satisfied:
   - $X \cap (\Sigma \times [0, \epsilon]) = \{P_1, \ldots, P_m\} \times [0, \epsilon]$.
   - $X \cap (\Sigma \times [1-\epsilon, 1]) = \{P_1, \ldots, P_m\} \times [1-\epsilon, 1]$.

A component of a braid is sometimes called a *strand*.

Remark. Our definition of a braid differs somewhat from the definitions in the standard textbooks [**Bir1974**, Chapter 1.1], [**MK1999**, p. 3], [**KT2008**, Definition 1.4] and [**BZH2014**, p. 23]. In particular, in contrast to the above references, we work with smooth

[698] In fact we introduced the concept of a braid before, namely in Exercise 44.7. Using the claim in the proof of the Braid Group Lemma 64.4 one can easily verify that the notion of a braid in Exercise 44.7 is equivalent to the notion of a braid in Definition 64.1.

[699] In fact this condition is a consequence of (2) and (3). But didactically it seems to us that it is better to make this explicit.

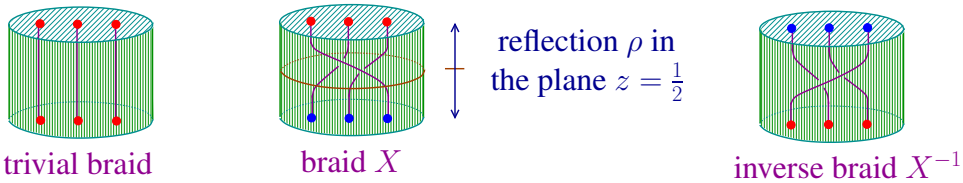


instead of polygonal or topological objects. We leave it to the reader to show that the different definitions lead to equivalent theories. □

Example. We continue with the notation from the above Definition 64.1.

(1) In this setting $T := \{P_1, \ldots, P_m\} \times [0,1]$ is clearly an m-strand braid. We refer to it as the *trivial braid*.

(2) Let X be a braid in $(\Sigma, \{P_1, \ldots, P_m\})$. We denote by $\rho\colon \Sigma \times [0,1] \to \Sigma \times [0,1]$ the reflection in $\Sigma \times \{\frac{1}{2}\}$ that is given by $\rho(x,t) = \rho(x, 1-t)$. Clearly $X^{-1} := \rho \circ X$ is again a braid. We refer to it as the *inverse of* X.



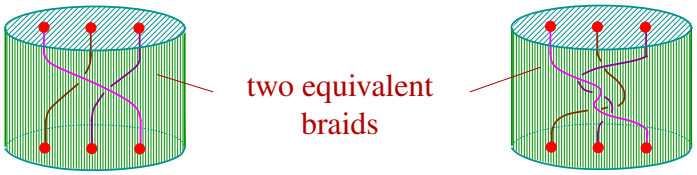
The above naming conventions will make sense in a few minutes. □

Similar to knots and links we do not care about braids "on the nose", but only up to a suitable form of deformation. This thought leads us to the following definition:

Definition 64.2. Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \ldots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points.

(1) Two braids X_0, X_1 in $(\Sigma, \{P_1, \ldots, P_m\})$ are called equivalent if there exists a proper smooth isotopy $\Phi\colon (\{1, \ldots, m\} \times [0,1]) \times [0,1] \to (\Sigma \setminus \partial\Sigma) \times [0,1]$ with the following properties:
- For $t \in \{0,1\}$ we have $\Phi_t(\{1, \ldots, m\} \times [0,1]) = X_t$.
- Each $\Phi_t(\{1, \ldots, m\} \times [0,1])$ is a braid in $(\Sigma, \{P_1, \ldots, P_m\})$.

(2) We denote by $\mathrm{B}^{P_1,\ldots,P_m}(\Sigma)$ the set of equivalence classes of braids in $(\Sigma, \{P_1, \ldots, P_m\})$.



Note that pictorially we can compose two braids $X * Y$ by "stacking X on top of Y".[700] The following definition makes this idea precise:[701]

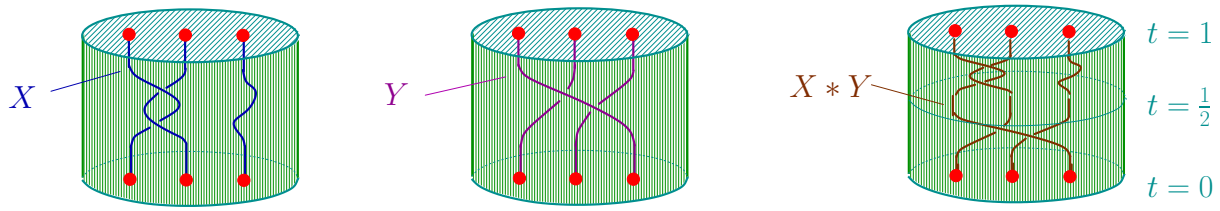
[700] One could also stack X *below* Y and one would again get a group. The "on top" convention is consistent with all the usual references [**Art1925a**, p. 48], [**Bir1974**, Chapter 1.1], [**MK1999**, p. 12] and [**KT2008**, p. 8].

[701] Note that the following loosely resembles the definition of a "combination of two homotopies" as given in the Homotopy Combination Lemma 18.8.

Definition 64.3. Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points. Given two braids X, Y in $(\Sigma, \{P_1, \dots, P_m\})$ we define the composition

$$X * Y \;:=\; \{(P, \tfrac{t+1}{2}) \,|\, (P,t) \in X\} \cup \{(Q, \tfrac{t}{2}) \,|\, (Q,t) \in Y\}.$$

Using Condition (3) from Definition 64.1 one can easily verify that $X * Y$ is again a braid in $(\Sigma, \{P_1, \dots, P_m\})$.



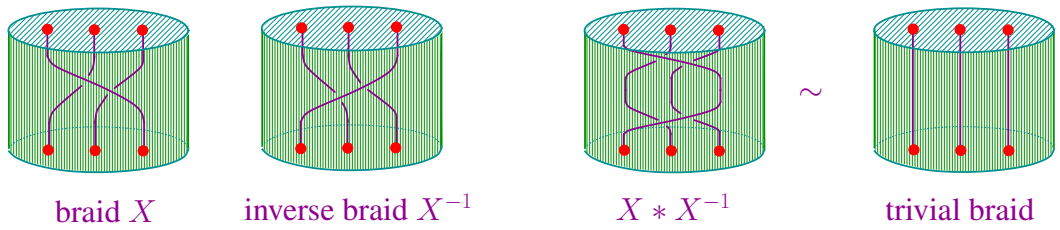
Not surprisingly the composition of braids leads to a group structure:

Lemma 64.4. (Braid Group Lemma) Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points. The map

$$\begin{array}{rcl} \mathrm{B}^{P_1,\dots,P_m}(\Sigma) \times \mathrm{B}^{P_1,\dots,P_m}(\Sigma) & \to & \mathrm{B}^{P_1,\dots,P_m}(\Sigma) \\ ([X],[Y]) & \mapsto & [X * Y] \end{array}$$

is well-defined and it defines a group structure on $\mathrm{B}^{P_1,\dots,P_m}(\Sigma)$. This group structure has the following two features:

- The trivial element is represented by the trivial braid from page 1394.
- Given a braid X the inverse of $[X]$ is represented by the inverse braid X^{-1} from page 1394.



Remark. The m-strand braid groups $\mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2)$ were first introduced and studied by Emil Artin [**Art1925a**, p. 48]. □

Proof. We start out with the following claim:

Claim. We denote by $\mathrm{pr}_{[0,1]} \colon \Sigma \times [0,1] \to [0,1]$ the projection. There exists a diffeomorphism $\varphi \colon \{1, \dots, m\} \times [0,1] \to X$ such that for each $(k,t) \in \{1, \dots, m\} \times [0,1]$ we have the equality $\mathrm{pr}_{[0,1]}(\varphi(k,t)) = t$.

Proof. Let $X_1, \dots, X_m$ be the m components of X. It follows easily from Definition 64.1 and the Inverse Mapping Theorem 30.1 that for each $i \in \{1, \dots, m\}$ the projection map $\mathrm{pr}_{[0,1]} \colon X_i \to [0,1]$ is a diffeomorphism.
Clearly the inverses of these m diffeomorphisms $X_i \to [0,1]$ give us the desired diffeomorphism $\varphi \colon \{1, \dots, m\} \times [0,1] \to X$. ⊞



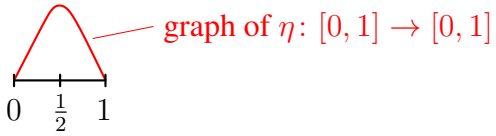
Now we turn to the actual verification of the axioms of a group that we formulated in Definition 2.1:

(1) Let $X \subseteq \Sigma \times [0,1]$ be an m-strand braid. As before let $T \subseteq \Sigma \times [0,1]$ be the trivial m-strand braid. We need to show that $T * X \sim X$ and that $X * T \sim X$. We pick a diffeomorphism $\varphi\colon \{1,\ldots,m\} \times [0,1] \to X$ as in the claim. We consider the following map:

$$\begin{array}{rcl} \Phi\colon \{1,\ldots,m\} \times [0,1] \times [0,1] & \to & \Sigma \times [0,1] \\ (i,t,s) & \mapsto & \begin{cases} (P_i,t) & \text{if } t \in [\frac{1+s}{2},1], \\ (\mathrm{pr}_\Sigma(\varphi(i,t\cdot\frac{2}{1+s})),t) & \text{if } t \in [0,\frac{1+s}{2}]. \end{cases} \end{array}$$

One can easily verify that this map is the desired smooth isotopy which shows that $T * X \sim X$. The same way one shows that $X * T \sim X$.

(2) Let $X \subseteq \Sigma \times [0,1]$ be an m-strand braid. We need to show that $X * X^{-1} \sim T$ and that $X^{-1} * X \sim T$. We pick a diffeomorphism $\varphi\colon \{1,\ldots,m\} \times [0,1] \to X$ as in the claim and we pick $\epsilon > 0$ as in Definition 64.1. It follows easily from the Smooth Transition Function Lemma 24.9 that there exists a smooth map $\eta\colon [0,1] \to [0,1]$ with $\eta(t) = 2t$ for $t \in [0,\frac{1-\epsilon}{2}]$, with $\eta(t) = 2-2t$ for $t \in [\frac{1+\epsilon}{2},1]$ and such that $\eta(t) \geq 1-\epsilon$ for all $t \in [\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}]$.



We consider the map

$$\begin{array}{rcl} \Phi\colon \{1,\ldots,m\} \times [0,1] \times [0,1] & \to & \Sigma \times [0,1] \\ ((k,t),s) & \mapsto & (\varphi(k,\eta(t)\cdot(1-s)),t). \end{array}$$

Using the Smooth Pasting Proposition 25.19 one can easily verify that this map is smooth. It is now straightforward to verify that Φ is the desired smooth isotopy that shows that $X^{-1} * X \sim T$. The same way one shows that $X * X^{-1} \sim T$.

(3) Similar to the proof of the Path-Homotopy Product Proposition 88.10 one can easily show that for m-braids X, Y, Z we have $(X * Y) * Z \sim X * (Y * Z)$. We leave it to the reader to fill in the details. ■

The next lemma shows that the braid group is functorial under smooth embeddings of pointed surfaces:

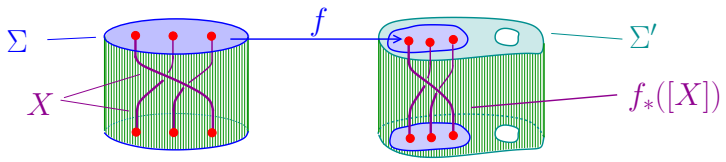
Lemma 64.5. (Braid Group–Functor Lemma) Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points.

(1) Let Σ' be a connected 2-dimensional smooth manifold. Every smooth embedding $f\colon \Sigma \to \Sigma'$ induces a group homomorphism

$$\begin{array}{rcl} f_*\colon \mathrm{B}^{P_1,\dots,P_m}(\Sigma) & \to & \mathrm{B}^{f(P_1),\dots,f(P_m)}(\Sigma') \\ [X] & \mapsto & [\underbrace{\mathrm{diag}(f, \mathrm{id}_{[0,1]})}_{\text{map } \Sigma\times[0,1]\to\Sigma'\times[0,1]} \circ X]. \end{array}$$

If $f\colon \Sigma \to \Sigma'$ is a diffeomorphism, then f_* is an isomorphism.

(2) The isomorphism type of the group $\mathrm{B}^{P_1,\dots,P_m}(\Sigma)$ only depends on Σ and m.



Proof.

(1) This statement follows easily from the definitions.

(2) Let $P_1, \dots, P_m$ and $Q_1, \dots, Q_m$ two families of pairwise distinct points in Σ. Recall that it follows from the Homogeneous-via-Diffeotopies Proposition 44.6 that there exists a diffeomorphism $f\colon \Sigma \to \Sigma$ such that for $i = 1, \dots, m$ we have $f(P_i) = Q_i$. By (1) we thus obtain an isomorphism $\mathrm{B}^{P_1,\dots,P_m}(\Sigma) \to \mathrm{B}^{Q_1,\dots,Q_m}(\Sigma)$. ∎

In the following we will often use the Braid Group–Functor Lemma 64.5 (2) to ignore the precise choice of points $P_1, \dots, P_m \in \Sigma$ and to suppress them from the notation. This thought leads us to the following notation:

Notation 64.6.

(1) Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points. In the following we refer to $\mathrm{B}_m(\Sigma) \coloneqq \mathrm{B}^{P_1,\dots,P_m}(\Sigma)$ as the m-strand braid group of Σ.

(2) The group

$$\mathrm{B}_m \coloneqq \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)$$

is called the (standard) m-strand braid group.

The following lemma gives us a convenient alternative characterization of the m-strand braid groups $\mathrm{B}_m \coloneqq \mathrm{B}^{(0,0),(1,0),\dots,(m,0)}(\mathbb{R}^2)$.

Lemma 64.7. (Braid Group–Finite Radius Lemma) Let $m \in \mathbb{N}_0$. We denote by $\iota\colon \overline{B}^2_{m+1} \to \mathbb{R}^2$ the inclusion. The induced map

$$\iota_*\colon \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\overline{B}^2_{m+1}) \to \underbrace{\mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)}_{=:\text{standard braid group } \mathrm{B}_m}$$

is an isomorphism.

Proof. We leave it to the reader to construct, say using the Smooth Transition Function Lemma 24.9, a smooth isotopy $F\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ with the following properties:

(1) We have $F_0 = \mathrm{id}$ and $F_1(\mathbb{R}^2) \subseteq \overline{B}^2_{m+1}$.

(2) We have $F(\overline{B}^2_{m+1} \times [0,1]) \subseteq \overline{B}^2_{m+1}$.

(3) We have $F_t(i,0) = (i,0)$ for all $i \in \{1,\dots,m\}$ and $t \in [0,1]$.

We now consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2_{m+1}) & \xrightarrow{\iota_*} & \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2) & & \\
& \searrow^{(F_1|_{\overline{B}^2_{m+1}})_* = (F_0|_{\overline{B}^2_{m+1}})_* = \mathrm{id}} & \downarrow{\scriptstyle F_{1*}} & \searrow^{F_{1*}=F_{0*}=\mathrm{id}} & \\
& & \mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2_{m+1}) & \xrightarrow{\iota_*} & \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2).
\end{array}$$

We make two clarifications and observations:

- As we will see in Exercise 64.8, two smooth embeddings that are smoothly isotopic rel the endpoints of the braids induce the same group homomorphisms between the braid groups. Thus in the above diagram we have $F_{1*} = F_{0*}$ and $(F_1|_{\overline{B}^2_{m+1}})_* = (F_0|_{\overline{B}^2_{m+1}})_*$.
- The induced map on braids is clearly covariantly functorial in the smooth embedding. This implies that the above diagram commutes.

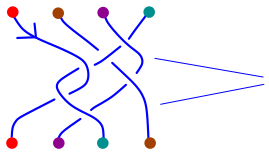
It follows from these observations that $F_{1*}\colon \mathrm{B}(\mathbb{R}^2) \to \mathrm{B}(\overline{B}^2_{m+1})$ is an inverse to ι_*. ■

64.2. **The pure braid group.** The following question arises: How can we show that an element of a braid group $\mathrm{B}^{P_1,\dots,P_m}(\Sigma)$ is non-trivial? The following lemma gives us an elementary epimorphism onto the permutation group S_m, which at least sometimes can be used to show that a specific element of a braid group is non-trivial.

Lemma 64.8. (Braid Group–Permutation Lemma) Let Σ be a connected non-empty 2-dimensional smooth manifold, let $m_0 \in \mathbb{N}$ and let $P_1,\dots,P_m \in \Sigma\backslash\partial\Sigma$ be pairwise distinct points. The map

$$\begin{array}{rcl}
\epsilon\colon \mathrm{B}^{P_1,\dots,P_m}(\Sigma) & \to & \mathrm{Bij}(\{P_1,\dots,P_m\}) \\
X & \mapsto & \left(\begin{array}{rcl} \{P_1,\dots,P_m\} & \to & \{P_1,\dots,P_m\} \\ & & \text{unique } P_j \text{ such that there} \\ P_i & \mapsto & \text{exists a component } X_i \text{ of } X \\ & & \text{with } \partial X_i = \{(P_i,1)\} \cup \{(P_j,0)\} \end{array} \right)
\end{array}$$

is a homomorphism and it is an epimorphism. We refer to it as the endpoint permutation homomorphism.



a braid defines a bijection from $(P_1,1),\dots,(P_m,1)$ to $(P_1,0),\dots,(P_m,0)$

Proof. First note that it is elementary to verify that the map is a homomorphism.[702] Next note that it follows easily from the Homogeneous-via-Diffeotopies Proposition 44.6 (which we can apply since Σ is connected and since $P_1,\dots,P_m \in \Sigma \setminus \partial\Sigma$), that the given map $\epsilon\colon \mathrm{B}^{P_1,\dots,P_m}(\Sigma) \to \mathrm{Bij}(\{P_1,\dots,P_m\})$ is an epimorphism. ■

[702] Note though that here it is important that the various conventions and definitions match: The fact that in Definition 64.3 we defined $X * Y$ by stacking X on top of Y implies that the map we interested in is given by $\{P_1,\dots,P_m\} \times \{1\} \to \{P_1,\dots,P_m\} \times \{0\}$.

For the record we provide the following definition, which we will barely use:

Definition 64.9. (1) Let Σ be a connected non-empty 2-dimensional smooth manifold, let $m_0 \in \mathbb{N}$ and let $P_1, \dots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points.
   (a) Any braid in the kernel of $\epsilon\colon \mathrm{B}^{P_1,\dots,P_m}(\Sigma) \to \mathrm{Bij}(\{P_1, \dots, P_m\})$ is called a pure braid.
   (b) The kernel of the homomorphism $\epsilon\colon \mathrm{B}^{P_1,\dots,P_m}(\Sigma) \to \mathrm{Bij}(\{P_1, \dots, P_m\})$ is called the pure braid group $\mathrm{P}^{P_1,\dots,P_m}(\Sigma)$.
(2) The group
$$\mathrm{P}_m \coloneqq \mathrm{P}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)$$
is called the (standard) m-strand pure braid group.

64.3. **Interlude: Presentations of groups.** To provide a clean algebraic description of braid groups we need to introduce some language and notions from group theory. More precisely we want to introduce the notion of a free group and a presentation of a group. These notions will be discussed in much greater detail later in Chapter 98 and Chapter 100.

To keep the exposition concise we make the conscious choice of being at times a trifle imprecise and vague. We refer to Chapter 98 and Chapter 100 for the precise definitions and statements.

64.3.1. **Free groups.** We start out with a (slightly vague) definition of a free group:

Definition 64.10. We consider a set with distinct elements $s_1, \dots, s_k$. We refer to
$$\langle s_1, \dots, s_k \rangle \coloneqq \{s_{r_1}^{n_1} \cdot \ldots \cdot s_{r_m}^{n_m} \,|\, r_i \neq r_{i+1} \text{ for } i = 1, \dots, m-1 \text{ and } s_i \neq 0 \text{ for } i = 1, \dots, m\}$$
as the free group on the generators $s_1, \dots, s_k$. Here the multiplication is given by the obvious multiplication.[703]

Example. If we consider a set with a single element s, then we see easily that the map
$$\begin{array}{ccc} \mathbb{Z} & \to & \langle s \rangle \\ n & \mapsto & s^n \end{array}$$
is an isomorphism of groups. □

The following lemma shows that $\langle s_1, \dots, s_k \rangle$ is a free group on the set $\{s_1, \dots, s_k\}$ in the more general sense of Definition 98.6:

Lemma 64.11. (Free Group Lemma) We consider a set with distinct elements $s_1, \dots, s_k$. The group $\langle s_1, \dots, s_k \rangle$ has the following universal property: For any group G and any choice of $g_1, \dots, g_k \in G$ there exists a *unique homomorphism* $\psi\colon \langle s_1, \dots, s_k \rangle \to G$ such that for each $i \in \{1, \dots, k\}$ we have $\psi(s_i) = g_i$.

Proof. The lemma follows essentially almost immediately from the definitions. We refer to the Free Group–Existence Lemma 98.9 for a few more details. ■

We conclude this subsection with the following definition:

[703] We explain this multiplication in greater detail in Definition 98.1 and we will show in the Free Product Lemma 98.3 that this multiplication does indeed define a group structure.

Definition 64.12. Let G be a group and let $S = \{s_1, \dots, s_k\} \subseteq G$ be a subset. We say that S is a generating set for the group G if the natural homomorphism $\langle x_1, \dots, x_k \rangle \to G$ with $\psi(x_i) = s_i$ is an epimorphism.

Example. Let $n \in \mathbb{N}$. We consider the permutation group S_n. A generating set for S_n is given by any transposition $(i \;\; i{+}1)$ and the cycle $(1 \;\; 2 \;\; \dots \;\; n)$. This fact surely gets taught in any reasonable course on elementary group theory. □

64.3.2. **Presentations.** Of course most groups are not isomorphic to free groups. To describe more general groups we will need to introduce some extra notation. We start out with the following general notion:

Definition 64.13. Let G be a group and let $S \subseteq G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } S \;\coloneqq\; \langle\langle S \rangle\rangle \;\coloneqq\; \bigcap_{S \subseteq H \lhd G} H.$$

In other words, $\langle\langle S \rangle\rangle$ is the smallest normal subgroup of G that contains S.

This definition allows us to introduce a new type of groups:

Definition 64.14. Let X be a set with distinct elements $x_1, \dots, x_k$. Furthermore suppose that we are given $r_1, \dots, r_l \in \langle x_1, \dots, x_k \rangle$. We write

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\coloneqq\; \langle x_1, \dots, x_k \rangle / \underbrace{\langle\langle \{r_1, \dots, r_l\} \rangle\rangle}_{\text{see Definition 64.13}}.$$

Example. For $k \in \mathbb{N}_0$ one can easily show by hand that $\langle g \,|\, g^k \rangle$ is isomorphic to $\mathbb{Z}/k\mathbb{Z}$. □

The following lemma plays the role of the Free Group Lemma 64.11:

Lemma 64.15. (Presentation–Homomorphism Lemma) Let π be a group with a finite presentation $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ and let G be a group. If $g_1, \dots, g_k \in G$ satisfy the relations $r_1, \dots, r_l$ (i.e. if each $r_i(x_1, \dots, x_k) \in G$ is trivial[704]), then there exists a unique homomorphism

$$\beta \colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; G$$

with $\beta(x_i) = g_i$ for $i = 1, \dots, k$.

Proof. We proceed with a few easy steps:

- It follows from the Free Group Lemma 64.11 that there exists a unique homomorphism $\alpha \colon \langle x_1, \dots, x_k \rangle \to G$ such that for each $i \in \{1, \dots, k\}$ we have $\alpha(x_i) = g_i$.
- Since the $g_1, \dots, g_k \in G$ satisfy the relations $r_1, \dots, r_l$ we see that $r_1, \dots, r_l$ lie in the kernel of α.
- Since the kernel of α is a normal subgroup we obtain from Definition 64.13 that $\langle\langle r_1, \dots, r_l \rangle\rangle$ also lies in the kernel of α.
- It follows from the definition (or universal property) of a quotient group that we get an induced homomorphism. ■

[704] We can express this a little more formally: We say that the elements $g_1, \dots, g_k$ satisfy the relation $r \in \langle x_1, \dots, x_k \rangle$ if the natural homomorphism $\langle x_1, \dots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \dots, k$ sends r to the trivial element.

Example. It follows from the Presentation–Homomorphism Lemma 64.15 that there exists a unique homomorphism

$$\langle x, y \mid [x, y] \rangle \ \to \ \mathbb{Z}^2$$

which sends x to $(1, 0)$ and y to $(0, 1)$. Later in the Free Abelian Group-Presentation Lemma 100.14 we will see that this map is in fact an isomorphism. □

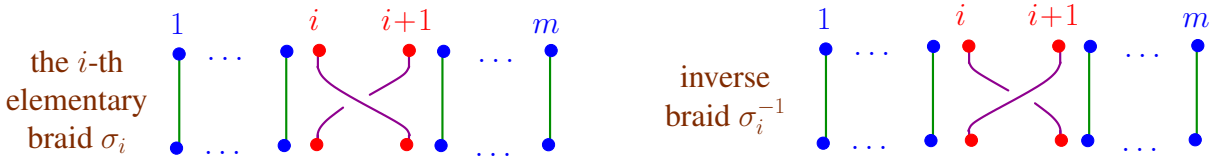
We conclude this purely algebraic section with the following harmless definition:

Definition 64.16. Let G be a group. A presentation for G is a set $\{x_1, \dots, x_k\}$ together with elements $r_1, \dots, r_l \in \langle x_1, \dots, x_k \rangle$ and together with an isomorphism

$$\langle X \mid R \rangle \ \xrightarrow{\cong} \ G.$$

64.4. **A presentation for the braid group.** Our goal is to gain a better understanding of the braid groups $\mathrm{B}_m = \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2)$. A first step towards understanding a group is to find a (finite) generating set. To do so we introduce the following notation:

Notation 64.17. Let $m \in \mathbb{N}$. Given $i \in \{1, \dots, m-1\}$ we denote, following [**Art1925a**, p. 49][705], by $\sigma_i \in \mathrm{B}_m = \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)$ the braid that is shown in the figure below to the left. We refer to it as the i-th elementary braid.[706]



Proposition 64.18. (Braid Group–Generating Set Proposition) Let $m \in \mathbb{N}$. The elementary braids $\sigma_1, \dots, \sigma_{m-1}$ form a generating set for $\mathrm{B}_m = \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)$.

Sketch of proof. Clearly it suffices to prove the following claim:

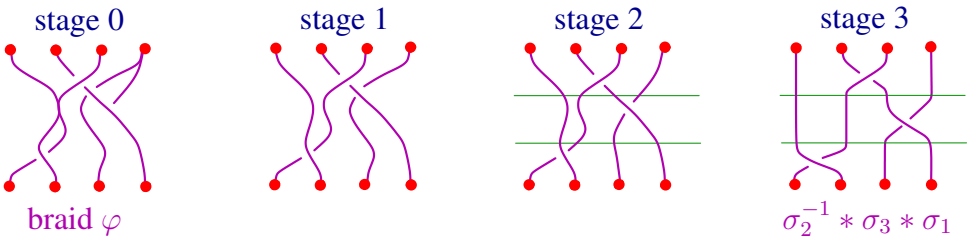
Claim. Any m-strand braid X in $\mathbb{R}^2$ is equivalent to an m-strand braid that is the composition of the elementary braids $\sigma_1, \dots, \sigma_{m-1}$ and their inverses.

Proof. We proceed in several stages:

(1) Similar to the Transversality Theorem 50.9 we can find a smooth isotopy that turns the braid into a new braid such that the projections of the components of the braid to $\mathbb{R} \times \{0\} \times [0, 1]$ are transverse to one another.
(2) After a further smooth isotopy we can assume that beyond the property stated in (1) there exists also an $n \in \mathbb{N}$ such that each band $\mathbb{R} \times \{0\} \times [\frac{i-1}{n}, \frac{i}{n}]$ contains a single crossing.
(3) After a further smooth isotopy we obtain a braid that is the composition of n elementary braids (or their inverses).

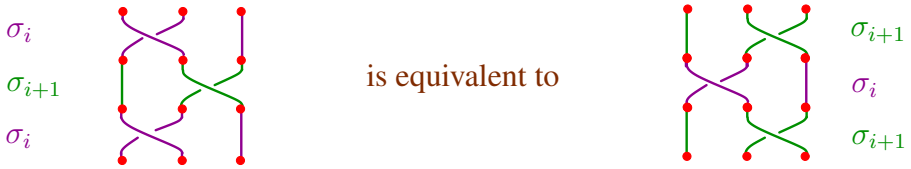
We leave it to the reader to turn these ideas into a proper proof. ■

[705]This convention is also used say in [**BZH2014**, Section 10.A] and [**PrS1997**, p. 47]. On the other hand the opposite convention is used in [**KT2008**, p. 16]. The discrepancy comes from the fact that our convention is the time-honored convention, but the opposite convention has the advantage that σ_i corresponds to a positive crossing.
[705]To unclutter the pictures we henceforth no longer show the surrounding cylinder.



It is not hard to see that the elementary braids $\sigma_1, \ldots, \sigma_{m-1}$ do not generate a free group. In fact one quickly identifies two relations among the elementary braids $\sigma_1, \ldots, \sigma_{m-1}$:

- Braid relation: For any $i \in \{1, \ldots, m-2\}$ we have $[\sigma_i * \sigma_{i+1} * \sigma_i] = [\sigma_{i+1} * \sigma_i * \sigma_{i+1}]$.



- Far commutativity: For any $i, j \in \{1, \ldots, m-1\}$ with $|i-j| \geq 2$ we have $[\sigma_i * \sigma_j] = [\sigma_j * \sigma_i]$.



The following proposition says that we have essentially found all relations among the elementary braids σ_i:

Proposition 64.19. (Braid Group–Presentation) Let $m \in \mathbb{N}$. The map

$$\left\langle \sigma_1, \ldots, \sigma_{m-1} \;\middle|\; \begin{array}{ll} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} & \text{for } i = 1, \ldots, m-2 \\ \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i & \text{for } |i-j| \geq 2 \end{array} \right\rangle \to \mathrm{B}_m$$
$$\sigma_i \mapsto [\sigma_i]$$

is well-defined and it is an isomorphism.

Remark. A presentation for the pure braid group P_n is given in [**KT2008**, p. 21]. □

Proof. First of all note that the discussion preceding the proposition, together with the Presentation–Homomorphism Lemma 64.15, shows that the given map is well-defined. Next note that it follows from the Braid Group–Generating Set Proposition 64.18 that the map is an epimorphism. Finally we point out that the much more interesting result that the map is a monomorphism was first proved by Emil Artin [**Art1925a**, Satz 1]. We also refer to [**Bir1974**, Theorem 1.8], [**MK1999**, Theorem 2.2] and [**KT2008**, Theorem 1.12] for alternative proofs. ■

For convenience we introduce the following convention:

Convention 64.20. We will use the Braid Group–Presentation 64.19 to identify the braid group B_m with the above presentation. In particular, we will denote by σ_i indiscriminately a generator of the presentation, the corresponding group element and the i-th elementary braid.

We conclude this section with the following proposition that was first proved by Wei-Lang Chow [**Cho1948**, Theorem III]:

Proposition 64.21. (Braid Group–Center Proposition) Let $m \in \mathbb{N}_{\geq 3}$. The center of the braid group B_m is generated by $(\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{m-1})^m$.



Proof. In Exercise 64.9 we will verify that $(\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{m-1})^m$ lies indeed in the center of the braid group B_m. The much more tricky proof that the center of the braid group B_m is generated by $(\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{m-1})^m$ is given in [**Cho1948**, Theorem III], [**Bir1974**, Corollary 1.8.4], [**KT2008**, Theorem 1.24] or [**BZH2014**, Proposition 10.13]. Since we will not really make use of this result we will not provide a proof of this statement. ■

64.5. **Adding and deleting strands.** Our next goal is to find homomorphisms between the braid groups on different number of strands:

Lemma 64.22. (Strand Deletion-Addition Lemma) Let $m_0 \in \mathbb{N}$. We consider the following subgroup of the braid group $\mathrm{B}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2})$:

$$\widetilde{\mathrm{B}}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2}) := \{X \in \mathrm{B}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2}) \mid \underset{\substack{\uparrow \\ \text{permutation that we} \\ \text{defined in the Braid Group} \\ \text{Permutation Lemma 64.8}}}{\epsilon(X)}(m+1) = (m+1)\}.$$

(Note that in particular the pure braid group $\mathrm{P}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2})$ is a subgroup of $\widetilde{\mathrm{B}}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2})$.) With this notation we can formulate the two statements of the lemma:

(1) The "strand addition map"

$$\begin{array}{rcl} \alpha\colon \mathrm{B}^{(1,0),\ldots,(m,0)}(\overline{B}^2_{m+1}) & \to & \widetilde{\mathrm{B}}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2}) \\ [X] & \mapsto & [X \sqcup \underbrace{(\{(m+1,0)\} \times [0,1])}_{\substack{\text{extra strand which is} \\ \text{disjoint from } X \subseteq B^2_{m+1}}}] \end{array}$$

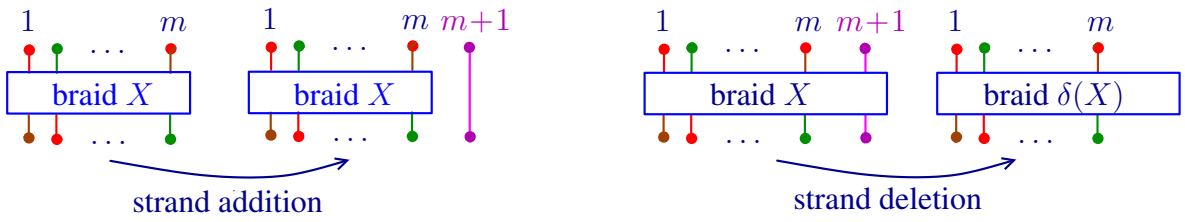
is group homomorphism and it is injective.

(2) The "strand deletion map"

$$\begin{array}{rcl} \delta\colon \widetilde{\mathrm{B}}^{(1,0),\ldots,(m,0),(m+1,0)}(\overline{B}^2_{m+2}) & \to & \mathrm{B}^{(1,0),\ldots,(m,0)}(\overline{B}^2_{m+2}) \\ [X] & \mapsto & \left[\begin{array}{l} \text{only the components of } X \text{ that} \\ \text{contain the points } (P_1,0),\ldots,(P_m,0) \end{array}\right] \end{array}$$

is a group homomorphism and it is surjective.

Proof. One can easily verify that both maps are well-defined group homomorphisms. It remains to show that the "strand addition map" α is a monomorphism and that the "strand



deletion map" δ is an epimorphism. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2_{m+1}) & \xrightarrow{\alpha} & \widetilde{\mathrm{B}}^{(1,0),\dots,(m,0),(m+1,0)}(\overline{B}^2_{m+2}) & \xrightarrow{\delta} & \mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2_{m+2}) \\ \cong\downarrow & & & & \downarrow\cong \\ \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2) & & \xrightarrow{\quad\mathrm{id}\quad} & & \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2). \end{array}$$

Note that the vertical maps are the isomorphisms from the Braid Group–Finite Radius Lemma 64.7. It follows easily from the definitions that the diagram commutes. It is now clear that α is a monomorphism and that δ is an epimorphism. ■

The above discussion lets us introduce a monomorphism $\alpha\colon \mathrm{B}_m \to \mathrm{B}_{m+1}$:

Notation 64.23. Let $m \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{B}_m := \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}^2) & \xrightarrow{=:\iota} & \mathrm{B}_{m+1} := \mathrm{B}^{(1,0),\dots,(m,0),(m+1,0)}(\mathbb{R}^2) \\ \cong\uparrow & & \cong\uparrow \\ \mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2_{m+1}) & \xhookrightarrow{\alpha} & \mathrm{B}^{(1,0),\dots,(m,0),(m+1,0)}(\overline{B}^2_{m+2}). \end{array}$$

Here the vertical maps are given by the Braid Group–Finite Radius Lemma 64.7 and the bottom horizontal monomorphism α is given by Strand Addition Lemma 64.22 (1). We refer to the monomorphism $\iota\colon \mathrm{B}_m \to \mathrm{B}_{m+1}$ that we just defined as the standard monomorphism.

Example. Let $m \in \mathbb{N}_0$ and let $i \in \{1, \dots, m-1\}$. It follows easily from Notation 64.23 that under the monomorphism $\iota\colon \mathrm{B}_m \to \mathrm{B}_{m+1}$ we have $\iota(\sigma_i) = \sigma_i$. □

64.6. Exercises for Chapter 64.

Exercise 64.1. In Definition 64.1 and throughout this chapter we restricted ourselves to the setting that Σ is a 2-dimensional smooth manifold. In principle we could also allow smooth manifolds of other dimensions. What can we say about the braid group in these more general setups?

Exercise 64.2. Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points. Given two braids X, Y in $(\Sigma, \{P_1, \dots, P_m\})$ we introduced in Definition 64.3 the product $X * Y$ by stacking X on top of Y. We could also introduce a product $X \star Y$ by stacking X below Y. What is the relationship between the groups $(\mathrm{B}^{P_1,\dots,P_m}(\Sigma), *)$ and $(\mathrm{B}^{P_1,\dots,P_m}(\Sigma), \star)$?

Exercise 64.3. Let $m \in \mathbb{N}_{\geq 2}$ and let $i, j \in \{1, \dots, m-1\}$. Give a purely topological proof that σ_i and σ_j are conjugate in B_m.

Exercise 64.4. Let $m \in \mathbb{N}_{\geq 2}$.

(a) Show that the braid group

$$\mathrm{B}_m = \left\langle \sigma_1, \ldots, \sigma_{m-1} \;\middle|\; \begin{array}{ll} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} & \text{for } i = 1, \ldots, m-2 \\ \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i & \text{for } |i-j| \geq 2 \end{array} \right\rangle$$

admits an epimorphism onto $\mathbb{Z}$.

(b) Show that B_m admits a unique epimorphism onto $\mathbb{Z}$ (up to sign).

(c) Show that every epimorphism $\mathrm{B}_m \to A$ to an abelian group A factors through the epimorphism from (a).

Exercise 64.5. What is the smallest $m \in \mathbb{N}_0$ such that the corresponding braid group $\mathrm{B}_m = \mathrm{B}^{(1,0),\ldots,(m,0)}(\mathbb{R}^2)$ is not abelian?

Exercise 64.6. Let $m, n \in \mathbb{N}_0$. Show that there exists a monomorphism

$$\mathrm{B}_m \times \mathrm{B}_n \;\to\; \mathrm{B}_{m+n}.$$

Exercise 64.7. Let Σ be a connected non-empty 2-dimensional smooth manifold. We consider the corresponding braid group $\mathrm{B}_m(\Sigma)$ from Definition 64.1. Show that for every $m \in \mathbb{N}$ there exists an epimorphism $\mathrm{B}_m(\Sigma) \to \pi_1(\Sigma)$.

Exercise 64.8. Let Σ be a connected non-empty 2-dimensional smooth manifold and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \ldots, P_m \in \Sigma \setminus \partial\Sigma$ be pairwise distinct points. Finally let $f_0, f_1 \colon \Sigma \to \Sigma'$ be two smooth embeddings into a connected 2-dimensional smooth manifold Σ'. Show that if f_0, f_1 are smoothly isotopic rel $\{P_1, \ldots, P_m\}$, then we have the equality $f_{0*} = f_{1*} \colon \mathrm{B}^{P_1,\ldots,P_m}(\Sigma) \to \mathrm{B}^{f_0(P_1),\ldots,f_0(P_m)}(\Sigma')$.

Exercise 64.9. Let $m \in \mathbb{N}$. Show that $(\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{m-1})^m$ lies in the center of the braid group B_m.

Exercise 64.10. Given $m \in \mathbb{N}$ we consider the pure braid group

$$\mathrm{P}_m \;\coloneqq\; \mathrm{P}^{(1,0),(2,0),\ldots,(m,0)}(\mathbb{R}^2)$$

from Definition 64.9. A very slight modification of the Strand Deletion-Addition Lemma 64.22 gives us an epimorphism

$$\delta \colon \mathrm{P}_m \;\to\; \mathrm{P}_{m-1}$$

which is given by deleting the last strand. Describe the isomorphism type of the kernel.

Exercise 64.11. By the Braid Group–Presentation 64.19 we know that we have an isomorphism

$$\langle \sigma_1, \sigma_2 \,|\, \sigma_1 \cdot \sigma_2 \cdot \sigma_1 = \sigma_2 \cdot \sigma_1 \cdot \sigma_1 \rangle \;\xrightarrow{\cong}\; \mathrm{B}_3.$$

Show that we have a well-defined homomorphism

$$\begin{array}{rcl} \langle \sigma_1, \sigma_2 \,|\, \sigma_1 \cdot \sigma_2 \cdot \sigma_1 = \sigma_2 \cdot \sigma_1 \cdot \sigma_1 \rangle & \to & \mathrm{SL}(2, \mathbb{Z}) \\ \sigma_1 & \mapsto & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \sigma_2 & \mapsto & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \end{array}$$

Remark. It follows from the discussion on page 1912 that this map is an epimorphism. Furthermore, with some effort one can also deduce from the discussion on page 1912 that the

kernel of this epimorphism is infinite cyclic and it is generated by $(\sigma_1 \cdot \sigma_2 \cdot \sigma_1)^4$. Alternatively we refer to [**Miln1978**, Theorem 10.5] for details.

Exercise 64.12. By the Braid Group–Presentation 64.19 we know that we have an isomorphism

$$\langle \sigma_1, \sigma_2 \,|\, \sigma_1 \cdot \sigma_2 \cdot \sigma_1 = \sigma_2 \cdot \sigma_1 \cdot \sigma_1 \rangle \xrightarrow{\cong} \mathrm{B}_3.$$

(a) Show that we have a well-defined homomorphism

$$\begin{array}{rcl} \langle x, y \,|\, x^2 = y^3 \rangle & \to & \langle \sigma_1, \sigma_2 \,|\, \sigma_1 \cdot \sigma_2 \cdot \sigma_1 = \sigma_2 \cdot \sigma_1 \cdot \sigma_1 \rangle \\ x & \mapsto & \sigma_1 \cdot \sigma_2 \cdot \sigma_1 \\ y & \mapsto & \sigma_1 \cdot \sigma_2 \end{array}$$

(b) Show that the map in (a) is an isomorphism.

Exercise 64.13. Let $m \in \mathbb{N}$. What is the relationship between the braid groups $\mathrm{B}_k(S^2)$ and the braid groups B_l?

CHAPTER 65

The Burau representation of the braid group

In the previous Chapter 64 we introduced for every $m \in \mathbb{N}_0$ the braid group B_m and we studied some basic properties of braid groups. In this chapter we will introduce the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ and we will introduce the closely related reduced Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$.

This chapter is organized as follows:

- In Section 65.1 we will introduce the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$.
- In Section 65.2 we will introduce a modification of the Burau representations, namely the *reduced* Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$.
- In Section 65.3 we will see that the Burau representations are unitary in a suitable sense.
- In Section 65.4 we will provide exercises for Chapter 65.

In the next Chapter 66 we will explore the relationship between braids and links.

65.1. **The Burau representation.** One way to understand a given group G is by finding homomorphisms from G to well-known groups, e.g. permutation groups or matrix groups.

In Section 64.2 we showed that every braid group $\mathrm{B}^{P_1,\ldots,P_m}(\Sigma)$ admits an epimorphism onto the permutation group S_m. In this section we restrict ourselves to *the* braid group $\mathrm{B}_m = \mathrm{B}^{(1,0),\ldots,(m,1)}(\mathbb{R}^2)$. The following is historically the first interesting homomorphism from braid groups to matrix groups:

Proposition 65.1. (Burau Representation Proposition) Let $m \in \mathbb{N}_{\geq 2}$. Given any $i \in \{1, \ldots, m-1\}$ we consider the matrix

$$P_i := \begin{pmatrix} \mathrm{id}_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{m-i-2} \end{pmatrix}.$$

The map

$$\mathrm{B}_m = \left\langle \sigma_1, \ldots, \sigma_{m-1} \;\middle|\; \begin{array}{ll} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} & \text{for } i = 1, \ldots, m-2 \\ \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i & \text{for } |i-j| \geq 2 \end{array} \right\rangle \to \mathrm{GL}(n, \mathbb{Z}[t^{\pm 1}])$$
$$\sigma \mapsto P_i$$

is a well-defined homomorphism.

The Burau Representation Proposition 65.1 leads us to the following definition:

Definition 65.2. Let $m \in \mathbb{N}_{m \geq 2}$. We refer to the representation

$$\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$$

from the Burau Representation Proposition 65.1 as the Burau representation.[706] We extend this definition to $m = 1$ by defining $\beta_1 \colon \mathrm{B}_1 \to \mathrm{GL}(1, \mathbb{Z}[t^{\pm 1}])$ as the unique, namely the trivial representation.

Example. We consider the pure braid that is shown in the figure below.

braid $X = \sigma_1 * \sigma_2^{-1} * \sigma_2^{-1} * \sigma_1$



We calculate that[707]

$$\begin{aligned}
\beta_3(X) &= \beta_3(\sigma_1) \cdot \beta_3(\sigma_2)^{-1} \cdot \beta_3(\sigma_2)^{-1} \cdot \beta_3(\sigma_1) \\
&= \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-1} & 1-t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-1} & 1-t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} t^2-2t+2 & t-t^2 & t-1 \\ 1-t & t & 0 \\ t^{-1}-t^{-2} & 0 & 1-t^{-1}+t^{-2} \end{pmatrix}.
\end{aligned}$$

This shows that the braid $X = \sigma_1 * \sigma_2^{-1} * \sigma_2^{-1} * \sigma_1$ represents a non-trivial element in the braid group B_3. □

Proof. First note that one can easily verify that the matrices P_i lie in $\mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$. It now follows from the Presentation–Homomorphism Lemma 64.15 together with the Braid Group–Presentation 64.19 that it suffices to prove the following claim:

Claim.

(1) For any $i \in \{1, \dots, m-2\}$ we have $P_i \cdot P_{i+1} \cdot P_i = P_{i+1} \cdot P_i \cdot P_{i+1}$.
(2) For any two $i, j \in \{1, \dots, m-1\}$ with $|i-j| \geq 2$ we have $P_i \cdot P_j = P_j \cdot P_i$.

Proof.

(1) Let $i \in \{1, \dots, m-2\}$. The desired result follows from the following elementary calculation:

$$\begin{aligned}
P_i \cdot P_{i+1} \cdot P_i &= \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-t & t-t^2 & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
P_{i+1} \cdot P_i \cdot P_{i+1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-t & t-t^2 & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

(2) This statement is essentially clear. ■

For the record we mention the following elementary lemma:

[706] The Burau representation was introduced by Werner Burau [**Bur1935**].

[707] Here we use that $\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ t^{-1} & 1-t^{-1} \end{pmatrix}$.

Lemma 65.3. (Burau Representation–Strand Addition Lemma) Let $m \in \mathbb{N}_0$. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{B}_m & \xrightarrow{\beta_m} & \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}]) \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\ \mathrm{B}_{m+1} & \xrightarrow{\beta_{m+1}} & \mathrm{GL}(m+1, \mathbb{Z}[t^{\pm 1}]). \end{array}$$

Here the left vertical map is the monomorphism from Notation 64.23.

Proof. The lemma follows immediately from the definitions. ∎

The following question arises naturally: Is the Burau representation faithful? In other words, is the homomorphism $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ a monomorphism? It turns out that the answer is complicated:

- ($m \leq 2$) As we will see in Exercise 65.1, the cases $m = 1$ and $m = 2$ are essentially trivial.
- ($m = 3$) The case $m = 3$ was first proved by Wilhelm Magnus and Ada Peluso [**MP1969**], see also [**Bir1974**, Theorem 3.15].
- ($m = 4$) The case $m = 4$ was recently proved by Vasudha Bharathram, Joan Birman and Tara Brendle [**BBB2026**]. According to [**Big2002**, Proposition 1.5] this result gives some positive evidence towards the much more interesting Question 70.21, whether the Jones polynomial detects the unknot.
- ($m \geq 5$) Stephen Bigelow [**Big1999**, Theorem 1.2] showed that the Burau representation is not faithful for $m = 5$. It follows from the Burau Representation–Strand Addition Lemma 65.3 that this implies that the Burau representation is not faithful for any $m \geq 5$. This extends earlier results in [**Mood1991**, p. 380] and [**LP1993**, Theorem 1.9]. The case $m \geq 6$ is dealt with in detail in [**KT2008**, Chapter 3.3].

65.2. **The reduced Burau representation.** In representation theory one usually tries to decompose representations into lower-dimensional representations. For example note that the Burau representation has the property that for any $X \in \mathrm{B}_m$ we have the equality $\beta_m(X) \cdot (1, \ldots, 1) = (1, \ldots, 1)$. In other words, the Burau representation restricts to the trivial 1-dimensional representation on $D := \mathbb{Z}[t^{\pm 1}] \cdot (1, \ldots, 1) \subseteq \mathbb{Z}[t^{\pm 1}]^m$. Thus the Burau descends to a representation on the $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}[t^{\pm 1}]^m / D$. This representation is called the *reduced Burau representation*.

In Exercise 65.6 we will see that the above reduced Burau representation is equivalent to the reduced Burau representation that we provide in the following proposition:

Proposition 65.4. (Reduced Burau Representation Proposition) Let $m \in \mathbb{N}$. We set

$$S := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}]).$$

There exists a unique representation

$$\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$$

such that for each $X \in \mathrm{B}_m$ we have

$$S^{-1} \cdot \beta_m(X) \cdot S \;=\; \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 \\ * & 1 \end{pmatrix}.$$

In fact this representation can be described as follows:

($m=1$) The representation β_1^{red} is the unique representation $\mathrm{B}_1 \to \mathrm{GL}(0, \mathbb{Z}[t^{\pm 1}]) = \{\mathrm{id}_0\}$.

($m=2$) The representation β_2^{red} is given by $\beta_2^{\mathrm{red}}(\sigma_1) = -t$.

($m \geq 3$) The representation β_m^{red} is given by

$$\beta_m^{\mathrm{red}}(\sigma_1) \;=\; \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \mathrm{id}_{m-3} \end{pmatrix},$$

$$\text{for } i = 2, \ldots, m-2 \text{ we have } \beta_m^{\mathrm{red}}(\sigma_i) \;=\; \begin{pmatrix} \mathrm{id}_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id}_{m-i-2} \end{pmatrix},$$

$$\beta_m^{\mathrm{red}}(\sigma_{m-1}) \;=\; \begin{pmatrix} \mathrm{id}_{m-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}.$$

Definition 65.5. Let $m \in \mathbb{N}$. We refer to the representation

$$\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \;\to\; \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$$

from the Reduced Burau Representation Proposition 65.4 as the reduced Burau representation.

Remark. The reduced Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ is rather charming, since it contains most of the essential information of the original Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$, while sticking to slightly smaller matrices. The key disadvantage of the reduced Burau representation is that there is no straightforward analog of the Burau Representation–Strand Addition Lemma 65.3. □

Proof. In this proof we consider the following subset of $\mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$:

$$G \;:=\; \left\{ \begin{pmatrix} A & 0 \\ v & 1 \end{pmatrix} \;\middle|\; A \in \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}]) \text{ and } v^{\mathrm{T}} \in \mathbb{Z}[t^{\pm 1}]^{m-1} \right\}.$$

We start out with the following claim:

Claim. For each $i \in \{1, \ldots, m-1\}$ we have $S^{-1} \cdot \beta_m(\sigma_i) \cdot S \in G$.

Proof. First note that

$$S^{-1} \;=\; \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

An elementary calculation shows that for each $i \in \{2, \dots, m-2\}$ we have

$$S^{-1} \cdot \beta_m(\sigma_i) \cdot S = \begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id}_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id}_{m-i-1} \end{pmatrix} \cdot \begin{pmatrix} \ddots & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{id}_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id}_{m-i-1} \end{pmatrix}$$

The same way we see that

$$S^{-1} \cdot \beta_m(\sigma_1) \cdot S = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \mathrm{id}_{m-2} \end{pmatrix} \quad \text{and} \quad S^{-1} \cdot \beta_m(\sigma_{m-1}) \cdot S = \begin{pmatrix} \mathrm{id}_{m-3} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

These calculations show that in all cases we have $S^{-1} \cdot \beta_m(\sigma_i) \cdot S \in G$. ⊞

By the claim we know that for each $i = 1, \dots, m-1$ we have $S^{-1} \cdot \beta_m(\sigma_i) \cdot S \in G$. Note that G is clearly a subgroup of $\mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$. It thus follows from the claim together with the Braid Group–Generating Set Proposition 64.18 that for each $X \in \mathrm{B}_m$ we have $S^{-1} \cdot \beta_m(X) \cdot S \in G$. In other words, for each $X \in \mathrm{B}_m$ there exists a unique matrix $\beta_m^{\mathrm{red}}(X) \in \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ such that

$$S^{-1} \cdot \beta_m(X) \cdot S = \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 \\ * & 1 \end{pmatrix}.$$

It follows from elementary linear algebra, and the fact that $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ is a homomorphism, that $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ is also a homomorphism. Finally note that the promised explicit description of the reduced Burau representation follows from the calculations in the proof of the claim. ■

Finally recall that on page 1409 we discussed the question whether the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ is faithful. The following slightly proposition shows in particular that the reduced Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ is faithful if and only if the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ is faithful:

Proposition 65.6. (Burau–Last Row Proposition) Let $m \in \mathbb{N}$.

(1) Let $X \in \mathrm{B}_m$. Using the notation from the Reduced Burau Representation Proposition 65.4 we write

$$S^{-1} \cdot \beta_m(X) \cdot S =: \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 \\ r & 1 \end{pmatrix}. \quad \text{We also write} \quad \beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1} =: \begin{pmatrix} a_1 \\ \vdots \\ a_{m-1} \end{pmatrix}.$$

Then[708]
$$-(1 + t + \cdots + t^{m-1}) \cdot r = \sum_{i=1}^{m-1} \cdot (1 + t + \cdots + t^{i-1}) \cdot a_i.$$

(2) We have $\ker(\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])) = \ker(\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}]))$.

Proof. [709] Let $m \in \mathbb{N}$.

(1) Let $X \in \mathrm{B}_m$. We consider the row vector $v := (1, t, t^2, \dots, t^{m-1})$.

Claim. $v \cdot \beta_m(X) \;=\; v.$

Proof. First note that it follows almost immediately from Definition 65.2 that for any $i \in \{1, \dots, m-1\}$ we have $v \cdot \beta_m(\sigma_i) = v$. The claim now follows from the Braid Group–Generating Set Proposition 64.18. ⊞

We set $w := v \cdot S = (1, 1+t, 1+t+t^2, \dots, 1+t+\cdots+t^{m-1})$. Note that

$$w \cdot \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 \\ r & 1 \end{pmatrix} \;=\; v \cdot S \cdot S^{-1} \cdot \beta_m(X) \cdot S \;=\; v \cdot \beta_m(X) \cdot S \underset{\text{by the claim}}{\overset{\uparrow}{=}} v \cdot S \;=\; w.$$

If we subtract w from both sides we obtain the equality

$$w \cdot \begin{pmatrix} \beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1} & 0 \\ r & 0 \end{pmatrix} \;=\; 0. \quad \text{In other words:} \quad w \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_{m-1} \\ r \end{pmatrix} \;=\; 0.$$

Since $w = (1, 1+t, \dots, 1+t+\cdots+t^{m-1}, 1+t+\cdots+t^{m-1})$ this gives us the desired result.

(2) This statement follows easily from (1). ■

65.3. **The Burau representations are unitary.** In this last section we will see that the Burau representation and its reduced sibling are unitary, in the sense that they preserve some inner product. To formulate the result precisely we need to introduce the following piece of notation:

Notation 65.7. Given $m \in \mathbb{N}_0$ we consider the following matrix in $\mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$:

$$\Omega_m \;:=\; \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1-t & 1 & 0 & \dots & 0 \\ 1-t & 1-t & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & 1-t & \dots & 1 \end{pmatrix}.$$

Lemma 65.8. (Burau Unitarian Lemma) For any $m \in \mathbb{N}_0$ and any $X \in \mathrm{B}_m$ we have the equality[710] [711]

$$\overline{\beta_m(X)} \cdot \Omega_m \cdot \beta_m(X)^{\mathrm{T}} \;=\; \Omega_m.$$

Proof. Note that given an invertible matrix $A \in \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ we have $\overline{A} \cdot \Omega_m \cdot A^{\mathrm{T}} = \Omega_m$ if and only if we have $\overline{A}^{-1} \cdot \Omega_m \cdot (A^{-1})\mathrm{T} = \Omega_m$. It follows from this observation that we only need to prove the desired equality for the generators σ_i, $i = 1, \dots, m-1$. But for

[708] This statement is a little technical. But it will be an essential ingredient in the proof of the Markov Invariance Lemma 67.3 (2).

[709] Here we follow the argument from [**KT2008**, Lemma 3.10].

[710] Here, given a matrix A over $\mathbb{Z}[t^{\pm 1}]$ we denote by $\overline{A}$ the matrix that we obtain by applying the ring homomorphism $t \mapsto t^{-1}$ to all entries.

[711] This result shows that the Burau representation β_m preserves the sesquilinear form on $\mathbb{Z}[t^{\pm 1}]^m$ that is given by $(v, w) \mapsto \overline{w} \cdot \Omega_m \cdot v^{\mathrm{T}}$ where we think of v and w as row vectors and where β acts by right multiplication on the row vectors.

these the equality can be verified easily by hand. For more details we refer to [**KT2008**, Lemma 3.1]. ■

Next we turn to the reduced setting:

Notation 65.9. Given $m \in \mathbb{N}$ we consider the following matrix in $\mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$:

$$\Omega_m^{\mathrm{red}} := \begin{pmatrix} 1+t & -1 & 0 & \dots & 0 \\ -t & 1+t & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & -t & 1+t & -1 \\ 0 & \dots & 0 & -t & 1+t \end{pmatrix}.$$

We conclude this chapter with the following analog of the Burau Unitarian Lemma 65.8:

Lemma 65.10. (Reduced Burau Unitarian Lemma) For any $m \in \mathbb{N}_0$ and any braid $X \in \mathrm{B}_m$ we have the equality

$$\overline{\beta_m^{\mathrm{red}}(X)} \cdot \Omega_m^{\mathrm{red}} \cdot \beta_m^{\mathrm{red}}(X)^{\mathrm{T}} = \Omega_m^{\mathrm{red}}.$$

Proof. As in the Reduced Burau Representation Proposition 65.4 we set

$$S := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}]).$$

By the Reduced Burau Representation Proposition 65.4 we know that for each $X \in \mathrm{B}_m$ we have

$$S^{-1} \cdot \beta_m(X) \cdot S = \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 \\ * & 1 \end{pmatrix}.$$

Note that for all $X \in \mathrm{B}_m$ we have

$$\begin{aligned} \overline{S^{-1} \cdot \beta_m(X) \cdot S} \cdot S^{-1} \cdot \Omega_m \cdot (S^{-1})^{\mathrm{T}} \cdot (S^{-1} \cdot \beta_m(X) \cdot S)^{\mathrm{T}} &= \\ = S^{-1} \cdot \overline{\beta_m(X)} \cdot S \cdot S^{-1} \cdot \Omega_m \cdot (S^{-1})^{\mathrm{T}} \cdot (S^{\mathrm{T}} \cdot \beta_m(X)^{\mathrm{T}} \cdot (S^{-1})^{\mathrm{T}} &= \\ = S^{-1} \cdot \overline{\beta_m(X)} \cdot \Omega_m \cdot \beta_m(X)^{\mathrm{T}} \cdot (S^{-1})^{\mathrm{T}} &= \\ = S^{-1} \cdot \Omega_m \cdot (S^{-1})^{\mathrm{T}}. & \end{aligned}$$

$\uparrow$ by the Burau Unitarian Lemma 65.8

The desired statement follows easily from the above discussion together with the calculation

$$\begin{aligned} S^{-1} \cdot \Omega_m \cdot (S^{-1})^{\mathrm{T}} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1-t & 1 & 0 & \dots & 0 \\ 1-t & 1-t & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & 1-t & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+t & -1 & 0 & \dots & 0 \\ -t & 1+t & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & -t & 1+t & -1 \\ 0 & \dots & 0 & -t & 1+t \end{pmatrix}. \end{aligned}$$

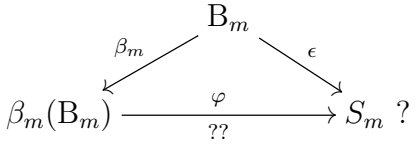
■

65.4. Exercises for Chapter 65.

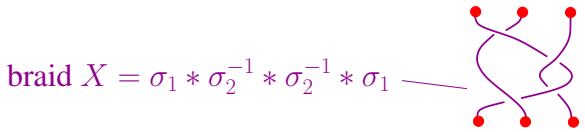
Exercise 65.1. Show that the Burau representation $\beta_2 \colon \mathrm{B}_2 \to \mathrm{GL}(2, \mathbb{Z}[t^{\pm 1}])$ is a monomorphism.

Exercise 65.2. Show that there exists an injective homomorphism $\mathrm{B}_3 \to \mathrm{GL}(2, \mathbb{C})$.

Exercise 65.3. Let $m \in \mathbb{N}_{\geq 2}$. We consider the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z})$ and we consider the homomorphism $\epsilon \colon \mathrm{B}_m \to S_m$ from the Braid Group–Permutation Lemma 64.8. Does there exist a homomorphism $\varphi \colon \beta_m(\mathrm{B}_m) \to S_m$ such that the following diagram commutes:



Exercise 65.4. As on page 1408 we consider the braid $X = \sigma_1 * \sigma_2^{-1} * \sigma_2^{-1} * \sigma_1$. Show that it represents an element of infinite order in B_3.
Hint. On page 1408 we calculated $\beta_3(X) \in \mathrm{GL}(3, \mathbb{Z}[t^{\pm 1}])$. Show that this matrix has infinite order. You could for example apply a ring homomorphism $\mathbb{Z}[t^{\pm 1}] \to \mathbb{C}$ to get to a matrix in $\mathrm{GL}(3, \mathbb{C})$.



Exercise 65.5. Let $m \in \mathbb{N}_{\geq 3}$.

(a) Show that the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ does not send the center of the braid group to the center of $\mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$.
(b) Show that the reduced Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ sends the center of the braid group to the center of $\mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$.

Remark. Evidently you should use the Braid Group–Center Proposition 64.21.

Exercise 65.6. Let $m \in \mathbb{N}$.

(a) Show that for every $X \in \mathrm{B}_m$ we have $\beta_m(X) \cdot (1, \ldots, 1) = (1, \ldots, 1)$.
(b) We consider the $\mathbb{Z}[t^{\pm 1}]$-module $D := \mathbb{Z}[t^{\pm 1}] \cdot (1, \ldots, 1) \subseteq \mathbb{Z}[t^{\pm 1}]^m$. Show that the Burau representation $\beta_m \colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\cong} \mathrm{Aut}_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}]^m)$ descends to a representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{Aut}_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}]^m / D)$.
(c) Show that $\mathbb{Z}[t^{\pm 1}]^m / D$ is a free $\mathbb{Z}[t^{\pm 1}]$-module of rank $m-1$.
(d) Provide a basis for the free $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}[t^{\pm 1}]^m / D$ such that the corresponding representation $\widetilde{\beta}_m \colon \mathrm{B}_m \to \mathrm{Aut}_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}]^m / D) \xrightarrow{\cong} \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ is the reduced Burau representation from the Reduced Burau Representation Proposition 65.4.

CHAPTER 66

Closures of braids and Markov's Theorem

In Chapter 64 we introduced the braid groups $\mathrm{B}_m = \mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2)$. These have a distinctly "knotty" flavor. In this chapter we will make this connection between braids and links precise. In particular we will show that every oriented link is the closure of a braid.

This chapter is organized as follows:

- In Section 66.1 we will see that every oriented link is the closure of a braid.
- In Section 66.2 we will state the Markov Theorem 66.9 which says that the closures of two braids are smoothly isotopic if and only if the braids are related by a sequence of conjugations and (de-) stabilizations.
- In Section 66.3 we will provide exercises for Chapter 66.

In the next Chapter 67 we will use the results of this chapter, together with the reduced Burau representation, to introduce an invariant of oriented links, namely the Alexander polynomial.

66.1. **Closures of braids.** Braids remind us of knots and links. In fact in this section we will see that there is indeed a close relationship between braids and oriented links.

In Notation 64.6 we introduced the braid group $\mathrm{B}_m := \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)$. An argument as in the proof of the Braid Group–Finite Radius Lemma 64.7 shows that the inclusion $\iota\colon \mathbb{R}_{>0}\times\mathbb{R}\to\mathbb{R}^2$ induces an isomorphism

$$\iota_*\colon \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}_{>0}\times\mathbb{R}) \;\to\; \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}^2)$$

of groups. For the purpose of this chapter it is better to work with the braid group to the left. So we make the following identification:

Identification 66.1. Given any $m\in\mathbb{N}_0$ we make the identification

$$\mathrm{B}_m \;=_{\mathrm{i}}\; \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(\mathbb{R}_{>0}\times\mathbb{R}).$$

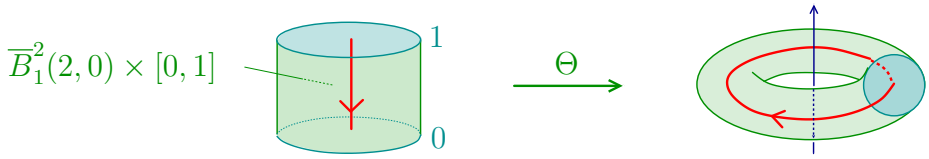
The following notation already explains why we made the above identification:

Notation 66.2. Throughout this chapter we will consider the map

$$\begin{array}{rcl}\Theta\colon \mathbb{R}_{>0}\times\mathbb{R}\times[0,1] & \to & \mathbb{R}^3\\ ((x,z),t) & \mapsto & \underbrace{\begin{pmatrix}\cos(2\pi t) & -\sin(2\pi t) & 0\\ \sin(2\pi t) & \cos(2\pi t) & 0\\ 0 & 0 & 1\end{pmatrix}}_{\text{rotation around the } z\text{-axis}}\cdot\begin{pmatrix}x\\0\\z\end{pmatrix}.\end{array}$$

Loosely speaking this map describes the rotation of the "right xz-half plane" around the z-axis.

Now we can introduce the key player of this chapter:



Definition 66.3.

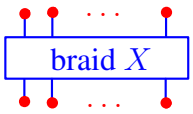
(1) Let X be an m-strand braid in $(\mathbb{R}_{>0} \times \mathbb{R}, \{(1,0),(2,0),\dots,(m,0)\})$. We define the closure of the braid X as the link[712]

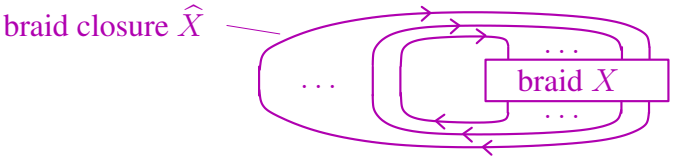
$$\widehat{X} \;:=\; \Theta(X) \;\subseteq\; \mathbb{R}^3.$$

We give $\widehat{X} = \Theta(X)$ the orientation that is induced by the convention that the strands of the braid X go "downward". It follows easily from the definitions that equivalent braids lead to smoothly isotopic links. In other words we have a well-defined map

$$\begin{array}{ccc} \mathrm{B}_m & \to & \texttt{Link}^{\mathrm{or}}(\mathbb{R}^3) \\ [X] & \mapsto & [\widehat{X}]. \end{array}$$

(2) If $\widehat{X}$ is smoothly isotopic to an oriented link L, then we refer to X as a braid representative of L.

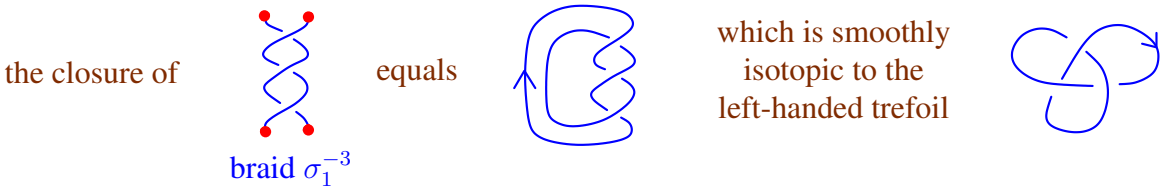




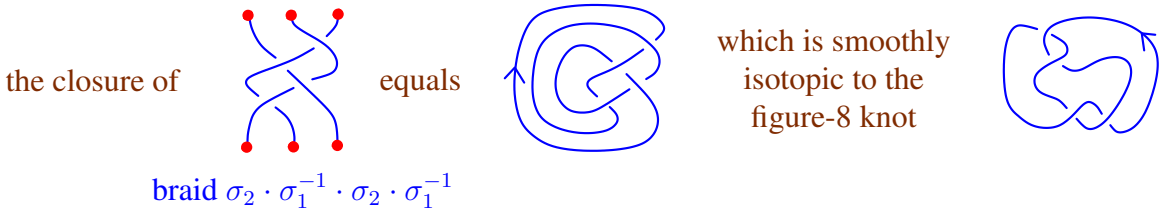
Example.

(1) Clearly the closure of the trivial m-strand braid is just the m-component trivial link.

(2) In the following figure we show a braid whose closure is the left-handed trefoil.



In the following figure we show a braid whose closure is the figure-8 knot.

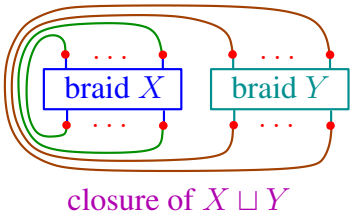


(3) Let $X \in \mathrm{B}_m$ and $Y \in \mathrm{B}_n$.

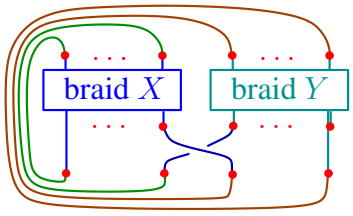
(a) We consider the "obvious" juxtaposition $X \sqcup Y \in \mathrm{B}_{m+n}$. Then $\widehat{X \sqcup Y}$ is clearly the split union of $\widehat{X}$ and $\widehat{Y}$.

[712] It follows easily from Condition (3) in Definition 64.1 that $\Theta(X)$ is indeed a smooth submanifold.

(b) We consider the braid $(X \sqcup Y) * \sigma_m^{\pm 1} \in \mathrm{B}_{m+n}$. One can easily verify that the closure of $(X \sqcup Y) * \sigma_m^{\pm 1}$ is given by the connected sum of the two knots $\widehat{X}$ and $\widehat{Y}$, as introduced in Definition 63.6.



closure of $X \sqcup Y$



closure of $(X \sqcup Y) * \sigma_m$

More examples of braid closures will be considered in Exercise 66.2 and 66.7. □

The following result was first proved, at least implicitly, by James Alexander [**Ale1923a**]:

Proposition 66.4. (Alexander–Braid Closure Proposition) Every oriented link $L \subseteq \mathbb{R}^3$ is smoothly isotopic to the closure of a braid.

Remark. There are many other proofs of the Alexander–Braid Closure Proposition 66.4 in the literature, see e.g. [**Bir1974**, Theorem 2.1], [**Kaw1996**, Theorem 1.2.2] and [**BZH2014**, Proposition 2.12]. Our proof is a smooth analog of the proof of [**KT2008**, Theorem 2.3]. For practical purposes Vogel's algorithm [**Voge1990**] is usually considered the most efficient. □

Before we head to the proof of the Alexander–Braid Closure Proposition 66.4 let us point out one almost immediate corollary:

Proposition 66.5. (Link Set Cardinality Proposition) The set $\mathtt{Link}^{\mathrm{or}}(\mathbb{R}^3)$ of smooth isotopy classes of oriented links is countable.

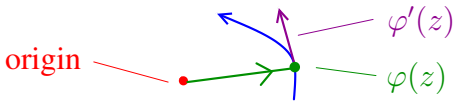
Proof of Proposition 66.5 assuming Proposition 66.4. It follows from the Braid Group–Generating Set Proposition 64.18 that the braid groups B_m are countable. It now follows from the Alexander–Braid Closure Proposition 66.4 together with the Countability Lemma 1.37 (2) and (4) that $\mathtt{Link}^{\mathrm{or}}(\mathbb{R}^3)$ is countable. ■

The proof of the Alexander–Braid Closure Proposition 66.4 will require the remainder of this section. In the next Section 66.2 we will state the Markov Theorem 66.9 which tells us when two braids have smoothly isotopic closures.

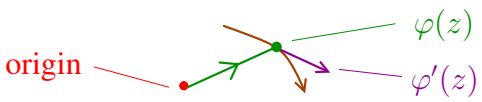
We start out by introducing some language:

Definition 66.6. Let $\varphi\colon S^1 \to \mathbb{R}^2$ be an immersion and let $I \subseteq S^1$ be some subset.

- We say φ is counterclockwise on I if for each $z \in I$ we have $\det(\varphi(z)\ \varphi'(z)) > 0$.[713]
- We say φ is clockwise on I if for each $z \in I$ we have $\det(\varphi(z)\ \varphi'(z)) < 0$.



counterclockwise



clockwise

[713]Note that by the discussion on page 803 this condition is equivalent to saying that $\varphi(z)$ and $\varphi'(z)$ form a positive basis of $\mathbb{R}^2$.

The proof of the Alexander–Braid Closure Proposition 66.4 will make use of the following lemma:

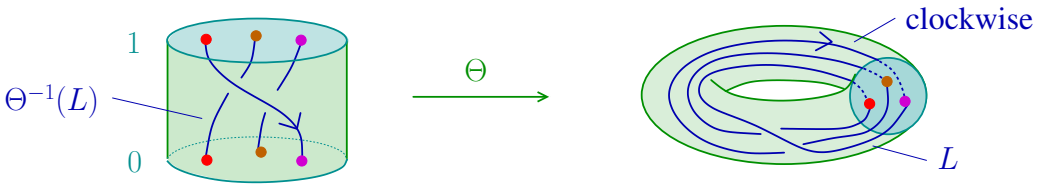
Lemma 66.7. (Link–Braid Closure Criterion Lemma) Let $L \subseteq \mathbb{R}^3$ be a link that admits a link diagram D such that the associated immersion $\varphi\colon \{1,\dots,m\} \times [0,1] \to \mathbb{R}^2$ is clockwise. Then L is smoothly isotopic to the closure of a braid.



Proof of the Link–Braid Closure Criterion Lemma 66.7. As in Notation 66.2 we consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{R}_{>0} \times \mathbb{R} \times [0,1] & \to & \mathbb{R}^3 \\ ((x,z),t) & \mapsto & \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) & 0 \\ \sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}. \end{array}$$

Recall that we assume that L corresponds to a diagram that is clockwise. It follows easily that for each $t \in [0,1]$ the half-space $\Theta(\mathbb{R}_{>0} \times \mathbb{R} \times \{t\})$ is actually transverse to L.[714]



It now follows fairly easily that there exist $P_1,\dots,P_m \in \mathbb{R}_{>0} \times \mathbb{R} \times [0,1]$ such that the preimage $Y := \Theta^{-1}(L) \subseteq \overline{B}^2 \times [0,1]$ is a braid in $(\mathbb{R}_{>0} \times \mathbb{R}, \{P_1,\dots,P_m\})$. Furthermore, the hypothesis that the diagram is clockwise implies that the orientation conventions match.[715]

Similar to the proof of the Braid Group–Functor Lemma 64.5 we now apply the Homogeneous-via-Diffeotopies Proposition 44.6 and we see that there exists a diffeotopy $F\colon \mathbb{R}_{>0} \times \mathbb{R} \times [0,1] \to \mathbb{R}_{\geq 0} \times \mathbb{R}$ with $F_0 = \mathrm{id}$ and with $F_1(P_i) = (i,0)$ for $i = 1,\dots,m$. Finally we set $X := \mathrm{diag}(F_1, \mathrm{id}_{[0,1]})(Y) \subseteq \mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}_{>0} \times \mathbb{R} \times [0,1])$. Clearly $\Theta(X)$ is smoothly isotopic to $\Theta(Y) = L$. ∎

Sketch of proof of Alexander–Braid Closure Proposition 66.4. Let $L \subseteq \mathbb{R}^3$ be an oriented link. By the Link–Braid Closure Criterion Lemma 66.7 it suffices to prove the following claim:

Claim. There exists a link diagram D for the link L such that the associated immersion $\varphi\colon \{1,\dots,m\} \times S^1 \to \mathbb{R}^2$ is clockwise.

Proof. We proceed in several stages:

(1) By the Link Diagram Existence Proposition 112.7 we know that L admits a diagram. After a translation we can arrange that the origin does not lie on the diagram.

(2) After possibly applying another smooth isotopy we can find a decomposition of the domain $\{1,\dots,m\} \times S^1$ into finitely many non-trivial circle segments such that in

[714] In fact this conclusion also holds if the diagram is counterclockwise.

[715] This statement does not hold if the diagram is counterclockwise.

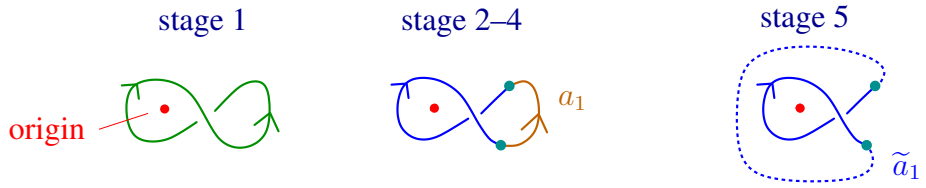
the interior on each circle segment the immersion φ is either clockwise ("blue") or counterclockwise ("brown"). We can also arrange that the crossings are in the interiors of the intervals.

(3) We divide each "brown" counterclockwise circle segment into finitely many circle segments such that each new circle segment contains at most one crossing.

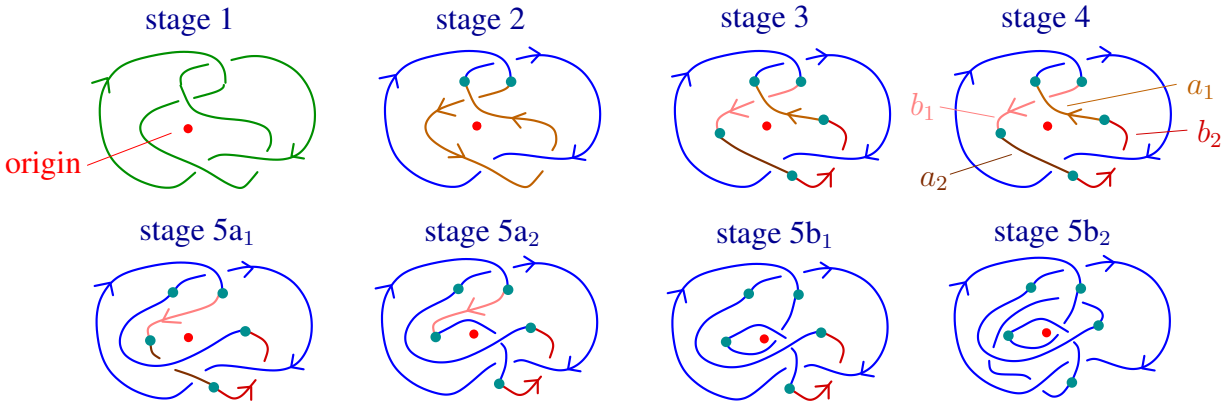
(4) (a) We pick an ordering $a_1, \dots, a_k$ of the brown counterclockwise circle segments which contain an overcrossing.
(b) We pick an ordering $b_1, \dots, b_l$ of the brown counterclockwise circle segments that contain an undercrossing.

(5) (a) We take the brown overcrossing segments $a_1, \dots, a_k$ and drag them across the origin (and over all other pieces of the diagram, except for the earlier modified overcrossing brown segments $\widetilde{a}_1, \dots, \widetilde{a}_i$) to turn them into clockwise segments $\widetilde{a}_1, \dots, \widetilde{a}_k$.
(b) We take the undercrossing brown segments $b_1, \dots, b_l$ and drag them across the origin (and under all other pieces of the diagram, except for the earlier modified undercrossing brown segments) to turn them into clockwise segments.

We first give a simplified example which shows some of the above stages:



All the five stages are illustrated in the following more complicated figure:



We have absolutely no intention to fill in the details for the above steps. ■

66.2. **The Markov Theorem.** It is not hard to come up with examples of non-equivalent braids that lead to smoothly isotopic links. Playing around with examples one is eventually lead to the following concepts:

Definition 66.8. In the following let $m \in \mathbb{N}$. Furthermore let X be an m-strand braid in $(\mathbb{R}_{>0} \times \mathbb{R}, \{(1,0), \dots, (m,0)\})$.

(M1) Given any m-strand braid Y in $(\mathbb{R}_{>0} \times \mathbb{R}, \{(1,0), \dots, (m,0)\})$ we refer to $Y * X * Y^{-1}$ as a conjugation of X.

(M2) Let $\iota(X) \in \mathrm{B}_m$ be the $(m+1)$-braid that, as in the Strand Deletion-Addition Lemma 64.22 (1), is given by adding a trivial $(m+1)$-st strand.
- We refer to the $(m+1)$-strand braid $\iota(X) * \sigma_m$ as a positive stabilization of X.
- We refer to the $(m+1)$-strand braid $\iota(X) * \sigma_m^{-1}$ as a negative stabilization of X.

These two operations, and their inverses, are called Markov moves.

positive stabilization of X | negative stabilization of X



$X * \sigma_m$ | $X * \sigma_m^{-1}$

The following is the key theorem that relates the theory of braids to the theory of oriented links:

Theorem 66.9. (Markov Theorem) Let $X \in \mathrm{B}_m$ and $X' \in \mathrm{B}_{m'}$ be two braids. Then

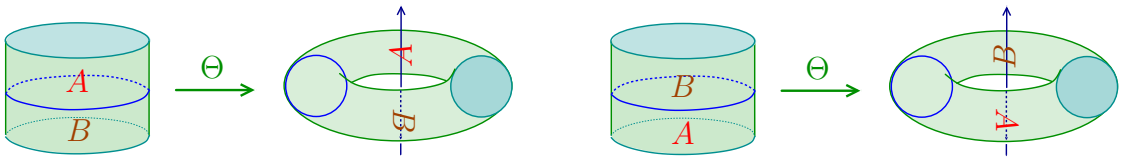
$\widehat{X}$ is smoothly isotopic to $\widehat{X'}$ $\iff$ X and X' are related by a sequence of smooth isotopies and the Markov moves
(M1) conjugation
(M2) stabilization
and their inverses.

Proof. We first provide the proof of the much easier "$\Leftarrow$"-direction:

(M1) We start out with the following claim:

Claim. Let A, B be two braids in $\mathrm{B}^{(1,0),\dots,(m,0)}(\mathbb{R}_{>0} \times \mathbb{R} \times [0,1])$. Then $\widehat{A * B}$ is smoothly isotopic to $\widehat{B * A}$.

Proof. If one looks carefully at Definition 64.3, Notation 66.2 and Definition 66.3 one sees that $\widehat{A * B}$ differs from $\widehat{B * A}$ by rotation by π around the z-axis.



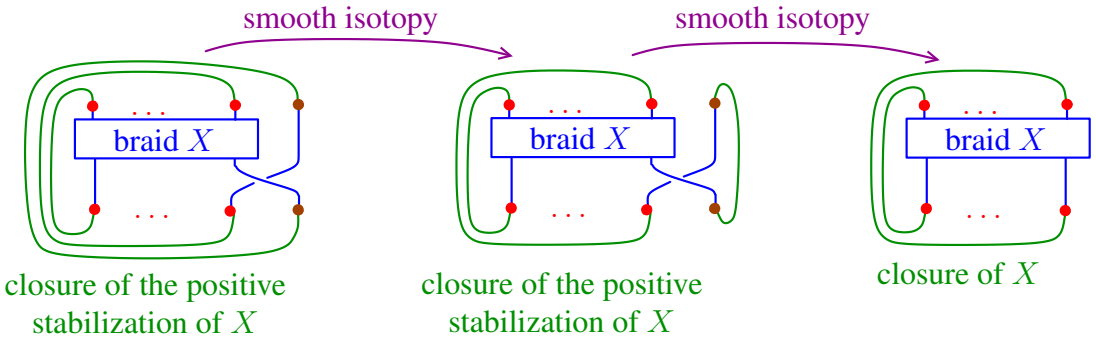
It follows that $\widehat{A * B}$ is smoothly isotopic to $\widehat{B * A}$. ⊞

Now let $X, Y \in \mathrm{B}_m$. We see that we have the following equalities in $\mathtt{Link}^{\mathrm{or}}(\mathbb{R}^3)$:

$$\begin{array}{lcl} \text{closure of } Y * X * Y^{-1} & = & \text{closure of } (Y * X) * Y^{-1} \\ & = & \text{closure of } (Y * Y^{-1}) * X \\ & & \uparrow \text{ by the Braid Group Lemma 64.4} \end{array} \qquad \begin{array}{lcl} & & \overset{\text{by the claim}}{\downarrow} \\ & = & \text{closure of } Y^{-1} * (Y * X) \\ & = & \text{closure of } X. \\ & & \uparrow \text{ by the Braid Group Lemma 64.4} \end{array}$$

(M2) Let $X \in \mathrm{B}_m$. We consider the stabilizations $X_\pm = \iota(X) * \sigma_m^{\pm}$. It follows from the discussion on page 1416 that $\widehat{X_\pm}$ is smoothly isotopic to the connected sum of $\widehat{X}$ with a trivial knot, which is (as we saw on page 1391) smoothly isotopic to $\widehat{X}$ itself.

Alternatively the following figure should make it clear that the closure of a stabilization is smoothly isotopic to the closure of the original braid.



The much more interesting "$\Rightarrow$"-direction of the theorem was first formulated by Andrey Markov [**Mark1936**, p. 75]. It seems like the first detailed proof of the theorem was given in [**Bir1974**, Theorem 2.3]. Alternative textbook proofs are given in [**MK1999**, Chapter 9], [**Mant2004**, Chapter 11], [**BZH2014**, Theorem 10.23] and [**KT2008**, Theorem 2.1]. ■

66.3. Exercises for Chapter 66.

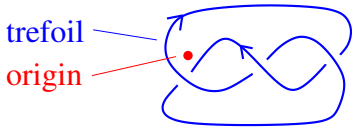
Exercise 66.1. Let $m \in \mathbb{N}$ and let $\beta \in \mathrm{B}_m$. Let $\sigma \in S_m$ be the corresponding permutation from the Braid Group–Permutation Lemma 64.8. Recall that every permutation can be written uniquely as a product of cycles. Show that the number of cycles for σ equals the number of components of the closure $\widehat{\beta}$.

Exercise 66.2. In the figure below we show three knots.

(a) In each case provide an explicit braid X such that the closure $\widehat{X}$ is smoothly isotopic to the knot.
(b) In each case compute the Alexander polynomial from Definition 67.2.



Exercise 66.3. Apply the algorithm in the proof of the Alexander–Braid Closure Proposition 66.4 to the diagram of the trefoil that is shown below.



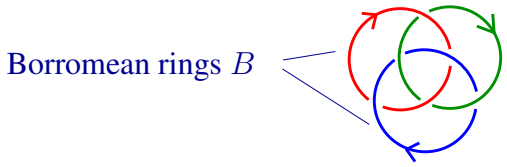
Exercise 66.4. Let $n \in \mathbb{N}$ be odd. We consider the torus knot $T(2, n)$ from Definition 111.13.

(a) Provide an explicit braid X such that the closure $\widehat{X}$ is smoothly isotopic to the torus knot $T(2, n)$.
(b) Compute the Alexander polynomial from Definition 67.2.

Exercise 66.5. Let $L \subseteq \mathbb{R}^3$ be an oriented link. We consider the mirror L^{mir} and the reverse L^{rev}. In this exercise we consider the Alexander polynomial from Definition 67.2.

(a) What is the relationship between $\Delta_L(t)$ and $\Delta_{L^{\text{mir}}}(t)$?
(b) What is the relationship between $\Delta_L(t)$ and $\Delta_{L^{\text{rev}}}(t)$?

Exercise 66.6. Provide an explicit braid X such that the closure $\widehat{X}$ is smoothly isotopic to the Borromean rings.



Exercise 66.7. Let $p, q \in \mathbb{Z}$ be coprime. Find an explicit braid representative for the torus knot $T(p, q)$ from Definition 111.13.

Exercise 66.8. Let $L \subseteq \mathbb{R}^3$ be an oriented link. We pick an m-strand braid B such that $\widehat{B}$ is smoothly isotopic to L. We now consider the Burau representation instead of the reduced Burau representation. Thus we consider

$$f(t) \;=\; (-1)^{m-1} \cdot \frac{1-t}{1-t^m} \cdot \det(\beta_m(B) - \mathrm{id}) \;\in\; \mathbb{Z}[t^{\pm 1}].$$

(a) Show that $f(t)$ is independent of the choice of the braid B.
(b) Why do we not consider the link invariant $f_L(t) := f(t)$?

Exercise 66.9. We consider the Alexander polynomial $\Delta_L(t)$ from Definition 67.2.

(a) Show that the Alexander polynomial of a split union of two non-empty links is zero.
(b) Show that the Alexander polynomials for oriented knots is multiplicative under the connected sum operation.

Hint. Use the discussion on page 1416.

Exercise 66.10. Use the Markov Theorem 66.9 to give a new definition of the linking number of an oriented 2-component link, that only makes use of concepts from braids.

Exercise 66.11. Is the figure-8 knot smoothly isotopic to a torus knot $T(p, q)$?

Exercise 66.12. Let $L \subseteq \mathbb{R}^3$ be an oriented link and let X be a braid whose closure $\widehat{X}$ is smoothly isotopic to L. We consider the mirror L^{mir} from Definition 62.11 and the reverse L^{rev} from Definition 62.10.

(a) Find a braid whose closure is smoothly isotopic to L^{mir}.
(b) Find a braid whose closure is smoothly isotopic to L^{rev}.

Exercise 66.13. Let $\Phi\colon \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ be the ring homomorphism that is given by $t \mapsto -1$. Given $X \in \mathrm{B}_m$ we consider the abelian group

$$H(X) \;:=\; \mathbb{Z}^m / (\Phi(\beta(X)) - \mathrm{id}_m) \cdot \mathbb{Z}^m.$$

Show that the isomorphism type of the abelian group $H(X)$ is invariant under conjugation of X and stabilization of X.

Remark. Note that this exercise, together with the Markov Theorem 66.9, implies that for an oriented link $L \subseteq \mathbb{R}^3$ we get a well-defined isomorphism type $H(L) := H(\widehat{X})$ where $X \in \mathrm{B}_m$ is chosen such that $L = \widehat{X}$.

Exercise 66.14. Given an oriented link L we define its *braid index* $\beta(L)$ as follows:

$$\beta(L) \;:=\; \min\{m \,|\, \text{there exists a braid } X \in \mathrm{B}_m \text{ with } \widehat{X} = L\}.$$

(a) Let $L \subseteq \mathbb{R}^3$ be an oriented link. We consider the abelian group $H(L)$ from Exercise 66.13. Show that the abelian group $H(L)$ is generated by $\beta(L)$ elements.

(b) Show that there exist oriented knots with arbitrarily large braid index.

CHAPTER 67

The Alexander polynomial via the Burau representation

In the previous Chapter 64 we introduced the braid groups $\mathrm{B}_m = \mathrm{B}^{(1,0),\dots,(m,0)}(\overline{B}^2)$. These have a distinctly "knotty" flavor. In this chapter we will make this connection between braids and links precise. In particular we will show that every oriented link is the closure of a braid. This fact, together with the reduced Burau representation, can be used to (re-) introduce an invariant of oriented links, namely the Alexander polynomial.

This chapter is organized as follows:

- In Section 67.1 we will use the reduced Burau representation and the Markov Theorem 66.9 to (re-) introduce an invariant of oriented links in $\mathbb{R}^3$, namely the Alexander polynomial.
- In Section 67.2 we will provide a proof of the Reduced Burau Representation–Link Invariant Theorem 67.1.
- In Appendix 67.3 we will see that one can also calculate the Alexander polynomial via the original Burau representation.
- In Section 67.4 we will provide exercises for Chapter 67.

In the next Chapter 68 we will use the Burau representation to define the Conway polynomial of an oriented link. The Conway polynomial is a very clever refinement of the Alexander polynomial.

67.1. **The Alexander polynomial via the Burau representation.** The Markov Theorem 66.9 gives us a new approach to defining invariants of oriented links: We "just" need to find an invariant of braids that is invariant under the Markov moves, i.e. that is invariant under conjugations and stabilizations.

The following theorem shows that this idea can work:

Theorem 67.1. (Reduced Burau Representation–Link Invariant Theorem) Let $L \subseteq \mathbb{R}^3$ be an oriented link and let X be an m-strand braid in $\mathbb{R}_{>0} \times \mathbb{R}$ such that its closure $\widehat{X}$ is smoothly isotopic to the oriented link L. We consider the reduced Burau representation $\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \to \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ from Definition 65.5. Then

$$\Delta_L(t) \;:=\; (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}) \;\in\; \mathbb{Z}[t^{\pm 1}]$$

is, up to multiplication by a power of t, independent of the choice of the braid X.[716]

Proof. We outsource the proof of the theorem to the next Section 67.2. ■

The Reduced Burau Representation–Link Invariant Theorem 67.1 allows us to introduce the following definition:

[716] In Definition 68.5 we will see how we can fine-tune the definition to dispose of the "power of t"-indeterminacy.

Definition 67.2. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. By the Alexander–Braid Closure Proposition 66.4 we can pick an m-strand braid X in $\mathbb{R}_{>0} \times \mathbb{R}$ such that $\widehat{X}$ is smoothly isotopic to L. We refer to

$$\Delta_L(t) \;:=\; (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}) \;\in\; \mathbb{Z}[t^{\pm 1}]$$

as the (single-variable) Alexander polynomial of L. As we discussed in the Burau Representation–Link Invariant Theorem 67.1, this is an invariant of the smooth isotopy classes of links, that is well-defined up to multiplication by a power of t.[717]

Example.

(1) (a) Let L be the trivial m-component link. As we discussed on page 1416, it is the closure of the trivial m-strand braid T. We see that

$$\Delta_L(t) \overset{\text{by Definition 67.2}}{\downarrow} = (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\underbrace{\beta_m^{\mathrm{red}}(T)}_{=\mathrm{id}_{m-1}} - \mathrm{id}_{m-1})$$

$$= (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(0_{(m-1)\times(m-1)}) \qquad = \begin{cases} 1, & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases}$$

(b) Now let U be the unknot. But this time we think of it as the closure of $\sigma_1^{-1} \in \mathrm{B}_2$. Then we see that

$$\Delta_L(t) \overset{\text{by Definition 67.2}}{\downarrow} = (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\underbrace{\beta_m^{\mathrm{red}}(\sigma_1^{-1})}_{=-t^{-1}} - \mathrm{id}_{m-1}) \;=\; -\tfrac{1-t}{1-t^2} \cdot (-t^{-1} - 1) \;=\; t^{-1}.$$

Thus we do not get exactly the same answer as in (a), but the answers only differ by a power of t.

(2) Let K be the left-handed trefoil. On page 1416 we showed that K is the closure of the 2-strand braid $\sigma_1^{-3} \in \mathrm{B}_2$. We can now calculate the Alexander polynomial of K:

$$\Delta_K(t) \underset{\text{by Definition 67.2}}{\uparrow} = (-1)^1 \cdot \tfrac{1-t}{1-t^2} \cdot \det(\beta_2^{\mathrm{red}}(\sigma_1^{-3}) - \mathrm{id}_1) \underset{\text{by the Reduced Burau Representation Proposition 65.4}}{\uparrow} = -\tfrac{1-t}{1-t^2} \cdot \det((-t)^{-3} - 1)$$

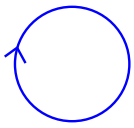
$$= -\tfrac{1-t}{1-t^2} \cdot (-t^{-3} - 1) \qquad \underset{\text{by elementary algebra}}{\uparrow} = t^{-1} - t^{-2} + t^{-3}.$$

(3) Let K be the figure-8 knot. On page 1416 we showed that K is the closure of the 3-strand braid $\sigma_2 \cdot \sigma_1^{-1} \cdot \sigma_2 \cdot \sigma_1^{-1} \in \mathrm{B}_3$. This time we see that

$$\begin{aligned}
\Delta_K(t) &\overset{\text{by Definition 67.2}}{=} (-1)^2 \cdot \tfrac{1-t}{1-t^3} \cdot \det(\beta_3^{\mathrm{red}}(X) - \mathrm{id}_2) \\
&= \tfrac{1-t}{1-t^3} \cdot \det\left(\beta_3^{\mathrm{red}}(\sigma_2) \cdot \beta_3^{\mathrm{red}}(\sigma_1)^{-1} \cdot \beta_3^{\mathrm{red}}(\sigma_2) \cdot \beta_3^{\mathrm{red}}(\sigma_1)^{-1} - \mathrm{id}_2\right) \\
&\overset{\text{since } \beta_3^{\mathrm{red}} \text{ is a homomorphism}}{=} \tfrac{1-t}{1-t^3} \cdot \det\left(\begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix} \cdot \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix} \cdot \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix}^{-1} - \mathrm{id}_2\right) \\
&\overset{\text{by the Reduced Burau Representation Proposition 65.4}}{=} -\tfrac{1}{t^2} \cdot (t^2 - 3t + 1) \\
&\overset{\text{by elementary algebra which is best left to a computer algebra system}}{=} t^2 + 3t - 1.
\end{aligned}$$

(since the Alexander polynomial is only well-defined up to a power of t)

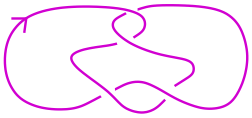
Note that the above calculations show that the Alexander Polynomial for the trivial knot, the trefoil and the figure-8 knot are pairwise different (even if we take into account a multiplication by a power of t).



$\Delta_K(t) = 1$



$\Delta_K(t) = t^2 - t + 1$



$\Delta_K(t) = t^2 - 3t + 1$

It follows that the three knots are pairwise not smoothly isotopic. Also note that in all cases we observe an uncanny similarity to earlier calculations of (other flavors of) the Alexander polynomial. □

67.2. Proof of the Reduced Burau Representation–Link Invariant Theorem 67.1.

The following lemma contains all the hard bits of the proof of the Reduced Burau Representation–Link Invariant Theorem 67.1.

Lemma 67.3. (Markov Invariance Lemma) Let $m \in \mathbb{N}$ and let $X \in \mathrm{B}_m$.

(1) $(-1)^{m-1} \cdot \frac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id})$ is unchanged under conjugation.

(2) (a) A positive stabilization does not change $(-1)^{m-1} \cdot \frac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id})$.

(b) A negative stabilization changes $(-1)^{m-1} \cdot \frac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id})$ by a factor t^{-1}.

Proof of Theorem 67.1 assuming Lemma 67.3. The theorem is an immediate consequence of the Markov Invariance Lemma 67.3 together with the Markov Theorem 66.9. ■

Proof of the Markov Invariance Lemma 67.3 (1). Let $X, Y \in \mathrm{B}_m$. The desired statement follows from the following elementary calculation:

$$\begin{array}{rcl} \det(\beta_m^{\mathrm{red}}(Y \cdot X \cdot Y^{-1}) - \mathrm{id}_m) & \overset{\text{since } \beta_m^{\mathrm{red}} \text{ is a homomorphism}}{\downarrow} & \\ & = & \det(\beta_m^{\mathrm{red}}(Y) \cdot \beta_m^{\mathrm{red}}(X) \cdot \beta_m^{\mathrm{red}}(Y)^{-1} - \mathrm{id}_m) \\ & = & \det(\beta_m^{\mathrm{red}}(Y) \cdot (\beta_m^{\mathrm{red}}(X) - \mathrm{id}_m) \cdot \beta_m^{\mathrm{red}}(Y)^{-1}) \\ & = & \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_m). \\ & \underset{\text{by the multiplicativity of the determinant}}{\uparrow} & \end{array}$$

■

Proof of the Markov Invariance Lemma 67.3 (2a). [718] Let $X \in \mathrm{B}_m$. In the following we consider its positive stabilization $X_+ = \iota(X) \cdot \sigma_m \in \mathrm{B}_{m+1}$. Note that we need to prove the following:

Statement.

$$\frac{1-t^m}{1-t} \cdot \det(\beta_{m+1}^{\mathrm{red}}(X_+) - \mathrm{id}_m) \;=\; -\frac{1-t^{m+1}}{1-t} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}).$$

Before we turn to the proof of the statement let us introduce some notation. Let $l \in \mathbb{N}$.

(i) We write

$$S_l \;:=\; \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \text{Note that } S_l^{-1} \;=\; \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) Given $Y \in \mathrm{B}_l$ we write, using the notation from the Reduced Burau Representation Proposition 65.4:

$$S_l^{-1} \cdot \beta_l(Y) \cdot S_l \;=:\; \begin{pmatrix} \beta_l^{\mathrm{red}}(Y) & 0 \\ r(Y) & 1 \end{pmatrix}.$$

Now let $X \in \mathrm{B}_m$. Evidently we need to relate $\beta_{m+1}^{\mathrm{red}}(X_+)$ to $\beta_m^{\mathrm{red}}(X)$. This goal is achieved with our first claim:

Claim 1. We have the following equality:

$$\begin{pmatrix} \beta_{m+1}^{\mathrm{red}}(X_+) & 0 \\ r(X_+) & 1 \end{pmatrix} \;=\; \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 & 0 \\ r(X) & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id}_{m-2} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1-t & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Proof. Since $X_+ = \iota(X) * \sigma_m$ it makes sense to first consider $\iota(X)$. In this setting we see that

$$\beta_{m+1}(\iota(X)) \;\underset{\text{by the Burau Representation Strand Addition Lemma 65.3}}{\uparrow}\; \begin{pmatrix} \beta_m(X) & 0 \\ 0 & 1 \end{pmatrix} \;\underset{\text{by (ii)}}{\uparrow}\; \begin{pmatrix} S_m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_m^{\mathrm{red}}(X) & 0 & 0 \\ r(X) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_m^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that

[718] Here we follow the argument from [**KT2008**, Lemma 3.12].

$$\begin{pmatrix} \beta^{\mathrm{red}}_{m+1}(X_+) & 0 \\ r(X_+) & 1 \end{pmatrix} \overset{\text{follows from (ii)}}{\downarrow} = S^{-1}_{m+1} \cdot \beta_{m+1}(X_+) \cdot S_{m+1} \overset{\text{since } X_+ = \iota(X)\cdot\sigma_m}{\downarrow} = S^{-1}_{m+1} \cdot \beta_{m+1}(\iota(X)) \cdot \beta_{m+1}(\sigma_m) \cdot S_{m+1}$$

$$= S^{-1}_{m+1} \cdot \begin{pmatrix} S_m & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 & 0 \\ r(X) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} S^{-1}_m & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id}_{m-1} & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} \cdot S_{m+1}$$

$\uparrow$ by the above and by Definition 65.2 of $\beta_{m+1}(\sigma_m)$

$$= \begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 & 0 \\ r(X) & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id}_{m-2} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1-t & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$\uparrow$ by a direct calculation of the blue and purple products

⊞

We now write

$$\begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 \\ r(X) & 1 \end{pmatrix} =: \begin{pmatrix} A_{(m-2)\times(m-2)} & B_{(m-2)\times 1} & 0 \\ C_{1\times(m-2)} & T & 0 \\ P & Q & 1 \end{pmatrix}.$$

With this notation we can now express $\beta^{\mathrm{red}}_m(X) - \mathrm{id}_{m-1}$ and $\beta^{\mathrm{red}}_{m+1}(X) - \mathrm{id}_{m-1}$:

Claim 2.

(a) $$\beta^{\mathrm{red}}_m(X) - \mathrm{id}_{m-1} = \begin{pmatrix} A - \mathrm{id}_{m-2} & B \\ C & T-1 \end{pmatrix},$$

(b) $$\beta^{\mathrm{red}}_{m+1}(X_+) - \mathrm{id}_m = \begin{pmatrix} A - \mathrm{id}_{m-2} & B & t\cdot B \\ C & T-1 & t\cdot T \\ P & Q & t\cdot Q - t - 1 \end{pmatrix}.$$

Proof. The first statement is clear. The second statement follows from the observation that Claim 1 implies that

$$\begin{pmatrix} \beta^{\mathrm{red}}_{m+1}(X_+) & 0 \\ r(X_+) & 1 \end{pmatrix} = \begin{pmatrix} A & B & 0 & 0 \\ C & T & 0 & 0 \\ P & Q & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id}_{m-2} & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1-t & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & B & t\cdot B & 0 \\ C & T & t\cdot T & 0 \\ P & Q & t\cdot Q - t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

⊞

Claim 3.

$$-\tfrac{1-t^m}{1-t} \cdot (P\ Q) = \sum_{i=1}^{m-1} \tfrac{1-t^{i+1}}{1-t} \cdot i\text{-th row of } \begin{pmatrix} A - \mathrm{id}_{m-2} & B \\ C & T-1 \end{pmatrix}.$$

Proof. Recall that by (ii) and Claim (2a) we have

$$S^{-1}_m \cdot \beta_m(X) \cdot S_m = \begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 \\ r(X) & 0 \end{pmatrix} \quad \text{and} \quad \beta^{\mathrm{red}}_m(X) - \mathrm{id}_{m-1} = \begin{pmatrix} A - \mathrm{id}_{m-2} & B \\ C & T-1 \end{pmatrix}.$$

Also recall that by definition we have $(P,\, Q) = r(X)$. Thus we see that the claim is just a reformulation of the Burau–Last Row Proposition 65.6 applied to $\beta^{\mathrm{red}}_m(X)$. ⊞

We now see that

$$\begin{aligned}
\frac{1-t^m}{1-t}\cdot\det(\beta^{\mathrm{red}}_{m+1}(X_+)-\mathrm{id}_m) &\overset{(a)}{=} \frac{1-t^m}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & t\cdot B \\ C & T-1 & t\cdot T \\ P & Q & t\cdot Q-t-1\end{pmatrix}\\
&\overset{(b)}{=} \frac{1-t^m}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & 0 \\ C & T-1 & t \\ P & Q & -t-1\end{pmatrix}\\
&\overset{(c)}{=} \frac{1-t^m}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & 0 \\ C & T-1 & t \\ 0 & 0 & \frac{1-t^m}{1-t}\cdot(-t-1)+t\cdot\frac{1-t^{m-1}}{1-t}\end{pmatrix}\\
&\overset{(d)}{=} -\frac{1-t^{m+1}}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & 0 \\ C & T-1 & t \\ 0 & 0 & 1\end{pmatrix}\\
&\overset{(e)}{=} -\frac{1-t^{m+1}}{1-t}\cdot\det(\beta^{\mathrm{red}}_m(X)-\mathrm{id}_{m-1})
\end{aligned}$$

And here are the justifications:

(a) This is just Claim 2a.
(b) We just subtract t times the second-to-last column from the last column.
(c) First we multiply the last row by $\frac{1-t^m}{1-t}$. Then for $i=1,\dots,m=1$ we add $\frac{1-t^{i+1}}{1-t}$ times the i-th row to the last row. It follows from Claim 3 that in the last row all but the last entry are now zero.
(d) We just use that $\frac{1-t^m}{1-t}\cdot(-t-1)+t\cdot\frac{1-t^{m-1}}{1-t}=-\frac{1-t^{m+1}}{1-t}$.
(e) This is just Claim 2b. ■

Proof of the Markov Invariance Lemma 67.3 (2b). [719] Let $X\in \mathrm{B}_m$. In the following we consider its positive stabilization $X_-=\iota(X)\cdot\sigma_m\in\mathrm{B}_{m+1}$. Note that we need to prove the following:

Statement.

$$\frac{1-t^m}{1-t}\cdot\det(\beta^{\mathrm{red}}_{m+1}(X_-)-\mathrm{id}_m) \;=\; -\frac{1-t^{m+1}}{1-t}\cdot\det(\beta^{\mathrm{red}}_m(X)-\mathrm{id}_{m-1}).$$

The proof is of course very similar to the proof for the positive calculation. But since the statement is not entirely symmetric (this time we pick a power of t), we want to go through the main steps again.

Claim 1. We have the following equality:

$$\begin{pmatrix}\beta^{\mathrm{red}}_{m+1}(X_-) & 0\\ r(X_-) & 1\end{pmatrix} = \begin{pmatrix}\beta^{\mathrm{red}}_m(X) & 0 & 0\\ r(X) & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\mathrm{id}_{m-2} & 0 & 0 & 0\\ 0 & 1 & t & 0\\ 0 & 0 & 1-t & 1\\ 0 & 0 & 1 & 1\end{pmatrix}.$$

Proof. Note that

[719] Here we follow the argument from [**KT2008**, Lemma 3.12].

$$\begin{pmatrix} \beta^{\mathrm{red}}_{m+1}(X_-) & 0 \\ r(X_-) & 1 \end{pmatrix} \overset{\text{follows from (ii)}}{\downarrow} = S_{m+1}^{-1}\cdot\beta_{m+1}(X_-)\cdot S_{m+1} \overset{\text{since } X_- = \iota(X)\cdot\sigma_m^{-1}}{\downarrow} = S_{m+1}^{-1}\cdot\beta_{m+1}(\iota(X))\cdot\beta_{m+1}(\sigma_m^{-1})\cdot S_{m+1}$$

$$= S_{m+1}^{-1}\cdot\begin{pmatrix} S_m & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 & 0 \\ r(X) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} S_m^{-1} & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \mathrm{id}_{m-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-1} & 1-t^{-1} \end{pmatrix}\cdot S_{m+1}$$

$\uparrow$ by the calculation on page 1427 and by Definition 65.2 of $\beta_{m+1}(\sigma_m^{-1})$

$$= \begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 & 0 \\ r(X) & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \mathrm{id}_{m-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -t^{-1} & t^{-1} \\ 0 & 0 & t^{-1} & 1-t^{-1} \end{pmatrix}.$$

⊞

Again we write

$$\begin{pmatrix} \beta^{\mathrm{red}}_m(X) & 0 \\ r(X) & 1 \end{pmatrix} =: \begin{pmatrix} A_{(m-2)\times(m-2)} & B_{(m-2)\times 1} & 0 \\ C_{1\times(m-2)} & T & 0 \\ P & Q & 1 \end{pmatrix}.$$

Claim 2.

(a) $$\beta^{\mathrm{red}}_m(X) - \mathrm{id}_{m-1} = \begin{pmatrix} A-\mathrm{id}_{m-2} & B \\ C & T-1 \end{pmatrix},$$

(b) $$\beta^{\mathrm{red}}_{m+1}(X_-) - \mathrm{id}_m = \begin{pmatrix} A-\mathrm{id}_{m-2} & B & t\cdot B \\ C & T-1 & t\cdot T \\ P & Q & t\cdot Q - t - 1 \end{pmatrix}.$$

Proof. The first statement is clear. The second statement follows from the observation that Claim 1 implies that

$$\begin{pmatrix} \beta^{\mathrm{red}}_{m+1}(X_-) & 0 \\ r(X_-) & 1 \end{pmatrix} = \begin{pmatrix} A & B & 0 & 0 \\ C & T & 0 & 0 \\ P & Q & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \mathrm{id}_{m-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -t^{-1} & t^{-1} \\ 0 & 0 & t^{-1} & 1-t^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A & B & 0 & -B \\ C & T & 0 & -T \\ P & Q & 0 & -Q \\ 0 & 0 & t^{-1} & 1-t^{-1} \end{pmatrix}.$$

⊞

Claim 3. $$-\tfrac{1-t^m}{1-t}\cdot(P\ Q) = \sum_{i=1}^{m-1}\tfrac{1-t^{i+1}}{1-t}\cdot i\text{-th row of }\begin{pmatrix} A-\mathrm{id}_{m-2} & B \\ C & T-1 \end{pmatrix}.$$

Proof. This claim is unchanged from Claim 3 on page 1428. ⊞

We now see that

$$\begin{aligned}
\frac{1-t^m}{1-t}\cdot\det(\beta^{\mathrm{red}}_{m+1}(X_-)-\mathrm{id}_m) &\overset{(a)}{=} \frac{1-t^m}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & t\cdot B\\ C & T-1 & t\cdot T\\ P & Q & t\cdot Q-t-1\end{pmatrix}\\
&\overset{(b)}{=} \frac{1-t^m}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & 0\\ C & T-1 & t\\ P & Q & -t-1\end{pmatrix}\\
&\overset{(c)}{=} \frac{1-t^m}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & 0\\ C & T-1 & t\\ 0 & 0 & \frac{1-t^m}{1-t}\cdot(-t-1)+t\cdot\frac{1-t^{m-1}}{1-t}\end{pmatrix}\\
&\overset{(d)}{=} -\frac{1-t^{m+1}}{1-t}\cdot\det\begin{pmatrix} A-\mathrm{id}_{m-2} & B & 0\\ C & T-1 & t\\ 0 & 0 & 1\end{pmatrix}\\
&\overset{(e)}{=} -\frac{1-t^{m+1}}{1-t}\cdot\det(\beta^{\mathrm{red}}_m(X)-\mathrm{id}_{m-1})
\end{aligned}$$

And here are the justifications:

(a) This is just Claim 2a.
(b) We just subtract t times the second-to-last column from the last column.
(c) First we multiply the last row by $\frac{1-t^m}{1-t}$. Then for $i=1,\dots,m=1$ we add $\frac{1-t^{i+1}}{1-t}$ times the i-th row to the last row. It follows from Claim 3 that in the last row all but the last entry are now zero.
(d) We just use that $\frac{1-t^m}{1-t}\cdot(-t-1)+t\cdot\frac{1-t^{m-1}}{1-t}=-\frac{1-t^{m+1}}{1-t}$.
(e) This is just Claim 2b. ■

67.3. **Appendix: The Alexander polynomial via the Burau representation.** In Definition 67.2 we used the reduced Burau representation to define the Alexander polynomial of an oriented link in $\mathbb{R}^3$. In this appendix we will see that one can also calculate the Alexander polynomial in terms of the original Burau representation.

To formulate the result we will need the following piece of Definition 2.20:

Notation 67.4. Given a ring R, a matrix $A\in \mathrm{M}(m\times m,R)$ and $i,j\in\{1,\dots,m\}$ we denote by $A_{\widehat{i},\widehat{j}}$ the matrix that is obtained from A by removing the i-th row and the j-th column.

We can now formulate the main result of this appendix:

Proposition 67.5. (Alexander Polynomial-via-Burau Representation Proposition) Let L be an oriented non-empty link and let X be an m-strand braid in $\mathbb{R}_{>0}\times\mathbb{R}$ such that $\widehat{X}$ is smoothly isotopic to L. Furthermore let $i,j\in\{1,\dots,m\}$. Then (up to multiplication by a power of t) we have the equality

$$\Delta_L(t) \;=\; (-1)^{m-1+i+j}\cdot\det((\beta_m(X)-\mathrm{id}_m)_{\widehat{i},\widehat{j}}).$$

Remark. One can also use the right-hand side of Alexander Polynomial-via-Burau Representation Proposition 67.5 to define the Alexander polynomial. This approach is worked out in [**MK1999**, p. 170]. □

All the hard work for the proof of the Alexander Polynomial-via-Burau Representation Proposition 67.5 is done in the following technical lemma:

Lemma 67.6. (Burau–Reduced Burau–Determinant Lemma) In the following let $m \in \mathbb{N}_0$ and let $X \in \mathrm{B}_m$. For any $i, j \in \{1, \dots, m\}$ we have the equality

$$\det((\beta_m(X) - \mathrm{id}_m)_{\widehat{i},\widehat{j}}) \;=\; (-1)^{i+j} \cdot t^{i-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}).$$

Proof of Proposition 67.5 assuming Lemma 67.6. Let L be an oriented non-empty link and let X be an m-strand braid in $\mathbb{R}_{>0} \times \mathbb{R}$ such that $\widehat{X}$ is smoothly isotopic to L. Furthermore let $i, j \in \{1, \dots, m\}$. Then we have the following equalities:

$$\Delta_L(t) \;\underset{\text{see Definition 67.2}}{\overset{\uparrow}{=}}\; (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}) \;\underset{\text{by the Burau–Reduced Burau Determinant Lemma 67.6}}{\overset{\uparrow}{=}}\; (-1)^{m-1+i+j} \cdot \det((\beta_m(X) - \mathrm{id}_m)_{\widehat{i},\widehat{j}}).$$

■

Proof of the Burau–Reduced Burau–Determinant Lemma 67.6. [720] First note that the case $m = 1$ follows easily from the fact, pointed out in Footnote 29, that the determinant of a (0×0)-matrix is 1.

So we now assume $m \geq 2$. Let $X \in \mathrm{B}_m$. Throughout the remainder of this proof we write

$$\begin{aligned} A &:= \beta_m(X) - \mathrm{id}_m \\ C &:= \beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}. \end{aligned}$$

With this notation we need to prove that

$$\det(A_{\widehat{i},\widehat{j}}) \;=\; (-1)^{i+j} \cdot t^{i-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(C).$$

Claim 1. We consider

$$u := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}[t^{\pm 1}]^m \qquad \text{and} \qquad v := (1 \;\; t \;\; t^2 \;\; \dots \;\; t^{m-1}) \in \mathbb{Z}[t^{\pm 1}]^{1 \times m}.$$

With this notation we have $A \cdot u = 0$ and $v \cdot A = 0$.

Proof. We set

$$P := \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}.$$

Clearly

$$P \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad (t^{i-1} \;\; t^i) \cdot P \;=\; (t^{i-1} \;\; t^i).$$

It follows easily from Definition 65.2 that

$$\beta_m(\sigma_i) \cdot u \;=\; u \qquad \text{and} \qquad v \cdot \beta_m(\sigma_i) \;=\; v.$$

By the Braid Group–Generating Set Proposition 64.18 we know that the σ_i generate B_m. It thus follows from the above that for our $X \in \mathrm{B}_m$ we have

$$\beta_m(X) \cdot u \;=\; u \qquad \text{and} \qquad v \cdot \beta_m(X) \;=\; v.$$

By definition of $A = \beta_m(X) - \mathrm{id}_m$ this implies the desired equalities

$$A \cdot u \;=\; 0 \qquad \text{and} \qquad v \cdot A \;=\; 0.$$

⊞

Next we introduce some notation:

- We set

[720] This argument is due to Ziang Li.

$$M \;:=\; S^{-1}\cdot A\cdot S \;\underset{\text{Definition 65.5}}{\uparrow}\; \begin{pmatrix} C & 0 \\ w & 0 \end{pmatrix}$$

where $w \in \mathbb{Z}[t^{\pm 1}]^{1\times(m-1)}$.

- For $k \in \{1,\dots,m\}$ we set

$$\begin{aligned} q_k &:= 1 + t + \cdots + t^{k-1} \\ z &:= (q_1 \;\; q_2 \;\; \dots \;\; q_m). \end{aligned}$$

Claim 2. We have the following equality in $\mathbb{Q}(t)$:

$$q_m \cdot w \;=\; -z\cdot C.$$

Proof. First note that we have the following equalities:

$$\begin{aligned} (z\cdot C + q_m\cdot w \;\; 0) \;&=\; \underbrace{(z \;\; q)}_{=v\cdot S}\cdot \underbrace{\begin{pmatrix} C & 0 \\ w & 0 \end{pmatrix}}_{=M} \;=\; v\cdot S\cdot M \\ &=\; v\cdot \underbrace{(S\cdot M\cdot S^{-1})}_{=A}\cdot S \;=\; v\cdot A\cdot S \;\underset{\text{by Claim 1}}{\uparrow}\; 0. \end{aligned}$$

We have thus shown that $z\cdot C + q_m\cdot w = 0$. ⊞

Claim 3. We consider the adjugate $\operatorname{adj}(M)$ of the matrix M as introduced in Definition 2.20. We have the equality

$$\operatorname{adj}(M) \;=\; \frac{\det(C)}{q_m}\cdot \mathfrak{e}_m \cdot (q_1 \;\; q_2 \;\; \dots \;\; q_m).$$

Proof. Since the last column of M is zero we see that all but the last row of the adjugate $\operatorname{adj}(M)$ are indeed zero. We now turn to the calculation of the last row. The promised statement follows from the following two calculations:

- Let $j \in \{1,\dots,m-1\}$. We denote the rows of C by $C_1,\dots,C_{m-1}$. With this notation we have the following equalities:

$$\begin{aligned} \det(M_{\hat{j},\hat{m}}) \;&\overset{(1)}{=}\; (-1)^{m+j}\cdot \det\begin{pmatrix} C_1 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_{m-1} \\ w \end{pmatrix} \;\overset{(2)}{=}\; -(-1)^{m+j}\cdot \frac{1}{q_m}\cdot \sum_{k=1}^{m-1} q_k \cdot \det\begin{pmatrix} C_1 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_{m-1} \\ C_k \end{pmatrix} \\ &\overset{(3)}{=}\; -(-1)^{m+j}\cdot \frac{q_j}{q_m}\cdot \det\begin{pmatrix} C_1 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_{m-1} \\ C_j \end{pmatrix} \;\overset{(4)}{=}\; -\frac{q_j}{q_m}\cdot \det\begin{pmatrix} C_1 \\ \vdots \\ C_{j-1} \\ C_j \\ C_{j+1} \\ \vdots \\ C_{m-1} \end{pmatrix} \;=\; \frac{q_j}{q_m}\cdot \det(C). \end{aligned}$$

Here are the justifications:
(1) This follows from the definition of $M = \begin{pmatrix} C & 0 \\ w & 0 \end{pmatrix}$ and Definition 2.20 of the adjugate.
(2) This follows from Claim 2.
(3) This follows from the fact that for $k \neq j$ two rows are the same.

(4) The extra sign appears from the permutation of two rows.

- Finally note that by definition we have $\operatorname{adj}(M)_{m,m} = \det(C)$. ⊞

Claim 4. We have the equality

$$\operatorname{adj}(A) = \frac{\det(C)}{q_m} \cdot \underbrace{u \cdot v}_{(m \times m)\text{-matrix}}.$$

Proof. We have the following equalities:

$$\begin{aligned} \operatorname{adj}(A) &\overset{\text{definition of } M}{=} \operatorname{adj}(S \cdot M \cdot S^{-1}) \\ &= \frac{\det(C)}{q_m} \cdot \underbrace{S \cdot \mathfrak{e}_m}_{=u} \cdot \underbrace{(q_1 \ q_2 \ \ldots \ q_m) \cdot S^{-1}}_{=v} \\ &\overset{\text{follows from Claim 3}}{=} \frac{\det(C)}{q_m} \cdot u \cdot v. \end{aligned}$$

$$\overset{\text{follows from elementary linear algebra and the fact that } \det(S)=1}{=} S \cdot \operatorname{adj}(M) \cdot S^{-1}$$

⊞

We now see that

$$\begin{aligned} (-1)^{i+j} \cdot \det(A_{\widehat{i},\widehat{j}}) &\overset{\text{by Definition 2.20 of the adjugate}}{=} \operatorname{adj}(A)_{j,i} \\ &= \frac{\det(C)}{q_m} \cdot t^{i-1} \\ &\overset{\text{by definition of } u, v}{=} \end{aligned}$$

$$\begin{aligned} &\overset{\text{by Claim 4}}{=} \frac{\det(C)}{q_m} \cdot (u \cdot v)_{j,i} \\ &= \det(C) \cdot \tfrac{1-t}{1-t^m} \cdot t^{i-1}. \\ &\overset{\text{since } q_m \cdot (1-t) = 1 - t^m}{=} \end{aligned}$$

$$\overset{\text{by definition of the } (m \times m)\text{-matrix } u \cdot v}{=} \frac{\det(C)}{q_m} \cdot (u_j \cdot v_i)$$

■

67.4. Exercises for Chapter 67.

Exercise 67.1. Let $m \in \mathbb{N}$ and let $\beta \in \mathrm{B}_m$. Let $\sigma \in S_m$ be the corresponding permutation from the Braid Group–Permutation Lemma 64.8. Recall that every permutation can be written uniquely as a product of cycles. Show that the number of cycles for σ equals the number of components of the closure $\widehat{\beta}$.

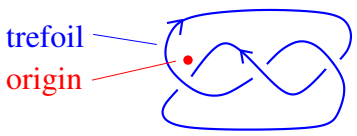
Exercise 67.2. In the figure below we show three knots.

(a) In each case provide an explicit braid X such that the closure $\widehat{X}$ is smoothly isotopic to the knot.

(b) In each case compute the Alexander polynomial from Definition 67.2.



Exercise 67.3. Apply the algorithm in the proof of the Alexander–Braid Closure Proposition 66.4 to the diagram of the trefoil that is shown below.



Exercise 67.4. Let $n \in \mathbb{N}$ be odd. We consider the torus knot $T(2,n)$ from Definition 111.13.

(a) Provide an explicit braid X such that the closure $\widehat{X}$ is smoothly isotopic to the torus knot $T(2,n)$.

(b) Compute the Alexander polynomial from Definition 67.2.

Exercise 67.5. Does there exist a braid $X \in \overline{B}_2$ such that the closure $\widehat{X}$ is smoothly isotopic to the figure-8 knot.

Exercise 67.6. Let $L \subseteq \mathbb{R}^3$ be an oriented link. We consider the mirror L^{mir} from Definition 62.11 and the reverse L^{rev} from Definition 62.10. In this exercise we consider the Alexander polynomial from Definition 67.2.

(a) What is the relationship between $\Delta_L(t)$ and $\Delta_{L^{\text{mir}}}(t)$?

(b) What is the relationship between $\Delta_L(t)$ and $\Delta_{L^{\text{rev}}}(t)$?

Remark. Make use of Exercise 66.12.

Exercise 67.7. Compute the Alexander polynomial of the Borromean rings.
Remark. Make use of your solution to Exercise 66.6.

Exercise 67.8. Let $p, q \in \mathbb{Z}$ be coprime. Find an explicit braid representative for the torus knot $T(p,q)$ from Definition 111.13 and compute the Alexander polynomial.

Exercise 67.9. Let $L \subseteq \mathbb{R}^3$ be an oriented link. We pick an m-strand braid X such that $\widehat{X}$ is smoothly isotopic to L. We now consider the Burau representation instead of the reduced Burau representation. Thus we consider

$$f(t) \;=\; (-1)^{m-1} \cdot \frac{1-t}{1-t^m} \cdot \det(\beta_m(B) - \mathrm{id}_m) \;\in\; \mathbb{Z}[t^{\pm 1}].$$

(a) Show that $f(t)$ is independent of the choice of the braid X.

(b) Why do we not consider the link invariant $f_L(t) := f(t)$?

Exercise 67.10. We consider the Alexander polynomial $\Delta_L(t)$ from Definition 67.2.

(a) Show that the Alexander polynomial of a split union of two non-empty links is zero, i.e. show that for any two oriented non-empty links L_1 and L_2 we have

$$\Delta_{L_1 \sqcup L_2}(t) \;=\; 0.$$

(b) Show that the Alexander polynomials for oriented knots is multiplicative under the connected sum operation, i.e. show that for any two oriented knots K_1 and K_2 we have

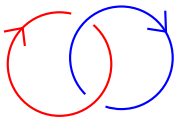
$$\Delta_{K_1 \# K_2}(t) \;=\; \Delta_{K_1}(t) \cdot \Delta_{K_2}(t).$$

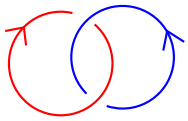
Hint. Use the discussion on page 1416.

Exercise 67.11. Use the Markov Theorem 66.9 to give a new definition of the linking number of an oriented 2-component link, that only makes use of concepts from braids.

Exercise 67.12. Is the figure-8 knot smoothly isotopic to a torus knot $T(p,q)$?

Exercise 67.13. As on page 1383 we consider the two Hopf links H^+ and H^- in $\mathbb{R}^3$ that are shown in the figure below. Compute their Alexander polynomials and show that the two links are not smoothly isotopic.

positive Hopf link H^- 

negative Hopf link H^+ 

CHAPTER 68

The Conway polynomial via the Burau representation

Recall that in Definition 67.2 we used the family of reduced Burau representations $\{\beta_m^{\mathrm{red}}\colon \mathrm{B}_m \to \mathrm{GL}(m-1,\mathbb{Z}[t^{\pm 1}])\}_{m\in\mathbb{N}}$ to define the Alexander polynomial $\Delta_L(t)$ of any oriented link $L \subseteq \mathbb{R}^3$. In this chapter we will introduce a remarkable refinement of the Alexander polynomial, namely the Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$. As we will see, the Conway polynomial satisfies a "skein relation". Later in Chapter 69 we will see that the Conway polynomial is in fact completely characterized by the value of the unknot and the skein relation.

This chapter is organized as follows:

- In Section 68.1 we will use the reduced Burau representation to introduce the Conway polynomial.
- In Section 68.2 we will formulate the Alexander-Conway Skein Theorem 68.7 which says that the Conway polynomial satisfies a "skein relation".
- In Section 68.3 we will prove the Alexander-Conway Skein Theorem 68.7.
- In Appendix 68.4 we will provide the elementary proof of the Rewriting Polynomial Lemma 68.4.
- In Section 68.5 we will provide exercises for Chapter 68.

In the next Chapter 69 we will provide a characterization of the Conway polynomial by two simple axioms, namely the value for the unknot and the skein relation.

68.1. **Definition of the Conway polynomial.** We start out by introducing some notation:

Notation 68.1. Let $m \in \mathbb{N}$. It follows from the Presentation–Homomorphism Lemma 64.15 that the map

$$\begin{array}{rcl} \epsilon\colon \mathrm{B}_m = \left\langle \sigma_1,\dots,\sigma_{m-1} \;\middle|\; \begin{array}{ll} \sigma_i\cdot\sigma_{i+1}\cdot\sigma_i = \sigma_{i+1}\cdot\sigma_i\cdot\sigma_{i+1} & \text{for } i=1,\dots,m-2 \\ \sigma_i\cdot\sigma_j = \sigma_j\cdot\sigma_i & \text{for } |i-j|\geq 2 \end{array} \right\rangle & \to & \mathbb{Z} \\ \sigma_i & \mapsto & 1 \end{array}$$

is a well-defined homomorphism to $\mathbb{Z}$.

The following theorem is a variation on the Reduced Burau Representation–Link Invariant Theorem 67.1:

Theorem 68.2. (Reduced Burau Representation–Link Invariant Theorem) Let $L \subseteq \mathbb{R}^3$ be an oriented link and let X be an m-strand braid in $\mathbb{R}_{>0}\times\mathbb{R}$ such that the closure $\widehat{X}$ (from Definition 66.3) is smoothly isotopic to L. We consider the reduced Burau representation $\beta_m^{\mathrm{red}}\colon \mathrm{B}_m \to \mathrm{GL}(m-1,\mathbb{Z}[t^{\pm 1}])$ from Definition 65.5. The rational

function[721]

$$f_L(s) \;:=\; (-1)^{m-1}\cdot s^{-\epsilon(X)}\cdot\frac{s-s^{-1}}{s^m-s^{-m}}\cdot\det(\beta_m^{\mathrm{red}}(X)-\mathrm{id}_{m-1})\big|_{t=s^2}$$

is independent of the choice of the braid X.[722]

Example. In the following we return to the examples that we studied on page 1425:

(1) Let L be the trivial m-component link. It is the closure of the trivial m-braid T. Thus we see that

$$f_L(s) \;\underset{\uparrow}{=}\; (-1)^{m-1}\cdot s^0\cdot\tfrac{s-s^{-1}}{s^m-s^{-m}}\cdot\det(\underbrace{\beta_m^{\mathrm{red}}(T)}_{=\mathrm{id}_{m-1}}-\mathrm{id}_{m-1})\big|_{t=s^2} = \begin{cases}1, & \text{if } m=1,\\ 0, & \text{if } m>1.\end{cases}$$

see the Burau Representation Link Invariant Theorem 68.2

(2) Let T be the left-handed trefoil. It is the closure of σ_1^{-3}. Thus we see that

$$f_T(s) \;=\; (-1)^1\cdot s^3\cdot\tfrac{s-s^{-1}}{s^2-s^{-2}}\cdot\det(\underbrace{\beta_2^{\mathrm{red}}(\sigma_1)^{-3}-\mathrm{id}_1}_{-t^3-1,\text{ see page 1425}})\big|_{t=s^2} \;=\; s^2-1+s^{-2}.$$

(3) Let K be the figure-8 knot. It is the closure of the 3-strand braid $\sigma_2\cdot\sigma_1^{-1}\cdot\sigma_2\cdot\sigma_1^{-1}\in\mathrm{B}_3$. Thus we see that

$$f_K(s) \;=\; (-1)^2\cdot\tfrac{s-s^{-1}}{s^3-s^{-3}}\cdot\underbrace{\det(\beta_3^{\mathrm{red}}(\sigma_2\cdot\sigma_1^{-1}\cdot\sigma_2\cdot\sigma_1^{-1})-\mathrm{id}_2)}_{=t^2-3t+1,\text{ see page 1426}}\big|_{t=s^2} \;=\; -s^2+3-s^{-2}.$$

(4) (a) Let H^- be the negative Hopf link. It is the closure of $\sigma_1^2\in\mathrm{B}_2$. Thus we see that

$$f_{H^-}(s) \;=\; (-1)^1\cdot s^{-2}\cdot\tfrac{s-s^{-1}}{s^2-s^{-2}}\cdot\det(\underbrace{\beta_2^{\mathrm{red}}(\sigma_1)^2}_{=-t}-\mathrm{id}_1)\big|_{t=s^2} \;=\; s^{-1}-s.$$



(b) Let H^+ be the positive Hopf link. It is the closure of $\sigma_1^{-2}\in\mathrm{B}_2$. Thus we see that

$$f_{H^+}(s) \;=\; (-1)^1\cdot s^2\cdot\tfrac{s-s^{-1}}{s^2-s^{-2}}\cdot\det(\underbrace{\beta_2^{\mathrm{red}}(\sigma_1)^2}_{=-t^{-1}}-\mathrm{id}_1)\big|_{t=s^2} \;=\; s-s^{-1}.$$

□

Proof. The proof of the theorem is almost identical to the proof of the Burau Representation–Link Invariant Theorem 67.1: By the Markov Theorem 66.9 we just need to show that the expression is unchanged under conjugation and stabilization. The proof of these two statements follows from almost the same calculations as in the Markov Invariance Lemma 67.3. We leave the details to the reader. ■

The Reduced Burau Representation–Link Invariant Theorem 68.2 gives us an invariant of oriented links without any indeterminacy. But before we officially introduce a new link

[721] Here the notation $g(t)|_{t=s^2}$ is short hand for the following procedure: We apply the ring homomorphism $\mathbb{Z}[t^{\pm1}]\to\mathbb{Z}[s^{\pm1}]$ that is given by $t\mapsto s^2$ to $g(t)\in\mathbb{Z}[t^{\pm1}]$.

[722] Note that in contrast to the Reduced Burau Representation–Link Invariant Theorem 67.1 we do not have an indeterminacy at all.

invariant, let us first prove the following lemma, which is somewhat similar to the earlier Polynomial Anti-Symmetry Lemma 295.2:

Lemma 68.3. (Burau–Polynomial Anti-Symmetry Lemma) Let $m \in \mathbb{N}$ and let $X \in \mathrm{B}_m$ be a braid. The rational function

$$f(s) \;=\; (-1)^{m-1} \cdot s^{-\epsilon(X)} \cdot \frac{s - s^{-1}}{s^m - s^{-m}} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})\big|_{t=s^2}$$

from the Reduced Burau Representation–Link Invariant Theorem 68.2 satisfies $f(-s^{-1}) = f(s)$.

Proof. Let $m \in \mathbb{N}$ and let $X \in \mathrm{B}_m$ be a braid. First note that

$$\frac{s - s^{-1}}{s^m - s^{-m}}\Big|_{s=-s^{-1}} \;=\; (-1)^{m-1} \cdot \frac{s - s^{-1}}{s^m - s^{-m}}.$$

Thus it suffices to prove the following claim:

Claim. The Laurent polynomial

$$g(s) \;=\; s^{-\epsilon(X)} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})\big|_{t=s^2}$$

has the property that $g(s^{-1}) = (-1)^{m-1} \cdot g(-s)$.

Proof. In the following we denote by $\Omega_m^{\mathrm{red}} \in \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}])$ the matrix from Notation 65.9. The precise definition of Ω_m^{red} is irrelevant for us, we just need to know the following two facts:

(i) By the Reduced Burau Unitarian Lemma 65.10 we know that we have the equality

$$\overline{\beta_m^{\mathrm{red}}(X)} \cdot \Omega_m^{\mathrm{red}} \cdot \beta_m^{\mathrm{red}}(X)^{\mathrm{T}} \;=\; \Omega_m^{\mathrm{red}}.$$

(ii) In Exercise 68.1 we will see that $\det(\Omega_m^{\mathrm{red}}) \neq 0$.

It follows that

$$\begin{aligned}
g(s^{-1}) \;&=\; s^{\epsilon(X)} \cdot \det(\overline{\beta_m^{\mathrm{red}}(X)} - \mathrm{id}_{m-1})\big|_{t=s^2} \\
&=\; s^{\epsilon(X)} \cdot \det(\Omega_m^{\mathrm{red}} \cdot (\beta_m^{\mathrm{red}}(X)^{\mathrm{T}})^{-1} \cdot (\Omega_m^{\mathrm{red}})^{-1} - \mathrm{id}_{m-1})\big|_{t=s^2} \\
&\quad\uparrow \\
&\quad\text{follows from (i) and (ii)} \\
&=\; s^{\epsilon(X)} \cdot \det(\Omega_m^{\mathrm{red}} \cdot ((\beta_m^{\mathrm{red}}(X)^{\mathrm{T}})^{-1} \cdot (\mathrm{id}_{m-1} - \beta_m^{\mathrm{red}}(X)^{\mathrm{T}})) \cdot (\Omega_m^{\mathrm{red}})^{-1})\big|_{t=s^2} \\
&=\; s^{\epsilon(X)} \cdot \underbrace{\det(\beta_m^{\mathrm{red}}(X))^{-1}\big|_{t=s^2}}_{=(-t)^{\epsilon(X)}} \cdot (-1)^{m-1} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})\big|_{t=s^2} \\
&\quad\uparrow \\
&\quad\text{follows from the multiplicativity of the determinant,} \\
&\quad\text{the fact that } \det((B^{\mathrm{T}})^{-1}) = \det(B)^{-1} \text{ and} \\
&\quad\text{the fact that } \det(-B_{(m-1)\times(m-1)}) = (-1)^{m-1} \cdot \det(B) \\
&=\; (-1)^{m-1} \cdot s^{\epsilon(X)} \cdot (-s^2)^{-\epsilon(X)} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})\big|_{t=(-s)^2} \\
&=\; (-1)^{m-1} \cdot (-s)^{-\epsilon(X)} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})\big|_{t=(-s)^2} \\
&=\; (-1)^{m-1} \cdot g(-s).
\end{aligned}$$

■

The above lemma brings us to the following elementary lemma:

[722] Recall that given a matrix A over $\mathbb{Z}[t^{\pm 1}]$ we denote by $\overline{A}$ the matrix that we obtain by applying the ring homomorphism $t \mapsto t^{-1}$ to all entries.

Lemma 68.4. (Rewriting Polynomial Lemma) We consider the map

$$\begin{array}{rccc} \Theta\colon & \mathbb{Z}[z] & \to & \mathbb{Z}[w^{\pm 1}] \\ & p(z) & \mapsto & p(w-w^{-1}). \end{array}$$

This map has the following properties:

(1) Θ is a ring homomorphism.
(2) The map Θ is injective.
(3) The image of Θ is given by all $f(w) \in \mathbb{Z}[w^{\pm 1}]$ with $f(-w^{-1}) = f(w)$.

Proof. To keep the conversation going we outsource the elementary proof of the lemma to Appendix 68.4. ■

The above results lead us to the following definition, which is a variation on Definition 67.2:

Definition 68.5. Let $L \subseteq \mathbb{R}^3$ be an oriented link. By the Alexander–Braid Closure Proposition 66.4 we can pick an m-strand braid X in $\mathbb{R}_{>0} \times \mathbb{R}$ such that $\widehat{X}$ is smoothly isotopic to L. It follows from the Burau–Polynomial Anti-Symmetry Lemma 68.3 together with the Rewriting Polynomial Lemma 68.4 that there exists a unique $\nabla_L(z) \in \mathbb{Q}(z)$ such that

$$\nabla_L(s-s^{-1}) \;=\; (-1)^{m-1} \cdot s^{-\epsilon(X)} \cdot \frac{s-s^{-1}}{s^m - s^{-m}} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})\big|_{t=s^2}.$$

We refer to $\nabla_L(z)$ as the Conway polynomial of L. It follows from the Reduced Burau Representation–Link Invariant Theorem 68.2 that $\nabla_L(z)$ does not depend on the choice of X.

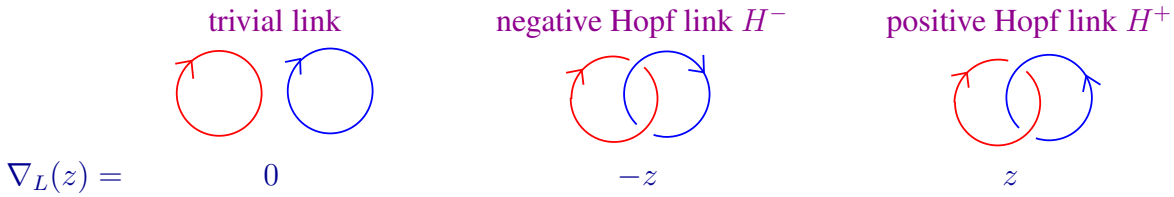
Example. In the following we return to the examples that we studied on page 1438:

(1) For the unknot U we have $f_U(t) = 1$, thus $\nabla_U(z) = 1$.
(2) For the left-handed trefoil T we saw that $f_T(s) = s^2 - 1 + s^{-2}$. Thus $\nabla_T(z) = z^2 + 1$.
(3) For the figure-8 knot K we see that $f_K(s) = -s^2 + 3 - s^{-2}$. Thus $\nabla_K(z) = -z^2 + 1$.



Next we turn to the links:

(4) Let $m \in \mathbb{N}_{\geq 2}$. For the trivial m-component link L we have $f_L(t) = 0$, thus $\nabla_L(z) = 0$.
(5) (a) For the negative Hopf link H^- we saw that $f_{H^-}(s) = s^{-1} - s$. Thus $\nabla_{H^-}(z) = -z$.
(b) For the positive Hopf link H^+ we saw that $f_{H^+}(s) = s - s^{-1}$. Thus $\nabla_{H^+}(z) = z$.



We will consider more examples in Exercise 68.3. □

For completeness sake let us formulate the following almost obvious lemma:

Lemma 68.6. (Conway–Alexander Polynomial Lemma) Let $L \subseteq \mathbb{R}^3$ be an oriented link. We have, up to a power of s, the following equality

$$\nabla_L(s - s^{-1}) \;=\; \Delta_L(s^2).$$

In particular the Conway polynomial determines the Alexander polynomial.

Proof. As we will now see, the promised equality follows almost immediately from the definitions:

$$\begin{array}{rcl}
\nabla_L(s-s^{-1}) & \overset{\text{Definition 68.5}}{=} & (-1)^{m-1}\cdot s^{-\epsilon(X)}\cdot \underbrace{\frac{s-s^{-1}}{s^m-s^{-m}}}_{(-1)^{m-1}\cdot s^{1-m}\cdot\frac{1-s^2}{1-s^{2m}}} \cdot \det(\beta_m^{\mathrm{red}}(X)-\mathrm{id}_{m-1})\big|_{t=s^2} \\
& = & s^{-\epsilon(X)}\cdot(-1)^{m-1}\cdot s^{1-m}\cdot\big((-1)^{m-1}\cdot\tfrac{1-t}{1-t^m}\cdot\det(\beta_m^{\mathrm{red}}(X)-\mathrm{id}_{m-1})\big)\big|_{t=s^2} \\
& = & s^{-\epsilon(X)}\cdot(-1)^{m-1}\cdot s^{1-m}\cdot\Delta_L(t=s^2). \\
& \overset{\text{Definition 67.2}}{=} &
\end{array}$$

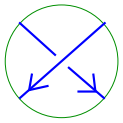
■

68.2. **The skein relation for the Conway polynomial.** The following theorem is an analog of the Alexander-Conway Skein Theorem 68.7:

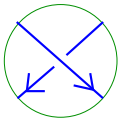
Theorem 68.7. (Alexander-Conway Skein Theorem) The Conway polynomial has the following property: Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same *diagram*, except that in a smoothly embedded closed 3-ball they are (up to smooth isotopy) of the form shown below. Then we have the Alexander-Conway skein relation[723]

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) \;=\; z\cdot\nabla_{L_0}(z).$$

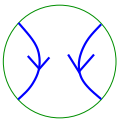
Proof. We will provide the proof of the Alexander-Conway Skein Theorem 68.7 in the next Section 68.3. ■



L_+



L_-



L_0

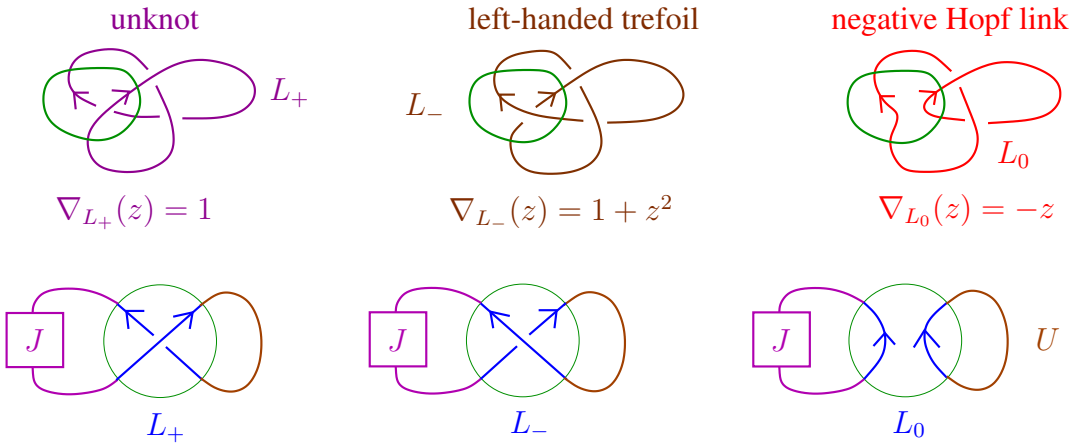
$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\cdot\nabla_{L_0}(z)$$

Example.

(1) We consider the oriented links L_+, L_- and L_0 that are shown below. We also show the Conway polynomials of these knots and links which conveniently enough we already calculated on page 5651.
We see that we indeed have $\underbrace{\nabla_{L_+}(z)}_{=1} - \underbrace{\nabla_{L_-}(z)}_{=1+z^2} \;=\; z\cdot\underbrace{\nabla_{L_0}(z)}_{-1}$. Magic!

(2) As a "proof of concept" let us consider the split union L_0 of a non-empty link J together with an unknot U as shown below to the right.

[723]The mnemonic for L_+ and L_- is that L_+ follows the right-hand rule ("if the thumb points into the direction of the upper strand, then the fingers point into the direction of the lower-strands"), whereas L_- does not follow the right-hand rule.



We now see that

$$z \cdot \nabla_{J \sqcup U}(z) \;=\; z \cdot \nabla_{L_0}(z) \;\underset{\uparrow}{=}\; \nabla_{L_+}(z) - \nabla_{L_-}(z) \;\underset{\uparrow}{=}\; \nabla_J(z) - \nabla_J(z) \;=\; 0.$$

by the Alexander-Conway Skein Theorem 68.7 — since $L_\pm$ are both smoothly isotopic to J

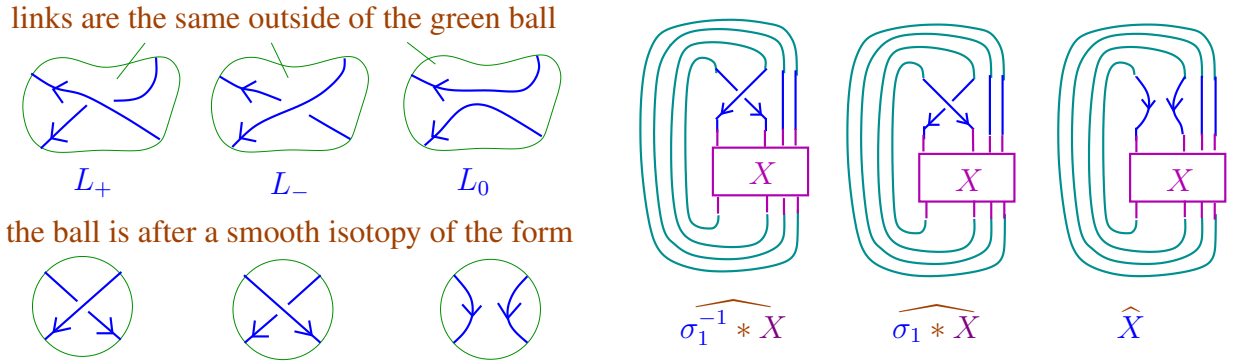
This implies that $\nabla_{J \sqcup U}(z) = 0$. In particular we have thus shown that for any trivial link with ≥ 2 components the Conway polynomial vanishes. □

68.3. **Proof of the Alexander-Conway Skein Theorem 68.7.** This section is of course dedicated to the proof of the Alexander-Conway Skein Theorem 68.7. In the next Chapter 69 we will give a characterization of the Conway polynomial in terms of two axioms.

The key ingredient in the proof of the Alexander-Conway Skein Theorem 68.7 is the following proposition, which allows us to translate our hypothesis into the setting of braids:

Proposition 68.8. (Skein Relation–Braid Proposition) Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same *diagram*, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown below. Then there exists an $m \in \mathbb{N}$ and a braid $X \in \mathrm{B}_m$ such that (up to smooth isotopy) we have[724]

$$L_+ \;=\; \widehat{\sigma_1^{-1} * X}, \quad L_- \;=\; \widehat{\sigma_1 * X} \quad \text{and} \quad L_0 \;=\; \widehat{X}.$$



[724]The fact that $L_+ = \widehat{\sigma_1^{-1} * X}$ and $L_- = \widehat{\sigma_1 * X}$ shows that the sign conventions are not completely consistent. But we are using the historically established convention for the definition of σ_i and L_+, L_-, L_0.

Sketch of proof of the Skein Relation–Braid Proposition 68.8. First recall that in the proof of the Alexander–Braid Closure Proposition 66.4 we gave a fairly explicit algorithm for turning a diagram into a diagram that is closure of a link. The idea now is to get our current situation into the setting into a position such that the algorithm of the proof of the Alexander–Braid Closure Proposition 66.4 does almost all of the hard work.

More precisely:

(1) (a) It follows from our hypothesis and the Smooth Ball Embedding Theorem 45.10 that after applying a smooth isotopy we can assume that the smoothly embedded closed 3-ball is just a standard Euclidean 3-ball in $\mathbb{R}^3$ and that the links in this standard Euclidean 3-ball are of the standard form.

(b) After a translation we can assume that the origin is just to the left of the above standard ball and that there is no part of the link between the origin and the standard ball.

(2) We now apply the algorithm from the proof of the Alexander–Braid Closure Proposition 66.4 to the above situation. Note that the pieces of the links in the 3-ball are clockwise, so they are all left unchanged.

(3) The end result will be that we find braids X' and X'' such that

$$L_+ \;=\; \widehat{X' * \sigma_1^{-1} * X''}, \quad L_- \;=\; \widehat{X' * \sigma_1 * X''} \quad \text{and} \quad L_0 \;=\; \widehat{X' * X''}.$$

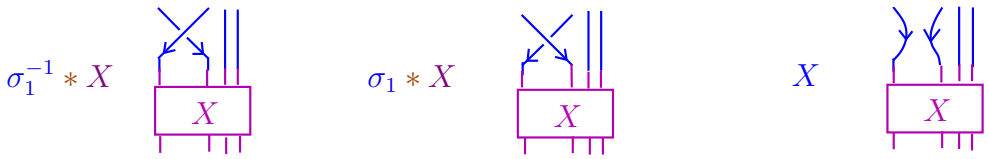
(4) Using the Markov move (M1) (i.e. using conjugation of braids by X') we see that we can arrange that X' is a trivial braid.

We leave it to the reader to turn this sketch into a formal proof. ■

Proof of the Alexander-Conway Skein Theorem 68.7. We start out with two preparations:

- Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same *diagram*, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown above. By the Skein Relation–Braid Proposition 68.8 there exists an $m \in \mathbb{N}$ and a braid $X \in \mathrm{B}_m$ such that (up to smooth isotopy) we have

$$L_+ \;=\; \widehat{\sigma_1^{-1} * X}, \quad L_- \;=\; \widehat{\sigma_1 * X} \quad \text{and} \quad L_0 \;=\; \widehat{X}.$$



- Given a braid $Y \in \mathrm{B}_m$ we write

$$\Theta(Y) \;:=\; \det(\beta_m^{\mathrm{red}}(Y) - \mathrm{id}_{m-1}) \;\in\; \mathbb{Z}[t^{\pm 1}].$$

It follows easily from the above and Definition 68.5 that it suffices to prove that

$$s \cdot \Theta(\sigma_1^{-1} * X)\big|_{t=s^2} - s^{-1} \cdot \Theta(\sigma_1 * X)\big|_{t=s^2} \;=\; (s - s^{-1}) \cdot \Theta(X)\big|_{t=s^2}.$$

In fact multiplying by s and substituting $t = s^2$ we see that the above equality is equivalent to the following claim:

Claim. We have the equality

$$t \cdot \Theta(\sigma_1^{-1} * X) - \Theta(\sigma_1 * X) \;=\; (t-1) \cdot \Theta(X).$$

Proof. We write

$$\beta_m^{\mathrm{red}}(X) \;=:\; \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix}$$

where M is an $(m-3) \times (m-3)$-matrix and all the other symbols are matrices of the obvious size. Now we see that

$$\Theta(X) \;=\; \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}) \;=\; \det \underbrace{\begin{pmatrix} a-1 & b & x \\ c & d-1 & y \\ p & q & M - \mathrm{id}_{m-3} \end{pmatrix}}_{=:A_0}.$$

Next we see that

$$\beta_m^{\mathrm{red}}(\sigma_1^{-1} * X) \;=\; \underset{\substack{\uparrow \\ \text{see Definition 65.5}}}{} \begin{pmatrix} -t^{-1} & 0 & 0 \\ t^{-1} & 1 & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix} \;=\; \begin{pmatrix} -t^{-1} \cdot a & -t^{-1} \cdot b & -t^{-1} \cdot x \\ t^{-1} \cdot a + c & t^{-1} \cdot b + d & t^{-1} \cdot x + y \\ p & q & M \end{pmatrix}.$$

We now subtract id_{m-1}, we add the first row from the second row and then we pull out $-t^{-1}$ from the first row. This way we see that

$$\Theta(\sigma_1^{-1} * X) \;=\; \det(\beta_m^{\mathrm{red}}(\sigma_1^{-1} * X) - \mathrm{id}_{m-1}) \;=\; -t^{-1} \cdot \det \underbrace{\begin{pmatrix} a+t & b & x \\ c-1 & d-1 & y \\ p & q & M - \mathrm{id}_{m-3} \end{pmatrix}}_{=:A_+}.$$

Similarly we see that

$$\beta_m^{\mathrm{red}}(\sigma_1 * X) \;=\; \underset{\substack{\uparrow \\ \text{see Definition 65.5}}}{} \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix} \;=\; \begin{pmatrix} -t \cdot a & -t \cdot b & -t \cdot x \\ a + c & b + d & x + y \\ p & q & M \end{pmatrix}.$$

We now subtract id_{m-1}, we pull out $-t$ from the first row and then we subtract the first row from the second row. This way we see that

$$\Theta(\sigma_1 * X) \;=\; \det(\beta_m^{\mathrm{red}}(\sigma_1 * X) - \mathrm{id}_{m-1}) \;=\; -t \cdot \det \underbrace{\begin{pmatrix} a+t^{-1} & b & x \\ c-t^{-1} & d-1 & y \\ p & q & M - \mathrm{id}_{m-3} \end{pmatrix}}_{=:A_-}.$$

We make the following observations and preparations:

(i) The matrices A_0, A_-, A_+ are the same, except for the first column.
(ii) We denote by A_0', A_-', A_+' the first columns of the above matrices, i.e. we have

$$A_+' \;=\; \begin{pmatrix} a+t \\ c-1 \\ p \end{pmatrix}, \quad A_-' \;=\; \begin{pmatrix} a+t^{-1} \\ c-t^{-1} \\ p \end{pmatrix}, \quad \text{and} \quad A_0' \;=\; \begin{pmatrix} a-1 \\ c \\ p \end{pmatrix}.$$

and we denote by A'' the collection of the remaining columns.
(iii) We have $-A'_+ + t \cdot A'_- = (t-1) \cdot A'_0$.

Now we see that

$$\begin{array}{rcl} t \cdot \Theta(\sigma_1^{-1} * X) - \Theta(\sigma_1 * X) & \overset{\text{by the above calculations}}{=} & t \cdot (-t^{-1}) \cdot \det(A'_+ \ A'') - (-t) \cdot \det(A'_- \ A'') \\ & = & \det(-A'_+ + t \cdot A'_- \ A'') \\ & \overset{\text{multilinearity of the determinant}}{=} & (t-1) \cdot \det(A'_0 \ A'') \end{array} \quad \begin{array}{l} = \det((1-t) \cdot A'_0 \ A'') \\ \overset{\text{follows from (iii)}}{=} (t-1) \cdot \Theta(X). \end{array} \quad \blacksquare$$

68.4. **Appendix: Rewriting antisymmetric polynomials.** The sole purpose of this appendix is to prove the following lemma:

Lemma 68.4. (Rewriting Polynomial Lemma) We consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}[z] & \to & \mathbb{Z}[w^{\pm 1}] \\ p(z) & \mapsto & p(w - w^{-1}). \end{array}$$

This map has the following properties:
(1) Θ is a ring homomorphism.
(2) The map Θ is injective.
(3) The image of Θ is given by all $f(w) \in \mathbb{Z}[w^{\pm 1}]$ with $f(-w^{-1}) = f(w)$.

Proof. First of all it is clear that the map Θ is a ring homomorphism. Next we point out that, by considering the highest z-power of a polynomial in z, one can easily verify that the map Θ is injective. It remains to determine the image of Θ. For the purpose of this proof we consider the subset

$$A \ = \ \{f(w) \in \mathbb{Z}[w^{\pm 1}] \,|\, f(-w^{-1}) = f(w)\}$$

of anti-symmetric Laurent polynomials in $\mathbb{Z}[w^{\pm 1}]$. We need to show that the image of Θ equals A. Using the observation that $(-w^{-1})^{-1} - (-w^{-1})^{-1} = w - w^{-1}$ one can easily verify that the image of Θ is contained in A. Thus it remains to prove the following claim:

Claim. Given any $f(w) \in A$ there exists a $p(z) \in \mathbb{Z}[z]$ with $\Theta(p(z)) = f(w)$, i.e. such that $p(w - w^{-1}) = f(w)$.

Proof. First note that for any non-zero $f(w) \in A$ we can write

$$f(w) \ = \ \sum_{k=-m}^{m} a_k \cdot w^k$$

where for each k we have $a_{-k} = (-1)^k \cdot a_k$ and where $a_m \neq 0$. We set $\varphi(f(w)) := m$. For convenience we also set $\varphi(0) := -1$. We now prove the claim by induction on $\varphi(f(w))$. The case $\varphi(f(w)) = -1$ is clear. So suppose we now have an $m \in \mathbb{N}_0$ such that we know the statement for all $f(w) \in A$ with $\varphi(f(w)) < m$. Next let

$$f(w) \ = \ \sum_{k=-m}^{m} a_k \cdot w^k$$

be an element of A with $a_m \neq 0$. Note that $a_{-m} = (-1)^m \cdot a_m$. We consider

$$\widetilde{f}(w) \;:=\; f(w) - a_m \cdot \underbrace{(w - w^{-1})^m}_{\substack{\text{by the binomial formula}\\ \text{this term is of the form}\\ (-1)^m \cdot w^{-m} + \ldots + w^m}}.$$

An elementary calculation, using the binomial formula, shows that $\varphi(\widetilde{f}(w)) < m$. Therefore our induction hypothesis applies and we see that there exists a $\widetilde{p}(z) \in \mathbb{Z}[z]$ such that $\widetilde{p}(w - w^{-1}) = \widetilde{f}(w)$. Finally note that the polynomial $p(z) = \widetilde{p}(z) + a_m \cdot z^m$ clearly satisfies $p(w - w^{-1}) = f(w)$. ■

68.5. Exercises for Chapter 68.

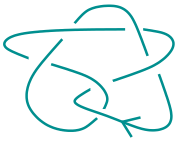
Exercise 68.1. Let $m \in \mathbb{N}$. As in Notation 65.9 we consider the $(m-1) \times (m-1)$-matrix

$$\Omega_m^{\text{red}} := \begin{pmatrix} 1+t & -1 & 0 & \ldots & 0 \\ -t & 1+t & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & -t & 1+t & -1 \\ 0 & \ldots & 0 & -t & 1+t \end{pmatrix}$$

over $\mathbb{Z}[t^{\pm 1}]$). Show that $\det(\Omega_m^{\text{red}}) \neq 0$.

Exercise 68.2. Let $L \subseteq \mathbb{R}^3$ be an oriented link. In Definition 68.5 we introduced the Conway polynomial $\nabla_L(z)$ which is a priori an element of $\mathbb{Q}(z)$. Show that $\nabla_L(z)$ always lies in $\mathbb{Z}[z] \subseteq \mathbb{Q}(z)$.
Hint. You could take inspiration from the proof of the Link Invariant–Uniqueness Proposition 69.6.

Exercise 68.3. In the figure below we show three knots.
(a) In each case provide an explicit braid X such that the closure $\widehat{X}$ is smoothly isotopic to the knot.
(b) In each case compute the Conway polynomial from Definition 68.5.
Remark. This exercise can be viewed as a continuation of Exercise 66.2.

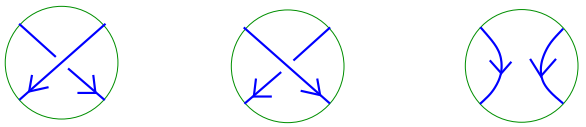
cinquefoil  three-twist knot  stevedore knot 

CHAPTER 69

A characterization of the Conway polynomial

Let $L \subseteq \mathbb{R}^3$ be an oriented link. In Definition 68.5 we introduced the Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$. We saw that the Conway polynomial has the following two properties:

- On page 1440 we saw that the value for the unknot equals 1.
- By the Alexander-Conway Skein Theorem 68.7 we know that if we are given three oriented links L_+, L_-, L_0 that are the same, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown below, then
$$\nabla_{L_+}(z) - \nabla_{L_-}(z) \;=\; z \cdot \nabla_{L_0}(z).$$



$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \cdot \nabla_{L_0}(z)$$

In this chapter we will see that the above two properties fully characterize the Conway polynomial. This fact allows us in particular to prove all results about Conway polynomial just using the above two properties.

This chapter is organized as follows:

- In Section 69.1 we will formulate the Unlinking Proposition 69.2 which says in particular that every link diagram admits an "unlinking sequence".
- In Section 69.2 we will use unlinking sequences to show that the Conway polynomial is fully characterized by the value of the trivial knot and the Alexander-Conway skein relation.
- In Section 69.3 we will discuss several consequences of the axiomatic characterization of the Conway polynomial.
- In Section 69.4 we will see that the Conway polynomial of an oriented 2-component link determines the linking number of the link.
- In Section 69.5 we will provide a proof of the Unlinking Proposition 69.2.
- In Section 69.6 we will provide exercises for Chapter 69.

In the next Chapter 70 we will introduce the Jones polynomial of any oriented link.

69.1. **Interlude: Unlinking sequences.** Our first goal of this chapter is to show that the Conway polynomial is actually completely characterized by the value of the trivial knot together with the skein relation from the Alexander-Conway Skein Theorem 68.7. Before we can get to the proof of that statement we will need to study unlinking sequences of diagrams. At this point it is convenient to introduce the following concept:

Definition 69.1. We say a link $L \subseteq \mathbb{R}^3$ is totally splittable if L is smoothly isotopic to the split union, in the sense of Definition 63.1, of its components.[725]

This definition allows us to formulate the one and only result of this section:

Proposition 69.2. (Unlinking Proposition) Let D be a link diagram in the sense of Definition 112.4.

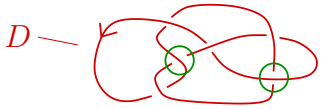
(1) There exists a sequence of crossing changes (which always involve two different components) that turns the diagram D into a diagram of a link that is totally splittable in the sense of Definition 69.1.
(2) There exists a sequence of crossing changes that turns the diagram D into a diagram of a trivial link.

Proof. To keep the discussion flowing we will relegate the proof of the Unlinking Proposition 69.2 to Appendix 69.5. ■

The Unlinking Proposition 69.2 allows us to introduce the following analog of Definition 293.3:

Definition 69.3. Let D be a link diagram. By the Unlinking Proposition 69.2 (2) we can define the unlinking number $u(D) \in \mathbb{N}_0$ of the diagram as the minimal number of crossing changes in the diagram that are needed to turn the diagram into the diagram of a trivial link.

Example. We consider the diagram D that is shown in the figure below.



changing these two crossings
produces the diagram of an unknot

If we perform crossing changes at the two green crossings, then we obtain a diagram of an unknot. Thus $u(D) \leq 2$. By going through every single crossing of D one sees that a single crossing change will not trivialize the corresponding knot. Thus $u(D) \geq 2$. In summary: $u(D) = 2$. □

Remark. Let $L \subseteq \mathbb{R}^3$ be an oriented link in $\mathbb{R}^3$. It follows easily from the argument on page 5615 that

$$u(L) \;=\; \min\{u(D) \,|\, D \text{ is a diagram of } L\}.$$

For better or worse it does not suffice to just consider any single diagram of L to determine $u(L)$. For example in the above diagram we have $u(D) = 2$. But a priori there might be a different diagram D' which represents the same knot, but with $u(D') < 2$. In fact strange things can happen: In [**Ble1984**, p. 470] it is shown that there exists a knot K such that $u(K) = 2$ but such that for all minimal crossing number diagrams D for K we have $u(D) = 3$. □

69.2. **A characterization of the Conway polynomial.** We start out by introducing a slight variation on Notation 108.20:

Notation 69.4. We denote by $\texttt{Link}^{\text{or}}_{\neq\varnothing}(\mathbb{R}^3)$ the set of smooth isotopy classes of oriented non-empty links in $\mathbb{R}^3$.

[725]It follows easily from Definition 63.1 (see also the discussion in Definition 109.4) that an m-component link $L = L_1 \cup \cdots \cup L_m \subseteq \mathbb{R}^3$ is totally splittable precisely if there exist m smooth embeddings $\Phi_i \colon \overline{B}^3 \to \mathbb{R}^3$, $i = 1, \ldots, m$ with disjoint images such that for each $i \in \{1, \ldots, m\}$ the component L_i is contained in $\Phi_i(B^3)$.

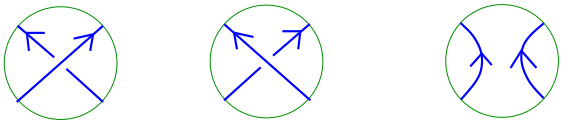
Now we can finally provide the following fascinating theorem which was first observed by John Conway [**Conw1970**, p. 338]:

Theorem 69.5. (Conway Polynomial-Characterization Theorem) The Conway polynomial from Definition 68.5 is the only map[726]

$$\mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \;\to\; \mathbb{Q}(z)$$

which satisfies the following two Conway axioms:

(C1) The invariant of the unknot equals 1.
(C2) The invariant satisfies the skein relation from the Alexander-Conway Skein Theorem 68.7, which is also summarized in the figure below.



(C1) $\nabla_{\mathrm{unknot}}(z) = 1$
(C2) $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \cdot \nabla_{L_0}(z)$

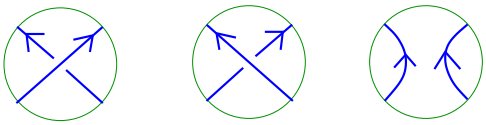
The proof of the uniqueness statement of the Conway Polynomial-Characterization Theorem 69.5 rests on the shoulders of the following slightly more general proposition:

Proposition 69.6. (Link Invariant–Uniqueness Proposition) Let $\mathbb{K}$ be a field and let $a_+, a_-, a_0, u \in \mathbb{K}$ such that $a_0 \neq 0$. There exists at most one map

$$\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \;\to\; \mathbb{K}$$

which satisfies the following two axioms:

(A1) The invariant of the unknot equals u.
(A2) The invariant satisfies the skein relation which is summarized in the figure below.



(A1) $\Theta(\mathrm{unknot}) = u$
(A2) $a_+ \cdot \Theta(L_+) + a_- \cdot \Theta(L_-) = a_0 \cdot \Theta(L_0)$

Proof of the Link Invariant–Uniqueness Proposition 69.6. Let $\mathbb{K}$ be a field and let $\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \to \mathbb{K}$ be a map which satisfies the axioms (A1) and (A2). We need to show that Θ is completely determined by (A1) and (A2). By the Link Diagram Existence Proposition 112.7 every oriented link is smoothly isotopic to a link that is associated to a diagram. Thus it suffices to show that for every non-empty link diagram the value of Θ for the associated link is completely determined by (A1) and (A2).

Given a link diagram D we introduce the following:

$$\begin{aligned} c(D) &:= \text{number of crossings of } D\ , \\ u(D) &:= \text{unknotting number of } D, \text{ see Definition 69.3.} \end{aligned}$$

We consider the complexity

$$\chi(D) \;:=\; (c(D), u(D)) \;\in\; \mathbb{N}_0 \times \mathbb{N}_0.$$

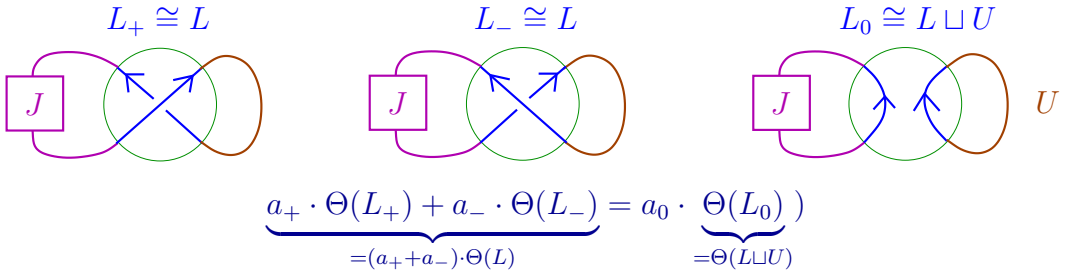
[726]Here we denote by $\mathbb{Q}(z)$ the quotient field of $\mathbb{Z}[z]$, i.e. $\mathbb{Q}(z)$ is the field of rational functions in z.

As usual we give $\mathbb{N}_0 \times \mathbb{N}_0$ the lexicographic ordering. The desired statement now follows easily from the following two claims together with a double induction: First on the crossing number and then (for fixed crossing number) on the unknotting number.

Claim 1. Let D be a non-empty link diagram with complexity $\chi(D) = (c, 0)$. The invariant Θ for any corresponding link is determined by (A1) and (A2).

Proof. Let D be a link diagram with complexity $\chi(D) = (c, 0)$. This means that $u(D) = 0$. By definition this means that the associated link L is actually a trivial link.

- If L has a single component, then (A1) is by definition satisfied.
- We now suppose that L has ≥ 2 components. The argument on page 1442 shows that addition of a trivial component changes Θ by a factor of $\frac{a_+ + a_-}{a_0}$ (Here we use that a_0 is non-zero.)



Thus $\Theta(L) = u_0 \cdot (\frac{a_+ + a_-}{a_0})^{m-1}$. ⊞

Claim 2. Suppose that Θ is determined by (A1) and (A2) for all links associated to diagrams of complexity $\leq (c, u) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then it is also determined for all links associated to diagrams of complexity $(c, u + 1)$.

Proof. Let D be a diagram of complexity $(c, u + 1)$. By the Unlinking Proposition 69.2 we can turn D into the diagram of a trivial link by flipping $u(D) = u + 1$ crossings of D. We now flip just a single such crossing. We denote by D' the resulting diagram. Note that $u(D') = u(D) - 1$ and note that $c(D') = c(D)$. Thus $\chi(D') < \chi(D)$. Let D_0 be the diagram that we obtain by resolving the crossing. Evidently D_0 has one crossing less than D. Thus we also have an inequality $\chi(D_0) < \chi(D)$.

Let L, L', L_0 be the links associated to D, D', D_0. We obtain from the above discussion and our hypothesis that $\Theta(L')$ and $\Theta(L_0)$ are determined by (A1) and (A2). But $\Theta(L)$ is determined by $\Theta(L')$ and $\Theta(L_0)$ together with (A2). Thus we see that $\Theta(L)$ is also determined by (A1) and (A2). ■

Proof of the Conway Polynomial-Characterization Theorem 69.5. First note that it follows from the calculation on page 1440 and the Alexander-Conway Skein Theorem 68.7 that the Conway polynomial satisfies Properties (C1) and (C2).

The uniqueness statement follows of course from the Link Invariant–Uniqueness Proposition 69.6 applied to $\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \to \mathbb{Q}(z)$ with $a_+ = 1$, $a_- = -1$, $a_0 = z$ and with $u = 1$. ■

69.3. **Consequences of the Alexander-Conway Skein Theorem 68.7.** The Conway Polynomial-Characterization Theorem 69.5 invites us to (re-) prove the properties of the

Conway polynomial just using the Conway axioms (C1) and (C2). To formulate our first result we introduce the following notation:

Definition 69.7. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. Given $i \in \mathbb{N}_0$ we write

$$a_i(L) \ := \ \text{the } z^i\text{-coefficient of } \nabla_L(z).$$

For peace of mind we extend this definition to $a_i(L) = 0$ for all $i < 0$.

Now we can formulate our first result:

Proposition 69.8. (Conway Polynomial–Low Coefficients Proposition) Given an oriented non-empty link $L \subseteq \mathbb{R}^3$ we denote, as in Definition 69.7, by $a_i(L)$ the z^i-coefficient of the Conway polynomial $\nabla_L(z)$. The following statements hold:

(1) For any oriented knot $K \subseteq \mathbb{R}^3$ we have $a_0(K) = 1$.
(2) Let $m \in \mathbb{N}$ and let L be an oriented m-component link in $\mathbb{R}^3$. We have $a_i(L) = 0$ for $i = 0, \dots, m-2$.

Sketch of proof.

(1) Let $K \subseteq \mathbb{R}^3$ be an oriented knot. We need to show that $a_0(K) = 1$. We pick a diagram D for K. Recall that it follows from the Unlinking Proposition 69.2 (2) that there exists a sequence of diagrams $D = D_0, D_1, \dots, D_m$ such that D_m is the diagram of an unknot and such that each D_i, $i = 1, \dots, m$ is obtained from D_{i-1} by a crossing change. Given $i \in \{1, \dots, m\}$ we denote by K_i the knot that is represented by D_i. We now see that

$$a_0(K) \ = \ a_0(K_0) \ \underset{\uparrow}{=} \ a_0(K_1) \ \underset{\uparrow}{=} \ \dots \ \underset{\uparrow}{=} \ a_o(K_m) \ \underset{\uparrow}{=} \ 1.$$

these equalities follow immediately from the Conway Axiom (C2)

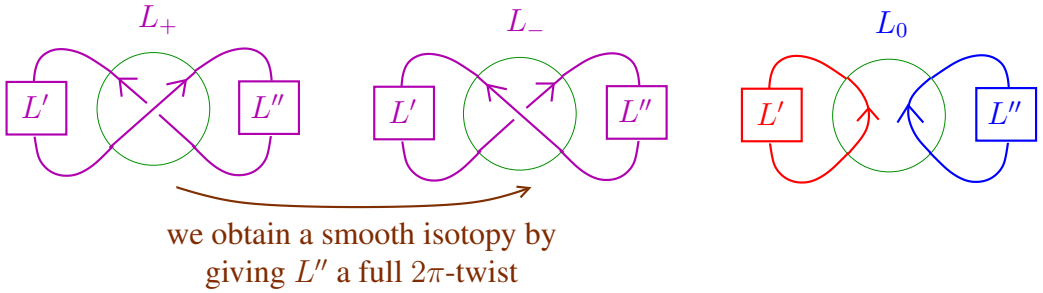
by (C1), since K_m is an unknot

(2) This statement can be proved iteratively for links with $2, 3, 4, \dots$ components. For a fixed number of components the statement gets proved, somewhat similar to the proof of (1), by using the Unlinking Proposition 69.2 (2), the Conway axiom (C2) and the fact that the Conway polynomial of a link with ≥ 2 components is zero. The details will be filled in Exercise 69.13. ■

Next we turn to split unions and connected sums:

Proposition 69.9. (Conway Polynomial–Splittable Proposition) Let $L \subseteq \mathbb{R}^3$ be an oriented link. If L is splittable in the sense of Definition 109.5, i.e. if L is the split union of two non-empty links, then $\nabla_L(z) = 0$.

Proof. Let $L \subseteq \mathbb{R}^3$ be an oriented splittable link. It is pretty clear that L is smoothly isotopic to a link L_0 that is given by a diagram as shown in the figure below to the right.



Keeping the above pretty picture in mind we now see that

$$z \cdot \nabla_L(z) \quad = \quad z \cdot \nabla_{L_0}(z) \quad \underset{\uparrow}{=} \quad \nabla_{L_+}(z) - \nabla_{L_-}(z) \quad \underset{\uparrow}{=} \quad 0.$$

by the Alexander-Conway Skein Theorem 68.7 — since L_+ and L_- are smoothly isotopic

Of course this calculation implies that $\nabla_L(z) = 0$. ■

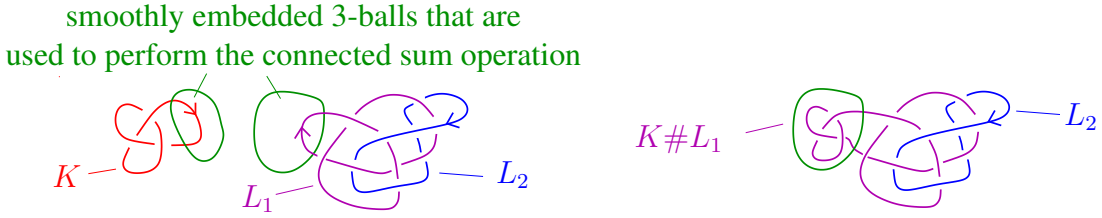
We conclude this section by considering connected sums:

Proposition 69.10. (Connected Sum–Conway Polynomial Proposition) Let K and $\widetilde{K}$ be two oriented knots in $\mathbb{R}^3$. The following equality holds:

$$\nabla_{K\#\widetilde{K}}(z) \;=\; \nabla_K(z) \cdot \nabla_{\widetilde{K}}(z) \;\in\; \mathbb{Z}[z].$$

Sketch of proof. We fix an oriented knot $K \subseteq \mathbb{R}^3$. We start out with two initial observations:

- The Conway Polynomial–Low Coefficients Proposition 69.8 tells us that $\nabla_K(z) \neq 0$.
- Given an oriented non-empty link $L = L_1 \cup L_2 \cup \cdots \cup L_m$ we can consider, as in Definition 63.6, the oriented link $K\#L := (K\#L_1) \cup L_2 \cup \cdots \cup L_m$.



These two observations allow us to consider the map

$$\begin{aligned} \Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) &\to \mathbb{Q}(z) \\ L &\mapsto \frac{\nabla_{K\#L}(z)}{\nabla_K(z)}. \end{aligned}$$

Claim. The above map $\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \to \mathbb{Q}(z)$ satisfies the axioms (C1) and (C2) from the Conway Polynomial-Characterization Theorem 69.5.

Proof.

(C1) Since the connected sum of the knot K with the trivial knot U is smoothly isotopic to K we see that $\Theta(U) = 1$.

(C2) Suppose that we are given three oriented links L_+, L_-, L_0 as in the formulation of (C2). By performing the connected sum of a diagram of K with the given diagrams[727] we obtain again a triple that satisfies the compatibility conditions and we obtain from (C2) the equality

$$\nabla_{K\#L_+}(z) - \nabla_{K\#L_-}(z) \;=\; z \cdot \nabla_{K\#L_0}(z).$$

Dividing by ∇_K we see that Θ satisfies (C2). ⊞

[727] The meticulous reader will have noticed that we did not introduce this concept. We leave it to the meticulous reader to make sense of this notion.

It follows from the claim together with the Conway Polynomial-Characterization Theorem 69.5 that for any oriented knot $\widetilde{K} \subseteq \mathbb{R}^3$ we have

$$\frac{\nabla_{K\#\widetilde{K}}(z)}{\nabla_K(z)} \;=\; \nabla_{\widetilde{K}}(z).$$

This gives us the desired property. ■

69.4. **The Conway polynomial and the linking number.** The main goal of this section is to prove the following slightly surprising relationship between the Conway polynomial and the linking number:

Proposition 69.11. (Conway Polynomial–Linking Number Proposition) In the following let $L = L_1 \cup L_2$ be an oriented 2-component link in $\mathbb{R}^3$. As in Definition 69.7 we consider the coefficients $a_i(L)$ of $\nabla_L(z)$. We have the equality

$$\underbrace{z\text{-coefficient of } \nabla_L(z)}_{=a_1(L_1\cup L_2)} \;=\; \mathrm{lk}(L_1, L_2).$$

Proof. Let $L = L_1 \cup L_2$ be an oriented 2-component link. It follows from the Link Diagram Existence Proposition 112.7 that we can assume that L is associated to a diagram D.

By the Unlinking Proposition 69.2 (1) there exists a sequence of crossing changes (which always involve two different components) that turns D into the diagram of an oriented link that is splittable in the sense of Definition 109.5, i.e. into a link where the two components are contained in two disjoint smoothly embedded 3-balls. The proposition thus follows almost immediately from the following two claims:

Claim 1. If $L_1 \cup L_2$ is a splittable link, then $a_1(L_1 \cup L_2) = \mathrm{lk}(L_1, L_2)$.

Proof. We prove the claim by showing that both sides are equal to zero. First note that it follows from the Conway Polynomial–Splittable Proposition 69.9 that $a_1(L_1 \cup L_2) = 0$.

It remains to show that $\mathrm{lk}(L_1, L_2) = 0$. By Definition 109.5 there exists a smooth embedding $\Phi\colon \overline{B}^3 \to \mathbb{R}^3 \setminus L_1$ such that $L_2 \subseteq \Phi(B^3)$. We consider the following little diagram:

$$\begin{array}{ccc} & \overset{[L_2]}{\mathrm{H}_1(L_2)} & \\ \swarrow & & \searrow \\ \underbrace{\mathrm{H}_1(\Phi(\overline{B}^3))}_{=0} & \longrightarrow & \mathrm{H}_1(\mathbb{R}^3 \setminus L_1) \end{array}$$

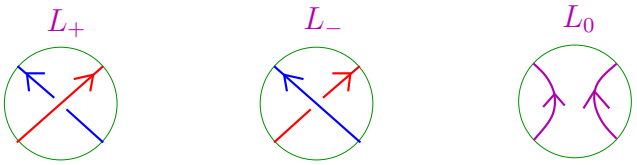
where all maps are induced by inclusions. Since $\mathrm{H}_1(\overline{B}^3) = 0$ we see that the right diagonal map is zero. But by Definition 284.1 this implies that $\mathrm{lk}(L_1, L_2) := \Phi_{L_1}([L_2]) = 0$.[728] [729] ⊞

Claim 2. Let D and D' be two 2-component diagrams that differ by a single crossing change between two different components. We denote by L and L' the corresponding two oriented 2-component links. Then $a_1(L) = \mathrm{lk}(L_1, L_2)$ if and only if $a_1(L') = \mathrm{lk}(L'_1, L'_2)$.

[728] This provides a solution to Exercise 284.3.

[729] One could also use the Linking Number-via-Crossings Proposition 284.8 to prove that the linking number is zero.

Proof. In the following let D_+ be the unique diagram among D, D' where the crossing under consideration is positive and let D_- be the other diagram, evidently it is the unique diagram among D, D' where the crossing is negative. We denote by L_+ and L_- the corresponding two oriented 2-component links. As usual we denote by L_0 the link that we obtain by resolving the crossing.



$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \cdot \nabla_{L_0}(z)$$
$$a_1(L_+) - a_1(L_-) = a_0(L_0)$$

We make one key observation: Since the crossing change takes place between the two different components the link L_0 has one component less than L_- and L_+, which in our case means, that L_0 is actually a knot. We now see that

$$\mathrm{lk}(L_{+,1}, L_{+,2}) - \mathrm{lk}(L_{-,1}, L_{-,2}) \underset{\text{follows easily from the Linking Number-via-Crossings Proposition 284.8}}{=} 1 \overset{\text{by the Conway Polynomial–Low Coefficients Proposition 69.8 (1), since } L_0 \text{ is a knot}}{\doteq} a_0(L_0) \underset{\text{by the Alexander-Conway Skein Theorem 68.7}}{=} a_1(L_{+,1} \cup L_{+,2}) - a_1(L_{-,1} \cup L_{-,2}).$$

Note that it follows from this equality that $a_1(L_{+,1} \cup L_{+,2}) = \mathrm{lk}(L_{+,1}, L_{+,2})$ if and only if $a_1(L_{-,1} \cup L_{-,2}) = \mathrm{lk}(L_{-,1}, L_{-,2})$. ■

The Conway Polynomial–Linking Number Proposition 69.11 allows us to prove a result on Alexander polynomials which so far had not even been on our radar:

Proposition 69.12. (Alexander Polynomial–Linking Number Proposition) Let $L = L_1 \cup L_2$ be an oriented 2-component link in $\mathbb{R}^3$. The single-variable Alexander polynomial $\Delta_L(t)$ is divisible by $t-1$, i.e. there exists a $g(t) \in \mathbb{Z}[t^{\pm 1}]$ with $\Delta_L(t) = (t-1) \cdot g(t)$. Furthermore we have the equality

$$\underbrace{\frac{\Delta_L(t)}{t-1}}_{=g(t) \in \mathbb{Z}[t^{\pm 1}]} \Big|_{t=1} = \pm \mathrm{lk}(L_1, L_2).$$

Proof. Let $L = L_1 \cup L_2$ be an oriented 2-component link in $\mathbb{R}^3$. We start out with a few observations:

(a) It follows almost immediately from the Conway Polynomial–Linking Number Proposition 69.11 and the Conway Polynomial–Symmetry Proposition 295.7 that there exists a $q(z) \in \mathbb{Z}[z]$ with $q(0) = \mathrm{lk}(L_1, L_2)$ and such that $\nabla_L(z) = z \cdot q(z^2)$.
(b) The Alexander–Conway Polynomial Lemma 295.4 tells us that $\Delta_L(t^2) \doteq \nabla_L(t - t^{-1})$.
(c) Note that

$$\Delta_L(t^2) \underset{\text{by (b)}}{\doteq} \nabla_L(t - t^{-1}) \underset{\text{by (a)}}{=} (t - t^{-1}) \cdot q((t - t^{-1})^2) \underset{\text{elementary algebra}}{\doteq} (t^2 - 1) \cdot q(t^2 - 2 + t^{-2}).$$

(d) It follows from (c) and some elementary algebra that with $g(t) := q(t-2+t^{-1}) \in \mathbb{Z}[t^{\pm 1}]$ we have the desired equality $\Delta_L(t) \doteq (t-1) \cdot g(t)$.

Finally we see that

$$\frac{\Delta_L(t)}{t-1}\Big|_{t=1} \underset{\text{by (d)}}{=} \pm g(1) \underset{\substack{\text{follows from} \\ g(t)=q(t-2+t^{-1})}}{=} \pm q(0) \underset{\text{by (b)}}{=} \pm \operatorname{lk}(L_1, L_2).$$

■

69.5. **Appendix: Proof of the Unlinking Proposition 69.2.** This appendix is completely dedicated to the proof of the following result:

Proposition 69.2. (Unlinking Proposition) Let D be a link diagram.

(1) There exists a sequence of crossing changes (which always involve two different components) that turns the diagram D into a diagram of a link that is totally splittable in the sense of Definition 69.1.

(2) There exists a sequence of crossing changes that turns the diagram D into a diagram of a trivial link.

This appendix is organized as follows:

- In Section 69.5.1 we will recall the definition of a link diagram.
- In Section 69.5.2 we will provide the proof of the Unlinking Proposition 69.2.

69.5.1. **Link diagrams.** The Unlinking Proposition 69.2 is evidently a statement about link diagrams. In many instances it suffices to work with a naive notion of a link diagram. But for the current purpose it is best to work carefully with the precise definitions. For the reader's convenience we will use this subsection to recall the relevant notions. First of all, to make sense of a link diagram we need the notion of a diagrammatic map:

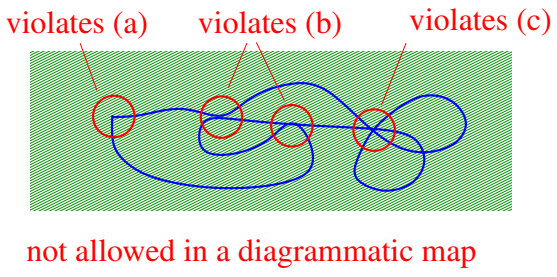
Definition 112.2. A map $\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2$ is called diagrammatic if the following conditions are satisfied:

(a) The map is an immersion in the sense of Definition 30.9, in our case this just means that γ is smooth and that for each $z \in S^1_1 \sqcup \cdots \sqcup S^1_m$ we have $\gamma'(z) \neq 0$.[730]

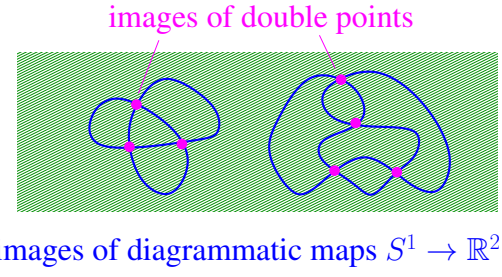
(b) If $z \neq w \in S^1_1 \sqcup \cdots \sqcup S^1_m$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.

(c) For every $P \in \gamma\big(S^1_1 \sqcup \cdots \sqcup S^1_m\big)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Any $z \in S^1_1 \sqcup \cdots \sqcup S^1_m$ for which there exists a $w \neq z \in S^1_1 \sqcup \cdots \sqcup S^1_m$ with $\gamma(z) = \gamma(w)$ is called a double point of γ.



not allowed in a diagrammatic map



images of diagrammatic maps $S^1 \to \mathbb{R}^2$

[730] We refer to Notation 112.1 for the definition of $\gamma'(z)$.

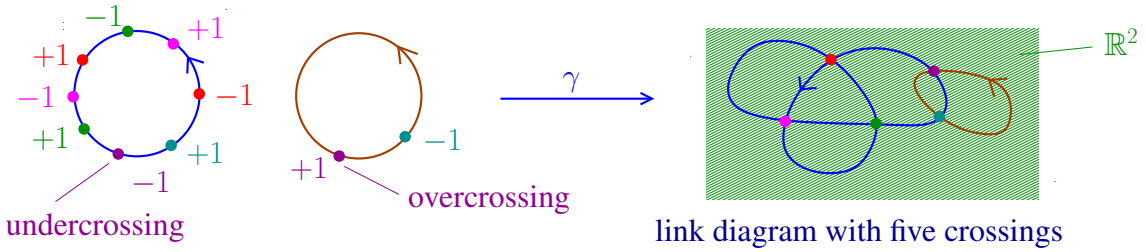
Definition 112.4.

(1) An m-component link diagram is a diagrammatic map $\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2$ together with a map
$$c\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$$
which has the property that for $z \neq w \in S^1_1 \sqcup \cdots \sqcup S^1_m$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.

(2) A 1-component link diagram is called a knot diagram.

(3) Let (γ, c) be a link diagram. Let z be a double point of γ. If $c(z) = +1$, then we say that the double point is an overcrossing, otherwise we call it an undercrossing.

(4) We refer to the images of the double points as the crossings of the link diagram.



The following reminds us that link diagrams give rise to smooth isotopy classes of links:

Lemma 112.5. (Diagram-to-Link Lemma) Let $(\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2, c)$ be a link diagram.

(1) There exists a smooth map $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ that satisfies the following condition:
   ($\star$) For every $P, Q \in S^1_1 \sqcup \cdots \sqcup S^1_m$ with $\gamma(P) = \gamma(Q)$ and with $c(P) = 1$ (and thus necessarily $c(Q) = -1$) we have $\eta(P) > \eta(Q)$.

(2) For any smooth map $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ that satisfies ($\star$) the image of the map $(\gamma, \eta)\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2 \times \mathbb{R}$ is a link.

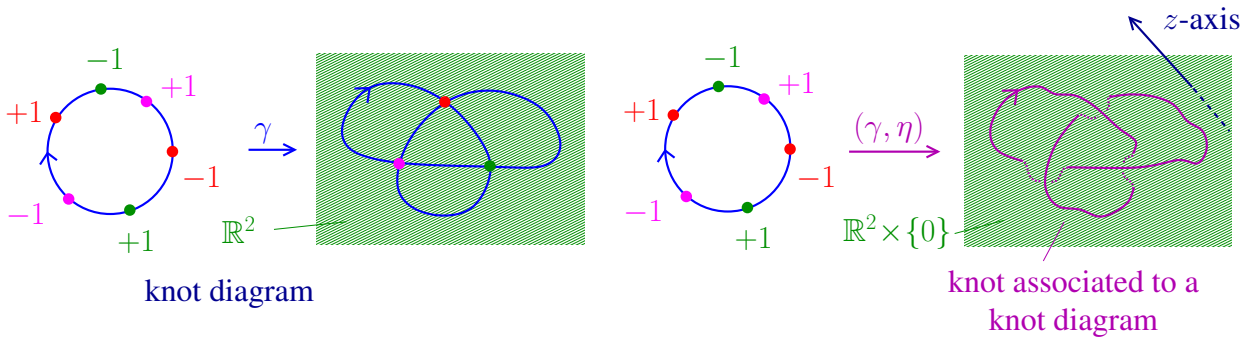
(3) The smooth isotopy type of the link in (b) does not depend on the choice of η, as long as the condition in ($\star$) is satisfied.

If we equip $S^1_1 \sqcup \cdots \sqcup S^1_m$ with the standard orientation, then in the above statements we can also replace "link" by "oriented link".

Definition 69.13. The Diagram-to-Link Lemma 112.5 shows that a link diagram gives rise to an essentially unique (oriented) link. We refer to this link as the link associated to the link diagram.

Example. In the figure below we show to the left a knot diagram. To the right we show the associated knot.
It is pretty clear that the resulting knot is smoothly isotopic to the trefoil. □

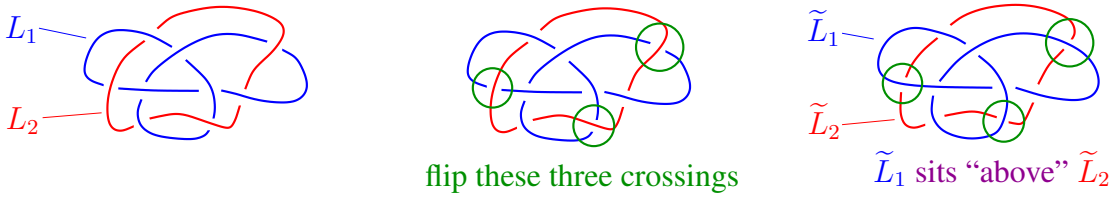
69.5.2. **Proof of the Unlinking Proposition 69.2.** After the preparations from the last subsection we can finally provide a proof of the Unlinking Proposition 69.2:



Proof of the Unlinking Proposition 69.2 (1). [731] Throughout this proof let $D = (\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2, c)$ be a link diagram. We define a new map $\widetilde{c}\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$ as follows:

- For each $1 \leq i < j \leq m$ and $P_i \in S^1_i$ and $Q_j \in S^1_j$ with $\gamma(P_i) = \gamma(Q_j)$ we set $\widetilde{c}(P_i) := +1$ and $\widetilde{c}(Q_j) := -1$.
- All other values of $\widetilde{c}$ equal the values of c.[732]

As a next step we use the Diagram-to-Link Lemma 112.5 (1) to pick for each $i \in \{1, \ldots, m\}$ a smooth function $\widetilde{\eta}_i\colon S^1 \to \mathbb{R}$ which satisfies condition $(\star)$ for the i-th component. By adding constants to the various functions we can arrange that for $i < j$ the minimum of $\widetilde{\eta}_i\colon S^1 \to \mathbb{R}$ is larger than the maximum of $\widetilde{\eta}_j\colon S^1 \to \mathbb{R}$. Note that the resulting function $\widetilde{\eta}\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ satisfies Condition $(\star)$. Thus $\widetilde{L} := (\gamma, \widetilde{\eta})(S^1_1 \sqcup \cdots \sqcup S^1_m)$ is the link associated to the diagram $\widetilde{D}$.



Note that by construction the different components of $\widetilde{L}$ lie in pairwise disjoint "bands" $\mathbb{R}^2 \times [a_i, b_i]$. It now follows easily that the resulting link $\widetilde{L}$ is totally splittable. ∎

Proof of the Unlinking Proposition 69.2 (2). [733] We start out with a special case of the desired statement:

Claim. Let D be a knot diagram. There exists a sequence of crossing changes that turns the diagram D into a diagram for the unknot.

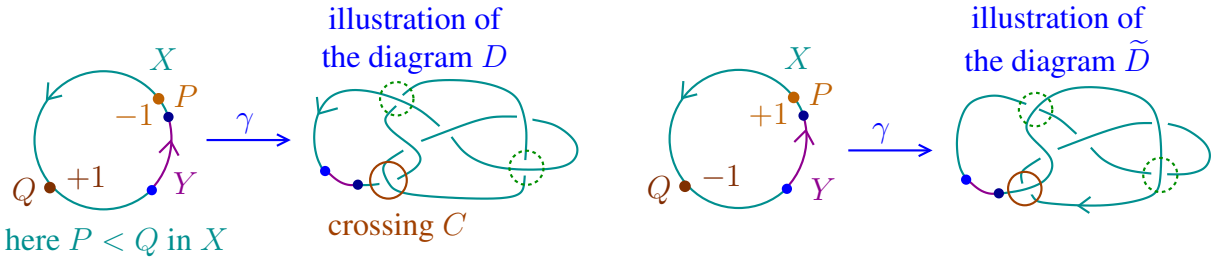
Proof. By Definition 112.4 the diagram D consists of a diagrammatic map $\gamma\colon S^1 \to \mathbb{R}^2$ together with a map $c\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$ which has the property that for $z \neq w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.

[731] The following arguments give in particular a solution to Exercise 113.12 (c).
[732] Colloquially speaking we changed the crossings between the components such that the earlier component always crosses over a later component.
[733] The following arguments give in particular a solution to Exercise 113.12 (a).

We make two preparations:

(a) It follows easily from the Diagram–Double Point Lemma 112.3 that we can pick two non-trivial circle segments X and Y on S^1 (in the sense of Definition 2.51) such that $S^1 = X \cup Y$, such that $X \cap Y = \partial X = \partial Y$ and such that X does not contain a double point of $\gamma\colon S^1 \to \mathbb{R}^2$.
(b) Since X and Y are circle segments we have an obvious notion of "$\leq$" on X and Y that comes from the standard orientation on S^1.

For each crossing $C \in \mathbb{R}^2$ of the diagram we do the following: We consider the preimage of $\gamma^{-1}(C) \subseteq S^1$. By our choice of Y we know that Y does not contain a double point. Thus $\gamma^{-1}(C) \subseteq X$. Since X has a total ordering it makes sense to name the preimages of C by P, Q where $P < Q$. We set $\widetilde{c}(P) := +1$ and $\widetilde{c}(Q) := -1$. This way we obtain a new diagram $\widetilde{D} = (\gamma, \widetilde{c})$.[734]

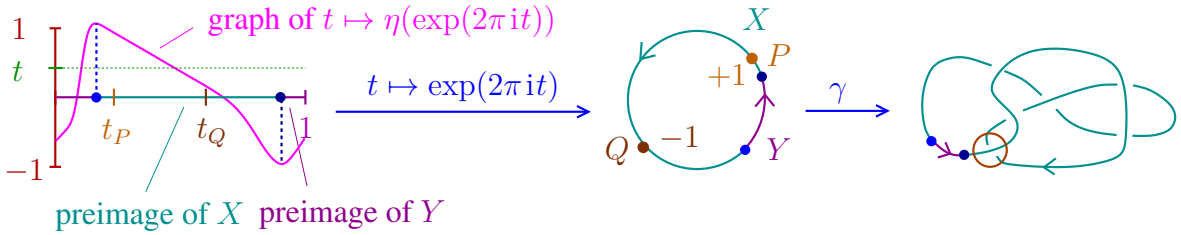


It remains to prove the following subclaim:

Subclaim. The knot associated to the knot diagram $\widetilde{D} = (\gamma, \widetilde{c})$ is the trivial knot.

Proof of subclaim. It follows fairly easily from our choice of crossings, and using the Smooth Transition Function Lemma 24.9, that we can find a smooth map $\eta\colon S^1 \to [0,1]$ with the following properties:

(c) The minimal value of η equals -1 and the maximal value of η equals 1.
(d) The map η is strictly monotonously decreasing on X.[735]
(e) The map η is strictly monotonously increasing on Y.
(f) The map η satisfies Condition $(\star)$ from the Diagram-to-Link Lemma 112.5 (1) for the knot diagram $\widetilde{D} = (\gamma, \widetilde{c})$.

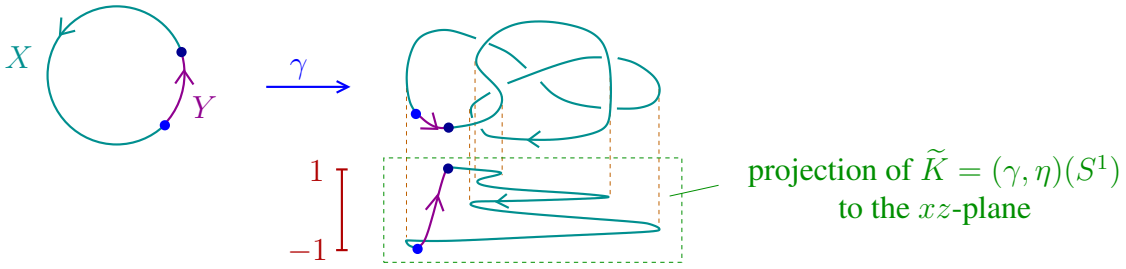


[734] Colloquially what we just did can be described as follows: We start at the starting point of X and we walk along the diagram. Each time we hit a crossing for the first time we turn it into an overcrossing. Thus the second time we transverse it, it will be an undercrossing.
[735] Recall that we introduced a total order $<$ on X, thus it makes sense to say that $\eta\colon X \to \mathbb{R}$ is strictly monotonously decreasing.

Note that it follows from (f) and the Diagram-to-Link Lemma 112.5 (1) that $\widetilde{K} := (\gamma, \eta)(S^1)$ is the knot associated to the knot diagram $\widetilde{D} = (\gamma, \widetilde{c})$. So we need to show that $\widetilde{K}$ is the unknot.

To prove that $\widetilde{K} := (\gamma, \eta)(S^1)$ is the unknot we start out with an observation: If follows from (c), (d) and (e) that for each $t \in [-1, 1]$ there exists a unique $x_t \in X$ with $\eta(x_t) = t$ and a unique $y_t \in Y$ with $\eta(y_t) = t$. We make the following observation:

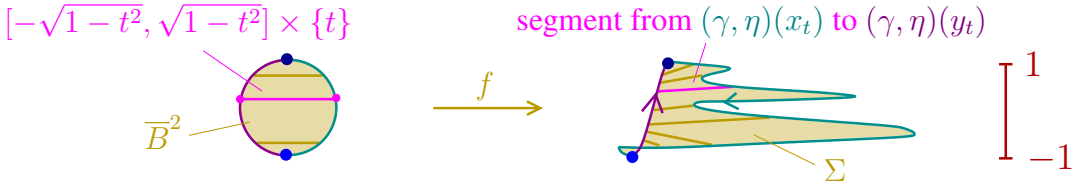
(∗) It follows from (a) that for $x_t \neq y_t$ we have $\gamma(x_t) \neq \gamma(y_t)$.



We now denote by $\Sigma \subseteq \mathbb{R}^3$ the subset that is given by considering for each $t \in [-1, 1]$ the Euclidean line segment from $(\gamma, \eta)(x_t)$ to $(\gamma, \eta)(y_t)$. Furthermore we consider the map

$$f \colon \overline{B}^2 \to \Sigma$$

that is given by sending each horizontal segment $[-\sqrt{1-t^2}, \sqrt{1-t^2}] \times \{t\}$ affine linearly to the Euclidean segment from $(\gamma, \eta)(x_t)$ to $(\gamma, \eta)(y_t)$.



We leave it to the reader to verify that the map $f \colon \overline{B}^2 \to \Sigma$ is continuous. The map is clearly surjective and it follows easily from (∗) that the map is injective. It now follows from the Compact-Hausdorff Proposition 4.23 (3) that $f \colon \overline{B}^2 \to \Sigma$ is a homeomorphism. Note that essentially by construction we have $f(S^1) = \widetilde{K}$.

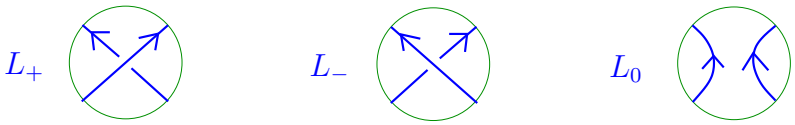
We leave it to the reader to verify that $f \colon \overline{B}^2 \to \mathbb{R}^3$ is in fact a smooth embedding. It then follows from the Smooth Embedding Theorem 30.15 that $\Sigma = f(\overline{B}^2)$ is actually a smooth submanifold of $\mathbb{R}^3$ with boundary $\partial\Sigma = \widetilde{K}$. It now follows from the Genus-Zero Knots Proposition 121.10 that $\widetilde{K}$ is indeed a trivial knot. ⊞

Now we turn to the general case of the statement. Thus let D be a link diagram. We apply the crossing changes from (1) to obtain a new diagram $\widetilde{D}$ and then we apply the above claim to each component of the diagram $\widetilde{D}$. It is pretty clear that the end result is a diagram of a trivial link. ■

69.6. Exercises for Chapter 69.

Exercise 69.1. Let D be a diagram with c crossings. Show that for the unknotting number $u(D)$ we have the inequality $u(D) \leq \frac{c}{2}$.

Exercise 69.2. We consider the by now familiar setup of three links that are the same except that in a ball they differ as shown in the figure below. Is it possible that L_+, L_- and L_0 have the same number of components?



Exercise 69.3. Show that in the formulation of the Link Invariant–Uniqueness Proposition 69.6 we cannot drop the hypothesis that $a_0 \neq 0$.

Exercise 69.4. Let $m \in \mathbb{N}$ and let $L \subseteq \mathbb{R}^3$ be an oriented link with m components. Show, only using the axioms (C1) and (C2) of the Conway polynomial, that

$$\nabla_L(-z) \;=\; (-1)^{m-1} \cdot \nabla_L(z).$$

Remark. This gives a new proof of the Conway Polynomial–Symmetry Proposition 295.7.

Exercise 69.5. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. Show, only using the axioms (C1) and (C2) of the Conway polynomial, that

$$\nabla_{L^{\mathrm{rev}}}(z) \;=\; \nabla_L(z).$$

Remark. This gives a new proof of the Conway Polynomial–Reverse Lemma 295.11.

Exercise 69.6. Let $m \in \mathbb{N}$. Show, only using the axioms (C1) and (C2) of the Conway polynomial, that for any oriented m-component link $L \subseteq \mathbb{R}^3$ we have the following equality in $\mathbb{Z}[z]$:

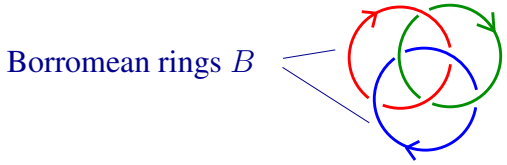
$$\nabla_{L^{\mathrm{mir}}}(z) \;=\; \nabla_L(-z) \;=\; (-1)^{m-1} \cdot \nabla_L(z).$$

Remark. This gives a new proof of the Conway Polynomial–Mirror Lemma 295.12

Exercise 69.7. Use the Conway axioms to calculate the Conway polynomial for the following oriented links:

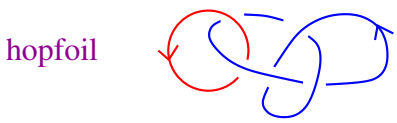
(a) The trivial 2-component link.
(b) The positive and the negative Hopf link.
(c) The left-handed trefoil.
(d) The figure-8 knot.

Exercise 69.8. Use the Conway axioms to calculate the Conway polynomial of the Borromean rings that are shown in the figure below.



Exercise 69.9.

(a) Compute the Conway polynomial of the hopfoil that is shown in the figure below.
(b) Let K be any oriented knot and let L be the oriented link that we obtaining by "adding an earring" to K. Determine the Conway polynomial L in terms of the Conway polynomial of K.

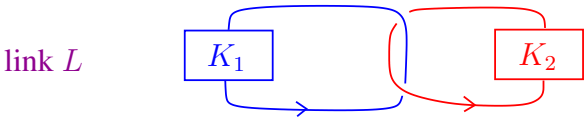


Exercise 69.10. Let K_1, K_2 be two oriented knots. We consider the oriented link L that is shown below.

(a) Determine the Conway polynomial of the oriented link L in terms of the Conway polynomials of the oriented knots K_1 and K_2.

(b) Determine the single-variable Alexander polynomial of the oriented link L in terms of the single-variable Alexander polynomials of the oriented knots K_1 and K_2.

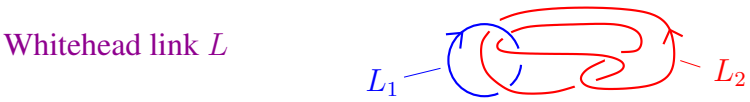
Remark. This exercise is a generalization of Exercise 69.9.



Exercise 69.11. We consider the Whitehead link $L = L_1 \cup L_2$ that is shown below.

(a) How many coefficients of the Conway polynomial of the Whitehead link L do you get "for free", i.e. from general properties of the Conway polynomial?

(b) Determine the Conway polynomial of the Whitehead link L.



Exercise 69.12. In Definition 295.6 we defined for any oriented non-empty link L its determinant as $\det(L) = \nabla_L(2\mathrm{i})$. Give an axiomatic characterization of the determinant of oriented non-empty links.

Exercise 69.13. Let $m \in \mathbb{N}$ and let L be an oriented m-component link in $\mathbb{R}^3$. Use the Conway axioms (C1) and (C2) to show that $a_i(L) = 0$ for $i = 0, \ldots, m-2$.

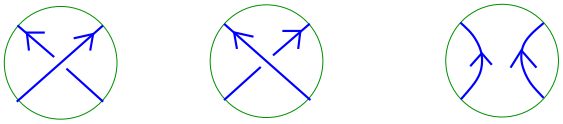
Hint. Prove the statement iteratively for $m = 1, 2, 3, \ldots$. For a fixed m, use the Unlinking Proposition 69.2 (2), the Conway axiom (C2) and the fact that the Conway polynomial of a link with ≥ 2 components is zero.

Remark. This gives a new proof of the Conway Polynomial–Low Coefficients Proposition 69.8 (2).

CHAPTER 70

The Jones polynomial

In the previous Chapter 69 we introduced the Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$ of an oriented link $L \subseteq \mathbb{R}^3$. We saw that it is completely characterized by the value for the trivial knot and by the following "skein relation":



(C1) $\nabla_{\text{unknot}}(z) = 1$
(C2) $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \cdot \nabla_{L_0}(z)$

This makes us wonder: Are there other invariants of oriented links that are characterized in that way? The miracle of knot theory is, yes, such an invariant does exist, namely the Jones polynomial. The Jones polynomial was discovered by Vaughan Jones [**JonV1985, JonV1987**] in 1984 and came completely out of the blue.

This chapter is organized as follows:

- In Section 70.1 we will remind the reader of Reidemeister moves on diagrams.
- In Section 70.2 we will introduce the Kauffman polynomial of a diagram.
- In Section 70.3 we will introduce the writhe of a diagram.
- In Section 70.4 we will introduce the Jones polynomial $V_L(t)$ of a link.
- In Section 70.5 we will formulate and prove the Jones skein relation.
- In Section 70.6 we will give a characterization of the Jones polynomial via two axioms, which are somewhat similar to the above axioms for the Conway polynomial.
- In Section 70.7 we will use the axiomatic characterization of the Jones polynomial to prove several properties of the Jones polynomial.
- In Section 70.8 we will see that the Jones polynomial determines the determinant of a knot K.
- In Section 70.9 we will provide exercises for Chapter 70.

In the next Chapter 64 we will introduce the braid group and the Burau representation.

70.1. **Reidemeister moves.** In the earlier chapters we defined invariants of an oriented link $L \subseteq \mathbb{R}^3$ by considering the knot complement $\mathbb{R}^3 \setminus L$ or by considering Seifert surfaces for L. This way we defined in particular the Alexander polynomial and the Conway polynomial. In the Conway Polynomial-Characterization Theorem 69.5 we showed that the Conway polynomial is completely characterized by the value of the trivial knot and a "skein relation" on diagrams.

In this chapter we will follow a different route. We will first define an invariant of oriented diagrams and afterwards we will see that it descends to an invariant of oriented links. To make this approach work we need to explain the relationship between oriented links in $\mathbb{R}^3$ and link diagrams. First of all we recall Notation 69.4:

Notation 70.1.

- Let $\texttt{Link}_{\neq\varnothing}(\mathbb{R}^3)$ be the set of smooth isotopy classes of non-empty links in $\mathbb{R}^3$.
- Let $\texttt{Link}^{\text{or}}_{\neq\varnothing}(\mathbb{R}^3)$ be the set of smooth isotopy classes of oriented non-empty links in $\mathbb{R}^3$.

Next we introduce the corresponding notation for link diagrams:

Notation 70.2.

- Let $\texttt{Diagram}_{\neq\varnothing}$ be the set of smooth isotopy classes of non-empty link diagrams.[736]
- Let $\texttt{Diagram}^{\text{or}}_{\neq\varnothing}$ be the set of smooth isotopy classes of oriented non-empty link diagrams.

Recall that in the Diagram-to-Link Lemma 112.5 we associated to any (oriented) link diagram a smooth isotopy class of an (oriented) link. This defines a well-defined map

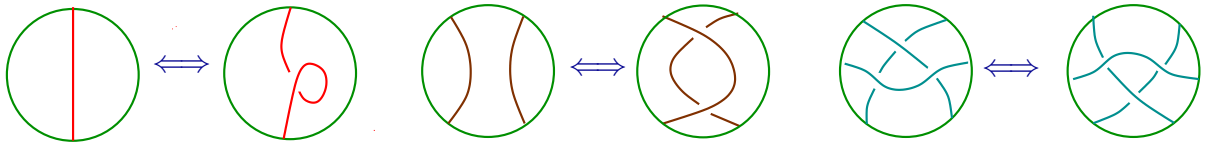
$$\texttt{Diagram}_{\neq\varnothing} \;\to\; \texttt{Link}_{\neq\varnothing}(\mathbb{R}^3).$$

Similarly we get a map in the oriented setting. Now we can formulate the precise relationship between link diagrams and links:

Theorem 70.3. (Link–Link Diagram Theorem) We consider the Reidemeister Moves I, II and III on diagrams that are shown below. The map

$$\begin{array}{rcl} \texttt{Diagram}_{\neq\varnothing}/\text{R-I, R-II, R-III} & \to & \texttt{Link}_{\neq\varnothing}(\mathbb{R}^3) \\ [D] & \mapsto & [\text{associated link}] \end{array}$$

is a bijection. A similar statement holds in the oriented setting.



Reidemeister Move I Reidemeister Move II Reidemeister Move III

in each of the three Reidemeister Moves it is understood that the link diagram outside of the disk stays unchanged

Proof. We showed in the Link Diagram Existence Proposition 112.7 that the map is a surjection. It follows from the Reidemeister Moves Theorem 112.10 that the map is an injection. ■

70.2. **The Kauffman bracket of a diagram.** In this section we will introduce the Kauffman bracket of a diagram. As we will see, the Kauffman bracket admits an elementary definition and it comes surprisingly close to being a link invariant.

First of all we need the following notation:

Notation 70.4.

(a) We denote by $\bigcirc$ any diagram that consists of a single smoothly embedded closed curve with no crossings.

(b) Given a diagram D we denote by $D \sqcup \bigcirc$ any diagram of the form $D \sqcup U$ that admits a smoothly embedded disk $\Sigma \subseteq \mathbb{R}^2$ such that $\Sigma \cap D = \varnothing$ and $\partial\Sigma = U$.

[736] We refer to Definition 112.4 for the precise definition of a link diagram.

Now we are already ready to introduce the Kauffman bracket:

Proposition 70.5. (Kauffman Bracket Proposition) There exists a unique map

$$\begin{array}{rcl} \texttt{Diagram}_{\neq\varnothing} & \to & \mathbb{Z}[A] \\ D & \mapsto & \langle A \rangle \end{array}$$

with the following three properties:[737]

(K1) $\langle \bigcirc \rangle = 1.$

(K2) $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2) \cdot \langle D \rangle.$

(K3) $\langle \times \rangle = A \cdot \langle)(\rangle + A^{-1} \cdot \langle \asymp \rangle.$

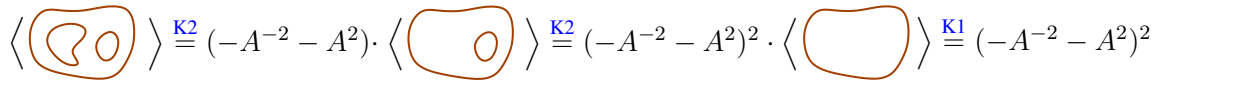
This map is called the Kauffman bracket of D.[738]

Example.

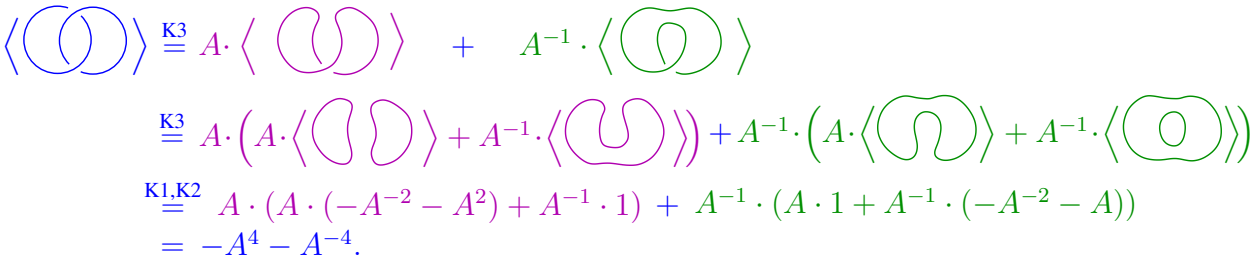
(1) Let $m \in \mathbb{N}$. Note that

$$\langle \underbrace{\bigcirc \dots \bigcirc}_{m \text{ copies}} \rangle \overset{K2}{=} (-A^{-2} - A^2) \cdot \langle \underbrace{\bigcirc \dots \bigcirc}_{m-1 \text{ copies}} \rangle \overset{K2}{=} \dots \overset{K2}{=} (-A^{-2} - A^2)^{m-2} \cdot \langle \bigcirc \rangle \overset{K1}{=} (-A^{-2} - A^2)^{m-1}.$$

(2) Let D be a diagram with m components and without a single crossing. By the Fill-in-Disk Proposition 107.15 we know that each component of the diagram bounds a disk in $\mathbb{R}^2$. Starting with an innermost disk we see iteratively, as in (1), that $\langle D \rangle = (-A^{-2} - A^2)^{m-1}$.



(3) In the figure below we calculate the Kauffman bracket of the standard diagram of the Hopf link:



This calculation should convince the reader that calculating the Kauffman bracket of a diagram is doable. But clearly that is a very time consuming process. □

Proof. It is clear that, given a diagram D, one can, as in the above examples, iteratively apply (K1), (K2) and (K3) to calculate the Kauffman bracket of D. This shows that there exists at most one map with the desired properties. It remains to prove the existence of such a map. A first idea might be to just apply the axioms (K1), (K2) and (K3) many

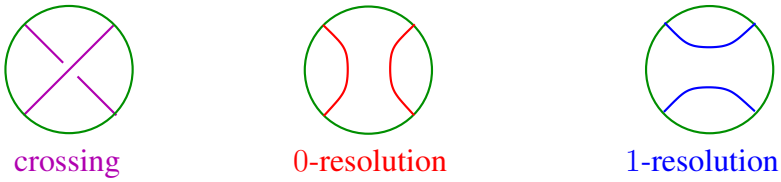
[737] To keep the exposition fluid we leave it to the reader to fill in the precise quantifiers for the three statements.

[738] The Kauffman bracket was introduced by Louis Kauffman [**Kau1987b**].

times to a given diagram D and define $\langle D\rangle$ as the end result. But the following question arises: Is this approach independent of the order in which we deal with the crossings? To get around this issue we will define the Kauffman bracket in a slightly different way, by considering all possible ways to resolve all crossings at once.

To make sense of this idea we need to introduce the following language and notation:

(a) In the figure below we show the 0-resolution and the 1-resolution of a crossing in a diagram.



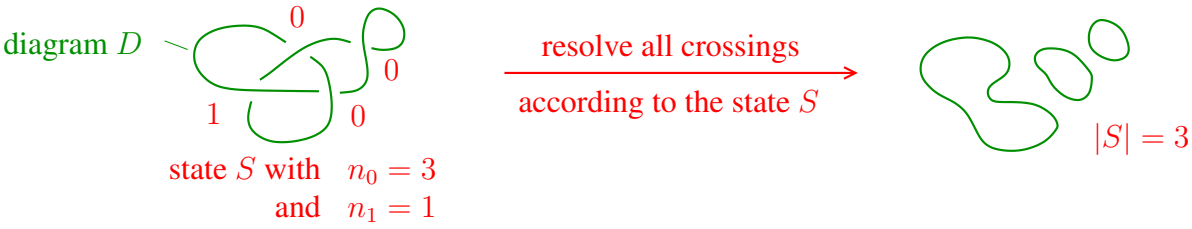
Slightly less pictorially the resolutions are given as follows:

- In an 0-resolution we turn left when we go along the upper strand towards the center.
- In a 1-resolution we turn right when we go along the upper strand towards the center.

(b) A *state* of a diagram D is a choice of one of the two types of resolution for each crossing. We denote by $\mathrm{State}(D)$ the set of states of D. Note that a diagram with c crossings has 2^c states, i.e. $\#\,\mathrm{state}(D) = 2^c$.

(c) Given a state S of a diagram D we introduce the following two notions:

- We write $\langle D\,|\,S\rangle := t^{n_0-n_1}$ where n_0 is the number of 0-resolutions and where n_1 is the number of 1-resolutions in the state.
- We denote by $|S|$ the number of components that we obtain by resolving all crossings of D according to the state S.



Given a diagram D we set, inspired by the above example (2) and the Kauffman axiom (K3):

$$\langle D\rangle := \sum_{S\in\mathrm{State}(D)} \langle D\,|\,S\rangle \cdot (-A^{-2}-A^2)^{|S|-1}.$$

We claim that the map $D \mapsto \langle D\rangle$ has all the desired properties. First note that it follows immediately from the definition that $\langle D\rangle$ is a smooth isotopy invariant, i.e. it defines a map $\mathtt{Diagram}_{\neq\varnothing} \to \mathbb{Z}[A]$. In the following claim we will deal with the three Kauffman axioms:

Claim.

(K1) $\langle \bigcirc \rangle = 1.$

(K2) $\langle D \sqcup \bigcirc \rangle = (-A^{-2}-A^2)\cdot\langle D\rangle.$

(K3) $\langle \times \rangle = A\cdot\langle)(\rangle + A^{-1}\cdot\langle \asymp \rangle.$

Proof. All three statements follow from elementary observations:

(K1) The trivial one-component diagram has precisely one state S. Clearly this state has zero 0-resolutions and zero 1-resolutions and it satisfies $|S| = 1$. The statement now follows immediately from the above definition of $\langle D \rangle$.

(K2) This statement follows from the observation that $D \sqcup \bigcirc$ has the same states as D, but when we resolve the crossings we always get one extra component.

(K3) Let $\text{State}_0\left(\times\right)$ be the states of the original diagram where the resolution of the crossing under consideration is given by a 0-resolution. Similarly we define $\text{State}_1\left(\times\right)$. The statement follows easily from the definition of $\langle D \rangle$ and the observation that

$$\begin{aligned} \text{State}\left(\times\right) &= \text{State}_0\left(\times\right) \sqcup \text{State}_1\left(\times\right), \\ \text{State}\left(\supset\subset\right) &= \text{State}_0\left(\times\right), \\ \text{State}\left(\cup\cap\right) &= \text{State}_1\left(\times\right). \end{aligned}$$

■

The real question is, whether the Kauffman bracket polynomial is invariant under the Reidemeister moves. The following proposition shows that this is a near miss:

Proposition 70.6. (Kauffman Bracket–Reidemeister Move Proposition)

(I) If a diagram D' is obtained from a diagram D by the "left-to-right" Reidemeister I move which adds a crossing, then

$$\langle D' \rangle \;=\; (-A^3) \cdot \langle D \rangle.$$

(II) The Kauffman bracket is unchanged under any Reidemeister II move.

(III) The Kauffman bracket is unchanged under any Reidemeister III move.

Proof. We consider the three Reidemeister Moves separately:

(I) In the following calculation we see that the Kauffman axioms (K2) and (K3), and the smooth isotopy invariance of the Kauffman polynomial, imply that the "left-to-right" Reidemeister I move leads to an extra factor of $-A^3$:

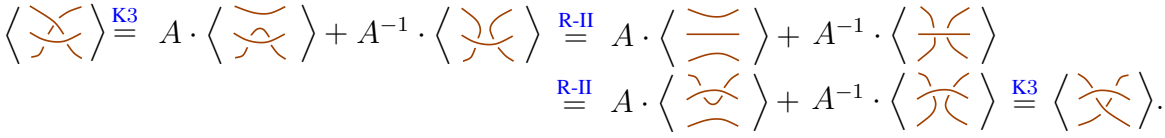
$$\begin{aligned} \left\langle \right\rangle &\overset{\text{K3}}{=} A \cdot \left\langle \right\rangle + A^{-1} \cdot \left\langle \right\rangle \\ &\overset{\text{K2}}{=} A \cdot (-A^2 - A^{-2}) \cdot \left\langle \right\rangle + A^{-1} \cdot \left\langle \right\rangle \;=\; -A^3 \cdot \left\langle \right\rangle. \end{aligned}$$

(II) In the following calculation we see that the Kauffman axioms (K2) and (K3), and the smooth isotopy invariance of the Kauffman polynomial, imply that the Kauffman bracket is unchanged under a Reidemeister II move:

$$\begin{aligned} \left\langle \right\rangle &\overset{\text{K3}}{=} A \cdot \left\langle \right\rangle + A^{-1} \cdot \left\langle \right\rangle \\ &\overset{\text{K3}}{=} A \cdot \left(A \cdot \left\langle \right\rangle + A^{-1} \cdot \left\langle \right\rangle \right) + A^{-1} \cdot \left(A \cdot \left\langle \right\rangle + A^{-1} \cdot \left\langle \right\rangle \right) \\ &\overset{\text{K2}}{=} A^2 \cdot \left\langle \right\rangle + \left\langle \right\rangle + (-A^2 - A^{-2}) \cdot \left\langle \right\rangle + A^{-2} \cdot \left\langle \right\rangle \\ &= \left\langle \right\rangle. \end{aligned}$$

(II) In the following calculation we see that the Kauffman axiom (K3), together with the invariance of the Kauffman bracket under Reidemeister II moves (which we just

established), implies that the Kauffman bracket is unchanged under a Reidemeister III move:



We have thus dealt with all three types of Reidemeister moves. ■

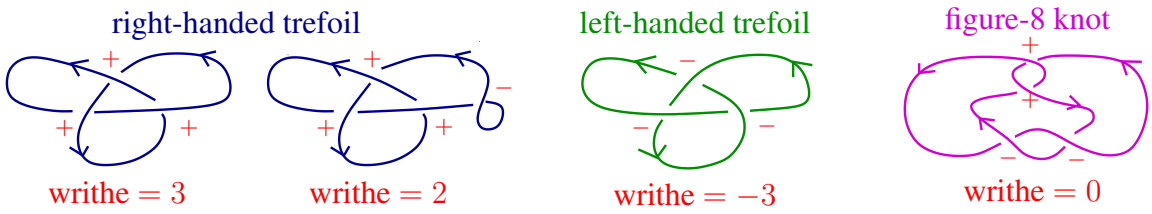
The Kauffman Bracket–Reidemeister Move Proposition 70.6 tells us already something quite interesting: Up to multiplication by a power of $-A^3$ the Kauffman bracket gives us a well-defined invariant of links. But we want something even better, namely an invariant that is well-defined without any indeterminacies. The key idea to achieving that goal is the notion of the writhe of an oriented link diagram which we will introduce in the next section.

70.3. **The writhe of an oriented link diagram.** Let D be an *oriented* link diagram. Given a crossing of the oriented diagram we introduced in Definition 113.1 what it means for the crossing to be positive or negative. We will not repeat the completely formal definition of the sign of a crossing. Loosely speaking a crossing is positive if one has to turn counterclockwise to go from the direction of the upper strand to the direction of the lower strand. Hopefully the following figure will remind the reader of the concept:



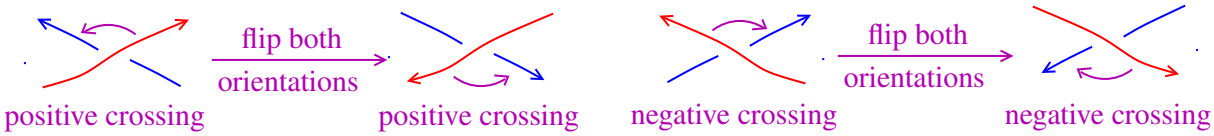
To compensate for the fact that the Kauffman bracket is not invariant under the Reidemeister I move we now introduce the following definition, which we first encountered in Exercise 112.11:

Definition 70.7. Given an oriented link diagram D we define the writhe $w(D)$ of D as the sum of the signs of the crossings. Note that the writhe is clearly invariant under smooth isotopies, it thus defines a map $w\colon \mathtt{Diagram}^{\mathrm{or}}_{\neq\varnothing} \to \mathbb{Z}$.



Lemma 70.8. (Writhe Orientation Lemma) Let D be an oriented link diagram. If we swap the orientation of *all* components, then the writhe is unchanged.

Proof. It follows easily from the definitions, or the figure below, that swapping the orientation of both strands in a crossing does not change the sign of a crossing.[739]



It now follows immediately from Definition 70.7 that the writhe is unchanged if we swap the orientations of all components of the diagram. ■

The following lemma shows that with respect to Reidemeister moves the writhe behaves similarly to the Kauffman bracket polynomial:

Lemma 70.9. (Writhe–Reidemeister Move Lemma)

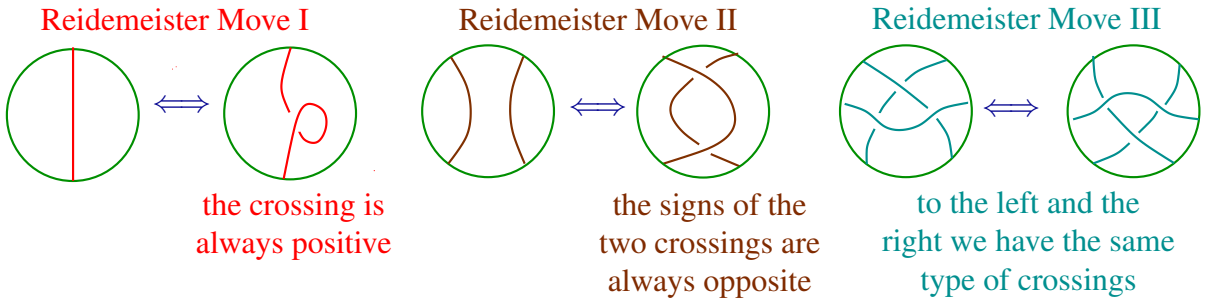
(I) If a diagram D' is obtained from a diagram D by the "left-to-right" Reidemeister I move which adds a crossing, then

$$w(D') \;=\; w(D) + 1.$$

(II) The writhe is unchanged under any Reidemeister II move.

(III) The writhe is unchanged under any Reidemeister III move.

Proof. We start out by recalling the unoriented Reidemeister moves:



We now turn to the discussion of the three Reidemeister moves:

(I) Regardless of the orientation of the strand, the sign of the extra crossing is always going to be $+1$.

(II) Regardless of the orientations of the strands the two extra crossings will invariably have opposite signs. So the contribution to the writhe is zero.

(III) Regardless of the orientations of the strands, the crossings to the left and right have matching signs. So the contribution to the writhe is unchanged. ■

70.4. **The Jones polynomial of an oriented link.** In the Kauffman Bracket–Reidemeister Move Proposition 70.6 and the Writhe–Reidemeister Move Lemma 70.9 we showed that the Kauffman bracket and the writhe are both unchanged under Reidemeister moves II and III and that both change in a very predictable way under a Reidemeister I move.

The following theorem shows that the writhe can be used to magically cancel the deficiencies of the Kauffman bracket:

[739] The formal proof of this fact, which really refer to Definition 113.1, boils down to the observation that $\det(-\mathrm{id}_2) = (-1)^2 = 1$.

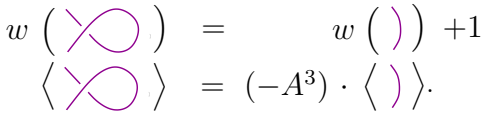
Theorem 70.10. (Jones Polynomial Existence Theorem) Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. We pick a diagram D for L. The expression

$$(-A^3)^{-w(D)} \cdot \langle D \rangle$$

does not depend on the choice of the diagram D for the link L.[740]

Proof. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. Let D and D' be two diagrams for L. By the Link–Link Diagram Theorem 70.3 we know that D and D' are related by a finite sequence of Reidemeister moves and smooth isotopies. By the Writhe–Reidemeister Move Lemma 70.9 and the Kauffman Bracket–Reidemeister Move Proposition 70.6 we know that the two factors of $(-A^3)^{-w(D)} \cdot \langle D \rangle$ are unchanged under Reidemeister Moves II and III.

Finally we turn to the slightly more interesting Reidemeister Move I. For the reader's convenience we recall the relevant statements of the Writhe–Reidemeister Move Lemma 70.9 and the Kauffman Bracket–Reidemeister Move Proposition 70.6:



It is now clear that under a Reidemeister I Move the changes in the two factors of the term $(-A^3)^{-w(D)} \cdot \langle D \rangle$ cancel. This shows that the product $(-A^3)^{-w(D)} \cdot \langle D \rangle$ is actually invariant under any Reidemeister I Move. ■

The Jones Polynomial Existence Theorem 70.10 leads us to the following definition:

Definition 70.11. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. By the Link Diagram Existence Proposition 112.7 we know that L admits a diagram D. It follows from the Jones Polynomial Existence Theorem 70.10 that[741] [742] [743]

$$V_L(t) \ := \ ((-A^3)^{-w(D)} \cdot \langle D \rangle)|_{t^{\frac{1}{4}}=A^{-1}} \ \in \ \mathbb{Z}[t^{\pm\frac{1}{4}}]$$

is a well-defined smooth isotopy invariant of L. We refer to it as the Jones polynomial of the oriented link L.

Example.

(1) Let $m \in \mathbb{N}$. For the trivial m-component link $L \subseteq \mathbb{R}^3$ we take the standard diagram D. Clearly this diagram has writhe $w(D) = 0$. Furthermore recall that on page 1464

[740] Given an oriented diagram D we could also define the *reduced writhe* $\widetilde{w}(D)$ as the sum of the signs of the crossings between strands of the *same* component. The argument of the proof of the Jones Polynomial Existence Theorem 70.10 shows that the expression $(-A)^{-3\widetilde{w}(D)} \cdot \langle D \rangle$ also defines a link invariant. The reason why we prefer $w(D)$ over $\widetilde{w}(D)$ will become apparent in the proof of the Jones Polynomial-Characterization Theorem 70.16.

[741] The notation $\mathbb{Z}[t^{\pm\frac{1}{4}}]$ is suggestive, but also a little awkward. For us $\mathbb{Z}[t^{\pm\frac{1}{4}}]$ is the Laurent polynomial ring on a single-variable, which happens to be called $t^{\frac{1}{4}}$, and we use the obvious notation that $(t^{\frac{1}{4}})^2 = t^{\frac{1}{2}}$, that $(t^{\frac{1}{4}})^4 = t$ and so on.

[742] The notation $V_L(t)$ is in honor of Vaughan Jones, who introduced the Jones polynomial [**JonV1985, JonV1987**] through a totally different approach. The notation $V_L(t)$ might also have been chosen since, at least in a hand-written form, it fits well with the notation $\Delta_L(t)$ and $\nabla_L(t)$ for the Alexander and the Conway polynomial.

[743] In 1990 Vaughan Jones was awarded a Fields medal for the discovery of the Jones polynomial.

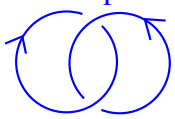
we showed that $\langle D\rangle = (-A^{-2}-A^2)^{m-1}$. With the substitution $t^{\frac{1}{4}} = A^{-1}$ we see that

$$V_{\text{trivial } m\text{-component link}}(t) \;=\; (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{m-1}.$$

Recall that in contrast to this calculation we showed on page 1442 that for $m \in \mathbb{N}_{\geq 2}$ the Conway polynomial of the trivial m-component link is zero. Thus already in this basic example we see that the Jones polynomial is dramatically different from the Conway polynomial.

(2) In the following we consider the positive Hopf link H^+. Using the calculation of the Kauffman bracket on page 1464 we can now easily calculate the Jones polynomial of the positive Hopf link H^+:

positive Hopf link H^+



$$\langle D\rangle = -A^4 - A^{-4}$$
$$w(D) = 2$$
$$\Longrightarrow\quad (-A)^{-3w(D)} \cdot \langle D\rangle = (-A)^{-3\cdot 2} \cdot (-A^4 - A^{-4}) = -A^{-2} - A^{-10}$$
$$\Longrightarrow\quad V_{H^+}(t) = -t^{\frac{1}{2}} - t^{\frac{5}{2}}$$

Since this is so much fun we also calculate the Jones polynomial of the negative Hopf link:

negative Hopf link H^-



$$\langle D\rangle = -A^4 - A^{-4}$$
$$w(D) = -2$$
$$\Longrightarrow\quad (-A)^{-3w(D)} \cdot \langle D\rangle = (-A)^{-3\cdot(-2)} \cdot (-A^4 - A^{-4}) = -A^{10} - A^2$$
$$\Longrightarrow\quad V_{H^-}(t) = -t^{-\frac{5}{2}} - t^{-\frac{1}{2}}$$

This shows that the Jones polynomial can distinguish between these two Hopf links.

In principle we have all the tools for calculating more Jones polynomials. But the approach via the Kauffman bracket is rather time consuming, since a diagram with c crossings leads to 2^c summands. In the next section we will get to know a more efficient way to calculate the Jones polynomial. □

We conclude this section with two elementary observations on the Jones polynomial. First of all we have the following analog of the Conway Polynomial–Reverse Lemma 295.11:

Lemma 70.12. (Jones Polynomial–Reverse Lemma) For any oriented non-empty link $L \subseteq \mathbb{R}^3$ we have

$$V_{L^{\text{rev}}}(t) \;=\; V_L(t).$$

This shows in particular that the Jones is an invariant of *unoriented* knots.

Proof. This lemma follows almost immediately from the fact that the Kauffman bracket is an invariant of unoriented diagrams together with the Writhe Orientation Lemma 70.8. ■

The following lemma is an analog of the Conway Polynomial–Mirror Lemma 295.12:

Lemma 70.13. (Jones Polynomial–Mirror Lemma) For any oriented non-empty link $L \subseteq \mathbb{R}^3$ we have the following equality in $\mathbb{Z}[t^{\pm 1}]$:

$$V_{L^{\text{mir}}}(t) \;=\; V_L(t^{-1}).$$

Proof. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. We pick a diagram D for L. We denote by D^{mir} the diagram where we flipped all crossings of D. Clearly D^{mir} is a diagram for L^{mir}. It follows easily from Definition 70.11 that it suffices to prove the following claim:

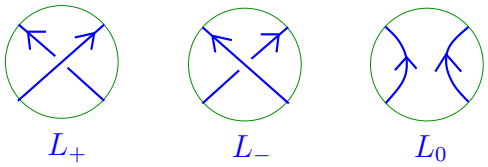
Claim.
(1) $\langle D^{\mathrm{mir}}\rangle = \langle D\rangle|_{A=A^{-1}}$.
(2) $w(D^{\mathrm{mir}}) = -w(D)$.

Proof.
(1) This statement follows immediately from the uniqueness statement of the Kauffman Bracket Proposition 70.5 and the observation that the map $D \mapsto \langle D\rangle|_{A=A^{-1}}$ clearly also satisfies the three axioms (K1), (K2) and (K3).
(2) This statement follows easily from Definition 70.7 and the observation that flipping a crossing flips the sign of the crossing. ■

70.5. **The Jones skein relation.** The following theorem is the analog of the Alexander-Conway Skein Theorem 68.7:

Theorem 70.14. (Jones Skein Theorem) The Jones polynomial has the following property: Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same *diagram*, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown below. Then we have the Jones skein relation

$$t^{-1}\cdot V_{L_+}(t) - t\cdot V_{L_-}(t) \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\cdot V_{L_0}(t).$$



Remark. In the literature there are several alternative, but essentially equivalent, definitions of the Jones polynomial, which can in particular lead to slightly different skein relations. For example in [**JonV1985**, Theorem 12] the definition of the Jones polynomial leads to the skein relation $t^{-1}\cdot V(\,\times\,) - t\cdot V(\,\times\,) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\cdot V(\,)(\,)$. □

Proof. Let $\times$, $\times$ and $)($ be the given oriented diagrams for L_+, L_-, L_0. We also consider the corresponding unoriented diagrams $\times$, $\times$ and $)($. Note that by Definition 70.11 we need to prove the following claim:

Claim.
$A^4\cdot(-A^3)^{-w(\times)}\cdot\langle\,\times\,\rangle - A^{-4}\cdot(-A^3)^{-w(\times)}\cdot\langle\,\times\,\rangle \;=\; (A^{-2}-A^2)\cdot(-A^3)^{-w()()}\cdot\langle\,)(\,\rangle.$

Proof. We add to the above diagrams the unoriented diagram $\smile\!\frown$.[744] By the Kauffman Bracket Proposition 70.5 we have the following equalities:

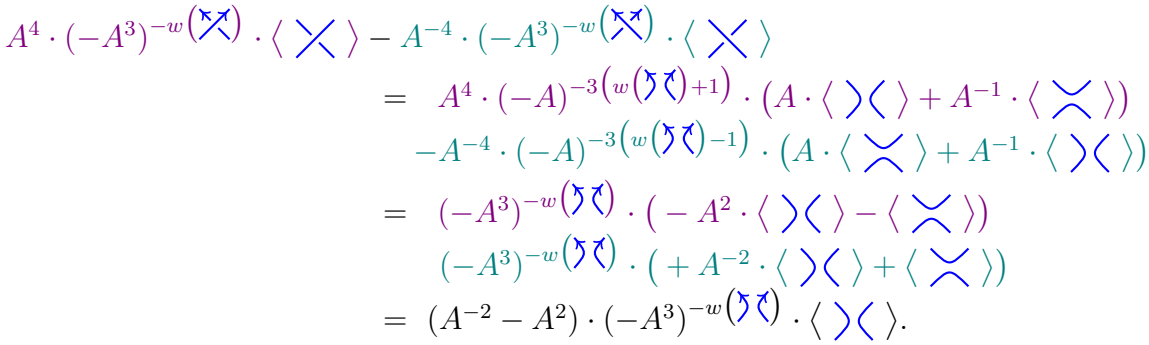
$$\begin{aligned}\langle\,\times\,\rangle &= A\cdot\langle\,)(\,\rangle + A^{-1}\cdot\langle\,\smile\!\frown\,\rangle,\\ \langle\,\times\,\rangle &= A\cdot\langle\,\smile\!\frown\,\rangle + A^{-1}\cdot\langle\,)(\,\rangle.\end{aligned}$$

Furthermore it follows almost immediately from Definition 70.7 that

$$\begin{aligned}w(\times) &= w()() + 1,\\ w(\times) &= w()() - 1.\end{aligned}$$

[744]Note that we cannot give $\smile\!\frown$ an orientation that is coherent with the orientations of the other diagrams. But this issue is irrelevant for the argument that follows.

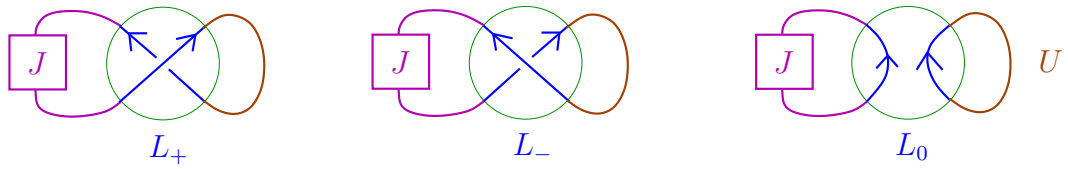
Now we just need to combine all these facts and we are done. Indeed, we have the following equalities:



$$A^4 \cdot (-A^3)^{-w(\ )} \cdot \langle \ \rangle - A^{-4} \cdot (-A^3)^{-w(\ )} \cdot \langle \ \rangle$$
$$= A^4 \cdot (-A)^{-3(w(\ )+1)} \cdot (A \cdot \langle \ \rangle + A^{-1} \cdot \langle \ \rangle)$$
$$-A^{-4} \cdot (-A)^{-3(w(\ )-1)} \cdot (A \cdot \langle \ \rangle + A^{-1} \cdot \langle \ \rangle)$$
$$= (-A^3)^{-w(\ )} \cdot (-A^2 \cdot \langle \ \rangle - \langle \ \rangle)$$
$$(-A^3)^{-w(\ )} \cdot (+A^{-2} \cdot \langle \ \rangle + \langle \ \rangle)$$
$$= (A^{-2} - A^2) \cdot (-A^3)^{-w(\ )} \cdot \langle \ \rangle. \qquad \blacksquare$$

Example.

(1) We consider the split union L_0 of a non-empty link J together with an unknot U. It is essentially clear that we can find a diagram for L_0 as shown at the bottom right.



We now see that

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V_{L_0}(t) \underset{\substack{\uparrow \\ \text{by the Jones} \\ \text{Skein Theorem 70.14}}}{=} t^{-1} \cdot V_{L_+}(t) - t \cdot V_{L_-}(t) \underset{\substack{\uparrow \\ \text{since } L_\pm \text{ are both} \\ \text{smoothly isotopic to } J}}{=} (t^{-1} - t) \cdot \nabla_J(t).$$

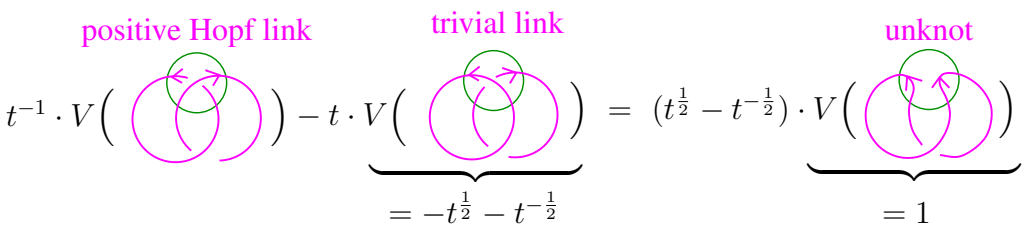
Solving for $V_{J \sqcup U}(t) = V_{L_0}(t)$ we see that

$$V_{J \sqcup U}(t) = \frac{t^{-1} - t}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \cdot V_J(t) = (-t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V_J(t).$$

On page 1470 we showed that the Jones polynomial of the unknot equals 1. If we apply the above logic iteratively to trivial links with $2, 3, 4, \ldots$ components we see, as on page 1470, that

$$V_{\text{trivial } m\text{-component link}}(t) = (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{m-1}.$$

(2) In the following we want to apply the Jones Polynomial-Characterization Theorem 70.16 to (re-) calculate the Jones polynomial of the positive Hopf link. Using the above calculation of the Jones polynomial for the trivial links we obtain the following equality:



Solving for the Jones polynomial of the positive Hopf link we see, as on page 1470, that

$$V_{\text{positive Hopf link}}(t) = t \cdot (t \cdot (-t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})) = -t^{\frac{5}{2}} - t^{\frac{1}{2}}.$$

Using the same type of argument, or appealing to the Jones Polynomial–Mirror Lemma 70.13, we also obtain that

$$V_{\text{negative Hopf link}}(t) \;=\; -t^{-\frac{5}{2}} - t^{-\frac{1}{2}}.$$

(3) Similar to (2), and using the above calculations of the Jones polynomial of the trivial links and the negative Hopf link we obtain the following equality:

unknot left-handed trefoil negative Hopf link

$$t^{-1}\cdot \underbrace{V\Big(\quad\Big)}_{=1} \;-\; t\cdot V\Big(\quad\Big) \;=\; (t^{\frac{1}{2}}-t^{-\frac{1}{2}})\cdot \underbrace{V\Big(\quad\Big)}_{=-t^{-\frac{5}{2}}-t^{-\frac{1}{2}}}$$

Solving for the Jones polynomial of the left-handed trefoil we see that

$$V_{\text{left-handed trefoil}}(t) \;=\; t^{-1}\cdot(t^{-1}-\underbrace{(t^{\frac{1}{2}}-t^{-\frac{1}{2}})\cdot(-t^{-\frac{5}{2}}-t^{-\frac{1}{2}})}_{=-t^{-2}-1+t^{-3}+t^{-1}}) \;=\; -t^{-4}+t^{-3}+t^{-1}.$$

Note that it follows from this calculation together with the Jones Polynomial–Mirror Lemma 70.13 that the Jones polynomial of the right-handed trefoil is given by $-t^4+t^3+t^1$. This shows that the Jones polynomial can, in contrast to the Alexander polynomial, distinguish the left-handed trefoil from the right-handed trefoil.

(4) Similar to (2) we now want to calculate the Jones polynomial of the figure-8 knot:

figure-8 knot unknot negative Hopf link

$$t^{-1}\cdot V\Big(\quad\Big) - t\cdot \underbrace{V\Big(\quad\Big)}_{=1} \;=\; (t^{\frac{1}{2}}-t^{-\frac{1}{2}})\cdot \underbrace{V\Big(\quad\Big)}_{=-t^{-\frac{5}{2}}-t^{-\frac{1}{2}}}$$

Solving for the Jones polynomial of the figure-8 knot we see that[745]

$$V_{\text{figure-8 knot}}(t) \;=\; t\cdot(t+\underbrace{(t^{\frac{1}{2}}-t^{-\frac{1}{2}})\cdot(-t^{-\frac{5}{2}}-t^{-\frac{1}{2}})}_{=-t^{-2}-1+t^{-3}+t^{-1}}) \;=\; t^2-t+1-t^{-1}+t^{-2}.$$

□

We conclude this section with a proposition, which tells us in particular that the Jones polynomials is always non-zero:[746]

Proposition 70.15. (Jones Polynomial-at-1 Proposition) Let $m\in\mathbb{N}$. Furthermore let $L\subseteq\mathbb{R}^3$ be an oriented m-component link. Then

$$V_L(1) \;=\; (-2)^{m-1}.$$

Proof. The proposition follows immediately from the following claim:

[745]Recall that on page 2107 we showed that the figure-8 knot is amphichiral. This observation, together with the Jones Polynomial–Mirror Lemma 70.13, gives a theoretical reason why the Jones polynomial of the figure-8 knot just had to be symmetric.

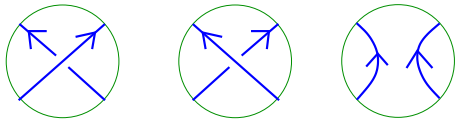
[746]The proposition can be viewed as an analog of the Alexander Polynomial-at-1 Proposition 289.17 and of the Conway Polynomial–Low Coefficients Proposition 69.8.

Claim.
(1) The equality holds for all trivial links.
(2) The evaluation of the Jones polynomial at $t = 1$ is unaffected by any crossing change.
(3) Any diagram can be turned into a diagram of a trivial link by crossing changes.

Proof.
(1) Recall that on page 1470 and on page 1472 we showed that for any $m \in \mathbb{N}$ we have the equality:
$$V_{\text{trivial } m\text{-component link}}(t) \;=\; (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{m-1}.$$
Plugging in $t = 1$ gives us the desired result.
(2) The Jones Skein Theorem 70.14 tells us, with the obvious notation, that
$$t^{-1} \cdot V(\;) - t \cdot V(\;) \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V(\;).$$
Thus, plugging in $t = 1$ we see that
$$V(\;)|_{t=1} - V(\;)|_{t=1} \;=\; 0.$$
(3) This statement is the content of the Unlinking Proposition 69.2. ■

70.6. **A characterization of the Jones polynomial.** The following theorem is an analog of the Conway Polynomial-Characterization Theorem 69.5:

Theorem 70.16. (Jones Polynomial-Characterization Theorem) There exists precisely one map
$$\texttt{Link}^{\text{or}}_{\neq\varnothing}(\mathbb{R}^3) \;\to\; \mathbb{C}(t^{\frac{1}{4}})$$
which satisfies the following two Jones axioms:
(J1) The invariant of the unknot equals 1.
(J2) The invariant satisfies the Jones skein relation that is shown below.



(J1) $V_{\text{unknot}}(t) \;=\; 1$
(J2) $t^{-1} \cdot V_{L_+}(t) - t \cdot V_{L_-}(t) \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V_{L_0}(t)$

Remark. The Conway Polynomial-Characterization Theorem 69.5 and the Jones Polynomial-Characterization Theorem 70.16 can be summarized as follows: The Conway polynomial is completely determined by the following axioms

(C1) $\nabla(\;) \;=\; 1$ (C2) $\nabla(\;) \;-\; \nabla(\;) \;=\; z \cdot \nabla(\;)$,

and the Jones polynomial is completely determined by the following axioms:[747]

(J1) $V(\;) \;=\; 1$ (J2) $t^{-1} \cdot V(\;) \;-\; t \cdot V(\;) \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V(\;)$. □

Proof. First note that it follows from the calculation on page 1472 and the Jones Skein Theorem 70.14 that the Jones polynomial satisfies Properties (J1) and (J2). This gives us the existence statement of the theorem.

[747]The properties (C1) and (C2) were essentially known to James Alexander [**Ale1927**]. It is thus quite stunning that it took 60 years to come up with the definition of the Jones polynomial [**JonV1985, JonV1987**].

The uniqueness statement follows of course from the Link Invariant–Uniqueness Proposition 69.6 applied to $\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \to \mathbb{C}(t^{\frac{1}{4}})$ with $a_+ = t^{-1}$, $a_- = -t$, $a_0 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and with $u = 1$. ■

70.7. **Consequences of the Jones Polynomial-Characterization Theorem 70.16.** In this section we will use the Jones Polynomial-Characterization Theorem 70.16 to prove several statements about the Jones polynomial.

First of all, the following proposition, which can be viewed as an analog of the Conway Polynomial–Zero Coefficients Proposition 295.9, implies in particular that the Jones polynomial of an oriented knot is an honest-to-G*d Laurent polynomial in t:

Proposition 70.17. (Jones Polynomial–Symmetry Proposition) Let L be an oriented non-empty m-component link.

(1) If m is odd, then $V_L(t) \in \mathbb{Z}[t^{\pm 1}]$.

(2) If m is even, then $V_L(t) \in t^{\frac{1}{2}} \cdot \mathbb{Z}[t^{\pm 1}]$.

Proof. Let L be an oriented non-empty m-component link. Recall that the Jones polynomial lies in $\mathbb{Z}[t^{\pm\frac{1}{4}}]$, which is a Laurent polynomial ring in the variable $t^{\frac{1}{4}}$. Thus it makes sense to set[748]

$$\widetilde{V}_L(t^{\frac{1}{4}}) \;:=\; (-1)^{m-1} \cdot V_L(\mathrm{i}t^{\frac{1}{4}}) \;\in\; \mathbb{C}(t^{\frac{1}{4}}).$$

Elementary algebra shows that our desired statements are equivalent to the following:[749]

Statement. For every oriented non-empty m-component link $L \subseteq \mathbb{R}^3$ we have the equality

$$\underbrace{(-1)^{m-1} \cdot V_L(\mathrm{i}t^{\frac{1}{4}})}_{=\widetilde{V}_L(t^{\frac{1}{4}})} \;=\; V_L(t^{\frac{1}{4}}).$$

It follows from the Jones Polynomial-Characterization Theorem 70.16 that to prove the above statement it suffice to prove the following claim:

Claim. The invariant $\widetilde{V}_L(t^{\frac{1}{2}}) \in \mathbb{Q}(t^{\frac{1}{2}})$ satisfies the Jones axioms (J1) and (J2).

Proof.

(J1) For the unknot U we have $\widetilde{V}_U(t^{\frac{1}{2}}) = (-1)^{1-1} \cdot V_U(-\mathrm{i}t^{\frac{1}{2}}) = 1$.

(J2) An elementary argument shows that in the usual setting of crossing change the following statements hold:

- L_- and L_+ have the same number of components.
- The number of components of L_0 differs from the number of components of $L_\pm$ by *exactly one*.

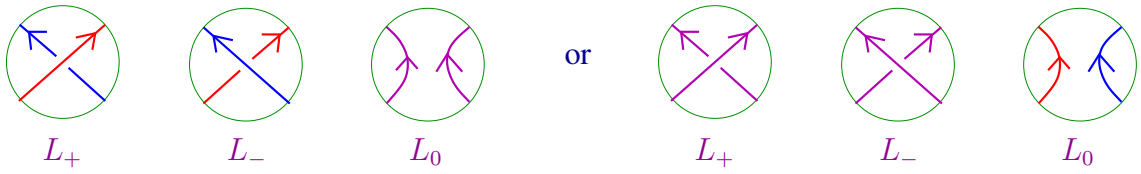
It follows easily from this observation and elementary algebra that $\widetilde{V}_L(t^{\frac{1}{2}}) \in \mathbb{C}(t^{\frac{1}{4}})$ satisfies the Jones axioms (J2). ■

[748] The author does not claim responsibility for introducing variables of the type $t^{\frac{1}{2}}$ and $t^{\frac{1}{4}}$.

[749] Since this step is confusing, let us fill in some details. Let $f(z) = \sum_{k\in\mathbb{Z}} a_k \cdot z^k$ be a polynomial in $\mathbb{C}[z]$.

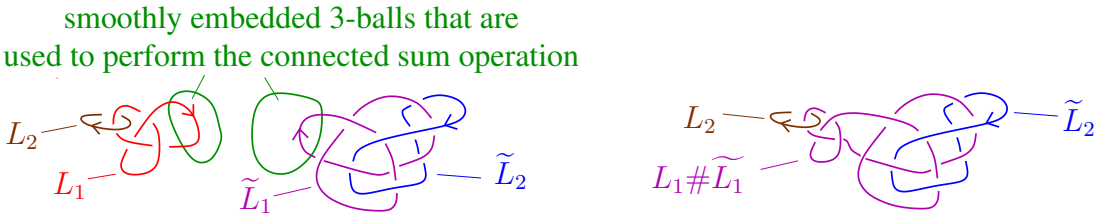
(1) If $f(\mathrm{i}z) = f(z)$, then we claim that f is a polynomial in z^4. This can can be seen as follows: We have $f(\mathrm{i}z) = \sum_{k\in\mathbb{Z}} (\mathrm{i}^k \cdot a_k) \cdot z^k$. Thus if $f(\mathrm{i}z) = f(z)$, then we have for all $k \in \mathbb{Z}$ that $a_k = \mathrm{i}^k \cdot a_k$, which implies that $a_k = 0$ unless $4|k$.

(2) If $f(\mathrm{i}z) = -f(z)$, then we claim that $z^2 \cdot f(z)$ is a polynomial in z^4. This can can be seen as follows: As in (1) we see that for all $k \in \mathbb{Z}$ we have $a_k = -\mathrm{i}^k \cdot a_k$, which implies that $a_k = 0$ unless $4|(k+2)$.



In Definition 63.6 we introduced the connected sum of two oriented knots. Using the more general set up from Definition 56.17 we can generalize this concept to links as long as we specify which components we pick out:

Definition 70.18. Let $L \subseteq \mathbb{R}^3$ be an oriented link with $m \in \mathbb{N}$ components and let $\widetilde{L} \subseteq \mathbb{R}^3$ be an oriented link with $\widetilde{m} \in \mathbb{N}$ components We perform the connected sum, as in Definition 56.17, along the first components of L and $\widetilde{L}$ to obtain an oriented $(m+\widetilde{m}-1)$-component link $L\#\widetilde{L}$.



The following proposition is the analog of the Connected Sum–Conway Polynomial Proposition 69.10. (For the purpose of the current discussion it is better to look at the formulation on page 1452.)

Proposition 70.19. (Connected Sum–Jones Polynomial Proposition) Let L and $\widetilde{L}$ be two oriented non-empty links in $\mathbb{R}^3$. The following equality holds:

$$V_{L\#\widetilde{L}}(t) \;=\; V_L(t)\cdot V_{\widetilde{L}}(t) \;\in\; \mathbb{Z}[t^{\pm 1}].$$

Proof. [750] We fix an oriented non-empty link $\widetilde{L} \subseteq \mathbb{R}^3$. Recall that the Jones Polynomial-at-1 Proposition 70.15 tells us that $V_{\widetilde{L}}(t) \neq 0$. Thus we can consider the map

$$\begin{aligned} \Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) &\to \mathbb{C}(t^{\frac{1}{4}}) \\ L &\mapsto \frac{V_{L\#\widetilde{L}}(t)}{V_{\widetilde{L}}(t)}. \end{aligned}$$

Claim. The above map $\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \to \mathbb{C}(t^{\frac{1}{4}})$ satisfies the axioms (J1) and (J2) from the Jones Polynomial-Characterization Theorem 70.16.

Proof.

(J1) Since the connected sum of the link $\widetilde{L}$ with the trivial knot U is smoothly isotopic to $\widetilde{L}$ we see that $\Theta(U) = 1$.

(J2) Suppose that we are given three oriented links L_+, L_-, L_0 as in the formulation of (J2). By performing the connected sum of a diagram of $\widetilde{L}$ with the given diagrams we obtain again a triple that satisfies the compatibility conditions and we obtain from

[750] As we will see, the proof follows the same logic as the proof of the Connected Sum–Conway Polynomial Proposition 69.10 that we gave on page 1452.

(J2) the equality

$$t^{-1} \cdot V_{L_+ \# \widetilde{L}}(t) - t \cdot V_{L_- \# \widetilde{L}}(t) \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V_{L_0 \# \widetilde{L}}(t).$$

Dividing by $V_{\widetilde{L}}$ we see that Θ satisfies (J2). ⊞

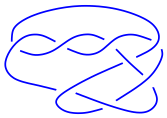
Note that it follows from the claim together with the Jones Polynomial-Characterization Theorem 70.16 that for any oriented non-empty link $\widetilde{L} \subseteq \mathbb{R}^3$ we have

$$\frac{V_{L \# \widetilde{L}}(t)}{V_{\widetilde{L}}(t)} \;=\; V_L(t).$$

This gives us the desired property. ■

Example. In the following figure we see the connected sum $K_1 = 4_1 \# 4_1$ of two copies of the figure-8 knot and we show the knot $K_2 = 8_9$.

$K_1 = 4_1 \# 4_1$ 

$K_2 = 8_9$ 

$V_{K_1}(t) = (t^{-2} - t^{-1} + 1 - t + t^2)^2$

$\Delta_{K_1}(t) = (t^{-1} - 3 + t)^2$

$V_{K_2}(t) = (t^{-2} - t^{-1} + 1 - t + t^2)^2$

$\Delta_{K_2}(t) = -t^{-3} + 3t^{-2} - 5t^{-1} + 7 - 5t + 3t^2 - t^3$

It follows from the calculation on page 1473 together with the Connected Sum–Jones Polynomial Proposition 70.19 we see that

$$V_{4_1 \# 4_1}(t) \;=\; V_{4_1}(t)^2 \;=\; (t^{-2} - t^{-1} + 1 - t + t^2)^2.$$

A heroic calculation, which we gladly leave to the reader, shows that the Jones polynomial of 8_9 is the same. Thus the knots $K_1 = 4_1 \# 4_1$ and $K_2 = 8_9$ have the *same Jones polynomial*. On the other hand the *Alexander polynomials are different*. This example shows that the Jones polynomial does not determine the Alexander polynomial. □

70.8. **The Jones polynomial and the determinant.** First we recall the following definition:

Definition 295.6. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. We refer to

$$\det(L) \;:=\; \nabla_L(-2\mathrm{i})$$

as the determinant of the link L.

Proposition 70.20. (Jones Polynomial–Determinant Proposition) For any oriented non-empty link $L \subseteq \mathbb{R}^3$ we have

$$\det(L) \;=\; V_L(t^{\frac{1}{2}} = \mathrm{i}).$$

Proof. We start out with the following characterization of the determinant:

Claim 1. The determinant $\det\colon \mathtt{Link}^{\mathrm{or}}_{\neq \varnothing}(\mathbb{R}^3) \to \mathbb{Q}$

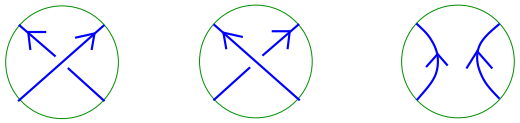
is the only map that satisfies the following two axioms:

(D1) The invariant of the unknot equals 1.

(D2) The invariant satisfies the skein relation that is shown below.

Proof. We first verify that the determinant has these two properties:

(D1) By the Conway axiom (C1) we have $\nabla_{\mathrm{unknot}}(t) = 1$. Thus $\nabla_{\mathrm{unknot}}(-2\mathrm{i}) = 1$.



(D1) $\det(\text{unknot}) = 1$
(D2) $\det(L_+) - \det(L_-) = -2\mathrm{i}\det(L_0)$

(D2) By the Conway axiom (C2) we have

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) \;=\; z \cdot \nabla_{L_0}(z).$$

This gives us

$$\nabla_{L_+}(-2\mathrm{i}) - \nabla_{L_-}(-2\mathrm{i}) \;=\; -2\mathrm{i} \cdot \nabla_{L_0}(-2\mathrm{i}).$$

By Definition 295.6 this shows that the determinant satisfies axioms (D1) and (D2).

The uniqueness statement follows from the Link Invariant–Uniqueness Proposition 69.6 applied to $\Theta\colon \mathtt{Link}^{\mathrm{or}}_{\neq\varnothing}(\mathbb{R}^3) \to \mathbb{Q}$ with $a_+ = 1$, $a_- = -1$, $a_0 = -2\mathrm{i}$ and with $u = 1$. ⊞
It remains to prove the following claim:

Claim 2. The invariant $V_L(t^{\frac{1}{2}} = \mathrm{i})$ satisfies the axioms (D1) and (D2).

Proof.

(D1) By the Jones axiom (J1) we have $V_{\text{unknot}}(t) = 1$. Thus $V_{\text{unknot}}(t^{\frac{1}{2}} = \mathrm{i}) = 1$.
(D2) By the Jones axiom (J2) we have the equality

$$t^{-1} \cdot V_{L_+}(t) - t \cdot V_{L_-}(t) \;=\; (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot V_{L_0}(t).$$

With $t^{\frac{1}{2}} = \mathrm{i}$ this gives us

$$(-1) \cdot V_{L_+}(t) - (-1) \cdot V_{L_-}(t) \;=\; (\mathrm{i} + \mathrm{i}) \cdot V_{L_0}(t).$$

It follows from the Jones Polynomial-Characterization Theorem 69.5 that the map $L \mapsto V(q^{\frac{1}{2}} = \mathrm{i})$ also satisfies the Axioms (D1) and (D2). So we are done.

■

We conclude this chapter with the following *open* question:

Question 70.21. Does there exist a non-trivial knot K such that $V_K(t) = 1$?

70.9. Exercises for Chapter 70.

Exercise 70.1. Show, just using the Kauffman Bracket Proposition 70.5 that for any m-component link diagram D we have

$$\langle D\rangle|_{A=1} \;=\; (-2)^{m-1}.$$

Hint. Prove the statement by induction on the number of crossings of D.

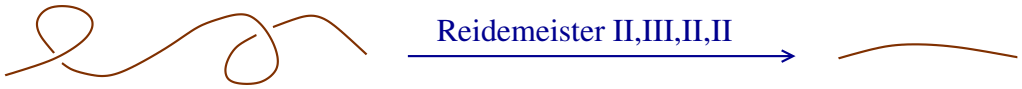
Exercise 70.2. Let D, D' be two knots diagrams. We can form the disjoint union $D \sqcup D'$ and the connected sum $D\#D'$. (We will not bore the reader by coming up with precise definitions.) Just using the Kauffman Bracket Proposition 70.5 determine

(a) $\langle D \sqcup D'\rangle$ in terms of $\langle D\rangle$ and $\langle D'\rangle$.
(b) $\langle D\#D'\rangle$ in terms of $\langle D\rangle$ and $\langle D'\rangle$.

Exercise 70.3. In the figure below we see an excerpt of a diagram with two crossings which contribute opposite signs to the writhe. Show that one can get rid of the two crossings via a sequence of

- Reidemeister II move.
- Reidemeister III move.
- Reidemeister II move.
- Reidemeister II move.

Remark. This exercise is similar to Exercise 49.1.



Exercise 70.4. Let D and $\widetilde{D}$ be two oriented diagrams for the same knot $K \subseteq \mathbb{R}^3$. We suppose that the writhes of D and $\widetilde{D}$ agree and we suppose that the rotation numbers from Definition 49.8 agree. Show that n D and $\widetilde{D}$ are related by a sequence of Reidemeister moves II and III.
Hint. Use the Whitney-Graustein Theorem 49.11 and use Exercise 70.3. *Remark.* This exercise can be viewed as a converse to the Writhe–Reidemeister Move Lemma 70.9.

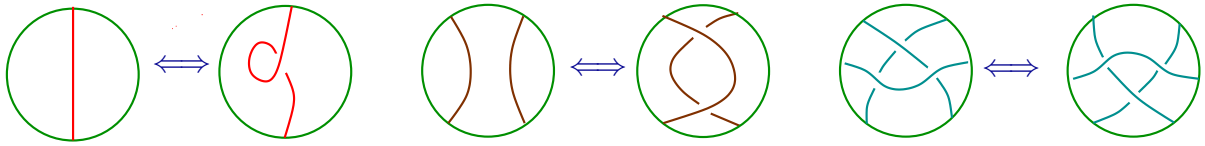
Exercise 70.5. Let D be an oriented diagram. This gives rise to an oriented knot K and another knot K' which is parallel to K in the obvious sense. We will not make this precise, but we refer to the figure below for a hopefully instructive example. Show that

$$\mathrm{lk}(K, K') \;=\; \text{writhe } w(D).$$

Hint. Use the Linking Number-via-Crossings Proposition 284.8.
Remark. This exercise reminds us of the lesson from page 5436 where we saw that the "obvious" candidate for a longitude is not necessarily a longitude.

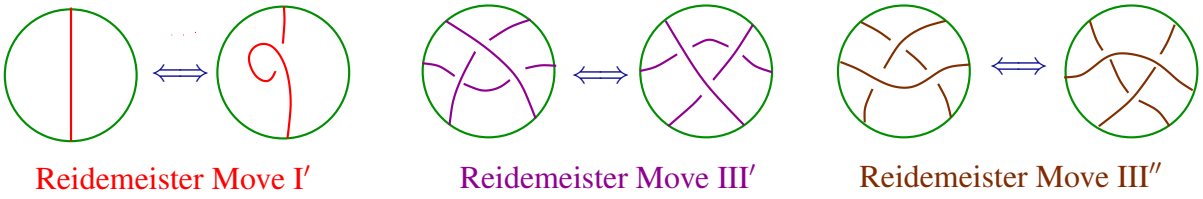


Exercise 70.6. In the following we show, for the reader's convenience, the original Reidemeister moves:



Show that the following alternative Reidemeister moves are a consequence of our original Reidemeister moves.

Exercise 70.7. Let $m \in \mathbb{N}$ and let $L = L_1 \cup \cdots \cup L_m \subseteq \mathbb{R}^3$ be an oriented m-component link. Let $\widetilde{L}$ be the same link except that we flipped the orientation of the i-th component.
(a) Show that



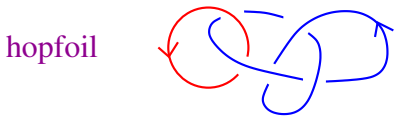
$$V_{\widetilde{L}}(t) \;=\; t^{-3\cdot\sum\limits_{k\neq i}\mathrm{lk}(L_k,L_i)}\cdot V_L(t).$$

(b) Get a reality check on (a) by testing out the Hopf link.

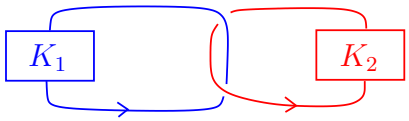
Exercise 70.8. Let K_1 and K_2 be two oriented knots. We consider the split union $K_1 \sqcup K_2$ that we introduced in Definition 109.4. Determine $V_{K_1\sqcup K_2}(t)$ in terms of $V_{K_1}(t)$ and $V_{K_2}(t)$.

Exercise 70.9.

(a) Compute the Jones polynomial of the hopfoil that is shown in the figure.
(b) Let K be any oriented knot and let L be the oriented link that we obtaining by "adding an earring" to K. Determine the Jones polynomial L in terms of the Jones polynomial of K.



Exercise 70.10. Let K_1, K_2 be two oriented knots. We consider the oriented link L that is shown below. Determine the Jones polynomial of the oriented link L in terms of the Jones polynomials of the oriented knots K_1 and K_2.
Remark. This exercise is a generalization of Exercise 70.9.



Exercise 70.11. Let $L \subseteq \mathbb{R}^3$ be an oriented non-empty link. Show, only using the axioms (J1) and (J2) of the Jones polynomial, that

$$V_{L^{\mathrm{rev}}}(t) \;=\; V_L(t).$$

Remark. This gives a new proof of the Jones Polynomial–Reverse Lemma 70.12.

Exercise 70.12. Let $m \in \mathbb{N}$. Show, only using the axioms (J1) and (J2) of the Jones polynomial, that for any oriented m-component link $L \subseteq \mathbb{R}^3$ we have the following equality in $\mathbb{Z}[t^{\frac{1}{2}}]$:

$$V_{L^{\mathrm{mir}}}(t) \;=\; V_L(t^{-1}).$$

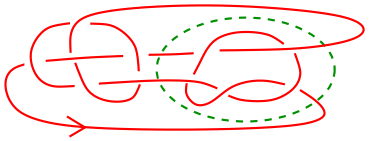
Remark. This gives a new proof of the Jones Polynomial–Mirror Lemma 70.13

Exercise 70.13. Let $m \in \mathbb{N}$ and let $L \subseteq \mathbb{R}^3$ be an oriented m-component link. Show that

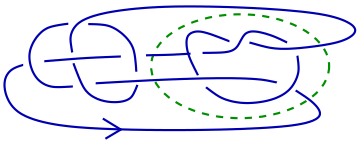
$$V_L(t^{\frac{1}{2}} = \exp(2\pi \mathrm{i}/6)) \;=\; (-1)^{m-1}.$$

Hint. Use the Jones Skein Theorem 70.14 and the Link Invariant–Uniqueness Proposition 69.6.

Exercise 70.14. In the figure below we consider the Conway knot and the Kinoshita-Terasaka knot. Show that they have the same Jones polynomial.
Hint. The polynomials are "mutants" of one another, i.e. we can obtain one from the other by taking the green 2-sphere, which intersects each knot in four points, and by flipping the inside.



Kinoshita-Terasaka knot



Conway knot

Part IX

Differential forms and de Rham cohomology

In the first chapters of this part of the lecture notes we introduce differential forms on smooth manifolds and we will see that they can be integrated over compact oriented smooth manifolds. We will then state and prove Stokes' Theorem, which is easily one of the most important influential theorems in all of mathematics.

Afterwards we will introduce de Rham cohomology of smooth manifolds. For each $k \in \mathbb{N}_0$ de Rham cohomology defines a contravariant functor $M \mapsto \mathrm{H}^k_{\mathrm{dR}}(M)$ from the category of smooth manifolds to the category of real vector spaces. De Rham cohomology groups have many uses. For example we can use them to show that for $g \neq h$, the surfaces of genus g and genus h are not homotopy equivalent, let alone homeomorphic or diffeomorphic.

To preserve the author's and the reader's sanity we apply throughout this part of the lecture notes the following two standard conventions.

Convention 70.22. Let M be a smooth submanifold of some $\mathbb{R}^n$.

(1) We will use the natural isomorphism from the Tangent Space-Natural isomorphism Proposition 29.24 to denote the physical tangent space $\widetilde{\mathrm{T}}_P M$ also by $\mathrm{T}_P M$. In other words, we will not distinguish in our notation between the physical and the abstract tangent space.

(2) If M is actually an open subset of $\mathbb{R}^n$, then we will use the Differential-Equals-Differential Lemma 29.6 as an excuse for not distinguishing between the various notions of differentials.

Convention 70.23. Let M be a smooth manifold.

(1) Let W be a submanifold of M. We denote by $i\colon W \to M$ the inclusion map. By the Physical Tangent Space-Inclusion Proposition 29.20 we know that for each $P \in W$ the map $\mathrm{D}i_P\colon \mathrm{T}_P W \to \mathrm{T}_P M$ is a monomorphism. On many occasions we will just identify $\mathrm{T}_P W$ with its image under $\mathrm{D}i_P$ in $\mathrm{T}_P M$. In particular we will view $\mathrm{T}_P W$ as a vector subspace of $\mathrm{T}_P M$.

(2) Let $P \in \partial M$. By the same logic as in (1) we will view $\mathrm{T}_P\, \partial M$ as a vector subspace of $\mathrm{T}_P M$.

This convention brings us in line with corresponding situation for physical tangent spaces as described in the Physical Tangent Space-Inclusion Proposition 29.5 (1) and (2).

This part of the lecture notes is organized as follows:

- In Chapter 71 we will introduce, as a warm-up, the fairly harmless notion of a 1-form on a smooth manifold and we will see that one can integrate 1-forms on compact oriented 1-dimensional smooth manifolds.
- In Chapter 72 we will take a little break from the theory of smooth manifolds to recall some concepts from (multi-) linear algebra.
- In Chapter 73 we will extend our little break from the theory of smooth manifolds to recall some concepts from integration theory.
- In Chapter 74 we will generalize the concepts from Chapter 71: More precisely, for any $k \in \mathbb{N}_0$ we will use the concepts from Chapter 72 to introduce the notion of a k-form on a smooth manifold. Furthermore, using the results from Chapter 73, we will see that one can integrate k-forms on compact oriented k-dimensional smooth manifolds.
- In Chapter 75 we will state and prove Stokes' Theorem 75.11.

- In Chapter 76 we will use the high-end Stokes' Theorem 75.11 to prove several classical versions of Stokes' Theorem 75.11, namely the Green's Theorem 76.1, the Kelvin-Stokes Theorem 76.3 and the Gauß Divergence Theorem 76.26. These three theorems might be the only theorems of the lecture notes that have any practical applications.
- In Chapter 77 we will give a short introduction into the algebraic theory of cochain complexes over a field. Later on, when we introduce singular cohomology groups, we will discuss cochain complexes in greater detail.
- In Chapter 78 we will introduce the de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M)$ of a smooth manifold M. For each $k \in \mathbb{N}_0$ these define a contravariant functor from the category of smooth manifolds to the category of real vector spaces. We will show, using Stokes' Theorem 75.11, that for any closed oriented non-empty k-dimensional smooth manifold the k-th de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(M)$ is non-zero.
- In Chapter 79 we will show that if two maps between two smooth manifolds are homotopic, then they induce the same map on de Rham cohomology.
- In Chapter 80 we will learn about the Mayer–Vietoris sequence in de Rham cohomology. This is arguably the most important tool for calculating de Rham cohomology groups. It will allow us to calculate the de Rham cohomology groups of spheres, the torus and the Klein bottle.
- In Chapter 81 we will show that every smooth manifold admits a "smoothly good cover". In the De Rham Cohomology-Finiteness Proposition 81.10 we will use these covers to show that the de Rham cohomology groups of any compact smooth manifold are finite-dimensional. We will use Mayer-Vietoris sequences and smoothly good covers to prove the Künneth-Theorem 81.21 which allows us to calculate the de Rham cohomology groups of a product $M \times N$ in terms of the de Rham cohomology groups of M and N.
- In Chapter 82 we will show that open star-shaped subsets of $\mathbb{R}^n$ are diffeomorphic to $\mathbb{R}^n$. This result will play a crucial role later in the proof of Poincaré Duality for de Rham cohomology.
- In Chapter 83, given a smooth manifold M, we will introduce the de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR,c}}(M)$ *with compact support*. For compact smooth manifolds these equal the usual de Rham cohomology groups, but we will see that for non-compact smooth manifolds these are dramatically different from the ordinary de Rham cohomology groups.
- In Chapter 84 we will state and prove the De Rham-Poincaré Duality Theorem 84.5 which, for an oriented n-dimensional smooth manifold relates the k-th de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(M)$ to the de Rham cohomology group $\mathrm{H}^{n-k}_{\mathrm{dR,c}}(M)$ with compact support in the opposite dimension. The proof of the De Rham-Poincaré Duality Theorem 84.5 will make use of smoothly good covers.
- In Chapter 85 we will use the De Rham-Poincaré Duality Theorem 84.5 to calculate the de Rham cohomology groups and rings of various smooth manifolds, e.g. the complex and real projective spaces.

CHAPTER 71

Differential forms of degree 1

The key object in the definition of de Rham cohomology are differential forms. At the first encounter these are often considered confusing. Therefore, as a warm-up, in this chapter we start out with the easiest examples of differential forms, namely differential forms of degree 1. Only once we have mastered this concept will we move to the differential forms of higher degree. This will happen in the next Chapter 74.

This chapter is organized as follows:

- In Section 71.1 we will define differential forms of degree 1 (usually just called "1-forms").
- In Section 71.2 we will introduce the notion of a *smooth* 1-form. We will state the Smooth 1-Form Propositions 71.9 which gives a characterization of smooth 1-forms.
- In Section 71.3 we will prove the Smooth 1-Form Propositions 71.9.
- In Section 71.4 we will introduce the integral of a smooth 1-form along a compact oriented 1-dimensional smooth manifolds.
- In Section 71.5 we will introduce the notion of an *exact* 1-form.
- In Section 71.6 we will provide exercises for Chapter 71.

In the next Chapter 72 we will introduce some concepts from multilinear algebra.

71.1. **Definition of differential forms of degree 1.** We start out gently with the following easy definition.

Definition 71.1.

(1) Let M be a smooth manifold and let $P \in M$. We write

$$\mathrm{T}^*_P M \;:=\; \mathrm{Hom}_{\mathbb{R}}(\mathrm{T}_P M, \mathbb{R}).$$

We refer to elements of $\mathrm{T}^*_P M$ as covectors.

(2) Given a smooth map $f\colon M \to N$ between two smooth manifolds and given $P \in M$ we obtain the induced homomorphism $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ and we obtain the dual homomorphism

$$\begin{array}{rcl} \mathrm{T}^*_{f(P)} N & \to & \mathrm{T}_P M \\ [\varphi\colon \mathrm{T}_{f(P)} N \to \mathbb{R}] & \mapsto & [\varphi \circ \mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}] \end{array}$$

Remark. As in Definition 29.8 let $\mathcal{PSmMfd}$ be the category of pointed smooth manifolds and let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces. It follows from the Physical Tangent Space-Functor Proposition 29.15 and the discussion on page 238 that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{PSmMfd}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ (M,P) & \mapsto & \mathrm{T}^*_P M, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}((M,P),(N,Q)) & \mapsto & \mathrm{Hom}_{\mathbb{R}}(\mathrm{T}^*_Q N, \mathrm{T}^*_P M) \\ f & \mapsto & f^* = \underbrace{(\mathrm{D}f_P)^*} \end{array}$$

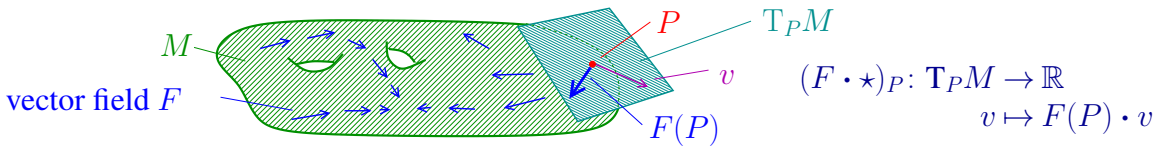
define a contravariant functor. □

Definition 71.2. A differential form of degree 1 on a smooth manifold M is a map ω that assign to every point P in M a covector $\omega_P \in \mathrm{T}_P M$. Usually, we just say 1-form instead of differential form of degree 1.

Example.

(1) Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold and let $F\colon M \to \mathbb{R}^n$ be a continuous map.[751] We denote by $F \cdot \star$ the differential 1-form which assigns to $P \in M$ the covector

$$\begin{array}{rcl} \mathrm{T}_P M & \to & \mathbb{R} \\ v & \mapsto & \underbrace{F(P) \cdot v}_{\substack{\text{standard inner} \\ \text{product in } \mathbb{R}^n}}. \end{array}$$



(2) Let M be a smooth submanifold of $\mathbb{R}^n$. Given $i \in \{1, \dots, n\}$ we denote by dx^i the 1-form which assigns to $P \in M$ the covector

$$\begin{array}{rcl} dx^i\colon \mathrm{T}_P M & \to & \mathbb{R} \\ (x_1, \dots, x_n) & \mapsto & x_i. \end{array}$$

(3) Let $\omega_1, \dots, \omega_k$ be 1-forms on a smooth manifold M. Given maps $g_1, \dots, g_k\colon M \to \mathbb{R}$ we denote by $g_1 \cdot \omega_1 + \dots + g_k \cdot \omega_k$ the 1-form which assigns to $P \in M$ the covector

$$\begin{array}{rcl} \mathrm{T}_P M & \to & \mathbb{R} \\ v & \mapsto & g_1(P) \cdot \underbrace{\omega_1(P)(v)}_{\in \mathbb{R}} + \dots + g_k(P) \cdot \underbrace{\omega_k(P)(v)}_{\in \mathbb{R}}. \end{array}$$

□

We introduce our next example as a proper definition.

Definition 71.3. Let $f\colon M \to \mathbb{R}$ be a smooth function on a smooth manifold M. The total differential df of f is the 1-form that assigns to $P \in M$ the covector

$$\begin{array}{rcl} \mathrm{T}_P M & \to & \mathbb{R} = \mathrm{T}_{f(P)}\mathbb{R} \\ v & \mapsto & \mathrm{D}f_P(v). \end{array}$$

Example. Given $i \in \{1, \dots, n\}$ we denote by

$$\begin{array}{rcl} x^i\colon \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \dots, x_n) & \mapsto & x_i \end{array}$$

the projection onto the i-th coordinate. If $M \subseteq \mathbb{R}^n$ is a smooth submanifold, then it follows from the fact that x^i is linear that the total differential of the function $x^i\colon M \to \mathbb{R}$ on M is precisely the 1-form dx^i on M that we had introduced above. □

[751] In most cases F will actually be a vector field on M.

Definition 71.4. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of codimension zero. Furthermore let $f\colon M \to \mathbb{R}$ be a smooth map. Note that by definition we have the partial derivatives $\frac{\partial f}{\partial x_i}\colon \mathring{M} \to \mathbb{R}$. It follows from the Submanifold Smooth Structure Lemma 25.23 (4), the Codimension-Zero Submanifold Proposition 25.49 and the Regular Closed–∂_*-Lemma 24.5 that these partial derivatives extend uniquely to smooth maps $\frac{\partial f}{\partial x_i}\colon M \to \mathbb{R}$.

The next lemma shows that on codimension-zero submanifolds we can express any 1-form uniquely in terms of the "elementary" 1-forms dx^i:

Lemma 71.5. (1-Form-via-dx^i-Lemma) Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of codimension zero.

(1) For every 1-form ω on M there exist uniquely determined maps $g_1, \ldots, g_n\colon M \to \mathbb{R}$ such that
$$\omega \;=\; g_1 \cdot dx^1 + \cdots + g_n \cdot dx^n.$$
(2) For every smooth function $f\colon M \to \mathbb{R}$ we have the following equality of 1-forms on M:
$$df \;=\; \frac{\partial f}{\partial x_1} \cdot dx^1 + \cdots + \frac{\partial f}{\partial x_n} \cdot dx^n.$$

Example. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold (here we actually do not demand that M is of codimension zero) and let $F = (F_1, \ldots, F_n)\colon M \to \mathbb{R}^n$ be a continuous map. Given any $P \in M$ and any $v = (v_1, \ldots, v_n) \in \mathrm{T}_P M = \mathbb{R}^n$ we have the following equality:

$$(F \cdot \star)_P(v) \underset{\substack{\uparrow\\ \text{definition of } F\cdot\star\\ \text{on page 1487}}}{=} F(P) \cdot v \underset{\substack{\uparrow\\ \text{definition of } \cdot}}{=} \sum_{i=1}^{n} F_i(P) \cdot v_i \underset{\substack{\uparrow\\ \text{follows from } dx^i(v) = v_i}}{=} (F_1 \cdot dx^1 + \cdots + F_n \cdot dx^n)_P(v).$$

Therefore $F \cdot \star = F_1 \cdot dx^1 + \cdots + F_n \cdot dx^n$, as 1-forms on M. □

Proof. Let $M \subseteq \mathbb{R}^n$ be a 1-form on a smooth submanifold of codimension zero. Note that since M is a submanifold of $\mathbb{R}^n$ of codimension zero we see that for any $P \in M$ we have $\mathrm{T}_P M = \mathbb{R}^n$.

(1) Let ω be a 1-form on M. As usual we denote by $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ the standard basis vectors of $\mathbb{R}^n$. Given $P \in M$ and given $i \in \{1, \ldots, n\}$ we set $g_i(P) \coloneqq \omega_P(\mathfrak{e}_i)$. Given any $v = (v_1, \ldots, v_n) \in \mathrm{T}_P M = \mathbb{R}^n$ we see that
$$\omega_P(v) \;=\; \omega_P\Big(\sum_{i=1}^{n} \mathfrak{e}_i \cdot v_i\Big) \underset{\substack{\uparrow\\ \text{since } \omega_P \text{ is linear}}}{=} \sum_{i=1}^{n} \omega_P(\mathfrak{e}_i) \cdot v_i \underset{\substack{\uparrow\\ \text{by definition of } g_i \text{ and}\\ \text{since } dx^i(v) = v_i}}{=} \sum_{i=1}^{n} g_i(P) \cdot dx^i(v) \;=\; \Big(\sum_{i=1}^{n} g_i \cdot dx^i\Big)_P(v).$$
Next we must show that the functions $g_1, \ldots, g_n\colon M \to \mathbb{R}$ are unique. Thus let $\widetilde{g}_1, \ldots, \widetilde{g}_n\colon M \to \mathbb{R}$ be functions with $g_1 \cdot dx^1 + \cdots + g_n \cdot dx^n = \widetilde{g}_1 \cdot dx^1 + \cdots + \widetilde{g}_n \cdot dx^n$. Let $i \in \{1, \ldots, n\}$. Since $\mathfrak{e}_i$ lies in $\mathrm{T}_P M = \mathbb{R}^n$ we can apply the above equality to $\mathfrak{e}_i$ and we obtain that
$$g_i(P) \underset{\substack{\uparrow\\ \text{since } dx^j(\mathfrak{e}_i) = \delta_{ij}\text{, where}\\ \delta_{ij} \text{ denotes the Kronecker delta}}}{=} (g_1 \cdot dx^1 + \cdots + g_n \cdot dx^n)_P(\mathfrak{e}_i) \underset{\substack{\uparrow\\ \text{follows from the above equality}\\ \text{applied to } \mathfrak{e}_i \in \mathrm{T}_P M = \mathbb{R}^n}}{=} (\widetilde{g}_1 \cdot dx^1 + \cdots + \widetilde{g}_n \cdot dx^n)_P(\mathfrak{e}_i) \underset{\substack{\uparrow\\ \text{since } dx^j(\mathfrak{e}_i) = \delta_{ij}}}{=} \widetilde{g}_i(P).$$
We have also showed that for $i = 1, \ldots, n$ we have: $g_i = \widetilde{g}_i$.

(2) Let $f\colon M \to \mathbb{R}$ be a smooth map. Recall that it follows from the Differential-Equals-Differential Lemma 29.6 that at each point $P \in M$ the 1-form df equals the map $\mathrm{T}_P M = \mathbb{R}^n \to \mathbb{R}$ that is given by multiplication with the differential $\mathrm{D}f_P \in \mathrm{M}(1\times n, \mathbb{R})$. Now let $P \in M$ and let $v = (v_1, \dots, v_n) \in \mathrm{T}_P M = \mathbb{R}^n$. We see that

$$\begin{aligned} df_P(v) &\overset{\text{by the above discussion}}{=} \mathrm{D}f_P \cdot v \overset{\text{definition of } \mathrm{D}f_P}{=} \Big(\tfrac{\partial f}{\partial x_1}(P) \dots \tfrac{\partial f}{\partial x_n}(P)\Big) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(P) \cdot v_i \\ &= \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(P) \cdot dx^i(v) = \Big(\tfrac{\partial f}{\partial x_1} \cdot dx^1 + \dots + \tfrac{\partial f}{\partial x_n} \cdot dx^n\Big)(v). \end{aligned}$$

(definition of dx^i)

■

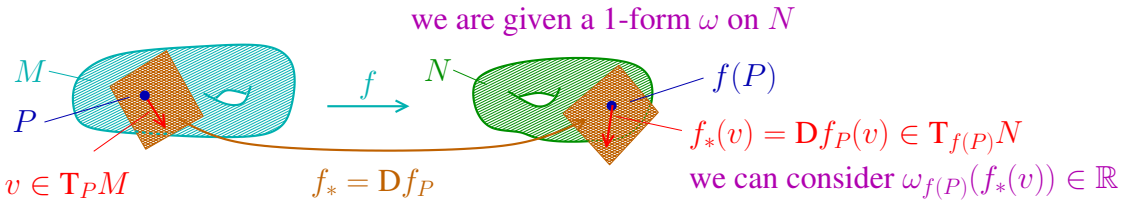
Definition 71.6. Let $f\colon M \to N$ be a smooth map between two smooth manifolds and let ω be a 1-form on N. We denote by $f^*(\omega)$ the 1-form on M that at any point $P \in M$ is given by the covector

$$(f^*(\omega))_P\colon \mathrm{T}_P M \xrightarrow{f_* = \mathrm{D}f_P} \mathrm{T}_{f(P)} N \xrightarrow{\omega_{f(P)}} \mathbb{R}.$$

In other words, $f^*(\omega)$ is defined by the property that for any $P \in M$ and $v \in \mathrm{T}_P M$ we have

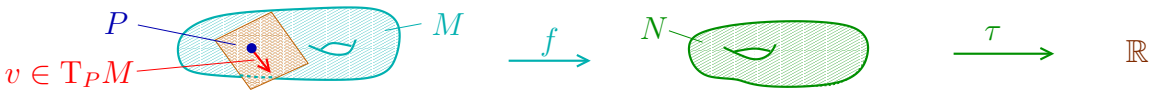
$$(f^*(\omega))_P(v) = \omega_{f(P)}(\underbrace{f_*(v)}_{\in \mathrm{T}_{f(P)} N}).$$

We refer to $f^*(\omega)$ as the pullback of ω along f.



Lemma 71.7. (Pullback-Differential Lemma) Let $f\colon M \to N$ be a smooth map between smooth manifolds. Given any smooth map $\tau\colon N \to \mathbb{R}$ we have the following equality of 1-forms on M:

$$f^*(d\tau) = d(\tau \circ f).$$



Proof. Let $P \in M$ and let $v \in \mathrm{T}_P M$. Then

$$(f^*(d\tau))_P(v) \overset{\text{by definition of } f^*}{=} d\tau_{f(P)}((\mathrm{D}f_P)(v)) \overset{\text{definition of } d\tau}{=} \mathrm{D}\tau_{f(P)}((\mathrm{D}f_P)(v)) \overset{\text{Chain Rule}}{=} \mathrm{D}(\tau \circ f)_P(v) \overset{\text{by definition of } d(\tau\circ f)}{=} (d(\tau \circ f))_P(v).$$

■

71.2. **Smooth 1-forms.** The reader might have noticed that we did not pose any continuity or smoothness restrictions at all on 1-forms. In this section we will finally provide the definition of what it means for a 1-form to be smooth:

Definition 71.8.

(1) Let $W \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ or an open subset of the upper half-space $\mathbb{H}^n$. We say that a 1-form[752] $\omega = f_1 \cdot dx^1 + \cdots + f_k \cdot dx^n$ on W is smooth, if the maps $f_1, \ldots, f_k \colon W \to \mathbb{R}$ are smooth in the sense of Definition 24.1.

(2) Let ω be a 1-form on a smooth manifold M.

(a) Let $\Phi \colon U \to V$ be a chart for M. We say that ω is smooth with respect to the chart Φ, if $(\Phi^{-1})^*(\omega)$ is a smooth 1-form on V.

(b) We say that ω is smooth on M, if ω is smooth with respect to every chart Φ of M.[753]



The following proposition plays a role similar to the Smooth-via-Atlas Lemma 25.16:

Proposition 71.9. (Smooth 1-Form Proposition) Let ω be a 1-form on a smooth manifold M. The following statements are equivalent:

(1) The 1-form ω is smooth.

(2) There exists an atlas[754] $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ of M such that ω is smooth with respect to every chart Φ_i of the atlas.

(3) Given any smooth vector field[755] v on M the following function is smooth:

$$\begin{array}{ccc} M & \to & \mathbb{R} \\ P & \mapsto & \underbrace{\underbrace{\omega_P}_{\in \mathrm{T}_P^* M}\,(\,\underbrace{v(P)}_{\in \mathrm{T}_P M}\,)}_{\in \mathbb{R}}. \end{array}$$

Proof. We postpone the technical proof of the Smooth 1-Form Proposition 71.9 to the next Section 71.3. ∎

Before we can formulate the next proposition we need to introduce the following bit of notation:

Notation 71.10. Given a smooth manifold M we set

$$\Omega^1(M) \;:=\; \text{vector space of all smooth 1-forms on } M.$$

Proposition 71.11. (Pullback Smooth 1-Form Proposition)

(1) Let $f \colon M \to N$ be a smooth map between two smooth manifolds and let ω be a smooth 1-form on N. Then the pullback $f^*(\omega)$ is a smooth 1-form on M.

---

[752] As we pointed out in the 1-Form-via-dx^i-Lemma 71.5, every 1-form on W can be uniquely written in this form. Here implicitly we use that an open subset of $\mathbb{H}^n$ is also a smooth submanifold of $\mathbb{R}^n$, potentially with non-empty boundary.

[753] The same way one could also define the notion of a continuous 1-form on a smooth manifold. Some of our later results will also work for continuous 1-forms. But for the main purpose of these chapters we will need to work with smooth 1-forms and for us there is no need to strive for maximal generality.

[755] Here and throughout the discussion of smooth manifolds it is understood that the charts come from the smooth structure of M.

[755] We refer to Definition 29.29 for the definition of a smooth vector field on a smooth manifold.

(2) The maps
$$\begin{array}{rcl} \text{smooth manifold } M & \mapsto & \Omega^1(M), \\ \text{smooth map } f\colon M \to N & \mapsto & (f^*\colon \Omega^1(N) \to \Omega^m(M)) \end{array}$$
define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

Proof. We also postpone the proof of the Pullback Smooth 1-Form Proposition 71.11 to the next Section 71.3. ■

The following lemma doubles as an interesting example:

Lemma 71.12. (Dot Product-Smooth 1-Form Lemma) Let M be a smooth submanifold of $\mathbb{R}^n$.

(1) Given any smooth map $F\colon M \to \mathbb{R}^n$ the corresponding 1-form $F \cdot \star$, which we introduced on page 1487, is smooth.
(2) Given any smooth 1-form ω on M there exists a unique smooth vector field[756] F on M with $\omega = F \cdot \star$.

Proof.

(1) Let $F\colon M \to \mathbb{R}^n$ be a smooth map. We use the "(3)⇒(1)" criterion of the Smooth 1-Form Proposition 71.9. Let v be a smooth vector field on M. We consider the map
$$\begin{array}{rcl} M & \to & \mathbb{R} \\ P & \mapsto & (F \cdot \star)_P(v(P)) = F(P) \cdot v(P) = \sum\limits_{i=1}^{n} F_i(P) \cdot v_i(P). \end{array}$$
Since the maps $F\colon M \to \mathbb{R}$ and $v\colon M \to \mathbb{R}$ are smooth, the map $P \mapsto F(P) \cdot v(P)$ is also smooth. Therefore it follows from the "(3)⇒(1)"-direction of the Smooth 1-Form Proposition 71.9 that $F \cdot \star$ is a smooth 1-form.
(2) Except for the next remark we will not make use of this statement. Therefore we outsource this proof to Exercise 71.1. ■

Remark. Given any smooth manifold M we set
$$\mathfrak{X}(M) \;:=\; \text{the vector space of all smooth vector fields on } M.$$
This is evidently a real vector space. If M is a smooth submanifold of some $\mathbb{R}^n$, then it follows from the Dot Product-Smooth 1-Form Lemma 71.12 that the map
$$\begin{array}{rcl} \mathfrak{X}(M) & \to & \Omega^1(M) \\ v & \mapsto & v \cdot \star \end{array}$$
is an isomorphism of vector spaces. We had just pointed out that the maps $M \mapsto \Omega^1(M)$ and $f \mapsto f^*$ form a contravariant functor from the category $\mathcal{SmMfd}$ of pointed smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces. One can ask, whether the maps $M \mapsto \mathfrak{X}(M)$ also form, in a reasonable way, a covariant (or contravariant?) functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces. But this is not the case, since a smooth map $f\colon M \to N$ induces neither a map $\mathfrak{X}(M) \to \mathfrak{X}(N)$ nor a map $\mathfrak{X}(N) \to \mathfrak{X}(M)$. □

[756] Recall that according to Definition 29.27 a smooth vector field on the smooth submanifold M of $\mathbb{R}^n$ is a smooth map $v\colon M \to \mathbb{R}^n$ such that for each $P \in M$ we have $v(P) \in \mathrm{T}_P M$.

71.3. **Proof of Propositions 71.9 and 71.11.** In this section we will provide the proofs of the Smooth 1-Form Proposition 71.9 and of the Pullback Smooth 1-Form Proposition 71.11. Readers who are not obsessed with technical proofs are advised to move on to the next section 71.4.

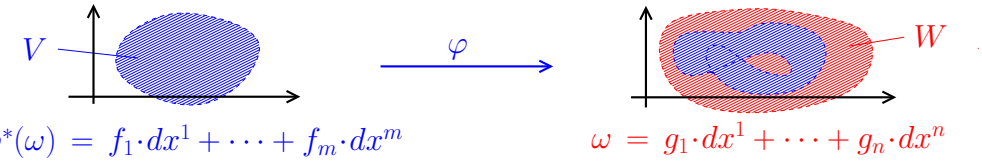
For the proof of the Smooth 1-Form Proposition 71.9 we will need the following lemma:

Lemma 71.13. (Basic-Pullback 1-Form Lemma) Let $\varphi\colon V \to W$ be a smooth map from a codimension-zero submanifold[757] $V \subseteq \mathbb{R}^m$ to a codimension-zero submanifold of $\mathbb{R}^n$. Given any smooth 1-form ω on W the pullback 1-form $\varphi^*(\omega)$ on V is also smooth.

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_n)\colon V \to W$ be a smooth map from a codimension-zero submanifold $V \subseteq \mathbb{R}^m$ to a codimension-zero submanifold of $\mathbb{R}^n$ and let ω be a smooth 1-form on W. Since V and W are codimension-zero submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$ we obtain from the 1-Form-via-dx^i-Lemma 71.5 that we can write

$$\begin{aligned} \omega &= g_1 \cdot dx^1 + \cdots + g_n \cdot dx^n \\ \varphi^*(\omega) &= f_1 \cdot dx^1 + \cdots + f_m \cdot dx^m, \end{aligned}$$

where $g_1, \ldots, g_n\colon W \to \mathbb{R}$ and $f_1, \ldots, f_m\colon V \to \mathbb{R}$ are functions. By hypothesis $g_1, \ldots, g_n$ are smooth. We must show that $f_1, \ldots, f_m$ are also smooth.



Now let $k \in \{1, \ldots, m\}$. Given $P \in V$ we see that

$$\begin{aligned} f_k(P) &\underset{\text{since } dx^i(\mathfrak{e}_j) = \delta_{ij}}{=} \big(f_1(P) \cdot dx^1 + \cdots + f_m(P) \cdot dx^m\big)(\mathfrak{e}_k) \\ &= \underbrace{\Big(\sum_{i=1}^n (g_i \circ \varphi)(P) \cdot dx^i\Big)}_{=\omega_{\varphi(P)}} \underbrace{\Big(\sum_{j=1}^n \frac{\partial \varphi_j}{\partial x_k}(P) \cdot \mathfrak{e}_j\Big)}_{=(\mathrm{D}\varphi_P)(\mathfrak{e}_k)=\varphi_*(\mathfrak{e}_k)} \\ &= (\varphi^*(\omega))_P(\mathfrak{e}_k) \underset{\text{by Definition 71.6 of } \varphi^*(\omega)}{=} \omega_{\varphi(P)}(\varphi_*(\mathfrak{e}_k)) \\ &\underset{\text{since } dx^i(\mathfrak{e}_j) = \delta_{ij}}{=} \sum_{i=1}^n (g_i \circ \varphi)(P) \cdot \frac{\partial \varphi_i}{\partial x_k}(P). \end{aligned}$$

Thus we see that f_k is a composition of smooth maps. Therefore f_k itself is smooth. ∎

Now we can already provide the proof of the Pullback Smooth 1-Form Proposition 71.11.

Proof of the Pullback Smooth 1-Form Proposition 71.11. We need to prove the following claim:

Claim.

(0) For any two smooth maps $f\colon A \to B$ and $g\colon B \to C$ between smooth manifolds and any 1-form ω on C have $(g \circ f)^*(\omega) = g^*(f^*(\omega))$.

(1) Let $f\colon M \to N$ be a smooth map between two smooth manifolds and let ω be a smooth 1-form on N. Then the pullback $f^*(\omega)$ is a smooth 1-form on M.

(2) The maps

$$\begin{aligned} \text{smooth manifold } M &\mapsto \Omega^1(M), \\ \text{smooth map } f\colon M \to N &\mapsto (f^*\colon \Omega^1(N) \to \Omega^m(M)) \end{aligned}$$

[757] In our applications V will be an open subset of $\mathbb{R}^m$ or of $\mathbb{H}^m$.

define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

Proof.

(0) This statement follows easily from the Tangent Space-Functor Proposition 29.15 and the fact that the functor $V \mapsto \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is contravariant.

(1) Let $f\colon M \to N$ be a smooth map between two smooth manifolds and let ω be a smooth 1-form on N. We need to show that the pullback $f^*(\omega)$ is a smooth 1-form on M.

Let $\Phi\colon U \to \widetilde{U}$ be a chart for M. We need to show that $(\Phi^{-1})^*(f^*(\omega))$ is a smooth 1-form on U. Let $\{\Psi_i\colon V_i \to \widetilde{V}_i\}_{i\in I}$ be an atlas for N. Note that $\{\Phi^{-1}(f^{-1}(V_i))\}_{i\in I}$ is an open cover of U. As in the Smooth Pasting Proposition 24.7 it suffices to show that for each $i \in I$ the restriction of $(\Phi^{-1})^*(f^*(\omega))$ to $\Phi^{-1}(f^{-1}(V_i))$ is smooth. Thus let $i \in I$. Note that on $f^{-1}(V_i)$ we have

$$\begin{array}{rcccl} (\Phi^{-1})^*(f^*(\omega)) & \overset{\text{follows from (0)}}{\overset{\downarrow}{=}} & (f\circ\Phi^{-1})^*(\omega) & = & (\Psi_i^{-1}\circ\Psi_i\circ f\circ\Phi^{-1})^*(\omega) \\ & = & (\Psi_i\circ f\circ\Phi^{-1})^*((\Psi_i)^{-1})^*(\omega)). & & \end{array}$$

Since ω is smooth we know that $(\Psi_i^{-1})^*(\omega)$ is smooth. Since $\Psi_i \circ f \circ \Phi^{-1}$ is smooth we obtain from the Basic-Pullback 1-Form Lemma 71.13 that $(\Phi^{-1})^*(f^*(\omega))$ is indeed smooth.

(2) This statement follows easily from (0) and (1). ∎

The following lemma proves the statement of the Smooth 1-Form Proposition 71.9 in a special case.

Lemma 71.14. (Euclidean 1-Form Lemma) Let ω be a 1-form on an open subset W of $\mathbb{R}^k$ or $\mathbb{H}^k$. The following statements are equivalent:

(1) The 1-form ω is smooth,

(2) For every diffeomorphism $\varphi\colon V \to W$ the 1-form $\varphi^*(\omega)$ on V is smooth.

(3) For all smooth vector fields $v\colon W \to \mathbb{R}^k$ the following map is smooth:

$$\begin{array}{rcl} W & \to & \mathbb{R} \\ P & \mapsto & \omega_P(v(P)). \end{array}$$

Proof. Let $\omega = g_1 \cdot dx^1 + \cdots + g_k \cdot dx^k$ be a 1-form on an open subset W of $\mathbb{R}^k$ or $\mathbb{H}^k$. The statement "(1)⇒(2)" follows from the Basic-Pullback 1-Form Lemma 71.13. The converse "(2)⇒(1)" is of course trivial, we just need to set $\varphi := \mathrm{id}$.

It remains to prove that "(1)⇔(3)". We make the following preliminary remark: Given an arbitrary smooth vector field $v = (v_1, \ldots, v_k)\colon W \to \mathbb{R}^k$ we have

$$\omega_P(v(P)) \quad = \quad \underbrace{\Big(\sum_{i=1}^k g_i(P)\cdot dx^i\Big)}_{=\omega_P}\underbrace{\Big(\sum_{j=1}^k v_j(P)\cdot \mathfrak{e}_j\Big)}_{=v(P)} \quad \overset{\text{since } dx^i(\mathfrak{e}_j)=\delta_{ij}}{\overset{\uparrow}{=}} \quad \sum_{i=1}^k g_i(P)\cdot v_i(P).$$

From this calculation it follows that if the functions $g_1, \ldots, g_k$ are smooth, then the map $P \mapsto \omega_P(v(P))$ is also smooth. This argument proves the "(1) ⇒ (3)" implication. Conversely, if we consider the constant vector field $G(P) \equiv \mathfrak{e}_i$, then we obtain that $g_i(P) = \omega_P(G(P))$. By hypothesis the function $P \mapsto \omega_P(G(P))$ is smooth. It follows that g_i is also smooth. This proves the "(3)⇒(1)"-implication. ∎

We turn to the actual proof of the Smooth 1-Form Proposition 71.9.

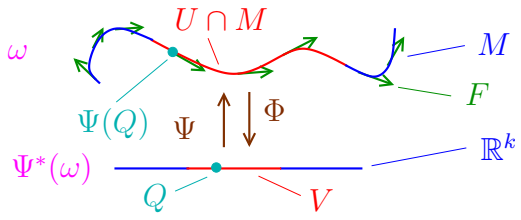
Proof of the Smooth 1-Form Proposition 71.9. Let ω be a 1-form on a k-dimensional smooth manifold M. To simplify the notation a little bit we assume that $\partial M = \varnothing$.

The statement "(1)⇒(2)" is of course trivial. The converse "(2)⇒(1)" can be proved, similar to the proof of Smooth-via-Atlas Lemma 25.16, by reducing it to the special case that we considered in the Euclidean 1-Form Lemma 71.14. We leave it to the reader to fill in the few remaining details.

We now prove the statement "(1)⇒(3)". Thus let F be a smooth vector field on M. Given $P \in M$ we define $g(P) := \omega_P(F(P)) \in \mathbb{R}$. We must show that the function g on the smooth manifold M is smooth. To do we use the definition of "smooth" from Definition 25.15. Thus let $\Phi\colon U \to V$ be a chart around P. We denote by $\Psi\colon V \to U$ the inverse map. We must show that the function $g \circ \Psi\colon V \to \mathbb{R}$ is smooth. Given $Q \in V$ we have the following equalities:

$$\begin{aligned}
(g \circ \Psi)(Q) &\overset{\text{definition of } g}{=} \omega_{\Psi(Q)}(F(\Psi(Q))) \\
&\overset{\text{since } \Psi \circ \Phi = \mathrm{id}}{=} ((\Psi \circ \Phi)^*(\omega))_{\Psi(Q)}(F(\Psi(Q))) \\
&= (\Phi^*(\Psi^*(\omega)))_{\Psi(Q)}(F(\Psi(Q))) \\
&\overset{\text{since } (\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*}{=} (\Psi^*(\omega))_Q\big(\Phi_*(F(\Psi(Q)))\big). \\
&\overset{\text{by Definition 71.6 of } \Phi^*(\omega)}{}
\end{aligned}$$

By hypothesis $\Psi^*(\omega)$ is a smooth 1-form on V and furthermore $Q \mapsto \Phi_*(F(\Psi(Q)))$ is a smooth vector field on V. It now follows from the Euclidean 1-Form Lemma 71.14 that the function $g \circ \Psi$ is smooth.



We now prove the statement "(3)⇒(2)". Thus we suppose that for all smooth vector fields F on M the function $x \mapsto \omega_x(F(x))$ is smooth. We must prove that given any $P \in M$ there exists a chart $\Phi\colon S \to T$ around P such that $(\Phi^{-1})^*(\omega)$ is a smooth 1-form on T. In fact the proof of the "(3)⇒(1)"-implication in the Euclidean 1-Form Lemma 71.14 shows that it suffices to prove the following statement:

Statement. Given any $P \in M$ there exists a chart $\Phi\colon S \to T$ around P such that for each $i \in \{1, \dots, k\}$ the map $y \mapsto ((\Phi^{-1})^*(\omega))_y(\mathfrak{e}_i)$ on T is smooth.

Let $P \in M$ and let $\Phi\colon U \to V$ be an arbitrary chart around P. We denote by $\Psi\colon V \to U$ the inverse map. Let $i \in \{1, \dots, k\}$. Note that $P \mapsto \Psi_*(\mathfrak{e}_i)$ is a vector field on U. Unfortunately in general this map cannot be extended to a smooth vector field on M, so it is not entirely clear how we can use our hypothesis on ω. To get around this problem we need the following claim:

Claim. There exists an open neighborhood S of $P \in M$ together with a smooth function $\tau\colon M \to [0,1]$ with the following properties:
(1) $\tau(x) = 1$ for all $x \in S$,
(2) $\operatorname{supp}(\tau) \subseteq U$.

Proof. This claim can be proved easily using the Smooth Transition Function Lemma 24.9. We leave it to the reader to fill in the details. ⊞

We now consider on M the vector field that is defined by

$$F(x) \;=\; \begin{cases} \tau(x) \cdot \Psi_*(\mathfrak{e}_i), & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the obvious analog of the Smooth Pasting Proposition 25.19 that this is a smooth vector field on M. We set $T := \Phi(S)$. Given $y \in T$ we have

$$((\Phi^{-1})^*(\omega))_y(\mathfrak{e}_i) \;=\; (\Psi^*(\omega))_y(\mathfrak{e}_i) \underset{\substack{\uparrow \\ \text{by Definition 71.6 of } \Psi^*(\omega)}}{=} \omega_{\Psi(y)}(\Psi_*(\mathfrak{e}_i)) \underset{\substack{\uparrow \\ \text{since } \Psi(y) \in S \text{ we} \\ \text{have } \tau(y) = 1}}{=} \omega_{\Psi(y)}(F(\Psi(y))).$$

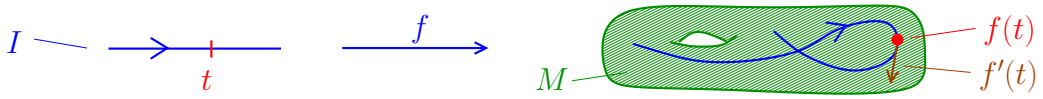
By hypothesis the map $x \mapsto \omega_x(F(x))$ is smooth on M, in particular on S. Since Ψ is smooth it follows from the above equality that the function $y \mapsto ((\Phi^{-1})^*(\omega))_y(\mathfrak{e}_i)$ is smooth on T. We have thus proved the above statement. ■

71.4. **Integration of differential forms of degree 1.** In this section we will see that 1-forms are secretly designed to be integrated over compact oriented 1-dimensional smooth manifolds. Before we get to integrals we need to introduce two definitions:

Definition 71.15. Let $I \subseteq \mathbb{R}$ be an interval with non-empty interior, let M be a smooth manifold and let $f\colon I \to M$ be a smooth map. Given $t \in I$ we write[758]

$$f'(t) \;=\; \mathrm{D}f_t\underset{\substack{\uparrow \\ \text{where } 1 \in \mathbb{R} = \mathrm{T}_t\mathbb{R}}}{(1)} \;\in\; \mathrm{T}_{f(t)}M.$$

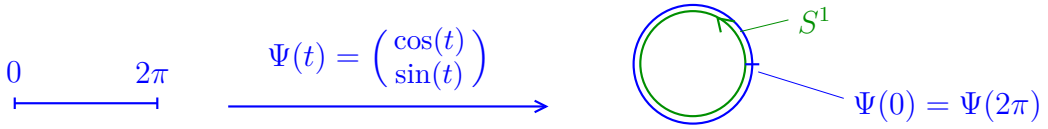
If M is a smooth submanifold of some $\mathbb{R}^n$, then this definition corresponds to the definition of $f'(t)$ in Notation 24.8.



Definition 71.16. Let M be an oriented 1-dimensional smooth manifold. A compact parametrization for M[759] is a continuous map $\Psi\colon [a,b] \to M$ with the following properties:
(1) We have $\Psi([a,b]) = M$.
(2) The restriction of Ψ to (a,b) is injective.
(3) The map $\Psi\colon [a,b] \to M$ is smooth in the sense of Definition 24.3.
(4) For every $P \in (a,b)$ the rank of the differential $\mathrm{D}\Psi_P$ equals one.
(5) For every $P \in (a,b)$ the differential[760] $\mathrm{D}\Psi_P\colon \mathbb{R} = \mathrm{T}_P(a,b) \to \mathrm{T}_{\Psi(P)}M$ is orientation-preserving.

[758] It follows easily from the hypothesis that the interval has non-empty interior and the Regular Closed-∂_*-Lemma 24.5 that this definition also makes sense on the endpoints of the interval I.

Example. We consider the smooth manifold S^1 with the standard orientation defined on page 817.



One can easily verify that the map $\Psi\colon [0,2\pi] \to S^1$ given by $\Psi(t) = (\cos(t), \sin(t))$ is a compact parametrization in the above sense. □

Proposition 71.17. (1-Form Integration Proposition) Let M be a compact oriented connected non-empty 1-dimensional smooth manifold.

(1) M admits a compact parametrization.

(2) Let $\Psi\colon [a,b] \to M$ be a compact parametrization and let ω be a smooth 1-form on M. The following map is smooth, in particular continuous:

$$\begin{array}{rcl} [a,b] & \to & \mathbb{R} \\ t & \mapsto & \underbrace{\underbrace{\omega_{\Psi(t)}}_{\in \mathrm{T}^*_{\Psi(t)}M}\ \underbrace{(\Psi'(t))}_{\in \mathrm{T}_{\Psi(t)}M}}_{\in \mathbb{R}}. \end{array}$$

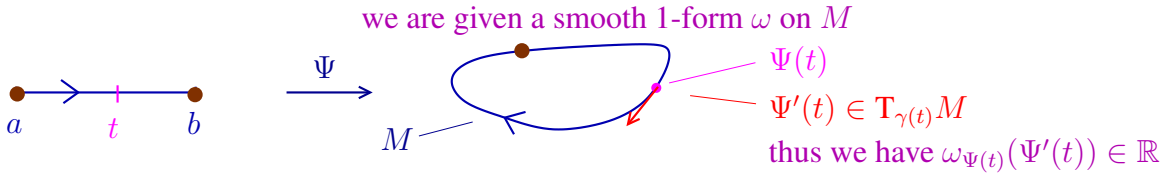
(3) The integral

$$\int_M \omega \ := \ \underbrace{\int_{t=a}^{t=b} \omega_{\Psi(t)}(\Psi'(t))\,dt}_{\substack{\text{Riemann integral,} \\ \text{defined since integrand} \\ \text{is continuous by (2)}}}$$

does not depend on the choice of a compact parametrization.

(4) If, as usual, we denote by $-M$ the same smooth manifold but with opposite orientation, then

$$\int_{-M} \omega \ = \ -\int_M \omega.$$



Example. As above we consider the smooth manifold S^1 with the standard orientation. Furthermore we consider the vector field $F(x,y) = (-y,x)$ on S^1. Using the above compact

[759]The name "compact parametrization" is a little awkward. The reason for the choice of this name is that will use the word "parametrization" for a related, but different concept, see Definition 205.1, where the domains will be open subsets of some $\mathbb{R}^n$.

[760]Here and throughout these lecture notes we equip any interval with the standard orientation.

parametrization $\Psi\colon [0,2\pi] \to S^1$ with $\Psi(t) = (\cos(t), \sin(t))$ we calculate that

$$\int_{S^1} F \cdot \star \overset{\text{definition of the integral}}{=} \int_{t=0}^{t=2\pi} (F \cdot \star)_{\Psi(t)}(\Psi'(t))\,dt \overset{\text{definition of the 1-form } F\cdot\star \text{ on page 1487}}{=} \int_{t=0}^{t=2\pi} F(\Psi(t)) \cdot \Psi'(t)\,dt$$
$$= \int_{t=0}^{t=2\pi} \underbrace{\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}}_{=F(\Psi(t))} \cdot \underbrace{\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}}_{\Psi'(t)} dt = \int_{t=0}^{t=2\pi} 1\,dt = 2\pi.$$

□

The 1-Form Integration Proposition 71.17 allows us to introduce the following definition:

Definition 71.18. Let M be a compact oriented 1-dimensional smooth manifold and let ω be a smooth 1-form on M. We denote by $M_1, \dots, M_d$ the components of M. For $i = 1, \dots, d$ we pick a compact parametrization $\Psi_i\colon [a,b] \to M_i$ and we set

$$\int_M \omega := \sum_{i=1}^{d} \int_{t=a}^{t=b} \omega_{\Psi(t)}(\Psi'(t))\,dt.$$

Proof. Let M be a compact oriented connected non-empty 1-dimensional smooth manifold.

(1) First note that the statement is clear for $[0,1]$ and for S^1. Next note that it follows almost immediately from the Smooth 1-dimensional Manifold Classification Theorem 28.8 that any compact oriented connected non-empty 1-dimensional smooth manifold M admits an orientation-preserving diffeomorphism either to $[0,1]$ or to S^1. Thus the desired statement also holds in full generality.

(2) We denote by $e = 1$ the unit vector in $\mathbb{R}$. Given $t \in [a,b]$ we see that

$$\omega_{\Psi(t)}(\Psi'(t)) \underset{\text{by definition of } \Psi'(t)}{=} \omega_{\Psi(t)}(\Psi_*(e)) \underset{\text{by Definition 71.6 of } \Psi^*(\omega)}{=} (\Psi^*(\omega))_t(e).$$

By hypothesis the 1-form ω on M is smooth. It follows from the Pullback Smooth 1-Form Proposition 71.11 that the pullback 1-form $\Psi^*(\omega)$ on $[a,b]$ is also smooth. Furthermore the constant vector field e on $\mathbb{R}$ is evidently smooth. Thus it follows from the "(1)⇒(3)"-implication of the Smooth 1-Form Proposition 71.9 that the function $t \mapsto (\Psi^*(\omega))_t(e)$ is smooth.

(3) Let ω be a smooth 1-form on M. Furthermore let $\Phi\colon [a,b] \to M$ and $\Psi\colon [c,d] \to M$ be two compact parametrizations. In the following we first deal with the case that M is diffeomorphic to $[0,1]$.



It follows easily from the Inverse Mapping Theorem 30.1 together with the Smooth Pasting Proposition 25.19 that $\Theta := \Psi^{-1} \circ \Phi\colon (a,b) \to (c,d)$ is an orientation-preserving

diffeomorphism. It follows that

$$\begin{aligned}
\int\limits_{M \text{ via } \Phi} \omega &= \int\limits_{t=a}^{t=b} \omega_{\Phi(t)}(\Phi'(t))\,dt && = \int\limits_{t=a}^{t=b} \omega_{(\Psi\circ\Theta)(t)}\big((\Psi\circ\Theta)'(t)\big)\,dt \\
&= \int\limits_{t=a}^{t=b} \omega_{\Psi(\Theta(t))}\big(\underbrace{\Psi'(\Theta(t))}_{\in \mathrm{T}_{\Psi(\Theta(t))}M} \cdot \underbrace{\Theta'(t)}_{\in\mathbb{R}}\big)\,dt && = \int\limits_{t=a}^{t=b} \omega_{\Psi(\Theta(t))}\big(\Psi'(\Theta(t))\big)\cdot\Theta'(t)\,dt \\
&\quad\uparrow \text{ Chain Rule} && \quad\uparrow \text{ since } \omega_{\Psi(\Theta(t))} \colon \mathrm{T}_{\Psi(\Theta(t))}M \to \mathbb{R} \text{ is linear} \\
&= \int\limits_{s=\Theta(a)}^{s=\Theta(b)} \omega_{\Psi(s)}(\Psi'(s))\,ds && = \int\limits_{s=c}^{s=d} \omega_{\Psi(s)}(\Psi'(s))\,ds = \int\limits_{M \text{ via } \Psi} \omega. \\
&\quad\uparrow \text{ substitution } s=\Theta(t) && \quad\uparrow \text{ we have } \Theta(a)=c \text{ and } \Theta(b)=d \text{ since } \Theta \text{ is orientation-preserving}
\end{aligned}$$

The case that M is diffeomorphic to S^1 is a little bit more difficult, since we have to break the intervals (a,b) and (c,d) into two open subintervals to get diffeomorphisms. We leave it to the reader to fill in the details.

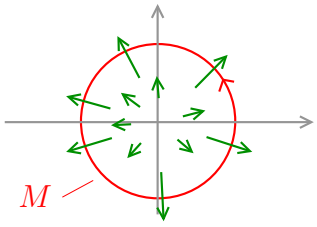
(4) This statement follows almost immediately from the observation that if we swap the two endpoints of a Riemann integral, then we pick up a minus sign. Again we leave it to the reader to fill in the details. ■

Example. Let F be a smooth vector field on $\mathbb{R}^n$. As above we define the 1-form $F \cdot \star$ on $\mathbb{R}^n$ by $(F\cdot\star)_P(v) := F(P)\cdot v$. Furthermore let $M \subseteq \mathbb{R}^n$ be a compact oriented 1-dimensional smooth submanifold. We denote by $i\colon M \to \mathbb{R}^n$ the inclusion. We thus obtain a 1-form $i^*(F\cdot\star)$ on M. We pick a compact parametrization $\Psi\colon [a,b] \to M$. Then

$$\begin{aligned}
\int\limits_{\Psi} i^*(F\cdot\star) &\overset{\text{Definition 71.18}}{=} \int\limits_{t=a}^{t=b} (i^*(F\cdot\star))_{\Psi(t)}(\Psi'(t))\,dt \overset{\text{Definition 71.6 of } i^*}{=} \int\limits_{t=a}^{t=b} (F\cdot\star)_{\Psi(t)}(i_*(\Psi'(t)))\,dt \overset{\text{definition of } F\cdot\star \text{ on page 1487}}{=} \\
&= \int\limits_{t=a}^{t=b} \underbrace{F(\Psi(t))\cdot\Psi'(t)}_{\substack{\text{standard inner product of the vectors} \\ F(\Psi(t)) \text{ and } \Psi'(t)}}\,dt.
\end{aligned}$$

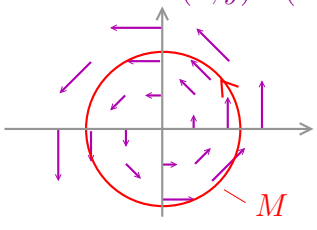
We illustrate this calculation in three different cases:

vector field $F(x,y)=(x,y)$



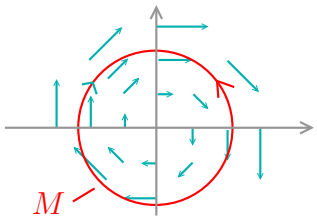
$\int\limits_M i^*(F\cdot\star) = 0,$
since $F(\Psi(t))\cdot\Psi'(t) = 0$

vector field $G(x,y)=(-y,x)$



$\int\limits_M i^*(G\cdot\star) > 0,$
since $G(\Psi(t))\cdot\Psi'(t) > 0$

vector field $H(x,y)=(y,x)$



$\int\limits_M i^*(H\cdot\star) = 0,$
since $H(\Psi(t))\cdot\Psi'(t) < 0$

We see that integral depends on the standard inner product of the vector field with the tangent vectors. □

71.5. **Exact differential forms of degree 1.** The following proposition shows that nothing is easier than integrating total differentials.

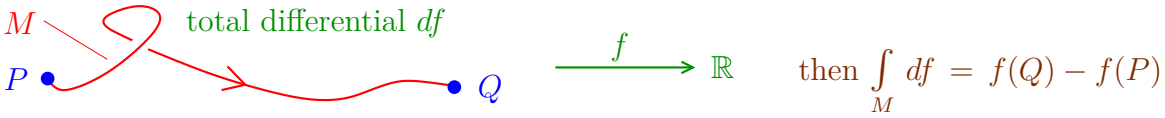
Proposition 71.19. (Total Differential-Integration Proposition) Let M be a compact oriented connected 1-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. We denote by $df \in \Omega^1(M)$ the total differential from Definition 71.3.

(1) We assume that M has non-empty boundary. We equip ∂M with the orientation provided by the Boundary Orientation Lemma 31.38. Then we can write $\partial M = -\{P\} \cup \{Q\}$ for points $P, Q \in \partial M$ and we have the equality

$$\int_M df \;=\; f(Q) - f(P).$$

(2) If M is closed, then

$$\int_M df \;=\; 0.$$



Proof. Let $\Psi\colon [a,b] \to M$ be a compact parametrization.

(1) Note that if M has non-empty boundary, then it follows from the Smooth 1-dimensional Manifold Classification Theorem 28.8 and the discussion on page 825 that we have $\partial M = -\{P\} \cup \{Q\} = -\{\Psi(b)\} \cup \{\Psi(a)\}$. We see that

$$\int_M df \;=\; \int_{t=a}^{t=b} \underbrace{(df_{\Psi(t)})(\Psi'(t))}_{=(f\circ\Psi)'(t),\text{ by the Chain Rule}}\, dt \;=\; \int_{t=a}^{t=b} \tfrac{d}{dt} f(\Psi(t))\, dt \;=\; f(\underbrace{\Psi(b)}_{=Q}) - f(\underbrace{\Psi(a)}_{=P}).$$

The last equality follows from the Fundamental Theorem of Calculus 73.27.

(2) This calculation is almost identical to the calculation in (1), the only difference is that now $\Psi(b) = \Psi(a)$. ■

The above proposition motivates the following definition:

Definition 71.20. Let M be a smooth manifold and let ω be a smooth 1-form on M. We say that ω is exact if there exists a smooth function $f\colon M \to \mathbb{R}$ with $\omega = df$.

Example. We consider the smooth manifold S^1 with the standard orientation and we consider the vector field $G(x,y) = (-y,x)$ on S^1. Recall that on page 1497 we calculated that $\int_{S^1} G \cdot \star \neq 0$. It follows from the Total Differential-Integration Proposition 71.19 that the smooth 1-form $G \cdot \star$ is *not* exact. □

Lemma 71.21. (Pullback Exact 1-Form Lemma) Let $f\colon M \to N$ be a smooth map between smooth manifolds and let ω be a smooth 1-form on N. If ω is exact, then so is $f^*(\omega)$.

Proof. This statement follows immediately from the Pullback-Differential Lemma 71.7. Indeed, if ω is exact, then there exists a smooth function $\tau\colon N \to \mathbb{R}$ with $d\tau = \omega$.



By the Pullback-Differential Lemma 71.7 we have $f^*(d\tau) = d(\tau \circ f)$. From this observation we obtain that $f^*(\omega) = f^*(d\tau) = d(\tau \circ f)$, which implies that $f^*(\omega)$ is exact. ■

Example. We consider the smooth manifold $M = \mathbb{R}^2$. We denote by $i\colon S^1 \to M = \mathbb{R}^2$ the inclusion. Furthermore we consider the smooth vector fields

$$\begin{array}{rcl} F\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x,y) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (-y,x) \end{array}$$

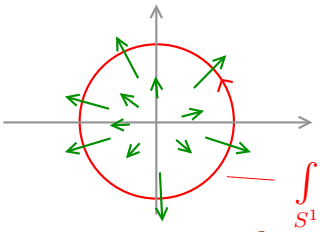
Note that

$$F \cdot \star \underset{\uparrow}{=} x\cdot dx + y\cdot dy = \frac{\partial \frac{1}{2}(x^2+y^2)}{\partial x}\cdot dx + \frac{\partial \frac{1}{2}(x^2+y^2)}{\partial y}\cdot dy \underset{\uparrow}{=} d\big(\tfrac{1}{2}(x^2+y^2)\big).$$

see page 1488 — by the 1-Form-via-dx^i-Lemma 71.5 (2)

This shows that $F \cdot \star$ is exact.

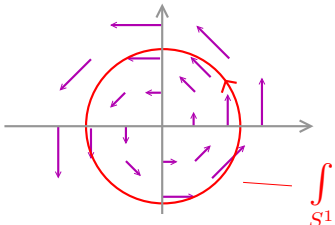
vector field $F(x,y) = (x,y)$



$\int_{S^1} i^*(F \cdot \star) = 0$

the 1-form $F \cdot \star$ on $\mathbb{R}^2$ is exact

vector field $G(x,y) = (-y,x)$



$\int_{S^1} i^*(G \cdot \star) \neq 0$

the 1-form $G \cdot \star$ on $\mathbb{R}^2$ is not exact

On the other hand we just saw that the 1-form $i^*G \cdot \star$ on S^1 is not exact, thus it follows from the Pullback Exact 1-Form Lemma 71.21 that the 1-form $G \cdot \star$ on $\mathbb{R}^2$ is not exact. □

71.6. Exercises for Chapter 71.

Exercise 71.1. Let V be a real vector space. Recall that we say that an $\mathbb{R}$-bilinear map $\varphi\colon V \times V \to \mathbb{R}$ is *non-singular*, if the $\mathbb{R}$-linear map

$$\begin{array}{rcl} f\colon V & \to & V^* = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \\ v & \mapsto & \begin{pmatrix} V & \to & \mathbb{R} \\ w & \mapsto & \varphi(v,w) \end{pmatrix} \end{array}$$

is an isomorphism.

(a) Let $V \subseteq \mathbb{R}^n$ be a vector subspace. Show that the following $\mathbb{R}$-bilinear map is non-singular:

$$\begin{array}{rcl} V \times V & \to & \mathbb{R} \\ (v,w) & \mapsto & \underbrace{\langle v,w\rangle}_{\substack{\text{standard inner} \\ \text{product on } \mathbb{R}^n}}. \end{array}$$

(b) Let M be a submanifold of $\mathbb{R}^n$. Show that given any smooth 1-form μ on M there exists a unique vector field F on M such that $\mu = F \cdot \star$.
(c) Show that the vector field in (b) is smooth.

Exercise 71.2. Let $U \subseteq \mathbb{R}^2$ be an open subset and let $f, g\colon U \to \mathbb{R}$ be smooth maps. We consider the 1-form $\omega = f \cdot dx + g \cdot dy$ on U.
(a) Show that if ω is exact, then $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.
(b) Show that if $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ and if $U \subseteq \mathbb{R}^2$ is convex, then ω is exact.
(c) Does the conclusion of (b) also hold, if we drop the convexity hypothesis?

Exercise 71.3.
(a) Let $\varphi\colon \mathbb{R}_{>0} \to \mathbb{R}$ be a smooth map. We consider the map

$$\begin{array}{rcl} \varphi\colon \mathbb{R}^n \setminus \{0\} & \to & \mathbb{R} \\ x = (x_1, \ldots, x_n) & \mapsto & \varphi(\|x\|) = \varphi\big(\sqrt{x_1^2 + \cdots + x_n^2}\big). \end{array}$$

Express the total differential $d\varphi$ as a linear combination $g_1 \cdot dx^1 + \cdots + g_n \cdot dx^n$.
(b) We say that a vector field $G\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ is *radial*, if there exists a continuous map $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$G(x) \;=\; f(\|x\|) \cdot \tfrac{x}{\|x\|} \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Show that for every smooth radial vector field $G\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ the corresponding 1-form $G \cdot \star$ is exact.

Exercise 71.4. Given $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rcl} g_k\colon S^1 & \to & S^1 \\ z & \mapsto & z^k. \end{array}$$

Furthermore we consider the vector field $F(x, y) = (-y, x)$ on S^1 and we consider the corresponding smooth 1-form $F \cdot \star$ on S^1. Show that

$$\int_{S^1} g_k^*(F \cdot \star) \;=\; 2\pi \cdot k.$$

Exercise 71.5. Let M be a connected 1-dimensional smooth manifold with non-empty boundary. Show that every smooth 1-form on M is exact.

Exercise 71.6. We consider the smooth manifold $S^1_{\geq 0} = \{(x, y) \in S^1 \mid y \geq 0\}$. We endow $S^1_{\geq 0}$ with the standard orientation "pointing counterclockwise". Determine the integral of the 1-form $y \cdot dx$ on $S^1_{\geq 0}$.

CHAPTER 72

Interlude: Multilinear algebra

In this chapter we recall some concepts from (multi-) linear algebra, namely alternating forms. The reader who needs a refresher on this material should stay here. The reader who is confident about their linear algebra abilities can move to Chapter 74 where we finally return to smooth manifolds.

This chapter is organized as follows:

- In Section 72.1 we will introduce alternating multilinear forms.
- In Section 72.2 we will introduce the wedge product of alternating forms.
- In Section 72.3 we will provide exercises for Chapter 72.

In the next Chapter 73 we will recall some concepts from integration theory.

72.1. **Definition of alternating multilinear forms.** We start out with some concepts from linear algebra:

Definition 72.1. Let V and W be real vector spaces and let

$$\begin{aligned} \Xi\colon V^k = \overbrace{V \times \cdots \times V}^{k\text{-times}} &\to W \\ (v_1, \dots, v_k) &\mapsto \Xi(v_1, \dots, v_k) \end{aligned}$$

be a map.

(i) We say that Ξ is multilinear if Ξ is $\mathbb{R}$-linear in every of its k arguments. More precisely, we require that for all $v, v' \in V$ and all $\lambda \in \mathbb{R}$ we have[761]

$$\begin{aligned} \Xi(\dots, v + v', \dots) &= \Xi(\dots, v, \dots) + \Xi(\dots, v', \dots) \\ \Xi(\dots, \lambda \cdot v, \dots) &= \lambda \cdot \Xi(\dots, v, \dots). \end{aligned}$$

(ii) We say that Ξ is alternating if for all $v, v' \in V$ we have

$$\Xi(\dots, v, \dots, v', \dots) = -\Xi(\dots, v', \dots, v, \dots).$$

We recall the following characterization of determinants:

Proposition 2.18 (Determinant–Characterization Proposition) Let $n \in \mathbb{N}_0$. There exists a *unique* map, called the determinant,

$$\det\colon \mathrm{M}(n \times n, \mathbb{R}) = (\mathbb{R}^n)^n \to \mathbb{R}$$

with the following properties:

(i) The map is multilinear and alternating.

(ii) The map is normed, i.e. if $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ denotes the standard basis of $\mathbb{R}^n$, then we have $\det(\mathfrak{e}_1 \dots \mathfrak{e}_n) = 1$. In other words, we have $\det(\mathrm{id}_n) = 1$.

[761] The non-specified arguments on the left and right are of course understood to be the same.

The determinant has furthermore the following two properties:

(iii) Given any $n \in \mathbb{N}$ and given any $A = (a_{ij})_{i,j=1,\dots,n} \in \mathrm{M}(n \times n, \mathbb{R})$ we have

$$\det(A) \;=\; \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot a_{1,\sigma(1)} \cdot \ldots \cdot a_{n,\sigma(n)} \qquad \text{(Leibniz Formula)}$$

where $\operatorname{sign}(\sigma) \in \{\pm 1\}$ denotes the sign of the permutation. It follows in particular that the map $\det\colon \mathrm{M}(n \times n, \mathbb{R}) \to \mathbb{R}$ is smooth.

(iv) The determinant is multilinear and alternating in the rows.

In the following we consider generalizations of the determinant:

Definition 72.2. Let V be a real vector space. An alternating k-form on V is a map

$$\begin{array}{rcl} \Xi\colon V^k & \to & \mathbb{R} \\ (v_1, \dots, v_k) & \mapsto & \Xi(v_1, \dots, v_k), \end{array}$$

which is multilinear and alternating.

Example. In the following we give many examples of alternating k-forms on real vector spaces:

(1) Let $\lambda \in \mathbb{R}$. It follows from the Determinant–Characterization Proposition 2.18 that

$$\begin{array}{rcl} (\mathbb{R}^n)^n & \to & \mathbb{R} \\ (v_1, \dots, v_n) & \mapsto & \lambda \cdot \det \underbrace{\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}}_{n \times n - \text{matrix}} \end{array}$$

is an alternating n-form on $\mathbb{R}^n$. In fact, as we will see in Exercise 72.1, it follows pretty easily from the uniqueness statement of the Determinant–Characterization Proposition 2.18 that every alternating n-form on $\mathbb{R}^n$ is of this form.

(2) An alternating 1-form is by definition a homomorphism $V \to \mathbb{R}$, i.e. an element in the dual vector space $V^* := \operatorname{Hom}(V, \mathbb{R})$. An alternating 1-form is also called a linear form.

(3) Let $w \in \mathbb{R}^n$ be a vector. It follows easily from the Cross Product Lemma 31.26 that the map

$$\begin{array}{rcl} \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n-1 \text{ copies}} & \to & \mathbb{R} \\ (v_1, \dots, v_{n-1}) & \mapsto & w \cdot \underbrace{(v_1 \times \cdots \times v_{n-1})}_{\substack{\text{cross product, see} \\ \text{Definition 31.25}}} \end{array} \quad \begin{array}{c} \text{Leibniz Formula 2.18} \\ \downarrow \\ = \end{array} \quad \det \underbrace{\begin{pmatrix} w & v_1 & \dots & v_{n-1} \end{pmatrix}}_{n \times n - \text{matrix}}$$

is an alternating $(n-1)$-form on $\mathbb{R}^n$.

(4) Let V be a real vector space. By the Product-of-Sets-and-Groups Lemma 6.4 we have $V^0 = \{0\}$. Note that in this setting every map $V^0 \to \mathbb{R}$ is multilinear and alternating. Thus an alternating 0-form is the same as an arbitrary map $\{0\} \to \mathbb{R}$.

(5) Let V be a real vector space and let $k \in \mathbb{N}_0$.

- The 0-map $V^k \to \mathbb{R}$ is of course an alternating k-form.
- The sum of two alternating k-form Ξ, Ξ' on V is defined as the alternating k-form $\Xi + \Xi'$, which is given by

$$(\Xi + \Xi')(v_1, \dots, v_k) \;=\; \Xi(v_1, \dots, v_k) + \Xi'(v_1, \dots, w_k).$$

In the same fashion we can also define the product of an alternating k-form with a real number.

Note that these operations turn the set of all alternating k-forms on V into a real vector space. □

Lemma 72.3. (Alternating Criterion)

(1) Let V be a real vector space and let $\Xi\colon V^k \to \mathbb{R}$ be a multilinear map. The following statements are equivalent:
   (a) The map Ξ is alternating.
   (b) Given any $v_1, \dots, v_k \in V$ such that two v_i are the same we have $\Xi(v_1, \dots, v_k) = 0$.
   (c) For every set $v_1, \dots, v_k \in V$ of k linearly dependent vectors we have the equality $\Xi(v_1, \dots, v_k) = 0$.

(2) Let V be an n-dimensional real vector space. Given any $k > n$ there exists only a single alternating k-form on V, namely the zero map.

Proof.

(1) Let $\Xi\colon V^k \to \mathbb{R}$ be a multilinear map. First we prove (a)⇒(b), thus we assume that $\Xi\colon V^k \to \mathbb{R}$ is alternating. Given $v \in V$ we have

$$\Xi(\dots v \dots v \dots) \underset{\text{since } \Xi \text{ is alternating}}{=} -\Xi(\dots v \dots v \dots).$$

Therefore we have $2 \cdot \Xi(\dots v \dots v \dots) = 0$, which implies that $\Xi(\dots v \dots v \dots) = 0$.

Next we prove (b)⇒(a). Thus we suppose that $\Xi(v_1, \dots, v_k) = 0$ whenever two v_i are equal. We need to show that we always have

$$\Xi(\dots v \dots w \dots) = -\Xi(\dots w \dots v \dots).$$

Indeed we have:

$$\begin{aligned} 0 &\underset{\text{by hypothesis}}{=} \Xi(\dots v + w \dots v + w \dots) \\ &= \underbrace{\Xi(\dots v \dots v \dots)}_{=0 \text{ by hypothesis}} + \Xi(\dots v \dots w \dots) + \Xi(\dots w \dots v \dots) + \underbrace{\Xi(\dots w \dots w \dots)}_{=0 \text{ by hypothesis}}. \\ &\underset{\text{since } \Xi \text{ is multilinear}}{\uparrow} \end{aligned}$$

We have thus shown that $0 = \Xi(\dots v \dots w \dots) + \Xi(\dots w \dots v \dots)$. But this means precisely that the desired equality $\Xi(\dots v \dots w \dots) = -\Xi(\dots w \dots v \dots)$ holds.

We prove (b)⇒(c). Thus we suppose that $\Xi(v_1, \dots, v_k) = 0$ whenever two v_i are the same. Now let $v_1, \dots, v_k \in V$ be linearly dependent vectors. This means that there exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ with $\lambda_1 \cdot v_1 + \dots + \lambda_k \cdot v_k = 0$ such that not all $\lambda_i = 0$. We pick $i \in \{1, \dots, k\}$ with $\lambda_i \neq 0$. Then we see that

$$\begin{aligned} \Xi(v_1 \dots v_i \dots v_k) &\underset{\text{since } \lambda_i \neq 0}{=} \Xi\big(v_1 \dots \tfrac{1}{\lambda_i} \cdot \lambda_i \cdot v_i \dots v_k\big) \\ &= \tfrac{1}{\lambda_i} \cdot \Xi\Big(v_1 \dots \cdot -\sum_{j \neq i} \lambda_j \cdot v_j \dots v_k\Big) \\ &\underset{\text{since } \Xi \text{ is multilinear}}{=} -\tfrac{1}{\lambda_i} \cdot \sum_{j \neq i} \underbrace{\Xi(v_1 \dots v_j \dots v_k)}_{=0, \text{ since the } i\text{-th and the } j\text{-th entry} = v_j} = 0. \end{aligned}$$

It remains to prove (b)⇒(c). But this statement is trivial, since if two entries are the same, then the vectors are of course linearly dependent.

(2) This statement follows immediately from (a)⇒(c), since for $k > \dim(V)$ any k vectors in V are necessarily linear dependent. ■

The following lemma, which will play a crucial role later on, let's us practice the definitions:

Lemma 72.4. (Form–Matrix Multiplication Lemma) Let Ξ be an alternating k-form on $\mathbb{R}^k$ and let A be a $(k \times k)$-matrix over $\mathbb{R}$. For all $v_1, \dots, v_k \in \mathbb{R}^k$ we have

$$\Xi(A \cdot v_1, \dots, A \cdot v_k) \;=\; \det(A) \cdot \Xi(v_1, \dots, v_k).$$

Proof. By the discussion on page 1503 we know that there exists a $\lambda \in \mathbb{R}$ such that

$$\Xi(v_1, \dots, v_k) \;=\; \lambda \cdot \det(v_1 \;\dots\; v_k), \qquad \text{for all } v_1, \dots, v_k \in \mathbb{R}^k.$$

Given any $v_1, \dots, v_k \in \mathbb{R}^k$ we obtain that

$$\begin{aligned} \Xi(A \cdot v_1, \dots, A \cdot v_k) &= \lambda \cdot \det(A \cdot v_1 \;\dots\; A \cdot v_k) &&= \lambda \cdot \det(A \cdot (v_1 \;\dots\; v_k)) \\ &= \lambda \cdot \det(A) \cdot \det(v_1 \;\dots\; v_k) &&= \det(A) \cdot \Xi(v_1, \dots, v_k). \\ &\uparrow \\ &\text{multiplicativity of the determinant} \end{aligned}$$

■

72.2. **The wedge product of alternating forms.** In this section we will introduce the wedge product of alternating forms. It allows us to construct new alternating forms out of given alternating forms. We start out with the definition of the wedge product of alternating 1-forms:

Definition 72.5. Let V be a real vector space and let $\varphi_1, \dots, \varphi_k \in V^*$ be linear forms. We define

$$\begin{aligned} \varphi_1 \wedge \dots \wedge \varphi_k \colon V^k &\to \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \det \underbrace{\begin{pmatrix} \varphi_1(v_1) & \dots & \varphi_1(v_k) \\ \vdots & & \vdots \\ \varphi_k(v_1) & \dots & \varphi_k(v_k) \end{pmatrix}}_{\text{real } (k \times k)\text{-matrix}}. \end{aligned}$$

It follows easily from the multilinearity of the determinant that $\varphi_1 \wedge \dots \wedge \varphi_k$ is an alternating k-form. We refer to it as the wedge product of $\varphi_1, \dots, \varphi_k$.[762]

Example. We consider the vector space $V = \mathbb{R}^n$. Given $i = 1, \dots, n$ we consider again the map

$$\begin{aligned} dx^i \colon \mathbb{R}^n &\to \mathbb{R} \\ (w_1, \dots, w_n) &\mapsto w_i. \end{aligned}$$

Given vectors $v_i = \begin{pmatrix} v_{i1} \\ \vdots \\ v_{in} \end{pmatrix} \in \mathbb{R}^n$, $i = 1, \dots, n$ we calculate that

$$\begin{aligned} (dx^1 \wedge \dots \wedge dx^n)(v_1, \dots, v_n) &= \det \begin{pmatrix} dx^1(v_1) & \dots & dx^1(v_n) \\ \vdots & & \vdots \\ dx^n(v_1) & \dots & dx^n(v_n) \end{pmatrix} &&= \det \begin{pmatrix} v_{11} & \dots & v_{n1} \\ \vdots & & \vdots \\ v_{1n} & \dots & v_{nn} \end{pmatrix} \\ &&&= \det \underbrace{(v_1 \;\dots\; v_n)}_{n \times n - \text{matrix}}. \end{aligned}$$

We have thus shown that the alternating n-form

$$dx^1 \wedge \dots \wedge dx^n \colon (\mathbb{R}^n)^n = \mathrm{M}(n \times n, \mathbb{R}) \;\to\; \mathbb{R}$$

[762] On page 1508 we will justify the weird notation $\varphi_1 \wedge \dots \wedge \varphi_k$.

corresponds precisely to the determinant. □

On page 1504 we see that the alternating k-forms on any real vector space form a vector space. It is now time to give this vector space a name:

Notation 72.6. Let V be a real vector space. We denote by $\wedge^k V^*$ the real vector space of all alternating k-forms on V.

Remark. The notation $\wedge^k V^*$ is standard, but arguably misleading. In fact in Exercise 72.2 we will see that not every alternating k-form can be written as the wedge product of k linear forms. □

The following proposition allows us to give a basis for the vector spaces $\wedge^k V^*$:

Proposition 72.7. (Alternating Forms-Basis Proposition) Let V be an n-dimensional real vector space and let $\varphi_1, \dots, \varphi_n$ be a basis for V^*. Let $k \in \{0, \dots, n\}$.

(1) The wedge products
$$\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \quad \text{with} \quad i_1 < i_2 < \dots < i_k$$
form a basis of $\wedge^k V^*$.

(2) We have $\dim(\wedge^k V^*) = \binom{n}{k}$.

Proof. Statement (2) is an immediate consequence of Statement (1). Thus it remains to prove Statement (1). To be more precise, it remains to prove the following claim:

Claim. Let $\Xi \in \wedge^k V^*$.

(a) We can Ξ write as a linear combination $\Xi = \sum_{i_1 < \dots < i_k} c_{i_1 \dots i_k} \cdot \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$.

(b) The coefficients $c_{i_1 \dots i_k}$ are unique.

Proof. Let $b_1, \dots, b_n$ be a basis of V which is dual to the basis $\varphi_1, \dots, \varphi_n$ of V^*, i.e. we have
$$\varphi_i(b_j) \;=\; \delta_{ij} \;=\; \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
From the definition of the wedge product it follows easily that for $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$ we have

(∗) $$(\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k})(b_{j_1}, \dots, b_{j_k}) \;=\; \begin{cases} 1, & \text{if } (i_1, \dots, i_k) = (j_1, \dots, j_k), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Xi \in \wedge^k V^*$.

(a) Given $i_1 < \dots < i_k$ we set
$$c_{i_1 \dots i_k} \;:=\; \Xi(b_{i_1}, \dots, b_{i_k}) \in \mathbb{R}.$$
We consider
$$\Theta \;:=\; \sum_{i_1 < \dots < i_k} c_{i_1 \dots i_k} \cdot \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}.$$
We want to show that $\Theta = \Xi$.

(i) It follows from (∗) that Ξ and Θ have the same values on all $(b_{j_1}, \dots, b_{j_k}) \in V^k$ with $j_1 < \dots < j_k$.

(ii) Since Θ and Ξ are alternating it follows from (i) that Ξ and Θ have the same values on $(b_{j_1}, \dots, b_{j_k}) \in V^k$ for arbitrary $j_1, \dots, j_k$.

(iii) Since Θ and Ξ are multilinear it follows from (ii) that Ξ and Θ have the same values on $(v_1, \dots, v_k) \in V^k$ for arbitrary $v_1, \dots, v_k \in V$.[763]

(b) From $(*)$ we obtain easily the uniqueness statement, since $c_{i_1 \dots i_k} = \Xi(b_{j_1}, \dots, b_{j_k})$. ■

In the following proposition we introduce the wedge product of alternating forms of arbitrary degree:

Proposition 72.8. (Wedge Product Proposition) Let V be a finite-dimensional real vector space and let $k, l \in \mathbb{N}_0$. There exists a unique map

$$\begin{array}{rcl} \wedge^k V^* \times \wedge^l V^* & \to & \wedge^{k+l} V^* \\ (\sigma, \tau) & \mapsto & \sigma \wedge \tau \end{array}$$

with the following properties:

(1) The map is $\mathbb{R}$-bilinear, i.e. for all $\sigma, \sigma_1, \sigma_2 \in \wedge^k V^*$, for all $\tau, \tau_1, \tau_2 \in \wedge^l V^*$ and for all $\lambda \in \mathbb{R}$ we have

$$\begin{array}{rcl} (\sigma_1 + \sigma_2) \wedge \tau & = & \sigma_1 \wedge \tau + \sigma_2 \wedge \tau, \\ \sigma \wedge (\tau_1 + \tau_2) & = & \sigma \wedge \tau_1 + \sigma \wedge \tau_2 \end{array}$$

and

$$(\lambda \cdot \sigma) \wedge \tau \;=\; \lambda \cdot (\sigma \wedge \tau) \;=\; \sigma \wedge (\lambda \cdot \tau).$$

(2) Given any linear forms $\psi_1, \dots, \psi_k, \eta_1, \dots, \eta_l \in V^*$ we have

$$\underbrace{(\psi_1 \wedge \dots \wedge \psi_k)}_{\text{see Definition 72.5}} \wedge \underbrace{(\eta_1 \wedge \dots \wedge \eta_l)}_{\text{see Definition 72.5}} \;=\; \underbrace{\psi_1 \wedge \dots \wedge \psi_k \wedge \eta_1 \wedge \dots \wedge \eta_l}_{\text{see Definition 72.5}}.$$

Definition 72.9. In the following, given $\sigma \in \wedge^k V^*$ and $\eta \in \wedge^l V^*$ we refer to the alternating $(k+l)$-form $\sigma \wedge \tau \in \wedge^{k+l} V^*$ as the wedge product of σ and τ.

Sketch of proof. Let $\varphi_1, \dots, \varphi_n$ be a basis of V^*. Let $\sigma \in \wedge^k V^*$ and let $\tau \in \wedge^l V^*$. By the Alternating Forms-Basis Proposition 72.7 we can write

$$\sigma \;=\; \sum_{i_1 < \dots < i_k} a_{i_1, \dots, i_k} \cdot \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \quad \text{and} \quad \tau \;=\; \sum_{j_1 < \dots < j_l} b_{j_1, \dots, j_l} \cdot \varphi_{j_1} \wedge \dots \wedge \varphi_{j_l},$$

where the coefficients $a_{i_1, \dots, i_k} \in \mathbb{R}$ and $b_{j_1, \dots, j_l} \in \mathbb{R}$ are uniquely determined. We set

$$\sigma \wedge \tau \;:=\; \sum_{\substack{i_1 < \dots < i_k \\ dred j_1 < \dots < j_l}} a_{i_1, \dots, i_k} \cdot b_{j_1, \dots, j_l} \cdot \underbrace{\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \dots \wedge \varphi_{j_l}}_{\text{see Definition 72.5}}.$$

One can now easily show that this map has the properties (1) and (2). Furthermore it also elementary to show that this is the unique map that satisfies (1) and (2). ■

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[763] Indeed, if for $i = 1, \dots, k$ we write $v_i = \sum_{j=1}^{n} v_{ij} \cdot b_j$, then we see that

$$\Xi(v_1, \dots, v_k) = \Xi\Big(\sum_{j_1=1}^{n} v_{1j_1} \cdot b_{j_1}, \dots, \sum_{j_k=1}^{n} v_{1j_k} \cdot b_{j_k}\Big) \overset{\substack{\text{multilinearity applied to the entries } 1, \dots, k \\ \downarrow}}{=} \sum_{j_1, \dots, j_k = 1}^{n} v_{1j_1} \dots v_{kj_k} \cdot \Xi(b_{j_1}, \dots, b_{j_k})$$

$$\underset{\substack{\uparrow \\ \text{see (b) above}}}{=} \sum_{j_1, \dots, j_k = 1}^{n} v_{1j_1} \dots v_{kj_k} \cdot \Xi(b_{j_1}, \dots, b_{j_k}) \underset{\substack{\uparrow \\ \text{same argument backwards}}}{=} \Theta(v_1, \dots, v_k).$$

The following proposition can be viewed as a sequel to the Wedge Product Proposition 72.8. The proposition says that the wedge product is associative, and that it is commutative "up to a suitable sign".

Lemma 72.10. (Wedge Product Properties Lemma) The wedge product on a finite-dimensional real vector space V has the following properties:

(1) Given any $\alpha \in \wedge^k V^*, \beta \in \wedge^l V^*$ and $\gamma \in \wedge^m V^*$ we have

$$(\alpha \wedge \beta) \wedge \gamma \;=\; \alpha \wedge (\beta \wedge \gamma).$$

It follows this observation and the discussion on page 115 that we can write wedge products without parentheses.

(2) Given any $\alpha \in \wedge^k V^*$ and $\beta \in \wedge^l V^*$ we have

$$\beta \wedge \alpha \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)} \cdot \alpha \wedge \beta.$$

Example.

(1) Let V be a finite-dimensional real vector space and let $\varphi_1, \dots, \varphi_k \in V^*$. It follows from the Wedge Product Properties Lemma 72.10 that we can form unambiguously the wedge product $\varphi_1 \wedge \cdots \wedge \varphi_k$. Furthermore note that it follows from iteratively applying the Wedge Product Proposition 72.8 (2) that this wedge product agrees with the definition of $\varphi_1 \wedge \cdots \wedge \varphi_k$ from Definition 72.5.

(2) Given any two linear forms φ and ψ in V^* we obtain from the Wedge Product Properties Lemma 72.10 (2) the following equality of alternating 2-forms:

$$\varphi \wedge \psi \;=\; -\psi \wedge \varphi.$$

Given a linear form φ we see in particular that $\varphi \wedge \varphi = 0$. □

Proof. Statement (1) follows easily from the Wedge Product Proposition 72.8 (2) together with the Alternating Forms-Basis Proposition 72.7 and the Wedge Product Proposition 72.8 (1). Statement (2) follows from the Alternating Forms-Basis Proposition 72.7, from the Wedge Product Proposition 72.8 (2) and from the following claim:

Claim. Let $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_l \in V^*$. Then:

$$\psi_1 \wedge \cdots \wedge \psi_l \wedge \varphi_1 \wedge \cdots \wedge \varphi_k \;=\; (-1)^{kl} \cdot \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \psi_1 \wedge \cdots \wedge \psi_l.$$

Proof. We first deal with the case $k = l = 1$. Thus let $\varphi, \psi \in V^*$. Then for any two vectors $v, w \in V$ we have

$$(\psi \wedge \varphi)(v,w) \underset{\text{Definition 72.5}}{=} \det \begin{pmatrix} \psi(v) & \psi(w) \\ \varphi(v) & \varphi(w) \end{pmatrix} \underset{\substack{\text{since by the Determinant Characterization Proposition 2.18 we know that determinants are alternating in the rows}}}{=} -\det \begin{pmatrix} \varphi(v) & \varphi(w) \\ \psi(v) & \psi(w) \end{pmatrix} \underset{\text{Definition 72.5}}{=} -(\varphi \wedge \psi)(v,w).$$

Now let $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_l \in V^*$. From the above discussion we obtain that for any choice of $i \in \{1, \dots, k\}$, $j \in \{1, \dots, l\}$ we have $\psi_i \wedge \varphi_j = -\varphi_j \wedge \psi_i$. Using this observation we obtain that

$$\begin{array}{rcl} \psi_1 \wedge \cdots \wedge \psi_l \wedge \varphi_1 \wedge \cdots \wedge \varphi_k & \overset{\text{since we require } k \text{ transpositions to "push" } \psi_l \text{ to the back}}{=} & (-1)^k \cdot \psi_1 \wedge \cdots \wedge \psi_{l-1} \wedge \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \psi_l \\ & = & (-1)^{kl} \cdot \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \psi_1 \wedge \cdots \wedge \psi_l. \\ & & \uparrow \text{we apply the same argument altogether } l \text{ times} \end{array}$$

■

We conclude this discussion of linear algebra with the following definition and a straightforward lemma regarding the new definition:

Definition 72.11. Let $f\colon U \to V$ be a homomorphism between two finite-dimensional real vector spaces. For every $k \in \mathbb{N}_0$ we obtain the induced homomorphism

$$\begin{array}{rcl} f^*\colon \wedge^k V^* & \to & \wedge^k U^* \\ (\Xi\colon V^k \to \mathbb{R}) & \mapsto & \left(\begin{array}{rcl} f^*\Xi\colon U^k & \to & \mathbb{R} \\ (u_1, \ldots, u_k) & \mapsto & \Xi(f(u_1), \ldots, f(u_k)) \end{array} \right). \end{array}$$

Example. Let U be an n-dimensional real vector space and let $f\colon U \to U$ be an endomorphism of U. In Exercise 72.8 we will see that the induced $\wedge^n U^* \to \wedge^n U^*$ is given by multiplication by $\det(f)$. □

Lemma 72.12. (Alternating Form-Functoriality Lemma)

(1) The maps

$$\begin{array}{rcl} V & \mapsto & \wedge^k V^* \\ f & \mapsto & f^* \end{array}$$

define a contravariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces.

(2) Let $f\colon U \to V$ be a homomorphism between two finite-dimensional real vector spaces. The map $f^*\colon \wedge^k V^* \to \wedge^k U^*$ respects the wedge product, i.e. for any $\alpha \in \wedge^k V^*$ and $\beta \in \wedge^l V^*$ we have

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta \qquad \in \wedge^{k+l} U^*.$$

In other words, the wedge product is natural.

Proof.

(1) This statement follows immediately from the definitions.

(2) For wedge products of linear forms the statement follows immediately from Definition 72.5. The general case follows from the bilinearity of the wedge product and the Alternating Forms-Basis Proposition 72.7. ■

72.3. Exercises for Chapter 72.

Exercise 72.1. Show that for every alternating n-form ω on $\mathbb{R}^n$ there exists an $\lambda \in \mathbb{R}$ such that ω is given by

$$\begin{array}{rcl} (\mathbb{R}^n)^n & \to & \mathbb{R} \\ (v_1, \ldots, v_n) & \mapsto & \lambda \cdot \det \underbrace{\begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}}_{n \times n - \text{matrix}} \end{array}$$

Exercise 72.2.

(a) Let V be a vector space and let $\alpha \in V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ be a linear form. Show that $\alpha \wedge \alpha = 0$.

(b) We consider the vector space $\mathbb{R}^4$. For $i = 1, 2, 3, 4$ we denote by $dx^i \in \mathrm{Hom}(\mathbb{R}^4, \mathbb{R})$ the linear form that is given by

$$dx^i(a_1, a_2, a_3, a_4) \;\coloneqq\; a_i.$$

Show that there are no linear forms $\varphi, \psi \colon \mathbb{R}^4 \to \mathbb{R}$ such that

$$\varphi \wedge \psi \;=\; dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

Hint. Given an alternating 2-form ω consider the "square" $\omega \wedge \omega$.

(c) We consider the vector space $\mathbb{R}^6$. Does there exist an alternating 2-form ω such that $\omega \wedge \omega \wedge \omega \neq 0$?

Exercise 72.3. For $i = 1, 2, 3$ we denote by $dx^i \colon \mathbb{R}^3 \to \mathbb{R}$ the usual projection onto the i-th coordinate. These are linear forms on the vector space $\mathbb{R}^3$. It follows from the Alternating Forms-Basis Proposition 72.7 that $dx^1 \wedge dx^2$, $dx^1 \wedge dx^3$ and $dx^2 \wedge dx^3$ form a basis of the vector space $\wedge^2(\mathbb{R}^3)^*$ of alternating 2-forms. Now let $w = (w_1, w_2, w_3) \in \mathbb{R}^3$. The map

$$\begin{array}{rcl} \omega \colon \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R} \\ (v_1, v_2) & \mapsto & w \cdot (v_1 \times v_2) \end{array}$$

is an alternating 2-form on $\mathbb{R}^3$. Express this alternating 2-form in terms of the above basis $dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3$ of $\wedge^2(\mathbb{R}^3)^*$.

Exercise 72.4. Let V be a real vector space. Show that the map

$$\begin{array}{rcl} (V^*)^k = \overbrace{V^* \times \cdots \times V^*}^{k\text{-times}} & \to & \wedge^k V^* \\ (\varphi_1, \ldots, \varphi_k) & \mapsto & \varphi_1 \wedge \cdots \wedge \varphi_k \end{array}$$

is multilinear and alternating.

Exercise 72.5. Let V be a finite-dimensional real vector space. We say a 2-form ω on V is *non-degenerate* if the map

$$\begin{array}{rcl} V & \to & \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ v & \mapsto & \left(\begin{array}{rcl} V & \to & \mathbb{R} \\ w & \mapsto & \varphi(v, w) \end{array} \right) \end{array}$$

is an isomorphism.

(a) Show that if V admits a non-degenerate 2-form, then $\dim_{\mathbb{R}}(V)$ is even.

(b) Now assume that V is $2n$-dimensional and that φ is a non-degenerate 2-form on V. Show that $\varphi^n = \underbrace{\varphi \wedge \cdots \wedge \varphi}_{n \text{ times}} \neq 0$.

Exercise 72.6. Let V be the vector space of continuous maps $\mathbb{R} \to \mathbb{R}$.

(a) Does there exist an epimorphism $\varphi \colon V \to \mathbb{R}^2$?

(b) Does there exist an alternating 2-form $\omega \in \wedge^2 V^*$ which is non-zero?

Exercise 72.7. Let $k \in \{1, \ldots, n\}$. For every subset $I \subseteq \{1, \ldots, n\}$ with k elements we consider the map

$$\begin{array}{rcl} \varphi_I \colon (\mathbb{R}^n)^k & \to & \mathbb{R} \\ (v_1, \ldots, v_k) & \mapsto & \begin{array}{c} \text{determinant of the matrix which we obtain from} \\ \text{the } n \times k\text{-matrix } (v_1 \ldots v_k) \text{ by deleting all rows in } I. \end{array} \end{array}$$

These are alternating k-forms on $\mathbb{R}^n$. Do these alternating k-forms form a basis of $\wedge^k(\mathbb{R}^n)^*$?

Exercise 72.8. Let U be an n-dimensional real vector space and let $f\colon U \to U$ be an endomorphism of U. Show that the induced $\wedge^n U^* \to \wedge^n U^*$ is given by multiplication by $\det(f)$.

Exercise 72.9. Let V be a finite-dimensional real vector space. Show that an orientation of V, in the sense of Definition 31.3, is naturally the same as choosing one of the two elements of the set $((\wedge^{\dim(V)} V^*) \setminus \{0\})/(\mathbb{R} \setminus \{0\})$.
Remark. This definition of an orientation has the nice feature that it also works for 0-dimensional real vector spaces.

CHAPTER 73

Interlude: Integration theory

In Chapter 71 we introduced the notion of a smooth 1-form on a smooth manifold and we showed (using the classical Riemann integral) that one can integrate smooth 1-forms over compact oriented 1-dimensional smooth manifolds. In the previous Chapter 72 we introduced the notion of an alternating k-form. In the next Chapter 74 we will use this linear algebraic concept to introduce smooth k-forms on smooth manifolds. Furthermore, we will see that one can integrate smooth k-forms over compact oriented k-dimensional smooth manifolds.

To introduce this integral we will make use of the Lebesgue integral, a popular concept from real analysis. In this chapter we will recall some basic definitions, concepts and results from integration theory. The reader who is confident about integration theory can move straight to the next Chapter 74.

This chapter is organized as follows:

- In Section 73.1 we will define the notion of a Borel σ-algebra.
- In Section 73.2 we will introduce the notion of a measurable function.
- In Section 73.3 we will introduce the Lebesgue measure.
- In Section 73.4 we will introduce the Lebesgue integral.
- In Section 73.5 we will recall properties of the Lebesgue integral.
- In Section 73.6 we will discuss the relationship between integration and differentiation.
- In Section 73.7 we will provide exercises for Chapter 73.

In the next Chapter 74 we will introduce differential forms of arbitrary degree.

73.1. The Borel σ-algebra.

We start out with the following set theoretic definition:

Definition 73.1. Let Ω be a set. We say that $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra if the following statements hold:

(1) $\Omega \in \mathcal{A}$.

(2) $A \in \mathcal{A} \implies$ the complement $A^c := \Omega \setminus A$ is also contained in $\mathcal{A}$.

(3) The union of *countably* many sets in $\mathcal{A}$ lies again in $\mathcal{A}$. In other words, if $\{A_i\}_{i \in I}$ is a *countable* family of sets in $\mathcal{A}$, then $\bigcup_{i \in I} A_i \in \mathcal{A}$.

Example. For any set Ω the power set $\mathcal{P}(\Omega)$ is evidently a σ-algebra. □

Lemma 73.2. (σ-Algebra–Intersection Lemma) Let Ω be a set and let $\{\mathcal{A}_i\}_{i \in I}$ be a non-empty family of σ-algebras on Ω. The intersection $\bigcap_{i \in I} \mathcal{A}_i$ is also a σ-algebra on Ω.

Proof. The lemma follows easily from the definitions. ■

Definition 73.3. Let Ω be a set and let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$. We define[764]

$$\sigma(\mathcal{E}) \;:=\; \text{intersection of all } \sigma\text{-algebras which contain } \mathcal{E},$$

and we refer to $\sigma(\mathcal{E})$ as the σ-algebra generated by $\mathcal{E}$. Furthermore we call $\mathcal{E}$ a generating set of the σ-algebra $\sigma(\mathcal{E})$.

The following definition gives us the only σ-algebras which we care about in these lecture notes:

Definition 73.4. Let $n \in \mathbb{N}$.

(1) In the following we refer to the σ-algebra on $\mathbb{R}^n$ that is generated by all closed cuboids $[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ as the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ on $\mathbb{R}^n$. We call the sets in $\mathcal{B}(\mathbb{R}^n)$ measurable.

(2) Given any subset $A \subseteq \mathbb{R}^n$ we refer to $\mathcal{B}(A) := \{A \cap E \,|\, E \in \mathcal{B}(\mathbb{R}^n)\}$ as the Borel σ-algebra $\mathcal{B}(A)$ on A.[765]

Proposition 73.5. (Borel σ-Algebra Proposition)

(1) Every closed and every open subset of $\mathbb{R}^n$ is measurable.

(2) The open subsets of $\mathbb{R}^n$ are a generating set for the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$.

Proof.

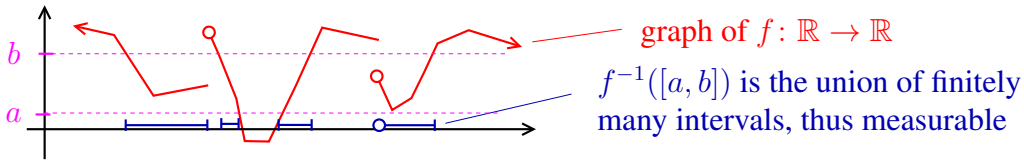
(1) (a) Similar to the proof of the Second Countability Lemma 13.2 one can easily show that every open subset of $\mathbb{R}^n$ is the union of countably many closed cuboids. Since by definition each closed cuboid lies in $\mathcal{B}(\mathbb{R}^n)$ and since $\mathcal{B}(\mathbb{R}^n)$, as a σ-algebra, is closed under countable unions, we see that each open subset of $\mathbb{R}^n$ is contained in $\mathcal{B}(\mathbb{R}^n)$.

(b) In (a) we just showed that open subsets lie in $\mathcal{B}(\mathbb{R}^n)$. But as a σ-algebra we know that $\mathcal{B}(\mathbb{R}^n)$ is closed under taking complements. It follows that all closed subsets of $\mathbb{R}^n$ lie in $\mathcal{B}(\mathbb{R}^n)$.

(2) We will prove this statement in Exercise 73.3. ■

73.2. **Measurable functions.** Eventually our goal is to introduce the Lebesgue-integral of "suitable" functions. The following definition makes it precise what we mean by a "suitable" function:

Definition 73.6. Let Ω be a set and let $\mathcal{A}$ be a σ-algebra on Ω. A function $f\colon \Omega \to \mathbb{R}$ is called measurable, if for all $A \subseteq \mathcal{B}(\mathbb{R})$ we have $f^{-1}(A) \in \mathcal{A}$.



The following proposition shows that all functions we care about, are in fact measurable:

[764] Since $\mathcal{P}(\Omega)$ is a σ-algebra, we see that $\mathcal{E}$ is contained in at least one σ-algebra, namely $\mathcal{P}(\Omega)$. Therefore it follows from the σ-Algebra–Intersection Lemma 73.2 that $\sigma(\mathcal{E})$ is indeed defined.

[765] It follows from the Intersection–Distributive Lemma 1.5 that this is indeed a σ-algebra.

Proposition 73.7. (Continuous-Measurable Proposition) Let $A \subseteq \mathbb{R}^n$ be a measurable subset. Every continuous function $f\colon A \to \mathbb{R}$ is measurable.

The proof of the Continuous-Measurable Proposition 73.7 rests on the following lemma:

Lemma 73.8. (Generating Set–Measurable Lemma) Let Ω be a set and let $\mathcal{A}$ be a σ-algebra on Ω. Let $\mathcal{E}$ be a generating set for $\mathcal{B}(\mathbb{R})$. For any function $f\colon \Omega \to \mathbb{R}$ we have

$$f \text{ is measurable} \iff f^{-1}(E) \in \mathcal{A} \quad \text{for alle } E \in \mathcal{E}.$$

Proof of the Generating Set–Measurable Lemma 73.8. The "$\Rightarrow$"-direction follows immediately from the definitions. We turn to the proof of the "$\Leftarrow$"-direction. We set

$$\mathcal{M} := \{M \subseteq \mathbb{R} \mid f^{-1}(M) \in \mathcal{A}\}.$$

By hypothesis we have $\mathcal{E} \subseteq \mathcal{M}$. We need to show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$. First note that it follows easily from the Image-Preimage Lemma 1.9 that $\mathcal{M}$ is a σ-Algebra. We now see that

$$\mathcal{B}(\mathbb{R}) \underset{\substack{\uparrow\\ \text{since } \mathcal{E} \text{ is a generating set of } \mathcal{B}(\mathbb{R})}}{=} \sigma(\mathcal{E}) \underset{\substack{\uparrow\\ \text{definition of } \sigma(\mathcal{E})}}{:=} \text{intersection of all } \sigma\text{-algebras which contain } \mathcal{E} \underset{\substack{\uparrow\\ \text{since we just saw that } \mathcal{M} \text{ is a } \sigma\text{-algebra that contains } \mathcal{E}}}{\subseteq} \mathcal{M}.$$

■

Proof of the Continuous-Measurable Proposition 73.7. Let $A \subseteq \mathbb{R}^n$ be a measurable subset and let $f\colon A \to \mathbb{R}$ be a continuous map. We need to show that f is measurable. By the Borel σ-Algebra Proposition 73.5 (2) we know that the open subsets form a generating set of $\mathcal{B}(\mathbb{R})$. Thus it follows from the Generating Set–Measurable Lemma 73.8 that it suffices to prove the following claim:

Claim. For every open subset $U \subseteq \mathbb{R}$ the preimage $f^{-1}(U)$ is measurable in A.

Proof. Let $U \subseteq \mathbb{R}$ be an open subset. Since f is continuous we know that $f^{-1}(U)$ is an open subset of A. By definition of the subspace topology on A this means that there exists an open subset $V \subseteq \mathbb{R}^n$ such that $f^{-1}(U) = A \cap V$. By the Borel σ-Algebra Proposition 73.5 (2) we know that $V \in \mathcal{B}(\mathbb{R}^n)$. It now follows from the definition of $\mathcal{B}(A)$ that we also have $f^{-1}(U) = A \cap V \in \mathcal{B}(A)$. ■

73.3. **The Lebesgue measure.** We move on to one of the most essential definitions from measure theory:

Definition 73.9. Let Ω be a set and let $\mathcal{A}$ be a σ-algebra on Ω.

(1) A map $\mu\colon \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is called σ-additive, if for every *countable* family $\{A_i\}_{i \in I}$ of pairwise disjoint sets in the σ-algebra $\mathcal{A}$ the following equality holds:

$$\mu\Big(\underbrace{\bigsqcup_{i \in I} A_i}_{\substack{\in \mathcal{A}, \text{ since } \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \Big) = \sum_{i \in I} \mu(A_i).$$

(2) A σ-additive map $\mu\colon \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ with $\mu(\varnothing) = 0$ is called a measure on $\mathcal{A}$.

In the following proposition we collect a few essential properties of measures:

Proposition 73.10. (Measure Proposition) Let Ω be a set, let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra and let $\mu\colon \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measure.

(1) Let $A, B \in \mathcal{A}$.

(a) $$\mu(A \cup B) \;=\; \mu(A) + \mu(B) - \mu(A \cap B).$$

(b) $$A \subseteq B \quad\Longrightarrow\quad \mu(A) \leq \mu(B).$$

(2) For any countable family $\{A_i\}_{i\in I}$ of subsets in $\mathcal{A}$ we have

$$\mu\Big(\bigcup_{i\in I} A_i\Big) \;\leq\; \sum_{i\in I} \mu(A_i) \qquad (\sigma\text{-subadditivity}).$$

(3) Let $A_i \in \mathcal{A}$, $i \in \mathbb{N}$ and let $A \subseteq \Omega$. We write $A_i \nearrow A$ if for each $i \in \mathbb{N}$ we have $A_i \subseteq A_{i+1}$ and if $\bigcup_{i=1}^{\infty} A_i = A$. With this notation we have

$$A_i \nearrow A \quad\Longrightarrow\quad A \in \mathcal{A} \text{ and } \lim_{i\to\infty} \mu(A_i) \;=\; \mu(A).$$

Proof. All statements follow easily from the definition of a measure. We will fill in the details in Exercise 73.5. ■

Now we move on to the only measure we will care about in these notes:

Theorem 73.11. (Lebesgue Measure Theorem) Let $n \in \mathbb{N}$. There exists a unique measure

$$\lambda^n\colon \mathcal{B}(\mathbb{R}^n) \;\to\; \mathbb{R}_{\geq 0} \cup \{\infty\}$$

on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ such that for every closed cuboid $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with $a_i \leq b_i$ the following equality holds:

$$\lambda^n([a_1, b_1] \times \cdots \times [a_n, b_n]) \;=\; (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n).$$

Proof. This theorem is (implicitly) proved in every reasonable book on measure and integration theory. We now explain how one can deduce the theorem from [**Fol1999**]:

(1) The case $n = 1$ is dealt with in [**Fol1999**, Theorem 1.16] and on [**Fol1999**, p. 37].

(2) Let $n \in \mathbb{N}$. It follows from [**Fol1999**, Proposition 1.3] that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ is the "product" $\mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R})$ of the 1-dimensional σ-algebras. The promised statement now follows from [**Fol1999**, p. 64] and [**Fol1999**, Theorem 1.14]. ■

Definition 73.12. Let $n \in \mathbb{N}$.

(1) The measure $\lambda^n\colon \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ from the Lebesgue Measure Theorem 73.11 is called the Lebesgue measure λ^n. If n is clear from the context, then often we write also λ instead of λ^n.

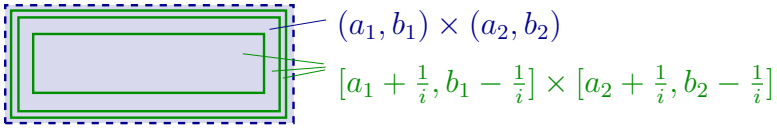
(2) As any other reasonable person we often say "volume" instead of "Lebesgue measure" and for subsets of $\mathbb{R}^2$ we often speak of the "area". In the same vein we also at times write vol instead of λ^n.

Example.

(1) We consider the open cuboid $(a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$. We see that:

$$\lambda^n((a_1,b_1)\times\cdots\times(a_n,b_n)) \overset{\text{this follows from the Measure Proposition 73.10 (3), since } [a_1+\frac{1}{i},b_1-\frac{1}{i}]\times\ldots\times[a_n+\frac{1}{i},b_n-\frac{1}{i}] \nearrow (a_1,b_1)\times\ldots\times(a_n,b_n)}{=} \lim_{i\to\infty} \lambda^n([a_1+\tfrac{1}{i},b_1-\tfrac{1}{i}]\times\cdots\times[a_n+\tfrac{1}{i},b_n-\tfrac{1}{i}])$$
$$\overset{\text{by definition of } \lambda^n}{=} \lim_{i\to\infty} (b_1-\tfrac{2}{i}-a_1)\cdot\ldots\cdot(b_n-\tfrac{2}{i}-a_n) = (a_1-b_1)\cdot\ldots\cdot(a_n-b_n).$$

The analogous statement also holds for "half-open" cuboids $[a_1, b_1) \times \cdots \times [a_n, b_n)$.



(2) Let $U \subseteq \mathbb{R}^n$ be an open subset.



In Exercise 73.6 we will see that we can write U as the disjoint union of a countable family $\{R_i\}_{i\in I}$ of half-open cuboids. It follows from the σ-additivity of the Lebesgue-measure and the calculation in (1) that $\lambda^n(A)$ is just the series over the "usual naive" measure of the cuboids R_i.

(3) Let $K \subseteq \mathbb{R}^n$ be a compact subset. By the Heine-Borel Theorem 3.44 we know that K is closed and that there exists an $r > 0$ such that $K \subseteq [-r, r] \times \cdots \times [-r, r]$. It follows from the Borel σ-Algebra Proposition 73.5 and the Measure Proposition 73.10 (1b) that K is measurable with finite Lebesgue measure.

(4) Let $X \subseteq \mathbb{R}^n$ be measurable subset. If X is a subset of measure zero in the sense of Definition 37.1, then it follows easily from the definition of the Lebesgue measure and the Measure Proposition 73.10 (1b) and (2) that $\mathrm{vol}(A) = 0$. □

We conclude this discussion of the Lebesgue measure with the following pretty proposition:

Proposition 73.13. (Determinant-Volume Proposition) Let $X \subseteq \mathbb{R}^n$ be measurable with $\mathrm{vol}(X) < \infty$ and let $A \in \mathrm{M}(n \times n, \mathbb{R})$. Then the subset $A \cdot X \subseteq \mathbb{R}^n$ is also measurable and we have the equality $\mathrm{vol}(A \cdot X) = |\det(A)| \cdot \mathrm{vol}(X)$.

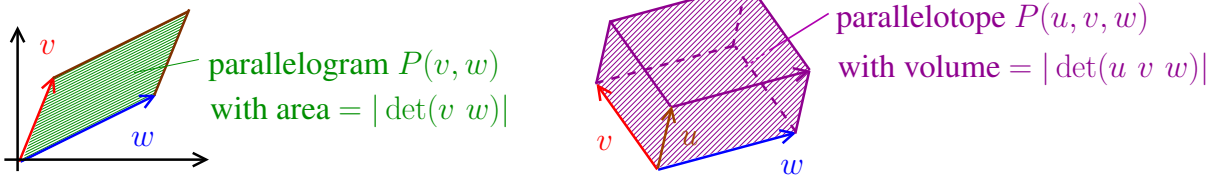
Proof. This theorem is proved in [**Fol1999**, Theorem 2.44]. ■

Example. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be n vectors. We consider the corresponding parallelotope

$$P(v_1,\ldots,v_n) = \left\{ \sum_{i=1}^n \lambda_i \cdot v_i \,\middle|\, \lambda_1,\ldots,\lambda_n \in [0,1] \right\}.$$

We set $A := (v_1 \ldots v_n) \in \mathrm{M}(n \times n, \mathbb{R})$. Then we see that

$$\mathrm{vol}(P(v_1,\ldots,v_n)) = \mathrm{vol}(P(A\cdot\mathfrak{e}_1,\ldots,A\cdot\mathfrak{e}_n)) = \mathrm{vol}(A\cdot P(\mathfrak{e}_1,\ldots,\mathfrak{e}_n))$$
$$\overset{\text{by the Determinant-Volume Proposition 73.13}}{=} |\det(A)| \cdot \underbrace{\mathrm{vol}(\mathfrak{e}_1,\ldots,\mathfrak{e}_n)}_{=1} = |\det(v_1 \ldots v_n)|.$$



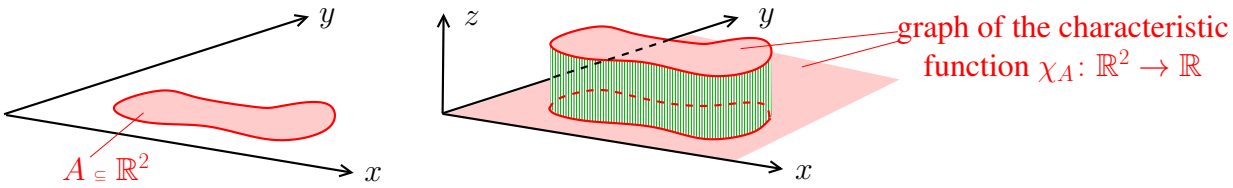
This calculation gives a very nice geometric interpretation of the absolute value of the determinant of a matrix. □

73.4. **The Lebesgue integral.** Now that we have studied the Lebesgue measure in great detail, it is time to move on to the definition of the Lebesgue integral. We start out with the following elementary definition.

Notation 73.14. Let $A \subseteq \mathbb{R}^n$ be a subset. We refer to

$$\begin{aligned} \chi_A \colon \Omega &\to \mathbb{R} \\ x &\mapsto \chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases} \end{aligned}$$

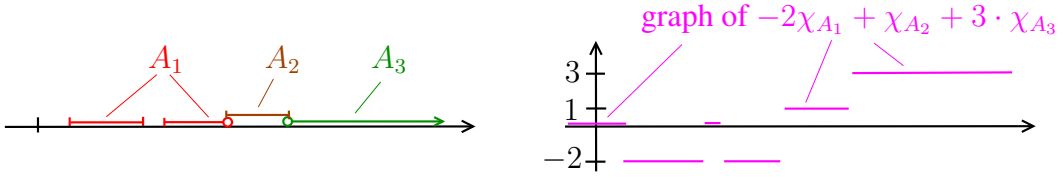
as the characteristic function of A.



Definition 73.15. A step function on $\mathbb{R}^n$ is a function of the Form

$$f = \sum_{i=1}^{m} c_i \cdot \chi_{A_i},$$

where $c_1, \ldots, c_m \in \mathbb{R}$ and where $A_1, \ldots, A_m$ are pairwise disjoint measurable sets.[766]



Next we introduce the integral of a non-negative step function :

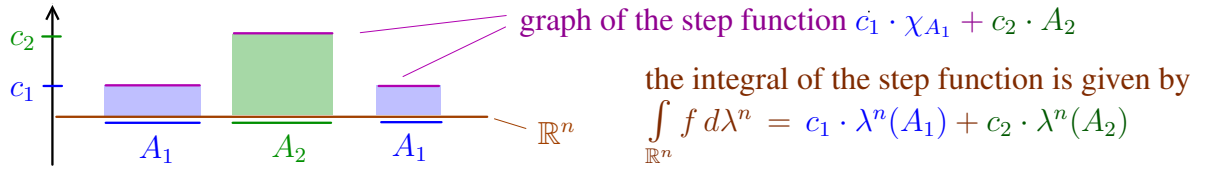
Definition 73.16. Let $f \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a non-negative step function. We write

$$f = \sum_{i=1}^{m} c_i \cdot \chi_{A_i},$$

[766] In other words: A step function is a linear combination of characteristic functions of pairwise disjoint measurable subsets.

where $c_1, \dots, c_m \in \mathbb{R}_{>0}$ and $A_1, \dots, A_m$ are pairwise disjoint measurable sets. We now set[767] [768]

$$\int_{\mathbb{R}^n} f \, d\lambda^n := \sum_{i=1}^{m} c_i \cdot \lambda^n(A_i) \quad \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$



We move on to arbitrary non-negative measurable functions:

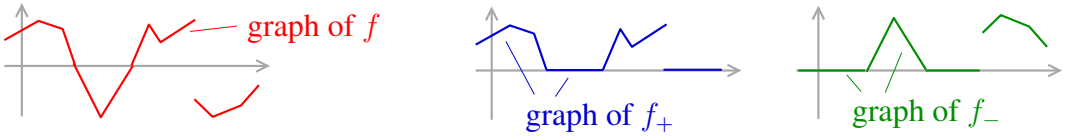
Definition 73.17. Let $f \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a non-negative, measurable function. We set

$$\int_{\mathbb{R}^n} f \, d\lambda^n := \sup\Big\{ \underbrace{\int_{\mathbb{R}^n} g \, d\lambda^n}_{\text{defined above}} \;\Big|\; g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ a non-negative step function with } g \leq f \Big\}.$$

Finally we want to consider measurable functions, which can assume positive and negative values. We introduce the following piece of notation:

Notation 73.18. For a function $f \colon \mathbb{R}^n \to \mathbb{R}$ we write

$$f_+ := \max\{f, 0\} \quad \text{and} \quad f_- := -\min\{f, 0\}.$$



If f is measurable, then it follows from an elementary argument that f_- and f_+ are also measurable.[769] Thus we can write $f = f_+ - f_-$ as the difference of two measurable non-negative functions. This allows us to introduce the general notion of the Lebesgue-integral:

Definition 73.19. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ a measurable function. We consider the non-negative measurable functions $f_+, f_- \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. If

$$\underbrace{\int_{\mathbb{R}^n} f_+ \, d\lambda^n}_{\text{defined above}} < \infty \quad \text{and} \quad \underbrace{\int_{\mathbb{R}^n} f_- \, d\lambda^n}_{\text{defined above}} < \infty,$$

then we say that f is Lebesgue-integrable and we refer to

$$\int_{\mathbb{R}^n} f \, d\lambda^n := \int_{\mathbb{R}^n} f_+ \, d\lambda^n - \int_{\mathbb{R}^n} f_- \, d\lambda^n$$

as the Lebesgue-Integral of f.

---

[767] A priori there are several ways how we can write a step function as such a linear combination. But it follows easily from the σ-additivity of Lebesgue measure that this definition not depend on the choice of the $A_1, \dots, A_m$ and the $c_1, \dots, c_m$.

[768] Note that this integral can assume the value $+\infty$.

[769] This can be seen as follows: Let $A \in \mathcal{B}(\mathbb{R})$. If $0 \notin A$, then $f_-^{-1}(A) = (f^{-1})(A \cap \mathbb{R}_{>0})$ and if $0 \in A$, then $f_-^{-1}(A) = (f^{-1})(A \cup \mathbb{R}_{\leq 0})$.

Example. Let $A \subseteq \mathbb{R}^n$ be a measurable subset. If $\lambda^n(A) < \infty$, then it follows immediately from the definitions that the characteristic function $\chi_A \colon \mathbb{R}^n \to \mathbb{R}$ is Lebesgue-integrable with

$$\int_{\mathbb{R}^n} \chi_A \, d\lambda^n \;=\; \lambda^n(A).$$

□

Before we turn to other examples, let us first introduce the Lebesgue-integral of functions on measurable subsets of $\mathbb{R}^n$:

Definition 73.20. Let $A \subseteq \mathbb{R}^n$ be a measurable subset and let $f \colon A \to \mathbb{R}$ be a measurable function.[770] We define $\widetilde{f} \colon \mathbb{R}^n \to \mathbb{R}$ to be the extension of f that is given by assigning to every point outside of A the value 0. We say f is Lebesgue-integrable if $\widetilde{f}$ is, and in the case of integrability we define

$$\int_A f \, d\lambda^n \;:=\; \int_{\mathbb{R}^n} \widetilde{f} \, d\lambda^n.$$

The following proposition is the only Lebesgue integrability result that we will need in these notes.

Proposition 73.21. (Lebesgue Integrability Proposition) Let $A \subseteq \mathbb{R}^n$ be a measurable subset with finite Lebesgue measure. (For example by the discussion on page 1516 we know that A could be a compact subset of $\mathbb{R}^n$.) Every bounded continuous map $f \colon A \to \mathbb{R}$ is Lebesgue integrable.

Sketch of proof. Let $A \subseteq \mathbb{R}^n$ be a measurable subset with finite Lebesgue measure and let $f \colon A \to \mathbb{R}$ be a continuous map. We assume that f is bounded, i.e. we assume that there exists a $C \in \mathbb{R}$ such that $|f(a)| \leq C$ for all $a \in A$.

Let $\widetilde{f} \colon \mathbb{R}^n \to \mathbb{R}$ be the extension of f. It follows from the Continuous-Measurable Proposition 73.7 that f is measurable. As we pointed out above, it follows that $\widetilde{f}$ and $\widetilde{f}_\pm$ are also measurable. It is elementary to see that the integrals of $\widetilde{f}_-, \widetilde{f}_+$ are bounded by $C \cdot \lambda^n(A) \in \mathbb{R}_{\geq 0}$. It follows that $\widetilde{f}$ (and thus also f) is Lebesgue integrable. ■

73.5. **Properties of the Lebesgue-integral.** The following proposition summarizes some basic statements about the Lebesgue-integral:

Proposition 73.22. (Lebesgue Integral Proposition) Let $U \subseteq \mathbb{R}^n$ be a measurable subset.

(1) The Lebesgue integral is monotonous in the sense that if $f, g \colon U \to \mathbb{R}$ are two Lebesgue integrable functions such that for every $x \in \mathbb{R}^n$ we have $f(x) \leq g(x)$, then

$$\int_U f \, d\lambda^n \;\leq\; \int_U f \, d\lambda^n.$$

(2) (a) The Lebesgue integral is linear, i.e. it is a homomorphism from the real vector space of Lebesgue-integrable functions on U to $\mathbb{R}$.

(b) Let $A, B \subseteq \mathbb{R}^n$ be measurable subsets and let $f \colon A \cup B \to \mathbb{R}$ be Lebesgue integrable map. Then

$$\int_{A \cup B} f \, d\lambda^n \;=\; \int_A f \, d\lambda^n + \int_B f \, d\lambda^n - \int_{A \cap B} f \, d\lambda^n.$$

[770] One can easily verify that $\widetilde{f}$ is measurable.

(3) (a) Let $f\colon U \to \mathbb{R}$ be a Lebesgue integrable map, then

$$\left| \int_U f \, d\lambda^n \right| \leq \operatorname{vol}(U) \cdot \underbrace{\sup\{|f(x)| \,|\, x \in U\}}_{\in \mathbb{R}_{\geq 0} \cup \{\infty\}}.$$

(b) If $Z \subseteq \mathbb{R}^n$ is measurable with $\operatorname{vol}(Z) = 0$, then for any Lebesgue integrable function $f\colon Z \to \mathbb{R}$ the Lebesgue integral equals 0.

(4) Let $f, g\colon U \to \mathbb{R}$ be two measurable functions on a measurable subset $U \subseteq \mathbb{R}^n$. If f and g differ only on a measurable subset Z with $\operatorname{vol}(Z) = 0$, then f is Lebesgue integrable if and only if g is Lebesgue integrable and in case they are both Lebesgue integrable, the Lebesgue integrals agree.

Proof.

(1) This statement follows easily from the definition, see [**Fol1999**, p. 50].

(2) (a) This statement follows from [**Fol1999**, Proposition 2.21].

(b) This statement follows from (a).

(3) (a) This statement follows easily from the definitions.

(b) This statement follows immediately from (a).

(4) This statement follows immediately from (2) and (3b). ■

With a great sigh of relief we now learn that in the 1-dimensional setting the Riemann-integral and the Lebesgue-integral agree on all reasonable functions:

Proposition 73.23. (Riemann-Lebesgue Integral Proposition) Every Riemann integrable function $f\colon [a, b] \to \mathbb{R}$ is Lebesgue integrable and

$$\underbrace{\int_{x=a}^{x=b} f(x)\, dx}_{\text{Riemann integral}} = \underbrace{\int_{[a,b]} f \, d\lambda}_{\text{Lebesgue integral}}.$$

Proof. This proposition is proved in [**Fol1999**, Theorem 2.28]. ■

Before we turn to the remaining two theorems of this section, let us introduce the following notation:

Definition 73.24. Let $U \subseteq \mathbb{R}^n$ be a measurable subset and let $f\colon U \to \mathbb{R}$ be a measurable function. Depending on the context we denote the Lebesgue integral by

$$\int_U f \, d\lambda = \int_{x \in U} f(x)\, dx = \int_{y \in U} f(y)\, dy.$$

The following Fubini Theorem often makes it possible to reduce the calculation of a multi-dimensional Lebesgue integral to the popular one-dimensional case, which we can handle using the Riemann-Lebesgue Integral Proposition 73.23. The Fubini Theorem, as stated below, is not formulated in the most general setting, but it is instead formulated with actual calculations in mind.

Theorem 73.25. (Fubini Theorem) Let $A \subseteq \mathbb{R}^k \times \mathbb{R}^m$ be a compact subset and suppose that we are given a continuous map

$$\begin{array}{rcl} f\colon A & \to & \mathbb{R} \\ (x,y) & \mapsto & f(x,y) \end{array}$$

We set

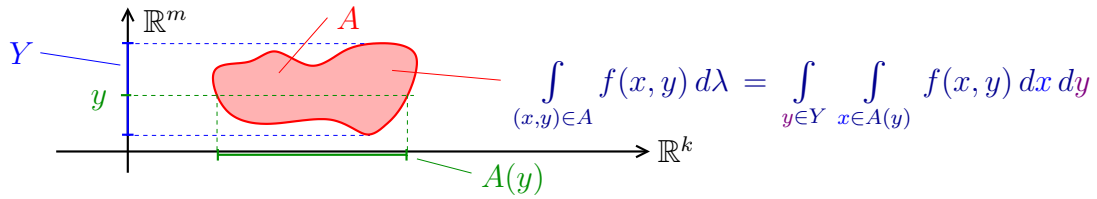
$$Y \;:=\; \{y \in \mathbb{R}^k \,|\, \text{there exists an } x \in \mathbb{R}^m \text{ with } (x,y) \in A\},$$

and for $y \in Y$ we set

$$A(y) \;:=\; \{x \in \mathbb{R}^m \,|\, (x,y) \in A\}.$$

Then

$$\int\limits_{(x,y)\in A} f(x,y)\,d\lambda \;=\; \int\limits_{y\in Y}\int\limits_{x\in A(y)} f(x,y)\;dx\;\,dy.$$



Proof. This theorem follows from [**Fol1999**, Theorem 2.37]. Note that the fact that A is compact and f is continuous means, by the Lebesgue Integrability Proposition 73.21 and Exercise 7.6 (a) that the terms on the right hand are Lebesgue integrable. ■

Example. We consider the triangle $A = \{(x,y) \in \mathbb{R}^2 \,|\, x \in [0,2] \text{ and } y \in [0,x]\}$ and a continuous map $f\colon A \to \mathbb{R}$. It follows from the Lebesgue Integrability Proposition 73.21 that f is Lebesgue integrable. In this case

$$Y \;=\; \text{all possible } y\text{-coordinates of points in } A \;=\; [0,2].$$

Furthermore, given $y \in [0,2]$ we have

$$A(y) \;=\; \text{all } x\text{-coordinates of points in } A \text{ whose } y\text{-coordinate equals } y \;=\; [y,2].$$

We see that

$$\int\limits_A f(x,y)\,d\lambda \;=\; \underset{\uparrow}{\int\limits_{y\in[0,2]}\int\limits_{x\in[y,2]}} f(x,y)\,dx\;\,dy \;=\; \underset{\uparrow}{\int\limits_{y=0}^{y=2}\int\limits_{x=y}^{x=2}} f(x,y)\,dx\;\,dy.$$

by the Fubini Theorem 73.25 — by the Riemann-Lebesgue Integral Proposition 73.23

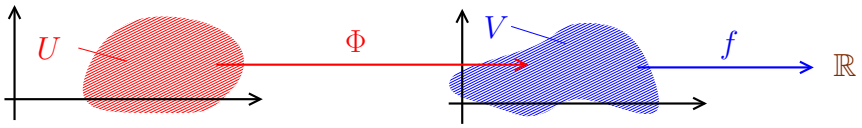
We have thus reduced the calculation of the Lebesgue integral to the calculation of two Riemann integrals, which is hopefully doable. □

We conclude with one of the hardest and most useful theorem on Lebesgue integrals:

Theorem 73.26. (Lebesgue Transformation Theorem) Let $n \in \mathbb{N}_0$ and let U and V be open subsets of $\mathbb{R}^n$ or of $\mathbb{H}^n$. Furthermore let $\Phi\colon U \to V$ be a diffeomorphism and let $f\colon V \to \mathbb{R}$ be a continuous map. Then

$$\int\limits_{x\in U} f(\Phi(x)) \cdot |\det \mathrm{D}\Phi(x)|\,d\lambda^n \;=\; \int\limits_{y\in V} f(y)\,d\lambda^n,$$

in the sense that one side is Lebesgue integrable if and only if the other side is Lebesgue integrable, and if both are Lebesgue integrable, then the Lebesgue integrals agree.



$$\int_{x\in U} f(\Phi(x)) \cdot |\det \mathrm{D}\Phi(x)|\, d\lambda^n \;=\; \int_{y\in V} f(y)\, d\lambda^n$$

Proof. This theorem is proved by a heroic argument that eventually reduces the theorem to the Determinant-Volume Proposition 73.13. We refer to [**Fol1999**, Theorem 2.47] for details in the case that U and V are open subsets of $\mathbb{R}^n$. The case that U and V are open subsets of $\mathbb{H}^n$ is proved in a similar fashion. ∎

73.6. **Integration and differentiation.** In this final section we consider the interplay between integration and differentiation. We start out with the familiar Fundamental Theorem of Calculus 73.27 (FTC):

Theorem 73.27. (Fundamental Theorem of Calculus (FTC)) Let $I \subseteq \mathbb{R}$ be an intervall.

(1) Let $f\colon I \to \mathbb{R}$ be a smooth function. For any $a \in I$ the function

$$\begin{aligned} I &\to \mathbb{R} \\ s &\mapsto \int_{t=a}^{t=s} f(t)\, dt \end{aligned}$$

is also smooth with derivative at $s \in I$ given by $f(s)$.

(2) Let $g\colon I \to \mathbb{R}$ be a smooth function. For any $a, b \in I$ we have

$$\int_{t=a}^{t=b} g'(t)\, dt \;=\; g(b) - g(a).$$

Proof.

(1) This statement is proved in every proof-based course on real analysis. (This might be a great moment to have a look at [**Frie2023**].)

(2) This statement follows easily from Statement (1). ∎

The final theorem of this chapter says that in the higher-dimensional setup we can interchange integration and differentiation:

Theorem 73.28. (Leibniz Integral Rule) Let $n \in \mathbb{N}$, let $U \subseteq \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}$ be a smooth function. Furthermore let $V \subseteq \mathbb{R}^{n-1}$ be an open subset and let $I \subseteq \mathbb{R}$ be a compact interval such that $V \times I \subseteq U$. We consider the function

$$\begin{aligned} V &\to \mathbb{R} \\ v &\mapsto \int_{t\in I} f(v,t)\, d\lambda. \end{aligned}$$

This function is smooth and for any $i \in \{1, \dots, n-1\}$ the i-th partial derivative is given by

$$\frac{\partial}{\partial x_i} \int_{t\in I} f(v,t)\, d\lambda \;=\; \int_{t\in I} \frac{\partial f}{\partial x_i}(v,t)\, d\lambda.$$

Proof. We refer to [**PedS2015**, Theorem 9.2.11] for a proof of this theorem. ■

73.7. Exercises for Chapter 73.

Exercise 73.1. Let Ω be a set and let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra.

(a) Show that if $\{A_i\}_{i\in I}$ is a countable family of sets in $\mathcal{A}$, then the intersection also lies in $\mathcal{A}$, i.e. we have $\bigcap\limits_{i\in I} A_i \in \mathcal{A}$.

(b) Show that if $A, B \in \mathcal{A}$, then we also have $A \setminus B \in \mathcal{A}$.

Exercise 73.2.

(a) Show that cardinality of $\mathcal{B}(\mathbb{R})$ equals the cardinality of $\mathbb{R}$.
(b) First write down a (complicated) subset of $\mathbb{R}$ and then show that it is measurable.
(c) Challenge your friends to produce an explicit non-measurable subset of $\mathbb{R}$.

Remark. Together with Cantor's Theorem 1.39 we obtain from (a) that "almost all" subsets of $\mathbb{R}$ are *not* measurable. Fortunately (b) shows that basically every subset of $\mathbb{R}$ that one cares about, is actually measurable.

Exercise 73.3.

(a) Let Ω be a set and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\Omega)$. As in Definition 73.3 we denote by $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ the σ-algebras that are generated by $\mathcal{A}$ and $\mathcal{B}$. Show that if $\mathcal{A} \subseteq \sigma(\mathcal{B})$ and $\mathcal{B} \subseteq \sigma(\mathcal{A})$, then $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.
(b) Show that open rectangles form a generating set for the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$.
(c) Show that open subsets of $\mathbb{R}^n$ form a generating set for the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$.
(d) Show that compact subsets of $\mathbb{R}^n$ form a generating set for the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$.

Exercise 73.4. Let $A \subseteq \mathbb{R}^n$ be a measurable subset. Shows that the characteristic function $\chi_A \colon \mathbb{R}^n \to \mathbb{R}$ is measurable.

Exercise 73.5. Let Ω be a set, let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra and let $\mu \colon \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measure.

(1) Let $A, B \in \mathcal{A}$. Prove the following two statements:
   (a) $$\mu(A \cup B) \;=\; \mu(A) + \mu(B) - \mu(A \cap B).$$
   (b) $$A \subseteq B \quad\Longrightarrow\quad \mu(A) \leq \mu(B).$$
   Remark. The careful reader will notice that Exercise 73.1 (b) needs to be applied.
(2) Show that for any countable family $\{A_i\}_{i\in I}$ of subsets in $\mathcal{A}$ we have
$$\mu\Big(\bigcup_{i\in I} A_i\Big) \;\leq\; \sum_{i\in I} \mu(A_i).$$
(3) Let $A_i \subseteq \mathcal{A}$, $i \in \mathbb{N}$ and let $A \subseteq \Omega$. We write $A_i \nearrow A$ if for each $i \in \mathbb{N}$ we have $A_i \subseteq A_{i+1}$ and if $\bigcup\limits_{i=1}^{\infty} A_i = A$. Show that
$$A_i \nearrow A \quad\Longrightarrow\quad A \in \mathcal{A} \text{ and } \lim_{i\to\infty} \mu(A_i) \;=\; \mu(A).$$

Exercise 73.6. Let $U \subseteq \mathbb{R}^n$ be an open subset. Show that we can write U as the disjoint union of a countable family $\{R_i\}_{i\in I}$ of half-open cuboids, i.e. we can write U as the disjoint union of a countable family of subsets of the form $[a_1, b_1) \times \cdots \times [a_n, b_n)$.

Exercise 73.7. Show that the volume of the closed 3-ball of radius r equals $\frac{4}{3}\pi \cdot r^3$.

Exercise 73.8. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and let $f_k\colon \mathbb{R}^n \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of functions. We write $f_k \nearrow f$, if the following two conditions are satisfied:

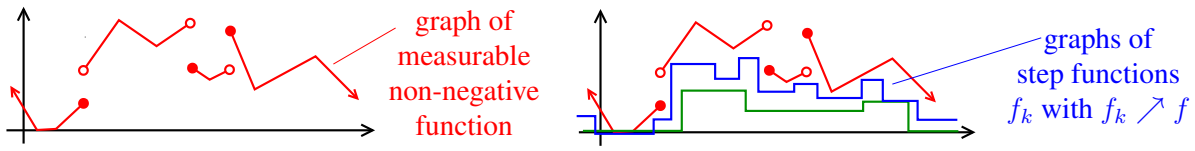
(i) For every $x \in \Omega$ the sequence $f_k(x)$ is monotonously increasing.
(ii) For every $x \in \Omega$ we have $\lim\limits_{k\to\infty} f_k(x) = f(x)$.

Let $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a non-negative measurable function.

(a) Show that there exists a sequence $\{f_k\}_{k\geq 1}$ of non-negative step functions with $f_k \nearrow f$.
Hint. Define f_k by rounding down.

(b) Assume we are in the setting of (a). Show that

$$\int_\Omega f\, d\lambda^n \;:=\; \lim_{k\to\infty} \underbrace{\int_\Omega f_k\, d\lambda^n}_{\text{defined in Definition 73.16, this is a monotonously increasing sequence, thus the limit exists in } \mathbb{R}_{\geq 0}\cup\{\infty\}} \;\in\; \mathbb{R}_{\geq 0}\cup\{\infty\}.$$



CHAPTER 74

Differential forms of arbitrary degree

In Chapter 71 we introduced the notion of a (smooth) 1-form on a smooth manifold and we learned in particular how to integrate those 1-forms over a compact oriented 1-dimensional smooth manifold. In this chapter we will generalize these concepts from dimension 1 to arbitrary dimensions. More precisely, given any $k \in \mathbb{N}_0$ we will use the concept we learned about in Chapter 72 to introduce the notion of (smooth) k-form on a smooth manifold. Furthermore, using the tricks we learned in Chapter 73, we will see that it makes sense to integrate those k-forms over a compact oriented k-dimensional smooth manifold.

This chapter is organized as follows:

- In Section 74.1 we will introduce differential forms on smooth manifolds.
- In Section 74.2 we will introduce the integral of a smooth n-form on a compact oriented n-dimensional smooth manifold.
- In Section 74.3 we will learn how to calculate integrals of n-forms via compact parametrizations.
- In Section 74.4 we will study the interplay between groups actions and differential forms.
- In Section 74.5 we will provide exercises for Chapter 74.

In the next Chapter 75 we will state and prove Stokes' Theorem 75.11.

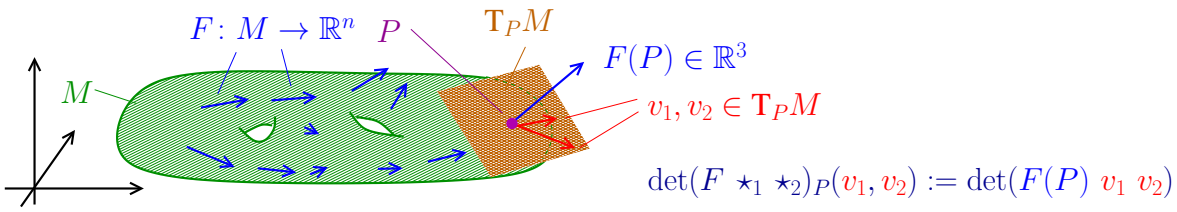
74.1. **Differential forms on smooth manifolds.** In this section we finally return to the study of smooth manifolds. The algebraic notions from the previous sections allow us to introduce the following definition:

Definition 74.1. Let M be a smooth manifold.

(1) A differential form ω of degree k on M (or k-form ω on M for short) assigns to any point $P \in M$ an alternating k-form ω_P on the tangent space $\mathrm{T}_P M$.
(2) Given two differential forms ω and σ on M we define the differential form $\omega \wedge \sigma$ pointwise, i.e. for every $P \in M$ we set $(\omega \wedge \sigma)_P \coloneqq \omega_P \wedge \sigma_P$.

Example. Using the examples of alternating k-forms on page 1503 we can easily construct interesting examples of k-forms on smooth manifolds.

(1) Since $\wedge^1 V^* = V^*$ we see that the above definition of a k-form for $k = 1$ agrees with Definition 71.2.
(2) On page 1503 we saw that $\wedge^0 V^* =_{\mathrm{i}} \mathbb{R}$. Therefore a 0-form on a smooth manifold M is the same as a map $f\colon M \to \mathbb{R}$.
(3) Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. Furthermore let $F\colon M \to \mathbb{R}^n$ be a map.



We obtain an $(n-1)$-form $\det(F \star_1 \cdots \star_{n-1})$ on M which assigns to each $P \in M$ the following alternating $(n-1)$-form:[771]

$$\begin{array}{rcl} \det(F \star_1 \cdots \star_{n-1})_P\colon (\mathrm{T}_P M)^{n-1} & \to & \mathbb{R} \\ (v_1, \ldots, v_{n-1}) & \mapsto & \det \underbrace{(F(P)\ v_1\ \ldots\ v_{n-1})}_{n \times n - \text{matrix}}. \end{array}$$

(4) For $M = \varnothing$ and $k \in \mathbb{N}_0$ there exists a unique map as in the definition of a k-form. Thus there exists a unique k-form on $\varnothing$, namely the zero form. □

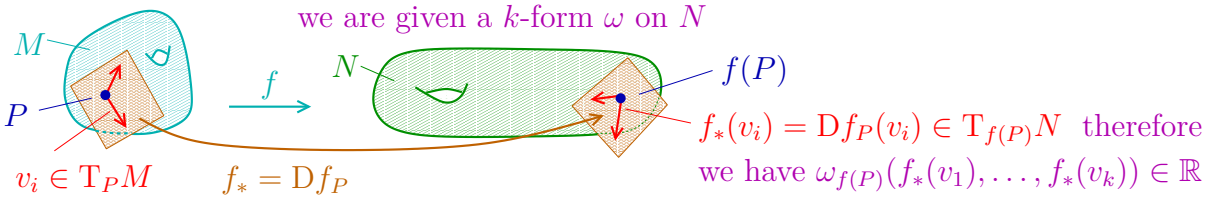
In Definition 71.8 we defined what it means for a 1-form on a smooth manifold to be smooth. In the following discussion we will generalize this notion to differential forms of any degree.

Definition 74.2. Let $f\colon M \to N$ be a smooth map between smooth manifolds.

(1) Let $P \in M$. As usual we denote by $f_*\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ the induced map. Given $k \in \mathbb{N}_0$ we consider the induced map

$$\begin{array}{rcl} f^*\colon \wedge^k \mathrm{T}^*_{f(P)} N & \to & \wedge^k \mathrm{T}^*_P M \\ \omega & \mapsto & \left(\begin{array}{rcl} f^*(\omega)\colon (\mathrm{T}_P M)^k & \to & \mathbb{R} \\ (v_1, \ldots, v_k) & \mapsto & \omega(f_*(v_1), \ldots, f_*(v_k)) \end{array} \right). \end{array}$$

(2) Given a k-form ω on N we denote by $f^*(\omega)$ the k-form on M which is given by the alternating forms $\{(f^*(\omega))_P\}_{P \in M}$. We refer to $f^*(\omega)$ as the pullback of the k-form ω.



Example. Let N be a smooth manifold, let M be a smooth submanifold of N and let $i\colon M \to N$ be the inclusion. Given any k-form ω on N we refer to the pullback $i^*(\omega)$ as the *restriction* of the k-form to M. □

Next we want to define what it means for a differential form on a smooth manifold M to be smooth. Given differential forms of degree one we had introduced this notion in Definition 71.8. The definition for differential forms of arbitrary degree is completely analogous:

[771] Our notation is hopefully suggestive, but certainly non-standard. In general, given a smooth manifold M, given a vector field F on M and given an n-form ω on M one can consider the $(n-1)$-form $F \lrcorner \omega$ on M which at $P \in M$ is given by $(F \lrcorner \omega)(v_1, \ldots, v_{n-1}) = \omega(F(P), v_1, \ldots, v_{n-1})$. With this notation our form can be written as $F \lrcorner (dx^1 \wedge \cdots \wedge dx^n)$.

Definition 74.3.

(1) Let ω be an m-form on an open subset $W \subseteq \mathbb{R}^k$ respectively an open subset $W \subseteq \mathbb{H}^k$. By the Alternating Forms-Basis Proposition 72.7 the wedge products $dx^{i_1} \wedge \cdots \wedge dx^{i_m}$ with $i_1 < i_2 < \cdots < i_m$ form a basis of $\wedge^m (\mathbb{R}^k)^*$. Therefore we can write ω as a linear combination

$$\omega \;=\; \sum_{i_1<\cdots<i_m} f_{i_1\ldots i_m} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_m},$$

where $f_{i_1\ldots i_m} \colon W \to \mathbb{R}$ are uniquely determined real-valued functions on W. We say that the m-form ω is smooth if all the functions $f_{i_1\ldots i_m}$ are smooth.

(2) Let ω be an m-form on a k-dimensional smooth manifold M.

(a) Let $\Phi \colon U \to V$ be a chart for M. We say that ω is smooth with respect to the chart Φ, if the m-form $(\Phi^{-1})^*(\omega)$ is smooth on V.

(b) We say that ω is smooth if ω is smooth with respect to every chart of M.



The following proposition is a generalization of the Smooth 1-Form Proposition 71.9.

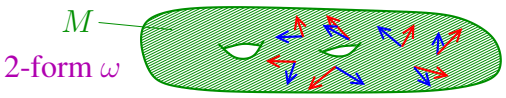
Proposition 74.4. (Smooth Form Proposition) Let ω be an m-form on a smooth manifold M. The following statements are equivalent:

(1) The m-form ω is smooth on M.

(2) There exists an atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ of M such that the m-form is smooth with respect to every chart Φ_i of the atlas.

(3) For every choice of smooth vector fields $F_1, \ldots, F_m$ on M the following function is smooth:

$$\begin{aligned} M &\to \mathbb{R} \\ P &\mapsto \omega_P(F_1(P), \ldots, F_m(P)). \end{aligned}$$



vector field F_1

vector field F_2

we consider the function $P \mapsto \omega_P(F_1(P), F_2(P))$

Proof. The proof of the proposition is very similar to the proof of the Smooth 1-Form Proposition 71.9 which we gave in Section 71.3. We leave it to the reader to fill in the details. ∎

The following notation generalizes the notation from Notation 71.10:

Notation 74.5. Given a non-empty smooth manifold M and $m \in \mathbb{N}_0$ we set

$$\Omega^m(M) \;:=\; \text{the real vector space of all smooth } m\text{-forms on } M.$$

Example.

(1) It follows easily from the discussion on page 1503 of alternating 0-forms that $\Omega^0(M)$ is essentially the same as the vector space of all smooth functions $M \to \mathbb{R}$.

[771] If we had defined an m-form as an element of the product $\prod_{P\in M} \wedge^m (\mathrm{T}^*_P M)$, then the case $M = \varnothing$ would have been taken care of by the Product-of-Sets-and-Groups Lemma 6.4.

(2) Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold and let $F\colon M \to \mathbb{R}^n$ be a smooth map. We consider the $(n-1)$-form $\det(F \star_1 \cdots \star_{n-1})$ on M which assigns to any point $P \in M$ the following alternating $(n-1)$-form

$$\begin{array}{rcl} (\mathrm{T}_P M)^{n-1} & \to & \mathbb{R} \\ (v_1, \ldots, v_{n-1}) & \mapsto & \det \underbrace{(F(P) \;\; v_1 \;\ldots\; v_{n-1})}_{n\times n-\text{matrix}}. \end{array}$$

Using the "(3)⇒(1)" implication of the Smooth Form Proposition 74.4 and using the Leibniz Formula 2.18 one can easily show that this a smooth $(n-1)$-form on M.

(3) Note that it follows immediately from the discussion on page 1526 that for all $m \in \mathbb{N}_0$ we have $\Omega^m(\varnothing) = \{0\}$. □

Lemma 74.6. (Wedge Product Smooth Lemma) Let M be a smooth manifold and let $k, l \in \mathbb{N}_0$. Given $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ the $(k+l)$-form $\omega \wedge \eta$ which is given by $(\omega \wedge \eta)_P \coloneqq \omega_P \wedge \eta_P$ is smooth.

Proof. The lemma follows from the following observations:

(1) Let U be an open subset of some $\mathbb{R}^n$ or $\mathbb{H}^n$. If ω and η are given by wedges of 1-forms, then the statement boils down to the fact that products and sums of smooth functions $U \to \mathbb{R}$ are again smooth. The case of general differential forms on U then follows from the Alternating Forms-Basis Proposition 72.7.

(2) The case of a general smooth manifold follows from (1), the definitions and the Alternating Form-Functoriality Lemma 72.12 (2).

We leave it to the reader to fill in the details. ■

The following proposition is a generalization of the Pullback Smooth 1-Form Proposition 71.11.

Proposition 74.7. (Pullback Smooth Form Proposition)

(1) Let $f\colon M \to N$ be a smooth map between two smooth manifolds and let ω be an m-form on N.
   (a) If ω is smooth, then the pullback m-form $f^*(\omega)$ on M is also smooth.
   (b) For any l-form η on N we have $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

(2) The maps

$$\begin{array}{rcl} \text{smooth manifold } M & \mapsto & \Omega^m(M), \\ \text{smooth map } f\colon M \to N & \mapsto & (f^*\colon \Omega^m(N) \to \Omega^m(M)) \end{array}$$

define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

Proof.

(1) (a) The proof is completely analogous to the proof of the Pullback Smooth 1-Form Proposition 71.11. We leave it to the reader to make the necessary adaptations.
   (b) This statement follows from the Alternating Form-Functoriality Lemma 72.12.

(2) It follows from the Tangent Space-Functor Proposition 29.15 together with the Alternating Form-Functoriality Lemma 72.12 that the maps

$$\begin{array}{rcl}(M,P) & \mapsto & \wedge^k \mathrm{T}^*_P M \\ (f\colon (M,P)\to(N,Q)) & \mapsto & \big(f^*\colon \wedge^k \mathrm{T}^*_Q N \to \wedge^k \mathrm{T}^*_P M\big)\end{array}$$

form a contravariant functor from the category $\mathcal{PSmMfd}$ of pointed smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces. The desired statement follows from this observation together with (1). ■

The following proposition is an analog of the Smooth Pasting Proposition 25.19 in the present context:

Proposition 74.8. (Smooth Form–Pasting Proposition) Let M be a smooth manifold and let ω be a k-form on M. If there exists an open cover $\{U_i\}_{i\in I}$ of M such that for each $i \in I$ the restriction $\omega|_{U_i}$ is a smooth form on U_i, then ω itself is smooth.

Proof. The proof of this proposition is essentially the same as the elementary proof of the Smooth Pasting Proposition 25.19. We leave it to the reader to fill in the details. ■

As an example we will now consider a frequently used construction of smooth forms:

Definition 74.9. Let M be a smooth manifold, let $U \subseteq M$ be an open subset and let ω be a smooth k-form on U. We denote by $\widetilde{\omega}$ the k-form on M which is given by extending ω by zero outside of U. We refer to $\widetilde{\omega}$ as the extension of ω to M.

Our next goal is to give a criterion for the extension of a smooth form to be smooth again. To do so we will need the following definition, which is a variation on the definition of the support of a map $f\colon X \to \mathbb{R}$ that we gave in Definition 14.14.

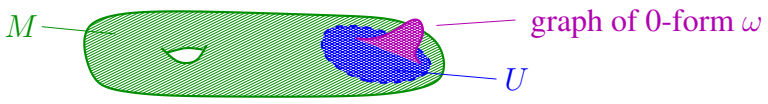
Definition 74.10. Let M be a smooth manifold and let ω be a differential form on M. We define the support of the differential form ω as

$$\operatorname{supp}(\omega) \;=\; \text{the closure of the set of all points } P \in M \text{ such that } \omega_P \neq 0.$$

Now we can formulate the promised result.

Lemma 74.11. (Extension-of-Form–Smooth Lemma) Let M be a smooth manifold, let $U \subseteq M$ be an open subset and let ω be a smooth k-form on U. If ω has compact support in U, then the extension of ω to M is also smooth.

Proof. We denote by $\widetilde{\omega}$ the extension of ω to M. First note that the form $\widetilde{\omega}$ is zero on $M \setminus \operatorname{supp}(\omega)$.



Since $\operatorname{supp}(\omega)$ is compact we know by the Compact-Closed Lemma 3.39 (2), and using the ever-convenient fact that smooth manifolds are Hausdorff, that $M \setminus \operatorname{supp}(\omega)$ is an open subset of M. Furthermore the form $\widetilde{\omega}$ is evidently smooth on the open subset U. Since $M \setminus \operatorname{supp}(\omega)$ and U are an open cover of M, we obtain from the Smooth Form–Pasting Proposition 74.8 that $\widetilde{\omega}$ is smooth. ■

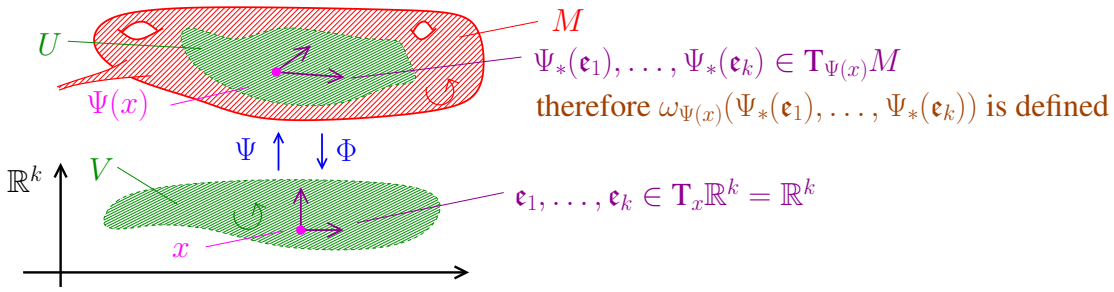
74.2. **Integration of differential forms.** In the 1-Form Integration Proposition 71.17 we introduced the integral of a smooth 1-form on a compact oriented connected non-empty 1-dimensional smooth manifold. Our next goal is to generalize this notion to the higher-dimensional setting. More precisely, given an oriented k-dimensional smooth manifold and given a smooth k-form ω with compact support on M we want to introduce a reasonable integral $\int_M \omega \in \mathbb{R}$.

We will first discuss a special case, before we deal with the general setting, namely we will first consider the case that we are given a smooth k-form ω such that the support $\operatorname{supp}(\omega)$ is compact and contained in the domain of a chart:

Proposition 74.12. (Integral-of-Form-in-Chart Proposition) Let M be an oriented k-dimensional smooth manifold and let ω be a smooth k-form on M. We suppose that $\operatorname{supp}(\omega)$ is compact and we suppose that there exists a chart $\Phi\colon U \to V$ such that $\operatorname{supp}(\omega) \subseteq U$ and which is either orientation-preserving or orientation-reversing[772]. We denote by $\Psi\colon V \to U$ the inverse map of Φ.

(1) The map[773]
$$\begin{array}{ccc} V & \to & \mathbb{R} \\ x & \mapsto & \underbrace{\omega_{\Psi(x)}}_{\in \wedge^k \mathrm{T}^*_{\Psi(x)}M}\Big(\underbrace{\Psi_*(\mathfrak{e}_1),\dots,\Psi_*(\mathfrak{e}_k)}_{k \text{ vectors in } \mathrm{T}_{\Psi(x)}M}\Big). \end{array}$$
is a smooth map with compact support.

(2) We set $\epsilon := +1$ if Φ is orientation-preserving and we set $\epsilon := -1$ if Φ is orientation-reversing. The value of the Lebesgue-integral[774]
$$\epsilon \cdot \int_{x\in V} \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1),\dots,\Psi_*(\mathfrak{e}_k))\, d\lambda^k$$
does not depend on the choice of the chart.



Proof.

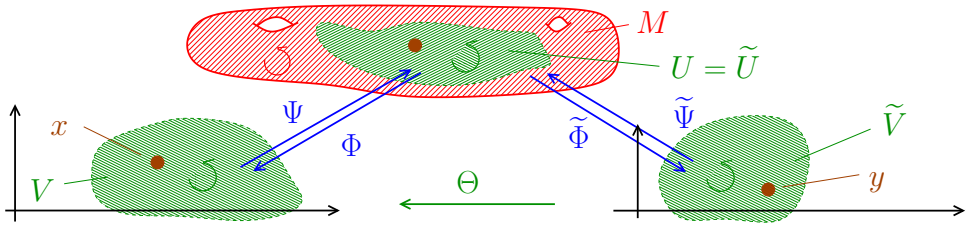
(1) Given $x \in V$ we have $\omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1),\dots,\Psi_*(\mathfrak{e}_k)) = (\Psi^*(\omega))_x(\mathfrak{e}_1,\dots,\mathfrak{e}_k)$. By the Pullback Smooth Form Proposition 74.7 we know that the k-form $\Psi^*(\omega)$ on V is smooth. It follows from the Smooth Form Proposition 74.4 that the map
$$\begin{array}{ccc} V & \to & \mathbb{R} \\ x & \mapsto & \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1),\dots,\Psi_*(\mathfrak{e}_k)) \end{array}$$

[772] We equip any open subset of $\mathbb{R}^k$ with the standard orientation that we defined on page 812.
[773] As always $\mathfrak{e}_1,\dots,\mathfrak{e}_k$ is the standard basis of $\mathbb{R}^k$.
[774] It follows from (1) and the Lebesgue Integrability Proposition 73.21 that the Lebesgue-integral of this function $V \to \mathbb{R}$ is defined.

is smooth. Recall that we assume that $\operatorname{supp}(\omega)$ is compact. It follows from the Compact Image Lemma 4.15 that $\Phi(\operatorname{supp}(\omega))$ is also compact. Note that the support of the map $x \mapsto \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1), \dots, \Psi_*(\mathfrak{e}_k))$ is contained in $\Phi(\operatorname{supp}(\omega))$. It follows from these observations and the Compact-Closed Lemma 3.39 (1) that the support of the map $x \mapsto \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1), \dots, \Psi_*(\mathfrak{e}_k))$ is indeed compact.

(2) First suppose that we are given *orientation-preserving* charts $\Phi\colon U \to V$ and $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ with $\operatorname{supp}(\omega) \subseteq U \cap \widetilde{U}$. We denote the inverses of Φ and $\widetilde{\Phi}$ by Ψ and $\widetilde{\Psi}$. We need to show that

$$\underbrace{\int\limits_{x\in V} \omega_{\Psi(x)}\big(\Psi_*(\mathfrak{e}_1), \dots, \Psi_*(\mathfrak{e}_k)\big)\, d\lambda^n}_{=:I(\Psi)} \;=\; \underbrace{\int\limits_{y\in \widetilde{V}} \omega_{\widetilde{\Psi}(y)}\big(\widetilde{\Psi}_*(\mathfrak{e}_1), \dots, \widetilde{\Psi}_*(\mathfrak{e}_k)\big)\, d\lambda^n}_{=:I(\widetilde{\Psi})}.$$



By passing to the common open subset $U \cap \widetilde{U}$ we might as well suppose that $U = \widetilde{U}$. We denote by Θ the diffeomorphism $\Theta := \Phi \circ \widetilde{\Psi}\colon \widetilde{V} \to V$. We perform the following calculation:

$$\begin{aligned} I\big(\widetilde{\Psi}\big) &= \int\limits_{y\in\widetilde{V}} \omega_{\widetilde{\Psi}(y)}\big(\mathrm{D}\widetilde{\Psi}_y(\mathfrak{e}_1), \dots, \mathrm{D}\widetilde{\Psi}_y(\mathfrak{e}_k)\big)\, d\lambda^n \\ &= \int\limits_{y\in\widetilde{V}} \omega_{\widetilde{\Psi}(y)}\big(\mathrm{D}\Psi_{\Theta(y)}\big(\mathrm{D}\Theta_y \cdot \mathfrak{e}_1\big), \dots, \mathrm{D}\Psi_{\Theta(y)}\big(\mathrm{D}\Theta_y \cdot \mathfrak{e}_k\big)\big)\, d\lambda^n \\ &\uparrow \quad \text{from the Chain Rule 24.17 it follows that for } y \in \widetilde{V} \text{ we have:} \\ &\qquad \mathrm{D}\widetilde{\Psi}_y = \mathrm{D}(\Psi\circ\Theta)_y = \mathrm{D}\Psi_{\Theta(y)} \cdot \mathrm{D}\Theta_y \\ &= \int\limits_{y\in\widetilde{V}} \omega_{\Psi(\Theta(y))}\big(\mathrm{D}\Psi_{\Theta(y)}(\mathfrak{e}_1), \dots, \mathrm{D}\Psi_{\Theta(y)}(\mathfrak{e}_k)\big) \cdot \det(\mathrm{D}\Theta_y)\, d\lambda^n. \\ &\uparrow \quad \text{by the Form–Matrix Multiplication Lemma 72.4 applied to the alternating } k\text{-form} \\ &\qquad (v_1, \dots, v_k) \mapsto \eta(v_1, \dots, v_k) := \omega_{\widetilde{\Psi}(y)}\big(\mathrm{D}\Psi_{\Theta(y)}(v_1), \dots, \mathrm{D}\Psi_{\Theta(y)}(v_k)\big) \\ &\qquad \text{and the } (k\times k)\text{-matrix } \mathrm{D}\Theta_y \end{aligned}$$

On the other hand we have

$$\begin{aligned} I(\Psi) &= \int\limits_{x\in V} \omega_{\Psi(x)}\big((\mathrm{D}\Psi_x)(\mathfrak{e}_1), \dots, (\mathrm{D}\Psi_x)(\mathfrak{e}_k)\big)dx \\ &= \int\limits_{x\in\widetilde{V}} \omega_{\Psi(\Theta(y))}\big((\mathrm{D}\Psi_{\Theta(y)})(\mathfrak{e}_1), \dots, (\mathrm{D}\Psi_{\Theta(y)})(\mathfrak{e}_k)\big) \cdot |\det \mathrm{D}\Theta_y|\, d\lambda^n. \\ &\uparrow \quad \text{the Lebesgue Transformation Theorem 73.26 says that for a function } g \text{ on } V \text{ we have} \\ &\qquad \int\limits_{x\in V} g(x)\, d\lambda^n = \int\limits_{y\in\widetilde{V}} g(\Theta(y)) \cdot |\det \mathrm{D}\Theta_y|\, d\lambda^n \end{aligned}$$

This shows that the integrals $I(\Psi)$ and $I(\widetilde{\Psi})$ agree if $\det(\mathrm{D}\Theta_y) > 0$ for every $y \in \widetilde{V}$. But fortunately, this is the case, since Ψ and $\widetilde{\Psi}$ are both assumed to be orientation-preserving, which implies that $\Theta\colon V \to \widetilde{V}$ is orientation-preserving which in turn implies by the Orientation Checking-via-differentials Lemma 31.30 that $\det(\mathrm{D}\Theta_y) > 0$ for every $y \in \widetilde{V}$.

All the other cases, where the charts have different orientation behavior, are dealt with in almost the same way. ■

Now we are in a position to define the integral of any smooth k-form with compact support on an oriented k-dimensional smooth manifold:

Definition 74.13. Let M be an oriented k-dimensional smooth manifold and let ω be a smooth k-form on M with compact support. We pick finitely many charts $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,r}$ with the following two properties:

(1) We have $\operatorname{supp}(\omega) \subseteq U_1 \cup \dots \cup U_r$.

(2) Each chart is either orientation-preserving or orientation-reversing.[775]

It follows easily from the Manifold–Partitions-of-Unity Theorem 27.2 that there exist smooth functions $\{\mu_i\colon M \to [0,1]\}_{i=1,\dots,r}$ with the following two properties:

(a) For every $x \in \operatorname{supp}(\omega)$ we have $\mu_1(x) + \dots + \mu_r(x) = 1$.

(b) For every $i \in \{1,\dots,r\}$ we have $\operatorname{supp}(\mu_i) \subseteq U_i$.

We set[776]

$$\int_M \omega \coloneqq \sum_{i=1}^{r} \underbrace{\int_M \mu_i \cdot \omega}_{\substack{\text{defined in the} \\ \text{Integral-of-Form-in-Chart} \\ \text{Proposition 74.12}}}.$$

We will see in the Integral-of-Form–Well-Defined Lemma 74.14 below that this definition does not depend on any of the choices.

Example.

(1) Let M be a compact oriented zero-dimensional smooth manifold. Recall that this means that M consists of finitely many points $P_1,\dots,P_k$ together with signs $\epsilon_1,\dots,\epsilon_k$ in$\{-1,1\}$. If one goes carefully through the definitions (as given in the Integral-of-Form-in-Chart Proposition 74.12) one sees that for any 0-form ω on M (which is really just a function $\omega\colon P \to \mathbb{R}$) we have

$$\int_M \omega = \sum_{i=1}^{k} \epsilon_i \cdot \omega(P_i).$$

(2) On page 1537 we will see that for compact 1-dimensional smooth manifolds the above definition agrees with the original definition that we gave in the 1-Form Integration Proposition 71.17. □

[775] Evidently there exists an atlas such that all charts are connected. In this case every chart is orientation-preserving or orientation-reversing. It now follows from the hypothesis that $\operatorname{supp}(\omega)$ is compact that such a finite collection of charts exist.

[776] Note that $\operatorname{supp}(\mu_i \cdot \omega)$ is a closed subset of the compact subspace $\operatorname{supp}(\omega)$. Thus it follows from the Compact-Closed Lemma 3.39 (1) that $\operatorname{supp}(\mu_i \cdot \omega)$ is compact, which implies that the integral of $\mu_i \cdot \omega$ over M is indeed already defined.

The following lemma says that the above definition does not depend on the choice of the partition of unity:

Lemma 74.14. (Integral-of-Form–Well-Defined Lemma) The integral of a smooth k-form with compact support on an oriented k-dimensional smooth manifold is well-defined.

Proof. Let M be an oriented k-dimensional smooth manifold M and let ω be a smooth k-form with compact support on M. By the Integral-of-Form-in-Chart Proposition 74.12 it remains to show that the definition of the integral does not depend on the choice of the smooth functions μ_i satisfying the above conditions (a) and (b). Let $\mu_1, \dots, \mu_r \colon M \to [0,1]$ and $\nu_1, \dots, \nu_s \colon M \to [0,1]$ be two such choices of smooth maps. Then

$$\sum_{i=1}^{r} \int_M \mu_i \cdot \omega \;\underset{\text{since } \sum_{j=1}^{s} \nu_j = 1 \text{ on } \operatorname{supp}(\omega)}{=} \sum_{i=1}^{r} \int_M \Big(\sum_{j=1}^{s} \nu_j \Big) \mu_i \cdot \omega \;=\; \sum_{i=1}^{r} \int_M \sum_{j=1}^{s} \mu_i \cdot \nu_j \cdot \omega \;\underset{\text{by the linearity of the integral}}{=} \sum_{i=1}^{r} \sum_{j=1}^{s} \int_M \mu_i \cdot \nu_j \cdot \omega \;\underset{\text{same argument backwards}}{=} \sum_{j=1}^{s} \int_M \mu_j \cdot \omega.$$

∎

In the following proposition we summarize a few basic statements about the integrals of forms:

Proposition 74.15. (Form Integration Proposition) Let M be an oriented k-dimensional smooth manifold.[777]

(1) The integral of smooth k-forms with compact support on M is linear in the smooth k-forms with compact support.

(2) Let M be an open subset of $\mathbb{R}^k$ or of $\mathbb{H}^k$ and let $f \colon M \to \mathbb{R}$ is a smooth map. If we equip M with the standard orientation, then

$$\underbrace{\int_M f \cdot dx^1 \wedge \cdots \wedge dx^k}_{\text{integral of a } k\text{-form}} \;=\; \underbrace{\int_{x \in M} f(x)\, d\lambda^k}_{\text{Lebesgue-integral}}.$$

(3) If U is an open subset of M and if ω is a smooth k-form on M with $\operatorname{supp}(\omega) \subseteq U$, then

$$\int_U \omega \;=\; \int_M \omega.$$

(4) Let N be an oriented k-dimensional smooth manifold and let $f \colon M \to N$ be an orientation-preserving diffeomorphism. Given any smooth k-form ω with compact support on N we have

$$\int_M f^*(\omega) \;=\; \int_N \omega.$$

(5) Given any smooth k-form ω with compact support on M the following equality holds:

$$\int_{-M} \omega \;=\; -\int_M \omega.$$

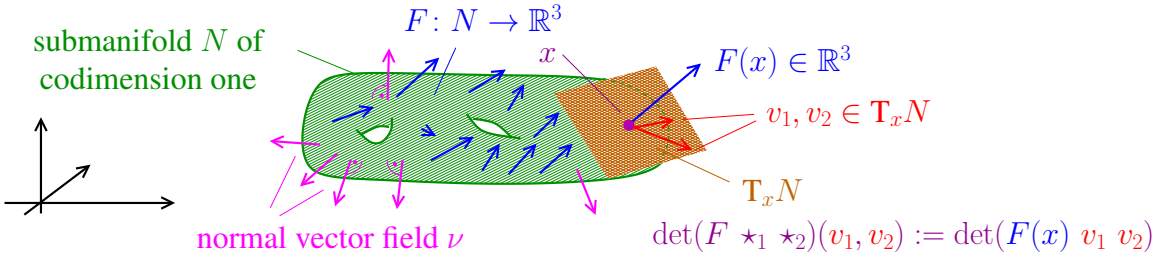
Sketch of proof. In the following we hopefully provide enough justifications for the reader to be able to fill in the details, if the desire for more details arises.

(1) This statement follows immediately from the linearity of the Lebesgue integral.

[777] Note that the integral of smooth k-forms with compact support on oriented k-dimensional smooth manifolds is completely characterized by Properties (1)–(5).

(2) This statement follows immediately from the definition by using the identity chart and using the fact that $(dx^1 \wedge \cdots \wedge dx^k)(\mathfrak{e}_1, \ldots, \mathfrak{e}_k) = 1$.
(3) This statement again follows easily from the definitions.
(4) This statement follows easily from the definitions.
(5) This statement follows again easily from the definitions and the observation that an orientation-preserving chart for M is automatically an orientation-reversing chart for $-M$ and vice versa. ■

Example. Let $n \in \mathbb{N}_{\geq 2}$ and let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of codimension 1. Furthermore let $\nu\colon M \to \mathbb{R}^n$ be a unit normal field in the sense of Definition 31.24, i.e. for each $P \in M$ we have $\|\nu(P)\| = 1$ and we have $\nu(P) \perp \mathrm{T}_P M$. Furthermore let $F\colon M \to \mathbb{R}^n$ be a smooth map. Recall that by the Orientations-via-Normal Vector Fields Proposition 31.27 the unit normal field ν defines an orientation on M.



We denote by $\det(F \star_1 \cdots \star_{n-1})$ the smooth $(n-1)$-form on M which assigns to any point $P \in M$ the alternating $(n-1)$-form

$$\begin{aligned} \det(F \star_1 \cdots \star_{n-1})_P\colon (\mathrm{T}_P M)^{n-1} &\to \mathbb{R} \\ (v_1, \ldots, v_{n-1}) &\mapsto \det(F(P)\ v_1\ \ldots\ v_{n-1}). \end{aligned}$$

The integral $\int_M \det(F \star_1 \cdots \star_{n-1})$ is called the flow of F through (M, ν).[778] □

To practice our new definitions we now prove the following lemma, which will play a role on several occasions:

Lemma 74.16. (Non-Zero Integral Existence Lemma) Given any oriented non-empty k-dimensional smooth manifold M there exists $\omega \in \Omega^k(M)$ with compact support, such that $\int_M \omega \neq 0$.

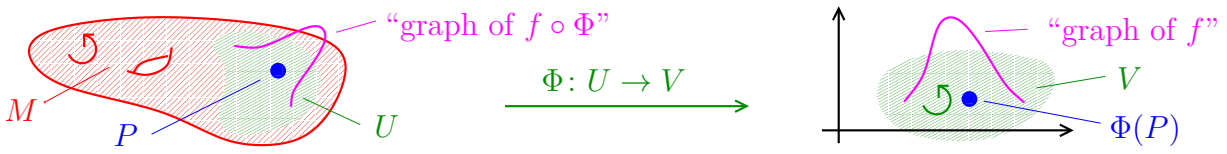
Proof. Since M is non-empty there exists a point $P \in M$. We pick an orientation-preserving chart $\Phi\colon U \to V$. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth map $f\colon V \to \mathbb{R}_{\geq 0}$ with $f(\Phi(P)) > 0$ and with compact support. Next we consider the k-form ω on M which is given by

$$\omega = \Phi^*(f \cdot dx^1 \wedge \cdots \wedge dx^k) \quad \text{on } U$$

and

$$\omega = 0 \quad \text{outside of } U.$$

[778] In case the reader has seen a different definition of the flow of $F\colon M \to \mathbb{R}^{n+1}$ through (M, ν) we leave it as a very enlightening exercise to show that the two notions agree.



Using the Smooth Form–Pasting Proposition 74.8 one can show easily that ω is a smooth k-form on M. Finally note that

$$\int_M \omega \overset{\text{by the Form Integration Proposition 74.15 (3)}}{=} \int_U \omega \overset{\text{follows from the Form Integration Proposition 74.15 (4), since } \Phi \text{ is orientation-preserving}}{=} \int_U \Phi^*(f \cdot dx^1 \wedge \cdots \wedge dx^k) \overset{\text{by the Form Integration Proposition 74.15 (2)}}{=} \int_V f \cdot dx^1 \wedge \cdots \wedge dx^k = \int_V f \, d\lambda^k \overset{\text{since the map } f \text{ is non-negative, continuous and since for } \Phi(P) \text{ the map is positive}}{>} 0. \qquad \blacksquare$$

We conclude this section with a more technical proposition:

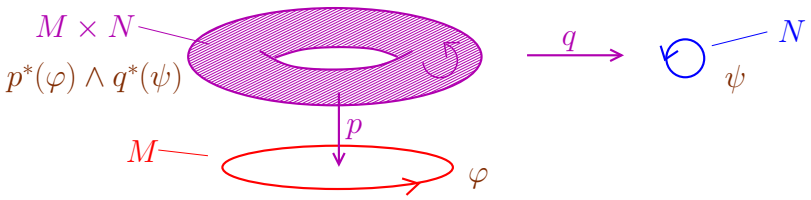
Proposition 74.17. (Product-Pullback-Integral Proposition) Let M and N be two closed oriented smooth manifolds. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the obvious projections. Let $\varphi \in \Omega^r(M)$ and let $\varphi \in \Omega^s(N)$ with $r + s = \dim(M) + \dim(N)$.

(1) If $r = \dim(M)$ and $s = \dim(N)$, then

$$\int_{M\times N} p^*(\varphi) \wedge q^*(\psi) = \int_M p^*(\varphi) \cdot \int_N q^*(\psi).$$

(2) If $r \neq \dim(M)$ (which is equivalent to $s \neq \dim(N)$), then

$$\int_{M\times N} p^*(\varphi) \wedge q^*(\psi) = 0.$$



Proof.

(1) Statement (1) follows easily from the Fubini Theorem 73.25. Filling in the details is the task of Exercise 74.3.

(2) We suppose that $r \neq \dim(M)$. If $r > \dim(M)$, then $\varphi = 0$. If $r < \dim(M)$, then $s > \dim(M)$, thus $\psi = 0$. In either case we see that the integrand is zero. $\blacksquare$

74.3. **Calculations via compact parametrizations.** If one actually wants to calculate the integral of k-forms, then Definition 74.13 using the μ_i is pretty much useless. In this section we will state and prove a proposition which in practice can be very convenient for calculating integrals of k-forms on compact oriented k-dimensional smooth manifolds.

Definition 74.18. Let M be a k-dimensional smooth manifold. A compact parametrization of M is a map $\Psi\colon W \to M$ such that

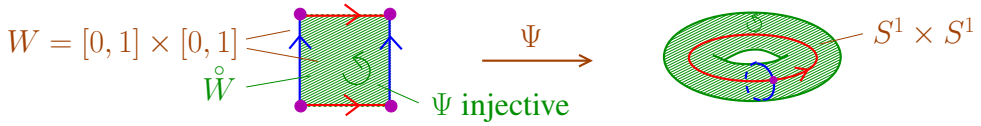
(0) (a) $W \subseteq \mathbb{R}^k$ is a compact subset.
(b) W is regular closed, i.e. W equals the closure of the interior of W. (Note that this implies that for every $P \in \mathring{W}$ we have $\mathrm{T}_P W = \mathbb{R}^k$.)
(c) ∂W is a finite union of submanifolds of $\mathbb{R}^k$ of codimension ≥ 1.
(1) We have $\Psi(W) = M$.
(2) The restriction of Ψ to $\mathring{W}$ is injective.
(3) The map Ψ is smooth in the sense of Definition 24.3.
(4) For every $P \in \mathring{W}$ the rank of the differential $\mathrm{D}\Psi_P$ equals k.
(5) For every $P \in \mathring{W}$ the differential $\mathrm{D}\Psi_P\colon \mathbb{R}^k = \mathrm{T}_P W \to \mathrm{T}_{\Psi(P)} M$ is orientation-preserving.

Example.

(1) One can easily verify that for connected 1-dimensional smooth manifolds the above notion of a compact parametrization equals the notion that we introduced in Definition 71.16.
(2) It is straightforward to show that the map

$$\begin{array}{rcl} [0,1]\times[0,1] & \to & S^1 \times S^1 \\ (s,t) & \mapsto & (\exp(2\pi\,\mathrm{i}s, 2\pi\,\mathrm{i}t)) \end{array}$$

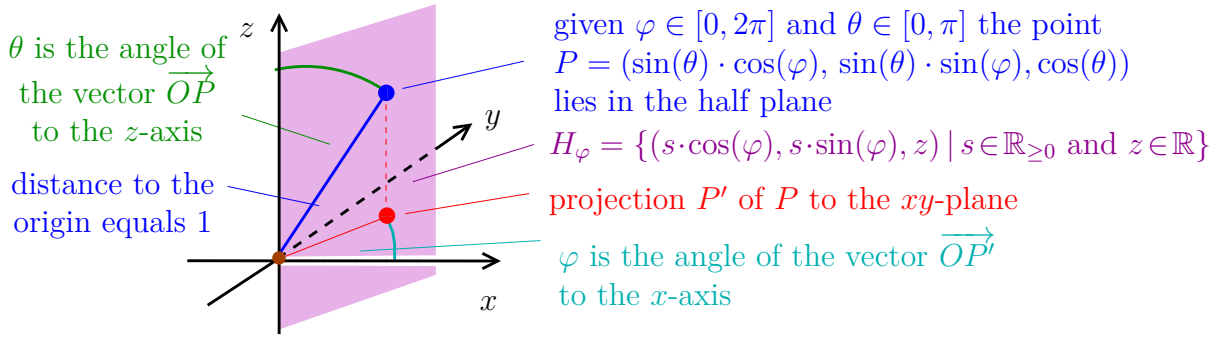
is a compact parametrization of the 2-dimensional torus $S^1 \times S^1$.



(3) We consider the smooth manifold S^2 with the standard orientation from page 817, i.e. a basis (v, w) of $\mathrm{T}_P S^2$ is positive if $\det(P\ v\ w) > 0$. One can now easily verify that the map[779]

$$\begin{array}{rcl} \Psi\colon [0,\pi]\times[0,2\pi] & \to & S^2 \\ (\theta,\varphi) & \mapsto & \begin{pmatrix} \sin(\theta)\cdot\cos(\varphi) \\ \sin(\theta)\cdot\sin(\varphi) \\ \cos(\theta) \end{pmatrix} \end{array}$$

is a compact parametrization of S^2.



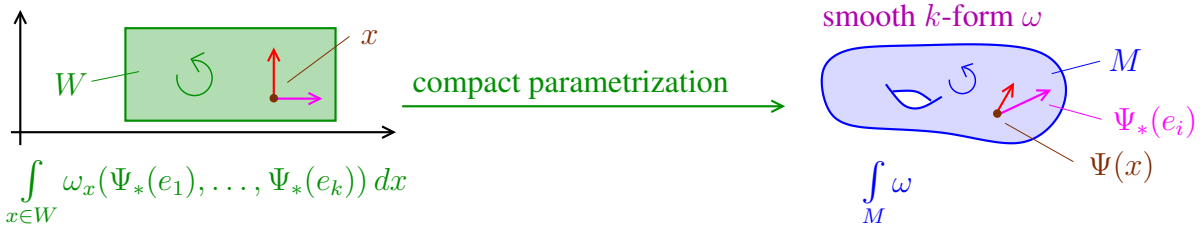
[779] Note that if we swapped the order of φ and θ then the map would no longer be orientation-preserving.

(4) Let $M \subseteq \mathbb{R}^k$ be a compact submanifold of codimension zero. It follows easily from the Codimension-Zero Submanifold Proposition 25.49 that $\mathrm{id}\colon M \to M$ is a compact parametrization. □

The next proposition shows what compact parametrizations are good for:

Proposition 74.19. (Integral-via-Parametrization Proposition) Let M be a compact oriented k-dimensional smooth manifold and let $\Psi\colon W \to M$ be a compact parametrization of M. For every smooth k-form ω on M we have[780]

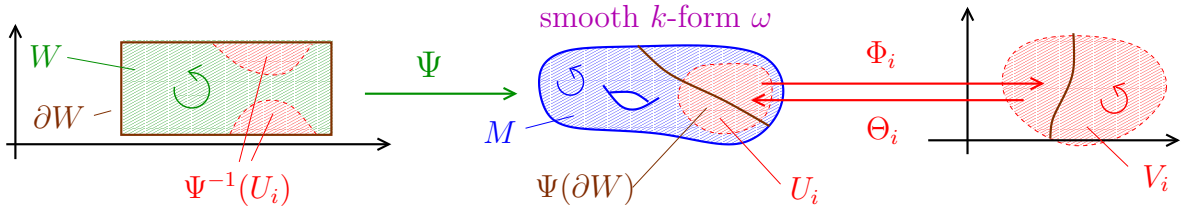
$$\int_M \omega \;=\; \int_{x\in W} \omega_{\Psi(x)}(\mathrm{D}\Psi_x(\mathfrak{e}_1),\dots,\mathrm{D}\Psi_x(\mathfrak{e}_k))\,dx.$$



Remark. Note that it follows from the Integral-via-Parametrization Proposition 74.19 that for compact connected 1-dimensional smooth manifolds the definition of the integral of a smooth 1-form that we gave in the 1-Form Integration Proposition 71.17 agrees with Definition 74.13. □

Proof. Let M be a compact oriented k-dimensional smooth manifold, let $\Psi\colon W \to M$ be a compact parametrization of M and let ω be a smooth k-form on M. The case $k = 0$ follows easily from the discussion on page 1532, so we might as well assume that $k > 0$.

Since $k > 0$ we can pick an atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,r}$ for M such that each chart is orientation-preserving. For $i = 1,\dots,r$ we write $\Theta_i := \Phi_i^{-1}$.



It follows from the Manifold–Partitions-of-Unity Theorem 27.2 that we can pick a smooth partition of unity $\{\mu_i\colon M \to [0,1]\}_{i=1,\dots,r}$ such that for every $i \in \{1,\dots,r\}$ we have

[780]Since we assume that W is regular closed it follows from the argument of the Regular Closed–∂_*-Lemma 24.5 that it makes sense to talk of $\Psi_*(\mathfrak{e}_i) = \mathrm{D}\Psi_x(\mathfrak{e}_i)$ for all points $x \in W$.

$\operatorname{supp}(\mu_i) \subseteq U_i$. Then we see that

$$\int_M \omega \overset{\text{by Definition 74.13 and by the the Integral-of-Form-in-Chart Proposition 74.12}}{\overset{\downarrow}{=}} \sum_{i=1}^{r} \int_M \mu_i \cdot \omega \overset{\text{by the Measure-Zero Properties Proposition 37.3 together with the Lebesgue Integral Proposition 73.22 (2b),(3b) and the facts that } \Psi\colon W \to M \text{ is smooth and that } \partial W \text{ is a finite union of submanifolds of codimension} \geq 1}{\overset{\downarrow}{=}} \sum_{i=1}^{r} \int_{y \in V_i} \mu_i(\Theta_i(y)) \cdot \omega_{\Theta_i(y)}(\Theta_{i*}(\mathfrak{e}_1), \ldots, \Theta_{i*}(\mathfrak{e}_k))\, dy \overset{\downarrow}{=}$$

$$= \sum_{i=1}^{r} \int_{y \in V_i \setminus \Phi_i(\Psi(\partial W))} \mu_i(\Theta_i(y)) \cdot \omega_{\Theta_i(y)}(\Theta_{i*}(\mathfrak{e}_1), \ldots, \Theta_{i*}(\mathfrak{e}_k))\, dy$$

$$= \sum_{i=1}^{r} \int_{x \in \Psi^{-1}(U_i) \setminus \partial W} \mu_i(\Psi(x)) \cdot \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1), \ldots, \Psi_*(\mathfrak{e}_k))\, dx$$

$\uparrow$ follows, by the same calculation as in the proof of the Integral-of-Form-in-Chart Proposition 74.12, from the Lebesgue Transformation Theorem 73.26 applied to the orientation-preserving diffeomorphism $\Phi_i \circ \Psi\colon \Psi^{-1}(U_i) \setminus \partial W \to V_i \setminus \Phi_i(\Psi(\partial W))$

$$= \int_{x \in W \setminus \partial W} \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1), \ldots, \Psi_*(\mathfrak{e}_k))\, dx$$

$\uparrow$ since the μ_i add up to one

$$= \int_{x \in W} \omega_{\Psi(x)}(\Psi_*(\mathfrak{e}_1), \ldots, \Psi_*(\mathfrak{e}_k))\, dx.$$

$\uparrow$ by the Lebesgue Integral Proposition 73.22 (4) and the fact that ∂W is a finite union of submanifolds of codimension ≥ 1 ■

Example. We consider S^2 together with the smooth 2-form ω that at each $P \in S^2$ is given by[781]

$$\begin{aligned} \omega\colon \mathrm{T}_P S^2 \times \mathrm{T}_P S^2 &\to \mathbb{R}^3 \\ (v, w) &\mapsto \det(P \;\; v \;\; w). \end{aligned}$$

We consider the compact parametrization

$$\begin{aligned} \Psi\colon [0,\pi] \times [0, 2\pi] &\to S^2 \\ (\theta, \varphi) &\mapsto \begin{pmatrix} \sin(\theta) \cdot \cos(\varphi) \\ \sin(\theta) \cdot \sin(\varphi) \\ \cos(\theta) \end{pmatrix} \end{aligned}$$

from page 1536 and we calculate that

$$\int_{S^2} \omega \overset{\text{by the Integral-via-Parametrization Proposition 74.19 and the Fubini Theorem 73.25}}{\overset{\downarrow}{=}} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\theta=\pi} \det(\Psi(\theta,\varphi) \;\; \mathrm{D}\Psi_{(\theta,\varphi)} \cdot \mathfrak{e}_1 \;\; \mathrm{D}\Psi_{(\theta,\varphi)} \cdot \mathfrak{e}_2)\, d\varphi\, d\theta$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\theta=\pi} \underbrace{\det \begin{pmatrix} \sin(\theta) \cdot \cos(\varphi) & \cos(\theta) \cdot \cos(\varphi) & -\sin(\theta) \cdot \sin(\varphi) \\ \sin(\theta) \cdot \sin(\varphi) & \cos(\theta) \cdot \sin(\varphi) & \sin(\theta) \cdot \cos(\varphi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{pmatrix}}_{=\sin(\theta)} d\theta\, d\varphi \;=\; 4\pi.$$

□

We conclude with one application of the Integral-via-Parametrization Proposition 74.19:

[781] If $F\colon S^2 \to \mathbb{R}^3$ is the map that is given by $P \mapsto P$, then the 2-form we are considering right now is what we called $\det(F \star_1 \cdots \star_{n-1})$ on page 1526.

Lemma 74.20. (Integral-n-Form-on-$\mathbb{R}^n$-Lemma) Let $M \subseteq \mathbb{R}^n$ be a compact smooth submanifold of codimension zero.

(1) Given any smooth n-form ω on M we have

$$\underbrace{\int_M \omega}_{\text{integral of } n-\text{form}} = \underbrace{\int_{x\in M} \omega_x(\mathfrak{e}_1, \ldots, \mathfrak{e}_n)\, dx.}_{\text{integral of function}}$$

(2) Given any smooth function $f\colon M \to \mathbb{R}$ we have

$$\underbrace{\int_M f \cdot dx^1 \wedge \cdots \wedge dx^n}_{\text{integral of } n-\text{form}} = \underbrace{\int_{x\in M} f(x)\, dx.}_{\text{integral of function}}$$

Proof. Statement (1) follows from the Integral-via-Parametrization Proposition 74.19 applied to the compact parametrization $\mathrm{id}\colon M \to M$. Statement (2) follows from Statement (1) together with the calculation on page 1506 which implies that we have the equality $(dx^1 \wedge \cdots \wedge dx^n)(\mathfrak{e}_1, \ldots, \mathfrak{e}_n) = 1$. ∎

74.4. **Groups actions and differential forms.** In this last section we will introduce a tool to construct smooth forms on quotient manifolds. First we recall a few facts on group actions on smooth manifolds: Let G be a group that acts freely, properly and smoothly on a smooth manifold M.

(1) In the Manifold–Quotient Proposition 25.39 we saw that there exists a unique smooth structure on M/G such that the projection $p\colon M \to M/G$ is a local diffeomorphism.
(2) In the Orientation-Action Proposition 31.44 we saw that if M is oriented and the G-action preserves the orientation, then M/G admits a unique orientation such that $p\colon M \to M/G$ is orientation-preserving.

We can now extend this discussion to smooth differential forms on M. First of all we have the following mostly self-explanatory definition.

Definition 74.21. Let G be a group that acts smoothly on a smooth manifold M. Given $g \in G$ we denote by $\phi_g\colon M \to M$ the diffeomorphism, which is given by $x \mapsto g \cdot x$. We say that a differential form ω on M is G-invariant, if for all $g \in G$ we have $\phi_g^*(\omega) = \omega$.

Lemma 74.22. (G-Invariant Form Lemma) Let M be a smooth manifold and let G be a group that acts on M freely, properly and smoothly. We denote by $p\colon M \to M/G$ the projection, which is a local diffeomorphism by the Manifold–Quotient Proposition 25.39.

(1) Let ω be a k-form on M. If ω is G-invariant, then there exists a unique smooth k-form $\overline{\omega}$ on the smooth manifold M/G with

$$\omega = p^*(\overline{\omega}).$$

(2) Given two G-invariant differential forms η and ω on M we have

$$\overline{\eta \wedge \omega} = \overline{\eta} \wedge \overline{\omega}.$$

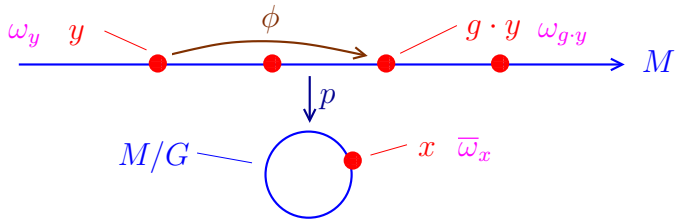
Proof. Let M be a smooth manifold and let G be a group that acts on M freely, properly and smoothly. We denote by $p\colon M \to M/G$ the projection. First note that Statement (2)

follows almost immediately from the uniqueness part of Statement (1) together with the Pullback Smooth Form Proposition 74.7 (1b). Thus it remains to prove Statement (1).

Let ω be a k-form on M which is G-invariant. Let x be a point in M/G. We pick $y \in M$ with $p(y) = x$. As we mentioned above, by the Manifold–Quotient Proposition 25.39 we know that the projection $p\colon M \to M/G$ is a local diffeomorphism. By the Tangent Space-Inclusion Proposition 29.20 (3) this implies that $\mathrm{D}p_y\colon \mathrm{T}_yM \to \mathrm{T}_x(M/G)$ is an isomorphism. It follows from the Alternating Form-Functoriality Lemma 72.12 and the Isomorphism Preservation Lemma 5.26 that $(\mathrm{D}p_y)^*\colon \wedge^k \mathrm{T}_x(M/G) \to \wedge^k \mathrm{T}_yM$ is an isomorphism. This observation allows us to set $\overline{\omega}_x \coloneqq ((\mathrm{D}p_y)^{-1})^*(\omega_y) \in \wedge^k \mathrm{T}_x(M/G)$.

Claim. The definition of $\overline{\omega}_x$ does not depend on the choice of y.

Proof. Note that any other point in the preimage of x is of the form $g \cdot y$ for some $g \in G$. We denote by $\phi\colon M \to M$ the map that is given by $x \mapsto g \cdot x$.



The claim now follows from the following calculation:

$$
\begin{array}{lll}
((\mathrm{D}p_y)^{-1})^*(\overbrace{\omega_y}^{\in \wedge^k \mathrm{T}_yM}) & \overset{\text{since } \omega \text{ is } G\text{-invariant}}{=} & ((\mathrm{D}p_y)^{-1})^*((\mathrm{D}\phi_y)^*(\omega_{g\cdot y})) \\
& = & ((\mathrm{D}p_{g\cdot y} \circ \mathrm{D}\phi_y)^{-1})^*((\mathrm{D}\phi_y)^*(\omega_{g\cdot y})) \\
& \overset{\text{since } p\circ\phi = p}{=} & ((\mathrm{D}(p\circ\phi)_y)^{-1})^*((\mathrm{D}\phi_y)^*(\omega_{g\cdot y})) \\
& = & ((\mathrm{D}p_{g\cdot y})^{-1})^*(\underbrace{\omega_{g\cdot y}}_{\in \wedge^k \mathrm{T}_{g\cdot y}M})
\end{array}
$$

by the Physical Tangent Space Functor Proposition 29.15 (2) — by functoriality

The above argument shows that we get a well-defined k-form $\overline{\omega}$ on M/G. By definition we have $\omega = p^*(\overline{\omega})$. Since the maps $\mathrm{D}p_y$ are isomorphisms we also see that $\overline{\omega}$ is uniquely determined. Finally note that since $p\colon M \to M/G$ is a local diffeomorphism and since ω is smooth we obtain from the Smooth Form–Pasting Proposition 74.8 that $\overline{\omega}$ is also smooth. ∎

Example.

(1) We denote by dx the usual 1-form on $\mathbb{R}$, which at every $P \in \mathbb{R}$ is given by

$$
\begin{array}{rccc}
dx_P\colon \widetilde{\mathrm{T}}_P\mathbb{R} = \mathbb{R} & \to & \mathbb{R} \\
v & \mapsto & v
\end{array}
$$

This 1-form is invariant under the additive action of $\mathbb{Z}$ on $\mathbb{R}$. Thus it follows from the Tangent Space-Isomorphism Proposition 29.24 together with the G-Invariant Form Lemma 74.22 (1) that it induces a 1-form $\overline{dx}$ on the smooth manifold $\mathbb{R}/\mathbb{Z}$. We consider the map $\Psi\colon [0,1] \to \mathbb{R}/\mathbb{Z}$ that is given by $\Psi(t) = [p(t)]$. One can easily verify that this

map is a compact parametrization in the sense of Definition 71.16. Note that

$$\int_{\mathbb{R}/\mathbb{Z}} \overline{dx} \;=\; \int_{[0,1]} \Psi^*(\overline{dx})(1)\,d\lambda \;=\; \int_{t=0}^{t=1} \Psi^*(\overline{dx})(1)\,dt \;=\; \int_{t=0}^{t=1} dx(1)\,dt \;=\; \int_{t=0}^{t=1} 1\,dt \;=\; 1.$$

by the Integral-via-Parametrization Proposition 74.19 — by the Riemann-Lebesgue Integral Proposition 73.23

(2) We denote by dx and dy the usual 1-forms on $\mathbb{R}^2$. As in (1) we see that it follows from the G-Invariant Form Lemma 74.22 (1) that they induce 1-forms $\overline{dx}$ and $\overline{dy}$ on $\mathbb{R}^2/\mathbb{Z}^2$. Note that

$$\int_{\mathbb{R}^2/\mathbb{Z}^2} \overline{dx} \wedge \overline{dy} \;=\; \int_{\mathbb{R}^2/\mathbb{Z}^2} \overline{dx \wedge dy} \;=\; 1.$$

by the G-Invariant Form Lemma 74.22 (2) — as in (1), using the obvious compact parametrization $[0,1]\times[0,1] \to \mathbb{R}^2/\mathbb{Z}^2$ □

74.5. Exercises for Chapter 74.

Exercise 74.1. We consider the 2-dimensional smooth manifold

$$M \;:=\; \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1 \text{ and } z \geq \tfrac{1}{2}\},$$

with the orientation which is given by the unit normal field $\nu(P) = P$.

(a) Provide a compact parametrization for M.

(b) We consider the 1-forms α and β on M which for $P \in M$ are given as follows:

$$\begin{array}{rcl} \alpha\colon \mathrm{T}_P M & \to & \mathbb{R} \\ (x,y,z) & \mapsto & 2x \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \mathrm{T}_P M & \to & \mathbb{R} \\ (x,y,z) & \mapsto & -z \end{array}.$$

Determine the integral of the 2-form $\alpha \wedge \beta$ on M.

Exercise 74.2. Provide a compact parametrization for each sphere S^n.

Exercise 74.3. Let M and N be two closed oriented smooth manifolds. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the obvious projections. Let $\varphi \in \Omega^r(M)$ and let $\varphi \in \Omega^s(N)$ with $r+s = \dim(M) + \dim(N)$. Show that if $r = \dim(M)$ and $s = \dim(N)$, then

$$\int_{M\times N} p^*(\varphi) \wedge q^*(\psi) \;=\; \int_M p^*(\varphi) \cdot \int_N q^*(\psi).$$

Hint. Use the Fubini Theorem 73.25.

Remark. In other words, the task of the exercise is to provide a proof of the Product-Pullback-Integral Proposition 74.17 (1).

Exercise 74.4. Let M be a compact oriented n-dimensional smooth manifold, let ω be a smooth n-form on M and let $p\colon \widetilde{M} \to M$ be a finite covering. What is the relationship between

$$\int_M \omega \qquad \text{and} \qquad \int_{\widetilde{M}} p^*(\omega)\,?$$

Exercise 74.5. Let M be an n-dimensional smooth manifold.

(a) Show that if M is orientable, then there exists a smooth n-form which is non-zero everywhere.

(b) Does the converse to (a) hold?

CHAPTER 75

Stokes' Theorem

In this chapter we will state and prove Stokes' Theorem, which is easily one of the most famous theorem in mathematics, and which has many wonderful applications.

This chapter is organized as follows:

- In Section 75.1 we will define the differential of differential forms on open subsets of $\mathbb{R}^n$.
- In Section 75.2 we will define the differential of differential forms on smooth manifolds.
- In Section 75.3 we will state Stokes' Theorem 75.11 and we will discuss some applications.
- In Section 75.4 we will prove Stokes' Theorem 75.11.
- In Appendix 75.5 we will use Stokes' Theorem 75.11 to provide a proof of the Cauchy Theorem 39.8.
- In Section 75.6 we will provide exercises for Chapter 75.

In the next Chapter 76 we will consider some classical versions of Stokes' Theorem 75.11.

75.1. **The differential of differential forms I.** In this section we will introduce the differential of differential forms. As usual we first introduce this notion for differential forms on open subsets of $\mathbb{R}^k$ and $\mathbb{H}^k$. Afterwards we will use charts to define the notion for differential forms on arbitrary smooth manifolds.

Definition 75.1. Let U be an open subset of $\mathbb{R}^k$ or $\mathbb{H}^k$. Furthermore let ω be a smooth m-form on U. By the Alternating Forms-Basis Proposition 72.7 the wedge products $dx^{i_1} \wedge \cdots \wedge dx^{i_m}$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq k$ form a basis of $\wedge^m (\mathbb{R}^k)^*$. Thus we can consider ω as a linear combination, i.e. we can write

$$\omega \quad = \quad \sum_{i_1 < \cdots < i_m} f_{i_1 \ldots i_m} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_m},$$

where $f_{i_1 \ldots i_m}$ are uniquely defined real-valued smooth functions on U. We define the differential of ω[782] to be the $(m+1)$-form

$$d\omega \quad := \quad \sum_{i_1 < \cdots < i_m} \underbrace{df_{i_1 \ldots i_m}}_{\text{the total differential of the function } f_{i_1 \ldots i_m} : U \to \mathbb{R}, \text{ recall that for a smooth function } f : U \to \mathbb{R} \text{ we know by the 1-Form-via-}dx^i\text{-Lemma 71.5 that the total differential is given by } df = \frac{\partial f}{\partial x_1} \cdot dx^1 + \cdots + \frac{\partial f}{\partial x_k} \cdot dx^k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_m}.$$

Example.

(1) If $\omega \in \Omega^0(U)$, i.e. if ω is a smooth function on U, then $d\omega$ is by definition just the total differential of this smooth function.

(2) On $\mathbb{R}^2$ we consider the 1-form $\omega = 3x^2y \cdot dx + x^3 \cdot dy$. We perform the following calculation:

$$\begin{aligned} d\omega &= d(3x^2y\cdot dx + x^3\cdot dy) \overset{\text{above definition of the differential of a 1-form}}{=} d(3x^2y)\wedge dx + d(x^3)\wedge dy \\ &= \left(\frac{\partial(3x^2y)}{\partial x}\cdot dx + \frac{\partial(3x^2y)}{\partial y}\cdot dy\right)\wedge dx + \left(\frac{\partial(x^3)}{\partial x}\cdot dx + \frac{\partial(x^3)}{\partial y}\cdot dy\right)\wedge dy \\ &\overset{\text{by the 1-Form-via-}dx^i\text{-Lemma 71.5}}{=} 6xy\cdot \underbrace{dx\wedge dx}_{=0} + 3x^2\cdot \underbrace{dy\wedge dx}_{=-dx\wedge dy} + 3x^2\cdot dx\wedge dy + 0\cdot \underbrace{dy\wedge dy}_{=0} = 0. \end{aligned}$$

(3) On $U=\mathbb{R}^n$ we consider the 2-form $\omega = dx^i\wedge dx^j$. Then:

$$d\omega = d(dx^i\wedge dx^j) = d(1\cdot dx^i\wedge dx^j) = \underbrace{d\,1}_{\substack{=0,\text{ since}\\ \text{all } \frac{\partial 1}{\partial x_i}=0}}\wedge dx^i\wedge dx^j = 0.$$

□

In the following proposition we summarize most of the essential statements about differentials of smooth forms:

Proposition 75.2. (Euclidean Differential Proposition) Let U be an open subset of $\mathbb{R}^k$ or of $\mathbb{H}^k$. Furthermore let $\omega,\omega'\in\Omega^l(U)$, let $\sigma\in\Omega^m(U)$ and let $\lambda\in\mathbb{R}$. Then the following statements hold:

(1) $d(\omega+\omega') = d\omega + d\omega'$ and $d(\lambda\cdot\omega) = \lambda\cdot d\omega$.
(2) The $(l+1)$-form $d\omega$ is a smooth form.
(3) We have the following Product Rule for the differential:

$$d(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^{\deg(\omega)}\cdot w\wedge d\sigma.$$

(4) $d(d\omega) = 0$.

Example. On $\mathbb{R}^2$ we consider the function $f(x,y)=x^3y$. It follows from the 1-Form-via-dx^i-Lemma 71.5 that $df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = 3x^2y\cdot dx + x^3\cdot dy$. This is just the 1-form on $\mathbb{R}^2$, which we considered above. Note that it follows directly from the Euclidean Differential Proposition 75.2 (4) that $d\omega = d(df) = 0$. □

Example. Let U be an open subset of $\mathbb{R}^k$ or $\mathbb{H}^k$. We consider Statements (3) and (4) of the Euclidean Differential Proposition 75.2 for 0-forms, i.e. for real-valued function, on U.

(3) Let $f,g\colon U\to\mathbb{R}$ be two smooth functions. We can view f and g as 0-forms. Then

$$\begin{aligned} d(f\cdot g) = df\cdot g + f\cdot dg &\overset{\text{this follows from the 1-Form-via-}dx^i\text{-Lemma 71.5}}{\Longleftrightarrow} \sum_{i=1}^k \frac{\partial(f\cdot g)}{\partial x_i}\cdot dx^i = \left(\sum_{i=1}^k \frac{\partial f}{\partial x_i}\cdot dx^i\right)\cdot g + f\cdot\left(\sum_{i=1}^k \frac{\partial g}{\partial x_i}\cdot dx^i\right) \\ &\Longleftrightarrow \sum_{i=1}^k \frac{\partial(f\cdot g)}{\partial x_i}\cdot dx^i = \sum_{i=1}^k \left(\frac{\partial f}{\partial x_i}\cdot g + f\cdot\frac{\partial g}{\partial x_i}\right)\cdot dx^i \\ &\Longleftrightarrow \frac{\partial(f\cdot g)}{\partial x_i} = \frac{\partial f}{\partial x_i}\cdot g + f\cdot\frac{\partial g}{\partial x_i} \quad \text{for all } i=1,\dots,k. \end{aligned}$$

(since the dx^i form a basis of $\mathrm{Hom}(\mathbb{R}^k,\mathbb{R})$)

Thus we have shown that for 0-forms the statement of the Euclidean Differential Proposition 75.2 (3) corresponds precisely to the Product Rule for partial derivatives.

(4) Let $f\colon U \to \mathbb{R}$ be a smooth function. Then

$$\begin{aligned} d(df) &= d\Big(\sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \cdot dx^i\Big) && = \sum_{i=1}^{k} d\Big(\frac{\partial f}{\partial x_i}\Big) \wedge dx^i \\ &\uparrow && \uparrow \\ &\text{by the 1-Form-via-}dx^i\text{-Lemma 71.5} && \text{Definition 75.1 of } d \\ &= \sum_{i=1}^{k} \Big(\sum_{j=1}^{k} \frac{\partial^2 f}{\partial x_j \partial x_i} \cdot dx^j\Big) \wedge dx^i && = \sum_{i<j} \Big(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j}\Big) \cdot dx^i \wedge dx^j. \\ &\uparrow && \uparrow \\ &\text{by the 1-Form-via-}dx^i\text{-Lemma 71.5} && \text{since } dx^i \wedge dx^j = -dx^j \wedge dx^i \\ & && \text{and } dx^i \wedge dx^i = 0 \end{aligned}$$

By the Alternating Forms-Basis Proposition 72.7 we know that the 2-forms $dx^i \wedge dx^j$ with $i < j$ are linearly independent. Thus we see that the statement $d(df) = 0$ is *equivalent* to the Schwarz Theorem 24.2 that says that one can swap the order in which partial derivatives are taken. □

Before we turn to the proof of the Euclidean Differential Proposition 75.2, let us introduce the following notation:

Notation 75.3. Let $n \in \mathbb{N}_0$.

(1) Given an increasing sequence $I = 1 \le i_1 < \cdots < i_k \le n$ of natural numbers we write

$$dx^I \;:=\; dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We denote by $|I| := k$ the number of elements in the sequence. We refer to $|I|$ as the length of the sequence.

(2) Let U be an open subset of $\mathbb{R}^k$ or of $\mathbb{H}^k$. It follows from the Alternating Forms-Basis Proposition 72.7 that any k-form ω can be written as

$$\omega \;=\; \sum_{|I|=k} f_I \cdot dx^I,$$

where we sum over all increasing sequences $I = 1 \le i_1 < \cdots < i_k \le n$ of length k.

Proof of Proposition 75.2. Statement (1) of the Euclidean Differential Proposition 75.2 is trivial. Statement (2) follows easily from the definitions and from the 1-Form-via-dx^i-Lemma 71.5.

We turn to Statement (3). In the above example we had seen that the Product Rule for 0-forms is just the usual Product Rule for partial derivatives. We will now reduce the general case to this special case. Thus let $\omega \in \Omega^l(U)$ and let $\sigma \in \Omega^m(U)$. We write

$$\omega \;=\; \sum_{|I|=l} f_I \cdot dx^I \qquad \text{and} \qquad \sigma \;=\; \sum_{|J|=m} g_J \cdot dx^J$$

and we obtain

$$d\omega \;=\; \sum_{|I|=l} df_I \wedge dx^I \qquad \text{and} \qquad d\sigma \;=\; \sum_{|J|=m} dg_J \wedge dx^J.$$

Now we calculate that

$$\begin{aligned} d(\omega\wedge\sigma) &= d\Big(\sum_{|I|=l} f_I\cdot dx^I \wedge \sum_{|J|=m} g_J\cdot dx^J\Big) \quad \text{by the multilinearity of wedge products} \\ &= d\Big(\sum_{I,J} f_I\cdot g_J\cdot dx^I\wedge dx^J\Big) \\ &= \sum_{I,J} d(f_I\cdot g_J)\cdot dx^I\wedge dx^J \qquad = \sum_{I,J}(g_J\cdot df_I + f_I\cdot dg_J)\wedge dx^I\wedge dx^J \\ &\quad \uparrow \text{Definition 75.1 of the differentials} \qquad \uparrow \text{by the Product Rule for products of 0-forms, which we had discussed in the above example} \\ &= \sum_{I,J}\big(df_I\wedge dx^I\big)\wedge g_J\cdot dx^J + (-1)^{\deg(\omega)}\cdot\sum_{I,J} f_I\cdot dx^I\wedge\big(dg_J\wedge dx^J\big) \\ &\quad \uparrow \text{by the Wedge Product Properties Lemma 72.10 we have } dg^J\wedge dx^I = -(1)^l\cdot dx^I\wedge dg^J \\ &= d\omega\wedge\sigma + (-1)^{\deg(\omega)}\cdot\omega\wedge d\sigma. \end{aligned}$$

Finally we turn to the proof of Statement (4). In the above example we had seen that the statement for 0-forms corresponds to the Schwarz Theorem 24.2, which says that it does not matter in which order one applies partial derivatives of a smooth function. Again we will reduce the general case to this special case. Thus let $\omega\in\Omega^l(U)$. As above we write $\omega = \sum_{|I|=l} f_I\cdot dx^I$. Then:

$$d(d(\omega)) = d\Big(\sum_{|I|=l} df_I\wedge dx^I\Big) = \sum_{|I|=l}\big(\overbrace{d(df_I)}^{=0 \text{ by the example on page 1544}}\wedge dx^I + (-1)\cdot df_I\wedge \underbrace{d(dx^I)}_{d(dx^I)=d(1\cdot dx^I)=d1\wedge dx^I=0}\big) = 0.$$

$\uparrow$ by the Product Rule from Statement (4)

■

We also have the following essential proposition:

Proposition 75.4. (Differential-Natural Proposition) Let $U\subseteq\mathbb{R}^m$ and $V\subseteq\mathbb{R}^n$ be open subsets and let $\Theta\colon U\to V$ be a smooth map. Given any smooth k-form ω on V we have the following equality of smooth $(k+1)$-forms on U:

$$d(\Theta^*(\omega)) = \Theta^*(d\omega).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \Omega^{k+1}(V) & \xleftarrow{\quad d\quad} & \Omega^k(V) \\ \downarrow\Theta^* & & \downarrow\Theta^* \\ \Omega^{k+1}(U) & \xleftarrow{\quad d\quad} & \Omega^k(U). \end{array}$$

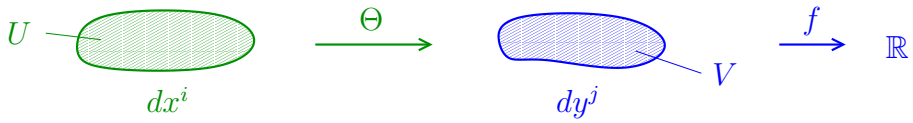
The analogous statement also holds for open subsets $U\subseteq\mathbb{H}^m$ and/or $V\subseteq\mathbb{H}^n$.

Example. In the proof of the Differential-Natural Proposition 75.4 will we see that for a 0-form ω the statement is just a reformulation of the usual Chain Rule 24.17. □

Proof. Let U be an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$ and let V be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Furthermore let $\Theta=(\Theta_1,\ldots,\Theta_n)\colon U\to V$ be a smooth map. First we consider the special case of a smooth 0-form, i.e. of a smooth function $f\colon V\to\mathbb{R}$.

Claim. Given any smooth function $f\colon V\to\mathbb{R}$ we have $d(\Theta^*f)=\Theta^*(df)$.

Proof. We denote by $dx^1, \dots, dx^m$ the usual 1-forms on $U \subseteq \mathbb{R}^m$ and we denote by $dy^1, \dots, dy^n$ the usual 1-forms on $V \subseteq \mathbb{R}^n$.



Then

$$d(\Theta^* f) \;=\; d(f \circ \Theta) \;\overset{\text{by the 1-Form-via-}dx^i\text{-Lemma 71.5 applied to } f\circ\Theta}{=}\; \sum_{i=1}^{m} \frac{\partial (f \circ \Theta)}{\partial x_i} \cdot dx^i \;\overset{\text{by the Chain Rule 24.17}}{=}\; \sum_{i=1}^{m} \underbrace{\sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} \circ \Theta \right)}_{\text{independent of } i} \cdot \frac{\partial \Theta_j}{\partial x_i}\, dx^i$$

$$= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} \circ \Theta \right) \cdot d\Theta_j \;=\; \Theta^* \left(\sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \cdot dy^j \right) \;=\; \Theta^*(df).$$

(first $=$: by the 1-Form-via-dx^i Lemma 71.5 applied to Θ_j; second $=$: follows from the equality $(g \circ \Theta) \cdot d\Theta_j = \Theta^*(g \cdot dy^j)$ [783]; third $=$: by the 1-Form-via-dx^i Lemma 71.5) ⊞

Now let ω be an arbitrary smooth k-form on V. As we discussed in Notation 75.3, we can uniquely write

$$\omega \;=\; \sum_{|I|=k} f_I \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

We calculate that

$$d(\Theta^*(\omega)) \;=\; d\Big(\Theta^*\Big(\sum_{|I|=k} f_I \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}\Big)\Big) \;\overset{\text{by the Pullback Smooth Form Proposition 74.7 (1b) we always have } \Theta^*(\alpha\wedge\beta) \;=\; \Theta^*\alpha \wedge \Theta^*\beta}{=}\; d\Big(\sum_{|I|=k} \Theta^* f_I \cdot \Theta^* dx^{i_1} \wedge \dots \wedge \Theta^* dx^{i_k}\Big)$$

$$= d\Big(\sum_{|I|=k} \Theta^* f_I \cdot d(\Theta^* x^{i_1}) \wedge \dots \wedge d(\Theta^* x^{i_k})\Big) \;\overset{\text{follows from the Euclidean Differential Proposition 75.2 (3) and (4)}}{=}\; \sum_{|I|=k} d(\Theta^* f_I) \cdot d(\Theta^* x^{i_1}) \wedge \dots \wedge d(\Theta^* x^{i_k})$$

(first $=$: from the claim, applied to $f = x^i$, it follows that $\Theta^* dx^i = d(\Theta^* x^i)$)

$$= \sum_{|I|=k} \Theta^* df_I \cdot \Theta^* dx^{i_1} \wedge \dots \wedge \Theta^* dx^{i_k}$$

(follows from the claim applied to $f = f_I$)

$$= \Theta^*\Big(\sum_{|I|=k} df_I \cdot dx^{i_1} \wedge \dots \wedge dx^{i_k}\Big) \;=\; \Theta^*(d\omega).$$

(since by the Pullback Smooth Form Proposition 74.7 (1b) we have $\Theta^*(\alpha\wedge\beta) \;=\; \Theta^*\alpha \wedge \Theta^*\beta$) ■

75.2. **The differential of differential forms II: Smooth manifolds.** Next we want to introduce the differential of differential forms on smooth manifolds. As usual, the idea is to use charts to reduce the definition to the case considered above, namely of differential forms that are defined on open subsets of $\mathbb{R}^k$ and on open subsets of $\mathbb{H}^k$:

Definition 75.5. Let M be a k-dimensional smooth manifold and let $\omega \in \Omega^l(M)$.

- (i) First let $\Phi\colon U \to V$ be a chart for M of type (i).[784] We denote by $\Psi\colon V \to U$ the inverse map of Φ. Then $\Psi^*(\omega)$ is an l-form on V, i.e. on open subset of $\mathbb{R}^k$. We define $d\omega$ on the open subset U as follows:

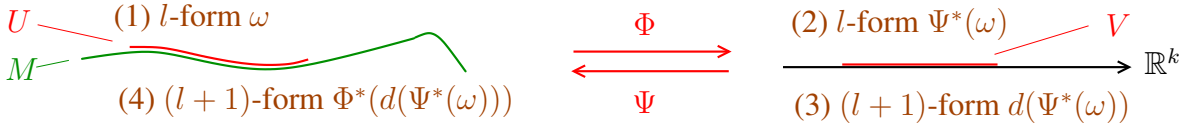
$$d\omega|_U \;:=\; \Phi^*\left(d(\Psi^*(\omega))\right).$$

[783] This follows from $(g \circ \Theta) \cdot d\Theta_j = (g \circ \Theta) \cdot d(y^j \circ \Theta) = (g \circ \Theta) \cdot dy^j \circ \Theta^* = \Theta^* g \cdot \Theta^* dy^j = \Theta^*(g \cdot dy^j)$.

(It follows from the Differential-Well Defined Lemma 75.6 below that this definition does not depend on the choice of the chart.)

(ii) Similarly we define $d\omega|_U$ for charts $\Phi\colon U \to V$ of type (ii).

- Thus we have defined $d\omega$ on the domains of charts of M. In summary we have defined $d\omega$ on all of M. We refer to the $(l+1)$-form $d\omega$ on M as the differential of ω.

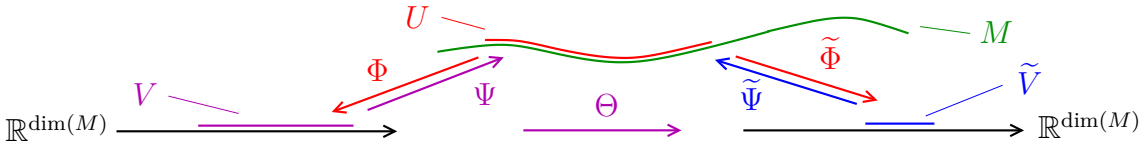


To make sense of the previous definition, we still owe the reader the following lemma:

Lemma 75.6. (Differential-Well Defined Lemma) Let M be a smooth manifold and let $\omega \in \Omega^l(M)$. Let $\Phi\colon U \to V$ and $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ be two charts for M with inverse maps Ψ and $\widetilde{\Psi}$. On $U \cap \widetilde{U}$ we have the following equality:

$$\Phi^*\left(d(\Psi^*(\omega))\right) \;=\; \widetilde{\Phi}^*\big(d\big(\widetilde{\Psi}^*(\omega)\big)\big).$$

Proof. For simplicity we only consider the case that Φ and $\widetilde{\Phi}$ are both charts of type (i). After replacing U and $\widetilde{U}$ by $U \cap \widetilde{U}$ we might as well suppose that $U = \widetilde{U}$. We denote by Θ the diffeomorphism $\Theta := \widetilde{\Phi} \circ \Psi\colon V \to \widetilde{V}$.



Then, using the Pullback Smooth Form Proposition 74.7 (2), we obtain the following equalities on $U = \widetilde{U}$:

$$\begin{array}{ccccc} \Phi^*(d(\Psi^*(\omega))) & \overset{\text{since } \widetilde{\Phi}^*\circ\widetilde{\Psi}^* = (\widetilde{\Psi}\circ\widetilde{\Phi})^* = \mathrm{id}}{=} & \Phi^*(d(\Psi^*(\widetilde{\Phi}^*\widetilde{\Psi}^*(\omega)))) & \overset{\text{since } \Theta^* = \Psi^*\circ\widetilde{\Phi}^*}{=} & \Phi^*(d(\Theta^*(\widetilde{\Psi}^*(\omega)))) \\ & = & \Phi^*(\Psi^*(\widetilde{\Phi}^*(d(\widetilde{\Psi}^*(\omega))))) & = & \widetilde{\Phi}^*(d(\widetilde{\Psi}^*(\omega))). \end{array} \quad \overset{\text{by the Differential Natural Proposition 75.4}}{=} \; \Phi^*(\Theta^* d(\widetilde{\Psi}^*(\omega)))$$

■

Example. Let $\omega \in \Omega^0(M)$. Recall that this means that ω is a smooth function on M. In this case $d\omega \in \Omega^1(M)$ is exactly the 1-form which we introduced in Definition 71.3. [785] □

Some of our earlier results on the differential on open subsets of $\mathbb{R}^k$ and $\mathbb{H}^k$ have obvious generalizations to the more general setting of smooth forms on smooth manifolds:

[784] By Definition 23.2 a chart of type (i) is a diffeomorphism to an open subset of $\mathbb{R}^k$.

[785] This can be seen as follows: It suffices to verify this statement for 0-forms ω whose support is contained in the domain of a chart $\Phi\colon U \to V$. In this case we see that for every $P \in U$ and every $v \in \mathrm{T}_P M$ we have

$$\begin{array}{ccccc} d\omega(v) & \overset{\text{definition of } d \text{ of a } k\text{-form on } M}{=} & \Phi^*(d(\Psi^*(\omega)))(v) & \overset{\text{by definition of } \Phi^*}{=} & d(\Psi^*(\omega))(\Phi_*(v)) \\ & \overset{\text{since } d = \mathrm{D} \text{ for functions on } \mathbb{R}^k}{=} & \mathrm{D}(\omega\circ\Psi)(\Phi_*(v)) & \overset{\text{by the Chain Rule}}{=} & \mathrm{D}\omega(\Psi_*(\Phi_*(v))) \end{array} \quad \begin{array}{c} \overset{\text{since } \Psi^*(\omega) = \omega\circ\Psi}{=} \; \mathrm{D}(\Psi^*(\omega))(\Phi_*(v)) \\ \overset{\text{since } \Psi\circ\Phi = \mathrm{id}}{=} \; \mathrm{D}\omega_P(v). \end{array}$$

Proposition 75.7. (Differentials-on-Manifolds Proposition) Let M be a smooth manifold, let $\omega, \omega' \in \Omega^l(M)$, let $\sigma \in \Omega^m(M)$ and let $\lambda \in \mathbb{R}$. Then the following statements hold:

(1) $d(\omega + \omega') = d\omega + d\omega'$ and $d(\lambda \cdot \omega) = \lambda \cdot d\omega$.
(2) (a) The $(l+1)$-form $d\omega$ is a smooth form.
    (b) If ω has compact support, so has $d\omega$.
(3) We have the following Product Rule for the differential:

$$d(\omega \wedge \sigma) \;=\; d\omega \wedge \sigma + (-1)^{\deg(\omega)} \cdot w \wedge d\sigma.$$

(4) $d(d\omega) = 0$.

Proposition 75.8. (Differential-Natural Proposition) Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds. Given any smooth k-form ω on N we have

$$d(\varphi^*(\omega)) \;=\; \varphi^*(d\omega).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \Omega^{k+1}(N) & \xleftarrow{\;d\;} & \Omega^k(N) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} \\ \Omega^{k+1}(M) & \xleftarrow{\;d\;} & \Omega^k(M). \end{array}$$

Proof of Propositions 75.7 and 75.8. Basically all statements follow easily from the Euclidean Differential Proposition 75.2 and the Differential-Natural Proposition 75.4. The only statement which requires a little bit of thought is the statement about compact support. Thus let ω be a k-form with compact support. It follows easily from the definitions that $\operatorname{supp}(d\omega) \subseteq \operatorname{supp}(\omega)$. Furthermore by definition $\operatorname{supp}(d\omega)$ is a closed subset of M. Thus it follows from the Compact-Closed Lemma 3.39 that $\operatorname{supp}(d\omega)$ is indeed compact. ■

At times we will use the following language:

Definition 75.9.

(1) A smooth form ω with $d\omega = 0$ is called closed.
(2) A smooth form ω for which there exists a smooth form μ with $d\mu = \omega$ is called exact.

Note that it follows from the Differentials-on-Manifolds Proposition 75.7 (4) that an exact form is also closed.

Example. We consider S^1. Note that it follows from the Alternating Criterion 72.3 (2) that $\Omega^2(S^1) = 0$. Thus we see that any smooth 1-form on S^1 is closed. On the other hand, on page 1499 we gave an explicit smooth 1-form on S^1 which is not exact. This shows that not every closed form is also exact. □

Lemma 75.10. (Pullback-Closed-Exact Lemma) Let $f\colon M \to N$ be a smooth map and let $\omega \in \Omega^k(N)$.

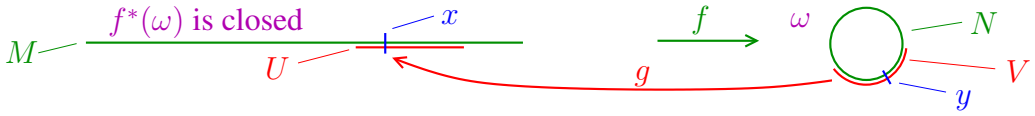
(1) If ω is closed, then so is $f^*(\omega)$.
(2) If ω is exact, then so is $f^*(\omega)$.
(3) If f is a local diffeomorphism, then ω is closed if and only if $f^*(\omega)$ is closed.[786]

[786] On page 1551 we will see that the analog for "exact" does not hold.

Example. Let M be a smooth manifold and let G be a group that acts on M freely, properly and smoothly. Furthermore let ω be a G-invariant k-form on M. By the G-Invariant Form Lemma 74.22 (1) there exists a smooth k-form $\overline{\omega}$ on M/G with $\omega = p^*(\overline{\omega})$. It follows from the Pullback-Closed-Exact Lemma 75.10 (3) that ω is closed if and only if $\overline{\omega}$ is closed. □

Proof. Statements (1) and (2) follow easily from the Differential-Natural Proposition 75.8. It remains to prove Statement (3). Thus we now suppose that $f\colon M \to N$ is a local diffeomorphism. By (1) it remains to show that if $f^*(\omega)$ is closed, then so is ω.

Thus we suppose that $d(f^*(\omega)) = 0$. We want to show that $d\omega = 0$. By the Differential-Natural Proposition 75.8 it suffices to show that for each $y \in N$ there exists an open neighborhood V of y such that $d\omega|_V = 0$. Now let $y \in N$. We pick $x \in M$ with $f(x) = y$. Since f is a local diffeomorphism there exists an open neighborhood U of x and an open neighborhood V of y such that $f|_U\colon U \to V$ is a diffeomorphism. We denote the inverse by $g\colon V \to U$. We denote by $i\colon U \to M$ and $j\colon V \to N$ the inclusions.



Then we see that

$$d\omega|_V = j^*(d\omega) \overset{\text{by the Differential-Natural Proposition 75.8}}{=} d(j^*(\omega)) = d((\overbrace{f \circ i \circ g}^{=\mathrm{id}_V} \circ j)^*(\omega)) = d(j^*(g^*(i^*(f^*(\omega))))) = j^*(g^*(i^*(d(f^*(\omega))))) = j^*(g^*(i^*(0))) = 0.$$

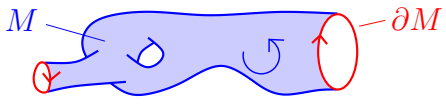
(by the Differential Natural Proposition 75.8; since $f^*(\omega)$ is closed)

■

75.3. **Stokes' Theorem.** Now we can formulate by far the most interesting result on differential forms on smooth manifolds:

Theorem 75.11. (Stokes' Theorem) Let M be an oriented k-dimensional smooth manifold. Furthermore let ω be a smooth $(k-1)$-form on M with compact support. We denote by $i\colon \partial M \to M$ the inclusion. Then[787] [788]

$$\int_M d\omega = \int_{\partial M} i^*(\omega).$$



$$\int_M d\omega = \int_{\partial M} i^*(\omega)$$

Remark.

[787] In the Differentials-on-Manifolds Proposition 75.7 we saw that $d\omega$ has compact support. For peace of mind let us also explain why the $(k-1)$-form $i^*(\omega)$ on ∂M has compact support, so that it actually makes sense to talk of the integral of $i^*(\omega)$: First note that the support of $i^*(\omega)$ is a closed subset that is contained in the intersection of the support of ω and ∂M. But by the Smooth Manifold–Boundary Proposition 25.27 (2) we know that ∂M is a closed subset of M. Thus we obtain from the Compact-Closed Lemma 3.39 (2) that the support of $i^*(\omega)$ is a compact subset of ∂M.

[788] We equip ∂M with the orientation introduced in Convention 31.39.

(1) The above formulation of Stokes' Theorem was first given by Elie Cartan [**Carta1945**, p. 40]. It can be viewed as a generalization of various classical versions of Stokes' Theorem, namely Green's Theorem 76.1, the Classical Stokes' Theorem 75.11 and the Gauß Divergence Theorem 76.26, which we will discuss in the next chapter. We refer to [**Katz1979**] for an enlightening history of Stokes' Theorem in its various incarnations. Stokes' Theorem, in its many incarnations, is also the hero of a novel by Michèle Audin [**Aud2016**].
(2) In many applications the object at hand is not quite a smooth manifold, for example one might be interested in a cube or a half-ball or a cylinder. To deal with these cases one needs a sufficiently flexible notion of a smooth manifold "with corner" and then prove a generalized version of Stokes' Theorem 75.11. This program is carried out in [**Lee2002**, Theorem 16.25]. □

Example. Let M be a compact oriented connected 1-dimensional smooth manifold with non-empty boundary $\partial M = -\{P\} \cup \{Q\}$ and let $f\colon M \to \mathbb{R}$ be a smooth function. We denote by $i\colon \partial M \to M$ the inclusion.



We see that

$$\int_M df \;=\; \int_{\partial M} i^*f \;=\; \int_{-\{P\}\cup\{Q\}} i^*f \;=\; f(Q) - f(P).$$

by Stokes' Theorem 75.11 — see page 1532

This shows that in the case $k = 1$ the above Stokes' Theorem 75.11 basically agrees with our earlier Total Differential-Integration Proposition 71.19. □

The remainder of this chapter is organized as follows:

(1) In the remainder of this section we discuss a few special cases and consequences of Stokes' Theorem 75.11.
(2) In Section 75.4 we will provide a proof of Stokes' Theorem 75.11.
(3) In Appendix 75.5 we will use Stokes' Theorem 75.11 to provide a proof of the Cauchy Theorem 39.8 in complex analysis.

Before we do anything else, it is worth stating the following important special case of Stokes' Theorem 75.11:

Corollary 75.12. (Stokes-on-Closed Corollary) Let M be an oriented k-dimensional smooth manifold and let η be a smooth k-form on M. If M is closed and if η is exact, then

$$\int_M \eta \;=\; 0.$$

Proof of Corollary 75.12. Let M be a *closed* oriented k-dimensional smooth manifold and let η be an *exact* smooth k-form on M. By definition the latter condition means that there exists a smooth $(k-1)$-form ω on M with $\eta = d\omega$. Since M is in particular compact we see that the support of ω is compact. Now we see that

$$\int_M \eta \;=\; \int_M d\omega \;=\; \int_{\partial M} i^*(\omega) \;=\; 0$$

by Stokes' Theorem 75.11 — since $\partial M = \varnothing$

■

Example. We denote by dx the usual 1-form on $\mathbb{R}$, which at every $P \in \mathbb{R}$ is given by

$$\begin{array}{rccc} dx\colon \mathrm{T}_P\mathbb{R} = \mathbb{R} & \to & \mathbb{R} \\ v & \mapsto & v. \end{array}$$

Note that this 1-form is exact, since it is the differential of the identity function $x \mapsto x$. In particular dx is closed. Next we consider the usual $\mathbb{Z}$-action on $\mathbb{R}$ and we denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the projection. As we saw on page 1541, the 1-form dx is invariant under the additive action of $\mathbb{Z}$ on $\mathbb{R}$. Thus it follows from the G-Invariant Form Lemma 74.22 (1) that it induces a 1-form $\overline{dx}$ on the smooth manifold $\mathbb{R}/\mathbb{Z}$, i.e. there exists a form $\overline{dx}$ on $\mathbb{R}/\mathbb{Z}$ with $dx = p^*(\overline{dx})$. Since p is a local diffeomorphism we obtain from the Pullback-Closed-Exact Lemma 75.10 (3) that $\overline{dx}$ is closed. On page 1541 we also saw that

$$\int_{\mathbb{R}/\mathbb{Z}} \overline{dx} \;=\; 1.$$

Note that this calculation, together with the observation that $\mathbb{R}/\mathbb{Z}$ is a closed smooth manifold and the Stokes-on-Closed Corollary 75.12, shows that the 1-form $\overline{dx}$ on $\mathbb{R}/\mathbb{Z}$ is *not* exact. This shows again that the analog of the Pullback-Closed-Exact Lemma 75.10 for "exact" does not hold. □

We conclude this discussion of Stokes' Theorem 75.11 with the following pretty proposition:

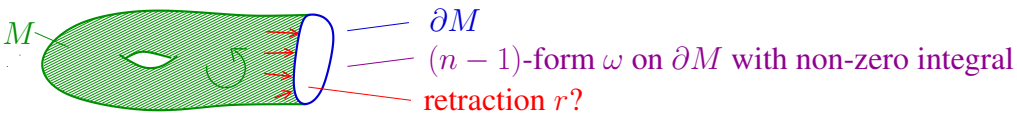
Proposition 75.13. (Boundary Non-Retraction Proposition) Let M be a *compact*[789] orientable non-empty smooth manifold. There is no retraction from M to ∂M, i.e. there is no continuous map $r\colon M \to \partial M$ such that $r(x) = x$ for all $x \in \partial M$.

Remark.

(1) In the Boundary Non-Retraction Proposition 75.13 we used very different techniques to prove a slightly stronger version of the above statement, where we do not need to assume that M is orientable. Also note that in the Kernel-of-Boundary Inclusion Corollary 185.20 and the Topological Fundamental Class-of-Boundary Corollary 194.14 we will give more proofs of the Boundary Non-Retraction Proposition 75.13 and we will extend the statement to topological manifolds.

(2) As we explain in Section 38.4, the Boundary Non-Retraction Proposition 75.13 has many applications. In particular it allows us to prove the Brouwer Fixed Point Theorem 38.7. □

Proof. Let M be a compact orientable non-empty n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion. We now suppose that there exists a retraction $M \to \partial M$. Since M is non-empty this implies that $\partial M \neq \varnothing$. It follows from the Replace-by-Smooth Retraction 36.4 (1) that there exists in fact a smooth retraction $r\colon M \to \partial M$. Recall that this means that r is a smooth map with $r \circ i = \mathrm{id}_{\partial M}$.



[789] The adjective "compact" is of course necessary, for example the smooth manifold $M = [0, \infty)$ admits a retraction onto its boundary $\partial M = \{0\}$.

Since M is orientable we can pick an orientation for M. Furthermore, since ∂M is non-empty it follows from the Non-Zero Integral Existence Lemma 74.16 that there exists a smooth $(n-1)$-form on ∂M with $\int_{\partial M} \omega \neq 0$. We now see that

$$0 \neq \int_{\partial M} \omega = \int_{\partial M} i^*(r^*(\omega)) = \int_M d(r^*(\omega)) = \int_M r^*(d\omega) = \int_M 0 = 0.$$

- $\int_{\partial M} \omega = \int_{\partial M} i^*(r^*(\omega))$: from $r \circ i = \mathrm{id}_{\partial M}$ it follows that $i^* \circ r^* = \mathrm{id}_{\Omega_{n-1}(M)}$
- $\int_{\partial M} i^*(r^*(\omega)) = \int_M d(r^*(\omega))$: by Stokes' Theorem 75.11
- $\int_M d(r^*(\omega)) = \int_M r^*(d\omega)$: follows from the Differential-Natural Proposition 75.8
- $\int_M r^*(d\omega) = \int_M 0$: the Alternating Criterion 72.3 implies that $\Omega_n(\partial M) = 0$, which implies that $d\omega = 0$

We have thus obtained a contradiction. ∎

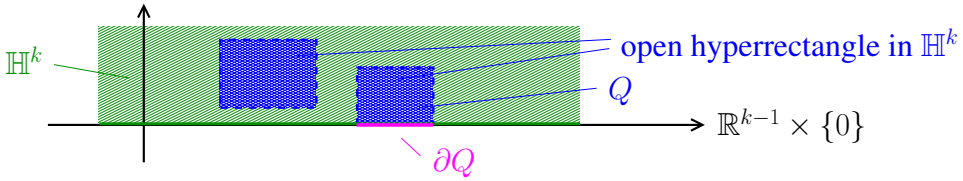
75.4. **Proof of Stokes' Theorem 75.11.** In this section we provide a proof of Stokes' Theorem 75.11. The plan is to first consider the very specific setting of a "open hyperrectangle", where we can do an explicit calculation. Afterwards we will prove the general case using charts and using a partition of unity.

First we need to define what we mean by an open hyperrectangle:

Definition 75.14.

(1) An open hyperrectangle in $\mathbb{R}^k$ is a subset of $\mathbb{R}^k$ of the form $(a_1, b_1) \times \cdots \times (a_k, b_k)$ where $-\infty < a_i < b_i < \infty$.

(2) A subset Q of the upper half-space $\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}$ is called an open hyperrectangle in $\mathbb{H}^k$ if Q is the intersection of an open hyperrectangle in $\mathbb{R}^k$ with $\mathbb{H}^k$. Note that it follows easily from the definitions that Q is a smooth manifold with $\partial Q = Q \cap (\mathbb{R}^{k-1} \times \{0\})$.



The next lemma is a special case of Stokes' Theorem 75.11. As we will see soon, we will reduce the general case to this special case:

Lemma 75.15. (Hyperrectangle–Stokes Lemma) Let Q be an open hyperrectangle in $\mathbb{H}^k$. As usual we denote by $i\colon \partial Q \to Q$ the inclusion. For any smooth $(k-1)$-form ω on Q with compact support in Q we have

$$\int_Q d\omega = \int_{\partial Q} i^*(\omega).$$

Proof of Lemma 75.15. Let Q be an open hyperrectangle in $\mathbb{H}^k$ and let ω be a smooth $(k-1)$-form on Q with compact support. We extend ω to a $(k-1)$-form $\widetilde{\omega}$ on $\mathbb{H}^k$ by assigning to every point outside of Q the zero $(k-1)$-form and by assigning to every point in Q the given $(k-1)$-form ω. Note that it follows from the Extension-of-Form–Smooth Lemma 74.11 that this new $(k-1)$-form $\widetilde{\omega}$ is smooth.

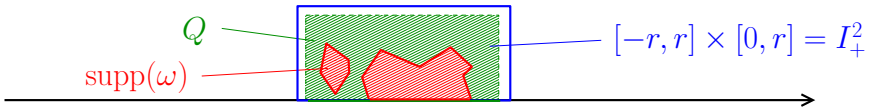
By the Alternating Forms-Basis Proposition 72.7 we can write

$$\widetilde{\omega} \;=\; \sum_{i=1}^{k} f_i \cdot dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k$$

where $f_1, \ldots, f_k \colon \mathbb{H}^k \to \mathbb{R}$ are smooth functions. It is worth recalling that we are supposed to prove the following claim:

Claim. $$\int\limits_Q d\omega \;=\; \int\limits_{\partial Q} i^*(\omega).$$

Proof. Our goal is to calculate the two sides of the claim separately and to compare the results. To get to these calculations we need to make one more preparation: Since hyperrectangles are bounded we can pick an $r > 0$ such that $Q \subseteq (-r,r)^{n-1} \times [0,r)$. We set $I := [-r,r]$ and given $k \in \mathbb{N}_0$ we write $I^k_+ := I^{k-1} \times [0,r]$.



Let us first calculate the left hand side. To do so we need to calculate the differential of $d\omega$. So let us do so:

$$d\widetilde{\omega} \;\overset{\text{see Definition 75.1}}{\downarrow}\; = \sum_{i=1}^{k} df_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k \;\overset{\text{by the 1-Form-via-}dx^i \text{ Lemma 71.5}}{\downarrow}\; = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial f_i}{\partial x_j} \cdot dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i} \cdot dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^k.$$

$\uparrow$ follows from $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$

Note that we see in particular that for each $x \in \mathbb{H}^k$ we have

$$(d\widetilde{\omega})_x(\mathfrak{e}_1, \ldots, \mathfrak{e}_k) \;\overset{\text{by the above calculation}}{\uparrow}\; = \sum_{i=1}^{k} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x) \cdot \underbrace{(dx^1 \wedge \cdots \wedge dx^k)(\mathfrak{e}_1, \ldots, \mathfrak{e}_k)}_{=1} = \sum_{i=1}^{k} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

Now we see that

$$\int\limits_Q d\omega \;\overset{\text{calculated via the identity chart, which is obviously orientation-preserving}}{\downarrow}\; = \int\limits_{x \in Q} (d\omega)_x(\mathfrak{e}_1, .., \mathfrak{e}_k) d\lambda \;\overset{\substack{\text{since } Q \subseteq I^{k-1} \times [0,r], \\ \text{since } \widetilde{\omega} = \omega \text{ on } Q \text{ and} \\ \text{since } \widetilde{\omega} = 0 \text{ outside of } Q}}{\downarrow}\; = \int\limits_{x \in I^{k-1} \times [0,r]} (d\widetilde{\omega})_x(\mathfrak{e}_1, .., \mathfrak{e}_k)\, d\lambda \;\overset{\text{by the above calculation}}{\downarrow}\; = \sum_{i=1}^{k} \int\limits_{x \in I^{k-1} \times [0,r]} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x)\, d\lambda$$

$$= \sum_{i=1}^{k-1} \int\limits_{(a,b) \in I^{i-1} \times I^{k-1-i}_+} (-1)^{i-1} \cdot \underbrace{\int_{x_i=-r}^{x_i=r} \frac{\partial f_i}{\partial x_i}(a, x_i, b)\, dx_i}_{= f_i(a,r,b) - f_i(a,-r,b) = 0,\ \text{by the FTC 73.27 and since } f_i = 0 \text{ outside of } Q} d\lambda + \int\limits_{a \in I^{k-1}} (-1)^{k-1} \cdot \underbrace{\int_{x_k=0}^{x_k=r} \frac{\partial f_k}{\partial x_k}(a, x_k)\, dx_k}_{= f_k(a,r) - f_k(a,0) = -f(a,0),\ \text{by the FTC 73.27 and since } f_k = 0 \text{ outside of } Q} d\lambda$$

$\uparrow$ by the Fubini Theorem 73.25 applied to $I^{i-1} \times [-r,r] \times I^{k-1-t}_+$ for $i = 1, .., k-1$ and to $I^{k-1} \times [0,r]$ for $i = k$

$$= (-1)^k \cdot \int\limits_{a \in I^{k-1}} f(a,0)\, d\lambda.$$

Now we turn to the calculation of the right hand side of the claim. We consider the map

$$\begin{array}{rcl} \Phi\colon \partial Q & \to & V \coloneqq \{x \in \mathbb{R}^{k-1} \mid (x,0) \in Q\} \\ (a,0) & \to & a. \end{array}$$

One can easily verify that this map is a chart for the $(k-1)$-dimensional smooth manifold ∂Q.[790] Furthermore note that it follows from the explicit definition of the boundary orientation in Convention 31.39 that Φ is an orientation-preserving if $(-1)^k$ is even. We calculate

$$\begin{aligned} \int_{\partial Q} i^*(\omega) &\overset{\text{by the above chart } \Phi \text{ and since } \Phi_*^{-1}(\mathfrak{e}_i) = \mathfrak{e}_i}{\downarrow =} (-1)^k \cdot \int_{a \in V} \omega_{(a,0)}(\mathfrak{e}_1, \dots, \mathfrak{e}_{k-1})\, d\lambda \\ &\overset{\text{since } \widetilde{\omega} = \omega \text{ on } Q \text{ and since } \widetilde{\omega} = 0 \text{ outside of } Q}{\downarrow =} (-1)^k \cdot \int_{a \in I^{k-1}} \widetilde{\omega}_{(a,0)}(\mathfrak{e}_1, \dots, \mathfrak{e}_{k-1})\, d\lambda \\ &= (-1)^k \cdot \int_{a \in I^{k-1}} \sum_{i=1}^{k} f_i(a,0) \cdot \underbrace{(dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k)(\mathfrak{e}_1, \dots, \mathfrak{e}_{k-1})}_{\begin{array}{l} = 0, \quad \text{if } i \neq k, \\ = 1, \quad \text{if } i = k \end{array}} d\lambda \\ &= (-1)^k \cdot \int_{a \in I^{k-1}} f_k(a,0)\, d\lambda. \end{aligned}$$

Thus we have shown that the left and the right-hand side of the claim agree. ■

Proof of Stokes' Theorem 75.11. Let M be an oriented k-dimensional smooth manifold and let ω be a smooth $(k-1)$-form on M with compact support. We denote by $i\colon \partial M \to M$ the inclusion. We need to show that

$$\int_M d\omega = \int_{\partial M} i^*(\omega).$$

In the following we will assume that $k \geq 2$. The case $k = 1$ can be proved by a slight variation on the argument, or alternatively, one can argue as on page 1550 that it follows from the Total Differential-Integration Proposition 71.19.

Claim. There exist charts $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,r}$ such that the following statements hold:
(a) For each $i \in \{1, \dots, r\}$ the set V_i is an open hyperrectangle in $\mathbb{H}^k$.
(b) For each $i \in \{1, \dots, r\}$ the map $\Phi_i\colon U_i \to V_i$ is orientation-preserving.
(c) We have $\operatorname{supp}(\omega) \subseteq U_1 \cup \dots \cup U_r$.

Proof. First we show that for every $P \in M$ there exists a chart $\Phi_P\colon U_P \to V_P$ around P with the desired properties (a) and (b). Indeed, let $P \in M \setminus \partial M$. We pick a chart $\Phi_P\colon U_P \to V_P$ around P. After possibly restricting ourselves to a smaller open neighborhood U_P of P and after possibly composing Φ_P with a translation we can suppose that V_P is an open hyperrectangle in $\mathbb{H}^k$. Furthermore, since $k \geq 2$ we can arrange, after possibly composing Φ_P with a reflection in the $(x_1 = 0)$-hyperplane, that Φ_P is orientation-preserving.

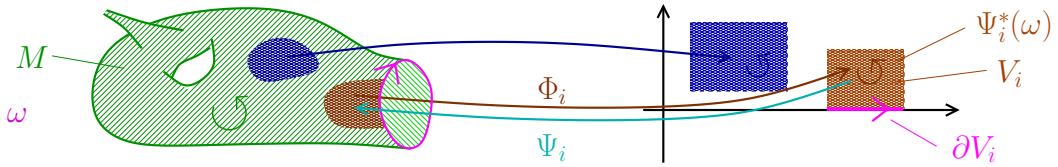
Since $\operatorname{supp}(\omega)$ is compact we can cover it by finitely many of the charts. These then have the desired properties. ⊞

It follows easily from the Manifold–Partitions-of-Unity Theorem 27.2 that there exist smooth functions $\{\mu_i\colon M \to [0,1]\}_{i=1,\dots,r}$ with the following two properties:
(1) For every $x \in \operatorname{supp}(\omega)$ we have $\mu_1(x) + \dots + \mu_r(x) = 1$.

[790] Note that our current discussion of the right-hand side of the claim works if $\partial Q \neq \varnothing$ and also if $\partial Q = \varnothing$.

(2) For every $i \in \{1, \dots, r\}$ we have $\operatorname{supp}(\mu_i) \subseteq U_i$.

From (1) it follows that $\omega = \mu_1 \cdot \omega + \cdots + \mu_r \cdot \omega$. Since both terms of Stokes' Theorem 75.11 are additive it suffices it to prove Stokes' Theorem for each summand $\mu_i \cdot \omega$. In other words, we can suppose that there exists an i such that $\operatorname{supp}(\omega) \subseteq U_i$. We set $\Psi_i := \Phi_i^{-1}$.



Now we perform the following calculation:

$$\int_M d\omega \overset{\text{since } \operatorname{supp}(\omega) \subseteq U_i}{=} \int_{U_i} d\omega = \int_{V_i} \Psi_i^*(d\omega) \overset{\text{by the Differential-Natural Proposition 75.8 we can swap } d \text{ with induced maps}}{=} \int_{V_i} d(\Psi_i^*(\omega))$$

$$= \int_{\partial V_i} \Psi_i^*(\omega) \overset{\text{definition of the integral of a form, see the Integral-of-Form-in-Chart Proposition 74.12}}{=} \int_{U_i \cap \partial M} \omega = \int_{\partial M} \omega.$$

by hypothesis we have $\operatorname{supp}(\omega) \subseteq U_i$, therefore the equality follows from the Hyperrectangle–Stokes Lemma 75.15, (here we use that V_i is an open hyperrectangle)

since $\operatorname{supp}(\omega) \subseteq U_i$

■

75.5. **Appendix: Proof of the Cauchy Theorem 39.8.** In Chapter 39 we already stated the Cauchy Theorem 39.8, which is arguably the foundational result in complex analysis. In this appendix we will use Stokes' Theorem 75.11 to provide a proof of the Cauchy Theorem 39.8. Since we will not really make use of the Cauchy Theorem 39.8 we relegate the proof to this appendix.

First we recall Definition 39.3:

Definition 75.16. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a map.

(1) We say that f is holomorphic in $z_0 \in U$, if the limit

$$\frac{d}{dz} f(z_0) := f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$$

exists. We refer to $\frac{d}{dz} f(z_0) = f'(z_0)$ as the derivative of f at z.

(2) We say that f is holomorphic if f is holomorphic in each $z_0 \in U$.

We continue with recalling Definition 39.6:

Definition 75.17.

(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a continuous map $\gamma\colon [a, b] \to U$.

(2) Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a continuous map and let $\gamma\colon [a,b] \to U$ be a smooth path. We define the path integral of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} \underbrace{f(\gamma(t))}_{\in\mathbb{C}} \cdot \underbrace{\gamma'(t)}_{\in\mathbb{C}}\,dt \;\in \mathbb{C}.$$

product of complex numbers

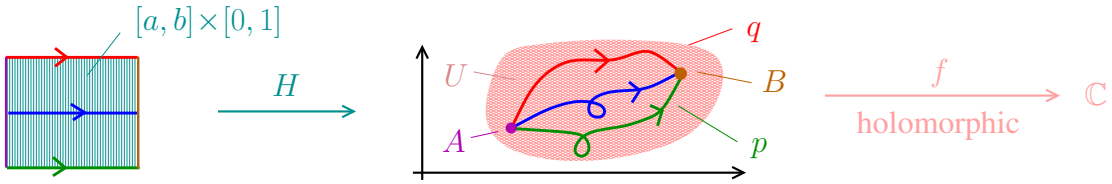
Now we can recall the following theorem, which is one of the most foundational results in complex analysis:

Theorem 39.8. (Cauchy Theorem) Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $p, q\colon [a,b] \to U$ be two smooth paths from $A = p(a) = q(a)$ to $B = p(b) = q(b)$. If p and q are path-homotopic, i.e. if there exists a continuous map $H\colon [a,b] \times [0,1] \to U$ with the following properties:

(a) for every $t \in [a,b]$ we have $H(t,0) = p(t)$ and $H(t,1) = q(s)$,
(b) for every $s \in [0,1]$ we have $H(a,s) = A$ and $H(b,s) = B$,

then

$$\int_p f(z)\,dz \;=\; \int_q f(z)\,dz.$$

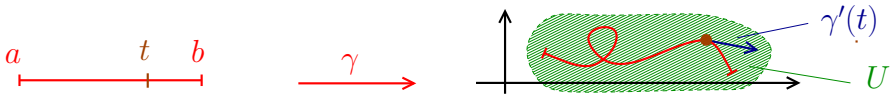


The idea of course is to reduce the Cauchy Theorem 39.8 to Stokes' Theorem 75.11. To do so we need to relate the above path integrals to integrals of 1-forms. This leads us to the following definition, which is a variation on the definition given in the 1-Form Integration Proposition 71.17:

Definition 75.18. Let $U \subseteq \mathbb{R}^n$ be an open subset and let ω be a smooth 1-form on U. Given a smooth path $\gamma\colon [a,b] \to \mathbb{R}^n$ we refer to

$$\int_\gamma \omega \;:=\; \int_{t=a}^{t=b} \omega_{\gamma(t)}(\gamma'(t))\,dt.$$

as the integral of the 1-form ω along γ.[791]



The following lemma relates the path integral of a complex valued function to integrals of 1-forms along paths:

[791] Evidently this integral is the same as $\int_{[a,b]} \gamma^*(\omega)$, as defined in the 1-Form Integration Proposition 71.17.

Lemma 75.19. (Path Integral–Reformulation Lemma) Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a smooth map and let $\gamma\colon [a,b] \to U$ be a smooth path. We write $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Then

$$\underbrace{\int_\gamma f(z)\,dz}_{\substack{\text{path integral, see}\\ \text{Definition 39.6}}} = \underbrace{\int_\gamma (u\cdot dx - v\cdot dy)}_{\substack{\text{integral of 1-form}\\ \text{as defined above}}} + \mathrm{i}\underbrace{\int_\gamma (v\cdot dx + u\cdot dy)}_{\substack{\text{integral of 1-form}\\ \text{as defined above}}}.$$

Proof. We write $\alpha \coloneqq \mathrm{Re}(\gamma)$ and $\beta \coloneqq \mathrm{Im}(\gamma)$. We perform the following calculation:

$$\begin{aligned}
\text{real part of } \int_\gamma f(z)\,dz &= \text{real part of } \int_{t=a}^{t=b} f(\gamma(t))\cdot\gamma'(t)\,dt \\
&= \text{real part of } \int_{t=a}^{t=b} \underbrace{(u(\gamma(t))+\mathrm{i}v(\gamma(t)))}_{=f(\gamma(t))}\cdot\underbrace{(\alpha'(t)+\mathrm{i}\beta'(t))}_{=\gamma'(t)}\,dt \\
&= \int_{t=a}^{t=b} u(\gamma(t))\cdot\underbrace{\alpha'(t)}_{=dx(\gamma'(t))} - v(\gamma(t))\cdot\underbrace{\beta'(t)}_{=dy(\gamma'(t))}\,dt \\
&= \int_{t=a}^{t=b} u(\gamma(t))\cdot dx(\gamma'(t)) - v(\gamma(t))\cdot dy(\gamma'(t))\,dt \underset{\substack{\uparrow\\ \text{by the above definition}}}{=} \int_\gamma (u\cdot dx - v\cdot dy).
\end{aligned}$$

A very similar calculation gives us the promised result for the imaginary part of the path integral. ∎

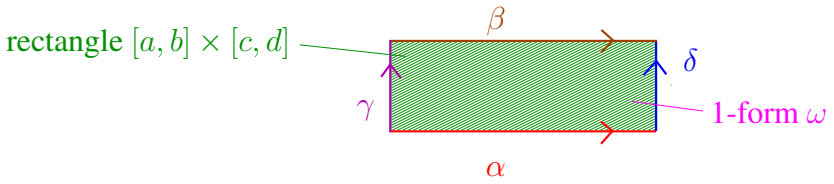
The key to our proof of the Cauchy Theorem 39.8 is the following "Stokes like" result:

Lemma 75.20. (Stokes-for-Rectangles Lemma) Let ω be a smooth 1-form on the rectangle $[a,b]\times[c,d]$. Then[792]

$$\int_{[a,b]\times[c,d]} d\omega = \int_\alpha \omega - \int_\beta \omega - \int_\gamma \omega + \int_\delta \omega$$

where $\alpha, \beta, \gamma, \delta$ are the paths

$$\begin{aligned}
\alpha\colon [a,b] &\to \mathbb{R}^2 & \beta\colon [a,b] &\to \mathbb{R}^2 & \gamma\colon [a,b] &\to \mathbb{R}^2 & \delta\colon [a,b] &\to \mathbb{R}^2\\
t &\mapsto (t,c) & t &\mapsto (t,d) & t &\mapsto (a,t) & t &\mapsto (b,t).
\end{aligned}$$



Proof of Lemma 75.20. The proof of the lemma is basically identical to the proof of the Hyperrectangle–Stokes Lemma 75.15. We will fill in the details in Exercise 75.13. ∎

[792] Strictly speaking we did not define smooth forms on rectangles, but we expect that the reader will have no troubles with making sense of the statement of the lemma.

Proof of the Cauchy Theorem 39.8. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a holomorphic function. We set $u := \operatorname{Re}(f)$ and $v := \operatorname{Im}(f)$. Note that we now have $f = u + \mathrm{i}v\colon U \to \mathbb{C}$. On U we consider the 1-forms

$$\mu \;:=\; u \cdot dx - v \cdot dy \qquad \text{and} \qquad \nu \;:=\; v \cdot dx + u \cdot dy.$$

We start our proof with the following claim:

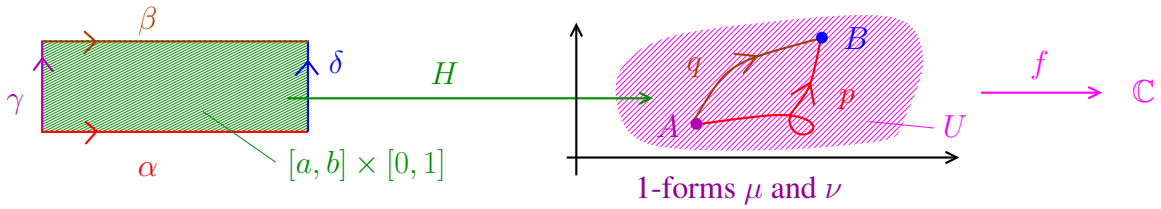
Claim. We have $d\mu = 0$ and $d\nu = 0$.

Proof. First we show that $d\mu = 0$. We perform the following calculation:

$$\begin{aligned} d\mu \;=\; d(u \cdot dx - v \cdot dy) &\overset{\substack{\text{Definition 75.1 of the differential}\\ \downarrow}}{=} \frac{\partial u}{\partial x} \cdot \underbrace{dx \wedge dx}_{=0} + \frac{\partial u}{\partial y} \cdot \underbrace{dy \wedge dx}_{=-dx\wedge dy} - \frac{\partial v}{\partial x} \cdot dx \wedge dy - \frac{\partial v}{\partial y} \cdot \underbrace{dy \wedge dy}_{=0} \\ &\underset{\substack{\uparrow\\ \text{by the 1-Form-via-}dx^i\text{-Lemma 71.5}}}{=} \Big(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \Big) \cdot dx \wedge dy \underset{\substack{\uparrow\\ \text{since } f \text{ is holomorphic we know by the}\\ \text{Smooth-Complex Differentiability}\\ \text{Proposition 39.5 that } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}}{=} 0. \end{aligned}$$

Now we want to show that $d\nu = 0$. By the Smooth-Complex Differentiability Proposition 39.5 we know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. This observation implies, by almost the same calculation as above, that $d\nu = 0$. ⊞

Now let $p, q\colon [a,b] \to U$ be two smooth paths from $A = p(a) = q(a)$ to $B = p(b) = q(b)$. We assume that there exists a path-homotopy $H\colon [a,b] \times [0,1] \to U$ between p and q. Using a slight variation on the Replace-by-Smooth Homotopy Proposition 36.6, see Exercise 47.1, one can show that there exists in fact a smooth path-homotopy H.



Now we perform the following calculation:

$$\begin{aligned} 0 \;=\; \int_{[a,b]\times[0,1]} 0 &\overset{\substack{\text{by the claim we}\\ \text{have } d\mu = 0\\ \downarrow}}{=} \int_{[a,b]\times[0,1]} H^*(d\mu) \overset{\substack{\text{as in the Differential-Natural}\\ \text{Proposition 75.4}\\ \downarrow}}{=} \int_{[a,b]\times[0,1]} d(H^*\mu) \overset{\substack{\text{by the Stokes-for-Rectangles}\\ \text{Lemma 75.20}\\ \downarrow}}{=} \int_\alpha H^*\mu - \int_\beta H^*\mu - \int_\gamma H^*\mu + \int_\delta H^*\mu \\ &= \underbrace{\int_{H\circ\alpha} \mu}_{\substack{\text{note that}\\ H\circ\alpha = p}} - \underbrace{\int_{H\circ\beta} \mu}_{\substack{\text{note that}\\ H\circ\beta = q}} - \underbrace{\int_{H\circ\gamma} \mu}_{\substack{=0 \text{ since } H\circ\gamma\\ \text{is constant}}} + \underbrace{\int_{H\circ\delta} \mu}_{\substack{=0 \text{ since } H\circ\delta\\ \text{is constant}}} = \int_p \mu - \int_q \mu \underset{\substack{\uparrow\\ \text{by the Path Integral–Reformulation}\\ \text{Lemma 75.19}}}{=} \text{real part of } \Big(\int_p f(z)\,dz - \int_q f(z)\,dz \Big). \end{aligned}$$

Of course basically the same argument also shows that the imaginary parts of the path integrals agree. ■

75.6. Exercises for Chapter 75.

Exercise 75.1. Let $U \subseteq \mathbb{R}^n$ be an open set and let $f_1, \dots, f_n \colon U \to \mathbb{R}$ be smooth functions. Show that
$$df_1 \wedge \dots \wedge df_n \;=\; \det\Big(\underbrace{\Big(\frac{\partial f_i}{\partial x_j}\Big)_{i,j=1,\dots,n}}_{=\mathrm{D}(f_1,\dots,f_n)}\Big) \cdot dx^1 \wedge \dots \wedge dx^n.$$

Exercise 75.2. We consider the smooth map
$$\begin{array}{rcl} f \colon \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x^2, xy, 0, y-z). \end{array}$$
We consider the 2-form $\omega = \exp(x) \cdot dy \wedge dz + 2dx \wedge dz + xy \cdot dx \wedge dy$. Compute the pullback $f^*(\omega)$.

Exercise 75.3. Let U be an open subset of $\mathbb{R}^n$.

(a) Show that if $\varphi \in \Omega^k(U)$ is exact and if $\psi \in \Omega^l(U)$ is closed, then $\varphi \wedge \psi$ is exact.
(b) Let $\psi \in \Omega^l(U)$ be a closed form. We consider the form $dx^1 \wedge \psi$. Give an explicit $(l+1)$-form μ with $d\mu = dx^1 \wedge \psi$.

Exercise 75.4. Let $U \subseteq \mathbb{R}^3$ be an open subset and let $G = \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix} \colon U \to \mathbb{R}^3$ be a smooth map. Show that
$$\det(G \star_1 \star_2) \;=\; G_x \cdot (dy \wedge dz) + G_y \cdot (dz \wedge dx) + G_z \cdot (dx \wedge dy).$$
Hint. Show that both forms give the same result for any two vectors $v, w \in \mathbb{R}^3$.

Exercise 75.5. Recall that given $i = 1, \dots, n+1$ we write
$$dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1} \;\coloneqq\; dx^1 \wedge \dots dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^{n+1}.$$
On S^n we consider the n-form
$$\omega \;=\; \sum_{i=1}^{n+1} (-1)^i \cdot x_i \cdot dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}.$$
(a) Determine $\int\limits_{S^n} \omega$.
(b) Show that there is no smooth $(n-1)$-form η on S^n with $d\eta = \omega$.

Exercise 75.6. Does there exist a smooth $(k-1)$-form on $\mathbb{R}^k$ with compact support and with the property that $\int_{\mathbb{R}^k} d\omega \neq 0$?

Exercise 75.7. Let M be an n-dimensional smooth manifold and let $\Phi \colon U \to V$ be a chart. For $i = 1, \dots, n$ we denote by $\pi_i \colon V \to \mathbb{R}$ the projection onto the i-th coordinate and we write $\Phi^i \coloneqq \pi_i \circ \Phi \colon U \to \mathbb{R}$. The smooth functions $\Phi^1, \dots, \Phi^n \colon V \to \mathbb{R}$ are often called *local coordinates*. Furthermore, for a smooth function $f \colon U \to \mathbb{R}$ and $P \in U$ we write
$$\frac{\partial f}{\partial \Phi^i}|_P \;\coloneqq\; \frac{\partial}{\partial x_i}(f \circ \Phi^{-1})|_{\Phi(P)}.$$
Finally let ω be a smooth m-form on V. Show that we can write
$$\omega \;=\; \sum_{i_1 < \dots < i_m} f_{i_1 \dots i_m} \cdot d\Phi^{i_1} \wedge \dots \wedge d\Phi^{i_m},$$

where $f_{i_1 \dots i_m}$ are uniquely defined real-valued smooth functions on V. Furthermore show that

$$d\omega \;:=\; \sum_{i_1<\dots<i_m} \sum_{j=1}^{n} \frac{\partial f_{i_1,\dots,i_m}}{\partial \Phi^j}\, d\Phi^j \wedge d\Phi^{i_1} \wedge \dots \wedge d\Phi^{i_m}.$$

Exercise 75.8. Find an explicit 2-form ω on $\mathbb{R}^3$ with $d\omega = dx \wedge dy \wedge dz$.

Exercise 75.9. Let M be a closed oriented k-dimensional smooth manifold, let ω be a smooth $(k-1)$-form on M and let $f\colon M \to \mathbb{R}$ be a smooth map. Which of the following two statements always holds?

(a) $\int_M f \cdot d\omega = 0$.

(b) $\int_M f \cdot d\omega = -\int_M df \wedge \omega$.

Exercise 75.10.

(a) Let M be a $(k+1)$-dimensional smooth manifold and let ω be a k-form on M. Show that if ω is exact, then for all closed oriented k-dimensional smooth manifolds W and all smooth maps $f\colon W \to M$ we have $\int_W f^*(\omega) = 0$.

(b) On $M = \mathbb{R}^2 \setminus \{(0,0)\}$ we consider the 1-form ω which is given by

$$\omega(x,y) \;=\; \frac{y \cdot dx - x \cdot dy}{x^2+y^2}.$$

(i) Is the 1-form ω closed?

(ii) Is the 1-form ω exact?

Exercise 75.11. Let M be a compact oriented n-dimensional smooth manifold and let $f\colon M \to S^{n-1}$ be a smooth map. Does there exist a smooth $(n-1)$-form on S^{n-1} with $\int_M d(f^*(\omega)) \neq 0$?

Exercise 75.12. Let $\omega = f \cdot dx + g \cdot dy$ be a smooth 1-form on $\mathbb{R}^2$.

(a) Show that if $d\omega = 0$, then $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$.

(b) Show that if $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$, then there exists a smooth 0-form φ on $\mathbb{R}^2$ with $d\varphi = \omega$.

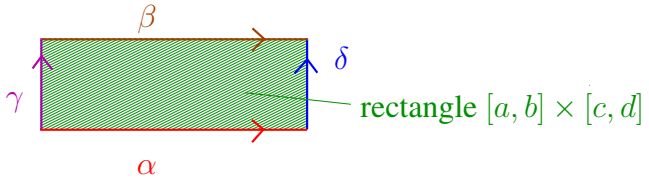
Exercise 75.13. Let ω be a smooth 1-form on $[a,b] \times [c,d]$. Show that

$$\int_{[a,b]\times[c,d]} d\omega \;=\; \int_\alpha \omega - \int_\beta \omega - \int_\gamma \omega + \int_\delta \omega$$

where $\alpha, \beta, \gamma, \delta$ are the paths

$$\begin{array}{llll} \alpha\colon [a,b] \to \mathbb{R}^2 & \beta\colon [a,b] \to \mathbb{R}^2 & \gamma\colon [a,b] \to \mathbb{R}^2 & \delta\colon [a,b] \to \mathbb{R}^2 \\ \quad t \mapsto (t,c) & \quad t \mapsto (t,d) & \quad t \mapsto (a,t) & \quad t \mapsto (b,t). \end{array}$$

Hint. Modify the proof of the Hyperrectangle–Stokes Lemma 75.15.



Exercise 75.14. Let M be a closed orientable non-empty n-dimensional smooth manifold.
(a) Show that there exists a smooth n-form on M that is not exact.
(b) In (a), can we drop the hypothesis that M is closed?
(c) In (b), can we drop the hypothesis that M is orientable?

Exercise 75.15. Let M be a closed oriented n-dimensional smooth manifold, let N be a smooth manifold and let $F\colon M \times [0,1] \to N$ be a smooth homotopy. Show that for any smooth n-form α on N we have

$$\int_M F_0^*(\alpha) \;=\; \int_M F_1^*(\alpha).$$

Exercise 75.16. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a holomorphic function. We say that a path $\gamma\colon [a,b] \to U$ is *piecewise smooth* if there exist $a = s_0 < s_1 < \cdots < s_k = b$ such that each $\gamma|_{[t_i,t_{i+1}]}$ is smooth. We define the corresponding *path integral* of f along γ as follows:

$$\int_\gamma f(z)\,dz \;\coloneqq\; \sum_{i=0}^{k-1} \int_{t=s_i}^{t=s_{i+1}} f(\gamma(t)) \cdot \gamma'(t)\,dt \;\in \mathbb{C}.$$

Show that if $\gamma, \delta\colon [a,b] \to U$ are two piecewise smooth paths that are path-homotopic, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

Hint. Modify the proof of the Cauchy Theorem 39.8.

CHAPTER 76

Classical versions of Stokes' Theorem 75.11

In the last chapter we proved Stokes' Theorem 75.11, which has an extremely elegant formulation. But it is perhaps initially not entirely clear, what Stokes' Theorem tries to tell us. In this chapter we will formulate three classical versions of Stokes' Theorem 75.11, namely:

(1) Green's Theorem 76.1.
(2) The Kelvin-Stokes Theorem 76.3.
(3) The Gauß Divergence Theorem 76.26.

We will not make use of any these statements later on. Thus this chapter can be viewed as a highlight of these notes, or as a complete distraction, best to be avoided.

Throughout this chapter we will only work with smooth submanifolds of some $\mathbb{R}^n$. Note that by the Submanifold Smooth Structure Lemma 25.23 we can view every k-dimensional smooth submanifold of $\mathbb{R}^n$ as a k-dimensional smooth manifold in its own right. When in the following we talk of a "chart" of a smooth submanifold M of $\mathbb{R}^n$ we actually mean "chart" of M viewed as a smooth manifold in its own right.

As we will see, Green's Theorem 76.1 and the Kelvin-Stokes Theorem 75.11 are straightforward to state with our present language. The discussion of the Gauß Divergence Theorem 76.26 will require a major, but interesting, detour. In particular we will need to introduce the integral of a continuous real-valued function on a compact smooth submanifold of some $\mathbb{R}^n$.

This chapter is organized as follows:

- In Section 76.1 we will state and prove Green's Theorem 76.1.
- In Section 76.2 we will state and prove the Kelvin-Stokes Theorem 76.3.
- In Section 76.3 we will state and prove the Gauß Divergence Theorem 76.26.
- In Section 76.4 we will provide exercises for Chapter 76.

In the next Chapter 77 we will consider the purely algebraic notion of cochain complexes over a field.

76.1. **Green's Theorem.** The most basic non-trivial application of Stokes' Theorem 75.11 is Green's Theorem, which has its origins in the work of George Green [**Green1828**]:

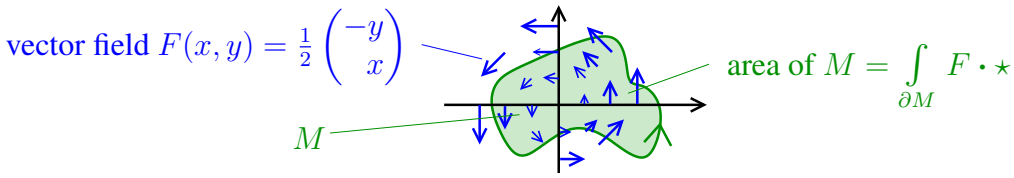
Theorem 76.1. (Green's Theorem) Let $M \subseteq \mathbb{R}^2$ be a compact 2-dimensional smooth manifold and let $u, v \colon M \to \mathbb{R}$ be smooth functions. Then[793 794]

$$\int_{\partial M} \underbrace{(v \cdot dx + u \cdot dy)}_{\text{smooth 1-form on } \partial M} = \int_M \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\substack{\text{real-valued} \\ \text{function on } M}} d\lambda^2.$$

Example. Let $M \subseteq \mathbb{R}^2$ be a compact 2-dimensional smooth manifold. We consider the vector field $F(x,y) = \frac{1}{2} \cdot \begin{pmatrix} -y \\ x \end{pmatrix}$. As on page 1487 we consider the 1-form $F \bullet \star$ which assigns to $P \in M$ the covector $v \mapsto F(P) \bullet v$. Then

$$\int_{\partial M} F \bullet \star \underset{\uparrow}{=} \int_{\partial M} -\tfrac{1}{2}y \cdot dx + \tfrac{1}{2}x \cdot dy \underset{\uparrow}{=} \int_M \frac{\partial}{\partial x}\left(\tfrac{1}{2}x\right) - \frac{\partial}{\partial y}\left(-\tfrac{1}{2}y\right) d\lambda^2 = \int_M 1 \, d\lambda^2 = \text{area of } M.$$

see page 1488 — by Green's Theorem 76.1



Thus we can calculate the area of M by an integral of a 1-form along the boundary of M. In fact this principle is implemented by a planimeter, which in real life can be used to determine the area of such M. □

Proof. Let $M \subseteq \mathbb{R}^2$ be a compact 2-dimensional smooth manifold. We equip M with the standard orientation. Furthermore let $u, v \colon M \to \mathbb{R}$ be smooth functions. We perform the following calculation:

$$\begin{aligned}
\int_{\partial M} v \cdot dx + u \cdot dy &\underset{\downarrow}{=} \int_M d(v \cdot dx + u \cdot dy) \\
&= \int_M \frac{\partial v}{\partial x} \cdot \underbrace{dx \wedge dx}_{=0} + \frac{\partial v}{\partial y} \cdot \underbrace{dy \wedge dx}_{=-dx \wedge dy} + \frac{\partial u}{\partial x} \cdot dx \wedge dy + \frac{\partial u}{\partial y} \cdot \underbrace{dy \wedge dy}_{=0} \\
&\underset{\uparrow}{=} \int_M \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \cdot dx \wedge dy \underset{\uparrow}{=} \int_M \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right).
\end{aligned}$$

by Stokes' Theorem 75.11

by Definition 75.1 of the differential

by the 1-Form-via-dx^i-Lemma 71.5

by the Integral-n-Form-on-$\mathbb{R}^n$ Lemma 74.20

■

76.2. **The Kelvin-Stokes Theorem.** In this section we formulate and prove the Kelvin-Stokes' Theorem which plays a huge role in physics.

[793] We refer to Definition 71.18 for the definition of the left-hand side.

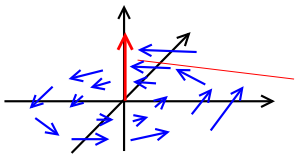
[794] As always we equip M with the orientation coming from the standard orientation of $\mathbb{R}^2$ and we equip ∂M with the corresponding boundary orientation.

Definition 76.2. Let $U \subseteq \mathbb{R}^3$ be an open subset and let $F = (F_x, F_y, F_z)$ be a smooth vector field on U. The curl of F is defined as the vector field on U that is given by

$$\operatorname{curl} F := \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}}_{\substack{\text{suggestive notation} \\ \text{using the cross product on } \mathbb{R}^3 \\ \text{from Definition 31.25}}} := \begin{pmatrix} \frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y \\ -\frac{\partial}{\partial x} F_z + \frac{\partial}{\partial z} F_x \\ \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \end{pmatrix}.$$

Example. Given $F\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ we have $\operatorname{curl} F\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

Vektorfeld $F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$



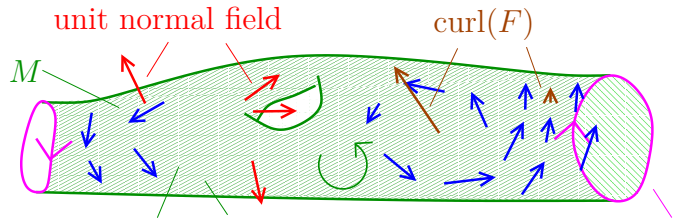
$\operatorname{curl} F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

This calculation explains perhaps the intuition, that at any point in $\mathbb{R}^3$ the vector $\operatorname{curl} F$ indicates along which "axis the vector field is turning". □

The classical Kelvin-Stokes Theorem can now be formulated as follows:

Theorem 76.3. (Kelvin-Stokes Theorem) Let $M \subseteq \mathbb{R}^3$ be a compact oriented 2-dimensional smooth submanifold, let U be an open neighborhood of $M \subseteq \mathbb{R}^3$ and let $F\colon U \to \mathbb{R}^3$ be a smooth vector field on U. Then

$$\int_M \underbrace{\det(\operatorname{curl} F \ \star_1 \ \star_2)}_{\substack{\text{the 2-form that is given by} \\ (v_1, v_2) \mapsto \det(\operatorname{curl}(F)\, v_1\, v_2)}} = \int_{\partial M} \underbrace{F \cdot \star_{\cdot}}_{\substack{\text{the 1-form that} \\ \text{is given by} \\ v \mapsto F(P) \cdot v}}$$



the Kelvin-Stokes' Theorem says that $\int_M \det(\operatorname{curl}(F) \ \star_1 \ \star_2) = \int_{\partial M} F \cdot \star$

the integral $\int_M \det(\operatorname{curl}(F) \ \star_1 \ \star_2)$ is positive since the scalar product of curl and the normal field is > 0

the integral $\int_{\partial M}(F \cdot \star)$ is positive, since the scalar product of the vector field and the orientation is > 0

Remark. According to [**Katz1979**] this theorem was first mentioned in letters by Lord Kelvin and George Stokes in the 1850s. Seemingly the first time it appeared in print was as an exam problem at Cambridge. □

Proof. We will prove the classical Kelvin-Stokes' Theorem 76.3 using the general Stokes' Theorem 75.11. Thus let $M \subseteq \mathbb{R}^3$ be a compact oriented 2-dimensional smooth submanifold

and let $F = (F_x, F_y, F_z)$ be a smooth vector field on some open neighborhood U of $M \subseteq \mathbb{R}^3$. Recall that Stokes' Theorem 75.11 says:

$$\int_M d(F \bullet \star) \;=\; \int_{\partial M} F \bullet \star.$$

The classical theorem of Kelvin-Stokes now follows from the following claim:

Claim. The differential $d(F \bullet \star)$ of the 1-form $F \bullet \star$ is the 2-form $\det(\operatorname{curl}(F) \star_1 \star_2)$.

Proof. Since is F defined on U we can view $F \bullet \star$ as a 1-form on U and we might as well prove the desired equality on U. Recall that on page 1488 we saw that

$$F \bullet \star \;=\; F_x \cdot dx + F_y \cdot dy + F_z \cdot dz.$$

Next we calculate

$$\begin{aligned} d(F \bullet \star) &\overset{\text{by the above}}{=} d(F_x \cdot dx + F_y \cdot dy + F_z \cdot dz) \overset{\text{Definition 75.1 of of the differential}}{=} dF_x \wedge dx + dF_y \wedge dy + dF_z \wedge dz \overset{\text{by the 1-Form-via-}dx^i\text{-Lemma 71.5 and since } dx^i \wedge dx^i = 0}{=} \\ &= \left(\frac{\partial F_x}{\partial y} dy + \frac{\partial F_x}{\partial z} dz\right) \wedge dx + \left(\frac{\partial F_y}{\partial x} dx + \frac{\partial F_y}{\partial z} dz\right) \wedge dy + \left(\frac{\partial F_z}{\partial x} dx + \frac{\partial F_z}{\partial y} dy\right) \wedge dz \\ &\overset{\text{since } dx^i \wedge dx^j = -dx^j \wedge dx^i \text{ and since } dx^i \wedge dx^i = 0}{=} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) dx \wedge dz + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) dy \wedge dz. \end{aligned}$$

We want to show that $d(F \bullet \star) = \det(\operatorname{curl}(F) \star_1 \star_2)$ as 2-forms on U. We prove this equality by showing that both sides give the same results when applied to two vectors in $\mathbb{R}^3$. Thus let $v_1 = (v_{1x}, v_{1y}, v_{1z})$ and $v_2 = (v_{2x}, v_{2y}, v_{2z})$ be two vectors in $\mathbb{R}^3$. Using the above calculation we see that

$$\begin{aligned} d(F \bullet \star)\left(\begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix}, \begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix}\right) &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)(v_{1x}v_{2y} - v_{1y}v_{2x}) \\ &\quad + \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)(v_{1x}v_{2z} - v_{1z}v_{2x}) \\ &\quad + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)(v_{1y}v_{2z} - v_{1z}v_{2y}) \\ &= \det\begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} & v_{1x} & v_{2x} \\ -\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z} & v_{1y} & v_{2y} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} & v_{1z} & v_{2z} \end{pmatrix} = \underbrace{\det\big(\operatorname{curl} F(P)\; v_1\; v_2\big)}_{=\det(\operatorname{curl}(F) \star_1 \star_2)(v_1, v_2)}. \end{aligned}$$

■

76.3. **The Gauß Divergence Theorem.** In this last section of the chapter we will state and prove a third classical consequence of Stokes' Theorem 75.11, namely the Gauß Divergence Theorem 76.26. Unfortunately even just stating the theorem requires quite some preparations. This section is organized as follows:

- In Subsection 76.3.1 we will introduce the purely linear algebraic concept of a Gram determinant of any real matrix.
- In Subsection 76.3.2 we will recall the cross product of $n-1$ vectors in $\mathbb{R}^n$ and we use the Gram determinant to give a geometric interpretation of the cross product.

- In Subsection 76.3.3 we will use the Gram determinant to define the integral of a continuous function on a submanifold of $\mathbb{R}^n$. For codimension-one submanifolds we can rewrite this integral in terms of cross products.
- In Subsection 76.3.4 we will introduce the flux of a vector field through an oriented codimension-one submanifold of $\mathbb{R}^n$.
- In Subsection 76.3.5 we will state and prove the Gauß Divergence Theorem 76.26 which says that for a compact codimension-zero submanifold M of $\mathbb{R}^n$ and a smooth vector field F on M the integral of the "divergence of F" over M equals the flux of F through the boundary of M.

76.3.1. **The Gram determinant.** To state the Gauß Divergence Theorem we will need to introduce the notion of the integral of a continuous function on a compact submanifold of some $\mathbb{R}^n$. To get to this point we need to introduce the following, initially rather unintuitive, definition:

Definition 76.4. Given $V \in \mathrm{M}(n \times k, \mathbb{R})$ we define its Gram determinant as follows:[795]

$$\mathrm{gram}(V) \;:=\; \det(\underbrace{V^{\mathrm{T}} \cdot V}_{(k \times k)\text{-matrix}}).$$

Example. Let $W \in \mathrm{M}(k \times k, \mathbb{R})$. For $n \geq k$ we consider the $(n \times k)$-matrix $V = \begin{pmatrix} 0 \\ W \end{pmatrix}$. Then

$$\begin{aligned} \mathrm{gram}(V) \;=\; \det(V^{\mathrm{T}} \cdot V) \;=\; \det\left(\begin{pmatrix} 0 & W^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 0 \\ W \end{pmatrix} \right) \;&=\; \det(W^{\mathrm{T}} \cdot W) \\ &=\; \det(W^{\mathrm{T}}) \cdot \det(W) \;=\; \det(W)^2. \end{aligned}$$

$\uparrow$ since W is a square matrix

□

In a second we will give a geometric interpretation of the Gram determinant. The key to doing so will be the following, technical looking lemma:

Lemma 76.5. (Orthogonal Matrix–Gram Determinant Lemma) Given any matrix $V \in \mathrm{M}(n \times k, \mathbb{R})$ and given any $A \in \mathrm{O}(n)$ we have

$$\mathrm{gram}(A \cdot V) \;=\; \mathrm{gram}(V).$$

Proof. The lemma follows from the following little calculation:

$$\mathrm{gram}(A \cdot V) \;=\; \det((A \cdot V)^{\mathrm{T}} \cdot (A \cdot V)) \;=\; \det(V^{\mathrm{T}} \cdot (\underbrace{A^{\mathrm{T}} \cdot A}_{=\,\mathrm{id},\ \text{since } A \in \mathrm{O}(n)}) \cdot V) \;=\; \det(V^{\mathrm{T}} \cdot V) \;=\; \mathrm{gram}(V).$$

■

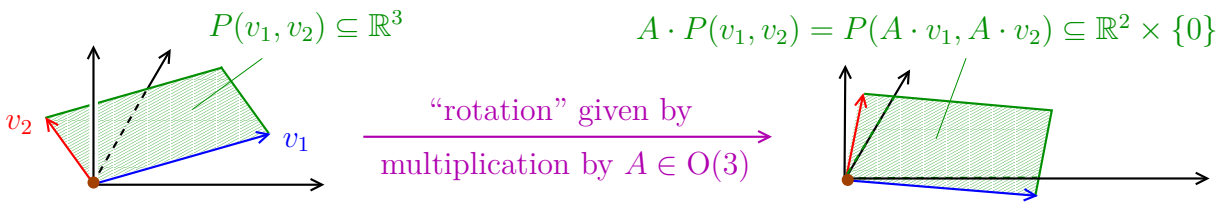
To state the main example for the Gram determinant we need the following notation:

Notation 76.6. Given $u_1, \ldots, u_k \in \mathbb{R}^n$ we consider the corresponding parallelotope

$$P(u_1, \ldots, u_k) \;=\; \left\{ \sum_{i=1}^{k} \lambda_i \cdot u_i \;\middle|\; \lambda_1, \ldots, \lambda_k \in [0,1] \right\} \qquad \subseteq \mathbb{R}^n.$$

Example. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. We denote by $V = (v_1 \ldots v_k)$ the $n \times k$-matrix whose columns are $v_1, \ldots, v_k$. By the Orthogonal Action Lemma 2.37 there exists an orthogonal matrix $A \in \mathrm{O}(n)$ such that $A \cdot v_1, \ldots, A \cdot v_k \in \mathbb{R}^k \times \{0\}$. In other words, there exist $w_1, \ldots, w_k \in \mathbb{R}^k$ such that $A \cdot v_i = \begin{pmatrix} 0 \\ w_i \end{pmatrix}$ for $i = 1, \ldots, k$.

[795] Note that V^{T} is an $(k \times n)$-matrix, thus $V^{\mathrm{T}} \cdot V$ is a $(k \times k)$-matrix.



We see that

$$\begin{array}{ccccccc} \mathrm{gram}(v_1 \dots v_k) & = & \mathrm{gram}(V) & \overset{\text{by the Orthogonal Matrix–Gram Determinant Lemma 76.5}}{=} & \mathrm{gram}(A \cdot V) & = & \mathrm{gram}\left(\begin{pmatrix} 0 \\ w_1 \end{pmatrix} \dots \begin{pmatrix} 0 \\ w_k \end{pmatrix}\right) \\ & & & \overset{\text{by the above example}}{=} & \det(w_1 \ \dots \ w_k)^2 & \overset{\text{see page 1516}}{=} & \mathrm{vol}(P(w_1, \dots, w_k))^2. \end{array}$$

Thus we see that $\sqrt{\mathrm{gram}(v_1 \dots v_k)}$ can be viewed as the "k-dimensional volume of the k-dimensional parallelotope $P(v_1, \dots, v_k) \subseteq \mathbb{R}^n$." □

76.3.2. **The cross product.** We start out by recalling Definition 31.25:

Definition 76.7. Let $v_1, \dots, v_n \in \mathbb{R}^{n+1}$. We define the cross product

$$v_1 \times \cdots \times v_n \ := \ \sum_{k=1}^{n+1} (-1)^{k+1} \cdot \det \left(\begin{array}{c} (n \times n)\text{-matrix obtained from} \\ \text{the } (n+1) \times n\text{-matrix } (v_1 \ \dots \ v_n) \\ \text{by crossing out the } k\text{-th row} \end{array} \right) \cdot \underbrace{\mathfrak{e}_k}_{\in \mathbb{R}^{n+1}} \ \in \mathbb{R}^{n+1}.$$

Using the gram determinant we can now provide a geometric interpretation of the cross product:

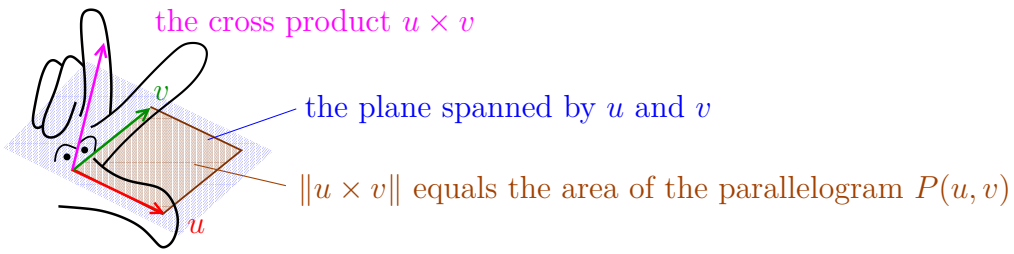
Proposition 76.8. (Cross Product Characterization Proposition) Suppose we are given $v_1, \dots, v_n \in \mathbb{R}^{n+1}$. The cross product $v_1 \times \cdots \times v_n \in \mathbb{R}^{n+1}$ has the following properties:

(a) $\det \Big(\underbrace{v_1 \times \cdots \times v_n \ \ v_1 \ \dots \ v_n}_{(n+1) \times (n+1)\text{-matrix}} \Big) \geq 0.$

(b) $v_1 \times \cdots \times v_n$ is orthogonal to $v_1, \dots, v_n$.

(c) $\|v_1 \times \cdots \times v_n\| = \sqrt{\mathrm{gram}(v_1, \dots, v_n)}$.

Furthermore the cross product is the only vector in $\mathbb{R}^{n+1}$ that satisfies (a), (b) and (c).



The proof of the Cross Product Characterization Proposition 76.8 relies on the following two lemmas.

Lemma 76.9. (Scalar–Cross Product Lemma) Let $v_1, \dots, v_n \in \mathbb{R}^{n+1}$. Given any $u \in \mathbb{R}^{n+1}$ we have

$$\det(u\ v_1\ \dots\ v_n) \ = \ \underbrace{\langle u, v_1 \times \cdots \times v_n \rangle}_{\text{standard inner product}}.$$

Proof of Lemma 76.9. The equality follows almost immediately from the Leibniz expansion of the determinant of $(u\ v_1\ \dots\ v_n)$ by the first row. ∎

Lemma 76.10. (Cross Product–Matrix Lemma) Let $v_1, \dots, v_n \in \mathbb{R}^{n+1}$. Given any $A \in \mathrm{O}(n+1)$ we have

$$Av_1 \times \cdots \times Av_n \ = \ \det(A) \cdot A(v_1 \times \cdots \times v_n) \qquad \in \mathbb{R}^{n+1}.$$

Proof of Lemma 76.10. We start out with the following easy observation:

(∗) Let $a, b \in \mathbb{R}^{n+1}$. If for every $i \in \{1, \dots, n+1\}$ we have $\langle a, \mathfrak{e}_i \rangle = \langle b, \mathfrak{e}_i \rangle$, then $a = b$.

Now we turn to the proof of the actual statement. To simplify the notation we deal with the case $n = 2$. The general case is proved the same way. Thus let $u, v \in \mathbb{R}^3$ be two vectors and let $A \in \mathrm{O}(3)$. By (∗) it suffices to show that for $i = 1, 2, 3$ we have

$$\langle \mathfrak{e}_i, Au \times Av \rangle \ = \ \langle \mathfrak{e}_i, \det(A) \cdot A(u \times v) \rangle.$$

Thus let $i \in \{1, 2, 3\}$. We calculate that

$$\begin{array}{lll}
\langle \mathfrak{e}_i, Au \times Av \rangle & \overset{\text{by the Scalar–Cross Product Lemma 76.9}}{\downarrow} = \det(AA^{-1}\mathfrak{e}_i\ Au\ Av) & \overset{\text{by the definition of matrix multiplication}}{\downarrow} = \det(A \cdot (A^{-1}\mathfrak{e}_i\ u\ v)) \\
\overset{\text{by the Scalar–Cross Product Lemma 76.9}}{\uparrow} = \det(A) \cdot \langle A^{-1}\mathfrak{e}_i, u \times v \rangle & \overset{\text{since } A \text{ is orthogonal we have } \langle Ax, Ay \rangle = \langle x, y \rangle}{\uparrow} = \det(A) \cdot \langle \mathfrak{e}_i, A(u \times v) \rangle & \overset{\text{multiplicativity of the determinant}}{\downarrow} = \det(A) \cdot \det(A^{-1}\mathfrak{e}_i\ u\ v) \\
 & & = \langle \mathfrak{e}_i, \det(A) \cdot A(u \times v) \rangle.
\end{array}$$

∎

Proof of the Cross Product Characterization Proposition 76.8. Let $v_1, \dots, v_n$ be vectors in $\mathbb{R}^{n+1}$. First we show that the cross product $v_1 \times \cdots \times v_n$ has indeed all three properties:[796]

(a) Note that $\det(v_1 \times \cdots \times v_n\ v_1\ \dots\ v_n) \underset{\text{by the Scalar–Cross Product Lemma 76.9}}{\uparrow} = \langle v_1 \times \cdots \times v_n, v_1 \times \cdots \times v_n \rangle \ \geq \ 0.$

(b) Let $i \in \{1, \dots, n\}$. We see that

$$\langle v_i, v_1 \times \cdots \times v_n \rangle \underset{\text{follows from the Scalar–Cross Product Lemma 76.9}}{\uparrow} = \det(v_i\ v_1\ \dots\ v_n) \underset{\text{since two columns are the same}}{\uparrow} = 0.$$

[796]Note that in the Cross Product Lemma 31.26 we already showed that the cross product satisfies (a) and (b). For the reader's convenience we recall the straightforward argument.

(c) By the Orthogonal Action Lemma 2.37 there exists an orthogonal matrix $A \in \mathrm{O}(n+1)$ and $w_1, \dots, w_n \in \mathbb{R}^n$ such that $A \cdot v_i = \begin{pmatrix} 0 \\ w_i \end{pmatrix}$ for $i = 1, \dots, n$. Now we see that

$$\|v_1 \times \cdots \times v_n\|^2 \overset{\text{by the Cross Product-Matrix Lemma 76.10}}{=} \|Av_1 \times \cdots \times Av_n\|^2 = \left\| \begin{pmatrix} 0 \\ w_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 \\ w_n \end{pmatrix} \right\|^2$$
$$\overset{\text{by the definition of the cross product}}{=} \left\| \begin{pmatrix} \det(w_1 \dots w_n) \\ 0 \end{pmatrix} \right\|^2 = \det(w_1 \dots w_n)^2$$
$$\overset{\text{see page 1566}}{=} \operatorname{gram} \begin{pmatrix} 0 & \dots & 0 \\ w_1 & \dots & w_n \end{pmatrix} = \operatorname{gram}(Av_1 \dots Av_n) \overset{\text{by the Orthogonal Matrix–Gram Determinant Lemma 76.5}}{=} \operatorname{gram}(v_1 \dots v_n).$$

Finally note that it follows from elementary considerations that the cross product is nailed down by Properties (a), (b) and (c). ■

76.3.3. **Integral of functions on submanifolds of $\mathbb{R}^n$.** In this subsection we will introduce the integral of a continuous real-valued function on a compact submanifold of some $\mathbb{R}^n$. As for the integral of a differential form, which we introduced in Section 74.2, we first consider the case of a function whose support is contained in the domain of a chart. The following proposition is the analog of the Integral-of-Form-in-Chart Proposition 74.12:

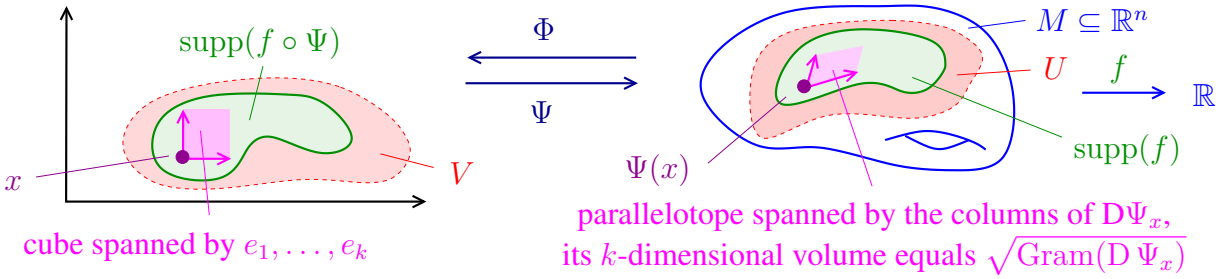
Proposition 76.11. (Integration-in-Chart Proposition) Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a continuous map. We suppose that $\operatorname{supp}(f)$ is compact and we suppose that there exists a chart $\Phi\colon U \to V$ with $\operatorname{supp}(f) \subseteq U$. We denote by $\Psi\colon V \to U$ the inverse map of Φ. The function

$$\begin{array}{ccc} V & \to & \mathbb{R} \\ x & \mapsto & f(\Psi(x)) \cdot \sqrt{\operatorname{gram}(\mathrm{D}\Psi_x)} \end{array}$$

has compact support and is continuous. It follows from the Lebesgue Integrability Proposition 73.21 that this function is Lebesgue-integrable. We define[797]

$$\int_M f \coloneqq \int_{x \in V} f(\Psi(x)) \cdot \sqrt{\operatorname{gram}(\mathrm{D}\Psi_x)}\, dx.$$

This integral does not depend on the choice of the chart.



[797]The idea behind the definition is that the term $\sqrt{\operatorname{gram}(\mathrm{D}\Psi_x)}$ measures to what degree the map Ψ "deforms the k-dimensional volume".

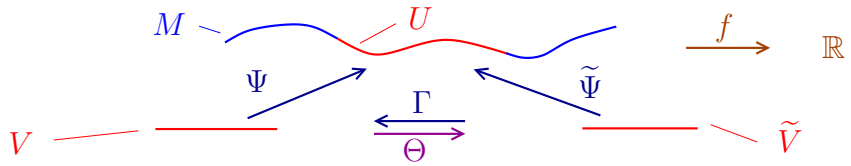
The proof of the Integration-in-Chart Proposition 76.11 makes use of the following elementary lemma:

Auxiliary Lemma 76.12. Given any $V \in \mathrm{M}(n \times k, \mathbb{R})$ and any $A \in \mathrm{M}(k \times k, \mathbb{R})$ we have the following equality:
$$\mathrm{gram}(\underbrace{V \cdot A}_{n \times k\text{-matrix}}) \;=\; \det(A)^2 \cdot \mathrm{gram}(V).$$

Proof of Lemma 76.12. The lemma follows from the following calculation:
$$\begin{array}{rcll} \mathrm{gram}(V \cdot A) & = & \det((V \cdot A)^{\mathrm{T}} \cdot V \cdot A) & = \det(A^{\mathrm{T}} \cdot \overbrace{V^{\mathrm{T}} \cdot V}^{(k \times k)\text{-matrix}} \cdot A) \\ & = & \det(A^{\mathrm{T}}) \cdot \det(V^{\mathrm{T}} \cdot V) \cdot \det(A) & = \det(A) \cdot \mathrm{gram}(V) \cdot \det(A). \\ & & \uparrow & \uparrow \\ & & \text{by the multiplicativity of the determinant, here} & \text{since } \det(A^{\mathrm{T}}) = \det(A) \\ & & \text{we use that } A^{\mathrm{T}},\ V^{\mathrm{T}} \cdot V \text{ and } A \text{ are } (k \times k)\text{-matrices} & \end{array}$$
■

Proof of Proposition 76.11. Let $f\colon M \to \mathbb{R}$ be a continuous function on a k-dimensional submanifold M of some $\mathbb{R}^n$. Furthermore let $\Phi\colon U \to V$ and $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ be two charts which both contain $\mathrm{supp}(f)$. We must show that with Φ and $\widetilde{\Phi}$ we obtain the same answer.

By passing to $U \cap \widetilde{U}$ we might as well assume that $U = \widetilde{U}$. We denote by Ψ and $\widetilde{\Psi}$ the inverses of Φ and $\widetilde{\Phi}$. We consider the diffeomorphisms $\Theta = \widetilde{\Phi} \circ \Psi\colon V \to \widetilde{V}$ and $\Gamma = \Phi \circ \widetilde{\Psi}\colon \widetilde{V} \to V$.



The proposition now follows from the following calculation:
$$\begin{array}{rcl} \int\limits_M f \text{ defined using } \Phi & = & \int\limits_{x \in V} f(\Psi(x)) \cdot \sqrt{\mathrm{gram}(\mathrm{D}\Psi_x)}\, dx \\ & = & \int\limits_{y \in \widetilde{V}} f(\Psi(\Gamma(y))) \cdot \sqrt{\mathrm{gram}_{\Phi}(\mathrm{D}\Psi_{\Gamma(y)})} \cdot |\det \mathrm{D}\Gamma_y|\, dy \\ & & \uparrow \\ & & \text{by the Lebesgue Transformation Theorem 73.26} \\ & & \text{applied to } \Gamma = \Phi \circ \widetilde{\Psi}\colon \widetilde{V} \to V \\ & = & \int\limits_{y \in \widetilde{V}} f\big(\widetilde{\Psi}(y)\big) \cdot \sqrt{\mathrm{gram}\,\big(\mathrm{D}\widetilde{\Psi}_y \cdot \mathrm{D}\Theta_{\Gamma(y)}\big)} \cdot |\det \mathrm{D}\Gamma_y|\, dy \\ & & \uparrow \\ & & \text{by the Chain Rule 24.17 applied} \\ & & \text{to } \Psi = \widetilde{\Psi} \circ \Theta \\ & = & \int\limits_{y \in \widetilde{V}} f\big(\widetilde{\Psi}(y)\big) \cdot \sqrt{\mathrm{gram}\,\big(\mathrm{D}\widetilde{\Psi}_y\big)} \cdot \underbrace{|\det(\mathrm{D}\Theta_{\Gamma(y)})| \cdot |\det \mathrm{D}\Gamma_y|}_{=1,\ \text{since } \Theta \circ \Gamma = \mathrm{id}}\, dy \\ & & \uparrow \\ & & \text{by the above Lemma 76.12 applied} \\ & & \text{to the } (k \times k)\text{-matrix } \mathrm{D}\Theta_{\Gamma(y)} \\ & = & \int\limits_{y \in \widetilde{V}} f\big(\widetilde{\Psi}(y)\big) \cdot \sqrt{\mathrm{gram}\,\big(\mathrm{D}\widetilde{\Psi}_y\big)}\, dy \;=\; \int\limits_M f \text{ defined using } \widetilde{\Phi}. \end{array}$$
■

Using partitions of unity we can now remove the restriction that we require our continuous functions to have support in the domain of a chart:

Definition 76.13. Let M be a k-dimensional smooth manifold of some $\mathbb{R}^n$/ Furthermore let $f\colon M \to \mathbb{R}$ be a continuous map with compact support. We pick finitely many charts $\{\Phi_i\colon U_i \to V_i\}_{i=1,\ldots,r}$ with $\operatorname{supp}(f) \subseteq U_1 \cup \cdots \cup U_r$. It follows easily from the Manifold–Partitions-of-Unity Theorem 27.2 that there exists smooth functions $\{\mu_i\colon M \to [0,1]\}_{i=1,\ldots,r}$ with the following two properties:

(1) For every $x \in \operatorname{supp}(f)$ we have $\mu_1(x) + \cdots + \mu_r(x) = 1$.

(2) For every $i \in \{1,\ldots,r\}$ we have $\operatorname{supp}(\mu_i) \subseteq U_i$.

We set

$$\int_M f \coloneqq \sum_{i=1}^{r} \underbrace{\int_M \mu_i \cdot f}_{\substack{\text{defined in the Integral} \\ \text{-of-Form-in-Chart} \\ \text{Proposition 74.12}}}.$$

The same argument as in the proof of the Integral-of-Form–Well-Defined Lemma 74.14 shows that this definition does not depend on any of the choices.

One particular example deserves its own definition:

Definition 76.14. Given any compact k-dimensional smooth submanifold $M \subseteq \mathbb{R}^n$ we refer to

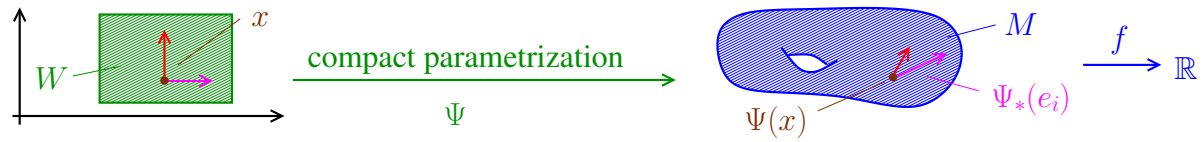
$$\operatorname{vol}_k(M) \coloneqq \int_M 1$$

as the k-dimensional volume of M. If $k = 1$, then we also write "length" and when $k = 2$, then we also write "area".

If one wants to calculate a "real life" example, then the above definition is not very helpful. Fortunately we have the proposition.

Proposition 76.15. ($\mathbb{R}$-Integral-via-Parametrization Proposition) Let M be a compact k-dimensional smooth submanifold of $\mathbb{R}^n$ and let $\Psi\colon W \to \mathbb{R}^n$ be a compact parametrization of M in the sense of Definition 74.18. For every continuous map $f\colon M \to \mathbb{R}$ we have

$$\int_M f = \int_{x \in W} f(\Psi(x)) \cdot \sqrt{\operatorname{gram}(\mathrm{D}\Psi_x)}\, dx.$$



Sketch of proof. The proof of this proposition is very similar to the proof of the Integral-via-Parametrization Proposition 74.19. We leave it to the reader to make the necessary changes. ∎

Our next goal is to provide an even more user friendly version of the $\mathbb{R}$-Integral-via-Parametrization Proposition 76.15 for codimension-one submanifolds of some $\mathbb{R}^n$. To do so we will need the following notation:

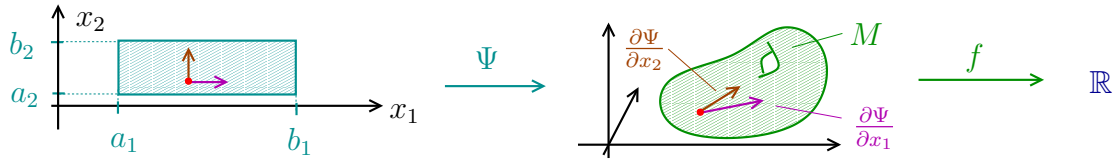
Notation 76.16. Let $W \subseteq \mathbb{R}^k$ be a subspace that is regular closed, i.e. W equals the closure of the interior of W. Let $\Psi\colon W \to \mathbb{R}^n$ be a smooth map. Given $i \in \{1,\ldots,k\}$ we write[798]

$$\frac{\partial \Psi}{\partial x_i} \coloneqq \begin{pmatrix} \frac{\partial \Psi_1}{\partial x_i} \\ \vdots \\ \frac{\partial \Psi_n}{\partial x_i} \end{pmatrix}.$$

Furthermore we set $\mathrm{D}\Psi = \left(\frac{\partial \Psi}{\partial x_1} \dots \frac{\partial \Psi}{\partial x_k}\right)$.

Proposition 76.17. (ℝ-Integral-on-Hypersurface Proposition) Let $M \subseteq \mathbb{R}^{n+1}$ be a codimension-one submanifold. Furthermore let $\Psi\colon [a_1, b_1] \times \dots \times [a_n, b_n] \to M$ be a compact parametrization of M. Given any continuous function $f\colon M \to \mathbb{R}$ we have

$$\int\limits_M f \;=\; \int\limits_{x_1=a_1}^{x_1=b_1} \dots \int\limits_{x_n=a_n}^{x_n=b_n} f(\Psi(x_1, \dots, x_n)) \cdot \left\| \frac{\partial \Psi}{\partial x_1} \times \dots \times \frac{\partial \Psi}{\partial x_n} \right\| dx_n \dots dx_1.$$



Proof. To simplify the notation we assume that $n = 2$. Now we see that:

$$\int\limits_M f \;\underset{\text{by the Integral-via-Parametrization Proposition 74.19}}{=}\; \int\limits_{[a_1,a_2]\times[b_1,b_2]} f(\Psi(x)) \cdot \sqrt{\operatorname{Gram}(\mathrm{D}\Psi_x)}\, d\lambda \;\underset{\substack{\text{follows from the Fubini Theorem 73.25,} \\ \text{the Riemann-Lebesgue Integral Proposition 73.23} \\ \text{and the Cross Product Characterization Proposition 76.8} \\ \text{applied to } \mathrm{D}\Psi_{(a,b)} = \left(\frac{\partial \Psi}{\partial x_1}|_{(a,b)} \; \frac{\partial \Psi}{\partial x_2}|_{(a,b)}\right)}}{=}\; \int\limits_{x_1=a_1}^{x_1=b_1} \int\limits_{x_2=a_2}^{x_2=b_2} f(\Psi(x_1, x_2)) \cdot \left\| \frac{\partial \Psi}{\partial x_1} \times \frac{\partial \Psi}{\partial x_2} \right\| dx_2\, dx_1.$$

■

We formulate our main example as a lemma:

Lemma 76.18. (Sphere–Integration Lemma) For any continuous function $f\colon S^2 \to \mathbb{R}$ we have

$$\int\limits_{S^2} f \;=\; \int\limits_{\varphi=0}^{\varphi=2\pi} \int\limits_{\theta=0}^{\theta=\pi} f\underbrace{\begin{pmatrix} \cos(\varphi) \cdot \sin(\theta) \\ \sin(\varphi) \cdot \sin(\theta) \\ \cos(\theta) \end{pmatrix}}_{\substack{\text{point with spherical} \\ \text{coordinates } (\varphi, \theta)}} \cdot \sin(\theta)\, d\theta\, d\varphi.$$

Proof. On page 1536 we saw that the map

$$\begin{aligned} S\colon [0, \pi] \times [0, 2\pi] &\to \mathbb{R}^3 \\ (\theta, \varphi) &\mapsto \begin{pmatrix} \cos(\varphi) \cdot \sin(\theta) \\ \sin(\varphi) \cdot \sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{aligned}$$

is a compact parametrization of S^2. The lemma now follows from the ℝ-Integral-on-Hypersurface Proposition 76.17 together with a dull calculation. ■

[798] Here we use the Regular Closed–∂_*-Lemma 24.5 to make sense of the partial derivatives on W.

Example. Let us calculate the area of S^2:

$$\text{area of } S^2 \underset{\text{by Definition 76.14}}{\uparrow} = \int_{S^2} 1 \underset{\text{by the Sphere–Integration Lemma 76.18}}{\uparrow} = \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} 1 \cdot \sin(\theta)\, d\theta\, d\varphi = \int_{\varphi=0}^{\varphi=2\pi} 2\, d\varphi = 4\pi.$$

□

76.3.4. **The flux of a vector field.** We are inching towards the formulation of the Gauß Divergence Theorem 76.26. The theorem deals in particular with the "flux of a vector field" through an oriented codimension-one submanifold of $\mathbb{R}^n$. In this subsection we will explain what this means.

First we recall Definition 31.24:

Definition 76.19. Let N be a codimension-one smooth submanifold of $\mathbb{R}^n$. A unit normal field is a continuous map

$$v\colon N \to \mathbb{R}^n$$

such that the following conditions are satisfied:

(1) For each $P \in N$ we have $\|v(P)\| = 1$.
(2) For each $P \in N$ we have $v(P) \perp \mathrm{T}_P N$.

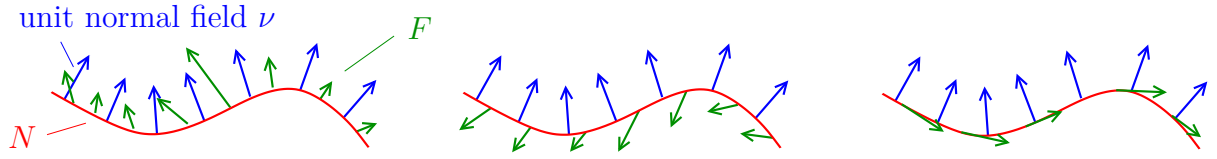


We move on to a new definition, which plays a major role in physics:

Definition 76.20. Let N be a compact submanifold of $\mathbb{R}^n$ of codimension one and let $\nu\colon N \to \mathbb{R}^n$ be a unit normal field. Furthermore let $F\colon N \to \mathbb{R}^n$ be a continuous map. The map

$$\begin{aligned} F \cdot \nu\colon N &\to \mathbb{R} \\ P &\mapsto \underbrace{F(P) \cdot \nu(P)}_{\text{standard inner product}} \end{aligned}$$

is a continuous function on the compact submanifold N, thus integrable. We refer to the integral $\int_N F \cdot \nu$ as the flux of F through (N, ν).



the flux is positive, since $F \cdot \nu > 0$ | the flux is negative, since $F \cdot \nu < 0$ | the flux is zero, since $F \cdot \nu = 0$

The following proposition shows that we can express the flux as an integral of a suitable differential form:

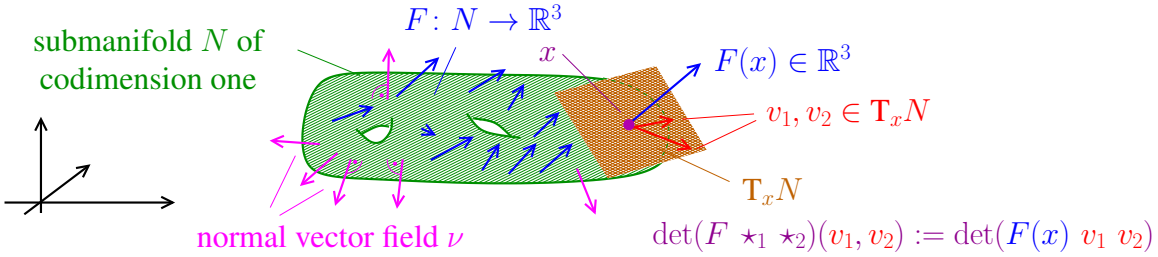
Proposition 76.21. (Flux-via-Forms Proposition) Let N be a compact submanifold of $\mathbb{R}^n$ of codimension one and let $\nu\colon N \to \mathbb{R}^n$ be a unit normal field. By the Orientations-via-Normal Vector Fields Proposition 31.27 this endows N with a natural orientation. Let $F\colon N \to \mathbb{R}^n$ be a smooth map. As on page 1526 we denote by $\det(F \star_1 \cdots \star_{n-1})$ the

smooth $(n-1)$-Form on N which at each $x \in N$ is given by

$$\begin{aligned} (\mathrm{T}_x N)^{n-1} &\to \mathbb{R} \\ (v_1, \dots, v_{n-1}) &\mapsto \det(F(x)\ v_1\ \dots\ v_{n-1}). \end{aligned}$$

Then

$$\underbrace{\int_N \det(F \star_1 \cdots \star_{n-1})}_{\text{integral of } (n-1)\text{-Form}} = \underbrace{\int_N F \cdot \nu.}_{\text{flux of } F \text{ through } (N,\nu)}$$



Proof. Let N be a compact submanifold of $\mathbb{R}^n$ of codimension one, let $\nu\colon N \to \mathbb{R}^n$ be a unit normal field and let $F\colon N \to \mathbb{R}^n$ be a smooth map. It follows easily from the definitions that it suffices to consider the case that there exists an orientation-preserving chart $\Phi\colon U \to V$ with $\operatorname{supp}(F) \subseteq U$. We denote by $\Psi\colon V \to U$ the inverse of Φ.

Let $x \in V$. We write $\widetilde{F} := F(\Psi(x))$ and $\widetilde{\nu} = \nu(\Psi(x))$. Furthermore, for $i = 1, \dots, n-1$ we write $v_i := (\mathrm{D}\Psi_x)(\mathfrak{e}_i)$. We see that

$$\begin{aligned} \int_N \det(F \star_1 \cdots \star_{n-1}) &\overset{\text{since } \operatorname{supp}(F) \subseteq U}{=} \int_{x \in V} \det(\widetilde{F} \star_1 \cdots \star_{n-1})(v_1, \dots, v_{n-1})\, d\lambda^n \\ &= \int_{x \in V} \widetilde{F} \cdot (v_1 \times \cdots \times v_{n-1})\, d\lambda^n \\ &= \int_{x \in V} \widetilde{F} \cdot \widetilde{\nu} \cdot \sqrt{\operatorname{gram}(v_1, \dots, v_{n-1})}\, d\lambda^n \\ &\overset{\text{definition of } \det(\widetilde{F} \star_1 \cdots \star_{n-1})}{=} \int_{x \in V} \det(\widetilde{F}\ v_1\ \dots\ v_{n-1})\, d\lambda^n \\ &= \int_N F \cdot \nu\, d\lambda^n. \end{aligned}$$

by the Scalar–Cross Product Lemma 76.9

by the Cross Product Characterization Proposition 76.8 we have $v_1 \times \cdots \times v_{n-1} = \widetilde{\nu} \cdot \sqrt{\operatorname{gram}(v_1, \dots, v_{n-1})}$[799]

by the definition in the Integration-in-Chart Proposition 76.11 ∎

76.3.5. **The Gauß Divergence Theorem.** We just need two more definitions, and we can finally state the Gauß Divergence Theorem 76.26. First we need to recall Definition 29.1:

Definition 76.22. Let M be a submanifold of $\mathbb{R}^n$ and let $P \in \partial M$. We say that a tangent vector $v \in \mathrm{T}_P M$ points outward if the following two statements hold:

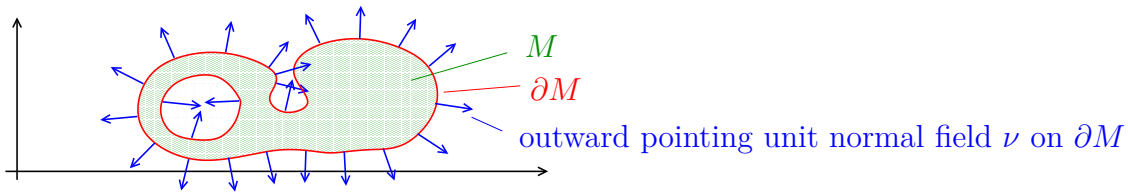
- There exists an $\epsilon < 0$ and a smooth map $\gamma\colon (\epsilon, 0] \to M$ with $v = \gamma'(0)$.
- We have $v \notin \mathrm{T}_P \partial M$.

[799] Indeed: Both vectors evidently have properties (b) and (c) of the Cross Product Characterization Proposition 76.8. It remains to show that $\det(\widetilde{\nu}\ v_1, \dots, v_{n-1}) > 0$. This can be seen as follows. Since $\mathfrak{e}_1, \dots, \mathfrak{e}_{n-1}$ is a positive basis of $\mathbb{R}^{n-1}$ and since the chart is orientation-preserving we know that $v_1, \dots, v_{n-1}$ is a positive basis of the tangent space $\mathrm{T}_{\Psi(x)} M$. By definition of the orientation, see the Orientations-via-Normal Vector Fields Proposition 31.27, this means precisely that $\det(\widetilde{\nu}\ v_1, \dots, v_{n-1}) > 0$.



Proposition 76.23. (Outward Pointing Vector Field Proposition) Let M be a codimension-zero submanifold of $\mathbb{R}^n$. There exists a unique unit normal field v on ∂M such that for each $P \in \partial M$ the vector $v(P)$, viewed as a tangent vector to M, points outward.

Definition 76.24. Let M be a codimension-zero submanifold of $\mathbb{R}^n$. We refer to the unit normal field v on ∂M that we introduced in the Outward Pointing Vector Field Proposition 76.23 as the outward pointing unit normal field on ∂M.



Proof. One can easily show, say using the trichotomy of Exercise 29.2, that for every $P \in \partial M$ there exists a unique $v \in \mathrm{T}_P M$ with $\|v(P)\| = 1$, with $v(P) \perp \mathrm{T}_P M$ and which points outward. It remains to show that these vectors define a continuous map.

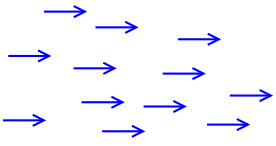
By the convention on page 819 the standard orientation on $\mathbb{R}^n$ endows M with an orientation. In the Boundary Orientation Lemma 31.38 we equipped ∂M with a natural orientation. It follows from the Orientations-via-Normal Vector Fields Proposition 31.27 that this natural orientation gives rise to a unit normal field v on ∂M. The proposition now follows from the Tangent Space-Isomorphism Proposition 29.24. ■

Definition 76.25. Let U be a codimension-zero submanifold of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^n$ be a smooth vector field. We refer to[800]

$$\begin{aligned} \operatorname{div} F\colon U &\to \mathbb{R} \\ P &\mapsto \operatorname{div} F(P) := \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}(P). \end{aligned}$$

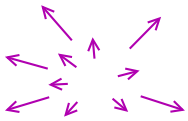
as the divergence of F.

vector field $F(x,y) = (1,0)$



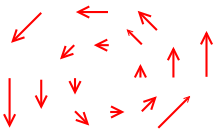
vector field with $\operatorname{div}(F) = \frac{\partial 1}{\partial x} + \frac{\partial 0}{\partial y} = 0$

vector field $G(x,y) = (x,y)$



vector field with $\operatorname{div}(G) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2$

vector field $H(x,y) = (-y,x)$



vector field with $\operatorname{div}(H) = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0$

Now we can finally formulate the Gauß Divergence Theorem:

[800]This definition looks initially quite arbitrary. But note that $\operatorname{div} F(P) = \operatorname{trace}(\mathrm{D}F_P)$. This point of view at least combines concepts that we are used to.

Theorem 76.26. (Gauß Divergence Theorem) Let M be a compact codimension zero submanifold of $\mathbb{R}^n$. We denote by ν the outward pointing unit normal field on ∂M. Given any smooth map $F\colon M \to \mathbb{R}^n$ we have

$$\underbrace{\int_{\partial M} F \cdot \nu}_{\substack{\text{the flux of} \\ F \text{ through } \partial M}} = \int_M \operatorname{div} F.$$

Remark. Special cases of the Gauß Divergence Theorem 76.26 were formulated and proved by Carl Friedrich Gauß [**Gaus1813**].

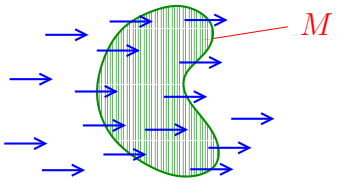


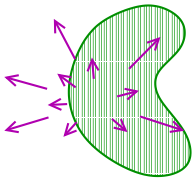
Mikhail Ostrogradsky (1801-1862), stamp from 1967

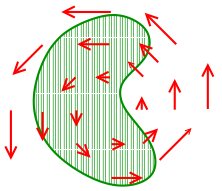
The first proof of the general case was given by Mikhail Ostragradsky [**Ostr1826**]. □

Example. As above we consider the following vector fields F, G and H. Furthermore let $M \subseteq \mathbb{R}^2$ be a compact non-empty 2-dimensional submanifold.

vector field $F(x,y) = (1,0)$ | vector field $G(x,y) = (x,y)$ | vector field $H(x,y) = (-y,x)$







vector field with $\operatorname{div}(F) = 0$ the flux through ∂M is zero | vector field with $\operatorname{div}(G) = 2$ the flux through ∂M is zero | vector field with $\operatorname{div}(H) = 0$ the flux through ∂M is zero

It follows from the Gauß Divergence Theorem 76.26 that the flux of F and H through ∂M is zero, whereas the flux of G through ∂M is positive. □

Example. We consider the closed 3-ball $\overline{B}^3$. Let $F\colon \overline{B}^3 \to \mathbb{R}^3$ be the map that is given by $F(x,y,z) = (x,y,z)$. We consider the corresponding 2-form $\det(F \star_1 \star_2)$. Now we see that

$$\int_{S^2} \det(F \star_1 \star_2) \underset{\substack{\uparrow \\ \text{by the Flux-via-Forms} \\ \text{Proposition 76.21}}}{=} \int_{S^2} F \cdot \nu \underset{\substack{\uparrow \\ \text{by the Gauß Divergence} \\ \text{Theorem 76.26}}}{=} \int_{\overline{B}^3} \operatorname{div}(F) = \int_{\overline{B}^3} 3 = 3 \cdot \operatorname{vol}(\overline{B}^3) \underset{\substack{\uparrow \\ \text{see Exercise 73.7}}}{=} 3 \cdot \tfrac{4}{3} \cdot \pi = 4\pi.$$

Fortunately this matches the calculation of the left-hand side from page 1538. □

We will prove the Gauß Divergence Theorem 76.26 using Stokes' Theorem 75.11. The key to reducing our situation to the general setting of Stokes' Theorem 75.11 is the following lemma:

Auxiliary Lemma 76.27. (Differential-is-Divergence Lemma) Let $U \subseteq \mathbb{R}^n$ be an open subset and let $F\colon U \to \mathbb{R}^n$ be a smooth map. We have the following equality of n-forms on U:[801]

$$d\underbrace{\det(F\star_1\cdots\star_{n-1})}_{(n\text{-}1)\text{-form on } U} = \operatorname{div} F\cdot dx^1\wedge\cdots\wedge dx^n.$$

Proof. Let U be an open subset of $\mathbb{R}^n$ and let $F=(F_1,\dots,F_n)\colon U\to\mathbb{R}^n$ be a smooth vector field on U. We need to determine the n-form $d\det(F\star_1\cdots\star_{n-1})$ on U. To do so we need to get $\det(F\star_1\cdots\star_{n-1})$ into a form such that we can apply the definition of the differential. We do so in the next claim:

Claim.
$$\det(F\star_1\cdots\star_{n-1}) = \sum_{i=1}^{n}(-1)^{i-1}\cdot F_i\cdot dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n.$$
($\widehat{dx^i}$: this term is skipped)

Proof. We need to show that the left-hand side and the right-hand side return the same real number when we apply them to arbitrary vectors $v_1,\dots,v_{n-1}\in\mathbb{R}^n$. This is indeed the case, since for $v_i=\begin{pmatrix}v_{i1}\\ \vdots\\ v_{in}\end{pmatrix}$, $i=1,\dots,n-1$ we have the following equalities:

$$\det(F\star_1\cdots\star_{n-1})(v_1,\dots,v_{n-1}) = \det\begin{pmatrix}F_1 & v_{11} & \dots & v_{n-1,1}\\ \vdots & \vdots & & \vdots\\ F_n & v_{1n} & \dots & v_{nn}\end{pmatrix} \overset{\text{Laplace expansion by the first column}}{=} \sum_{i=1}^{n}(-1)^{i-1}\cdot F_i\cdot\det\begin{pmatrix}v_1\ \dots\ v_{n-1}\\ \text{with } i\text{-th row}\\ \text{removed}\end{pmatrix}$$
$$= \sum_{i=1}^{n}(-1)^{i-1}\cdot F_i\cdot(dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n)(v_1,\dots,v_{n-1}).$$
(= follows immediately from the definition of dx^j and the definition of the wedge product)

It follows that

$$d\det(F\star_1\cdots\star_{n-1}) \overset{\text{by the claim and the linearity of } d}{=} \sum_{i=1}^{n} d\big((-1)^{i-1}\cdot F_i\cdot dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\big) \overset{\text{by Definition 75.1}}{=}$$
$$= \sum_{i=1}^{n}(-1)^{i-1}\cdot dF_i\wedge dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n$$
$$= \sum_{i=1}^{n}(-1)^{i-1}\cdot\Big(\sum_{j=1}^{n}\frac{\partial F_i}{\partial x_j}\cdot dx^j\Big)\wedge dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n$$
(= by the 1-Form-via-dx^i-Lemma 71.5)
$$= \sum_{i=1}^{n}(-1)^{i-1}\cdot\frac{\partial F}{\partial x_i}\cdot(-1)^{i-1}\cdot dx^1\wedge\cdots\wedge dx^{i-1}\wedge dx^i\wedge dx^{i+1}\wedge\cdots\wedge dx^n$$
(= since $dx^j\wedge dx^j=0$ and $dx^j\wedge dx^k=-dx^k\wedge dx^j$)
$$= \operatorname{div}(F)\cdot dx^1\wedge\cdots\wedge dx^n.$$

Proof of the Gauß Divergence Theorem 76.26. Let M be a compact codimension-zero smooth submanifold of $\mathbb{R}^n$. We denote by ν the outward pointing unit normal field on

[801] One could use the equality of the lemma also as a definition of the divergence, which is arguably a little less "random".

∂M. Let $F\colon M \to \mathbb{R}^n$ be a smooth vector field. We now see that

$$\begin{array}{ccccccc} & & \text{by the Integral-}n\text{-Form-on-}\mathbb{R}^n\text{-Lemma 74.20} & & & & \\ & & \downarrow & & & & \\ \int\limits_M \operatorname{div} F & = & \int\limits_M \operatorname{div} F \cdot dx^1 \wedge \cdots \wedge dx^n & & & & \\ & = & \int\limits_M d \det(F \star_1 \cdots \star_{n-1}) & = & \int\limits_{\partial M} \det(F \star_1 \cdots \star_{n-1}) & = & \int\limits_{\partial M} F \cdot \nu. \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{by the Differential-is-Divergence Lemma 76.27} & & \text{by Stokes Theorem 75.11} & & \text{by the Flux-via-Forms Proposition 76.21} \end{array}$$

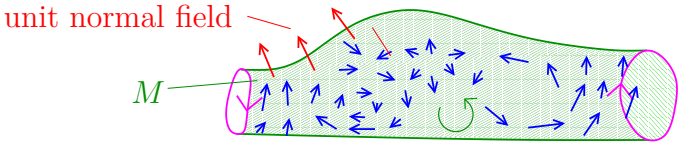
■

76.4. Exercises for Chapter 76.

Exercise 76.1. In the figure below we show an oriented 2-dimensional smooth manifold M together with a map $F\colon \mathbb{R}^3 \to \mathbb{R}$.

(a) Sketch $\operatorname{curl} F$ on M.

(b) At which points on M is the integrand of $\int\limits_M \det(\operatorname{curl}(F) \star_1 \star_2)$ positive?



Exercise 76.2. Let $V = (v_1, \ldots, v_k) \in \mathrm{M}(n \times k, \mathbb{R})$. Show that

$$\operatorname{gram} \underbrace{(v_1 \ \ldots \ v_k)}_{n\times k-\text{matrix}} \ = \ \det\Big(\underbrace{(v_i \overset{\text{standard inner product}}{\overset{\downarrow}{\cdot}} v_j)_{i,j=1,\ldots,k}}_{k\times k-\text{matrix}} \Big).$$

Exercise 76.3. Let $v_1, \ldots, v_k \in \mathbb{R}^n$. Show that $v_1, \ldots, v_k$ are linearly dependent if and only if $\operatorname{Gram}(v_1 \ldots v_k) = 0$.

Exercise 76.4. Let $f\colon S^n \to \mathbb{R}$ be a continuous function and let $A \in \mathrm{O}(n+1)$. Show that

$$\int\limits_{x\in S^n} f(x) \ = \ \int\limits_{x\in S^n} f(A \cdot x).$$

Exercise 76.5.

(a) We consider the cylinder

$$Z \coloneqq \{(x, y, z) \,|\, x^2 + z^2 = 2 \text{ and } y \in [0, 3]\}.$$

Calculate the integral of the function $(x, y) \mapsto x^2 \cdot y$ on Z, i.e. determine $\int\limits_Z x^2 \cdot y$.

(b) Let

$$M \coloneqq \{(x, y, x^2 + y^2) \,|\, x^2 + y^2 \leq 1\}$$

be the graph of the function $f(x, y) = x^2 + y^2$ over the closed disk of radius 1. Determine the area of M.

Exercise 76.6.

(a) Determine the length of the helix $H \coloneqq \{(\cos(t), \sin(t), t) \,|\, t \in [0, 6\pi]\}$.

(b) Determine the 3-dimensional volume of $S^3 \subseteq \mathbb{R}^4$.

Exercise 76.7. Determine the flux of the vector field

$$F(x,y,z) \;=\; \begin{pmatrix} x \cdot z \\ y \cdot z \\ z \end{pmatrix}$$

through the cylinder $Z \;:=\; \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2+y^2 = 4 \text{ and } z \in [0,3]\right\}$ with "outward pointing" unit normal field.

Exercise 76.8.

(a) Does there exist a smooth vector field

$$F\colon A := \{P \in \mathbb{R}^n \mid \|P\| \leq 2\} \;\to\; \mathbb{R}^n$$

with $\operatorname{div}(F) \equiv 0$, with $F(1,0,\ldots,0) = (1,\ldots,1)$ and such that for all $P \in A$ with $\|P\| = 2$ we have $F(P) = P$?

(b) Does there exist a smooth vector field

$$F\colon B := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, 0 < x^2+y^2 \leq 1\right\} \;\to\; \mathbb{R}^2$$

with $\operatorname{div}(F) \equiv 0$ and such that for all $P \in B$ with $\|P\| = 1$ we have $F(P) = P$?

CHAPTER 77

Cochain complexes over a field

In this purely algebraic chapter we give a short introduction into the algebraic theory of cochain complexes over a field. As a standalone chapter the material is rather dry, but it will come alive right afterwards, when we apply the definitions and results to differential forms on smooth manifolds. In the following we will not thrive to achieve maximal generality but we only want to introduce concepts and prove statements that we need for the discussion of de Rham cohomology.

Later on, in Chapter 197, we will discuss the theory of cochain complexes in much greater detail.

This chapter is organized as follows:

- In Section 77.1 we will introduce the cohomology of a cochain complex.
- In Section 77.2 we will study exact sequences of cochain complexes.
- In Section 77.3 we will state and prove the Evaluation-Isomorphism Theorem 77.18.
- In Section 77.4 we will provide exercises for Chapter 77.

In the next Chapter 78 we will introduce de Rham cohomology of smooth manifolds.

77.1. **Cohomology of a cochain complex.** We start out by introducing the key player of this chapter:

Definition 77.1. Let $\mathbb{F}$ be a field.

(1) A cochain complex (D^*, δ_*) of $\mathbb{F}$-vector spaces is a sequence[802]

$$\ldots \leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \ldots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \xleftarrow{\delta_{-1}} 0$$

of homomorphisms between $\mathbb{F}$-vector spaces such that the composition of any two successive homomorphisms is zero, i.e. for all $i \in \mathbb{N}$ we have $\delta_n \circ \delta_{n-1} = 0$. We refer to the maps δ_* as coboundary maps.

(2) We define the n-th cohomology group of a cochain complex (D^*, δ_*) of $\mathbb{F}$-vector spaces to be the $\mathbb{F}$-vector space[803]

$$\mathrm{H}^n(D) \coloneqq \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

Note that the cohomology groups are in fact $\mathbb{F}$-vector spaces.

As always we care not only about objects, but also about morphisms between them. This leads us to the following definition:

[802] For whatever reason cochain complexes will get written from right-to-left. In the lecture notes we will frequently work with the closely related notion of a chain complex. These usually get written left-to-right.

[803] Note that since we are dealing with a cochain complex we have for every $n \in \mathbb{N}_0$ the inclusion $\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n) \subseteq \ker(\delta_n \colon D^n \to D^{n+1})$.

Definition 77.2. Let $\mathbb{F}$ be a field. A cochain map $f\colon C^* \to D^*$ between two cochain complexes of $\mathbb{F}$-vector spaces consists of a family $\{f_n\colon C^n \to D^n\}_{n\in\mathbb{N}_0}$ of homomorphisms that commute with the coboundary maps, i.e. such that for all $n \in \mathbb{N}_0$ we have

$$\underset{\uparrow}{\delta_n} \circ f_n \;=\; f_{n+1} \circ \underset{\uparrow}{\delta_n}\colon C^n \;\to\; D^{n+1}.$$
$$\delta_n\colon D^n \to D^{n+1} \qquad \delta_n\colon C^n \to C^{n+1}$$

In other words, the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longleftarrow & C^{n+1} & \xleftarrow{\delta_n} & C^n & \xleftarrow{\delta_{n-1}} & C^{n-1} & \longleftarrow & \ldots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \longleftarrow & D^{n+1} & \xleftarrow[\delta_n]{} & D^n & \xleftarrow[\delta_{n-1}]{} & D^{n-1} & \longleftarrow & \ldots
\end{array}$$

The above definition naturally leads us to the following definition:

Definition 77.3. Let $\mathbb{F}$ be a field. We refer to category $\mathcal{CoChCplx}_{\mathbb{F}}$ that is given by

$$\mathrm{Ob}(\mathcal{CoChCplx}_{\mathbb{F}}) \;:=\; \text{all cochain chain complexes of } \mathbb{F}\text{-vector spaces,}$$

and with morphisms

$$\mathrm{Mor}(C^*, D^*) \;:=\; \text{all cochain maps from } C^* \text{ to } D^*$$

with the obvious composition of morphisms, as the category of cochain complexes of $\mathbb{F}$-vector spaces.

This definition allows us to formulate our first lemma:

Lemma 77.4. (Cochain Complex-H*-Lemma) Let $\mathbb{F}$ be a field.

(1) If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^* of $\mathbb{F}$-vector spaces, then the map
$$\begin{array}{rcl} f_*\colon \mathrm{H}^n(C) & \to & \mathrm{H}^n(D) \\ [c] & \mapsto & [f_n(c)] \end{array}$$
is a well-defined homomorphism of $\mathbb{F}$-vector spaces.

(2) Given $n \in \mathbb{N}_0$ the maps
$$\begin{array}{rcl} C^* & \mapsto & \mathrm{H}^n(C^*) \\ (f\colon C^* \to D^*) & \mapsto & (f^*\colon \mathrm{H}^n(C^*) \to \mathrm{H}^n(D^*)) \end{array}$$
define a covariant functor from the category $\mathcal{CoChCplx}_{\mathbb{F}}$ of cochain complexes of $\mathbb{F}$-vector spaces to the category $\mathcal{Vec}_{\mathbb{F}}$ of $\mathbb{F}$-vector spaces.

Proof.

(1) Let $c \in \ker(\delta_n\colon C^n \to C^{n+1})$. Then

$$\delta_n(f_n(c)) \underset{\substack{\uparrow \\ \text{Definition 77.2} \\ \text{of a cochain map}}}{=} f_{n+1}(\delta_n(c)) \underset{\substack{\uparrow \\ \text{since } \delta_n(c)=0}}{=} f_n(0) \;=\; 0.$$

This shows that $f_n(c)$ represents indeed an element in $\mathrm{H}^n(C)$. We still need to show that the given map f_* does not depend on the choice of the representative. Therefore let $c, \widetilde{c} \in \ker(\delta_n\colon C^n \to C^{n+1})$ be two elements which represent the same element in $\mathrm{H}^n(C)$.

We have to show that $[f_n(c)] = [f_n(\widetilde{c})]$. Indeed we have

$$[f_n(c)] - [f_n(\widetilde{c})] \;=\; [f_n(c-\widetilde{c})] \underset{\uparrow}{=} [f_n(\delta_{n-1}(e))] \underset{\uparrow}{=} [\delta_{n+1}(f_{n+1}(e))] \;=\; 0 \;\in \mathrm{H}^n(D).$$

since $c, \widetilde{c}$ represent the same element in $\mathrm{H}^n(C)$ there exists an $e \in C^{n-1}$ with $\delta_{n-1}(e) = c - \widetilde{c}$

by Definition 77.2 of a cochain map

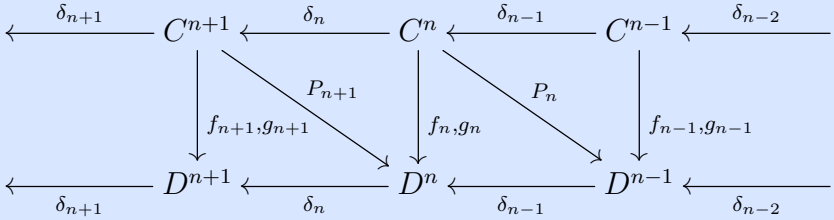
(2) This statement follows easily from the definitions. ∎

We now want a criterion for two cochain maps to induce the same map on cohomology. This leads us to the following definition:

Definition 77.5. Let $\mathbb{F}$ be a field. Let $f, g\colon C^* \to D^*$ be two cochain maps between two cochain complexes C^* and D^* of $\mathbb{F}$-vector spaces. A cochain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n\colon C^n \to D^{n-1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n \;=\; f_n - g_n.$$

The maps are illustrated in the following diagram:



If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are cochain homotopic.[804]

Lemma 77.6. (Cochain Homotopy–H^*-Lemma) Two cochain maps that are cochain homotopic induce the same map on cohomology groups.

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a cochain homotopy between two cochain maps f and g from C^* to D^*. Then for any $z \in \ker(\delta_n\colon C^n \to C^{n+1})$ we get the equalities

$$\begin{aligned} f_n(z) - g_n(z) \;=\; (f_n - g_n)(z) \;&=\; (\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n)(z) \\ &=\; \delta_{n-1}(P_n(z)) + P_{n+1}(\underbrace{\delta_n(z)}_{=0}) \;=\; \delta_{n-1}(P_n(z)), \end{aligned}$$

i.e. $f_n(z)$ and $g_n(z)$ represent the same element in $\mathrm{H}^n(D) = \frac{\ker(\delta_n)}{\mathrm{im}(\delta_{n-1})}$. ∎

Finally the following lemma says that cohomology commutes with direct products.

Lemma 77.7. (Product–H^*-Lemma) Let $\mathbb{F}$ be a field. If $\{C_a\}_{a\in A}$ is a family of cochain complexes of $\mathbb{F}$-vector spaces, then the natural projections $\prod_{a\in A} C_a \to C_b$, $b \in A$ (which come from the definition of a product) induce for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \;\xrightarrow{\cong}\; \prod_{a\in A} \mathrm{H}^n(C_a).$$

[804] The argument of the Chain Homotopy-Lemma 144.6 shows of course that being cochain homotopic is an equivalence relation on the set of all cochains $C^* \to D^*$.

Proof. Let $n \in \mathbb{N}_0$. Note that for each $b \in A$ the projection $\prod_{a\in A} C_a \to C_b$ is clearly a chain map and that it induces a homomorphism

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \;\to\; \mathrm{H}^n(C_b).$$

It follows from the definition of a product, see Definition 6.1, that we get a well-defined homomorphism

$$\begin{array}{ccc} \mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) & \to & \prod_{b\in A} \mathrm{H}^n(C_b) \\ [(c_a)_{a\in A}] & \mapsto & ([c_a])_{a\in A}. \end{array}$$

One can easily verify that an inverse is given by

$$\begin{array}{ccc} \prod_{b\in A} \mathrm{H}^n(C_b) & \to & \mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \\ ([c_a])_{a\in A} & \mapsto & [(c_a)_{a\in A}]. \end{array}$$

■

77.2. **Exact sequences of cochain complexes.** Again we start out with a basic definition:

Definition 77.8. Let $\mathbb{F}$ be a field. A sequence of homomorphisms of $\mathbb{F}$-vector spaces

$$\ldots\, A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

is called exact, if for each n we have

$$\ker(f_n) \;=\; \mathrm{im}(f_{n+1}).$$

This definition allows us to formulate the following very popular lemma:

Lemma 77.9. (Five Lemma) Let $\mathbb{F}$ be a field and let

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle a} & & \cong\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \cong\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

be a diagram of homomorphisms of $\mathbb{F}$-vector spaces such that the horizontal sequences are exact and such that every square commutes up to a fixed unit in $\mathbb{F}$. If a is an epimorphism, if b and d are isomorphisms, and if e is a monomorphism, then c is an isomorphism.

Proof. The proof of the lemma is elementary, but impossible to read, when written down by somebody else. Thus we outsource the proof to Exercise 77.2.[805] ■

We now introduce a notion of a particularly elementary exact sequence:

Definition 77.10. Let $\mathbb{F}$ be a field. A short exact sequence is an exact sequence of the form

$$0 \;\to\; C \;\to\; B \;\to\; A \;\to\; 0,$$

which consists of at most three non-trivial $\mathbb{F}$-vector spaces.

Remark. Let $\mathbb{F}$ be a field.

(1) Let $f\colon A \to B$ be a homomorphism. It follows easily from the definitions that the homomorphisms

$$0 \;\to\; \ker(f) \;\to\; A \;\to\; \mathrm{im}(f) \;\to\; 0$$

[805] Later we will state an analog of the lemma, namely the Five Lemma 145.11. The later lemma will have the same minimalist proof in the lecture notes.

form a short exact sequence.

(2) Let
$$0 \to C \to B \xrightarrow{\beta} A \to 0,$$
be a short exact sequence. Note that β is an epimorphism. Since we are dealing with vector spaces we can pick a homomorphism $\sigma\colon A \to B$ with $\beta \circ \varphi = \mathrm{id}_A$. One can now easily verify that the map
$$\begin{array}{ccc} A \times C & \to & B \\ (a, c) & \mapsto & \alpha(a) + \sigma(c) \end{array}$$
is an isomorphism. □

Definition 77.11. A short exact sequence of cochain complexes
$$0 \to C^* \xrightarrow{i} B^* \xrightarrow{p} A^* \to 0$$
consists of cochain complexes C^*, B^*, A^* and cochain maps $i\colon C^* \to B^*$ and $p\colon B^* \to A^*$ such that for each $n \in \mathbb{N}_0$ the maps
$$0 \to C^n \xrightarrow{i} B^n \xrightarrow{p} A^n \to 0$$
form a short exact sequence.

The following definition is arguably the most fruitful definition in (co-) homological algebra:

Definition 77.12. Let $\mathbb{F}$ be a field. Let
$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$
be a short exact sequence of cochain complexes over $\mathbb{F}$. In particular for each $n \in \mathbb{N}_0$ we have a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{n+1} & \xrightarrow{f_{n+1}} & B^{n+1} & \xrightarrow{g_{n+1}} & A^{n+1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \delta_n^C} & & \uparrow{\scriptstyle \delta_n^B} & & \uparrow{\scriptstyle \delta_n^A} & & \\ 0 & \longrightarrow & C^n & \xrightarrow{f_n} & B^n & \xrightarrow{g_n} & A^n & \longrightarrow & 0 \end{array}$$
where the horizontal sequences are exact. Now we will define a connecting homomorphism
$$\delta_n\colon \mathrm{H}^n(A) \to \mathrm{H}^{n+1}(C).$$
To do we perform the following steps:

(1) Let $z \in \mathrm{H}^n(A)$.
(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n^B(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) \coloneqq [\gamma] \in \mathrm{H}^{n+1}(C)$.

The following proposition explains the name "connecting homomorphism":

Proposition 77.13. (Cohomology–LES Proposition)

(1) The connecting homomorphisms are well-defined and they are indeed homomorphisms.

(2) The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of cochain complexes give rise to commuting diagrams of cohomology groups involving the connecting homomorphisms.
(3) If
$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$
is a short exact sequence of cochain complexes, then
$$\ldots \to \mathrm{H}^n(C) \xrightarrow{f_*} \mathrm{H}^n(B) \xrightarrow{g_*} \mathrm{H}^n(A) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(C) \xrightarrow{f_*} \mathrm{H}^{n+1}(B) \xrightarrow{g_*} \ldots$$
forms an exact sequence, which is usually called the Long Exact Sequence (LES) in cohomology.

Proof. In the Connecting Homomorphism Lemma 145.9 and the LES-Proposition 145.10 we will provide the proof of slightly more general statements. As the reader will see, the proofs are not very enlightening, and in a way each step of the proof is forced onto oneself, since at each step there is only one thing one can. Thus it is perhaps better to go with the following famous exercise of Serge Lang [**Lang1965**, p. 105] and [**Lang1984**, p. 175]: *Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book.* ■

77.3. **The Evaluation-Isomorphism Theorem.** We start out with the following definition, which is just a slight variation on the concept of a cochain complex that we introduced in Definition 77.1.

Definition 77.14. Let $\mathbb{F}$ be a field.
(1) A chain complex of $\mathbb{F}$-vector spaces is a sequence
$$\ldots \to D_n \xrightarrow{\partial_n} D_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \to 0$$
of homomorphisms between $\mathbb{F}$-vector spaces such that the composition of any two successive homomorphisms is zero, i.e. for all $i \in \mathbb{N}_0$ we have $\partial_n \circ \partial_{n+1} = 0$. We refer to the maps ∂_* as boundary maps.
(2) We define the n-th homology group of a chain complex (D_*, ∂_*) of $\mathbb{F}$-vector spaces to be the $\mathbb{F}$-vector space
$$\mathrm{H}_n(D) := \frac{\ker(\partial_n \colon D_n \to D_{n-1})}{\mathrm{im}(\partial_{n+1} \colon D_{n+1} \to D_n)}.$$
Note that the homology groups are in fact $\mathbb{F}$-vector spaces.

As we will see, there are two ways to go from cochain complexes to chain complexes and back. The first one is completely formal, the second is a little more interesting. So let us start with the formal lemma:

Lemma 77.15. (Index Reversal Lemma) Let $\mathbb{F}$ be a field and let
$$0 \leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \ldots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$
be a cochain complex of $\mathbb{F}$-vector spaces. Given $i \in \{0, \ldots, n\}$ we set $D'_i := D^{n-i}$ and $\partial_i := \delta_{n-i}$. Then
$$0 \to D'_n \xrightarrow{\partial_n} D'_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} D'_1 \xrightarrow{\partial_1} D'_0 \to 0$$
is a chain complex of $\mathbb{F}$-vector spaces.

Proof. This lemma is of course trivial. ■

To formulate our second way to relate cochain complexes to chain complexes we need the following piece of notation:

Notation 77.16. Let $\mathbb{F}$ be a field. Given a homomorphism

$$\varphi\colon V \to W$$

of $\mathbb{F}$-vector spaces we introduce the dual homomorphism

$$\begin{array}{rcl} \varphi^*\colon \mathrm{Hom}_{\mathbb{F}}(W,\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F}) \\ (\psi\colon W \to \mathbb{F}) & \mapsto & (\psi \circ \varphi\colon V \to \mathbb{F}). \end{array}$$

Lemma 77.17. (Dual Cochain Complex Lemma) Let $\mathbb{F}$ be a field and let

$$\ldots \to D_n \xrightarrow{\partial_n} D_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \to 0$$

be a chain complex of $\mathbb{F}$-vector spaces. Then

$$\ldots \leftarrow \mathrm{Hom}_{\mathbb{F}}(D_2,\mathbb{F}) \xleftarrow{(\partial_2)^*} \mathrm{Hom}_{\mathbb{F}}(D_1,\mathbb{F}) \xleftarrow{(\partial_1)^*} \mathrm{Hom}_{\mathbb{F}}(D_0,\mathbb{F}) \leftarrow 0$$

is a cochain complex of $\mathbb{F}$-vector spaces.

Proof. As we discussed on page 238, the two maps $V \mapsto \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ and $\varphi \mapsto \varphi^*$ define a contravariant functor from the category $\mathcal{V}ec_{\mathbb{F}}$ of $\mathbb{F}$-vector spaces to itself. This implies in particular that for any two homomorphisms $\varphi\colon U \to V$ and $\psi\colon V \to W$ of $\mathbb{F}$-vector spaces we have the equality $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. In the setting of the lemma this implies that we have $\partial_n^* \circ \partial_{n-1}^* = (\partial_{n-1} \circ \partial_n)^* = 0^* = 0$. ■

Now we can formulate the main result of this section:

Theorem 77.18. (Evaluation-Isomorphism Theorem) Let $\mathbb{F}$ be a field and let (V_*, ∂_*) be a chain complex of $\mathbb{F}$-vector spaces. The map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}}\colon \mathrm{H}^k(\mathrm{Hom}_{\mathbb{F}}(V_*,\mathbb{F})) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*),\mathbb{F}). \\ [\varphi\colon V_k \to \mathbb{F}] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_k(V_*) & \to & \mathbb{F} \\ [v] & \mapsto & \varphi(v) \end{array} \right) \end{array}$$

is well-defined, it is natural, and it is an isomorphism of $\mathbb{F}$-vector spaces.

Remark. In the Algebraic Universal Coefficient-H*-Theorem 201.22 we will prove an analog of the Evaluation-Isomorphism Theorem 77.18 where, instead of $\mathbb{F}$-vector spaces, we consider $\mathbb{Z}$-modules. □

The proof of the Evaluation-Isomorphism Theorem 77.18 rests on the shoulders of the following lemma:

Lemma 77.19. (Exactness Preservation Lemma) Let $\mathbb{F}$ be a field. Furthermore let $0 \to B \xrightarrow{i} Z \xrightarrow{p} H \to 0$ be a short exact sequence of $\mathbb{F}$-vector spaces. Then the sequence

$$0 \to \mathrm{Hom}_{\mathbb{F}}(H,\mathbb{F}) \xrightarrow{p^*} \mathrm{Hom}_{\mathbb{F}}(Z,\mathbb{F}) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{F}}(B,\mathbb{F}) \to 0$$

is also exact.

Sketch of proof of Lemma 77.19. We perform the following steps:

(1) First we show that $p\colon Z \to H$ admits a right inverse $\varphi\colon H \to \mathbb{Z}$. To do so we pick a basis $\mathcal{B}$ for H and for each $b \in \mathcal{B}$ we pick a $z_b \in Z$ with $p(z_b) = b$. Since $\mathcal{B}$ is a basis there exists a unique homomorphism $\varphi\colon H \to Z$ such that $\varphi(b) = z_b$ for all $b \in \mathcal{B}$. Thus $p \circ \varphi = \mathrm{id}_H$.

(2) Using the exactness of the given sequence one can easily verify that the map

$$\begin{array}{rcl} \Theta\colon B\times H & \to & Z \\ (b,h) & \mapsto & i(b)+\varphi(h) \end{array}$$

is an isomorphism.

(3) We denote by $j\colon B\to B\times Z$ the natural inclusion and we denote by $q\colon B\times Z\to Z$ the natural projection.

(4) We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(H,\mathbb{F}) & \xrightarrow{p^*} & \mathrm{Hom}_{\mathbb{F}}(Z,\mathbb{F}) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathbb{F}}(B,\mathbb{F}) & \longrightarrow & 0. \\ & & & {\scriptstyle q^*}\searrow & \cong\downarrow{\scriptstyle \Theta^*} & \nearrow{\scriptstyle j^*} & & & \\ & & & & \mathrm{Hom}_{\mathbb{F}}(B\times H,\mathbb{F}) & & & & \end{array}$$

It follows from $p\circ\Theta=q$ and $\Theta\circ j=i$ that the diagram commutes.

(5) I follows from (2) that the vertical map is an isomorphism.

(6) It is elementary to verify that the lower sequence is exact.

(7) It follows from all of the above that the top horizontal sequence is exact.

We leave it to the reader to fill in the few remaining details. ■

Proof of the Evaluation-Isomorphism Theorem 77.18. Let $\mathbb{F}$ be a field and let (V_*,∂_*) be a chain complex of $\mathbb{F}$-vector spaces. First note that it follows easily from the definitions that the map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}}\colon \mathrm{H}^k(\mathrm{Hom}_{\mathbb{F}}(V_*,\mathbb{F})) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V),\mathbb{F}). \\ [(\varphi\colon V_k\to\mathbb{F})] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_k(V_*) & \to & \mathbb{F} \\ [v] & \mapsto & \varphi(v) \end{array}\right) \end{array}$$

is well-defined and that it is natural. So it remains to show that it is an isomorphism.

For each $n\in\mathbb{N}_0$ we write $Z_n\coloneqq\ker(\partial_n\colon V_n\to V_{n-1})$ and $B_n\coloneqq\mathrm{im}(\partial_{n+1}\colon V_{n+1}\to V_n)$. We consider the following diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \xrightarrow{\iota} & V_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \xrightarrow{\iota} & V_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Note that the vertical maps define three chain complexes, that the horizontal sequences are exact and that the diagram commutes. In other words, we have a short exact sequence of vertical chain complexes.

We apply the contravariant functor $A\mapsto\mathrm{Hom}_{\mathbb{F}}(A,\mathbb{F})$ to the above short exact sequence of chain complexes. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" chain cocomplexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}) & \xrightarrow{\partial_n^*} & \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F}) & \xrightarrow{\iota^*} & \mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(B_{n-2},\mathbb{F}) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}_{\mathbb{F}}(V_{n-1},\mathbb{F}) & \xrightarrow{\iota^*} & \mathrm{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Dual Cochain Complex Lemma 77.17 that this is a sequence of cochain complexes.

(2) Since we work over a field we obtain from the Exactness Preservation Lemma 77.19 that the above horizontal sequences of cochain complexes are still exact. ⊞

We apply the Cohomology–LES Proposition 77.13 to the short exact sequence of cochain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F})}_{=\mathrm{H}^n(Z_*;\mathbb{F})} \to \underbrace{\mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F})}_{=\mathrm{H}^n(B_*;\mathbb{F})}.$$

In fact, it follows easily from Definition 77.12 that the connecting homomorphism $\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F})$ is the map that is induced by the inclusion $i_n\colon B_n \to Z_n$. By the Cohomology–LES Proposition 77.13 this connecting homomorphism gives us the following long exact sequence:

$$\begin{aligned}
\cdots \to \mathrm{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) \xrightarrow{i_n^*} \mathrm{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}) &\to \mathrm{H}^n(\mathrm{Hom}(V_*,\mathbb{F})) \\
&\to \mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \xrightarrow{i_{n-1}^*} \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F}) \to \dots
\end{aligned}$$

It follows easily from the definitions that we now obtain the following short exact sequence

$$0 \to \overbrace{\mathrm{coker}(\mathrm{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}))}^{\substack{=0 \text{ by the Exactness Preservation Lemma 77.19} \\ \text{applied to the short exact} \\ \text{sequence } 0 \to B_n \to Z_n \to \mathrm{H}_n(V_*) \to 0}} \to \mathrm{H}^n(\mathrm{Hom}_{\mathbb{F}}(V_*,\mathbb{F})) \to \underbrace{\ker(\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F}))}_{\substack{\text{isomorphism to } \mathrm{Hom}(\mathrm{H}_n(V_*),\mathbb{F}) \text{ given by} \\ \text{the Exactness Preservation Lemma 77.19} \\ \text{applied to the short exact sequence} \\ 0 \to B_n \to Z_n \to \mathrm{H}_l(V_*) \to 0}} \to 0.$$

Finally note that if one goes carefully through the definition of the explicit isomorphism $\mathrm{H}^n(\mathrm{Hom}_{\mathbb{F}}(V_*,\mathbb{F})) \to \ker(\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F}))$, then one sees that it is given by the evaluation homomorphism. ■

We conclude this algebraic chapter with the following generalization of the Exactness Preservation Lemma 77.19:

Lemma 77.20. (Generalized Exactness Preservation Lemma) Let $\mathbb{F}$ be a field. If

$$\dots \xrightarrow{d_{k+2}} A_{k+1} \xrightarrow{d_{k+1}} A_k \xrightarrow{d_k} A_{k-1} \xrightarrow{d_{k-1}} \dots$$

is an exact sequence of homomorphisms between $\mathbb{F}$-vector spaces, then so is

$$\dots \xrightarrow{d^*_{k+2}} A^*_{k+1} \xrightarrow{d^*_{k+1}} A^*_k \xrightarrow{d^*_k} A^*_{k-1} \xrightarrow{d^*_{k-1}} \dots$$

$$\dots \xrightarrow{d^*_{k+2}} \mathrm{Hom}_{\mathbb{F}}(A_{k+1},\mathbb{F}) \xrightarrow{d^*_{k+1}} \mathrm{Hom}_{\mathbb{F}}(A_k,\mathbb{F}) \xrightarrow{d^*_k} \mathrm{Hom}_{\mathbb{F}}(A_{k-1},\mathbb{F}) \xrightarrow{d^*_{k-1}} \dots$$

Proof. One can either prove this lemma by hand or one can observe that it follows immediately from the Evaluation-Isomorphism Theorem 77.18. ■

77.4. Exercises for Chapter 77.

Exercise 77.1. Let $\mathbb{F}$ be a field.

(a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of homomorphisms between finite-dimensional $\mathbb{F}$-vector spaces. Show that $\dim(B) = \dim(A) + \dim(C)$.

(b) Let $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$ be an exact sequence of homomorphisms between finite-dimensional $\mathbb{F}$-vector spaces. Show that

$$\sum_{i=0}^{k} (-1)^i \cdot \dim_{\mathbb{F}}(C_i) \;=\; 0.$$

Exercise 77.2. Let $\mathbb{F}$ be a field and let

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle a} & & \cong\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \cong\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

be a diagram of homomorphisms of $\mathbb{F}$-vector spaces such that the horizontal sequences are exact and such that every square commutes up to a fixed unit in $\mathbb{F}$. Show that if a is an epimorphism, if b and d are isomorphisms, and if e is a monomorphism, then c is an isomorphism.

Exercise 77.3. Let $\mathbb{F}$ be a field. For the following conditions (a)–(d) provide a short exact sequence of cochain complexes $0 \to A^* \to B^* \to C^* \to 0$ of $\mathbb{F}$-vector spaces such that the connecting homomorphism $\mathrm{H}^0(C^*) \to \mathrm{H}^1(A^*)$ is

(a) . . . the zero homomorphism,
(b) . . . an isomorphism,
(c) . . . a monomorphism but not an epimorphism,
(d) . . . an epimorphism but not a monomorphism.

Exercise 77.4. Let $\mathbb{F}$ be a field. We consider the following commutative diagram of homomorphisms between $\mathbb{F}$-vector spaces.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow \\
0 & \longrightarrow & \widetilde{A} & \longrightarrow & \widetilde{B} & \longrightarrow & \widetilde{C} & \longrightarrow & 0.
\end{array}$$

We suppose that the horizontal sequences are exact and that the maps a and c to the left and right are the zero map. Does it follow that the middle map b is also the zero map?

Exercise 77.5. Let $\mathbb{F}$ be a field and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of homomorphisms between $\mathbb{F}$-vector spaces. Show that there exists an exact sequence

$$\begin{aligned} 0 \longrightarrow \ker(a) \xrightarrow{\varphi} \ker(b) \xrightarrow{\psi} \ker(c) \\ \longrightarrow \operatorname{coker}(a) \xrightarrow{\varphi'} \operatorname{coker}(b) \xrightarrow{\psi'} \operatorname{coker}(c) \longrightarrow 0. \end{aligned}$$

This statement is sometimes referred to as the "Snake Lemma".

Exercise 77.6. Let $\mathbb{F}$ be a field and let

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$$

be a short exact sequence of chain complexes of $\mathbb{F}$-vector spaces. We consider the following diagram

$$\begin{array}{ccccccccc} \to \mathrm{H}^k(\mathrm{Hom}_{\mathbb{F}}(C_*,\mathbb{F})) & \xrightarrow{g^*} & \mathrm{H}^k(\mathrm{Hom}_{\mathbb{F}}(B_*,\mathbb{F})) & \xrightarrow{f^*} & \mathrm{H}^k(\mathrm{Hom}_{\mathbb{F}}(A_*,\mathbb{F})) & \xrightarrow{(\partial_k)^*} & \mathrm{H}^{k+1}(\mathrm{Hom}_{\mathbb{F}}(C_*,\mathbb{F})) \to \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} \\ \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(C_*),\mathbb{F}) & \xrightarrow{g^*} & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(B_*),\mathbb{F}) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(A_*),\mathbb{F}) & \xrightarrow{(\partial_k)^*} & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}^{k+1}(C_*),\mathbb{F}) \to \end{array}$$

where the vertical maps are given by the Evaluation-Isomorphism Theorem 77.18. Show that the diagram commutes.

CHAPTER 78

De Rham cohomology of smooth manifolds

In this chapter we can finally introduce the key player of this part of the lecture notes, namely the de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M)$ of a smooth manifold M. The study of the de Rham cohomology groups will be the main topic of the remainder of this part of the lecture notes.

This chapter is organized as follows:

- In Section 78.1 we will define de Rham cohomology of a smooth manifold.
- In Section 78.2 we will perform a few basic calculations of de Rham cohomology.
- In Section 78.3 we will discuss the relationship between De Rham cohomology and integration.
- In Section 78.4 we will see that the wedge product on differential forms turns de Rham cohomology into a ring.
- In Section 78.5 we will provide exercises for Chapter 78.

In the next Chapter 79 we will study the interplay between de Rham cohomology and homotopies.

78.1. **Definition of de Rham cohomology of a smooth manifold.** We start out by recalling some notation:

Notation 78.1.

(1) Given a smooth manifold M and given $m \in \mathbb{N}_0$ we set, as in Notation 74.5,

$$\Omega^m(M) \;:=\; \text{vector space of all smooth } m\text{-forms on } M.$$

For convenience we also set $\Omega^{-1}(M) := \{0\}$.

(2) In Definition 75.5 we introduced the differential

$$d_k \colon \Omega^k(M) \;\to\; \Omega^{k+1}(M).$$

Often, when the subscript k is understood from the context, we just write d instead of d_k.

Definition 78.2. Let M be a smooth manifold. By the Differentials-on-Manifolds Proposition 75.7 (4) we know that the differentials $d_k \colon \Omega^k(M) \to \Omega^{k+1}(M)$ have the property that $d_{k+1} \circ d_k = 0$. Thus we obtain a cochain complex

$$\ldots \;\leftarrow\; \Omega^2(M) \;\xleftarrow{d_1}\; \Omega^1(M) \;\xleftarrow{d_0}\; \Omega^0(M) \;\leftarrow\; 0$$

of real vector spaces in the sense of the discussion in Definition 77.1. For $k \in \mathbb{N}_0$ the k–th de Rham cohomology group[812] of M is defined as the cohomology of this cochain complex, i.e. it is defined as

$$\mathrm{H}^k_{\mathrm{dR}}(M) \coloneqq \frac{\ker\left(d_k\colon \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\mathrm{im}\left(d_{k-1}\colon \Omega^{k-1}(M) \to \Omega^k(M)\right)}.$$

Note that $\mathrm{H}^k_{\mathrm{dR}}(M)$ is a real vector space.

At this point it is worth recalling Definition 75.9:

Definition 78.3. Let M be a smooth manifold.

(1) A smooth form ω with $d\omega = 0$ is called closed.

(2) A smooth form ω for which there exists a smooth form μ with $d\mu = \omega$ is called exact.[813]

Note that with this language we have

$$\mathrm{H}^k_{\mathrm{dR}}(M) = \frac{\text{vector space of closed } k\text{-forms}}{\text{vector space of exact } k\text{-forms}}.$$

We have now associated real vector spaces to smooth manifolds. The real goal is to define contravariant functors from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces. It is thus opportune to recall the following notation:

Notation 78.4. Let $f\colon M \to N$ be a smooth map between smooth manifolds. In Definition 74.2 we saw that every m-form ω on N defines a pullback m-form $f^*(\omega)$ on M. In the Pullback Smooth Form Proposition 74.7 we saw that the pullback of a smooth differential form is again smooth. In other words, we have a homomorphism $f^*\colon \Omega^m(N) \to \Omega^m(M)$.

Proposition 78.5. ($\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition)

(1) If $f\colon M \to N$ is a smooth map between two smooth manifolds, then for every $k \in \mathbb{N}_0$

$$\begin{array}{rcl} f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) & \to & \mathrm{H}^k_{\mathrm{dR}}(M) \\ [\omega] & \mapsto & [f^*(\omega)] \end{array}$$

is a well-defined homomorphism of $\mathbb{R}$-vector spaces.

(2) Recall that we denote by $\mathcal{SmMfd}$ the category of smooth manifolds that is given by

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{SmMfd}) & \coloneqq & \text{all smooth manifolds,} \\ \mathrm{Mor}(M,N) & \coloneqq & \text{all smooth maps from } M \text{ to } N. \end{array}$$

For every $k \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} M & \mapsto & \mathrm{H}^k_{\mathrm{dR}}(M) \\ (f\colon M \to N) & \mapsto & (f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)) \end{array}$$

define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

Proof.

(1) Statement (1) is an immediate consequence of the Differential-Natural Proposition 75.8 (which says that the differential commutes with pulling back differential forms) together with the Cochain Complex-H^*-Lemma 77.4.

[812] It would make more sense to call these objects "de Rham cohomology vector spaces", but for the sake of readability we stick to the more concise "de Rham cohomology groups".

[813] The suspicious reader will have noticed that we now introduced the notion "exact" twice. These notions are related: The cochain complex given by the $\Omega^i(M)$ is exact in the sense of Definition 77.8 if and only if each closed form is exact in the sense of the current definition.

(2) This statement follows easily from the definitions. ■

Example. Let $f\colon M \to N$ be a *diffeomorphism* between manifolds. It follows immediately from the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 78.5 and the elementary Isomorphism Preservation Lemma 5.26 that for all $k \in \mathbb{N}_0$ the induced map $f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ is an isomorphism. We will often make use of this observation without mentioning the logic explicitly. □

78.2. **First calculations of de Rham cohomology.** The de Rham cohomology groups are initially quite mysterious. In this section we will perform a couple of elementary calculations to get a sense for what is happening. We start out with the most basic calculation, which often gets used implicitly without referring to it:

Lemma 78.6. (Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) For the empty smooth manifold $\varnothing$ we have $\mathrm{H}^k_{\mathrm{dR}}(\varnothing) = 0$ for all $k \in \mathbb{N}_0$.

Proof. This lemma follows immediately from the observation that in this case we know, by the discussion on page 98, that $\Omega^k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$. ■

The following lemma implies that for calculations of de Rham cohomology groups one can usually restrict oneself to connected smooth manifolds:

Lemma 78.7. (Path-Component-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) Let M be a smooth manifold and let $\{M_s\}_{s\in S}$ be the path-components[814] of M. Given $s \in S$ we denote by $i_s\colon M_s \to M$ the inclusion map. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \prod\limits_{s\in S} \iota_s^*\colon \mathrm{H}^k_{\mathrm{dR}}(M) & \to & \prod\limits_{s\in S} \mathrm{H}^k_{\mathrm{dR}}(M_s) \\ \varphi & \mapsto & (i_s^*(\varphi))_{s\in S} \end{array}$$

is an isomorphism.

Proof. It follows from the Product–H^*-Lemma 77.7 that it suffices to prove the following claim:

Claim. The maps

$$\begin{array}{rcl} \prod\limits_{s\in S} \iota_s^*\colon \Omega^k(M) & \to & \prod\limits_{s\in S} \Omega^k(M_s) \\ \omega & \mapsto & (i_s^*(\omega))_{s\in S} \end{array}$$

define an isomorphism of cochain complexes.

Proof. It is basically clear that this map is a cochain map between cochain complexes. Since $M = \bigsqcup_{s\in S} M_s$ and since $\iota_s^*\colon \Omega^k(M) \to \Omega^k(M_s)$ is the restriction map we see that for each $k \in \mathbb{N}_0$ the map is injective. It remains to show that the map is surjective. So suppose that for each $s \in S$ we are given a smooth k-form ω_s on M_s. Since $M = \bigsqcup_{s\in S} M_s$ there exists a unique k-form ω on M such that for each s we have $\omega|_{M_s} = \omega_s$. Now note that it follows from the Manifold–Local Properties Lemma 23.8 (3) that all components M_s are open subsets of M. Thus it follows from the Smooth Form–Pasting Proposition 74.8 that ω is in fact smooth. ■

We move on to a somewhat more interesting calculation:

[814] Often we will make (sub-) consciously use of the Manifold–Local Properties Lemma 23.8 (3a) which implies that for smooth manifolds the components are precisely the path components.

Proposition 78.8. ($\mathrm{H}^0_{\mathrm{dR}}$-Proposition)

(1) Given any smooth manifold M the map

$$\begin{array}{rcl} \zeta_M\colon \mathbb{R} & \to & \mathrm{H}^0_{\mathrm{dR}}(M) \\ r & \mapsto & [\text{constant function } x \mapsto r] \end{array}$$

is a well-defined homomorphism and it is natural. Here by "natural" we mean that for any smooth map $f\colon M \to N$ between smooth manifolds the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} \\ \downarrow{\scriptstyle \zeta_N} & & \downarrow{\scriptstyle \zeta_M} \\ \mathrm{H}^0_{\mathrm{dR}}(N) & \xrightarrow{f^*} & \mathrm{H}^0_{\mathrm{dR}}(M). \end{array}$$

(2) If M is a path-connected non-empty smooth manifold, then the map $\zeta_M\colon \mathbb{R} \to \mathrm{H}^0_{\mathrm{dR}}(M)$ from (1) is an isomorphism. For any point $P \in M$ an inverse is given by the map

$$\begin{array}{rcl} \mathrm{H}^0_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [f] & \mapsto & f(P). \end{array}$$

Example. Let $f\colon M \to N$ be a smooth map between path-connected non-empty smooth manifolds. It follows from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (1) and (2) that the induced map $f^*\colon \mathrm{H}^0_{\mathrm{dR}}(N) \to \mathrm{H}^0_{\mathrm{dR}}(M)$ is an isomorphism. □

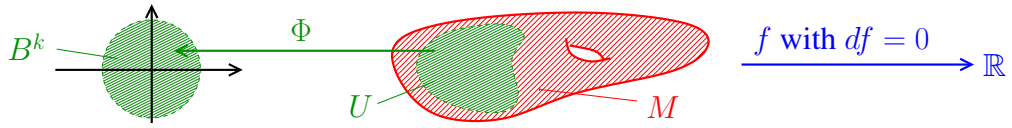
Proof. Let M be a k-dimensional smooth manifold. Note that

$$\mathrm{H}^0_{\mathrm{dR}}(M) \underset{\substack{\uparrow \\ \text{since } \Omega^{-1}(M)=0}}{=} \ker\{\Omega^0(M) \to \Omega^1(M)\} \underset{\substack{\uparrow \\ \text{since } \Omega^0(M)=\{\text{smooth maps } M\to\mathbb{R}\}}}{=} \{f\colon M \to \mathbb{R} \mid df = 0\}.$$

The key to proving the lemma is the following claim:

Claim. The smooth maps $f\colon M \to \mathbb{R}$ with $df = 0$ are precisely the locally constant[815] maps.

Proof. Let $f\colon M \to \mathbb{R}$ be a smooth map. Let $\Phi\colon U \to V$ be chart.



We make the following observations:

(a) f is constant on U if and only if $f \circ \Phi^{-1}$ is constant on V.

(b) $$df = 0 \text{ on } U \underset{\substack{\uparrow \\ \text{since } \Phi \text{ is a} \\ \text{diffeomorphism}}}{\iff} \Phi^*(df) = 0 \underset{\substack{\uparrow \\ \text{by the Differential-Natural} \\ \text{Proposition 75.8}}}{\iff} d(\Phi^* f) = 0 \underset{\substack{\uparrow \\ \text{Definition 74.2 of } \Phi^* f}}{\iff} d(f \circ \Phi^{-1}) = 0.$$

(c) Let $g\colon V \to \mathbb{R}$ be a smooth map. By the 1-Form-via-dx^i-Lemma 71.5 we have $dg = 0$ if and only if all partial derivatives of g vanish.

Since every point on M admits a chart to B^k or to $B^k_{\geq 0} := B^k \cap \mathbb{H}^k$ we obtain from (a) and (b) that it suffices to prove the claim for smooth maps on the open k-ball $V = B^k$ and

[815] Note that according to Definition 4.61 a map $f\colon M \to \mathbb{R}$ is locally constant if every $P \in M$ admits an open neighborhood U such that $f|_U$ is constant.

on the open half-ball $V = B^k_{\geq 0}$. Given a constant map $g\colon V \to \mathbb{R}$ it follows from (c) that $dg = 0$. Conversely, let $g\colon V \to \mathbb{R}$ be a smooth map with $dg = 0$. By (c) this means that all partial derivatives of g vanish. It follows easily from the Mean Value Theorem from calculus and the fact that we are considering the open ball B^k or the half-open $B^k_{\geq 0}$ that this means that g has to be constant. ⊞

We have thus shown that

$$\mathrm{H}^0_{\mathrm{dR}}(M) \;=\; \{\text{all locally constant maps } M \to \mathbb{R}\}.$$

We now finally turn to the proofs of the two statements:

(1) It follows from the claim that the map $\zeta_M\colon \mathbb{R} \to \mathrm{H}^0_{\mathrm{dR}}(M)$ is well-defined. It is clear that this map is natural.

(2) Now we assume that M is path-connected. It follows easily that the right-hand side of the above description of $\mathrm{H}^0_{\mathrm{dR}}(M)$ is just the set of all constant maps $M \to \mathbb{R}$. Finally if M is non-empty, then it follows basically immediately from the above description of $\mathrm{H}^0_{\mathrm{dR}}(M)$ that $\zeta_M\colon \mathbb{R} \to \mathrm{H}^0_{\mathrm{dR}}(M)$ is an isomorphism and that for any $P \in M$ the given evaluation map $\mathrm{H}^0_{\mathrm{dR}}(M) \to \mathbb{R}$ is an inverse. ■

At the other end of the dimension spectrum we have an even more elementary lemma:

Lemma 78.9. (High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) Let M be a k-dimensional smooth manifold.

(1) (a) For $i > k$ we have $\Omega^i(M) = 0$.
(b) For $i > k$ we have $\mathrm{H}^i_{\mathrm{dR}}(M) = 0$.

(2) For every $\omega \in \Omega^k(M)$ we have $d\omega = 0$, i.e. every $\omega \in \Omega^k(M)$ represents a de Rham cohomology class.

Proof. Let M be a k-dimensional smooth manifold.

(1) (a) By the Tangent Space-Dimension Proposition 29.17 we know that each tangent space $\mathrm{T}_P M$ is a real vector space of dimension k. Now let $i > k$. It follows from the Alternating Criterion Lemma 72.3 that for all $P \in M$ we have $\wedge^i \mathrm{T}^*_P M = 0$. In particular we have $\Omega^i(M) = 0$.
(b) This statement follows immediately from (a).

(2) Conveniently enough this statement also follows immediately from (1a). ■

The results we have obtained so far now allow us to calculate the de Rham cohomology of singletons, i.e. of smooth manifolds consisting of a single point. This sounds like a modest achievement, but later we will see, that this calculation will be step 0 in several induction arguments.

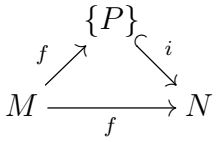
Lemma 78.10. (Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) If $M = \{*\}$ is a smooth manifold that consists of a single point $*$, then

$$\mathrm{H}^k_{\mathrm{dR}}(\{*\}) \;\cong\; \begin{cases} 0, & \text{if } k \geq 1, \\ \mathbb{R}, & \text{if } k = 0. \end{cases}$$

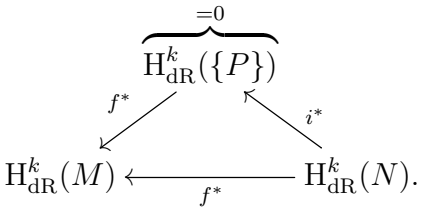
Proof. Since $\{*\}$ is a connected non-empty 0-dimensional smooth manifold we see that the statement follows immediately from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (2) together with the High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.9 (1b). ■

Example. Let M and N be non-empty smooth manifolds. Let $f\colon M \to N$ be a smooth map. We suppose that f is constant, i.e. we suppose that there exists a $P \in N$ such that

$f(M) = \{P\}$. We denote by $i\colon \{P\} \to N$ the inclusion. Finally let $k \in \mathbb{N}$. We consider the commutative diagram



which, by the functoriality of de Rham cohomology, gives rise to the commutative diagram



By the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10 we know that $\mathrm{H}^k_{\mathrm{dR}}(\{P\}) = 0$. Thus we see that the induced map $f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ factors through the zero vector space, i.e. the map $f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ is the zero map. On many occasions we will make use of this observation without referring to this discussion. □

78.3. **De Rham cohomology and integration.** The following proposition is a very useful tool for showing that certain de Rham cohomology classes are non-zero:

Proposition 78.11. (Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition)
(1) For every closed oriented k-dimensional smooth manifold the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \displaystyle\int_M \omega \end{array}$$

is well-defined.
(2) If M is non-empty, then the map from (1) is an epimorphism.

Proof.

(1) It follows immediately from the Stokes-on-Closed Corollary 75.12 that the homomorphism is well-defined.
(2) Next we assume that M is non-empty. Note that it follows from the High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.9 (2) that every $\omega \in \Omega^k(M)$ represents a de Rham cohomology class. Fortunately we already showed in the Non-Zero Integral Existence Lemma 74.16 that on any oriented non-empty k-dimensional smooth manifold we can find a smooth k-form whose integrals is non-zero. So we are done. ■

By playing our cards well we can also use this proposition to show that certain cohomology classes of lower degree are also non-zero:

Example. We consider the torus $S^1 \times S^1$ together with the projections

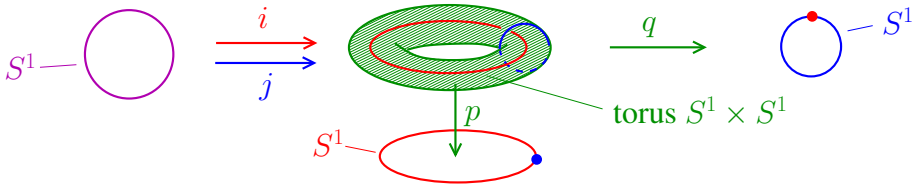
$$\begin{array}{rcl} p\colon S^1 \times S^1 & \to & S^1 \\ (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon S^1 \times S^1 & \to & S^1 \\ (x,y) & \mapsto & y. \end{array}$$

Note that it follows from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 (2) that there exists a non-zero cohomology class $\theta \in \mathrm{H}^1_{\mathrm{dR}}(S^1)$. We consider the cohomology classes $p^*(\theta), q^*(\theta) \in \mathrm{H}^2_{\mathrm{dR}}(S^1 \times S^1)$. We claim that these are linearly independent. To show this we consider the smooth embeddings

$$\begin{array}{rcl} i\colon S^1 & \to & S^1 \times S^1 \\ z & \mapsto & (z,1) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon S^1 & \to & S^1 \times S^1 \\ z & \mapsto & (1,z) \end{array}$$

These allow us to define the following map:

$$\begin{array}{rcl}\Phi\colon \mathrm{H}^1_{\mathrm{dR}}(S^1\times S^1) & \to & \mathrm{H}^1_{\mathrm{dR}}(S^1)\times \mathrm{H}^1_{\mathrm{dR}}(S^1)\\ \rho & \mapsto & (i^*(\rho), j^*(\rho)).\end{array}$$



Note that $p\circ i = \mathrm{id}_{S^1}$, $q\circ j = \mathrm{id}_{S^1}$ and that $p\circ j$ and that $q\circ i$ are constant maps. It follows from these observations together with the discussion on page 1596 and the functoriality of de Rham cohomology that $\Phi(p^*(\theta)) = (\theta, 0)$ and $\Phi(q^*(\theta)) = (0,\theta)$. It follows that the cohomology classes $p^*(\theta), q^*(\theta) \in \mathrm{H}^2_{\mathrm{dR}}(S^1\times S^1)$ are indeed linearly independent. □

78.4. **The wedge product on de Rham cohomology.** De Rham cohomology has one more ace up its sleeve, namely it has a natural product:

Proposition 78.12. (Wedge-on-de Rham Cohomology Proposition) Let M be a smooth manifold. For any $k, l \in \mathbb{N}_0$ the map

$$\begin{array}{rcl}\wedge\colon \mathrm{H}^k_{\mathrm{dR}}(M)\times \mathrm{H}^l_{\mathrm{dR}}(M) & \to & \mathrm{H}^{k+l}_{\mathrm{dR}}(M)\\ ([\omega],[\sigma]) & \mapsto & [\omega]\wedge[\sigma] \coloneqq [\underset{\uparrow}{\omega\wedge\sigma}]\end{array}$$

see Definition 74.1

is well-defined. It has the following properties:

(1) The map $\wedge\colon \mathrm{H}^k_{\mathrm{dR}}(M)\times \mathrm{H}^l_{\mathrm{dR}}(M) \to \mathrm{H}^{k+l}_{\mathrm{dR}}(M)$ is $\mathbb{R}$-bilinear.

(2) For every $a \in \mathrm{H}^k_{\mathrm{dR}}(M)$, $b \in \mathrm{H}^l_{\mathrm{dR}}(M)$ and $c \in \mathrm{H}^m_{\mathrm{dR}}(M)$ we have

$$a\wedge(b\wedge c) \;=\; (a\wedge b)\wedge c.$$

(3) For every $a \in \mathrm{H}^k_{\mathrm{dR}}(M)$ and $b \in \mathrm{H}^l_{\mathrm{dR}}(M)$ we have

$$b\wedge a \;=\; (-1)^{\deg(a)\cdot\deg(b)}\cdot a\wedge b.$$

(4) Let $1_M \in \mathrm{H}^0_{\mathrm{dR}}(M)$ be the cohomology class that is defined by the constant function $x \mapsto 1$.[816] For any $\omega \in \mathrm{H}^k_{\mathrm{dR}}(M)$ we have

$$1_M\wedge\omega \;=\; \omega\wedge 1_M \;=\; \omega.$$

Proof. Let M be a smooth manifold. First we show that the given map is well-defined. To do so, recall that the Product Rule from the Differentials-on-Manifolds Proposition 75.7 (3) says that given $\omega \in \Omega^k(M)$ and $\sigma \in \Omega^l(M)$ we have

$$(*)\qquad d(\omega\wedge\sigma) \;=\; d\omega\wedge\sigma + (-1)^k\cdot\omega\wedge d\sigma.$$

Now let $\omega \in \Omega^k(M)$ and $\sigma \in \Omega^l(M)$ be smooth forms. We make the following observations:

(a) It follows immediately from $(*)$ that $\omega\wedge\sigma$ is again closed.

[816] Recall that we know by the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 that if the smooth manifold M is connected and non-empty, then $\mathrm{H}^0_{\mathrm{dR}}(M) = \mathbb{R}\cdot 1_M$.

(b) If $\omega' = \omega + d\tau$ is another representative of $[\omega]$, then

$$\omega' \wedge \sigma - \omega \wedge \sigma \underset{\text{since "}\wedge\text{" is } \mathbb{R}\text{-bilinear}}{=} (\omega' - \omega) \wedge \sigma = d\tau \wedge \sigma \underset{\text{by } (*) \text{ and since } d\sigma = 0}{=} d(\tau \wedge \sigma).$$

This shows that $\omega \wedge \sigma$ and $\omega' \wedge \sigma$ represent the same de Rham cohomology class, i.e. the wedge product does not depend on the choice of the representative of the de Rham cohomology class of the first entry.

(c) The proof that the wedge product does not depend on the choice of the representative of the second de Rham cohomology class is basically the same as the proof of (b).

Finally note that Statements (1), (2), (3) and (4) follow immediately from the definitions and the corresponding statements in the Wedge Product Proposition 72.8 and the Wedge Product Properties Lemma 72.10. ■

Example. We denote by dx and dy the usual 1-forms on $\mathbb{R}^2$. Note that these are closed. As on page 1541 we know that these forms are invariant under the usual $\mathbb{Z}^2$-action on $\mathbb{R}^2$. It follows from the G-Invariant Form Lemma 74.22 (1) together with the Pullback-Closed-Exact Lemma 75.10 (3) that dx and dy induce closed 1-forms $\overline{dx}$ and $\overline{dy}$ on $\mathbb{R}^2/\mathbb{Z}^2$. As on page 1540 we see that

$$\int_{\mathbb{R}^2/\mathbb{Z}^2} \overline{dx} \wedge \overline{dy} \underset{\substack{\text{by the } G\text{-Invariant} \\ \text{Form Lemma 74.22 (2)}}}{=} \int_{\mathbb{R}^2/\mathbb{Z}^2} \overline{dx \wedge dy} \underset{\substack{\text{as on page 1541, using the obvious compact} \\ \text{parametrization } [0,1]\times[0,1] \to \mathbb{R}^2/\mathbb{Z}^2}}{=} 1.$$

It follows from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 that $\overline{dx} \wedge \overline{dy}$ represents a non-trivial element in $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{R}^2/\mathbb{Z}^2)$. This shows that in general the wedge product on de Rham cohomology is non-trivial. □

The wedge product motivates us to consider all de Rham cohomology groups at once. This thought leads us to the following definition:

Definition 78.13. Given an n-dimensional smooth manifold M we set[817]

$$\mathrm{H}^*_{\mathrm{dR}}(M) = \bigoplus_{i=0}^{n} \mathrm{H}^i_{\mathrm{dR}}(M).$$

Furthermore given $a_i, b_i \in \mathrm{H}^i_{\mathrm{dR}}(M)$, $i = 0, \dots, n$ we set

$$\left(\sum_{i=0}^{n} a_i\right) \wedge \left(\sum_{i=0}^{n} b_i\right) := \sum_{i=0}^{n} \sum_{j=0}^{n} \underbrace{a_i \wedge b_j}_{\in \mathrm{H}^{i+j}_{\mathrm{dR}}(M)}.$$

It follows easily from the Wedge-on-de Rham Cohomology Proposition 78.12 that with this multiplication $\mathrm{H}^*_{\mathrm{dR}}(M)$ is a (possibly non-commutative) ring, in the sense of Definition 2.6.

We conclude this introductory chapter on de Rham cohomology groups with the following rather formal proposition:

[817] Given vector spaces $V_0, \dots, V_n$ we denote, as in the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 (3) by $\bigoplus_{i=0}^{n} V_i := V_0 \oplus \cdots \oplus V_n := \{(v_1, \dots, v_n) \mid v_i \in V_i\}$ the *direct sum* of $V_0, \dots, V_n$.

Proposition 78.14. ($\mathrm{H}^*_{\mathrm{dR}}$-Ring Functor Proposition) The maps

$$\begin{array}{rcl} M & \mapsto & \mathrm{H}^*_{\mathrm{dR}}(M) \\ (f\colon M \to N) & \mapsto & (f^*\colon \mathrm{H}^*_{\mathrm{dR}}(N) \to \mathrm{H}^*_{\mathrm{dR}}(M)) \end{array}$$

define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Ring}$ of rings.

Proof. By the Alternating Form-Functoriality Lemma 72.12 we know that for a homomorphism $f\colon U \to V$ between vector spaces and for all $\alpha \in \wedge^k V^*$ and $\beta \in \wedge^l V^*$ we have the equality

$$f^*(\alpha \wedge \beta) \;=\; f^*(\alpha) \wedge f^*(\beta).$$

The proposition follows easily from this observation together with the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 78.5. ∎

78.5. Exercises for Chapter 78.

Exercise 78.1. Let M be a connected n-dimensional smooth manifold. Show that if $\mathrm{H}^n_{\mathrm{dR}}(M) \neq 0$, then M is orientable.

Exercise 78.2. Let $\omega = f \cdot dx + g \cdot dy$ be a smooth 1-form on $\mathbb{R}^2$.

(a) Show that if $d\omega = 0$, then $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$.

(b) Show that if $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$, then there exists a smooth 0-form φ on $\mathbb{R}^2$ with $d\varphi = \omega$.

(c) Show that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}^2) = 0$.

Exercise 78.3.

(a) Let $P \in S^2$. Show that $\mathrm{H}^1_{\mathrm{dR}}(S^2 \setminus \{P\}) = 0$.
Hint. Use Exercise 78.2.

(b) Let $Q \in S^1 \times S^1$. Show that $\mathrm{H}^1_{\mathrm{dR}}((S^1 \times S^1) \setminus \{Q\}) \neq 0$.
Hint. Use that $\mathrm{H}^1_{\mathrm{dR}}(S^1) \neq 0$ and use functoriality of de Rham cohomology.

(c) Use (a) and (b) to show that S^2 and $S^1 \times S^1$ are not diffeomorphic.

Exercise 78.4. On $\mathbb{R}^3 \setminus \{(0,0,0)\}$ we consider the 2-form

$$\omega \;=\; \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot (x\,dy \wedge dx + y\,dz \wedge dx + z\,dx \wedge dy).$$

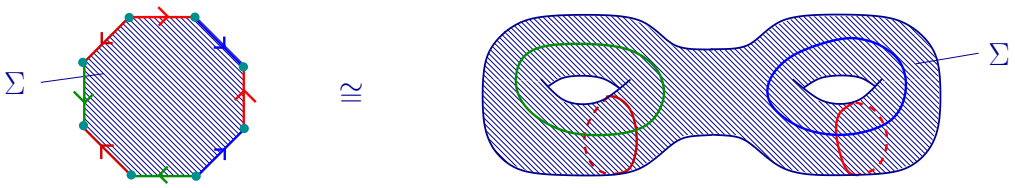
(a) Show that the restriction of ω to S^2 represents a non-zero element of $\mathrm{H}^2_{\mathrm{dR}}(S^2)$.

(b) Show that the 2-form ω on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ is closed.

(c) Is the cohomology class $[\omega] \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{R}^3 \setminus \{(0,0,0)\})$ zero or non-zero?

Exercise 78.5. We consider the surface of genus 2 as shown in the figure below in two different guises. Show that $\dim(\mathrm{H}^1_{\mathrm{dR}}) \geq 4$.
Hint. Proceed as in the case of the torus which we dealt with on page 1597. More precisely, in the figure we show the images of smooth embeddings $i_1, i_2, i_3, i_4\colon S^1 \to \Sigma$. Show that there exist smooth maps $p_1, p_2, p_3, p_4\colon \Sigma \to S^1$ such that $p_j \circ i_j = \mathrm{id}_{S^1}$ and such that for $j \neq k$ the map $p_j \circ i_k$ is a constant map.

Exercise 78.6. Let $n \in \mathbb{N}_0$. Show that there exist n de Rham cohomology classes $\theta_1, \ldots, \theta_n \in \mathrm{H}^1_{\mathrm{dR}}((S^1)^n)$ such that $\theta_1 \wedge \cdots \wedge \theta_n$ is non-zero.



Exercise 78.7. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n\}$. Show that

$$\dim_{\mathbb{R}} \left(\mathrm{H}^k_{\mathrm{dR}}(\text{the } n\text{-torus } (S^1)^n) \right) \ \geq \ \binom{n}{k}.$$

Hint. Use that by the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 we know that $\dim_{\mathbb{R}}(\mathrm{H}^k_{\mathrm{dR}}((S^1)^k)) \geq 1$ and apply the functoriality of de Rham cohomology to inclusions and projections.

Exercise 78.8. We consider the 0-dimensional smooth manifold $\mathbb{Z}$. Calculate $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{Z})$. It is pretty clear that this is an infinite-dimensional real vector space. But what can one say about the cardinality of a basis? Is it countable or uncountable?

Exercise 78.9. Let M be a smooth manifold of dimension $2n$. A *symplectic structure on* M is a 2-form $\omega \in \Omega_2(M)$ that is closed and such that $\omega^n \in \Omega_{2n}(M)$ is non-zero everywhere. A *symplectic manifold* is a pair (M, ω) as above.

(a) Show that $\mathbb{R}^{2n}$ admits a symplectic structure.
(b) Show that if M admits a symplectic structure, then M is orientable.
(c) Show that if M is closed and it admits a symplectic structure, then the de Rham cohomology groups $\mathrm{H}^{2i}_{\mathrm{dR}}(M)$ are non-zero for $i = 0, \ldots, n$.

CHAPTER 79

De Rham cohomology and homotopies

In this chapter we will show that if two smooth maps $f_0, f_1 \colon M \to N$ between two smooth manifolds M and N are homotopic, then the induced maps $f_0^*, f_1^* \colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ are the same. This result will allow us to show that if two smooth manifolds are homotopy equivalent, then their de Rham cohomology groups are isomorphic. This result in turn will give us the, not overly exciting, calculations of the de Rham cohomology groups of $\mathbb{R}^n$, B^k and $\overline{B}^k$.

The results of this chapter will develop their full potential in the next chapter where will prove the existence of "Mayer-Vietoris sequences" for de Rham cohomology groups. These sequences, in combination with the material of the current chapter, will then allow us to calculate the de Rham cohomology groups of large classes of smooth manifolds.

This chapter is organized as follows:

- In Section 79.1 we will discuss the relationship between De Rham cohomology and homotopic maps.
- In Section 79.2 we will show that the De Rham cohomology groups of homotopy equivalent smooth manifolds are isomorphic.
- In Appendix 79.3 we will discuss applications to classical vector calculus.
- In Section 79.4 we will provide exercises for Chapter 79.

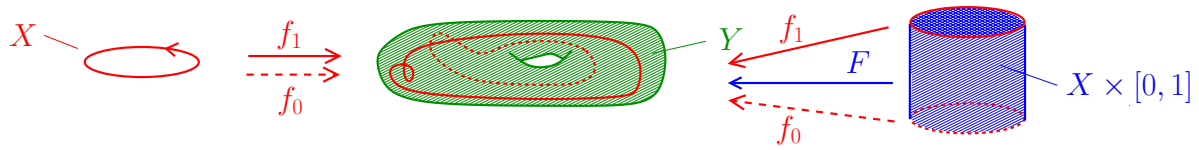
In the next Chapter 80 we will consider Mayer–Vietoris sequences in de Rham cohomology.

79.1. De Rham cohomology and homotopic maps.

It is worth recalling Definition 18.2:

Definition 79.1. Let X and Y be two topological spaces and let $f_0, f_1 \colon X \to Y$ be two continuous maps.

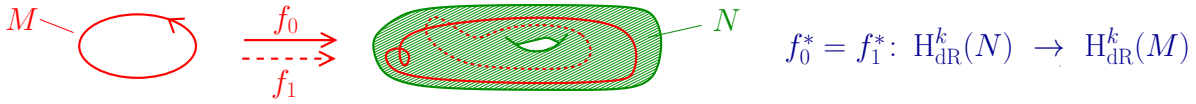
(1) A homotopy between the maps f_0 and f_1 is a continuous map $F \colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$.
(2) If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are homotopic, and we write $f_0 \simeq f_1$.



We know that smooth maps between smooth manifolds induce maps between the corresponding de Rham cohomology groups. The following proposition makes the bold claim that two smooth maps which are homotopic, induce the same map on de Rham cohomology:

Proposition 79.2. (Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition) Let f_0 and f_1 be two smooth maps between two smooth manifolds M and N. If f_0 and f_1 are homotopic, then for any $k \in \mathbb{N}_0$ we have

$$f_0^* = f_1^* \colon \mathrm{H}^k_{\mathrm{dR}}(N) \ \to \ \mathrm{H}^k_{\mathrm{dR}}(M).$$



This chapter is organized as follows:

(1) The remainder of this section is occupied with the proof of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2.
(2) In the next Section 79.2 we will consider applications of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2. In particular we will calculate the de Rham cohomology groups of contractible smooth manifolds.
(3) In Appendix 79.3 we will discuss several applications of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 to classical vector calculus.

The key to the proof of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 is the following lemma:

Lemma 79.3. (Cylinder Inclusion Lemma) Let M be a smooth manifold. For $r = 0, 1$ we denote by $\iota_r \colon M \to M \times [0,1]$ the embedding that is given by $i_r(z) = (z, r)$. For any $k \in \mathbb{N}_0$ we have[818] [819]

$$\iota_0^* \ = \ \iota_1^* \colon \mathrm{H}^k_{\mathrm{dR}}(M \times [0,1]) \ \to \ \mathrm{H}^k_{\mathrm{dR}}(M).$$

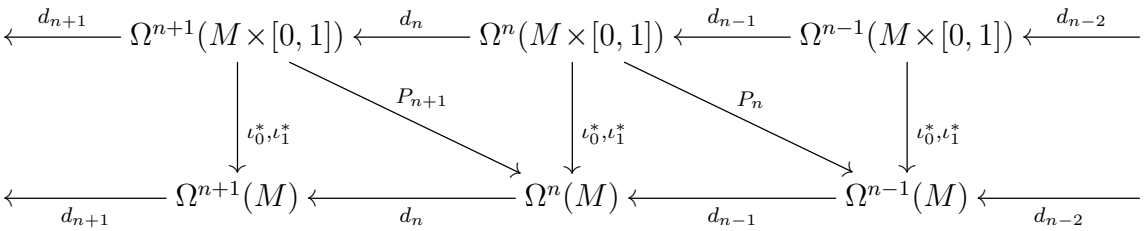
Proof of the Cylinder Inclusion Lemma 79.3. Let M be an m-dimensional smooth manifold. We need to show that for all $k \in \mathbb{N}_0$ we have $\iota_0^* = \iota_1^* \colon \mathrm{H}^k_{\mathrm{dR}}(M \times [0,1]) \to \mathrm{H}^k_{\mathrm{dR}}(M)$. By the Cochain Homotopy–H^*-Lemma 77.6 it suffices to show that there exists a cochain homotopy

$$P_* \colon \Omega^*(M \times [0,1]) \ \to \ \Omega^{*-1}(M)$$

from ι_0^* to ι_1^*, i.e. we have to find homomorphisms $\{P_n \colon \Omega^n(M \times [0,1]) \ \to \ \Omega^{n-1}(M)\}_{n \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$ we have

$$d_{n-1} \circ P_n + P_{n+1} \circ d_n \ = \ \iota_0^* - \iota_1^* \colon \Omega^n(M \times [0,1]) \ \to \ \Omega^n(M).$$

For orientation we summarize in the following diagram all objects arising in the definition of a cochain homotopy:



[818] Note that if $\partial M \neq \varnothing$, then it follows from the Smooth Manifold Product Proposition 26.12 that $M \times [0,1]$ is a smooth manifold with corner $\angle(M \times [0,1]) = \partial M \times \{0,1\}$. Technically speaking we did not introduce k-forms, de Rham cohomology etc. for smooth manifolds with corner. But all definitions go through without any real change. We leave it to the reader to fill in the details.

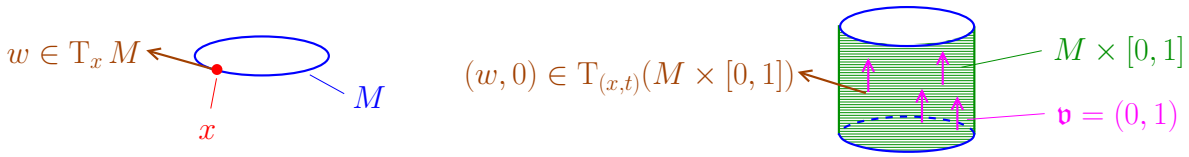
[819] Note that the smooth structure of the smooth manifold $M \times [0,1]$ from the Smooth Manifold Product Propositions 25.34 and 26.12 has the property that ι_0 and ι_1 are smooth maps. This implies that the maps ι_0^* and ι_1^* are actually defined.

We start out with a few preparations. First recall that it follows from the Product Orientation Proposition 31.48 (3) that for $(x,t) \in M \times [0,1]$ we have a natural isomorphism

$$\mathrm{T}_x M \times \mathrm{T}_t[0,1] \xrightarrow{\cong} \mathrm{T}_{(x,t)}(M \times [0,1])$$

which is induced by the smooth embedding $M \to M \times [0,1]$ that is given by $y \mapsto (y,t)$ and the smooth embedding $[0,1] \to M \times [0,1]$ that is given by $s \mapsto (x,s)$. We use this natural isomorphism to make the identification $\mathrm{T}_x M \oplus \mathrm{T}_t[0,1] =_{\mathrm{i}} \mathrm{T}_{(x,t)}(M \times [0,1])$. This identification leads to the following notation:

- Given $w \in \mathrm{T}_x M$ we obtain the corresponding tangent vector $(w,0) \in \mathrm{T}_{(x,t)}(M \times [0,1])$.
- We consider the "vertical" vector $\mathfrak{v} := (0,1) \in \mathrm{T}_x M \oplus \underbrace{\mathrm{T}_t([0,1])}_{=\mathbb{R}} =_{\mathrm{i}} \mathrm{T}_{(x,t)}(M \times [0,1])$.



Next, given $n \in \mathbb{N}_0$ we consider the map

$$\begin{aligned} P_n \colon \Omega^n(M \times [0,1]) &\to \Omega^{n-1}(M) \\ \omega &\mapsto \begin{cases} \text{the } (n-1)\text{-form that at a point } x \in M \text{ is given by} \\ \underbrace{(w_1, \dots, w_{n-1})}_{n-1 \text{ vectors in } \mathrm{T}_x M} \mapsto (-1)^{n+1} \cdot \int_{t=0}^{t=1} \omega_{(x,t)}(\underbrace{(w_1,0), \dots, (w_{n-1},0), \mathfrak{v}}_{n \text{ vectors in } \mathrm{T}_{(x,t)}(M \times [0,1])})\, dt. \end{cases} \end{aligned}$$

It follows easily from the Leibniz Integral Rule 73.28 and the definitions that this map $P_n \colon \Omega^n(M \times [0,1]) \to \Omega^{n-1}(M)$ is well-defined, i.e. that $P_n(\omega)$ is a smooth $(n-1)$-form on M. Also note that P_n is evidently $\mathbb{R}$-linear. We now claim that these maps P_n define the desired cochain homotopy between ι_0^* and ι_1^*, i.e. we claim that for all $n \in \mathbb{N}_0$ we have

$$(*) \qquad d_{n-1} \circ P_n + P_{n+1} \circ d_n = \iota_1^* - \iota_0^* \colon \Omega^n(M \times [0,1]) \to \Omega^n(M).$$

The proof will be carried out in the following three steps:

(1) First we show that the maps P_n are natural.
(2) Then we prove the equality $(*)$ for open subsets of $\mathbb{R}^m$ and open subsets of $\mathbb{H}^m$.
(3) Finally will use partitions of unity and (1) to reduce the general case to the cases dealt with in (2).

These three steps are the content of the following three claims:

Claim 1. The maps $P_n \colon \Omega^n(M \times [0,1]) \to \Omega^{n-1}(M)$ are natural in the sense that for any smooth map $f \colon U \to V$ between smooth manifolds the following diagram commutes:

$$\begin{array}{ccc} \Omega^n(U \times [0,1]) & \xrightarrow{P_n} & \Omega^{n-1}(U) \\ {\scriptstyle (f \times \mathrm{id}_{[0,1]})^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \Omega^n(V \times [0,1]) & \xrightarrow{P_n} & \Omega^{n-1}(V). \end{array}$$

Proof. Let $\omega \in \Omega^n(V \times [0,1])$. We need to show that $P_n((f \times \mathrm{id})^*(\omega)) = f^*(P_n(\omega))$. This follows from the observation that for any $x \in U$, any $t \in [0,1]$ and any vectors

$w_1, \dots, w_{n-1} \in \mathrm{T}_x U$ we have the following equalities:

$$\begin{aligned} P_n((f \times \mathrm{id})^*(\omega))_x(w_1, \dots, w_{n-1}) &\overset{\text{definition of } P_n}{=} (-1)^{n+1} \cdot \int_{t=0}^{t=1} (f \times \mathrm{id})^*(\omega)_{(x,t)}((w_1, 0), \dots, (w_{n-1}, 0), \mathfrak{v}_U)\, dt \\ &\overset{\text{Definition 74.2 of } (f \times \mathrm{id})^*(\omega)}{=} (-1)^{n+1} \cdot \int_{t=0}^{t=1} \omega_{(f(x),t)}((f \times \mathrm{id})_*(w_1, 0), \dots, (f \times \mathrm{id})_*(w_{n-1}, 0), (f \times \mathrm{id})_*(0, 1))\, dt \\ &\overset{\text{since } d(f \times \mathrm{id}) = df \times d\,\mathrm{id}}{=} (-1)^{n+1} \cdot \int_{t=0}^{t=1} \omega_{(f(x),t)}((f_*(w_1), 0), \dots, (f_*(w_{n-1}), 0), (0, 1))\, dt \\ &\overset{\text{definition of } P_n}{=} (-1)^{n+1} \cdot P_n(\omega)_{f(x)}(f_*(w_1), \dots, f_*(w_{n-1})) \overset{\text{Definition 74.2 of } f^*}{=} f^*(P_n(\omega))_x(w_1, \dots, w_{n-1}). \end{aligned}$$

⊞

Claim 2. Let V be an open subset of $\mathbb{R}^m$ or of $\mathbb{H}^m$. For any $n \in \mathbb{N}_0$ we have

$$d_{n-1} \circ P_n + P_{n+1} \circ d_n = \iota_1^* - \iota_0^* \colon \Omega^n(V \times [0,1]) \to \Omega^n(V).$$

Proof. We first prove the claim for $n \geq 1$. We introduce the following notation and conventions:

- Since V is an open subset of $\mathbb{R}^m$ or of $\mathbb{H}^m$ we obtain from the Physical Tangent Spaces–Lemma 29.2 (1) t that for every $x \in V$ we can make the identification $\mathrm{T}_x V = \mathbb{R}^m$.
- We denote the standard basis of the dual space of $\mathbb{R}^m$ by $dx^1, \dots, dx^m$.
- Since $V \times [0,1]$ is an open subset of $\mathbb{R}^m \times [0,1]$ or of $\mathbb{H}^m \times [0,1]$ we obtain from the Physical Tangent Spaces–Lemma 29.2 (1) that for every $(x,t) \in V \times [0,1]$ we can make the identification $\mathrm{T}_x(V \times [0,1]) = \mathbb{R}^{m+1}$.
- We denote the standard basis of the dual space of $\mathbb{R}^{m+1}$ by $dy^1, \dots, dy^m, dt$.

It follows from the Alternating Forms-Basis Proposition 72.7 that for $n \geq 1$ every differential n-form on $V \times [0,1]$ is a real linear combination of forms of the following two types:

(α) Forms $\alpha = f \cdot dy^{j_1} \wedge \dots \wedge dy^{j_n}$ where $f \colon V \times [0,1] \to \mathbb{R}$ is a smooth function.
(β) Forms $\beta = f \cdot dy^{j_1} \wedge \dots \wedge dy^{j_{n-1}} \wedge dt$ where $f \colon V \times [0,1] \to \mathbb{R}$ is a smooth function.

By the linearity of the terms in the claim it suffices to prove the claim for forms of type (α) and (β). Before we prove the claim for these two types of forms we make two general remarks:

(α) Given a d-form $\alpha = f \cdot dy^{j_1} \wedge \dots \wedge dy^{j_d}$ on $V \times [0,1]$ we see that for any $x \in V$ and any $h_1, \dots, h_{d-1} \in \mathrm{T}_x V$ we have

$$P_d(\alpha)_x(h_1, \dots, h_{d-1}) \overset{\text{definition of } (f \times \mathrm{id})^*}{=} (-1)^{d+1} \cdot \int_{t=0}^{t=1} f(x,t) \cdot \underbrace{(dy^{j_1} \wedge \dots \wedge dy^{j_d})((h_1, 0), \dots, (h_{d-1}, 0), \mathfrak{v})}_{\substack{= 0, \text{ since all } dy^i(\mathfrak{v}) = 0, \\ \text{(see also Definition 72.5 of the wedge product)}}} dt = 0.$$

(β) Given a d-form $\beta = f \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_{d-1}} \wedge dt$ on $V \times [0,1]$ we see that for any $x \in V$ and any $h_1, \ldots, h_{d-1} \in \mathrm{T}_x V$ we have

$$P_d(\beta)_x(h_1,\ldots,h_{d-1}) \overset{\text{definition of } (f\times \mathrm{id})^*}{=} (-1)^{d+1} \cdot \int_{t=0}^{t=1} f(x,t) \cdot \underbrace{(dy^{j_1} \wedge \cdots \wedge dy^{j_{d-1}} \wedge dt)((h_1,0),\ldots,(h_{d-1},0),\mathfrak{v})}_{= (dx^{j_1} \wedge \cdots \wedge dx^{j_{d-1}})(h_1,\ldots,h_{d-1}),\ \text{since } dy^j(w,0) = dx^j(w),\ dt(h_i) = 0 \text{ and since } dt(\mathfrak{v}) = 1} dt$$

$$= \left((-1)^{d+1} \cdot \left(\int_{t=0}^{t=1} f(x,t)\, dt \right) \cdot dx^{j_1} \wedge \cdots \wedge dx^{j_{d-1}} \right)(h_1,\ldots,h_{d-1}).$$

Now we turn to the actual verification that the desired equality $d_{n-1} \circ P_n + P_{n+1} \circ d_n = \iota_1^* - \iota_0^*$ holds for n-forms of the type (α) and (β):

(α) We first consider an n-form $\alpha = f \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_n}$ on $V \times [0,1]$ as above. We just discussed that $P_n(\alpha) = 0$. From this we obtain that $(d_{n-1} \circ P_n)(\alpha) = 0$. Thus it remains to show that $(P_{n+1} \circ d_n)(\alpha) = (\iota_1^*)(\alpha) - (\iota_0^*)(\alpha)$. We now verify that this equality holds. Thus let $x \in V$. We see that:

$$(P_{n+1} \circ d_n)(\alpha) \overset{\text{definition of } d\alpha}{=} P_{n+1}(df \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_n}) \overset{\text{follows from the 1-Form-via-}dy^i\text{-Lemma 71.5 (2)}}{=}$$

$$= P_{n+1}\left(\left(\sum_{i=1}^{m} \frac{\partial f}{\partial y_i} \cdot dy^i + \frac{\partial f}{\partial t} \cdot dt \right) \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_n} \right)$$

$$= \sum_{i=1}^{m} P_{n+1}\Big(\underbrace{\frac{\partial f}{\partial y_i} \cdot dy^i \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_n}}_{\text{of type } (\alpha),\ \text{thus it follows from the above discussion that } P_{n+1} = 0} \Big) + P_{n+1}\left(\frac{\partial f}{\partial t} \cdot dt \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_n} \right)$$

$$\overset{\text{by the Wedge Product Properties Lemma 72.10 (2) we pick up a } (-1)^n \text{ while pushing } dt \text{ to the back}}{=} (-1)^n \cdot P_{n+1}\left(\frac{\partial f}{\partial t} \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_n} \wedge dt \right) \overset{\text{by the above discussion of } P_{n+1} \text{ of a form of type } (\beta)}{=}$$

$$= (-1)^n \cdot (-1)^{n+2} \cdot \left(\int_0^1 \frac{\partial f}{\partial t}(x,t)\, dt \right) \cdot dx^{j_1} \wedge \cdots \wedge dx^{j_n}$$

$$\overset{\text{by the Fundamental Theorem of Calculus 73.27}}{=} f(x,1) \cdot dx^{j_1} \wedge \cdots \wedge dx^{j_n} - f(x,0) \cdot dx^{j_1} \wedge \cdots \wedge dx^{j_n} \overset{\text{follows from } \iota_t^*(dy^j) = dx^j \text{ and the Alternating Form-Functoriality Lemma 72.12}}{=} (\iota_1^*)(\alpha) - (\iota_0^*)(\alpha).$$

We have thus proved the desired equality for α.

(β) Now we consider an n-form $\beta = f \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_{n-1}} \wedge dt$ on $V \times [0,1]$ as above. First note that $\iota_0^*(dt) = \iota_1^*(dt) = 0$. This implies that $\iota_0^*(\beta) = \iota_1^*(\beta) = 0$. Thus it remains to show that $(d_{n-1} \circ P_n)(\beta) + (P_{n+1} \circ d_n)(\beta) = 0$. Let us turn to the calculation of the

first summand. Given $x \in V$ we calculate that

$$\begin{aligned}
(d_{n-1} \circ P_n)(\beta) &\overset{\text{by the above discussion of } P_n \text{ of a form of type } (\beta)}{=} d_{n-1}\Big((-1)^{n+1} \cdot \Big(\int_{t=0}^{t=1} f(x,t)\,dt\Big) \cdot dx^{j_1} \wedge \cdots \wedge dx^{j_{n-1}}\Big) \overset{\text{definition of } d_{n-1}}{=} \\
&= (-1)^{n+1} \cdot \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\Big(\int_{t=0}^{t=1} f(x,t)\,dt\Big) \cdot dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-1}} \\
&= (-1)^{n+1} \cdot \sum_{i=1}^{m} \Big(\int_{t=0}^{t=1} \frac{\partial f}{\partial x_i}(x,t)\,dt\Big) \cdot dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-1}}. \\
&\quad \uparrow \text{by the Leibniz Integral Rule 73.28 we can interchange integration and partial differentiation}
\end{aligned}$$

Furthermore given $x \in V$ we calculate that

$$\begin{aligned}
(P_{n+1} \circ d_n)(\beta) &\overset{\text{by definition of } d_n, \text{ note that the summand for } \frac{\partial f}{\partial t} \cdot dt \text{ vanishes since } dt \wedge dt = 0}{=} P_{n+1}\Big(\underbrace{\sum_{i=1}^{m} \frac{\partial f}{\partial y_i}(x,t) \cdot dy^i \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_{n-1}} \wedge dt}_{\text{again form of type } (\beta)}\Big) \\
&= (-1)^{n+2} \cdot \sum_{i=1}^{m} \Big(\int_{t=0}^{t=1} \frac{\partial f}{\partial y_i}(x,t)\,dt\Big) \cdot dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-1}}. \\
&\quad \uparrow \text{by the above discussion of } P_{n+1} \text{ of a form of type } (\beta)
\end{aligned}$$

It follows from these calculations that we indeed have $(d_{n-1} \circ P_n)(\beta) + (P_{n+1} \circ d_n)(\beta) = 0$.

We have thus proved the claim for $n \geq 1$. It remains to prove the claim for $n = 0$. This follows immediately from the calculation that for any smooth function $f \colon V \to \mathbb{R}$ we have the following equalities:

$$\underbrace{(d_{-1} \circ P_0)(f)}_{=0} + (P_1 \circ d_0)(f) \overset{\text{follows from the the 1-Form-via-}dx^i\text{-Lemma 71.5}}{=} P_1\Big(\underbrace{\sum_{i=1}^{m} \frac{\partial f}{\partial y_i} \cdot dy^i}_{\text{type } (\alpha)} + \underbrace{\frac{\partial f}{\partial t} \cdot dt}_{\text{type } (\beta)}\Big) \overset{\text{by the calculations on page 1604}}{=} \int_{t=0}^{t=1} \frac{\partial f}{\partial t} dt = (\iota_1^*)(f) - (\iota_0^*)(f).$$

$\uparrow$ by the Fundamental Theorem of Calculus 73.27 ⊞

The next claim gives us finally the desired result:

Claim 3. For every every smooth manifold M and any $k \in \mathbb{N}_0$ we have

$$\iota_0^* = \iota_1^* \colon \mathrm{H}^k_{\mathrm{dR}}(M \times [0,1]) \to \mathrm{H}^k_{\mathrm{dR}}(M).$$

Proof. Let M be an m-dimensional smooth manifold. First we pick a smooth atlas $\{\Phi_j \colon U_j \to V_j\}_{j \in J}$ for the smooth manifold M. For each $j \in J$ we denote by $i_j \colon U_j \to M$

the inclusion. We consider the diagram

$$\begin{array}{ccc}
\Omega^n(M\times[0,1]) & \xrightarrow{(d_{n-1}\circ P_n + P_{n+1}\circ d_n)-(\iota_1^*-\iota_0^*)} & \Omega^n(M) \\
\downarrow \prod i_j^* & & \downarrow \prod i_j^* \\
\prod_{j\in J}\Omega^n(U_j\times[0,1]) & \xrightarrow{\prod(d_{n-1}\circ P_n + P_{n+1}\circ d_n)-(\iota_1^*-\iota_0^*)} & \prod_{j\in J}\Omega^n(U_j) \\
\cong\uparrow \prod \Phi_j^* & & \cong\uparrow \prod \Phi_j^* \\
\prod_{j\in J}\Omega^n(V_j\times[0,1]) & \xrightarrow{\prod(d_{n-1}\circ P_n + P_{n+1}\circ d_n)-(\iota_1^*-\iota_0^*)} & \prod_{j\in J}\Omega^n(V_j).
\end{array}$$

We make the following observations:

- By Claim 1 we know that the maps P_n are natural. It is also clear that the maps ι_0^* and ι_1^* are natural, i.e. they commute with induced maps given by smooth maps between smooth manifolds. This observation, together with the Differential-Natural Proposition 75.8 implies that the horizontal maps are natural in the above sense. It follows that the diagram commutes.
- Recall that the inclusion induced maps on n-forms are just restrictions. It follows from this observation and the fact that the U_j form an open cover of M that the two top vertical maps are monomorphisms.
- The two bottom vertical maps are isomorphisms since the Φ_j are diffeomorphisms.
- Since each V_j is diffeomorphic to an open subset $\mathbb{R}^m$ or of $\mathbb{H}^m$ we obtain from Claim 2 that the bottom map is the zero map.

It follows from this discussion that the top map is also the zero map. But this is precisely what we wanted to show. ∎

Now we finally have most of the tools to prove the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2. Thus let f_0 and f_1 be two smooth maps between two smooth manifolds M and N. We assume that f_0 and f_1 are homotopic. We need to show that for any $k\in\mathbb{N}_0$ we have

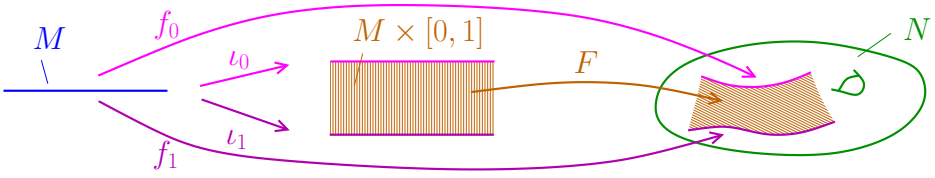
$$f_0^* = f_1^* \colon \mathrm{H}^k_{\mathrm{dR}}(N) \;\to\; \mathrm{H}^k_{\mathrm{dR}}(M).$$

We prove the proposition in three steps:

(1) First we consider the case that the maps are smoothly homotopic.
(2) Then we consider the case that the smooth manifold M has no boundary.
(3) Finally we consider the general case.

Proof of Proposition 79.2 if the maps are smoothly homotopic. Let f_0 and f_1 be two smooth maps between two smooth manifolds M and N. We assume that they are smoothly homotopic, i.e. we assume that there exists a smooth homotopy $F\colon M\times[0,1]\to N$ between the maps $f_0, f_1\colon M\to N$. We need to show that $f_0^* = f_1^*\colon \mathrm{H}^k_{\mathrm{dR}}(N)\to\mathrm{H}^k_{\mathrm{dR}}(M)$.

For $r=0,1$ let $\iota_r\colon M\to M\times[0,1]$ be the embedding that is given by $\iota_r(z)=(z,r)$.



Then

$$f_0^* = (F \circ \iota_0)^* \underset{\text{by functoriality}}{=} \iota_0^* \circ F^* \overset{\text{by the Cylinder Inclusion Lemma 79.3}}{=} \iota_1^* \circ F^* \underset{\text{by functoriality}}{=} (F \circ \iota_1)^* = f_1^*.$$

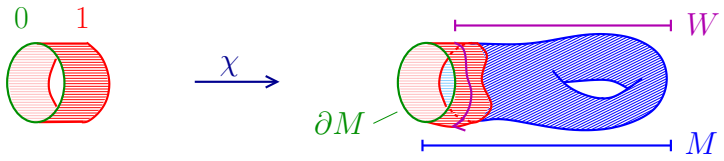
We have thus shown that the two maps f_0^* and f_1^* agree. ■

Proof of Proposition 79.2 if $\partial M = \varnothing$. Let M and N be smooth manifolds. We now assume that $\partial M = \varnothing$. Furthermore let $f_0, f_1 \colon M \to N$ be smooth maps that are homotopic. Since we assume that $\partial M = \varnothing$ we can apply the Replace-by-Smooth Homotopy Proposition 36.6 which tells us that f_0 and f_1 are in fact smoothly homotopic. But this case we just dealt with. So we are done. ■

Before we can deal with the general case of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 we need to prove the following technical lemma:

Lemma 79.4. (Remove Boundary-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) Let M be a smooth manifold. For any $k \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{dR}}(M \setminus \partial M)$ is an isomorphism.

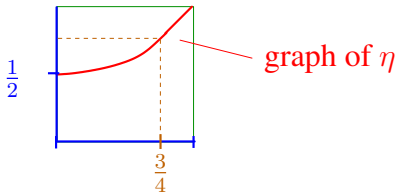
Proof. First note that by the Smooth Collar Neighborhood Theorem 35.6 we can find a smooth collar $\chi \colon [0,1] \times \partial M \to M$. To simplify the notation we identify $[0,1] \times \partial M$ with its image in M under the smooth embedding χ. We set $W := M \setminus \chi([0, \frac{1}{2}) \times \partial M)$.



Our first goal is to prove the following claim:

Claim 1. We denote by $i \colon W \to M$ the inclusion. The corresponding inclusion induced map $i^* \colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{dR}}(W)$ is an isomorphism.

Proof. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a diffeomorphism $\eta \colon [0,1] \to [\frac{1}{2}, 1]$ such that $\eta(t) = t$ for $t \in [\frac{3}{4}, 1]$.



We use the map $\eta \colon [0,1] \to [\frac{1}{2}, 1]$ to define the map

$$\begin{aligned} f \colon M &\to W \\ P &\mapsto \begin{cases} (\eta(t), x), & \text{if } P = (t,x) \in [0,1] \times \partial M, \\ P, & \text{otherwise.} \end{cases} \end{aligned}$$

We claim that $i \circ f \colon M \to M$ is smoothly homotopic to id_M and that $f \circ i \colon W \to W$ is smoothly homotopic to id_W. To do so we consider the smooth homotopies

$$\begin{aligned} F \colon ([0,1] \times \partial M) \times [0,1] &\to [0,1] \times \partial M \\ ((t,x),s) &\mapsto (\eta(t) \cdot (1-s) + t \cdot s, x) \end{aligned}$$

and
$$\begin{array}{rccl} G\colon ([\frac{1}{2},1]\times\partial M)\times[0,1] & \to & [\frac{1}{2},1]\times\partial M \\ ((t,x),s) & \mapsto & (\eta(t)\cdot(1-s)+t\cdot s,x). \end{array}$$

Note that by construction of η these homotopies are stationary on $[\frac{3}{4},1]\times\partial M$. It follows from the Smooth Pasting Proposition 25.19 that both homotopies extend to smooth homotopies on M and W in the obvious way. These extensions are now the desired smooth homotopies from $i\circ f$ to id_M and from $f\circ i$ to id_W.

Next recall that we already proved the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 for smooth homotopies. It thus follows that $f^*\circ i^* = (i\circ f)^* = (\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{H}^k_{\mathrm{dR}}(M)}$. Similarly we see that $i^*\circ f^* = \mathrm{id}_{\mathrm{H}^k_{\mathrm{dR}}(W)}$. This argument shows that $i^*\colon \mathrm{H}^k_{\mathrm{dR}}(M)\to\mathrm{H}^k_{\mathrm{dR}}(W)$ is, as promised, an isomorphism. ⊞

Claim 2. We denote by $i'\colon W\to M\setminus\partial M$ the inclusion. The corresponding inclusion induced map $(i')^*\colon \mathrm{H}^k_{\mathrm{dR}}(M\setminus\partial M)\to\mathrm{H}^k_{\mathrm{dR}}(W)$ is an isomorphism.

Proof. If in the proof of Claim 1 we replace the interval $[0,1]$ by $(0,1]$ we immediately get the desired statement. ⊞

Finally we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(M) & \xrightarrow{\quad\iota^*\quad} & \mathrm{H}^k_{\mathrm{dR}}(M\setminus\partial M) \\ & {\scriptstyle\cong}\searrow^{i^*} \quad \swarrow_{(i')^*}{\scriptstyle\cong} & \\ & \mathrm{H}^k_{\mathrm{dR}}(W). & \end{array}$$

We make the following clarifications and observations:

- We denote by $\iota\colon M\setminus\partial M\to M$ the inclusion.
- The diagram commutes by the functoriality of de Rham cohomology.
- We just showed that the diagonal maps are isomorphisms.

It follows from this discussion that the top horizontal map is indeed an isomorphism. ■

Proof of Proposition 79.2 in the general case. Now we allow the case that $\partial M\neq\varnothing$. The idea is to reduce the proof to the case of smooth manifolds without boundary, which we just dealt with. Thus let $f_0,f_1\colon M\to N$ be two smooth maps that are homotopic. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(N) & \xrightarrow{(f_0\circ\iota)^*,(f_1\circ\iota^*)} & \mathrm{H}^k_{\mathrm{dR}}(M\setminus\partial M) \\ & {\scriptstyle f_0^*,f_1^*}\searrow \quad \nearrow_{\iota^*}{\scriptstyle\cong} & \\ & \mathrm{H}^k_{\mathrm{dR}}(M) & \end{array}$$

We make the following clarifications and observations:

- The map $\iota\colon M\setminus\partial M\to M$ is the inclusion.
- The diagram commutes by the functoriality of de Rham cohomology.
- It follows from the Remove Boundary-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 79.4 that the right diagonal map is an isomorphism.
- Evidently $M\setminus\partial M$ is a smooth manifold with empty boundary. Since $f_0,f_1\colon M\to N$ are homotopic we see that $f_0\circ\iota, f_1\circ\iota\colon M\setminus\partial M\to N$ are also homotopic. Above we already proved the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 in the case of empty boundary. Thus it follows that the two horizontal maps on top agree.

It follows from these observations that the two left diagonal maps are also the same. ■

79.2. **De Rham cohomology and homotopy equivalences.** Let M and N be two smooth manifolds. In the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 78.5 we saw that any *smooth* map $f\colon M \to N$ induces a map on de Rham cohomology. We will now see that even every *continuous* map induces a map on de Rham cohomology:

Proposition 79.5. ($\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition)

(1) Let $f\colon M \to N$ be a continuous map between two smooth manifolds. By the Whitney Approximation Theorem 36.1 we know that $f\colon M \to N$ is homotopic to a smooth map $\widetilde{f}\colon M \to N$. For every $k \in \mathbb{N}_0$ the map

$$f^* := \widetilde{f}^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)$$

is independent of the choice of $\widetilde{f}$. For smooth maps this definition agrees with the definition given in the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 78.5.

(2) We denote by $\mathcal{SmMfd}^{\mathrm{cont}}$ the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfd}^{\mathrm{cont}}) &:= \text{all smooth manifolds,} \\ \mathrm{Mor}(M,N) &:= \text{all } \textit{continuous} \text{ maps from } M \text{ to } N. \end{aligned}$$

For every $k \in \mathbb{N}_0$ the maps

$$\begin{aligned} M &\mapsto \mathrm{H}^k_{\mathrm{dR}}(M) \\ (f\colon M \to N) &\mapsto (f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)) \end{aligned}$$

define a contravariant functor from the category $\mathcal{SmMfd}^{\mathrm{cont}}$ to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

Sketch of proof.

(1) Note that if $\widetilde{f}, \widetilde{f}'\colon M \to M$ are two smooth maps that are homotopic to f, then it follows from the Homotopy-Equivalence Relation Lemma 18.9 that $\widetilde{f}$ and $\widetilde{f}'$ are also homotopic. Thus it follows from the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 that $\widetilde{f}^* = (\widetilde{f}')^*$. This shows that the map f^* is independent of the choice of $\widetilde{f}$. Finally, if f is smooth, then we take $\widetilde{f} := f$ and we see that we obtain the usual induced map on de Rham cohomology.

(2) We now verify the two properties of a contravariant functor:

- The identity $\mathrm{id}\colon M \to M$ is smooth, hence it follows from the discussion in (1) that $\mathrm{id}^* = \mathrm{id}_{\mathrm{H}^k_{\mathrm{dR}}(M)}$.
- Let $f\colon L \to M$ and $g\colon M \to N$ be continuous maps. We pick a smooth map $\widetilde{f}\colon L \to M$ which is homotopic to f and we pick a smooth map $\widetilde{g}\colon M \to N$ which is homotopic to g. It follows from the Homotopy Composition Lemma 18.10 (2) that $\widetilde{g} \circ \widetilde{f}$ is homotopic to $g \circ f$. It follows that

$$(g \circ f)^* := (\widetilde{g} \circ \widetilde{f})^* \underset{\substack{\uparrow \\ \text{by the } \mathrm{H}^n_{\mathrm{dR}}\text{-Functor} \\ \text{Proposition 78.5}}}{=} \widetilde{f}^* \circ \widetilde{g}^* =: f^* \circ g^*.$$

■

We can now formulate the following generalization of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2:

Proposition 79.6. (Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition) Let f_0 and f_1 be two continuous maps between two smooth manifolds M and N. If f_0 and f_1 are homotopic, then for any $k \in \mathbb{N}_0$ we have

$$f_0^* = f_1^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M).$$

Proof. We pick a smooth map $\widetilde{f_0}\colon M \to N$ that is homotopic to f_0 and we pick a smooth map $\widetilde{f_1}\colon M \to N$ that is homotopic to f_1. It follows from the Homotopy-Equivalence Relation Lemma 18.9 that $\widetilde{f_0}, \widetilde{f_1}$ are homotopic. Thus we see that

$$f_0^* := \widetilde{f_0}^* \underset{\uparrow}{=} \widetilde{f_1}^* =: f_1^*.$$

by the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2

■

Next we recall Definition 19.9:

Definition 79.7. Let X and Y be topological spaces.

(1) A continuous map $f\colon X \to Y$ is a homotopy equivalence between X and Y, if there exists a homotopy inverse to f, i.e. if there exists a continuous map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

(2) If there exists a homotopy equivalence between X and Y, then we say that the topological spaces X and Y are homotopy equivalent.

(3) We say X is contractible, if X is homotopy equivalent to a singleton, i.e. if X is homotopy equivalent to a topological space which consists of precisely one point.

Now we obtain the following corollary to the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2:

Proposition 79.8. (Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition) Let M and N be smooth manifolds and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon M \to N$ is a homotopy equivalence, then $f^*\colon \mathrm{H}^n_{\mathrm{dR}}(N) \to \mathrm{H}^n_{\mathrm{dR}}(M)$ is an isomorphism.

(b) Suppose that M is a submanifold of N. If M is a deformation retract of N, then the inclusion-induced map $\mathrm{H}^n_{\mathrm{dR}}(N) \to \mathrm{H}^n_{\mathrm{dR}}(M)$ is an isomorphism.

(2) If M and N are homotopy equivalent, then $\mathrm{H}^n_{\mathrm{dR}}(M) \cong \mathrm{H}^n_{\mathrm{dR}}(N)$.

Proof.

(1) (a) Let $f\colon M \to N$ be a homotopy equivalence and let $g\colon N \to M$ be a homotopy inverse of f, i.e. g is a continuous map such that $f \circ g \simeq \mathrm{id}_N$ and $g \circ f \simeq \mathrm{id}_M$. It follows from the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.6 and the above $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 that $f^* \circ g^* = (g \circ f)^* = \mathrm{id}_{\mathrm{H}^n_{\mathrm{dR}}(M)}$ and similarly $g^* \circ f^* = \mathrm{id}_{\mathrm{H}^n_{\mathrm{dR}}(N)}$. But that means precisely that $f^*\colon \mathrm{H}^n_{\mathrm{dR}}(N) \to \mathrm{H}^n_{\mathrm{dR}}(M)$ is an isomorphism.

(b) This statement follows immediately from (1a) and the Deformation Retract–Homotopy Equivalence Lemma 19.10.

(2) This statement follows immediately from (1a). ■

Corollary 79.9. (Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary) Let M be a smooth manifold. If M is contractible (i.e. if M is homotopy equivalent to a singleton), then we have

$$\mathrm{H}^n_{\mathrm{dR}}(M) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

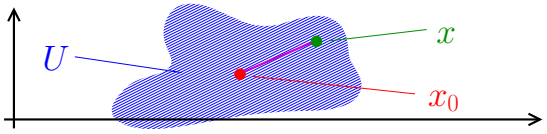
Remark. The statement of the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9, or variations thereof, is sometimes called the Poincaré Lemma, since Henri Poincaré proved in [**Poi1886**, p. 203] (for $M = \mathbb{R}^3$) the grandfather of all such results. □

Proof. We perform the following calculation:

$$\mathrm{H}^n_{\mathrm{dR}}(M) \underset{\text{by the Homotopy Equivalence-}\mathrm{H}^*_{\mathrm{dR}}\text{ Proposition 79.8 (2)}}{\cong} \mathrm{H}^n_{\mathrm{dR}}(\{*\}) \underset{\text{by the Singleton }\mathrm{H}^*_{\mathrm{dR}}\text{-Lemma 78.10}}{\cong} \begin{cases} \mathbb{R}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

■

Example. Recall that we say that a subset U of $\mathbb{R}^n$ is *star-shaped* if there exists a point $x_0 \in U$ such that for every $x \in U$ the Euclidean segment from x_0 to x lies in U.



As on page 531 we see that the map

$$\begin{aligned} f\colon U \times [0,1] &\to U \\ (x,t) &\mapsto x_0 \cdot (1-t) + x \cdot t \end{aligned}$$

is a homotopy from id_U to a constant map. By the Contractability Criterion Lemma 19.20 this shows that U is contractible. It now follows from the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 that for every smooth manifold that is homeomorphic to a star-shaped subset we have

$$\mathrm{H}^n_{\mathrm{dR}}(U) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

This applies of course in particular to $M = \overline{B}^m$, $M = B^m$ and $M = \mathbb{R}^m$. □

79.3. **Appendix: Applications to classical vector calculus.** In this appendix we apply the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 to classical vector calculus. We will not make use of this discussion otherwise.

Definition 79.10. Let $U \subseteq \mathbb{R}^3$ be an open subset. We denote by

$$C^\infty(U) \coloneqq \text{all smooth maps } U \to \mathbb{R} \quad \text{and} \quad \mathcal{V}(U) \coloneqq \text{all smooth maps } U \to \mathbb{R}^3$$

the set of all smooth real-valued functions respectively smooth vector fields on U. We recall that there are three classical maps relating smooth functions and smooth vector fields:

$$\begin{aligned} \mathrm{grad}\colon C^\infty(U) &\to \mathcal{V}(U) & \mathrm{curl}\colon \mathcal{V}(U) &\to \mathcal{V}(U) & \mathrm{div}\colon \mathcal{V}(U) &\to C^\infty(U) \\ f &\mapsto \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} & \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} &\mapsto \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ -\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} & \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} &\mapsto \tfrac{\partial F_x}{\partial x} + \tfrac{\partial F_y}{\partial y} + \tfrac{\partial F_z}{\partial z}. \end{aligned}$$

Given $f \in C^\infty(U)$ we refer to $\mathrm{grad}(f)$ as the gradient of f and given $F \in \mathcal{V}(U)$ we refer to $\mathrm{curl}(F)$ as the curl of f and we refer to $\mathrm{div}(F)$ as the divergence of F.

It is straightforward to verify that $\mathrm{curl} \circ \mathrm{grad} = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$. Put another way, the following sequence of maps is in fact a chain complex:

$$0 \to C^\infty(U) \xrightarrow{\mathrm{grad}} \mathcal{V}(U) \xrightarrow{\mathrm{curl}} \mathcal{V}(U) \xrightarrow{\mathrm{div}} C^\infty(U) \to 0.$$

Now we will relate the above discussion to forms. A smooth function is by definition the same as a differential 0-form. A smooth function also gives rise to a 3-form, more precisely we have the following simple-minded isomorphism

$$\begin{array}{rcl} C^\infty(U) & \to & \Omega^3(U) \\ f & \mapsto & f \cdot dx \wedge dy \wedge dz. \end{array}$$

More interestingly, a smooth vector field gives rise to a differential 1-form on U and also a 2-form on U. More precisely we have the following maps:

$$\begin{array}{rcl} \mathcal{V}(U) & \to & \Omega^1(U) \\ F & \mapsto & \left\{ \begin{array}{l} \text{differential 1-form that} \\ \text{at a point } x \in U \text{ is given by} \\ \mathrm{T}_xU = \mathbb{R}^3 \to \mathbb{R} \\ \qquad v \mapsto \underbrace{F \cdot v}_{\text{standard inner product}} \end{array} \right. \end{array}$$

$$\begin{array}{rcl} \mathcal{V}(U) & \to & \Omega^2(U) \\ F & \mapsto & \left\{ \begin{array}{l} \text{differential 2-form that at a} \\ \text{point } x \in U \text{ is given by} \\ (\mathrm{T}_xU)^2 = (\mathbb{R}^3)^2 \to \mathbb{R} \\ \qquad (v_1, v_2) \mapsto F \cdot \underbrace{(v_1 \times v_2)}_{\text{cross product}}. \end{array} \right. \end{array}$$

For fans of the private notation of this author the maps are just the maps $F \mapsto F \cdot \star$ and $F \mapsto \det(F \star_1 \star_2)$ which we introduced on page 1487 and 1526. It is straightforward to see that these maps are well-defined and that they are isomorphisms.

Lemma 79.11. (Grad-Curl-Div–Differential Lemma) The following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\infty(U) & \xrightarrow{\text{grad}} & \mathcal{V}(U) & \xrightarrow{\text{curl}} & \mathcal{V}(U) & \xrightarrow{\text{div}} & C^\infty(U) \longrightarrow 0 \\ & & \cong\downarrow \text{id} & & \cong\downarrow F \mapsto F\cdot\star & & \cong\downarrow F \mapsto \det(F \star_1 \star_2) & & \cong\downarrow f \mapsto f\cdot dx\wedge dy\wedge dz \\ 0 & \longrightarrow & \Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \longrightarrow 0. \end{array}$$

commutes.

Proof. In the following three steps we deal with the three squares from left to right:

(1) It follows easily from the 1-Form-via-dx^i-Lemma 71.5 and the various definitions that the first square commutes.

(2) Implicitly we already showed this on page 1565. But for the reader's convenience we prove the statement again. We start out with the following claim:[820]

Claim. For every $G = \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix} \in \mathcal{V}(U)$ we have the following equality:

$$\det(G \star_1 \star_2) = G_x \cdot (dy \wedge dz) + G_y \cdot (dz \wedge dx) + G_z \cdot (dx \wedge dy).$$

[820] Note that this claim gives the solution to Exercise 75.4.

Proof. Let $v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ and $w = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$ be two vectors. Then we see that

$$\begin{aligned} \det(G \star_1 \star_2)(v,w) &= \det \begin{pmatrix} G_x & v_x & w_z \\ G_y & v_y & w_y \\ G_z & v_z & w_z \end{pmatrix} \\ &= G_x \cdot \det \begin{pmatrix} v_y & w_y \\ v_z & w_z \end{pmatrix} - G_y \cdot \det \begin{pmatrix} v_x & w_x \\ v_z & w_z \end{pmatrix} + G_z \cdot \det \begin{pmatrix} v_x & w_x \\ v_y & w_y \end{pmatrix} \\ &= G_x \cdot (dy \wedge dz)(v,w) + G_y \cdot (dz \wedge dx)(v,w) + G_z \cdot (dx \wedge dy)(v,w) \\ &\quad \uparrow \\ &\quad \text{see Definition 72.5 of the wedge product of 1-forms} \\ &= (G_x \cdot (dy \wedge dz) + G_y \cdot (dz \wedge dx) + G_z \cdot (dx \wedge dy))(v,w). \end{aligned}$$ ⊞

Now let $F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \in \mathcal{V}(U)$. We need to show that $d(F \cdot \star) = \det(\operatorname{curl}(F) \star_1 \star_2)$. Using the claim we can now indeed verify that

$$\begin{aligned} d(F \cdot \star) &= d(F_x \cdot dx + F_y \cdot dy + F_z \cdot dz) \\ &= \left(\frac{\partial F_x}{\partial y} \cdot dy + \frac{\partial F_x}{\partial z} \cdot dz\right) \wedge dx + \left(\frac{\partial F_y}{\partial x} \cdot dx + \frac{\partial F_y}{\partial z} \cdot dz\right) \wedge dy + \left(\frac{\partial F_z}{\partial x} \cdot dx + \frac{\partial F_z}{\partial y} \cdot dy\right) \wedge dz \\ &\quad \uparrow \\ &\quad \text{follows from Definition 75.1, the 1-Form-via-}dx^i\text{-Lemma 71.5,} \\ &\quad \text{and the fact that } dx \wedge dx = dy \wedge dy = dz \wedge dz = 0 \\ &= \underbrace{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)}_{\substack{\text{first coordinate} \\ \text{of curl}(F)}} \cdot (dy \wedge dz) + \underbrace{\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)}_{\substack{\text{second coordinate} \\ \text{of curl}(F)}} \cdot (dz \wedge dx) + \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}_{\substack{\text{third coordinate} \\ \text{of curl}(F)}} \cdot (dx \wedge dy) \\ &= \det(\operatorname{curl}(F) \star_1 \star_2). \\ &\quad \uparrow \\ &\quad \text{by the claim} \end{aligned}$$

(3) It follows from the Differential-is-Divergence Lemma 76.27 that the third square commutes. ■

We can now formulate the main result of this appendix:

Theorem 79.12. (Star-Shaped–Rotation-Curl-Divergence Theorem) Let U be a star-shaped open subset of $\mathbb{R}^3$.

(1) Every smooth vector field on U with vanishing curl is the gradient of a smooth function.
(2) Every smooth vector field on U with vanishing divergence is the curl of a smooth vector field.
(3) Every smooth function on U is the divergence of a smooth vector field.

Proof. Let U be a star-shaped open subset of $\mathbb{R}^3$. We consider the following diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^\infty(U) & \xrightarrow{\text{grad}} & \mathcal{V}(U) & \xrightarrow{\text{curl}} & \mathcal{V}(U) & \xrightarrow{\text{div}} & C^\infty(U) & \longrightarrow & 0 \\ & & \cong\downarrow \text{id} & & \cong\downarrow F \mapsto F\cdot\star & & \cong\downarrow F \mapsto \det(F \star_1 \star_2) & & \cong\downarrow f \mapsto f\cdot dx\wedge dy\wedge dz & & \\ 0 & \longrightarrow & \Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) & \longrightarrow & 0. \end{array}$$

We make the following observations:

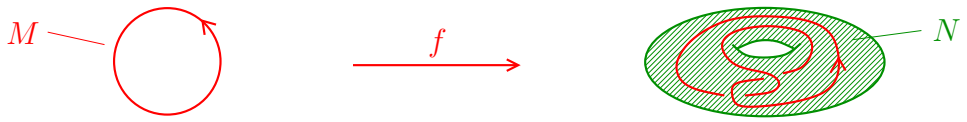
(1) By the Grad-Curl-Div–Differential Lemma 79.11 we know that the diagram commutes.
(2) Since U is star-shaped it follows from the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 that the cohomology groups of the lower sequence vanish for $k = 1, 2, 3$.
(3) As discussed above, all vertical maps are isomorphisms.

It follows from the above that the corresponding cohomology groups of the top vertical sequence also vanish. The three statements now follow from these three vanishing statements. ■

79.4. Exercises for Chapter 79.

Exercise 79.1. Let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in A$ be a point. Since $\mathbb{R}^n \setminus A$ and $\mathbb{R}^n \setminus \{x\}$ are both open subsets of $\mathbb{R}^n$ we can view both of them as smooth manifolds. Show that the inclusion $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus \{x\}$ induces for each $i \in \mathbb{N}_0$ an isomorphism $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{R}^n \setminus \{x\}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathbb{R}^n \setminus A)$ for all i.

Exercise 79.2. We consider the smooth manifolds $M = S^1$ and let $N = \overline{B}^2 \times S^1$. We consider the smooth map $f\colon M \to N$ that is shown in the figure below. Show that for every $i \in \mathbb{N}$ the induced map $f^*\colon \mathrm{H}^i_{\mathrm{dR}}(N) \to \mathrm{H}^i_{\mathrm{dR}}(M)$ is the zero map.



Exercise 79.3. For which of the following smooth manifolds M does there exists a smooth map $f\colon M \to M$ which is not homotopic to a constant map?

(a) $M = S^2$.
(b) $M = S^1 \times S^1$.
(c) $M = \overline{B}^2$.
(d) $M = S^1 \times \overline{B}^2$.
(e) $M =$ Klein bottle.

Exercise 79.4. For which $n \in \mathbb{Z}$ is the map

$$\begin{aligned} f_n\colon S^1 &\to S^1 \\ z &\mapsto z^n \end{aligned}$$

homotopic to the identity?
Hint. Make use of the integration map

$$\begin{aligned} \mathrm{H}^1_{\mathrm{dR}}(S^1) &\to \mathbb{R} \\ [\omega] &\mapsto \int_{S^1} \omega. \end{aligned}$$

CHAPTER 80

Mayer–Vietoris sequences in de Rham cohomology

In this chapter we will formulate and prove the existence of Mayer-Vietoris sequences in de Rham cohomology. In a nutshell these sequences allow us, with a little bit of luck, to calculate the de Rham cohomology groups of a smooth manifold M in terms of the de Rham cohomology groups of open subsets. In this chapter we will use these Mayer–Vietoris sequences to calculate the de Rham cohomology groups of many smooth manifolds. In the following chapters these Mayer–Vietoris sequences will also play an essential role in more theoretical arguments.

This chapter is organized as follows:

- In Section 80.1 we will introduce Mayer–Vietoris sequences.
- In Section 80.2 we will use Mayer-Vietoris sequences to determine the de Rham cohomology groups of spheres.
- In Section 80.3 we will study self-maps of spheres and the corresponding induced maps on de Rham cohomology.
- In Section 80.4 we will use Mayer-Vietoris sequences to compute the de Rham cohomology groups of a product $N \times S^k$ in terms of the de Rham cohomology groups of N.
- In Section 80.5 we will use Mayer-Vietoris sequences to calculate the de Rham cohomology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$.
- In Section 80.6 we will use Mayer-Vietoris sequences to calculate the de Rham cohomology of mapping tori. In particular we will calculate the de Rham cohomology groups of the Klein bottle.
- In Appendix 80.7 we will use our calculation of the de Rham cohomology groups of spheres to give a new proof of the Hairy Ball Theorem 46.18.
- In Appendix 80.8 we will show that if non-empty open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are *homeomorphic*, then $m = n$.
- In Section 80.9 we will provide exercises for Chapter 80.

In the next Chapter 81 we will show that smooth manifolds admit "smoothly good covers".

80.1. Mayer–Vietoris sequence for de Rham cohomology.

We adopt the following notation:

Notation 80.1. Given a topological space D and a subspace $C \subseteq D$ we denote the corresponding inclusion map $C \to D$ by i_C^D. To avoid pandemonium with subscripts we often drop the "$*$" when we deal with inclusion-induced maps on de Rham cohomology, e.g. we will write $i_C^D \colon \mathrm{H}^n_{\mathrm{dR}}(D) \to \mathrm{H}^n_{\mathrm{dR}}(C)$ instead of the more accurate $(i_C^D)^* \colon \mathrm{H}^n_{\mathrm{dR}}(D) \to \mathrm{H}^n_{\mathrm{dR}}(C)$.

The following theorem gives us the promised exact sequences:

Theorem 80.2. (Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem) Let M be a smooth manifold. Given any two open subsets U, V of M with $M = U \cup V$ the following two statements hold:

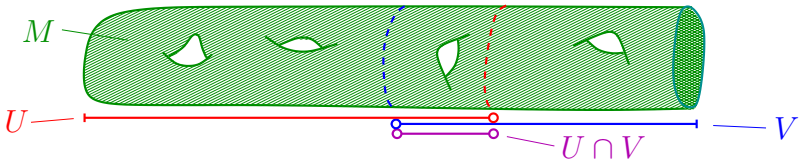
(1) The following sequence of cochain complexes is exact

$$0 \;\to\; \Omega^*(M) \;\xrightarrow{(i_U^M, -i_V^M)}\; \Omega^*(U) \oplus \Omega^*(V) \;\xrightarrow{i_{U\cap V}^U \oplus i_{U\cap V}^V}\; \Omega^*(U \cap V) \;\to\; 0.$$

(2) The connecting homomorphisms $\delta_k \colon \mathrm{H}^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR}}(M)$ that arise from (1) together with the Cohomology–LES Proposition 77.13 are natural and they give rise to an exact sequence of the form

$$\cdots \to \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{(i_U^M, -i_V^M)} \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \mathrm{H}^k_{\mathrm{dR}}(V) \xrightarrow{i_{U\cap V}^U \oplus i_{U\cap V}^V} \mathrm{H}^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR}}(M) \to \dots$$

This exact sequence is called the Mayer-Vietoris sequence in de Rham cohomology.



Remark.

(1) In the Compact–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 85.12 we will prove a variation on the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2.

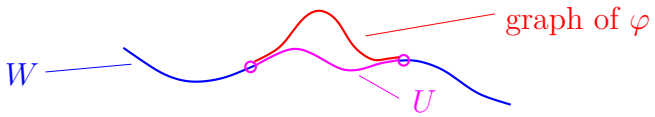
(2) In the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 we will prove the following: The combination of the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2, the Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.6, the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10, the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2 and the slightly more esoteric Exhaustions-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 230.13 essentially determines de Rham cohomology. □

Our proof of the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 makes use of the following elementary lemma:

Lemma 80.3. (Extend Smooth Forms Lemma) Let W be a smooth manifold, let $U \subseteq W$ be an open subset, let ω be a smooth form on U and let $\varphi \colon W \to \mathbb{R}$ be a smooth map. We define

$$\epsilon(\varphi \cdot \omega, W) \;:=\; \text{the form on } W \text{ that is given by } \varphi \cdot \omega \text{ on } U \text{ and that is zero on } W \setminus U.$$

If $\mathrm{supp}(\varphi) \subseteq U$, then $\epsilon(\varphi \cdot \omega, W)$ is a smooth form on W.



Proof of the Extend Smooth Forms Lemma 80.3. It follows almost immediately from the Extension-of-Form–Smooth Lemma 74.11 that $\epsilon(\varphi \cdot \omega, W)$ is indeed a smooth form on W. ■

Proof of Theorem 80.2. Let M be a smooth manifold and let U, V be two open subsets of M with $M = U \cup V$.

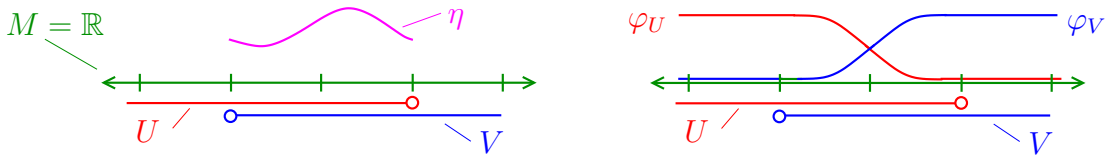
(1) Recall that we need to show that the following sequence of cochain complexes is exact

$$0 \to \Omega^*(M) \xrightarrow{(i_U^M, -i_V^M)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{i_{U\cap V}^U \oplus i_{U\cap V}^V} \Omega^*(U \cap V) \to 0.$$

First note that it follows easily from $M = U \cup V$ that the left-hand map $(i_U^M, -i_V^M)$ is a monomorphism. Next note that it is straightforward to verify that the composition of the two maps is zero. Furthermore note that one can easily verify that the image of the left-hand map $(i_U^M, -i_V^M)$ is the kernel of $i_{U\cap V}^U \oplus i_{U\cap V}^V$. Thus it remains to prove the following claim:

Claim. For each $k \in \mathbb{N}_0$ the map $i_{U\cap V}^U \oplus i_{U\cap V}^V \colon \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ is an epimorphism.

Proof. Let $\eta \in \Omega^k(U \cap V)$. Since U and V is an open cover of M we can apply the Manifold–Partitions-of-Unity Theorem 27.2 which tells us that there exists a smooth partition of unity $\varphi_U \colon M \to [0,1], \varphi_V \colon M \to [0,1]$ subordinate to the open cover of M that is given by U and V. In other words, there exist two smooth maps $\varphi_U \colon M \to [0,1]$, $\varphi_V \colon M \to [0,1]$ such that $\varphi_U + \varphi_V \equiv 1$ and such that $\operatorname{supp}(\varphi_U) \subseteq U$ and $\operatorname{supp}(\varphi_V) \subseteq V$.



Note that we have the inclusion $\operatorname{supp}(\varphi_V|_U) \subseteq U \cap \operatorname{supp}(\varphi_V) \subseteq U \cap V$ and that we have the inclusion $\operatorname{supp}(\varphi_U|_V) \subseteq \operatorname{supp}(\varphi_U) \cap V \subseteq U \cap V$. Thus we obtain from the Extend Smooth Forms Lemma 80.3 the two smooth forms $\epsilon(\eta \cdot \varphi_V, U)$ and $\epsilon(\eta \cdot \varphi_U, V)$. Now we see that

$$\begin{aligned}(i_{U\cap V}^U \oplus i_{U\cap V}^V)(\epsilon(\eta \cdot \varphi_V, U), \epsilon(\eta \cdot \varphi_U, V)) &\overset{\text{by definition of } i_{U\cap V}^U \text{ and } i_{U\cap V}^V}{=} \epsilon(\eta \cdot \varphi_V, U)|_{U\cap V} + \epsilon(\eta \cdot \varphi_U, V)|_{U\cap V} \\ &\overset{\text{by definition of } \varphi_U \text{ and } \varphi_V}{=} \eta \cdot \varphi_V + \eta \cdot \varphi_U \overset{\text{since } \varphi_U + \varphi_V \equiv 1}{=} \eta.\end{aligned}$$

⊞

(2) It follows immediately from Statement (1) together with the Cohomology–LES Proposition 77.13 that an open cover $\{U, V\}$ of a smooth manifold M gives rise to natural connecting homomorphisms $\delta_k \colon \mathrm{H}_{\mathrm{dR}}^k(U \cap V) \xrightarrow{\delta_k} \mathrm{H}_{\mathrm{dR}}^{k+1}(M)$ which in turn give rise to the exact sequence of Statement (2). ■

80.2. **The de Rham cohomology groups of spheres.** Using the Mayer–Vietoris-$\mathrm{H}_{\mathrm{dR}}^*$-Theorem 80.2 and some of our earlier results we can now compute many de Rham cohomology groups. The following proposition gives us the arguably most fundamental calculation of non-trivial de Rham cohomology groups:

Proposition 80.4. (Sphere–$\mathrm{H}_{\mathrm{dR}}^*$–Proposition) Let $n \in \mathbb{N}$.

(1) Given $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The homomorphism

$$\begin{aligned} \mathrm{H}^n_{\mathrm{dR}}(S^n) &\to \mathbb{R} \\ [\omega] &\mapsto \int_{S^n} \omega \end{aligned}$$

from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 is an isomorphism.

Proof. First note that in the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 we showed that the map in Statement (2) is an epimorphism. It is thus clear that it would follow from Statement (1) that it is an isomorphism. This discussion shows that it remains to prove Statement (1).

In fact, by induction we will prove the following, ever so slightly more general version of Statement (1):

Claim. For any $n \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have an isomorphism

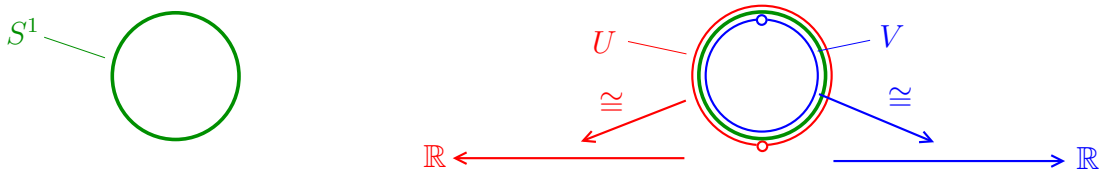
$$\mathrm{H}^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{R}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We will prove the claim by induction. First note that for $n = 0$ the statement holds by the Path-Component-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.7 and the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10.

Before we discuss the induction step we introduce some notation. Given S^n we write

$$U := \{(x_1, \dots, x_{n+1}) \in S^n \,|\, x_{n+1} \neq -1\} \quad \text{and } V = \{(x_1, \dots, x_{n+1}) \in S^n \,|\, x_{n+1} \neq +1\}.$$

Recall that the stereographic projection, that we defined in the Stereographic Projection Lemma 4.24, gives diffeomorphisms from U and V to $\mathbb{R}^n$.



We obtain that

$$\mathrm{H}^k_{\mathrm{dR}}(U) \cong \mathrm{H}^k_{\mathrm{dR}}(V) \cong \mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n) \underset{\uparrow}{\cong} \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

by the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9

Next we consider the map

$$\begin{aligned} \iota \colon S^{n-1} &\to U \cap V \\ P &\mapsto (P, 0). \end{aligned}$$

One can easily show that this map is a homotopy equivalence.[821] It follows from the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 that ι induces isomorphisms of de Rham cohomology groups.

[821] Indeed, this can be seen as follows: First of all it follows easily from the Compact-Hausdorff Proposition 4.23 that the map $\iota \colon S^{n-1} \to S^{n-1} \times \{0\} \subseteq U \cap V$ is a homeomorphism. Furthermore one can easily verify that the map

$$\begin{aligned} (U \cap V) \times [0,1] &\to U \cap V \\ ((x, x_{n+1}), t) &\mapsto \left(x \cdot \tfrac{1}{\|x\|} \cdot \sqrt{1 - x_{n+1}^2 \cdot (1-t)^2}, x_{n+1} \cdot (1-t)\right) \end{aligned}$$

where $\underset{\uparrow}{x} \in \underset{\uparrow}{\mathbb{R}^n}$ and $x_{n+1} \in \mathbb{R}$

Now we can finally perform the induction step from $n = 0$ to $n = 1$: We consider the long exact Mayer–Vietoris sequence coming from the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 that corresponds to the decomposition $S^1 = U \cup V$:

$$0 \to \underbrace{\mathrm{H}^0_{\mathrm{dR}}(S^1)}_{\cong\mathbb{R}} \to \underbrace{\mathrm{H}^0_{\mathrm{dR}}(U)}_{\cong\mathbb{R}} \oplus \underbrace{\mathrm{H}^0_{\mathrm{dR}}(V)}_{\cong\mathbb{R}} \to \mathrm{H}^0_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta_0} \mathrm{H}^1_{\mathrm{dR}}(S^1) \to \underbrace{\mathrm{H}^1_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{\mathrm{H}^1_{\mathrm{dR}}(V)}_{=0} \to \dots$$
$$\iota^* \big\downarrow \cong$$
$$\underbrace{\mathrm{H}^0_{\mathrm{dR}}(S^0)}_{\cong\mathbb{R}^2}.$$

First note that it follows from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 that $\mathrm{H}^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}$. Next recall that it follows from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (1) and (2) that inclusion-induced maps between connected non-empty smooth manifolds induce isomorphisms on 0-th de Rham cohomology. Using these observations and using the Mayer–Vietoris sequence it is now straightforward to show that the de Rham cohomology of S^1 is precisely as predicted.[822]

The induction step for $n \geq 2$ from $n-1$ to n is even easier, since one does not really have to worry about 0-th de Rham cohomology. More precisely, for $k \geq 1$ the above Mayer–Vietoris sequence now looks as follows:

$$\dots \to \underbrace{\mathrm{H}^k_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{\mathrm{H}^k_{\mathrm{dR}}(V)}_{=0} \to \mathrm{H}^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR}}(S^n) \to \underbrace{\mathrm{H}^{k+1}_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{\mathrm{H}^{k+1}_{\mathrm{dR}}(V)}_{=0} \to \dots$$
$$\iota^* \big\downarrow \cong$$
$$\underbrace{\mathrm{H}^k_{\mathrm{dR}}(S^{n-1})}_{\text{know this by induction}}.$$

It is now straightforward to prove the induction step. ∎

80.3. **The degree of a self-map of spheres.** Our goal is "to understand de Rham cohomology". This involves two aspects:

(1) Ideally we would like to be able to calculate the de Rham cohomology groups of all smooth manifolds that we are interested in.
(2) We would like to calculate the induced maps on de Rham cohomology for interesting maps between smooth manifolds.

As we will see in the coming chapters, these two aspects cross fertilize one another. In the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we just calculated the de Rham cohomology groups of spheres. The above work program suggests that for self-maps of spheres we should now try to understand the induced maps on de Rham cohomology groups.

We start out with the following naive algebraic concept:

Definition 80.5. Let V be a 1-dimensional real vector space and let $\Phi\colon V \to V$ be a homomorphism. Note that Φ is given by multiplication by a unique real number. We refer to this real number as the degree $\deg(\Phi)$.[823]

---

is a deformation retraction from $U \cap V$ to $S^{n-1} \times \{0\}$. It follows easily from this discussion together with the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $\iota\colon S^{n-1} \to U \cap V$ is a homotopy equivalence.

[822] Alternatively one can also use Exercise 77.1 (b) to deduce from this exact sequence that $\mathrm{H}^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$.

Now we can introduce the main concept of this section:

Definition 80.6. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a continuous map. By the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we know that $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is isomorphic to $\mathbb{R}$. Thus we can define the degree of f:

$$\deg(f) \coloneqq \deg\left(f^*\colon \mathrm{H}^n_{\mathrm{dR}}(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)\right).$$

Remark. Let $n \in \mathbb{N}$. Recall that in Definition 46.5 we used quite different techniques to define the degree of a continuous map $f\colon S^n \to S^n$. In the De Rham Degree–Local Data Proposition 86.6 we will see that the two definitions of the degree agree. □

Our goal for the remainder of this section is to explore this concept of the degree of a self-map of a sphere. We start out with an essential calculation:

Lemma 80.7. (Degree-of-Power Lemma) Given $k \in \mathbb{Z}$ we have

$$\deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{pmatrix} = k.$$

Proof. Given $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rccc} g_k\colon & S^1 & \to & S^1 \\ & z & \mapsto & z^k. \end{array}$$

Next we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^1_{\mathrm{dR}}(S^1) & \xrightarrow{\quad g_k^* \quad} & \mathrm{H}^1_{\mathrm{dR}}(S^1) \\ \cong \downarrow {\scriptstyle [\omega]\mapsto\int_{S^1}\omega} & & \cong \downarrow {\scriptstyle [\omega]\mapsto\int_{S^1}\omega} \\ \mathbb{R} & \xrightarrow[\cdot \deg(g_k)]{} & \mathbb{R}. \end{array}$$

First note that it follows from the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 (2) that the vertical maps are isomorphisms. Next note that it follows almost immediately from the definition of the degree that the diagram commutes. It follows from this discussion that the lemma follows from the following claim:

Claim. We consider the vector field $F(x,y) = (-y,x)$ on S^1 and we consider the corresponding smooth 1-form $F \cdot \star$ on S^1, that we introduced on page 1487.

(1) We have $\displaystyle\int_{S^1} F \cdot \star = 2\pi.$

(2) For $k \in \mathbb{Z}$ we have[824] $\displaystyle\int_{S^1} g_k^*(F \cdot \star) = 2\pi \cdot k.$

Proof. Note that $g_1 = \mathrm{id}_{S^1}$. Thus we see that Statement (1) is just a special case of Statement (2). Now let $k \in \mathbb{Z}$. We consider the compact parametrization $\Psi\colon [0, 2\pi] \to S^1$ that is given by $\Psi(t) = \exp(\mathrm{i}t) = (\cos(t), \sin(t))$. Similar to the discussion on page 1497

[823] The degree is really just the determinant of the endomorphism. But for the sake of consistency with concepts that we introduce later, see e.g. the discussion in Definition 151.1, we prefer the name "degree".

[824] This claim is essentially the content of Exercise 71.4.

we calculate:

see Definition 71.18n of the integral of a 1-form

we use the definition on Definition 71.6 of the pullback of a 1-form, we also use that by the Chain Rule we have $g_{k*}(\Psi'(t)) = (g_k \circ \Psi)'(t)$

$$\int_{S^1} g_k^*(F \cdot \star) \overset{\downarrow}{=} \int_{t=0}^{t=2\pi} g_k^*(F \cdot \star)_{\Psi(t)}(\Psi'(t))\,dt \overset{\downarrow}{=} \int_{t=0}^{t=2\pi} (F \cdot \star)_{(g_k \circ \Psi)(t)}((g_k \circ \Psi)'(t))\,dt$$

$$\overset{\uparrow}{=} \int_{t=0}^{t=2\pi} F((g_k \circ \Psi)(t)) \cdot (g_k \circ \Psi)'(t)\,dt$$

definition of the 1-form $F \cdot \star$ on page 1487

$$\overset{\uparrow}{=} \int_{t=0}^{t=2\pi} \begin{pmatrix} -\sin(k \cdot t) \\ \cos(k \cdot t) \end{pmatrix} \cdot \begin{pmatrix} -k \cdot \sin(k \cdot t) \\ k \cdot \cos(k \cdot t) \end{pmatrix} dt = \int_{t=0}^{t=2\pi} k\,dt = 2\pi \cdot k.$$

since we have the equalities $(g_k \circ \Psi)(t) = \exp(\mathrm{i}t)^k = \exp(\mathrm{i}k \cdot t) = (\cos(k \cdot t), \sin(k \cdot t))$ ■

The following lemma, which can be viewed as an analog of the Degree Homotopy Invariance Theorem 46.6 and the Degree Multiplicativity Theorem 46.7, summarizes several basic properties of the degree of continuous self-maps of spheres:

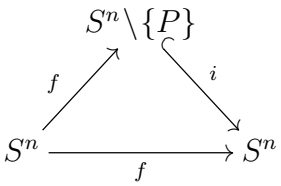
Lemma 80.8. (Sphere–De Rham-Degree Lemma) Let $n \in \mathbb{N}$ and let $f, g\colon S^n \to S^n$ be two continuous maps.

(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$.
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) If f is a self-homeomorphism of S^n, then $\deg(f^{-1}) = \deg(f)^{-1}$.
(6) If f is a self-diffeomorphism of S^n, then

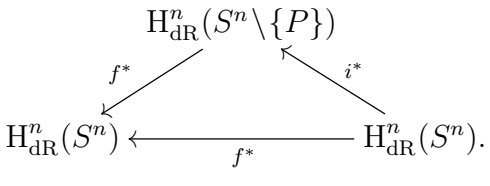
$$\deg(f) = \begin{cases} 1, & \text{if } f \text{ is orientation-preserving} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof.

(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\mathrm{H}^n_{\mathrm{dR}}(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) We suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram



which by the functoriality of de Rham cohomology gives rise to the commutative diagram



Next note that the stereographic projection "starting at $P \in S^n$" gives a diffeomorphism from $S^n \setminus \{P\}$ to $\mathbb{R}^n$. This observation together with the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 implies that $\mathrm{H}^n_{\mathrm{dR}}(S^n \setminus \{P\}) = 0$. But this means that $f^*\colon \mathrm{H}^n_{\mathrm{dR}}(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ factors through the zero vector space, i.e. f^* is the zero map.
(3) If f is homotopic to g, then it follows from the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.6 that $\deg(f) = \deg(g)$.

(4) We consider the following diagram:

$$\mathrm{H}^n_{\mathrm{dR}}(S^n) \xrightarrow[\cdot\deg(g)]{g^*} \mathrm{H}^n_{\mathrm{dR}}(S^n) \xrightarrow[\cdot\deg(f)]{f^*} \mathrm{H}^n_{\mathrm{dR}}(S^n).$$
$$\qquad\qquad\qquad

Example.

(1) If we take $N = \{*\}$, then the combination of the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10 and the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.10 gives us a new proof of the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4.

(2) It follows easily from the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.10 together with the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 that

$$\mathrm{H}^k_{\mathrm{dR}}(S^2 \times S^2) \cong \begin{cases} 0, & \text{if } k \geq 5, \\ \mathbb{R}, & \text{if } k = 4, \\ 0, & \text{if } k = 3, \\ \mathbb{R}^2, & \text{if } k = 2, \\ 0, & \text{if } k = 1, \\ \mathbb{R}, & \text{if } k = 0. \end{cases}$$

□

Proof. We prove the proposition by induction on $k \in \mathbb{N}_0$. For $k = 0$ we have $S^0 = \{-1, 1\}$ and the statement follows almost immediately from the Path-Component-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.7.

Now we assume that the statement holds for some $k - 1 \in \mathbb{N}_0$. As in the proof of the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we set

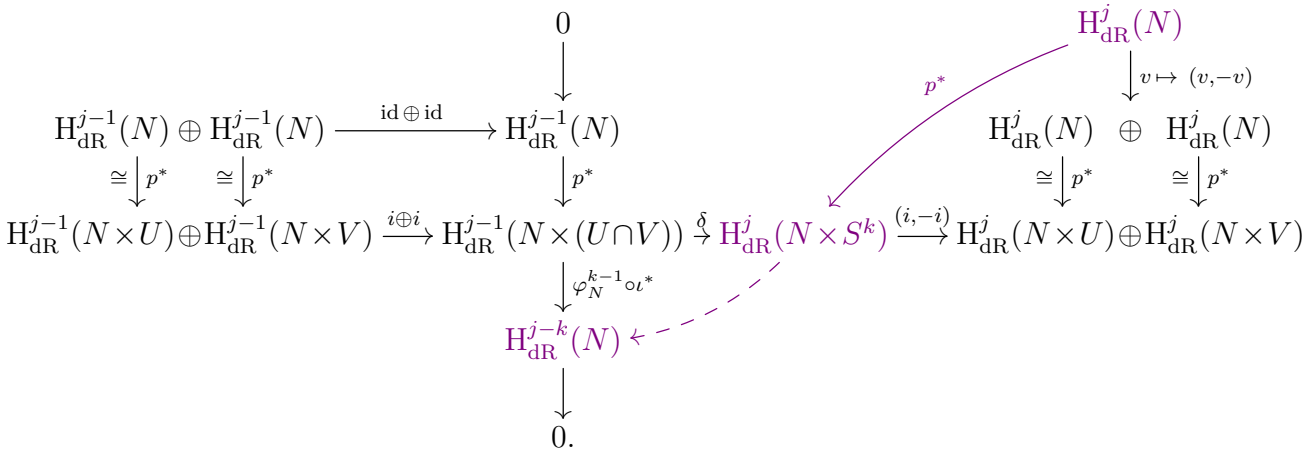
$$U := \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} \neq -1\} \quad \text{and } V = \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} \neq +1\}.$$

In the proof of the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we already made the following observations:

(1) U and V are diffeomorphic to $\mathbb{R}^k$. In particular they are contractible.

(2) The map $\iota\colon S^{k-1} \to U \cap V$ that is given by $x \mapsto (x, 0)$ is a homotopy equivalence.

Next, given $j \in \mathbb{N}_0$, we consider the following diagram:



We make the following clarifications and observations:

(3) We denote by i the obvious inclusion induced maps.

[825]Here "natural" means "natural in N". More precisely, a smooth map $M \to N$ between smooth manifolds gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^j_{\mathrm{dR}}(N) & \xrightarrow{p^*} & \mathrm{H}^j_{\mathrm{dR}}(N \times S^k) & \xrightarrow{\varphi_N^k} & \mathrm{H}^{j-k}_{\mathrm{dR}}(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f \times \mathrm{id}_{S^k})^*} & & \downarrow{\scriptstyle f^*} & & \\ 0 & \longrightarrow & \mathrm{H}^j_{\mathrm{dR}}(M) & \xrightarrow{p^*} & \mathrm{H}^j_{\mathrm{dR}}(M \times S^k) & \xrightarrow{\varphi_M^k} & \mathrm{H}^{j-k}_{\mathrm{dR}}(M) & \longrightarrow & 0. \end{array}$$

(4) The horizontal sequence comes from the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2.
(5) Given any subspace $X \subseteq S^k$ we denote by $p\colon N \times X \to N$ the natural projection. Note that it follows easily from (1) and the Homotopy-Equivalence Basics Lemma 19.14 (5) that the projections $p\colon N \times U \to N$ and $p\colon N \times V \to V$ are homotopy equivalences. In particular they induce isomorphisms of de Rham cohomology groups.
(6) It follows from the functoriality of de Rham cohomology that the diagram commutes.
(7) We claim that the vertical sequence in the middle is exact. To see this we consider the following diagram:

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\mathrm{H}^{j-1}_{\mathrm{dR}}(N) & \xrightarrow{p^*} & \\
{\scriptstyle p^*}\downarrow & & \\
\mathrm{H}^{j-1}_{\mathrm{dR}}(N \times (U \cap V)) & \xrightarrow[(\mathrm{id}_N \times \iota)^*]{\cong} & \mathrm{H}^{j-1}_{\mathrm{dR}}(N \times S^{k-1}) \\
{\scriptstyle \varphi_N^{k-1} \circ (\mathrm{id}_N \times \iota)^*}\downarrow & \swarrow{\scriptstyle \varphi_N^{k-1}} & \\
\mathrm{H}^{j-k}_{\mathrm{dR}}(N) & & \\
\downarrow & & \\
0. & &
\end{array}$$

It follows from (2) together with the Homotopy-Equivalence Basics Lemma 19.14 (5) that $\mathrm{id}_N \times \iota\colon N \times S^{k-1} \to N \times (U \cap V)$ is a homotopy equivalence. It is now a consequence of the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 that the horizontal map is an isomorphism. By induction we know that the zig-zag vertical sequence is exact. Hence the straight vertical sequence is also exact.
(8) It follows from (5) and (6) that the image of $i \oplus i$ in $\mathrm{H}^{j-1}_{\mathrm{dR}}(N \times (U \cap V))$ equals the image of $p^*\colon \mathrm{H}^{j-1}_{\mathrm{dR}}(N) \to \mathrm{H}^{j-1}_{\mathrm{dR}}(N \times (U \cap V))$.

The proposition follows from these observations and a mild diagram chase, which we leave to the reader. ∎

Using the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.10 we can now easily calculate the de Rham cohomology groups of (higher dimensional) tori:

Lemma 80.11. (Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) Let $n \in \mathbb{N}$.
(1) For any $k \in \mathbb{N}_0$ we have[826]

$$\dim(\mathrm{H}^k_{\mathrm{dR}}(n\text{-dimensional torus } (S^1)^n)) \;=\; \binom{n}{k}.$$

(2) Let $\theta \in \mathrm{H}^1_{\mathrm{dR}}(S^1)$ be a non-zero de Rham cohomology class. For $i = 1, \dots, n$ we denote by $p_i\colon (S^1)^n \to S^1$ the projection onto the i-th factor. The classes $p_1^*(\theta), \dots, p_n^*(\theta)$ represent a basis for $\mathrm{H}^1_{\mathrm{dR}}((S^1)^n) \cong \mathbb{R}^n$.

Proof.
(1) We prove this statement by induction on n. For $n = 0$ this statement follows from the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10. The induction step is a straightforward consequence of

[826] We refer to Definition 1.43 for the definition of $\binom{n}{k}$.

the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.10 together with the recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

from page 111.

(2) For $j = 1, \dots, n$ we consider the smooth embedding

$$\begin{array}{rcl} i_j \colon S^1 & \to & (S^1)^n \\ z & \mapsto & (1, \dots, 1, \underset{\substack{\uparrow \\ j\text{-th coordinate}}}{z}, 1, \dots, 1). \end{array}$$

We now consider the maps

$$\begin{array}{rcl} \Phi \colon \prod_{j=1}^n \mathrm{H}^1_{\mathrm{dR}}(S^1) & \to & \mathrm{H}^1_{\mathrm{dR}}((S^1)^n) \\ (\omega_1, \dots, \omega_n) & \mapsto & p_1^*(\omega_1) + \dots + p_n^*(\omega_n) \end{array} \qquad \begin{array}{rcl} \Psi \colon \mathrm{H}^1_{\mathrm{dR}}((S^1)^n) & \to & \prod_{j=1}^n \mathrm{H}^1_{\mathrm{dR}}(S^1) \\ \omega & \mapsto & (i_1^*(\omega), \dots, i_n^*(\omega)). \end{array}$$

Note that $p_k \circ i_l$ is the identity if $k = l$ and the constant map otherwise. It follows easily from this observation and the discussion on page 1596 that $\Psi \circ \Phi = \mathrm{id}$. This shows that Φ is a monomorphism. Thus $p_1^*(\theta), \dots, p_n^*(\theta)$ are linearly independent. But by (1) we know that $\dim(\mathrm{H}^1_{\mathrm{dR}}((S^1)^n)) = n$. Thus $p_1^*(\theta), \dots, p_n^*(\theta)$ form in fact a basis of $\mathrm{H}^1_{\mathrm{dR}}((S^1)^n)$. ■

Example. We consider the smooth manifold $\mathbb{R}^2/\mathbb{Z}^2$. On page 1541 we introduced two closed 1-forms $\overline{dx}$ and $\overline{dy}$. We claim that

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^2/\mathbb{Z}^2) = \begin{cases} \mathbb{R} \cdot [\text{constant} = 1 \text{ function}], & \text{if } k = 0, \\ \mathbb{R} \cdot [\overline{dx}] \oplus \mathbb{R} \cdot [\overline{dy}], & \text{if } k = 1, \\ \mathbb{R} \cdot [\overline{dx} \wedge \overline{dy}], & \text{if } k = 2. \end{cases}$$

Indeed: By the Torus-as-Quotient Lemma 25.42 we know that $\mathbb{R}^2/\mathbb{Z}^2$ is diffeomorphic to $S^1 \times S^1$. Thus it follows from the Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 80.11 that it suffices to show that the given de Rham cohomology classes are linearly independent. But this can be verified easily using the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11. □

80.5. **De Rham cohomology of the complex projective spaces.** The only goal of this section is to prove the following proposition:

Proposition 80.12. ($\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition)

(1) For any $n \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

(2) Given any $m \leq n$ we denote by

$$\begin{array}{rcl} \iota \colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \dots : z_m] & \mapsto & [z_0 : \dots : z_m : 0] \end{array}$$

the usual embedding which we first introduced on page 338. The induced map

$$\iota^* \colon \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^m)$$

is an isomorphism for $k \leq 2m$.

Remark. The references [**MTo1997**, Theorem 14.8] and [**CdS2001**, p. 96] give examples of explicit closed $2k$-forms that are generators of $\mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$. □

Proof.

(1) We prove this statement by induction on $n \in \mathbb{N}_0$. For $n = 0$ we have $\mathbb{C}\mathrm{P}^0 = \{*\}$. Thus this case was successfully dealt with by the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10. We now assume that the statement holds for some $n-1 \in \mathbb{N}_0$. We need to prove the statement for n. To do so we introduce the following notation:

- We consider again the usual embedding
$$\begin{array}{rcl} \iota\colon \mathbb{C}\mathrm{P}^{n-1} & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_{n-1}] & \mapsto & [z_0 : \cdots : z_{n-1} : 0]. \end{array}$$
- We write $U := \mathbb{C}\mathrm{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$ and $V := \mathbb{C}\mathrm{P}^n \setminus \iota(\mathbb{C}\mathrm{P}^{n-1})$. It follows easily from the definition of the topology on $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ that U and V are open subsets of $\mathbb{C}\mathrm{P}^n$ and that $U \cup V = \mathbb{C}\mathrm{P}^n$.
- We denote by $j_U\colon U \to \mathbb{C}\mathrm{P}^n$, $j_V\colon V \to \mathbb{C}\mathrm{P}^n$, $i_U\colon U \cap V \to U$ and $i_V\colon U \cap V \to V$ the inclusion maps.

In the following figure we illustrate analogs of these subsets for $\mathbb{R}\mathrm{P}^2$ instead of $\mathbb{C}\mathrm{P}^2$. Perhaps it helps.



Claim.
(a) The map $\iota\colon \mathbb{C}\mathrm{P}^{n-1} \to U$ is a homotopy equivalence.
(b) V is diffeomorphic to $\mathbb{C}^n$.
(c) $U \cap V$ is homotopy equivalent to S^{2n-1}.

Proof.
(a) We consider the map
$$\begin{array}{rcl} F\colon U \times [0,1] & \to & U \\ ([z_0 : \cdots : z_n : z_{n+1}], t) & \mapsto & [z_0 : \cdots : z_n : z_{n+1} \cdot (1-t)]. \end{array}$$
It follows from the definition of U that in our setting the one of the first $n+1$-entries of $(z_0, \ldots, z_n, z_{n+1} \cdot (1-t))$ is never the zero vector, i.e. the map F is actually well-defined and takes values in U. Using the Homotopy Quotient Theorem 18.11 one can easily verify that this map F is continuous. It is now clear that the map is a deformation retraction from U to $\iota(\mathbb{C}\mathrm{P}^n)$. Since ι is an embedding it now follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $\iota\colon \mathbb{C}\mathrm{P}^{n-1} \to U$ is a homotopy equivalence.
(b) In the proof of the Projective Space–Manifold Proposition 25.13 we already showed that the map $\mathbb{C}^n \to V$ that is given by $(z_1, \ldots, z_n) \mapsto [z_1 : \cdots : z_n : 1]$ is a diffeomorphism.
(c) It follows easily from (b) that $U \cap V$ is diffeomorphic to $\mathbb{C}^n \setminus \{0\}$. Evidently this topological space is homotopy equivalent to S^{2n-1}. ⊞

We now consider the following diagram:

$$\begin{array}{ccccccccc} \ldots \to \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) & \xrightarrow{(j_U^*,-j_V^*)} & \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \overbrace{\mathrm{H}^k_{\mathrm{dR}}(V)}^{=0 \text{ for } k\neq 0} & \xrightarrow{i_U^*\oplus i_V^*} & \underbrace{\mathrm{H}^k_{\mathrm{dR}}(U\cap V)}_{\cong \mathrm{H}^k_{\mathrm{dR}}(S^{2n-1})} & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \to \ldots \\ & \searrow^{\iota^*} & \cong \downarrow \iota^* & & & & \\ & & \underbrace{\mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})}_{\substack{\text{we know this}\\ \text{by induction}}} & & & & \end{array}$$

We make the following clarifications and observations:

- The top horizontal sequence is the long exact sequence provided by the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2.
- It follows easily from the functoriality of de Rham cohomology that the triangle commutes.
- It follows from (a) and the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 (1b) that the vertical maps $\iota^*\colon \mathrm{H}^k_{\mathrm{dR}}(U) \to \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})$ are all isomorphisms.
- It follows from (b) and the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 that $\mathrm{H}^k_{\mathrm{dR}}(V) \cong \mathbb{R}$ for $k = 0$ and that $\mathrm{H}^k_{\mathrm{dR}}(V) = 0$ for $k \neq 0$.
- It follows from (c), the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 (1a) and the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 that $\mathrm{H}^k_{\mathrm{dR}}(U\cap V) \cong \mathrm{H}^k_{\mathrm{dR}}(S^{2n-1}) \cong \mathbb{R}$ for $k = 0, 2n-1$, and $\mathrm{H}^k_{\mathrm{dR}}(U\cap V) = 0$ otherwise.

Now let us first consider the case $k > 1$. It follows easily from the above sequence that the inclusion induced maps $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})$ are isomorphisms for $k \leq 2n-1$. The most interesting part of the above sequence now looks as follows:

$$\underbrace{\mathrm{H}^{2n-2}_{\mathrm{dR}}(S^{2n-1})}_{\substack{=0, \quad \text{for } n\neq 1,\\ \cong\mathbb{R}, \quad \text{for } n=1}} \to \mathrm{H}^{2n-1}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \to \underbrace{\mathrm{H}^{2n-1}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})}_{=0, \text{ by induction}} \to \underbrace{\mathrm{H}^{2n-1}_{\mathrm{dR}}(S^{2n-1})}_{\cong\mathbb{R}} \to \mathrm{H}^{2n}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \to \underbrace{\mathrm{H}^{2n}_{\mathrm{dR}}(S^{2n-1})}_{=0}$$

The desired statement for $k > 1$ now follows almost immediately from the above discussion.

For $k = 0, 1$ one needs to think just a little harder, using the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (1) and (2), to get the desired result. We leave it to the reader to fill in the few remaining details.

(2) First note that for $m = n-1$ we just showed implicitly in the proof of Statement (1) that for $k = 0, \ldots, m-1$ the inclusion induced map

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \;\to\; \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^m)$$

is an isomorphism. For an arbitrary $m \leq n$ the desired statement follows easily from the functoriality of de Rham cohomology and by considering iteratively the cases $m, m+1, \ldots, n-1, n$. Again we leave it to the reader to fill in the details. ∎

80.6. **De Rham cohomology groups of mapping tori.** In this section we will study the de Rham cohomology groups of mapping tori. We start out by recalling the following lemma:

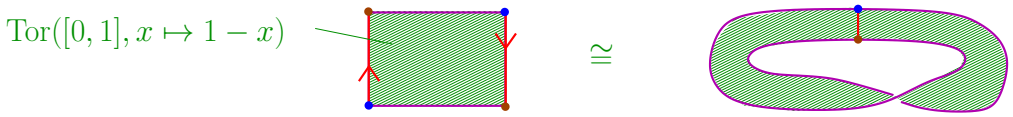
Lemma 25.36. (Mapping Torus-of-Manifold Lemma) Let M be a k-dimensional smooth manifold M and let $f\colon M \to M$ be a self-diffeomorphism. The mapping torus

$$\operatorname{Tor}(M,f) \;:=\; (M \times [0,1])/(x,0) \sim (f(x),1)$$

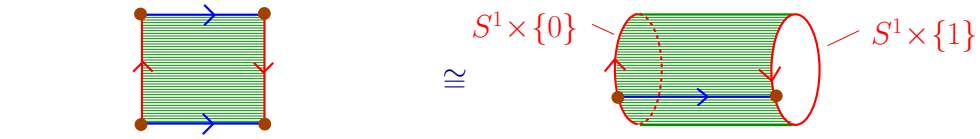
is naturally a $(k+1)$-dimensional smooth manifold.

Examples.

(1) We consider the smooth manifold $[0,1]$. Let $f\colon [0,1] \to [0,1]$ be the self-diffeomorphism of $[0,1]$ that is given by $f(x) = 1-x$. It follows immediately from the definitions that the corresponding mapping torus $\operatorname{Tor}([0,1],f)$ is precisely the Möbius band as defined on page 310.



(2) Let $M = S^1$ and let $f\colon S^1 \to S^1$ be the self-diffeomorphism that is given by $f(z) = \overline{z}$.



Klein bottle $([0,1] \times [0,1])/\sim$ mapping torus $\operatorname{Tor}(S^1, z \mapsto \overline{z})$

As we discussed on page 575, the resulting mapping torus is homeomorphic to the Klein bottle.[827] □

Now we can formulate the main result of this section:

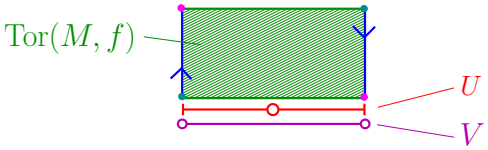
Proposition 80.13. (Mapping Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition) Let M be a smooth manifold and let $f\colon M \to M$ be a self-diffeomorphism. We denote by $i\colon M \to \operatorname{Tor}(M,f)$ the natural map that is given by $x \mapsto [(x,0)]$. For every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\delta_n\colon \mathrm{H}^n_{\mathrm{dR}}(M) \;\to\; \mathrm{H}^{n+1}_{\mathrm{dR}}(\operatorname{Tor}(M,f))$$

such that the following sequence is exact:

$$\ldots \to \mathrm{H}^n_{\mathrm{dR}}(\operatorname{Tor}(M,f)) \xrightarrow{i^*} \mathrm{H}^n_{\mathrm{dR}}(M) \xrightarrow{f^*-\mathrm{id}} \mathrm{H}^n_{\mathrm{dR}}(M) \xrightarrow{\delta_n} \mathrm{H}^{n+1}_{\mathrm{dR}}(\operatorname{Tor}(M,f)) \xrightarrow{i^*} \mathrm{H}^{n+1}_{\mathrm{dR}}(M) \to \ldots$$

Sketch of proof. The idea is quite simple: Let $p\colon M \times [0,1] \to \operatorname{Tor}(M,f)$ be the natural projection. We consider $U := p(M \times ([0,\frac{1}{2}) \cup (\frac{1}{2},1]))$ and $V := p(M \times (0,1))$.



These sets form an open cover of $\operatorname{Tor}(M,f)$ and we can thus apply the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2. Next we make the following observations:

(1) The map $M \to U$ given by $x \mapsto [(x,\frac{1}{4})]$ is a homotopy equivalence.

[827] In fact, if one goes carefully through our definitions of the smooth structures one can easily show that the mapping torus is in fact in fact diffeomorphic to the Klein bottle.

(2) The map $M \to V$ given by $x \mapsto [(x, \frac{1}{2})]$ is a homotopy equivalence.
(3) We have $U \cap V = p((0, \frac{1}{2}) \times M) \sqcup p((\frac{1}{2}, 1) \times M)$. Thus $U \cap V$ is homotopy equivalent to $M \sqcup M$.

We apply these homotopy equivalences, together with the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2, to the Mayer-Vietoris sequence. After some algebraic manipulations one obtains the desired result.

In the proof of the Mapping Torus-H_*-Proposition 153.6, which we will provide on page 2909, we will work out the details in the context of "singular homology". We leave it to the reader to modify the later argument to deal with the current setting. ■

The Mapping Torus-H_*-Proposition 153.6 provides us with the tools for computing the homology groups of the Klein bottle:

Lemma 80.14. (Klein Bottle-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma) Let K be the Klein bottle. We have isomorphisms

$$\mathrm{H}^n_{\mathrm{dR}}(K) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider $M = S^1$ and we consider the self-diffeomorphism $f\colon S^1 \to S^1$ that is given by $f(z) = \overline{z} = z^{-1}$. As we just discussed, the Klein bottle K is homeomorphic to the mapping torus $\mathrm{Tor}(S^1, f)$. It follows from the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 that it suffices to calculate the de Rham cohomology groups of the mapping torus $\mathrm{Tor}(S^1, f)$.

Next note that it follows from the Mapping Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.13 that we get the exact sequence

$$\begin{aligned} 0 \longrightarrow\ & \mathrm{H}^0_{\mathrm{dR}}(\mathrm{Tor}(S^1, f)) \xrightarrow{i^*} \mathrm{H}^0_{\mathrm{dR}}(S^1) \xrightarrow{f^*-\mathrm{id}} \mathrm{H}^0_{\mathrm{dR}}(S^1) \xrightarrow{\delta^0} \dots \\ \xrightarrow{\delta^0}\ & \mathrm{H}^1_{\mathrm{dR}}(\mathrm{Tor}(S^1, f)) \xrightarrow{i^*} \mathrm{H}^1_{\mathrm{dR}}(S^1) \xrightarrow{f^*-\mathrm{id}} \mathrm{H}^1_{\mathrm{dR}}(S^1) \xrightarrow{\delta^1} \\ \xrightarrow{\delta^1}\ & \mathrm{H}^2_{\mathrm{dR}}(\mathrm{Tor}(S^1, f)) \xrightarrow{i^*} \mathrm{H}^2_{\mathrm{dR}}(S^1) \xrightarrow{f^*-\mathrm{id}} \mathrm{H}^2_{\mathrm{dR}}(S^1) \xrightarrow{\delta^2} \end{aligned}$$

To get our hands on the de Rham cohomology groups of $K \cong \mathrm{Tor}(S^1, f)$ we make the following observations:

(1) By the Sphere-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.4 we know that $\mathrm{H}^i_{\mathrm{dR}}(S^1) \cong \mathbb{R}$ for $i = 0, 1$ and that $\mathrm{H}^i_{\mathrm{dR}}(S^1) = 0$ for $i \geq 2$.
(2) Since S^1 is connected it follows from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (1) and (2) that the induced map $f^*\colon \mathrm{H}^0_{\mathrm{dR}}(S^1) \to \mathrm{H}^0_{\mathrm{dR}}(S^1)$ is the identity map.
(3) It follows from the Degree-of-Power Lemma 80.7 (or alternatively the Sphere–Degree–Matrix Lemma 80.9) that the induced homomorphism $f^*\colon \mathrm{H}^1_{\mathrm{dR}}(S^1) \to \mathrm{H}^1_{\mathrm{dR}}(S^1)$ is given by multiplication by -1.

It follows easily from these observations that the above exact sequence simplifies to an exact sequence of the following form:

$$\begin{aligned} 0 \longrightarrow\ & \mathrm{H}^0_{\mathrm{dR}}(\mathrm{Tor}(S^1, f)) \to \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{\delta^0} \dots \\ \xrightarrow{\delta^0}\ & \mathrm{H}^1_{\mathrm{dR}}(\mathrm{Tor}(S^1, f)) \to \mathbb{R} \xrightarrow{\cdot(-2)} \mathbb{R} \xrightarrow{\delta^1} \\ \xrightarrow{\delta^1}\ & \mathrm{H}^2_{\mathrm{dR}}(\mathrm{Tor}(S^1, f)) \to 0 \xrightarrow{0} 0 \xrightarrow{\delta^2} \end{aligned}$$

Elementary linear algebra now gives us the promised de Rham cohomology groups. ■

80.7. **Appendix: The Hairy Ball Theorem.** In this chapter we (re-) introduced the degree of a continuous self-map of a sphere. For the reader's entertainment we recall in this appendix several statements one can prove using this concept. First of all let us recall the following theorem:

Theorem 46.15. (Sphere-Degree Fixed Point Theorem) Let $f\colon S^n \to S^n$ be a continuous map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Proof. [828] Let $f\colon S^n \to S^n$ be a continuous map that does not admit a fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\operatorname{id}$.

It follows easily from our hypothesis that f has no fixed point that for all $t \in [0,1]$ and all $x \in S^n$ we have $(1-t)\cdot f(x) \neq t \cdot x$ Therefore we can consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-t)\cdot f(x) - t\cdot x}{\|(1-t)\cdot f(x) - t\cdot x\|}. \end{array}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \underset{\uparrow}{=} \deg(-\operatorname{id}) \underset{\uparrow}{=} \det(-\operatorname{id}_{n+1}) = (-1)^{n+1}.$$

by the Sphere–De Rham Degree Lemma 80.8 (3) — by the Sphere–Degree–Matrix Lemma 80.9

■

We use this opportunity to reprove the following result:

Proposition 46.17. (Free Action-on-Sphere Proposition) If a finite group G acts freely and continuously on an even-dimensional sphere, then G is either trivial or $G \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. The case of a 0-dimensional sphere is of course trivial. Now suppose that G is a group that acts freely and continuously on a sphere S^n with $n \geq 1$. We consider the map

$$\begin{array}{rcl} G & \to & \mathbb{Z} \\ g & \mapsto & \deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & g\cdot x \end{pmatrix}. \end{array}$$

It follows from the Sphere–De Rham-Degree Lemma 80.8 (5) that this map takes values in $\{-1,1\}$ and it follows from the Sphere–De Rham-Degree Lemma 80.8 (4) that the map $G \to \{-1,1\}$ is actually a group homomorphism.

Now if n is even, then it follows from the Sphere-Degree Fixed Point Theorem 46.15 (which we had just reproved) and our hypothesis that G acts freely that the above map $G \to \{-1,1\}$ is actually a monomorphism. ■

Next we recall Definition 29.27 of a vector field on a submanifold of $\mathbb{R}^n$:

Definition 80.15. Let $M \subseteq \mathbb{R}^n$ be a submanifold. A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for every $P \in M$ we have $v(P) \in \mathrm{T}_P M$.

Examples.

[828] We proved the Sphere-Degree Fixed Point Theorem 46.15 on page 1080 using the original notion of the degree of a continuous map $f\colon S^n \to S^n$. The proof we now provide is almost identical to the proof on page 1080.

(1) Let P be a point on S^n. By the Physical Tangent Spaces–Lemma 29.2 we have

$$\mathrm{T}_P S^n \;=\; \{v \in \mathbb{R}^{n+1} \,|\, P \cdot v = 0\}.$$

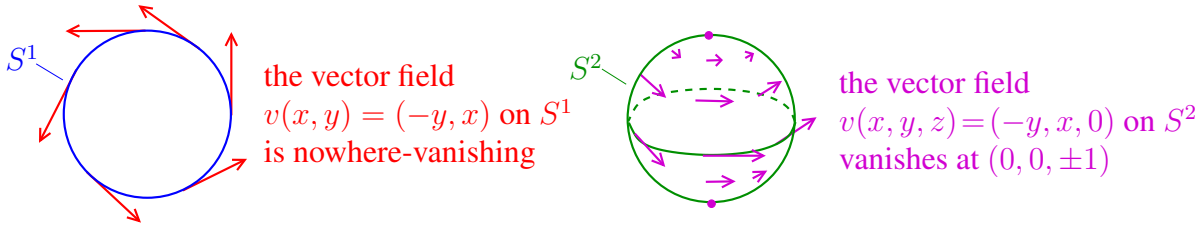
(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{array}$$

This vector field has the property that it is nowhere-vanishing.

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$



This vector field vanishes at the "North Pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "South Pole" $(0, \ldots, 0, -1)$. □

In Question 29.28 we asked whether it is possible to find a nowhere-vanishing vector field on even-dimensional spheres. The following theorem says that this is not possible:

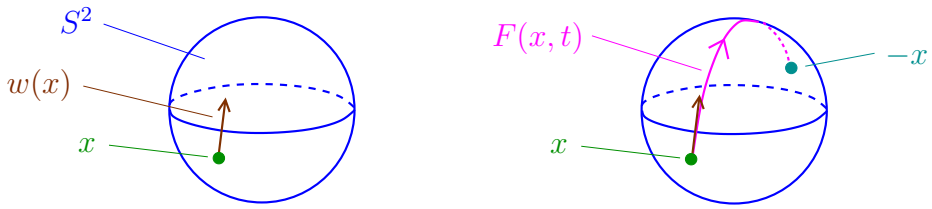
Theorem 46.18. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.

Proof. We proved the Hairy Ball Theorem 46.18 on page 1082 using the original notion of the degree of a smooth map $f\colon S^n \to S^n$. In an another shameless case of self-plagiarism the proof we now provide is again almost identical to our earlier proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. Since v is nowhere vanishing we can consider the vector field w on S^n that is given by rescaling v to length one, i.e. we consider the vector field that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) \coloneqq \frac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$



Note that the map F is a homotopy between the two maps $F_0 = \mathrm{id}$ and $F_1 = -\mathrm{id}$. It follows that

$$1 \underset{\uparrow}{=} \deg(\mathrm{id}) \underset{\uparrow}{=} \deg(-\mathrm{id}) \underset{\uparrow}{=} \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$

by the Sphere–de Rham-Degree Lemma 80.8 (1) and (3)

by the Sphere–Degree–Matrix Lemma 80.9

But this means that n is odd. ∎

80.8. **Appendix: Topological invariance of domain.** Recall that in the Differential-of-Diffeomorphism Proposition 24.19 we showed the following: If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are non-empty open subsets which are *diffeomorphic*, then $m = n$. Using the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we can now upgrade this statement from "diffeomorphic" to "homeomorphic":

Proposition 80.16. (Topological Invariance-of-Domain Proposition) Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be non-empty open subsets. If U and V are *homeomorphic*, then $m = n$.

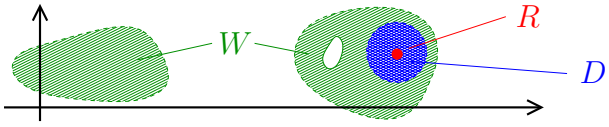
Proof. In the following we deal with the case that $m, n \geq 2$. The other cases can be dealt with, with minor adjustments (using the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8) of the argument below. We leave these cases to the reader.

We start out with the following claim:

Claim. Let $d \in \mathbb{N}_{\geq 2}$, let W be an open subset of $\mathbb{R}^d$ and let $R \in W$. We denote by $\iota\colon W \setminus \{R\} \to W$ the inclusion. For any $k \in \mathbb{N}$ we have

$$\mathrm{coker}(\mathrm{H}^k_{\mathrm{dR}}(W) \xrightarrow{\iota^*} \mathrm{H}^k_{\mathrm{dR}}(W\setminus\{R\})) \oplus \ker(\mathrm{H}^{k+1}_{\mathrm{dR}}(W) \xrightarrow{\iota^*} \mathrm{H}^{k+1}_{\mathrm{dR}}(W\setminus\{R\})) \cong \begin{cases} \mathbb{R}, & \text{if } k = d-1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since W is an open subset of $\mathbb{R}^d$ we can pick an $\epsilon > 0$ with $D := B^d_\epsilon(R) \subseteq W$.



It follows from the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 that for $k \geq 1$ we obtain a long exact sequence of the following form:

$$\cdots \to \mathrm{H}^k_{\mathrm{dR}}(W) \to \mathrm{H}^k_{\mathrm{dR}}(W \setminus \{R\}) \oplus \underbrace{\mathrm{H}^k_{\mathrm{dR}}(D)}_{=0 \text{ since } k \geq 1} \to \underbrace{\mathrm{H}^k_{\mathrm{dR}}(D \setminus \{R\})}_{\cong \mathbb{R}, \text{ if } k = d-1, \; = 0, \text{ else}} \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR}}(W) \to \dots$$

The claim now follows immediately from the following clarifications and observations:

(1) All maps, except for the connecting homomorphism, are induced by the inclusions.
(2) Since D is contractible we obtain from the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 that for all $k \geq 1$ we have $\mathrm{H}^k_{\mathrm{dR}}(D) = 0$.
(3) Note that $D \setminus \{R\}$ admits a deformation retraction to the sphere $S^{d-1}_{\frac{\epsilon}{2}}(R)$. It follows that for any $k \in \mathbb{N}$ we have isomorphisms

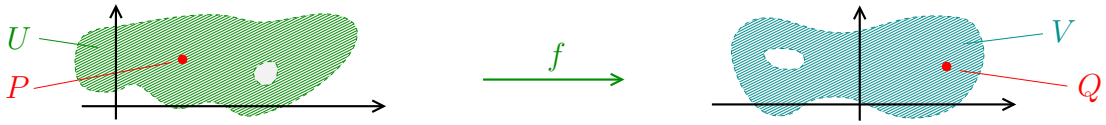
$$\mathrm{H}^k_{\mathrm{dR}}(D \setminus \{R\}) \underset{\uparrow}{\cong} \mathrm{H}^k_{\mathrm{dR}}(S^{d-1}_{\frac{\epsilon}{2}}(R)) \underset{\uparrow}{\cong} \begin{cases} \mathbb{R}, & \text{if } k = d-1, \\ 0, & \text{otherwise.} \end{cases}$$

by the Homotopy Equivalence $\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8

by the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4

⊞

Now let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be non-empty open subsets. We assume that there exists a homeomorphism $f\colon U \to V$. Since U is non-empty we can pick $P \in U$. We set $Q := f(P)$.



We denote by $i\colon U \setminus \{P\} \to U$ and $j\colon V \setminus \{Q\} \to Q$ the inclusions. It follows from the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 and elementary arguments that for any $k \in \mathbb{N}_0$ we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(j^*) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(V) & \xrightarrow{j^*} & \mathrm{H}^k_{\mathrm{dR}}(V \setminus \{Q\}) & \longrightarrow & \operatorname{coker}(j^*) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle f^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f^*} & & \downarrow & & \\
0 & \longrightarrow & \ker(i^*) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(U) & \xrightarrow{i^*} & \mathrm{H}^k_{\mathrm{dR}}(U \setminus \{P\}) & \longrightarrow & \operatorname{coker}(i^*) & \longrightarrow & 0.
\end{array}$$

Since f is a homeomorphism we obtain from the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 (2) that the middle two vertical maps are isomorphisms. It follows from elementary arguments (e.g. the Five Lemma 77.9) that vertical maps to the left and right are also isomorphisms. In particular we see that the kernels and cokernels on top and bottom are also isomorphic. It now follows almost immediately from the claim that $m = n$. ■

80.9. Exercises for Chapter 80.

Exercise 80.1. Let $\mathbb{F}$ be a field and let $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$ be an exact sequence of homomorphisms between finite-dimensional $\mathbb{F}$-vector spaces. Show that

$$\sum_{i=0}^{k} (-1)^k \cdot \dim_{\mathbb{F}}(C_i) \;=\; 0.$$

Remark. This exercise is almost the same as Exercise 77.1 (b).

Exercise 80.2. Let $d \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. Compute the de Rham cohomology groups of the smooth manifold $(S^d)^n$.

Exercise 80.3. Let $X \subseteq \mathbb{R}^n$ be a non-empty compact subset. As in the Stereographic Projection Lemma 4.24 we make the identification $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ of the sphere with the space $\mathbb{R}^n$ together with a "point at infinity".

(a) Show that for every $k \neq n-1$ the inclusion-induced map

$$\mathrm{H}^k_{\mathrm{dR}}(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=_{\mathrm{i}} S^n} \setminus X) \;\to\; \mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n \setminus X)$$

is an isomorphism.

(b) Show that there exists a short exact sequence

$$0 \;\to\; \mathrm{H}^{n-1}_{\mathrm{dR}}((\mathbb{R}^n \cup \{\infty\}) \setminus X) \;\to\; \mathrm{H}^{n-1}_{\mathrm{dR}}(\mathbb{R}^n \setminus X) \;\to\; \mathbb{R} \;\to\; 0.$$

Exercise 80.4. Let $P_1, \dots, P_n \in \mathbb{R}^2$ be pairwise distinct points.

(a) Compute the isomorphism types of all of the de Rham cohomology groups of the smooth manifold $M := \mathbb{R}^2 \setminus \{P_1, \dots, P_n\}$.

(b) Give explicit closed 1-forms on M which define a basis for $\mathrm{H}^1_{\mathrm{dR}}(M)$.

Exercise 80.5. Let $X \subseteq \mathbb{R}^k$ be a finite set with m elements. Determine the de Rham cohomology groups $\mathrm{H}^j_{\mathrm{dR}}(\mathbb{R}^k \setminus X)$.

Exercise 80.6. We consider the torus $S^1 \times S^1$ and the two projections $p, q\colon S^1 \times S^1 \to S^1$. Furthermore we consider the embeddings $i, j\colon S^1 \to S^1 \times S^1$ that are given by $i(z) = (z, 1)$ and $j(z) = (1, z)$.

(a) Show that the maps

$$\begin{array}{rccc} \Phi\colon & \mathrm{H}^1_{\mathrm{dR}}(S^1) \times \mathrm{H}^1_{\mathrm{dR}}(S^1) & \to & \mathrm{H}^1_{\mathrm{dR}}(S^1 \times S^1) \\ & (a, b) & \mapsto & p^*(a) + q^*(b) \end{array}$$

and

$$\begin{array}{rccc} \Psi\colon & \mathrm{H}^1_{\mathrm{dR}}(S^1 \times S^1) & \to & \mathrm{H}^1_{\mathrm{dR}}(S^1) \times \mathrm{H}^1_{\mathrm{dR}}(S^1) \\ & c & \mapsto & (i^*(c), j^*(c)) \end{array}$$

are inverses of one another.

(b) Let $\omega \in \mathrm{H}^1_{\mathrm{dR}}(S^1)$ be a non-zero element and let $a, b, c, d \in \mathbb{Z}$. We consider the map

$$\begin{array}{rccc} \Theta\colon & S^1 \times S^1 & \to & S^1 \times S^1 \\ & (x, y) & \mapsto & (x^a \cdot y^b, x^c \cdot y^d). \end{array}$$

Determine $\Theta^*\colon \mathrm{H}^1_{\mathrm{dR}}(S^1 \times S^1) \to \mathrm{H}^1_{\mathrm{dR}}(S^1 \times S^1)$ with respect to the basis $p^*(\omega), q^*(\omega)$.
Hint. Use the Degree-of-Power Lemma 80.7.

(c) Provide an example of a closed 3-dimensional smooth manifold M with $\mathrm{H}^1_{\mathrm{dR}}(M) \cong \mathbb{R}^2$.
Hint. Use the Mapping Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.13.

Exercise 80.7. Let $n \in \mathbb{N}_0$. Let $\omega \in \Omega^1(S^1)$ be a smooth 1-form with $\int_{S^1} \omega = 1$. For $i = 1, \dots, n$ we denote by $p_i\colon (S^1)^n \to S^1$ the projection onto the i-th factor. We set $\theta_i := p_i^*([\omega])$.

(a) Show that there exists a unique ring homomorphism

$$\Phi\colon \mathbb{R}[x_1, \dots, x_n]/(x_i^2 = 0, x_i \cdot x_j - x_j \cdot x_i = 0) \to \mathrm{H}_{\mathrm{dR}}((S^1)^n)$$

such that for each $i \in \{1, \dots, n\}$ we have $\Phi(x_i) = \theta_i$.

(b) Show that the ring homomorphism Φ is a monomorphism.

(c) Show that the ring homomorphism Φ is an epimorphism.

Remark. The case $n = 2$ was mostly dealt with on page 1626.

Exercise 80.8. Compute the de Rham cohomology groups of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ where $z \sim -z$ for $z \in S^1$.
Hint. Use a Mayer–Vietoris argument and use the Degree-of-Power Lemma 80.7.

Exercise 80.9.

(a) Show that for any $k \in \mathbb{Z}$ there exists a continuous self map of S^2 of degree k.
Hint. We view S^2 as a subspace of $\mathbb{C} \times \mathbb{R}$. Given $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rccc} h_k\colon & S^2 & \to & S^2 \\ & (z = x + \mathrm{i}y, x_3) & \mapsto & (z^k, x_3). \end{array}$$

Now make use of the naturality of the connecting homomorphism in the Mayer–Vietoris sequence to reduce the calculation of the degree to the Degree-of-Power Lemma 80.7.

(b) Show that for any $d \in \mathbb{N}$ and any $k \in \mathbb{Z}$ there exists a continuous self map of S^2 of degree k.

Exercise 80.10. Show that for any $k \in \mathbb{Z}$ there exists a smooth map $f\colon S^1 \times S^1 \to S^1 \times S^1$ such that the induced map on second de Rham cohomology is given by multiplication by k.

Exercise 80.11. Let $T = S^1 \times S^1$, let $P \in T$ be a point and let $\Phi\colon \overline{B}^2 \to T$ be a smooth embedding with $\Phi(0) = P$. We set $W := T \setminus \{P\}$ and we set $D := \Phi(B^2)$. Finally we denote by $p, q\colon T \to S^1$ the projection onto the first and second factor.

(a) Show that the map

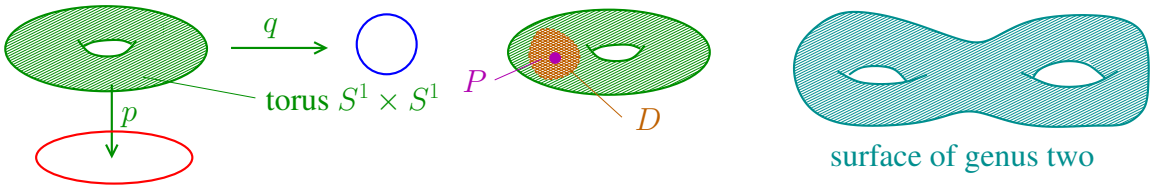
$$\begin{array}{rcl} \mathrm{H}^1_{\mathrm{dR}}(S^1) \times \mathrm{H}^1_{\mathrm{dR}}(S^1) & \to & \mathrm{H}^1_{\mathrm{dR}}(W) \\ (\alpha, \beta) & \mapsto & (p|_W)^*(\alpha) + (q|_W)^*(\beta) \end{array}$$

is an isomorphism.

(b) Show that the inclusion induced map $\mathrm{H}^1_{\mathrm{dR}}(T) \to \mathrm{H}^1_{\mathrm{dR}}(W)$ is an isomorphism.
Hint. Apply the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 to $T = W \cup D$ and use (a).

(c) Show that the inclusion induced map $\mathrm{H}^1_{\mathrm{dR}}(W) \to \mathrm{H}^1_{\mathrm{dR}}(D \setminus \{P\})$ is the zero map.

(d) Compute the de Rham cohomology groups of the surface Σ of genus 2.
Hint. Show that Σ can be obtained from two copies of W and apply the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2.

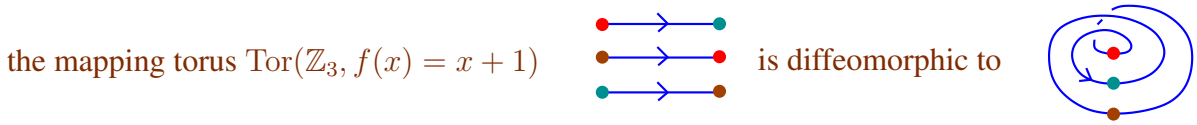


Exercise 80.12. Let $g \in \mathbb{N}$.

(a) Show that the torus admits an atlas consisting of three charts whose domains are diffeomorphic to the 2-balls B^2.

(b) Show that the torus does not admit an atlas with two charts such that each domain is diffeomorphic to the 2-ball B^2.

Exercise 80.13. We consider the 0-dimensional smooth manifold $M = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. Let $f\colon M \to M$ be the diffeomorphism that is given by $f(x) = x + 1 \bmod 3$. As we illustrate in the figure below, the mapping torus $\mathrm{Tor}(M, f)$ is diffeomorphic to S^1. Determine in this context all the de Rham cohomology groups and maps in the Mapping Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.13.



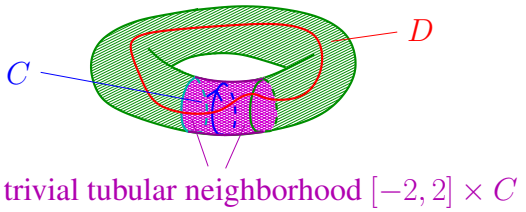
Exercise 80.14. Let M be a smooth manifold. What is the content of the Mapping Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.13 for $f = \mathrm{id}_M$?

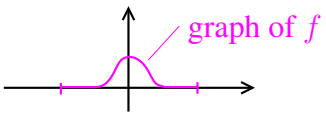
Exercise 80.15. Let M be a closed 2-dimensional smooth manifold and let $f\colon M \to M$ be a diffeomorphism. Show that $\dim(\mathrm{H}^1_{\mathrm{dR}}(\mathrm{Tor}(M,f))) = \dim(\mathrm{H}^2_{\mathrm{dR}}(\mathrm{Tor}(M,f)))$.

Exercise 80.16. Let Σ be the surface of genus 2 equipped with some orientation. Let $C \subseteq \Sigma$ be an oriented submanifold that is diffeomorphic to S^1. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can pick a tubular neighborhood $[-2,2]\times C$ for C. We denote by dt the usual differential 1-form on $[-2,2] \subseteq \mathbb{R}$. Let $f\colon [-2,2] \to \mathbb{R}_{\geq 0}$ be a smooth function with $\mathrm{supp}(f) \subseteq [-1,1]$ and $\int_{t=0}^{t=1} f(t)\,dt = 1$. We denote by $p\colon [-2,2]\times C \to [-2,2]$ the projection. Finally denote by ω the smooth closed differential 1-form on Σ that is given by $p^*(f\cdot dt)$ on $[-2,2]\times C$ and that is extended to all of Σ by setting it zero outside of $C\times[-2,2]$.

Now let D be another oriented 1-dimensional submanifold of Σ. We denote by $\iota\colon D \to \Sigma$ the inclusion. We obtain the integral $\int_D \iota^*(\omega)$.

(a) Show that the integral is an integer.
(b) What is the geometric interpretation of this integer?



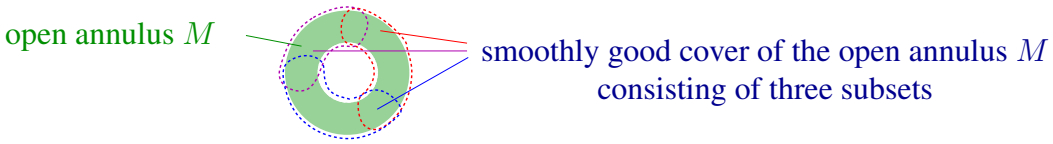


CHAPTER 81

Smoothly good covers of smooth manifolds

We start out with the following definition:

Definition 81.1. Let M be an n-dimensional smooth manifold and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of M. We say that the cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M is smoothly good if for every choice of $i_1, \ldots, i_m \in I$ the intersection $U_{i_1} \cap \cdots \cap U_{i_m}$ is diffeomorphic to a (possibly empty) open convex subset of $\mathbb{R}^n$.[829]



In this chapter we will do the following:

- In Section 81.1 we will show that every compact smooth manifold admits a finite smoothly good cover.
- In Section 81.2 we will use such covers to prove the De Rham Cohomology-Finiteness Proposition 81.10 which says that the de Rham cohomology groups of compact smooth manifolds are always finite-dimensional.
- In the purely algebraic Section 81.3 we will introduce the tensor product of two $\mathbb{F}$-vector spaces and we state several basic properties of the tensor product. The reader who is familiar with the tensor product might want to skip this section.
- In Section 81.4 we will state the Künneth Theorem for de Rham cohomology which allows us to express the de Rham cohomology groups of a product $M \times N$ in terms of tensor products of the de Rham cohomology groups of M and N.
- In Section 81.5 we will prove the Künneth Theorem for de Rham cohomology.
- In Section 81.6 we will provide exercises for Chapter 81.

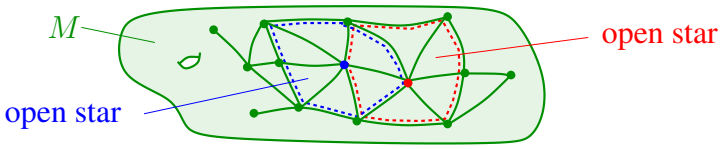
In the next Chapter 82 we will show that every open star-shaped subset of $\mathbb{R}^n$ (thus in particular every open convex non-empty subset of $\mathbb{R}^n$) is diffeomorphic to $\mathbb{R}^n$. This fact, together with the existence of smoothly good open covers, will play a key role in the proof of the De Rham-Poincaré Duality Theorem 84.5.

81.1. **Existence of smoothly good covers of smooth manifolds.** The following theorem is the main technical results of this chapter:

[829] In Definition 178.1 we will introduce a related concept for any topological space X, namely we will say that a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X is *good* if for every choice of $i_1, \ldots, i_m \in I$ the intersection $U_{i_1} \cap \cdots \cap U_{i_m}$ is either empty or contractible. Evidently every convex open non-empty subset of $\mathbb{R}^n$ is contractible. This explains hopefully why we call the current notion "smoothly good".

Theorem 81.2. (Smoothly Good Cover Theorem)
(1) Every closed smooth manifold admits a finite smoothly good open cover.
(2) Every smooth manifold with empty boundary admits a locally finite smoothly good[830] open cover.

Remark.
(1) In the Smooth Manifold-Simplicial Structure Theorem 179.5 we will see that every compact smooth manifold M admits a "smooth simplicial structure". It follows from the Open Stars-Good Cover Lemma 178.2 that the corresponding "open stars" of the vertices form a finite open cover of M such that any intersection of finitely many sets of the open cover is contractible or empty.



This statement is somewhat weaker than the statement of the Smoothly Good Cover Theorem 81.2, but for many purposes, e.g. for the proofs of the De Rham Cohomology-Finiteness Proposition 81.10 and of the Künneth-H^*_{dR}-Theorem 81.21, this weaker statement is good enough.
(2) The statement of the Smoothly Good Cover Theorem 81.2 falls under the category of "well-known to the experts". But it is not that easy to find reliable proofs in the literature. The proof that we will give is an adaptation of the argument given in [**GuH2019**, Appendix C]. On page 1663 we will mention some related literature. □

The proof of the Smoothly Good Cover Theorem 81.2 will require the remainder of this section. In the next Section 81.2 we will use the Smoothly Good Cover Theorem 81.2 to show that the de Rham cohomology groups of compact smooth manifolds are finite-dimensional.

81.1.1. **Convex functions.** Before we can provide the proof of the Smoothly Good Cover Theorem 81.2 we need to prepare the ground:

Definition 81.3. We say that a matrix $H \in \mathrm{M}(n \times n, \mathbb{R})$ is positive-definite if for all $v \in \mathbb{R}^n \setminus \{0\}$ we have $v^{\mathrm{T}} \cdot H \cdot v > 0$.

Soon we will make use of the following basic lemma:

Lemma 81.4. (Positive Definite–Open Condition Lemma) For any $n \in \mathbb{N}_0$ the positive-definite matrices form an open subset of $\mathrm{M}(n \times n, \mathbb{R})$.

Proof of the Positive Definite–Open Condition Lemma 81.4. By the JH-Openness Criterion 3.3 it suffices to prove the following claim:

Claim. If $H \in \mathrm{M}(n \times n, \mathbb{R})$ is positive-definite, then there exists an open neighborhood of H which consists only of positive-definite matrices.

[830] Recall that according to Definition 3.19 we say that a family $\{U_i\}_{i \in I}$ of subsets of M is *locally finite* if each $x \in M$ admits an open neighborhood W such that $\{i \in I \,|\, U_i \cap W \neq \varnothing\}$ is finite.

Proof. First note that it follows from elementary arguments that a matrix $A \in \mathrm{M}(n\times n, \mathbb{R})$ is positive-definite if and only if for all $v \in S^{n-1}$ we have $v^{\mathrm{T}} \cdot A \cdot v > 0$. Now let $H \in \mathrm{M}(n\times n, \mathbb{R})$ be positive-definite. We consider the map

$$\begin{array}{rccc} f\colon \mathrm{M}(n\times n,\mathbb{R}) \times S^{n-1} & \to & \mathbb{R} \\ (A, v) & \mapsto & v^{\mathrm{T}} \cdot A \cdot v. \end{array}$$

Clearly this map is continuous. Since H is positive-definite we now see that $f^{-1}(\mathbb{R} \setminus \{0\})$ is an open neighborhood of $\{H\} \times S^{n-1}$. Since S^{n-1} is compact it follows easily from the Tube Lemma 7.12 that there exists an open neighborhood U of $H \in \mathrm{M}(n \times n, \mathbb{R})$ such that $U \times S^{n-1} \subseteq f^{-1}(\mathbb{R} \setminus \{0\})$. It follows from the observation at the beginning of the proof that all matrices in U are positive-definite. We have thus found the desired open neighborhood U of H. ∎

We move on to a definition from analysis:

Definition 81.5. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $\varphi\colon U \to \mathbb{R}$ be a smooth function.

(1) Given $P \in U$ we refer to

$$\mathrm{Hess}(\varphi)_P \coloneqq \left(\frac{\partial^2 \varphi}{\partial x_i\, \partial x_j}\bigg|_P\right)_{i,j=1,\dots,n} \in \mathrm{M}(n\times n, \mathbb{R})$$

as the Hessian[837] matrix of φ at P.[838]

(2) We say that $\varphi\colon U \to \mathbb{R}$ is convex if for every $P \in U$ the Hessian matrix $\mathrm{Hess}(\varphi)_P$ is positive-definite

The notion of a convex function allows us to formulate the following proposition:

Proposition 81.6. (Convex Function–Preimage Proposition) Let $U \subseteq \mathbb{R}^n$ be an open *convex* subset and let $\varphi\colon U \to \mathbb{R}$ be a smooth function. If φ is convex, then for every $c \in \mathbb{R}$ the set

$$U_c \coloneqq \underbrace{\{x \in U \,|\, \varphi(x) < c\}}_{=\varphi^{-1}((-\infty,c))}$$

is convex.

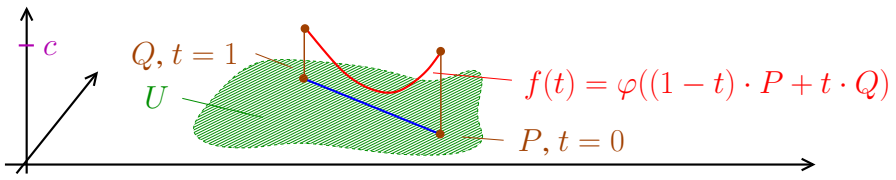
Proof of the Convex Function–Preimage Proposition 81.6. Let $c \in \mathbb{R}$. We want to show that $U_c \coloneqq \{x \in U \,|\, \varphi(x) < c\}$ is convex. Let $P \neq Q \in U_c$. In other words, let $P, Q \in U$ be two distinct points with $\varphi(P) < c$ and $\varphi(Q) < c$. We need to show that the Euclidean segment $\{(1-t)\cdot P + t \cdot Q \,|\, t \in [0,1]\}$ is contained in U_c. To do so we consider the map

$$\begin{array}{rccc} f\colon [0,1] & \to & \mathbb{R} \\ t & \mapsto & \varphi(\underbrace{(1-t)\cdot P + t\cdot Q}_{\in U \text{ since } U \text{ is convex}}). \end{array}$$

It suffices to show that $f(t) < c$ for all $t \in [0,1]$. Since $[0,1]$ is compact it follows from the Compact Image Lemma 4.15 (2) that the function $f\colon [0,1] \to \mathbb{R}$ assumes a maximum at some $t_0 \in [0,1]$. It now remains to show that $f(t_0) < c$. The key to doing so is the following claim:

[837] The Hessian matrix is named after the German mathematician Otto Hesse (1811-1874). Despite appearances the name "Hessian" does not refer to the state of "Hesse".

[838] Implicitly we use throughout that the Hessian matrix is symmetric. The fact that given a smooth function the partial derivatives commute is variously known (depending on the geographic location of the author) as the Schwarz Theorem, the Young Theorem or the Clairaut Theorem.



Claim. For every $t \in (0,1)$ we have $f''(t) > 0$.

Proof. We write $P = (P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$. Note that for any $t \in (0,1)$ we have

$$f'(t) \underset{\uparrow}{=} \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}((1-t)\cdot P + t \cdot Q) \cdot (P_j - Q_j).$$

follows from the Chain Rule applied to $f(t) = \varphi((1-t)\cdot P + t \cdot Q)$

and we have

$$f''(t) = (f')'(t) \underset{\uparrow}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \overbrace{\frac{\partial^2 \varphi}{\partial x_i \, \partial x_j}((1-t)\cdot P + t \cdot Q)}^{ij\text{-entry of the Hessian matrix}} \cdot (P_i - Q_i) \cdot (P_j - Q_j)$$

follows from the Chain Rule applied to the above expression for $f'(t)$

$$\underset{\uparrow}{=} (P-Q)^{\mathrm{T}} \cdot \mathrm{Hess}(\varphi)_{(1-t)\cdot P + t\cdot Q} \cdot (P-Q) \underset{\uparrow}{>} 0.$$

by Definition 81.5 of the Hessian

since the Hessian of φ at the point $(1-t)\cdot P + t \cdot Q \in U$ is assumed to be positive-definite and since $P \neq Q$ ⊞

Now assume that $f(t_0) \geq c$. Since $f(0) = \varphi(P) < c$ and $f(1) = \varphi(Q) < c$ we see that $t_0 \in (0,1)$. But by elementary calculus this implies that $f'(t_0) = 0$ and $f''(t_0) \leq 0$. But the latter inequality is a contradiction to the above claim. ■

We need to introduce one last piece of notation:

Notation 81.7. Let M be an n-dimensional smooth submanifold of some $\mathbb{R}^d$. Given $P \in M$ and $r \in \mathbb{R}$ we write as usual

$$B_r^M(P) := \{Q \in M \mid \underset{\uparrow}{\|Q - P\|} < r\} = B_r^n(P) \cap M.$$

usual Euclidean norm

Note that by definition of the subspace topology $B_r^M(P) \subseteq \mathbb{R}^d$ is an open neighborhood of the point $P \in M$.

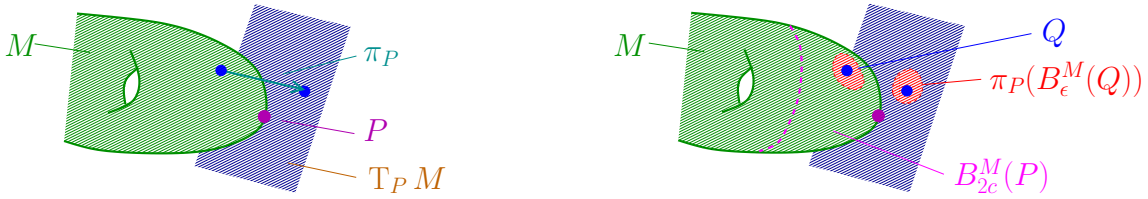
The following lemma shows in particular that the images of "small" balls under projections are again convex:

Lemma 81.8. (Convex Projection Lemma) Let M be a smooth submanifold of some $\mathbb{R}^d$. We assume that $\partial M = \varnothing$. Given $P \in M$ we denote by

$$\pi_P \colon M \to \mathrm{T}_P M$$

the orthogonal projection onto the vector subspace $\mathrm{T}_P M \subseteq \mathbb{R}^d$. There exists a $c \in (0,1)$ with the following two properties:

(1) The restriction of π_P to $B^M_{2c}(P) \to \mathrm{T}_P M$ is a smooth embedding.
(2) For every $Q \in B^M_c(P)$ and every $\epsilon < c$ the image of $B^M_\epsilon(Q)$ under π_P is an open convex subset of $\mathrm{T}_P M$.



Proof of the Convex Projection Lemma 81.8. Let M be an n-dimensional smooth submanifold of some $\mathbb{R}^d$ with $\partial M = \varnothing$. Furthermore let $P \in M$. It follows from the Orthogonal Action Lemma 2.37 that, possibly after an application of an orthogonal matrix, we can assume that $\mathrm{T}_P M = \mathbb{R}^n \times \{0\}$. Note that in this setting the orthogonal projection

$$\begin{array}{rccl} & \pi_P \colon M & \to & \mathrm{T}_P M = \mathbb{R}^n \times \{0\} \\ \text{is given by} & (x,y) & \mapsto & (x,0). \end{array}$$

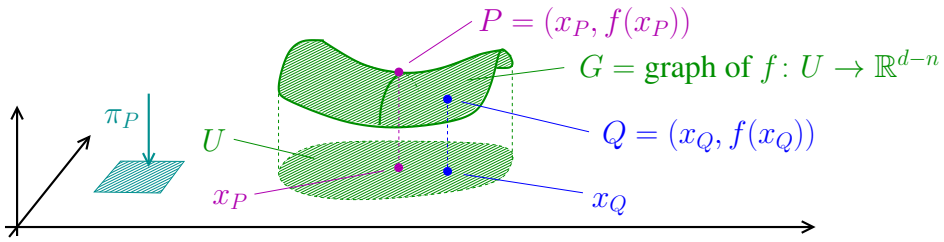
To simplify the notation a little bit we now think of π_P as the map $\pi_P \colon M \to \mathbb{R}^n$ that is the projection onto the first n coordinates. We write $x_P := \pi_P(P) \in \mathbb{R}^n$.

Next note that it follows from our hypothesis that $\partial M = \varnothing$ and the Local Graph Lemma 30.5 that there exists an open convex neighborhood U of $x_P \in \mathbb{R}^n$ and a smooth map $f \colon U \to \mathbb{R}^{d-n}$ such that

$$G := \text{graph of } f \colon U \to \mathbb{R}^{d-n} := \{(x, f(x)) \,|\, x \in U\}$$

is a neighborhood of $P = (x_P, f(x_P)) \in M$. Thus for the remainder of the proof we can replace M by G. Next given any $Q = (x_Q, f(x_Q)) \in G$ we consider the smooth function

$$\begin{array}{rccl} \varphi_Q \colon & U & \to & \mathbb{R} \\ & x & \mapsto & \|(x, f(x)) - (x_Q, f(x_Q))\|^2. \end{array}$$



Of course eventually we want to apply the Convex Function–Preimage Proposition 81.6. This idea leads us to the following claim:

Claim 1.

(a) $$\mathrm{Hess}(\varphi_P \colon U \to \mathbb{R})_{x_P} = \left(\frac{\partial^2 \varphi_P}{\partial x_i \, \partial x_j}\Big|_{x_P} \right)_{i,j=1,\dots,n} = \mathrm{id}_n .$$

(b) There exists a $c \in (0,1)$ such that $B^n_{2c}(x_P) \subseteq U$ and such that for every $Q \in B^G_c(P)$ and every $x \in B^n_{2c}(x_P)$ the Hessian $\mathrm{Hess}(\varphi_Q \colon U \to \mathbb{R})_x$ is positive-definite.

Proof.

(a) First note that for every $x \in U$ we have

$$\begin{aligned}\varphi_P(x) &= \|(x, f(x)) - (x_P, f(x_P))\|^2 = \|\underbrace{x - x_P}_{\in \mathbb{R}^n}\|^2 + \|\underbrace{f(x) - f(x_P)}_{\in \mathbb{R}^{d-n}}\|^2 \\ &= \underbrace{\sum_{i=1}^{n} (x_i - x_{P,i})^2}_{\substack{\text{Hessian at } P \text{ is the} \\ \text{identity matrix}}} + \underbrace{\sum_{j=1}^{d-n} (f_j(x) - f_j(x_P))^2}_{\substack{\text{Hessian at } P \text{ is the} \\ \text{zero matrix}}}.\end{aligned}$$

We want to calculate the Hessian matrices of the two summands. First note that a direct calculation shows that the Hessian of the first summand at P is the identity matrix.

The calculation of the second Hessian matrix requires slightly more thought: Note that an elementary argument, which we carried out in Exercise 29.1, shows that for any $x \in U$ we have

$$\mathrm{T}_{(x,f(x))}G = \text{span of the columns of the } d \times n\text{-matrix } \begin{pmatrix} \mathrm{id}_n \\ \mathrm{D}f_x \end{pmatrix}.$$

It follows from this calculation and $\mathrm{T}_P G = \mathrm{T}_P M = \mathbb{R}^n \times \{0\}$ that for all $i = 1, \dots, n$ and for all $j = 1, \dots, d-n$ we have $\frac{\partial f_j}{\partial x_i}(x_P) = 0$. It follows easily from this observation and the Chain Rule, that the Hessian matrix of the second summand at P is the zero matrix.

(b) We consider the map

$$\begin{aligned} H \colon U \times U &\to \mathrm{M}(n \times n, \mathbb{R}) \\ (w, x) &\mapsto \mathrm{Hess}(\varphi_{(w,f(w))} \colon U \to \mathbb{R})_x. \end{aligned}$$

We claim that this map is continuous. This can be seen as follows: First note that, as in (a), one can explicitly writing down $\varphi_{(w,f(w))}$. Now, using the Chain Rule, one sees that H can be written as an expression of the partial derivatives of f. Since f is smooth we see that $H \colon U \times U \to \mathrm{M}(n \times n, \mathbb{R})$ is indeed continuous.

Next note that by (a) we know that $H(x_P, x_P) = \mathrm{id}$, in particular we see that the matrix is positive-definite. It follows from the Positive Definite–Open Condition Lemma 81.4 that there exists an open neighborhood W of $(x_P, x_P) \in U \times U$ such that all corresponding matrices are positive-definite. By Definition 7.3 of the product topology there exists a $c \in (0, 1)$ with $B_c^n(x_P) \times B_{2c}(x_P) \subseteq W$. One can now easily verify that this $c \in (0, 1)$ has the desired properties. ⊞

The following claim shows that this $c \in (0, 1)$ has the desired properties:

Claim 2.

(a) The restriction of π_P to $B_{2c}^G(P) \to \mathbb{R}^n$ is a smooth embedding.
(b) For every $Q \in B_c^G(P)$ and every $\epsilon < c$ the image of $B_\epsilon^G(Q)$ under π_P is an open convex subset of $\mathrm{T}_P G$.

Proof.

(1) Note that $\pi_P \colon G \to U$ is a diffeomorphism since the inverse is given by $x \mapsto (x, f(x))$ which is of course smooth. Thus for any open subset $V \subseteq G$ the map $\pi_P \colon V \to \pi_P(V)$ is also a diffeomorphism. This applies in particular to $V = B_{2c}^G(P)$.

(2) Let $Q = (x_Q, f(x_Q)) \in B^G_c(P)$ and let $\epsilon < c$. It follows from the following four observations that $\pi_P(B^G_\epsilon(Q))$ is indeed convex:

(i) We have the following equalities:

$$\pi_P(B^G_\epsilon(Q)) \underset{\substack{\uparrow \\ \text{by the definition of } G \text{ as} \\ \text{the graph of } f\colon U \to \mathbb{R}^{d-n}}}{=} \{x \in U \,|\, (x, f(x)) \in B^G_\epsilon(Q)\} \underset{\substack{\uparrow \\ \text{since by definition we have} \\ \varphi_Q(x) = \|(x, f(x)) - (x_Q, f(x_Q))\|^2}}{=} \{x \in U \,|\, \varphi_Q(x) < \epsilon^2\}.$$

(ii) By our choice of $c \in (0,1)$ we know that for every $x \in B^n_{2c}(x_P)$ the Hessian $\mathrm{Hess}(\varphi_Q\colon U \to \mathbb{R})_x$ is positive-definite.

(iii) We have

$$\pi_P(B^G_\epsilon(Q)) \underset{\substack{\uparrow \\ \text{follows from the inequality} \\ \|\pi_P(x) - \pi_P(y)\| \le \|x - y\|}}{\subseteq} B^n_\epsilon(x_Q) \underset{\substack{\uparrow \\ \text{by the triangle} \\ \text{inequality}}}{\subseteq} B_{c+\epsilon}(x_P) \underset{\substack{\uparrow \\ \text{since } \epsilon < c}}{\subseteq} B_{2c}(x_P).$$

(iv) It follows from (ii) and (iii) and the Convex Function–Preimage Proposition 81.6, applied to $\varphi_Q\colon B_{2c}(x_P)$ that $\{x \in U \,|\, \varphi_Q(x) < \epsilon^2\}$ is convex. ∎

81.1.2. **Existence of smoothly good covers.** The next proposition contains basically all the meat of the Smoothly Good Cover Theorem 81.2:

Proposition 81.9. (Submanifold–Smoothly Good Cover Proposition) Suppose that $M \subseteq \mathbb{R}^d$ is an n-dimensional smooth submanifold with empty boundary. There exists a map $\epsilon\colon M \to (0, \frac{1}{2})$ such that the open subsets $\{B^M_{\epsilon(P)}(P)\}_{P \in M}$ form a smoothly good open cover of M.

Proof of Proposition 81.9. Let $M \subseteq \mathbb{R}^d$ be an n-dimensional smooth submanifold with empty boundary. Given $P \in M$ we denote again by

$$\pi_P\colon M \to \mathrm{T}_P M$$

the orthogonal projection onto the vector subspace $\mathrm{T}_P M \subseteq \mathbb{R}^d$. Now recall that by the Convex Projection Lemma 81.8 we know that for each $P \in M$ there exists a $c(P) \in (0,1)$ with the following two properties:

(1) The restriction of π_P to $B^M_{2c(P)}(P) \to \mathrm{T}_P M$ is a smooth embedding.

(2) For every $Q \in B^M_{c(P)}(P)$ and every $\epsilon < c(P)$ the image of $B^M_\epsilon(Q)$ under π_P is an open convex subset of $\mathrm{T}_P M$.

Next for each $P \in M$ we set

$$\epsilon(P) \coloneqq \tfrac{1}{2} \cdot c(P) \qquad \text{and} \qquad U_P \coloneqq B^M_{\epsilon(P)}(P) \coloneqq \{Q \in M \,|\, \|Q - P\| < \epsilon(P)\}.$$

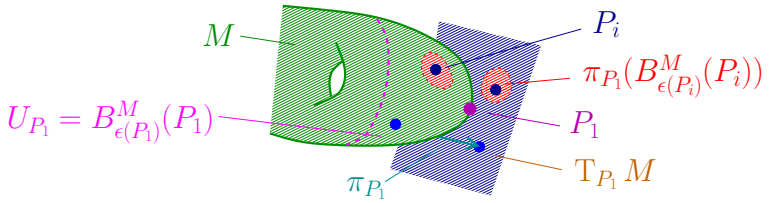
Evidently each U_P is an open subset of M. Thus it remains to show that the family $\{U_P\}_{P \in M}$ is a smoothly good open cover of M.

Let $P_1, \dots, P_k \in M$ such that the intersection $U_{P_1} \cap \cdots \cap U_{P_k}$ is non-empty. We need to show that $U_{P_1} \cap \cdots \cap U_{P_k}$ is diffeomorphic to an open convex subset of $\mathbb{R}^n$. Note that after possibly reordering the P_i's we can assume that $\epsilon(P_1) \ge \epsilon(P_i)$ for all $i = 1, \dots, k$.

Claim. Let $i \in \{1, \dots, k\}$.

(a) We have $U_{P_i} \subseteq B^M_{2c(P_1)}(P_1)$.

(b) The set $U_{P_i} = B^M_{\epsilon(P_i)}(P_i)$ gets mapped under $\pi_{P_1} \colon M \to \mathrm{T}_{P_1}M$ onto an open convex subset of $\mathrm{T}_{P_1}M$.



Proof. Let $i \in \{1, \ldots, k\}$.

(a) Since $U_{P_1} \cap \cdots \cap U_{P_k}$ is non-empty we can pick a point $x \in U_{P_1} \cap U_{P_i}$. We see that

$$\|P_i - P_1\| \underset{\text{triangle inequality}}{\leq} \|P_i - x\| + \|P_1 - x\| \underset{\text{since } x \in U_{P_i} \text{ and } x \in U_{P_1}}{<} \epsilon(P_i) + \epsilon(P_1) \underset{\text{since } \epsilon(P_i) \leq \epsilon(P_1)}{\leq} 2 \cdot \epsilon(P_1) \underset{\text{definition of } \epsilon(P_1)}{=} c(P_1).$$

We have thus shown that $P_i \in B^M_{c(P_1)}(P_1)$. Since $\epsilon(P_i) \leq \epsilon(P_1) = \frac{1}{2} \cdot c(P_i)$ we deduce from the triangle inequality that $U_{P_i} = B^M_{\epsilon(P_i)}(P_i) \subseteq B^M_{2c(P_1)}(P_1)$.

(b) It follows from the definition of $c(P_1)$ and the fact that $\epsilon(P_i) \leq \epsilon(P_1) < c(P_1)$ that the set $U_{P_i} = B^M_{\epsilon(P_i)}(P_i)$ gets mapped under π_{P_1} onto an open convex subset of $\mathrm{T}_{P_1}M$. ⊞

Note that it follows from Statement (a) of the claim that $U_{P_1} \cup \cdots \cup U_{P_k} \subseteq B^M_{2c(P_1)}(P_1)$. It follows from Property (1) of $c(P_1)$ that $\pi_{P_1} \colon U_{P_1} \cup \cdots \cup U_{P_k} \to \mathrm{T}_{P_1}M$ is a smooth embedding. Now we see that

$$U_{P_1} \cap \cdots \cap U_{P_k} \underset{\text{since } \pi_{P_1} \colon U_{P_1} \cup \cdots \cup U_{P_k} \to \mathrm{T}_{P_1}M \text{ is a smooth embedding}}{\overset{\cong}{\to}} \pi_{P_1}(U_{P_1} \cap \cdots \cap U_{P_k}) = \pi_{P_1}(U_{P_1}) \cap \cdots \cap \pi_{P_1}(U_{P_k}).$$

By Statement (b) of the claim we know that each $\pi_{P_1}(U_{P_i})$ is an open convex subset of $\mathrm{T}_{P_1}M$. Evidently the intersection of finitely many open convex subsets of $\mathrm{T}_{P_1}M$ is again an open convex subset. ■

For the reader's convenience let us recall the theorem that we actually want to prove:

Theorem 81.2. (Smoothly Good Cover Theorem)

(1) Every closed smooth manifold admits a finite smoothly good open cover.
(2) Every smooth manifold with empty boundary admits a locally finite smoothly good open cover.

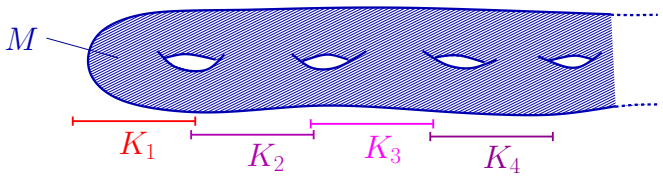
Proof of the Smoothly Good Cover Theorem 81.2. Let M be a smooth manifold with empty boundary.

(1) We first assume that M is compact. It follows from the Euclidean Embedding Theorem 33.2 together with the Smooth Embedding Theorem 30.15 that we can view M as a submanifold of some $\mathbb{R}^d$. It now follows from the Submanifold–Smoothly Good Cover Proposition 81.9 that M admits a smoothly good open cover. Since M is compact we see that M is covered by finitely many of these sets and we are done.

(2) Now we now assume that M is non-compact. Since $\partial M = \varnothing$ it follows from the Diffeomorphic-to-Euclidean Submanifold Corollary 33.3 that we can assume that M is a proper smooth submanifold of some $\mathbb{R}^d$. Note that by Definition 40.11 the adjective "proper" means in particular that M is a closed subset of $\mathbb{R}^d$. By the Compact-Closed

Lemma 3.39 this implies that for each $m \in \mathbb{N}$ the set $K_m := M \cap (\overline{B}^d_m(0) \setminus B^d_{m-1}(0))$ is compact.

It follows from the Submanifold–Smoothly Good Cover Proposition 81.9 and the Convex Projection Lemma 81.8 that there exists a map $\epsilon\colon M \to (0, \frac{1}{2})$ such that all the open subsets $U_P := B^M_{\epsilon(P)}(P)$ together form a smoothly good open cover of M.



As we mentioned above, for each $m \in \mathbb{N}$ we know that K_m is compact. Thus there exists for each $m \in \mathbb{N}$ a finite set $I_m \subseteq M$ such that $K_m \subseteq \bigcup_{P \in I_m} U_P$. Since the U_P have diameter $< \frac{1}{2}$ we see that for $|m-n| \geq 2$ and $P \in I_m$ and $Q \in I_n$ we have $U_P \cap U_Q = \varnothing$.

We set $I := \bigcup_{m \in \mathbb{N}} I_m$. It follows easily from the above discussion that $\{U_P\}_{P \in I}$ is a locally finite smoothly good open cover of M. ■

81.2. **Finite dimensionality of de Rham cohomology.** As we already mentioned in the introduction to the chapter, we will use the Smoothly Good Cover Theorem 81.2 to prove the following result:

Proposition 81.10. (De Rham Cohomology-Finiteness Proposition) For every compact smooth manifold M and every $j \in \mathbb{N}_0$ the de Rham cohomology group $\mathrm{H}^j_{\mathrm{dR}}(M)$ is a finite-dimensional real vector space.

The proof of the De Rham Cohomology-Finiteness Proposition 81.10 makes use of the following elementary lemma from linear algebra:

Lemma 81.11. (Dimension Squeezing Lemma) Let $\mathbb{F}$ be a field. Suppose that we are given an exact sequence of $\mathbb{F}$-vector spaces of the form

$$A \xrightarrow{\alpha} B \to C \to D \xrightarrow{\delta} E.$$

If the "outer" vector spaces B and D are finite-dimensional, then so is the middle vector space C.

Proof of Lemma 81.11. It follows from the elementary discussion on page 1584 that

$$C \;\cong\; \operatorname{coker}(\alpha\colon A \to B) \oplus \ker(\delta\colon D \to E).$$

Since B and D are finite-dimensional we see that the two summands on the right, and thus also C, are finite-dimensional. ■

Proof of Proposition 81.10 if M is closed. By the Smoothly Good Cover Theorem 81.2 we know that M admits a finite smoothly good open cover $\{U_i\}_{i=1,\ldots,k}$. Thus it suffices to prove the following claim:

Claim. Let $k \in \mathbb{N}_0$. For every smooth manifold that admits a smoothly good open cover with $\leq k$ sets all the de Rham cohomology groups are finite-dimensional.

Proof. The way the claim is formulated it is clear that we want to prove the claim by induction on $k \in \mathbb{N}_0$. The case $k = 0$ is a consequence of the elementary Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.6. Now suppose that the claim holds for some $k \in \mathbb{N}_0$. Let M be a smooth manifold that admits a smoothly good open cover consisting of open subsets $U_1, \dots, U_{k+1}$.

We set $V \coloneqq U_1 \cup \dots \cup U_k$. Since the U_i are open we can apply the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 which tells us that there exists an exact sequence of the form

$$\dots \to \mathrm{H}^{j-1}_{\mathrm{dR}}(V) \oplus \mathrm{H}^{j-1}_{\mathrm{dR}}(U_{k+1}) \to \mathrm{H}^{j-1}_{\mathrm{dR}}(V \cap U_{k+1}) \to \mathrm{H}^j_{\mathrm{dR}}(M) \to \mathrm{H}^j_{\mathrm{dR}}(V) \oplus \mathrm{H}^j_{\mathrm{dR}}(U_{k+1}) \to \dots$$



We collect our knowledge on the various groups involved:

- By induction hypothesis we know that all of the $\mathrm{H}^*_{\mathrm{dR}}(V)$ are finite-dimensional.
- It follows from the definition of a good open cover that U_{k+1} is diffeomorphic to an open convex subset of $\mathbb{R}^n$, which implies by the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 that the Rham cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(U_{k+1})$ are finite-dimensional.
- Note that
$$V \cap U_{k+1} = \left(\bigcup_{i=1}^{k} U_i\right) \cap U_{k+1} = \bigcup_{i=1}^{k} (U_i \cap U_{k+1}).$$
It now follows easily from the definitions that the k sets $U_i \cap U_k$ form a smoothly good open cover of $V \cap U_{k+1}$. Thus we obtain from our induction hypothesis that the de Rham cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(V \cap U_{k+1})$ are also finite-dimensional.

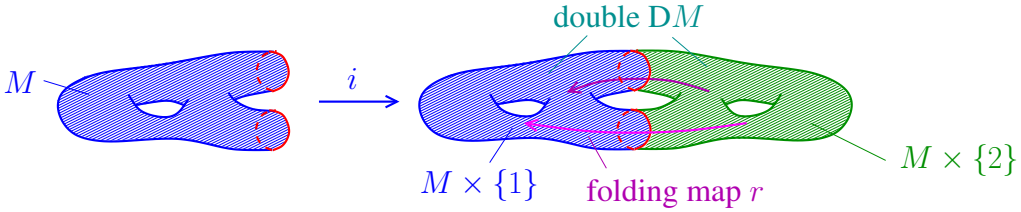
It now follows from this discussion, the above exact sequence and the Dimension Squeezing Lemma 81.11 that the de Rham cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(M)$ are finite-dimensional. ∎

Proof of Proposition 81.10 in the general setting. Let M be a compact smooth manifold and let $j \in \mathbb{N}_0$. As in Definition 54.8 we consider the double

$$\mathrm{D}M \coloneqq ((M \times \{1\}) \sqcup (M \times \{2\}))/\sim \qquad \text{where } (w,1) \sim (w,2) \text{ for all } w \in \partial M.$$

Furthermore we consider the embedding

$$\begin{array}{rcl} i\colon M & \to & \mathrm{D}M \\ P & \mapsto & [(P,1)] \end{array} \qquad \text{and we consider the folding map} \qquad \begin{array}{rcl} r\colon \mathrm{D}M & \to & M \\ [(P,j)] & \mapsto & P. \end{array}$$



By the Doubling-Manifold Lemma 54.10 we know the following:

(a) We can equip DM with the structure of a closed smooth manifold.
(b) The maps $i\colon M \to \mathrm{D}M$ and $r\colon \mathrm{D}M \to M$ are continuous.[839]

[839]Note that the map $i\colon M \to \mathrm{D}M$ is in fact smooth. Also note that if $\partial M \neq \varnothing$, then the folding map r is *not* smooth. But this is not an issue, since we established functoriality for *continuous* maps between smooth manifolds.

We continue with the following observations:

(c) By definition we have $r \circ i = \mathrm{id}_M$.
(d) It follows from (c) and the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 that $i^* \circ r^* = \mathrm{id}_{\mathrm{H}^j_{\mathrm{dR}}(M)}$. In particular we see that $i^* \colon \mathrm{H}^j_{\mathrm{dR}}(\mathrm{D}M) \to \mathrm{H}^j_{\mathrm{dR}}(M)$ is an epimorphism.
(e) Since $\mathrm{D}M$ is a closed smooth manifold we know by the above discussion that $\mathrm{H}^j_{\mathrm{dR}}(\mathrm{D}M)$ is finite-dimensional.
(f) It follows from (d) and (e) that $\mathrm{H}^k_{\mathrm{dR}}(M)$ is indeed finite-dimensional. ■

81.3. **Interlude: The tensor product over fields.** In this purely algebraic section we will introduce the tensor product of two vector spaces over the same field.[840] We will make use of this notion in the statement of the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21 which calculates the de Rham cohomology groups of products of smooth manifolds. The reader who is well-acquainted with tensor products of vector spaces might want to move straight to Section 81.4.

We start out with the following harmless piece of notation:

Notation 81.12. Let $\mathbb{F}$ be a field. Given a set M write

$$\mathbb{F}^{(M)} := \begin{array}{l}\text{the set of all maps } \varphi \colon M \to \mathbb{F} \text{ such that} \\ \varphi(m) = 0 \text{ for all but finitely many } m \in M.\end{array}$$

This set has an obvious $\mathbb{F}$-vector space structure. Given $m \in M$ we denote by m also the map $M \to \mathbb{F}$ which is given by $m \mapsto 1_{\mathbb{F}}$ and $n \mapsto 0$ for all $m \neq n$.

With this notation we can provide the definition of the tensor product of two vector spaces:

Definition 81.13. Let $\mathbb{F}$ be a field and let V and W be $\mathbb{F}$-vector spaces.
(1) We consider the $\mathbb{F}$-vector space $\mathbb{F}^{(V \times W)}$ generated by the set $V \times W$. We define the tensor product $V \otimes_{\mathbb{F}} W$ to be the following $\mathbb{F}$-vector space:

$$V \otimes_{\mathbb{F}} W := \mathbb{F}^{(V \times W)} / N(V, W)$$

where $N(V, W)$ is the vector subspace of $\mathbb{F}^{(V \times W)}$ generated by the elements of the form

$$((f \cdot v + f' \cdot v'), w) - (f \cdot (v, w) + f' \cdot (v', w))$$

and

$$(v, (f \cdot w + f' \cdot w')) - (f \cdot (v, w) + f' \cdot (v, w'))$$

with $v, v' \in V$, $w, w' \in W$ and $f, f' \in \mathbb{F}$.
(2) For $v \in V$ and $w \in W$ we write $v \otimes w := [(v, w)] \in \mathbb{F}^{(V \times W)} / N(V, W) = V \otimes_{\mathbb{F}} W$.

As it stands the definition of a tensor product is very ad hoc and seems to be rather random. Our first goal is thus to show that the tensor product is characterized by a universal property. To formulate this universal property we need the following definition:

Definition 81.14. Let $\mathbb{F}$ be a field and let V, W and Z be $\mathbb{F}$-vector spaces. We say that a map $\varphi \colon V \times W \to Z$ is $\mathbb{F}$-bilinear, if for all $v, v' \in V$, $w, w' \in W$ and $f, f' \in \mathbb{F}$ we have

$$\begin{array}{rcll} \varphi(f \cdot v + f' \cdot v', w) & = & f \cdot \varphi(v, w) + f' \cdot \varphi(v', w) & \text{and} \\ \varphi(v, f \cdot w + f' \cdot w') & = & f \cdot \varphi(v, w) + f' \cdot \varphi(v, w') & \end{array}$$

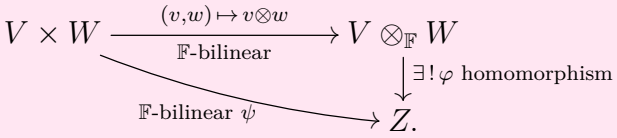
[840] In Chapter 159 we will discuss the tensor product of abelian groups and in Section 312.2 we will introduce the tensor product of modules over an arbitrary ring R.

Lemma 81.15. (Tensor Product–Universal Property Lemma) Let $\mathbb{F}$ be a field and let V and W be $\mathbb{F}$-vector spaces.

(1) The map
$$\begin{array}{rcl} f\colon V\times W & \to & V\otimes_{\mathbb{F}} W \\ (v,w) & \mapsto & v\otimes w \end{array}$$
is $\mathbb{F}$-bilinear. In other words, for all $v,v'\in V$, $w,w'\in W$ and $f,f'\in\mathbb{F}$ we have
$$\begin{array}{rcl} (f\cdot v+f'\cdot v')\otimes w & = & f\cdot(v\otimes w)+f'\cdot(v'\otimes w) \ \in\ V\otimes_{\mathbb{F}} W \\ v\otimes(f\cdot w+f'\cdot w') & = & f\cdot(v\otimes w)+f'\cdot(v\otimes w') \ \in\ V\otimes_{\mathbb{F}} W. \end{array}$$

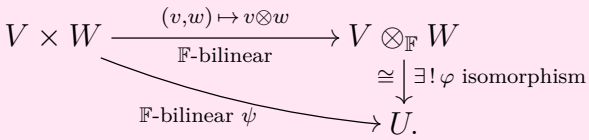
(2) (a) Any element in $V\otimes_{\mathbb{F}} W$ is of the form $\sum_{i=1}^{n} v_i\otimes w_i$ with $v_1,\dots,v_n\in V$ and $w_1,\dots,w_n\in W$.
(b) Given any $v\in V$ and $w\in W$ we have $v\otimes 0=0=0\otimes w\in V\otimes_{\mathbb{F}} W$.

(3) Given any $\mathbb{F}$-vector space Z and given any $\mathbb{F}$-bilinear map $\psi\colon V\times W\to Z$ there exists a unique homomorphism $\varphi\colon V\otimes_{\mathbb{F}} W\to Z$ of $\mathbb{F}$-vector space such that the following diagram commutes:



In fact this homomorphism is given by
$$\begin{array}{rcl} \varphi\colon V\otimes_{\mathbb{F}} W & \to & Z \\ \sum_{i=1}^{n} v_i\otimes w_i & \mapsto & \sum_{i=1}^{n}\psi(v_i,w_i). \end{array}$$

(4) Let U be an $\mathbb{F}$-vector space and let $\psi\colon V\times W\to U$ be an $\mathbb{F}$-bilinear map. If this $\mathbb{F}$-bilinear map satisfies the universal property of (3), then there exists a unique isomorphism $\varphi\colon V\otimes_{\mathbb{F}} W\to U$ of $\mathbb{F}$-vector spaces such that the following diagram commutes:



Proof. Let V and W be two $\mathbb{F}$-vector spaces.

(1) This statement is an immediate consequence of the definition of the tensor product of two vector spaces.

(2) (a) This statement follows easily from the definitions and the fact that by (1) we know that for any $v\in V$, $w\in W$ and $f\in\mathbb{F}$ we have $f\cdot(v\otimes w)=(f\cdot v)\otimes w$.
(b) Let $v\in V$ and $w\in W$. It follows from (1) that we have the following equality in the $\mathbb{F}$-vector space $V\otimes_{\mathbb{F}} W$: $v\otimes 0_W=v\otimes(0_{\mathbb{F}}\cdot 0_W)=0_{\mathbb{F}}\cdot(v\otimes 0_W)=0_{V\otimes_{\mathbb{F}} W}$. The same way we see that $0_V\otimes w=0_{V\otimes_{\mathbb{F}} W}$.

(3) Let Z be an $\mathbb{F}$-vector space and let $\psi\colon V\times W\to Z$ be an $\mathbb{F}$-bilinear map.

Claim. The map
$$\begin{array}{rcl} V\otimes_{\mathbb{F}} W & \to & Z \\ \sum_{i=1}^{n} v_i\otimes w_i & \mapsto & \sum_{i=1}^{n}\psi(v_i,w_i) \end{array}$$

is a well-defined homomorphism and it is the unique homomorphism $\varphi\colon V \otimes_{\mathbb{F}} W \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{(v,w)\mapsto v\otimes w} & V \otimes_{\mathbb{F}} W \\ & \searrow^{\psi} & \downarrow \varphi \\ & & Z. \end{array}$$

Proof. First note that it follows from the fact that $V \times W$ is a basis for the $\mathbb{F}$-vector space $\mathbb{F}^{(V\times W)}$ that there exists a *unique* homomorphism

$$\mathbb{F}^{(V\times W)} \to Z$$

which sends each $(v, w) \in V \times W$ to $\psi(v, w)$. Since ψ is $\mathbb{F}$-bilinear we see that the subgroup $N(V, W)$ lies in the kernel of the above homomorphism. It follows easily from the Quotient Group Lemma 2.4 that the above homomorphism descends to a well-defined homomorphism

$$\begin{array}{rcl} \varphi\colon V \otimes_{\mathbb{F}} W := \mathbb{F}^{(V\times W)}/N(V,W) & \to & Z \\ \left[\sum_{i=1}^{n} (v_i, w_i)\right] = \sum_{i=1}^{n} v_i \otimes w_i & \mapsto & \sum_{i=1}^{n} \psi(v_i, w_i). \end{array}$$

Next note that it is clear that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{(v,w)\mapsto v\otimes w} & V \otimes_{\mathbb{F}} W \\ & \searrow^{\psi} & \downarrow \varphi \\ & & Z. \end{array}$$

Finally note that it follows easily from the fact that $\mathbb{F}^{(V\times W)} \to V \otimes_{\mathbb{F}} W$ is an epimorphism that φ is uniquely determined. ⊞

(4) This statement follows from the usual argument involving universal properties. We leave it to the reader to fill in the details. ■

We move on to induced maps between tensor products:

Definition 81.16. Let $\mathbb{F}$ be a field and let $f\colon V \to V'$ and $g\colon W \to W'$ be homomorphisms between $\mathbb{F}$-vector spaces. It follows easily from the Tensor Product–Universal Property Lemma 81.15 (1) that the map

$$\begin{array}{rcl} V \times W & \to & V' \otimes_{\mathbb{F}} W' \\ (v_i, w_i) & \mapsto & f(v_i) \otimes g(w_i) \end{array}$$

is $\mathbb{F}$-bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 81.15 (3) that the map

$$\begin{array}{rcl} f \otimes g\colon V \otimes_{\mathbb{F}} W & \to & V' \otimes_{\mathbb{F}} W' \\ \sum_{i=1}^{n} v_i \otimes w_i & \mapsto & \sum_{i=1}^{n} f(v_i) \otimes g(w_i) \end{array}$$

is a well-defined homomorphism of $\mathbb{F}$-vector spaces. We refer to $f \otimes g$ as the tensor product of the homomorphisms f and g.

The following lemma summarizes three key properties of the tensor product of two homomorphisms:

Lemma 81.17. (Tensor–Map Composition Lemma) Let $\mathbb{F}$ be a field.

(1) Given any $\mathbb{F}$-vector spaces V and W we have $\mathrm{id}_V \otimes_{\mathbb{F}} \mathrm{id}_W = \mathrm{id}_{V \otimes_{\mathbb{F}} W}$.

(2) Let $f\colon V \to V'$, $f'\colon V' \to V''$, $g\colon W \to W'$ and $g'\colon W' \to W''$ be homomorphisms between $\mathbb{F}$-vector spaces. Then we have

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon V \otimes_{\mathbb{F}} W \;\to\; V'' \otimes_{\mathbb{F}} W''.$$

(3) Let $f, g\colon V \to W$ be homomorphisms between $\mathbb{F}$-vector spaces and let Z be an $\mathbb{F}$-vector space. Then

$$(f+g) \otimes \mathrm{id}_Z \;=\; f \otimes \mathrm{id}_Z + g \otimes \mathrm{id}_Z\colon V \otimes_{\mathbb{F}} Z \;\to\; W \otimes_{\mathbb{F}} Z.$$

The same statement also holds if we swap the roles of the two sides.

Proof. All the statements follow immediately from the above explicit description of the tensor product of homomorphisms. Alternatively, as we will see in Exercise 81.8, one can also deduce the statements from the universal property stated in the Tensor Product–Universal Property Lemma 81.15 (3). ∎

The next lemma gives us an important, albeit slightly dull, calculation of a tensor product:

Lemma 81.18. (Tensor-with-$\mathbb{F}$–Lemma) Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. The maps

$$\begin{array}{rcl} \Theta\colon \mathbb{F} \otimes_{\mathbb{F}} V & \to & V \\ \sum_{i=1}^m f_i \otimes v_i & \mapsto & \sum_{i=1}^m f_i \cdot v_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Omega\colon V & \to & \mathbb{F} \otimes_{\mathbb{F}} V \\ v & \mapsto & 1_{\mathbb{F}} \otimes v \end{array}$$

are well-defined homomorphisms, they are natural and they are inverses to one another. In particular both are natural isomorphisms. Here "natural" means that for any homomorphism $\varphi\colon V \to W$ of $\mathbb{F}$-vector spaces the following diagrams commute:

$$\begin{array}{ccc} \mathbb{F} \otimes_{\mathbb{F}} V & \xrightarrow[\cong]{\Theta} & V \\ {\scriptstyle \mathrm{id}_{\mathbb{F}} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{F} \otimes_{\mathbb{F}} W & \xrightarrow[\cong]{\Theta} & W \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V & \xrightarrow[\cong]{v \mapsto 1_{\mathbb{F}} \otimes v} & \mathbb{F} \otimes_{\mathbb{F}} V \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{F}} \otimes \varphi} \\ W & \xrightarrow[\cong]{w \mapsto 1_{\mathbb{F}} \otimes w} & \mathbb{F} \otimes_{\mathbb{F}} W. \end{array}$$

The analogous statements also hold if the $\mathbb{F}$-term is on the right instead of the left.

Proof. Note that the map

$$\begin{array}{rcl} \mathbb{F} \times V & \to & V \\ (f, v) & \mapsto & f \cdot v \end{array}$$

is $\mathbb{F}$-bilinear. It thus follows from the Tensor Product–Universal Property Lemma 81.15 (3) that the given map

$$\begin{array}{rcl} \Theta\colon \mathbb{F} \otimes_{\mathbb{F}} V & \to & V \\ \sum_{i=1}^m f_i \otimes v_i & \mapsto & \sum_{i=1}^m f_i \cdot v_i \end{array}$$

is a well-defined homomorphism. It follows easily from the definitions that this map is natural.

It is clear that $\Omega\colon V \to \mathbb{F} \otimes_{\mathbb{F}} V$ is a well-defined natural homomorphism. Furthermore it is clear that $\Theta \circ \Omega\colon V \to V$ is the identity. It remains to prove that $\Omega \circ \Theta\colon \mathbb{F} \otimes_{\mathbb{F}} V \to \mathbb{F} \otimes_{\mathbb{F}} V$ is the identity. This follows easily from the definitions together with the following claim:

Claim. Any element in $\mathbb{F} \otimes_{\mathbb{F}} V$ is represented by an element of the form $1_{\mathbb{F}} \otimes v$ for some $v \in V$.

Proof. By the Tensor Product–Universal Property Lemma 81.15 (2) any element of $\mathbb{F} \otimes_{\mathbb{F}} V$ is of the form $\sum_{i=1}^{m} f_i \otimes v_i \in \mathbb{F} \otimes_{\mathbb{F}} V$. We see that

$$\sum_{i=1}^{m} f_i \otimes v_i \;=\; \sum_{i=1}^{m} (f_i \cdot 1_{\mathbb{F}}) \otimes v_i \;\underset{\text{by the Tensor Product–Universal Lemma 81.15 (1)}}{=}\; \sum_{i=1}^{m} 1_{\mathbb{F}} \otimes (f_i \cdot v_i) \;\underset{\text{by the Tensor Product–Universal Lemma 81.15 (1)}}{=}\; 1_{\mathbb{F}} \otimes \Big(\sum_{i=1}^{m} (f_i \cdot v_i) \Big).$$

⊞

If the $\mathbb{F}$-term is on the right, then the argument is evidently almost identical. ■

We move on to the study of the relationship between tensor products and direct sums:

Lemma 81.19. (Tensor–Direct Sum Lemma) Let $\mathbb{F}$ be a field, let $\{V_i\}_{i \in I}$ be a family of $\mathbb{F}$-vector spaces and let W be an $\mathbb{F}$-vector space. Given $j \in I$ let $\iota_j \colon V_j \to \bigoplus_{i \in I} V_i$ be the natural homomorphism. Note that it follows from the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 (3) that the homomorphisms $\iota_j \otimes \mathrm{id}_W \colon V_j \otimes_{\mathbb{F}} W \to \left(\bigoplus_{i \in I} V_i \right) \otimes_{\mathbb{F}} W$ give rise to a natural homomorphism

$$\Xi \colon \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W) \;\to\; \left(\bigoplus_{i \in I} V_i \right) \otimes_{\mathbb{F}} W.$$

This homomorphism is a natural isomorphism of $\mathbb{F}$-vector spaces.

Sketch of proof. We need to show that the above homomorphism

$$\Phi \colon \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W) \;\to\; \left(\bigoplus_{i \in I} V_i \right) \otimes_{\mathbb{F}} W$$

has an inverse. In the following let $w \in W$. For each index $j \in I$ we consider the homomorphism $V_j \to V_j \otimes_{\mathbb{F}} W$ that is given by $v \mapsto v \otimes w$. We compose this homomorphism with the natural homomorphism $V_j \otimes_{\mathbb{F}} W \to \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W)$ to obtain a homomorphism $V_j \to \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W)$. By the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 (3) we get an induced homomorphism

$$\bigoplus_{j \in I} V_j \;\to\; \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W).$$

In summary we have now constructed a map

$$\left(\bigoplus_{j \in I} V_j \right) \times W \;\to\; \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W).$$

One can easily verify that this map is $\mathbb{F}$-bilinear. It follows from the Tensor Product–Universal Property Lemma 81.15 (3) that we get an induced homomorphism

$$\Psi \colon \left(\bigoplus_{j \in I} V_j \right) \otimes_{\mathbb{F}} W \;\to\; \bigoplus_{i \in I} (V_i \otimes_{\mathbb{F}} W).$$

Note that the two homomorphisms $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are both the identity on the generating set $\left\{ v \otimes w \,\middle|\, v \in \bigcup_{j \in I} V_j \text{ and } w \in W \right\}$. Thus we see that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. This implies that Φ is indeed an isomorphism. Finally note that one can easily verify that Φ is natural. ■

Example. Let $\mathbb{F}$ be a field and let V and W be two $\mathbb{F}$-vector spaces. Furthermore let A be a basis of V and let B be a basis of W. Note that

$$V \otimes_{\mathbb{F}} W \underset{\text{since } A \text{ is a basis of } V \text{ and } B \text{ is a basis of } W}{\overset{\cong}{\longleftarrow}} \left(\bigoplus_{a \in A} \mathbb{F} \right) \otimes_{\mathbb{F}} \left(\bigoplus_{b \in B} \mathbb{F} \right) \underset{\text{by the Tensor–Direct Sum Lemma 81.19}}{\overset{\cong}{\longrightarrow}} \bigoplus_{a \in A} \left(\mathbb{F} \otimes_{\mathbb{F}} \left(\bigoplus_{b \in B} \mathbb{F} \right) \right) \underset{\text{by the Tensor-with-}\mathbb{F} \text{ Lemma 81.18}}{\overset{\cong}{\longrightarrow}} \bigoplus_{a \in A} \bigoplus_{b \in B} \mathbb{F} \overset{\cong}{\longrightarrow} \bigoplus_{(a,b) \in A \times B} \mathbb{F}.$$

In fact, if one goes carefully through the definitions of the isomorphisms one concludes that the set $\{a \otimes b \,|\, a \in A, b \in B\}$ forms a basis for the tensor product $V \otimes_{\mathbb{F}} W$. In particular we obtain from this discussion the equality

$$\dim_{\mathbb{F}}(V \otimes_{\mathbb{F}} W) \;=\; \#(A \times B) \;=\; \#A \cdot \#B \;=\; \dim_{\mathbb{F}}(V) \cdot \dim_{\mathbb{F}}(W). \qquad \square$$

We conclude this short introduction to tensor products of vector spaces with the following lemma:

Lemma 81.20. (Tensor-Exactness Lemma) Let $\mathbb{F}$ be a field. If

$$\ldots \;\to\; W_{m+1} \xrightarrow{\varphi_{m+1}} W_m \xrightarrow{\varphi_m} W_{m-1} \xrightarrow{\varphi_{m-1}} \ldots$$

is an exact sequence of $\mathbb{F}$-vector spaces, then for any $\mathbb{F}$-vector space V the sequence

$$\ldots \;\to\; V \otimes_{\mathbb{F}} W_{m+1} \xrightarrow{\mathrm{id}_V \otimes \varphi_{m+1}} V \otimes_{\mathbb{F}} W_m \xrightarrow{\mathrm{id}_V \otimes \varphi_m} V \otimes_{\mathbb{F}} W_{m-1} \xrightarrow{\mathrm{id}_V \otimes \varphi_{m-1}} \ldots$$

is also exact.

Remark. On page 3093 we will see that in related settings, e.g. when we consider the tensor product of abelian groups, tensoring does in general *not* preserve exactness. So we should consider ourselves lucky, that in our current setting, tensoring does preserve exactness. $\square$

Proof. We start out with the following criterion for a sequence to be exact:

Claim. Let $\ldots \;\to\; C_{m+1} \xrightarrow{\varphi_{m+1}} C_m \xrightarrow{\varphi_m} C_{m-1} \xrightarrow{\varphi_{m-1}} \ldots$ be a sequence of homomorphisms between $\mathbb{F}$-vector spaces. The sequence is exact if and only if the following holds: For each index i we have $\mathbb{F}$-vector spaces A_i and B_i, an isomorphism $\Theta_i \colon A_i \oplus B_i \to C_i$ and an isomorphism $\psi \colon B_i \to A_{i-1}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{m+1} & \xrightarrow{\varphi_{m+1}} & C_m & \xrightarrow{\varphi_m} & C_{m-1} & \longrightarrow & \ldots \\
 & & \Theta_{m+1}\uparrow\cong & & \Theta_m\uparrow\cong & & \Theta_{m-1}\uparrow\cong & & \\
\ldots & \longrightarrow & A_{m+1} \oplus B_{m+1} & \xrightarrow{\begin{pmatrix} 0 & \psi_{m+1} \\ 0 & 0 \end{pmatrix}} & A_m \oplus B_m & \xrightarrow{\begin{pmatrix} 0 & \psi_m \\ 0 & 0 \end{pmatrix}} & A_{m-1} \oplus B_{m-1} & \longrightarrow & \ldots
\end{array}$$

Proof. The "if"-direction is essentially clear, since the bottom sequence (if it exists) is clearly exact. We turn to the proof of the "only if"-direction. For each index i we set $A_i := \ker(\varphi_i \colon C_i \to C_{i-1})$. Since we are dealing with vector spaces we know that there exists a vector subspace B_i of C_i with $C_i = A_i + B_i$ and $A_i \cap B_i = \{0\}$. For each i we denote by $\Theta_i \colon A_i \oplus B_i \to C_i$ the obvious map and for each i we denote by $\psi_i \colon B_i \to A_{i-1}$ the composition of the following maps:

$$B_i \xrightarrow{b \mapsto (0,b)} A_i \oplus B_i \xrightarrow{\Theta_i} C_i \xrightarrow{\varphi_i} C_{i-1} \xrightarrow{\Theta_{i-1}^{-1}} A_{i-1} \oplus B_{i-1} \xrightarrow{(a,b) \mapsto a} A_{i-1}.$$

It follows easily from the exactness of the original sequence that these objects have all the desired properties. $\boxplus$

Now we turn to the actual proof of the lemma. Let

$$\ldots \;\to\; W_{m+1} \xrightarrow{\varphi_{m+1}} W_m \xrightarrow{\varphi_m} W_{m-1} \xrightarrow{\varphi_{m-1}} \ldots$$

be an exact sequence of $\mathbb{F}$-vector spaces. It follows from the claim that for each index i we have $\mathbb{F}$-vector spaces A_i and B_i, an isomorphism $\Theta_i \colon A_i \oplus B_i \to W_i$ and an isomorphism

$\psi\colon B_i \to A_{i-1}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & W_{m+1} & \xrightarrow{\varphi_{m+1}} & W_m & \xrightarrow{\varphi_m} & W_{m-1} & \longrightarrow & \dots \\
 & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \Theta_{m+1}} & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \Theta_m} & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \Theta_{m-1}} & & \\
\dots & \longrightarrow & A_{m+1}\oplus B_{m+1} & \xrightarrow{\left(\begin{smallmatrix}0 & \psi_{m+1}\\ 0 & 0\end{smallmatrix}\right)} & A_m\oplus B_m & \xrightarrow{\left(\begin{smallmatrix}0 & \psi_m\\ 0 & 0\end{smallmatrix}\right)} & A_{m-1}\oplus B_{m-1} & \longrightarrow & \dots.
\end{array}$$

We take this diagram and we tensor it with V "from the left". This leads us to the following diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & V\underset{\mathbb{F}}{\otimes}W_{m+1} & \xrightarrow{\mathrm{id}\otimes\varphi_{m+1}} & V\underset{\mathbb{F}}{\otimes}W_m & \xrightarrow{\mathrm{id}\otimes\varphi_m} & V\underset{\mathbb{F}}{\otimes}W_{m-1} & \longrightarrow & & \\
 & \uparrow{\scriptstyle \cong}\,{\scriptstyle \mathrm{id}\otimes\Theta_{m+1}} & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \mathrm{id}\otimes\Theta_m} & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \mathrm{id}\otimes\Theta_{m-1}} & & & \\
\longrightarrow & V\underset{\mathbb{F}}{\otimes}(A_{m+1}\oplus B_{m+1}) & \xrightarrow{\mathrm{id}\otimes\left(\begin{smallmatrix}0 & \psi_{m+1}\\ 0 & 0\end{smallmatrix}\right)} & V\underset{\mathbb{F}}{\otimes}(A_m\oplus B_m) & \xrightarrow{\mathrm{id}\otimes\left(\begin{smallmatrix}0 & \psi_m\\ 0 & 0\end{smallmatrix}\right)} & V\underset{\mathbb{F}}{\otimes}(A_{m-1}\oplus B_{m-1}) & \longrightarrow & & \\
 & \uparrow{\scriptstyle \cong}\,{\scriptstyle \Xi} & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \Xi} & & \uparrow{\scriptstyle \cong}\,{\scriptstyle \Xi} & & & \\
\to & V\underset{\mathbb{F}}{\otimes}A_{m+1}\oplus V\underset{\mathbb{F}}{\otimes}B_{m+1} & \xrightarrow{\left(\begin{smallmatrix}0 & \mathrm{id}\otimes\psi_{m+1}\\ 0 & 0\end{smallmatrix}\right)} & V\underset{\mathbb{F}}{\otimes}A_m\oplus V\underset{\mathbb{F}}{\otimes}B_m & \xrightarrow{\left(\begin{smallmatrix}0 & \mathrm{id}\otimes\psi_m\\ 0 & 0\end{smallmatrix}\right)} & V\underset{\mathbb{F}}{\otimes}A_{m-1}\oplus V\underset{\mathbb{F}}{\otimes}B_{m-1} & \to & &
\end{array}$$

We make the following clarifications and observations:

(a) It follows from Tensor–Map Composition Lemma 81.17 (2) that the top part of the diagram commutes again.

(b) The bottom vertical isomorphisms are provided by the Tensor–Direct Sum Lemma 81.19. It follows from the naturality of these isomorphisms that the bottom part of the diagram also commutes.

(c) Note that it follows from the above Tensor–Map Composition Lemma 81.17 (2) that the homomorphisms $\mathrm{id}_V\otimes\Theta_i\colon V\otimes_{\mathbb{F}}(A_i\oplus B_i)\to V\otimes_{\mathbb{F}}W_i$ and also the homomorphisms $\mathrm{id}_V\otimes\psi_i\colon V\otimes_{\mathbb{F}}B_i\to V\otimes_{\mathbb{F}}A_{i-1}$ are isomorphisms.

(d) It follows from (c) and the claim that the bottom horizontal sequence is exact.

(e) It follows from (a), (b), (c) and (d) that the top sequence is, as desired, exact. ∎

81.4. **The statement of the Künneth Theorem for de Rham cohomology.** Now that we are familiar with the tensor product of vector spaces we can state the last major result of this chapter:

Theorem 81.21. (Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem) Let M and N be two smooth manifolds. We denote by $p\colon M\times N\to M$ and $q\colon M\times N\to N$ the projections. If one of M or N is a closed smooth manifold, then for each $k\in\mathbb{N}_0$ the map[841]

$$\begin{array}{rcl}
\times\colon \bigoplus_{i=0}^{k}\mathrm{H}^i_{\mathrm{dR}}(M)\otimes_{\mathbb{R}}\mathrm{H}^{k-i}_{\mathrm{dR}}(N) & \to & \mathrm{H}^k_{\mathrm{dR}}(M\times N)\\
\sum_{j=1}^{m}\varphi_j\otimes\psi_j & \mapsto & \underbrace{p^*(\varphi_j)}_{\in\mathrm{H}^i_{\mathrm{dR}}(M\times N)}\wedge\underbrace{q^*(\psi_j)}_{\in\mathrm{H}^{k-i}_{\mathrm{dR}}(M\times N)}
\end{array}$$

is an isomorphism.

[841] The map $\times$ is sometimes called the "cross product" on de Rham cohomology.

Proof. We postpone the proof to the next Section 81.5. ■

Example.

(1) Let N be a smooth manifold and let $k \in \mathbb{N}$. We denote by $p\colon N \times S^k \to N$ and $q\colon N \times S^k \to S^k$ the natural projections. We pick a non-zero de Rham cohomology class $\omega \in \mathrm{H}^k_{\mathrm{dR}}(S^k)$. Let $j \in \mathbb{N}_0$. It follows from the calculation of the de Rham cohomology groups of the sphere (see the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4) together with the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21 that the following following map is an isomorphism:

$$\begin{array}{rcl} \mathrm{H}^j_{\mathrm{dR}}(N) \otimes_{\mathbb{R}} \overbrace{\mathrm{H}^0_{\mathrm{dR}}(S^k)}^{=\mathbb{R}\cdot 1_{S^k}} \oplus \mathrm{H}^{j-k}_{\mathrm{dR}}(N) \otimes_{\mathbb{R}} \overbrace{\mathrm{H}^k_{\mathrm{dR}}(S^k)}^{=\mathbb{R}\cdot\omega} & \to & \mathrm{H}^j_{\mathrm{dR}}(N \times S^k) \\ (\varphi \otimes \psi, \varphi' \otimes \psi') & \mapsto & p^*(\varphi) \wedge q^*(\psi) + p^*(\varphi') \wedge q^*(\psi'). \end{array}$$

It now follows from the Tensor-with-$\mathbb{F}$–Lemma 81.18 that the following map is an isomorphism:

$$\begin{array}{rcl} \mathrm{H}^j_{\mathrm{dR}}(N) \oplus \mathrm{H}^{j-k}_{\mathrm{dR}}(N) & \to & \mathrm{H}^j_{\mathrm{dR}}(N \times S^k) \\ (\varphi, \psi) & \mapsto & p^*(\varphi) + p^*(\psi) \wedge q^*(\omega). \end{array}$$

This gives us a new proof of the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.10.

(2) Let $n \in \mathbb{N}$. We consider the product $S^n \times S^n$. Let $p, q\colon S^n \times S^n \to S^n$ be the natural projection onto the first (respectively second) factor. Let $\theta \in \mathrm{H}^n_{\mathrm{dR}}(S^n)$ be a non-zero element. It follows easily from the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21 together with the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4, the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 and the Wedge-on-de Rham Cohomology Proposition 78.12 (4) that $p^*(\theta)$ and $q^*(\theta)$ form a basis of $\mathrm{H}^n_{\mathrm{dR}}(S^n \times S^n)$. This approach can be used to give a new proof of the Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 80.11 (2).

(3) In Exercise 81.11 we will see that in the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21 we cannot completely drop the requirement that one of the two smooth manifolds is closed. □

81.5. **The proof of the Künneth Theorem for de Rham cohomology.** Throughout this section we adopt the following almost self-explanatory notation:

Notation 81.22. Given any two smooth manifolds M and N we denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the two natural projections.

In this section we will prove the following proposition:

Proposition 81.23. (Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition) Let M be a smooth manifold. Let $s \in \mathbb{N}_0$. For every smooth manifold N that admits a smoothly good open cover consisting of s sets and every $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \times\colon \bigoplus\limits_{i \in \mathbb{N}_0} \mathrm{H}^i_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathrm{H}^{k-i}_{\mathrm{dR}}(N) & \to & \mathrm{H}^k_{\mathrm{dR}}(M \times N) \\ \sum\limits_{j=1}^{m} \varphi_j \otimes \psi_j & \mapsto & p^*(\varphi_j) \wedge q^*(\psi_j) \end{array}$$

is an isomorphism.

Proof of Theorem 81.21 assuming Proposition 81.23. Let M and N be two smooth manifolds. We assume that one of these two smooth manifolds is closed. Since the statement of the theorem is essentially symmetric in the two factors we can assume that N is closed. It follows from the Smoothly Good Cover Theorem 81.2 (1) that N admits a finite smoothly good open cover. It now follows from the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 81.23 that for every

$k \in \mathbb{N}_0$ the map

$$\begin{aligned} \times \colon \bigoplus_{i \in \mathbb{N}_0} \mathrm{H}^i_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathrm{H}^{k-i}_{\mathrm{dR}}(N) &\to \mathrm{H}^k_{\mathrm{dR}}(M \times N) \\ \sum_{j=1}^{m} \varphi_j \otimes \psi_j &\mapsto p^*(\varphi_j) \wedge q^*(\psi_j) \end{aligned}$$

is an isomorphism. It follows from the High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.9 that this corresponds to the desired result. ■

Proof of the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 81.23. Let M be any smooth manifold. Evidently we prove the statement of the lemma by induction on $s \in \mathbb{N}_0$.

First note that if $s \leq 1$, then $N = \varnothing$ or N is diffeomorphic to an open convex subset of $\mathbb{R}^n$. These are dealt with in the following claim:

Claim 0. The desired statement holds if:
(1) $N = \varnothing$.
(2) N is diffeomorphic to an open convex subset of $\mathbb{R}^n$.

Proof. We now deal with these two special cases:

(1) If $N = \varnothing$, then we obtain from the Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.6 together with the Tensor Product–Universal Property Lemma 81.15 that both sides of the wannabe isomorphism are in fact zero.
(2) We now assume that N is diffeomorphic to an open convex subset of $\mathbb{R}^n$. Recall that this implies in particular that N is contractible. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \overbrace{\mathrm{H}^0_{\mathrm{dR}}(N)}^{=\mathbb{R}\cdot 1_N} & \xrightarrow{\varphi \otimes \psi \mapsto p^*(\varphi) \wedge q^*(\psi)} & \mathrm{H}^k_{\mathrm{dR}}(M \times N) \\ & \searrow^{\cong}_{\varphi \otimes r \cdot 1_N \mapsto r \cdot \varphi} \quad \nearrow_{\cong}^{p^*} & \\ & \mathrm{H}^k_{\mathrm{dR}}(M). & \end{array}$$

We make the following observations:
(a) Since N is contractible it follows easily from the Homotopy-Equivalence Basics Lemma 19.14 (5) that the map $p \colon M \times N \to M$ is a homotopy equivalence. It thus follows from the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 that the right diagonal map is an isomorphism.
(b) It follows from the Tensor-with-$\mathbb{F}$–Lemma 81.18 that the left diagonal map is an isomorphism.
(c) It follows from the Wedge-on-de Rham Cohomology Proposition 78.12 (4) that the diagram commutes.
(d) Since N is contractible it follows from the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10 together with the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 that all de Rham cohomology groups of N vanish except for $\mathrm{H}^0_{\mathrm{dR}}(N) = \mathbb{R} \cdot 1_{\mathbb{R}}$.

It follows from (a), (b) and (c) that the top horizontal map is indeed an isomorphism. By (d) this is indeed the desired result. ⊞

Now we perform the induction step from s to $s+1$. Thus let N' be a smooth manifold that admits a smoothly good open cover consisting of sets $U_1, \dots, U_s, V$. We now set $U := U_1 \cup \dots \cup U_s$.

Claim 1. Given $j \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccc}
\dots \longrightarrow & \mathrm{H}^{j-1}_{\mathrm{dR}}(U \cap V) & \xrightarrow{\delta_{j-1}} & \mathrm{H}^{j}_{\mathrm{dR}}(N) & \xrightarrow{\binom{\iota}{-\iota}} & \begin{matrix}\mathrm{H}^{j}_{\mathrm{dR}}(U)\\ \mathrm{H}^{j}_{\mathrm{dR}}(V)\end{matrix} & \xrightarrow{\iota\oplus\iota} \dots \\
& \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & \\
\dots \longrightarrow & \mathrm{H}^{j-1}_{\mathrm{dR}}(M \times (U \cap V)) & \xrightarrow{\delta_{j-1}} & \mathrm{H}^{j}_{\mathrm{dR}}(M \times N) & \xrightarrow{\binom{\iota}{-\iota}} & \begin{matrix}\mathrm{H}^{j}_{\mathrm{dR}}(M \times U)\\ \mathrm{H}^{j}_{\mathrm{dR}}(M \times V)\end{matrix} & \xrightarrow{\iota\oplus\iota} \dots
\end{array}$$

We make the following clarifications:

- The vertical "stacks" of vector spaces are supposed to be read as direct sums.
- For the sake of brevity all inclusion induced maps are called ι.
- The top horizontal sequence is the exact sequence provided by the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 for the open cover $\{U, V\}$ of N.
- The bottom horizontal sequence is the exact sequence provided by the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 for the open cover $\{M \times U, M \times V\}$ of $M \times N$.

The following statements hold:

(a) Both horizontal sequences are exact.
(b) The diagram commutes.

Proof.

(a) It follows from the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2, applied to the open cover $\{U, V\}$ of $N = U \cup V$ and applied to the open cover $\{M \times U, M \times V\}$ of $M \times N$, that the two horizontal sequences are exact.

(b) The diagram commutes by the naturality of the connecting homomorphism, which we established in the Cohomology–LES Proposition 77.13 (2). ⊞

In the following two claims we will build on the above commutative diagram from Claim 1 to obtain a proof of the proposition:

Claim 2. Let $i, j \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccccccc}
\to \mathrm{H}^{i}_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{j-1}_{\mathrm{dR}}(U \cap V) & \xrightarrow{\mathrm{id}\otimes\delta_{j-1}} & \mathrm{H}^{i}_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{j}_{\mathrm{dR}}(N) & \xrightarrow{\binom{\mathrm{id}\otimes\iota}{-\mathrm{id}\otimes\iota}} & \begin{matrix}\mathrm{H}^{i}_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{j}_{\mathrm{dR}}(U)\\ \mathrm{H}^{i}_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{j}_{\mathrm{dR}}(V)\end{matrix} & \xrightarrow{\mathrm{id}\otimes\iota\oplus\mathrm{id}\otimes\iota} \\
\downarrow{\scriptstyle \substack{\varphi\otimes\psi \\ \downarrow \\ p^*(\varphi)\wedge q^*(\psi)}} & & \downarrow{\scriptstyle \substack{\varphi\otimes\psi \\ \downarrow \\ p^*(\varphi)\wedge q^*(\psi)}} & & \downarrow{\scriptstyle \substack{\varphi\otimes\psi \\ \downarrow \\ p^*(\varphi)\wedge q^*(\psi)}} & & \\
\to \mathrm{H}^{i+j-1}_{\mathrm{dR}}(M \times (U \cap V)) & \xrightarrow{\delta_{i+j-1}} & \mathrm{H}^{i+j}_{\mathrm{dR}}(M \times N) & \xrightarrow{\binom{\iota}{-\iota}} & \begin{matrix}\mathrm{H}^{i+j}_{\mathrm{dR}}(M \times U)\\ \mathrm{H}^{i+j}_{\mathrm{dR}}(M \times V)\end{matrix} & \xrightarrow{\iota\oplus\iota} &
\end{array}$$

The following statements hold:

(a) Both horizontal sequences are exact.
(b) The diagram commutes.

Proof.

(a) By Claim 1 (a) we know that the bottom sequence is exact. Furthermore note that it follows also easily Claim 1 (a), the Tensor-Exactness Lemma 81.20 and the natural isomorphisms from the Tensor–Direct Sum Lemma 81.19 that the top horizontal sequence is exact.

(b) For the squares that do not involve the connecting homomorphisms the commutativity is a straightforward consequence of Claim 1 (b) and the $\mathrm{H}^*_{\mathrm{dR}}$-Ring Functor Proposition 78.14. We now turn to the square that involves the connecting homomorphisms. It follows from the Tensor Product–Universal Property Lemma 81.15 (2a) that it suffices to prove the following subclaim:

Subclaim. For each closed i-form $\varphi \in \Omega^i(M)$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^{j-1}_{\mathrm{dR}}(U \cap V) & \xrightarrow{\delta_{j-1}} & \mathrm{H}^j(N) \\ \downarrow{\scriptstyle \psi \mapsto p^*([\varphi]) \wedge q^*(\psi)} & & \downarrow{\scriptstyle \psi \mapsto p^*([\varphi]) \wedge q^*(\psi)} \\ \mathrm{H}^{i+j-1}_{\mathrm{dR}}(M \times (U \cap V)) & \xrightarrow{\delta_{i+j-1}} & \mathrm{H}^{i+j}_{\mathrm{dR}}(M \times N). \end{array}$$

Proof of subclaim. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(N) & \xrightarrow{\binom{\iota}{-\iota}} & \begin{matrix}\Omega^*(U)\\ \Omega^*(V)\end{matrix} & \xrightarrow{\iota \oplus \iota} & \Omega^*(U \cap V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi \mapsto p^*(\varphi) \wedge q^*(\psi)} & & \downarrow{\scriptstyle \psi \mapsto p^*(\varphi) \wedge q^*(\psi)} & & \downarrow{\scriptstyle \psi \mapsto p^*(\varphi) \wedge q^*(\psi)} & & \\ 0 & \longrightarrow & \Omega^*(M \times N) & \xrightarrow{\binom{\iota}{-\iota}} & \begin{matrix}\Omega^*(M \times U)\\ \Omega^*(M \times V)\end{matrix} & \xrightarrow{\iota \oplus \iota} & \Omega^*(M \times (U \cap V)) & \longrightarrow & 0. \end{array}$$

Using the Pullback Smooth Form Proposition 74.7 (2b) one can easily verify that the diagram commutes. It now follows from the naturality of connecting homomorphisms, see Definition 77.12, that the connecting homomorphisms in de Rham cohomology also form a commutative diagram. ⊞

Claim 3. Given $k \in \mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccccccc}
\to \bigoplus_{i\in\mathbb{N}_0} \mathrm{H}^i_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{k-i-1}_{\mathrm{dR}}(U\cap V) & \longrightarrow & \bigoplus_{i\in\mathbb{N}_0} \mathrm{H}^i_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{k-i}_{\mathrm{dR}}(N) & \longrightarrow & \bigoplus_{i\in\mathbb{N}_0} \begin{matrix} \mathrm{H}^i_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{k-i}_{\mathrm{dR}}(U) \\ \mathrm{H}^i_{\mathrm{dR}}(M) \underset{\mathbb{R}}{\otimes} \mathrm{H}^{k-i}_{\mathrm{dR}}(V) \end{matrix} & \longrightarrow \\
\downarrow{\scriptstyle \begin{matrix}\varphi\otimes\psi \\ \downarrow \\ p^*(\varphi)\wedge q^*(\psi)\end{matrix}} & & \downarrow{\scriptstyle \begin{matrix}\varphi\otimes\psi \\ \downarrow \\ p^*(\varphi)\wedge q^*(\psi)\end{matrix}} & & \downarrow{\scriptstyle \begin{matrix}\varphi\otimes\psi \\ \downarrow \\ p^*(\varphi)\wedge q^*(\psi)\end{matrix}} & \\
\to \mathrm{H}^{k-1}_{\mathrm{dR}}(M\times(U\cap V)) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(M\times N) & \longrightarrow & \begin{matrix} \mathrm{H}^k_{\mathrm{dR}}(M\times U) \\ \mathrm{H}^k_{\mathrm{dR}}(M\times V) \end{matrix} & \longrightarrow
\end{array}$$

where the top horizontal sequence is given by the direct sum of the top horizontal sequences from Claim 2. The following statements hold:

(a) Both horizontal sequences are exact.
(b) The diagram commutes.
(c) The outer vertical maps are isomorphisms.
(d) The middle vertical maps are isomorphisms. (This is the result we had been shooting for.)

Proof.

(a) This statement is an immediate consequence of Claim 2 (a).
(b) It follows easily from Claim 2 (b) that the diagram commutes.
(c) Since $U \cap V$, U and V admit a smoothly good open cover of size $s-1$ respectively of size 1 we see that the statement follows from induction and the base case $s = 1$.
(d) This statement follows from (a), (b), (c) and the Five Lemma 77.9. ■

81.6. Exercises for Chapter 81.

Exercise 81.1. Let $n \in \mathbb{N}$. Does there exist an n-dimensional smooth manifold M that has an open cover $M = U \cup V$ such that U and V are both diffeomorphic to $\mathbb{R}^n$ and with $\dim(\mathrm{H}^1_{\mathrm{dR}}(M)) \geq 3$?

Exercise 81.2. Let $m \in \mathbb{N}$. What is the lowest $C(m) \in \mathbb{N}$ that you can give such that for every smooth manifold M that admits a smoothly good open cover consisting of m subsets we have

$$\sum_{k\in\mathbb{N}_0} \dim_{\mathbb{R}}(\mathrm{H}^k_{\mathrm{dR}}(M)) \;\leq\; C(m)\,?$$

Exercise 81.3. Show that S^2 does not admit a smoothly good open cover with less than four subsets.

Exercise 81.4. Provide an explicit finite smoothly good open cover of the surface of genus two.

Exercise 81.5. Show that S^2 does not admit a smoothly good open cover with less than four subsets.

Exercise 81.6. We consider the smooth manifold $M = \mathbb{R}^2 \setminus (\mathbb{Z} \times \{0\})$.

(a) Show that $\mathrm{H}^1_{\mathrm{dR}}(M)$ is not finite-dimensional.

(b) Show that $\mathrm{H}^1_{\mathrm{dR}}(M)$ has uncountable dimension.
Remark. In Exercise 78.8 we gave an example of a smooth manifold such that $\mathrm{H}^0_{\mathrm{dR}}(M)$ has uncountable dimension.

Exercise 81.7. Let $\mathbb{F}$ be a field. Give an example of $\mathbb{F}$-vector spaces V and W such that not every element of $V \otimes_{\mathbb{F}} W$ can be written as $v \otimes w$ for some $v \in V$ and $w \in W$.

Exercise 81.8. Let $\mathbb{F}$ be a field. Use the universal property formulated in the Tensor Product–Universal Property Lemma 81.15 (3) to prove the following statements:

(a) Given any two $\mathbb{F}$-vector spaces V and W we have $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes_{\mathbb{F}} W}$.

(b) Let $f\colon V \to V'$, $f'\colon V' \to V''$, $g\colon W \to W'$ and $g'\colon W' \to W''$ be homomorphisms between $\mathbb{F}$-vector spaces. Then

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon V \otimes_{\mathbb{F}} W \;\to\; V'' \otimes_{\mathbb{F}} W''.$$

(c) Let $f, g\colon V \to W$ be homomorphisms between $\mathbb{F}$-vector spaces and let Z be an $\mathbb{F}$-vector space. Then

$$(f+g) \otimes \mathrm{id}_Z \;=\; f \otimes \mathrm{id}_Z + g \otimes \mathrm{id}_Z\colon V \otimes_{\mathbb{F}} Z \;\to\; W \otimes_{\mathbb{F}} Z.$$

Exercise 81.9. Given a closed n-dimensional smooth manifold M we define its de Rham Euler characteristic as

$$\chi_{\mathrm{dR}}(M) \;:=\; \sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{R}}(\mathrm{H}^i_{\mathrm{dR}}(M)).$$

(a) Let M be a closed smooth manifold and let $f\colon M \to M$ be a diffeomorphism. Show that

$$\chi_{\mathrm{dR}}(\underbrace{\mathrm{Tor}(M,f)}_{\text{mapping torus}}) \;=\; 0.$$

(b) Show that for two closed smooth manifolds M and N we have the following equality:

$$\chi_{\mathrm{dR}}(M \times N) \;=\; \chi_{\mathrm{dR}}(M) \cdot \chi_{\mathrm{dR}}(N).$$

Remark. It follows from the De Rham-Isomorphism Theorem 231.16, the Universal Coefficient-H^*–Field Theorem 202.2, the Homology-χ-Proposition 156.2 and the Euler Characteristics–Equality Corollary 223.6 (see also the discussion on page 4287) that the de Rham Euler characteristic equals the smooth Euler characteristic which we introduced in Definition 53.4 and that it equals the "usual" Euler characteristic which we will define in Definition 156.1.

Exercise 81.10. Let W be a compact smooth manifold and let $f\colon W \to W$ be a continuous map. We define the Lefschetz number

$$\Lambda(W,f) \;:=\; \sum_{j \in \mathbb{N}_0} (-1)^j \cdot \mathrm{tr}(f^*(\mathrm{H}^j_{\mathrm{dR}}(W) \to \mathrm{H}^j_{\mathrm{dR}}(W)).$$

Let $n \in \mathbb{N}$ and let $\rho\colon S^n \to S^n$ be the reflection in a hyperplane. Given any compact smooth manifold M determine the Lefschetz number of

$$\begin{aligned} f = (\mathrm{id}, \rho)\colon M \times S^n &\to M \times S^n \\ (x,y) &\mapsto (x, \rho(y)). \end{aligned}$$

Note that the result might involve the de Rham Euler characteristic that we introduced in Exercise 81.9.
Hint. Make use of the naturality of the isomorphism provided by the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21.

Remark. It follows from the De Rham-Isomorphism Theorem 231.16 and the Universal Coefficient-H^*–Field Theorem 202.2 that the above Lefschetz number agrees with the more standard definition provided in Definition 182.10.

Exercise 81.11. In this exercise we consider the two non-compact 0-dimensional smooth manifolds $M := N := \mathbb{N}$. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the natural projections. Show that the map

$$\begin{array}{rcl} \mathrm{H}^0_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathrm{H}^0_{\mathrm{dR}}(N) & \to & \mathrm{H}^0_{\mathrm{dR}}(M \times N) \\ \varphi \otimes \psi & \mapsto & p^*(\varphi) \wedge q^*(\psi) \end{array}$$

is *not* an isomorphism.

Remark. This shows that in the statement of the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21 we cannot completely drop the hypothesis that one of the two smooth manifolds needs to be closed.

CHAPTER 82

Open star-shaped subsets of $\mathbb{R}^n$

In this short chapter we will prove that every open star-shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. This result is interesting in its own right, and it will play an essential role in our proof of the De Rham-Poincaré Duality Theorem 84.5.

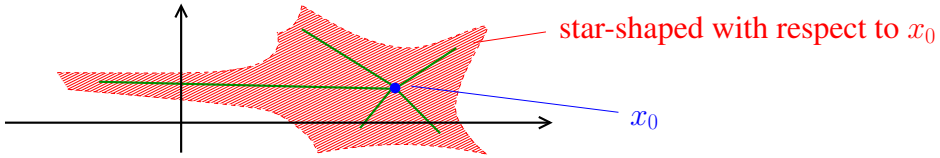
This chapter is organized as follows:

- In Section 82.1 we will show that every open star-shaped subsets of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.
- In Section 82.2 we will consider closed star-shaped subsets of $\mathbb{R}^n$.
- In Section 82.3 we will provide exercises for Chapter 82.

In the next Chapter 83 we will introduce de Rham cohomology with compact support.

82.1. **Closed star shaped subsets of $\mathbb{R}^n$.** First let us recall what we mean by a star-shaped subset:

Definition 82.1. We say that a subset X of $\mathbb{R}^n$ is star-shaped if there exists a point $x_0 \in X$ such that for every $x \in X$ the Euclidean segment $\{x_0 \cdot (1-t) + x \cdot t \,|\, t \in [0,1]\}$ from x_0 to x lies in X.



Example. Evidently every non-empty convex subset of $\mathbb{R}^n$ is star-shaped. □

Now we can already formulate the main result of this chapter:

Theorem 82.2. (Star Shaped-Diffeomorphism Type Theorem) Every open star-shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

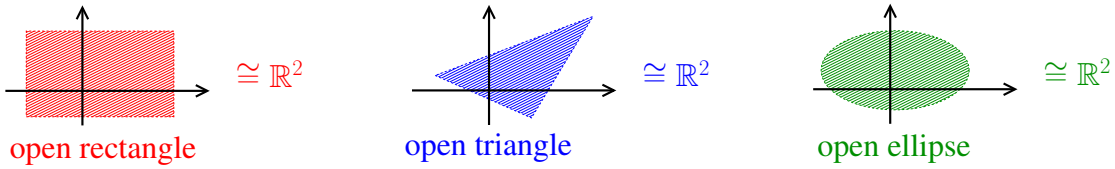
Examples.

(1) Let $n \in \mathbb{N}$. Evidently B^n is an open star-shaped subset of $\mathbb{R}^n$. On page 652 we already saw that the maps

$$\begin{array}{ccc} B^n & \to & \mathbb{R}^n \\ x & \mapsto & \dfrac{x}{\sqrt{1-\|x\|^2}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{R}^n & \to & B^n \\ x & \mapsto & \dfrac{x}{\sqrt{1+\|x\|^2}}. \end{array}$$

are inverse diffeomorphisms. Note that the two diffeomorphisms are given by "stretching" each radial Euclidean segment to a radial ray. The proof of the Star Shaped-Diffeomorphism Type Theorem 82.2 will follow the same approach, but the stretching will be done in a much more clever way.

(2) It follows from the Star Shaped-Diffeomorphism Type Theorem 82.2 that open rectangles, open triangles and open ellipses are diffeomorphic to $\mathbb{R}^2$.



As we will see in Exercise 82.1, even in these simple situations it is surprisingly difficult to write down an *explicit* diffeomorphism to $\mathbb{R}^2$. □

Remark.

(1) Even though this theorem is very well known, there seem to be very few published proofs. A proof is given in [**Grom1990**, Conclusion 1.4], in [**vGl1985**, Theorem 2] and in [**Feru2007**, Proposition 237]. A proof is sketched in [**GuH2019**, Exercises 5.3i-iv] and in [**BJ1982**, p. 86]. Our proof follows the argument of [**GT1998**, p. 60].

(2) The Smoothly Good Cover Theorem 81.2 together with the Star Shaped-Diffeomorphism Type Theorem 82.2 implies that every compact n-dimensional smooth manifold admits a finite open cover, such that the intersection of any collection of sets from the open cover is either empty or diffeomorphic to $\mathbb{R}^n$. This statement is proved at various levels of detail in [**BoT1982**, Theorem 5.1], [**KaL1987**, Chapter VI.3] and [**Lee2009**, Proposition 10.16] using methods from differential geometry.[842] A self-contained elementary proof is given in [**vGl1985**].

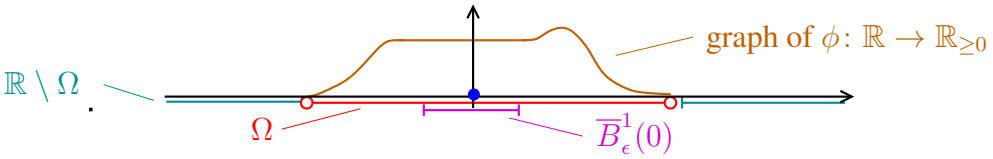
(3) Recall that in the Convex-to-Ball Proposition 4.28 we showed in particular that every bounded open convex non-empty subset U of $\mathbb{R}^n$ is *homeomorphic* to B^n, and thus, using the above explicit diffeomorphism $B^n \to \mathbb{R}^n$, we see that U is homeomorphic to $\mathbb{R}^n$. The Star Shaped-Diffeomorphism Type Theorem 82.2 is a significant generalization of that earlier result. □

The proof of the Star Shaped-Diffeomorphism Type Theorem 82.2 will require most of the remainder of this chapter. We start out with the following lemma:

Lemma 82.3. (Zero Outside Lemma) Let Ω be an open subset of $\mathbb{R}^n$ with $0 \in \Omega$. There exists an $\epsilon > 0$ and a smooth function $\phi\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with the following two properties:

(a) We have $\mathbb{R}^n \setminus \Omega = \phi^{-1}(0)$.

(b) On $\overline{B}^n_\epsilon(0)$ we have $\phi \equiv 1$.



[842] These proofs all seem to rely on the statement that in a Riemannian manifold any open convex subset (where the notion "convex" is defined in terms of "geodesics" on the Riemannian manifold) is diffeomorphic to $\mathbb{R}^n$. The references for that statement seem even more obscure than for the Star Shaped-Diffeomorphism Type Theorem 82.2.

Proof of Lemma 82.3. Since $\Omega \subseteq \mathbb{R}^n$ is open we know that the complement $\mathbb{R}^n \setminus \Omega$ is closed. It follows from this observation, together with the Zero Set-Realization Proposition 27.15, that there exists a smooth function $\psi\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\mathbb{R}^n \setminus \Omega = \psi^{-1}(0)$. Since Ω is open and since $0 \in \Omega$ there exists an $\epsilon > 0$ such that $B^n_{2\epsilon}(0) \subseteq \Omega$. It follows from the Smooth Transition Function Lemma 24.9 that there exists a smooth function $\tau\colon \mathbb{R}_{\geq 0} \to [0,1]$ such that $\tau \equiv 1$ on $[0,\epsilon]$ and such that $\tau \equiv 0$ on $[2\epsilon, \infty)$. One can now easily verify that the function

$$\begin{array}{rcl} \phi\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \tau(\|x\|) + (1 - \tau(\|x\|)) \cdot \psi(x) \end{array}$$

has the desired properties. ■

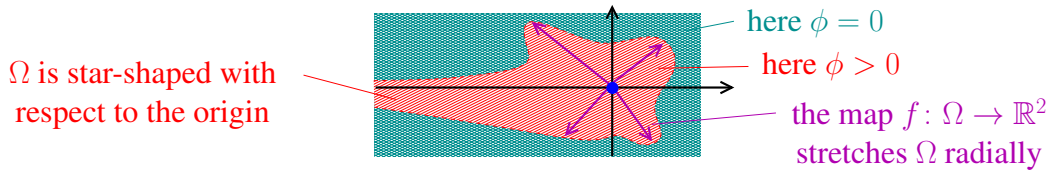
Proof of the Star Shaped-Diffeomorphism Type Theorem 82.2. Let Ω be an open star-shaped subset of $\mathbb{R}^n$. Without loss of generality we can assume that Ω is star-shaped with respect to the origin. We need to show that Ω is diffeomorphic to $\mathbb{R}^n$.

It follows from the Zero Outside Lemma 82.3 that there exists an $\epsilon > 0$ and a smooth function $\phi\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with the following two properties:

(a) We have $\mathbb{R}^n \setminus \Omega = \phi^{-1}(0)$.
(b) On $\overline{B}^n_\epsilon(0)$ we have $\phi \equiv 1$.

We continue with a simple observation: Since Ω is star-shaped with respect to $0 \in \Omega$ we know that for every $x \in \Omega$ and every $s \in [0,1]$ we have $s \cdot x \in \Omega$. In particular we always have $\phi(s \cdot x) \neq 0$. This observation allows us to consider the map

$$\begin{array}{rcl} f\colon \Omega & \to & \mathbb{R}^n \\ x & \mapsto & \underbrace{\left(1 + \left(\int\limits_{s=0}^{s=1} \frac{1}{\phi(s \cdot x)} \cdot \|x\|\, ds\right)^2\right)}_{=:\lambda(x)} \cdot x. \end{array}$$



Our goal is to show that this map $f\colon \Omega \to \mathbb{R}^n$ is a diffeomorphism. We will do so in the following three claims:

Claim 1. The map $f\colon \Omega \to \mathbb{R}^n$ is smooth.

Proof. First note that it follows easily from the fact that $\phi \equiv 1$ on $B^n_\epsilon(0)$ that for any point $x = (x_1, \ldots, x_n) \in B^n_\epsilon(0)$ we have $f(x) = (1 + \|x\|^2) \cdot x = (1 + x_1^2 + \cdots + x_n^2) \cdot x$. This shows that f is smooth on $B^n_\epsilon(0)$. Furthermore note that it follows easily from the fact that $x \mapsto \|x\|$ is smooth on $\Omega \setminus \{0\}$, the Product Rule and the Leibniz Integral Rule 73.28 that f is smooth on $\Omega \setminus \{0\}$. It now follows from these two observations together with the Smooth Pasting Proposition 24.7 that $f\colon \Omega \to \mathbb{R}^n$ is smooth. ⊞

Next we want to show that the map $f\colon \Omega \to \mathbb{R}^n$ is a bijection. We start out with the following two observations:

(1) Note that for every $x \in \Omega$ we have $f(x) = \lambda(x) \cdot x$ with $\lambda(x) \geq 0$. In other words, the image of $\Omega \cap (\mathbb{R}_{\geq 0} \cdot x)$ under the map f is contained in the ray $\mathbb{R}_{\geq 0} \cdot x$.

(2) Given $y \in S^{n-1}$ we set

$$A(y) \;:=\; \sup \underbrace{\{t > 0 \,|\, t \cdot y \in \Omega\}}_{\neq \varnothing \text{ since } \Omega \text{ is open and } 0 \in \Omega} \quad \in \mathbb{R}_{>0} \cup \{\infty\}.$$

Note that it follows from the fact that Ω is open and that $0 \in \Omega$ that $A(y) > 0$. Also note that it follows easily from the definitions and the fact that Ω is open that $[0, A(y)) \cdot y = \Omega \cap \mathbb{R}_{\geq 0} \cdot y$.

Claim 2.
(a) For every $y \in S^{n-1}$ the map $f\colon \Omega \to \mathbb{R}^n$ sends the Euclidean segment $[0, A(y)) \cdot y$ bijectively to the ray $\mathbb{R}_{\geq 0} \cdot y$.
(b) The map $f\colon \Omega \to \mathbb{R}^n$ is a bijection.

Proof.

(a) Let $y \in S^{n-1}$. We consider the map

$$\begin{aligned} g\colon [0, A(y)) &\to \mathbb{R}_{\geq 0} \\ r &\mapsto \left(1 + \left(\int_{s=0}^{s=1} \frac{1}{\phi(s \cdot r \cdot y)} \cdot r \, ds\right)^2\right) \cdot r \;\underset{\substack{\uparrow \\ \text{substitution } t = r \cdot s \text{ in} \\ \text{the integral}}}{=}\; \left(1 + \left(\int_{t=0}^{t=r} \frac{1}{\phi(t \cdot y)} \, dt\right)^2\right) \cdot r \end{aligned}$$

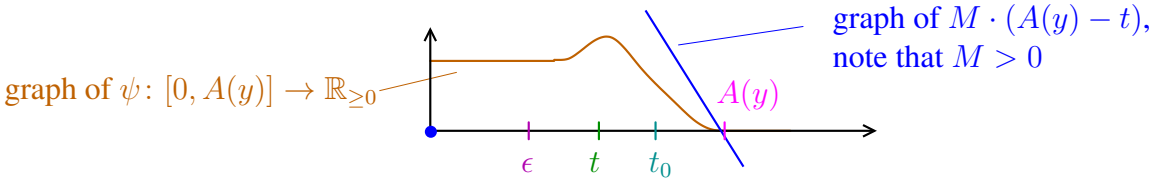
Note that for any $y \in S^{n-1}$ and $r \in [0, A(y))$ we have $f(r \cdot y) = g(r) \cdot y$. It follows that it suffices to show that the map $g\colon [0, A(y)) \to \mathbb{R}_{\geq 0}$ is a bijection.

First note that evidently we have $g(0) = 0$. Next note that it follows easily from the Product Rule and the Fundamental Theorem of Calculus 73.27 that $g'(r) > 1$ for all $r \in [0, A(y))$. This implies in particular that the given map $g\colon [0, A(y)) \to \mathbb{R}_{\geq 0}$ is injective. Thus it remains to show that $\lim\limits_{r \to A(y)} g(r) = \infty$. We distinguish two cases:

- First we consider the case that $A(y) = \infty$. Recall that for all $r \in [0, A(y)) = [0, \infty)$ we have $g'(r) > 1$. It follows from elementary real analysis that we have the desired equality $\lim\limits_{r \to A(y)} g(r) = \lim\limits_{r \to \infty} g(r) = \infty$.
- We now consider the case $A(y) < \infty$.

  Subclaim. There exists an $M > 0$ such that for every $t \in [0, A(y))$ we have
  $$\phi(t \cdot y) \;\leq\; M \cdot (A(y) - t).$$

  Proof of subclaim. First note that, as we already pointed out above, it follows easily from the definition of $A(y)$ and the fact that Ω is open, that $A(y) \cdot y \notin \Omega$. Since (by the definition of ϕ) we have $\phi^{-1}(0) = \mathbb{R}^n \setminus \Omega$ this implies that $\phi(A(y) \cdot y) = 0$. We now consider the smooth function $\psi\colon [0, A(y)] \to \mathbb{R}$ that is given by $\psi(t) := \phi(t \cdot y)$. Note that $\psi(0) = \phi(0) = 1$ and $\psi(A(y)) = \phi(A(y) \cdot y) = 0$.
  Next we set $M := \max\{-\psi'(t) \,|\, t \in [0, A(y)]\}$. We claim that this M has the desired property.



First note that it follows from $\psi(0) = \phi(0) = 1$ and $\psi(A(y)) = \phi(A(y) \cdot y) = 0$, together with the Mean Value Theorem [**Frie2023**, Satz 11.2][843], that we have $M > 0$. Now suppose that we are given a $t \in [0, A(y)]$. It follows from the Mean Value Theorem, applied to the smooth function $\psi|_{[t,A(y)]}\colon [t, A(y)] \to \mathbb{R}$, that there exists a $t_0 \in [t, A(y)]$ such that the following equality holds:

$$\underbrace{\psi(A(y))}_{=0} - \underbrace{\psi(t)}_{=\phi(t \cdot y)} = \psi'(t_0) \cdot (A(y) - t).$$

It follows that

$$\phi(t \cdot y) \underset{\text{definition of } \psi}{=} \psi(t) \underset{\text{by the above}}{=} -\psi'(t_0) \cdot (A(y) - t) \underset{\text{definition of } M}{\leq} M \cdot (A(y) - t).$$

□

We now see that

$$\begin{aligned}
\lim_{r \to A(y)} g(r) &= \lim_{r \to A(y)} \left(1 + \left(\int_{t=0}^{t=r} \frac{1}{\phi(t \cdot y)}\, dt\right)^2\right) \cdot r \\
&\underset{\text{by the inequality from the subclaim}}{\geq} \lim_{r \to A(y)} \left(1 + \left(\frac{1}{M} \cdot \underbrace{\int_{t=0}^{t=r} \frac{1}{A(y) - t}\, dt}_{= -\ln(A(y)-r) + \ln(A(y))}\right)^2\right) \cdot r \\
&= \lim_{r \to A(y)} \left(1 + \frac{1}{M^2} \cdot (\ln(A(y)) - \underbrace{\ln(A(y) - r)}_{\to -\infty \text{ as } r \to A(y)})^2\right) \cdot r = \infty.
\end{aligned}$$

(b) We need to show that the map $f\colon \Omega \to \mathbb{R}^n$ is a bijection. This follows easily from the following three observations:

- We have $f(0) = 0$.
- By (a) we know that for every $y \in S^{n-1}$ the map $f\colon \Omega \to \mathbb{R}^n$ sends the half-open Euclidean segment $[0, A(y)) \cdot y$ bijectively to the ray $\mathbb{R}_{\geq 0} \cdot y$.
- Since $\Omega \subseteq \mathbb{R}^n$ is an open subset we see that for every $y \in S^{n-1}$ we have the equality $\Omega \cap (\mathbb{R}_{\geq 0} \cdot y) = [0, A(y)) \cdot y$.

⊞

We have now shown that the map $f\colon \Omega \to \mathbb{R}^n$ is smooth and a bijection. It remains to prove the following claim:

Claim 3. The inverse $f^{-1}\colon \mathbb{R}^n \to \Omega$ is smooth.

Proof. By the Smooth Pasting Proposition 25.19 together with Inverse Function Theorem 24.21 it suffices to prove that for every $x \in \Omega$ the corresponding differential $\mathrm{D}f_x$ is invertible.

[843] Since there are various results called Mean Value Theorem let us formulate the statement we have in mind: If $f\colon [a, b] \to \mathbb{R}$ is smooth, then there exists a $\xi \in (a, b)$ such that $f'(\xi) = \frac{f(b) - f(a)}{b - a}$.

Suppose that this is not the case. This means that there exists an $x \in \Omega$ and an $h \in \mathbb{R}^n \setminus \{0\}$ such that $\mathrm{D}f_x \cdot h = 0$.

We start out with a general calculation: For any $v \in \mathbb{R}^n$ we have

$$\mathrm{D}f_x \cdot v \underset{\substack{\uparrow \\ \text{definition of } f \text{ on page 1664}}}{=} \mathrm{D}\left(\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda(y) \cdot y_1 \\ \vdots \\ \lambda(y) \cdot y_n \end{pmatrix}\right)_x \cdot v \underset{\substack{\uparrow \\ \text{this follows easily from the Product Rule for partial derivatives}}}{=} \overbrace{\left(\begin{pmatrix} \mathrm{D}\lambda_x \cdot x_1 \\ \vdots \\ \mathrm{D}\lambda_x \cdot x_n \end{pmatrix} + \lambda(x) \cdot \mathrm{id}_n\right)}^{(n \times n)\text{-matrix}} \cdot v$$

$$= (\mathrm{D}\lambda_x \cdot v) \cdot x + \lambda(x) \cdot v.$$

From this calculation and the fact that $\mathrm{D}f_x \cdot h = 0$ it follows that $(\mathrm{D}\lambda_x \cdot h) \cdot x + \lambda(x) \cdot h = 0$. Since $h \neq 0$ we now see that $h = \mu \cdot x$ for some $\mu \neq 0$ and we see that $x \neq 0$. This implies that $\mathrm{D}\lambda_x \cdot x + \lambda(x) = 0$. In summary we see that $\mathrm{D}\lambda_x \cdot x = -\lambda(x) < 0$.

On the other hand we consider the function

$$h(t) := \lambda(t \cdot x) \underset{\substack{\uparrow \\ \text{see definition of } \lambda \text{ on page 1664}}}{=} 1 + \left(\int_{s=0}^{s=1} \frac{1}{\phi(s \cdot t \cdot x)} \cdot \|t \cdot x\| \, ds\right)^2 \underset{\substack{\uparrow \\ \text{substitution } w = s \cdot t \cdot \|x\|}}{=} 1 + \left(\int_{w=0}^{w=t \cdot \|x\|} \frac{1}{\phi(w \cdot \frac{x}{\|x\|})} \, ds\right)^2.$$

It follows that

$$\mathrm{D}\lambda_{t \cdot x} \cdot x \underset{\substack{\uparrow \\ \text{by the Chain Rule}}}{=} \frac{d}{ds}(s \mapsto \lambda(s \cdot x))|_{s=t} = h'(1) \underset{\substack{\uparrow \\ \text{follows from the above description of } h(t) \text{ and the Fundamental Theorem of Calculus 73.27}}}{>} 0.$$

We have thus shown that $\mathrm{D}\lambda_x \cdot x = h'(1) > 1$. This now gives us the desired contradiction. ■

82.2. **Closed star shaped subsets of $\mathbb{R}^n$.** For the record we also state the following corollary to the Star Shaped-Diffeomorphism Type Theorem 82.2. We will note make use of this corollary.

Corollary 82.4. (Closed Star Shaped-Homeomorphism Type Corollary) Let A be a bounded closed subset of $\mathbb{R}^n$ that is star-shaped and which has non-empty interior. There exists a homeomorphism $f\colon A \to \overline{B}^n$ such that the restriction of f to the interiors $\mathring{A} \to B^n$ is a diffeomorphism.

Remark.

(1) Note that the Closed Star Shaped-Homeomorphism Type Corollary 82.4 is optimal in the following sense: We do not demand that A is a submanifold of $\mathbb{R}^n$ (e.g. A could be the cube $[-1,1]^n$), so in general it makes no sense to demand that $f\colon A \to \overline{B}^n$ is a diffeomorphism.

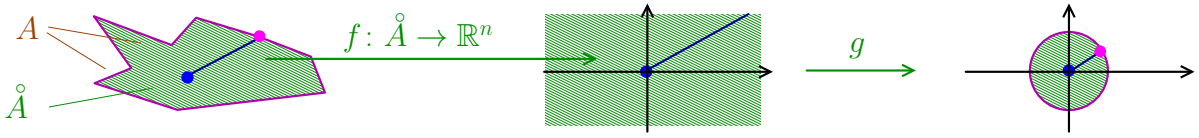
(2) In the Smooth Convex-to-Ball Proposition 30.22 we showed that every compact non-empty codimension-zero smooth *submanifold* of $\mathbb{R}^n$ is diffeomorphic to $\overline{B}^n$. □

Sketch of proof. Since we will not make use of the corollary we will only provide a sketch of the argument. After these apologies, let A be a bounded closed subset of $\mathbb{R}^n$ that is

star-shaped and which has non-empty interior. After a translation we can assume that A is star-shaped with respect to the origin. We leave it to the reader to verify that the interior of A is still star-shaped with respect to the origin. Let $f\colon \mathring{A} \to \mathbb{R}^n$ be one of the diffeomorphisms which we constructed in the proof of the Star Shaped-Diffeomorphism Type Theorem 82.2. Next we consider the diffeomorphism

$$\begin{aligned} g\colon \mathbb{R}^n &\to B^n \\ x &\mapsto \sqrt{1-\|x\|^2}\cdot x \end{aligned}$$

from page 652.



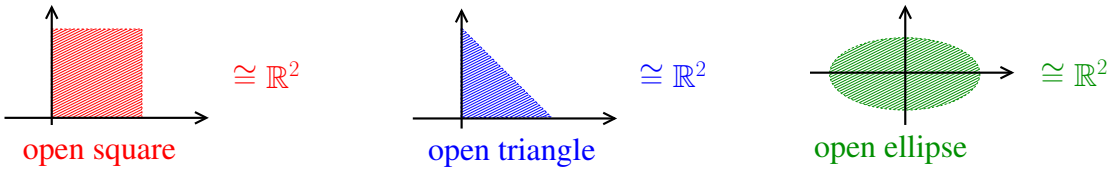
It follows from the descriptions of f and g that for each $v \in S^{n-1}$ the diffeomorphism $g \circ f\colon \mathring{A} \to B^n$ sends the radial half-open Euclidean segment $\mathring{A} \cap (v \cdot \mathbb{R}_{\geq 0})$ to $[0,1) \cdot v$. Thus the map extends in an obvious way to a map $h\colon A \to \overline{B}^n$. One can easily verify that h is a bijection. In Exercise 82.3 we will see that h is continuous. It now follows from the Compact-Hausdorff Proposition 4.23 (3) that $h\colon A \to \overline{B}^n$ is a homeomorphism. The author is fully aware that a few details got skipped along the way. We leave it to the reader to fill in the details. ■

82.3. Exercises for Chapter 82.

Exercise 82.1. We consider the following three open convex star-shaped subsets of $\mathbb{R}^2$:

(a) The open square $U = (0,1) \times (0,1)$.
(b) The open triangle $V = \{(x,y) \in \mathbb{R}^2 \,|\, x > 0, y > 0, x + y \leq 1\}$.
(c) The open ellipse $W = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + 2 \cdot y^2 < 1\}$.

Provide in each case an *explicit* diffeomorphism to $\mathbb{R}^2$.



Exercise 82.2. Let $M \subseteq \mathbb{R}^n$ be a compact codimension-zero submanifold of $\mathbb{R}^n$ such that $0 \in M \setminus \partial M$. Use the Product Tubular Neighborhood–Existence Theorem 58.4 (3) to show that there exists an $\epsilon > 0$ and a smooth function $\phi\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with the following two properties:

(a) We have $\mathbb{R}^n \setminus M = \phi^{-1}(0)$.
(b) On $\overline{B}^n_\epsilon(0)$ we have $\phi \equiv 1$.

Remark. Thus the goal is to avoid the Zero Set-Realization Proposition 27.15.

Exercise 82.3. Let $n \in \mathbb{N}_0$, let $A \subseteq \mathbb{R}^n$ be a bounded closed subset with $0 \in \mathring{A}$ and let $f\colon \mathring{A} \to B^n$ be a continuous map such that for each $v \in S^{n-1}$ the map f restricts to a

bijection $\mathring{A} \cap ([0,\infty) \times \{v\}) \to [0,1) \times \{v\}$. Show that the map $f\colon \mathring{A} \to B^n$ extends to a homeomorphism $A \to \overline{B}^n$.

CHAPTER 83

De Rham cohomology with compact support

In this chapter we will introduce de Rham cohomology with compact support. More precisely, given a smooth manifold M we define real vector spaces $\mathrm{H}^k_{\mathrm{dR},c}(M)$, which are defined the same way as the usual de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M)$, except that we now restrict ourselves to smooth forms "with compact support". For compact smooth manifolds this restriction has no effect. But for non-compact smooth manifolds the effect is surprisingly dramatic.

De Rham cohomology with compact support is interesting in its own right. For us the main application is that we will need it to formulate the De Rham-Poincaré Duality Theorem 84.5. Even if only cares about the De Rham-Poincaré Duality Theorem 84.5 for closed smooth manifolds, we will need to make use of de Rham cohomology with compact support in the proof of the De Rham-Poincaré Duality Theorem 84.5, since even for closed smooth manifolds the proof goes via an induction argument which will involve non-compact smooth manifolds.

This chapter is organized as follows:

- In Section 83.1 we will define de Rham cohomology with compact support.
- In Section 83.2 we will discuss functoriality of de Rham cohomology with compact support. In particular we will see that in suitable settings de Rham cohomology with compact support is *covariantly* functorial.
- In Section 83.3 we will perform various calculations de Rham cohomology with compact support. In particular we will give the (as we think somewhat surprising) calculation for $\mathbb{R}^n$.
- In Section 83.4 we will prove the existence of a Mayer–Vietoris sequence for de Rham cohomology with compact support.
- In Section 83.5 we will provide exercises for Chapter 83.

In the next Chapter 84 we will use de Rham cohomology with compact support to formulate and prove the De Rham-Poincaré Duality Theorem 84.5.

83.1. **Definition of de Rham cohomology with compact support.** In this section we will see that, by tweaking the definition of de Rham cohomology a little bit, we suddenly get a new invariant for non-compact smooth manifolds.

Definition 83.1. Let M be a smooth manifold.

(1) Let ω be a differential form on M. As in Definition 74.10 we define the support of the differential form ω as

$$\mathrm{supp}(\omega) \;=\; \text{the closure of the set of all points } P \in M \text{ such that } \omega_P \neq 0.$$

(2) Given a smooth manifold M and $m \in \mathbb{N}_0$ we set

$$\Omega_c^m(M) \;:=\; \text{real vector space}^{844} \text{ of all smooth } m\text{-forms with compact support on } M.$$

For safety we also set $\Omega_c^{-1}(M) := 0$.

Note that it follows from the Differentials-on-Manifolds Proposition 75.7 (2b) that the differential $d\colon \Omega^m(M) \to \Omega^{m+1}(M)$ restricts to a map $d\colon \Omega_c^m(M) \to \Omega_c^{m+1}(M)$. Thus these maps form a cochain complex in the sense of Definition 77.1 and we get a new invariant:

Definition 83.2. Let M be a smooth manifold and let $k \in \mathbb{N}_0$. The *k–th de Rham cohomology group with compact support of M* is defined as the real vector space

$$\mathrm{H}^k_{\mathrm{dR},c}(M) \;:=\; \frac{\ker\left(d_k\colon \Omega_c^k(M) \to \Omega_c^{k+1}(M)\right)}{\mathrm{im}\left(d_{k-1}\colon \Omega_c^{k-1}(M) \to \Omega_c^k(M)\right)}.$$

Example. Let M be a compact smooth manifold. It follows from the Compact-Closed Lemma 3.39 (1) that the condition that the support is compact has no impact. Thus for compact smooth manifolds Rham cohomology with compact support equals the usual de Rham cohomology. □

In the following we will explore de Rham cohomology with compact support. In particular we want to see whether our earlier results on ordinary de Rham cohomology have analogs in the new setting. In particular we want to do the following:

- We want to study functoriality of de Rham cohomology with compact support.
- We want to determine de Rham cohomology with compact support in degree 0.
- We want to determine de Rham cohomology with compact support of $\mathbb{R}^n$.
- We want to prove a Mayer–Vietoris style theorem for de Rham cohomology with compact support.

83.2. **Functoriality of de Rham cohomology with compact support.** In the following let $f\colon M \to N$ be a smooth map between two smooth manifolds and let ω be a smooth differential form on N. In the Pullback Smooth Form Proposition 74.7 we saw that the pullback $f^*(\omega)$ is again smooth. As we pointed out in the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 78.5, it follows from this fact and the Differential-Natural Proposition 75.8 that the following map is well-defined:

$$\begin{array}{rcl} f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) & \to & \mathrm{H}^k_{\mathrm{dR}}(M) \\ [\omega] & \mapsto & [f^*(\omega)]. \end{array}$$

This approach to defining an induced map on de Rham cohomology with compact support *fails* since given a smooth form ω on N with compact support there is no reason why $f^*(\omega)$ should have compact support. To get at least some degree of functoriality it is worth recalling Definition 4.63:

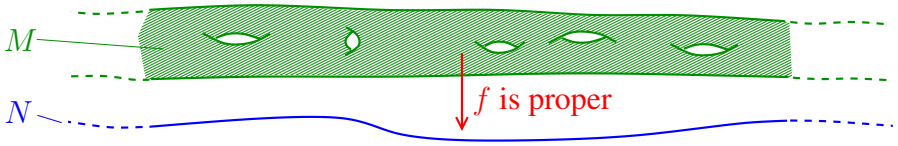
Definition 83.3. A continuous map $f\colon X \to Y$ between topological spaces is called proper if the preimage of every compact subset of Y is a compact subset of X.[845]

[844] By the Compact–Union-Intersection Lemma 3.41 we know that the union of two compact sets is again compact. Using this fact one can easily show that the forms with compact support do indeed form a real vector space.

Example.

(1) Let X be a topological space and let $A \subseteq X$. If A is closed, then it follows from the Compact-Closed Lemma 3.39 (1) that the inclusion $A \to X$ is proper.

(2) The map $f\colon M \to N$ shown in the figure below is proper.

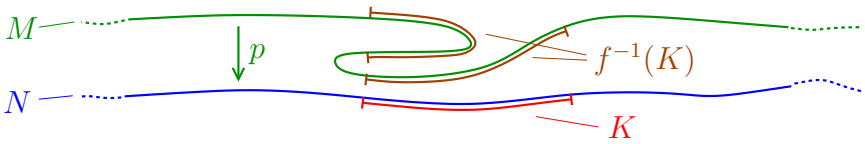


(3) It is clear that any continuous map from a non-compact space to a compact space is not proper. □

The following lemma explains our interest in proper maps:

Lemma 83.4. (Compact Support-Pullback Lemma) Let $f\colon M \to N$ be a smooth map between smooth manifolds. If f is proper, then for every differential form ω on N with compact support the pullback $f^*(\omega)$ also has compact support.

Proof. Let $K \subseteq N$ be the support of ω. Evidently $f^*(\omega)$ vanishes outside of $f^{-1}(K)$.



Since f is proper we know that $f^{-1}(K)$ is a compact subspace of M. By definition of the support we know that the support of $f^*(\omega)$ is a closed subset of $f^{-1}(K)$. It follows from the Compact-Closed Lemma 3.39 (1) that the support of $f^*(\omega)$ is indeed compact. ■

This discussion motivates the following definition:

Definition 83.5. We refer to the category $\mathcal{SmMfdPropMap}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfdPropMap}) &:= \text{all smooth manifolds,} \\ \mathrm{Mor}(M,N) &:= \text{all proper smooth maps from } M \text{ to } N, \end{aligned}$$

with the usual composition of maps as the category of smooth manifolds with proper smooth morphisms. Note that this is a subcategory of the category $\mathcal{SmMfd}$ of smooth manifolds.

The Compact Support-Pullback Lemma 83.4 allows us to formulate the following analog of the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 78.5:

Proposition 83.6. ($\mathrm{H}^n_{\mathrm{dR,c}}$-Functor Proposition)

(1) If $f\colon M \to N$ is a proper smooth map between two smooth manifolds, then for every $k \in \mathbb{N}_0$ the map

[845] In Definition 30.7 we defined what it means for a continuous map between two topological *manifolds* to be proper. As we explain in the Manifold–Proper Map Lemma 30.8, these two notions of "proper" coincide in many settings. In this and the following chapters we will use the above notion of "proper", and not the notion which we introduced specifically for continuous maps between topological manifolds. We apologize for any confusion that might arise.

$$\begin{array}{rcl} f^*\colon \mathrm{H}^k_{\mathrm{dR,c}}(N) & \to & \mathrm{H}^k_{\mathrm{dR,c}}(M) \\ [\omega] & \mapsto & [f^*(\omega)] \end{array}$$

is a well-defined homomorphism of $\mathbb{R}$-vector spaces.

(2) For every $k \in \mathbb{N}_0$ the maps[846]

$$\begin{array}{rcl} M & \mapsto & \mathrm{H}^k_{\mathrm{dR,c}}(M) \\ (f\colon M \to N) & \mapsto & (f^*\colon \mathrm{H}^k_{\mathrm{dR,c}}(N) \to \mathrm{H}^k_{\mathrm{dR,c}}(M)) \end{array}$$

define a contravariant functor from the above category $\mathcal{SmMfdPropMap}$ of smooth manifolds with proper smooth morphisms to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

(3) Diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups with compact support.

Proof. Statement (1) follows again from the Compact Support-Pullback Lemma 83.4 together with the fact, shown in the Differential-Natural Proposition 75.8, that the differential commutes with pulling back differential forms. The other statements follow easily from the definitions. ■

83.3. **Calculations of de Rham cohomology with compact support.** As we mentioned on page 1671, if M is a compact smooth manifold, then de Rham cohomology with compact support equals the usual de Rham cohomology. We will now see that for non-compact smooth manifolds there are big differences. For example in the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (2) we saw that for every connected non-empty smooth manifold M we have $\mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. Now the behavior is quite different:

Proposition 83.7. (Non-Compact–$\mathrm{H}^0_{\mathrm{dR,c}}$-Proposition) Let M be a connected smooth manifold M. If M is non-compact, then $\mathrm{H}^0_{\mathrm{dR,c}}(M) = 0$.

Proof. For any smooth manifold M we have

$$\mathrm{H}^0_{\mathrm{dR,c}}(M) \underset{\substack{\uparrow \\ \text{since } \Omega^{-1}_{\mathrm{c}}(M)=0}}{=} \{f \in \Omega^0_{\mathrm{c}}(M) \,|\, df = 0\} \underset{\substack{\uparrow \\ \text{see the proof of the} \\ \mathrm{H}^0_{\mathrm{dR}}\text{-Proposition 78.8}}}{=} \begin{array}{c}\text{all smooth maps } f\colon M \to \mathbb{R} \text{ that are locally} \\ \text{constant and that have compact support.}\end{array}$$

Now assume that M is non-compact and connected. In this case the zero function is the only locally constant function with compact support. Thus we have shown the desired equality $\mathrm{H}^0_{\mathrm{dR,c}}(M) = 0$. ■

Remark. In the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 (2) we saw that homotopy equivalent smooth manifolds have isomorphic de Rham cohomology groups. Now we already know that this statement does *not* hold for de Rham cohomology groups with compact support. Indeed, the smooth manifolds $M = \{0\}$ and $N = \mathbb{R}$ are homotopy equivalent. Since $M = \{0\}$ is compact we obtain from the above observation and the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (2) that $\mathrm{H}^0_{\mathrm{dR,c}}(M) \cong \mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. On the other hand, $N = \mathbb{R}$ is non-compact, thus we obtain from the Non-Compact–$\mathrm{H}^0_{\mathrm{dR}}$-Proposition 83.7 that $\mathrm{H}^0_{\mathrm{dR,c}}(N) = 0$. □

[846] Recall that for usual de Rham cohomology we extended in the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 functoriality from smooth maps to continuous maps. By working with "proper homotopies" one could do so also for de Rham cohomology with compact support. Since we will not need this extension, we will not work out the details.

We continue with the following analog of the High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.9:

Lemma 83.8. (High-Dimensional-$\mathrm{H}^*_{\mathrm{dR,c}}$-Lemma) Let M be a k-dimensional smooth manifold.

(1) (a) For $i > k$ we have $\Omega^i_{\mathrm{c}}(M) = 0$.
(b) For $i > k$ we have $\mathrm{H}^i_{\mathrm{dR,c}}(M) = 0$.
(2) For every $\omega \in \Omega^k(M)$ we have $d\omega = 0$, i.e. every $\omega \in \Omega^k(M)$ with compact support represents a class in $\mathrm{H}^k_{\mathrm{dR,c}}(M)$.

Proof. The proof of this lemma is basically the same as the (very elementary) proof of the corresponding High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.9. ∎

We can now give our first complete calculation of de Rham cohomology with compact support of a non-compact smooth manifold:

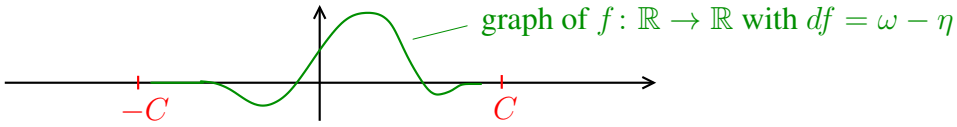
Proposition 83.9. ($\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R})$-Proposition)

(1) For $i \neq 1$ we have $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}) = 0$.
(2) The map
$$\begin{array}{ccc} \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int\limits_{\mathbb{R}} \omega \end{array}$$
is well-defined and it is an isomorphism.

Proof. By the Non-Compact–$\mathrm{H}^0_{\mathrm{dR}}$-Proposition 83.7 we know that $\mathrm{H}^0_{\mathrm{dR,c}}(\mathbb{R}) = 0$ and by the High-Dimensional-$\mathrm{H}^*_{\mathrm{dR,c}}$-Lemma 83.8 we know that $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}) = 0$ for $i > 1$. Thus it remains to prove Statement (2).

First recall that in Definition 74.13 we defined the integral of any smooth form *with compact support* on a smooth manifold. Thus it makes sense to consider the integral of any $\omega \in \Omega^1_{\mathrm{c}}(\mathbb{R})$ over $\mathbb{R}$.

- First we show that the map $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) \to \mathbb{R}$ is well-defined. Thus suppose that we have $\omega, \eta \in \Omega^1_{\mathrm{c}}(\mathbb{R})$ that represent the same element of $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R})$. This means that there exists a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ *with compact support* such that $df = \omega - \eta$. We pick a $C \geq 0$ such that $\mathrm{supp}(f) \subseteq [-C, C]$.



Now we see that

$$\int_{\mathbb{R}} \omega - \int_{\mathbb{R}} \eta \;=\; \int_{\mathbb{R}} (\omega - \eta) \;=\; \int_{\mathbb{R}} df \;\overset{\uparrow}{=}\; \int_{t=-C}^{t=C} f'(t)\,dt \;\overset{\downarrow}{=}\; f(C) - f(-C) \;=\; 0 - 0 \;=\; 0.$$

follows from the definition in the Integral-of-Form-in-Chart Proposition 74.12 and the Riemann-Lebesgue Integral Proposition 73.23

follows from the Fundamental Theorem of Calculus 73.27

This shows that the given map $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) \to \mathbb{R}$ is actually well-defined.

- It remains to show that the map $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) \to \mathbb{R}$ is an isomorphism. First note that the map is clearly a homomorphism. Next let us show that the map is an *epimorphism*. We only need to show that the image is non-zero. By the Smooth Transition Function

Lemma 24.9 there exists a smooth map $\varphi\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ with compact support and with $\varphi(0) > 0$. We now consider the 1-form $\varphi \cdot dx$. It follows easily from the Lebesgue Integral Proposition 73.22 (1) that $\int_{\mathbb{R}} \varphi \cdot dx > 0$.

Finally we need to show that the map $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) \to \mathbb{R}$ is a *monomorphism*. Now suppose that we are given $\omega \in \Omega^1_{\mathrm{c}}(\mathbb{R})$ with $\int_{\mathbb{R}} \omega = 0$. We can write $\omega = \varphi \cdot dx$ where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a smooth map. Since the 1-form ω has compact support there exists a $C \in \mathbb{R}_{\geq 0}$ such that $\mathrm{supp}(\varphi) \subseteq [-C, C]$. Next we consider the map

$$\begin{array}{rccc} f\colon \mathbb{R} & \to & \mathbb{R} \\ s & \mapsto & \int_{t=-C}^{t=s} \varphi(t)\, dt. \end{array}$$

It follows easily from the fact that $\int_{\mathbb{R}} \omega = 0$ that the support of f is contained in $[-C, C]$, in particular f has compact support. Finally note that we have the following equality of 1-forms on the smooth manifold $\mathbb{R}$:

$$df \underset{\uparrow}{=} \frac{\partial f}{\partial x} \cdot dx = \frac{\partial}{\partial x} \int_{t=-C}^{t=s} \varphi(t)\, dt \cdot dx \underset{\uparrow}{=} \varphi \cdot dx = \omega.$$

by the 1-Form-via-dx^i Lemma 71.5

by the Fundamental Theorem of Calculus 73.27

We have thus shown that $df = \omega$. This implies that $[\omega] = 0 \in \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R})$. ∎

In the following we will generalize some of the tricks that we used in the proof of the $\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R})$-Proposition 83.9.

Proposition 83.10. (Top Dimension-$\mathrm{H}^n_{\mathrm{dR,c}}$-Integration Proposition) Let M be an oriented n-dimensional smooth manifold with empty boundary.

(1) For any $\eta \in \Omega^{n-1}_{\mathrm{c}}(M)$ we have $\int_M d\eta = 0$.

(2) The map

$$\begin{array}{rccc} & \mathrm{H}^n_{\mathrm{dR,c}}(M) & \to & \mathbb{R} \\ & [\omega] & \mapsto & \int_M \omega \end{array}$$

is well-defined and it is an epimorphism if M is non-empty.

Proof.

(1) Let $\eta \in \Omega^{n-1}_{\mathrm{c}}(M)$. Since the support of η is compact we obtain from the Compact Submanifold Exhaustion Proposition 38.2 together with the Compact Subspace Covering Lemma 3.38 that there exists a compact smooth submanifold $W \subseteq M$ of codimension zero such that $\mathrm{supp}(\eta) \subseteq W \setminus \partial W$. We denote by $i\colon W \to M$ the inclusion. Then we see that

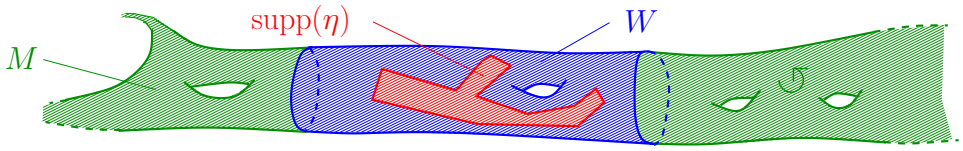
$$\int_M d\eta \underset{\uparrow}{=} \int_W i^* d\eta \overset{\downarrow}{=} \int_W d(i^*\eta) \underset{\uparrow}{=} \int_{\partial W} i^*\eta \underset{\uparrow}{=} 0.$$

this follows easily from $\mathrm{supp}(\omega) \subseteq W \setminus \partial W$

by the Differential-Natural Proposition 75.8

follows from Stokes' Theorem 75.11

since $i^*\eta = 0$ on ∂W



(2) It follows from (1) that the given map $\mathrm{H}^n_{\mathrm{dR,c}}(M) \to \mathbb{R}$ is well-defined. It is clear that the map is a homomorphism. Finally basically the same argument as in the proof of the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 shows that if M is non-empty then there exists a smooth n-form ω on M with compact support such that $\int_M \omega \neq 0$. ■

We move on to a subtle definition, which gives us the rather shocking fact, that de Rham cohomology with compact support is *covariantly* functorial under inclusions of open subsets:

Definition 83.11. Let M be a smooth manifold and let $U \subseteq M$ be an open subset.

(1) Given a smooth k-form $\omega \in \Omega^k_{\mathrm{c}}(U)$ with compact support we denote by $\widetilde{\omega}$ the k-form on M which is given by extending ω by zero outside of U. We refer to $\widetilde{\omega}$ as the extension of ω to M. It follows from the hypotheses that $U \subseteq M$ is open and that ω has compact support, together with the Extension-of-Form–Smooth Lemma 74.11, that $\widetilde{\omega}$ is a smooth k-form.

(2) We denote by $i\colon U \to M$ the inclusion. It follows easily from the definitions and the Compact Image Lemma 4.15 that the map

$$\begin{array}{rcl} i_*\colon \mathrm{H}^k_{\mathrm{dR,c}}(U) & \to & \mathrm{H}^k_{\mathrm{dR,c}}(M) \\ [\omega] & \mapsto & [\widetilde{\omega}] \end{array}$$

is well-defined. Note that this makes de Rham cohomology with compact support *covariantly* functorial under inclusions.

We continue with a generalization of the $\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R})$-Proposition 83.9. In the proof of this generalization we will make use of the definitions that we just introduced.

Proposition 83.12. ($\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R}^n)$-Proposition) Let $n \in \mathbb{N}$.

(1) For $i \neq n$ we have $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$.

(2) The map

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR,c}}(\mathbb{R}^n) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int_{\mathbb{R}^n} \omega \end{array}$$

from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR,c}}$-Integration Proposition 83.10 is an isomorphism.

Remark. We will provide an indirect approach to proving the $\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R}^n)$-Proposition 83.12, avoiding almost all analysis. A more direct analytic proof is given in [**BoT1982**, p. 38]. □

Proof. We start out with the following observations:

(a) It follows from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR,c}}$-Integration Proposition 83.10 that the map given in (2) is an epimorphism.

(b) We consider S^n and the North Pole $N := (0, \ldots, 0, 1) \in S^n$. By definition of the smooth structure on S^n we know that the stereographic projection $\varphi\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a diffeomorphism. It follows from the $\mathrm{H}^n_{\mathrm{dR,c}}$-Functor Proposition 83.6 (3) that φ^* induces an isomorphism on de Rham cohomology with compact support.

(c) It follows from the Non-Compact–$\mathrm{H}^0_{\mathrm{dR}}$-Proposition 83.7 that $\mathrm{H}^0_{\mathrm{dR,c}}(S^n \setminus \{N\}) = 0$.

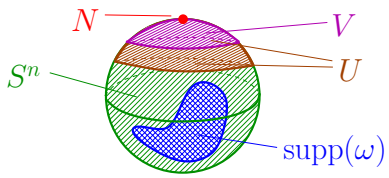
(d) By the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we know that $\mathrm{H}^k_{\mathrm{dR}}(S^n) = 0$ for $k \neq 0, n$ and we know that $\mathrm{H}^n_{\mathrm{dR}}(S^n) \cong \mathbb{R}$.
(e) Since S^n is compact we see that $\mathrm{H}^*_{\mathrm{dR,c}}(S^n) = \mathrm{H}^n_{\mathrm{dR}}(S^n)$.
(f) We denote by $i\colon S^n \setminus \{N\} \to S^n$ the inclusion. Since $S^n \setminus \{N\}$ is an open subset of S^n we obtain from the above discussion, preceding the proposition, an induced map $i_*\colon \mathrm{H}^k_{\mathrm{dR,c}}(S^n \setminus \{N\}) \to \mathrm{H}^k_{\mathrm{dR,c}}(S^n)$.

It follows easily from these facts that it suffices to prove the following claim:

Claim. For every $k \in \mathbb{N}$ the map $i_*\colon \mathrm{H}^k_{\mathrm{dR,c}}(S^n \setminus \{N\}) \to \mathrm{H}^k_{\mathrm{dR,c}}(S^n)$ is a monomorphism.

Proof. Let ω be a smooth k-form on $S^n \setminus \{N\}$ with compact support. As above we consider the extension $\widetilde{\omega}$ to S^n. We now assume that $i_*([\omega]) = [\widetilde{\omega}] = 0 \in \mathrm{H}^k_{\mathrm{dR,c}}(S^n)$. This means that there exists a smooth $(k-1)$-form η on S^n such that $\widetilde{\omega} = d\eta$. Since $\mathrm{supp}(\omega) \subseteq S^n \setminus \{N\}$ and since $\mathrm{supp}(\omega)$ is compact we see that there exists an open contractible neighborhood U of $N \in S^n$ with $U \cap \mathrm{supp}(\omega) = \varnothing$.

Next note that $d\eta|_U = \omega|_U = 0$. In other words, we see that $\eta|_U$ is closed. Since $k \in \mathbb{N}$ and since U is contractible we obtain from the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 that there exists a smooth $(k-2)$-form μ on U such that $d\mu = \eta|_U$. Using the Smooth Transition Function Lemma 24.9 we can find a smooth map $\rho\colon U \to [0,1]$ with compact support and such that $\rho \equiv 1$ in a neighborhood V of N. Thus $\rho \cdot \mu$ defines a smooth $(k-2)$-form on U with compact support and which agrees with μ in the neighborhood V of $N \subseteq U$.



Note that it follows again from the Extension-of-Form–Smooth Lemma 74.11 that we can extend $\rho \cdot \mu$ to a smooth $(k-2)$-form $\widetilde{\rho \cdot \mu}$ on S^n. Finally we consider the following smooth $(k-1)$-form on S^n:

$$\xi \;:=\; \eta - d(\widetilde{\rho \cdot \mu}).$$

We are done once we have shown that $d\xi = \omega$ and that $\xi|_{S^n \setminus \{N\}}$ has compact support.

(a) First note that

$$d\xi \;=\; d(\eta - d(\widetilde{\rho \cdot \mu})) \;=\; d\eta - dd(\widetilde{\rho \cdot \mu}) \;=\; d\eta \;=\; \omega.$$

(b) Note that ξ vanishes on the neighborhood V of N. Next note that this implies that $\mathrm{supp}(\xi) \subseteq S^n \setminus V \subseteq S^n \setminus \{N\}$. Thus $\xi|_{S^n \setminus \{N\}}$ has compact support. ∎

83.4. Mayer–Vietoris sequence for de Rham cohomology with compact support.

In this section we will formulate and prove a Mayer–Vietoris sequence for de Rham cohomology with compact support. The casual reader might think that it looks the same as the sequence given in the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2. But this is not the case, since the induced maps go *the other direction.*

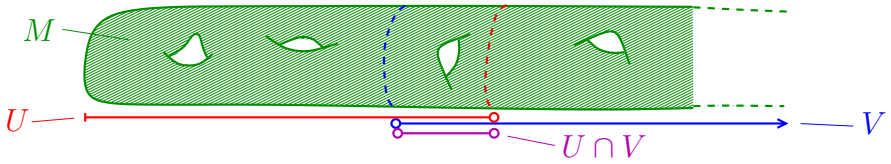
Theorem 83.13. (Mayer–Vietoris Theorem) Let M be a smooth manifold and let U, V be open subsets of M with $M = U \cup V$. Let $j_U \colon U \to M$, $j_V \colon V \to M$, $i_U \colon U \cap V \to U$ and $i_V \colon U \cap V \to V$ be the inclusion maps. Then the following two statements hold:

(1) The following sequence of cochain complexes is exact[847]

$$0 \leftarrow \Omega^*_c(M) \xleftarrow{j_{U*} \oplus -j_{V*}} \Omega^*_c(U) \oplus \Omega^*_c(V) \xleftarrow{i_{U*} + i_{V*}} \Omega^*_c(U \cap V) \leftarrow 0.$$

(2) The connecting homomorphisms $\delta_k \colon \mathrm{H}^k_{\mathrm{dR,c}}(M) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR,c}}(U \cap V)$ that arise from (1) together with the Cohomology–LES Proposition 77.13 are natural and they give rise to an exact sequence of the form

$$\leftarrow \mathrm{H}^k_{\mathrm{dR,c}}(M) \xleftarrow{j_{U*} \oplus -j_{V*}} \mathrm{H}^k_{\mathrm{dR,c}}(U) \oplus \mathrm{H}^k_{\mathrm{dR,c}}(V) \xleftarrow{i_{U*} + i_{V*}} \mathrm{H}^k_{\mathrm{dR,c}}(U \cap V) \xleftarrow{\delta_{k-1}} \mathrm{H}^{k-1}_{\mathrm{dR,c}}(M) \leftarrow$$

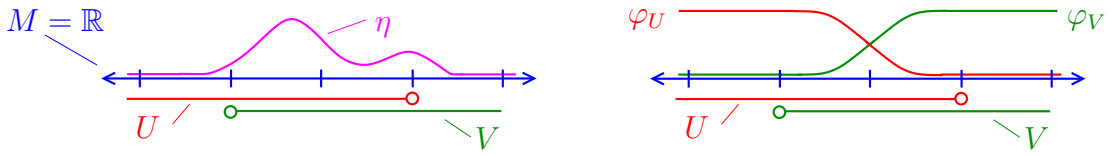


Proof of the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR,c}}$-Theorem 83.13. It turns out that the proof of the present Mayer–Vietoris Theorem is even easier than the proof of the earlier Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2:

(1) Recall that we need to prove that the following sequence of cochain complexes is exact

$$0 \leftarrow \Omega^*_c(M) \xleftarrow{j_{U*} \oplus -j_{V*}} \Omega^*_c(U) \oplus \Omega^*_c(V) \xleftarrow{i_{U*} + i_{V*}} \Omega^*_c(U \cap V) \leftarrow 0.$$

The only statement which requires at least a little bit of thought is that the map $j_{U*} \oplus -j_{V*} \colon \Omega^*_c(U) \oplus \Omega^*_c(V) \to \Omega^*_c(M)$ is an epimorphism. Thus let $\eta \in \Omega^*_c(M)$. Since U, V form an open cover of M we obtain from the Manifold–Partitions-of-Unity Theorem 27.2 that there exists a smooth partition of unity $\varphi_U, \varphi_V \colon M \to [0,1]$ such that $\mathrm{supp}(\varphi_U) \subseteq U$ and $\mathrm{supp}(\varphi_V) \subseteq V$. Then $\varphi_U \cdot \eta$ and $-\varphi_V \cdot \eta$ define smooth forms in U and V with compact support and these get mapped to η under the given map.



(2) It follows immediately from Statement (1) together with the Cohomology–LES Proposition 77.13 that an open cover $\{U, V\}$ of a smooth manifold M gives rise to natural connecting homomorphisms $\delta_k \colon \mathrm{H}^k_{\mathrm{dR,c}}(M) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR,c}}(U \cap V)$ which in turn give rise to the exact sequence of Statement (2). ∎

83.5. Exercises for Chapter 83.

Exercise 83.1. For every $k \in \mathbb{N}_0$ provide an example of a smooth map $f \colon M \to N$ between two smooth manifolds and a smooth k-form $\omega \in \Omega^k_c(N)$ such that $f^*(\omega)$ does not have compact support.

[847] Note that here we use again that (as we discussed in Definition 83.11) forms with compact support and de Rham cohomology with compact support are *covariantly* functorial under inclusions of open subsets.

Exercise 83.2. Let M be a smooth manifold and let $\{M_s\}_{s\in S}$ be the path-components of M. Given $s \in S$ we denote by $i_s\colon M_s \to M$ the inclusion map. Finally let $k \in \mathbb{N}_0$. What is the relationship between $\mathrm{H}^k_{\mathrm{dR,c}}(M)$ and $\mathrm{H}^k_{\mathrm{dR,c}}(M_i)$?
Remark. The goal of this exercise is to figure out what the correct analog of the Path-Component-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.7 might be.

Exercise 83.3. Let $X \subseteq \mathbb{R}^k$ be a finite set with m elements. Determine $\mathrm{H}^j_{\mathrm{dR,c}}(\mathbb{R}^k \setminus X)$.

Exercise 83.4. We consider the torus $T = S^1 \times S^1$ and a point $P \in T$. Determine $\mathrm{H}^j_{\mathrm{dR,c}}(T \setminus \{P\})$.

Exercise 83.5. Show that de Rham cohomology with compact support is not contravariantly functorial. More precisely show that, given $k \in \mathbb{N}_0$, there is no possibility to assign to each smooth map $f\colon M \to N$ a homomorphism $f^*\colon \mathrm{H}^k_{\mathrm{dr,c}}(N) \to \mathrm{H}^k_{\mathrm{dr,c}}(M)$ such that the de Rham cohomology groups in degree k and these homomorphisms define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds (where we only allow smooth maps) to the category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces.
Remark. On page 1671 we pointed out that the obvious attempt at defining an induced map fails. But a priori it could be that a more clever idea works.

Exercise 83.6. Show that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k_{\mathrm{dR,c}}(S^n \times \mathbb{R}) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Hint. Consider the decomposition $S^{n+1} = U_+ \cup U_-$ where $U_\pm = S^{n+1} \setminus \{(0,\ldots,0,\pm 1)\}$.[848]

Exercise 83.7. Given any smooth manifold M and given any $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dr,c}}(M) & \to & \mathrm{H}^k_{\mathrm{dR}}(M) \\ [\omega] & \mapsto & [\omega] \end{array}$$

is clearly well-defined. Provide an example where this map is neither injective nor surjective.

[848]Note that this hint is not a typo.

CHAPTER 84

The Poincaré Duality Theorem for de Rham cohomology

In this chapter we will prove the De Rham-Poincaré Duality Theorem 84.5. This theorem is easily one of the most interesting theorems in the theory of de Rham cohomology group. As we will see in this and the following chapters, this theorem has a sheer endless list of applications and consequences.

This chapter is organized as follows:

- In Section 84.1 we will recall a few elementary concepts from linear algebra which we will need throughout this chapter.
- In Section 84.2 we will state the De Rham-Poincaré Duality Theorem 84.5 and we will discuss a few straightforward consequences.
- In Section 84.3 we will provide the (surprisingly short) proof of the De Rham-Poincaré Duality Theorem 84.5.
- In Section 84.4 we will provide exercises for Chapter 84.

In the next Chapter 85 we will use the De Rham-Poincaré Duality Theorem 84.5 to perform several non-trivial calculations.

84.1. **Non-singular pairings.** In this short section we will recall and introduce some elementary definitions from linear algebra. First we recall Definition 2.24 and Definition 2.25:

Definition 84.1. Let R be a commutative ring and let V and W be R-modules.

(1) A pairing on $V \times W$ is a map

$$\varphi\colon V \times W \to R$$

such that for all $v, v', w, w' \in W$ we have

$$\begin{aligned} \varphi(v+v', w) &= \varphi(v,w) + \varphi(v',w), \quad \text{and} \\ \varphi(v, w+w') &= \varphi(v,w) + \varphi(v,w'). \end{aligned}$$

(2) A pairing is called R-bilinear if for all $\lambda \in R$, all $v \in V$ and all $w \in W$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v,w)$ and $\varphi(v, \lambda \cdot w) = \lambda \cdot \varphi(v,w)$.

(3) We say that a pairing $\varphi\colon V \times V \to S$ is symmetric if for all $v, w \in V$ we have $\varphi(v,w) = \varphi(w,v)$.

We add one definition:

Definition 84.2. Let R be a commutative ring. An R-bilinear pairing $\langle\ ,\ \rangle\colon V \times W \to R$ is called non-singular if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,R) \\ v & \mapsto & \begin{pmatrix} W & \to & R \\ w & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,R) \\ w & \mapsto & \begin{pmatrix} V & \to & R \\ v & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array}$$

are isomorphisms. Otherwise we say that the R-bilinear pairing is singular.

We will soon need the following elementary lemma from linear algebra:

Lemma 84.3. (Non Singular Form-over-Field Lemma) Let $\mathbb{F}$ be a field and let

$$\langle\ ,\ \rangle\colon V \times W \ \to\ \mathbb{F}$$

be an $\mathbb{F}$-bilinear pairing. We assume that W is finite-dimensional and that the above map

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_{\mathbb{F}}(W,\mathbb{F}) \\ v & \mapsto & \left(\begin{array}{ccc} W & \to & \mathbb{F} \\ w & \mapsto & \langle v,w\rangle \end{array}\right) \end{array}$$

is an isomorphism. Then the following statements hold:

(1) We have $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(W)$.
(2) The pairing is non-singular, i.e. the above map $W \to \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ is also an isomorphism.

Proof.

(1) By hypothesis we have an isomorphism $V \to \mathrm{Hom}_{\mathbb{F}}(W,\mathbb{F})$. Since W is finite-dimensional it follows from elementary linear algebra that $\mathrm{Hom}_{\mathbb{F}}(W,\mathbb{F})$ is isomorphic to W. In summary it follows that $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(W,\mathbb{F})) = \dim_{\mathbb{F}}(W)$.
(2) Since $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})) \in \mathbb{N}_0$ it suffices to show that the given map $W \to \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ is a monomorphism. So suppose that we have $w \in W \setminus \{0\}$. We proceed with a few elementary steps:
   (a) It follows from the fact that any vector space has a basis that there exists an $\mathbb{F}$-linear homomorphism $\varphi\colon W \to \mathbb{F}$ with $\varphi(w) \neq 0$.
   (b) Since the natural map $V \to \mathrm{Hom}_{\mathbb{F}}(W,\mathbb{F})$ is (by hypothesis) an isomorphism we see that there exists a $v \in V$ such that $\varphi(w') = \langle v, w'\rangle$ for all $w' \in V$.
   (c) It follows from (b) that $\langle v, w\rangle = \varphi(w) \neq 0$.
   (d) Note that it follows from (c) that the homomorphism in $\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ that is given by $v \mapsto \langle v, w\rangle$ is non-zero. We have thus shown that the map $W \to \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ is a monomorphism. ■

84.2. **The Poincaré Duality Theorem.** Before we can state the Poincaré Duality Theorem we need to make one very last preparation, namely we need to prove the following lemma:

Lemma 84.4. ($\mathrm{H}^*_{\mathrm{dR}}$-Pairing Lemma) Let M be an oriented n-dimensional smooth manifold with empty boundary. Given any $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \displaystyle\int_M \omega \wedge \eta \end{array}$$

is a well-defined $\mathbb{R}$-bilinear pairing.

Proof. We start out with the following claim:

Claim. If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{n-k}_{\mathrm{c}}(M)$, then $\omega \wedge \eta$ also has compact support.

Proof. It follows immediately from the definitions that $\mathrm{supp}(\omega \wedge \eta)$ is a closed subset of the compact subset $\mathrm{supp}(\eta)$. It thus follows from the Compact-Closed Lemma 3.39 (1) that $\mathrm{supp}(\omega \wedge \eta)$ is compact. ⊞

It follows from this claim, the hypothesis that M has empty boundary and the Top Dimension-$\mathrm{H}^n_{\mathrm{dR,c}}$-Integration Proposition 83.10 that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \int\limits_M \omega \wedge \eta \end{array}$$

is well-defined. Finally note that it is clear that this map is $\mathbb{R}$-bilinear. ■

Finally we are ready to formulate the main result of this part of the lecture notes:

Theorem 84.5. (De Rham-Poincaré Duality Theorem) Let M be an oriented n-dimensional smooth manifold.

(1) If M admits a finite smoothly good open cover (in the sense of Definition 81.1), then for any $k \in \mathbb{N}_0$ the following $\mathbb{R}$-bilinear pairing is non-singular:

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \int\limits_M \omega \wedge \eta. \end{array}$$

(2) If M is closed, then for any $k \in \mathbb{N}_0$ the following $\mathbb{R}$-bilinear pairing is non-singular:

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \int\limits_M \omega \wedge \eta. \end{array}$$

We will provide the proof of the De Rham-Poincaré Duality Theorem 84.5 in the next Section 84.3. In the reminder of this section we will consider a few examples and consequences of the De Rham-Poincaré Duality Theorem 84.5.

Example. We consider $M = \mathbb{R}^n$. Evidently $M = \mathbb{R}^n$ has a smoothly good cover consisting of one subset. Using our earlier calculations we can verify the statement of the De Rham-Poincaré Duality Theorem 84.5 in this context:

(1) For $k \neq 0$ we know by the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9 and the $\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R}^n)$-Proposition 83.12 that $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ and $\mathrm{H}^{n-k}_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$. Thus the desired statement holds trivially.

(2) For $k = 0$ it follows from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (2) and the $\mathrm{H}^*_{\mathrm{dR,c}}(\mathbb{R}^n)$-Proposition 83.12 that

$$\begin{array}{rcl} \mathbb{R} & \to & \mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}^n) \\ r & \mapsto & [\text{constant function} = r] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR,c}}(\mathbb{R}^n) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int\limits_{\mathbb{R}^n} \omega \end{array}$$

are both isomorphisms. Using these isomorphisms one can easily verify that the statement of the De Rham-Poincaré Duality Theorem 84.5 holds also for $M = \mathbb{R}^n$ with $k = n$. □

Remark.

(1) The proof of the De Rham-Poincaré Duality Theorem 84.5 is fairly topological in natural and it is close in spirit to our proof of the later Poincaré Duality Theorem 210.1 for singular (co-) homology and compact oriented topological manifolds. There is a different, very elegant, approach to proving the De Rham Poincaré Duality Theorem 84.5,

via the Hodge Theorem. This proof requires much more analysis though. We refer to [**War1983**, Chapter 6] for details.

(2) Note that the conclusion of the De Rham-Poincaré Duality Theorem 84.5 holds in fact for any oriented smooth manifold: It is not necessary to assume that the smooth manifold has a finite smoothly good open cover. We refer to [**Mic2008**, Theorem 12.16] for details. □

Before we turn to the proof of the De Rham-Poincaré Duality Theorem 84.5, let us state two almost immediate corollaries:

Corollary 84.6. (De Rham Cohomology Symmetry Corollary) For any closed orientable n-dimensional smooth manifold M and any $k \in \mathbb{N}_0$ we have[849]

$$\dim_{\mathbb{R}}(\mathrm{H}^k_{\mathrm{dR}}(M)) \;=\; \dim_{\mathbb{R}}(\mathrm{H}^{n-k}_{\mathrm{dR}}(M)).$$

Proof. Let M be a closed orientable n-dimensional smooth manifold and let $k \in \mathbb{N}_0$. We pick an orientation for M. The De Rham-Poincaré Duality Theorem 84.5 (2) says that the pairing

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \int\limits_M \omega \wedge \eta \end{array}$$

is non-singular. Recall that by the De Rham Cohomology-Finiteness Proposition 81.10 we know that $\mathrm{H}^k_{\mathrm{dR}}(M)$ is finite-dimensional. Thus the promised statement follows from the Non Singular Form-over-Field Lemma 84.3 (1). ■

Conclude this section with another interesting consequence:

Proposition 84.7. (Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition) If M is a closed oriented[850] connected non-empty n-dimensional smooth manifold, then the homomorphism

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\eta] & \mapsto & \int\limits_M \eta \end{array}$$

from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 is an isomorphism.

Proof. Since M is a closed oriented smooth manifold we obtain from the De Rham-Poincaré Duality Theorem 84.5 (2) that the map

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR}}(M) & \to & \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^0_{\mathrm{dR}}(M),\mathbb{R}) \\ [\eta] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}^0_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int\limits_M \omega \wedge \eta \end{array} \right) \end{array}$$

is an isomorphism. Since M is connected and non-empty we furthermore know by the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 (2) that $\mathrm{H}^0_{\mathrm{dR}}(M) = \mathbb{R} \cdot [\text{constant} = 1 \text{ function}]$. The proposition follows from these two observations. ■

The Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 allows us to introduce the following definition:

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[849] Of course this implies that $\mathrm{H}^k_{\mathrm{dR}}(M)$ and $\mathrm{H}^{n-k}_{\mathrm{dR}}(M)$ are isomorphic. We formulated the corollary in terms of dimensions to highlight the fact that there is no natural isomorphism between these de Rham cohomology groups.

[850] We will deal with the case of non-orientable smooth manifolds in the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 85.8.

Definition 84.8. Let M be a closed oriented connected non-empty n-dimensional smooth manifold. By the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 there exists a unique de Rham cohomology class $[M]^*_{\mathrm{dR}} \in \mathrm{H}^n_{\mathrm{dR}}(M)$ with

$$\int_M [M]^*_{\mathrm{dR}} \;=\; 1.$$

We refer to $[M]^*_{\mathrm{dR}}$ as the de Rham fundamental class of M. Note that it follows from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 that $\mathrm{H}^n_{\mathrm{dR}}(M) = \mathbb{R} \cdot [M]^*_{\mathrm{dR}}$.

84.3. **The proof of the De Rham-Poincaré Duality Theorem 84.5.** This section is only concerned with the proof of the De Rham-Poincaré Duality Theorem 84.5. In the next Chapter 85 we will use the De Rham-Poincaré Duality Theorem 84.5 to perform various calculations of de Rham cohomology groups.

We start out by introducing the following standard notation from linear algebra:

Notation 84.9. Let $\mathbb{F}$ be a field.

(1) Given an $\mathbb{F}$-vector space V we write $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$.

(2) Given an $\mathbb{F}$-homomorphism $\varphi\colon V \to W$ we denote by

$$\begin{array}{rcl} \varphi^*\colon W^* & \to & V^* \\ (f\colon W \to \mathbb{F}) & \mapsto & (f \circ \varphi\colon V \to \mathbb{F}) \end{array}$$

the induced map.

We continue with a notation which is very specific to our current purpose:

Notation 84.10. Let W be an oriented n-dimensional smooth manifold. We consider the homomorphism

$$\begin{array}{rcl} \Theta_W\colon \Omega^k(W) & \to & (\Omega^{n-k}_{\mathrm{c}}(W))^* \\ \alpha & \mapsto & \left(\begin{array}{rcl} \Omega^{n-k}_{\mathrm{c}}(W) & \to & \mathbb{R} \\ \beta & \mapsto & \int\limits_W \alpha \wedge \beta \end{array} \right). \end{array}$$

Note that it follows easily from the $\mathrm{H}^*_{\mathrm{dR}}$-Pairing Lemma 84.4 and the Product Rule from the Differentials-on-Manifolds Proposition 75.7 (3) that we get an induced homomorphism

$$\Theta_W\colon \mathrm{H}^i(\Omega^k(W)) \;\to\; \mathrm{H}^i((\Omega^{n-k}_{\mathrm{c}}(W))^*).$$

With this notation we can formulate the next two lemmas, which are key ingredients in the proof of the De Rham-Poincaré Duality Theorem 84.5.

Lemma 84.11. (Form-to-Compact Support Form-Lemma) Let Z be an oriented n-dimensional smooth manifold and let $Y \subseteq Z$ be an open subset. We denote by $j\colon Y \to Z$ the inclusion map.

(1) The following diagram commutes:

$$\begin{array}{ccc} \Omega^k(Z) & \xrightarrow{\quad j^* \quad} & \Omega^k(Y) \\ \downarrow{\scriptstyle \Theta_Z} & & \downarrow{\scriptstyle \Theta_Y} \\ \Omega^{n-k}_{\mathrm{c}}(Z)^* & \xrightarrow{\quad (j_*)^* \quad} & \Omega^{n-k}_{\mathrm{c}}(Y)^*. \end{array}$$

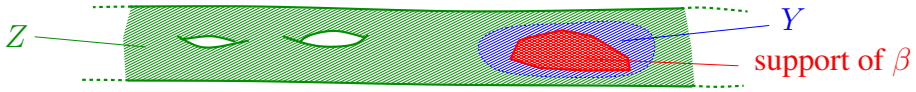
Here $j_*\colon \Omega_c^{n-k}(Y) \to \Omega_c^{n-k}(Z)$ is the inclusion induced homomorphism that we introduced in Definition 83.11.

(2) The following diagram commutes up to multiplication by $(-1)^{k+1}$:

$$\begin{array}{ccc} \Omega^k(Z) & \xrightarrow{\quad d_k \quad} & \Omega^{k+1}(Z) \\ \downarrow{\scriptstyle \Theta_Z} & & \downarrow{\scriptstyle \Theta_Z} \\ \Omega_c^{n-k}(Z)^* & \xrightarrow{\quad (d_{n-k-1})^* \quad} & \Omega_c^{n-k-1}(Z)^*. \end{array}$$

Proof.

(1) Let $\alpha \in \Omega^k(Z)$ and let $\beta \in \Omega_c^{n-k}(Y)$.



The following calculation shows that the given diagram does indeed commute:

$$\begin{array}{lcccl} & & \text{definition of } (j_*)^* & \text{definition of } \Theta_Z & \text{since } \alpha \wedge j_*(\beta) \text{ vanishes outside of } Y \\ & & \downarrow & \downarrow & \downarrow \\ ((j_*)^*(\Theta_Z(\alpha)))(\beta) & = & \Theta_Z(\alpha)(j_*(\beta)) & = \int_Z \alpha \wedge j_*(\beta) & = \int_Y j^*(\alpha \wedge j_*(\beta)) \\ & = & \int_Y j^*\alpha \wedge j^*(j_*(\beta)) & = \int_Y j^*\alpha \wedge \beta & = \Theta_Y(j^*\alpha)(\beta). \\ & & \uparrow & \uparrow & \uparrow \\ & & \text{follows from the Pullback Smooth Smooth Form Proposition 74.7 (1b)} & \text{since } j^*(j_*(\beta)) = \beta & \text{definition of } \Theta_Y \end{array}$$

(2) Let $\alpha \in \Omega^k(Z)$ and let $\beta \in \Omega_c^{n-k-1}(Z)$. The following calculation shows that the given diagram does indeed commute up to multiplication by $(-1)^{k+1}$:

$$\begin{array}{lcccl} & & \text{definition of } (d_{n-k-1})^* & \text{definition of } \Theta_Z & \\ & & \downarrow & \downarrow & \\ (d_{n-k-1})^*(\Theta_Z(\alpha))(\beta) & = & \Theta_Z(\alpha)(d_{n-k-1}(\beta)) & = \int_Z \alpha \wedge d_{n-k-1}(\beta) + (-1)^{k+1} \cdot \overbrace{\int_Z d_{n-1}(\alpha \wedge \beta)}^{\substack{=0 \text{ by the Top Dimension} \\ \mathrm{H}^n_{\mathrm{dR},c}\text{-Integration} \\ \text{Proposition 83.10}}} & \\ & = & (-1)^{k+1} \cdot \int_Z d_k(\alpha) \wedge \beta & = (-1)^{k+1} \cdot \Theta_Z(d_k(\alpha))(\beta). & \\ & & \uparrow & & \\ & & \text{by the Product Rule from the Differentials-on-Manifolds Proposition 75.7 (3) we have } d_{n-1}(\alpha \wedge \beta) = d_k(\alpha) \wedge \beta + (-1)^k \cdot \alpha \wedge d_{n-k-1}(\beta) & & \end{array}$$

■

To formulate the next lemma it is convenient to recall the notation and the statement of the following purely algebraic theorem, which we proved a while ago:

Theorem 77.18 (Evaluation-Isomorphism Theorem) Let $\mathbb{F}$ be a field and let (V_*, ∂_*) be a chain complex of $\mathbb{F}$-vector spaces. The evaluation map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^k(\mathrm{Hom}_{\mathbb{F}}(V_*, \mathbb{F})) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*), \mathbb{F}). \\ [\varphi\colon V_k \to \mathbb{F}] & \mapsto & \begin{pmatrix} \mathrm{H}_k(V_*) & \to & \mathbb{F} \\ [v] & \mapsto & \varphi(v) \end{pmatrix} \end{array}$$

is well-defined, it is natural, and it is an isomorphism of $\mathbb{F}$-vector spaces.

The evaluation map allows us to formulate our second new ingredient in the proof of the Mayer-Vietoris Compatibility Lemma 84.12:

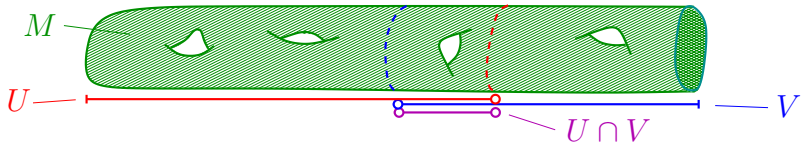
Lemma 84.12. (Mayer-Vietoris Compatibility Lemma) Let M be an oriented n-dimensional smooth manifold and let U, V be open subsets with $M = U \cup V$. Let $j_U \colon U \to M$, $j_V \colon V \to M$, $i_U \colon U \cap V \to U$ and $i_V \colon U \cap V \to V$ be the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(M) & \xrightarrow{(j_U^*, -j_V^*)} & \mathrm{H}^k_{\mathrm{dR}}(U) \oplus \mathrm{H}^k_{\mathrm{dR}}(V) & \xrightarrow{i_U^* \oplus i_V^*} & \mathrm{H}^k_{\mathrm{dR}}(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{\mathrm{dR}}(M) \longrightarrow \dots \\
 & & \downarrow{\scriptstyle \mathrm{ev} \circ \Theta_M} & & \downarrow{\scriptstyle \mathrm{diag}(\mathrm{ev} \circ \Theta_U, \mathrm{ev} \circ \Theta_V)} & & \downarrow{\scriptstyle \mathrm{ev} \circ \Theta_{U \cap V}} & & \downarrow{\scriptstyle \mathrm{ev} \circ \Theta_M} \\
\dots & \to & \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M)^* & \longrightarrow & \mathrm{H}^{n-k}_{\mathrm{dR,c}}(U)^* \oplus \mathrm{H}^{n-k}_{\mathrm{dR,c}}(V)^* & \longrightarrow & \mathrm{H}^{n-k}_{\mathrm{dR,c}}(U \cap V)^* & \xrightarrow{\delta^*} & \mathrm{H}^{n-k-1}_{\mathrm{dR,c}}(M)^* \to \dots
\end{array}$$

Here we consider the following horizontal sequences:

- The top sequence is given by the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2.
- The bottom sequence is given by the *dual* of the exact sequence given by the Mayer–Vietoris Theorem 83.13 for de Rham cohomology with compact support.

The statement of the lemma is that the first two squares commute and that the third square commutes up to multiplication by $(-1)^{k+1}$.



Proof of the Mayer-Vietoris Compatibility Lemma 84.12. Let M be an oriented n-dimensional smooth manifold and let U, V be open subsets with $M = U \cup V$. We consider the following expanded diagram:

$$\begin{array}{ccccccccc}
 & & \overbrace{\mathrm{H}^k(\Omega_*(M))}^{=\mathrm{H}^k_{\mathrm{dR}}(M)} & \xrightarrow{(j_U^*, -j_V^*)} & \overbrace{\mathrm{H}^k(\Omega^*(U))}^{=\mathrm{H}^k_{\mathrm{dR}}(U)} \oplus \overbrace{\mathrm{H}^k(\Omega^*(V))}^{=\mathrm{H}^k_{\mathrm{dR}}(V)} & \xrightarrow{i_U^* \oplus i_V^*} & \overbrace{\mathrm{H}^k(\Omega^*(U \cap V))}^{=\mathrm{H}^k_{\mathrm{dR}}(U \cap V)} & \xrightarrow{\delta} & \overbrace{\mathrm{H}^{k+1}(\Omega^*(M))}^{=\mathrm{H}^{k+1}_{\mathrm{dR}}(M)} \\
 & & \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \mathrm{diag}(\Theta_U, \Theta_V)} & & \downarrow{\scriptstyle \Theta_{U \cap V}} & & \downarrow{\scriptstyle \Theta_M} \\
\mathrm{H}_{n-k}(\Omega_{\mathrm{c}}^{n-*}(M)^*) & \to & \mathrm{H}_{n-k}(\Omega_{\mathrm{c}}^{n-*}(U)^*) & \oplus & \mathrm{H}_{n-k}(\Omega_{\mathrm{c}}^{n-*}(V)^*) & \to & \mathrm{H}_{n-k}(\Omega_{\mathrm{c}}^{n-*}(U \cap V)^*) & \xrightarrow{\delta^*} & \mathrm{H}_{n-k-1}(\Omega_{\mathrm{c}}^{n-*}(M)^*) \\
 & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{diag}(\mathrm{ev}, \mathrm{ev})} & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} \\
 & & \underbrace{\mathrm{H}^{n-k}(\Omega_{\mathrm{c}}^*(M))^*}_{=\mathrm{H}^{n-k}_{\mathrm{dR,c}}(M)^*} & \longrightarrow & \underbrace{\mathrm{H}^{n-k}_{\mathrm{dR,c}}(\Omega_{\mathrm{c}}^*(U))^*}_{=\mathrm{H}^{n-k}_{\mathrm{dR,c}}(U)^*} \oplus \underbrace{\mathrm{H}^{n-k}_{\mathrm{dR,c}}(\Omega_{\mathrm{c}}^*(V))^*}_{=\mathrm{H}^{n-k}_{\mathrm{dR,c}}(V)^*} & \longrightarrow & \underbrace{\mathrm{H}^{n-k}_{\mathrm{dR,c}}(\Omega_{\mathrm{c}}^*(U \cap V))^*}_{=\mathrm{H}^{n-k}_{\mathrm{dR,c}}(U \cap V)^*} & \xrightarrow{\delta^*} & \underbrace{\mathrm{H}^{n-k-1}_{\mathrm{dR,c}}(\Omega_{\mathrm{c}}^*(M))^*}_{=\mathrm{H}^{n-k-1}_{\mathrm{dR,c}}(M)^*}
\end{array}$$

Note that the composition of the two vertical maps are precisely the vertical maps from the Mayer-Vietoris Compatibility Lemma 84.12. Therefore it remains to prove the following claim:

Claim. All squares in the diagram commute, except for the square to the top right, which commutes up to the sign $(-1)^{k+1}$.

Proof. In the following we first consider the first layer of squares and then we consider the second layer of squares:

(1) We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k(M) & \xrightarrow{(j_U^*,-j_V^*)} & \Omega^k(U)\oplus\Omega^*(V) & \xrightarrow{i_U^*\oplus i_V^*} & \Omega^k(U\cap V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \mathrm{diag}(\Theta_U,\Theta_V)} & & \downarrow{\scriptstyle \Theta_{U\cap V}} & & \\ 0 & \to & (\Omega_c^{n-k}(M))^* & \xrightarrow{((j_{U*})^*,-(j_{V*})^*)} & (\Omega_c^{n-k}(U))^*\oplus(\Omega_c^{n-k}(V))^* & \xrightarrow{(i_{U*})^*\oplus(i_{V*})^*} & (\Omega_c^{n-k}(U\cap V))^* & \to & 0. \end{array}$$

We make the following observations:

(a) It follows from the Form-to-Compact Support Form-Lemma 84.11 (1) that the diagram commutes.

(b) Evidently $(\Omega^*(M), d)$ is a cochain complex. It follows from the Index Reversal Lemma 77.15 together with the equally elementary Dual Cochain Complex Lemma 77.17 that $(\Omega_c^{n-*}(M), d^*)$ is also a cochain complex.

(c) The Form-to-Compact Support Form-Lemma 84.11 (2) can be reinterpreted as saying that the vertical maps are chain maps "up to multiplication by $(-1)^{k+1}$".

For both short exact sequences of (co-) chain complexes we can now consider the corresponding long exact sequence of cohomology groups. It follows from (a) that the first two squares in the diagram involving cohomology groups commutes.

Next recall that by the Cohomology–LES Proposition 77.13 (2) we know that connecting homomorphisms are natural. By (c) this result does not quite apply in our context, so we have to work a little harder. Fortunately the desired statement follows from the following two observations:

- Let $\mathfrak{e}_k$ be the coboundary maps at the bottom. If instead we work with the coboundary maps $\widetilde{e}_k := (-1)^{k+1}\cdot\mathfrak{e}_k$, then the cohomology groups are unchanged and by (c) the vertical maps are now cochain maps. It thus follows from the Cohomology–LES Proposition 77.13 that we get a commuting diagram involving connecting homomorphisms.
- It follows immediately from the definition of the connecting homomorphism, see Definition 77.12, that the connecting homomorphisms for the cochain complexes defined by $\mathfrak{e}_k$ and defined by $\widetilde{e}_k := (-1)^{k+1}$ differ precisely by the factor $(-1)^{k+1}$.

(2) It follows from the naturality of the evaluation map that the first two squares commute. Finally note that it follows from the elementary Exercise 77.6 that the square involving the connecting homomorphisms also commutes. ∎

Now we are finally ready to prove the De Rham-Poincaré Duality Theorem 84.5. For the reader's convenience we recall the statement:

Theorem 84.5. (De Rham-Poincaré Duality Theorem) Let M be an oriented n-dimensional smooth manifold.

(1) If M admits a finite smoothly good open cover, then for any $k \in \mathbb{N}_0$ the following $\mathbb{R}$-bilinear pairing is non-singular:

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M)\times\mathrm{H}^{n-k}_{\mathrm{dR,c}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \displaystyle\int_M \omega\wedge\eta. \end{array}$$

(2) If M is closed, then for any $k \in \mathbb{N}_0$ the following $\mathbb{R}$-bilinear pairing is non-singular:

$$\begin{aligned} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) &\to \mathbb{R} \\ ([\omega],[\eta]) &\mapsto \int_M \omega \wedge \eta. \end{aligned}$$

Proof of the De Rham-Poincaré Duality Theorem 84.5. Note that Statement (2) is an immediate consequence of Statement (1) together with the following two observations:

- By the Smoothly Good Cover Theorem 81.2 we know that every closed smooth manifold admits a locally finite smoothly good open cover.
- As we pointed out on page 1671, for a compact smooth manifold de Rham cohomology with compact support equals ordinary de Rham cohomology.

Thus it remains to prove Statement (1). We rephrase Statement (1) in such a way that we can prove it by induction. To cut a long story short, Statement (1) follows immediately from the following claim:

Claim. Let $m \in \mathbb{N}_0$. If M is an oriented n-dimensional smooth manifold that admits a finite smoothly good open cover consisting of $\leq m$ open subsets, then for any $k \in \mathbb{N}_0$ the $\mathbb{R}$-bilinear pairing

$$\begin{aligned} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M) &\to \mathbb{R} \\ ([\omega],[\eta]) &\mapsto \int_M \omega \wedge \eta \end{aligned}$$

is non-singular.

Proof. First we consider the cases $m = 0$ and $m = 1$. If $M = \varnothing$, then it follows from the Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.6 that the form is non-singular. If $M \neq \varnothing$, then we need to deal with an oriented n-dimensional smooth manifold M that is diffeomorphic to an open convex non-empty subset of $\mathbb{R}^n$. In this case it follows from the Star Shaped-Diffeomorphism Type Theorem 82.2 that f admits a diffeomorphism $f\colon M \to \mathbb{R}^n$. After possibly composing f with a reflection in a hyperplane we can assume that f is actually orientation-preserving. As we discussed on page 1682, the desired statement holds for $\mathbb{R}^n$. We will now see that it then also holds for M. To do so we consider the following diagram

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n) & \xrightarrow{[\omega] \mapsto \left([\eta] \mapsto \int_{\mathbb{R}^n} \omega \wedge \eta\right)} & \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^{n-k}_{\mathrm{dR,c}}(\mathbb{R}^n), \mathbb{R}) \\ \uparrow{\scriptstyle \omega} \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f^{-1})^{**}} \\ \mathrm{H}^k_{\mathrm{dR}}(M) & \xrightarrow[[\omega] \mapsto \left([\eta] \mapsto \int_M \omega \wedge \eta\right)]{} & \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^{n-k}_{\mathrm{dR,c}}(M), \mathbb{R}). \\ & & \uparrow{\scriptstyle \eta} \end{array}$$

By the discussion on page 1682 the top horizontal map is an isomorphism. The vertical maps are isomorphisms since f is a diffeomorphism. Thus it now suffices to show that the diagram commutes. This is indeed the case since for any $[\omega] \in \mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n)$ and $[\eta] \in \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M)$ we calculate that

$$\int_{\mathbb{R}^n} \omega \wedge (f^{-1})^*(\eta) \;\underset{\text{by the Form Integration Proposition 74.15, since } f \text{ is an orientation-preserving diffeomorphism}}{=} \int_M f^*(\omega \wedge (f^{-1})^*(\eta)) \;\overset{\text{this follows from the Alternating Form-Functoriality Lemma 72.12}}{=} \int_M f^*(\omega) \wedge \underbrace{f^*((f^{-1})^*\eta)}_{=(f^{-1}\circ f)^*(\eta)=\eta} = \int_M f^*(\omega) \wedge \eta.$$

This concludes the cases of $m = 0$ and $m = 1$.

Now assume that the statement of the claim holds for some $m \in \mathbb{N}$. We need to prove it for $m + 1$. Let M be an oriented n-dimensional smooth manifold that admits a finite smoothly good open cover consisting of $m + 1$ open subsets $U_1, \ldots, U_{m+1}$. We set $V := U_1 \cup \cdots \cup U_m$. Next we consider the following essential diagram:

$$\begin{array}{ccccccccc}
.. \to & \mathrm{H}^k_{\mathrm{dR}}(M) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(U_{m+1}) \oplus \mathrm{H}^k_{\mathrm{dR}}(V) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(U_{m+1} \cap V) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{\mathrm{dR}}(M) & \to .. \\
& \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \mathrm{diag}(\Theta_{U_{m+1}}, \Theta_V)} & & \downarrow{\scriptstyle \Theta_{U_{m+1}\cap V}} & & \downarrow{\scriptstyle \Theta_M} & \\
.. \to & \mathrm{H}^{n-k}_{\mathrm{dR,c}}(M)^* & \longrightarrow & \mathrm{H}^{n-k}_{\mathrm{dR,c}}(U_{m+1})^* \oplus \mathrm{H}^{n-k}_{\mathrm{dR,c}}(V)^* & \longrightarrow & \mathrm{H}^{n-k}_{\mathrm{dR,c}}(U_{m+1} \cap V)^* & \xrightarrow{\delta^*} & \mathrm{H}^{n-k-1}_{\mathrm{dR,c}}(M)^* & \to ..
\end{array}$$

We make the following clarifications and observations:

(1) The top sequence is given by the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2.

(2) The bottom sequence is given by the dual of the exact sequence given by the Mayer–Vietoris Theorem 83.13 for de Rham cohomology with compact support. Note that it follows from the Generalized Exactness Preservation Lemma 77.20 that the bottom sequence is indeed exact.

(3) It follows from the Mayer-Vietoris Compatibility Lemma 84.12 that the diagram commutes up to a sign.

(4) We make the following observations:
- U_{m+1} admits a smoothly good open cover with one subset.
- V admits a smoothly good open cover with m subsets.
- Note that
$$U_{m+1} \cap V \;=\; U_{m+1} \cap \Big(\bigcup_{i=1}^{k} U_i \Big) \;=\; \bigcup_{i=1}^{k} (U_{m+1} \cap U_i).$$
It now follows easily from the definition of a smoothly good open cover, see Definition 81.1, that the m sets $U_{m+1} \cap U_i$ form a smoothly good open cover of $U_{m+1} \cap V$.

It follows from these observations, the fact that we already dealt with the case $m = 0, 1$ and the induction hypothesis that in the above diagram the corresponding vertical maps $\mathrm{diag}(\Theta_{U_{m+1}}, \Theta_V)$ and $\Theta_{U_{m+1}\cap V}$ are isomorphisms.

(5) It follows from (1)–(4) together with the Five Lemma 77.9 that the desired vertical maps Θ_M are also isomorphisms. ∎

84.4. Exercises for Chapter 84.

Exercise 84.1. Let M be a closed oriented $2n$-dimensional smooth manifold. By the De Rham-Poincaré Duality Theorem 84.5 we know that the $\mathbb{R}$-bilinear pairing

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR}}(M) \times \mathrm{H}^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\omega], [\eta]) & \mapsto & \displaystyle\int_M \omega \wedge \eta \end{array}$$

is non-singular. By the Wedge-on-de Rham Cohomology Proposition 78.12 we know that this pairing is symmetric if n is even and we know that it is anti-symmetric if n is odd.

(a) Classify non-singular $\mathbb{R}$-bilinear symmetric forms over a finite-dimensional real vector space up to isometry.

(b) Classify non-singular $\mathbb{R}$-bilinear anti-symmetric forms over a finite-dimensional real vector space up to isometry.

Exercise 84.2.

(a) Let M be a closed oriented smooth manifold. Show that M does not admit a deformation retraction onto a submanifold $N \subsetneq M$.

(b) Does the conclusion in (a) also hold, if we drop the hypothesis that M is oriented?

Exercise 84.3. Given a closed n-dimensional smooth manifold M we define, as in Exercise 81.9, its de Rham Euler characteristic as

$$\chi_{\mathrm{dR}}(M) \coloneqq \sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{R}}(\mathrm{H}^i_{\mathrm{dR}}(M)).$$

(a) Show that the Euler characteristic of every closed oriented odd-dimensional smooth manifolds is zero.

(b) Show that the Euler characteristic of every closed oriented $(4k+2)$-dimensional smooth manifolds is even.

Hint. Make use of Exercise 84.1.

Exercise 84.4. Let $m \neq n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Use the De Rham-Poincaré Duality Theorem 84.5 to show that

$$p^*([S^m]^*) \wedge q^*([S^n]^*) \;=\; \pm[S^m \times S^n]^*.$$

CHAPTER 85

Calculations using the Poincaré Duality Theorem

In this chapter we will use the De Rham-Poincaré Duality Theorem 84.5 from the previous chapter to calculate the de Rham cohomology groups and ring of various smooth manifolds, in particular the real and complex projective spaces and of knot complements.

This chapter is organized as follows:

- In Section 85.1 we will use the De Rham-Poincaré Duality Theorem 84.5 to calculate the de Rham cohomology *ring* of the complex projective spaces $\mathbb{C}\mathrm{P}^n$.
- In Section 85.2 we will introduce the notion of a finite covering $p\colon \widetilde{M} \to M$ of smooth manifolds and we will introduce the transfer map $p_*\colon \mathrm{H}^k_{\mathrm{dR}}(\widetilde{M}) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ which, rather confusingly, "goes the wrong way".
- In Section 85.3 we will present several applications of the transfer map. In particular we will use them to calculate the de Rham cohomology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$. (Spoiler alert: they are not very interesting).
- In Section 85.4 we will calculate the de Rham cohomology groups of knot complements.
- In Appendix 85.5 we will prove a variation on the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2, which we made use of in Section 85.4.
- In Section 85.6 we will provide exercises for Chapter 85.

In the next Chapter 86 we will discuss more consequences of the De Rham-Poincaré Duality Theorem 84.5.

85.1. **De Rham cohomology of the complex projective spaces.** In the $\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.12 we managed to calculate the de Rham cohomology *groups* of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In this section we will use the De Rham-Poincaré Duality Theorem 84.5 to calculate the de Rham cohomology *ring* of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. We start out by recalling the calculation of the de Rham cohomology groups:

Proposition 80.12 ($\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition)

(1) For any $n \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

(2) Given any $m \leq n$ we denote by

$$\begin{array}{rcl} \iota\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_m] & \mapsto & [z_0 : \cdots : z_m : 0] \end{array}$$

the usual embedding which we first introduced on page 338. The induced map

$$\iota^*\colon \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^m)$$

is an isomorphism for $k \leq 2m$.

$$\begin{array}{c} 0 \\ \mathrm{H}^6_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^3) \cong \mathbb{R} \\ \mathrm{H}^4_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^3) \cong \mathbb{R} \\ \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^3) \cong \mathbb{R} \end{array} \quad \begin{array}{c} \cong \quad \iota^* \\ \longrightarrow \\ \longrightarrow \\ \cong \quad \iota^* \end{array} \quad \begin{array}{c} 0 \\ 0 \\ \mathrm{H}^4_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^2) \cong \mathbb{R} \\ \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^2) \cong \mathbb{R} \end{array}$$

We also introduce the following notation:

Notation 85.1.

(1) Let $n \in \mathbb{N}_0$. In the Projective Space–Manifold Proposition 25.13 and the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we showed that $\mathbb{C}\mathrm{P}^n$ is a closed $2n$-dimensional smooth manifold that comes with a natural orientation. Thus we will always view $\mathbb{C}\mathrm{P}^n$ as an *oriented* smooth manifold. In particular it comes with the canonical de Rham fundamental class $[\mathbb{C}\mathrm{P}^n]^*_{\mathrm{dR}} \in \mathrm{H}^{2n}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ that we introduced in Definition 84.8.

(2) For any $n \in \mathbb{N}$ we denote by $\iota\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the above embedding. By the $\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 213.5 we can define $x := (\iota^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*_{\mathrm{dR}}) \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$. We refer to $x \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ as the standard generator.

Now we have the tools to determine the wedge product of all complex projective spaces:

Proposition 85.2. (Wedge Product-$\mathbb{C}\mathrm{P}^n$-Proposition)

(1) Let $n \in \mathbb{N}_0$. For every $k \in \mathbb{N}_0$ with $k \leq n$ the k-th power[851] $x^k \in \mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{R}$ is non-zero.

(2) Let $n \in \mathbb{N}_0$. The obvious map

$$\mathbb{R}[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n), \wedge)$$

is an isomorphism of $\mathbb{R}$-algebras.

Proof.

(1) We prove this statement by induction on n. Evidently the statement holds for $n = 0$ and $n = 1$. Now suppose that it holds for some $n-1$ with $n \geq 2$. We need to show that for every $k \in \{0, \ldots, n\}$ the k-th power $x^k \in \mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{R}$ is non-zero. In the following two claims we deal with the cases $k \in \{0, \ldots, n-1\}$ and $k = n$ separately:

Claim 1. For $k \in \{0, \ldots, n-1\}$ the k-th power $x^k \in \mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ is a generator.

Proof. Let $\iota\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ be the usual embedding and let $\widetilde{x} \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})$ and $x \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ be the standard generators. It follows easily from the functoriality of de Rham cohomology, applied to the commutative diagram

$$\begin{array}{ccc} & \mathbb{C}\mathrm{P}^1 & \\ \swarrow & & \searrow \\ \mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}^n \end{array}$$

[851] Let M be a smooth manifold and let $\varphi \in \mathrm{H}^d_{\mathrm{dR}}(M)$. As in any ring we define $\varphi^0 = 1_M$ and given any $k \in \mathbb{N}$ we define $\varphi^k := \underbrace{\varphi \wedge \cdots \wedge \varphi}_{k \text{ times}} \in \mathrm{H}^{kd}_{\mathrm{dR}}(M)$.

that we have $\iota^*(x) = \widetilde{x}$. We consider the following diagram:

$$\begin{array}{ccc} \overset{\widetilde{x}^k}{\mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})} & \xleftarrow[\cong]{\iota^*} & \overset{x^k}{\mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n})} \\ {\scriptstyle \varphi \mapsto \varphi^k}\uparrow & & \uparrow{\scriptstyle \varphi \mapsto \varphi^k} \\ \mathrm{H}^{2}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1}) & \xleftarrow[\cong]{\iota^*} & \mathrm{H}^{2}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n}). \\ {\scriptstyle \widetilde{x}} & & {\scriptstyle x} \end{array}$$

It follows from the Pullback Smooth Form Proposition 74.7 that the diagram commutes and it follows from the $\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.12 (2) that the horizontal maps are isomorphisms. Since $k \leq n-1$ we obtain that $\widetilde{x}^k$ is by induction hypothesis a non-zero element of $\mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1})$. We now see that

$$\begin{array}{ccccccc} i^*(x^k) & \underset{\uparrow}{=} & (i^*(x))^k & = & \widetilde{x}^k & \underset{\uparrow}{\neq} & 0 \qquad \in \mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n-1}). \\ & \text{by the Pullback Smooth} & & & & \text{since } k \leq n-1 \text{ this follows} & \\ & \text{From Proposition 74.7} & & & & \text{from our induction hypothesis} & \end{array}$$

It follows that x^k is a non-zero element of $\mathrm{H}^{2k}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$. ⊞

It remains to prove the following claim:

Claim 2. The n-th power $x^n \in \mathrm{H}^{2n}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ is non-zero.

Proof. We consider the pairing

$$\begin{array}{rcl} \overbrace{\mathrm{H}^{2}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n})}^{=\mathbb{R}\cdot x} \times \overbrace{\mathrm{H}^{2n-2}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^{n})}^{=\mathbb{R}\cdot x^{n-1}} & \to & \mathbb{R} \\ ([\varphi],[\psi]) & \mapsto & \displaystyle\int_{\mathbb{C}\mathrm{P}^n} \varphi \wedge \psi. \end{array}$$

As we discussed above, $\mathbb{C}\mathrm{P}^n$ is a closed orientable $2n$-dimensional smooth manifold. It thus follows from the De Rham-Poincaré Duality Theorem 84.5 (2) that the above pairing is non-singular. It follows from this observation, the fact that $x \in \mathrm{H}^2_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ and $x^{n-1} \in \mathrm{H}^{2n-2}_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ are basis elements and the Non Singular Form-over-Field Lemma 84.3 that $x^n = x \wedge x^{n-1} \neq 0$. ⊞

(2) First note that it follows immediately from the definitions and the Cup Product-Commutativity Proposition 203.20 that the objects in (2) are $\mathbb{R}$-algebras. This observation, together with the $\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.12 shows that Statement (2) is a consequence of (1). ■

The reader will now wonder, what about the de Rham cohomology groups of the real projective spaces? We will answer this question in the next section.

85.2. **The transfer map.** In this section we will introduce the notion of a covering $p\colon \widetilde{M} \to M$ of smooth manifolds. Given a covering $p\colon \widetilde{M} \to M$ of smooth manifolds of finite degree we will introduce the transfer maps $p_*\colon \mathrm{H}^k_{\mathrm{dR}}(\widetilde{M}) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ which go the "wrong way". As we will see in the next section, these maps, together with the De Rham-Poincaré Duality Theorem 84.5, will allow us to compute the n-th de Rham cohomology groups of *non-orientable* n-dimensional smooth manifold and they will allow us to calculate the de Rham cohomology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$.

We start out with the definition of a covering of smooth manifolds:

Definition 85.3. Let $p\colon \widetilde{M} \to M$ be a smooth map between smooth manifold.

(1) An open subset $U \subseteq M$ is called uniformly covered, if $p^{-1}(U) \subseteq \widetilde{M}$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of $\widetilde{M}$ with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a diffeomorphism.

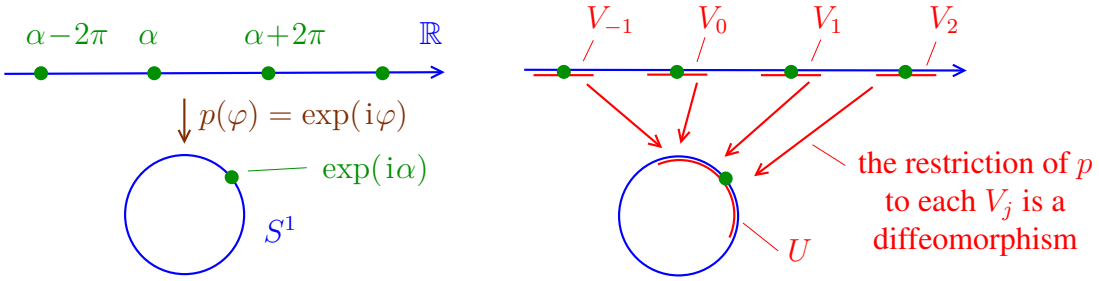
(2) We say that the map $p\colon \widetilde{M} \to M$ is a covering of smooth manifolds, if p is surjective and if for every $x \in M$ there exists an open neighborhood U of $x \in M$ which is uniformly covered.

(3) Let $\widetilde{M} \to M$ be a covering of smooth manifolds. If M is connected, then we will see in Exercise 85.8 that the cardinality of $p^{-1}(x)$ is independent of the choice of $x \in M$. We refer to this cardinality as the degree of the covering. If the degree is $n \in \mathbb{N}$, then we refer to p as an n-fold covering.

Remark. We will discuss this notion of a covering in much greater detail in Section 91.2 and in Chapter 95. □

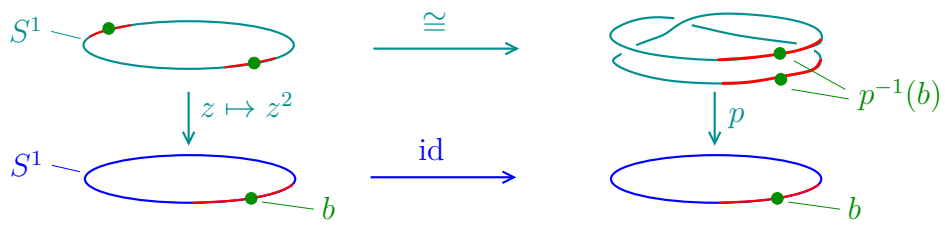
Example.

(1) The map
$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$
is a covering. Indeed, let $z = \exp(\mathrm{i}\alpha)$ be a point in S^1. We pick the open neighborhood
$$U := \big\{ \exp(\mathrm{i}\varphi) \,\big|\, \varphi \in \big(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\big)\big\}.$$
Then
$$p^{-1}(U) = \bigsqcup_{j\in\mathbb{Z}} \underbrace{\big(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\big)}_{=:V_j},$$
and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a diffeomorphism.



(2) Similar to (1) one can easily show that for any $n \in \mathbb{N}$ the map
$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$
is an n-fold covering of smooth manifolds. In the figure below we do our best to illustrate this smooth covering for $n = 2$.

(3) It follows easily from Exercise 25.10 that for each $n \in \mathbb{N}_0$ the map
$$\begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n \\ x & \mapsto & [x] \end{array}$$
is a 2-fold covering of smooth manifolds. □



The notion of a covering allows us to introduce the rather weird concept of a transfer map:

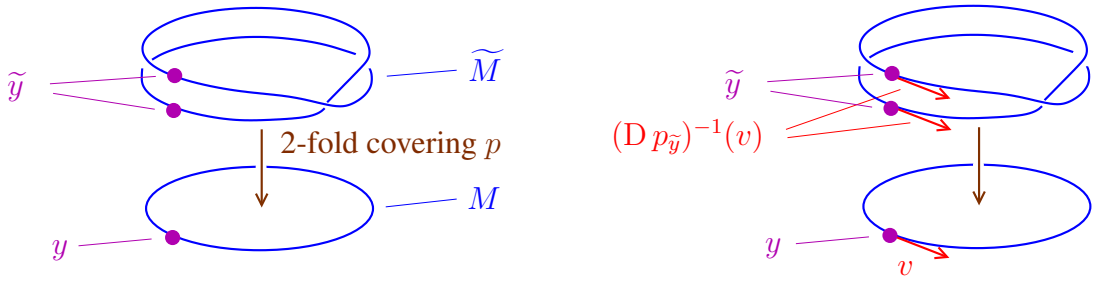
Definition 85.4. Let $p\colon \widetilde{M} \to M$ be a finite covering of a smooth manifold M. Furthermore let $\widetilde{\omega} \in \Omega_k(\widetilde{M})$ be a smooth k-form on $\widetilde{M}$. We now define a corresponding smooth form $p_*(\widetilde{\omega}) \in \Omega_k(M)$ as follows: Given $y \in M$ and given $v_1, \dots, v_k \in \mathrm{T}_y M$ we set[852]

$$p_*(\widetilde{\omega})_y(v_1, \dots, v_k) := \sum_{\substack{\widetilde{y} \in \widetilde{M} \text{ with} \\ p(\widetilde{y}) = y}} \widetilde{\omega}(\underbrace{(\mathrm{D}p_{\widetilde{y}})^{-1}(v_1)}_{\in \mathrm{T}_{\widetilde{y}}\widetilde{M}}, \dots, \underbrace{(\mathrm{D}p_{\widetilde{y}})^{-1}(v_k)}_{\in \mathrm{T}_{\widetilde{y}}\widetilde{M}}).$$

We refer to the map[853]

$$\begin{aligned} p_*\colon \Omega_k(\widetilde{M}) &\to \Omega_k(M) \\ \widetilde{\omega} &\mapsto p_*(\widetilde{\omega}) \end{aligned}$$

as the transfer map.



The following lemma shows in particular the surprising fact that the transfer map is actually a cochain map:

Lemma 85.5. (De Rham–Transfer Map Lemma) Let $p\colon \widetilde{M} \to M$ be an m-fold covering of a smooth manifold M.

(1) The transfer map $p_*\colon \Omega_k(\widetilde{M}) \to \Omega_k(M)$ is a cochain map.
(2) The map $p_* \circ p^*\colon \Omega^k(M) \to \Omega^k(M)$ is given by multiplication by m.

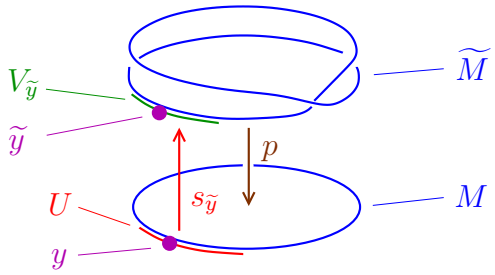
Proof.

(1) Let $\widetilde{\omega} \in \Omega_k(\widetilde{M})$. We need to show that $p^*(d\widetilde{\omega}) = d(p^*\widetilde{\omega}) \in \Omega_{k+1}(M)$. It suffices to prove the following claim:

[852]Note that it follows from the definition of a covering of a smooth manifold that $p\colon \widetilde{M} \to M$ is a local diffeomorphism. It thus follows from the Tangent Space-Inclusion Proposition 29.20 (3) that for each $\widetilde{y} \in \widetilde{M}$ the differential $\mathrm{D}p_{\widetilde{y}}\colon \mathrm{T}_{\widetilde{y}}\widetilde{M} \to \mathrm{T}_y M$ is an isomorphism.
[853]It follows easily from the Smooth Form–Pasting Proposition 74.8 that $p_*(\widetilde{\omega})$ is indeed a smooth k-form on M.

Claim 1. For each $y \in M$ there exists an open neighborhood of $y \in M$ such that we have the desired equality $p^*(d\widetilde{\omega})|_U = d(p^*\widetilde{\omega})|_U \in \Omega_{k+1}(U)$.

Proof. Let $y \in M$. Since $p\colon \widetilde{M} \to M$ is a finite covering of a smooth manifold we know that there exists an open neighborhood U of y and a family of open subsets $\{V_{\widetilde{y}}\}_{\widetilde{y} \in p^{-1}(y)}$ of $\widetilde{M}$ such that for each $\widetilde{y} \in p^{-1}(y)$ the set $V_{\widetilde{y}}$ is an open neighborhood of $\widetilde{y} \in \widetilde{M}$ and such that the map $p_{\widetilde{y}} := p|_{V_{\widetilde{y}}}\colon V_{\widetilde{y}} \to U$ is a diffeomorphism. We write $s_{\widetilde{y}} := p_{\widetilde{y}}^{-1}\colon U \to V_{\widetilde{y}}$.



Next note that it follows easily from the definitions that for any $\widetilde{\eta} \in \Omega_l(\widetilde{M})$ we have the following equality:

$$p_*(\widetilde{\eta})|_U \;=\; \sum_{\widetilde{y} \in p^{-1}(y)} s_{\widetilde{y}}^*(\widetilde{\eta}|_{V_{\widetilde{y}}}).$$

We now see that

$$p_*(d\widetilde{\omega})|_U \;\underset{\substack{\uparrow\\ \text{follows from the}\\ \text{above equality}}}{=}\; \sum_{\widetilde{y} \in p^{-1}(y)} s_{\widetilde{y}}^*(d(\widetilde{\omega}|_{V_{\widetilde{y}}})) \;\underset{\substack{\uparrow\\ \text{follows from the Differential}\\ \text{Natural Proposition 75.4}}}{=}\; \sum_{\widetilde{y} \in p^{-1}(y)} d(s_{\widetilde{y}}^*(\widetilde{\omega}|_{V_{\widetilde{y}}})) \;\underset{\substack{\uparrow\\ \text{this also follows from}\\ \text{the above equality}}}{=}\; d((p_*\widetilde{\omega})|_U).$$

(2) Let $\omega \in \Omega^k(M)$. It suffices to prove the following claim:

Claim 2. For each $y \in M$ there exists an open neighborhood of y such that we have the desired equality $(p_* \circ p^*)(\omega)|_U = m \cdot \omega|_U$.

Proof. Let $y \in M$. We use the notation from the proof of Claim 1. With this notation we see that

$$(p_* \circ p^*)(\omega)|_U \;\underset{\substack{\uparrow\\ \text{by the description of the}\\ \text{transfer map in Claim 1}\\ \text{and since } p^*(\omega)|_{V_{\widetilde{y}}} = p_{\widetilde{y}}^*(\omega|_U)}}{=}\; \sum_{\widetilde{y} \in p^{-1}(y)} s_{\widetilde{y}}^*(p_{\widetilde{y}}^*(\omega|_U)) \;\underset{\substack{\uparrow\\ \text{since de Rham cohomology}\\ \text{is contravariant}}}{=}\; \sum_{\widetilde{y} \in p^{-1}(y)} \underbrace{(p_{\widetilde{y}} \circ s_{\widetilde{y}})}_{=\mathrm{id}}{}^*(\omega|_U) \;=\; \underbrace{\#p^{-1}(y)}_{=m} \cdot \omega|_U.$$

■

The De Rham–Transfer Map Lemma 85.5 (1) allows us to introduce the following definition:

Definition 85.6. Let $p\colon \widetilde{M} \to M$ be a finite covering of a smooth manifold M and let $k \in \mathbb{N}_0$. It follows from the above De Rham–Transfer Map Lemma 85.5 (1) that the transfer map $p_*\colon \Omega_*(\widetilde{M}) \to \Omega_*(M)$ induces a homomorphism

$$p_*\colon \mathrm{H}^k_{\mathrm{dR}}(\widetilde{M}) \;\to\; \mathrm{H}^k_{\mathrm{dR}}(M)$$

which we also call transfer map.

The following proposition summarizes the two main properties of the transfer map on de Rham cohomology:

Proposition 85.7. (De Rham-Transfer–Projection Proposition) Let $p\colon \widetilde{M} \to M$ be an m-fold covering of a smooth manifold M and let $k \in \mathbb{N}_0$. The following statements hold:

(1) The map $p_* \circ p^*\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ is given by multiplication by m.
(2) The projection induced map $p^*\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{dR}}(\widetilde{M})$ is a monomorphism.

Proof. Let $p\colon \widetilde{M} \to M$ be an m-fold covering of a smooth manifold M and let $k \in \mathbb{N}_0$.

(1) This statement follows immediately from the De Rham–Transfer Map Lemma 85.5 (2).
(2) Since multiplication by $m \neq 0$ is an isomorphism of any real vector space we obtain from (1) that $p_* \circ p^*\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{dR}}(M)$ is an isomorphism. In particular the first map, namely p^*, is a monomorphism. ■

85.3. **Applications of the transfer map.** In the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 we showed that for any closed connected *orientable* non-empty n-dimensional smooth manifold M we have $\mathrm{H}^n_{\mathrm{dR}}(M) \cong \mathbb{R}$. We will now use the transfer map from the previous section to prove the following proposition which tells us what happens, if we replace "orientable" by "non-orientable":

Proposition 85.8. (Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition II) Let M be a closed connected n-dimensional smooth manifold. If M is non-orientable, then $\mathrm{H}^n_{\mathrm{dR}}(M) = 0$.

Proof. We denote by $p\colon \widetilde{M} \to M$ the orientation covering from the Orientation Covering–Existence Proposition 95.8. Since M is non-orientable we obtain from the Orientation Covering–Properties-Proposition 95.10 (2) that $\widetilde{M}$ is a closed, oriented and connected n-dimensional smooth manifold. By the Orientation Covering–Properties-Proposition 95.10 (3) we know that there exists an orientation-reversing diffeomorphism $\Xi\colon \widetilde{M} \to \widetilde{M}$ which has the property that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Xi} & \widetilde{M} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & M. & \end{array}$$

We consider the corresponding homomorphisms on de Rham cohomology groups:

$$\begin{array}{ccc} \mathrm{H}^n_{\mathrm{dR}}(\widetilde{M}) & \xleftarrow[\cong]{\Xi^*} & \mathrm{H}^n_{\mathrm{dR}}(\widetilde{M}) \\ & {\scriptstyle p^*}\nwarrow \quad \nearrow{\scriptstyle p^*} & \\ & \mathrm{H}^n_{\mathrm{dR}}(M). & \end{array}$$

We make the following observations:

(1) The diagram commutes by functoriality of de Rham cohomology.
(2) Since $\widetilde{M}$ is connect orientable and non-empty we obtain from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 that $\mathrm{H}^n_{\mathrm{dR}}(\widetilde{M})$ is isomorphic to $\mathbb{R}$.
(3) We claim that the top horizontal map is given by multiplication by -1. This can be seen as follows: We consider the following maps:

$$\mathbb{R} \xleftarrow[\cong]{\int_{\widetilde{M}} \omega \leftarrow\!\shortmid [\omega]} \mathrm{H}^n_{\mathrm{dR}}(\widetilde{M}) \xleftarrow[\cong]{\Xi^*} \mathrm{H}^n_{\mathrm{dR}}(\widetilde{M}) \xrightarrow[\cong]{[\omega] \mapsto \int_{\widetilde{M}} \omega} \mathbb{R}.$$

By the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 we know that the outer maps are isomorphisms. Since $\Xi\colon \widetilde{M} \to M$ is orientation-reversing it follows from the Form Integration Proposition 74.15 (4) and (5) that the map from left to right is given by multiplication by -1. Since the outer maps are the same and since they are isomorphisms, we see that the middle map is also given by multiplication by -1.

(4) It follows from the De Rham-Transfer–Projection Proposition 85.7 (2) that the diagonal maps are monomorphisms.

It follows from these observations that there is only possibility left for $\mathrm{H}^n_{\mathrm{dR}}(M)$, namely it has to be trivial. ■

In the $\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.12 we calculated the de Rham cohomology groups of complex projective spaces. We can now give the, rather dull, calculation of the de Rham cohomology groups of real projective spaces:

Proposition 85.9. ($\mathbb{R}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition) Let $n \in \mathbb{N}$.

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ or if } k = n \text{ and } n \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For $k \neq 0$ and $k \neq n$ this statement follows from the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 together with the De Rham-Transfer–Projection Proposition 85.7 (2).

Next note that for $k = 0$ the statement follows from the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8. Finally note that we showed in the Real Projective Space Orientability Lemma 31.46 that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd. Thus for $k = n$ the promised statement follows from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 and the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition II 85.8. ■

85.4. **The de Rham cohomology of knot complements.** The main goal of this section is to calculate the de Rham cohomology groups of complements of knots in S^3:

Proposition 85.10. (Knot Complement-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition) Let $K \subseteq S^3$ be an oriented knot, i.e. K is an oriented 1-dimensional smooth submanifold that is diffeomorphic to S^1. For $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n_{\mathrm{dR}}(S^3 \setminus K) \xrightarrow{\cong} \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{R}, & \text{if } n = 0, 1. \end{cases}$$

Furthermore, if μ_K denotes a meridian of K, as introduced in Definition 109.1, then the inclusion induced map $\mathrm{H}^1_{\mathrm{dR}}(S^3 \setminus K) \to \mathrm{H}^1_{\mathrm{dR}}(\mu_K)$ is an isomorphism.



Our proof of the Knot Complement-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 85.10 relies on the following lemma, which is arguably interesting in its own right:

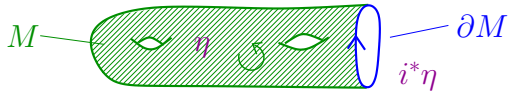
Lemma 85.11. ($\mathrm{H}^*_{\mathrm{dR}}$-Kernel-of-Boundary Inclusion Lemma) Let M be a compact orientable n-dimensional smooth manifold manifold with connected non-empty boundary. We denote by $i\colon \partial M \to M$ the inclusion map. The map

$$i^*\colon \mathrm{H}^{n-1}_{\mathrm{dR}}(M) \to \mathrm{H}^{n-1}_{\mathrm{dR}}(\partial M)$$

is the zero map.

Proof of the $\mathrm{H}^*_{\mathrm{dR}}$-Kernel-of-Boundary Inclusion Lemma 85.11. Let M be a compact orientable n-dimensional smooth manifold manifold with connected non-empty boundary. We pick an orientation for M and we equip ∂M with orientation provided by the Boundary Orientation Lemma 31.38. Since ∂M is connected and non-empty we obtain from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 (applied to the closed oriented $(n-1)$-dimensional smooth manifold ∂M) that it suffices to prove the following claim:

Claim. The map
$$\begin{array}{rcl} \mathrm{H}^{n-1}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\eta] & \mapsto & \int\limits_{\partial M} i^*(\eta) \end{array}$$
is the zero map.



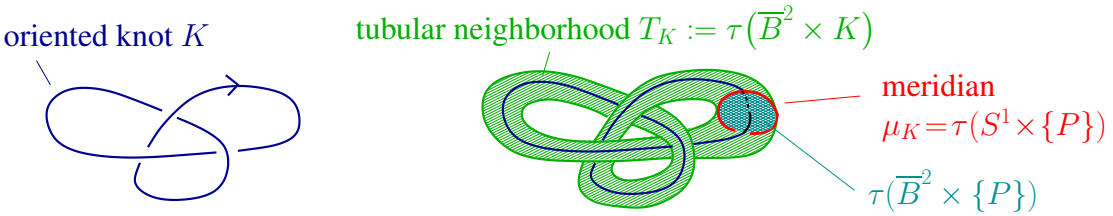
Proof. Let $\eta \in \Omega^{n-1}(M)$ be a closed form. We see that
$$\int\limits_{\partial M} i^*(\eta) \underset{\substack{\uparrow \\ \text{by Stokes' Theorem 75.11}}}{=} \int\limits_M d(\eta) \underset{\substack{\uparrow \\ \text{since } \eta \text{ is closed}}}{=} \int\limits_M 0 = 0.$$
■

Our proof of the Knot Complement-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 85.10 also relies on the following theorem, which is a variation on the usual Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2, and which we will prove in the appendix to this chapter.

Theorem 85.12. (Submanifold–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem) Let M be a closed smooth manifold and let $R, S \subseteq M$ be two codimension zero submanifolds such that the following hold:
(1) $M = R \cup S$.
(2) R and S are compact.
(3) $R \cap S = \partial R = \partial S$.
There exist homomorphisms $\delta_k \colon \mathrm{H}^k_{\mathrm{dR}}(R \cap S) \to \mathrm{H}^{k+1}_{\mathrm{dR}}(M)$ such that the following sequence is exact:
$$\cdots \to \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{(i^M_R, -i^M_S)} \mathrm{H}^k_{\mathrm{dR}}(R) \oplus \mathrm{H}^k_{\mathrm{dR}}(S) \xrightarrow{i^R_{R\cap S} \oplus i^S_{R\cap S}} \mathrm{H}^k_{\mathrm{dR}}(R \cap S) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR}}(M) \to \ldots$$

Proof of the Knot Complement-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 85.10. Let $K \subseteq S^3$ be an oriented knot. By the Meridian Proposition 109.2 (4) we know that there exists a smooth embedding $\tau \colon \overline{B}^2 \times K \to S^3$ with $\tau(0, x) = x$ for all $x \in K$ and such that $\mu_K = \tau(S^1 \times \{P\})$ for some $P \in K$. We set $X_K := S^3 \backslash \tau(B^2 \times K)$ and we set $T_K := \tau(\overline{B}^2 \times K)$. Note that it follows from the Smooth Embedding Theorem 30.15, together with the Codimension-Zero Submanifold Proposition 25.49, that X_K and T_K are compact codimension-zero submanifolds of S^3 with $X_K \cap T_K = \partial X_K = \partial T_K$.
We start out with the following claim:

Claim. The inclusion $X_K \to S^3 \setminus K$ induces an isomorphism on all de Rham cohomology groups.



Proof. We consider the following map:

$$\begin{array}{rcl} (S^3 \setminus K) \times [0,1] & \to & X_K = S^3 \setminus \tau(B^2 \times K) \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \in S^3 \setminus \tau(\overline{B}^2 \times K), \\ \tau(\frac{w}{1-t+t\cdot|z|}, z), & \text{if } P = \tau(w,z) \in \tau((\overline{B}^2 \setminus \{0\}) \times K). \end{cases} \end{array}$$

Note that this map is a deformation retraction from $S^3 \setminus K$ to X_K.[854] The claim follows from this observation together with the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 (1b). ⊞

By the claim it remains to determine the de Rham cohomology groups of X_K. From the above Submanifold–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 85.12 we obtain the following exact sequence:

$$\begin{array}{rcccccccccc} 0 & \leftarrow & \mathrm{H}^3_{\mathrm{dR}}(\partial T_K) & \xleftarrow{i^* \oplus i^*} & \mathrm{H}^3_{\mathrm{dR}}(X_K) & \oplus & \mathrm{H}^3_{\mathrm{dR}}(T_K) & \xleftarrow{(i^*, -i^*)} & \mathrm{H}^3_{\mathrm{dR}}(S^3) & \xleftarrow{\delta_2} \\ & \leftarrow & \mathrm{H}^2_{\mathrm{dR}}(\partial T_K) & \xleftarrow{i^* \oplus i^*} & \mathrm{H}^2_{\mathrm{dR}}(X_K) & \oplus & \mathrm{H}^2_{\mathrm{dR}}(T_K) & \xleftarrow{(i^*, -i^*)} & \mathrm{H}^2_{\mathrm{dR}}(S^3) & \xleftarrow{\delta_1} \\ & \leftarrow & \mathrm{H}^1_{\mathrm{dR}}(\partial T_K) & \xleftarrow{i^* \oplus i^*} & \mathrm{H}^1_{\mathrm{dR}}(X_K) & \oplus & \mathrm{H}^1_{\mathrm{dR}}(T_K) & \xleftarrow{(i^*, -i^*)} & \mathrm{H}^1_{\mathrm{dR}}(S^3) & \xleftarrow{\delta_0} \\ & \leftarrow & \mathrm{H}^0_{\mathrm{dR}}(\partial T_K) & \xleftarrow{i^* \oplus i^*} & \mathrm{H}^0_{\mathrm{dR}}(X_K) & \oplus & \mathrm{H}^0_{\mathrm{dR}}(T_K) & \xleftarrow{(i^*, -i^*)} & \mathrm{H}^0_{\mathrm{dR}}(S^3) & \leftarrow 0. \end{array}$$

We now need to write down all known de Rham cohomology groups that appear in the above sequence:

- In the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$–Proposition 80.4 we determined the de Rham cohomology groups of spheres. In particular we saw that $\mathrm{H}^k_{\mathrm{dR}}(S^k) = \mathbb{R} \cdot [S^k]^*_{\mathrm{dR}}$.
- We denote by $p\colon \partial T_K = \tau(S^1 \times K) \to S^1$ and $q\colon \partial T_K = \tau(S^1 \times K) \to K$ the two obvious projections. It follows from the Torus-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 80.11 (2) that $p^*([S^1]^*_{\mathrm{dR}})$ and $q^*([K]^*_{\mathrm{dR}})$ form a basis of $\mathrm{H}^1_{\mathrm{dR}}(\partial T_K)$.[855]
- By a slight abuse of notation we denote by $q\colon T_K = \tau(\overline{B}^2 \times K) \to K$ also the obvious projection. It follows from the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 that $q^*([K]_{\mathrm{dR}})$ is a basis of $\mathrm{H}^1_{\mathrm{dR}}(T_K)$.

[854] Note that it follows easily from the Homotopy Combination Lemma 18.8 (1) and the Codimension-Zero Smooth Embedding Proposition 30.21 that this map is indeed continuous.

[855] As we discussed on page 1655, this conclusion can also be deduced from the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21.

We now obtain the following exact sequence:

$$
\begin{array}{ccccccccccc}
0 & \leftarrow & \underbrace{\mathrm{H}^3_{\mathrm{dR}}(\partial T_K)}_{=0} & \xleftarrow{i^*\oplus i^*} & \mathrm{H}^3_{\mathrm{dR}}(X_K) & \oplus & \underbrace{\mathrm{H}^3_{\mathrm{dR}}(T_K)}_{=0} & \xleftarrow{(i^*,-i^*)} & \underbrace{\mathrm{H}^3_{\mathrm{dR}}(S^3)}_{=\mathbb{R}\cdot[S^3]^*_{\mathrm{dR}}} & \xleftarrow{\delta_2} & \\
& \leftarrow & \underbrace{\mathrm{H}^2_{\mathrm{dR}}(\partial T_K)}_{=\mathbb{R}\cdot[\partial T_K]^*_{\mathrm{dR}}} & \xleftarrow{i^*\oplus i^*} & \mathrm{H}^2_{\mathrm{dR}}(X_K) & \oplus & \underbrace{\mathrm{H}^2_{\mathrm{dR}}(T_K)}_{=0} & \xleftarrow{(i^*,-i^*)} & \underbrace{\mathrm{H}^2_{\mathrm{dR}}(S^3)}_{=0} & \xleftarrow{\delta_1} & \\
& & & \uparrow & & & & & & & \\
& & & \text{by the } \mathrm{H}^*_{\mathrm{dR}}\text{-Kernel-of-Boundary Inclusion Lemma 85.11} & & & & & & & \\
& & & \text{we know that both maps are the zero map} & & & & & & & \\
& \leftarrow & \underbrace{\mathrm{H}^1_{\mathrm{dR}}(\partial T_K)}_{=\mathbb{R}\cdot p^*([S^1]^*_{\mathrm{dR}})\oplus\mathbb{R}\cdot q^*([K]^*_{\mathrm{dR}})} & \xleftarrow{i^*\oplus i^*} & \mathrm{H}^1_{\mathrm{dR}}(X_K) & \oplus & \underbrace{\mathrm{H}^1_{\mathrm{dR}}(T_K)}_{=\mathbb{R}\cdot q^*([K]^*_{\mathrm{dR}})} & \xleftarrow{(i^*,-i^*)} & \underbrace{\mathrm{H}^1_{\mathrm{dR}}(S^3)}_{=0} & \xleftarrow{\delta_0} & \\
& \leftarrow & \underbrace{\mathrm{H}^0_{\mathrm{dR}}(\partial T_K)}_{=\mathbb{R}\cdot\mathrm{const}_1} & \xleftarrow{i^*\oplus i^*} & \mathrm{H}^0_{\mathrm{dR}}(X_K) & \oplus & \underbrace{\mathrm{H}^0_{\mathrm{dR}}(T_K)}_{=\mathbb{R}\cdot\mathrm{const}_1} & \xleftarrow{(i^*,-i^*)} & \underbrace{\mathrm{H}^0_{\mathrm{dR}}(S^3)}_{=\mathbb{R}\cdot\mathrm{const}_1} & \leftarrow & 0.
\end{array}
$$

We make the following observations:

(3) First note that we obtain from the above discussion that the connecting homomorphism $\delta_2\colon \mathbb{R}\cdot[\partial T_K]^* = \mathrm{H}^2_{\mathrm{dR}}(\partial T_K) \to \mathrm{H}^3_{\mathrm{dR}}(S^3) = \mathbb{R}\cdot[S^3]^*_{\mathrm{dR}}$ is an epimorphism. Since every epimorphism $\mathbb{R}\to\mathbb{R}$ is necessarily an isomorphism we see that the connecting homomorphism $\delta_2\colon \mathrm{H}^2_{\mathrm{dR}}(\partial T_K) \to \mathrm{H}^3_{\mathrm{dR}}(S^3)$ is in fact an isomorphism. It follows that $\mathrm{H}^3_{\mathrm{dR}}(X_K) = 0$.

(2) The discussion in (3) and the exact sequence also gives us that $\mathrm{H}^2_{\mathrm{dR}}(X_K) = 0$.

(1) Next we consider the following maps:

$$
\mathrm{H}^1_{\mathrm{dR}}(X_K) \oplus \underbrace{\mathrm{H}^1_{\mathrm{dR}}(T_K)}_{=\mathbb{R}\cdot q^*([K]^*_{\mathrm{dR}})} \xrightarrow[\cong]{i^*\oplus i^*} \underbrace{\mathrm{H}^1_{\mathrm{dR}}(\partial T_K)}_{=\mathbb{R}\cdot p^*([S^1]^*_{\mathrm{dR}})\oplus\mathbb{R}\cdot q^*([K]^*_{\mathrm{dR}})} \xrightarrow{j^*} \underbrace{\mathrm{H}^1_{\mathrm{dR}}(\mu_K)}_{=\mathbb{R}\cdot[\mu_K]^*_{\mathrm{dR}}},
$$

where $j\colon \mu_K \to \partial T_K$ is the inclusion. We make the following observations:

(a) The lower part of the above exact sequence tells us that the homomorphism to the left is an isomorphism.

(b) Since $q\circ i = q\colon T_K \to K$ we see that $i^*(q^*([K]^*_{\mathrm{dR}})) = q^*([K]^*_{\mathrm{dR}})$.

(c) It follows from (a) and (b) that $\mathrm{H}^1_{\mathrm{dR}}(X_K) \cong \mathbb{R}$ and that the projection of $i^*(\mathrm{H}^1_{\mathrm{dR}}(X_K))$ to $\mathbb{R}\cdot p^*([S^1]^*_{\mathrm{dR}})$ is non-zero.

(d) The map $p\circ j\colon \mu_K \to S^1$ is an orientation-preserving diffeomorphism. Hence $j^*(p^*([S^1]^*_{\mathrm{dR}})) = [\mu_K]^*_{\mathrm{dR}}$. Furthermore the map $q\circ j\colon K\to S^1$ is a constant map. Hence $j^*(q^*([K]^*_{\mathrm{dR}})) = 0$.

(e) It follows from (c) and (d) that the inclusion $i\circ j\colon \mu_K \to X_K$ induces an isomorphism on first de Rham cohomology.

(0) Finally note that the very bottom of the above exact sequence implies immediately that $\mathrm{H}^0_{\mathrm{dR}}(X_K) \cong \mathbb{R}$. ∎

85.5. **Appendix: The Mayer–Vietoris sequence for submanifolds.** Let M be a smooth manifold. In the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 we saw that a cover $M = U\cup V$ by *open subsets* of M gives rise to an exact sequence of de Rham cohomology groups. In many situations one is given a cover $M = R\cup S$ by *compact codimension-zero submanifolds*. In this appendix we will see that such a cover also gives rise to an exact sequence of de Rham cohomology groups. More precisely, we will prove the following theorem:

Theorem 85.12. (Submanifold–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem) Let M be a smooth manifold and let $R, S \subseteq M$ be two codimension zero submanifolds such that the following hold:

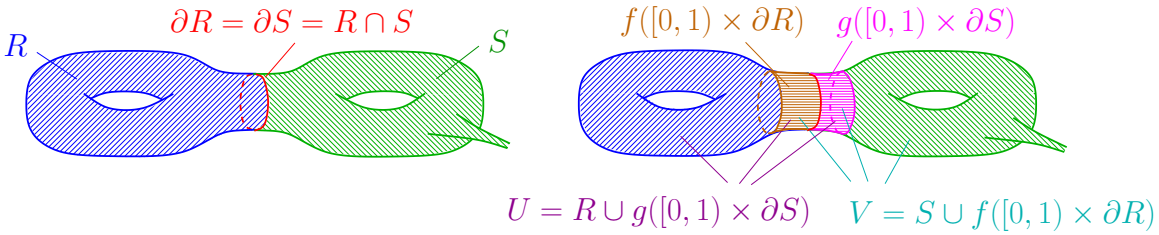
(1) $M = R \cup S$.
(2) R and S are closed subsets of M.[856]
(3) $R \cap S$ is a union of components of ∂R and it is a union of components of ∂S.

There exist natural connecting homomorphisms $\delta_k \colon \mathrm{H}^k_{\mathrm{dR}}(R \cap S) \to \mathrm{H}^{k+1}_{\mathrm{dR}}(M)$ such that the following sequence is exact:

$$\cdots \to \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{(i^M_R, -i^M_S)} \mathrm{H}^k_{\mathrm{dR}}(R) \oplus \mathrm{H}^k_{\mathrm{dR}}(S) \xrightarrow{i^R_{R\cap S} \oplus i^S_{R\cap S}} \mathrm{H}^k_{\mathrm{dR}}(R \cap S) \xrightarrow{\delta_k} \mathrm{H}^{k+1}_{\mathrm{dR}}(M) \to \dots$$

Proof. Let M be a smooth manifold and let $R, S \subseteq M$ be codimension-zero submanifolds such that the conditions (1), (2) and (3) are satisfied. In fact, to simplify the notation a little bit we assume that $R \cap S = \partial R = \partial S$. By the Smooth Collar Neighborhood Theorem 35.6 we can pick smooth collars $f \colon [0,1] \times \partial R \to R$ and $g \colon [0,1] \times \partial S \to S$. We set

$$U \coloneqq R \cup \underbrace{g([0,1) \times \partial S)}_{\subseteq S} \quad \text{and} \quad V \coloneqq S \cup \underbrace{f([0,1) \times \partial R)}_{\subseteq R}.$$



In the following claim we see that U and V are "open approximations" of R and S:

Claim.

(a) U is an open subset of M and R is a deformation retract of U.
(b) V is an open subset of M and S is a deformation retract of V.
(c) $U \cap V$ is an open subset of M and $R \cap S$ is a deformation retract of $U \cap V$.

Proof.

(a) (a) First we show that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (2) closed subsets of M we obtain from the Cover-Open Subset Lemma 3.20 (2a) that it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R. Furthermore note that it follows immediately from the Basics-of-Collars Lemma 35.7 that $U \cap S = g([0,1) \times \partial S)$ is open in S.

(b) We need to show that R is a deformation retract of U. To do we consider the following map:

$$\begin{aligned} U \times [0,1] &\to U \\ (u,t) &\mapsto \begin{cases} u, & \text{if } u \in R, \\ g(s \cdot t, y), & \text{if } u = g(s,y) \in g([0,1) \times \partial S). \end{cases} \end{aligned}$$

[856] Note that by the Compact-Closed Lemma 3.39 (2) this condition is satisfied if R and S are compact.

It remains to show that this map is continuous. By the Homotopy Combination Lemma 18.8 (1) it suffices to show that R and $g([0,1)\times\partial S)$ are closed subsets of U. We see this as follows: By our hypothesis R and S are closed subsets of M. It follows from the Subspace-Characterization Lemma 3.17 (2) that $R = R\cap U$ and $g([0,1)\times\partial S) = S\cap U$ are closed subsets of U.

(b) The proof of this statement is of course almost identical to the proof of (a).

(c) The proof of this statement is very similar to the proof of (a). ⊞

Next we consider the following diagram:

$$\begin{array}{ccccccccccc}
\ldots \to & \mathrm{H}^k_{\mathrm{dR}}(M) & \xrightarrow{(i_U^M,-i_V^M)} & \mathrm{H}^k_{\mathrm{dR}}(U)\oplus\mathrm{H}^k_{\mathrm{dR}}(V) & \xrightarrow{i^U_{U\cap V}\oplus i^V_{U\cap V}} & \mathrm{H}^k_{\mathrm{dR}}(U\cap V) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}_{\mathrm{dR}}(M) & \to\ldots \\
& \downarrow{\scriptstyle \mathrm{id}} & & \cong\downarrow{\scriptstyle i_R^U} \quad {\scriptstyle i_S^V}\downarrow\cong & & \cong\downarrow{\scriptstyle i^{U\cap V}_{R\cap S}} & & \downarrow{\scriptstyle \mathrm{id}} & \\
\ldots \to & \mathrm{H}^k_{\mathrm{dR}}(M) & \xrightarrow{(i_R^M,-i_S^M)} & \mathrm{H}^k_{\mathrm{dR}}(R)\oplus\mathrm{H}^k_{\mathrm{dR}}(S) & \xrightarrow{i^R_{R\cap S}\oplus i^S_{R\cap S}} & \mathrm{H}^k_{\mathrm{dR}}(R\cap S) & \xrightarrow{=:\delta_k} & \mathrm{H}^{k+1}_{\mathrm{dR}}(M) & \to\ldots
\end{array}$$

We make the following clarifications and observations:

(a) All the vertical maps are induced by inclusions. It follows from the functoriality of de Rham cohomology that the two squares to the left commute.

(b) By the claim we know that U and V form an open cover of M. Thus we can apply the original Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 to the decomposition $M = U\cup V$ and we obtain natural connecting homomorphisms $\delta_k\colon \mathrm{H}^k_{\mathrm{dR}}(U\cap V)\to\mathrm{H}^{k+1}_{\mathrm{dR}}(M)$ that make the top sequence exact.

(c) It follows from the claim and the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 (1) that the vertical maps are all isomorphisms.

(d) Since the vertical maps are isomorphisms we see that we can define connecting homomorphisms $\delta_k\colon \mathrm{H}^k_{\mathrm{dR}}(R\cap S)\to\mathrm{H}^{k+1}_{\mathrm{dR}}(M)$ that make the bottom sequence exact. ■

85.6. Exercises for Chapter 85.

Exercise 85.1. We recall the following facts:

- Each $\mathbb{R}\mathrm{P}^n$ is a closed connected non-empty n-dimensional smooth manifold.
- By the Real Projective Space Orientability Lemma 31.46 we know that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd.
- For a closed closed connected non-empty n-dimensional smooth manifold M we know by the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Propositions 84.7 and 85.8 that

$$\mathrm{H}^n_{\mathrm{dR}}(M) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } M \text{ is orientable,} \\ 0, & \text{if } M \text{ is non-orientable.} \end{cases}$$

Use these facts together with the analog of the Mayer–Vietoris sequences in the proof of the $\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.12 to show that for any $n\in\mathbb{N}$ we have

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k=0 \text{ or } k=n \text{ and } n \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Remark. This exercise gives an alternative (and arguably a more illuminating) proof of the $\mathbb{R}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 85.9.

Exercise 85.2. We consider the following closed oriented 6-dimensional smooth manifolds:
- The product $S^2 \times S^4$.
- The complex projective space $\mathbb{C}\mathrm{P}^3$.

Prove the following two statements:
(a) The de Rham cohomology groups of $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are isomorphic.
(b) Use wedge products to show that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not diffeomorphic.

Exercise 85.3. Let $n \in \mathbb{N}_0$. Show that the smooth manifold $\mathbb{C}\mathrm{P}^n$ admits an orientation-reserving diffeomorphism if and only if n is odd.

Exercise 85.4. We first introduce or recall the following definition: Let X be a topological space and let $A \subseteq X$ be a subspace. We say that A is a *retract* of X if there exists a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$.
(a) Show that S^{n-1} is not a retract of $\overline{B}^n$.
(b) Let $m < n$. We denote by $\iota\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the standard embedding. Show that $\iota(\mathbb{C}\mathrm{P}^m)$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Exercise 85.5. Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. Calculate the de Rham cohomology groups of the connected $M\#N$ in terms of the de Rham cohomology groups of M and N.
Hint. Use the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 and use the $\mathrm{H}^*_{\mathrm{dR}}$-Kernel-of-Boundary Inclusion Lemma 85.11.

Exercise 85.6.
(a) Use Exercise 85.5 to show that the de Rham cohomology groups of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $S^2 \times S^2$ are isomorphic.
(b) Show that the de Rham cohomology *rings* of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $S^2 \times S^2$ are *not* isomorphic.

Exercise 85.7. We consider the cone on $\mathbb{C}\mathrm{P}^2$, i.e. we consider the topological space

$$\mathrm{Cone}(\mathbb{C}\mathrm{P}^2) \;=\; (\mathbb{C}\mathrm{P}^2 \times [0,1])/\sim \qquad \text{where } (x,0) \sim (y,0) \text{ for all } x, y \in \mathbb{C}\mathrm{P}^2.$$

Show that $\mathrm{Cone}(\mathbb{C}\mathrm{P}^2)$ does not admit the structure of a closed 5-dimensional smooth manifold.
Hint. Assume there exists a smooth manifold structure. What can you say about the de Rham cohomology groups?

Exercise 85.8. Let $\widetilde{M} \to M$ be a covering of smooth manifolds. Show that if M is connected, then for any $x, y \in M$ there exists a bijection between $p^{-1}(x)$ and $p^{-1}(y)$.

Exercise 85.9. Let $p, q \in \mathbb{N}$ be coprime. In Definition 91.4 we will introduce the corresponding lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$. For example it follows easily from the definitions that $L(2,1)$ is diffeomorphic to $\mathbb{R}\mathrm{P}^3$. If you are comfortable with lens spaces, then compute the isomorphism types of their de Rham cohomology groups.

Exercise 85.10. Let M be a compact oriented n-dimensional smooth manifold manifold with connected non-empty boundary. Use the $\mathrm{H}^*_{\mathrm{dR}}$-Kernel-of-Boundary Inclusion Lemma 85.11 to give a two-line proof that ∂M is not a retract of M.
Remark. Note that it follows from the Replace-by-Smooth Retraction Proposition 36.4 (1) that if there exists a retraction, then there exists also a smooth retraction.

Exercise 85.11. Let M be a closed oriented connected n-dimensional smooth manifold.

(a) Let $\Phi\colon \overline{B}^n \to M$ be a smooth embedding. Show that $\mathrm{H}^n_{\mathrm{dR}}(M \setminus \Phi(B^n)) = 0$.
Hint. Use the $\mathrm{H}^*_{\mathrm{dR}}$-Kernel-of-Boundary Inclusion Lemma 85.11 and use the Submanifold–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 85.12.

(b) Show that for any $P \in M$ we have $\mathrm{H}^n_{\mathrm{dR}}(M \setminus \{P\}) = 0$.

CHAPTER 86

The degree of a map between smooth manifolds

In Definition 46.5 we used regular values and Sard's Theorem 38.1 to introduce the notion of a degree of a smooth map between closed oriented smooth manifolds. In this chapter we will (re-) introduce the notion of the degree using de Rham cohomology. We will show that the two notions of a degree lead to the same result. We will also see that our new approach has several interesting consequences, for example we will be able to express the linking number of two disjoint oriented knots via an integral.

This chapter is organized as follows:

- In Section 86.1 we will use de Rham cohomology to (re-) introduce the degree of a continuous map between closed oriented connected non-empty smooth manifolds.
- In Section 86.2 we will state and prove the Gauß Linking number Theorem 86.13.
- In Section 86.3 we will state and prove the Gauß-Bonnet Theorem 86.19.
- In Section 86.4 we will provide exercises for Chapter 86.

86.1. **The degree of a map between smooth manifolds.** In this section we will use the de Rham cohomology groups to (re-) introduce the fundamental concept of the degree of a continuous map between closed oriented connected non-empty smooth manifolds of the same dimension. To define this notion it is convenient to recall Definition 84.8.

Definition 86.1. Let M be a closed oriented connected non-empty n-dimensional smooth manifold. By the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 there exists a unique de Rham cohomology class $[M]^*_{\mathrm{dR}} \in \mathrm{H}^n_{\mathrm{dR}}(M)$ with

$$\int_M [M]^*_{\mathrm{dR}} \;=\; 1.$$

We refer to $[M]^*_{\mathrm{dR}}$ as the de Rham fundamental class of M. Note that it follows from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 that $\mathrm{H}^n_{\mathrm{dR}}(M) = \mathbb{R} \cdot [M]^*_{\mathrm{dR}}$.

Example. Let $n \in \mathbb{N}_0$. We consider the closed oriented connected smooth manifold $\mathbb{R}^n/\mathbb{Z}^n$. It follows immediately from the calculation on page 1541 that the de Rham fundamental class is represented by the smooth n-form $\overline{dx^1} \wedge \cdots \wedge \overline{dx^n}$. □

Definition 86.2. Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional smooth manifolds. Since $\mathrm{H}^n_{\mathrm{dR}}(M) = \mathbb{R} \cdot [M]^*_{\mathrm{dR}}$ there exists a unique $\deg(f) \in \mathbb{R}$ with[857]

$$f^*([N]^*_{\mathrm{dR}}) \;=\; \deg(f) \cdot [M]^*_{\mathrm{dR}} \qquad \in\; \mathrm{H}^n_{\mathrm{dR}}(M).$$

We refer to $\deg(f)$ as the degree of the map $f\colon M \to N$.

Example. For self-maps of spheres this notion of a degree equals the notion that we introduced in Definition 80.6. □

In Definition 46.5 we already introduced the notion of a degree of a continuous map between closed oriented smooth manifolds. We will see shortly that these two notions agree. But first let us explore the definition of the degree as stated above. The following lemma gives a convenient way to calculate the degree of a smooth map.

Lemma 86.3. (Degree-Integration Lemma) Let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds.

(1) If ω is a smooth n-form on N with $\int_N \omega = 1$, then
$$\deg(f) \;=\; \int\limits_M f^*(\omega).$$
(2) We have $\deg(f) = \int\limits_M f^*([N]^*_{\mathrm{dR}})$.

Proof. Statement (2) follows of course from Statement (1), the definition of $[N]^*_{\mathrm{dR}}$ and the definition of the integral of a de Rham cohomology class. Statement (1) in turn follows from the following observations:

(a) As we discussed in the High-Dimensional-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.9 (2), every smooth n-form on N defines a cohomology class. In particular ω defines a cohomology class.
(b) It follows from (a) and the definition of $[N]^*_{\mathrm{dR}}$ that $[\omega] = [N]^*_{\mathrm{dR}}$.
(c) It follows from the definition of $[M]^*_{\mathrm{dR}}$ that $f^*([\omega]) = [f^*(\omega)] = \int\limits_M f^*(\omega) \cdot [M]^*_{\mathrm{dR}}$.
(d) It follows from (c) that $\deg(f) = \int\limits_M f^*(\omega)$. ∎

The following proposition collects some of the key properties of the degree:

Proposition 86.4. (Smooth Manifold–Degree Proposition)

(1) Let $f_0, f_1\colon M \to N$ be continuous maps between two closed oriented connected non-empty smooth manifolds of the same dimension. If f_0 and f_1 are homotopic, then
$$\deg(f_0) \;=\; \deg(f_1).$$
(2) If $f\colon L \to M$ and $g\colon M \to N$ are continuous maps between closed oriented connected non-empty smooth manifolds of the same dimension, then
$$\deg(g \circ f) \;=\; \deg(g) \cdot \deg(f).$$
(3) Let $f\colon M \to N$ be a diffeomorphism closed oriented connected non-empty smooth manifolds of the same dimension. Then
$$\deg(f) \;=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof.

(1) Since $f, g\colon M \to N$ are homotopic we know by the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.6 that $f^* = g^*\colon \mathrm{H}^n_{\mathrm{dR}}(N) \to \mathrm{H}^n_{\mathrm{dR}}(M)$. But this implies of course that $\deg(f) = \deg(g)$.
(2) Note that

[857] Recall that in the $\mathrm{H}^n_{\mathrm{dR}}$-Functor Proposition 79.5 we associated to a continuous map an induced map on de Rham cohomology.

$$\begin{aligned}\deg(g\circ f)\cdot[N]^*_{\mathrm{dR}} &= (g\circ f)^*([N]^*_{\mathrm{dR}}) \overset{\text{by the } \mathrm{H}^n_{\mathrm{dR}}\text{-Functor Proposition 79.5}}{=} f^*(g^*([N]^*_{\mathrm{dR}})) &&= f^*(\deg(g)\cdot[M]^*_{\mathrm{dR}})\\ &= \deg(g)\cdot f^*([M]^*_{\mathrm{dR}}) &&= \deg(g)\cdot\deg(f)\cdot[L]^*_{\mathrm{dR}}.\end{aligned}$$

Thus it follows that $\deg(g)\cdot\deg(f)=\deg(g\circ f)$.

(3) Let ω be a smooth n-form on N with $\int_N\omega=1$. The desired statement follows from the following calculation:

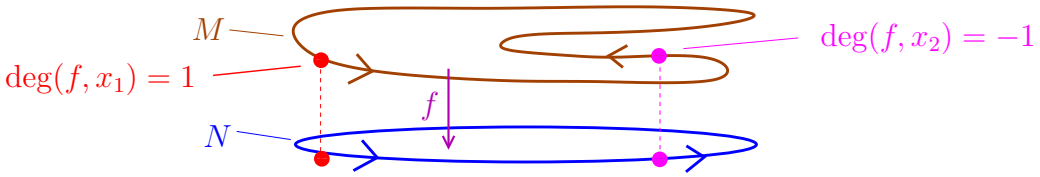
$$\deg(f) \underset{\substack{\uparrow\\ \text{by the Degree Integration Lemma 86.3}}}{=} \int_M f^*(\omega) \underset{\substack{\uparrow\\ \text{by the Form Integration Proposition 74.15}}}{=} \epsilon\cdot\int_N\omega \quad \text{where} \begin{cases}\epsilon=+1, & \text{if } f \text{ is orientation-preserving}\\ \epsilon=-1, & \text{if } f \text{ is orientation-reversing.}\end{cases}$$

■

In the following we will see that the degree defined above agrees with the degree from Definition 46.5. To get to this point it is convenient to recall Definition 46.1:

Definition 86.5. Let $f\colon M\to N$ be a smooth map between two closed oriented n-dimensional smooth manifolds and let x. If f is a local diffeomorphism at x, then we define the local degree of f at the point x as

$$\deg(f,x) := \begin{cases}+1, & \text{if } f \text{ is orientation-preserving at } x\\ -1, & \text{if } f \text{ is orientation-reversing at } x.\end{cases}$$

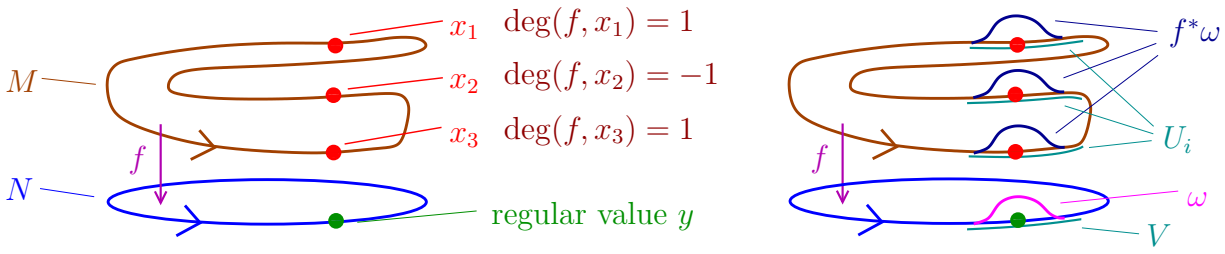


Proposition 86.6. (De Rham Degree–Local Data Proposition) Let $f\colon M\to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds. Let $y\in N$ be a regular value of f in the sense of Definition 32.1. Then[858]

$$\deg(f) = \sum_{x\in f^{-1}(y)}\deg(f,x).$$

Proof. Let $f\colon M\to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds and let $y\in N$ be a regular value of f. It follows from the Stack-of-Records Lemma 32.3 that $f^{-1}(y)$ consists of finitely many points $x_1,\dots,x_k$ and that there exists disjoint open neighborhoods $U_1,\dots,U_k$ of $x_1,\dots,x_k$ and an open neighborhood V of y such that for each $i\in\{1,\dots,k\}$ the map $f\colon U_i\to V$ is a diffeomorphism.

[858]By Sard's Theorem 38.1 we know that there exists a regular value y of f. It follows that the degree is an integers, which was a priori not clear from the definition.



As we saw in the proof of the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 there exists a smooth n-form ω on N with $\int_N \omega = 1$ and with $\operatorname{supp}(\omega) \subseteq V$. Now note that

$$\deg(f) \overset{\text{by the Degree-Integration Lemma 86.3}}{=} \int_M f^*(\omega) = \sum_{i=1}^k \int_{U_i} f^*(\omega) \overset{\text{by the Form Integration Proposition 74.15}}{=} \sum_{i=1}^k \deg(f,x_i) \cdot \underbrace{\int_V \omega}_{=1} = \sum_{i=1}^k \deg(f,x_i).$$

since we have the inclusion $\operatorname{supp}(f^*(\omega)) \subseteq U_1 \cdots \cup \cdots \cup U_k$

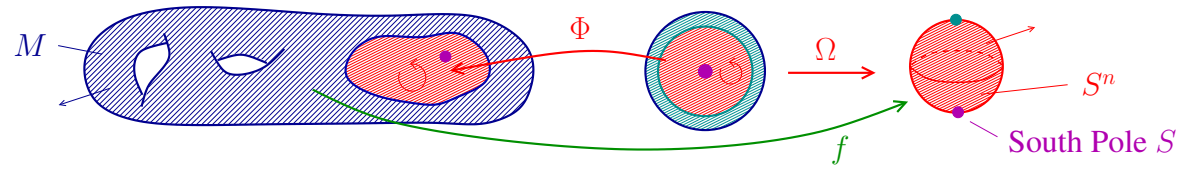
■

Example. Let M be a closed oriented connected non-empty n-dimensional smooth manifold. As in the Degree Realization Proposition 46.19 we will now show that M admits a smooth map $f\colon M \to S^n$ of degree 1: First, since M is non-empty we can pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n \to M$. By the Smooth Transition Function Lemma 24.9 there exists a monotonously increasing smooth function $\tau\colon [0,1] \to [0,1]$ such that $\tau(t) = t$ for $t \in [0, \frac{1}{2}]$ and such that $\tau(t) = 1$ for $t \in [\frac{3}{4}, 1]$. Next, similar to the setting of the Ball Quotient-Sphere Lemma 8.10 we consider the map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n & \to & S^n \\ r \cdot v & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \cdots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \cdots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi \cdot \tau(r)) \\ \sin(\pi \cdot \tau(r)) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}} \end{array} \quad \text{with } r \in [0,1] \text{ and } v \in S^{n-1}.$$

Finally we consider the map

$$\begin{array}{rcl} f\colon M & \to & S^n \\ P & \mapsto & \begin{cases} \Omega(Q), & \text{if } P = \Phi(Q) \text{ for some } Q \in \overline{B}^n, \\ (0,\ldots,0,1), & \text{if } P \notin \Phi(B^n). \end{cases} \end{array}$$



Using the Smooth Pasting Proposition 25.19 one can easily show that f is smooth and that the South Pole $(0,\ldots,0,-1)$ is a regular value with a single preimage, namely $\Phi(0)$. Furthermore one can easily verify that the map f is orientation-preserving at $\Phi(0)$. It follows from the De Rham Degree–Local Data Proposition 86.6 that $\deg(f) = 1$. □

We can now prove easily that the two notions of "degree" that we have encountered mercifully agree:

Proposition 86.7. (Degrees Agree Proposition) Given any continuous map between closed oriented connected non-empty n-dimensional smooth manifolds the degree from Definition 46.5 agrees with the degree defined above in Definition 86.2.

Remark. The two definitions of the degree in Definition 46.5 and in Definition 86.2 have complementary advantages and disadvantages:

(1) The Definition 46.5 is great for performing actual calculations. In particular the results from Section 46.3, 46.4 and 46.5 are now available to us, even if we did not study the earlier definition of degree.
(2) On the other hand the more formal Definition 86.2 is much more suitable for proving theoretical statements. For example compare the ease with which we proved the Smooth Manifold–Degree Proposition 86.4 to the effort we had to put into proving the corresponding statements in the Degree Homotopy Invariance Theorem 46.6 and the Degree Multiplicativity Theorem 46.7. □

Proof. Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional smooth manifolds. By the Whitney Approximation Theorem 36.1 we know that f is homotopic to a smooth map $g\colon M \to N$. By Sard's Theorem 38.1 there exists a regular value y of g. Now we see that

$$\underbrace{\deg(f)}_{\text{Definition 46.5}} := \deg(g) := \sum_{x\in g^{-1}(x)} \deg(g,x) \overset{\text{by the De Rham Degree–Local Data Proposition 86.6}}{=} \deg(g) =: \underbrace{\deg(f).}_{\text{Definition 86.2}}$$

■

On page 1709 we pointed out that it is easy to find smooth maps of degree one to spheres. It is less clear how one can define interesting maps of degree one out of spheres. This observation motivated the following question:

Question 46.21. Does there exist a smooth map $S^2 \to S^1 \times S^1$ of degree one?

Now that we have more tools regarding degrees we can finally give a negative answer to the above question. In fact the following proposition even proves a much stronger result:

Proposition 86.8. (Sphere-to-Torus Proposition) Every smooth map $S^2 \to S^1 \times S^1$ has degree zero.

In our proof of the Sphere-to-Torus Proposition 86.8 we will make use of the following lemma, which might be of independent interest:

Lemma 86.9. (Wedge-Integration-Degree Lemma) Let $f\colon M \to N$ be a smooth between two closed oriented connected smooth manifolds of the same dimension and let $\varphi \in \mathrm{H}^k_{\mathrm{dR}}(N)$ and $\psi \in \mathrm{H}^{n-k}_{\mathrm{dR}}(N)$. Then

$$\int_M f^*\varphi \wedge f^*\psi \;=\; \deg(f) \cdot \int_N \varphi \wedge \psi.$$

Proof of the Wedge-Integration-Degree Lemma 86.9. Since $\mathrm{H}^n_{\mathrm{dR}}(N) = \mathbb{R}\cdot[N]^*_{\mathrm{dR}}$ we see that $\varphi\wedge\psi = r\cdot[N]^*_{\mathrm{dR}}$ for some $r\in\mathbb{R}$. We now see that

$$\int_M f^*\varphi\wedge f^*\psi \;=\; \int_M f^*(\varphi\wedge\psi) \;=\; r\cdot\int_M f^*([N]^*_{\mathrm{dR}}) \;\underset{\substack{\uparrow\\ \text{by the Degree-Integration}\\ \text{Lemma 86.3}}}{=}\; r\cdot\deg(f) \;=\; \deg(f)\cdot\int_N \varphi\wedge\psi.$$

■

Proof of the Sphere-to-Torus Proposition 86.8. Let $f\colon S^2\to S^1\times S^1$ be a smooth map. By the Torus-as-Quotient Lemma 25.42 we know that there exists a diffeomorphism $g\colon S^1\times S^1\to\mathbb{R}^2/\mathbb{Z}^2$. We denote by $\overline{dx}$ and $\overline{dy}$ the closed 1-forms on $\mathbb{R}^2/\mathbb{Z}^2$ that we introduced on page 1541. Recall that on page 1541 we saw that

$$\int_{\mathbb{R}^2/\mathbb{Z}^2} [\overline{dx}]\wedge[\overline{dy}] \;=\; 1.$$

Now we see that

$$\deg(f) \;\overset{\substack{\text{by Smooth Manifold–Degree}\\ \text{Proposition 86.4}}}{\overset{\downarrow}{=}}\; \pm\cdot\deg(g\circ f) \;\overset{\substack{\text{by the Wedge-Integration-Degree}\\ \text{Lemma 86.9}}}{\overset{\downarrow}{=}}\; \pm\cdot\deg(g\circ f)\cdot\int_{S^2}(g\circ f)^*([\overline{dx}])\wedge(g\circ f)^*([\overline{dy}])$$
$$\underset{\substack{\uparrow\\ \text{since by the Sphere–}\mathrm{H}^*_{\mathrm{dR}}\text{–Proposition 80.4 we}\\ \text{know that } \mathrm{H}^1_{\mathrm{dR}}(S^2)=0}}{=}\; \pm\deg(g\circ f)\cdot\int_{S^2} 0\wedge 0 \;=\; 0.$$

■

86.2. **Gauß Linking number Theorem.** Recall that in Definition 52.2 we introduced the linking number for suitable submanifolds of some $\mathbb{R}^n$. For the purpose of this section we will only consider a special case:

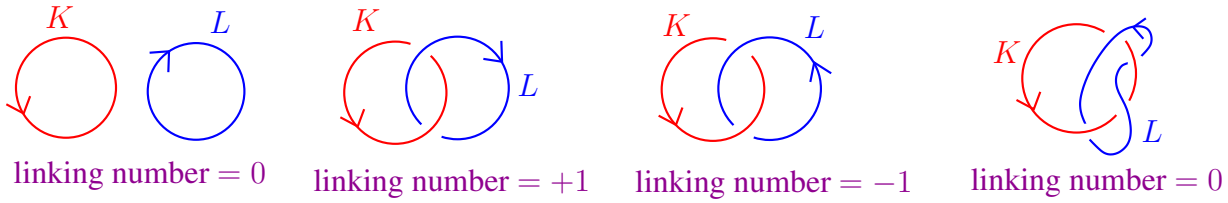
Definition 86.10.

(1) An oriented knot in $\mathbb{R}^3$ is an oriented submanifold of $\mathbb{R}^3$ that is diffeomorphic to S^1.

(2) Let K and L be disjoint oriented knots in $\mathbb{R}^3$. We define the linking number $\mathrm{lk}(K,L)$ as[859]

$$\mathrm{lk}(K,L) \;:=\; -\deg\begin{pmatrix} K\times L & \to & S^2\\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|}\end{pmatrix}$$

In the figure below we show some linking numbers that we calculated on page 1186



Remark. Note that it follows immediately from the Degrees Agree Proposition 86.7 that Definition 86.10 is consistent with the earlier Definition 52.2 of the linking number. □

Next we recall two basic properties of linking numbers:

[859]The minus sign is a little awkward, but we picked it so that it is consistent with the definition that is usually used in knot theory.

Proposition 86.11. (Linking Number–Isotopy Proposition) Let K and L be two disjoint oriented knots in $\mathbb{R}^3$. If $F\colon (K \sqcup L) \times [0,1] \to \mathbb{R}^3$ is a smooth isotopy, then,

$$\mathrm{lk}(F_0(K), F_0(L)) \;=\; \mathrm{lk}(F_1(K), F_1(L)).$$

Proposition 86.12. (Linking Number–Symmetry Proposition) Let K and L be two disjoint oriented knots in $\mathbb{R}^3$. We have the equality

$$\mathrm{lk}(L, K) \;=\; \mathrm{lk}(K, L).$$

Proof. These two propositions are in fact a special cases of the Linking Number–Isotopy Proposition 52.6 and the Linking Number–Symmetry Proposition 52.7. As we saw back then, both statements follow from elementary statements of the degree, which in this chapter were summarized in the Smooth Manifold–Degree Proposition 86.4. ■

Now we can formulate the following remarkable theorem:

Theorem 86.13. (Gauß Linking Number Theorem) Let K and L be two disjoint oriented knots in $\mathbb{R}^3$. Let $f\colon [0,2\pi] \to K$ and $g\colon [0,2\pi] \to L$ be two compact parametrizations in the sense of Definition 71.16. Then

$$\underbrace{\mathrm{lk}(K,L)}_{\text{see Definition 86.10}} \;=\; \frac{1}{4\pi} \int_{s=0}^{s=2\pi} \int_{t=0}^{t=2\pi} \frac{\det \overbrace{\big((f(s)-g(t)) \;\; f'(s) \;\; g'(t)\big)}^{3\times 3\text{–matrix}}}{\|f(s)-g(t)\|^3}\, dt\, ds.$$

Remark.

(1) The linking number of an oriented 2-component link was initially introduced by Gauß in 1833 by the integral in the Gauß Linking Number Theorem 86.13. We refer to the figure below for a scan of the relevant paragraphs in [**Gaus1877**, p. 605].

Von der *Geometria Situs*, die Leibnitz ahnte und in die nur einem Paar Geometern (Euler und Vandermonde) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts.

Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie x, y, z; der zweiten x', y', z' und

$$\iint \frac{(x'-x)(\mathrm{d}y\,\mathrm{d}z' - \mathrm{d}z\,\mathrm{d}y') + (y'-y)(\mathrm{d}z\,\mathrm{d}x' - \mathrm{d}x\,\mathrm{d}z') + (z-z')(\mathrm{d}x\,\mathrm{d}y' - \mathrm{d}y\,\mathrm{d}x')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{\frac{3}{2}}} = V$$

dann ist dies Integral durch beide Linien ausgedehnt

$$= 4m\pi$$

und m die Anzahl der Umschlingungen.

The term *geometria situs* is hereby the earlier name for topology. We refer to [**Epp1999**, p. 73] and [**RN2011**, Section 5] for more information on the work of Gauß regarding the linking number.

(2) The statement of the Gauß Linking Number Theorem 86.13 is really quite remarkable. For example it is not at all clear why the integral in the Gauß Linking Number Theorem 86.13 is an integer and why it is invariant under a smooth isotopy of $K \sqcup L$.

(3) The Gauß Linking Number Theorem 86.13 is also proved in [**RN2011**, Section 5] and [**tD2000**, p. 284]. □

Proof. Our proof of the Gauß Linking Number Theorem 86.13 requires the following four little preparations:

(1) Let ω be the smooth 2-form that at each $P \in S^2$ is given by

$$\begin{array}{rcl} \mathrm{T}_P S^2 \times \mathrm{T}_P S^2 & \to & \mathbb{R} \\ (v,w) & \mapsto & \frac{1}{4\pi} \cdot \det(P\ v\ w). \end{array}$$

On page 1538, or alternatively on page 1576, we showed that $\int_{S^2} \omega = 1$.

(2) The map

$$\begin{array}{rcl} f \times g \colon [0,2\pi] \times [0,2\pi] & \to & K \times L \\ (s,t) & \mapsto & (f(s), f(t)) \end{array}$$

is a compact parametrization in the sense of Definition 74.18.

(3) We introduce the following two maps:

$$\begin{array}{rcl} \tau \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (v,w) & \mapsto & v - w \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu \colon \mathbb{R}^n \setminus \{0\} & \to & S^{n-1} \\ x & \mapsto & \frac{x}{\|x\|}. \end{array}$$

(4) A straightforward calculating using the Chain Rule, shows that

$$\mathrm{D}\Big(\underbrace{\nu \circ \tau \circ (f,g)}_{\substack{\text{this is a smooth map} \\ [0,2\pi] \times [0,2\pi] \to S^{n-1}}} \Big)_{(s,t)} = \frac{1}{\|f(s) - g(t)\|} \cdot \begin{pmatrix} f'(s) & -g'(s) \end{pmatrix}.$$

Now we calculate that

$$\begin{array}{rcl} \mathrm{lk}(K,L) & \overset{\text{by Definition 86.10}}{=} & -\deg((\nu \circ \tau)|_{K \times L}) \overset{\substack{\text{by the Degree-Integration} \\ \text{Lemma 86.3 and by (1)}}}{=} -\int\limits_{K \times L} ((\nu \circ \tau)|_{K \times L})^*(\omega) \overset{\substack{\text{by the Integral-via-Parametrization} \\ \text{Proposition 74.19 and by (2)}}}{=} \\ & = & -\int\limits_{s=0}^{s=2\pi} \int\limits_{t=0}^{t=2\pi} ((\nu \circ \tau)|_{K \times L})^*(\omega)_{(f(s),g(t))}(f'(s), g'(t))\, dt\, ds \\ & = & -\int\limits_{s=0}^{s=2\pi} \int\limits_{t=0}^{t=2\pi} \frac{1}{4\pi} \det \begin{pmatrix} \frac{f(s)-g(t)}{\|f(s)-g(t)\|} & \frac{f'(s)}{\|f(s)-g(t)\|} & \frac{-g'(t)}{\|f(s)-g(t)\|} \end{pmatrix} dt\, ds. \\ & \overset{\substack{\text{follows from the definition of } \omega \\ \text{together with (4)}}}{=} & \end{array}$$

But this expression equals of course the expression given in the theorem. ■

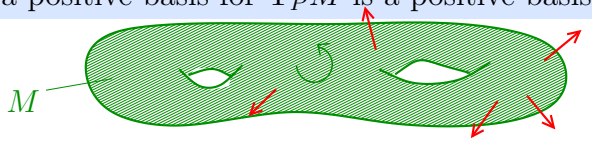
Neither the definition of the linking number in Definition 86.10 nor the integral in the Gauß Linking Number Theorem 86.13 are particularly useful for calculating the linking number in an explicit situation. Fortunately in Chapter 52 we already saw several ways for how

to compute linking numbers. Also note that in Section 284.1 we will return to the study of linking numbers where we will do more calculations. Finally note that in the Linking Number Reformulation Proposition 285.8 we will see that there are at least five different ways for how to think about the linking number.

86.3. **The Gauß-Bonnet Theorem.** We start out by introducing a new piece of notation:

Notation 86.14. Let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented smooth submanifold of codimension one. Recall that by the Orientations-via-Normal Vector Fields Proposition 31.27 the orientation of M gives rise to a unit normal field $v_M \colon M \to \mathbb{R}^{n+1}$, which at each $P \in M$ is characterized by the following properties:[860]

(1) $v_M(P)$ is normal to the tangent space $\mathrm{T}_P M \subseteq \mathbb{R}^{n+1}$,
(2) $\|v_M(P)\| = 1$,
(3) $v_M(P)$ followed by a positive basis for $\mathrm{T}_P M$ is a positive basis for $\mathbb{R}^{n+1}$.



Example. We equip S^n with the standard orientation which we introduced on page 817. It follows immediately from the definitions that for each $P \in S^n$ we have $v_{S^n}(P) = P$. □

Notation 86.15.

(1) Given a smooth submanifold $M \subseteq \mathbb{R}^{n+1}$ of codimension one and given a smooth map $F \colon M \to \mathbb{R}^{n+1}$ we denote, as on page 1526, by $\det(F \star_1 \cdots \star_n)$ the n-form that at each $P \in M$ is given by

$$\begin{array}{rcl} (\mathrm{T}_P M)^n & \to & \mathbb{R} \\ (v_1, \ldots, v_n) & \mapsto & \det(F(P)\ v_1\ \ldots\ v_n). \end{array}$$

Note that if $F(P) \notin \mathrm{T}_P M$, then elementary linear algebra implies that the alternating n-form $\det(F \star_1 \cdots \star_n)_P$ is non-zero and thus a basis for the 1-dimensional vector space $\wedge^* \mathrm{T}_P M$.

(2) If M is an *oriented* smooth submanifold of codimension one, then we write

$$\Xi(M) \ := \ \det(v_M \star_1 \cdots \star_n).$$

By the discussion on (1) we know that for each $P \in M$ the n-form $\Xi(M)$ is a basis for $\wedge^* \mathrm{T}_P M$.

Example. As above we equip S^n with the standard orientation which we introduced on page 817. It follows immediately from the definitions and the above example that for any $P \in S^n$ and $w_1, \ldots, w_n \in \mathrm{T}_P S^n$ we have

$$\Xi(S^n)_P(w_1, \ldots, w_n) \ = \ \det(P\, w_1\ \ldots\ w_n).$$ □

Now we are finally able to introduce the main definition of this section:

Definition 86.16. Let $M \subseteq \mathbb{R}^{n+1}$ be an oriented smooth submanifold of codimension one. Since v_M is non-zero everywhere we know that at each $P \in M$ the alternating n-form

[860] The map $v_M \colon M \to S^n$ is precisely the Gauß map from Definition 53.31.

$\Xi(M)_P$ is a basis of $\wedge^* \mathrm{T}_P M$. This means that there exists a unique $K_M(P) \in \mathbb{R}$ such that

$$\underbrace{v_M^*(\Xi(S^n))_P}_{\substack{\text{pullback of the } n\text{-form} \\ \Xi(S^n) \text{ under } v_M \colon M \to S^n}} = K_M(P) \cdot \Xi(M)_P \quad \in \wedge^* \mathrm{T}_P M = \mathbb{R} \cdot \Xi(M)_P.$$

As in [**GuH2019**, p. 163] we refer to $K_M(P)$ as the total curvature of M at P.

Remark. The name "total curvature" strongly suggests that there is a different definition of $K_M(P)$ that has more immediate geometric content. In fact in [**doC2016**, Chapter 6] the connection of $K_M(P)$ to the curvature of Riemann manifolds is explained. Since these are not lecture notes on differential geometry we will not go further into this direction. But we hope that the examples below give some intuition for what is going on. □

Example. Let $n \in \mathbb{N}_0$. We consider $S^n \subseteq \mathbb{R}^{n+1}$. We claim that $K_{S^n}(P) = 1$ for each $P \in S^n$: Thus let $P \in S^n$. By definition we need to show that $v_{S^n}^*(\Xi(S^n))_P = 1 \cdot \Xi(S^n)_P$. But this follows from the observation that on page 1714 we saw that $v_{S^n} \colon S^n \to S^n$ is just the identity map. □

The following lemma shows that an orientation-preserving isometry does not affect the total curvature:

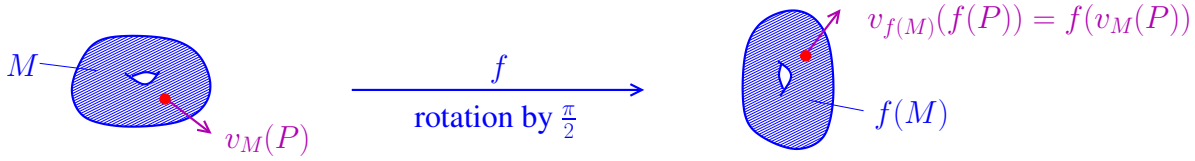
Lemma 86.17. (Total Curvature–Isometry Lemma) Let $M \subseteq \mathbb{R}^{n+1}$ be an oriented smooth submanifold of codimension one. Furthermore let $A \in \mathrm{SO}(n+1)$ and let $v \in \mathbb{R}^{n+1}$. We consider the map

$$\begin{array}{rccc} f \colon & \mathbb{R}^{n+1} & \to & \mathbb{R}^{n+1} \\ & x & \mapsto & A \cdot x. \end{array}$$

For any $P \in M$ we have $K_{f(M)}(f(P)) = K_M(P)$.

Proof. We start out with two observations:

(a) It follows easily from the definitions and elementary linear algebra that for any $P \in M$ we have $v_{f(M)}(f(P)) = f(v_M(P))$. In particular we have $v_{f(M)} \circ f = f \circ v_M \colon M \to S^n$.
(b) It follows easily from (a) that $f^*(\Xi(f(M))_{f(P)}) = \Xi(M)_P \in \wedge^* \mathrm{T}_P M$.



We now see that

$$\begin{array}{rcl} f^*(K_{f(M)}(f(P)) \cdot \Xi(f(M))_{f(P)}) & \overset{\text{Definition 86.16 of } K_{f(M)}(f(P))}{=} & f^*(v_{f(M)}^*(\Xi(S^n))_{f(P)}) \\ & = & v_M^*(\Xi(S^n)_P) \\ & \overset{\text{follows from an elementary calculation, since } A \in \mathrm{SO}(n+1)}{=} & f^*(K_M(P) \cdot \Xi(f(M))_{f(P)}). \end{array}$$

$$\begin{array}{rcl} \overset{\text{by (a) we have } v_{f(M)} \circ f = f \circ v_M \text{ thus we have } f^* \circ v_{f(M)}^* = v_M^* \circ f^*}{=} & v_M^*(f^*(\Xi(S^n))_P) \\ = & K_M(P) \cdot \Xi(M)_P \\ \overset{\text{Definition 86.16 of } K_M(P)}{=} & f^*(K_M(P) \cdot \Xi(f(M))_{f(P)}). \end{array}$$

$$\overset{\text{follows from (b) and since } f^* \text{ is } \mathbb{R}\text{-linear}}{=}$$

It follows that $K_{f(M)}(f(P)) = K_M(P)$. ■

The following proposition gives us a non-trivial calculation of total curvatures:

Proposition 86.18. (Graph–Total Curvature Proposition) Let $U \subseteq \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}$ be a smooth function and let $x_0 \in U$ such that $\mathrm{D}f_{x_0} = 0$. We consider the graph

$$M \;:=\; \{(x, f(x)) \,|\, x \in U\}$$

which we give the orientation such that the projection to U is orientation-preserving. As in Definition 81.5 we consider the Hessian matrix

$$\mathrm{Hess}(f)_{x_0} \;:=\; \left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\Big|_0\right)_{i,j=1,\ldots,n} \;\in\; \mathrm{M}(n \times n, \mathbb{R}).$$

With this notation we have[861]

$$K_M(x_0, f(x_0)) \;=\; \det(-\mathrm{Hess}(f)_{x_0}).$$

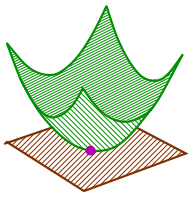
Example.

(1) Let $a_1, \ldots, a_n \in \mathbb{R}$. We consider the smooth function

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & \sum_{i=1}^{n} a_i \cdot x_i^2 \end{array}$$

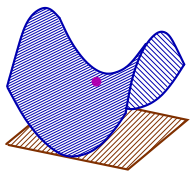
and we consider the corresponding graph M.

graph of $x_1^2 + x_2^2$



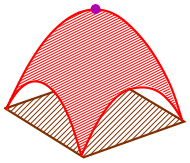
curvature 4

graph of $x_1^2 - x_2^2$



curvature -4

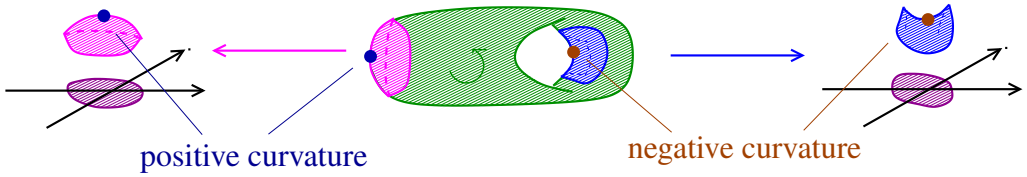
graph of $-x_1^2 - x_2^2$



curvature 4

Clearly all partial derivatives of f vanish at 0. An elementary calculation shows that the Hessian $\mathrm{Hess}(f)_0 \in \mathrm{M}(n \times n, \mathbb{R})$ is the diagonal matrix with diagonal entries $2a_1, \ldots, 2a_n$. It thus follows from the Graph–Total Curvature Proposition 86.18 that $K_M(0, \ldots, 0) = (-1)^n \cdot 2a_1 \cdot \ldots \cdot 2a_n$.

(2) Let $M \subseteq \mathbb{R}^{n+1}$ be an oriented smooth submanifold of codimension one and let $P \in M$. It follows from the Orthogonal Action Lemma 2.37 (4) and the Local Graph Lemma 30.5 that there exists an open neighborhood U of P and an $A \in \mathrm{SO}(n+1)$ such that $A \cdot U$ is a graph.



[861] By the Schwarz Theorem 24.2 we know that $\mathrm{Hess}(f)_{x_0}$ is a symmetric matrix. It follows from the Spectral Theorem 2.35 that $\mathrm{Hess}(f)_{x_0}$ is diagonalizable over $\mathbb{R}$. The eigenvalues $a_1, \ldots, a_n$ (counted with multiplicities) of $\mathrm{Hess}(f)_{x_0}$ are sometimes called the *principal curvatures*. With this language we see that $K_M(x_0, f(x_0))$ is the product of the principal curvatures.

By the Total Curvature–Isometry Lemma 86.17 we know that the application of A does not change the total curvature. Since the total curvature is clearly a local property we see that we can use the Graph–Total Curvature Proposition 86.18 to determine the total curvature at P. □

Proof. Let $U \subseteq \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}$ be a smooth function and let $x_0 \in U$ such that $\mathrm{D}f_{x_0} = 0$. We consider the graph $M := \{(x, f(x)) \mid x \in U\}$ and we set $y_0 := f(x_0)$. To shorten the notation we write $v := v_M\colon M \to \mathbb{R}^{n+1}$. Note that by the definition of the total curvature the statement of the proposition is equivalent to the following statement:

Statement. For any $w_1, \dots, w_n \in \mathrm{T}_{(x_0,y_0)}M$ we have

$$v^*(\Xi(S^n))_{(x_0,y_0)}(w_1, \dots, w_n) \;=\; \det(-\mathrm{Hess}(f)_{x_0}) \cdot \Xi(M)_{(x_0,y_0)}(w_1, \dots, w_n).$$

This is the statement we will prove. To prove this statement we first of all need to understand the map $v\colon M \to \mathbb{R}^{n+1}$:

Claim 1. For each $P = (x, f(x)) \in M$ we have

$$v(P) \;=\; \frac{(-\operatorname{grad} f_x, 1)}{\|(-\operatorname{grad} f_x, 1)\|}.$$

In particular $v(x_0, y_0) = \mathfrak{e}_{n+1}$.

Proof. An elementary argument, see Exercise 29.1, shows that for each $(x, f(x)) \in M$ we have

$$\mathrm{T}_{(x,f(x))}M \;=\; \text{span of the columns of the } ((n+1)\times n)\text{-matrix } \begin{pmatrix} \mathrm{id}_n \\ \mathrm{D}f_x \end{pmatrix}.$$

One can easily verify that the given $v(P)$ is orthogonal to $\mathrm{T}_{(x,f(x))}M$. Clearly $v(P)$ has length one and we leave it to the reader to verify that the orientation conventions work out. ⊞

Claim 2. For any $w \in \mathbb{R}^n$ we have

$$\mathrm{D}v_{(x_0,y_0)} \cdot \begin{pmatrix} w \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} -\mathrm{Hess}(f)_{x_0} \cdot w \\ 0 \end{pmatrix}.$$

Proof. We consider the basis $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ of $\mathrm{T}_{(x_0,y_0)}M = \mathbb{R}^n \times \{0\} = \mathrm{T}_{\mathfrak{e}_{n+1}}S^n$. It suffices to show that with respect to this basis (of the domain and also of the target) the homomorphism $\mathrm{D}v_{(x_0,y_0)}$ is represented by the matrix $\mathrm{Hess}(f)_{x_0}$. Let $i, j \in \{1, \dots, n\}$. We denote by $\pi_j\colon S^n \to \mathbb{R}$ the projection onto the j-th coordinate. It follows from the above discussion that it suffices to show that $\pi_j(\mathrm{D}v_{(x_0,y_0)}(\mathfrak{e}_i)) = -\frac{\partial^2 f}{\partial x_i\, \partial x_j}|_{x_0}$. This desired statement is verified in the following calculation:

$$\pi_j(\mathrm{D}v_{(x_0,y_0)}(\mathfrak{e}_i)) \;\underset{\text{follows from the chain rule and since } \pi \text{ is linear}}{\uparrow}\; \mathrm{D}(\pi_j \circ v)_{(x_0,y_0)}(\mathfrak{e}_i) \;\underset{\text{follows easily from the Differential-from-Partial Derivatives Proposition 24.16}}{\downarrow}\; \frac{\partial}{\partial x_i}\left(x \mapsto \frac{-\frac{\partial f}{\partial x_j}(x)}{\|(-\operatorname{grad} f_x, 1)\|}\right)\Big|_{x_0} \;\underset{\text{follows from an elementary calculation involving the quotient and the product rule, it also uses that } \operatorname{grad} f_{x_0} = 0}{\uparrow}\; -\frac{\partial^2 f}{\partial x_i\, \partial x_j}\Big|_{x_0}.$$

⊞

The following claim now gives us the desired statement:

Claim 3. For any $w_1, \dots, w_n \in \mathrm{T}_{(x_0,y_0)}M = \mathbb{R}^n \times \{0\}$ we have

$$v^*(\Xi(S^n))_{(x_0,y_0)}(w_1, \dots, w_n) \;=\; \det(-\mathrm{Hess}(f)_{x_0}) \cdot \Xi(M)_{(x_0,y_0)}(w_1, \dots, w_n).$$

Proof. Let $\begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_n \\ 0 \end{pmatrix} \in \mathrm{T}_{(x_0,y_0)}M = \mathbb{R}^n \times \{0\}$.

We calculate that

$$\begin{aligned} v^*(\Xi(S^n))_{(x_0,y_0)}\left(\begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_n \\ 0 \end{pmatrix}\right) &\overset{\text{by definition of } v^*}{=} \Xi(S^n)_{v(x_0,y_0)}\left(\mathrm{D}v_{(x_0,y_0)} \cdot \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \mathrm{D}v_{(x_0,y_0)} \cdot \begin{pmatrix} w_n \\ 0 \end{pmatrix}\right) \\ &\overset{\text{by the discussion on page 1714 together with Claim 2}}{=} \det\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\mathrm{Hess}(f)_{x_0} \cdot w_1 \\ 0 \end{pmatrix} \dots \begin{pmatrix} -\mathrm{Hess}(f)_{x_0} \cdot w_n \\ 0 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} 0 & -\mathrm{Hess}(f)_{x_0} \cdot (w_1 \dots w_n) \\ 1 & 0 \end{pmatrix} \\ &= \det(-\mathrm{Hess}(f)_{x_0}) \cdot \det\begin{pmatrix} 0 & (w_1 \dots w_n) \\ 1 & 0 \end{pmatrix} \\ &\overset{\text{follows easily from the definition of } \Xi \text{ and from } v(x_0,y_0) = \mathfrak{e}_{n+1}}{=} \det(-\mathrm{Hess}(f)_{x_0}) \cdot \Xi(M)_{(x_0,y_0)}\left(\begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} w_n \\ 0 \end{pmatrix}\right). \end{aligned}$$

∎

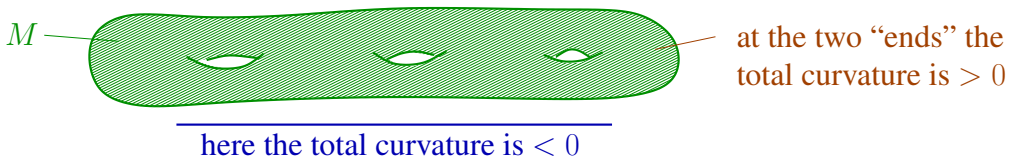
Now we can formulate the main result of this section which was first formulated by Carl Friedrich Gauß and Pierre Bonnet [**Bonn1848**].

Theorem 86.19. (Gauß-Bonnet Theorem) Let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. We have the following two equalities:

(1) $$\int\limits_{x \in M} K_M(x)\, dx = (n+1) \cdot \mathrm{vol}(\overline{B}^{n+1}) \cdot \deg(v_M \colon M \to S^n).$$

(2) If n is even, then[862] $$\int\limits_{x \in M} K_M(x)\, dx = \tfrac{n+1}{2} \cdot \mathrm{vol}(\overline{B}^{n+1}) \cdot \chi^{\mathrm{sm}}(M).$$

Example. We consider the case $n = 2$. Thus let $M \subseteq \mathbb{R}^3$ be a closed oriented connected 2-dimensional smooth submanifold.



In Exercise 73.7 we showed that $\mathrm{vol}(B^3) = \frac{4}{3}\pi$. The Gauß-Bonnet Theorem 86.19 (2) thus says that

$$\int\limits_{x \in M} K_M(x)\, dx = 2\pi \cdot \chi^{\mathrm{sm}}(M).$$

□

Proof.

[862] Here $\chi^{\mathrm{sm}}(M)$ is the smooth Euler characteristic which we introduced in Definition 53.4. It follows from the De Rham-Isomorphism Theorem 231.16, the Universal Coefficient-H*–Field Theorem 202.2, the Homology-χ-Proposition 156.2 and the Euler Characteristics–Equality Corollary 223.6 (see also the discussion on page 4287) that the smooth Euler characteristic equals the de Rham Euler characteristic $\sum\limits_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{R}}(\mathrm{H}^i_{\mathrm{dR}}(M))$ and that it equals the "usual" Euler characteristic which we will define in Definition 156.1.

(1) We start out with the following claim:

Claim. We have the following equality:

$$\int_{S^n} \Xi(v_{S^n}) \;=\; \int_{S^n} \det(v_{S^n} \star_1 \cdots \star_n) \;=\; (n+1)\cdot \mathrm{vol}(\overline{B}^{n+1}).$$

Proof. [863] Recall that on page 1714 we saw that $v_{S^n}(P) = P$ for all $P \in S^n$. By a slight abuse of notation we denote by $v\colon \overline{B}^{n+1} \to \mathbb{R}^{n+1}$ also the map that is given by $v(P) = P$. Now we see that

$$\int_{S^n} \det(v \star_1 \ldots \star_n) \;=\; \int_{S^n} v \cdot \nu \;=\; \int_{\overline{B}^{n+1}} \mathrm{div}(v) \;=\; \int_{\overline{B}^{n+1}} (n+1) \;=\; (n+1)\cdot \mathrm{vol}(\overline{B}^{n+1}).$$

(first $=$: by the Flux-via-Forms Proposition 76.21; second $=$: by the Gauß Divergence Theorem 76.26)

We can now perform the desired calculation:

$$\int_{x\in M} K_M(x)\,dx = \int_{x\in M} \underbrace{(K_M(x)\cdot v_M(x))\cdot v_M(x)}_{= K_M(x) \text{ since } \|v_M(x)\| = 1}\,dx \;\overset{\text{by the Flux-via-Forms Proposition 76.21}}{=}\; \int_M \det(K_M \cdot v_M \star_1 \ldots \star_n)$$

$$= \int_M K\cdot \Xi(M) \;=\; \int_M (v_M)^*(\Xi(S^n)) \;=\; (n+1)\cdot \mathrm{vol}(B^{n+1})\cdot \deg(v_M\colon M \to S^n).$$

(since det is $\mathbb{R}$-multilinear; Definition 86.16 of K_M; by the Degree-Integration Lemma 86.3 and since by the claim we know that $[\Xi(v_{S^n})] = (n+1)\cdot \mathrm{vol}(\overline{B}^{n+1})\cdot [S^n]^*_{\mathrm{dR}}$)

(2) We have the following equalities:

$$\int_{x\in M} K_M(x)\,dx \;=\; (n+1)\cdot \mathrm{vol}(\overline{B}^{n+1})\cdot \deg(v_M\colon M \to S^n) \;=\; \tfrac{1}{2}\cdot(n+1)\cdot \mathrm{vol}(\overline{B}^{n+1})\cdot \chi^{\mathrm{sm}}(M).$$

(by Statement (1); first note that we know by the Degrees Agree Proposition 86.7 that the degree from Definition 46.5 agrees with the degree from Definition 86.2; since n is even, and since v_M is by definition the Gauß map, see Definition 53.31, we obtain this equality from the Gauß Map-Degree Theorem 53.32)

86.4. Exercises for Chapter 86.

Exercise 86.1. Show that every continuous map $S^4 \to \mathbb{C}\mathrm{P}^2$ has degree zero.

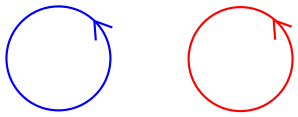
Exercise 86.2. Let $g < h$. Show that every continuous map $\Sigma_g \to \Sigma_h$ has degree zero.

Exercise 86.3. Show that every continuous map $f\colon S^2 \times S^2 \to \mathbb{C}\mathrm{P}^2$ has even degree.

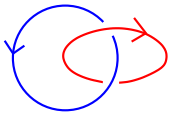
Exercise 86.4. For which $d \in \mathbb{Z}$ can you find a continuous map $S^2 \times S^2 \times S^2 \to \mathbb{C}\mathrm{P}^3$ of degree d?

Exercise 86.5. Use the right-hand side of the Gauß Linking Number Theorem 86.13 to compute the linking numbers of the unlink and the Hopf link.

[863] On page 1576 we already proved this statement for $n = 2$. The proof of the general case is of course essentially the same.



unlink



Hopf link

Exercise 86.6. Let $n \in \mathbb{N}$ and let $f\colon M \to N$ be a continuous map between two closed oriented connected non-empty n-dimensional smooth manifolds. Show that if f is not surjective, then $\deg(f) = 0$.
Hint. Make use of Exercise 85.11.

Exercise 86.7. Show that for any $n \in \mathbb{N}$ and any $m \in \mathbb{Z}$ there exists a continuous map $f\colon S^n \to S^n$ of degree m.
Hint. It might be helpful to start out with the case $n = 1$.

Part X

Fundamental Groups I - Basics

In this part, and most of the subsequent parts, of the lecture notes we enter the realm of algebraic topology. In a nutshell algebraic topology is all about constructing functors from the category of topological spaces to some algebraic category, e.g. the category of (abelian) groups, vector spaces or rings.

The discussion of this part rests on Part I and we will make use of Chapter 18 on homotopies and Chapter 19 on homotopy equivalences. But otherwise it is mostly independent of the other previous chapters. At times some familiarity with topological and smooth manifolds will come in handy.

This part of the lecture notes is organized as follows:

- In Chapter 87 we will give a short motivation and introduction to the techniques that we will develop in these lecture notes.
- In Chapter 88 we will introduce the fundamental group $\pi_1(X, x_0)$ of a topological space together with a base point x_0. We will calculate the fundamental group of the circle S^1 and of the higher-dimensional spheres S^n, $n \geq 2$.
- In Chapter 89 we study first basic applications of fundamental groups. For example we will give a proof of the Fundamental Theorem of Algebra using fundamental groups.
- In Chapter 90 we will see that fundamental groups form a covariant functor from the category of pointed topological spaces to the category of groups. The functoriality is the key for developing the full potential of fundamental groups.
- In Chapter 91 we will introduce the notion of a covering of a topological space and we will study the relationship between coverings and fundamental groups.
- In Chapter 92 we will study the relationship between coverings and fundamental groups. This discussion will allow us to determine the fundamental groups of many interesting topological spaces.
- In Chapter 93 we will use the results from Chapter 91 to prove the Action-π_1-Theorem 93.1 which explains the relationship between group actions and fundamental groups. This theorem allows us to calculate the fundamental groups of tori, Klein bottles, real projective spaces and lens spaces.
- In Chapter 94 we will study semi-serious applications of fundamental groups to "real life problems".
- In Chapter 95 we will study coverings of smooth and topological manifolds. This chapter can certainly be skipped at a first reading.
- In Chapter 96 we will introduce wedges of pointed topological spaces and we will introduce topological graphs.
- In Chapter 97 we will recall some basic concepts and facts about abelian groups.
- In Chapter 98 we turn to the purely algebraic study of (possibly non-abelian) groups. In particular we will introduce the notion of free and amalgamated products of groups.
- In Chapter 99 we will state the Seifert–van Kampen Theorem 99.1 which says that, under suitable hypothesis, the fundamental group of a union of two topological spaces can be written as an amalgamated product of the fundamental groups of the two given topological spaces.
- In Chapter 100 we will learn about presentations of groups. These allow us to express many of the groups we are interested in, in finite data.
- In Chapter 101 we will calculate the fundamental groups of surfaces and we will state the classification of compact 2-dimensional manifolds.

- In Chapter 102 we will prove a variation on the Seifert–van Kampen Theorem 99.1 which is especially suitable for manifolds. We will use it to show that given any finitely presented group π and given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.
- In Chapter 103 we will discuss decision problems. For example we will discuss the question whether it is possible to show that two finitely presented groups are isomorphic.

CHAPTER 87

How to show that topological spaces are (not) homeomorphic?

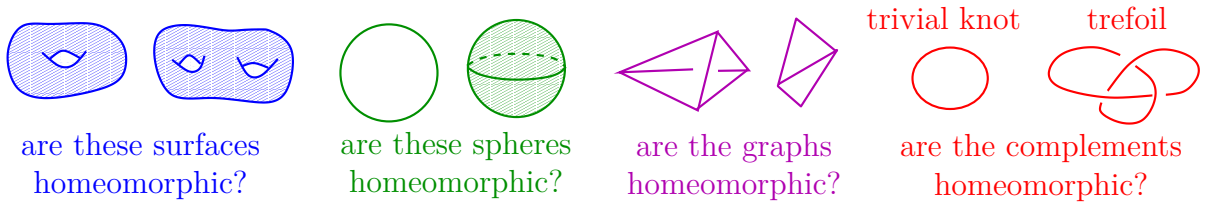
In the next Chapter 88 we will introduce the fundamental group of a pointed topological space. This chapter collects some ideas to motivate the definition of the fundamental group.

In the previous chapters we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.

On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.

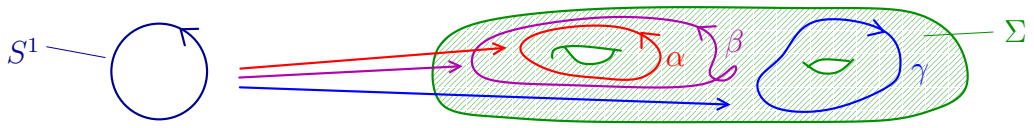
But these criteria are of no use when it comes to studying more subtle examples and questions. For example they do not suffice to address the following questions:

Question 87.1.
(1) Can surfaces with different genera be homeomorphic?
(2) Given $k \neq l \in \mathbb{N}$, is it possible that the spheres S^k and S^l are homeomorphic?
(3) Are the two graphs shown below homeomorphic?
(4) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?
In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?



Here is another type of question that we cannot really address right now.

Question 87.2. We consider the maps $S^1 \to \Sigma$ from S^1 to the surface Σ that are shown in the figure below. The curves α and β can be deformed into one another. But it does not look like one can deform α into γ. But how can one prove the non-existence of a deformation?



We have already seen some ideas for tackling these problems:

(a) In Chapter 78 we introduced the de Rham cohomology groups of smooth manifolds.
(b) In the Cauchy Theorem 39.8 we showed the path integrals of holomorphic functions along two smooth paths α and β agree, if α and β are path-homotopic. This can be used to show that smooth paths in subsets of $\mathbb{C}$ are not path-homotopic.

First let's talk about approach (a). Recall that to each smooth manifold M and each $k \in \mathbb{Z}_{\geq 0}$ we can assign the de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(M)$. In the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8 (2) we saw that if two smooth manifolds are homeomorphic, then their de Rham cohomology groups are isomorphic. These invariants have several disadvantages:

(1) The de Rham cohomology groups are only defined for smooth manifolds.
(2) At least just starting from the definition it is not clear how one can calculate the de Rham cohomology groups of a given smooth manifold.

Later on, in Chapter 198, we will introduce "singular cohomology" which will take care of all of these three issues.

We turn to the second approach from complex analysis. We recall some definitions that are familiar to the reader who has taken a course on complex analysis.

Definition 87.3.

(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a continuous map $\gamma\colon [a,b] \to U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a loop.
(2) Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called path-homotopic in U, if there exists a homotopy rel $\{a,b\}$ from γ_0 to γ_1. In other words, if there exists a continuous map

$$\begin{array}{rcl} \Gamma\colon [a,b] \times [0,1] & \to & U \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

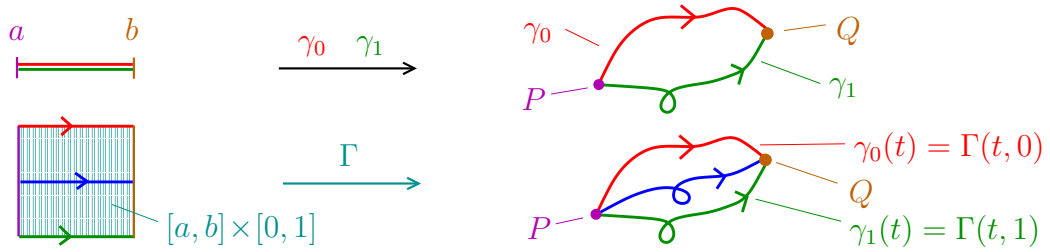
with the following properties:

(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,
(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

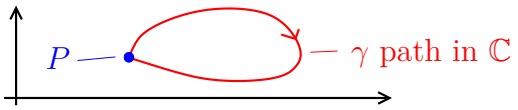
In the figure below we illustrate a path-homotopy between two paths.
(3) A loop $\gamma\colon [a,b] \to U$ is called null-homotopic if it is path-homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.
(4) We say U is simply connected if U is non-empty, if U is path-connected and if each loop is null-homotopic.

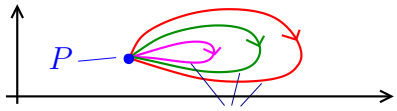
Example. Every loop in $\mathbb{C}$ is null-homotopic. Indeed, we let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop with starting and endpoint P. Then

$$\begin{array}{rcl} \Gamma\colon [a,b] \times [0,1] & \to & \mathbb{C} \\ (t,s) & \mapsto & \gamma(t) \cdot (1-s) + P \cdot s \end{array}$$



is a path-homotopy between the loop γ and the constant path at P.





path-homotopy from γ to a constant path

In particular $\mathbb{C}$ is simply connected.

We recall the Definition 39.6 of the path integral:

Definition 87.4. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a continuous map. Furthermore let $\gamma\colon [a,b] \to U$ a smooth path. We define the path integral of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} \underbrace{f(\gamma(t))}_{\in\mathbb{C}} \cdot \underbrace{\gamma'(t)}_{\in\mathbb{C}}\,dt \;\in \mathbb{C}.$$

product of complex numbers

Examples.

(1) Let $U = \mathbb{C}\setminus\{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ be the loop that is given by $\gamma(t) = \exp(2\pi\,\mathrm{i}t)$, i.e. γ is a loop that "goes once around the origin". Then

$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \frac{1}{\exp(2\pi\,\mathrm{i}t)}\cdot\exp(2\pi\,\mathrm{i}t)\cdot 2\pi\,\mathrm{i}\,dt \;=\; 2\pi\,\mathrm{i}.$$

(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero. □

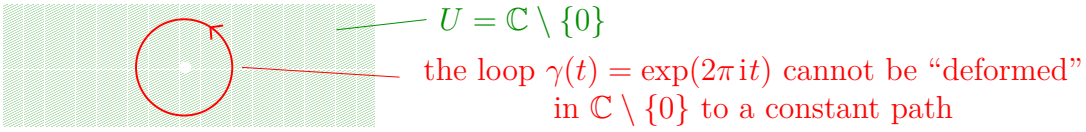
We also recall the following theorem:

Theorem 39.8. (Cauchy Theorem)[864] Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a *holomorphic* function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

[864] We will not really make use of the Cauchy Theorem 950. But it motivates us to study path-homotopies in general topological spaces.

Corollary 87.5. The loop $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ given by $\gamma(t) = \exp(2\pi i t)$ is *not* path-homotopic to a constant path, i.e. γ is not null-homotopic.



Proof. In the above example we saw that $\int_\gamma \frac{1}{z}\,dz \neq 0$ and we saw that for a constant loop δ in $\mathbb{C} \setminus \{0\}$ we have $\int_\delta \frac{1}{z}\,dz = 0$. Thus it follows from the Cauchy Theorem 39.8 that the loop γ is not path-homotopic to a constant loop. ■

Example. If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. It follows from the discussion on page 1726 and Corollary 87.5 that $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ are not homeomorphic. □

This second approach to studying topological spaces, namely through path-homotopies of loops, also has several disadvantages:

(1) A priori these notions are defined only for subsets of $\mathbb{C}$.
(2) This approach does not allow us to show that the two subsets $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0,1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definition of being simply connected makes sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

Regarding the second objection, here is one idea for distinguishing $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0,1\}$: naively one could say that V has "more" loops that are not null-homotopic than U. In the Chapter 88 we will in particular make precise what "more" means.

CHAPTER 88

The fundamental group

In this chapter we assign to a topological space X together with a base point $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$. In this chapter we will calculate the fundamental group for the circle and all spheres. In the later chapters we will develop the tools to calculate the fundamental group for more interesting topological spaces.

This chapter is organized as follows:

- In Section 88.1 we will introduce the notion of a path-homotopy between two paths with the same starting and endpoint.
- In Section 88.2 we will introduce the fundamental group $\pi_1(X, x_0)$ of a pointed topological space (X, x_0).
- In Section 88.3 we will discuss how a change of base points affects the fundamental group.
- In Section 88.4 we will calculate the fundamental group of the circle.
- In Section 88.5 we will calculate the fundamental group of the spheres S^n for $n \geq 2$.
- In Section 88.6 we will give an alternative description of fundamental groups.
- In Appendix 88.7 we will discuss homotopy classes of loops, which do not keep a point fixed.
- In Section 88.8 we will provide exercises for Chapter 88.

In the next Chapter 89 we will study some first applications of fundamental groups.

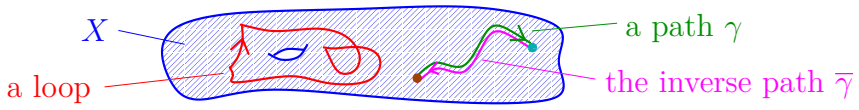
88.1. **Path-homotopy.** In this chapter we extend some of the ideas and notions about paths that we introduced in the previous section to the more general setting of paths in topological spaces. The following definition is a slight generalization of the concept of a path that we initially introduced in Definition 4.30.

Definition 88.1. Let X be a topological space.

(1) We define a path in X as a continuous map $\gamma\colon [a, b] \to X$ where $a < b$ are real numbers. We call $\gamma(a)$ the starting point of γ and we call $\gamma(b)$ the endpoint of γ. Often we say that γ is a path from $\gamma(a)$ to $\gamma(b)$.

(2) A loop is a path for which the starting point and the endpoint coincide.

(3) Given a path $\gamma\colon [a, b] \to X$ we refer to the path $\overline{\gamma}\colon [a, b] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(b + a - t)$ as the inverse path of γ.



Definition 88.2. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1 \colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A path-homotopy between the paths γ_0 and γ_1 is a homotopy rel $\{a,b\}$ between γ_0 and γ_1. In other words, spelling out Definition 18.4, a path-homotopy is a continuous map

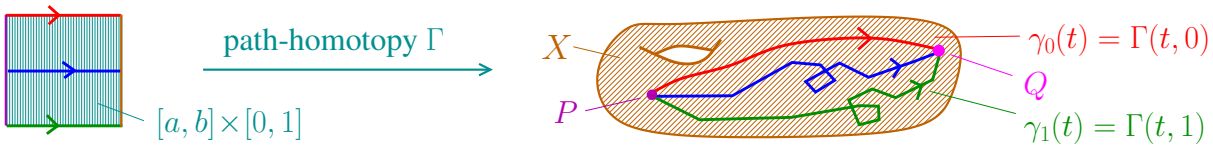
$$\begin{array}{rcl} \Gamma \colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties:

(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,
(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are path-homotopic, and we write $\gamma_0 \simeq_{\mathrm{p}} \gamma_1$.

(3) We say that a loop $\gamma \colon [a,b] \to X$ is null-homotopic, if it is path-homotopic to the constant path δ given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.



Example.

(A) Let C be a convex subset of $\mathbb{R}^n$ and let $\gamma_0, \gamma_1 \colon [a,b] \to C$ be two paths in C with same starting point and the same endpoint. As in the Convex Homotopy Lemma 18.3 we see that

$$\begin{array}{rcl} \Gamma \colon [a,b] \times [0,1] & \to & C \\ (t,s) & \mapsto & \underbrace{\gamma_0(t) \cdot (1-s) + \gamma_1(t) \cdot s}_{\in C, \text{ since } C \text{ is convex}} \end{array}$$

is a path-homotopy between γ_0 and γ_1.[865]

(B) Let $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$. We consider the loop

$$\begin{array}{rcl} \gamma \colon [0,1] & \to & X \\ t & \mapsto & \exp(2\pi \mathrm{i} t) = (\cos(2\pi t), \sin(2\pi t)). \end{array}$$

In Corollary 87.5 we showed that the path is not null-homotopic in X. We will soon give a different proof of this statement.

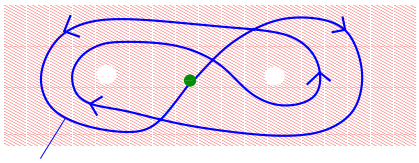
(C) We consider the two loops in the figure below. In both cases it is hard to see how these loops could possibly be path-homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.[866] We will return to this example on page 1936.

(D) We consider the real projective space $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2 / \sim$ where $P \sim -P$ for $P \in S^1 = \partial \overline{B}^2$. Note that

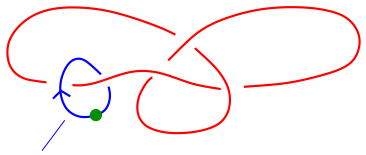
$$\begin{array}{rcl} \gamma \colon [0,1] & \to & \overline{B}^2 / \sim \\ t & \mapsto & [(2t-1, 0)] \end{array}$$

[865] We had already illustrated this fact in the figure on page 1726.

[866] For example the given loop in $\mathbb{C} \setminus \{0,1\}$ is actually null-homotopic in $\mathbb{C} \setminus \{0\}$ and it is null-homotopic in $\mathbb{C} \setminus \{1\}$. In particular our trick of using Corollary 87.5 does not work in this case.

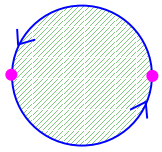


loop in $\mathbb{C} \setminus \{0,1\}$

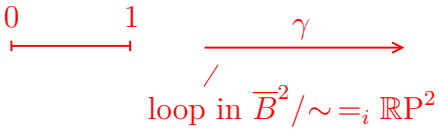


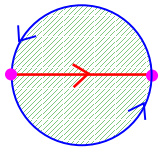
loop in the complement of a knot

is a loop in $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$.[867] But is it null-homotopic?



$\overline{B}^2/\sim =_i \mathbb{R}\mathrm{P}^2$





(E) On page 172 we considered the pseudocircle, namely the set $X = \{A, B, C, D\}$ with a slightly unusual topology. Furthermore we gave an example of a continuous non-constant loop $f\colon S^1 \to X$. The question arises, is this loop null-homotopic? □

Our first lemma of this chapter will provide us with a rich source of path-homotopies. To formulate the lemma we need the following definition:

Definition 88.3. Let $\gamma\colon [a,b] \to X$ be a path in a topological space X. Furthermore d let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to the composition $\gamma \circ \varphi\colon [a,b] \to X$ as a reparametrization of γ.

The following lemma says that reparametrizing does not change the path-homotopy class of a path.

Lemma 88.4. (Reparametrization-Homotopy Lemma) Let $\gamma\colon [a,b] \to X$ be a path in a topological space X. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.

Proof. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$, $\varphi(b) = b$. A path-homotopy between γ and $\gamma \circ \varphi$ is given by[868]

$$\begin{array}{rcl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t \cdot s + \varphi(t) \cdot (1-s)). \end{array}$$

■

The following lemma is a special case of the Homotopy-Equivalence Relation Lemma 18.9.

Lemma 88.5. (Path-Homotopy–Equivalence Relation Lemma) Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.

Proof. This lemma is just the special case of the Homotopy-Equivalence Relation Lemma 18.9 with $X = [a,b]$ and $A = \{a,b\}$. For the reader's convenience we provide the

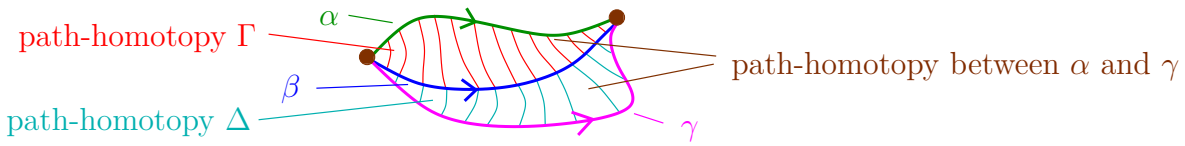
[867] Why is it a loop?

[868] Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?

straightforward proof also in this context. We have to show that "path-homotopy" satisfies the three properties of an equivalence relation, as introduced in Definition 1.16. Let $\alpha, \beta, \gamma\colon [a,b] \to X$ be three paths from x to y.

(1) The "constant path-homotopy" given by $\Gamma(t,s) \coloneqq \gamma(t)$ is a path-homotopy between γ and γ.
(2) Let Γ be a path-homotopy from α to β. It is basically clear that the map Γ' defined by $\Gamma'(t,s) \coloneqq \Gamma(t, 1-s)$ is a path-homotopy from β to α.
(3) Let Γ be a path-homotopy from α to β and let Δ be a path-homotopy from β to γ. We consider the map

$$\begin{array}{rcl} [a,b]\times[0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \Gamma(t,2s), & \text{if } s \in \left[0,\frac{1}{2}\right], \\ \Delta(t,2s-1), & \text{if } s \in \left(\frac{1}{2},1\right] \end{cases} \end{array}$$



It follows from the Homotopy Combination Lemma 18.8 (1) that this map is continuous. It is now clear that this map is a path-homotopy from α to γ. ■

The Path-Homotopy–Equivalence Relation Lemma 88.5 allows us to make the following definition:

Definition 88.6. Given a path $f\colon [a,b] \to X$ in a topological space X we denote by[869] $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the path-homotopy class of f.

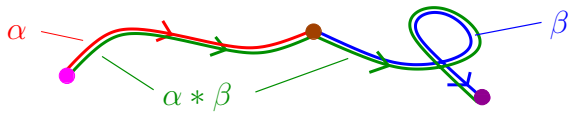
We now recall Definition 4.38:

Definition 88.7. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha*\beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a+2t(b-a)), & \text{if } t \in \left[0,\frac{1}{2}\right] \\ \beta(c+(2t-1)(d-c)), & \text{if } t \in \left[\frac{1}{2},1\right]. \end{cases} \end{array}$$

Similarly we also define the concatenation of a finite number of paths.

The concatenation of two paths is thus given by "first running along α" and then "running along β". Here we reparametrized the path so that the domain of $\alpha * \beta$ is $[0,1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0,1]$.

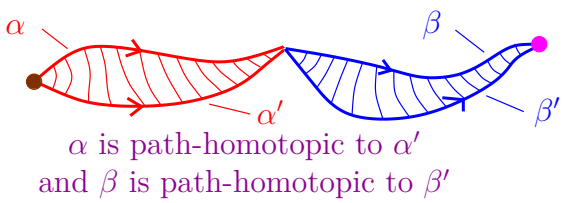


[869] Some basic symbols, like $[-]$ are oversubscribed in mathematics. For example, if one goes carefully through the notes, one sees that we denote by $[f]$ the homotopy class and the path-homotopy class of f. But in practice that should not really be an issue.

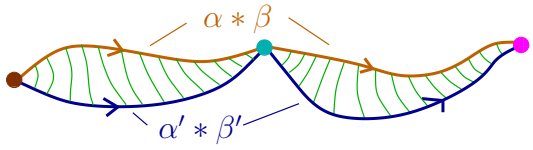
Lemma 88.8. (Path-Homotopy–Concatenation Lemma) Let X be a topological space. Let $\alpha, \alpha' \colon [a,b] \to X$ and $\beta, \beta' \colon [c,d] \to X$ be paths with $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then we have the implication

$$\alpha \simeq_{\mathrm{p}} \alpha' \text{ and } \beta \simeq_{\mathrm{p}} \beta' \implies \alpha * \beta \simeq_{\mathrm{p}} \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.



α is path-homotopic to α'
and β is path-homotopic to β'



then $\alpha * \beta$ is path-homotopic to $\alpha' * \beta'$

Proof. We leave it to the reader to provide the fairly elementary proof. ■

Definition 88.9. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the[870] product of the equivalence classes $[\alpha]$ and $[\beta]$ as follows:

$$[\alpha] \cdot [\beta] \;:=\; [\alpha * \beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 88.10. (Path-Homotopy Product Proposition) Let X be a topological space. Given any three paths $\alpha, \beta, \gamma \colon [0,1] \to X$ the following hold:

(1) If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then[871]

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) \;=\; ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

(2) For $x \in X$ let e_x be the constant path that is given by $e_x(t) := x$. The following equality holds:

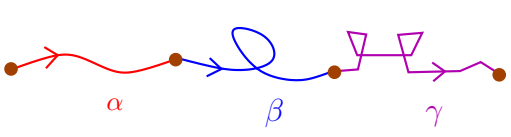
$$[e_{\alpha(0)}] \cdot [\alpha] \;=\; [\alpha] \;=\; [\alpha] \cdot [e_{\alpha(1)}].$$

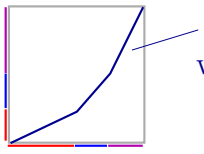
(3) We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold:

$$[\alpha] \cdot [\overline{\alpha}] \;=\; [e_{\alpha(0)}] \quad \text{and} \quad [\overline{\alpha}] \cdot [\alpha] \;=\; [e_{\alpha(1)}].$$

Proof. Let $\alpha, \beta, \gamma \colon [0,1] \to X$ be three paths in a topological space X.

(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. We consider the map $\varphi \colon [0,1] \to [0,1]$ that is shown in the figure below. It follows immediately from the definitions that $(\alpha * (\beta * \gamma)) \circ \varphi = (\alpha * \beta) * \gamma$. Thus it follows from the Reparametrization-Homotopy Lemma 88.4 that $\alpha * (\beta * \gamma) \simeq_{\mathrm{p}} (\alpha * \beta) * \gamma$, which is what we need to show.





graph of $\varphi \colon [0,1] \to [0,1]$
which is used to reparametrize
the path $\alpha * (\beta * \gamma)$ to
obtain the path $(\alpha * \beta) * \gamma$

[870] According to the Path-Homotopy–Concatenation Lemma 88.8 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

[871] Put differently, the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are path-homotopic. Note that these paths are similar in the sense that both paths "run through" the points on α, β and γ, but at different speeds. In general the paths are not equal, for example we have $(\alpha * (\beta * \gamma))(\frac{1}{2}) = \alpha(1)$ and $((\alpha * \beta) * \gamma)(\frac{1}{2}) = \beta(1)$.

(2) We consider the map

$$\begin{aligned}\varphi\colon [0,1] &\to [0,1]\\ t &\mapsto \begin{cases} 0, & \text{if } t \in [0,\tfrac{1}{2}],\\ 2t-1, & \text{if } t \in [\tfrac{1}{2},1].\end{cases}\end{aligned}$$

Then

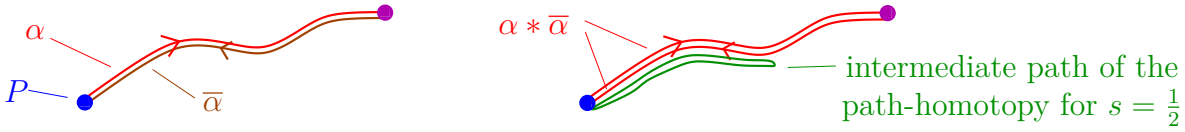
$$e_{\alpha(0)} * \alpha \underset{\text{by Definition 88.7}}{=} \alpha \circ \varphi \underset{\text{by the Reparametrization Homotopy Lemma 88.4}}{\simeq_{\mathrm{p}}} \alpha.$$

This shows that $[e_{\alpha(0)}] \cdot [\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha] \cdot [e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map

$$\begin{aligned}F\colon [0,1]\times[0,1] &\to X\\ (t,s) &\mapsto \begin{cases} \alpha(2t\cdot s), & \text{if } t \in [0,\tfrac{1}{2}],\\ \alpha(2s-2t\cdot s), & \text{if } t \in [\tfrac{1}{2},1].\end{cases}\end{aligned}$$

It follows from the Homotopy Combination Lemma 18.8 (1) that this map is indeed continuous.



For $s = 0$ we obtain the constant path at P and for $s = 1$ we obtain the path $\alpha * \overline{\alpha}$. For any $s \in [0,1]$ we furthermore have $F(0,s) = F(1,s) = P$. Thus F is a path-homotopy from the constant path to $\alpha * \overline{\alpha}$. This implies that $[e_{\alpha(0)}] = [\alpha * \overline{\alpha}] = [\alpha] \cdot [\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha} * \alpha] = [e_{\alpha(1)}]$. ∎

88.2. **The fundamental group of a pointed topological space.** In the Path-Homotopy Product Proposition 88.10 we saw that the concatenation of path-homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the path-homotopy classes do not form a group, since the concatenation of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.

We will resolve this problem by restricting our attention to loops that start and end at a fixed base point.

Definition 88.11. Let X be a topological space and let $x_0 \in X$ be a point. A loop in (X, x_0) is a path $f\colon [0,1] \to X$[872] with $f(0) = f(1) = x_0$, i.e. x_0 is the starting as well as the endpoint of f. If X is understood, then often we just say "loop in x_0".

Proposition 88.12. (π_1–Group Proposition) Let X be a topological space and $x_0 \in X$. The set

$$\pi_1(X,x_0) \coloneqq \{\text{path-homotopy classes of loops in } (X,x_0)\}$$

together with the product map $[\alpha] \cdot [\beta] \coloneqq [\alpha * \beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

Proof. It is clear that the concatenation of any two loops in (X, x_0) is defined. According to the Path-Homotopy–Concatenation Lemma 88.8 the concatenation of paths descends to a well-defined map

[872] Note that part of the definition of "loop in (X, x_0)" is that the domain is the interval $[0,1]$.

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha]\cdot[\beta] := [\alpha * \beta]. \end{array}$$

The Path-Homotopy Product Proposition 88.10 (1)–(3) gives us the following:

(1) The above product map satisfies the associativity axiom.
(2) The path-homotopy class of the constant loop $[e_{x_0}]$ is a neutral element.
(3) The inverse of the path-homotopy class of a loop $\gamma\colon [0,1]\to X$ in x_0 is given by the path-homotopy class of the inverse loop defined by $\overline{\gamma}(t) := \gamma(1-t)$, $t\in[0,1]$. ■

Definition 88.13. Let (X,x_0) be a pointed topological space. We call

$$\pi_1(X,x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X,x_0)\}$$

the fundamental group of X with respect to the base point x_0. If $\pi_1(X,x_0)$ is the trivial group, then we usually write $\pi_1(X,x_0)=0$.[873]

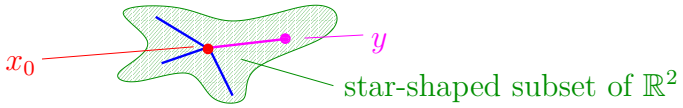
One of our main goals in the intermediate future will be to determine the fundamental group of the topological spaces that we introduced in the earlier chapters, for example spheres, surfaces, the projective spaces $\mathbb{R}\mathrm{P}^n$ and so on. We start with the simplest example.

Example.

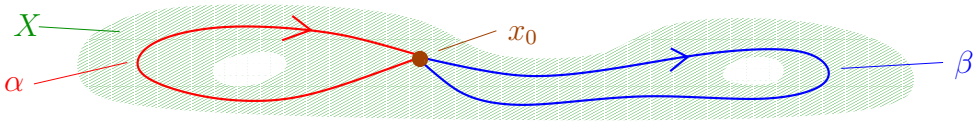
(1) We say that a subset $X\subseteq\mathbb{R}^n$ is star-shaped with respect to $x_0\in X$ if for all $y\in X$ the Euclidean segment $\{x_0\cdot(1-t)+y\cdot t\,|\,t\in[0,1]\}$ lies in X. As on page 1729 we see that all loops in (X,x_0) are null-homotopic. More precisely, if $\gamma\colon[0,1]\to\mathbb{R}^n$ is a loop in (X,x_0), then

$$\begin{array}{rcl} F\colon[0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s)+x_0\cdot s \end{array}$$

is a path-homotopy in X between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X,x_0)=0$.



(2) Let X be a convex subset of $\mathbb{R}^n$. (For example open and closed balls and open and closed rectangles are convex.) Note that X is star-shaped with respect to any $x_0\in X$, thus we obtain from (1) that $\pi_1(X,x_0)=0$ for all $x_0\in X$.
(3) The π_1–Group Proposition 88.12 says that $\pi_1(X,x_0)$ is a group, but it is not clear whether the group is necessarily abelian. In fact, in the figure below we see two loops α, β in (X,x_0) where it is not clear at all whether $[\alpha]\cdot[\beta]\overset{?}{=}[\beta]\cdot[\alpha]\in\pi_1(X,x_0)$.



In fact with the Wedge-π_1-Proposition 99.10 we will be able to show that the two sides are not equal. □

[873] This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X,x_0)$, and it would therefore make more sense to write $\pi_1(X,x_0)=1$ if the group $\pi_1(X,x_0)$ is trivial.

88.3. **Change of base points.** The definition of the fundamental group of a topological space X relies on the choice of a base point x_0. The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

Proposition 88.14. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon [0,1] \to X$ be a path from x_0 to x_1.

(1) (a) The map

$$\begin{array}{rcl} p^*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}\,] \end{array}$$

is well-defined, it only depends on the path-homotopy class of p and it is a group isomorphism.

(b) If $x_0 = x_1$, then the above map p^* is just conjugation by $[p] \in \pi_1(X,x_0)$, i.e. for every $g \in \pi_1(X,x_0) = \pi_1(X,x_1)$ we have

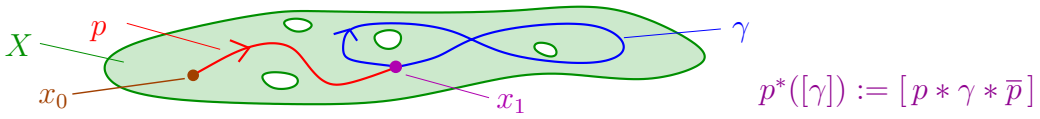
$$p^*(g) \;=\; [p]\cdot g\cdot [p]^{-1} \;\in\; \pi_1(X,x_0).$$

(2) (a) If p is a constant path, then $p^* = \mathrm{id}$.

(b) If $q\colon [0,1] \to X$ is a path from x_1 to another point x_2, then

$$(p*q)^* \;=\; p^*\circ q^*\colon \pi_1(X,x_2) \;\to\; \pi_1(X,x_0).$$

(3) If $\pi_1(X,x_0)$ is abelian, then for any two paths $p,q\colon [0,1]\to X$ from x_0 to x_1 we have $p^* = q^*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$.



Proof. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1]\to X$ from x_0 to x_1.

(1) (a′) Let γ be a loop in (X,x_1). Then $p*\gamma*\overline{p}$ is evidently a loop in (X,x_0). From the Path-Homotopy–Concatenation Lemma 88.8 and the Path-Homotopy Product Proposition 88.10 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}\,] \end{array}$$

is well-defined. Furthermore for two loops γ,δ in (X,x_1) we have

$$\begin{array}{rcccl} \Phi([\gamma])\cdot\Phi([\delta]) & = & [p*\gamma*\overline{p}\,]\cdot[p*\delta*\overline{p}\,] & = & [p*\gamma*\overline{p}*p*\delta*\overline{p}] \\ & = & [p*\gamma*e_{x_1}*\delta*\overline{p}\,] & = & [p*(\gamma*\delta)*\overline{p}\,] \;=\; \Phi([\gamma*\delta]) \;=\; \Phi([\gamma]\cdot[\delta]). \\ & \uparrow & & \uparrow & \end{array}$$

the two statements follow from the Path-Homotopy Concatenation Proposition 88.10 (3) and (2) together with the Path-Homotopy–Concatenation Lemma 88.8

Thus we have shown that Φ is a group homomorphism. Next note that it follows from the Path-Homotopy–Concatenation Lemma 88.8 that $\Phi = p^*$ only depends

[873]The notation p^* is motivated by Exercise 88.16 where we will see that the fundamental groups are in a suitable sense contravariant in the base point.

on the path-homotopy class of p. Finally, below Statement (2) we will show that Φ is even a group isomorphism.

(b) This statement follows easily from the definitions.

(2) These two statements follow easily from the Path-Homotopy Product Proposition 88.10 and the fact that the induced map p^* only depends on the path-homotopy class of p.

(1) (a″) Let $p\colon [0,1] \to x_0 \to X_1$ be a path. It follows from the Path-Homotopy Product Proposition 88.10, the fact that the induced map p^* only depends on the path-homotopy class of p together with Statements (2a) and (2b) that $\overline{p}^*\colon \pi_1(X,x_0) \to \pi_1(X,x_1)$ is an inverse to $p^*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$.

(4) This statement is a straightforward consequence of (1b) and (2). We will provide the proof in Exercise 88.9. ■

We obtain the following lemma as a straightforward corollary to the Change-of-Base Point Proposition 88.14:

Lemma 88.15. (Base Point Independence Lemma) Let X be a topological space and let $x_0, x_1 \in X$. If X is path-connected, then the groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic.

Proof. This statement follows easily from the Change-of-Base Point Proposition 88.14. ■

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we saw that there exists an $x_0 \in X$ such that $\pi_1(X,x_0) = 0$. It now follows from the Base Point Independence Lemma 110.2 that $\pi_1(X,x_1) = 0$ for any point $x_1 \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial. □

If X is a path-connected topological space, then the Base Point Independence Lemma 110.2 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. This leads us to the following definition:

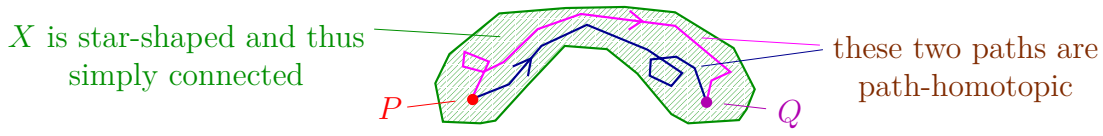
Definition 88.16. Let X be a path-connected non-empty topological space.

(1) We denote by $\pi_1(X)$ the isomorphism type of the fundamental group.

(2) If $\pi_1(X) = 0$, then we say that X is simply connected.[874]

Example. On page 1734 we saw that star-shaped subsets of $\mathbb{R}^n$ are simply connected. □

Lemma 88.17. (Paths-in-Simply Connected Space Lemma) Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths $\alpha, \beta\colon [0,1] \to X$ from P to Q are path-homotopic.



Proof. The proof is precisely the content of Exercise 88.2. ■

In the following two sections we will first calculate the isomorphism type of the fundamental groups of the circle S^1 and then we will calculate the isomorphism type of the fundamental

[874] It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of "simply connectedness" as in Definition 87.3.

groups of the spheres S^n with $n \geq 2$. Both calculations require some effort. But the effort will pay off in multiple ways:

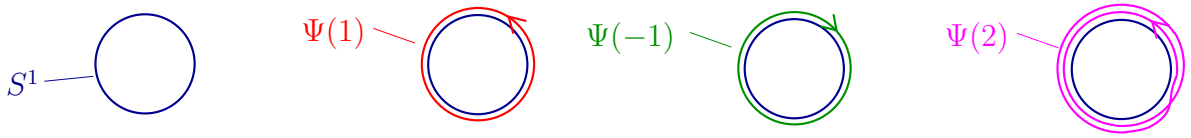
(1) Using these calculations we can already prove several interesting results, like the Fundamental Theorem of Algebra and (with a little bit of extra theory) the Borsuk-Ulam Theorem for S^2.
(2) The techniques we will employ in these special cases will give us the ideas for tackling much more general computational problems later on.

88.4. **The fundamental group of the circle.** The goal of this section is to prove the following important theorem, which gives us our first calculation of a non-trivial fundamental group.

Theorem 88.22. (Circle-π_1-Theorem) The map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1,1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}\cdot n\cdot t)\end{array}\right] \end{array}$$

is an isomorphism of groups.

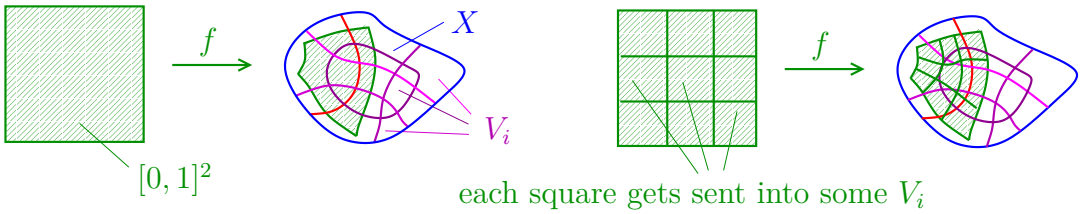


Remark. With some massaging one can deduce the Circle-π_1-Theorem 88.22 from the Hopf Degree Theorem 46.22. We will not go down this route, instead we will provide an independent argument. □

As is the case for many other calculations of fundamental groups, the proof of the Circle-π_1-Theorem 88.22 makes use of the Lebesgue Lemma, which we now recall for the reader's convenience.

Lemma 10.9. (Lebesgue Lemma) Let $f\colon [0,1]^m \to X$ be a continuous map from the cube $[0,1]^m$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. There exists an $N \in \mathbb{N}$ such that for every $a_1,\dots,a_m \in \{0,\dots,N-1\}$ there exists an $i \in I$ with

$$f\big(\big[\tfrac{a_1}{N},\tfrac{a_1+1}{N}\big]\times\cdots\times\big[\tfrac{a_m}{N},\tfrac{a_m+1}{N}\big]\big) \;\subseteq\; V_i.$$



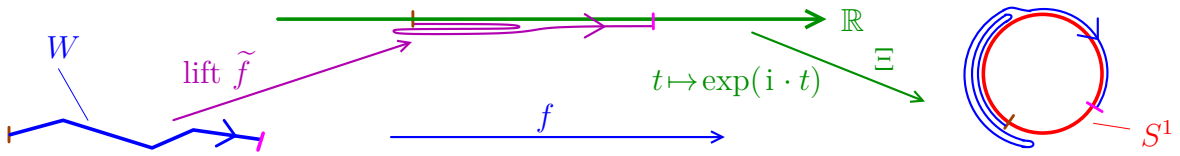
The next definition is the key tool to studying the fundamental group of S^1:

Definition 88.18.

(1) We work with the map

$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

(2) Let W be a topological space and let $f\colon W \to S^1$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon W \to \mathbb{R}$ such that $\Xi \circ \widetilde{f} = f$, in other words, such that for all $w \in W$ we have
$$f(w) \;=\; \exp\big(\mathrm{i} \cdot \widetilde{f}(w)\big).$$



Example. For $n \in \mathbb{Z}$ we consider the map $f\colon [0,1] \to S^1$ with $f(t) \coloneqq \exp(2\pi\,\mathrm{i} \cdot n \cdot t)$. A lift of f is given by the map
$$\begin{array}{rcl} \widetilde{f}\colon [0,1] & \to & \mathbb{R} \\ t & \mapsto & 2\pi \cdot n \cdot t. \end{array}$$
This lift is not unique though since we can "shift" the lift by any $k \cdot 2\pi$ with $k \in \mathbb{Z}$. □

Lemma 88.19. (Lift Uniqueness Lemma) Let W be a connected topological space, let $f\colon W \to S^1$ be a continuous map and let $\widetilde{\varphi}, \widetilde{\psi}\colon W \to \mathbb{R}$ be two lifts of f. If $\widetilde{\varphi}$ and $\widetilde{\psi}$ agree at some point $w_0 \in W$, then they agree everywhere.

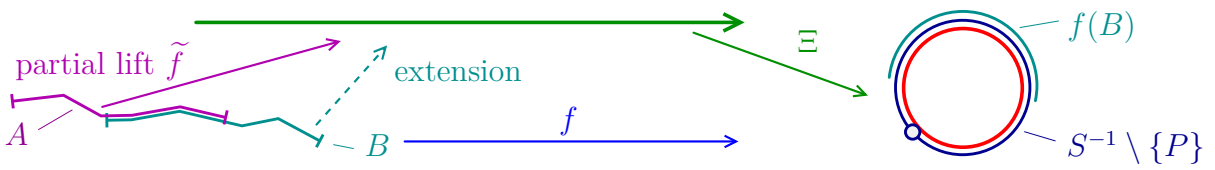
Proof. [875] We consider the map
$$\begin{array}{rcl} \theta\colon W & \to & \mathbb{R} \\ w & \mapsto & \theta(w) \coloneqq \widetilde{\varphi}(w) - \widetilde{\psi}(w). \end{array}$$
By hypothesis we have $\theta(w_0) = 0$. Next note that for each $w \in W$ we have
$$\underbrace{\exp\big(\mathrm{i} \cdot (\widetilde{\varphi}(w) - \widetilde{\psi}(w))\big)}_{=\exp(\mathrm{i}\cdot\theta(w))=\Xi(\theta(w))} \;=\; \exp\big(\mathrm{i} \cdot \widetilde{\varphi}(w)\big) \cdot \exp\big(\mathrm{i} \cdot \widetilde{\psi}(w)\big)^{-1} \;\underset{\substack{\uparrow \\ \text{since } \widetilde{\varphi},\widetilde{\psi}\colon W \to \mathbb{R} \text{ are} \\ \text{lifts of } f}}{=}\; f(w) \cdot f(w)^{-1} \;=\; 1.$$
This shows that the continuous map $w \mapsto \theta(w) \coloneqq \widetilde{\varphi}(w) - \widetilde{\psi}(w)$ takes values in the discrete subspace $2\pi \cdot \mathbb{Z} = \Xi^{-1}(1) \subseteq \mathbb{R}$. Since W is connected it follows from the Connected-to-Discrete Lemma 4.37 that the map θ is constant. Since the map θ vanishes at w_0 we see that it is zero everywhere. This implies that $\widetilde{\varphi} = \widetilde{\psi}$. ■

Our next goal is to prove the existence of lifts of continuous maps $X \to S^1$ in suitable settings. The key tool for constructing lifts is the following lemma:

Lemma 88.20. (Lift-Extension Lemma) Let X be a topological space. We suppose that we have a decomposition $X = A \cup B$ such that A and B are closed subsets. Let $f\colon X \to S^1$ be a continuous map and let $\widetilde{f}\colon A \to \mathbb{R}$ be a lift of $f|_A$. We suppose that the following statements hold:
(1) There exists a $P \in S^1$ with $f(B) \subseteq S^1 \setminus \{P\}$.
(2) The intersection $A \cap B$ is connected and non-empty.
Then there exists a lift $\widetilde{f}\colon X \to \mathbb{R}$ of f that agrees with the given lift on A.

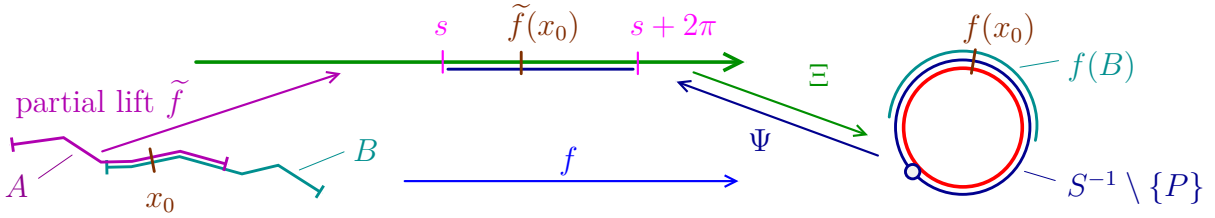
Proof. We start with a few preparations:
(1) We pick $x_0 \in A \cap B$ and we pick $s \in \mathbb{R}$ with $\Xi(s) = P$.

[875] The proof is almost identical to the proof of the uniqueness statement of the original Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14.



(2) Since $f(x_0) \in f(A\cap B) \subseteq f(B) \subseteq S^1\setminus\{P\}$ and since $\widetilde{f}\colon A\to\mathbb{R}$ is a lift of $f\colon A\to S^1$ we see that $\widetilde{f}(x_0)\notin \Xi^{-1}(P)$.
(3) It follows from (2) that there exists an $m\in\mathbb{Z}$ such that $\widetilde{f}(x_0)\in(s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi)$.
(4) One can easily verify that the restriction of Ξ to $(s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi)\to S^1\setminus\{P\}$ is a homeomorphism.[876] We denote by Ψ the inverse.

We set
$$\begin{aligned}\widetilde{f}\colon X &\to \mathbb{R}\\ x &\mapsto \begin{cases}\widetilde{f}(x), & \text{if } x\in A,\\ \Psi(\underbrace{f(x)}_{\in S^1\setminus\{P\}}), & \text{if } x\in B.\end{cases}\end{aligned}$$



Note that on $A\cap B$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{A\cap B}$ and by design they agree on x_0. By hypothesis the intersection $A\cap B$ is connected. Thus we obtain from the Lift Uniqueness Lemma 88.19 that the two maps actually agree on the overlap $A\cap B$. This shows that $\widetilde{f}\colon X\to\mathbb{R}$ is well-defined.

Next note that since A and B are closed it follows from the Pasting Proposition 4.7 that this map $\widetilde{f}\colon X\to\mathbb{R}$ is continuous. It is now clear that it is a lift of $f\colon X\to\mathbb{R}$. ■

The following lemma is a slightly weaker version of the original Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14.

Lemma 88.21. (Lift-from-S^1-to-$\mathbb{R}$ Lemma) Given any continuous map $f\colon[0,1]\to S^1$ and given any $\widetilde{y}_0\in\mathbb{R}$ with $f(0)=\exp(\mathrm{i}\cdot\widetilde{y}_0)=:\Xi(\widetilde{y}_0)$ there exists a *unique* lift $\widetilde{f}\colon[0,1]\to\mathbb{R}$ of f with $\widetilde{f}(0)=\widetilde{y}_0$.

Proof of Lemma 88.21. Let $f\colon[0,1]\to S^1$ be a continuous map and let $\widetilde{y}_0\in\mathbb{R}$ such that $f(0)=\exp(\mathrm{i}\cdot\widetilde{y}_0)$. We need to show that there exists a *unique* lift $\widetilde{f}\colon[0,1]\to\mathbb{R}$ of f with $\widetilde{f}(0)=\widetilde{y}_0$.

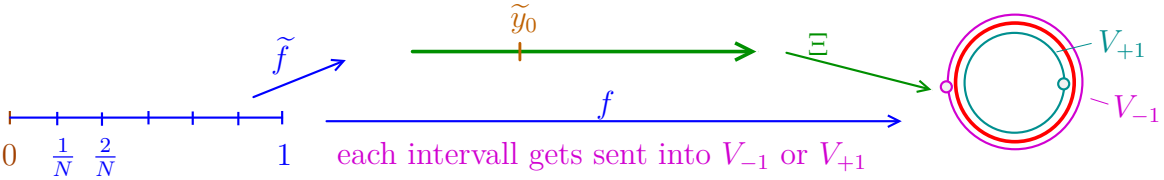
The uniqueness statement follows immediately from the Lift Uniqueness Lemma 88.19. Thus it remains to show the existence statement.

[876] Indeed: It is clear that the map is a bijection and continuous. Using the Subbasis-Open Map Lemma 4.14 one can easily verify that the map is also open. Thus the map is a homeomorphism by the Open-Closed–Injective Lemma 4.22.

The idea is quite simple: We want to apply the Lift-Extension Lemma 88.20. To do so we need to decompose the interval $[0,1]$ into smaller intervals, such that on each subinterval the image of f misses at least one point in S^1.

We set $V_{\pm 1} := S^1 \setminus \{\pm 1\}$. By the Lebesgue Lemma 10.9 there exists an $N \in \mathbb{N}$ such that for every $i \in \{0, \dots, N-1\}$ the image $f([\frac{i}{N}, \frac{i+1}{N}])$ is contained in V_{-1} or V_{+1}. It remains to prove the following claim:

Claim. For each $i \in \{0, \dots, N\}$ there exists a lift $\widetilde{f}\colon [0, \frac{i}{N}] \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{y}_0$.



Proof. For $i = 0$ we just set $\widetilde{f}(0) := \widetilde{y}_0$. Now suppose that we have already defined $\widetilde{f}$ on some interval $[0, \frac{i}{N}]$ with $i < N$. Note that by construction there exists an $\epsilon \in \{\pm 1\}$ with $f([\frac{i}{N}, \frac{i+1}{N}]) \in V_\epsilon = S^1 \setminus \{\epsilon\}$. Thus it follows from the Lift-Extension Lemma 88.20[877] that we can extend the lift f to the interval $[0, \frac{i+1}{N}]$. ∎

We can now formulate and prove the full version of the Circle-π_1-Theorem 88.22.

Theorem 88.22. (Circle-π_1-Theorem) Let $z_0 = \exp(2\pi \mathrm{i} \cdot t_0) \in S^1$. Given any continuous map $\gamma\colon [0,1] \to S^1$ with $\gamma(0) = z_0$ we denote by $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}$ the unique lift[878] with $\widetilde{\gamma}(0) = t_0 \in \mathbb{R}$. The maps

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1, z_0) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & z_0 \cdot \exp(2\pi \mathrm{i} \cdot n \cdot t) \end{array}\right] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Theta\colon \pi_1(S^1, z_0) & \to & \mathbb{Z} \\ [\gamma\colon [0,1] \to S^1] & \mapsto & \frac{1}{2\pi} \cdot (\widetilde{\gamma}(1) - \widetilde{\gamma}(0)) \end{array}$$

are well-defined, they are isomorphisms and they are inverses of one another.

The proof of the Circle-π_1-Theorem 88.22 rests on the following lemma, which is also of independent interest.

Lemma 88.23. (Square Lifting Lemma) Let $F\colon [0,1] \times [0,1] \to S^1$ be a continuous map. Given any $c \in \mathbb{R}$ with $\Xi(c) = F(0,0)$ there exists a unique lift $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}$ of F with $\widetilde{F}(0,0) = c$.

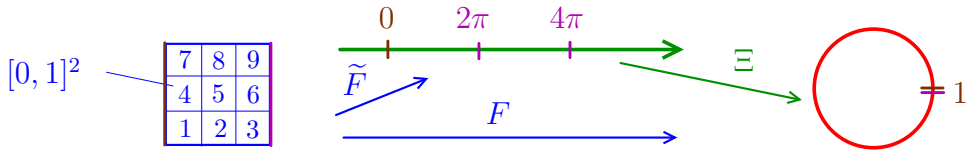
Proof of the Square Lifting Lemma 88.23. Let $F\colon [0,1] \times [0,1] \to S^1$ be a continuous map and let $c \in \mathbb{R}$ with $\Xi(c) = F(0,0)$. As in the proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21 we set $V_{\pm 1} := S^1 \setminus \{\pm 1\}$. By the Lebesgue Lemma 10.9 there exists an $N \in \mathbb{N}$ such that for every $i, j \in \{0, \dots, N-1\}$ there exists an $\epsilon \in \{\pm 1\}$ with

$$f\big([\tfrac{i}{N}, \tfrac{i+1}{N}] \times [\tfrac{j}{N}, \tfrac{j+1}{N}]\big) \subseteq V_\epsilon.$$

The same way as we lifted the map $[0,1] \to S^1$ to $\mathbb{R}$ by lifting the map "intervall-by-intervall" starting from the left, we now lift the map $F\colon [0,1]^2 \to S^1$ to $\mathbb{R}$ by lifting the

[877] Here we use that the intersection of two adjacent intervals is connected and non-empty.

map "square-by-square" starting from the "bottom left".[879] We leave it to the reader to fill in the details.



The uniqueness statement of the lemma is just a special case of the Lift Uniqueness Lemma 88.19. ■

Proof of the Circle-π_1-Theorem 88.22. To simplify the notation we only consider the case that $z_0 = 1$ and $t_0 = 0$. The following claim shows that $\Theta\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is well-defined.

Claim 1. Let $\alpha, \beta\colon [0,1] \to S^1$ be two loops in $(S^1, 1)$. If α and β are path-homotopic, then $\widetilde{\alpha}(1) = \widetilde{\beta}(1) \in \mathbb{R}$.

Proof. Let $F\colon [0,1] \times [0,1] \to S^1$ be a path-homotopy between α and β. By the Square Lifting Lemma 88.23 we know that F lifts to a continuous map $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}$.

Next note that the fact that F is a path-homotopy means that $F(1,t) = 1$ for all $t \in [0,1]$. This in turn implies that $\widetilde{F}(1,t) \in 2\pi \cdot \mathbb{Z}$ for all $t \in [0,1]$. It follows from the Connected-to-Discrete Lemma 4.37 that the map $t \mapsto \widetilde{F}(1,t)$ is constant. Therefore we have shown that $\widetilde{\alpha}(1) = \widetilde{F}(1,0) = \widetilde{F}(1,1) = \widetilde{\beta}(1)$. ⊞

Next note that it follows almost immediately from the definitions that $\Psi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ is a homomorphism. Furthermore note that it follows easily from the definitions[880] that $\Theta \circ \Psi\colon \mathbb{Z} \to \mathbb{Z}$ is the identity. Thus it remains to prove the following claim:

Claim 2. The map $\Theta\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is a bijection.

Proof. Since $\Theta \circ \Psi = \mathrm{id}_{\mathbb{Z}}$ we see that Θ is surjective. It remains to show that Θ is injective. So suppose that we are given $[\alpha], [\beta] \in \pi_1(S^1, 1)$ with $\Theta([\alpha]) = \Theta([\beta])$, i.e. with $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Note that the maps $\widetilde{\alpha}, \widetilde{\beta}\colon [0,1] \to \mathbb{R}$ have the same starting point and the same endpoint in $\mathbb{R}$. *Since* $\mathbb{R}$ *is convex* we see, as in the Convex Homotopy Lemma 18.3, that

$$\begin{aligned} F\colon [0,1] \times [0,1] &\to \mathbb{R} \\ (t,s) &\mapsto \widetilde{\alpha}(t) \cdot (1-s) + \widetilde{\beta}(t) \cdot s \end{aligned}$$

is a path-homotopy in $\mathbb{R}$ between $\widetilde{\alpha}$ and $\widetilde{\beta}$. But this implies that

$$\begin{aligned} \Xi \circ F\colon [0,1] \times [0,1] &\to S^1 \\ (t,s) &\mapsto \Xi(\widetilde{\alpha}(t) \cdot (1-s) + \widetilde{\beta}(t) \cdot s) \end{aligned}$$

is a path-homotopy in S^1 between $\alpha = \Xi \circ \widetilde{\alpha}$ and $\beta = \Xi \circ \widetilde{\beta}$. Thus $[\alpha] = [\beta] \in \pi_1(S^1, 1)$. ■

88.5. The fundamental group of spheres.

In the Circle-π_1-Theorem 88.22 we just calculated the fundamental group of S^1. In this section we will deal with the higher-dimensional spheres.

[879] The key observation is that if for any square that we add to the domain the intersection with the previous domain is always connected and non-empty.

[880] To be slightly more precise, this follows from the fact that the lift of the loop $t \mapsto \exp(2\pi \mathrm{i} \cdot n \cdot t)$ is given by $t \mapsto 2\pi \cdot n \cdot t$.

Proposition 88.24. (Sphere-π_1-Proposition) For any $n \geq 2$ the sphere S^n is simply connected, i.e. we have $\pi_1(S^n) = 0$.

In the proof of the Sphere-π_1-Proposition 88.24 we will need the following lemma:

Lemma 88.25. (Sphere-Minus-Point Lemma) For every $P \in S^n$ the topological space $S^n \setminus \{P\}$ is simply connected.

Proof of Lemma 88.25. Recall that on page 1734 we saw that $\mathbb{R}^n$ is simply connected. Thus it suffices to show that $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

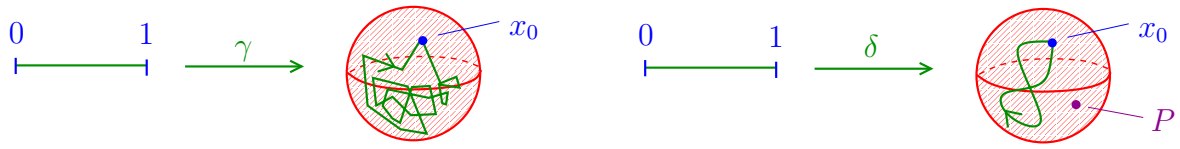
Recall that in the Stereographic Projection Lemma 4.24 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ with $f^{-1}(\infty) = N := (0, \ldots, 0, 1)$. Thus f restricts to a homeomorphism $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$. Now let P be any other point on S^n. It follows from the Orthogonal Action Lemma 2.37 (4) that there exists a homeomorphism $g\colon S^n \to S^n$ with $g(P) = N$. Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. ■

Now we can turn to the proof of the Sphere-π_1-Proposition 88.24.

> Here is a very short "proof": Let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ. By the Sphere-Minus-Point Lemma 88.25 the topological space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n.
>
> This sounds convincing, except for one crucial gap: we did not justify why there exists a point P that does not lie in the image of γ. In fact, as we saw in the Space-Filling Curve Theorem 10.21, such a point does not need to exist, since there are surjective continuous maps from an interval to any S^n.

We provide two approaches to fixing this problem. The first approach uses the Whitney Approximation Theorem 36.1. The second approach is completely self-contained and considerably longer.

First proof of the Sphere-π_1-Proposition 88.24. Let $n \geq 2$, let $x_0 \in S^n$ and furthermore let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. It follows immediately from the Whitney Approximation Theorem 36.1, applied to the closed subset[881] $A = \{0,1\} \subseteq [0,1]$, that there exists a homotopy rel $\{0,1\}$ from $\gamma\colon [0,1] \to S^n$ to a smooth map $\delta\colon [0,1] \to S^n$ with $\delta(0) = \delta(1) = x_0$. It follows from the Measure-Zero Properties Proposition 37.3 (4), the Full Measure-Density Proposition 37.4 (1) and the hypothesis that $n \geq 2$ that the smooth map $\delta\colon [0,1] \to S^n$ is not surjective. We pick $P \in S^n \setminus \delta([0,1])$.



By the Sphere-Minus-Point Lemma 88.25 we know that δ is homotopic rel $\{0,1\}$ to the constant map. It follows from the Path-Homotopy–Equivalence Relation Lemma 88.5 that

[881] We use the observation that it follows immediately from the definition on Definition 25.20 that the restriction of any map $[0,1] \to M$ to a smooth manifold is smooth on the subset $\{0,1\}$.

γ is also homotopic rel $\{0,1\}$ to the constant map. We have thus shown that δ represents the trivial element in $\pi_1(S^n, x_0)$. ■

Second proof of the Sphere-π_1-Proposition 88.24. Let $n \geq 2$. We write

$$N \coloneqq (0,\dots,0,1) \quad \text{“North Pole”} \quad \text{and} \quad U \coloneqq S^n \setminus \{N\},$$

and also

$$S \coloneqq (0,\dots,0,-1) \quad \text{“South Pole”} \quad \text{and} \quad V \coloneqq S^n \setminus \{S\}.$$

We fix a base point $x_0 \in U \cap V$. In the following we will only use the following three facts about U and V:

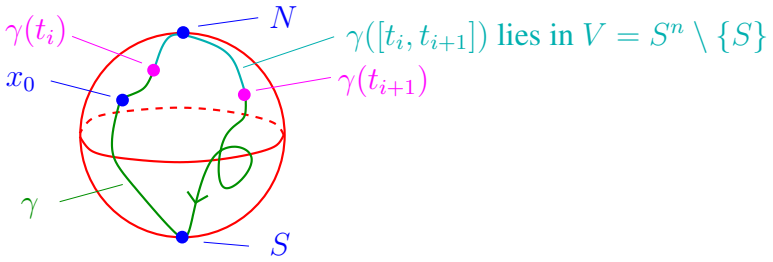
(I) U and V form an open cover of S^n.
(II) By the Sphere-Minus-Point Lemma 88.25 we know that U and V are simply connected.
(III) Our hypothesis $n \geq 2$ implies that the intersection $U \cap V = S^n \setminus \{N, S\}$ is path-connected.[882]

It follows from (II) together with the Path-Homotopy–Equivalence Relation Lemma 88.5 that it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in U. Thus let $\gamma\colon [0,1] \to S^n$ be a loop in (S^n, x_0).

Claim 1. There exists a subdivision

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1,$$

such that the following hold:
(1) for each $i \in \{0,\dots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subseteq U$ or $\gamma([t_i, t_{i+1}]) \subseteq V$, and
(2) for each $i \in \{1,\dots,k-1\}$ we have $\gamma(t_i) \in U \cap V$.



Proof. By (I) we can apply the Lebesgue Lemma 10.9 to $S^n = U \cup V$ and to $\gamma\colon [0,1] \to S^n$. Thus there exists a $k \in \mathbb{N}$ such that for every $i \in \{0,\dots,k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0,\dots,k$ satisfies condition (1).

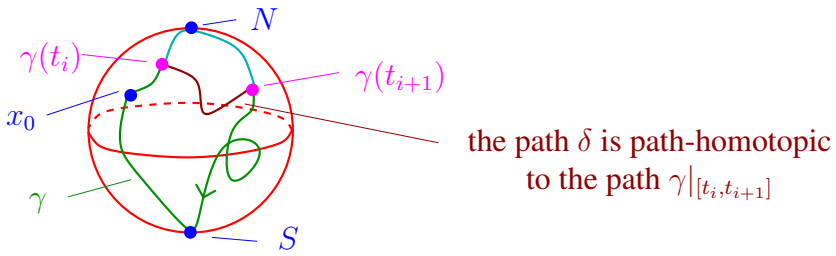
Now suppose that there exists an $i \in \{1,\dots,k-1\}$ with $\gamma(t_i) \notin U$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ are contained in U. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subseteq V$. But then we can remove t_i from the subdivision. The same way we deal with the case that $\gamma(t_i) \notin V$. Iterating this process we can assume that for each every i we have $\gamma(t_i) \in U \cap V$. ⊞

As mentioned before, it now, suffices to prove the following claim:

Claim 2. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path in U.

[882]This can be seen as follows: $S^n \setminus \{N, S\}$ is homeomorphic to $S^{n-1} \times (-1, 1)$, which in turn is clearly path-connected.

Proof. We pick a subdivision of $[0,1]$ as in Claim 1. Suppose that there exists an $i \in \{0,\dots,k-1\}$ such that $\gamma([t_i,t_{i+1}])$ is not contained in U. By the choice of our subdivision we know that this implies that $\gamma([t_i,t_{i+1}])$ lies entirely in V. Since $U \cap V$ is path-connected and since $\gamma(t_i)$ and $\gamma(t_{i+1})$ both lie in $U \cap V$ we can pick a path $\delta\colon [t_i,t_{i+1}] \to U \cap V$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i,t_{i+1}]}$ and δ both lie in the simply connected set V they are path-homotopic by the Paths-in-Simply Connected Space Lemma 88.17. In the loop γ we now replace $\gamma|_{[t_i,t_{i+1}]}$ by δ.



It is straightforward to see that this way we obtain a new loop that is path-homotopic to the original loop and that stays within $U \cap V \subseteq U$ on the interval $[t_i,t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that is contained in U. ∎

88.6. **Alternative description of fundamental groups.** In this short section we will present a slightly different way to think about fundamental groups, namely we will see that we can think of fundamental groups also as defined via continuous maps $S^1 \to X$. We start out with the following identification.

Identification 88.26. As we discussed on page 314 the map

$$\begin{array}{rcl} [0,1]/0\sim 1 & \to & S^1 \\ [t] & \mapsto & \exp(2\pi \mathrm{i}\cdot t) \end{array}$$

is a homeomorphism. We use it to make the identification $S^1 =_{\mathrm{i}} [0,1]/0\sim 1$.

This identification allows us to introduce the following convention:

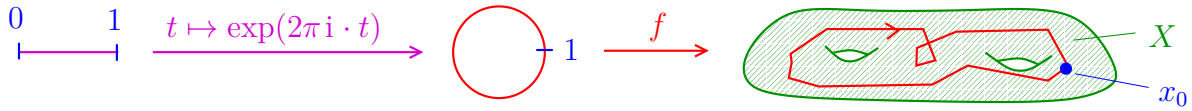
Convention 88.27. Let X be a topological space and let $x_0 \in X$.

(1) Using the above identification, using the Topological-Quotient Proposition 8.2 (1b) and using the Homotopy Quotient Theorem 18.11 we see that we can make the natural identification

$$\pi_1(X,x_0) \;=_{\mathrm{i}}\; \{\text{all continuous maps } (S^1,1)\to(X,x_0)\}/\sim$$

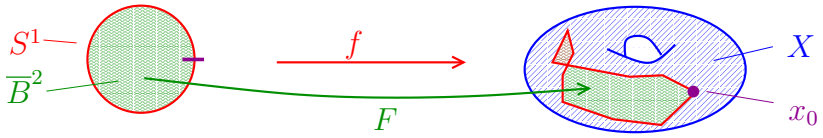
where $f \sim g$ if there exists a homotopy $F\colon S^1 \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $F_t(1) = x_0$.

(2) We refer to continuous maps $(S^1,1) \to (X,x_0)$ also as loops in (X,x_0).



In this new language the following lemma gives a useful criterion for showing that a loop is null-homotopic.

Lemma 88.28. (Trivial-in-π_1-Lemma) Let X be a topological space, let $x_0 \in X$ and let $f\colon (S^1, 1) \to (X, x_0)$ be a continuous map. The following statements are equivalent:
(1) The map $f\colon S^1 \to X$ is homotopic to a constant map.
(2) There exists a continuous map $F\colon \overline{B}^2 \to X$ so that $F|_{S^1} = f$.
(3) The map f represents the trivial element in $\pi_1(X, x_0)$.



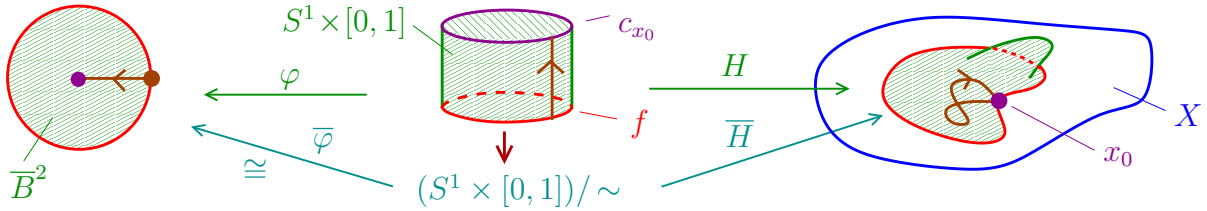
Proof.
(1)⇒(2). We assume that $f\colon S^1 \to X$ is homotopic to a constant map c_{x_0}. In other words, we assume that there exists a homotopy $H\colon S^1 \times [0,1] \to X$ with $H_0 = f$ and $H_1 = c_{x_0}$. We make a few preparations:

(a) Let $\sim$ be the equivalence relation on $S^1 \times [0,1]$ given by $(P,1) \sim (Q,1)$ for all $P, Q \in S^1$.
(b) We consider the map

$$\begin{array}{rcl} \varphi\colon S^1 \times [0,1] & \to & \overline{B}^2 \\ (P,t) & \mapsto & x \cdot (1-t) \cdot P. \end{array}$$

(c) Note that $\varphi(P) = \varphi(Q)$ and $H(P) = H(Q)$ whenever $P \sim Q$.
(d) From the Topological-Quotient Proposition 8.2 (1b) we obtain induced continuous maps $\overline{\varphi}\colon (S^1 \times [0,1])/\sim \to \overline{B}^2$ and $\overline{H}\colon (S^1 \times [0,1])/\sim \to X$.
(e) Using the Compact-Hausdorff Proposition 4.23 (3) we see that $\overline{\varphi}\colon (S^1 \times [0,1])/\sim \to \overline{B}^2$ is actually a homeomorphism.

The promised map $F\colon \overline{B}^2 \to X$ is now given by $F := \overline{H} \circ \overline{\varphi}^{-1}$.



(2)⇒(3). Let $F\colon \overline{B}^2 \to X$ be a continuous map with $F|_{S^1} = f$. We consider the map

$$\begin{array}{rcl} \Theta\colon S^1 \times [0,1] & \to & \overline{B}^2 \\ ((x,y),t) & \mapsto & (t \cdot x, t \cdot y + (1-t)). \end{array}$$

One can now easily verify that the desired homotopy of maps $(S^1, *) \to (X, x_0)$ is given by $H = F \circ \Theta\colon S^1 \times [0,1] \to X$.



(3)⇒(1). This direction is basically obvious. Indeed, the fact that f represents the trivial element in $\pi_1(X, x_0)$ means that there exists a homotopy of maps of pairs $(S^1, 1) \to (X, x_0)$

from f to the constant map. But this means in particular that f is homotopic to the constant map c_{x_0}. ■

88.7. **Appendix: Homotopy classes of loops.** In the last section of this chapter we want to study the question, when are two continuous maps $S^1 \to X$ to a topological space homotopic, if we do not care about base points at all. In fact the following proposition reduces this question to a question about fundamental groups.

Proposition 88.29. (Homotopies-of-Loops Proposition) Let X be a topological space and let $f_0, f_1 \colon S^1 \to X$ be two continuous maps. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.

(1) The following statement holds:[883]

$$f_0 \text{ and } f_1 \text{ are homotopic}^{884} \iff \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0).\end{array}$$

(2) If $x_0 = x_1$, then the following statement holds:

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff [f_0] \text{ and } [f_1] \text{ are conjugate in } \pi_1(X, x_0).$$



Proof. Note that Statement (2) is an immediate consequence of Statement (1) and the observation that if $x_0 = x_1$, then any path from x_0 to $x_1 = x_0$ is in fact a loop and thus defines an element in $\pi_1(X, x_0)$. Therefore it suffices to prove Statement (1).

Let X be a topological space. Let $f_0, f_1 \colon S^1 \to X$ be two continuous maps. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$. As before we consider the map $\Xi \colon [0,1] \to S^1$ that is given by $\Xi(t) = \exp(2\pi \mathrm{i} \cdot t)$.

"$\Rightarrow$" We suppose that $f_0, f_1 \colon S^1 \to X$ are homotopic. This implies that there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $F(0,t) = F(1,t)$ and such that $F_0 = f_0 \circ \Xi$ and $F_1 = f_1 \circ \Xi$. We denote by $\gamma \colon [0,1] \to X$ the path that is given by $\gamma(t) = F(0,t)$ for $t \in [0,1]$. We consider the surjective continuous map $\Phi \colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is shown in the figure below. The map $F \circ \Phi$ is easily seen to be a path-homotopy between the loops f_0 and $\gamma * f_1 * \overline{\gamma}$ in (X, x_0). But that is exactly what we needed to show.



"$\Leftarrow$" In the following we now suppose that there exists a path $\gamma \colon [0,1] \to X$ from x_0 to x_1 such that $[\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0)$. This implies that there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ such that the following hold:

(a) For all $t \in [0,1]$ we have $F(0,t) = F(1,t) = x_0$.

[883] We use the convention of page 1744 to view $f_0, f_1 \colon S^1 \to X$ as loops $f_0, f_1 \colon [0,1] \to X$.

[884] Sometimes two maps $S^1 \to X$ that are homotopic are called *freely homotopic* to stress the fact that no base points are fixed.

(b) We have $F_0 = f_0 \circ \Xi$ and $F_1 = (\gamma * f_1 * \overline{\gamma}) \circ \Xi$.

We consider the continuous map $\Psi\colon [0,1] \times [0,1] \to [0,1] \times [0,1]$ that is shown in the figure below. The map $F \circ \Psi\colon [0,1] \times [0,1] \to X$ is a homotopy such that the following hold:

(a) For all $t \in [0,1]$ we have $(F \circ \Psi)(0,t) = (F \circ \Psi)(1,t) = \gamma(t)$.

(b) We have $(F \circ \Psi)_0 = c_{x_0} * (f_0 \circ \Xi) * c_{x_0}$ and $(F \circ \Psi)_1 = f_1 \circ \Xi$.



Using the Path-Homotopy Product Proposition 88.10 (3) we can in fact find a homotopy $G\colon [0,1] \times [0,1] \to X$ such that the following hold:

(a) For all $t \in [0,1]$ we have $G(0,t) = G(1,t)$.

(b) We have $G_0 = f_0 \circ \Xi$ and $G_1 = f_1 \circ \Xi$.

This homotopy G descends to a homotopy $S^1 \times [0,1] \to X$ from f_0 to f_1. ■

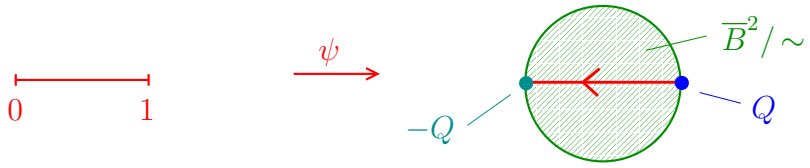
88.8. Exercises for Chapter 88.

Exercise 88.1. We denote by "$\sim$" the equivalence relation on $\overline{B}^n$ that is generated by $x \sim -x$ for $x \in S^{n-1}$. We set $Q = (1, 0, \ldots, 0)$. We consider the loop

$$\begin{array}{rccl} \psi\colon & [0,1] & \to & \overline{B}^n/\!\sim \\ & t & \mapsto & [Q \cdot (1-2t)]. \end{array}$$

Show that the loop $\psi * \psi$ is null-homotopic.

Remark. We will later see that ψ itself is not null-homotopic.

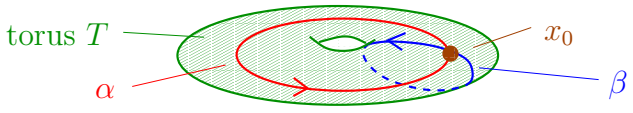


Exercise 88.2. Let X be a topological space and let P, Q be two points in X. We suppose that X is simply connected. Show that any two paths from P to Q are path-homotopic.

Exercise 88.3. Provide an example of a topological space X and two points $x_0, x_1 \in X$ such that $\pi_1(X, x_0)$ is not isomorphic to $\pi_1(X, x_1)$.

Exercise 88.4. Let X be a set with four points that is equipped with the trivial topology from page 136. Furthermore let $x_0 \in X$. Determine $\pi_1(X, x_0)$.

Exercise 88.5. We consider the torus $T = S^1 \times S^1$ and the two loops α and β in (T, x_0) that are shown in the figure below. Does the equality $[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$ hold in $\pi_1(T, x_0)$?



Exercise 88.6. Let (X, x_0) be a pointed topological space and let $\alpha, \beta\colon [0,1] \to X$ be two loops in (X, x_0). We consider the subspace $(S^1 \times \{1\}) \cup (\{1\} \times S^1) \subseteq S^1 \times S^1$ and the map

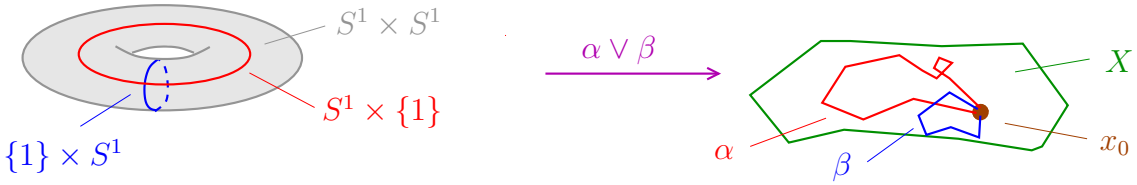
$$\begin{array}{rcl} \alpha \vee \beta\colon (S^1 \times \{1\}) \cup (\{1\} \times S^1) & \to & X \\ (\exp(2\pi \mathrm{i} s), \exp(2\pi \mathrm{i} t) & \mapsto & \begin{cases} \alpha(s), & \text{if } t = 0, \\ \beta(t), & \text{is } s = 0. \end{cases} \end{array}$$

Show that the following two statements are equivalent:

(a) The elements $[\alpha], [\beta] \in \pi_1(X, x_0)$ commute.

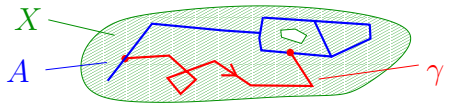
(b) We can extend $\alpha \vee \beta\colon (S^1 \times \{1\}) \cup (\{1\} \times S^1) \to X$ to a continuous map on all of $S^1 \times S^1$.

Remark. One can also reformulate this in terms of the torus $([0,1] \times [0,1])/\sim$.



Exercise 88.7. Does every continuous map $f\colon S^1 \to S^1$ admit a lift to $\mathbb{R}$? In other words, does there always exists a continuous map $\widetilde{f}\colon S^1 \to \mathbb{R}$ such that $\exp(\widetilde{f}(z) \cdot \mathrm{i}) = f(z)$ for all $z \in S^1$?

Exercise 88.8. Let X be a topological space, let $A \subseteq X$ be a path-connected subspace and let $x_0 \in A$. We denote by $i\colon A \to X$ the inclusion map. We suppose that the map $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an epimorphism. Finally let $\gamma\colon [0,1] \to X$ be a path with $\gamma(0) \in A$ and $\gamma(1) \in A$. Show that there exists a homotopy $F\colon [0,1] \times [0,1] \to X$ rel $\{0,1\}$ such that $F_0 = \gamma$ and such that F_1 takes values in A.



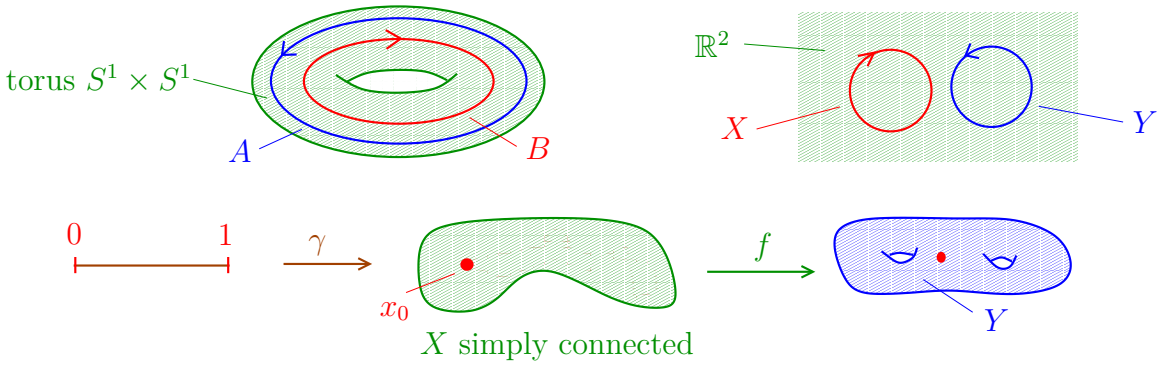
Exercise 88.9. Let X be a topological space and let x_0 and x_1 be two points in X. Furthermore let $p, q\colon [0,1] \to X$ be two paths from x_0 to x_1. We suppose that $\pi_1(X, x_0)$ is abelian. Show that $p_* = q_*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$.

Exercise 88.10.

(a) Show that the two oriented 1-dimensional submanifolds A and B of $S^1 \times S^1$ that are shown in the figure on the left are not smoothly isotopic.

(b) We consider $\mathbb{R}^2$ with the two oriented 1-dimensional submanifolds X and Y that are shown in the figure on the right. Are these X and Y smoothly isotopic?
Hint. You could make use of the Isotopy Extension Theorem 44.2.

Exercise 88.11. Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $\gamma\colon [0,1] \to X$ be a loop in a point $x_0 \in X$. Show that if X is simply connected, i.e. if every loop in X is null-homotopic, then the loop $f \circ \gamma\colon [0,1] \to Y$ is null-homotopic.



Exercise 88.12. Show that the map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(\mathbb{C}\setminus\{0\},1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{C}\setminus\{0\} \\ t & \mapsto & \exp(2\pi\,\mathrm{i}\cdot n\cdot t) \end{array}\right] \end{array}$$

is an isomorphism of groups. There are at least two approaches one can take:

(a) Using the map $f\colon \mathbb{C}\setminus\{0\} \to S^1$ given by $f(z) = \frac{z}{|z|}$ one can try to reduce the problem to the Circle-π_1-Theorem 88.22.

(b) One can modify the proof of the Circle-π_1-Theorem 88.22 by replacing the map

$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(t\cdot \mathrm{i}) \end{array} \qquad \text{by the map} \qquad \begin{array}{rcl} \widehat{\Xi}\colon \mathbb{R}\times\mathbb{R}_{>0} & \to & \mathbb{C}\setminus\{0\} \\ (t,r) & \mapsto & r\cdot\exp(t\cdot\mathrm{i}) \end{array}$$

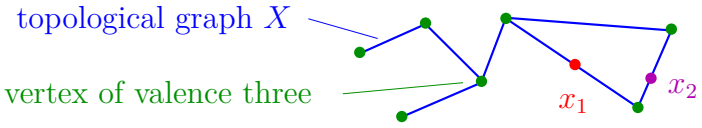
Exercise 88.13. Let $k \in \mathbb{N}$ and let X be a topological manifold. We refer to

$$\mathrm{Conf}_k(X) \;:=\; \{(x_1,\ldots,x_k) \in X^k \mid \text{for all } i \neq j \text{ we have } x_i \neq x_j\}$$

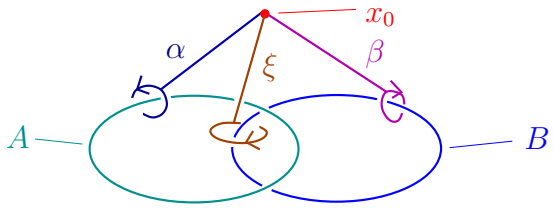
as the *configuration space of k points in X*. Let $X \subseteq \mathbb{R}^2$ be a connected topological graph that has at least one vertex of valence three. Show that $\pi_1(\mathrm{Conf}_2(X))$ admits an epimorphism onto $\mathbb{Z}$.

Hint. Consider the map

$$\begin{array}{rcl} \mathrm{Conf}_2(X) & \to & S^1 \\ (x_1,x_2) & \mapsto & \frac{x_2-x_1}{\|x_2-x_1\|}. \end{array}$$



Exercise 88.14. Let $A \sqcup B \subseteq \mathbb{R}^3$ be the subspace that is shown in the figure below. Show that $[\xi] \in \pi_1(\mathbb{R}^3 \setminus (A \sqcup B), x_0)$ is a product of copies of $[\alpha], [\beta]$ and their inverses.



Exercise 88.15. Let X be a topological space and let $f\colon X \to X$ be a continuous map. As in Definition 21.19 we call

$$\operatorname{Tor}(X,f) \;\coloneqq\; (X \times [0,1])\,/\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

the *mapping torus of* (X,f) and we refer to the map

$$\begin{aligned} p\colon\ \operatorname{Tor}(X,f) = (X\times[0,1])/(x,0)\sim(f(x),1) &\to S^1 \\ [(x,t)] &\mapsto \exp(2\pi\,\mathrm{i}t) \end{aligned}$$

as *the natural projection onto* S^1. We pick $x_0 \in X$.

(a) Show that if the topological space X is path-connected, then the induced homomorphism $p_*\colon \pi_1(\operatorname{Tor}(X,f),[(x_0,0)]) \to \pi_1(S^1,1)$ is an epimorphism.

(b) Show that in general in (a) we cannot drop the hypothesis that X is path-connected.

Exercise 88.16. Given a topological space X the *fundamental groupoid* is defined as the category $\Pi(X)$ where

- The objects are points of X, i.e. $\operatorname{Ob}(\Pi(X)) = X$.
- Given $x,y \in \operatorname{Ob}(\Pi(X)) = X$ we define $\operatorname{Mor}_{\Pi(X)}(x,y)$ as the set of path-homotopy classes of paths from x to y. The composition of a morphism $[\alpha]$ from x to y and a morphism $[\beta]$ from y to z is hereby given by $[\beta]\circ[\alpha] := [\alpha * \beta]$.

Now we can formulate the following exercises:

(a) Show that $\Pi(X)$ is indeed a category.

(b) Show that for any $x \in X$ the fundamental group $\pi_1(X,x)$ is isomorphic to $\operatorname{Mor}_{\Pi(X)}(x,x)$.

(c) Show that we have a *contravariant* functor $\Pi(X) \to \mathit{Group}$ which on the level objects is given by $x \mapsto \pi_1(X,x)$.

CHAPTER 89

Applications of fundamental groups

In this short chapter we present a few applications of the results on fundamental groups that we proved in the previous chapter:

(1) We will prove the Fundamental Theorem of Algebra.
(2) We prove the Borsuk-Ulam Theorem 89.1 for S^2.
(3) We will show that path-integrals give a complete path-homotopy invariant for loops in $\mathbb{C} \setminus \{0\}$.

We will not use any of the results of this chapter later on. The reader who is eager to move forward with the development of the theory can safely skip this chapter.

This chapter is organized as follows:

- In Section 89.1 we will use the calculation $\pi_1(S^1) \cong \mathbb{Z}$ from the previous chapter to give another proof of the Fundamental Theorem of Algebra.
- In Section 89.2 we will use the calculation $\pi_1(S^1) \cong \mathbb{Z}$ from the previous chapter to prove the 2-dimensional Borsuk-Ulam Theorem 89.1.
- In Section 89.3 we will discuss the relationship between fundamental groups and path integrals.
- In Section 89.4 we will provide exercises for Chapter 89.

In the next Chapter 90 we will study the functoriality of fundamental groups.

89.1. **The Fundamental Theorem of Algebra.** As a first application of our result we give another proof of the Fundamental Theorem of Algebra:

Theorem 32.4. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Remark. Other proofs of the Fundamental Theorem 32.4 of Algebra are provided on page 842 and on page 2898. □

Proof. We start out with some notation: Let $q(z)$ be a polynomial with complex coefficients. If $q(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = r$, then we can consider the path

$$\begin{array}{rccl} \gamma_r^{q(z)}\colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(r \cdot \exp(2\pi\,\mathrm{i} \cdot s))}{\left|q(r \cdot \exp(2\pi\,\mathrm{i} \cdot s))\right|} \cdot \dfrac{|q(r)|}{q(r)}. \end{array}$$

Note that $\gamma_r^{q(z)}(0) = \gamma_r^{q(z)}(1) = 1$, i.e. $\gamma_r^{q(z)}$ is a loop in $(S^1, 1)$.

Now we start out with the actual proof of the theorem. Thus let

$$p(z) \;=\; a_n \cdot z^n + a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0$$

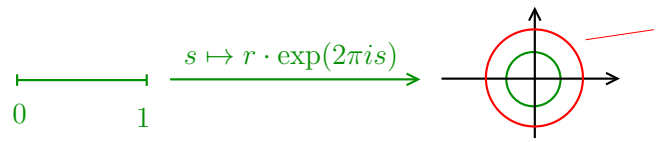
be a polynomial with $a_n \neq 0$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. We need to show that $n = 0$. For $t \in [0,1]$ we also consider the polynomial

$$p_t(z) \;:=\; a_n \cdot z^n + t \cdot (a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0).$$

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n \cdot z^n$. Since $a_n \neq 0$ there exists an $R > 0$ such that[885]

$$|a_n \cdot z^n| \;>\; |\underbrace{a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0}_{=0 \text{ if } n=0}|$$

for any $z \in \mathbb{C}$ with $|z| = R$. This inequality implies that for any $t \in [0,1]$ and any $z \in \mathbb{C}$ with $|z| = R$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = R$.



for $|z| = R$ we have $|a_n \cdot z^n| > |a_{n-1} \cdot z^{n-1} + \cdots + a_0|$, thus for any $t \in [0,1]$ we have $p_t(z) \neq 0$ on the circle $|z| = R$

We obtain the following sequence of path-homotopic loops:

$$\text{constant loop at } 1 \overset{\text{follows from the definition}}{=} \gamma_0^{p(z)} \underset{\text{p}}{\simeq} \gamma_R^{p(z)} = \gamma_R^{p_1(z)} \underset{\text{p}}{\simeq} \gamma_R^{p_0(z)} \overset{\text{follows from } p_0(z) = a_n z^n}{=} \text{the loop } \begin{array}{l}[0,1] \to S^1 \\ t \mapsto \exp(2\pi i n t).\end{array}$$

maps $\gamma_r^{p(z)}$ with $r \in [0,R]$ are well-defined, since the polynomials $p(z)$ has no zeros, they form a path-homotopy from $\gamma_0^{p(z)}$ to $\gamma_R^{p(z)}$

maps $\gamma_R^{p_t(z)}$ with $t \in [0,1]$ are well-defined, since polynomials $p_t(z)$ have no zeros on $|z|=R$, and and they form a path-homotopy from $\gamma_R^{p_1(z)}$ to $\gamma_R^{p_0(z)}$

This shows that the constant loop is path-homotopic to the loop on the right-hand side. By the Circle-π_1-Theorem 88.22 this is only possible if $n = 0$. ■

89.2. **The 2-dimensional Borsuk-Ulam Theorem.** The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984). It was first proved in 1932.

Theorem 89.1. (2D-Borsuk-Ulam Theorem) For every continuous map $f\colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Remark. The statement of the 2D-Borsuk-Ulam Theorem 89.1 also makes sense for the dimension 2 replaced by any dimension $n \in \mathbb{N}_0$. For $n = 0, 1$ one can prove the statement using elementary methods.[886] We proved the case $n \in \mathbb{N}_0$ in the Borsuk-Ulam Theorem 47.12 in Chapter 47 using the notion of the degree of a continuous map between closed oriented smooth manifolds. In Section 165.3 we will give an alternative proof for the case $n \in \mathbb{N}_0$ using singular homology groups. □

Example.

(1) It follows from the 2D-Borsuk-Ulam Theorem 89.1 that there exists a pair of antipodal points on earth with the same temperature and the same barometric pressure.
(2) The 2D-Borsuk-Ulam Theorem 89.1 implies in particular that there is no injective continuous map from $S^2 \to \mathbb{R}^2$.

[885] In fact we can write it down explicitly: it follows easily from the triangle inequality that any $R \in \mathbb{R}$ with $R \cdot |a_n| > |a_{n-1}| + \cdots + |a_0|$ will work.
[886] How would you prove the statement?

(3) In Chapter 48 we give many applications of the general Borsuk-Ulam Theorem 47.12. Most of these applications specialize to, less impressive applications, for $n = 2$. □

Our proof of the 2D-Borsuk-Ulam Theorem 89.1 makes use of the concept of the degree of a continuous map $S^1 \to S^1$. In the following we develop this concept, which is of independent interest. At the end of this section we return to the proof of the 2D-Borsuk-Ulam Theorem 89.1.

Definition 89.2.

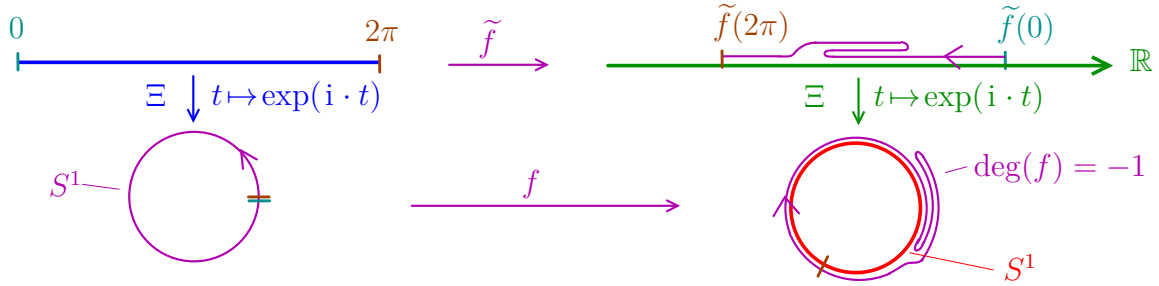
(1) As in Definition 88.18 we work with the map

$$\begin{array}{rccc} \Xi\colon & \mathbb{R} & \to & S^1 \\ & t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

(2) Let $f\colon S^1 \to S^1$ be a continuous map. By the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21 we can lift the path[887] $f \circ \Xi\colon [0, 2\pi] \to S^1$ to a path $\widetilde{f}\colon [0, 2\pi] \to \mathbb{R}$. We refer to

$$\deg(f) \;:=\; \tfrac{1}{2\pi}\cdot\big(\widetilde{f}(2\pi) - \widetilde{f}(0)\big) \;\in\; \mathbb{Z}$$

as the degree of f.



Lemma 89.3. (Degree–Well-Defined Lemma) The above degree of a continuous map $S^1 \to S^1$ is well-defined.

Proof. Let $f\colon S^1 \to S^1$ be a continuous map. Let $\widetilde{f}_1, \widetilde{f}_2\colon [0, 2\pi] \to \mathbb{R}$ be two lifts of the map $f \circ \Xi\colon [0, 2\pi] \to S^1$. Since $\exp(\mathrm{i}\widetilde{f}_1(0)) = \Xi(\widetilde{f}_1(0)) = f(\Xi(0)) = \Xi(\widetilde{f}_2(0)) = \exp(\mathrm{i}\widetilde{f}_2(0))$ we see that $\widetilde{f}_2(0) - \widetilde{f}_1(0) \in 2\pi\cdot\mathbb{Z}$. It follows that the map $t \mapsto \widetilde{f}_1(t) + \widetilde{f}_2(0) - \widetilde{f}_1(0)$ is also a lift of $f \circ \Xi$. Since this lift has the same starting point as $\widetilde{f}_2$ it follows from the uniqueness statement of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21 that for all $t \in [0, 2\pi]$ we have $\widetilde{f}_2(t) = \widetilde{f}_1(t) + \widetilde{f}_2(0) - \widetilde{f}_1(0)$. Using this observation we now see that

$$\tfrac{1}{2\pi}\cdot\big(\widetilde{f}_2(2\pi) - \widetilde{f}_2(0)\big) \underset{\text{by the above discussion}}{\overset{\uparrow}{=}} \tfrac{1}{2\pi}\cdot\big(\underbrace{\widetilde{f}_1(2\pi) + \widetilde{f}_2(0) - \widetilde{f}_1(0)}_{=\widetilde{f}_2(2\pi)} - \widetilde{f}_2(0)\big) \underset{\text{simplifying}}{\overset{\uparrow}{=}} \tfrac{1}{2\pi}\cdot\big(\widetilde{f}_1(2\pi) - \widetilde{f}_1(0)\big).$$

This shows that the definition of the degree of f is independent of the choice of the lift of $f \circ \Xi$. ■

Remark. For continuous self-maps of S^1 we now have two notions of degree, namely the one we introduced above and the one we introduced in Definition 46.5. Fortunately we already showed implicitly in the Degree-via-Lifts Lemma 46.14 that these two notions of a degree coincide. □

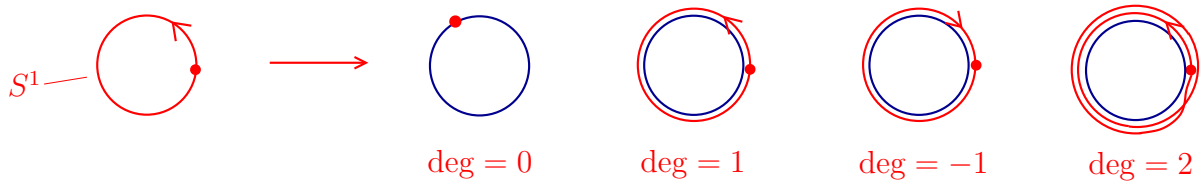
[887] The fact that $1 \neq 2\pi$ is one of the major curses of mathematics.

Now that we know that the notion of a degree is well-defined it is worth looking at a few examples.

Example.

(1) Let $c\colon S^1 \to S^1$ be a constant map. Since any lift of the constant map $c \circ \Xi\colon [0, 2\pi] \to S^1$ is also constant we deduce that the degree of the constant map $c\colon S^1 \to S^1$ is zero.

(2) Let $k \in \mathbb{Z}$. We consider the self-map of S^1 that is given by $z \mapsto z^k$.



A lift of the map $(z \mapsto z^k) \circ \Xi$ is given by the map $t \mapsto 2\pi \cdot t \cdot k$. We deduce that $\deg(z \mapsto z^k) = k$. □

The following lemma[888] provides us with one of the key properties of degrees.

Lemma 89.4. (Degree Homotopy Lemma) If $f, g\colon S^1 \to S^1$ are two homotopic maps, then $\deg(f) = \deg(g)$.

Proof. Let $F\colon S^1 \times [0,1] \to S^1$ be a homotopy between two maps $f, g\colon S^1 \to S^1$. By the Square Lifting Lemma 88.23 we know that the map $F \circ (\Xi \times \mathrm{id}_{[0,1]})\colon [0, 2\pi] \times [0,1] \to S^1$ lifts to a continuous map $\widetilde{F}\colon [0, 2\pi] \times [0,1] \to \mathbb{R}$. Next note that for any $t \in [0, 2\pi]$ we have $\Xi(\widetilde{F}_t(0)) = F_t(\Xi(0)) = F_t(1) = F_t(\Xi(2\pi)) = \Xi(\widetilde{F}_t(2\pi))$. This implies that for each $t \in [0, 2\pi]$ we have $\widetilde{F}_t(2\pi) - \widetilde{F}_t(0) \in 2\pi \cdot \mathbb{Z}$. It follows from the Connected-to-Discrete Lemma 4.37 that the map $t \mapsto \widetilde{F}_t(2\pi) - \widetilde{F}_t(0)$ is constant. We now see that

$$\deg(f) \underset{\substack{\uparrow \\ \text{by definition,} \\ \text{since } F_0 = f}}{=} \tfrac{1}{2\pi} \cdot \big(\widetilde{F}_0(2\pi) - \widetilde{F}_0(0)\big) \underset{\substack{\uparrow \\ \text{by the above discussion} \\ \text{discussion}}}{=} \tfrac{1}{2\pi} \cdot \big(\widetilde{F}_1(2\pi) - \widetilde{F}_1(0)\big) \underset{\substack{\uparrow \\ \text{by definition,} \\ \text{since } F_1 = g}}{=} \deg(g).$$

■

In the proof of the 2D-Borsuk-Ulam Theorem 89.1 we will need the following notion of an odd self-map of S^1:

Definition 89.5. A map $f\colon S^1 \to S^1$ is called odd, if $f(-z) = -f(z)$ for all $z \in S^1$.

Example. Let $k \in \mathbb{Z}$. The map $S^1 \to S^1$ that is given by $z \mapsto z^k$ is odd if and only if k is odd. □

The statement of the following proposition is mercifully easy to internalize.[889]

Proposition 89.6. (Odd Degree Proposition) Let $n \in \mathbb{N}$ and let $f\colon S^1 \to S^1$ be a continuous map. If f is odd, then $\deg(f)$ is odd.

Proof. Let $f\colon S^1 \to S^1$ be a continuous odd map. We pick a lift $\widetilde{f}\colon [0, 2\pi] \to \mathbb{R}$ of the map $f \circ \Xi\colon [0, 2\pi] \to S^1$. Recall that this means that $\Xi(\widetilde{f}(t)) = f(\Xi(t))$ for any $t \in [0, 2\pi]$.

[888] Note that the following lemma can be viewed as a special case of the Degree–Homotopy Proposition 46.4.

[889] The following proposition is closely related to the Smooth Odd Map–Odd Degree Proposition 47.14. Furthermore in the Odd Map–Odd Degree Proposition 165.6 we will generalize the proposition to all dimensions.

Note that for any $t \in [0,\pi]$ we have

$$\Xi(\widetilde{f}(t+\pi)+\pi) \overset{\text{by definition of } \Xi}{=} -\Xi(\widetilde{f}(t+\pi)) \overset{\text{since } \widetilde{f} \text{ is a lift}}{=} -f(\Xi(t+\pi)) \overset{\text{by definition of } \Xi}{=} -f(-\Xi(t)) \overset{\text{since } f \text{ is odd}}{=} -(-f(\Xi(t))) = f(\Xi(t)).$$

We have thus shown that $t \mapsto \widetilde{f}(t+\pi)+\pi$ is also a lift of $f|_{[0,\pi]}$. It follows from the uniqueness statement of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21 that there exists an $n \in \mathbb{Z}$ such that for all $t \in [0,\pi]$ we have $\widetilde{f}(t+\pi)+\pi = \widetilde{f}(t)+n\cdot 2\pi$. We now see that

$$\deg(f) = \tfrac{1}{2\pi}\cdot\big(\widetilde{f}(2\pi)-\widetilde{f}(0)\big) = \tfrac{1}{2\pi}\cdot\Big(\underbrace{\widetilde{f}(2\pi)-\widetilde{f}(\pi)}_{=n\cdot 2\pi-\pi,\text{ by the above applied to } t=\pi} + \underbrace{\widetilde{f}(\pi)-\widetilde{f}(0)}_{=n\cdot 2\pi-\pi,\text{ by the above applied to } t=0}\Big) = 2n-1.$$

We have thus shown that $\deg(f)$ is odd. ■

Using the above results on degrees we can now give a concise proof of the 2D-Borsuk-Ulam Theorem 89.1.

Proof of the 2D-Borsuk-Ulam Theorem 89.1. [890] Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Recall that we need to show that there exists an $x \in S^2$ with $f(-x) = f(x)$.

Suppose that there exists no pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$. Put differently, suppose that for every $x \in S^2$ we have $f(x) \neq f(-x)$. We consider the composition of the following two maps

$$\begin{array}{rccccl} \Phi\colon S^1\times[0,1] & \to & S^2_{\geq 0} & \to & S^1. \\ (y,t) & \mapsto & (y\cdot\sqrt{1-t^2},t) & & \\ & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}}_{\text{defined since } f(x)\neq f(-x)} \end{array}$$

A quick look at the definitions shows that Φ_0 is odd and it shows that Φ_1 is a constant map. Thus we see that

$$\text{odd number} \underset{\uparrow}{=} \deg(\Phi_0) \overset{\text{by the Degree Homotopy Lemma 89.4, since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{=} \deg(\Phi_1) \underset{\uparrow}{=} \deg(\text{constant map}) \underset{\uparrow}{=} 0.$$

since Φ_0 is odd we obtain from the Odd Degree Proposition 89.6 that $\deg(\Phi_0)$ is odd

by definition of Φ_1

on page 1754 we saw that the degree of any constant map is zero

We have thus obtained a contradiction. ■

89.3. **Interlude: Fundamental groups and path integrals.** In Chapter 87 we saw that path integrals can play a role when studying smooth paths in open subsets of $\mathbb{C}$. In this section we will see that path-integrals can be used to study fundamental groups of open subsets of $\mathbb{C}$. This section is charming since it connects two fields of mathematics,

[890] The following proof is almost verbatim the same as the proof of the Borsuk-Ulam Theorem that we gave on page 1107. For the most part we just need to replace the Smooth Odd Map–Odd Degree Proposition 47.14 by the Odd Degree Proposition 89.6.

that at first glance are quite different. Nonetheless, we will not make use of any of the results of this section, thus the section can be skipped safely.

Let us recall the definition of path integrals from Definition 87.4:

Definition 89.7. Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a continuous map. Furthermore let $\gamma\colon [a,b] \to U$ be a smooth path. We define the path integral of f along γ as follows:

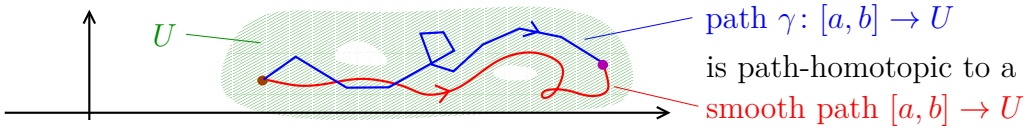
$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} f(\gamma(t)) \cdot \gamma'(t)\,dt \;\in \mathbb{C}.$$

We also recall the following theorem about the (non-) effect on path-homotopies on path integrals.

Theorem 39.8. (Cauchy Theorem) Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a *holomorphic* function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

Proposition 89.8. (Smooth Path-Connectivity Proposition) Let $U \subseteq \mathbb{R}^n$ be an open subset. Every path $\gamma\colon [a,b] \to U$ is path-homotopic to a smooth map.



Proof. This statement is a special case of the Smooth Path-Connectivity Proposition 25.32. We will also provide a down-to-earth proof in Exercise 89.7. ∎

Now we can extend our definition of path integrals, at least for the case that f is holomorphic, from smooth paths to arbitrary paths.

Definition 89.9. Let $U \subseteq \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a *holomorphic* function and let $\gamma\colon [a,b] \to U$ be a path. By the Smooth Path-Connectivity Proposition 89.8 we know that γ is path-homotopic to a smooth path $\widetilde{\gamma}\colon [a,b] \to U$. We set

$$\int_\gamma f(z)\,dz \;:=\; \int_{\widetilde{\gamma}} f(z)\,dz.$$

(It follows from the Cauchy Theorem 39.8 together with the Homotopy-Equivalence Relation Lemma 18.9 that this definition does not depend on the choice of $\widetilde{\gamma}$.)

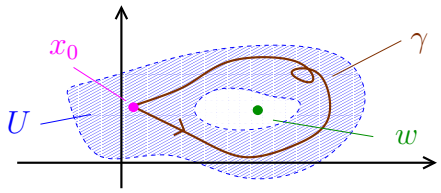
Theorem 89.10. (Path Integral-Homomorphism Theorem) Let $U \subseteq \mathbb{C}$ be an open subset and let $x_0 \in U$. For every $w \in \mathbb{C} \setminus U$ the map

$$\begin{array}{rcl} \pi_1(U,x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi\,\mathrm{i}} \int_\gamma \frac{1}{z-w}\,dz \end{array}$$

is a well-defined homomorphism.

Proof. We start out with the following claim:

Claim. The map $\pi_1(U,x_0) \to \mathbb{Z}$ is well-defined.



the map
$$\begin{array}{rcl} \pi_1(U,x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi\,\mathrm{i}} \int\limits_\gamma \frac{1}{z-w}\,dz \end{array}$$
is a homomorphism

Proof. Suppose we are given loops $\gamma, \delta\colon [0,1] \to U$ in x_0 which represent the same class in $\pi_1(U,x_0)$. In other words, we assume that γ and δ are path-homotopic. We pick a smooth loop $\widetilde{\gamma}$ that is path-homotopic to γ and we pick a smooth loop $\widetilde{\delta}$ that is path-homotopic to δ. It follows from the Path-Homotopy–Equivalence Relation Lemma 88.5 that $\widetilde{\gamma}$ and $\widetilde{\delta}$ are also path-homotopic. Thus their path-integrals agree, but these path-integrals are by definition the path-integrals of γ and δ. ⊞

Finally note that it follows easily from the definitions and a little calculation that the given map $\pi_1(U,x_0) \to \mathbb{Z}$ is a homomorphism. We will fill in the details in Exercise 89.8. ■

Now we can formulate the main result of this section.

Proposition 89.11. (Path Integral–Isomorphism Proposition) The map
$$\begin{array}{rcl} \Theta\colon \pi_1(\mathbb{C}\setminus\{0\},1) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi\,\mathrm{i}} \int\limits_\gamma \frac{1}{z}\,dz \end{array}$$
is an isomorphism.

Proof. In Exercise 88.12 we showed, using the Circle-π_1-Theorem 88.22, that the map
$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(\mathbb{C}\setminus\{0\},1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{C}\setminus\{0\} \\ t & \mapsto & \exp(2\pi\,\mathrm{i}\cdot n\cdot t) \end{array}\right] \end{array}$$
is an isomorphism of groups. It is straightforward to see that $\Theta\circ\Psi\colon \mathbb{Z} \to \pi_1(\mathbb{C}\setminus\{0\},1) \to \mathbb{Z}$ is the identity. Since Ψ is an isomorphism we see that $\Theta = \Psi^{-1}$ is also an isomorphism. ■

89.4. Exercises for Chapter 89.

Exercise 89.1. Let $f\colon S^2 \to S^2$ be a continuous map such that $f(x) \neq f(-x)$ for all $x \in S^2$. Show that f is surjective.

Exercise 89.2.

(a) Does the analog of the 2D-Borsuk-Ulam Theorem 89.1 hold for the torus $S^1 \times S^1$? More precisely: Is it true that for every continuous map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ there exists a point $(z,w) \in S^1 \times S^1$ with $f(z,w) = f(-z,-w)$?

(b) Show that given any continuous map $f\colon S^1 \times S^1 \to \mathbb{R}$ there exists an $(x,y) \in S^1 \times S^1$ with $f(-x,-y) = f(x,y)$.

Exercise 89.3. Let $f, g\colon S^1 \to S^1$ be continuous maps.

(1) Show that if f is not surjective, then $\deg(f) = 0$.

(2) Show that $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Exercise 89.4. Let $f\colon S^1 \to S^1$ be a map. We say that f is *even* if for every $z \in S^1$ we have $f(-z) = f(z)$. Shows that the degree of any even continuous map is even.

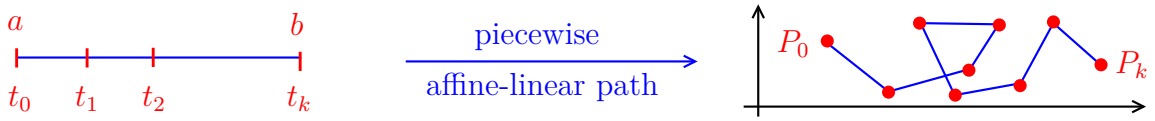
Exercise 89.5. Let $f\colon S^1 \to S^1$ be a continuous map. Use results from elementary real analysis to show that if $\deg(f) \neq 1$, then f has a fixed point.

Exercise 89.6. Let $A \in \mathrm{O}(2)$. Show that $\deg(z \mapsto A \cdot z) = \det(A)$.

Exercise 89.7. Let $U \subseteq \mathbb{R}^n$ be an open subset. We say that a path $\delta\colon [a,b] \to U$ is *piecewise affine-linear* if there exist $a = t_0 < t_1 < \cdots < t_k = b$ and $P_0, \ldots, P_k \in U$ such that for every $s \in [t_i, t_{i+1}]$ we have
$$\delta(s) \;=\; P_i \cdot \tfrac{t_{i+1}-s}{t_{i+1}-t_i} + P_{i+1} \cdot \tfrac{s-t_i}{t_{i+1}-t_i}.$$

(1) Show that every path $\beta\colon [a,b] \to U$ is path-homotopic in U to a piecewise-affine linear path $\gamma\colon [a,b] \to U$.

(2) Show that every piecewise affine-linear path $\gamma\colon [a,b] \to U$ is path-homotopic in U to a smooth path $\delta\colon [a,b] \to U$.
Hint. We want to find a smooth map δ, but we do not demand that $\delta'(s) \neq 0$ for all s. In particular the image of δ does not have to look smooth to the human eye.



Exercise 89.8. Let $U \subseteq \mathbb{C}$ be an open subset and let $w \in \mathbb{C} \setminus U$. Show that for any two paths $\alpha, \beta\colon [0,1] \to U$ with $\alpha(1) = \beta(0)$ we have
$$\int\limits_{\alpha * \beta} \tfrac{1}{z-w}\, dz \;=\; \int\limits_{\alpha} \tfrac{1}{z-w}\, dz + \int\limits_{\beta} \tfrac{1}{z-w}\, dz.$$

CHAPTER 90

Functoriality of fundamental groups

To unlock the full power of fundamental groups we need to view the fundamental group as a covariant functor from the category of pointed topological spaces to the category of groups.

This chapter is organized as follows:

- In Section 90.1 we will see that fundamental groups are a functor.
- In Section 90.2 we will use functoriality to show that S^1 is not a retract of $\overline{B}^2$.
- In Section 90.3 we will show that the fundamental groups of homotopy equivalent topological spaces are isomorphic.
- In Section 90.4 we will determine the fundamental group of the product of a family topological spaces.
- In Appendix 90.5 we will prove that the fundamental groups of topological groups are abelian.
- In Appendix 90.6 we will use functoriality to draw a few conclusions on the fundamental group of the surface of genus 2. In particular we will show that the torus is not homeomorphic to the surface of genus 2.
- In Section 90.7 we will provide exercises for Chapter 90.

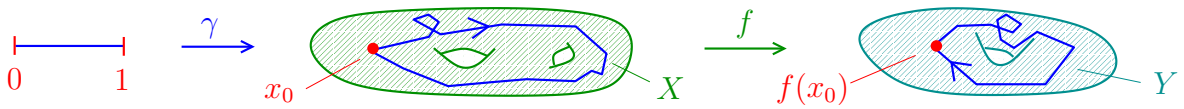
In the next Chapter 91 we will introduce the concept of a covering of a topological space.

90.1. **The fundamental group as functor.** In the following definition we will see that a continuous map induces maps on fundamental groups:

Definition 90.1. Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma\colon [0,1] \to X$ is a loop in the point x_0, then $f \circ \gamma\colon [0,1] \to Y$ is a loop in the point $f(x_0)$. It follows from the Homotopy Composition Lemma 18.10 that

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the path-homotopy class. We call f_* the induced map.



Example.

(1) If $f\colon X \to Y$ is a constant map, then the induced map $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is the trivial group homomorphisms.

(2) Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. We want to understand the map $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$. Here is one possible answer: It follows immediately from the definitions that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(S^1,1) & \xrightarrow{\quad f_* \quad} & \pi_1(S^1,1) \\ {\scriptstyle k \mapsto [t \mapsto \exp(2\pi \mathrm{i} k t)]}\uparrow \cong & & \cong \uparrow {\scriptstyle k \mapsto [t \mapsto \exp(2\pi \mathrm{i} k t)]} \\ \mathbb{Z} & \xrightarrow{\quad g \mapsto n \cdot g \quad} & \mathbb{Z}. \end{array}$$

(Here the vertical isomorphisms are given by the Circle-π_1-Theorem 88.22.) □

The induced maps look they should define a covariant functor. We just need to figure out what the categories are supposed to be. This leads us to the following definition:

Definition 90.2. A pointed topological space is a pair (X, x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PTop}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & := & \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) & := & \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{array}$$

with the usual composition of maps as the category of pointed topological spaces.

Proposition 90.3. (π_1-Functor Proposition)

(1) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category of groups. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Group}) \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}((X,x_0),(Y,y_0)) & \to & \mathrm{Mor}(\pi_1(X,x_0),\pi_1(Y,y_0)) \\ (f\colon (X,x_0) \to (Y,y_0)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array} \right) \end{array}$$

form a covariant functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{Group}$ of groups.

(2) If $f\colon X \to Y$ is a homeomorphism between two topological spaces, then for any $x_0 \in X$ the induced map $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is a group isomorphism.

Example. On page 183 we saw that the upper and lower hemispheres $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are homeomorphic to the closed ball $\overline{B}^n$. It follows from the π_1-Functor Proposition 90.3 and the discussion on page 1734 that the hemispheres are simply connected. □

Proof. Statement (2) follows immediately from Statement (1) together with the Isomorphism Preservation Lemma 5.26. Thus it remains to prove Statement (1).

Let $f\colon (X,x_0) \to (Y,y_0)$ be a continuous map between pointed topological spaces. It follows easily from the definitions that for any two loops γ and δ in (X,x_0) we have $(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta)$. But this means that $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is a group homomorphism. Furthermore one can easily verify that

$$\begin{array}{rcll} (\mathrm{id}_X)_* & = & \mathrm{id}_{\pi_1(X,x_0)}, & \text{for all pointed pairs } (X,x_0), \\ (g \circ f)_* & = & g_* \circ f_*, & \text{for all continuous maps } f\colon (X,x_0) \to (Y,y_0), g\colon (Y,y_0) \to (Z,z_0). \end{array}$$

But this means precisely that the given maps define a covariant functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{Group}$ of groups. ■

In many applications we will want to know whether a given map between topological spaces induces an isomorphism of fundamental groups. The following proposition, which can be viewed as an addendum to the Change-of-Base Point Proposition 88.14, says that in most settings the property of being an isomorphism is independent of the choice of base point.

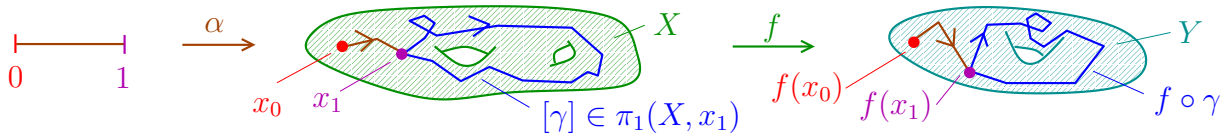
Proposition 90.4. (Change-of-Base Point Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$.

(1) If $\alpha\colon [0,1] \to X$ is a path from x_0 to some point x_1, then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{f_*} & \pi_1(Y,f(x_1)) \\ \cong\downarrow{\scriptstyle \alpha_*} & & \cong\downarrow{\scriptstyle (f\circ\alpha)_*} \\ \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,f(x_0)). \end{array}$$

(The vertical maps are the isomorphisms that we introduced in the Change-of-Base Point Proposition 88.14 (1).)

(2) If $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $f_*\colon \pi_1(X,x_1) \to \pi_1(Y,f(x_1))$ is also an isomorphism (epimorphism, monomorphism).



Proof.

(1) This statement follows immediately from the definitions.

(2) This statement follows from (1) and the fact that by the Change-of-Base Point Proposition 88.14 (1) the vertical maps are isomorphisms. ■

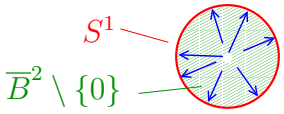
90.2. **Retractions and fundamental groups.** In this section we will see that the functoriality of the fundamental groups can be helpful for showing the non-existence of retractions. Let us recall Definition 19.1:

Definition 90.5. Let X be a topological space and let $A \subseteq X$ be a subspace. We say that A is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

Examples.

(1) The circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$, in fact a retraction is given by $r(z,w) = (z,1)$.

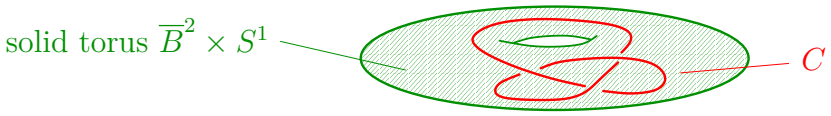
(2) On page 523 we saw that S^{n-1} is a retract of $\overline{B}^n \setminus \{0\}$.



$r(x) := \frac{x}{\|x\|}$ defines a retraction from $\overline{B}^2 \setminus \{0\}$ to S^1

(3) The set $S^0 = \{-1, 1\}$ is not a retract of $\overline{B}^1 = [-1, 1]$. Indeed, since $[-1, 1]$ is connected and since S^0 is discrete it follows from the Connected-to-Discrete Lemma 4.37 that any continuous map from $[-1, 1]$ to S^0 has to be constant. Thus there does not exist a continuous map $f\colon [-1, 1] \to S^0$ with $f(-1) = -1$ and $f(1) = 1$.

(4) Finally we consider the knotted curve C in the solid torus $\overline{B}^2 \times S^1$ that is shown in the figure below:



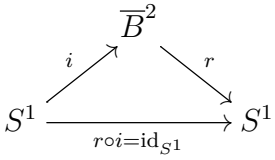
We invite the reader to think about the question whether or not C is a retract of $\overline{B}^2 \times S^1$. We will answer the question in Section 305.1. □

Now we can push the second-to-last example up by one dimension.

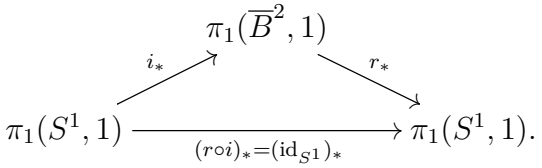
Proposition 90.6. (Circle–Non Retract Proposition) The circle S^1 is not a retract of $\overline{B}^2$.

Remark. In the Boundary Non-Retraction Propositions 38.6 and 75.13 we used quite different techniques to prove the generalization of the Circle–Non Retract Proposition 90.6 to spheres of all dimensions. We will give an alternative proof of the generalization, using singular homology groups, in the Sphere-Non Retract Proposition 149.9. □

Proof. Let $i\colon S^1 \to \overline{B}^2$ be the inclusion map. We now suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition of a retraction this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of continuous maps between topological spaces



which induces



It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we also know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 1734 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in the Circle-π_1-Theorem 88.22 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group. Thus we have obtained a contradiction. ■

Lemma 90.7. (Retraction-π_1-Lemma) Let X be a topological space and let $A \subseteq X$ be a retract of X. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.

(1) There exists an epimorphism $\varphi\colon \pi_1(X, a) \to \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.

(2) The inclusion-induced map $\pi_1(A, a) \to \pi_1(X, a)$ is a monomorphism.

Proof. Almost exactly the same argument as in the proof of the Circle–Non Retract Proposition 90.6 also provides a proof of the lemma. We will fill in the details in Exercise 90.7. ■

Example. We had just seen on page 1761 that the circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$. Furthermore we saw in the Circle-π_1-Theorem 88.22 that $\pi_1(S^1, 1)$ is not the trivial group. By the Retraction-π_1-Lemma 90.7 the map $\pi_1(S^1 \times \{1\}, (1,1)) \to \pi_1(S^1 \times S^1, (1,1))$ is a monomorphism. It follows from this discussion that the fundamental group of the torus is non-trivial. In the Sphere-π_1-Proposition 88.24 we saw that $\pi_1(S^2) = 0$. Thus we see that the torus $S^1 \times S^1$ and the sphere S^2 are not homeomorphic. □

90.3. **The fundamental groups of homotopy equivalent topological spaces.** In the following two propositions we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. On numerous occasions this fact will greatly simplify the calculation of fundamental groups.

Proposition 90.8. (Homotopy-π_1-Proposition) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then there exists a path $\alpha\colon [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} & \xrightarrow{g_*} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*}\ {\scriptstyle [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]} \\ & \xrightarrow{f_*} & \pi_1(Y, f(x_0)). \end{array}$$

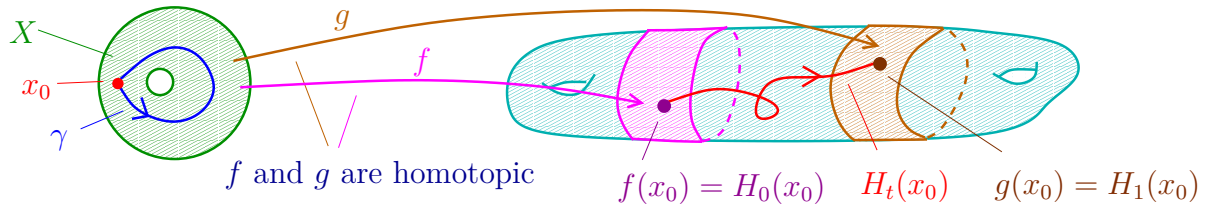
(Here the vertical map is the isomorphism given by the Change-of-Base Point Proposition 88.14 (1).) In particular

$$\begin{array}{c} f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism". Furthermore, the following two refinements hold:

(a) If f and g are homotopic rel x_0, then $f_* = g_*$.[891]

(b) If $f(x_0) = g(x_0)$ and if the two maps f and g are homotopic, then there exists an $\alpha \in \pi_1(Y, f(x_0))$ such that $f_*(z) = \alpha \cdot g_*(z) \cdot \alpha^{-1}$ for every $z \in \pi_1(X, x_0)$.



Proposition 90.9. (Homotopy Equivalence-π_1-Proposition)

(1) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$. If f is a homotopy equivalence, then the induced map

$$f_*\colon \pi_1(X, x_0) \ \to \ \pi_1(Y, f(x_0))$$

is an isomorphism.

[891] This statement can be viewed as a special case of the Homotopy Composition Lemma 18.10.

(b) Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map
$$i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$$
is an isomorphism.

(2) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

Example. For any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

on page 523 we saw that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$, thus we get an isomorphism from the Homotopy Equivalence-π_1-Proposition 90.9 (1b)

$$\pi_1(\mathbb{R}^n \setminus \{Q\}) \cong \pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

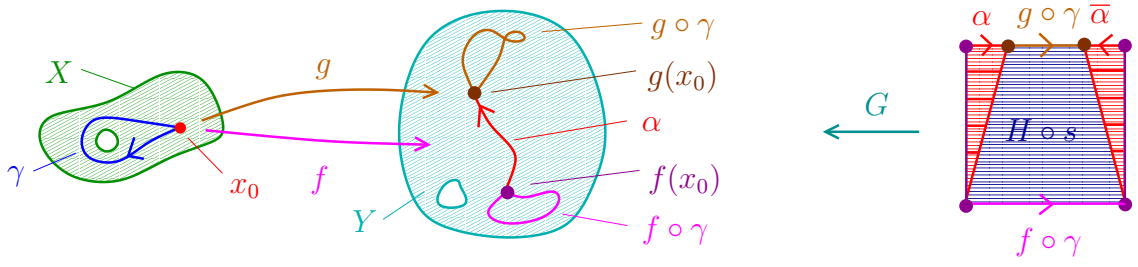
follows from the π_1-Functor Proposition 90.3 (2) since $\mathbb{R}^n \setminus \{Q\}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\}$

by the Circle-π_1-Theorem 88.22 and the Sphere-π_1-Proposition 88.24 □

Proof of the Homotopy-π_1-Proposition 90.8. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces, let $x_0 \in X$ and let $H\colon X \times [0,1] \to Y$ be a homotopy between f and g. Let
$$\begin{aligned} \alpha\colon [0,1] &\to Y \\ t &\mapsto H(x_0, t) \end{aligned}$$
be the path with which the homotopy H connects the two points $H(x_0, 0) = f(x_0)$ and $H(x_0, 1) = g(x_0)$.

Claim. The maps
$$\begin{aligned} f_*\colon \pi_1(X, x_0) &\to \pi_1(Y, f(x_0)) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned} \quad \text{and} \quad \begin{aligned} \pi_1(X, x_0) &\to \pi_1(Y, f(x_0)) \\ [\gamma] &\mapsto [\alpha * (g \circ \gamma) * \overline{\alpha}] \end{aligned}$$
are identical.

Proof. Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). We need to show that the loops $f \circ \gamma$ and $\alpha * (g \circ \gamma) * \overline{\alpha}$ are path-homotopic. We consider the map
$$\begin{aligned} G\colon [0,1] \times [0,1] &\to Y \\ (r,t) &\mapsto \begin{cases} \alpha(4r), & \text{if } r \in \left[0, \tfrac{1}{4}t\right], \\ H\left(\gamma\left(\frac{r - \frac{1}{4}t}{1 - \frac{1}{2}t}\right), t\right), & \text{if } r \in \left[\tfrac{1}{4}t, 1 - \tfrac{1}{4}t\right], \\ \overline{\alpha}(4r - 3), & \text{if } r \in \left[1 - \tfrac{1}{4}t, 1\right]. \end{cases} \end{aligned}$$



Using the Pasting Proposition 4.7 (2) it is straightforward to verify that G is indeed a continuous map. Furthermore it follows easily from the definition that the map G satisfies

$G_0(t) = G_1(t) = f(x_0)$ for all $t \in [0,1]$ and that it satisfies

$$G(r,0) = (f \circ \gamma)(r) \quad \text{and} \quad G(r,1) = (\alpha * (g \circ \gamma) * \overline{\alpha})(r) \quad \text{for all } r \in [0,1].$$

This observation shows that G is in fact a path-homotopy between the loops $f \circ \gamma$ and $\alpha * (g \circ \gamma) * \overline{\alpha}$[892] in $(Y, f(x_0))$. ⊞

As in the statement of the proposition we consider the following diagram:

$$\begin{array}{ccc} & \xrightarrow{g_*} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*} \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}] \\ & \xrightarrow[f_*]{} & \pi_1(Y, f(x_0)). \end{array}$$

We make the following observations:

(i) It follows from the claim that the diagram commutes.
(ii) By the Change-of-Base Point Proposition 88.14 (1) we know that the vertical map is an isomorphism.

It follows from (i) and (ii) that f_* is an isomorphism if and only if g_* is an isomorphism. Evidently the analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism". Finally we turn to the proofs of Statements (a) and (b).

(a) Now suppose that f and g are homotopic rel x_0. By definition this means that there exists a homotopy H with $H(x_0, t) = f(x_0)$ for all $t \in [0,1]$. In other words, the path $\alpha(t) \coloneqq H(x_0, t)$ is a constant path. It follows from the above that $f_* = g_*$.
(b) If $f(x_0) = g(x_0)$, then α is a loop in $(Y, f(x_0))$, i.e. it defines an element in $\pi_1(Y, y_0)$. Thus we have $[\alpha * (g \circ \gamma) * \overline{\alpha}] = [\alpha] \cdot g_*(\gamma) \circ [\alpha]^{-1}$. By the above commutative diagram we obtain the desired equality. ■

Proof of the Homotopy Equivalence-π_1-Proposition 90.9.

(1) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We pick a homotopy inverse $g\colon Y \to X$ of f. By definition of a homotopy inverse this means that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Next note that it follows from the Homotopy-π_1-Proposition 90.8 and the functoriality of fundamental groups that the induced maps

$$\begin{aligned} \alpha \coloneqq g_* \circ f_* = (g \circ f)_*\colon \pi_1(X, x_0) &\to \pi_1(X, g(f(x_0)) \\ \beta \coloneqq f_* \circ g_* = (f \circ g)_*\colon \pi_1(Y, f(x_0)) &\to \pi_1(Y, f(g(f(x_0)))) \end{aligned}$$

are isomorphisms. This shows that f_* has the left inverse $\alpha^{-1} \circ g_*$ and it has the right inverse $g_* \circ \beta^{-1}$. It follows from the Inverse Uniqueness Lemma 5.10 (1) that f_* is an isomorphism. Similarly we see that g_* is an isomorphism.

(b) This statement follows immediately from (a) and the Deformation Retract–Homotopy Equivalence Lemma 19.10.

(2) If $X = \{x_0\}$ is a singleton, then there exists only a unique map $[0,1] \to X$, thus we see that in this case the fundamental group is trivial. The case that X is contractible, i.e. the case that X is homotopy equivalent to a singleton, now follows from this calculation together with Statement (1). ■

[892] To be precise, G is a homotopy between $f \circ \gamma$ and a suitable parametrization of $\alpha * (g \circ s) * \overline{\alpha}$.

In the $\mathbb{R}$-$\mathbb{R}^2$-Non Homeomorphism Lemma 4.36 we already saw that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 90.10. ($\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma) The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

Remark. Not surprisingly, the obvious generalization of the $\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma 90.10 holds: If $k \neq l$, the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. In fact we proved this statement already in the Topological Invariance-of-Domain Proposition 80.16 using de Rham cohomology. In the $\mathbb{R}^k$-$\mathbb{R}^l$-Non Homeomorphism–Proposition 149.6 we will provide an alternative proof of this statement using singular homology. □

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Note that f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\mathbb{Z} \underset{\text{page 1764}}{\cong} \pi_1(\mathbb{R}^2 \setminus \{P\}) \underset{\text{since } f \text{ is a homeomorphism}}{\cong} \pi_1(\mathbb{R}^3 \setminus \{Q\}) \underset{\text{page 1764}}{=} 0.$$

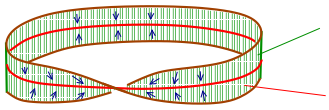
Thus we obtained a contradiction. ■

We conclude this section with one more calculation of a fundamental group.

Corollary 90.11. (Möbius Band-π_1-Corollary) We have

$$\pi_1(\text{Möbius band }) \cong \mathbb{Z}.$$

Proof. On page 527 we saw that the Möbius band admits a deformation retraction onto the "central circle" C.



Thus we see that $\pi_1(\text{Möbius band }) \underset{\text{by the Homotopy Equivalence } \pi_1\text{-Proposition 90.9}}{\cong} \pi_1(C) \cong \pi_1(S^1) \underset{\text{by the Circle-}\pi_1 \text{ Theorem 88.22}}{\cong} \mathbb{Z}.$ ■

90.4. **The fundamental group of the product of topological spaces.** In Chapters 7 and 8 we got to know several ways to construct new (pointed) topological spaces out of given (pointed) topological spaces. It is natural to ask how these constructions affect the fundamental groups. In this section we consider the product of two pointed topological spaces.

Before we state the relevant proposition it is worth recalling the definition of the direct product of two groups.

Definition 90.12. Let G and H two groups. We can form the Cartesian product

$$G \times H := \{(g,h) \,|\, g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1, h_1) \cdot (g_2, h_2) \coloneqq (g_1 \cdot g_2, h_1 \cdot h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the direct product of the groups G and H.

Now we can formulate the promised proposition.

Proposition 90.13. (Product-π_1-Proposition)

(1) Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j \colon \prod_{i \in I} X_i \to X_j$ the natural projection. The induced natural homomorphism[893]

$$\begin{array}{rcl} \Phi \colon \pi_1\Big(\prod_{i \in I} X_i, (x_i)_{i \in I}\Big) & \to & \prod_{i \in I} \pi_1(X_i, x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i \in I} \end{array}$$

is an isomorphism.

(2) If we are given a finite family $(X_1, x_1), \ldots, (X_k, x_k)$ of pointed topological spaces, then the inverse to the map from (1) is given by the natural homomorphism

$$\begin{array}{rcl} \Psi \colon \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_k, x_k) & \to & \pi_1(X_1 \times \cdots \times X_k, (x_1, \ldots, x_k)) \\ (g_1, \ldots, g_k) & \mapsto & \iota_{1*}(g_1) \cdot \ldots \cdot \iota_{k*}(g_k) \end{array}$$

where $\iota_i \colon X_i \to X_1 \times \cdots \times X_k$ is given by $y \mapsto (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$.

Example.

(1) It follows immediately from the Product-π_1-Proposition 90.13 together with the Circle-π_1-Theorem 88.22 that for any $n \in \mathbb{N}$ we have an isomorphism $\pi_1((S^1)^n) \cong \mathbb{Z}^n$. We will discuss this example in greater detail in the Torus-π_1-Proposition 90.14.

(2) From the Product-π_1-Proposition 90.13 together with the Circle-π_1-Theorem 88.22 we also obtain that $\pi_1((S^1)^{\mathbb{N}}) \cong \mathbb{Z}^{\mathbb{N}}$. This result is interesting, since by Tychonoff's Theorem 7.14 we know that $(S^1)^{\mathbb{N}}$ is compact. Since $\mathbb{Z}^{\mathbb{N}}$ is uncountable we have shown that there exists a compact topological space with uncountable fundamental group. □

Remark.

(1) The Product-π_1-Proposition 90.13 can be summarized as saying that the π_1-functor turns a product in the category of pointed topological spaces into a product in the category *Group* of groups. Put more succinctly: the fundamental group preserves products.

(2) In most cases we apply the Product-π_1-Proposition 90.13 to the case of two pointed topological spaces (A, a_0) and (B, b_0). Let us spell out this special case. We consider the projections

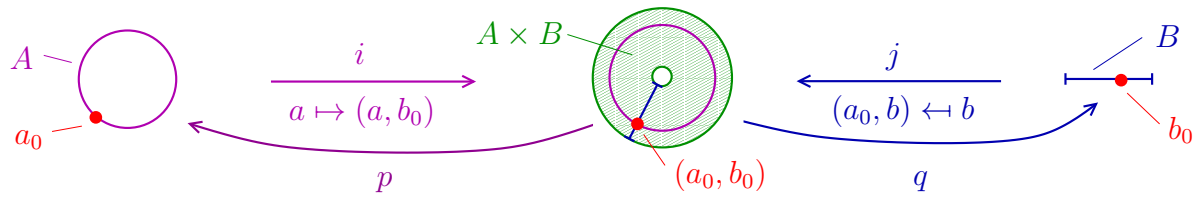
$$\begin{array}{rcl} p \colon A \times B & \to & A \\ (a, b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q \colon A \times B & \to & B \\ (a, b) & \mapsto & b \end{array}$$

and the inclusions

$$\begin{array}{rcl} i \colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j \colon B & \to & A \times B \\ b & \mapsto & (a_0, b). \end{array}$$

The maps

[893] The projections p_i induce homomorphisms on fundamental groups which in turn, by the Product-of-Sets-and-Groups Lemma 6.4, induce the given homomorphism.



$$\begin{array}{rcl}\Phi\colon \pi_1(A\times B,(a_0,b_0)) & \to & \pi_1(A,a_0)\times\pi_1(B,b_0)\\ \gamma & \mapsto & (p_*(\gamma),q_*(\gamma))\\ i_*(\alpha)\cdot j_*(\beta) & \leftarrow\!\shortmid & (\alpha,\beta)\end{array}$$

are both natural[894] isomorphisms. □

Proof.

(1) First we introduce some general notation:

- Given two pairs of topological spaces (W,A) and (X,B) we denote by $C((W,A),(X,B))$ the set of all continuous maps $f\colon W\to X$ with $f(A)\subseteq B$.
- We say that two continuous maps $f,g\in C((W,A),(X,B))$ are *homotopic* if there exists a homotopy $H\colon W\times[0,1]\to X$ with $H_0=f$ and $H_1=g$ such that for all $t\in[0,1]$ we have $H_t(A)\subseteq B$.
- We denote by $[(W,A),(X,B)]$ the set of all homotopy classes of continuous maps $(W,A)\to(X,B)$.
- By definition we have $\pi_1(Z,z_0)=[([0,1],\{0,1\}),(Z,\{z_0\})]$ for any pointed topological space (Z,z_0).

Let $(X_i,x_i)_{i\in I}$ be a family of pointed topological spaces. We set $x:=(x_i)_{i\in I}$. It follows from the Topological-Product Proposition 7.2 (1) that the map

$$\begin{array}{rcl}C\Big(([0,1],\{0,1\}),\Big(\prod\limits_{i\in I}X_i,x\Big)\Big) & \to & \prod\limits_{i\in I}C(([0,1],\{0,1\}),(X_i,x_i))\\ f & \mapsto & (p_i\circ f)_{i\in I}\end{array}$$

is a bijection. The analogous statement holds if we replace the pair $([0,1],\{0,1\})$ by the pair $([0,1]\times[0,1],\{0,1\}\times[0,1])$. This shows that the map

$$\begin{array}{rcl}\pi_1\Big(\prod\limits_{i\in I}X_i,x\Big)=\Big[([0,1],\{0,1\}),\Big(\prod\limits_{i\in I}X_i,x\Big)\Big] & \to & \prod\limits_{i\in I}[([0,1],\{0,1\}),(X_i,x_i)]=\prod\limits_{i\in I}\pi_1(X_i,x_i)\\ [f] & \mapsto & \underbrace{[(p_i\circ f)_{i\in I}]}_{=p_{i*}(f)}\end{array}$$

is a bijection. But this is precisely the homomorphism that we are considering.

(2) To simplify the notation we just work with two pointed topological spaces (X,x_0) and (Y,y_0) and we denote the two inclusions by $i\colon X\to X\times Y$ and $j\colon Y\to X\times Y$. Since

---

[894]Here "natural" means the following. If $f\colon(A,a_0)\to(\widetilde{A},\widetilde{a}_0)$ and $g\colon(B,b_0)\to(\widetilde{B},\widetilde{b}_0)$ are continuous maps of pointed topological spaces, then it follows easily from the definitions that the following diagrams commute:

$$\begin{array}{ccc}\pi_1(A\times B,(a_0,b_0)) & \xrightarrow{\Phi} & \pi_1(A,a_0)\times\pi_1(B,b_0)\\ \downarrow{\scriptstyle (f\times g)_*} & & \downarrow{\scriptstyle f_*\times g_*}\\ \pi_1(\widetilde{A}\times\widetilde{B},(\widetilde{a}_0,\widetilde{b}_0) & \xrightarrow{\Phi} & \pi_1(\widetilde{A},\widetilde{a}_0)\times\pi_1(\widetilde{B},\widetilde{b}_0))\end{array}$$

and

$$\begin{array}{ccc}\pi_1(A,a_0)\times\pi_1(B,b_0) & \xrightarrow{\Psi} & \pi_1(A\times B,(a_0,b_0))\\ \downarrow{\scriptstyle f_*\times g_*} & & \downarrow{\scriptstyle (f\times g)_*}\\ \pi_1(\widetilde{A},\widetilde{a}_0)\times\pi_1(\widetilde{B},\widetilde{b}_0) & \xrightarrow{\Psi} & \pi_1(\widetilde{A}\times\widetilde{B},(\widetilde{a}_0,\widetilde{b}_0)).\end{array}$$

by (1) we already know that Φ is an isomorphism it remains to prove the following claim:

Claim. We have $\Phi \circ \Psi = \mathrm{id}$ as self-maps of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Given $(\alpha, \beta) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ we obtain the following equalities in the group $\pi_1(X, x_0) \times \pi_1(Y, y_0)$:

$$\begin{array}{rcl} (\Phi \circ \Psi)(\alpha, \beta) & \overset{\text{definition of } \Psi}{=} & \Phi(i_*(\alpha) \cdot j_*(\beta)) \overset{\text{definition of } \Phi}{=} (p_*(i_*(\alpha) \cdot j_*(\beta)), q_*(i_*(\alpha) \cdot j_*(\beta))) \overset{\text{since } p_* \text{ and } q_* \text{ are homomorphisms}}{=} \\ & = & (\underbrace{p_*(i_*(\alpha))}_{=(p\circ i)_*=\mathrm{id}_*} \cdot \underbrace{p_*(j_*(\beta))}_{=(p\circ j)_*=0}, \underbrace{q_*(i_*(\alpha))}_{=(q\circ i)_*=0} \cdot \underbrace{q_*(j_*(\beta))}_{=(q\circ j)_*=\mathrm{id}_*}) = (\alpha, \beta). \end{array}$$

■

We record one particularly interesting application of the Product-π_1-Proposition 90.13:

Proposition 90.14. (Torus-π_1-Proposition) Let $n \in \mathbb{N}$. We consider the n-dimensional torus $(S^1)^n = S^1 \times \cdots \times S^1$. Given $j \in \{1, \ldots, n\}$ we consider the loop

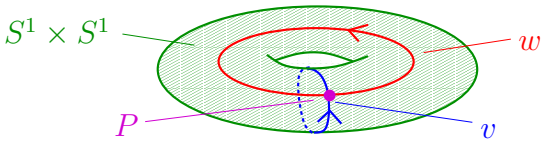
$$\begin{array}{rcl} v_j \colon [0,1] & \to & S^1 \times \cdots \times S^1 \\ t & \mapsto & (1, \ldots, 1, \underbrace{\exp(2\pi \mathrm{i} \cdot t)}_{\uparrow}, 1, \ldots, 1). \end{array}$$

$$j\text{-th coordinate}$$

The map

$$\begin{array}{rcl} \mathbb{Z}^n & \mapsto & \pi_1(S^1 \times \cdots \times S^1, (1, \ldots, 1)) \\ (r_1, \ldots, r_n) & \mapsto & [v_1]^{r_1} \cdot \ldots \cdot [v_n]^{r_n} \end{array}$$

is an isomorphism.



the map $\begin{array}{rcl} \mathbb{Z}^2 & \to & \pi_1(S^1 \times S^1, P) \\ (r, s) & \mapsto & [v]^r \cdot [w]^s \end{array}$ is an isomorphism

Example. Let $n \geq 2$. It follows immediately from the π_1-Functor Proposition 90.3 (2), the fact, shown in the Sphere-π_1-Proposition 88.24 that for $n \geq 2$ we have $\pi_1(S^n) = 0$ and the Torus-π_1-Proposition 90.14 that the n-dimensional torus $(S^1)^n$ is not homeomorphic to the n-dimensional sphere S^n. □

Proof. The statement follows immediately from the Product-π_1-Proposition 90.13 and the explicit isomorphism $\Psi \colon \mathbb{Z} \to \pi_1(S^1, 1)$ given in the Circle-π_1-Theorem 88.22. ■

90.5. **Appendix I: Fundamental groups of topological groups.** In this very short section we consider the fundamental groups of topological groups. We will prove one little proposition whose proof is related to the proof of the Product-π_1-Proposition 90.13. First we recall Definition 7.30:

Definition 90.15. A topological group is a topological space X together with a group structure, such that the two maps

$$\begin{array}{rcl} X \times X & \to & X \\ (x, y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous. (Evidently we equip $X \times X$ with the product topology.)

Example. As we saw in the Matrices–Topological Group Lemma 7.31, the usual matrix groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SO}(n)$, $\mathrm{SL}(n,\mathbb{R})$ etc. are topological groups. □

Proposition 90.16. (Topological Group-π_1-Proposition) The fundamental group of any path-connected topological group is abelian.[895]

Proof. Let X be a path-connected topological group with neutral element e. We want to show that $\pi_1(X,e)$ is abelian. Thus let $f,g\colon [0,1]\to X$ be two loops in (X,e). We need to show that $[f]\cdot[g]=[g]\cdot[f]\in\pi_1(X,e)$. We consider the multiplication map

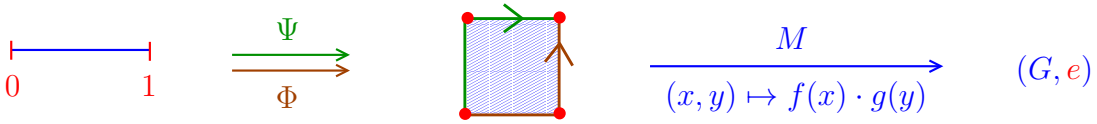
$$\begin{array}{rccc} M\colon & [0,1]\times[0,1] & \to & X \\ & (x,y) & \mapsto & f(x)\cdot g(y) \end{array}$$

and we consider the two paths

$$\Phi\colon [0,1]\to[0,1]\times[0,1],\quad s\mapsto\begin{cases}(2s,0), & \text{if } s\in[0,\tfrac12],\\ (1,2s-1), & \text{if } s\in[\tfrac12,1]\end{cases}$$

and

$$\Psi\colon [0,1]\to[0,1]\times[0,1],\quad s\mapsto\begin{cases}(0,2s), & \text{if } s\in[0,\tfrac12],\\ (2s-1,1), & \text{if } s\in[\tfrac12,1].\end{cases}$$



Since $[0,1]\times[0,1]$ is convex we see that there exists a path-homotopy H between the two paths Φ and Ψ, which both go from $(0,0)$ to $(1,1)$. Now we see that

$$[f]\cdot[g] = [f*g] = [M\circ\Phi] \underset{\uparrow}{=} [M\circ\Psi] = [g*f] = [g]\cdot[f] \in \pi_1(X,e).$$

a path-homotopy between $M\circ\Phi$ and $M\circ\Psi$ is given by $M\circ H$, note that here we use that the map M is continuous

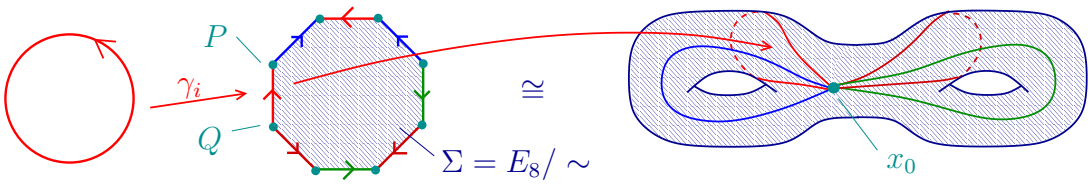
■

90.6. **Appendix II: The surface of genus 2.** We conclude this chapter with an extended example to practice some of the methods from this chapter. Note that as of right now we lack the tools to calculate the fundamental groups of all surfaces. But with the current tricks we can already prove some statements, in particular we can already show that the surface of genus 2 is not homeomorphic to the torus.

Lemma 90.17. (Surface-Retraction-to-Curve Lemma) Let E_8 be the regular octagon from the definition of the surface of genus 2 and let $\Sigma := E_8/\sim$ be the surface of genus 2, see page 342 for details. We denote by $x_0\in\Sigma$ the point corresponding to the vertices of E_8. Let $\gamma_1,\gamma_2,\gamma_3,\gamma_4\colon S^1\to\Sigma=E_8/\sim$ be the maps given by the four pairs of sides.

(1) Each map $\gamma_i\colon S^1\to\Sigma=E_8/\sim$ is an embedding. We set $C_i := \gamma_i(S^1)$.
(2) For each $i\in\{1,\dots,4\}$ there exists a retraction $\rho_i\colon \Sigma=E_8/\sim\to C_i$ such that for $j\neq i$ the map $\rho_i\circ\gamma_j\colon S^1\to C_i$ is a constant map.
(3) For each $i\in\{1,\dots,4\}$ there exists an epimorphism $\pi_1(\Sigma,x_0)\to\mathbb{Z}$ that sends $[\gamma_i]$ to 1[896] and all other $[\gamma_j]$ to zero.
(4) There exists an epimorphism $\pi_1(\Sigma)\to\mathbb{Z}^4$.

[895] In the H-Space-π_1-Proposition 299.27 we will generalize the result from topological groups to "H-spaces".
[896] Recall that by the bijection on page 1744 we view maps $f\colon S^1\to\Sigma$ with $f(1)=x_0$ as elements in $\pi_1(\Sigma,x_0)$.



Sketch of proof.

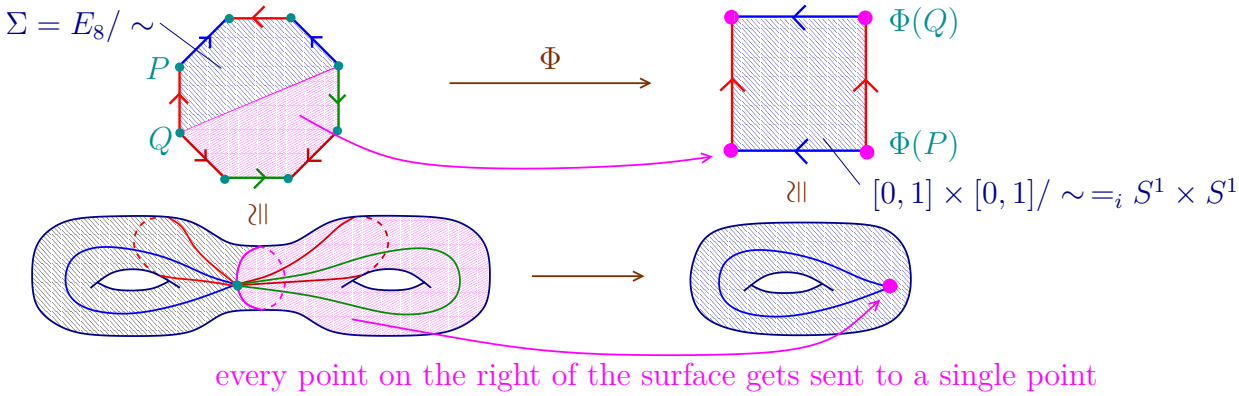
(1) Let $i \in \{1, \dots, 4\}$ and let P_i, Q_i be the vertices of the corresponding edge of E_8. The map γ_i is given by

$$\begin{array}{rcl} \gamma_i \colon S^1 =_{\mathrm{i}} [0,1]/\sim & \to & E_8/\sim \\ \exp(2\pi\,\mathrm{i}\,t) = [t] & \mapsto & [t \cdot P_i + (1-t) \cdot Q_i]. \end{array}$$

It is clear that γ_i is continuous and injective. By the Surface-is-Topological Manifold Proposition 23.7 we know that $\Sigma = E_8/\sim$ is Hausdorff. It follows from the Compact-Hausdorff Proposition 4.23 (2) that γ_i is a closed embedding.

(2) We consider the case $i = 1$. The other cases are dealt with in a similar fashion.

(a) We consider the map $\Phi \colon E_8/\sim \to S^1 \times S^1$ that is sketched in the figure below on the top. It is given by sending all points in the lower pentagon to a single point, such a map can be extended continuously to the upper pentagon in such a way that the remaining four sides of the upper pentagon get sent homeomorphically to the four sides of the square.[897] [898]



(b) The desired retraction is now given by the concatenation of the maps

$$\Sigma \xrightarrow{\Phi} S^1 \times S^1 \xrightarrow{(z,w)\mapsto z} S^1.$$

(3) Let $i \in \{1, \dots, 4\}$. The induced map $\rho_{i*} \colon \pi_1(\Sigma, x_0) \to \pi_1(C_i, x_0)$ has the desired properties.

(4) It follows from (3) that the map

[897] It is clearly slightly painful to write down this map explicitly. It is easier to write down a similar map, which maps a square X onto a triangle Y in such a way that one side of the square gets sent to a vertex of the triangle and such that the remaining three sides of the square get sent to the three sides of the triangle. For example the map

$$\begin{array}{rcl} X \coloneqq [0,1] \times [0,1] & \to & Y \coloneqq \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1], y \in [0,x]\} \\ (x,y) & \mapsto & (x, x \cdot y) \end{array}$$

is of that form.

[898] In the lower part of the above figure we illustrate the map for the surfaces viewed as subsets of $\mathbb{R}^3$. There the map Φ is given by crushing the right-hand half of the surface of genus 2 to a single point.

$$\begin{array}{rcl} \pi_1(\Sigma, x_0) & \to & \mathbb{Z}^4 \\ g & \mapsto & (\rho_{1*}(g), \dots, \rho_{4*}(g)) \end{array}$$

is an epimorphism. ■

Corollary 90.18. The surface Σ of genus 2 is not homeomorphic to $S^1 \times S^1$.

Proof. In the Torus-π_1-Proposition 90.14 we saw that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$. Elementary algebra shows that there is no epimorphism from $\mathbb{Z}^2$ onto $\mathbb{Z}^4$. On the other hand we just saw in the Surface-Retraction-to-Curve Lemma 90.17 that there exists an epimorphism $\pi_1(\Sigma) \to \mathbb{Z}^4$. It follows that $\pi_1(S^1 \times S^1)$ is not isomorphic to $\pi_1(\Sigma)$. By the π_1-Functor Proposition 90.3 this implies that $S^1 \times S^1$ is not homeomorphic to Σ. ■

90.7. Exercises for Chapter 90.

Exercise 90.1.

(a) Let $f\colon X \to Y$ be a surjective continuous map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ necessarily an epimorphism?

(b) Let $f\colon X \to Y$ be an injective continuous map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ necessarily a monomorphism?

Exercise 90.2. Let $P_1, \dots, P_k \in \mathbb{R}^2$ be k distinct points. Show that given any base point $x_0 \in \mathbb{R}^2 \setminus \{P_1, \dots, P_k\}$ there exists an epimorphism from $\pi_1(\mathbb{R}^2 \setminus \{P_1, \dots, P_k\}, x_0)$ onto $\mathbb{Z}^k$.

Exercise 90.3. We consider the cylinder $Z = S^1 \times [0, 1]$. Let $P_1, \dots, P_k \in S^1 \times [0, 1]$ be distinct points. What is the maximal $m \in \mathbb{N}_0$ such that there exists an epimorphism $\pi_1(Z \setminus \{P_1, \dots, P_k\}) \to \mathbb{Z}^m$? Does the answer depend on the choice of the P_i?
Remark. At this point you cannot give a complete answer to the exercise, but you can still try to guess the largest possible m.

Exercise 90.4. Given a group π we define its *automorphism group* $\mathrm{Aut}(\pi)$ as the group of all self-isomorphisms of π. We define the *inner automorphism group* as

$$\mathrm{Inn}(\pi) \coloneqq \{\varphi \in \mathrm{Aut}(\pi) \mid \text{ there exists an } h \in \pi \text{ such that } \varphi(g) = hgh^{-1} \text{ or all } g \in G\}.$$

One can easily verify that $\mathrm{Inn}(\pi)$ is a normal subgroup of $\mathrm{Aut}(\pi)$. Thus we can define the *outer automorphism group* to be the group

$$\mathrm{Out}(\pi) \coloneqq \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi).$$

Now let X be a path-connected topological space, let $x_0 \in X$ and let $n \in \mathbb{N}$.

(a) We denote by $\mathrm{Homeo}(X)$ the group of self-homeomorphisms of X. Show that the map

$$\begin{array}{rcl} \Theta\colon \mathrm{Homeo}(X) & \to & \mathrm{Out}(\pi_1(X, x_0)) \\ (f\colon X \to X) & \mapsto & (\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(X, f(x_0)) \xrightarrow[\uparrow]{\gamma_*} \pi_1(X, x_0)) \end{array}$$

pick any path γ from x_0 to $f(x_0)$ and consider the corresponding isomorphism given by the Change-of-Base Point Proposition 88.14 (1)

is well-defined, i.e. independent of the choice of the paths.

(b) Show that $\Theta\colon \mathrm{Homeo}(X) \to \mathrm{Out}(\pi_1(X, x_0))$ is a group homomorphism.

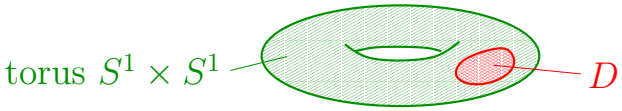
Exercise 90.5. Recall that we denote by $\mathcal{PTop}$ the category of pointed topological spaces. Suppose we are given $(X, x_0), (Y, y_0) \in \mathrm{Ob}(\mathcal{PTop})$. Show that the direct product of (X, x_0) and (Y, y_0) in $\mathcal{PTop}$ is given by $(X \times Y, (x_0, y_0))$.
Hint. This exercise is very similar to Exercise 6.6.

Exercise 90.6. Let $f\colon X \to Y$ be a continuous map between two topological spaces. Let $x_0 \in X$ be a base point. We write $y_0 = f(x_0)$ and we denote by $i\colon f(X) \to Y$ the inclusion map. Suppose that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Does it follow that $i_*\colon \pi_1(f(X), y_0) \to \pi_1(Y, y_0)$ is also an isomorphism?

Exercise 90.7. Let X be a topological space and let $A \subseteq X$ be a retract. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.

(a) Show that there exists an epimorphism $\varphi\colon \pi_1(X, a) \to \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.
(b) Show that the inclusion-induced map $i_*\colon \pi_1(A, a) \to \pi_1(X, a)$ is a monomorphism.

Exercise 90.8. We consider the torus $S^1 \times S^1$. Let D be the image of a smooth embedding $\overline{B}^2 \to S^1 \times S^1$. Is D a retract of $S^1 \times S^1$?



Exercise 90.9.

(a) Is $\mathrm{SL}(n, \mathbb{C})$ a retract of $\mathrm{GL}(n, \mathbb{C})$?
(b) Is $\mathrm{SL}(n, \mathbb{C})$ a deformation retract of $\mathrm{GL}(n, \mathbb{C})$?

Hint. Consider interesting maps of $\mathbb{C} \setminus \{0\}$ into and out of $\mathrm{GL}(n, \mathbb{C})$.

Exercise 90.10. We consider the Möbius band M and its boundary ∂M. Note that the boundary ∂M is homeomorphic to S^1 and recall that on page 527 we saw that M is homotopy equivalent to S^1. Is ∂M a retract of M?



Exercise 90.11. Show that every continuous map $\overline{B}^2 \to \overline{B}^2$ has a fixed point.
Hint. Show that if there was a continuous map $f\colon \overline{B}^2 \to \overline{B}^2$ without fixed points, then there would exist a retraction $\overline{B}^2 \to S^1$.

Exercise 90.12. Show that there exists a homeomorphism of the torus $S^1 \times S^1$ that does not extend to a homeomorphism of the solid torus $S^1 \times \overline{B}^2$.

Exercise 90.13. Let $X \subseteq \mathbb{R}^2$ be a non-empty subset such that $\mathbb{R}^2 \setminus X$ is path-connected and non-empty.

(a) Show that if X is compact, then $\pi_1(\mathbb{R}^2 \setminus X)$ is non-trivial.
Hint. Since X is non-empty there exists a point P. What is $\pi_1(\mathbb{R}^2 \setminus \{P\})$?
(b) Does the conclusion of (a) also hold if we just assume that X is closed?

Exercise 90.14. Let $a, b, c, d \in \mathbb{Z}$. We consider the map

$$\begin{array}{rcl} f\colon S^1 \times S^1 & \to & S^1 \times S^1 \\ (z, w) & \mapsto & (z^a \cdot w^b, z^c \cdot w^d). \end{array}$$

In the Torus-π_1-Proposition 90.14 we gave an explicit isomorphism $\Theta\colon \mathbb{Z}^2 \to \pi_1(S^1 \times S^1, (1,1))$.

(a) Determine the map $\Theta^{-1} \circ f_* \circ \Theta\colon \mathbb{Z}^2 \to \mathbb{Z}^2$.

(b) For which choices of $a, b, c, d \in \mathbb{Z}$ is the map f a homeomorphism?

Exercise 90.15. Let K be a path-compact connected non-empty topological space. Is $\pi_1(K)$ necessarily countable?

Exercise 90.16. Let G be a path-connected topological group and let (X, x_0) be a path-connected topological space. Let

$$F\colon G \times X \to X$$

be an action of G on X such that F itself is continuous. We consider the evaluation map

$$\begin{array}{rcl} F_{x_0}\colon G & \to & X \\ g & \mapsto & g \cdot x_0. \end{array}$$

Show that for every $\gamma \in \pi_1(G, g_0)$ the element $F_{x_0 *}(\gamma)$ lies in the center of $\pi_1(X, x_0)$.

CHAPTER 91

Coverings

Arguably our greatest success so far in the theory of fundamental groups was the calculation of $\pi_1(S^1)$ in the Circle-π_1-Theorem 88.22. The calculation relied on a detailed study of the map $\Xi\colon \mathbb{R} \to S^1$ that is given by $\Xi(t) = \exp(\mathrm{i}t)$. In particular this setup had two interesting features:

(1) Given any $z \in S^1$ we could find an open neighborhood U (called $V_\pm$) such that the preimage of the open neighborhood consists of disjoint open subsets $\{U_i\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ such that for each $i \in \mathbb{Z}$ the map Ξ restricts to a homeomorphism $U_i \to U$.
(2) $\mathbb{R}$ is simply connected.

The idea now is to generalize the above approach of calculating $\pi_1(S^1, 1)$ to more general settings. The key to doing so is the concept of a "covering" $p\colon X \to B$ between topological spaces. In this chapter we will introduce coverings and we will discuss many examples. In the next Chapter 92 and the follow-up Chapter 93 we will relate coverings to fundamental groups. In particular in Chapter 93 we will use coverings to determine the fundamental groups of the real projective spaces and of the Klein bottle.

This chapter is organized as follows:

- In Section 91.1 we will study group actions on topological spaces.
- In Section 91.2 we will introduce coverings of topological spaces.
- In Section 91.3 we will discuss a few basic properties of coverings.
- In Section 91.4 we will provide exercises for Chapter 91.

In the next Chapter 92 we will discuss the relationship between coverings and fundamental groups.

91.1. **Group actions.** The main topic of this chapter are "coverings". As we will see shortly, a main source of "coverings" is the already familiar concept of group actions. In this section we consider group actions in their own right. In particular we will give a new description of the Klein bottle and we will introduce new, fascinating topological spaces, namely the "lens spaces". In the subsequent sections we turn to "coverings" and their interaction with groups actions and fundamental groups.

We recall two definitions about actions from Definition 2.36 and Definition 9.4 and we introduce one new definition.

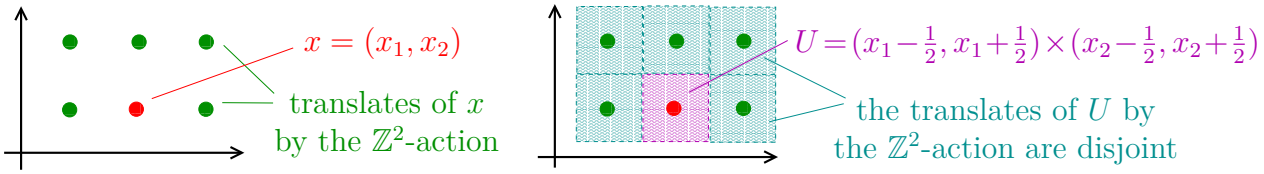
Definition 91.1. Let G be a group that acts on a topological space X.

(1) We say that G acts freely if $g \cdot x = x$ for some $g \in G$ and some $x \in X$ implies that $g = e$.
(2) We G acts properly if for every two points[899] x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, g \cdot U \cap V \neq \varnothing\}$ is finite.

(3) We say that G acts discretely, if for each $x \in X$ there exists an open neighborhood U of x so that[900 901 902]
$$U \cap g \cdot U \;=\; \varnothing \quad \text{for all } g \neq e.$$

Examples.

(1) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the open subset $U := (x_1 - \frac{1}{2}, x_1 + \frac{1}{2}) \times \cdots \times (x_n - \frac{1}{2}, x_n + \frac{1}{2})$ has the desired property.



(2) The action
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$
of the group $\mathbb{Z}$ on the topological space $\mathbb{R} \times [-1,1]$ is discrete. Indeed, for any (x,y) in X the open neighborhood $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times [-1,1]$ has the desired property.

(3) The map
$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$
defines an action of the group $\{\pm 1\}$ on S^n that is discrete. Indeed, for any $P \in S^n$ the open hemisphere $U := \{Q \in S^n \mid \underbrace{P \cdot Q}_{\text{standard inner product}} > 0\}$ has the desired property.

(4) Evidently every discrete action is free. But the converse does not hold. For example the action of the group $(\mathbb{R}, +)$ on $\mathbb{R}$ by addition is clearly free but one can easily verify that it is not discrete. □

We consider our last two types of examples in greater detail.

Lemma 91.2. (Klein Bottle-as-Quotient Lemma)[903] We consider the following two self-homeomorphisms of $\mathbb{R}^2$:
$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$
We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that
$$G \;:=\; \begin{array}{l} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written as} \\ \text{a finite composition of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$
The group G acts by definition continuously on $\mathbb{R}^2$.

(1) The elements A and B of G do not commute, in particular the group G is non-abelian.

(2) The action of G on $\mathbb{R}^2$ is proper and discrete.

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[900] Note that if G acts discretely, then it also acts freely.

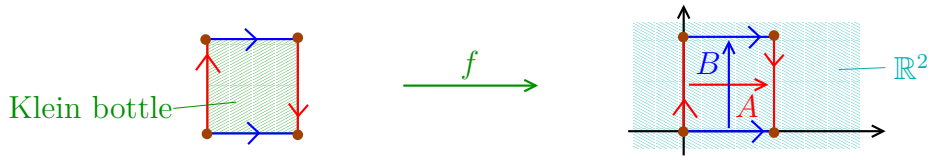
[901] It follows easily from the properties of an action that if $U \cap g \cdot U = \varnothing$ for all $g \neq e$, then $g \cdot U \cap k \cdot U = \varnothing$ for all $k \neq g$. On several occasions we will make use of this observation.

[902] In Exercise 91.1 we will see that there exists an action that is discrete but not proper.

(3) We consider the Klein bottle $K = ([0,1] \times [0,1])/\sim$ as defined on page 312. The map

$$\begin{array}{rcl} f\colon K = ([0,1] \times [0,1])/\sim & \to & \mathbb{R}^2/G \\ [(x,y)] & \mapsto & [(x,y)] \end{array}$$

is a homeomorphism.



Proof. In the group G we use the product symbol "$\cdot$" to indicate the group structure instead of the "$\circ$" symbol.

(1) We calculate that

$$(A \cdot B)(0,0) \;=\; A(0,1) \;=\; (1,0) \;\neq\; (1,2) \;=\; B(1,1) \;=\; (B \cdot A)(0,0).$$

This shows that A and B do not commute in G. This shows that G is not abelian.

(2) We start out with the following claim:

Claim 1. Given $\epsilon, \eta \in \{-1,1\}$ we have $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$.

Proof. First let $\epsilon = 1$. We perform the following calculation:

$$\begin{array}{lllll} (A\cdot B)(x,y) & = A(x,y+1) & = (x+1,-y) & = B^{-1}(x+1,1-y) & = (B^{-1}\cdot A)(x,y), \\ (A\cdot B^{-1})(x,y) & = A(x,y-1) & = (x+1,2-y) & = B(x+1,1-y) & = (B\cdot A)(x,y). \end{array}$$

We just showed that $A \cdot B^\eta = B^{-\eta} \cdot A$. By taking inverses on both sides we get the promised equalities $A^{-1} \cdot B^\eta = B^{-\eta} \cdot A^{-1}$ for $\epsilon = -1$. ⊞

Claim 2.

(a) Any $g \in G$ is of the form $A^k \cdot B^l$ for some $k, l \in \mathbb{Z}$.

(b) For $k, l \in \mathbb{Z}$ we have

$$(A^k \cdot B^l)(x,y) \;=\; \begin{cases} (x+k, y+l), & \text{if } k \text{ is even,} \\ (x+k, 1-y-l), & \text{if } k \text{ is odd.} \end{cases}$$

Proof.

(a) Let $g \in G$. By definition we can write g as a product of powers of A and B. Using the equality $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$ from Claim 1 we can swap the order of powers of A with powers of B, at the price of changing the exponent of B. This means that we can bring all $A^{\pm 1}$ to the left and get the desired result.

(b) This statement follows from an elementary straightforward calculation. ⊞

Claim 3. The group G acts properly on $\mathbb{R}^2$.

Proof. Let $P, Q \in \mathbb{R}^2$. We set $U := B^2_1(P)$ and $V := B^2_1(Q)$. It follows easily from Claim 2 (a) and (b) that the set $\{g \in G \mid g \cdot U \cap V \neq \varnothing\}$ is finite. ⊞

Claim 4. The group G acts discretely on $\mathbb{R}^2$.

Proof. Let $(x,y) \in \mathbb{R}^2$. We set $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times (y - \frac{1}{2}, y + \frac{1}{2})$. It follows easily from Claim 2 (a) and (b) that for any non-trivial $g \in G$ we have $U \cap g \cdot U = \varnothing$. ⊞

[903] This lemma is basically the solution to Exercise 31.10.

(3) Using Claim 2 from the proof of Statement (2) one can easily show that the map $f\colon ([0,1]\times[0,1])/\sim \to \mathbb{R}^2/G$ is a bijection. It follows from the Quotient-to-Quotient Lemma 8.6 that f is continuous. By the Quotient Topology-Properties Proposition 8.5 (1) we know that $([0,1]\times[0,1])/\sim$ is compact. By (2) we know that the action of G is proper. Thus it follows from the Group Action-Hausdorff Quotient Proposition 9.5 that $\mathbb{R}^2/\sim$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 4.23 (3) that f is a homeomorphism. ■

We turn to our final example of a group acting on a topological space.

Lemma 91.3. (Lens Space-Discrete Action Lemma) We consider the 3-dimensional sphere $S^3 = \{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_1|^2+|z_2|^2 = 1\}$. Let $p\in\mathbb{Z}\setminus\{0\}$ and let $q\in\mathbb{Z}$ with $\gcd(p,q)=1$. We set $\xi := \exp(2\pi\,\mathrm{i}/p)$.

(1) The map
$$\begin{array}{rcl} \mathbb{Z}/p\mathbb{Z}\times S^3 & \to & S^3 \\ (k+p\mathbb{Z},(z_1,z_2)) & \mapsto & (\xi^k\cdot z_1, \xi^{q\cdot k}\cdot z_2)\end{array}$$
defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}/p\mathbb{Z}$ on S^3.

(2) The quotient space $S^3/(\mathbb{Z}/p\mathbb{Z})$ is a closed 3-dimensional topological manifold that has a unique structure of an oriented smooth manifold such that the natural projection $S^3\to S^3/(\mathbb{Z}/p\mathbb{Z})$ is a local diffeomorphism and orientation-preserving.

Proof.

(1) It is straightforward to verify that the map $\mathbb{Z}/p\mathbb{Z}\times S^3\to S^3$ does indeed define an action of the group $\mathbb{Z}/p\mathbb{Z}$ on the topological space S^3. It follows from the Smooth Manifold–Boundary Proposition 25.27 (1c) that the action is smooth and a direct calculation shows that the action is orientation-preserving.

Claim. The action is discrete.

Proof. Let $(z_1,z_2)\in S^3$. First we assume that $z_1\neq 0$. We set
$$U := \{(u_1,u_2)\in S^3 \,|\, u_1 = z_1\cdot r\cdot\exp(2\pi\,\mathrm{i}t) \text{ with } r\in(0,\infty), t\in(-\tfrac{1}{2p},\tfrac{1}{2p})\}.$$
One can easily verify that U is an open neighborhood of $(z_1,z_2)\in S^3$.[904] Next note that $\xi^k = 1$ if and only if $k\in p\cdot\mathbb{Z}$ and note that our hypothesis $\gcd(p,q)=1$ implies that $\xi^{k\cdot q}=1$ if and only if $k\in p\cdot\mathbb{Z}$. Using these two observations one can easily show that all non-trivial translates of U are disjoint.[905]

If $z_2\neq 0$, then we just swap the roles of the first and second coordinate in the definition of U. ⊞

(2) Since S^3 is a closed oriented 3-dimensional smooth manifold, since the action is smooth and orientation-preserving and since any action of a finite group is automatically proper we obtain from the Manifold–Quotient Proposition 25.39 and the Orientation-Action Proposition 31.44 that $L(p,q)$ is a closed 3-dimensional topological manifold that has a unique structure of an oriented smooth manifold such that the projection $S^3\to S^3/(\mathbb{Z}/p\mathbb{Z})$ is a local diffeomorphism and orientation-preserving. ■

[904] We set $\widetilde{U} = \{(u_1,u_2)\in\mathbb{C}^2 \,|\, u_1 = z_1\cdot r\cdot\exp(2\pi\,\mathrm{i}t) \text{ with } r\in(0,\infty), t\in(-\tfrac{1}{2p},\tfrac{1}{2p})\}$. Note that $\widetilde{U}$ is an open neighborhood of $(z_1,z_2)\in\mathbb{C}^2$. Thus $U=\widetilde{U}\cap S^3$ is an open neighborhood of $(z_1,z_2)\in S^3$.

[905] Alternatively one can use the Free-Proper Action Lemma 25.44 to show that the action is discrete.

Definition 91.4. Let p $in\mathbb{N}$ and let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$.[906] We refer to

$$L(p,q) \coloneqq S^3/(\mathbb{Z}/p\mathbb{Z})$$

(↑ action from the Lens Space Discrete Action Lemma 91.3)

as a lens space.[907] We will always view $L(p,q)$ as an oriented 3-dimensional smooth manifold.

Example. We consider two special cases of lens spaces.

(1) For $p = 1$ we only have the action of the trivial group on S^3 and the quotient space $L(1,q)$ is therefore naturally homeomorphic to the original sphere S^3.
(2) For $p = 2$ we have q is odd, so for $k = 0$ we have $\exp(2\pi\,\mathrm{i}k/2) = \exp(2\pi\,\mathrm{i}kq/2) = 1$ and for $k = 1$ we have $\exp(2\pi\,\mathrm{i}k/2) = \exp(2\pi\,\mathrm{i}kq/2) = -1$. Put differently, the action of $\mathbb{Z}/2\mathbb{Z}$ on S^3 is the same as the action of $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ that we had used in the definition of the real projective space $S^2/\sim =_{\mathrm{i}} \mathbb{R}\mathrm{P}^3$. This shows that $L(2,q) = S^3/\sim =_{\mathrm{i}} \mathbb{R}\mathrm{P}^3$. □

For $p \in \mathbb{N}$ and $q \in \mathbb{N}$ with $\gcd(p,q) = 1$ we just introduced a new topological space, namely the lens space $L(p,q)$. It is natural to ask: For which parameters $(p,q), (\widetilde{p},\widetilde{q})$ are these topological spaces homeomorphic? We will now provide a partial answer:

Lemma 91.5. (Lens Space-Diffeomorphism Lemma) Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$.
(1) If $q \equiv r^{\pm 1} \bmod p$,[908] there is an orientation-preserving diffeomorphism $L(p,q) \to L(p,r)$.
(2) If $q \equiv -r^{\pm 1} \bmod p$, there is an orientation-reversing diffeomorphism $L(p,q) \to L(p,r)$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We pick $s \in \mathbb{Z}$ with $r \cdot s \equiv 1 \bmod p$. We suppose that $q \equiv \pm r \bmod p$ or that $q \equiv \pm s \bmod p$.

Claim. The map

$$\begin{aligned} \Phi\colon S^3 &\to S^3 \\ (z_1, z_2) &\mapsto \begin{cases} (z_1, z_2), & \text{if } q \equiv r \bmod p, \\ (z_1, \overline{z_2}), & \text{if } q \equiv -r \bmod p, \\ (z_2, z_1), & \text{if } q \equiv s \bmod p, \\ (z_2, \overline{z_1}), & \text{if } q \equiv -s \bmod p \end{cases} \end{aligned}$$

descends to a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ and this map is a bijection.

Proof. We set $\xi \coloneqq \exp(2\pi\,\mathrm{i}/p)$. If $q \equiv \pm r \bmod p$, then one can easily verify that Φ descends to a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$. Now let us consider the case that

[906] By definition we have $L(p, q + k \cdot p) = L(p,q)$. It follows that any lens space is of the form $L(p,q)$ with $q \in \{0, \ldots, p-1\}$.
[907] The name "lens space" comes from an alternative description of these smooth manifolds.
[908] By our hypothesis we have $\gcd(p,r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s = 1 \bmod p$. We write $r^{-1} \coloneqq s$.

$q \equiv s \bmod p$. Given $(z_1, z_2), (\widetilde{z}_1, \widetilde{z}_2) \in S^3$ we see that :

$$\begin{aligned}
[(z_1, z_2)] = [(\widetilde{z}_1, \widetilde{z}_2)] \in L(p,q) &\implies \text{there exists } k \in \mathbb{Z} \text{ with } (\xi^k \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z}_1, \widetilde{z}_2) \\
&\overset{r\cdot s \equiv 1 \bmod p}{\implies} \text{we have } (\xi^{k\cdot r\cdot s} \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z}_1, \widetilde{z}_2) \\
&\overset{s \equiv q \bmod p}{\implies} \text{we have } (\xi^{k\cdot r\cdot q} \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z}_1, \widetilde{z}_2) \\
&\implies \text{with } l = k \cdot q \text{ we have } (\xi^{r\cdot l} \cdot z_1, \xi^{l} \cdot z_2) = (\widetilde{z}_1, \widetilde{z}_2) \\
&\implies \text{with } l = k \cdot q \text{ we have } (\xi^{l} \cdot z_2, \xi^{r\cdot l} \cdot z_1) = (\widetilde{z}_2, \widetilde{z}_1) \\
&\implies [(z_2, z_1)] = [(\widetilde{z}_2, \widetilde{z}_1)] \in L(p,r) \\
&\implies [\Phi(z_1, z_2)] = [\Phi(\widetilde{z}_1, \widetilde{z}_2)] \in L(p,r).
\end{aligned}$$

This argument shows that we get a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$. Evidently the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is surjective. Furthermore, running the above argument backwards one see that the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is injective. In summary we see that the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is a bijection. Finally we note that the case $r \equiv -s \bmod p$ is dealt with in a similar fashion. ⊞

Next we make the following three observations:

(1) The map $\Phi\colon S^3 \to S^3$ is a diffeomorphism.
(2) Using the Boundary Orientation Lemma 31.38 one can easily show that the diffeomorphism $\Phi\colon S^3 \to S^3$ is orientation-preserving if $q \equiv r^{\pm 1} \bmod p$ and that it is orientation-reversing if $q \equiv -r^{\pm 1} \bmod p$.
(3) The projections $S^3 \to L(p,q)$ and $S^3 \to L(p,r)$ are by definition local diffeomorphisms and orientation-preserving.

It follows from (1) and (3) that the map $L(p,q) \to L(p,r)$ is a local diffeomorphism. Since it is a bijection it follows from the Smooth Pasting Proposition 25.19 that the map is a diffeomorphism. The statement regarding the orientations follows from (2) and (3). ■

In the proof of the Lens Space-Diffeomorphism Lemma 91.5 it was quite straightforward to write down the homeomorphism. This raises the question whether there are homeomorphisms between more lens spaces that are harder to find, or whether all other pairs of lens spaces are non-homeomorphic. More precisely, we have the following question.

Question 91.6. Let $p, \widetilde{p} \in \mathbb{N}$ and let $q, \widetilde{q} \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(\widetilde{p}, \widetilde{q}) = 1$. If $L(p,q)$ and $L(\widetilde{p}, \widetilde{q})$ are homeomorphic, does it follow that $p = \widetilde{p}$ and $q \equiv \pm \widetilde{q}^{\pm 1} \bmod p$?

This question will keep us busy for a very long time. We will eventually provide the answer in the Lens Space-Homeomorphism Classification Theorem **??**. Furthermore, in the Lens Space-Homotopy Classification Theorem **??** we will give a classification of the lens spaces up to homotopy equivalence.

We had just seen on page 1776 that the group $\{\pm 1\}$ acts freely and continuously on any sphere. We had also seen that any finite cyclic group $\mathbb{Z}/p\mathbb{Z}$ acts freely and continuously on the 3-sphere. This raises the question, whether such an action by $\mathbb{Z}/p\mathbb{Z}$ also exists on the 2-sphere. More generally we can ask the following question:

Question 91.7. Given $n \in \mathbb{N}$, which finite cyclic groups can act freely and continuously on the n-dimensional sphere S^n ?[909]

We will give a complete answer to this question once we have introduced homology groups. We refer to the Free Action-on-Sphere Proposition 46.17 for details.

91.2. **Coverings of topological spaces.** After this short excursion into group actions on topological spaces we turn to the main topic of this chapter, namely coverings:

Definition 91.8. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subseteq B$ is called uniformly covered, if $p^{-1}(U) \subseteq X$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of X with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.[910]

(2) We say that the map $p\colon X \to B$ is a[911] covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of $b \in B$ which is uniformly covered.

(3) Sometimes we call X a covering space of B.

Remark.

(1) In the literature there can be subtle differences in the definition of a covering. Our definition agrees with the definition in [**Bre1993**, p. 139] and [**Jä2005**, Chapter 9]. But in [**Hat2002**, p. 56] and [**Mun1975**, p. 336] the covering $p\colon X \to B$ does not need to be surjective. As usual, whichever convention one picks, one is going to regret it at some point.

(2) If U is a uniformly covered subset of B, then clearly any open subset of U is also uniformly covered. For example it follows that if $p\colon X \to B$ is a covering and if B is locally (path-) connected, then any point $b \in B$ admits also a (path)- connected uniformly covered neighborhood. □

Example.

(1) The map

$$\begin{array}{rccc} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$

is a covering. Indeed, let $P = \exp(\mathrm{i}\alpha)$ be a point in S^1. We pick the open neighborhood

$$U \;:=\; \left\{ \exp(\mathrm{i}\varphi) \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right)\right\}.$$

Then

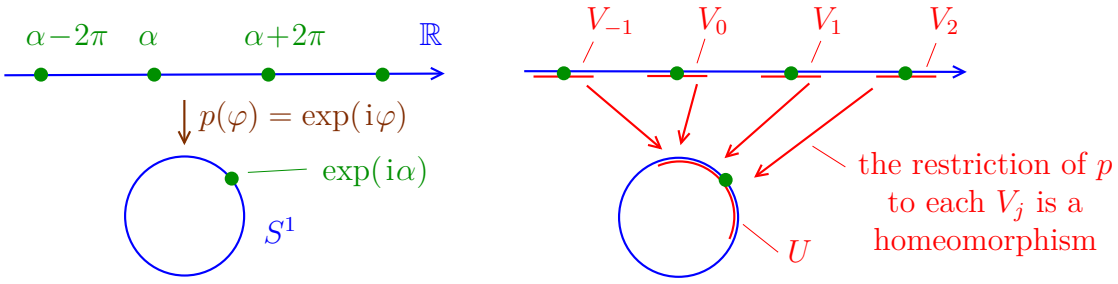
$$p^{-1}(U) \;=\; \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism.

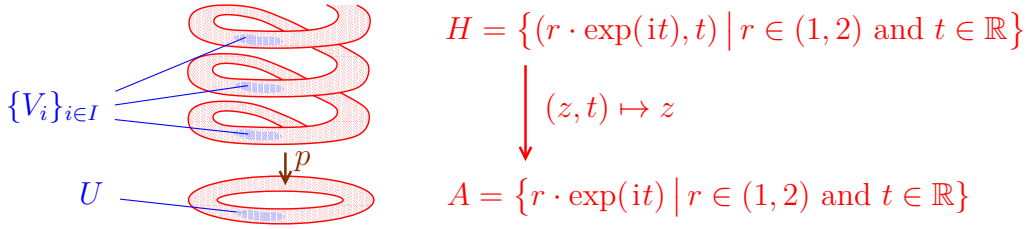
[909] Any cyclic group $\mathbb{Z}/k\mathbb{Z}$, $k \neq 0$ admits a non-trivial action on S^n where $j \in \mathbb{Z}/k\mathbb{Z}$ acts by rotation around the x-axis by the angle $2\pi\frac{j}{k}$. But this action is not free, since the points $(-1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ are fixed.

[910] Let $b \in B$. It follows immediately from the definition that each $x \in p^{-1}(b)$ is contained in precisely one V_i. This implies that we can arrange that the index set is $p^{-1}(b)$. We will often make use of this observation.

[911] In Definition 3.19 we introduced the notion of a "cover". As we mentioned in Definition 3.19, such a "cover" is sometimes also called a "covering". Now we introduce the notion of a "covering". Rather maddeningly, in the literature this notion often also gets called a "cover". Fortunately, from the context, it is usually clear what is meant by "cover" and "covering". These issues do not arise in other languages, e.g. in German and French what we call "cover" is referred to as "Überdeckung" and "recouvrement" and what we call "covering" is referred to as "Überlagerung" and "revêtement".



(2) In the figure below we illustrate another covering:



This "infinite helix" which gives a good pictorial way to think about coverings. □

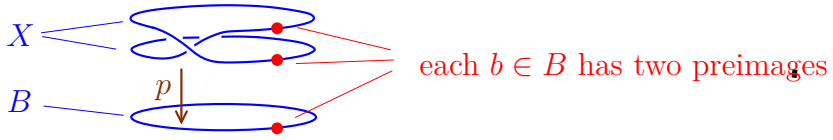
Lemma 91.9. (Covering–Preimage Lemma) Let $p\colon X \to B$ be a covering of topological spaces. If B is a connected topological space, then for any two points P and Q in B we have[912]

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Proof. The proof of the lemma is the content of Exercise 91.3. ■

Example.

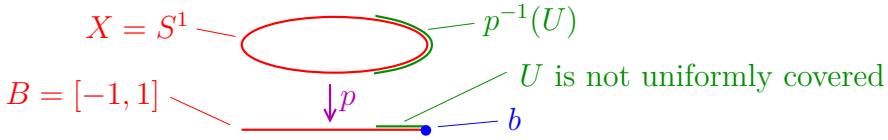
(1) We consider the map $p\colon X \to B$ that is shown in the figure.[913] This map is a covering and every $b \in B$ has precisely two preimages.



(2) The map

$$\begin{array}{rcl} p\colon S^1 & \to & [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

is *not* a covering since we see that the conclusion of the Covering–Preimage Lemma 91.9 does not hold.



In fact one can show that $b = 1 \in [-1,1]$ does not admit a neighborhood that is uniformly covered. □

[912] Recall that given a set S we denote by $\#S$ the cardinality of S, as defined on Definition 1.36. In particular $\#p^{-1}(P) = \#p^{-1}(Q)$ means that there exists a bijection between the sets $p^{-1}(P)$ and $p^{-1}(Q)$.

[913] A mathematically precise description would be to consider the Möbius band $([0,1] \times [0,1])/\sim$ and to consider the map from the boundary to the central circle that is given by $[(x,y)] \mapsto [(x, \frac{1}{2})]$.

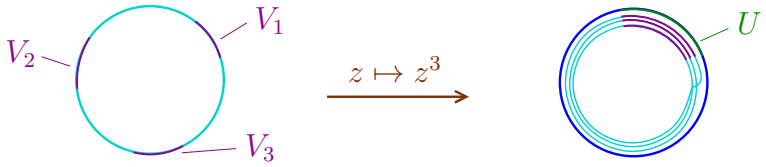
Definition 91.10. Let $p\colon X \to B$ be a covering of a connected non-empty topological space.

(1) We define the degree $[X : B]$ of p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.[914]
(2) If the degree is a finite number n, then we say that p is an n-fold covering. Otherwise we say that p is an infinite covering. If the cardinality is countable, then sometimes we call it a countable covering.
(3) If $p\colon X \to B$ is a covering over a possibly disconnected topological space, then we say that p is a finite covering if for each component C of B the covering $p\colon p^{-1}(C) \to C$ is a finite covering in the sense of (2).

For completeness sake we extend this definition to the case that B, and thus also X, are the empty space. In this case we define the degree to be 1.

Examples.

(1) It follows immediately from the Bijective-Local Homeomorphism Lemma 4.62 that a covering $p\colon X \to B$ has degree one if and only if p is a homeomorphism.
(2) As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Given any $n \in \mathbb{N}$ the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is an n-fold covering.[915]



(3) Let $n \in \mathbb{N}$. Note that basically the same argument as before shows that the map $p\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $p(z) = z^n$ is an n-fold covering.
(4) We consider the map

$$\begin{array}{rcl} p\colon \overbrace{(\mathbb{R} \times [-1,1])/(x,y) \sim (x+1,y)}^{\text{annulus}} & \to & \overbrace{(\mathbb{R} \times [-1,1])/(x,y) \sim (x+\frac{1}{2},-y)}^{\text{Möbius band}} \\ [(x,y)] & \mapsto & [(x,y)]. \end{array}$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band.
(5) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1 \sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a 2-fold covering from the torus to the Klein bottle.
(6) In the figure below we sketch three coverings of topological graphs:

---

[914] By the Covering–Preimage Lemma 91.9 this definition does not depend on the choice of b.

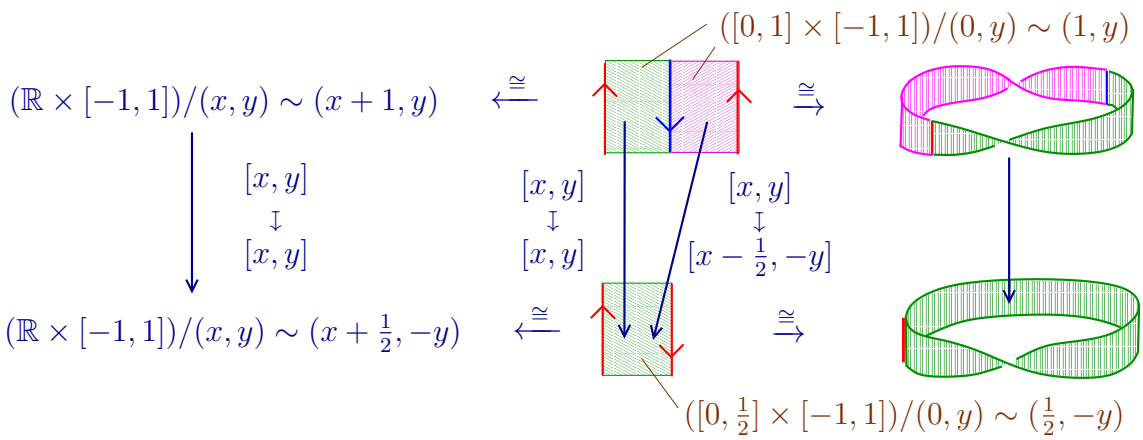
[915] This can be seen as follows: Let $z = \exp(2\pi \mathrm{i}\varphi)$ be a point in S^1. We set

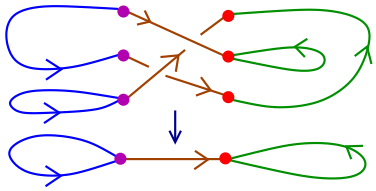
$$U := \big\{\exp(2\pi \mathrm{i}\psi) \,\big|\, \psi \in \big(\varphi - \tfrac{1}{4}, \varphi + \tfrac{1}{4}\big)\big\}.$$

Note that

$$p^{-1}(U) = \bigsqcup_{j=1}^{n} \underbrace{\big\{\exp(2\pi \mathrm{i}\psi) \,\big|\, \psi \in \big(\tfrac{1}{n}(\varphi - \tfrac{1}{4}) + \tfrac{j}{n}, \tfrac{1}{n}(\varphi + \tfrac{1}{4}) + \tfrac{j}{n}\big)\big\}}_{=:V_j}.$$

One can easily verify that the restriction of p to each V_j is a homeomorphism.





The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.[916] □

The following proposition connects the notion of discrete actions to the theory of coverings.

Proposition 91.11. (Action–Covering Proposition) Let X be a topological space together with a discrete and continuous action by a group G.

(1) The natural projection $p\colon X\to X/G$ is a covering where the degree is given by $\#G$.
(2) Let H be a subgroup of G. The natural projection

$$\begin{array}{rcl} p\colon X/H & \to & X/G \\ [x] & \mapsto & [x] \end{array}$$

is a covering of degree $[G:H]$[917].

Example. The Action–Covering Proposition 91.11 (1), applied to the two examples on page 1776, shows that

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x]=x+\mathbb{Z}^n \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n=_{\mathrm{i}} S^n/x\sim -x \\ x & \mapsto & [x] \end{array}$$

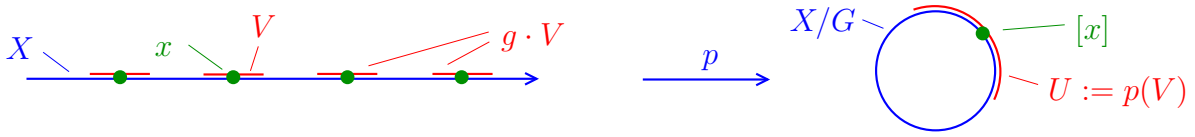
is an infinite respectively 2-fold covering. □

Proof of Proposition 91.11. In the following we prove Statement (1). We leave it to the reader to generalize the proof to the more general setting of Statement (2).

Let G be a group that acts discretely and continuously on a topological space X. We want to show that the projection $p\colon X\to X/G$ is a covering. Evidently the map is

[916] It is a good exercise to convince oneself, that the indicated maps are in fact coverings.
[917] Here $[G:H]$ denotes the index of the subgroup $H\subseteq G$, i.e. the cardinality of G/H. We define this notion in detail in Definition 92.9.

surjective. Next let $[x] \in X/G$. Since the group acts discretely there exists, by definition, an open neighborhood V of $x \in X$ such that $V \cap g{\cdot}V = \varnothing$ for all $g \neq e$. We put $U := p(V)$.



Claim. U is a uniformly covered open neighborhood of $[x]$.

Proof. It follows from the Group Action-Projection Lemma 9.8 (1) that $p\colon X \to X/G$ is an open map. Thus it follows that the set U is an open neighborhood of $[x] = p(x)$.

Next note that the preimage $p^{-1}(U)$ is the disjoint union of the subsets $g \cdot V, g \in G$. Since G acts continuously we know that each $g \cdot V$ is again open. The restriction of the map p to any $g \cdot V$ is a continuous and bijective map $g \cdot V \to U$. Since $p\colon X \to X/G$ is open and since $g \cdot V \subseteq X$ is open we see that the restriction of p to $g \cdot V$ is also open. It follows from the Open-Closed–Injective Lemma 4.22 that the map $p|_{g\cdot V}\colon g \cdot V \to U$ is in fact a homeomorphism. Thus we have shown that U is uniformly covered. ⊞

Since this holds for any point in X/G we have shown that the projection $p\colon X \to X/G$ is a covering. Finally note that $[X : X/G] = p^{-1}([x]) = \#G$. ■

Example.

(1) Let $X = \mathbb{R} \times [-1,1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & \big(x + \frac{1}{2}n, (-1)^n \cdot y\big). \end{array}$$

Let $H = 2\mathbb{Z}$. According to the Action–Covering Proposition 91.11 (2) the projection $p\colon X/H \to X/G$ is a 2-fold covering. Put differently, all the horizontal maps in the following commutative diagram are coverings:

$$\begin{array}{ccc} (\mathbb{R}\times[-1,1])/2\mathbb{Z} & \longrightarrow & (\mathbb{R}\times[-1,1])/\mathbb{Z} \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ (\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}2k, (-1)^{2k}\cdot y) & \longrightarrow & (\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}k, (-1)^k\cdot y) \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ \underbrace{(\mathbb{R}\times[-1,1])/(x,y) \sim (x+k,y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}k, (-1)^k\cdot y)}_{=\text{Möbius band}}. \end{array}$$

Thus we recover the result from page 1783 that the annulus is a 2-fold covering of the Möbius band.

(2) We consider again the group G of self-homeomorphisms of $\mathbb{R}^2$ that is generated by

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$

We let H be the subgroup of G generated by A^2 and B. Note that A^2 is the map that is given by

$$\begin{array}{rcccl} A^2\colon \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) & \mapsto & \underbrace{((x+1)+1, (1-(1-y))}_{=(x+2,y)}. \end{array}$$

Note that A^2 and B are two homeomorphisms that commute, so any element in H is in fact of the form

$$\begin{array}{rcl} A^{2k} \circ B^l \colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x + 2k, y + l) \end{array}$$

for some $k, l \in \mathbb{Z}$. By the Action–Covering Proposition 91.11 (2) the projection $p \colon \mathbb{R}^2/H \to \mathbb{R}^2/G$ is a 2-fold covering. We already know that $\mathbb{R}^2/G$ is the Klein bottle. It follows immediately from the above description of H, $\mathbb{R}^2/H = \mathbb{R}^2/(2\mathbb{Z} \oplus \mathbb{Z})$, so $\mathbb{R}^2/H$ is just the 2-dimensional torus. As on page 1783 we see that we can view the 2-dimensional torus as a 2-fold covering of the Klein bottle. □

91.3. **Basic properties of coverings.** The following lemma summarizes some basic facts about coverings.

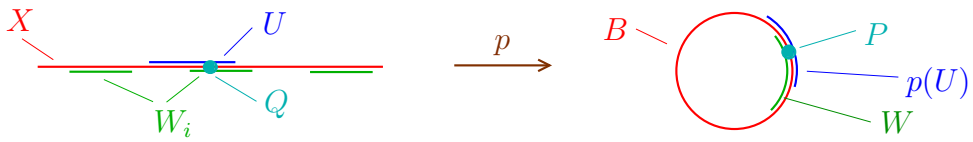
Lemma 91.12. (Covering–Basics Lemma) Given any covering $p \colon X \to B$ of topological spaces the following statements hold:

(1) The map $p \colon X \to B$ is open.[918]
(2) For every $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.
(3) If B is Hausdorff, then so is X.[919]
(4) The following two statements are equivalent:
  (a) X is compact.
  (b) p is a finite covering and B is compact.
(5) If B is connected and locally connected, then for any component X_0 of X the restriction of $p \colon X_0 \to B$ is also a covering.

Proof. Let $p \colon X \to B$ be a covering of topological spaces.

(1) Let U be an open subset of X. We need to show that $p(U)$ is open. By the JH-Openness Criterion 3.3 it suffices to show that for each $P \in p(U)$ there exists an open neighborhood V such that $P \in V \subseteq p(U)$. Since p is a covering there exists an open neighborhood W of P such that $p^{-1}(W)$ is the union of disjoint open subsets $\{W_i\}_{i \in I}$ with the property that each $p|_{W_i} \colon W_i \to W$ is a homeomorphism.

Since $P \in p(U)$ there exists a $Q \in U$ with $p(Q) = P$. There exists an $i \in I$ with $Q \in W_i$. Since W_i is open and since U is open the set $U \cap W_i$ is also open in X. Since $p \colon W_i \to W$ is a homeomorphism and since $U \cap W_i$ is open in W_i the set $p(U \cap W_i)$ is also open in $p(W_i) = W$. But since W is open it follows from the Open-Closed Inclusion Lemma 4.13 that $V := p(U \cap W_i)$ is also an open subset of B. It is the desired open neighborhood of P with $P \in V \subseteq p(U)$.



(2) Let $b \in B$. Since $p \colon X \to B$ is a covering there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property that each $p|_{V_i} \colon V_i \to U$ is a homeomorphism. For each $i \in I$ there exists a unique point $x_i \in V_i$ with $p(x_i) = b$ and the preimage $p^{-1}(b)$ consists of precisely the x_i, $i \in I$. Since

[918] Recall that this means that the image of each open subset in X is open in B.
[919] In Exercise 91.2 we will see that the converse does not necessarily hold.

each V_i is open in X it follows from the definition of the subspace topology that each $x_i \in p^{-1}(b)$ is an open subset of $p^{-1}(b)$. Put differently, the subspace topology on $p^{-1}(b)$ is the discrete topology. But this is equivalent to saying that $p^{-1}(b)$ is a discrete subset of X.

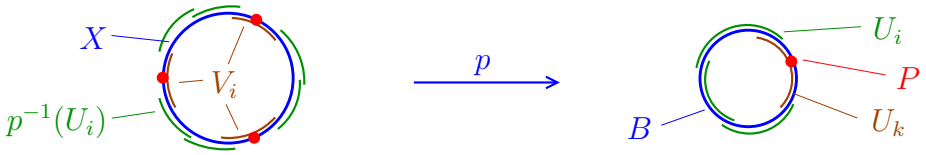
(3) This statement is relatively easy to prove. We will provide the details in Exercise 91.12.

(4) We first prove the(a)⇒(b)-direction. Thus we assume that X is compact. By definition the covering $p\colon X \to B$ is surjective and continuous. Thus it follows from the Compact Image Lemma 4.15 that B is also compact.

It remains to show that the covering has finite degree.[920] By definition we can cover the topological space B by open subsets that are uniformly covered. Since B is compact we can cover B by finitely many open subsets $U_1, \ldots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \cdots \cup U_{k-1}$.[921] Now let P be a point in $U_k \setminus (U_1 \cup \cdots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that each $p|_{V_i}\colon V_i \to U_k$ is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open cover

$$X \;=\; p^{-1}(U_1) \cup \ldots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i\in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open subsets V_i, $i \in I$ from the open cover. Since X is compact, we see that I is finite. But $\#I$ is precisely the degree of the covering. Thus we have shown that the covering has finite degree.



We turn to the proof of the (b)⇒(a)-direction. Let $p\colon X \to B$ be an n-fold covering of a compact space B. We need to show that X is compact. Thus let $\{U_i\}_{i\in I}$ be an open cover of X.

We start out with some preparations: Let $b \in B$. We pick an open neighborhood M of b that is uniformly covered. Recall that by definition this means that $p^{-1}(M)$ is the disjoint union of open subsets $\{N_x\}_{x\in p^{-1}(b)}$ with the following two properties:

- For each $x \in p^{-1}(b)$ we have $x \in N_x$.
- The restriction of p to each N_x is a homeomorphism.

For each $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We set

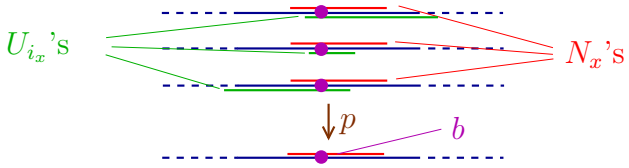
$$W_b \;:=\; \bigcap_{x\in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

[920] If B is Hausdorff the argument is quite simple: Let $b \in B$. By the Points-in-Hausdorff Lemma 3.32 we know that $\{b\} \subseteq B$ is a closed subset. Since p is continuous we see that $p^{-1}(b)$ is a closed subset of X. By (2) we know that $p^{-1}(b)$ is a discrete subset. Since X is compact it now follows from the Compact-Discrete Lemma 3.40 that $[X : B] := p^{-1}(b)$ is finite.

[921] Indeed, if U_k was contained in $U_1 \cup \cdots \cup U_{k-1}$, then we could remove U_k from the collection of open subsets and we continue with the collection $U_1, \ldots, U_{k-1}$.

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open cover $\{U_i\}_{i\in I}$.

Proof. Let $x \in p^{-1}(b)$. We denote by $p_x := p|_{N_x} \colon N_x \to M$ the homeomorphism that is given by restricting p to N_x. We write $F := p^{-1}(b)$.



We see that:

$$p^{-1}(W_b) \underset{\text{since } p^{-1}(W_b) \subseteq \bigcup_{y\in F} N_y}{=} \bigcup_{y\in F} p_y^{-1}\Big(\bigcap_{x\in F} p_x(U_{i_x}\cap N_x)\Big) \underset{\text{by the Image-Preimage Lemma 1.9 (7)}}{=} \bigcup_{y\in F} \overbrace{\bigcap_{x\in F} \underbrace{p_y^{-1}(p_x(U_{i_x}))}_{=U_{iy} \text{ if } y=x} \cap \underbrace{p_y^{-1}(p_x(N_x))}_{=N_y}}^{\subseteq U_{i_y}\cap N_y} \subseteq \bigcup_{y\in F} U_{i_y}.$$

⊞

By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is the intersection of finitely many open subsets of B, thus W_b is an open subset of B. Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by finitely many of these open subsets, i.e. there exist $b_1, \dots, b_k \in B$ with $B = W_{b_1} \cup \dots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \dots, p^{-1}(W_{b_k})$ cover X. Note that by the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open cover $\{U_i\}_{i\in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i\in I}$.

(5) We now assume that B is connected and locally connected. Let X_0 be a component of X_0. Since B is locally connected and since $p\colon X \to B$ is a local homeomorphism we see that X is also locally connected. It now follows from the Locally Path-Connected Lemma 4.57 (3) that X_0 and $X \setminus X_0$ are open subsets of X.

- First we show that $p|_{X_0} \colon X_0 \to X$ is surjective. By (1) and the above discussion we know that $p(X_0)$ and $p(X \setminus X_0)$ are open subsets of B. But since B is connected and X_0 is non-empty we now see that $p(X_0) = B$.
- Now let $b_0 \in B$. Since B is locally connected it follows, see also the remark on page 1781, that b_0 admits an open connected neighborhood U that is uniformly covered. Since X_0 and $X \setminus X_0$ are open subsets of X one can now easily verify that U is also uniformly covered with respect to the restriction of p to X_0. ■

91.4. Exercises for Chapter 91.

Exercise 91.1. Let $m \in \mathbb{N}$ and let $a_1, \dots, a_m \in \mathbb{Z} \setminus \{0\}$. We consider the continuous action

$$\begin{array}{ccc} \mathbb{Z}^m \times \mathbb{R}^m & \to & \mathbb{R}^m \\ ((k_1, \dots, k_m), (x_1, \dots, x_m)) & \mapsto & (a_1^{k_1} \cdot x_1, \dots, a_m^{k_m} \cdot x_m) \end{array}$$

of the group $(\mathbb{Z}^m, +)$ on the topological space $\mathbb{R}^m$.

(a) For which parameters $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \mathbb{Z} \setminus \{0\}$ is the above action discrete in the sense of Definition 91.1?

(b) For which parameters $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \mathbb{Z} \setminus \{0\}$ is the above action proper in the sense of Definition 9.4?

Remark. In particular at the end you should have found parameters such that the action is discrete but not proper.

Exercise 91.2. We consider $X = \mathbb{R}^2 \setminus \{(0,0)\}$ together with the action of $G = \mathbb{Z}$ that is given by $n \cdot (x, y) = (2^n \cdot x, 2^{-n} \cdot y)$.

(a) Show that this action is discrete in the sense of Definition 91.1.

(b) Show that this is action is not proper in the sense of Definition 9.4.

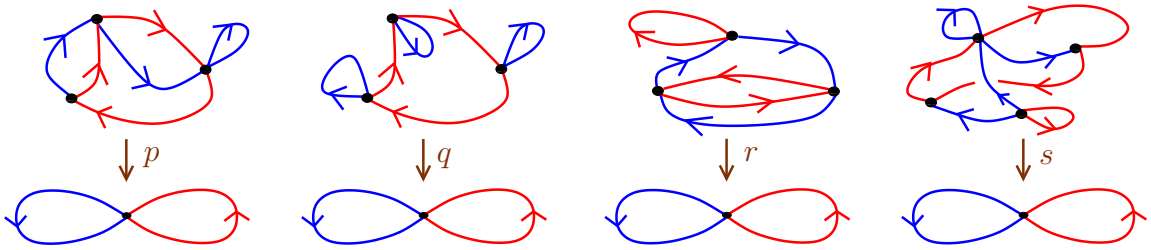
(c) Show that the quotient space $X/\mathbb{Z}$ is *not* Hausdorff.

Remark. It follows from the Action–Covering Proposition 91.11 that the natural projection $p\colon X \to X/\mathbb{Z}$ is a covering. We have now found an example where the covering space is Hausdorff, but the base space $X/\mathbb{Z}$ is not Hausdorff. This shows that the converse to the Covering–Basics Lemma 91.12 (3) does not hold.

Exercise 91.3. Let $p\colon X \to B$ be a covering of topological spaces. We suppose that B is a connected topological space. Show that for any two points P and Q in B we have

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Exercise 91.4. Which of the maps between topological graphs that are shown in the figure below are covering maps?



Exercise 91.5.

(a) Show that the map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $z \mapsto z^2$ is a 2-fold covering.

(b) Show that

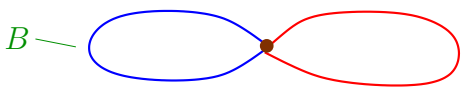
$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$

is a covering.

Exercise 91.6. Let $p\colon X \to B$ be a covering of topological spaces. In the Covering–Basics Lemma 91.12 (1) we showed that p is an open map. Is p also necessarily a closed map?

Exercise 91.7. Let M be the Möbius band. Does there exist a covering map $p\colon M \to M$ of degree 3?

Exercise 91.8. We consider the topological space B that is shown in the figure below. Show that for each $n \in \mathbb{N}$ there exists an n-fold covering $p\colon X \to B$ such that X is connected.



Exercise 91.9. Let $p\colon X \to B$ be a continuous map between topological spaces that is two-to-one, i.e. such that for every $b \in B$ the preimage $p^{-1}(b)$ consists of precisely two points. Is p necessarily a covering?

Exercise 91.10.

(a) We consider the topological space A in the figure below. Find a covering $p\colon X \to A$ such that X is simply connected.

(b) We consider the topological space B in the figure below. Find a covering $p\colon Y \to B$ such that Y is simply connected.

Remark. Perhaps we lack the tools to prove that X and Y are simply connected, but that should not prevent the reader from giving it a go.



Exercise 91.11.

(a) Show that there exists a continuous map $p\colon X \to B$ between topological spaces that is a local homeomorphism and that is surjective but which is not a covering map.

(b) Let $p\colon X \to B$ be a continuous map between topological spaces that is a local homeomorphism and that is surjective. Furthermore we suppose that p is proper, i.e. we assume that the preimage of each compact set is compact. Is p necessarily a covering map?

Exercise 91.12. Let $p\colon X \to B$ be a covering of topological spaces.

(a) Show that if B is Hausdorff, then so is X.

(b) Show that if p is a finite covering and if X is Hausdorff, then so is B.

(c) Does (b) also hold for infinite coverings?

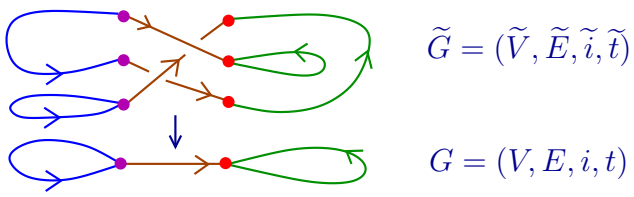
Exercise 91.13. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. As in Definition 21.19 we consider the mapping torus $\mathrm{Tor}(X,f) = (X\times[0,1])/(x,0)\sim(f(x),1)$. Show that the map

$$\begin{array}{rccc} q\colon & X\times\mathbb{R} & \to & \mathrm{Tor}(X,f) \\ & (x,t) & \mapsto & [(x,t)] \end{array}$$

is a covering.

Exercise 91.14. Given an abstract graph $G = (V,E,i,t)$ and given any vertex $v \in V$ we set $I(v) := \{e \in E \,|\, i(e) = v\}$ and $T(v) := \{e \in E \,|\, t(e) = v\}$. Now let $\widetilde{G} = (\widetilde{V},\widetilde{E},\widetilde{i},\widetilde{t})$ and $G = (V,E,i,t)$ be two abstract graphs and let $\widetilde{G} \to G$ be a morphism in the sense of Definition 11.3. This means that we have maps $\alpha\colon \widetilde{V} \to V$ and $\beta\colon \widetilde{E} \to E$ such that $i\circ\beta = \alpha\circ\widetilde{i}\colon \widetilde{E} \to V$ and $t\circ\beta = \alpha\circ\widetilde{t}\colon \widetilde{E} \to V$.

We assume that for each $\widetilde{v} \in \widetilde{V}$ the maps $\alpha\colon I(\widetilde{v}) \to I(v)$ and $\alpha\colon I(\widetilde{v}) \to I(v)$ are bijections. Show that corresponding map $|\widetilde{G}| \to |G|$ between the topological realizations is a covering.



Exercise 91.15. Let X, Y and Z be topological spaces and let $p\colon X \to Y$ and $q\colon Y \to Z$ be continuous maps.

(a) Which of the following three statements are true?
  (i) If p and q are coverings of finite degree, then $q \circ p$ is also a covering.
  (ii) If $q \circ p$ is a covering of finite degree, then p is also a covering.
  (iii) If $q \circ p$ is a covering of finite degree, then q is also a covering.

(b) If p, q and $q \circ p$ are coverings of finite degree, what is the relationship between the degrees of p, q and $q \circ p$?

Exercise 91.16. Let $p\colon X \to B$ be a surjective continuous map between topological spaces. Let $f\colon [0,1] \to B$ be a map. Does there always exist a lift from f to X, i.e. does there exist a continuous map $g\colon [0,1] \to X$ such that $p \circ g = f$?

Exercise 91.17. We consider the 5-fold covering map

$$\begin{array}{rccc} p\colon X = S^1 & \to & B = S^1 \\ z & \mapsto & z^5. \end{array}$$

Let $f\colon [0,1] \to B = S^1$ be the loop that is given by $t \mapsto \exp(2\pi \mathrm{i} k t)$. By the Path-Lifting Proposition 92.6 this loop lifts to a path $g\colon [0,1] \to X = S^1$. For which k is the lift g again a loop?

Exercise 91.18. Let $p\colon \widetilde{X} \to X$ be a covering of topological spaces.

(a) Show that for $A \subseteq X$ we have $\quad$ closure of $p^{-1}(A) \;=\; p^{-1}(\text{closure of } A)$

(b) Show that for $A \subseteq X$ we have $\quad$ interior of $p^{-1}(A) \;=\; p^{-1}(\text{interior of } A)$.

Exercise 91.19. Show that there exists a 2-fold covering $p\colon G \to F$ from the surface G of genus 3 to the surface F of genus 2.

Hint. Consider the closed curve $C \subseteq F$ shown in the figure below. The preimage of C under p could consist of two closed curves D_1, D_2 such that $G \setminus (D_1 \cup D_2)$ consists of two components which both are homeomorphic to $F \setminus C$.



Exercise 91.20. Let $n \in \mathbb{N}$ and let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix with non-zero determinant. Show that the map

$$\begin{array}{rccc} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [A \cdot v] \end{array}$$

is a covering map of degree $|\det(A)|$.

Remark. This can be viewed as a generalization of Exercise 25.13.

Exercise 91.21. Show that there is no covering map from the sphere S^2 to the torus $S^1 \times S^1$.
Hint. Use the Covering-Degree Proposition 92.10.

Exercise 91.22. Let (X, d) be a metric space and let $p\colon \widetilde{X} \to X$ be a covering. Given $x, y \in \widetilde{X}$ we define

$$\widetilde{d}(x,y) := \inf \left\{ \sum_{i=0}^{n-1} d(w_i, w_{i+1}) \;\middle|\; \begin{array}{c} \text{with } w_0, \ldots, w_n \in X,\ p(w_0) = x,\ p(w_n) = y \text{ and such} \\ \text{that for each } i \in \{0, \ldots, n-1\} \text{ there exists a uniformly} \\ \text{covered open subset that contains } w_i \text{ and } w_{i+1} \end{array} \right\}.$$

(a) Show that $\widetilde{d}$ defines a metric on $\widetilde{X}$ and show that the corresponding topology equals the topology of $\widetilde{X}$.
(b) Show that $p\colon \widetilde{X} \to X$ is a local isometry.

Remark Recall that a topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. The exercise shows in particular that every covering of a metrizable topological space is again metrizable.

CHAPTER 92

Coverings and fundamental groups

In the previous Chapter 91 we introduced the notion of a covering between topological spaces. In this chapter we will prove several results that relate coverings to fundamental groups. In the following Chapter 93 we will use the results of this chapter to calculate in particular the fundamental groups of the real projective spaces and the Klein bottle.

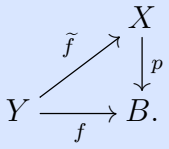
This chapter is organized as follows:

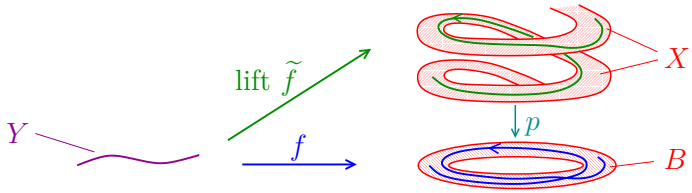
- In Section 92.1 we will discuss (uniqueness of) lifts of maps to coverings.
- In Section 92.2 we will see that paths can always be lifted and we will explore the relationship between coverings and fundamental groups. For example we will prove the coverings always induce a monomorphism of fundamental groups.
- In Section 92.3 we will discuss pullbacks of coverings.
- In Section 92.4 we will provide exercises for Chapter 92.

In the next Chapter 93 we will discuss the interaction between group actions and fundamental groups.

92.1. **Lifts of maps.** The following definition generalizes the notion of a lift that we introduced in Definition 88.18.

Definition 92.1. Let $p\colon X \to B$ be a covering and let $f\colon Y \to B$ be a continuous map between topological spaces. A lift of f to X is a continuous map $\widetilde{f}\colon Y \to X$ such that $p \circ \widetilde{f} = f$, i.e. such that the following diagram of maps commutes:





Similar to the discussion in Section 88.4 we want to prove the existence and uniqueness of lifts of paths $p\colon [0,1] \to B$ or more generally, of continuous maps $[0,1]^m \to B$. We start out with the following proposition which is a generalization of the Lift Uniqueness Lemma 88.19.

Proposition 92.2. (Lift Uniqueness Proposition) Let $p\colon X \to B$ be a covering and let Y be a connected topological space. Furthermore let $f\colon Y \to B$ be a continuous map

and let $\widetilde{f}, \widehat{f} \colon Y \to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $y_0 \in Y$, then they agree everywhere.

Proof. We set

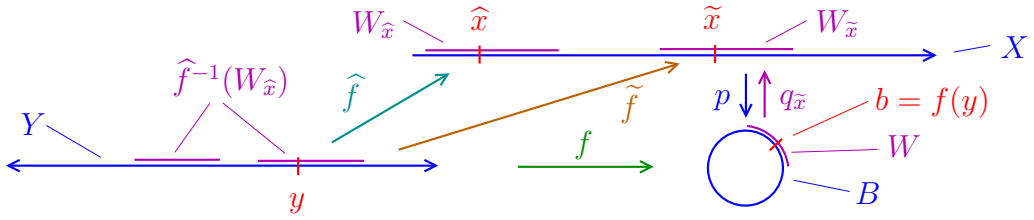
$$\begin{aligned} U &:= \{y \in Y \mid \widetilde{f}(y) = \widehat{f}(y)\} &\subseteq Y \\ V &:= \{y \in Y \mid \widetilde{f}(y) \neq \widehat{f}(y)\} &\subseteq Y. \end{aligned}$$

Note that by hypothesis $U \neq \varnothing$. We want to show that $U = Y$. Since Y is connected it suffices to prove that U and V are open subsets of Y. By the JH-Openness Criterion 3.3 it suffices to prove the following claim:

Claim. Every $y \in Y$ admits an open neighborhood N such that $N \subseteq U$ or $N \subseteq V$.

Proof. Let $y \in Y$. We set $b := f(y)$ and we set $\widetilde{x} := \widetilde{f}(y)$ and $\widehat{x} := \widehat{f}(y)$. Since $p \colon X \to B$ is a covering we know that b admits an open neighborhood W that is uniformly covered, i.e. we know that we can write $p^{-1}(W) = \bigsqcup_{x \in p^{-1}(b)} W_x$ such that for each $x \in p^{-1}(b)$ the following statements hold:

(1) W_x is an open neighborhood of $x \in X$.
(2) The map $p|_{W_x} \colon W_x \to W$ is a homeomorphism. We denote by $q_x \colon W \to W_x$ the inverse.



Note that on $\widetilde{f}^{-1}(W_{\widetilde{x}})$ we have

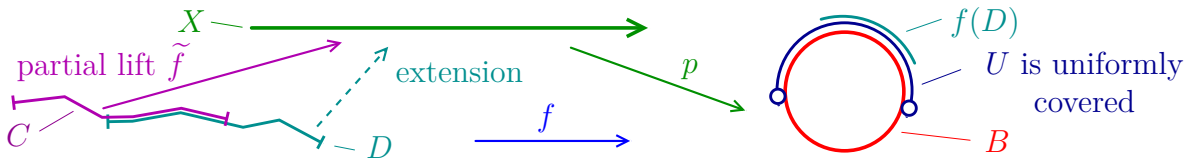
$$\widetilde{f} = \underbrace{q_{\widetilde{x}} \circ p|_{W_{\widetilde{x}}}}_{=\mathrm{id}_{W_{\widetilde{x}}}} \circ \widetilde{f} = q_{\widetilde{x}} \circ \underbrace{p|_{W_{\widetilde{x}}} \circ \widetilde{f}}_{=f} = q_{\widetilde{x}} \circ f \colon \widetilde{f}^{-1}(W_{\widetilde{x}}) \to W_{\widetilde{x}}.$$

Similarly we see that on $\widehat{f}^{-1}(W_{\widehat{x}})$ we have $\widehat{f} = q_{\widehat{x}} \circ f$. The claim follows from the following observations:

- If $y \in U$, i.e. if $\widetilde{x} = \widehat{x}$, then we see that $\widetilde{f} = \widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$, i.e. we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subseteq U$.
- If $y \notin U$, i.e. if $\widetilde{x} \neq \widehat{x}$, then we see that the images of $\widetilde{f}$ and $\widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$ are disjoint, thus we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subseteq V$. ∎

As in Section 88.4 our next goal is to prove the existence of lifts in suitable settings. The key tool for constructing lifts is the following generalization of the Lift-Extension Lemma 88.20.

Lemma 92.3. (Lift-Extension Lemma) Let $p \colon X \to B$ be a covering and let Y be a topological space. We suppose that we have a decomposition $Y = C \cup D$ such that C and D are closed subsets. Let $f \colon Y \to B$ be a continuous map and let $\widetilde{f} \colon C \to X$ be a lift of $f|_C$. We suppose that the following statements hold:
(1) The image $f(D)$ is contained in a uniformly covered open subset of B.
(2) The intersection $C \cap D$ is connected and non-empty.
Then there exists a lift $\widetilde{f} \colon Y \to X$ of f that agrees with the given lift on C.



Proof. We start with a few preparations:

(a) Since $C \cap D \neq \varnothing$ we can pick $x_0 \in C \cap D$.

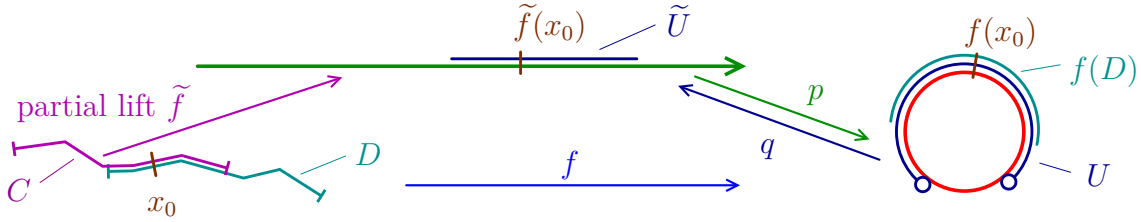
(b) We pick a uniformly covered open subset U of B that contains $f(D)$.

(c) Since U is a uniformly covered open subset it follows from $p(\widetilde{f}(x_0)) = f(x_0) \in f(D)$ that there exists an open neighborhood $\widetilde{U}$ of $\widetilde{f}(x_0)$ such that $p\colon \widetilde{U} \to U$ is a homeomorphism.

(d) We denote by $q\colon U \to \widetilde{U}$ the inverse of $p\colon \widetilde{U} \to U$.

We set

$$\begin{aligned} \widetilde{f}\colon Y &\to X \\ x &\mapsto \begin{cases} \widetilde{f}(x), & \text{if } x \in C, \\ q(\underbrace{f(x)}_{\in U}), & \text{if } x \in D. \end{cases} \end{aligned}$$



Note that on $C \cap D$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{C\cap D}$ and by design they agree on x_0. By hypothesis the intersection $C \cap D$ is connected. Thus we obtain from the Lift Uniqueness Proposition 92.2 that the two maps actually agree on the overlap $C \cap D$. This shows that $\widetilde{f}\colon Y \to X$ is well-defined.

Finally note that since C and D are closed it follows from the Pasting Proposition 4.7 that this map $\widetilde{f}\colon Y \to X$ is continuous. It is now clear that it is a lift of $f\colon Y \to X$. ■

The following lemma is in particular a generalization of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21.

Lemma 92.4. (Lift-Existence Lemma) Let $p\colon X \to B$ be a covering of topological spaces. Given any continuous map $f\colon [0,1]^m \to B$ and given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1]^m \to X$ with $\widetilde{f}(0) = x$.

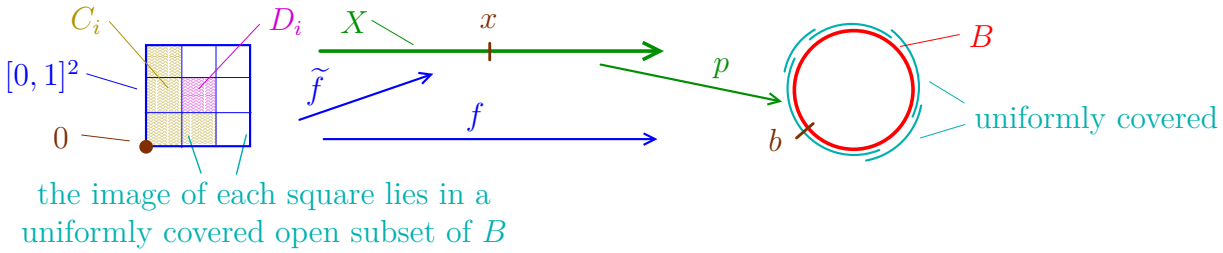
Remark. We will generalize the above proposition in the Map Lifting Criterion 127.5. □

Proof. [922] Let $p\colon X \to B$ be a covering of topological spaces, let $f\colon [0,1]^m \to B$ be a continuous map and let $x \in X$ with $p(x) = f(0)$. Since $p\colon X \to B$ is a covering there exists an open cover of B by uniformly covered open subsets of B. It follows from the Lebesgue Lemma 10.9 that there exists an $N \in \mathbb{N}$ such that for every $(i_1, \ldots, i_m) \in \{0, \ldots, N-1\}^m$ the image $f([\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}])$ is contained in a uniformly covered open subset of B. The idea is now to lift the map $f\colon [0,1]^m \to B$ as follows:

(1) We set $\widetilde{f}(0) \coloneqq x$.

[922] The proof of the proposition is a generalization of the proof of Claim 1 in the proof of the Circle-π_1-Theorem 88.22.

(2) We apply the Lift-Extension Lemma 92.3 iteratively to the N^m cubes of $[0,1]^m$ starting from the "bottom left".



We now carry out the technical details. We introduce the following notation:

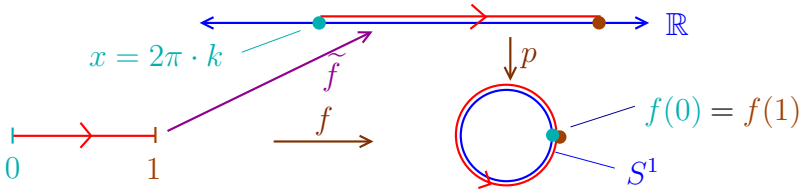
(a) We write $I := \{0,\dots,N-1\}^m$. We give I the lexicographic order.
(b) Given $i := (i_1,\dots,i_m) \in I$ we write $D_i := [\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}]$.
(c) Given $i \in I$ we write $C_i := \bigcup_{h<i} D_h$.

We set $\widetilde{f}(0) := x$. Now we use the Lift-Extension Lemma 92.3 to add the cubes $\{D_i\}_{i\in I}$ in lexicographic order to the domain of $\widetilde{f}$. This is possible since the image of each D_i is contained in a uniformly covered open subset of B and since given any $i \in I$ the intersection $C_i \cap D_i$ is easily seen to be non-empty and connected. ■

92.2. **Lifts of paths.** We now want to apply the results from the previous sections to paths and path-homotopies. It is convenient to introduce the following definition:

Definition 92.5. Let $p\colon X \to B$ be a covering and let $f\colon [0,1] \to B$ be a path. Let $x \in X$ be a point with $p(x) = f(0)$. A path $\widetilde{f}\colon [0,1] \to X$ is called a lift of f to the starting point x, if $\widetilde{f}$ is a lift of f with $\widetilde{f}(0) = x$.

Example. Again we consider the covering $p\colon \mathbb{R} \to S^1$ that is given by $t \mapsto \exp(it)$. We consider the path $f\colon [0,1] \to S^1$ that is given by $t \mapsto \exp(2it)$. Given any $k \in \mathbb{Z}$ the map $\widetilde{f}\colon [0,1] \to \mathbb{R}$ given by $t \mapsto 2\pi \cdot k + 2t$ is a lift of f to the starting point $2\pi \cdot k$.



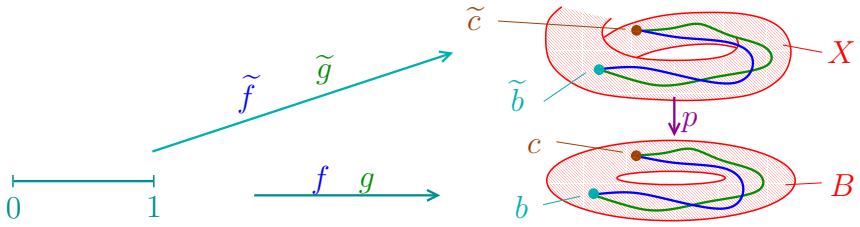
Note that the path $f\colon [0,1] \to S^1$ is a loop but the lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ is not a loop. □

Proposition 92.6. (Path-Lifting Proposition) Let $p\colon X \to B$ be a covering.

(1) Let $f\colon [0,1] \to B$ be a path. Given any $\widetilde{b} \in X$ with $p(\widetilde{b}) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to X$ of f to the starting point $\widetilde{b}$.
(2) Let $f, g\colon [0,1] \to B$ be two paths and let $\widetilde{f}, \widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.

Proof.

(1) This statement follows immediately from the Lift-Existence Lemma 92.4.



(2) We set $b \coloneqq f(0) = g(0)$ and $c \coloneqq f(1) = g(1)$. Furthermore let $\widetilde{b} \coloneqq \widetilde{f}(0) = \widetilde{g}(0)$ be the common starting point of the lifts. Let

$$\begin{array}{rccc} H\colon & [0,1] \times [0,1] & \to & B \\ & (t,s) & \mapsto & H(t,s) \end{array}$$

be a path-homotopy between the paths f and g. By the Lift-Existence Lemma 92.4 we can lift H to a continuous map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ with $\widetilde{H}(0,0) = \widetilde{f}(0)$.

Claim. The map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. In particular the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree.

Proof.

(a) For each $s \in [0,1]$ we have $p(\widetilde{H}(0,s)) = H(0,s) = b$, thus the map $s \mapsto \widetilde{H}(0,s)$ takes values in $p^{-1}(b)$. By the Covering–Basics Lemma 91.12 (2) we know that $p^{-1}(b)$ is a discrete subspace. It follows from the Connected-to-Discrete Lemma 4.37 that the map $s \mapsto \widetilde{H}(0,s)$ is constant. Similarly we see that the $s \mapsto \widetilde{H}(1,s)$ is constant. Thus we have shown that $\widetilde{H}$ is a path-homotopy.

(b) Since $\widetilde{H}_0$ and $\widetilde{f}$ are both lifts of $H_0 = f$ to the starting point $\widetilde{b}$ we obtain from the Lift Uniqueness Proposition 92.2 that $\widetilde{H}_0 = \widetilde{f}$.

(c) It follows from (a) that $\widetilde{H}_1$ and $\widetilde{g}$ are both lifts of $H_1 = g$ to the starting point $\widetilde{b}$. Thus we obtain again from the Lift Uniqueness Proposition 92.2 that $\widetilde{H}_1 = \widetilde{g}$.

In summary, we have shown that $\widetilde{H}$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. ■

Now we can formulate the first interesting result relating the fundamental group to coverings:

Proposition 92.7. (Covering-π_1-Monomorphism Proposition) Let $p\colon X \to B$ be a covering. Given any $x_0 \in X$ the induced map $p_*\colon \pi_1(X, x_0) \to \pi_1(B, p(x_0))$ is a monomorphism.

Example. In the figure below we see one of the coverings from page 1784:



Even though we cannot compute the fundamental groups of X and B it is clear that the fundamental group of X is "bigger" in a suitable sense than the fundamental group of B. But counterintuitively the Covering-π_1-Monomorphism Proposition 92.7 shows that there exists a monomorphism, namely p_*, from $\pi_1(X, x_0)$ into $\pi_1(B, p(x_0))$. □

Proof. We set $b_0 \coloneqq p(x_0)$. We see that

$$[f] \in \ker(p_*\colon \pi_1(X,x_0) \to \pi_1(B,b_0)) \implies [p\circ f] = e \in \pi_1(B,b_0) \implies p\circ f \simeq_{\mathrm{p}} \mathrm{const}_{b_0}$$
$$\implies f \simeq_{\mathrm{p}} \mathrm{const}_{x_0} \implies [f] = e \in \pi_1(X,x_0).$$

this follows from the Path-Lifting Proposition 92.6 (2), since f is the lift of $p\circ f$ to the starting point x_0 and const_{x_0} is the lift of const_{b_0} to the starting point x_0

■

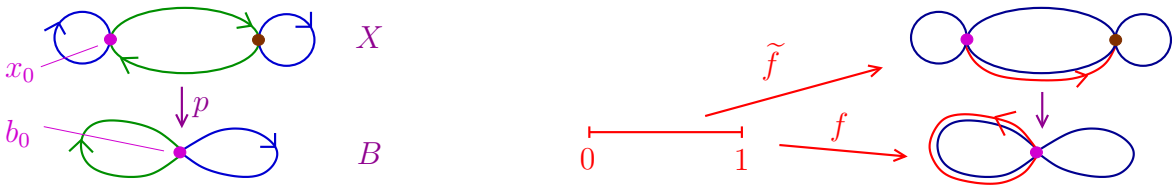
Example. By the Sphere-π_1-Proposition 88.24 we know that $\pi_1(S^2) = 0$ whereas we saw in the Torus-π_1-Proposition 90.14 that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is non-trivial. It follows from the Covering-π_1-Monomorphism Proposition 92.7 that there is no covering map from $S^1 \times S^1$ to S^2. □

Proposition 92.8. (Loop–Lifting Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. Let $f\colon [0,1] \to B$ be a loop in (B,b_0). We denote by $\widetilde{f}\colon [0,1] \to X$ the lift[923] of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X,x_0) \iff [f] \text{ lies in } p_*(\pi_1(X,x_0)) \subseteq \pi_1(B,b_0).$$

In particular, if $\widetilde{f}$ is not a loop in (X,x_0), then $[f]$ is a non-trivial element of $\pi_1(B,b_0)$.

Example. We consider the pointed topological space (B,b_0) and the loop $f\colon [0,1] \to B$ that are shown in the figure below. We also show a covering $p\colon X \to B$ together with the lift $\widetilde{f}$ of f to the starting point x_0:



Since $\widetilde{f}$ is not a loop it follows from the Loop–Lifting Proposition 92.8 that $[f] \in \pi_1(B,b_0)$ is non-trivial. In Exercise 92.1 we will use the same approach to show that $\pi_1(B,b_0)$ is actually non-abelian. □

Proof. Let f be a loop in (B,b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0.

"⇒" If $\widetilde{f}$ is a loop in (X,x_0), then it defines an element in $\pi_1(X,x_0)$. Therefore we see that $[f] = [p\circ\widetilde{f}] = p_*([\widetilde{f}]) \in p_*(\pi_1(X,x_0))$.

"⇐" We assume that $[f]$ lies in $p_*(\pi_1(X,x_0)) \subseteq \pi_1(B,b_0)$. We make two observations:

- The hypothesis that $[f] \in p_*(\pi_1(X,x_0))$ implies that there exists a loop g in (X,x_0) with $[f] = p_*([g]) = [p\circ g]$. In particular we see that f and $p\circ g$ are path-homotopic.
- Note that $\widetilde{f}$ is a lift of f and that g is a lift of $p\circ g$ and note that these two lifts have the same starting point x_0.

It follows from these two observations and the Path-Lifting Proposition 92.6 (2) that the endpoint of $\widetilde{f}$ agrees with the endpoint of $\widetilde{g}$, which is x_0. Thus we have shown that $\widetilde{f}$ is a loop in (X,x_0). ■

The following definition should be familiar from a (linear) algebra course.

[923] Note that such a lift exists, and is unique, by the Path-Lifting Proposition 92.6 (1).

Definition 92.9. Let G be a group and let H be a subgroup of G.

(1) We get an equivalence relation $\sim$ on G by setting $g_1 \sim g_2$ if there exists an $h \in H$ with $g_1 \cdot h = g_2$. We denote by G/H the set of equivalence classes.
(2) If in (1) we use left multiplication by $h \in H$, then also get an equivalence relation and we denote the set of equivalence classes by $H\backslash G$. Note that the map $[g] \mapsto [g^{-1}]$ defines a bijection $H\backslash G \to G/H$.
(3) We write $[G : H] := \#G/H = \#H\backslash G$.

Proposition 92.10. (Covering-Degree Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then[924]

$$\underbrace{[X : B]}_{\substack{\text{degree as defined} \\ \text{in Definition 91.10}}} = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))].$$

Example. Let $n \geq 2$.

(1) On page 1784 we saw that we have a 2-fold covering $p\colon S^n \to S^n/\{\pm 1\} =_{\mathrm{i}} \mathbb{R}\mathrm{P}^n$.
(2) It follows from the Covering-Degree Proposition 92.10 that $p_*(\pi_1(S^n))$ is subgroup of index two of $\pi_1(S^n/\{\pm 1\})$.
(3) Since $n \geq 2$ we know by the Sphere-π_1-Proposition 88.24 that $\pi_1(S^n) = 0$.
(4) It follows from (1)–(3) that $\pi_1(S^n/\{\pm 1\})$ is a group with two elements.
(5) It is elementary to see that (4) implies that $\pi_1(\mathbb{R}\mathrm{P}^n) = \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}/2\mathbb{Z}$.

In the next chapter we will return to this calculation in a more systematic fashion. □

As we will see shortly, the Covering-Degree Proposition 92.10 follows almost immediately from the following lemma:

Lemma 92.11. (Covering–Bijection Lemma) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then the map

$$\begin{aligned} \Theta\colon p^{-1}(b_0) &\to p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \\ y_0 &\mapsto [\, p \circ (\underbrace{\text{path } f \text{ in } X \text{ from } x_0 \text{ to } y_0}_{\text{exists since } X \text{ is path-connected}}) \,] \end{aligned}$$

is well-defined and it is a bijection.



[924] Recall that the left-hand side and the right-hand side are defined as cardinalities of sets.

Proof of Lemma 92.11. Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We assume that X is path-connected.

(1) First we show that the map Θ is well-defined. Thus let $f, g\colon [0,1] \to X$ be two paths from x_0 to y_0. Note that

$$\underbrace{[p \circ f]}_{\in \pi_1(B,b_0)} = [p \circ f * \underbrace{p \circ \overline{g} * p \circ g}_{\text{null-homotopic by the Path-Homotopy Product Proposition 88.10}}] = [p \circ (\underbrace{f * \overline{g}}_{\text{loop in } (X,x_0) \text{ since the endpoint of } f \text{ equals the starting point of } \overline{g}})] \cdot [p \circ g] = p_*(\underbrace{[f * \overline{g}]}_{\in \pi_1(X,x_0)}) \cdot \underbrace{[p \circ g]}_{\in \pi_1(B,b_0)}.$$



(2) Next we show that the map Θ is surjective. Thus let $[f] \in \pi_1(B, b_0)$. By the Path-Lifting Proposition 92.6 (1) there exists a lift $\widetilde{f}$ of f to the starting point x_0. Note that $p(\widetilde{f}(1)) = f(1) = b_0$. Thus $\widetilde{f}(1) \in p^{-1}(b_0)$. Evidently $\widetilde{f}$ is a path in X from x_0 to $\widetilde{f}(1)$. Thus we see that $\Theta(\widetilde{f}(1)) = [p \circ \widetilde{f}] = [f]$.

(3) Finally we show that the map Θ is injective. Suppose we have $y_0, z_0 \in p^{-1}(b_0)$ such that $\Theta(y_0) = \Theta(z_0)$.

(a) We pick a path $f_0\colon [0,1] \to X$ from x_0 to y_0 and we pick a path $g_0\colon [0,1] \to X$ from x_0 to z_0.



(b) The fact that $\Theta(y_0) = \Theta(z_0) \in p_*(\pi_1(X,x_0))\backslash\pi_1(B,b_0)$ implies that there exists a loop $h\colon [0,1] \to X$ in (X,x_0) such that $[p \circ h] \cdot [p \circ f_0] = [p \circ g_0]$.

(c) Note that $h * f_0$ is defined since the endpoint of h equals x_0, which is the starting point of f_0.[925] Now we see that

$$[p \circ (h * f_0)] = [(p \circ h) * (p \circ f_0)] = [p \circ h] \cdot [p \circ f_0] = [p \circ g_0].$$

(d) By (c) we know that $p \circ (h * f_0) \simeq_{\mathrm{p}} p \circ g_0$.

(e) Since $h * f_0$ is a lift of $p \circ (h * f_0)$ to the starting point x_0 and since g_0 is a lift of $p \circ g_0$ to the starting point x_0 we obtain from (d) and the Path-Lifting Proposition 92.6 (2) that the endpoints of $h * f_0$ and g_0 agree. But these are precisely y_0 and z_0. ∎

Proof of Proposition 92.10. Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We assume that X is path-connected. We see that

$$[X : B] \underset{\text{Definition 91.10 of degree}}{=} \#p^{-1}(b_0) \underset{\text{by the Covering–Bijection Lemma 92.11}}{=} \#p_*(\pi_1(X,x_0))\backslash\pi_1(B,b_0) \underset{\text{Definition 92.9 of index}}{=} [\pi_1(B,b_0) : p_*(\pi_1(X,x_0))].$$

∎

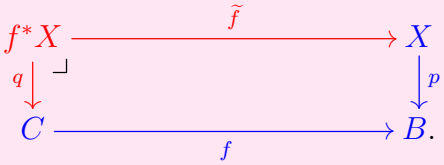
[925] This is the reason why we work with right cosets.

92.3. **Pullbacks of coverings.** We conclude this chapter with a lemma on pullbacks of coverings. This lemma will play a minor role here and there, but it can also be skipped at a first reading.

Lemma 92.12. (Pullback-Covering Lemma) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. As in the Topological-Pullback Proposition 7.22 we consider the pullback

$$f^*X \;:=\; \{(c,x) \in C \times X \mid f(c) = p(x)\}$$

which gives us the following commutative diagram:



If $p\colon X \to B$ is in fact a covering, then the following statements hold:

(1) The map $q\colon f^*X \to C$ is also a covering.

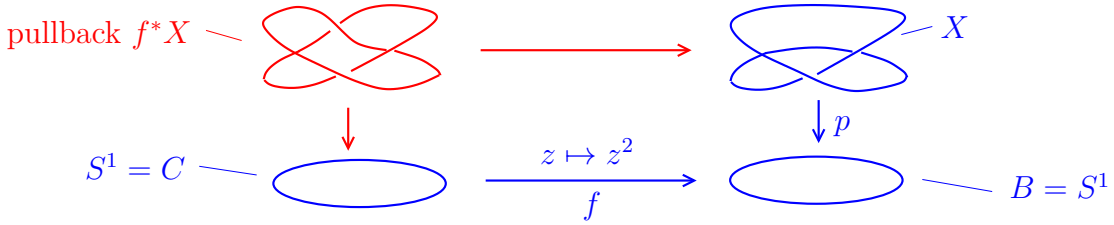
(2) If B and C are connected, then

$$\deg(q\colon f^*X \to C) \;=\; \deg(p\colon X \to B).$$

(3) Let $c \in C$. We write $b = f(c)$ and we pick[926] $x \in X$ such that $p(x) = b$. The following equality holds:

$$q_*(\pi_1(f^*X, (c,x))) \;=\; f_*^{-1}(p_*(\pi_1(X,x))) \;\subseteq\; \pi_1(C,c).$$

Example. Similar to the discussion on page 292 we consider the covering $p\colon X \to B = S^1$ that is shown in the figure below and we consider the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$.



Note that in this case the pullback covering $f^*X \to C = S^1$ is a product, i.e. there exists a homeomorphism $f^*X \cong S^1 \times \{-1, 1\}$ that commutes with the projections to S^1. □

Proof. Let $p\colon X \to B$ be a covering. We write $Z := f^*X$ and $z := (c,x)$. Finally we denote by $q\colon f^*X \to C$ the map that is given by $(c,x) \mapsto c$.

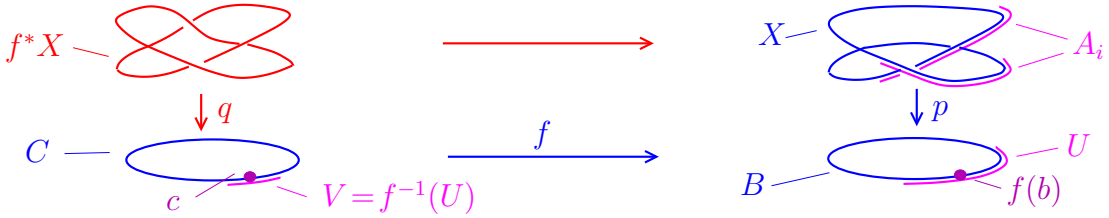
(1) First we show that $q\colon f^*X \to C$ is surjective. Thus let $c \in C$. Since $p\colon X \to B$ is in particular surjective we see that there exists an $x \in X$ with $p(x) = f(c)$. Note that $(c,x) \in f^*X$. Since $q(c,x) = c$ we have shown that q is surjective.

Next let $c \in C$. We need to show that c admits a uniformly covered neighborhood. Since p is a covering the point $f(c) \in B$ admits a uniformly covered neighborhood U.

[926] Recall that covering maps are in particular surjective, in particular such x exists.

This means that $p^{-1}(U) = \bigsqcup_{i\in I} A_i$ such that each A_i is open in X and such that each $p|_{A_i}\colon A_i \to U$ is a homeomorphism.

We set $V := f^{-1}(U)$. We claim that V is uniformly covered. Given $i \in I$ we consider $B_i := \{(d,x) \in f^*X \mid d \in V, x \in A_i\}$. Since $B_i = f^*X \cap (V \times A_i)$ we see that B_i is an open subset of f^*X. Finally note that one can easily verify that each restriction $q|_{B_i}\colon B_i \to V$ is a homeomorphism.



(2) Since B and C are connected it makes sense to speak of the degrees of p and q. For any $c \in C$ we have a bijection

$$q^{-1}(c) = \{(c,x) \in C \times X \mid f(c) = p(x)\} \xrightarrow{\text{bijection}} p^{-1}(f(c)) = \deg(p).$$

This shows that the degree of q equals the degree of p.

(3) We need to show that $q_*(\pi_1(Z,z)) = f_*^{-1}(\pi_1(X,x))$. We show the two inclusions "$\subseteq$" and "$\supseteq$" separately.

"$\subseteq$" Let $\widetilde{f}\colon Z \to X$ be the map that is given by $(c,x) \to x$. By construction we have $f \circ q = \widetilde{f} \circ p$. But this implies immediately that $f_*(q_*(\pi_1(Z,z)))$ is contained in $p_*(\pi_1(X,x))$.

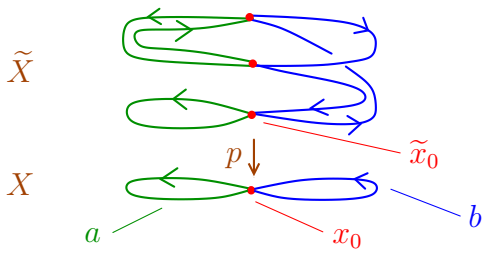
"$\supseteq$" Let $\gamma\colon [0,1] \to C$ be a loop in (C,c) with $[f\circ\gamma] = f_*([\gamma]) \in p_*(\pi_1(X,x))$. We write $g := f \circ \gamma$. Let $\widetilde{g}\colon [0,1] \to X$ be the lift with $\widetilde{g}(0) = x$. Since $[g] \in p_*(\pi_1(X,x))$ we obtain from the Loop–Lifting Proposition 92.8 that $\widetilde{g}$ is actually a loop. Thus the map $\alpha\colon [0,1] \to C \times X$ that is given by $t \mapsto (\gamma(t), \widetilde{g}(t))$ is a loop. By construction the loop takes values in Z and evidently we have $q_*([\alpha]) = [\gamma]$. This shows that $[\gamma] \in q_*(\pi_1(Z,z))$. ■

92.4. Exercises for Chapter 92.

Exercise 92.1. We consider the 3-fold covering $p\colon \widetilde{X} \to X$ shown in the figure below and we consider the loops $a, b \in (X, x_0)$.

(a) Determine the lifts of $a * b$ and $b * a$ to $(\widetilde{X}, \widetilde{x}_0)$.

(b) Use (a) to show that $\pi_1(X, x_0)$ is non-abelian.



Exercise 92.2.

(a) Consider the following maps:

$$\begin{array}{ccc} & & (S^1,1) \\ & \text{¿¿¿lift } \widetilde{f}\,?? \nearrow & \downarrow{\scriptstyle p}\ {\scriptstyle z\mapsto z^2} \\ (S^1,1) & \xrightarrow[f]{z\mapsto z} & (S^1,1). \end{array}$$

Show that the map f does not lift to a continuous map $\widetilde{f}$.

(b) Let $p\colon \widetilde{X} \to X$ be a finite covering and let $f\colon X \to X$ be a continuous map. Does there necessarily exist a continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ that makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\quad f \quad} & X\,? \end{array}$$

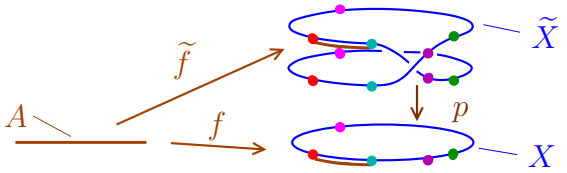
Exercise 92.3. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a topological space X.

(a) Show that the map

$$\begin{array}{rccl} \Xi\colon & \widetilde{X} & \to & \widetilde{X} \\ & x & \mapsto & \text{the unique element in } p^{-1}(p(\{x\})) \setminus \{x\} \end{array}$$

is continuous.

(b) Let A be a topological space and let $f\colon A \to X$ be a continuous map. Show that if $\widetilde{f}\colon A \to \widetilde{X}$ is a lift of f, then $\Xi \circ \widetilde{f}$ is the unique other lift of f.



$\Xi\colon \widetilde{X} \to \widetilde{X}$ swaps the points of the same color

Exercise 92.4. Let K be the Klein bottle. Does there exist a 2-fold covering $p\colon K \to K$?

Exercise 92.5. Let $p\colon X \to B$ be a covering of topological spaces and let $b_0 \in B$. We assume that X is path-connected. Let $f\colon [0,1] \to B$ be a loop in (B, b_0). Furthermore let $\widetilde{f}, \widetilde{g}$ be two lifts of f to different starting points. Does the following statement hold:

$$\widetilde{f} \text{ is a loop} \iff \widetilde{g} \text{ is a loop.}$$

Exercise 92.6. We consider the pointed topological space (B, b_0) and the two loops α, β in (B, b_0) that are shown in the figure below.

(a) Provide a covering $p\colon X \to B$ and a point $x_0 \in X$ such that the lift of $\overline{\alpha} * \beta * \alpha * \alpha$ to the starting point x_0 is not a loop.

(b) Provide a covering $p\colon X \to B$ and a point $x_0 \in X$ such that the lift of $\alpha * \alpha * \beta * \overline{\alpha} * \overline{\alpha} * \overline{\beta}$ to the starting point x_0 is not a loop.

(c) For which concatenations of $\alpha, \overline{\alpha}, \beta, \overline{\beta}$ can you find a covering $p\colon X \to B$ and a point $x_0 \in X$ such that the lift of the given loop to the starting point x_0 is not a loop?
Remark. It is perhaps difficult at this stage to give a complete answer, but it is nonetheless worth trying.



Exercise 92.7. Let $n \in \mathbb{N}_{\geq 2}$. We set $Q \coloneqq (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. On page 1799 we saw that $\pi_1(\mathbb{R}\mathrm{P}^n, [Q]) \cong \mathbb{Z}/2\mathbb{Z}$. Give an explicit loop γ in $(\mathbb{R}\mathrm{P}^n, [Q])$ such that $[\gamma]$ represents the unique non-trivial element of $\pi_1(\mathbb{R}\mathrm{P}^n, [Q])$.

Exercise 92.8. Show that given any $n \geq 3$ there exists a closed non-orientable connected n-dimensional smooth manifold M with $\pi_1(M) \cong \mathbb{Z}$.

Exercise 92.9. Let $p\colon X \to B$ be a covering of pointed topological spaces. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We suppose that X is simply connected. Let f and g loops in (B, b_0) and let $\widetilde{f}$ and $\widetilde{g}$ be the lifts of f and g to the starting point x_0. Show that the following holds:

$$f \text{ and } g \text{ are path-homotopic loops} \iff \text{the endpoints of } \widetilde{f} \text{ and } \widetilde{g} \text{ agree.}$$

CHAPTER 93

Group actions and fundamental groups

In this chapter we will build on the techniques and results of Chapter 91 and Chapter 92 to calculate the fundamental groups of many topological spaces, in particular of the real projective spaces, lens spaces and the Klein bottle. We will also consider mapping tori and their fundamental groups.

This chapter is organized as follows:

- In Section 93.1 we will state and prove the Action–π_1-Theorem 93.1 which often allows us to calculate fundamental groups of quotients of simply connected topological spaces. We will also discuss several examples, for example we will determine the fundamental groups of tori.
- In Section 93.2 we will use the Action–π_1-Theorem 93.1 to calculate many fundamental groups, in particular we will calculate the fundamental groups of lens spaces and of the real projective spaces $\mathbb{R}\mathrm{P}^n$.
- In Section 93.3 we will use the Action–π_1-Theorem 93.1 to calculate the fundamental group of mapping tori.
- In Section 93.4 we will provide exercises for Chapter 93.

In the next Chapter 94 we will discuss some applications of fundamental groups to "real life" problems.

93.1. **The main theorem on group actions and fundamental groups.** The following theorem will allow us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous chapter.

Theorem 93.1. (Action–π_1-Theorem) Let X be a topological space and let G be a group that acts continuously and discretely[927] on X. We choose an $x_0 \in X$. We denote by $p\colon X \to X/G$ the natural projection.

(1) If X is simply connected, then the map

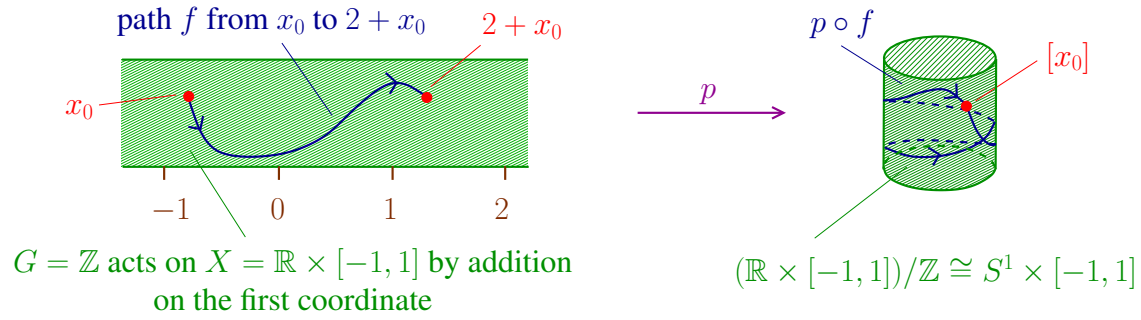
$$\begin{array}{rcl} \Theta_{x_0}\colon G & \to & \pi_1(X/G, [x_0]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined natural isomorphism of groups. In particular we have a natural isomorphism $\pi_1(X/G) \cong G$.

(2) [928] If X is path-connected (but not necessarily simply connected), then $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$ and the map

$$\begin{array}{rcl} \Theta_{x_0}\colon G & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined natural isomorphism of groups.



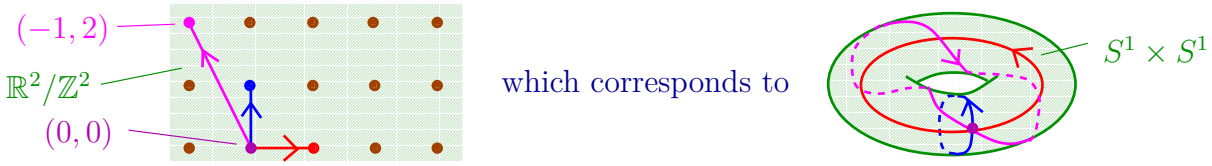
Before we provide the proof of the Action–π_1-Theorem 93.1 it is perhaps motivating to consider a few basic examples. In the following examples we will see that the Action–π_1-Theorem 93.1 can be used to recover some of our calculations, namely we can use it to calculate the fundamental groups of tori and the Möbius band. After the proof we will give more interesting applications.

Example.

(1) Let $n\in\mathbb{N}$. It follows from the discussion on page 1734 that $\mathbb{R}^n$ is simply connected. We saw on page 1776 that the usual continuous action of $(\mathbb{Z}^n,+)$ on $\mathbb{R}^n$ is discrete. It follows from the Action–π_1-Theorem 93.1 (1) that the map

$$\begin{array}{rcl}\mathbb{Z}^n & \to & \pi_1(\mathbb{R}^n/\mathbb{Z}^n,[(0)])\\ v & \mapsto & \left[\begin{array}{rcl}[0,1] & \to & \mathbb{R}^n/\mathbb{Z}^n\\ t & \mapsto & [t\cdot v]\end{array}\right]\end{array}$$

is an isomorphism. Using the homeomorphism $(S^1)^n\to\mathbb{R}^n/\mathbb{Z}^n$ from the Torus-as-Quotient Lemma 9.2 this also gives a new calculation of the fundamental group of $(S^1)^n$. In particular we obtain a new proof of the Circle-π_1-Theorem 88.22 and the Torus-π_1-Proposition 90.14.



(2) We consider $X=\mathbb{R}\times[-1,1]$ with the continuous $\mathbb{Z}$-action

$$\begin{array}{rcl}\mathbb{Z}\times(\mathbb{R}\times[-1,1]) & \to & \mathbb{R}\times[-1,1]\\ (n,(x,y)) & \mapsto & (x+n,(-1)^n\cdot y).\end{array}$$

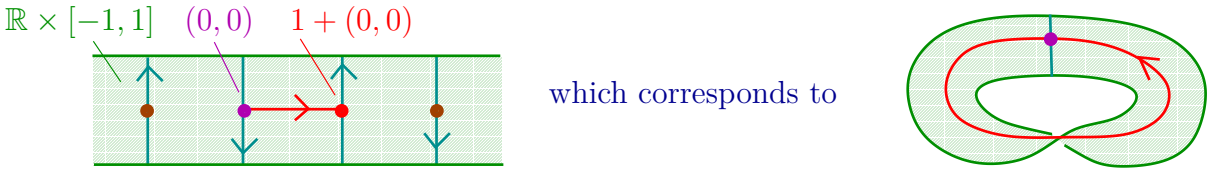
On page 1776 we saw that this action is discrete. Note that on page 1734 we saw that $\mathbb{R}\times[-1,1]$ is simply connected. It follows from the Action–π_1-Theorem 93.1 (1) that the map

$$\begin{array}{rcl}\mathbb{Z} & \to & \pi_1((\mathbb{R}\times[-1,1])/\mathbb{Z},([(0,0)])\\ n & \mapsto & \left[\begin{array}{rcl}[0,1] & \to & (\mathbb{R}\times[-1,1])/\mathbb{Z}\\ t & \mapsto & [(n\cdot t,0)]\end{array}\right]\end{array}$$

[927]Recall that according to Definition 91.1 we say that G acts discretely, if for each $x\in X$ there exists an open neighborhood U such that $U\cap g\cdot U=\varnothing$ for all $g\neq e$.

[928]Statement (2) is a generalization of Statement (1). For many, but not all, applications Statement (1) is good enough.

is an isomorphism. Recall that in the Möbius Band-as-Quotient Lemma 9.6 we saw that $(\mathbb{R} \times [-1,1])/\mathbb{Z}$ is homeomorphic to the Möbius band.



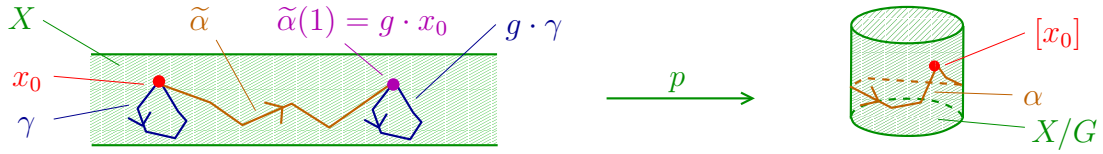
Thus we obtain a new proof of the Möbius Band-π_1-Corollary 90.11. □

Now we finally provide the proof of the Action–π_1-Theorem 93.1

Proof of the Action–π_1-Theorem 93.1. Note that Statement (1) is just a user-friendly special case of Statement (2). Thus it suffices to prove Statement (2). Let X be a path-connected topological space and let G be a group that acts continuously and discretely on X. Note that it follows from the Action–Covering Proposition 91.11 that the natural projection $p\colon X \to X/G$ is a covering. Finally let $x_0 \in X$. We start out with the following claim:

Claim 1. The image $p_*(\pi_1(X, x_0)) \subseteq \pi_1(X/G, [x_0])$ is a normal subgroup.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$ and let $[\alpha] \in \pi_1(X/G, [x_0])$. We denote by $\widetilde{\alpha}\colon [0,1] \to X$ the lift of α with the starting point x_0. Note that $p(\widetilde{\alpha}(1)) = \alpha(1) = [x_0]$. This shows that there exists a $g \in G$ with $\widetilde{\alpha}(1) = g \cdot x_0$. We denote by $g \cdot \gamma$ the loop in X that is given by $t \mapsto g \cdot \gamma(t)$. Note that we can form the loop $\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}$ since the endpoints always match the starting point.



Now we see that

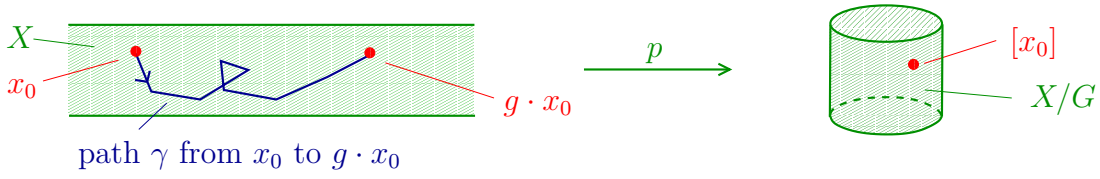
since $p(g \cdot x) = p(x) \in X/G$ for all $x \in X$

$$[\alpha] \cdot p_*([\gamma]) \cdot [\alpha]^{-1} = [p \circ \widetilde{\alpha}] \cdot [p \circ \gamma] \cdot [p \circ \overline{\widetilde{\alpha}}] \overset{\downarrow}{=} [p \circ \widetilde{\alpha}] \cdot [p \circ (g \cdot \gamma)] \cdot [p \circ \overline{\widetilde{\alpha}}]$$
$$= [p \circ (\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\text{defined since the endpoints always match the starting points}})] = p_*([\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\text{loop in } (X, x_0)}]) \in p_*(\pi_1(X, x_0)).$$

Thus we have shown that $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$. ⊞

Similar to the Covering–Bijection Lemma 92.11 we consider the map

$$\begin{aligned} \Phi\colon G &\to \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g &\mapsto [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)]. \end{aligned}$$



Claim 2. The map Φ is well-defined and it is a bijection.

Proof. We consider the maps

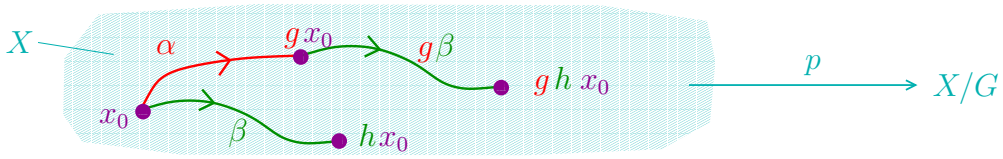
$$\begin{array}{ccccc} G & \to & p^{-1}([x_0]) & \to & \pi_1(X/G,[x_0])/p_*(\pi_1(X,x_0)) \\ g & \mapsto & g\cdot x_0 & & \\ & & y_0 & \mapsto & [p\circ(\text{path in } X \text{ from } x_0 \text{ to } y_0)]. \end{array}$$

The map to the left is clearly a bijection. The claim now follows from the observation that we showed in the Covering–Bijection Lemma 92.11 that the map to the right is well-defined and that it is a bijection. ⊞

It remains to prove the following claim:

Claim 3. The map $\Phi\colon G\to \pi_1(X/G,[x_0])/p_*(\pi_1(X,x_0))$ is a group homomorphism.

Proof. Let $g,h\in G$. We pick a path α from x_0 to $g\cdot x_0$ and we pick a path β from x_0 to $h\cdot x_0$. Then $g\cdot\beta$[929] is a path from $g\cdot x_0$ to $(g\cdot(h\cdot x_0))=gh\cdot x_0$ and $\alpha * g\cdot\beta$ is a path from x_0 to $gh\cdot x_0$.



It follows that

$$\begin{array}{ccccccc} \Phi(g\cdot h) & = & [p\circ(\alpha * g\cdot\beta)] & \overset{\text{since } p\circ(\gamma*\delta)=p\circ\gamma*p\circ\delta}{=} & [(p\circ\alpha)*(p\circ g\cdot\beta)] & \overset{\text{since for all } x\in X \text{ we have } p(g\cdot x)=p(x)\in X/G}{=} & [(p\circ\alpha)*(p\circ\beta)] \\ & & \uparrow & & & & = [p\circ\alpha]\cdot[p\circ\beta] = \Phi(g)\cdot\Phi(h). \\ & & \text{since } \alpha*g\cdot\beta \text{ is a path from } x_0 \text{ to } gh\cdot x_0 & & & & \uparrow \\ & & & & & & \text{definition of } \cdot \text{ in } \pi_1(X/G,[x_0]) \end{array}$$

■

93.2. **Applications of the Action–π_1-Theorem 93.1.** Using the Action–π_1-Theorem 93.1 we can now determine the fundamental groups of many of the topological spaces that we have encountered so far.

We start out with the mysterious lens spaces which we introduced in Definition 91.4. The following proposition gives us in particular a first partial answer to Question 91.7.

Proposition 93.2. (Lens Space-π_1-Proposition)

(1) Let p *in*$\mathbb{N}$ and let $q\in\mathbb{Z}$ with $\gcd(p,q)=1$. If $L(p,q)=S^3/(\mathbb{Z}/p\mathbb{Z})$ is a lens space as introduced in Definition 91.4, then

$$\pi_1(L(p,q)) \;\cong\; \mathbb{Z}/p\mathbb{Z}.$$

(2) Let $p,\widetilde{p}\in\mathbb{Z}\setminus\{0\}$ and let $q,\widetilde{q}\in\mathbb{Z}$ with $\gcd(p,q)=1$ and $\gcd(\widetilde{p},\widetilde{q})=1$. If $L(p,q)$ is homotopy equivalent (e.g. homeomorphic) to $L(\widetilde{p},\widetilde{q})$, then $p=\widetilde{p}$.[930]

Proof.

[929] Here $g\cdot\beta$ denotes the path $[0,1]\to X$ that is given by $t\mapsto g\cdot\beta(t)$.

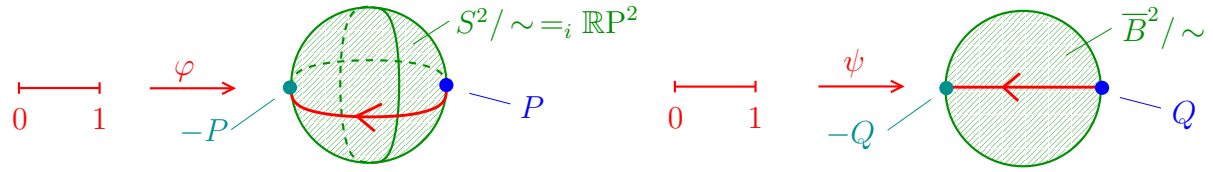
[930] This statement gives a partial answer to Question 91.6. Unfortunately as of right now we cannot say anything about the parameters q and $\widetilde{q}$. It will take a very long time to get a grip on these parameters.

(1) First we note that by the Sphere-π_1-Proposition 88.24 we know that S^3 is simply connected. Next note that in the Lens Space-Discrete Action Lemma 91.3 we showed that the action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 is discrete. Thus the statement follows from the Action–π_1-Theorem 93.1 (1).
(2) This statement follows from (1) and the Homotopy Equivalence-π_1-Proposition 90.9 (1a). ■

We move on to the only slightly less bewildering real projective spaces. The following proposition can be viewed as a refinement of the argument on page 1799.

Proposition 93.3. (Real Projective Space–π_1-Proposition) Let $n \in \mathbb{N}_{\geq 2}$.[931]

(1) We consider the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$.
  (a) $\pi_1(\text{real projective space } \mathbb{R}\mathrm{P}^n) =_{\mathrm{i}} \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}/2\mathbb{Z}$.
  (b) We set $P := (1, 0, \ldots, 0) \in S^n$. The unique non-trivial element of the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^n, [P]) =_{\mathrm{i}} \pi_1(S^n/\{\pm 1\}, [P])$ is represented by the loop
$$\begin{array}{rcl} \varphi\colon [0,1] & \to & \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\} \\ t & \mapsto & [(\cos(\pi t), \sin(\pi t), 0, \ldots, 0)]. \end{array}$$
(2) Let "$\sim$" be the equivalence relation on $\overline{B}^n$ that is generated by $x \sim -x$ for $x \in S^{n-1}$.
  (a) $\pi_1(\overline{B}^n/\sim) \cong \mathbb{Z}/2\mathbb{Z}$.
  (b) With $Q = (1, 0, \ldots, 0) \in \overline{B}^n$ the unique non-trivial element of $\pi_1(\overline{B}^n/\sim), [Q])$ is represented by the loop
$$\begin{array}{rcl} \psi\colon [0,1] & \to & \overline{B}^n/\sim \\ t & \mapsto & [2t-1, 0, \ldots, 0)]. \end{array}$$



Proof.

(1) These two statements follow almost immediately from the following observations:
  (i) We consider the usual continuous action of $\{\pm 1\}$ on S^n. On page 1776 we saw that the action of $\{\pm 1\}$ on S^n is discrete. We denote by $p\colon S^n \to S^n/\{\pm 1\}$ the projection.
  (ii) Since $n \geq 2$ we obtain from the Sphere-π_1-Proposition 88.24 that the sphere S^n is simply connected.
  (iii) Note that it follows from (i) and (ii) together with the Action–π_1-Theorem 93.1 (1) that the map
$$\begin{array}{rcl} \{\pm 1\} & \to & \pi_1(S^n/\sim) \\ \epsilon & \mapsto & [p \circ (\text{path in } S^n \text{ from } P \text{ to } \epsilon \cdot P)] \end{array}$$
  is an isomorphism. Since $t \mapsto (\cos(\pi t), \sin(\pi t), 0, \ldots, 0)$ is precisely a path from P to $-P$ we see that φ represents the non-trivial element of $\pi_1(S^n/\{\pm 1\}, [P])$.
  (iv) Recall that in the Projective Space-Sphere Quotient Lemma 9.13 we saw that the inclusion $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeomorphism $S^n/\{\pm 1\} \to \mathbb{R}\mathrm{P}^n$.

[931] Recall that in the Projective Space-Avatar Lemma 9.16 (or alternatively on page 338) we showed that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1. Thus it follows from the Circle-π_1-Theorem 88.22 that $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$.

(2) In the Projective Space-Avatar Lemma 9.16 (2) we showed that the map

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ \underbrace{[(v, x_{n+1})]}_{v\in\mathbb{R}^n,\, x_{n+1}\in\mathbb{R}} & \mapsto & \begin{cases} [v], & \text{if } x_{n+1}\geq 0, \\ [-v], & \text{if } x_{n+1}\leq 0 \end{cases} \end{array}$$

is a homeomorphism with $f([P]) = [Q]$. It follows from this observation together with the results in (1) that $\pi_1(\overline{B}^n/\sim) \cong \mathbb{Z}/2\mathbb{Z}$ and that the unique non-trivial of $\pi_1(\overline{B}^n/\sim, [Q])$ is presented by the loop $f\circ\varphi$, i.e. by the loop $t\mapsto[(\cos(\pi t), \sin(\pi t), 0, \dots, 0)]$. It is straightforward to see that this loop is path-homotopic to the loop ψ. ■

Proposition 93.4. (SO(3)–π_1**-Proposition)**
(1) $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}/2\mathbb{Z}$.
(2) The unique non-trivial element is represented by the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}}_{\text{rotation around the } x\text{-axis}} \end{array}$$

at the base point $\mathrm{id}\in\mathrm{SO}(3)$.

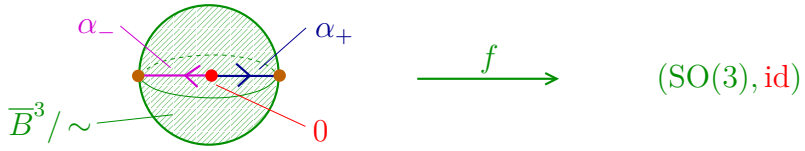
Remark.
(1) On page 1829 we will give a somewhat different proof of the SO(3)–π_1-Proposition 93.4.
(2) In Chapter 94 we will provide several interesting applications of the above calculation of $\pi_1(\mathrm{SO}(3))$. □

Proof.
(1) (a) Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x\sim -x$ for $x\in S^2$.
(b) We set $Q := (1,0,0)\in\overline{B}^3$.
(c) In the Projective Space-Avatar Lemma 9.16 (3) we saw that the map

$$\begin{array}{rcl} f\colon \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ \underbrace{[r\cdot v]}_{r\in[0,1] \text{ and } v\in S^2} & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi\cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0}\cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. Note that $f(0) = \mathrm{id}$.
(d) Recall that in the Real Projective Space–π_1-Proposition 93.3 (2) we showed that $\pi_1(\overline{B}^3/\sim, [Q]) \cong \mathbb{Z}/2\mathbb{Z}$.
(e) It follows from (c) and (d) that $\pi_1(\mathrm{SO}(3), f([Q])) \cong \mathbb{Z}/2\mathbb{Z}$.
(2) We still need to show that γ represents the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$.
(f) Given $\epsilon\in\{-1,1\}$ let $\alpha_\epsilon\colon [0,1]\to\overline{B}^3$ be the path that is given by $t\mapsto[t\cdot Q\cdot\epsilon]$.
(g) In the Real Projective Space–π_1-Proposition 93.3 (2) we showed that the unique non-trivial element of $\pi_1(\overline{B}^3/\sim, [Q])$ is represented by the loop $\overline{\alpha}_+ * \alpha_-$.
(h) Note that α_+ is a path from 0 to Q. It follows from (d) and the Change-of-Base Point Proposition 88.14 that the unique non-trivial element of $\pi_1(\overline{B}^3/\sim, 0)$ is represented by $\alpha_+ * (\overline{\alpha}_+ * \alpha_-) * \overline{\alpha}_+ \simeq_{\mathrm{p}} \alpha_- * \overline{\alpha}_+$.

(i) It follows from (c) and (h) and the fact that $f(0) = \mathrm{id}$ that the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$ is represented by $\varphi := f \circ (\overline{\alpha}_+ * \alpha_-)$.



(j) The loop that we gave in the statement of the proposition is precisely $\overline{\varphi}$. But if φ represents a non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$, then so does $\overline{\varphi}$. ■

The SO(3)–π_1-Proposition 93.4 also raises the following question:

Question 93.5. Let $n \geq 4$. What is $\pi_1(\mathrm{SO}(n))$?[932] [933]

We will answer the question in the Matrix Group-π_1-Proposition 239.15. In that proposition we will also compute the fundamental groups of many other matrix groups.

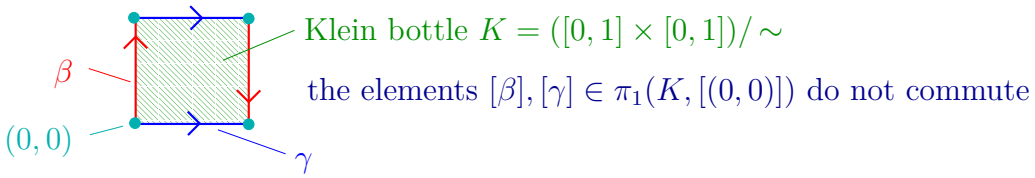
We continue with our last application of the Action–π_1-Theorem 93.1 (1), namely we will now compute the fundamental group of the Klein bottle.

Proposition 93.6. (Klein Bottle–π_1-Proposition) As in the Klein Bottle-as-Quotient Lemma 91.2 we consider the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses.

(1) The fundamental group of the Klein bottle $K = ([0,1] \times [0,1])/\sim$ is isomorphic to G.
(2) The fundamental group of the Klein bottle K is non-abelian, in particular the two elements $[\beta]$ and $[\gamma]$ shown in the figure below do not commute.



Remark. We just showed that fundamental groups can be non-abelian groups. In fact, as we will see later in Chapter 99, many, actually arguably "most" fundamental groups are non-abelian. This will pose several major problems. For example, what is a good way to describe a (non-abelian) group? How can we decide whether or not two (non-abelian) groups are isomorphic? □

Proof.

(1) In the Klein Bottle-as-Quotient Lemma 91.2 we showed the following two facts:
  (a) The group G acts discretely on $\mathbb{R}^2$ in the sense of Definition 91.1.
  (b) The inclusion-induced map $f\colon ([0,1] \times [0,1])/\sim \to \mathbb{R}^2/G$ is a homeomorphism.

[932] Recall that the Matrix Group-π_0-Proposition 4.54 (3) says that $\mathrm{SO}(n)$ is path-connected.
[933] What is $\pi_1(\mathrm{SO}(2))$?

It follows from (a) together with the Action–π_1-Theorem 93.1 (1) that the map

$$\begin{array}{rcl} \Theta\colon G & \to & \pi_1(\mathbb{R}^2/G,[(0,0)]) \\ g & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}^2/G \\ t & \mapsto & [t\cdot \underbrace{g(0,0)}_{\in\mathbb{R}^2}\,] \end{array}\right] \end{array}$$

is an isomorphism. If we combine this calculation with (b) we see that the fundamental group of the Klein bottle is isomorphic to G.

(2) (a) We set $C := B^{-1}\cdot A$. An easy calculation shows that the two homeomorphisms $B(x,y)=(x,y+1)$ and $C(x,y)=(x+1,-y)$ do not commute.

(b) It follows from (a) that $(f_*)^{-1}(\Theta(B)),(f_*)^{-1}(\Theta(C)) \in \pi_1(K,[(0,0)])$ do not commute.

(c) We consider the two loops β,γ in $(K,[(0,0)])$ that are shown in the above figure. Note that it follows easily from the definitions that $(f_*)^{-1}(\Theta(B)) = [\beta]$ and that $(f_*)^{-1}(\Theta(C)) = [\gamma]$.

(d) It follows from (b) and (c) that $[\beta]$ and $[\gamma]$ do not commute. ■

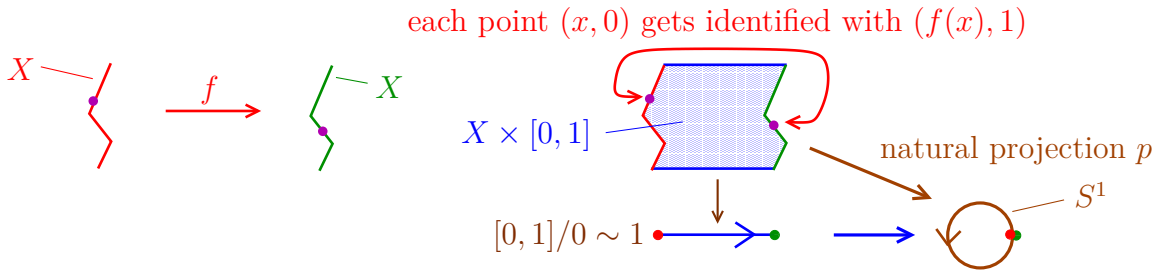
93.3. **The fundamental group of mapping tori.** We recall Definition 21.19:

Definition 93.7. Let X be a topological space and let $f\colon X\to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X,f) := (X\times[0,1])/\!\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X,f). We introduce two maps:

$$\underbrace{\begin{array}{rcl} X & \to & \mathrm{Tor}(X,f) \\ x & \mapsto & [(x,0)] \end{array}}_{\text{the natural injection}} \quad\text{and}\quad \underbrace{\begin{array}{rcl} \mathrm{Tor}(X,f) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi i t). \end{array}}_{\text{the natural projection}}$$



Examples.

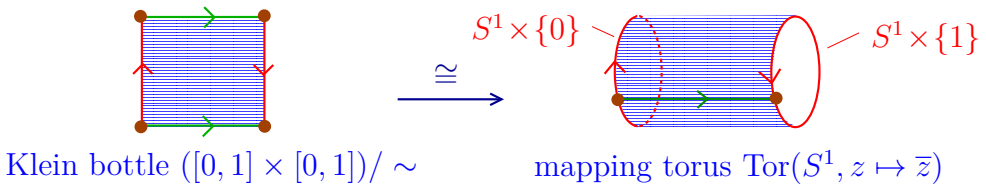
(1) Let X be a topological space. In the Mapping Torus-of-id-Lemma 21.22 we showed that the map

$$\begin{array}{rcl} \Phi\colon \mathrm{Tor}(X,\mathrm{id}) & \to & X\times S^1 \\ [(x,t)] & \mapsto & (x,\exp(2\pi i t)) \end{array}$$

is a homeomorphism.

(2) We consider the homeomorphism $f\colon S^1\to S^1$ that is given by $z\mapsto \overline{z}$. We also consider the Klein bottle $([0,1]\times[0,1])/\!\sim$, as defined on page 312.

[933] It follows from the Topological-Quotient Proposition 8.2 (1b) that the natural projection is indeed continuous.



On page 575 we showed that the map

$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & \mathrm{Tor}(S^1, z \mapsto \overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi \mathrm{i} x), y)] \end{array}$$

is a homeomorphism. □

Our goal in this section is to determine the fundamental groups of mapping tori $\mathrm{Tor}(X, f)$, where $f\colon X \to X$ is a homeomorphism. The key to doing so is to give an alternative description of such mapping tori. This is provided by the following lemma:

Lemma 93.8. (Mapping Torus-of-Homeomorphism Lemma) Let X be a topological space and let $f\colon X \to X$ be a *homeomorphism*. Then the following hold:

(1) The map[934]

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x,t)) & \mapsto & (f^n(x), t+n) \end{array}$$

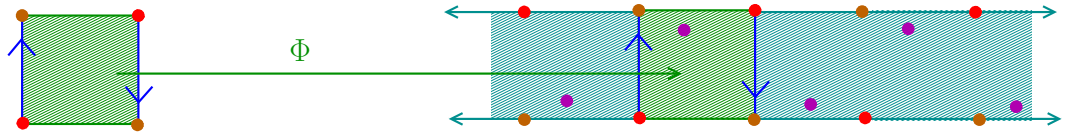
defines a continuous, proper and discrete action of the group $\mathbb{Z}$ on $X \times \mathbb{R}$.

(2) The map

$$\begin{array}{rcl} \mathrm{Tor}(X,f) = (X \times [0,1])/\sim & \to & (X \times \mathbb{R})/\mathbb{Z} \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$

is a homeomorphism.[935]

Example. We consider the case that $X = [-1,1]$ and $f(x) = -x$. The corresponding $\mathbb{Z}$-action on $[-1,1] \times \mathbb{R}$ is given by $n \cdot (x,t) := ((-1)^n \cdot x, t+n)$.



In the figure we attempt to illustrate the mapping torus $\mathrm{Tor}([-1,1], x \mapsto -x)$ and we illustrate $[-1,1] \times \mathbb{R}$ with the $\mathbb{Z}$-action. □

Proof.

---

[934]Here for $n \geq 1$ we write $f^n(x) = (\overbrace{f \circ \cdots \circ f}^{n\text{-times}})(x)$, for $n = 0$ we write $f^0(x) = x$ and for $n \leq -1$ we write $f^n(x) = (\underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{|n|\text{-times}})(x)$.

[935]It follows from the Mapping Torus-of-Homeomorphism Lemma 93.8 together with the Manifold–Quotient Proposition 25.39 that if X is an n-dimensional topological (respectively) smooth manifold and f is a homeomorphism (respectively diffeomorphism), then the corresponding mapping torus $\mathrm{Tor}(X,f) = (X \times \mathbb{R})/(\mathbb{Z}/f\mathbb{Z})$ is an $(n+1)$-dimensional topological (respectively smooth) manifold. This gives new proofs of the Mapping Torus-of-Manifold Lemma 25.36 and Manifold-Mapping Torus Lemma 59.11.

(1) This statement follows easily from the definitions. We leave it to the reader to work out the details.

(2) We need to show that the map

$$\begin{array}{rccc} \Phi\colon & \mathrm{Tor}(X,f) = (X\times[0,1])/\!\sim & \to & (X\times\mathbb{R})/\mathbb{Z} \\ & [(x,t)] & \mapsto & [(x,t)] \end{array}$$

is a homeomorphism. First note that it follows easily from the Quotient-to-Quotient Lemma 8.6 that the map is continuous. An elementary argument shows that the map is a bijection. By the Open-Injective Lemma 4.22 it remains to prove the following claim:

Claim. The map Φ is closed.

Proof. Let $A \subseteq \mathrm{Tor}(X,f)$ be a closed subset. In the proof of this claim we denote by $p\colon X\times[0,1] \to (X\times[0,1])/\!\sim$ and $q\colon X\times\mathbb{R} \to (X\times\mathbb{R})/\mathbb{Z}$ the projections. By the Topological-Quotient Proposition 8.2 (0b) we know that $p^{-1}(A) \subseteq X\times[0,1]$ is closed. An elementary argument shows that $q^{-1}(\Phi(A)) = \bigcup_{n\in\mathbb{Z}} f^n(p^{-1}(A))$. Next note that it follows easily from the Cover-Open Subset Lemma 3.20, applied to the locally finite closed cover $\{X\times[n,n+1]\}_{n\in\mathbb{Z}}$ of $X\times\mathbb{R}$, that $q^{-1}(\Phi(A))$ is a closed subset of $X\times\mathbb{R}$. It follows from the Topological-Quotient Proposition 8.2 (0b) that $\Phi(A) \subseteq (X\times\mathbb{R})/\mathbb{Z}$ is closed. ■

In the following discussion we adopt the following notation:

Notation 93.9. Given a topological space X and a homeomorphism $f\colon X \to X$ we denote the corresponding quotient space by $(X\times\mathbb{R})/\mathbb{Z}/f\mathbb{Z}$ and we will identify the mapping torus $\mathrm{Tor}(X,f)$ with $(X\times\mathbb{R})/(\mathbb{Z}/f\mathbb{Z})$ using the homeomorphism of the Mapping Torus-of-Homeomorphism Lemma 93.8.

The Mapping Torus-of-Homeomorphism Lemma 93.8 allows us to prove the following lemma, which we will also use on other occasions:

Lemma 93.10. (Mapping Torus Covering Lemma) Let X be a topological space and let $f\colon X \to X$ be a homeomorphism.

(1) The map

$$\begin{array}{ccc} X\times\mathbb{R} & \to & (X\times\mathbb{R})/(\mathbb{Z}/f\mathbb{Z}) = \mathrm{Tor}(X,f) \\ (x,t) & \mapsto & [(x,t)] \end{array}$$

is a covering of infinite degree.

(2) For every $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} \mathrm{Tor}(X,f^k) = (X\times\mathbb{R})/(\mathbb{Z}/f^k\mathbb{Z}) & \to & (X\times\mathbb{R})/(\mathbb{Z}/f\mathbb{Z}) = \mathrm{Tor}(X,f) \\ [(x,t)] & \mapsto & [(x,k\cdot t)] \end{array}$$

is a k-fold covering.

Proof. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism.

(1) Statement (1) is an immediate consequence of the Mapping Torus-of-Homeomorphism Lemma 93.8 and Proposition 91.11 (2).

(2) Let $k \in \mathbb{N}$. We consider the following commutative diagram of maps:

$$\begin{array}{ccc} (X \times \mathbb{R})/(\mathbb{Z}/f^k\mathbb{Z}) & \xrightarrow{[(x,t)]\mapsto[(x,k\cdot t)]} & (X \times \mathbb{R})/(\mathbb{Z}/f\mathbb{Z}). \\ & {\scriptstyle [(x,t)]\mapsto[(x,k\cdot t)]} \searrow \quad \nearrow {\scriptstyle [(x,t)]\mapsto[(x,t)]} & \\ & (X \times \mathbb{R})/(k \cdot \mathbb{Z})_f & \end{array}$$

The left diagonal map is easily seen to be well-defined and to be a homeomorphism. By Proposition 91.11 (2) the right diagonal map is a k-fold covering. It follows that the top horizontal map is also a k-fold covering. ■

Examples.

(1) We apply the Mapping Torus Covering Lemma 93.10 to $X = [0,1]$ and the homeomorphism that is given by $f(x) := 1 - x$. Note that for all x we have

$$f^2(x) \;=\; f(f(x)) \;=\; f(1-x) \;=\; 1-(1-x) \;=\; x.$$

This shows that $f^2 = \mathrm{id}$. Thus we recover the statement from page 1783 that the annulus $\mathrm{Tor}([0,1], \mathrm{id})$ is a 2-fold covering of the Möbius band $\mathrm{Tor}([0,1], f(x) = 1-x)$.

(2) We apply the Mapping Torus Covering Lemma 93.10 to $X = S^1$ and the self-homeomorphism of $X = S^1$ that is given by $f(z) := \overline{z}$. As in the previous example we see that $f^2 = \mathrm{id}$. Thus we get, after the discussion on page 1786, a new proof that the torus $\mathrm{Tor}(S^1, \mathrm{id})$ is a 2-fold covering of the Klein bottle $\mathrm{Tor}(S^1, f(z) = \overline{z})$. □

Now let us finally return to our main goal, namely the calculation of fundamental groups of mapping tori. For this we need to introduce a new concept from group theory:

Definition 93.11. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$.[936] We define the semidirect product of N and $\mathbb{Z}/k\mathbb{Z}$ with respect to φ as the group[937] $N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z}$ where the underlying set is given by the direct product $N \times \mathbb{Z}/k\mathbb{Z}$,[938] but where the group multiplication[939] is given by

$$(h,m) \cdot (h',m') \;:=\; (h \cdot \varphi^m(h'), m + m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}/k\mathbb{Z}$ instead of $N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z}$.

Examples.

(1) If N is a group and if $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.

(2) Let $N = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form the semidirect product $\mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$, which is called the n-th dihedral group D_n. It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to

[936] Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \mathrm{id}$.

[937] In Exercise 93.6 we will see that $N = \{(h,0) \mid h \in N\} \subseteq N \rtimes \mathbb{Z}/k\mathbb{Z}$ is a normal subgroup of $N \rtimes \mathbb{Z}/k\mathbb{Z}$. Put differently, we have $N \lhd N \rtimes \mathbb{Z}/k\mathbb{Z}$. This is the reason for the symbol "$\rtimes$" in the notation of the semidirect product.

[938] In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}/k\mathbb{Z}$ is given by $|N| \cdot k$.

[939] It is an elementary argument to show that $N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z}$ with the given multiplication is indeed a group.

the permutation group S_3 via the isomorphism [940]

$$\begin{array}{rcl}\Psi\colon \mathbb{Z}/3\mathbb{Z}\rtimes\mathbb{Z}/2\mathbb{Z} & \to & S_3\\ (h,m) & \mapsto & (1\ 2\ 3)^h\cdot(1\ 2)^m.\end{array}$$

□

The following lemma shows that many groups can be written as a semidirect product $N\rtimes\mathbb{Z}$.

Lemma 93.12. (Semidirect Product Lemma) Let G be a group, let $\xi\colon G\to\mathbb{Z}$ be an epimorphism and let $t\in G$ with $\xi(t)=1$. We write $N:=\ker(\xi)$ and we denote by φ the isomorphism

$$\begin{array}{rcl}\varphi\colon N & \to & N\\ h & \mapsto & t\cdot h\cdot t^{-1}.\end{array}$$

Then the map[941]

$$\begin{array}{rcl}\Theta\colon G & \to & N\rtimes_\varphi\mathbb{Z}\\ g & \mapsto & \big(g\cdot t^{-\xi(g)},\xi(g)\big)\end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Θ is injective and surjective. It remains to show that Θ is a homomorphism: Given $g,h\in G$ we have indeed

$$\begin{aligned}\Theta(g\cdot h) &= (g\cdot h\cdot t^{-\xi(g\cdot h)},\xi(g\cdot h)) = (g\cdot t^{-\xi(g)}\cdot\underbrace{t^{\xi(g)}\cdot h\cdot t^{-\xi(h)}}_{=\varphi^{\xi(g)}(h)}\cdot t^{-\xi(g)},\xi(g)+\xi(h))\\ &= (g\cdot t^{-\xi(g)},\xi(g))\cdot(h\cdot t^{-\xi(h)},\xi(h)) = \Theta(g)\cdot\Theta(h).\end{aligned}$$

■

The following proposition shows that the fundamental group of a mapping torus $\mathrm{Tor}(X,f)$ is isomorphic to some semidirect product $\pi_1(X)\rtimes\mathbb{Z}$.

Proposition 93.13. (Mapping Torus-π_1-Proposition) Let (X,x_0) be a pointed path-connected topological space, let $f\colon X\to X$ be a homeomorphism and let γ be a path from x_0 to $f(x_0)$. We denote by $\gamma_*\colon\pi_1(X,f(x_0))\to\pi_1(X,x_0)$ the isomorphism given by the Change-of-Base Point Proposition 88.14 and we define

$$\varphi:=\gamma_*\circ f_*\colon \pi_1(X,x_0)\xrightarrow[\cong]{f_*}\pi_1(X,f(x_0))\xrightarrow[\cong]{\gamma_*}\pi_1(X,x_0).$$

Let $i\colon X\to\mathrm{Tor}(X,f)$ be the natural inclusion and let $p\colon\mathrm{Tor}(X,f)\to S^1$ be the natural projection. There exists a natural isomorphism

$$\Theta\colon \pi_1(\mathrm{Tor}(X,f),[(x_0,0)])\xrightarrow{\cong}\pi_1(X,x_0)\rtimes_\varphi\mathbb{Z}$$

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[940] First note that a straightforward calculation shows that $(1\,2)^{-1}\cdot(1\,2\,3)\cdot(1\,2)=(1\,3\,2)=(1\,2\,3)^{-1}$. This implies that also for any $m\in\mathbb{Z}/2\mathbb{Z}$ and $s\in\mathbb{Z}$ we have $(1\,2)^{-m}\cdot(1\,2\,3)^s\cdot(1\,2)^m=(1\,2\,3)^{s(-1)^m}$. Now for any $(h,m),(h',m')\in\mathbb{Z}/n\mathbb{Z}\rtimes_\varphi\mathbb{Z}/2\mathbb{Z}$ we have

$$\begin{aligned}\Psi(h,m)\Psi(h',m') &= (1\,2\,3)^h\cdot(1\,2)^m\cdot(1\,2\,3)^{h'}\cdot(1\,2)^{m'} = (1\,2\,3)^h\cdot(1\,2)^m\cdot(1\,2)^{-m}\cdot(1\,2\,3)^{(-1)^m h'}\cdot(1\,2)^m\cdot(1\,2)^{m'}\\ &= (1\,2\,3)^{h+(-1)^m h'}\cdot(1\,2)^{m+m'} = \Psi(h+(-1)^m h',m+m') = \Psi((h,m)\cdot(h',m')).\end{aligned}$$

We have thus shown that Ψ is a homomorphism. It is now straightforward to verify that Ψ is in fact an isomorphism.

[941] Note that $\xi(g\cdot t^{-\xi(g)})=\xi(g)+\xi(t)(-\xi(g))=0$, i.e. $g\cdot t^{-\xi(g)}$ does indeed lie in $N=\ker(\xi)$.

such that the following diagram commutes: [942]

$$\begin{array}{ccccc} \pi_1(X,x_0) & \xrightarrow{i_*} & \pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) & \xrightarrow{p_*} & \pi_1(S^1,1) \\ \downarrow \mathrm{id} & & \cong \downarrow \Theta & & \downarrow \cong \\ \pi_1(X,x_0) & \xrightarrow{g\mapsto(g,0)} & \pi_1(X,x_0) \rtimes_\varphi \mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z}. \end{array}$$

(Here the vertical isomorphism to the right is the standard isomorphism from the Circle-π_1-Theorem 88.22.)

Proof. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. Recall that in the Mapping Torus-of-Homeomorphism Lemma 93.8 we showed that the following two statements hold:

(1) The map
$$\begin{array}{rcl} \mathbb{Z}\times(X\times\mathbb{R}) & \to & X\times\mathbb{R} \\ (n,(x,t)) & \mapsto & (f^n(x),t+n) \end{array}$$
defines a continuous, proper and discrete action of the group $\mathbb{Z}$ on $X\times\mathbb{R}$.

(2) The map
$$\begin{array}{rcl} \Psi\colon \mathrm{Tor}(X,f) & \to & (X\times\mathbb{R})/\mathbb{Z} \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$
is a homeomorphism.

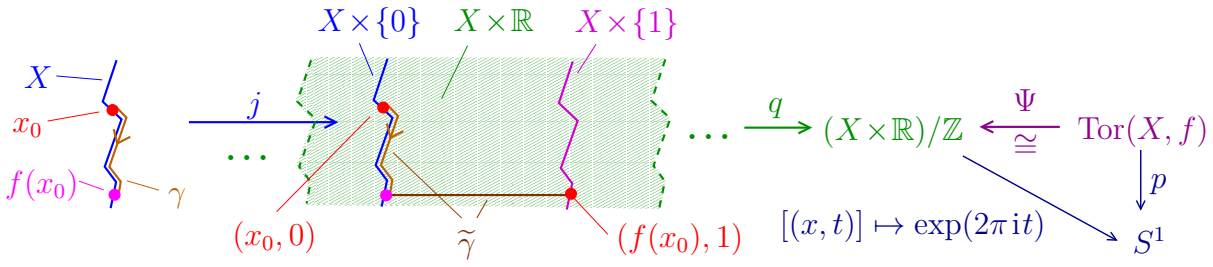
We introduce the following notation:

(a) We denote by $q\colon X\times\mathbb{R}\to(X\times\mathbb{R})/\mathbb{Z}$ the natural projection.

(b) We denote by $j\colon X\to X\times\mathbb{R}$ the inclusion that is given by $x\mapsto(x,0)$. Using the fact that $\{0\}$ is a deformation retract of $\mathbb{R}$ and using the Deformation Retract–Homotopy Equivalence Lemma 19.10 one can easily show that $j\colon X\to X\times\mathbb{R}$ is a homotopy equivalence.

(c) We denote by $\widetilde{\gamma}\colon[0,1]\to X\times\mathbb{R}$ the path from $(x_0,0)$ to $(f(x_0),1))$ that is given by the concatenation of the path $i\circ\gamma$ and the "horizontal" path $t\mapsto(x_0,t)$.

It follows from (1) and the Action–π_1-Theorem 93.1 (2) that the map
$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)])/q_*(\pi_1(X\times\mathbb{R},(x_0,0))) \\ n & \mapsto & [(q\circ\widetilde{\gamma})^n] \end{array}$$
is an isomorphism. This map gives rise to an epimorphism $\xi\colon\pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)])\to\mathbb{Z}$ with kernel $q_*(\pi_1(X\times\mathbb{R},(x_0,0)))$. Note that $\xi([q\circ\widetilde{\gamma}])=1$. From the Semidirect Product Lemma 93.12 we obtain an explicit isomorphism
$$\Theta\colon \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)]) \to q_*(\pi_1(X\times\mathbb{R},(x_0,0)))\rtimes\mathbb{Z}$$



[942] Note that both maps on the bottom are group homomorphisms.

In summary we get the following diagram:

$$\begin{array}{ccccc}
\pi_1(X,x_0) & \xrightarrow{i_*} & \pi_1(\mathrm{Tor}(X),f),[(x_0,0)]) & \xrightarrow{q_*} & \pi_1(S^1,1) \\
\cong\downarrow j_* & & \cong\downarrow \Psi_* & & \downarrow \mathrm{id} \\
\pi_1(X\times\mathbb{R},(x_0,0)) & \longrightarrow & \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)]) & \xrightarrow[[(x,t)]\mapsto \exp(2\pi i t)]{\text{induced by}} & \pi_1(S^1,1) \\
\downarrow \mathrm{id} & & \cong\downarrow \Theta & & \uparrow n\mapsto[t\mapsto\exp(2\pi i n)] \\
\pi_1(X\times\mathbb{R},(x_0,0)) & \xrightarrow{g\mapsto(g,0)} & q_*(\pi_1(X\times\mathbb{R},(x_0,0)))\rtimes\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z} \\
\cong\uparrow j_* & & \cong\uparrow \text{induced by } \psi_*{}^{945} & & \cong\downarrow \mathrm{id} \\
\pi_1(X,x_0) & \xrightarrow{g\mapsto(g,0)} & \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z}.
\end{array}$$

It follows fairly easily from the definitions that the diagram commutes.[947] Therefore the composition of the three middle vertical isomorphisms gives us the promised isomorphism $\pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) \to \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z}$. ■

We conclude this section with a discussion of several examples.

Example.

(1) Let $K = \mathrm{Tor}(S^1, f(z) \coloneqq \overline{z})$ be the Klein bottle. It follows from the discussion on page 1760 that the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the Mapping Torus-π_1-Proposition 93.13 that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z}\rtimes_\varphi\mathbb{Z}.$$

(2) Let X be a path-connected topological space. We calculate that

$$\pi_1(X\times S^1) \underset{\text{by the Mapping Torus-of-id Lemma 21.22}}{\overset{\uparrow}{\cong}} \pi_1(\mathrm{Tor}(X,\mathrm{id}_X)) \underset{\text{by the Mapping Torus-}\pi_1\text{-Proposition 93.13}}{\overset{\uparrow}{\cong}} \pi_1(X)\rtimes_{\mathrm{id}}\mathbb{Z} = \pi_1(X)\times\mathbb{Z} = \pi_1(X)\times\pi_1(S^1).$$

So in this special case we obtain the same result as in the Product-π_1-Proposition 90.13.

(3) Let $A\in \mathrm{GL}(n,\mathbb{Z})$. In Exercise 93.8 we will prove the following two statements:
- The map $f(A)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ given by $[v]\mapsto[Av]$ is a homeomorphism.
- We use the Action–π_1-Theorem 93.1 to make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) = \mathbb{Z}^n$. We will see that the induced map $f(A)_*$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) = \mathbb{Z}^n$ is given by multiplication by A.

It follows from these two facts together with the Mapping Torus-π_1-Proposition 93.13 that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n\rtimes_A\mathbb{Z}$. □

93.4. Exercises for Chapter 93.

Exercise 93.1. Let $X = \{A,B,C,D\}$ be the pseudocircle from page 172.

(a) Use the Action–π_1-Theorem 93.1 to show that the fundamental group of the pseudocircle X is isomorphic to $\mathbb{Z}$.

[946] Note that it follows from the Covering-π_1-Monomorphism Proposition 92.7 that the homomorphism $\psi_*\colon \pi_1(X,x_0)\to q_*(\pi_1(X\times\mathbb{R},(x_0,0)))$ is in fact an isomorphism.

[947] Perhaps the most opaque statement is that the two semidirect products match up. But this also follows once one writes down the definitions in detail.

(b) Let $f\colon S^1 \to X$ be the map that we introduced on page 172. Show that the induced map $f_*\colon \pi_1(S^1, 1) \to \pi_1(X, A)$ is an isomorphism.

Exercise 93.2. Let X be a topological space and let G be a group that acts continuously, properly and freely on X.

(a) Show that the natural projection $X \to X/G$ is a local homeomorphism.

(b) We denote by $\sim$ the equivalence relation on the topological space $X \times [0,1]$ that is generated by $(x,t) \sim (g \cdot x, t)$ for $g \in G$ and $(x,t) \in X \times [0,1]$. Show that the map

$$\begin{array}{rcl} (X \times [0,1])/\sim & \to & X/G \times [0,1] \\ [(x,t)] & \mapsto & ([x],t) \end{array}$$

is a homeomorphism.

Remark. This is a special case of the Product-Quotient Mixing Lemma 17.8, which relies on the tricky Whitehead Quotient Theorem 17.6. But in our setting, using (a), one can give an elementary proof.

(c) Suppose that $F\colon X \times [0,1] \to X$ is a homotopy that is G-equivariant in the sense that for any $(x,t) \in X \times [0,1]$ and any $g \in G$ we have $F(g \cdot x, t) = g \cdot F(x,t)$. Show that the map

$$\begin{array}{rcl} X/G \times [0,1] & \to & X/G \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Remark. This is a special case of the Homotopy Quotient Theorem 18.11. But in our setting one can deduce it from (b).

Exercise 93.3. Let X be a path-connected topological space and let $p\colon X \to B$ be a covering. Does there exist a group G that acts continuously and discretely on X and a homeomorphism $\Theta\colon B \to X/G$ such that $\Theta \circ p\colon X \to X/G$ is the natural projection?

Exercise 93.4. Let $p\colon \mathbb{R} \to S^1$ be the usual covering map that is given by $t \mapsto \exp(2\pi \mathrm{i} t)$. Let $\epsilon \in \{-1, 1\}$ We say that a map $f\colon \mathbb{R} \to \mathbb{R}$ is ϵ-equivariant if for every $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have $f(n+t) = \epsilon \cdot n + f(t)$.

(a) Show that any self-homeomorphism of S^1 lifts to a self-homeomorphism of $\mathbb{R}$ that is ϵ-equivariant for some $\epsilon \in \{-1, 1\}$.

(b) Show that any self-homeomorphism of $\mathbb{R}$ is either monotonously increasing or monotonously decreasing.

(c) Let $g\colon \mathbb{R} \to \mathbb{R}$ be an ϵ-equivariant self-homeomorphism. Show that there exists an isotopy

$$G\colon \mathbb{R} \times [0,1] \to \mathbb{R}$$

from g to the map that is given by $x \mapsto \epsilon \cdot x$ and such that each G_t is ϵ-equivariant.

(d) Show that any self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.

Hint. Use Exercise 93.2.

Remark. This gives a new proof of Exercise 18.9 (b).

Exercise 93.5.

(a) An *affine linear automorphism of* $\mathbb{R}^n$ is a map of the form

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R}^n \\ v & \mapsto & A \cdot v + w \end{array}$$

where $A \in \mathrm{GL}(n,\mathbb{R})$ and $w \in \mathbb{R}^n$. Let G be a finite group that acts via affine linear automorphisms on $\mathbb{R}^n$. Show that there exists an $x \in \mathbb{R}^n$ with $g \cdot x = x$ for all $g \in G$.
Hint. Pick any $p \in \mathbb{R}^n$ and construct a new point out of p, using the G action.

(b) Let G be a group that acts freely and via affine linear automorphisms on $\mathbb{R}^n$. Show that G is torsion-free, i.e. show that every non-trivial $g \in G$ has infinite order.

(c) Use (b) to show that the fundamental group of the Klein bottle is torsion-free.

Exercise 93.6. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$. As in Definition 93.11 we define the *semidirect product of N and $\mathbb{Z}/k\mathbb{Z}$ with respect to φ* as the group $N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z}$ where the underlying set is given by the direct product $N \times \mathbb{Z}/k\mathbb{Z}$, and where the multiplication is given by

$$(h,m) \cdot (h',m') \coloneqq (h \cdot \varphi^m(h'), m + m').$$

(a) Show that $N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z}$ is indeed a group.

(b) Show that $N = \{(h,0) \in N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z} \,|\, h \in N\}$ is a normal subgroup of $N \rtimes \mathbb{Z}/k\mathbb{Z}$.

(c) Is $\mathbb{Z}/k\mathbb{Z} = \{(0,n) \in N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z} \,|\, n \in \mathbb{Z}/k\mathbb{Z}\} \subseteq N \rtimes_\varphi \mathbb{Z}/k\mathbb{Z}$ necessarily a normal subgroup?

(d) Suppose that $k = 0$. Let $n \in \mathbb{N}$ and let K be a subgroup of N of some finite index m. Show that if K is preserved by φ, then $K \rtimes n\mathbb{Z}$ is a subgroup of $N \rtimes \mathbb{Z}$ of index $m \cdot n$.

Exercise 93.7. We consider the torus $S^1 \times S^1$. In the Torus-π_1-Proposition 90.14 and on page 1806 we gave an explicit isomorphism $\Phi\colon \mathbb{Z}^2 \to \pi_1(S^1 \times S^1, (1,1))$. For which $(m,n) \in \mathbb{Z}^2$ can we find an embedded circle in $S^1 \times S^1$ that represents $\Phi(m,n)$?
Remark. We might lack the tools to give a complete answer, but you can still strive to find the maximal set of (m,n) for which $\Phi(m,n)$ can be represented by an embedded circle.

Exercise 93.8. Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix.

(a) Show that the map

$$\begin{array}{rccc} f(A)\colon & \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ & v & \mapsto & A \cdot v \end{array}$$

is continuous and well-defined.

(b) We identify $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0)$ with $\mathbb{Z}^n$ as in the Action–π_1-Theorem 93.1. Show that the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) & \xrightarrow{f(A)_*} & \pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle =} \\ \mathbb{Z}^n & \xrightarrow{v \mapsto A \cdot v} & \mathbb{Z}^n \end{array}$$

commutes.

(c) Show that $f(A)$ is a homeomorphism if and only if $A \in \mathrm{GL}(n,\mathbb{Z})$.

Exercise 93.9. Does there exist a covering map $p\colon K \to T$ from the Klein bottle to the 2-dimensional torus?

Exercise 93.10.

(a) Let (X,x_0) be a pointed topological space and let $b \in \pi_1(X,x_0)$. Does there necessarily exist a continuous map $f\colon (S^1,1) \to (X,x_0)$ with $b \in f_*(\pi_1(S^1,1))$?

(b) Let (X,x_0) be a pointed topological space and let $b,c \in \pi_1(X,x_0)$. Does there necessarily exist a continuous map $f\colon (S^1 \times S^1,(1,1)) \to (X,x_0)$ such that we have $b,c \in f_*(\pi_1(S^1 \times S^1,(1,1)))$?

Exercise 93.11. Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\varphi} & \pi & \xrightarrow{\psi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & \widetilde{K} & \xrightarrow{\widetilde{\varphi}} & \widetilde{\pi} & \xrightarrow{\widetilde{\psi}} & \widetilde{G} & \longrightarrow & 1 \end{array}$$

be a commutative diagram of group homomorphisms with the following properties:

(1) φ and $\widetilde{\varphi}$ are monomorphisms.
(2) ψ and $\widetilde{\psi}$ are epimorphisms.
(3) $\operatorname{im}(\varphi) = \ker(\psi)$ and $\operatorname{im}(\widetilde{\varphi}) = \ker(\widetilde{\psi})$.

Show that if two out of the three vertical maps α, β and γ are isomorphisms, then so is the third one.

Exercise 93.12. Let $n \in \mathbb{N}$. We set $\xi := \exp(2\pi \mathrm{i}/n)$. We consider the following two diffeomorphisms of S^3:

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (w,z) \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (\xi \cdot z, \xi^{-1} \cdot w). \end{array}$$

We denote by G the subgroup of the diffeomorphisms of S^3 that is generated by f and g, i.e. that is given by all compositions of f, g, f^{-1}, g^{-1} of finite length.

(a) Show that G is non-abelian.
(b) Determine the isomorphism type of the group G.
(c) Show that S^3/G is a closed orientable 3-dimensional smooth manifold.

Exercise 93.13. Let $p \in \mathbb{N}$.

(a) Let $A \in \mathrm{O}(4)$ be a matrix of order p such that the resulting action

$$\begin{array}{rcl} \mathbb{Z}/p\mathbb{Z} \times S^3 & \to & S^3 \\ ([k], v) & \mapsto & A^k \cdot v \end{array}$$

is discrete. Show that there exist $q_1, q_2 \in \mathbb{Z}$, both coprime to p, such that A is conjugate to the matrix

$$\begin{pmatrix} \exp(2\pi \mathrm{i} q_1/p) & 0 \\ 0 & \exp(2\pi \mathrm{i} q_2/p) \end{pmatrix}$$

in $\mathrm{U}(2) \subseteq \mathrm{SO}(4)$.

(b) We introduce two concepts:

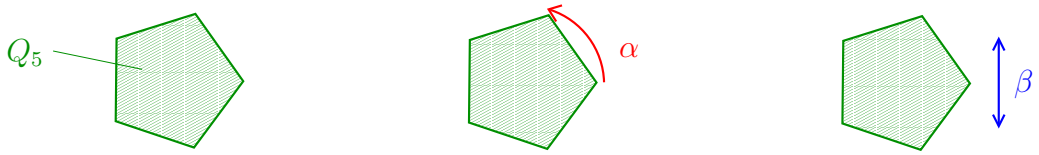
- We say that two actions of a group G on a topological space X are *equivalent* if there exists an isomorphism $\Phi\colon G \to G$ and a homeomorphism $\Psi\colon X \to X$ such that for all $g \in G$ and $x \in X$ we have $g \cdot_1 \Psi(x) = \Psi(\Phi(g) \cdot_2 x)$.
- We say that an action of a group G on S^3 is *isometric*, if for each $g \in G$ the action is given by multiplication by an orthogonal matrix. In other words, the action is given by a group homomorphism $G \to \mathrm{O}(4)$.

Show that any discrete action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 is equivalent to one of the actions from the Lens Space-Discrete Action Lemma 91.3.

Exercise 93.14. Let $n \geq 3$. Show that there exists a closed orientable connected non-empty n-dimensional with non-abelian fundamental group.

Exercise 93.15. Let G be a finite group. Show that there exists a finite connected topological graph X (in the sense of Definition 11.9) such that G acts freely and continuously on X.
Remark. You could start with $G = \mathbb{Z}/n\mathbb{Z}$.

Exercise 93.16. Let $n \in \mathbb{N}_{\geq 3}$. Let Q_n be the regular n-gon in $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ that is given by the convex hull of the points $\exp(2\pi \mathrm{i} k/n)$, $k = 1, \ldots, n$. Let $\alpha\colon Q_n \to Q^n$ be the counterclockwise rotation by the angle $2\pi/n$ and let β be the reflection of Q_n in the x-axis. Let G be the group of self-homeomorphisms of Q_n generated by α and β. What is the isomorphism type of this group?



Exercise 93.17. Let $k \in \mathbb{N}_0$, let G be a group and let $\varphi\colon G \to \mathbb{Z}/k\mathbb{Z}$ be an epimorphism. If $k = 0$, then we saw in the Semidirect Product Lemma 93.12 that G is isomorphic to a semidirect product $N \rtimes \mathbb{Z}/k\mathbb{Z}$. Does the statement also hold for $k \neq 0$?

Exercise 93.18. Let G be a group and let $\varphi\colon G \to G$ be an isomorphism. We consider the corresponding semidirect product $G \rtimes_\varphi \mathbb{Z}$. Is it true that $G \rtimes_\varphi \mathbb{Z}$ is abelian if and only if $G = \mathrm{id}$?

Exercise 93.19. Let G and π be groups and let $\varphi\colon \pi \to \mathrm{Aut}(G)$ be a homomorphism. We consider the set $G \times \pi$ with the multiplication

$$(g, a) \cdot (h, b) \;:=\; (g \cdot \varphi(a)(h), a \cdot b).$$

(a) Show that $G \times \pi$ with the above multiplication is a group. This group is denoted by $G \rtimes_\varphi \pi$ and it is called the *semidirect product of G and π with respect to φ*.
(b) In what sense does this definition of the semidirect product of two groups generalize the concept introduced in Definition 93.11?
(c) Let K be a group. Furthermore let $\alpha\colon K \to \pi$ be an epimorphism onto a group π. We suppose that α admits a right inverse, i.e. we suppose that there exists a homomorphism $\varphi\colon \pi \to K$ with $\alpha \circ \varphi = \mathrm{id}_\pi$. Show that K is isomorphic to a semidirect product of the form $\ker(\alpha) \rtimes \pi$.
(d) (i) Let $n \in \mathbb{N}$. Show that $\mathrm{GL}(n, \mathbb{R})$ is isomorphic to a semidirect product of the form $\mathrm{SL}(n, \mathbb{R}) \rtimes (\mathbb{R} \setminus \{0\})$.
(ii) For which $n \in \mathbb{N}$ is $\mathrm{GL}(n, \mathbb{R})$ isomorphic to a direct product $\mathrm{SL}(n, \mathbb{R}) \rtimes (\mathbb{R} \setminus \{0\})$?
(iii) Repeat (i) and (ii) with $\mathbb{C}$ instead of $\mathbb{R}$.

Exercise 93.20. Let $K = ([0,1] \times [0,1])/\sim$ be the Klein bottle. Let G be the group that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x, y+1) \end{array}$$

and their inverses. In the Klein Bottle–π_1-Proposition 93.6 we saw that $\pi_1(K)$ is isomorphic to G and on page 1818 we saw that $\pi_1(K)$ is isomorphic to the semidirect product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}$

where $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ is given by multiplication by -1. Provide an explicit isomorphism between G and $\mathbb{Z} \rtimes_\varphi \mathbb{Z}$.

Exercise 93.21. Let A be an abelian group and let $\varphi\colon A \to A$ be an isomorphism. Show that if A is torsion-free, then the semidirect product $A \rtimes_\varphi \mathbb{Z}$ is also torsion-free.
Remark. It follows from this exercise and the discussion on page 1818 that the fundamental group of the Klein bottle is torsion-free.

Exercise 93.22. Use the Mapping Torus-π_1-Proposition 93.13 to give another calculation of the fundamental group of the Möbius band.

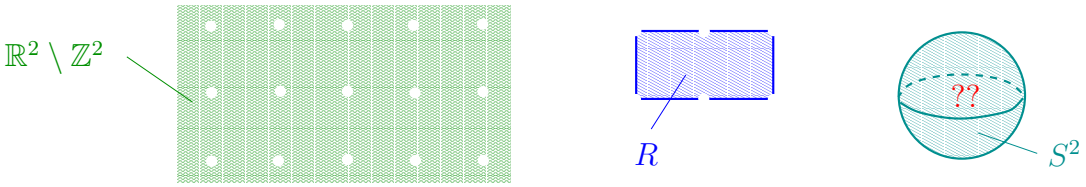
Exercise 93.23. We consider the self-map $\varphi\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ of the torus $\mathbb{R}^2/\mathbb{Z}^2$ that is given by $\varphi([(x,y)]) = [(x+2y, y)]$.
(a) Show that φ is a homeomorphism.
(b) Determine the induced map $\varphi_*\colon \pi_1(\mathbb{R}^2/\mathbb{Z}^2, [(0,0)]) \to \pi_1(\mathbb{R}^2/\mathbb{Z}^2, [(0,0)])$.
We consider the mapping torus $\mathrm{Tor}(\mathbb{R}^2/\mathbb{Z}^2, \varphi)$ and we consider the self-map that is given by $f([(x,y,t)]) = [(-x-y+t, -y, t)]$.
(c) Show that f is not homotopic to the identity.
(d) Show that $f^2 = f \circ f$ is isotopic to the identity.

Exercise 93.24. An *m-component link* is a submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1. For which m can you find a link $L \subseteq S^3$ such that $\pi_1(S^3 \setminus L)$ is abelian?
Remark. You do not have the tools to provide a complete answer but you can try to find as many m as possible.

Exercise 93.25. Let G be the subgroup of homeomorphisms of $\mathbb{R}^2 \setminus \mathbb{Z}^2$ that is generated by the maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 \setminus \mathbb{Z}^2 & \to & \mathbb{R}^2 \setminus \mathbb{Z}^2 \\ (x,y) & \mapsto & (x+2, y) \end{array} \qquad \begin{array}{rcl} \beta\colon \mathbb{R}^2 \setminus \mathbb{Z}^2 & \to & \mathbb{R}^2 \setminus \mathbb{Z}^2 \\ (x,y) & \mapsto & (x, y+2) \end{array} \qquad \begin{array}{rcl} \gamma\colon \mathbb{R}^2 \setminus \mathbb{Z}^2 & \to & \mathbb{R}^2 \setminus \mathbb{Z}^2 \\ (x,y) & \mapsto & (-x, -y). \end{array}$$

(a) Show that G acts discretely and continuously on $\mathbb{R}^2 \setminus \mathbb{Z}^2$.
(b) We set $R := ([0,2] \times [0,1]) \setminus \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)\}$.
  (i) Find an equivalence relation on R such that the quotient space $(\mathbb{R}^2 \setminus \mathbb{Z}^2)/G$ is homeomorphic to $R/\!\sim$.
  (ii) The quotient $R/\!\sim$ is homeomorphic to a subspace of S^2. Which one?



CHAPTER 94

Applied topology

In the Projective Space-Avatar Lemma 9.16 (3) we showed that SO(3) is homeomorphic to $\mathbb{R}\mathrm{P}^3$ and in the SO(3)–π_1-Proposition 93.4 we showed that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$. In this chapter we first give an alternative proof of both statements. We will then draw several consequences of the calculation of $\pi_1(\mathrm{SO}(3))$. In particular we will discuss some "real life" applications of our results, in particular we will discuss the incomparable "belt trick" and the "plate trick" and we discuss some applications of topology to robotics.

This chapter is organized as follows:

- In Section 94.1 we will discuss the matrix groups SO(3) and SU(2). In particular we will show that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
- In Section 94.2 we will discuss the "belt trick" which ultimately relies on the fact that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
- In Section 94.3 we will present the "plate trick" which is a variation on the "belt trick" and also rests on the fact that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
- In Section 94.4 we will discuss an application of fundamental groups to topological robotics.
- In Section 94.5 we will discuss linkages.
- In Section 94.6 we will provide exercises for Chapter 94.

In the next Chapter 95 we will discuss coverings of smooth and topological manifolds.

94.1. **The matrix groups SO(3) and SU(2).** We recall that the Matrix Group-Smooth Manifold Proposition 32.10 implies in particular that we can view SO(3) and SU(2) as closed 3-dimensional smooth manifolds. In this section we will see that the smooth manifolds SO(3) and SU(2) are diffeomorphic to smooth manifolds we are already very familiar with. We start out our discussion of SO(3) and SU(2) with some definitions.

Definition 94.1. We consider the set of matrices[948 949]

$$T \coloneqq \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SU}(2) = \left\{B \in \mathrm{SL}(2,\mathbb{C}) \,\middle|\, \mathrm{tr}(B) = 0 \text{ and } B = -\overline{B}^{\mathrm{T}}\right\}.$$

Note that T is a 3-dimensional real vector space. We equip it with the ordered basis given by the so-called "Pauli matrices"[950]

$$E_1 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

and we equip T with the inner product given by

$$\begin{array}{rcl} T \times T & \to & \mathbb{R} \\ (A,B) & \mapsto & g(A,B) \coloneqq \frac{1}{2}\,\mathrm{tr}\left(A \cdot \overline{B}^{\mathrm{T}}\right). \end{array}$$

We consider the isomorphism

$$\begin{array}{rcl} \Phi\colon \mathbb{R}^3 & \to & T \\ e_i & \mapsto & E_i. \end{array}$$

It follows from a straightforward calculation that Φ defines an isometry from $(\mathbb{R}^3, \langle\,,\,\rangle)$ to (T, g).[951] Next that it follows from this observation that Φ induces an isomorphism $\mathrm{SO}(3) \to \mathrm{SIsom}(T, g)$, where $\mathrm{SIsom}(T, g)$ denotes the group of isometries of (T, g) of determinant $+1$. We use this isomorphism to identify these two groups.

Lemma 94.2. ($\mathbf{SU}(2)$-Action Lemma) The map

$$\begin{array}{rccc} q\colon & \mathrm{SU}(2) & \to & \mathrm{SIsom}(T,g) = \mathrm{SO}(3) \\ & A & \mapsto & \begin{pmatrix} T & \to & T \\ B & \mapsto & A \cdot B \cdot A^{-1}) \end{pmatrix} \end{array}$$

has the following properties:

(1) The map is well-defined in the sense that it takes values in $\mathrm{SIsom}(T, g)$.
(2) The map is a group homomorphism.
(3) For $A, A' \in \mathrm{SU}(2)$ we have $\quad q(A) = q(A') \iff A = \pm A'$.
(4) The map is a local diffeomorphism.

Example. Given $t \in \mathbb{R}$ we consider the matrix

$$A = \begin{pmatrix} \exp(-\mathrm{i}t) & 0 \\ 0 & \exp(\mathrm{i}t) \end{pmatrix} \in \mathrm{SU}\,(2).$$

We want to determine $q(A) \in \mathrm{SO}(3)$. This means that we have to express the linear map $T \to T$ that is given by $B \mapsto A \cdot B \cdot A^{-1}$ with respect to the ordered basis (E_1, E_2, E_3). A straightforward calculation shows that

$$\begin{aligned} A \cdot E_1 \cdot A^{-1} &= E_1, \\ A \cdot E_2 \cdot A^{-1} &= \begin{pmatrix} 0 & \exp(-2\mathrm{i}t) \\ \exp(2\mathrm{i}t) & 0 \end{pmatrix} = \cos(2t) \cdot E_2 + \sin(2t) \cdot E_3, \\ & \qquad \uparrow \text{ since } \exp(\mathrm{i}s) = \cos(s) + \mathrm{i}\sin(s) \\ A \cdot E_3 \cdot A^{-1} &= \begin{pmatrix} 0 & -\mathrm{i} \cdot \exp(-2\mathrm{i}t) \\ \mathrm{i} \cdot \exp(2\mathrm{i}t) & 0 \end{pmatrix} = -\sin(2t) \cdot E_2 + \cos(2t) \cdot E_3. \end{aligned}$$

But this implies that the homomorphism $T \to T$ given by conjugation with A, with respect to the ordered basis (E_1, E_2, E_3) is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix} \quad \text{put differently, we have} \quad q\begin{pmatrix} \exp(-\mathrm{i}t) & 0 \\ 0 & \exp(\mathrm{i}t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}.$$

[948] Recall that the trace $\mathrm{tr}(A)$ of a matrix is defined as the sum of all the diagonal entries.

[948] In the literature, see e.g. [**Nab2011**, Appendix], one often also works with the vector space

$$\widetilde{T} := \{B \in \mathrm{SL}(2,\mathbb{C}) \,|\, \mathrm{tr}(B) = 0 \text{ and } B = \overline{B}^{\mathrm{T}}\}.$$

The map $A \mapsto \mathrm{i} \cdot A$ defines an isomorphism $T \to \widetilde{T}$ and the discussion of T that we give can also be done, with minor modifications, with $\widetilde{T}$ instead.

[949] Wolfgang Pauli (1900–1958) was Swiss-American theoretical physicist and one of the pioneers of quantum physics. These matrices were also independently introduced by Paul Dirac. Wikipedia writes:
"In 1928, building on 2×2-spin matrices which he [Paul Dirac] discovered independently of Wolfgang Pauli's work on non-relativistic spin systems, [...] he proposed the Dirac equation as a relativistic equation of motion for the wave function of the electron. This work led Dirac to predict the existence of the positron, the electron's antiparticle [...]."

[950] In the calculation we only have to show that $g(E_i, E_j) = \frac{1}{2}\,\mathrm{tr}(E_i \cdot \overline{E}_j^{\mathrm{T}}) = \delta_{ij} = \langle e_i, e_j\rangle$ for all choices of $i, j \in \{1, 2, 3\}$. But this can be verified easily.

□

Proof of the SU(2)-Action Lemma 94.2.

(2) It follows easily from the definition that q is a group homomorphism.

(1) We have to show that for $A \in \mathrm{SU}(2)$ and $B \in T$ we also have $A \cdot B \cdot A^{-1} \in T$. This is indeed the case since

$$\mathrm{tr}(A{\cdot}B{\cdot}A^{-1}) \,=\, \mathrm{tr}(B) \,=\, 0 \quad \text{and} \quad \big(\overline{A{\cdot}B{\cdot}A^{-1}}\big)^{\mathrm{T}} \,=\, (\overline{A}^{-1})^{\mathrm{T}}{\cdot}\overline{B}^{\mathrm{T}}{\cdot}\overline{A}^{\mathrm{T}} \underset{\substack{\uparrow\\ \text{follows the observation}\\ \text{that } B=\overline{B}^{\mathrm{T}} \text{ and } A^{-1}=\overline{A}^{\mathrm{T}}}}{=} A{\cdot}B{\cdot}A^{-1}.$$

We also have to show that for $A \in \mathrm{SU}(2)$ we get indeed an isometry of (T, g). So let $B, C \in T$. Then we have indeed

$$\begin{aligned} g(q(A)(B), q(A)(C)) \,&=\, \tfrac{1}{2}\,\mathrm{tr}\left(A{\cdot}B{\cdot}A^{-1}{\cdot}\overline{A{\cdot}C{\cdot}A^{-1}}^{\mathrm{T}}\right) \,=\, \tfrac{1}{2}\,\mathrm{tr}\left(A{\cdot}B\overbrace{\cdot A^{-1}{\cdot}(\overline{A}^{-1})^{\mathrm{T}}}^{=\mathrm{id}}{\cdot}\overline{C}^{\mathrm{T}}{\cdot}\overline{A}^{\mathrm{T}}\right) \\ &=\, \tfrac{1}{2}\,\mathrm{tr}\left(A{\cdot}B{\cdot}\overline{C}^{\mathrm{T}}{\cdot}\overline{A}^{\mathrm{T}}\right) \underset{\substack{\uparrow\\ \text{since } \mathrm{tr}(A\cdot P)=\mathrm{tr}(P\cdot A)}}{=} \tfrac{1}{2}\,\mathrm{tr}\left(B{\cdot}\overline{C}^{\mathrm{T}}{\cdot}\underbrace{\overline{A}^{\mathrm{T}}{\cdot}A}_{=\mathrm{id}}\right) \,=\, \tfrac{1}{2}\,\mathrm{tr}\left(B{\cdot}\overline{C}^{\mathrm{T}}\right). \end{aligned}$$

This shows that q defines a group homomorphism $\mathrm{SU}(2) \to \mathrm{O}(3)$. By the Matrix Group-$\pi_0$-Proposition 4.54 we know that $\mathrm{SU}(2)$ is path-connected. The image $q(\mathrm{SU}(2))$ is a path-connected subset of $\mathrm{O}(3)$ that contains $q(\mathrm{id}) = \mathrm{id}$. It follows again from the Matrix Group-π_0-Proposition 4.54 that $q(\mathrm{SU}(2))$ is contained in $\mathrm{SO}(3)$.

(3) Let $A, A' \in \mathrm{SU}(2)$. If $A = -A'$ then we clearly have $q(A) = q(A')$. Now suppose that conversely we have $q(A) = q(A')$. We then have in particular $q(A)(E_i) = q(A')(E_i)$ for $i = 1, 2, 3$, i.e. we have $A \cdot E_i \cdot A^{-1} = A' \cdot E_i \cdot (A')^{-1}$ for $i = 1, 2, 3$. These equalities give us equations relating the four entries of the matrix A to the four entries of the matrix A'. An elementary calculation shows that the only solutions are $A = \pm A'$. We leave the details to the reader.

(4) It is convenient to "enlarge" the map $q\colon \mathrm{SU}(2) \to \mathrm{SIsom}(T, g)$ as follows. More precisely, we consider the map[951]

$$\begin{aligned} \widetilde{q}\colon V \coloneqq \Big\{ \underbrace{\begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix}}_{=:M(a,b,c,d)} \Big| \, a,b,c,d \in \mathbb{R} \Big\} \;&\to\; \mathrm{End}(T) \underset{\substack{\downarrow\\ \text{via the basis } E_1,E_2,E_3 \text{ of } T}}{=} \mathrm{M}(3\times 3, \mathbb{R}) \\ A \;&\mapsto\; \begin{pmatrix} T & \to & T \\ B & \mapsto & A\cdot B\cdot \overline{A}^{\mathrm{T}} \end{pmatrix}. \end{aligned}$$

Recall that for $A \in \mathrm{SU}(2)$ we have $A^{-1} = \overline{A}^{\mathrm{T}}$. Thus we see that the restriction of the map $\widetilde{q}\colon V \to \mathrm{End}(T)$ to $\mathrm{SU}(2)$ equals our previous map. By [**Nab2011**, Appendix] an elementary, but slightly painful calculation shows that

$$\widetilde{q}\begin{pmatrix} a+b\mathrm{i} & c+d\mathrm{i} \\ -c+d\mathrm{i} & a-b\mathrm{i} \end{pmatrix} \,=\, \begin{pmatrix} a^2-b^2-c^2+d^2 & 2ab+2cd & -2ac+2bd \\ -2ab+2cd & a^2-b^2+c^2-d^2 & 2ad+2bc \\ 2ac-2bd & 2bc-2ad & a^2+b^2-c^2-d^2 \end{pmatrix}.$$

[951] A straightforward calculation shows that for $A \in V$ we have $A \cdot \overline{A}^{\mathrm{T}} = \lambda \cdot \mathrm{id}_2$ for some $\lambda \in \mathbb{R}$. Using this calculation one can easily show that $\widetilde{q}(A)$ does indeed define an endomorphism of T.

This shows that the above map $\widetilde{q}\colon V \to \mathrm{End}(T)$ is smooth. It follows from the Submanifold Smooth Structure Lemma 25.23 that our original map $q\colon \mathrm{SO}(2) \to \mathrm{SO}(3)$ is also smooth.

It remains to show that $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a local diffeomorphism. By the Inverse Mapping Theorem 30.1 it suffices to show that for every $A \in \mathrm{SU}(2)$ the differential $\mathrm{D}q_A\colon \widetilde{\mathrm{T}}_A\mathrm{SU}(2) \to \widetilde{\mathrm{T}}_{q(A)}\mathrm{SO}(3)$ is an isomorphism. We start out with the point $A = \mathrm{id}$.

Claim 1. The differential $\mathrm{D}q_{\mathrm{id}}\colon \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SU}(2) \to \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SO}(3)$ is an isomorphism.

Proof. In the Matrix Group-Smooth Manifold Proposition 32.10 we showed that

$$\begin{aligned}\widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SU}(2) &= \{B \in \mathrm{M}(2\times 2,\mathbb{C}) \,|\, B = -\overline{B}^{\mathrm{T}} \text{ and } \mathrm{tr}(B) = 0\}\\ \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SO}(3) &= \{B \in \mathrm{M}(3\times 3,\mathbb{R}) \,|\, B = -B^{\mathrm{T}} \text{ and } \mathrm{tr}(B) = 0\}.\end{aligned}$$

Note that both are 3-dimensional real vector spaces. Thus it suffices to show that $\mathrm{D}q_{\mathrm{id}}$ is a monomorphism. Since $q = \widetilde{q}$ on $\mathrm{SU}(2)$ it suffices to show that the restriction of $\mathrm{D}\widetilde{q}_{\mathrm{id}}$ to $\widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SU}(2)$ is a monomorphism. We calculate $\mathrm{D}\widetilde{q}_{\mathrm{id}}\colon \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SU}(2) \to \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SO}(3)$ for the basis vectors

$$v_1 = \begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix},\quad v_2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \text{ and } \quad v_3 = \begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$$

of $\widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SU}(2)$. We see that

$$\begin{aligned}\mathrm{D}\widetilde{q}_{\mathrm{id}}(v_1) &= \mathrm{D}\widetilde{q}_{\mathrm{id}}(\underbrace{\tfrac{d}{dt}M(1,t,0,0)|_{t=0}}_{=v_1}) \overset{\text{definition of the differential}}{=} \tfrac{d}{dt}q(M(1,t,0,0))|_{t=0} \overset{\text{by the above calculation of } \widetilde{q}}{=} \begin{pmatrix}0 & 2 & 0\\ 2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\\ \mathrm{D}\widetilde{q}_{\mathrm{id}}(v_2) &= \mathrm{D}\widetilde{q}_{\mathrm{id}}(\tfrac{d}{dt}M(1,0,t,0)|_{t=0}) = \tfrac{d}{dt}q(M(1,0,t,0))|_{t=0} = \begin{pmatrix}0 & 0 & -2\\ 0 & 0 & 0\\ -2 & 0 & 0\end{pmatrix},\\ \mathrm{D}\widetilde{q}_{\mathrm{id}}(v_3) &= \mathrm{D}\widetilde{q}_{\mathrm{id}}(\tfrac{d}{dt}M(1,0,0,t)|_{t=0}) = \tfrac{d}{dt}q(M(1,0,0,t))|_{t=0} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix}.\end{aligned}$$

This calculation shows that $\mathrm{D}\widetilde{q}_{\mathrm{id}}$ is a monomorphism. ⊞

Claim 2. For any $A \in \mathrm{SU}(2)$ the map $\mathrm{D}q_A\colon \widetilde{\mathrm{T}}_{q(A)}\mathrm{SU}(2) \to \widetilde{\mathrm{T}}_{\mathrm{id}}\mathrm{SO}(3)$ is an isomorphism.

Proof. Let $A \in \mathrm{SU}(2)$. Since q is a group homomorphism we see that the following diagram commutes:

$$\begin{array}{ccc}\mathrm{SU}(2) & \xrightarrow{q} & \mathrm{SO}(3)\\ \downarrow{\scriptstyle A\cdot} & & \downarrow{\scriptstyle q(A)\cdot}\\ \mathrm{SU}(2) & \xrightarrow{q} & \mathrm{SO}(3).\end{array}$$

It follows from the Chain Rule, the fact that multiplication by A and by $q(A)$ is a diffeomorphism and Claim 1 that $\mathrm{D}q_A\colon \widetilde{\mathrm{T}}_A\mathrm{SU}(2) \to \widetilde{\mathrm{T}}_{q(A)}\mathrm{SO}(3)$ is an isomorphism. ■

The following theorem is the main result of this section.

Theorem 94.3. (SU(2)-SO(3)-Covering Theorem) Let $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the map from the SU(2)-Action Lemma 94.2. The following three statements hold:

(1) The following map is a diffeomorphism:

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z,w) & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}. \end{array}$$

(2) There exists a unique diffeomorphism $g\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$ such that the following diagram commutes:

$$\begin{array}{ccc} S^3 & \xrightarrow[\cong]{f} & \mathrm{SU}(2) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{R}\mathrm{P}^3 & \xrightarrow[\cong]{g} & \mathrm{SO}(3). \end{array}$$

(Here we denote by $p\colon S^3 \to S^3/\{\pm 1\} =_{\mathrm{i}} \mathbb{R}\mathrm{P}^3$ the usual projection.)

(3) The map $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a 2-fold covering.

Remark. In the Projective Space-Avatar Lemma 9.16 we gave a somewhat different homeomorphism between $\mathbb{R}\mathrm{P}^3$ and SO(3). In future discussions we will prefer the homeomorphism from the SU(2)-SO(3)-Covering Theorem 94.3. □

Convention 94.4. We will use the diffeomorphism from the SU(2)-SO(3)-Covering Theorem 94.3 to make the identifications $S^3 =_{\mathrm{i}} \mathrm{SU}(2)$ and $\mathbb{R}\mathrm{P}^3 =_{\mathrm{i}} \mathrm{SO}(3)$.

Proof of the SU(2)-SO(3)-Covering Theorem 94.3. One can easily show that the two maps[952]

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z,w) & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{f}\colon \mathrm{SU}(2) & \to & S^3 \\ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} & \mapsto & (z,w) \end{array}$$

are inverses of one another. Furthermore, using the Submanifold Smooth Structure Lemma 25.23 one can show that both maps are smooth. This implies that both maps, in particular f, is a diffeomorphism. This concludes the proof of Statement (1).

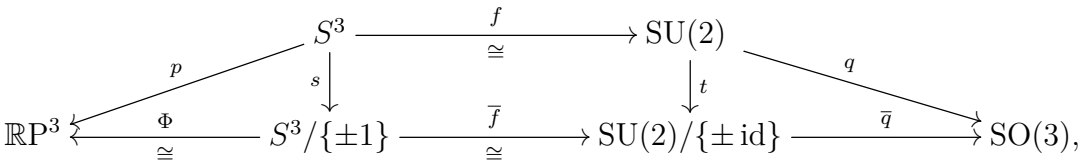
The proof of Statements (2) and (3) starts out with the following claim:

Claim. The map $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective.

Proof. By the SU(2)-Action Lemma 94.2 we know that q is a local diffeomorphism. It follows that $q(\mathrm{SU}(2))$ is an open subset of SO(3). Since SU(2) is compact, since q is continuous and since SO(3) is Hausdorff we obtain from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $q(\mathrm{SU}(2)$ is also a closed subset of SO(3). By the Matrix Group-π_0-Proposition 4.54 (3) we know that SO(3) is connected. It follows that $q(\mathrm{SU}(2) = \mathrm{SO}(3)$. ⊞

[952] In the Quaternion Unit Proposition 171.5 (2) we will see that the map f arises naturally if we think of S^3 as the "unit quaternions".

Next we consider the following diagram:



We make the following observations and clarifications:

(a) We denote by $\Phi\colon S^3/\{\pm 1\} \to \mathbb{R}\mathrm{P}^3$ the standard diffeomorphism and we set $s := \Phi^{-1} \circ t$.
(b) One can easily verify that the group $\{\pm 1\}$ acts discretely and smoothly on $\mathrm{SU}(2)$. We thus can use the Manifold–Quotient Proposition 25.39 to equip $\mathrm{SU}(2)/\{\pm 1\}$ with a smooth manifold structure such that the projection $t\colon \mathrm{SU}(2) \to \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ is a local diffeomorphism.
(c) Note that $f(-P) = -f(P)$. It follows that the diffeomorphism $f\colon S^3 \to \mathrm{SU}(3)$ descends to a bijection $\overline{f}\colon S^3/\{\pm 1\} \to \mathrm{SU}(2)/\{\pm \mathrm{id}\}$.
(d) Since s, f and t are local diffeomorphisms we see that $\overline{f}$ is a local diffeomorphism. Since $\overline{f}$ is a bijection we obtain from the Smooth Pasting Proposition 25.19 that $\overline{f}$ is actually a diffeomorphism.
(e) By the SU(2)-Action Lemma 94.2 (4) we have $q(A) = q(A')$ if and only if $A = \pm A'$. It follows that q descends to an injective map $\overline{q}\colon \mathrm{SU}(2)/\{\pm \mathrm{id}\} \to \mathrm{SO}(3)$.
(f) By the claim we also know that $\overline{q}\colon \mathrm{SU}(2)/\{\pm \mathrm{id}\} \to \mathrm{SO}(3)$ is surjective.
(g) By the SU(2)-Action Lemma 94.2 we know that q is a local diffeomorphism.
(h) As in (d) we obtain from (e), (f) and (g) that $\overline{q}\colon \mathrm{SU}(2)/\{\pm \mathrm{id}\} \to \mathrm{SO}(3)$ is actually a diffeomorphism.
(i) It is clear that $g := \overline{q} \circ \overline{f} \circ \Phi\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$ makes the diagram commute and by (d) and (h) we know that $g\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$ is a diffeomorphism. Since the diagram commutes and since $p\colon S^3 \to \mathbb{R}\mathrm{P}^3$ is surjective we see that g is uniquely determined.
(j) Since $p\colon S^3 \to \mathbb{R}\mathrm{P}^3$ is a 2-fold covering and since $g := \overline{q} \circ \overline{f} \circ \Phi\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$ is a diffeomorphism we see that $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a 2-fold covering. ■

We can now give a new proof of the SO(3)–π_1-Proposition 93.4.

Proposition 93.4. **(SO(3)–π_1-Proposition)**
(1) We have $\pi_1(\mathrm{SU}(2)) = 0$ and $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
(2) The unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$ is represented by the loop

$$\begin{aligned} \gamma\colon [0,1] &\to \mathrm{SO}(3) \\ t &\mapsto \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}}_{\text{rotation around the } x\text{-axis}}. \end{aligned}$$

Proof.

(1) In the SU(2)-SO(3)-Covering Theorem 94.3 we showed that SU(2) is homeomorphic to S^3 and we showed that SO(3) is homeomorphic to $\mathbb{R}\mathrm{P}^3$. Thus the statements follow from the Sphere-π_1-Proposition 88.24

(2) The calculation on page 1825 shows that a lift of the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ to a path $[0,1] \to \mathrm{SU}(2)$ to the starting point $\mathrm{id} \in \mathrm{SU}(2)$ is given by

$$\begin{array}{rcl} \widetilde{\gamma}\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} \exp(-\mathrm{i}\pi t) & 0 \\ 0 & \exp(\mathrm{i}\pi t) \end{pmatrix}. \end{array}$$

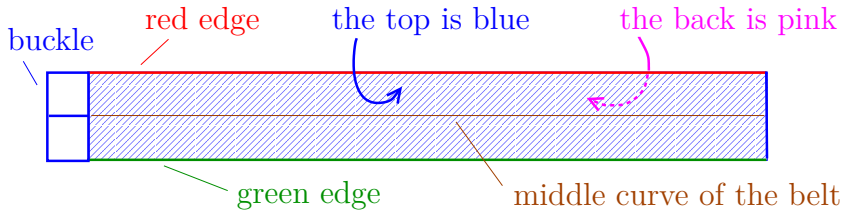
Note that $\widetilde{\gamma}(1) = -\mathrm{id}$, i.e. $\widetilde{\gamma}$ is *not* a loop. Since $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a covering and since $\mathrm{SU}(2)$ is simply connected it follows from the Loop–Lifting Proposition 92.8 that $[\gamma]$ is non-trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id})$. ■

94.2. **The belt trick.** In this section we will see that one can visualize the non-trivial element in $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$ using a belt. We proceed as follows.

(1) First of all note that a matrix in $\mathrm{SO}(3)$ is precisely given by an ordered orthonormal basis of $\mathbb{R}^3$ that obeys the "right hand rule", i.e. by a choice of an ordered set of three orthogonal vectors v_1, v_2, v_3 of length 1 in $\mathbb{R}^3$ with $\det(v_1\ v_2\ v_3) = 1$.

(2) We equip $\mathrm{SO}(3)$ with the base point given by the identity matrix, i.e. by the standard basis $\{e_1, e_2, e_3\}$.

(3) Now suppose that we are given a belt with a buckle, a red edge and a green edge, with a blue top side and a pink lower side.



(4) Any point on a twisted belt defines a triple of orthogonal vectors:
  (a) The first vector lies in the plane defined by the belt and points away from the buckle.
  (b) The second vector also lies in the plane defined by the belt and points towards the red edge.
  (c) The third is orthogonal to the belt and it is attached to the top side of the belt.

(5) In the remainder of the discussion we fix the buckle in the xy-plane such that it points in the x-direction, so that to a point at the buckle we assign the identity matrix.

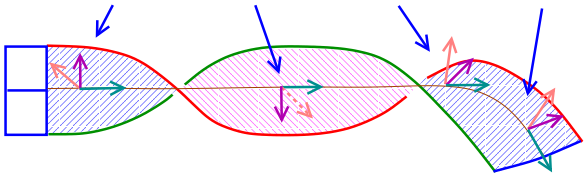
(6) We say that the end of the belt is in *standard position* if it lies in a plane parallel to the plane of the buckle and if it points into the x-direction. Put differently, the end of the belt is in standard position precisely if the matrix assigned to the endpoint is the identity matrix.

(7) When we walk on the middle curve of the belt from the buckle towards the other end of the belt, then we obtain a path $[0,1] \to \mathrm{SO}(3)$. In particular, if the end of the belt is in standard position, then the path defined by the belt is in fact a loop in $\mathrm{SO}(3)$ at the base point id.

(8) When we hold the belt tightly without twisting it, then we assign the identity matrix to every point on the belt, i.e. the belt defines the constant loop $t \mapsto \mathrm{id} \in \mathrm{SO}(3)$.

(9) Now we give the belt, that is still fixed at the buckle, a full twist. As in (7) this defines a loop $\gamma\colon [0,1] \to \mathrm{SO}(3)$, in fact it is straightforward to see that this loop is precisely

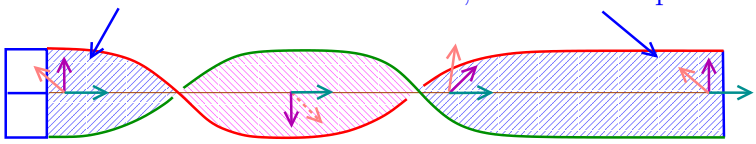
every point on a twisted belt defines a matrix in SO(3)



moving along the middle curve of the belt we get a path $[0,1] \to \mathrm{SO}(3)$

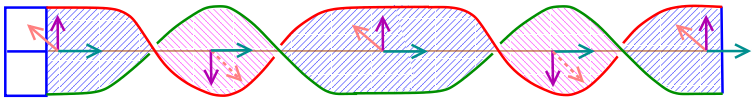
the loop that we considered in the SO(3)–π_1-Proposition 93.4 (2). In particular we have visualized the non-trivial element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id}) \in \mathbb{Z}/2\mathbb{Z}$.

both points define the standard basis of $\mathbb{R}^3$, i.e. the base point $\mathrm{id} \in \mathrm{SO}(3)$



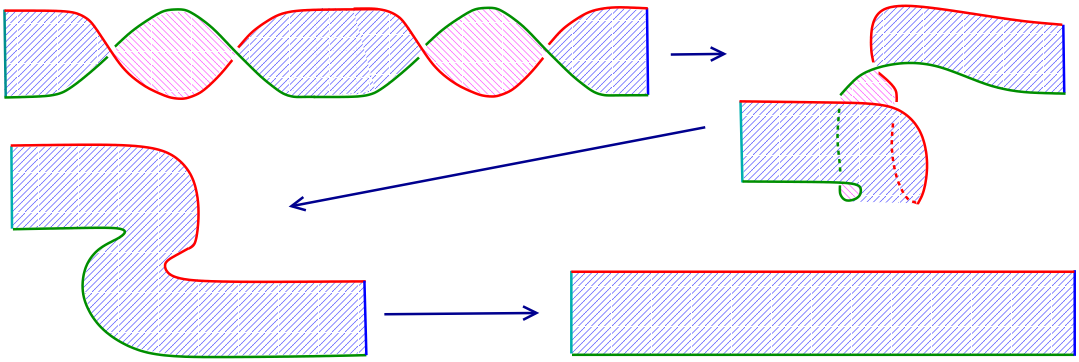
the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ defined by the matrices along the middle curve is a loop and thus defines an element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) The loop $\gamma * \gamma\colon [0,1] \to \mathrm{SO}(3)$ is given by performing another full twist along the belt. It follows from the SO(3)–π_1-Proposition 93.4 that $[\gamma * \gamma] = [\gamma]^2$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $\gamma * \gamma$ is null-homotopic.



the double full twist defines the loop $\gamma * \gamma$ which represents $[\gamma]^2 \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) One can see that the loop $\gamma * \gamma$ is null-homotopic as follows: one can untwist the belt while keeping the end in standard position, put differently, we can find a path-homotopy from the loop $\gamma * \gamma$ to the trivial loop. This homotopy is illustrated in the figure below.



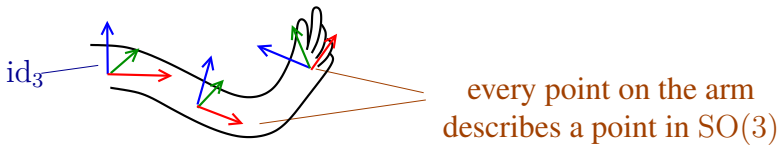
94.3. **The plate trick.** The plate trick is a variation on the belt trick. The plate trick has the great advantage that the only tool it requires is the body of a volunteer. The disadvantage is that the mathematical explanation is arguably a little more confusing.

The basic idea is as in the belt trick. Namely we want to describe paths and loops in SO(3) by the placement of a physical object in $\mathbb{R}^3$. But this time we make use of our arm instead of the belt.

First note that each point on the arm defines a point in SO(3):

(a) The first vector lies on the arm and it points from the shoulder to the hand.
(b) The second vector also lies on the arm and it points from one side to the other.
(c) The third vector points away from the body.

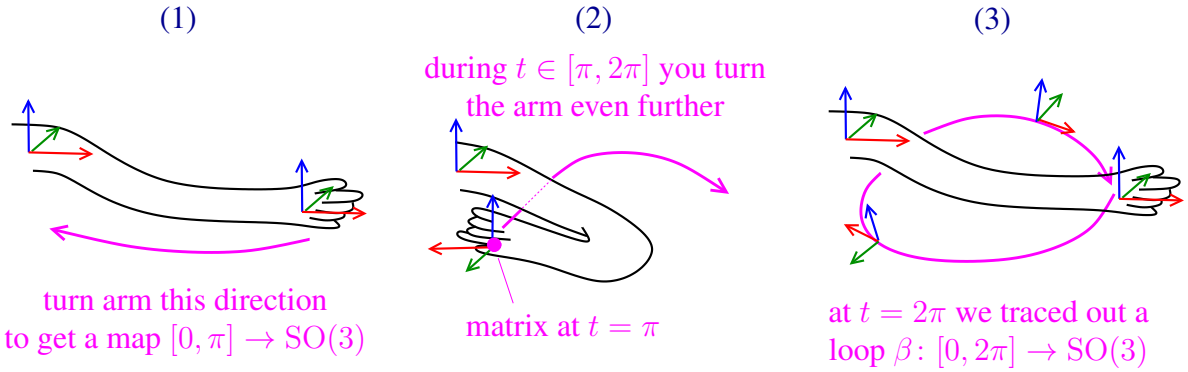
After picking the right coordinate system we can assume that the shoulder corresponds to the identity matrix $\mathrm{id}_3 \in \mathrm{SO}(3)$.



Now you, or your volunteer, does the following:

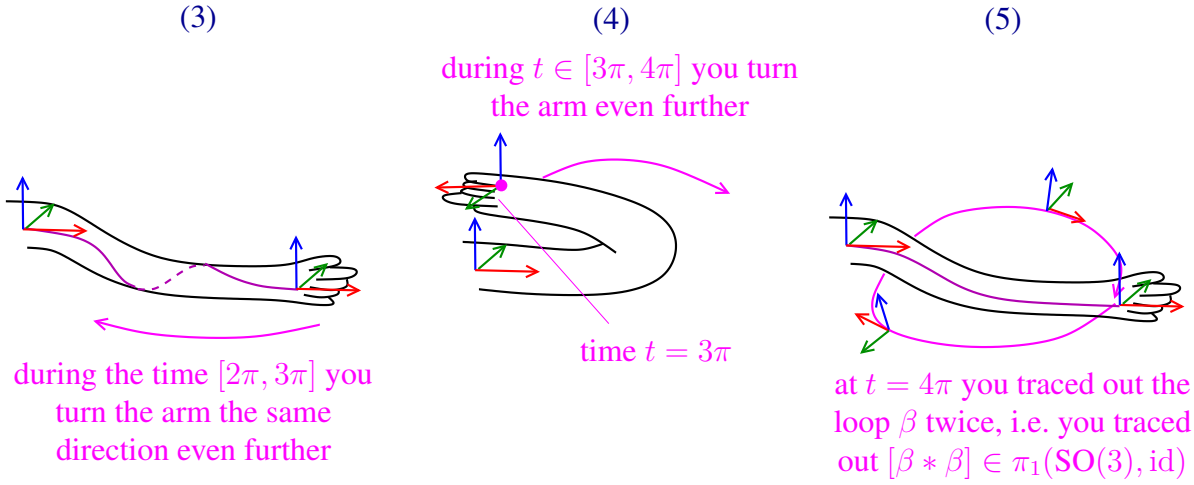
(1) You stretch your left arm and hands outward such that the fingers are parallel to the shoulder. If you feel like it, then you put a plate on your hand for illustration purposes.[953]
(2) You rotate the left arm counterclockwise. The physiology of most arms will force the forearm to go under the arm. During this movement the hand (or the plate) describes for each time t a point in SO(3).
(3) You continue till at $t = 2\pi$ your hand has done a 2π turn and the left arm is outstretched again.[954] This means that at $t = 2\pi$ your hand defines the same matrix as at $t = 0$. In other words, during this time interval $[0, 2\pi]$ the hand movement describes a loop β in SO(3). In fact this loop can be described explicitly, namely it is given by

$$\begin{array}{rcl} \beta\colon [0,2\pi] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{array}$$



[953] If you are courageous you put a glass of water on your hand. If you don't spill any water along the way, you performed the plate trick correctly.
[954] But the arm is twisted and hurts slightly.

(4) Now you rotate the left arm even further in the counterclockwise direction. The usual physiology will force the movement of the forearm to go over the arm.
(5) Again you continue till the hand has done another 2π turn. Your arm now traced out the loop β for a second time. In other words, in the time interval $[0, 4\pi]$ your hand traced out the loop $\beta\colon [0, 4\pi] \to \mathrm{SO}(3)$. Note that at the time $t = 4\pi$ your arm is finally again in the original, relaxed, position.



So what is happening this time? To answer this question we now parametrize the arm by $r \in [0, 1]$ where $r = 0$ corresponds to shoulder and $r = 1$ corresponds to the hand. As above we parametrize time by the interval $t \in [0, 4\pi]$ where time $t = 0$ corresponds to the starting time, $t = 2\pi$ corresponds to the time at our first full twist and $t = 4\pi$ corresponds to the moment when we completed the second full twist and your arm is back to the starting point.

The arm thus describes a continuous map

$$\begin{array}{rcl} \Theta\colon [0,1] \times [0,4\pi] & \to & \mathrm{SO}(3) \\ (r,t) & \mapsto & \text{matrix in } \mathrm{SO}(3) \text{ described by the point on the arm} \\ & & \text{with position } r \text{ and at time } t \end{array}$$

Since the arm at $t = 4\pi$ is back in the starting position from $t = 0$ and since the starting point at $r = 0$ is fixed we see that Θ descends to a well-defined map

$$\begin{array}{rcl} \overline{\Theta}\colon ([0,1] \times [0,4\pi])/\sim & \to & \mathrm{SO}(3) \\ [(r,t)] & \mapsto & \Theta(r,t) \end{array}$$

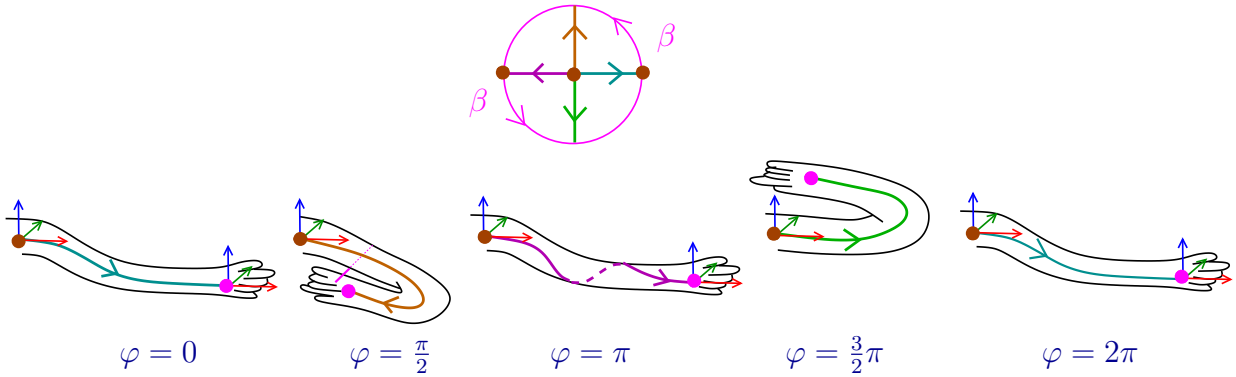
where $\sim$ is the equivalence relation on $[0, 1] \times [0, 4\pi]$ that is generated by $(0, t) \sim (0, t')$ and by $(r, 0) \sim (r, 4\pi)$. Note that it follows from the Topological-Quotient Proposition 8.2 (1b) that the map $\overline{\Theta}\colon ([0, 1] \times [0, 4\pi])/\sim \to \mathrm{SO}(3)$ is also continuous. Using the Compact-Hausdorff Proposition 4.23 (3) one can easily show that the map

$$\begin{array}{rcl} f\colon ([0,1] \times [0,4\pi])/\sim & \to & \overline{B}^2 \\ [(r,\varphi)] & \mapsto & r \cdot \exp(\frac{1}{2}\varphi \mathrm{i}) \end{array}$$

is a homeomorphism. In summary we see that the arm defines the continuous map

$$\begin{aligned} \Xi := \overline{\Theta} \circ f^{-1} \colon \overline{B}^2 &\to \mathrm{SO}(3) \\ r \cdot \exp(\varphi \mathrm{i}) &\mapsto \begin{array}{c}\text{matrix in SO(3) described by the point on the arm} \\ \text{with position } r \text{ and at time } 2\varphi\end{array} \end{aligned}$$

Note that the restriction $\Xi|_{S^1} \colon S^1 \to \mathrm{SO}(3)$ describes the loop $\beta * \beta \colon S^1 \to \mathrm{SO}(3)$. The map $\Xi \colon \overline{B}^2 \to \mathrm{SO}(3)$ gives a continuous extension of this map to a map on $\overline{B}^2$. It thus follows from the Trivial-in-π_1-Lemma 88.28 that $[\beta]^2 = [\beta * \beta]$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}_3)$. In other words, we have shown that β represents an element of order at most 2 in $\pi_1(\mathrm{SO}(3), \mathrm{id}_3)$.[955]

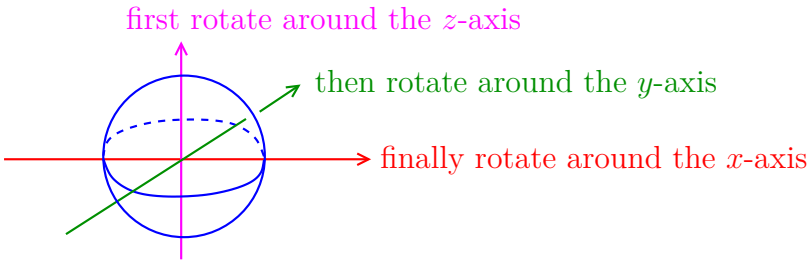


Evidently there are many videos of the plate trick. Some of the most charming videos can be found here:

`http://ariwatch.com/VS/Algorithms/PlateTrick.htm`

94.4. **Topological robotics.** In this section we want to discuss a "real-life" application of the SO(3)–π_1-Proposition 93.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

(1) first we can rotate the ball around the z-axis by an angle $\gamma \in S^1$,
(2) then we can rotate the ball around the y-axis by an angle $\beta \in S^1$, finally
(3) we can rotate the ball around the x-axis by an angle $\alpha \in S^1$.



Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma:

[955]There is a slightly different way to perform and explain the plate trick: If we start out with an arm that is stretched out away from the body and parallel to the ground, then the map $\Theta \colon [0,1] \times [0,4\pi] \to \mathrm{SO}(3)$ has the extra property that $\Theta(t,0) = \Theta(t,4\pi) = \Theta(0,s) = \mathrm{id}$ for all $t \in [0,1]$ and $s \in [0,4\pi]$. After reparametrizing 4π to 1 we see that Θ defines a path-homotopy $([0,1],\{0,1\}) \to (\mathrm{SO}(3), \mathrm{id})$ from the loop $\beta * \beta \colon [0,1] \to \mathrm{SO}(3)$ to the trivial loop $[0,1] \to \mathrm{SO}(3)$.

Lemma 94.5. (Torus-to-SO(3)-Lemma) The map

$$\begin{array}{rcl} \Phi\colon S^1\times S^1\times S^1 & \to & \mathrm{SO}(3) \\ (\alpha,\beta,\gamma) & \mapsto & \begin{array}{c}\text{rotation by } \alpha\\ \text{around the } x\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \beta\\ \text{around the } y\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \gamma\\ \text{around the } z\text{-axis}\end{array} \end{array}$$

is a surjection.[956]

Sketch of proof. Let $A \in \mathrm{SO}(3)$ be matrix with columns v_1, v_2, v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1 = e_1$. Since $v_1 = e_1, v_2, v_3$ is an orthonormal basis we see that v_2, v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2 = e_2$ and $v_3 = e_3$. ■

This lemma gives rise to the following practical problem: Given a matrix $P \in \mathrm{SO}(3)$, can we continuously find parameters $\alpha(P), \beta(P), \gamma(P) \in S^1$ such that these parameters give back the matrix P? More precisely, we have the following question:

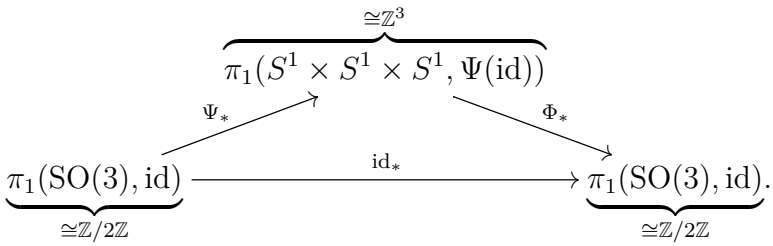
Question 94.6. Does there exist a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1\times S^1\times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$?

It turns out that the answer is negative. More precisely, we have the following proposition:

Proposition 94.7. (No Splitting-SO(3)-Torus Proposition) There is no continuous map $\Psi\colon \mathrm{SO}(3) \to S^1\times S^1\times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$.

Remark. This proposition falls into the area of "topological robotics", more information on this example and on topological robotics can be found in [**Gh2014**, p. 76]. Other aspects of topological robotics are discussed in the monograph [**Far2008**]. □

Proof. Let us suppose that there exists a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1\times S^1\times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram:



By the SO(3)–π_1-Proposition 93.4 we have that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$ and by the Torus-π_1-Proposition 90.14 we know that $\pi_1(S^1\times S^1\times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}/2\mathbb{Z}$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon \pi_1(\mathrm{SO}(3)) \to \pi_1(S^1\times S^1\times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. ■

94.5. **Linkages.** We conclude this chapter with a somewhat unrelated but fun topic that also appears in topological robotics. To do so we recall Definition 11.12:

[956] The map is also continuous. Why is that?

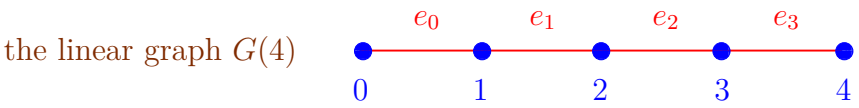
Definition 94.8.

(1) An abstract graph G is a triple (V, E, λ) where V is a non-empty set, E is a set and λ is a map

$$\lambda\colon E \ \to \ \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called vertices of G and the elements of E are called the edges of G. Furthermore, given $e \in E$ the points in $\lambda(e)$ are called the endpoints of e.

(2) Given $n \in \mathbb{N}$ we define the linear graph of length n to be the abstract graph $G(n)$ which is given by the vertex set $V = \{0, \dots, n\}$ and the edge set $\{e_0, \dots, e_{n-1}\}$ with $\lambda(e_i) = \{i, i+1\}$ for $i = 0, \dots, n-1$.



Definition 94.9.

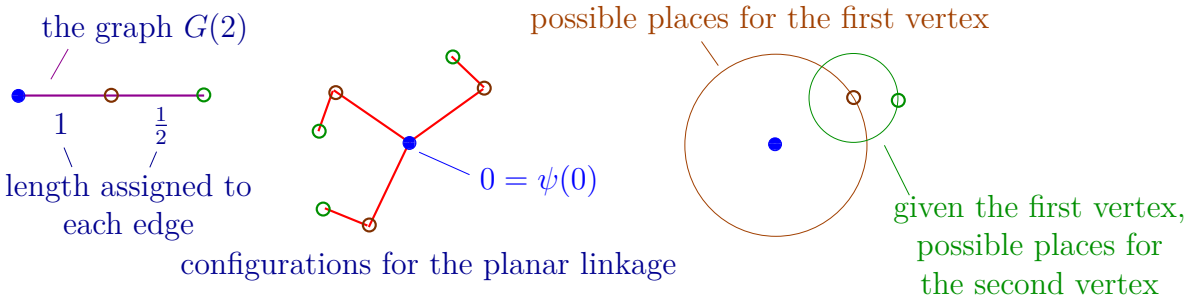
(1) A planar linkage is a triple $(G = (V, E), \lambda, \psi)$ consisting of a finite abstract graph $G = (V, E)$, a function $\lambda\colon E \to \mathbb{R}_{>0}$ and a continuous map $\psi\colon W \to \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ where W is a subset of V. Put differently, λ assigns a length to each edge and ψ represents a subset of the vertices by points in $\mathbb{C}$.

(2) A configuration of a planar linkage $(G = (V, E), \lambda, \psi)$ is a map $f\colon V \to \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ such that $f|_W = \psi$ and such that for every edge e the distance between the endpoints of e is $\lambda(e)$.[957]

(3) The set $C(G = (V, E), \lambda, \psi)$ of all configurations is called the configuration space of the linkage. We have an injective map

$$\begin{array}{ccc} C(G = (V, E), \lambda, \psi) & \to & \mathbb{C}^{|V|} := \mathrm{maps}(V \to \mathbb{C}) \\ f & \mapsto & (v \mapsto f(v)) \end{array}$$

thus we can view the configuration space $C(G = (V, E), \lambda, \psi)$ as a topological space.



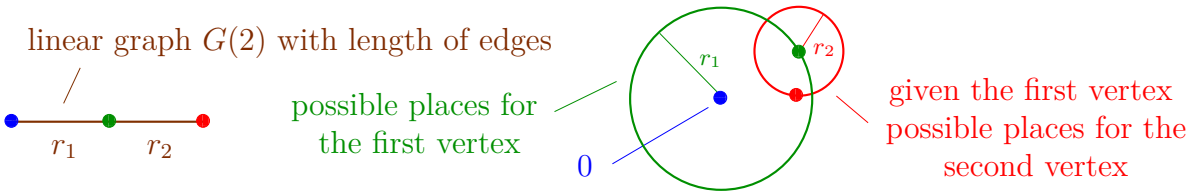
In some cases the topological shape of the configuration space of a planar linkage is easy to describe. More precisely, we have the following lemma:

[957] In particular if e has only one endpoint, then we need $\lambda(e) = 0$.

Lemma 94.10. Let $G(m)$ be the linear graph with m edges. We consider a planar linkage of the form $(G(m), \lambda, \psi(0) = 0)$. For $i = 1, \ldots, m$ we write $r_i = \lambda(e_i)$. Then the map

$$\begin{array}{rcl} (S^1)^m = S^1 \times \cdots \times S^1 & \to & C(G(m), \lambda, \psi) \\ (z_1, \ldots, z_m) & \mapsto & \left(\begin{array}{rcl} \{0, \ldots, m\} & \to & \mathbb{C} \\ k & \mapsto & \sum_{i=1}^{k} r_i \cdot z_i \end{array} \right) \end{array}$$

is a homeomorphism.



Proof. The lemma is a straightforward consequence of the definitions and the Compact-Hausdorff Proposition 4.23 (3). We leave it to the reader to provide the proof. ■

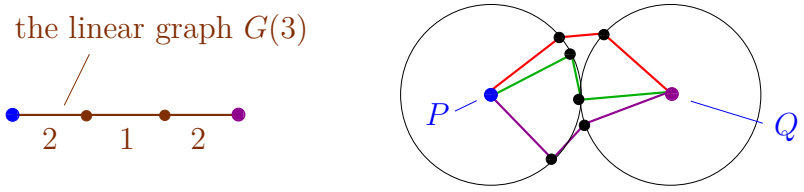
For general linkages the topology of the configuration spaces are much harder to describe than for free planar linkages.

(1) Given $r_1, \ldots, r_m > 0$ we denote by $C(r_1, \ldots, r_m, P, Q)$ the configuration space corresponding to the planar linkage defined by $G(m)$, $\lambda(e_i) = r_i$ and by the conditions that $\psi(0) = P$ and $\psi(m) = Q$.

(a) Using elementary arguments one can easily show that

$$C(r_1, r_2, P, Q) = \begin{cases} \varnothing, & \text{if } r_1 + r_2 < d(P, Q), \\ \text{one point}, & \text{if } r_1 + r_2 = d(P, Q), \\ \text{two points}, & \text{if } r_1 + r_2 > d(P, Q). \end{cases}$$

(b) Kevin Walker [**WalkK1985**, p. 35], see also [**Gh2014**, p. 12], showed in 1985 in his bachelor thesis that the surfaces of genus $0, 1, \ldots, 4$ are homeomorphic to configuration spaces of the type $(r_1, r_2, r_3, r_4, P, Q)$ for suitable values of $r_1, \ldots, r_4$ and points P and Q.



configurations for the planar linkage

(2) Michael Kapovich and John Millson [**KaM2002**] showed that given any compact smooth manifold M there exists a planar linkage such that each component of the configuration space is homeomorphic to M. In particular this shows that any compact smooth manifold can be represented by a 2-dimensional physical object.

94.6. **Exercises for Chapter 94.**

Exercise 94.1. In the $\mathrm{SU}(2) - \mathrm{SO}(3)$-Covering Theorem 94.3 we gave an explicit homeomorphism $g\colon \mathbb{R}\mathrm{P}^3 \to \mathrm{SO}(3)$. We consider $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \{[(x,0)] \in \mathbb{R}\mathrm{P}^3 \,|\, x \in \mathbb{R}^3 \setminus \{0\}\}$. What is the image $g(\mathbb{R}\mathrm{P}^2) \subseteq \mathrm{SO}(3)$?

Exercise 94.2. We consider the following loop:

$$\begin{array}{rcl} \varphi\colon S^1 & \to & \mathrm{SO}(3) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{array}$$

Provide an explicit homotopy from $\varphi * \varphi$ to the constant loop.

Exercise 94.3. Let $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the homomorphism that we defined in the SU(2)-Action Lemma 94.2. Show that

$$q\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ q\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ q\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Remark. This calculation will actually turn out to be useful in Section 183.4.

CHAPTER 95

Coverings of smooth and topological manifolds

As the title of the chapter says, in this chapter we consider coverings of smooth and topological manifolds. In particular we will see that coverings of manifolds are again manifolds. We will also construct the orientation covering of any non-orientable connected smooth manifold.

This chapter is organized as follows:

- In Section 95.1 we will show that fundamental groups of topological manifolds are always countable.
- In Section 95.2 we will show that a covering of a (smooth) manifold has naturally the structure of a (smooth) manifold. structures on coverings of manifolds
- In Section 95.3 we will introduce the 2-fold orientation covering of a non-orientable smooth manifold.
- In Section 95.4 we will introduce the orientation character $\pi_1(M, x_0) \to \{\pm 1\}$ of a non-orientable smooth manifolds.
- In Section 95.5 we will introduce the notion of orientation-preserving and orientation-reversing loops on a non-orientable smooth manifold.
- In Section 95.6 we show that non-orientable smooth manifolds admit orientation flipping diffeotopies.
- In Section 95.7 we will provide exercises for Chapter 95.

In the next Chapter 96 we will introduce wedges of topological spaces and we will introduce topological graphs.

95.1. **Fundamental groups of topological manifolds.** Our only goal in this section is to prove a proposition, which will help us with dealing with a technical issue in the following section. But the proposition is also interesting in its own right and it provides a good way to practice what we learned in the last chapters.

Proposition 95.1. (Manifold–Countable-π_1-Proposition)

(1) Let X be a topological space and let $x_0 \in X$. If X is second-countable and locally simply connected, then $\pi_1(X, x_0)$ is countable.
(2) If X is a topological manifold and if $x_0 \in X$, then $\pi_1(X, x_0)$ is countable.

Remark.

(1) In the Smooth Manifolds-Invariants Proposition 179.10 (1) and the Topological Manifolds-Invariants Proposition 189.14 (1a) we will give an alternative proof of the Manifold–Countable-π_1-Proposition 95.1 for smooth respectively topological manifolds.
(2) In the Smooth Manifolds-Invariants Proposition 179.10 (1b), the Topological Manifolds-Invariants Proposition 189.14 (1b) and on page 4851 we will see that fundamental groups of compact topological manifolds are in fact "finitely generated". □

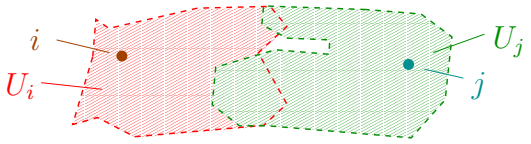
Proof. Statement (2) follows almost immediately from Statement (1). Indeed, first note that topological manifolds are by definition second-countable. Furthermore note that in the Manifold–Local Properties Lemma 23.8 we showed that any n-dimensional topological manifold is locally contractible. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 (2) that topological manifolds are locally simply connected.

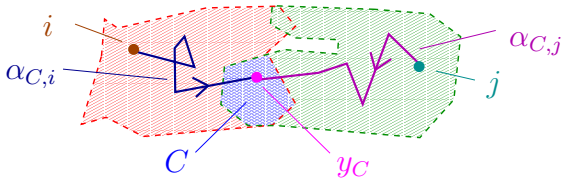
Now we turn to the proof of Statement (1). Thus let X be a topological space that is second-countable and locally simply connected. Furthermore let $x_0 \in X$. Since X is locally simply connected we know that for each $x \in X$ there exists an open neighborhood $U_x \subseteq X$ of x that is simply connected. Since X is second-countable we know by the Countable Cover Lemma 13.4 that there exists a countable subset $I \subseteq X$ such that $X = \bigcup_{i \in I} U_i$. We pick $i_0 \in I$ in the path-component of x_0. By the Change-of-Base Point Proposition 88.14 (1) it suffices to show that $\pi_1(X, i_0)$ is countable.

Claim 1. For each $i, j \in I$ the intersection $U_i \cap U_j$ has only countably many path-components, i.e. $\pi_0(U_i \cap U_j)$ is countable.

Proof. Let $\{W_k\}_{k \in K}$ be the path-components of $U_i \cap U_j$. For each $k \in K$ we pick a point z_k. Since U_i and U_j are open we know that $U_i \cap U_j$ is also open. Since X is locally simply connected we see that there exists a simply connected neighborhood C_k of z_k with $C_k \subseteq U_i \cap U_j$. Recall that "simply connected" means in particular that C_k is path-connected. Therefore it follows from the Path-Component Lemma 4.44 that $C_k \subseteq W_k$. Thus we see that the open subsets C_k, $k \in K$ are disjoint. Now we obtain from our hypothesis that X is second-countable, together with say the Countable Cover Lemma 13.4, that the index set K is countable. ⊞

Let $i, j \in I$. Given any $C \in \pi_0(U_i \cap U_j)$ we pick a point $y_C \in C_{i,j}$. Since U_i and U_j are in particular path-connected we can pick a path $\alpha_{C,i}$ from i to y_C and we can pick a path $\alpha_{C,j}$ from j to y_C. Note that it follows from Claim 1 together with the Countability Lemma 1.37 that the set $\bigcup_{i \in I} \bigcup_{j \in J} \pi_0(U_i \cap U_j)$ is countable. Thus it remains to prove the following claim:





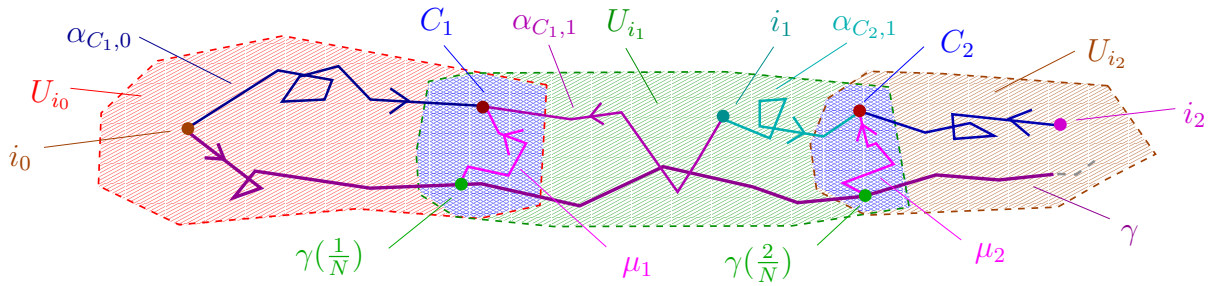
Claim 2. Every loop in i_0 is path-homotopic to a composition of the paths $\alpha_{C,i}$ and their inverses.

Proof. Let $\gamma\colon [0,1] \to X$ be a loop in i_0. We start out with a few preparations:

- Since the sets $\{U_i\}_{i \in I}$ form an open cover and since $[0,1]$ is compact we obtain easily from the Lebesgue Lemma 10.9 that there exists an $N \in \mathbb{N}$ and $i_1, \ldots, i_{N-1} \in I$ such that for each $k \in \{0, \ldots, N-1\}$ we have $\gamma([\frac{k}{N}, \frac{k+1}{N}]) \subseteq U_{i_k}$.
- Given $k \in \{0, \ldots, N-1\}$ we define γ_k to be the restriction of γ to the interval $[\frac{k}{N}, \frac{k+1}{N}]$.
- For $k = 1, \ldots, N-1$ we have $\gamma(\frac{k}{N}) \in U_{i_{k-1}} \cap U_{i_k}$. We denote by C_k the path-component of $U_{i_{k-1}} \cap U_{i_k}$ that contains $\gamma(\frac{k}{N})$. Since C_k is path-connected we know that there exists a path $\mu_k\colon [0,1] \to C_k$ from $\gamma(\frac{k}{N})$ to y_{C_k}.

Next we reuse some notation that we had introduced earlier on.

- Given a path $\delta\colon [a,b] \to X$ we denote, as in Definition 88.6, by $[\delta]$ the path-homotopy class of δ.
- Given paths $\delta\colon [a,b] \to X$ and $\epsilon\colon [c,d] \to X$ with $\delta(b) = \epsilon(c)$ we denote, as in Definition 88.7, by $\delta * \epsilon\colon [0,1] \to X$ the concatenation.
- Given a path $\delta\colon [0,1] \to X$ we denote by $\overline{\delta}\colon [0,1] \to X$ the inverse path that is given by $\overline{\delta}(t) = \delta(1-t)$.



Now we can provide the actual proof of the claim. By constantly using the Path-Homotopy Product Proposition 88.10 and the Path-Homotopy–Concatenation Lemma 88.8 we obtain that

$$
\begin{aligned}
[\gamma] &= [\gamma_0 * \gamma_1 * \ldots * \gamma_{N-1}] \\
&= [\underbrace{\gamma_0 * \mu_1}_{\substack{\text{same endpoints} \\ \text{as } \alpha_{C_1,0}}} * \underbrace{\overline{\mu_1} * \gamma_1 * \mu_2}_{\substack{\text{same endpoints} \\ \text{as } \overline{\alpha_{C_1,1}} * \alpha_{C_2,1}}} * \overline{\mu_2} * \ldots * \mu_{N-1} * \underbrace{\overline{\mu_{N-1}} * \gamma_{N-1}}_{\substack{\text{same endpoints} \\ \text{as } \overline{\alpha_{C_{N-1},N}}}}] \\
&= [\alpha_{C_1,0} * \overline{\alpha_{C_1,1}} * \alpha_{C_1,1} * \ldots * \overline{\alpha_{C_{N-1},N}}] \\
&\quad \uparrow \\
&\quad \text{here we use the Paths-in-Simply Connected Space Lemma 88.17 and} \\
&\quad \text{we use that the } U_{i_k} \text{ are simply connected}
\end{aligned}
$$

We have thus proved the desired statement. ∎

95.2. **Manifold structures on coverings of manifolds.** We start out by recalling Definition 91.8:

Definition 95.2. Let $p\colon X \to B$ be a continuous map between two topological spaces. A map $p\colon X \to B$ is called a covering, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of $b \in B$ which is uniformly covered, i.e. $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism.

In the context of smooth and complex manifolds it is appropriate to introduce the following variation.

Definition 95.3.

(1) Let $p\colon X \to B$ be a continuous map between smooth manifolds. We say that the map $p\colon X \to B$ is a covering of smooth manifolds, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property, that the restriction of p to each subset V_i is a diffeomorphism.

(2) We have the analogous definition of a covering between complex manifolds, we just need to replace "diffeomorphism" by "biholomorphism".

Remark. Note that a covering map between smooth manifolds is in particular a local diffeomorphism, which implies by the Smooth Pasting Proposition 25.19 that it is smooth. The analogous statement also holds for coverings of complex manifolds. □

Example.

(1) Let M be a smooth manifold and let G be a group that acts freely, properly and smoothly on M. In the Manifold–Quotient Proposition 25.39 we equipped M/G with a smooth structure. The argument of the Action–Covering Proposition 91.11 shows that the projection $M \to M/G$ is a covering of smooth manifolds in the sense of Definition 95.3. Together with Exercise 25.10 this shows, for example, that the natural projection $S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ is a covering of smooth manifolds.

(2) We consider the smooth manifold $M = \mathbb{R}$ and the map

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3. \end{array}$$

This map is a homeomorphism, in particular it is a 1-fold covering between topological spaces. On the other hand this map is not a (local) diffeomorphism, hence it is not a covering between smooth manifolds. □

Proposition 95.4. (Manifold Covering Proposition) Let X be an n-dimensional topological manifold and let $p\colon \widetilde{X} \to X$ be a covering.

(0) If X and $\widetilde{X}$ are connected, then $p\colon \widetilde{X} \to X$ is a countable covering.[958]

Now suppose that $p\colon \widetilde{X} \to X$ is a countable covering.

(1) The covering space $\widetilde{X}$ is also an n-dimensional topological manifold.

(2) (a) If X admits a smooth atlas, then $\widetilde{X}$ admits a smooth atlas such that $p\colon \widetilde{X} \to X$ is a covering of smooth manifolds.

(b) If X is equipped with a smooth structure[959] then $\widetilde{X}$ admits a *unique* smooth structure such that $p\colon \widetilde{X} \to X$ is a covering of smooth manifolds.

(3) (a) Regardless of whether we work with smooth or topological manifolds we have the equality $\partial\widetilde{X} = p^{-1}(\partial X)$.

(b) If $p\colon \widetilde{X} \to X$ is a finite covering and if X is a *closed* smooth or topological manifold, then so is $\widetilde{X}$.

(4) If X is an oriented smooth manifold, then $\widetilde{X}$ admits a unique orientation such that $p\colon \widetilde{X} \to X$ is orientation-preserving.

(5) If X is equipped with a complex structure[960], then $\widetilde{X}$ admits a unique complex structure such that $p\colon \widetilde{X} \to X$ is a covering of complex manifolds.

Convention 95.5. If M is a smooth manifold, i.e. if M is a topological manifold equipped with a smooth structure, and if we are given a covering $p\colon \widetilde{M} \to M$, then usually we equip

[958] Recall that a covering $p\colon \widetilde{X} \to X$ of a connected space is called *countable* if the degree, i.e. the cardinality of the preimage of a point in X, is countable.

[959] We refer to Definition 25.5 for the definition of a smooth structure.

[960] We refer to Definition 40.3 for the definition of a complex structure.

$\widetilde{M}$ with the unique smooth structure given by the Manifold Covering Proposition 95.4 (2b). The same convention also applies to coverings of oriented smooth manifolds and to coverings of complex manifolds.

In the proof of the Manifold Covering Proposition 95.4 we will need the following technical lemma.

Lemma 95.6. (Second Countable Cover Lemma) Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. If X is second-countable, then $\widetilde{X}$ is also second-countable.

Proof of Lemma 95.6. Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. Recall that we say $U \subseteq X$ is uniformly covered, if $p^{-1}(U)$ is the union of disjoint open subsets $\{\widetilde{U}_i\}_{i\in I}$ with the property that the restriction of p to each subset $\widetilde{U}_i$ is a homeomorphism. If U is uniformly covered, then it follows from the remark on page 1781 that every open subset of U is also uniformly covered.

Let $\mathcal{B}$ be a countable basis for the topology of X. We denote by $\mathcal{U} = \{U_j\}_{j\in J}$ the family of all open subsets of X that are uniformly covered. Since p is a covering we have $\bigcup_{j\in J} U_j = X$. It follows from the Small Basis Lemma 3.54 that

$$\mathcal{B}' \coloneqq \{B \in \mathcal{B} \mid \text{ there exists a } j \in J \text{ with } B \subseteq U_j\}$$

is also a basis for the topology of X.

Now let $B \in \mathcal{B}'$. Since open subsets of uniformly covered subsets are uniformly covered we know that $p^{-1}(B)$ is the union of disjoint open subsets B_i, $i \in I_B$ such that the restriction of p to each B_i is a homeomorphism. Since p is a countable covering the index set I_B is countable.

Claim. $$\widetilde{\mathcal{B}} \coloneqq \{B_i \mid B \in \mathcal{B}' \text{ and } i \in I_B\}$$ is a basis for the topology of $\widetilde{X}$.

Proof. It is straightforward to see that $\widetilde{\mathcal{B}}$ has the basis property. Every set in $\widetilde{\mathcal{B}}$ is of course open in $\widetilde{X}$. By the Union-of-Basis Sets Lemma 3.53 it suffices to show that given any open subset W of $\widetilde{X}$ and any point $x \in W$ there exists a set B_i in $\widetilde{\mathcal{B}}$ with $x \in B \subseteq W$. So let $x \in W$. Since $\mathcal{B}'$ is a basis for the topology of X there exists a $B \in \mathcal{B}'$ with $p(x) \in B \subseteq p(W)$. But then there exists an $i \in I_B$ with $x \in B_i \subseteq W$. ⊞

Since $\mathcal{B}'$ is countable and since each I_B is countable we know by the Countability Lemma 1.37 that $\widetilde{\mathcal{B}}$ consists of countably many open subsets. This shows that $\widetilde{X}$ is second-countable. ■

Proof of the Manifold Covering Proposition 95.4. Let X be an n-dimensional topological manifold and let $p\colon \widetilde{X} \to X$ be a covering.

(0) Suppose that X and $\widetilde{X}$ are path-connected. If X is the empty set there is nothing to show. If X is non-empty, we pick $x_0 \in X$. Furthermore we pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. By the Manifold–Countable-π_1-Proposition 95.1 we know that $\pi_1(X, x_0)$ is countable. Next note that it follows from the Covering-Degree Proposition 92.10 that $[\widetilde{X} : X] = |\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))|$. It follows from the fact that $\pi_1(X, x_0)$ is countable and the Countability Lemma 1.37 that $[\widetilde{X} : X]$ is countable.

Now we assume that $p\colon \widetilde{X} \to X$ is a countable covering.

(1) First we note that it follows from the hypothesis that X is a topological manifold that X is Hausdorff. It follows from the Covering–Basics Lemma 91.12 (3) that $\widetilde{X}$ is also Hausdorff. Next note that it follows from our hypothesis that X is a topological manifold and that p is a countable covering, together with the Second Countable Cover Lemma 95.6, that $\widetilde{X}$ is second-countable.

Since $p\colon \widetilde{X} \to X$ is a covering there exists a family of open subsets $\{U_i\}_{i\in I}$ of X such that each U_i is uniformly covered. Furthermore, since X is an n-dimensional topological manifold there exists an atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. After possibly replacing each V_j by the family $\{V_j \cap U_i\}_{i\in I}$ we can without loss of generality assume that each V_j is uniformly covered. Thus for each j the preimage $p^{-1}(V_j)$ is the union of disjoint open subsets $\{\widetilde{V}_{ji}\}_{i\in I_j}$ with the property, that the restriction of p to each subset $\widetilde{V}_{ji}$ is a homeomorphism. It is clear that $\{\Phi_j \circ p\colon \widetilde{V}_{ji} \to W_j \mid j \in J \text{ and } i \in I_j\}$ is an n-dimensional atlas for $\widetilde{X}$.

(2) (a) The transition maps of the atlas from (1) are precisely given by restrictions to open subsets of the transition maps of the atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. Thus if we start out with a smooth atlas for X, then the atlas we just constructed for $\widetilde{X}$ is also smooth.

Since $p\colon \widetilde{X} \to X$ is a covering we know that each $x \in X$ admits an open neighborhood U such that $p^{-1}(U)$ is the union of a family $\{V_i\}_{i\in I}$ of disjoint open subsets such that each $p\colon V_i \to U$ is a homeomorphism. By the definition of the smooth structure on $\widetilde{X}$ these maps are now diffeomorphisms. We have thus shown that $p\colon \widetilde{X} \to X$ is a covering of smooth manifolds.

(b) This statement follows immediately from (a), together with the rather straightforward Exercise 25.31.

(3) (a) Let X be a topological manifold. Since $p\colon \widetilde{X} \to X$ is a local homeomorphism we obtain easily from the Boundary–Local Homeomorphism Lemma 23.14 that $\partial\widetilde{X} = p^{-1}(\partial X)$. If X is in fact a smooth manifold, then the statement follows from the fact that now $p\colon \widetilde{X} \to X$ is a local diffeomorphism together with the Boundary–Diffeomorphism Lemma 25.29.

(b) This statement follows from (a) together with the Covering–Basics Lemma 91.12 (4).

(4) This statement follows immediately from the fact that p is a local diffeomorphism together with the Manifold–Orientation Basics Lemma 31.31 (6).

(5) The proof of this statement is almost identical to the proof of (2). We leave it to the reader to fill in the details. ■

95.3. **The orientation covering of a non-orientable smooth manifold.** We saw that many of our examples of non-orientable smooth manifolds admit a 2-fold covering such that the covering space is an orientable smooth manifold:

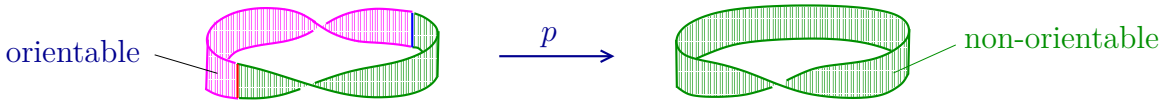
Examples.

(1) By the discussion on page 1784 the projection $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\{\pm 1\}$ is a 2-fold covering. This shows that the real projective plane (which is non-orientable) is covered by the sphere (which is orientable).

(2) On page 1783 we considered the map

$$\begin{array}{rcl} p\colon \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\frac{1}{2},1-y)}^{\text{Möbius band}} \\ [(x,y)] & \to & [(x,y)] \end{array}$$

and we remarked that this is a 2-fold covering. Thus the non-orientable Möbius band admits a 2-fold covering such that the covering space is orientable.



(3) Almost the same map as in (2) also shows that there exists a 2-fold covering map from the torus to the Klein bottle. □

In this section we will see that every non-orientable smooth manifold admits a 2-fold covering such that the covering space is orientable. Furthermore we will see that this 2-fold covering is in a suitable sense unique.

First let us introduce and recall some notation which we will use throughout this section.

Notation 95.7.

(1) Given an orientation $\mathcal{O}$ for a real vector space we denote by $-\mathcal{O}$ the unique other orientation.

(2) Given an isomorphism $\varphi\colon V\to W$ between vector spaces and given an orientation $\mathcal{O}$ on V we denote by $\varphi(\mathcal{O})$ the corresponding orientation on W.

Proposition 95.8. (Orientation Covering–Existence Proposition) Let M be a smooth manifold. We consider the set

$$\widetilde{M} \;=\; \{(Q,\mathcal{O}) \mid Q\in M \text{ and } \mathcal{O} \text{ is an orientation of } \mathrm{T}_QM\}$$

together with the map

$$\begin{array}{rcl} p=p_M\colon \widetilde{M} & \to & M \\ (Q,\mathcal{O}) & \mapsto & Q. \end{array}$$

(1) There exists a unique topology on $\widetilde{M}$ such that $p_M\colon \widetilde{M}\to M$ is a 2-fold covering.

(2) There exists a unique smooth structure on the topological space $\widetilde{M}$ such that the projection $p_M\colon \widetilde{M}\to M$ is a covering of smooth manifolds.

(3) Let $\widetilde{x}=(x,\mathcal{O})\in\widetilde{M}$. By (2) the map $p\colon \widetilde{M}\to M$ is a local diffeomorphism, which implies that $\mathrm{D}p_{\widetilde{x}}\colon \mathrm{T}_{\widetilde{x}}\widetilde{M}\to \mathrm{T}_xM$ is an isomorphism. Now we equip $\mathrm{T}_{\widetilde{x}}\widetilde{M}$ with the orientation $(\mathrm{D}p_{\widetilde{x}})^{-1}(\mathcal{O})$. These orientations of the tangent spaces of $\widetilde{M}$ define an orientation of the smooth manifold $\widetilde{M}$.

The Orientation Covering–Existence Proposition 95.8 leads us to the following definition:

Definition 95.9. Let M be a smooth manifold. The covering $p_M\colon \widetilde{M}\to M$ from the Orientation Covering–Existence Proposition 95.8 is called the orientation covering of M. We always equip $\widetilde{M}$ with the topology, smooth structure and orientation from the Orientation Covering–Existence Proposition 95.8.

Remark. In the Orientation Covering–Properties-Proposition 191.29 we will prove an analog of the Orientation Covering–Existence Proposition 95.8 for topological manifolds. □

Proof. Let M be an n-dimensional smooth manifold.

(1) Let $\Phi\colon U \to V$ be a chart of the smooth atlas of M and let $\epsilon \in \{-,+\}$. We introduce the following notation:
- We denote by $i\colon U \to M$ the inclusion map.
- For $x \in V \subseteq \mathbb{R}^n$ we denote by $\mathcal{O}_{V,x}$ the standard orientation of $\mathrm{T}_x V$ that we introduced on page 812.

We set
$$B_\epsilon(\Phi) \coloneqq \{(Q, \underbrace{\epsilon \cdot (\mathrm{D}i_Q \circ (\mathrm{D}\Phi_Q)^{-1})(\mathcal{O}_{V,\Phi(Q)})}_{\text{orientation of } \mathrm{T}_Q M}) \mid Q \in U\} \subseteq \widetilde{M}.$$

It is straightforward to show that the collection of these sets $B_\pm(\Phi)$ satisfies the basis properties (B1) and (B2) from Definition 3.52. Now we endow $\widetilde{M}$ with the topology generated by the sets $B_\pm(\Phi)$.

Claim 1. The projection $p\colon \widetilde{M} \to M$ is continuous.

Proof. Note that for each chart $\Phi\colon U \to V$ of M we have $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. This shows that $p^{-1}(U)$ is in fact an open subset of $\widetilde{M}$. Next note that it follows from the Basis Characterization Lemma 3.57 that the domains of the charts for M have the basis property for the topological space M. Combining these two observations with the Subbasis-Continuity Proposition 4.4 we see that p is indeed continuous. ⊞

Claim 2. The projection $p\colon \widetilde{M} \to M$ is a 2-fold covering.

Proof. Let $Q \in M$. We pick a chart $\Phi\colon U \to V$ around Q such that U is connected. Note that $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. It follows from the Manifold–Orientation Basics Lemma 31.31 (1), applied to the connected smooth manifold U, that $B_-(\Phi)$ and $B_+(\Phi)$ are disjoint. Evidently the maps $p\colon B_\pm(\Phi) \to U$ are bijections.

Let $\epsilon \in \{-,+\}$. Using the Subbasis-Open Map Lemma 4.14 one sees easily that the map $p\colon B_\epsilon(\Phi) \to U$ is open. We already know that the map $p\colon B_\epsilon(\Phi) \to U$ is continuous. Thus it follows from the Open-Closed–Injective Lemma 4.22 that the map $p\colon B_\epsilon(\Phi) \to U$ is a homeomorphism.

In summary we have shown that U is uniformly covered by the two open subsets $B_\pm(\Phi)$. This implies that p is a 2-fold covering. ⊞

Finally note that the uniqueness of the topological structure on $\widetilde{M}$ with the desired properties is an almost immediate consequence of the Bijective-Local Homeomorphism Lemmalem:bijective-local-homeo.

(2) This statement is an immediate consequence of (1) and the Manifold Covering Proposition 95.4 (2).

(3) It follows easily from the definition of the smooth structure on $\widetilde{M}$, see the Manifold Covering Proposition 95.4 (2), that a smooth atlas for $\widetilde{M}$ is given by the maps $\Phi \circ p\colon B_\epsilon(\Phi) \to V$, where Φ runs over all smooth charts $\Phi\colon U \to V$ of M with U *connected*, and where $\epsilon \in \{-,+\}$. By the Orientations-via-Atlas Lemma 31.21 it suffices to consider the continuity of the orientation with respect to the above atlas. So let $\Phi\colon U \to V$ be a chart of M such that U is connected and let $\epsilon \in \{-,+\}$. We denote by $\Phi_\epsilon = \Phi \circ p\colon B_\pm(\Phi) \to V$ the corresponding chart of $\widetilde{M}$. But then by construction we know that Φ_ϵ is orientation-preserving for all points if $\epsilon = +$ and it is orientation-reversing for all points if $\epsilon = -$. ■

The following proposition summarizes the key properties of the orientation coverings.

Proposition 95.10. (Orientation Covering–Properties-Proposition)

(1) Let $f\colon M \to N$ be a local diffeomorphism between smooth manifolds. The map

$$\begin{array}{rcl} \widetilde{f}\colon \widetilde{M} & \to & \widetilde{N} \\ (P, \mathcal{O}) & \mapsto & (f(P), \mathrm{D}f_P(\mathcal{O})). \end{array}$$

where $\mathcal{O}$ is the orientation of $\mathrm{T}_P M$ and $\mathrm{D}f_P(\mathcal{O})$ is the orientation of $\mathrm{T}_{f(P)}N$.

is the unique orientation-preserving local diffeomorphism $\widetilde{M} \to \widetilde{N}$ that has the property that the following diagram commutes:[961]

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\tilde{f}} & \widetilde{N} \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow{f} & N. \end{array}$$

(2) Let M be a connected smooth manifold.

(a) If M is oriented, then there exists a unique orientation-preserving diffeomorphism $\Theta\colon \widetilde{M} \to M \sqcup -M$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Theta} & M \sqcup -M \\ & {\scriptstyle p_M}\searrow \quad \swarrow{\scriptstyle q} & \\ & M & \end{array}$$

where q is the obvious map that is given by the identity on each component of $M \sqcup -M$.

(b) If M is non-orientable, then $\widetilde{M}$ is connected.

(3) Let M be a smooth manifold. The map

$$\begin{array}{rcl} \Xi_M\colon \widetilde{M} & \to & \widetilde{M} \\ (Q, \mathcal{O}) & \mapsto & (Q, -\mathcal{O}) \end{array}$$

is the unique orientation-reversing self-diffeomorphism $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ which has the property that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Xi_M} & \widetilde{M} \\ & {\scriptstyle p_M}\searrow \quad \swarrow{\scriptstyle p_M} & \\ & M. & \end{array}$$

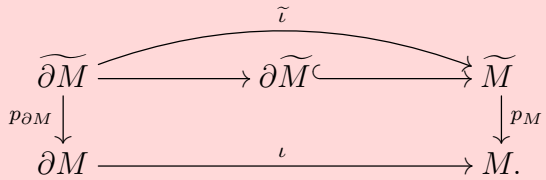
Furthermore, given a local diffeomorphism $f\colon M \to N$ we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(4) Let M be a non-orientable connected smooth manifold. Suppose that $q\colon \widehat{M} \to M$ is a 2-fold covering of smooth manifolds such that $\widehat{M}$ is a connected orientable smooth manifold. There exists a unique diffeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\Phi} & \widehat{M} \\ & {\scriptstyle p_M}\searrow \quad \swarrow{\scriptstyle q} & \\ & M. & \end{array}$$

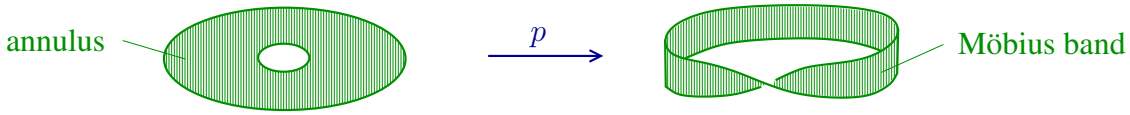
If $\widehat{M}$ is in fact oriented, then there exists a unique orientation-preserving diffeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ that makes the above diagram commute.

(5) Let M be a smooth manifold. We denote by $\iota\colon \partial M \to M$ the inclusion. There exists a unique natural diffeomorphism $\partial\widetilde{M} \to \widetilde{\partial M}$ such that the following diagram commutes:



Remark. Note that in the Orientation Covering–Properties-Proposition 191.29 we will prove an analog of the Orientation Covering–Properties-Proposition 95.10 for topological manifolds. □

Example. It follows from the uniqueness statement of the Orientation Covering–Properties-Proposition 95.10 (4) and the discussion on page 1844 that the orientation covering of the Möbius band is diffeomorphic to the annulus and that the orientation covering of the Klein bottle is diffeomorphic to the torus.



Similarly one sees that for even n the orientation covering of $\mathbb{R}\mathrm{P}^n$ is diffeomorphic to S^n.□

The proof of the Orientation Covering–Properties-Proposition 95.10 relies on the following lemma, which is of independent interest.

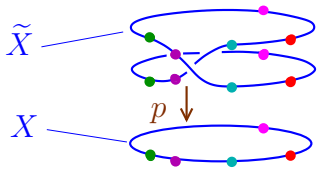
Lemma 95.11. (Preimage–Swapping Lemma) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a topological space X.

(1) The map

$$\begin{array}{rccl} \Xi\colon & \widetilde{X} & \to & \widetilde{X} \\ & x & \mapsto & \text{the unique element in } p^{-1}(p(\{x\})) \setminus \{x\} \end{array}$$

is continuous.

(2) Let A be a topological space and let $f\colon A \to X$ be a continuous map. We say that a map $\widetilde{f}\colon A \to \widetilde{X}$ is a *lift of* f, if $p \circ \widetilde{f} = f$. If $\widetilde{f}\colon A \to \widetilde{X}$ is a lift of f, then $\Xi \circ \widetilde{f}$ is the unique other lift of f.



$\Xi\colon \widetilde{X} \to \widetilde{X}$ swaps the points of the same color

Proof of Lemma 95.11. We proved the lemma in Exercise 92.3. ■

Sketch of proof of Proposition 95.10.

[961] We leave it to the reader to express this statement in categorical language.

(1) It is clear that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[=:\widetilde{f}]{(Q,\mathcal{O})\mapsto(f(Q),\mathrm{D}f_Q(\mathcal{O}))} & \widetilde{N} \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

From the fact that p_M, p_N and f are local diffeomorphisms it follows easily that the top map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ is also a local diffeomorphism. It follows immediately from the definitions of the orientations on $\widetilde{M}$ and $\widetilde{N}$ that $\widetilde{f}$ is indeed orientation-preserving. The proof of the fact that $\widetilde{f}$ is the unique map with the stated properties follows easily from the Preimage–Swapping Lemma 95.11 (2).

(2) Let M be a connected smooth manifold.

(a) We suppose that M is oriented. One can easily verify that the map

$$\begin{array}{rcl} \Theta\colon \widetilde{M} & \to & M \sqcup -M \\ (P,\mathcal{O}) & \mapsto & \begin{cases} P \in M, & \text{if } \mathcal{O} \text{ is a positive orientation,} \\ P \in -M, & \text{if } \mathcal{O} \text{ is a negative orientation} \end{cases} \end{array}$$

is an orientation-preserving diffeomorphism $\Theta\colon \widetilde{M} \to M \sqcup -M$ that has the desired properties. Uniqueness can once again be deduced from the Preimage–Swapping Lemma 95.11 (2).

(b) Now we suppose that M is non-orientable. We need to show that $\widetilde{M}$ is connected. If $\widetilde{M}$ was not connected, then $\widetilde{M}$ would consist of at least two components $\{\widetilde{M}_i\}_{i\in I}$. Since M is connected it follows from the Covering-Basics Lemma 91.12 (5) that the restriction of each $p\colon \widetilde{M} \to M$ to $p_i\colon \widetilde{M}_i \to M$ is also a covering of smooth manifolds. Clearly we have the equalities $2 = [\widetilde{M} : M] = \sum_{i\in I} [\widetilde{M}_i : M]$. It follows that $\#I = 2$ and that each $p_i\colon \widetilde{M}_i \to M$ is a covering of degree one, i.e. each p_i is a diffeomorphism. But that is not possible since M was assumed to be non-orientable whereas $\widetilde{M}_i$ is orientable.

(3) It is straightforward to show that Ξ_M is a local diffeomorphism. It is basically clear that it is orientation-preserving. The uniqueness follows easily from the Preimage–Swapping Lemma 95.11 (2). Finally, if we are given given a local diffeomorphism $f\colon M \to N$, then it follows immediately from the definitions that we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(4) Let M be a non-orientable connected smooth manifold. Suppose we are given a 2-fold covering $q\colon N \to M$ of smooth manifolds where N is a connected orientable smooth manifold and where q is a local diffeomorphism. Since q is a local diffeomorphism we can consider the corresponding local diffeomorphism $\widetilde{q}\colon \widetilde{N} \to \widetilde{M}$ from (1) which makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\quad \widetilde{q} \quad} & \widetilde{M} \\ {\scriptstyle p_N}\downarrow & & \downarrow{\scriptstyle p_M} \\ N & \xrightarrow{\quad q \quad} & M. \end{array}$$

It follows easily from the fact that q, p_M and p_N are 2-fold coverings of smooth manifolds that $\widetilde{q}\colon \widetilde{N} \to \widetilde{M}$ is also a 2-fold covering of smooth manifolds. From the fact that N is

connected and orientable together with (2) we obtain that $\widetilde{N}$ consists of two components $\widetilde{N}_1$ and $\widetilde{N}_2$. As in (2) we see that the restriction of p_N to $\widetilde{N}_1$ and the restriction of $\widetilde{q}$ to $\widetilde{N}_2$ are 1-fold coverings of smooth manifolds. In other words $p_N\colon \widetilde{N}_1 \to N$ is in fact a diffeomorphism. One can now easily verify that $\widetilde{q} \circ (p_N|_{\widetilde{N}_1})^{-1}\colon N \to \widetilde{M}$ has the desired properties.

If N is oriented, then, after possibly composing the diffeomorphism $N \to \widetilde{M}$ with the orientation-reversing diffeomorphism $\Xi\colon \widetilde{M} \to \widetilde{M}$ from (3), we can arrange that the diffeomorphism $N \to \widetilde{M}$ is orientation-preserving. Once again one can deduce from the Preimage–Swapping Lemma 95.11 (2) that the orientation-preserving diffeomorphism $\Phi\colon N \to \widetilde{M}$, that commutes with the projections to M, is unique.

(5) We will not make use of this statement. Thus we have no qualms with delegating the proof of this statement to Exercise 95.10. ■

95.4. **The orientation character.** We start out with an elementary lemma in group theory.

Lemma 95.12. (Index 2–Subgroup Lemma)
(1) Let π be a group. Any subgroup of index 2 is normal.
(2) If G is a group with two elements, then there exists a unique isomorphism $G \to \mathbb{Z}/2\mathbb{Z}$.

Proof. We outsource the proof to Exercise 95.1. ■

Convention 95.13. Let G be a group with two elements. It follows from the Index 2–Subgroup Lemma 95.12 (2) that we can safely make the identification $G =_{\mathrm{i}} \mathbb{Z}/2\mathbb{Z}$.

Proposition 95.14. (Orientation Character Proposition) Let M be a non-orientable connected smooth manifold and let $x_0 \in M$. We denote by $p\colon \widetilde{M} \to M$ the orientation cover. Finally we pick a point $\widetilde{x}_0 \in \widetilde{M}$ with $p(\widetilde{x}_0) = x_0$.
(1) (a) The subgroup $p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \subseteq \pi_1(M, x_0)$ is a normal subgroup of index 2.
(b) The subgroup $p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \subseteq \pi_1(M, x_0)$ does not depend on the choice of $\widetilde{x}_0$.
(2) There exists a unique isomorphism $\pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \to \mathbb{Z}/2\mathbb{Z}$.
(3) The projection

$$w_1^{x_0}\colon \pi_1(M, x_0) \;\to\; \pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \underset{\uparrow}{=_{\mathrm{i}}} \mathbb{Z}/2\mathbb{Z}$$

see above convention

is an epimorphism that does not depend on the choice of $\widetilde{x}_0$.
(4) Let $x_1 \in M$ be some other point and let $\gamma\colon [0,1] \to M$ be a path from x_0 to x_1. We denote by

$$\begin{array}{rcl} \gamma_*\colon \pi_1(M, x_1) & \to & \pi_1(M, x_0) \\ [\varphi] & \mapsto & [\gamma * \varphi * \overline{\gamma}] \end{array}$$

the isomorphism that we introduced in the Change-of-Base Point Proposition 88.14 (1). The following diagram commutes:

$$\begin{array}{ccc} \pi_1(M, x_1) & \xrightarrow{w_1^{x_1}} & \\ \downarrow{\scriptstyle \gamma_*} & & \mathbb{Z}/2\mathbb{Z}. \\ \pi_1(M, x_0) & \xrightarrow{w_1^{x_0}} & \end{array}$$

Proof. Let M be a non-orientable connected smooth manifold, let $x_0 \in M$ and let $\widetilde{x}_0 \in \widetilde{M}$ with $p(\widetilde{x}_0) = x_0$.

(1) (a) Note that it follows from the Orientation Covering–Existence Proposition 95.8 that the orientation covering $p\colon \widetilde{M} \to M$ has degree 2. Furthermore, since M is non-orientable we know by the Orientation Covering–Properties-Proposition 95.10 (2b) that $\widetilde{M}$ is connected Therefore it follows from the Covering-Degree Proposition 92.10 that $p_*(\pi_1(\widetilde{M}))$ is an index 2 subgroup of $\pi_1(M)$. By the Index 2–Subgroup Lemma 95.12 (1) we know that every index 2 subgroup of a group is normal. In particular $p_*(\pi_1(\widetilde{M}))$ is a normal subgroup of $\pi_1(M)$ of index 2.

(b) Let $\widetilde{x}_0'$ be the other point in the preimage of x_0. Since M is non-orientable we know that $\widetilde{M}$ is path-connected, thus there exists a path $\widetilde{\gamma}$ from $\widetilde{x}_0'$ to $\widetilde{x}_0$. We write $\gamma := p \circ \widetilde{\gamma}$. Note that γ is a loop in x_0. It then follows easily from the Change-of-Base Point Proposition 88.14 that $p_*(\pi_1(\widetilde{M}, \widetilde{x}_0')) = g \cdot p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \cdot g^{-1}$. But by (a) we know that $p_*(\pi_1(\widetilde{M}, \widetilde{x}_0))$ is a normal subgroup. Thus we see that $p_*(\pi_1(\widetilde{M}, \widetilde{x}_0')) = g \cdot p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \cdot g^{-1}$.

(2) This statement is an immediate consequence of the Index 2–Subgroup Lemma 95.12 (2).

(3) By the above it remains to show that the epimorphism $w_1^{x_0}$ does not depend on the choice of $\widetilde{x}_0$. So let $\widehat{x}_0$ be the unique other point in $p^{-1}(x_0)$. By the Orientation Covering–Properties-Proposition 95.10 (3) there exists a homeomorphism $\Xi\colon \widetilde{M} \to \widetilde{M}$ with $\Xi(\widetilde{x}_0) = \widehat{x}_0$ and with $p = p \circ \Xi$. We see that

$$\begin{aligned} p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) &\overset{\text{since } p = p\circ\Xi}{=} (p \circ \Xi)_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) \overset{\text{functoriality of } \pi_1}{=} p_*(\Xi_*(\pi_1(\widetilde{M}, \widetilde{x}_0))) \\ &= p_*(\pi_1(\widetilde{M}, \Xi(\widetilde{x}_0))) \\ &\overset{\text{since } \Xi \text{ is a homeomorphism and since } \pi_1 \text{ is functorial}}{=} p_*(\pi_1(\widetilde{M}, \widehat{x}_0)). \end{aligned}$$

(4) Let $\gamma\colon [0,1] \to M$ be a path from x_0 to x_1. By the Path-Lifting Proposition 92.6 (1) there exists a unique lift $\widetilde{\gamma}\colon [0,1] \to M$ of γ to the starting point $\widetilde{x}_0$. By the Change-of-Base Point Proposition 90.4 (1) the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\widetilde{M}, \gamma(1)) & \xrightarrow{p_*} & \pi_1(M, x_1) \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ \pi_1(\widetilde{M}, \widetilde{x}_0) & \xrightarrow{p_*} & \pi_1(M, x_0). \end{array}$$

But this implies immediately the promised statement. ∎

The Orientation Character Proposition 95.14 allows us to introduce the following definition:

Definition 95.15. Let M be a connected smooth manifold and let $x_0 \in M$.

(1) If M is non-orientable, then we consider the orientation covering $p\colon \widetilde{M}^{\mathrm{or}} \to M$ and we pick $\widetilde{x}_0 \in \widetilde{M}^{\mathrm{or}}$ with $p(\widetilde{x}_0) = x_0$. We then refer to the epimorphism

$$w_1^{x_0}(M) := w_1^{x_0}\colon \pi_1(M, x_0) \;\to\; \pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}^{\mathrm{or}}, \widetilde{x}_0)) =_{\mathrm{i}} \mathbb{Z}/2\mathbb{Z},$$

as the orientation character of M. Note that it follows from the e Orientation Character Proposition 95.14 (1b) that this definition does not depend on the choice of the point $\widetilde{x}_0$.

(2) If M is orientable, then we define the orientation character of M to be the trivial homomorphism $w_1^{x_0}\colon \pi_1(M,x_0) \to \mathbb{Z}/2\mathbb{Z}$.[962]

It is worth formulating the following immediate corollary to the Orientation Character Proposition 95.14.

Corollary 95.16. (Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary)

(1) If M is a non-orientable connected non-empty smooth manifold, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$.

(2) Every simply connected smooth manifold is orientable.

We conclude this section with the observation that the orientation character is in a suitable sense contravariantly functorial:

Lemma 95.17. (w_1-Functorial Lemma) Let $f\colon M \to N$ be a local diffeomorphism between connected smooth manifolds and let $x_0 \in M$. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(M,x_0) & \xrightarrow{\quad f_* \quad} & \pi_1(N,f(x_0)) \\ & {\scriptstyle w_1^{x_0}(M)} \searrow \quad \swarrow {\scriptstyle w_1^{f(x_0)}(N)} & \\ & \mathbb{Z}/2\mathbb{Z}. & \end{array}$$

Sketch of proof. This statement follows easily from the naturality statement of the Orientation Covering–Properties-Proposition 95.10 (1). We leave it to the reader to fill in the details. ■

95.5. **Orientation-preserving and orientation-reversing loops.** We start out with the following frequently used definition:

Definition 95.18. Let M be a connected smooth manifold and let $\gamma\colon [0,1] \to M$ be a loop in a point x_0. We say that γ is orientation-preserving if $w_1^{x_0}(\gamma) = 0$ otherwise we say that γ is orientation-reversing.

Remark. In Definition 45.3 we already introduced the notion of orientation-preserving and orientation-reversing loops in a more restrictive setup. It follows immediately from the Orientation–Smooth Isotopy Proposition 95.20 below that the two definitions agree on the type of loops that are covered by both definitions. □

The following lemma gives a practical criterion for determining whether or not a loop is orientation-preserving.

Lemma 95.19. (Orientation–Lift-of-Loop Lemma) Let M be a connected smooth manifold. As before we denote by $p\colon \widetilde{M} \to M$ the orientation covering provided by

[962] The notation $w_1^{x_0}$ comes from the fact that later, on page 5295, we will relate the orientation character $w_1^{x_0}\colon \pi_1(M,x_0) \to \mathbb{Z}/2\mathbb{Z}$ to the Stiefel-Whitney class $w_1(\mathrm{T}M) \in \mathrm{H}^1(M;\mathbb{F}_2)$ which we will introduce in Definition 274.14. We will see that there is not only one Stiefel-Whitney class, but there is in fact a Stiefel-Whitney class $w_k(\mathrm{T}M) \in \mathrm{H}^k(M;\mathbb{F}_2)$ for each $k \in \mathbb{N}_0$.

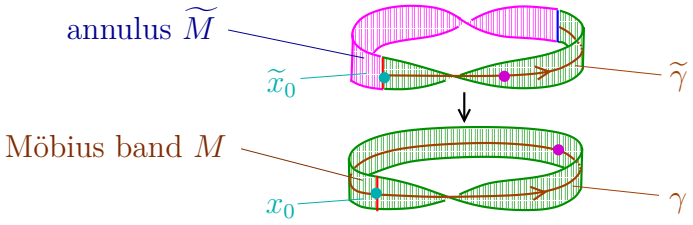
the Orientation Covering–Existence Proposition 95.8. Let $x_0 \in M$ be a point and let $\gamma\colon [0,1] \to M$ be a loop in x_0. We pick $\widetilde{x}_0 \in p^{-1}(x_0)$ and we denote by $\widetilde{\gamma}\colon [0,1] \to \widetilde{M}$ the unique lift of γ, provided by the Path-Lifting Proposition 92.6 (1), to the starting point x_0. Then

$$\begin{aligned} \gamma \text{ is } \textit{orientation-preserving} &\iff \widetilde{\gamma}(1) = \widetilde{\gamma}(0) \\ \gamma \text{ is } \textit{orientation-reversing} &\iff \widetilde{\gamma}(1) \neq \widetilde{\gamma}(0). \end{aligned}$$

Proof. This lemma follows immediately from the Loop–Lifting Proposition 92.8. ■

Example.

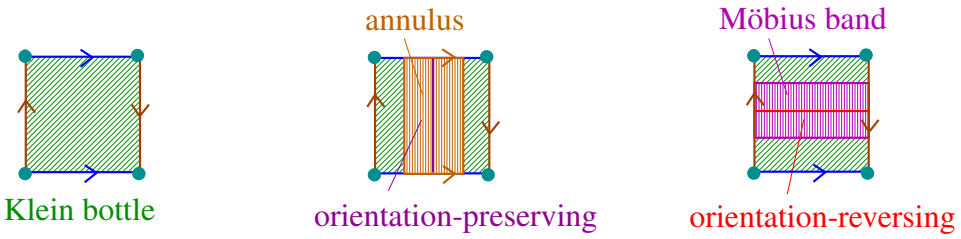
(1) Let M be the Möbius band and let $\widetilde{M}$ be the annulus. We consider the 2-fold covering $p\colon \widetilde{M} \to M$ that is defined on page 1783. Note that by the Orientation Covering–Properties-Proposition 95.10 (4) we know that this covering is just an incarnation of the orientation covering. We pick a point in M and we denote by γ a loop that goes "once around M".



The lift of γ is not a loop, hence it follows from the Orientation–Lift-of-Loop Lemma 95.19 that γ is orientation-reversing.

(2) Let Σ be a 2-dimensional smooth manifold and let $\gamma \subseteq \Sigma$ be a smoothly embedded loop. From the discussion in (1) and the w_1-Functorial Lemma 95.17 we obtain the following useful observation:

- If the tubular neighborhood of γ is an annulus, then γ is orientation-preserving.
- If the tubular neighborhood of γ is a Möbius band, then γ is orientation-reversing.



A similar discussion also holds in higher dimensions, we just need to do the following:[963] [964]

- We need to replace the annulus by the product $\overline{B}^{n-1} \times S^1$.

[963] In fact it follows easily from the Tubular Neighborhood–Existence Theorem 57.11 together with the Linear $\overline{B}^k$-Bundle-Isomorphism Lemma 58.6 that any smoothly embedded loop in an n-dimensional smooth manifold admits a tubular neighborhood that is of one of these two forms.

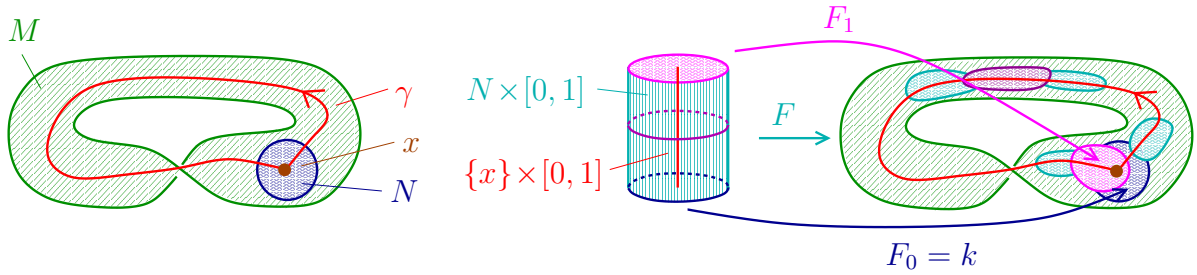
[964] Also note that it follows from the Deform-to-Embedding Theorem **??** that in dimension $n \geq 3$ any element of $\pi_1(M, x_0)$ can be represented by a smoothly embedded loop. This discussion shows that an n-dimensional smooth manifold M is orientable if and only if for each smoothly embedded loop $\gamma \subseteq M$ the tubular neighborhood is diffeomorphic to a product $\overline{B}^{n-1} \times \gamma$. The reader can easily verify that the statement also holds in dimension 2.

- We need to replace the Möbius band by the "twisted" $\overline{B}^{n-1}$-bundle $(\overline{B}^{n-1} \times [0,1])/\sim$ where $(x,0) \sim (A \cdot x, 1)$ for some $A \in \mathrm{O}(n-1)$ with $\det(A) = -1$. □

The following proposition hopefully justifies the names "orientation-preserving" and "orientation-reversing" for loops.

Proposition 95.20. (Orientation–Smooth Isotopy Proposition) Let M be a smooth manifold, let $\gamma\colon [0,1] \to M$ be a smooth loop at a point $x \in M$ and let N be a codimension-zero submanifold of M with $x \in N \setminus \partial N$. We denote by $k\colon N \to M$ the inclusion map. Finally let $F\colon N \times [0,1] \to M$ be a smooth isotopy with $F_0 = k$ such that for each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$. Then[965]

$$(\mathrm{D}k_x)^{-1} \circ \mathrm{D}(F_1)_x\colon \mathrm{T}_xN \to \mathrm{T}_xN \text{ is orientation-preserving} \iff \gamma \text{ is orientation-preserving.}$$



The proof of the Orientation–Smooth Isotopy Proposition 95.20 rests squarely on the shoulders of the following rather technical lemma.

Auxiliary Lemma 95.21. Let M be a smooth manifold. We denote by $p\colon \widetilde{M} \to M$ the orientation cover. Let $\gamma\colon [0,1] \to M$ be a smooth loop at some point $x \in M$ and let N be a codimension-zero submanifold of M with $x \in N \setminus \partial N$. We denote by $k\colon N \to M$ the inclusion map. Let $F\colon N \times [0,1] \to M$ be a smooth isotopy with $F_0 = k$ such that for each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$. We pick an orientation $\mathcal{O}_x$ of T_xN and we denote by $\widetilde{\gamma}\colon [0,1] \to \widetilde{M}$ the unique lift of $\gamma\colon [0,1] \to M$ to the starting point $(x, \mathrm{D}\,k_x(\mathcal{O}_x))$. Then for each $t \in [0,1]$ the map $(\mathrm{D}p_{\widetilde{\gamma}(t)})^{-1} \circ \mathrm{D}(F_t)_x\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M}$ is orientation-preserving.

Proof of Auxiliary Lemma 95.21. We consider the following two subsets of $[0,1]$:

$$\begin{aligned} A &:= \{t \in [0,1] \mid (\mathrm{D}p_{\widetilde{\gamma}(t)})^{-1} \circ \mathrm{D}(F_t)_x\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M} \text{ is orientation-preserving}\} \\ B &:= \{t \in [0,1] \mid (\mathrm{D}p_{\widetilde{\gamma}(t)})^{-1} \circ \mathrm{D}(F_t)_x\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M} \text{ is orientation-reversing}\}. \end{aligned}$$

We want to show that $A = [0,1]$. By the choice of the lift $\widetilde{\gamma}$ and since $F_0 = k$ we know that $0 \in A$. By the Interval-Connected Proposition 4.32 we know that the interval $[0,1]$ is connected. Thus it remains to show that A and B are both open subsets of $[0,1]$.

In the following we will show that A is open. The proof that B is open is basically the same. Thus let $t \in A$. Since $p\colon \widetilde{M} \to M$ is a local diffeomorphism we can pick an open neighborhood $\widetilde{U}$ of $\widetilde{\gamma}(t) \in \widetilde{M}$ such that $p(U)$ is an open neighborhood of $\gamma(t)$ and such that the projection $p\colon \widetilde{U} \to U := p(\widetilde{U})$ is a diffeomorphism.

(1) We denote by $q\colon U \to \widetilde{U}$ the inverse of $p\colon \widetilde{U} \to U$.

[965] In Definition 31.17 we defined what it means for an *automorphism* of a finite-dimensional real vector space to be orientation-preserving.

(2) We denote by $j\colon U \to M$ and $\widetilde{j}\colon \widetilde{U} \to \widetilde{M}$ the inclusion maps.
(3) We equip $\widetilde{U}$ with the standard codimension-zero submanifold orientation from page 819. In other words, we equip $\widetilde{U}$ with the unique orientation such that for each $y \in \widetilde{U}$ the map $\mathrm{D}\widetilde{j}_y\colon \mathrm{T}_y\widetilde{U} \to \mathrm{T}_y\widetilde{M}$ is orientation-preserving.

Since $F\colon N \times [0,1] \to M$ is continuous and since U is an open subset of M we know that $F^{-1}(U)$ is an open neighborhood of (x,t). Using the fact that $x \notin \partial N$ we see that we can pick a product neighborhood $W \times [a,b]$ of (x,t) with the following properties:

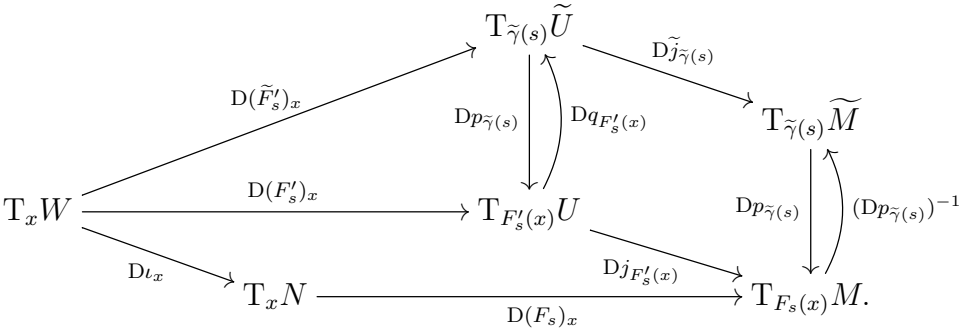
(1) We have $(x,t) \in W \times [a,b] \subseteq F^{-1}(U)$.
(2) $W \subseteq N \setminus \partial N$ is an open subset that is diffeomorphic to an open ball.
(3) $[a,b] \subseteq [0,1]$ is a neighborhood of $t \in [0,1]$.

Next we introduce a few more players:

- We denote by $\iota\colon W \to N$ the inclusion map.
- Since W is diffeomorphic to an open ball we know that W is orientable. We equip it with the unique orientation such that $\mathrm{D}\iota_x\colon \mathrm{T}_xW \to \mathrm{T}_xN$ is orientation-preserving.
- We denote by $F'\colon W \times [a,b] \to U$ the restriction of $F\colon N \times [0,1] \to M$ to $W \times [a,b]$.
- We consider the map
$$\begin{aligned} \widetilde{F}'\colon W \times [a,b] &\to \widetilde{U} \\ (w,s) &\mapsto q(F'(w,s)) = (q \circ F'_s)(w). \end{aligned}$$
Since q is a diffeomorphism we see that $\widetilde{F}'$ is a smooth isotopy.

Claim. For $s \in [a,b]$ we have
$$\text{the map } \mathrm{D}\widetilde{F}'_s\colon \mathrm{T}_xW \to \mathrm{T}_{\widetilde{\gamma}(s)}\widetilde{U} \text{ is orientation-preserving} \iff s \in A.$$

Proof. To prove the claim we consider the following diagram:



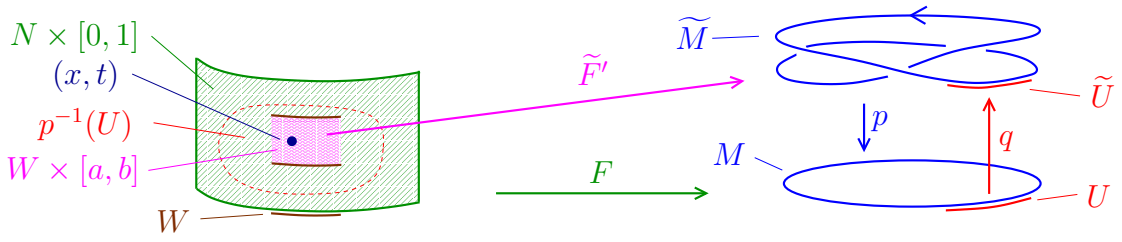
Note that the above diagram commutes by the functoriality of the differential and the fact that $\widetilde{F}'_s = q \circ F'_s\colon W \to \widetilde{U}$, $j \circ F'_s = F_s \circ \iota$, $p \circ \widetilde{j} = j \circ p\colon \widetilde{U} \to M$ and $q = p^{-1}\colon \widetilde{U} \to U$. It follows from the commutativity of the diagram that

$$\underbrace{\mathrm{D}\widetilde{j}_{\widetilde{\gamma}(s)}}_{\substack{\text{orientation-preserving}\\ \text{by definition of the}\\ \text{orientation on } \widetilde{U}}} \circ \mathrm{D}(\widetilde{F}'_s)_x \circ \underbrace{(\mathrm{D}\iota_x)^{-1}}_{\substack{\text{orientation-preserving}\\ \text{by definition of the}\\ \text{orientation of } \mathrm{T}_xW}} = \underbrace{(\mathrm{D}p_{\widetilde{\gamma}(s)})^{-1} \circ \mathrm{D}(F_s)_x}_{\substack{\text{orientation-preserving}\\ \text{precisely if } s \in A}}\colon \mathrm{T}_xN \to \mathrm{T}_{\widetilde{\gamma}(t)}\widetilde{M}.$$

The claim follows immediately from this observation. ⊞

Now we can easily conclude the proof of the lemma.

(1) Since $t \in A$ we obtain from the claim that $\widetilde{F}'_t\colon W \to \widetilde{U}$ is orientation-preserving at x.



(2) It follows from the Manifold–Orientation Basics Lemma 31.31 that $\widetilde{F}'_t \colon W \to \widetilde{U}$ is orientation-preserving.
(3) It follows from the Smooth Isotopy-Orientation Proposition 42.4 that $\widetilde{F}'_s \colon W \to \widetilde{U}$ is orientation-preserving for all $s \in [a,b]$.
(4) It follows from (3) that the differential $\mathrm{D}\widetilde{F}'_s \colon \mathrm{T}_x W \to \mathrm{T}_{\widetilde{\gamma}(s)}\widetilde{U}$ is orientation-preserving for all $s \in [a,b]$.
(5) It follows from (4) and the claim that $[a,b] \subseteq A$.
(6) Since $[a,b]$ is a neighborhood of $t \in [0,1]$ we now deduce from the JH-Openness Criterion 3.3 that A is indeed an open subset of $[0,1]$. ∎

Proof of the Orientation–Smooth Isotopy Proposition 95.20. Let M be a smooth manifold. Furthermore let $\gamma \colon [0,1] \to M$ be a smooth loop at a point $x \in M$ and let N be a codimension-zero submanifold of M with $x \in N \setminus \partial N$. We denote by $k \colon N \to M$ the inclusion map. Finally let $F \colon N \times [0,1] \to M$ be a smooth isotopy with $F_0 = k$ such that for each $t \in [0,1]$ we have $F_t(x) = \gamma(t)$. We set $\Phi = F_1 \colon N \to M$ and we set

$$\epsilon \;:=\; \begin{cases} +1, & \text{if } \gamma \text{ is orientation-preserving,} \\ -1, & \text{if } \gamma \text{ is orientation-reversing.} \end{cases}$$

We need to show that

$$(\mathrm{D}k_x)^{-1} \circ \mathrm{D}\Phi_x \colon \mathrm{T}_x N \to \mathrm{T}_x N \text{ is orientation-preserving} \iff \epsilon = 1.$$

We pick an orientation $\mathcal{O}_x$ of $\mathrm{T}_x N$. Let $\widetilde{\gamma} \colon [0,1] \to \widetilde{M}$ be the unique lift of $\gamma \colon [0,1] \to M$ to the starting point $(x, \mathrm{D}\,k_x(\mathcal{O}_x))$. Now we see that

$$\begin{aligned} \mathrm{D}\Phi_x(\mathcal{O}_x) &= \mathrm{D}p_{(x,\epsilon\cdot \mathrm{D}k_x(\mathcal{O}_x))} \circ (\mathrm{D}p_{(x,\epsilon\cdot \mathrm{D}k_x(\mathcal{O}_x))})^{-1} \circ \mathrm{D}\Phi_x(\mathcal{O}_x) \\ &\underset{\uparrow}{=} \mathrm{D}p_{(x,\epsilon\cdot \mathrm{D}k_x(\mathcal{O}_x))}(\text{orientation of } \widetilde{M} \text{ at } (x, \epsilon \cdot \mathrm{D}k_x(\mathcal{O}_x))) \underset{\uparrow}{=} \epsilon \cdot \mathrm{D}k_x(\mathcal{O}_x). \end{aligned}$$

by Lemma 95.21 (1) applied to $t = 1$, note that $\gamma(1) = (x, \epsilon \cdot \mathrm{D}k_x(\mathcal{O}_x))$ and $\Phi = F_1$

by the Orientation Covering Existence Proposition 95.8

But this is precisely what we needed to show. ∎

95.6. **Orientation flipping diffeotopies II.** In this section we will give a new proof of the Orientation-Flipping Proposition 45.6. Before we can restate and reprove said proposition let us recall Definition 41.18:

Definition 95.22. Let X be a topological space. In the following we say that a homotopy $F \colon X \times [0,1] \to X$ has compact support if there exists a compact subset $K \subseteq X$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

Now we can recall the Orientation-Flipping Proposition 45.6.

Proposition 45.6. (Orientation-Flipping Proposition) Let M be a non-orientable connected smooth manifold and let $X, Y \subseteq M \setminus \partial M$ be two disjoint finite subsets. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $X \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) For each $x \in X$ the map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

Remark. Our first proof of the Orientation-Flipping Proposition 45.6 rested on the Isotopy Extension Theorem 44.2. Our new proof only requires the existence of the orientation cover. Our new proof also has the big advantage that it can be generalized to topological manifolds. We refer to page 3701 for a few more details. □

Our new proof of the Orientation-Flipping Proposition 45.6 is in many ways similar to the previous proof. In particular it rests on elementary two auxiliary lemmas which we now recall.

Auxiliary Lemma 45.7. Let X be a topological manifold, let $A \subseteq X$ be a closed subset and let $F\colon X \times [0,1] \to X$ be a homotopy rel A. If F has compact support, then there exists an open neighborhood U of A such that F is a homotopy rel U.

Auxiliary Lemma (Extend Diffeotopy Lemma) 45.8. Let M be a smooth manifold and let U be an open subset of M. Furthermore let $G\colon U \times [0,1] \to U$ be a diffeotopy with compact support K. Then

$$\begin{array}{rcl} \widetilde{G}\colon M \times [0,1] & \to & M \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x \in K, \\ x, & \text{if } x \notin K \end{cases} \end{array}$$

is a diffeotopy of M.

Again all the work for the proof of the Orientation-Flipping Proposition 45.6 is now done in the proof of the following auxiliary lemma, which deals with the case that $\#X = 1$.

Auxiliary Lemma 45.9. Let M be a connected *non-orientable* smooth manifold. Furthermore let $Y \subseteq M$ be a finite set and let $x \in M \setminus \partial M$ be a point with $x \notin Y$. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel $\partial M \cup Y$.
(4) $\Phi := F_1$ is the identity on $\{x\} \cup Y$.
(5) The map $\Phi := F_1$ has the following properties:
  (a) The map $\mathrm{D}\Phi_x\colon \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation-reversing.
  (b) For each $y \in Y$ the map $\mathrm{D}\Phi_y\colon \mathrm{T}_y M \to \mathrm{T}_y M$ is orientation-preserving.

New proof of Lemma 45.9. Let M be a connected *non-orientable* smooth manifold. By the Smooth 1-Manifold Classification Theorem 28.8 we know that $\dim(M) \geq 2$. We denote

by $p\colon \widetilde{M} \to M$ the orientation covering from Definition 95.9. We recall that by the Orientation Covering–Existence Proposition 95.8 and the Orientation Covering Proposition 95.10 (2b) this covering has the following properties:

(O1) The map p is a local diffeomorphism. In particular for each $a \in \widetilde{M}$ the differential $\mathrm{D}p_a\colon \mathrm{T}_a\widetilde{M} \to \mathrm{T}_{p(a)}M$ is an isomorphism.
(O2) Our hypothesis that M is non-orientable implies that $\widetilde{M}$ is connected.

Now let $Y \subseteq M$ be a finite set and let $x \in M \setminus \partial M$ with $x \notin Y$.

(a) We pick an orientation $\mathcal{O}_x$ for T_xM.
(b) We set $W := M \setminus Y$ and $\widetilde{W} := p^{-1}(M \setminus Y)$. Since Y is finite we see that Y and $p^{-1}(Y)$ are closed subsets. It follows from the Open Subset-Submanifold Lemma 25.8 and the Open Subset-Submanifold Boundary Lemma 25.11 that W and $\widetilde{W}$ are smooth manifolds with $\partial W = \partial M$ and $\partial\widetilde{W} = \partial\widetilde{M}$.
(c) Since $p^{-1}(Y)$ is a finite set and since $\dim(\widetilde{W}) = \dim(\widetilde{M}) = \dim(M) \geq 2$ we know by the elementary Exercise 23.6 that $\widetilde{W}$ is also path-connected.
(d) It follows from the Smooth Path-Connectivity Proposition 25.32 together with the Push Away-from-Boundary Lemma 23.17 that there exists in fact a smooth path $\widetilde{\gamma}\colon [0,1] \to \widetilde{W} \setminus \partial\widetilde{W}$ from $(x, \mathcal{O}_x)$ to $(x, -\mathcal{O}_x)$.

By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $F\colon W \times [0,1] \to W$ with the following properties:

(0) For each $t \in [0,1]$ we have $F_t(x) = (p \circ \widetilde{\gamma})(t)$.
(1) $F_0 = \mathrm{id}$.
(2) F has compact support.
(3) F is a diffeotopy rel ∂W.

Note that it follows from (2), together with the Extend Diffeotopy Lemma 45.8, that we can extend F to a diffeotopy $F\colon M \times [0,1] \to M$ rel Y on all of M by setting $F(x,t) = x$ for $x \notin W$. Finally we set $\Phi := F_1\colon M \to M$.

By design F has properties (1), (2) and (3). Furthermore note that by Lemma 45.7 we know that F is actually a diffeotopy rel an open neighborhood of Y. This implies that for each $y \in Y$ we have $\mathrm{D}\Phi_y = \mathrm{id}_{\mathrm{T}_yM}$. Thus (5b) is also satisfied. Since $(p \circ \widetilde{\gamma})(1) = x$ we also see that $\Phi(x) = x$. Together with the above discussion this shows that Φ is the identity on $\{x\} \cup Y$. We thus verified that (4) holds.

It remains to show that (5a) holds. This statement is fortunately taken care of by the Orientation–Smooth Isotopy Proposition 95.20. Indeed, we just need to apply the Orientation–Smooth Isotopy Proposition 95.20 to the smooth manifold $N = M$ and to the smooth loop $p \circ \widetilde{\gamma}\colon [0,1] \to M$, which is orientation-reversing by the Orientation–Lift-of-Loop Lemma 95.19. ∎

New proof of the Orientation-Flipping Proposition 45.6. Let $x_1, \dots, x_k$ be the elements of X. It follows from the ever useful Smooth Isotopy-Equivalence Relation Proposition 41.5 that it suffices to flip the signs of the points $x_1, \dots, x_k$ one-by-one. But this can be done by applying Lemma 45.9 altogether k times, namely to the points x_i and the sets $Y_i := Y \cup \{x_1, \dots, \widehat{x_i}, \dots, x_k\}$. ∎

95.7. Exercises for Chapter 95.

Exercise 95.1.

(a) Let π be a group. Show that every index 2 subgroup of π is normal.
(b) Give an example of a group π together with a subgroup of index 3 that is not normal in π.
(c) Show that if G is a group with two elements, then there exists a *unique* isomorphism $G \to \mathbb{Z}/2\mathbb{Z}$.
(d) What is the smallest $n \in \mathbb{N}$ such that there exists a group with n elements which does not admit a *unique* isomorphism $G \to \mathbb{Z}/n\mathbb{Z}$?
(e) What is the smallest $n \in \mathbb{N}$ such that there exists a group G with n elements which does not admit an isomorphism $G \to \mathbb{Z}/n\mathbb{Z}$?

Exercise 95.2. Let $f\colon S^1 \to S^1$ be a continuous map with $f(1) = 1$. We denote by $p\colon \mathbb{R} \to S^1$ the covering that is given by $t \mapsto \exp(\mathrm{i}t)$.

(a) Show that there exists a unique continuous map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{f} & S^1. \end{array}$$

(b) Suppose that f is a homeomorphism. Show that the map $\widetilde{f}$ from (a) is also a homeomorphism.
(c) Suppose that f is a diffeomorphism. Show that the map $\widetilde{f}$ from (a) is also a diffeomorphism.

Exercise 95.3. Let $f\colon M \to N$ be a smooth map between two smooth manifolds. We assume that f is a covering of topological spaces and we assume that f is a local diffeomorphism. Does it follow that f is a covering of smooth manifolds?

Exercise 95.4.

(a) We consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N \end{array}$$

of continuous maps between smooth manifolds $\widetilde{M}$, M, $\widetilde{N}$ and N. We assume that p and q are covering maps of smooth manifolds and we assume that p and q have the same finite degree. Show that f is a diffeomorphism if and only if $\widetilde{f}$ is a diffeomorphism.
(b) Let M and N be smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We denote by $p\colon \widetilde{M} \to M$ and $q\colon \widetilde{M} \to M$ the orientation coverings of M and N. Show that there exists an orientation-preserving diffeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the

following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{f} & N. \end{array}$$

Exercise 95.5. Let K be the Klein bottle and let $x \in K$. Describe an explicit self-diffeomorphism $f\colon K \to K$ with $f(x) = x$ such that $\mathrm{D}f_x\colon \mathrm{T}K_x \to \mathrm{T}K_x$ is orientation-preserving.

Exercise 95.6. Let $p\colon T \to K$ be the 2-fold covering of the Klein bottle by the torus $T = S^1 \times S^1$ that we discussed on page 1783.

(a) Give an example of a smooth map $f\colon K \to K$ such that there is no continuous map $\widetilde{f}\colon T \to T$ which makes the following diagram commute:

$$\begin{array}{ccc} T & \xrightarrow{\widetilde{f}} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{f} & K. \end{array}$$

Remark. This exercise shows that not every continuous self-map of a smooth manifold gives rise to a continuous self-map of its orientation cover.

(b) Provide an explicit orientation-reversing diffeomorphism $f\colon T \to T$ such that $p \circ f = p$.

Exercise 95.7. Let M be an oriented n-dimensional smooth manifold, let N be a non-orientable connected n-dimensional smooth manifold and let $f\colon M \to N$ be a smooth embedding. We denote by $p\colon \widetilde{N} \to N$ the orientation-covering from the Orientation Covering–Properties-Proposition 95.10. Show that there exists an orientation-preserving smooth embedding $g\colon M \to \widetilde{N}$ with $p \circ g = f$.

Exercise 95.8. Let $f\colon M \to N$ be an immersion between two smooth manifolds with empty boundary and of the same dimension.

(a) We suppose that M and N are compact, non-empty and connected. Show that f is a covering map.
Hint. Use the Inverse Mapping Theorem 30.1.

(b) Does the conclusion of (a) still hold if instead we only assume that M and N are connected and that f is surjective?

Exercise 95.9. Let M and $\widetilde{M}$ be closed oriented connected non-empty smooth manifolds and let $p\colon \widetilde{M} \to M$ be a finite covering. We denote by $\deg(p) \in \mathbb{Z}$ the degree from Definition 46.3. Show that $\deg(p) = [\widetilde{M} : M]$.

Exercise 95.10. Let M be a smooth manifold. We denote by $\iota\colon \partial M \to M$ the inclusion. Show that there exists a unique natural diffeomorphism $\partial\widetilde{M} \to \widetilde{\partial M}$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \overset{\widetilde{\iota}}{\frown} & & \\ \widetilde{\partial M} & \longrightarrow & \partial\widetilde{M} & \hookrightarrow & \widetilde{M} \\ {\scriptstyle p_{\partial M}}\downarrow & & & & \downarrow{\scriptstyle p_M} \\ \partial M & & \xrightarrow{\iota} & & M. \end{array}$$

Exercise 95.11. Let $k \in \mathbb{N}$. In Definition 9.19 we introduced the non-orientable surface N_k of genus k. The orientation covering of N_k is diffeomorphic to a surface of some genus g. Try to determine g in terms of k.

Exercise 95.12. Recall that in Definition 54.12 we introduced the connected sum $M\#N$ of connected non-empty smooth manifolds M and N. Also recall that if M and N are both orientable, then we need to specify an orientation to get a well-defined connected sum.

(a) Let $M = (\mathbb{R}\mathrm{P}^2 \times S^1)\#(S^1 \times S^2)$. Describe the orientation covering of M as a connected sum of known *oriented* smooth manifolds.

(b) Let $M = \mathbb{R}\mathrm{P}^4\#\mathbb{C}\mathrm{P}^2$. Describe the orientation covering of M as a connected sum of known *oriented* smooth manifolds.

Exercise 95.13. Let M be a smooth manifold and let $p\colon \widetilde{M} \to M$ be the orientation covering. Show that

$$\begin{aligned} \#\,\text{boundary components of } \widetilde{M} \;\; = \;\; & \#\,\text{non-orientable boundary components of } M \\ & +2 \cdot \#\,\text{orientable boundary components of } M. \end{aligned}$$

Exercise 95.14. Let M and N be two connected smooth manifolds, let $f\colon M \to N$ be a local diffeomorphism and finally let $x_0 \in M$. We denote by $w_1^{x_0}(M) \in \mathrm{Hom}(\pi_1(M, x_0), \mathbb{Z}/2\mathbb{Z})$ and $w_1^{f(x_0)}(N) \in \mathrm{Hom}(\pi_1(N, f(x_0)), \mathbb{Z}/2\mathbb{Z})$ the corresponding orientation characters. Show that

$$w_1^{x_0}(M) \;=\; w_1^{f(x_0)}(N) \circ (f_*\colon \pi_1(M, x_0) \to \pi_1(N, f(x_0))) \;\in\; \mathrm{Hom}(\pi_1(M, x_0) \to \mathbb{Z}/2\mathbb{Z}).$$

Exercise 95.15. Let M be an n-dimensional smooth manifold and let

$$\widetilde{M} \;=\; \{(Q, \mathcal{O}) \mid Q \in M \text{ and } \mathcal{O} \text{ is an orientation of } \mathrm{T}_Q M\}$$

be the orientation-covering from the Orientation Covering–Existence Proposition 95.8. We consider the topological space

$$\widehat{M} \;:=\; (\widetilde{M} \times \mathbb{R})/((Q, \mathcal{O}), t) \sim /((Q, -\mathcal{O}), -t).$$

(a) Show that $\widehat{M}$ is an $(n+1)$-dimensional topological manifold and show that we can equip it with a smooth structure.

(b) Show that $\widehat{M}$ is orientable.

(c) Show that there exists a smooth embedding $M \to \widehat{M}$.

This exercise so far shows that any n-dimensional smooth manifold can be smoothly embedded into an orientable $(n+1)$-dimensional smooth manifold.

(d) Show that any compact n-dimensional smooth manifold can be smoothly embedded into a closed orientable $(n+1)$-dimensional smooth manifold.

Exercise 95.16. Does there exist a closed non-orientable 3-dimensional smooth manifold with $\pi_1(M) \cong \mathbb{Z}$.

Exercise 95.17. Let us assume for a minute that our universe admits an orientation-reversing loop and that you are left-handed. If you walk once along the loop, are you still left-handed?

Remark. This exercise is of course not entirely serious, but it is perhaps worth thinking about it and discussing it with friends.

CHAPTER 96

Wedges and topological graphs

In the previous chapters we used coverings and group actions to calculate fundamental groups of many topological spaces, e.g. of the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$, of the real projective spaces $\mathbb{R}\mathrm{P}^n$, of the Klein bottle and the matrix group $\mathrm{SO}(3)$. There are still many topological spaces that elude us, for example so far we cannot calculate the fundamental groups of the surfaces of genus $g \geq 2$ or of topological graphs.

The next big goal will be to formulate and prove the Seifert–van Kampen Theorem which will allow us to do many more calculations. In this intermediate chapter we will introduce the wedge of pointed topological spaces, which is the same as the coproduct of pointed topological spaces. Furthermore we will discuss topological graphs and we will relate topological graphs to wedges of circles. Later in the Seifert–van Kampen Theorem we will then learn how to calculate fundamental groups of wedges and of topological graphs.

This chapter is organized as follows:

- In Section 96.1 we will introduce the wedge of a family of pointed topological spaces.
- In Section 96.2 we will discuss the relationship between abstract graphs and wedges of circles.
- In Section 96.3 we will prove the Graph-mod-Tree Proposition 96.11.
- In Section 96.4 we will give a short summary of where we stand in terms of calculating fundamental groups, and what still needs to be done.
- In Section 96.5 we will provide exercises for Chapter 96.

In the next Chapter 97 we will recall some basic facts about abelian groups.

96.1. **The wedge of pointed topological spaces.** In Section 7.6 we introduced the disjoint union of topological spaces and we saw that the disjoint union is the coproduct in the category of topological spaces. In this section we now consider analogous constructions and results in the context of pointed topological spaces.

Definition 96.1. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

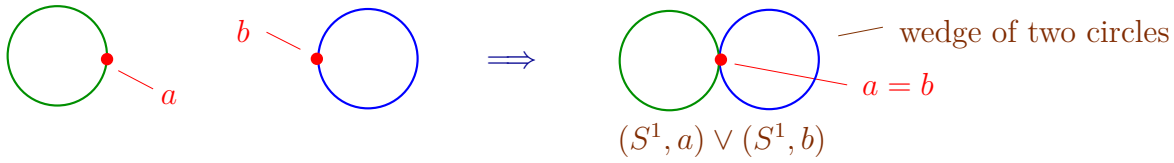
(1) We refer to[966]
$$\bigvee_{i\in I} (X_i, x_i) \coloneqq \Big(\bigsqcup_{i\in I} X_i\Big)\Big/\bigsqcup_{i\in I} \{x_i\}$$
as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point. If we do not say anything else it will be denoted by $*$. We equip the wedge with this base point to view the wedge as a pointed topological space.

(2) Given $j \in I$ we refer to the map
$$\begin{aligned} \iota_j \colon X_j &\to \bigvee_{i\in I} (X_i, x_i) \\ x &\mapsto [x] \end{aligned}$$
as the natural injection.

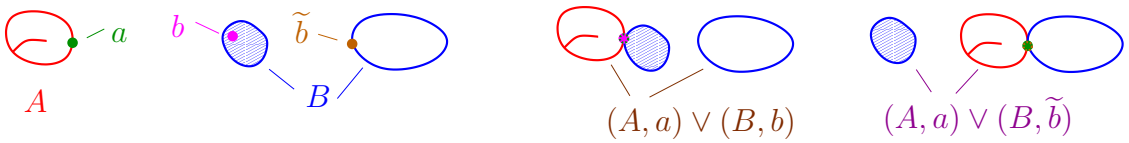
Example.

(1) It follows from the Definition 8.3 of the quotient topological space in Definition 8.8 that the wedge of the empty family of pointed topological spaces is $\varnothing/\varnothing$, which is defined as the topological space consisting of a single point $\{\varnothing\}$.

(2) The wedge $(A,a)\vee(B,b)$ of two pointed topological spaces (A,a) and (B,b) should be viewed as the result of gluing A and B along the points a and b.



(3) In general the homeomorphism type of the wedge depends on the choice of base points. This becomes particularly clear if we consider disconnected spaces. For example it is pretty clear that the wedges $(A,a)\vee(B,b)$ and $(A,a)\vee(B,\widetilde{b})$ considered in the figure below are not homeomorphic.



(4) If (A,a) is a pointed topological space and $(\{*\},*)$ is a pointed topological space consisting of a point, then it follows from completely formal reasons, see Exercise 5.19, that the natural injection $A\to(A,a)\vee(\{*\},*)$ is a homeomorphism. □

The following proposition can be viewed as an analog of the Topological-Coproduct Proposition 7.26.

Proposition 96.2. (Wedge Proposition) Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed topological spaces. We consider the wedge

$$\bigvee_{i\in I}(X_i,x_i) \;:=\; \Big(\bigsqcup_{i\in I}X_i\Big)/\!\sim \qquad \text{where } x_i\sim x_j \text{ for all } i,j\in I.$$

(0) A subset $U\subseteq\bigvee_{i\in I}(X_i,x_i)$ is open (closed) if and only if for each $i\in I$ the preimage $\iota_i^{-1}(U)$ is an open (closed) subset of X_i.

(1) (a) Let $j\in J$. The natural injection $\iota_j\colon X_j\to\bigvee_{i\in I}(X_i,x_i)$ is continuous and it is an embedding.[967]

(b) If (Z,z_0) is a pointed topological space and if $\{f_i\colon(X_i,x_i)\to(Z,z_0)\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{aligned}\bigvee_{i\in I}f_i\colon\bigvee_{i\in I}(X_i,x_i) &\to Z\\ [(x,i)] &\mapsto f_i(x)\end{aligned}$$

is continuous and it sends the wedge point to z_0.

(∗) A map $f\colon\bigvee_{i\in I}(X_i,x_i)\to Z$ is continuous if and only if each $f\circ\iota_i\colon X_i\to Z$ is continuous.

[966] We use obvious adaptions of the notation for finite families of pointed topological spaces.

(2) The pointed topological space $\left(\bigvee_{i\in I}(X_i,x_i),*\right)$ together with the natural injections ι_i is a coproduct of $\{(X_i,x_i)\}_{i\in I}$ in the category $\mathcal{PTop}$ of pointed topological spaces from Definition 90.2.



Proof. Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(0) Both statements follow easily from the Topological-Coproduct Proposition 7.26 (0) together with the Topological-Quotient Proposition 8.2 (0).

(1) (a) Let $j\in I$. It follows immediately from the Topological-Quotient Proposition 8.2 (1a) and the Topological-Coproduct Proposition 7.26 (1a) that the natural injection $\iota_j\colon X_j\to X:=\bigvee_{i\in I}(X_i,x_i)$ is continuous. It also pretty clear that this map is injective. It remains to prove the following claim:

Claim. The map $\iota_j\colon X_j\to\iota_j(X_j)$ is open.

Proof. Let $U\subseteq X_j$ be an open subset. First suppose that $x_j\notin U$. Using (0) one can now easily verify that $\iota_j(U)$ is in fact open in X, in particular it is open in $\iota_j(X_j)$. Finally we suppose that $x_j\in U$. We set $V:=\iota_j(U)\cup\bigcup_{i\neq j}\iota_i(X_i)$. Using (0) one can again easily verify that V is an open subset of X. Furthermore it follows easily from the definitions that $\iota_j(U)=\iota_j(X_j)\cap V$. Thus we have shown that $\iota_j(U)$ is indeed open in $\iota_j(X_j)$. ⊞

(b) This statement follows immediately from the Topological-Quotient Proposition 8.2 (1b) and the Topological-Coproduct Proposition 7.26 (1b).

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from (1a) and (1b). We leave it to the reader to fill in the few remaining details. ■

Example.

(1) Let $(X_i,x_i)_{i\in I}$ be a family of pointed topological spaces. Let $J\subseteq I$. Using the Wedge Proposition 96.2 one can easily show that the natural map $\bigvee_{j\in J}(X_j,x_j)\to\bigvee_{i\in I}(X_i,x_i)$ is an embedding. Furthermore we see that a retraction from $\bigvee_{i\in I}(X_i,x_i)$ to the subspace $\bigvee_{i\in J}(X_j,x_j)$ is given by sending all points in X_i with $i\neq j$ to the wedge point.

(2) Let $G=(V=\{v\},E,i,t)$ be an abstract graph with a *single* vertex. We consider the bijection

$$f\colon|G|=\Big(V\sqcup\bigsqcup_{e\in E}[0,1]\Big)/\sim\ \to\ \bigvee_{e\in E}(S^1_e,1)$$
$$[x]\ \mapsto\begin{cases}*, & \text{if } x=v,\\ \exp(2\pi\,\mathrm{i}t)\in S^1_e, & \text{if } x=(e,t) \text{ with } e\in E,\ t\in[0,1].\end{cases}$$

First note that it follows from the Abstract Graph-Topology Lemma 11.6 (3) that this map is continuous. Next, using the Wedge Proposition 96.2, and using the fact

[967] In most cases this statement allows us to safely view each X_j as a subspace of the wedge $\bigvee_{i\in I}(X_i,x_i)$.

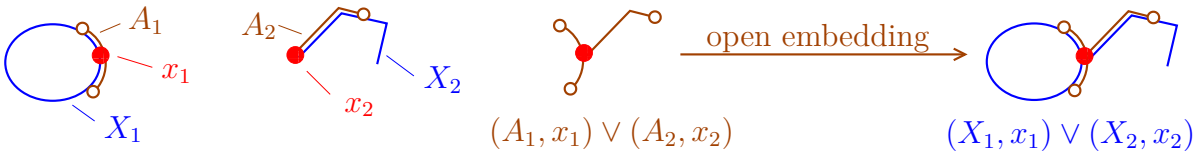
that the map $[0,1]/0 \sim 1 \to S^1$ given by $[t] \mapsto \exp(2\pi i t)$ is a homeomorphism, one can show that the inverse f^{-1} is also continuous. In summary, the above map f is a homeomorphism. □

The following lemma shows that taking wedges behaves well with respect to the subspace topology. This lemma gets used frequently, often subconsciously.

Lemma 96.3. (Wedge–Subspace Lemma)[968] Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. Let $J \subseteq I$ be a subset. Suppose that for each $j \in J$ we are given a subspace $A_j \subseteq X_j$ with $x_j \in A_j$.

(1) The natural map $\bigvee_{j\in J} (A_j, x_j) \to \bigvee_{i\in I} (X_i, x_i)$ is an embedding.

(2) If each $A_j \subseteq X_j$ is open (closed), then the map in (1) is an open (closed) embedding.



Proof. The lemma follows fairly easily from the Wedge Proposition 96.2. We outsource the details to Exercise 96.1. ■

The following lemma plays the role of the Disjoint Union Topology-Properties Lemma 7.28. It is entirely possible that later on we use the lemma below without giving it credit.

Lemma 96.4. (Wedge–Properties Lemma) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(1) If I is finite and if each X_i is compact, then $\bigvee_{i\in I} (X_i, x_i)$ is compact.[969]

(2) If each X_i is (path)-connected, then the wedge $\bigvee_{i\in I} (X_i, x_i)$ is (path)-connected.

(3) The wedge $\bigvee_{i\in I} (X_i, x_i)$ is Hausdorff if and only if each X_i is Hausdorff.

Proof.

(1) This statement follows immediately from the Disjoint Union Topology-Properties Lemma 7.28 (1) together with the Quotient Topology-Properties Proposition 8.5 (1).

(2) This statement follows immediately from the Union Connected Lemma 4.39 together with the Wedge Proposition 96.2 (1a).

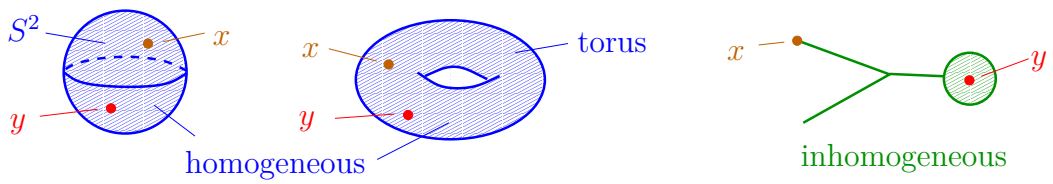
(3) This statement can be proved easily using the Wedge Proposition 96.2 (0a). We will fill in the details in Exercise 96.5. ■

On page 1863 we saw that in general the homeomorphism type of the wedge of two topological spaces depends on the choice of the base points. Fortunately we will now see that in many settings the choice of base point is largely irrelevant. Let us recall Definition 41.17:

Definition 96.5. We say that a topological space X is homogeneous if given any two points $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$.

[968] The statement of the lemma is closely related to Exercise 8.6.

[969] In Exercise 96.6 we will discuss the converse of this statement.



Example.

(1) It is clear that the topological space $\mathbb{R}^n$ is homogeneous. Furthermore it follows from the Orthogonal Action Lemma 2.37 (3) that the sphere S^n is homogeneous.

(2) In the Manifold–Homogeneous Proposition 41.21 we showed that every connected topological manifold with empty boundary is homogeneous. □

Lemma 96.6. (Wedging Point–Irrelevance Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. If each X_i is homogeneous, then different choices of base points still lead to homeomorphic wedges.

Proof. Suppose that for each $i \in I$ we have base points $x_i, y_i \in X_i$. Since each X_i is homogeneous we know that for each $i \in I$ there exists a homeomorphism $f_i\colon X_i \to X_i$ with $f_i(x_i) = y_i$. It follows from the Wedge Proposition 96.2 (1) that the maps

$$\bigvee_{i\in I}(X_i, x_i) \;\to\; \bigvee_{i\in I}(X_i, y_i) \qquad \text{and} \qquad \bigvee_{i\in I}(X_i, y_i) \;\to\; \bigvee_{i\in I}(X_i, x_i)$$

induced by the maps f_i and by the maps f_i^{-1} are continuous. Clearly they are inverses of one another, so they are homeomorphisms. ■

This lemma leads us to the following slightly sloppy definition.

Definition 96.7. Let $\{X_i\}_{i\in I}$ be a family of non-empty homogeneous topological spaces (e.g. a family of spheres). For each $i \in I$ we pick a point $x_i \in X_i$ and we write

$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I}(X_i, x_i).$$

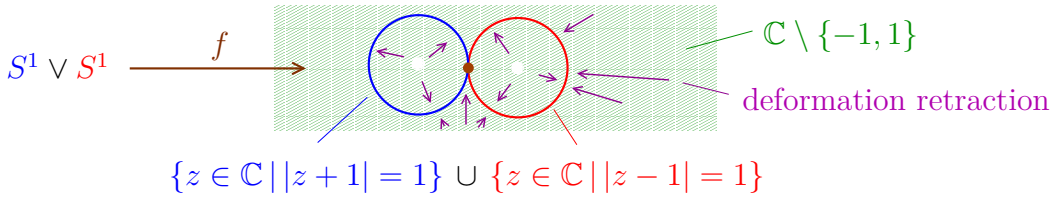
By the Wedging Point–Irrelevance Lemma 96.6 the homeomorphism type of $\bigvee_{i\in I} X_i$ is well-defined.

Example. We consider the wedge $S^1 \vee S^1$ of two circles. (By the Wedging Point–Irrelevance Lemma 96.6 there is no real need to specify which points in the two copies of S^1 get identified.) By the Wedge–Properties Lemma 96.4 (1) we know that $S^1 \vee S^1$ is compact. Using the Compact-Hausdorff Proposition 4.23 (3) one can easily write down a homeomorphism

$$S^1 \vee S^1 \;=\; (S^1, 1) \vee (S^1, -1) \;\xrightarrow{\cong}\; \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}.$$

It is also pretty elementary to show that the topological space to the right is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.
It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $S^1 \vee S^1$ is homotopy equivalent to $\mathbb{C} \setminus \{\pm 1\}$. □

We conclude this section with a technical but important lemma which can safely be skipped at a first reading.



Lemma 96.8. (Wedge–Homotopy Lemma) Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces and let (Z, z_0) be a pointed topological space. Furthermore suppose that for each $i \in I$ we are given a homotopy $F_i \colon X_i \times [0,1] \to Z$ such that for each $t \in [0,1]$ we have $F_i(x_i, t) = z_0$. Then the map

$$\begin{array}{rcl} \left(\bigvee\limits_{i \in I} (X_i, x_i) \right) \times [0,1] & \to & Z \\ ([(x,i)], t) & \mapsto & F_i(x,t) \end{array}$$

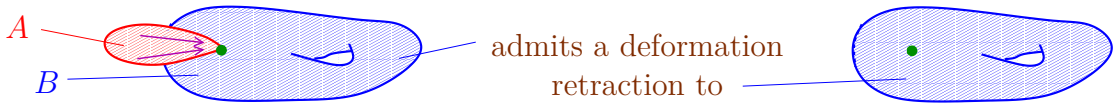
is a homotopy.

Proof. We need to show that the given map is continuous. Note that it follows from Exercise 7.20 that the map

$$\begin{array}{rcl} \left(\bigsqcup\limits_{i \in I} X_i \right) \times [0,1] & \to & Z \\ ((x,i), t) & \mapsto & F_i(x,t) \end{array}$$

is continuous. It follows from the Homotopy Quotient Theorem 18.11 that this homotopy on the disjoint union of the X_i descends to a homotopy on the wedge of the (X_i, x_i). ■

Example. Let (A, a_0) and (B, b_0) be two pointed topological spaces. We consider the wedge $A \vee B = (A, a_0) \vee (B, b_0)$.



Note that if $\{a_0\}$ is a deformation retract of A, then it follows easily from the Wedge–Homotopy Lemma 96.8 that[970] the subspace $B \xrightarrow{\cong} (\{a_0\}, a_0) \vee (B, b_0)$ is a deformation retract of the wedge $(A, a_0) \vee (B, b_0)$. □

96.2. **Abstract graphs and wedges of circles.** In this section we have another look at abstract graphs and their topological incarnations. First we recall the following definitions from Definition 11.21 and Definition 11.29.

Definition 96.9. Let $G = (V, E)$ be an abstract graph.

(1) If G is finite, i.e. if V and E are finite, then we refer to

$$\chi(G) \;:=\; \#V \,-\, \#E \;=\; \text{number of vertices} \,-\, \text{number of edges}$$

as the Euler characteristic of G.

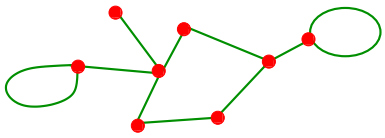
(2) We say that G is a tree if G is non-empty, if it does not contain a loop and if given any $v, w \in V$ there exists a unique injective path from v to w. If G is finite, then it follows from the Tree Characterization Proposition 11.28 that G is a tree if and only if G is connected and if $\chi(G) = 1$.

[970] On page 1863 we saw that the natural map $B \to (\{a_0\}, a_0) \vee (B, b_0)$ is a homeomorphism.

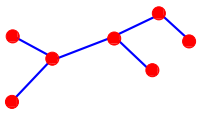
(3) The valence of a vertex v is defined as

$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

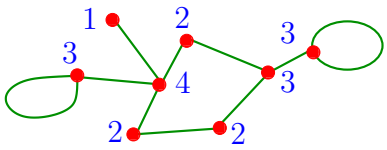
Given a topological graph $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ we define the properties of X to be the properties of G, e.g. we define $\chi(X) := \chi(G)$ and we say that X is a tree if G is a tree.[971]



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$



example of a tree



valences of the vertices

Proposition 96.10. (Finite Tree–Contractible Proposition) Let T be an abstract graph. If T is a finite tree, then the following hold:

(1) The topological realization $|T|$ admits a deformation retraction to a vertex, in particular it is contractible.
(2) The fundamental group $\pi_1(|T|)$ is trivial.

Remark.

(1) The statement of the Finite Tree–Contractible Proposition 96.10 (1) is basically the content of Exercise 19.19.
(2) In the Tree–π_1-Proposition 99.18 we will generalize the statement of the Finite Tree–Contractible Proposition 96.10 (2) to infinite trees and we will see that fundamental groups characterize trees. □

Proof. Note that Statement (2) is an immediate consequence of Statement (1) together with the Homotopy Equivalence-π_1-Proposition 90.9 (2). Thus it suffices to prove Statement (1).

We will prove that any tree admits a deformation retraction to a vertex. We will prove this statement by induction on the number of edges. If a tree has zero edges, then it follows from the connectedness of T that T consists of precisely one vertex. So we are done with this special case.

Now suppose that we know that all trees with n edges admit a deformation retraction to a vertex. Let $T = (V, E)$ be a tree with $n + 1$ edges. By the Tree–Valence One Edge Lemma 11.30 we know that T admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We set $T' = (V \setminus \{v\}, E \setminus \{e\})$. Note that T' is a non-empty connected abstract subgraph of T. It follows from the Tree–Basics Lemma 11.22 (1) that T' is again a tree.

Claim. $|T'|$ is a deformation retract of $|T|$.

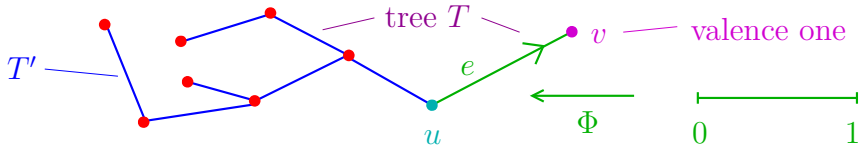
Proof. We denote by u the other vertex of e. Since $u \neq v$ we obtain from the Abstract Graph-Topology Lemma 11.6, see also the discussion on page 381, that there exists a closed

[971] On many occasions we will use the fact, established in the Topological Realization–Properties Lemma 11.8 (2), that a topological graph is connected if and only if the underlying abstract graph is connected.

embedding $\Phi\colon [0,1] \to |T|$ with $\Phi(0) = u$ and $\Phi(1) = v$. We consider the map

$$\begin{array}{rcl} H\colon |T| \times [0,1] & \to & |T| \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in |T'|, \\ \Phi(t \cdot y), & \text{if } x = \Phi(y) \text{ for some } y \in [0,1]. \end{cases} \end{array}$$

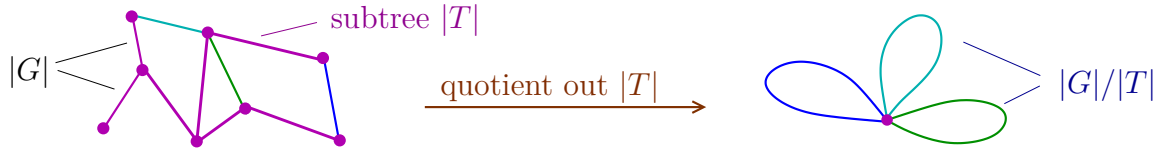
It follows from the Abstract Graph-Topology Lemma 11.6 that $|T'|$ and $|e| = \Phi([0,1])$ are closed subsets of $|T|$. It thus follows from the Homotopy Combination Lemma 18.8 that the map H is continuous.



It is now clear that H is a deformation retraction from $|T|$ to $|T'|$. ⊞
We have thus shown that $|T'|$ is a deformation retract of $|T'|$. By our induction hypothesis $|T'|$ admits a deformation retraction to a vertex $v \in T'$. So it follows from the Iterated Deformation Retract Lemma 19.5 that $|T|$ also admits a deformation retraction to the vertex v. Finally it follows from this fact together with the Point Deformation Retract Lemma 19.19 that T is contractible. ■

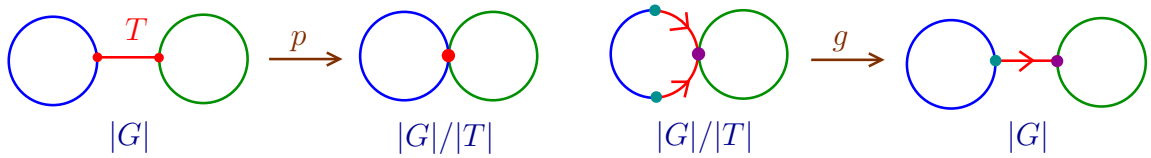
We conclude this discussion of abstract and topological graphs with the following proposition and a corresponding corollary.

Proposition 96.11. (Graph-mod-Tree Proposition) Let G be an abstract graph. For every finite subtree T of G the projection $|G| \to |G|/|T|$ is a homotopy equivalence.



Proof. We outsource the proof to the next section. ■

Example. We consider the abstract graph G and the subtree T that are shown in the figure below.



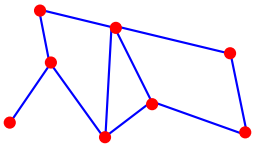
Recall that it follows from the Graph-mod-Tree Proposition 96.11 that the projection $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. In fact recall that in Exercise 19.11 we already saw that the map $g\colon |G|/|T| \to |G|$ that is also sketched in the figure is a homotopy inverse to p. The Graph-mod-Tree Proposition 96.11 can be viewed as a notationally slightly heroic generalization of this example. □

Remark.

(1) The content of the Graph-mod-Tree Proposition 96.11 is basically identical to the content of the Graph-mod-Tree Proposition 22.13. We will now give a somewhat different self-contained proof.
(2) In the 1-Complex-mod-Tree Proposition 137.7 and the Quotient-by-Contractible Corollary 140.4 we will provide two generalizations of the Graph-mod-Tree Proposition 96.11. □

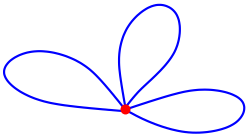
The statement of the following corollary is almost identical to the statement of the Graph-Homotopy Type Corollary 22.14.

Corollary 96.12. (Graph-Homotopy Type Corollary)
(1) Let $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ be a finite topological graph. If X is connected and non-empty, then the topological graph X is homotopy equivalent to the wedge of $1 - \chi(X)$ circles.
(2) Two finite connected topological graphs with the same Euler characteristic are homotopy equivalent.



topological graph of Euler characteristic $8 - 10 = -2$

is homotopy equivalent to

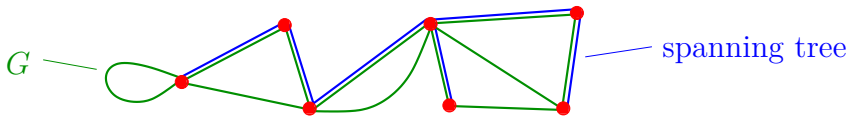


wedge of three circles

Remark. Later, in the Graph-π_1-Proposition 99.13 we will study the question, whether the converse to Statement (3) holds: If two finite connected topological graphs are homotopy equivalent, does it follow that the Euler characteristics are the same? □

Proof. Let G be a finite abstract graph.

(1) By the Spanning Tree–Existence Proposition 11.26 (1) we know that G admits a spanning tree T. Next note that by the Graph-mod-Tree Proposition 96.11 we know that the map $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. By the Abstract Graph–Quotient Lemma 11.32 (2) we know that $|G|/|T|$ is homeomorphic to $|G/T|$. Thus it remains to show that $|G/T|$ is homeomorphic to the wedge of $1 - \chi(G)$ circles. We make the following observation:
   (a) Note that by definition the spanning tree T contains all vertices of G. In other words, the abstract graph G/T contains only a single vertex.
   (b) Note that
$$\chi(G/T) \underset{\text{by the Abstract Graph Quotient Lemma 11.32 (1)}}{=} \chi(G) - \chi(T) + 1 \underset{\text{by the Tree Characterization Proposition 11.28}}{=} \chi(G) - 1 + 1 = \chi(G).$$
   (c) By (a) and (b) we know that G/T has $1 - \chi(G)$ edges.
   (d) It follows from (a) and (c), together with the discussion on page 1865 that $|G/T|$ is indeed homeomorphic to the wedge of $1 - \chi(G)$ circles.



(2) The last statement is an immediate consequence of (1) and the Homotopy-Equivalence Basics Lemma 19.14. ■

96.3. **Proof of the Graph-mod-Tree Proposition 96.11.** The sole purpose of this section is to provide the following proof:

Proof of the Graph-mod-Tree Proposition 96.11. As we mentioned above, the content of the Graph-mod-Tree Proposition 96.11 is basically identical to the content of the Graph-mod-Tree Proposition 22.13. The proof of the Graph-mod-Tree Proposition 22.13 relied on the language of cofibrations and, more seriously, on the Homotopy Extension Theorem 140.2. We will now give a self-contained proof.

Let $G = (V, E, i, t)$ be an abstract graph and let $T = (W, F)$ be a finite subtree of G. Recall that in Definition 11.31 we introduced the notion of the quotient abstract graph G/T. In the Abstract Graph–Quotient Lemma 11.32 we introduced the map

$$p\colon |G| = (V \sqcup (E \times [0,1]))/\overset{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)}{\overset{\downarrow}{\sim}} \;\to\; |G/T| = (V/\sim \sqcup((E\setminus F)\times[0,1]))/\overset{\text{similarly } (e,0)\sim \rho(i(e)) \text{ and } (e,1)\sim \rho(t(e))}{\overset{\downarrow}{\sim}}$$

$$[x] \mapsto \begin{cases} [*], & \text{if } x \in |T|, \\ [x], & \text{if } x \in V, \\ [(e,t)], & \text{if } x = (e,t) \text{ with } e \in E\setminus F \text{ and } t \in [0,1]. \end{cases}$$

We also showed that the induced map $p\colon |G|/|T| \to |G/T|$ is a homeomorphism. Note that p fits into the following commutative diagram:

$$\begin{array}{ccc} |G| & \xrightarrow{\quad p \quad} & |G/T| \\ & {\scriptstyle x\mapsto[x]}\searrow \quad \nearrow & \\ & |G|/|T|. & \end{array}$$

We need to show that the projection $|G| \to |G|/|T|$ to the left is a homotopy equivalence. As we just mention, the diagonal map to the right is a homeomorphism. Thus it suffices to show that the horizontal map $p\colon |G| \to |G/T|$ is a homotopy equivalence.

Note that by the Finite Tree–Contractible Proposition 96.10 we know that $|T|$ admits a deformation retraction $R\colon |T| \times [0,1] \to |T|$ to some vertex $t_0 \in T$. Put differently, there exists a homotopy $R\colon |T| \times [0,1] \to |T|$ such that $R_1 = \mathrm{id}_{|T|}$ and such that $R_0 = \mathrm{const}_{t_0}$ is the constant map. We extend R somewhat by setting $R(v,t) \coloneqq v$ for all vertices v that do not lie in $|T|$.

Let f be an edge of G that is not contained in T.

- If $i(f) = t(f)$, then it follows from the discussion on page 381 that the characteristic map of f gives rise to a continuous map $\Phi_f\colon [-2,2] \to |G|$ that induces a homeomorphism $[-2,2]/-2 \sim 2 \to |f|$.
- If $i(f) \neq t(f)$, then the same discussion on page 381 shows that there exists a continuous map $\Phi_f\colon [-2,2] \to |G|$ which restricts to a homeomorphism $[-2,2] \to |f|$.[972]

We write $a_f = \Phi_f(-2)$ and $b_f = \Phi_f(2)$.

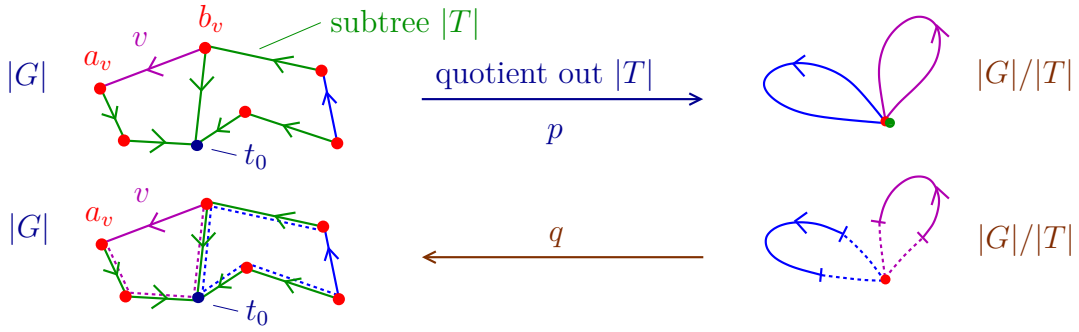
Next we consider the quotient graph G/T. Let $p\colon |G| \to |G/T|$ be the projection. For every edge f of G/T, i.e. for every $f \in E \setminus F$, we denote by $\Psi_f\colon [-2,2] \to |G/T|$ the map

[972] In other words, the map Φ_f is just a rescaling of the characteristic maps of f.

$p \circ \Phi_f$. We consider the map

$$q\colon |G/T| \to |G|$$
$$x \mapsto \begin{cases} R(a_f, -s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1], \\ \Psi_f(2 \cdot s), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1,1], \\ R(b_f, s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [1,2]. \end{cases}$$

Using the Abstract Graph-Topology Lemma 11.6 one can show that q is continuous.



It remains to prove the following claim:

Claim. The maps p and q are homotopy inverses of one another.

Proof. The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$H\colon |G/T| \times [0,1] \to |G/T|$$
$$(x,t) \mapsto \begin{cases} *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Psi_f(\frac{2s}{1+t}), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1-t,1+t], \\ *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [1+t,2]. \end{cases}$$

Using the Abstract Graph-Topology Lemma 11.6 and the Homotopy Combination Lemma 18.8 one can easily show that H is continuous. It follows immediately from the definitions that $H_0 = p \circ q$ and $H_1 = \mathrm{id}_{|G/T|}$. Next we consider the following map

$$\widetilde{H}\colon |G| \times [0,1] \to |G|$$
$$(x,t) \mapsto \begin{cases} R(x,t), & \text{if } x \in T, \\ R(a_f, s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Phi_f(\frac{2s}{1+t}), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-1-t,1+t], \\ R(b_f, -s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [1+t,2]. \end{cases}$$

Again, using the Abstract Graph-Topology Lemma 11.6 and the Homotopy Combination Lemma 18.8 one can show that $\widetilde{H}$ is continuous. Furthermore it follows again immediately from the definitions that $\widetilde{H}_0 = q \circ p$ and $\widetilde{H}_1 = \mathrm{id}_{|G|}$. ∎

96.4. **Outlook.** Summarizing we have now finally managed to calculate the fundamental groups of the following topological spaces:

(1) the circle S^1 and all spheres S^n with $n \geq 2$,
(2) the n-dimensional torus $(S^1)^n \cong \mathbb{R}^n/\mathbb{Z}^n$,
(3) the Klein bottle,
(4) the real projective spaces $\mathbb{R}\mathrm{P}^n$,

(5) the lens spaces $L(p, q)$,
(6) trees,
(7) products of the above.

Nonetheless, the list of topological spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

(1) topological graphs,
(2) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C} \setminus \{0, 1\}$,
(3) surfaces of genus ≥ 2,
(4) knot complements,
(5) wedges of topological spaces.

In the following two chapters we will first build up our knowledge of group theory. Then we will state and prove the Seifert-van Kampen Theorem in various flavors which will help us with calculating the open fundamental groups.

96.5. Exercises for Chapter 96.

Exercise 96.1. Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. Let $J \subseteq I$ be a subset. Suppose that for each $j \in J$ we are given a subspace $A_j \subseteq X_j$ with $x_j \in A_j$.

(1) Show that the natural map $\bigvee_{j\in J} (A_j, x_j) \to \bigvee_{i\in I} (X_i, x_i)$ is an embedding.
(2) Suppose that each $A_j \subseteq X_j$ is open (closed). Show that the map in (a) is an open (closed) embedding.

Exercise 96.2. Is the wedge $\bigvee_{i\in\mathbb{N}} S^1_i$ homeomorphic to a subspace of $\mathbb{R}^2$?

Exercise 96.3. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(a) If the wedge $\bigvee_{i\in I} (X_i, x_i)$ is compact, does it follow that each X_i is compact?
(b) If the wedge $\bigvee_{i\in I} (X_i, x_i)$ is compact, does it follow that all but finitely many X_i consist of a single point?

Exercise 96.4. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. If the wedge $\bigvee_{i\in I} (X_i, x_i)$ is connected, does it follow that each X_i is connected?
Remark. In other words, the question is whether the converse to the Wedge–Properties Lemma 96.4 (2) holds.

Exercise 96.5. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each X_i is Hausdorff. Show that $\bigvee_{i\in I} (X_i, x_i)$ is also Hausdorff.

Exercise 96.6. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a point $x_i \in X_i$. We use these points to form the wedge $\bigvee_{i\in I} X_i$. Let $K \subseteq \bigvee_{i\in I} X_i$ be a compact subset.

(a) We suppose that each X_i is Hausdorff. Show that there exists a finite subset $J \subseteq I$ such that $K \subseteq \bigvee_{i\in J} X_i$.
Hint. For any $i \in I$ with $K \cap (X_i \setminus \{x_i\}) \neq \varnothing$ pick some point $a_i \in K \cap (X_i \setminus \{x_i\})$. We denote by A the union of these points. Show that A is a discrete subset of K.

(b) Show that in general the conclusion of (a) does not hold if we do not assume that the X_i are Hausdorff.

Exercise 96.7. Let $X := \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$. We denote by $\|-\|$ the usual Euclidean norm on X. We denote by $P=(0,0)$ the center of X. The *SNCF metric* on X is given by

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (A,B) & \mapsto & \begin{cases} \|A-P\|+\|B-P\|, & \text{if } A,B,P \text{ do not lie on a line,} \\ \|A-B\|, & \text{if } A,B,P \text{ do lie on a line.} \end{cases} \end{array}$$

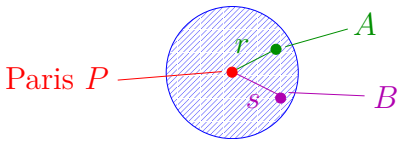
This metric takes its name from the fact that all (OK, many) train lines in France go towards Paris. So if you want to go from A to B you invariably (OK, often) have to go through Paris.

(a) Show that the map

$$\begin{array}{rcll} \varphi\colon \bigvee\limits_{z\in S^1} ([0,1]_z, 0) & \to & X & \\ [r] & \mapsto & r\cdot z & \text{if } r \in [0,1]_z. \end{array}$$

is not a homeomorphism.

(b) Does there exist a homeomorphism between $\bigvee\limits_{z\in S^1} ([0,1]_z, 1)$ and X?



in the Paris metric the distance between P and Q equals $r+s$

Exercise 96.8. Let (X,x_0) and (Y,y_0) be two pointed topological spaces. We use the base points to define the wedge $X \vee Y$.

(a) Show that the map

$$\begin{array}{rcl} f\colon X \vee Y & \to & X \times Y \\ [P] & \mapsto & \begin{cases} (P,y_0), & \text{if } P \in X, \\ (x_0,P), & \text{if } P \in Y \end{cases} \end{array}$$

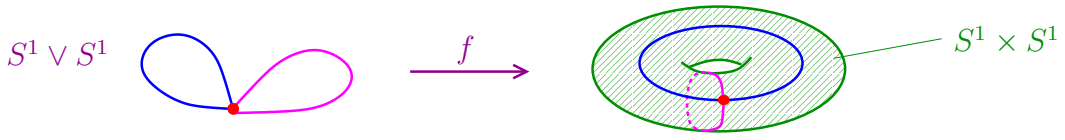
is an embedding.

Remark. We do not assume that X and Y have any nice properties.

(b) Using (a) we can view $X \vee Y$ as a subspace of $X \times Y$. We refer to the quotient

$$X \wedge Y \;:=\; (X,x_0) \wedge (Y,y_0) \;:=\; (X \times Y)/(X \vee Y)$$

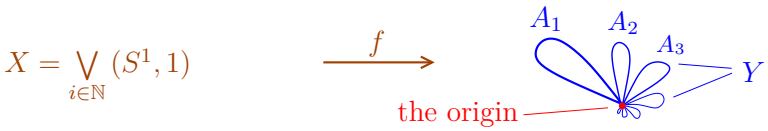
as the *smash product of* (X,x_0) *and* (Y,y_0). Show that for any $m,n \in \mathbb{N}_0$ the smash product $S^m \wedge S^n$ is homeomorphic to S^{m+n}.



Exercise 96.9. Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each X_i is locally contractible and we suppose that each $\{x_i\}$ is a closed subset of X_i. Show that the wedge $\bigvee\limits_{i\in I} (X_i,x_i)$ is also locally contractible.

Exercise 96.10. We consider $X = \bigvee_{i\in\mathbb{N}} (S^1, 1)$. We also consider the subset Y of $\mathbb{R}^2$ that is shown in the figure below. It is the union of infinitely many circles $A_i \subseteq \mathbb{R}^2$, $i \in \mathbb{N}$ that meet in the origin. There exists a fairly obvious bijection $f\colon X \to Y$ which sends the i-th circle to A_i.

(a) Is the map f continuous?
(b) Is the map f^{-1} continuous?



Exercise 96.11. In the following we consider the wedge $X := \bigvee_{i\in\mathbb{N}_0} [0, 1]_i$ given by "wedging" the points $0 \in [0, 1]_i$, $i \in \mathbb{N}_0$.

(a) Show that the sequences of points $x_i := \frac{1}{2^i} \in [0, 1]_i \subseteq X$, $i \in \mathbb{N}_0$ does not converge to the wedge point.
(b) Show that X is not metrizable, i.e. show that it does not admit a metric such that the topology given by the metric agrees with the topology of X.

Exercise 96.12. Show that every topological graph is locally contractible.
Remark. If you get away without using the Homotopy Quotient Theorem 18.11, then you should go carefully over your continuity arguments.

Exercise 96.13. Let T be a tree and let $v \in T$ be any vertex. Show that T admits a deformation retraction to v.
Remark. This is a slight refinement of the statement of the Finite Tree–Contractible Proposition 96.10 (2).

Exercise 96.14. Let X be a topological space. We say that two coverings $p\colon Y \to X$ and $q\colon Z \to X$ are *equivalent* if there exists a homeomorphism $f\colon Y \to Z$ with $p = q \circ f$. How many inequivalent coverings of $X = S^1 \vee S^1$ of degree 2 can you find?

CHAPTER 97

Basics of group theory I: Abelian groups

In the Product-π_1-Proposition 90.13 we showed that for any two pointed topological spaces (X, x_0) and (Y, y_0) we have an isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. In more formal terms, we showed the following:

(∗) The fundamental group of the product of two pointed topological spaces is isomorphic to the product of the fundamental groups of the two pointed topological spaces.

In the Wedge Proposition 96.2 we just determined the coproduct of two pointed topological spaces. It is natural to ask, whether in (∗) we can replace product by coproduct.

To answer this question we first need to understand the coproduct in the category *Group* of groups. Also, more generally it is worthwhile to build up our understanding of groups.

Since general groups are quite difficult we will first build up our understanding of *abelian* groups. More precisely, in this chapter we study the coproduct in the category *AbGroup* of abelian groups, we introduce free abelian groups, we study the rank of (free) abelian groups and we will state and prove the classification of finitely generated abelian groups.

In the next chapter we will then see to what degrees these concepts and results on abelian groups generalize to the context of (potentially non-abelian) groups.

This chapter is organized as follows:

- In Section 97.1 we will introduce the direct sum of a family of abelian groups.
- In Section 97.2 we will discuss free abelian groups.
- In Section 97.3 we will see that every subgroup of a free abelian group is again a free abelian group.
- In Section 97.4 we will introduce the rank of abelian groups.
- In Section 97.5 we will use the Smith Normal Form Theorem 97.22 to give the classification of finitely generated abelian groups.
- In Appendix 97.6 we will provide a proof of the Smith Normal Form Theorem 97.22.
- In Appendix 97.7 we will introduce group rings.
- In Section 97.8 we will provide exercises for Chapter 97.

In the next Chapter 98 we will turn to the theory of (potentially) non-abelian groups. In particular we will introduce free groups and amalgamated product of groups.

97.1. **The direct sum of abelian groups.** We start out by recalling the following definitions from the Product-of-Sets-and-Groups Lemma 6.4 and from Notation 6.9.

Definition 97.1. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) We define the product of the X_s to be the group

$$\prod_{s\in S} X_s := \left\{ f\colon S \to \bigcup_{s\in S} X_s \;\middle|\; \text{for every } s \in S \text{ we have } f(s) \in X_s \right\},$$

where the group structure is given by the obvious "componentwise multiplication".[973]

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s \coloneqq \left\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \,\middle|\, \text{all but finitely many } g_s \text{ are trivial} \right\}.$$

(3) Given $t \in S$ we refer to

$$\begin{array}{rcl} \iota_t \colon X_t & \to & \bigoplus_{s\in S} X_s \\ x & \mapsto & \left(\begin{array}{lcl} S & \to & \bigcup_{s\in S} X_s \\ s & \mapsto & \begin{cases} x, & \text{if } s = t, \\ 0_{X_t}, & \text{if } s \neq t. \end{cases} \end{array} \right) \end{array}$$

as the natural homomorphism.

By the Product-of-Sets-and-Groups Lemma 6.4 we know that $\prod_{s\in S} X_s$ is the product of the X_s in the category *AbGroup* of abelian groups and by the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 we know that $\bigoplus_{s\in S} X_s$ is the coproduct of the X_s in the category *AbGroup* of abelian groups. For the reader's convenience we recall this second statement in greater detail:

Lemma 97.2. (Coproduct-of-Abelian Groups Lemma) Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) Let Z be an abelian group and let $\{f_s \colon X_s \to Z\}_{s\in S}$ be a family of homomorphisms. The map

$$\begin{array}{rcl} \bigoplus_{s\in S} f_s \colon \bigoplus_{s\in S} X_s & \to & Z \\ (g_s)_{s\in S} & \mapsto & \sum_{\substack{s\in S \text{ with} \\ g_s \neq e}} f_s(g_s) \end{array}$$

is the unique homomorphism[974] such that for each $s \in S$ we have $f_s = \left(\bigoplus_{s\in S} f_s\right) \circ \iota_s$.

(2) The coproduct in the category *AbGroup* of *abelian* groups of $\{X_s\}_{s\in S}$ is given by $\bigoplus_{s\in S} X_s$ together with the above natural homomorphisms $\iota_t \colon X_t \to \bigoplus_{s\in S} X_s$.

Proof. This statement is basically the content of the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 (2). But of course one can also prove it easily by hand. ∎

Convention 97.3.

(1) Let $\{X_s\}_{s\in S}$ be a family of abelian groups. For $t \in S$ we use the natural monomorphism $\iota_t \colon X_t \to \bigoplus_{s\in S} X_s$ from the Coproduct-of-Abelian Groups Lemma 97.2 to view X_t as a subgroup of the direct sum $\bigoplus_{s\in S} X_s$.

[973] Let $\{X_i\}_{i\in\varnothing}$ be the empty family of groups. It follows from the discussion on page 98 that the product contains precisely one element, so the product is the trivial group.

[974] Note that in the verification of that assertion that Φ is a homomorphism it is essential that Z is abelian.

(2) Let G be an abelian group and let $\{X_s\}_{s\in S}$ be a family of subgroups. The inclusions $X_s \to G$ define a natural homomorphism

$$\bigoplus_{s\in S} X_s \ \to \ G.$$

If this natural homomorphism is a monomorphism, then we refer to the image also as the direct sum of the X_s.[975]

Example. Let G be an abelian group and let A and B be two subgroups. If $A\cap B = \{0\}$ then the natural homomorphism $A\oplus B \to G$ is easily seen to be a monomorphism. Thus we can view the direct sum $A\oplus B$ as a subgroup of G. □

Theorem 97.4. (Chinese Remainder Theorem) If $r, s \in \mathbb{N}$ are coprime, then the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}/(r\cdot s)\mathbb{Z} & \to & \mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}/s\mathbb{Z} = \mathbb{Z}/r\mathbb{Z}\oplus\mathbb{Z}/s\mathbb{Z} \\ [k] & \mapsto & ([k],[k]) \end{array}$$

is an isomorphism.

Proof. Since $r, s \in \mathbb{N}$ are coprime there exist $a, b \in \mathbb{Z}$ with $a\cdot r + b\cdot s = 1$. We consider the map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}/r\mathbb{Z}\times\mathbb{Z}/s\mathbb{Z} & \to & \mathbb{Z}/(r\cdot s)\mathbb{Z} \\ ([e]_{\mathbb{Z}/r\mathbb{Z}}, [f]_{\mathbb{Z}/s\mathbb{Z}}) & \mapsto & [e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}/rs\mathbb{Z}}. \end{array}$$

One can easily verify that this map is well-defined. We claim that it is an inverse to Ψ. Indeed, we have

$$\begin{array}{rcl} \Psi(\Phi([k]_{\mathbb{Z}/rs\mathbb{Z}})) = \Psi([k]_{\mathbb{Z}/r\mathbb{Z}}, [k]_{\mathbb{Z}/s\mathbb{Z}}) = [k\cdot b\cdot s + k\cdot a\cdot r]_{\mathbb{Z}/rs\mathbb{Z}} & = & [k\cdot \overbrace{(b\cdot s + a\cdot r)}^{=1}]_{\mathbb{Z}/rs\mathbb{Z}} \\ & = & [k]_{\mathbb{Z}/rs\mathbb{Z}} \end{array}$$

and we have

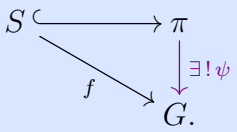
$$\begin{array}{rcl} \Phi(\Psi([e]_{\mathbb{Z}/r\mathbb{Z}}, [f]_{\mathbb{Z}/r\mathbb{Z}})) & = & \Phi([e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}/rs\mathbb{Z}}) \\ & = & ([e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}/r\mathbb{Z}}, [e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}/s\mathbb{Z}}) \\ & = & ([e\cdot \underbrace{b\cdot s}_{=1-a\cdot r}]_{\mathbb{Z}/r\mathbb{Z}}, [f\cdot \underbrace{a\cdot r}_{=1-b\cdot s}]_{\mathbb{Z}/s\mathbb{Z}}) = ([e]_{\mathbb{Z}/r\mathbb{Z}}, [f]_{\mathbb{Z}/s\mathbb{Z}}). \end{array}$$

■

97.2. **Free abelian groups.** We start out with the following definition, which in principle can be modified to many different settings.

Definition 97.5.

(1) We say that a group π is a free abelian group on a subset $S\subseteq\pi$, if the following universal property holds: For any *map* $f\colon S\to G$ of the set S to an abelian group G there exists a unique *homomorphism* $\psi\colon \pi\to G$ that makes the following diagram commute:



We call S a basis of the free abelian group π.

[975]Sometimes in the literature this image is called the *internal direct sum*. In practice it becomes a mess to distinguish in the notation between the "direct sum" and the "internal direct sum", thus we will not try to distinguish the two concepts.

(2) We say that a group π is free abelian if it is a free abelian group on some subset S.

Example. One can easily show that the group

$$\mathbb{Z}^\infty := \{(x_1, x_2, x_3, \dots) \mid x_1, x_2, \dots \in \mathbb{Z} \text{ and all but finitely many } x_i \text{ are zero}\}$$

is a free abelian group on the subset $\{e_1, e_2, \dots\}$ where $e_i := (0, \dots, 0, 1, 0, \dots)$ has a unique 1 in the i-th entry. □

As always, it is easy to prove a suitable uniqueness of an object that is described by a universal property:

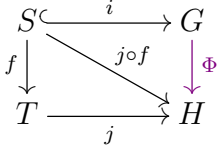
Lemma 97.6. (Free Abelian Group–Uniqueness Lemma) Let G be a free abelian group on a subset $S \subseteq G$ and let H be a free abelian group on a subset $T \subseteq H$. Given any bijection $f\colon S \to T$ there exists a unique isomorphism $\psi\colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Example.

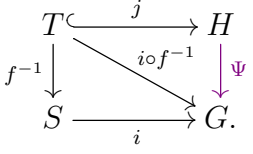
(1) Let $m \in \mathbb{N}_0$. One can easily verify that $\mathbb{Z}^m$ is a free abelian group on the subset $\{\mathfrak{e}_1, \dots, \mathfrak{e}_m\}$ given by the m standard basis vectors of $\mathbb{Z}^m$. It follows from the Free Abelian Group–Uniqueness Lemma 97.6 that if G is a free abelian group on a finite set S, then G is isomorphic to $\mathbb{Z}^{\#S}$.

(2) Let π be a free abelian group with basis S. If $\Phi\colon \pi \to \pi$ is a homomorphism such that $\Phi(s) = s$ for all $s \in S$, then it follows immediately from the uniqueness statement of the above definition, that $\Phi = \mathrm{id}$. □

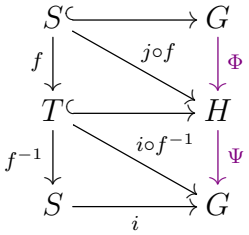
Proof of Lemma 97.6. We denote by $i\colon S \to G$ and $j\colon T \to H$ the inclusion maps. Furthermore we consider the maps $f\colon S \to T$ and $f^{-1}\colon T \to S$. Since G is a free abelian group on the subset $S \subseteq G$ and since H is a free abelian group on the subset $T \subseteq H$ we see that there exist homomorphisms $\Phi\colon G \to H$ and $\Psi\colon H \to G$ that make the following diagrams commute:



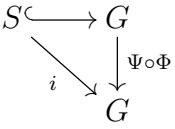
and



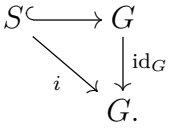
In particular we get the commutative diagram



which simplifies to



and of course we also have the commutative diagram



It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\Psi \circ \Phi = \mathrm{id}_G$. The same way one shows that $\Phi \circ \Psi = \mathrm{id}_H$. In particular Φ and Ψ are isomorphisms. ■

To prove existence of free abelian groups it is useful to specialize the definition of direct sums from the previous section to the case that $X_s = \mathbb{Z}$ for all $s \in S$. This leads us to the following definition, which will stay with us throughout these notes.

Definition 97.7. Let S be a set.

(1) We define[976]

$$\mathbb{Z}^{(S)} := \bigoplus_{s\in S} \mathbb{Z} = \{\text{all maps } S \to \mathbb{Z} \text{ that are zero for all but finitely many } s \in S\}.$$

(2) Let $s \in S$. The map

$$\begin{array}{rcl} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

lies in $\mathbb{Z}^{(S)}$ and it is also called s. Thus we can view S naturally as a subset of $\mathbb{Z}^{(S)}$.

(3) Given elements $s_1, \dots, s_k \in S$ and $n_1, \dots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 \cdot s_1 + \cdots + n_k \cdot s_k$ of $\mathbb{Z}^{(S)}$. We refer to $n_1 \cdot s_1 + \cdots + n_k \cdot s_k$ as a formal linear combination of elements in S. Every element of $\mathbb{Z}^{(S)}$ can be written in a unique way as such a formal linear combination of finitely many elements of S. Indeed, for $f \in \mathbb{Z}^{(S)}$ we can uniquely write[977]

$$f = \sum_{s\in S \text{ with } f(s)\neq 0} f(s) \cdot s.$$

The following lemma will not really be a surprise to the alert reader.

Lemma 97.8. (Free Abelian Group–Existence Lemma) Let S be a set. The group $\mathbb{Z}^{(S)}$ is a free abelian group on the subset $S \subseteq \mathbb{Z}^{(S)}$.

The Free Abelian Group–Existence Lemma 97.8 suggests the following definition:

Definition 97.9. Let S be a set. We refer to $\mathbb{Z}^{(S)}$ as *the* free abelian group generated by S.

Proof of Lemma 97.8. Let $f\colon S \to G$ be a map to an abelian group. We consider the map[978]

$$\begin{array}{rcl} \psi\colon \mathbb{Z}^{(S)} & \to & G \\ \underbrace{\sum_{i=1}^{k} n_i \cdot s_i}_{\substack{\text{we saw above that} \\ \text{every element of } \mathbb{Z}^{(S)} \text{ is} \\ \text{uniquely of that form}}} & \mapsto & \sum_{i=1}^{k} n_i \cdot f(s_i). \end{array}$$

It is straightforward to verify that ψ is in fact a homomorphism, that ψ has the desired property and that ψ is the unique such homomorphism. ∎

For fans of category theory we have the following lemma on offer:

Lemma 97.10. (Free Abelian–Forgetful Functor Lemma) Let $\mathcal{Set}$ be the category of sets and let $\mathcal{AbGroup}$ be the category of abelian groups. The two functors

$$\begin{array}{rcl} F\colon \mathcal{Set} & \to & \mathcal{AbGroup} \\ S & \mapsto & \mathbb{Z}^{(S)} \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{AbGroup} & \to & \mathcal{Set} \\ (G, +) & \mapsto & G \end{array}$$

form an adjoint pair in the sense of Definition 6.33.

[976] Note that, if one looks carefully at Definition 97.1 (1) and (2) one sees that the second equality is, slightly surprisingly, an *equality* of sets and also of abelian groups.

[977] It is easy to see that both sides define the same map $S \to \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

[978] Here we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$ and we would need to replace the sum by a product.

Proof. It follows easily from the Free Abelian Group–Existence Lemma 97.8 and Definition 97.5 that for a set S and a group G the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{S}et}(S, \overbrace{G}^{=G(G,\,\cdot\,)}) & \to & \mathrm{Mor}_{\mathcal{G}}(\overbrace{\mathbb{Z}^{(S)}}^{=F(S)}, (G,\ \cdot\ )) \\ (\text{map } h\colon S \to G \text{ of sets}) & \mapsto & \begin{array}{l}\text{the unique homomorphism } \mathbb{Z}^{(S)} \to (G,\ \cdot\ ) \\ \text{that restricts to } h \text{ on the subset } S \subseteq \mathbb{Z}^{(S)}\end{array} \end{array}$$

is a natural bijection. ■

The following lemma gives a typical application of free abelian groups.

Lemma 97.11. (Free Abelian Group–Splitting Lemma) Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) If A is a free abelian group, then there exists a right inverse $\sigma\colon A \to G$ of φ, this means that there exists a homomorphism $\sigma\colon A \to G$ with $\varphi \circ \sigma = \mathrm{id}_A$.[979]
(2) If $\sigma\colon A \to G$ is a right inverse of φ, then $G = \ker(\varphi) \oplus \sigma(A)$.[980]

Proof. Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) We assume that A is a free abelian group. We pick a basis B for the free abelian group A. Since $\varphi\colon G \to A$ is an epimorphism we can choose for each $b \in B$ an element $\widetilde{b} \in G$ with $\varphi(\widetilde{b}) = b$.

Since A is a free abelian group on B there exists a unique homomorphism $\sigma\colon A \to G$ with $\sigma(b) = \widetilde{b}$ for all $b \in B$. For each $b \in \widetilde{B}$ we have $(\varphi \circ \sigma)(b) = \varphi(\widetilde{b}) = b$. It follows from the discussion on page 1879 that $\varphi \circ \sigma = \mathrm{id}_A$.

(2) Let $\sigma\colon A \to G$ be a right inverse of φ. We start out with the following claim:

Claim. We have $\ker(\varphi) \cap \sigma(A) = \{0\}$.

Proof. If $g = \sigma(a) \in \ker(\varphi) \cap \sigma(A)$, then

$$g = \sigma(a) = (\sigma \circ \mathrm{id}_A)(a) \underset{\substack{\uparrow \\ \text{since } \varphi\circ\sigma = \mathrm{id}_A}}{=} \sigma(\varphi(\sigma(a))) \underset{\substack{\uparrow \\ \text{since } g = \sigma(a)}}{=} \sigma(\varphi(g)) \underset{\substack{\uparrow \\ \text{since } g \in \ker(\varphi)}}{=} \sigma(0) = 0.$$ ⊞

As we discussed on page 1878, it follows from the above claim that the natural homomorphism

$$\begin{array}{rcl} \Theta\colon \ker(\varphi) \oplus \sigma(A) & \to & G \\ g \oplus h & \mapsto & g + h \end{array}$$

is a monomorphism. It remains to show that Θ is an epimorphism. Let $g \in G$. Note that

$$\varphi(g - \sigma(\varphi(g))) = \varphi(g) - \varphi(\sigma(\varphi(g))) \underset{\substack{\uparrow \\ \text{since } \varphi\circ\sigma = \mathrm{id}_A}}{=} \varphi(g) - \varphi(g) = 0.$$

This shows that $g - \sigma(\varphi(g)) \in \ker(\varphi)$. It follows that $g \in \mathrm{im}(\sigma) \oplus \ker(\varphi)$. ■

Theorem 97.12. (Basis Cardinality Theorem) Let F be an abelian group. If F is free on two subsets S and T, i.e. if S and T are bases for F, then $\#S = \#T$.

[979] Note that any such right inverse $\sigma\colon A \to G$ is in particular a monomorphism.
[980] Here we use the Convention 97.3 to view the direct sum $\ker(\varphi) \oplus \sigma(A)$ as a subgroup of G.

Proof. The proof of this statement is basically the same as the proof that in a vector space any two bases have the same cardinality. This proof is given in detail in [**Loe2014**, Chapter 16.7]. We will only make use of the case that one of the sets is finite. This special case is also dealt with in Exercise 97.4. ■

The Basis Cardinality Theorem 97.12 allows us to make the following definition:

Definition 97.13. If F is a free abelian group on a subset S, then we define[981] [982]

$$\operatorname{rank}(F) \coloneqq \#S \quad \in \mathbb{N}_0 \cup \{\infty\}.$$

Example.

(1) The rank of the free abelian group $\mathbb{Z}^m$ is m.
(2) Let G be a free abelian group on a subset S and let H be a free abelian group on a subset T. It follows easily from the definitions and the Coproduct-of-Abelian Groups Lemma 97.2, that $G \oplus H$ is a free abelian group of rank $\operatorname{rank}(G) + \operatorname{rank}(H) \in \mathbb{N}_0 \cup \{\infty\}$.

□

97.3. **Subgroups of free abelian groups.** The only goal of this section is to prove following non-trivial theorem.

Theorem 97.14. (Free Abelian Subgroup Theorem) Let F be a free abelian group. Every subgroup H of F is a free abelian subgroup with $\operatorname{rank}(H) \leq \operatorname{rank}(F)$.

The discussion in the previous section of free abelian groups extends in an obvious way to more general settings, like free modules. The Free Abelian Subgroup Theorem 97.14 is quite different, since it does not generalize to other settings, and we really have to work specifically with properties of $\mathbb{Z}$.

We first provide a proof in the special case that F is a free abelian group of finite rank. Afterwards we provide the proof of the general case, which makes use of the esoteric Well-Ordering Theorem 1.32.

Proof of Theorem 97.14 if F has finite rank. Let F be an abelian group of rank n and let H be a subgroup of F. We need to show that H is free of rank $\leq n$.

We prove the statement by induction on n. If $n = 1$, then $F \cong \mathbb{Z}$. Since $\mathbb{Z}$ is a PID (principal ideal domain) we know that every subgroup of $\mathbb{Z}$ is of the form $k \cdot \mathbb{Z}$ for some $k \in \mathbb{N}_0$. Thus we see that every subgroup is a free abelian group of rank ≤ 1.

Now we suppose that the statement holds for all free abelian groups of rank $< n$ with $n \in \mathbb{N}_{\geq 2}$. Let F be a free abelian group of rank n and let $H \subseteq F$ be a subgroup. Let $t_1, \ldots, t_n$ be a basis for F. Let $F_n \subseteq F$ be the free abelian group on $t_1, \ldots, t_{n-1}$. We set $H_n \coloneqq H \cap F_n$. By induction H_n is a free abelian group of rank $\leq n - 1$. It remains to prove the following claim:

Claim. Either $H = H_n$ or there exists an $x \in H \setminus H_n$ with[983] $H = H_n \oplus \mathbb{Z} \cdot x$.

---

[981] Here we denote by $\#S$ the cardinality of the set S, see Definition 1.36.

[982] For simplicity we only view the cardinalities as elements of $\mathbb{N}_0 \cup \{\infty\}$. In other words, for the rank of an abelian group we do not distinguish between the different infinite cardinalities. This simplification is good enough for our purpose and it prevents us from having to deal with thorny set theoretic issues.

[983] It follows from the above example, that $\operatorname{rank}(H_n \oplus \mathbb{Z} \cdot x) = \operatorname{rank}(H_n) + \operatorname{rank}(\mathbb{Z} \cdot x) = n - 1 + 1 = n$.

Proof. We consider the following sequence of homomorphisms:

$$\begin{array}{ccccccc} H/H_n & = & H/(H\cap F_n) & \xrightarrow{\cong} & (H+F_n)/F_n & \subseteq & F/F_n \\ & & [h] & \mapsto & [h] & & [m\cdot t_n] \end{array} \quad \begin{array}{ccc} \xleftarrow{\cong} & \mathbb{Z}. \\ & m. \end{array}$$

If the resulting homomorphism $H/H_n \to \mathbb{Z}$ is trivial, then $H = H_n$, and we are done. Now suppose that the homomorphism $\varphi\colon H/H_n \to \mathbb{Z}$ is non-trivial. Then there exists an $r \in \mathbb{N}$ such that the map $\varphi\colon H/H_n \to \mathbb{Z}\cdot r$ is an epimorphism. It follows from the Free Abelian Group–Splitting Lemma 97.11, applied to the free abelian group $\mathbb{Z}\cdot r$, that $H = H_n \oplus \mathbb{Z}\cdot x$ for some $x \in H \setminus H_n$. ∎

Proof of Theorem 97.14 for any F. Let F be a free abelian group and let H be a subgroup of F. We pick a basis T for F. By the Well-Ordering Theorem 1.32 we can equip T with a well-order $\leq$. Recall that by Definition 1.31 this means the following: $\leq$ is a partial order with the following two properties:

(a) For every $a, b \in T$ we have $a \leq b$ or $b \leq a$.
(b) Every non-empty subset $S \subseteq T$ has a minimal element, i.e. given any non-empty subset $S \subseteq T$ there exists an $s \in S$ such that for all $\widetilde{s} \in S$ we have $s \leq \widetilde{s}$.

Given $t \in T$ we set

$$\begin{array}{rclcrcl} F_t & \coloneqq & \sum\limits_{s<t} \mathbb{Z}\cdot s, & & \overline{F}_t & \coloneqq & \sum\limits_{s\leq t} \mathbb{Z}\cdot s, \\ H_t & \coloneqq & H\cap F_t, & & \overline{H}_t & \coloneqq & H\cap \overline{F}_t. \end{array}$$

Evidently $F = \bigcup\limits_{t\in T} \overline{F}_t$ and $H = \bigcup\limits_{t\in T} \overline{H}_t$. Note that

$$\begin{array}{ccccccc} & & \text{since } H_t = H\cap F_t = (H\cap \overline{F}_t)\cap F_t = \overline{H}_t \cap F_t & & & & \\ & & \downarrow & & & & \\ \overline{H}_t/H_t & = & \overline{H}_t/(\overline{H}_t\cap F_t) & \xrightarrow{\cong} & (\overline{H}_t+F_t)/F_t & \subseteq & \overline{F}_t/F_t \\ & & [h] & \mapsto & [h] & & m\cdot t \end{array} \quad \begin{array}{ccc} \xleftarrow{\cong} & \mathbb{Z}. \\ & m. \end{array}$$

As in the above proof of the case that F has finite rank we obtain from the Free Abelian Group–Splitting Lemma 97.11 that $\overline{H}_t = H_t$ or that $\overline{H}_t = H_t \oplus \mathbb{Z}\cdot x_t$ for some $x_t \in \overline{H}_t \setminus H_t$.

We set $U \coloneqq \{t \in T \mid \overline{H}_t \neq H_t\}$. Note that $\#U \leq T$. It follows that it remains to prove the following claim:

Claim.
(1) The group H is generated by the set $\{x_u\}_{u\in U}$.
(2) The group H is free abelian on the set $\{x_u\}_{u\in U}$.

Proof.

(1) Let H' be the subgroup of H generated by the set $\{x_u\}_{u\in U}$. We suppose that $H \neq H'$.
  (a) Let $h \in H$. Since $F = \bigcup\limits_{t\in T} \overline{F}_t$ we see in particular that each $h \in H$ lies in some $\overline{F}_t$. Since we have a well-order on T there exists a least $\mu(h) \in T$ such that $h \in \overline{F}_{\mu(h)}$.
  (b) We consider $J \coloneqq \{\mu(h) \mid h \in H \text{ and } h \notin H'\} \subseteq T$.
  (c) Since $H \neq H'$ we see that J is non-empty. Since T is well-ordered we can consider the least element $j \in J$.
  (d) We pick $h' \in H$ with $\mu(h') = j$ and $h' \notin H'$. It follows from $\mu(h') = j$ that $h' \in \overline{F}_j$. Thus we see that $h' \in H \cap \overline{F}_j = \overline{H}_j$.
  (e) In the discussion preceding the claim we saw that $h' \in \overline{H}_j$ implies that $h' = a + m\cdot x_j$ for some $a \in H_j$ and some $m \in \mathbb{N}$. It follows that $a = h' - b \in H$.

(f) Note that $h' \in H'$ and $m \cdot x_j \in H'$. This implies that $a \notin H'$.
(g) Note that $a \in H_j$. This implies that $\mu(a) < j$.
(h) By (e), (f) and (g) we have thus obtained a contradiction to the choice of j. It follows that $H = H'$.

(2) By (1) it remains to show that the x_u, $u \in U$ are linearly independent. If this was not the case, then there would exist a non-trivial linear combination

$$\underbrace{m_1 \cdot x_{u_1} + \cdots + m_{n-1} \cdot x_{u_{n-1}}}_{\in H_{u_t}} + \underbrace{m_n \cdot x_{u_n}}_{\in \mathbb{Z} \cdot x_{u_n}} = 0 \quad \text{where } u_1 < \cdots < u_n \text{ and } m_n \neq 0.$$

Then $m_n \cdot x_{u_n} \in \mathbb{Z} \cdot x_{u_n} \cap H_{u_n}$. Since $\mathbb{Z} \cdot x_{u_n} \cap H_{x_{u_n}} = 0$ we have now obtained a contradiction. ■

97.4. **Rank of abelian groups.** Following [**Rob1995**, p. 96] we now define the rank of an abelian group as follows:

Definition 97.15. Let G be an abelian group.

(1) Let $S \subseteq G$ be a subset. By the Free Abelian Group–Existence Lemma 97.8 there exists a unique homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to G$ that makes the following diagram commute:

$$\begin{array}{ccc} S & \xhookrightarrow{i} & \mathbb{Z}^{(S)} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \Xi_S} \\ & & G \end{array}$$

where $i \colon S \to G$ is the inclusion map. We call Ξ_S the natural homomorphism $\mathbb{Z}^{(S)} \to G$.

(2) We define the $\operatorname{rank}(G) \in \mathbb{N}_0 \cup \{\infty\}$ as the largest cardinality[984] of a subset $S \subseteq G$ such that the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to G$ is a monomorphism.

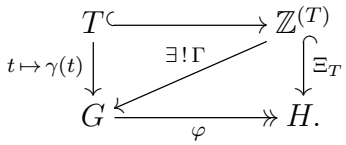
Lemma 97.16. (Rank Comparison Lemma)

(1) If G is a free abelian group on a set S, then $\operatorname{rank}(G) = \#S$.
(2) If $\varphi \colon G \to H$ is a monomorphism of abelian groups, then $\operatorname{rank}(G) \leq \operatorname{rank}(H)$.
(3) If $\varphi \colon G \to H$ is an epimorphism of abelian groups, then $\operatorname{rank}(G) \geq \operatorname{rank}(H)$.

Proof.

(1) Since G is a free abelian group generated by S we see that $\operatorname{rank}(G) \geq \#S$. It follows from the Free Abelian Subgroup Theorem 97.14 and the Basis Cardinality Theorem 97.12 that $\operatorname{rank}(G) \leq \#S$.
(2) This statement follows immediately from the definitions.
(3) Let $\varphi \colon G \to H$ be an epimorphism of abelian groups. Let $T \subseteq H$ be a subset such that the natural homomorphism $\Xi_T \colon \mathbb{Z}^{(T)} \to H$ is a monomorphism. Since $\varphi \colon G \to H$ is an epimorphism we see that for each $t \in T$ there exists a $\gamma(t) \in G$ with $\gamma(\varphi(t)) = t$. By the Free Abelian Group–Existence Lemma 97.8 there exists a unique homomorphism $\Gamma \colon \mathbb{Z}^{(T)} \to G$ with $\Gamma(t) = \gamma(t)$ for all $t \in T$. In summary we have the following commutative diagram:

[984] For simplicity we again only view the cardinalities as elements of $\mathbb{N}_0 \cup \{0\}$.



Note that we have $\varphi(\Gamma(t)) = \varphi(\gamma(t)) = t$ for all $t \in T$. It follows from the discussion on page 1879 that $\varphi \circ \Gamma = \mathrm{id}$. In particular we see that Γ is a monomorphism. It follows that $\mathrm{rank}(G) \geq \#T = \mathrm{rank}(H)$. ■

The following definition will be used frequently:

Definition 97.17. A short exact sequence of abelian groups is a sequence of homomorphisms[985]

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

between abelian groups A, B and C such that $\alpha\colon A \to B$ is a monomorphism, such that $\beta\colon B \to C$ is an epimorphism and such that $\mathrm{im}(\alpha) = \ker(\beta)$.

Proposition 97.18. (Rank Additivity Proposition) Let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

be a short exact sequence of abelian groups. If at least two out of the three groups have finite rank, then so has the third and we have the equality in $\mathbb{N}_0 \cup \{\infty\}$:

$$\mathrm{rank}(B) \;=\; \mathrm{rank}(A) + \mathrm{rank}(C).$$

Example. Let $B = A \oplus C$ be a direct sum of two abelian groups of finite rank. We get the short exact sequence $A \xrightarrow{a \mapsto a \oplus 0} B = A \oplus C \xrightarrow{b \oplus c \mapsto c} C$.

By the Rank Additivity Proposition 97.18 we have $\mathrm{rank}(A \oplus C) = \mathrm{rank}(A) + \mathrm{rank}(C)$. □

Sketch of a proof of Proposition 97.18. It follows from the Rank Comparison Lemma 97.16 that it remains to prove the following two statements:

(1) If A and C have finite rank, then so does B.
(2) We have the equality $\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C)$.

Let be a finite subset of A such that the natural homomorphism $\Xi_S\colon \mathbb{Z}^{(S)} \to A$ is a monomorphism and let $T = \{t_1, \ldots, t_l\}$ be a finite subset of C such that the natural homomorphism $\Xi_T\colon \mathbb{Z}^{(T)} \to C$ is a monomorphism. Since β is an epimorphism we can pick for each $i \in \{1, \ldots, l\}$ a $\widetilde{t}_i \in B$ with $\beta(\widetilde{t}_i) = t_i$. We set $\widetilde{T} := \{\widetilde{t}_1, \ldots, \widetilde{t}_l\}$. One can easily verify that $S \cap \widetilde{T} = \varnothing$. Using the fact that we are dealing with a short exact sequence it is now elementary to show that the natural homomorphism $\Xi_{S \cup \widetilde{T}}\colon \mathbb{Z}^{(S \cup \widetilde{T})} \to B$ is a monomorphism. This shows that $\mathrm{rank}(B) \geq \mathrm{rank}(A) + \mathrm{rank}(C)$.

Let $Y \subseteq B$ be a maximal subset such that the natural homomorphism $\Xi_Y\colon \mathbb{Z}^{(Y)} \to B$ is a monomorphism. Let $Z \subseteq Y$ be a maximal subset such that $\beta \circ \Theta_Z\colon \mathbb{Z}^{(Z)} \to C$ is a monomorphism. By definition we have $\#Z \leq \mathrm{rank}(C)$. We set $X := Y \setminus Z$. Let $x \in X$. It follows easily from the maximality of Z that the image of $\beta(x)$ in $C/\beta(\mathbb{Z}^{(Z)})$ is torsion. Thus there exists an $n_x \in \mathbb{N}$ such that $n_x \cdot x \in \ker(\beta)$. We set $\widetilde{x} := n_x \cdot x$. It follows from $\ker(\beta) = \mathrm{im}(\alpha)$ that $\widetilde{x} \in \mathrm{im}(\alpha)$. We set $\widetilde{X} := \{\widetilde{x}\}_{x \in X}$. One can easily verify that

[985] In Definition 145.1 we will get to know the general concept of an "exact sequence" and in Definition 145.7 we will then define again the notion of a "short exact sequence". At that point it will be clearer where the name "short exact sequence" comes from.

$\#\widetilde{X} = \#X$ and that the natural homomorphism $\Xi_{\widetilde{X}}\colon \mathbb{Z}^{(\widetilde{X})} \to A$ is a monomorphism. By definition we have $\#\widetilde{X} \leq \operatorname{rank}(A)$. In summary we have shown that

$$\operatorname{rank}(B) \;=\; \#Y \;=\; \#X + \#Z \;=\; \#\widetilde{X} + \#Z \;\leq\; \operatorname{rank}(A) + \operatorname{rank}(C). \qquad \blacksquare$$

97.5. **Finitely generated abelian groups.** In this section we will introduce the notion of a "finitely generated abelian group" and we will see that these are among the rare classes of mathematical objects which are completely classified.

Definition 97.19.

(1) Given an abelian group G we say that $S \subseteq G$ is a generating set if the natural homomorphism $\Xi_S\colon \mathbb{Z}^{(S)} \to G$ is an epimorphism.
(2) We call an abelian group finitely generated if it admits a finite generating set.

Lemma 97.20. (Finite Generation Lemma) Let $\varphi\colon A \to B$ be a homomorphism of abelian groups.

(1) If φ is an epimorphism and if A is finitely generated, then so is B.
(2) If φ is a monomorphism and if B is finitely generated, then so is A.
(3) Let
$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$
be a short exact sequence of abelian groups. If two of these groups are finitely generated, then so is the third.

Proof.

(1) Let $\varphi\colon A \to B$ be an epimorphism of abelian groups. If S is a finite generating set for A, then clearly the finite set $\varphi(S)$ is a generating set for B.
(2) We only need to show that if B is a finitely generated group, then so is every subgroup A of B. Let T be a finite generating set for B. We denote by $\Xi_T\colon \mathbb{Z}^{(T)} \to B$ the natural epimorphism. Note that $\Xi_T^{-1}(A)$ is a subgroup of $\mathbb{Z}^{(T)}$. We can summarize the maps in the following commutative diagram:
$$\begin{array}{ccc} \Xi_T^{-1}(A) & \hookrightarrow & \mathbb{Z}^{(T)} \\ {\scriptstyle \Xi_T}\downarrow & & \downarrow{\scriptstyle \Xi_T} \\ A & \hookrightarrow & B. \end{array}$$
By the Free Abelian Subgroup Theorem 97.14 we know that $\Xi_T^{-1}(A)$ is a free abelian group of rank $\leq \#T$. Let S be a basis for $\Xi_T^{-1}(A)$. It follows immediately from the definitions that $\Xi_T(S)$ is a generating set for A. Since $\#\Xi_T(S) \leq \#S \leq \#T$ we see that A is finitely generated.
(3) If B is finitely generated then it follows from (1) and (2) that A and C are also finitely generated. So it remains to prove that if A and C are finitely generated, then so is B. Let $\{s_1, \ldots, s_k\}$ be a finite generating set for A and let $\{t_1, \ldots, t_l\}$ be a finite generating set for C. Since β is an epimorphism we know that for each $t_i \in T$ we can pick a $\widetilde{t}_i \in B$ with $\beta(\widetilde{t}_i) = t_i$. We leave it to the reader to verify that $\{s_1, \ldots, s_k, \widetilde{t}_1, \ldots, \widetilde{t}_l\}$ is a generating set for B. $\blacksquare$

Lemma 97.21. (Matrix Quotient Lemma)

(1) Let H be an abelian group with finite generating set $S = \{s_1, \dots, s_m\}$. There exists an $n \in \mathbb{N}_0$ and an $m \times n$–matrix A over $\mathbb{Z}$ such that the map

$$\begin{array}{rcl} \mathbb{Z}^m/A \cdot \mathbb{Z}^n & \to & H \\ \left[\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}\right] & \mapsto & \sum_{i=1}^{m} v_i \cdot s_i \end{array}$$

is a well-defined isomorphism.

(2) If $A \in \mathrm{M}(n \times n, \mathbb{Z})$ is a diagonal matrix with entries $d_1, \dots, d_n \in \mathbb{Z}$, then

$$\mathbb{Z}^n/A \cdot \mathbb{Z}^n \cong \bigoplus_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}.$$

(3) Let A be an $m \times n$–matrix over $\mathbb{Z}$. For matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ the map

$$\begin{array}{rcl} \mathbb{Z}^m/A \cdot \mathbb{Z}^n & \cong & \mathbb{Z}^m/PAQ \cdot \mathbb{Z}^n \\ [v] & \mapsto & [P \cdot v] \end{array}$$

is well-defined and it is an isomorphism.

Proof.

(1) Let H be an abelian group with finite generating set $S = \{s_1, \dots, s_m\}$. By definition the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to H$ is an epimorphism. Let $\Theta \colon \mathbb{Z}^m \to \mathbb{Z}^{(S)}$ be the natural isomorphism given by $\Theta(e_i) = s_i$. Note that $\Xi_S \circ \Theta \colon \mathbb{Z}^m \to H$ is also an epimorphism. By the Free Abelian Subgroup Theorem 97.14 we know that $\ker(\Xi_s) \subseteq \mathbb{Z}^m$ is a finitely generated free abelian subgroup. Let $a_1, \dots, a_n \in \mathbb{Z}^m$ be a basis for this free abelian group. We set $A \coloneqq (a_1 \ \dots \ a_n) \in \mathrm{M}(m \times n)$. One can easily verify that this matrix has the desired properties.

(2) This statement follows easily from the definitions.

(3) First note that for $Q \in \mathrm{GL}(n, \mathbb{Z})$ we have $Q \cdot \mathbb{Z}^n = \mathbb{Z}^n$. It follows that the given map is well-defined. An inverse is given by $[w] \mapsto [P^{-1} \cdot w]$. This shows that the given map is an isomorphism. ■

Theorem 97.22. (Smith Normal Form Theorem) Let A be an $m \times n$–matrix over $\mathbb{Z}$. There exist two matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ and $d_1, \dots, d_t \in \mathbb{N}$ such that[986]

$$P \cdot A \cdot Q = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & d_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \in \mathrm{M}(m \times n, \mathbb{Z}).$$

The matrix on the right is sometimes called a Smith normal form of A.

Remark. In Exercise 97.12 we will discuss a more refined version of the Smith Normal Form Theorem 97.22, which then also gives us a uniqueness statement for the Smith normal form. □

Proof. We postpone the proof to Appendix 97.6. Alternatively we refer to [**NewM1972**, Chapter II.15] and [**Nor2012**, Chapter 1] for a proof. ■

As a straightforward corollary we can now prove the following result:

[986] Here the last row and column of zeros can be arbitrarily large.

Lemma 97.23. (Finite Matrix-Quotient Lemma) Given any $A \in \mathrm{M}(n \times n, \mathbb{Z})$ we have

$$\mathbb{Z}^n/A \cdot \mathbb{Z}^n \text{ is finite} \iff \det(A) \neq 0.$$

Furthermore, if $\det(A) \neq 0$, then $\mathbb{Z}^n/A \cdot \mathbb{Z}^n$ is a finite group with $|\det(A)|$ elements.

Proof. It follows from the Smith Normal Form Theorem 97.22 and the Matrix Quotient Lemma 97.21 (3) that it suffices to deal with the case that $A \in \mathrm{M}(n \times n, \mathbb{Z})$ is a diagonal matrix. But in this special case the statement follows easily from the Matrix Quotient Lemma 97.21 (2). ■

Now we can state and prove one of the best-loved theorems in mathematics.

Theorem 97.24. (Abelian Group-Classification Theorem) Let H be a finitely generated abelian group. There exist prime powers[987] $q_1, \ldots, q_k$ such that

$$H \cong \mathbb{Z}^{\mathrm{rank}(H)} \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/q_i\mathbb{Z}.$$

Furthermore the prime powers $q_1, \ldots, q_k$ are uniquely determined up to permutation.

Definition 97.25. Given a finitely generated abelian group A we denote the direct sum of the cyclic groups on the right-hand side of the Abelian Group-Classification Theorem 97.24 as the canonical form of A.

Remark. In Exercise 97.13 we will formulate an alternative classification of finitely generated abelian groups. □

Sketch of proof. [988] Let H be a finitely generated abelian group. We first prove the existence of the promised isomorphism. Note that it follows from the Matrix Quotient Lemma 97.21 (1), the Smith Normal Form Theorem 97.22 and the Matrix Quotient Lemma 97.21 (2) and (3) that there exist $r \in \mathbb{N}$ and $d_1, \ldots, d_t \in \mathbb{N}$ such that

$$H \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}/d_i\mathbb{Z}.$$

The existence of the promised decomposition follows from the following two observations:

(1) It follows from the Rank Additivity Proposition 97.18 and the Rank Comparison Lemma 97.16 that $r = \mathrm{rank}(H)$.
(2) It follows from the Chinese Remainder Theorem 97.4 that each summand $\mathbb{Z}/d_i\mathbb{Z}$ is isomorphic to the direct sum of quotients of $\mathbb{Z}$ by prime powers.

It remains to prove the uniqueness of the prime power q_i up to permutation. One can do so by introducing the following invariant of abelian groups: Given a prime p, given $k \in \mathbb{N}$ and given an abelian group V we consider

$$T_{p,k}(V) := \#\{v \in V \mid p^k \cdot v = 0\}.$$

These numbers are clearly isomorphism invariants of abelian groups. By studying these invariants one can easily prove that the q_i are unique up to permutation. We will fill in the details in Exercise 97.14 (a). ■

[987] By a prime power we mean any natural number of the form p^s where p is a prime and $s \in \mathbb{N}$.
[988] We provide a sketch of a proof, full details are for example given in [**Hun1980**, Chapter II.2].

Proposition 97.26. (Cancellation Proposition) Let A and C be finitely generated abelian groups and let B be an abelian group. Then

$$A \oplus C \cong B \oplus C \implies A \cong B.$$

Proof. Let A and C be finitely generated abelian groups and let B be an abelian group. We assume that $A \oplus C \cong B \oplus C$. First note that it follows from the hypothesis that A and C are finitely generated, together with the Finite Generation Lemma 97.20 (2) and (3), that B is also finitely generated.

By the Abelian Group-Classification Theorem 97.24 and the Chinese Remainder Theorem 97.4 we know that the finitely generated group C is isomorphic to the direct sum of finitely many cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/q^l\mathbb{Z}$, where q is a prime. By induction it remains to prove the Cancellation Proposition for $C = \mathbb{Z}$ and for $C = \mathbb{Z}/q^l\mathbb{Z}$. This special case can be proved for example using the group invariants $T_{p,k}(V)$ that we introduced in the proof of the Abelian Group-Classification Theorem 97.24. We will fill in the details in Exercise 97.14 (b). ■

We continue our discussion of finitely generated abelian groups with the following pleasant lemma.

Lemma 97.27. (Epi-Automorphism Lemma) Let T be a finitely generated abelian group. If $\varphi\colon T \to T$ is an epimorphism, then it is already an isomorphism.

Proof. We outsource the proof to Exercise 97.15. ■

For completeness' sake we introduce the following definition, which will play a role every now and then.

Definition 97.28. Let A be an abelian group. We refer to

$$\mathrm{Tor}(A) := \{a \in A \mid \text{ there exists an } n \in \mathbb{N} \text{ with } n \cdot a = 0\}$$

as the torsion subgroup of A.[989] If $\mathrm{Tor}(A)$ is the trivial subgroup, then we say that A is torsion-free.

It follows from the Abelian Group-Classification Theorem 97.24 that any torsion-free finitely generated abelian group is in fact a free abelian group.

Abelian groups that are not finitely generated are not classified and they tend to be significantly harder to deal with. For example the following lemma shows that, in contrast to the finitely generated case, not every torsion-free abelian group is a free abelian group.

Lemma 97.29. ($\mathbb{Q}$-not-Free Lemma) The abelian group $(\mathbb{Q}, +)$ is torsion-free but not free abelian.

Proof. Clearly $(\mathbb{Q}, +)$ is torsion-free. If it was free abelian, then there would exist an epimorphism to $\mathbb{Z}$. In Exercise 97.6 we will show that such an epimorphism cannot exist. ■

97.6. Appendix: Proof of the Smith Normal Form Theorem 97.22.

In this appendix we will provide a sketch of the proof of the Smith Normal Form Theorem 97.22.

[989] In Exercise 98.14 we will see that A is indeed a subgroup.

Proof of the Smith Normal Form Theorem 97.22. First note that if an $m \times n$–matrix P is obtained from an $m \times n$–matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $(m \times m)$–matrix of the form[990]

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$–matrix Q by adding the k-th row to the l-th row, then P is obtained by left-multiplying Q by an $(m \times m)$–matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

where the extra 1 sits in the $(1,k)$-entry. Both these $(m \times m)$-matrices lie in $\mathrm{GL}(m,\mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\mathrm{GL}(n,\mathbb{Z})$ as above.

Now let A be an $m \times n$–matrix over $\mathbb{Z}$. It follows from the above discussion that it suffices to show that we can turn A into the desired form using row and column operations.

If A is the zero matrix, then there is nothing to show. So suppose that A is not the zero matrix. We perform the following steps:

(a) After swapping rows and columns we can arrange that the absolute value of the $(1,1)$-entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1,1)$-entry v_{11} is positive.
(b) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1,1)$-entry, lie between 0 and $v_{11} - 1$.
(c) The same way as in (2) we can arrange that all entries in the first column, except for the $(1,1)$-entry, lie between 0 and $v_{11} - 1$.

Now we have to distinguish two cases:

(1) If there exists outside of the $(1,1)$-entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1,1)$-entry will be less than before. So this process will come to a halt after finitely many steps.
(2) If all entries in the first row and column, except for the $(1,1)$-entry are zero, then

$$A = \begin{pmatrix} v_{11} & 0 \\ 0 & A' \end{pmatrix}$$

for some $(m-1) \times (n-1)$–matrix A' and we perform the same operations on the smaller matrix A'. ∎

[990] Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

97.7. **Appendix: Group rings.** In this chapter we learned about the free abelian group generated by a set. If we look at the free abelian group generated by group, we actually get a new and interesting mathematical object, namely a ring:

Definition 97.30. Let G be a group and let R be a commutative ring (in most cases $R = \mathbb{Z}$). We denote the free R-module generated[991] by the set G by $R[G]$. Put differently, with Definition 97.7 we have

$$R[G] \coloneqq \left\{ \sum_{i=1}^{m} r_i \cdot g_i \,\middle|\, r_1, \dots, r_m \in R \text{ and } g_1, \dots, g_m \in G \right\}.$$

We equip $R[G]$ with the multiplication that is given by[992]

$$\left(\sum_{i=1}^{m} r_i \cdot g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \coloneqq \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_R s_j}_{\in R} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $R[G]$ as the group ring of G with R-coefficients. (For $R = \mathbb{Z}$ we refer to $\mathbb{Z}[G]$ as the group ring of G.)

The following lemma summarizes a few basic facts about group rings.

Lemma 97.31. (Group Ring Lemma) Let G be a group and let R be a commutative ring.

(1) The group ring $R[G]$ is a ring in the sense of Definition 2.6. The multiplicatively neutral element is given by $1_R \cdot e$ where e denotes the trivial element in G.
(2) If R is not the zero ring, then the group ring $R[G]$ is commutative if and only if G is an abelian group.
(3) If R is a field, then $R[G]$ is an algebra over the field R.
(4) Given a unit $r \in R$ and an element $g \in G$ the product $r \cdot g$ is a unit in $R[G]$.
(5) If G has a torsion element, i.e. if there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$, then $R[G]$ has zero divisors.

Proof. All the statements are verified easily. Perhaps the most interesting statement is (5). We will not rob the reader of the pleasure of proving this in Exercise 97.19. ■

Example. Let R be any commutative ring.

(1) We consider the infinite cyclic group $\langle t \rangle \coloneqq \{\dots t^{-2}, t^{-1}, 1, t, t^2, \dots\}$. In this case the group ring $R[\langle t \rangle]$ is isomorphic to the ring $R[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map

$$\begin{array}{ccc} R[t^{\pm 1}] & \to & R[\langle t \rangle] \\ \sum_{i=r}^{i=s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{i=s} a_i \cdot t^i \end{array}$$

is clearly a ring isomorphism.
(2) If H is a free abelian group of rank m, then any choice of a basis $t_1, \dots, t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ to the group ring $R[H]$. □

Subconsciously or consciously one constantly makes use of the following lemma:

[991] The free R-module generated by a set is defined in the same way as we defined the free abelian group generated by a set.
[992] The multiplication is basically given by naive "multiplying out".

Lemma 97.32. (Group Ring–Universal Property Lemma) Let R be a commutative ring and let G be a group. Furthermore let S be an R-algebra, i.e. S is an R-module with a compatible ring structure (not necessarily commutative). We denote by $S^\times$ the group of multiplicative units in S. Given any group homomorphism $\varphi\colon G \to S^\times$ there exists a unique ring R-algebra homomorphism $R[G] \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\quad\varphi\quad} & S^\times \\ \downarrow{\scriptstyle g\mapsto 1_R\cdot g} & & \downarrow \\ R[G] & \xrightarrow[R\text{-algebra homomorphism}]{\exists!} & S. \end{array}$$

In fact this R-algebra homomorphism is given by

$$\begin{array}{rcl} R[G] & \to & S \\ \sum\limits_{g\in G} r_g \cdot g & \mapsto & \sum\limits_{g\in G} r_g \cdot \varphi(g). \end{array}$$

Proof. The lemma follows easily from the definitions. ∎

Lemma 97.33. (Group Ring Functor Lemma) Let R be a commutative ring.

(1) The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{G}roup) & \to & \mathrm{Ob}(\mathcal{R}ing) \\ G & \mapsto & R[G] \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{G}roup}(G,H) & \to & \mathrm{Mor}_{\mathcal{R}ing}(R[G],R[H]) \\ (\varphi\colon G\to H) & \mapsto & \left(\begin{array}{rcl} \varphi_*\colon R[G] & \to & R[H] \\ \sum\limits_{i=1}^m r_i\cdot g_i & \mapsto & \sum\limits_{i=1}^m r_i\cdot\varphi(g_i) \end{array}\right) \end{array}$$

define a covariant functor from the category $\mathcal{G}roup$ of groups to the category $\mathcal{R}ing$ of rings.

(2) The group ring functor is a left adjoint functor in the sense of Definition 6.33. In particular we obtain from the Left Adjoint–Colimit Proposition 6.35 that the group ring functor preserves all colimits.

Proof. Let R be a commutative ring.

(1) One can easily verify that the induced map $\varphi_*\colon R[G] \to R[H]$ is indeed a ring homomorphism. All other statements are basically trivial.

(2) We consider the two functors

$$\begin{array}{rcl} \alpha\colon \mathcal{G}roup & \to & \mathcal{R}ing \\ G & \mapsto & R[G] \\ (G \xrightarrow{\varphi} H) & \mapsto & (R[G] \xrightarrow{\varphi_*} R[H]) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \mathcal{R}ing & \to & \mathcal{G}roup \\ R[G] & \mapsto & R[G]^\times \\ (R[G] \xrightarrow{\varphi} R[H]) & \mapsto & (R[G]^\times \xrightarrow{\varphi} R[H]^\times) \end{array}$$

It follows almost immediately from the Group Ring–Universal Property Lemma 97.32 that for any group X and any ring Y we have a natural bijection

$$\mathrm{Mor}_{\mathcal{R}ing}(\underbrace{\alpha(X)}_{=R[X]}, Y) \to \mathrm{Mor}_{\mathcal{G}roup}(X, \underbrace{\beta(Y)}_{=R[Y]^\times}).$$

This shows that α is a left-adjoint of the functor β. ∎

We continue with the following elementary lemma:

Lemma 97.34. (Group Ring–Module Lemma) Let G be a group and let A be an abelian group (with additive notation for the group structure). Suppose we are given a G-action on the group A.[993] Then the map[994]

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^{m} r_i \cdot g_i, a \right) & \mapsto & \sum_{i=1}^{m} r_i \cdot (\underbrace{g_i \cdot a}_{\in A}) \end{array}$$

defines a natural[995] $\mathbb{Z}[G]$-left module[996] structure on A.

Proof. On a more formal level this lemma can be rephrased as follows: We are given a group homomorphism $G \to \mathrm{Aut}(A)$. From the Group Ring–Universal Property Lemma 97.32 we obtain a corresponding ring homomorphism $\mathbb{Z}[G] \to \mathrm{End}(A)$. Since A is evidently an $\mathrm{End}(A)$-left module we get an induced $\mathbb{Z}[G]$-left module structure on A. ■

As a side remark we mention some (partially obsolete) conjectures. These say (or said) in particular that in some sense the statements of the Group Ring Lemma 97.31 (4) and (5) are optimal.

Conjecture 97.35. (Kaplansky Conjecture) Let G be a torsion-free group and let $\mathbb{F}$ be a field.

(1) The elements 0 and 1 are the only idempotents of $\mathbb{F}[G]$, i.e. they are the only elements x in $\mathbb{F}[G]$ with $x^2 = x$.
(2) The group ring $\mathbb{F}[G]$ is a domain, i.e. it has no non-trivial zero divisors.

Historic Conjecture 97.35. (Kaplansky Conjecture) Let G be a torsion-free group and let $\mathbb{F}$ be a field.

(3) The only units in $\mathbb{F}[G]$ are the trivial units, i.e. the elements of the form $f \cdot g$ with $f \in \mathbb{F} \setminus \{0\}$ and $g \in G$.

Conjecture 97.36. (Skew Field Conjectures) Given any torsion-free group G the group ring $\mathbb{C}[G]$ can be embedded in a skew field.

Remark.

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[993] Recall that according to Convention 2.40 this means that G acts by automorphisms of the abelian group A.

[994] Here we use a somewhat perilous notation, namely we use the symbol "$\cdot$" with two different meanings. First of all, for $g \in G$ and $a \in A$ we denote by $g \cdot a \in A$ the element defined by the G-action on A. Secondly, for $r \in \mathbb{Z}$ and $b \in A$ we write, as usual, $r \cdot b = \underbrace{b + \cdots + b}_{r\text{-times}}$ for $r \geq 0$ and $r \cdot b = \underbrace{-b - \cdots - b}_{(-r)\text{-times}}$ for $r < 0$.

[995] We leave it to the reader to ponder what "natural" might mean in this context.

[996] If R is a non-commutative ring, then one needs to be a little careful when one talks about "R-modules". More precisely, there are now two different concepts one can consider: Given an abelian group M one can demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = r \cdot (s \cdot m)$ in which case one says that M is an "R-left module". Alternatively one could demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = s \cdot (r \cdot m)$ in which case one says that M is an "R-right module". This discussion is the analog of the discussion on page 127. The notion of left and right modules will be discussed in great detail in Chapter 309.

(1) The three conjecture in the Kaplansky Conjecture 97.35 are named after the Canadian mathematician Irving Kaplansky [**Kapy1957, Kapy1970**]. The Kaplansky Conjecture 97.35 (1) is often also referred to as the *zero divisor conjecture*. These conjectures are formulated in [**Lü2002**, Conjecture 10.14], [**Lam1991**, p. 95] and [**FrG2005**, p. 729]. It is elementary to see that the Skew Field Conjectures 97.36 implies the Kaplansky Conjecture 97.35 (2) and that the Kaplansky Conjecture 97.35 (2) implies the Kaplansky Conjecture 97.35 (1).

(2) The Kaplansky Conjecture 97.35 (3) is by now obsolete since Giles Gardam [**Garda2021**, Theorem A][**Garda2023**, Theorem A] and Alan Murray [**Murr2021**] showed that for any characteristic there exists a field $\mathbb{F}$ and a torsion-free group G such that $\mathbb{F}[G]$ has non-trivial units.

(3) The Kaplansky Conjectures are very attractive since they are easy to state. But it is usually very very hard to prove non-trivial group-theoretic statements that are valid for all (torsion-free) groups. In particular the Kaplansky Conjectures 97.35 (1) and (2) are still wide open. We refer to [**Lü2009**, Chapter 4.5] for a summary of what is known regarding these two conjectures. □

97.8. Exercises for Chapter 97.

Exercise 97.1. Let $\varphi\colon G \to A$ be an epimorphism between abelian groups. Is it necessarily true that $G \cong \ker(\varphi) \oplus \sigma(A)$?

Exercise 97.2. An abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. An abelian group G is called *torsion*, if given any $g \in G$ there exists an $n \in \mathbb{N}$ with $n \cdot g = 0$. Does there exist a non-trivial abelian group that is divisible and torsion?

Exercise 97.3. Let A be an abelian group and let $\mathcal{B} \subseteq A$ be a subset. We recall the following two concepts from linear algebra:

- We define $\operatorname{span}(\mathcal{B})$ as the smallest abelian subgroup of A that contains $\mathcal{B}$. We say $\mathcal{B}$ spans A if $\operatorname{span}(\mathcal{B}) = A$. One can easily verify that this is equivalent to saying that every element of A is a linear combination of elements of $\mathcal{B}$.
- We say $\mathcal{B}$ is *linearly independent* if for every choice of pairwise distinct $b_1, \dots, b_k$ in $\mathcal{B}$ and every choice of $n_1, \dots, n_k \in \mathbb{Z}$ we have

$$n_1 \cdot b_1 + \cdots + n_k \cdot b_k = 0 \implies n_1 = \cdots = n_k = 0.$$

Show that the following two statements are equivalent:

(i) $\mathcal{B}$ is a basis of A in the sense of Definition 97.5.
(ii) $\mathcal{B}$ spans A and $\mathcal{B}$ is linearly independent.

Exercise 97.4. Let F be an abelian group that is free on two subsets S and T. Show that if S is finite, then so is T and $\#S = \#T$.
Hint. Recall that on page 1882 we provided an elementary proof of the Free Abelian Subgroup Theorem 97.14 in the special case that F is a free abelian group of finite rank. A modification of that argument can also be used to solve this exercise.

Exercise 97.5.

(a) What is the rank of the abelian group $\mathbb{Q}$?
(b) What is the rank of the abelian group $\mathbb{R}$?

Exercise 97.6. Show that $\mathbb{Q}$ does not admit an epimorphism onto $\mathbb{Z}$.

Exercise 97.7. Given $n \in \mathbb{Z}$ we consider the matrix $B(n) = \begin{pmatrix} 0 & 3 \\ 3 & n \end{pmatrix}$ and we consider the abelian group $A_n := \mathbb{Z}^2/B(n) \cdot \mathbb{Z}^2$. Complete the following sentence: Given $m, n \in \mathbb{Z}$ the abelian groups A_m and A_n are isomorphic if and only if ....

Exercise 97.8. Let G be an abelian group and let $S \subseteq G$ be a subset. Show that the following two statements are equivalent:

(1) S is a generating set of G.
(2) Any subgroup H of G that contains S is already equal to G.

Exercise 97.9.

(a) Show that there exists an abelian group G with $G \oplus G \cong G$.
(b) Let A be a finitely generated abelian group. Show that if B is an abelian group with $B \oplus A \cong A$, then $B \cong 0$.
Remark. A stronger statement holds: If A is a finitely generated abelian group and if B, C are abelian groups with $B \oplus A \cong C \oplus A$, then $B \cong C$. This stronger statement is proved in [**CohnP1956, WalkE1956**].

Exercise 97.10. Give an example of finitely generated abelian groups A and B and an abelian group C such that $A \oplus C \cong B \oplus C$ but such that A and B are not isomorphic.

Exercise 97.11. Let G be an additive abelian group. We say that an element $g \in G$ is *primitive* if there does not exist an $h \in G$ and an $n \geq 2$ with $n \cdot h = g$.

(a) Suppose that G is finitely generated. Let $g \in G$. Show that the following statements are equivalent:
(i) g is a primitive element.
(ii) We can write $G = \mathbb{Z} \cdot g \oplus H$ for some abelian group G.
(iii) There exists a homomorphism $\varphi\colon G \to \mathbb{Z}$ with $\varphi(g) = 1$.
(b) Which of the implications in (a) also hold for abelian groups that are not finitely generated?

Exercise 97.12.

(a) Let $p, q \in \mathbb{N}$. Show that if p and q are coprime, then there exist $X, Y \in \mathrm{GL}(2, \mathbb{Z})$ with
$$X \cdot \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \cdot Y = \begin{pmatrix} p \cdot q & 0 \\ 0 & 1 \end{pmatrix}.$$
(b) Let $A \in \mathrm{M}(m \times n; \mathbb{Z})$. By the Smith Normal Form Theorem 97.22 we know that there exist matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ such that $P \cdot A \cdot Q$ is of the form
$$P \cdot A \cdot Q = \begin{pmatrix} D_{k\times k} & 0_{k\times(n-k)} \\ 0_{k\times(m-k)} & 0_{(n-k)\times(m-k)} \end{pmatrix}$$
where $D = \mathrm{diag}(d_1, \ldots, d_k)$ is a diagonal $(k \times k)$-matrix such that the diagonal entries $d_1, \ldots, d_k$ lie in $\mathbb{N}$. Show that we can find P and Q such that for each $i \in \{1, \ldots, k-1\}$ we have $d_i | d_{i+1}$.

(c) Show that $d_1, \dots, d_k \in \mathbb{N}$ with $d_i | d_{i+1}$ as in (b) are unique.

Exercise 97.13. Let A be a finitely generated abelian group.

(a) Show that there exist $a_1, \dots, a_k \in \mathbb{N}$ with $a_i | a_{i+1}$ for $i = 1, \dots, k-1$ and an $r \in \mathbb{N}_0$ such that
$$A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/a_i\mathbb{Z}.$$

(b) Show that $a_1, \dots, a_k$ and r are uniquely determined by A.

Remark. This exercise can be viewed as an alternative formulation of the Abelian Group-Classification Theorem 97.24.

Exercise 97.14. Given a prime p and $k \in \mathbb{N}$ and given an abelian group V we set
$$T_{p,k}(V) := \#\{v \in V \mid p^k \cdot v = 0\}.$$

(a) Let $r, s \in \mathbb{N}_0$ and let $a_1, \dots, a_k$ and $b_1, \dots, b_l$ be prime powers. Suppose that for all primes p and all $k \in \mathbb{N}$ we have
$$T_{p,k}(\mathbb{Z}^r \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_k\mathbb{Z}) = T_{p,k}(\mathbb{Z}^s \oplus \mathbb{Z}/b_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_l\mathbb{Z}).$$
Show that $\mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_k\mathbb{Z} \cong \mathbb{Z}/b_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_l\mathbb{Z}$.

(b) Let A and B be finitely generated abelian groups and let q be a prime power.
(i) Show that if $A \oplus \mathbb{Z}/q\mathbb{Z} \cong B \oplus \mathbb{Z}/q\mathbb{Z}$, then $A \cong B$.
(ii) Show that if $A \oplus \mathbb{Z} \cong B \oplus \mathbb{Z}$, then $A \cong B$.

Exercise 97.15. Let T be an abelian group and let $\varphi\colon T \to T$ is an epimorphism.

(a) Show that if T is finitely generated, then $\varphi\colon T \to T$ is an isomorphism.

(b) Is the hypothesis that T is finitely generated in (a) necessary?

Exercise 97.16. What is the minimal $n \in \mathbb{N}$ for which there exist at least 5 pairwise non-isomorphic abelian groups of order n?

Exercise 97.17. An abelian group H is called *torsion* if for every $h \in H$ there exists an $n \in \mathbb{N}$ with $n \cdot h = 0$. We say that H is *torsion-free* if for every non-zero h and any $n \in \mathbb{N}$ we have $n \cdot h \neq 0$. We consider the group $G = \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

(1) How many maximal torsion-free subgroups does G contain?

(2) How many maximal torsion subgroups does G contain?

(Here by "maximal" we means that it is not contained in a larger subgroup with the same property.)

Exercise 97.18.

(a) Let $\varphi\colon G \to \mathbb{Z}$ be an epimorphism. Does φ admit a left-right inverse? In other words, does there exist a homomorphism $\phi\colon \mathbb{Z} \to G$ with $\varphi \circ \phi = \mathrm{id}_{\mathbb{Z}}$?

(b) Let G be an abelian group and let H be a torsion-free abelian group. Let $\varphi\colon G \to H$ be an epimorphism. Does φ admit a left-right inverse? In other words, does there exist a homomorphism $\phi\colon H \to G$ with $\varphi \circ \phi = \mathrm{id}_H$?

Exercise 97.19. Let G be a group and let R be a commutative ring. Suppose that there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$. Show that $R[G]$ has a zero divisor, i.e. show that there exist non-zero elements $p, q \in R[G]$ with $p \cdot q = 0$.
Hint. First consider the case that $G = \langle t \mid t^n \rangle$ is a finite cyclic group with $n \geq 2$ elements.

Exercise 97.20. We consider the cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ which we think of as a multiplicative group with generator x. Show that $1 - x - x^{-1}$ is a unit in group group ring $\mathbb{Z}[G]$.

CHAPTER 98

Basics of group theory II: The free & amalgamated product of groups

In the previous chapter we discussed abelian groups in great detail. We recall the definition of free abelian groups and we recalled the classification of finitely generated abelian groups. In this chapter we now turn to the study of general groups, which are not necessarily abelian.

This chapter is organized as follows:

- In Section 98.1 we will introduce the free product of a family of groups.
- In Section 98.2 we will introduce free groups.
- In Section 98.3 we will introduce the abelianization of a group.
- In Section 98.4 we will introduce the amalgamated product of groups.
- In Appendix 98.5 we will state (but not prove) the Normal Form Theorem for amalgamated products.
- In Appendix 98.6 we will state (and again not prove) the Grushko-Neumann Theorem 98.31.
- In Section 98.7 we will provide exercises for Chapter 98.

In the next Chapter 99 we will state and prove the Seifert–van Kampen Theorem 99.1.

98.1. **The free product of groups.** In the Coproduct-of-Abelian Groups Lemma 97.2 we saw that the coproduct of a family $\{X_s\}_{s\in S}$ of *abelian* groups in the category $\mathcal{A}b\mathcal{G}roup$ of *abelian* groups is given by the direct sum $\bigoplus_{s\in S} X_s$. In this section we want to determine the coproduct of any family of groups in the category $\mathcal{G}roup$ of all groups. The following definition is the key to doing so.

Definition 98.1. Let $\{X_s\}_{s\in S}$ be a family of groups. (After possibly replacing the family $\{X_s\}_{s\in S}$ by the family $\{X_s \times \{s\}\}_{s\in S}$ we can and will assume that the groups X_s are pairwise disjoint.)

(1) A reduced word in $\{X_s\}_{s\in S}$ is a finite sequence $(x_1, \dots, x_m)$ such that the following conditions are satisfied:
   (a) each x_i lies in one of the groups X_s,[997]
   (b) no x_i is the neutral element of any X_s,
   (c) any two consecutive x_j's lie in two different groups.
   Note that we also allow the "empty sequence" ().

(2) We denote by $\ast_{s\in S} X_s$ the set of all reduced words.

(3) Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be two elements in $\ast_{s \in S} X_s$. We define the product of these two elements as follows:[998]
   (a) First suppose that x_m and y_1 lie in different groups. Then we define
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \coloneqq (x_1, \dots, x_m, y_1, \dots, y_n).$$
   (b) Now suppose that x_m and y_1 lie in the same group. Let $j \in \{1, \dots, \min\{m, n\}\}$ be the maximal element such that x_{m+1-i} and y_i lie in the same group and such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$. If $j < \min\{m, n\}$, then we set[999]
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \coloneqq (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$
   If $j = m$, we set $(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \coloneqq (y_{m+1}, \dots, y_n)$.
   We proceed similarly if $j = n$.
(4) We refer to $\ast_{s \in S} X_s$ together with this product structure as the free product of the X_s. For a finite family $X_1, \dots, X_k$ of groups we denote the free product by $X_1 * \cdots * X_k$.

To consider our first example of a free product we introduce one new definition.

Definition 98.2. Let t be a symbol. We define
$$\langle t \rangle \coloneqq \{\dots, t^{-2}, t^{-1}, e, t, t^2, \dots\}.$$
The set $\langle t \rangle$ admits an obvious group structure that is given by $t^i \cdot t^j \coloneqq t^{i+j}$. This group $\langle t \rangle$ is called the infinite cyclic group generated by t. The map $\mathbb{Z} \to \langle t \rangle$, $n \mapsto t^n$ is evidently an isomorphism.

Example. We consider the free product $\langle s \rangle * \langle t \rangle$. Examples of elements in this free product are (s^3, t^{-5}, s^2) and $(s^{-2}, t^4, s^7, t^{-1})$. The product of these two elements is given by
$$(s^3, t^{-5}, s^2) \cdot (s^{-2}, t^4, s^7, t^{-1}) = (s^3, \underbrace{t^{-5} \cdot t^4}_{=t^{-1}}, s^7, t^{-1}).$$
Also note that $(s) \cdot (t) = (s, t) \neq (t, s) = (t) \cdot (s)$. Thus we see that the free product $\langle s \rangle * \langle t \rangle$ is very much non-abelian. □

We have the following important lemma, which allows us to view each free product of groups as a group in its own right.

Lemma 98.3. (Free Product Lemma) Let $\{X_s\}_{s \in S}$ be a family of groups.
(1) The free product $\ast_{s \in S} X_s$ with the above product is again a group.

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[997] Since the groups X_s are disjoint we know that for each x_i there exists a unique $s \in S$ with $x_i \in X_s$.

[998] A more informal way of stating the product formula is as follows: Given two reduced words $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ we put them next to another to obtain $(x_1, \dots, x_m, y_1, \dots, y_n)$ and we do the following:
   (a) We delete any occurrence of a subsequence of the form a, a^{-1} for $a \in X_s$.
   (b) If there exists a subsequence of the form a, b with $a, b \in X_s$ and $a \neq b^{-1}$, then we replace it by $a \cdot b \in X_s$.
This way we end up with a reduced word. The problem with this informal definition is that it is not entirely clear that mixing (a) and (b) we have a well-defined procedure.

[999] Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j} \cdot y_{j+1}$.

(2) The neutral element of $\underset{s\in S}{*} X_s$ is given by the empty sequence ().

(3) The inverse of an element $(x_1,\dots,x_m)\in \underset{s\in S}{*} X_s$ is given by $(x_m^{-1},\dots,x_1^{-1}) \in \underset{s\in S}{*} X_s$.

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1,\dots,x_m)\in \underset{s\in S}{*} X_s$ is given by $(x_m^{-1},\dots,x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

We will therefore follow a different approach. We write $W := \underset{s\in S}{*} X_s$ and we denote by $(\mathrm{Bij}(W),\circ)$ the group of bijections $W\to W$, where the group structure is given by composition of maps. For $g\in X_s$ we consider the map[1000]

$$\begin{array}{rcl} \lambda_g\colon W & \to & W \\ (x_1,\dots,x_m) & \mapsto & \begin{cases} (g,x_1,x_2,\dots,x_m), & \text{if } x_1\notin X_s, \\ (g\cdot x_1,x_2,\dots,x_m), & \text{if } x_1\in X_s \text{ and } g\neq x_1^{-1}, \\ (x_2,\dots,x_m), & \text{if } g=x_1^{-1}. \end{cases} \end{array}$$

We make the following two observations:

(a) If $()\in W$ is the empty word, then $\lambda_{()}=\mathrm{id}$.

(b) It is straightforward to see that for any $g,g'\in X_s$ we have $\lambda_g\circ\lambda_{g'}=\lambda_{g\cdot g'}\colon W\to W$.

It follows from (a) and (b) that for any $g\in X_s$ we have $\lambda_{g^{-1}}\circ\lambda_g=\mathrm{id}_W$. This shows in particular that each λ_g is a bijection. Next we consider the map

$$\begin{array}{rcl} \lambda\colon W & \to & \mathrm{Bij}(W) \\ (x_1,\dots,x_n) & \mapsto & \lambda_{x_1}\circ\cdots\circ\lambda_{x_n}. \end{array}$$

This map is injective since $\lambda(x_1,\dots,x_n)$ applied to the empty word () gives the word $(x_1,\dots,x_n)$. Thus different elements of W define different bijections of W.

Using the above two observations (a) and (b) it is straightforward to verify that the map $\lambda\colon W\to\mathrm{Bij}(W)$ satisfies $\lambda(x\cdot y)=\lambda_x\circ\lambda_y$. Put differently, λ defines an injective map $\lambda\colon (W,\ \cdot\ )\to(\mathrm{Bij}(W),\circ)$ that preserves the product structure. But since $(\mathrm{Bij}(W),\circ)$ satisfies the associativity law we now see that $(W,\ \cdot\ )$ also satisfies the associativity law. ■

Example. Let G and H be two groups. We consider the corresponding free product $G*H$. Let $g\in G$ and $h\in H$ be two non-trivial elements. Note that in $G*H$ we have $(g)\cdot(h)=(g,h)$ and $(h)\cdot(g)=(h,g)$. These are different elements of the free product $G*H$. This shows, similar to the discussion on page 1899, that the free product of at least two non-trivial groups is non-abelian. □

The following lemma, which can be viewed as an analog of the Coproduct-of-Abelian Groups Lemma 97.2, summarizes the main properties of the free product of groups.

Lemma 98.4. (Coproduct-of-Groups Lemma) Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) (a) Let $t\in S$. The map

[1000] It is straightforward to see that the sequences on the right do indeed lie in $\underset{s\in S}{*} X_s$.

$$\begin{aligned} \iota_t\colon X_t &\to \mathop{\ast}_{s\in S} X_s \\ x &\mapsto \begin{cases} (x), & \text{if } x \neq e, \\ (), & \text{if } x = e \end{cases} \end{aligned}$$

is a monomorphism.

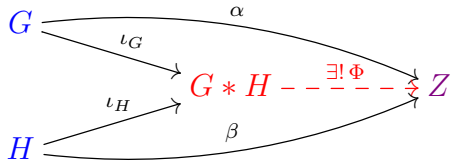
(b) Let Z be a group and let $\{f_s\colon X_s \to Z\}_{s\in S}$ be a family of homomorphisms. The map

$$\begin{aligned} \Phi\colon \mathop{\ast}_{s\in S} X_s &\to Z \\ (x_1, \ldots, x_m) &\mapsto \underbrace{f_{s_1}(x_1) \cdot \ldots \cdot f_{s_m}(x_m)}_{\text{where } x_i \in X_{s_i}} \end{aligned}$$

is a homomorphism and it is the unique homomorphism such that for each $s \in S$ we have $f_s = \Phi \circ \iota_s$.

(2) The coproduct in the category *Group* of groups of $\{X_s\}_{s\in S}$ is given by $\mathop{\ast}_{s\in S} X_s$ together with the homomorphisms $\iota_t\colon X_t \to \mathop{\ast}_{s\in S} X_s$ from (1a).

Example. It is worth considering the case of the free product of two groups G and H. Let $\alpha\colon G \to Z$ and $\beta\colon H \to Z$ be two group homomorphisms to some group Z. The above Coproduct-of-Groups Lemma 98.4 (2) says that there exists a unique group homomorphism $\Phi\colon G * H \to Z$ such that the following diagram commutes:



For example, using $Z = G \times H$ with the obvious homomorphisms from G and H to $G \times H$ we get a natural homomorphism $G * H \to G \times H$. □

Proof.

(1) (a) Let $t \in T$. It is clear that the map $\iota_t\colon X_t \to \mathop{\ast}_{s\in S} X_s$ is injective and it follows immediately from the definition of the group structure on $\mathop{\ast}_{s\in S} X_s$ that the map is a group homomorphism.

(b) Let Z be a group and let $\{f_s\colon X_s \to Z\}_{s\in S}$ be a family of homomorphisms. One can easily verify that the given map $\Phi\colon \mathop{\ast}_{s\in S} X_s \to Z$ is a homomorphism and that it satisfies $f_s = \Phi \circ \iota_s$ for each $s \in S$. It remains to show the uniqueness of Φ. Any element in $\mathop{\ast}_{s\in S} X_s$ is a product of elements of the form $\iota_t(x)$ with $x \in X_t$. It follows that any homomorphism $\delta\colon \mathop{\ast}_{s\in S} X_s \to Z$ is already uniquely determined by the values of δ on all elements of the form $\iota_t(x)$ with $x \in X_t$. Put differently, there is at most one homomorphism $\delta\colon \mathop{\ast}_{s\in S} X_s \to Z$ with $\delta(\iota_t(x)) = f_t(\iota_t(x)) = (f_t \circ \iota_t)(x)$ for all $t \in T$ and all $x \in X_t$. This implies the uniqueness of Φ.

(2) This statement follows immediately from (1a) and (1b) and the definition of the coproduct in Definition 6.5. ■

Convention 98.5. Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) Let $t \in T$. We refer to

$$\begin{aligned} \iota_t \colon X_t &\to \mathop{\ast}_{s\in S} X_s \\ x &\mapsto \begin{cases} (x), & \text{if } x \neq e, \\ (), & \text{if } x = e \end{cases} \end{aligned}$$

as the natural monomorphism $\iota_t \colon X_t \to \mathop{\ast}_{s\in S} X_s$.

(2) For $t \in S$ we use the natural monomorphism $\iota_s \colon X_t \to \mathop{\ast}_{s\in S} X_s$ to view X_t as a subgroup of the free product $\mathop{\ast}_{s\in S} X_s$.

(3) Let $(x_1, \dots, x_m) \in \mathop{\ast}_{s\in S} X_s$. Since $(x_1, \dots, x_m) = (x_1)\cdot\ldots\cdot(x_m)$ we can use (2) to denote this element by $x_1 \cdot \ldots \cdot x_m$.

Example. We consider again the free product $\langle s\rangle * \langle t\rangle$. Using the above convention we now write $s^3 \cdot t^{-5} \cdot s^2 = (s^3, t^{-5}, s^2)$ and $s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1} = (s^{-2}, t^4, s^2, t^{-1})$. The product of these two elements is given by very reasonable looking

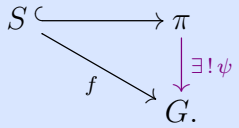
$$(s^3 \cdot t^{-5} \cdot s^2) \cdot (s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1}) \;=\; s^3 \cdot \underbrace{t^{-5} \cdot t^4}_{=t^{-1}} \cdot s^2 \cdot t^{-1}.$$

Also note that the discussion on page 1899 shows that $s \cdot t \neq t \cdot s$, in particular we see that the free product $\langle s\rangle * \langle t\rangle$ is non-abelian. □

98.2. **Free groups.** In Section 97.2 we introduced the notion of a free *abelian* group. In this section we will modify the definition to introduce the notion of a free group. We start out with the following definition which is the analog of Definition 97.5.

Definition 98.6.

(1) We say that a group π is a free group on a subset $S \subseteq \pi$, if the following universal property holds: For any *map* $f \colon S \to G$ of the set S to a group G there exists a unique *homomorphism* $\psi \colon \pi \to G$ that makes the following diagram commute:



We call S a basis of the free group π.

(2) We say that a group π is free if it is a free group on some subset S.

Lemma 98.7. (Free Group–Uniqueness Lemma) Let G be a free group on a subset $S \subseteq G$ and let H be a free group on a subset $T \subseteq H$. Given any bijection $f \colon S \to T$ there exists a unique isomorphism $\psi \colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Proof. The proof of this lemma is basically identical to the proof of the Free Abelian Group–Uniqueness Lemma 97.6. ■

Definition 98.8. For a set S we write

$$\langle S\rangle \;:=\; \mathop{\ast}_{s\in S} \langle s\rangle.$$

The lemma below allows us to call $\langle S\rangle$ the free group on the set S. The map $S \to \langle S\rangle$ that is given by $s \mapsto (s)$ is an injection and we use it to view S as a subset of $\langle S\rangle$.

The following lemma is now the analog of the Free Abelian Group–Existence Lemma 97.8.

Lemma 98.9. (Free Group–Existence Lemma) Let S be a set. The group $\langle S\rangle$ is a free group on the subset $S \subseteq \langle S\rangle$.

Proof. Let G be a group and let $f\colon S \to G$ be a map. For $s \in S$ the map

$$\begin{array}{rcl} \psi_s\colon \langle s\rangle & \to & G \\ s^n & \mapsto & f(s)^n \end{array}$$

is evidently the unique homomorphism $\psi_s\colon \langle s\rangle \to G$ with $\psi_s(s) = f(s)$. It now follows from the Coproduct-of-Groups Lemma 98.4 (2) that the map

$$\begin{array}{rcl} \psi\colon \langle S\rangle = \underset{s\in S}{\ast} \langle s\rangle & \to & G \\ s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k} & \mapsto & f(s_{i_1})^{n_1} \cdot \ldots \cdot f(s_{i_k})^{n_k} \end{array}$$

is the unique homomorphism $\langle S\rangle \to G$ with $\psi(s) = \psi_s(s) = f(s)$ for $s \in S$. ■

The Free Group–Existence Lemma 98.9 motivates the following definition:

Definition 98.10.

(1) For a set S we refer to $\langle S\rangle = \underset{s\in S}{\ast} \langle s\rangle$ as *the* free group on the set S.

(2) Let $s_1, \ldots, s_k$ be distinct elements of a set. We refer to

$$\langle s_1, \ldots, s_k\rangle \;\coloneqq\; \langle s_1\rangle * \cdots * \langle s_k\rangle$$

as the free group on the generators $s_1, \ldots, s_k$. Note that each element of the free group $\langle s_1, \ldots, s_k\rangle$ is uniquely of the form

$$s_{r_1}^{n_1} \cdot \ldots \cdot s_{r_m}^{n_m}$$

with $r_i \neq r_{i+1}$ for $i = 1, \ldots, m-1$ and $s_i \neq 0$ for $i = 1, \ldots, m$.

(3) Sometimes we denote the isomorphism type of a free group on k generators just by F_k.

Example. For any set S there exists by the Free Group–Existence Lemma 98.9 a unique homomorphism $\Xi_S\colon \langle S\rangle \to \mathbb{Z}^{(S)}$ that sends $s \in S$ to s viewed as an element in $\mathbb{Z}^{(S)}$. Since the set $S \subseteq \mathbb{Z}^{(S)}$ generates $\mathbb{Z}^{(S)}$ we see that this homomorphism is an epimorphism. □

In the Free Abelian Subgroup Theorem 97.14 we showed that every subgroup of a free *abelian* group is again a free abelian group. It it is natural to ask whether the analogous statement holds for free groups? We record this as a question for future reference.

Question 98.11. Is every subgroup of a free group again a free group?

We will answer this question in the Countable Free Subgroup Proposition 130.5 and the Free Subgroup Proposition 138.6.

We conclude this section with the following analog of the Free Abelian–Forgetful Functor Lemma 97.10:

Lemma 98.12. (Free–Forgetful Functor Lemma) Let *Set* be the category of sets and let *Group* be the category of groups. The two functors

$$\begin{array}{rcl} F\colon \mathit{Set} & \to & \mathit{Group} \\ S & \mapsto & \langle S\rangle \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathit{Group} & \to & \mathit{Set} \\ (G,\ \cdot\ ) & \mapsto & G \end{array}$$

form an adjoint pair in the sense of Definition 6.33.

Proof. It follows easily from the Free Group–Existence Lemma 98.9 and Definition 98.6 that for a set S and a group G the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{S}}(S, \overbrace{G}^{=G(G,\,\cdot\,)}) & \to & \mathrm{Mor}_{\mathcal{G}}(\overbrace{\langle S\rangle}^{=F(S)}, (G,\ \cdot\ )) \\ (\text{map } h\colon S \to G \text{ of sets}) & \mapsto & \text{the unique homomorphism } \langle S\rangle \to (G,\ \cdot\ ) \\ & & \text{that restricts to } h \text{ on the subset } S \subseteq \langle S\rangle \end{array}$$

is a natural bijection. ∎

98.3. **The abelianization of a group.** It is pretty clear that abelian groups are much easier to handle than non-abelian groups. In this section we will get to know the abelianization of a group, which allows us in many settings to turn questions about arbitrary groups into questions about abelian groups.

Definition 98.13. Let G be a group.

(1) Given $x, y \in G$ we refer to $[x,y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$ as the commutator of x and y.

(2) We refer to

$$[G,G] := \left\{ \prod_{i=1}^{n} [x_i, y_i] \;\middle|\; x_1, y_1, \ldots, x_n, y_n \in G \right\}$$

as the commutator subgroup of G.

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $G = \langle a,b,c,d\rangle$ is the free group on four generators a,b,c,d, then we have $[a,b]\cdot[c,d] \in [G,G]$. In Exercise 98.2 we will see that $[a,b]\cdot[c,d]$ is not a commutator in G. □

Lemma 98.14. (Commutator Subgroup Lemma) Let G be a group.

(1) The commutator subgroup $[G,G]$ is a normal subgroup of G.

(2) The group $G_{\mathrm{ab}} := G/[G,G]$ is abelian.

(3) Let $\alpha\colon G \to H$ be a homomorphism to an abelian group. There exists a unique homomorphism $\beta\colon G_{\mathrm{ab}} = G/[G,G] \to H$ such that the following diagram commutes:[1001]

$$\begin{array}{ccc} G & & \\ {\scriptstyle g\mapsto[g]}\downarrow & \searrow^{\alpha} & \\ G_{\mathrm{ab}} = G/[G,G] & \xrightarrow[\exists!\,\beta]{} & H \text{ abelian}. \end{array}$$

This homomorphism β is given by $\beta([g]) = \alpha(g)$.

[1001] Another way of formulating this statement is to say that for any group G and any abelian group H the map

$$\begin{array}{rcl} \mathrm{Hom}(G_{\mathrm{ab}}, H) & \to & \mathrm{Hom}(G,H) \\ (\alpha\colon G_{\mathrm{ab}} \to H) & \mapsto & \left(G \xrightarrow{g\,\mapsto\,[g]} G_{\mathrm{ab}} \xrightarrow{\alpha} H\right) \end{array}$$

is a bijection.

Proof. Let G be a group.

(1) First we show that $[G,G]$ is a subgroup of G:
- Evidently the subset $[G,G]$ is closed under products.
- With $n=0$ we see that $e \in [G,G]$.
- The inverse of a commutator is again a commutator; indeed given $x,y \in G$ we have
$$[x,y]^{-1} = (xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1} = [y,x].$$
It follows easily that $[G,G]$ is closed under taking inverses.

We have thus shown that $[G,G]$ is a subgroup of G. It remains to show that $[G,G]$ is a normal subgroup of G. The key to doing so is the following claim:

Claim. A conjugate of a commutator is again a commutator.

Proof. Given $x,y,g \in G$ we have
$$\begin{aligned} g[x,y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1},gyg^{-1}]. \end{aligned}$$ ⊞

Given $x_1,\dots,x_n,y_1,\dots,y_n \in G$ and given $g \in G$ we now see that
$$g \cdot \left(\prod_{i=1}^{n}[x_i,y_i]\right) g^{-1} \underset{\uparrow}{=} \prod_{i=1}^{n} g\cdot[x_i,y_i]\cdot g^{-1} \underset{\uparrow}{=} \prod_{i=1}^{n}[gx_ig^{-1},gy_ig^{-1}] \in [G,G].$$
follows from the equality $g(ab)g^{-1} = gag^{-1}\cdot gbg^{-1}$ — by the claim

We have thus shown that $[G,G]$ is a normal subgroup.

(2) For any two elements $x,y \in G$ we have
$$xy\,[G,G] = yx\cdot x^{-1}y^{-1}xy\,[G,G] = yx\cdot \underbrace{[x^{-1},y^{-1}]}_{\in[G,G]}\,[G,G] = yx\,[G,G].$$
This shows that $G_{\mathrm{ab}} = G/[G,G]$ is indeed abelian.

(3) Let G be a group and let $\alpha\colon G \to H$ be a homomorphism to an abelian group. Using the fact that H is abelian one can easily verify that $[G,G]$ lies in the kernel of α. The desired statement now follows immediately from the Quotient Group Lemma 2.4 (1b). ■

The Commutator Subgroup Lemma 98.14 leads us to the following definition:

Definition 98.15. Let G be a group.

(1) We call $G_{\mathrm{ab}} := G/[G,G]$ the abelianization of G.[1002]

(2) We denote by
$$\begin{aligned} \Phi_G\colon G &\to G_{\mathrm{ab}} = G/[G,G] \\ g &\mapsto [g] \end{aligned}$$
the natural projection.

(3) Let $\alpha\colon G \to H$ be a homomorphism. By the Commutator Subgroup Lemma 98.14 (3), applied to the abelian group H_{ab} and the homomorphism $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$, there exists a unique homomorphism $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \Phi_G}\downarrow & & \downarrow{\scriptstyle \Phi_H} \\ G_{\mathrm{ab}} & \xrightarrow{\alpha_*} & H_{\mathrm{ab}}. \end{array}$$

We refer to α_* as the induced homomorphism. Note that α_* is given by $\alpha_*([g]) = [\alpha(g)]$.

Example.

(1) Let G be an abelian group. In this case every commutator is evidently trivial, thus $[G,G] = \{e\}$. This implies that the natural homomorphism $\Phi_G \colon G \to G_{\mathrm{ab}} \to G/[G,G]$ is an isomorphism.

(2) Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subseteq S_k$ the alternating group, i.e. the group of all permutations with positive sign. In basically every standard algebra book (see [**Rob1995**, p. 69] or [**Bog2008**, Theorem 11.3]) it is shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself. We just saw in the Commutator Subgroup Lemma 98.14 (1) that the commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group. □

Proposition 98.16. (Abelianization Functor Proposition)

(1) The maps
$$\begin{array}{rcl} G & \mapsto & G_{\mathrm{ab}} = G/[G,G] \\ (\varphi \colon G \to H) & \mapsto & (\varphi_* \colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}) \end{array}$$
define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

(2) The abelianization functor is a left adjoint functor in the sense of Definition 6.33. In particular we obtain from the Left Adjoint–Colimit Proposition 6.35 that the abelianization functor preserves all colimits.

Proof.

(1) This statement follows easily from the definitions.

(2) We consider the two functors
$$\begin{array}{rcl} \alpha \colon \mathit{Group} & \to & \mathit{AbGroup} \\ G & \mapsto & G_{\mathrm{ab}} = G/[G,G] \\ (\varphi \colon G \to H) & \mapsto & (\varphi_* \colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta \colon \mathit{AbGroup} & \to & \mathit{Group} \\ G & \mapsto & G \\ (\varphi \colon G \to H) & \mapsto & (\varphi \colon G \to H) \end{array}$$
Note that it follows easily from the Commutator Subgroup Lemma 98.14 (3) that for any group X and any abelian group Y we have a natural bijection
$$\mathrm{Mor}_{\mathit{AbGroup}}(\underbrace{\alpha(X)}_{=X_{\mathrm{ab}}}, Y) \to \mathrm{Mor}_{\mathit{Group}}(X, \underbrace{\beta(Y)}_{=Y}).$$
This shows that α is a left-adjoint of the functor β. ■

The following lemma will be used on many occasions.

[1002] In the literature the abelianization of a group π is sometimes also written as $\mathrm{H}_1(\pi)$. The reason for that somewhat strange looking notation will become clear later on when we formulate the Hurewicz Theorem 166.10 and when we consider the homology groups, see Chapter 317.

Lemma 98.17. (Abelianization-Epimorphism Lemma) If $\alpha\colon G \to H$ is an epimorphism, then the induced map $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an epimorphism.

Proof. For fun we give two arguments:

(1) By definition of α_* we have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \Phi_G}\downarrow & & \downarrow{\scriptstyle \Phi_H} \\ G_{\mathrm{ab}} & \xrightarrow{\alpha_*} & H_{\mathrm{ab}}. \end{array}$$

By hypothesis $\alpha\colon G \to H$ is an epimorphism. It follows that $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$ is an epimorphism, which implies that α_* is an epimorphism.

(2) The statement follows for formal reasons from the Abelianization Functor Proposition 98.16 (2) together with the Left Adjoint–Colimit Proposition 6.35 and the Cokernel-as-Coequalizer 6.29. ∎

The following proposition allows us to calculate the abelianization in several basic situations.

Proposition 98.18. (Abelianization Coproduct Proposition)

(1) Let $\{X_s\}_{s\in S}$ be a family of groups. We have natural homomorphisms

$$\begin{array}{rcl} \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} & \to & \bigoplus_{s\in S} (X_s)_{\mathrm{ab}} \\ [(x_1,\ldots,x_k)] & \mapsto & \sum_{i=1}^{k} [x_i] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \bigoplus_{s\in S} (X_s)_{\mathrm{ab}} & \to & \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} \\ \sum_{i=1}^{k} [x_i] & \mapsto & [x_1 \cdot \ldots \cdot x_k]. \end{array}$$

These are inverses of one another, in particular they are isomorphisms.

(2) Let S be a set. The map

$$\begin{array}{rcl} \langle S\rangle_{\mathrm{ab}} & \to & \mathbb{Z}^{(S)} \\ [s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k}] & \mapsto & \sum_{i=1}^{k} n_i \cdot s_i \end{array}$$

is a natural isomorphism.

Proof of the Abelianization Coproduct Proposition 98.18.

(1) It is worth explaining where the two maps in the proposition come from.

(a) By the Coproduct-of-Abelian Groups Lemma 97.2 (1a) we have for each $s \in S$ a natural homomorphism $X_s \to (X_s)_{\mathrm{ab}} \to \bigoplus_{s\in S}(X_s)_{\mathrm{ab}}$. By the Coproduct-of-Groups Lemma 98.4 these give rise to a homomorphism $\underset{s\in S}{*} X_s \to \bigoplus_{s\in S} (X_s)_{\mathrm{ab}}$. Since the target is abelian we obtain from the Commutator Subgroup Lemma 98.14 (3) an induced homomorphism $\left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} \to \bigoplus_{s\in S} (X_s)_{\mathrm{ab}}$.

(b) By the Coproduct-of-Groups Lemma 98.4 (1a) we have for each $s \in S$ a natural homomorphism $X_s \to \underset{s\in S}{*} X_s$. This homomorphism induces a homomorphism $(X_s)_{\mathrm{ab}} \to \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}}$. By the Coproduct-of-Abelian Groups Lemma 97.2 (1b) these homomorphisms gives rise to a homomorphism $\bigoplus_{s\in S} (X_s)_{\mathrm{ab}} \to \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}}$.

If one goes through the explicit descriptions of the homomorphisms one sees that the above two homomorphisms are precisely the homomorphisms we wrote down in the

proposition. This shows that the maps of the proposition are well-defined and that they are homomorphisms. Now that we know that the given maps are well-defined it is pretty clear that they are inverses of one another.

Alternatively one could also argue that the statement essentially follows from the Abelianization Functor Proposition 98.16 (2) together with the Left Adjoint–Colimit Proposition 6.35 and the fact that coproducts are by definition colimits.

(2) This statement follows almost immediately from the elementary discussion on page 1906 and Statement (1) together with the observation that by definition we have $\langle S\rangle = \mathop{\ast}_{s\in S} \langle s\rangle$ and $\mathbb{Z}^{(S)} = \bigoplus_{s\in\mathbb{Z}} \mathbb{Z}$. ■

We conclude this section with a proposition which be viewed as an analog of the Basis Cardinality Theorem 97.12 for free abelian groups.

Proposition 98.19. (Basis Cardinality Proposition) Let G be a group that is free on a subset S and let H be a group that is free on a subset $T \subseteq H$. If G and H are isomorphic, then $\#S = \#T$.

Proof. It follows immediately from the Free Group–Uniqueness Lemma 98.7 that we can assume that $G = \langle S\rangle$ and that $H = \langle T\rangle$.

(a) Let $\alpha\colon \langle S\rangle \to \langle T\rangle$ be an isomorphism.

(b) It follows from the Abelianization Functor Proposition 98.16 together with the Isomorphism Preservation Lemma 5.26 that $\alpha_*\colon \langle S\rangle_{\mathrm{ab}} \to \langle T\rangle_{\mathrm{ab}}$ is also an isomorphism.

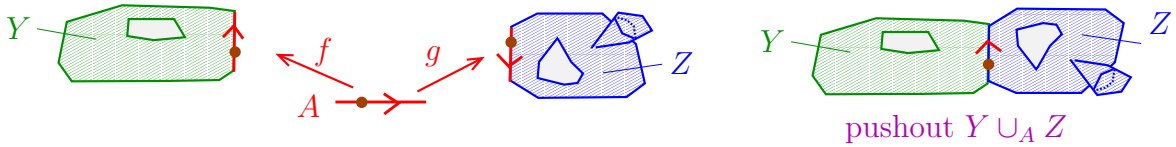
(c) It follows from (b) and the Abelianization Coproduct Proposition 98.18 (2) that $\mathbb{Z}^{(S)}$ and $\mathbb{Z}^{(T)}$ are isomorphic.

(d) It follows from (c) and the Basis Cardinality Theorem 97.12 that $\#S = \#T$. ■

98.4. **The amalgamated product of groups.** Suppose we are given two continuous maps $f\colon (A, a_0) \to (Y, y_0)$ and $g\colon (A, a_0) \to (Z, z_0)$ between pointed topological spaces. Note that by the same argument as in the Topological Pushout Proposition 8.12 we see that the topological space

$$Y \cup_A Z \;\coloneqq\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

together with the base points $[y_0] = [z_0]$ is the pushout of the two maps in the category of pointed topological spaces.



In Section 8.4 we saw that the pushout is a useful tool to construct and describe (pointed) topological spaces. Two questions now arise:

(1) Does the fundamental group preserve pushouts?

(2) What on earth is the pushout in the category *Group* of groups?

In the remainder of this section we will answer Question (2). We will return to Question (1) in the next chapter, when we will formulate and prove the Seifert–van Kampen Theorem 99.1.

The construction of the pushout in the category *Group* of groups will be formally quite similar to the above explicit description of the pushout in the category of pointed topological

spaces. We first take a coproduct, afterwards we take a quotient. One problem arises: Given a group G and a subgroup A the quotient G/A is in general not a group. Only if A is a normal subgroup, then G/A is in fact a group. This leads us to the following definition:

Definition 98.20. Let G be a group and let $A \subseteq G$ be a subset.

(1) We define

$$\text{subgroup of } G \text{ generated by } A \;:=\; \langle A \rangle \;:=\; \bigcap_{\substack{A \subseteq H \subseteq G \\ H \text{ subgroup}}} H,$$

i.e. $\langle A \rangle$ is the intersection of all subgroups H of G that contain A. This is again a subgroup[1003] of G, namely the smallest[1004] subgroup of G that contains A.

(2) We define

$$\text{subgroup of } G \text{ normally generated by } A \;:=\; \langle\langle A \rangle\rangle \;:=\; \bigcap_{\substack{A \subseteq H \lhd G \\ H \text{ normal subgroup}}} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all normal subgroups H of G that contain A. This is again a normal[1005] subgroup of G.

Remark. Let G be a group and let $S \subseteq G$ be a subset.

(1) It is not hard to see that the subgroup $\langle S \rangle$ equals the set of all elements of G that can be written as products of elements in S and their inverses.

(2) Similarly one can easily show that the subgroup $\langle\langle S \rangle\rangle$ equals the set of all elements of G that can be written as products of conjugates of elements in S and their inverses, i.e.

$$\langle\langle S \rangle\rangle \;=\; \left\{ g_1 s_1^{\epsilon_1} g_1^{-1} \dots g_k s_k^{\epsilon_k} g_k^{-1} \,|\, s_1, \dots, s_k \in S, g_1 \dots, g_k \in G \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\} \right\}. \square$$

Lemma 98.21. (Normal Closure–Factorization Lemma) Let G be a group and let $A \subseteq G$ be a subset. Let $\alpha \colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. Then there exists a unique group homomorphism $\beta \colon G/\langle\langle A \rangle\rangle \to \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow^{\alpha} & \\ G/\langle\langle A \rangle\rangle & \xrightarrow[\exists ! \, \beta]{} & \Gamma. \end{array}$$

Proof. Let G be a group, let $A \subseteq G$ be a subset and let $\alpha \colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subseteq \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup of G it follows immediately from the definition of $\langle\langle A \rangle\rangle$ that $\langle\langle A \rangle\rangle \subseteq \ker(\alpha)$. The lemma follows from the Quotient Group Lemma 2.4 (1b). ■

The following proposition can be viewed as an analog of the Topological Pushout Proposition 8.12 and the above discussion of the pushout in the category of pointed topological spaces.

[1004]Here we use the following observation: If $\{H_i\}_{i \in I}$ is a family of (normal) subgroups of G, then it is straightforward to see that the intersection $\bigcap_{i \in I} H_i$ is again a (normal) subgroup of G.

[1005]Here "smallest" means that if B is another subgroup that contains A, then $\langle\langle A \rangle\rangle \subseteq B$. This statement is obvious, since B appears in the definition of $\langle\langle A \rangle\rangle$ as one of the H's.

Proposition 98.22. (Group Pushout Proposition) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We consider the group

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle$$

and the maps

$$\begin{array}{rcl}\mu\colon A & \to & A *_G B \\ a & \mapsto & [(a)]\end{array} \qquad \text{and} \qquad \begin{array}{rcl}\nu\colon B & \to & A *_G B \\ b & \mapsto & [(b)]\end{array}$$

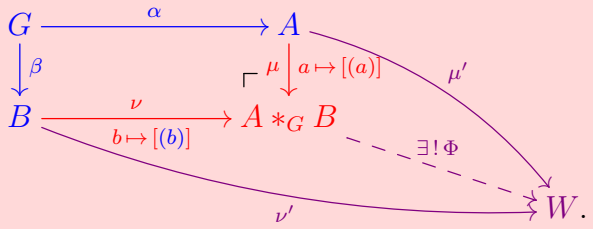
The following statements hold:

(1) (a) The maps $\mu\colon A \to A *_G B$ and $\nu\colon B \to A *_G B$ are homomorphisms.

(b) Suppose we are given two homomorphisms $\mu'\colon A \to W$ and $\nu'\colon B \to W$ to some group W such that $\mu' \circ \alpha = \nu' \circ \beta\colon G \to W$. By the Coproduct-of-Groups Lemma 98.4 the homomorphisms μ' and ν' induce a homomorphism $A * B \to W$. This homomorphism descends to a homomorphism

$$\Phi\colon A *_G B \;\to\; W.$$

This homomorphism is the unique homomorphism $\Phi\colon A *_G B \to W$ with $\Phi \circ \mu = \mu'$ and $\Phi \circ \nu = \nu'$. In particular we obtain the following commutative diagram:



(2) The group $A *_G B$ with the natural homomorphisms $\alpha\colon A \to A *_G B$ and $\beta\colon B \to A *_G B$ is the pushout, in the sense of Definition 6.23, in the category *Group* of groups.

Definition 98.23. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle$$

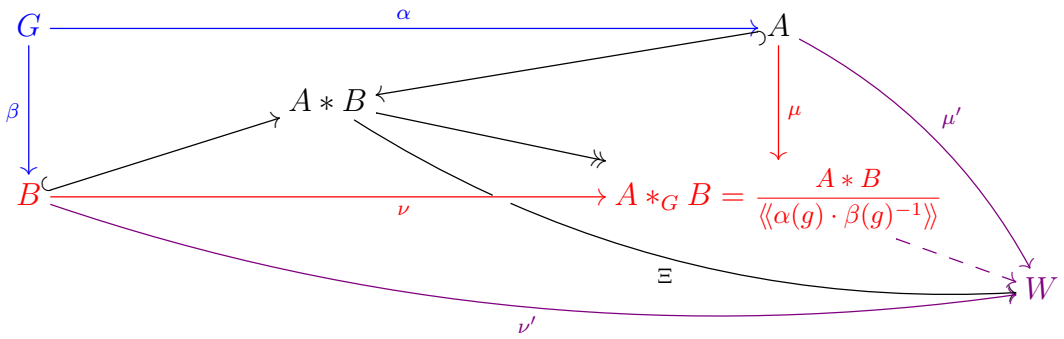
and we refer to $A *_G B$ as the amalgamated product of A and B with amalgam G.[1006] We refer to the obvious homomorphisms $A \to A *_G B$ and $B \to A *_G B$ as the natural homomorphisms.

Proof.

(1) (a) The statement follows from the Coproduct-of-Groups Lemma 98.4 (1a).

[1006]The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

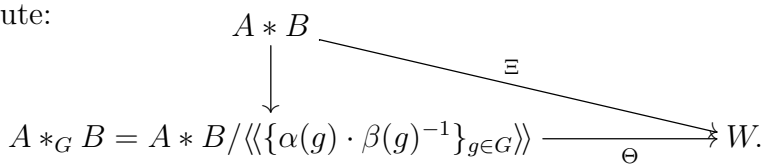
(b) Suppose we are given two homomorphisms $\mu' \colon A \to W$ and $\nu' \colon B \to W$ to some group W such that $\mu' \circ \alpha = \nu' \circ \beta \colon G \to W$. We consider the following diagram:



where $\Xi \colon A * B \to W$ is the unique homomorphism determined by ν' and μ'. Note that for any $g \in G$ we have

$$\Xi(\underbrace{\alpha(g) \cdot \beta(g)^{-1}}_{\in A*B}) = \Xi(\alpha(g)) \cdot \Xi(\beta(g))^{-1} \underset{\text{definition of } \Xi}{=} \mu'(\alpha(g)) \cdot \nu'(\beta(g))^{-1} \underset{\substack{\text{this follows from} \\ \mu' \circ \alpha = \nu' \circ \beta}}{=} e.$$

Put differently, all elements of the form $\alpha(g) \cdot \beta(g)^{-1}$ in $A * B$ lie in the kernel of Ξ. By the Normal Closure–Factorization Lemma 98.21 there exists a unique homomorphism $\Theta \colon A * B / \langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle \to W$ that makes the following diagram commute:

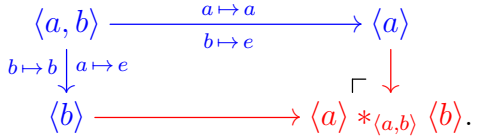


This proves the existence of Θ. The uniqueness follows from the observation that another choice of Θ would give another choice for Ξ, but this homomorphism is unique.

(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ■

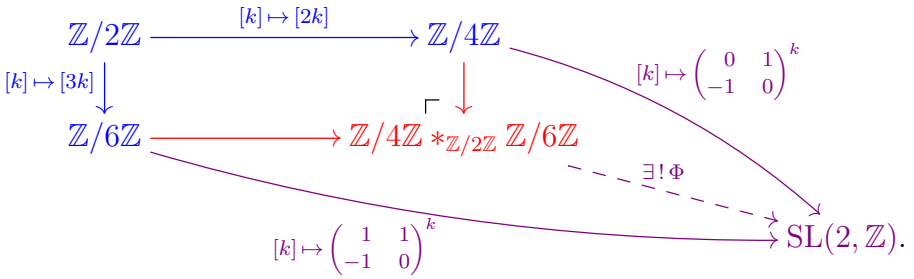
Example.

(1) We consider the following pushout diagram



It follows easily from the universal property of a pushout or from the explicit description of the amalgamated product that $\langle a \rangle *_{\langle a,b \rangle} \langle b \rangle = \{e\}$.

(2) We consider the following diagram:



Since the two maps from $\mathbb{Z}/2\mathbb{Z}$ to $\mathrm{SL}(2,\mathbb{Z})$ agree we obtain from the Group Pushout Proposition 98.22 (1b) a unique homomorphism $\Phi\colon \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \to \mathrm{SL}(2,\mathbb{Z})$ that makes the diagram commute. More surprisingly, this homomorphism is in fact an isomorphism. We refer to [**Rosb1994**, p. 218] and [**Ser1980**, p. 35] for a proof. □

The following lemma summarizes some formal properties of the amalgamated product.

Lemma 98.24. (Amalgamated Product Lemma)

(1) For two groups A and B and the trivial group $G = \{e\}$, the corresponding amalgamated product is the usual free product, i.e.

$$A *_{\{e\}} B \ = A * B.$$

(2) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ B & \longrightarrow & A *_G B. \end{array}$$

The following two statements hold:

(a) If the left vertical map β is an isomorphism, then the right vertical map is also an isomorphism and the inverse $A *_G B \to A$ is given by the identity on A and $\beta^{-1}\colon B \to G$.

(b) If the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.[1007]

The analogous statements of course hold with the roles of "A" and "B" swapped.

(3) Let A, B, C and G, H be groups and let $\alpha\colon G \to A$, $\beta\colon G \to B$, $\widetilde{\beta}\colon H \to B$ and $\gamma\colon H \to C$ be group homomorphisms. Then the universal properties of the amalgamated products define an isomorphism[1008] [1009]

$$(A *_G B) *_H C \ \xrightarrow{\cong}\ A *_G (B *_H C).$$

(4) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms and let $\varphi\colon H \to G$ be an epimorphism. The natural map $A *_G B \to A *_H B$ is an isomorphism.

[1007] In Exercise 98.26 we will see that the analogous statement for "monomorphism" does not hold.

[1008] In an attempt to lessen the burden on the reader we refrained from writing down the super-precise statement.

[1009] This statement is actually quite useful since it allows us to form iterated amalgamated products without having to worry about parentheses.

Proof. As we will now see, all statements follow from formal properties of pushouts:

(1) This statement follows from the fact that $\{e\}$ is the initial object of the category *Group* of groups and together with the Pushout–Coproduct Lemma 6.24.
(2) These two statements follow from the Pushout–Opposite Lemma 6.26 (a) and (b).
(3) This statement follows from the Pushout Composition Lemma 6.25.
(4) This statement follows from Exercise 6.18. ■

We conclude this discussion of amalgamated products with the following which is specific to the group theoretic setting.[1010]

Lemma 98.25. (Amalgamated Product–Quotient Lemma) If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to \{e\}$ is the trivial homomorphism, then the projection $A \to A/\langle\langle \alpha(G)\rangle\rangle$ induces a natural isomorphism

$$A *_G \{e\} \xrightarrow{\cong} A/\langle\langle \alpha(G)\rangle\rangle.$$

Proof. The statement follows immediately from the definition of the amalgamated product and the natural isomorphism $A * \{e\} \xrightarrow{\cong} A$. ■

98.5. **Appendix I: The Normal Form Theorem.** In this appendix we will state and outline proofs of a few more advanced properties of amalgamated products. It is quite reasonable to skip this appendix at a first reading and to return to it, when the need arises.

Recall that on page 1962 we saw that the amalgamated product of two non-trivial groups can be trivial. The following proposition gives a criterion for the amalgamated product to be non-trivial.

Proposition 98.26. (Amalgamated Product-Monomorphism Proposition) Suppose we are given two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon G \to B$. If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.

Convention 98.27. If we are in the setting of the Amalgamated Product-Monomorphism Proposition 98.26, then we will use the two natural monomorphisms $A \to A *_G B$ and $B \to A *_G B$ to identify A, B and G with the images in $A *_G B$ under these monomorphisms.

Remark. In Exercise 98.26 we will see that there exists an injective group homomorphism $\alpha\colon G \to A$ and a non-injective group homomorphism $\beta\colon G \to B$ such that the amalgamated product $A *_G B$ is trivial. This shows that there is no obvious way how one can weaken the hypothesis in the Amalgamated Product-Monomorphism Proposition 98.26 that *both* maps are monomorphisms. □

The proof of the Amalgamated Product-Monomorphism Proposition 98.26 relies on the following proposition:

Theorem 98.28. (Normal Form Theorem) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. We identify G with its image in A and B. Let $S \subseteq A$ be a set of coset representatives for $G\backslash A$ and let $T \subseteq B$ be a set of coset representatives for $G\backslash B$.

[1010] Note thought that the lemma is analogous to the discussion on page 318 where we discussed the pushout in the category of topological spaces where one of the two maps is given by the unique map $A \to \{*\}$ to the terminal object in the category *Top* of topological spaces.

Then any element h in $A *_G B$ we can be written *uniquely* as a product $h = g \cdot c_1 \cdots c_k$ where $g \in G$ and $c_1, \ldots, c_k \in S \sqcup T$.

Proof of the Normal Form Theorem 98.28. This theorem is proved in [**Ser1980**, p. 3], or alternatively [**vdW1948**], [**LyS1977**, Theorem IV.2.6] and [**CgRR2008**, Satz 5.2]. ■

Proof of the Amalgamated Product-Monomorphism Proposition 98.26. Suppose we are given two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon G \to B$. Let S be a set of coset representatives for $G \backslash A$ and let T be a set of coset representatives for $G \backslash B$. We have the following modest commutative diagram

$$\begin{array}{ccc} G \times S & \xrightarrow{(g,s)\mapsto g\cdot s} & A \\ & \searrow^{(g,s)\mapsto g\cdot s} & \downarrow \text{natural homomorphism} \\ & & A *_G B. \end{array}$$

The top map is a bijection by definition of S. Furthermore the diagonal map is an injection by the Normal Form Theorem 98.28. Thus the vertical map is a monomorphism. ■

98.6. **Appendix II: The Grushko-Neumann Theorem.** Since appendices are so much fun, here is one more. We will state the Grushko-Neumann Theorem and we will use it to prove the Grushko-Decomposition Theorem. Both deal with free products of groups. They are "nice to have", but they will not play an essential role in these notes.

To state the two theorem we will need the following definition, which is modelled on the definition of a finitely generated abelian group which we gave in Definition 97.19.

Definition 98.29.
(1) Given a group G we say that $S \subseteq G$ is a generating set if the natural homomorphism[1011] $\langle S \rangle \to G$ is an epimorphism.
(2) We say that a group G is finitely generated if it admits a finite generating set.

Example. It follows from the Abelianization-Epimorphism Lemma 98.17 together with the example on page 1906 that any finitely generated abelian group in the sense from Definition 97.19 is also a finitely generated group in the above sense. □

We continue with the following natural definition.

Definition 98.30. Let G be a finitely generated group. We denote by $d(G) \in \mathbb{N}_0$ the minimal number of elements in a generating set for G.

Remark. It is an unfortunate fact of life that in the literature the number $d(G)$ is usually referred to as the rank of the group G. But for abelian groups the invariants $\operatorname{rank}(G)$ (see Definition 97.15) and $d(G)$ can differ quite dramatically. For example for $G = (\mathbb{Z}/2\mathbb{Z})^n$ we have $\operatorname{rank}(G) = 0$ but $d(G) = n$. □

[1011] Here we mean by "natural homomorphism $\langle S \rangle \to G$" the unique homomorphism $\psi\colon \langle S \rangle \to G$ from the Free Group–Existence Lemma 98.9 that makes the following diagram commute:

$$\begin{array}{ccc} S & \hookrightarrow & \langle S \rangle \\ & \searrow^{\iota} & \downarrow \psi \\ & & G \end{array}$$

where $\iota\colon S \to G$ is the inclusion map.

The following theorem was proved by Igor Grushko [**Gru1940**] in 1940 and independently by Bernhard Neumann [**NeumB1943**] in 1940.

Theorem 98.31. (Grushko-Neumann Theorem) Given any two finitely generated groups A and B we have $$d(A * B) \;=\; d(A) + d(B).$$

Proof. The proof is evidently given in the above two references, but it can also be found in any self-respecting book on combinatorial group theory, see e.g. [**LyS1977**, Corollay IV.1.9] and [**MKS1976**, p. 192]. ■

We saw that some groups can be written as a direct product of two groups in different ways. For example it follows from the Chinese Remainder Theorem 97.4 that the groups $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/35\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}/21\mathbb{Z}$ are isomorphic. The next theorem says that the situation is much more rigid for free products. Before we can state the theorem we have to introduce the following definition:

Definition 98.32. A group G is indecomposable if it is not isomorphic to the free product of two non-trivial groups.

Lemma 98.33. (Indecomposability Lemma) Abelian groups and finite groups are indecomposable.

Proof. The lemma will be proved in Exercise 98.32. ■

Now we can state the Grushko Decomposition Theorem.

Theorem 98.34. (Grushko Decomposition Theorem) Let G be a finitely generated group.

(1) The group G is isomorphic to the free product $G = A_1 * \cdots * A_k$ of non-trivial indecomposable groups.[1012]

(2) The isomorphism types of $A_1, \ldots, A_k$ are unique up to permutation. More precisely, if $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are indecomposable finitely generated groups such that
$$A_1 * \cdots * A_k \;\cong\; B_1 * \cdots * B_l,$$
then $k = l$ and furthermore there exists a permutation $\sigma \in S_k$ such that $A_i \cong B_{\sigma(i)}$ for $i = 1, \ldots, k$.

Proof.

(1) This statement follows easily from induction on $d(G)$ and the Grushko-Neumann Theorem 98.31. Indeed, if $d(G) = 0$ then there is nothing to prove. Next suppose that the statement is known for all groups G with $d(G) < k$. Let G be a group with $d(G) = k$. If G is indecomposable, we are done. If G is not indecomposable, then we can write G as the free product of two non-trivial groups A and B. By the Grushko-Neumann Theorem 98.31 we have $d(G) = d(A) + d(B)$. Since A and B are non-trivial we have $d(A) \geq 1$ and $d(B) \geq 1$. Thus we have $d(A) < k$ and $d(B) < k$ and by induction we can find decompositions for A and B. These two decompositions put together give the desired decomposition for $G = A * B$.

(2) The uniqueness statement was proved by Aleksandr Kurosh in 1934, see [**Kuro1934**, Isomorphismsatz] and [**Kuro1960**, p. 27]. Alternative proofs are given in [**Stal1975**,

[1012] If G is trivial then $k = 0$ and the free product of zero groups is by definition the trivial group.

p. 168f] and [**Nel**, Theorem 4.5]. Also see [**CM1982**, Chapter II.4] for more context. ■

Example. It follows from the Indecomposability Lemma 98.33 and from the Grushko Decomposition Theorem 98.34 that the free product $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/35\mathbb{Z}$ is not isomorphic to the free product $\mathbb{Z}/5\mathbb{Z} * \mathbb{Z}/21\mathbb{Z}$. □

98.7. Exercises for Chapter 98.

Exercise 98.1. Let F be a free group. Show that the center of F is the trivial group.

Exercise 98.2. We consider the free group $G = \langle a, b, c, d\rangle$ on the four generators a, b, c, d. Show that $[a, b] \cdot [c, d]$ is not a commutator.

Exercise 98.3. We consider the homomorphism $\varphi\colon \langle x, y\rangle \to \langle x, y\rangle$ that is uniquely determined by $\varphi(x) = x \cdot y^{-1} \cdot x^{-1}$ and $\varphi(y) = x \cdot y$.

(a) Show that φ is an isomorphism.
(b) There exists an $n \in \mathbb{N}$ such that $\varphi^n = \mathrm{id}$. Determine the minimal such n.

Exercise 98.4. Let $k \in \mathbb{N}$ and let $(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$ be two n-tuples of distinct points in $\mathbb{R}^k$. For which n does there exist an action by a free group F on $\mathbb{R}^k$ such that for every $i \in \{1, \ldots, n\}$ there exists a $g \in F$ with $g \cdot P_i = Q_i$?

Exercise 98.5. Let F be a free group on a set with at least two elements. Show that there exists an epimorphism $\varphi\colon F \to G$ onto a finite non-abelian group G.

Exercise 98.6.

(a) Let $\{X_s\}_{s\in S}$ be a family of groups. Let $g \in \mathop{\ast}_{s\in S} X_s$. Show that if g is not conjugate to an element of any of subgroups X_s, then g has infinite order, i.e. for any $n \in \mathbb{N}$ we have $g^n \neq e$.
(b) Let A and B be two non-trivial groups. Show that the free product $A * B$ is infinite.
(c) Let S be a set. Show that every non-trivial element in the free group $\langle S\rangle$ has infinite order.

Exercise 98.7. Let $p, q, r \in \mathbb{N}_{\geq 2}$ be coprime. By the Chinese Remainder Theorem 97.4 we know that $\mathbb{Z}/pq\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/qr\mathbb{Z}$. Show that $\mathbb{Z}/pq\mathbb{Z} * \mathbb{Z}/r\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/qr\mathbb{Z}$.
Hint. Make use of Exercise 98.6.

Exercise 98.8. Let $\langle T\rangle$ be the free group on a generating set T. We denote by $\mathbb{Z}[\langle T\rangle]$ the group ring from Definition 97.30. Show that given any ring S and given any map $g\colon T \to S$ of sets there exists a unique ring homomorphism $\varphi\colon \mathbb{Z}[\langle T\rangle] \to S$ with $\psi(t) = g(t)$ for all $t \in T$.

Exercise 98.9. Let G be a group and let $g \in G$. The *centralizer* of $g \in G$ is defined as

$$C_G(g) \coloneqq \{h \in G \mid gh = hg\}.$$

(a) Let A and B be groups and let $g \in A * B$. We view A and B as subgroups of $A * B$ in the obvious way. Let $g \in A * B$. Show that

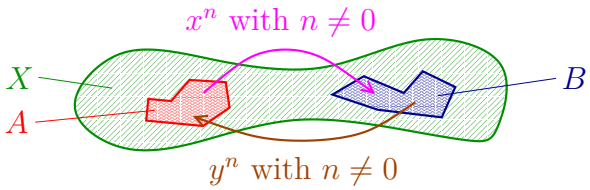
$$C_G(g) = \begin{cases} C_A(g), & \text{if } g \in A, \\ C_B(g), & \text{if } g \in B, \\ \langle h \rangle, \quad \text{where } h \in A * B \text{ of infinite order} & \text{otherwise.} \end{cases}$$

(b) Let S be a set. Show that every abelian subgroup of the free group $\langle S \rangle$ is isomorphic to $\mathbb{Z}$.

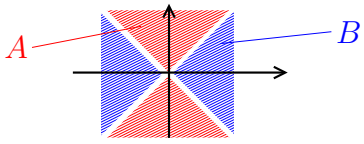
Exercise 98.10. Let G be a group and let $x, y \in G$. Suppose that there exists an action of G on some set X and two disjoint non-empty subsets A and B of X such that the following two conditions are satisfied:

(1) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $x^n \cdot A \subseteq B$, and
(2) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $y^n \cdot B \subseteq A$.

Show that x and y form the basis of a free subgroup of G, i.e. show that the natural homomorphism $\langle x, y \rangle \to G$ is a monomorphism.
Hint. Show that any non-trivial element $(g_1, \ldots, g_k) \in \langle x, y \rangle$ acts non-trivially on X.
Remark. This statement is known as the *Ping-Pong Lemma*.



Exercise 98.11. We consider the group $G = \mathrm{SL}(2, \mathbb{Z})$ and we consider the two matrices $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Use the Ping-Pong Lemma from Exercise 98.10 to show that x and y form a basis of a free subgroup of G.
Hint. Consider the set $X = \mathbb{R}^2$ together with the two subsets $A = \{(x, y) \in \mathbb{R}^2 \,|\, |y| > |x|\}$ and $B = \{(x, y) \in \mathbb{R}^2 \,|\, |x| > |y|\}$.



Exercise 98.12. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha \colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

(a) Show that for any $n \in \mathbb{N}$ the group $\mathrm{GL}(n, \mathbb{Z})$ is residually finite.
(b) Show that the free group on two generators is residually finite.
Remark. Use Exercise 98.11.
(c) Show that every finitely generated free group is residually finite.
(d) Show that every free group is residually finite.

Exercise 98.13. As in Exercise 98.12 we say that a group π is *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

(a) We want to show that every finitely generated free group $\langle x_1, \dots, x_k \rangle$ is residually finite. We proceed as follows:
- Let $g \in \langle x_1, \dots, x_k \rangle$ be a non-trivial element. We pick $d \in \mathbb{N}$ such that g can be written as a word of length d.
- Let S be the set of all elements of $\langle x_1, \dots, x_k \rangle$ that can be written as a word of length d.
- Find a homomorphism $\varphi\colon \langle x_1, \dots, x_k \rangle \to \text{Bij}(S)$ such that $\varphi(g) \neq \text{id}$.

(b) Show that every free group is residually finite.

Remark. This

Exercise 98.14. Let G be a group. We consider the subset

$$\text{Tor}(G) \coloneqq \{g \in G \mid \text{ there exists an } n \in \mathbb{N} \text{ with } g^n = e\}.$$

(a) Suppose that G is abelian. Show that $\text{Tor}(G)$ is a subgroup of G. We refer to it as the *torsion subgroup of* G.

(b) Give an example of a group G such that $\text{Tor}(G)$ is not a subgroup of G.

Exercise 98.15. We consider the free group $\langle x, y \rangle$. Let $a = (a_1, a_2, \dots, a_m)$ be a reduced word in $\langle x \rangle$ and $\langle y \rangle$. We denote by $\overline{a} = (a_m, \dots, a_1)$ the reverse word. Given any reduced word $z = (z_1, \dots, z_n)$ we write

$$|z|_a = \text{number of non-overlapping subsequences of } z_1, \dots, z_n \text{ that agree with } a_1, \dots, a_m.$$

Now we consider the map

$$\begin{aligned} \varphi_a\colon \langle x, y \rangle &\to \mathbb{Z}_{\geq 0} \\ z &\mapsto |z|_a - |z|_{\overline{a}}. \end{aligned}$$

(a) Show that in general φ_a is not a homomorphism.

(b) Show that there exists a $D > 0$ such that for every $g, h \in \langle x, y \rangle$ we have

$$|\varphi_a(g \cdot h) - \varphi_a(g) - \varphi_a(h)| \leq D.$$

Remark. Any map $\varphi\colon G \to \mathbb{R}$ from a group G to $\mathbb{R}$ that admits a D as above is called a *quasi-morphism*. The above quasi-morphism is called the *Brooks quasi-morphism* [**Brooks1981**].

Exercise 98.16. Let $\alpha\colon \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ and $\beta\colon \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the obvious epimorphisms. How many elements does the amalgamated product $\mathbb{Z}/3\mathbb{Z} *_{\mathbb{Z}/6\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ contain?

Exercise 98.17. Let S and T be two sets. Show that if there exists an epimorphism $\langle S \rangle \to \langle T \rangle$, then $\#S \geq \#T$.

Exercise 98.18. Let $A \in \text{GL}(n, \mathbb{Z})$ be a matrix. We consider the semidirect product $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ from Definition 93.11. What is the abelianization of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$?

Exercise 98.19. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. What is the abelianization of the amalgamated product $A *_G B$?

Exercise 98.20. Let $k \geq 5$. Determine the abelianization of the permutation group S_k.

Exercise 98.21. Let G be an abelian group, let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be epimorphisms. Show that $A *_G B$ is also abelian.

Exercise 98.22. Given a group G we denote by G_{ab} its abelianization. Give an example of a monomorphism $\varphi\colon A \to B$ such that the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is not a monomorphism.

Exercise 98.23. Let G be a group and let A be a subgroup. What is the relationship between the abelianizations G_{ab}, A_{ab} and $(G/A)_{\mathrm{ab}}$).

Exercise 98.24. Show that the following two functors

$$\begin{array}{rccc} F\colon \mathcal{AbGroup} & \to & \mathcal{Group} \\ G & \mapsto & G \end{array} \qquad \text{and} \qquad \begin{array}{rccc} G\colon \mathcal{Group} & \to & \mathcal{AbGroup} \\ G & \mapsto & G_{\mathrm{ab}} \end{array}$$

form an adjoint pair in the sense of Definition 6.33.

Exercise 98.25. Let $\varphi\colon G \to A$ and $\psi\colon G \to B$ group monomorphisms which we use to define the amalgamated product $A *_G B$. Suppose that the map $A \to A *_G B$ is an isomorphism. Show that ψ is an isomorphism.

Exercise 98.26.

(a) Let G be a group. Let $\alpha\colon G \to A$ be a monomorphism to a simple group A and let $\beta\colon G \to B$ be an epimorphism that is not an isomorphism. Show that the amalgamated product $A *_G B$ is trivial.
(b) Give an example for the setup in (a).

Exercise 98.27. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ B & \longrightarrow & A *_G B. \end{array}$$

Show, just using the explicit description of the amalgamated product, that if the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.

Exercise 98.28. Let A, B, G be groups and let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We suppose that we have a pushout diagram of the following form:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ B & \longrightarrow & \{e\}. \end{array}$$

Show that A is normally generated by $\alpha(\ker(\beta))$.
Remark. In many applications one has $G = B \times C$ for some third group C where the map $\beta\colon B \times C \to B$ is the projection. Then the hypothesis just says that A is normally generated by $\alpha(\{1\} \times C)$.

Exercise 98.29. We consider the category $\mathcal{Group}$ of groups. Let $\varphi\colon X \to Y$ be a group homomorphism. Show that the coequalizer of φ and the trivial homomorphism is naturally isomorphic to the cokernel $Y/\langle\langle \mathrm{im}(\varphi)\rangle\rangle$.
Remark. This exercise is the analog of the Cokernel-as-Coequalizer 6.29.

Exercise 98.30. Let A and G be groups and let $\varphi\colon A \to G$ be an epimorphism. Furthermore let G' be a copy of G. We consider the amalgamated product $G *_A G'$ given by using twice the epimorphism $\varphi\colon A \to G$ and $\varphi\colon A \to G' = G$. Show that the natural homomorphism $G \to G *_A G'$ is an isomorphism.

Exercise 98.31.

(a) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms and let $\varphi\colon H \to G$ be an epimorphism. Show that the natural map $A *_G B \to A *_H B$ is an isomorphism.

(b) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We assume that β admits a right-inverse φ, i.e. we assume that there exists a homomorphism $\varphi\colon G \to B$ such that $\beta \circ \varphi = \mathrm{id}_B$. Then there exists a natural isomorphism

$$A *_G B \xrightarrow{\cong} A/\langle\langle \alpha(\ker(\beta)) \rangle\rangle.$$

Remark. For $B = \{e\}$ this statement is precisely the statement of the Amalgamated Product–Quotient Lemma 98.25.

Exercise 98.32. Show that abelian groups and finite groups are indecomposable.

Exercise 98.33.

(a) Let H be a group and let $A \subseteq H$ be a subgroup. We denote by $\iota\colon A \to H$ the inclusion homomorphism. Let $\Theta\colon H \to \widetilde{H}$ be an isomorphism to some group $\widetilde{H}$. We use the homomorphisms $\iota\colon A \to H$ and $\Theta \circ \iota\colon A \to \widetilde{H}$ to form the amalgamated product $H *_A \widetilde{H}$. Let $\alpha\colon H \to H *_A \widetilde{H}$ and $\widetilde{\alpha}\colon H \to \widetilde{H} \to H *_A \widetilde{H}$ be the natural homomorphisms. Show that for $h \in H$ we have $\alpha(h) = \widetilde{\alpha}(h)$ if and only if $h \in A$.
Hint. Use the Normal Form Theorem 98.28.

(b) Let $\mathcal{C}$ be a category. We recall that a morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called an *epimorphism* if for every two morphisms $h_1, h_2 \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$. Show that every epimorphism $\varphi\colon G \to H$ in the category $\mathcal{G}roup$ of groups is surjective.
Remark. This gives a new proof of the Epimorphism Proposition 5.15 (1).

Exercise 98.34. Let $\varphi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\varphi(x) = (1, 0)$ and $\varphi(y) = (0, 1)$. Show that the kernel $\ker(\varphi)$ is normally generated by the commutator $[x, y] = xyx^{-1}y^{-1}$.

Exercise 98.35. Let $\varphi\colon G \to H$ be a group epimorphism. We write $K := \ker(\varphi)$. We consider its abelianization K_{ab}.

(a) Let $h \in G$. We pick $\widetilde{h} \in G$ with $\varphi(\widetilde{h}) = h$. Show that the map

$$\begin{array}{ccc} K_{\mathrm{ab}} & \to & K_{\mathrm{ab}} \\ [k] & \mapsto & [\widetilde{h} \cdot k \cdot \widetilde{h}^{-1}] \end{array}$$

is a group isomorphism and show that this map does not depend on the choice of $\widetilde{h}$.

(b) What goes wrong in (a) if we work with K instead of K_{ab}?

CHAPTER 99

The Seifert–van Kampen Theorem

We are still lacking the computation of fundamental groups of even fairly simple topological spaces, for example we do not know the fundamental groups of wedges of circles, of topological graphs and of surfaces. In this chapter we will take a new approach to calculating fundamental groups, namely we will formulate and prove the Seifert–van Kampen Theorem which allow us to calculate the fundamental group of a given topological space in terms of the fundamental groups of suitable subspaces and in terms of inclusion-induced maps. Using the Seifert–van Kampen Theorem we will be able to calculate many of the missing fundamental groups.

This chapter is organized as follows:

- In Section 99.1 we will state the Seifert–van Kampen Theorem 99.1.
- In Section 99.2 we will prove that the map in the Seifert–van Kampen-Theorem 99.1 is surjective.
- In Section 99.3 we will discuss an alternative definition of the free product of groups.
- In Section 99.4 we will use the purely algebraic discussion from Section 99.3 to show that the map in the Seifert–van Kampen-Theorem 99.1 is injective.
- In Section 99.5 we will discuss the fundamental groups of wedges of topological spaces.
- In Section 99.6 we will provide a proof of the Wedge-π_1-Proposition 99.10.
- In Section 99.7 we will determine the fundamental groups of topological graphs.
- In Section 99.8 we will calculate the fundamental group of mapping cones.
- In Section 99.9 we will discuss exhaustions of topological spaces.
- In Section 99.10 we will provide exercises for Chapter 99.

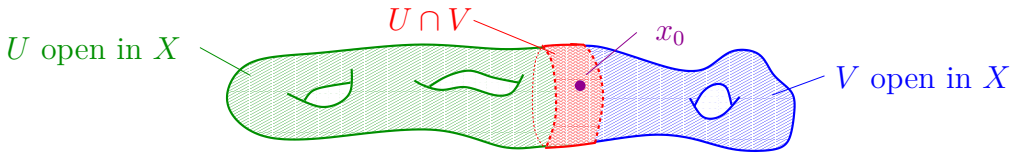
In the next Chapter 100 we will introduce presentations of groups.

99.1. **The formulation of the general Seifert–van Kampen Theorem.** The idea of the Seifert–van Kampen theorem is to reduce the calculation of the fundamental group of a topological space X to the determination of fundamental groups of "simpler subsets" of X.

More precisely: Let X be a topological space and let $U, V \subseteq X$ be two subsets such that $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{array}{ccc} U \cap V & \xrightarrow{i} & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow[l]{} & X = U \cup V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i_*} & \pi_1(U, x_0) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle k_*} \\ \pi_1(V, x_0) & \xrightarrow[l_*]{} & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{array}$$

In the Regluing Lemma 8.17 we saw that if U and V are *open* subsets of X, then the diagram to the left is a pushout. The following theorem now says that if U and V are open and if $U \cap V$ is path-connected, then the diagram to the right is also a pushout.

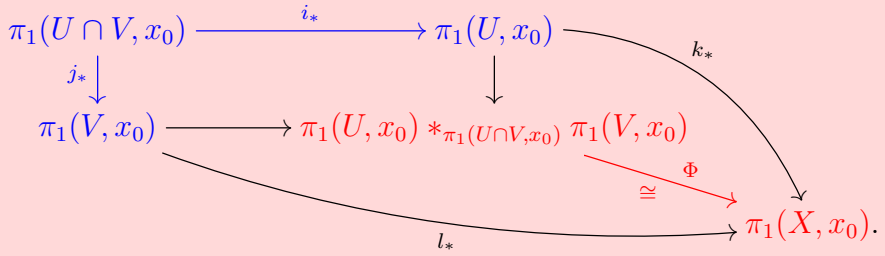


Theorem 99.1. (Seifert–van Kampen Theorem)[1013] Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two *open* subsets U and V such that $U \cap V$ is non-empty and *path-connected*. Let $x_0 \in U \cap V$. We denote by $i\colon U \cap V \to U$ and $j\colon U \cap V \to V$ the inclusions.

(1) (a) The inclusions $k\colon U \to X$ and $l\colon V \to X$ induce an isomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

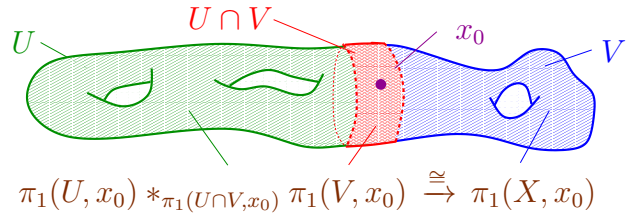
which has the property that the following diagram commutes:

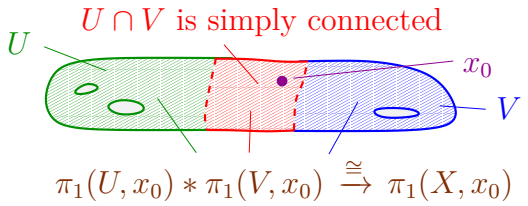


Here the "undecorated" maps are natural homomorphism to the amalgamated product.

(b) The group $\pi_1(X, x_0)$ is a pushout.

(2) If $U \cap V$ is simply connected, then the inclusions $U \to X$ and $V \to X$ induce an isomorphism $\Phi\colon \pi_1(U, x_0) * \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$



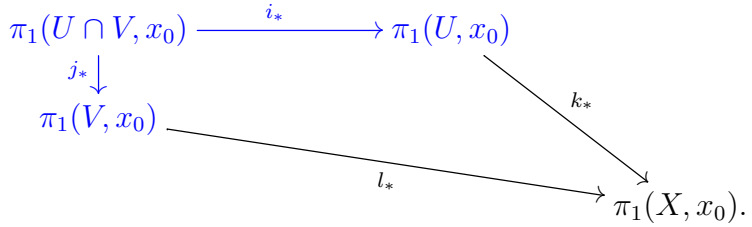


Remark. In the Manifold–Seifert-van Kampen Theorem 102.1 and the CW-Seifert–van Kampen Theorem 137.10 we will prove two very useful variations on the Seifert–van Kampen Theorem 99.1. □

Summary of proof. First note that Statement (2) is a special case of Statement (1a). Furthermore note that Statement (1b) follows from Statement (1a) and the Group Pushout Proposition 98.22 (2),

[1013] The theorem was proved independently by Herbert Seifert (1907-1996) and Egbert van Kampen (1908-1942).

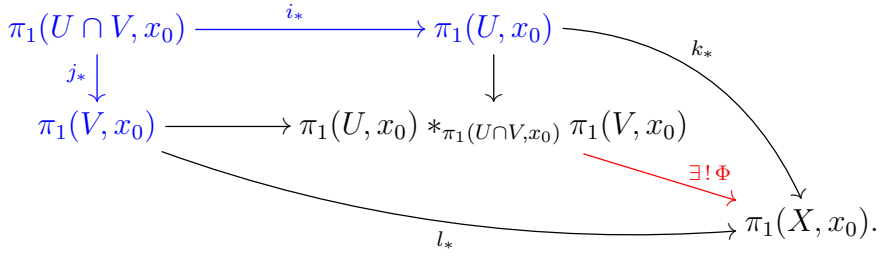
Thus it remains to prove Statement (1a). Note that the various inclusions of topological spaces give rise to the following commutative diagram:



By the Group Pushout Proposition 98.22 (1b) there exists a unique homomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

such that the following diagram commutes:



It remains to show that Φ is surjective and that Φ is injective. The plan is as follows:

(a) In Section 99.2 we will see that Φ is surjective.

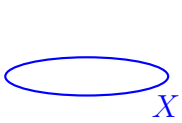
(b) In Section 99.4 we will show that Φ is injective by showing that there exists a map $\Psi\colon \pi_1(X, x_0) \to \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$ with $\Psi \circ \Phi = \mathrm{id}$. To construct the map Ψ it is convenient to work with an alternative description of free and amalgamated products. We will explain this alternative point of view in Section 99.3. ■

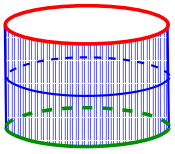
As we outlined above, we will provide the longish proof of the Seifert–van Kampen Theorem 99.1 in the next three sections. But let us first consider an example to get a feel for the Seifert–van Kampen Theorem 99.1. Next we recall Definition 20.21:

Definition 99.2. Let X be a non-empty topological space. We define the suspension of X to be the topological space

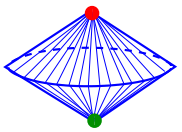
$$\Sigma(X) \;:=\; (X \times [-1, 1])/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point.





$X \times [-1, 1]$



suspension $\Sigma(X)$

As a first application of the Seifert–van Kampen Theorem 99.1 we can now prove the following lemma:

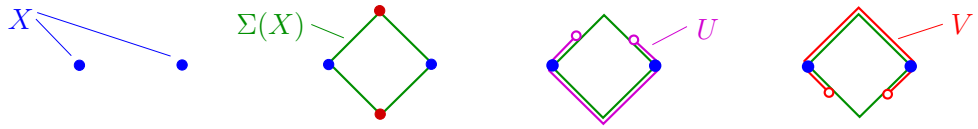
Lemma 99.3. (Suspension-π_1-Lemma) If X is a path-connected[1014] non-empty topological space, then its suspension $\Sigma(X) = (X \times [-1,1])/\sim$ is simply connected.

Example. On page 556 we saw that the suspension $\Sigma(S^n)$ of the n-dimensional sphere is homeomorphic to the $(n+1)$-dimensional sphere S^{n+1}.

(1) By the Ball-Sphere-Connected Lemma 4.35 we know that for $n \geq 1$ the sphere S^n is path-connected. Thus it follows from the Suspension-π_1-Lemma 99.3 that for $n \geq 1$ we have $\pi_1(S^{n+1}) = 0$. In other words, we have obtained a new proof of the Sphere-π_1-Proposition 88.24.

(2) The example $n = 0$ together with the Circle-π_1-Theorem 88.22 shows that in the Suspension-π_1-Lemma 99.3 we cannot drop the hypothesis that the topological space X is path-connected. □

Proof. Let X be a path-connected non-empty topological space. In the following we denote by $p\colon X \times [-1,1] \to \Sigma(X)$ the natural projection. We consider the two subsets $U = p(X \times [-1, \frac{1}{2}))$ and $V = p(X \times (-\frac{1}{2}, 1])$. It follows from the Definition 8.3 of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X \times \{0\} \subseteq \Sigma(X)$.



We calculate that

$$\pi_1(\Sigma(X), x) \underset{\uparrow}{\cong} \pi_1(U,x) *_{\pi_1(U\cap V, x)} \pi_1(V,x) \underset{\uparrow}{=} \begin{array}{c}\text{amalgamated product}\\ \text{of trivial groups}\end{array} = \text{trivial group.}$$

by the Seifert–van Kampen -Theorem 99.1 which we can apply since $U \cap V \cong X \times (-\frac{1}{2}, \frac{1}{2})$ is path-connected

as in the Cone Contractability Lemma 20.20 we see that U and V are contractible, hence their fundamental groups are trivial by the Homotopy Equivalence-π_1-Proposition 90.9 (2)

We have thus shown that $\Sigma(X)$ is indeed simply connected. ■

In the following three sections we provide the proof of the Seifert–van Kampen-Theorem 99.1. Afterwards we will consider many more applications of the Seifert–van Kampen-Theorem 99.1.

99.2. Proof of the Seifert–van Kampen-Theorem 99.1: Surjectivity.

Proof of the Seifert–van Kampen-Theorem 99.1: Surjectivity. Let X be a topological space and let $U, V \subseteq X$ be two open subsets with $X = U \cup V$ and such that $U \cap V$ is path-connected and non-empty. We choose a base point $x_0 \in U \cap V$. We need to show that the map

$$\Phi\colon \pi_1(U,x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V,x_0) \to \pi_1(X,x_0)$$

from page 1923 is surjective. We consider the following commutative diagram of homomorphisms:

[1014] In Exercise 106.5 we will consider the fundamental group of topological spaces that are not path-connected.

$$\begin{array}{ccc} \pi_1(U,x_0) * \pi_1(V,x_0) & & \\ \downarrow & \overset{\Xi}{\searrow} & \\ \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) & \xrightarrow[\Phi]{} & \pi_1(X,x_0) \end{array}$$

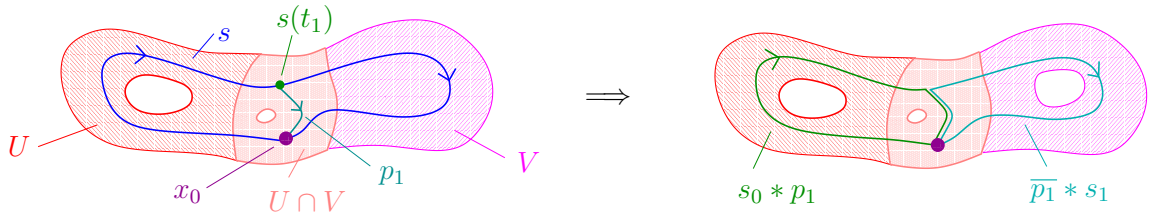
where the horizontal homomorphism Ξ and the left diagonal homomorphism are given by the Coproduct-of-Groups Lemma 98.4 (1b) and where the vertical homomorphism is given by the Quotient Group Lemma 2.4 (1b). It remains to prove that the horizontal homomorphism $\Xi\colon \pi_1(U,x_0) * \pi_1(V,x_0) \to \pi_1(X,x_0)$ is surjective. Clearly it suffices to prove the following claim:

Claim. Every loop $s\colon [0,1] \to X = U \cup V$ in x_0 is path-homotopic to a loop which is the concatenation of finitely many loops, each of which lies in (U,x_0) or in (V,x_0).

Proof. Let $s\colon [0,1] \to X = U \cup V$ be a loop in x_0. Since U and V are *open* we can use the Lebesgue Lemma 10.9 to find a subdivision

$$0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = 1$$

of the interval $[0,1]$ such that the image of each $s_i := s|_{[t_i,t_{i+1}]}$ lies in U or in V. The same argument as in the proof of the Sphere-π_1-Proposition 88.24 shows that we can assume that each $s(t_i)$ lies in $U \cap V$.[1015] The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U \cap V$ is path-connected we can find for each $i \in \{1,\ldots,k\}$ a path p_i in $U \cap V$ that connects $s(t_i) \in U \cap V$ to the base point $x_0 \in U \cap V$.



Now we have the following path-homotopies:

$$s \overset{\text{by the Reparametrization–Homotopy Lemma 88.4, since } s \text{ is a reparametrization of } s_0 * \cdots * s_k}{\simeq_{\mathrm{p}}} s_0 * \cdots * s_k \overset{\text{since by the Path-Homotopy Product Proposition 88.10 (3) we know that each } p_i * \overline{p_i} \text{ is null-homotopic}}{\simeq_{\mathrm{p}}} s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k$$

$$\simeq_{\mathrm{p}} \underbrace{(\, s_0 * p_1 \,)}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}} * \underbrace{(\, \overline{p_1} * s_1 * p_2 \,)}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}} * \underbrace{(\, \overline{p_2} * s_2 * p_3 \,)}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}} * (\overline{p_3} * \ldots * (\overline{p_k} * s_k).$$

Thus we have shown that s is path-homotopic to a concatenation of loops, each of which lies in (U,x_0) or in (V,x_0). ∎

[1015] Indeed, if $s(t_i)$ lies in $U \setminus V$, then $s([t_{i-1},t_i])$ and $s([t_i,t_{i+1}])$ cannot lie in V, hence they both lie in U. This implies that $s_{([t_{i-1},t_{i+1}])}$ lies in U, and therefore we can remove t_i from the subdivision. We proceed similarly if $s(t_i)$ lie in $V \setminus U$.

99.3. **Interlude: An alternative definition of the free product of groups.** In our proof of the injectivity of the map in the Seifert–van Kampen-Theorem 99.1 it is convenient to work with a different model for the free and the amalgamated product of two groups. We will present this alternative approach in this purely algebraic section. We return to the proof of the Seifert–van Kampen-Theorem 99.1 in the next section.

Definition 99.4. Let A and B be two groups.

(1) We set[1016]
$$S(A,B) \;:=\; \{\text{all finite sequences } (g_1,\dots,g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}.$$
We equip $S(A,B)$ with the product structure given by
$$(x_1,\dots,x_k)\cdot(y_1,\dots,y_l) \;:=\; (x_1,\dots,x_k,y_1,\dots,y_l).$$
This turns $S(A,B)$ into a monoid in the sense of Definition 2.1.[1017]

(2) Let $\alpha\colon G\to A$ and $\beta\colon G\to B$ be two group homomorphisms. We denote by $\sim_G$ the equivalence relation on $S(A,B)$ that is generated by the relations

(a) $$(x_1,\dots,x_l,e,x_{l+2},\dots,x_k) \;\sim_G\; (x_1,\dots,x_l,x_{l+2},\dots,x_k)$$
(b) $$(x_1,\dots,x_l,x_{l+1},\dots,x_k) \;\sim_G\; (x_1,\dots,x_l\cdot x_{l+1},\dots,x_k).$$
$\uparrow$ if both x_l and x_{l+1} lie in A or both lie in B
(c) $$(x_1,\dots,x_l,\alpha(g),x_{l+2},\dots,x_k) \;\sim_G\; (x_1,\dots,x_l,\beta(g),x_{l+2},\dots,x_k).$$
$\uparrow$ for any $g\in G$

If $G=\{e\}$, then we just write $\sim$ instead of $\sim_G$.

(3) The monoid structure on $S(A,B)$ descends to a well-defined map
$$\begin{array}{rcl} S(A,B)/\!\sim_G \;\times\; S(A,B)/\!\sim_G & \to & S(A,B)/\!\sim_G \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)]. \end{array}$$
One can easily verify that this defines a group structure on $S(A,B)/\!\sim_G$.[1018]

Proposition 99.5. (Alternative–Amalgamated Product Proposition) Let A and B be two groups and let $\alpha\colon G\to A$ and $\beta\colon G\to B$ be two group homomorphisms. The map[1019]
$$\begin{array}{rcl} \Phi\colon S(A,B)/\!\sim_G & \to & A *_G B = (A*B)/\langle\!\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\!\rangle \\ [\underbrace{(x_1,\dots,x_k)}_{\in S(A,B)}] & \mapsto & [\underbrace{x_1\cdot\ldots\cdot x_k}_{\in A*B}] \end{array}$$
is well-defined and it is a group isomorphism.

In the proof of the Alternative–Amalgamated Product Proposition 99.5 we will need the following little lemma.

[1016] The definition of $S(A,B)$ is similar to the definition of $A*B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.

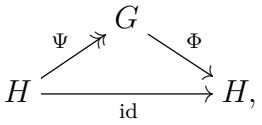
[1017] Loosely speaking a monoid is like a group, except that we do not demand the existence of inverses. In our case the neutral element is again represented by the empty sequence $(\,)$.

[1018] Since we have a monoid structure on $S(A,B)$ we see that this multiplication defines a monoid structure on $S(A,B)/\!\sim_G$. Furthermore it follows from (2a) and (2b) that the inverse of $[(g_1,\dots,g_k)]$ is given by $[(g_k^{-1},\dots,g_1^{-1})]$. Thus we have in fact a group structure.

[1019] Recall that we use the natural monomorphisms from the Coproduct-of-Groups Lemma 98.4 (1a) to view A and B as subgroups of $A*B$. If we want to be more precise, then we could say that Φ is given by $[(x_1,\dots,x_k)]\mapsto[(x_1)\cdot\ldots\cdot(x_k)]$.

Auxiliary Lemma 99.6. Let $\Phi\colon G \to H$ and $\Psi\colon H \to G$ be two group homomorphisms such that $\Phi \circ \Psi = \mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.

The lemma can be summarized as follows: If we are given a commutative diagram of the form



then Φ is an isomorphism.

Proof of Lemma 99.6. It follows from $\Phi \circ \Psi = \mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g \in \ker(\Phi)$. Since Ψ is an epimorphism there exists an $h \in H$ with $\Psi(h) = g$. Since $\Phi \circ \Psi = \mathrm{id}_H$ we see that $h = \Phi(\Psi(h)) = \Phi(g) = e$. Thus $h = e$ and thus $g = \Psi(h) = \Psi(e) = e$. ∎

Proof of Proposition 99.5. We start out with the following claim:

Claim 1. The map
$$\begin{array}{rcl}\Phi\colon S(A,B)/\sim_G & \to & A *_G B = (A * B)/\langle\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle \\ [(x_1,\dots,x_k)] & \mapsto & [x_1\cdot\ldots\cdot x_k]\end{array}$$
is well-defined, it is a homomorphism and it is surjective.

Proof. We consider the map
$$\begin{array}{rcl}\widetilde{\Phi}\colon S(A,B) & \to & A *_G B \\ (x_1,\dots,x_k) & \mapsto & [x_1\cdot\ldots\cdot x_k].\end{array}$$
As above we view $S(A,B)$ as a monoid. With this structure it is clear that $\widetilde{\Phi}$ is a monoid homomorphism in the sense of Definition 2.2. Since $A * B$ is a subset of $S(A,B)$ we see that $\widetilde{\Phi}$ is surjective.

It remains to show that $\widetilde{\Phi}$ descends to a well-defined map on $S(A,B)/\sim_G$. The first two relations in the definition of $\sim_G$ do not pose a problem. But let us consider relation (c). Let $w = (x_1,\dots,x_l)$, $\widetilde{w} = (x_{l+2},\dots,x_k)$ in $S(A,B)$ and let $g \in G$. Note that we have the following equality in $A *_G B$:
$$\widetilde{\Phi}(\underbrace{(w,\alpha(g),\widetilde{w})}_{\in S(A,B)}) \overset{\substack{\text{since } \widetilde{\Phi} \text{ is a monoid}\\ \text{homomorphism}}}{=} \widetilde{\Phi}(w)\cdot\alpha(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(w)\cdot\overbrace{\alpha(g)\cdot\beta(g)^{-1}}^{=e\in A*_G B}\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w})$$
$$= \widetilde{\Phi}(w)\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(\underbrace{(w,\beta(g),\widetilde{w})}_{\in S(A,B)}).$$
We have thus verified that $\widetilde{\Phi}$ factors through the relation given by (d). ⊞

Claim 2. The map
$$\begin{array}{rcl}\Psi\colon A *_G B & \to & S(A,B)/\sim_G \\ \underbrace{[(x_1,\dots,x_k)]}_{\in A*B} & \mapsto & \underbrace{[(x_1,\dots,x_k)]}_{\in S(A,b)}\end{array}$$
is well-defined and it is a homomorphism.

Proof. First we consider the map
$$\begin{array}{rcl}\widetilde{\Psi}\colon A * B & \to & S(A,B)/\sim_G \\ (x_1,\dots,x_k) & \mapsto & [(x_1,\dots,x_k)].\end{array}$$

One can easily verify that this is in fact a group homomorphism. It follows immediately from the definitions that for any $g \in G$ we have

$$\widetilde{\Psi}(\alpha(g)\cdot\beta(g)^{-1}) = [(\alpha(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g)\cdot\beta(g)^{-1})] \underset{\uparrow}{=} [()] = e.$$

follows from part (c), (b) and (a) of the definition of $\sim_G$

Thus it follows from the Normal Closure–Factorization Lemma 98.21 that $\widetilde{\Psi}$ descends to a well-defined homomorphism $\Psi\colon A *_G B \to S(A,B)/\sim_G$. ⊞

It is clear that $\Phi \circ \Psi = \mathrm{id}_{A*_G B}$. It is not so obvious that $\Psi \circ \Phi = \mathrm{id}_{S(A,B)/\sim_G}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 99.6, to show that $\Psi\colon A *_G B \to S(A,B)/\sim_G$ is surjective. This in turn follows from the following claim:

Claim 3. Any element in $S(A,B)$ is equivalent, via $\sim_G$, to an element in $A * B$.

Proof. We prove the claim by induction on the length of a sequence in $S(A,B)$. Clearly the claim holds for the unique sequence in $S(A,B)$ of length 0. Suppose we already know the claim for all sequences of length $\leq k-1$. Let $(x_1,\dots,x_k) \in S(A,B)$ be a sequence of length k. If $(x_1,\dots,x_k)$ is a reduced word then by definition it lies in $A * B$ and we are done. Now we suppose that $(x_1,\dots,x_k)$ is not reduced. Then, by Definition 98.1, one of the following occurs:

(1) there exists an $i \in \{1,\dots,k\}$ with $x_i = e$, or
(2) there exists an $i \in \{1,\dots,k-1\}$ with $x_i, x_{i+1} \in A$, or
(3) there exists an $i \in \{1,\dots,k-1\}$ with $x_i, x_{i+1} \in B$.

In the first case the sequence is equivalent to $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)$. In the second and third case the sequence is equivalent to $(x_1,\dots,x_{i-1}\cdot x_{i+1},\dots,x_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. ■

Remark. Let A and B be two groups. We have now seen that there are two ways to think about the free product $A * B$. We can either take the original approach by viewing elements of $A * B$ as reduced words. Or we can use the Alternative–Amalgamated Product Proposition 99.5 to identify $A * B$ with the group $S(A,B)/\sim$. Both approaches to defining the free product of A and B have their advantages:

(1) In the group $S(A,B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B.
(2) On the other hand in the group $A * B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a,b) and (b,a) of $A * B$ are by definition different. But from the definition of $S(A,B)/\sim$ it is not immediately clear that $[(a,b)]$ and $[(b,a)]$ are indeed different. □

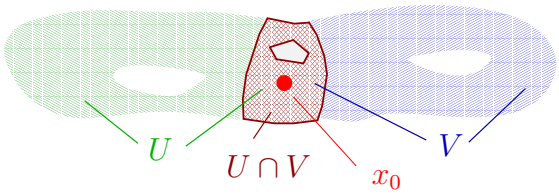
99.4. **Proof of the Seifert–van Kampen-Theorem 99.1: Injectivity.** We return to the proof of the Seifert–van Kampen-Theorem 99.1. We recall the setup. Let X be a topological space and let $U, V \subseteq X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$ such that $U \cap V$ is non-empty. We choose a base point $x_0 \in U \cap V$. We write $A \coloneqq \pi_1(U,x_0)$, $B \coloneqq \pi_1(V,x_0)$ and $G \coloneqq \pi_1(U \cap V, x_0)$. On page 1923 we defined a

homomorphism

$$\Phi\colon A *_G B = \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0).$$

In Section 99.2 we showed that this homomorphism is surjective. It remains to show that this homomorphism is injective. This is what we will do now:

Proof of the Seifert–van Kampen-Theorem 99.1: Injectivity. We want to show that the map $\Phi\colon A *_G B = \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$ which is induced by the inclusions $i\colon U \to X$ and $j\colon V \to X$ is injective.



To prove injectivity of Φ it is enough to show that there exists a map $\Psi\colon \pi_1(X, x_0) \to A *_G B$ such that $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$. We will prove the existence of such a map Ψ in three steps:

(1) First we construct a map $\overline{\Psi}\colon \{\text{loops in } (X, x_0)\} \to A *_G B$.
(2) Then we show that $\overline{\Psi}$ descends to a well-defined map

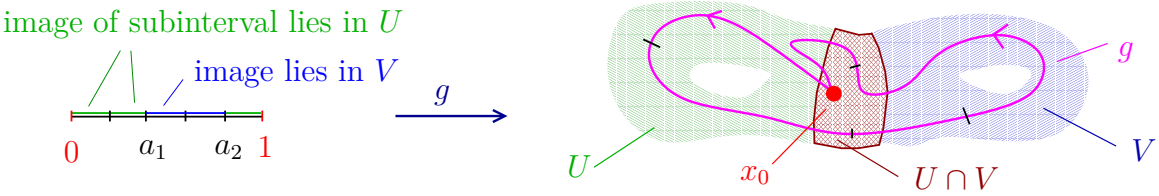
$$\begin{aligned} \Psi\colon \pi_1(X, x_0) &\to A *_G B \\ [g] &\mapsto \overline{\Psi}(g). \end{aligned}$$

(3) Finally we show that $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$.

For the remainder of the proof we will identify $A *_G B$ with $S(A, B)/\sim_G$ using the isomorphism given by the Alternative–Amalgamated Product Proposition 99.5.

Step (1). Let $g\colon [0, 1] \to X$ be a loop in (X, x_0). We apply the Lebesgue Lemma 10.9 to the map $g\colon [0, 1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \dots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

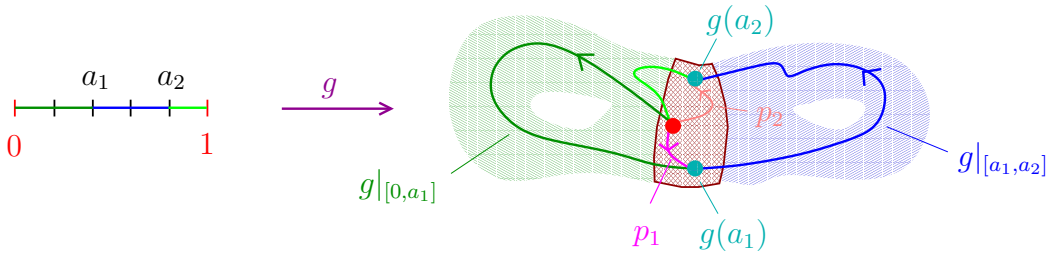
(a) We say that a subinterval of $[0, 1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say that an interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.
(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$.
(c) Given a coloring α we denote by $0 < a_1 < \cdots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.



(d) A *basing* for a coloring α is a collection of paths $p_1, \dots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.[1020]

Given a coloring α and a basing $p_1, \dots, p_{k-1}$ we set[1021] [1022]

$$\overline{\Psi}(g) \;\coloneqq\; \big[\big[\underbrace{g|_{[0,a_1]} * \overline{p}_1}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \big[\underbrace{p_1 * g|_{[a_1,a_2]} * \overline{p}_2}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \dots, \big[\underbrace{p_{k-1} * g|_{[a_{k-1},1]}}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big]\big] \;\in\; S(A,B)/\sim_G \,= A *_G B.$$



Our next goal is to prove the following claim:

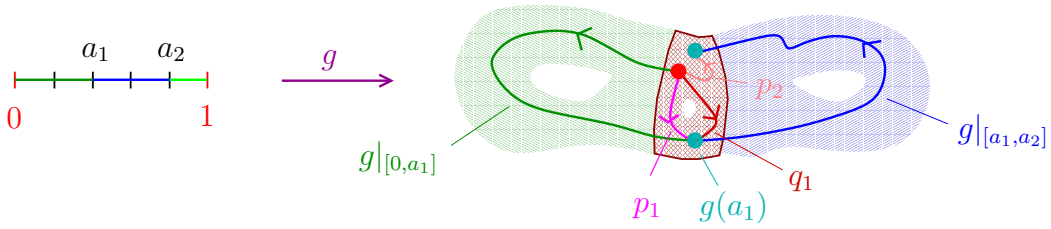
Claim 1. The definition of $\overline{\Psi}(g) \in S(A,B)/\sim_G$ does not depend on any of the choices made.

Proof. We need to prove the following three statements:

(a) Given N and a coloring the definition does not depend on the choice of the basings.
(b) Given N the definition does not depend on the choice of the coloring.
(c) The definition does not depend on the choice of N.

We now prove these three statements:

(a) We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. Thus let us pick a different path q_1 from x_0 to $g(a_1)$.



[1020] Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.
[1021] Each path $p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}$ is a loop in (U,x_0) or (V,x_0), so each path represents an element in $\pi_1(U,x_0)$ or $\pi_1(V,x_0)$. Thus we obtain a sequence of elements in $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$. The equivalence class of this sequence defines an element in $S(A,B)/\sim_G = A *_G B$.
[1022] This definition is almost the same as in the proof on page 1925.

In this case we have the following equality in $S(A,B)/\sim_G$:

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} \;=\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1 * \overbrace{(p_1 * \overline{q}_1) * (\underbrace{q_1 * \overline{p}_1}_{\text{loop in } U\cap V})}^{\text{null-homotopic loop}}],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1 * (p_1 * \overline{q}_1)],\ [(q_1 * \overline{p}_1) * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\qquad\uparrow\\
&\qquad\text{note that } q_1 * \overline{p}_1 \text{ is a loop in } U\cap V, \text{ thus it defines elements in } G=\pi_1(U\cap V, x_0),\\
&\qquad\text{thus we get this equality in } S(A,B)/\sim_G\\
&\quad= \big[[g|_{[0,a_1]} * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * \overline{q}_1)],\ [q_1 * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{q}_1],\ [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \text{definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}.
\end{aligned}$$

Basically the same argument (with slightly more messy notation) shows that the choices of the other paths p_i are also irrelevant.

(b) Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single $i \in \{0, \frac{1}{N}, \dots, 1-\frac{1}{N}\}$ where the values for $\alpha(i)$ and $\beta(i)$ differ.

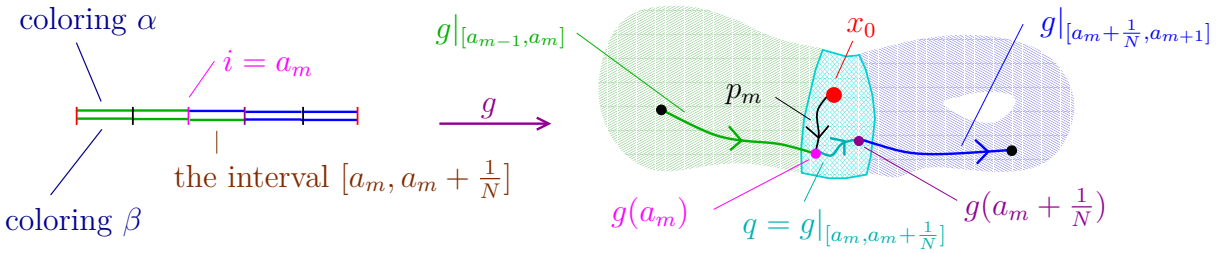
We consider the case, sketched in the figure below, that $\alpha(i)$ = blue, $\beta(i)$ = green and such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.[1023]

Let $0 = a_0 < a_1 < \dots < a_k = 1$ be the decomposition corresponding to the coloring α. We pick a basing $p_1, \dots, p_{k-1}$ for α. There exists a unique $m \in \{1, \dots, k-1\}$ with $a_m = i$. The corresponding decomposition for the coloring β is then given by $0 = a_0 < a_1 < \dots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m+\frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \;=\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m]} * \overline{p}_m], [p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}], \dots\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m+\frac{1}{N},a_{m+1}]} * \overline{p}_{m+1}], \dots\big]\\
&\qquad\uparrow\\
&\qquad\text{since } g_{[a_{m-1},a_m]} \simeq_{\mathrm{p}} g_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} \text{ and}\\
&\qquad\text{since } g_{[a_m,a_m+\frac{1}{N}]} \simeq_{\mathrm{p}} q * g_{[a_m+\frac{1}{N},a_{m+1}]}\\
&\quad= \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \dots, p_{m-1}, p_m * q, p_{m+1}, \dots, p_{k-1}.
\end{aligned}$$

(c) First suppose that instead of dividing $[0,1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the

[1023] In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it to the reader to perform that argument.



definitions that we end up with the same element of $A *_G B$. Now if we had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A *_G B$. This concludes the proof of (c). ⊞

Step (2). We need to prove the following claim:

Claim 2. If $g, g' \colon [0,1] \to X$ are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Proof. Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{array}{rccc} H \colon & [0,1] \times [0,1] & \to & X \\ & (t,s) & \mapsto & H(t,s) \end{array}$$

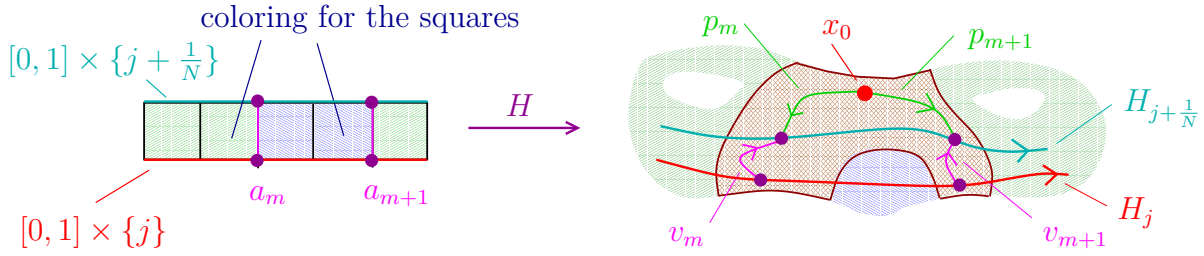
from the loop g to the loop g'. This time we apply the Lebesgue Lemma 10.9 to the map $H \colon [0,1] \times [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an $N \in \mathbb{N}$ such that for every choice of $i, j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H lies in U or in V. Given $a \in [0,1]$ we introduce the two paths

$$\underbrace{\begin{array}{rccc} H_a \colon & [0,1] & \to & X \\ & t & \mapsto & H(t,a) \end{array}}_{\text{“horizontal path”}} \quad \text{and} \quad \underbrace{\begin{array}{rccc} V_a \colon & [0,1] & \to & X \\ & t & \mapsto & H(a,t). \end{array}}_{\text{“vertical path”}}$$

Note that each H_a is a loop in x_0 and that $H_0 = g$ and $H_1 = g'$. Therefore it suffices to show that for every $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(H_j) = \overline{\Psi}(H_{j+\frac{1}{N}})$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

As above we say that a subset of $[0,1] \times [0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say that the subset can be *colored blue* if the image lies in V. We pick a map $\alpha \colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i + \frac{1}{N}] \times [\frac{j}{N}, j + \frac{1}{N}]$ can be colored. Note that α also defines a coloring for the loops H_j and $H_{j+\frac{1}{N}}$.

We define $0 = a_0 < a_1 < \cdots < \dots a_k = 1$ as above for the coloring α. We pick a basing p_i, $i = 1, \dots, k-1$ for the loop H_j. For $i = 0, \dots, k$ we write $v_i = V_{a_i}|_{[j, j+\frac{1}{N}]}$. Put differently, v_j connects the point $H_j(a_i)$ to the corresponding point $H_{j+\frac{1}{N}}(a_i)$.



Now we have the following equality in $A *_G B$:

$$\begin{aligned}
\overline{\Psi}(H_j) &= \overline{\Psi}(H_j) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\
&= \big[\big[\dots\big] * [p_m * H_j|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}] * \big[\dots\big]\big] \\
&= \big[\big[\dots\big] * [p_m * v_m * \overline{v}_m * H_j|_{[a_m,a_{m+1}]} * v_{m+1} * \overline{v}_{m+1} * \overline{p}_{m+1}] * \big[\dots\big]\big] \\
&\quad \uparrow \\
&\quad \text{since each } v_m * \overline{v}_m \text{ is null-homotopic} \\
&= \big[\big[\dots\big] * [p_m * v_m * H_{j+\frac{1}{N}}|_{[a_m,a_{m+1}]} * \overline{v}_{m+1} * \overline{p}_{m+1}] * \big[\dots\big]\big] \\
&\quad \uparrow \\
&\quad \text{since } H \text{ gives rise to path-homotopies from } \overline{v}_m * H_j|_{[a_m,a_{m+1}]} * v_{m+1} \text{ to} \\
&\quad H_{j+\frac{1}{N}}|_{[a_m,a_{m+1}]} \text{ for } m = 0, \dots, k-1 \\
&= \overline{\Psi}(H_{j+\frac{1}{N}}) \text{ defined using } \alpha \text{ and the basing } p_1 * v_1, \dots, p_{k-1} * v_{k-1}.
\end{aligned}$$

This concludes the proof of the claim and thus of part (2) of our three step program. ⊞
Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{aligned}
\Psi\colon \pi_1(X, x_0) &\to A *_G B \\
[g] &\mapsto \big[\overline{\Psi}(g)\big].
\end{aligned}$$

Step (3). It remains to prove the following claim:

Claim 3. We have $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$.

Proof. So let $[y_1] \cdot \ldots \cdot [y_k] \in A *_G B$ where $y_1, \dots, y_k$ are loops in (U, x_0) or (V, x_0). Then

$$\begin{aligned}
(\Psi \circ \Phi)([y_1] \cdot \ldots \cdot [y_k]) &\overset{\text{definition of the map } \Phi\colon A *_G B = \pi_1(U,x_0) * \pi_1(V,x_0) \to \pi_1(X,x_0)}{\downarrow} \\
&= \Psi([y_1 * \cdots * y_k]) = \big[\overline{\Psi}(y_1 * \cdots * y_k)\big] = [y_1 * \cdots * y_k] \\
&= [y_1] \cdot \ldots \cdot [y_k]. \qquad \uparrow \\
&\qquad \text{by definition of } \overline{\Psi}, \text{ where we use } N = k \text{ and where} \\
&\qquad \text{we can use the constant paths as basings since} \\
&\qquad y_1, \dots, y_k \text{ are already loops in } (U, x_0) \text{ or } (V, x_0)
\end{aligned}$$

This concludes the proof of the claim and thus also the proof of the Seifert–van Kampen-Theorem 99.1 ■

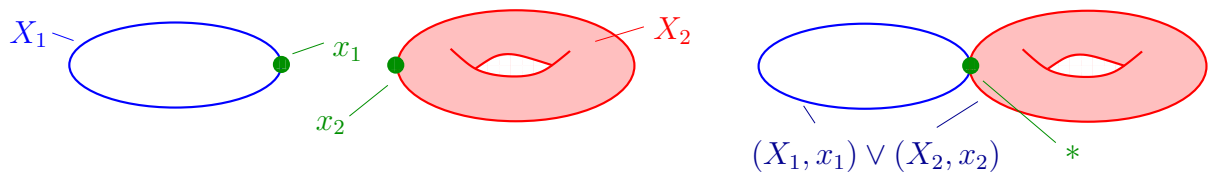
99.5. **Fundamental groups of wedges.** We start out by recalling Definition 96.1:

Definition 99.7. Let $\{(X_i, x_i)\}_{i \in I}$ be a y family of pointed topological spaces.

(1) We refer to $$\bigvee_{i \in I} (X_i, x_i) := \Big(\bigsqcup_{i \in I} X_i\Big) \Big/ \bigsqcup_{i \in I} \{x_i\}$$ as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point and we denote it by $*$.

(2) Let $j \in I$. We refer to $$\begin{aligned} \iota_j\colon X_j &\to \bigvee_{i \in I} (X_i, x_i) \\ x &\mapsto [x] \end{aligned}$$ as the natural inclusion. Note that in the Wedge Proposition 96.2 we showed that each natural inclusion is an embedding, thus we can view each X_j as a subspace of the wedge.

Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. The natural inclusions ι_{i*} induce a homomorphism

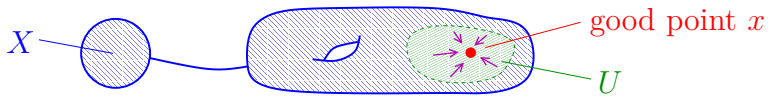


$$\underbrace{\ast_{i\in I}\, \pi_1(X_i, x_i)}_{\text{coproduct of the fundamental groups}} \quad \to \quad \pi_1\Big(\underbrace{\bigvee_{i\in I} (X_i, x_i)}_{\text{coproduct of the } (X_i, x_i)} , * \Big).$$

It seems reasonable to expect that this homomorphism is an isomorphism. For simplicity let us restrict ourself for a second to the case that we are dealing with just two topological spaces X_1 and X_2. It is now very tempting to apply the Seifert–van Kampen-Theorem 99.1 to the decomposition $X_1 \vee X_2 = X_1 \cup X_2$. Unfortunately this approach hits a snag: in the Seifert–van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. But in most cases X_1 and X_2 will certainly not be open subspaces of $X_1 \vee X_2$.

The idea now is to replace each X_i by an *open* subset U_i which is "close" to X_i. This idea leads us to the following somewhat technical definition.

Definition 99.8. We say that a point x in a topological space X is good, if the following two conditions are satisfied:
(1) $\{x\}$ is a closed subset of X[1024].
(2) There exists an open neighborhood U of x such that x is a deformation retract of U.



Remark. What we call a "good point" is sometimes called a "non-degenerate point", see e.g. [**Lee2000**, p. 212]. On the other hand, the term "non-degenerate point" is defined in a subtly different way, see e.g. [**Spa1995**, p. 380] or [**Bre1993**, p. 435]. Also note that a point $x \in X$ is good if the pair $(X, \{x\})$ is good in the sense of [**Hat2002**, p. 114]. □

The following lemma gives us two important types of examples of good points, which will take care of most examples that we will have to deal with.

Lemma 99.9. (Good Point-Lemma)
(1) Every point in a topological manifold (e.g. on a sphere) is good.
(2) Let (X_i, x_i) be a family of pointed topological spaces. If each $x_i \in X_i$ is good, then the wedge point $*$ is a good point in $\bigvee_{i\in I} (X_i, x_i)$.

Proof.

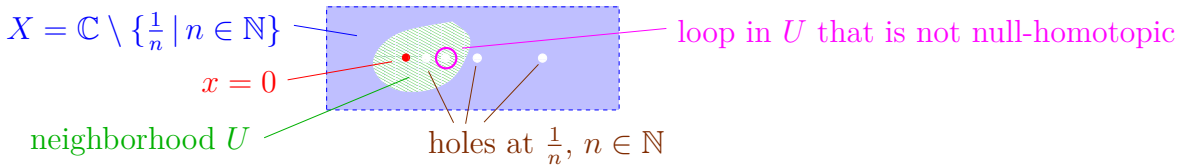
[1024]Recall that if X is Hausdorff, then we know by the Points-in-Hausdorff Lemma 3.32 (1) that any subset consisting of a single point is closed. Thus the first condition in the definition of a good point is often satisfied "for free".



(1) This statement follows immediately from the Points-in-Hausdorff Lemma 3.32 (here we use that topological manifolds are by definition Hausdorff) and the Manifold–Local Properties Lemma 23.8.

(2) Let (X_i, x_i) be a family of pointed topological spaces such that each $x_i \in X_i$ is good. It follows easily from the Wedge Proposition 96.2 (0) that the wedge point $*$ defines a closed subset of the wedge. For each $i \in I$ let U_i be an open neighborhood of $x_i \in X_i$ such that x_i is a deformation retract of U_i. By the Wedge–Subspace Lemma 96.3 we can think of $U \coloneqq \bigvee_{i\in I} U_i$ as an open neighborhood of the wedge point $*$. Furthermore it follows from the Wedge–Homotopy Lemma 96.8 that U admits a deformation retraction to $*$. ■

Non-Example. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Morally speaking the point $x = 0$ is not good, "since any open neighborhood of 0 has non-trivial fundamental group".



In Exercise 99.2 we will provide a proper proof that $x = 0$ is not a good point of X. □

We can now state the main result of this section:

Proposition 99.10. (Wedge-π_1-Proposition) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each x_i is a good point in X_i. Given $j \in I$ we denote by $\iota_j \colon X_j \to \bigvee_{i\in I} X_i$ the natural inclusion. We denote by $*$ the wedge point of $\bigvee_{i\in I} (X_i, x_i)$. The map

$$\ast_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} (X_i, x_i), *\Big),$$

which is induced by the inclusions ι_i together with the Coproduct-of-Groups Lemma 98.4 (1b), is an isomorphism.

Proof. We postpone the proof of the Wedge-π_1-Proposition 99.10 to the next section. ■

Using the Wedge-π_1-Proposition 99.10 we can now provide the long overdue calculation of the fundamental groups of the wedge of circles:

Lemma 99.11. (Wedge-of-Circles–π_1-Lemma) Let S be a set. The map[1025]

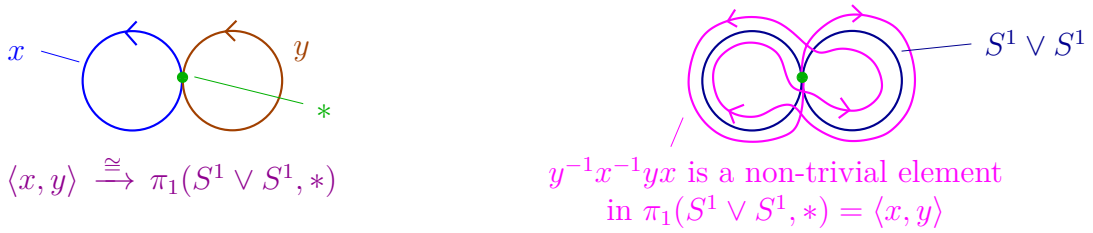
$$\langle S \rangle \;\to\; \pi_1\Big(\bigvee_{s\in S} (S^1, 1), *\Big)$$

that is determined by

$$s \;\mapsto\; \left[\begin{array}{rcl} [0,1] & \to & \bigvee_{s\in S} (S^1, 1) \\ t & \mapsto & [\exp(2\pi \mathrm{i} t), s] \end{array}\right]$$

is an isomorphism.

[1025]Here $\langle S \rangle$ denotes the free group on the set S as introduced in Definition 98.8.



Proof. We consider the following maps:

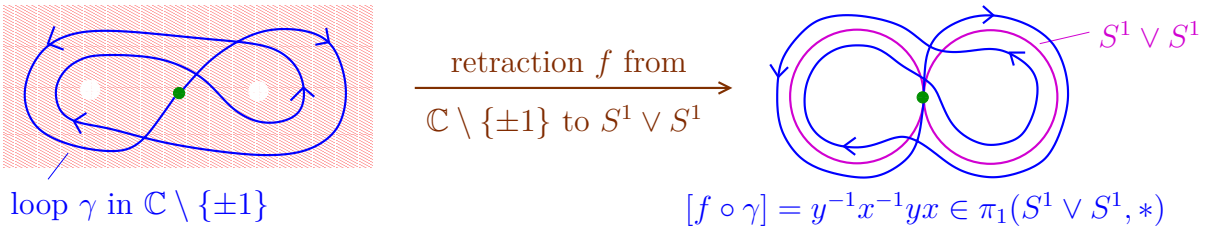
$$\langle S\rangle \underset{\text{(a)}}{=} \ast_{s\in S}\langle s\rangle \xrightarrow[\text{(b)}]{\cong} \ast_{s\in S}\pi_1(S^1,1) \xrightarrow[\text{(c)}]{\cong} \pi_1\Big(\bigvee_{s\in S}(S^1,1),*\Big)$$

We now clarify what we mean by the above maps:

(a) By Definition 98.8 we have $\langle S\rangle = \ast_{s\in S}\langle s\rangle$ where $\langle s\rangle = \{\dots, s^{-1}, e, s, s^2, \dots\}$.
(b) It follows from the Circle-π_1-Theorem 88.22 that for each $s \in S$ we have an isomorphism from the infinite cyclic group $\langle s\rangle$ to $\pi_1(S^1, 1)$ by sending the generator s of $\langle s\rangle$ to the path-homotopy class of the loop given by $t \mapsto \exp(2\pi\,\mathrm{i}t)$.
(c) By the Good Point-Lemma 99.9 (or alternatively by a completely elementary argument) we know that $1 \in S^1$ is a good point. The isomorphism is thus given by the Wedge-π_1-Proposition 99.10.

We have now seen that all of the above maps are indeed isomorphisms. The composition is precisely the map that we claimed was an isomorphism. ■

Example. We consider the loop γ from page 1729 in $\mathbb{C}\setminus\{\pm1\}$ that is also illustrated in the figure below.



We had already pointed out on page 1866 that there exists a deformation retraction

$$f\colon \mathbb{C}\setminus\{\pm1\} \;\to\; S^1\vee S^1 \;\overset{\downarrow}{=}\; \underbrace{\{z\in\mathbb{C}\mid |z+1|=1\}\cup\{z\in\mathbb{C}\mid |z-1|=1\}}_{\text{two circles that meet at the origin}}.$$

(we use the homeomorphism from page 1729 to make this identification)

Note that $f_*([\gamma]) = [f\circ\gamma] = y^{-1}x^{-1}yx \in \langle x,y\rangle \underset{\uparrow}{=} \pi_1(S^1\vee S^1,*)$,

(we use the Wedge-of-Circles–π_1-Lemma 99.11 to make this identification)

which is non-trivial. Thus we see that $f\circ\gamma$ is not null-homotopic in $S^1\vee S^1$. This implies that γ is not null-homotopic in $\mathbb{C}\setminus\{\pm1\}$. □ □

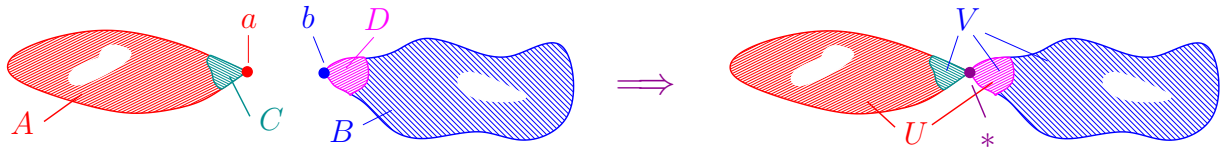
99.6. **Proof of the Wedge-π_1-Proposition 99.10.** In this section we provide the proof of the Wedge-π_1-Proposition 99.10. We proceed as follows:

(1) In the Finite Wedge-π_1-Lemma 99.12 we first deal with the case of the wedge of two pointed topological spaces. By applying this lemma iteratively we get the desired statement also for the wedge of finitely many pointed topological spaces.
(2) We reduce the general case of the Wedge-π_1-Proposition 99.10 via a compactness argument to the case of the wedge of finitely many pointed topological spaces.

Lemma 99.12. (Finite Wedge-π_1-Lemma) Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. The inclusion maps induce an isomorphism[1026]
$$\pi_1(A,a) * \pi_1(B,b) \xrightarrow{\cong} \pi_1((A,a) \vee (B,b), *).$$

Proof. Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. All wedges below will be with respect to these two base points, thus, to simplify the notation we will drop them from the notation. In this spirit we write $X = A \vee B$. As usual we denote by $*$ the wedge point in $A \vee B$. By the Wedge Proposition 96.2 (1b) we can view A and B as subspaces of $X = A \vee B$ in the obvious way.

Since a is a good point we can pick an open neighborhood C in A of a that deformation retracts to a. We can also pick an open neighborhood D of b in B that deformation retracts to b. We consider $U := A \vee D$ and $V := B \vee C$.



Next we consider the following diagram:
$$\begin{array}{ccc} \pi_1(A,a) * \pi_1(B,b) & \longrightarrow & \pi_1(X,*) \\ \downarrow & & \downarrow \text{id} \\ \pi_1(U,*) * \pi_1(V,*) & \longrightarrow & \pi_1(X,*). \end{array}$$

We make the following observations:

(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) By the Wedge–Subspace Lemma 96.3 we can view $U = A \vee D$ and $V = B \vee C$ as open subsets of $X = A \vee B$.
(3) Note that it follows from the Wedge–Homotopy Lemma 96.8 that A is a deformation retract of $U = A \vee D$.
(4) As in (3) we see that B is a deformation retract of $V = B \vee C$.
(5) As in (3), we see that $A \cap B = \{*\}$ is a deformation retract of $U \cap V = C \vee D$. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1b) that $U \cap V$ is simply connected.

[1026] In the definition of a "good point" we demanded that the one-point subset given by the point is closed. Strictly speaking we do not need this fact in the proof of the Finite Wedge-π_1-Lemma 99.12.

(6) By (2) and (5) we can apply the Seifert–van Kampen-Theorem 99.1 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.
(7) It follows from (3), (4) and the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (7) together with the Homotopy Equivalence-π_1-Proposition 90.9 (1) that the two inclusion-induced maps $\pi_1(A, a) \to \pi_1(U, *)$ and $\pi_1(B, b) \to \pi_1(V, *)$ are isomorphisms. This implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We obtain from (1), (6) and (8) that the top horizontal map is, as desired, also an isomorphism. ∎

Proof of the Wedge-π_1-Proposition 99.10. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces such that each x_i is a good point in X_i. We use these points to form the wedge $\bigvee_{i\in I} X_i$. Recall that the fact that x_i is good means that
(1) $\{x_i\}$ is a closed subset of X_i and
(2) that there exists an open neighborhood U_i of x_i in X_i such that x_i is a deformation retract of U_i.

We need to show that the homomorphism

$$\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$$

that is induced by the natural inclusions ι_i is an isomorphism. We will show that the homomorphism Φ is surjective and injective. In both cases the trick is to reduce the statement to the case of the Finite Wedge-π_1-Lemma 99.12.

Claim 1. The map $\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i\Big)$ is injective.

Proof. Given a finite subset $J \subseteq I$ we consider the following diagram:

$$\begin{array}{ccc} \mathop{\ast}_{i\in J} \pi_1(X_i, x_i) & \xrightarrow{\;\cong\;} & \pi_1\Big(\bigvee_{i\in J} X_i, *\Big) \\ \downarrow & & \downarrow{\scriptstyle \iota_*} \\ \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) & \longrightarrow & \pi_1\Big(\bigvee_{i\in I} X_i, *\Big). \end{array}$$

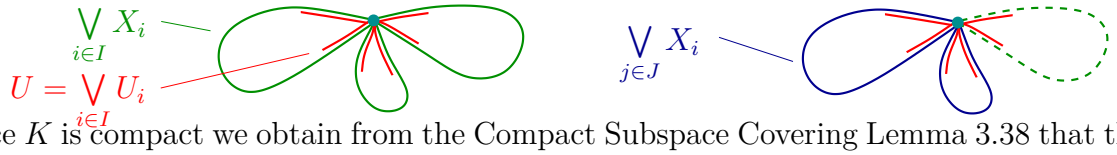
We make the following clarifications and observations:
(1) The maps are induced by the various obvious inclusion. It follows easily from the definitions that the diagram commutes.
(2) Since J is finite we know by the Finite Wedge-π_1-Lemma 99.12, applied finitely many times, that the top horizontal homomorphism is an isomorphism.
(3) Note that as on page 1864 we see that $\bigvee_{i\in J} X_i \subseteq \bigvee_{i\in I} X_i$ is a retract. It follows from the Retraction-π_1-Lemma 90.7 that the right vertical map is a monomorphism.
(4) Note that it follows easily from the explicit description of a free product, that given any $g = (g_1, \ldots, g_m) \in \mathop{\ast}_{i\in I} \pi_1(X_i, x_i)$ there exists a finite subset $J \subseteq I$ such that g lies in the image of the left vertical map.

(5) The claim follows from the above observations. ⊞

The proof that Φ is also surjective requires more of an effort.

Claim 2. We set $U := \bigcup_{i\in I} U_i$. Given any compact subset $K \subseteq \bigvee_{i\in I} X_i$ there exists a finite subset $J \subseteq I$ with $K \subseteq U \cup \bigvee_{i\in J} X_i$.

Proof. Given $i \in I$ we set $V_i := X_i \setminus \{x_i\}$. Since x_i is a good point we see that V_i is an open subset of X_i. Using the Wedge Proposition 96.2 (0) one can now easily see that V_i is in fact an open subset of $\bigvee_{i\in I} X_i$. Evidently the V_i together with U form an open cover of $\bigvee_{i\in I} X_i$.



Since K is compact we obtain from the Compact Subspace Covering Lemma 3.38 that there exists a finite subset $J \subseteq I$ with $K \subseteq U \cup \bigcup_{j\in J} V_j$. But this set is precisely $U \cup \bigvee_{i\in J} X_i$. ⊞

Claim 3. The map

$$\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$$

is surjective.

Proof. Given a finite subset $J \subseteq I$ we consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathop{\ast}_{j\in J} \pi_1(X_j, *) & \xrightarrow[\Phi_J]{\cong} & \pi_1\Big(\bigvee_{i\in J} X_i, *\Big) & \xrightarrow{\cong} & \\
\downarrow & & \downarrow & & \pi_1\Big(U \cup \bigvee_{i\in J} X_i, *\Big) \\
\mathop{\ast}_{i\in I} \pi_1(X_i, x_i) & \xrightarrow{\Phi} & \pi_1\Big(\bigvee_{i\in I} X_i, *\Big). & \leftarrow &
\end{array}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the natural maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.

(2) Since J is finite we obtain, from applying the Finite Wedge-π_1-Lemma 99.12 finitely many times, that the top horizontal map is an isomorphism.

(3) It follows easily from the Wedge–Subspace Lemma 96.3 together with the Wedge–Homotopy Lemma 96.8 that $\bigvee_{i\in J} X_i$ is a deformation retract of $U \cup \bigvee_{i\in J} X_i$. Thus it follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1b) that the top diagonal map is an isomorphism.

Now we turn to the actual proof that the above map Φ is surjective. So suppose that we are given some $[\gamma] \in \pi_1\Big(\bigvee_{i\in J} X_i, *\Big)$. It follows from the Compact Image Lemma 4.15 together with Claim 2 that there exists a finite subset $J \subseteq I$ such that the image of γ is contained in $U \cup \bigvee_{i\in J} X_i$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\gamma]$ actually lies in the image of the bottom horizontal map Φ. ■

99.7. **Fundamental groups of topological graphs.** Now we have all the tools to calculate the fundamental groups of the topological realizations of finite abstract graphs.

Proposition 99.13. (Graph-π_1-Proposition)

(1) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph and let T be a spanning tree[1027] of G. Let $e_1, \dots, e_n$ be the edges not contained in T. We pick some $v \in V$. For $j = 1, \dots, n$ we pick a path[1028] $\alpha_j \colon [0,1] \to |T|$ from v to $i(e_j)$, we denote by $\beta_j \colon [0,1] \to |e_j| \subseteq |G|$ the obvious path from $i(e_j)$ to $t(e_j)$, and we pick a path $\gamma_j \colon [0,1] \to |T|$ from $t(e_j)$ to v. The map

$$\Theta_G \colon \langle x_1, \dots, x_n \rangle \;\to\; \pi_1(|G|, v)$$

that is given by

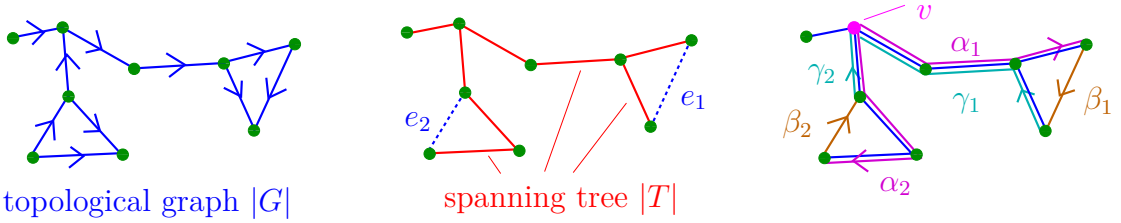
$$x_j \;\mapsto\; [\alpha_j * \beta_j * \gamma_j]$$

is an isomorphism.

(2) Let X be a finite connected non-empty topological graph[1029] with v vertices and e edges. Then

$$\pi_1(X) \;\cong\; \text{free group on } 1 - \chi(X) = 1 + e - v \text{ generators.}$$

(3) Two finite connected topological graphs are homotopy equivalent if and only if they have the same Euler characteristic.



Proof.

(1) We consider the maps

$$\langle x_1, \dots, x_n \rangle \xrightarrow{\Theta_G} \pi_1(|G|, [v]) \to \pi_1(|G|/|T|, v) \xleftarrow{\cong} \pi_1\Big(\bigvee_{j=1}^{n} S^1_j, *\Big) \xleftarrow{\cong} \langle y_1, \dots, y_n \rangle.$$

isomorphism induced by the homeomorphism on page 1865

isomorphism by the Graph-Homotopy Type Corollary 96.12 (1) and the Homotopy Equivalence-π_1-Proposition 90.9 (1)

isomorphism given by the Wedge-of-Circles–π_1-Lemma 99.11

Note that it follows fairly easily from the explicit descriptions of the various maps that for each $j \in \{1, \dots, n\}$ the images of x_j and y_j in $\pi_1(|G|/|T|, v)$ are represented by path-homotopic loops, in particular the images of x_j and y_j in $\pi_1(|G|/|T|, [v])$ are the same. It follows from the above and the fact that all other maps are isomorphisms that Θ_G itself is indeed an isomorphism.

(2) Basically by definition we only need to prove the statement if $X = |G|$ is the topological realization of a finite connected non-empty abstract graph $G = (V, E, i, t)$. By the Spanning Tree–Existence Proposition 11.26 we know that G admits a spanning tree

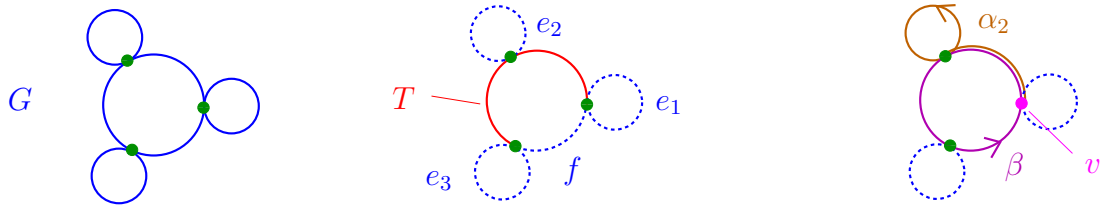
[1027] Recall that according to Definition 11.23 a spanning tree for G is a subtree of G that contains all vertices of G.

[1028] Note that it follows from the Abstract Graph-Topology Lemma 11.6 that $|T|$ is path-connected.

$T = (W, F)$. By definition T contains all vertices of V. Furthermore, since T is a tree we know by the Tree Characterization Proposition 11.28 that $\#W - \#F = 1$. Thus it follows that there are precisely $\#E - \#F = e - (\#W - 1) = e - v + 1$ edges of G that are not contained in T. The desired statement now follows from (1).

(3) Let X and X' be two finite connected topological graphs. If $\chi(X) = \chi(X')$, then we already saw in the Graph-Homotopy Type Corollary 96.12 (2) that X and X' are homotopy equivalent. On the other hand, if $\chi(X) \neq \chi(X')$, then it follows from (1) together with the Basis Cardinality Proposition 98.19 that X and X' are not homotopy equivalent. ■

Example. We look at the graph G shown in the figure below. We also show a spanning tree T and a "base vertex v". We make the following observations:



(1) The three small loops e_1, e_2, e_3 and the edge f are not contained in T.

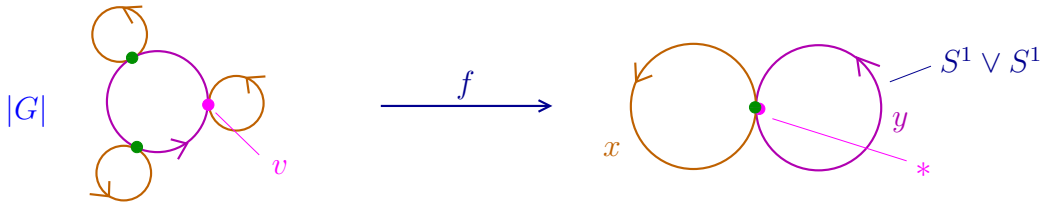
(2) Each loop e_i gives rise to a loop α_i with base point v, by traveling along T, going once around e_i and going back to v along T.

(3) The edge f gives rise to a loop β by going once around the central circle.

It follows from the Graph-π_1-Proposition 99.13 that the map

$$\begin{array}{rcl} \langle a_1, a_2, a_3, b\rangle & \to & \pi_1(|G|, v) \\ a_i & \mapsto & [\alpha_i] \\ b & \mapsto & [\beta] \end{array}$$

is an isomorphism. Next we do something much more interesting: We consider the map $f\colon |G| \to S^1 \vee S^1$ shown in the figure below.



One can easily verify that f is a covering of degree 3. We use the Wedge-of-Circles–π_1-Lemma 99.11 to make the usual identification $\pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle$. By working directly with the definition of f_* one can see that f_* is given by

$$\begin{array}{rcl} f_*\colon \langle a_1, a_2, a_3, b\rangle = \pi_1(|G|, v) & \to & \pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle \\ a_i & \mapsto & y^{i-1} \cdot x \cdot y^{-(i-1)} \\ b & \mapsto & y^3. \end{array}$$

Since $f\colon |G| \to S^1 \vee S^1$ is a covering of degree 3 we obtain from the Covering-π_1-Monomorphism Proposition 92.7 together with the Covering-Degree Proposition 92.10 that the induced map f_* is a monomorphism and that the image of f_* is a subgroup of index three.

In summary, we have now shown that the free group $\langle x, y\rangle$ on two generators contains the subgroup $f_*(\langle a_1, a_2, a_2, b\rangle) = \langle x, yxy^{-1}, y^2xy^{-2}, y\rangle$ which is a free group on four generators. Thus we see that the free group of rank two can contain "larger" free groups. This is in complete contrast to the situation in free abelian groups which we studied in the Free Abelian Subgroup Theorem 97.14. □

99.8. **The fundamental group of mapping cones.** In this final section we will use results from this chapter to calculate the fundamental groups of mapping cones. If the reader's interest in mapping cones is rather muted, then the reader can also move on to the next chapter. Let us first recall Definition 21.7:

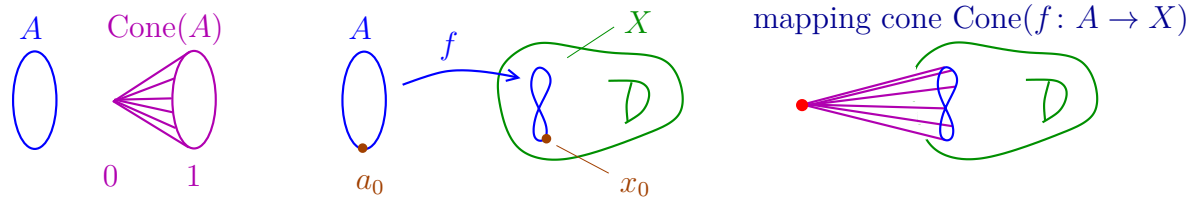
Definition 99.14.

(1) Given a topological space A we define the corresponding cone as

$$\mathrm{Cone}(A) \;:=\; (A \times [0,1])/(A \times \{0\}).$$

(2) Given a continuous map $f\colon A \to X$ between topological spaces the corresponding mapping cone is defined as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \sqcup X)/\sim \text{ where } [(a,1)] \sim f(a) \text{ for all } a \in A.$$

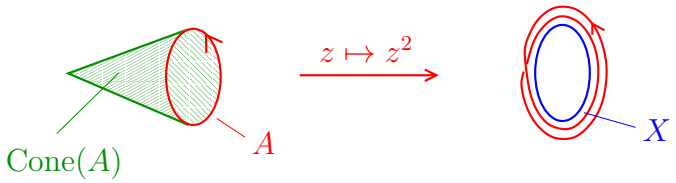


Example. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^2$. In Exercise 21.7 we showed that the corresponding mapping cone $\mathrm{Cone}(f\colon S^1 \to S^1)$ is homeomorphic to the real projective plane $\mathbb{R}\mathrm{P}^2$. □

Lemma 99.15. (Mapping Cone-π_1-Lemma) Let $f\colon A \to X$ be a continuous map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. If A is path-connected, then the inclusion $X \to \mathrm{Cone}(f\colon A \to X)$ induces an isomorphism

$$\pi_1(X, x_0)/\langle\langle f_*(\pi_1(A, a_0))\rangle\rangle \;\to\; \pi_1(\mathrm{Cone}(f\colon A \to X), [x_0]).$$

Example. Let $n \in \mathbb{N}$ and let $A = X = S^1$. We consider the map $f\colon A = S^1 \to X = S^1$ that is given by $f(z) = z^n$.



It follows from the Mapping Cone-π_1-Lemma 99.15 and the Circle-π_1-Theorem 88.22, perhaps counter-intuitively, that $\pi_1(\mathrm{Cone}(f\colon A \to X)) \cong \mathbb{Z}/n\mathbb{Z}$. □

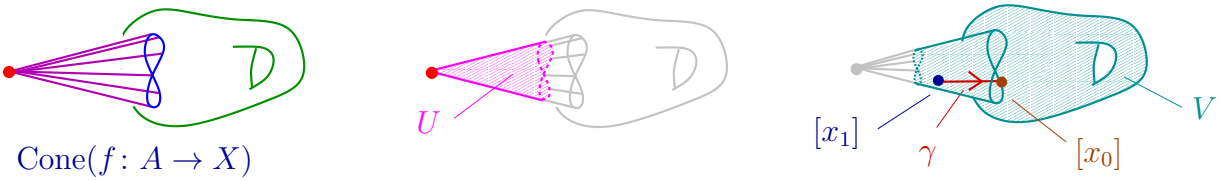
Proof. Let (A, a_0) be a pointed path-connected topological space and let $f\colon A \to X$ be a continuous map to some topological space. We set $x_0 := f(a_0)$.

The key idea of the proof is to apply the Seifert–van Kampen-Theorem 99.1 to the following two subsets of $\mathrm{Cone}(f\colon A \to X)$:

$$\begin{aligned} U &= \left\{[(a,t)] \in \mathrm{Cone}(A) \,\middle|\, t < \tfrac{3}{4}\right\} \\ V &= \mathrm{Cone}(f\colon A \to X) \setminus \left\{[(a,t)] \in \mathrm{Cone}(A) \,\middle|\, t \leq \tfrac{1}{4}\right\}. \end{aligned}$$

To make this idea work we need to introduce a few more objects:

- We denote by $F\colon A \times (\frac{1}{4}, 1] \to \mathrm{Cone}(f\colon A \to X)$ the map that is given by $(a,t) \mapsto [(a,t)]$.
- We set $x_1 := (a_0, \frac{1}{2})$.
- We denote by $\gamma\colon [0,1] \to \mathrm{Cone}(f\colon A \to X)$ the "straight path" from $[x_1]$ to $[x_0]$ that is given by $t \mapsto [(a_0, \frac{1}{2}(t+1))]$.



Next we consider the following diagram of homomorphisms:

$$\begin{array}{ccc}
\pi_1(X,x_0)/\langle\langle f_*(\pi_1(A,a_0))\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_0]) \\
\cong\downarrow & & \downarrow\mathrm{id} \\
\pi_1(V,[x_0])/\langle\langle f_*(\pi_1(A,a_0))\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_0]) \\
\mathrm{id}\downarrow & & \downarrow\mathrm{id} \\
\pi_1(V,[x_0])/\langle\langle F_*(\pi_1(A\times(\frac{1}{4},1],(a_0,1)))\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_0]) \\
\cong\downarrow\gamma_* & & \gamma_*\downarrow\cong \\
\pi_1(V,[x_1])/\langle\langle F_*(\pi_1(A\times(\frac{1}{4},1],x_1))\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_1]) \\
\mathrm{id}\uparrow & & \uparrow\mathrm{id} \\
\pi_1(V,[x_1])/\langle\langle F_*(\pi_1(A\times(\frac{1}{4},\frac{3}{4}),x_1))\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_1]) \\
\mathrm{id}\downarrow & & \downarrow\mathrm{id} \\
\pi_1(V,[x_1])/\langle\langle \pi_1(U\cap V,[x_1])\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_1]) \\
\cong\uparrow & \text{Amalgamated Product–Quotient Lemma 98.25} & \downarrow\mathrm{id} \\
\pi_1(V,[x_1]) *_{\pi_1(U\cap V,[x_1])} \underbrace{\pi_1(U,[x_1])}_{=\{e\}} & \xrightarrow[\text{Seifert–van Kampen}]{\cong} & \pi_1(\mathrm{Cone}(f\colon A \to X),[x_1]).
\end{array}$$

We want to show that the top horizontal map is an isomorphism. It suffices to prove the following claim:

Claim.

(1) All vertical maps are isomorphisms.
(2) The diagram commutes.
(3) The bottom horizontal map is an isomorphism.

Proof.

(1) We work our way from the top to the bottom.
   (a) Using the Homotopy Pushout Lemma 18.13 one can easily show that X is a deformation retract of V. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1b) that the top left vertical map is an isomorphism.
   (b) Since the map $\iota\colon A \to A \times (\frac{1}{4}, 1]$ given by $a \mapsto (a, 1)$ is a homotopy equivalence and since $F \circ \iota = f$ we see that $f_*(\pi_1(A, a_0)) = F_*(\pi_1(A \times \frac{1}{4}, 1], (a_0, 1)))$. Thus the second left vertical map is indeed the identity.
   (c) The maps indicated by γ_* are isomorphisms by the Change-of-Base Point Proposition 88.14.
   (d) The inclusion $A \times (\frac{1}{4}, \frac{3}{4})$ to $A \times (\frac{1}{4}, 1]$ is a homotopy equivalence. It follows that the fourth left vertical map is the identity.
   (e) The restriction of F to $A \times (\frac{1}{4}, \frac{3}{4})$ is in fact an embedding with image $U \cap V$. Thus we see that the fifth left vertical map is also the identity.
   (f) The argument of the Cone Contractability Lemma 20.20 shows that U is contractible, which implies by the Homotopy Equivalence-π_1-Proposition 90.9 that $\pi_1(U, [x_1])$ is the trivial group. It follows from this observation together with the Amalgamated Product–Quotient Lemma 98.25 that the bottom left vertical map is an isomorphism.

(2) The two vertical maps γ_* from the third to the fourth level are the isomorphisms induced by the choice of base point coming from the path γ from $[x_0]$ to $[x_1]$. It follows from the Change-of-Base Point Proposition 90.4 (1) that the third rectangle commutes.

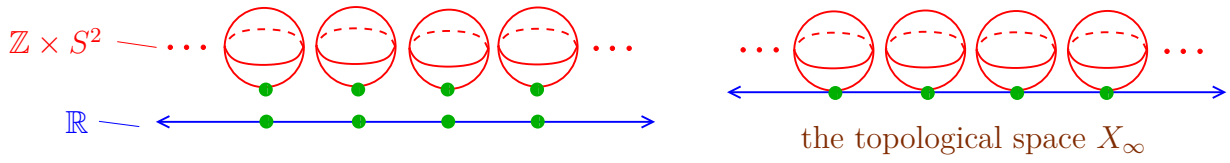
   All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It is thus clear that the remaining rectangles commute.

(3) It follows almost immediately from the Definition 8.3 of the quotient topology that U and V are open subsets of $\mathrm{Cone}(f\colon A \to X)$. Furthermore note that it follows from the hypothesis that A is path-connected and the Quotient Topology-Properties Proposition 8.5 (3) that $U \cap V \cong A \times (\frac{1}{4}, \frac{3}{4})$ is path-connected. These two observations show that we can apply the Seifert–van Kampen-Theorem 99.1 and we obtain that the bottom horizontal map is an isomorphism. ∎

99.9. **Exhaustions of topological spaces.** In many situations one is dealing with non-compact topological spaces where one would be tempted to apply the Seifert–van Kampen-Theorem 99.1 or the Wedge-π_1-Proposition 99.10 "infinitely many times", which does not really make sense. For example let us consider the topological space

$$X_\infty \;=\; \big(\mathbb{R} \sqcup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n, (0, 0, -1)).$$

The topological space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. We would like to show that X_∞ is simply connected. The following lemma lets us get around the issue of "infinite operations".



Proposition 99.16. (Exhaustion–Simply Connected Proposition) Let X be a topological space and let
$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$
be a sequence of subspaces of X such that the following three conditions are satisfied:
(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.
(2) For each $i \in \mathbb{N}$ we have $X_i \subseteq \mathring{X}_{i+1}$.
(3) Each X_i is simply connected.
Then for each $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

The proof of the Exhaustion–Simply Connected Proposition 99.16 rests on the following lemma which will come in handy on several occasions.

Lemma 99.17. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let
$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$
be a sequence of subspaces of X such that the following two conditions are satisfied:
(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subseteq \mathring{X}_{i+1}$.
Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subseteq X_i$.

Proof of the Compact Subspace–Exhaustion Lemma 99.17. We start out with the following little claim.

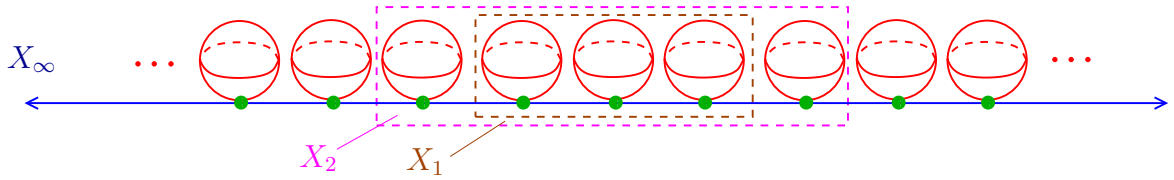
Claim. We have
$$\bigcup_{i \in \mathbb{N}_0} \mathring{X}_i = X.$$

Proof. We have the following equalities and inclusions
$$X \underset{\text{by Hypothesis (1)}}{=} \bigcup_{i \in \mathbb{N}_0} X_i \underset{\text{by Hypothesis (2)}}{=} \bigcup_{i \in \mathbb{N}_0} \mathring{X}_i \subseteq X.$$
We see that the inclusion to the right is in fact an equality. ⊞

Now let $K \subseteq X$ be a compact subspace. Recall that by the Interior-Closure Lemma 3.22 (1) we know that interiors are always open subsets. It follows from the Compact Subspace Covering Lemma 3.38 and the claim that K is contained in the union of finitely many of the $\mathring{X}_i$. It follows from our hypothesis on the X_i that K is in fact contained in a single X_i. ■

Proof of the Exhaustion–Simply Connected Proposition 99.16. Let $x_0 \in X$ and let $f\colon [0,1] \to X$ be a loop in x_0. By the Compact Image Lemma 4.15 we know that $f([0,1])$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 99.17 we know that $f([0,1])$ is contained in a single X_i. It follows from Hypothesis (3) that f is null-homotopic in X_i. It is clear that this implies that f is also null-homotopic in X. ■

Example. We return to the topological space X_∞ from the beginning of the section. Given $k \in \mathbb{N}$ we set
$$X_k = \big((-k-1, k+1) \sqcup (\{-k, \dots, k\} \times S^2)\big) / n \sim (n, (0,0,-1)).$$
Using the Wedge-π_1-Proposition 99.10 one can easily show that each X_k is simply connected. It follows from the Exhaustion–Simply Connected Proposition 99.16 that X_∞ is simply connected. □



Using the Exhaustion–Simply Connected Proposition 99.16 we can now give a characterization of arbitrary trees in terms of their fundamental groups.

Proposition 99.18. (Tree–π_1-Proposition) Let $T = (V, E, \varphi)$ be a non-empty connected abstract graph. The following statements are equivalent:[1029]

(1) T is a tree.
(2) $\pi_1(|T|)$ is the trivial group.

Remark. In the Tree–Contractible Proposition 137.4 we will give another characterization of trees. □

Proof. Let $T = (V, E, \varphi)$ be a non-empty connected abstract graph.

We start out with a proof of the (1)⇒(2)-direction. Since T is non-empty we can pick a vertex $v \in V$. We consider the subgraph given by $T_0 := \{v\}$. Iteratively we build a sequence of subgraphs $T_0, T_1, T_2, \dots$ of T as follows: given T_i we add all edges (and the corresponding vertices) that have at least one vertex in common with T_i. We make the following observations:

(a) One can easily verify that each T_i is connected.
(b) It follows from (a) and the Tree–Basics Lemma 11.22 (1) that each T_i is a tree.
(c) It follows from (b) and Finite Tree–Contractible Proposition 96.10 that each T_i is simply connected.
(d) One can easily verify that for each $i \in \mathbb{N}_0$ we have $T_i \subseteq \mathring{T}_{i+1}$.
(e) We have $T = \bigcup_{i \in \mathbb{N}_0} T_i$.

It follows from the above together with the Exhaustion–Simply Connected Proposition 99.16 that T is simply connected.

We turn to the proof of the (2)⇒(1)-direction. By the Spanning Tree–Existence Proposition 11.26 we know that T admits a spanning tree S. Note that it remains to show that if $S \neq T$, then $\pi_1(|T|)$ is non-trivial. Thus suppose that there exists an edge $e \in T \setminus S$. We denote by $|e| \subseteq |T|$ the corresponding subspace. By definition there exist $v, w \in V$ with $\varphi(e) = \{v, w\}$.[1030] By definition of the topological realization in Definition 11.17 there exists a continuous map $\alpha\colon [0,1] \to |e|$ with $\alpha(0) = v$ and $\alpha(1) = w$ such that α restricts to a homeomorphism $(0,1) \to |e| \setminus \{v, w\}$. Since S is a spanning tree we know that $v, w \in S$. Also note that trees are connected. Thus it follows from the Abstract Graph-Topology Lemma 11.6 that $|S|$ is connected. Therefore there exists a continuous map $\beta\colon [0,1] \to |S|$ with $\beta(0) = v$ and $\beta(1) = w$. Next we consider the map

$$\begin{array}{rcl} f\colon |T| & \to & S^1 \\ x & \mapsto & \begin{cases} 1, & \text{if } x \notin \alpha([0,1]), \\ \exp(2\pi \mathrm{i} t), & \text{if } x = \alpha(t) \text{ for some } t \in [0,1]. \end{cases} \end{array}$$

[1030] Note that it is possible that $v = w$.

It follows easily from the Abstract Graph-Topology Lemma 11.6 (3) that f is continuous.



Note that $\alpha * \beta$ is a loop in v and note that $f \circ (\alpha * \beta)\colon [0,1] \to S^1$ is path-homotopic to the loop $t \mapsto \exp(2\pi\, i t)$. It follows from these observations together with the Circle-π_1-Theorem 88.22 that $f_*([\beta * \alpha]) \in \pi_1(S^1, 1)$ is non-trivial. But this shows that $\pi_1(|T|, v)$ is non-trivial. ■

99.10. Exercises for Chapter 99.

Exercise 99.1.

(a) Show that in the statement of the Seifert–van Kampen Theorem 99.1 we cannot just drop the hypothesis that U and V are open.
(b) Show that in the statement of the Seifert–van Kampen Theorem 99.1 we cannot just drop the hypothesis that $U \cap V$ is path-connected.

Exercise 99.2. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that the point $x = 0$ is not good, i.e. show that there does not exist an open neighborhood U of $x = 0$ such that $x = 0$ is a deformation retract of U.

Exercise 99.3. Let M be an n-dimensional topological manifold and let P be a point in $M \setminus \partial M$ and let $x_0 \in M \setminus \{P\}$ be a base point. Let $i\colon M \setminus \{P\} \to M$ be the inclusion. We consider the induced map

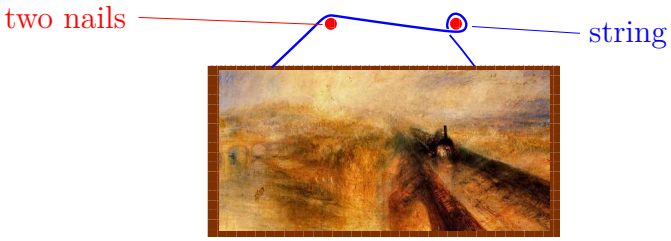
$$i_*\colon \pi_1(M \setminus \{P\}, x_0) \;\to\; \pi_1(M, x_0).$$

(a) Show that if $n = 2$, then i_* is an epimorphism.
(b) Show that if $n > 2$, then i_* is an isomorphism.

Exercise 99.4.

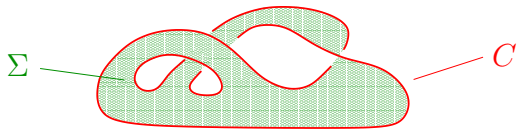
(a) How can one hang a painting with two nails provided, such that if one removes either of the two nails, the painting falls down?
(b) What is a solution to (a) if we use three nails instead of two nails?
(c) Bonus Question: Who is the artist?

Remark. There is even a whole paper [**DDMMRP2014**] that is dedicated to this problem and variations on it.



Exercise 99.5. Let Σ be the 2-dimensional smooth manifold that is shown in the figure below. We denote by C the boundary of Σ.

(a) What is the homeomorphism type of the topological space that we obtain from gluing a disk $\overline{B}^2$ to Σ along the boundary C?
(b) Is C a retract of Σ?



Exercise 99.6.

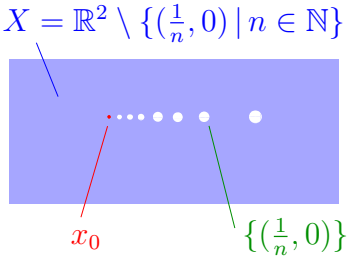
(a) Let $n \geq 3$ and furthermore let $K \subseteq \mathbb{R}^n$ be a compact subset. We view K also as a subset of $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$. Let $x_0 \in \mathbb{R}^n \setminus K$. Show that the inclusion-induced map $\pi_1(\mathbb{R}^n \setminus K, x_0) \to \pi_1(S^n \setminus K, x_0)$ is an isomorphism.
(b) Does the conclusion of (a) also hold for $n = 2$?
(c) Does the conclusion of (a) also hold for non-compact subsets of $\mathbb{R}^n$?

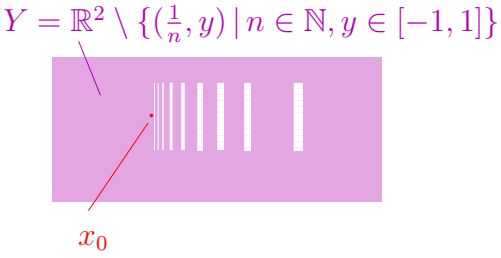
Exercise 99.7. We consider the two topological spaces

$$X = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, 0) \,|\, n \in \mathbb{N}\} \text{ and } \quad \text{and } Y = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, y) \,|\, n \in \mathbb{N}, y \in [-1, 1]\}.$$

We write $x_0 = (0, 0)$. We refer to the figure below for an illustration.

(a) Show that $\pi_1(X, x_0)$ is uncountable.
Hint. For $n \in \mathbb{N}$ we write $Z_n \coloneqq \mathbb{R}^2 \setminus \{\frac{1}{n}, 0\}$ and we denote by $f_n \colon X \to Z_n$ the inclusion map. Use the induced maps $f_{n*} \colon \pi_1(X, x_0) \to \pi_1(Z_n, x_0) \cong \mathbb{Z}$ to find a homomorphism from $\pi_1(X, x_0)$ to a group such that the image is uncountable.
(b) Show that $\pi_1(Y, x_0)$ is countable.
Hint. Use Exercise 4.52 as an input to find a monomorphism from $\pi_1(Y, x_0)$ into a countable group.





Exercise 99.8. Let $k \in \mathbb{N}_0$. We pick an isomorphism $\Theta \colon \langle x_1, \ldots, x_k \rangle \to \pi_1\Big(\bigvee_{i=1}^{k} S^1, *\Big)$. Show that for any homomorphism $\varphi \colon \langle x_1, \ldots, x_k \rangle \to \langle x_1, \ldots, x_k \rangle$ there exists a continuous map $f \colon \Big(\bigvee_{i=1}^{k} S^1, *\Big) \to \Big(\bigvee_{i=1}^{k} S^1, *\Big)$ such that $\Theta^{-1} \circ f_* \circ \Theta = \varphi$.

Exercise 99.9. We consider the wedge $S^1 \vee S^1$ of two circles and we denote by $*$ the wedge point. We consider the map $f \colon S^1 \vee S^1 \to S^1 \vee S^1$ that is shown in the figure below.

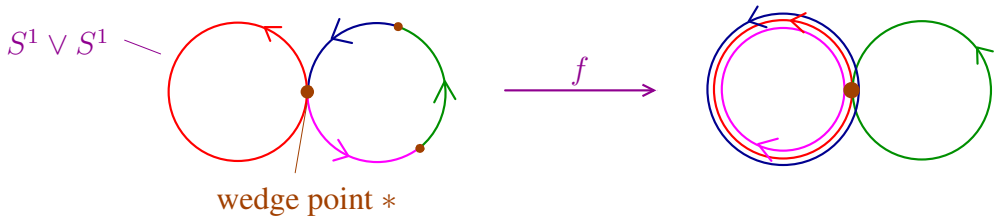
(a) Show that f is a homotopy equivalence.
(b) Show that every homeomorphism of $S^1 \vee S^1$ preserves the wedge point $*$.

(c) Determine

$$\{\varphi \in \mathrm{Aut}(\pi_1(S^1 \vee S^1, *)) \mid \text{ there exists a homeomorphism } g \text{ with } \varphi = g_*\}.$$

(d) Show that f is not homotopic to a self-homeomorphism of $S^1 \vee S^1$.

Remark. In Proposition **??** we will show that there exist self-homotopy equivalences of smooth manifolds that are not homotopic to homeomorphisms.



Exercise 99.10. Let Σ be the surface of genus two and let X be the wedge of two circles.

(a) Show that there exists an embedding $f\colon X \to \Sigma$ such that $f(X)$ is a retract of Σ.

(b) Show that $\pi_1(\Sigma)$ is non-abelian.



Exercise 99.11. Given $k \in \mathbb{N}$ we denote by X_k the topological space that we obtain from removing k distinct points from S^2. What is the minimal $k \in \mathbb{N}$ such that $\pi_1(X_k)$ is non-abelian?

Exercise 99.12. Let $g \in \mathbb{N}_0$. We denote by Σ_g the surface of genus g.

(a) Let $P \in \Sigma_g$. What is the fundamental group of $\pi_1(\Sigma_g \setminus \{P\})$?

(b) Let $g \neq h$. Show that the surface Σ_g of genus g is not homeomorphic to the surface Σ_h of genus h.

Remark. Recall that it follows from the Manifold–Homogeneous Proposition 41.21 that given any $x, y \in \Sigma_g$ there exists a homeomorphism $f\colon \Sigma_g \to \Sigma_g$ with $f(x) = y$.

Exercise 99.13.

(a) Let $\phi\colon \langle x, y\rangle \to \mathbb{Z}$ be the homomorphism given by $\phi(x) = 1$ and $\phi(y) = 0$. Show that $\ker(\phi)$ is not finitely generated.
Remark. In fact in [**LyS1977**, Proposition I.3.12] and in [**dlH2000**, p. 45] it is shown that any non-trivial *normal* subgroup of infinite index of a free group is a free group on an infinite generating set.

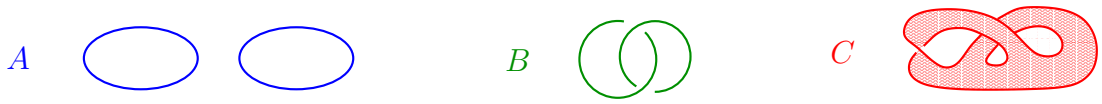
(b) Let $\psi\colon \langle s, t\rangle \times \langle x, y\rangle \to \mathbb{Z}$ be the homomorphism given by $\psi(s) = 1$, $\psi(t) = 0$, $\psi(x) = 1$ and $\psi(y) = 0$. Show that $\ker(\psi)$ is finitely generated.

Exercise 99.14. Let G and H be two groups.

(a) Suppose that there exist monomorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Does it follow that G and H are isomorphic?

(b) Suppose that there exist epimorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Does it follow that G and H are isomorphic?

Exercise 99.15. We consider the subsets A, B, C of $\mathbb{R}^3$ that are shown in the figure below. What is the isomorphism type of $\pi_1(\mathbb{R}^3 \setminus A)$, $\pi_1(\mathbb{R}^3 \setminus B)$ and $\pi_1(\mathbb{R}^3 \setminus C)$?

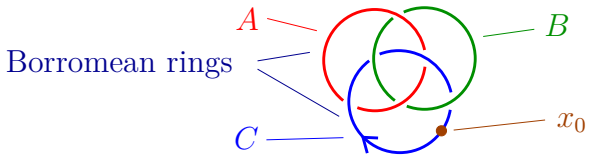


Exercise 99.16. Let X and Y be two path-connected non-empty topological spaces. We consider the join $X * Y$ from Definition 20.1. Show that the join $X * Y$ is simply connected. *Hint.* Let $\pi\colon X \times [0,1] \times Y \to X * Y$ be the natural projection. In the following we set $U := \pi^{-1}(X \times [0,1) \times Y)$ and we set $V := \pi^{-1}(X \times (0,1] \times Y)$. Determine the fundamental groups of U, V and $U \cap V$. At some point you should make use of the Product-π_1-Proposition 90.13.

Exercise 99.17. We consider the Borromean rings $A \cup B \cup C \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ which are shown in the figure below.

(a) Determine $\pi_1(S^3 \setminus (A \cup B), x_0)$.

(b) Does C define a non-trivial element in $\pi_1(S^3 \setminus (A \cup B), x_0)$?



Exercise 99.18. Let M and M' be two oriented connected n-dimensional smooth manifolds with non-empty boundary. We consider the boundary connected sum $M \#_{\mathrm{b}} M'$ that we introduced in Definition 61.10. Let $x_0 \in M \cap M' \subseteq M \#_{\mathrm{b}} M'$. Show that the inclusion maps induce an isomorphism

$$\pi_1(M, x_0) * \pi_1(M', x_0') \xrightarrow{\cong} \pi_1(M \#_{\mathrm{b}} M', x_0).$$

Remark. Use the Boundary Connected Sum Proposition 61.11 (3) and collars or tubular neighborhoods.

Exercise 99.19. Let M be a connected smooth manifold, let T be a component of ∂M, let $x_0 \in T$ be a base point and let $F\colon T \times [0,1] \to T$ be a diffeotopy such that $F_0 = F_1 = \mathrm{id}_T$. By the Smooth Collar Neighborhood Theorem 35.6 we know that there exists a smooth collar $\chi\colon T \times [0,1] \to M$. We consider the homeomorphism

$$\begin{array}{rcl} \Theta_F\colon M & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \chi(T \times [0,1]), \\ \chi(F(x,t),t), & \text{if } P = \chi(F,t) \in \chi(T \times [0,1]). \end{cases} \end{array}$$

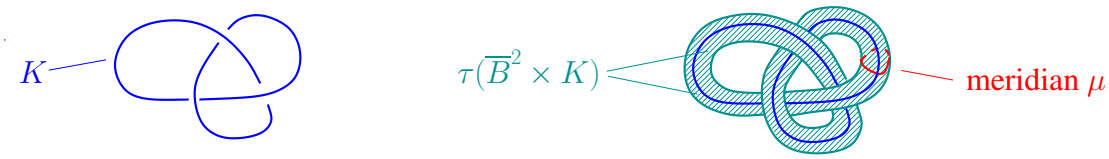
Finally we consider the loop $\gamma\colon [0,1] \to M$ that is given by $\gamma(t) = F_t(x_0)$. Show that the induced homomorphism

$$\Theta_{F*}\colon \pi_1(M, *) \to \pi_1(M, x_0)$$

is given by conjugation by $[\gamma] \in \pi_1(M, x_0)$.

Exercise 99.20. Let $n \in \mathbb{N}_0$ and let $K \subseteq S^n$ be a closed oriented connected non-empty submanifold of codimension 2 which admits a tubular map $\tau\colon \overline{B}^2 \times K \to S^n$. For any

$x \in K$ we refer to $\tau(S^1 \times \{x\})$ as a *meridian* of K. Show that $\pi_1(S^n \setminus K)$ is normally generated (in the sense of Definition 98.20) by any meridian of K.
Hint. Make use of Exercise 98.28.



CHAPTER 100

Presentations of groups

In the Seifert–van Kampen Theorem 99.1 we just learned how to express fundamental groups of suitable topological spaces as amalgamated products. But it is not always clear in what sense we calculated a fundamental group and what such a description as an amalgamated product is trying to tell us.

In this chapter we introduce the notion of a (finite) presentation of a group. We will see in this and the following chapters that we can describe (fundamental) groups as finitely presented groups, i.e. we can give a finite description of these groups.

This chapter is organized as follows:

- In Section 100.1 we will introduce generating sets of groups.
- In Section 100.2 we will introduce presentations of groups.
- In Section 100.3 we will discuss the effect of various constructions of groups on their presentations.
- In Section 100.4 we will discuss Tietze transformations.
- In Appendix 100.5 we will show a simple method which often can be used to turn a given presentation of a group into a presentation with fewer generators.
- In Appendix 100.6 we will discuss the Andrews-Curtis Conjecture 100.34 in group theory.
- In Section 100.7 we will provide exercises for Chapter 100.

In the next Chapter 101 we will state and discuss the classification of compact 2-dimensional manifolds.

100.1. **Generating sets of groups.** In Definition 98.1 we introduced the notion of a "reduced word", which begs the question, what is a "word". Here is the answer.

Definition 100.1.

(1) Let S be a set. A word in S is a finite sequence $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ with $s_1, \dots, s_k \in S$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. We refer to k as the length of the word.

(2) Let G be a group and let $S \subseteq G$ be a subset. A word $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ defines an element $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ in G. In casual language we also refer to an expression of the form $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ as a word.

We recall the following definition that we briefly mentioned in Definition 64.12 and Definition 98.29.

Definition 100.2.

(1) Let G be a group and let $S \subseteq G$ be a subset. We denote by $\Xi_S \colon \langle S \rangle \to G$ the natural homomorphism (given by the Free Group–Existence Lemma 98.9) from the free group $\langle S \rangle$ to G that is given by $s \mapsto s$.

(2) Given a group G we say that $S \subseteq G$ is a generating set if the natural homomorphism $\Xi_S \colon \langle S \rangle \to G$ is an epimorphism.
(3) We say that a group G is finitely generated if it admits a finite generating set.

Lemma 100.3. (Generating Set–Characterization Lemma) Let G be a group and let $S \subseteq G$ be a subset. The following statements are equivalent:

(1) S is a generating set for G.
(2) Every element of G can be represented by a word in S, i.e. we have
$$G = \left\{ s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k} \mid s_1, \ldots, s_k \in S \text{ and } \epsilon_1, \ldots, \epsilon_k \in \{-1, 1\} \right\}.$$
(3) If H is a subgroup of G that contains S, then $H = G$.

Proof.
(1)⇒(2): This statement follows immediately from the definitions and the trivial observation that for $s \in S$ and $n = \epsilon \cdot m$ with $\epsilon \in \{-1, 1\}$ and $m \in \mathbb{N}$ we have $s^{\epsilon \cdot m} = (s^\epsilon)^m$.
(2)⇒(3): This statement follows from the observation that if $H \subseteq G$ is subgroup that contains S, then H also contains all words in S.
(3)⇒(1): This statement follows from the observation that the image of the natural homomorphism $\Xi_S \colon \langle S \rangle \to G$ is a subgroup of G that contains S. ■

Example. It follows from the Abelianization-Epimorphism Lemma 98.17 together with the Abelianization Coproduct Proposition 98.18 (1) that any finitely generated abelian group in the sense from Definition 97.19 is also a finitely generated group in the above sense. □

To formulate our next example let us remind the reader of the following definition from algebra:

Definition 100.4. A domain R is called Euclidean if there exists a map $\varphi \colon R \to \mathbb{N}_0$ such that for every $a, b \in R \in \{0\}$ there exist $g, r \in R$ with $a = g \cdot b + r$ such that either $r = 0$ or $\varphi(r) < \varphi(b)$.

Lemma 100.5. (Matrix Group-Generating Set Lemma)

(1) Let R be a commutative Euclidean domain. Recall that the elementary matrices are the matrices of the following form
$$\underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(I)}}, \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(II)}} \text{ and } \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(III)}},$$
with u a unit in R and $r \in R$. The set of elementary matrices form generating set for $\mathrm{GL}(n, R)$.
(2) The group $\mathrm{GL}(n, \mathbb{Z})$ is finitely generated.

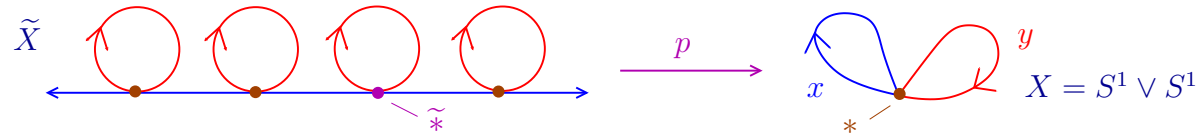
Proof.
(1) We outsource the proof to Exercise 100.3.
(2) This statement follows immediately from (1). ■

Lemma 100.6. (Generating Set–Lemma)
(1) If G is a free group on a subset S, then S is a generating set for G.
(2) If $\varphi\colon G \to H$ is an epimorphism between two groups and if S is a generating set for G, then $\varphi(S)$ is a generating set of H. In particular, if G is finitely generated, then so is H.

Proof. Both statements follow easily from the definitions. ■

Example. First recall that it follows from the Wedge-of-Circles–π_1-Lemma 99.11 that we can make the identification $\pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle$. In the figure below we show a covering $p\colon \widetilde{X} \to S^1 \vee S^1$. It follows from the Covering-π_1-Monomorphism Proposition 92.7 that $\pi_1(\widetilde{X}, \widetilde{*})$ is isomorphic to a subgroup of $\pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle$.

Using the Graph-π_1-Proposition 99.13 one can show that $\pi_1(\widetilde{X}, \widetilde{*})$ admits epimorphisms onto free groups of any rank $n \in \mathbb{N}$. Thus it follows from the Generating Set–Lemma 100.6 that $\pi_1(\widetilde{X}, \widetilde{*})$ is not finitely generated.



We have thus shown that there exists a finitely generated group that admits a generated subgroup that is not finitely generated. This is in contrast to the situation in *abelian* groups, where in the Finite Generation Lemma 97.20 (2) we showed that every subgroup of a finitely generated *abelian* subgroup is again finitely generated. □

Lemma 100.7. (Generating Set Properties–Lemma) Let G be a group and let S be a generating set for G.
(1) If $\varphi\colon G \to G$ is a homomorphism such that $\varphi(s) = s$ for all $s \in S$, then $\varphi = \mathrm{id}$.
(2) If for all $s, t \in S$ we have $s \cdot t = t \cdot s$, then G is abelian.
(3) If $\varphi\colon H \to G$ is a group homomorphism with $S \subseteq \varphi(H)$, then φ is an epimorphism.

Proof.

(1) Let $\varphi\colon G \to G$ be a homomorphism such that $\varphi(s) = s$ for every $s \in S$. We consider $H := \{g \in G \,|\, \varphi(g) = g\}$. It is straightforward to verify that H is a subgroup of G. Since H contains S it follows from the Generating Set–Characterization Lemma 100.3 (1)⇒(3) that $H = G$, i.e. $\varphi = \mathrm{id}$.
(2) We assume that for all $s, t \in S$ we have $s \cdot t = t \cdot s$. We want to show that G is abelian. We start out with the following claim:

Claim. For all $s, t \in S$ and all $\mu, \nu \in \{-1, 1\}$ we have $s^\mu \cdot t^\nu = t^\nu \cdot s^\mu$.

Proof. First we consider the case $\mu = 1$ and $\nu = -1$:

$$s \cdot t^{-1} = \underbrace{t^{-1} \cdot t}_{=e} \cdot s \cdot t^{-1} \underset{\substack{\uparrow \\ \text{since } s\cdot t = t\cdot s}}{=} t^{-1} \cdot s \cdot t \cdot t^{-1} = t^{-1} \cdot s.$$

By swapping the roles of s and t we see that the above also takes care of the case $\mu = -1$ and $\nu = 1$. Thus it remains to deal with the case $\mu = -1$ and $\nu = -1$. By

hypothesis we have $t \cdot s = s \cdot t$. Taking inverses we see that $(t \cdot s)^{-1} = (s \cdot t)^{-1}$. But this means precisely that $s^{-1} \cdot t^{-1} = t^{-1} \cdot s^{-1}$. ⊞

Now let $g, h \in G$. By the Generating Set–Characterization Lemma 100.3 (1)⇒(2) we can write $g = s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ and $h = t_1^{\tau_1} \cdot \ldots \cdot t_l^{\tau_l}$ for some $s_1, \ldots, s_k, t_1, \ldots, t_l \in S$ and $\epsilon_1, \ldots, \epsilon_k, \tau_1, \ldots, \tau_l \in \{-1, 1\}$. By applying the above claim altogether $k \cdot l$ times we see that $g \cdot h = h \cdot g$.

(3) Let $\varphi\colon H \to G$ be a group homomorphism with $S \subseteq \varphi(G)$. Note that $\varphi(H)$ is a subgroup of G. Since $S \subseteq \varphi(G)$ it thus follows from the Generating Set–Characterization Lemma 100.3 (1)⇒(3) that $\varphi(H) = G$. ■

100.2. **Presentations of groups.** In this section we introduce presentations of groups. Before we do so it is convenient to recall Definition 64.13 and Definition 98.20:

Definition 100.8. Let G be a group and let $S \subseteq G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } S \;:=\; \langle\langle S \rangle\rangle \;:=\; \bigcap_{S \subseteq H \lhd G} H.$$

If $\langle\langle S \rangle\rangle = G$, then we say that S normally generates G

Example.

(1) Let $\varphi\colon \langle x, y \rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\varphi(x) = (1, 0)$ and $\varphi(y) = (0, 1)$. In Exercise 98.34 we showed that $\ker(\varphi)$ is normally generated by a single element, namely the commutator $xyx^{-1}y^{-1}$.

(2) Let $\varphi\colon \pi \to G$ be a group epimorphism where π is finitely generated and G is finitely presented. In the Kernel–Normal Generation Proposition **??** we will see that $\ker(\varphi)$ is finitely normally generated. □

Lemma 100.9. (Normal Generator Lemma) Let G be a group and let $S \subseteq G$ be a subset.

(1) If S is a normal subgroup, then $\langle\langle S \rangle\rangle = S$.

(2) For any $T \subseteq G$ with $S \subseteq T \subseteq \langle\langle S \rangle\rangle$ we have $\langle\langle T \rangle\rangle = \langle\langle S \rangle\rangle$.

(3) Suppose that for each $s \in S$ we are given $g_s \in G$. Then $\langle\langle \{g_s \cdot s \cdot g_s^{-1}\}_{a \in S} \rangle\rangle = \langle\langle S \rangle\rangle$.

(4) Let G be a group, let $H \subseteq G$ be a subgroup and let $S \subseteq H$. If S, viewed as a subset of H, normal normally generates H, then S and H normally generate the same subgroup of G.

Proof. The proof follows easily from the definitions. We leave it to the reader to fill in the details. ■

Now we can provide the main definition of this section:

Definition 100.10. [1031]Let X be a set and let R be a subset of the free group $\langle X \rangle$ generated by the set X. We write

$$\langle X \mid R \rangle \;:=\; \langle X \rangle \,/\, \langle\langle R \rangle\rangle.$$

If $X = \{x_1, \ldots, x_k\}$ and $R = \{r_1, \ldots, r_l\}$, then we use the convenient shorthand

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;:=\; \langle \{x_1, \ldots, x_k\} \,|\, \{r_1, \ldots, r_l\} \rangle.$$

On many occasions we will use that the Generating Set–Lemma 100.6 implies that X is a generating set of $\langle X \mid R \rangle$.

Examples.

(1) For $k \in \mathbb{N}_0$ we have
$$\langle g \mid g^k \rangle \;=\; \langle g \rangle / \langle\langle g^k \rangle\rangle \;\underset{\uparrow}{=}\; \langle g \rangle / \{ g^{k \cdot n} \mid n \in \mathbb{Z} \} \;\cong \mathbb{Z}/k\mathbb{Z}.$$
this follows easily from the observation that $\langle g \rangle$ is abelian

(2) The group $\langle x_1, \ldots, x_k \mid \rangle$ with $R = \varnothing$ is just the free group on $x_1, \ldots, x_k$. □

Convention 100.11. Given a group $\langle X \mid R \rangle$ and given $g \in \langle X \rangle$ we denote the images of the g in the quotient $\langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ again by g.

As we will now see, finite presentations of groups are useful for finding group homomorphisms. To formulate the corresponding lemma we need the following definition:

Definition 100.12. Let $r \in \langle x_1, \ldots, x_k \rangle$. Furthermore let G be a group and let $g_1, \ldots, g_k$ be elements of G. We say that the elements $g_1, \ldots, g_k$ satisfy the relation r if the natural homomorphism[1032] $\langle x_1, \ldots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \ldots, k$ sends r to the trivial element.[1033]

Example.

(1) We consider the word $r = [x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1} \in \langle x_1, x_2 \rangle$. If a group G is abelian, then any two elements $g_1, g_2 \in G$ satisfy the relation $x_1 x_2 x_1^{-1} x_2^{-1}$.

(2) Let G be a finite group with d elements. It follows from Lagrange's Theorem, see e.g. [**KuS2004**, p. 8], that every element in G satisfies the relation $x^d \in \langle x \rangle$. □

The following lemma, which is a very mild generalization of the Presentation–Homomorphism Lemma 64.15, is the key to finding homomorphisms out of finitely presented groups:

Lemma 100.13. (Presentation–Homomorphism Lemma) Let π be a group with a finite presentation $\pi = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ and let G be a group.

(1) If $g_1, \ldots, g_k \in G$ satisfy the relations $r_1, \ldots, r_l$, then there exists a unique homomorphism
$$\beta \colon \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle \;\to\; G$$
with $\beta(x_i) = g_i$ for $i = 1, \ldots, k$.

(2) Every homomorphism $\langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle \to G$ is of the form described in (1).

Remark. An equivalent formulation of the Presentation–Homomorphism Lemma 100.13 is to say that the map
$$\begin{array}{ccc} \mathrm{Hom}(\pi, G) & \to & \{\text{all } (g_1, \ldots, g_k) \in G^k \text{ that satisfy } r_1, \ldots, r_l\} \\ f & \mapsto & (f(x_1), \ldots, f(x_k)) \end{array}$$
is a bijection. Note that if G is a finite group, then the right-hand side is a finite set. At least in this case it is usually straightforward, albeit tedious, to determine the right-hand side. □

Proof.

[1031] In fact we encountered this notion already in Definition 100.10.

[1032] The fact that there exists a unique such homomorphism is a consequence of the Free Group–Existence Lemma 98.9.

[1033] More colloquially speaking: $g_1, \ldots, g_k$ satisfy a relation r if replacing the x_i by g_i in the word r returns the trivial element in G.

(1) Let $g_1, \dots, g_k \in G$. By the Free Group–Existence Lemma 98.9 there exists a unique homomorphism $\alpha\colon \langle x_1, \dots, x_k\rangle \to G$ with $\alpha(x_i) = g_i$ for $i = 1, \dots, k$. If $g_1, \dots, g_k \in G$ satisfy the relations $r_1, \dots, r_l$, then it follows from the Normal Closure–Factorization Lemma 98.21 that α descends to a unique homomorphism $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle \to G$, which clearly has the desired properties.
(2) This statement is basically obvious. ∎

Example.

(1) We consider the group $\mathrm{Homeo}(\mathbb{R}^2)$ of self-homeomorphisms of $\mathbb{R}^2$ and we consider the following two elements of this group:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s, t+1). \end{array}$$

We consider the free group $\langle x, y\rangle$. Note that there exists a unique homomorphism $\alpha\colon \langle x, y\rangle \to \mathrm{Homeo}(\mathbb{R}^2)$ with $\alpha(x) = A$ and $\alpha(y) = B$. By the Klein Bottle-as-Quotient Lemma 91.2 and the Generating Set–Characterization Lemma 100.3 (2)⇒(1) we know that $G \coloneqq \mathrm{im}(\alpha)$ is isomorphic to the fundamental group of the Klein bottle.

Note that $\alpha\colon \langle x, y\rangle \to G$ is not an isomorphism since for any $(s,t) \in \mathbb{R}^2$ we have

$$\begin{aligned} \alpha(xyx^{-1}y)(s,t) &= ABA^{-1}B(s,t) = ABA^{-1}(s,t+1) = AB(s-1, 1-(t+1)) \\ &= A(s-1, 1-t) = s,t), \end{aligned}$$

in other words $xyx^{-1}y \in \ker(\alpha)$. By the Presentation–Homomorphism Lemma 100.13 this means that α descends to an epimorphism

$$\beta\colon \langle x, y\rangle / \langle\langle xyx^{-1}y\rangle\rangle = \langle x, y \mid xyx^{-1}y\rangle \to G.$$

Is this homomorphism β also a monomorphism? We will come back to this question later on page 1977.

(2) We consider the groups $\langle a, b \,|\, aba^{-1}b\rangle$ and $\langle u, v \,|\, u^2 = v^2\rangle$. It follows from the Presentation–Homomorphism Lemma 100.13 that there exist unique homomorphisms

$$\begin{array}{rcl} \Phi\colon \langle a, b \,|\, aba^{-1}b\rangle & \to & \langle u, v \,|\, u^2 = v^2\rangle \\ a & \mapsto & v \\ b & \mapsto & v^{-1} \cdot u \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \langle u, v \,|\, u^2 = v^2\rangle & \to & \langle a, b \,|\, aba^{-1}b\rangle \\ u & \mapsto & ab \\ v & \mapsto & a. \end{array}$$

Using the Generating Set Properties–Lemma 100.7 (1) and (2) one can easily show that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$, thus both are isomorphisms. □

The following lemma provides us with a particularly interesting example of a presentation:

Lemma 100.14. (Free Abelian Group-Presentation Lemma) Let $m \in \mathbb{N}$. There exists a unique isomorphism[1034]

$$\alpha\colon \langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } 1 \le i < j \le m\rangle \xrightarrow{\cong} \mathbb{Z}^m$$

which sends each x_i to the i-th standard basis vector e_i of $\mathbb{Z}^m$.

Proof. We set $X = \{x_1, \dots, x_m\}$ and $R = \{[x_i, x_j] \,|\, 1 \le i < j \le m\}$. Since $\mathbb{Z}^m$ is abelian we obtain from the Presentation–Homomorphism Lemma 100.13 that there exists a unique homomorphism $\alpha\colon \langle X \,|\, R\rangle \to \mathbb{Z}^m$ with $\alpha(x_i) = e_i$. Since the image of α contains

[1034] In Definition 98.13 we introduced for two elements g, h of a group G the commutator $[x, y] \coloneqq xyx^{-1}y^{-1}$.

the generating set $\{e_1, \dots, e_m\}$ of $\mathbb{Z}^m$ we obtain from the Generating Set Properties–Lemma 100.7 (3) that α is an epimorphism. Thus it remains to show that α is a monomorphism.

Claim. The group $\langle X \mid R \rangle$ is abelian.

Proof. Recall that it follows from the Generating Set–Lemma 100.6 that $X = \{x_1, \dots, x_m\}$ is a generating set for $\langle X \mid R \rangle$. Let $i < j$. We have the following equality in $\langle X \mid R \rangle$:

$$x_j \cdot x_i \;=\; \underbrace{x_i \cdot x_j \cdot x_i^{-1} \cdot x_j^{-1}}_{=[x_i,x_j]=e\in\langle X \mid R\rangle} \cdot x_j \cdot x_i \;=\; x_i \cdot x_j.$$

This shows that the elements of the generating set commute. Thus it follows from the Generating Set Properties–Lemma 100.7 (2) that the group $\langle X \mid R \rangle$ is abelian. ⊞

As we mentioned on page 1879 (2), the group $\mathbb{Z}^m$ is free abelian on the set $\{\mathfrak{e}_1, \dots, \mathfrak{e}_m\}$. We just learned in the above claim that $\langle X \mid R \rangle$ is abelian. This implies in particular that there exists a unique homomorphism $\beta\colon \mathbb{Z}^m \to \langle X \mid R \rangle$ with $\beta(\mathfrak{e}_i) = x_i$. Note that for each $i \in \{1, \dots, m\}$ we have $(\beta \circ \alpha)(x_i) = \beta(\alpha(x_i)) = \beta(\mathfrak{e}_i) = x_i$. Thus it follows from the Generating Set Properties–Lemma 100.7 (1) that $\beta \circ \alpha = \mathrm{id}$. We have thus shown that $\alpha\colon \langle X \mid R \rangle \to \mathbb{Z}^m$ is a monomorphism. ■

We continue with the following definitions, which generalize some of the concepts from Definition 64.16.

Definition 100.15.

(1) A presentation is a pair (X, R) consisting of a set X together with a subset $R \subseteq \langle X \rangle$. We call the elements of X the generators and we call the elements of R the relators[1035] of the presentation.

(2) Let (X, R) be a representation. As in Definition 100.10 we consider the corresponding group

$$\langle X \mid R \rangle \;\coloneqq\; \langle X \rangle \,/\, \langle\langle R \rangle\rangle.$$

On many occasions we will just use the notation $\langle X \mid R \rangle$ to denote a presentation and its corresponding group.

(3) Let π be a group. A presentation for π is a presentation (X, R) together with an isomorphism

$$\langle X \mid R \rangle \;\xrightarrow{\cong}\; \pi.$$

Very often one just says $\langle X \mid R \rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

(4) We say that a group π is finitely presented, if π admits a presentation with finitely many generators and finitely many relators.

Example.

(1) Every group admits a presentation. Indeed, let π be a group. Then we set $X \coloneqq \pi$, i.e. X is the set of elements of π. By the Free Group–Existence Lemma 98.9 there exists a unique homomorphism $\alpha\colon \langle X \rangle \to \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set

[1035] The relators are often also called relations.

$R \coloneqq \ker(\alpha)$. Then

$$\langle X \mid R\rangle \;=\; \langle X\rangle/\langle\langle R\rangle\rangle \;=\; \langle X\rangle/\langle\langle \ker(\alpha)\rangle\rangle \;\underset{\uparrow}{=}\; \langle X\rangle/\ker(\alpha) \;\xrightarrow[\alpha]{\cong}\; \pi.$$

by the Normal Generator Lemma 100.9 (1), since $R = \ker(\alpha)$ is a normal subgroup

(2) Using a modification of the argument in (1) one sees that a group π is finitely generated, in the sense of Definition 100.2, if and only if π admits a presentation $\langle X \mid R\rangle \to \pi$ such that X is finite.

(3) We had just seen in the Free Abelian Group-Presentation Lemma 100.14 that

$$\mathbb{Z}^n \;\cong\; \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ with } 1 \le i < j \le n\rangle.$$

This shows that $\mathbb{Z}^n$ is a finitely presented group.

(4) There exist groups that are finitely generated but not finitely presented. In fact it follows easily from the Countability Lemma 1.37 that there are only countably many isomorphism types of finitely presented groups. On the other hand it is shown in [**Rob1995**, Chapter 14] that there are uncountably many isomorphism types of finitely generated groups.[10361037]

(5) We saw on page 1957 that the fundamental group of the Klein bottle is finitely generated. But is it also finitely presented? □

Let us recall Definition 98.15:

Definition 100.16. Let π be a group. The commutator subgroup of π is defined as

$$[\pi,\pi] \;\coloneqq\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \;\middle|\; x_1, y_1, \dots, x_n, y_n \in \pi \right\}$$

We call $\pi_{\mathrm{ab}} \coloneqq \pi/[\pi,\pi]$ the abelianization of π.

Proposition 100.17. (Abelianization-of-Presentation Proposition) Suppose we are given a finitely presented group $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$. We denote by

$$\Phi\colon \langle x_1, \dots, x_m\rangle \;\to\; \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism[1038]

$$\Psi\colon \underbrace{\pi/[\pi,\pi]}_{=\pi_{\mathrm{ab}}} \;\xrightarrow{\cong}\; \mathbb{Z}^m/\operatorname{span}\{\Phi(r_1), \dots, \Phi(r_n)\}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\langle x_1, \dots, x_m\rangle & \xrightarrow[\Phi]{x_i \mapsto e_i} & \mathbb{Z}^m \\
\downarrow & & \downarrow \\
\pi = \langle x_1, \dots, x_m\rangle/\langle\langle r_1, \dots, r_n\rangle\rangle & & \\
\downarrow & & \downarrow \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow[\cong]{\Psi} & \mathbb{Z}^m/\operatorname{span}\{\Phi(r_1), \dots, \Phi(r_n)\}.
\end{array}$$

[1036]The idea is that in a free group $\langle x_1, \dots, x_k\rangle$ there are uncountably many infinite sequences of elements $\{r_1, r_2, \dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

[1037]Note that this also shows that any free group on ≥ 2 generators contains uncountably many different subgroups.

Example. We consider the group $\pi = \langle x, \mid x^2y^{-2}x, y^{-1}x^3yx\rangle$.

Let $\Phi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1, 0)$ and $\Phi(y) = (0, 1)$. It follows from the Abelianization-of-Presentation Proposition 100.17 that the abelianization of π is given by

$$\pi_{\mathrm{ab}} = \mathbb{Z}^2/\operatorname{span}\{\underbrace{\Phi(x^2y^{-2}x)}_{\binom{3}{-2}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\binom{4}{0}}\} = \mathbb{Z}^2/\begin{pmatrix}3 & 4\\ -2 & 0\end{pmatrix}\mathbb{Z}^2 \cong \mathbb{Z}/8\mathbb{Z}.$$

can be calculated using the Smith normal form of the matrix □

Proof. It follows from the Quotient Group Lemma 2.4 there exists a unique homomorphism

$$\Psi\colon \pi/[\pi,\pi] \to \mathbb{Z}^m/\operatorname{span}\{\Phi(r_1), \dots, \Phi(r_n)\}$$

with $\Psi([x_i]) = [e_i]$, i.e. that makes the diagram in the proposition commute. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1, \dots, x_m\rangle$ and we write $G = \langle y_1, \dots, y_n\rangle$. We consider the following diagram:

$$\begin{array}{ccc}
G_{\mathrm{ab}} = G/[G,G] & \xrightarrow[\cong]{[y_i]\mapsto e_i} & \mathbb{Z}^n \\
{\scriptstyle [y_i]\mapsto [r_i]}\downarrow & & \downarrow{\scriptstyle e_i\mapsto \Phi(r_i)} \\
F_{\mathrm{ab}} = F/[F,F] & \xrightarrow[\cong]{[x_i]\mapsto e_i} & \mathbb{Z}^m \\
\twoheaddownarrow & & \twoheaddownarrow \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow{\Psi} & \mathbb{Z}^m/\operatorname{span}\{\Phi(r_1), \dots, \Phi(r_n)\}.
\end{array}$$

Note that the diagram commutes. It follows from the Abelianization Coproduct Proposition 98.18 (2) that the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram chase" that the bottom horizontal map is also an isomorphism.[1039] We leave the details to the reader. ■

100.3. **Presentations of groups and constructions of groups.** In this section we will see how various familiar constructions of groups interact with presentations.

Lemma 100.18. (Quotient Presentation–Lemma) Let $G = \langle X \mid R\rangle$ be a group and let $\varphi\colon G \to H$ be an epimorphism. If $S \subseteq \langle X\rangle$ is a subset, such that $\ker(\varphi) = \langle\langle\{[s] \mid s \in S\}\rangle\rangle$, then

[1038] By $\operatorname{span}\{v_1, \dots, v_n\}$ we mean the set of all linear combinations $\sum_{i=1}^{n} \lambda_i \cdot v_i$ with $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$.

[1039] More precisely, we use the following fact: suppose that we are given a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\cong} & A' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\cong} & B' \\
\twoheaddownarrow & & \twoheaddownarrow \\
C & \longrightarrow & C'
\end{array}$$

of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a "diagram chase". This statement can also be viewed as "one half" of the Five Lemma 145.11.

$$\begin{array}{rcl} \langle X \mid R \cup S \rangle & \to & H \\ x & \mapsto & \varphi([x]) \end{array}$$

is a presentation for H.

Proof. We consider the following diagram:

$$\begin{array}{ccccc} \langle X \rangle & & \xrightarrow{\quad\mathrm{id}\quad} & & \langle X \rangle \\ \downarrow & & & & \downarrow \\ & & & & \langle X \mid R \rangle \\ & & & & \downarrow \\ \langle X \mid R \cup S \rangle & & \longrightarrow & \langle X \mid R \rangle / \ker(\varphi) & \xrightarrow{[g] \mapsto \varphi(g)} H. \end{array}$$

It follows easily from the Quotient Group Lemma 2.4 that the identity homomorphism on the top (in both directions) descends to homomorphisms (in both directions) between the groups on the bottom left. It follows from the Generating Set Properties–Lemma 100.7 (1) that the two homomorphisms are inverses of one another, in particular the homomorphism from the left to the center is an isomorphism. The lemma now follows from the observation that the homomorphism to the bottom right is an isomorphism, since φ is an epimorphism. ■

Before we give our next example, and before we formulate further results on propositions let us introduce the following intuitive notation.

Notation 100.19. Given a set X we sometimes write a relator in the form $a = b$ for $a, b \in \langle X \rangle$, this just means that the relator is given by $a \cdot b^{-1} \in \langle X \rangle$.

Example. Let A be a finitely generated abelian group. By the Abelian Group-Classification Theorem 97.24 there exist $a_1, \ldots, a_n \in \mathbb{N}_0$ such that $A \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z}$. In other words, we can write A as a quotient of $\mathbb{Z}^n$. Using the Abelianization-of-Presentation Proposition 100.17 and the Quotient Presentation–Lemma 100.18 one can now easily verify that

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}/a_i\mathbb{Z} \cong \langle x_1, \ldots, x_n \mid [x_i, x_j] = e \text{ for } 1 \leq i < j \leq n \text{ and } x_i^{a_i} = e \text{ for } i = 1, \ldots, n \rangle.$$

This shows that every finitely generated abelian group is also finitely presented. □

The following lemma gives us a convenient way to find a presentation of the amalgamated product of two groups.

Lemma 100.20. (Amalgamated Product–Presentation Lemma) Let $A = \langle X \mid R \rangle$ and $B = \langle Y \mid S \rangle$ be two groups with $X \cap Y = \varnothing$. Furthermore let G be a group with generating set Z and let $\alpha \colon G \to A$ and $\beta \colon G \to B$ be two group homomorphisms. There is a natural isomorphism

$$\langle X \sqcup Y \mid R \sqcup S \sqcup \{\alpha(z) = \beta(z) \mid z \in Z\} \rangle \xrightarrow{\cong} A *_G B.$$

In particular A and B are finitely presented and if G is finitely generated, then $A *_G B$ is finitely presented.

Example.

(1) The case $G = \{e\}$ in the Amalgamated Product–Presentation Lemma 100.20, together with the Amalgamated Product Lemma 98.24 (1), gives us presentations of free products of two groups.
(2) Let $A = \langle a \rangle$ and let $B = \langle b \rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y] \rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha\colon G \to A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta\colon G \to B$ that is given by $\beta(x) = e$ and $\beta(y) = b$. Then, similar to the discussion on page 1911, we see that

$$A *_G B \underset{\substack{\uparrow \\ \text{by the Amalgamated Product} \\ \text{-Presentation Lemma 100.20}}}{=} \langle a, b \mid \alpha(x) = \beta(x), \alpha(y) = \beta(y) \rangle = \langle a, b \mid a = e, e = b \rangle = \langle a, b \mid a, b \rangle = \{e\}.$$

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial. □

Proof of the Amalgamated Product–Presentation Lemma 100.20. We first consider the setting of the free product $\langle X \mid R \rangle * \langle Y \mid S \rangle$. We consider the following diagram:

$$\begin{array}{ccccc} \langle X \sqcup Y \rangle & \longrightarrow & \langle X \rangle * \langle Y \rangle & \longrightarrow & \langle X \sqcup Y \rangle \\ \downarrow & & \downarrow & & \downarrow \\ \langle X \sqcup Y \mid R \cup S \rangle & \xrightarrow{\alpha} & \langle X \mid R \rangle * \langle Y \mid S \rangle & \xrightarrow{\beta} & \langle X \sqcup Y \mid R \cup S \rangle. \end{array}$$

We make the following observations and clarifications:
(1) The top horizontal maps are given by the universal properties of free groups and free products.
(2) The vertical maps are (induced by) the natural projections.
(3) The bottom left horizontal map is induced by the Normal Closure–Factorization Lemma 98.21.
(4) It follows from the Normal Closure–Factorization Lemma 98.21 that the homomorphism $\langle X \rangle \to \langle X \sqcup Y \mid R \cup S \rangle$ descends to a homomorphism starting at $\langle X \mid R \rangle$. The same way we obtain a homomorphism starting at $\langle Y \mid S \rangle$. These two homomorphisms give us the bottom right horizontal homomorphism.
(5) Using the Generating Set–Lemma 100.6 and the Generating Set Properties–Lemma 100.7 (1) one can easily verify that $\beta \circ \alpha = \mathrm{id}$ and $\alpha \circ \beta = \mathrm{id}$.

We have thus proved the desired statement for the case of a free product. The case of a non-trivial amalgamated product follows easily from the above discussion together with the Quotient Presentation–Lemma 100.18. ■

Next we want to find presentations for semidirect products.

Lemma 100.21. (Semidirect Product-Presentation Lemma) Let $\pi = \langle X \mid R \rangle$ be a presentation and let $\varphi\colon \pi \to \pi$ be an isomorphism. A presentation for the semidirect product[1040] $\pi \rtimes_\varphi \mathbb{Z}$ is given by the isomorphism[1041]

$$\begin{array}{rcl} \Phi\colon \langle X, t \mid R, t \cdot x \cdot t^{-1} = \varphi(x) \text{ for all } x \in X \rangle & \to & \pi \rtimes_\varphi \mathbb{Z} \\ x & \mapsto & (x, 0) \\ t & \mapsto & (0, 1). \end{array}$$

In particular if π admits a finite presentation, then so does $\pi \rtimes_\varphi \mathbb{Z}$.

Example.

(1) Let K be the Klein bottle. On page 1818 we showed that $\pi_1(K) \cong \langle x \rangle \rtimes_\varphi \mathbb{Z}$ where $\varphi(x) = x^{-1}$. Thus we obtain from the Semidirect Product-Presentation Lemma 100.21 that a presentation for $\pi_1(K)$ is given by $\langle x, t \,|\, t \cdot x \cdot t^{-1} = x^{-1} \rangle$.

(2) Let $n \in \mathbb{N}_0$. As on page 1815 we consider the dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$ where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by multiplication by -1 on $\mathbb{Z}/n\mathbb{Z}$. Note that it follows easily from the Semidirect Product-Presentation Lemma 100.21 that a presentation for the dihedral group D_n is given by $\langle x, t \,|\, x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle$. □

Proof. Recall that the multiplication in the semidirect product $\pi \rtimes_\varphi \mathbb{Z}$ is given by

$$(g, m) \cdot (g', m') \coloneqq (g \cdot \varphi^m(g'), m + m').$$

One can now easily verify, using the Presentation–Homomorphism Lemma 100.13, that the homomorphism Φ in the lemma does actually exist. Next we consider the map

$$\begin{array}{rcl} \Psi \colon \pi \rtimes_\varphi \mathbb{Z} & \to & \langle X, t \,|\, R, txt^{-1} = \varphi(x) \text{ for all } x \in X \rangle \\ (g, n) & \mapsto & g \cdot t^n. \end{array}$$

Again one can show, with a little bit of an effort, that Ψ is a homomorphism. It is straightforward to verify that $\Phi \circ \Psi = \mathrm{id}$. Finally note that it follows almost immediately from the Generating Set Properties–Lemma 100.7 (1) that $\Psi \circ \Phi = \mathrm{id}$. ■

We conclude this section with an interesting little group-theoretic lemma. It will play an important role later in the proof of the Topological Manifolds-Invariants Proposition 189.14 (4b). But it is also of independent interest. For the formulation of the lemma we need the following harmless definition.

Definition 100.22. Let G and H be groups. We say that H is a retract of G if there exist a group homomorphism $\varphi \colon H \to G$ and $\rho \colon G \to H$ such that $\rho \circ \varphi = \mathrm{id}_H$.

Lemma 100.23. (Finite Presentation-Retract Lemma) Every retract of a finitely presented group is also finitely presented.

Proof. Let H be a group, let G be a finitely presented group and let $\varphi \colon H \to G$ and $\rho \colon G \to H$ be group homomorphisms such that $\rho \circ \varphi = \mathrm{id}_H$. We need to show that H is also finitely presented.

To do so we pick a finite presentation $G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ for the group G. We write $F = \langle g_1, \dots, g_k \rangle$. In the following we denote by $p \colon F \to G$ the obvious epimorphism. For each $i \in \{1, \dots, k\}$ we pick $w_i \in F$ with $p(w_i) = (\varphi \circ \rho \circ p)(g_i) \in G$. It suffices to prove the following claim:

Claim. A finite presentation for H is given by $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l, g_1^{-1} w_1, \dots, g_k^{-1} w_k \rangle$.

Proof. Let L be the group that is defined by the finite presentation of the claim. We denote by $u \colon G \to L$ and $q \colon F \to L$ the natural projections. Note that for each $j \in \{1, \dots, l\}$ we

[1040] We refer to Definition 93.11 for the definition of the semidirect product.

[1041] To be more precise, for $x \in X$ we pick an $\widetilde{x} \in \langle X \rangle$ such that $[\widetilde{x}] = \varphi(x)$ and the relation is given by $txt^{-1} = \widetilde{x}$. In practice our shorthand should not pose a problem.

have $\rho(p(r_i)) = \rho(e) = e$ and for each $i \in \{1, \dots, k\}$ we have

$$\rho(p(g_i^{-1} w_i)) \;=\; \rho(p(g_i^{-1})) \cdot \rho(p(w_i)) \;\underset{\substack{\uparrow \\ \text{since } p(w_i) = (\varphi \circ \rho \circ p)(g_i) \text{ and} \\ \text{since } \rho \circ \varphi = \mathrm{id}_H}}{=}\; \rho(p(g_i^{-1})) \cdot \rho(p(g_i)) \;=\; e.$$

It follows from the Normal Closure–Factorization Lemma 98.21 that the homomorphism $p \circ \varphi \colon F \to H$ factors through a homomorphism $v \colon L \to H$.

We want to show that $u \circ \varphi$ is an isomorphism. First note that

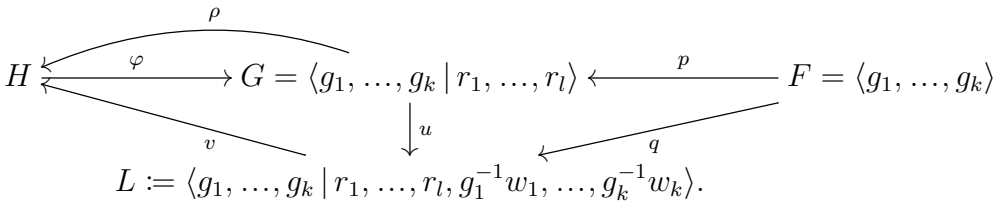
$$v \circ (u \circ \varphi) \;=\; (v \circ u) \circ \varphi \;\underset{\substack{\uparrow \\ \text{since } \rho = v \circ u}}{=}\; \rho \circ \varphi \;=\; \mathrm{id}_H .$$

By Lemma 99.6 it remains to show that $u \circ \varphi$ is an epimorphism. Note that for $i = 1, \dots, k$ we have

$$(u \circ \varphi)(\underbrace{(\rho \circ p)(w_i)}_{\in H}) \;\underset{\substack{\uparrow \\ \text{since } \varphi \circ \rho = \mathrm{id}_H}}{=}\; u(p(w_i)) \;=\; q(w_i) \;\underset{\substack{\uparrow \\ \text{since } q(w_i) = g_i \in L}}{=}\; g_i \;\in\; L.$$

Since $g_1, \dots, g_k$ is a generating set for L we see that $u \circ \varphi$ is indeed an epimorphism.

For convenience we summarize all objects in the following diagram:



■

100.4. **Tietze transformations.** We have now seen that many groups can be described by finite presentations. The question arises, to what degree are finite presentations unique? To answer this question we introduce the following definition:

Definition 100.24. Let

$$\phi \colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\xrightarrow{\cong}\; \pi$$

be a presentation of a group π. We obtain new presentations of π as follows:

(T1) For any element s of $\langle\langle r_1, \dots, r_l \rangle\rangle \subseteq \langle X \rangle$ we obtain the presentation

$$\langle x_1, \dots, x_k \mid r_1, \dots, r_l, s \rangle \;\underset{\substack{\uparrow \\ \text{it follows from } s \in \langle\langle r_1, \dots, r_l \rangle\rangle \text{ that} \\ \text{the two groups are identical}}}{=}\; \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle \;\xrightarrow{\phi}\; \pi.$$

(T2) For any $x \notin \{x_1, \dots, x_k\}$ and any two words $s, t \in \langle x_1, \dots, x_k \rangle$ we obtain the presentation[1042]

$$\begin{aligned} \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, s \cdot x \cdot t \rangle \;&\to\; \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle \;\xrightarrow{\phi}\; \pi. \\ x_i \;&\mapsto\; x_i \\ x \;&\mapsto\; s^{-1} \cdot t^{-1} \end{aligned}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called Tietze transformations.[1043]

[1043] First note that the map to the left is well-defined. This can be seen using the Presentation–Homomorphism Lemma 100.13 and the observation that under the assignment $x_i \mapsto x_i$ and $x \mapsto s^{-1} \cdot t^{-1}$

Example. We want to show that the group $\langle a, b, c \,|\, (ab)^2ab^2 \rangle$ is a free group on two generators. Borrowing an argument from [**MKS1976**, p. 50] we see this as follows:

$$\begin{aligned}
\langle a, b, c \,|\, (ab)^2ab^2 \rangle &\overset{\text{(T2),(T2)}}{\cong} \langle a, b, c, x, y \,|\, (ab)^2ab^2, x = ab, y = ab^2 \rangle \\
&\overset{\text{(T1)}}{\cong} \langle a, b, c, x, y \,|\, (ab)^2ab^2, x = ab, y = ab^2, x^2y \rangle \\
&\overset{\text{(T1)}^{-1}}{\cong} \langle a, b, c, x, y \,|\, x = ab, y = ab^2, x^2y \rangle \\
&\overset{\text{(T1),(T1)}}{\cong} \langle a, b, c, x, y \,|\, x = ab, y = ab^2, x^2y, b = x^{-1}y, a = xy^{-1}x \rangle \\
&\overset{\text{(T2)}^{-1},\text{(T2)}^{-1}}{\cong} \langle c, x, y \,|\, x = xy^{-1}x \cdot x^{-1}y, y = xy^{-1}x(x^{-1}y)(x^{-1}y), x^2y \rangle \\
&\overset{\text{(T2)}^{-1}}{\cong} \langle c, x, y \,|\, x^2y \rangle \overset{\text{(T1),(T1)}^{-1}}{\cong} \langle c, x, y \,|\, y = x^{-2} \rangle \overset{\text{(T2)}^{-1}}{\cong} \langle c, x \,|\, \rangle.
\end{aligned}$$

As we can see, working with Tietze transformations leads too long and unreadable arguments. □

The above Tietze transformations show that from a given finite presentation we can obtain infinitely many new finite presentations. This fact is not that surprising. What is much more interesting is the following theorem which was first proved by Heinrich Tietze [**Tie1908**] in 1908.

Theorem 100.25. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

The remainder of this section is occupied with the proof of the Tietze Theorem 100.25. The proof of the Tietze Theorem 100.25 builds on the following four elementary auxiliary lemmas.

Auxiliary Lemma 100.26. Let $\theta\colon G \to H$ be a group homomorphism and let X be a generating set for G. The subgroup of $G * H$ that is normally generated by the set $\{g \cdot \theta(g^{-1}) \,|\, g \in G\}$ equals the subgroup of $G * H$ that is normally generated by the set $\{x \cdot \theta(x^{-1}) \,|\, x \in X\}$.

Proof of Lemma 100.26. We set $N \coloneqq \{x \cdot \theta(x^{-1}) \,|\, x \in X\}$. We only need to show that given any $g \in G$ we have $g \cdot \theta(g^{-1}) \in \langle\langle N \rangle\rangle$. This statement follows easily by induction on the length of g as a word in the generating set X and the observation that for $x, y \in X$ we have the following equality:

$$xy \cdot \theta((xy)^{-1}) \;=\; xy \cdot \theta(y^{-1}) \cdot \theta(x^{-1}) \;=\; x \cdot \underbrace{y \cdot \theta(y)^{-1}}_{\in N} \cdot x^{-1} \cdot \underbrace{x \cdot \theta(x^{-1})}_{\in N}.$$

■

Auxiliary Lemma 100.27. Let G and H be two groups, let $\theta\colon G \to H$ be an isomorphism and let $\alpha\colon G \to \pi$ be an isomorphism. We consider the homomorphism

$$\Psi\colon G * H \;\to\; \pi$$

defined by α on G and by $\alpha \circ \theta^{-1}$ on H. The kernel of the homomorphism Ψ is generated by $\{g \cdot \theta(g^{-1}) \,|\, g \in G\}$.

---

the relation $s \cdot x \cdot t$ gets sent to $s \cdot s^{-1} \cdot t^{-1} \cdot t = e$. The map to the left is also an isomorphism. In fact, as one can verify using the Generating Set Properties–Lemma 100.7 (1), an inverse is given by $x_i \mapsto x_i$.

Proof of Lemma 100.27. We set $M \coloneqq \{g \cdot \theta(g^{-1}) \,|\, g \in G\}$. We need to show that the kernel of the homomorphism Ψ is generated by M. In the following we consider elements $a_1 \cdot \ldots \cdot a_k \in \ker(\Psi) \subseteq G * H$ where the a_i alternate between elements of G and elements of H. If $k = 1$, then it follows from the fact that α and θ are isomorphisms that a_1 is trivial and we are done. Now suppose that $k \geq 2$. If $a_1 \in G$, then we can write

$$\underbrace{a_1 \cdot a_2 \cdot a_3 \cdot \ldots a_k}_{k \text{ terms}} = \underbrace{a_1 \cdot \theta({a_1}^{-1})}_{\in M} \cdot \underbrace{\underbrace{\theta(a_1) \cdot a_2}_{\in H} \cdot a_3 \cdot \ldots a_k}_{k-1 \text{ terms}}.$$

In basically the same we proceed if $a_1 \in H$. Iterating this procedure we can reduce the situation to the case $k = 1$, which we have already dealt with. ∎

Auxiliary Lemma 100.28. Let

$$\phi\colon \langle x_1, \ldots, x_k \,|\, r_1(x_1, \ldots, x_k), \ldots, r_l(x_1, \ldots, x_k)\rangle \;\to\; \pi$$

be a presentation. For pairwise different $y_1, \ldots, y_k$ the presentation

$$\begin{aligned} \langle y_1, \ldots, y_k \,|\, r_1(y_1, \ldots, y_k), \ldots, r_l(y_1, \ldots, y_k)\rangle &\;\to\; \pi \\ y_i &\;\mapsto\; \phi(x_i) \end{aligned}$$

can be obtained from the original one by Tietze transformations.

Proof of Lemma 100.28. We consider the Tietze transformations

$$\begin{aligned} \langle x_1, \ldots, x_k \,|\, r_1(x_1, \ldots, x_k), \ldots\rangle &\overset{(T2)}{\rightsquigarrow} \langle x_1, \ldots, x_k, y_1 \,|\, r_1(x_1, \ldots, x_k), \ldots, y_1 \cdot x_1^{-1}\rangle \\ &\overset{(T1)}{\rightsquigarrow} \langle x_1, \ldots, x_k, y_1 \,|\, r_1(x_1, \ldots, x_k), \ldots, y_1 \cdot x_1^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ &\overset{(T1)}{\rightsquigarrow} \langle x_1, \ldots, x_k, y_1 \,|\, y_1 \cdot x_1^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ &\overset{(T1)}{\rightsquigarrow} \langle x_1, \ldots, x_k, y_1 \,|\, x_1 \cdot {y_1}^{-1}, y_1 \cdot x_1^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ &\overset{(T1)}{\rightsquigarrow} \langle x_1, \ldots, x_k, y_1 \,|\, x_1 \cdot {y_1}^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ &\overset{(T2)}{\rightsquigarrow} \langle x_2, \ldots, x_k, y_1 \,|\, r_1(y_1, \ldots, x_k), \ldots\rangle. \end{aligned}$$

Iterating this process leads to the desired sequence of Tietze transformations. ∎

Auxiliary Lemma 100.29. Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be group epimorphisms. Let $X \subseteq A$ be a subset such that $\ker(\alpha) = \langle\langle X\rangle\rangle$ and let $Y \subseteq A$ be a subset such that $\ker(\beta) = \langle\langle \alpha(Y)\rangle\rangle$. Then $\ker(\beta \circ \alpha\colon A \to C) = \langle\langle X \cup Y\rangle\rangle$.

Proof of Lemma 100.29. It is clear that $X \cup Y \subseteq \ker(\beta \circ \alpha\colon A \to C)$. To prove the reverse inclusion let $g \in \ker(\beta \circ \alpha\colon A \to C)$. We have $\alpha(g) \in \ker(\beta) = \langle\langle \alpha(Y)\rangle\rangle$. Since α is an epimorphism the map $\alpha\colon \langle\langle Y\rangle\rangle \to \langle\langle \alpha(Y)\rangle\rangle$ is an epimorphism. Thus there exists an $h \in \langle\langle Y\rangle\rangle$ with $\alpha(g) = \alpha(h)$. But this implies that $g \cdot h^{-1} \in \ker(\alpha) = \langle\langle X\rangle\rangle$. It follows that $g \in \langle\langle X \cup Y\rangle\rangle$. ∎

Proof of the Tietze Theorem 100.25. Let π be a group and let $\phi_1\colon \langle X_1 \,|\, R_1\rangle \xrightarrow{\cong} \pi$ and $\phi_2\colon \langle X_2 \,|\, R_2\rangle \xrightarrow{\cong} \pi$ be two finite presentations for π. We need to show that they are related by a finite sequence of Tietze transformations. First note that it follows from Lemma 100.28 that by applying a sequence of Tietze transformations we can arrange that $X_1 \cap X_2 = \varnothing$.

We denote by $p_1\colon \langle X_1\rangle \to \langle X_1 \,|\, R_1\rangle$ and $p_2\colon \langle X_2\rangle \to \langle X_2 \,|\, R_2\rangle$ the natural epimorphisms. Let

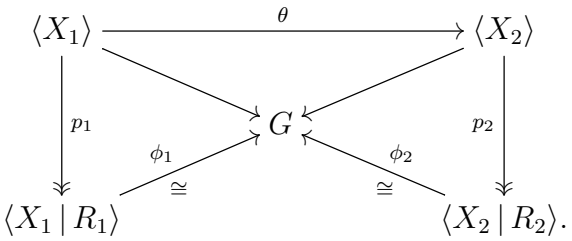
$$\Psi\colon \langle X_1\rangle * \langle X_2\rangle = \langle X_1 \sqcup X_2\rangle \;\to\; \pi$$

be the homomorphism that is induced by $\phi_1 \circ p_1$ on $\langle X_1\rangle$ and $\phi_2 \circ p_2$ on $\langle X_2\rangle$.

Note that the restriction of Ψ to $\langle X_2\rangle$ is an epimorphism. Therefore we can pick for each $x \in X_1$ a $w_x \in \langle X_2\rangle$ with $\Psi(x) = \Psi(w_x)$. We write $S_1 := \{x \cdot w_x^{-1} \,|\, x \in X_1\} \subseteq \langle X_1 \sqcup X_2\rangle$. The same way we define $S_2 \subseteq \langle X_1 \sqcup X_2\rangle$.

Claim. We have $\ker(\Psi) = \langle\langle R_2 \cup S_1\rangle\rangle$.

Proof. First note that it follows almost immediately from Lemmas 100.26, 100.27 and 100.29 that $\ker(\Psi) = \langle\langle R_1 \cup R_2 \cup S_1\rangle\rangle$. Thus it remains to show that $\langle\langle R_1 \cup R_2 \cup S_1\rangle\rangle = \langle\langle R_2 \cup S_1\rangle\rangle$. Put differently, given $r_1 \in R_1$ we need to show that $r_1 \in \langle\langle R_2 \cup S_1\rangle\rangle$. Let $\theta\colon \langle\langle X_1\rangle\rangle \to \langle\langle X_2\rangle\rangle$ be the homomorphism that is given by $x \mapsto w_x$ for $x \in X_1$. Note that we have the following commutative diagram:



In particular we have $(\phi_2 \circ p_2)(\theta(r_1)) = (\phi_1 \circ p_1)(r_1) = 1$. This implies that $\theta(r_1) \in \langle\langle R_2\rangle\rangle$. On the other hand it follows from Lemma 100.26 that $r_1 \cdot \theta(r_1)^{-1} \in \langle\langle S_1\rangle\rangle$. Thus we see that $r_1 = r_1 \cdot \theta(r_1)^{-1} \cdot \theta(r_1) \in \langle\langle S_1 \cup R_2\rangle\rangle$. ⊞

After these preparations we turn to the actual proof of the theorem.

(a) First note that we can turn the presentation $\langle X_2 \,|\, R_2\rangle$ into $\langle X_1 \sqcup X_2 \,|\, R_2 \cup S_1\rangle$ by a sequence of Tietze transformations (T2), where we add one by one the elements of X_1 and for each element of X_1 we add the corresponding relation in S_1.

(b) Note that $R_1 \cup S_2 \subseteq \ker(\Psi)$. By the claim this implies that $R_1 \cup S_2 \subseteq \langle\langle R_2 \cup S_1\rangle\rangle$. It follows immediately that, using Tietze transformations (T1) we can turn the presentation $\langle X_1 \sqcup X_2 \,|\, R_2 \cup S_1\rangle$ into the presentation $\langle X_1 \sqcup X_2 \,|\, R_1 \cup S_2 \cup R_2 \cup S_1\rangle$.

(c) By exactly the same reasoning as in (a) and (b) we can also turn the presentation $\langle X_1 \,|\, R_1\rangle$ into the presentation $\langle X_1 \sqcup X_2 \,|\, R_1 \cup S_2 \cup R_2 \cup S_1\rangle$ via a sequence of Tietze transformations.

Since $\langle X_1 \,|\, R_1\rangle$ and $\langle X_2 \,|\, R_2\rangle$ can be turned into precisely the same presentation using Tietze transformations, the two presentations are in fact related by Tietze transformations. ■

Remark. Alternative proofs of the Tietze Theorem 100.25 can be found in [**Tie1908**] and also [**LyS1977**, Proposition II.2.2] and [**MKS1976**, Theorem 1.5]. □

100.5. **Appendix: Reducing the number of generators.** For theoretical purposes a finite presentation is often as good as any other. For calculations though it is usually convenient to work with presentations with a "small" number of generators. In practice the following elementary lemma can often be used to turn a given presentation into a presentation with fewer generators:

Lemma 100.30. (Generator Elimination Lemma) Suppose that we are given a presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle$ where $r_i = r_i(x_1, \ldots, x_k) \in \langle x_1, \ldots, x_k\rangle$. If there exists a

generator x_m and a relator r_n such that x_m appears only once in r_n, i.e. such that r_n is of the form $r_n = y \cdot x_m^{\epsilon} \cdot z$ where $\epsilon \in \{-1, 1\}$ and where $y, z \in \langle x_1, \dots, \widehat{x_m}, \dots, x_k \rangle$, then the map

$$\begin{array}{rcl} \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle & \to & \langle x_1, \dots, \widehat{x_m}, \dots, x_l \,|\, r_i(x_1, \dots, (y^{-1} \cdot z^{-1})^{\epsilon}, \dots, x_k), i \neq m \rangle \\ x_i & \mapsto & \begin{cases} x_i, & \text{if } i \neq m, \\ (y^{-1} \cdot z^{-1})^{\epsilon}, & \text{if } i = m \end{cases} \end{array}$$

is well-defined and it is an isomorphism.

Example. We consider the presentation

$$\langle x_1, x_2, x_3 \,|\, \overbrace{x_2 \cdot x_1 \cdot x_2^{-1}}^{=r_1}, \overbrace{x_3 \cdot x_2 \cdot x_1^{-1}}^{=r_2}, \overbrace{{x_2}^2 \cdot x_1^2}^{=r_3} \rangle.$$

We consider the generator x_2 and the relator $r_2 = x_3 \cdot x_2 \cdot x_1^{-1}$. It follows from the Generator Elimination Lemma 100.30 that the map

$$\langle x_1, x_2, x_3 \,|\, \overbrace{x_2 \cdot x_1 \cdot x_2^{-1}}^{=r_1(x_1,x_2,x_3)}, \overbrace{x_3 \cdot x_2 \cdot x_1^{-1}}^{=r_2}, \overbrace{x_2^2 \cdot x_1^2}^{=r_3(x_1,x_2,x_3)} \rangle \to \langle x_1, x_3 \,|\, \overbrace{(\underbrace{x_3^{-1} \cdot x_1}_{\text{“}= x_2\text{”}}) \cdot x_1 \cdot (\underbrace{x_3^{-1} \cdot x_1}_{\text{“}= x_2\text{”}})^{-1}}^{=r_1(x_1, x_3^{-1} \cdot x_1, x_3)}, \overbrace{(\underbrace{x_3^{-1} \cdot x_1}_{\text{“}= x_2\text{”}})^2 \cdot x_1^2}^{=r_3(x_1, x_3^{-1} \cdot x_1, x_3)} \rangle$$

$$\begin{array}{rcl} x_1 & \mapsto & x_1 \\ x_2 & \mapsto & x_3^{-1} \cdot x_1 \\ x_3 & \mapsto & x_3 \end{array}$$

is an isomorphism. □

Proof. First note that by the definition of a free group there exist unique homomorphisms

$$\begin{array}{rcl} \langle x_1, \dots, x_k \rangle & \to & \langle x_1, \dots, \widehat{x_m}, \dots, x_k \rangle \\ x_i & \mapsto & \begin{cases} x_i, & \text{if } i \neq m, \\ (y^{-1} \cdot z^{-1})^{\epsilon}, & \text{if } i = m \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta \colon \langle x_1, \dots, \widehat{x_m}, \dots, x_k \rangle & \to & \langle x_1, \dots, x_k \rangle \\ x_i & \mapsto & x_i. \end{array}$$

For $i \neq m$ we set $s_i := r_i(x_1, \dots, (y^{-1} \cdot z^{-1})^{\epsilon}, \dots, x_k) := \alpha(r_i)$. Next we consider the following diagram:

$$\begin{array}{ccc} \langle x_1, \dots, x_k \rangle & \underset{\alpha}{\overset{\beta}{\rightleftarrows}} & \langle x_1, \dots, \widehat{x_m}, \dots, x_k \rangle \\ \downarrow & & \downarrow \\ A := \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle & \underset{=:\Psi}{\overset{=:\Phi}{\rightleftarrows}} & \langle x_1, \dots, \widehat{x_m}, \dots, x_k \,|\, s_i, i \neq m \rangle =: B, \end{array}$$

where the vertical maps are the natural epimorphisms. It follows easily from the Presentation–Homomorphism Lemma 100.13 that α and β descend to well-defined homomorphisms at the bottom. The homomorphism $\Phi \colon A \to B$, going from left to right, is the one we are interested in. It remains to prove the following claim:

Claim. We have $\Phi \circ \Psi = \mathrm{id}_B$ and $\Psi \circ \Phi = \mathrm{id}_A$.

Proof. It follows from the Generating Set–Lemma 100.6 that $x_1, \dots, \widehat{x_m}, \dots, x_k$ is a generating set for B. By definition we have $(\Phi \circ \Psi)(x_i) = x_i$ for $i \neq m$. It follows from the Generating Set Properties–Lemma 100.7 (1) that $\Phi \circ \Psi = \mathrm{id}_B$.

Next we want to show that $\Psi \circ \Phi = \mathrm{id}_A$. Since we already know that $\Phi \circ \Psi = \mathrm{id}_B$ it follows from the Inverse Uniqueness Lemma 5.10 that it remains to show that Ψ is an isomorphism. By the above we already know that Ψ is a monomorphism. Thus it remains to show that $\Psi\colon B \to A$ is an epimorphism. By the Generating Set–Lemma 100.6 we know that $x_1, \ldots, x_k$ is a generating set for A. It now follows from the Generating Set Properties–Lemma 100.7 (3) that it suffices to show that $x_1, \ldots, x_k$ lie in the image of Ψ. It is clear that $x_1, \ldots, \widehat{x_m}, \ldots, x_k$ lie in the image of Ψ. Furthermore, since we have the equality $x_m = (y^{-1} \cdot z^{-1})^\epsilon \in \langle x_1, \ldots, \widehat{x_m}, \ldots, x_k \rangle$, and since the image of Ψ is a *subgroup*, we also see that x_m lies in the image of Ψ. ∎

100.6. **Appendix: The Andrews-Curtis Conjecture.** In this appendix we want to introduce Nielsen transformations and we want to present the Andrews-Curtis Conjecture. Both are mostly peripheral to the content of these notes. But for general culture it is perhaps interesting to know that in this supposedly fairly elementary context of finite presentations there are difficult conjectures lurking in the background.

First we introduce the following language.

Definition 100.31. Given a set S and $n \in \mathbb{N}_0$ we refer to an element $(g_1, \ldots, g_n) \in S^n$ as an ordered set of elements in S.[1044]

Now we can introduce the definition we are really interested in.

Definition 100.32. Let π be a group and let $(g_1, \ldots, g_n)$ be an ordered set of elements in π. A Nielsen operation is one of the following three ways of obtaining a new ordered set of elements in π:

(1) $(g_1, \ldots, g_i, \ldots, g_n) \rightsquigarrow (g_1, \ldots, g_i^{-1}, \ldots, g_n)$

(2) $(g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) \rightsquigarrow (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_n)$

(3) $(g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) \rightsquigarrow (g_1, \ldots, g_i \cdot g_j, \ldots, g_j, \ldots, g_n)$.

The following theorem was proved first by Jakob Nielsen [**Nie1924**].

Theorem 100.33. (Nielsen Equivalence Theorem) Let $F = \langle x_1, \ldots, x_n \rangle$ be a free group. If $y_1, \ldots, y_n \in F$ form a basis[1045] for the free group F, then $(y_1, \ldots, y_n)$ can be obtained from $(x_1, \ldots, x_n)$ through a finite sequence of Nielsen operations (1), (2) and (3).

Proof. We refer to [**LyS1977**, Proposition I.4.1] and [**MKS1976**, Theorem 3.2] for a proof of the theorem. ∎

Remark.

(1) Let $F = \langle g_1, \ldots, g_n \rangle$ be a finitely generated free group. It follows from the Free Group–Existence Lemma 98.9 that any Nielsen operation, applied to $(g_1, \ldots, g_n)$, gives rise to an endomorphism of the free group F. One can easily show that these endomorphisms are in fact automorphisms of the free group. We refer to any such automorphism as a *Nielsen automorphism*. The Nielsen Equivalence Theorem 100.33 implies almost immediately that any automorphism of the free group is in fact a composition of finitely

[1044] In other words, an ordered set of elements in S is just an element of some S^n.

[1045] Recall that by the definition in Definition 98.6 the elements $y_1, \ldots, y_n$ form a basis if the natural map $\langle y_1, \ldots, y_n \rangle \to F$ is an isomorphism.

many Nielsen automorphisms. In other words, the group $\mathrm{Aut}(F)$ is generated by the Nielsen transformations.

(2) The fact that the group $\mathrm{Aut}(F)$ is generated by the Nielsen operations is similar in spirit to the fact, established in the Matrix Group-Generating Set Lemma 100.5 that the group $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z})$ is generated by elementary matrices. □

The following conjecture was formulated by James Andrews and Morton Curtis [**AC1965**] in 1965.

Conjecture 100.34. (Andrews-Curtis Conjecture) Let $\langle g_1, \dots, g_n \,|\, r_1, \dots, r_n \rangle$ be a finite presentation. If the corresponding group is the trivial group, then $(r_1, \dots, r_n)$ can be obtained from $(g_1, \dots, g_n)$ through a finite sequence of operations where we allow the Nielsen operations (1), (2), (3) and the following fourth operation:

(4) $\qquad (g_1, g_2, g_3, \dots, g_n) \;\rightsquigarrow\; (hg_1h^{-1}, g_2, g_3, \dots, g_n) \quad$ for some $h \in \pi$.

The Andrews-Curtis Conjecture is notoriously difficult. It seems like most experts believe the conjecture to be false, but not much progress has been made over the last decades. Finally note that the Andrews-Curtis Conjecture will raise its head once more in Section 256.2.

100.7. Exercises for Chapter 100.

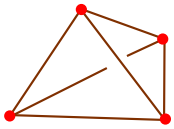
Exercise 100.1. Is the group $\langle x, y \,|\, x^2, y^3, xy^{-1}x^{-1}y, (xy)^7 \rangle$ trivial?

Exercise 100.2. Let (V, E, i, t) be an abstract graph. We consider the following group

$$\langle V \,|\, \{[i(e), t(e)] \,|\, e \in E\} \rangle.$$

In other words, the generators are the vertices of the graph and each edge gives us the relation that the corresponding vertices commute. In the figure below we see several graphs. All these are familiar groups. Which ones?

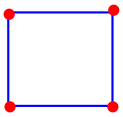
Remark. Groups that arise from a graph with the above construction are called *right-angled Artin groups.*.



graph A



graph B



graph C

Exercise 100.3. Let R be a commutative Euclidean domain. Show that every matrix in $\mathrm{GL}(n, R)$ can be written as a product of elementary matrices, i.e. of matrices of the following form:

$$\underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(I)}}, \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(II)}} \quad \text{and} \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(III)}}, \quad \text{with } r \text{ a unit in } R.$$

Exercise 100.4. Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n \in \mathbb{N}$.

(a) Let $a, b \in K \setminus \{0\}$. Calculate the commutator of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ in the group $\mathrm{SL}(2, K)$.

(b) Show that the abelianization of $\mathrm{SL}(n, \mathbb{K})$ is the trivial group.
Hint. Use the Matrix Group-Generating Set Lemma 100.5 and try to write the elementary matrices as commutators.

(c) Determine the abelianization of $\mathrm{GL}(n, \mathbb{K})$.

Remark. A group with trivial abelianization is sometimes called *perfect*. We have thus shown that the groups $\mathrm{SL}(n, \mathbb{K})$ are perfect.

Exercise 100.5. We consider the permutation group S_5.

(a) Let $\tau = (1\ 2) \in S_5$. What is $\langle\langle \tau \rangle\rangle$?

(b) Let $\sigma = (1\ 2\ 3) \in S_5$. What is $\langle\langle \sigma \rangle\rangle$?

Exercise 100.6. Show that the group $\langle x, y \,|\, x^2 = y^3 \rangle$ admits an epimorphism onto the permutation group S_3.

Exercise 100.7. We consider the permutation group S_3. Show that a presentation for S_3 is given by $\langle \sigma, \tau \,|\, \sigma^3, \tau^2, \sigma\tau\sigma\tau^{-1} \rangle$.

Exercise 100.8. Let π be a group and let $A, B \subseteq \pi$ be two subsets of π. Show that there exists a natural isomorphism between $(\pi / \langle\langle A \rangle\rangle) / \langle\langle B \rangle\rangle$ and $\pi / \langle\langle A \cup B \rangle\rangle$.

Exercise 100.9. Let π be a finitely generated group. Show that every finite-index subgroup is also finitely generated.
Remark. In the Finite-Index Subgroup–Presentation Proposition 138.3 we will prove the more subtle result that every finite-index subgroup of a finitely presented group is finitely presented.

Exercise 100.10. Let π be a group and let $\phi\colon \pi \times \langle t \rangle \to \mathbb{Z}$ be an epimorphism. We suppose that $k := \phi(t) \neq 0$. We set $\Gamma := \ker(\pi \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z})$. Since ϕ is an epimorphism there exists an $x \in \pi$ such that $\gcd(\phi(x), k) = 1$. Let $\mu\colon \Gamma \to \Gamma$ be the automorphism that is given by $g \mapsto xgx^{-1}$. Show that the map

$$\begin{array}{rcl} \Gamma \rtimes_\mu \mathbb{Z} & \to & \pi \times \mathbb{Z} \\ (g, n) & \mapsto & (gx^n, n + \phi(g)) \end{array}$$

is an isomorphism.
Hint. You could use Exercise 93.11 together with Exercise 93.6 (d).
Remark.

(1) By Exercise 100.9 and the Finite-Index Subgroup–Presentation Proposition 138.3 we know that if π is finitely generated (presented), then so is Γ. The exercise thus says that if a finitely presented group π admits an epimorphism onto $\mathbb{Z}$, then we can write a product $\pi \times \mathbb{Z}$ in many interesting and perhaps surprising ways as the semidirect product of a finitely presented group.

(2) We will prove a manifold analog in the Product-as-Mapping Torus Proposition 233.8.

Exercise 100.11. Let π be a finitely generated group and let G be a finite group. Show that the set $\mathrm{Hom}(\pi, G)$ of homomorphisms from π to G is a finite set.

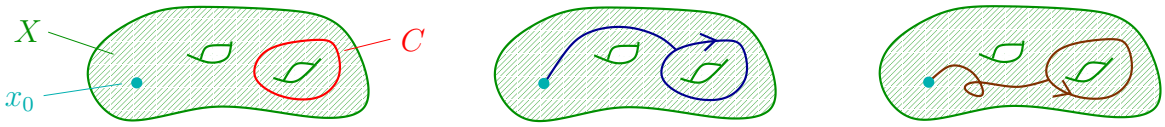
Exercise 100.12. Let $A = \langle X \,|\, R\rangle$ and $B = \langle Y \,|\, S\rangle$ be two groups. Determine a presentation for the direct product $A \times B$.

Exercise 100.13. Let π be a group that admits a finite presentation and let $\langle X \,|\, R\rangle$ be a presentation where X is finite. Show that there exists a finite subset $S \subseteq R$ such that $\langle X \,|\, R\rangle = \langle X \,|\, S\rangle$.

Exercise 100.14. Let (X, x_0) be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism with $f(x_0) = x_0$. Show that there exists an isomorphism of the following form:
$$\pi_1(\mathrm{Tor}(X, f))_{\mathrm{ab}} \;\cong\; \mathbb{Z} \oplus \mathrm{coker}\left(\pi_1(X, x_0)_{\mathrm{ab}} \xrightarrow{f_* - \mathrm{id}} \pi_1(X, x_0)_{\mathrm{ab}}\right).$$
Remark. Make use of the Mapping Torus-π_1-Proposition 93.13.

Exercise 100.15. Given any $l, m, n \in \mathbb{N}$ we consider the von Dyck group
$$T(l, m, n) \;:=\; \langle a, b \,|\, a^l, b^m, (ab)^n\rangle.$$
Show that $T(l, m, n)_{\mathrm{ab}} \cong \mathbb{Z}/\gcd(l, m, n)\mathbb{Z}$.
Remark. It follows in particular that if l, m, n are coprime, then the abelianization is trivial. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then it follows from the Van Dyck Group-Cardinality Corollary 132.32 together with the Van Dyck Group-Presentation Lemma 132.31 that the von Dyck group $T(l, m, n)$ is nonetheless infinite.

Exercise 100.16. Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $C \subseteq X$ be a subset that is homeomorphic to S^1. We pick a homeomorphism $\varphi\colon S^1 \to C$ and a path $\gamma\colon [0, 1] \to X$ from x_0 to $\varphi(1)$. We denote by $\widetilde{\varphi}\colon [0, 1] \to X$ the map that is given by $t \mapsto \varphi(\exp(2\pi\, i t))$. Show that the subgroup
$$\langle\langle C\rangle\rangle \;:=\; \langle\langle \gamma * \varphi * \widetilde{\gamma}\rangle\rangle \;\subseteq\; \pi_1(X, x_0)$$
neither depends on the choice of φ nor on the choice of γ.



Exercise 100.17. Let π be a group and let S be a generating set for π. We set $V := \pi$, we set $E := V \times S$ and we define the initial map $i\colon E = V \times S \to V$ by $i((v, s)) = v$ and we define the terminal map $t\colon E = V \times S \to V$ by $t((v, s)) = v \cdot s$. Note that (V, E, i, t) is an abstract graph. It is called the *Cayley graph* $\Gamma(\pi, S)$ of the group π with respect to the generating set S.

(a) We consider $\pi = \langle x, y \,|\, xyx^{-1}y^{-1}\rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S)|$ as a subspace of $\mathbb{R}^2$.

(b) We consider $\pi = \langle x, y \,|\, \varnothing\rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S)|$ as a subspace of $\mathbb{R}^2$.

(c) We consider $\pi = \langle x, y \,|\, x^2, xyx^{-1}y^{-1}\rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S|$ as a subspace of $\mathbb{R}^3$.

Exercise 100.18. Let N be a group and let $\varphi\colon N \to N$ be an isomorphism.

(a) Let $k \in \mathbb{N}$ and let $G \subseteq N$ be a subgroup with $\varphi^k(G) = G$. Show that $G \rtimes_\varphi k\mathbb{Z}$ is a subgroup of $N \rtimes_\varphi \mathbb{Z}$.

(b) Show that in general not every subgroup of $N \rtimes_\varphi \mathbb{Z}$ is of the form $G \rtimes_\varphi k\mathbb{Z}$.
(c) Let π be a finite-index subgroup of $N \rtimes_\varphi \mathbb{Z}$.
  (i) Show that there exists a $k \in \mathbb{N}$ and a normal subgroup G of N with $\varphi^k(G) = G$ such that $[N \rtimes_\varphi \mathbb{Z} : \pi] = k \cdot [G : \pi]$ and such that π is *isomorphic* to $G \rtimes_{\varphi^k} \mathbb{Z}$.
  (ii) We suppose that N is finitely generated. Show that there exists a $k \in \mathbb{N}$ and a normal finite-index subgroup G of N such that $G \rtimes_\varphi k\mathbb{Z} \subseteq \pi$.

Exercise 100.19. Let R be a ring, let $r \in R$ and let $n \geq 3$. Show that the elementary matrix

$$\begin{pmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}$$

is a commutator in $\mathrm{GL}(n, R)$.

Exercise 100.20. Let R be a commutative Euclidean domain in the sense of Definition 100.4 and let $n \in \mathbb{N}_{\geq 3}$. Show that the map

$$\begin{array}{rcl} \det\colon \mathrm{GL}(n,R) & \to & R^\times := \text{units in } (R,\cdot) \\ A & \mapsto & \det(A) \end{array}$$

is an isomorphism.
Hint. Use the Matrix Group-Generating Set Lemma 100.5 and make use of Exercise 100.19.

Exercise 100.21. Recall that on page 1912 we showed that we have an isomorphism

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \;\to\; \mathrm{SL}(2,\mathbb{Z})$$

where the homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ are the unique monomorphisms. Use this fact to determine the abelianization of $\mathrm{SL}(2,\mathbb{Z})$.
Remark. This shows that in Exercise 100.20 we do indeed have to assume that $n \geq 3$.

Exercise 100.22. We consider the group $(\mathbb{Z}[\frac{1}{2}], +)$. Let $\varphi\colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$ be the group isomorphism that is given by multiplication by 2. We consider the corresponding semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes_\varphi \mathbb{Z}$. Show that a presentation for this semidirect product is given by $\langle x, y \,|\, xyx^{-1} = y^2 \rangle$.

Exercise 100.23. We consider the *Heisenberg group*

$$H \;:=\; \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{Z}) \;\middle|\; x, y, z \in \mathbb{Z} \right\}.$$

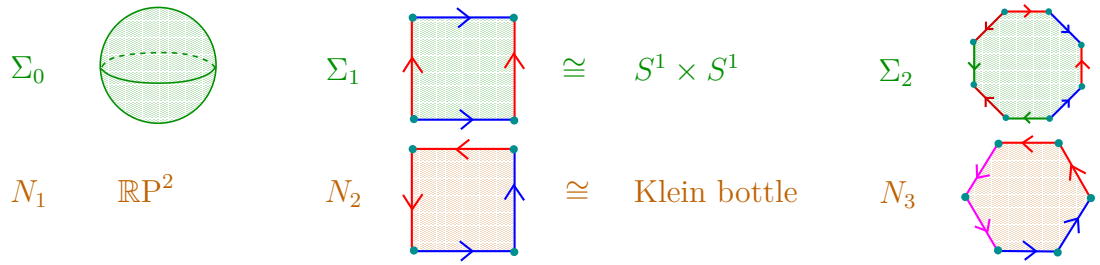
(a) Show that there exists an epimorphism $\langle a, b \,|\, [a,[a,b]], [b,[a,b]] \rangle \to H$.
(b) Show that in (a) we can upgrade from "epimorphism" to "isomorphism".

Exercise 100.24. Let P be a property of groups. A group G is called *locally* P if every finitely generated subgroup has property P.
(a) Is every locally abelian group in fact an abelian group?
(b) Is every locally free group in fact a free group?
(c) Is every locally finite group in fact a finite group?

CHAPTER 101

The classification of compact 2-dimensional manifolds I

Given $g \in \mathbb{N}_0$ we introduced in Definition 9.18 the surface Σ_g of genus g and given $g \in \mathbb{N}$ we introduced in Definition 9.19 the non-orientable surface N_g of genus g.[1046]



In this chapter we will first use the Seifert–van Kampen Theorem 99.1 to provide the long overdue calculation of the fundamental groups of the (non-orientable) surfaces Σ_g and N_g. Afterwards, in Section 101.5 we will use these calculations to give the classification of compact 2-dimensional smooth and topological manifolds.

This chapter is organized as follows:

- In Section 101.1 we will determine presentations for the fundamental groups of the orientable surfaces Σ_g of genus g.
- In Section 101.2 we will determine presentations for fundamental groups of the non-orientable surfaces N_g.
- In Section 101.3 we will prove a technical, but important, result about the boundary of 2-dimensional topological manifolds.
- In Section 101.4 we will calculate the fundamental groups of surfaces with non-empty boundary.
- In Section 101.5 we will state the Surface Classification Theorem 101.16. Our calculations of the fundamental groups will allow us to prove one of the two directions of the theorem. The proof of the other direction will rely on the Diffeomorphism-to-Standard Surfaces Theorem 255.14, which we will prove much later, with very different tools.
- In Appendix 101.6 we will discuss *non-compact* 2-dimensional smooth manifolds.
- In Section 101.7 we will provide exercises for Chapter 101.

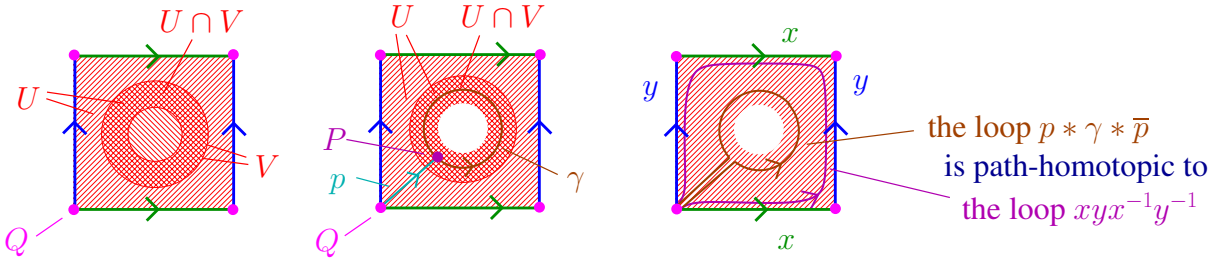
In the next Chapter 102 we will formulate a version of the Seifert–van Kampen Theorem that is particularly suitable for manifolds.

101.1. **The fundamental groups of the surfaces Σ_g.** We want to use the Seifert–van Kampen-Theorem 99.1 to determine the fundamental groups of the surfaces Σ_g of genus

[1046] In these notes, the notion of "(non-orientable) surface of genus g" is properly defined. At times we will also use the word "surface" in a colloquial sense, even though as a standalone term we never introduced it.

$g \geq 2$. Before we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.

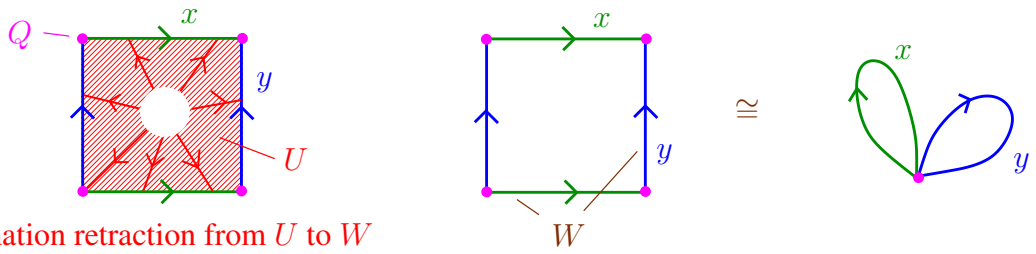
(1) We first consider the torus $T = ([0,1] \times [0,1])/\sim$. We will use the following notation:
   (a) We denote by Q the point on T corresponding to one of the four vertices of the square.
   (b) We set
$$U = T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2},\tfrac{1}{2})} \subseteq T \quad \text{and } V = B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) \subseteq T,$$
   i.e. U is the complement of a closed disk and V is an open disk. Note that $T = U \cup V$ and note that $U \cap V$ is an open annulus, i.e. homeomorphic to $S^1 \times (0,1)$.
   (c) We denote by $i\colon U \cap V \to U$ the inclusion.
   (d) We let $P = (\frac{1}{3},\frac{1}{3}) \in U \cap V$ and we denote by p the direct path from Q to P.
   (e) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus". Since $U \cap V$ has an obvious deformation retraction to a circle we obtain from the Circle-π_1-Theorem 88.22 together with the Homotopy Equivalence-π_1-Proposition 90.9 (1b) that γ represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.
   (f) We denote by x and y the two loops in (T, Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T, Q)$ by x and y as well.



Now we see that

$$\begin{array}{ccccc}
\pi_1(T,P) & \overset{\cong}{\underset{\uparrow}{}} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) & = & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\} \\
 & & \text{by the Seifert–van Kampen Theorem 99.1} & & V \text{ is a disk, hence } \pi_1(V,P) = \{e\} \\
 & = & \pi_1(U,P)/\langle\langle i_*(\pi_1(U\cap V,P))\rangle\rangle & = & \pi_1(U,P)/\langle\langle [\gamma]\rangle\rangle \\
 & & \text{by the Amalgamated Product Quotient Lemma 98.25 we have } A *_G \{e\} = A/\langle\langle \alpha(G)\rangle\rangle & & \text{since } \pi_1(U\cap V,P) = \langle [\gamma]\rangle \\
 & \overset{\cong}{\underset{\uparrow}{\to}} & \pi_1(U,Q)/\langle\langle [p*\gamma*\overline{p}]\rangle\rangle & = & \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \quad = \quad (*). \\
 & & \text{by the Change-of-Base Point Proposition 88.14 (1) the path } p \text{ gives rise to an isomorphism } p_* & & \text{since } p*\gamma*\overline{p} \text{ is path-homotopic the path to } xyx^{-1}y^{-1}
\end{array}$$

One can easily show that $W := \partial([0,1]\times[0,1])/\sim$ is a deformation retract of U[1047] and that W is homeomorphic to the wedge of two circles[1048]. By the Wedge-of-Circles–π_1-Lemma 99.11 we can therefore make the identification $\pi_1(W,P) =_{\mathrm{i}} \langle x, y\rangle$.



Now we continue with the above calculation of $\pi_1(T,P)$:

$$(*) \quad = \quad \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \quad \overset{\cong}{\to} \quad \pi_1(W,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle$$

by the Homotopy Equivalence-π_1-Proposition 90.9 (1b), since W is a deformation retract of U

$$= \quad \langle x,y\rangle/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \quad = \quad \langle x,y \mid [x,y]\rangle \quad \cong \quad \mathbb{Z}^2.$$

by the Wedge-of-Circles–π_1-Lemma 99.11 since W is the wedge of the circles x and y

by the Free Abelian Group Presentation Lemma 100.14

Thus we have reproved the fact, first shown in the Torus-π_1–Corollary 90.14, that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.

(2) We turn to the Klein bottle $K = ([0,1]\times[0,1])/\sim$ with base point $Q = [(0,0)]$. The approach to calculating the fundamental group of the Klein bottle is almost identical to the above approach to calculating the fundamental group of the torus. We just replace the situation illustrated in the figure above by the situation illustrated in the figure below. The same argument as above shows that[1049] [1050]

$$\pi_1(K,Q) \;\cong\; \langle x,y \mid xyx^{-1}y\rangle.$$

In the Klein Bottle–π_1-Proposition 93.6 we gave a very different calculation of $\pi_1(K,Q)$. More precisely, we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s,t+1) \end{array}$$

[1047] We consider the map

$$\begin{array}{rcl} r\colon U & \to & \partial([0,1]\times[0,1]) \\ x & \mapsto & \text{unique point in } \mathbb{R}_{>0}\cdot x \cap \partial([0,1]\times[0,1]). \end{array}$$

A deformation retraction is given by

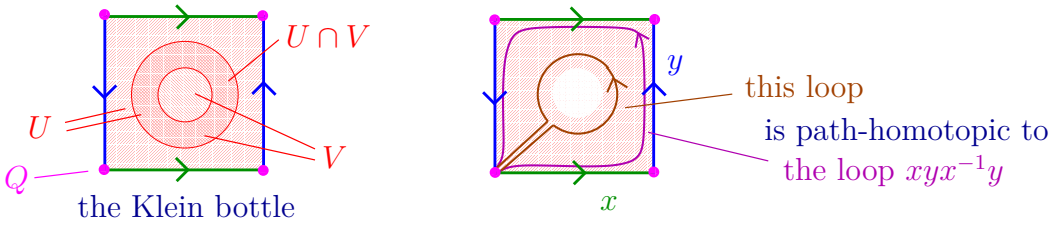
$$\begin{array}{rcl} U\times[0,1] & \to & U \\ ([P],t) & \mapsto & [P\cdot(1-t) + r(P)\cdot t]. \end{array}$$

(This map is continuous by the popular Homotopy Quotient Theorem 18.11.)

[1048] A homeomorphism is given by sending the point on W that corresponds to the vertices to the wedge point of the two circles, and by sending the edges that get identified to the same circle. This idea can be made precise using the Quotient-to-Quotient Lemma 8.6 and the Compact-Hausdorff Proposition 4.23 (3).

[1049] This shows that $\pi_1(K,Q)$ is finitely presented. This answers the question we had raised on page 1959.

[1050] Using this presentation and using the Abelianization-of-Presentation Proposition 100.17 we see that the abelianization of $\pi_1(K,Q)$ is given by $\pi_1(K,Q)_{\mathrm{ab}} \;\cong\; \langle x,y \mid xyx^{-1}y\rangle_{\mathrm{ab}} \;\cong\; \mathbb{Z}^2/\binom{0}{2}\cdot\mathbb{Z} \;\cong\; \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.



and their inverses and we showed that there exists an isomorphism $\varphi\colon G \to \pi_1(K,Q)$. On page 1957 we showed that the map $\Phi\colon \langle x,y \mid xyx^{-1}y\rangle \to G$ given by $\Phi(x) = A$ and $\Phi(y) = B$ is an epimorphism. We will now show that this homomorphism is in fact an isomorphism. We consider the diagram

$$\langle x,y \mid xyx^{-1}y\rangle \xrightarrow{\cong} \pi_1(K,Q) \xleftarrow[\varphi]{\cong} G.$$
$$\underset{x\mapsto A,\ y\mapsto B}{\overset{\Phi}{\curvearrowright}}$$

Here the isomorphisms to $\pi_1(K,Q)$ are the ones given by the two calculations of $\pi_1(K,Q)$. If one goes carefully through the definitions of the two horizontal maps, then one can see that the diagram commutes. Since two of the three maps are isomorphisms and since the diagram commutes it follows that Φ is also an isomorphism.

(3) Next we consider again the real projective space $\mathbb{R}\mathrm{P}^2$. Here we use the description of $\mathbb{R}\mathrm{P}^2$ from the Projective Space-Avatar Lemma 9.16 (2) as

$$\mathbb{R}\mathrm{P}^2 \ =_{\mathrm{i}}\ \overline{B}^2/\!\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

As in the previous two examples we decompose $\mathbb{R}\mathrm{P}^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{R}\mathrm{P}^2$ and we denote by α the loop

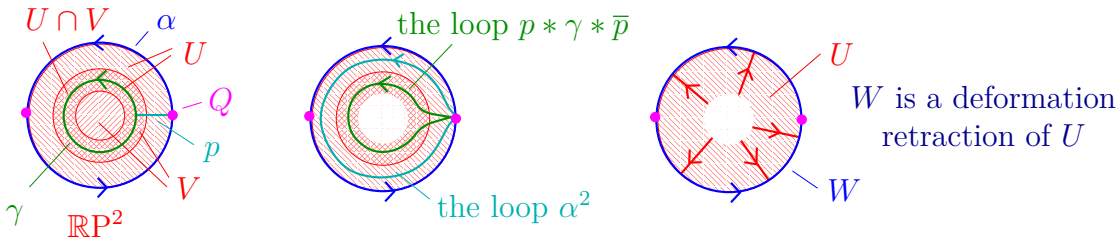
$$\begin{aligned} \alpha\colon [0,1] &\to \mathbb{R}\mathrm{P}^2 \\ t &\mapsto [\exp(\mathrm{i}\pi t)]. \end{aligned}$$

By a rather slight abuse of notation we denote by α also the corresponding element in $\pi_1(\mathbb{R}\mathrm{P}^2,Q)$. Furthermore we denote by W the image of α in $\mathbb{R}\mathrm{P}^2$. One can easily verify that W is homeomorphic to S^1. It follows from the Circle-π_1-Theorem 88.22 that α is a generator of $\pi_1(W,Q)$. With similar arguments as in the torus case, we see that

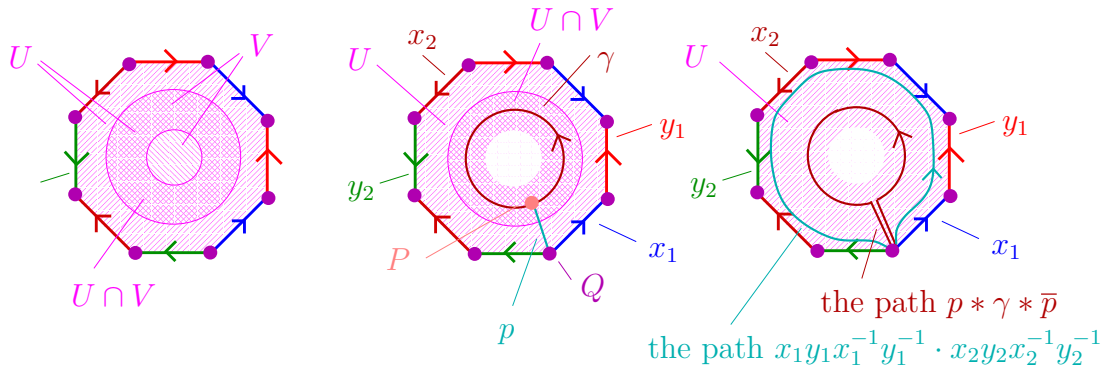
$$\pi_1(\mathbb{R}\mathrm{P}^2,Q) \underset{\text{by the Seifert–van Kampen-Theorem 99.1 and the Amalgamated Product–Quotient Lemma 98.25}}{\overset{\uparrow}{\cong}} \pi_1(U,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle \underset{\text{by the Homotopy Equivalence-}\pi_1\text{-Proposition 90.9 (1b), since } W \text{ is a deformation retract of } U}{\overset{\downarrow}{\cong}} \pi_1(W,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle \underset{\text{since } \pi_1(W,Q) = \langle\alpha\rangle \text{ and } [p*\gamma*\overline{p}] = \alpha^2}{\overset{\uparrow}{\cong}} \langle\alpha\rangle/\langle\langle\alpha^2\rangle\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

In particular we obtain the same result, as we should, as in the Real Projective Space–π_1-Proposition 93.3.

(4) Now that we have gained some confidence that the approach of determining fundamental groups using the Seifert–van Kampen-Theorem 99.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.



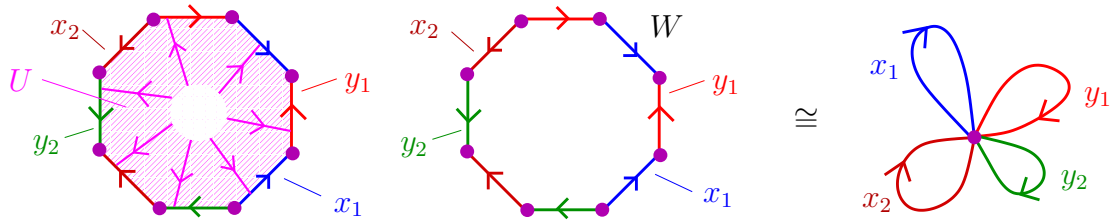
So let us consider the surface $F = E_8/\sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk such that $U \cap V$ is an annulus. Similar to the torus case we write $W = \partial E_8/\sim$. As in the torus case we see that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. Finally almost



the same argument as in the torus case shows that

$$\begin{aligned}\pi_1(E_8/\sim, P) &\cong \pi_1(U,Q)/\langle\langle \overbrace{\underbrace{x_1y_1x_1^{-1}y_1^{-1} \cdot x_2y_2x_2^{-1}y_2^{-1}}_{=[x_1,y_1]\cdot[x_2,y_2]}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\ &\cong \pi_1(W,Q)/\langle\langle [x_1,y_1]\cdot[x_2,y_2]\rangle\rangle \\ &\cong \langle x_1,y_1,x_2,y_2\rangle/\langle\langle [x_1,y_1]\cdot[x_2,y_2]\rangle\rangle = \langle x_1,y_1,x_2,y_2 \mid [x_1,y_1]\cdot[x_2,y_2]\rangle.\end{aligned}$$

To summarize: we have finally succeeded in calculating the fundamental group of the surface of genus 2.



Proposition 101.1. (Surface-π_1-Proposition)

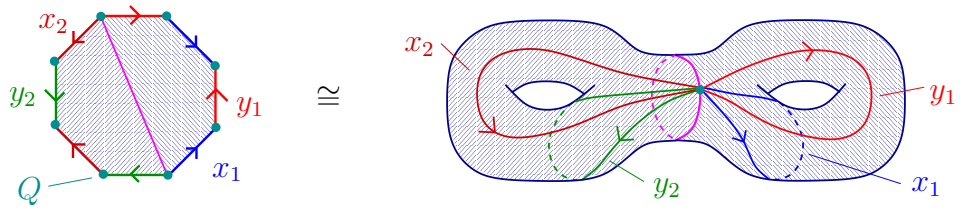
(1) Let $g \in \mathbb{N}$. Let Q be the point in $\Sigma_g = E_{4g}/\sim$ corresponding to the vertex of E_{4g}. We have an isomorphism [1051]

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

where the $x_1, y_1, \ldots, x_g, y_g$ correspond to edges of E_{4g}. Furthermore we have

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q)_{\text{ab}} \;\cong\; \mathbb{Z}^{2g}.$$

(2) For $g \neq h$ the surfaces of genus g and h are not homotopy equivalent, in particular they are not homeomorphic.



Proof.

(1) Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from the Sphere-π_1-Proposition 88.24 and for $g = 1$ we obtain the presentation from the discussion on page 1976. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition. Finally the stated abelianization of the fundamental group is an immediate consequence of the Abelianization-of-Presentation Proposition 100.17.

(2) For $g \neq h$ we obtain from the Abelian Group-Classification Theorem 97.24 that the abelianizations of the fundamental groups of the surfaces of genus $g \neq h$ are not isomorphic. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 that the surfaces are not homotopy equivalent. ■

Let $g \geq 2$ and let π be the fundamental group of the surface of genus g. We have just shown that $\pi_{\text{ab}} \cong \mathbb{Z}^{2g}$. This raises the question, whether perhaps π is isomorphic to $\mathbb{Z}^{2g}$ or to the free group F_{2g} on $2g$ generators. The following lemma settles one of the two questions.

Lemma 101.2. (Surface-π_1-Nonabelian Lemma) Given any $g \geq 2$ the fundamental group $\pi_1(\Sigma_g) \cong \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$ is non-abelian.

Proof. We will prove the lemma in Exercise 101.1. ■

The lemma says in particular that $\pi_1(\Sigma_g)$ is not isomorphic to $\mathbb{Z}^{2g}$. It is much harder to determine whether or not $\pi_1(\Sigma_g)$ is a free group. We record this as a question for later.

Question 101.3. Is $\pi_1(\Sigma_g)$ isomorphic to the free group F_{2g} on $2g$ generators?

We will give an answer to the question in the Surface-π_1–Not Free Proposition 130.21 and also on page 2999.

In Exercise 101.4 we will see that if $g \geq 2$, then $\pi_1(\Sigma_g)$ contains a free group on two generators as a subgroup. This raises the question, what other groups can occur as subgroups of π? Let us record this as a question.

[1051] This statement also makes sense for $g = 0$.

Question 101.4. What isomorphism types of subgroups of $\pi_1(\Sigma_g)$ can appear? Is every subgroup again a free group or the fundamental group of a surface?

We will provide the answer to Question 101.4 in the Subgroups-of-Surface Groups Proposition 130.7.

101.2. **The fundamental groups of the non-orientable surfaces N_g.** In this section we will discuss the fundamental groups of the non-orientable surfaces N_g of some genus $g \in \mathbb{N}$. The following proposition is the analog of the Surface-π_1-Proposition 101.1.

Proposition 101.5. (Surface-π_1-Proposition II)

(1) For any $g \in \mathbb{N}$ we have

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g) \cong \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle$$

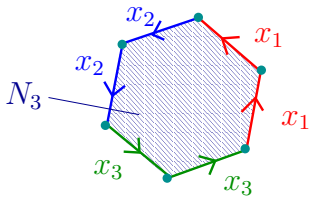
and

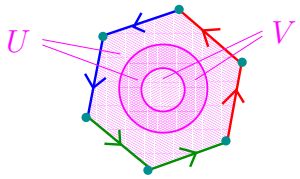
$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g)_{\text{ab}} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

(2) For $g \neq h$ the non-orientable surfaces of genus g and h are not homeomorphic.

Proof.

(1) Let $g \in \mathbb{N}$. Recall that by the definition on Definition 9.19 the non-orientable surface of genus one is the real projective plane. In the Real Projective Space–π_1-Proposition 93.3 (or alternatively above on page 1977) we saw that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z} \cong \langle x \mid x^2 \rangle$. The case $g \geq 2$ follows from a straightforward modification of the calculation of the fundamental group of the surface of genus 2 that we gave on page 1978. Using the figure below the reader should have no troubles with assembling a full argument.





Finally note that

$$\pi_1(N_g)_{\text{ab}} \underset{\text{by (1)}}{\cong} \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle_{\text{ab}} \underset{\substack{\text{by the Abelianization-of-Presentation} \\ \text{Proposition 100.17}}}{\cong} (\mathbb{Z}^g)/\mathbb{Z} \cdot \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

(2) For $g \neq h$ we obtain from the above calculation and from the Abelian Group-Classification Theorem 97.24 that the abelianizations of the fundamental groups of the non-orientable surfaces of genus g and h are not isomorphic. Hence the non-orientable surfaces of genus $g \neq h$ are not homeomorphic. ∎

We conclude this section with a short discussion of the properties of fundamental groups of compact 2-dimensional smooth manifolds. We recall that the fundamental group of the real projective space $\mathbb{RP}^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, therefore it is torsion and in particular it is not torsion-free. It is less clear whether or not the fundamental groups of the other 2-dimensional smooth manifolds are torsion-free. More precisely, we have the following question.

Question 101.6.

(1) Let $g \in \mathbb{N}$. Is the fundamental group

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

torsion-free?

(2) Let $k \geq 2$. Is the fundamental group

$$\pi_1(k \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x_1, \ldots, x_k \mid x_1^2 \cdot \ldots \cdot x_k^2 \rangle$$

torsion-free?

We will come back to this question in the Surface-π_1-Properties Proposition 105.8, the Surface Group–Torsion Free Corollary 130.8 and the Surface Group–Torsion Free Proposition 318.8.

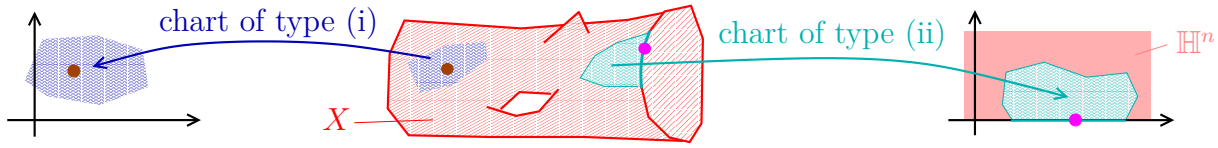
101.3. **Interlude: The boundary of 2-dimensional topological manifolds.** In the next section we will study the fundamental groups of 2-dimensional topological manifolds with boundary. Before we get there we need to prove the subtle statement that the boundary of a 2-dimensional topological manifold is what you think it should be. To make sense of this it is perhaps worth reminding us of Definition 23.2.

Definition 101.7. Let X be a topological space.

(a) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the upper half-space $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.
  In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii).

(b) We say that X is an n-dimensional topological manifold if every point admits a chart and if X is Hausdorff and second countable.

(c) We say that a point x on a topological manifold X is a boundary point if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.



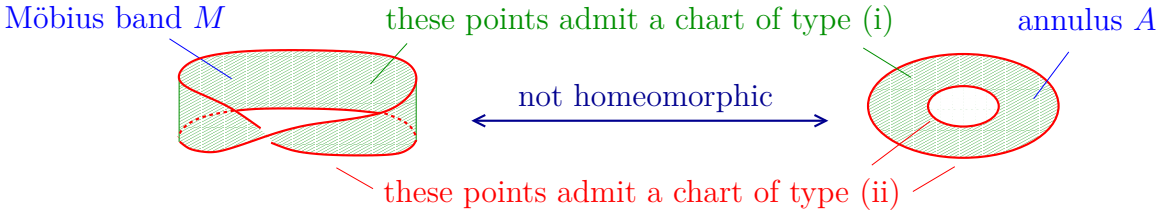
The following proposition now settles the indeterminacy from the Unique Type-of-Chart Proposition 23.13 for 2-dimensional topological manifolds. The proposition below is formulated as an upgrade to the Unique Type-of-Chart Proposition 23.13:

Proposition 101.8. (Unique 2D-Type-of-Chart Proposition Proposition) Let F be a 2-dimensional topological manifold. The following statements hold:

(i) (a) Every point on F admits either a chart of type (i) or a chart of type (ii).
  (b) If F is equipped with a smooth atlas, i.e. if we view F as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of F viewed as a smooth manifold.

Remark. It might come as a surprise that we do not yet know the statement of the proposition. In fact in the Unique Type-of-Chart Proposition 25.26 we proved the analog of Statement (1) for charts from a smooth atlas. But at the moment we do not have the tools to prove Statement (i) for topological manifolds of dimension greater than two. We will deal with the case of higher-dimensional topological manifolds in the Unique Type-of-Chart Proposition 150.11 once we have introduced and studied homology groups. □

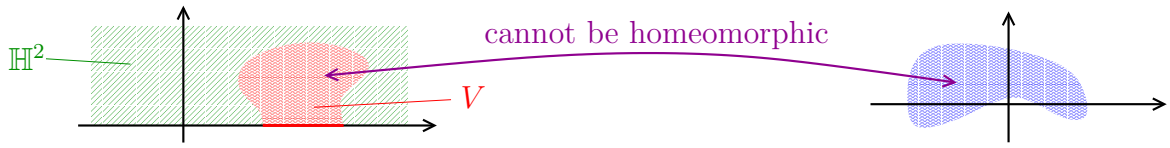
Example. In the figure below we show the Möbius band M and the annulus A.



The Unique 2D-Type-of-Chart Proposition Proposition 101.8 says that the boundary of the Möbius band M and the annulus A, *as a topological manifold*, is exactly what we think it should be, namely precisely the points that "obviously" admit a chart of type (ii). In particular we see that ∂M consists of one component whereas ∂A consists of two components. Since homeomorphisms of topological manifolds induce homeomorphisms of their boundaries and since ∂M is not homeomorphic to ∂A we see that the Möbius band M is not homeomorphic to the annulus A. □

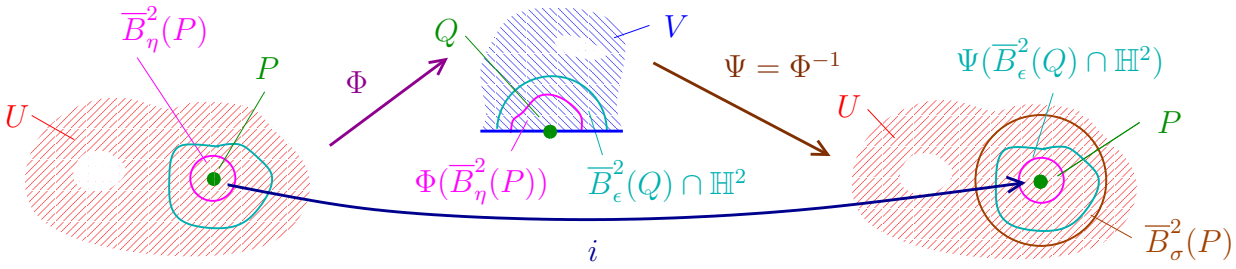
The remainder of this section deals with the proof of the Unique 2D-Type-of-Chart Proposition Proposition 101.8. The reader-in-a-hurry can also just skip the proof and move to the next section. The key ingredient in the proof of the Unique 2D-Type-of-Chart Proposition Proposition 101.8 is the following lemma.

Lemma 101.9. (2D-Boundary Preservation Lemma) Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half-space in $\mathbb{R}^2$. If V is an open subset of $\mathbb{H}^2$ with $V \cap \partial\mathbb{H}^2 \neq \varnothing$, then V is not homeomorphic to an open subset of $\mathbb{R}^2$.



Proof of Lemma 101.9. Let V be an open subset of the upper half-space $\mathbb{H}^2$ which contains at least one point $Q \in \partial\mathbb{H}^2 = \{(x, 0) \mid x \in \mathbb{R}\}$. Furthermore let U be an open subset of $\mathbb{R}^2$. Suppose that there exists a homeomorphism $\Phi\colon U \to V$. We write $\Psi := \Phi^{-1}$ and we write $P = \Psi(Q)$.

Since U is open in $\mathbb{R}^2$ we can pick a $\sigma > 0$ with $\overline{B}^2_\sigma(P) \subseteq U$. Furthermore, since V is open in $\mathbb{H}^2$ and since Ψ is continuous there exists an $\epsilon > 0$ with $\overline{B}^2_\epsilon(Q) \cap \mathbb{H}^2 \subseteq V$ and with $\Psi(\overline{B}^2_\epsilon(Q)) \subseteq \overline{B}^2_\sigma(P)$. Finally, by the continuity of Φ there exists an $\eta > 0$ with $\Phi(\overline{B}^2_\eta(P)) \subseteq \overline{B}^2_\epsilon(Q))$.



We consider the following commutative diagram of maps:

$$\begin{array}{ccc} & (\overline{B}^2_\epsilon(Q) \setminus \{Q\}) \cap \mathbb{H}^2 & \\ {\scriptstyle \Phi}\nearrow & & \searrow{\scriptstyle \Psi = \Phi^{-1}} \\ \overline{B}^2_\eta(P) \setminus \{P\} & \xrightarrow[i]{} & \overline{B}^2_\sigma(P) \setminus \{P\}, \end{array}$$

where $i\colon \overline{B}^2_\eta(P) \setminus \{P\} \to \overline{B}^2_\sigma(P) \setminus \{P\}$ denotes the inclusion map. We pick a base point $x_0 \in \overline{B}^2_\eta(P) \setminus \{P\}$. By the functoriality of fundamental groups, the above commutative diagram of continuous maps gives rise to the following commutative diagram:

$$\begin{array}{ccc} & \pi_1((\overline{B}^2_\epsilon(Q) \setminus \{Q\}) \cap \mathbb{H}^2, \Phi(x_0)) & \\ {\scriptstyle \Phi_*}\nearrow & & \searrow{\scriptstyle \Psi_* = \Phi_*^{-1}} \\ \pi_1(\overline{B}^2_\eta(P) \setminus \{P\}, x_0) & \xrightarrow[i_*]{} & \pi_1(\overline{B}^2_\sigma(P) \setminus \{P\}, x_0). \end{array}$$

We make the following observations:

(1) The topological spaces $\overline{B}^2_\eta(P) \setminus \{P\}$ and $\overline{B}^2_\sigma(P) \setminus \{P\}$ on the left and on the right are homotopy equivalent to S^1. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 together with the Circle-π_1-Theorem 88.22 that the fundamental groups are non-trivial.

(2) First note that it is straightforward to see that $\overline{B}^2_\eta(P) \setminus \{P\}$ is a deformation retract of $\overline{B}^2_\sigma(P) \setminus \{P\}$.[1052] Thus it follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1b) that the inclusion-induced map $i_*\colon \pi_1(\overline{B}^2_\eta(P) \setminus \{P\}, x_0) \to \pi_1(\overline{B}^2_\sigma(P) \setminus \{P\}, x_0)$ is an isomorphism.

(3) The topological space $(\overline{B}^2_\epsilon(Q) \setminus \{Q\}) \cap \mathbb{H}^2$ is star-shaped[1053], which implies by the discussion on page 1736 that $\pi_1((\overline{B}^2_\epsilon(Q) \setminus \{Q\}) \cap \mathbb{H}^2, \Phi(x_0)) = 0$.

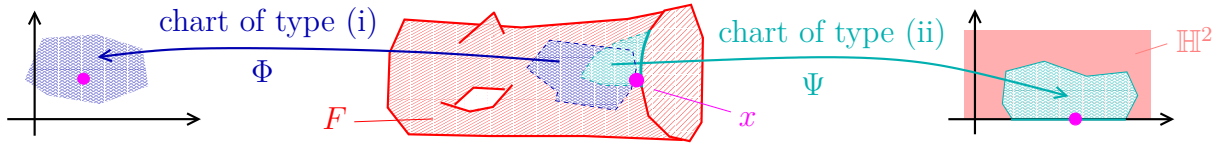
Summarizing we just showed that an isomorphism of non-trivial groups factors through the trivial group. This is not possible and hence we have obtained the desired contradiction. ■

Now we can turn to the proof of the Unique 2D-Type-of-Chart Proposition Proposition 101.8.

[1052] This is the reason why we work with closed disks instead of open disks.
[1053] Why is that?

Proof of the Unique 2D-Type-of-Chart Proposition Proposition 101.8. Let F be a 2-dimensional topological manifold.

(1) (a) Let x be a point on F. We need to show that x admits either a chart of type (i) or it admits a chart of type (ii). Suppose that x admits a chart $\Phi\colon U \to V$ of type (i) and also a chart $\Psi\colon \widetilde{U} \to \widetilde{V}$ of type (ii). Then $\Phi \circ \Psi^{-1}\colon \Psi(U \cap \widetilde{U}) \to \Phi(U \cap \widetilde{U})$ would give rise to a homeomorphism from an open subset of $\mathbb{H}^2$ containing a point on $\partial\mathbb{H}^2 = \{(x,0) \mid x \in \mathbb{R}\}$ to an open subset in $\mathbb{R}^2$. But by the 2D-Boundary Preservation Lemma 101.9 such a homeomorphism cannot exist.



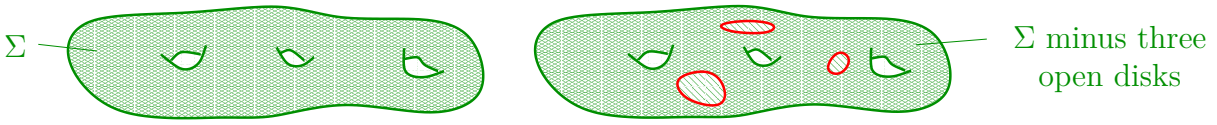
(b) Now suppose that F is equipped with a smooth atlas. For the purpose of this proof we write $\partial^{\text{top}}F$ and $\partial^{\text{sm}}F$ for the boundaries of F viewed as a topological respectively a smooth manifold. By the definitions on Definition 23.2 and Definition 25.10 we have

$\partial^{\text{top}}F$ = all points that do not admit a chart of type (i)
$\partial^{\text{sm}}F$ = all points that do not admit a chart of type (i) from the smooth atlas.

By definition we have $\partial^{\text{top}}F \subseteq \partial^{\text{sm}}F$. Now let $P \in \partial^{\text{sm}}F$. This means that P does not admit a chart of type (i) from the smooth atlas. By definition of a smooth manifold it admits a chart of type (ii). But by (1) this means that it cannot admit any chart of type (i). Thus we see that $P \in \partial^{\text{sm}}F$. ■

101.4. **The fundamental groups of surfaces with boundary.** As mentioned above, one goal of any classification theorem is to provide a "complete" list of objects. We already have a good list of closed surfaces. Now we need to add surfaces with non-empty boundary to our arsenal.

Definition 101.10. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \ldots, \varphi_n\colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2))$ as Σ minus n open disks.



The following lemma captures some of the main features of the above definition.

Lemma 101.11. (Surface-Minus-Open Disks Lemma) Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.

(1) The topological space Σ minus n open disks is a compact connected 2-dimensional topological manifold with n boundary components, which are given by the boundaries of the closed disks whose interiors were removed.

(2) Σ minus n open disks is a submanifold of the smooth manifold Σ. It follows from the Submanifold Smooth Structure Lemma 25.23 that Σ minus n open disks has a standard smooth structure.

(3) The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.
(4) Σ minus n open disks is orientable if and only if Σ is orientable.

Sketch of proof.

(1),(2) Let $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ be smooth embeddings with disjoint images. Note that we obtain from applying the Codimension-Zero Smooth Embedding Proposition 30.21 (2) iteratively altogether n times that $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ is a compact connected submanifold of Σ. In particular we see that it is a smooth and topological manifold in its own right. The fact that the boundary, as a topological manifold, is precisely given by $\varphi_1(S^1), \dots, \varphi_n(S^1)$ follows from the Smooth Manifold–Boundary Proposition 25.27 (2a) and the Unique 2D-Type-of-Chart Proposition Proposition 101.8.

(3) This statement follows easily from the Smooth Ball Embedding Theorem 45.10 and the trivial observation, that in our context it makes no difference if we precompose a smooth embedding $\varphi_i \colon \overline{B}^2 \to \Sigma$ with an orientation-reversing self-diffeomorphism of $\overline{B}^2$.

(4) If Σ is orientable, then the statement follows from the Manifold–Orientation Basics Lemma 31.31 (6b). If Σ is non-orientable, then the statement follows from Exercise 31.19. ■

The Surface-Minus-Open Disks Lemma 101.11 allows us to introduce the following notation:

Notation 101.12. Given $g, n \in \mathbb{N}_0$ we write

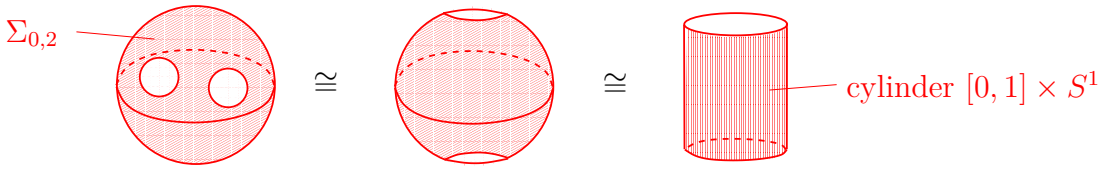
$$\Sigma_{g,n} \coloneqq \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

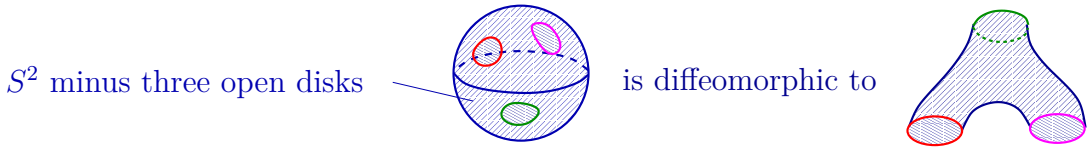
$$N_{g,n} \coloneqq \text{the non-orientable surface of genus } g \text{ minus } n \text{ open disks.}$$

Example.

(1) The surface $\Sigma_{0,2}$ is the result of removing the interiors of two disjoint closed disks from the 2-sphere. It is pretty elementary to show that the resulting smooth manifold is diffeomorphic to the "cylinder" $[0,1] \times S^1$.



(2) A smooth manifold that is diffeomorphic to S^2 minus three open disks is often called a *pair of pants*. As we will see in Exercise 101.17, pairs of pants can be viewed as a "building block" for almost all compact connected 2-dimensional smooth manifolds.



(3) We saw in the figure on page 693 that $\mathbb{R}\mathrm{P}^2$ minus an open disk is diffeomorphic to the Möbius band, i.e. $N_{1,1}$ is diffeomorphic to the Möbius band. □

The following lemma shows that the fundamental groups of our new friends are rather dull and relatively uninformative.

Lemma 101.13. (Surface-with-Boundary–π_1-Lemma)

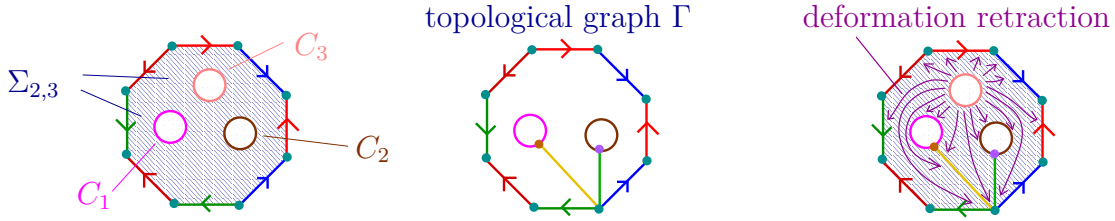
(1) Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ the surface $\Sigma_{g,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-2g-n$. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g+n-1$ generators.

(2) Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the surface $N_{k,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-k-n$. In particular $\pi_1(N_{k,n})$ is a free group on $k+n-1$ generators.

Remark. In the Surface-with-Boundary–π_1-Lemma 107.8 we will prove a refinement of the above Surface-with-Boundary–π_1-Lemma 101.13. □

Sketch of proof.

(1) We denote by $C_1, \dots, C_n$ the boundary components of $\Sigma_{g,n}$. Let Γ be the topological space that is given by the wedge of $2g$-circles (which is obtained from identifying as usual the edges of the $4g$-gon pairwise), together with the boundary components $C_1, \dots, C_{n-1}$ and together with edges $e_1, \dots, e_{n-1}$ that connect the wedge point to $C_1, \dots, C_{n-1}$. The deformation retraction from $\Sigma_{g,n}$ to Γ is given by pushing the remaining boundary component C_n "inward". We illustrate this in the figure below. (This is one of the rare occasions where we provide a "proof by picture". It is of course possible to give a completely rigorous proof, but one does not gain anything from doing so.)



We equip Γ with the obvious structure of a topological graph which has $n = 1+(n-1)$ vertices and $2g+2(n-1)$ edges. Note that $\chi(\Gamma) = 2-2g-n$. Finally we see that

$$\pi_1(\Sigma_{g,n}) \underset{\uparrow}{\cong} \pi_1(\Gamma) \cong \pi_1\Big(\begin{array}{c}\text{connected topological graph } \Gamma \\ \text{with } \chi(\Gamma) = 2-2g-n\end{array}\Big) \underset{\uparrow}{\cong} \begin{array}{c}\text{free group on} \\ 2g+n-1 \text{ generators}\end{array}$$

by the above together with the Homotopy Equivalence-π_1-Proposition 90.9 (1b)

by the Graph-π_1 Proposition 99.13

(2) The proof of this statement is almost identical to the proof of Statement (1). ■

In the following we will see that we can "recover" the closed surfaces Σ_g and N_g from any $\Sigma_{g,n}$ respectively any $N_{g,n}$. The key to doing so is the following definition:

Definition 101.14. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

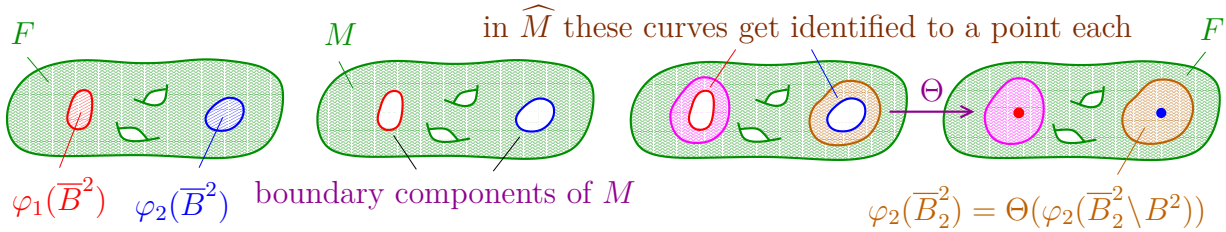


The following lemma shows in particular that $\widehat{\Sigma}_{g,n}$ is homeomorphic to Σ_g and that $\widehat{N}_{g,n}$ is homeomorphic to N_g.

Lemma 101.15. (Surface Recovery Lemma) Let F be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. If we set M to be F minus n open disks, then $\widehat{M}$ is homeomorphic to F.

Proof. By the Surface-Minus-Open Disks Lemma 101.11 we can work with any smooth embeddings. Thus let $\varphi_1, \dots, \varphi_n \colon \overline{B}^2_2 \to F$ be smooth embeddings of the closed disks *of radius* 2 with disjoint images. We need to show that for $M := F \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ there exists a homeomorphism $\widehat{M} \to F$. We consider the following map:[1054]

$$\begin{array}{rccl} \Theta \colon \widehat{M} & \to & F & \\ [P] & \mapsto & \begin{cases} \varphi_i\big(\frac{Q}{\|Q\|} \cdot 2 \cdot (\|Q\| - 1)\big), & \text{if } P = \varphi_i(Q) \text{ for some } i \in \{1, \dots, n\} \text{ and } Q \in \overline{B}^2_2 \setminus B^2, \\ [P], & \text{otherwise.} \end{cases} & \end{array}$$



It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 (1), the Pasting Proposition 4.7 (2) and the Topological-Quotient Proposition 8.2 (1b) that Θ is continuous. Note that by the Compact Image Lemma 4.15 we know that $\widehat{M}$ is compact. Now it follows almost immediately from the Compact-Hausdorff Proposition 4.23 (3) that the given map $\Theta \colon \widehat{M} \to F$ is indeed a homeomorphism. ∎

101.5. **The Surface Classification Theorem.** One of the goals of mathematicians is to classify objects of one's favorite category $\mathcal{C}$. We could for example consider the category of (finitely presented) groups, the category of (finite-dimensional) $\mathbb{K}$-vector spaces or the category of (compact) n-dimensional topological (or smooth) manifolds. But what does "classify" mean? One interpretation would be the following:

(I) Provide a collection $\{X_i\}_{i \in I}$ of objects with the following two properties:
  (a) Any other object of $\mathcal{C}$ is isomorphic to some X_i.
  (b) If $i \neq j$, then X_i and X_j are not isomorphic.
(II) Provide an algorithm which determines, given an object in $\mathcal{C}$, to which X_i it is isomorphic to.

[1054] Note that discretely we are using the Surface-Minus-Open Disks Lemma 101.11 (1) which says that the boundary of M, viewed as a topological manifold, is indeed what we think it should be, namely it is given by $\varphi_1(S^1) \sqcup \dots \sqcup \varphi_n(S^1)$.

For example in the category of finite-dimensional $\mathbb{K}$-vector spaces a complete list is given by $\{\mathbb{K}^i\}_{i\in\mathbb{N}_0}$ and the calculation of the dimension gives us (II). Furthermore the Abelian Group-Classification Theorem 97.24 gives a classification of the objects in the category of finitely generated abelian groups.

In these notes we are mostly interested in the classification of connected n-dimensional smooth and topological manifolds. For $n = 1$ we have obtained such a classification with the Classification Theorems 28.1 and 28.8. In this setting a complete list is given by $\{S^1, [0,1], \mathbb{R}, [0,1)\}$ and given a connected non-empty 1-dimensional topological or smooth manifold we only need to determine whether it is compact and whether it has non-empty boundary to determine to which of the four it corresponds.

We turn our gaze to the 2-dimensional setting. We will state the classification theorem for compact connected non-empty 2-dimensional topological and smooth manifolds and we will prove (I b) and (II) in the above setting. For the proof of (I a) we will need some patience though, we will only provide it in Chapter 255.

Theorem 101.16. (Surface Classification Theorem) Every compact connected non-empty 2-dimensional topological manifold is homeomorphic to either

(1) the surface $\Sigma_{g,n}$ for *unique* $g, n \in \mathbb{N}_0$ or
(2) to the surface $N_{k,n}$ for *unique* $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Furthermore, if we are given a smooth manifold, then we can upgrade "homeomorphic" to "diffeomorphic".

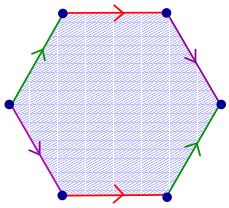
Remark.

(1) At the end of the section, in the Surface Recognition Proposition 101.21, we will state a slightly more precise version of the Surface Classification Theorem 101.16 that gives us Goal (II) from page 1987.
(2) The surfaces Σ_g, and thus also the surfaces $\Sigma_{g,n}$, come with a canonical orientation and each of these surfaces admits an orientation-reversing self-diffeomorphism given by a reflection. Thus, if we are given a compact *oriented* connected 2-dimensional smooth manifold, then we can in fact find an *orientation-preserving* diffeomorphism to the corresponding $\Sigma_{g,n}$. □

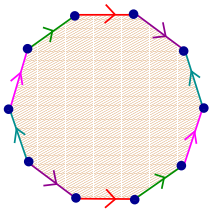
Example.

(1) At least for closed orientable connected 2-dimensional smooth manifolds it is perhaps tempting to dismiss the Surface Classification Theorem 101.16 (1) as "obvious". After all, what other closed orientable connected 2-dimensional smooth manifold could there possibly be? It is thus instructive to consider the figure below. It shows a hexagon and a decagon where *opposite sides* are identified. The quotient spaces are closed orientable 2-dimensional smooth manifolds.[1055] To which of the standard surfaces Σ_g are they diffeomorphic to? We will come back to this question on page 2988.

[1055]Indeed, basically the same proof (see the Standard Smooth Structures Proposition 25.6) that showed that the $4g$-gon, with appropriate sides identified, is a closed orientable 2-dimensional smooth manifold also shows that the topological spaces shown in the figure are closed orientable 2-dimensional smooth manifolds.



hexagon with opposite sides identified



dodecagon with opposite sides identified

(2) Let $g \in \mathbb{N}$. We consider the surface Σ_g of genus g. The connected sum $\Sigma_g \# \mathbb{RP}^2$ contains a Möbius band as a submanifold of codimension zero,[1056] hence $\Sigma_g \# \mathbb{RP}^2$ is non-orientable. It follows from the Surface Classification Theorem 101.16 that $\Sigma_g \# \mathbb{RP}^2$ is diffeomorphic to $k \cdot \mathbb{RP}^2$ for some unique $k \in \mathbb{N}$. In the next chapter, in Exercise 102.1, we will determine k as a function of g. □

The Surface Classification Theorem 101.16 is an immediate consequence of the following three theorems.

Theorem 255.14. (Diffeomorphism-to-Standard Surfaces Theorem) Let Σ be a compact connected non-empty 2-dimensional smooth manifold.

(1) If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.
(2) If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 101.17. (Surface-Smooth Structure Existence Theorem) Every compact 2-dimensional topological manifold admits a smooth structure.[1057]

Theorem 101.18. (Surfaces-Non Homeomorphic Theorem) The topological spaces $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic.

We now turn to the discussion of the proofs of the three theorems.

Proof of the Diffeomorphism-to-Standard Surfaces Theorem 255.14. We provide the proof of Theorem 255.14 in Chapter 255. Actually the proof is not as advanced as it may sound. The only ingredient one needs is the notion and existence of a handle decomposition which are given in Chapter 250. We will also sketch an alternative approach to proving the Theorem 255.14 in Exercise **??**. Also note that there are several other approaches to proving Theorem 255.14. For example [**Mata2002**, Chapter 5.2] and [**Hirs1976**, Chapter 9.3] give a proof using "Morse theory".[1058] ■

[1056] This follows from the observation that $\Sigma_g \# \mathbb{RP}^2$ contains $\mathbb{RP}^2 \setminus \frac{1}{2}B^2$, which by the figure on page 693 is a Möbius band.

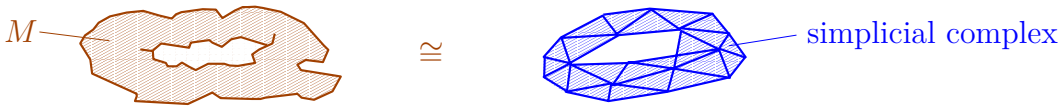
[1057] In **??** we will discuss the question, which topological manifolds admit a smooth structure. In the PL-Smoothing Theorem **??** we will see that there exists a compact 8-dimensional topological manifold that does not admit a smooth structure.

[1058] We will provide an introduction to Morse Theory in Chapter 250 but we will not discuss how Morse theory can be used directly to prove Theorem 255.14. We will only use Morse theory indirectly, since it allows us to study smooth manifolds via handle decompositions, which is the key to our proof of Theorem 255.14.

Proof of the Surface-Smooth Structure Existence Theorem 101.17. Let M be a compact 2-dimensional topological manifold. There are two approaches to showing that M admits a smooth structure.

(1) In 1922 Tibor Radó [**Rad1926**] showed that M is homeomorphic to a finite 2-dimensional simplicial complex. Alternative proofs are given in [**Moi1977**, p. 60], [**DM1968**, Corollary 1], [**Tn1992**, Theorem 4.1], [**MTh2001**, Theorem 3.1.1], [**AS1960**, p. 107] and [**GX2013**, Appendix E]. A sketch of the proof is provided in [**Lee2000**, Theorem 5.12]. In all of the proofs a key ingredient is given by the Schoenflies Theorem 170.6 which we will formulate later, but which we will not prove.

We will give the definition of a simplicial complex on Definition 174.31. For the purpose of this proof it suffices to know that this means that M is homeomorphic to a topological space that is obtained from gluing finitely many triangles linearly along their sides.



Recall that in the Standard Smooth Structures Proposition 25.6 we showed that the surface $E_{4g}/\sim$ of genus g admits a smooth structure. The proof can be modified without too many problems to show that any compact 2-dimensional topological manifold that admits a simplicial structure, also admits a smooth structure.

(2) In [**Hat2013**, Theorem A] a direct proof is given for the fact that M admits a smooth structure. Note that this approach does not rely on the Schoenflies Theorem 170.6. ■

Since we did not really prove the first two theorems we should at least provide full details for the third theorem. The idea is to find a complete set of invariants to distinguish the $\Sigma_{g,m}$ and $N_{h,n}$ pairwise. The first invariant we want to use is of course the (abelianization) of the fundamental group. But since this invariant is not enough we have to look for further invariants. One idea would be to use the number of boundary components. But even that is not good enough: In the following figure we see two compact connected 2-dimensional topological manifolds such that the fundamental groups are both isomorphic to the free group on two generators and such that both have one boundary component.



It is rather unlikely though that they are homeomorphic.[1059] This example strongly suggests that the fundamental group and the number of boundary components is not enough to distinguish all examples. To resolve this conundrum let us recall Definition 101.14.

Notation 101.19. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

The following proposition contains all invariants that we will need.

[1059] As a smooth manifold X is orientable and Y is non-orientable. But as of right now we do not know whether orientability is preserved under homeomorphisms.

Proposition 101.20. (Surface–Full Invariants Proposition) For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:

X	Σ_g	$\Sigma_{g,m}$ with $m \geq 1$	N_k	$N_{k,n}$ with $n \geq 1$
isomorphism type of $\pi_1(X)_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{k+n-1}$
number of boundary components	0	m	0	n
isomorphism type of $\pi_1(\widehat{X})_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. The number of boundary components can be determined using the Unique 2D-Type-of-Chart Proposition Proposition 101.8. The abelianizations of the fundamental groups of X are given by the Surface-π_1-Propositions 101.1 and 101.5 and the Surface-with-Boundary–π_1-Lemma 101.13. Finally the abelianizations of the fundamental of $\widehat{X}$ are given by the above calculation and the fact, shown in the Surface Recovery Lemma 101.15, that $\widehat{\Sigma_{g,m}} \cong \Sigma_g$ and $\widehat{N_{k,n}} \cong N_k$. ■

Now we can prove the Surfaces-Non Homeomorphic Theorem 101.18.

Proof of the Surfaces-Non Homeomorphic Theorem 101.18. It follows immediately from the Surface–Full Invariants Proposition 101.20 that the $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic. For completeness we point out that in Exercise 155.22 we will use (relative) homology groups to give an alternative proof of the Theorem 101.18. Also note that in the Surfaces-Non Diffeomorphic Corollary 53.28 we showed the weaker statement that the surfaces Σ_g are pairwise non-diffeomorphic. ■

As we will now see, the Surface–Full Invariants Proposition 101.20 also gives us a practical way to determine whether two surfaces are homeomorphic respectively diffeomorphic. More precisely, we have the following proposition, which gives a hopefully satisfying answer to Goal (II) from page 1987.

Proposition 101.21. (Surface Recognition Proposition) Let M and N be two compact connected non-empty 2-dimensional topological (respectively smooth) manifolds. The following are equivalent:

(1) M and N are homeomorphic (respectively diffeomorphic),
(2) (a) $\pi_1(M)_{\text{ab}} \cong \pi_1(N)_{\text{ab}}$,
(b) M and N have the same number of boundary components,
(c) $\pi_1(\widehat{M})_{\text{ab}} \cong \pi_1(\widehat{N})_{\text{ab}}$.
In the smooth setting we can replace (c) by
(c′) M and N have the same orientability.

Remark.

(1) Only much later, in the Smooth-Topological Orientation Proposition 191.25, will we see that homeomorphisms between smooth manifolds preserve orientability. If we factor this proposition in, then we can replace (c) by (c′) also in the topological setting.
(2) In the Surface-π_1–Not Free Proposition 130.21 we will see that the fundamental group of the surface Σ_g is not isomorphic to the free group on $2g$ generators. Once we know this fact, one can replace (a) and (b) by $\pi_1(M) \cong \pi_1(N)$. □

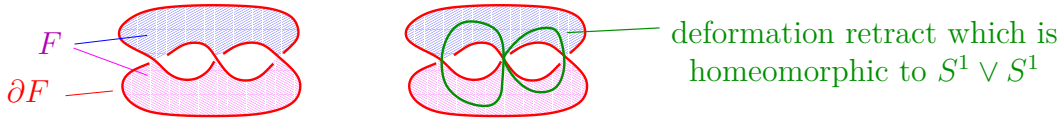
Proof. In the topological setting this proposition follows immediately from the Surface–Full Invariants Proposition 101.20 and the Surface Classification Theorem 101.16. In the

smooth setting the proposition follows from the above, together with the fact that on page 822 we showed that the $\Sigma_{g,m}$ are orientable and that in the Surface Non-Orientability Lemma 31.45 we showed that the $N_{k,n}$ are non-orientable. ■

The Surface Classification Theorem 101.16 allows us to make the following definition:

Definition 101.22. Let M be a compact connected non-empty 2-dimensional topological manifold. By the Surface Classification Theorem 101.16 there exists a unique $g \in \mathbb{N}_0$ such that M is homeomorphic to some $\Sigma_{g,n}$ or to some $N_{g,n}$. We refer to g as the genus(M) of M.

Example. We consider the compact 2-dimensional smooth manifold F shown in the figure below. We indicate that it is homotopy equivalent to the wedge of two circles.



It follows that $\pi_1(F)_{\text{ab}} \cong \langle x, y \rangle_{\text{ab}} \cong \mathbb{Z}^2$. Since F is orientable and since it has one boundary component we obtain from the Surface Recognition Proposition 101.21 that F is homeomorphic to $\Sigma_{1,1}$, i.e. F is homeomorphic to the torus minus one open disk. □

By now we might have the impression that we have a solid understanding of surfaces. The following question provides a good test for our true understanding.

Question 101.23. Let M be the surface of genus $g \in \mathbb{N}_0$ and let $p\colon \widetilde{M} \to M$ be a covering of finite degree d such that $\widetilde{M}$ itself is connected. By the Manifold Covering Proposition 95.4 and the Covering–Basics Lemma 91.12 (4) we know that $\widetilde{M}$ is a closed orientable connected 2-dimensional smooth manifold. Can we determine the genus of $\widetilde{M}$ in terms of g and d?

We will answer this question later on in the Subgroups-of-Surface Groups Proposition 130.7. It is worth trying to come up with an answer with our current knowledge.

We conclude this discussion of the classification theorem with a long remark on various other approaches to giving the topological classification of 2-dimensional topological manifolds.

Remark. The Surface Classification Theorem 101.16 says in particular that every compact connected non-empty 2-dimensional topological (respectively smooth) manifold is homeomorphic (respectively diffeomorphic) to one of the $\Sigma_{g,n}$ or one of the $N_{k,n}$. Basically all proofs in the literature that deal with the purely topological setting use the fact, mentioned on page 1990, that a 2-dimensional topological manifold admits a simplicial structure. The idea then is to modify the simplicial structure to turn it into a simplicial structure of one of $\Sigma_{g,n}$ or $N_{k,m}$. The traditional account of this proof is given in [**ST1934, ST1980**] or alternatively in [**Mass1991**, Theorem 5.1]. A variation on this proof, which is usually referred to as Conway's Zip Proof, is outlined in [**FW1999, Wild2010**], with full details given in [**Gle**] and [**SchubJ2017**]. Other proofs are given in [**Tn1992**], [**Meie**] and [**AKM2023**]. There is also a proof due to Eric Zeeman, see [**Zee1966, Put2**]. The proof given in [**Arm1988**, Chapter 7] is somewhat different, its basic idea is related to the approach

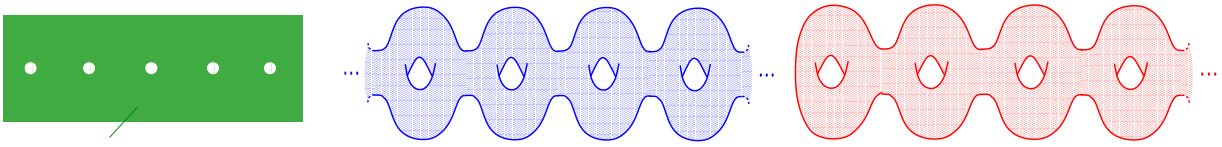
taken in Exercise **??**. Finally we point to [**GX2013**] for a book that is completely dedicated to the proof of the Surface Classification Theorem 101.16 in the topological setting. □

101.6. **Appendix: Non-compact 2-dimensional smooth manifolds.** Just for kicks we also want to consider *non-compact* 2-dimensional smooth manifolds. A short moment's thought there is a plethora of examples:

(1) Let Σ be a connected 2-dimensional smooth manifold Σ and let A be a closed non-empty discrete subset $A \subsetneq \Sigma$. Since A is closed and $\varnothing \neq A \subsetneq \Sigma$ we know by Exercise 23.4 that $\Sigma \setminus A$ is non-compact. Furthermore, since A is closed and discrete and since Σ is 2-dimensional we obtain from Exercise 23.6 that $\Sigma \setminus A$ is connected. Since $\Sigma \setminus A$ is an open subset of Σ we see that $\Sigma \setminus A$ is in fact a non-compact connected 2-dimensional smooth manifold. For example, this shows that $W = \mathbb{R}^2 \setminus \{(n, 0) \,|\, n \in \mathbb{Z}\}$ is a non-compact connected 2-dimensional smooth manifold.

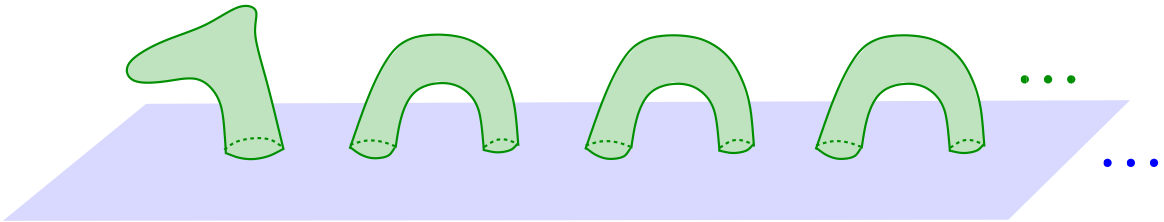
(2) In the figure below we show two non-compact 2-dimensional smooth manifolds X and Y for which the name "infinite genus" seems appropriate.

In Exercises 101.22 and Exercise 101.23 we will show, with some effort, that the non-compact 2-dimensional smooth manifolds W, X and Y are pairwise not diffeomorphic. This already shows that the classification of non-compact connected 2-dimensional smooth manifolds can get a little tricky. In fact the classification of non-compact orientable connected 2-dimensional smooth manifolds with empty boundary was first given by Ian Richards [**Ric1963**] in 1963, building on earlier work of Béla Kerékjártó [**Kere1923**] from 1923. In [**Barro1974**], [**BroM1979**, Theorem 2.2], [**Ton1980**, Proposition A.3] and [**PM2007**] this result was extended to the case of 2-dimensional smooth manifolds with non-empty boundary.



W surface X of "infinite genus" surface Y of "infinite genus"

Remark. As an aside we point out that the surface Y of infinite genus shown in the figure above is sometimes called the *Loch Ness monster*. The reason is shown in the figure below: the visible part of an infinite Loch Ness monster, together with the surface of the infinite Loch Ness form a 2-dimensional smooth manifold that is diffeomorphic to Y.



the infinite Loch Ness monster produces a surface of infinite genus

More information on the mathematical Loch Ness monster and an explicit rigorous construction are given in [**ARM2017**]. □

101.7. Exercises for Chapter 101.

Exercise 101.1. Let $g \in \mathbb{N}_{\geq 2}$. We consider the group

$$\pi \;:=\; \pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle.$$

(a) Show that π admits an epimorphism onto a free group of rank ≥ 2.
(b) What is the largest $k \in \mathbb{N}$ for which you manage to find an epimorphism onto a free group of rank k?

Exercise 101.2. Let $g \in \mathbb{N}_{\geq 2}$. We consider the group

$$\pi \;:=\; \pi_1(\text{non-orientable surface of genus } g) \;\cong\; \langle x_1, \ldots, x_g \,|\, x_1^2 \cdot \ldots x_g^2 \rangle.$$

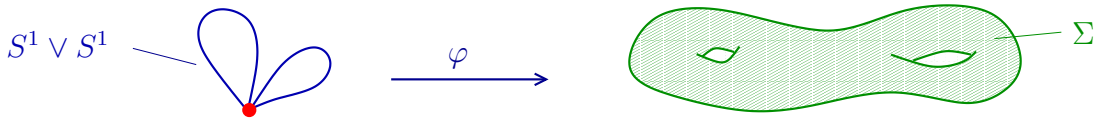
Show that for $g \geq 2$ this group is non-abelian.
Hint. Show that there exists an epimorphism onto the free product of two non-trivial groups.

Exercise 101.3. We consider the group

$$\pi \;:=\; \pi_1(\text{surface of genus } 2) \;\cong\; \langle x_1, x_2, x_3, x_4 \,|\, [x_1, x_2] \cdot [x_3, x_4] \rangle.$$

Show for $i \neq j$ none of the generators x_i and x_j commute.
Hint. For $i \neq j$ find a suitable epimorphism of π onto a free group.

Exercise 101.4. Let π be the fundamental group of the surface Σ of genus g. Show that π contains a free subgroup of rank 2.
Hint. You could make use of your solution to Exercise 101.1 or you could show that there exists an embedding $\varphi\colon S^1 \vee S^1 \to \Sigma$ such that $\varphi(S^1 \vee S^1)$ is a retract of Σ. We refer to the figure below for an illustration.



Exercise 101.5. Let K be the Klein bottle. Recall that on page 1976 we showed that $\pi_1(K) \cong \langle x, y \mid xyx^{-1}y \rangle$. In Exercise 9.17 we showed that K is homeomorphic to the non-orientable surface of genus 2. Also recall that in the Surface-π_1-Proposition 101.5 we showed that $\pi_1(N_2) \cong \langle x, y \,|\, x^2y^2 \rangle$. Give a purely algebraic proof that the groups $\langle x, y \mid xyx^{-1}y \rangle$ and $\langle x, y \,|\, x^2y^2 \rangle$ are isomorphic.

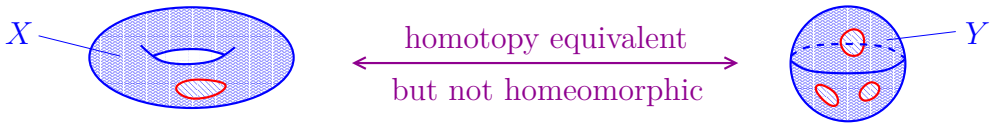
Exercise 101.6. Let $g \in \mathbb{N}$. Recall that by the Surface-π_1-Proposition 101.5 we have an isomorphism

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g) \;\cong\; \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle.$$

Show that if $g \geq 2$, then the group is non-abelian.
Recall. Recall that on page 1900 we showed that a free product $A * B$ of two non-trivial groups is non-abelian.

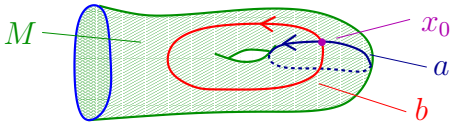
Exercise 101.7. Let X be the torus minus an open disk and let Y be the sphere minus three open disks.
(a) Show that X and Y are homotopy equivalent.

(b) Show that X and Y are not homeomorphic.



Exercise 101.8. We consider the 2-dimensional smooth manifold M and the two oriented loops a and b in the point $x_0 \in M$ that are shown in the figure below.

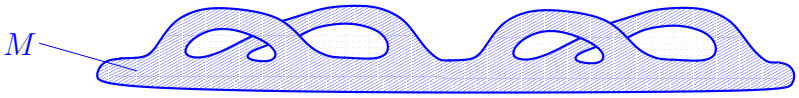
(a) Sketch an *embedded* loop $g\colon S^1 \to M$ that represents the element $a^3 \cdot b \in \pi_1(M, x_0)$.
(b) Sketch an *embedded* loop $g\colon S^1 \to M$ that represents the commutator $[a, b] \in \pi_1(M, x_0)$.



Exercise 101.9. Let $g \in \mathbb{N}$. We denote by Σ_g the surface of genus g.

(a) Does there exist a smooth map $\Sigma_g \to S^2$ that is a local diffeomorphism?
(b) Let M be the smooth manifold that we obtain from removing a single point from the surface of genus g. Does there exist a smooth map $M \to S^2$ that is a local diffeomorphism?

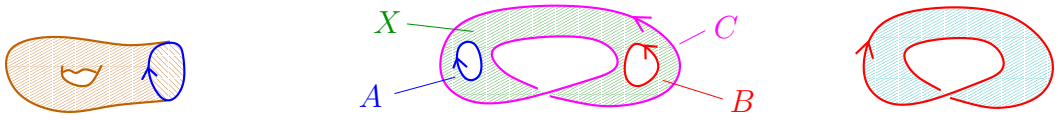
Exercise 101.10. Determine the genus of the surface M shown in the figure below.



Exercise 101.11. Let X be the Möbius band minus two disks. We denote by A, B, C the three boundary components of X. We glue a torus minus one open disk and a Möbius band to the boundary components of X. By the Gluing-Smooth Manifolds-Proposition 54.5 we obtain a compact 2-dimensional smooth manifold.

(a) Do the resulting diffeomorphism types depend on the gluing?
(b) How do the resulting smooth manifolds fit into the classification scheme of the Surface Classification Theorem 101.16?

We refer to the figure below for an illustration.



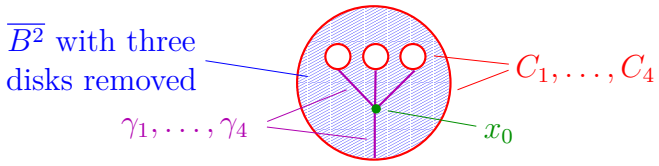
torus minus one disk Möbius band minus two disks Möbius band

Exercise 101.12. Recall that in the figure on page 693 we saw that $N_{0,1}$ is a Möbius band. In particular we can view $N_{0,1}$ as a submanifold of $\mathbb{R}^3$.

(a) Is $N_{2,1}$ diffeomorphic to a submanifold of $\mathbb{R}^3$?
(b) Show that $N_{3,1}$ is diffeomorphic to a submanifold of $\mathbb{R}^3$.

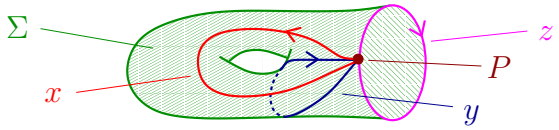
(c) For which $k \in \mathbb{N}$ is $N_{k,1}$ diffeomorphic to a submanifold of $\mathbb{R}^3$?

Exercise 101.13. Let F be the sphere minus k open disks. (Alternatively, let F be the disk $\overline{B}^2$ minus $k-1$ open disks.) We denote by $C_1, \dots, C_k$ the boundary components of F. We pick a base point $x_0 \in F$. Furthermore for each $i \in \{1, \dots, k\}$ we pick a path γ_i from a point on C_i to x_0. Finally let $\Gamma \subsetneq \pi_1(F, x_0)$ be a proper normal subgroup. Show that there exists an $i \in \{1, \dots, k\}$ such that C_i, viewed as an element in $\pi_1(F, x_0)$ via the path γ_i, does not lie in Γ.
Hint. You might want to start out with your preferred base point x_0 and your preferred paths $\gamma_1, \dots, \gamma_k$, e.g. as shown in the figure below.



Exercise 101.14.

(a) Let Σ be the torus minus one disk and let $P \in \partial\Sigma$ be a base point. Furthermore let x, y, z be the closed curves shown in the figure below. We denote by x, y, z also the elements in $\pi_1(\Sigma, P)$ represented by these curves. Convince yourself of the fact that $\pi_1(\Sigma, P) = \langle x, y, z \,|\, [x, y] = z \rangle$.
(b) We continue with the notation from (a). Show that $z \in \pi_1(\Sigma, P)$ is not a proper power, i.e. show that there is no $w \in \pi_1(\Sigma, P)$ such that $z = w^d$ for some $d \in \mathbb{N}_{\geq 2}$.
(c) Let $\Sigma_{g,1}$ be the surface of genus g minus one open disk. What is a suitable generalization of the statement (a)?
(d) Let g, h. Let $\Sigma_{g,1}$ and $\Sigma_{h,1}$ be the surface of genus g (respectively h) minus one open disk. Show that if $g \geq h$, then there exists a continuous map $\Sigma_{g,1} \to \Sigma_{h,1}$ that is a homeomorphism on the boundary.

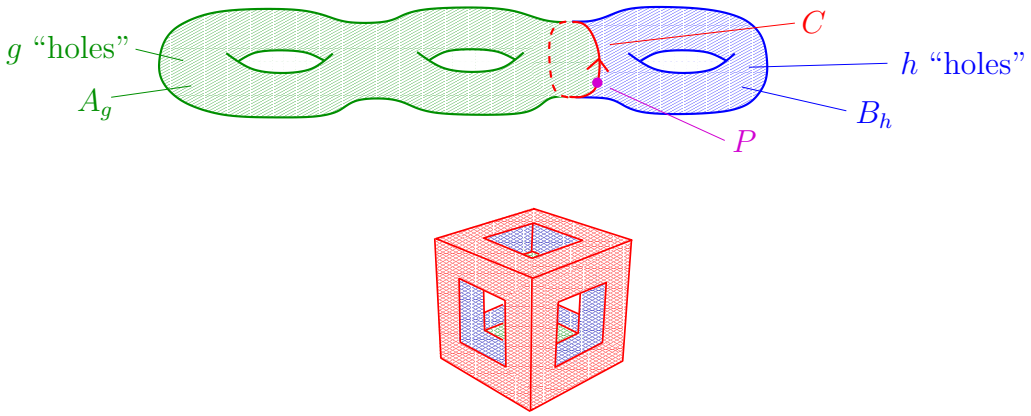


Exercise 101.15. Let $g, h \in \mathbb{N}_0$. In the following we consider the closed curve C on the surface $\Sigma_{g+h} = A_g \cup B_h$ that is shown in the figure below.

(a) Show that C represents the trivial element in the abelianization $\pi_1(\Sigma_{g+h}, P)_{\text{ab}}$.
(b) For which values of g and h is C a retract of Σ_{g+h}?
(c) For which values of g and h are A_g and or B_h a retract of Σ_{g+h}?
(d) For which values of g and h does C represent a non-trivial element in $\pi_1(\Sigma_{g+h}, P)$?

Remark. Some results from Exercise 101.14 might come in handy.
Remark. It might be that for some questions we cannot give complete answers for the precise values of g and h. Try to give the maximal number of possible values for g and h.

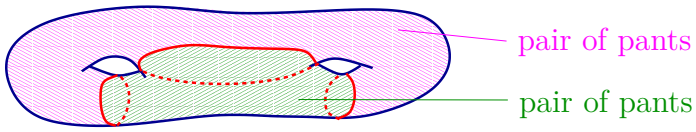
Exercise 101.16. What is the genus of the surface shown in the figure below.



Exercise 101.17. We refer to $\Sigma_{0,3}$, i.e. the sphere minus three open disks, as a *pair of pants*. We say that a 2-dimensional smooth manifold M admits a *pair of pants decomposition* if there exist submanifolds $P_1, \dots, P_k$ such that each P_i is a pair of pants and such that for each $i \neq j$ the intersection $P_i \cap P_j$ is a (possibly empty) union of boundary components of P_i and also of P_j.

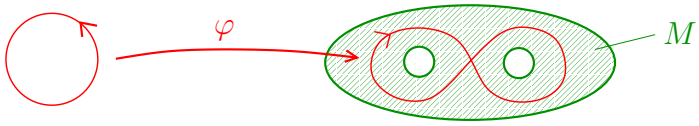
(a) Determine all $g, n \in \mathbb{N}_0$ such that $\Sigma_{g,n}$ admits a pair of pants decomposition.

(b) Determine all $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ for which $N_{k,n}$ admits a pair of pants decomposition.



Exercise 101.18. We consider the 2-dimensional smooth manifold M together with the map $\varphi\colon S^1 \to M$ that is shown in the figure below. Show that φ is not homotopic to a smooth embedding.

Remark. At this stage it is a challenging, but by no means impossible, task to provide a completely rigorous proof.



Exercise 101.19.

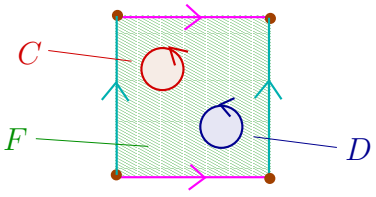
(a) Let $k \in \mathbb{N}$. By the Orientation Covering–Existence Proposition 95.8 we know that the non-orientable surface N_k admits an essentially unique 2-fold covering $\Sigma \to N_k$ that is orientable. What is the genus of Σ?

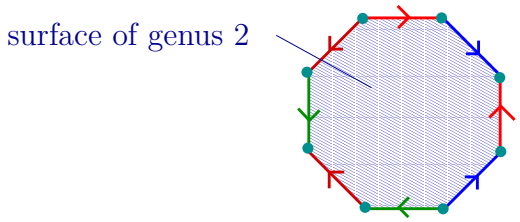
Hint. First deal with the cases $N_1 \cong \mathbb{R}\mathrm{P}^2$ and $N_2 \cong \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$. Given $k \in \mathbb{N}_{\geq 2}$ you can write N_k as the connected sum of an orientable smooth manifold with $\mathbb{R}\mathrm{P}^2$ or with $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$.

(b) Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. By the Orientation Covering–Existence Proposition 95.8 we now know that the non-orientable surface $N_{k,m}$ admits an essentially unique 2-fold covering $\Sigma \to N_{k,m}$ that is orientable. What is the genus of Σ and what is the number of boundary components of Σ?

Hint. In Exercise 156.7 we will give a different, perhaps less ad hoc approach to dealing with these two questions.

Exercise 101.20. Let F be the torus minus two open disks. We pick an orientation-reversing diffeomorphism $\varphi\colon C \to D$ between the two boundary components. Show that $F/c \sim \varphi(c)$ is diffeomorphic to the surface of genus two.

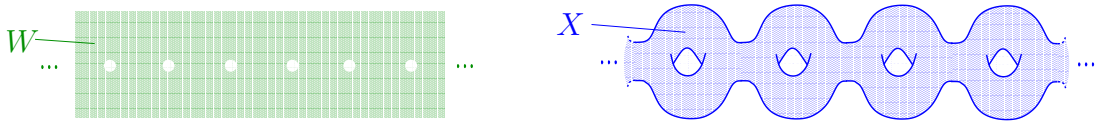




Exercise 101.21. For which values of $g \in \mathbb{N}$ and $n \in \mathbb{N}$ is every boundary component of $N_{g,n}$ a retract of $N_{g,n}$?



Exercise 101.22. Let $W = \mathbb{R}^2 \setminus \{(n,0) \mid n \in \mathbb{Z}\}$ and let X be the surface of infinite genus that is shown in the figure below. Show that W and X are not diffeomorphic.
Hint. What compact submanifolds can occur?
Remark. The same argument also shows that W is not diffeomorphic to the other surfaces of "infinite genus" that are shown in the figure below.
Remark. Later in Exercise 209.8 we will see that W and X are not homeomorphic. With the present knowledge it is at best quite challenging to give a watertight proof that W and X are not homeomorphic.

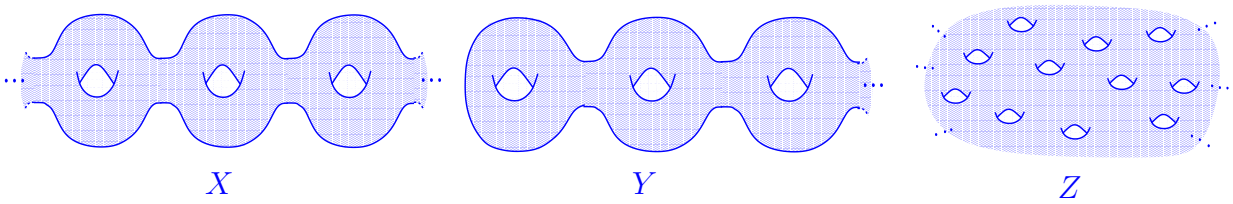


Exercise 101.23. Let X be a topological space. Given a compact subset $K \subseteq X$ we define $m(K)$ to be the number of non-compact path-components of $X \setminus K$ that are *not* contained in a compact subset of X. As in Exercise 4.34 we refer to

$$\mathrm{End}(X) \;:=\; \max\{m(K) \mid K \text{ is a compact subset of } X\} \in \mathbb{N}_0 \cup \{\infty\}$$

as the *number of ends of* X.

(a) Determine the number of ends for each of the three non-compact surfaces shown in the figure below.
(b) Which (if any) of the three "surfaces of infinite genus" shown in the figure below are homeomorphic?
(c) Is it true that given any $n \in \mathbb{N}$ there exists a simply connected 2-dimensional smooth manifold such that the number of ends equals n?



(d) Is it true that there exists a connected 2-dimensional smooth manifold such that the number of ends is infinite?

Exercise 101.24. We set

$$\begin{aligned} H &:= \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} \qquad \text{the open half-plane} \\ A &:= \{(x,0) \,|\, x < -1\} \\ B &:= \{(x,0) \,|\, x > 1\}. \end{aligned}$$

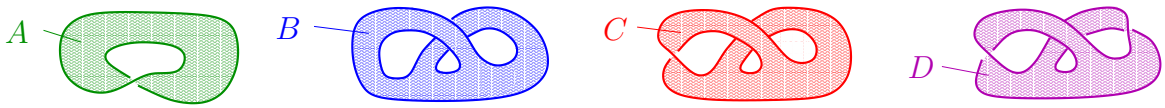
We consider the smooth manifold $M = H \cup A \cup B$. A *track* from A to B is an oriented connected proper 1-dimensional smooth manifold T of M such that $\partial T = \{-P\} \cup \{Q\}$ where $P \in A$ and $Q \in B$.

(a) Show that if S and T are two tracks from A to B, then there exists a proper smooth isotopy from S to a track S' and from T to a track T' such that S' and T' are disjoint.

(b) Show that all tracks from A to B are smoothly isotopic.



Exercise 101.25. We consider the 2-dimensional topological manifolds A, B, C and D in the figure below. What is the homeomorphism type of $\widehat{A}, \widehat{B}, \widehat{C}$ and $\widehat{D}$?



Exercise 101.26. Let Σ_2 be the surface of genus 2.

(a) Does there exist a covering $\Sigma_2 \to S^1 \times S^1$?

(b) Does there exist a covering $S^1 \times S^1 \to \Sigma_2$?

Exercise 101.27. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \ldots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. In Definition 101.10 we defined

$$\Sigma \text{ minus } n \text{ open disks } := \Sigma \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2)).$$

Why did we not just pick n smooth embeddings of the *open* disk with disjoint images?

Exercise 101.28. Let X be a compact connected 2-dimensional smooth manifold. Does every self-homeomorphism of ∂X extend to a self-homeomorphism of X?

Exercise 101.29. Does there exist a finitely presented group which is not isomorphic to the fundamental group of a compact connected non-empty 2-dimensional topological manifold?

Exercise 101.30. Show that a 2-dimensional smooth manifold M is non-orientable if and only if it contains the Möbius band as a submanifold.

Exercise 101.31. We consider the topological space $X = (S^1 \times [0,1])/\sim$ where $(z,1) \sim (-z,1)$ for all $z \in S^1$. Show that X is homeomorphic to a topological space that we are familiar with.

CHAPTER 102

The Seifert–van Kampen Theorem applied to manifolds

In this chapter we will first prove a variation on the Seifert–van Kampen-Theorem 99.1 which is more suitable to smooth manifolds. We will use this variation to prove the Manifold-π_1-Realization Proposition 102.5 which says, in a nutshell, that every finitely presented group occurs as the fundamental group of a closed orientable smooth manifold of dimension ≥ 4.

This chapter is organized as follows:

- In Section 102.1 we will use smooth collars to prove the Manifold–Seifert-van Kampen Theorem 102.1.
- In Section 102.2 we will determine the fundamental groups of connected sums of smooth manifolds.
- In Section 102.3 we will show that any finitely presented group is the fundamental groups of a smooth manifold of dimension ≥ 4.
- In Section 102.4 we will provide exercises for Chapter 102.

In the next Chapter 103 we will discuss decision problems that arise in group theory and topology.

102.1. The Seifert-van Kampen Theorem for manifolds.

For many applications the following theorem, which is a variation on the Seifert–van Kampen-Theorem 99.1, is very useful.

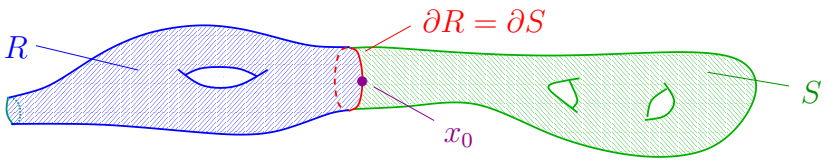
Theorem 102.1. (Manifold–Seifert-van Kampen Theorem) Let M be a smooth or topological manifold and let $R, S \subseteq M$ be two codimension zero submanifolds such that the following hold:[1060]

(1) $M = R \cup S$.
(2) R and S are closed subsets of M. (Note that by the Compact-Closed Lemma 3.39 (2) this condition is satisfied if R and S are compact.)
(3) $R \cap S$ is a single component of ∂R and it is a single component of ∂S.[1061 1062]

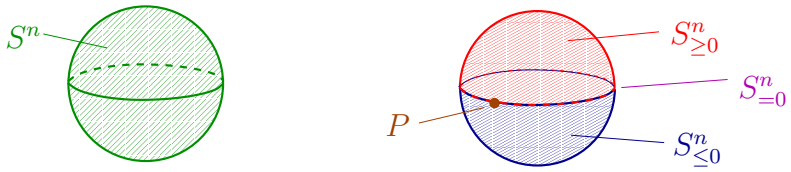
For every base point $x_0 \in R \cap S$ the inclusion-induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \ \to \ \pi_1(M, x_0)$$

is an isomorphism.



Example. Given $n \in \mathbb{N}_0$ we denote as usual by $S^n_{\geq 0}$ the upper hemisphere of S^n and we denote by $S^n_{\leq 0}$ the lower hemisphere of S^n.



Clearly the two hemispheres are closed subsets and one can show easily, say using the Preimage-of-Interval Lemma 38.3, that both are codimension-zero submanifolds of S^n. We denote by $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$ the "equator" of S^n, which is of course homeomorphic to S^{n-1}. Given any $n \in \mathbb{N}_{\geq 2}$ and any $P \in S^n_{=0}$ we can perform the following calculation:

$$\pi_1(S^n, P) \underset{\uparrow}{\cong} \pi_1(S^n_{\leq 0}, P) *_{\pi_1(S^n_{=0}, P)} \pi_1(S^n_{\geq 0}, P) \underset{\downarrow}{\cong} \{e\} *_{\pi_1(S^{n-1}, P)} \{e\} \underset{\uparrow}{\cong} \{e\}.$$

by the Manifold–Seifert-van Kampen Theorem 102.1, note that we use that our hypothesis $n \in \mathbb{N}_{\geq 2}$ implies that S^{n-1} is path-connected

by page 183 the hemispheres are homeomorphic to $\overline{B}^n$, hence the fundamental groups are trivial

by definition or by the Amalgamated Product–Quotient Lemma 98.25

This calculation gives another proof of the Sphere-π_1-Proposition 88.24.[1061] □

Of course the idea is to prove the Manifold–Seifert-van Kampen-Theorem 102.1 by reducing it to the Seifert-van Kampen-Theorem 99.1. But note that we cannot apply the Seifert-van Kampen-Theorem 99.1 directly, since R and S are not open. We will apply the same trick as in the proof of the Wedge-π_1-Proposition 99.10, namely we will expand R and S to open subsets U and V to which we can apply the Seifert-van Kampen-Theorem 99.1. The key to carrying out this idea is the following theorem.

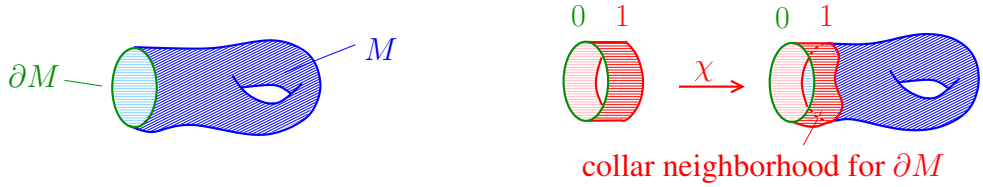
[1060]For smooth manifolds there is also a slight and useful variation: If M is a smooth manifold and if R, S are codimension-zero submanifolds with corner (in the sense of Definition 26.14) and if $R \cap S$ is a smooth boundary of R and S (in the sense of Definition 59.4), then the desired statements also hold. The only difference is that in the proof we need to replace the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7.

[1060]Since R and S are closed subsets and since they are codimension-zero submanifolds we know by the Codimension-Zero Submanifold Proposition 25.49 that the boundary of R and S viewed as subsets of M is actually given by boundary components of R and S viewed as smooth manifolds in their own right.

[1060]The fact that $R \cap S$ is a single component of ∂R implies, by the Smooth Manifold–Boundary Proposition 25.27 (1), the Manifold–Local Properties Lemma 23.8 (2), the Locally Path-Connected Lemma 4.57 (1b) and the Path-Component Lemma 4.44 (2) that $R \cap S$ is path-connected.

[1061]This argument is at the end quite similar to the proof of the Suspension-π_1-Lemma 99.3.

Theorem 102.2. (Smooth Collar Neighborhood Theorem) Given any smooth manifold M and given any union R of components of ∂M there exists a smooth an embedding $\chi\colon [0,1]\times R\to M$ with the following two properties:
(1) For every $P\in R$ we have $\chi(0,P)=P$.
(2) The image of $[0,1)\times R$ is an open subset of M.
The image of χ is sometimes called a collar neighborhood of R.



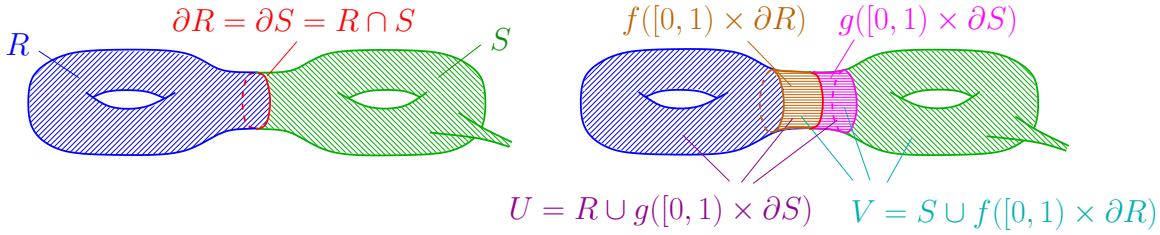
Proof of the Smooth Collar Neighborhood Theorem 102.2. Recall that in the Smooth Collar Neighborhood Theorem 35.6 we showed that there exists a smooth embedding $\chi\colon [0,1]\times\partial M\to M$ that satisfies (1) and such that $[0,1]\times\partial M$ is a closed subset of M. In the Basics-of-Collars Lemma 35.7 we saw that for each union R of components of ∂M the image of $[0,1)\times R$ is an open subset of M. ■

Proof of Theorem 102.1. [1062] First let M be a *smooth* manifold. Let $R,S\subseteq M$ be two codimension-zero submanifolds such that the following hold:

(1) $M=R\cup S$.
(2) R and S are closed subsets of M.
(3) $R\cap S$ is a component of ∂R and it is a component of ∂S.

In fact, to simplify the notation a little bit we assume that $R\cap S=\partial R=\partial S$. Let $x_0\in R\cap S$. We pick smooth embeddings $f\colon [0,1]\times\partial R\to R$ and $g\colon [0,1]\times\partial S\to S$ that are provided by the Smooth Collar Neighborhood Theorem 102.2. We set

$$U\ :=\ R\cup\underbrace{g([0,1)\times\partial S)}_{\subseteq S}\quad\text{and}\quad V\ :=\ S\cup\underbrace{f([0,1)\times\partial R)}_{\subseteq R}.$$



In the following claim we see that U and V are "open approximations" of R and S:

Claim.
(a) U is an open subset of M and R is a deformation retract of U.
(b) V is an open subset of M and S is a deformation retract of V.
(c) $U\cap V$ is an open subset of M and $R\cap S$ is a deformation retract of $U\cap V$.

[1062] The proof of the theorem is formally quite similar to the proof of the Compact–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 85.12.

Proof.

(a) (a) First we show that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (2) closed subsets of M we obtain from the Cover-Open Subset Lemma 3.20 (2a) that it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R. Furthermore $U \cap S = g([0,1) \times \partial S)$ is open in S by the second property of g, as stated in the Smooth Collar Neighborhood Theorem 102.2.

(b) We need to show that R is a deformation retract of U. To do we consider the following map:

$$\begin{array}{rcl} U \times [0,1] & \to & U \\ (u,t) & \mapsto & \begin{cases} u, & \text{if } u \in R, \\ g(s \cdot t, y), & \text{if } u = g(s,y) \in g([0,1) \times \partial S). \end{cases} \end{array}$$

It remains to show that this map is continuous. By the Homotopy Combination Lemma 18.8 (1) it suffices to show that R and $g([0,1) \times \partial S)$ are closed subsets of U. We see this as follows: By our hypothesis R and S are closed subsets of M. It follows from the Subspace-Characterization Lemma 3.17 (2) that $R = R \cap U$ and $g([0,1) \times \partial S) = S \cap U$ are closed subsets of U.

(b) The proof of this statement is of course almost identical to the proof of (a).

(c) The proof of this statement is very similar to the proof of (a). ⊞

We consider the following diagram:

$$\begin{array}{ccc} \pi_1(R,x_0) *_{\pi_1(R\cap S,x_0)} \pi_1(S,x_0) & \longrightarrow & \pi_1(M,x_0) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) & \longrightarrow & \pi_1(M,x_0). \end{array}$$

The following steps are similar to the ones we performed in the proof of the Wedge-π_1-Proposition 99.10.

(a) Since the vertical maps are induced by inclusions we see that the diagram commutes.

(b) We consider the following diagram:

$$\begin{array}{ccccc} \pi_1(R,x_0) & \longleftarrow & \pi_1(R\cap S,x_0) & \longrightarrow & \pi_1(S,x_0) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(U,x_0) & \longleftarrow & \pi_1(U\cap V,x_0) & \longrightarrow & \pi_1(V,x_0). \end{array}$$

Here all maps are induced by inclusions. It follows that the diagram commutes. Furthermore we obtain from the claim and the Homotopy Equivalence-π_1-Proposition 90.9 (1) that the vertical maps are all isomorphisms. It follows that the induced map between amalgamated products, shown in the initial diagram to the left, is an isomorphism.

(c) It follows from the claim that we can apply the Seifert–van Kampen-Theorem 99.1 to the decomposition $M = U \cup V$. Thus we see that the bottom horizontal map of the initial commutative diagram is an isomorphism.

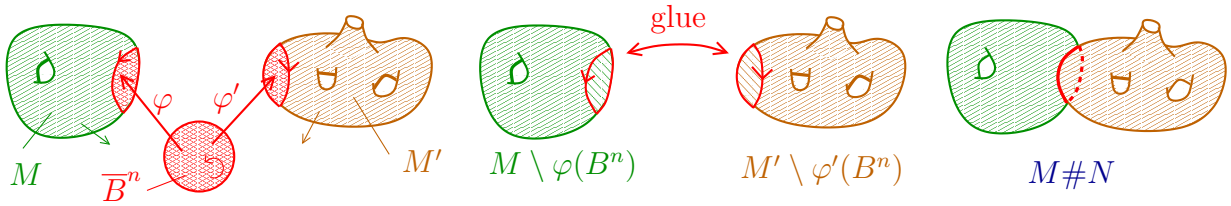
(d) Since the vertical maps are isomorphisms we see that the top horizontal map of the initial diagram is also an isomorphism.

Now we assume that M is a topological manifold. The argument is basically identical, we just need to replace the Smooth Collar Neighborhood Theorem 35.6 by the Topological Collar Neighborhood Theorem 35.14. ■

102.2. **Connected sum of smooth manifolds.** We start out by recalling Definition 54.12:

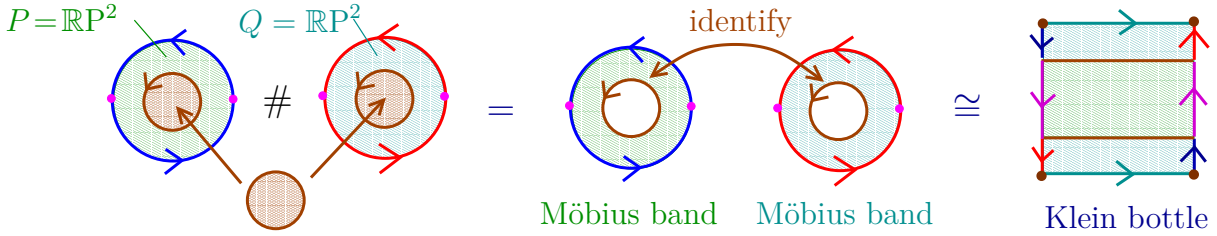
Definition 102.3. Let $n \in \mathbb{N}_0$. Let M and M' be two connected non-empty n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings. (If M and M' are both orientable, then we demand that M and M' are oriented, that φ is orientation-preserving and that φ' is orientation-reversing.) We define the connected sum of the smooth manifolds M and M' as[1063]

$$M\#M' \coloneqq \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big)/\sim \quad \text{where } \varphi(P)\sim\varphi'(P) \text{ for all } P\in S^{n-1}.$$



Example. In the Surface–Connected Sum Lemma 54.15 (1) we showed that the connected sum of g tori is diffeomorphic to the surface Σ_g of genus g. Furthermore in the Surface–Connected Sum Lemma 54.15 (2) we showed that the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface N_g of genus g. □

Example. It follows easily from the Möbius Band-in-Surface Lemma 25.51 that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle K.[1064] Just for fun and for practice we provide the direct calculation $\pi_1(K) \cong \pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2)$ using the Manifold–Seifert-van Kampen Theorem 102.1. In order to distinguish the two copies of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ we write $P = \mathbb{R}\mathrm{P}^2$ and $Q = \mathbb{R}\mathrm{P}^2$.



We view $\frac{1}{2} \in \overline{B}^2 \subseteq \mathbb{C}$ as a common base point. We have

$$\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2, \tfrac{1}{2}) = \pi_1(P\#Q, \tfrac{1}{2}) \underset{\uparrow}{\cong} \pi_1\big(P \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big) *_{\pi_1(\frac{1}{2}S^1, \frac{1}{2})} \pi_1\big(Q \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big).$$

by the definition of $P\#Q$ and the Manifold–Seifert-van Kampen Theorem 102.1

We pick a generator g for $\pi_1(\frac{1}{2}S^1, \frac{1}{2})$. In the calculation of $\pi_1(\overline{B}^2/\sim)$ on page 1977 we saw that $\pi_1(P \setminus \frac{1}{2}B^2, \frac{1}{2}) \cong \langle x\rangle$ and that the inclusion-induced map $\pi_1(\frac{1}{2}S^1) \to \pi_1(P \setminus \frac{1}{2}B^2, \frac{1}{2})$

[1063] In the following we will frequently use that by the Codimension-Zero Smooth Embedding Proposition 30.21 together with the Gluing-Smooth Manifolds-Proposition 54.5 we know that $M\#M'$ is naturally an n-dimensional smooth manifold, that the diffeomorphism type of $M\#M'$ is independent of the choice of φ and φ' (this is where the confusing condition on orientations enter), that $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$ are codimension-zero submanifolds of $M\#M'$ and that they are both closed subsets.

[1064] This statement also follows from the Surface–Connected Sum Lemma 54.15 (2) together with Exercise 9.17.

is given by $g \mapsto x^2$. Evidently the same holds for Q with $\pi_1(Q \setminus \frac{1}{2}B^2) \cong \langle y \rangle$. Summarizing we see that

$$\begin{aligned}\pi_1(K) \;\cong\; \pi_1(\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2, \tfrac{1}{2}) \;&\cong\; \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1, \frac{1}{2})} \pi_1\big(Q \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big) \\ &\cong \text{ amalgamated product of } \begin{pmatrix} \langle g \rangle & \to & \langle x \rangle \\ g & \mapsto & x^2 \end{pmatrix} \text{ and } \begin{pmatrix} \langle g \rangle & \to & \langle y \rangle \\ g & \mapsto & y^2 \end{pmatrix} \\ &\underset{\uparrow}{\cong} \;\; \langle x, y \mid x^2 = y^2 \rangle \;=\; \langle x, y \mid x^2 \cdot y^{-2} \rangle.\end{aligned}$$

by the Amalgamated Product Presentation Lemma 100.20

□

We now turn to the study of the fundamental groups of the connected sum of two smooth manifolds of dimension ≥ 3.

Proposition 102.4. (Connected Sum-π_1-Proposition) Let $n \geq 3$ and let M and M' be two connected non-empty n-dimensional smooth manifolds. Then[1065]

$$\pi_1(M \# M') \;\cong\; \pi_1(M) * \pi_1(M').$$

Example.

(1) Let (p, q) and (r, s) be two pairs of coprime integers. We consider the corresponding lens spaces $L(p, q)$ and $L(r, s)$. Note that

$$\pi_1(L(p,q) \# L(r,s)) \underset{\uparrow}{\cong} \pi_1(L(p,q)) * \pi_1(L(r,s)) \underset{\uparrow}{\cong} \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/r\mathbb{Z}.$$

by the Connected Sum-π_1 Proposition 102.4 — by the Lens Space-π_1 Proposition 93.2

Let us consider the connected sums $L(3,1) \# L(35,1)$ and $L(5,1) \# L(21,1)$. It follows from the Chinese Remainder Theorem 97.4 that the abelianizations of the fundamental groups are isomorphic. On the other hand it follows from the Indecomposability Lemma 98.33 and the Grushko Decomposition Theorem 98.34 that the fundamental groups themselves are not isomorphic, thus the two topological spaces $L(3,1) \# L(35,1)$ and $L(5,1) \# L(21,1)$ are not homeomorphic.

(2) Let $n \geq 3$ and let M and N be two oriented connected n-dimensional smooth manifolds. We denote by $-N$ the same smooth manifold as N but with the opposite orientation. It follows from the Connected Sum-π_1-Proposition 102.4 that $\pi_1(M \# N) \cong \pi_1(M \# (-N))$. Thus the fundamental groups cannot distinguish between $M \# N$ and $M \# (-N)$, even though there is a priori no reason why these should be homeomorphic.

(3) Let $m \geq 2$. We consider the connected sum of k copies of $S^1 \times S^m$. We see that:

$$\pi_1((S^1 \times S^m) \# \ldots \# (S^1 \times S^m)) \underset{\uparrow}{\cong} \pi_1(S^1 \times S^m) * \ldots * \pi_1(S^1 \times S^m) \underset{\uparrow}{\cong} \overbrace{\langle g_1 \rangle * \ldots * \langle g_k \rangle}^{= \langle g_1, \ldots, g_k \rangle}.$$

by the Connected Sum-π_1 Proposition 102.4, since $m \geq 2$ — by the Product-π_1-Proposition 90.13 we know that $\pi_1(S^1 \times S^m) \cong \langle g_i \rangle$

□

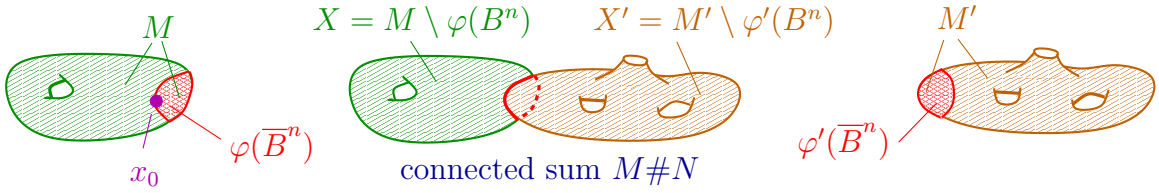
Proof of the Connected Sum-π_1-Proposition 102.4. Let M and M' be two connected non-empty n-dimensional smooth manifolds, and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two

[1065] Note that by the Smooth Connected Sum-Proposition 54.13 (0b) we know that $M \# M'$ is also connected, thus we do not need to worry about base points.

smooth embeddings. Recall that the connected sum of M and M' is defined as

$$M\#M' \coloneqq \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'}\big) / \sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for } P \in S^{n-1}.$$

We pick a base point x_0 that lies in the intersection of the images of X and X' in $M\#M'$.



Claim. The inclusion-induced maps $\pi_1(X, x_0) \to \pi_1(M, x_0)$ and $\pi_1(X', x_0) \to \pi_1(M', x_0)$ are isomorphisms.

Proof. First note that by symmetry it suffices to show that the inclusion-induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that X and $\varphi(\overline{B}^n)$ are n-dimensional submanifolds of M and that they are closed subsets of M. Note that $X \cap \varphi(\overline{B}^n) = \varphi(S^{n-1})$. From the Manifold–Seifert-van Kampen Theorem 102.1 we obtain the following pushout diagram:

$$\begin{array}{ccc} \pi_1(\varphi(S^{n-1}), x_0) & \longrightarrow & \pi_1(X, x_0) \\ \downarrow & & \ulcorner \downarrow \\ \pi_1(\varphi(\overline{B}^n), x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

Since $n \geq 3$ we obtain from the Sphere-π_1-Proposition 88.24 that $\pi_1(\varphi(S^{n-1}), x_0) = \{e\}$. Thus the vertical map to the left is an isomorphism. It follows from the Amalgamated Product Lemma 98.24 (2a) that the right vertical map is also an isomorphism. ⊞

Now we see that

$$\pi_1(M\#M', x_0) \underset{\text{by the Manifold–Seifert-van Kampen Theorem 102.1, since } X \cap X' \cong S^{n-1} \text{ is simply connected}}{\overset{\cong}{\leftarrow}} \pi_1(X, x_0) * \pi_1(X', x_0) \underset{\text{by the above claim}}{\overset{\cong}{\rightarrow}} \pi_1(M, x_0) * \pi_1(M', x_0).$$

■

102.3. **Fundamental groups of high-dimensional smooth manifolds.** Later on, in the Smooth Manifolds-Invariants Proposition 179.10 we will see that the fundamental group of a compact connected smooth manifold is necessarily finitely presented. In this section we want to study the converse: which finitely presented groups can occur as fundamental groups of compact smooth manifolds?

First note that it follows from the Topological 1-Manifold Classification Theorem 28.1 that only the trivial group and $\mathbb{Z}$ occur as the fundamental group of a compact connected 1-dimensional topological manifold.

Similarly it follows from the Surface Classification Theorem 101.16, the Surface-π_1-Propositions 101.1 and 101.5 and the Surface-with-Boundary–π_1-Lemma 101.13 that only "few" finitely presented groups occur as the fundamental group of compact connected 2-dimensional topological manifolds. In contrast we will now see that once we go to dimension

≥ 4 any finitely presented group occurs as the fundamental group of a closed orientable smooth manifold.

Proposition 102.5. (Manifold-π_1-Realization Proposition) Let π be a finitely presented group. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

Remark.

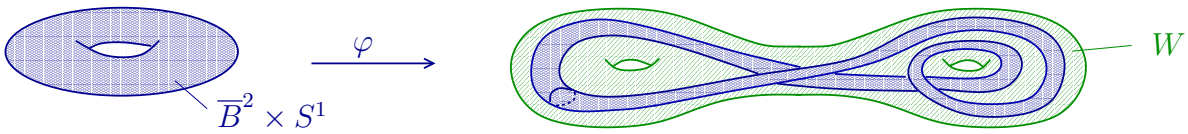
(1) We will give an alternative proof of the Manifold-π_1-Realization Proposition 102.5 on page 4933.
(2) The Manifold-π_1-Realization Proposition 102.5 is a key ingredient in the proof of the Manifold–Non-Classification Theorem 103.4 which says that closed smooth manifolds of a fixed dimension ≥ 4 cannot be classified. □

The idea of the proof is as follows. Given a finite presentation $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle$ we start out with the connected sum $X = (S^1 \times S^{n-1}) \# \ldots \# (S^1 \times S^{n-1})$ of k copies of $S^1 \times S^{n-1}$. Above we showed that the fundamental group of X is isomorphic to the free group $\langle g_1, \ldots, g_k \rangle$. We will modify X by removing "solid tori $\overline{B}^{n-1} \times S^1$" and gluing in copies of $S^{n-2} \times \overline{B}^2$ to modify the fundamental group.

To carry out this idea we need to formulate and prove two lemmas. The first lemma is actually quite interesting in its own right.

Lemma 102.6. (Drilling Out–π_1–Lemma) Let W be an n-dimensional smooth manifold. Let $k \in \mathbb{N}$ and let $\varphi \colon \overline{B}^{n-k} \times S^k \to W \setminus \partial W$ be a smooth embedding.

(1) If $n \geq k+3$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion-induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an isomorphism.
(2) If $n = k+2$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion-induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an epimorphism.



Proof of the Drilling Out–π_1–Lemma 102.6. First note that it follows from the Change-of-Base Point Proposition 90.4 that it suffices to prove the statement for some base point $w_0 \in \varphi(S^{n-k-1} \times S^k)$. We consider the following pushout diagram:

$$\begin{array}{ccc} \pi_1(\varphi(S^{n-k-1} \times S^k), w_0) & \longrightarrow & \pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \\ \downarrow & & \downarrow \\ \pi_1(\varphi(B^{n-k} \times S^k), w_0) & \longrightarrow & \pi_1(W, w_0) \end{array}$$

It follows from the Manifold–Seifert-van Kampen Theorem 102.1 (which we can apply by the Codimension-Zero Smooth Embedding Proposition 30.21) that this is a pushout diagram.

If $n \geq k+3$, then $n-k-1 \geq 2$. Thus we obtain from the Sphere-π_1-Proposition 88.24 that $\pi_1(S^{n-k-1})$ is trivial. It follows from the Product-π_1-Proposition 90.13 that the left

vertical map is an isomorphism[1066], thus it follows from the Amalgamated Product Lemma 98.24 (2a) that the right vertical map is – as promised – an isomorphism.

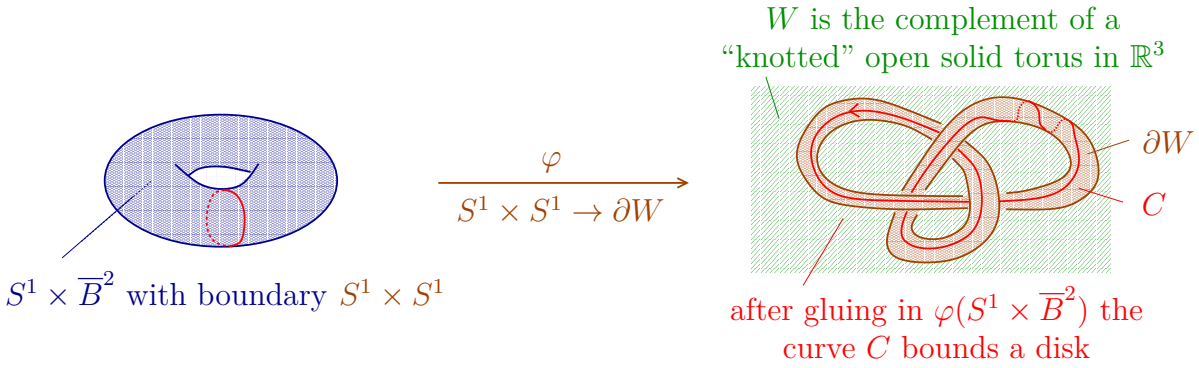
If $n = k+2$, then the left vertical map is an epimorphism. It follows from the Amalgamated Product Lemma 98.24 (2b) that the right vertical map is also an epimorphism. ■

Lemma 102.7. (Fill-in-Solid Torus-π_1-Lemma) Let W be an n-dimensional smooth manifold. Let $\varphi\colon S^{n-2}\times S^1 \to \partial W$ be a diffeomorphism onto a component of ∂W. We set

$$X \;:=\; (W \sqcup (S^{n-2}\times \overline{B}^2))/\sim \qquad \text{where } \varphi(x,y)\sim(x,y) \text{ for } (x,y)\in S^{n-2}\times S^1.$$

We pick $x\in S^{n-2}$ and we set $C := \varphi(\{x\}\times S^1)\subseteq \partial W$. If $n\geq 3$, then for any $w_0\in W$ the inclusion-induced map $\pi_1(W,w_0)\to\pi_1(X,w_0)$ induces an isomorphism[1067]

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\xrightarrow{\cong}\; \pi_1(X,w_0).$$



Proof of the Fill-in-Solid Torus-π_1-Lemma 102.7. We will only need the case $n\geq 4$. Thus in the following we will provide the (slightly easier) proof of the special case $n\geq 4$. We outsource the proof of the case $n=3$ to Exercise 102.5. Note that it follows from the Change-of-Base Point Proposition 90.4 that it suffices to prove the statement for some base point $w_0\in\varphi(S^{n-2}\times S^1)$. Next note that we have the following two isomorphisms:

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\underset{\uparrow}{\xrightarrow{\cong}}\; \pi_1(W,w_0)*_{\pi_1(S^{n-2}\times S^1,w_0)}\pi_1\big(S^{n-2}\times\overline{B}^2,w_0\big) \;\underset{\uparrow}{\xrightarrow{\cong}}\; \pi_1(X,w_0).$$

an isomorphism by the Amalgamated Product–Quotient Lemma 98.25, here we use that $\pi_1(S^{n-2}\times\overline{B}^2,w_0)=0$ and that $\pi_1(S^{n-2}\times S^1,w_0)=\langle C\rangle$, note that for the latter statement we used that $n\geq 4$

an isomorphism by the Manifold–Seifert-van Kampen Theorem 102.1 which we can apply by the Gluing-Smooth Manifolds-Proposition 54.5

Since the horizontal maps are induced by the inclusions we see that the inclusion-induced map $\pi_1(W,w_0)/\langle\langle C\rangle\rangle\to\pi_1(X,w_0)$ is indeed an isomorphism. ■

Now we are fully equipped for proving the Manifold-π_1-Realization Proposition 102.5.

Proof of the Manifold-π_1-Realization Proposition 102.5. Throughout the proof we will mostly ignore base points. But at the end of the proof we will argue, why being a little sloppy with base points in between does not affect the outcome.

Let $\pi = \langle g_1,\dots,g_k \,|\, r_1,\dots,r_l\rangle$ be a finitely presented group. Furthermore let $n\geq 4$. We consider

[1066] Note that this argument also applies for $k=1$

[1067] Here we use that Exercise 100.16 tells us that $C=\{0\}\times C$ gives us a well-defined normal subgroup $\langle\langle C\rangle\rangle$.

$$X \coloneqq \underbrace{(S^1\times S^{n-1})\#\ldots\#(S^1\times S^{n-1})}_{k \text{ copies of } S^1\times S^{n-1}}.$$

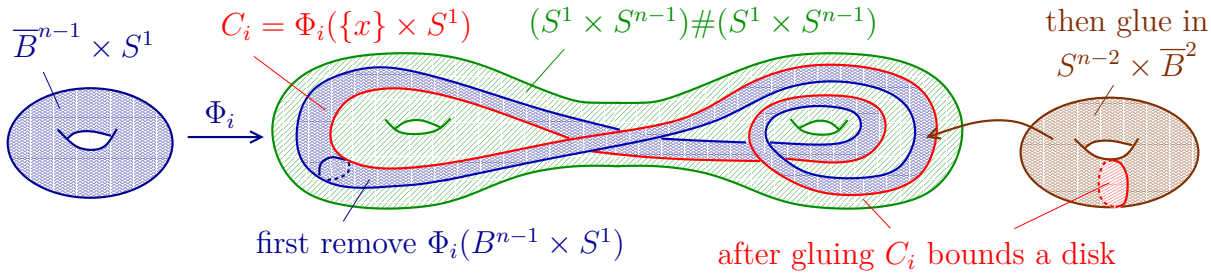
It follows basically from the Smooth Connected Sum-Proposition 54.13 that X is a closed orientable connected n-dimensional smooth manifold. As on page 2006 we see that

$$\pi_1(\overbrace{(S^1\times S^{n-1})\#\ldots\#(S^1\times S^{n-1})}^{=X}) \underset{\substack{\uparrow\\ \text{by the Connected Sum-}\pi_1\\ \text{Proposition 102.4,}\\ \text{since } n\geq 3}}{\cong} \pi_1(S^1\times S^{n-1})*\ldots*\pi_1(S^1\times S^{n-1}) \underset{\substack{\uparrow\\ \text{by the Product-}\pi_1\text{-Proposition 90.13}\\ \text{and since } n\geq 3 \text{ we know}\\ \text{that } \pi_1(S^1\times S^{n-1})\cong\mathbb{Z}}}{\cong} \overbrace{\langle g_1\rangle *\ldots* \langle g_k\rangle}^{=\langle g_1,\ldots,g_k\rangle}.$$

We use this isomorphism to make the identification $\pi_1(X,x_0)=\langle g_1,\ldots,g_k\rangle$. Next we pick loops $b_1,\ldots,b_l\colon (S^1,*)\to(X,x_0)$ that represent $r_1,\ldots,r_l\in\pi_1(X,x_0)=\langle g_1,\ldots,g_k\rangle$.

Since $\dim(X)=n\geq 3$ we can apply the Curve–Smooth Embedding Theorem **??** and we obtain that the map $b_1\sqcup\cdots\sqcup b_l\colon S^1\sqcup\cdots\sqcup S^1\to X$ is in fact homotopic to a smooth embedding $c_1\sqcup\cdots\sqcup c_l\colon S^1\sqcup\cdots\sqcup S^1\to X$. It follows from the Smooth Embedding Theorem 30.15 and the Product Tubular Neighborhood–Existence Theorem 58.4 (2)[1068] that there exist smooth embeddings $\Phi_i\colon \overline{B}^{n-1}\times S^1\to X$, $i=1,\ldots,l$ with disjoint images such that for each $i\in\{1,\ldots,l\}$ and each $y\in S^1$ we have $\Phi_i(0,y)=c_i(y)$. Next we consider[1069]

$$Y \coloneqq X\setminus\bigcup_{i=1}^{l}\Phi_i(B^{n-1}\times S^1) \qquad \text{and} \qquad Z \coloneqq \Big(Y\sqcup\bigsqcup_{i=1}^{l}(S^{n-2}\times\overline{B}^2)\Big)\Big/\underset{\substack{\uparrow\\ \text{where } \Phi_i(x,y)\sim(x,y) \text{ for}\\ x\in S^{n-2} \text{ and } y\in S^1}}{\sim}$$



[1068] Note that we can apply this theorem since X is orientable.

[1069] The smooth manifold Y is effectively obtained from $(S^1\times S^{n-1})\#\ldots\#(S^21\times S^{n-2})$ by removing copies of $\overline{B}^{n-1}\times S^1$ and gluing in copies of $S^{n-2}\times\overline{B}^2$. This operation is frequently called a surgery. We will discuss surgeries in greater detail in **??**.

It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 and the Gluing-Smooth Manifolds-Proposition 54.5 that Z is indeed a closed orientable connected n-dimensional smooth manifold. Finally we see that

the Fill-in-Solid Torus-π_1-Lemma 102.7 applied l times, hereby we set $C_i \coloneqq \Phi_i(\{x\} \times S^1)$ for some $x \in S^{n-2}$

the Drilling Out–π_1–Lemma 102.6 applied k times

$$\begin{array}{ccccc} \pi_1(Z) & \overset{\cong}{\leftarrow} & \pi_1(Y)/\langle\langle C_1, \dots, C_l \rangle\rangle & \overset{\cong}{\leftarrow} & \pi_1(X)/\langle\langle C_1, \dots, C_l \rangle\rangle \\ & \cong & \langle g_1, \dots, g_k \rangle / \langle\langle x_1 r_1 x_1^{-1}, \dots, x_l r_l x_l^{-1} \rangle\rangle & = & \underbrace{\langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_l \rangle\rangle}_{= \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle} . \\ & \uparrow & & \uparrow & \end{array}$$

by the above, the x_i are necessary since we lost track of the base points

by the Normal Generator Lemma 100.9 (3)

We have thus found the promised smooth manifold. ■

In the proof of the Manifold-π_1-Realization Proposition 102.5 we used on several occasions that we are dealing with dimensions ≥ 4. The following question arises naturally.

Question 102.8. Is every finitely presented group the fundamental group of a closed (or at least compact) orientable connected 3-dimensional smooth manifold?

We will answer Question 102.8 in the 3-Manifold–π_1–Abelian Theorem 257.7.

102.4. Exercises for Chapter 102.

Exercise 102.1. Let $g \in \mathbb{N}$. By the Surface Classification Theorem 101.16 we know that $\Sigma_g \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to $k \cdot \mathbb{R}\mathrm{P}^2$ for some unique $k \in \mathbb{N}$. Determine k as a function of g.

Exercise 102.2.

(a) Let M be an oriented connected m-dimensional smooth manifold and let N be a 1-dimensional submanifold *that is diffeomorphic to* S^1. Let $x_0 \in M \setminus N$. Suppose that $m \geq 4$. Show that the inclusion-induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is an isomorphism.
Hint. Use the Product Tubular Neighborhood–Existence Theorem 58.4 (2).

(b) Show that the conclusion (a) does not hold for $m = 3$, i.e. give an example of a connected 3-dimensional smooth manifold M and a connected 1-dimensional submanifold N such that the inclusion-induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is not an isomorphism.
Remark. The smooth manifold M in the example does not have to be closed.

Exercise 102.3. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We consider the map

$$\begin{array}{rccc} \varphi \colon & S^1 \times S^1 & \to & S^1 \times S^1 \\ & (z, w) & \mapsto & (z^q \cdot w^p, z^s r \cdot w^r). \end{array}$$

We set

$$M \coloneqq ((S^1 \times \overline{B}^{-2}) \sqcup (S^1 \times \overline{B}^2)) / \sim \qquad \text{where } (z, w) \sim \varphi(z, w).$$

Compute $\pi_1(M)$.

Exercise 102.4. Let M be a compact oriented connected non-empty smooth manifold of dimension $n \geq 3$ such that $\pi_1(M)$ admits an epimorphism onto a finite non-trivial group

G. Given $k \in \mathbb{N}$ we denote by $k \cdot M$ the connected sum of k copies of M. Show that the smooth manifolds $k \cdot M$, $k \in \mathbb{N}$ are pairwise non-diffeomorphic.
Hint. Consider the set of epimorphisms of a fundamental group onto G.

Exercise 102.5. Let W be a 3-dimensional smooth manifold. Let $\varphi\colon S^1 \times S^1 \to \partial W$ be a diffeomorphism onto a component of ∂W. We set

$$X \;:=\; (W \sqcup (S^1 \times \overline{B}^2))/\sim \qquad \text{where } \varphi(x,y) \sim (x,y) \text{ for } (x,y) \in S^1 \times S^1.$$

We pick $x \in S^1$ and we set $C := \varphi(\{x\} \times S^1) \subseteq \partial W$. Show that for any $w_0 \in W$ the map

$$\pi_1(W, w_0)/\langle\langle C \rangle\rangle \;\xrightarrow{\cong}\; \pi_1(X, w_0)$$

is an isomorphism.
Remark. Make use of Exercise 98.31 (b).

Exercise 102.6. Let $g \in \mathbb{N}$ and let Σ be the surface of genus g minus one open disk. We pick a diffeomorphism $\varphi\colon S^1 \to \partial\Sigma$. Let $\psi\colon S^1 \times \partial\Sigma \to \partial\Sigma \times S^1$ be the diffeomorphism given by $(z,w) \mapsto (\varphi(z), \varphi^{-1}(w))$. We consider

$$((S^1 \times \partial\Sigma) \sqcup (\partial\Sigma \times S^1))/\sim \qquad \text{where } (z,w) \sim (\varphi(z), \varphi^{-1}(w)) \text{ for } (z,w) \in S^1 \times \partial\Sigma.$$

We consider the map

$$\begin{array}{rcl} f\colon ((S^1 \times \partial\Sigma) \sqcup (\partial\Sigma \times S^1))/\sim & \to & \Sigma \times \Sigma \\ [(z,w)] & \mapsto & \begin{cases} (z, \varphi(w)), & \text{if } (z,w) \in S^1 \times \partial\Sigma, \\ (\varphi(z), w), & \text{if } (z,w) \in \partial\Sigma \times S^1. \end{cases} \end{array}$$

Show that for any $(z,w) \in S^1 \times \partial\Sigma$ the induced map

$$f_*\colon \pi_1(((S^1 \times \partial\Sigma) \sqcup (\partial\Sigma \times S^1))/\sim, (z,w)) \;\to\; \pi_1(\Sigma \times \Sigma, (z,w))$$

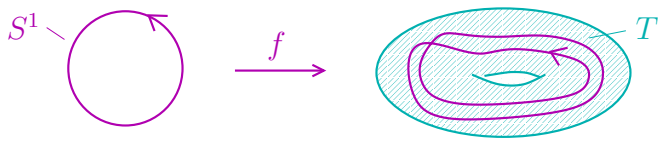
has non-trivial kernel.
Remark. You might want to use the Normal Form Theorem 98.28.

Exercise 102.7. Show that condition (2) in the statement of the Manifold–Seifert-van Kampen Theorem 102.1 cannot be dropped. More precisely, give an example of an m-dimensional smooth manifold and codimension-zero submanifolds R and S such that conditions (1) and (3) are satisfied, but such that the amalgamated product $\pi_1(R, x_0) *_{\pi_1(R \cap S, x_0)} \pi_1(S, x_0)$ is not isomorphic to $\pi_1(M, x_0)$.
Hint. The dimension m does not have to be taken particularly large.

Exercise 102.8. Let $n \in \mathbb{N}$, let M_n be S^2 minus n open disks and let C be a closed curve in $M_n \setminus \partial M_n$.

(a) Show that C is separating.
(b) Let $n = 3$. Show that C either bounds a disk or that it is isotopic to a boundary component of M_n.
(c) Show that the conclusion of (b) is in general not true for $n \geq 4$.

Exercise 102.9. Show that the map $f\colon S^1 \to T = S^1 \times S^1$ that is shown in the figure below is not homotopic to a smooth embedding.
Remark. This settles an issue that arose on page **??**.



Exercise 102.10. Let M be a topological space and let $R, S \subseteq M$ be subspaces such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M.
(3) R is a smooth manifold and S is a smooth manifold.
(4) $R \cap S$ is non-empty and path-connected.
(5) $R \cap S$ is a codimension-zero submanifold of ∂R and $R \cap S$ is a codimension-zero submanifold of ∂S.

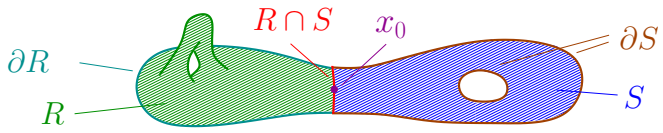
Then for every base point $x_0 \in R \cap S$ the inclusion-induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.

Hint. Apply the Smooth Collar Neighborhood Theorem 35.14 (several times) and make use of the Codimension-Zero Submanifold Proposition 25.49.

Remark. The case that $R \cap S$ is a component of ∂R and a component of ∂S is just a slight reformulation of the Manifold–Seifert-van Kampen Theorem 102.1. The more general case arises for example in the boundary connected sum construction from Definition 61.10.



Exercise 102.11. We consider the following two subspaces of $\mathbb{R}^2$:

$$U \;:=\; ([0,2] \times \{0\}) \cup \bigcup_{k=0}^{\infty} \{(x,y) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \,|\, \|(x - \tfrac{3}{2^{k+1}}, y)\| = \tfrac{1}{2^{k+1}}\},$$
$$V \;:=\; ([0,2] \times \{0\}) \cup \bigcup_{k=0}^{\infty} \{(x,y) \in \mathbb{R} \times \mathbb{R}_{\leq 0} \,|\, \|(x - \tfrac{3}{2^{k+1}}, y)\| = \tfrac{1}{2^{k+1}}\}.$$

Thus loosely speaking U and V are obtained by attaching infinitely many, ever smaller, half-circles to the interval $[0,2]$.

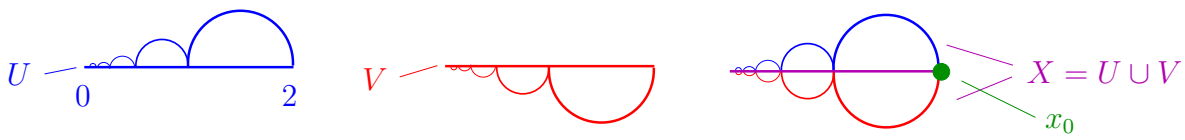
(a) Show that $X := U \cup V$ is the pushout of U and V.
Hint. Use the Regluing Lemma 8.17.

(b) We consider the base point $x_0 = (2,0) \in U \cap V$. Show that the inclusion-induced map

$$\pi_1(U, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

is *not* an isomorphism.

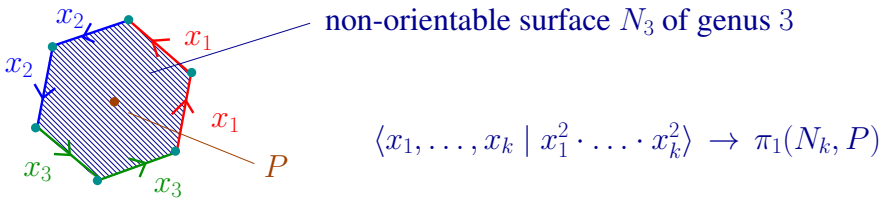
Remark. This shows that in the Seifert–van Kampen-Theorem 99.1 (or in the Manifold–Seifert-van Kampen Theorem 102.1) one cannot replace the stated hypothesis by the natural sounding hypothesis that U and V should form a push-out.



Exercise 102.12. Let $k \in \mathbb{N}_{\geq 2}$. We consider the non-orientable surface N_k of genus k from Definition 9.19. In the Surface-π_1-Propositions 101.5 we gave an explicit isomorphism

$$\langle x_1, \ldots, x_k \mid x_1^2 \cdot \ldots \cdot x_k^2 \rangle \xrightarrow{\cong} \pi_1(\text{non-orientable surface } N_k \text{ of genus } k).$$

In Definition 95.15 we introduced the orientation character $w_1^P \colon \pi_1(N_k, P) \to \mathbb{Z}/2\mathbb{Z}$. Determine this orientation character explicitly, i.e. determine $w_1^P(x_i)$, $i = 1, \ldots, k$.
Note. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian we can mostly ignore base point issues.



CHAPTER 103

Decision problems

In the last chapters we developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of topological spaces, e.g. topological graphs, surfaces and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most "nice" spaces. Furthermore we will see later in the Smooth Manifolds-Invariants Proposition 179.10 (1b) and the Topological Manifolds-Invariants Proposition 189.14 (1b) that the fundamental group of every compact topological manifold admits a *finite* presentation.

The question arises, do we now have the tools to classify our favorite objects? This question leads to the question whether problems in group theory and topological are algorithmically decidable. In this chapter we will see that all too often this is not the case.

This chapter is organized as follows:

- In Section 103.1 we will discuss decision problems in group theory.
- In Section 103.2 we will discuss decision problems in topology.
- In Section 103.3 we will provide exercises for Chapter 103.

In the next Chapter 104 we will introduce direct colimits.

103.1. **Decision problems in group theory.** The question that arises after all these calculations of fundamental groups is the following: What information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 103.1.

(1) Triviality Problem: Does there exist an algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents the trivial group?
(2) Isomorphism Problem: Does there exist an algorithm that can decide whether or not two given finite presentations $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ represent isomorphic groups?
(3) Word Problem: Does there exist an algorithm that can determine, given
    (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
    (b) a word w in $g_1, \dots, g_k$, i.e. an element $w \in \langle g_1, \dots, g_k \rangle$,
    whether or not w represents the trivial element in the group π?
(4) Conjugacy Problem: Does there exist an algorithm that can determine, given
    (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
    (b) and two words v and w in $g_1, \dots, g_k$,
    whether or not v and w represent conjugate elements in the group π?

Remark.

(1) These above questions were first posed by Max Dehn [**Deh1911**] in 1911. In fact shortly afterwards, in 1912 Max Dehn [**Deh1912**] gave positive answers to the word problem and the conjugacy problem for the fundamental groups of compact 2-dimensional smooth manifolds.
(2) The above questions are formulated in a slightly informal way. After all, what is an "algorithm" supposed to be? The questions are made more precise in [**CZ1993**, Chapter 7], [**Stil1982**] and [**MillC1992**]. □

In Definition 100.24 we introduced the Tietze transformations on presentations. In the Tietze Theorem 100.25 we showed that any two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle | \rangle$. The problem is that at times it can be very hard to find the correct Tietze transformations.

Example. We consider the presentation

$$\pi \;=\; \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1} \rangle.$$

This is a presentation of the trivial group as can be seen as follows:[1070] [1071]

(1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
(2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,
(3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
(4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
(5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
(6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
(7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
(8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement. □

Now suppose that we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. If we are given a finite presentation of the trivial group and if we systematically run through all finite sequences of Tietze transformations, then sooner or later we end up with the empty presentation.

But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation? Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?

So the real question is, does there exist an algorithm, that given a presentation which defines a *non-trivial* group, actually verifies that the group is *non-trivial*?

Unfortunately the answer to this question and in fact all of the above questions is quite sobering.

[1070] We leave it to the reader to show that these steps (1)–(7) can be realized by Tietze transformations.
[1071] For elements g, h of $\langle a, b, c \rangle$ we write $g \sim h$ if they represent the same element in π.

Theorem 103.2. (Groups–Non-Solvability Theorem) The answer to all four questions raised in Question 103.1 is no.

Remark. Here "no" means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm. □

Proof.

(1) Sergei Adyan [**Ady1955**] and Michael Rabin [**Rabi1958**] showed in 1955 that the answer to the first question is no. Alternatively see [**MillC1992**, Theorem 3.3] and [**Gor1995**, Theorem 3] for a proof.

(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle | \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no.

(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either.

Historically, it was first proved by Sergey Novikov [**Nov1955**] and William Boone [**Boo1958**] that (3) has a negative answer before a negative answer to (1) was given.

(4) An element in a group is trivial if and only if it is conjugate to the trivial element. Hence the fact that (3) has a negative answer leads to a negative answer to (4). ■

Remark.

(1) There are many other non-existence results in the vein of the Groups–Non-Solvability Theorem 103.2. For example, [**MillC1992**, Corollary 3.4], see also Exercise 103.2, says the following:
   (a) it is undecidable whether a finitely presented group is free,
   (b) it is undecidable whether a finitely presented group is abelian,
   (c) it is undecidable whether a finitely presented group is nilpotent,
   (d) it is undecidable whether a finitely presented group is solvable,
   (e) it is undecidable whether a finitely presented group is simple,
   (f) it is undecidable whether a finitely presented group is torsion-free.

(2) The statement of the Groups–Non-Solvability Theorem 103.2 that the answer to all four questions of Question 103.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example Mikhail Gromov [**Grom1987**][1072] introduced the notion of a "word hyperbolic group". Examples of word hyperbolic groups are given by
   (a) finite groups,
   (b) free groups, fundamental groups of surfaces of genus ≥ 2,
   (c) fundamental groups of hyperbolic smooth manifolds of any dimension,
   (d) fundamental groups of "Riemannian manifolds with negative sectional curvature".

[1072] Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as "geometric objects", giving rise to the subject of "geometric group theory".

In fact in a precise sense a "random" finitely presented group is word hyperbolic, see e.g. [**Oll2005**] for more details. If we restrict ourselves to the class of all word hyperbolic groups, then it is known that the answer to all four questions posed in Question 103.1 turns out to be yes. We refer to [**BrHa1999, Löh2015, DG2011, Sel1995**] for an introduction to word hyperbolic groups and for details on the various algorithms that deal with the problems in the setting of word hyperbolic groups. □

103.2. **Decision problems in topology.** Of course our main goal is not to classify groups, but our goal is to understand "nice" topological spaces. For example ideally we would like to classify compact topological and smooth manifolds. The following theorem gives us some hope that this task is not completely hopeless.

Theorem 103.3. (Compact Manifold–Isomorphism Type Theorem) Let $n \in \mathbb{N}_0$.

(1) There exist only countably many homeomorphism classes of compact n-dimensional topological manifolds.
(2) There exist only countably many diffeomorphism classes of compact n-dimensional smooth manifolds.

Remark. When it comes to noncompact topological manifolds the situation is much worse. In [**McMi1962**, Theorem 2], [**Gla1967**, Theorem 3] and [**CuK1965**, Theorem 5.1] it is shown that for any dimension $n \geq 3$ there are exist uncountably many pairwise non-homeomorphic contractible n-dimensional topological manifolds with empty boundary. We refer to [**AO2021**] for more references. □

Proof. Statement (1) is proved in [**ChK1970**]. Statement (2) follows from [**Pets1984**, p. 77] and the fact that every smooth manifold admits a Riemannian structure, see e.g. the Inner Product-on-Vector Bundle Proposition 269.10. For $n \geq 6$ Statement (2) can also be deduced from the fact that every compact n-dimensional smooth manifold admits a finite simplicial structure, see the Smooth Manifold-Simplicial Structure Theorem 179.5, the fact that there are only countably many isomorphism classes of finite simplicial complexes, see Exercise 174.4, and the (very difficult) High-Dimensional Manifold–Smooth Structure Theorem **??** which says that for $n \geq 6$ every compact topological manifold admits at most finitely many smooth structures. ■

Given $n \in \mathbb{N}$ we ideally would like to classify closed orientable connected n-dimensional smooth manifolds up to diffeomorphism. For $n = 1$ we did so in the Smooth 1-Manifold Classification Theorem 28.8 and for $n = 2$ we did so in the Surface Classification Theorem 101.16. In particular we saw that in these dimensions only "few groups" appear as fundamental groups.

We also proved the following result which shows that in dimension ≥ 4 the situation is dramatically different.

Proposition 102.5. (Manifold-π_1-Realization Proposition) Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

The Non-Solvability Theorem 103.2 and the Manifold-π_1-Realization Proposition 102.5 are at the heart of the proof of the following theorem:

Theorem 103.4. (Manifold–Non-Classification Theorem) Let $n \geq 4$.

(1) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are homeomorphic.
(2) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are diffeomorphic.

Proof. Statement (1) was first proved by Aleksandr Markov [**Mark1958**]. We refer to [**BHP1968**, Theorem 1] and [**Hak1973**] for a detailed discussion of the proofs of both statements. In particular these references also explain in more detail how one can describe a closed smooth manifold with finite data. We also refer to [**Stil1993**, Chapter 9.4.2], [**CZ1993**, Chapter 7.2] and [**Gor2021**] for more information and variations on the above results. ■

This leaves us with the case of 3-dimensional smooth manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

Theorem 103.5. (Smooth 3-Manifold Recognition Theorem) There exists an algorithm that can decide whether or not two closed orientable connected 3-dimensional smooth manifolds are diffeomorphic.

Proof. The proof of this theorem is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [**Thu1982**] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 103.5. We refer to [**AFW2015**, Theorem 4.27] for details and precise references. ■

103.3. Exercises for Chapter 103.

Exercise 103.1.

(a) Show that every finite group is isomorphic to some permutation group S_n.
(b) Let $\pi = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finitely presented group. Suppose that we know that π is finite. Show that there exists an algorithm which determines an injective homomorphism $\varphi\colon \pi \to S_n$ for some $n \in \mathbb{N}$.
(c) Let $\pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ and $\Gamma = \langle h_1, \ldots, h_m \mid s_1, \ldots, s_n \rangle$ be two finitely presented groups. Suppose we know that π and Γ are finite. Show that there exists an algorithm that can determine whether or not π and Γ are isomorphic.

Exercise 103.2.

(a) Show that it is undecidable whether a finitely presented group is free.
(b) Show that it is undecidable whether a finitely presented group is abelian.
(c) Show that it is undecidable whether a finitely presented group is nilpotent.
(d) Show that it is undecidable whether a finitely presented group is solvable.

Somewhat more precisely, for each group property show that there is no algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ represents a group that has said property.

Exercise 103.3.

(a) Is it decidable whether a finitely presented group is isomorphic to $\mathbb{Z}$?
(b) Let $G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finitely presented group. Is it decidable whether two elements $a, b \in \langle g_1, \dots, g_k \rangle$ represent commuting elements in G?

Exercise 103.4. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha \colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial. Show that the word problem is solvable in a finitely generated group that is residually finite.
Remark. A solution to this exercise can be found in [**MillC1992**, Theorem 5.2].

Part XI

Fundamental Groups II - Advanced topics

This part of the lecture notes contains several bonus chapters on fundamental groups, which are nice to have, but which are also not essential for the main story line of these notes.

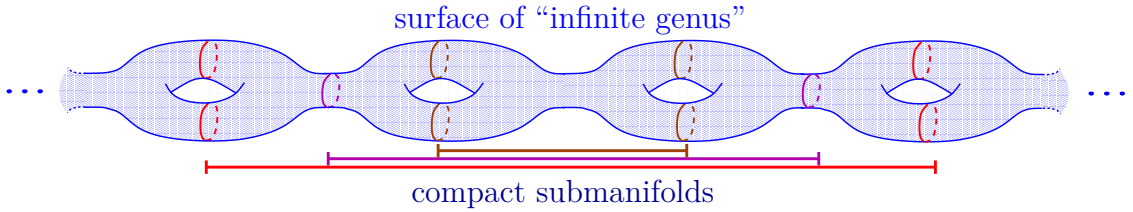
This part of the lecture notes is organized as follows:

- In Chapter 104 we will discuss the categorical notion of direct colimits.
- In Chapter 105 we will use direct colimits to determine fundamental groups of non-compact topological spaces that can be written as the union of compact topological spaces.
- In Chapter 106 we will prove the HNN-Seifert–van Kampen Theorem 106.7 which will allow us to calculate the fundamental group of a topological space that is obtain by gluing together two subspaces of a given topological space.
- In Chapter 107 we will study several topics on curves and surfaces.

CHAPTER 104

Direct colimits

The goal of this and the next chapter is to determine the fundamental group of "unbounded spaces", e.g. we would like to determine the fundamental group of the surface of "infinite genus" illustrated in the figure below.[1073] One idea might be to view the surface of "infinite genus" as a "limit" of a direct sequence of compact surfaces. But then we need to make precise what we mean by a "limit of a direct sequence of topological spaces" and "limit of a direct sequence of (fundamental) groups". We will introduce these notions in this chapter. In the next chapter we will use this language of limits to calculate, among many other things, the fundamental group of the surface of infinite genus.



The results of this and the following chapter are also essential ingredients in the proof of the HNN-Seifert-van Kampen Theorem 106.7 which is a very useful variation on the Seifert-van Kampen Theorem 99.1.

This chapter is organized as follows:

- In Section 104.1 we will introduce directed sets.
- In Section 104.2 we will introduce direct systems and direct colimits.
- In Section 104.3 we will study direct colimits in the category of sets and (abelian) groups.
- In Section 104.4 we will study the direct colimit in the category of topological category.
- In Section 104.5 we will consider examples and properties of the direct colimit in the category of topological spaces.
- In Appendix 104.6 we will study the interplay between direct colimits and products.
- In Section 104.7 we will provide exercises for Chapter 104.

In the next Chapter 105 we will discuss the relationship between fundamental groups and direct colimits.

104.1. **Directed sets.** The following definition partly recalls and extends some notions that we introduced on page 100.

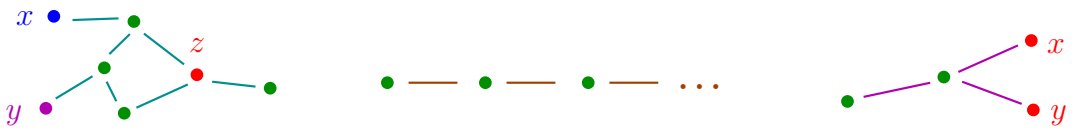
Definition 104.1.

[1073] Actually we also need to make precise what we mean by the surface of "infinite genus".

(1) Let X be a set and let $\leq$ be a relation on X.
   (a) The relation is called reflexive if for every $x \in X$ we have $x \leq x$.
   (b) The relation is called transitive if for every $x, y, z \in X$ we have
$$x \leq y \text{ and } y \leq z \implies x \leq z.$$
   (c) The relation is called antisymmetric if $x \leq y$ and $y \leq x$ implies $x = y$.
   (d) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(2) A directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) We consider the three partial orders that are illustrated in the figure below. Here it is understood that $P \leq Q$ if P is connected to Q via a path that goes throughout from left to right. We see that the two partial orders to the left form directed sets whereas the one on the right does not form a directed set.



directed set directed set not a directed set

(2) Let X be a subset of $\mathbb{R}$. If we denote by $\leq$ the usual "less or equal" relation on the real numbers, then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
(3) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. We consider the relation that is given by the inclusion "$\subseteq$". It follows easily from the definitions that $(\mathcal{P}(M), \subseteq)$ is a directed set. □

104.2. **Direct systems and direct colimits.** Now that we know what a directed set is, we can define the notion of a direct system in a category:

Definition 104.2. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over $(I \leq)$ is a diagram (in the sense of Definition 6.10) in $\mathcal{C}$ over the directed set $(I, \leq)$. In other words, a direct system in $\mathcal{C}$ over $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}_{i \leq j}$ such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,
(2) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

For the most part we are interested in the following special case of a direct system:

Definition 104.3. Let $\mathcal{C}$ be a category and let $w \in \mathbb{Z}$. A direct sequence[1074] in the category $\mathcal{C}$ consists of a family $\{X_i\}_{i \in \mathbb{Z}_{\geq w}}$ of objects and a family of morphisms $\{\varphi_i \colon X_i \to X_{i+1}\}_{i \in \mathbb{Z}_{\geq w}}$. Evidently we usually write a direct sequence as follows:
$$X_w \xrightarrow{\varphi_w} X_{w+1} \xrightarrow{\varphi_{w+1}} X_{w+2} \xrightarrow{\varphi_{w+2}} \dots$$
A direct sequence gives rise to a direct system over the directed set $(\mathbb{Z}_{\geq w}, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} := \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i \colon X_i \to X_j$.

[1074] Later, in Definition 229.3 we will introduce the notion of an "inverse sequence".

Examples.

(A) In the category $\mathcal{AbGroup}$ of abelian groups we consider the direct sequence

$$\mathbb{Z} \xrightarrow{v\mapsto(v,0)} \mathbb{Z}^2 \xrightarrow{v\mapsto(v,0)} \mathbb{Z}^3 \xrightarrow{v\mapsto(v,0)} \dots.$$

(B) In the category $\mathcal{Group}$ of groups we consider the direct sequence

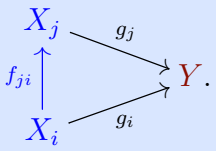
$$\langle x_1\rangle \hookrightarrow \langle x_1, x_2\rangle \hookrightarrow \langle x_1, x_2, x_3\rangle \hookrightarrow \dots$$

of free groups where the morphisms are the obvious inclusion homomorphisms. □

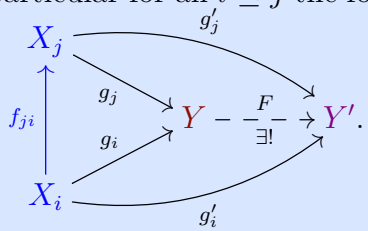
The experienced reader will by now have anticipated the following definition:

Definition 104.4. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ in the category $\mathcal{C}$. A direct colimit[1075] of the direct system is a colimit[1076], in the sense of Definition 6.21, of the direct system viewed as a diagram. In other words, a direct colimit is an object Y in $\mathcal{C}$ together with a family of morphisms $\{g_i\colon X_i \to Y\}_{i\in I}$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ji} = g_i\colon X_i \to Y$, i.e. for all $i \leq j$ the following diagram commutes:



(2) If we are given another object Y' and a family of morphisms $\{g'_i\colon X_i \to Y'\}_{i\in I}$ that satisfy (1), then there exists a unique morphism $F\colon Y \to Y'$ such that for all $i \in I$ we have $g'_i = F \circ g_i\colon X_i \to Y'$. In particular for all $i \leq j$ the following diagram commutes:



By the Colimit Uniqueness Lemma 6.22 we know that if a direct colimit exists, then it is unique in the usual sense. We denote the direct colimit, if it exists, by $\varinjlim X_i$ or sometimes by $\varinjlim_I X_i$ or $\varinjlim_i X_i$. [1077]

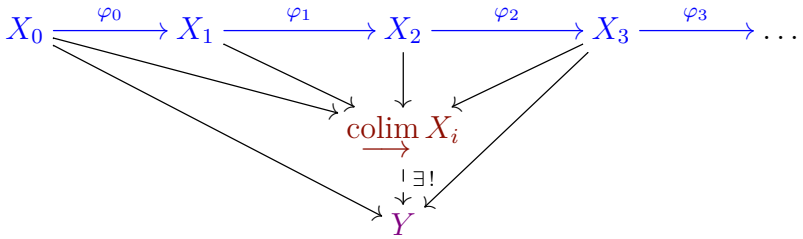
Remark.

[1075] Somewhat confusingly a "direct colimit" is often also called an "inductive limit".

[1076] It is somewhat unfortunate that what historically is called a direct colimit is actually a colimit. It is important to remember this, since as a general principle colimits tend to commute with colimits, but colimits do not commute with limits. For example we will see in the Direct Colimit–Product–Non-Commuting Lemma 104.23 that in general colimits do not commute with products.

[1077] Here we ignore the maps f_{ji} and g_i in the notation.

(1) In the case of a direct sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct colimit can be summarized in the following diagram:



More precisely, a direct colimit $\operatorname{colim} X_i$, if it actually exists, comes with morphisms $X_i \to \operatorname{colim} X_i$ which form commutative diagrams, furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\operatorname{colim} X_i \to Y$ which makes everything commute.

(2) In Definition 229.2 we will introduce the notion of an inverse system, which is the same as a direct system, except that the directions of the arrows are reversed. This leads in Definition 229.4 to the notion of an inverse limit of an inverse system. □

Examples. We discuss the direct colimits of the direct systems we had discussed above.

(A) As on page 2025 we consider in the category *AbGroup* of abelian groups the direct sequence of monomorphisms $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \dots$ that is given by the inclusion maps $\varphi_i(v) := (v, 0)$. We claim that

$$\operatorname{colim} \mathbb{Z}^i \;=\; \mathbb{Z}^\infty \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

where the maps $g_j \colon \mathbb{Z}^j \to \operatorname{colim} \mathbb{Z}^i = \mathbb{Z}^\infty$, $j \in \mathbb{N}$ are the obvious monomorphisms. Indeed, if we are given an abelian group Y' and homomorphisms $g'_j \colon \mathbb{Z}^j \to Y'$ that satisfy $g'_{j+1} \circ \varphi_j = g'_j$ for all $j \in \mathbb{N}$, then we set

$$\begin{aligned} F \colon \operatorname{colim} \mathbb{Z}^i \;=\; \mathbb{Z}^\infty \;&\to\; Y' \\ (x_1, \dots, x_j, 0, \dots) \;&\mapsto\; g'_j(x_1, \dots, x_j). \end{aligned}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of the x_j.[1078] It is clear that for all $i \in \mathbb{N}$ we have $g'_i = F \circ g_i$. Furthermore, since any $x \in \mathbb{Z}^\infty$ is contained in the image of some g_j, we see that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have shown that the direct colimit is the free abelian group on an infinite countable generating set.

(B) As on page 2025 we consider in the category *Group* of groups the direct sequence of monomorphisms $\langle x_1 \rangle \to \langle x_1, x_2 \rangle \to \dots$. Similar to the argument in (A) one can show that

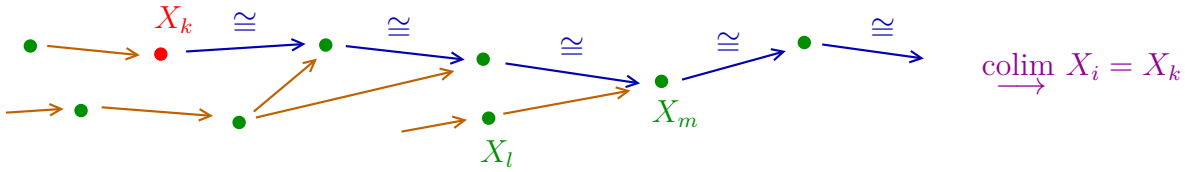
$$\operatorname{colim} \langle x_1, x_2, \dots, x_i \rangle \;=\; \langle x_1, x_2, \dots \rangle \;=\; \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \operatorname{colim} F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious homomorphisms. Put differently, we have shown that the direct colimit is the free group on an infinite countable generating set. □

[1078] Note that we did not demand that $x_j \neq 0$.

The following lemma gives a simple but frequently applicable approach to calculating a direct colimit.

Lemma 104.5. (Direct Colimit–Stabilization Lemma) Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system in the category $\mathcal{C}$. Suppose that there exists a $k \in I$ such that for all $l \geq k$ the morphism $f_{lk}\colon X_k \to X_l$ is in fact an isomorphism. Then $\varinjlim X_i$ exists and it is naturally isomorphic to X_k. More precisely, a direct colimit is given by the object $\varinjlim X_i := X_k$ and where for each $l \in I$ the morphism $X_l \to X_k$ is given by $f_{mk}^{-1} \circ f_{ml}\colon X_l \to X_k$ where $m \in I$ is chosen such that $m \geq k$ and $m \geq l$.



Proof. We will provide the proof in Exercise 104.8. ■

Example.

(1) If $(I, \leq)$ is a finite directed set, then there exists a maximal element $i_{\max}$. It follows from the Direct Colimit–Stabilization Lemma 104.5 that the direct colimit is given by $X_{i_{\max}}$.

(2) We have

$$\varinjlim \left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \dots\right) \underset{\text{by the Direct Colimit Stabilization Lemma 104.5}}{\cong} \mathbb{Z}.$$

□

104.3. **Direct colimits in the category of sets and (abelian) groups.** In the following we prove the existence of direct colimits in some of our favorite categories. We start out with the category of sets, which in its own might not be so interesting, but which will serve as a guide for other categories.

Proposition 104.6. (Sets–Direct Colimit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category $\mathcal{S}et$ of sets.

(1) The direct colimit is given by the set

$$\varinjlim X_i := \left(\bigsqcup_{i\in I} X_i\right)/\sim \quad \text{where}^{1079}\ x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i$$

together with the natural maps from each X_i into $\varinjlim X_i$.

(2) If all of the maps $f_{ji}\colon X_i \to X_j$ are inclusion maps, i.e. if for each $i \leq j$ we have $X_i \subseteq X_j$ and $f_{ji}\colon X_i \to X_j$ is the inclusion, then we can also take

$$\varinjlim X_i := \bigcup_{i\in I} X_i$$

together with the natural maps from each X_i into $\bigcup_{i\in I} X_i$.

[1079] Of course, as always, we mean by "$\sim$" the equivalence relation that is generated, in the sense of Definition 1.22, by the relation $x \sim f_{ji}(x)$.

Proof. Both statements follow easily from the definitions and elementary arguments. For peace of mind we carry out the argument in detail.

(1) This statement follows easily from the Colimit-via-Coproduct+Coequalizer Lemma 6.30 (1) and the Relation–Coequalizer Lemma 6.28.

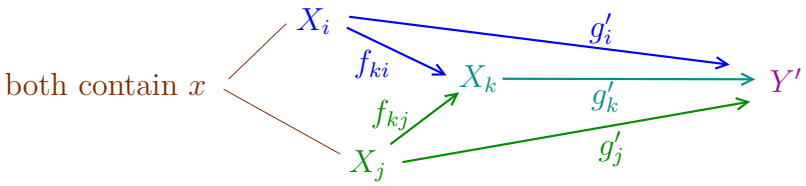
(2) We now assume that all of the maps $f_{ji}\colon X_i \to X_j$ are inclusion maps. For $i \in I$ we denote by $g_i\colon X_i \to \bigcup_{i\in I} X_i$ the natural inclusion. Since we only deal with inclusion maps it is clear that for any $i \leq j$ we have $g_j \circ f_{ji} = g_i$. Now suppose that we are given a set Y' and maps $g'_i\colon X_i \to Y'$ such that for all $i \leq j$ we have $g'_j \circ f_{ji} = g'_i\colon X_i \to Y$.

Claim. The map

$$\begin{aligned} F\colon \bigcup_{i\in I} X_i &\to Y' \\ x &\mapsto g'_i(x) \qquad \text{if } x \in X_i \end{aligned}$$

is well-defined.

Proof. Let $x \in \bigcup_{i\in I} X_i$. Suppose that $x \in X_i$ and $x \in X_j$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq i$ and $k \geq j$.



Now we see that

$$g'_i(x) = (g'_k \circ f_{ki})(x) = g'_k(f_{ki}(x)) \underset{\substack{\uparrow \\ \text{since } f_{ki} \text{ and } f_{kj} \text{ are} \\ \text{inclusion maps}}}{=} g'_k(x) \underset{\uparrow}{=} g'_k(f_{kj}(x)) = (g'_k \circ f_{kj})(x) = g'_j(x).$$

⊞

It follows immediately from the claim that the map $F\colon \bigcup_{i\in I} X_i \to Y'$ has the property that for all $i \in I$ we have $g'_i = F \circ g_i\colon X_i \to Y'$.

It remains to prove the uniqueness of F. Let $x \in \bigcup_{i\in I} X_i$. We pick $i \in I$ with $x \in X_i$. We necessarily have $F(x) = (F \circ g_i)(x) = g'_i(x)$. ■

Next we turn to our favorite algebraic categories. The Sets–Direct Colimit Existence Proposition 104.6 suggests that one can construct direct colimits as quotients of coproducts. This is indeed the case.

Proposition 104.7. (Algebraic–Direct Colimit Existence Proposition)

(1) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the following two categories:
   (a) the category *Group* of groups,
   (b) the category *AbGroup* of abelian groups.
   Let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$. A direct colimit is given by[1080] [1081]

$$\text{(a)} \quad \varinjlim X_i = \Big(\ast_{i\in I} X_i\Big) / \langle\langle f_{ji}(x) \cdot x^{-1} \mid \ i \leq j \text{ and } x \in X_i \rangle\rangle,$$

$$\text{(b)} \quad \varinjlim X_i = \Big(\bigoplus_{i\in I} X_i\Big) / \langle f_{ji}(x) - x \mid \ i \leq j \text{ and } x \in X_i \rangle,$$

with the obvious algebraic structure. Alternatively in each of the two categories a direct colimit can be chosen whose underlying set is given by

$$\operatorname*{colim}_{\longrightarrow} X_i \;=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i.$$

In particular in either of the two categories any direct system admits a direct colimit.

(2) Suppose we are in the setting of (1). If all of the homomorphisms $f_{ji}\colon X_i \to X_j$ are inclusion maps, then as underlying set we can always take

$$\operatorname*{colim}_{\longrightarrow} X_i \;=\; \bigcup_{i\in I} X_i.$$

Proof. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the two given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$.

(1) We consider the two types of descriptions of the colimit:

- For the category *Group* of groups it follows easily from the Coproduct-of-Groups Lemma 98.4, Exercise 98.29 and the Colimit-via-Coproduct+Coequalizer Lemma 6.30 (1) that

  (a) $$\operatorname*{colim}_{\longrightarrow} X_i \;=\; \Big(\mathop{\ast}_{i\in I} X_i\Big)/\langle\langle f_{ji}(x)\cdot x^{-1} \mid \ i \leq j \text{ and } x \in X_i\rangle\rangle.$$

  Similarly, for the category *AbGroup* of abelian groups it follows easily from the Coproduct-of-Abelian Groups Lemma 97.2, the Cokernel-as-Coequalizer 6.29 and the Colimit-via-Coproduct+Coequalizer Lemma 6.30 (1) that

  (b) $$\operatorname*{colim}_{\longrightarrow} X_i \;=\; \Big(\bigoplus_{i\in I} X_i\Big)/\langle f_{ji}(x) - x \mid \ i \leq j \text{ and } x \in X_i\rangle.$$

- Next we turn to the second description of colimits. We set

  $$\operatorname*{colim}_{\longrightarrow} X_i \;\coloneqq\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i$$

  and we denote by $g_j\colon X_j \to \operatorname*{colim}_{\longrightarrow} X_i$, $j \in I$, the natural maps. If we work in the category of (abelian) groups, then $\operatorname*{colim}_{\longrightarrow} X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$ is naturally an (abelian) group as follows: Let $a, b \in \operatorname*{colim}_{\longrightarrow} X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$. This means that there exist $i, j \in I$ and $\widetilde{a} \in X_i$ and $\widetilde{b} \in X_j$ with $a = [\widetilde{a}]$ and $b = [\widetilde{b}]$. By definition of a directed set there exists a $k \in I$ with $i \leq k$ and $j \leq k$. Using multiplicative notation we set

  $$a \cdot b \;\coloneqq\; \big[\underbrace{f_{ki}(\widetilde{a})}_{\in X_k} \cdot \underbrace{f_{kj}(\widetilde{b})}_{\in X_k}\big] \;\in\; \operatorname*{colim}_{\longrightarrow} X_i.$$

  It follows easily from the definition of a direct system that this definition of the product is independent of the choice of k and that this defines indeed an (abelian) group structure on $\operatorname*{colim}_{\longrightarrow} X_i$.

(2) This statement follows easily from the definitions. ∎

The following lemma allows us often to get a handle on elements in direct colimits:

[1080] Here given a group G and a subset A of G we denote by $\langle\langle A\rangle\rangle \subseteq G$ the subgroup normally generated by A, see Definition 98.20 and $\langle A\rangle$ denotes the subgroup generated by A. If G is abelian, then basically by definition we have $\langle\langle A\rangle\rangle = \langle A\rangle$.

[1081] Note that it follows from these explicit descriptions, together with the Countability Lemma 1.37, that the direct colimit of a countable family of countable (abelian or non-abelian) groups is again countable.

Lemma 104.8. (Elements-in-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in the category $\mathcal{AbGroup}$ of abelian groups or the category $\mathcal{Group}$ of groups. Then the following three statements hold:

(1) Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.
(2) Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{lj}(a_j) = f_{lk}(a_k) \in X_l$.
(3) Let $a_j \in X_j$. The image of a_j in $\varinjlim X_i$ is zero if and only if there exists a $k \geq j$ such that $f_{kj}(a_j) = 0 \in X_k$.

Proof.

(1) Statement (1) follows immediately from the explicit description of the direct colimit given in the Algebraic–Direct Colimit Existence Proposition 104.7.
(2) Statement (2) will be proved in Exercise 104.5. For the exercise it is useful to note that the symbol "$\sim$" in the definition of $\varinjlim X_i$ in the Algebraic–Direct Colimit Existence Proposition 104.7 is only a relation and we really have to work with the equivalence relation generated by this relation, see Definition 1.22 for more information.
(3) Statement (3) is a special case of Statement (2). ■

We conclude this section with a general definition:

Definition 104.9. Let $(I, \leq)$ be a directed set. Furthermore let $(\{K_i\}_{i\in I}, \{\chi_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems in a category $\mathcal{C}$.

(1) We define a morphism between the two direct systems to be a family $\{f_i\colon K_i \to L_i\}_{i\in I}$ of morphisms that has the property that for any $i \leq j$ the following diagram commutes:
$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle \chi_{ji}}\downarrow & & \downarrow{\scriptstyle \lambda_{ji}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$
We say that the morphism is an isomorphism if all f_i are isomorphisms.
(2) By the universal property of limits the morphisms $K_i \to L_i \to \varinjlim L_i$ give rise to a uniquely defined morphism
$$\varinjlim f_i\colon \varinjlim K_i \to \varinjlim L_i.$$

Remark. In Definition 209.11 we will consider a more general notion of a morphism between two direct systems. □

Example. We consider the two direct systems that are given by $\{\mathbb{Z}^i\}_{i\in\mathbb{N}}$ and $\{\mathbb{Z}^i \oplus 0\}_{i\in\mathbb{N}}$ where $\mathbb{Z}^i \oplus 0 \subseteq \mathbb{Z}^\infty$. We have the following isomorphisms:
$$\varinjlim \mathbb{Z}^i \quad \underset{\uparrow}{\cong} \quad \varinjlim (\mathbb{Z}^i \oplus 0) \quad \underset{\uparrow}{\cong} \quad \mathbb{Z}^\infty.$$
since the two direct systems are isomorphic — by the Algebraic–Direct Colimit Existence Proposition 104.7

We have thus obtained the result from page 2026 in a more systematic fashion. □

104.4. **The direct colimit in the topological category.** We now turn to the direct colimit in the topological category. In this setting we will only be interested in the case of a direct sequence of embeddings. We treat this special case in the following proposition,

which is formulated along the lines of the Topological-Product Proposition 7.2, the Topological-Pullback Proposition 7.22, the Topological-Coproduct Proposition 7.26, the Topological-Quotient Proposition 8.2, and the Topological-Pushout Proposition 8.12.

Proposition 104.10. (Topological-Direct Colimit Proposition) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding.

(0) We consider the set

$$\varinjlim X_i \coloneqq \Big(\bigsqcup_{i \in \mathbb{N}_0} X_i \Big) / \sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

Given $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \varinjlim X_i$ the natural map.

(a) We say that $U \subseteq \varinjlim X_i$ is open if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is an open subset of X_i. This defines a topology on $\varinjlim X_i$. In the remainder of this proposition we equip $\varinjlim X_i$ with this topology.

(b) A subset $U \subseteq \varinjlim X_i$ is closed with respect to the above topology if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is a closed subset of X_i.

(1) (a) For each $i \in \mathbb{N}_0$ the natural map $g_i \colon X_i \to \varinjlim X_i$ is an embedding.

(b) Let Y' be a topological space. Given a family of continuous maps $g_i' \colon X_i \to Y'$ with the property that for any $i \in \mathbb{N}_0$ we have $g_{i+1}' \circ \varphi_i = g_i'$, the map

$$\begin{aligned} G' \colon \varinjlim X_i &\to Y' \\ x &\mapsto g_i'(x_i) \qquad \text{if } x = g_i(x_i) \text{ for some } x_i \in X_i \end{aligned}$$

is well-defined and it is continuous.

(∗) A map $h \colon \varinjlim X_i \to Y'$ to some topological space is continuous if and only if each map $h \circ g_i \colon X_i \to Y'$ is continuous.

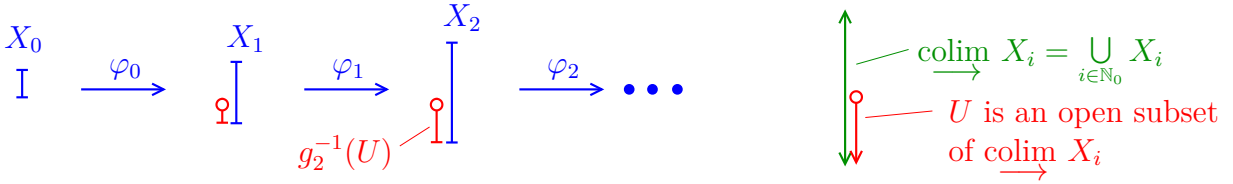
(2) The topological space $\varinjlim X_i$, together with the natural inclusions $g_i \colon X_i \to \varinjlim X_i$, is the direct colimit of the above direct sequence in the category of topological spaces.

If all the embeddings $\varphi_i \colon X_i \to X_{i+1}$ are in fact inclusion maps, i.e. if for each $i \in \mathbb{N}_0$ we have $X_i \subseteq X_{i+1}$ and $\varphi_i \colon X_i \to X_{i+1}$ is the inclusion, then we can replace (0) by

(0′) We consider the set

$$\varinjlim X_i \coloneqq \bigcup_{i \in \mathbb{N}_0} X_i.$$

We say that $U \subseteq \varinjlim X_i$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap X_i$ is an open subset of X_i. This defines a topology on X which again satisfies (1a), (1b) and (2).



Proof. [1082] Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

[1082] In this proposition and its proof we give an ad hoc explicit description of the colimit. Essentially the same description can also be obtained from the Colimit-via-Coproduct+Coequalizer Lemma 6.30 together with the Topological-Coproduct Proposition 7.26 and the Topological Relation–Coequalizer Lemma 8.19.

be a direct sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. We set

$$\varinjlim X_i \coloneqq \Big(\bigsqcup_{i\in\mathbb{N}_0} X_i\Big)/\sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

For each $i \in \mathbb{N}_0$ we consider the natural inclusion $g_i\colon X_i \to \varinjlim X_i$.

(0) (a) We set

$$\mathcal{T} \coloneqq \big\{U \subseteq \varinjlim X_i \,\big|\, \text{for each } i \in \mathbb{N} \text{ the preimage } g_i^{-1}(U) \text{ is open in } X_i\big\}.$$

One can easily verify, say using the Image-Preimage Lemma 1.9, that $\mathcal{T}$ defines a topology on $\varinjlim X_i$.

(b) This statement follows immediately from the definition of being closed and the observation that by the Image-Preimage Lemma 1.9 (8) we always have the equality $g_i^{-1}(\varinjlim X_i \setminus U) = g_i^{-1}(\varinjlim X_i) \setminus g_i^{-1}(U) = X_i \setminus g_i^{-1}(U)$.

(1) (a) Let $j \in \mathbb{N}_0$. We need to show that the natural inclusion $g_j\colon X_j \to \varinjlim X_i$ is an embedding. The map is evidently injective. It follows immediately from the definition of $\mathcal{T}$ that $g_j\colon X_j \to \varinjlim X_i$ is continuous. To show that $g_j\colon X_j \to X$ is actually an embedding it remains, by the Open-Closed–Injective Lemma 4.22 (2), to prove the following claim:

Claim. Given any open subset $U \subseteq X_j$ there exists an open subset $V \subseteq \varinjlim X_i$ with $V \cap g_j(X_j) = g_j(U)$.

Proof. Let $U_j \subseteq X_j$ be an open subset. Since $\varphi_j\colon X_j \to X_{j+1}$ is an embedding there exists an open subset $U_{j+1} \subseteq X_{j+1}$ such that $\varphi_j(X_j) \cap U_{j+1} = \varphi_j(U_j)$. We iterate this procedure to obtain subsets $U_j, U_{j+1}, U_{j+2}, \ldots$. We set $V \coloneqq \bigcup_{k\geq j} g_k(U_k)$. Evidently we have $V \cap g_j(X_j) = g_j(U_j)$. It remains to show that V is open in $\varinjlim X_i$. This means that we need to show that for every $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. For $i \geq j$ this follows immediately from the construction of V. It remains to consider the case $i < j$. First note that we have $g_{j-1}^{-1}(V) = (g_j \circ \varphi_{j-1})^{-1}(V) = \varphi_{j-1}^{-1}(g_j^{-1}(V)) = \varphi_{j-1}^{-1}(U_j)$. Since $U_j \subseteq X_j$ is open and since $\varphi_{j-1}\colon X_{j-1} \to X_j$ is continuous we see that $g_{j-1}^{-1}(U_j)$ is indeed an open subset of X_{j-1}. Iterating this argument we see that for all $i \in \{0, \ldots, j-1\}$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. ⊞

(b) As in the proof of the Sets–Direct Colimit Existence Proposition 104.6 we see that the map $G'\colon \varinjlim X_i \to Y'$ is well-defined. By construction we have $G' \circ g_i = g_i'$ for each $i \in \mathbb{N}_0$. It remains to show that the map G' is continuous. Thus let V be an open subset of Y'. We need to show that $(G')^{-1}(V)$ is an open subset of $\varinjlim X_i$. By definition of the topology on $\varinjlim X_i$ this means that we have to show that for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}((G')^{-1}(V))$ is an open subset of X_i. But by construction of the homomorphism G' we have $G' \circ g_i = g_i'$. Thus we see that $g_i^{-1}((G')^{-1}(V)) = (g_i')^{-1}(V)$, which is of course an open subset of X_i since g_i' is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows from (1a) and (1b) together with the observation, which we proved in the Sets–Direct Colimit Existence Proposition 104.6, that the map G in (1b) is the unique map which satisfies $G \circ g_i = g'_i$ for all $i \in \mathbb{N}_0$.

Finally note that if all the maps $f_{ji} \colon X_i \to X_j$ are in fact inclusion maps, then it follows by the same argument as in the Sets–Direct Colimit Existence Proposition 104.6 (2) that we can work with $\varinjlim X_i \coloneqq \bigcup_{i \in \mathbb{N}_0} X_i$. We leave it to the reader to make the few adjustments to finish off this proof. ∎

In many settings the following lemma gives us a convenient way to think about the direct colimit of topological spaces:

Lemma 104.11. (Direct Colimit-Characterization Lemma) Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ be a sequence of subspaces such that $X = \bigcup_{i \in \mathbb{N}_0} X_i$.

Then the following three statements are equivalent:

(a) We have $X = \varinjlim X_i$.

(b) For every subset $U \subseteq X$ we have

$$A \subseteq X \text{ is open} \iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap X_i \text{ is open in } X_i$$

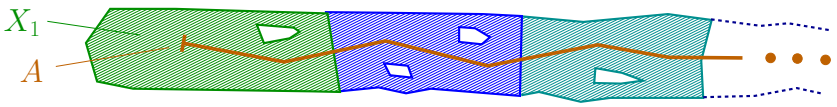
(c) For every subset $A \subseteq X$ we have

$$A \subseteq X \text{ is closed} \iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap X_i \text{ is closed in } X_i$$

Proof of Lemma 104.11. First of all note that it follows from the Iterated Subspace Lemma 3.18 that each inclusion $X_i \to X_{i+1}$ is an embedding. We now see that the equivalence of (a) and (b) is just a reformulation of the Topological-Direct Colimit Proposition 104.10 (0′).

Next we prove the (b)⇒(c)-direction. For each subset $A \subseteq X$ we now see that

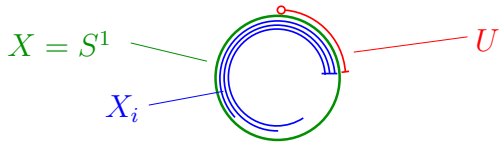
$$\begin{aligned} A \subseteq X \text{ is closed in } X &\iff X \setminus A \text{ is open in } X \\ &\iff \text{for all } i \in \mathbb{N} \text{ the subset } (X \setminus A) \cap X_i \text{ is open in } X_i \\ &\uparrow \\ &\text{since we assume that (b) holds} \\ &\iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap X_i \text{ is closed in } X_i \\ &\uparrow \\ &\text{follows from } (X \setminus A) \cap X_i = X_i \setminus (A \cap X_i) \text{ and} \\ &\text{the definition of closed subsets} \end{aligned}$$



The proof of the (c)⇒(b)-direction is basically the same as the proof of the (b)⇒(c)-direction, since a subset is open if and only if its complement is closed. ∎

Example. The Topological-Direct Colimit Proposition 104.10 raises the following question: If we are given a topological space X and subspaces $X_1 \subseteq X_2 \subseteq \ldots$ with $\bigcup_{i \in \mathbb{N}} X_i = X$, does the direct colimit topology $\varinjlim X_i$ necessarily agree with the original topology on X? It turns out that in general this is *not* the case. For example we consider the circle $X = S^1$ and the subspaces $X_i \coloneqq \{\exp(2\pi\,\mathrm{i}t) \mid t \in [0, 1 - \frac{1}{n})\}$.

For $U = \{\exp(2\pi\,\mathrm{i}t) \mid t \in [0, \frac{1}{4})\}$ we see that each $U \cap X_i$ is an open subset of X_i, but $U \subseteq S^1$ is of course not open.



Thus it follows from the Direct Colimit-Characterization Lemma 104.11 (a)$\Longleftrightarrow$(b) that X is not the direct colimit of the X_i. □

In contrast to the above example, the following two lemmas give us useful criteria for when we have $X = \varinjlim X_i$.

Criterion 104.12. (Union-equals-Direct Colimit Criterion) Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup_{i\in\mathbb{N}_0} X_i = X$.

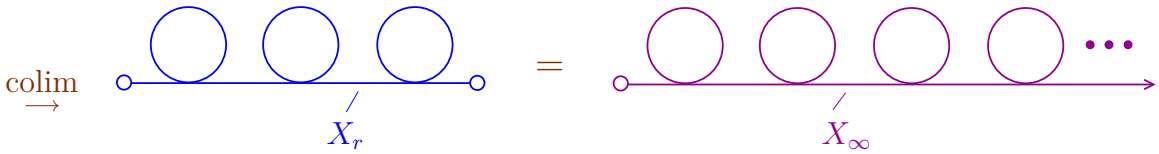
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subseteq \mathring{X}_{i+1}$.

Then $X = \varinjlim X_i$ as topological spaces.

Example. Given $r \in \mathbb{N}_0$ we set

$$X_r \;:=\; \big(0, r+\tfrac{1}{2}\big) \;\cup\; \bigcup_{j=1}^{r} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j+\tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j+\frac{1}{4}\mathrm{i} \,\in\, \mathbb{C}} \;\subseteq\; \mathbb{C}.$$

In other words, for $r \in \mathbb{N}_0$ the set X_r is the subspace of $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ that is given by the interval $(0, r+\frac{1}{2})$ with r circles attached. We obtain a direct sequence of continuous maps by considering the inclusions $X_r \to X_{r+1}$.



It follows fairly easily from the Union-equals-Direct Colimit Criterion 104.12 that the direct colimit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\varinjlim X_r \;=\; X_\infty \;:=\; \big(0,\infty\big) \;\cup\; \bigcup_{j=1}^{\infty} \big\{z \in \mathbb{C} \,\big|\, \big|z - (j+\tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\} \;\subseteq\; \mathbb{C},$$

where the maps $X_s \to \varinjlim X_r = X_\infty$, $s \in \mathbb{N}_0$ are the obvious inclusion maps. □

Proof. By the Direct Colimit-Characterization Lemma 104.11 (a)$\Leftrightarrow$(b) it suffices to prove the following claim:

Claim. For each subset $U \subseteq X$ we have

$$U \subseteq X \text{ is open} \quad\Longleftrightarrow\quad \text{for each } i \in \mathbb{N}_0 \text{ the subset } U \cap X_i \subseteq X_i \text{ is open.}$$

Proof. First we prove the "$\Rightarrow$"-direction. Thus let $U \subseteq X$ be open. But then, by definition of the subspace topology, each $U \cap X_i$ is open in X_i.

We turn to the proof of the "$\Leftarrow$"-direction. We now assume that each $U \cap X_i$ is open in X_i. By definition of the subspace topology this means that for each $i \in \mathbb{N}_0$ there exists

an open subset $V_i \subseteq X$ such that $V_i \cap X_i = U \cap X_i$. Now we see that

$$\begin{array}{ccccccc} \bigcup\limits_{i\in\mathbb{N}_0}(V_i\cap \mathring{X}_i) & \subseteq & \bigcup\limits_{i\in\mathbb{N}_0}(V_i\cap X_i) & \overset{\text{since } V_i\cap X_i = U\cap X_i}{=} & \bigcup\limits_{i\in\mathbb{N}_0}(U\cap X_i) & = & U\cap\bigcup\limits_{i\in\mathbb{N}_0} X_i \overset{\text{follows from (1)}}{=} U \\ & = & U\cap\bigcup\limits_{i\in\mathbb{N}_0}\mathring{X}_i & = & \bigcup\limits_{i\in\mathbb{N}_0}(U\cap\mathring{X}_i) & = & \bigcup\limits_{i\in\mathbb{N}_0}(V_i\cap\mathring{X}_i). \end{array}$$

since it follows from (1) and (2) that $X = \bigcup\limits_{i\in\mathbb{N}_0}\mathring{X}_i$

since $V_i \cap X_i = U \cap X_i$ and since $\mathring{X}_i \subseteq X_i$

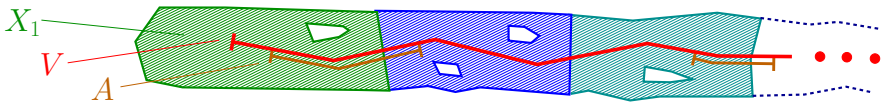
We see that all inclusions are equalities. This implies that $U = \bigcup\limits_{i\in\mathbb{N}_0}(V_i\cap\mathring{X}_i)$. Thus we have shown that U is the union of open subsets, which implies that U itself is open. ∎

Lemma 104.13. (Direct Colimit–Subspace Lemma) Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ be closed subsets of X such that $X = \operatorname{colim} X_i$. Then for each closed[1083] subset $V \subseteq X$ we have $V = \operatorname{colim}(V \cap X_i)$ as topological spaces.

Proof. Let X be a topological space, let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ be closed subsets of X such that $X = \operatorname{colim} X_i$ and let $V \subseteq X$ be a closed subset. For $i \in \mathbb{N}_0$ we set $V_i := V \cap X_i$. We want to show that $V = \operatorname{colim} V_i$. By the Direct Colimit-Characterization Lemma 104.11 (a)⟺(c) it suffices to prove the following claim:

Claim. For every subset $A \subseteq V$ the following statement holds:

$$A \subseteq V \text{ is closed} \iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap V_i \text{ is closed in } V_i$$



Proof. The "⇒"-direction is a consequence of the Iterated Subspace Lemma 3.18. Now we turn to the "⇐"-direction. We see that:

for all $i \in \mathbb{N}$ the subset $A \cap V_i$ is closed in V_i

$\Longrightarrow$ for all $i \in \mathbb{N}$ the subset $A \cap (V \cap X_i)$ is closed in $V \cap X_i$

$\Longrightarrow$ for all $i \in \mathbb{N}$ the subset $A \cap (V \cap X_i) = (A \cap V) \cap X_i$ is closed in X

↑ follows from the Open-Closed Inclusion Lemma 4.13, here we use that it follows from the hypothesis on V and X_i that $V \cap X_i$ is a closed subset of X

$\Longrightarrow$ the subset $A \cap V$ is closed in X

↑ by the Direct Colimit-Characterization Lemma 104.11 (a)⟺(c) applied to $X = \operatorname{colim} X_i$

$\Longrightarrow$ the subset V is closed in A.

↑ follows from the Subspace-Characterization Lemma 3.17 (2) ∎

We conclude this section with the following lemma, which we will use later on to build open subsets in direct colimits:

[1083] In Exercise 104.14 we will see that we cannot just drop the hypothesis that V is closed.

Lemma 104.14. (Direct Colimit–Open Subset Lemma) Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be subspaces of X with $\varinjlim X_i = X$. Suppose that we are given $w \in \mathbb{N}_0$ and suppose that for each $i \geq w$ we are given an open subset $U_i \subseteq X_i$. If for each $i \geq w$ we have $U_i \subseteq U_{i+1}$, then $\bigcup_{i \geq w} U_i$ is an open subset of X.

Proof. We set $U \coloneqq \bigcup_{i \geq w} U_i$. Since $X = \varinjlim X_i$ It follows from the Direct Colimit-Characterization Lemma 104.11 (a)$\Longleftrightarrow$(b) that it suffices to prove the following claim:

Claim. For each $j \in \mathbb{N}_0$ the intersection $X_j \cap U$ is open in X_j.

Proof. Thus let $j \in \mathbb{N}_0$. We set $m \coloneqq \max\{j, w\}$. Note that

$$X_j \cap U \;=\; X_j \cap \bigcup_{i \geq w} U_i \;\underset{\substack{\uparrow \\ \text{since for each } i \geq w \text{ we} \\ \text{have } U_i \subseteq U_{i+1}}}{=}\; X_j \cap \bigcup_{i \geq m} U_i \;=\; \bigcup_{i \geq m} (X_j \cap U_i).$$

Next note that for $i \geq m$ it follows from the fact that U_i is open in X_i together with the Iterated Subspace Lemma 3.18 that $X_j \cap U$ is open in X_j. In summary we have shown that $X_j \cap U_i$ is a union of open subsets of X_j, thus $X_j \cap U_i$ is an open subset of X_j. ■

104.5. **Examples and properties of the direct colimit of topological spaces.** In this section we study several direct colimits of natural direct sequences of embeddings of topological spaces which lead to new and interesting topological spaces. Along the way we also develop the theory of direct colimits of direct sequences of topological spaces a little further.

Definition 104.15.

(1) We consider the *set*

$$\mathbb{R}^\infty \;\coloneqq\; \mathbb{R}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R} \text{ but only finitely many } x_i\text{'s are non-zero}\}$$

(2) Given a subset $X \subseteq \mathbb{R}^k$ we write

$$X \times \{0\} \;\coloneqq\; \{(x, 0, 0, \dots) \mid x \in X\} \;\subseteq\; \mathbb{R}^\infty.$$

We have an obvious bijection of $X \times \{0\}$ to X and we use it to equip $X \times \{0\}$ with a topology.

(3) Unless we explicitly say something else we equip the set $\mathbb{R}^\infty$ with the topology given by $\mathbb{R}^\infty = \varinjlim (\mathbb{R}^i \times \{0\})$. In other words, $U \subseteq \mathbb{R}^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (\mathbb{R}^i \times \{0\})$ is open in $\mathbb{R}^i \times \{0\}$.

(4) Similarly we define $\mathbb{C}^\infty \coloneqq \varinjlim (\mathbb{C}^i \times \{0\})$ as a topological space.

Example.

(1) To practice the above definition we will show that any linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the above topology. The proof proceeds in two steps:

(a) By the Topological-Direct Colimit Proposition 104.10 (1b) it suffices to that show that for each $n \in \mathbb{N}_0$ the map

$$\begin{aligned} f_n \colon \mathbb{R}^n \times \{0\} &\to \mathbb{R}^\infty \\ x &\mapsto f(x, 0) \end{aligned}$$

is continuous.

(b) Let $n \in \mathbb{N}_0$. Note that there exists a $k \in \mathbb{N}$ such that $f(e_1), \dots, f(e_n) \in \mathbb{R}^k \times \{0\}$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n \times \{0\} & \xrightarrow{x \mapsto f(x,0)} & \mathbb{R}^\infty \\ \downarrow{\scriptstyle (x,0)\mapsto x} & & \uparrow{\scriptstyle y\mapsto (y,0)} \\ \mathbb{R}^n & \xrightarrow{\widetilde{f}} & \mathbb{R}^k \end{array}$$

for a suitable linear map $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}^k$. We make three observations:

- The left vertical map is a homeomorphism.
- Each linear map $\mathbb{R}^n \to \mathbb{R}^k$ is continuous.
- By the Topological-Direct Colimit Proposition 104.10 (1a) we know that the right vertical map is continuous.

It follows that the top horizontal map is continuous.

Note that we could equip the set $\mathbb{R}^\infty$ with the obvious generalization of the usual Euclidean metric. In Exercise 104.15 we will see that with the corresponding metric topology it is *not* true that every linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous.

(2) We consider the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, \dots) & \mapsto & (x_1, y_1, x_2, y_2, \dots). \end{array}$$

This map has an obvious inverse. Using the approach from (1) one can easily show that both maps are continuous, thus they are both homeomorphisms. At times we will use these homeomorphisms to make the identification $\mathbb{C}^\infty =_{\mathrm{i}} \mathbb{R}^\infty$ □

The above definition motivates the following related definition.

Definition 104.16. We consider the topological space

$$S^\infty := \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\middle|\, \sum_{j=1}^{\infty} x_j^2 = 1\right\} = \varinjlim (S^{i-1} \times \{0\}).$$

Thus $U \subseteq S^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (S^{i-1} \times \{0\})$ is an open subset of $S^{i-1} \times \{0\}$.

The set S^∞ comes with two natural topologies, the following lemma shows that they are the same.

Lemma 104.17. (Topology-on-S^∞-Lemma) We consider the set S^∞. The direct colimit topology $\varinjlim (S^{i-1} \times \{0\})$ on S^∞ agrees with the subspace topology coming from $\mathbb{R}^\infty$.

Proof. [1084] The lemma follows from the following two claims.

Claim 1. If $V \subseteq \mathbb{R}^\infty$ is an open subset, then $S^\infty \cap V$ is open in S^∞.

Proof. Recall that by definition of the topology on S^∞ we need to show that for each $i \in \mathbb{N}_0$ the intersection $(S^{i-1} \times \{0\}) \cap V$ is open in $S^{i-1} \times \{0\}$. Thus let $i \in \mathbb{N}_0$. Since V is open in $\mathbb{R}^\infty$ we know that $(\mathbb{R}^i \times \{0\}) \cap V$ is an open subset of $\mathbb{R}^i \times \{0\}$. Since S^{i-1} is equipped with the subspace topology coming from $\mathbb{R}^i$ we see that the intersection $(S^{i-1} \times \{0\}) \cap V = (S^{i-1} \times \{0\}) \cap ((\mathbb{R}^i \times \{0\}) \cap V)$ is indeed open in $S^{i-1} \times \{0\}$. ⊞

[1084] The proof is very ad hoc and very specific to the given setting. Unfortunately this author did not manage to prove a reasonable lemma along the lines that "suitable embeddings $A_i \to X_i$ give rise to an embedding $\varinjlim A_i \to \varinjlim X_i$" which would apply to the given setting.

Claim 2. Given any open subset U of S^∞ there exists an open subset $V \subseteq \mathbb{R}^\infty$ with $S^\infty \cap V = U$.

Proof. Let $U \subseteq S^\infty$ be an open subset. We consider

$$V \;:=\; \{r \cdot x \,|\, r \in \mathbb{R}_{>0} \text{ and } x \in U\} \;\subseteq\; \mathbb{R}^\infty.$$

Evidently we have $S^\infty \cap V = U$. We need to show that V is open in $\mathbb{R}^\infty$. By definition of the topology on $\mathbb{R}^\infty$ we need to show that for each $i \in \mathbb{N}_0$ the intersection $(\mathbb{R}^i \times \{0\}) \cap V$ is open in $\mathbb{R}^i \times \{0\}$. Thus let $i \in \mathbb{N}_0$. It follows easily from the explicit definition of V that

$$(\mathbb{R}^i \times \{0\}) \cap V \;=\; \{r \cdot x \,|\, r \in \mathbb{R}_{>0} \text{ and } x \in (S^{i-1} \times \{0\}) \cap U\}.$$

Since U is open in S^∞ we know that $U \cap (S^{i-1} \times \{0\})$ is open in $S^{i-1} \times \{0\}$. The claim now follows from the general observation that if $W \subseteq S^{i-1}$ is an open subset of S^{i-1}, then $\{r \cdot x \,|\, r \in \mathbb{R}_{>0} \text{ and } x \in W\}$ is open in $\mathbb{R}^i$. ∎

Even though we did not show it yet, we strongly expect that the finite-dimensional spheres S^n are not contractible. This raises the following question:

Question 104.18. Is the infinite-dimensional sphere S^∞ contractible?

We will answer this question in the S^∞–Contractibility Lemma 104.25.

Before we can formulate our next lemma it is helpful to introduce the following definition:

Definition 104.19. Let G be a group that acts on two sets X and Y. A map $f\colon X \to Y$ is called G-equivariant if given any $x \in X$ and $g \in G$ we have $f(g \cdot x) = g \cdot f(x)$.

Lemma 104.20. (Direct Colimit–Action Lemma) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. Let G be a group. Suppose that for each $i \in \mathbb{N}_0$ we are given a continuous action of G on X_i such that each φ_i is G-equivariant. It follows almost immediately from this data that G also acts continuously on $\operatorname{colim} X_i$.

(1) The natural maps $g_i\colon X_i \to \operatorname{colim} X_i$ are G-equivariant.
(2) The resulting natural map $\operatorname{colim} (X_i/G) \to \big(\operatorname{colim} X_i\big)/G$ is a homeomorphism.

Sketch of proof. For $g \in G$ let $\theta_i^g\colon X_i \to X_i$ be the continuous map that is given by multiplication by g. We consider the continuous maps $X_i \xrightarrow{\theta_i^g} X_i \to \operatorname{colim} X_i$. By our hypothesis on the actions and the universal property of direct colimits we get a unique induced continuous map

$$\theta^g\colon \operatorname{colim} X_i \;\to\; \operatorname{colim} X_i.$$

Using the uniqueness of the universal property one can easily verify that these maps define a continuous G-action on $\operatorname{colim} X_i$ and that the natural inclusions $X_i \to \operatorname{colim} X_i$ are G-equivariant. Thus we get induced maps

$$X_i/G \;\to\; (\operatorname{colim} X_i)/G.$$

It follows from the Quotient-to-Quotient Lemma 8.6 that these maps are continuous. By the universal property of direct colimits we obtain a continuous map

$$\Xi\colon \operatorname{colim} (X_i/G) \;\to\; \big(\operatorname{colim} X_i\big)/G.$$

It follows from the explicit description of the direct colimit in the category of topological spaces that we gave in the Topological-Direct Colimit Proposition 104.10 that this map Ξ is a bijection. We want to show that Ξ is a homeomorphism. By the Open-Closed–Injective Lemma 4.22 it remains to prove the following claim:

Claim. The map $\Xi\colon \varinjlim (X_i/G) \to \big(\varinjlim X_i\big)/G$ is an open map.

Proof. For $i \in \mathbb{N}_0$ we denote by $p_i\colon X_i \to X_i/G$ the natural projection. Furthermore we denote by $q\colon \varinjlim X_i \to \big(\varinjlim X_i\big)/G$ the natural projection. Let $U \subseteq \varinjlim (X_i/G)$ be an open subset. By definition of the direct colimit topology this means that for each $i \in \mathbb{N}_0$ the intersection $U_i \coloneqq (X_i/G) \cap U$ is an open subset of X_i/G. Note that by Definition 8.3 of the quotient topology this means that $\widetilde{U}_i \coloneqq p_i^{-1}(U_i)$ is an open subset of X_i. We set $\widetilde{U} \coloneqq \bigcup\limits_{i\in\mathbb{N}_0} \widetilde{U}_i \subseteq \varinjlim X_i$. It follows easily from the definition of the direct colimit topology that $\widetilde{U}$ is an open subset of $\varinjlim X_i$. Next note that it follows from the Group Action-Projection Lemma 9.8 that $q(\widetilde{U})$ is an open subset of $\big(\varinjlim X_i\big)/G$. Finally note that it follows easily from the various definitions that $\Xi(U) = q(\widetilde{U})$. We have thus shown that Ξ is indeed an open map. ∎

We consider one last example of an interesting direct sequence of topological spaces.

Definition 104.21.

(1) By the discussion on page 338 we can consider the direct sequence

$$\mathbb{R}\mathrm{P}^0 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^1 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^2 \xrightarrow{[x]\mapsto[x:0]} \dots$$

of embeddings. We refer to

$$\mathbb{R}\mathrm{P}^\infty \coloneqq \varinjlim \mathbb{R}\mathrm{P}^i$$

as the infinite-dimensional real projective space.

(2) The same way we also define the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$.

At this point there is not much to say about the infinite-dimensional projective spaces, except that we can show that there exist several different ways to describe these topological spaces:

Lemma 104.22. (Infinite Projective Space-Sphere Quotient Lemma)

(1) The obvious action of the group $\{\pm 1\}$ on S^∞ is continuous and discrete.

(2) There exists a homeomorphism $S^\infty/\{\pm 1\} \to \mathbb{R}\mathrm{P}^\infty$.

Similarly we see that we have homeomorphisms $\mathbb{R}^\infty/(\mathbb{R}\setminus\{0\}) \cong \mathbb{R}\mathrm{P}^\infty$, $S^\infty/S^1 \cong \mathbb{C}\mathrm{P}^\infty$ and $\mathbb{C}^\infty/(\mathbb{C}\setminus\{0\}) \cong \mathbb{C}\mathrm{P}^\infty$.

Proof.

(1) We leave the verification of this statement to the reader.

(2) We consider the following commutative diagram:

$$\begin{array}{ccc} \varinjlim (S^n/\{\pm 1\}) & \longrightarrow & \varinjlim \mathbb{R}\mathrm{P}^n \\ \downarrow & & \downarrow \\ S^\infty/\{\pm 1\} & \longrightarrow & \mathbb{R}\mathrm{P}^\infty. \end{array}$$

It follows from the Projective Space-Sphere Quotient Lemma 9.13 that the top horizontal map is a homeomorphism. The right vertical map is a homeomorphism by definition. Finally the left vertical map is a homeomorphism by the Direct Colimit–Action Lemma 104.20. It follows that the bottom map is a homeomorphism.

The other statements are proved the same way using the Direct Colimit–Action Lemma 104.20. ■

104.6. **Appendix: Direct colimits and products.** Let

$$X_1 \xrightarrow{\varphi_0} X_2 \xrightarrow{\varphi_1} X_3 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding. For each $i \in \mathbb{N}$ we denote by $g_i \colon X_i \to \operatorname{colim} X_i$ the natural map. Furthermore let Y be a topological space. By definition of the direct colimit the continuous maps

$$g_i \times \mathrm{id}_Y \colon X_i \times Y \;\to\; (\operatorname{colim} X_i) \times Y$$

induce a continuous map

$$\operatorname{colim} (X_i \times Y) \;\to\; \big(\operatorname{colim} X_i\big) \times Y.$$

It follows easily from the explicit description of the direct colimits, which we gave in the Topological-Direct Colimit Proposition 104.10, that this map is a bijection. The following lemma shows that in general this map is *not* a homeomorphism.

Lemma 104.23. (Direct Colimit–Product–Non-Commuting Lemma) For each $i \in \mathbb{N}$ let $X_i := \bigvee_{j=1}^{i} ([0,1],0)$ be the wedge of i intervals and for each $i \in \mathbb{N}$ let $\varphi_i \colon X_i \to X_{i+1}$ be the obvious embedding. In this setting the above continuous bijective map

$$\Theta \colon \operatorname{colim} (X_i \times \mathbb{Q}) \;\to\; \big(\operatorname{colim} X_i\big) \times \mathbb{Q}$$

is *not* a homeomorphism.

Proof. We will show that there exists a closed subset $A \subseteq \operatorname{colim} (X_i \times \mathbb{Q})$ such that $\Theta(A)$ is not a closed subset of $\big(\operatorname{colim} X_i\big) \times \mathbb{Q}$.

We construct A as follows. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number. For each $j \in \mathbb{N}$ we consider the subset

$$A_j \;:=\; \{(x,y) \in [0,1] \times \mathbb{Q} \,|\, y \in [\tfrac{\alpha}{j}, \tfrac{\alpha}{j} + x]\}.$$

This subset has two properties which are relevant to us:

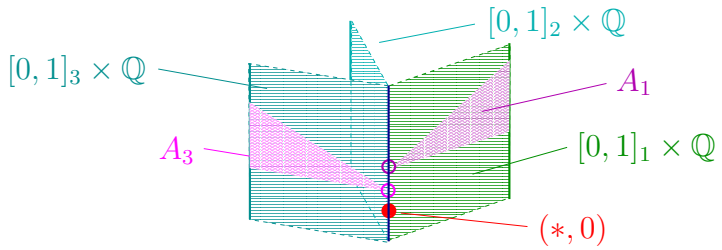
(a) The subset $A_j \subseteq [0,1] \times \mathbb{Q}$ is closed.[1085]
(b) Since α is irrational we see that $A_j \cap (\{0\} \times \mathbb{Q}) = \varnothing$.

For $j \geq i$ we denote by $\iota_i \colon [0,1] \to X_j$ the natural inclusion. We also consider the map $\iota_i \times \mathrm{id}_{\mathbb{Q}} \colon [0,1] \times \mathbb{Q} \to X_j \times \mathbb{Q}$. For $j \in \mathbb{N}$ we denote by $f_j \colon X_j \times \mathbb{Q} \to \operatorname{colim} (X_i \times \mathbb{Q})$ the natural map. Finally we set

$$A \;:=\; \bigcup_{i \in \mathbb{N}} f_i((\iota_i \times \mathrm{id}_{\mathbb{Q}})(A_i)) \;\subseteq\; \operatorname{colim} (X_i \times \mathbb{Q}).$$

It remains to prove the following claim:

[1085] There is an obvious closed subset of $[0,1] \times \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$ whose intersection with $[0,1] \times \mathbb{Q}$ equals A_j. It follows from the Product Topology-Basics Lemma 7.6 (1a) that A_j is indeed a closed subset of $[0,1] \times \mathbb{Q}$.



Claim.
(1) A is a closed subset of $\operatorname{colim}(X_i \times \mathbb{Q})$.
(2) $\Theta(A)$ is not a closed subset of $\left(\operatorname{colim} X_i\right) \times \mathbb{Q}$.

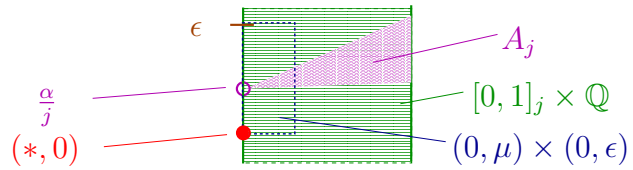
Proof. We turn to the proof of the two statements:

(1) By the Topological-Direct Colimit Proposition 104.10 (0a) it suffices to show that each $f_j^{-1}(A)$ is a closed subset of $X_j \times \mathbb{Q}$. Note that

$$\begin{aligned} f_j^{-1}(A) &= f_j^{-1}\Big(\bigcup_{i\in\mathbb{N}} f_i((\iota_i \times \mathrm{id}_{\mathbb{Q}})(A_i))\Big) \overset{\text{by the Image-Preimage Lemma 1.9 (6)}}{=} \bigcup_{i\in\mathbb{N}} f_j^{-1}(f_i((\iota_i \times \mathrm{id}_{\mathbb{Q}})(A_i))) \\ &= \bigcup_{i=0}^{j} f_j^{-1}(f_i((\iota_i \times \mathrm{id}_{\mathbb{Q}})(A_i))) \quad \text{(this follows from (b))} \\ &= \bigcup_{i=0}^{j} (\iota_i \times \mathrm{id}_{\mathbb{Q}})(A_i) \subseteq X_j \times \mathbb{Q}. \end{aligned}$$

It follows from the Wedge–Subspace Lemma 96.3 (2) and the Product-of-Subsets Lemma 7.8 (1) that each $\iota_i \times \mathrm{id}_{\mathbb{Q}} \colon [0,1] \times \mathbb{Q} \to X_j \times \mathbb{Q}$ is a closed embedding. By (a) we know that $A_i \subseteq [0,1] \times \mathbb{Q}$ is closed. In summary we have shown that $f_j^{-1}(A)$ is the union of $j+1$ closed subsets of $X_j \times \mathbb{Q}$, hence $f_j^{-1}(A)$ itself is a closed subset of $X_j \times \mathbb{Q}$.

(2) We denote by $*$ the wedge point of $\bigvee_{i\in\mathbb{N}} [0,1]$. It follows from (b) that $(*,0) \notin \Theta(A)$. To show that $\Theta(A)$ is not a closed subset of $\left(\operatorname{colim} X_i\right) \times \mathbb{Q}$ it suffices to show that each open neighborhood of $(*,0) \in \left(\operatorname{colim} X_i\right) \times \mathbb{Q}$ has a non-empty intersection with $\Theta(A)$. By Definition 7.3 of the product topology it suffices to show that for each open neighborhood U of $* \in \operatorname{colim} X_i$ and each $\epsilon > 0$ we have $(U \times (-\epsilon, \epsilon)) \cap \Theta(A) \neq \varnothing$. Given such U and ϵ we pick $i \in \mathbb{N}$ with $\frac{\alpha}{j} < \frac{1}{2\epsilon}$. Since U is an open neighborhood of $* \in \operatorname{colim} X_i$ we see that there exists a $\mu > 0$ such that $[0,\mu) \subseteq U \cap [0,1]_j$.



It is clear that $((0,\mu)\times(0,\epsilon)) \cap A_j \neq \varnothing$. This implies that $(U \times (-\epsilon,\epsilon)) \cap \Theta(A) \neq \varnothing$. ■

In contrast to this awkward example we have the following proposition:

Proposition 104.24. (Direct Colimit–Product Topology Proposition) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding. For each $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \varinjlim X_i$ the natural map. Furthermore let Y be a topological space. If Y is regionally compact, then the natural map

$$\varinjlim (X_i \times Y) \;\to\; \left(\varinjlim X_i\right) \times Y$$

is a homeomorphism.

Remark. A very important special case of the Direct Colimit–Product Topology Proposition 104.24 is given by $Y = [0,1]$, since this allows us to study homotopies on direct colimits. In many important settings this special case is also dealt with in the Interval CW-Product Proposition 137.1. □

Sketch of proof. Given two topological spaces A and B we denote by $A^B \coloneqq \mathrm{Mor}_{\mathcal{T}op}(A,B)$ the set of continuous maps from A to B. We equip A^B with the compact-open topology that we introduced in Definition 16.2. Let X, Y and Z be topological spaces. If Y is regionally compact, then we know by the Cylinder–Path Space Adjunction Lemma 16.11 that the two functors

$$\begin{array}{rcl} F \colon \mathcal{T}op & \to & \mathcal{T}op \\ Z & \mapsto & Z \times Y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G \colon \mathcal{T}op & \to & \mathcal{T}op \\ X & \mapsto & X^Y \end{array}$$

form an adjoint pair. It now follows from the Left Adjoint–Colimit Proposition 6.35 that the functor F preserves all colimits, in particular all direct colimits. ■

As a typical application of the Direct Colimit–Product Topology Proposition 104.24 we will now show that the infinite-dimensional sphere S^∞

$$S^\infty \;\coloneqq\; \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \;\middle|\; \sum_{j=1}^{\infty} x_j^2 = 1\right\} \;=\; \varinjlim S^i$$

from Definition 104.16 is contractible:

Lemma 104.25. (S^∞–Contractibility Lemma) The infinite-dimensional sphere S^∞ is contractible.

Proof. We want to show that S^∞ is contractible. Note that by the Contractability Criterion Lemma 19.20 it suffices to show that there exists a homotopy H from id_{S^∞} to a constant map. Our goal is to find such H. We set

$$X \;\coloneqq\; \{(0, x_2, x_3, \dots) \in \mathbb{R}^\infty \,|\, x_2, x_3, \dots \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{array}{rcl} F \colon \mathbb{R}^\infty \times [0,1] & \to & \mathbb{R}^\infty \\ (P,t) & \mapsto & \underbrace{P \cdot (1-t) + (0,P) \cdot t}_{\neq 0 \text{ if } P \neq 0} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G \colon X \times [0,1] & \to & \mathbb{R}^\infty \\ (Q,t) & \mapsto & \underbrace{Q \cdot (1-t) + (1,0,\dots) \cdot t}_{\neq 0 \text{ if } Q \neq 0}. \end{array}$$

Note that F_1 takes values in X and note that G_1 is a constant map. For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t) \in S^\infty$ and $\widetilde{G}(P,t) \in S^\infty$ to be the normalizations of $F(P,t)$

and $G(P,t)$ to length one. Next we consider the map[1086]

$$\begin{array}{rcl} H\colon S^\infty \times [0,1] & \to & S^\infty \\ (P,t) & \mapsto & \begin{cases} \widetilde{F}(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ \widetilde{G}(\underbrace{\widetilde{F}(P,1)}_{\in X}, 2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

Clearly $H_0 = \mathrm{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show that H is continuous. We will do so in two steps:

Claim 1. For each $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} H\colon S^n \times [0,1] & \to & S^\infty \\ (x,t) & \mapsto & H((x,0),t) \end{array}$$

is continuous.

Proof. Let $n \in \mathbb{N}_0$. Note that $H(S^n \times [0,1]) \subseteq S^{n+1}$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} S^n \times [0,1] & \xrightarrow{H} & S^\infty \\ & \searrow^{H} & \uparrow \\ & & S^{n+1} \end{array} .$$

By the Topological-Direct Colimit Proposition 104.10 (1a) we know that the vertical map is a continuous. It follows from this discussion that it suffices to show that for each $n \in \mathbb{N}_0$ the diagonal map is a continuous.

By the Homotopy Combination Lemma 18.8 (1) it suffices to show that for each $n \in \mathbb{N}_0$ the restriction of $H\colon S^n \times [0,1] \to S^{n+1}$ to the subspaces $S^n \times [0,\frac{1}{2}]$ and $S^n \times [\frac{1}{2},1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ⊞

Claim 2. The map H is continuous.

Proof. This statement now follows immediately from Claim 1, the Direct Colimit–Product Topology Proposition 104.24 and the Topological-Direct Colimit Proposition 104.10 (1b). ■

104.7. Exercises for Chapter 104.

Exercise 104.1. Compute the isomorphism type of the direct colimit of the following direct sequence of homomorphisms of abelian groups:

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \dots$$

Exercise 104.2. Let $(I,\leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category *AbGroup* of abelian groups.

(a) If all of the X_i are torsion, is the direct colimit $\varinjlim X_i$ torsion?
(b) If all of the X_i are torsion-free, is the direct colimit $\varinjlim X_i$ torsion-free?
(c) If all of the X_i are free abelian, is the direct colimit $\varinjlim X_i$ free abelian?
(d) If all of the X_i are finite, is the direct colimit $\varinjlim X_i$ finite?

[1086] One can easily verify that the map is well-defined for $t = \frac{1}{2}$.

(e) If all of the X_i are non-trivial, is the direct colimit $\varinjlim X_i$ non-trivial?

Exercise 104.3. We consider the direct sequence $\mathrm{GL}(1,\mathbb{R}) \to \mathrm{GL}(2,\mathbb{R}) \to \mathrm{GL}(3,\mathbb{R}) \to \dots$ of groups, where the maps are given by the group homomorphisms

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

What is the relationship between $\varinjlim \mathrm{GL}(i,\mathbb{R})$ and the group $\mathrm{GL}(\mathbb{R}^\infty)$ of isomorphisms of $\mathbb{R}^\infty$?

Exercise 104.4.

(a) We consider the following direct sequence of homomorphisms

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{\cdot 5} \dots$$

Show that the corresponding direct colimit $\varinjlim \mathbb{Z}$ is isomorphic to $(\mathbb{Q},+)$.

(b) Provide a direct sequence of homomorphisms of finitely generated abelian groups such that the direct colimit is isomorphic to $\mathbb{Z}[\frac{1}{2}]$.

(c) Provide a direct sequence of homomorphisms of finitely generated abelian groups such that the direct colimit is isomorphic to $\mathbb{Q}^2$.

Exercise 104.5. Let $(I,\leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system of abelian groups over I. Let $a_j \in X_j$ and let $a_k \in X_k$. Prove the following statement: If the images of a_j and a_k in $\varinjlim X_i$ agree, then there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in X_l$.

Exercise 104.6. Let $(P,\leq)$ be a partially ordered set as in Definition 1.25. We fix a set $\{*\}$ consisting of a single element. As in Definition 5.5 we define the *order category* $\mathcal{Order}_{(P,\leq)}$ to be the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{Order}_{(P,\leq)}) &= P \\ \mathrm{Mor}(x,y) &= \begin{cases} \{*\}, & \text{if } x \leq y, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

together with the only possible way to compose morphisms. Characterize which direct systems in the order category have direct colimits.

Exercise 104.7. Let G be an (abelian) group. We denote by $\{G_i\}_{i\in I}$ the set of finitely generated subgroups of G. We write $i \leq j$ if and only if $G_i \subseteq G_j$.

(a) Show that the groups $\{G_i\}_{i\in I}$ together with the inclusion homomorphisms form a direct system in the category $\mathcal{Group}$ of groups (category $\mathcal{AbGroup}$ of abelian groups).

(b) Show that we have a natural isomorphism

$$\varinjlim G_i \cong G.$$

Exercise 104.8. Let $(I,\leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system in the category $\mathcal{C}$. Suppose that there exists an $i \in I$ such that $f_{ji}\colon X_i \to X_j$ is an isomorphism for all $j \geq i$. Show that $\varinjlim X_n$ exists and show that it is naturally isomorphic to X_i.

Remark. In other words, the exercise is to prove the Direct Colimit–Stabilization Lemma 104.5.

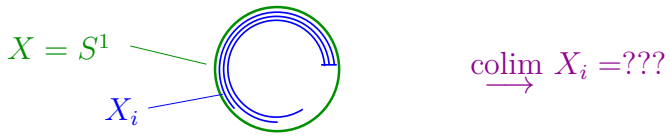
Exercise 104.9. Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of topological spaces. Suppose that for each $i \in \mathbb{N}$ we are given an injective continuous map $f_i\colon X_i \to X_{i+1}$. Show the map $X_i \to \varinjlim X_i$ is not necessarily an embedding.

Exercise 104.10.

(a) For $i \in \mathbb{N}$ we define $F_i = \langle x_i, x_{i+1}\rangle$. For $i \in \mathbb{N}$ we denote by $g_i\colon F_i \to F_{i+1}$ the homomorphism that is given by $g_i(x_i) = e, g_i(x_{i+1}) = x_{i+1}$. Furthermore for $i < j$ we define $f_i\colon F_i \to F_j$ to be the homomorphism $g_{j-1} \circ \cdots \circ g_{i+1} \circ g_i$. What is the direct colimit $\varinjlim F_i$ in the category *Group* of groups?

(b) Given $k \in \mathbb{N}$ we define $X_k = \mathbb{N}$ to be the topological space where the open subsets are $\{1,\dots,k\}, \{k+1\}, \{k+2\}, \dots$. For $k \le l$ we denote by $f_{lk}\colon X_k \to X_l$ the identity map. (Note that with the choice of our topologies on X_k and X_l this map is in fact continuous.) What is the direct colimit $\varinjlim X_k$ in the category of topological spaces?
Remark. As a set the direct colimit is again $\mathbb{N}$, but what is the topology?

Exercise 104.11. We consider the circle $X = S^1$ and for each $i \in \mathbb{N}$ we consider the subspace $X_i := \{\exp(2\pi\,\mathrm{i}t) \mid t \in [0, 1-\frac{1}{n})\}$. What is the homeomorphism type of $\varinjlim X_i$?



Exercise 104.12. Show that $S^\infty \setminus \{(1,0,\dots)\}$ is homeomorphic to $\mathbb{R}^\infty$.
Hint. What does the stereographic projection do for you?

Exercise 104.13. Show that $S^\infty \setminus \{(1,0,\dots)\}$ is homeomorphic to $\mathbb{R}^\infty$.
Hint. What does the stereographic projection do for you?

Exercise 104.14. For each $n \in \mathbb{N}$ we consider the following subspace of $\mathbb{R}^2$:

$$X_n \;:=\; \{(0,0)\} \cup \{(\tfrac{1}{i}, 0) \mid i \in \mathbb{N}\} \cup \{(\tfrac{1}{j}, \tfrac{1}{k}) \mid k \in \mathbb{N} \text{ and } j \in \{1,\dots,n\}\}.$$

We set $X := \varinjlim X_n$.

(a) Show that each X_n is a closed subset of X.

(b) Show that in X the diagonal sequence $\{(\frac{1}{n}, \frac{1}{n})\}_{n\in\mathbb{N}}$ does not converge to $(0,0)$.

(c) We set $V := \{(0,0)\} \cup \{(\frac{1}{j}, \frac{1}{k}) \mid j,k \in \mathbb{N}\}$ and we consider $A := \{(\frac{1}{j}, \frac{1}{k}) \mid j,k \in \mathbb{N}\}$. Show that A is not closed in V and show that for each $n \in \mathbb{N}$ the set $A \cap (V \cap X_n)$ is closed in $V \cap X_n$.
Remark. By the Direct Colimit-Characterization Lemma 104.11 (a)$\Longleftrightarrow$(c) this discussion shows that V is not the direct colimit of the $V \cap X_n$.

Remark. This exercise shows that in the Direct Colimit–Subspace Lemma 104.13 we cannot just drop the hypothesis that V is closed.

Exercise 104.15. We consider the set

$$\mathbb{R}^\infty \;:=\; \mathbb{R}^{(\mathbb{N})} \;:=\; \{(x_1, x_2, \dots) \mid \text{only finitely many } x_i \text{ are non-zero}\}.$$

Furthermore we consider the following two topologies on the set $\mathbb{R}^\infty$:

(1) As in the finite-dimensional case we can equip $\mathbb{R}^\infty$ with the Euclidean metric

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \coloneqq \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

By the Metric-Topology Lemma 3.7 the metric d defines a topology on $\mathbb{R}^\infty$, more precisely

$$\mathcal{M} \coloneqq \{U \subseteq \mathbb{R}^\infty \mid \text{for every } P \in U \text{ there exists an } \epsilon > 0 \text{ such that } B_\epsilon^d(P) \subseteq U\}$$

is a topology on $\mathbb{R}^\infty$. We refer to it as the *metric topology on* $\mathbb{R}^\infty$.

(2) As in Definition 104.15 we consider $\mathbb{R}^\infty$ with the topology that is given by

$$\operatorname*{colim}_{\longrightarrow} (\underbrace{\mathbb{R}^n \times \{0\}}_{\subseteq \mathbb{R}^\infty}) = \mathbb{R}^\infty,$$

i.e. with the topology

$$\mathcal{L} \coloneqq \{U \subseteq \mathbb{R}^\infty \mid \text{for every } n \in \mathbb{N}_0 \text{ the set } U \cap (\mathbb{R}^n \times \{0\}) \text{ is open in } \mathbb{R}^n \times \{0\}\}.$$

We refer to this as the *limit topology on* $\mathbb{R}^\infty$.

In the following we will see that the two topologies behave quite differently.

(a) Show that $\mathcal{M} \subseteq \mathcal{L}$.
(b) Show that the limit topology $\mathcal{L}$ is not contained in the metric topology $\mathcal{M}$.
(c) We equip $\mathbb{R}^\infty$ with the metric topology $\mathcal{M}$. Does the Heine-Borel Theorem hold in $\mathbb{R}^\infty$? More precisely, is every bounded closed subset of $\mathbb{R}^\infty$ compact in $\mathcal{M}$?
(d) Provide an example of a linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ that is not continuous with respect to the metric topology.

Exercise 104.16. We can view the set $\mathbb{R}^\infty$ as a subset of $\prod_{i \in \mathbb{N}} \mathbb{R}$. We equip $\prod_{i \in \mathbb{N}} \mathbb{R}$ with the product topology and we equip $\mathbb{R}^\infty$ with the corresponding subspace topology $\mathcal{S}$. We also consider the limit topology $\mathcal{L}$ and the metric topology $\mathcal{M}$ from Exercise 104.15.

(a) Is $\mathcal{S} \subseteq \mathcal{L}$?
(b) Is $\mathcal{L} \subseteq \mathcal{S}$?
(c) Is $\mathcal{S} \subseteq \mathcal{M}$?
(d) Show that $\mathcal{M} \subseteq \mathcal{L}$ (if you have not done so in Exercise 104.15.)

Exercise 104.17. Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding. Furthermore suppose that for each $i \in \mathbb{N}_0$ we have a subspace $A_i \subseteq X_i$ such that for each $i \in \mathbb{N}_0$ we have $\varphi_i(A_i) = A_{i+1}$.

(a) Show that if each $A_i \to X_i$ is an open embedding, then the natural map $A_i \to \operatorname*{colim}_{\longrightarrow} X_i$ is an open embedding.
(b) Give an example where each $A_i \to X_i$ is a closed embedding, but where the natural map $A_i \to \operatorname*{colim}_{\longrightarrow} X_i$ is *not* an embedding.

Exercise 104.18. In this exercise we want to show that given any compact subspace K of $\mathbb{R}^\infty$ there exists an $n \in \mathbb{N}$ with $K \subseteq \mathbb{R}^n \times \{0\}$. We proceed as follows. Let $K \subseteq \mathbb{R}^\infty$ be

a compact subspace. Suppose that there is no $n \in \mathbb{N}$ such that $K \subseteq \mathbb{R}^n \times \{0\}$. Then there exists an infinite increasing direct sequence $n_1, n_2, n_3, \dots$ of natural numbers and points $x_1, x_2, x_3, \dots$ in K such that $x_i \in (\mathbb{R}^{n_i} \times \{0\}) \setminus (\mathbb{R}^{n_i - 1} \times \{0\})$. We denote by X the set whose elements are precisely the x_i.

(a) Show that X is a discrete subset of $\mathbb{R}^\infty$.
(b) Show that X is a closed subset of K.
(c) Use the Compact-Discrete Lemma 3.40 (2) to obtain a contradiction.

Exercise 104.19. Let G be an abstract graph and let $G_0 \subseteq G_1 \subseteq G_2 \subseteq G_3 \dots$ be a direct sequence of subgraphs such that $\bigcup_{n \in \mathbb{N}} G_n = G$. Show that for the corresponding topological realizations we have a natural homeomorphism

$$\operatorname*{colim}_{\longrightarrow} |G_n| \xrightarrow{\cong} |G|.$$

Hint. Use the Abstract Graph-Topology Lemma 11.6 (3).

Exercise 104.20.

(1) Does every direct system over the category $\mathcal{CRing}$ of commutative rings have a direct colimit?
(2) Does every direct system over the category $\mathcal{Field}$ of fields have a direct colimit?

Exercise 104.21. Let I be a set and let $\leq$ be a relation that is reflexive and transitive. Show that any diagram in the category $\mathcal{Group}$ of groups over $(I, \leq)$ admits a colimit.

Exercise 104.22. Show that the action

$$\begin{array}{rcl} \{\pm 1\} \times S^\infty & \to & S^\infty \\ (\epsilon, v) & \mapsto & \epsilon \cdot v \end{array}$$

of the group $\{\pm 1\}$ on the topological space S^∞ is continuous and discrete.

Exercise 104.23. We consider the subspaces $X_k = \frac{1}{k!} \cdot \mathbb{Z}$, $k \in \mathbb{N}$. Does the direct colimit topology on $\operatorname*{colim}_{\longrightarrow} X_k$ agree with the subspace topology on $\bigcup_{k \in \mathbb{N}} X_k = \mathbb{Q} \subseteq \mathbb{R}$?

Exercise 104.24. Does there exist a direct sequence of open proper subsets $U_n \subsetneq S^1$, $n \in \mathbb{N}$, with $U_0 \subseteq U_1 \subseteq U_2 \subseteq \dots$ such that $S^1 = \operatorname*{colim}_{\longrightarrow} U_n$?

Exercise 104.25. We consider the topological space $\mathbb{R}^\infty = \operatorname*{colim}_{\longrightarrow} \mathbb{R}^k \times \{0\}$.

(a) We consider the direct sequence $a_k \coloneqq (0, \dots, 0, \underset{\substack{\uparrow \\ k\text{-th entry}}}{\frac{1}{k}}, 0, \dots)$

Does this direct sequence converge in the topological space $\mathbb{R}^\infty$? (We refer to Definition 3.33 for the definition of convergence of a direct sequence in a topological space.)
(b) Can you give a characterization of convergent direct sequences in $\mathbb{R}^\infty$?

Exercise 104.26.

(a) Let X be a topological space that is Hausdorff and regionally compact. Show that given any disjoint compact subspaces K and L of X there exist disjoint open neighborhoods U of K and V of L such that the closures $\overline{U}$ and $\overline{V}$ are compact.
Remark. This statement is similar in spirit to the Compact Hausdorff-Normal Lemma 3.49.

(b) Let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a direct sequence of topological spaces. Show that if each X_i is regionally compact and Hausdorff, then $\operatorname{colim}\limits_{\longrightarrow} X_i$ is also Hausdorff.

Exercise 104.27.

(a) Let $P, Q \in S^\infty$. Show that there exists a homeomorphism h of S^∞ with $h(P) = Q$.
(b) Let $P, Q \in \mathbb{R}\mathrm{P}^\infty$. Show that there exists a homeomorphism h of $\mathbb{R}\mathrm{P}^\infty$ with $h(P) = Q$.

CHAPTER 105

Fundamental groups and direct colimits

We already showed that fundamental groups behave well with respect to several operations:[1087]

(1) In the Product-π_1-Proposition 90.13 we showed that fundamental groups preserve products.
(2) In the Wedge-π_1-Proposition 99.10 we showed that, under slight technical hypotheses, fundamental groups preserve coproducts.
(3) In the Seifert–van Kampen Theorem 99.1 we showed that, again under fairly mild technical hypotheses, fundamental groups preserve pushouts.

This discussion, and the definitions from the last chapter, raise the question, whether fundamental groups preserve direct colimits. In this chapter we will see that, under suitable hypotheses, this is indeed the case.

In the second half of this chapter we will use this result to calculate the fundamental group of the surface of infinite genus and to prove several results on fundamental groups of finite genus.

This chapter is organized as follows:

- In Section 105.1 we will discuss to what degree fundamental groups commute with direct colimits.
- In Section 105.2 we will discuss the fundamental group of the surface of "infinite genus".
- In Section 105.3 we will provide exercises for Chapter 105.

In the next Chapter 106 we will state and prove the HNN-Seifert–van Kampen Theorem 106.7.

105.1. **Fundamental groups and direct colimits.** The following proposition gives the promised statement about the relationship between fundamental groups and direct colimits.

Proposition 105.1. (Direct Colimit-π_1-Proposition) Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of pointed topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding with $\varphi_i(w_i) = w_{i+1}$. For each $j \in \mathbb{N}_0$ we denote by $g_j \colon X_j \to \varinjlim X_i$ the natural map. If for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subseteq \operatorname{int}(g_{i+1}(X_{i+1}))$, then we have a natural isomorphism

$$\varinjlim \pi_1(X_i, w_i) \xrightarrow{\cong} \pi_1\big(\varinjlim X_i, g_0(w_0)\big).$$

Before we turn to the proof of the Direct Colimit-π_1-Proposition 105.1 we state a corollary, which is the result which in practice will be more relevant.

[1087] To smoothen the exposition we are a little ambiguous about whether we are dealing with the category of topological spaces or the category of pointed topological spaces.

Proposition 105.2. (Exhaustion-π_1-Proposition) Let X be a topological space and let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$$

be a sequence of subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in I} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$.

Then for any $w \in X_0$ we have a natural isomorphism

$$\operatorname*{colim}_{\longrightarrow} \pi_1(X_i, w) \xrightarrow{\cong} \pi_1(X, w).$$

Example. We assume that we are in the setting of the Exhaustion-π_1-Proposition 105.2. If all the X_i are simply connected, then we see that

$$\begin{array}{ccccc} \pi_1(X,w) & \xleftarrow{\cong} & \operatorname*{colim}_{\longrightarrow} \pi_1(X_i,w) & \cong & \operatorname*{colim}_{\longrightarrow} \{e\} \quad = \quad \{e\}. \\ \uparrow & & \uparrow & & \uparrow \\ \text{by the Exhaustion-}\pi_1 & & \text{since we assume that the} & & \text{by the Direct Colimit} \\ \text{Proposition 105.2} & & X_i \text{ are simply connected} & & \text{Stabilization Lemma 104.5} \end{array}$$

This discussion shows that we just reproved the statement of the Exhaustion–Simply Connected Proposition 99.16. □

Proof of Proposition 105.2 assuming Proposition 105.1. We consider the following natural diagram:

$$\begin{array}{ccc} \operatorname*{colim}_{\longrightarrow} \pi_1(X_i, w) & \longrightarrow & \pi_1\big(\operatorname*{colim}_{\longrightarrow} X_i, w\big) \\ & \searrow \quad \swarrow & \\ & \pi_1(X, w). & \end{array}$$

First note that it follows from the universal property of direct colimits that the diagram commutes. Next note that it follows from our hypothesis that we can apply the Direct Colimit-π_1-Proposition 105.1, which then tells us, that the horizontal map is an isomorphism. Since we know that for each $i \in \mathbb{N}_0$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$ we obtain from the Union-equals-Direct Colimit Criterion 104.12 that we have a natural homeomorphism $\operatorname*{colim}_{\longrightarrow} X_i \to X$. Thus the right diagonal map is an isomorphism. It follows from this discussion that the left diagonal map is an isomorphism. ■

The proof of the Direct Colimit-π_1-Proposition 105.1 rests on the following elementary lemma from earlier on.

Lemma 99.17. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$$

be a sequence of subspaces of X such that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subseteq X_i$.

Now we can provide the proof of the Direct Colimit-π_1-Proposition 105.1.

Proof of the Direct Colimit-π_1-Proposition 105.1. Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding. For each $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \varinjlim X_i$ the natural map. We assume that for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subseteq \operatorname{int}(g_{i+1}(X_{i+1}))$. We set $X \coloneqq \varinjlim X_i$. It follows from the Topological-Direct Colimit Proposition 104.10 that the natural maps $g_i \colon X_i \to X$ are embeddings and that the union of the images of the X_i is all of X. Thus we can and will view each X_i as a subspace of X. In summary we can now assume that we have a sequence

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$

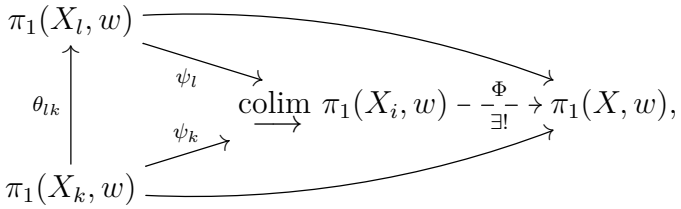
of subspaces of X such that the following two conditions are satisfied:

(a) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(b) For each $i \in \mathbb{N}_0$ we have $X_i \subseteq \mathring{X}_{i+1}$.

Let $w \in X_0$. Given $k < l$ we denote by $\theta_{lk} \colon \pi_1(X_k, w) \to \pi_1(X_l, w)$ the inclusion-induced homomorphism. Given $k \in \mathbb{N}_0$ we denote by $\psi_k \colon \pi_1(X_k, w) \to \varinjlim \pi_1(X_i, w)$ the natural homomorphism.

Note that by the universal property of a direct colimit there exists a unique homomorphism $\Phi \colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)$ that makes the following diagram commute: For any $k < l$:



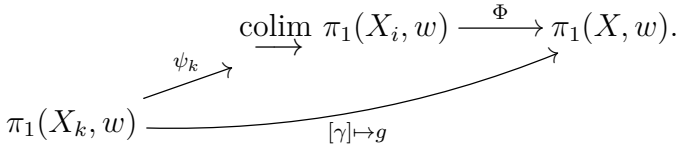
where the curved maps are induced by the inclusions $X_k \to X$ and $X_l \to X$. We need to show that Φ is an isomorphism.

(1) First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$.

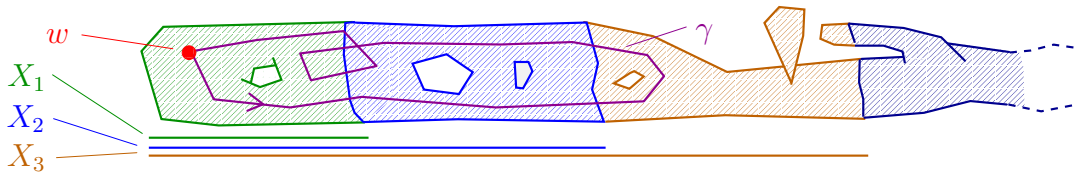
Claim. There exists a compact subset K of X with $w \in K$ such that g lies in the image of the inclusion-induced map $\pi_1(K, w) \to \pi_1(X, w)$.

Proof. We pick a loop $\gamma \colon [0, 1] \to X$ that represents $g \in \pi_1(X, w)$. By the Compact Image Lemma 4.15 the image $\gamma([0, 1])$ is compact. Thus $K \coloneqq \gamma([0, 1])$ has the desired property. ⊞

Note that it follows from the claim together with the Compact Subspace–Exhaustion Lemma 99.17 that there exists a $k \in \mathbb{N}_0$ such that g lies in the image of the inclusion-induced map $\pi_1(X_k, w) \to \pi_1(X, w)$. We consider the following commutative diagram:



By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.



(2) Now we show that Φ is injective. So let $g \in \ker\big(\Phi\colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)\big)$. By the Algebraic–Direct Colimit Existence Proposition 104.7 we can write

$$\varinjlim \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} \pi_1(X_i, w)\Big)/\sim \quad \text{where } h \sim \theta_{lk}(h) \text{ for all } k \le l,\ h \in \pi_1(X_k, w).$$

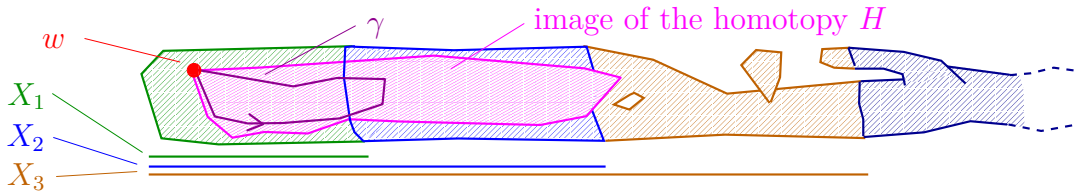
It follows in particular from this explicit description that there exists a $k \in \mathbb{N}_0$ and a $g_k \in \pi_1(X_k, w)$ with $g = \psi_k(g_k)$. In summary, we have the following situation:

$$\begin{array}{ccccc} & & \varinjlim \pi_1(X_i, w) & \xrightarrow[g\mapsto e]{\Phi} & \pi_1(X, w). \\ & \overset{\psi_k}{\nearrow}{\scriptstyle g_k \mapsto g} & & & \\ \pi_1(X_k, w) & & \xrightarrow[g_k \mapsto e]{} & & \end{array}$$

Claim. There exists a compact subset K of X such that g_k lies in the kernel of the inclusion-induced map $\pi_1(X_k, w) \to \pi_1(X_k \cup K, w)$.

Proof. We pick a loop $\gamma\colon [0,1] \to X_k$ that represents g_k. By the above we know that $[\gamma] \in \ker(\pi_1(X_k, w) \to \pi_1(X, w))$. By definition this means that there exists a path-homotopy $H\colon [0,1]\times[0,1] \to X$ from the path γ to the constant path. We appeal again to the Compact Image Lemma 4.15 to deduce that $K := H([0,1]\times[0,1])$ is a compact subset of X. Evidently K has the desired property. ⊞

It follows from the claim, the Compact Subspace–Exhaustion Lemma 99.17 and the fact that we always have $X_i \subseteq X_{i+1}$ that there exists an $l \ge k$ such that $\theta_{lk}([\gamma])$ is trivial in $\pi_1(X_l, w)$.



We obtain the following diagram:

$$\begin{array}{ccccc} \pi_1(X_l, w) & & \xrightarrow{\psi_l} & & \\ {\scriptstyle g_k \mapsto e}\uparrow{\scriptstyle \theta_{lk}} & & & & \\ \pi_1(X_k, w) & \xrightarrow[g_k \mapsto g]{\psi_k} & \varinjlim \pi_1(X_i, w) & \xrightarrow[g\mapsto e]{\Phi} & \pi_1(X, w). \end{array}$$

It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. ■

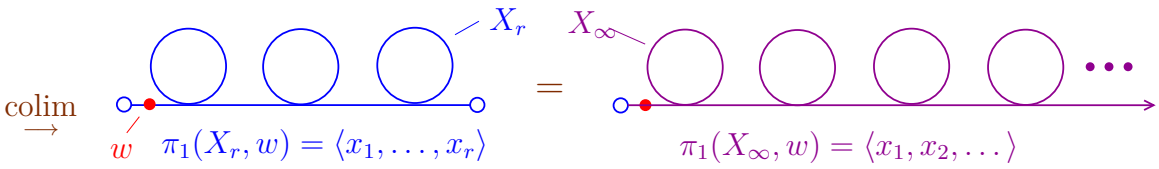
Example.

(1) As on page 2034 we denote by X_∞ the interval $(0,\infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We obtain that [1088]

$$\pi_1\big(X_\infty, w\big) \underset{\uparrow}{=} \pi_1\big(\operatorname*{colim}_{\longrightarrow} X_r, w\big) \underset{\uparrow}{\overset{\cong}{\longleftarrow}} \operatorname*{colim}_{\longrightarrow} \pi_1\big(X_r, w\big) \overset{\text{by the Graph-}\pi_1\text{-Proposition 99.13}}{\underset{\cong}{\downarrow}} \operatorname*{colim}_{\longrightarrow} \langle x_1, \dots, x_r\rangle \underset{\uparrow}{=} \langle x_1, x_2, \dots\rangle.$$

see page 2026 — by the Direct Colimit-π_1 Proposition 105.1 — see page 2026

Furthermore the proof of the Graph-π_1-Proposition 99.13 and the above argument show that the generators $x_1, x_2, \dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n, $n \in \mathbb{N}$, corresponds to a loop which goes "once around the n-th circle".



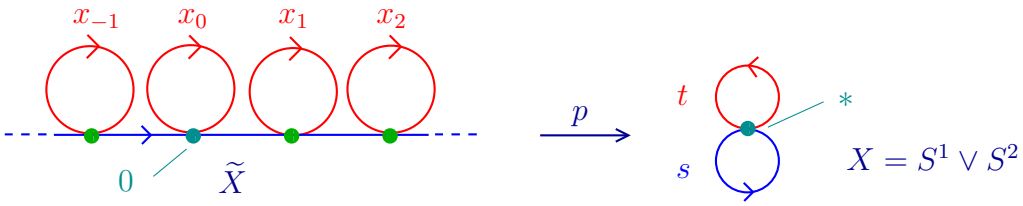
(2) We consider

$$\widetilde{X} \;:=\; \mathbb{R} \;\cup\; \bigcup_{j\in\mathbb{Z}} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}\mathrm{i} \in \mathbb{C}} \;\subseteq\; \mathbb{C}.$$

Put differently, $\widetilde{X}$ is the real line with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\widetilde{X}, 0)$ is the free group on infinitely many generators $\{x_n\}_{n\in\mathbb{Z}}$. In the figure below we illustrate a covering map $p\colon \widetilde{X} \to X = S^1 \vee S^1$. By the Covering-$\pi_1$-Monomorphism Proposition 92.7 the induced homomorphism $p_*\colon \pi_1(\widetilde{X}, 0) \to \pi_1(X, *)$ is a monomorphism. We had just seen that $\pi_1(\widetilde{X})$ is a free group on infinitely many generators and by the Wedge-of-Circles-π_1-Lemma 99.11 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate orientations and conventions this shows that the homomorphism

$$\begin{array}{ccccccc} \pi_1(\widetilde{X}, 0) & \overset{\cong}{\longleftarrow} & \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\rangle & \to & \langle s, t\rangle & \overset{\cong}{\longrightarrow} & \pi_1(X, *) \\ & & x_n & \mapsto & s^n \cdot t \cdot s^{-n} & & \end{array}$$

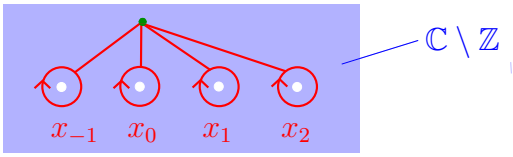
is a monomorphism. In particular we see that the free group $\langle s, t\rangle$ on two generators contains a free group on infinitely many generators.



[1088] The third equality follows from the Graph-π_1-Proposition 99.13, to be more precise, it follows from the fact that for any $r \le s$ the inclusion map $\iota_{sr}\colon X_r \to X_s$ and the isomorphisms of the Graph-π_1-Proposition 99.13 give rise to a commutative diagram

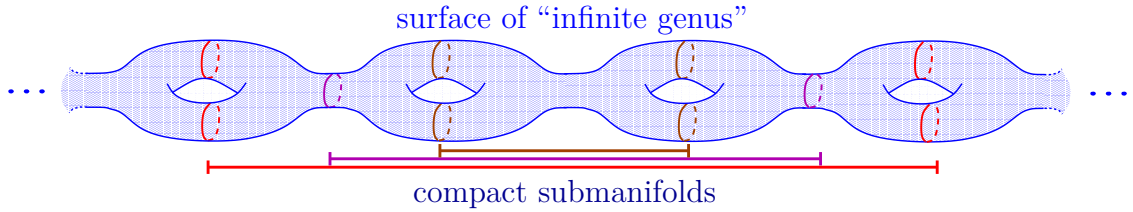
$$\begin{array}{ccc} \pi_1(X_r, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_r\rangle \\ {\scriptstyle (\iota_{sr})_*}\downarrow & & \downarrow{\scriptstyle x_i \mapsto x_i} \\ \pi_1(X_s, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_s\rangle. \end{array}$$

(3) An argument similar to (1) shows that $\pi_1(\mathbb{C}\setminus\mathbb{Z})$ is a free group generated by $x_n, n\in\mathbb{Z}$ where x_n is a loop which goes "once around the n-th hole". We leave the task of



providing the details to the reader. □

105.2. **The surface of "infinite genus".** In this section we return to the initial goal of the previous chapter, namely we want to determine the fundamental group of the surface of "infinite genus" that we illustrate in the figure below. Before we can calculate its fundamental group we have to give a precise description of the surface of infinite genus.



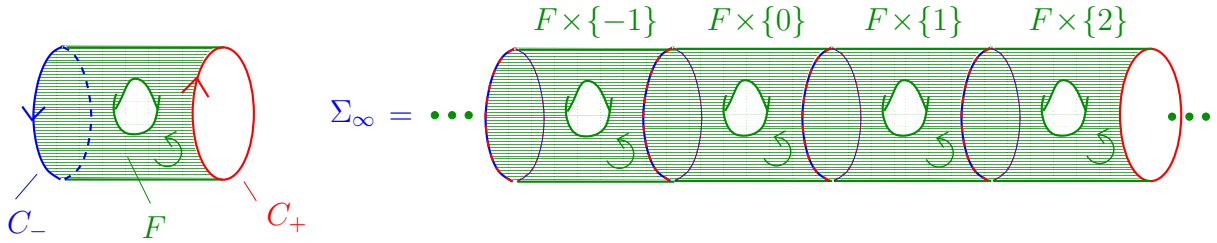
Definition 105.3. We write

$$F \;:=\; \text{the torus minus two open disks.}$$

We equip F with the standard orientation. We label the two boundary components of F by C_- and C_+. We pick an orientation-reversing diffeomorphism $\varphi_-\colon S^1\to C_-$ and we pick an orientation-preserving diffeomorphism $\varphi_+\colon S^1\to C_+$. We define

$$\Sigma_\infty \;:=\; \Big(\bigsqcup_{i\in\mathbb{Z}}(F\times\{i\})\Big)\big/\sim \qquad \text{where } (\varphi_+(x),i)\sim(\varphi_-(x),i+1) \text{ for } x\in S^1 \text{ and } i\in\mathbb{Z}.$$

We refer to Σ_∞ as the surface of infinite genus. It should be clear that this definition gives a rigorous description of the surface that we sketched above.



Lemma 105.4. (Surface-of-Infinite Genus Lemma)

(1) The surface Σ_∞ is an orientable 2-dimensional smooth manifold with empty boundary.

(2) Given $k\in\mathbb{N}$ we write

$$X_k \;:=\; \Big(\bigsqcup_{i=-k+1}^{k}(F\times\{i\})\Big)\big/\sim \qquad \begin{array}{l}\text{where } (\varphi_+(x),i)\sim(\varphi_-(x),i+1) \text{ for } x\in S^1\\ \text{and } i\in\{-k+1,\dots,k-1\}.\end{array}$$

The obvious maps $X_k\to\Sigma_\infty$, $k\in\mathbb{N}_0$, are smooth embeddings and they define a homeomorphism

$$\varinjlim X_i \;\xrightarrow{\cong}\; \Sigma_\infty.$$

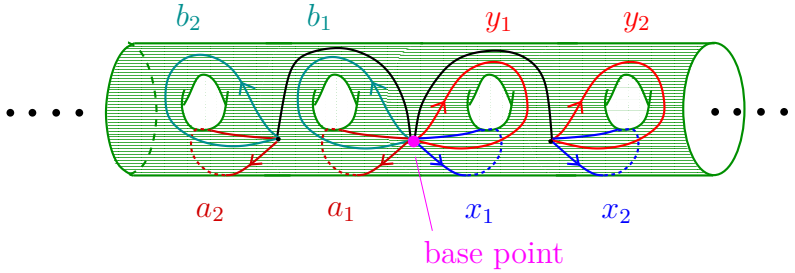
Sketch of proof. We equip F with an orientation. Note that it follows from the Gluing-Smooth Manifolds-Proposition 54.5 that each X_k is an oriented 2-dimensional smooth manifold, such that the obvious map $X_k \to X_{k+1}$ is an orientation-preserving smooth embedding and such that the image of each X_k is contained in $X_{k+1} \setminus \partial X_{k+1}$.

(1) It follows from the Second Countability Lemma 13.2 (3b) that Σ_∞ is second-countable. Using the above discussion of the X_k one can easily show that Σ_∞ is Hausdorff. Furthermore one can show that Σ_∞ has a smooth structure with the property that each of the obvious maps $X_k \to \Sigma_\infty$ is a smooth embedding and such that each point of Σ_∞ admits a chart to an open subset of $\mathbb{R}^2$. It follows from the above discussion of the orientations that Σ_∞ can be equipped with an orientation that matches the orientations of the X_k.

(2) We just saw in (1) that for each $k \in \mathbb{Z}$ the obvious map $X_k \to \Sigma_\infty$ is an embedding. Thus we can view each X_k as a subspace of Σ_∞. It follows from the above discussion that $X_k \subseteq \mathring{X}_{k+1}$. Thus it follows from the Union-equals-Direct Colimit Criterion 104.12 that the induced map $\operatorname{colim}_{\longrightarrow} X_i \to \Sigma_\infty$ is a homeomorphism. ■

In the following lemma we compute the fundamental group of the surface of infinite genus.

Lemma 105.5. (Surface-of-Infinite Genus–π_1-Lemma) The fundamental group of the surface of infinite genus is a free group on a countable infinite set.



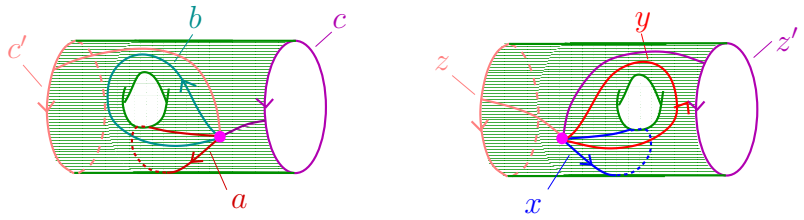
Remark. Later, in the Non-Compact Surface-π_1-Proposition 130.6 we will see that the fundamental group of any non-compact orientable 2-dimensional smooth manifold is a free group. □

Proof. We continue with the notation from the Surface-of-Infinite Genus Lemma 105.4. By the Surface-of-Infinite Genus Lemma 105.4 (2) it suffices to show that the fundamental group of $\operatorname{colim}_{\longrightarrow} X_i$ is a free group on infinitely many generators. By the Surface-with-Boundary–π_1-Lemma 107.8 we have[1089]

$$\pi_1(F) \;=\; \langle a, b, c, c' \mid c = [a, b] \cdot c' \rangle \;=\; \langle x, y, z, z' \mid z = [x, y] \cdot z' \rangle$$

where the generators z, z' and c, c' correspond to the two boundary components. An induction argument, using the Manifold–Seifert-van Kampen Theorem 102.1 together with the Amalgamated Product–Presentation Lemma 100.20 and the above presentations of $\pi_1(F)$

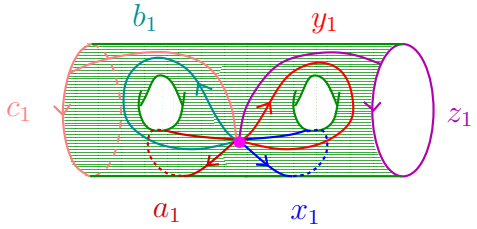
[1089]This is indeed a consequence of the Surface-with-Boundary–π_1-Lemma 107.8, the only change is that we reversed the orientation of one of the boundary curves.



shows that for any $l \geq k$ we have a commutative diagram

$$\begin{array}{ccc} \pi_1(X_k) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1, y_1, \ldots, x_k, y_k \\ a_1, b_1, \ldots, a_k, b_k \end{matrix} z_k, c_k \;\middle|\; \prod_{j=1}^{k} [a_j, b_j] \cdot c_k = \prod_{j=1}^{k} [x_j, y_j] \cdot z_k \right\rangle \\ \Big\downarrow & & \Big\downarrow \scriptstyle c_k \mapsto \prod_{j=k+1}^{l} [a_j,b_j]\cdot c_l \;\; z_k \mapsto \prod_{j=k+1}^{l} [x_j,y_j]\cdot z_l \\ \pi_1(X_l) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1, y_1, \ldots, x_l, y_l \\ a_1, b_1, \ldots, a_l, b_l \end{matrix} z_l, c_l \;\middle|\; \prod_{j=1}^{l} [a_j, b_j] \cdot c_l = \prod_{j=1}^{l} [x_j, y_j] \cdot z_l \right\rangle \end{array}$$

where the left vertical map is induced by the inclusion $X_k \to X_l$. We leave the details to the energetic reader. For any $l \geq k$ we also have the following commutative diagram of



homomorphisms:

$$\begin{array}{ccc} \left\langle \begin{matrix} x_1, y_1, \ldots, x_k, y_k \\ a_1, b_1, \ldots, a_k, b_k \end{matrix} p \right\rangle & \xrightarrow[p \mapsto \prod_{j=1}^{k} [a_j,b_j]\cdot c_k]{\cong} & \left\langle \begin{matrix} x_1, y_1, \ldots, x_k, y_k \\ a_1, b_1, \ldots, a_k, b_k \end{matrix} z_k, c_k \;\middle|\; \prod_{j=1}^{k} [a_j, b_j] \cdot c_k = \prod_{j=1}^{k} [x_j, y_j] \cdot z_k \right\rangle \\ \Big\downarrow & & \Big\downarrow \scriptstyle c_k \mapsto \prod_{j=k+1}^{l} [a_j,b_j]\cdot c_l \;\; z_k \mapsto \prod_{j=k+1}^{l} [x_j,y_j]\cdot z_l \\ \left\langle \begin{matrix} x_1, y_1, \ldots, x_l, y_l \\ a_1, b_1, \ldots, a_l, b_l \end{matrix} p \right\rangle & \xrightarrow[p \mapsto \prod_{j=1}^{l} [a_j,b_j]\cdot c_l]{\cong} & \left\langle \begin{matrix} x_1, y_1, \ldots, x_l, y_l \\ a_1, b_1, \ldots, a_l, b_l \end{matrix} z_l, c_l \;\middle|\; \prod_{j=1}^{l} [a_j, b_j] \cdot c_l = \prod_{j=1}^{l} [x_j, y_j] \cdot z_l \right\rangle \end{array}$$

The horizontal homomorphisms are in fact isomorphisms, this is for example a consequence of the Tietze transformations from Definition 100.24.[1090]

[1090]Indeed, we write $X = \{x_1, y_1, \ldots, x_m, y_m, a_1, b_1, \ldots, a_m, b_m\}$, $u = \prod_{j=1}^{m} [a_j, b_j]$, $v = \prod_{j=1}^{m} [x_j, y_j]$, $c = c_m$ and $z = z_m$, then

$$\langle X, z, c \mid uc = vz \rangle = \langle X, z, c \mid z = v^{-1}uc \rangle \xleftarrow{\cong} \langle X, c \rangle \xrightarrow{\cong} \langle X, c, p \mid p = uc \rangle = \langle X, c, p \mid X, c = u^{-1}p \rangle \xleftarrow{\cong} \langle X, p \rangle$$

where all three isomorphisms are the isomorphisms of the Tietze transformation (T2). It is straightforward to see that the concatenation equals for $m = k, l$ precisely the horizontal maps in the diagram.

We denote by Σ_∞ the surface of infinite genus. By the Surface-of-Infinite Genus Lemma 105.4 we can view each X_k as a subspace of Σ_∞. As we saw in the proof of the Surface-of-Infinite Genus Lemma 105.4, we have $X_k \subseteq \mathring{X}_{k+1}$. We see that for any $x_0 \in X_1$ we have

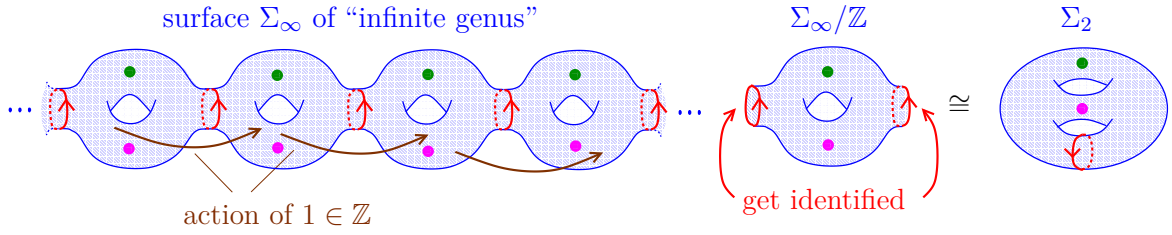
$$\pi_1(\Sigma_\infty, x_0) \underset{\text{by Exhaustion-}\pi_1\text{ Proposition 105.2}}{\uparrow} = \varinjlim \pi_1(X_i, x_0) \underset{\text{by the above two commutative diagrams}}{\uparrow} = \varinjlim \left\langle \begin{matrix} x_1, y_1, \dots, x_i, y_i \\ a_1, b_1, \dots, a_i, b_i \end{matrix} \; p \right\rangle \underset{\text{see the example on page 2026}}{\uparrow} = \left\langle \begin{matrix} x_1, y_1, \dots \\ a_1, b_1, \dots \end{matrix} \; p \right\rangle.$$

This concludes the proof of the lemma. ∎

In the final part of this section we will see that we can actually use the Surface-of-Infinite Genus–π_1-Lemma 105.5 to draw several interesting conclusions on fundamental groups of surfaces of *finite* genus. First we have the following lemma:

Lemma 105.6. (Surface-of-Infinite Genus–Covering Lemma) Let $g \geq 2$. We denote by Σ_∞ the surface of infinite genus and we denote by Σ_g the surface of genus g.

(1) There exists an action of $\mathbb{Z}$ on Σ_∞ that is discrete and continuous and such that the quotient $\Sigma_\infty/\mathbb{Z}$ is homeomorphic to Σ_g.
(2) There exists a covering $p\colon \Sigma_\infty \to \Sigma_g$ of infinite degree.



Sketch of proof.

(1) We continue with the Definition 105.3 and the Surface-of-Infinite Genus Lemma 105.4. Let $g \in \mathbb{N}_{\geq 2}$. As before we consider

$$\Sigma_\infty := \Big(\bigcup_{i \in \mathbb{Z}} (F \times \{i\}) \Big) / \sim \quad \text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \text{ and } i \in \mathbb{Z}.$$

Given $n \in \mathbb{Z}$ and $[(x,k)] \in \Sigma_\infty$ we define $n \cdot [(x,k)] := [(x, k+(g-1)\cdot n)]$. It follows from Exercise 105.3[1091] that this action is continuous and proper. Furthermore we obtain from Exercise 105.3 that the quotient $\Sigma_\infty/\mathbb{Z}$ is homeomorphic to the topological space that is given by identifying the two boundary components of

$$\Big(\bigcup_{i=0}^{g-1} (F \times \{i\}) \Big) / \sim \qquad \begin{matrix} \text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \\ \text{and } i \in \{0, \dots, g-1\}. \end{matrix}$$

The experienced reader will surely be convinced that this topological space is homeomorphic to the surface of genus g. We leave it to the scrupulous reader to fill in the details.[1092]

[1091] It follows from the Smooth Collar Neighborhood Theorem 35.6 together with the Basics-of-Collars Lemma 35.7 that there exist disjoint open neighborhoods of the two boundary components of F, thus the extra hypothesis for Exercise 105.3 (c) is in fact satisfied.

[1092] If all other ideas fail, then one can always appeal to the Surface Classification Theorem 101.16.

(2) This statement follows immediately from (1) together with the Action–Covering Proposition 91.11. ∎

Using the above results we can prove the following proposition, which says that we can write the fundamental group of a surface of genus ≥ 2 as a semidirect product of $\mathbb{Z}$ with a free group on an infinite generating set.

Proposition 105.7. (Surface-π_1–Semidirect Product Proposition) Let $g \geq 2$. We consider the automorphism

$$\varphi\colon \underbrace{\langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle}_{\text{free group of infinite rank}} \to \underbrace{\langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle}_{\text{same free group of infinite rank}}$$

that is induced by

$$\begin{aligned} x_{j,i} &\mapsto x_{j,i+1} \\ y_{j,i} &\mapsto y_{j,i+1}. \end{aligned}$$

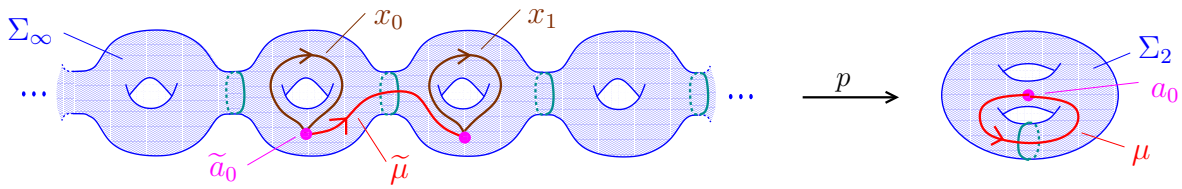
There exists an isomorphism

$$\pi_1(\Sigma_g) \cong \langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle \rtimes_\varphi \mathbb{Z}.$$

Sketch of proof. We consider the surface Σ_∞ of infinite genus together with the $\mathbb{Z}$-action that is provided by the Surface-of-Infinite Genus–Covering Lemma 105.6 (1). By the Action–Covering Proposition 91.11 (2) we know that the projection $p\colon \Sigma_\infty \to \Sigma_\infty/\mathbb{Z} = \Sigma_g$ is a covering.

We pick a base point $a_0 \in \Sigma_g$. Next let $\widetilde{a}_0 \in \Sigma_\infty$ be a point with $p(\widetilde{a}_0) = a_0$. It follows from the Covering-π_1-Monomorphism Proposition 92.7 that $p_*\colon \pi_1(\Sigma_\infty, \widetilde{a}_0) \to \pi_1(\Sigma_g, a_0)$ is a monomorphism. We will use this monomorphism to view $\pi_1(\Sigma_\infty, \widetilde{a}_0)$ as a subgroup of $\pi_1(\Sigma_g, a_0)$. We pick a path $\widetilde{\mu} \in \Sigma_\infty$ from $a_0 = 0 \cdot a_0$ to $1 \cdot a_0$. We write $\mu := p \circ \widetilde{\mu}$ and we consider the isomorphism

$$\begin{aligned} \psi\colon \pi_1(\Sigma_\infty, \widetilde{a}_0) &\to \pi_1(\Sigma_\infty, \widetilde{a}_0) \\ g &\mapsto [\mu] \cdot g \cdot [\mu]^{-1}. \end{aligned}$$



Now we have the following isomorphisms:

$$\pi_1(\Sigma_g, a_0) \xrightarrow{\cong} \pi_1(\Sigma_\infty, \widetilde{a}_0) \rtimes_\psi \mathbb{Z} \xrightarrow{\cong} \langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle \rtimes_\varphi \mathbb{Z}.$$

by the Surface-of-Infinite Genus–π_1-Lemma 105.5 we have an explicit isomorphism $\pi_1(\Sigma_\infty, \widetilde{a}_0) \cong \langle \{x_{1,i}, y_{1,i}, \ldots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle$, we leave it to the reader to verify that ψ and φ match up

by the Action–π_1-Theorem 93.1 (2) we have an epimorphism $\gamma\colon \pi_1(\Sigma_g, a_0) \to \mathbb{Z}$ with $\gamma([\mu]) = 1$ and with kernel $\pi_1(\Sigma_\infty, \widetilde{a}_0)$, thus we obtain the isomorphism from the Semidirect Product Lemma 93.12

We have thus found the promised isomorphism. ∎

Using the Surface-π_1–Semidirect Product Proposition 105.7 we can prove the following proposition:

Proposition 105.8. (Surface-π_1-Properties Proposition) Let $g \in \mathbb{N}_0$.
(1) If $g \geq 2$, then every non-trivial abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.
(2) $\pi_1(\Sigma_g)$ is torsion-free.

Proof.

(1) This statement is a pleasant application of the Surface-π_1–Semidirect Product Proposition 105.7 and Exercise 98.9. We will provide the details in Exercise 105.4.
(2) Let $x \in \pi_1(\Sigma_g)$ be a non-trivial element. The image of the natural homomorphism $\mathbb{Z} \to \pi_1(\Sigma_g)$ given by $n \mapsto x^n$ is an abelian subgroup of $\pi_1(\Sigma_g)$. It follows from (1) that the image is infinite, which implies that no power of x is trivial. We have thus shown that $\pi_1(\Sigma_g)$ is torsion-free. ■

105.3. Exercises for Chapter 105.

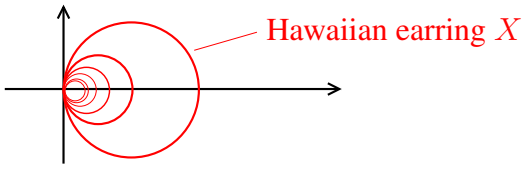
Exercise 105.1. We consider the following subset of $\mathbb{R}^2$:

$$X \coloneqq \bigcup_{n\in\mathbb{N}} \text{circle of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n}, 0) = \bigcup_{n\in\mathbb{N}} \{(x,y) \in \mathbb{R}^2 \,|\, (x-\tfrac{1}{n})^2 + y^2 = \tfrac{1}{n^2}\}.$$

This subset, equipped with the subspace topology is colloquially referred to as the *Hawaiian earring*, see the figure below for an illustration.

(a) Let $A = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. Show that X is homeomorphic to $[0,1]/A$.
Remark. We encountered the topological space $[0,1]/A$ in Exercise 8.14.
(b) Show that X is compact.
(c) Is X homeomorphic to the wedge of countably infinitely many circles?
(d) Show that $\pi_1(X)$ is not finitely generated.
(e) Even better than (d): Show that $\pi_1(X)$ is uncountably generated, i.e. show that there is no countable generating set for $\pi_1(X)$.
Hint. Use continuous maps $\{f_i \colon X \to S^1\}_{i\in\mathbb{N}}$ to define an epimorphism $\pi_1(X) \to \mathbb{Z}^{(\mathbb{N})}$.

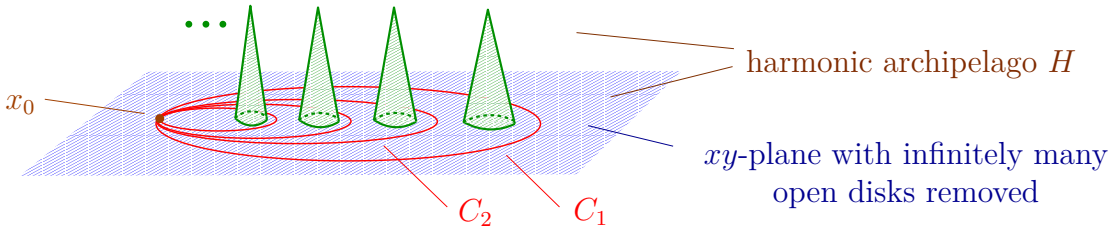
The combination of (b) and (e) implies, perhaps somewhat surprisingly, that there exist compact spaces whose fundamental groups are uncountably generated.
Remark. The precise determination of $\pi_1(X)$ is actually rather delicate, we refer to [**MoM1986**] and [**Smit1992**] for details. In particular, despite appearances, it is shown in [**Smit1992**] that $\pi_1(X)$ is not a free group.



Exercise 105.2. For $n \in \mathbb{N}$ let C_n be the circle in $\mathbb{R}^2 \times \{0\}$ of radius $\frac{1}{2n}$ with center $(\frac{1}{n}, 0)$. The *harmonic archipelago* H is the topological space that is obtained from $\mathbb{R}^2$ as follows:

For each $n \in \mathbb{N}$ we delete an open disk between C_n and C_{n+1} and we glue on a cone such that the z-coordinate of the cone equals 1. We refer to the figure below for an illustration.

(a) Give a precise description of the harmonic archipelago H.
Remark. You might want to skip this exercise, unless you are inclined to super-rigorous mathematics.
(b) Show that the harmonic archipelago H is non-compact.
(c) Let $x_0 = (0,0)$ be the origin. Show that for each $n \in \mathbb{N}$ the inclusion-induced map $\pi_1(C_n, x_0) \to \pi_1(H, x_0)$ is a monomorphism.
(d) As in Exercise 105.1 we refer to $X := \bigcup_{n\in\mathbb{N}} C_n$ as the Hawaiian earring. Show that the inclusion-induced map $\pi_1(X, x_0) \to \pi_1(H, x_0)$ is not a monomorphism.
(e) Is $\pi_1(H, x_0)$ countable or uncountable?



Exercise 105.3. Let X be a topological space, let A and B be two subspaces of X and let $f\colon A \to B$ be a homeomorphism. We define

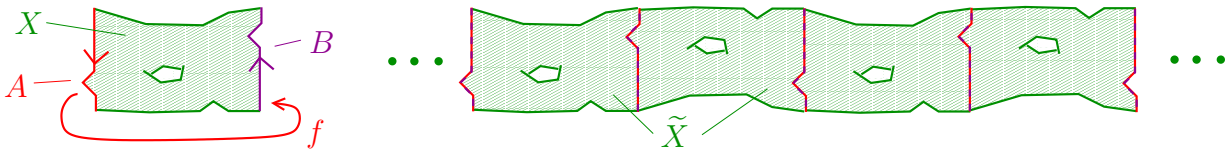
$$\widetilde{X} := \Big(\bigcup_{i\in\mathbb{Z}} (X \times \{i\}) \Big) / (a,i) \sim (f(a), i-1) \quad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

We consider the action of $\mathbb{Z}$ on $\widetilde{X}$ that is given by $n\cdot[(x,i)] := [(x,i+n)]$ for $(x,i) \in X\times\{i\}$.

(a) Show that this action is continuous.
(b) We write $X(f) = X/\!\sim$ where $a \sim f(a)$ for $a \in A$. Show that the maps

$$\begin{array}{rcl} \Phi\colon X(f) & \to & \widetilde{X}/\mathbb{Z} \\ [x] & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \widetilde{X}/\mathbb{Z} & \to & X(f) \\ [(x,i)] & \mapsto & [x] \end{array}$$

are continuous and inverses of one another. (This implies of course that both maps are homeomorphisms.)
(c) Suppose that there exist disjoint open neighborhoods U of A and V of B. Show that the action is discrete.



Exercise 105.4. Let $g \geq 2$. We denote by Σ_g the surface of genus g. Show that every abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.
Hint. Make use of the Surface-π_1–Semidirect Product Proposition 105.7 and Exercise 98.9.

Exercise 105.5. Let $\pi = \langle \{g_i\}_{i\in\mathbb{N}} \,|\, \{r_i\}_{i\in\mathbb{N}} \rangle$ be a presentation with countably many generators and relators. Show that given any $n \geq 4$ there exists an orientable connected

n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$.
Hint. Start out with $X = (\mathbb{R}^2 \setminus \{(n,0) \mid n \in \mathbb{N}\}) \times (0,1) \times B^{n-3}$ and modify X using ideas from the proof of the Manifold-π_1-Realization Proposition 102.5.

Exercise 105.6. Provide an example of a topological space X and a sequence

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$

of subspaces of X such that for each $i \in \mathbb{N}_0$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$ and such that for a base point $w \in X_0$ the sequence

$$\pi_1(X_0,w) \to \pi_1(X_1,w) \to \pi_1(X_2,w) \to \pi_1(X_3,w) \to \dots$$

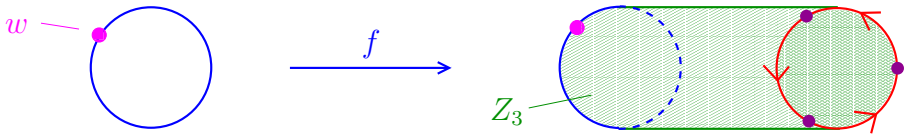
is of the form $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \dots$

Exercise 105.7.

(a) Let $n \in \mathbb{N}$. We consider the topological space

$$Z_n \coloneqq (S^1 \times [0,1])/(z,1) \sim (z \cdot \exp(2\pi\,\mathrm{i}/n), 1).$$

(i) Show that Z_n is homotopy equivalent to S^1.
(ii) We consider the map $f\colon S^1 \to Z_n$ that is given by $z \mapsto (z,0)$. Let $w \in S^1$. What is the induced map $f_*\colon \pi_1(S^1,w) \to \pi_1(Z_n,(w,0))$?



(b) Show that given any sequence of natural numbers $n_1, n_2, \dots$ there exists a topological space X and a sequence

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$

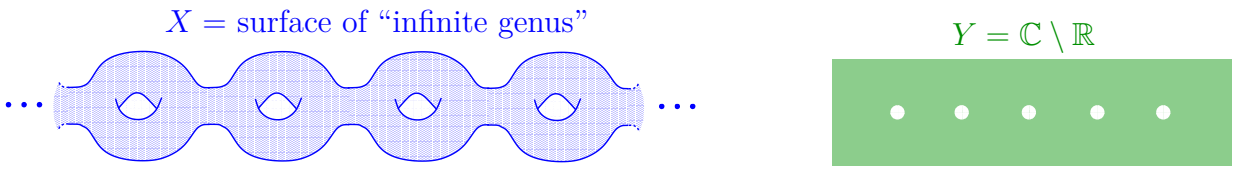
of subspaces of X such that for each $i \in \mathbb{N}_0$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$ and such that for every base point $w \in X_0$ the sequence

$$\pi_1(X_0,w) \to \pi_1(X_1,w) \to \pi_1(X_2,w) \to \pi_1(X_3,w) \to \dots$$

is of the form $\mathbb{Z} \xrightarrow{\cdot n_1} \mathbb{Z} \xrightarrow{\cdot n_2} \mathbb{Z} \xrightarrow{\cdot n_3} \dots$

(c) Show that there exists a path-connected topological space whose fundamental group is isomorphic to $\mathbb{Q}$.

Exercise 105.8. Let X be the surface of infinite genus and let $Y = \mathbb{C} \setminus \mathbb{R}$. Both are non-compact 2-dimensional smooth manifolds with boundary By the discussion on page 2054 and the Surface-of-Infinite Genus–π_1-Lemma 105.5 we know that the fundamental groups are isomorphic. Are X and Y homeomorphic?



in both cases the fundamental group is a free group $\langle x_1, x_2, x_3, \dots \rangle$

Exercise 105.9. We consider the topological spaces $\mathbb{R}^2$ and $\mathbb{Z}$. Show that there exist two embeddings $\varphi, \psi\colon \mathbb{Z} \to \mathbb{R}^2$ such that the fundamental groups $\pi_1(\mathbb{R}^2 \backslash \varphi(\mathbb{Z}))$ and $\pi_1(\mathbb{R}^2 \backslash \psi(\mathbb{Z}))$ are not isomorphic.
Hint. Groups are not isomorphic if one is countable and the other one is uncountable.

Exercise 105.10.

(a) Show that the center of any free group with at least two generators is the trivial group.
Remark. This exercise is identical to Exercise 98.1.

(b) Let $g \in \mathbb{N}_{\geq 2}$. We consider the surface Σ_g of genus g. Show that the center of $\pi_1(\Sigma_g)$ is the trivial group.
Hint. Make use of the Surface-of-Infinite Genus–π_1-Lemma 105.5.

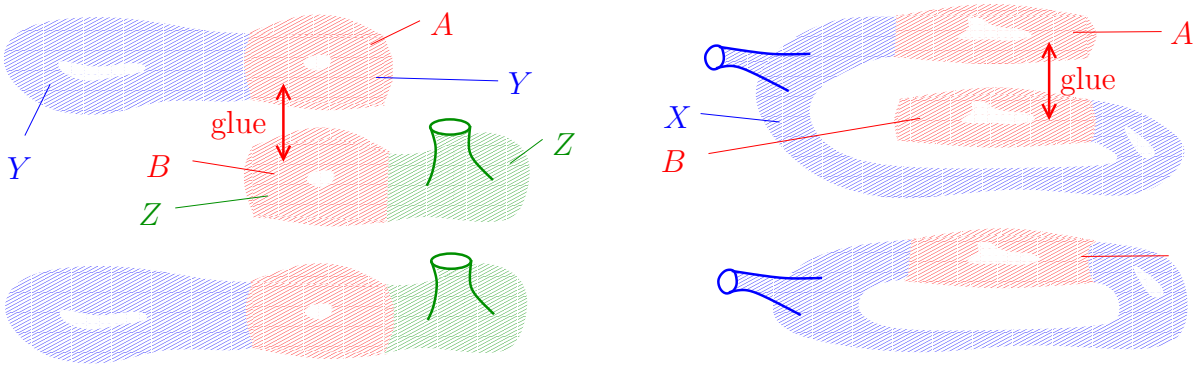
CHAPTER 106

The HNN-Seifert–van Kampen Theorem

One of our main goals at the moment is to calculate fundamental groups of interesting topological spaces. As we have discovered, one of the most powerful tools is the Seifert–van Kampen-Theorem 99.1.

In light of Exercise 8.23 (c) we can say, modulo technical details, that the Seifert–van Kampen-Theorem 99.1 allows us to calculate the fundamental group of a topological space $X = Y \cup_f Z$ that is obtained from gluing two path-connected topological spaces Y and Z along path-connected open subsets $A \subseteq Y$ and $B \subseteq Z$. This situation is illustrated in the figure below, to the left.

Sometimes one is given the situation that is shown in the figure below, to the right. Namely we are given a path-connected topological space X and we glue two disjoint path-connected open subsets A and B together. In this chapter we will prove the HNN-Seifert–van Kampen Theorem 106.7 which will allow us to calculate the fundamental group in the latter situation.



$\pi_1(Y \cup_f Z) = \pi_1(Y) *_{\pi_1(A)} \pi_1(Z)$ $\qquad$ $\pi_1 = ???$

This chapter is organized as follows:

- In Section 106.1 we will introduce HNN-extensions
- In Section 106.2 we will state and prove the HNN-Seifert–van Kampen Theorem 106.7.
- In Section 106.3 we will study fundamental groups of punctured spheres.
- In Section 106.4 we will provide exercises for Chapter 106.

In the next Chapter 107 we will prove several statements about curves on surfaces.

106.1. HNN-extensions.

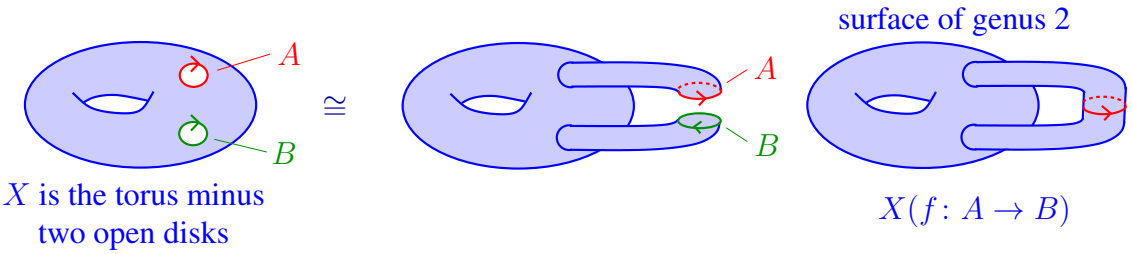
We start out with the following notation:

Notation 106.1. Let X be a topological space, let A and B be subspaces of X and finally let $f\colon A \to B$ be a homeomorphism. We write

$$X(f) \;:=\; X(f\colon A \to B) \;:=\; X/\!\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A.$$

Examples.

(1) Let X be the torus minus two open disks. We denote by A and B the two boundary components and we endow them with the orientation coming from a fixed orientation of the torus. Let $f\colon A \to B$ be an orientation-reversing diffeomorphism. Then one sees that $X(f\colon A \to B)$ is diffeomorphic[1093] to the surface of genus 2. This statement is illustrated in the figure below.[1094]



(2) Let Y be a topological space and let $f\colon Y \to Y$ be a homeomorphism. We consider $X = Y \times [0,1]$. We denote by $f_{01}\colon Y \times \{0\} \to Y \times \{1\}$ the map that is defined by $f(y,0) = (f(y),1)$. It follows immediately from the definitions that

$$X(f_{01}) \;=\; (Y \times [0,1])(f_{01}) \;=\; (Y \times [0,1])/\!\sim \;=\; \mathrm{Tor}(X,f),$$

where $\mathrm{Tor}(X,f)$ denotes the mapping torus that we introduced in Definition 21.19. □

Our next goal is to determine $\pi_1(X(f))$ in terms of the fundamental groups of X, A and the induced map $f_*\colon \pi_1(A) \to \pi_1(B)$. To formulate the Seifert–van Kampen-Theorem 99.1 we needed to introduce the notion of an amalgamated product of groups. So it does perhaps now come as a surprise that we need a new construction from group theory to formulate the desired result.

Definition 106.2. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. We refer to

$$\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\!\langle \{\alpha(g) \cdot t \cdot \beta(g)^{-1} \cdot t^{-1}\}_{g \in \Gamma} \rangle\!\rangle$$

as the HNN-extension corresponding to (π, α, β).[1095]

Example. Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t \rangle$. □

In the following lemmas we summarize some basic properties of HNN-extensions. The first lemma says in particular that a semidirect product is a special case of an HNN-extension:

Lemma 106.3. (HNN–Semidirect Product Lemma) Let $\pi = \Gamma$ and let $\alpha\colon \pi \to \pi$ and $\beta\colon \pi \to \pi$ be isomorphisms. We write $\varphi = \beta \circ \alpha^{-1}\colon \pi \to \pi$. In this setting the map[1096]

$$\begin{array}{rcl} \Phi\colon \pi \rtimes_\varphi \mathbb{Z} & \to & \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \\ (g,n) & \mapsto & g \cdot t^{-n} \end{array}$$

[1093] Note that by the Gluing-Smooth Manifolds-Proposition 54.5 we know that $X(f\colon A \to B)$ is indeed a smooth manifold.

[1094] What topological space do we obtain if f is chosen to be orientation-preserving?

[1095] The concept of such a group was introduced by Graham Higman, Bernhard Neumann and Hannah Neumann [**HNN1950**] in 1950, whence the name.

is an isomorphism.

Proof. We first verify that the given map is indeed a homomorphism. Let (g, m) and (h, n) be two elements in $\pi \rtimes_\varphi \mathbb{Z}$. First we assume that $m \geq 0$. Then the following equalities hold in $\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle$:

since α is an isomorphism we can write $h = \alpha(\alpha^{-1}(h)) = t \cdot \beta(\alpha^{-1}(h)) \cdot t^{-1} = t \cdot \varphi(h) \cdot t^{-1} = t \cdot t \cdot \varphi(\varphi(h)) \cdot t^{-1} \cdot t^{-1} = \cdots = t^m \cdot \varphi^m(h) \cdot t^{-m}$

$$\begin{aligned} \Phi(g,m) \cdot \Phi(h,n) &= g \cdot t^{-m} \cdot h \cdot t^{-n} \\ &= g \cdot t^{-m} \cdot t^m \cdot \varphi^m(h) \cdot t^{-m} \cdot t^{-n} \\ &= \Phi(g \cdot \varphi^m(h), m+n) \end{aligned} \qquad \begin{aligned} &= g \cdot \varphi^m(h) \cdot t^{-m-n} \\ &= \Phi((g,m)(h,n)). \end{aligned}$$

see Definition 93.11 for the description of the product structure on $\pi \rtimes_\varphi \mathbb{Z}$

If $m < 0$ then a very similar calculation, using that β is an isomorphism, shows again that $\Phi(g, m) \cdot \Phi(h, n) = \Phi((g, m)(h, n))$. This shows that Φ is indeed a homomorphism. By the Coproduct-of-Groups Lemma 98.4 (2) there exists a unique homomorphism

$$\Psi \colon \pi * \langle t \rangle \ \to \ \pi \rtimes_\varphi \mathbb{Z}$$

with $\Psi(g) = (g, 0)$ for every $g \in \pi$ and with $\Psi(t) = (0, 1)$. Using the Normal Closure–Factorization Lemma 98.21 and using an elementary calculation one can easily show that Ψ descends to a map $\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \to \pi \rtimes_\varphi \mathbb{Z}$. Since $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on the generating sets π and t we obtain from the Generating Set Properties-Lemma 100.7 that both homomorphisms are the identity. In particular Φ is an isomorphism. ■

The following lemma can be viewed as an analog of the Amalgamated Product–Presentation Lemma 100.20.

Lemma 106.4. (HNN-Presentation Lemma) Let $\pi = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ be a finitely presented group and let $g_1, \ldots, g_m$ be a generating set for Γ. A presentation for the HNN-extension of (π, α, β) is given by

$$\langle x_1, \ldots, x_k, t \mid r_1, \ldots, r_l, \alpha(g_i) = t \cdot \beta(g_i) \cdot t^{-1} \text{ for } i = 1, \ldots, m \rangle.$$

Proof. We apply the Normal Closure–Factorization Lemma 98.21 twice to obtain the desired isomorphism and its inverse. ■

Lemma 106.5. (HNN-Epimorphism Lemma) Let π and Γ be two groups and let $\alpha, \beta \colon \Gamma \to \pi$ be two homomorphisms. Then there exists an epimorphism

$$\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \ \to \ \mathbb{Z}$$

that is uniquely determined by $g \mapsto 0$ if $g \in \pi$ and $t^n \mapsto n$. We refer to it as the natural epimorphism onto $\mathbb{Z}$.

Proof. This statement is once again an immediate consequence of the Normal Closure–Factorization Lemma 98.21. ■

The following proposition is an analog of the Amalgamated Product-Monomorphism Proposition 98.26.

[1096] We refer to Definition 93.11 for the definition of the semidirect product $\pi \rtimes_\varphi \mathbb{Z}$.

Proposition 106.6. (HNN-Extension Monomorphism Proposition) Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. If α and β are monomorphisms, then the obvious inclusion map

$$\begin{array}{ccc} \pi & \to & \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \coloneqq (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g) \cdot t \cdot \beta(g)^{-1} \cdot t^{-1}\}_{g \in \Gamma} \rangle\rangle \\ h & \mapsto & h \end{array}$$

is also a monomorphism.

Proof. The proposition is proved in [**Ser1980**, p. 45]. Alternatively the proposition follows from the Normal Form Theorem for HNN-extensions which is formulated and proved in [**LyS1977**, Theorem IV.2.1] or alternatively in [**CgRR2008**, Satz 6.3] and [**Stil1993**, Chapter 9.2]. ■

106.2. **The HNN-Seifert–van Kampen Theorem.** Now we return to topological spaces. The following theorem can be considered as a sibling of the much used Seifert–van Kampen-Theorem 99.1 and the Manifold–Seifert-van Kampen Theorem 102.1.

Theorem 106.7. (HNN-Seifert–van Kampen Theorem) Suppose we are in one of the following two settings:

(1) We are given a path-connected topological space X, two disjoint open subsets A and B of X and a homeomorphism $f\colon A \to B$.
(2) We are given a path-connected topological manifold[1097] X, two distinct components A and B of ∂X and a homeomorphism $f\colon A \to B$.

We pick a base point $x_0 \in A$ and we pick a path γ in X from x_0 to $f(x_0)$. Furthermore we write $\Gamma = \pi_1(A, x_0)$. Finally we denote by

$$\alpha\colon \Gamma = \pi_1(A, x_0) \to \pi_1(X, x_0)$$

the inclusion-induced map and we denote by

$$\beta\colon \Gamma = \pi_1(A, x_0) \xrightarrow{f_*} \pi_1(B, f(x_0)) \to \pi_1(X, f(x_0)) \xrightarrow{\gamma_*} \pi_1(X, x_0)$$

the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from the Change-of-Base Point Proposition 88.14 (1), using the path γ. Then there exists an isomorphism

$$\Phi\colon \pi_1(X(f), x_0) \xrightarrow{\cong} \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle$$

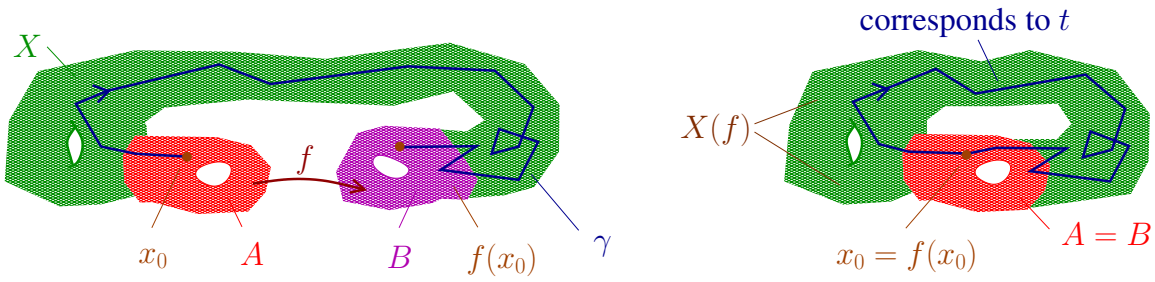
which has the following two properties:

(1) The diagram

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ \swarrow & & \searrow \\ \pi_1(X(f), x_0) & \xrightarrow{\quad\Phi\quad} & \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \end{array}$$

with the obvious diagonal maps commutes.
(2) The closed loop in $X(f)$ defined by γ corresponds under the isomorphism Φ to t.

[1098] For smooth manifolds there is also a slight and useful variation: If X is a smooth manifold and if A, B are disjoint smooth boundaries of X (in the sense of Definition 59.4), then the desired statements also hold. The only difference is that in the proof we need to replace the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7.



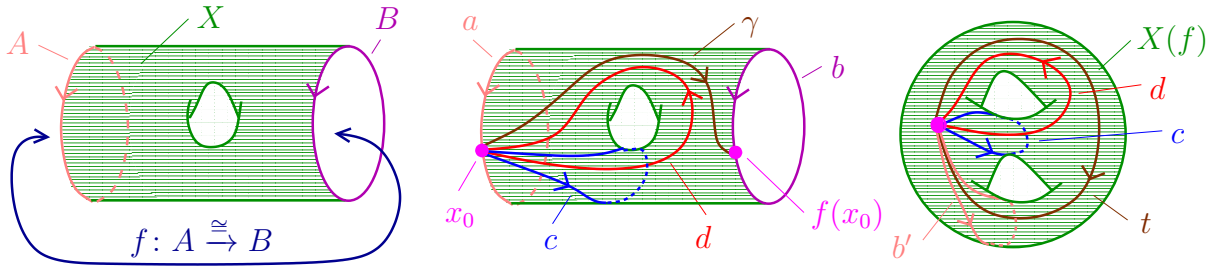
Remark.

(1) The HNN-Seifert–van Kampen Theorem 106.7 is a frequently used tool in the calculation of fundamental groups of topological spaces. The popularity of the result is in contrast to the sparsity of proofs in the literature. A sketch of an outline of a proof is given in [**ScW1979**]. A much more general statement is proved in [**Bou2016**, Proposition IV.5.5 on p. 431].

(2) In the HNN–Gluing Theorem 113.23 we will state and prove a variation on the HNN-Seifert–van Kampen Theorem 106.7 which in many cases is easier to apply than the above HNN-Seifert–van Kampen Theorem 106.7. □

Example. As on page 2064 we consider the surface X that is given by removing two open disks from a torus. We refer to the two boundary components as A and B. We equip A and B with the orientations given in the figure below. We pick a diffeomorphism $f\colon A \to B$ that preserves the orientation. As discussed on page 2064, the resulting topological space $X(f)$ is homeomorphic to the surface of genus two.

We pick a base point $x_0 \in A$. We pick a loop a in (A, x_0), we pick loops c, d in (X, x_0), a path γ in X from x to $f(x)$ and a loop b in $(B, f(x))$ as indicated in the figure below. Finally we write $\widetilde{b} = \gamma * b * \overline{\gamma}$.



By the Surface-with-Boundary–π_1-Lemma 107.8 and as in the proof of the Surface-of-Infinite Genus–π_1-Lemma 105.5 we have an isomorphism

$$\pi_1(X, x_0) \xrightarrow{\cong} \langle c, d, a, \widetilde{b} \mid a = [c, d] \cdot \widetilde{b} \rangle.$$

We write $\Gamma = \pi_1(A, x_0)$. Note that $\Gamma = \langle a \rangle$ and $\alpha(a) = a$ and $\beta(a) = \widetilde{b}$. We see that

$$\pi_1(X(f), x_0) \underset{\text{by the HNN-Seifert-van Kampen Theorem 106.7 (2)}}{\cong} \langle \pi_1(X), t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \underset{\text{by the HNN-Presentation Lemma 106.4}}{=} \langle c, d, a, \widetilde{b}, t \mid a = [c,d] \cdot \widetilde{b}, a = t \cdot \widetilde{b} \cdot t^{-1} \rangle$$

$$\underset{\text{by the Tietze transformations from Definition 100.24}}{=} \langle c, d, \widetilde{b}, t \mid [c,d] \cdot \widetilde{b} = t\widetilde{b}t^{-1} \rangle \quad = \quad \langle c, d, \widetilde{b}, t \mid \underbrace{[c,d] \cdot \widetilde{b} \cdot t \cdot \widetilde{b}^{-1} \cdot t^{-1}}_{=[c,d][\widetilde{b},t]} \rangle.$$

Thus we have now obtained the same presentation that we had already obtained in the Surface-π_1-Proposition 101.1. □

In the proof of Theorem 106.7 (1) we will need one extra piece of notation and one more lemma.

Notation 106.8. Let π and Γ be groups and let $\alpha, \beta \colon \Gamma \to \pi$ be homomorphisms. Given $i \in \mathbb{Z}$ we write $\pi_i \coloneqq \pi$ and $\Gamma_i \coloneqq \Gamma$. We write[1099]

$$\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots \coloneqq \varinjlim \Big(\underbrace{\pi_{-i} \underset{\Gamma_{-i+1}}{*} \ldots \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \ldots \underset{\Gamma_i}{*} \pi_i}_{\text{iterated amalgamated product defined by } \alpha \text{ and } \beta} \Big).$$

By a slight abuse of language we refer to this group as an infinite amalgamated product. By the universal property of the direct colimit there exists an automorphism Ξ of this infinite amalgamated product that shifts all indices by one to the right.

Lemma 106.9. (HNN-as-Semidirect Product Lemma) With the above notation there exists a well-defined map

$$\begin{aligned} \Big(\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots \Big) \rtimes_\Xi \mathbb{Z} &\xrightarrow{\cong} \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \\ (g_i, n) &\mapsto t^i \cdot g \cdot t^{-i} \cdot t^{-n}, \quad \text{where } g_i \in \pi_i \end{aligned}$$

which is an isomorphism.

Example. If α and β are isomorphisms, then it follows easily from the Amalgamated Product Lemma 98.24 (2a) that the "infinite amalgamated product" is isomorphic to π. We see that in this special case the statement of the HNN-as-Semidirect Product Lemma 106.9 is basically the same as the statement of the HNN–Semidirect Product Lemma 106.3. □

Sketch of proof of Lemma 106.9. It follows easily from the universal property of the direct colimit and of the amalgamated product that there exists a well-defined homomorphism[1100]

$$\begin{aligned} \ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots &\to \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \\ g_i &\mapsto t^i \cdot g \cdot t^{-i}, \quad \text{where } g_i = g \in \pi_i = \pi. \end{aligned}$$

[1099] By the Amalgamated Product Lemma 98.24 (3) we can form the iterated amalgamated product without having to worry about parentheses.

[1100] This can be seen as follows: Given $\gamma_i \in \Gamma_i$ we have in the infinite amalgamated product that $\alpha(\gamma_i) \in \pi_{i-1}$ equals $\beta(\gamma_i) \in \pi_i$. Since $t^{i-1} \cdot \alpha(\gamma_i) \cdot t^{-(i-1)} = t^{i-1} \cdot t \cdot \beta(\gamma_i) \cdot t^{-1} \cdot t^{-(i-1)} = t^i \cdot \beta(\gamma_i) \cdot t^{-i}$ we see that these get sent to the same element on the right-hand side.

A straightforward calculation as in the proof of the HNN–Semidirect Product Lemma 106.3 shows that we get an induced homomorphism

$$\begin{array}{rcl}\left(\dots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots\right) \rtimes_\Xi \mathbb{Z} & \xrightarrow{\cong} & \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \\ (g_i, n) & \mapsto & t^i \cdot g \cdot t^{-i} \cdot t^{-n}, \quad \text{where } g_i \in \pi_i \end{array}$$

on the semidirect product. This shows that the map given in the lemma actually makes sense. It remains to show that the map is an isomorphism. Note that using the Normal Closure–Factorization Lemma 98.21 one can easily show that there exists a well-defined homomorphism from right to left which is an inverse to our given map. We leave it to the reader to fill in the details. ■

Proof of Theorem 106.7 (1). [1101] Let X be a path-connected topological space, let A and B be two disjoint open path-connected subsets of X and let $f\colon A \to B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. Furthermore we set

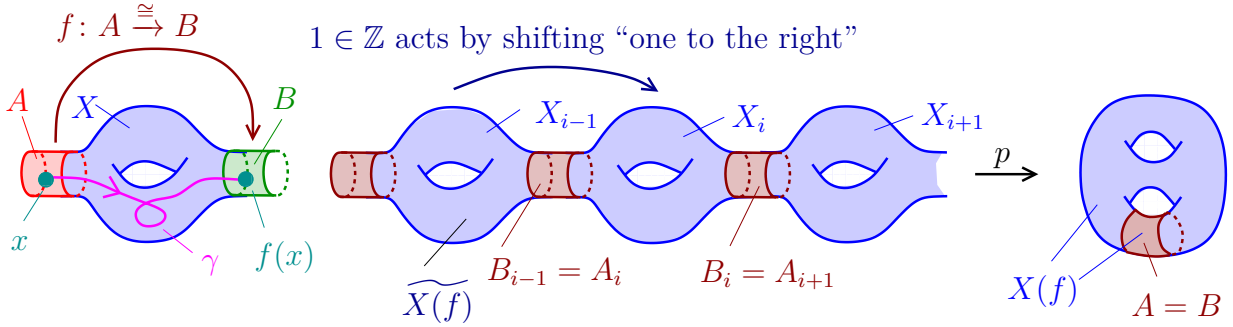
$$\widetilde{X(f)} \coloneqq \left(\bigcup_{i \in \mathbb{Z}} (X \times \{i\}) \right) / (a, i) \sim (f(a), i-1) \quad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

We equip $\widetilde{X(f)}$ with the base point $\widetilde{x}$ which is represented by $(x, 0)$. There is an obvious action of $\mathbb{Z}$ on $\widetilde{X(f)}$ given by $n \cdot [(P, i)] \coloneqq [(P, i+n)]$ for $(P, i) \in X \times \{i\}$. Since A and B are open and disjoint we can apply Exercise 105.3. We obtain that this action is discrete and continuous and that the map

$$\begin{array}{rcl} \Phi\colon X(f) & \to & \widetilde{X(f)}/\mathbb{Z} \\ [x] & \mapsto & [(x, 0)] \end{array}$$

is a homeomorphism.

For $i \in \mathbb{Z}$ we now write $X_i \coloneqq X \times \{i\}$, $A_i \coloneqq A \times \{i\}$, $B_i \coloneqq B \times \{i\}$ and $x_i \coloneqq \{x\} \times \{i\}$. Furthermore we write $\pi_i \coloneqq \pi_1(X_i, x_i)$ and $\Gamma_i = \pi_1(A_i, x_i)$.



[1101] As we saw on page 2064 and in the HNN–Semidirect Product Lemma 106.3, we can view Theorem 106.7 as a generalization of the Mapping Torus-π_1-Proposition 93.13. In fact the subsequent proof is partly modelled on the proof of the Mapping Torus-π_1-Proposition 93.13.

Claim. Given any $i \in \mathbb{Z}$ we consider the homomorphism

$$\pi_i \;=\; \pi_1(X_i, x_i) \;\underset{\uparrow}{\longrightarrow}\; \pi_1(\widetilde{X(f)}, x_i) \;\underset{\uparrow}{\longrightarrow}\; \pi_1(\widetilde{X(f)}, \widetilde{x}).$$

induced by the inclusion $X_i \to \widetilde{X(f)}$ — induced by the Change-of-Base Point Proposition 88.14 (1) where we use i translates of the path γ to connect x_i to $\widetilde{x} = x_0$

These homomorphisms induce an isomorphism

$$\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots \;\xrightarrow{\cong}\; \pi_1(\widetilde{X(f)}, \widetilde{x}).$$

Here given $i \in \mathbb{Z}$ the maps in the amalgamated product are defined as follows:

$$\overbrace{\pi_1(X_{i-1}, x_{i-1})}^{=\pi_{i-1}} \;\underset{\uparrow}{\xleftarrow{\gamma_*}}\; \pi_1(X_{i-1}, f(x_{i-1})) \;\underset{\uparrow}{\leftarrow}\; \pi_1(B_{i-1}, f(x_i)) \;\xleftarrow{f_*}\; \overbrace{\pi_1(A_i, x_i)}^{=\Gamma_i} \;\underset{\uparrow}{\rightarrow}\; \overbrace{\pi_i(X_i, x_i)}^{=\pi_i}.$$

given by the Change-of-Base Point Proposition 88.14 (1) — induced by the inclusion $B_{i-1} \to X_{i-1}$ — induced by the inclusion $A_i \to X_i$

Proof. To prove the claim we need to introduce one more gadget. Namely, given $n \in \mathbb{N}$ we write

$$\widetilde{Y}_n \;:=\; \Big(\bigcup_{i=-n}^{n} X_i \Big) \,/\, (a,i) \sim (f(a), i-1) \qquad \text{for } a \in A, i \in \{-n+1, \ldots, n\}.$$

It follows from the Seifert–van Kampen-Theorem 99.1, applied iteratively $2n$ times, that

$$\pi_1\big(\widetilde{Y}_n, \widetilde{x}\big) \;\xleftarrow{\cong}\; \pi_{-n} \underset{\Gamma_{-n+1}}{*} \pi_{-n+1} \ldots \pi_{n-1} \underset{\Gamma_n}{*} \pi_n.$$

Now we see that

$$\pi_1\big(\widetilde{X(f)}, \widetilde{x}\big) \;=\; \pi_1\big(\operatorname*{colim}_{\longrightarrow} \widetilde{Y}_n, \widetilde{x}\big) \;\underset{\uparrow}{\xleftarrow{\cong}}\; \operatorname*{colim}_{\longrightarrow} \pi_1\big(\widetilde{Y}_n, \widetilde{x}\big) \;\underset{\uparrow}{\xleftarrow{\cong}}\; \ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots$$

since the $\widetilde{Y}_n$ are open in $\widetilde{X(f)}$ we can apply the Exhaustion-π_1-Proposition 105.2 — by the above calculation and the definition of the right-hand side ⊞

In summary we obtain isomorphisms

by the above homeomorphism — by the Action–π_1-Theorem 93.1 (2) and the Semidirect Product Lemma 93.12, here the automorphism Θ is given by conjugation by $[\gamma] \in \pi_1(X(f), x)$

$$\begin{array}{ccc}
\pi_1(X(f), x) & \cong & \pi_1\big(\widetilde{X(f)}/\mathbb{Z}, [\widetilde{x}]\big) \\
& \cong & \pi_1(\widetilde{X(f)}, \widetilde{x}) \rtimes \mathbb{Z} \\
& \underset{\uparrow}{\cong} & \big(\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots \big) \rtimes_{\Xi} \mathbb{Z}
\end{array}$$

by the Covering-π_1-Monomorphism Proposition 92.7 — here, as in Notation 106.8, we denote by Ξ the automorphism of the infinite amalgamated product that "shifts every term by one to the right", the isomorphism is then given by the above claim and the verification that under this isomorphism the automorphisms Θ and Ξ match up

$$\underset{\uparrow}{\xrightarrow{\cong}}\; \langle \pi_1(X(f), x), t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle.$$

by the HNN-as-Semidirect Product Lemma 106.9

We leave it to the reader to verify that the given isomorphism has all the properties that were promised. ■

Sketch of the proof of Theorem 106.7 (2). The proof of Theorem 106.7 (2) is basically the same as the proof of Theorem 106.7 (1), we just need to replace the Seifert–van Kampen-Theorem 99.1 by the Manifold–Seifert-van Kampen Theorem 102.1. We leave it to the reader to fill in the details and to make the necessary changes to the proof. ■

In many situations the following corollary to the HNN-Seifert–van Kampen Theorem 106.7 is quite useful.

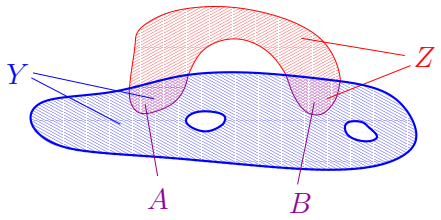
Corollary 106.10. (Add Generator-π_1-Corollary) Let X be a topological space. We suppose that it can be written as a union $X = Y \cup Z$ such that the following conditions are satisfied:

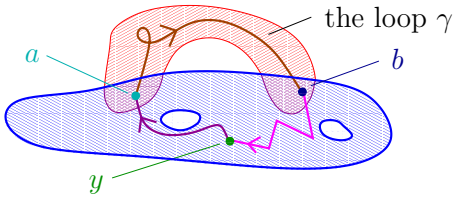
(1) Y and Z are open subsets,
(2) Y is path-connected,
(3) Z is simply connected,
(4) $Y \cap Z$ consists of two simply connected path components A and B, each of which is open in X.

Let y be a point in Y and let γ be the concatenation of a path in Y from y to a point $a \in A$, a path in Z from a to a point $b \in B$ and a path in Y from b to y. Then the map

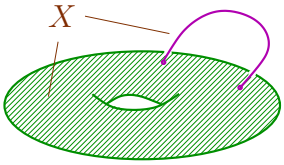
$$\pi_1(Y, y) * \langle t \rangle \;\; \to \;\; \pi_1(X, y)$$

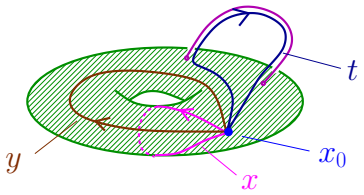
that is given by the inclusion-induced map $\pi_1(Y, y) \to \pi_1(X, y)$ and given by $t \mapsto [\gamma]$ is an isomorphism.





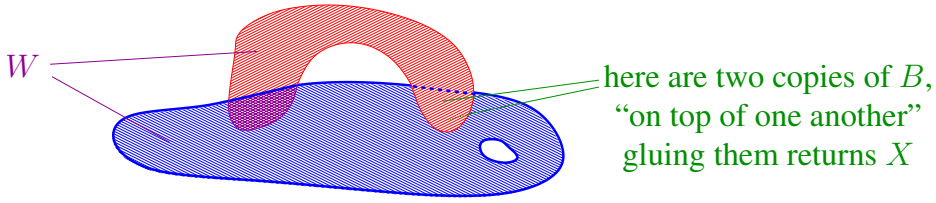
Example. We consider the topological space X that is obtained from "gluing an interval" to the torus. Using the notation from the figure below one can show quite easily, using the Add Generator-π_1-Corollary 106.10 that $\pi_1(X, x_0) \cong \langle x, y, t \,|\, [x, y] \rangle \cong \mathbb{Z}^2 * \langle t \rangle$.





In Exercise 106.1 we will work out the details of this example. □

Proof. Let Z' be a disjoint copy of Z (e.g. we could set $Z' = Z \times \{0\}$). We consider $W = (Y \sqcup Z')/\widetilde{a} \sim \widetilde{a}'$ where we identify each point $\widetilde{a} \in A$ with the corresponding point in Z'. We denote by B' the copy of B in Z' and thus also the copy in W. Thus we see



that B and its doppelgänger B' are contained in W. We denote by $f \colon B' \to B$ the obvious

homeomorphism. We obtain the following isomorphisms:[1102]

$$\pi_1(X,y) \underset{\text{by the Regluing Lemma 8.17 the obvious map } X \to W(f) \text{ is a homeomorphism}}{\cong} \pi_1(W(f),y) \underset{\text{by the HNN-Seifert–van Kampen Theorem 106.7 and since } B \text{ is simply connected}}{\cong} \pi_1(W,y) * \langle t \rangle \underset{\text{by the Seifert–van Kampen Theorem 99.1}}{\cong} (\pi_1(Y,y) *_{\pi_1(A,y)} \pi_1(Z',y)) * \langle t \rangle \underset{\text{since } A \text{ and } Z' \text{ are simply connected}}{=} \pi_1(Y,y) * \langle t \rangle.$$

One can easily verify that this is precisely the map in the statement of the corollary. ■

106.3. **Fundamental groups of punctured spheres.** We conclude this chapter with the following technical lemma which we will need once at a later occasion.

Lemma 106.11. (Punctured Sphere-π_1-Lemma) Let $n \in \mathbb{N}$ and let $P_1, \dots, P_n \in S^2$ be pairwise distinct points. Clearly we can pick smooth embeddings $\Phi_i \colon \overline{B}^2 \to S^2$, $i = 1, \dots, n$ with pairwise disjoint images such that for each $i \in \{1, \dots, n\}$ we have $\Phi_i(0) = P_i$. We set $M := S^2 \setminus (\Phi_1(B^2) \cup \dots \cup \Phi_n(B^2))$ and we pick a base point $* \in M \setminus \partial M$.

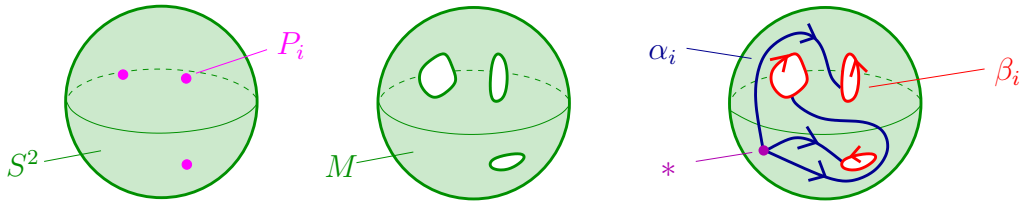
(1) The inclusion induced map $\pi_1(M,*) \to \pi_1(S^2 \setminus \{P_1, \dots, P_n\}, *)$ is an isomorphism.
(2) For $i = 1, \dots, n$ we write $C_i = \Phi_i(S^1)$ and pick a path $\beta_i \colon [0,1] \to C_i$ that goes once around C_i. We pick paths $\alpha_1, \dots, \alpha_n \colon [0,1] \to M$ with the following properties:
   (a) each α_i is a smooth embedding,
   (b) each α_i is a path from $*$ to the point $\beta_i(0)$,
   (c) for each i we have $\alpha_i([0,1)) \subseteq M \setminus \partial M$,
   (d) the images of the paths only intersect at $*$.
   For any $j \in \{1, \dots, n\}$ the homomorphism

$$\langle y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n \rangle \to \pi_1(M,*)$$

   that is uniquely determined by

$$y_i \mapsto [\alpha_i * \beta_i * \overline{\alpha_i}]$$

   is an isomorphism.



Remark.

(1) This proposition should ring a bell, it sounds quite like the Surface-with-Boundary–π_1-Lemma 107.8. The difference is that in the Surface-with-Boundary–π_1-Lemma 107.8 we only said that there exist certain paths β_i such that the given map is an isomorphism. The above lemma gives a more precise statement what type of paths work.

[1102] It follows from our hypothesis that A, Y, Z are open in X that in the settings of the HNN-Seifert–van Kampen Theorem 106.7 and the Seifert–van Kampen-Theorem 99.1 we do in fact deal with open subsets.

(2) By the Stereographic Projection Lemma 4.24 we know that $S^2 \setminus \{y\}$ is homeomorphic to $\mathbb{R}^2$. Thus the Punctured Sphere-π_1-Lemma 106.11 has a fairly obvious analog for $\mathbb{R}^2$ minus a finite collection of points or open disks.
(3) On page 2213 we will present a different approach to finding an explicit basis for $\pi_1(M, *)$.
(4) One might think that perhaps one can use any old paths α_i to obtain a basis for the free group $\pi_1(M, *)$. But as we will see in Exercise 106.8, this is not the case. □

We now provide a sketch of a proof of the Punctured Sphere-π_1-Lemma 106.11. Perhaps the most friendly words that can be uttered regarding the proof are that the proof shows that it can be quite hard to write down rigorous arguments once one leaves the friendly world of submanifolds and instead tries to deal with more general subsets.

Sketch of a proof.

(1) Clearly $M := S^2 \setminus (\Phi_1(B^2) \cup \cdots \cup \Phi_n(B^2))$ is a deformation retract of $S^2 \setminus \{P_1, \ldots, P_n\}$. It thus follows from the Homotopy-π_1-Proposition 90.8 that the inclusion induced map $\pi_1(M, *) \to \pi_1(S^2 \setminus \{P_1, \ldots, P_n\}, *)$ is an isomorphism.
(2) Without loss of generality we can assume that $j = n$. We consider $N := M \setminus \alpha_n([0, 1])$. This is an open subset of the smooth manifold M, which implies that N itself is a smooth manifold. Furthermore by Exercise 57.8 we know that the inclusion $M \setminus \alpha_n([0, 1]) \to M$ is a homotopy equivalence. For $i = 1, \ldots, n-1$ we write $X_i := \alpha_i((0, 1]) \subseteq N$. It follows from (a) and the Smooth Embedding Theorem 30.15 (1) that $X = X_1 \cup \cdots \cup X_{n-1}$ is a submanifold of N. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can pick a tubular map $X \times [-1, 1] \to N$. We write $W := N \setminus ((-1, 1) \times X)$, we pick a base point $\star \in W$ and we pick a path p in M from $*$ to $\star$.

Using the Add Generator-π_1-Corollary 106.10 one can prove[1103] the following:

(i) there exist loops $\eta_1, \ldots, \eta_{n-1}$ in $(N, \star)$ such that the obvious map

$$\pi_1(W, \star) * \langle [\eta_1], \ldots, [\eta_{n-1}] \rangle \ \to \ \pi_1(N, \star)$$

is an isomorphism,

(ii) for each $i \in \{1, \ldots, n-1\}$ the loop $p * \eta_i * \overline{p}$ is path-homotopic to $\alpha_i * \beta_i * \overline{\alpha_i}$.

Putting everything together we see that we have an isomorphism

$$\pi_1(W) * \langle [\alpha_1 * \beta_1 * \overline{\alpha_1}], \ldots, [\alpha_{n-1} * \beta_{n-1} * \overline{\alpha_{n-1}}] \rangle \ \xrightarrow{\cong} \ \pi_1(M).$$

By the Surface-with-Boundary–π_1-Lemma 107.8 we know that the right-hand side is a free group of rank $n-1$. It follows from the Grushko-Neumann Theorem 98.31 that $\pi_1(W)$ is trivial. We have thus obtained the desired isomorphism.[1104] ■

The following lemma shows that boundary curves with arbitrary paths to the base point always give a *normal* generating set.

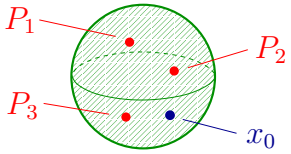
Lemma 106.12. (Punctured Sphere–Normal Generators Lemma) Let $P_1, \ldots, P_n$ be pairwise distinct points on S^2 and let $x_0 \in S^2 \setminus \{P_1, \ldots, P_n\}$ be a base point. Let

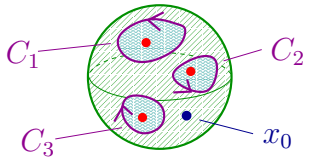
[1103] Here is the point where the demanding reader might feel that not enough details are provided.
[1104] The esthetically-inclined reader might object to the use of the Grushko-Neumann Theorem 98.31. But any other approach also opens up a can of worms. For example one might be tempted to show that W is contractible.

$\Phi_1, \ldots, \Phi_n \colon \overline{B}^2 \to S^2 \setminus \{x_0\}$ be smooth embeddings with disjoint images such that for each $i \in \{1, \ldots, n\}$ we have $\Phi_i(0) = P_i$. For $i = 1, \ldots, n$ we set $C_i := \Phi_i(S^1)$. We pick any paths connecting these loops to x_0 and we denote by $C_1, \ldots, C_n$ also the resulting elements in the fundamental groups.

(1) The elements $C_1, \ldots, C_{n-1} \in \pi_1(S^2 \setminus \{P_1, \ldots, P_n\}, x_0)$ form a *normal* generating set of $\pi_1(S^2 \setminus \{P_1, \ldots, P_n\}, x_0)$.

(2) In fact, for any $i \in \{1, \ldots, n\}$ the elements[1105] $C_1, \ldots, \widehat{C_i} \ldots, C_n$ also form a *normal* generating set of $\pi_1(S^2 \setminus \{P_1, \ldots, P_n\}, x_0)$.





Proof. After reordering the points we might as well assume that $i = n$. The lemma follows from the following little calculation:

$$\{e\} = \pi_1(\mathbb{R}^2) \overset{\text{since } S^2 \setminus \{P_n\} \cong \mathbb{R}^2}{\cong} \pi_1(S^2 \setminus \{P_n\}) \cong \pi_1((S^2 \setminus \{P_1, \ldots, P_n\}) \cup \Phi_1(B^2) \cdots \cup \Phi_{n-1}(\overline{B}^2)) \cong \pi_1((S^2 \setminus \{P_1, \ldots, P_n\}))/\langle\langle [C_1], \ldots, [C_{n-1}] \rangle\rangle.$$

follows from applying the Seifert-van Kampen Theorem 102.1 altogether $n-1$ times and since $\Phi_i(S^1)$ is a deformation retract of $\Phi_i(\overline{B}^2) \setminus \{P_i\}$ ∎

106.4. Exercises for Chapter 106.

Exercise 106.1. Give full details for the calculation of the fundamental group of the topological space shown in the figure on page 2071.

Exercise 106.2. Let $\pi = \langle a \rangle$, let $\Gamma = \langle t \rangle$ and let $\alpha, \beta \colon \Gamma \to \pi$ be the homomorphisms given by $\alpha(t) = a$ and $\beta(t) = a^2$. Show that the corresponding HNN-extension is isomorphic to the semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes_\varphi \mathbb{Z}$ where $\varphi \colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$ is the group homomorphism given by multiplication by 2.

Exercise 106.3. Let X be a path-connected topological space. Furthermore let $f \colon X \to X$ be a homeomorphism. In the Mapping Torus-π_1-Proposition 93.13 we saw that there exists an isomorphism $\psi \colon \pi_1(\mathrm{Tor}(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$. Give an alternative proof of this statement using the HNN-Seifert–van Kampen Theorem 106.7.

Exercise 106.4. Let X be the surface of genus g minus two open disks. We denote by C and D the resulting boundary components. We pick orientation-preserving diffeomorphisms $\varphi \colon S^1 \to C$ and $\psi \colon S^1 \to D$.

(a) We consider the homeomorphism

$$\begin{array}{rcl} f \colon C \times S^1 & \to & D \times S^1 \\ (c, z) & \mapsto & (\psi(z), \varphi^{-1}(c)). \end{array}$$

[1105] Note that by the Loop-π_1-Lemma 111.15 we know that the submanifolds $C_1, \ldots, C_n$ give rise to elements of $\pi_1(S^2 \setminus \{P_1, \ldots, P_n\}, x_0)$ which are well-defined up to conjugation.

What is the abelianization of the fundamental group of $(X \times S^1)(f)$?

(b) Does the answer to (a) change if we pick φ to be orientation-preserving and ψ to be orientation-reversing?

Exercise 106.5. Let X be a topological space. In the Suspension-π_1-Lemma 99.3 we showed that if X is non-empty and path-connected, then the fundamental group of the suspension $\Sigma(X)$ is trivial.

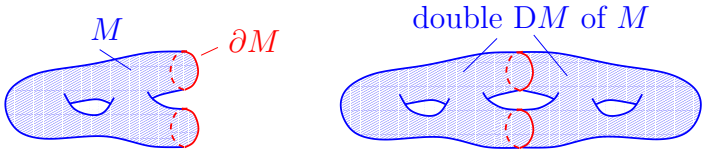
(a) Suppose that X has two path-components. Show that $\pi_1(\Sigma(X)) \cong \mathbb{Z}$.

(b) Suppose that X has k path-components with $k \geq 1$. Show that $\pi_1(\Sigma(X))$ is isomorphic to a free group on $k-1$ generators.

Exercise 106.6. Let M be a compact connected n-dimensional smooth manifold such that ∂M has $k \in \mathbb{N}$ components. As in Definition 54.8 we refer to

$$\mathrm{D}M \;:=\; (M \times \{1\}) \cup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$$

as the double of M. Show that $\pi_1(\mathrm{D}M)$ admits an epimorphism onto a free group of rank $k-1$.



Exercise 106.7. Use the Smoothly Good Cover Theorem 81.2 together with the Finite Presentation-Retract Lemma 100.23, the Seifert-van Kampen Theorem 99.1 and the HNN–Seifert-van Kampen Theorem 106.7 to show that the fundamental group of every compact connected non-empty smooth manifold is finitely presented.

Exercise 106.8. We consider the permutation group S_4 where given $\alpha, \beta \in S_4$ we write $\alpha \cdot \beta = \alpha \circ \beta$. We consider the cycles[1106]

$$\sigma = (1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu = (1\,2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \text{and } \nu = (1\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

(a) Show that σ and μ form a generating set for S_4.

(b) Show that σ and ν do not form a generating set for S_4.

(c) Show that $(\mu \cdot \sigma) \cdot \mu \cdot (\mu \cdot \sigma)^{-1} = \nu$.

(d) Show that the endomorphism $\langle x, y \rangle \to \langle x, y \rangle$ that is given by $x \mapsto (yx)^{-1} x (yx)$ and $y \mapsto y$ is not an isomorphism.

(e) We consider M which is defined as the disk $\overline{B}^2$ minus two open disks[1107] We consider the base point $*$ and the closed curves β_1 and β_2 as shown in the figure below. Show that

[1106] Recall that for pairwise disjoint $a_1, \ldots, a_k \in \{1, \ldots, n\}$ we denote by $(a_1\ a_2\ \ldots\ a_k) \in S_n$ the permutation that is given by

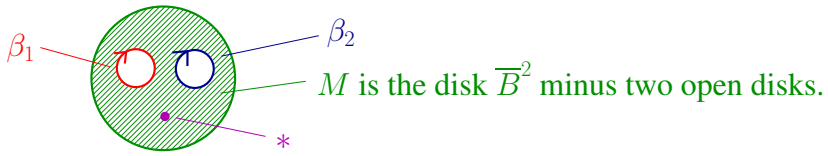
$$\begin{array}{rcl} \{1,\ldots,n\} & \to & \{1,\ldots,n\} \\ m & \mapsto & \begin{cases} a_{i+1}, & \text{if } m = a_i \text{ for some } i \in \{1,\ldots,k-1\}, \\ a_1 & \text{if } m = a_k, \\ m, & \text{otherwise.} \end{cases} \end{array}$$

Put differently, the numbers $(a_1, \ldots, a_k)$ get cyclically permuted and all other numbers stay the same.

[1107] On can show easily, say using stereographic projection, that M is diffeomorphic to S^2 minus three open disks.

for $i = 1, 2$ there exist paths α_i from $*$ to the starting point of β_i such that $[\alpha_1 * \beta_1 * \overline{\alpha_1}]$ and $[\alpha_2 * \beta_2 * \overline{\alpha_2}]$ do *not* form a generating set for $\pi_1(M, *)$.

Remark. Using Whitehead's Algorithm [**WhdJ1936**] (see also [**LyS1977**, p. 33] and [**MKS1976**, p. 166]) one can determine when conjugates of a basis of a free group form a basis again.



CHAPTER 107

Topics on curves and surfaces

In this chapter we turn to the study of surfaces and curves on surfaces. More precisely, in this chapter we will do the following:

(1) In the Curves-on-Surfaces-Classification Theorem 107.7 we classify curves on compact oriented connected 2-dimensional smooth manifolds up to diffeomorphism.
(2) In the Surface-with-Boundary–π_1-Lemma 107.8 we prove a refined version of the Surface-with-Boundary–π_1-Lemma 101.13. We will use this refinement to prove the Surface Boundary–Retraction Proposition 107.11 which deals with retractions of surfaces to boundary components.
(3) In the Not Null-Homotopic Curve Proposition 107.14 we will give criteria for curves on surfaces to be not null-homotopic.

This chapter is organized as follows:

- In Section 107.1 we will explain what we mean by a curve (with various adjectives) in these notes.
- In Section 107.2 we will state and prove the Curves-on-Surfaces-Classification Theorem 107.7.
- In Section 107.3 we will state and prove the Surface-with-Boundary–π_1-Lemma 107.8.
- In Section 107.4 we will state and prove the Not Null-Homotopic Curve Proposition 107.14.
- In Section 107.5 we will provide exercises for Chapter 107.

107.1. **Curves in topological spaces and smooth manifolds.** Throughout these notes we will use the words curves and loops with various adjectives. We now try to fix our conventions, and we hope that we manage to stick to them throughout:

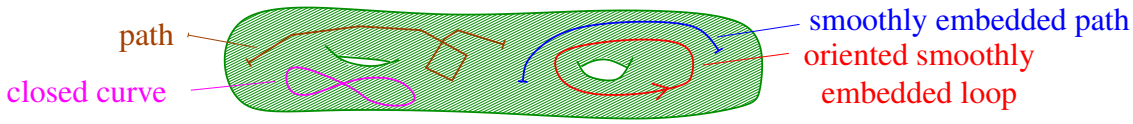
Definition 107.1. [1108] Let X be a topological space.

(1) A curve in X is a connected non-empty 1-dimensional manifold I together with a continuous map $I \to X$.
(2) If $I = [a, b]$ is a compact interval, then we speak of a path in X.
(3) If $I = S^1$, then we speak of a closed curve or a loop in X.
(4) Sometimes, when $I = [a, b]$ with $a < b$ and we have a continuous map $f\colon [a, b] \to X$ with $f(a) = f(b)$, then we also refer to f as a loop.[1109]

Let M be a smooth manifold.

(3) A smoothly embedded curve in M is the *image* of a smooth embedding of a 1-dimensional smooth manifold. Note that by the Smooth Embedding Theorem 30.15 we know that a smoothly embedded curve in M is the same as a 1-dimensional submanifold of M. If the smooth submanifold is oriented, then we speak of an oriented smoothly embedded curve.

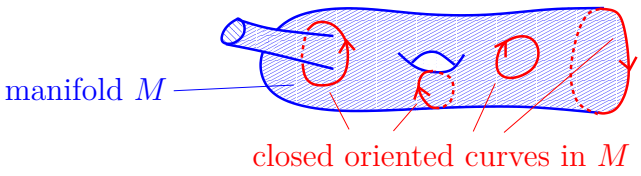
(4) We define (oriented) smoothly embedded paths and loops in the obvious way.



The word "curve" gets used in the literature with many different meanings. Here is the definition we will (try to) use consistently in this chapter:

Definition 107.2. Let M be a smooth manifold. A closed (oriented) curve in M is a closed (oriented) connected 1-dimensional submanifold either of ∂M or of M.[1110]

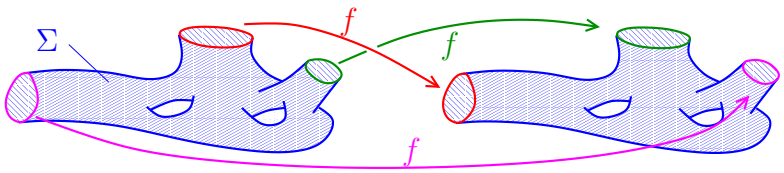
Example. Note that the boundary ∂M of a smooth manifold M is *not* a submanifold of M.



In particular, if M is a 2-dimensional smooth manifold and if C is a closed boundary component, then C is not a submanifold of M, but it is a closed curve in M in the above sense. □

107.2. **Curves on surfaces.** In this section we will use the Surface Classification Theorem 101.16 to classify curves on suitable surfaces. This classification result relies on the following lemma which we think is interesting in its own right.

Lemma 107.3. (Boundary Component-Permutation Lemma) Let Σ be a compact connected 2-dimensional smooth manifold. We denote the boundary components of Σ by $C_1, \dots, C_n$. Given any permutation $\sigma \in S_n$ there exists a diffeomorphism f of Σ with $f(C_i) = C_{\sigma(i)}$, $i = 1, \dots, n$. Furthermore, if Σ is oriented, then we can choose f to be orientation-preserving.



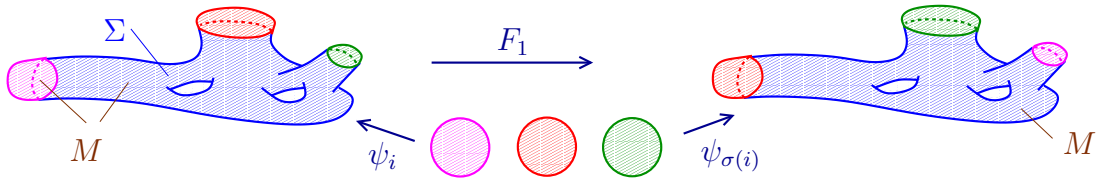
Proof. In the proof we deal with the setting that Σ is oriented. The case that Σ is non-orientable is dealt with in almost the same way. Since Σ is oriented we can equip the boundary components $C_1, \dots, C_n$ with the corresponding orientations that are given by the Boundary Orientation Lemma 31.38. Furthermore we pick orientation-reversing diffeomorphisms $\varphi_1, \dots, \varphi_n \colon S^1 \to C_i$ and we consider

$$M \;\coloneqq\; (\Sigma \sqcup (\overline{B}_1^2 \sqcup \cdots \sqcup \overline{B}_n^2))/\sim \qquad \text{where } \varphi_i(x) \sim x \text{ for } x \in S^1 \text{ and } i \in \{1, \dots, n\}.$$

[1108] It is quite tricky to have a consistent use of these terms, which often get used colloquially. Note though that these definitions are consistent with Definitions 29.1 and 88.1.

[1109] This convention is justified by the observation that $[a, b]/a \sim b$ is homeomorphic to S^1.

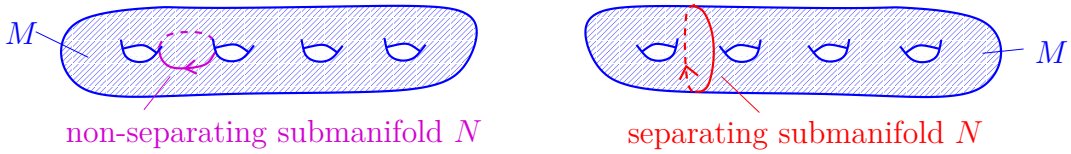
Note that it follows from the Codimension-Zero Submanifold Proposition 25.49 together with the Gluing-Smooth Manifolds-Proposition 54.5 that M is a closed oriented connected 2-dimensional smooth manifold with obvious orientation-preserving smooth embeddings $\psi_i \colon \overline{B}^2 \to M$. Now let $\sigma \in S_n$ be a permutation. It follows from the Smooth Ball Embedding Theorem 45.10 that there exists a diffeotopy $F \colon M \times [0,1] \to M$ with $F_0 = \mathrm{id}$ such that for each $i \in \{1, \ldots, n\}$ we have $F_1 \circ \psi_i = \psi_{\sigma(i)} \colon \overline{B}^2 \to M$.



Since $F_0 = \mathrm{id}$ is orientation-preserving it follows from the Smooth Isotopy-Orientation Proposition 42.4 that F_1 is orientation-preserving. The desired diffeomorphism is now given by the restriction of F_1 to Σ. ■

We move on to the following general definition.

Definition 107.4. Let M be a connected smooth manifold and let $N \subseteq M$ be a closed submanifold of codimension one. We say that N is non-separating if $M \setminus N$ is connected, otherwise we see that N is separating.



Next we recall that in Definition 55.6 we introduced the notion of cutting an orientable smooth manifold along a proper codimension-one submanifold. In the Regluing Proposition 55.10 and the Recutting Proposition 55.11 we saw that this gluing operation is the inverse of the operation of gluing two boundary components that we introduced in the Gluing-Smooth Manifolds-Proposition 54.5. We will not recall the precise definitions but we hope that the figure below will bring back fond memories.

The following lemma explains how gluing and cutting affects surfaces.

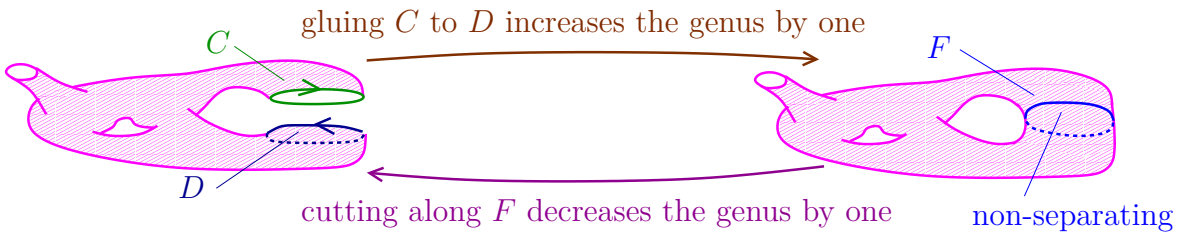
Lemma 107.5. (Surface-Gluing-Cutting Lemma) Let Σ be a compact oriented connected 2-dimensional smooth manifold.

(1) Let C and D be two distinct boundary components of Σ. We equip C and D with the orientation coming from the Boundary Orientation Lemma 31.38. Gluing C to D via an orientation-reversing diffeomorphism increases the genus[1111] by one and it decreases the number of boundary components by two.

(2) Let F be a closed non-separating smoothly embedded closed curve on Σ. Cutting along F decreases the genus by one and it increases the number of boundary components by two. (In particular this implies that the genus of Σ is ≥ 1.)

Sketch of proof.

[1111] We refer to Definition 101.22 for the definition of the genus.



(1) By the Surface Classification Theorem 101.16 we can assume that $\Sigma = \Sigma_{g,n}$ for some $g \in \mathbb{N}_0$ and some $n \in \mathbb{N}_{\geq 2}$. We denote by M the result of gluing the boundary components C and D via an orientation-reversing diffeomorphism. Note that it follows from the Gluing-Smooth Manifolds-Proposition 54.5 that M is a compact oriented connected 2-dimensional smooth manifold. In particular it has some genus h. Also note that it follows from the Boundary Component-Permutation Lemma 107.3 that the diffeomorphism type of M does not depend on the choice of the two boundary components C and D.

First note that evidently the number of boundary components under the gluing operation went down by two. There are different ways to show that gluing C to D via an orientation-reversing diffeomorphism increases the genus by one:

(a) One can prove this statement by providing an explicit diffeomorphism. We have dealt with a special case in Exercise 101.20. The general case is not much harder.

(b) We saw in the Surface–Full Invariants Proposition 101.20 that the genus of M can be read off directly from suitable invariants. Using the HNN-Seifert–van Kampen Theorem 106.7 one can calculate $\pi_1(M)$. This calculation, together with the Surface–Full Invariants Proposition 101.20 implies that $h = g + 1$.

(c) One can use the Euler characteristic, which we will introduce in Chapter 156, together with the Union-χ-Proposition 156.10 and the Surface-χ-Lemma 156.15 to see once again that $h = g + 1$.

(2) If we cut along a non-separating smoothly embedded closed curve, then the resulting 2-dimensional smooth manifold is again connected. Thus we see that Statement (2) is an immediate consequence of Statement (1) together with the Recutting Proposition 55.11. ■

We move on to the following definition:

Definition 107.6. Let Σ be an oriented 2-dimensional smooth manifold and let C and D be two oriented smoothly embedded closed curves in Σ. We say that C and D are equivalent if there exists an orientation-preserving self-diffeomorphism f of Σ that restricts to an orientation-preserving diffeomorphism $f\colon C \to D$.

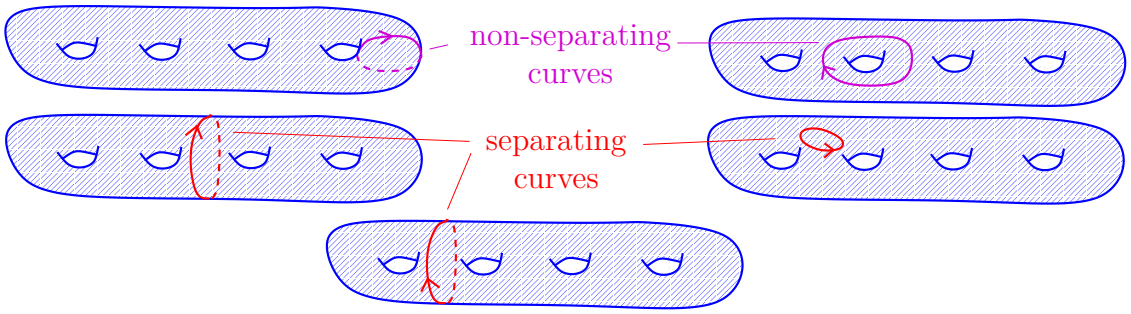
It is clear that if two oriented smoothly embedded closed curves are equivalent, then either both are separating or both are non-separating. Furthermore if they are both separating, then the results of cutting the 2-dimensional smooth manifold along the oriented smoothly embedded closed curves are diffeomorphic. The following theorem gives us the converse to these two observations.

Theorem 107.7. (Curves-on-Surfaces-Classification Theorem) Let Σ be a compact oriented connected 2-dimensional smooth manifold and let C and D be two oriented smoothly embedded closed curve in $\Sigma \setminus \partial\Sigma$.

(1) If C and D are non-separating, then C and D are equivalent.
(2) Suppose that C and D are separating.
   - We cut Σ along C in the sense of Definition 55.6. We denote by $\Sigma_C^{\pm}$ the component of Σ cut along C that contains $C^{\pm}$.
   - Similarly we define $\Sigma_D^{\pm}$.

   If Σ_C^- is diffeomorphic to Σ_D^- and if Σ_C^+ is diffeomorphic to Σ_D^+, then C and D are equivalent.

Example. In the figure below we consider the surface of genus 4 with different (non-) separating oriented smoothly embedded closed curves.



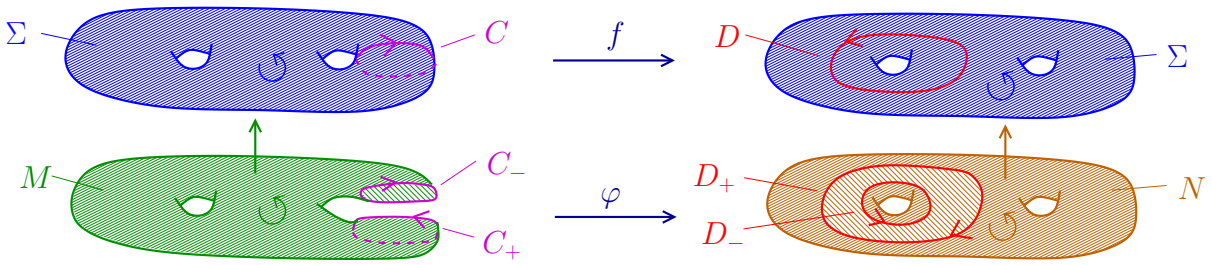
It follows from the Curves-on-Surfaces-Classification Theorem 107.7 (1) that the two non-separating oriented smoothly embedded closed curves shown are actually equivalent. Furthermore it follows easily from the Curves-on-Surfaces-Classification Theorem 107.7 (2) and the future Exercise 156.9 that any separating oriented smoothly embedded closed curve on the surface of genus four is equivalent to one of the separating oriented smoothly embedded closed curves shown in the figure. □

Remark. In Exercise 107.3 we will generalize the Curves-on-Surfaces-Classification Theorem 107.7 to certain pairs of oriented smoothly embedded closed curves. □

Sketch of proof. Let Σ be a compact oriented connected 2-dimensional smooth manifold.

(1) Let C and D be two non-separating oriented smoothly embedded closed curves on Σ. We denote by M respectively N the smooth manifold obtained from Σ by cutting along C respectively D. We denote by $C_{\pm}$ and $D_{\pm}$ the boundary components arising from C and D. Since C and D are non-separating it follows from the Surface-Gluing-Cutting Lemma 107.5 (2), the Surface Classification Theorem 101.16 and the Boundary Component-Permutation Lemma 107.3 that there exists an orientation-preserving diffeomorphism $\varphi\colon M \to N$ that restricts to orientation-preserving diffeomorphisms $\varphi\colon C_{\pm} \to D_{\pm}$. Using the Diffeotopies-of-S^1-Proposition 42.8 and using a collar neighborhood (as provided by the Smooth Collar Neighborhood Theorem 35.6) one can arrange that under the obvious identifications $C_- = C_+$ and $D_- = D_+$ the maps $\varphi\colon C_- \to D_-$ and $\varphi\colon C_+ \to D_+$ agree. It follows from the Gluing-Smooth Manifolds-Proposition 54.5 (1b) that $\varphi\colon M \to N$ descends to an orientation-preserving diffeomorphism $\overline{\varphi}\colon M/C_- = C_+ \to N/D_- = D_+$. If we combine this diffeomorphism with the

diffeomorphisms $\Sigma \to M/C_- = C_+$ and $\Sigma \to N/D_- = D_+$ from the Recutting Proposition 55.11 we obtain the desired orientation-preserving diffeomorphism $f\colon \Sigma \to \Sigma$ such that f restricts to an orientation-preserving diffeomorphism $f\colon C \to D$.



(2) The argument is similar to the proof of Statement (1), except that we do not need to appeal to the Surface-Gluing-Cutting Lemma 107.5 (2), instead we can use our hypothesis directly. We leave it to the reader to fill in the details. ■

107.3. **More on fundamental groups of surfaces with boundary.** In the Surface-with-Boundary–π_1-Lemma 101.13 we calculated the isomorphism types of the fundamental groups of $\Sigma_{g,n}$ and $N_{g,n}$. These calculations though are not optimal, for example it is unclear what maps on fundamental groups the inclusion of a boundary component induces. In this section we will formulate and prove a refined (and necessarily more technical) version of the Surface-with-Boundary–π_1-Lemma 101.13 (1). Towards the end of this section we will use this stronger version to study the question, whether components of the boundary are retracts of $\Sigma_{g,n}$ and $N_{g,n}$.

Lemma 107.8. (Surface-with-Boundary–π_1-Lemma) Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \ldots, C_n$. The following statements hold:

(1) Let $* \in \Sigma_{g,n}$ be the standard base point given by the "vertex" of $E_{4g}/\sim$. We pick an orientation for $\Sigma_{g,n}$. There exists a presentation

$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \cdot z_1 \cdot \ldots \cdot z_n \rangle$$

with the following properties:

(a) The elements $x_1, y_1, \ldots, x_g, y_g$ are represented by the edges of the $4g$-gon in the "usual way".

(b) For any $i \in \{1, \ldots, n\}$ the element z_i is represented by a loop of the form $\alpha * \gamma * \overline{\alpha}$ where α is a path from $*$ to a point on C_i and γ is a path that goes once around C_i in an orientation-preserving way.[1112]

(2) For every $i \in \{1, \ldots, n\}$ the obvious map[1113]

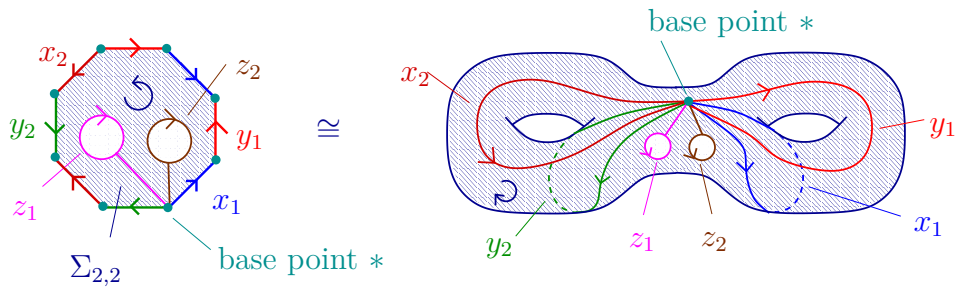
$$\underbrace{\langle x_1, y_1, .., x_g, y_g, z_1, .., \widehat{z_i}, .., z_n \rangle}_{\text{free group}} \;\to\; \langle x_1, y_1, .., x_g, y_g, z_1, .., z_n \mid \underbrace{[x_1, y_1] \ldots [x_g, y_g] \cdot z_1 \ldots z_n}_{\text{relation}} \rangle$$

is an isomorphism. The inverse is induced by the obvious assignments together with

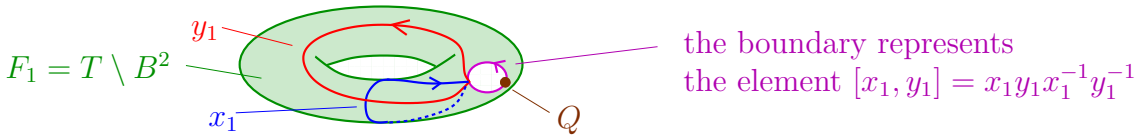
$$z_i \;\mapsto\; {z_{i-1}}^{-1} \cdot \ldots \cdot {z_1}^{-1} \cdot [y_g, x_g] \cdot \ldots \cdot [y_1, x_1] \cdot {z_n}^{-1} \cdot \ldots {z_{i+1}}^{-1}.$$

In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

(3) If $g \geq 1$ or if $n \geq 2$, then for every $i \in \{1, \ldots, n\}$ the inclusion-induced homomorphism $\pi_1(C_i) \to \pi_1(\Sigma_{g,n})$ is a monomorphism.



Example. In the Surface–Connected Sum Lemma 54.15 we showed that we can view the surface Σ of genus 2 as the connected sum of two copies of the torus T. We can use this point of view together with the Surface-with-Boundary–π_1-Lemma 107.8 to give a new calculation of the fundamental group of the surface Σ of genus 2.



Indeed: Note that $\Sigma = F_1 \cup F_1$ with $F_1 = T \setminus B^2$ and $F_2 = T \setminus B^2$ where $F_1 \cap F_2 = \partial F_1 = \partial F_2 \cong S^1$. For each $Q \in F_1 \cap F_2$ we obtain that[1114]

$$\pi_1(\Sigma, Q) \overset{\text{by the Manifold Seifert-van Kampen Theorem 102.1}}{\cong} \pi_1(F_1, Q) *_{\pi_1(F_1 \cap F_2, Q)} \pi_1(F_2, Q) \overset{\text{by the Surface-with-Boundary–}\pi_1\text{-Lemma 107.8 where } t \mapsto [x_1, y_1] \text{ and } t \mapsto [x_2, y_2]}{\cong} \langle x_1, y_1 \rangle *_{\langle t \rangle} \langle x_2, y_2 \rangle$$
$$\cong \langle x_1, x_2, y_1, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle.$$
$$\uparrow \text{ by the Amalgamated Product Presentation Lemma 100.20}$$

□

Sketch of the proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

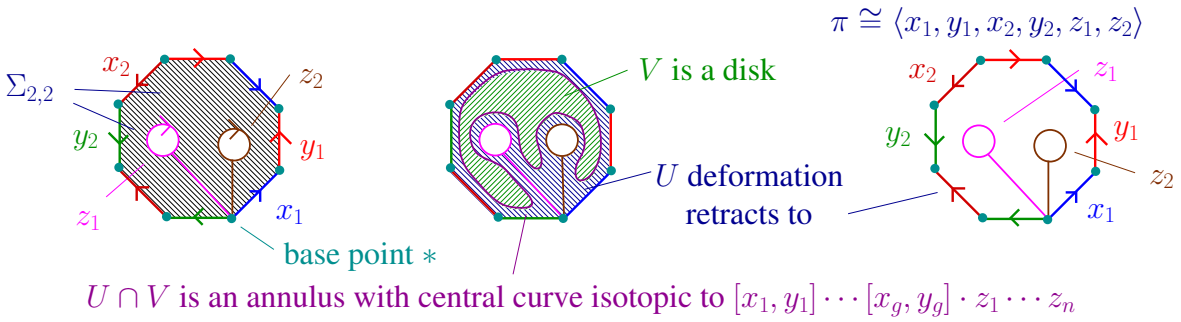
(1) The following argument is very similar to the argument on the page 1976 and 1978 and it relies heavily on the notation from the figure below. We have

$$\pi_1(\Sigma_{g,n}) \overset{\text{by the Manifold–Seifert-van Kampen-Theorem 102.1}}{=} \pi_1(U) *_{\pi_1(U \cap V)} \underbrace{\pi_1(V)}_{=\{e\}} \overset{\text{by the Amalgamated Product–Quotient Lemma 98.25}}{\cong} \pi_1(U) / \langle\langle \pi_1(U \cap V) \rangle\rangle$$
$$= \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \rangle / \langle\langle [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \cdot z_1 \cdot \ldots \cdot z_n \rangle\rangle.$$
$$\uparrow \text{ this follows from the deformation retraction shown in the figure below, together with the Homotopy Equivalence-}\pi_1\text{-Proposition 90.9 (1b)}$$

[1112] Since we picked an orientation for $\Sigma_{g,n}$ we obtain from the Boundary Orientation Lemma 31.38 that each boundary component inherits an orientation.

[1113] Here $\widehat{z_i}$ means that we actually leave out z_i.

[1114] Why is this group isomorphic to the group $\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] \cdot [x_2, y_2] \rangle$ that we obtained in the Surface-π_1-Proposition 101.1?



(2) It follows immediately from the discussion in Definition 100.24 that for any $i \in \{1, \ldots, n\}$ the obvious map

$$\langle x_1, y_1, .., x_g, y_g, z_1, .., \widehat{z_i}, .., z_n \rangle \to \langle x_1, y_1, .., x_g, y_g, z_1, .., z_n \mid [x_1, y_1] \ldots [x_g, y_g] \cdot z_1 .. z_n \rangle$$

is an isomorphism. The map that we give in the inverse direction is evidently a converse.

(3) Now suppose that $g \geq 1$ or that $n \geq 2$. Let $i \in \{1, \ldots, n\}$. The image of $\pi_1(C_i)$ in $\pi_1(\Sigma_{g,n})$ is the subgroup generated by z_i. It suffices to show that z_i is an element of infinite order in $\pi_1(\Sigma_{g,n})$. We apply (2) to i. We obtain an isomorphism

$$\begin{array}{rcl} \pi_1(\Sigma_{g,n}) & \to & \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, \widehat{z_i}, \ldots, z_n \rangle \\ z_i & \mapsto & {z_{i-1}}^{-1} \cdot \ldots \cdot {z_1}^{-1} \cdot [y_g, x_g] \cdot \ldots \cdot [y_1, x_1] \cdot \ldots {z_n}^{-1} \cdot \ldots \cdot {z_{i+1}}^{-1}. \end{array}$$

Since $g \geq 1$ or $n \geq 2$ we see that the element on the right-hand side is represented by a non-empty reduced word. Thus by definition it represents a non-trivial element in a free group. By Exercise 98.6 we know that every non-trivial element of a free group has infinite order. Thus we have shown that z_i is an element of infinite order of $\pi_1(\Sigma_{g,n})$. This implies that the inclusion-induced homomorphism $\pi_1(C_i) \to \pi_1(\Sigma_{g,n})$ is a monomorphism. ■

The following technical lemma can also be viewed as a refinement of the Surface-with-Boundary–π_1-Lemma 101.13 (1).

Lemma 107.9. (Surface-with-Boundary–Retraction Lemma) Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

(1) We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \ldots, C_n$. Given any $i \in \{1, \ldots, n\}$ there exists the following:
   (a) A topological graph $G \subseteq \Sigma_{g,n}$, which contains all C_j with $i \neq j$ and which contains a point on C_j,
   (b) a deformation retraction $r \colon \Sigma_{g,n} \to G$,
   (c) a spanning tree $T \subseteq G$,
   (d) a homeomorphism[1115]

$$\Theta \colon G/T \to C_1 \vee \cdots \vee \widehat{C_i} \vee \cdots \vee C_n \vee \underbrace{S^1 \vee \cdots \vee S^1}_{2g \text{ copies}}$$

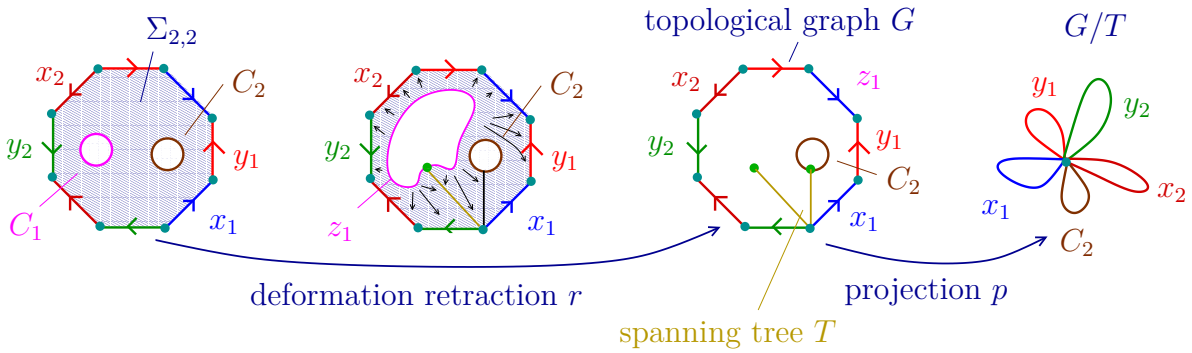
such that the following diagram commutes for all $j \neq i$:

$$\begin{array}{ccccc} & & C_j & \longrightarrow & C_1 \vee \ldots \vee \widehat{C_i} \vee \ldots \vee C_n \vee S^1 \vee \ldots \vee S^1 \\ & \swarrow & \downarrow & & \Theta \downarrow \cong \\ \Sigma_{g,n} & \xrightarrow{r} & G & \xrightarrow{p} & G/T. \end{array}$$

Note that $(p \circ r)_* \colon \pi_1(\Sigma_{g,n}) \to \pi_1(G/T)$ is an isomorphism by the Homotopy Equivalence-π_1-Proposition 90.9 and the Graph-mod-Tree Proposition 96.11.

(2) The fundamental group $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.[1116]

Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ completely analogous statements also hold for $N_{k,n}$.



Sketch of proof.

(1) In the figure above we show that, starting from any boundary component C_i of $\Sigma_{g,n}$, we can push "inward" to obtain a deformation retraction[1117] to a subspace that is a topological graph G that can be described as

$$G \;=\; \big(C_1 \sqcup \cdots \sqcup \widehat{C_i} \sqcup \cdots \sqcup C_n \sqcup \underbrace{S^1 \vee \cdots \vee S^1}_{2g \text{ copies}}\big) \text{ together with a spanning tree } T.$$

The vertical map Θ to the right is the homeomorphism given by the discussion on page 1865. It follows immediately from the construction that the diagram commutes.

(2) This statement follows from (1) together with the Wedge-of-Circles–π_1-Lemma 99.11.

The proof for $N_{k,n}$ is almost identical to the above proof for $\Sigma_{g,n}$. We leave it to the reader to figure out the precise statements and to write down the proof. ∎

Next we recall Definition 90.5:

Definition 107.10. Let X be a topological space. We say that a subset A of X is a retract of X if there exists a retraction $r \colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

For the reader's convenience we recall the statement and the proof of the Circle–Non Retract Proposition 90.6.

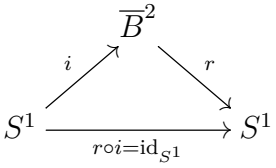
Proposition 90.6. (Circle–Non Retract Proposition) The circle S^1 is not a retract of $\overline{B}^2$.

[1115] Here $\widehat{C_i}$ means that we leave out C_i.

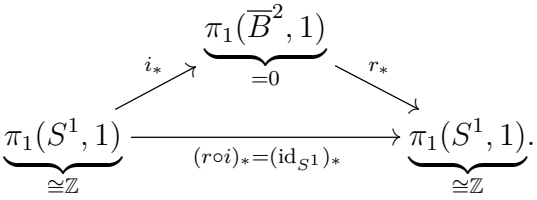
[1116] Of course we just obtained the same result in the Surface-with-Boundary–π_1-Lemma 107.8.

[1117] We leave it to others with more time and energy to write down a rigorous argument. One possibility would be to apply several times the deformation retractions that we found in Exercise 19.24.

Proof. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$, i.e. a continuous map with $r \circ i = \mathrm{id}_{S^1}$. We obtain the commutative diagram



which induces the commutative diagram



But a commutative diagram as given on the right-hand side cannot exist since an isomorphism of a non-trivial group cannot factor through the trivial group. ∎

The question arises, what is the correct generalization of the Circle–Non Retract Proposition 90.6 to arbitrary compact connected 2-dimensional topological manifolds? The following lemma gives an answer to this question.

Proposition 107.11. (Surface Boundary–Retraction Proposition) Let F be a compact connected 2-dimensional topological manifold with c boundary components.
(1) If $c = 1$, then the unique boundary component of F is not a retract of F.
(2) For $c \geq 2$, every boundary component of F is a retract of F.

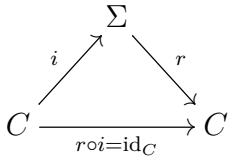
Proof. It follows from the Surface Classification Theorem 101.16 that it suffices to deal with the case that $F = \Sigma_{g,c}$ or $F = N_{g,c}$. In the following we deal with the case $F = \Sigma_{g,c}$. The other case is dealt with in almost the same way. Now let $g \in \mathbb{N}_0$.

(1) The case $g = 0$ was dealt with in the Circle–Non Retract Proposition 90.6. Now we assume that $g \geq 1$. We denote by C the unique boundary component of $\Sigma := \Sigma_{g,1}$. We have just seen in the Surface-with-Boundary–π_1-Lemma 107.8 (3) that $\pi_1(C) \to \pi_1(\Sigma)$ is a monomorphism if $g \geq 1$. So we cannot apply the same argument as in the Circle–Non Retract Proposition 90.6 where we studied the induced map on the level of fundamental groups. The idea now is to study instead the induced map on the *abelianizations* of the fundamental groups.

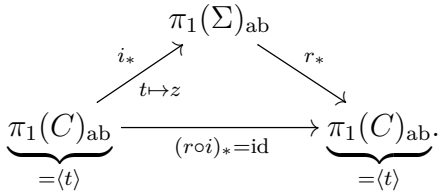
We denote by $i\colon C \to \Sigma$ the inclusion map and we suppose that there exists a retraction $r\colon \Sigma \to C$, i.e. a continuous map with $r \circ i = \mathrm{id}_C$. We write $\pi_1(C) = \langle t \rangle$ and we use the isomorphism from the Surface-with-Boundary–π_1-Lemma 107.8 (1) to make the identification

$$\pi_1(\Sigma) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \cdot z \rangle$$

where z corresponds to t under the inclusion $C = \partial\Sigma \to \Sigma$. We obtain the commutative diagram



which induces by the Abelianization Functor Proposition 98.16 the commutative diagram

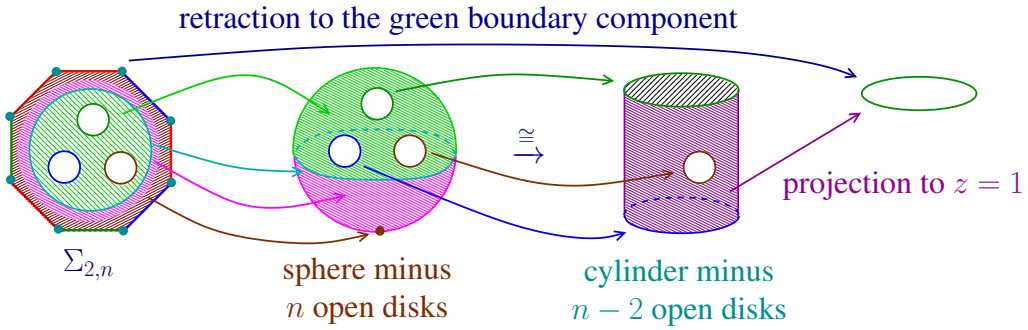


In $\pi_1(\Sigma)$ we have

$$z \;=\; ([x_1, y_1] \cdot \ldots \cdot [x_g, y_g])^{-1} \;\in\; \pi_1(\Sigma).$$

This shows that z lies in the commutator subgroup of $\pi_1(\Sigma)$, i.e. z represents the trivial element in $\pi_1(\Sigma)_{\text{ab}}$. Thus we have shown that the left diagonal map is the zero map. But the bottom horizontal map is an isomorphism of a non-trivial group and the diagram commutes. We have thus obtained a contradiction.

(2) This statement follows immediately from the Surface-with-Boundary–Retraction Lemma 107.9 since we can project the wedge of circles onto any circle. Alternatively, a retraction from $\Sigma_{2,n}$ to a boundary component is sketched in the figure below. The general case is treated the same way.[1118]



The proof for $N_{k,n}$ is almost identical to the above proof for $\Sigma_{g,n}$. We leave it to the reader to work out the details. ■

107.4. **Closed oriented curves and fundamental groups.** Oriented smoothly embedded closed curves in a given smooth manifold M have the nice feature that they are easy to draw. But strictly speaking they do not represent elements in $\pi_1(M)$ since elements in $\pi_1(M)$ are represented by *maps*[1119] $S^1 \to M$ and not just the images of such maps. Thus we need to clarify the connection between oriented smoothly embedded closed curves and homotopy classes of continuous maps.

Corollary 107.12. (Oriented Submanifold–Element-of-π_1-Corollary) Let M be a smooth manifold and let $C \subseteq M$ be an oriented smoothly embedded closed curve in M. Furthermore let $x_0 \in C$. For any two orientation-preserving diffeomorphisms $\alpha, \beta \colon S^1 \to C$ with $\alpha(1) = \beta(1) = x_0$ we have $[\alpha] = [\beta] \in \pi_1(M, x_0)$.

Proof. It is fairly straightforward to prove the corollary using the Diffeotopies-of-S^1-Proposition 42.8 and the Reparametrization-Homotopy Lemma 88.4. We leave the task of filling in the details to the industrious reader. ■

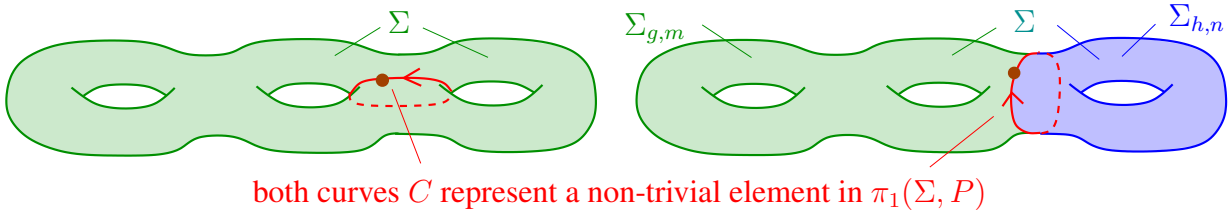
Convention 107.13. Let M be a smooth manifold and let $C \subseteq M$ be oriented smoothly embedded closed curve in M. Throughout these notes we use that the Oriented Submanifold–Element-of-π_1-Corollary 107.12 allows us to view $[C]$ unambiguously as an element in $\pi_1(M, x_0)$ for any base point $x_0 \in C$.

[1118] The proof is somewhat similar to the proof of the Surface-Retraction-to-Curve Lemma 90.17.

[1119] Here of course we take the point of view from the convention on page 1744 where we identified $\pi_1(X, x_0)$ with homotopy classes of continuous maps $(S^1, 1) \to (X, x_0)$

Proposition 107.14. (Not Null-Homotopic Curve Proposition) Let Σ be a compact orientable connected 2-dimensional smooth manifold and let C be an oriented smoothly embedded closed curve on Σ.

(1) If C is non-separating, then $[C]$ is not null-homotopic.
(2) If C is separating, and if neither of the two components of Σ cut along C is diffeomorphic to a disk, then $[C]$ is not null-homotopic.



Proof. Let Σ be a compact 2-dimensional smooth manifold and let C be an oriented smoothly embedded closed curve on Σ.

(1) First assume that C is non-separating. By the Surface Classification Theorem 101.16 we can assume that $\Sigma = \Sigma_{g,n}$ for some $g \in \mathbb{N}_0$ and some $n \in \mathbb{N}_0$. Since C is non-separating we know by the Surface-Gluing-Cutting Lemma 107.5 (2) that $g \geq 1$. Next note that by the Curves-on-Surfaces-Classification Theorem 107.7 we only need to show that there exists a single non-separating oriented smoothly embedded closed curve on $\Sigma_{g,n}$ that is not null-homotopic. By the Surface-with-Boundary–π_1-Lemma 107.8 we have a presentation

$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \cdot z_1 \cdot \ldots \cdot z_n \rangle$$

where each x_i is represented by a non-separating oriented smoothly embedded closed curve. But the x_i represent non-trivial elements in the abelianization of $\pi_1(\Sigma_{g,n})$, in particular the corresponding oriented smoothly embedded closed curves are not null-homotopic.

(2) Now we suppose that C is separating. We cut Σ along C and we obtain two connected surfaces. By the Surface Classification Theorem 101.16 we can assume that the two surfaces are of the form $\Sigma_{g,m}$ and $\Sigma_{h,n}$. By hypothesis neither surface is a disk, so we have $g > 0$ or $m > 1$ and we have $h > 0$ or $n > 1$. To simplify the notation we now assume that $g = 1$ and $h = 1$ and $m = 2$ and $n = 2$. We perform the following calculation:

$$\begin{aligned}
\pi_1(\Sigma) &\overset{\text{by the Manifold–Seifert-van Kampen Theorem 102.1}}{\cong} \pi_1(\Sigma_{1,2}) \underset{\pi_1(C)}{*} \pi_1(\Sigma_{1,2}) \overset{\text{by the Surface-with-Boundary–}\pi_1\text{-Lemma 107.8, where } C \text{ corresponds to } c_1 \text{ and to } z_1^{-1}}{=} \langle a, b, c_1, c_2 \,|\, [a,b]\cdot c_1 \cdot c_2 \rangle \underset{\pi_1(C)}{*} \langle x, y, z_1, z_2 \,|\, [x,y]\cdot z_1 \cdot z_2 \rangle \\
&= \langle a, b, c_1, c_2, x, y, z_1, z_2 \,|\, [a,b]\cdot c_1 \cdot c_2, [x,y]\cdot z_1 \cdot z_2, c_1 = z_1^{-1} \rangle.
\end{aligned}$$

by the Amalgamated Product Presentation Lemma 100.20

We need to show that c_1 represents a non-trivial element in $\pi_1(\Sigma)$. In our present setting we see that c_1 represents a non-trivial element in the abelianization, but this is *not* the case if $m = 1$. So we need to come up with an alternative argument to show that c_1 is non-trivial. Note that it follows from the Presentation–Homomorphism

Lemma 100.13 that there exists a homomorphism

$$\varphi\colon \langle a, b, c_1, c_2, x, y, z_1, z_2 \mid [a,b]\cdot c_1 \cdot c_2, [x,y]\cdot z_1 \cdot z_2, c_1 = z_1^{-1}\rangle \ \to \ \langle u, v\rangle$$

with[1120]

$$a \mapsto u, \quad b \mapsto v, \quad c_1 \mapsto [v,u], \quad c_2 \mapsto e, \quad x \mapsto v, \quad y \mapsto u, \quad z_1 \mapsto [u,v], \quad z_2 \mapsto e.$$

Finally note that $\varphi(c_1) = [u,v]$ is a non-trivial element of the free group $\langle u, v\rangle$. It follows that C is not null-homotopic. ■

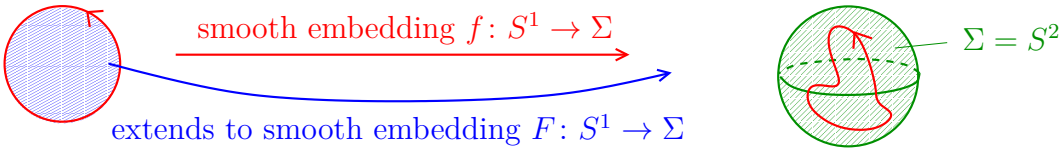
Remark. What is the "right" proof for a statement is often a matter of taste. For example an alternative, arguably more conceptual, proof for the Not Null-Homotopic Curve Proposition 107.14 (1) can be obtained by the combination of the Surface Classification Theorem 101.16 together with the HNN-Seifert–van Kampen Theorem 106.7, the Surface-with-Boundary–π_1-Lemma 107.8 (3) and the HNN-Extension Monomorphism Proposition 106.6. Similarly an alternative proof for the Not Null-Homotopic Curve Proposition 107.14 (2) can be obtained by the combination of the Surface Classification Theorem 101.16 with the Manifold–Seifert-van Kampen Theorem 102.1, the Surface-with-Boundary–π_1-Lemma 107.8 (3) and the Amalgamated Product-Monomorphism Proposition 98.26. □

We conclude this section with the following consequence of the Not Null-Homotopic Curve Proposition 107.14.

Proposition 107.15. (Fill-in-Disk Proposition)

(1) Let Σ be a compact orientable 2-dimensional smooth manifold and let $f\colon S^1 \to \Sigma$ be a smooth embedding. If f is null-homotopic, then there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.

(2) Let $\Sigma = S^2$ or $\Sigma = \mathbb{R}^2$. Given any smooth embedding $f\colon S^1 \to \Sigma$ there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.



Remark. In the Schoenflies Theorem 170.6 we will state a topological analog of the Fill-in-Disk Proposition 107.15. Furthermore, in the High-Dimensional Smooth Schoenflies Theorem 264.10 and the Low Dimensional Smooth Schoenflies Theorem 264.16 we will discuss high-dimensional analogs of the Fill-in-Disk Proposition 107.15. □

Proof.

(1) Let $f\colon S^1 \to \Sigma$ be a smooth embedding that is null-homotopic. By the Not Null-Homotopic Curve Proposition 107.14 (1) we know that $f(S^1)$ is separating. Let A and B be the two components of Σ cut along $f(S^1)$. By the Not Null-Homotopic Curve Proposition 107.14 (2) we know that at least one of A and B is diffeomorphic to a disk.

[1120]Recall that $[s,t]$ denotes the commutator of s and t, i.e. we have $[s,t] = sts^{-1}t^{-1}$. The verification that the relations are indeed satisfied relies on the trivial but useful observation that $[s,t]^{-1} = [t,s]$.

Let us assume that A is diffeomorphic to $\overline{B}^2$. It follows easily from the Diffeotopies-of-S^1-Proposition 42.8, see also the Diffeomorphisms-of-S^1-Extension Corollary 42.9, that there exists such a diffeomorphism $F\colon \overline{B}^2 \to A$ with $F|_{S^1} = f$.

(2) (a) For $\Sigma = S^2$ this statement follows immediately from (1) and the fact, shown in the Sphere-π_1-Proposition 88.24 that $\pi_1(S^2) = 0$.

(b) Now let $\Sigma = \mathbb{R}^2$ and let $f\colon S^1 \to \mathbb{R}^2$ be a smooth embedding. Since S^1 is compact there exists an $r > 0$ such that $f(S^1) \subseteq B^2_r$. We now apply (1) to the compact smooth manifold $\overline{B}^2_r$ and the smooth embedding $f\colon S^1 \to \overline{B}^2_r$. We obtain the desired extension $F\colon \overline{B}^2 \to \overline{B}^2_r \subseteq \mathbb{R}^2$. ■

We will prove more statements about curves on surfaces in the exercises to Chapter 156, once we have developed the notion of the Euler characteristic.

107.5. Exercises for Chapter 107.

Exercise 107.1. Redo the calculation of the fundamental group of the surface of genus g using the Manifold–Seifert-van Kampen-Theorem 102.1 and the Surface-with-Boundary–π_1-Lemma 107.8.

Exercise 107.2. Let Σ be a compact connected 2-dimensional smooth manifold and let C and D be two closed oriented non-separating oriented smoothly embedded closed curves in $\Sigma \setminus \partial\Sigma$. In the Curves-on-Surfaces-Classification Theorem 107.7 we showed that if Σ is orientable, then there exists a self-diffeomorphism φ of Σ with $\varphi(C) = D$.

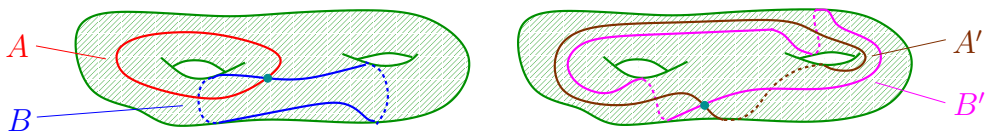
(a) Show that this conclusion does not necessarily hold if we drop the hypothesis that Σ is orientable.

(b) Where in the proof of the Curves-on-Surfaces-Classification Theorem 107.7 did we really make use of the hypothesis that Σ is orientable?

Exercise 107.3. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $\{A, B\}$ and $\{A', B'\}$ be pairs of closed smoothly embedded closed curves on Σ such that A intersects B transversally in a single point and such that A' intersects B' transversally in a single point.

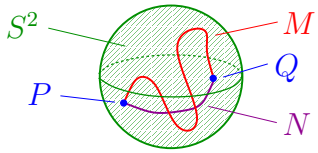
(a) Show that there exists a diffeomorphism $f\colon \Sigma \to \Sigma$ with $f(A) = A'$ and $f(B) = B'$.
Remark. The statement of this exercise, and of the closely related Curves-on-Surfaces-Classification Theorem 107.7, are proved and extended, in [**FaM2011**, Chapter 1.3]. They refer to statements of this type as the *change of coordinates principle*.

(b) Suppose that all the closed curves are oriented. Can we find f such that the maps $f\colon A \to A'$ and $f\colon B \to B'$ are both orientation-preserving?



Exercise 107.4. Let $P \neq Q$ be two points on S^2 and let M and N be two connected 1-dimensional smooth submanifolds of S^2 such that $\partial M = \{P, Q\}$ and $\partial N = \{P, Q\}$. Show that M and N are smoothly isotopic rel $\partial M = \partial N$.

Hint. First arrange that $\widetilde{\mathrm{T}}_P M \neq \widetilde{\mathrm{T}}_P N$ and $\widetilde{\mathrm{T}}_Q M \neq \widetilde{\mathrm{T}}_Q N$ and that the algebraic intersection number of M and N is zero. Afterwards use the Transversality Theorem 50.9 to make M and N transverse. Finally apply the Fill-in-Disk Proposition 107.15.
Remark. To preserve sanity one has to interpret a few results generously and ignore corner issues.



Exercise 107.5. Let $f\colon S^1 \to S^2$ be a smooth embedding.

(a) Show that there exist two smooth embeddings $F_1, F_2\colon \overline{B}^2 \to S^2$ with $F_1(\overline{B}^2) \cup F_2(\overline{B}^2) = S^2$, with $F_1(\overline{B}^2) \cap F_2(\overline{B}^2) = f(S^1)$ and such that $F_i|_{S^1} = f$.

(b) Show that there exists a diffeomorphism $\varphi\colon S^2 \to S^2$ such that $\varphi \circ f\colon S^1 \to S^2$ is the "equatorial embedding" given by $z \mapsto (z, 0)$.



Exercise 107.6.

(a) Let $\varphi\colon S^1 \to A := [-1,1] \times S^1$ be a smooth embedding with $\varphi(S^1) \subseteq (-1,1) \times S^1$ that represents a non-trivial element in $\pi_1(A)$. Show that φ is smoothly isotopic to the smooth embedding $z \mapsto (z, 0)$.
Remark. Cut A along $\varphi(S^1)$. What surface do you get?

(b) Let $\varphi, \psi\colon S^1 \to T := S^1 \times S^1$ be smooth embeddings with disjoint images such that both represent non-trivial elements in $\pi_1(A)$. Show that $\varphi(S^1)$ and $\psi(S^1)$ are smoothly isotopic.
Remark. Cut T along $\varphi(S^1)$ and make use of (a).

Remark. Later, in Exercise **??** we will prove an addendum to this exercise.

Exercise 107.7. Let $U \subseteq \mathbb{R}^2$ be an open connected non-empty subset. Show that $\pi_1(U)$ is a free group.
Hint. Use the Compact Submanifold Exhaustion Proposition 38.2 together with the Exhaustion-π_1-Proposition 105.2 and the Surface-with-Boundary–π_1-Lemma 107.8.

Part XII

Knots and links in S^3

In Part VIII we introduced knots and links in $\mathbb{R}^3$ and we used several clever ad hoc methods to define the Alexander polynomial, the Jones polynomial and the HOMFLY-PT polynomial of oriented links.

Now that we have acquired our first set of tools from algebraic topology, namely the fundamental groups, we will apply these to the study of oriented knots and links. For our current purpose it is convenient to slightly change our focus, namely instead of developing the theory of links in $\mathbb{R}^3$ we now mostly work with the equivalent theorem of links in S^3.

In this part of the lecture notes we will use fundamental groups and "Fox calculus" to define the (twisted) Alexander polynomials of knots and links. This puts the lecture notes in the tradition of the classical book by Crowell and Fox [**CF1977**]. In contrast to most other books on knot theory we will eschew homology and cohomology for the time being.

These lecture notes are organized as follows:

- In Chapter 108 we will introduce knots and links in arbitrary 3-manifolds (not just in $\mathbb{R}^3$), we will discuss basic examples, concepts and constructions and we will present some basic results.
- In Chapter 109 we will learn about several constructions of new knots and links out of given links. In particular we will introduce the meridian of a knot and we will (re-) introduce the disjoint union of links and the connected sum of two oriented knots.
- In Chapter 110 we will discuss topological various invariants of link complements and we will point out that most standard invariants are useless, except for the fundamental group. We will then give a crash course on fundamental groups to prepare the reader for the coming chapters.
- In Chapter 111 we will study the fundamental group $\pi_1(S^3 \setminus L)$ of a link. In particular we will calculate this group if L is a torus knot.
- In Chapter 112 we will introduce link diagrams and we will show that every link admits a diagram. Furthermore we will formulate the Reidemeister moves that can be used to relate any two diagrams of a given link.
- In Chapter 113 we will present an algorithm, which takes as input any diagram of a link L gives, and which gives as output the so-called Wirtinger presentations of $\pi_1(S^3 \setminus L)$, which is a presentation of deficiency one. This calculation allows us in particular to show that for any oriented m-component link the abelianization of $\pi_1(S^3 \setminus L)$ is naturally isomorphic to $\mathbb{Z}^m$.
- In Chapter 114 we will use the fact that for an oriented knot K we have a natural isomorphism $\pi_1(S^3 \setminus K)_{\text{ab}} \xrightarrow{\cong} \mathbb{Z}$ to introduce the linking number $\text{lk}(K, J)$ with any oriented knot J in the complement of K. Using the Wirtinger presentation we can give a diagrammatic way to calculate the linking number.
- In Chapter 115 we use the linking number to introduce the longitude of an oriented knot and we will show that it is essentially unique. We will use the meridian and the longitude of an oriented link to introduce the fairly simple minded coloring polynomials of an oriented knot, which will turn out to be surprisingly effective in distinguishing knots.
- In Chapter 116 we will introduce the purely algebraic concept of the Alexander function $\Delta_{\pi,\Phi} \in \mathbb{Q}(t_1, \dots, t_m)$ of a deficiency-one group π and an epimorphism $\Phi\colon \pi \to \mathbb{Z}^m$ onto a free abelian group of rank $m \geq 1$.

- In Chapter 117 we will study the Alexander function of fundamental groups of link complements. A very mild variation on this concept leads us to the Alexander polynomial of oriented knots and links.
- In Chapter 118 we will show that the Alexander polynomial of an oriented link is always symmetric.
- In Chapter 119 we will provide an elementary way to calculate the Alexander polynomial of an oriented link from a diagram.
- In Chapter 120 we will calculate the fundamental groups of closures of braids and show that the two definitions of the Alexander polynomial via Fox calculus and via the Burau representation agree.
- In Chapter 121 we will introduce the concept of a Seifert surface of a knot and we will use Seifert's algorithm to show that every knot admits a Seifert surface.
- In Chapter 122 we will introduce the notion of a fibered knot and we will show that torus knots are fibered.
- In Chapter 123 we will show that degrees of Alexander polynomials give lower bounds on the genus of a knot and we will show that Alexander polynomials of fibered knots have specific properties.
- In Chapter 124 we will introduce twisted Alexander functions of groups. This concept generalizes the above "untwisted" notion. We will outline how some of the "untwisted" results generalize to the "twisted" setup.
- In Chapter 125 we will use the twisted Alexander function to define, not surprisingly, the twisted Alexander polynomials of knots and links and we will see that these again give lower bounds on the knot genus and obstructions to fiberedness. In general these twisted invariants give better information than the original Alexander polynomial.
- In Chapter 126 we will compare many different notions of equivalence of links.

CHAPTER 108

Introduction to knots and links

In this chapter we will (re-) introduce knots and links and we will discuss several examples, basic results and notions.

This chapter is organized as follows:

- In Section 108.1 we will (re-) introduce knots and links.
- In Section 108.2 we will introduce the notion of a smooth isotopy between two links. In the Smooth Isotopy–Diffeomorphism Proposition 108.11 we will give a characterization for two links to be smoothly isotopic.
- In Section 108.3 we will provide a proof of the Smooth Isotopy–Diffeomorphism Proposition 108.11.
- In Section 108.4 we will see that the theory of knots in S^3 is essentially the same as the theory of knots in $\mathbb{R}^3$ and the same as the theory of knots in $\overline{B}^3$.
- In Section 108.5 we will discuss orientations of knots and links.
- In Section 108.6 we will introduce the mirror of a link.
- In Section 108.7 we will study tubular maps for links. We will use these to introduce the exterior of a link.
- In Section 108.8 we will provide exercises for Chapter 108.

In the next Chapter 109 we will introduce several constructions of knots and links.

108.1. **Knots and links.** The first thought would be to introduce knots and links as "objects" in the Euclidean space $\mathbb{R}^3$. For various reasons the theory of knots and links becomes a little neater if we think of knots and links as objects in the 3-dimensional sphere S^3.

To explain the relationship between $\mathbb{R}^3$ and S^3 we recall the following 3-dimensional version of the Stereographic Projection Lemma 4.24.

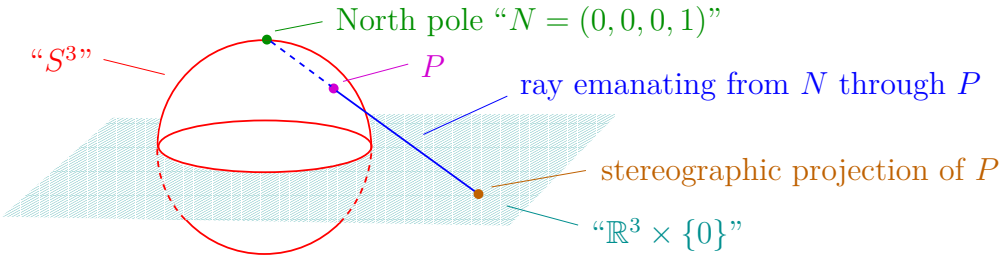
Lemma 108.1. (Stereographic Projection Lemma) We refer to the map[1121]

$$\begin{array}{rcl} \Phi\colon S^3 & \to & \mathbb{R}^3 \cup \{\infty\} \\ (x_1,x_2,x_3,x_4) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right), & \text{if } x_4 < 1, \\ \infty, & \text{if } x_4 = 1. \end{cases} \end{array}$$

as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N := (0,0,0,1) \in S^3$ to ∞.
(2) For any $P \in S^3$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^3$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^3 \times \{0\}$ in $(\Phi(P), 0)$.
(3) For any $(v_1,v_2,v_3,0) \in S^3 \cap (\mathbb{R}^3 \times \{0\})$ we have $\Phi(v_1,v_2,v_3,0) = (v_1,v_2,v_3)$.

(4) The map Φ is a homeomorphism.
(5) (a) The restriction of Φ to a map $S^3 \setminus \{N\} \to \mathbb{R}^3$ is an orientation-preserving diffeomorphism.
(b) The restriction of Φ to a map $S^n_{\leq 0} \to \overline{B}^n$ is a diffeomorphism and the restriction of Φ to a map $S^n_{\geq 0} \to (\mathbb{R}^n \cup \{\infty\}) \setminus B^n$ is a homeomorphism.



Proof. The statements of the lemma follow from the Stereographic Projection Lemma 4.24 together with the discussion on page 785 and Exercise 31.8. ■

Identification 108.2. By definition and by the Stereographic Projection Lemma 108.1 we have identifications

$$S^3 =_{\mathrm{i}} \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2+|z|^2 = 1\right\} =_{\mathrm{i}} \left\{(w,x,y,z) \,\middle|\, w^2+x^2+y^2+z^2 = 1\right\} \underset{\uparrow}{=_{\mathrm{i}}} \mathbb{R}^3 \cup \{\infty\}.$$

via the stereographic projection as defined in the Stereographic Projection Lemma 108.1 note that in Exercise 31.8 we saw that this map is (lucky us!) orientation-preserving

We will go back and forth between these models without mentioning these maps. In particular we will use the Stereographic Projection Lemma 108.1 (5) to view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$.

In Definition 62.1 we will introduce the notion of a knot in $\mathbb{R}^3$. We now ext this definition to the notion of a knot in an arbitrary 3-dimensional smooth manifolds:

Definition 108.3. Let M be a 3-dimensional smooth manifold, e.g. let $M = \mathbb{R}^3$ or let $M = S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. A knot in M is a submanifold of that is diffeomorphic to S^1.[1122] If we do not specify M, then it is understood that we are dealing with a knot in S^3.

We continue with the following definition of the trivial knot in S^3, which is consistent with Definition 62.2.

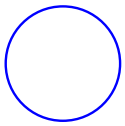
Definition 108.4. We define

$$\text{trivial knot} \;:=\; \{(x,y,0) \in \mathbb{R}^3 \,|\, x^2+y^2 = 1\} \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3.$$

In the literature the trivial knot is often also called the unknot.

Example. In the figure below we show the trivial knot together with three other examples of knots, namely the trefoil, the figure-8 knot[1123] and stevedore knot.

[1121] We equip $\mathbb{R}^3 \cup \{\infty\}$ with the topology that we introduced on page 152.
[1122] Sometimes, to distinguish this knot from the higher dimensional siblings that we will introduce on Definition 111.27 and Definition 258.1, we also refer to such a knot as a *classical knot*.
[1123] The figure-8 knot is also known as Listing's knot [**Lis1847**].


trivial knot


(left-handed) trefoil

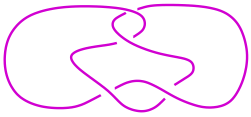


figure-8 knot


stevedore knot

More precisely, we as on page 1381, we show four 1-dimensional submanifolds of $\mathbb{R}^3$ that we view as 1-dimensional submanifolds of $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$. □

Remark. The definition of a knot is supposed to model the "physical objects" that we have in mind and that are sketched in the figure above. It is therefore perhaps at first not clear why we consider knots in $S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ instead of knots in $\mathbb{R}^3$. There are several reasons:

- For the most part topologists prefer, if possible, to work with compact spaces. In particular the compact space $S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ is often strongly preferable to the non-compact space $\mathbb{R}^3$.
- Sometimes the complement of a knot or link in S^3 is easier to describe than the complement in $\mathbb{R}^3$. For example we will see in Trivial Knot–Complement Proposition 111.11 and Exercise 111.4 that the complement of the trivial knot in S^3 is much neater than the complement of the trivial knot in $\mathbb{R}^3$.

For peace of mind we will show in the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17 that the theory of knots in $\mathbb{R}^3$ is essentially the same as the theory of knots in S^3. □

As in Definition 62.3 we see that the notion of a knot can be generalized to the notion of a link:

Definition 108.5. Let $m \in \mathbb{N}_0$ and let M be a 3-dimensional smooth manifold.

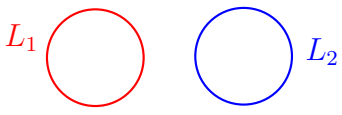
(1) An m-component link in M is a submanifold of M that is diffeomorphic to the disjoint union of m copies of S^1 together with an ordering of the components of L, i.e. together with a bijection from $\{1, \dots, m\}$ to the set of components of L.
(2) Given an m-component link $L \subseteq S^3$ and given $i \in \{1, \dots, m\}$ we denote by L_i the i-th component of L.

The next definition gives us an essential class of links:

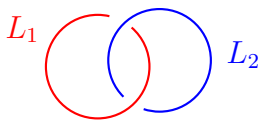
Definition 108.6. Let $m \in \mathbb{N}_0$. We define

$$\text{trivial } m\text{-component link} := \bigsqcup_{i=1}^{m} \{(x, y + 3(i-1), 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$$

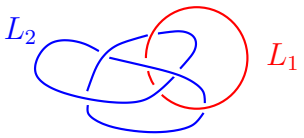
with the obvious numbering of the components of L.



trivial 2-component link



Hopf link



link where one component is a trefoil and one is the trivial knot

Remark. It follows from the Smooth 1-Manifold Classification Theorem 28.8 together with the Manifold–Local Properties Lemma 23.8 (3b) that a link in a 3-dimensional smooth

manifold M is the same as a 1-dimensional submanifold of M that is closed as a smooth manifold (together with a numbering of the components). □

108.2. **Links up to smooth isotopy.** As we already mentioned in Section 62.2, the main concept in knot theory is that we do not want to study knots and links as totally rigid submanifolds of S^3, but we would like to say that two links are "the same" if one can be "deformed" into the other. Fortunately we already have the language to turn this idea into formal mathematical language, namely the notion of a smooth isotopy between (oriented) submanifolds that we introduced in Definition 41.10 is exactly what we need. For the reader's convenience we repeat the definition of "smoothly isotopic" in our context.

Definition 108.7. We say that two m-component links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_m$ in S^3 are smoothly isotopic if there exists a smooth isotopy from L to $\widetilde{L}$, i.e. if there exists a smooth map

$$\begin{array}{rccc} F\colon L \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:[1124]

(1) For each $t \in [0,1]$ the map $F_t\colon L \to S^3$ is a smooth embedding.
(2) We have $F_0 = \mathrm{id}$ and for each $i \in \{1,\dots,m\}$ we have $F_1(L_i) = \widetilde{L}_i$.

We are now lead to the following variation on Notation 62.6:

Notation 108.8. Note that it follows from the Smooth Isotopy-Equivalence Relation Proposition 41.5 that the property of being smoothly isotopic defines an equivalence relation on the set of links. Thus it makes sense to define $\texttt{Knot}(S^3)$ and $\texttt{Link}(S^3)$ as the set of smooth isotopy classes of knots respectively links.

Example. One can easily convince oneself that the three knots shown in the figure below are smoothly isotopic.



Usually we do not distinguish two links if they are smoothly isotopic. For example, any knot that is smoothly isotopic to a trivial knot is called *trivial knot*. Sometimes we also say that a knot that is smoothly isotopic to a trivial knot is *trivial*. □

Playing around with pictures for some time shows that it might be quite difficult to show that the trefoil is smoothly isotopic to the trivial knot. This arouses the suspicion, that the trefoil is in fact not smoothly isotopic to the trivial knot. This raises the following question.

Question 108.9. How can we show that the trefoil is not smoothly isotopic to the trivial knot?

We will answer this question in the Trefoil–Non-Trivial Proposition 111.19. We also want to discuss the following generalization of Question 108.9.

[1124] As usual, given a map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by $F_t\colon X \to Y$ the map that is given by $x \mapsto F(x,t)$.

Question 108.10. Given a knot K[1125], how can we determine whether or not K is smoothly isotopic to the trivial knot?

To address Question 108.10 it is convenient to have an alternative characterization for two links to be smoothly isotopic:

Proposition 108.11. (Smooth Isotopy–Diffeomorphism Proposition) Let L and $\widetilde{L}$ be two m-component links in S^3. The following two statements are equivalent:

(1) The links L and $\widetilde{L}$ are smoothly isotopic.
(2) There exists an orientation-preserving self-diffeomorphism $f\colon S^3 \to S^3$ such that for each $i \in \{1, \dots, m\}$ we have $f(L_i) = \widetilde{L}_i$.

We postpone the proof of the Smooth Isotopy–Diffeomorphism Proposition 108.11 to the next Section 108.3.

We will now use the Smooth Isotopy–Diffeomorphism Proposition 108.11 to prove the following proposition, which allows us to relate these two questions to the question, whether two topological spaces are diffeomorphic.

Proposition 108.12. (Smooth Isotopy–Link Complement Proposition) Let L and $\widetilde{L}$ be two links in S^3. If the links L and $\widetilde{L}$ are smoothly isotopic, then there exists an orientation-preserving diffeomorphism between the link complements $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$.[1126]

Proof. Let L and $\widetilde{L}$ be two links in S^3. If the links L and $\widetilde{L}$ are smoothly isotopic, then it follows from the Smooth Isotopy–Diffeomorphism Proposition 108.11 that there exists an orientation-preserving self-diffeomorphism $f\colon S^3 \to S^3$ with $f(L) = \widetilde{L}$. Evidently f restricts to an orientation-preserving diffeomorphism between the link complements $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$. ■

108.3. **Proof of the Smooth Isotopy–Diffeomorphism Proposition 108.11.** This section is totally dedicated to the proof of the Smooth Isotopy–Diffeomorphism Proposition 108.11 and it consists of two subsections:

- In Subsection 108.3.1 we will prove the (1)⇒(2)-direction of Proposition 108.11.
- In Subsection 108.3.2 we will prove the (2)⇒(1)-direction of Proposition 108.11.

In the next Section 108.4 we will discuss the relationship between knot theory in $\mathbb{R}^3$ and in S^3.

108.3.1. **Proof of Proposition 108.11 (1)⇒(2).** At this point it is convenient to recall the following theorem:

Theorem 44.2. (Isotopy Extension Theorem) Let M be a smooth manifold with empty boundary and let K be a compact smooth manifold. Given any smooth isotopy $F\colon K \times [0,1] \to M$ there exists a diffeotopy $G\colon M \times [0,1] \to M$ with the following properties:

(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon K \times [0,1] \to M$, in other words, we have

$$G(F_0(x), t) \;=\; F(x,t) \quad \text{for all } x \in K \text{ and } t \in [0,1].$$

[1125] The phrase "given a knot K" is consciously slightly vague.
[1126] Note that this course is basically a special case of the Smooth Isotopies-Complements Proposition 44.5.

Proof of the Smooth Isotopy–Diffeomorphism Proposition 108.11 (1)⇒(2). Let

$$\begin{array}{rcl} F\colon L\times[0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

be a smooth isotopy from a link L to a knot $\widetilde{L}$. By the Isotopy Extension Theorem 44.2 we can extend the smooth isotopy F to a diffeotopy of S^3. This means in particular that there exists a diffeotopy

$$\begin{array}{rcl} G\colon S^3\times[0,1] & \to & S^3 \\ (z,t) & \mapsto & G(z,t) \end{array}$$

from the identity to a diffeomorphism $\Phi = G_1\colon S^3 \to S^3$ with $\Phi|_K = F_1$, in particular with $\Phi(L) = \widetilde{L}$. By the Smooth Isotopy-Orientation Proposition 42.4 the diffeomorphism Φ is orientation-preserving. ■

108.3.2. **Proof of Proposition 108.11 (2)⇒(1).** In this subsection we will prove the following generalization of the Smooth Isotopy–Diffeomorphism Proposition 108.11 (1)⇒(2):

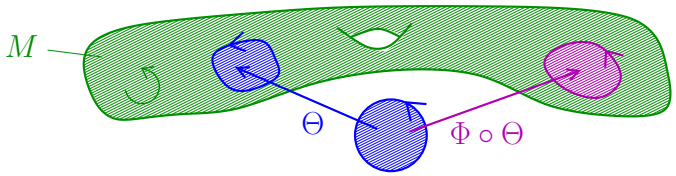
Proposition 108.13. (Submanifold–Diffeomorphism-Isotopy Proposition) In the following let $n \in \mathbb{N}$, let X be a compact smooth manifold and let $f_0, f_1\colon X \to S^n$ be smooth embeddings which are not surjective. If there exists an orientation-preserving diffeomorphism $\Phi\colon S^n \to S^n$ with $\Phi \circ f_0 = f_1$, then f_0 and f_1 are smoothly isotopic.

Our proof of the Submanifold–Diffeomorphism-Isotopy Proposition 108.13 relies on the following lemma, which might be of independent interest (and which gives a solution to Exercise 45.4):

Lemma 108.14. (Diffeotopic-to-Identity-on-Disk Lemma) Let M be an oriented connected n-dimensional smooth manifold, let $\Phi\colon M \to M$ be an orientation-preserving diffeomorphism and let $P \in M \setminus \partial M$. There exists an orientation-preserving smooth embedding $\Theta\colon \overline{B}^n \to M \setminus \partial M$ with $\Theta(0) = P$ and a diffeotopy from Φ to a diffeomorphism $\Psi\colon M \to M$ such that $\Psi|_{\Theta(\overline{B}^n)} = \mathrm{id}_{\Theta(\overline{B}^n)}$.

Proof of the Diffeotopic-to-Identity-on-Disk Lemma 108.14. Let M be an oriented connected n-dimensional smooth manifold, let $\Phi\colon M \to M$ be an orientation-preserving diffeomorphism and let $P \in M \setminus \partial M$.

First note that it follows easily from the fact that $P \in M \setminus \partial M$, and the argument of the Chart-to-Balls Lemma 23.5, that we can pick an orientation-preserving smooth embedding $\Theta\colon \overline{B}^n \to M \setminus \partial M$ with $\Theta(0) = P$. Since Φ is an orientation-preserving diffeomorphism we observe that $\Phi \circ \Theta\colon \overline{B}^n \to M \setminus \partial M$ is also an orientation-preserving smooth embedding.



Since M is connected it follows from the Smooth Ball Embedding Theorem 45.10 below that there exists a diffeotopy $H\colon M \times [0,1] \to M$ with $H_0 = \mathrm{id}$ and with $H_1 \circ \Phi \circ \Theta = \Theta$.

[1126]Recall that according to Definition 41.3 a *diffeotopy* of a smooth manifold W is a smooth map $G\colon W \times [0,1] \to W$ such that for each $t \in [0,1]$ the map $G_t\colon W \to W$, given by $w \mapsto G(w,t)$, is a diffeomorphism.

The map

$$\begin{array}{rcl} M \times [0,1] & \to & M \\ (x,t) & \mapsto & H(\Phi(x),t) \end{array}$$

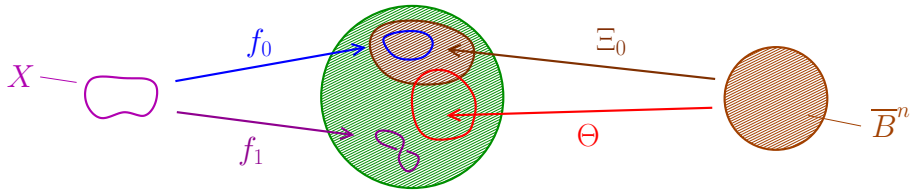
is a diffeotopy from Φ to a diffeomorphism that is the identity on $\Theta(\overline{B}^n)$. ■

Proof of the Submanifold–Diffeomorphism-Isotopy Proposition 108.13. Let $n \in \mathbb{N}$, let X be a compact smooth manifold, let $f_0, f_1 \colon X \to S^n$ be two smooth embeddings which are not surjective and let $\Phi \colon S^n \to S^n$ be an orientation-preserving diffeomorphism with $\Phi \circ f_0 = f_1$. By the Diffeotopic-to-Identity-on-Disk Lemma 108.14 we know that there exists an orientation-preserving smooth embedding $\Theta \colon \overline{B}^n \to S^n$ and a diffeotopy from Φ to a diffeomorphism $\Psi \colon M \to M$ that is the identity on $\Theta(\overline{B}^n)$.

Claim. There exists a smooth embedding $\Xi_0 \colon \overline{B}^n \to S^n$ with $f_0(X) \subseteq \Xi_0(B^n)$.

Proof. [1127] Since $f_0 \colon X \to S^n$ is not surjective we can pick $P \in S^n \setminus f_0(X)$. Since X is compact we obtain from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 (2) that the image $f_0(X)$ is compact and hence closed. This implies that there exists an $\epsilon > 0$ such that $\overline{B}^{n+1}_\epsilon(P) \cap S^n \subseteq S^n \setminus f_0(X)$. Finally note that, as in the Stereographic Projection Lemma 4.24, one can easily show that $S^n \setminus B^{n+1}_\epsilon(P)$ is diffeomorphic to $\overline{B}^n$. ⊞

Next note that by the Smooth Ball Embedding Theorem 45.10 we know that the orientation-preserving smooth embeddings $\Xi_0 \colon \overline{B}^n \to S^n$ and $\Theta \colon \overline{B}^n \to S^n$ and also $\Theta \colon \overline{B}^n \to S^n$ and $\Xi_1 \colon \overline{B}^n \to S^n$ are smoothly isotopic.



We introduce one last piece of notation: Given two smooth embeddings $g_0, g_1 \colon Y \to S^n$ from a smooth manifold Y to S^n we now write $g_0 \simeq_{\text{sm}} g_1$ if g_0 and g_1 are smoothly isotopic. By the Smooth Isotopy-Equivalence Relation Proposition 41.5 we know that this is indeed an equivalence relation.

We now see that we have the following smooth isotopies of smooth embeddings $X \to S^n$:

$$\begin{array}{lcl} f_0 & = & \Xi_0 \circ \Xi_0^{-1} \circ f_0 \\ & & \quad \downarrow \text{ since } \Xi_0 \simeq_{\text{sm}} \Theta \\ & & \simeq_{\text{sm}} \Theta \circ \Xi_0^{-1} \circ f_0 \\ & & \simeq_{\text{sm}} \Phi|_{\Theta(\overline{B}^n)} \circ \Theta \circ \Xi_0^{-1} \circ f_0 \\ & & \quad \uparrow \text{ since } \Psi \simeq_{\text{sm}} \Phi \\ & & \simeq_{\text{sm}} \Phi \circ \Xi_0 \circ \Xi_0^{-1} \circ f_0 \\ & & \quad \uparrow \text{ since } \Theta \simeq_{\text{sm}} \Xi_0 \end{array} \qquad \begin{array}{l} \text{since } \Psi|_{\Theta(\overline{B}^n)} = \text{id} \\ \downarrow \\ \simeq_{\text{sm}} \Psi|_{\Theta(\overline{B}^n)} \circ \Theta \circ \Xi_0^{-1} \circ f_0 \\ \simeq_{\text{sm}} \Phi \circ \Theta \circ \Xi_0^{-1} \circ f_0 \\ \\ = \Phi \circ f_0 = f_1. \end{array}$$

■

108.4. **Knots in $\overline{B}^3$, $\mathbb{R}^3$ and S^3.** The following proposition says says that the theories of "links up to smooth isotopy" in the three smooth manifolds $\overline{B}^3$, $\mathbb{R}^3$ and S^3 are essentially the same.

[1127] The proof of this claim is also the content of Exercise 30.10.

Proposition 108.15. (Link-in-$\overline{B}^3$-$\mathbb{R}^3$-S^3-Proposition) The obvious smooth embedding $\overline{B}^3 \to \mathbb{R}^3$ and the obvious smooth embedding $\mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ induce bijections

$$\begin{array}{c}\text{links in } \overline{B}^3 \text{ up to}\\ \text{smooth isotopy}\end{array} \xrightarrow{\cong} \begin{array}{c}\text{links in } \mathbb{R}^3 \text{ up to}\\ \text{smooth isotopy}\end{array} \xrightarrow{\cong} \begin{array}{c}\text{links in } S^3 \text{ up to}\\ \text{smooth isotopy.}\end{array}$$

In other words, with the obvious notation, the smooth embeddings induce bijections

$$\mathtt{links}(\overline{B}^3) \xrightarrow{\cong} \mathtt{links}(\mathbb{R}^3) \xrightarrow{\cong} \mathtt{links}(S^3).$$

Proof. The Link-in-$\overline{B}^3$-$\mathbb{R}^3$-S^3-Proposition 108.15 is an immediate consequence of the Link-in-$\overline{B}^3$-$\mathbb{R}^3$-Lemma 108.16 and the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17 which we will formulate and prove below. ■

Lemma 108.16. (Link-in-$\overline{B}^3$-$\mathbb{R}^3$-Lemma)

(1) Every link in $\mathbb{R}^3$ is smoothly isotopic to a link in $\overline{B}^3$.
(2) Let $L, \widetilde{L}$ be two links in $\overline{B}^3$. If L and $\widetilde{L}$ are smoothly isotopic in $\mathbb{R}^3$, then they are also smoothly isotopic in $\overline{B}^3$.

Proof. In the proof we adopt the following notation: Given a smooth submanifold $X \subseteq \mathbb{R}^3$ and given $r \in (0, \infty)$ we write $r \cdot X := \{r \cdot x \,|\, x \in X\}$. It is clear that X and $r \cdot X$ are smoothly isotopic in $\mathbb{R}^3$.

(1) Let $L \subseteq \mathbb{R}^3$ be a smooth link. Since L is compact there exists an $r > 0$ such that $L \subseteq B^3_r$. But this implies that L is smoothly isotopic to the link $\frac{1}{r} \cdot L \subseteq \overline{B}^3$.
(2) Let $L, \widetilde{L} \subseteq \overline{B}^3$ be links and let $F\colon L \times [0,1] \to \mathbb{R}^3$ be a smooth isotopy in $\mathbb{R}^3$ from L to $\widetilde{L}$. Again, since $L \times [0,1]$ is compact we see that there exists an $r > 1$ such that $F(L \times [0,1]) \subseteq B^3_r$. We now see that in the ball $\overline{B}^3$ we have obvious smooth isotopies from L to $\frac{1}{r} \cdot L$, from $\frac{1}{r} \cdot L$ to $\frac{1}{r} \cdot \widetilde{L}$ and from $\frac{1}{r} \cdot L$ to $\widetilde{L}$. It follows from the Smooth Isotopy-Equivalence Relation Proposition 41.5 that L is smoothly isotopic to $\widetilde{L}$. ■

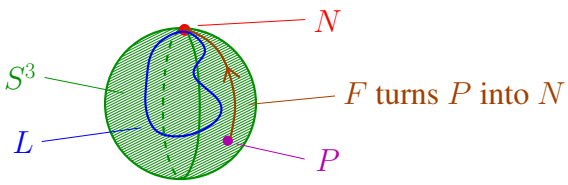
Lemma 108.17. (Link-in-$\mathbb{R}^3$-S^3-Lemma)[1128]

(1) Every link in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ is smoothly isotopic to a link in $\mathbb{R}^3$.
(2) Let $L, \widetilde{L}$ be two links in $\mathbb{R}^3$. If L and $\widetilde{L}$ are smoothly isotopic in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$, then they are also smoothly isotopic in $\mathbb{R}^3$.

Sketch of proof. Let $N = \mathfrak{e}_4$ be the North Pole of S^3. As usual we make the identification $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ where $\mathfrak{e}_4 = N = \infty$.

(1) Let L be a link in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. Since $\dim(L) = 1 < 3 = \dim(S^3)$ we know that $L \neq S^3$. Therefore we can pick $P \in S^3 \setminus L$. By the Orthogonal Action Lemma 2.37 (4) there exists an $A \in \mathrm{SO}(4)$ with $A \cdot P = N$. By the Diffeotopies-via-Matrices Lemma 41.12 (2) there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \mathrm{id}$ and such that F_1 is given by multiplication by A.
Clearly F defines a smooth isotopy from $L = F_0(L)$ to $F_1(L) \subseteq S^3 \setminus \{N\} = \mathbb{R}^3$.
(2) Let $L, \widetilde{L}$ be links in $\mathbb{R}^3$ and let $H\colon L \times [0,1] \to S^3$ be a smooth isotopy from L to $\widetilde{L}$. We start out with the following rather technical claim:

[1128] This lemma is basically the content of Exercise 41.9.



Claim. There exists a smooth embedding $\Phi\colon \overline{B}^3 \to \mathrm{SO}(4)$ such that $\Phi(0) = \mathrm{id}$ and such that $\Phi(\overline{B}^3)\cdot \mathfrak{e}_4 = \{\Phi(x)\cdot \mathfrak{e}_4 \,|\, x \in \overline{B}^3\}$ is a neighborhood of $\mathfrak{e}_4 = \infty$ in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$.

Proof. We leave it to the reader to verify that the following map has all the desired properties:

$$\begin{aligned} \Phi\colon \overline{B}^3 &\to \mathrm{SO}(4) \\ x &\mapsto \text{Gram-Schmidt orthogonalization process (starting with the last column)} \\ &\qquad \text{applied to the matrix } \underbrace{(\mathfrak{e}_1 \;\; \mathfrak{e}_2 \;\; \mathfrak{e}_3 \;\; \mathfrak{e}_4 + (x,0))}_{\in \mathrm{GL}_+(4,\mathbb{R})}. \end{aligned}$$

⊞

Next note that it follows fairly easily from $\Phi(0) = \mathrm{id}$, from $L, \widetilde{L} \subseteq \mathbb{R}^3$ and from the Tube Lemma 7.12 , that there exists an $\epsilon \in (0,1)$ such that for every $x \in \overline{B}^3_\epsilon$ we have $\Phi(x)\cdot L \subseteq \mathbb{R}^3$ and $\Phi(x)\cdot \widetilde{L} \subseteq \mathbb{R}^3$. Furthermore note that it follows from the above claim, the Measure-Zero Properties Proposition 37.3 and the Full Measure-Density Proposition 37.4 that $\Phi(\overline{B}^3_\epsilon)\cdot \mathfrak{e}_4 \cap (S^3 \setminus H(L\times[0,1]))$ is a dense subset of $\Phi(\overline{B}^3_\epsilon)\cdot \mathfrak{e}_4$. This implies that we can pick $x \in \overline{B}^3_\epsilon$ with $\Phi(x)\cdot \mathfrak{e}_4 \in S^3 \setminus H(L\times[0,1])$. Note that this means that $(\Phi(x)^{-1} \circ H)(L\times[0,1]) \subseteq \mathbb{R}^3 = S^3 \setminus \{\mathfrak{e}_4\}$.

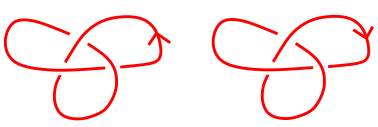
We now have the following three smooth isotopies

$$\begin{aligned} L\times[0,1] &\to \mathbb{R}^3 \\ (v,t) &\mapsto \Phi(t\cdot x)^{-1}(v) \end{aligned} \qquad \begin{aligned} L\times[0,1] &\to \mathbb{R}^3 \\ (v,t) &\mapsto \Phi(x)^{-1}\cdot H(v,t) \end{aligned} \qquad \begin{aligned} \widetilde{L}\times[0,1] &\to \mathbb{R}^3 \\ (v,t) &\mapsto \Phi((1-t)\cdot x)^{-1}(v). \end{aligned}$$

Combining these three smooth isotopies gives us the desired smooth isotopy from L to $\widetilde{L}$ in $\mathbb{R}^3$. ■

108.5. **Orientations.** In many settings it is also natural to equip knots and links with orientations. This leads us to the following generalization of Definition 62.7:

Definition 108.18. Let $m \in \mathbb{N}_0$. A link in 3-dimensional smooth manifold M is called oriented if it is oriented as a 1-dimensional smooth manifold.



trefoils with opposite orientation

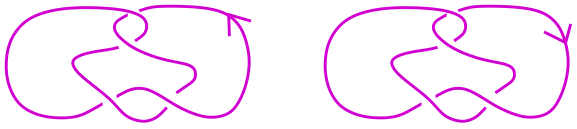


figure-8 knot with opposite orientations

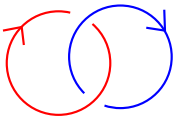
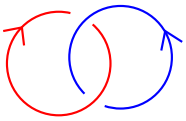
We move on to the obvious generalization of Definition 108.19:

Definition 108.19. Let M be a 3-dimensional smooth manifold. We say that two *oriented* links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_m$ in M are smoothly isotopic, if there exists a smooth isotopy $F\colon L\times[0,1] \to M$ from L to $\widetilde{L}$ such that for each $i \in \{1,\ldots,m\}$ the map $F_1\colon L_i \to \widetilde{L}_i$ is an orientation-preserving diffeomorphism.

This definition leads us to the following variation on Notation 108.8.

Notation 108.20. We denote by $\texttt{Knot}^{\text{or}}(S^3)$ and $\texttt{Link}^{\text{or}}(S^3)$ the set of smooth isotopy classes of oriented knots respectively oriented links.

Example. Similar to the discussion on page 1383 we consider the two Hopf links H^+ and H^- that are shown in the figure below.

positive Hopf link H^-  negative Hopf link H^+ 

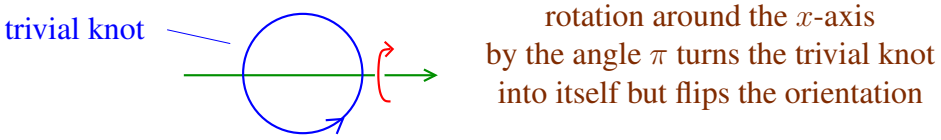
The difference is that we flipped the orientation of the blue component. In Exercise 111.6 we will show, using the fundamental group of a suitable topological space, that these two oriented Hopf links are not smoothly isotopic. □

In light of the above example we now restrict ourselves for a second to knots. For knots one can wonder whether the orientation of a knot really makes a difference. This suggests the following generalization of Definition 62.10:

Definition 108.21. Let M be a 3-dimensional smooth manifold.

(1) Let $K \subseteq M$ be an oriented knot. We define the reverse K^{rev} of K to be the knot K with the opposite orientation.
(2) A knot K is called reversible if the two orientations on K give smoothly isotopic oriented knots, otherwise we call the knot irreversible.

Example. As on page 1384 we see, using the figure below, that the trivial knot in $\mathbb{R}^3$ or also S^3 is reversible.



In Exercise 108.2 we will show that the trefoil and the figure-8 knot are also reversible. □

This leads us to the following question:

Question 108.22. Is every knot reversible?

We will answer this question in the Knot-without-Symmetries Proposition 115.16.

108.6. **The mirror of a link.** In Definition 62.11 we introduced the notion of a mirror of a link in $\mathbb{R}^3$. We now extend this definition to links in S^3:

Definition 108.23. Given a link $L \subseteq S^3$ the reflection of L in any hyperplane of $\mathbb{R}^4$ is called mirror of L and it is denoted by L^{mir}.

The first statement of the following lemma is a variation on the $\mathbb{R}^3$-Link Mirror Lemma 108.24 (1). The second statement shows that Definition 62.11 and Definition 108.23 of a mirror are consistent:

Lemma 108.24. (S^3-Link Mirror Lemma)

(1) Let $L \subseteq S^3$ be a link. The reflections in any two hyperplanes of $\mathbb{R}^4$ give rise to smoothly isotopic links.
(2) Let $L \subseteq \mathbb{R}^3$ be a link. As usual, using the identification $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$, we view L as a link in S^3. The image of L under a reflection of an affine hyperplane in $\mathbb{R}^3$ corresponds under this embedding to a mirror of L.

Sketch of proof.

(1) Let $L \subseteq S^3$ be a link and let H_0 and H_1 be two hyperplanes of $\mathbb{R}^4$. By the Orthogonal Action Lemma 2.37 there exists an $X \in \mathrm{SO}(4)$ with $X \cdot H_0 = H_1$. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(4)$ is path-connected. It follows from the Smooth Path-Connectivity Proposition 25.32 that there exists a smooth path $\gamma\colon [0,1] \to \mathrm{SO}(4)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = X$. We now consider the diffeotopy

$$\begin{array}{rcl} F\colon S^3 \times [0,1] & \to & S^3 \\ (x,t) & \mapsto & \text{reflection in the hyperplane } \gamma(t) \cdot H_0 \text{ applied to } x. \end{array}$$

The restriction of $F\colon S^3 \times [0,1] \to S^3$ to $L \times [0,1] \to S^3$ gives us the desired smooth isotopy between the reflections of L in H_0 and H_1.
(2) Let $L \subseteq \mathbb{R}^3$ be a link. We start out with the following claim, which closely resembles Statement (1).

Claim. The reflections of L in two affine hyperplanes A_0 and A_1 in $\mathbb{R}^3$ are smoothly isotopic.



Proof. Let A_0 and A_1 be two affine hyperplanes in $\mathbb{R}^3$. By definition we can write $A_0 = P_0 + H_0$ and $A_1 = P_1 + H_1$ where $P_0, P_1 \in \mathbb{R}^3$ and where H_0, H_1 are two hyperplanes in $\mathbb{R}^3$. As in (1) we see that there exists an $X \in \mathrm{SO}(3)$ with $X \cdot H_0 = H_1$ and we see that there exists a smooth path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = X$. We now consider the diffeotopy

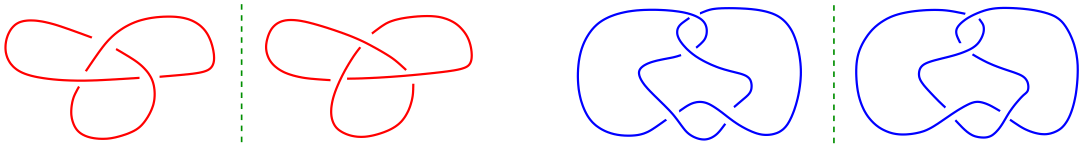
$$\begin{array}{rcl} F\colon S^3 \times [0,1] & \to & S^3 \\ & & \text{reflection in the affine hyperplane} \\ (x,t) & \mapsto & (1-t) \cdot P_0 + t \cdot P_1 + \gamma(t) \cdot H_0 \\ & & \text{applied to the point } x. \end{array}$$

The restriction of $F\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ to $L \times [0,1] \to \mathbb{R}^3$ gives us the desired smooth isotopy between the reflections of L in H_0 and H_1. ⊞

It follows from the claim that remains to show that there exists a single hyperplane H of $\mathbb{R}^3$ such that the reflection of L in H corresponds to the mirror of $L \subseteq \mathbb{R}^3 \subseteq S^3$ as defined above. Next note that the stereographic projection, that we used in the Stereographic Projection Lemma 108.1, commutes with the reflection in the $(x_0 = 0)$-hyperplane of $\mathbb{R}^4$ respectively $\mathbb{R}^3$. Thus we see that the hyperplane $(x_0 = 0)$ has the desired property. ■

Convention 108.25. Let $L \subseteq S^3$. Since usually we only care about links up to smooth isotopy we use the S^3-Link Mirror Lemma 108.24 to talk of "the mirror" of L instead of the slightly more correct "a mirror of L".

Example. Keeping the S^3-Link Mirror Lemma 108.24 (2) in mind, we see in the figure below the trefoil and its mirror and we also see the figure-8 knot and its mirror.



the trefoil K and its mirror K^{mir} the figure-8 knot J and its mirror J^{mir}

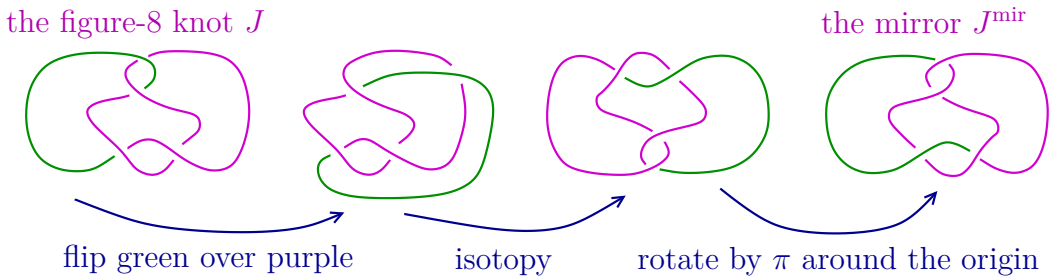
Note that the "double mirror" of a knot K is just the original knot. □

The above discussion lead us to the following variation on Definition 62.14:

Definition 108.26. We say that a link L in S^3 is amphichiral if it is smoothly isotopic to its mirror, otherwise we call the link L chiral.

Examples.

(1) The trivial knot equals of course its mirror, in particular the trivial knot is amphichiral.
(2) In the figure below we show the figure-8 knot J:



We also show a smoothly isotopy from J to its mirror J^{mir}. This implies that the figure-8 knot is also amphichiral.[1129] □

It is now natural to ask whether the trefoil is amphichiral. But endless hours of playing around with the trefoil do not lead to any success. So the suspicion arises that the following question should be answered in the negative:

Question 108.27. Is the trefoil amphichiral?

Much later, in the Trefoil-is-Chiral Proposition 115.15 and the Trefoil-is-Chiral Proposition **??** we will use two different approaches to give a negative answer to this question. In the first approach we use the fundamental group together with the "longitude", in the latter approach we use linking pairings of the "2-fold branched covering of L".

108.7. **Tubular maps and exteriors of links.** In this last section of this chapter we will introduce the "exterior of a link". The exterior X_L of a link L is a compact smooth

[1129] Note that this discussion just gave us a solution to Exercise 62.3.

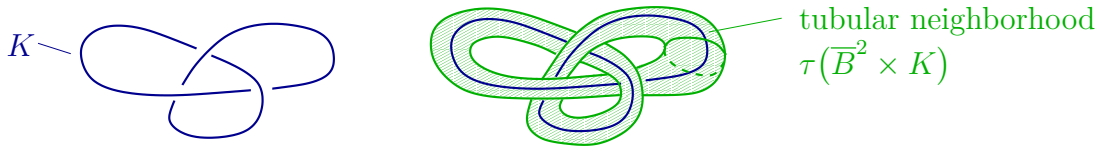
manifold that is homotopy equivalent to $S^3 \setminus L$. The fact that X_L is compact makes it an attractive object to work with.

To introduce the exterior of a link we need to provide the following definition, which is a special case of a more general concept that we introduced in Definition 58.2:

Definition 108.28. Let M be an n-dimensional smooth manifold with $\partial M = \varnothing$ and let $L \subseteq M$ be a closed k-dimensional smooth submanifold.

(1) A tubular map is a smooth embedding $\tau\colon \overline{B}^{n-k} \times L \to M$ such that for every $P \in L$ we have $\tau(0, P) = P$.

(2) The image of a tubular map is called a tubular neighborhood of L.



The following theorem says that tubular maps exist in many settings and that they are unique in a suitable sense:

Theorem 108.29. (Link Tubular Map Theorem) Let M be an oriented 3-dimensional smooth manifold with $\partial M = \varnothing$ and let $L \subseteq M$ be a closed oriented 1-dimensional smooth submanifold.

(1) There exists an orientation-preserving[1130] tubular map $\tau\colon \overline{B}^2 \times L \to M$.

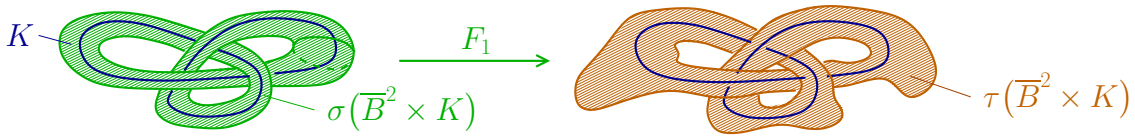
(2) Let $\sigma, \tau\colon \overline{B}^2 \times L \to M$ be two orientation-preserving tubular maps for L. There exists a diffeotopy $F\colon M \times [0,1] \to M$ and a smooth map $g\colon L \to \mathrm{SO}(2)$ with the following properties:

(a) $F_0 = \mathrm{id}$.

(b) F is a homotopy rel L.

(c) The diffeomorphism $F_1\colon M \to M$ restricts to the map

$$\begin{array}{rcl} \sigma(\overline{B}^2 \times L) & \to & \tau(\overline{B}^2 \times L) \\ \sigma(v, x) & \mapsto & \tau(g(x) \cdot v, x). \end{array}$$



Proof.

(1) This statement is a special case of the Product Tubular Neighborhood–Existence Theorem 58.4 (2) together with the Tubular Map Lemma 58.3.

(2) This statement is a special case of the Tubular Map–Uniqueness Theorem 58.5. ∎

[1130] We equip $\overline{B}^2$ with the standard orientation and we equip $\overline{B}^2 \times L$ with the product orientation that we introduced in the Product Orientation Proposition 31.48.

Definition 108.30. Let L be a link in S^3. By the Link Tubular Map Theorem 108.29 (1) there exists a tubular map $\tau\colon \overline{B}^2 \times L \to S^3$. We refer to $X_L \coloneqq S^3 \setminus \tau(B^2 \times L)$ as the exterior of L.

A priori the definition of the exterior depends on the choice of the tubular map. Fortunately the following lemma tells us that this is not a real issue:

Lemma 108.31. (Link Exterior Lemma) Let L be a link and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map.

(1) The exterior $X_L \coloneqq S^3 \setminus \tau(B^2 \times L)$ of L is a compact smooth submanifold of S^3 with $\partial X_L = \tau(S^1 \times L)$. In particular X_L inherits an orientation from S^3.
(2) The exterior X_L is well-defined up to an orientation-preserving diffeomorphism.
(3) If L and L' are smoothly isotopic links, then there exists an orientation-preserving diffeomorphism between the link exteriors X_L and $X_{L'}$.
(4) The inclusion $X_L \to S^3 \setminus L$ is a homotopy equivalence, in fact X_L is a deformation retract of $S^3 \setminus L$.

Proof.

(1) This statement follows almost immediately from the Smooth Embedding Theorem 30.15 and the Codimension-Zero Submanifold Proposition 25.49.
(2) This statement is an easy consequence of the Link Tubular Map Theorem 108.29 (2) and the Smooth Isotopy-Orientation Proposition 42.4.
(3) Let L and L' be two smoothly isotopic links and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map for L. Since L and L' are smoothly isotopic we know by the Smooth Isotopy–Diffeomorphism Proposition 108.11 that there exists an orientation-preserving self-diffeomorphism $f\colon S^3 \to S^3$ such that for each $i \in \{1, \dots, m\}$ we have $f(L_i) = \widetilde{L}_i$.

Note that f and τ give rise to a tubular map

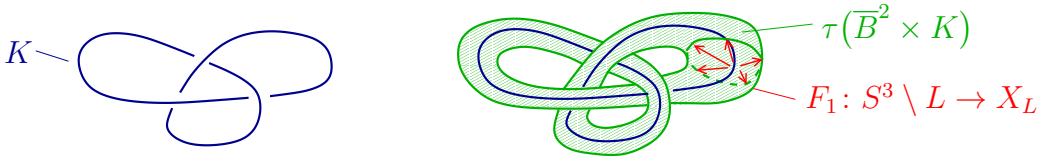
$$\begin{array}{rcl} \widetilde{\tau}\colon \overline{B}^2 \times \widetilde{L} & \to & S^3 \\ (x, \widetilde{y}) & \mapsto & f(\tau(x, \underbrace{f^{-1}(\widetilde{y})}_{\in L})) \end{array}$$

for $\widetilde{L}$ such that the map f restricts to a diffeomorphism from $X_L = S^3 \setminus \tau(B^2 \times L)$ to $X_{\widetilde{L}} = S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$.
(4) This statement is the content of the Basics-of-Tubular Neighborhood Lemma 57.10. For the reader's convenience we provide the argument, since in the current setting the proof is much easier than in the general setting.

Let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map. We consider the map that retracts $S^3 \setminus L$ radially to X_L, i.e. we consider the map

$$\begin{array}{rcl} (S^3 \setminus L) \times [0,1] & \xrightarrow{F} & S^3 \setminus L \\ (x,t) & \mapsto & \begin{cases} \tau(v \cdot ((1-t) + \frac{t}{\|v\|}), y), & \text{if } x = \tau(v,y) \text{ for } (v,y) \in (\overline{B}^2 \setminus \{0\}) \times L, \\ x, & \text{if } (x,t) \notin \tau(B^2 \times L). \end{cases} \end{array}$$

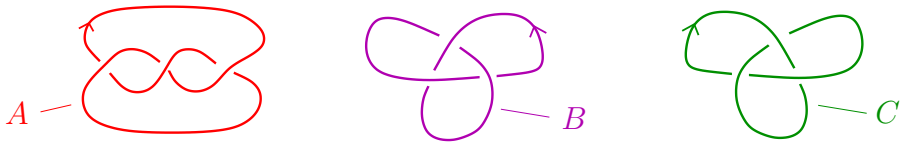
It follows from the Codimension-Zero Submanifold Proposition 25.49 and the Homotopy Combination Lemma 18.8 (1) that this map is continuous. It is now clear that F



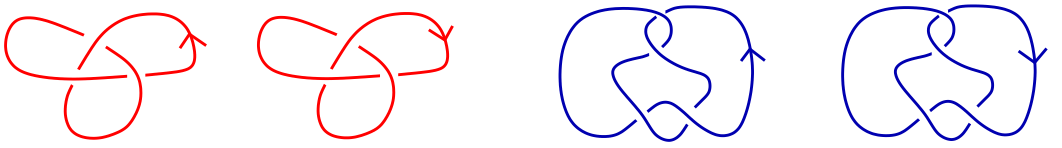
is a deformation retraction from $S^3 \setminus L$ to X_L. It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the inclusion $X_L \to S^3 \setminus L$ is a homotopy equivalence. ■

108.8. Exercises for Chapter 108.

Exercise 108.1. In the figure below we show to three oriented knots A, B and C. Precisely two of these are smoothly isotopic (you do not have to show this). Which two knots are smoothly isotopic?
Remark. This exercise is identical to Exercise 62.1.



Exercise 108.2. Show that the trefoil and the figure-8 knot are reversible. In other words, show that the two oriented knots shown in the figure to the left, respectively to the right, are smoothly isotopic.
Remark. This exercise is identical to Exercise 62.2.



trefoil with the two orientations figure-8 knot with the two orientations

Exercise 108.3. Let M be a 3-dimensional smooth manifold. For the purpose of this exercise we say that two knots K and $\widetilde{K}$ in M are *weakly equivalent* if there exists a continuous map

$$\begin{array}{rcl} F \colon K \times [0,1] & \to & M \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following two conditions hold:

(1) For each $t \in [0,1]$ the map $F_t \colon K \to M$ is a smooth embedding.
(2) We have $F_0 = \mathrm{id}$ and we have $F_1(K) = \widetilde{K}$.

If two knots are weakly equivalent, are they also smoothly isotopic in the sense of Definition 108.7?
Remark. This exercise is almost identical to Exercise 62.6.

Exercise 108.4. Is the Hopf link, that is shown in the figure below, amphichiral, i.e. is it smoothly isotopic to its mirror image?
Hint. Use the tools from algebraic topology that you have acquired since Chapter 62.



Exercise 108.5. Show that for any link $L \subseteq S^3$ the complement $S^3 \setminus L$ is path-connected.
Hint. Make use of the Link Tubular Map Theorem 108.29.

Exercise 108.6. Let $K \subsetneq S^3$ be a compact subspace. Show that there exists a smooth embedding $\varphi\colon \overline{B}^3 \to S^3$ with $K \subseteq \varphi(B^3)$.

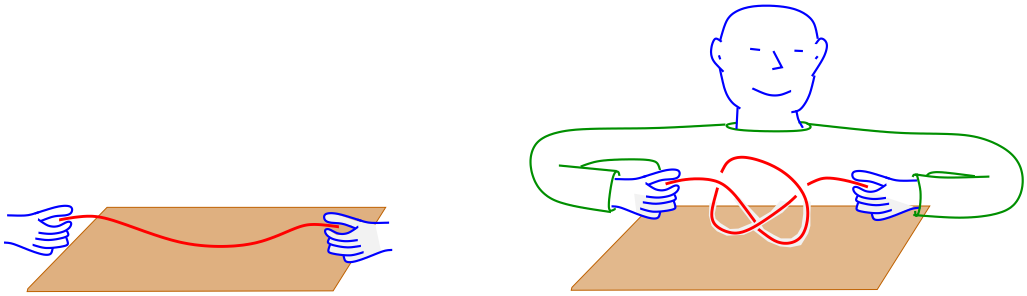
Exercise 108.7.

(a) Show that there exists a compact 2-dimensional smooth manifold and a closed 1-dimensional smooth manifold $L \subseteq M$ such that L does *not* admit a tubular map.

(b) Show that there exists a closed 2-dimensional smooth manifold and a closed 1-dimensional smooth manifold $L \subseteq M$ such that L does *not* admit a tubular map.

(c) Show that there exists a closed 3-dimensional smooth manifold and a closed 1-dimensional smooth manifold $L \subseteq M$ such that L does *not* admit a tubular map.
Remark. This shows that in the Link Tubular Map Theorem 108.29 we cannot drop the hypothesis that M is orientable.

Exercise 108.8. Can you pick up a loose unknotted string with both hands and turn it into a knotted string as shown in the figure below, while holding on to the ends of the string? Either way, which smooth isotopy types of knotted strings can you produce this way?
Remark. This exercise is identical to Exercise 62.10. But since this is the best exercise in knot theory, it is impossible to state it too often.



Since this chapter has a significant overlap with Chapter 62, most of the exercises from Section 62.6 are also perfectly suitable in the current context.

CHAPTER 109

Constructions of knots and links

In Chapter 109 we introduced the disjoint union and the connected sum of (oriented) knots in $\mathbb{R}^3$. In this chapter we will reintroduce these notions for knots and links in S^3. More importantly, we will fill in some of the technical details (e.g. well-definedness) of these notions, that we swept under the rug in the original discussion in Chapter 109. We will also introduce a few new concepts, like the meridian.

This chapter is organized as follows:

- In Section 109.1 we will recall the Smooth Ball Embedding Theorem 45.10 which says that smooth embeddings of balls into smooth manifolds are essentially unique. This very general theorem is an essential tool in differential topology that we will use several times in this chapter.
- In Section 109.2 we will introduce the meridian of an oriented knot.
- In Section 109.3 we will introduce the split union of two links.
- In Section 109.4 we will introduce the connected sum of oriented submanifolds and knots.
- In Section 109.5 we will describe the connected sum of "real life" knots.
- In Appendix 109.6 we will introduce the concept of a string knot. These give us another way to think about knots up to smooth isotopy.
- In Section 109.7 we will provide exercises for Chapter 109.

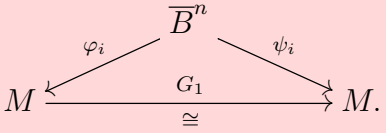
In the next Chapter 110 we will provide a "crash course" on fundamental groups.

109.1. **Uniqueness of smooth embeddings of balls.** In the following sections we will introduce several constructions of knots and links out of given knots and links. In each case the construction will depend on some choices and we will have to argue that the smooth isotopy of the new link that we constructed is independent of our choices. The key in each case will be (variations on) the following proposition:

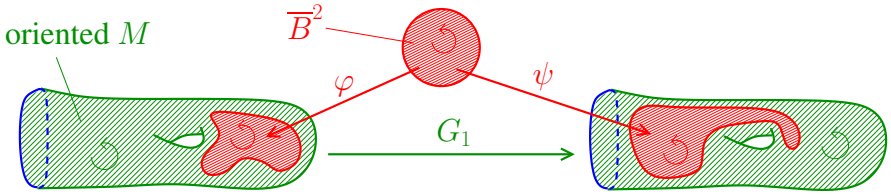
Theorem 45.10. (Smooth Ball Embedding Theorem) Let $n \geq 2$ and let M be a connected oriented n-dimensional smooth manifold. Let $\varphi_1, \dots, \varphi_k \colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_k \colon \overline{B}^n \to M \setminus \partial M$ be two sets of k orientation-preserving smooth embeddings of $\overline{B}^n$ with pairwise disjoint images. Then there exists a diffeotopy

$$G \colon M \times [0,1] \to M$$

from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1 \colon M \to M$ which has the property that for each $i \in \{1, \dots, k\}$ the following diagram commutes:



Finally, if $\varphi_i(0) = \psi_i(0)$, $i = 1, \dots, k$, then we can find a diffeotopy rel $\{\varphi_1(0), \dots, \varphi_k(0)\}$.

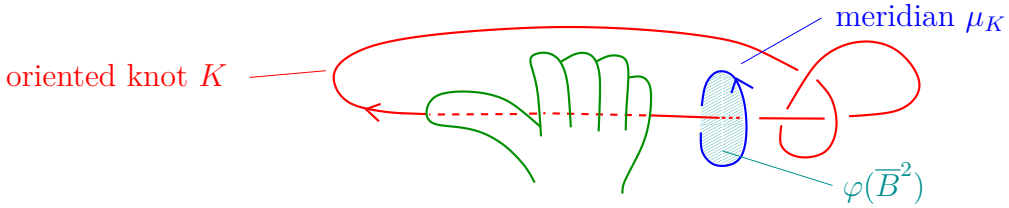


109.2. **The meridian of an oriented knot.** The following definition introduces one of the most important concepts in knot theory:

Definition 109.1. Let L be an oriented link in S^3 and let K be a component of L. Furthermore let $\mu_K \subseteq S^3 \setminus K$ be an oriented smoothly embedded loop in the sense of Definition 107.1. We say that μ_K is a meridian of K if there exists a smooth embedding $\varphi \colon \overline{B}^2 \to S^3$ with the following three properties:

(1) The map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \mu_K$.
(2) The image $\varphi(\overline{B}^2)$ intersects K transversally in a single point[1131] P and it does not intersect any of the other components of L.
(3) A positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ followed by a positive basis for $\mathrm{T}_P K$ gives a positive basis for $\mathrm{T}_P S^3$.

Remark. More casually speaking Definition 109.1 says the following: The meridian of an oriented knot K is any oriented smoothly embedded loop μ which "circles once around the knot K" where the orientation is given by the "right-hand rule".



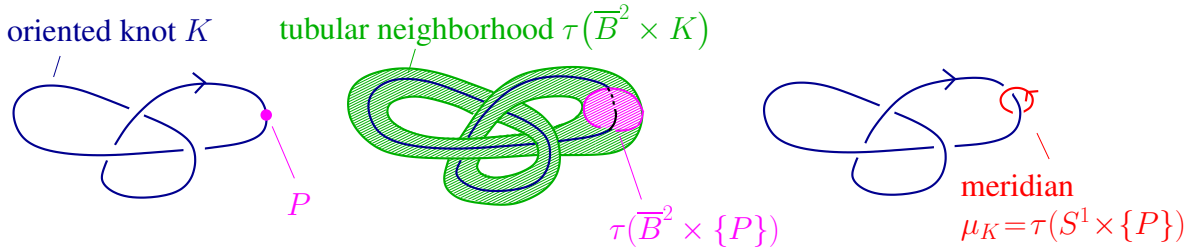
The "right-hand rule" means that if the thumb points into the direction of the knot, then the fingers point into the direction of the meridian. □

The following proposition says that meridians always exist and that they are essentially unique:

[1131] Here "intersects transversally in a single point" means, according to Definition 50.3, that $\varphi(\overline{B}^2) \cap K$ consists of a single point P and that $\mathrm{T}_P \varphi(\overline{B}^2) + \mathrm{T}_P K = \mathrm{T}_P S^3$.

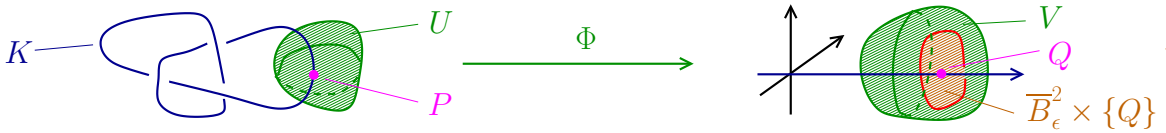
Proposition 109.2. (Meridian Proposition) Let L be an oriented link in S^3 and let K be a component of L.

(1) K admits a meridian.
(2) Let $\tau\colon \overline{B}^2 \times L \to S^3$ be an orientation-preserving tubular map. For any $P \in K$ the image $\tau(S^1 \times \{P\})$ is a meridian of K.
(3) Any two meridians for K are smoothly isotopic in $S^3 \setminus L$.
(4) Given any meridian μ of the component K there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ and a $P \in K$ with $\mu = \tau(S^1 \times \{P\})$.



Sketch of proof of the Meridian Proposition 109.2.

(1) Let K be a component of L. We pick $P \in K$. Since K is a closed 1-dimensional smooth manifold it follows almost immediately from the Definition 25.22 of a submanifold that there exists a chart $\Phi\colon U \to V$ around P with $\Phi(U \cap L) = V \cap (\{(0,0)\} \times \mathbb{R})$. We set $Q := \Phi(P)$. Since V is open there exists an $\epsilon > 0$ with[1132] $\overline{B}^2_\epsilon \times \{Q\} \subseteq V$. One can now easily verify that $\Phi^{-1}(S^1_\epsilon \times \{Q\})$, with a suitable orientation, is a meridian for K.



(2) Let $\tau\colon \overline{B}^2 \times L \to S^3$ be an orientation-preserving tubular map and let $P \in K$. Using the observation that $\tau(S^1 \times \{P\}) = \partial\tau(\overline{B}^2 \times \{P\})$ one can easily verify that $\tau(S^1 \times \{P\})$ is a meridian of K.
(3) By the Smooth Isotopy-Equivalence Relation Proposition 41.5 we know that being smoothly isotopic is an equivalence relation. Thus it suffices to show that any meridian is smoothly isotopic to a meridian that arises as in (2) from an orientation-preserving tubular map.

Let $\mu \subseteq S^3 \setminus K$ be a meridian of the knot K. We pick a corresponding smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ as in the definition of the meridian. We denote by P the unique intersection point of K and $\varphi(\overline{B}^2)$. Note that it follows immediately from the Link Tubular Map Theorem 108.29 that there exists an orientation-preserving smooth embedding $\tau\colon \overline{B}^2_2 \times K \to S^3$ such that $\tau(0,x) = x$ for all $x \in K$. It remains to prove the following claim:

Claim. There exists a smooth isotopy $H\colon S^1 \times [0,1] \to S^3 \setminus K$ with $H_0 = \varphi|_{S^1}$ and such that for all $x \in S^1$ we have $H_1(x) = \tau(x, P)$.

[1132] Here $\overline{B}^n_r$ denotes the closed n-ball around the origin of radius r.

Proof. We prove the claim in four steps:

(a) We pick a point $Q \neq P$ on K. Since $\tau(B^2_2 \times (K \setminus \{Q\}))$ is an open neighborhood of $P = \tau(0, P)$ there exists an $\epsilon > 0$ such that $\varphi(\overline{B}^2_\epsilon) \subseteq \tau(B^2_2 \times (K \setminus \{Q\}))$. After applying the smooth isotopy

$$\begin{array}{rcl} F\colon \overline{B}^2 \times [0,1] & \to & S^3 \\ (x,t) & \mapsto & \varphi(x \cdot ((1-t) + \epsilon \cdot t)) \end{array}$$

we can assume that $\varphi(\overline{B}^2)$ is contained in $\tau(B^2_2 \times (K \setminus \{Q\}))$.

(b) By (a) we can consider the map

$$\Theta\colon \overline{B}^2 \xrightarrow{\varphi} \tau(\overline{B}^2_2 \times (K \setminus \{Q\})) \xrightarrow{\tau(v,z) \mapsto \tau(v,P)} \tau(\overline{B}^2 \times \{P\}).$$

It follows easily from Property (2) of φ (as stated in the definition of a meridian) that the differential of Θ at 0 is an isomorphism. It follows from the Inverse Mapping Theorem 30.1 that there exists an $\epsilon > 0$ such that the restriction of Θ to $\overline{B}^2_\epsilon$ is a smooth embedding. As in (a) we can assume, after a smooth isotopy that is given by radial shrinking, that Θ is in fact a smooth embedding. Note that it follows from the fact that τ is orientation-preserving and Property (3) of φ that this smooth embedding $\Theta\colon \overline{B}^2 \to \tau(\overline{B}^2_2 \times \{P\})$ is actually orientation-preserving.

(c) Since $K \setminus \{Q\}$ is diffeomorphic to an open interval we can find a smooth deformation retraction

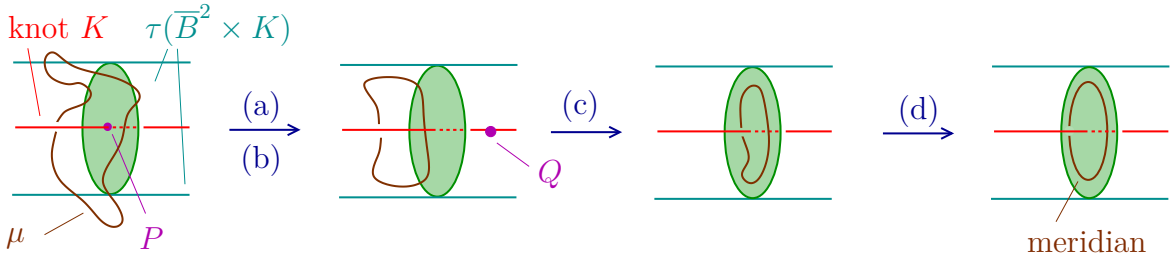
$$F\colon (K \setminus \{Q\} \times [0,1] \to K \setminus \{Q\}$$

from $K \setminus \{Q\}$ to $\{P\}$. This gives rise to a deformation retraction

$$(\mathrm{id}_{\overline{B}^2}, F)\colon \overline{B}^2 \times (K \setminus \{Q\}) \times [0,1] \to \overline{B}^2 \times (K \setminus \{Q\})$$

from $\overline{B}^2 \times (K \setminus \{Q\}$ to $\overline{B}^2 \times \{P\}$. Using this deformation retraction one can easily verify that the smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ from (b) is actually smoothly isotopic, rel $(0, P)$ to a smooth embedding $\overline{B}^2 \to \tau(B^2_2 \times \{P\})$.

(d) It follows from the Smooth Ball Embedding Theorem 45.10 that the smooth embedding $\overline{B}^2 \to \tau(B^2_2 \times \{P\})$ from (c) is smoothly isotopic, rel 0, to the obvious smooth embedding $\overline{B}^2 \to \tau(\overline{B}^2 \times \{P\})$. ⊞



(4) Let μ be a meridian of K. Recall that we need to show that μ arises from a suitable tubular map.

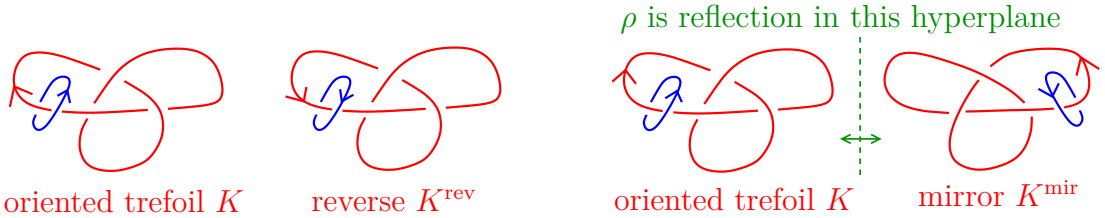
As above we obtain from the Link Tubular Map Theorem 108.29 an orientation-preserving smooth embedding $\tau\colon \overline{B}^2 \times K \to S^3$ such that $\tau(0, P) = P$ for all $P \in K$. We fix $P \in K$. By (2) we know that $\nu := \tau(S^1 \times \{P\})$ is a meridian of K. By (3) we know that there exists a smooth isotopy $F\colon \mu \times [0,1] \to S^3 \setminus L$ with $F_0 = \mathrm{id}$ and $F_1(\mu) = \nu$. We use the constant isotopy $L \times [0,1] \to L$ to extend F to a smooth isotopy of $\mu \cup L$. Finally we use the Isotopy Extension Theorem 44.2 to extend this

smooth isotopy of $\mu \cup L$ to a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ with $H_0 = \mathrm{id}$. It follows from the Smooth Isotopy-Orientation Proposition 42.4 that $H_1 \circ \tau\colon \overline{B}^2 \times L \to S^3$ is an orientation-preserving tubular map such that $(H_1 \circ \tau)(S^1 \times \{P\}) = F_1(\mu) = \nu$. ■

We conclude this chapter with the following lemma which tells us what effect reversing the orientation and taking mirror images have on meridians:

Lemma 109.3. (Meridian–Symmetries Lemma) Let $K \subseteq S^3$ be an oriented knot with meridian μ_K.

(1) A meridian for the reverse K^{rev} is given by $(\mu_K)^{\mathrm{rev}}$.
(2) Let $\rho\colon S^3 \to S^3$ be a reflection in a hyperplane. Then a meridian for the mirror $K^{\mathrm{mir}} = \rho(K)$ is given by $\rho(\mu_K)^{\mathrm{rev}}$.



Proof. We start out by recalling Definition 109.1 of a meridian. Let J be an oriented knot. We say that an oriented knot $\nu \subseteq S^3 \setminus J$ is a *meridian of* J if there exists a smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ with the following three properties:

(a) The map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \nu$.
(b) The image $\varphi(\overline{B}^2)$ intersects J transversally in a single point P.
(c) A positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ together with a positive basis for $\mathrm{T}_P J$ gives a positive basis for $\mathrm{T}_P S^3$.

In both cases of the lemma it is clear that the first two conditions of the definition of a meridian are satisfied. The third condition follows from some elementary linear algebra. We leave these considerations to the reader. ■

109.3. **The split union of links.** In this section we will introduce the "split union" of two links. The idea is quite simple: We want to define the "split union" of two links L and $\widetilde{L}$ by putting each link into a ball, and then embed the two balls disjointly into S^3. We now turn this idea into a precise mathematical concept:

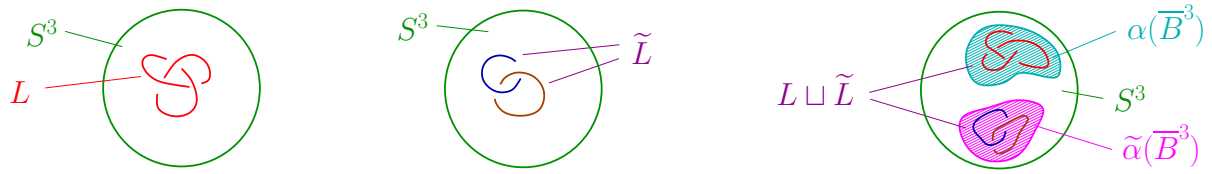
Definition 109.4. Let L and $\widetilde{L}$ be two oriented links in S^3.

- It follows from the Link-in-$\overline{B}^3$-$\mathbb{R}^3$-S^3-Proposition 108.15 that there exist oriented links $L' \subseteq \overline{B}^3$ and $\widetilde{L}' \subseteq \overline{B}^3$ (which are unique up to smooth isotopy) such that under the smooth embedding $\overline{B}^3 \to \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ the link L' is smoothly isotopic to L and such that the link $\widetilde{L}'$ is smoothly isotopic to $\widetilde{L}$.
- Clearly we can pick two orientation-preserving smooth embeddings $\alpha, \widetilde{\alpha}\colon \overline{B}^3 \to S^3$ with disjoint images.

We refer to

$$L \sqcup \widetilde{L} \;:=\; \alpha(L') \cup \widetilde{\alpha}(\widetilde{L}')$$

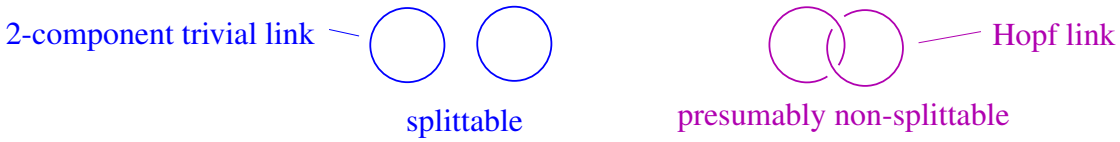
as the split union $L \sqcup \widetilde{L}$ of the links L and $\widetilde{L}$. It follows almost immediately from the Smooth Ball Embedding Theorem 45.10 that the smooth isotopy type of $\alpha(L') \cup \widetilde{\alpha}(\widetilde{L}')$ is well-defined and that it only depends on the smooth isotopy type of L and $\widetilde{L}$.[1133]



Definition 109.5. We say that a link is splittable if it is the split union of two non-empty links. Otherwise we call the link unsplittable.[1134]

Example.

(1) It follows quite easily from the definitions that the split union of two copies of the trivial knot is (smoothly isotopic) to the trivial 2-component link as defined in Definition 108.6. This shows in particular that the trivial 2-component link is splittable.

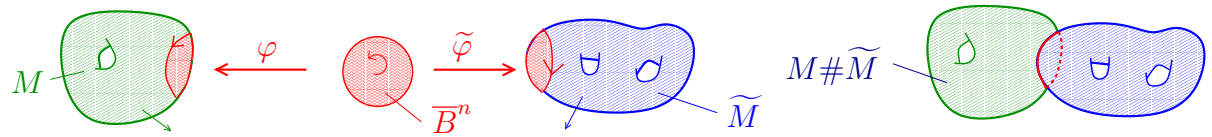


(2) It seems pretty obvious that the Hopf link is not splittable. But right now we lack the tools for formally proving that the Hopf link is indeed not splittable. □

109.4. **The connected sum of oriented submanifolds and knots.** In this section we will introduce the connected sum $K \# \widetilde{K}$ of two oriented knots $K, \widetilde{K}$ in S^3. The definition requires some preparation. Let us first recall the definition of the connected sum of smooth manifolds from Definition 54.12:

Definition 109.6. Let $n \in \mathbb{N}_0$ and let M and $\widetilde{M}$ be two oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$. Furthermore we pick an orientation-reversing smooth embedding $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M} \setminus \partial \widetilde{M}$. We define the connected sum of M and $\widetilde{M}$ as

$$M \# \widetilde{M} := \big(M \setminus \varphi(B^n)\big) \sqcup \big(\widetilde{M} \setminus \widetilde{\varphi}(B^n)\big) / \sim \quad \text{where } \varphi(P) \sim \widetilde{\varphi}(P) \text{ for all } P \in S^{n-1}.$$



Example. The above figure shows convincingly that the connected sum of a surface of genus g with a surface of genus h is a surface of genus $g + h$.[1135] □

[1133] This discussion shows that the current discussion is consistent with Definition 63.1.

[1134] It follows almost immediately from the definitions that a link is splittable if and only if there exist two disjoint smoothly embedded 3-balls in S^3 each of which contains at least one component of L.

[1135] In the Surface–Connected Sum Lemma 54.15 we gave a fairly formal proof of this statement.

The following proposition summarizes some key facts about the connected sum of smooth manifolds:

Proposition 109.7. (Manifold–Smooth Connected Sum-Proposition) Let $n \in \mathbb{N}_0$ and let M and $\widetilde{M}$ be two oriented connected n-dimensional smooth manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ be an orientation-preserving smooth embedding and finally suppose that $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M} \setminus \partial\widetilde{M}$ is an orientation-reversing smooth embedding. The following statements hold:

(0) The resulting connected sum $M\#\widetilde{M}$ is a smooth manifold with a unique orientation that coincides with the orientations of $M\setminus\varphi(B^n)$ and $\widetilde{M}\setminus\widetilde{\varphi}(B^n)$. Up to an orientation-preserving diffeomorphism the definition does not depend on the choice of φ and $\widetilde{\varphi}$.
(1) There exists an orientation-preserving diffeomorphism from $M\#S^n$ to M.
(2) The connected sum is commutative, i.e. $M\#\widetilde{M}$ admits an orientation-preserving diffeomorphism to $\widetilde{M}\#M$.
(3) If $n \geq 2$ or if all the manifolds are closed, then the connected sum is associative, i.e. $(M\#M')\#M''$ admits an orientation-preserving diffeomorphism to $M\#(M'\#M'')$.

Proof.

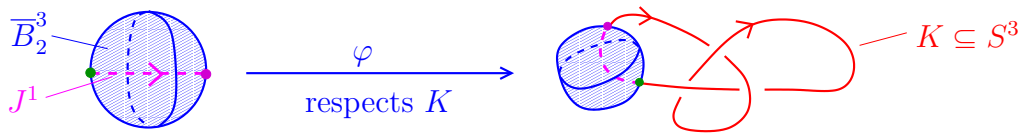
(0) The fact that $M\#\widetilde{M}$ is well-defined up to an orientation-preserving diffeomorphism is a pretty straightforward consequence of the Smooth Ball Embedding Theorem 45.10 and the Isotopy Extension Theorem 44.2. We refer to the proof of the Smooth Connected Sum-Proposition 54.13 (1) and (5) for details.
(1) This statement is proved in the Connected Sum-with-Sphere Lemma 54.16.
(2) This statement is proved in the Smooth Connected Sum-Proposition 54.13 (6a).
(3) For $n \geq 2$ this statement follows from the Smooth Connected Sum-Proposition 54.13 (6b). We turn to the case $n = 1$. We now assume that all manifolds are closed. Recall that the Smooth 1-Manifold Classification Theorem 28.8 says that every closed oriented connected non-empty 1-dimensional smooth manifold admits an orientation-preserving diffeomorphism to S^1. It is now clear that in the 1-dimensional case the connected sum is also associative.[1136] ∎

Next we introduce the connected sum of submanifolds. To do so we recall and expand on on Definition 45.14 and Definition 56.15:

Definition 109.8. Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$.

(1) We denote by $\overline{B}^n_2$ and $\overline{B}^k_2$ the closed balls of radius 2 in $\mathbb{R}^n$ and $\mathbb{R}^k$.
(2) We set $J^k := \{(x, 0) \in \overline{B}^n_2 \mid x \in \overline{B}^k_2\}$ and we view equip J^k with the obvious orientation.
(3) Let M be an oriented n-dimensional smooth manifold and let $K \subseteq M$ be an oriented k-dimensional submanifold. Furthermore let $\varphi\colon \overline{B}^n_2 \to M \setminus \partial M$ be a smooth embedding.
   (a) We say that φ respects K if $\varphi(J^k) = \varphi(\overline{B}^n_2) \cap K$, if $\varphi\colon \overline{B}^n_2 \to M$ is orientation-preserving and if $\varphi|_{J^k}\colon J^k \to \varphi(\overline{B}^n_2) \cap K$ is orientation-preserving.
   (b) We say that φ anti-respects K if the same conditions as in (a) hold, except that we now demand that both maps are orientation-reversing.

[1136] In the 1-dimensional case we need to restrict to closed manifolds, since otherwise (as we saw in Exercise 54.13) the triple connected sum $M\#(M'\#M'')$ is not necessarily well-defined.

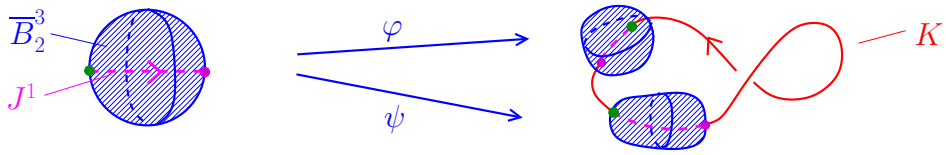


The following proposition, which is essentially the same as the Respectful Embedding Proposition 56.16, says that maps that respect a submanifold K always exist and that in a sense they are unique:

Proposition 109.9. (Respectful Embedding Proposition) Let $n \in \mathbb{N}$ and suppose we are given a $k \in \{1, \dots, n-1\}$. Let M be an oriented n-dimensional smooth manifold. Furthermore let $K \subseteq M$ be an oriented proper k-dimensional submanifold.

(1) If K is non-empty, then there exists a smooth embedding $\overline{B}^n_2 \to M$ that respects K.
(2) If K is connected, then for any two smooth embeddings $\varphi\colon \overline{B}^n_2 \to M$ and $\psi\colon \overline{B}^n_2 \to M$ that respect K there exists a diffeotopy $F\colon M \times [0,1] \to M$ such that $F_0 = \mathrm{id}$, such that $F_1 \circ \varphi = \psi$ and such that each $F_t \circ \varphi$ respects K.

The obvious analogs of (1) and (2) also hold for "anti-respect" instead of "respect".



Sketch of proof.

(1) This statement follows easily from the definition of a submanifold and the hypothesis that $k \in \{1, \dots, n-1\}$, which gives us enough freedom to arrange the orientation behavior by reflecting along hyperplanes.
(2) This statement follows immediately from the Ball-Submanifold Theorem 45.15. ■

Now we can define the connected sum of smooth submanifolds:

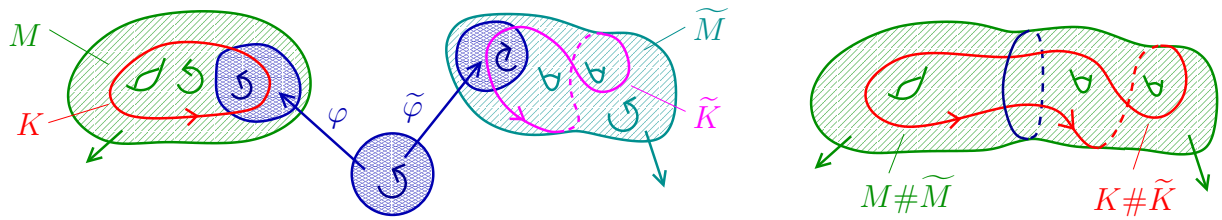
Definition 109.10. Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$. Let M be an oriented n-dimensional smooth manifold and let $K \subseteq M$ be an oriented proper k-dimensional submanifold. Furthermore let $\widetilde{M}$ be an oriented n-dimensional smooth manifold and let $\widetilde{K} \subseteq \widetilde{M}$ be an oriented proper k-dimensional submanifold. We proceed as follows:

- We pick a smooth embedding $\varphi\colon \overline{B}^n_2 \to M$ that respects K.
- We we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^n_2 \to \widetilde{M}$ that anti-respects $\widetilde{K}$.
- We use $\varphi\colon \overline{B}^n \to M$ and $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M}$ to define $M \# \widetilde{M}$ as in Definition 109.6.
- We use[1137] $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\varphi} K$ and $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\widetilde{\varphi}} \widetilde{K}$ to define the connected sum $K \# \widetilde{K}$.

We refer to the resulting pair $(M \# \widetilde{M}, K \# \widetilde{K})$ as the connected sum of (M, K) and $(\widetilde{M}, \widetilde{K})$.

The next proposition is identical to the Submanifold–Connected Sum Proposition 56.18 and it can be viewed as an analog of the Smooth Connected Sum-Proposition 54.13.

[1137] Note that here we deliberately restrict ourselves from balls of radius 2 to balls of radius 1.



Proposition 109.11. (Submanifold–Connected Sum Proposition) Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$. Let M be an oriented n-dimensional smooth manifold and let K be a closed oriented k-dimensional submanifold of M. Furthermore let $\widetilde{M}$ be an oriented n-dimensional smooth manifold and let $\widetilde{K}$ be a closed oriented k-dimensional submanifold of $\widetilde{M}$.

(1) The subset $K\#\widetilde{K}$ is a closed k-dimensional smooth submanifold of $M\#\widetilde{M}$.
(2) If M and $\widetilde{M}$ are both connected, then the oriented diffeomorphism type of the pair $(M\#\widetilde{M}, K\#\widetilde{K})$ does not depend on the choice of the map that respects K and the map that anti-respects $\widetilde{K}$.

Sketch of proof.

(1) We leave it to the reader to verify this statement. Note that here one needs to make use of the fact that the maps that (anti-) respect the submanifold give us control on $\overline{B}^n_2$ and not just on $\overline{B}^n$.
(2) This statement follows quite easily from the Respectful Embedding Proposition 109.9. ■

Now we can finally specialize to our beloved setting of knots in S^3. Following most books in knot theory, see e.g. [**Rolf1990**, p. 40] and [**Kaw1996**, p. 30], we introduce the connected sum of two oriented knots as follows:

Definition 109.12.

(1) Let K and $\widetilde{K}$ be two oriented knots in S^3. We proceed as follows:
- By the Respectful Embedding Proposition 109.9 we can pick a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we can pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3_2 \to S^3$ that anti-respects $\widetilde{K}$.
- We perform the connected sum $(S^3\#S^3, K\#\widetilde{K})$ as in Definition 109.10.
- By the Connected Sum-with-Sphere Lemma 54.16 we have an orientation-preserving diffeomorphism $S^3\#S^3 \to S^3$.

We use any such orientation-preserving diffeomorphism $S^3\#S^3 \to S^3$ to view $K\#\widetilde{K}$ as an oriented knot in S^3. We refer to this knot as the connected sum $K\#\widetilde{K}$ of K and $\widetilde{K}$. Note that it follows from the Smooth Isotopy–Diffeomorphism Proposition 108.11 that this definition of $K\#\widetilde{K} \subseteq S^3$ does not depend on the choice of an orientation-preserving diffeomorphism $S^3\#S^3 \to S^3$.

(2) We say that a knot K is prime[1138] if it is non-trivial[1139] and if it is not smoothly isotopic to the connected sum of two non-trivial knots.

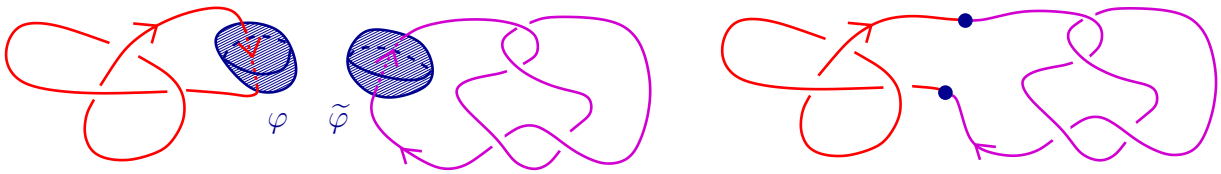
[1138] In principle it would be more suitable to call such a knot "irreducible". The name "prime" is justified by the Knot–Prime Decomposition Theorem 121.18.
[1139] We exclude the trivial knot for the same reason that 1 is not considered a prime number.

Remark.

(1) It will follow from the Knot-Standard Position Lemma 109.16 that the above definition is consistent with Definition 63.6.

(2) In Section 187.7 we will introduce the "internal connected sum" of two oriented submanifolds "via bands". In particular we will introduce the "internal connected sum" of two oriented knots which, for most intents and purposes, agrees with the connected sum of two oriented knots in S^3. □

Example. In the figure below we show the connected sum of the trefoil K and the figure-8 knot $\widetilde{K}$. Admittedly it can take a minute to connect the picture to the actual definition.



In the next Section 109.5 we will explain in greater detail how we come up with the picture of the connected sum to the right. □

The following proposition is an analog of the Manifold–Smooth Connected Sum-Proposition 109.7:

Proposition 109.13. (Oriented Knot–Monoid Proposition) As in Notation 108.8 we denote by $\texttt{Knot}^{\text{or}}(S^3)$ the set of smooth isotopy classes of oriented knots. With this notation the map

$$\begin{array}{rcl} \texttt{Knot}^{\text{or}}(S^3) \times \texttt{Knot}^{\text{or}}(S^3) & \to & \texttt{Knot}^{\text{or}}(S^3) \\ [(K_1, K_2)] & \mapsto & [K_1 \# K_2] \end{array}$$

is well-defined and it has the following properties:

(1) The sum operation has a neutral element, given by the trivial knot.

(2) The sum operation is associative.

(3) The sum operation is abelian.

Remark. The Oriented Knot–Monoid Proposition 109.13 allows us to unambiguously define the connected sum of any finite non-empty set of oriented knots. Similar to the discussion on page 1238 we define the connected sum of the empty set of oriented knots as the trivial knot. □

Sketch of proof. First note that it follows almost immediately from the Isotopy Extension Theorem 44.2 together with the Submanifold–Connected Sum Proposition 109.11 and the Smooth Isotopy–Diffeomorphism Proposition 108.11 that the connected sum operation is indeed well-defined on the set $\texttt{Knot}(S^3)$ of isotopy classes of oriented knots.

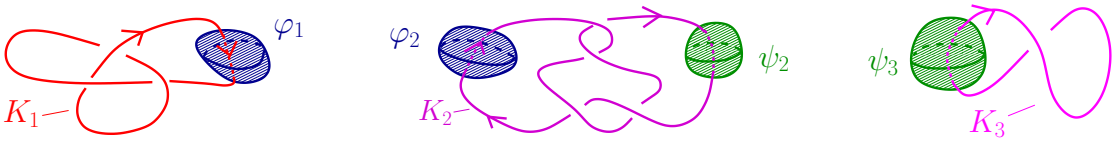
Now we turn to the proofs of the three statements:[1140]

(1) Let $K \subseteq S^3$ be the standard trivial knot. Note that the obvious smooth embedding $\varphi\colon \overline{B}^3 \to S^3_{\geq 0}$ respects K. If we use this map then it follows almost immediately

[1140] As we will see, the proof of Statements (1)–(3) is a slight variation on the proof of the corresponding statements of the Manifold–Smooth Connected Sum-Proposition 109.7. In other words, the arguments are similar to the proof of Connected Sum-with-Sphere Lemma 54.16 and of the Smooth Connected Sum-Proposition 54.13 (6a) and (6b).

from the definitions that for any other oriented knot $\widetilde{K}$ we have an obvious orientation-preserving diffeomorphism $(S^3, \widetilde{K}) \to (S^3 \# S^3, K \# \widetilde{K})$. This implies that $\widetilde{K}$ is smoothly isotopic to $K \# \widetilde{K}$.

(2) Let K_1, K_2, K_3 be three oriented knots in S^3. We do the following:
- We pick a smooth embedding φ_1 that respects K_1.
- We pick a smooth embedding φ_2 that anti-respects K_2.
- We pick a smooth embedding ψ_2 that respects K_2 and whose image is disjoint from the image of φ_2.
- We pick a smooth embedding ψ_3 that anti-respects K_3.



If we use these smooth embeddings, then we see that we have an obvious diffeomorphism $((S^3 \# S^3) \# S^3, (K_1 \# K_2) \# K_3) \to (S^3 \# (S^3 \# S^3), K_1 \# (K_2 \# K_3))$.

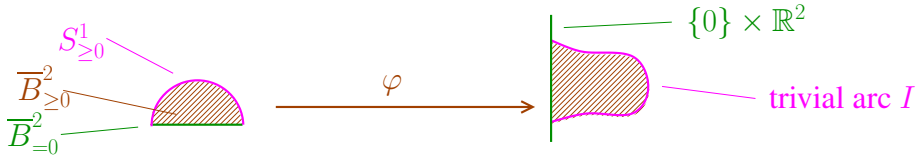
(3) For orientation purposes we write $\widetilde{S}^3$ for a second copy of S^3. Thus let $K \subseteq S^3$ and $\widetilde{K} \subseteq \widetilde{S}^3$ be two oriented knots. We pick a smooth embedding $\varphi\colon \overline{B}^3 \to S^3$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3 \to \widetilde{S}^3$ that anti-respects $\widetilde{K}$. Let $\tau\colon \overline{B}^3 \to \overline{B}^3$ be the reflection in the $(x_1 = 0)$-hyperplane. Note that $\widetilde{\varphi} \circ \tau\colon \overline{B}^3 \to S^3$ is a smooth embedding that respects $\widetilde{K}$ and that $\varphi \circ \tau\colon \overline{B}^3 \to S^3$ is a smooth embedding that anti-respects K. We use these smooth embeddings to define the connected sum $\widetilde{K} \# K \subseteq S^3 \# \widetilde{S}^3$. One can now easily verify that the obvious map $S^3 \# \widetilde{S}^3 \to \widetilde{S}^3 \# S^3$ is an (orientation-preserving) diffeomorphism that sends $\widetilde{K} \# K$ to $K \# \widetilde{K}$.

We leave it to the reader to fill in the remaining details. ■

109.5. **The connected sum of "real life" knots.** In Definition 109.12 we introduced the connected sum of two oriented knots K and $\widetilde{K}$ in S^3. The catch with the definition is that it is not particularly clear how this works "in practice". In this section we will try to show how one can perform this construction in "real life" examples.

Definition 109.14. Let $I \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ be a compact proper 1-dimensional smooth manifold that is diffeomorphic to $[-1, 1]$. We say that I is a trivial arc if there exists a smooth embedding $\varphi\colon \overline{B}^2_{\geq 0} \to \mathbb{R}_{\geq 0} \times \mathbb{R} \times \{0\}$ such that the following conditions hold:
- $\varphi(\overline{B}^2_{=0}) \subseteq \{0\} \times \mathbb{R}^2$.
- $\varphi(S^1_{\geq 0}) = I$.

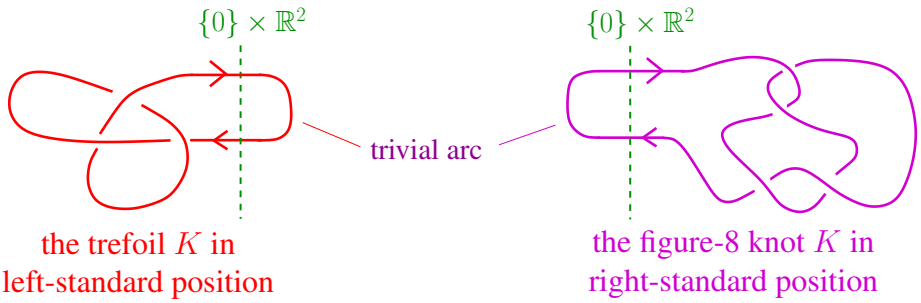
An argument as in the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 shows that any two trivial arcs are properly smoothly isotopic.



Definition 109.15. Let $K \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ be an oriented knot. We say that K is in left-standard position if there exists an $\epsilon > 0$ such that the following conditions hold:

(1) $K \cap ([-\epsilon, \epsilon] \times \mathbb{R}^2) = [-\epsilon, \epsilon] \times \{(\pm 1, 0)\}$.
(2) The orientation of K is given as follows:
   - The orientation of K points from $(-\epsilon, 1, 0)$ to $(\epsilon, 1, 0)$.
   - The orientation of K points from $(\epsilon, -1, 0)$ to $(-\epsilon, -1, 0)$.
(3) The intersection $K \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$ is a trivial arc.

Similarly we define what it means for K to be in right-standard position.



Next we will see that we can arrange any knot to be in standard position:

Lemma 109.16. (Knot-Standard Position Lemma) Every oriented knot K in the 3-sphere $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ is smoothly isotopic to a knot in left-standard position and it is smoothly isotopic to a knot in right-standard position.

Proof. Let $K \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ be an oriented knot. In the following we will provide the proof for "left-standard". The proof for "right-standard" is of course essentially the same.

Similar to Definition 109.8 we introduce the following notation:

(1) We denote by $\overline{B}^3_2$ the closed ball of radius 2 in $\mathbb{R}^3$.
(2) We set $J := \{(x, 0) \in \overline{B}^3_2 \,|\, x \in [-2, 2]\}$ and we equip J with the obvious orientation.
(3) We say a smooth embedding $\varphi\colon \overline{B}^n_2 \to S^3$ *respects* K if $\varphi(J) = \varphi(\overline{B}^n_2) \cap K$, if the smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ is orientation-preserving and if $\varphi|_J\colon J \to \varphi(\overline{B}^n_2) \cap K$ is orientation-preserving.

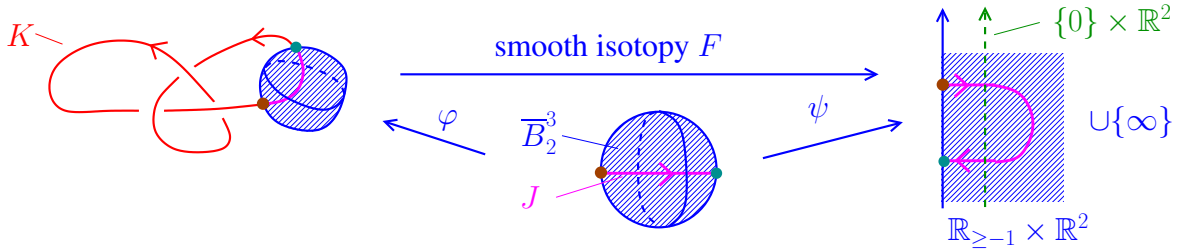
First note that it follows from the Respectful Embedding Proposition 109.9 that there exists a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K. Next note that the following claim gives us an alternative smooth embedding $\psi\colon \overline{B}^3_2 \to S^3$:

Claim. There exists an orientation-preserving smooth embedding $\psi\colon \overline{B}^3_2 \to S^3$ with the following properties:

(i) $\psi(\overline{B}^3_2) = (\mathbb{R}_{\geq -1} \times \mathbb{R}^2) \cup \{\infty\}$.
(ii) $\psi(\overline{B}^3) = (\mathbb{R}_{\geq 0} \times \mathbb{R}^2) \cup \{\infty\}$.
(iii) $\psi(x, 0, 0) = (x + 1, 1, 0)$ for all $x \in [-2, -\frac{1}{2}]$.
(iv) $\psi(x, 0, 0) = (1 - x, -1, 0)$ for all $x \in [\frac{1}{2}, 2]$.

Proof. Using stereographic projections we see, similar to Exercise 25.18, that we can find an explicit orientation-preserving diffeomorphism $\psi\colon \overline{B}^3_2 \to (\mathbb{R}_{\geq -1} \times \mathbb{R}^2) \cup \{\infty\}$. After some extra massaging (which we leave to the reader) one can easily arrange that the other properties hold. ⊞

Since φ and ψ are both orientation-preserving smooth embeddings $\overline{B}^3_2 \to S^3$ we obtain from the Smooth Ball Embedding Theorem 45.10 a smooth isotopy $F\colon \overline{B}^3_2 \times [0,1] \to S^3$ with $F_0 = \varphi$ and with $F_1 = \psi$.



Next note that it follows from the Isotopy Extension Theorem 44.2 that there exists a diffeotopy $G\colon S^3 \times [0,1] \to S^3$ with the following properties:

- $G_0 = \mathrm{id}$.
- For all $(x,t) \in \overline{B}^3_2 \times [0,1]$ we have $G(\varphi(x),t) = F(x,t)$.

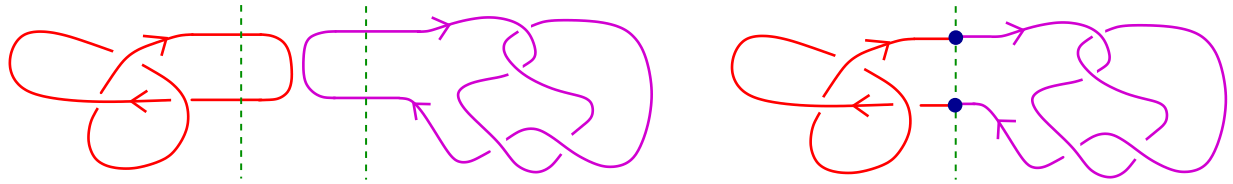
It is now clear that $G_1(K)$ is in left-standard form and that $G|_{K\times[0,1]}\colon K \times [0,1] \to S^3$ is the desired smooth isotopy. ■

The following lemma now gives us as a fairly hands-on description of the connected sum of two oriented knots in S^3:

Lemma 109.17. (Standard Connected Sum Lemma) Let K and $\widetilde{K}$ be two oriented knots in S^3. Furthermore let K^{st} be a knot in left-standard position that is smoothly isotopic to K and let $\widetilde{K}^{\mathrm{st}}$ be a knot in right-standard position that is smoothly isotopic to $\widetilde{K}$. Then (up to smooth isotopy) we have

$$K\#\widetilde{K} \;=\; \underbrace{(K^{\mathrm{st}} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2))}_{\subseteq S^3_{\leq 0}} \cup \underbrace{(\widetilde{K}^{\mathrm{st}} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2))}_{\subseteq S^3_{\geq 0}}.$$

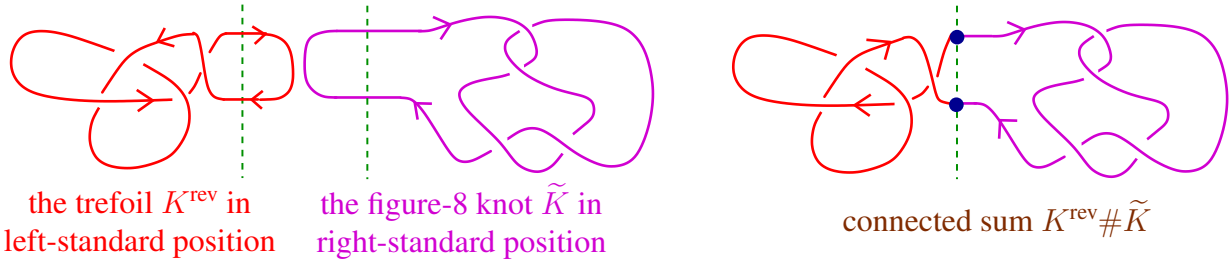
Example. In the figure below we show the trefoil K in left-standard position and we show the figure-8 knot $\widetilde{K}$ in right-standard position:



It now follows from the Standard Connected Sum Lemma 109.17 that the knot to the right-hand side is the connected sum $K\#\widetilde{K}$ of the trefoil K and the figure-8 knot $\widetilde{K}$.

Note though that the connected sum operation depends on the orientation of the knots. In the following figure we show the trefoil K^{rev} "with the opposite orientation" and again the figure-8 knot.

At least a priori there is no reason why $K\#\widetilde{K}$ and $K^{\mathrm{rev}}\#\widetilde{K}$ should be of the same smooth isotopy type. □



Proof. First of all note that it follows from the Oriented Knot–Monoid Proposition 109.13 that $K\#\widetilde{K}$ is smoothly isotopic to $K^{\text{st}}\#\widetilde{K}^{\text{st}}$. Thus we will really try to understand the latter connected sum. To do so we start out with the following claim:

Claim. There exists an orientation-preserving smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K^{st} with the following two properties:
(i) $\varphi(\overline{B}^3_2) = (\mathbb{R}_{\geq -1} \times \mathbb{R}^2) \cup \{\infty\}$.
(ii) $\varphi(\overline{B}^3) = (\mathbb{R}_{\geq 0} \times \mathbb{R}^2) \cup \{\infty\}$.

Proof. It follows easily from the uniqueness of trivial arcs and the discussion in the proof of the Knot-Standard Position Lemma 109.16 that any knot in left-standard position admits such a smooth embedding. ⊞

We can similarly find a smooth embedding $\psi\colon \overline{B}^3_2 \to S^3$ that respects $\widetilde{K}^{\text{st}}$ with the obvious properties. We now see that

$$K\#\widetilde{K} \overset{\text{by the Knot Connected Sum Proposition 109.13}}{\downarrow} = K^{\text{st}}\#\widetilde{K}^{\text{st}} \overset{\text{by Definition 109.12 and Definition 109.10 where we used the obvious diffeomorphism } S^3_{\leq 0} \cup S^3_{\geq 0} \to S^3}{\downarrow} = (K^{\text{st}} \setminus \varphi(B^3)) \cup (\widetilde{K}^{\text{st}} \setminus \widetilde{\varphi}(B^3)) \text{ in } \underbrace{(S^3 \setminus \varphi(B^3))}_{=S^3_{\leq 0}} \cup \underbrace{(S^3 \setminus \widetilde{\varphi}(B^3))}_{=S^3_{\geq 0}}$$

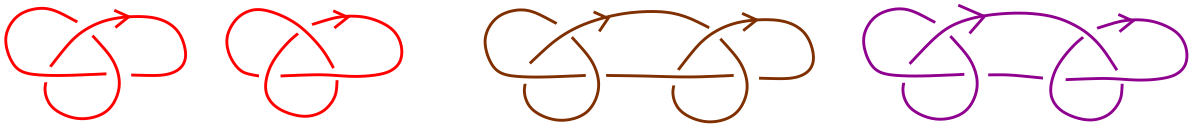
$$= (K^{\text{st}} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)) \cup (\widetilde{K}^{\text{st}} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)) \text{ in } S^3_{\leq 0} \cup S^3_{\geq 0} = S^3.$$

$\uparrow$ by the choice of φ and $\widetilde{\varphi}$

■

We conclude this section with one last example:

Example. In the figure below we show the trefoil and its "mirror image" K^{mir}, i.e. the reflection of K in the xy-hyperplane:



The connected sum $K\#K$ is usually called the *granny knot* and the connected sum $K\#K^{\text{mir}}$ is called the *square knot*. It is notoriously tricky to distinguish the square knot and the granny knot, even up to taking "mirror images". We will see later in Exercises 115.8 and **??** and on page 5604 that the granny knot and the square knot are indeed not smoothly isotopic. □

We conclude this section with the following lemma which says that taking mirror images and reverses are monoid homomorphisms:

Lemma 109.18. (Connected Sum–Mirror-Reverse Lemma) Let $(\texttt{Knot}^{\text{or}}(S^3), \#)$ be the monoid from the Oriented Knot–Monoid Proposition 109.13. The maps

$$\begin{array}{ccc} \texttt{Knot}^{\text{or}}(S^3) & \to & \texttt{Knot}^{\text{or}}(S^3) \\ [K] & \mapsto & [K^{\text{mir}}] \end{array} \quad \text{and} \quad \begin{array}{ccc} \texttt{Knot}^{\text{or}}(S^3) & \to & \texttt{Knot}^{\text{or}}(S^3) \\ [K] & \mapsto & [K^{\text{rev}}] \end{array}$$

are monoid homomorphisms in the sense of Definition 2.2.[1141]

Proof. First we point out that it is clear that taking mirrors and reverses preserves the trivial element of $\texttt{Knot}^{\text{or}}(S^3)$. It remains to prove that both operations are additive. Thus let K and $\widetilde{K}$ be two oriented knots in S^3. By the Knot-Standard Position Lemma 109.16 we can pick a knot K^{lst} in left-standard position that is smoothly isotopic to K and we can pick a knot $\widetilde{K}^{\text{rst}}$ in right-standard position that is smoothly isotopic to $\widetilde{K}$. We set $J := K^{\text{lst}} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$ and $\widetilde{J} := \widetilde{K}^{\text{rst}} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$.

(1) Let $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the $(x_1 = 0)$-plane. Note that $\rho(K^{\text{lst}})$ is again in left-standard position and that $\rho(\widetilde{K}^{\text{rst}})$ is again in right-standard position. Thus we see that

$$\begin{array}{rcl} [(K\#\widetilde{K})^{\text{mir}}] & = & [(K^{\text{lst}}\#\widetilde{K}^{\text{rst}})^{\text{mir}}] \overset{\text{by the Link Mirror Lemma 108.24}}{\underset{\downarrow}{=}} [\rho(K^{\text{lst}}\#\widetilde{K}^{\text{rst}})] \overset{\text{by the Standard Connected Sum Lemma 109.17}}{\underset{\downarrow}{=}} [\rho(J \cup \widetilde{J})] \\ & = & [\rho(J) \cup \rho(\widetilde{J})] \underset{\uparrow}{=} [\rho(K^{\text{lst}})\#\rho(\widetilde{K}^{\text{rst}})] \underset{\uparrow}{=} [(K^{\text{mir}})^{\text{lst}}]\#[(\widetilde{K}^{\text{mir}})^{\text{rst}}] \\ & & \text{by the Standard Connected Sum Lemma 109.17} \quad \text{by the Link Mirror Lemma 108.24} \\ & = & [K^{\text{mir}}]\#[\widetilde{K}^{\text{mir}}]. \end{array}$$

(2) Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation around $\mathbb{R} \times \{(0,0)\}$ by the angle π. Note that $\varphi((K^{\text{lst}})^{\text{rev}})$ is again in left-standard position and that $\varphi((\widetilde{K}^{\text{rst}})^{\text{rev}})$ is again in right-standard position. Thus we see that

$$\begin{array}{rcl} [(K\#\widetilde{K})^{\text{rev}}] & = & [(K^{\text{lst}}\#\widetilde{K}^{\text{rst}})^{\text{rev}}] \overset{\text{by the Standard Connected Sum Lemma 109.17}}{\underset{\downarrow}{=}} [(J \cup \widetilde{J})^{\text{rev}}] \\ & = & [\varphi((J \cup \widetilde{J})^{\text{rev}})] = [\varphi(J)^{\text{rev}} \cup \varphi(\widetilde{J})^{\text{rev}}] \\ & & \text{since } \varphi \text{ is isotopic to id} \\ & = & [\varphi((K^{\text{lst}})^{\text{rev}})]\#[\varphi((\widetilde{K}^{\text{rst}})^{\text{rev}})] \underset{\uparrow}{=} [(K^{\text{lst}})^{\text{rev}}]\#[(\widetilde{K}^{\text{rst}})^{\text{rev}}] = [K^{\text{rev}}]\#[\widetilde{K}^{\text{rev}}]. \\ & & \text{by the Standard Connected Sum Lemma 109.17, here we use} \\ & & \text{that } \varphi((K^{\text{lst}})^{\text{rev}}) \text{ is in left-standard position and} \\ & & \text{that } \varphi((\widetilde{K}^{\text{rst}})^{\text{rev}}) \text{ is again in right-standard position} \end{array}$$

■

109.6. **Appendix: String knots.** In this appendix we will introduce string knots. We will see the following:

- We will show that the set of string knots up to smooth isotopy is in a natural bijection to the set of oriented knots in S^3 up to smooth isotopy.

[1141] In plain English the lemma says that for two oriented knots K and $\widetilde{K}$ in S^3 the oriented knots $(K\#\widetilde{K})^{\text{mir}}$ is smoothly isotopic to $K^{\text{mir}}\#\widetilde{K}^{\text{mir}}$ and similarly for reverses.

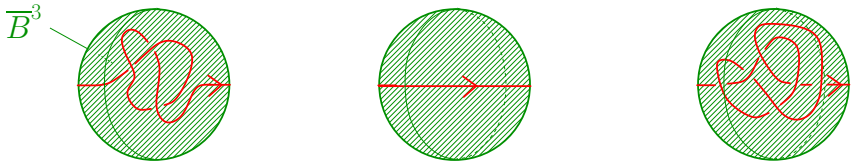
- We will show that smooth isotopy classes of string knots have an interesting monoid structure.

We think that the concepts and results of this appendix are interesting in their own right. But we will not really make use of them. First of all we have the following variation on Definition 108.3:

Definition 109.19. A string knot K in $\overline{B}^3$ is a 1-dimensional smooth submanifold of $\overline{B}^3$ that is diffeomorphic to $[-1,1]$ and that has the property that there exists an $\epsilon \in (0,1]$ such that

$$K \cap \{x \in \overline{B}^3 \,|\, \|x\| \geq 1-\epsilon\} \;=\; \{(x,0,0) \,|\, x \in [-1+\epsilon,-1] \cup [1-\epsilon,1]\}.$$

We always equip K with the orientation that goes from the left endpoint $(-1,0,0)$ to the right endpoint $(1,0,0)$.



trivial string knot

The following definition is an analog of Definition 108.4:

Definition 109.20. We define the

$$\text{trivial string knot} \;:=\; \{(x,0,0) \in \overline{B}^3 \,|\, x \in [-1,1]\}.$$

As always we only care about objects up to a suitable notion of a smooth isotopy. This leads us to the following variation on Definition 108.7:

Definition 109.21. We say that two string knots K and $\widetilde{K}$ are smoothly isotopic if there exists a smooth isotopy from K to $\widetilde{K}$ through string knot, i.e. if there exists a smooth map

$$\begin{aligned} F\colon K \times [0,1] &\to \overline{B}^3 \\ (z,t) &\mapsto F(z,t) \end{aligned}$$

such that the following hold:

(1) For each $t \in [0,1]$ the map $F_t\colon K \to \overline{B}^3$ is a smooth embedding such that $F_t(K)$ is a string knot.
(2) We have $F_0 = \mathrm{id}$ and we have $F_1(K) = \widetilde{K}$.

Notation 109.22. We denote by $\texttt{StKnot}(\overline{B}^3)$ the set of smooth isotopy classes of string knots.

Example. In the figure below we show a string knot that is smoothly isotopic to the trivial string knot.



As always the challenge is to determine whether a given string knot is smoothly isotopic to the trivial string knot. □

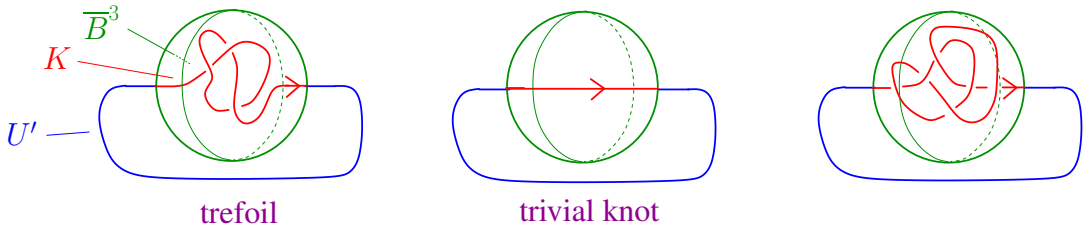
Definition 109.23. Let K be a string knot. In the following we consider the "horizontal arc" $U := \{(x,0,0) \,|\, |x| \geq 1\} \cup \{\infty\} \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ around "∞". We refer to

$$\mathrm{cl}(K) \;:=\; K \cup U \;\subseteq\; \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3.$$

as the closure of K. We give the closure $\mathrm{cl}(K)$ the orientation that comes from the orientation of K.

Example.

(1) In the figure below we show closures of the above string knots:



For illustration purposes we do not show the arc U around infinity, but we show an arc U' in $S^3 \setminus B^3$ that is smoothly isotopic to the arc $U \subseteq S^3 \setminus B^3$.

(2) One can easily verify that the closure of the trivial string knot is the trivial knot. □

Now two questions arise:

(1) What does "string knot theory" have to do with our previous knot theory?

(2) Why should be bother about string knots?

In the following we will try to answer these questions. First of all, recall that in the Link-in-$\overline{B}^3$-$\mathbb{R}^3$-S^3-Proposition 108.15 we showed that link theory in $\overline{B}^3$, $\mathbb{R}^3$ and S^3 is essentially the same. The following proposition now says that the theory of oriented knots is essentially the same as the theory of string knots up to smooth isotopy:

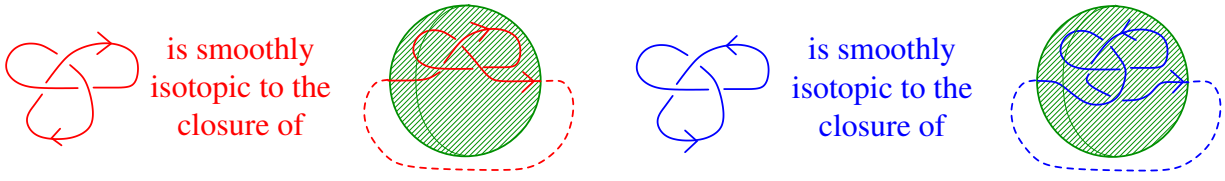
Proposition 109.24. (Knot–String Knot Proposition) The closure map $K \mapsto \mathrm{cl}(K)$ induces a bijection

$$\begin{array}{c}\text{string knots in } \overline{B}^3 \text{ up to} \\ \text{smooth isotopy}\end{array} \;\xrightarrow{\;\cong\;}\; \begin{array}{c}\text{oriented knot in } S^3 \text{ up} \\ \text{to smooth isotopy.}\end{array}$$

In other words, the closure map induces a bijection

$$\mathtt{StKnot}(\overline{B}^3) \;\xrightarrow{\;\cong\;}\; \mathtt{Knot}^{\mathrm{or}}(S^3).$$

Example. The Knot–String Knot Proposition 109.24 says in particular that every oriented knot is smoothly isotopic to the closure of a string knot. In practice it can sometimes be a little tricky to find a corresponding string knot.



For example in the figure above we show the trefoil with two different orientations and we show string knots whose closures are smoothly isotopic (as oriented knots) to the trefoil with the two orientations. □

Proof. It suffices to prove the following claim, which is an analog of the Link-in-$\overline{B}^3$-$\mathbb{R}^3$-Lemma 108.16 and the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17:

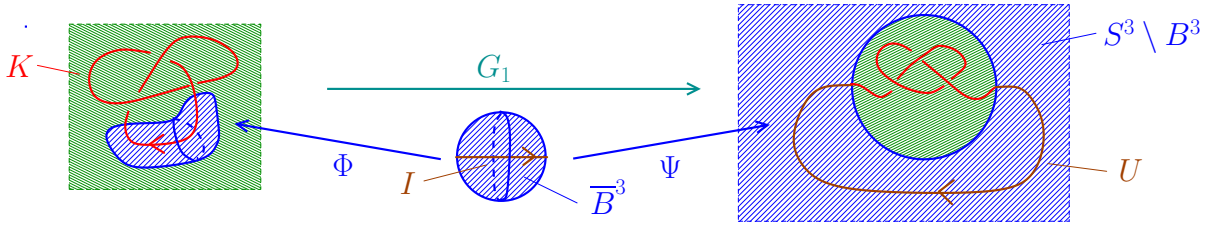
Claim.

(1) Every oriented knot in S^3 is smoothly isotopic to the closure of a string knot in $\overline{B}^3$.
(2) Let $K, \widetilde{K}$ be two string knots in $\overline{B}^3$. If the closures of K and $\widetilde{K}$ are smoothly isotopic in S^3 as oriented knots, then the string knots are also smoothly isotopic.

Proof.

(1) Let $K \subseteq S^3$ be an oriented knot. We set $I := \{(x,0,0) \in \overline{B}^3 \,|\, x \in [-1,1]\}$ and we equip I with the obvious orientation. We make two preparations:
   - Using the definition of a submanifold one can easily show that there exists an orientation-preserving smooth embedding $\Phi\colon \overline{B}^3 \to S^3$ such that Φ restricts to an orientation-preserving diffeomorphism $\Phi\colon I \to K \cap \Phi(\overline{B}^3)$.
   - As in Definition 109.23 we consider the horizontal arc U. We give it the orientation that points from $(0,0,1)$ to $(0,0,-1)$. Note that using the stereographic projection one can easily show that there exists an orientation-preserving diffeomorphism $\Psi\colon \overline{B}^3 \to S^3 \setminus B^3$ such that Ψ restricts to an orientation-preserving diffeomorphism $\Psi\colon I \to U$.

   It follows from the Respectful Embedding Proposition 109.9 that the smooth embeddings Φ and Ψ are both orientation-preserving it follows from the Smooth Ball Embedding Theorem 45.10 that there exists a diffeotopy $G\colon S^3 \times [0,1] \to S^3$ with $F_0 = \mathrm{id}$ and such that $G_1 \circ \Phi = \Psi$.



   The oriented knot K is thus smoothly isotopic to $G_1(K)$, which is the closure of the oriented knot $G_1(K) \cap \overline{B}^3$.
(2) To be added later. Seems surprisingly tricky.

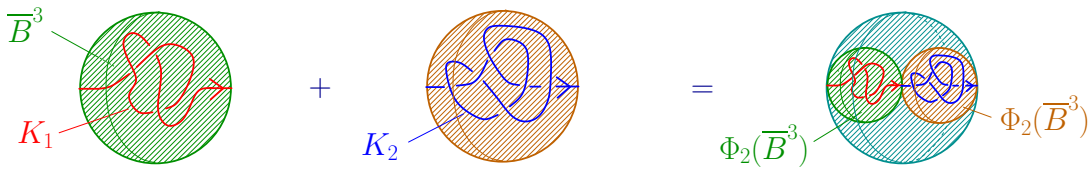
■

We will now see that string knots have a very nice additional structure:

Definition 109.25. Let $K_1, K_2 \subseteq \overline{B}^3$ be two string knots. In the following we denote by $\Phi_1\colon \overline{B}^3 \to \overline{B}^3_{\frac{1}{2}}(-\frac{1}{2},0,0)$ and $\Phi_2\colon \overline{B}^3 \to \overline{B}^3_{\frac{1}{2}}(\frac{1}{2},0,0)$ the obvious diffeomorphisms that are given by translation and radial shrinking. We set

$$K_1 + K_2 \;:=\; \Phi_1(K_1) \cup \Phi_2(K_2).$$

It follows easily from the definitions that $K_1 + K_2$ is again a string knot. We refer to it as the sum of K_1 and K_2.



The following proposition now shows that the sum operation turns $\texttt{StKnot}(\overline{B}^3)$ into an abelian monoid in the sense of Definition 2.1:

Proposition 109.26. (String Knot–Monoid Proposition) The map

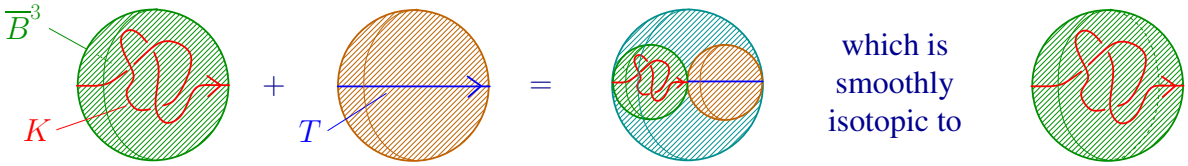
$$\begin{array}{rcl} \texttt{StKnot}(\overline{B}^3) \times \texttt{StKnot}(\overline{B}^3) & \to & \texttt{StKnot}(\overline{B}^3) \\ [(K_1, K_2)] & \mapsto & [K_1 + K_2] \end{array}$$

is well-defined and it has the following properties:

(1) The sum operation has a neutral element, given by the trivial string knot.
(2) The sum operation is associative.
(3) The sum operation is abelian.

Sketch of proof. It follows easily from the definitions that the sum operation is well-defined on the set $\texttt{StKnot}(\overline{B}^3)$. We now turn to the verification that is an abelian monoid:

(1) Let T be the trivial string knot. It is elementary to show that for any a string knot K the sums $T + K$ and $K + T$ are both smoothly isotopic to K. We will not bother with writing down details.

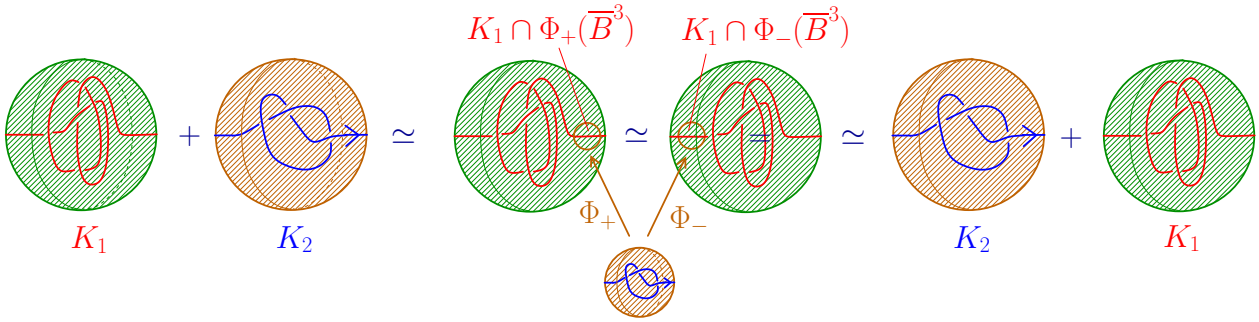


(2) It is now that to show that for three string knots K_1, K_2, K_3 the sums $(K_1 + K_2) + K_3$ and $K_1 + (K_2 + K_3)$ are smoothly isotopic.
(3) By the most interesting aspect of the sum operation is that it is abelian. Let us deal with claim in more detail than the previous claims. So let K_1, K_2 be two string knots. We need to show that $K_1 + K_2$ is smoothly isotopic to $K_2 + K_1$. Just for fun we provide two different arguments:
  (i) For the first prove we make the following preparations:
    (a) For the string knot K_1 we pick $\epsilon \in [0, 1)$ as in the Definition 109.19 of a string knot.
    (b) Let $\Phi_-\colon \overline{B}^3 \to \overline{B}^3_{-\frac{\epsilon}{2}}(-1 + \frac{\epsilon}{2}, 0, 0)$ and $\Phi_+\colon \overline{B}^3 \to \overline{B}^3_{\frac{\epsilon}{2}}(1 - \frac{\epsilon}{2}, 0, 0)$ be the obvious diffeomorphisms.
    (c) It follows from the Respectful Embedding Proposition 109.9 that the smooth embeddings Φ_- and Φ_+ are smoothly isotopic.

  We now see that we have the following smooth isotopies of string knots:

$$\begin{array}{rcl} K_1 + K_2 & \simeq & \text{string knot defined by replacing } K_1 \cap \Phi_+(\overline{B}^3) \text{ by } \Phi_+(K_2) \\ & \underset{\uparrow}{\simeq} & \text{string knot defined by replacing } K_1 \cap \Phi_-(\overline{B}^3) \text{ by } \Phi_-(K_2) \quad \simeq \quad K_2 + K_1. \end{array}$$

follows from (c)



(ii) We turn to the second proof that $K_1 + K_2$ is smoothly isotopic to $K_2 + K_1$, which is very short: One can easily verify that the closures $\mathrm{cl}(K_1 + K_2)$ and $\mathrm{cl}(K_2 + K_1)$ are smoothly isotopic.



By the Knot–String Knot Proposition 109.24 this implies that $K_1 + K_2$ is already smoothly isotopic to $K_2 + K_1$. ■

We conclude this appendix with the following lemma:

Lemma 109.27. (String Knot–Knot–Addition Lemma) The bijection

$$\begin{array}{ccc} \mathtt{StKnot}(\overline{B}^3) & \to & \mathtt{Knot}^{\mathrm{or}}(S^3) \\ [K] & \mapsto & [\mathrm{cl}(K)] \end{array}$$

from the Knot–String Knot Proposition 109.24 preserves the addition structure, i.e. for any two strings knots K_1, K_2 we have $[\mathrm{cl}(K_1 + K_2)] = [\mathrm{cl}(K_1) \# \mathrm{cl}(K_2)]$.

Proof. We leave it to the reader to fill in the details. ■

Remark. Note that the String Knot–Knot–Addition Lemma 109.27 together with the String Knot–Monoid Proposition 109.26 gives a new proof of the Oriented Knot–Monoid Proposition 109.13. □

109.7. Exercises for Chapter 109.

Exercise 109.1. Let $\varphi, \psi \colon \overline{B}^3 \to S^3$ be two smooth embeddings with disjoint images. Show that $S^3 \setminus (\varphi(B^3) \cup \psi(B^3))$ is diffeomorphic to $S^2 \times [0,1]$.

Exercise 109.2. Does there exist an oriented 2-component link $L_1 \cup L_2$ in S^3 such that L_1 is a meridian of L_2 and L_2 is a meridian of L_1?

Exercise 109.3. Let $K, J \subseteq S^3$ be oriented knots. We assume that there exists an orientation-preserving diffeomorphism $f \colon X_K \to X_J$ with $f(\mu_K) = \mu_J$. Show that there exists an orientation-preserving diffeomorphism $f \colon S^3 \to S^3$ with $f(K) = J$.
Hint. Use the Handle Attachment Proposition 246.3 to attach a 2-handle to X_K and X_J along $\mu_K \subseteq \partial X_K$ and $\mu_J \subseteq \partial X_J$.

Exercise 109.4.

(a) Let $X \sqcup Y$ be a subspace of S^3. We assume that there exists smooth embedding $\alpha, \beta \colon \overline{B}^3 \to S^3$ with disjoint images and such that $X \subseteq \alpha(B^3)$ and $Y \subseteq \beta(B^3)$. Show that $S^3 \setminus (X \sqcup Y)$ is homotopy equivalent to $(S^3 \setminus X) \vee S^2 \vee (S^3 \setminus Y)$.

(b) Show that (a) can be used to give a solution to Exercise 111.5.



Exercise 109.5.

(a) How can you turn a knot that is in left-standard position into a knot that is in right-standard position?

Remark. Beware of orientations.

(b) Let K and $\widetilde{K}$ be two oriented knots that are in left-standard position. Use (a) to show that $K\#\widetilde{K}$ and $\widetilde{K}\#K$ are smoothly isotopic.

Remark. This gives an alternative proof for the Oriented Knot–Monoid Proposition 109.13 (3).

Exercise 109.6. We consider the oriented trefoil K and the oriented figure-8 knot $\widetilde{K}$ as shown below. Is the connected sum $K\#\widetilde{K}$ smoothly isotopic to the connected sum $K^{\mathrm{rev}}\#\widetilde{K}$?



Exercise 109.7. We consider the oriented torus M and the two oriented submanifolds K_1 and K_2 of M that are shown in the figure below.

(a) Draw a figure for the connected sum $(M\#M, K_1\#K_1)$ from Definition 109.10.

(b) Draw a figure for the connected sum $(M\#M, K_1\#K_2)$ from Definition 109.10.

Remark. This exercise is identical to Exercise 56.5.



CHAPTER 110

Crash course on fundamental groups

Our goal is to find tools that allow us to show that two given knots or links are not smoothly isotopic. It follows from the Smooth Isotopy–Link Complement Proposition 108.12 that if two links L and $\widetilde{L}$ are smoothly isotopic, then the link complements $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$ are diffeomorphic. This shows that any diffeomorphism (or homeomorphism) invariant of smooth manifolds can in principle be used to address the smooth isotopy problem.

Depending on the reader's background, the reader might be aware of the following invariants:

(1) The homology groups $\mathrm{H}_*(X)$ of a topological space X.
(2) The cohomology groups $\mathrm{H}^*(X;G)$ of a topological space X with coefficients in an abelian group G.
(3) The de Rham cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(M)$ of a smooth manifold M.
(4) The fundamental group $\pi_1(X, x_0)$ of a pointed topological space (X, x_0).
(5) The higher homotopy groups $\pi_n(X, x_0)$, $n \geq 2$ of a pointed topological space (X, x_0).

In this chapter we will first inform the reader that the first three invariants are largely useless for the study of knots and links. In particular, if the reader is not aware of these invariants, it will not affect the reader's ability to follow the remainder of this part of the lecture notes.

Afterwards we will remind the reader of basic definitions and facts in the theory of fundamental groups. This discussion is meant as a refresher and to get all readers up to speed. For courageous readers it can also be viewed as a crash course on fundamental groups. We conclude this chapter with a very very quick discussion of higher homotopy groups.

This chapter is organized as follows:

- In Section 110.1 we will see that the (co-) homology groups of links complements are not very interesting.
- In Section 110.2, which is the heart of this chapter, we provide the promised refresher on fundamental groups.
- In Section 110.3 we will give a very short introduction to higher homotopy groups. Our goal is to provide just enough information to convince the reader that higher homotopy groups are largely useless in the study of knots and links.
- In Appendix 110.4 we will recall the abelianization of a group.
- In Section 110.5 we will provide exercises for Chapter 110.

In the next Chapter 111 we will study fundamental groups of link complements.

110.1. **(Co-) homology groups of link complements.** In case the reader is already aware of (co-) homology group we provide the following proposition, which shows that homology groups are not very effective at distinguishing links:

Proposition 110.1. (Link Complement-H_*–Proposition) In the following let $m \in \mathbb{N}$ and let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link in S^3. The homology groups of $S^3 \setminus L$ are given as follows:

$$\mathrm{H}_n(S^3 \setminus L) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}^{m-1}, & \text{if } n = 2, \\ \mathbb{Z}^m, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. This proposition is proved at larger and smaller levels of generality in the Knot Complement-H_*-Proposition 185.29, the Link Complement-H_*-Proposition 217.11, the Link Complement-H_*-Lemma 258.6 and the Link Complement-H_*–Basis-Proposition 283.4. ■

So what about (de Rham) cohomology groups? It turns out that for our purposes these are equally depressingly useless:

(1) First note that it follows from the Universal Coefficient-H^*-Theorem 201.23 that the singular cohomology groups of any topology space X are completely determined by the singular homology groups of X. In fact it follows easily from the Link Complement-H_*–Proposition 110.1, the Universal Coefficient-H^*-Theorem 201.23 and the Ext-Properties Lemma 201.19 (4) that for any abelian group G we have

$$\mathrm{H}^n(S^3 \setminus L; G) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ G^{m-1}, & \text{if } n = 2, \\ G^m, & \text{if } n = 1, \\ G, & \text{if } n = 0. \end{cases}$$

(2) In the De Rham-Isomorphism Theorem 231.16 we will see that the de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M)$ of any smooth manifold M are naturally isomorphic to the singular cohomology groups $\mathrm{H}^k(M; \mathbb{R})$.[1142] In particular the calculation in (1) for $G = \mathbb{R}$ gives us the de Rham cohomology groups of $S^3 \setminus L$.

This short discussion shows that (co-) homology groups are largely useless when it comes to the study of knots and links. Thus we bid farewell to (co-) homology groups. They will not reappear in this part of the lecture notes.

110.2. **The fundamental group refresher.** In this section we will recall the definition of fundamental groups and we will recall some basic properties and calculations. This section does not contain any proofs.

This section is organized as follows:

- In Subsection 110.2.1 we will give the definition of fundamental groups and we will present some basis calculations.
- In Subsection 110.2.2 we will see that fundamental groups define a covariant functor from the category of pointed topological spaces to the category of groups.

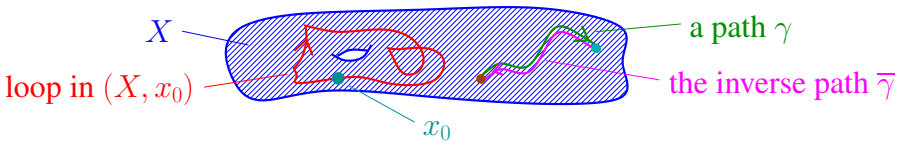
[1142] In the Knot Complement-H^*_{dR}-Proposition 85.10 we will also give a self-contained calculation of the de Rham cohomology groups of knot complements.

- In Subsection 110.2.3 we will introduce free groups and and we will introduce presentations of groups.
- In Subsection 110.2.4 we will state and discuss the Seifert-van Kampen Theorem 110.8.

110.2.1. **Definition of the fundamental group and basis calculations.** We start out by recalling some definitions:

Definition 88.1. Let X be a topological space.

(1) We define a path in X as a continuous map $\gamma\colon [0,1] \to X$. We call $\gamma(0)$ the starting point of γ and we call $\gamma(1)$ the endpoint of γ. Often we say that γ is a path from $\gamma(0)$ to $\gamma(1)$.
(2) Let $x_0 \in X$. A loop in (X, x_0) is a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.
(3) Given a path $\gamma\colon [0,1] \to X$ we refer to the path $\overline{\gamma}\colon [0,1] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(1-t)$ as the inverse path of γ.



Definition 88.2. Let X be a topological space.

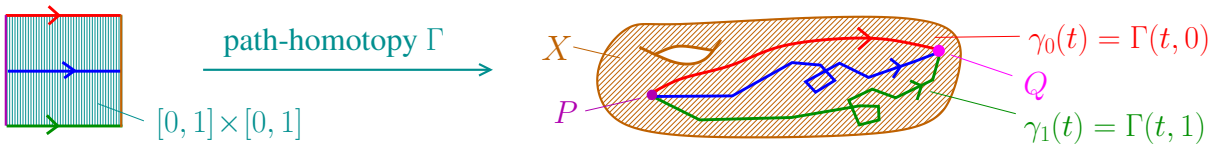
(1) Let $\gamma_0, \gamma_1\colon [0,1] \to X$ be two paths with the same starting point P and the same endpoint Q. A path-homotopy between the paths γ_0 and γ_1 is a continuous map

$$\begin{array}{rcl} \Gamma\colon [0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s) \end{array}$$

with the following properties:

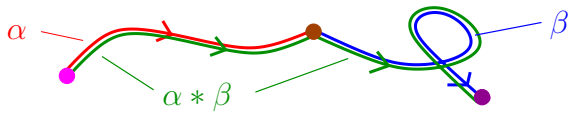
(a) for every $t \in [0,1]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,
(b) for every $s \in [0,1]$ we have $\Gamma(0,s) = P$ and $\Gamma(1,s) = Q$.

(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are path-homotopic.
(3) We say that a loop $\gamma\colon [0,1] \to X$ in x_0 is null-homotopic, if it is path-homotopic to the constant path δ given by $\delta(t) = x_0$ for all $t \in [0,1]$.



Definition 88.7. Let X be a topological space and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha*\beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \end{array}$$



Definition 88.13. Let (X, x_0) be a pointed topological space. We set

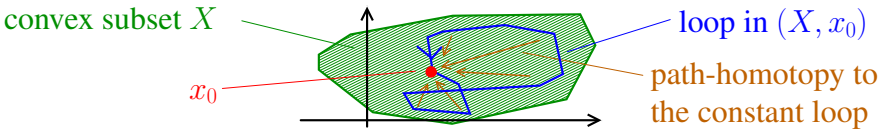
$$\pi_1(X, x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X, x_0)\}.$$

By the π_1–Group Proposition 88.12 we know that the map

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha * \beta] \end{array}$$

defines a group structure where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$. We refer to $\pi_1(X, x_0)$ as the fundamental group of X with respect to the base point x_0.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex subset of $\mathbb{R}^n$ and let $x_0 \in X$.



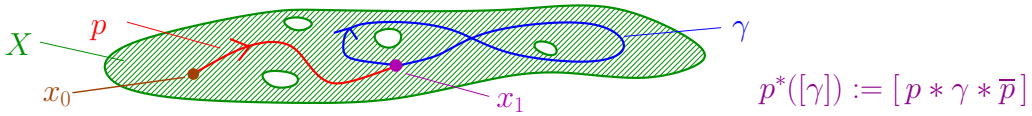
One can easily verify that each loop in (X, x_0) is path-homotopic to the constant loop in (X, x_0). It follows that $\pi_1(X, x_0)$ is the trivial group. □

The fundamental group depends a priori on the choice of a base point x_0. But the following lemma says that for many purposes the choice is largely irrelevant.

Lemma 110.2. (Base Point Independence Lemma) Let X be a topological space and let $x_0, x_1 \in X$. Any path $p\colon [0,1] \to X$ from x_0 to x_1 induces an isomorphism

$$\begin{array}{rcl} \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [\gamma] & \mapsto & [p * \gamma * \overline{p}]. \end{array}$$

In particular, if X is path-connected, then any two base points lead to isomorphic fundamental groups.



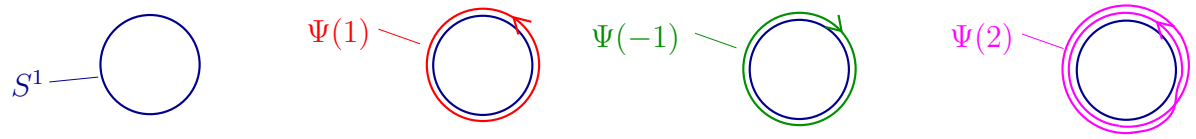
Definition 88.16. Let X be a path-connected non-empty topological space.
(1) We denote by $\pi_1(X)$ the isomorphism type of the fundamental group.
(2) If $\pi_1(X)$ is the trivial group, then we say that X is simply connected.

The following theorem gives us arguably the most important calculation of a fundamental group:

Theorem 88.22. (Circle-π_1-Theorem) The map

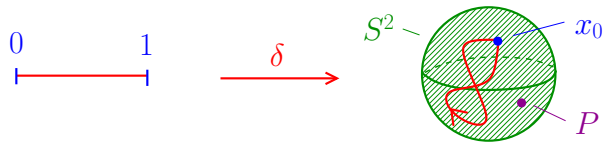
$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1,1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}\cdot n\cdot t) \end{array}\right] \end{array}$$

is an isomorphism of groups.



In contrast the following proposition says that higher-dimensional spheres are simply connected:

Proposition 88.24. (Sphere-π_1-Proposition) For any $n \geq 2$ the sphere S^n is simply connected, i.e. the fundamental group $\pi_1(S^n)$ is trivial.



one can show that every loop in (S^2, x_0) is path-homotopic to a loop which misses a point $P \in S^2$; since $S^2 \setminus \{P\} \cong \mathbb{R}^2$ the new loop is null-homotopic

110.2.2. **Functoriality of fundamental groups.** In this subsection we will see that fundamental groups form a functor. To make sense of this we first need to introduce a suitable category:

Definition 90.2. A pointed topological space is a pair (X, x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PTop}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & := & \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) & := & \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{array}$$

with the usual composition of maps as the category of pointed topological spaces.

Proposition 90.3. (π_1-Functor Proposition) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category of groups. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Group}) \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}((X,x_0),(Y,y_0)) & \to & \mathrm{Mor}(\pi_1(X,x_0),\pi_1(Y,y_0)) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ [\gamma\colon [0,1]\to X] & \mapsto & [f\circ\gamma\colon [0,1]\to Y] \end{array}\right) \end{array}$$

form a covariant functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{Group}$ of groups.

Example.

(1) Let $n \in \mathbb{Z}$. We consider the map $\begin{array}{rcl} f\colon S^1 & \to & S^1 \\ z & \mapsto & z^n. \end{array}$

As we discussed on page 1760 it follows easily from the Circle-π_1-Theorem 88.22 that the induced homomorphism $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$ is given by multiplication by n.

(2) It follows from the π_1-Functor Proposition 90.3 together with the Isomorphism Preservation Lemma 5.26 that any homeomorphism induces an isomorphism of fundamental groups. □

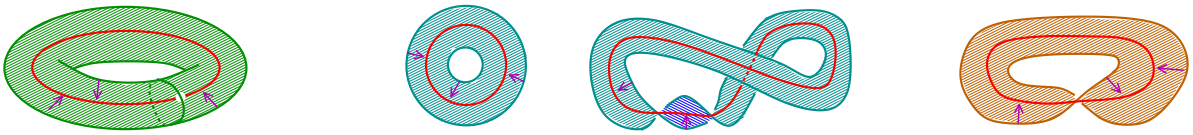
The following proposition gives a convenient criterion for a continuous map to induce an isomorphism of fundamental groups:

Proposition 90.9. (Homotopy Equivalence-π_1-Proposition) Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract[1143] of X, then for any $a_0 \in A$ the induced map

$$i_*\colon \pi_1(A, a_0) \;\to\; \pi_1(X, a_0)$$

is an isomorphism.

Example. In the figure below we show a solid torus, two annuli and the Möbius band.



solid torus annuli Möbius band

In each case there exists a deformation retraction to a subspace that is homeomorphic to S^1. It thus follows from the Homotopy Equivalence-π_1-Proposition 90.9 together with the Circle-π_1-Theorem 88.22 that in each case the fundamental group is isomorphic to $\mathbb{Z}$. □

We continue this discussion of functoriality with the following proposition:

Proposition 90.13. (Product-π_1-Proposition) Let $(X_1, x_1), \ldots, (X_n, x_n)$ be pointed topological spaces. For each $i \in \{1, \ldots, n\}$ we denote by $p_i\colon X_1 \times \cdots \times X_n \to X_i$ the natural projection. The induced natural homomorphism

$$\begin{array}{rcl} \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) & \to & \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n) \\ [f] & \mapsto & (p_{1*}([f]), \ldots, p_{n*}([f])). \end{array}$$

is an isomorphism.

We recall the following proposition, which at this stage is the most important application of the Product-π_1-Proposition 90.13:

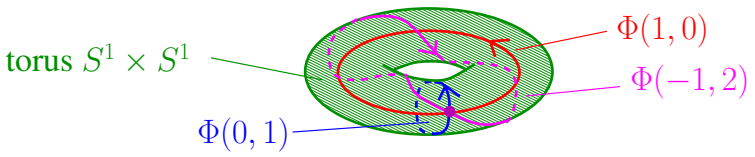
Proposition 90.14. (Torus-π_1-Proposition) The following map is an isomorphism:[1144]

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^2 & \to & \pi_1(\overbrace{S^1 \times S^1}^{\text{torus}}, (1,1)) \\ (m,n) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (\exp(2\pi\,\mathrm{i} m \cdot t), \exp(2\pi\,\mathrm{i} n \cdot t)) \end{array}\right]. \end{array}$$

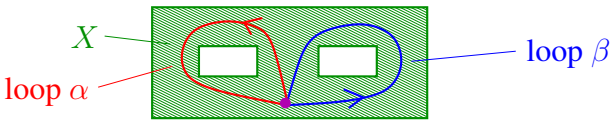
Analogous statements also hold for higher-dimensional tori.

[1143] Recall that according to Definition 19.3 A is a deformation retract of X if there exists a continuous map $R\colon X \times [0,1] \to X$ such that $R_0 = \mathrm{id}_X$, such that $R(a,t) = a$ for all $a \in A$ and $t \in [0,1]$ and such that $R(x,1) \in A$ for all $x \in X$.

[1144] Loosely speaking $(m,n) \in \mathbb{Z}^2$ corresponds to a loop that runs m times around the first S^1-factor and n times around the second factor.



110.2.3. **Presentations of groups.** So far we considered examples of topological spaces where the fundamental groups are isomorphic to the trivial group, or to $\mathbb{Z}$ or to $\mathbb{Z}^n$. In general fundamental groups of topological spaces can be all kinds of groups, and many of these fundamental group will be groups that we are not familiar with. For example, let us consider the topological space X that is given by removing the interiors of two open squares from a rectangle, and that is shown below:



In this subsection we will introduce some group theoretic language. [1145] These will in particular allow us to describe the fundamental group of the above topological space.

First we will give a (slightly informal) definition of the free group on a set:

Definition 110.3. Let X be a set.

(1) A reduced word in X is a formal expression of the form $x_1^{r_1} \cdot \ldots \cdot x_m^{r_m}$ such that the following conditions are satisfied:
   (a) Each x_i is an element of X and each r_i is a non-zero natural number.
   (b) For $i \in \{1, \ldots, m-1\}$ we have $x_i \neq x_{i+1}$.
   Note that we also allow the empty product which we denote by e.
(2) We denote by $\langle X \rangle$ the set of all reduced words in X.
(3) If $X = \{s_1, \ldots, s_k\}$, then we write $\langle s_1, \ldots, s_k \rangle := \langle \{s_1, \ldots, s_n\} \rangle$. [1146]
(4) We equip $\langle X \rangle$ with the "obvious multiplication" from Definition 98.1 which we illustrate below in an example. The argument from the Free Product Lemma 98.3 shows that this multiplication turns $\langle X \rangle$ into a group. We call it the free group generated by X.

Example.

(1) Let $X = \{x\}$ be a set that consists of a single element. Then $\langle x \rangle$ equals by definition all powers x^r with $r \in \mathbb{Z} \setminus \{0\}$ and the trivial element e. On this set the multiplication is the obvious one that is given by $x^i \cdot x^j := x^{i+j}$. This group $\langle x \rangle$ is at times called *the infinite cyclic group generated by* x. Evidently the map

$$\begin{array}{ccc} \mathbb{Z} & \to & \langle x \rangle \\ n & \mapsto & x^n \end{array}$$

is an isomorphism.

[1145] The following discussion can be viewed as a condensed version of Chapter 98. On a few occasions we will sacrifice rigor to keep the discussion concise.

[1146] One can easily see that this notation is consistent with Definitions 98.2 and 98.10.

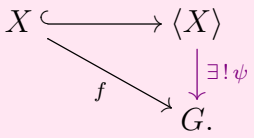
(2) We consider the set $X = \{s, t\}$. Examples of elements in the free group $\langle s, t\rangle$ are $s^3 \cdot t^{-5} \cdot s^2$ and $s^{-2} \cdot t^4 \cdot s^7 \cdot t^{-1}$. The product of these two elements is defined by

$$(s^3 \cdot t^{-5} \cdot s^2) \cdot (s^{-2} \cdot t^4 \cdot s^7 \cdot t^{-1}) := s^3 \cdot \underbrace{t^{-5} \cdot t^4}_{=t^{-1}} \cdot s^7 \cdot t^{-1} = s^3 \cdot t^{-1} \cdot s^7 \cdot t^{-1}.$$

Also note that $s \cdot t = s \cdot t \neq t \cdot s = t \cdot s$. Thus we see that the free group $\langle s, t\rangle$ is very much non-abelian. □

The following lemma explains where the name "free group" comes form:

Lemma 110.4. (Free Group Lemma) Let X be a set. The group $\langle X\rangle$ is a free group on the set X in the sense of Definition 98.6. In other words, $\langle X\rangle$ has the following universal property: For any *map* $f\colon X \to G$ of the set X to a group G there exists a unique *homomorphism* $\psi\colon \langle X\rangle \to G$ that makes the following diagram commute:



Proof. This lemma is just a reformulation of the Free Group–Existence Lemma 98.9.. ■

The free groups give us a few more examples of groups, which can be helpful. For example we will soon see that the fundamental group of the above topological space X is a free group on two generators. To get a really large range of groups we will have to look at quotients of free groups.

To do so we first recall Definition 98.20:

Definition 110.5. Let G be a group and let $S \subseteq G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } S := \langle\langle S\rangle\rangle := \bigcap_{S \subseteq H \lhd G} H.$$

More colloquially speaking, $\langle\langle S\rangle\rangle$ is the smallest normal subgroup of G that contains S.

Next we recall Definition 100.10:

Definition 110.6.

(1) Let $X = \{x_1, \dots, x_k\}$ be a set and let $r_1, \dots, r_l \in \langle x_1, \dots, x_k\rangle$. We write[1147]

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle := \langle x_1, \dots, x_k\rangle / \langle\langle \{r_1, \dots, r_l\}\rangle\rangle.$$

We say that we are given a finite presentation with generators $x_1, \dots, x_k$ and relators $r_1, \dots, r_l$.

(2) Let π be a group. A finite presentation for π is an isomorphism from π to a group as in (1).

Example.

(1) For $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k\rangle = \langle g\rangle / \langle\langle g^k\rangle\rangle \underset{\uparrow}{=} \langle g\rangle / \{g^{k\cdot n} \,|\, n \in \mathbb{Z}\} \cong \mathbb{Z}/k\mathbb{Z}.$$

this follows easily from the observation that $\langle g\rangle$ is abelian

[1147] Since $\langle\langle \{r_1, \dots, r_l\}\rangle\rangle$ is a normal subgroup of $\langle x_1, \dots, x_k\rangle$ we see that $\langle x_1, \dots, x_k\rangle / \langle\langle \{r_1, \dots, r_l\}\rangle\rangle$ is indeed a group.

(2) In the following we will often consider the presentation $\langle x_1, x_2 \,|\, [x_1, x_2]\rangle$ where the relator $[x_1, x_2] := x_1 x_2 {x_1}^{-1} x_2^{-1}$ is the commutator of x_1 and x_2.
(3) At times, given two elements $r, \widetilde{r} \in \langle x_1, \dots, x_k\rangle$ we write "$r = \widetilde{r}$" to indicate the relator $r \cdot \widetilde{r}^{-1} \in \langle x_1, \dots, x_k\rangle$. For example the expression $\langle x_1, x_2 \,|\, [x_1 \cdot x_2 = x_2 \cdot x_1\rangle$ is just a different way to write $\langle x_1, x_2 \,|\, \underbrace{x_1 \cdot x_2 \cdot {x_1}^{-1} \cdot x_2^{-1}}_{=[x_1,x_2]}\rangle$. □

In general it is difficult to get a good sense of finite presentations as above and it is hard to see what properties these groups have. One approach to probe a group is by trying to understand that group homomorphisms it admits.

In the following we will pursue this idea. More precisely, in the following we will see how one can find group homomorphisms from finite presentations to other groups. To formulate the corresponding lemma we need the following definition:

Definition 110.7. Let $r \in \langle x_1, \dots, x_k\rangle$. Furthermore let G be a group and let $g_1, \dots, g_k$ be elements of G. We say that the elements $g_1, \dots, g_k$ satisfy the relation r if the natural homomorphism $\langle x_1, \dots, x_k\rangle \to G$ from the Free Group Lemma 110.4 that is given by $x_i \mapsto g_i$, $i = 1, \dots, k$ sends r to the trivial element.[1148]

Example. As above we consider the word $[x_1, x_2] = x_1 x_2 {x_1}^{-1} x_2^{-1}$. If a group G is abelian, then it follows almost immediately from the definitions that any two elements $g_1, g_2 \in G$ satisfy the relation $[x_1, x_2] = x_1 x_2 {x_1}^{-1} x_2^{-1}$. □

Lemma 100.13. (Presentation–Homomorphism Lemma) Let π be a group with a finite presentation $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle$ and let G be a group.
(1) If $g_1, \dots, g_k \in G$ satisfy the relations $r_1, \dots, r_l$, then there exists a unique homomorphism
$$\beta\colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle \;\to\; G$$
with $\beta(x_i) = g_i$ for $i = 1, \dots, k$.
(2) Every homomorphism $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle \to G$ is of the form described in (1).

Remark. An equivalent formulation of the Presentation–Homomorphism Lemma 100.13 is to say that the map
$$\begin{array}{ccc} \mathrm{Hom}(\pi, G) & \to & \{\text{all } (g_1, \dots, g_k) \in G^k \text{ that satisfy } r_1, \dots, r_l\} \\ f & \mapsto & (f(x_1), \dots, f(x_k)) \end{array}$$
is a bijection. Note that if G is a finite group, then the right-hand side is a finite set. At least in this case it is usually straightforward, albeit tedious, to determine the right-hand side. □

We conclude this discussion of presentations of groups with a slightly non-trivial example:

Lemma 100.14. (Free Abelian Group-Presentation Lemma) There exists a unique isomorphism
$$\langle x_1, x_2 \mid [x_1, x_j]\rangle \;\xrightarrow{\cong}\; \mathbb{Z}^2$$
with $x_1 \mapsto (1, 0)$ and $x_2 \mapsto (0, 1)$.

[1148] More colloquially speaking: $g_1, \dots, g_k$ satisfy a relation r if replacing the x_i by g_i in the word r returns the trivial element in G.

110.2.4. **The Seifert-van Kampen Theorem.** Now, after this lengthy excursion into group theory we can return to topology. We can now formulate the following theorem, which is the key tool for determining fundamental groups in non-trivial settings:

Theorem 110.8. (Seifert-van Kampen Theorem) Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two subsets U and V such that $U \cap V$ is non-empty and *path-connected*. We denote by $i\colon U \cap V \to U$ and $j\colon U \cap V \to V$ the inclusions. We assume that one of the following two extra conditions is satisfied:

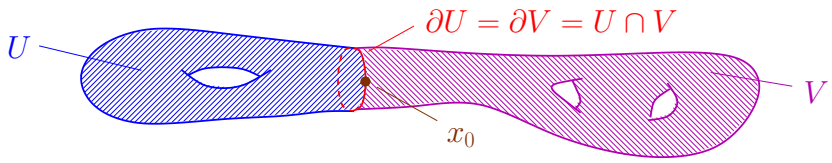
(1) U and V are both open subsets of X.
(2) Let X be a smooth manifold and let $U, V \subseteq X$ be two codimension zero submanifolds such that the following hold:[1149]
   (a) $X = U \cup V$.
   (b) U and V are closed subsets of X.
   (c) $U \cap V$ is a single component of ∂U and it is a single component of ∂V.

Under these conditions for every base point $x_0 \in U \cap V$ we can express the fundamental group of X in three different ways:[1150]

(i) We consider the amalgamated product $\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$ as introduced in Definition 98.23. The inclusion-induced map
$$\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$
is an isomorphism.

(ii) We have a pushout diagram
$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i_*} & \pi_1(U, x_0) \\ {\scriptstyle j_*}\downarrow & & \downarrow \\ \pi_1(V, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

(iii) We suppose that we have presentations of the form
$$\begin{aligned} \pi_1(U, x_0) &= \langle x_1, \ldots, x_a \,|\, r_1, \ldots, r_m \rangle \\ \pi_1(V, x_0) &= \langle y_1, \ldots, y_b \,|\, s_1, \ldots, s_n \rangle \end{aligned}$$
and we suppose that $\pi_1(U \cap V, x_0)$ is generated by[1151] $z_1, \ldots, z_c$. Then we have a presentation[1152]
$$\pi_1(X, x_0) = \langle x_1, \ldots, x_a, y_1, \ldots, y_b \,|\, r_1, \ldots, r_m, s_1, \ldots, s_n, \{ \underbrace{i_*(z_h)}_{\substack{\text{word in} \\ x_1, \ldots, x_a}} = \underbrace{j_*(z_h)}_{\substack{\text{word in} \\ y_1, \ldots, y_b}} \}_{h=1,\ldots,c} \rangle.$$



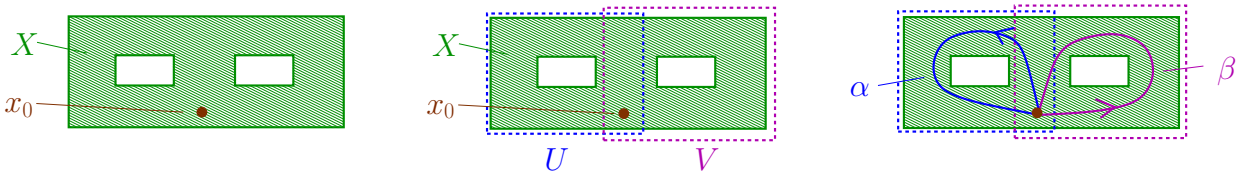
[1151] In this part of the course we will mostly care about formulation (iii).

[1152] Given a group G we say $g_1, \ldots, g_k$ form a generating set for G if the unique homomorphism $\langle x_1, \ldots, x_k \rangle \to G$ with $x_i \mapsto g_i$ (which we introduced in the Free Group Lemma 110.4) is an epimorphism.

Proof. This theorem combines features from the Seifert–van Kampen Theorem 99.1, the Manifold–Seifert-van Kampen Theorem 102.1 and the Amalgamated Product–Presentation Lemma 100.20. ■

The Seifert-van Kampen Theorem 110.8 will be our main tool for calculating fundamental groups. So we will practice using it in several examples which might or might not be familiar to the reader:

Example. We return to the example at the beginning of Subsection 110.2.3.



We consider the open subsets U and V that are given by the "left square" and the "right square". We make the following observations:

(1) Since U admits a deformation retraction to the α-circle we have $\pi_1(U, x_0) = \langle \alpha \rangle$.
(2) Since V admits a deformation retraction to the β-circle we have $\pi_1(V, x_0) = \langle \beta \rangle$.
(3) Since $U \cap V$ is contractible we see that $\pi_1(U \cap V, x_0) = \{e\}$.

Thus we see that

$$\pi_1(X, x_0) \underset{\uparrow}{=} \langle \alpha, \beta \rangle.$$

by the Seifert-van Kampen Theorem 110.8 and the above discussion

□

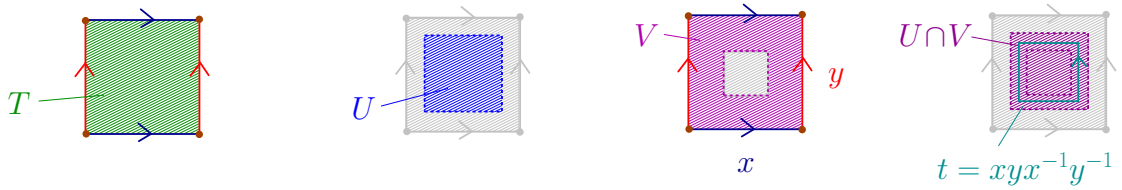
Example. We consider the torus $T := S^1 \times S^1$ which, by the discussion on page 310, we can identify with

$$T := ([0,1] \times [0,1])/\sim \qquad \text{where } (0,y) \sim (1,y) \text{ and } (x,0) \sim (x,1) \text{ for all } x, y \in [0,1].$$

In the following we will outline how one can use the Seifert-van Kampen Theorem 110.8 to give an alternative calculation of $\pi_1(T)$.[1153]

We consider the decomposition $T := U \cup V$ where $U := (\frac{1}{6}, \frac{5}{6}) \times (\frac{1}{6}, \frac{5}{6})$ and where $V := T \setminus ([\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}])$. Note that

$$\begin{aligned} \pi_1(U) &= \langle x, y \rangle && \text{since } U \text{ is homeomorphic to a rectangle minus two rectangles} \\ \pi_1(V) &= \langle \varnothing \rangle && \text{since } V \text{ is contractible} \\ \pi_1(U \cap V) &= \langle t \rangle, && \text{since the } t\text{-circle is a deformation retract of } U \cap V. \end{aligned}$$



Thus we see that

$$\pi_1(T, x_0) \underset{\uparrow}{=} \langle x, y \,|\, i_*(t) = j_*(t) \rangle \underset{\uparrow}{=} \langle x, y \,|\, [x,y] = e \rangle \underset{\uparrow}{\cong} \mathbb{Z}^2.$$

by the Seifert-van Kampen Theorem 110.8 and the above discussion

in the figure above we show that $i_*(t) = xyx^{-1}y^{-1} = [x,y]$

by the Free Abelian Group Presentation Lemma 100.14

[1153] In the discussion below we will skip several explanations. We refer to page 1976 for full details.

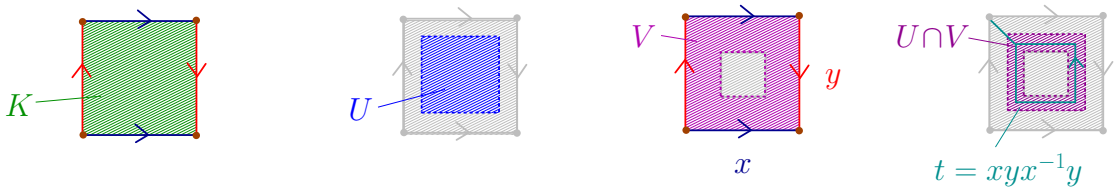
Rather reassuringly we obtain the same result as in the Torus-π_1-Proposition 90.14. □

Example. We consider the Klein bottle

$$K := ([0,1]\times[0,1])/\sim \text{ where } (0,y)\sim(1,1-y) \text{ and } (x,0)\sim(x,1) \text{ for all } x,y\in[0,1]$$

which we introduced on page 312. We consider the decomposition $K := U \cup V$ where $U := (\frac{1}{6},\frac{5}{6})\times(\frac{1}{6},\frac{5}{6})$ and where $V := K \setminus ([\frac{1}{3},\frac{2}{3}]\times[\frac{1}{3},\frac{2}{3}])$. Note that

$$\begin{aligned} \pi_1(U) &= \langle x,y\rangle && \text{since } U \text{ is homeomorphic to a rectangle minus two rectangles}\\ \pi_1(V) &= \langle\varnothing\rangle && \text{since } V \text{ is contractible}\\ \pi_1(U\cap V) &= \langle t\rangle, && \text{since the } t\text{-circle is a deformation retract.}\end{aligned}$$



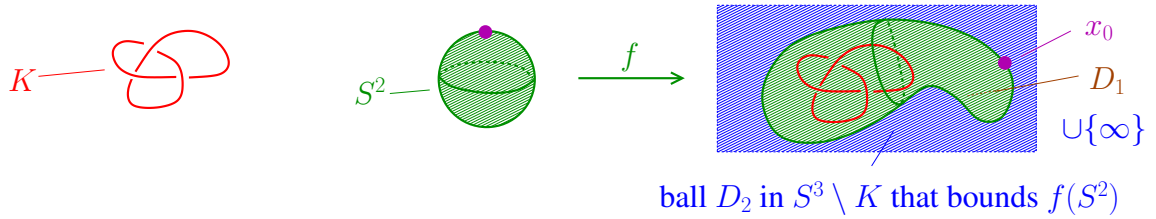
Thus we see that

$$\pi_1(K,x_0) \underset{\text{by the Seifert-van Kampen Theorem 110.8 and the above discussion}}{=} \langle x,y \,|\, i_*(t)=j_*(t)\rangle \underset{\text{in the figure above we show that } i_*(t)=xyx^{-1}y=[x,y]}{=} \langle x,y \,|\, xyx^{-1}y\rangle.$$

□

110.3. **Higher homotopy groups.** Let (X,x_0) again be a pointed topological space. In Definitions 134.3 and 134.9 we will introduce the corresponding homotopy groups $\pi_n(X,x_0)$, $n\in\mathbb{N}_{\geq 2}$ which are higher-dimensional analogs of the fundamental group $\pi_1(X,x_0)$. Loosely speaking elements of the homotopy group $\pi_n(X,x_0)$ are homotopy classes of continuous maps $(S^n,N)\to(X,x_0)$ where $N := (0,\dots,0,1)$ is the North Pole of S^n and where the homotopy fixes the North Pole N.

In the following we will explain why for any knot $K\subseteq S^3$ and any $x_0\in S^3\setminus K$) we have $\pi_2(S^3\setminus K,x_0)=0$. We do a proof by contradiction. We suppose that $\pi_2(S^3\setminus K,x_0)\neq 0$. It then follows from the (very deep!) Sphere Theorem **??** that there exists a smooth embedding $f\colon(S^2,N)\to(S^3\setminus K,x_0)$ such that $[f]$ is non-trivial in $\pi_2(S^3\setminus K,x_0)$.



Note that f is in particular a smooth embedding $S^2\to S^3$. It follows from the Generalized Smooth Schoenflies Theorem 170.7 that there exist two 3-dimensional balls D_1,D_2 in S^3 with $D_1\cup D_2=S^3$ and $D_1\cap D_2=f(S^2)$. Since K is connected and since $K\cap f(S^2)=\varnothing$ we see that K is contained in the interior of one of the two balls, say that it is contained in $\mathring{D}_1$. But this means that $D_2\subseteq S^3\setminus K$. Since $\pi_2(D_2)=0$ we see that $[f]=0\in\pi_2(D_2)$. But this implies also, by functoriality, that $[f]=0\in\pi_2(S^3\setminus K)$.

Finally we point out in the Knot Complement-π_n-Proposition **??** we will see that all higher homotopy groups of $S^3 \setminus K$ are also zero. So in our setting the only homotopy group that brings anything to the table is the fundamental group.

110.4. **Appendix: The abelianization of a group.** We just saw that in various settings that the fundamental group of a topological space can be non-abelian. The question arises, how can we get a handle on such groups? One elementary tool is the abelianization of a group, which we first introduced in Section 98.3. For the reader's convenience we recall the definition and the main features.

Definition 110.9. Let G be a group.

(1) Given $x, y \in G$ we refer to $[x, y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$ as the commutator of x and y.

(2) We refer to
$$[G, G] := \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \ldots, x_n, y_n \in G \right\}$$
as the commutator subgroup of G.

In the Commutator Subgroup Lemma 98.14 we showed that the commutator subgroup is a normal subgroup of G and that $G/[G, G]$ is an abelian group. This leads us to the following definition:

(3) We call
$$G_{\mathrm{ab}} := G/[G, G]$$
the abelianization of G.[1154]

Example. As we saw in the Abelianization Coproduct Proposition 98.18 (2) it follows easily from the definitions that the abelianization of a free group on a set X is just the free abelian group defined by X. □

In practice we will want to determine the abelianization of a group that is given by a presentation. This can be done using the following proposition:

Proposition 100.17. (Abelianization-of-Presentation Proposition) Suppose we are given a finitely presented group $\pi = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$. We denote by
$$\Phi \colon \langle x_1, \ldots, x_m \rangle \to \mathbb{Z}^m$$
the homomorphism that is given by $\Phi(x_i) = \mathfrak{e}_i$. There exists a unique isomorphism[1155]
$$\Psi \colon \underbrace{\pi/[\pi, \pi]}_{=\pi_{\mathrm{ab}}} \xrightarrow{\cong} \mathbb{Z}^m / \operatorname{span}\{\Phi(r_1), \ldots, \Phi(r_n)\}$$
such that the following diagram commutes:
$$\begin{array}{ccc} \langle x_1, \ldots, x_m \rangle & \xrightarrow[\Phi]{x_i \mapsto \mathfrak{e}_i} & \mathbb{Z}^m \\ \downarrow & & \downarrow \\ \pi = \langle x_1, \ldots, x_m \rangle / \langle\langle r_1, \ldots, r_n \rangle\rangle & & \\ \downarrow & & \\ \pi_{\mathrm{ab}} = \pi/[\pi, \pi] & \xrightarrow[\cong]{\Psi} & \mathbb{Z}^m / \operatorname{span}\{\Phi(r_1), \ldots, \Phi(r_n)\}. \end{array}$$

Loosely speaking the Abelianization-of-Presentation Proposition 100.17 says that the abelianization of a presentation is given by "abelianizing" the relators. The statement should become clearer by looking at examples:

Example.

(1) We consider the group

$$\pi \;=\; \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx\rangle.$$

In the following let $\Phi\colon \langle x,y\rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\Phi(x) = (1,0)$ and by $\Phi(y) = (0,1)$. It follows from the Abelianization-of-Presentation Proposition 100.17 that the abelianization of π is given by

$$\pi_{\mathrm{ab}} \;=\; \mathbb{Z}^2/\operatorname{span}\{\underbrace{\Phi(x^2y^{-2}x)}_{=\binom{3}{-2}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\binom{4}{0}}\} \;=\; \mathbb{Z}^2/\tbinom{3\ \ 4}{-2\ \ 0}\mathbb{Z}^2 \;\cong\; \mathbb{Z}/8\mathbb{Z}.$$

(the isomorphism $\cong$ can be calculated using the Smith normal form of the matrix)

(2) Let K be the Klein bottle. By the discussion on page 2144 we know that

$$\pi_1(K) \;=\; \langle x, y \mid xyx^{-1}y\rangle.$$

Again let $\Phi\colon \langle x,y\rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\Phi(x) = (1,0)$ and $\Phi(y) = (0,1)$. It follows from the Abelianization-of-Presentation Proposition 100.17 that the abelianization of $\pi_1(K)$ is given by

$$\pi_1(K)_{\mathrm{ab}} \;=\; \mathbb{Z}^2/\operatorname{span}\{\underbrace{\Phi(xyx^{-1}y)}_{=\binom{0}{2}}\} \;=\; \mathbb{Z}^2/\tbinom{0}{2}\mathbb{Z} \;\cong\; \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}.$$

This discussion, together with the Abelianization Functor Proposition 98.16 and the Isomorphism Preservation Lemma 5.26, shows in particular that $\pi_1(K)$ is not isomorphic to the fundamental group of the torus. □

110.5. **Exercises for Chapter 110.** The first two exercises of this chapter are only meant for readers who are familiar with homology groups. They require facts and tools that we will only provide in later chapters.

Exercise 110.1. Let $m \in \mathbb{N}$ and let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link. By the Link Complement-H$_*$–Proposition 110.1 we know that the homology groups of $S^3 \setminus L$ are given as follows:

$$\mathrm{H}_n(S^3 \setminus L) \;\cong\; \begin{cases} 0, & \text{if } n \geq 3,\\ \mathbb{Z}^{m-1}, & \text{if } n = 2,\\ \mathbb{Z}^m, & \text{if } n = 1,\\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For $n = 0, 1, 2$ make an educated guess what a basis of the homology group might be. Recall that every closed oriented n-dimensional smooth submanifold $M \subseteq S^3 \setminus L$ gives rise to a homology class $[M] \in \mathrm{H}_n(S^3 \setminus L)$.

Exercise 110.2. Let $m \in \mathbb{N}$ and let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link in $\mathbb{R}^3$. What are the homology groups of $\mathbb{R}^3 \setminus L$?

Remark. By the Link Complement-H$_*$–Proposition 110.1 we know the homology groups of $S^3 \setminus L$. The reader could try to address the following more general question: Given a

compact subspace $X \subseteq \mathbb{R}^3$, what is the relationship between the homology groups of $\mathbb{R}^3 \setminus X$ and $S^3 \setminus X$?

Exercise 110.3. If the reader is aware of the Euler characteristic, then calculate the Euler characteristic of the complement of an m-component link in S^3.

Exercise 110.4. Give an example of a topological space X and two base points x_0 and x_1 such that the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are *not* isomorphic.

Exercise 110.5. Show that the group $\langle x, y \,|\, xyx = yxy \rangle$ admits an epimorphism onto the permutation group S_3.

Exercise 110.6. Find an $r \in \langle x, y \rangle$ that is non-trivial such that $\langle x, y \,|\, r \rangle$ admits an epimorphism onto the permutation group S_5.

Exercise 110.7. Let K be the Klein bottle. Show that $\pi_1(K)$ is non-abelian.

Exercise 110.8. Determine the fundamental group of the real projective space $\mathbb{R}\mathrm{P}^2$.
Remark. As in the Projective Space-Avatar Lemma 9.16 (2) you can think of the real projective plane $\mathbb{R}\mathrm{P}^2$ as $\overline{B}^2/\sim$ where $z \sim -z$ for all $z \in \partial\overline{B}^2$. Now use the approach from page 2143.

Exercise 110.9. Let $L \subseteq S^3$ be a splittable link.

(a) Show that there exists a smooth embedding $f\colon S^2 \to S^3 \setminus L$ such that $f(S^2)$ is a retract of $S^3 \setminus L$.

(b) Show that if a link $L \subseteq S^3$ is splittable, then $\pi_2(S^3 \setminus L)$ is non-zero.

Exercise 110.10. Let $K \subseteq S^3$ be a knot. Show that $\pi_2(\mathbb{R}^3 \setminus K)$ non-zero.
Hint. Show that there exists a retraction from $\mathbb{R}^3 \setminus K$ to a 2-dimensional sphere in $\mathbb{R}^3 \setminus K$ and use the fact that $\pi_2(S^2) \cong \mathbb{Z}$.

Exercise 110.11. Let $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a presentation and let $w \in \langle x_1, \ldots, x_k \rangle$. Show that there exists an isomorphism

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \xrightarrow{\cong} \langle x_1, \ldots, x_k, y \,|\, r_1, \ldots, r_l, y = w \rangle.$$

Remark. This isomorphism is sometimes known as the "first Tietze move".

Exercise 110.12. Let $n \in \mathbb{N}$. What is the abelianization of the permutation group S_n?

Exercise 110.13. Let $p, q \in \mathbb{Z}$. What is the abelianization of the group $\langle x, y \,|\, x^p = y^q \rangle$?

CHAPTER 111

Fundamental groups of link complements

In this chapter we want to return to the study of links. In the last chapter we saw that the only invariant from algebraic topology that has any chance of being useful is the fundamental group. Thus, given a link $L \subseteq S^3$ we will study the fundamental group $\pi_1(S^3 \setminus L)$. In this chapter we will calculate this group for the trivial knot. Afterwards we will introduce the torus knots $T(p, q)$ (which are generalizations of the trefoil) and we will compute the fundamental groups $\pi_1(S^3 \setminus T(p, q))$ for all torus knots. We will use these calculations to show in particular that the trefoil is not smoothly isotopic to the trivial knot.

This chapter is organized as follows:

- In Section 111.1 we will see that the isomorphism type of the fundamental group of the complement of a link is a smooth isotopy invariant of a link.
- In Section 111.2 we will get to know one more way to describe spheres. In particular we will see that we can think of S^3 as the union of two solid tori.
- In Section 111.3 we will use the above description of S^3 to determine the fundamental groups of the complement of the trivial knot.
- In Section 111.4 we will introduce the torus knots $T(p, q)$.
- In Section 111.5 we will introduce some notation that will be convenient in the various calculations of fundamental groups in the later sections.
- In Section 111.6 we will determine the fundamental groups of the complement of the torus knot $T(p, q)$.
- In Section 111.7 we will determine the fundamental group of the split union of two links.
- In Section 111.8 we will give a formal definition of the Hopf link and we will calculate the fundamental group of the complement.
- In Section 111.9 we will determine the fundamental group of the connected sum of knots.
- In Appendix 111.10 we will consider fundamental groups of the complements of high-dimensional knots.
- In Section 111.11 we will provide exercises for Chapter 111.

In the next Chapter 112 we will introduce link diagrams. These will be useful for calculating fundamental groups of link complements.

111.1. **The fundamental group of the complement of a link.** Our idea now is to study the fundamental group of link complements. The following lemma, together with the Base Point Independence Lemma 110.2, says that we do not really have to worry about base points:

Lemma 111.1. (Link Complement–Connected Lemma) Given any link $L \subseteq S^3$ the complement $S^3 \setminus L$ is path-connected.

Proof. The lemma is a special case of the Complement-of-Submanifold Connected Corollary 50.11. Alternatively the lemma can be proved using the Link Tubular Map Theorem 108.29. Cleverly we already filled in the details in Exercise 108.5. ■

The Link Complement–Connected Lemma 111.1 together with the Base Point Independence Lemma 110.2 allows us to introduce the following definition:

Definition 111.2. Given a link $L \subseteq S^3$ we refer to the isomorphism type of $\pi_1(S^3 \setminus L)$ as the link group of L.

The following lemma shows that the link group has many different descriptions.

Lemma 111.3. (Link-π_1-Isomorphism Lemma)

(1) Given any link $L \subseteq S^3$ with exterior $X_L = S^3 \setminus \tau(B^2 \times L)$ the inclusion $X_L \to S^3 \setminus L$ induces an isomorphism
$$\pi_1(X_L) \xrightarrow{\cong} \pi_1(S^3 \setminus L).$$

(2) For a link $L \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ the inclusion $\mathbb{R}^3 \setminus L \to S^3 \setminus L$ induces an isomorphism[1156]
$$\pi_1(\mathbb{R}^3 \setminus L) \xrightarrow{\cong} \pi_1(S^3 \setminus L).$$

Proof.

(1) This statement follows immediately from the Link Exterior Lemma 108.31 together with the Homotopy Equivalence-π_1-Proposition 90.9 (4).

(2) This statement can be proved easily using the Seifert-van Kampen Theorem 110.8. We will work out the details in Exercise 111.1. ■

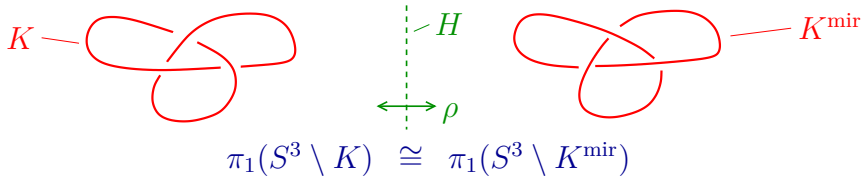
The following lemma is much more interesting, it shows that fundamental groups can (at least in principle) be used to distinguish knots and links up to smooth isotopy:

Lemma 111.4. (Isotopic Link-π_1-Lemma) Let $L, \widetilde{L}$ be links in S^3. If L and $\widetilde{L}$ are smoothly isotopic, then $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L})$.

Proof. We suppose that L and $\widetilde{L}$ are smoothly isotopic. It follows from the Smooth Isotopy–Link Complement Proposition 108.12 that the link complements $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$ are diffeomorphic. It follows from the π_1-Functor Proposition 90.3 that the groups $\pi_1(S^3 \setminus L)$ and $\pi_1(S^3 \setminus \widetilde{L})$ are isomorphic. ■

Before we head towards calculations, let us first state the following sobering lemma, which implies that just looking at the fundamental $\pi_1(S^3 \setminus L)$ is useless for distinguishing a link from its mirror:

Lemma 111.5. (Mirror Link-π_1-Lemma) Given any link L in S^3 there exists an isomorphism $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus L^{\mathrm{mir}})$.



[1156] As we will see shortly, somewhat surprisingly, technically it is often easier to work with $S^3 \setminus L$ instead of $\mathbb{R}^3 \setminus L$.

Proof. By definition of the mirror we have $L^{\text{mir}} = \rho(L)$ where $\rho\colon S^3 \to S^3$ is the reflection in some hyperplane $H \subseteq \mathbb{R}^4$. Note that ρ restricts to a diffeomorphism $S^3 \setminus L \to S^3 \setminus L^{\text{mir}}$. It follows that the induced map $\rho_*\colon \pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus L^{\text{mir}})$ is an isomorphism. ■

The name of the game is now to determine $\pi_1(S^3 \setminus L)$ for interesting knots and links, e.g. for the trivial knot and the trefoil. The trick is to find an elegant description of the complements of the trivial knot and the trefoil. For this purpose it is useful to introduce one more way to think about spheres. We will do so in the next section.

111.2. **One more avatar of spheres.** As we just mentioned, we want to get one more way to think about spheres. Even though in this part of the lecture notes we only really care about S^3, let us formulate the following lemma in a more general context:

Lemma 111.6. (Sphere–Solid Tori-Decomposition Lemma) Let $m, n \in \mathbb{N}_0$. We consider the following two maps:

$$\begin{aligned} \Phi\colon S^{m-1} \times \overline{B}^n &\to A \coloneqq \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|x\|^2 + \|y\|^2 = 1 \text{ and } \|y\|^2 \leq \tfrac{1}{2}\right\} \\ (a,b) &\mapsto \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right) \end{aligned}$$

and

$$\begin{aligned} \Psi\colon \overline{B}^m \times S^{n-1} &\to B \coloneqq \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|x\|^2 + \|y\|^2 = 1 \text{ and } \|x\|^2 \leq \tfrac{1}{2}\right\} \\ (a,b) &\mapsto \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right). \end{aligned}$$

These maps have the following properties:

(1) The map $\Phi\colon S^{m-1} \times \overline{B}^n \to A$ is an orientation-preserving[1157] diffeomorphism.
(2) The map $\Psi\colon \overline{B}^m \times S^{n-1} \to B$ is a diffeomorphism. Furthermore: It is orientation-preserving if and only if m is even.
(3) The map[1158]

$$\begin{aligned} \Theta\colon (S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1}) &\to S^{m+n-1} \\ [P] &\mapsto \begin{cases} \Phi(P), & \text{if } P \in S^{m-1} \times \overline{B}^n, \\ \Psi(P), & \text{if } P \in \overline{B}^m \times S^{n-1} \end{cases} \end{aligned}$$

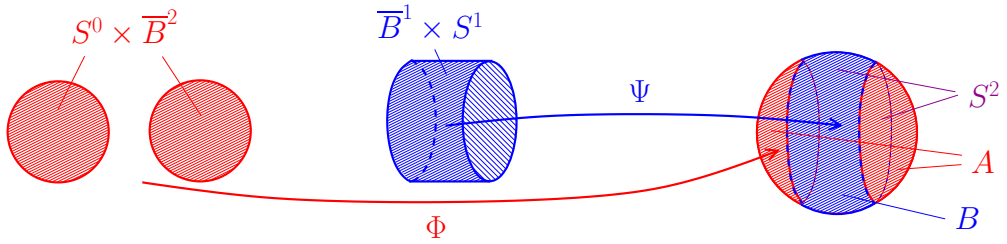
is well-defined and it is a diffeomorphism.

Convention 111.7. We use the diffeomorphism

$$\Theta\colon (S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1}) \to S^{m+n-1}$$

[1157] We equip each sphere with the standard orientation introduced on page 817. We equip A and B with the submanifold orientation and we use the Product Orientation Proposition 31.48 to equip products with natural orientations.

[1158] We equip the boundaries of $S^{m-1} \times S^n$ and $\overline{B}^m \times S^{n-1}$ with the standard collars that we introduced on page 883. We use these collars and the Gluing-Smooth Manifolds-Proposition 54.5 (1a) to equip the left-hand side with the structure of a smooth manifold. Furthermore we use the Gluing-Smooth Manifolds-Proposition 54.5 (4) to equip the left-hand side with the orientation, where the orientation matches the orientation of $S^{m-1} \times \overline{B}^n$ and which matches the orientation of $\overline{B}^m \times S^{n-1}$ if and only if m is even. It follows from (1) and (2) that with these choices of orientations the diffeomorphism Θ is orientation-preserving.



decomposition of S^2 with $m = 1$ and $n = 2$

from the Sphere–Solid Tori-Decomposition Lemma 111.6 (2) to add the smooth manifold $(S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1})$ to our list of avatars of S^{m+n-1}.

Sketch of proof.

(2) Since the orientations are a little more interesting in the second case let us treat the second case in detail. We consider the map

$$\begin{aligned} \widetilde{\Psi}\colon B = \big\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \;\big|\; \|x\|^2 + \|y\|^2 = 1 \;\&\; \|x\|^2 \leq \tfrac{1}{2}\big\} &\to \overline{B}^m \times S^{n-1} \\ (x,y) &\mapsto \Big(\frac{x}{\|y\|}, \frac{y}{\|y\|}\Big). \end{aligned}$$

One can easily verify that $\widetilde{\Psi}$ and Ψ are inverses of one another and that Ψ and $\widetilde{\Psi}$ are smooth. This observation implies that $\Psi\colon \overline{B}^m \times S^{n-1} \to B$ is a diffeomorphism. It remains to show that Ψ is orientation-preserving if and only if m is even. By the Manifold–Orientation Basics Lemma 31.31 it suffices to prove the following claim:

Claim. For $\mathfrak{e}_{m+1} = ((0,\dots,0),(1,0,\dots,0)) \in \overline{B}^m \times S^{n-1}$ the corresponding differential $\mathrm{D}\Psi_{\mathfrak{e}_{m+1}}\colon \widetilde{\mathrm{T}}_{\mathfrak{e}_{m+1}}(\overline{B}^m \times S^{n-1}) \to \widetilde{\mathrm{T}}_{\Psi(\mathfrak{e}_{m+1})}B$ is orientation-preserving if and only if m is even.

Proof. By the orientation conventions from page 817 and the Smooth Manifold Product Proposition 25.34 we know that $\{\mathfrak{e}_1,\dots,\mathfrak{e}_m,\mathfrak{e}_{m+2},\dots,\mathfrak{e}_{m+n}\}$ is a positive basis for $\widetilde{\mathrm{T}}_{\mathfrak{e}_{m+1}}(\overline{B}^m \times S^{n-1})$. First note that a straightforward calculation shows that the differential of Ψ at the point $\mathfrak{e}_{m+1}$ is actually the identity. Next note that $\Psi(\mathfrak{e}_{m+1}) = \mathfrak{e}_{m+1}$. Finally note that by definition $\{v_1,\dots,v_{m+n-1}\}$ is a positive basis for the tangent space $\widetilde{\mathrm{T}}_{\Psi(\mathfrak{e}_{m+1})}B = \widetilde{\mathrm{T}}_{\mathfrak{e}_{m+1}}S^{m+n-1} \subseteq \mathbb{R}^{m+n}$ if and only if $\det(\underbrace{\mathfrak{e}_{m+1}, v_1,\dots,v_{m+n-1}}_{(m+n)\times(m+n)\text{-matrix}}) > 0$. The claim follows almost immediately from these observations. ⊞

(1) The proof of this statement is almost identical to the above proof of (2).

(3) It is clear that Φ and Ψ agree on $S^{m-1} \times S^{n-1}$. It follows from the Topological-Quotient Proposition 8.2 (1b) that the given map

$$\Theta\colon (S^{m-1} \times \overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1}) \;\to\; S^{m+n-1}$$

is well-defined and continuous. Using (1) and (2) one can easily verify that the map is a bijection. It follows from the Compact-Hausdorff Proposition 4.23 (3) that the map Θ is a homeomorphism. Going through the definition of the smooth structure of the smooth manifold on the left, as defined in the Gluing-Smooth Manifolds-Proposition 54.5 (1a), shows that the map is actually a diffeomorphism. Finally it follows from (1) and (2) that the diffeomorphism is actually orientation-preserving. ■

Visualization. When it comes to the Sphere–Solid Tori-Decomposition Lemma 111.6 we are mostly interested in the case $m = n = 2$, i.e. we are interested in the diffeomorphism

$$\Theta\colon (S^1 \times \overline{B}^2) \cup_{S^1\times S^1} (\overline{B}^2 \times S^1) \xrightarrow{\cong} S^3.$$

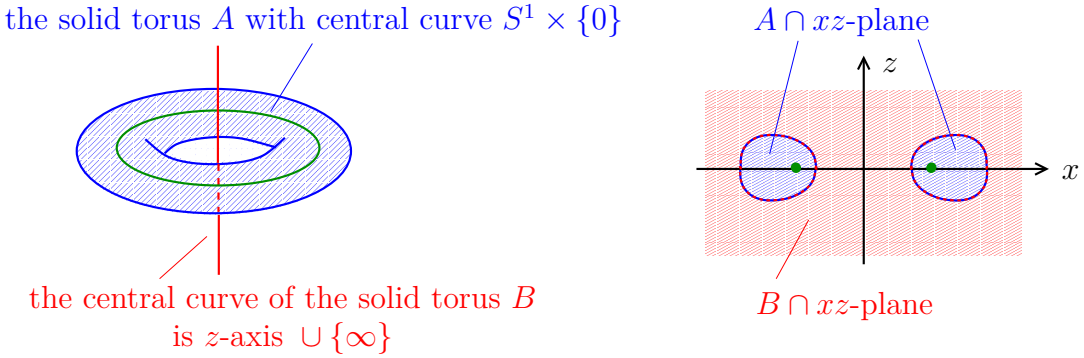
As we mentioned on page 2097 we have an identification $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ and we view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. In particular we can view the solid tori A and B (which we introduced in the Sphere–Solid Tori-Decomposition Lemma 111.6) as submanifolds of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. It follows easily from the definitions that in this setting we have

$$\text{“central curve of } A\text{”} \;=\; \Phi(S^1 \times \{0\}) \;=\; S^1 \times \{0\} \;\subseteq \mathbb{R}^2 \times \mathbb{R}$$

and

$$\text{“central curve of } B\text{”} \;=\; \Psi(\{0\} \times S^1) \;=\; \text{the } z\text{-axis} \cup \{\infty\}.$$

We visualize the solid torus A and the central curve of the solid torus B in the figure below.



The solid torus B is more difficult to illustrate. It consists of a closed disk attached to each point on the central curve. For example the closed disk attached to the origin is just the “obvious” closed disk in the xy-plane that touches the torus $\Phi(S^1 \times S^1)$.[1159] □

Next we recall that in the Torus Embedding Lemma 7.11 and on page 785 we saw that the map

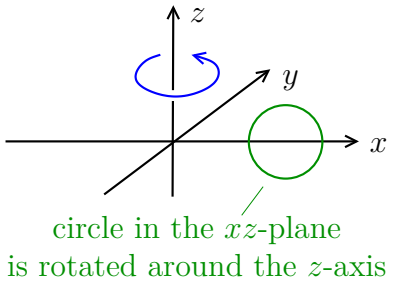
$$\begin{aligned} \Theta\colon \overline{B}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi)) &\mapsto \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in the } xz\text{-plane}}} \end{aligned}$$

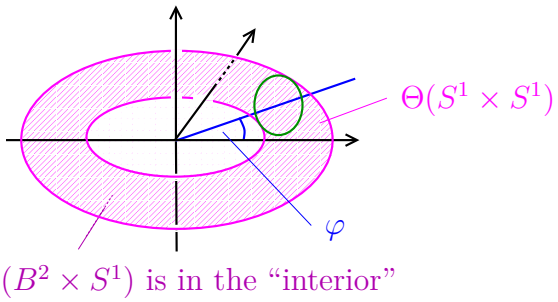
is a smooth embedding. This (or obvious variations thereof) is usually viewed as the standard smooth embedding of the solid torus into $\mathbb{R}^3$.

The above embedding and the Sphere–Solid Tori-Decomposition Lemma 111.6 give us two smooth embeddings of the solid torus $\overline{B}^2 \times S^1$ into $\mathbb{R}^3$. The following lemma says that these two smooth embeddings are essentially the same.

Lemma 111.8. (Torus Embeddings–Isotopic Lemma) Let $\Omega\colon S^3 \setminus \{N\} \to \mathbb{R}^3$ be the stereographic projection as defined in the Stereographic Projection Lemma 4.24. We

[1159] What is the disk attached to the point ∞?





consider the two smooth embeddings[1160] [1161]

$$\overline{B}^2\times S^1 \xrightarrow{\Theta} \mathbb{R}^3 \qquad \overline{B}^2\times S^1 \xrightarrow{\Xi} S^3\setminus\{N\} \xrightarrow{\Omega} \mathbb{R}^3$$

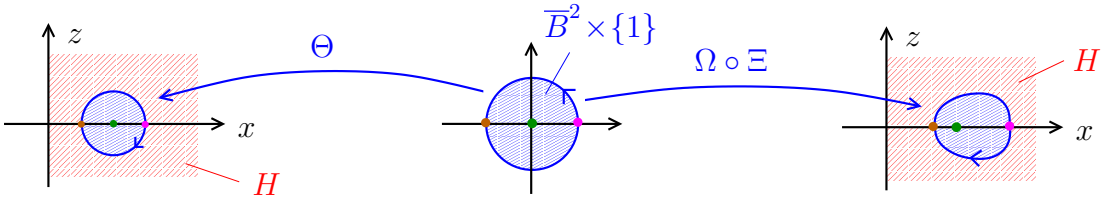
$$((x,y),e^{\mathrm{i}\varphi}) \mapsto \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1+\frac{x}{3}\\ 0\\ -\frac{y}{3}\end{pmatrix} \quad \text{and} \quad (z,w)\mapsto\left(\frac{w}{\sqrt{|z|^2+|w|^2}},\frac{z}{\sqrt{|z|^2+|w|^2}}\right)$$

provided by the Torus Embedding Lemma 7.11 and provided by the Sphere–Solid Tori-Decomposition Lemma 111.6. There exists a diffeotopy G of S^3 rel $\{(0,0)\}\times S^1$ with $G_0=\mathrm{id}$ and $G_1\circ\Theta=\Omega\circ\Xi$.

Convention 111.9. In all the future examples we will not distinguish between the two smooth embeddings of the solid torus $\overline{B}^2\times S^1\to\mathbb{R}^3$ that are given in the Torus Embeddings–Isotopic Lemma 111.8.

Proof. [1162] In the following we will give an explicit description of a diffeotopy from Θ to $\Omega\circ\Xi$. First note, that by the Isotopy Extension Theorem 44.2 it suffices to show that there exists a smooth isotopy rel $\{(0,0)\}\times S^1$ from Θ to $\Omega\circ\Xi$.

We write $H=\{(x,0,z)\in\mathbb{R}^3\,|\,x>0\}$. Let us first consider the restrictions of Θ and $\Omega\circ\Xi$ to the disk $\overline{B}^2\times\{1\}$. Note that in both cases the image lies in H.



It follows from the Isotopic-to-Differential Lemma 57.25, the Ball-Isotopic-to-Diffeomorphism Lemma 57.27 and the Smooth Isotopy-Equivalence Relation Proposition 41.5 that there exists a smooth isotopy $F\colon\overline{B}^2\times\{1\}\times[0,1]\to H$ rel $\{(0,0,1)\}$ from Θ to $\Omega\circ\Xi$.

[1160] Note that $\Xi\colon\overline{B}^2\times S^1\to S^3$ is just the composition of the swap $\overline{B}^2\times S^1\to S^1\times\overline{B}^2$ and the smooth embedding $\Phi\colon S^1\times\overline{B}^2\to S^3$.

[1161] As a reality check note that it follows from the Stereographic Projection Lemma 4.24 that Θ and $\Omega\circ\Xi$ agree on $\{(0,0)\}\times S^1$.

[1162] The lemma follows also from the Link Tubular Map Theorem 108.29 together with the observation that both smooth embeddings have "self-linking number = 0".

The desired isotopy is given by "rotating F" around the z-axis. Slightly more precisely it is given by the map

$$\begin{array}{rcl} \overline{B}^2 \times S^1 \times [0,1] & \to & \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi), t) & \mapsto & \text{rotation of } F((x,y),1,t) \in \mathbb{R}^3 \text{ by } \varphi \text{ around the } z\text{-axis.} \end{array}$$ ■

111.3. **Fundamental groups of complements of the trivial knot.** In this and the following sections we will use the description of S^3 from the previous section to give precise descriptions of the complements of the trivial knot and of torus knots (which contains the trefoil as a special case). These descriptions will allow us to determine the fundamental groups of complements of the trivial knot and of torus knots.

First let us recall Definition 108.4:

Definition 111.10. The trivial knot is defined as

$$\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2+y^2=1\} \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3.$$

Now we turn to the calculation of the fundamental groups of the complement of the trivial knot. We do this in the following two propositions:

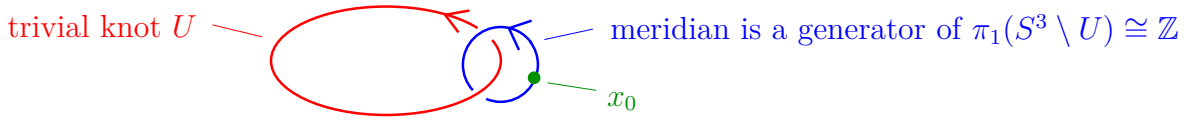
Proposition 111.11. (Trivial Knot–Complement Proposition) Let U be the trivial knot.

(1) Under the identification $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ from page 2097 the trivial knot corresponds to $U = \{(z,0) \in \mathbb{C}^2 \,|\, |z|^2 = 1\}$.
(2) $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}$.

Proposition 111.12. (Trivial Knot-π_1-proposition) Let U be the trivial knot. We have an isomorphism

$$\pi_1(S^3 \setminus U) \cong \mathbb{Z}.$$

In fact a generator of $\pi_1(S^3 \setminus U)$ is given by a meridian (see Definition 109.1 for the definition of a meridian).



Remark. Let $U \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be the trivial knot. In the Trivial Knot–Complement Proposition 111.11 we just showed that $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}$. On the other hand, in Exercise 111.4 we will see that $\mathbb{R}^3 \setminus U$ is homotopy equivalent to the wedge $S^1 \vee S^2$. Thus we see that it is much easier to understand the complement of the trivial knot in S^3 than the complement of the trivial knot in $\mathbb{R}^3$. This is one instance where it pays off to study knots in S^3 instead of $\mathbb{R}^3$. □

Proof of the Trivial Knot–Complement Proposition 111.11. Let U be the trivial knot.

(1) This statement follows immediately from the Stereographic Projection Lemma 108.1.
(2) Note that

$$\begin{array}{rcl} S^3 \setminus U & = & \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \setminus \{(z,0) \mid |z| = 1\} \\ & = & \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \text{ and } |w|^2 > 0\}. \end{array}$$

[1162]The trivial knot is often referred to as the "unknot". This explain why we call the trivial knot "U".

Now basically the same argument as in the proof of the Sphere–Solid Tori-Decomposition Lemma 111.6 shows that the map

$$\begin{array}{rcl} \Phi\colon S^1 \times \mathbb{C} & \to & S^3 \setminus U = \left\{(z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 > 0\right\} \\ (a,b) & \mapsto & \left(\frac{b}{\sqrt{|a|^2+|b|^2}}, \frac{a}{\sqrt{|a|^2+|b|^2}}\right) \end{array}$$

is a diffeomorphism. ■

Proof of the Trivial Knot-π_1-Proposition 111.12. Let U be the trivial knot. Furthermore let $f\colon S^1 \to S^1 \times \mathbb{C}$ be the embedding given by $z \mapsto (z,0)$. We now consider the following isomorphisms:

$$\pi_1(S^3 \setminus U) \xleftarrow[\uparrow]{\Phi_*} \pi_1(S^1 \times \mathbb{C}) \xleftarrow[\uparrow]{f_*} \pi_1(S^1) \xleftarrow[\uparrow]{\cong} \mathbb{Z}.$$

isomorphism since the map Φ from the Trivial Knot–Complement Proposition 111.11 is a homeomorphism	isomorphism by the Homotopy Equivalence-π_1-Proposition 90.9 since f is a homotopy equivalence	explicit isomorphism that is given by the Circle-π_1-Theorem 88.22

By the Circle-π_1-Theorem 88.22 we know that a generator of $\pi_1(S^1)$ is represented by the loop $[0,1] \to S^1$ that is given by $t \mapsto \exp(2\pi i t)$. One can easily verify, using the notation from the Trivial Knot–Complement Proposition 111.11 that the image $\Phi(f(S^1))$ is a meridian of U. ■

111.4. **Torus knots.** In this section we will introduce the family of torus knot, which as a special case contains the trefoil knot.

Definition 111.13.

(1) Let $p, q \in \mathbb{Z}$ be coprime. We refer to the image of the map

$$\begin{array}{rcl} \varphi_{p,q}\colon \mathbb{R} & \to & S^3 \\ t & \mapsto & \left(\frac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot p\cdot t), \frac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot q\cdot t)\right) \end{array}$$

as the[11631164] (p,q)-torus knot $T(p,q)$[1165]. We equip the torus knot with the orientation which turns the map $\varphi_{p,q}\colon \mathbb{R} \to T(p,q)$ into an orientation-preserving local diffeomorphism.[1166]

(2) We define the left-handed trefoil to be the $(2,3)$-torus knot and we define the right-handed trefoil to be the $(-2,3)$-torus knot.

Visualization.

(1) We give an alternative description of torus knots: Let $p, q \in \mathbb{Z}$ be coprime. We consider the oriented smoothly embedded closed curve $C = \{(p\cdot t, q\cdot t) \mid t \in \mathbb{R}\}$ on the torus $\Sigma = ([0,2\pi] \times [0,2\pi])/\sim$. By definition the (p,q)-torus knot $T(p,q)$ is the image of C

[1163] We will show in the Torus Knot Lemma 111.14 that $T(p,q)$ is indeed a knot.

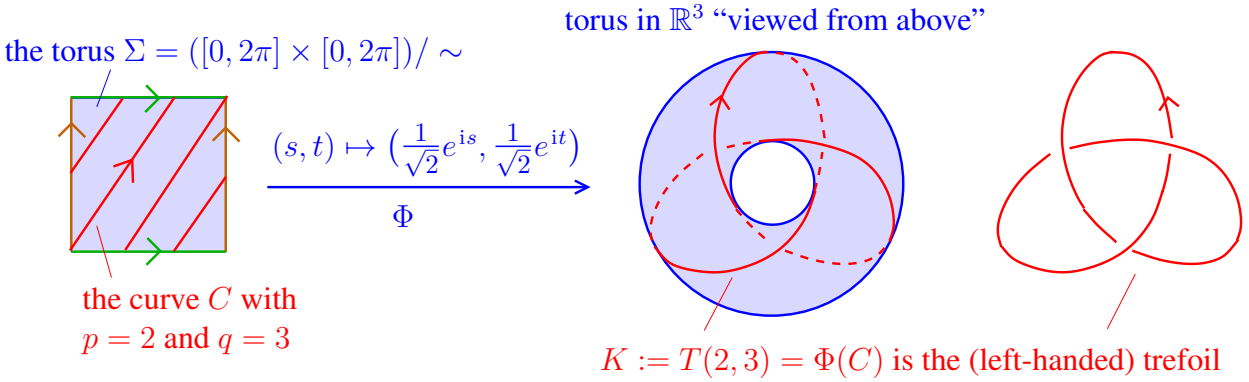
[1164] Some authors, see e.g. [**BZH2014**, Definition 3.36] exclude the trivial knot.

[1165] The torus knots lie on the standard torus $S^1 \times S^1 \subseteq S^3$, hence the name. In Exercise **??** we will see that any knot on the standard torus $S^1 \times S^1 \subseteq S^3$ is smoothly isotopic to a torus knot $T(p,q)$.

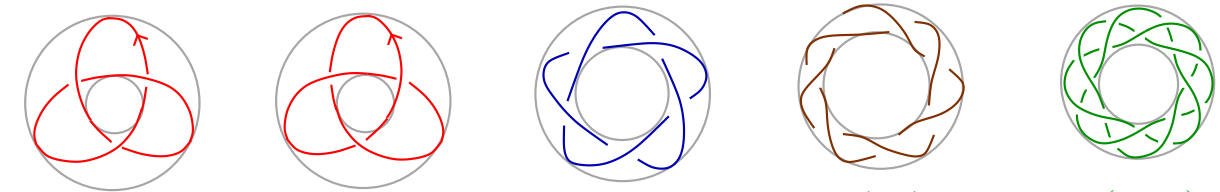
[1166] It follows from the Torus Embeddings–Isotopic Lemma 111.8 that this definition of the (p,q)-torus knot is consistent with Definition 62.16.

under the map

$$\begin{aligned} \Phi\colon \Sigma = ([0,2\pi]\times[0,2\pi])/\sim &\to S^3 \\ (s,t) &\mapsto \big(\tfrac{1}{\sqrt{2}}\exp(\mathrm{i}s), \tfrac{1}{\sqrt{2}}\exp(\mathrm{i}t)\big). \end{aligned}$$



(2) In the following figure we illustrate several examples of torus knots:[1167]



(3) A visualization of the (p,q)-torus knots is also given by Johannes Zander in

`https://www.desmos.com/3d/4ba6cb90d2`. □

The following lemma summarizes some key facts about torus knots:

Lemma 111.14. (Torus Knot Lemma)

(0) Each torus knot is indeed a knot.
(1) All torus knots $T(n,1)$ and $T(1,n)$ are smoothly isotopic to the trivial knot.
(2) For any coprime $p,q\in\mathbb{N}_0$ we have:
  (a) $T(-p,-q) = T(p,q)^{\mathrm{rev}}$.
  (b) $T(-p,q)$ and $T(p,-q)$ are mirrors of $T(p,q)$.
  (c) $T(p,q)$ is reversible, i.e. it is smoothly isotopic to $T(p,q)^{\mathrm{rev}}$.[1168]
  (d) $T(p,q)$ is smoothly isotopic to $T(q,p)$.

Proof.

[1167] The author knows, from personal experience, that it is quite tricky to draw the torus knots correctly. It is very easy to fall into the trap of drawing the mirror image or of getting the orientation wrong. It is also not clear, whether the literature is entirely consistent.

[1168] This shows in particular that the trefoil is reversible. We verified this statement before in Exercise 108.2.

(0) Let $p, q \in \mathbb{Z}$ be coprime. We consider again the map

$$\begin{array}{rcl} \varphi_{p,q}\colon \mathbb{R} & \to & S^3 \\ t & \mapsto & \Big(\frac{1}{\sqrt{2}} \exp(\mathrm{i} \cdot p \cdot t), \frac{1}{\sqrt{2}} \exp(\mathrm{i} \cdot q \cdot t) \Big). \end{array}$$

One can easily verify that this map is an immersion and that it descends to a smooth embedding $\mathbb{R}/2\pi\mathrm{i}\mathbb{Z} \to S^3$, whose image is precisely the (p,q)-torus knot $T(p,q)$. It follows from the Smooth Embedding Theorem 30.15 that the image $\varphi_{p,q}(\mathbb{R}^3) = T(p,q)$ is indeed a smooth submanifold of S^3 that is diffeomorphic to S^1. In other words, $T(p,q)$ is indeed a knot.

Using the Orientation-Action Proposition 31.44 (1) one can easily show that we can indeed equip the torus knot with a unique orientation which turns the above map $\varphi_{p,q}\colon \mathbb{R} \to T(p,q)$ into an orientation-preserving local diffeomorphism.

(1) We will prove this statement in Exercise 111.8.

(2) (a) It is clear that as unoriented submanifolds we have $T(-p,-q) = T(p,q)$. But the two maps $\varphi_{-p,-q}\colon \mathbb{R} \to S^3$ and $\varphi_{p,q}\colon \mathbb{R} \to S^3$ differ by a minus sign. This shows that as oriented knots we have $T(-p,-q) = T(p,q)^{\mathrm{rev}}$.

(b) First we consider the map

$$\begin{array}{rcl} g\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (\overline{z},w). \end{array}$$

First note that we have $T(-p,q) = g(T(p,q))$ as oriented knots. Next note that, with respect to real coordinates, g is just the reflection in the $(x_2 = 0)$-hyperplane. It follows that $T(-p,q) = g(T(p,q)) = T(p,q)^{\mathrm{mir}}$ as oriented knots. Similarly we see that $T(p,-q)$ is obtained from $T(p,q)$ by reflection in the $(x_4 = 0)$-hyperplane.

(c) For the purpose of this proof we denote by $\rho_i\colon S^3 \to S^3$ the reflection in the $(x_i = 0)$-hyperplane. Note that as oriented knots we have

$$\begin{array}{ccccccccc} & & \text{by (a)} & & \text{by the proof of (b)} & & \text{by the proof of (b)} & & \\ T(p,q)^{\mathrm{rev}} & \overset{\downarrow}{=} & T(-p,-q) & \overset{\downarrow}{=} & \rho_2(T(p,-q)) & \overset{\downarrow}{=} & \rho_2(\rho_4(T(p,q))) & & \\ & & & & & \underset{\uparrow}{\cong} & \rho_4(\rho_4(T(p,q))) & \underset{\uparrow}{=} & T(p,q). \\ & & & & & \text{smoothly isotopic by the } S^3\text{-Link Mirror Lemma 108.24} & & \text{since } \rho_4 \circ \rho_4 = \mathrm{id} & \end{array}$$

(d) We consider the map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (w,z) \end{array}$$

which with real coordinates we can write as

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ v & \mapsto & \underbrace{\begin{pmatrix} 0 & \mathrm{id}_2 \\ \mathrm{id}_2 & 0 \end{pmatrix}}_{=:A} \cdot v. \end{array}$$

Evidently $T(q,p) = f(T(p,q))$ as oriented knots. Note that $A \in \mathrm{SO}(4)$. By the Matrix Group-π_0-Proposition 4.54 we know $\mathrm{SO}(4)$ is path-connected. Using this observation we showed in the Diffeotopies-via-Matrices Lemma 41.12 that f is diffeotopic to the identity. It follows that $T(q,p)$ is smoothly isotopic to $T(p,q)$. ∎

111.5. **Interlude: Preparations for calculating fundamental groups.** In the following sections we will use the Seifert–van Kampen Theorem 110.8 to perform several calculations of fundamental groups. To prepare the ground for these calculations it is helpful to introduce some notation.

First of all we have the following fairly general lemma, which overlaps with the Oriented Submanifold–Element-of-π_1-Corollary 107.12:

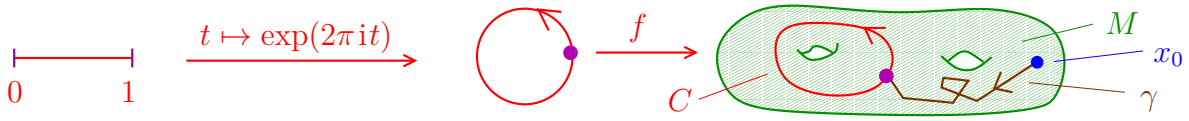
Lemma 111.15. (Loop-π_1-Lemma) Let M be a path-connected smooth manifold and let $x_0 \in M$.

(1) Let $C \subseteq M$ be an oriented smoothly embedded closed curve[1169] in M. We make two choices:
- We pick an orientation-preserving diffeomorphism $f\colon S^1 \to C$.
- Since M is path-connected we can pick a path $\gamma\colon [0,1] \to M$ from x_0 to $f(1) \in C$.

The element
$$[\ \underbrace{\gamma * (t \mapsto f(\exp(2\pi\,\mathrm{i}\,t))) * \overline{\gamma}}_{\substack{\text{the loop in } (M,x_0) \text{ that is given}\\ \text{by the concatenation of}\\ (1)\quad \text{the path } \gamma \text{ from } x_0 \text{ to } \gamma(1)\\ (2)\quad \text{the loop } t \mapsto f(\exp(2\pi\,\mathrm{i}\,t)) \text{ in } \gamma(1)\\ (3)\quad \text{the "inverse" path } \overline{\gamma} \text{ from } \gamma(1) \text{ to } x_0}}\ ]\ \in\ \pi_1(M,x_0)$$
is well-defined up to conjugation.

(2) Let $C, \widetilde{C} \subseteq M$ be two oriented submanifolds of M that are diffeomorphic to S^1. If C and $\widetilde{C}$ are smoothly isotopic as oriented smooth manifolds, then C and $\widetilde{C}$ define the same conjugacy class in $\pi_1(M,x_0)$.

(3) Let $C \subseteq M$ be an oriented submanifold of M that is diffeomorphic to S^1. If we denote by C^{rev} the same submanifold with opposite orientation, then C and C^{rev} define elements of $\pi_1(M,x_0)$ that are inverses of one another.



Sketch of proof. The first two statements of the lemma follow fairly easily from the Homotopies-of-Loops Proposition 88.29, the Diffeotopies-of-S^1-Proposition 42.8 and the Path-Homotopy–Concatenation Lemma 88.8. The last statement follows from the discussion of the group structure in the π_1-Group Proposition 88.12. We leave it to the reader to fill in the details. ∎

The Loop-π_1-Lemma 111.15 allows us to introduce the following slightly dangerous notation:

Notation 111.16. Let M be a path-connected smooth manifold, let $x_0 \in M$ be a base point and let $C \subseteq M$ be an oriented smoothly embedded closed curve in M. In the following we denote by $[C] \in \pi_1(M,x_0)$ the element, well-defined up to conjugation, that we introduced in the Loop-π_1-Lemma 111.15.

In the applications of Seifert–van Kampen Theorem 110.8, we will make use of the following almost self-explanatory notation:

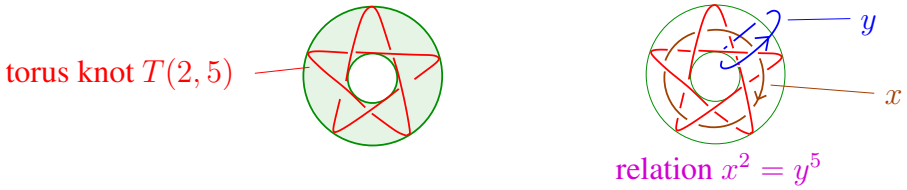
[1169] We refer to Definition 107.1 for the definition of an oriented closed curve.

Notation 111.17. Let G and $\widetilde{G}$ be two groups and let $\mu \in G$ and $\widetilde{\mu} \in \widetilde{G}$ be elements. We denote by $G *_{\mu=\widetilde{\mu}} \widetilde{G}$ the amalgamated product corresponding to the group homomorphisms $\mathbb{Z} \to G$ and $\mathbb{Z} \to \widetilde{G}$ that are given by $1 \mapsto \mu$ and $1 \mapsto \widetilde{\mu}$.

111.6. **Fundamental groups of complements of torus knots.** The following proposition shows that the fundamental groups of torus knot complements have very elegant and concise presentations:

Proposition 111.18. (Torus Knot-π_1-Proposition) For any coprime $p, q \in \mathbb{Z}$ we have

$$\pi_1(S^3 \setminus T(p,q)) \;\cong\; \langle x, y \mid x^p = y^q \rangle.$$



Example.

(1) As we pointed out above, the torus knots $T(b,1)$ and $T(1,b)$ are just the trivial knot in disguise. Using Exercise 110.11 one can easily show the groups $\langle x, y \mid x^b = y^1 \rangle$ and $\langle x, y \mid x^1 = y^b \rangle$ are both isomorphic to $\mathbb{Z}$. Thus the Torus Knot-π_1-Proposition 111.18 actually contains most of the results of the Trivial Knot-π_1-Proposition 111.12.

(2) Let $K = T(2,3)$ be the trefoil. It follows from the Torus Knot-π_1-Proposition 111.18 that $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$. □

Proof of the Torus Knot-π_1-Proposition 111.18. First recall that by the Sphere–Solid Tori-Decomposition Lemma 111.6 there exists a diffeomorphism

$$\Phi\colon S^3 \xrightarrow{\cong} \underbrace{(S^1 \times \overline{B}^2)}_{=:A} \cup_{S^1 \times S^1 = S^1 \times S^1} \underbrace{(\overline{B}^2 \times S^1)}_{=:B}$$

which is, up to scaling by a factor of $\sqrt{2}$, the identity on $T := S^1 \times S^1$. Thus it remains to prove the following claim:

Claim. For $K := \{(\exp(ps\mathrm{i}), \exp(qs\mathrm{i})) \mid s \in \mathbb{R}\} \subseteq T \subseteq A \cup B$ there exists an isomorphism $\pi_1((A \cup B) \setminus K) \cong \langle x, y \,|\, x^p = y^q \rangle$.

Proof. We start out with the following simple observations:

(a) A and B are codimension-zero submanifolds of $A \cup B$.
(b) A and B are closed subsets of $A \cup B$.
(c) We have $\partial A = \partial B = A \cap B = T$.

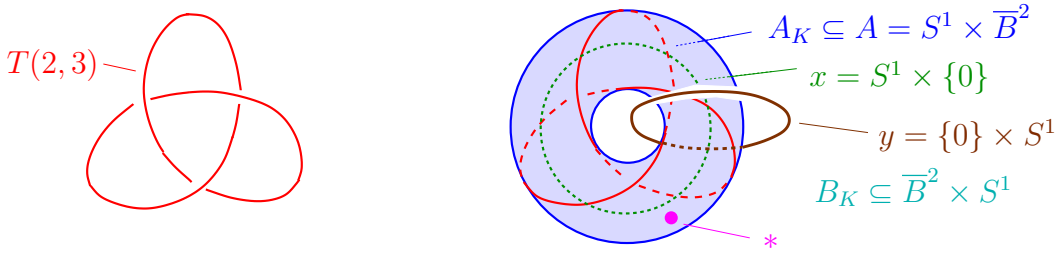
We write $A_K := A \setminus K, B_K := B \setminus K$ and $T_K := T \setminus K$. Since K is a closed subset of $A \cup B$ we obtain easily from the above that the following statements hold:

(a′) A_K and B_K are codimension-zero submanifolds of $\Sigma_K := (A \cup B) \setminus K$.
(b′) A_K and B_K are closed subsets of Σ_K.
(c′) $\partial A_K = \partial B_K = A_K \cap B_K = T_K$.

Next note that the obvious deformation retraction from $S^1 \times \overline{B}^2$ to $x := S^1 \times \{0\}$ restricts to a deformation retraction from A_K to $x = S^1 \times \{0\}$. Similarly we see that $y := \{0\} \times S^1$

is a deformation retract of B_K. Finally note that T_K admits a deformation retraction to a parallel copy of K. More precisely, for any $\epsilon > 0$ sufficiently small, there exists a deformation retraction from T_K to the loop[1170]

$$z \;=\; \{(\exp(ps\,\mathrm{i}), \exp((qs+\epsilon)\,\mathrm{i})) \mid s \in \mathbb{R}\} \;\subseteq\; T_K.$$



We pick a base point $* \in T_K$. As we discussed in Notation 111.16 the loops corresponding to the circles x, y and z with the obvious orientation define elements of the various fundamental groups. By an abuse of notation we denote these elements also by x, y and z. We consider the inclusion-induced homomorphisms

$$\begin{array}{ccccc} \pi_1(A_K,*) & \leftarrow & \pi_1(T_K,*) & \rightarrow & \pi_1(B_K,*) \\ i_*(g) & \leftarrow\!\shortmid & g & \shortmid\!\rightarrow & j_*(g). \end{array}$$

It follows easily from the Homotopy Equivalence-π_1-Proposition 90.9 together with the Circle-π_1-Theorem 88.22 and the discussion on page 1760 that these maps correspond to the maps

$$\begin{array}{ccccc} \langle x\rangle & \leftarrow & \langle z\rangle & \rightarrow & \langle y\rangle \\ x^p & \leftarrow\!\shortmid & z & \shortmid\!\rightarrow & y^q. \end{array}$$

Note that it follows from (a′), (b′) and (c′) that we can apply the Seifert–van Kampen Theorem 110.8 to $\Sigma_K = A_K \cup B_K$. We conclude this calculation in two different ways, using the two different formulations of the Seifert–van Kampen Theorem 110.8:

(i) Using Formulation (i) of the Seifert–van Kampen Theorem 110.8 we get:

$$\pi_1(\Sigma_K,*) \underset{\uparrow}{=} \pi_1(A_K,*) *_{\pi_1(T_K,*)} \pi_1(B_K,*) \underset{\uparrow}{\cong} \langle x\rangle *_{x^p=y^q} \langle y\rangle \underset{\uparrow}{=} \langle x,y \mid x^p = y^q\rangle.$$

by the Seifert–van Kampen Theorem 110.8 (i); by the above discussion; by the Amalgamated Product Presentation Lemma 100.20

(ii) Using Formulation (iii) of the Seifert–van Kampen Theorem 110.8 we get more succinctly:

$$\pi_1(\Sigma_K,*) \underset{\uparrow}{=} \langle x,y \mid x^p = y^q\rangle.$$

by the Seifert–van Kampen Theorem 110.8 (iii) together with the above discussion

■

Let $p, q \in \mathbb{Z}$ be coprime. In the Torus Knot-π_1-Proposition 111.18 we just showed that

$$\pi_1(S^3 \setminus T(p,q)) \;\cong\; \langle x,y \mid x^p = y^q\rangle.$$

[1170]This can be seen as follows: It is elementary to verify that the map

$$\begin{array}{ccc} \Theta\colon (\mathbb{R}/2\pi pq\mathbb{Z}) \times (0,1) & \rightarrow & T_K = (S^1 \times S^1) \setminus K \\ (s,t) & \mapsto & (\exp(ps\,\mathrm{i}), \exp(qs+t)\,\mathrm{i}) \end{array}$$

is a diffeomorphism. To the left we see that $(\mathbb{R}/2\pi pq\mathbb{Z}) \times \{\epsilon\}$ is a deformation retract of $(\mathbb{R}/2\pi pq\mathbb{Z}) \times (0,1)$. Hence $\Theta((\mathbb{R}/2\pi pq\mathbb{Z}) \times \{\epsilon\}) = \{(\exp(ps\,\mathrm{i}, \exp((qs+\epsilon)\,\mathrm{i})) \mid s \in \mathbb{R}\}$ is a deformation retract of T_K.

The question is, what is this result trying to tell us? So far we know two tricks for distinguishing groups:

(1) We can consider abelianizations, which we introduced in Definition 110.9.
(2) We can use the Presentation–Homomorphism Lemma 100.13 to study homomorphisms to known (finite) groups.

Let us first consider the abelianization of the group $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \mid x^p = y^q \rangle$. Using the Abelianization-of-Presentation Proposition 100.17 one can easily show (see Exercise 110.13) that for any coprime integers $p, q \in \mathbb{Z}$ the abelianization of the group $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \mid x^p = y^q \rangle$ is isomorphic to $\mathbb{Z}$.[1171] We see that, unfortunately, the abelianizations of these groups do not help us in our goal of distinguishing torus knots.

So let us turn to the second approach. For the time being we specialize to the task of distinguishing the trivial knot U from the trefoil $K = T(2,3)$. In the following lemma we will use finite quotients to show that the group $\pi_1(S^3 \setminus U) = \mathbb{Z}$ is not isomorphic to $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3 \rangle$.[1172]

Proposition 111.19. (Trefoil–Non-Trivial Proposition)
(1) The group $\langle x, y \mid x^2 = y^3 \rangle$ admits an epimorphism onto the permutation group S_3, in particular the group $\langle x, y \mid x^2 = y^3 \rangle$ is non-abelian and it is not isomorphic to $\mathbb{Z}$.
(2) The trefoil is not smoothly isotopic to the trivial knot.

Remark. The first proof of the Trefoil–Non-Trivial Proposition 111.19 using fundamental groups was given by Heinrich Tietze, see [**Tie1906**, p. 844] and [**Tie1908**, p. 82]. □

Proof.

(1) It follows from the Presentation–Homomorphism Lemma 100.13 and a straightforward calculation that there exists a unique homomorphism

$$\Phi\colon \langle x, y \mid x^2 = y^3 \rangle \quad \to \quad S_3 = \text{permutation group on three elements}$$

with

$$\Phi(x) \quad = \quad \sigma \quad \coloneqq \quad (1\,2) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and

$$\Phi(y) \quad = \quad \tau \quad \coloneqq \quad (1\,2\,3) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The permutations σ and τ do not commute, hence the image of Φ is a non-abelian subgroup of S_3.[1173] On the other hand the abelian group $\mathbb{Z}$ cannot admit an epimorphism onto a non-abelian group. This shows that the groups $\langle x, y \mid x^2 = y^3 \rangle$ and $\mathbb{Z}$ are not isomorphic.
(2) Evidently it follows from (1), the Trivial Knot-π_1-Proposition 111.12 and the Isotopic Link-π_1-Lemma 111.4 that the trefoil is not smoothly isotopic to the trivial knot. ■

Remark.

[1171] In fact that is not a coincidence: In the Link Group-Abelianization Proposition 113.9 we will see that for any knot J the abelianization of $\pi_1(S^3 \setminus J)$ is isomorphic to $\mathbb{Z}$.
[1172] Note that the following lemma gives the long overdue answer to Question 108.9.
[1173] In fact it is easy to see that the only non-abelian subgroup of S_3 is S_3 itself, i.e. Φ is an epimorphism onto S_3.

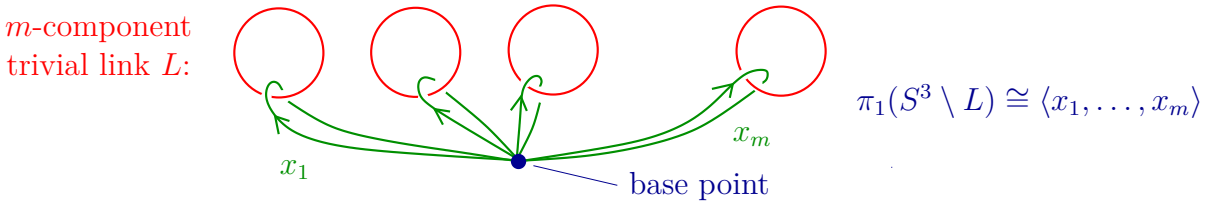
(1) In Exercise 111.17 we will get to know two ad hoc methods to distinguish all torus knots.
(2) In Chapter 117 we will introduce the Alexander polynomial which gives us a more systematic way to extract information from fundamental groups of link complements. We will then be able to distinguish all torus knots. □

111.7. **Splits unions and fundamental groups.** In this section we discuss the split union of links and the effect this construction has on fundamental groups of link complements:

Proposition 111.20. (Split Union-π_1-Proposition) Let L and $\widetilde{L}$ be two oriented links in S^3. We consider the split union $L \sqcup \widetilde{L}$ as introduced in Definition 109.5. There exists an isomorphism[1174]

$$\pi_1(S^3 \setminus (L \sqcup \widetilde{L})) \;\cong\; \pi_1(S^3 \setminus L) * \pi_1(S^3 \setminus \widetilde{L}).$$

Example. Let L be the trivial m-component link. Evidently L is the split union of m copies of the trivial knot.



It follows from the Split Union-π_1-Proposition 111.20 and the Trivial Knot-π_1-Proposition 111.12 that $\pi_1(S^3 \setminus L)$ is a free group on m generators. In the figure above we show generators. These can be obtained easily from the proof. □

Proof. We start out with a basic claim:

Claim. Let M be a simply connected 3-dimensional smooth manifold and let $A, B \subseteq M$ be two compact codimension-zero submanifolds such that $A \cap B = \partial A = \partial B$ is diffeomorphic to S^2. Then for any $x_0 \in A \cap B$ the inclusions $A \to M$ and $B \to M$ induce an isomorphism

$$\pi_1(A, x_0) * \pi_1(B, x_0) \;\to\; \pi_1(M, x_0).$$

Proof. We have

$$\pi_1(A, x_0) * \pi_1(B, x_0) \;\underset{\text{since } \pi_1(A \cap B) \cong \pi_1(S^2) = \{e\}}{\overset{\cong}{\uparrow}}\; \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \;\underset{\text{by the Seifert-van Kampen Theorem 110.8}}{\overset{\cong}{\uparrow}}\; \pi_1(M, x_0).$$

⊞

Now we turn to the actual proof of the proposition. Thus let L and $\widetilde{L}$ be two oriented links in S^3.

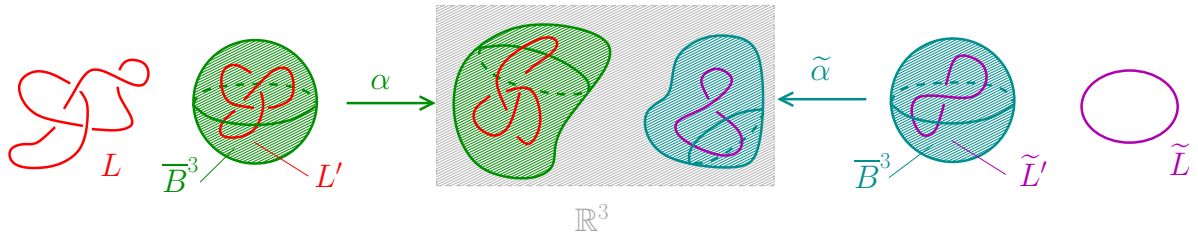
- First recall that it follows from the Link-in-$\overline{B}^3$-$\mathbb{R}^3$-S^3-Proposition 108.15 that there exist oriented links $L' \subseteq \overline{B}^3$ and $\widetilde{L}' \subseteq \overline{B}^3$ such that under the standard smooth embedding $\overline{B}^3 \to \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ the link L' is smoothly isotopic to L and such that the link $\widetilde{L}'$ is smoothly isotopic to $\widetilde{L}$.

[1174] It follows easily from the Manifold-with-Boundary-minus-Ball Proposition 247.14 that we have a homotopy equivalence $(S^3 \setminus (L \sqcup \widetilde{L})) \simeq (S^3 \setminus L) \vee (S^3 \setminus \widetilde{L}) \vee S^2$. This homotopy equivalence together with the Wedge-π_1-Proposition 99.10 gives an alternative proof of the Split Union-π_1-Proposition 111.20.

- Let $\alpha, \widetilde{\alpha}\colon \overline{B}^3 \to S^3$ be two orientation-preserving smooth embeddings with disjoint images.

By Definition 109.5 the split union is defined as

$$L \sqcup \widetilde{L} \;:=\; \alpha(L') \cup \widetilde{\alpha}(\widetilde{L}').$$



We now see that

$$\begin{aligned}
\pi_1(S^3 \setminus (L \sqcup \widetilde{L})) &= \pi_1(S^3 \setminus (\alpha(L') \cup \widetilde{\alpha}(\widetilde{L}'))) \\
&= \pi_1((S^3 \setminus (\alpha(B^3) \cup \alpha(B^3))) \cup_{\alpha(S^2)} \alpha(\overline{B}^3 \setminus L') \cup_{\widetilde{\alpha}(S^2)} \widetilde{\alpha}(\overline{B}^3 \setminus \widetilde{L}')) \\
&= \pi_1(S^3 \setminus (\alpha(B^3) \cup \widetilde{\alpha}(B^3))) * \pi_1(\alpha(\overline{B}^3 \setminus L')) * \pi_1(\widetilde{\alpha}(\overline{B}^3 \setminus \widetilde{L}')) \\
&\quad \uparrow \text{ by the claim, applied twice} \\
&\cong \pi_1(S^3) * \pi_1(S^3 \setminus L') * \pi_1(S^3 \setminus \widetilde{L}') \qquad \cong \pi_1(S^3 \setminus L) * \pi_1(S^3 \setminus \widetilde{L}). \\
&\quad \uparrow \text{ by the claim applied four times, here we use that by the Stereographic Projection Lemma 108.1 we know that } \pi_1(S^3 \setminus \overline{B}^3) = 0 \\
&\qquad\qquad \uparrow \text{ by the Link-}\pi_1\text{-Isomorphism Lemma 111.3 and since } \pi_1(S^3) = \{e\}
\end{aligned}$$

■

For the record we state the following theorem which can be viewed as a converse to the Split Union-π_1-Proposition 111.20.

Theorem 111.21. (Link Kneser Theorem) Let L be an oriented link in S^3. If there exist groups G_1 and G_2 and an isomorphism $\pi_1(S^3 \setminus L) \cong G_1 * G_2$, then we can write $L = L_1 \sqcup L_2$, where L_1 and L_2 are unions of components of L, such that $\pi_1(S^3 \setminus L) \cong G_i$ for $i = 1, 2$.

Proof. The theorem is a relative straightforward consequence of the more general Kneser Theorem **??** in 3-manifold topology. We will discuss the proof of the Link Kneser Theorem 111.21 on page **??**. ■

111.8. **The Hopf link.** In this very short section we will formally introduce the Hopf link and we will study a few of its basic properties:

Definition 111.22. We refer to

$$H \;:=\; (\{0\} \times S^1) \sqcup (S^1 \times \{0\}) \;\subseteq\; (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2) \;=_{\text{i}}\; S^3$$

as the Hopf link.

The following propositions are the analog of the Trivial Knot–Complement Proposition 111.11 and the Trivial Knot-π_1-Proposition 111.12:

Proposition 111.23. (Hopf Link-Complement-Proposition)

(1) The Hopf link H is smoothly isotopic to the link

$$\{(x,y,0)\in\mathbb{R}^3\,|\,x^2+y^2=1\}\cup\{(x,0,z)\in\mathbb{R}^3\,|\,(x-1)^2+z^2=1\}\ \subseteq\ \mathbb{R}^3\subseteq S^3.$$

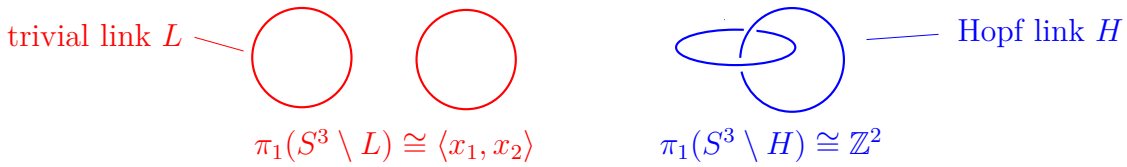
(2) $S^3\setminus H$ is diffeomorphic to $S^1\times S^1\times(-1,1)$.
(3) $\pi_1(S^3\setminus H)\cong\mathbb{Z}^2$.
(4) The Hopf link is not smoothly isotopic to the trivial 2-component link.

Proposition 111.24. (Hopf Link–π_1-Proposition)

(1) The Hopf link H is smoothly isotopic to the link

$$\{(x,y,0)\in\mathbb{R}^3\,|\,x^2+y^2=1\}\cup\{(x,0,z)\in\mathbb{R}^3\,|\,(x-1)^2+z^2=1\}\ \subseteq\ \mathbb{R}^3\subseteq S^3.$$

(2) $S^3\setminus H$ is diffeomorphic to $S^1\times S^1\times(-1,1)$.
(3) $\pi_1(S^3\setminus H)\cong\mathbb{Z}^2$.
(4) The Hopf link is not smoothly isotopic to the trivial 2-component link.



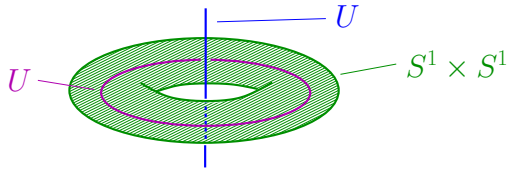
Sketch of proof of the Hopf Link-Complement-Proposition 111.23.

(1) This statement follows fairly easily from the identifications given by the Sphere–Solid Tori-Decomposition Lemma 111.6 and the Stereographic Projection Lemma 108.1.
(2) We will, just for fun, give two slightly different arguments:
- One can easily verify that the following map is a diffeomorphism:

$$\overbrace{((\overline{B}^2\setminus\{0\})\times S^1)\underset{S^1\times S^1}{\cup}(S^1\times(\overline{B}^2\setminus\{0\}))}^{=((\overline{B}^2\times S^1)\cup_{S^1\times S^1}(S^1\times\overline{B}^2))\setminus\text{Hopf link}}\ \to\ S^1\times S^1\times(-1,1)$$

$$[(z,w)]\ \mapsto\ \begin{cases}(\frac{z}{|z|},w,|z|-1), & \text{if } (z,w)\in(\overline{B}^2\setminus\{0\})\times S^1,\\ (z,\frac{w}{|w|},1-|w|), & \text{if } (z,w)\in S^1\times(\overline{B}^2\setminus\{0\}).\end{cases}$$

- We write $U:=\{0\}\times S^1$ and $\widetilde{U}:=S^1\times\{0\}$. Then we have $H:=U\cup\widetilde{U}$ in the sphere $S^3=_{\mathrm{i}}(\overline{B}^2\times S^1)\cup_{S^1\times S^1}(S^1\times\overline{B}^2)$.



We now see that

$$\begin{array}{ccccc} S^3\setminus H & = & S^3\setminus(U\sqcup\widetilde{U}) & = & (S^3\setminus U)\setminus\widetilde{U} \\ & & & \cong & S^1\times(\mathbb{R}^2\setminus\{0\}) \end{array} \quad \begin{array}{c} \\ \cong \\ \cong \\ \cong \end{array} \quad \begin{array}{c} \\ (S^1\times\mathbb{R}^2)\setminus(S^1\times\{0\}) \\ S^1\times S^1\times(-1,1). \end{array}$$

by the diffeomorphism from the proof of the Trivial Knot–Complement Proposition 111.11

since $\mathbb{R}^2\setminus\{0\}$ is diffeomorphic to $S^1\times(-1,1)$

■

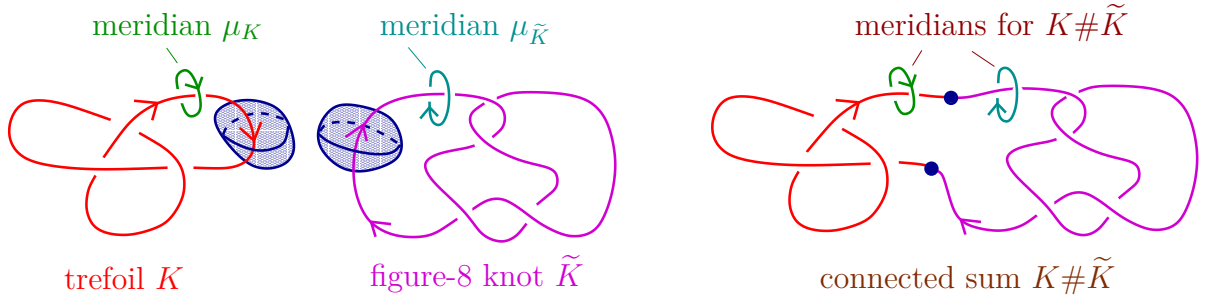
Proof of the Hopf Link–π_1-Proposition 111.24.

(1) We have the following isomorphisms:

$$\pi_1(S^3 \setminus H) \underset{\text{by the Hopf Link Complement Proposition 111.23}}{\cong} \pi_1(S^1 \times S^1 \times (-1,1)) \underset{\text{by the Homotopy Equivalence } \pi_1\text{-Proposition 90.9}}{\cong} \pi_1(S^1 \times S^1) \underset{\text{by the Torus-}\pi_1 \text{ Proposition 90.14}}{\cong} \mathbb{Z}^2.$$

(2) Let L be the trivial 2-component link. On page 2162 we showed that $\pi_1(S^3 \setminus L)$ is the free group on two generators. This group is non-abelian, so by the calculation in (1) it follows that $\pi_1(S^3 \setminus L)$ is not isomorphic to $\pi_1(S^3 \setminus H)$. It follows from the Isotopic Link-π_1-Lemma 111.4 that L is not smoothly isotopic to H. ■

111.9. **Connected sum of knots and fundamental groups.** Recall that in Definition 109.12 we introduced the connected sum $K\#\widetilde{K}$ of two oriented knots K and $\widetilde{K}$. For the reader's convenience we recall the definition (together with the meridians) in the figure below:



The goal in this section is to consider the effect of the connected sum operation of knots on fundamental groups. To formulate our main result we will need the following notation:

Notation 111.25. Let $K \subseteq S^3$ be an oriented knot. It follows from the Meridian Proposition 109.2 and the Loop-π_1-Lemma 111.15 that for any $x_0 \in S^3 \setminus K$ the meridian μ_K defines an element in $\pi_1(S^3 \setminus K, x_0)$ that is well-defined up to conjugation. If there is no danger of confusion, then we denote this element by $\mu_K \in \pi_1(S^3 \setminus K, x_0)$ as well.

Proposition 111.26. (Oriented Knot–Monoid-π_1-Proposition) Let K and $\widetilde{K}$ be two oriented knots in S^3. For any base points $x_0 \in S^3 \setminus K$ and $\widetilde{x}_0 \in S^3 \setminus \widetilde{K}$ and for any meridians μ_K and $\mu_{\widetilde{K}}$ there exists an isomorphism

$$\pi_1(S^3 \setminus (K\#\widetilde{K})) \cong \pi_1(S^3 \setminus K, x_0) *_{\mu_K = \mu_{\widetilde{K}}} \pi_1(S^3 \setminus \widetilde{K}, \widetilde{x}_0).$$

Example. Let K be the trefoil $K = T(2,3)$. Using the Torus Knot-π_1-Proposition 111.18 and the Oriented Knot–Monoid-π_1-Proposition 111.26 one can show, with enough patience, that the sets $\mathrm{Hom}(\pi_1(S^3 \setminus K), S_3)$ and $\mathrm{Hom}(\pi_1(S^3 \setminus (K\#K)), S_3)$ have different cardinalities. This implies that the trefoil K and the connected sum $K\#K$ are not smoothly isotopic. □

Sketch of proof. [1175] We start out with the following observation:

Claim 0. The statement of the proposition is independent of the choice of base points, choice of meridians and choice of paths connecting the meridians to the base points.

[1175] The logic of the proof is similar to the proof of the Connected Sum-π_1-Proposition 102.4.

Proof. The claim is an almost immediate consequence of the Loop-π_1-Lemma 111.15, the Change-of-Base Point Proposition 88.14 and the following group theoretic result (which in turn follows easily from the Group Pushout Proposition 98.22): If π, $\widetilde{\pi}$ are groups, if $\nu \in \pi$ and $\widetilde{\nu} \in \widetilde{\pi}$ and if $f\colon \pi \to \pi'$ is an isomorphism, then $\pi *_{\nu=\widetilde{\nu}} \widetilde{\pi}$ is isomorphic to $\pi' *_{f(\nu)=\widetilde{\nu}} \widetilde{\pi}$. ⊞

Next recall that in Definition 109.12 we defined the connected sum of two oriented knots in terms of string knots. Recall that string knots have two features:

(1) Given a string knot $J \subseteq \overline{B}^3$ we introduced in Definition 109.23 its closure $\mathrm{cl}(K) := K \cup U$ where we consider the "horizontal arc" $U := \{(x,0,0) \,|\, |x| \geq 1\} \cup \{\infty\} \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ around "∞".
(2) In Definition 109.25 we introduced the addition $J_1 + J_2$ of two string knots J_1 and J_2. The addition is essentially just given by stacking them next to one another.

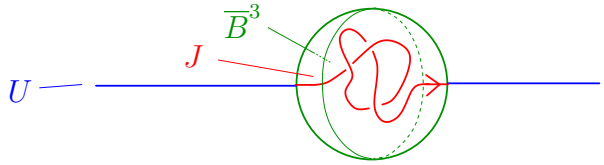
The connected sum of two oriented knots is defined using the above two constructions. Thus it makes sense to first consider the effect of these two constructions on fundamental groups of complements:

Claim 1. For any string knot $J \subseteq \overline{B}^3$ and any $x_0 \in S^2 \setminus J$ the inclusion induced map

$$\pi_1(\overline{B}^3 \setminus J, x_0) \;\to\; \pi_1(S^3 \setminus \mathrm{cl}(J), x_0)$$

is an isomorphism.

Proof. Let $J \subseteq \overline{B}^3$ be a string knot.



By the Seifert-van Kampen Theorem 110.8 we have the following pushout diagram:

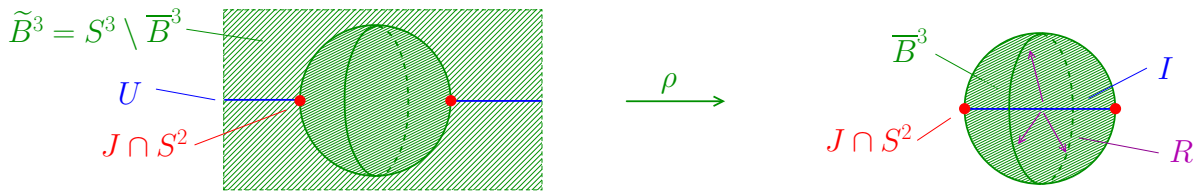
$$\begin{array}{ccc} \pi_1(S^2 \setminus J, x_0) & \longrightarrow & \pi_1(\overline{B}^3 \setminus J, x_0) \\ \downarrow & & \downarrow \\ \pi_1((S^3 \setminus B^3) \setminus J, x_0) & \longrightarrow & \pi_1(S^3 \setminus J, x_0). \end{array}$$

We make the following observations:

(a) We set $\widetilde{B}^3 := S^3 \setminus B^3$. Note that $J \cap \widetilde{B}^3 = U$. We claim that $S^2 \setminus U$ is a deformation retract of $\widetilde{B}^3 \setminus U$. This can be seen as follows: let $\rho\colon S^3 \to S^3$ be the inversion along S^2. It follows easily from the definitions that ρ restricts to a diffeomorphism $\rho\colon S^3 \setminus B^3 \to \overline{B}^3$ that is the identity on S^2 and that sends U to $I := \overline{B}^1 \times \{(0,0)\}$. The claim now follows from the observation that we have an "outward radial" deformation retraction R from $\overline{B}^3 \setminus I$ to $S^2 \setminus I = S^2 \setminus J$. In formulas this becomes the slightly unpleasant expression

$$\begin{array}{rcl} R\colon (\overline{B}^3 \setminus I) \times [0,1] & \to & \overline{B}^3 \setminus I \\ ((x,y,z),t) & \mapsto & \left(x, (y,z) \cdot \left((1-t) + t \cdot \sqrt{\frac{1-x^2}{y^2+z^2}}\right)\right). \end{array}$$

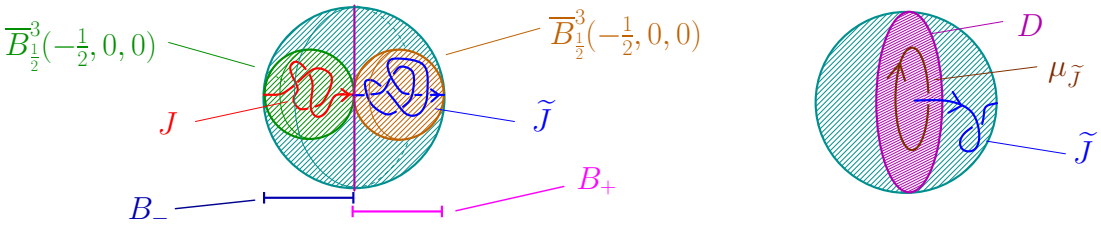
(b) It follows from (a) and the Homotopy Equivalence-π_1-Proposition 90.9 that the left vertical map is an isomorphism.



(c) It follows from (b) and the Pushout–Opposite Lemma 6.26 (a) that the right vertical map is also an isomorphism. ⊞

Claim 2. Let $J, \widetilde{J} \subseteq \overline{B}^3$ be two string knots. We denote by μ_J and $\mu_{\widetilde{J}}$ the corresponding meridians which are defined exactly as in Definition 109.1. We have an isomorphism

$$\pi_1(\overline{B}^3 \setminus (J + \widetilde{J})) \cong \pi_1(\overline{B}^3 \setminus J) *_{\mu_J = \mu_{\widetilde{J}}} \pi_1(\overline{B}^3 \setminus \widetilde{J}).$$



Proof. We introduce the following notation:

- By a slight abuse of notation we denote by $J, \widetilde{J}$ also the shrunk copies of $J, \widetilde{J}$ in the balls $\overline{B}^3_{\frac{1}{2}}(-\frac{1}{2},0,0)$ and $\overline{B}^3_{\frac{1}{2}}(\frac{1}{2},0,0)$.
- We write
$$\begin{aligned} B_- &:= \{(x,y,z) \in \overline{B}^3 \,|\, x \leq 0\}, \\ B_+ &:= \{(x,y,z) \in \overline{B}^3 \,|\, x \geq 0\}, \\ D &:= B_- \cap B_+. \end{aligned}$$

We now see that

$$\begin{aligned} \pi_1(\overline{B}^3 \setminus (J + \widetilde{J})) &\cong \pi_1(B_- \setminus J) *_{\pi_1(D \setminus \{(0,0,0)\})} \pi_1(B_+ \setminus \widetilde{J}) \\ &\cong \pi_1(\overline{B}^3_{\frac{1}{2}}(-\tfrac{1}{2},0,0) \setminus J) *_{\mu_J = \mu_{\widetilde{J}}} \pi_1(\overline{B}^3_{\frac{1}{2}}(\tfrac{1}{2},0,0) \setminus \widetilde{J}) \\ &\cong \pi_1(\overline{B}^3 \setminus J) *_{\mu_J = \mu_{\widetilde{J}}} \pi_1(\overline{B}^3 \setminus \widetilde{J}). \end{aligned} \qquad \begin{aligned} &\cong \pi_1(B_- \setminus J) *_{\mu_J = \mu_{\widetilde{J}}} \pi_1(B_+ \setminus \widetilde{J}) \end{aligned}$$

by the Seifert-van Kampen Theorem 110.8

since $D \setminus \{(0,0,0)\}$ admits a deformation retraction to $\mu_J = \mu_{\widetilde{J}}$

since $\overline{B}^3_{\frac{1}{2}}(-\frac{1}{2},0,0) \setminus J$ is a deformation retract of $B_- \setminus J$, and similar for $\widetilde{J}$

isomorphism induced by the shrinking diffeomorphisms ⊞

Now we can easily provide a proof of the proposition. So let $K, \widetilde{K}$ be two oriented knots in S^3. As in the Definition 109.12 of the connected sum $K \# \widetilde{K}$ we pick string knots $J \subseteq \overline{B}^3$ and $\widetilde{J} \subseteq \overline{B}^3$ such that the closure $\mathrm{cl}(J)$ is smoothly isotopic to K as an oriented knot and such the closure $\mathrm{cl}(\widetilde{J})$ is smoothly isotopic to $\widetilde{K}$ as an oriented knot.[1176] The connected sum is now defined as

$$K + \widetilde{K} := \mathrm{cl}(J + J').$$

[1176] We know by the Knot–String Knot Proposition 109.24 that such string knots exist.

We now see that

$$\begin{array}{lll} & \text{by definition} & \text{by Claim 1} \\ & \downarrow & \downarrow \\ \pi_1(S^3 \setminus (K\#\widetilde{K})) & \cong \pi_1(S^3 \setminus \mathrm{cl}(J+\widetilde{J})) & \cong \pi_1(\overline{B}^3 \setminus (J+\widetilde{J})) \\ & \cong \pi_1(\overline{B}^3 \setminus J) \underset{\mu_J=\mu_{\widetilde{J}}}{*} \pi_1(\overline{B}^3 \setminus \widetilde{J}) & \cong \pi_1(S^3 \setminus K, x_0) \underset{\mu_K=\mu_{\widetilde{K}}}{*} \pi_1(S^3 \setminus \widetilde{K}, \widetilde{x}_0) \\ & \uparrow & \uparrow \\ & \text{by Claim 2} & \text{by Claim 1, and since meridians of string knots are also meridians of the closures} \end{array}$$

■

111.10. **Appendix: High-dimensional knots.** Since the study of knots in S^3 was so much fun, it would be a shame not to consider high-dimensional knots.

Definition 111.27. An n-dimensional knot in S^k is a smooth submanifold of S^k that is diffeomorphic to S^n. For $n = 1$ and $k = 3$ we say that such a knot is classical. When we consider the case $n > 1$ and $k > 3$, then we say that the knot is high-dimensional.

The following definition, which doubles as an example, generalizes the notion of a trivial knot that we first introduced in Definition 108.4:

Definition 111.28. Let $n, k \in \mathbb{N}$ with $k \geq n$. We refer to

$$U \;:=\; \{(x,0) \in \mathbb{R}^{n+1} \times \mathbb{R}^{k-n-1} \,|\, x \in S^n\} \subseteq \mathbb{R}^k \cup \{\infty\} =_{\mathrm{i}} \; S^k$$

as the trivial n-dimensional knot in S^k.

The following definition is the analog of Definition 108.7:

Definition 111.29. We say that two n-dimensional knots K and $\widetilde{K}$ in S^k in S^n are smoothly isotopic if there exists a smooth isotopy from K to $\widetilde{K}$, i.e. if there exists a smooth map

$$\begin{array}{rccc} F\colon K \times [0,1] & \to & S^k \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:

(1) For each $t \in [0,1]$ the map $F_t\colon K \to S^k$ is a smooth embedding.
(2) We have $F_0 = \mathrm{id}$ and we have $F_1(K) = \widetilde{K}$.

As in the classical setting we only care about knots up to smooth isotopy. In particular any knot that is smoothly isotopic to the above trivial knot will also be called trivial.

We continue with the following unsurprising lemma:

Lemma 111.30. (Isotopic Link-π_1-Lemma) Let $n \in \mathbb{N}$. Let K and $\widetilde{K}$ be two n-dimensional knots in S^k. If K and $\widetilde{K}$ are smoothly isotopic, then $\pi_1(S^k \setminus K) \cong \pi_1(S^k \setminus \widetilde{K})$.

Proof. The proof of this statement is verbatim the same as the proof of the Isotopic Link-π_1-Lemma 111.4. ■

Our next result is an analog of the Trivial Knot–Complement Proposition 111.11 and the Trivial Knot-π_1-Proposition 111.12:

Proposition 111.31. (High Dimensional–Trivial Knot-π_1-Proposition) Let U be the trivial n-dimensional knot in S^k from Definition 111.28.

(1) $S^k \setminus U$ is diffeomorphic to $S^{k-n-1} \times \mathbb{R}^{n+1}$.
(2) (a) If $k = n+2$, then we have an isomorphism $\pi_1(S \setminus U) \cong \mathbb{Z}$.

(b) If $k \neq n+2$, then the fundamental group of $S^k \setminus U$, with respect to any base point, is trivial.

Proof. Let U be the trivial n-dimensional knot in S^k.

(1) We start out with the following claim:

Claim. We have a diffeomorphism

$$S^k \setminus U \;\cong\; (S^n \times (\overline{B}^{k-n} \setminus \{0\})) \cup_{S^{n-1}\times S^{k-n-1}} (\overline{B}^{n+1} \times S^{k-n-1}).$$

Proof. First we recall that in the Stereographic Projection Lemma 108.1 we gave an explicit diffeomorphism $\Phi\colon S^k \to \mathbb{R}^k \cup \{\infty\}$. Next we recall that in the Sphere–Solid Tori-Decomposition Lemma 111.6 we gave furthermore an explicit diffeomorphism $\Psi\colon (S^n \times \overline{B}^{k-n}) \cup_{S^{n-1}\times S^{k-n-1}} (\overline{B}^{n+1} \times S^{k-n-1}) \to S^k$. It follows immediately from the definitions that $\Phi(U) = \Psi(S^n \times \{0\})$. This shows the desired diffeomorphism is given by. ⊞

Somewhat similar to the proof of the Hopf Link-Complement-Proposition 111.23 we now see that the following map is a diffeomorphism:

$$\overbrace{(S^n \times (\overline{B}^{k-n} \setminus \{0\})) \cup (\overline{B}^{n+1} \times S^{k-n-1})}^{=(\overline{B}^2\times S^1)\cup_{S^1\times S^1}(S^1\times\overline{B}^2)\setminus\text{Hopf link}} \to S^{k-n-1} \times \mathbb{R}^{n+1}$$

$$[(z,w)] \mapsto \begin{cases} (\frac{w}{|w|}, \frac{z}{|w|}), & \text{if } (z,w) \in S^n \times (\overline{B}^{k-n} \setminus \{0\}), \\ (w,z), & \text{if } (z,w) \in \overline{B}^{n+1} \times S^{k-n-1}. \end{cases}$$

(2) This statement follows easily from (1) together with the Homotopy Equivalence-π_1-Proposition 90.9 together with the Circle-π_1-Theorem 88.22 and the Sphere-π_1-Proposition 88.24. ■

Evidently the question now is whether there exist high-dimensional knots that are not smoothly isotopic to the trivial knot. Or perhaps better, the question is whether we will manage to explicitly construct such examples. The good news is, we can indeed do so.

In the following we will present a construction, which has its origins in the work of Emil Artin [**Art1925b**] and Eric Zeeman [**Zee1965**]. This construction turns a codimension-two knot in S^{n+2} into a family of codimension-two knots in S^{n+3}.

Definition 111.32. We start out with the following preparations:

(1) As on page 319 we make the identification $S^{n+2} =_{\mathrm{i}} \overline{B}^{n+2}_+ \cup_{S^{n+1}} \overline{B}^{n+2}_-$.

(2) By the Sphere–Solid Tori-Decomposition Lemma 111.6 we can make the identification $S^{n+3} =_{\mathrm{i}} (S^1 \times \overline{B}^{n+2}) \cup_{S^1\times S^{n+1}} (\overline{B}^2 \times S^{n+1})$.

(3) We make the identification $\mathbb{R}^{n+2} = \mathbb{C} \times \mathbb{R}^n$. In particular we view $\overline{B}^{n+2}$ as a subspace of $\mathbb{C} \times \mathbb{R}^n$.

(4) Given $z \in S^1$ let

$$\begin{aligned} \rho_z\colon \overline{B}^{n+2} &\to \overline{B}^{n+2} \\ (w,x) &\mapsto (z \cdot w, x) \end{aligned}$$

be the rotation of $\overline{B}^{n+2}$ by the "angle" $z \in S^1$ around "the $0 \times \overline{B}^n$-axis".

Now let $K \subseteq S^{n+2}$ be an n-dimensional knot. It follows fairly easily from the Smooth Ball Embedding Theorem 45.10 and the fact that K is a smooth submanifold that, after possibly applying a diffeotopy of S^{n+2}, we can assume that $J_- := K \cap \overline{B}^{n+2}_- = \{(0,w) \mid w \in \overline{B}^n\}$

is the "trivial disk knot" in $\overline{B}^{n+2}$. We denote by $J_+ \coloneqq K \cap \overline{B}^{n+2}_+$ the "other disk knot". Next, given $m \in \mathbb{Z}$ we denote by Φ_m the diffeomorphism[1177]

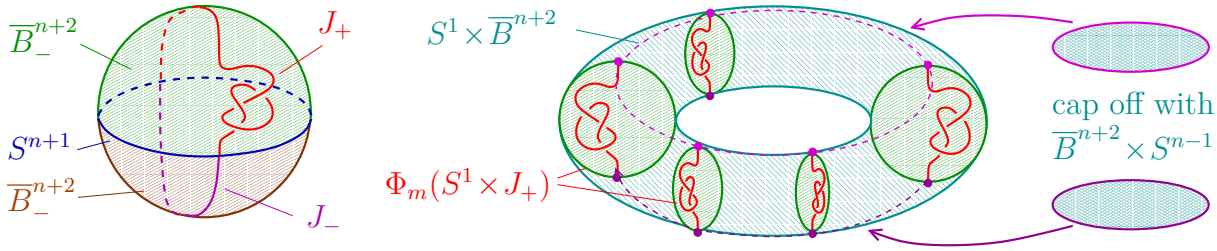
$$\begin{array}{rccc} \Phi_m \colon & S^1 \times \overline{B}^{n+2} & \to & S^1 \times \overline{B}^{n+2} \\ & (z, x) & \mapsto & (z, \rho_{z^m}(x)). \end{array}$$

Note that Φ_0 is the identity and also note that each Φ_m leaves the subset $S^1 \times S^{n-1}$ invariant as a subset. Thus we can define the m-twist spin $S_m(K)$ as follows:

$$S_m(K) \coloneqq \underbrace{\Phi_m(S^1 \times J_+)}_{\subseteq S^1 \times \overline{B}^{n+2}} \cup_{S^1 \times S^{n-1}} \underbrace{(\overline{B}^2 \times S^{n-1})}_{\subseteq \overline{B}^2 \times S^{n+1}} \subseteq \underbrace{(S^1 \times \overline{B}^{n+2}) \cup_{S^1 \times S^{n+1}} (\overline{B}^2 \times S^{n+1})}_{= S^{n+3}}.$$

It follows from the Gluing Submanifolds Proposition 56.12 that $S_m(K)$ is indeed a submanifold of S^{n+3}.

In the figure below we do our best to illustrate this construction for $n = 1$. Again informally speaking, $S_m(K)$ is given by spinning the disk knot J_+ around the S^1-direction, performing m spins around J_+ as you go around S^1, and then capping off the result by $\overline{B}^2 \times S^{n-1}$.



As we will see, it follows easily from the following proposition that, perhaps unsurprisingly, some twist spins are non-trivial. Perhaps more surprisingly the proposition also says that (± 1)-twist spins are always trivial.

Proposition 111.33. (Spun Knot-π_1-Proposition) Let $K \subseteq S^{n+2}$ be an n-dimensional knot.

(1) For the 0-twist spin there exists an isomorphism $\pi_1(S^{n+3} \setminus S_0(K)) \cong \pi_1(S^{n+2} \setminus K)$.
(2) For every $\epsilon \in \{-1, 1\}$ the ϵ-twist spin knot $S_\epsilon(K)$ is trivial.

[1178] Informally speaking Φ_m spins $\overline{B}^{n+2}$ altogether m times as we go around the S^1-direction.

Sketch of proof. Let $K \subseteq S^{n+2}$ be an n-dimensional knot.

(1) We have the following isomorphisms:[1179]

$$\begin{tikzcd}
\pi_1(S^{n+3} \setminus S_0(K)) & \pi_1(S^1 \times (\overline{B}^{n+2} \setminus J_+)) *_{\pi_1(S^1 \times (S^{n+1} \setminus S^{n-1}))} \pi_1(\overline{B}^2 \times (S^{n+1} \setminus S^{n-1})) \arrow[l, "\cong"', "\text{it follows from the Seifert–van Kampen Theorem 110.8, that this map is an isomorphism, note that here we use that } \Phi_0(S^1 \times J_+) = S^1 \times J_+"'] \arrow[d, "\cong"', "\text{Product-}\pi_1\text{-Proposition 90.13 isomorphisms given by the}"] \\
& (\langle t \rangle \times \pi_1(\overline{B}^{n+2} \setminus J_+)) *_{\langle s \rangle \times \langle t \rangle} \langle s \rangle \\
& \pi_1(\overline{B}^{n+2} \setminus J_+) \arrow[u, "\cong"', "\text{isomorphism by the group theoretic fact that } \Gamma \xrightarrow{\cong} (\langle t \rangle \times \Gamma) *_{\langle s \rangle \times \langle t \rangle} \langle s \rangle"] \arrow[d, "\cong"', "\text{isomorphism by the Seifert–van Kampen-Theorem 110.8}"] \\
& \pi_1(S^{n+2} \setminus K).
\end{tikzcd}$$

(2) This statement is proved implicitly in [**Zee1965**, p. 486] and explicitly in [**FO2015**, Corollary 2.1]. The reason for writing [**FO2015**] was that it felt easier to prove the statement, than to try to read the proofs written down by others. So we warmly recommend to try to prove the statement on your own instead of reading the proofs in the literature. ■

Now we can easily show that there exist non-trivial high-dimensional knots:

Corollary 111.34. (Non-Trivial High-Dimensional Knot Existence Corollary) Given any $n \in \mathbb{N}$ there exists an n-dimensional knot in S^{n+2} that is non-trivial.

Proof. Let $K \subseteq S^3$ be the trefoil and let $n \in \mathbb{N}$. We see that

$$\pi_1(S^{n+2} \setminus (n-1)\text{-st iterated 0-twist spin of } K) \underset{\substack{\uparrow \\ \text{by the Spun Knot} \\ \pi_1\text{-Proposition 111.33}}}{\cong} \pi_1(S^3 \setminus K) \overset{\substack{\text{by the Torus Knot-}\pi_1\text{-Proposition 111.18 and} \\ \text{the Trefoil–Non-Trivial Proposition 111.19}}}{\not\cong} \mathbb{Z} \underset{\substack{\uparrow \\ \text{by the High Dimensional} \\ \text{Trivial Knot-}\pi_1\text{-Proposition 111.31}}}{=} \pi_1(S^{n+2} \setminus U).$$

It follows from the Isotopic Link-π_1-Lemma 111.30 that the $(n-1)$-st iterated 0-twist spin of the trefoil is a non-trivial knot. ■

We conclude this appendix on high-dimensional knots with a short discussion of knots of other codimensions. In Exercise **??** we showed, building on the Embeddings-become-Isotopic Criterion **??**, that every n-dimensional knot in a sphere of dimension $\geq 2n+3$ is trivial. The following theorem is a refinement of this result:

[1179] By now the reader will be fully aware of the fact that it is supremely painful to completely keep track of all identifications, base points, induced maps and so on. Thus we also took some liberties to improve readability.

Theorem 111.35. (Haefliger's (Un-)knotting Theorem)
(1) Let $n, k \in \mathbb{N}$. If $2k > 3(n+1)$, then every n-dimensional knot in S^k is trivial.
(2) Given any $m \in \mathbb{N}$ there exists a $(4m-1)$-dimensional knot in S^{6m} that is non-trivial.

Example. From Haefliger's Unknotting Theorem 111.35, and an elementary calculation, we obtain the following two results:

(1) For $k \geq 4$ any 1-dimensional knot in S^k is trivial.
(2) For any $n \geq 2$ every n-dimensional knot in S^{2n+1} is trivial. As we saw in the Trefoil–Non-Trivial Proposition 111.19, this conclusion does not hold for $n = 1$. □

Proof. Statement (1) was proved by André Haefliger [**Hae1961a**, p. 47] in 1961. (An alternative proof is given in [**Wall2016**, p. 192].) Shortly afterwards André Haefliger [**Hae1962b**] also proved Statement (2). ■

Remark. We conclude this section with a few final remarks about high-dimensional knots and arbitrary codimensions:

(1) The Generalized Smooth Schoenflies Theorem 170.7 implies that, at least for $n \neq 4$, codimension one knots in S^n are trivial.
(2) As we remarked above, André Haefliger [**Hae1967, Hae1962a, Hae1961b, Hae1962b**] showed that in codimensions ≥ 3 there can be knots that are non-trivial. But Eric Zeeman [**Zee1963c**, p. 505] and John Stallings [**Stal1963**, Corollary 9.3] showed that knots of codimension ≥ 3 are "unknotted" in the "PL-category" and in the "topological category".

These remarks explain why topologists usually tend to stick to knots of codimension two. Nonetheless, we will make use of knots of larger codimension in Definition 258.1. □

111.11. Exercises for Chapter 111 .

Exercise 111.1.

(a) Let $n \geq 3$ and furthermore let $K \subseteq \mathbb{R}^n$ be a compact subset. We view K also as a subset of $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$. Let $x_0 \in \mathbb{R}^n \setminus K$. Show that the inclusion-induced map $\pi_1(\mathbb{R}^n \setminus K, x_0) \to \pi_1(S^n \setminus K, x_0)$ is an isomorphism.
(b) Does the conclusion of (a) also hold for $n = 2$?
(c) Does the conclusion of (a) also hold for non-compact subsets of $\mathbb{R}^n$?

Remark. This exercise is identical to Exercise 99.6.

Exercise 111.2. Given $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rccc} \varphi_k \colon & S^1 \times S^1 & \to & S^1 \times S^1 \\ & (w, z) & \mapsto & (w \cdot z^k, z). \end{array}$$

Show that $(S^1 \times \overline{B}^2) \cup_{\varphi_k} (S^1 \times \overline{B}^2)$ is homeomorphic to S^3 if and only if $k \in \{-1, 1\}$.
Hint. It might be helpful to compute the fundamental group or the first homology group of this topological space.

Exercise 111.3. Find an explicit smooth path in $\mathrm{SO}(4)$ from $A = \begin{pmatrix} 0 & \mathrm{id}_2 \\ \mathrm{id}_2 & 0 \end{pmatrix}$ to id_4.

Exercise 111.4.

(a) Let $K \subseteq \mathbb{R}^3$ be the trivial knot. Show that $\mathbb{R}^3 \setminus K$ is homotopy equivalent to $S^1 \vee S^2$.
(b) Let $L \subseteq \mathbb{R}^3$ be the m-component unlink. Show that $\mathbb{R}^3 \setminus L$ is homotopy equivalent to a wedge of spheres of suitable dimensions.
(c) Let $L \subseteq S^3$ be the m-component unlink. Show that $S^3 \setminus L$ is homotopy equivalent to a wedge of spheres of suitable dimensions.

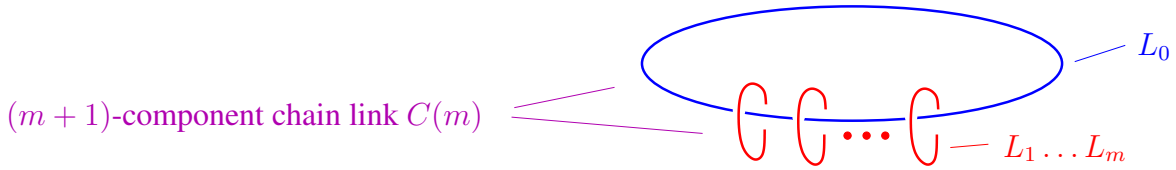
Exercise 111.5. Let $K \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be a knot.

(a) Show that there exists a smooth embedding $f\colon S^2 \to \mathbb{R}^3 \setminus K$ such that $f(S^2)$ is a retract of $\mathbb{R}^3 \setminus K$.
(b) Show that there exists a continuous map $g\colon \mathbb{R}^3 \setminus K \to S^3 \setminus K$ such that the induced map $g_*\colon \pi_1(\mathbb{R}^3 \setminus K) \to \pi_1(S^3 \setminus K)$ is an isomorphism.
(c) Show that $(S^3 \setminus K) \vee S^2$ is homotopy equivalent to $\mathbb{R}^3 \setminus K$.

Exercise 111.6. Show that the two Hopf links H_+ and H_-, that are shown in the figure below, are not smoothly isotopic.
Hint. Consider the fundamental group of the complement of the red component.

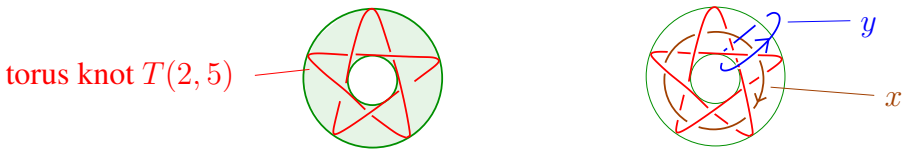


Exercise 111.7. Given $m \in \mathbb{N}$ we consider in this exercise the $(m+1)$-component link $C(m) = L_0 \cup L_1 \cup \cdots \cup L_m$ that is shown in the figure below. Determine $\pi_1(S^3 \setminus C(m))$, i.e. express $\pi_1(S^3 \setminus C_m)$ in terms of well-known groups.
Hint. Consider first the complement of L_0.
Remark. The link $C(m)$ is sometimes called a *chain link* or *key chain link*. Note that $C(0)$ is just a trivial knot and that $C(1)$ is smoothly isotopic to the Hopf link.

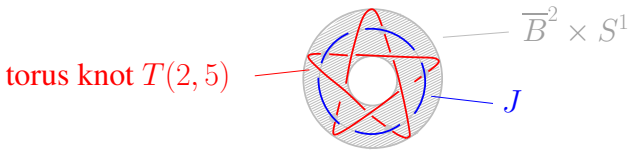


Exercise 111.8. Show that for every $n \in \mathbb{Z}$ the torus knots $T(n,1)$ and $T(1,n)$ are smoothly isotopic to the trivial knot.

Exercise 111.9. In the figure below we show the torus knot $T(2,5)$ and two oriented smoothly embedded loops x and y. Show directly (without worrying about base points) that $x^2 = y^5 \in \pi_1(S^3 \setminus T(2,5))$.



Exercise 111.10. Let $p, q \in \mathbb{Z}$ be coprime. We consider the corresponding torus knot $T(p,q) \subseteq (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$ and we consider the knot $J := \{0\} \times S^1$. Show that $\pi_1(S^3 \setminus (K \cup J))$ admits a presentation with three generators.



Exercise 111.11. Give an *explicit* diffeotopy from the map

$$\begin{array}{rccc} f\colon S^3 & \to & S^3 \\ v & \mapsto & \begin{pmatrix} 0 & \mathrm{id}_2 \\ \mathrm{id}_2 & 0 \end{pmatrix} \cdot v \end{array}$$

to the identity.

Exercise 111.12. As usual we consider the torus $S^1 \times S^1 \subseteq S^3 =_{\mathrm{i}} (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$. Given $m, n \in \mathbb{Z}$ we consider $L(m,n) := \{(y,z) \in S^1 \times S^1 \mid y^m = z^n\}$.

(a) What is the number of components of $L(m,n)$?
(b) For which $m, n \in \mathbb{Z}$ is $L(m,n)$ a torus knot?
(c) Draw $L(2,2)$ and $L(4,6)$.

Remark. The link $L(m,n)$ is called the *torus link* $L(m,n)$.

Exercise 111.13. Let $p, q \in \mathbb{N}$ coprime. Recall that in the proof of the Torus Knot-π_1-Proposition 111.18 we gave an explicit isomorphism $\langle x, y \mid x^p = y^q \rangle \to \pi_1(S^3 \setminus T(p,q))$.

(a) Show that $\pi_1(S^3 \setminus T(p,q))_{\mathrm{ab}} \cong \mathbb{Z}$.
(b) Find a word in x and y that represents a generator of $\pi_1(S^3 \setminus T(p,q))_{\mathrm{ab}}$.
(c) We consider the trefoil $T(2,3)$. Find a word in x and y that represents a meridian of the trefoil.
(d) Do (c) for all torus knots.
Remark. This might be a bit tricky.

Exercise 111.14. Let $p, q \in \mathbb{Z}$ be coprime with $p, q \notin \{-1, 1\}$. We consider the corresponding torus knot $T(p,q)$. Show that $\pi_1(T(p,q)) = \langle x, y \mid x^p = y^q \rangle$ admits an epimorphism onto a finite non-abelian group.

Exercise 111.15. Let $n \in \mathbb{N}_0$. As on pages 1815 and 1963 we consider the dihedral group

$$D_n \;=\; \underset{\uparrow}{\mathbb{Z}/n\mathbb{Z}} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z} \;\cong\; \langle x, t \mid x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle.$$

semidirect product where $\varphi\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is given by multiplication by -1

Distinguish the torus knots $T(2,p)$ by looking at epimorphisms onto dihedral groups.

Exercise 111.16. Use the discussion on page 1912 to show that there exists an epimorphism of $\pi_1(S^3 \setminus T(2,3)) \cong \langle x, y \mid x^2 \cdot y^3 \rangle$ onto $\mathrm{SL}(2,\mathbb{Z})$.

Exercise 111.17. Let $p, q \in \mathbb{Z}$ be coprime. We set $\Gamma(p,q) := \langle x, y \mid x^p = y^q \rangle$ and we consider the subgroup $C(p,q)$ of $\Gamma(p,q)$ that is generated by x^p and y^q. We denote by

$T(p,q)$ the (p,q)-torus knot. In the Torus Knot-π_1-Proposition 111.18 we showed that there exists an isomorphism $\pi_1(S^3 \setminus T(p,q)) \cong \Gamma(p,q)$.

(a) Show that $\Gamma(p,q)/C(p,q) \cong \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ where $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ denotes the free product of the groups $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$.

(b) The *center* of a group π is defined as $C(\pi) \coloneqq \{g \in \pi \mid gh = hg \text{ for all } h \in \pi\}$.
  (i) We assume that $p, q \neq 1$. Show that $C(\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}) = \{e\}$.
  (ii) Show that $C(\Gamma(p,q)) = C(p,q)$.

(c) Let (p,q) and (p',q') be two pairs of coprime natural numbers with $1 < p < q$ and $1 < p' < q'$. Show that if $(p,q) \neq (p',q')$, then the torus knots $T(p,q)$ and $T(p',q')$ are not smoothly isotopic.

Exercise 111.18. Let $n \in \mathbb{N}_0$. As in Exercise 111.15 we consider the dihedral group

$$D_n \;=\; \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z} \;\cong\; \langle x, t \mid x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle.$$

Let K_1 and K_2 be two oriented knots and let p be a prime. Use the Oriented Knot–Monoid-π_1-Proposition 111.26 to prove that the following two statements are equivalent:

(1) There exists an epimorphism $\pi_1(S^3 \setminus (K_1 \# K_2)) \to D_p$.
(2) There exists an $i \in \{1,2\}$ such that $\pi_1(S^3 \setminus K_i)$ admits an epimorphism onto D_p.

Exercise 111.19. Let K and $\widetilde{K}$ be oriented knots. Show that the fundamental groups of $S^3 \setminus (K \# \widetilde{K})$ and $S^3 \setminus (K \# (\widetilde{K}^{\mathrm{mir}})^{\mathrm{rev}})$ are isomorphic.

Exercise 111.20. We consider the two links L and $\widetilde{L}$ that are shown in the figure below.

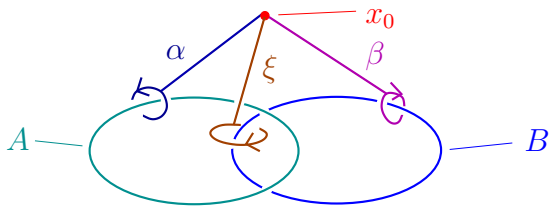
(a) Show that there exists an orientation-preserving diffeomorphism $X_L \to X_{\widetilde{L}}$ which preserves the ordering of the boundary components.

(b) Show that the two links L and $\widetilde{L}$ are not smoothly isotopic.



the two links are not smoothly isotopic but
the complements are diffeomorphic

Exercise 111.21. Let $A \sqcup B \subseteq S^3$ be the trivial 2-component link that is shown in the figure below. As we discussed on page 2162 it follows from the Split Union-π_1-Proposition 111.20 that $\pi_1(S^3 \setminus (A \sqcup B), x_0)$ is the free group on the meridians α and β. Express the loop ξ, that is shown in the figure, as a word in $[\alpha]$ and $[\beta]$.
Remark. This exercise is identical to Exercise 111.21.

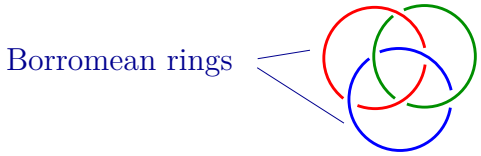


Exercise 111.22.

(a) Let $L \subseteq S^3$ be a link. Show that if L is splittable, then $\pi_1(S^3 \setminus L)$ is a non-abelian group.
(b) Show that the Hopf link is non-splittable.

Exercise 111.23. We consider the Borromean rings $L = L_1 \sqcup L_2 \sqcup L_3$ which are shown in the figure below.

(a) What element does on L_3 represent in $\pi_1(S^3 \setminus (L_1 \sqcup L_2))$?
Remark. Exercise 111.21 might be helpful.
(b) Show that the Borromean rings are non-trivial.



Exercise 111.24. We consider the picture that is shown below.

(a) Where was the picture taken?
(b) What is the corresponding link in $\mathbb{R}^3$?



Exercise 111.25. Let $K \subseteq S^3$ be a knot. We consider the "physical cone"

$$C \;\coloneqq\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subseteq\; \overline{B}^4.$$

(a) Show that C is homeomorphic to the disk $\overline{B}^2$.
(b) We suppose that K is smoothly isotopic to the trivial knot. Show that C is a topological submanifold of the topological manifold $\overline{B}^4$.

Exercise 111.26. Let $K \subseteq S^3$ be a knot such that $\pi_1(S^3 \setminus K) \ncong \mathbb{Z}$. As in Exercise 111.25 we consider the "physical cone"

$$C \;\coloneqq\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subseteq\; \overline{B}^4.$$

In Exercise 111.25 we showed that C is homeomorphic to the disk $\overline{B}^2$. In the following we will show that C is not a submanifold of the topological manifold $\overline{B}^4$.

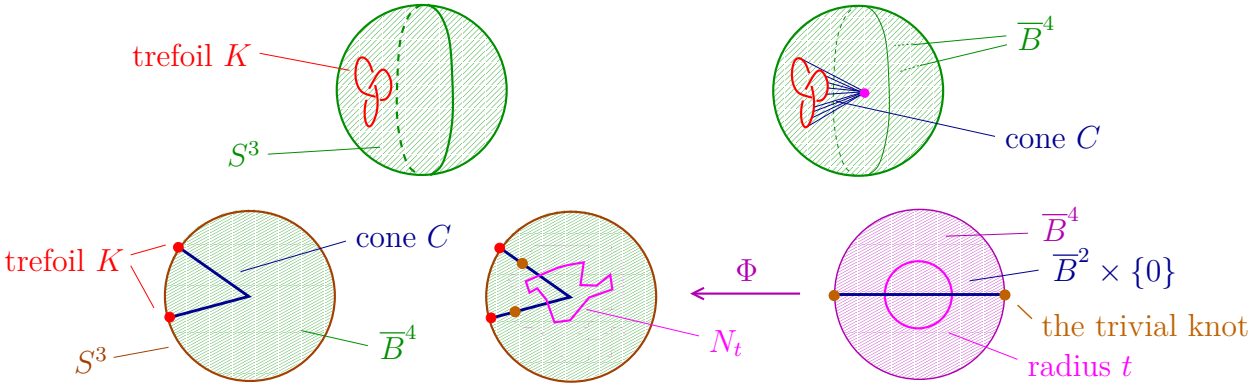
(a) Let $\alpha\colon \pi_1(S^3 \setminus K) \to \mathbb{Z}$ and $\beta\colon \mathbb{Z} \to \pi_1(S^3 \setminus K)$ be two homomorphisms. Show that the composition $\beta \circ \alpha\colon \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K)$ is not an isomorphism.
(b) Suppose that F is a proper 2-dimensional topological submanifold of $\overline{B}^4$. Show that given any point $P \in F \cap B^4$ there exists a continuous map $\Phi\colon \overline{B}^4 \to \overline{B}^4$ that is an embedding such that $\Phi(0) = P$ and such that $\Phi(\overline{B}^2 \times \{0\}) = \Phi(\overline{B}^4) \cap C$.
(c) Show that C is not a *topological submanifold* of $\overline{B}^4$.
Hint. Evidently the idea is to apply (c) to the point $P = 0 \in C$. The difficulty is

that the map Φ from (c) is not just a rescaling of the identity but it is potentially a completely different chart. To deal with the problem we introduce some notation.

(i) Given $I \subseteq [0,1]$ we write $D_I := \{v \in \overline{B}^4 \mid \|v\| \in I\}$.
(ii) Given $I \subseteq [0,1]$ we write $N_I := \Phi(D_I)$.
(iii) Given $t \in (0,1]$ we write $D_t := D_{\{t\}}$ and $N_t := N_{\{t\}}$.

Show that the following statements hold:

(1) Given any $t \in (0,1]$ there exists an $s \in (0,1)$ with $D_{[0,s]} \subseteq N_{[0,t)}$.
(2) Given any $s \in (0,1]$ there exists a $t \in (0,1)$ with $N_{[0,t]} \subseteq D_{[0,s)}$.
(3) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $D_r \setminus C \to D_{[r,s]} \setminus C$ and $D_s \setminus C \to D_{[r,s]} \setminus C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\pi_1(S^3 \setminus K)$.
(4) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $N_r \setminus C \to N_{[r,s]} \setminus C$ and $N_s \setminus C \to N_{[r,s]} \setminus C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\mathbb{Z}$.
(5) Use the above discussion together with (b) to show that such Φ cannot exist.



Exercise 111.27. Let $K \subseteq S^3$ be an oriented knot and let $m \in \mathbb{Z}$. As in Definition 111.32 we consider the m-twist spin $S_m(K) \subseteq S^4$. We view S^3 as a subset of S^4 via the embedding $z \mapsto (z, 0)$. The intersection $S_m(K) \cap S^3$ is a knot. What is the smooth isotopy type of this knot? A possible answer is that this knot is a connected sum of K with some suitable other knot.

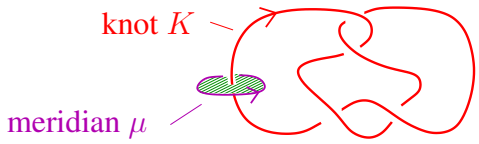
Exercise 111.28. We define an *m-component n-dimensional link in S^k* as a smooth submanifold of S^k that is diffeomorphic to the disjoint union of m copies of S^n. Now let $d \in \mathbb{N}$. Show that there exist two 2-component links $L = L_1 \sqcup L_2$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \widetilde{L}_2$ in S^{2d+1} with $\dim(L_1) = \dim(L_2) = \dim(\widetilde{L}_1) = \dim(\widetilde{L}_2) = d$ but such that L and $\widetilde{L}$ are not smoothly isotopic.

Exercise 111.29. Let $K \subseteq S^k$ be an n-dimensional knot. Show that if $n \leq k-3$, then $\pi_1(S^k \setminus K) = 0$.
Remark. You could use the Transversality Theorem 50.9 together with the Whitney Approximation Theorem 36.1.

Exercise 111.30. Let $K \subseteq S^3$ be an oriented knot. Show that $\pi_1(S^3 \setminus K)$ is normally generated (in the sense of Definition 110.5) by any meridian μ of K.
Hint. By the Meridian Proposition 109.2 (4) we can find a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$

such that $\mu = \tau(S^1 \times \{x\})$. We set $X_K = S^3 \setminus \tau(B^2 \times K)$. Recall that by the Link-π_1-Isomorphism Lemma 111.3 we know that $\pi_1(S^3 \setminus K) \cong \pi_1(X_K)$. Now apply the Manifold–Seifert-van Kampen Theorem 110.8 to $S^3 = X_K \cup_{\tau(S^1 \times K)} \tau(\overline{B}^2 \times K)$ and make use of Exercise 98.28.

Remark. This exercise is almost the same as Exercise 99.20.



Exercise 111.31. Let π be a group. We introduce the following notions:

- Given a subset $S \subseteq \pi$ we introduced in Definition 98.20 the subgroup $\langle\langle S \rangle\rangle$ as the intersection of all normal subgroups of π that contain S.
- We say that a subset $S \subseteq \pi$ *normally generates* π if $\langle\langle S \rangle\rangle = \pi$.
- We define the *weight of* π as

$$w(\pi) := \min\{\#S \mid S \text{ is a subset of } \pi \text{ that normally generates } \pi\} \in \mathbb{N}_0 \cup \{\infty\}.$$

The exercise is to determine the weights of various groups:

(a) Determine the weight of $\mathbb{Z}^m$.
(b) Determine the weight of the free group on m generators.
(c) Let $K \subseteq S^3$ be knot. Show that $w(\pi_1(S^3 \setminus K)) \leq 1$.
Remark. Here you can make use of Exercise 111.30.
(d) Let $L \subseteq S^3$ be an m-component link. Show that $w(\pi_1(S^3 \setminus L)) \leq m$.

CHAPTER 112

Link diagrams

In the previous Chapter 111 we gave a completely rigorous calculation of the isomorphism type of the fundamental group of the complement of torus links and we used this to show that the trefoil is not smoothly isotopic to the trivial knot.

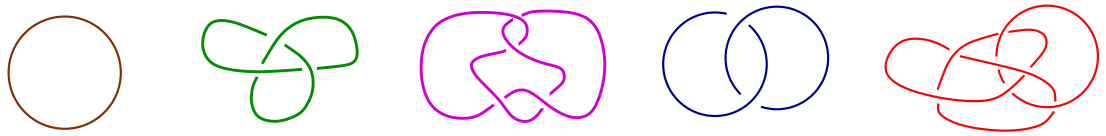
Our next goal is to calculate fundamental groups of complements of arbitrary links. But to calculate such fundamental groups we first need to find a way to describe all links in a reasonable way. In this chapter we will introduce the notion of a diagram of a link. We will see that diagrams are convenient ways to describe links. In the next chapter we will explain the Wirtinger algorithm which, given a link diagram for a link $L \subseteq S^3$, lets us determine a presentation of $\pi_1(S^3 \setminus L)$ from a diagram.

This chapter is organized as follows:

- In Section 112.1 we will introduce the notion of a link diagram and we will see that every link diagram defines the smooth isotopy type of an oriented link.
- In Section 112.2 we will state the Link Diagram Existence Proposition 112.7 which says that every smooth isotopy type of a link can be represented by a link diagram.
- In Section 112.3 we will see that any two diagrams of a given link are related by Reidemeister moves.
- In Section 112.4 we will prove the Link Diagram Existence Proposition 112.7.
- In Section 112.5 we will provide exercises for Chapter 112.

In the next Chapter 113 we will use link diagrams to describe the Wirtinger presentations of fundamental groups of link complements.

112.1. **Definition and existence of link diagrams.** In Chapter 108 we showed lots of 2-dimensional figures which present knots and links in $\mathbb{R}^3$. Some of these are shown in the figure below:



trivial knot trefoil figure-8 knot Hopf link whatever

Such 2-dimensional pictures, which clearly indicate a link, are usually called "diagrams". It takes some thought to say precisely what a "diagram" is supposed to be.[1180] In this chapter we will give a formal definition of a link diagram and we will see that every link admits such a diagram. In the course we will see on many occasions that link diagrams are a convenient tool for describing links. If the reader is comfortable with a naive interpretation of what a diagram is ("any picture like above"), then the reader can also move to the next

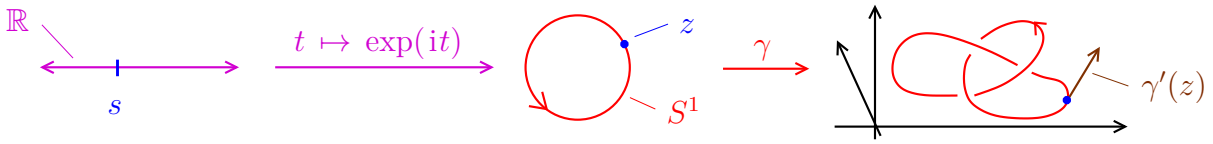
[1180]Unless one applies the popular all-purpose "I know it when I see it" -definition.

Chapter 113 where we will calculate the fundamental group $\pi_1(S^3 \setminus L)$ in terms of a diagram of the link.

Before we start with link theory we recall Definition 49.4:

Notation 112.1. Given a smooth map $\gamma\colon S^1 \to \mathbb{R}^n$ and $z = \exp(is) \in S^1$ we write

$\gamma'(z)$:= derivative at the point s of the function $\mathbb{R} \to \mathbb{R}^n$ given by $t \mapsto \gamma(\exp(it))$.



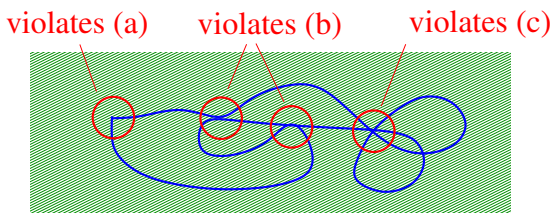
Definition 112.2. A map $\gamma\colon \bigsqcup_{i=1}^m S_i^1 \to \mathbb{R}^2$ is called diagrammatic if the following conditions are satisfied:

(a) The map is an immersion in the sense of Definition 30.9, in our case this just means that γ is smooth and that for each $z \in \bigsqcup_{i=1}^m S_i^1$ we have $\gamma'(z) \neq 0$.

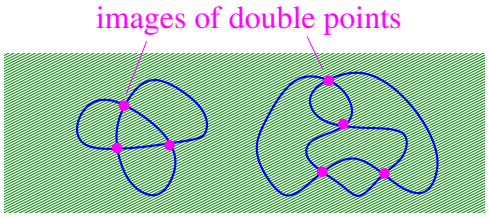
(b) If $z \neq w \in \bigsqcup_{i=1}^m S_i^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.

(c) For every $P \in \gamma\big(\bigsqcup_{i=1}^m S_i^1\big)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Any $z \in \bigsqcup_{i=1}^m S_i^1$ for which there exists a $w \neq z \in \bigsqcup_{i=1}^m S_i^1$ with $\gamma(z) = \gamma(w)$ is called a double point of γ.



not allowed in a diagrammatic map



images of diagrammatic maps $S^1 \to \mathbb{R}^2$

We start out with the following lemma which we will use subconsciously on many occasions.

Lemma 112.3. (Diagram–Double Point Lemma) Every diagrammatic map has only finitely many double points.

Sketch of proof. To simplify the notation a little bit we only deal with the case $m = 1$, i.e. we consider a diagrammatic map $\gamma\colon S^1 \to \mathbb{R}^2$. Recall that this means that γ has the following properties:

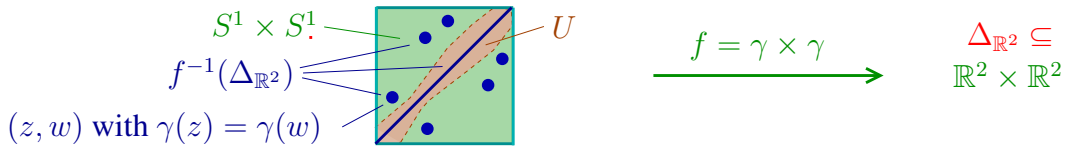
(a) For each $z \in S^1$ we have $\gamma'(z) \neq 0$.
(b) If $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.
(c) For every $P \in \gamma(S^1)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Next we consider the map

$$\begin{array}{rccc} f\colon & S^1 \times S^1 & \to & \mathbb{R}^2 \times \mathbb{R}^2 \\ & (z, w) & \mapsto & (\gamma(z), \gamma(w)). \end{array}$$

Given a set X we denote by $\Delta_X = \{(x,x) \mid x \in X\}$ the diagonal in $X \times X$. It follows easily from the definitions that

$$\#\text{double points of } \gamma \;=\; \tfrac{1}{2} \cdot \#(f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1}).$$



Thus it suffices to prove the following claim:

Claim. The set $f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1}$ is finite.

Proof. We make the following observations:

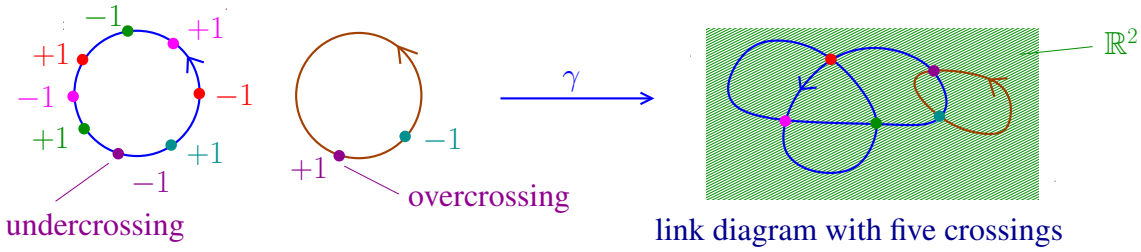
(1) Since $\Delta_{\mathbb{R}^2} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ is closed we see that $f^{-1}(\Delta_{\mathbb{R}^2}) \subseteq S^1 \times S^1$ is closed.
(2) It follows from (a) and the Local Smooth Embedding Theorem 30.13 that there exists an open neighborhood U of Δ_{S^1} such that $U \cap f^{-1}(\Delta_{\mathbb{R}^2}) = \Delta_{S^1}$.
(3) It follows from (b) and again the Local Smooth Embedding Theorem 30.13 that each $(z,w) \in f^{-1}(\Delta_{\mathbb{R}^2} \setminus \Delta_{S^1})$ is an isolated point in $f^{-1}(\Delta_{\mathbb{R}^2})$.

The above observations imply that $f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1}$ is a discrete subset of the compact topological space $(S^1 \times S^1) \setminus U$. It follows from the Compact-Discrete Lemma 3.40 that $f^{-1}(\Delta_{\mathbb{R}^2} \setminus \Delta_{S^1})$ is finite. ∎

Now we can finally introduce the main object of this chapter, namely the notion of a link diagram:

Definition 112.4.

(1) An m-component link diagram is a diagrammatic map $\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2$ together with a map
$$c\colon \{\text{double points of } \gamma\} \;\to\; \{\pm 1\}$$
which has the property that for $z \neq w \in \bigsqcup_{i=1}^{m} S^1_i$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.
(2) A 1-component link diagram is called a knot diagram.
(3) Let (γ, c) be a link diagram. Let z be a double point of γ. If $c(z) = +1$, then we say that the double point is an overcrossing, otherwise we call it an undercrossing.
(4) We refer to the images of the double points as the crossings of the link diagram.



The next lemma shows that link diagrams give rise to links:

Lemma 112.5. (Diagram-to-Link Lemma) Let $(\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2, c)$ be a link diagram.

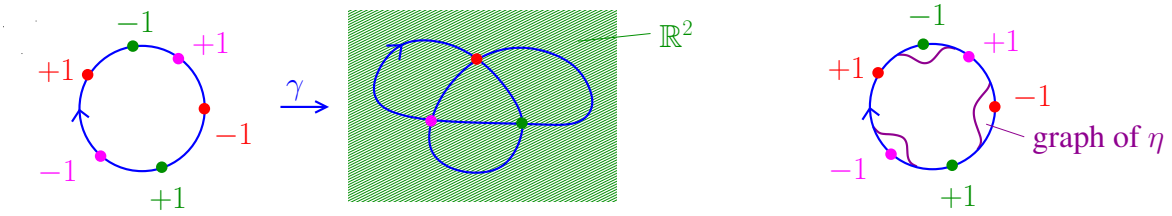
(1) There exists a smooth map $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ that satisfies the following condition:
($\star$) For every $P, Q \in S^1_1 \sqcup \cdots \sqcup S^1_m$ with $\gamma(P) = \gamma(Q)$ and with $c(P) = 1$ (and thus necessarily $c(Q) = -1$) we have $\eta(P) > \eta(Q)$.

(2) For any smooth map $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ that satisfies ($\star$) the image of the map $(\gamma, \eta)\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2 \times \mathbb{R}$ (together with the obvious ordering of the components) is an m-component link.

(3) The smooth isotopy type of the link in (2) does not depend on the choice of η, as long as the condition in ($\star$) is satisfied.

If we equip $S^1_1 \sqcup \cdots \sqcup S^1_m$ with the standard orientation, then in the above statements we can also replace "link" by "oriented link".

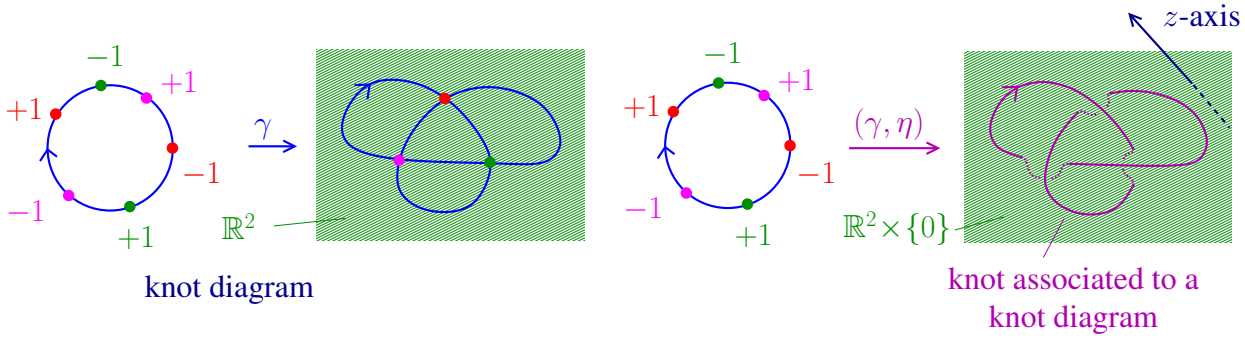
Example. In practice we often work with a map $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ that is equal to 0 for most points, except that it dips to -1 at the undercrossings.



The graph of such a smooth function is illustrated in the figure above. □

Definition 112.6. The Diagram-to-Link Lemma 112.5 shows that a link diagram gives rise to an essentially unique (oriented) link. We refer to this link as the link associated to the link diagram. At times we will rather blur the distinction between a link diagram and the associated link.

Example. In the figure below we show to the left a knot diagram. To the right we show the associated knot.



It is pretty clear that the resulting knot is smoothly isotopic to the trefoil. □

Proof.

(1) It follows quite easily from the Smooth Transition Function Lemma 24.9 that such a map η exists. We leave it to the reader to work out the details.[1181]

[1181] A slight refinement of this statement will be discussed in a later Diagram-to-Link Lemma 113.8.

(2) Let $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}$ be a smooth map that satisfies ($\star$). It is clear that the map

$$\begin{array}{rcl}(\gamma,\eta)\colon \eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m & \to & \mathbb{R}^3 \\ z & \mapsto & (\gamma(z),\eta(z))\end{array}$$

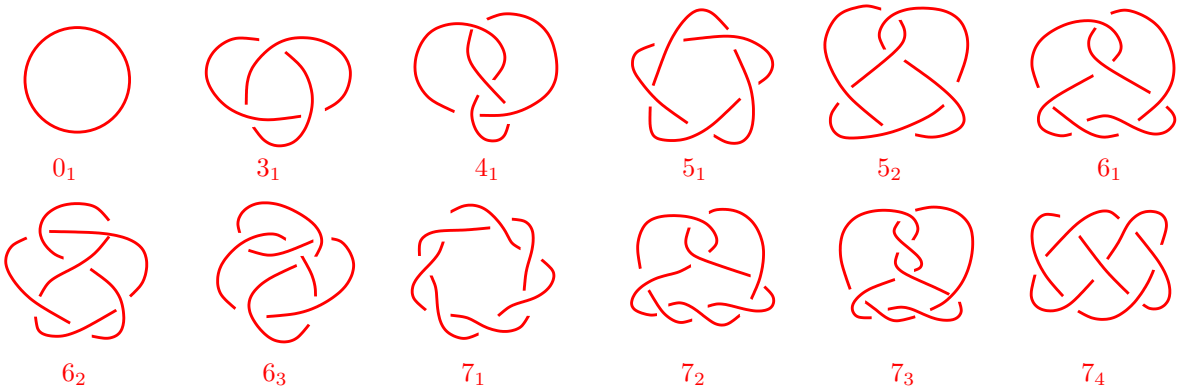
is injective. Since $S^1_1 \sqcup \cdots \sqcup S^1_m$ is compact and since $\mathbb{R}^3$ is Hausdorff it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map is actually an embedding. Since γ is an immersion it follows that the above map (γ,η) is also an immersion. In summary we see that the map (γ,η) in fact even a smooth embedding. Thus it follows from the Smooth Embedding Theorem 30.15 (1) that the image is a one-dimensional submanifold of $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ that is diffeomorphic to $S^1_1 \sqcup \cdots \sqcup S^1_m$. In other words, the image is a link.

(3) Now suppose that we are given two smooth maps η and $\eta^\dagger$ that satisfy ($\star$). We consider the map

$$\begin{array}{rcl}H\colon (S^1_1 \sqcup \cdots \sqcup S^1_m) \times [0,1] & \to & \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3 \\ (z,t) & \mapsto & (\gamma(z), \eta(z)\cdot(1-t) + \eta^\dagger(z)\cdot t).\end{array}$$

Clearly this map is smooth. By construction we know that for any two distinct points $z, w \in S^1_1 \sqcup \cdots \sqcup S^1_m$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta^\dagger(z) < \eta^\dagger(w)$. Using this observation one can easily verify that each H_t is a smooth embedding. In other words, the map H_t is a smooth isotopy between the two (oriented) links corresponding to η and $\eta^\dagger$. ∎

In practice, i.e. in future examples, we do what every other sane topologist does, namely we draw a suggestive picture and we interpret it as a link diagram and a corresponding link in the obvious way. For example in the figure below we show the trivial knot together with the first prime knots (ordered by crossing number) in the standard table of knots (where we do not distinguish between knots and their mirror image).



We refer to the standard textbooks on knot theory [**Rolf1990, BZH2014**] for a continuation of the table. But it should already be pretty clear from this table that knots provide a straightforward way to produce an almost endless list of interesting topological problems.

112.2. **Existence of link diagrams.** In the Diagram-to-Link Lemma 112.5 we saw that any link diagram gives rise to a link. The following proposition gives us the converse:

Proposition 112.7. (Link Diagram Existence Proposition)

(1) Given any (oriented) link $L \subseteq S^3$ there exists a link diagram such that L is smoothly isotopic to the link associated to the link diagram.
(2) Given any (oriented) link $L \subseteq \mathbb{R}^3$ there exists an $A \in \mathrm{SO}(3)$ such that the projection of $A \cdot L$ onto the xy-plane defines a link diagram for L.

Proof. The idea for the proof of the Link Diagram Existence Proposition 112.7 is quite easy: By the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17 we know that every link in S^3 is smoothly isotopic to a link in $\mathbb{R}^3$. Playing around with a physical model one sees that almost all projections of a link $L \subseteq \mathbb{R}^3$ onto a hyperplane lead to a diagram. Unfortunately it is technically quite tricky to show that this intuition is correct. Thus we postpone the rather lengthy proof of the Link Diagram Existence Proposition 112.7 to Section 112.4. ■

The Link Diagram Existence Proposition 112.7 allows us to introduce the following definition:

Definition 112.8. Let $L \subseteq S^3$ be a link. We define the crossing number $c(L)$ of L as the minimal number of crossings of any diagram of L.

Example. In the figure below we show to the left all knot diagrams with ≤ 2 crossings (up to smooth isotopy). Clearly all these diagrams represent the trivial knot.



We also show the usual diagram of the trefoil with three crossings. By the Trefoil–Non-Trivial Proposition 111.19 we know that the trefoil is not trivial. This discussion shows that the crossing number of the trefoil equals 3. □

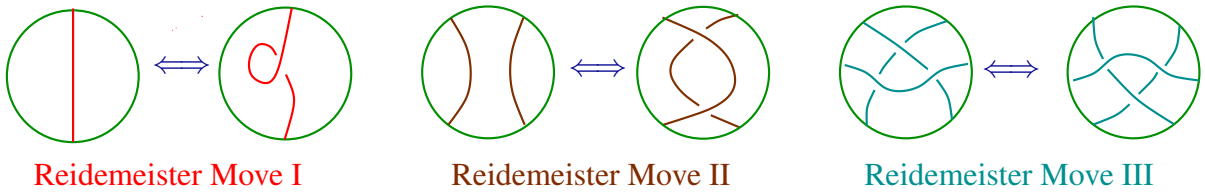
As an aside let us formulate the following open question:

Question 112.9. Is the crossing number additive under the connected sum operation? More precisely, if K and $\widetilde{K}$ are oriented knots, is $c(K\#\widetilde{K}) = c(K) + c(\widetilde{K})$?

112.3. **Reidemeister moves.** By the Link Diagram Existence Proposition 112.7 we now know that every (oriented) link arises, up to smooth isotopy, from an (oriented) link diagram. The question arises, when do two link diagrams give rise to smoothly isotopic links? The following theorem gives a complete answer. We formulate the theorem in a slightly informal way since we will not really make use of it.

Theorem 112.10. (Reidemeister Moves Theorem)
(1) Two link diagrams give rise to smoothly isotopic links if and only if the two diagrams are related by a finite sequence of smooth isotopies of $\mathbb{R}^2$ and Reidemeister moves. The three Reidemeister moves are illustrated in the figure below.
(2) Two oriented link diagrams give rise to smoothly isotopic links if and only if the two diagrams are related by a finite sequence of smooth isotopies of $\mathbb{R}^2$ and oriented Reidemeister moves. These are defined as below, except that we need to consider all consistent orientations.

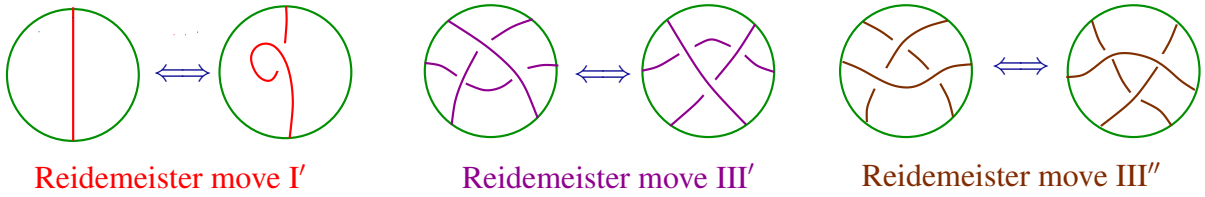
Proof.



(1) This statement goes back to the work of Kurt Reidemeister [**Rei1926**, p. 26], [**Rei1932**, Chapter 3] in 1932 and independently James Alexander and Garland Briggs [**AlB1927**, p. 563]. The formulations and the proofs in [**Rei1926, Rei1932, AlB1927**] are in terms of "polygonal links". The statement of the theorem is well known in our setting, where links are defined as submanifolds of S^3 and it appears in basically any textbook on link theory. Satisfactory proofs can be found in [**OSS2015**, Theorem B.1.1] and [**Que2025**].

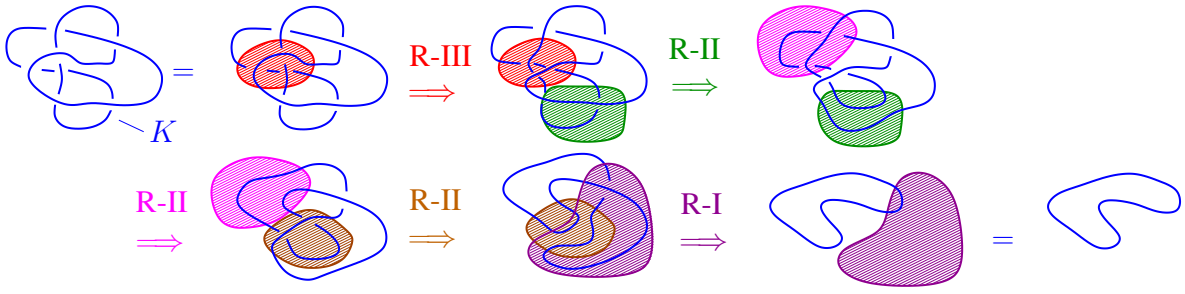
(2) Most of the references for (1) do not talk about the oriented case. But seemingly it is understood from the proofs from the unoriented setting also provide the desired statement in the oriented setting. ■

Example. In Exercise 112.10 we will see that the following variations on the above Reidemeister moves are actually consequences of the original Reidemeister moves.



To preserve our sanity we will not distinguish in our language between the original Reidemeister moves and the various variations. □

Example. In the figure below we show to the left a diagram of some knot K.



We also show how, using a sequence of Reidemeister moves, one can turn the diagram into the "trivial diagram". □

Remark. Throughout these lecture notes we will see on several occasions that the tools of algebraic topology, e.g. fundamental groups, linking pairings and Reidemeister torsion can be used fruitfully to study knots and links. The Link Diagram Existence Proposition 112.7

and the Reidemeister Moves Theorem 112.10 open up an alternative route to studying links. More precisely, these two results show that there exists a bijection

$$\begin{array}{c}\text{(oriented) link diagrams up to smooth isotopy}\\ \text{and Reidemeister moves}\end{array} \xrightarrow{\cong} \begin{array}{c}\text{(oriented) links up to}\\ \text{smooth isotopy.}\end{array}$$

This bijection lies at the heart of the definition of several link invariants:

- The Jones polynomial [**JonV1985**] that was introduced in 1984 and which we will discuss in Chapter 70.
- The HOMFLY-PT polynomial [**FYHLMO1985, PT1987**] that was discovered in 1985 and which we will discuss in **??**.

These polynomial invariants are quite different from any of the invariants that are obtained through algebraic-topological methods. We refer to [**Lic1997a**, Chapters 3 and 15] and [**BZH2014**, Chapter 17] for more information on these invariants. An even more powerful invariant that arises from link diagrams is given by Khovanov homology [**Kho2000**]. □

112.4. **Proof of the Link Diagram Existence Proposition 112.7.** This section, which occupies the remainder of this chapter, is completely dedicated to the proof of the Link Diagram Existence Proposition 112.7. In the next Chapter 113 we will use diagrams to calculate the fundamental groups of link complements.

The proof of the Link Diagram Existence Proposition 112.7 mostly rests on the next lemma. The formulation of this lemma requires the following notation:

Notation 112.11. Given $v \in S^2$ we denote by

$$\begin{array}{rcl}\pi_v \colon \mathbb{R}^3 & \to & v^\perp := \{w \in \mathbb{R}^3 \mid \langle v, w\rangle = 0\}\\ w & \mapsto & w - \langle v, w\rangle \cdot v\end{array}$$

the projection onto the orthogonal complement of v.

Now we can formulate the lemma.

Lemma 112.12. (Link Generic Projection Lemma) Let $L \subseteq \mathbb{R}^3$ be an m-component link and let $f \colon \bigsqcup_{i=1}^m S^1_i \to L$ be a diffeomorphism. We consider the following three properties of a vector $v \in S^2$:

(a) The map $\pi_v \circ f \colon \bigsqcup_{i=1}^m S^1_i \to v^\perp = \{w \in \mathbb{R}^3 \mid \langle v, w\rangle = 0\}$ is an immersion.

(b) Whenever $z \neq w \in \bigsqcup_{i=1}^m S^1_i$ satisfy $(\pi_v \circ f)(z) = (\pi_v \circ f)(w)$, then $(\pi_v \circ f)'(z)$ and $(\pi_v \circ f)'(w)$ are linearly independent.

(c) For every $P \in (\pi_v \circ f)\big(\bigsqcup_{i=1}^m S^1_i\big)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of either one or two points.

The set

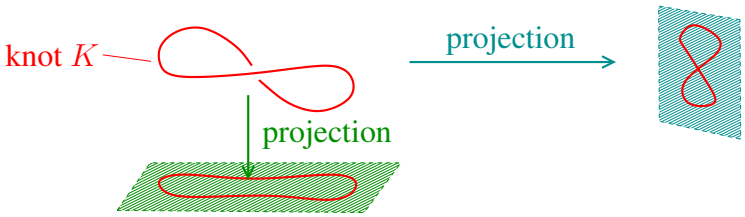
$$\{v \in S^2 \mid v \text{ satisfies (a), (b) and (c)}\}$$

has full measure in S^2.[1182]

Proof of the Link Generic Projection Lemma 112.12. [1183] To simplify the notation we only consider the case of a knot. The proof for links is essentially the same.

[1182] We refer to Definition 37.1 for the definition of a subset of full measure.

[1183] This argument was devised with Gerrit Herrmann.



Let $K \subseteq \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f\colon S^1 \to K$. We say that a subset of a smooth manifold W is *large* if it is open and if it has full measure in W. We make two observations:

- It follows from the Measure-Zero Properties Proposition 37.3 that the intersection of two large subsets is again large.
- It follows from the Measure-Zero Properties Proposition 37.3 that any self-diffeomorphism of S^2 sends large sets to large sets.

It is convenient to consider an extra property of vectors in S^2 which is somewhat weaker than the above property (c):

(c′) given any $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of finitely many points.

We write
$$V(a) \;:=\; \{v \in S^2 \mid v \text{ satisfies (a)}\}.$$
Similarly we define $V(b), V(c), V(a,b), V(a,b,c')$ and so on. Note that with this language we need to show that $V(a,b,c)$ has full measure.

Let $v \in S^2$. We start out with the following three elementary observations regarding the projections π_v:

(1) Given $u \in \mathbb{R}^3 \setminus \{0\}$ we have
$$\pi_v(u) = 0 \iff u \in \mathbb{R} \cdot v \iff \frac{u}{\|u\|} \in \{v, -v\}.$$
(2) The projection π_v is linear, hence for any $P \in \mathbb{R}^3$ we have $(\mathrm{D}\pi_v)_P = \pi_v$.
(3) It follows from (2) and the Chain Rule 24.17 that given any $z \in S^1$ we have the equality $(\pi_v \circ f)'(z) = \pi_v(f'(z))$.

Next we consider the three maps[1184] [1185]
$$\varphi\colon S^1 \to S^2,\quad z \mapsto \frac{f'(z)}{\|f'(z)\|} \qquad \psi\colon \overbrace{\{(z,w) \in S^1 \times S^1 \mid z \neq w\}}^{=:M} \to S^2,\quad (z,w) \mapsto \frac{f(z)-f(w)}{\|f(z)-f(w)\|} \qquad \&$$
$$\rho\colon S^2 \to S^2,\quad v \mapsto -v.$$

[1184]Since f is an immersion we see that the map φ is well-defined, i.e. we do not divide by zero. Similarly, since f is in particular injective we see that the map ψ is well-defined, i.e. once again we do not divide by zero.

[1185]The astute reader will notice that throughout the argument it might be more reasonable to work with maps φ and ψ that take values in $\mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ instead of taking values in S^2. We stick with the maps to S^2 since S^2 has the advantage that its tangent spaces can be described easily as vector subspaces of $\mathbb{R}^3$, which makes it easy to write down differentials. Also, the approach of working with maps to $\mathbb{R}\mathrm{P}^2$ instead creates an extra layer of notation which is as annoying as our notation which requires the extra map ρ.

We make the following observations:

(4) If X is a subset of full measure (respectively large subset) of S^2, then it follows from the above discussion that $X \cap \rho(X)$ is also a subset of full measure (respectively large).
(5) Since f is smooth we see that the map φ is also smooth.
(6) Given $v \in S^2$ it follows easily from (1) and (3) that $\pi_v \circ f \colon S^1 \to \mathbb{R}^2$ is an immersion if and only if neither v nor $-v$ lies in $\varphi(S^1)$. In other words, we have the equality $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus \varphi(S^1))$.
(7) Note that $M = \{(z,w) \in S^1 \times S^1 \mid z \neq w\}$ is an open subset of the smooth manifold $S^1 \times S^1$, thus it is a smooth manifold in a natural way. With this smooth manifold structure the map $\psi \colon M \to S^2$ is easily seen to be smooth.
(8) For each point $(x,y) \in S^1$ we make the identification[1186] $\mathrm{T}_{(x,y)}S^1 = \mathbb{R}$ via the basis vector $(-y,x)$. Given $(z,w) \in M$ we use the above identification to make the identification $\mathrm{T}_{(z,w)}M = \mathrm{T}_{(z,w)}(S^1 \times S^1) = \mathbb{R}^2$. Furthermore recall that given $v \in S^2$ we already saw in the Physical Tangent Spaces–Lemma 29.2 that we have the equality $\mathrm{T}_v S^2 = v^\perp = \{w \in \mathbb{R}^3 \mid \langle v, w\rangle = 0\} \subseteq \mathbb{R}^3$.
(9) Using the identifications from (8) a rather elementary calculation[1187] shows that for a point $(z,w) \in M$ we can write the differential $\mathrm{D}\psi_{(z,w)}$, viewed as a homomorphism $\mathrm{T}_{(z,w)}M = \mathbb{R}^2 \to \mathrm{T}_{\psi(z,w)}S^2 \subseteq \mathbb{R}^3$, in the following way as a (3×2)–matrix:

$$\mathrm{D}\psi_{(z,w)} \;=\; \frac{1}{\|f(z)-f(w)\|} \cdot \underbrace{\begin{pmatrix} \pi_{\psi(z,w)}(f'(z)) & -\pi_{\psi(z,w)}(f'(w)) \end{pmatrix}}_{(3\times 2)\text{–matrix, in particular two columns}}.$$

(10) It follows from (9) that $(z,w) \in M$ is a regular point of ψ if and only if the vectors $\pi_{\psi(z,w)}(f'(z))$ and $\pi_{\psi(z,w)}(f'(w))$ are linearly independent.
(11) Let $v \in S^2$. It follows immediately from (1) and the linearity of π_v that we have $\pi_v(f(z)) = \pi_v(f(w))$ if and only if $\psi(z,w) = v$ or $\psi(z,w) = -v$.
(12) We denote by $\sigma \colon S^1 \times S^1 \to S^1 \times S^1$ the diffeomorphism given by $(z,w) \mapsto (w,z)$. It follows immediately from the definitions that $\psi \circ \sigma = \rho \circ \psi$.
(13) It follows easily from (3), (10), (11) and (12) that

$$V(b) \;=\; \{\text{regular values of } \psi\}.$$

After these initial remarks we turn to our first claim:

Claim 1.
(α) The sets $V(a)$ and $V(a,b)$ are large.
(β) We have $V(a) = V(a,c')$ and $V(a,b) = V(a,b,c')$.

Proof. We prove the two statements of the claim in several steps.

(i) Since S^1 is compact we obtain from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $S^2 \setminus \varphi(S^1)$ is an open subset of S^2. Furthermore, since φ is smooth and since $\dim(S^1) < \dim(S^2)$ we obtain from the Full Measure-Density Proposition 37.4 (5) that $S^2 \setminus \varphi(S^1)$ is a subset of full measure. In summary, we have shown

[1186] As on many other occasions we use the Tangent Space-Isomorphism Proposition 29.24 to view the tangent space of a submanifold of some $\mathbb{R}^n$ as a vector subspace of $\mathbb{R}^n$.
[1187] This calculation can be performed easily using the Chain Rule 24.17 and by writing ψ as the composition of the map $(z,w) \mapsto f(z) - f(w)$ followed by the map $P \mapsto \frac{P}{\|P\|}$.

that $S^2 \setminus \varphi(S^1)$ is large. Note that it follows from (4) and (6) that we have the equality $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus (\varphi(S^1)))$ is large.

(ii) It follows from Sard's Theorem 38.1 that the set of regular values of ψ is a subset of full measure. It follows from (i) and (13), together with the Full Measure-Density Proposition 37.4 (1), that $V(a,b) = V(a) \cap V(b)$ has full measure.

(iii) Let $v \in V(a)$. It follows from (11), from the fact that S^1 is compact and from Exercise 30.7 that $\psi^{-1}(v)$ is finite. In particular we see that $V(a,c') = V(a)$ and thus also $V(a,b,c') = V(a,b)$.

(iv) It remains to show that $V(a,b)$ is open. Let $v \in V(a,b) = V(a) \cap V(b)$. From (13) and (iii), together with the Stack-of-Records Lemma 32.3 (2), we obtain an open neighborhood U of $v \in S^2$ that is still contained in $V(b)$. Since $V(a)$ is open we see that $U \cap V(a)$ is an open neighborhood of $v \in S^2$ that is still contained in $V(a,b)$. It follows from the JH-Openness Criterion 3.3 that $V(a,b)$ is indeed open. ⊞

Before we continue it is perhaps helpful to summarize what we have shown so far: In the claim we have seen that $V(a,b,c') = V(a,b)$ is large, in particular that it has full measure. But our actual goal is to show that $V(a,b,c)$ has full measure.

To do so we consider the map

$$\begin{array}{rccc} \Xi\colon & \overbrace{\{(x,y,z) \in S^1 \times S^1 \times S^1 \,|\, x \neq y, x \neq z \text{ and } y \neq z\}}^{=:N} & \to & S^2 \times S^2 \\ & (x,y,z) & \mapsto & (\psi(x,y), \psi(x,z)). \end{array}$$

We make the following simple observations:

(14) We view N as a smooth manifold in an obvious way. Evidently the map Ξ is smooth.

(15) It follows easily from (1) that we have $v \in V(c)$ if and only if none of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$.

(16) If one of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$, then so does (v,v). This can be seen as follows: If $\Xi(x,y,z) \in \{(\pm v, \pm v)\}$, then $f(x), f(y), f(z)$ lie on the line $\mathbb{R} \cdot v$. After possibly permuting x, y, z we can assume that $f(x) = f(y) + r \cdot v$ and that $f(x) = f(z) + s \cdot v$ with $r, s > 0$. But with this permutation we have $\Xi(x,y,z) = (v,v)$.

Claim 2. We consider the "partial diagonal"

$$\Delta \;:=\; \{(v,v) \in S^2 \times S^2 \,|\, v \in V(a,b)\}.$$

The complement of $\Xi(N) \cap \Delta$ has full measure in Δ.

Proof. In the previous claim we saw that $V(a,b)$ is an open subset of S^2. Since the diagonal $\{(w,w) \in S^2 \times S^2 \,|\, w \in S^2\}$ is a 2-dimensional submanifold of $S^2 \times S^2$ we now see that Δ is a 2-dimensional submanifold of $S^2 \times S^2$. Thus, by Exercise 50.3 it suffices to prove that if $(x,y,z) \in N$ satisfies $\Xi(x,y,z) = (v,v)$ for some $v \in V(a,b)$, then Ξ intersects Δ transversally, i.e. we have the equality

$$(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta \;=\; \mathrm{T}_{(v,v)}(S^2 \times S^2) \;\subseteq\; \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6.$$

We write $p = \pi_v(f'(x))$, $q = \pi_v(f'(y))$ and $r = \pi_v(f'(z))$ and we write $\mu = \frac{1}{\|f(x)-f(y)\|}$ and $\nu = \frac{1}{\|f(x)-f(z)\|}$. By (3), (10) and (13), by the fact that $v \in V(b)$ and by the fact that x, y, z are pairwise different we know that $\{p,q\}$, $\{p,r\}$ and $\{q,r\}$ are linearly independent pairs. In particular p and q form a basis for $\mathrm{T}_v S^2$. Note that this implies that $\begin{pmatrix} p \\ p \end{pmatrix}$ and $\begin{pmatrix} q \\ q \end{pmatrix}$

form a basis for $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. Thus, using (9) we see that

$$(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta \;=\; \text{span of the columns of } \begin{pmatrix} \mu\cdot p & -\mu\cdot q & 0 & p & q \\ \nu\cdot p & 0 & -\nu\cdot r & p & q \end{pmatrix}.$$

Note that

$$\operatorname{rank}\begin{pmatrix} \mu\cdot p & -\mu\cdot q & 0 & p & q \\ \nu\cdot p & 0 & -\nu\cdot r & p & q \end{pmatrix} \;=\; \operatorname{rank}\begin{pmatrix} \mu\cdot p & -\mu\cdot q & 0 & p & q \\ (\nu-\mu)\cdot p & \mu\cdot q & -\nu\cdot r & 0 & 0 \end{pmatrix}.$$

Since $\{q,r\}$ and $\{p,q\}$ are linearly independent pairs we see that the rank of the matrix is 4. Thus $(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta$ is a 4-dimensional vector subspace of $\mathbb{R}^6$. Since this vector subspace is contained in the 4-dimensional vector subspace $\mathrm{T}_{(v,v)}(S^2 \times S^2)$ we see that it equals $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. ⊞

We consider the "diagonal map" $d\colon V(a,b) \to \Delta \subseteq S^2 \times S^2$ that is given by $d(v) = (v,v)$. One can easily show that $d\colon V(a,b) \to \Delta$ is a diffeomorphism. It follows from (15) and (16) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$. By Claim 2 we know $\Delta \setminus \Xi(N)$ is a subset of full measure of Δ. Since d is a diffeomorphism we obtain from the Measure-Zero Properties Proposition 37.3 that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$ has full measure in $V(a,b,c)$. We saw above that $V(a,b)$ has full measure in S^2. It follows easily from the Measure-Zero Properties Proposition 37.3 that $\Delta \setminus \Xi(N)$ has full measure in S^2. ■

Proof of the Link Diagram Existence Proposition 112.7. Recall that in the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17 we showed that every link in S^3 is smoothly isotopic to a link in the subspace $\mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. It follows from the Diffeotopies-via-Matrices Lemma 41.12 (2) that it remains to prove the following claim:

Claim. Given any (oriented) link $L \subseteq \mathbb{R}^3$ there exists an $A \in \mathrm{SO}(3)$ such that the projection of $A \cdot L$ onto the xy-plane defines a link diagram for L.

Proof. Let $L \subseteq \mathbb{R}^3$ be an m-component link. We pick a diffeomorphism $f\colon \bigsqcup_{i=1}^{m} S^1_i \to L$ and we pick $v \in S^2$ as in the Link Generic Projection Lemma 112.12. It follows from the Orthogonal Action Lemma 2.37 (4) that, possibly after applying a matrix in $\mathrm{SO}(3)$ to L, we can assume that $v = \mathfrak{e}_3$. We define $\gamma \coloneqq \pi_{\mathfrak{e}_3} \circ f\colon S^1 \to \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$. It follows immediately from the fact that $v = \mathfrak{e}_3$ has the three properties (a), (b) and (c) stated in the Link Generic Projection Lemma 112.12 that γ is diagrammatic. Next we consider the function

$$\begin{aligned} \epsilon\colon S^1 &\to \mathbb{R} \\ w &\mapsto z\text{-coordinate of } f(w). \end{aligned}$$

Furthermore, given $z_1 \neq z_2 \in S^1$ with $\gamma(z_1) = \gamma(z_2)$ we define

$$c(z_i) \;\coloneqq\; \begin{cases} +1, & \text{if } \epsilon(z_i) > \epsilon(z_{3-i}), \\ -1, & \text{if } \epsilon(z_i) < \epsilon(z_{3-i}). \end{cases}$$

It is clear that (γ, c) is a link diagram. We pick a function $\eta\colon S^1 \to [-1,0]$ as in the Diagram-to-Link Lemma 112.5. We consider the map

$$\begin{aligned} H\colon S^1 \times [0,1] &\to \mathbb{R}^3 \\ (z,t) &\mapsto (\gamma(z), (1-t)\cdot\epsilon(t) + t\cdot\eta(t)). \end{aligned}$$

Clearly this map is smooth. Given any $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$. From this observation we deduce that each H_t is actually a smooth

embedding. Thus we see that H is a smooth isotopy from L to the link associated to the link diagram (γ, η). ■

112.5. Exercises for Chapter 112 .

Exercise 112.1. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be a smooth map such that the following condition is satisfied:

(∗) If $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.

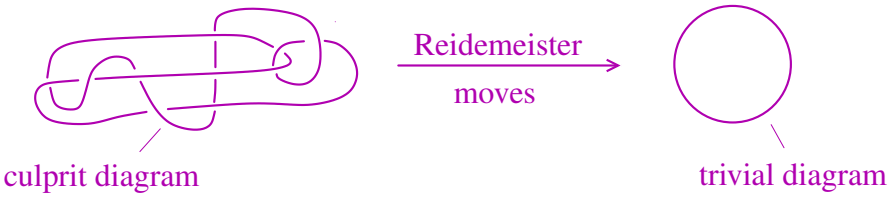
A point $z \in S^1$ for which there exists a $w \neq z \in S^1$ with $\gamma(z) = \gamma(w)$ is called a double point of γ. Does γ necessarily have finitely many double points?

Exercise 112.2. We consider the two knot diagrams shown in the figure below. Use Reidemeister moves to show that the associated knots are smoothly isotopic.

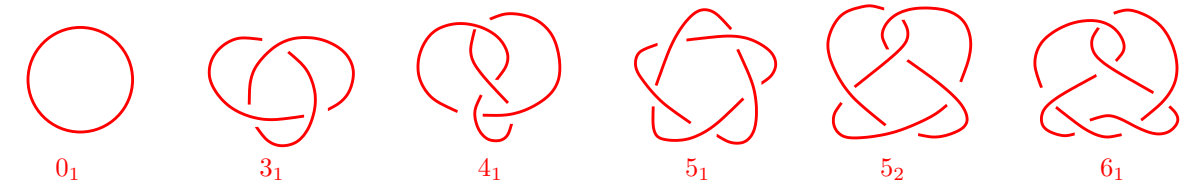


Exercise 112.3. Use Reidemeister moves to turn the diagram to the left into the trivial diagram on the right.

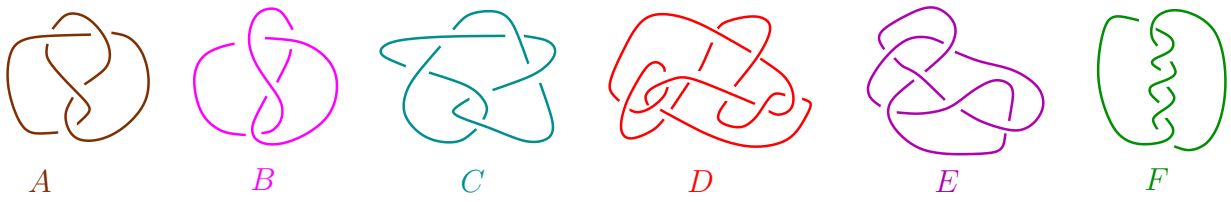
Remark. This diagram is sometimes called the *culprit diagram*, since one first needs to *increase* the number of crossings.



Exercise 112.4. We show again the list of knots from page 2183.



In the following we show six different knots. Each knot is smoothly isotopic (up to mirror image) to precisely one of the knots from the above table. Match the knots to the corresponding knot in the table.



Exercise 112.5. Let $L \subseteq S^3$ be a link. Recall that the crossing number $c(L)$ is defined as the smallest number of crossings of a link diagram for L.

(a) What are the knots with crossing number ≤ 4?
(b) What are the 2-component links with crossing number ≤ 4?
(c) What are the 3-component links with crossing number ≤ 4?
(d) What are the 4-component links with crossing number ≤ 4?

Exercise 112.6. Let $L \subseteq S^3$ be a link. Recall that the crossing number $c(L)$ is defined as the smallest number of crossings of a link diagram for L. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p, q)$, as defined on page Definition 111.13. What is the best *upper* bound on the crossing number of $T(p, q)$ that you can find?
Remark. It is usually quite hard to give a *lower* bound on the crossing number of a given knot or link.

Exercise 112.7. Let L be an oriented m-component link. Furthermore let $\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2$ and $c\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$ be two maps that form a diagram for L.

(a) What is a diagram for L^{mir}?
(b) What is a diagram for L^{rev}?

Exercise 112.8.

(a) Let D be a knot diagram. Show that there exists a sequence of crossing changes that turns the diagram into the unknot.
(b) Let D be a link diagram. Show that there exists a sequence of crossing changes that turns D into the diagram of a link that is totally splittable, i.e. into the diagram of a link where the components are contained in disjoint smoothly embedded 3-balls.
(c) Let D be a link diagram. Show that there exists a sequence of crossing changes that turns D into the diagram of a trivial link.

Exercise 112.9. We start out with the following definition:

- A *coloring of a link diagram* is a coloring of each strand (i.e. each circle segment between two undercrossings) by one of three colors, say red, green blue, such that at each crossing either all colors are the same or all three colors appear.
- The coloring is called *non-trivial*, if at least two colors get used in the diagram.
- A link diagram is called *colorable* if it admits a non-trivial coloring.

In the figure we show colorings of two diagrams of the trefoil.

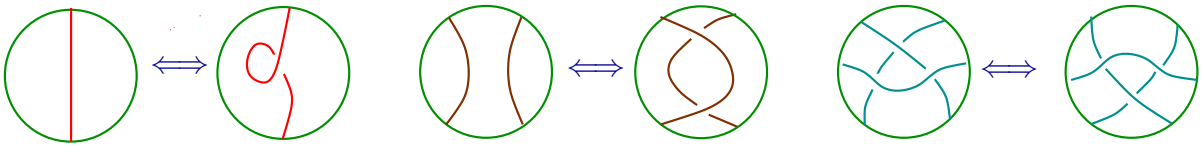
(a) Show that if two diagrams represent smoothly isotopic links, then either both are colorable or neither is colorable. In other words show that colorability is a link invariant and it makes sense to say that a link is colorable.
Hint. Use the Reidemeister Moves Theorem 112.10.



two diagrams for the right-handed trefoil

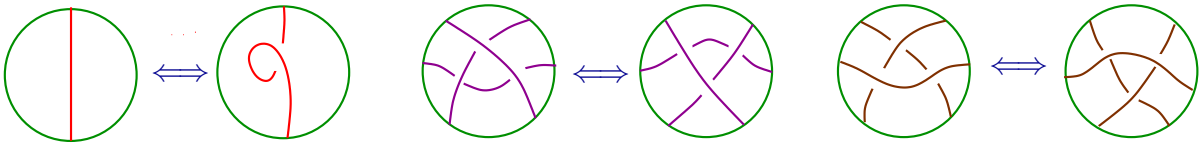
(b) Is the trivial knot colorable?
(c) Is the figure-8 knot colorable?
(d) Let K and $\widetilde{K}$ be two oriented knots. What is the connection between the colorability of K, $\widetilde{K}$ and the colorability of the connected sum $K\#\widetilde{K}$?

Exercise 112.10. In the following we show, for the reader's convenience, the original Reidemeister moves:



Reidemeister Move I Reidemeister Move II Reidemeister Move III

Show that the following alternative Reidemeister moves are a consequence of our original Reidemeister moves.

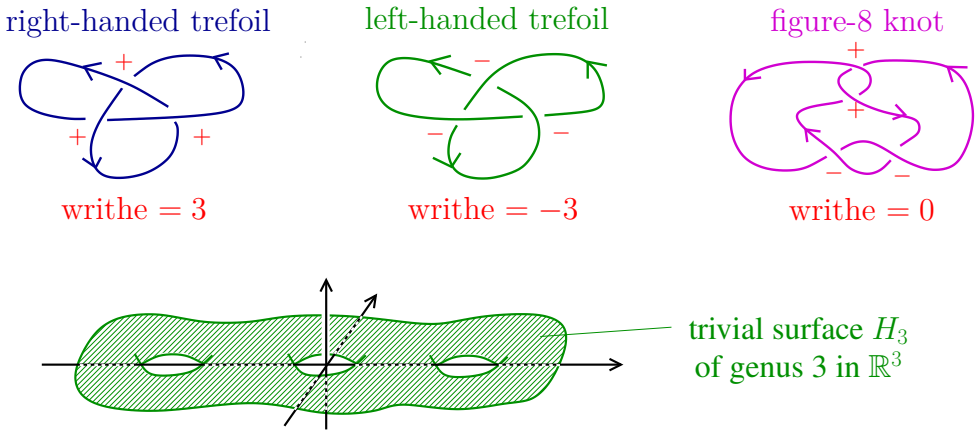


Reidemeister Move I′ Reidemeister Move III′ Reidemeister Move III″

Exercise 112.11. Given a diagram D of a knot K we pick an orientation and we define the writhe $w(D)$ of D as the sum of the signs of the crossings.

(a) Show that the writhe of a knot is independent of the orientation.
(b) Let D and $\widetilde{D}$ be two diagrams for the same knot $K \subseteq S^3$. Show that the following two statements are equivalent:
  (i) D and $\widetilde{D}$ are related by sequence of Reidemeister Moves II and III (and all the obvious variations thereof).
  (ii) The writhes of D and $\widetilde{D}$ agree.
  Remark. Of course you should make use of the Reidemeister Moves Theorem 112.10.

Exercise 112.12. Given $g \in \mathbb{N}_0$ we denote by H_g the "trivial" surface of genus g in $\mathbb{R}^3$ as shown in the figure below. (We decline to give a formal definition.) A torus knot $T(p, q)$ is, essentially by definition, contained on the trivial surface of genus 1. Show that every knot is contained in the trivial surface of some genus g.



Exercise 112.13. We consider the diagram shown below. Is the corresponding knot smoothly isotopic to a trefoil or to the figure-8 knot?



Exercise 112.14. We introduce the following language:

- An *arc* in 3-manifold M is a proper smooth submanifold that is diffeomorphic to $[0,1]$.
- We say that a disjoint collection $A_1, \dots, A_k \subseteq \overline{B}^3$ is *trivial* if it is properly smoothly isotopic to the collection of arcs $\overline{B}^3 \cap (\{(\frac{i}{2k}, 0)\} \times \mathbb{R})$, $i = 1, \dots, k$.
- We say that a link $L \subseteq S^3 = S^3_{\geq 0} \cup_{S^3_{=0}} S^3_{\leq 0} =_{\mathrm{i}} \overline{B}^3_+ \cup_{S^2} \overline{B}^3_-$ is in *bridge position* if L intersects S^2 transversally and if $L \cup \overline{B}^3_+$ and $L \cap \overline{B}^3_-$ is a trivial collection of arcs.

Now we can formulate the following exercises:

(a) Show that every link is smoothly isotopic to a link in bridge position.
Hint. Use diagrams.

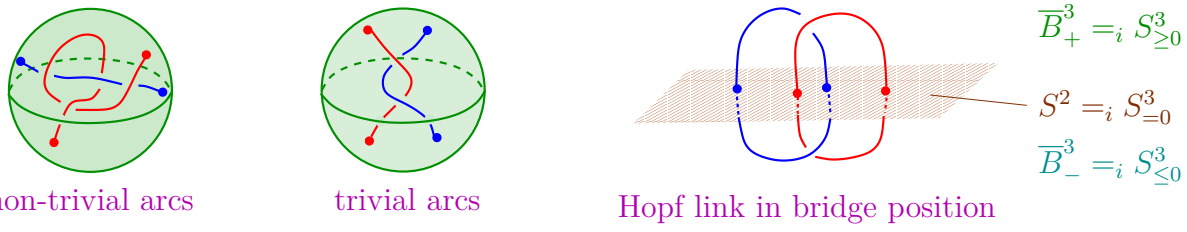
(b) Let L be an m-component link. Show that $\pi_1(S^3 \setminus L)_{\mathrm{ab}} \cong \mathbb{Z}^m$.

We define the *bridge number* $b(L)$ of a link L to be the minimal k such that $b(L)$ is smoothly isotopic to a link in bridge position and such that the number of arcs equals k.

(c) Show that $b(K) = 1$ if and only if K is the trivial knot.

(d) Show that the bridge number of the trefoil and the figure-8 knot equals 2.

(e) The *meridional rank* $m(L)$ of a link L is the minimal $k \in \mathbb{N}_0$ such that $\pi_1(S^3 \setminus L)$ is generated by k meridians. Show that $m(L) \leq b(L)$.
Remark. It is an open question whether $m(L) = b(L)$ holds for all links.



CHAPTER 113

Wirtinger presentations

In Chapter 111 we calculated the isomorphism types of the fundamental group of the complement of any torus knot. But that approach does not generalize to complements of other knots and links. In the previous chapter we learned about link diagrams and we saw that every link can be described by a link diagram.

In this chapter we will explain the Wirtinger algorithm which, given a link diagram for a link L, lets us determine a presentation of $\pi_1(S^3 \setminus L)$. In principle this allows us to calculate the fundamental group for any given link, provided we have a diagram. The problem is, that it is quite hard to extract meaningful information from the fundamental group.

In this chapter we will develop just enough tools to at least show that the trivial knot, the trefoil and the figure-8 knot are pairwise different, i.e. we will show that they are pairwise not smoothly isotopic. Soon, in Chapter 116, we will learn about the Alexander polynomial which will be a powerful way to extract invariants from fundamental groups of link complements.

This chapter is organized as follows:

- In Section 113.1 we will recall and introduce some elementary notions regarding diagrams.
- In Section 113.2 we will formulate the Wirtinger algorithm for obtaining a presentation of the group $\pi_1(S^3 \setminus L)$ from a link diagram.
- In Section 113.3 we will prove the Wirtinger Presentation Proposition 113.6.
- In Section 113.4 we will give more applications of the Wirtinger Presentation Proposition 113.6. In particular we will calculate the abelianization of link groups.
- In Section 113.5 we will formulate the Reidemeister-Schreier process and use it to extract more invariants from knot groups.
- In Appendix 113.6 we will formulate and prove the HNN-Gluing Theorem 113.6 which we used in the proof of the Wirtinger Presentation Proposition 113.6.
- In Section 113.7 we will provide exercises for Chapter 113.

In the next Chapter 114 we will use the calculation of the abelianization of knot groups to introduce the linking number of two disjoint oriented knots in S^3.

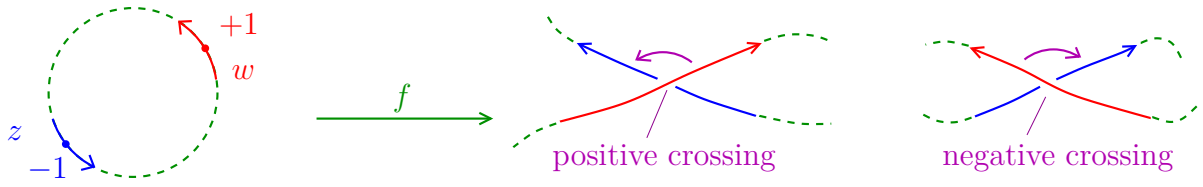
113.1. **Diagrams, crossings, strands.** In the previous chapter we showed in the Link Diagram Existence Proposition 112.7 that we can describe any link via a link diagram. In the next section we will present an explicit algorithm that, given a link diagram of a link L, produces a presentation for the fundamental group of the complement $S^3 \setminus L$.

Before we can state the promised algorithm we need to introduce a few more definitions.

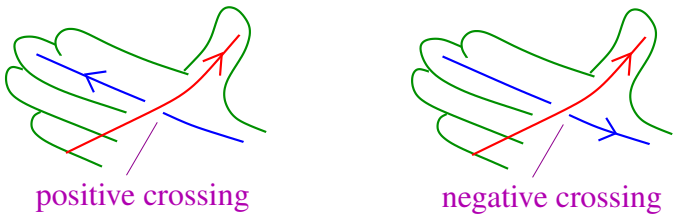
Definition 113.1. Let $\left(f\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram, in the sense of Definition 112.4.

(1) As in Definition 112.4 we refer to the images of the double points as the crossings of the link diagram.

(2) Let $x \in f\left(\bigsqcup_{i=1}^{m} S_i^1\right)$ be a crossing. Let $z, w \in \bigsqcup_{i=1}^{m} S_i^1$ with $f(z) = f(w) = x$ and such that $c(z) = -1$ and $c(w) = 1$, in other words, z is an undercrossing and w is an overcrossing. We say that c is a positive crossing if the ordered basis $(f'(w), f'(z))$ of derivatives[1188] is a positive basis[1189] for $\mathbb{R}^2$. Otherwise we say that x is a negative crossing.



Informally speaking the mnemonic for positive and negative crossing can be formulated as follows: At a crossing, place the right hand such that the thumb points into the direction of the "overstrand". If the fingers point into the direction of the "understrand", then we are dealing with a positive crossing. Otherwise we are dealing with a negative crossing.



Definition 113.2. Let $L \subseteq \mathbb{R}^2 \times [-1, 0] \subseteq \mathbb{R}^3$ be the oriented m-component link that is associated to the link diagram $\left(f\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$, in the sense of Definition 112.6.

(3) We refer to the components of $L \setminus (\mathbb{R}^2 \times \{-1\})$ as the strands of L.



We start out with two minor lemmas:

[1188] Here we equip $\mathbb{R}^2$ with the standard orientation from page 803. By definition a basis $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is a positive basis if $\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} > 0$. An elementary argument, see Exercise 113.1, shows that this holds precisely if $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is obtained from $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ by multiplication by some $\lambda > 0$ and counterclockwise rotation by some angle in $(0, \pi)$.

[1189] We refer to Notation 112.1 for the definition of the derivatives $f'(y)$ at a point $y \in S^1$.

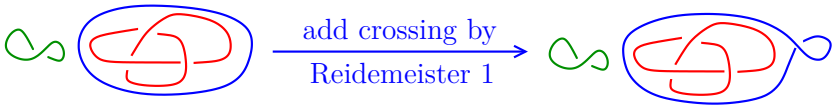
Lemma 113.3. (Crossing–Strand Lemma) If each component of a link diagram has at least one crossing, then the number of crossings equals the number of strands.

Proof. This lemma will be proved in Exercise 113.2. ■

The following lemma says that we can always arrange that the hypothesis of the Crossing–Strand Lemma 113.3 is satisfied:

Lemma 113.4. (One Crossing Exists Lemma) Every link admits a diagram such that each component has at least one crossing.

Proof. Let $L \subseteq S^3$ be a link. By the Link Diagram Existence Proposition 112.7 we know that L admits a link diagram.



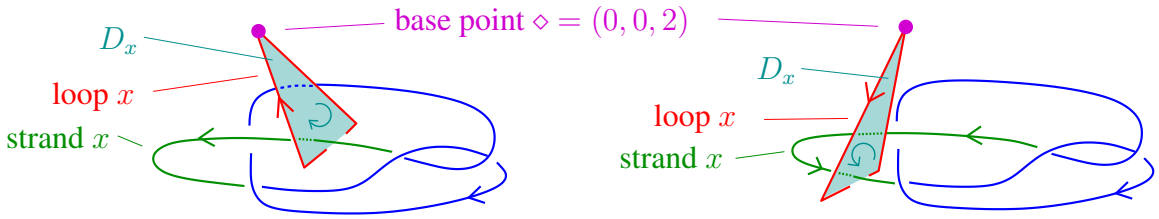
If we have a component without a crossing, then we just apply a Reidemeister 1 move to get a crossing. ■

113.2. **The Wirtinger presentation.** In this section we will see how a diagram of a link L gives rise to a presentation of $\pi_1(S^3 \setminus L)$. We start out by introducing some notation:

Notation 113.5. Let L be a link that is associated to a given link diagram.

(1) We work with the base point $\diamond := (0, 0, 2) \in \mathbb{R}^3$ which lies "above" the link L.

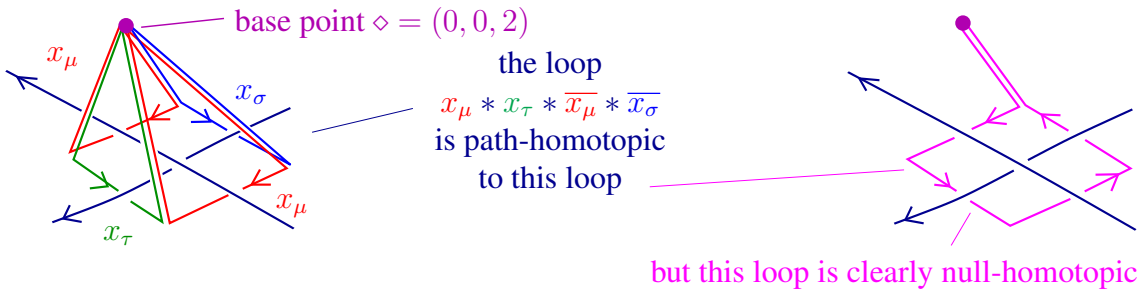
(2) Given a strand x we denote, by abuse of notation, by x also the boundary of any oriented triangle D_x that starts at $\diamond$ and that intersects x_i transversally in a unique point, which lies in the interior of D_x and such that the sign of the intersection point is $+1$.[1190] Note that this loop is path homotopic to a meridian "around the strand x". One can verify, with some patience, that the corresponding element $x \in \pi_1(S^3 \setminus L, \diamond)$ only depends on the choice of the strand.



In the following figure we show a positive crossing where the overstrand is called x_μ and the understrands are called x_σ and x_τ.

The illustration shows that the loop $x_\mu * x_\tau * \overline{x_\mu} * \overline{x_\sigma}$ is null-homotopic. In other words, in the fundamental group $\pi_1(S^3 \setminus L)$ the equality $x_\mu \cdot x_\tau \cdot x_\mu^{-1} \cdot x_\sigma^{-1} = e$ holds. This discussion motivates hopefully the following proposition, which shows how to determine the fundamental group of the complement of a link that is associated to a link diagram. We will sacrifice a little bit of rigor in the formulation in an attempt to make it more readable.

[1190] Loosely speaking the loops x "circles once around the strand x" according to the "right-hand rule".



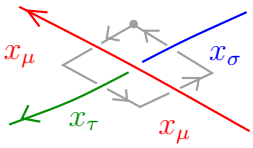
Proposition 113.6. (Wirtinger[1191] Presentation Proposition) Let $L \subseteq S^3$ be a link that is associated to a given link diagram such that each component has at least one crossing. We enumerate the strands by $x_1, \ldots, x_n$ and we enumerate the crossings by $1, \ldots, n$.[1192] [1193] For the i-th crossing we define a relation $r_i \in \langle x_1, \ldots, x_n \rangle$ as shown in the figure below.

(1) We consider the base point $\diamond = (0,0,2)$. We have an isomorphism

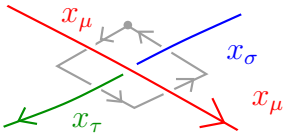
$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond)$$

where the generator x_i gets sent to the loop x_i that we introduced in Notation 113.5.

(2) In (1) we can drop any one of the relations and we still obtain an isomorphism.



the positive crossing gives
rise to the relation
$r_i = x_\mu \cdot x_\tau \cdot x_\mu^{-1} \cdot x_\sigma^{-1}$



the negative crossing gives
rise to the relation
$r_i = x_\mu^{-1} \cdot x_\tau \cdot x_\mu \cdot x_\sigma^{-1}$

Definition 113.7. Given a link L, any presentation with n generators and $n-1$ relators as obtained from the Wirtinger Presentation Proposition 113.6 (2) is called a Wirtinger presentation of $\pi_1(S^3 \setminus L)$.

We now proceed as follows:

(1) We will use the Wirtinger Presentation Proposition 113.6 to give a new calculation of the fundamental group of the complement of the trivial knot, of the trefoil and of the Hopf link.

(2) In the next Section 113.3 we will provide the proof of the Wirtinger Presentation Proposition 113.6.

[1191] The Wirtinger Presentation Proposition was first proved by the Austrian Mathematician Wilhem Wirtinger in 1905, it seems like this result was first published in [**Art1925a**]. We refer to [**Epp1999**, p. 253] for more on the history of this result.

[1192] It follows from the Crossing–Strand Lemma 113.3 that the number of crossings equals the number of strands.

[1193] For a knot it is customary to enumerate the strands cyclically $x_1, \ldots, x_n$, starting at a random strand and to say that the crossing r_i is between strand x_i and x_{i+1}.

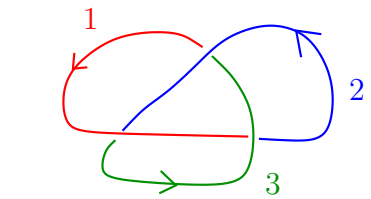
(3) In Section 113.4 we will consider more applications of the Wirtinger Presentation Proposition 113.6.

Examples.

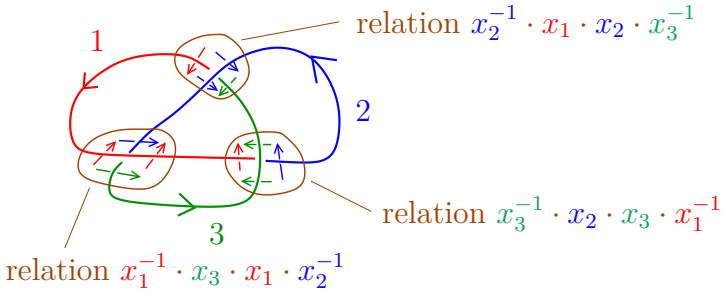
(1) We consider the knot diagram for the trivial knot U that is shown in the figure below. Note that it has one strand and one crossing, so the Wirtinger Presentation Proposition 113.6 applies and it returns that $\pi_1(S^3 \setminus U) \cong \langle x_1 \,|\, \rangle$, which is of course isomorphic to $\mathbb{Z}$. Fortunately this is consistent with the Trivial Knot-π_1-Proposition 111.12



(2) We consider the knot diagram for the trefoil K that is shown in the figure below to the left. We number the strands as shown. To the right we show how the three crossings give rise to three relations.



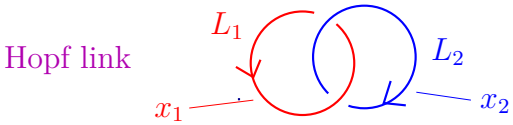
knot diagram for the trefoil with three negative crossings



It follows from this figure and the Wirtinger Presentation Proposition 113.6 (2) that there exists an isomorphism[1194]

$$\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1}, \underbrace{x_1^{-1} \cdot x_3 \cdot x_1 \cdot x_2^{-1}}_{\text{not needed}} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K).$$

(3) We consider the diagram of the Hopf link $H = L_1 \sqcup L_2$ that is shown below.



both crossings are positive, we get relations
$x_1 \cdot x_2 \cdot {x_1}^{-1} \cdot {x_1}^{-1}$
$x_2 \cdot x_1 \cdot {x_2}^{-1} \cdot {x_1}^{-1}$.

We see that

$$\pi_1(S^3 \setminus H) \xleftarrow[\uparrow]{\cong} \langle x_1, x_2 \,|\, x_1 \cdot x_2 \cdot {x_1}^{-1} \cdot {x_1}^{-1} \rangle = \langle x_1, x_2 \,|\, [x_1, x_2] \rangle \underset{\uparrow}{\cong} \mathbb{Z}^2.$$

by the Wirtinger Presentation Proposition 113.6 (2)

by the Free Abelian Group Presentation Lemma 100.14

Fortunately this calculation is consistent with our earlier calculation in the Hopf Link–π_1-Proposition 111.24 (1). □

113.3. **Proof of the Wirtinger Presentation Proposition 113.6.** This section is completely dedicated to the proof of the Wirtinger Presentation Proposition 113.6. In the next

[1194] In the Torus Knot-π_1-Proposition 111.18 we showed that $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$. In particular we have now shown that the two groups $\langle x_1, x_2, x_3 \,|\, x_3^{-1} x_2 x_3 x_1^{-1}, x_2^{-1} x_1 x_2 x_3^{-1} \rangle$ and $\langle x, y \mid x^2 = y^3 \rangle$ are isomorphic. In Exercise 113.5 we will give a purely algebraic proof of this statement.

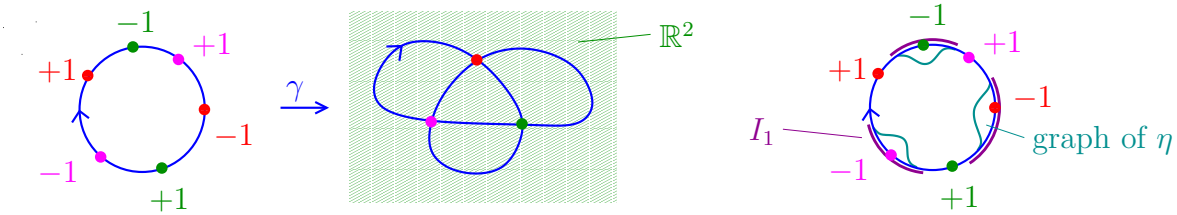
Section 113.4 we will return to the study of applications of the Wirtinger Presentation Proposition 113.6.

For the purpose of this proof it is convenient to work with the following variation on the Diagram-to-Link Lemma 112.5:

Lemma 113.8. (Diagram-to-Link Lemma) Let $\big(\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2, c\big)$ be a link diagram. We can pick disjoint circle segments $I_1, \ldots, I_k$ of $S^1_1 \sqcup \cdots \sqcup S^1_m$ and we can pick a smooth map $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to [-1, 0]$ with the following properties:

(a) Each circle segment I_j[1195] contains a unique undercrossing and each undercrossing is contained in some circle segment I_j.
(b) No circle segment I_j contains an overcrossing.
(c) The value of η outside union of the circle segments I_j is equal to 0.
(d) On each circle segment I_j the function η has a unique local minimum, namely at the corresponding undercrossing where the value of η is equal to -1.

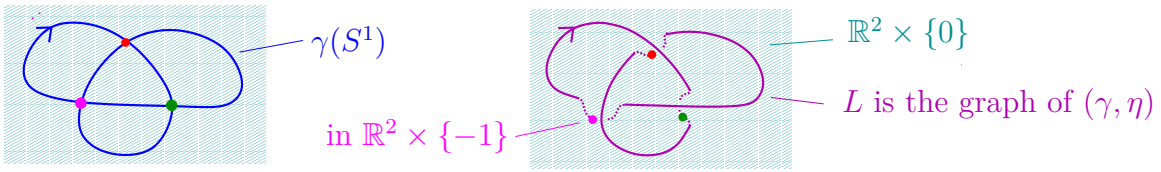
The image of $S^1_1 \sqcup \cdots \sqcup S^1_m$ under the map $(\gamma, \eta)\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^3$ is the link that we associated in Definition 112.6 to the link diagram.



Proof. It follows quite easily from the Smooth Transition Function Lemma 24.9 that such circle segments $I_1, \ldots, I_k$ and that such a smooth function $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to [-1, 0]$ exist. It follows from the Diagram-to-Link Lemma 112.5 (3) that the image of (γ, η) is indeed a link associated to the link diagram. ■

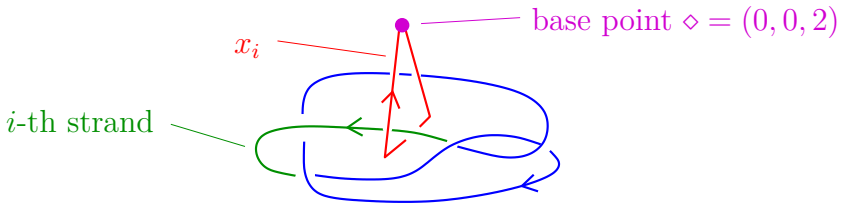
Now we start out by actually preparing the scene for the proof of the Wirtinger Presentation Proposition 113.6:

- Let $(\gamma\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^2, c)$ be a link diagram with n crossings such that each component has at least one crossing.
- We pick a smooth function $\eta\colon S^1_1 \sqcup \cdots \sqcup S^1_m \to [-1, 0]$ as in the statement of the Diagram-to-Link Lemma 113.8.
- We denote by $L \subseteq \mathbb{R}^2 \times [-1, 0]$ the image of the map $S^1_1 \sqcup \cdots \sqcup S^1_m \to \mathbb{R}^3$ that is given by $z \mapsto (\gamma(z), \eta(z))$.



[1195]Recall that according to Definition 2.51 a *circle segment* is a subspace of S^1 of the form $\{\exp(\mathrm{i}\varphi) \,|\, \varphi \in [a, b]\}$ where $[a, b] \subseteq \mathbb{R}$ is an interval of length $b - a < 2\pi$. In other words, a circle segment is a submanifold of S^1 that is diffeomorphic to a compact interval.

- We enumerate the strands by $x_1, \ldots, x_n$. It follows from the Crossing–Strand Lemma 113.3 that we have n crossings. We enumerate the crossings by $1, \ldots, n$.
- We work with the base point $\diamond = (0, 0, 2) \in \mathbb{R}^3$ which lies "above" the link L.
- For the i-th strand we denote, by abuse of notation, as in Notation 113.5 by x_i also the boundary of an oriented triangle such that x starts at $\diamond$ and "circles once around the i-th strand" according to the "right-hand rule". Note that this loop is path homotopic to a meridian "around the strand x_i".
- For the i-th crossing we define a relation $r_i \in \langle x_1, \ldots, x_n \rangle$ as on page 2198.



We break the proof of the statements of the Wirtinger Presentation Proposition 113.6 into three parts:

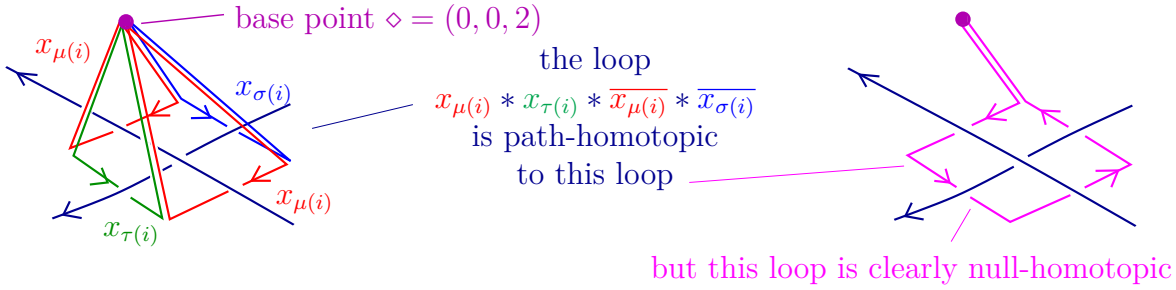
(1) (a) We construct an explicit homomorphism

$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle \;\to\; \pi_1(S^3 \setminus L, \diamond)$$

(b) We show that the homomorphism from (a) is an isomorphism.

(2) We show that in (1a) we can drop one relation.

Proof of the Wirtinger Presentation Proposition 113.6 (1a). Let $i \in \{1, \ldots, n\}$. We consider the i-th crossing. First we deal with the case that the crossing is positive. By "sliding" the various loops $x_{\sigma(i)}$ along the strands and by some further path-homotopies we can assume that we are in the situation in the figure below on the left. If one follows the loop $x_{\mu(i)} * x_{\tau(i)} * \overline{x_{\mu(i)}} * \overline{x_{\tau(i)}}$ on its journey one realizes that on three occasions one goes back and forth to the base point. Thus the loop is actually path-homotopic to the loop shown in the figure below on the right. But that loop is clearly null-homotopic. Almost the same logic applies to negative crossings.



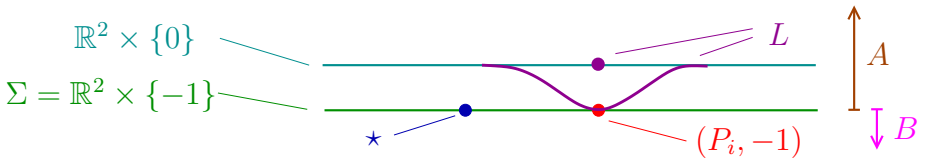
This discussion, together with the Presentation–Homomorphism Lemma 100.13, shows that there exists a unique group homomorphism

$$\langle x_1, \ldots, x_n \,|\, \text{relations corresponding to crossings} \rangle \;\to\; \pi_1(S^3 \setminus L, \diamond)$$

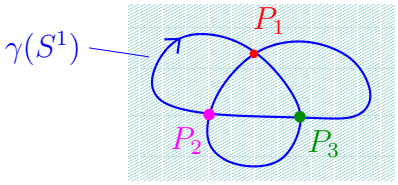
with $x_i \mapsto [x_i]$. ■

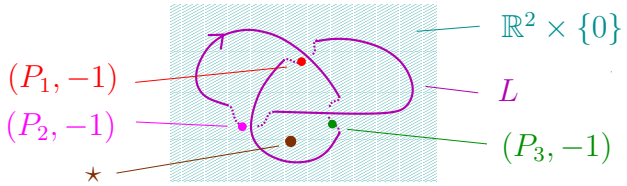
Proof for the Wirtinger Presentation Proposition 113.6 (1b). We continue with the above notation and we add some notation:

- Somewhat similar to the proof of Torus Knot-π_1-Proposition 111.18 we consider the following subspaces of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$:
$$\begin{aligned} A &:= \{(x,y,z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}, \\ B &:= \{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}, \\ \Sigma &:= \{(x,y,z) \in \mathbb{R}^3 \,|\, z = -1\} \cup \{\infty\} \;=\; A \cap B \end{aligned}$$
and we set $A_L := A \setminus L$, $B_L := B \setminus L$ and $\Sigma_L := \Sigma \setminus L$.
- We pick a base point $\star \in \Sigma_L$.
- We denote by $P_1, \ldots, P_n \subseteq \mathbb{R}^2$ the crossings of γ.



- Note that it follows from property (d) of the Diagram-to-Link Lemma 112.5 that
$$L \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) \;=\; L \cap (\mathbb{R}^2 \times \{-1\}) \;=\; \{(P_1,-1), \ldots, (P_n,-1)\}.$$





Claim 0. The inclusion maps induce an isomorphism
$$\pi_1(A_L, \star) *_{\pi_1(\Sigma_L, \star)} \pi_1(B_L, \star) \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus L, \star).$$

Proof. Using the Stereographic Projection Lemma 108.1 and elementary arguments one can make the following observations:

(a) A and B are codimension-zero submanifolds of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ with $A \cup B = S^3$.
(b) A and B are closed subsets of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$.
(c) We have $\partial A = \partial B = A \cap B = \Sigma$.

Since L is a closed subset of $A \cup B$ we obtain easily from the above that the following statements hold:

(a′) A_L and B_L are codimension-zero submanifolds of $S^3 \setminus L$ with $A_L \cup B_L = S^3 \setminus L$.
(b′) A_L and B_L are closed subsets of $S^3 \setminus L$.
(c′) $\partial A_L = \partial B_L = A_L \cap B_L = \Sigma_L$.

It follows from these observations that we can apply the Seifert–van Kampen Theorem 110.8, which gives us immediately the desired result. ⊞

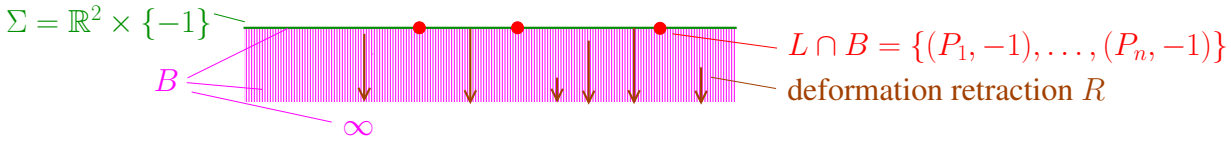
Our next goal is to understand the groups $\pi_1(A_L, \star)$, $\pi_1(B_L, \star)$ and $\pi_1(\Sigma_L, \star)$. We handle these three groups consecutively. We start out with the easiest group on our list, namely the group $\pi_1(B_L, \star)$:

Claim 1. The group $\pi_1(B_L, \star)$ is trivial.

Proof. It follows from the elementary Exercise 19.32 that the map

$$\begin{array}{rcl} R\colon B \times [0,1] & \to & B \\ (P,t) & \mapsto & \begin{cases} \infty, & \text{if } P = \infty \text{ or } t = 1, \\ P - (0,0,\frac{1}{1-t} - 1), & \text{otherwise} \end{cases} \end{array}$$

is continuous. Since R is continuous it is clear that R is a deformation retraction from B to $\{\infty\}$.



Note that this map restricts to a deformation retraction from B_L to $\{\infty\}$. This shows that B_L is contractible. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 (2) that $\pi_1(B_L, \star)$ is indeed trivial. ⊞

In the following we will describe the group $\pi_1(A_L, \diamond)$ where, as before, $\diamond = (0,0,2)$.

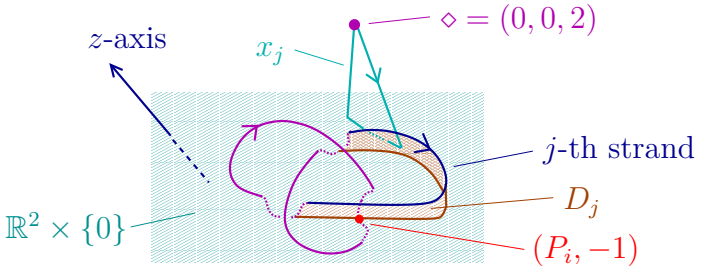
(1) We write

$$D \;:=\; \underbrace{\left\{(\gamma(z),w) \,\middle|\, z \in \bigsqcup_{i=1}^{m} S^1_i \text{ and } w \in [-1,\eta(z))\right\} \setminus \bigcup_{i=1}^{n} \{(P_i,-1)\}}_{\text{the points in } A_L \text{ that lie "below" } L} \;\subseteq\; \mathbb{R}^2 \times [-1,0].$$

(2) We enumerate the strands of the link.

(3) For $j \in \{1,\dots,n\}$ we denote by D_j the unique component of D that contains the j-th strand in its closure.

(4) For $j = 1,\dots,n$ we denote by x_j a loop in $\diamond = (0,0,2)$ that is the boundary of a triangle in $\mathbb{R}^3$ with the following properties:
- The boundary of the triangle intersects D_j in a single point.
- The intersection point is a transverse intersection point with positive sign.



Claim 2. The obvious map $\langle x_1,\dots,x_n\rangle \to \pi_1(A_L,\diamond)$ is an isomorphism.

Proof. We start out with three observations:

(1) Using the Smooth Embedding Theorem 30.15 one can easily verify that each D_j is a proper smooth submanifold of A_L. It follows from the Bicollar Neighborhood Theorem 55.2 that each D_j has a bicollar $\beta\colon D_j \times [-1,1] \to A_L$.

(2) We set $$Z \;:=\; A \setminus \left\{(\gamma(z),w) \,\middle|\, z \in \bigsqcup_{i=1}^{m} S^1_i \text{ and } w \in [-1,\eta(z)]\right\}.$$

Note that $Z \cup (D_1 \cup \cdots \cup D_n) = A_L$.

(3) Similar to Claim 1 we consider the deformation retraction

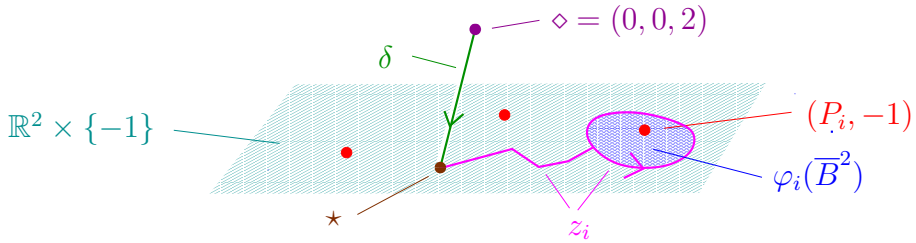
$$\begin{array}{rcl} A \times [0,1] & \to & A \\ (P,t) & \mapsto & \begin{cases} \infty, & \text{if } P = \infty \text{ or } t = 1, \\ P + (0,0,\frac{1}{1-t} - 1), & \text{otherwise} \end{cases} \end{array}$$

from A to $\{\infty\}$. It is straightforward to verify that this map restricts to a deformation retraction from Z to $\{\infty\}$.

(4) Each D_j is simply connected.

It follows from (1) that we can apply the HNN–Gluing Theorem 113.23 (which we formulate and prove in the Appendix 113.6) altogether n times. It follows from (3) and (4) that the obvious map $\langle x_1, \dots, x_n \rangle \to \pi_1(A_L, \diamond)$ is indeed an isomorphism. ⊞

Finally we turn to the group $\pi_1(\Sigma_L, \star)$. We pick orientation-preserving smooth embeddings $\varphi_i \colon \overline{B_L}^2 \to \mathbb{R}^2 \times \{-1\}$, $i = 1, \dots, n$ with disjoint images and such that for each i we have $\varphi_i(0) = (P_i, -1)$. As in the Loop-π_1-Lemma 111.15 we pick paths in Σ_L from $\star$ to a point on $\varphi_i(S^1)$ and we consider the loops in $\star$ that are given by the paths and by going "once around" $\varphi_i(S^1)$. We denote the resulting loops by $z_1, \dots, z_n$.[1196]



Claim 3. The loops $z_1, \dots, z_n$ form a normal generating set of $\pi_1(\Sigma_L, \star)$.

Proof. This statement follows immediately from the Punctured Sphere–Normal Generators Lemma 106.12 (1). ⊞

Let δ be a path in A_L from $\star \in \mathbb{R}^2 \times \{-1\}$ to $\diamond = (0,0,2)$. Now we can finally prove the desired statement. Indeed, we have the following isomorphisms

$$\begin{array}{lll} \pi_1(S^3 \setminus L, \star) & \overset{\text{by Claim 0}}{\underset{\cong}{\leftarrow}} & \pi_1(A_L, \star) *_{\pi_1(\Sigma_L, \star)} \pi_1(B_L, \star) \\ & \xrightarrow[\uparrow]{\delta_*} & \pi_1(A_L, \diamond)/\delta_*\langle\langle \pi_1(\Sigma_L, \star) \rangle\rangle \\ & = & \langle x_1, \dots, x_n \rangle / \langle\langle r_1, \dots, r_n \rangle\rangle. \end{array}$$

$$\begin{array}{lll} & \overset{\text{by Claim 0 and the Amalgamated Product Quotient Lemma 98.25}}{\underset{\cong}{\rightarrow}} & \pi_1(A_L, \star)/\langle\langle \pi_1(\Sigma_L, \star) \rangle\rangle \\ & \underset{\uparrow}{\overset{\cong}{\leftarrow}} & \langle x_1, \dots, x_n \rangle / \langle\langle \delta_*(z_1), \dots, \delta_*(z_n) \rangle\rangle \end{array}$$

isomorphism from the Change-of-Base Point Proposition 88.14

follows from Claims 2 and 3 and the Normal Generator Lemma 100.9 (2) and (4)

as on page 2201 we see that each $\delta_*(z_i)$ is conjugate to r_i, thus it follows from the Normal Generator Lemma 100.9 (3) that $\langle\langle z_1, \dots, z_n \rangle\rangle = \langle\langle r_1, \dots, r_n \rangle\rangle$

■

[1196] Regardless of what the figure might suggest, it is irrelevant where $\star$ lies and whether the paths from $\star$ to $\varphi_i(S^1)$ are embedded or whether they intersect the disks $\varphi_i(\overline{B}^2)$, as long as they lie in Σ_L.

Proof for the Wirtinger Presentation Proposition 113.6 (2). We just showed that we have a presentation

$$\langle x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond).$$

Recall that we need to show that we can drop any one of the relations in the above presentation. Therefore let $j \in \{1, \dots, n\}$. Note that by the Punctured Sphere–Normal Generators Lemma 106.12 (1) we know that the loops $z_1, \dots, \widehat{z_j}, \dots, z_n$ also form a normal generating set of $\pi_1(\Sigma_L, \star)$. Therefore we see that in the above equality we can drop the relation r_j. ■

113.4. **More applications of the Wirtinger Presentation Proposition 113.6 .** The following corollary to the Wirtinger Presentation Proposition 113.6 gives us the disappointing news that our old trick of using abelianizations of fundamental groups is not particularly useful for distinguishing links.

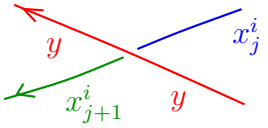
Proposition 113.9. (Link Group-Abelianization Corollary) Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an oriented m-component link in S^3. We denote by $\mu_1, \dots, \mu_m \subseteq S^3 \setminus L$ the meridians of the components of L. There exists a unique isomorphism

$$\Phi \colon \pi_1(S^3 \setminus L)_{\text{ab}} \to \mathbb{Z}^m$$

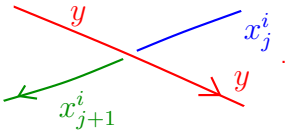
such that[1197] $\Phi(\mu_i) = \mathfrak{e}_i$ for $i = 1, \dots, m$.

Remark. Later, on page 3242, once we have introduced homology groups, we will give a different, arguably much more conceptual proof of the Link Group-Abelianization Proposition 113.9. □

Proof. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an oriented m-component link in S^3. It follows from the Link Diagram Existence Proposition 112.7, the One Crossing Exists Lemma 113.4 and the Link-π_1-Isomorphism Lemma 111.3 that without loss of generality we can assume that the link L is associated to a link diagram that has at least one crossing for each component. In the following we enumerate the strands by $x_1^1, \dots, x_{k_1}^1, x_1^2, \dots, x_{k_2}^2, \dots, x_1^m, \dots, x_{k_m}^m$ where for each $i \in \{1, \dots, m\}$ the strands $x_1^i, \dots, x_{k_i}^i$ are cyclically ordered strands of the i-th component. Furthermore we enumerate the crossings by $c_1^1, \dots, c_{k_1}^1, c_1^2, \dots, c_{k_2}^2, \dots, c_1^m, \dots, c_{k_m}^m$ where any crossing c_j^i separates the strands x_j^i and x_{j+1}^i (where we interpret $x_{k_i+1}^i$ as x_1^i.) Note that the relation r_j^i corresponding to the crossing c_j^i is of the form $y^\epsilon \cdot x_{j+1}^i \cdot y^{-\epsilon} \cdot (x_j^i)^{-1}$ where y is some generator and $\epsilon \in \{-1, 1\}$. In particular any two successive generators x_j^i and x_{j+1}^i are conjugate.



$r_j^i = y \cdot x_{j+1}^i \cdot y^{-1} \cdot (x_j^i)^{-1}$



$r_j^i = y^{-1} \cdot x_{j+1}^i \cdot y \cdot (x_j^i)^{-1}$

It follows from this discussion, the Wirtinger Presentation Proposition 113.6 (1) and the Abelianization-of-Presentation Proposition 100.17 that a presentation matrix for the abelian

[1197] As we discussed in Notation 111.25, the i-th meridian defines an element in $\pi_1(S^3 \setminus L)$ that is well-defined up to conjugation. It follows that the i-th meridian defines a well-defined element of $\pi_1(S^3 \setminus L)_{\text{ab}}$.

group $\pi_1(S^3 \setminus L)_{\text{ab}}$ is given by the block matrix

$$J := \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & J_m \end{pmatrix} \quad \text{where} \quad J_i := \underbrace{\begin{pmatrix} -1 & 0 & \dots & 1 \\ 1 & -1 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -1 \end{pmatrix}}_{\substack{(k_i \times k_i)\text{-matrix that corresponds} \\ \text{to the } i\text{-th component}}} .$$

We now see that

$$\pi_1(S^3 \setminus L)_{\text{ab}} \underset{\substack{\uparrow \\ \text{by the above}}}{\cong} \mathbb{Z}^{k_1+\cdots+k_m}/J \cdot \mathbb{Z}^{k_1+\cdots+k_m} \underset{\substack{\uparrow \\ \text{by the description} \\ \text{of the matrix } J}}{\cong} \bigoplus_{i=1}^{m} \mathbb{Z}^{k_i}/J_i \cdot \mathbb{Z}^{k_i} \underset{\substack{\uparrow \\ \text{by elementary} \\ \text{linear algebra}}}{\cong} \bigoplus_{i=1}^{m} \mathbb{Z} \cong \mathbb{Z}^m .$$

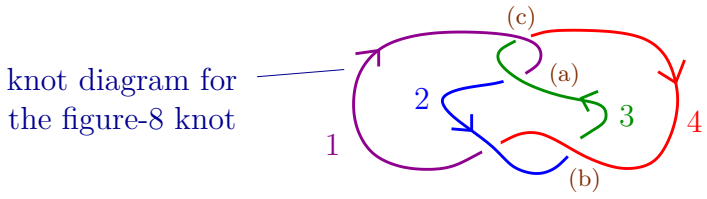
■

After this sobering corollary we move on to the next example.

Lemma 113.10. (Figure 8 Knot–π_1-Lemma) Let J be the oriented figure-8 knot that is shown in the figure below. There exists an isomorphism

$$\pi_1(S^3 \setminus J) \cong \langle x_1, x_2, x_3, x_4 \,|\, \underbrace{x_3 \cdot x_2 \cdot x_3^{-1} \cdot x_1^{-1}}_{\text{crossing (a)}}, \underbrace{x_4^{-1} \cdot x_3 \cdot x_4 \cdot x_2^{-1}}_{\text{crossing (b)}}, \underbrace{x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1}}_{\text{crossing (c)}} \rangle$$

where each x_i corresponds to a meridian of J.



Proof. The lemma follows immediately from the Wirtinger Presentation Proposition 113.6 (2) and staring at the above diagram. We leave it to the reader to provide full details. ■

On its own the Figure 8 Knot–π_1-Lemma 113.10 is not particularly enlightening. It is initially not clear whether this group is isomorphic to $\mathbb{Z}$ (the fundamental group of the complement of the trivial knot) or to the fundamental group of the complement of the trefoil. By the Link Group-Abelianization Proposition 113.9 we know that there is no hope that the abelianization will tell us something interesting.

So we need a different approach. Recall that in the Trefoil–Non-Trivial Proposition 111.19 we used the (non-) existence of an epimorphism onto the symmetric group S_3 to distinguish the trefoil from the trivial knot. The idea now is to repeat this trick with a different finite group. In fact using dihedral groups we can prove the following corollary:

Corollary 113.11. (Trefoil–Figure 8–Non-Isotopic Corollary) The trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.

Proof. Recall that on page 1815 and page 1963 we introduced for each $n \in \mathbb{N}$ the dihedral group $D_n = \langle x, t \,|\, x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle$. It is encouraging that $D_3 \cong S_3$ thus we try our

luck with D_5.[1198] Using the Presentation–Homomorphism Lemma 100.13 and enough time on our hands we obtain the following table:

	trivial knot	trefoil	figure-8 knot
epimorphism onto S_3	✗	✓	✗
epimorphism onto the dihedral group D_5	✗	✗	✓

The corollary follows from this table and the Isotopic Link-π_1-Lemma 111.4. ■

This approach of trying to distinguish knots by the (non-) existence of epimorphisms onto finite groups is rather ad hoc, but as we have seen, not without its merits. This motivates the following notation:

Notation 113.12. Let π be a group. We denote by $\mathcal{F}(\pi)$ the set of isomorphism classes of finite groups G for which there exists an epimorphism $\pi \to G$.[1199]

With this notation we can formulate the following generalization of the Trefoil–Figure 8–Non-Isotopic Corollary 113.11.

Proposition 113.13. (Trefoil–Figure 8–Detection Proposition) Let K be either the trivial knot, the figure-8 knot or a torus knot. Furthermore let $\widetilde{K}$ be another knot. If $\mathcal{F}(\pi_1(S^3 \setminus K) = \mathcal{F}(\pi_1(S^3 \setminus \widetilde{K}))$, then $\widetilde{K}$ is smoothly isotopic to K or to the mirror K^{mir}.

Proof. This proposition is proved in [**BoiF2020, BrR2020**] and [**Wilk2019**, Theorem B]. ■

We conclude this section with the following question which is still open.

Question 113.14. Let K and $\widetilde{K}$ be two knots in S^3 which are either trivial or prime. Suppose that we have an equality $\mathcal{F}(\pi_1(S^3 \setminus K)) = \mathcal{F}(\pi_1(S^3 \setminus \widetilde{K}))$, does it follow that K is smoothly isotopic to $\widetilde{K}$ or to $\widetilde{K}^{\text{mir}}$?

113.5. **The Reidemeister-Schreier process.** Even though the Link Group-Abelianization Proposition 113.9 was quite disappointing, using a clever jiu-jitsu move we will see in this section that we can actually turn the Link Group-Abelianization Proposition 113.9 to our advantage. The key to doing so is contained in the following definition and the subsequent lemma.

Definition 113.15. Let π be a group and let $n \in \mathbb{N}$. We say that a subgroup $\Gamma \subseteq \pi$ is cocyclic of order n if Γ is a normal subgroup such that π/Γ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Lemma 113.16. (Cocyclic Subgroup Lemma) If π is a group whose abelianization is isomorphic to $\mathbb{Z}$, then given any $n \in \mathbb{N}$ there exists a unique subgroup that is cocyclic of order n. It is given by the kernel of any epimorphism $\pi \to \mathbb{Z}/n\mathbb{Z}$.

Proof. Let π be a group whose abelianization is isomorphic to $\mathbb{Z}$ and let $n \in \mathbb{N}$. We denote by $\psi\colon \pi \to \pi_{\text{ab}}$ the natural projection and we pick an isomorphism $\varphi\colon \pi_{\text{ab}} \xrightarrow{\cong} \mathbb{Z}$. It is clear that $(\varphi \circ \psi)^{-1}(n \cdot \mathbb{Z})$ is a subgroup of π that is cocyclic of order n. On the other hand, suppose that Γ is a subgroup of π that is cocyclic of order n. We pick an isomorphism

[1198] The reader might ask, why we do not try our luck with D_4. But for reasons we cannot get into right now, given a knot K and an even number $2n$ there cannot be an epimorphism $\pi_1(S^3 \setminus K) \to D_{2n}$.

[1199] Note that this is indeed a set. For example every finite group with n elements is isomorphic to a subgroup of the permutation group S_n.

$\theta\colon \pi/\Gamma \cong \mathbb{Z}/n\mathbb{Z}$. It follows from the Commutator Subgroup Lemma 98.14 (3) that there exists a unique epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ and a unique isomorphism $\theta\colon \pi/\Gamma \to \mathbb{Z}/n\mathbb{Z}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \pi & \xrightarrow{\psi} & \pi_{\mathrm{ab}} & \xrightarrow{\varphi} & \mathbb{Z} \\ {\scriptstyle g\mapsto g\Gamma}\big\downarrow & \cong & & \cong & \big\downarrow{\scriptstyle \alpha} \\ \pi/\Gamma & & \xrightarrow[\cong]{\theta} & & \mathbb{Z}/n\mathbb{Z}. \end{array}$$

It follows immediately that $\Gamma = (\varphi \circ \psi)^{-1}(n \cdot \mathbb{Z})$. ■

The Link Group-Abelianization Proposition 113.9 and the Cocyclic Subgroup Lemma 113.16 allow us to make the following definition:

Definition 113.17. Given a knot K and $n \in \mathbb{N}$ we denote by $\pi(K, n)$ the unique subgroup of $\pi_1(S^3 \setminus K)$ that is cocyclic of order n.

Example. If K is the trivial knot, then we saw in the Trivial Knot-π_1-Proposition 111.12 that we have an isomorphism $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$. It follows easily that for any $n \in \mathbb{N}$ we have $\pi(K, n) \cong n \cdot \mathbb{Z}$, in particular $\pi(K, n)$ is isomorphic to $\mathbb{Z}$. □

We have now a new tool for trying to distinguish knots:

Lemma 113.18. (Cocyclic Subgroup–Invariant Lemma) Let K and J be two knots. If K and J are smoothly isotopic, then given any $n \in \mathbb{N}$ the groups $\pi(K, n)$ and $\pi(J, n)$ are isomorphic. In particular the abelianizations of these groups are isomorphic.

Proof. The lemma follows almost immediately from the Link Group-Abelianization Proposition 113.9. ■

The question arises, given a knot K and $n \in \mathbb{N}$, how can we determine $\pi(K, n)$ from a given presentation for $\pi_1(S^3 \setminus K)$? Fortunately the following proposition gives us an explicit algorithm. To facilitate the uptake we formulate the following proposition in a slightly informal way.

Proposition 113.19. (Reidemeister-Schreier Process) Let π be a group with presentation $\pi = \langle x_1, \dots, x_m, t \,|\, r_1, \dots, r_l \rangle$ and let $\phi\colon \pi \to \mathbb{Z}/n\mathbb{Z}$ be an epimorphism such that $\phi(x_1) = \cdots = \phi(x_m) = 0$ and $\phi(t) = 1$. Then a presentation for the subgroup $\ker(\phi\colon \pi \to \mathbb{Z}/n\mathbb{Z})$ is given by

$$\langle \underbrace{y_{1,0}, \dots, y_{1,n-1}, \dots, y_{m,0}, \dots, y_{m,n-1}, u}_{m \cdot n + 1 \text{ generators}} \,|\, \underbrace{s_1^0, \dots, s_1^{n-1}, \dots, s_l^0, \dots, s_l^{n-1}}_{l \cdot n \text{ relations}} \rangle$$

where each s_i^j is obtained as follows:

(1) For $i = 1, \dots, l$ and $j = 0, \dots, n-1$ we write $t^j \cdot r_i \cdot t^{-j}$ as an expression in

$$x_1, tx_1t^{-1}, \dots, t^{n-1}x_1t^{-n+1}, \dots, x_m, tx_mt^{-1}, \dots, t^{n-1}x_mt^{-n+1} \text{ and } t^n.$$

(2) We replace each term $t^j x_i t^{-j}$ by $y_{i,j}$ and we replace t^n by u. We obtain a word in the new generators and denote it by s_i^j.

The inclusion map $\ker(\phi\colon \pi \to \mathbb{Z}/n\mathbb{Z}) \to \pi$ is then given by sending each $y_{i,j}$ to $t^j x_i t^{-j}$ and by sending u to t^n.

Proof. This proposition is a special case of a solution to a more general problem: Namely, suppose that we are given a finitely presented group $\pi = \langle X \,|\, R\rangle$ and an epimorphism $\alpha\colon \pi \to G$ onto a finite group. How can we determine a presentation of $\ker(\alpha)$ from the given data? Later, in the Finite-Index Subgroup–Presentation Proposition 138.3 we will see that $\ker(\alpha)$ is finitely presented, but we will not provide an algorithm for finding a presentation.

Such an algorithm is provided from the Reidemeister-Schreier process that is explained in [**LyS1977**, Chapter II.4], [**MKS1976**, Chapter 2.3] or [**Bog2008**, Chapter 2.9]. Our proposition is a special case of this more general algorithm. ■

Instead of studying the proof of the Reidemeister-Schreier Proposition 113.19 our time is better spent trying to understand explicit examples.

Example. Let $K \subseteq S^3$ be the trefoil. We write $\pi = \pi_1(S^3 \setminus K)$. In the Torus Knot-$\pi_1$-Proposition 111.18 we showed that we have an isomorphism $\pi \cong \langle x, t \mid \underbrace{x^3 \cdot t^{-2}}_{=r}\rangle$. It follows from the Presentation–Homomorphism Lemma 100.13 that there exists a unique epimorphism $\phi\colon \pi \to \mathbb{Z}/2\mathbb{Z}$ with $\phi(x) = 0$ and $\phi(t) = 1$. Evidently $\pi(K,2) = \ker(\phi\colon \pi \to \mathbb{Z}/2\mathbb{Z})$. Now we will use the Reidemeister-Schreier Proposition 113.19 to find a presentation $\langle y_0, y_1, u \,|\, s_0, s_1\rangle$ for $\ker(\phi\colon \pi \to \mathbb{Z}/2\mathbb{Z})$. We write

$$\begin{aligned} t^0 \cdot r \cdot t^{-0} &= \underbrace{x^3}_{=y_0^3} \cdot \underbrace{t^{-2}}_{=u^{-1}} \\ t^1 \cdot r \cdot t^{-1} &= t^1 \cdot \underbrace{x^3 \cdot t^{-2}}_{=r} \cdot t^{-1} = t \cdot x^3 \cdot t^{-3} = (txt^{-1})^3 \cdot t^{-2} = (\underbrace{txt^{-1}}_{y_1^3})^3 \cdot \underbrace{t^{-2}}_{=u^{-1}}. \end{aligned}$$

Thus we see that

$$\pi(K,2) \cong \ker(\phi\colon \pi \to \mathbb{Z}/2\mathbb{Z}) \underset{\uparrow}{\cong} \langle y_0, y_1, u \,|\, y_0^3 \cdot u^{-1}, y_1^3 \cdot u^{-1}\rangle \underset{\uparrow}{\cong} \langle y_0, y_1 \,|\, y_0^3 = y_1^3\rangle.$$

by the Reidemeister-Schreier Proposition 113.19

the last relation implies that $y_1^3 = u$, the isomorphism follows from the Tietze transformations from Definition 100.24

Using the Abelianization-of-Presentation Proposition 100.17 we can now easily compute that the abelianization of the above group $\pi(K,2) = \ker(\phi\colon \pi \to \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

This calculation, together with the Cocyclic Subgroup–Invariant Lemma 113.18 and the above discussion of the trivial knot, gives a new proof of the Trefoil–Non-Trivial Proposition 111.19. □

Evidently we would like to continue with the figure-8 knot. But this case is harder than the trefoil. The case of the trefoil was fairly easily to handle, since we started out with a presentation where one generator got sent to one, and the other ones got sent to zero. Unfortunately this is not the case for Wirtinger presentations, so we will need to first "rearrange" the presentation. The key tool for doing so is the following lemma:

Lemma 113.20. (Presentation–Substitution Lemma) Let $X = \{x_1, \ldots, x_k\}$ be a finite set and furthermore let $r = \{r_1, \ldots, r_l\}$ be a finite subset of $\langle X\rangle = \langle x_1, \ldots, x_k\rangle$. Suppose that we can write each $r_i(x_1, \ldots, x_k)$ as a word $s_i(x_1, \ldots, x_{k-1}, x_k \cdot w)$ where w is

a word in $x_1, \dots, x_{k-1}$, then the homomorphism

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; \langle x_1, \dots, x_{k-1}, t \,|\, s_1, \dots, s_l \rangle$$

that is induced by $x_i \mapsto x_i$, $i = 1, \dots, k-1$, and $x_k \mapsto tw^{-1}$ is an isomorphism.

Definition 113.21. We refer to the isomorphism of the Presentation–Substitution Lemma 113.20 as the isomorphism given by the substitution $t = x_k \cdot w$.

Example. A picky reader might decide that the formulation of the Presentation–Substitution Lemma 113.20 is not completely precise. But the following example should make the meaning clear:

$$\langle a, b \,|\, a^{-1}ba^{-1}bab^{-1} \rangle \;=\; \langle a, b \,|\, \underbrace{a^{-1}b \cdot a^{-1}b \cdot a \cdot (a^{-1}b)^{-1} \cdot a^{-1}}_{=a^{-1}ba^{-1}bab^{-1}} \rangle \;\underset{\substack{\uparrow \\ \text{substitution } c = a^{-1}b}}{=}\; \langle a, c \,|\, c \cdot c \cdot a \cdot c^{-1} \cdot a^{-1} \rangle. \qquad \square$$

Proof. It follows from the Presentation–Homomorphism Lemma 100.13 that there exists in fact a unique homomorphism

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; \langle x_1, \dots, x_{k-1}, t \,|\, s_1, \dots, s_l \rangle$$

that has the property that $x_i \mapsto x_i$, $i = 1, \dots, k-1$ and that $x_k \mapsto t \cdot w^{-1}$. Furthermore, we have a map from the right to left by the reverse substitution $t \mapsto x_k \cdot w$. It is straightforward to verify that these homomorphisms are inverses of one another, hence both are isomorphisms. ■

In the following example we consider the figure-8 knot K and we carry out the first steps needed to calculate the isomorphism type of $\pi(K, 2)$.

Example. Let K be the figure-8 knot. In the Figure 8 Knot–π_1-Lemma 113.10 we saw that we have an isomorphism

$$\pi_1(S^3 \setminus K) \;\cong\; \langle y_1, y_2, y_3, y_4 \,|\, y_1^{-1}y_4y_1y_3^{-1}, y_3^{-1}y_2y_3y_1^{-1}, y_2^{-1}y_1y_2y_4^{-1} \rangle.$$

We denote by $\phi\colon \pi_1(S^3 \setminus K) \to \mathbb{Z}/2\mathbb{Z}$ the epimorphism that is given by sending each y_i to 1. We set $t := y_1$. The goal is to replace the generators y_1, y_2, y_3 by generators x_1, x_2, x_3 which satisfy $\phi(x_i) = 0$. In fact we have

$$\begin{aligned}
\pi_1(S^3 \setminus K) &= \langle t, y_2, y_3, y_4 \,|\, t^{-1}y_4ty_3^{-1}, y_3^{-1}y_2y_3t^{-1}, y_2^{-1}ty_2y_4^{-1} \rangle \\
&= \langle t, y_2, y_3, y_4 \,|\, \underbrace{t^{-1}y_4t^{-1}tty_3^{-1}}_{=t^{-1}y_4ty_3^{-1}}, \underbrace{t^{-1}ty_3^{-1}y_2t^{-1}ty_3t^{-1}}_{=y_3^{-1}y_2y_3t^{-1}}, \underbrace{t^{-1}ty_2^{-1}ty_2t^{-1}ty_4^{-1}}_{=y_2^{-1}ty_2y_4^{-1}} \rangle \\
&\underset{\substack{\uparrow \\ \text{substitution } x_i = y_it^{-1},\ x_i^{-1} = ty_i^{-1}, \text{ see} \\ \text{the Presentation–Substitution Lemma 113.20}}}{=} \langle t, x_2, x_3, x_4 \,|\, t^{-1}x_4tx_3^{-1}, t^{-1}x_3^{-1}x_2tx_3, t^{-1}x_2^{-1}tx_2x_4^{-1} \rangle.
\end{aligned}$$

At this point we have a presentation to which we can apply the Reidemeister-Schreier Proposition 113.19. A heroic calculation shows that $\pi(K, 2)_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$. Thus we obtain a new proof that the figure-8 knot and the trefoil are not smoothly isotopic. $\square$

The above procedure for distinguishing knots works fairly well in practice. For example, with enough time at one's hand, or alternatively a convenient computer implementation, one can use the above methods to show that all the knots shown in the figure on page 2183 are in fact pairwise different. In fact this is exactly what was done by James Alexander

and Garland Briggs [**AlB1927**, p. 563] when they first classified "small knots" (without the computer though.)

Nonetheless, the above approach is not overly satisfying. It would be much better to have invariants of knots that are much quicker to calculate. We will come back to this challenge in the next chapter.

113.6. **Appendix: The HNN-Gluing Theorem.**

In this appendix we will state and prove the very general HNN–Gluing Theorem 113.23 which we used in the proof of the Wirtinger Presentation Proposition 113.6, but which is also of independent interest.

It is worth recalling Definition 106.2:

Definition 113.22. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. We refer to

$$\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g) \cdot t \cdot \beta(g)^{-1} \cdot t^{-1}\}_{g\in\Gamma} \rangle\rangle$$

as the HNN-extension corresponding to (π, α, β).

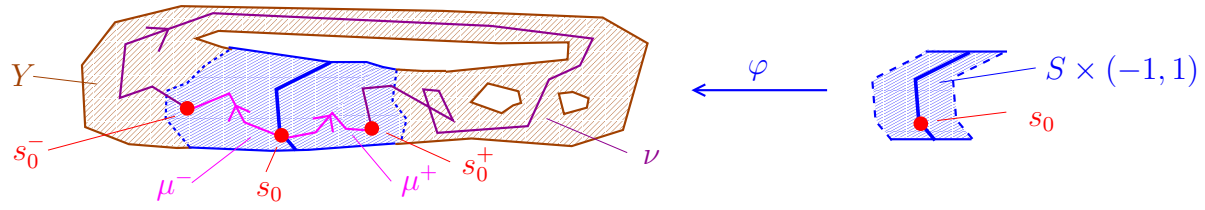
Example. Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t \rangle$. □

Theorem 113.23. (HNN–Gluing Theorem) Let Y be a topological space and let $S \subseteq Y$ be a closed path-connected non-empty subset such that $Y \setminus S$ is also path-connected. Furthermore let $\varphi\colon S \times (-1, 1) \to Y$ be an open embedding such that for each $s \in S$ we have $\varphi(s, 0) = s$. We introduce the following objects:

(1) We denote by $\iota_\pm\colon S \to Y \setminus S$ the map that is given by $x \mapsto \varphi(x, \pm\frac{1}{2})$.
(2) We pick $s_0 \in S$.
(3) We pick $s_0^- \in \varphi(S \times \frac{-1}{2})$ and we pick a path μ^- in $\varphi(S \times [-\frac{1}{2}, 0])$ from s_0 to s_0^-.
(4) We pick $s_0^+ \in \varphi(S \times \frac{1}{2})$ and we pick a path μ^+ in $\varphi(S \times [0, \frac{1}{2}])$ from s_0 to s_0^+.
(5) We pick a path ν in $Y \setminus S$ from s_0^- to s_0^+.
(6) We write $\alpha_- := \iota_{-*}\colon \Gamma = \pi_1(S, s_0) \to \pi_1(Y \setminus S, s_0^-)$.
(7) We write [1200] $\alpha_+ := \nu_* \circ \iota_{+*}\colon \pi_1(S, s_0) \to \pi_1(Y \setminus S, s_0^+) \to \pi_1(Y \setminus S, s_0^-)$.

The inclusion $Y \setminus S \to Y$ and the map $t \mapsto [\nu * \overline{\mu^+} * \mu^-] \in \pi_1(Y, s_0^-)$ induce an isomorphism

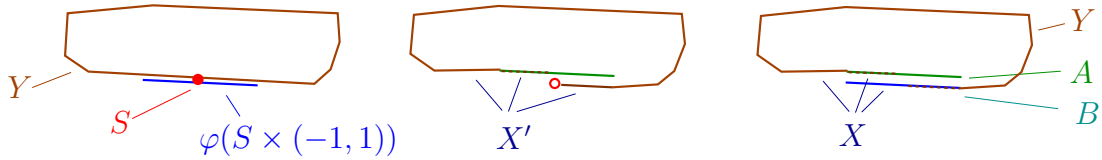
$$\langle \pi_1(Y \setminus S, s_0^-), t \mid \alpha_-(\pi_1(S, s_0)) = t \cdot \alpha_+(\pi_1(S, s_0)) \cdot t^{-1} \rangle \;\xrightarrow{\cong}\; \pi_1(Y, s_0^-).$$



[1200] The homomorphisms $\mu_*\colon \pi_1(Y \setminus S, s_0^+) \to \pi_1(Y \setminus S, s_0^-)$ is the isomorphism given by the Change-of-Base Point Proposition 88.14 (1).

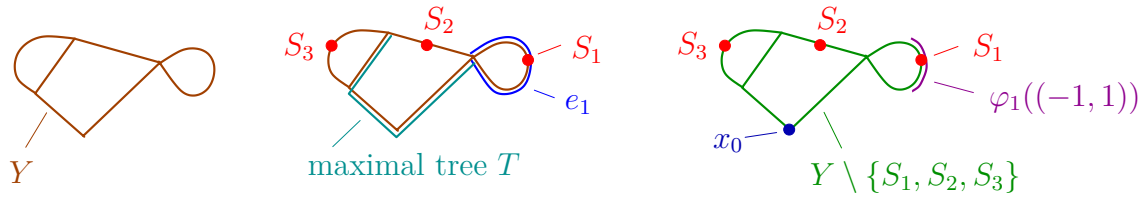
Sketch of proof. As we will see, this theorem follows fairly easily from the HNN-Seifert–van Kampen Theorem 106.7. Let us elaborate a little bit. We consider the topological space X which is given as follows:

(a) First we glue to $Y \setminus S$ the topological space $S \times (-1,1)$ via the given embedding $\varphi\colon S \times (-1,0) \to Y \setminus S$. We call the resulting topological space X'.
(b) Next we glue to X' a second copy of the topological space $S \times (-1,1)$ via the embedding $\varphi\colon S \times (0,1) \to Y \setminus S \subseteq X'$. We call the resulting topological space X.



Step (a) gives us a copy A of $S \times (-1,1)$ in X and Step (b) gives us a second copy B of $S \times (-1,1)$ in X. We denote by $f\colon A \to B$ the obvious homeomorphism. We denote by x_0 the image of s_0^- in X. Let $\gamma\colon [0,1] \to Y$ be the path $\nu * \overline{\mu^+} * \mu^-$. If we now apply the HNN-Seifert–van Kampen Theorem 106.7 to X, A, B, f, x_0 and γ we get, with minor massaging, the desired result. ■

Example. Let Y be a compact connected non-empty topological graph and let $T \subseteq Y$ be a maximal tree. We pick $x_0 \in T$. Let $e_1, \dots, e_n$ be the edges of Y that are not contained in T. For $i = 1, \dots, n$ we pick a point S_i in the interior of e_i. Note that for each $i \in \{1, \dots, n\}$ we can pick an open embedding $\varphi_i\colon (-1,1) \to e_i$ with $\varphi_i(0) = S_i$.



We now see that

$$\pi_1(Y, x_0) \underset{\uparrow}{\cong} \langle \pi_1(Y \setminus \{S_1, \dots, S_n\}, x_0), t_1, \dots, t_n \rangle \underset{\uparrow}{\cong} \langle t_1, \dots, t_n \rangle.$$

we apply the HNN–Gluing Theorem 113.23 altogether n times

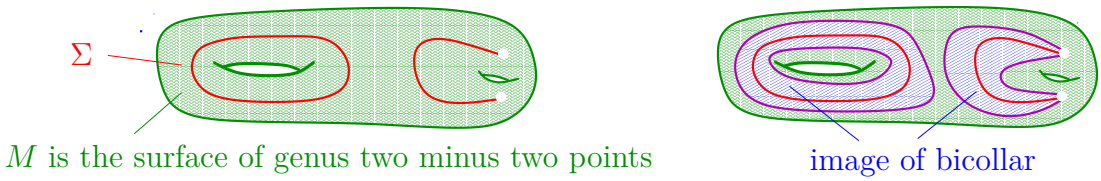
since $Y \setminus \{S_1, \dots, S_n\}$ is contractible

□

In most applications of the HNN–Gluing Theorem 113.23 we will deal with the case that Y is a smooth manifold and that S is a codimension one submanifold of Y. For such a case it is helpful to introduce the following definition, which is a slight generalization of the notion introduced in Definition 55.1.

Definition 113.24. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold of M with $\partial\Sigma = \varnothing$.[1201] A bicollar for the submanifold Σ is a smooth embedding $\beta\colon [-1,1] \times \Sigma \to M \setminus \partial M$ such that the following conditions are satisfied:
(1) For all $x \in \Sigma$ we have $\beta(0,x) = x$.
(2) $\beta([-1,1] \times \Sigma)$ is a closed subset of M.

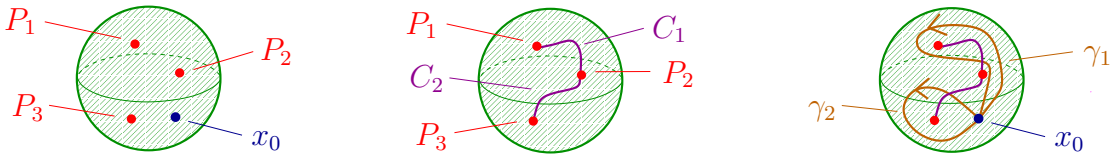
[1201] Since we assume that Σ has no boundary the condition that Σ is a proper submanifold just means that it is a closed subset of M.



We recall the following theorem, which we will use frequently in this part of the lecture notes:

Theorem 55.2. (Bicollar Neighborhood Theorem) Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a proper oriented codimension-one submanifold of M with $\partial\Sigma = \varnothing$. There exists an orientation-preserving bicollar $[-1,1] \times \Sigma \to M$.

Example. Let $P_1, \ldots, P_n$ be pairwise distinct points on S^2. Let $C = C_1 \cup \cdots \cup C_{n-1} \subseteq S^2$ be a 1-dimensional smooth submanifold such that each C_i is diffeomorphic to $[0,1]$ and such that $\partial C_i = \{P_i, P_{i+1}\}$. Let $x_0 \in S^2 \setminus \{P_1, \ldots, P_n\}$ be a base point. Furthermore let $\gamma_1, \ldots, \gamma_{n-1} \colon [0,1] \to S^2 \setminus \{P_1, \ldots, P_n\}$ be oriented smoothly embedded loops in x_0 such that γ_i intersects C_i transversally in a single point and such that $\gamma_i \cap C_j = \varnothing$ for $i \neq j$.



Note that using the Diffeotopies-of-$\mathbb{R}^n$-Proposition 42.5 (3) one can easily show that the topological space $S^2 \setminus (C_1 \cup \cdots \cup C_{n-1})$ is simply connected. It follows from the Bicollar Neighborhood Theorem 55.2, applied to the submanifolds $C_i \cap (S^2 \setminus \{P_1, \ldots, P_n\})$, that we can apply the HNN–Gluing Theorem 113.23 altogether $n-1$ times. We see that the oriented smoothly embedded loops $\gamma_1, \ldots, \gamma_{n-1}$ represent a basis for $\pi_1(S^2 \setminus \{P_1, \ldots, P_n\}, x_0)$ as a free group. □

113.7. Exercises for Chapter 113 .

Exercise 113.1. Recall that by the definition on page 803 a basis $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ of $\mathbb{R}^2$ is a positive basis of $\mathbb{R}^2$ if $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} > 0$. Show that this holds precisely if the second vector $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is obtained from the first vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ by multiplication by some $\lambda > 0$ and counterclockwise rotation by some angle in $(0, \pi)$.
Remark. This exercise is essentially the same as Exercise 31.4.

Exercise 113.2. Show that if a knot diagram has at least one crossing, then the number of crossings equals the number of strands.

Exercise 113.3. Let K_1 and K_2 be two oriented knots in S^3. Show that for $i = 1, 2$ there exists an epimorphism $\pi_1(S^3 \setminus (K_1 \# K_2)) \to \pi_1(S^3 \setminus K_i)$.
Remark. Evidently you should make use of the Oriented Knot–Monoid-π_1-Proposition 111.26. But what result from the current chapter do you also need?

Exercise 113.4. Given $m \in \mathbb{N}$ we denote by $m \cdot K$ the m-fold connected sum of the trefoil (with a fixed orientation). Show that for $m \neq n \in \mathbb{N}$ the knots $m \cdot K$ and $n \cdot K$ are not smoothly isotopic.
Hint. Use Exercise 113.3 and count the number of epimorphisms from fundamental groups onto the permutation group S_3.

Exercise 113.5. Give a purely algebraic proof that the two groups $\langle x, y \mid x^2 = y^3 \rangle$ and $\langle a, b, c \,|\, c^{-1}bca^{-1}, b^{-1}abc^{-1} \rangle$ are isomorphic.
Remark. You could use the Tietze transformations introduced in Definition 100.24 and you could use the Presentation–Substitution Lemma 113.20.

Exercise 113.6.

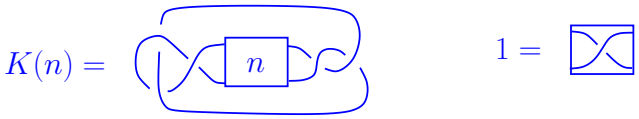
(a) Let K be an oriented knot with diagram D and let $\widetilde{K}$ be an oriented knot with diagram $\widetilde{D}$. How can you obtain a diagram for $K \# \widetilde{K}$ from D and $\widetilde{D}$?

(b) Let K and $\widetilde{K}$ be two oriented knots in S^3. Use the Wirtinger[1202] Presentation Proposition 113.6 to show that there exists an isomorphism

$$\pi_1(S^3 \setminus (K \# \widetilde{K})) \;\cong\; \pi_1(S^3 \setminus K, x_0) *_{\mu_K = \mu_{\widetilde{K}}} \pi_1(S^3 \setminus \widetilde{K}, \widetilde{x}_0).$$

Remark. This gives a new proof of the Oriented Knot–Monoid-π_1-Proposition 111.26.

Exercise 113.7. Given $n \in \mathbb{Z}$ we consider the generalized figure-8 knot $K(n)$ that is shown in the figure below. Show that $\pi_1(S^3 \setminus K(n))$ admits a presentation with two generators.



Exercise 113.8. Let W be the Whitehead link, let B the Borromean rings and for $n \in \mathbb{N}$ let U_n be the trivial n-component link. For which of these links L does there exist an epimorphism $\pi_1(S^3 \setminus L) \to S_3$ where S_3 is the permutation group on three elements?
Remark. You can use the Wirtinger Presentation Proposition 113.6 to determine presentations of the fundamental groups.

Exercise 113.9. We start out with two definitions:

- The *deficiency* of a finite presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ is defined to be $k - l$.
- Let π be a finitely presented group. We define the *deficiency* $\operatorname{def}(\pi)$ *of* π as the maximal deficiency of a finite presentation for π.

Let $m \in \mathbb{N}$. What can you say about the possible deficiencies of the groups $\pi_1(S^3 \setminus L)$ where L is a link with m components?
Remark. It is difficult to give lower bounds on deficiencies, so you can most likely only give approximate answers.

Exercise 113.10. Let $K \subseteq S^3$ be a knot. Show that $\pi_1(S^3 \setminus K)$ admits an epimorphism onto the permutation group S_3 if and only if K is colorable in the sense of Exercise 112.9

Exercise 113.11. Let $n \in \mathbb{N}$.

- An *n-coloring of a link diagram* is a map that assigns to each strand x_j (i.e. each circle segment between two undercrossings) an element $c_j \in \mathbb{Z}/n\mathbb{Z}$, called "color", such that

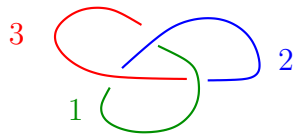
at each crossing the sum of the colors of the understrands equals twice the color of overstrand. In other words, if the understrands are x_i and x_j, and if the overstrand is x_k, then we have $c_i + c_j = 2c_k \in \mathbb{Z}/n\mathbb{Z}$.
- The n-coloring is called *non-trivial*, if it is not a constant function.

In the figure we show a non-trivial 3-coloring of a diagram of the trefoil.

(a) Show that, given a diagram, there exists a bijection between the set of (non-trivial) 3-colorings and the set of (non-trivial) colorings in the sense of Exercise 112.9.
(b) Draw your favorite diagram of the figure-8 knot and find an $n \in \mathbb{N}$ for which there exists a non-trivial n-coloring.
(c) Let $n \in \mathbb{N}$. Show that for a diagram D of a knot K there exists a bijection between following two sets:
  (i) The set of n-colorings of the diagram D.
  (ii) The set of non-trivial homomorphisms from the group $\pi_1(S^3 \setminus K)$ to the dihedral group $D_n := \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z} \cong \langle x, t \,|\, x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle$, which we introduced on page 1815.

Hint. Look at the Wirtinger Presentation Proposition 113.6.

Remark. This exercise shows that the number of n-colorings of a diagram of a knot is an invariant of the knot. This result can also be proved using the Reidemeister Moves Theorem 112.10.

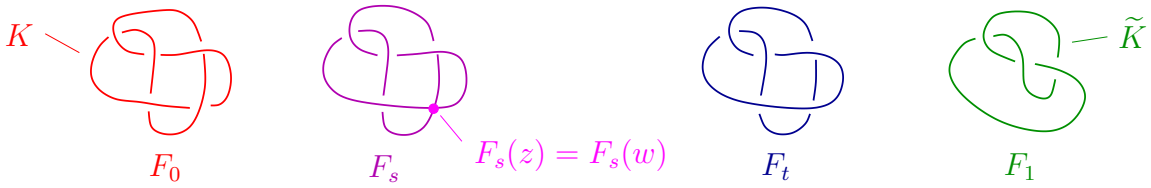


non-trivial 3-coloring of a diagram

Exercise 113.12. We say that two links L and $\widetilde{L}$ in $\mathbb{R}^3$ are related by a *crossing change* if there exists a smooth map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ and some $s \in (0,1)$ with the following properties:

(1) We have $F_0(S^1) = L$ and $F_1(S^1) = \widetilde{L}$.
(2) For every $t \neq s$ the map $F_t\colon S^1 \to \mathbb{R}^3$ is an embedding.
(3) The map $F_s\colon S^1 \to \mathbb{R}^3$ is an immersion and there exist $w, z \in S^1$ with the following properties:
  (i) the restriction of F_s to $S^1 \setminus \{w, z\}$ is an injection,
  (ii) $F_s(w) = F_s(z)$ and the vectors $F_s'(w)$ and $F_s'(z)$ are linearly independent.

In the figure below we illustrate the definition of a crossing change.



Given two links L and $\widetilde{L}$ the minimal number of crossing changes needed to turn L into $\widetilde{L}$ is called the *Gordian distance between L and $\widetilde{L}$*. The Gordian distance of a knot K

to the trivial knot is called the *unknotting number of* K. (In practice these numbers are notoriously difficult to calculate.)

(a) Let K be an oriented knot in S^3. Show that there exists a sequence of crossing changes that turns K into the unknot.
Hint. By the Link Diagram Existence Proposition 112.7 we know that K admits a diagram. Now make use of Exercise 112.8.

(b) Let K and K' be two oriented knots in S^3. Show that there exists a sequence of crossing changes that turns K into K'. In other words, show that the Gordian distance between two knots is always finite.

(c) Let L be an oriented link in S^3. Show that there exists a sequence of crossing changes that turns L into a link where the links are contained in disjoint smoothly embedded 3-balls.

(d) Let L and L' be two oriented links in S^3 with the same number of components. Show that there exists a sequence of crossing changes that turns L into L'. In other words, show that the Gordian distance between two links is always finite.

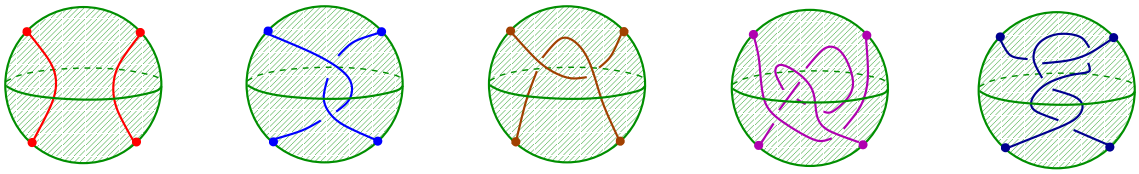
(e) What are the best lower and upper bounds you can find on the Gordian distance between the trefoil and the figure-8 knot?

Remark. For readers who prefer completely formal proofs, and who want to avoid diagrams, it can be a fun challenge to tweak the argument of the proof of the Embeddings-become-Isotopic Criterion **??** to obtain a proof of (b).

Exercise 113.13. A *tangle* is a proper 1-dimensional submanifold of $\overline{B}^3$. We consider the five tangles shown in the figure below. Note that the boundary of each of the five tangles is the same.

(a) For which pairs S, T of the five tangles does there exist an orientation-preserving diffeomorphism $\varphi\colon \overline{B}^3 \to \overline{B}^3$ with $\varphi(S) = T$?

(b) For which pairs S, T of the five tangles does there exist an orientation-preserving diffeomorphism $\varphi\colon \overline{B}^3 \to \overline{B}^3$ with $\varphi(S) = T$ and with $\varphi|_{S^2} = \mathrm{id}_{S^2}$?



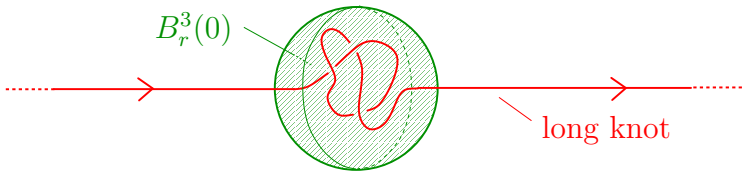
Exercise 113.14. We start out with a few definitions.

(1) A *long knot* is a submanifold K of $\mathbb{R}^3$ that is diffeomorphic to $\mathbb{R}$ and which has the property that there exists an $r > 0$ such that

$$K \cap (\mathbb{R}^3 \setminus B^3_r(0)) \;=\; (\mathbb{R} \times \{(0,0)\}) \cap (\mathbb{R}^3 \setminus B^3_r(0)).$$

A long knot comes with a natural orientation given by "going from $-\infty$ to ∞". We refer to $U \coloneqq \mathbb{R} \times \{0,0\}$ as the *trivial long knot*.

(2) In this exercise we say that two long knots K and J are *equivalent* if there exists a smooth isotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ such that $F_0(\mathbb{R}) = K$, $F_1(\mathbb{R}) = J$ and such that



there exists a $C > 0$ such that for each $t \in [0,1]$ and each $x \in \mathbb{R}$ with $|x| \geq C$ we have $F_t(x) = (x,0,0)$.[1203]

(3) Let L be a long knot. We pick an $r > 0$ such that

$$L \cap B^3_r(0) \subseteq \mathbb{R} \times \{(0,0)\}.$$

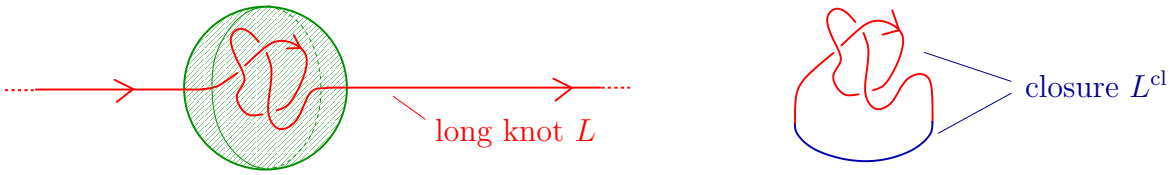
We denote by

$$\begin{array}{rcl} \Phi\colon \overline{B}^3_{2r}(0) & \to & S^3 \\ x & \mapsto & \left(\frac{x}{2r}, \sqrt{1-\left|\frac{x}{2r}\right|^2}\right) \end{array}$$

the obvious smooth embedding whose image is precisely the upper hemisphere. We refer to

$$L^{\mathrm{cl}} := \Phi\big(L \cap \overline{B}^3_{2r}(0)\big) \cup \left\{\left(x,0,0,\sqrt{1-x^2}\right) \mid x \in [-1,1]\right\}$$

as the *closure* L^{cl} *of the long knot* L. Note that the closure inherits an orientation from L.



Now we turn to the actual exercises:

(a) There is a pretty obvious definition of a composition $L\#L'$ of long knots L and L'. Turn this idea into a proper definition.

(b) Show that $L\#L'$ and $L'\#L$ are equivalent.

(c) Show that the fundamental groups of $\mathbb{R}^3 \setminus L$ and $S^3 \setminus L^{\mathrm{cl}}$ are isomorphic.

(d) Show that the closure L^{cl} is well-defined up to smooth isotopy and show that the map

$$\begin{array}{rcl} \{\text{long knots up to equivalence}\} & \to & \{\text{oriented knots up to smooth isotopy}\} \\ [L] & \mapsto & [L^{\mathrm{cl}}] \end{array}$$

is a bijection.

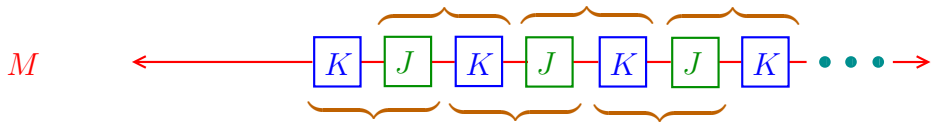
Exercise 113.15. We continue with the discussion of long knots from Exercise 113.15.

(a) Let K and J be two long knots such that $K\#J$ is equivalent to the trivial long knot. We consider the 1-dimensional smooth submanifold M of $\mathbb{R}^3$ that is shown in the figure below. We denote by $U = \mathbb{R} \times \{(0,0)\}$ the trivial long knot. Show that there exists a diffeomorphism $\Phi\colon (\mathbb{R}^3, M) \to (\mathbb{R}^3, U)$ and show that there exists a diffeomorphism $\Phi\colon (\mathbb{R}^3, M) \to (\mathbb{R}^3, K)$.

Hint. Make use of the fact that by Exercise 113.14 (b) we know that $J\#K$ is also equivalent to the long trivial knot.

Remark. The braces are supposed to give you an idea for the proof.

[1203] The discussion in Exercise 44.13 shows that we cannot replace this definition by demanding the existence of proper smooth isotopy.



(b) Let $K \subseteq S^3$ be an oriented knot. Suppose that there exists an oriented knot $J \subseteq S^3$ such that $K\#J$ is the trivial knot. Show that there exists a *homeomorphism* $\Theta\colon S^3 \to S^3$ such that $\Theta(K)$ equals the trivial knot.
Remark. It follows for example from the Trivial Knot-π_1-Recognition Proposition **??** that this implies that K is the trivial knot.

Remark. This argument is known as the "Mazur swindle" [**Maz1960**].

Exercise 113.16. Let $m \in \mathbb{N}$ be odd. We consider the torus knot $T(2,m)$. Show that there exists a presentation

$$\pi_1(S^3 \setminus T(2,m)) \;\cong\; \langle x, y \,|\, \underbrace{xyx \cdot \ldots \cdot xyx}_{\text{length } m} = \underbrace{yxy \cdot \ldots \cdot yxy}_{\text{length } m} \rangle.$$

Remark. This group is sometimes called the *m-the dihedral Artin group.*

Exercise 113.17.

(a) We consider the epimorphism

$$\phi\colon \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a, b \,|\, a^2, b^2 \rangle \;\to\; \mathbb{Z}/2\mathbb{Z}$$

that is given by $\phi(a) = \phi(b) = 1$. Show that $\ker(\phi)$ is isomorphic to $\mathbb{Z}$.
Hint. Use the Presentation–Substitution Lemma 113.20 and the Reidemeister-Schreier Process 113.19.

(b) We consider the connected sum $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ of two copies of the real projective space $\mathbb{R}\mathrm{P}^3$. Note that it follows from the Real Projective Space$-\pi_1$-Proposition 93.3 and the Connected Sum-π_1-Proposition 102.4 that $\pi_1(\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. What is the covering of $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ corresponding to $\ker(\phi)$?

Exercise 113.18. Let A and B be two finitely generated abelian groups such that we have an equality $\mathcal{F}(A) = \mathcal{F}(B)$. In other words, we assume that A and B have the same finite quotients. Are A and B necessarily isomorphic?

Exercise 113.19. Let K be the trefoil. Compute the abelianization of the group $\pi(K,3)$.

Exercise 113.20.

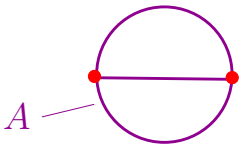
(a) Let K and $\widetilde{K}$ be two oriented knots. What is the connection between $\pi(K,2)$, $\pi(\widetilde{K},2)$ and $\pi(K\#\widetilde{K},2)$?

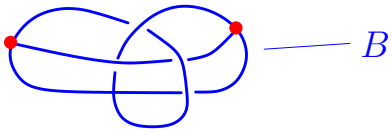
(b) Show that given any $n \in \mathbb{N}$ there exists a knot such that $\pi_1(S^3 \setminus K)$ has at least n generators.

Exercise 113.21. We consider the "spatial graphs" $A \subseteq \mathbb{R}^3$ and $B \subseteq \mathbb{R}^3$ that are shown in the figure below.

(a) (i) Determine a presentation for $\pi_1(\mathbb{R}^3 \setminus A)$.
(ii) What is the abelianization of $\pi_1(\mathbb{R}^3 \setminus A)$?

(b) Replace A by B and again solve (i) and (ii).





Exercise 113.22. Let π be a group. We introduce the following notions:

- Given a subset $S \subseteq \pi$ we introduced in Definition 98.20 the subgroup $\langle\langle S \rangle\rangle$ as the intersection of all normal subgroups of π that contain S.
- We say that a subset $S \subseteq \pi$ *normally generates* π if $\langle\langle S \rangle\rangle = \pi$.
- We define the *weight of* π as

$$w(\pi) := \min\{\#S \mid S \text{ is a subset of } \pi \text{ that normally generates } \pi\} \in \mathbb{N}_0 \cup \{\infty\}.$$

The exercise is to determine the weights of various groups:

(a) Determine the weight of $\mathbb{Z}^m$.
(b) Determine the weight of the free group on m generators.
(c) Let $L \subseteq S^3$ be an m-component link. Show that $w(\pi_1(S^3 \setminus L)) = m$.
Hint. Use Wirtinger presentations.

Remark. This exercise can be viewed as a continuation of Exercise 111.31.

Exercise 113.23. Use the HNN–Gluing Theorem 113.23 to calculate the fundamental groups of the torus and the Klein bottle, just using the fact that $\pi_1(S^1) \cong \mathbb{Z}$.

CHAPTER 114

Linking numbers

In this short chapter we introduce the linking number of two disjoint oriented knots. The linking number is one of the most basic invariants of topology, which we will encounter on several occasions and in different guises throughout these notes.

This chapter is organized as follows:

- In Section 114.1 we will introduce the linking number of two disjoint oriented knots K and $\widetilde{K}$ and we will prove same basic properties of linking numbers.
- In Section 114.2 we will see how one can calculate linking numbers from link diagrams. This will allow us to show that the linking number is symmetric in K and $\widetilde{K}$.
- In Section 114.3 we will provide exercises for Chapter 114.

In the next Chapter 115 we will use linking numbers to introduce the longitude of an oriented knot.

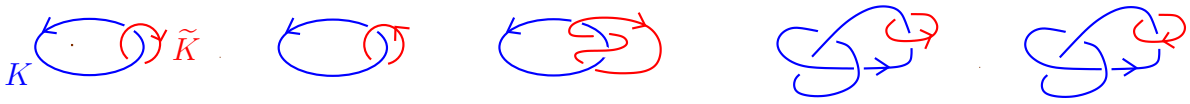
114.1. **Definition of Linking numbers and basic properties.** The Link Group-Abelianization Proposition 113.9 allows us to introduce the following basic and rather useful definition:

Definition 114.1. Let K and $\widetilde{K}$ be two disjoint oriented knots. By the Link Group-Abelianization Proposition 113.9 we have $\pi_1(S^3 \setminus K)_{\text{ab}} = \mathbb{Z} \cdot \mu_K$. Note that it follows from the Loop-π_1-Lemma 111.15 and the discussion in Notation 111.16 that $\widetilde{K}$ gives rise to a well-defined element of $\pi_1(S^3 \setminus K)_{\text{ab}}$ which we denote by $[\widetilde{K}]$. We now define the linking number $\text{lk}(K, \widetilde{K})$ to be the unique integer such that the following equality holds in $\pi_1(S^3 \setminus K)_{\text{ab}} = \mathbb{Z} \cdot \mu_K$:

$$[\widetilde{K}] \;=\; \text{lk}(K, \widetilde{K}) \cdot \mu_K \;\in\; \pi_1(S^3 \setminus K)_{\text{ab}} = \mathbb{Z} \cdot \mu_K.$$

Remark. In Definition 52.2 we already defined a linking number using a very different approach. Only much later, in the Linking Number Reformulation Proposition 285.8, will we see that the two definitions of linking numbers match. □

Example. In the figure below we show some basic examples of linking numbers:



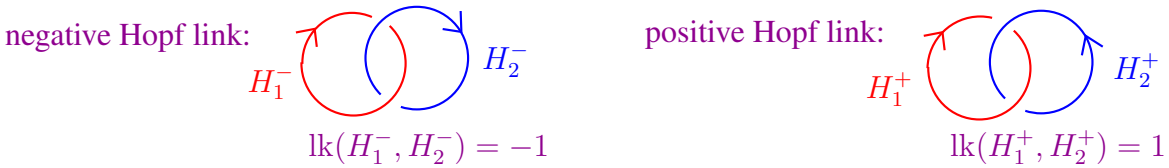
$\text{lk}(K, \widetilde{K}) = 1$ $\text{lk}(K, \widetilde{K}) = -1$ $\text{lk}(K, \widetilde{K}) = 2$ $\text{lk}(K, \widetilde{K}) = +1$ $\text{lk}(K, \widetilde{K}) = -1$

These result from the observation that for the trivial knot K we have $\pi_1(S^3 \setminus K) = \mathbb{Z} \cdot [\mu_K]$ and from the fact that for any oriented knot K we clearly have $\text{lk}(K, \mu_K) = 1$. □

In the following lemmas we collect some basic facts about linking numbers. The following lemma says that if we change the link formed by two disjoint oriented knots by a smooth isotopy, then the linking number does not change:

Lemma 114.2. (Linking Number–Isotopy-Lemma) Let K and J be a pair of two oriented disjoint knots and let $\widetilde{K}$ and $\widetilde{J}$ be another pair of two oriented disjoint knots. If the link $K \sqcup J$ is smoothly isotopic to the link $\widetilde{K} \sqcup \widetilde{J}$, then $\mathrm{lk}(K, J) = \mathrm{lk}(\widetilde{K}, \widetilde{J})$.

Example. In the figure below we consider the two Hopf links H^- and H^+:



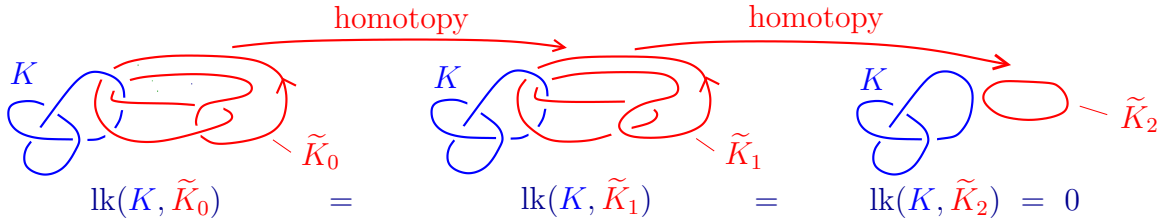
It follows from the Linking Number–Isotopy-Lemma 284.2 together with the calculation of the linking numbers in the figure preceding the lemma that the two Hopf links H^- and H^+ are not smoothly isotopic. □

Proof. This statement follows quite easily from Smooth Isotopy Characterization Proposition 283.1 (1)⇒(3) and the definitions. ■

The next lemma is perhaps a little more surprising. It says, that if we keep the first knot fixed, then we can change the second knot by a *homotopy* in the complement of the first knot:

Lemma 114.3. (Linking Number–Homotopy-Lemma) Let $K \subseteq S^3$ be an oriented knot and let $\widetilde{K}$ and $\widetilde{K}'$ be two oriented knots in the complement of K. If $\widetilde{K}$ and $\widetilde{K}'$ are homotopic in the complement of K, i.e. if there exists a continuous map $F\colon \widetilde{K} \times [0,1] \to S^3 \setminus K$ such that $F_0 = \mathrm{id}$ and such that $F_1\colon \widetilde{K} \to \widetilde{K}'$ is an orientation-preserving diffeomorphism, then

$$\mathrm{lk}(K, \widetilde{K}) \;=\; \mathrm{lk}(K, \widetilde{K}').$$



Proof. It follows easily from our hypothesis and the Homotopies-of-Loops Proposition 88.29 that $[\widetilde{K}] = [\widetilde{K}'] \in \pi_1(S^3 \setminus K)_{\mathrm{ab}}$. It follows immediately from this observation that $\mathrm{lk}(K, \widetilde{K}) = \mathrm{lk}(K, \widetilde{K}')$. ■

Lemma 114.4. (Linking Number–Orientation Lemma) For any disjoint oriented knots K and $\widetilde{K}$ we have

$$\mathrm{lk}(\overset{\downarrow}{K^{\mathrm{rev}}}, \widetilde{K}) \;=\; \mathrm{lk}(K, \overset{\downarrow}{\widetilde{K}^{\mathrm{rev}}}) \;=\; -\mathrm{lk}(K, \widetilde{K}).$$

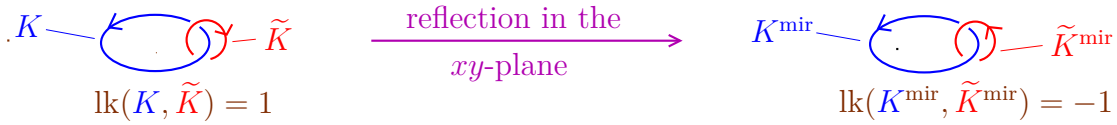
knots with orientation reversed

Proof. The lemma follows easily from the definitions and the following two facts:

(1) It follows from the Meridian–Symmetries Lemma 109.3 together with the Loop-π_1-Lemma 111.15 (3) that $[\mu_{K^{\mathrm{rev}}}] = [(\mu_K)^{\mathrm{rev}}] = -[\mu_K] \in \pi_1(S^3 \setminus K)_{\mathrm{ab}}$.
(2) It follows from the Loop-π_1-Lemma 111.15 (3) that $[\widetilde{K}^{\mathrm{rev}}] = -[\widetilde{K}] \in \pi_1(S^3 \setminus K)_{\mathrm{ab}}$. ■

Lemma 114.5. (Linking Number–Mirror Image Lemma) For any two disjoint oriented knots K and $\widetilde{K}$ we have the equality

$$\mathrm{lk}(K^{\mathrm{mir}}, \widetilde{K}^{\mathrm{mir}}) \;=\; -\mathrm{lk}(K, \widetilde{K}).$$



Proof. Let K and $\widetilde{K}$ be two disjoint oriented knots. Let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane. We have the following equalities in $\pi_1(S^3 \setminus K^{\mathrm{mir}})_{\mathrm{ab}}$:

$$\begin{aligned}
\mathrm{lk}(K^{\mathrm{mir}}, \widetilde{K}^{\mathrm{mir}}) \cdot \mu_{K^{\mathrm{mir}}} &\overset{\text{Definition 114.1 of } \mathrm{lk}(K^{\mathrm{mir}}, \widetilde{K}^{\mathrm{mir}})}{=} [\widetilde{K}^{\mathrm{mir}}] \\
&= [\mathrm{lk}(K, \widetilde{K}) \cdot \rho(\mu_K)] \\
&\overset{\text{Definition 108.23 of } K^{\mathrm{mir}}}{=} [\rho(\widetilde{K})] \\
&= \mathrm{lk}(K, \widetilde{K}) \cdot [\rho(\mu_K)] \\
&\overset{\text{Definition 114.1 of } \mathrm{lk}(K, \widetilde{K})}{=} [\rho(\mathrm{lk}(K, \widetilde{K}) \cdot \mu_K)] \\
&= \mathrm{lk}(K, \widetilde{K}) \cdot (-\mu_{K^{\mathrm{mir}}}).
\end{aligned}$$

by the Meridian–Symmetries Lemma 109.3 (2) we know that a meridian for $K^{\mathrm{mir}} = \rho(K)$ is given by $\rho(\mu_K)^{\mathrm{rev}}$, the equality now follows from the Loop-π_1-Lemma 111.15 (3)

From this calculation it follows that $\mathrm{lk}(K^{\mathrm{mir}}, \widetilde{K}^{\mathrm{mir}}) = -\mathrm{lk}(K, \widetilde{K})$. ■

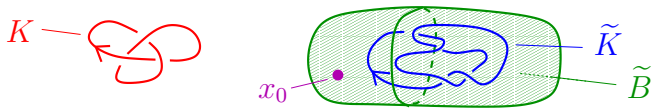
The following lemma implies that if the linking number of two disjoint knots is non-zero, then they form a non-splittable link:

Lemma 114.6. (Linking Number-of-Split Link Lemma) Let K and $\widetilde{K}$ be two disjoint oriented knots. If the link $K \cup \widetilde{K}$ is splittable, then $\mathrm{lk}(K, \widetilde{K}) = 0$.

Proof. Let K and $\widetilde{K}$ be two disjoint oriented knots such that the link $K \sqcup \widetilde{K}$ is splittable. It follows almost immediately from the definition of a splittable link in Definition 109.5 that we can write $S^3 = B \cup \widetilde{B}$ where B and $\widetilde{B}$ are smooth submanifolds with the following properties:

(a) B and $\widetilde{B}$ are diffeomorphic to $\overline{B}^3$.
(b) We have $B \cap \widetilde{B} = \partial B = \partial \widetilde{B}$.
(c) $K \subseteq B \setminus \partial B$ and $\widetilde{K} \subseteq \widetilde{B} \setminus \partial \widetilde{B}$.

Note that by (c) we have $S^3 \setminus K = (B^3 \setminus K) \cup \widetilde{B}$.

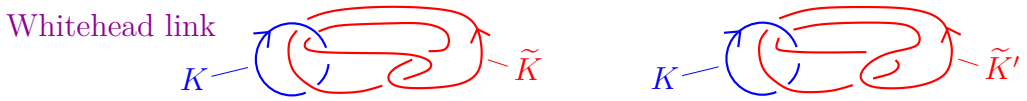


It now follows from (b) and the Manifold–Seifert-van Kampen Theorem 110.8 that for any $x_0 \in \partial B = \partial \widetilde{B}$ the inclusions $B^3 \setminus K \to S^3 \setminus K$ and $\widetilde{B} \to S^3 \setminus K$ induce an isomorphism

$$\pi_1(B^3 \setminus K, x_0) * \pi_1(\widetilde{B}, x_0) \;\to\; \pi_1(S^3 \setminus K, x_0).$$

Since $\pi_1(\widetilde{K}) \to \pi_1(\widetilde{B}) \cong \{e\}$ is the trivial map we now see that $\pi_1(\widetilde{K}) \to \pi_1(S^3 \setminus K)$ is the trivial map. This implies that $[\widetilde{K}]$ is the trivial element in $\pi_1(S^3 \setminus K, x_0)$. This in turn implies that $\mathrm{lk}(K, \widetilde{K}) = 0$. ■

Example. It is not completely unreasonable to ask whether the converse to the Linking Number-of-Split Link Lemma 114.6 holds. In the figure below we show two disjoint oriented knots K and $\widetilde{K}$. Note that in the complement of K the smoothly embedded closed curve $\widetilde{K}$ is *homotopic* to the smoothly embedded closed curve $\widetilde{K}'$ that is shown in the figure to the right.



As above on page 2221 it follows easily from the above figure and the Linking Number–Homotopy-Lemma 114.3 that $\mathrm{lk}(K, \widetilde{K}) = 0$. It seems unlikely that $K \cup \widetilde{K}$ is splittable. But as of right now we lack the tools to prove that the link is indeed not splittable. We will deal with this issue in Exercise 117.15. □

114.2. **Linking numbers via diagrams.** Our next goal is to find a practical way for calculating linking numbers. Before we can achieve this goal it is convenient to remind ourselves of the definition of positive and negative crossings from Definition 113.1:

Definition 114.7. A crossing in a link diagram is called positive (respectively negative) if the direction of the upper strand followed by the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$.



From the definition it is not clear how to calculate the linking number. To calculate the linking number we need a way to describe elements in a Wirtinger presentation. This leads us to the following lemma, which is of more general interest:

Lemma 114.8. (Element-in-Wirtinger Presentation Lemma) Let $L \subseteq S^3$ be an oriented link and let $J \subseteq S^3 \setminus L$ be an oriented knot. We assume that the link $L \cup J$ is associated to a link diagram such that each component has at least one crossing. We consider the base point $\diamond = (0, 0, 2)$. Let

$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond)$$

be the explicit isomorphism from the Wirtinger Presentation Proposition 113.6. We pick a base point P on J, away from a crossing. We do the following:

(a) We denote by $J^\diamond$ the loop that goes straight from $\diamond$ to P, goes once around J in the given direction, and that finally goes straight back to $\diamond$.

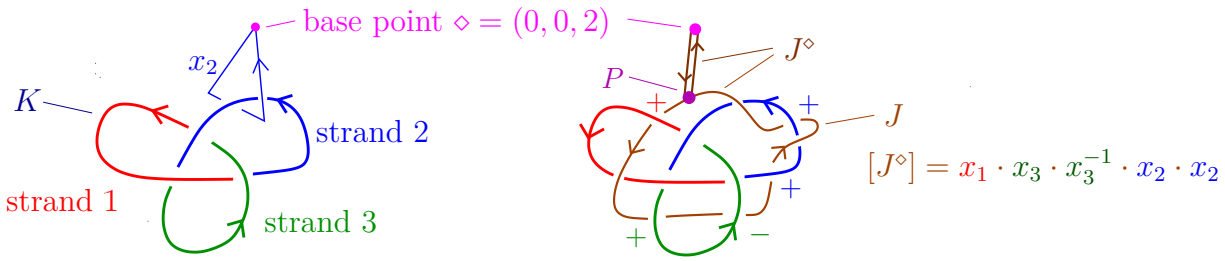
(b) We start with the empty word. We walk along J in the positive orientation, starting and ending at P. Whenever we go under L (at the strand i of L) we do the following:
- We right-multiply our word with x_i if it is a positive crossing.

- We right-multiply our word with x_i^{-1} if it is a negative crossing.

We denote the resulting word by $w_J \in \langle x_1, \dots, x_n \rangle$.

Under the above isomorphism the element $[w_J] \in \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle$ corresponds to $[J^\diamond] \in \pi_1(S^3 \setminus L, \diamond)$.

Example. In the following figure we consider the trefoil K with the usual diagram and we show an oriented knot $J \subseteq S^3 \setminus K$:



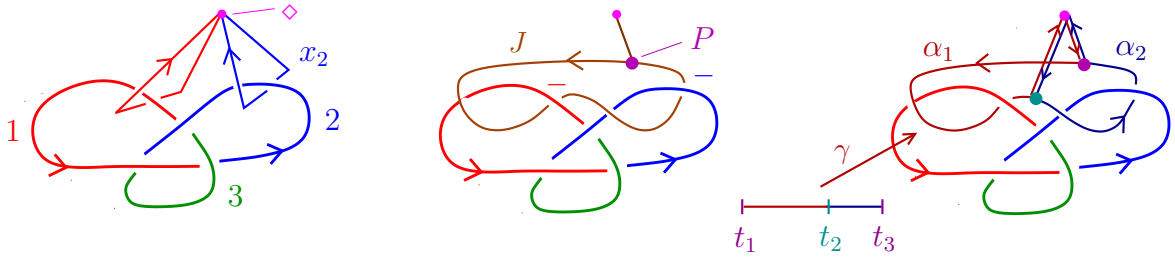
On page 2199 we showed that we have an isomorphism

$$\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K).$$

It follows almost immediately from the Element-in-Wirtinger Presentation Lemma 114.8 that $[J^\diamond] = x_1 \cdot x_3 \cdot x_3^{-1} \cdot x_2 \cdot x_2 = x_1 \cdot {x_2}^2$. □

Sketch of proof. We make the following preparations:

(1) For $i \in \{1, \dots, n\}$ we denote by x_i the loop in $\diamond$ that goes once around the i-th strand in the usual way.
(2) We pick a smooth map $\gamma\colon [0,1] \to S^3$ with the following properties:
  (a) We have $\gamma(0) = \gamma(1) = P$.
  (b) γ descends to a diffeomorphism $\gamma\colon [0,1]/0 \sim 1 \to J$.
(3) We pick $0 = t_1 < t_1 < \dots < t_{m+1} = 1$ such that for each $j \in \{1, \dots, m\}$ the image $\gamma([t_j, t_{j+1}]$ contains precisely one point where we go under L. Let $\sigma_j \in \{1, \dots, n\}$ be the corresponding strand. Furthermore we set $\epsilon_j := +1$ if the crossing is positive and we set $\epsilon_j := -1$ if the crossing is negative.
(4) For $j = 1, \dots, m$ we denote by α_j the loop that starts in $\diamond$, goes straight to $\gamma(t_j)$, goes along γ to $\gamma(t_{j+1})$ and then goes straight back to $\diamond$.



Now we see that in $\pi_1(S^3 \setminus L, \diamond)$ we have the following equalities:

$$[J^\diamond] \underset{\uparrow}{=} [\alpha_1 * \dots * \alpha_m] \underset{\uparrow}{=} [\alpha_1] \cdot \dots \cdot [\alpha_m] \underset{\uparrow}{=} [x_{\sigma_1}^{\epsilon_1}] \cdot \dots \cdot [x_{\sigma_m}^{\epsilon_m}] \underset{\uparrow}{=} [w_J].$$

since the loops $J^\diamond$ and $\alpha_1 * \dots * \alpha_m$ are homotopic

definition of the product in $\pi_1(S^3 \setminus L, \diamond)$

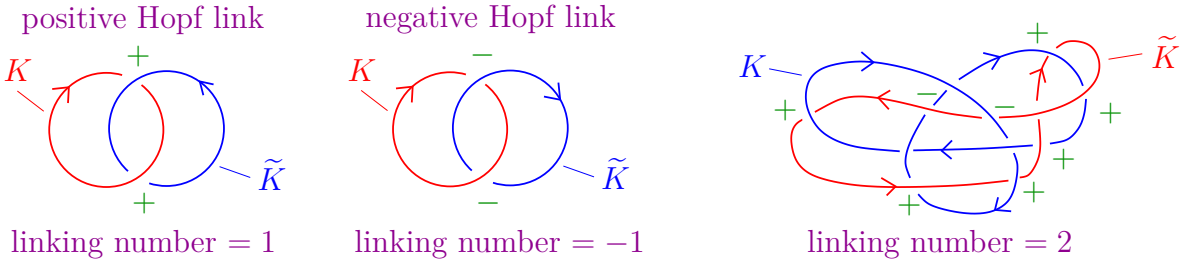
since the loop α_i is homotopic to $x_{\sigma_i}^{\epsilon_i}$

by definition we have $x_{\sigma_1}^{\epsilon_1} \cdot \dots \cdot x_{\sigma_m}^{\epsilon_m} = w_J$ ■

Using the Element-in-Wirtinger Presentation Lemma 114.8 we can now prove the following popular proposition:

Proposition 114.9. (Linking Number-via-Crossings Proposition) Let K and $\widetilde{K}$ be two disjoint oriented knots. If the link $K \cup \widetilde{K} \subseteq S^3$ is associated to a link diagram, then

$$\begin{aligned} \mathrm{lk}(K, \widetilde{K}) \;=\; & \tfrac{1}{2} \cdot \text{number of positive crossings between } K \text{ and } \widetilde{K} \\ - \; & \tfrac{1}{2} \cdot \text{number of negative crossings between } K \text{ and } \widetilde{K}. \end{aligned}$$



Proof. We introduce the following notation: We denote by $U(\widetilde{K}, K)$ the set of crossings where $\widetilde{K}$ crosses under K and we denote by $O(\widetilde{K}, K)$ the set of crossings where $\widetilde{K}$ crosses over K. For each crossing c of K and $\widetilde{K}$ we denote by $\epsilon_c \in \{-1, 1\}$ the sign of the crossing. We start out with the following claim:

Claim 1. We have $\mathrm{lk}(K, \widetilde{K}) \;=\; \sum_{c \in U(\widetilde{K}, K)} \epsilon_c.$

Proof. We consider the base point $\diamond = (0, 0, 2)$. Let

$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus K, \diamond)$$

be the explicit isomorphism from the Wirtinger Presentation Proposition 113.6. Recall that each x_i is a meridian for K. Now we want to apply the Element-in-Wirtinger Presentation Lemma 114.8 to $\widetilde{K}$. Thus we do the following: We pick a base point P on $\widetilde{K}$ and we walk along $\widetilde{K}$ in the positive orientation. At every undercrossing with K (at the strand i of K), we record x_i if it is a positive crossing and we record x_i^{-1} if it is a negative crossing. In summary we get a word of the form $x_{\sigma_1}^{\epsilon_1} \cdot \ldots \cdot x_{\sigma_m}^{\epsilon_m}$ with $\sigma_1, \ldots, \sigma_m \in \{1, \ldots, n\}$ and $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$. We now see that we have the following equalities in the abelian group $\pi_1(S^3 \setminus K; \diamond)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K$:

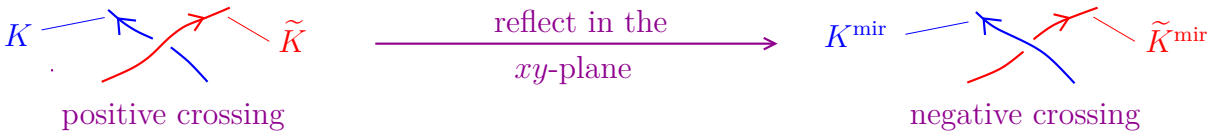
$$\mathrm{lk}(K, \widetilde{K}) \cdot \mu_K \;\underset{\text{by Definition 114.1 of } \mathrm{lk}(K, \widetilde{K})}{=}\; [\widetilde{K}^{\diamond}] \;\underset{\text{by the Element-in-Wirtinger Presentation Lemma 114.8}}{=}\; [x_{\sigma_1}^{\epsilon_1} \cdot \ldots \cdot x_{\sigma_m}^{\epsilon_m}] \;\overset{\text{we use additive notation}}{=}\; \sum_{j=1}^{m} \epsilon_j \cdot [x_{\sigma(j)}] \;\underset{\text{since each } x_i \text{ is a meridian}}{=}\; \sum_{j=1}^{m} \epsilon_j \cdot \mu_K \;=\; \Big(\sum_{c \in U(\widetilde{K}, K)} \epsilon_c \Big) \cdot \mu_K.$$

It follows that $\mathrm{lk}(K, \widetilde{K}) = \sum_{c \in U(\widetilde{K}, K)} \epsilon_c.$ ⊞

Claim 2. We have $\mathrm{lk}(K, \widetilde{K}) \;=\; \sum_{c \in O(\widetilde{K}, K)} \epsilon_c.$

Proof. Let K^{mir} and $\widetilde{K}^{\text{mir}}$ be the reflections of K and $\widetilde{K}$ in the xy-plane. Note that $K^{\text{mir}} \cup \widetilde{K}^{\text{mir}}$ has an obvious diagram with the following properties:

- The points where $\widetilde{K}$ crosses over K become points where $\widetilde{K}^{\text{mir}}$ crosses under K^{mir}.
- The signs of the crossings flip.



We now see that

$$\operatorname{lk}(K,\widetilde{K}) \underset{\uparrow}{=} -\operatorname{lk}(K^{\text{mir}},\widetilde{K}^{\text{mir}}) \underset{\uparrow}{=} -\sum_{c\in U(\widetilde{K}^{\text{mir}},K^{\text{mir}})}\epsilon_c \underset{\uparrow}{=} -\sum_{c\in O(\widetilde{K},K)}(-\epsilon_c) = \sum_{c\in O(\widetilde{K},K)}\epsilon_c.$$

by the Linking Number Mirror Image Lemma 114.5 — by Claim 1 — since reflections turns over- into undercrossings and since reflections flip signs of crossings ⊞

In summary it follows that

$$\begin{aligned}\operatorname{lk}(K,\widetilde{K}) &\underset{\text{by Claim 1 and Claim 2}}{\downarrow} \frac{1}{2}\cdot\Big(\sum_{c\,\in\, U(\widetilde{K},K)}\epsilon_c + \sum_{c\,\in\, O(\widetilde{K},K)}\epsilon_c\Big) = \frac{1}{2}\cdot\Big(\sum_{c\,\in\, U(\widetilde{K},K)\cup O(\widetilde{K},K)}\epsilon_c\Big)\\ &= \frac{1}{2}\cdot\begin{pmatrix}\text{number of positive crossings between } K \text{ and } \widetilde{K}\\ -\ \text{number of negative crossings between } K \text{ and } \widetilde{K}\end{pmatrix}.\end{aligned}$$

■

Using the Linking Number-via-Crossings Proposition 114.9 we can easily prove the following proposition:

Proposition 114.10. (Linking Number-Symmetry Proposition) For any disjoint oriented knots K and $\widetilde{K}$ we have

$$\operatorname{lk}(\widetilde{K},K) = \operatorname{lk}(K,\widetilde{K}).$$

Proof. Let K and $\widetilde{K}$ be two disjoint oriented knots. It follows from the Link Diagram Existence Proposition 112.7, the One Crossing Exists Lemma 113.4 and the Linking Number–Isotopy-Lemma 114.2 that we can assume that there exists a diagram for the link $K\cup\widetilde{K}$ such that each component has at least one crossing. We can thus use the Linking Number-via-Crossings Proposition 114.9 to calculate the linking numbers of K and $\widetilde{K}$. The symmetry of the linking numbers now follows from the observation that the formula in the Linking Number-via-Crossings Proposition 114.9 is symmetric in K and $\widetilde{K}$. ■

This concludes our short introduction to linking numbers. We will come across linking numbers on numerous occasions. Much later, in the Linking Number Reformulation Proposition 285.8 we will give another five alternative ways to define/calculate the linking number.

114.3. Exercises for Chapter 114 .

Exercise 114.1. Let K and K' be oriented knots and let J be a knot in the complement of K and K'. Show that if K and K' are homotopic in the complement of J, then

$$\operatorname{lk}(K,J) = \operatorname{lk}(K',J).$$

Exercise 114.2. Provide an example of an oriented 3-component link $L = L_1 \sqcup L_2 \sqcup L_3$ such that for every $i \neq j$ the following two statements hold:

(1) The linking number $\mathrm{lk}(L_i, L_j)$ is zero.
(2) The link $L_i \sqcup L_j$ is non-trivial.

Exercise 114.3. We consider the pairs of knots $K_1, \widetilde{K}_1$ and $K_2, \widetilde{K}_2$ shown in the figure below.

(a) Determine $\mathrm{lk}(K_1, \widetilde{K}_1)$.
(b) Determine $\mathrm{lk}(K_2, \widetilde{K}_2)$.



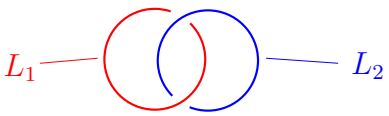
Exercise 114.4. For which matrices $A = (a_{ij}) \in \mathrm{M}(n \times n, \mathbb{Z})$ does there exist an oriented n-component link $L = L_1 \sqcup \cdots \sqcup L_n$ such that for every $i \neq j$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$?

Exercise 114.5. For which matrices $A = (a_{ij}) \in \mathrm{M}(n \times n, \mathbb{Z})$ does there exist an oriented n-component link $L = L_1 \sqcup \cdots \sqcup L_n$ such that for every $i \neq j$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$?

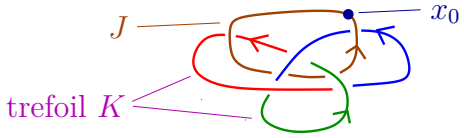
Exercise 114.6. We consider the unoriented Hopf link that is shown in the figure below. There are a priori eight different oriented links that we can produce out of the Hopf link:

- We have two choices of orientations for L_1.
- We have two choices of orientations for L_2.
- For any of the four choices we can take the mirror image of the oriented link.

How many different smooth isotopy types of oriented links do we get that way?



Exercise 114.7. We consider the trefoil K and the oriented knot $J \subseteq S^3 \setminus K$ that is shown in the figure below. Let $x_0 \in J$. Show that $[J] \in \pi_1(S^3 \setminus K, x_0)$ is non-trivial.
Hint. You could try to find an epimorphism $\varphi\colon \pi_1(S^3 \setminus K, x_0) \to G$ onto a finite group G such that $\varphi([J])$ is non-trivial.



Exercise 114.8. In the literature the Linking Number-via-Crossings Proposition 114.9 often gets used as a definition of the linking number. Use the Reidemeister Moves Theorem 112.10 (2) to make sense of this approach. More precisely, verify by hand that the right-hand side of the Linking Number-via-Crossings Proposition 114.9 is unchanged under any of the Reidemeister moves.

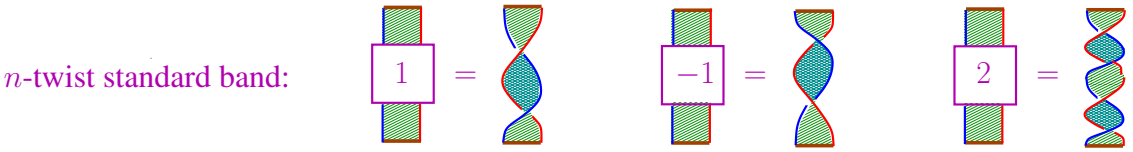
Exercise 114.9. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p,q) \subseteq S^1 \times S^1 \subseteq S^3$. Let T' be a knot on $S^1 \times S^1$ that runs parallel to $T(p,q)$ on $S^1 \times S^1$. To be more precise, T' is smoothly isotopic on $S^1 \times S^1$ to $T(p,q)$. Determine the linking number between $T(p,q)$ and T'.

Exercise 114.10. Let K, J be two disjoint oriented knots. We suppose that the link $K \sqcup J$ is given by a diagram in the xy-plane. Show that we have the following equality:

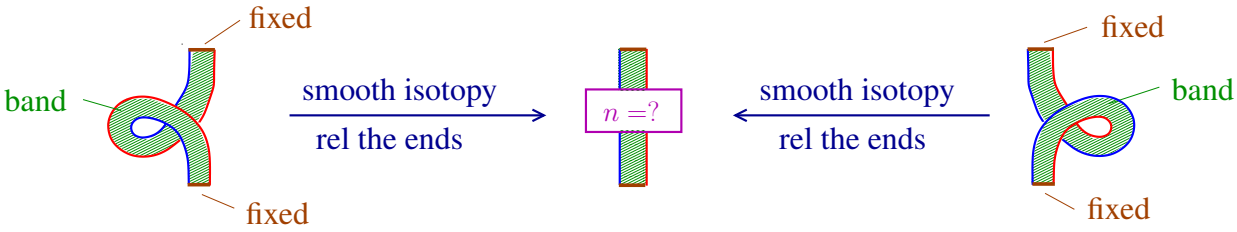
$$\begin{aligned} \operatorname{lk}(K,J) \;=\; & \text{number of positive crossings where } K \text{ crosses over } J \\ - \;& \text{number of negative crossings where } K \text{ crosses over } J. \end{aligned}$$

Remark. This formula give a more efficient version of the Linking Number-via-Crossings Proposition 284.8.

In the last two exercises we look at "bands" in $\mathbb{R}^3$ where the two brown ends are fixed. In the figure below we show the definition of the *n-twist standard band*:

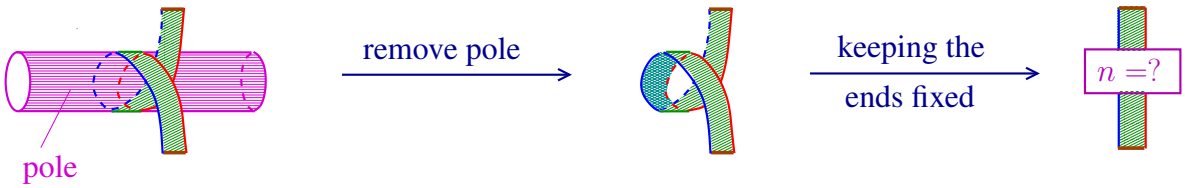


Exercise 114.11. We consider the two bands that are shown in the figure below to the left and right. For each of the two bands determine the $n \in \mathbb{Z}$ such that the band is smoothly isotopic (rel the ends) to the n-twist standard band.



Exercise 114.12. Suppose we are given a "horizontal pole". We want to wrap a band around the pole such that the ends are fixed as before, such that the band is "tight" and such that the band is tangent to the pole. We then remove the pole and we consider the question, whether the resulting band is twisted or not.

(a) For the band shown in the figure below, what is the corresponding number of twists in the standard band?
(b) How do you have to change the wrapping to get the opposite number of twists?
(c) Can you wrap the band around the pole several times in such a way that the band, which we obtain from removing the pole, is untwisted?


pole
remove pole
keeping the
ends fixed
$n = ?$

CHAPTER 115

Longitudes and coloring polynomials

In this chapter we will use the linking number, which we defined in the previous Chapter 114, to introduce the "longitude" of an oriented knot. We will use the pair of a meridian and a longitude to define "coloring polynomials" of oriented knots. These are fairly simply minded but surprisingly powerful invariants of knots.

This chapter is organized as follows:

- In Section 115.1 we will introduce the longitude of an oriented knot. We will formulate the Longitude Proposition 115.2 which says that longitudes always exist and that they are essentially unique.
- In Section 115.2 we will prove the Longitude Proposition 115.2.
- In Section 115.3 we will discuss the interaction between longitudes and symmetries of knots.
- In Section 115.4 we will introduce the coloring polynomials of oriented knots.
- In Section 115.5 we discuss the computability of the coloring polynomials of oriented knots. We will see that it is desirable to have presentations with a small number of generators. We will present a method for eliminating generators in a given presentation that works well in practice.
- In Section 115.6 we will provide exercises for Chapter 115.

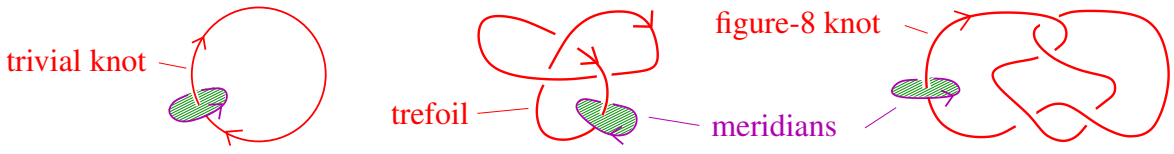
In the next Chapter 116 we will introduce the Alexander functions of groups.

115.1. **Longitudes.** We start out by recalling the definition of the meridian of an oriented knot:

Definition 109.1. Let L be an oriented link in S^3 and let K be a component of L. Let $\mu_K \subseteq S^3 \setminus K$ be an oriented smoothly embedded loop. We say that μ_K is a meridian of K if there exists a smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ with the following three properties:

(1) The map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \mu_K$.
(2) The image $\varphi(\overline{B}^2)$ intersects K transversally in a single point P^{1204} and it does not intersect any of the other components of L.
(3) A positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ followed by a positive basis for $\mathrm{T}_P K$ gives a positive basis for $\mathrm{T}_P S^3$.

By the Meridian Proposition 109.2 we know that meridians always exist and that they are unique up to smooth isotopy.



We now introduce the meridian's loyal sidekick, namely the longitude:

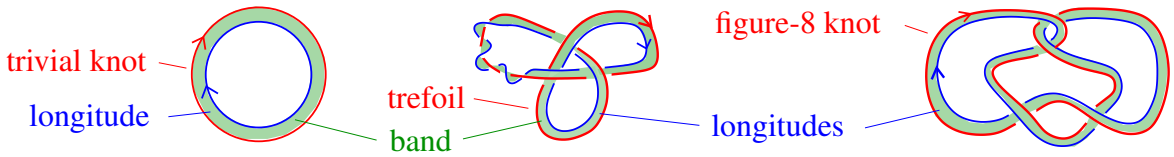
Definition 115.1. Let L be an oriented link in S^3 and let K be a component of L. Let $\lambda_K \subseteq S^3 \setminus K$ be an oriented smoothly embedded loop. We say that λ_K is a longitude of K if there exists a smooth embedding $\varphi\colon S^1 \times [0,1] \to S^3$ with the following four properties:

(1) The map[1205] $\varphi_0\colon S^1 \to S^3$ defines an orientation-preserving diffeomorphism $S^1 \to K$.
(2) The map $\varphi_1\colon S^1 \to S^3$ defines an orientation-preserving diffeomorphism $S^1 \to \lambda_K$.
(3) The image of φ does not intersect any other component of L.
(4) We have $\mathrm{lk}(K, \lambda_K) = 0$.

The image $\varphi(S^1 \times [0,1])$ is often called a smoothly embedded band.

Example.

(1) In the figure below we consider again the trinity of the trivial knot, the trefoil and the figure-8 knot.



Using the Linking Number-via-Crossings Proposition 114.9 one can easily verify that in the above figure the blue knots are indeed longitudes of the red knots.

(2) In practice one usually finds a longitude of a given knot K as follows: One draws a smoothly embedded closed curve that runs "parallel" to K and one computes the linking number via Linking Number-via-Crossings Proposition 114.9. If the linking number is non-zero, then one needs to add extra twists around K. □

The following proposition is an analog of the Meridian Proposition 109.2:

Proposition 115.2. (Longitude Proposition) Let L be an oriented link in S^3 and let K be a component of L.

(1) K admits a longitude.
(2) Let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism and let $\tau\colon \overline{B}^2 \times L \to S^3$ be an orientation-preserving tubular map. There exists a unique $m \in \mathbb{Z}$ such that the image of the map
$$\begin{array}{ccc} S^1 & \to & S^3 \\ z & \mapsto & \tau(z^m, \gamma(z)) \end{array}$$
is a longitude of K.
(3) Any two longitudes for K are smoothly isotopic in $S^3 \setminus L$.
(4) Given any longitude λ of the component K there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ such that λ is of the form described in (2).

The proof of the Longitude Proposition 115.2 shares some features with the proof of the Meridian Proposition 109.2. In particular it builds on the Link Tubular Map Theorem 108.29. We outsource the somewhat technical and lengthy proof of the Longitude Proposition 115.2 to Section 115.2. Afterwards we will continue with Section 115.3 where we will discuss the relationship between longitudes and symmetries of knots.

[1205] As usual, given a map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by $F_t\colon X \to Y$ the map that is given by $x \mapsto F(x,t)$.

115.2. **Proof of the Longitude Proposition 115.2.** The proof of the Longitude Proposition 115.2 will be given in the following two subsections:

- In Subsection 115.2.1 we provide the proof of the Longitude Proposition 115.2 (1), (2).
- In Subsection 115.2.2 we provide the proof of the Longitude Proposition 115.2 (3), (4).

115.2.1. **Proof of the Longitude Proposition 115.2 (1) and (2).** The following lemma is the key to proving the existence of longitudes:

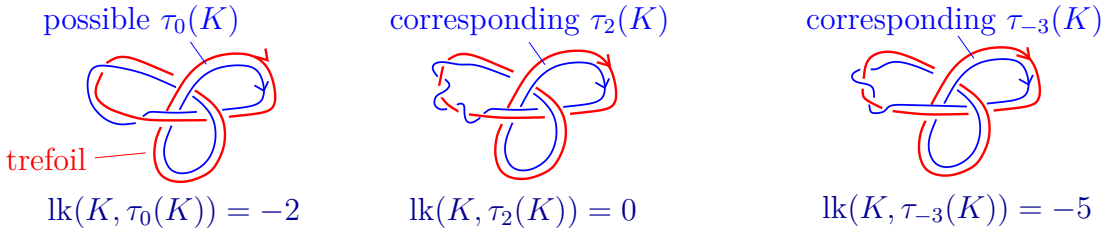
Lemma 115.3. (Sputnik Lemma) Let K be an oriented knot, let $\tau\colon \overline{B}^2 \times K \to S^3$ be an orientation-preserving tubular map and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism.

(1) Given any $m \in \mathbb{Z}$ we define the knot $\tau_m(K)$ as the image of the smooth embedding

$$\begin{array}{rcl} S^1 & \to & S^3 \\ z & \mapsto & \tau(z^m, \gamma(z)). \end{array}$$

For any $m, n \in \mathbb{Z}$ we have $\mathrm{lk}(K, \tau_n(K)) - \mathrm{lk}(K, \tau_m(K)) = n - m$.

(2) There exists a unique $m \in \mathbb{Z}$ with $\mathrm{lk}(K, \tau_m(K)) = 0$.



Proof of the Sputnik Lemma 115.3. Let $k \in \mathbb{Z}$. We introduce the following loops

$$\begin{array}{rcl} \gamma_k\colon [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (e^{2\pi i t \cdot k}, e^{2\pi i t}), \end{array} \qquad \begin{array}{rcl} \alpha\colon [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (e^{2\pi i t}, 1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (1, e^{2\pi i t}). \end{array}$$

Note that these loops define elements in $\pi_1(S^1 \times S^1, (1,1))$. Now let K be an oriented knot, let $\tau\colon \overline{B}^2 \times K \to S^3$ be an orientation-preserving tubular map and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism.

(1) Let $m, n \in \mathbb{Z}$. We have the following equalities in $\pi_1(S^3 \setminus K, \tau(1,1))_{\mathrm{ab}}$:

$$\begin{array}{lcl} \mathrm{lk}(K, \tau_n(K)) \cdot \mu_K \overset{\text{Definition 114.1 of } \mathrm{lk}(K,\tau_n(K))}{=} [\tau_n(K)] & = & [(\tau \circ \gamma_n)(S^1)] \\ & = & \tau_*([\alpha^{n-m}] \cdot [\alpha^m] \cdot [\beta]) \\ & = & \mu_K^{n-m} + [(\tau \circ \gamma_m)(S^1)] \\ & & \uparrow \text{Definition 109.1 of } \mu_K \end{array} \qquad \begin{array}{lcl} & = & \tau_*([\gamma_n]) \overset{\text{this equality in } \pi_1(S^1 \times S^1, (1,1)) \text{ follows from the Torus-}\pi_1\text{-Proposition 90.14}}{=} \tau_*([\alpha^n] \cdot [\beta]) \\ & = & \tau_*([\alpha^{n-m}]) + \tau_*([\gamma_m]) \\ & = & (n - m + \mathrm{lk}(K, \tau_m(K))) \cdot \mu_K. \\ & & \uparrow \text{Definition 114.1 of } \mathrm{lk}(K,\tau_m(K)) \end{array}$$

It follows that $\mathrm{lk}(K, \tau_n(K)) = n - m + \mathrm{lk}(K, \tau_m(K))$.

(2) This statement follows immediately from (1). ■

Now have no troubles proving Statements (1) and (2) of the Longitude Proposition 115.2:

Proof of the Longitude Proposition 115.2 (2). Let L be an oriented link in S^3, let K be a component of L, let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism and let $\tau\colon \overline{B}^2 \times L \to S^3$ be an orientation-preserving tubular map. Given any $m \in \mathbb{Z}$ we again

define the knot $\tau_m(K)$ as the image of the smooth embedding

$$\begin{array}{rcl} S^1 & \to & S^3 \\ z & \mapsto & \tau(z^m, \gamma(z)). \end{array}$$

By the Sputnik Lemma 115.3 (2) there exists a unique $m \in \mathbb{Z}$ such that $\mathrm{lk}(K, \tau_m(K))$. It remains to prove that $\tau_m(K)$ is indeed a longitude of K. To so we consider the map

$$\begin{array}{rcl} \varphi\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & \tau((t\cdot z)^m, \gamma(z)). \end{array}$$

One can easily verify that φ is a smooth embedding with all the desired properties. ■

Proof of the Longitude Proposition 115.2 (1). Let L be an oriented link in S^3 and let K be a component of L. By the Link Tubular Map Theorem 108.29 we know L admits an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$. It follows from the Longitude Proposition 115.2 (2) that K admits a longitude. ■

115.2.2. **Proof of the Longitude Proposition 115.2 (3) and (4).** Throughout this subsection we will make use of the following notation:

Notation 115.4.

(1) Let $p\colon \overline{B}^2 \times S^1 \to \overline{B}^2$ be the projection onto the first factor.

(2) We consider the homomorphism

$$\theta\colon \pi_1((\overline{B}^2 \setminus \{0\}, (1,1)) \times S^1) \xrightarrow{p_*} \pi_1(\overline{B}^2 \setminus \{0\}, 1) \xrightarrow{\pi_*} \pi_1(S^1, 1) =_{\mathrm{i}} \langle t \rangle$$

where the first map is induced by the projection $p\colon (\overline{B}^2 \setminus \{0\}) \times S^1 \to \overline{B}^2 \setminus \{0\}$ onto the first factor, the second map is induced by the map $\pi\colon \overline{B}^2 \setminus \{0\} \to S^1$ that is given by $\pi(z) = \frac{z}{|z|}$ and the third map is the standard identification given by the Circle-π_1-Theorem 88.22.

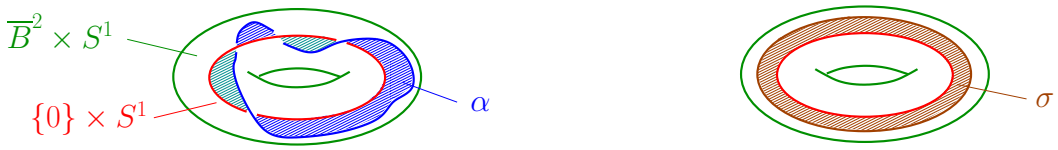
Now we can formulate the main technical lemma of this subsection:

Lemma 115.5. (Band-in-Solid Torus Lemma) Let $\alpha\colon S^1 \times [0,1] \to \overline{B}^2 \times S^1$ be a smooth embedding with the following properties:

(1) $\alpha(z,0) = (0,z)$ for all $z \in S^1$.

(2) The map $\theta \circ \alpha_{1*}\colon \pi_1(S^1,1) \to \langle t \rangle$ is the trivial map.

Then α is smoothly isotopic rel $S^1 \times \{0\}$ to the standard smooth embedding σ that is given by $(z,t) \mapsto (z, \frac{1}{2}t)$.



Sketch of proof. In the proof we adopt the following notation:

- As mentioned above we denote by $p\colon \overline{B}^2 \times S^1 \to \overline{B}^2$ the projection.
- Given a smooth map $h\colon S^1 \times [0,1] \to S^1$ and given a point $(z,t) \in S^1 \times [0,1]$ we write $h'(z,t) := \mathrm{D}h_{(z,t)}(0,1) \in \mathbb{R}^2$ where $(0,1) \in \mathrm{T}_{(z,t)}(S^1 \times [0,1]) = \mathrm{T}_z S^1 \times \mathbb{R}$.

As in the proof of the Meridian Proposition 109.2 (3) we now proceed in several stages, which are summarized in the following claim:

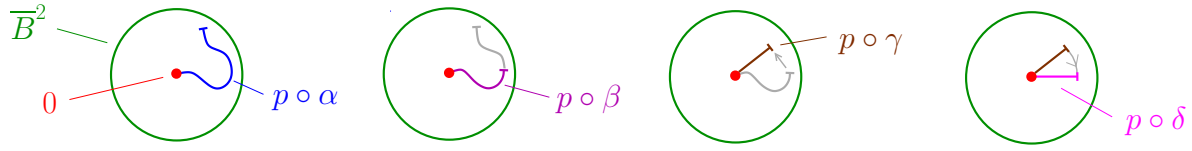
Claim.

(a) The smooth embedding α is smoothly isotopic rel $S^1 \times \{0\}$ to a smooth embedding β such that for all $(z,t) \in S^1 \times [0,1]$ we have $\langle (p \circ \beta)'(z,0), (p \circ \beta)'(z,t) \rangle \neq 0$.

(b) The smooth embedding β is smoothly isotopic rel $S^1 \times \{0\}$ to a smooth embedding γ such that for all $(z,t) \in S^1 \times [0,1]$ we have $(p \circ \gamma)(z,t) \in \mathbb{R}_{\geq 0} \cdot \mu(z)$ for some continuous map $\mu \colon S^1 \to S^1$.

(c) The smooth embedding γ is smoothly isotopic rel $S^1 \times \{0\}$ to a smooth embedding δ such that for all $(z,t) \in S^1 \times [0,1]$ we have $(p \circ \delta)(z,t) \in \mathbb{R}_{\geq 0} \cdot (1,0)$. (Here we finally use Hypothesis (2).)

(d) The smooth embedding δ is smoothly isotopic rel $S^1 \times \{0\}$ to the standard smooth embedding σ.



Proof.

(a) It follows easily from Hypothesis (1) and the fact that $\alpha \colon S^1 \times [0,1] \to \overline{B}^2 \times S^1$ is a smooth embedding that for any $z \in S^1$ we have $(p \circ \alpha)'(z,0) \neq 0$. In particular we have $\langle (p \circ \alpha)'(z,0), (p \circ \alpha)'(z,0) \rangle \neq 0$ for all $z \in S^1$. It follows from this observation and the Tube Lemma 7.12 that there exists an $\epsilon \in (0,1]$ such that $\langle (p \circ \alpha)'(z,0), (p \circ \alpha)'(z,t) \rangle \neq 0$ for all $z \in S^1$ and all $t \in [0,\epsilon]$. Clearly the smooth embedding $\beta \colon S^1 \times [0,1] \to \overline{B}^2 \times S^1$ that is given by $\beta(z,t) = \alpha(z, \epsilon \cdot t)$ has the desired properties.

(b) Given $z \in S^1$ we set $\mu(z) := \pi((p \circ \beta)'(z,0)) = \frac{(p \circ \beta)'(z,0)}{|(p \circ \beta)'(z,0)|} \in S^1$. Note that the extra property of β shows that the orthogonal projection of $(p \circ \beta)(\{z\} \times [0,1])$ onto $\mathbb{R}_{\geq 0} \cdot \mu(z)$ defines a smooth embedding. From this observation one gets easily the desired result.

(c) We consider the continuous map

$$\begin{array}{rcl} f \colon [0,1] & \to & S^1 \\ t & \mapsto & \mu(\exp(2\pi \mathrm{i} \cdot t)). \end{array}$$

We make a few observations:

(i) One can easily verify that the map $f \colon [0,1] \to S^1$ is a loop that is path-homotopic to the loop $t \mapsto (\pi \circ p \circ \gamma_1)(\exp(2\pi \mathrm{i} t))$. Note that the latter loop is null-homotopic, since we started out with a longitude. This shows that f is null-homotopic.

(ii) Note that it follows from the Lift-from-S^1-to-$\mathbb{R}$ Lemma 88.21 that there exists a continuous map $\widetilde{f} \colon [0,1] \to \mathbb{R}$ such that for all $t \in [0,1]$ we have the equality $\exp(2\pi \mathrm{i} \cdot \widetilde{f}(t)) = f(t) = \mu(\exp(2\pi \mathrm{i} \cdot t))$.

(iii) It follows from the Circle-π_1-Theorem 88.22 together with (i) that $\widetilde{f}(1) = \widetilde{f}(0)$.

(iv) It follows from (iii) that there exists a continuous map $\nu \colon S^1 \to \mathbb{R}$ such that $\nu(\exp(2\pi \mathrm{i} t)) = \widetilde{f}(t)$ for all $t \in [0,1]$.

In the following we denote by $q \colon \overline{B}^2 \times S^1 \to S^1$ the natural projection. One can now easily verify that the map

$$\begin{array}{rcl} H \colon (S^1 \times [0,1]) \times [0,1] & \mapsto & \overline{B}^2 \times S^1 \\ ((z,t),s) & \mapsto & ((p \circ \gamma)(z,t) \cdot \exp(2\pi \mathrm{i} \cdot ((1-s) - \nu(z) \cdot s)), (q \circ \gamma)(z,t)) \end{array}$$

is a smooth isotopy rel $S^1 \times \{0\}$ from $H_0 = \gamma$ to a smooth embedding $H_1 =: \delta$ with the desired properties.

(d) Note that δ and σ are both given by smooth embeddings $S^1 \times [0,1] \to (\mathbb{R}_{\geq 0} \cdot \mathfrak{e}_1 \cap \overline{B}^2)$. Thus we can view both as smooth embeddings $[0,1] \times S^1 \to S^1 \times [0,1]$ which on $\{0\} \times S^1$ are the obvious map. It follows from the Smooth Collar-Uniqueness Proposition 54.2 that δ and σ are smoothly isotopic in $S^1 \times [0,1]$ rel $\{0\} \times S^1$.[1206] ■

Proof of the Longitude Proposition 115.2 (3). To simplify the notation a little bit we will ignore all the components of the link except for K. Thus let K be an oriented knot in S^3. We need to show that any two longitudes for K in S^3 are smoothly isotopic. As in the proof of the Meridian Proposition 109.2 (3) we first pick a "model longitude":

Claim 1. There exists an orientation-preserving smooth embedding $\tau\colon \overline{B}^2 \times S^1 \to S^3$ such that $\tau(\{0\} \times S^1) = K$ and such that $\tau(\{1\} \times S^1)$ is a longitude for K.

Proof. We pick an orientation-preserving diffeomorphism $\gamma\colon S^1 \to K$. By the Link Tubular Map Theorem 108.29 together with the Longitude Proposition 115.2 we know that there exists an orientation-preserving tubular map $\sigma\colon \overline{B}^2 \times K \to S^3$ and an $m \in \mathbb{Z}$ such that the image of the map

$$\begin{array}{rcl} S^1 & \to & S^3 \\ z & \mapsto & \sigma(z^m, \gamma(z)) \end{array}$$

is a longitude for K. One can now easily verify that the map

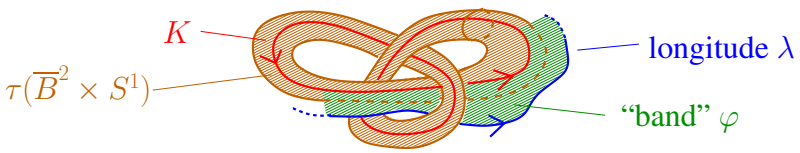
$$\begin{array}{rcl} \tau\colon \overline{B}^2 \times S^1 & \to & S^3 \\ (w,z) & \mapsto & \sigma(w \cdot z^m, \gamma(z)) \end{array}$$

has the desired properties. ⊞

By the Smooth Isotopy-Equivalence Relation Proposition 41.5 it suffices to show that any longitude λ of K is smoothly isotopic to the longitude $\tau(\{1\} \times S^1)$ from Claim 1. Thus let λ be a longitude of S^1. By Definition 115.1 of a longitude there exists a smooth embedding $\varphi\colon S^1 \times [0,1] \to S^3$ with the following three properties:

(1) The map $\varphi_0\colon S^1 \to S^3$ defines an orientation-preserving diffeomorphism $S^1 \to K$.
(2) The map $\varphi_1\colon S^1 \to S^3$ defines an orientation-preserving diffeomorphism $S^1 \to \lambda$.
(3) We have $\mathrm{lk}(K, \lambda) = 0$.

The idea for the proof is now rather obvious: We want to massage our setting such that we can employ the Band-in-Solid Torus Lemma 115.5.



Claim 2. The longitude λ is smoothly isotopic in $S^3 \setminus K$ to a longitude that is contained in $\tau(\overline{B}^2 \times S^1)$.

[1206] With the experience gained in the above arguments it is also not that hard to come with an argument: First one find a smooth isotopy from δ to a smooth embedding $\epsilon\colon S^1 \times [0,1] \to [0,1] \times S^1$ that "preserves intervals", afterwards one performs stretches on each interval to get the smooth isotopy to σ. We leave it to the reader to fill in the details.

Proof. Since S^1 is compact, since $\tau(B^2 \times S^1)$ is an open neighborhood of $K = \tau(\{0\} \times K)$ and since we have the inclusion $\varphi(S^1 \times \{0\}) \subseteq \tau(B^2 \times S^1)$ we obtain from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that $\varphi(S^1 \times [0, \epsilon]) \subseteq \tau(B^2 \times S^1)$. The claim follows from the observation that $\varphi_\epsilon(S^1)$ is clearly smoothly isotopic to $\lambda = \varphi_1(S^1)$ in $S^3 \setminus K = S^3 \setminus \varphi_0(S^1)$. ⊞

We consider the homomorphism

$$\theta\colon \pi_1((\overline{B}^2 \setminus \{0\}, (1,1)) \times S^1) \xrightarrow{p_*} \pi_1(\overline{B}^2 \setminus \{0\}, 1) \xrightarrow{\pi_*} \pi_1(S^1, 1) =_{\mathrm{i}} \langle t \rangle$$

where the first map is induced by the projection $p\colon (\overline{B}^2 \setminus \{0\}) \times S^1 \to \overline{B}^2 \setminus \{0\}$ onto the first factor, the second map is induced by the map $\pi\colon \overline{B}^2 \setminus \{0\} \to S^1$ that is given by $\pi(z) = \frac{z}{|z|}$ and the third map is the standard identification given by the Circle-π_1-Theorem 88.22.

It follows easily from the Band-in-Solid Torus Lemma 115.5 that it remains to prove the following claim:

Claim 3. The map $\psi := \tau^{-1} \circ \varphi\colon S^1 \times [0,1] \to \overline{B}^2 \times S^1$ is a smooth embedding with the following properties:
(1) $\psi(z, 0) = (0, z)$ for all $z \in S^1$.
(2) The map $\theta \circ \psi_{1*}\colon \pi_1(S^1, 1) \to \langle t \rangle$ is the trivial map.

Proof. Since φ takes values in $\tau(\overline{B}^2 \times S^1)$ and since φ and τ are smooth embeddings we see that $\tau^{-1} \circ \varphi$ is defined and that it is a smooth embedding. It follows easily from the definitions that (1) is satisfied. It remains to prove Statement (2). To do so we consider the homomorphism

$$\begin{aligned}\xi\colon \pi_1((\overline{B}^2 \setminus \{0\}) \times S^1, (1,1)) &\xrightarrow{\tau_*} \pi_1(\tau((\overline{B}^2 \setminus \{0\}) \times K), \tau(1,1)) \xrightarrow{\iota_*} \pi_1(S^3 \setminus K, \tau(1,1)) \\ &\to \pi_1(S^3 \setminus K, \tau(1,1))_{\mathrm{ab}} \xrightarrow[\Phi_K]{\cong} \langle t \rangle\end{aligned}$$

where $\iota\colon \tau((\overline{B}^2 \setminus \{0\}) \times K) \to \pi_1(S^3 \setminus K$ is the inclusion and $\Phi_K\colon \pi_1(S^3 \setminus K, \tau(1,1))_{\mathrm{ab}} \to \langle t \rangle$ is the isomorphism from the Link Group-Abelianization Proposition 113.9. We claim that $\theta = \xi$. This can be seen as follows: It follows from the Product-π_1-Proposition 90.13 that[1207] $[S^1 \times \{1\}]$ and $[\{1\} \times S^1]$ form a basis for $\pi_1((\overline{B}^2 \setminus \{0\}) \times S^1, (1,1))$. One can easily verify that $\xi(a) = \Phi_K(\mu_K) = \langle t \rangle = \theta(a)$ and that $\xi(b) = \Phi_K(\tau(\{1\} \times S^1)) = 1 = \theta(b)$ (here we used that $\tau(\{1\} \times S^1)$ is a longitude). It follows that $\theta = \xi$. Finally we see that

$$\begin{aligned}(\theta \circ \psi_{1*})([S^1]) &\overset{\text{since } \theta = \xi}{=} (\xi \circ \psi_{1*})([S^1]) \overset{\text{by the definition of } \xi \text{ and } \psi}{=} (\Phi_K \circ \iota_* \circ (\tau_* \circ (\tau_*)^{-1}) \circ \phi_{1*})([S^1]) \\ &= (\Phi_K \circ \iota_* \circ \varphi_{1*})([S^1]) = \Phi_K(\lambda) = 1.\end{aligned}$$

■

Proof of the Longitude Proposition 115.2 (4). Let L be an oriented link in S^3 and let K be a component of L. We need to show that given any longitude λ of the component K there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ such that λ is of the form described in (2). Fortunately this statement is an immediate consequence of the Statements (2) and (3), which we already proved. ■

[1207] Here we use Notation 111.16.

115.3. **Longitudes and symmetries.** We start out with recalling Definitions 108.21 and 108.23:

Definition 115.6. Let $K \subseteq S^3$ be an oriented knot.

(1) The reverse K^{rev} is defined as the knot K with the reverse orientation.
(2) The mirror K^{mir} is defined as the reflection of K in any hyperplane of $\mathbb{R}^4$.[1208] [1209] We give the mirror K^{mir} the orientation that turns the reflection into an orientation-preserving diffeomorphism $K \to K^{\text{mir}}$.
(3) The inverse K^{inv} is defined as the mirror of the reverse of K (which of course equals the reverse of the mirror).[1210]



The following lemma is an extension of the Meridian–Symmetries Lemma 109.3:

Lemma 115.7. (Meridian-Longitude–Symmetries Lemma) Let $K \subseteq S^3$ be an oriented knot with meridian μ_K and longitude λ_K. Furthermore let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane. We obtain the following table:

	meridian	longitude
original knot K	μ_K	λ_K
reverse K^{rev}	μ_K^{rev}	λ_K^{rev}
mirror $K^{\text{mir}} = \rho(K)$	$\rho(\mu_K)^{\text{rev}}$	$\rho(\lambda_K)$
inverse $K^{\text{inv}} = \rho(K)^{\text{rev}}$	$\rho(\mu_K)$	$\rho(\lambda_K)^{\text{rev}}$.

Proof. First note that the statements regarding the inverse follow immediately from the statements regarding the reverse and the mirror. We proved the statements regarding the meridians in the Meridian–Symmetries Lemma 109.3. The statement about the longitude of K^{rev} follows easily from the Linking Number–Orientation Lemma 114.4 and the statement about the longitude of K^{mir} follows equally easily from the Linking Number–Mirror Image Lemma 114.5. ■

Now the question arises whether an oriented knot is smoothly isotopic to any of its doppelgänger. This leads us to the following definitions from Definition 108.21 and Definition 108.26:

Definition 115.8. Let $K \subseteq S^3$ be an oriented knot.

(1) We say that K is reversible if K is smoothly isotopic to its reverse.
(2) We say that K is amphichiral if K is smoothly isotopic to its mirror. Otherwise we call the knot K chiral.[1211]

[1208] Note that we showed in the S^3-Link Mirror Lemma 108.24 that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.
[1209] Note that in Definition 108.23 we introduced the mirror of an unoriented knot. For the purpose of this section it is better to work with oriented knots.
[1210] The names "reverse" and "inverse" at times also get used with different meanings. We follow [**Conw1970**, p. 336] for our naming convention.

(3) We say that K is invertible if K is smoothly isotopic to its inverse.

Example. As we saw in the figure on page 2105, it is basically clear that the trivial knot is amphichiral and reversible. Furthermore, in the figure on page 2107 we showed that the figure-8 knot J is amphichiral. Also note that in Exercise 108.2 we showed that the trefoil and the figure-8 knot are reversible. □

Theses examples lead us to the following two questions:

Question 108.22. Is every knot reversible?

Question 108.27. Is the trefoil chiral?

Before we show how to answer both questions let us introduce the following simple-minded definition:

Definition 115.9.

(1) A group-pair system is a triple (G, g_1, g_2) consisting of a group G together with two elements $g_1, g_2 \in G$.
(2) An isomorphism between group-pair systems (G, g_1, g_2) and (H, h_1, h_2) is a group isomorphism $\varphi\colon G \to H$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$.

The following definition gives us the group-pair system we care about most:

Definition 115.10. Let $K \subseteq S^3$ be an oriented knot. By the Link Tubular Map Theorem 108.29 there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times K \to S^3$.

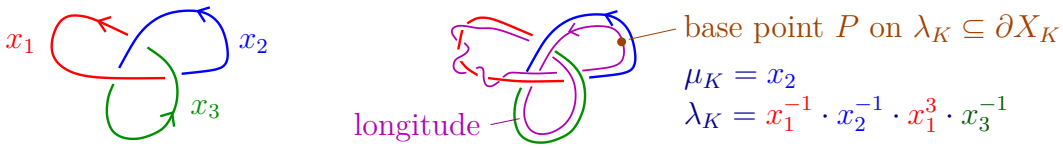
(1) As in Definition 108.30 we refer to $X_K := S^3 \setminus \tau(B^2 \times K)$ as the exterior of K.
(2) We use the tubular map to define the meridian $\mu_K \subseteq \partial X_K$ as in Definition 109.1 and the longitude $\lambda_K \subseteq \partial X_K$ as in Definition 115.1.
(3) Let $x \in K$. We refer to the group-pair system

$$(\pi_1(X_K, \tau(1,x)), \underset{\uparrow}{\mu_K}, \underset{\uparrow}{\lambda_K})$$

group elements by the Loop-π_1-Lemma 111.15

as the group-pair system of the oriented knot K. Note that it follows easily from the Link Tubular Map Theorem 108.29 (2) that the isomorphism type of this group-pair system is an invariant of the oriented knot K.

Example. We consider the following knot diagram for the trefoil K:



We pick a base point $P \in \lambda_K \subseteq \partial X_K$. As on page 2199 we get the presentation

$$\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K, P) \underset{\uparrow}{\xleftarrow{\cong}} \pi_1(X_K, P)$$

by the Link Exterior Lemma 108.31 (4)

where x_1, x_2, x_3 are meridians that correspond to the three strands. It follows easily from the discussion of the longitude of the trefoil on page 2231 and the Element-in-Wirtinger Presentation Lemma 114.8 that $\mu_K = x_2$ and $\lambda_K = x_1^{-1} \cdot x_2^{-1} \cdot x_1^3 \cdot x_3^{-1}$. In summary, the

group-pair system of K is isomorphic to the group-pair system

$$\Big(\underbrace{\langle x_1, x_2, x_3 \,|\, x_2^{-1}\cdot x_1\cdot x_2\cdot x_3^{-1}, x_3^{-1}\cdot x_2\cdot x_3\cdot x_1^{-1}\rangle}_{\cong \pi_1(S^3\setminus K,P)\cong \pi_1(X_K,P)}, \underbrace{x_2}_{=\mu_K}, \underbrace{x_1^{-1}\cdot x_2^{-1}\cdot x_1^3\cdot x_3^{-1}}_{=\lambda_K}\Big).$$

□

The following proposition can be viewed as a refinement of the Isotopic Link-π_1-Lemma 111.4 and the Mirror Link-π_1-Lemma 111.5:

Proposition 115.11. (Knot–Group-Pair System Proposition)

(1) Let K and J be two oriented knots. If K and J are smoothly isotopic, then the corresponding group-pair systems are isomorphic.

(2) Given an oriented knot K with a group-pair system $(\pi_K, \mu_K, \lambda_K)$ the following hold:

- (a) The group-pair system of the reverse K^{rev} is isomorphic to $(\pi_K, \mu_K^{-1}, \lambda_K^{-1})$.
- (b) The group-pair system of the mirror K^{mir} is isomorphic to $(\pi_K, \mu_K^{-1}, \lambda_K)$.
- (c) The group-pair system of the inverse K^{inv} is isomorphic to $(\pi_K, \mu_K\ , \lambda_K^{-1})$.

Sketch of proof.

(1) This statement follows easily from the Isotopy Extension Theorem 44.2.

(2) This statement follows basically immediately from the Meridian-Longitude–Symmetries Lemma 115.7 together with the Loop-π_1-Lemma 111.15 (3). ■

115.4. **Coloring polynomial.** The Knot–Group-Pair System Proposition 115.11 raises the following question: How we can show that two group-pair systems are not isomorphic? On page 2207 we saw that looking at finite quotients of groups can be a convenient tool for distinguishing groups. We now apply the same idea to group-pair systems. There are various ways of making this idea precise. We will use an approach due to Michael Eisermann [**Eiser2007**].

We start out with the following definition:

Definition 115.12. Let G be a group. We consider the group ring

$$\mathbb{Z}[G] \;:=\; \Big\{ \sum_{i=1}^{m} r_i\cdot g_i \;\Big|\; r_1,\dots,r_m\in\mathbb{Z} \text{ and } g_1,\dots,g_m\in G \Big\}.$$

We equip this abelian group with the involution that is induced by $g\mapsto g^{-1}$, more precisely, we set

$$\sum_{i=1}^{m} r_i\cdot g_i \;:=\; \sum_{i=1}^{m} r_i\cdot g_i^{-1}.$$

We can now introduce a new knot invariant:

Definition 115.13. Let G be a *finite* group and let $x\in G$.

(1) Given a group-pair system (π, g_1, g_2) where π is a finitely generated group we define

$$\mathrm{P}_G^x(\pi, g_1, g_2) \;:=\; \sum_{\{\varphi\in \mathrm{Hom}(\pi,G)\,|\,\varphi(g_1)=x\}}^{\text{finite sum by Exercise 115.3}} \varphi(g_2) \;\in\; \mathbb{Z}[G].$$

It is clear that isomorphic group-pair systems give the same element in $\mathbb{Z}[G]$.

(2) Let $K\subseteq S^3$ be an oriented knot. Following [**Eiser2007**, Definition 1.2] we refer to

$$\mathrm{P}_G^x(K) \;:=\; \mathrm{P}_G^x(\text{group-pair system } (\pi_1(X_K), \mu_K, \lambda_K) \text{ of } K) \;\in\; \mathbb{Z}[G]$$

as the (G,x)-coloring polynomial of K.

The following lemma collects a few basic facts about coloring polynomials:

Lemma 115.14. (Coloring Polynomial Lemma) Let G be a *finite* group and let $x \in G$.
(1) If K and $\widetilde{K}$ are smoothly isotopic oriented knots, then $\mathrm{P}_G^x(K) = \mathrm{P}_G^x(\widetilde{K})$.
(2) Let $K \subseteq S^3$ be an oriented knot.

(a) For the reverse K^{rev} we have $\mathrm{P}_G^x(K^{\mathrm{rev}}) = \overline{\mathrm{P}_G^{x^{-1}}(K)}$
(b) For the mirror K^{mir} we have $\mathrm{P}_G^x(K^{\mathrm{mir}}) = \mathrm{P}_G^{x^{-1}}(K)$.
(c) For the inverse K^{inv} we have $\mathrm{P}_G^x(K^{\mathrm{inv}}) = \overline{\mathrm{P}_G^x(K)}$.

Proof.
(1) This statement follows easily from the Knot–Group-Pair System Proposition 115.11 (1).
(2) In the following we perform the calculation for (a). The other statements are proved in a similar fashion. So let K be an oriented knot. We write $\pi_K := \pi_1(X_K)$. We see that

$$\overline{\mathrm{P}_G^{x^{-1}}(K)} = \overline{\sum_{\substack{\varphi \in \mathrm{Hom}(\pi_K, G) \\ \varphi(\mu_K) = x^{-1}}} \varphi(\lambda_K)} \underset{\text{since } \varphi \text{ is a homomorphism}}{=} \sum_{\substack{\varphi \in \mathrm{Hom}(\pi_K, G) \\ \varphi(\mu_K^{-1}) = x}} \varphi(\lambda_K^{-1}) \underset{\text{by the Knot–Group-Pair System Proposition 115.11 (2a)}}{=} \mathrm{P}_G^x(K^{\mathrm{rev}}).$$

■

The art now is to find, given a knot, a finite group G and element $x \in G$ such that the (G, x)-coloring polynomial has content. In the following examples and results we will see how this can be done in practice.

As a first application we can now finally answer Question 108.27 in the affirmative:

Proposition 115.15. (Trefoil-is-Chiral Proposition) The trefoil is chiral.

Proof. We equip the trefoil K with either orientation. We consider the alternating group $G = A_5$ and the element $x = (1\,2\,3\,4\,5) \in A_5$. An elementary (but somewhat lengthy) calculation shows that there exist precisely six epimorphisms $\pi_1(S^3 \setminus K) \to A_5$ which send the meridian to $x \in A_5$.

left handed trefoil K — $\mathrm{P}_{A_5}^x(K) = 1 + 5x$



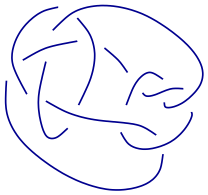


right handed trefoil K^{mir} — $\mathrm{P}_{A_5}^x(K^{\mathrm{mir}}) = 1 + 5x^{-1}$

One can then calculate that $\mathrm{P}_{A_5}^x(K) = 1+5x$ and[1212] $\mathrm{P}_{A_5}^x(K^{\mathrm{mir}}) = 1+5x^{-1}$ (see [**Eiser2007**, p. 307]). Since $x \neq x^{-1} \in A_5$ we obtain from the Coloring Polynomial Lemma 115.14 (1) that K is chiral. ■

Example. We consider the Conway knot and the Kinoshita-Terasaka knot that are shown in the figure below[1213]
This pair of knots is notoriously hard to distinguish. Now let G be the Mathieu group M_{11}, this is the unique simple group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and the smallest of the sporadic simple groups, see [**CCNPrW201985**] for details. As is explained in [**Eiser2007**, p. 315],

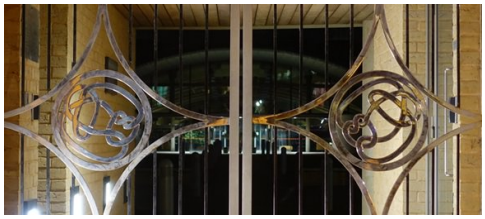
[1212] In fact, since by Exercise 108.2 we know that the trefoil K is reversible we see that the mirror is smoothly isotopic to the inverse, thus we see that $\mathrm{P}_{A_5}^x(K^{\mathrm{mir}}) = \mathrm{P}_{A_5}^x(K^{\mathrm{inv}}) = \overline{\mathrm{P}_{A_5}^x(K)} = \overline{1 + 5x} = 1+5x^{-1}$.
[1213] The knots are named after the mathematician John Conway [**Conw1970**] and after the mathematicians Shinichi Kinoshita and Hidetaka Terasaka [**KiT1957**].



Conway knot



Kinoshita-Terasaka knot



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one way of describing $G = M_{11}$ is to note that it is the subgroup of the alternating group A_{11} that is generated by

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 5 & 7 & 10 & 1 & 8 & 6 & 11 & 9 & 4 \end{pmatrix}.$$

Note that x has order 11 in A_{11} and thus also in $G = M_{11}$. By [**Eiser2007**, p. 317] we have the following table:[1214]

$$\begin{array}{lcllcl} \mathrm{P}^x_G(K) & = & 1 + 11x^3 + 11x^7, & \mathrm{P}^x_G(C) & = & 1 + 11x^3 + 11x^7, \\ \mathrm{P}^x_G(K^{\mathrm{rev}}) & = & 1 + 22x^3 + 11x^7, & \mathrm{P}^x_G(C^{\mathrm{rev}}) & = & 1 + 11x^3 + 11x^5 + 11x^7, \\ \mathrm{P}^x_G(K^{\mathrm{mir}}) & = & 1 + 11x^4 + 22x^8, & \mathrm{P}^x_G(C^{\mathrm{mir}}) & = & 1 + 11x^4 + 11x^6 + 11x^8, \\ \mathrm{P}^x_G(K^{\mathrm{inv}}) & = & 1 + 11x^4 + 11x^8, & \mathrm{P}^x_G(C^{\mathrm{inv}}) & = & 1 + 11x^4 + 11x^8. \end{array}$$

Since x has order 11 in G we see that the four elements in each of the two columns are different. This shows that the Kinoshita-Terasaka knot and the Conway knot are neither reversible, nor amphichiral nor invertible. Since the collection of four elements in $\mathbb{Z}[G]$ appearing in the two columns are different we also see that no flavor of the Kinoshita-Terasaka knot is smoothly isotopic to any flavor of the Conway knot. □

For the record let us formulate the following proposition, which gives in particular a negative answer to Question 108.22.

Proposition 115.16. (Knot-without-Symmetries Proposition) There exists an oriented knot that is neither reversible, nor amphichiral nor invertible.

Proof. This proposition follows immediately from the above example. ■

115.5. **Calculating the link coloring polynomial.** We have just seen that the knot coloring polynomial is quite powerful. This raises the question, how can calculate this invariant in practice? At this point it is convenient to recall Definition 100.12:

Definition 115.17. Let G be a group.

(1) Let $r \in \langle x_1, \ldots, x_k \rangle$ and let $g_1, \ldots, g_k$ be elements of G. We say that the elements $g_1, \ldots, g_k$ satisfy the relation r if the natural homomorphism $\langle x_1, \ldots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \ldots, k$ sends r to the trivial element.[1215]

[1214] Note that the Coloring Polynomial Lemma 115.14 implies that the coloring polynomial P^x_G of the inverse is determined by the coloring polynomial of the original knot. The lemma makes no such claim for the coloring polynomial of the mirror, in fact the following table shows that the coloring polynomial P^x_G of the mirror is not determined by the coloring polynomial P^x_G of the original knot.

(2) Let $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a presentation. If we are given $g_1, \dots, g_x \in G$ that satisfy the relations $r_1, \dots, r_l$, then it follows from elementary group theory, see, that there exists a unique homomorphism

$$\varphi_{g_1,\dots,g_l} \colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; G$$

such that for $i = 1, \dots, k$ we have $\varphi_{g_1,\dots,g_l}(x_i) = g_i$.

With this definition we can now formulate the following elementary lemma:

Lemma 115.18. (Knot Coloring Calculation Lemma) Let $K \subseteq S^3$ be an oriented knot and let $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a presentation for $\pi_1(S^3 \setminus K)$. Furthermore let $\mu \in \langle x_1, \dots, x_k \rangle$ be a word that represents the meridian and let $\lambda \in \langle x_1, \dots, x_k \rangle$ be a word that represents the longitude of K. For any finite group G and any $x \in G$ we have

$$P_G^x(K) \;=\; \sum_{\substack{(g_1,\dots,g_k) \in G^k \\ \text{that satisfy } r_1,\dots,r_l \\ \text{and with } \varphi_{(g_1,\dots,g_l)}(\mu) = x}} \varphi_{(g_1,\dots,g_l)}(\lambda).$$

Proof. By the Presentation–Homomorphism Lemma 100.13 we know that for any finite presentation $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ and any group G the map

$$\begin{array}{rcl} \mathrm{Hom}(\pi, G) & \to & \{\text{all } (g_1, \dots, g_k) \in G^k \text{ that satisfy } r_1, \dots, r_l\} \\ f & \mapsto & (f(x_1), \dots, f(x_k)) \end{array}$$

The lemma follows almost immediately from this completely general observation and the definition of the (G, x)-coloring polynomial of K. ∎

The Knot Coloring Calculation Lemma 115.18 suggests that the computational effort in calculating $P_G^x(K)$ grows *exponentially* in the number k of generators in the given presentation.

If we are given a diagram of a knot K with c crossings, then the Wirtinger Presentation Proposition 113.6 provides us with a presentation with c generators. On the other hand, in the Torus Knot-π_1-Proposition 111.18 we showed that for torus knots we can always find a presentation with just two generators. Thus there is some hope, that in many instances one can find presentations where the number of generators is significantly less than the number of crossings in a diagram.

The following lemma gives us a general technique for reducing the number of generators, which often works well in practice:

Lemma 100.30. (Generator Elimination Lemma) Suppose that we are given a presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ where $r_i = r_i(x_1, \dots, x_k) \in \langle x_1, \dots, x_k \rangle$. If there exists a generator x_m and a relator r_n such that x_m appears only once in r_n, i.e. such that r_n is of the form $r_n = y \cdot x_m^\epsilon \cdot z$ where $\epsilon \in \{-1, 1\}$ and where $y, z \in \langle x_1, \dots, \widehat{x_m}, \dots, x_k \rangle$, then the map

$$\begin{array}{rcl} \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle & \to & \langle x_1, \dots, \widehat{x_m}, \dots, x_l \,|\, r_i(x_1, \dots, (y^{-1} \cdot z^{-1})^\epsilon, \dots, x_k), i \neq m \rangle \\ x_i & \mapsto & \begin{cases} x_i, & \text{if } i \neq m, \\ (y^{-1} \cdot z^{-1})^\epsilon, & \text{if } i = m \end{cases} \end{array}$$

[1215] More colloquially speaking: $g_1, \dots, g_k$ satisfy a relation r if replacing the x_i by g_i in the word r returns the trivial element in G.

is well-defined and it is an isomorphism. The inverse is given by $x_i \mapsto x_i$ for $i \neq m$.

Example. On page 1968 we already gave an example for the Generator Elimination Lemma 100.30. We now consider a second example which is closer to our current purpose: Once again we consider the trefoil and we work with the following presentation from page 2199:

$$\langle x_1, x_2, x_3 \,|\, \overbrace{x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}}^{=r_1}, \overbrace{x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1}}^{=r_2} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K, P),$$

where x_1, x_2, x_3 are meridians. We consider the generator x_2 and we consider the relation $r_2 = x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1}$. It follows from the Generator Elimination Lemma 100.30 that the map

$$\langle x_1, x_2, x_3 \,|\, \overbrace{x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}}^{=r_1(x_1,x_2,x_3)}, \overbrace{x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1}}^{=r_2} \rangle \xrightarrow{\Phi} \langle x_1, x_3 \,|\, \overbrace{(\underbrace{x_3 \cdot x_1 \cdot x_3^{-1}}_{\text{“}=x_2\text{”}})^{-1} \cdot x_1 \cdot (\underbrace{x_3 \cdot x_1 \cdot x_3^{-1}}_{\text{“}=x_2\text{”}}) \cdot x_3^{-1}}^{=r_1(x_1, x_3 \cdot x_1 \cdot x_3^{-1}, x_3)} \rangle$$

$$\begin{aligned} x_1 &\mapsto x_1 \\ x_2 &\mapsto x_3 \cdot x_1 \cdot x_3^{-1} \\ x_3 &\mapsto x_3 \end{aligned}$$

is an isomorphism.

Finally, on page page 2239 we saw that in the initial presentation the longitude is given by $\lambda_K = x_1^{-1} \cdot x_2^{-1} \cdot x_1^3 \cdot x_3^{-1}$. Since we are given an explicit isomorphism Φ we can also describe the longitude in the new representation, namely it is given by

$$\Phi(\lambda_K) \;=\; \Phi(x_1^{-1} \cdot x_2^{-1} \cdot x_1^3 \cdot x_3^{-1}) \;=\; \underbrace{x_1^{-1} \cdot (x_3 \cdot x_1 \cdot x_3^{-1})^{-1} \cdot x_1^3 \cdot x_3^{-1}}_{= x_1^{-1} \cdot x_3 \cdot x_1^{-1} \cdot x_3^{-1} \cdot x_1^3 \cdot x_3^{-1}}.$$

□

115.6. Exercises for Chapter 115 .

Exercise 115.1. As in Exercise 112.11 we define the *writhe* of a knot diagram as the number of positive crossings minus the number of negative crossings.

(a) Determine the writhes of the standard diagrams of the two trefoils and of the figure-8 knot.

(b) Show that every knot admits a diagram of writhe zero.

(c) Let D be a knot diagram of writhe zero. Show that a longitude is given by taking a “parallel copy” of the diagram in $\mathbb{R}^2$.

Exercise 115.2. Find longitudes for the three oriented knots that are shown in the figure below.

cinquefoil  three-twist knot  stevedore knot 

Exercise 115.3. Let π be a finitely generated group and let G be a finite group. Show that the set $\mathrm{Hom}(\pi, G)$ of homomorphisms from π to G is a finite set.

Exercise 115.4. Let G be a group and $x \in G$. Determine P_G^x(trivial knot).

Exercise 115.5. Can it happen that the coloring polynomial $\mathrm{P}^x_G(K)$ of a knot is zero?

Exercise 115.6.

(a) Is the knot 5_1 chiral?
(b) Is the knot 5_1 reversible?

Exercise 115.7. Let G be a group. As we saw in Definition 97.30, and as we will see in Definition 116.1, we can equip $\mathbb{Z}[G]$ with a naive multiplication given by the multiplication in G and distributivity. Now let K and $\widetilde{K}$ be two oriented knots. We consider the connected sum $K\#\widetilde{K}$ from Definition 109.12. Let G be a finite group and let $x \in G$. Show that we have the following equality in the group ring $\mathbb{Z}[G]$:

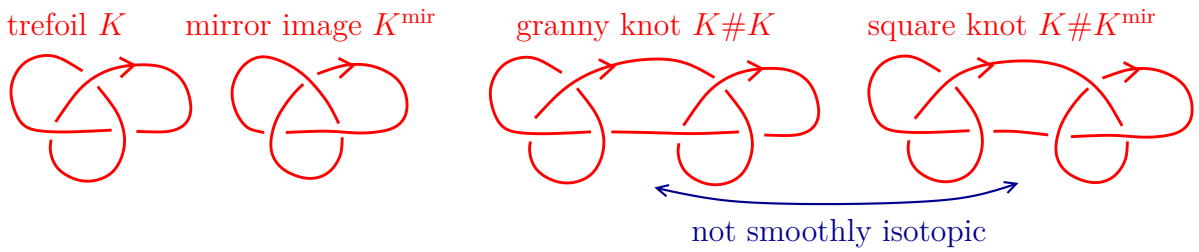
$$\mathrm{P}^x_G(K\#\widetilde{K}) \;=\; \mathrm{P}^x_G(K) \cdot \mathrm{P}^x_G(\widetilde{K}) \quad \in \mathbb{Z}[G].$$

Hint. Use the Oriented Knot–Monoid-π_1-Proposition 111.26.

Exercise 115.8. Let K be the trefoil.

(a) Show that the granny knot $K\#K$ is not smoothly isotopic to its mirror.
(b) Show that the square knot $K\#K^{\mathrm{mir}}$ is neither smoothly isotopic to the granny knot $K\#K$ nor to its mirror.

Hint. Use Exercise 115.7.
Remark. In Exercise **??** we will use a very different approach to prove the same statements.



Exercise 115.9. Let M be a closed 3-dimensional smooth manifold and let $\Phi\colon \overline{B}^3 \to M$ be a fixed smooth embedding. This smooth embedding gives us a map

$$\begin{array}{ccc} \text{set of smooth isotopy} \\ \text{classes of knots in } \overline{B}^3 & \to & \text{set of smooth isotopy} \\ & & \text{classes of knots in } M \\ K & \mapsto & \Phi(K). \end{array}$$

Is this map always injective?

Exercise 115.10. Let K be the Figure 8-knot. Show that $\pi_1(S^3 \setminus K)$ admits a presentation with two generators.

CHAPTER 116

The Alexander function of groups

Let π be a group that admits a presentation of deficiency one (e.g. a link group) and let $\Phi\colon \pi \to H$ be a homomorphism to a free abelian group (e.g. the abelianization of the link group). In this chapter we will, following the work of Ralph Fox [**Fox1953**] and Masaaki Wada [**Wad1994**], associate to (π, Φ) the corresponding Alexander function $\Delta_{\pi,\Phi} \in \mathbb{Q}(H)$ which is an element of the quotient field $\mathbb{Q}(H)$ of the group ring $\mathbb{Z}[H]$. In the next chapter we will use the group theoretic work of the current chapter to define the Alexander polynomial of knots and links.

This chapter is organized as follows:

- In Section 116.1 we will introduce the group ring $\mathbb{Z}[G]$ of a group G.
- In Section 116.2 we will introduce Fox derivatives and we will state some key properties.
- In Section 116.3 we will prove the Fox Derivative Proposition 116.3.
- In Section 116.4 we will introduce the Alexander function $\Delta_{\pi,\Phi}$ of a group with deficiency one together with a homomorphism $\Phi\colon \pi \to H$ to a free abelian group H.
- In Section 116.5 we will prove the Presentation Function Quotient Proposition 116.11, which says that for a given presentation the Alexander function is well-defined.
- In Section 116.6 we will prove the Alexander Function Theorem 116.12 which says that for a given group the Alexander function is independent of the choice of a deficiency one presentation.
- In Section 116.7 we will provide exercises for Chapter 116.

In the next Chapter 117 we will use the Alexander function of this chapter to introduce the Alexander polynomial of oriented knots and links.

116.1. **Group rings.** Before we can introduce the Alexander functions of groups we need to make a few preparations. First let us recall Definition 97.30:

Definition 116.1. Let G be a group. We consider the abelian group

$$\mathbb{Z}[G] \coloneqq \left\{ \sum_{i=1}^{m} r_i \cdot g_i \,\middle|\, r_1, \dots, r_m \in \mathbb{Z} \text{ and } g_1, \dots, g_m \in G \right\}$$

which we equip with the "obvious" multiplication that is given by

$$\left(\sum_{i=1}^{m} r_i \underset{\mathbb{Z}[G]}{\cdot} g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \coloneqq \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \underset{\mathbb{Z}}{\cdot} s_j}_{\in \mathbb{Z}} \cdot \underbrace{(g_i \underset{G}{\cdot} h_j)}_{\in G}.$$

We refer to $\mathbb{Z}[G]$ as the group ring of G. By the Group Ring Lemma 97.31 we know that $\mathbb{Z}[G]$ is a ring[1216] with multiplicatively neutral element $1 \coloneqq 1_{\mathbb{Z}} \cdot e_G$.

[1216]Recall that according to Definition 2.6 a ring is understood to have a multiplicative neutral element, to be associative with respect to multiplication and to be distributive. We do not demand that the multiplication is commutative.

Example.

(1) If H is a free abelian group of rank m, then any choice of a basis $t_1, \dots, t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ to the group ring $\mathbb{Z}[H]$.

(2) Let G be a group. Note that the natural injection $\mathbb{Z} \to \mathbb{Z}[G]$ that is given by $n \mapsto n \cdot e_G$ is a ring homomorphism. We use it to view $\mathbb{Z}$ as a subring of $\mathbb{Z}[G]$. □

The Group Ring Functor Lemma 97.33 allows us to introduce the following notation:

Notation 116.2. Let $\varphi\colon G \to H$ be a group homomorphism. As in the Group Ring Functor Lemma 97.33 we consider the ring homomorphism

$$\begin{array}{rcl} \varphi_*\colon \mathbb{Z}[G] & \to & \mathbb{Z}[H] \\ \sum_{i=1}^{m} r_i \cdot g_i & \mapsto & \sum_{i=1}^{m} r_i \cdot \varphi(g_i). \end{array}$$

116.2. **Fox calculus.** In this section we will study the purely algebraic notion of Fox derivatives, which was first introduced by Ralph Fox [**Fox1953**].

Proposition 116.3. (Fox Derivative Proposition) Let $\langle x_1, \dots, x_k \rangle$ be the free group on generators $x_1, \dots, x_k$. For $i \in \{1, \dots, k\}$ there exists a unique map

$$\frac{\partial}{\partial x_i}\colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \to \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$$

with the following properties:

(1) For any $i, j \in \{1, \dots, k\}$ we have $\frac{\partial}{\partial x_i} x_j = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$.

(2) For any $u, v \in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$ we have

$$\frac{\partial}{\partial x_i}(u \cdot v) = \frac{\partial}{\partial x_i}(u) + u \cdot \frac{\partial}{\partial x_i}(v) \qquad \text{(Leibniz Rule}^{1217}\text{)}.$$

(3) The map $\frac{\partial}{\partial x_i}$ is a homomorphism of abelian groups.

Example. In $\mathbb{Z}[\langle x_1, x_2, x_3 \rangle]$ we perform the following calculation:

$$\begin{aligned} \frac{\partial}{\partial x_2}(7 \cdot x_1 + 5 \cdot x_2 \cdot x_3 \cdot x_2) &\overset{\text{Property (3)}}{=} 7 \cdot \underbrace{\frac{\partial}{\partial x_2} x_1}_{=0} + 5 \cdot \frac{\partial}{\partial x_2}(x_2 \cdot x_3 \cdot x_2) \\ &\overset{\text{by the Leibniz Rule 116.3 (2)}}{=} 5 \cdot \underbrace{\frac{\partial}{\partial x_2} x_2}_{=1} + 5 x_2 \cdot \frac{\partial}{\partial x_2}(x_3 \cdot x_2) \\ &\overset{\text{by the Leibniz Rule 116.3 (2)}}{=} 5 + 5 \cdot x_2 \cdot \Big(\underbrace{\frac{\partial}{\partial x_2} x_3}_{=0} + x_3 \cdot \underbrace{\frac{\partial}{\partial x_2} x_2}_{=1} \Big) \\ &= 5 + 5 \cdot x_2 \cdot x_3. \end{aligned}$$

Note that we judiciously avoided negative exponents. We will only see below how to deal with those. □

Proof. We delay the proof of the Fox Derivative Proposition 116.3 to the next Section 116.3. ■

[1217] The name "Leibniz Rule" comes from the observation that this equality has a striking resemblance to the "Leibniz Rule", or product rule, for partial derivatives in analysis.

Definition 116.4.

(1) We refer to the map

$$\frac{\partial}{\partial x_i} \colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \to \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$$

from the Fox Derivative Proposition 116.3 as the i-th Fox derivative.

(2) At times we write $\frac{\partial r}{\partial x_i} := \frac{\partial}{\partial x_i} r$.

As we just mentioned, we postpone the proof of the Fox Derivative Proposition 116.3 to the next Section 116.3. In the following we will first state and prove two lemmas on Fox derivatives which allow us to get acquainted with the weird properties of Fox derivatives. The familiarity with the axioms will also later guide us in the proof of the Fox Derivative Proposition 116.3.

The following lemma should be read as an addendum to the Fox Derivative Proposition 116.3.

Lemma 116.5. (Fox Derivative Properties Lemma) Let $\langle x_1, \dots, x_k \rangle$ be the free group on generators $x_1, \dots, x_k$ and let $i \in \{1, \dots, k\}$.

(4) If $e \in \langle x_1, \dots, x_k \rangle$ is the trivial element, then

$$\frac{\partial}{\partial x_i} e = 0.$$

(5) Let $u \in \langle x_1, \dots, x_k \rangle$.

(a) For $n \in \mathbb{N}$ we have $\frac{\partial}{\partial x_i} u^n = (1 + u + \cdots + u^{n-1}) \cdot \frac{\partial u}{\partial x_i}$.

(b) $\frac{\partial}{\partial x_i} u^{-1} = -u^{-1} \cdot \frac{\partial u}{\partial x_i}$.

(6) Let $n \in \mathbb{Z}$.

(a) For $j \neq i$ we have $\frac{\partial}{\partial x_i} x_j^n = 0$.

(b) We have

$$\frac{\partial}{\partial x_i} x_i^n = \begin{cases} \sum_{k=0}^{n-1} x_i^k, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -\sum_{k=n}^{-1} x_i^k, & \text{if } n < 0. \end{cases}$$

In particular we have $\frac{\partial}{\partial x_i} x_i^{-1} = -x_i^{-1}$.

Proof.

(4) Note that

$$\frac{\partial}{\partial x_i} e = \frac{\partial}{\partial x_i} (e \cdot e) \underset{\text{by the Leibniz Rule 116.3 (2)}}{\uparrow} \frac{\partial}{\partial x_i} e + e \cdot \frac{\partial}{\partial x_i} e \underset{\text{since } 1 = 1 \cdot e = e}{\uparrow} 2 \cdot \frac{\partial}{\partial x_i} e.$$

It follows that $\frac{\partial}{\partial x_i} e = 0$.

(5) Let $u \in \langle x_1, \dots, x_k \rangle$.

(a) Given $n \in \mathbb{N}$ we see that

$$\frac{\partial}{\partial x_i} u^n = \frac{\partial}{\partial x_i} (u \cdot u^{n-1}) \underset{\text{by the Leibniz Rule 116.3 (2)}}{\uparrow} \frac{\partial}{\partial x_i} u + u \cdot \frac{\partial}{\partial x_i} u^{n-1} \underset{\text{iterating the argument}}{\uparrow} \frac{\partial u}{\partial x_i} \cdot \sum_{k=0}^{n-1} u^k.$$

(b) Note that

$$0 \underset{\uparrow}{=} \frac{\partial}{\partial x_i} e = \frac{\partial}{\partial x_i}(u \cdot u^{-1}) \underset{\uparrow}{=} \frac{\partial}{\partial x_i} u + u \cdot \frac{\partial}{\partial x_i} u^{-1}.$$

by (4) — by the Leibniz Rule 116.3 (2)

It follows that $\frac{\partial}{\partial x_i} u^{-1} = -u^{-1} \cdot \frac{\partial}{\partial x_i} u$.

(6) (a) This statement follows immediately from (4), (5a), (5b) and the fact that for $j \neq i$ we have $\frac{\partial}{\partial x_i} x_j = 0$.

(b) For $n \in \mathbb{N}$ this statement follows from (5a). For $n = 0$ this statement follows from (4). Finally note that for $n < 0$ we have

$$\frac{\partial}{\partial x_i} x_i^n = \frac{\partial}{\partial x_i}(x_i^{-1})^{-n} \underset{\uparrow}{=} \left(\sum_{k=0}^{-n-1} (x_i^{-1})^k \right) \cdot \frac{\partial x_i^{-1}}{\partial x_i} \underset{\uparrow}{=} \left(\sum_{k=n+1}^{0} x_i^k \right) \cdot (-x_i^{-1}) = -\sum_{k=n}^{-1} x_i^k.$$

by (5a) — by (5b) ■

Before we can formulate the next lemma we need to introduce the following harmless concept:

Definition 116.6. Let G be a group. We refer to the map

$$\begin{aligned} \epsilon := \epsilon_G \colon \mathbb{Z}[G] &\to \mathbb{Z} \\ \sum_{i=1}^{m} r_i \cdot g_i &\mapsto \sum_{i=1}^{m} r_i \end{aligned}$$

as the augmentation homomorphism. One can easily verify that the augmentation homomorphism is in fact a ring homomorphism.[1218]

The following theorem, which has a certain similarity with the Fundamental Theorem 73.27 of Calculus gives us a mysterious identity which will be crucial later on.

Theorem 116.7. (Fundamental Theorem-of-Fox Calculus) Let $k \in \mathbb{N}_0$. Given any $u \in \mathbb{Z}[\langle x_1, \ldots, x_k \rangle]$ we have the following equality:

$$\sum_{i=1}^{k} \frac{\partial u}{\partial x_i} \cdot (x_i - 1) = u - \underset{\uparrow}{\epsilon}(u).$$

augmentation homomorphism $\mathbb{Z}[\langle x_1, \ldots, x_k \rangle] \to \mathbb{Z}$

Example. We consider $u := x^3 \cdot y^{-2} \cdot x \in \mathbb{Z}[\langle x, y \rangle]$. We see that

$$\begin{aligned} \frac{\partial u}{\partial x} \cdot (x-1) + \frac{\partial u}{\partial y} \cdot (y-1) &\underset{\downarrow}{=} ((1 + x + x^2) + x^3 \cdot y^{-2}) \cdot (x-1) + x^3 \cdot (-y^{-2} - y^{-1}) \cdot (y-1) \\ &= \underset{\uparrow}{x^3 \cdot y^{-2} \cdot x - e} = u - \epsilon(u). \end{aligned}$$

by the Leibniz Rule 116.3 and the Fox Derivative Properties Lemma 116.5 (6) — worth checking! □

Proof. Since both sides are $\mathbb{Z}$-linear it suffices to prove the equality for $u \in \langle x_1, \ldots, x_k \rangle$. Note that in this case we have $\epsilon(u) = 1$. In other words, it suffices to prove the following claim:

[1218] In fact the augmentation homomorphism is essentially the same as the ring homomorphism from Notation 116.2 that corresponds to the group homomorphisms $G \to \{e\}$.

Claim. For every $u \in \langle x_1, \dots, x_k \rangle$ we have

$$\sum_{i=1}^{k} \frac{\partial u}{\partial x_i} \cdot (x_i - 1) \;=\; u - 1.$$

Proof. First note that if u is the trivial element, then it follows from the definitions and the Fox Derivative Properties Lemma 116.5 (4) that both sides are zero. In particular the equality holds.

It now remains to show that if the equality holds for some $u \in \langle x_1, \dots, x_k \rangle$, then it also holds for $u \cdot x_s$ and $u \cdot x_s^{-1}$, where $s \in \{1, \dots, k\}$ is arbitrary. We calculate

$$\begin{aligned}
\sum_{i=1}^{k} \frac{\partial}{\partial x_i}(u \cdot x_s) \cdot (x_i - 1) &\overset{\text{Leibniz Rule 116.3 (2)}}{=} \sum_{i=1}^{k} \Big(\frac{\partial}{\partial x_i} u + u \cdot \overbrace{\frac{\partial}{\partial x_i} x_s}^{=\delta_{is}} \Big) \cdot (x_i - 1) \\
&= \underbrace{\sum_{i=1}^{k} \frac{\partial}{\partial x_i} u \cdot (x_i - 1)}_{= u - 1, \text{ by the induction hypothesis}} + u \cdot (x_s - 1) \;=\; u - 1 + u \cdot x_s - u \\
&= u \cdot x_s - 1.
\end{aligned}$$

Since this is so much fun we also calculate

$$\begin{aligned}
\sum_{i=1}^{k} \frac{\partial}{\partial x_i}(u \cdot x_s^{-1}) \cdot (x_i - 1) &\overset{\text{Leibniz Rule 116.3 (2)}}{=} \sum_{i=1}^{k} \Big(\frac{\partial}{\partial x_i} u + u \cdot \overbrace{\frac{\partial}{\partial x_i} x_s^{-1}}^{=-\delta_{is} \cdot x_s^{-1}} \Big) \cdot (x_i - 1) \\
&= \underbrace{\sum_{i=1}^{k} \frac{\partial}{\partial x_i} u \cdot (x_i - 1)}_{= u - 1, \text{ by the induction hypothesis}} - u \cdot x_s^{-1} \cdot (x_s - 1) \;=\; u - 1 - u + u \cdot x_s^{-1} \\
&= u \cdot x_s^{-1} - 1.
\end{aligned}$$

■

116.3. **Proof of the Fox Derivative Proposition 116.3.** This section is solely dedicated to the proof of the Fox Derivative Proposition 116.3. In the next Section 116.4 we will use Fox derivatives to introduce the Alexander function of a group.

The idea of the proof is clear: Given an expression $x_{m_1}^{n_1} \cdot \ldots \cdot x_{m_p}^{n_p} \in \langle x_1, \dots, x_k \rangle$ we want to use the Leibniz Rule 116.5 and the formulas Fox Derivative Properties Lemma 116.5 to define the Fox derivative $\frac{\partial}{\partial x_i}(x_{m_1}^{n_1} \cdot \ldots \cdot x_{m_p}^{n_p})$ by going through this expression from "left to write". The only question is, why is well-defined? To write down a precise argument is technically a little delicate, which explains why the proof is a little longer than one might expect.

Proof of the Fox Derivative Proposition 116.3. We start out with some preparations:

(1) A *word in* $x_1, \dots, x_k$ is a finite sequence $w = (x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ where $s_1, \dots, s_t \in \{1, \dots, k\}$ and $\epsilon_1, \dots, \epsilon_t \in \mathbb{Z}$. We refer to $\ell(w) := t$ as the *length of the word* w.

(2) We denote by W the set of all words in $x_1, \dots, x_k$. We define a multiplication on W by "juxtaposition":

$$(x_{s_1}^{\mu_1}, \dots, x_{s_p}^{\mu_p}) \cdot (x_{t_1}^{\nu_1}, \dots, x_{t_q}^{\nu_q}) \;=\; (x_{s_1}^{\mu_1}, \dots, x_{s_p}^{\mu_p}, x_{t_1}^{\nu_1}, \dots, x_{t_q}^{\nu_q}).$$

Clearly this product turns W into a monoid.

(3) We say that a word $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ is *unreduced* if one of the following occurs:

(i) There exists an $i \in \{1, \dots, t\}$ with $\epsilon_i = 0$.

(ii) There exists a $j \in \{1, \dots, t-1\}$ with $s_j = s_{j+1}$.
Otherwise we say that the word is *reduced*. (Note that this matches the definition of a reduced word in Definition 98.1.)

(4) In Definition 98.10 we defined the free group $\langle x_1, \dots, x_k \rangle$ as the set of reduced words in $x_1, \dots, x_k$ with the group structure described in Definition 98.1.

(5) Note that it follows easily from the definitions of the multiplication maps that the map $\Theta\colon W \to \langle x_1, \dots, x_k \rangle$ given by $\Theta(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t}) := x_{s_1}^{\epsilon_1} \cdot \ldots \cdot x_{s_t}^{\epsilon_t}$ is a surjective monoid homomorphism in the sense of Definition 2.2.

After these preparations we start out with the actual proof. For the remainder of the proof we fix $i \in \{1, \dots, k\}$.

Claim 1. There exists a unique map[1219]

$$D_i\colon W \ \to \ \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$$

with the following properties:

(0) We have $D_i() = 0$.

(1) (a) For $j \neq i$ and any $n \in \mathbb{Z}$ we have $D_i(x_j^n) = 0$.

(b) For $n \in \mathbb{Z}$ we have $D_i(x_i^n) = \begin{cases} 1 + x_i + \dots + x_i^{n-1}, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -x_i^{-1} - \dots - x_i^n, & \text{if } n < 0. \end{cases}$

(2) For any $u, v \in W$ we have the following equality in $\mathbb{Z}[\langle x_1, \dots, x_k \rangle]$:

$$D_i(u, v) \ = \ D_i(u) + \Theta(u) \cdot D_i(v) \qquad \text{(Leibniz Rule 116.3 (2))}.$$

Proof. The three rules tell us iteratively in a unique way how to define D_i of a given word $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ "from left to right". More precisely, given $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t}) \in W$ we set

$$D_i(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t}) \ = \ \sum_{j=1}^{t} x_{s_1}^{\epsilon_1} \cdot \ldots \cdot x_{s_{j-1}}^{\epsilon_{j-1}} \cdot \begin{cases} 1 + x_{s_j} + \dots + x_{s_j}^{\epsilon_j - 1}, & \text{if } \epsilon_j > 0 \text{ and } s_j = i, \\ 0, & \text{if } \epsilon_j = 0 \text{ or } s_j \neq i, \\ -x_{s_j}^{-1} - \dots - x_{s_j}^{\epsilon_j}, & \text{if } \epsilon_j < 0 \text{ and } s_j = i. \end{cases}$$

Note that statements (0) and (1) hold by definition. Furthermore one can easily verify that (2) holds. We leave it to the reader to fill in the details. ⊞

Claim 2. For any $m, n \in \mathbb{Z} \setminus \{0\}$ we have

$$D_i(x_i^m, x_i^n) \ = \ D_i(x_i^{m+n}).$$

Proof. First note that this is a statement which holds in $\mathbb{Z}[x_i^{\pm 1}] \subseteq \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$. Next note that $\mathbb{Z}[x_i^{\pm 1}]$ can also be viewed as a subring of $\mathbb{Q}(x_i)$. Furthermore note that it follows easily from the definitions that in $\mathbb{Q}(x_i)$ we have for any $d \in \mathbb{Z}$ the equality

$$D_i(x_i^d) \ = \ \frac{1 - x_i^d}{1 - x_i}.$$

Note that in $\mathbb{Q}(x_i)$ we have the following equalities:

$$D_i(x_i^m, x_i^n) \underset{\text{by Claim 1 (2)}}{\uparrow} = D_i(x_i^m) + x_i^m \cdot D_i(x_i^n) \underset{\text{by the above trick}}{\uparrow} = \frac{1 - x_i^m}{1 - x_i} + x_i^m \cdot \frac{1 - x_i^n}{1 - x_i} \ = \ \frac{1 - x_i^{m+n}}{1 - x_i} \underset{\text{again the above trick}}{\uparrow} = D_i(x_i^{m+n}).$$

[1219] This map is of course motivated by the Fox Derivative Properties Lemma 116.5.

Since $\mathbb{Z}[x_i^{\pm 1}]$ is a subring of $\mathbb{Q}(x_i)$ we see that the desired equality actually also holds in $\mathbb{Z}[x_i^{\pm 1}] \subseteq \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$. ⊞

Claim 3. If we have $w, w' \in W$ with $\Theta(w) = \Theta(w') \in \langle x_1, \dots, x_k \rangle$, then $D_i(w) = D_i(w')$.

Proof. It follows by the above definition of the free group $\langle x_1, \dots, x_k \rangle$ that any element in $\langle x_1, \dots, x_k \rangle$ is represented by a *unique* reduced word. In particular, by reflexivity we can assume that w' is reduced. We now need to prove that for any unreduced word w in $x_1, \dots, x_k$ with $\Theta(w) = \Theta(w')$ we have $D_i(w) = D_i(w')$. We prove this statement by induction on the length of w. If $\ell(w) = 0$, then $\Theta(w) = e$ and thus $w' = ()$. It follows from Claim 1 (0) that $D_i(w) = e = D_i(w')$.

Now suppose that we have proved the statement for all w of length $< t$. Next let $w = (x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ be an unreduced word of length t. We have to distinguish two cases:

(i) There exists a $j \in \{1, \dots, t\}$ with $\epsilon_j = 0$. This means that we can write $w = (u, x_{s_j}^0, v)$ with $u, v \in W$. It follows that

$$D_i(w) = D_i(u, x_{s_j}^0, v) \underset{\text{by the Leibniz Rule 116.3 (2)}}{=} D_i(u) + u \cdot \underbrace{D_i(x_{s_j}^0)}_{=0, \text{ by (1)}} + \underbrace{u \cdot (x_{s_j}^0)}_{=u} \cdot D_i(v) \underset{\text{by the Leibniz Rule 116.3 (2)}}{=} D_i(u, v).$$

(ii) There exists a $j \in \{1, \dots, t-1\}$ with $s_j = s_{j+1}$. Note that this means that we can write $w = (u, x_s^\mu, x_s^\nu, v)$. We set[1220] $\widetilde{w} = (u, x_s^{\mu+\nu}, v)$. Evidently $\Theta(w) = \Theta(\widetilde{w})$. Since $\ell(\widetilde{w}) < \ell(w) = t$ it suffices, by induction, to prove that $D_i(w) = D_i(\widetilde{w})$. First we consider the case that $i = s$. In this case we have the following equalities in $\mathbb{Z}[\langle x_1, \dots, x_k \rangle]$:

$$\begin{aligned} D_i(w) = D_i(u, x_i^\mu, x_i^\nu, v) &\underset{\text{by the Leibniz Rule 116.3 (2)}}{=} D_i(u) + u \cdot (D_i(x_i^\mu, x_i^\nu) + x_i^{\mu+\nu} \cdot D_i(v)) \\ &\underset{\text{by Claim 2}}{=} D_i(u) + u \cdot (D_i(x_i^{\mu+\nu}) + x_i^{\mu+\nu} \cdot D_i(v)) \underset{\text{by the Leibniz Rule 116.3 (2)}}{=} D_i(u, x_i^{\mu+\nu}, v). \end{aligned}$$

The case $i \neq s$ is done in a similar fashion, but it is even simpler since it follows immediately from the definition of D_i that for $i \neq s$ we have $D_i(x_s^\mu, x_s^\nu) = 0$ and $D_i(x_s^{\mu+\nu}) = 0$. ⊞

Claim 4. The map

$$\begin{aligned} \frac{\partial}{\partial x_i} \colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] &\to \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \\ \sum_{j=1}^m a_j \cdot w_j &\mapsto \sum_{j=1}^m a_j \cdot D_i(\widetilde{w}_j) \\ &\qquad \uparrow \text{where } \Theta(\widetilde{w}_i) = w_i \end{aligned}$$

is well-defined, it has all the desired properties and it is the unique such homomorphism.

Proof. First note that it follows from Claim 3 that the given map is well-defined.

(1) It follows from Claim 1 that $\frac{\partial}{\partial x_i} x_j = D_i(x_j) = \delta_{ij}$.
(3) The map is by definition a homomorphism of abelian groups.

[1220] Note that it is possible that $\mu + \nu = 0$, which means that we could end up with a word that is unreduced in the sense of (i).

(2) Let $u, v \in \langle x_1, \dots, x_k \rangle$. We pick $\widetilde{u}, \widetilde{v} \in W$ with $\Theta(\widetilde{u}) = u$ and $\Theta(\widetilde{v}) = v$. Note that

$$\frac{\partial}{\partial x_i}(u \cdot v) \underset{\substack{\uparrow \\ \text{since } \Theta \text{ is a homomorphism we have } \Theta(\widetilde{u},\widetilde{v}) = \Theta(\widetilde{u}) \cdot \Theta(\widetilde{v}) = u \cdot v, \text{ the equality thus follows from from the Definition 116.4 of } \frac{\partial}{\partial x_i}}}{=} D_i(\widetilde{u}, \widetilde{v}) \underset{\substack{\uparrow \\ \text{by Claim 1}}}{=} D_i(\widetilde{u}) + \Theta(\widetilde{u}) \cdot D_i(\widetilde{v}) \underset{\substack{\uparrow \\ \text{by definition of } \frac{\partial}{\partial x_i}}}{=} \frac{\partial}{\partial x_i}(u) + u \cdot \frac{\partial}{\partial x_i}(v).$$

It follows easily from (3) that the desired equality also holds for all $u, v \in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$. Finally the uniqueness of $\frac{\partial}{\partial x_i}$ follows from the uniqueness statement of Claim 1. ■

116.4. **The Alexander function.** The reader might be wondering, where is this going? Why should one care about Fox derivatives? In this section we will give an answer to this pertinent question.

In the following we first recall Definitions 100.10 and 100.15:

Definition 116.8.

(1) Given any finite presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ we denote by

$$\mu = \mu_P \colon \langle x_1, \dots, x_k \rangle \to P$$

the natural projection.

(2) Let π be a group. A finite presentation for the group π consists of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ together with an isomorphism

$$\gamma \colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \xrightarrow{\cong} \pi.$$

(3) The deficiency of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ is defined to be $k - l$.

We continue with a weird definition, which is inspired by Analysis II:

Definition 116.9. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation.

(1) We define the Jacobi matrix $J(P)$ to be the $(l \times k)$-matrix

$$J(P) := \left\{ \mu_{P*} \Big(\underbrace{\frac{\partial r_i}{\partial x_j}}_{\in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]} \Big) \right\}_{\substack{i = 1, \dots, l \\ j = 1, \dots, k}} \in \mathrm{M}(l \times k, \mathbb{Z}[P]).$$

(2) Given a group homomorphism $\Phi \colon P \to H$ we write

$$J^{\Phi}(P) := \underset{\substack{\uparrow \\ \text{where } \Phi_* \text{ is the induced ring homomorphism } \mathbb{Z}[P] \to \mathbb{Z}[H] \text{ which we apply entry-wise}}}{\Phi_*}(J(P)) \in \mathrm{M}(l \times k, \mathbb{Z}[H]).$$

Before we can state the first result of the section we need to introduce just a little bit of extra notation:

Notation 116.10.

(1) Let H be a torsion-free abelian group. In Exercise 116.5 we will see that $\mathbb{Z}[H]$ is a domain. We denote by $\mathbb{Q}(H)$ the quotient field of the domain $\mathbb{Z}[H]$. Recall that $\mathbb{Q}(H) = \{\frac{f}{g} \,|\, f \in \mathbb{Z}[H] \text{ and } g \in \mathbb{Z}[H] \setminus \{0\}\}$.

(2) Let R be a ring and let $M \in \mathrm{M}(l \times k, R)$ be a matrix. Given $i \in \{1, \dots, k\}$ we denote by M_i the $l \times (k-1)$-matrix which we obtain by deleting the i-th column.

In the Wirtinger Presentation Proposition 113.6 (2) we saw that fundamental groups of link complements admit presentations of deficiency one. Since these are the groups we are mostly interested in, we now restrict ourselves to such groups.

The following proposition finally introduces the key object of this part of the lecture notes:

Proposition 116.11. (Presentation Function Quotient Proposition) Suppose we are given a presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle$ of *deficiency one*. Let $\Phi\colon P \to H$ be a non-trivial homomorphism to a free abelian group H.

(1) There exists an $i \in \{1, \dots, k\}$ such that $\Phi(x_i)$ is non-trivial.
(2) Let $i \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$. The term[1221·1223]

$$\Delta_{P,\Phi} \coloneqq \frac{\det(J^\Phi(P)_i)}{\Phi_*(x_i - 1)} \in \mathbb{Q}(H)$$

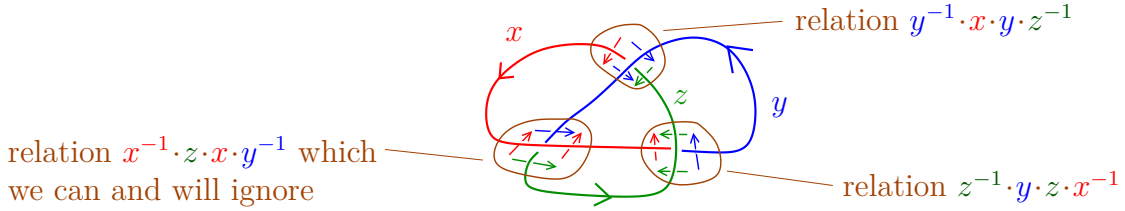
is well-defined (i.e. independent of the choice of i) up to multiplication by a sign.

Proof. In an attempt to keep the conversation flowing, without getting bogged down by technicalities, we postpone the proof of this proposition to Section 116.5. ■

Example. Let $K \subseteq S^3$ be the trefoil. We consider the Wirtinger presentation

$$P \coloneqq \langle x, y, z \,|\, \underbrace{y^{-1}\cdot x\cdot y\cdot z^{-1}}_{=r_1},\ \underbrace{z^{-1}\cdot y\cdot z\cdot x^{-1}}_{=r_2} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$$

from the discussion on page 2199.



Using the Presentation–Homomorphism Lemma 100.13 (or alternatively using the Link Group-Abelianization Proposition 113.9) one can easily verify that there exists a unique epimorphism $\Phi\colon P \to \langle t \rangle$ such that $\Phi(x) = \Phi(y) = \Phi(z) = t$. We now calculate that the Jacobi matrix $J(P) \in \mathrm{M}(2 \times 3, \mathbb{Z}[P])$ is given by

$$J(P) = \begin{pmatrix} \frac{\partial r_1}{\partial x} & \frac{\partial r_1}{\partial y} & \frac{\partial r_1}{\partial z} \\ \frac{\partial r_2}{\partial x} & \frac{\partial r_2}{\partial y} & \frac{\partial r_2}{\partial z} \end{pmatrix} = \begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & -y^{-1}\cdot x\cdot y\cdot z^{-1} \\ -z^{-1}\cdot y\cdot z\cdot x^{-1} & z^{-1} & z^{-1}\cdot y - z^{-1} \end{pmatrix}$$

this follows from the Fox Derivative Proposition 116.3 and the Fox Derivative Properties Lemma 116.5 applied to $r_1 = y^{-1}\cdot x\cdot y\cdot z^{-1}$ and $r_2 = z^{-1}\cdot y\cdot z\cdot x^{-1}$

$$= \begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & -1 \\ -1 & z^{-1} & z^{-1}\cdot y - z^{-1} \end{pmatrix}$$

since $y^{-1}\cdot x\cdot y\cdot z^{-1} = e$ and $z^{-1}\cdot y\cdot z\cdot x^{-1} = e$

[1222] Note that $J^\Phi(P)_i$ is a $(k-1)\times(k-1)$-matrix over $\mathbb{Z}[H]$. Thus it is a square matrix over a commutative ring, which allows us to consider the determinant.

[1223] The fact that $\Phi(x_i)$ is non-trivial and the hypothesis that H is a free abelian group implies that $\Phi_*(x_i - 1)$ is invertible in $\mathbb{Q}(H)$.

We obtain the following equalities in $\mathbb{Q}(t)$:

$$\frac{\det(J^\Phi(P)_1)}{\Phi_*(x-1)} = \frac{\det\left(\Phi_*\begin{pmatrix} \cancel{y^{-1}} & y^{-1}\cdot x - y^{-1} & -1 \\ \cancel{-1} & z^{-1} & z^{-1}\cdot y - z^{-1}\end{pmatrix}\right)}{\Phi_*(x-1)} = \frac{\det\begin{pmatrix} \cancel{t^{-1}} & 1-t^{-1} & -1 \\ \cancel{-1} & t^{-1} & 1-t^{-1}\end{pmatrix}}{t-1} = \frac{t^{-2}-t^{-1}+1}{t-1}$$

$$\frac{\det(J^\Phi(P)_2)}{\Phi_*(y-1)} = \frac{\det\left(\Phi_*\begin{pmatrix} y^{-1} & \cancel{y^{-1}\cdot x - y^{-1}} & -1 \\ -1 & \cancel{z^{-1}} & z^{-1}\cdot y - z^{-1}\end{pmatrix}\right)}{\Phi_*(y-1)} = \frac{\det\begin{pmatrix} t^{-1} & \cancel{1-t^{-1}} & -1 \\ -1 & \cancel{t^{-1}} & 1-t^{-1}\end{pmatrix}}{t-1} = \frac{-t^{-2}+t^{-1}-1}{t-1}$$

$$\frac{\det(J^\Phi(P)_3)}{\Phi_*(z-1)} = \frac{\det\left(\Phi_*\begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & \cancel{-1} \\ -1 & z^{-1} & \cancel{z^{-1}\cdot y - z^{-1}}\end{pmatrix}\right)}{\Phi_*(z-1)} = \frac{\det\begin{pmatrix} t^{-1} & 1-t^{-1} & \cancel{-1} \\ -1 & t^{-1} & \cancel{1-t^{-1}}\end{pmatrix}}{t-1} = \frac{t^{-2}-t^{-1}+1}{t-1}.$$

So all three terms do indeed agree up to a sign. □

Example. Since the previous example was so much fun, let us consider again the trefoil $K = T(2,3)$, but this time with a very different presentation. By the Torus Knot-π_1-Proposition 111.18 we know that we have a presentation $Q := \langle x, y \mid x^2 \cdot y^{-3}\rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$. Note that

$$J(Q) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\end{pmatrix} = \begin{pmatrix} 1+x & x^2\cdot(-y^{-1}-y^{-2}-y^{-3})\end{pmatrix}.$$

this follows from the Fox Derivative Proposition 116.3 and the Fox Derivative Properties Lemma 116.5 (5) applied to $r = x^2\cdot y^{-3}$

It follows from the Presentation–Homomorphism Lemma 100.13 that there exists a unique homomorphism $\Phi\colon Q \to \langle t\rangle$ with $\Phi(x) = t^3$ and $\Phi(y) = t^2$. Note that this is in fact an epimorphism onto $\langle t\rangle$. (In fact it follows from the Link Group-Abelianization Proposition 113.9 that up to a sign it is the unique such epimorphism.) We calculate

$$\frac{\det(J^\Phi(Q)_1)}{\Phi_*(x-1)} = \frac{\det(\Phi_*\begin{pmatrix}\cancel{1+x} & x^2\cdot(-y^{-1}-y^{-2}-y^{-3})\end{pmatrix})}{\Phi_*(x-1)} = \frac{\begin{pmatrix}\cancel{1+t^3} & t^6\cdot(-t^{-2}-t^{-4}-t^{-6})\end{pmatrix}}{t^3-1} = \frac{t^2-t+1}{t-1},$$

$$\frac{\det(J^\Phi(Q)_2)}{\Phi_*(y-1)} = \frac{\det(\Phi_*\begin{pmatrix}1+x & \cancel{x^2\cdot(-y^{-1}-y^{-2}-y^{-3})}\end{pmatrix})}{\Phi_*(y-1)} = \frac{\begin{pmatrix}1+t^3 & \cancel{t^6\cdot(-t^{-2}-t^{-4}-t^{-6})}\end{pmatrix}}{t^2-1} = -\frac{t^2-t+1}{t-1}.$$

So we get (up to a sign), the same result as with the Wirtinger presentation which we used above. Magic! □

The Presentation Function Quotient Proposition 116.11 is a little surprising and cute, but on its own it is not particularly useful. After all, we want to obtain invariants of groups, not of presentations. Fortunately the above two examples give us the hope that we have actually a group invariant at our hands.

Theorem 116.12. (Alexander Function Theorem) Let π be a group. Furthermore let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. If π admits a presentation $\gamma\colon P = \langle x_1, \dots, x_k \mid r_1, \dots, r_{k-1}\rangle \xrightarrow{\cong} \pi$ of deficiency one, then

$$\Delta_{\pi,\Phi} := \Delta_{P,\Phi\circ\gamma} = \underbrace{\frac{\det(J^\Phi(P)_i)}{\Phi_*(x_i-1)}}_{\text{where } \Phi(x_i) \text{ is non-trivial}} \in \mathbb{Q}(H)$$

is well-defined, i.e. independent of the choice of the presentation, up to multiplication by an element of the form $\pm h$ with $h \in H$.

Proof. We will provide the proof of the theorem in Section 116.6. ■

The indeterminacy statement in the Alexander Function Theorem 116.12 motivates the following notation:

Notation 116.13. Let H be a torsion-free abelian group. Given $p, q \in \mathbb{Q}(H)$ we write $p \doteq q$ if there exists an $\epsilon \in \{-1, 1\}$ and an $h \in H$ with $p = \epsilon \cdot h \cdot q$.

The Alexander Function Theorem 116.12 allows us to make the following definition:

Definition 116.14. Let π be a group and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. If π admits a presentation $\gamma\colon P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1}\rangle \to \pi$ of deficiency one, then we refer to

$$\Delta_{\pi,\Phi} \;:=\; \Delta_{P,\Phi\circ\gamma} \;=\; \underbrace{\frac{\det(J^{\Phi\circ\gamma}(P)_i)}{\Phi_*(x_i - 1)}}_{\text{where } \Phi(x_i) \text{ is non-trivial}} \;\in\; \mathbb{Q}(H)$$

as the[1224] Alexander function of (π, Φ). By the Alexander Function Theorem 116.12 the Alexander function is well-defined up to "$\doteq$".

Before we consider examples, let us get the following basic lemma out of the way:

Lemma 116.15. (Alexander Function–Functorial Lemma)

(1) Let π and $\widetilde{\pi}$ be two groups which admit presentations of deficiency one, let $f\colon \widetilde{\pi} \to \pi$ be an isomorphism and let $\Phi\colon \pi \to H$ be a non-trivial homomorphisms to a free abelian group H. Then
$$\Delta_{\widetilde{\pi},\Phi\circ f} \;=\; \Delta_{\pi,\Phi} \quad \in \mathbb{Q}(H).$$

(2) Let π be a group which admits a presentation of deficiency one. Let $\Phi\colon \pi \to G$ and $f\colon G \to H$ be two homomorphisms to free abelian groups such that $f \circ \Phi\colon \pi \to H$ is non-trivial. Then[1225]
$$f_*(\Delta_{\pi,\Phi}) \;=\; \Delta_{\pi,f\circ\Phi} \quad \in \mathbb{Q}(H).$$

Proof.

(1) This statement is an immediate consequence of the Alexander Function Theorem 116.12.

(2) By (1) we can assume that $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1}\rangle$. We pick $i \in \{1, \dots, k\}$ such that $(f \circ \Phi)(x_i) \neq 0$. We see that

$$f_*(\Delta_{\pi,\Phi}) \;=\; f_*\Big(\frac{\det(\Phi_*(J(\pi)_i))}{\Phi_*(x_i - 1)}\Big) \;=\; \frac{f_*(\det(\Phi_*(J(\pi)_i)))}{f_*(\Phi_*(x_i - 1))}$$
$$\underset{\uparrow}{=} \frac{\det((f\circ\Phi)_*(J(\pi)_i)))}{(f\circ\Phi)_*(x_i - 1))} \;=\; \Delta_{\pi,f\circ\Phi}.$$

since determinants commute with ring homomorphisms

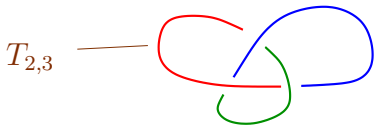
■

[1224] We call $\Delta_{\pi,\Phi}$ Alexander *function* since $\Delta_{\pi,\Phi} \in \mathbb{Q}(H)$ can be viewed as a rational function, i.e. the quotient of two polynomials. Later on we will introduce the Alexander *polynomial* of an oriented link, which is a Laurent polynomial, i.e. an element of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

[1225] Recall that $f\colon G \to H$ induces a ring homomorphism $f\colon \mathbb{Z}[G] \to \mathbb{Z}[H]$. It is clear that it also induces a ring homomorphism $f_*\colon \{\frac{r}{s} \in \mathbb{Q}(G) \,|\, r, s \in \mathbb{Z}[G], f_*(s) \neq 0\} \to \mathbb{Q}(H)$.

Example.

(1) Let $\pi = \pi_1(S^3 \setminus T_{2,3})$ be the fundamental group of the complement of the trefoil $T_{2,3}$. For at least one of the two epimorphisms $\pi \to \langle t \rangle$ we obtain from the calculations on page 2254 that $\Delta_{\pi,\Phi} \doteq \frac{t^2-t+1}{t-1}$.



for $\pi := \pi_1(S^3 \setminus T_{2,3})$ we have $\Delta_{\pi,\Phi} \doteq \frac{t^2-t+1}{t-1}$

In fact it follows from the Alexander Function–Functorial Lemma 116.15 that for the other epimorphism we get, "up to $\doteq$" the same result for the other epimorphism.

(2) Let $\pi = \mathbb{Z}$ and let $\Phi\colon \mathbb{Z} \to \langle t \rangle$ be the epimorphism with $\Phi(1) = t$. We consider the deficiency-one presentation that is given by the obvious map $\gamma\colon \langle s \,|\, \rangle \to \mathbb{Z}$. We see that

$$\Delta_{\mathbb{Z},\Phi} \;=\; \Delta_{\langle s\,|\,\rangle,\Phi\circ\gamma} \;=\; \frac{\det(J^{\Phi\circ\gamma}(\langle s \,|\, \rangle)_1)}{(\Phi\circ\gamma)(s)-1} \;=\; \frac{\det((0\times 0)\text{-matrix})}{\Phi(1)-1} \;\underset{\uparrow}{=}\; \frac{1}{t-1}.$$

by the Determinant–Characterization Proposition 2.18 (ii) the determinant of the (0×0)-matrix is 1

This calculation, together with the calculation in (1) and the Alexander Function–Functorial Lemma 116.15 gives us a new, and much more systematic, proof that the fundamental group of the complement of the trefoil is not isomorphic to $\mathbb{Z}$.

(3) Let $\pi = \mathbb{Z}^2$. Furthermore d let $\Phi\colon \mathbb{Z}^2 \to H$ be a non-trivial homomorphism to some free abelian group H. We consider the presentation that is given by the obvious isomorphism $\gamma\colon P := \langle x, y \,|\, \underbrace{xyx^{-1}y^{-1}}_{=:r} \rangle \to \mathbb{Z}^2$ from the Free Abelian Group-Presentation Lemma 100.14. Note that[1226]

$$J^{\Phi}(P) \;=\; \Phi_*\Big(\tfrac{\partial r}{\partial x} \;\; \tfrac{\partial r}{\partial y}\Big) \;=\; \Phi_*(1-\underbrace{xyx^{-1}}_{=y} \;\; x-\underbrace{xyx^{-1}y^{-1}}_{=1}) \;=\; (1-\Phi(y) \;\; \Phi(x)-1).$$

If $\Phi(x)$ is non-trivial, then

$$\Delta_{\mathbb{Z}^2,\Phi} \;=\; \Delta_{P,\Phi\circ\gamma} \;=\; \frac{\det(J^{\Phi}(P)_1)}{\Phi(x)-1} \;=\; \frac{\det(1-\cancel{\Phi(y)} \;\; \Phi(x)-1)}{\Phi(x)-1} \;\doteq\; 1 \;\in\; \mathbb{Q}(H).$$

Basically the same calculation also gives us the same conclusion if $\Phi(y)$ is non-trivial.

(4) Let $\pi =_{\mathrm{i}} \langle x, y \rangle$ be the free group on two generators and let $\Phi\colon \langle x, y \rangle \to H$ be a non-trivial homomorphism to a free abelian group H. We consider the deficiency-one presentation that is given by the obvious isomorphism $\langle x, y \,|\, e \rangle \xrightarrow{\cong} \langle x, y \rangle$. If $\Phi(x)$ is non-trivial, then we see that

$$\Delta_{\langle x,y\rangle,\Phi} \;=\; \frac{\det(J^{\Phi}(\langle x, y \,|\, e\rangle)_1)}{\Phi(x)-1} \;\underset{\uparrow}{=}\; \frac{\det(\Phi(0 \;\; 0)_1)}{\Phi(x)-1} \;=\; \frac{\det(\Phi(0))}{\Phi(x)-1} \;=\; 0 \;\in\; \mathbb{Q}(H).$$

by the Fox Derivative Properties Lemma 116.5 (4)

[1226] To simplify the notation we will ignore γ in our notation.

A very similar calculation also gives us the same conclusion if $\Phi(y)$ is non-trivial. Essentially the same argument shows that for any free group F of rank ≥ 2 and any non-trivial homomorphism $\Phi\colon F \to H$ to a free abelian group H we have $\Delta_{F,\Phi} = 0$. □

The remainder of the chapter is occupied with the proofs of the Presentation Function Quotient Proposition 116.11 and the Alexander Function Theorem 116.12. In the next chapter we will consider (a slight variation of) the Alexander function of fundamental groups of link complements. At that point we will consider examples ad nausea.

116.5. **Proof of the Presentation Function Quotient Proposition 116.11.** In this section we will provide the proof of the Presentation Function Quotient Proposition 116.11. Recall that in a nutshell the proposition says that given a deficiency-one presentation the Alexander function does not depend on the choice of the generator x_i.

The proof of that statement is mostly a consequence of the following two lemmas:

Lemma 116.16. (Jacobi Matrix–Vector Lemma) Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation. As before we denote by $J(P)$ the Jacobi matrix of P from Definition 116.9, which is an $(l \times k)$-matrix over $\mathbb{Z}[P]$. We have the following equality in the group ring $\mathbb{Z}[P]$:

$$J(P) \cdot \begin{pmatrix} 1-\mu_P(x_1) \\ \vdots \\ 1-\mu_P(x_k) \end{pmatrix} \;=\; 0.$$

Proof. We denote by $\mu := \mu_{P*}\colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \to \mathbb{Z}[P]$ the natural epimorphism. Note that for any $j \in \{1, \dots, l\}$ we have the following equalities in the group ring $\mathbb{Z}[P]$:

$$\begin{aligned} j\text{-th row of } J(P) \cdot \begin{pmatrix} 1-\mu(x_1) \\ \vdots \\ 1-\mu(x_k) \end{pmatrix} &\overset{\text{see Definition 116.9}}{=} \Big(\mu\big(\tfrac{\partial r_j}{\partial x_1}\big) \;\dots\; \mu\big(\tfrac{\partial r_j}{\partial x_k}\big)\Big) \cdot \begin{pmatrix} 1-\mu(x_1) \\ \vdots \\ 1-\mu(x_k) \end{pmatrix} \\ &= \sum_{i=1}^{k} \mu\big(\tfrac{\partial r_j}{\partial x_i}\big) \cdot (1-\mu(x_i)) \\ &\overset{\text{since } \mu \text{ is a ring homomorphism}}{=} \mu\Big(\sum_{i=1}^{k} \tfrac{\partial r_j}{\partial x_i} \cdot (1-x_i)\Big) \overset{\text{by the Fundamental Theorem 116.7 of Fox Calculus applied to } u = r_j}{=} \mu(r_j - 1) \;=\; \underbrace{\mu(r_j)}_{=e=1} - 1 \;=\; 0. \end{aligned}$$

We just showed that all rows are zero, hence the matrix under consideration is zero as well. ■

Our next lemma is essentially an amusing linear algebra exercise:

Lemma 116.17. (Matrix–Quotient Lemma) Let $\mathbb{F}$ be a field, let $M \in \mathrm{M}((k-1)\times k, \mathbb{F})$ be a matrix and let $w = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} \in \mathbb{F}^k$ such that $M \cdot w = 0$. For any $i, j \in \{1, \dots, k\}$ with

$w_i \neq 0$ and $w_j \neq 0$ we have the following equality in $\mathbb{F}$:[1227]

$$\frac{\det(M_i)}{w_i} = \pm \frac{\det(M_j)}{w_j}.$$

Proof. To simplify the notation we now assume that $i = 1$ and $j = 2$. (Note though that here we pick up a sign indeterminacy.) We denote the k columns of M by $c_1, \ldots, c_k$. Note that with this notation our hypothesis $M \cdot w = 0$ says that the following equality holds:

$$(*) \qquad c_1 \cdot w_1 + c_2 \cdot w_2 + c_3 \cdot w_3 + \cdots + c_k \cdot w_k = 0.$$

The lemma follows from the following calculation:

$$\begin{aligned}
\det(M_1) \cdot w_2 &= \det(c_2\ c_3\ \ldots\ c_k) \cdot w_2 \overset{\text{multilinearity of the determinant}}{=} \det\big(c_2 \cdot w_2\ c_3\ \ldots\ c_k\big) \\
&= \det\big((-c_1 \cdot w_1 - c_3 \cdot w_3 - \cdots - c_k \cdot w_k)\ c_3\ \ldots\ c_k\big) \\
&\overset{\text{follows from } (*)}{=} \det\big(-c_1 \cdot w_1\ c_3\ \ldots\ c_k\big) \overset{\text{follows from the multilinearity of the determinant}}{=} -\det(c_1\ c_3\ \ldots\ c_k) \cdot w_1 = -\det(M_2) \cdot w_1.
\end{aligned}$$

This gives us our desired result. ∎

Proof of the Presentation Function Quotient Proposition 116.11. As before we consider a presentation $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle$ of deficiency one and a non-trivial homomorphism $\Phi\colon P \to H$ to a free abelian group H.

(1) Since $x_1, \ldots, x_k$ is a generating set of P and since $\Phi\colon P \to H$ is non-trivial homomorphism we see that at least one $\Phi(x_i)$ is non-zero.

(2) Let $i \neq j \in \{1, \ldots, k\}$ with $\Phi(x_i) \neq 0$ and with $\Phi(x_j) \neq 0$. Note that it follows from the Jacobi Matrix–Vector Lemma 116.16 and the fact that $\Phi_*\colon \mathbb{Z}[P] \to \mathbb{Z}[H] \subseteq \mathbb{Q}(H)$ is a ring homomorphism that we have the following equality in $\mathbb{Q}(H)$:

$$J^{\Phi}(P) \cdot \begin{pmatrix} \Phi_*(1 - \mu_P(x_1)) \\ \vdots \\ \Phi_*(1 - \mu_P(x_k)) \end{pmatrix} = 0.$$

It now follows from the Matrix–Quotient Lemma 116.17 that

$$\frac{\det(J^{\Phi}(P)_i)}{\Phi_*(x_i - 1)} = \pm \frac{\det(J^{\Phi}(P)_j)}{\Phi_*(x_j - 1)} \in \mathbb{Q}(H).$$ ∎

116.6. **Proof of the Alexander Function Theorem 116.12.** We need to show that two deficiency-one presentations of a given group lead to essentially the same Alexander function $\Delta_{P,\Phi}$. The idea is to go between any two presentations using Tietze transformations, and to see that Tietze transformations do not change the invariant. Unfortunately this will require us to generalize our definition of the Alexander function to arbitrary finite presentations. This program is carried out in the following three subsections:

- In Subsection 116.6.1 we will introduce Tietze Transformations between presentations of a given group. We will see that we can relate any two finite presentations of a group

[1227] Recall that according to Notation 116.10 M_i and M_j are the matrices that we obtain from M by deleting the i-th respectively j-th column.

via Tietze transformations. The catch is, that if we start out with two presentations of deficiency one, we cannot necessarily relate one to another via a sequence of deficiency-one presentations.
- In Subsection 116.6.2 we will generalize generalize the notion of an Alexander function from deficiency-one presentations to arbitrary finite presentations.
- In Subsection 116.6.3 we will use Tietze transformations to show that the Alexander function of an arbitrary finitely presented group does not depend on the choice of the finite presentation. This statement implies in particular the Alexander Function Theorem 116.12.

116.6.1. **Tietze Transformations.** Let us remind the reader of the Tietze transformations which we introduced in Definition 100.24:

Definition 116.18. Let
$$\Xi\colon \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\cong} \pi$$
be a finite presentation of a group π. We obtain new finite presentations of π as follows:

(T1) For any element s of $\langle\langle r_1,\dots,r_l\rangle\rangle \subseteq \langle X\rangle$ we obtain the presentation
$$\langle x_1,\dots,x_k \mid r_1,\dots,r_l, s\rangle \underset{\uparrow}{=} \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\Xi} \pi.$$
since $s \in \langle\langle r_1,\dots,r_l\rangle\rangle$ the groups are identical

(T2) For any $x \notin \{x_1,\dots,x_k\}$ and any $s,t \in \langle x_1,\dots,x_k\rangle$ we obtain the presentation
$$\begin{array}{rcl}\langle x_1,\dots,x_k, x \mid r_1,\dots,r_l, s\cdot x\cdot t\rangle & \to & \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\Xi} \pi.\\ x_i & \mapsto & x_i\\ x & \mapsto & s^{-1}\cdot t^{-1}\end{array}$$

The above two methods, and their inverses, of obtaining new finite presentations out of a given one are called Tietze transformations.

Example. As an illustration of the Tietze transformations let us show three tricks:

(a) If we have a presentation, then we can replace any relator by a conjugate. For example, if $\langle x,y \,|\, r(x,y), s(x,y)\rangle$ is a presentation, then we have
$$\langle x,y \,|\, r(x,y)\rangle \overset{(T1)}{=} \langle x,y \,|\, r(x,y), x^{-1}\cdot r(x,y)\cdot x\rangle \overset{(T1)}{=} \langle x,y \,|\, x^{-1}\cdot r(x,y)\cdot x\rangle.$$

(b) One can "eliminate" a generator in suitable settings. For example, suppose we have a presentation of the form $P = \langle x,y,z \,|\, x\cdot v(y,z)^{-1}, w(x,y,z)\rangle$ where $v(y,z) \in \langle y,z\rangle$ and $w(x,y,z) \in \langle x,y,z\rangle$. We now see that
$$\begin{array}{rl}\langle x,y,z \,|\, x\cdot v(y,z)^{-1}, w(x,y,z)\rangle & \overset{(T1)}{=} \langle x,y,z \,|\, x\cdot v(y,z)^{-1}, w(x,y,z), w(v(y,z),y,z)\rangle\\ & \overset{(T1)}{=} \langle x,y,z \,|\, x\cdot v(y,z)^{-1}, w(v(y,z),y,z)\rangle\\ & \overset{(T2)}{=} \langle y,z \,|\, w(v(y,z),y,z)\rangle.\end{array}$$

(c) If $\langle x,y \,|\, r(x,y)\rangle$ is a presentation, then we can "substitute y by $\widetilde{y} = y^{-1}$" as follows:
$$\begin{array}{rl}\langle x,y \,|\, r(x,y)\rangle & \overset{(T2)}{=} \langle x,y,\widetilde{y} \,|\, r(x,y), \widetilde{y}\cdot y\rangle \overset{(T1)}{=} \langle x,y,\widetilde{y} \,|\, r(x,y), \widetilde{y}\cdot y, r(x,\widetilde{y}^{-1})\rangle\\ & \overset{(T1)}{=} \langle x,y,\widetilde{y} \,|\, \widetilde{y}\cdot y, r(x,\widetilde{y}^{-1})\rangle \overset{(T2)}{=} \langle x,\widetilde{y} \,|\, r(x,\widetilde{y}^{-1})\rangle.\end{array}$$

For example let $K \subseteq S^3$ be the trefoil. We now see that

$$\begin{array}{rcl} \pi_1(S^3 \setminus K) & \overset{\text{see page 2199}}{=} & \langle x,y,z \,|\, y^{-1}\cdot x\cdot y\cdot z^{-1},\ z^{-1}\cdot y\cdot z\cdot x^{-1}\rangle \\ & = & \langle y,z \,|\, z^{-1}\cdot y\cdot z\cdot y\cdot z^{-1}\cdot y^{-1}\rangle \\ & \overset{\text{apply (b) to } x\cdot y\cdot z^{-1}\cdot y^{-1}}{=} & \langle y,z \,|\, y\cdot z\cdot y = z\cdot y\cdot z\rangle. \end{array} \qquad \begin{array}{rcl} & \overset{\text{apply (a)}}{=} & \langle x,y,z \,|\, x\cdot y\cdot z^{-1}\cdot y^{-1},\ z^{-1}\cdot y\cdot z\cdot x^{-1}\rangle \\ & = & \langle y,z \,|\, y\cdot z\cdot y\cdot z^{-1}\cdot y^{-1}\cdot z^{-1}\rangle \\ & \overset{\text{apply (a)}}{=} & \end{array}$$

□

We can now recall the following theorem:

Theorem 100.25. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

The Tietze Theorem 100.25 will be the key ingredient in the proof of the Alexander Function Theorem 116.12, which we provide in the next section.

116.6.2. **Alexander functions for arbitrary presentations.** The catch with the idea of using the Tietze Theorem 100.25 to relate two deficiency-one presentations is that inbetween we might leave the cozy world of deficiency-one presentations. We thus need to generalize the notion of an Alexander function to arbitrary finite presentations.

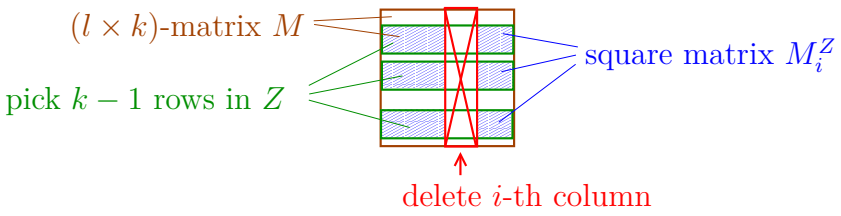
The idea, given a presentation $\langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle$, is as follows:

(1) We again look at the Jacobi matrix $J(P)$ from Definition 116.9.
(2) Again we delete a column from $J(P)$ and we obtain an $(l \times (k-1))$-matrix.
(3) To obtain a square matrix we now consider just $k-1$ rows at any given time.
(4) To get a well-defined invariant we need to consider all possibilities to consider $k-1$ rows and we use the greatest common divisor to get an invariant.

To put this idea into practice we need to introduce the following notation:

Notation 116.19. Let R be a ring and let $M \in \mathrm{M}(l \times k, R)$ be a matrix.

(1) Given $i \in \{1,\dots,k\}$ we denote by M_i the $l \times (k-1)$-matrix which is the result of deleting the i-th column.
(2) Given $Z \subseteq \{1,\dots,l\}$ with $\#Z = k-1$ we denote by M^Z the $(k-1) \times k$-matrix that is given by the rows[1228] in Z.
(3) Let $i \in \{1,\dots,k\}$ and let $Z \subseteq \{1,\dots,l\}$ with $\#Z = k-1$. We consider the corresponding $(k-1) \times (k-1)$-matrix $M_i^Z := (M^Z)_i$.



The following definition is surely familiar from an earlier algebra course:

Definition 116.20. Let R be a commutative ring.

(1) Given $r,s \in R$ we write $r|s$ if there exists a $t \in R$ with $s = t \cdot r$.

[1228] The mnemonic device here is that "Z" stands for "Zeilen". Unfortunately R for "rows" is already taken.

(2) Given a subset $M \subseteq R$ we say that $r \in R$ is a divisor of M if for every $m \in M$ we have $r|m$.
(3) Let R be a unique factorization domain (UFD). Let $M \subseteq R$ be a subset. There exists an element $\gcd_R(M) \in R$ such that $\gcd_R(M)$ is a divisor of M and such that if t is another divisor of $\gcd_R(M)$, then $t|\gcd_R(M)$. Note that $\gcd_R(M)$ is well-defined up to multiplication by a unit in R.

Example. Let R be a UFD. In Exercise 116.9 we will see that have $\gcd_R(R) = 1$ and that $\gcd_R(\varnothing) = 0$. □

The following lemma is a generalization of the Presentation Function Quotient Proposition 116.11.

Lemma 116.21. (Independence-of-Generator Lemma) We suppose that we are given a presentation let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$. Let $\Phi\colon P \to H$ be a non-trivial homomorphism to a free abelian group H.
(1) There exists an $i \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$.
(2) Let $i \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$. The term[1229] [1230]

$$\Delta_{P,\Phi} \coloneqq \frac{\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z \subseteq \{1,\dots,l\} \\ \#Z = k-1}}}{\Phi_*(x_i - 1)} \in \mathbb{Q}(H)$$

is well-defined (i.e. independent of the choice of i) up to multiplication by a sign.

Proof. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a presentation and let $\Phi\colon P \to H$ be a non-trivial homomorphism to a free abelian group H. The proof of Statement 1 is identical to the proof of the Presentation Function Quotient Proposition 116.11 (1). Thus let us turn to the proof of Statement 2. Let $i, j \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$ and $\Phi(x_j) \neq 0$. Furthermore let $Z \subseteq \{1, \dots, l\}$ with $\#Z = k - 1$. It follows easily from the Jacobi Matrix–Vector Lemma 116.16, the fact that $\Phi_*\colon \mathbb{Z}[P] \to \mathbb{Z}[H] \subseteq \mathbb{Q}(H)$ is a ring homomorphism and the Matrix–Quotient Lemma 116.17 that[1231]

$$\frac{\det(J^\Phi(P)_i^Z)}{\Phi_*(x_i - 1)} \doteq \frac{\det(J^\Phi(P)_j^Z)}{\Phi_*(x_j - 1)} \in \mathbb{Q}(H).$$

It follows that

$$\det(J^\Phi(P)_i^Z) \cdot \Phi_*(x_j - 1) \doteq \det(J^\Phi(P)_j^Z) \cdot \Phi_*(x_i - 1) \in \mathbb{Z}[H].$$

Taking greatest common divisors in $\mathbb{Z}[H]$ we obtain the following equality in $\mathbb{Z}[H]$:

$$\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z) \cdot \Phi_*(x_j - 1)\}_{\substack{Z \subseteq \{1,\dots,l\} \\ \#Z = k-1}} \doteq \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_j^Z) \cdot \Phi_*(x_i - 1)\}_{\substack{Z \subseteq \{1,\dots,l\} \\ \#Z = k-1}}.$$

[1229] Note that for a presentation of deficiency one we have to take $Z = \{1, \dots, k-1\}$ and we see that the definition agrees with the definition from the Presentation Function Quotient Proposition 116.11.
[1230] By the Laurent Polynomial Ring–UFD Proposition 287.12 we know that $\mathbb{Z}[H]$ is a unique factorization domain.
[1231] One might think that we are done, but since $\mathbb{Q}(H)$ is a field it does not make much sense to consider greatest common divisors at this point.

When we take greatest common divisors we can pull out common factors. Thus we obtain that

$$\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}} \cdot\Phi_*(x_j-1) \doteq \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_j^Z)\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}} \cdot\Phi_*(x_i-1).$$

From this we finally obtain the desired equality

$$\frac{\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}}}{\Phi_*(x_i-1)} \doteq \frac{\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_j^Z)\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}}}{\Phi_*(x_j-1)} \in \mathbb{Q}(H). \quad \blacksquare$$

116.6.3. **Proof of Theorem 116.12: Conclusion.** The following lemma is generalization of the Alexander Function Theorem 116.12.

Lemma 116.22. (Independence-of-Presentation Lemma) Let π be a group and let $\Phi\colon \pi\to H$ be a non-trivial homomorphism to a free abelian group H. If $\gamma\colon P\to\pi$ and $\delta\colon Q\to\pi$ are finite presentations, then

$$\Delta_{P,\Phi\circ\gamma} \doteq \Delta_{Q,\Phi\circ\delta} \in \mathbb{Q}(H).$$

Proof. By the Tietze Theorem 100.25 we know that the two presentations $\gamma\colon\pi\to P$ and $\delta\colon\pi\to Q$ are related by a finite sequence of Tietze transformations, as introduced in Definition 116.18. So it suffices to deal with the case that the presentations P and Q are related by a single Tietze transformation.

(T1) We consider the following situation: We have

$$\underbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l,s\rangle}_{=Q} = \underbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}_{=P} \xrightarrow{\gamma} \pi$$

where $s\in\langle\langle r_1,\dots,r_l\rangle\rangle\subseteq\langle X\rangle$. We see that the Jacobi matrix $J(Q)$ equals the Jacobi matrix $J(P)$, except that we are adding the row $\frac{\partial s}{\partial x_i}$, $i=1,\dots,k$. Recall that since $s\in\langle\langle r_1,\dots,r_l\rangle\rangle$. As we will see in Exercise 116.10 one obtains easily from the Fox Derivative Properties Lemma 116.5 (5) that there exist $\lambda_1,\dots,\lambda_l\in\mathbb{Z}[P]$ such that for every $i\in\{1,\dots,k\}$ we have

$$\frac{\partial s}{\partial x_i} = \sum_{j=1}^{l}\lambda_j\cdot\frac{\partial r_j}{\partial x_i}.$$

In other words the extra row of $J(Q)$ is just a left-linear combination of the previous rows.

Claim. Let $Z\subseteq\{1,\dots,l\}$ with $\#Z=k-2$. For any $i\in\{1,\dots,k\}$ we have

$$\det\left(J^\Phi(Q)_i^{Z\cup\{l+1\}}\right) = \sum_{\substack{j=1\\ j\notin Z}}^{l}\Phi_*(\lambda_j)\cdot\det\left(J^\Phi(Q)_i^{\{Z\cup\{j\}}\right).$$

Proof. First note that:

$$(l+1)\text{-st row of } J^\Phi(Q) \;=\; \Phi_*((l+1)\text{-st row of } J(Q)) \overset{\text{by the above discussion}}{=} \Phi_*\Big(\sum_{j=1}^{l} \lambda_j \cdot j\text{-th row of } J(Q)\Big)$$
$$\overset{\text{since } \Phi_* \text{ is a ring homomorphism}}{=} \sum_{j=1}^{l} \Phi_*(\lambda_j) \cdot \Phi_*(j\text{-th row of } J(Q)).$$

The claim follows easily from this calculation and the multilinearity of the determinant. ⊞

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(Q)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l+1\}\\ \#Z=k-1}} \overset{\text{follows easily from the above claim}}{\doteq} \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(Q)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}}$$
$$\doteq \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l\},\\ \#Z=k-1}}$$
$$\overset{\text{since the first } l \text{ rows of } J(Q) \text{ and } J(P) \text{ agree}}{\uparrow}$$

which then implies that $\Delta_{Q,\Phi\circ\gamma} \doteq \Delta_{P,\Phi\circ\gamma} \in \mathbb{Q}(H)$.

(T2) We consider the following situation: We have $x \notin \{x_1,\dots,x_k\}$, $s,t \in \langle x_1,\dots,x_k\rangle$ and the presentations

$$\overbrace{\langle x_1,\dots,x_k,x \mid r_1,\dots,r_l,s\cdot x\cdot t\rangle}^{=Q} \xrightarrow{\Theta} \overbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}^{=P} \xrightarrow{\gamma} \pi.$$
$$x_i \mapsto x_i$$
$$x \mapsto s^{-1}\cdot t^{-1}$$

We see that
$$J(Q) \;=\; \begin{pmatrix} J(P) & 0 \\ * & s \end{pmatrix}.$$

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(Q)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l+1\}\\ \#Z=k}} \overset{\text{the determinant is zero unless } l+1\in Z}{\doteq} \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(Q)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l+1\}\\ \#Z=k,\ l+1\in Z}}$$
$$\doteq \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z\subseteq\{1,\dots,l\},\\ \#Z=k-1}}$$
$$\overset{\text{since } \Phi(s) \text{ is a unit}}{\uparrow}$$

which then implies that $\Delta_{Q,\Phi\circ\gamma\circ\Theta} \doteq \Delta_{P,\Phi\circ\gamma} \in \mathbb{Q}(H)$. ■

Proof of the Alexander Function Theorem 116.12. We just proved the Independence-of-Presentation Lemma 116.22, which is a generalization of the Alexander Function Theorem 116.12. So we are done! ■

116.7. Exercises for Chapter 116 .

Exercise 116.1. Does there exist a variation on the Fox derivative which satisfies the "real" Leibniz Rule 116.3 (2) that we all know and love from calculus? More precisely: Let

$\langle x_1, \dots, x_k \rangle$ be the free group on generators $x_1, \dots, x_k$ and let $i \in \{1, \dots, k\}$. Does there exist a map

$$D_i \colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \ \to \ \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$$

with the following properties:

(1) We have $D_i(x_j) = \delta_{ij}$.
(2) For any $u, v \in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$ we have $D_i(u \cdot v) = D_i(u) \cdot v + u \cdot D_i(v)$.
(3) D_i is a homomorphism of abelian groups.

Exercise 116.2. Let $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a presentation. Show that

$$\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial r_l}{\partial x_1} & \cdots & \frac{\partial r_l}{\partial x_k} \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_k - 1 \end{pmatrix} \ = \ 0.$$

Exercise 116.3. Let $m, n \in \mathbb{Z}$. As in the Baumslag-Solitar Group Proposition 130.25 we consider the Baumslag-Solitar group

$$\mathrm{BS}(m,n) \ = \ \langle x, y \,|\, x^{-1} \cdot y^m \cdot x \ = \ y^n \rangle.$$

We consider the epimorphism $\Phi \colon \mathrm{BS}(m,n) \to \langle t \rangle$ that is given by $\Phi(x) = t$ and $\Phi(y) = e$. Determine the Alexander function $\Delta_{\mathrm{BS}(m,n),\Phi} \in \mathbb{Q}(t)$.

Exercise 116.4. Show that given any polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ there exists a group π of deficiency one and an epimorphism $\Phi \colon \pi \to \langle t \rangle$ such that $\Delta_{\pi,\Phi} \doteq \frac{f(t)}{t-1}$.

Exercise 116.5.

(a) Show that if R is a commutative ring that is a domain, then the Laurent polynomial ring $R[t^{\pm 1}]$ is also a domain.
(b) Let F be a finitely generated free abelian group. Show that $\mathbb{Z}[F]$ is a domain.
(c) Let H be a torsion-free abelian group. Show that $\mathbb{Z}[H]$ is a domain.
Hint. Reduce the problem to the case of a finitely generated subgroup of H.

Exercise 116.6. Let K be the trefoil. Show that given any $m \in \mathbb{Z}$ there exists a presentation P of deficiency one for $\pi_1(S^3 \setminus K)$ such that

$$\Delta_{P,\Phi} \ = \ \pm t^m \cdot \frac{t^2 - t + 1}{t - 1}$$

where Φ denotes one of the two epimorphisms $\pi_1(S^3 \setminus K) \to \langle t \rangle$.

Exercise 116.7. Let $T_{2,3}$ be the trefoil.

- By the Torus Knot-π_1-Proposition 111.18 we know that $\pi_1(S^3 \setminus T_{2,3}) \cong \langle x, y \,|\, x^2 \cdot y^{-3} \rangle$.
- On page 2260 we see that there exists a presentation $\pi_1(S^3 \setminus T_{2,3}) \cong \langle a, b \,|\, aba = bab \rangle$.

Give a purely algebraic proof that the groups $\langle x, y \,|\, x^2 \cdot y^{-3} \rangle$ and $\langle a, b \,|\, aba = bab \rangle$ are isomorphic.
Hint. Consider $x = bab$ and $y = ab$.
Remark 1. This exercise is very similar to Exercise 64.11.
Remark 2. Note that this exercise, together with the Braid Group–Presentation 64.19, shows that, rather curiously, $\pi_1(S^3 \setminus \text{trefoil})$ is isomorphic to the braid group B_3 from Notation 64.6.

Exercise 116.8. We consider the presentation $P = \langle a, b \,|\, aba = bab \rangle$. Show that there exist two epimorphisms $\Phi\colon P \to \mathbb{Z}$ and compute $\Delta_{P,\Phi}$.
Remark. The discussion on page 2260 shows P is also a presentation for the fundamental group of the trefoil complement. But it is fun to go through the exercise without making use of this fact.

Exercise 116.9. Let R be a UFD.

(a) Convince yourself that $\gcd_R(R) = 1$.
(b) Convince yourself that $\gcd_R(\varnothing) = 0$.

Exercise 116.10. We consider a free group $\langle x_1, \dots, x_k \rangle$. Let $r_1, \dots, l \in \langle x_1, \dots, x_k \rangle$. Finally let $s \in \langle\langle r_1, \dots, r_l \rangle\rangle \subseteq \langle X \rangle$. Show that there exist $\lambda_1, \dots, \lambda_l \in \mathbb{Z}[P]$ such that for every $i \in \{1, \dots, k\}$ we have

$$\frac{\partial s}{\partial x_i} \;=\; \sum_{j=1}^{l} \lambda_j \cdot \frac{\partial r_j}{\partial x_i}.$$

Exercise 116.11. Let π be a group that admits a finite presentation and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group. Let $\gamma\colon P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \to \pi$ be a finite presentation. Given $m \in \mathbb{N}_0$ we define $E_m(P)$ to be the ideal of $\mathbb{Z}[H]$ generated by the $(k-m) \times (k-m)$-minors of the Jacobi matrix $J^{\Phi\circ\gamma}(P) \in \mathrm{M}(l \times k, \mathbb{Z}[H])$.higher order

(a) Show that the ideal $E_m(P)$ does not depend on the choice of the presentation.

The greatest common divisor of $E_m(P)$ is sometimes called the *m-th Alexander polynomial of* (π, Φ).

(b) What is the relationship between the first Alexander polynomial and the Alexander function?
(c) Compute the Alexander polynomials for finitely generated free groups, for free abelian groups and for the groups $\pi_1(\Sigma_g)$ where Σ_g is the surface of genus g.
(d) Show that for $g \geq 2$ the surface group $\pi_1(\Sigma_g)$ is neither free nor free abelian.
Remark. This gives an alternative proof for the Surface-π_1–Not Free Proposition 130.21.

Remark. These invariants are discussed in greater detail in [**CF1977**, Chapter VII].

Exercise 116.12. Let π be a group that admits a presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle$ of deficiency one and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group. Let $i \in \{1, \dots, k\}$ such that $\Phi(x_i)$ is non-trivial. We introduce the following objects:

- Let $A \in \mathrm{M}(r \times r, \mathbb{Z}[H])$ be a square matrix. A *codimension-m minor of A* is the determinant of any square matrix that one obtains from A by crossing out m rows and m columns.
- We define the *k-th order Alexander function of* (π, Φ) as

$$\Delta^k_{\pi,\Phi} \;:=\; \frac{\gcd_{\mathbb{Z}[H]}\{\text{all codimension-}k\text{ minors of } J^{\Phi}(P)_i\}}{\Phi(x_i) - 1}.$$

  Note that by definition $\Delta^0_{\pi,\Phi} = \Delta_{\pi,\Phi}$.

Now we can formulate your tasks:

(a) Show that $\Delta^k_{\pi,\Phi}$ is well-defined up to "$\doteq$".
(b) Let $\pi = \mathbb{Z}^2 * \mathbb{Z}^2$ and let $\Phi\colon \pi \to \mathbb{Z}^2$ be the obvious homomorphism. Show that $\Delta_{\pi,\Phi} = 0$ and show that $\Delta^1_{\pi,\Phi} \neq 0$.

(c) Let $g \in \mathbb{N}$. Use the above higher order Alexander functions to show that the genus g surface group $\langle x_1, \ldots, x_{2g} \,|\, [x_1, x_2] \cdot \ldots \cdot [x_{2g-1}, x_{2g}] \rangle$ is not a free group.

Remark. This gives an alternative proof for the Surface-π_1–Not Free Proposition 130.21.

Remark. These invariants are closely related, but not quite the same, to the invariants from Exercise 116.11.

CHAPTER 117

The Alexander polynomial of knots and links

In this chapter we will use the results from the last Chapter 116 to finally introduce the Alexander polynomial of knots and links. The Alexander polynomial of a knot can be introduced in many different ways, a long list of different definitions is for example provided in [**BAr2021**]. The most common approach seems to be via Alexander modules and Seifert matrices. We will discuss this approach later in Chapter 286. In this part of the lecture notes we use an approach due to Crowell–Fox [**CF1977**] and Masaaki Wada [**Wad1994**]. The approach of this chapter has the great advantage that it requires only fundamental groups and it is completely self-contained.

This chapter is organized as follows:

- In Section 117.1 we will introduce Alexander polynomial of oriented knots.
- In Section 117.2 we will introduce the Alexander polynomial of oriented links with at least two components.
- In Section 117.3 we will discuss the role of orientations for the Alexander polynomials.
- In Section 117.4 we will introduce the Alexander polynomials of unoriented knots.
- In Section 117.5 we will discuss Alexander polynomials of mirrors.
- In Section 117.6 we will provide exercises for Chapter 117.

In the next Chapter 118 we will provide the proof for the symmetry theorem for Alexander polynomials.

117.1. **Alexander polynomial of oriented knots.** Before we turn to the definition of the Alexander polynomial of oriented knots and link let us recall, for the reader's convenience, the group theoretic Definition 116.14 we rely on:

Definition 117.1. Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group and let $\gamma\colon \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1}\rangle \to \pi$ be a presentation of deficiency 1. We pick $i \in \{1, \dots, n\}$ such that $(\Phi \circ \gamma)(x_i)$ is non-trivial. We set

$$\Delta_{\pi,\Phi} := \frac{\det\left((\Phi\circ\gamma)_*\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \cancel{\frac{\partial r_1}{\partial x_i}} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial r_{n-1}}{\partial x_1} & \cdots & \cancel{\frac{\partial r_{n-1}}{\partial x_i}} & \cdots & \frac{\partial r_{n-1}}{\partial x_n} \end{pmatrix}\right)}{(\Phi\circ\gamma)_*(x_i - 1)} \in \mathbb{Q}(H).$$

In the Presentation Function Quotient Proposition 116.11 and the Alexander Function Theorem 116.12 we showed that $\Delta_{\pi,\Phi}$ is well-defined, i.e. independent of the choice of the presentation and the choice of i, up to "$\doteq$", i.e. up to multiplication by an element of the form $\pm h$ with $h \in H$.

After this reminder we now turn to the definition of the Alexander polynomial of oriented knots:

Definition 117.2. Let $K \subseteq S^3$ be an oriented knot. As usual we denote by μ_K a meridian of K. By the Link Group-Abelianization Proposition 113.9 there exists a unique epimorphism

$$\Phi_K \colon \pi_1(S^3 \setminus K) \to \langle t \rangle$$

such that $\Phi_K(\mu_K) = t$. By the Wirtinger Presentation Proposition 113.6 we know that $\pi_1(S^3 \setminus K)$ admits a presentation of deficiency one. Thus it makes sense to define[1232] [1233]

$$\Delta_K = \Delta_K(t) := (t-1) \cdot \underbrace{\Delta_{\pi_1(S^3 \setminus K),\Phi_K}}_{\text{as defined above}} \in \mathbb{Q}(t).$$

We refer to $\Delta_K(t)$ as the Alexander polynomial of K.[1234]

Remark. The above definition of the Alexander polynomial of a knot seems to be rather unusual. Our approach follows the lead taken by Masaaki Wada [**Wad1994**]. A somewhat different (but ultimately equivalent) definition of the Alexander polynomial of a knot via Fox calculus is given by Crowell-Fox [**CF1977**, p. 119].

Most text books define the Alexander polynomial via Alexander modules [**Rolf1990**, p. 206], [**Lic1997a**, p. 55], [**Kaw1996**, p. 93] and [**BZH2014**, Definition 8.18]. Some text books also use a definition via Seifert matrices [**Crom2004**, p. 157] and [**Juh2023**, Defition 4.21]. We will explore these approaches in Chapter 286. A comparison of the different definitions is also given in [**BAr2021**]. □

Example.

(1) Let K be the trivial knot. We calculate

$$\Delta_K(t) \underset{\text{by the Trivial Knot-}\pi_1 \text{ Proposition 111.12 (2)}}{\overset{\uparrow}{\doteq}} (t-1) \cdot \Delta_{\mathbb{Z},\Phi_K} \underset{\text{see page 2256}}{\overset{\uparrow}{\doteq}} (t-1) \cdot \frac{1}{t-1} = 1.$$

(2) Let $K \subseteq S^3$ be the trefoil. We calculate

$$\Delta_K(t) \underset{\text{by Definition 117.2}}{\overset{\uparrow}{\doteq}} (t-1) \cdot \Delta_{\pi_1(S^3 \setminus K),\Phi_K} \underset{\text{see page 2254}}{\overset{\uparrow}{\doteq}} (t-1) \cdot \frac{t^2-t+1}{t-1} = t^2 - t + 1.$$

(3) Let K be the oriented figure-8 knot as in the figure below. By the Figure 8 Knot–π_1-Lemma 113.10 there exists an isomorphism

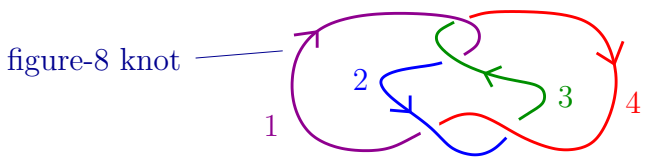
$$\gamma \colon P := \langle x_1, x_2, x_3, x_4 \mid \underbrace{x_3 \cdot x_2 \cdot x_3^{-1} \cdot x_1^{-1}}_{=r_1}, \underbrace{x_4^{-1} \cdot x_3 \cdot x_4 \cdot x_2^{-1}}_{=r_2}, \underbrace{x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1}}_{=r_3} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$$

such that the x_i correspond to meridians.

[1232] Here we use the obvious identification of the group ring $\mathbb{Z}[\langle t \rangle]$ with the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$. This way we end up with the quotient field $\mathbb{Q}(t)$.

[1233] In the proof of the Knot–Alexander Polynomial Proposition 117.3 it will become clear why we added the extra factor $t-1$.

[1234] In Chapter 120 we will see that this definition is consistent with Definition 67.2.



Note that

$$J^{\Phi_K\circ\gamma}(P) \;=\; (\Phi_K\circ\gamma)_*\Big(\tfrac{\partial r_i}{\partial x_j}\Big)_{\substack{i=1,2,3\\ j=1,2,3,4}}$$

$$= (\Phi_K\circ\gamma)_*\begin{pmatrix} -1 & x_3 & 1-x_1 & 0 \\ 0 & -1 & x_4^{-1} & x_4^{-1}x_3 - x_4^{-1} \\ 1-x_3 & 0 & -1 & x_1 \end{pmatrix} = \begin{pmatrix} -1 & t & 1-t & 0 \\ 0 & -1 & t^{-1} & 1-t^{-1} \\ 1-t & 0 & -1 & t \end{pmatrix}.$$

$\uparrow$ uses that for a relation $aba^{-1}d^{-1}$ we have $aba^{-1}d^{-1} = e$ and $aba^{-1} = d$

$\uparrow$ since the x_i are meridians we have $(\Phi_K\circ\gamma)(x_i) = t$

It follows that

$$\Delta_K(t) \;\doteq\; (t-1)\cdot\frac{\det(J^{\Phi_K\circ\gamma}(P)_1)}{(\Phi_K\circ\gamma)_*(x_i-1)} \;\doteq\; (t-1)\cdot\frac{\det\begin{pmatrix} t & 1-t & 0 \\ -1 & t^{-1} & 1-t^{-1} \\ 0 & -1 & t \end{pmatrix}}{t-1}$$

$$\doteq\; -1+3t-t^2. \qquad \square$$

The following proposition explains in particular why we called the above objects "polynomials".

Proposition 117.3. (Knot–Alexander Polynomial Proposition)

(1) The Alexander polynomial $\Delta_K(t)$ of an oriented knot K is well-defined up to "$\doteq$", i.e. it is well-defined up to multiplication by $\pm t^k$ for some $k \in \mathbb{Z}$.
(2) The Alexander polynomial Δ_K is an element of $\mathbb{Z}[t^{\pm 1}]$.
(3) If K and $\widetilde{K}$ are two oriented knots that are smoothly isotopic, then $\Delta_K \doteq \Delta_{\widetilde{K}}$.

Proof.

(1) This statement follows immediately from the Alexander Function Theorem 116.12.[1235]
(2) Let K and $\widetilde{K}$ be two oriented knots that are smoothly isotopic. As in the Isotopic Link-π_1-Lemma 111.4 we note that it follows almost immediately from the Smooth Isotopy–Link Complement Proposition 108.12 that there exists an orientation-preserving diffeomorphism $f\colon S^3 \setminus K \to S^3 \setminus \widetilde{K}$. It follows easily from the Meridian Proposition 109.2 that the image of a meridian of K is a meridian of $\widetilde{K}$. This shows that there exists an

[1235] There is an alternative approach to showing that we have a well-defined knot invariant: By the Reidemeister Moves Theorem 112.10 (2) we know that any two diagrams for an oriented link are related by a sequence of Reidemeister moves. This theorem can be used to show that any two Wirtinger presentations are related by a sequence of Wirtinger presentations, in particular they are related by a sequence of presentations of deficiency one. This way one can show that the Alexander polynomial (defined via Wirtinger presentations) is well-defined without going through presentations of uncertain deficiency. This approach is carried out in [**Wad1994**].

isomorphism $f_*\colon \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus \widetilde{K})$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(S^3 \setminus K) & \xrightarrow[\cong]{f_*} & \pi_1(S^3 \setminus \widetilde{K}) \\ & {\scriptstyle \Phi_K} \searrow \quad \swarrow {\scriptstyle \Phi_{\widetilde{K}}} & \\ & \langle t \rangle. & \end{array}$$

The desired statement now follows from the Alexander Function–Functorial Lemma 116.15.

(3) Let $K \subseteq S^3$ be an oriented knot and let $\Phi_K\colon \pi_1(S^3 \setminus K)_{\mathrm{ab}} \to \langle t \rangle$ be the unique epimorphism with $\Phi_K(\mu_K) = t$. By the Wirtinger Presentation Proposition 113.6 we know that there exists a presentation $\gamma\colon P \coloneqq \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$ where each x_i corresponds to a meridian. We see that

$$\Delta_K(t) \overset{\text{by Definition 117.2}}{\overset{\downarrow}{=}} (t-1) \cdot \Delta_{\pi_1(S^3 \setminus K), \Phi_K} \overset{\text{by Definition 116.14}}{\overset{\downarrow}{=}} (t-1) \cdot \frac{\det(J^{\Phi_K \circ \gamma}(P)_1)}{(\Phi_K \circ \gamma)(x_1) - 1}$$
$$= (t-1) \cdot \frac{\det(J^{\Phi_K \circ \gamma}(P)_1)}{t-1} = \underbrace{\det(J^{\Phi_K \circ \gamma}(P)_1)}_{\substack{\text{determinant of a matrix} \\ \text{with entries in } \mathbb{Z}[t^{\pm 1}]}} \in \mathbb{Z}[t^{\pm 1}].$$
$$\overset{\text{since } x_1 \text{ is a meridian}}{\uparrow}$$

■

Example. It follows from the calculations on page 2268 and the Knot–Alexander Polynomial Proposition 117.3 that the trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.



$\Delta_K \doteq 1$ $\qquad$ $\Delta_K \doteq 1 - t + t^2$ $\qquad$ $\Delta_K \doteq 1 - 3t + t^2$

Thus we recover the Trefoil–Figure 8–Non-Isotopic Corollary 113.11, but this time using a much more systematic approach. □

It is now natural to try to calculate the Alexander polynomial of more oriented knots, e.g. of torus knots. For the torus knots $T(p,q)$ we showed in the Torus Knot-π_1-Proposition 111.18 that $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \,|\, x^p \cdot y^{-q} \rangle$. The slight catch with this calculation is that the role of the orientation of $T(p,q)$ is not entirely clear and we do not know what element in the presentation corresponds to a meridian. Therefore we will first develop the theory of Alexander polynomials a little further, before we return to torus knots.

117.2. **Alexander polynomial of oriented links.** We move to the definition of the Alexander polynomial of an oriented link with ≥ 2 components:

Definition 117.4. Let $m \geq 2$ and let $L \subseteq S^3$ be an oriented m-component link. As usual we denote by $\mu_{L_1}, \dots, \mu_{L_m}$ the meridians of the m components of L. By the Link Group-Abelianization Proposition 113.9 there exists a unique epimorphism

$$\Phi_L\colon \pi_1(S^3 \setminus L) \to \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$$

such that for $i = 1, \dots, m$ we have $\Phi_L(\mu_{L_i}) = t_i$. By the Wirtinger Presentation Proposition 113.6 we know that $\pi_1(S^3 \setminus L)$ admits a presentation of deficiency one. Thus it makes sense to define[1236]

$$\Delta_L \;=\; \Delta_L(t_1, \dots, t_m) \;\coloneqq\; \underbrace{\Delta_{\pi_1(S^3 \setminus L), \Phi_L}}_{\text{see Definition 116.14}} \;\in\; \mathbb{Q}(t_1, \dots, t_m)$$

as the Alexander polynomial of L.

Example.

(1) We consider the trivial m-component link $L \subseteq S^3$ with $m \geq 2$ with any choice of orientation. We calculate

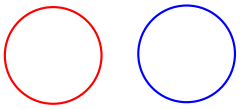
$$\Delta_L(t_1, \dots, t_m) \;\doteq\; \Delta_{\langle x_1, \dots, x_m \rangle, \Phi_L} \;\doteq\; 0.$$

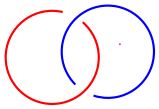
($\doteq$: see page 2162; $\doteq$: this follows from the calculation on page 2256 and the fact that $m \geq 2$)

(2) Let H be the Hopf link with any choice of orientation. We calculate

$$\Delta_H(t_1, t_2) \;\doteq\; \Delta_{\mathbb{Z}^2, \Phi_L} \;\doteq\; 1.$$

($\doteq$: by the Hopf Link–π_1 Proposition 111.24; $\doteq$: see page 2256)

trivial 2-component link L  $\Delta_L(t_1, t_2) = 0$

Hopf link H  $\Delta_H(t_1, t_2) \doteq 1$

This gives a systematic way to show that the Hopf link is non-trivial. □

We continue with the analog of the Knot–Alexander Polynomial Proposition 117.3.[1237]

Proposition 117.5. (Link–Alexander Polynomial Proposition) Let $m \in \mathbb{N}_{\geq 2}$.

(1) The Alexander polynomial $\Delta_L(t_1, \dots, t_m)$ of an oriented m-component link $L \subseteq S^3$ is well-defined up to "$\doteq$", i.e. it is well-defined up to a sign and up to multiplication by a monomial in $t_1, \dots, t_m$, i.e. it is well-defined up to multiplication by a factor of the form $\pm t_1^{r_1} \cdot \ldots \cdot t_m^{r_m}$ with $r_1, \dots, r_m \in \mathbb{Z}$.

(2) If L and $\widetilde{L}$ are two oriented m-component links that are smoothly isotopic, then $\Delta_L \doteq \Delta_{\widetilde{L}}$.

(3) The Alexander polynomial Δ_L is an element of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

Proof.

(1) This statement follows again immediately from the Alexander Function Theorem 116.12.

(2) Let $m \geq 2$ and let $L = L_1 \sqcup \dots \sqcup L_m \subseteq S^3$ be an oriented m-component link. We denote by $\Phi_L \colon \pi_1(S^3 \setminus L)_{\mathrm{ab}} \to \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$ the unique epimorphism with $\Phi_L(\mu_{L_i}) = t_i$. By the Wirtinger Presentation Proposition 113.6 we know that there exists a presentation

[1236] Here we use the obvious identification of the group ring $\mathbb{Z}[\langle t_1, \dots, t_m \rangle_{\mathrm{ab}}]$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This way we end up with the quotient field $\mathbb{Q}(t_1, \dots, t_m)$.

[1237] If one compares the proofs of the Knot–Alexander Polynomial Proposition 117.3 (3) and the Link–Alexander Polynomial Proposition 117.5 (2) one can see where the slight asymmetry between knots and links in the definition of the Alexander polynomial comes from.

$\gamma\colon P \coloneqq \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1}\rangle \xrightarrow{\cong} \pi_1(S^3 \backslash L)$ where each x_i is a meridian. Since $m \geq 2$ we can pick $i_1, i_2 \in \{1, \dots, n\}$ with $(\Phi_L \circ \gamma)(x_{i_1}) = t_1$ and with $(\Phi_L \circ \gamma)(x_{i_2}) = t_2$. Note that by definition we have the following equality in $\mathbb{Q}(t_1, \dots, t_m)$:

$$\Delta_L(t_1, \dots, t_m) \doteq \frac{\det(J^{\Phi_L \circ \gamma}(P)_{i_1})}{t_1 - 1} \doteq \frac{\det(J^{\Phi_L \circ \gamma}(P)_{i_2})}{t_2 - 1}.$$

This implies that we have the following equality in $\mathbb{Q}(t_1, \dots, t_m)$:

$$\underbrace{\det(J^{\Phi_L \circ \gamma}(P)_{i_1})}_{\in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]} \cdot (t_2 - 1) \doteq \underbrace{\det(J^{\Phi_L \circ \gamma}(P)_{i_2})}_{\in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]} \cdot (t_1 - 1).$$

We make the following observations:

(a) Note that $J^{\Phi_L \circ \gamma}(P)_{i_1}$ and $J^{\Phi_L \circ \gamma}(P)_{i_2}$ are matrices with entries in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. It follows that $\det(J^{\Phi_L \circ \gamma}(P)_{i_1})$ and $\det(J^{\Phi_L \circ \gamma}(P)_{i_2})$ both lie in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

(b) The proof of this statement is basically identical to the proof of the Knot–Alexander Polynomial Proposition 117.3 (2).

(c) We consider again the above equality

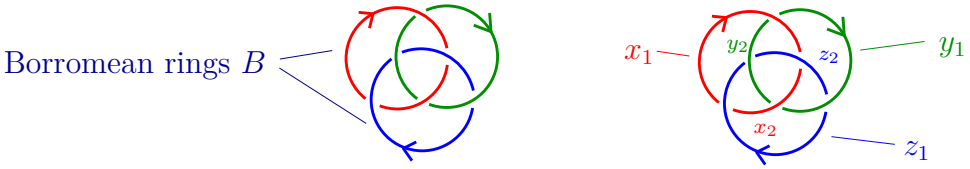
$$\det(J^{\Phi_L \circ \gamma}(P)_{i_1}) \cdot (t_2 - 1) \doteq \det(J^{\Phi_L \circ \gamma}(P)_{i_2}) \cdot (t_1 - 1).$$

By (a) we know that all factors lie in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Since $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ is a UFD and since $\gcd(t_1 - 1, t_2 - 1) = 1$ we see that $\det(J^{\Phi_L \circ \gamma}(P)_{i_1}) = (t_1 - 1) \cdot f$ for some $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

(d) It follows that

$$\Delta_L(t_1, \dots, t_m) \doteq \frac{\det(J^{\Phi_L \circ \gamma}(P)_{i_1})}{t_1 - 1} = \frac{(t_1 - 1) \cdot f}{t_1 - 1} = f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]. \blacksquare$$

Example. Let B be the Borromean ring as shown in the figure below.



From the diagram and the Wirtinger Presentation Proposition 113.6 we obtain a presentation with generators $x_1, x_2, y_1, y_2, z_1, z_2$ and relations $p_1, p_2, q_1, q_2, r_1, r_2$ where

$$\begin{aligned} p_1 &= x_1 \cdot y_1 \cdot x_1^{-1} \cdot y_2^{-1} \\ p_2 &= z_2^{-1} \cdot x_2 \cdot z_2 \cdot x_1^{-1} \end{aligned}$$

and where the other relations are given by cyclically permuting the letters x_i, y_i, z_i for fixed $i \in \{1, 2\}$. We consider the obvious epimorphism $\Phi_B\colon \pi_1(S^3 \setminus B) \to \langle x, y, z\rangle_{\text{ab}}$. Let P be the presentation we obtain by removing the last relation. Note that

$$\det(J^{\Phi_B}(P)_6) = \det \begin{pmatrix} 1 - y & 0 & x & -1 & 0 & \cancel{0} \\ -1 & z^{-1} & 0 & 0 & 0 & \cancel{-z^{-1} + z^{-1} \cdot x} \\ 0 & 0 & 1 - z & 0 & y & \cancel{-1} \\ 0 & -x^{-1} + x^{-1} \cdot y & -1 & x^{-1} & 0 & \cancel{0} \\ z & -1 & 0 & 0 & 1 - x & \cancel{0} \end{pmatrix}.$$

It follows that

$$\Delta_B \underset{\uparrow}{=} \frac{\det(J^{\Phi_B}(P))_6)}{\Phi_B(z_2)-1} \doteq \frac{(1-z)\cdot(1-x)\cdot(1-y)\cdot(1-z)}{z-1} \doteq (1-x)\cdot(1-y)\cdot(1-z).$$

the author would like to point out that he calculated $\det(J^{\Phi_B}(P)_6)$ by hand

□

117.3. **The role of orientations.** Note that we defined the Alexander polynomial for an *oriented* link. The orientation entered in the definition of the meridian which we then used to pick an epimorphism $\pi_1(S^3 \setminus L) \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$. In this section we will study to what degree the Alexander polynomial really depends on this choice. We start out with the following lemma:

Lemma 117.6. (Alexander Polynomial–Orientation Lemma) Let L be an oriented m-component link and let $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$. We denote by $L^{(\epsilon_1,\dots,\epsilon_m)}$ the link L but where we flipped the orientation of the i-th component precisely if $\epsilon_i = -1$. Then

$$\Delta_{L^{(\epsilon_1,\dots,\epsilon_m)}}(t_1, \dots, t_m) \doteq \Delta_L(t_1^{\epsilon_1}, \dots, t_m^{\epsilon_m}).$$

Example. In the figure we show Solomon's link S.



$\Delta_S(t_1, t_2) \doteq 1 + t_1 \cdot t_2$ $\quad$ $\Delta_{\widetilde{S}}(t_1, t_2) \doteq 1 + t_1^{-1} \cdot t_2$

We see that if we flip the orientation of the blue component, then the Alexander polynomials are not equivalent, so the oriented links are not smoothly isotopic. □

Proof. Let L be an oriented m-component link and let $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$. We make a few preparations:

(1) We write $L' \coloneqq L^{(\epsilon_1,\dots,\epsilon_m)}$.
(2) We write $\pi \coloneqq \pi_1(S^3 \setminus L) = \pi_1(S^3 \setminus L')$.
(3) (a) Let $\mu_1, \dots, \mu_m$ be meridians for the oriented link L.
    (b) Let $\mu'_1, \dots, \mu'_m$ be the corresponding meridians for the oriented link L'.
(4) It follows from the Meridian–Symmetries Lemma 109.3 that $\mu'_i = \mu_i$ if $\epsilon_i = 1$ and that $\mu'_i = \mu_i^{\text{rev}}$ if $\epsilon_i = -1$.
(5) As in Definition 117.4 we introduce the following notation:
    (a) We denote by $\Phi_L \colon \pi \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$ the epimorphism given by $\Phi_L(\mu_i) = t_i$.
    (b) We denote by $\Phi_{L'} \colon \pi \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$ the epimorphism given by $\Phi_{L'}(\mu'_i) = t_i$.
(6) We denote by $\Theta \colon \langle t_1, \dots, t_m\rangle_{\text{ab}} \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$ the isomorphism that is given by the assignment $t_i \mapsto t_i^{\epsilon_i}$. Note that it follows from (4) and the Loop-π_1-Lemma 111.15 (3) that the following diagram commutes:

$$\begin{array}{ccc} & \pi & \\ {\scriptstyle \Phi_L}\swarrow & & \searrow{\scriptstyle \Phi_{L'}} \\ \langle t_1, \dots, t_m\rangle_{\text{ab}} & \xrightarrow[t_i \mapsto t_i^{\epsilon_i}]{\Theta} & \langle t_1, \dots, t_m\rangle_{\text{ab}}. \end{array}$$

Now we turn to the actual proof of the promised equality. In the following we first consider the case $m \geq 2$. Sharp eyes will notice that the desired result is actually an immediate

consequence of the Alexander Function–Functorial Lemma 116.15 (2). Alternatively one can also do a direct calculation:

- We pick a presentation $P = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle$ of deficiency one for π.
- We pick an $i \in \{1, \dots, n\}$ such that $\Phi_L(x_i)$ is non-trivial.

We then see that

$$\begin{array}{ccccc} & \text{by Definition 117.4 of } \Delta_{L'} & & \text{since } \Phi_{L'} = \Theta \circ \Phi_L & \text{since determinants commute with ring homomorphisms} \\ & \downarrow & & \downarrow & \downarrow \\ \Delta_{L'}(t_1, \dots, t_m) & \doteq \dfrac{\det(\Phi_{L'*}(J(P)_i))}{\Phi_{L'*}(x_i - 1)} & & \doteq \dfrac{\det((\Theta \circ \Phi_L)_*(J(P)_i))}{(\Theta \circ \Phi_L)_*(x_i - 1)} & \doteq \Theta_*\Big(\dfrac{\det(\Phi_{L*}(J(P)_i))}{\Phi_{L*}(x_i - 1)}\Big) \\ & \doteq \Theta_*(\Delta_L(t_1, \dots, t_m)) & & \doteq \Delta_L(t_1^{\epsilon_1}, \dots, t_m^{\epsilon_m}). & \\ & \uparrow & & \uparrow & \\ & \text{by Definition 117.4 of } \Delta_L & & \text{by definition of } \Theta & \end{array}$$

The case $m = 1$ is basically identical, we just need to multiply the intermediate terms by $t - 1$ and we need to notice that $t^{-1} - 1 = (-t^{-1}) \cdot (t - 1) \doteq t - 1$. ■

A perspicacious reader might have noticed that all the Alexander polynomials that we have calculated so far are symmetric, up to "$\doteq$". It turns out that this is not a coincidence, this is in fact a general phenomenon. More precisely, we have the following theorem that was first proved by Guillermo Torres [**Tor1953**, p. 64]:

Theorem 117.7. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link $L \subseteq S^3$ we have

$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$

Proof. The proof of this theorem is much harder than all the other results on Alexander polynomials that we have seen so far. We outsource it to the next Chapter 118. ■

Next we recall Definition 108.21:

Definition 117.8. Let $L \subseteq S^3$ be an oriented m-component link. The reverse L^{rev} is defined as the link L with the reverse orientation. In other words, with the notation of the Alexander Polynomial–Orientation Lemma 117.6 we have $L^{\text{rev}} = L^{(-1,\dots,-1)}$.

The following lemma, which builds on the Alexander Polynomial–Symmetry Theorem 117.7, unfortunately shows that the Alexander polynomial cannot be used to distinguish a link from its reverse.

Lemma 117.9. (Alexander Polynomial–Reverse Lemma) For any oriented m-component link $L \subseteq S^3$ we have

$$\Delta_{L^{\text{rev}}}(t_1, \dots, t_m) \doteq \Delta_L(t_1, \dots, t_m).$$

Proof. We note that

$$\begin{array}{ccccc} \Delta_{L^{\text{rev}}}(t_1, \dots, t_m) & \doteq & \Delta_L(t_1^{-1}, \dots, t_m^{-1}) & \doteq & \Delta_L(t_1, \dots, t_m). \\ & \uparrow & & \uparrow & \\ & \text{follows from the Alexander Polynomial–Orientation Lemma 117.6} & & \text{by the Alexander Polynomial Symmetry Theorem 117.7} & \end{array}$$

■

117.4. **Alexander polynomials of unoriented knots.** The Alexander Polynomial–Reverse Lemma 117.9 implies that for knots the Alexander polynomial does not depend on the choice of an orientation. This is a double edged sword: On the negative side this has the

consequence that we cannot use Alexander polynomial to distinguish an oriented knot from its reverse. On the positive side it has the advantage that we get an invariant of unoriented knots and when calculating the invariant, we do not have to worry about orientations:

Definition 117.10. Let $K \subseteq S^3$ be an (unoriented) knot. We pick an epimorphism

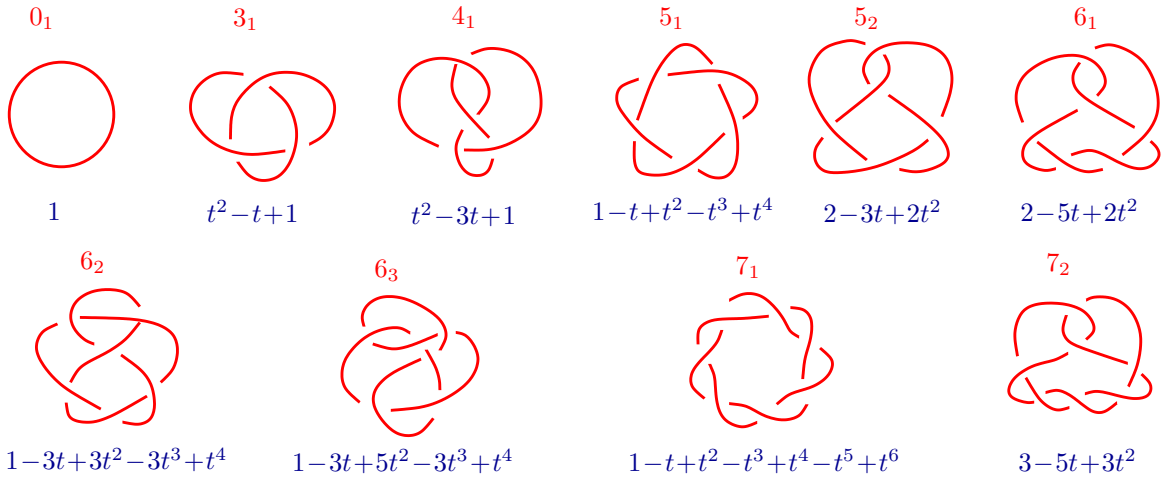
$$\Phi\colon \pi_1(S^3 \setminus K)_{\mathrm{ab}} \;\to\; \langle t\rangle.$$

We refer to

$$\Delta_K(t) \;:=\; (t-1)\cdot \Delta_{\pi_1(S^3\setminus K),\Phi} \;\in\; \mathbb{Z}[t^{\pm 1}]$$

as the Alexander polynomial of K. It follows from the Link Group-Abelianization Proposition 113.9 together with the Alexander Polynomial–Reverse Lemma 117.9 that this invariant is well-defined, i.e. it is independent of the choice of Φ.

Example. It is clear that with enough brain or machine power one can calculate the Alexander polynomial of any knot that is given by a diagram. For example we already calculated the Alexander polynomials of the trivial knot, the trefoil 3_1 and the figure-8 knot 4_1. On page 2183 we showed a list of small crossing knots.



Using heroic calculations one ends up with the above list of Alexander polynomials. □

The following proposition gives us the Alexander polynomials of an infinite family of knots, namely the family of torus knots which we introduced in Definition 111.13:

Proposition 117.11. (Torus Knot–Alexander Polynomial Proposition) Let $p, q \in \mathbb{Z}$ be coprime. For the torus knot $T(p,q)$ we have[1238]

$$\Delta_{T(p,q)}(t) \;\doteq\; \frac{(t-1)\cdot(t^{p\cdot q}-1)}{(t^p-1)\cdot(t^q-1)}.$$

Remark. We refer to [**EN1985**, Theorem 12.1] for a generalization of the above result to more general types of knots and links. □

Example. For the trivial knot $U = T_{(1,1)}$ we obtain from the Torus Knot–Alexander Polynomial Proposition 117.11 that $\Delta_U(t) \doteq 1$ and for the trefoil $K = T(2,3)$ we obtain that $\Delta_K(t) \doteq \frac{(t-1)\cdot(t^6-1)}{(t^2-1)\cdot(t^3-1)} \doteq t^2-t+1$.
This result recovers the results from page 2268. □

[1238] It is a nice algebra exercise to show that the right-hand side does indeed lie in $\mathbb{Z}[t^{\pm 1}]$.



Proof. Let $p, q \in \mathbb{Z}$ be coprime. Recall that by the Torus Knot-π_1-Proposition 111.18 we know that $\pi_1(S^3 \setminus K) \cong P := \langle x, y \mid \underbrace{x^p \cdot y^{-q}}_{=r} \rangle$. First note that

$$J(P) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix} \underset{\text{by the Fox Derivative Properties Lemma 116.5 (5)}}{\uparrow} = \begin{pmatrix} \sum_{i=0}^{p-1} x^i & -x^p \cdot \sum_{i=1}^{q} y^{-i} \end{pmatrix}.$$

Next note that it follows easily from the Presentation–Homomorphism Lemma 100.13 that, up to sign, the homomorphism $\Phi\colon P \to \langle t \rangle$ that is given by $\Phi(x) = t^q$ and $\Phi(y) = t^p$ is the unique epimorphism onto $\langle t \rangle$.

In the following we assume that $p, q \in \mathbb{N}$. The other cases can be dealt with in the same way, alternatively one can use Torus Knot Lemma 111.14 to reduce the general case to the case $p, q \in \mathbb{N}$. After this preamble we calculate:

$$\begin{aligned} \Delta_{T(p,q)}(t) &\underset{\text{by Definition 117.2}}{\downarrow} = (t-1) \cdot \frac{\det(J^\Phi(P)_2)}{\Phi_*(y-1)} = (t-1) \cdot \frac{\det\left(\Phi_*\left(\sum_{i=0}^{p-1} x^i \quad -x^p \cdot \sum_{i=1}^{q} y^{-i}\right)\right)}{\Phi_*(y-1)} \\ &= (t-1) \cdot \frac{\det\left(\sum_{i=0}^{p-1} t^{qi} \quad -t^{pq} \cdot \sum_{i=1}^{q} t^{-ip}\right)}{t^p - 1} = \frac{t-1}{t^p-1} \cdot \sum_{i=0}^{p-1} t^{qi} \underset{\text{elementary algebra}}{\uparrow} = \frac{t-1}{t^p-1} \cdot \frac{t^{pq}-1}{t^q-1}. \end{aligned}$$

■

Rather impressively the Torus Knot–Alexander Polynomial Proposition 117.11 allows us to completely classify the torus knots $T(p, q)$ with $p, q \in \mathbb{N}$[1239]:

Corollary 117.12. (Torus Knot–Classification Corollary) Let $p_1, q_1 \in \mathbb{N}$ be coprime and let $p_2, q_2 \in \mathbb{N}$ be coprime. We assume that $p_1 \neq 1$ and $q_1 \neq 1$. Then

$$T_{(p_1,q_1)} \text{ and } T_{(p_2,q_2)} \text{ are smoothly isotopic} \iff \{p_1, q_1\} = \{p_2, q_2\}.$$

Proof. First note that the"$\Leftarrow$"-direction was shown in the Torus Knot Lemma 111.14 (2d).

We turn to the proof of the"$\Rightarrow$"-direction. So suppose that we have coprime $p_1, q_1 \in \mathbb{N}$ and coprime $p_2, q_2 \in \mathbb{N}$ such that $T_{(p_1,q_1)}$ and $T_{(p_2,q_2)}$ are smoothly isotopic. It follows from the Knot–Alexander Polynomial Proposition 117.3 and the Torus Knot–Alexander Polynomial Proposition 117.11 that

$$\frac{(t-1)\cdot(t^{p_1\cdot q_1}-1)}{(t^{p_1}-1)\cdot(t^{q_1}-1)} \doteq \frac{(t-1)\cdot(t^{p_2\cdot q_2}-1)}{(t^{p_2}-1)\cdot(t^{q_2}-1)}.$$

A purely algebraic argument now shows that $\{p_1, q_1\} = \{p_2, q_2\}$. We outsource this argument to Exercise 117.5. ■

[1239] It follows from the Torus Knot Lemma 111.14 that the classification of torus knots (at least up to mirror images) reduced to the case of $p, q \in \mathbb{N}$.

The following proposition shows that the Alexander polynomial is multiplicative under the connected sum operation:

Proposition 117.13. (Connected Sum–Alexander Polynomial Proposition) Let J and K be two oriented knots. In Definition 109.12 we introduced the connected sum $J\#K$. The following equality holds:

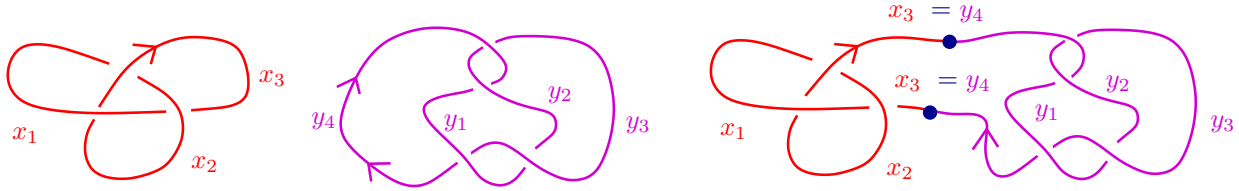
$$\Delta_{J\#K} \doteq \Delta_J \cdot \Delta_K \in \mathbb{Z}[t^{\pm 1}].$$

Proof. Let $J, K \subseteq S^3$ be two oriented knots. By the Wirtinger Presentation Proposition 113.6 we can find presentations

$$\begin{aligned} \pi_1(S^3 \setminus J) &= P \coloneqq \langle x_1, \ldots, x_j \,|\, r_1, \ldots, r_{j-1} \rangle \\ \pi_1(S^3 \setminus K) &= Q \coloneqq \langle y_1, \ldots, y_k \,|\, s_1, \ldots, s_{k-1} \rangle \end{aligned}$$

where each x_a is a meridian of J and where each y_b is a meridian of K. We now see that

$$\begin{aligned} \pi_1(S^3 \setminus (J\#K)) &\overset{\text{by the Oriented Knot Monoid-}\pi_1\text{-Proposition 111.26}}{\cong} \pi_1(S^3 \setminus J) *_{x_j = y_k} \pi_1(S^3 \setminus K) \\ &= \langle x_1, \ldots, x_j, y_1, \ldots, y_k \,|\, r_1, \ldots, r_{j-1}, s_1, \ldots, s_{k-1}, x_j \cdot y_k^{-1} \rangle \quad =: S. \\ &\underset{\text{by the Amalgamated Product Presentation Lemma 100.20}}{\uparrow} \end{aligned}$$



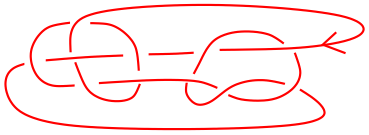
trefoil J figure-8 knot K connected sum $J\#K$

Note that all x_i, y_j are meridians. Thus, under the homomorphisms Φ_J, Φ_K and $\Phi_{J\#K}$ from Definition 117.2 all these get sent to t. We now see that

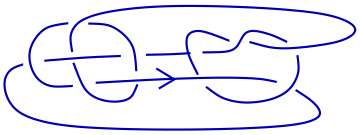
$$\begin{aligned}
\Delta_{J\#K} &= (t-1)\cdot\frac{\det(J^{\Phi_{J\#K}}(S)_{j+k})}{\det(\Phi_{J\#K}(y_k)-1)} \overset{\substack{\text{since the last generator, namely } y_k,\\ \text{corresponds to a meridian of } J\#K}}{=} (t-1)\cdot\frac{\det(J^{\Phi_{J\#K}}(S)_{j+k})}{t-1} = \det(J^{\Phi_{J\#K}}(S)_{j+k})\\
&= \det\Phi_{J\#K}\begin{pmatrix} \left(\frac{\partial r_a}{\partial x_b}\right)_{\substack{a=1,\dots,j-1\\ b=1,\dots,j-1}} & \left(\frac{\partial r_a}{\partial x_j}\right)_{a=1,\dots,j-1} & 0 & \cancel{0}\\ 0 & 0 & \left(\frac{\partial s_a}{\partial y_b}\right)_{\substack{a=1,\dots,k-1\\ b=1,\dots,k-1}} & \cancel{\left(\frac{\partial r_a}{\partial y_k}\right)_{a=1,\dots,k-1}}\\ 0 & \underset{\substack{\uparrow\\ j\text{-th column}}}{x_j} & 0 & \underset{\substack{\uparrow\\ (j+k)\text{-th column}}}{\cancel{-y_k^{-1}}}\end{pmatrix}\\
&= \det\begin{pmatrix} \Phi_J\left(\frac{\partial r_a}{\partial x_b}\right)_{\substack{a=1,\dots,j-1\\ b=1,\dots,j-1}} & \Phi_J\left(\frac{\partial r_a}{\partial x_j}\right)_{a=1,\dots,j-1} & 0\\ 0 & 0 & \Phi_K\left(\frac{\partial s_a}{\partial y_b}\right)_{\substack{a=1,\dots,k-1\\ b=1,\dots,k-1}}\\ 0 & \Phi_J(x_j) & 0\end{pmatrix}\\
&= (-1)^k\cdot\Phi_J(x_j)\cdot\underbrace{\det\left(\Phi_J\left(\frac{\partial r_a}{\partial x_b}\right)_{\substack{a=1,\dots,j-1\\ b=1,\dots,j-1}}\right)}_{=\Delta_J}\cdot\underbrace{\det\left(\Phi_K\left(\frac{\partial s_a}{\partial y_b}\right)_{\substack{a=1,\dots,k-1\\ b=1,\dots,k-1}}\right)}_{=\Delta_K}\\
&\doteq \Delta_J\cdot\Delta_K.
\end{aligned}$$

■

Example. The above calculations indicate that the Alexander polynomial is pretty effective at distinguishing knots. The Alexander polynomial is not perfect though. For example in the following figure we show the Conway knot and the Kinoshita-Terasaka knot which we already encountered on page 2240.



Kinoshita-Terasaka knot



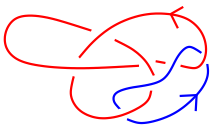
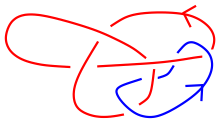
Conway knot

With enough effort one can show that in both case the Alexander polynomials are $\doteq 1$. But on page 2240 we already noticed that both knots are rather non-trivial. □

117.5. **Alexander polynomials and mirrors.** In this last section we return to the general study of links. In the Alexander Polynomial–Reverse Lemma 117.9 we showed that the Alexander polynomial of a link and its reverse coincide. In this section we will prove an analogous result for mirrors.

First let us recall Definition 108.23:

Definition 117.14. Let $L \subseteq S^3$ be an oriented link. The mirror L^{mir} is defined as the reflection of $L \subseteq S^3$ in any hyperplane of $\mathbb{R}^4$.[1240 1241] We give the mirror L^{mir} the orientation that turns the reflection into an orientation-preserving diffeomorphism $L \to L^{\text{mir}}$.

link L   the mirror L^{mir}

The following lemma shows that the Alexander polynomial also cannot be used to distinguish a link from its mirror.

Lemma 117.15. (Alexander Polynomial–Mirror Lemma) For any oriented m-component link $L \subseteq S^3$ we have

$$\Delta_{L^{\mathrm{mir}}}(t_1,\dots,t_m) \doteq \Delta_L(t_1^{-1},\dots,t_m^{-1}) \underset{\uparrow}{\doteq} \Delta_L(t_1,\dots,t_m).$$

by the Alexander Polynomial Symmetry Theorem 117.7

Proof. Let L be an oriented m-component link. We make a few preparations:

(1) We denote by $\rho\colon \mathbb{R}^4 \to \mathbb{R}^4$ the reflection in some hyperplane of $\mathbb{R}^4$.
(2) We write $L' := L^{\mathrm{mir}} := \rho(L)$.
(3) We write $\pi := \pi_1(S^3 \setminus L)$ and we write $\pi' = \pi_1(S^3 \setminus L')$. Note that ρ induces an isomorphism $\rho_*\colon \pi \to \pi'$.
(4) (a) Let $\mu_1,\dots,\mu_m$ be meridians for the oriented link L.
    (b) Let $\mu'_1,\dots,\mu'_m$ be the corresponding meridians of the oriented link L'.
(5) The Meridian–Symmetries Lemma 109.3 implies that for any i we have $\mu'_i = \rho(\mu_i)$[rev].
(6) As in Definition 117.4 we introduce the following notation:
    (a) We denote by $\Phi_L\colon \pi \to \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}$ the epimorphism given by $\Phi_L(\mu_i) = t_i$.
    (b) We denote by $\Phi_{L'}\colon \pi' \to \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}$ the epimorphism given by $\Phi_{L'}(\mu'_i) = t_i$.
(7) We denote by $\Theta\colon \langle t_1,\dots,t_m\rangle_{\mathrm{ab}} \to \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}$ the homomorphism that is given by $t_i \mapsto t_i^{-1}$.
(8) It follows easily from (5) and the Loop-π_1-Lemma 111.15 (3) that the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\rho_*} & \pi' \\ {\scriptstyle \Phi_L}\downarrow & & \downarrow{\scriptstyle \Phi_{L'}} \\ \langle t_1,\dots,t_m\rangle_{\mathrm{ab}} & \xrightarrow{\Theta} & \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}. \end{array}$$

Now we turn to the actual proof of the promised equality. In the following we first consider the case $m \geq 2$. Sharp eyes will once again notice that the desired result is actually an immediate consequence of the Alexander Function–Functorial Lemma 116.15 (1) together with the Alexander Polynomial Symmetry Theorem 117.7. Alternatively one can also do a direct calculation:

- We pick a presentation $\gamma\colon P = \langle x_1,\dots,x_n \,|\, r_1,\dots,r_{n-1}\rangle \to \pi$ of deficiency one for π. Note that it follows from (3) that $\rho_* \circ \gamma\colon \langle x_1,\dots,x_n \,|\, r_1,\dots,r_{n-1}\rangle \to \pi'$ is a presentation for π'.
- We pick an $i \in \{1,\dots,n\}$ such that $\Phi_L(x_i)$ is non-trivial.

We see that

[1240]Note that we showed in the S^3-Link Mirror Lemma 108.24 that, up to a smooth isotopy, the definition of L^{mir} does not depend on the choice of the hyperplane.
[1241]Note that in Definition 108.23 we introduced the mirror of an unoriented link. For the purpose of this section it is better to work with oriented links.

$$\begin{array}{lll}
 & \text{by Definition 117.4 of } \Delta_{L^{\mathrm{mir}}} & \text{by (8)} \\
\Delta_{L^{\mathrm{mir}}}(t_1,\dots,t_m) & \doteq \dfrac{\det(J^{\Phi_{L'}\circ\rho_*\circ\gamma}(P)_1)}{(\Phi_{L'}\circ\rho\circ\gamma)_*(x_i-1)} & \doteq \dfrac{\det((\Theta\circ\Phi_L\circ\gamma)_*(J(P)_1))}{(\Theta\circ\Phi_L\circ\gamma)_*(x_i-1)} \\
 & \doteq \Theta_*\Big(\dfrac{\det((\Phi_{L*}\circ\gamma)(J(P)_1))}{(\Phi_{L*}\circ\gamma)(x_i-1)}\Big) & \doteq \Theta_*(\Delta_L(t_1,\dots,t_m)) \\
 & \text{since determinants commute with ring homomorphisms} & \text{by Definition 117.4 of } \Delta_L \\
 & \doteq \Delta_L(t_1^{-1},\dots,t_m^{-1}) & = \Delta_L(t_1,\dots,t_m). \\
 & \text{by definition of } \Theta & \text{by the Alexander Polynomial Symmetry Theorem 117.7}
\end{array}$$

The case $m = 1$ is basically identical, we just need to multiply the intermediate terms by $t-1$ and we need to notice that $t^{-1}-1 = (-t^{-1})\cdot(t-1) \doteq t-1$. ■

117.6. Exercises for Chapter 117 .

Exercise 117.1. Given a non-zero Laurent polynomial $p(t) = \sum_{i=r}^{s} a_i \cdot t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its *degree* as

$$\deg(p(t)) \;=\; \deg\Big(\sum_{i=r}^{s} a_i \cdot t^i\Big) \;=\; s-r.$$

Show that for every knot K we have

$$\deg(\Delta_K(t)) - 1 \;\leq\; \text{crossing number of } K.$$

Exercise 117.2. Let π be a group that admits a deficiency-one presentation.

(a) Let $\Phi\colon \pi \to \langle t\rangle$ be an epimorphism. Show that $\Delta_{\pi,\Phi}\cdot(t-1) \in \mathbb{Z}[t^{\pm 1}]$.

(b) Let $m \geq 2$ and let $\Phi\colon \pi \to \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}$ be an epimorphism. Show that the corresponding Alexander function $\Delta_{\pi,\Phi}$ lies in $\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

Remark. This exercise can be viewed as a group theoretic generalization of the Knot–Alexander Polynomial Proposition 117.3 (3) and the Link–Alexander Polynomial Proposition 117.5 (3).

Exercise 117.3. Let D and D' be two m-component link diagrams and let L and L' be the oriented m-component links that are associated to D and D'. We use the corresponding Wirtinger presentations P and P' to define Alexander functions. Show, using the Reidemeister Moves Theorem 112.10 (2), that if L and L' are smoothly isotopic as oriented links, then the Alexander functions of P and P' in $\mathbb{Q}(t_1,\dots,t_m)$ are the same up to "$\doteq$".

Remark. In other words, the exercise is to show that one can get a link invariant without leaving the world of deficiency-one presentations.

Exercise 117.4.

(a) Let G_1, G_2 be groups that admit a presentation of deficiency one. Let $\Phi\colon G_2 \to H$ be a non-trivial homomorphism to a free abelian group H. Show that for any epimorphism $\gamma\colon G_1 \to G_2$ there exists a $p \in \mathbb{Z}[H]$ such that $\Delta_{G_1,\Phi\circ\gamma} = p\cdot\Delta_{G_2,\Phi}$.

Hint. You should make use of the flexibility provided by the Independence-of-Generator Lemma 116.21.

(b) We consider the following variation on (a): We now assume that there exists a monomorphism $\gamma\colon G_1 \to G_2$ such that $\Phi \circ \gamma\colon G_1 \to H$ is non-trivial. Can you find a relationship between $\Delta_{G_1,\Phi\circ\gamma}$ and $\Delta_{G_2,\Phi}$?

(c) Let K_1 and K_2 be two knots. We now suppose that there exists an epimorphism $\pi_1(S^3 \setminus K_1) \to \pi_1(S^3 \setminus K_2)$. Show that there exists a $p \in \mathbb{Z}[t^{\pm 1}]$ such that $\Delta_{K_1} = p \cdot \Delta_{K_2}$.

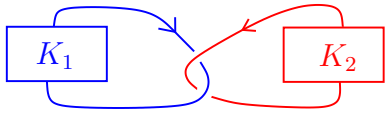
Exercise 117.5. Let $p_1, q_1 \in \mathbb{N}$ be coprime and let $p_2, q_2 \in \mathbb{N}$ be coprime. Show that if $T_{(p_1,q_1)}$ and $T_{(p_2,q_2)}$ are smoothly isotopic, then $\{p_1, q_1\} = \{p_2, q_2\}$.

Remark. By the discussion in the proof of the Torus Knot–Classification Corollary 117.12 it remains to prove the following statement:

$$\frac{(t-1)\cdot(t^{p_1\cdot q_1}-1)}{(t^{p_1}-1)\cdot(t^{q_1}-1)} \doteq \frac{(t-1)\cdot(t^{p_2\cdot q_2}-1)}{(t^{p_2}-1)\cdot(t^{q_2}-1)} \implies \{p_1, q_1\} = \{p_2, q_2\}.$$

Exercise 117.6. Is the figure-8 knot smoothly isotopic to a torus knot $T(p, q)$?

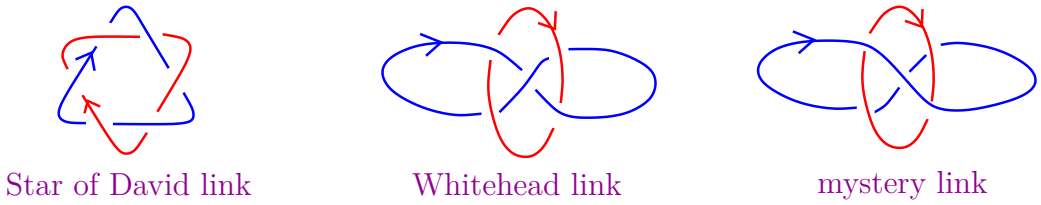
Exercise 117.7. We consider the oriented 2-component link L which is built out of oriented knots K_1 and K_2 as shown in the figure below. Determine $\Delta_L(t_1, t_2)$ in terms of the Alexander polynomials of K_1 and K_2.



Exercise 117.8. Compute the Alexander polynomials of the cinquefoil, the three-twist knot and the stevedore knot that are shown below.



Exercise 117.9. Compute the Alexander polynomial of the Star of David link, the Whitehead link and the mystery link shown in the figure below.



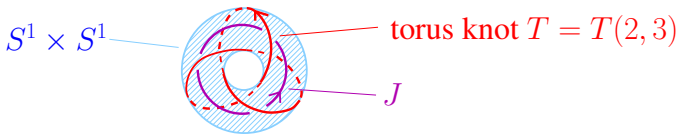
Exercise 117.10. Compute the Alexander polynomials of the figure-ain't knot and of the hopfoil that are shown below.



Exercise 117.11. Let $p, q \in \mathbb{N}$ be coprime. Let $L = T \cup J$ be the link that is given by the torus knot $T = T(p,q) \subseteq \overline{B}^2 \times S^1$ and the core curve $J = \{0\} \times S^1$ of the solid torus.

(a) Show that $\pi_1(S^3 \setminus L) \cong \langle x, y, z \,|\, [x,z], x^p \cdot y^{-q} \cdot z^q \rangle$.

(b) Calculate the Alexander polynomial of L.



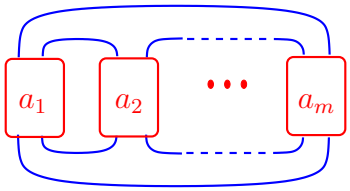
Exercise 117.12. Let $a_1, \ldots, a_m \in \mathbb{Z}$. We refer to the figure below for the definition of the *pretzel link* $P(a_1, \ldots, a_m)$:

(a) Show that $P(a_1, \ldots, a_m)$ is smoothly isotopic to $P(a_2, \ldots, a_m, a_1)$.

(b) Show that the mirror of a pretzel link is again a pretzel link.

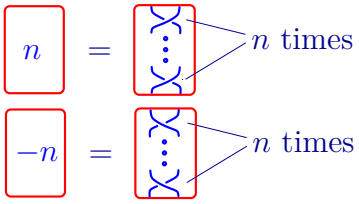
(c) For which $a_1, \ldots, a_m$ is $P(a_1, \ldots, a_m)$ a knot?
Hint. First show that the answer only depends on the parity of $a_1, \ldots, a_m$.

(d) Show that the Alexander polynomial of $P(-3, 5, 7)$ equals $\doteq 1$.
Hint. Try to find a presentation with few generators.

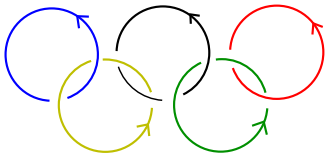


pretzel link $P(a_1, \ldots, a_m)$

where for $n \in \mathbb{N}_0$ we have



Exercise 117.13. Compute the Alexander polynomial of the Olympic rings.
Hint. Try to find a presentation with a small number of generators.



Exercise 117.14. As usual we consider the torus $S^1 \times S^1 \subseteq S^3 =_{\mathrm{i}} (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$. Let $m, n \in \mathbb{Z}$. We set $g := \gcd(m,n)$. As in Exercise 111.12 we consider the g-component torus link $L(m,n) := \{(y,z) \in S^1 \times S^1 \,|\, y^m = z^n\}$.

(a) Determine $\pi_1(S^3 \setminus L(m,n))$.

(b) Determine the Alexander polynomial of $L(m,n)$.

Exercise 117.15.

(a) Let L be a link that is splittable in the sense of Definition 109.5. Show that $\Delta_L = 0$.
Remark. Together with the calculation on page 2272 this gives us the not entirely trivial statement that the Borromean rings are not splittable.

(b) Show that the Whitehead link, which is shown in the figure below, is not splittable.
Remark. This confirms our suspicion from page 2223.



Exercise 117.16. In Exercise 116.12 we introduced higher order Alexander functions $\Delta^n_{\pi,\Phi}$ where π is a group that admits a presentation $P=\langle x_1,\dots,x_k\,|\,r_1,\dots,r_{k-1}\rangle$ of deficiency one, $\Phi\colon\pi\to H$ is a non-trivial homomorphism to a free abelian group H and $n\in\mathbb{N}_0$. With the obvious modifications to Definition 117.2 we can now introduce the higher order Alexander polynomials $\Delta^n_K(t)$ of an oriented knot. In the following we consider the two knots 6_1 and 9_{46}.

(a) Show that the 0-th Alexander polynomials equal $2-5t+2t^2$.

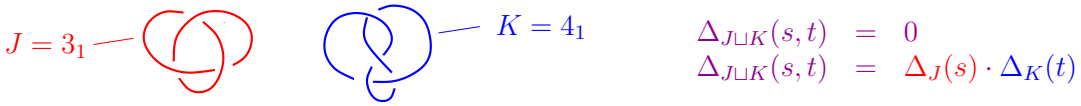
(b) Show that the first order Alexander polynomials of 6_1 and 9_{46} are different.



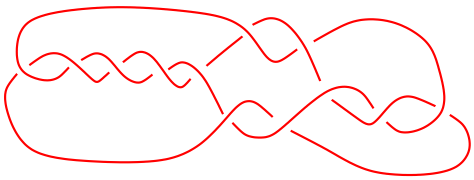
Exercise 117.17. In Exercise 116.12 we introduced higher order Alexander functions $\Delta^n_{\pi,\Phi}$ where π is a group that admits a presentation $P=\langle x_1,\dots,x_k\,|\,r_1,\dots,r_{k-1}\rangle$ of deficiency one, $\Phi\colon\pi\to H$ is a non-trivial homomorphism to a free abelian group and $n\in\mathbb{N}_0$. With the obvious modifications to Definition 117.4 we can now introduce the higher order Alexander polynomials $\Delta^n_L(t_1,\dots,t_m)$ of an oriented link with $m\geq 2$ components. Let $L=J\sqcup K$ be the split union of two oriented knots J and K that we introduced in Definition 109.4.

(a) If you did not solve Exercise 117.15, then show that $\Delta_L(s,t)=0$.

(b) Show that $\Delta^1_L(s,t)=\Delta_J(s)\cdot\Delta_K(t)$.



Exercise 117.18. Is the knot shown below trivial or not?



Exercise 117.19. An n-component link is called *Brunnian link* if each $(n-1)$-component sublink is trivial. For example the Borromean rings are Brunnian. Are there any other 3-component Brunnian links?

Exercise 117.20. Let $K\subseteq S^3$ be a knot. The number $\det(K):=|\Delta_K(-1)|$ is called the *determinant of* K.

(a) Show that the determinant is always non-zero.

(b) What is the determinant of the (p,q)-torus knot?

CHAPTER 118

The symmetry theorem for Alexander polynomials

In this chapter we will do the following:

(1) Given a diagram for a link L we introduce the "over-presentation" of $\pi_1(S^3 \setminus L)$ which is somewhat similar to the Wirtinger presentation, but in general it has fewer generators. It thus leads to more efficient calculations of Alexander polynomials.

(2) In the Alexander Polynomial–Symmetry Theorem 117.7 we prove the result, which we already mentioned in the previous chapter, that given any oriented m-component link $L \subseteq S^3$ we have
$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$
The key idea is to calculate the Alexander polynomial twice, namely once using the "over-presentation" and once using the "under-presentation". We will see that these two presentations are "dual" in a suitable sense, which allows us to obtain the symmetry result.

(3) In the Alexander Polynomial-at-1 Proposition 118.13 we will show that for any knot K we have $\Delta_K(1) = \pm 1$. Together with the Alexander Polynomial–Symmetry Theorem 117.7 this characterizes the Laurent polynomials that can arise as Alexander polynomials of knots.

(4) We conclude with two appendices on the Torres conditions for Alexander polynomials of links and on Alexander polynomials of alternating knots. These appendices do not contains proofs, but they contain results and open questions which are worth knowing.

This chapter is organized as follows:

- In Section 118.1 we will introduce the over-presentation of $\pi_1(S^3 \setminus K)$.
- In Section 118.2 we will use the over-presentation to prove that Alexander polynomials of links are symmetric.
- In Section 118.3 we will discuss the Alexander polynomial of a knot evaluated at $t = 1$.
- In Appendix 118.4 we will formulate the Torres condition.
- In Appendix 118.5 we will discuss Alexander polynomials of alternating knots.
- In Section 118.6 we will provide exercises for Chapter 118.

In the next Chapter 119 we will see how one can calculate the Alexander polynomial from diagrams.

118.1. **The over-presentation.** In this section we use a diagram of a link L to introduce the "over-presentation" for $\pi_1(S^3 \setminus L)$. In general this presentation has fewer generators than the Wirtinger presentation from the Wirtinger Presentation Proposition 113.6. This is useful since the effort to calculate Alexander polynomials of a link from a given deficiency one presentation grows rapidly with the number of generators.

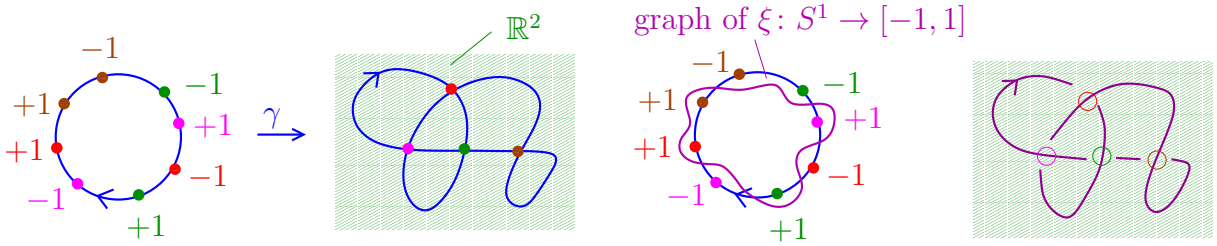
In the Diagram-to-Link Lemma 112.5 we showed that any link diagram gives rise to a link. In the following lemma we give a slightly different construction:

Lemma 118.1. (Alternative Diagram-to-Link Lemma) Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram such that each component has at least one crossing. We can pick a smooth map $\xi\colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1,1]$ with the following properties:

(a) $\xi^{-1}(1)$ is the set of overcrossings (as introduced in Definition 112.4).
(b) $\xi^{-1}(-1)$ is the set of undercrossings.
(c) Each component of $\xi^{-1}([0,1])$ contains an overcrossing.
(d) Each component of $\xi^{-1}([-1,0])$ contains an undercrossing.
(e) For each $z \in \bigsqcup_{i=1}^{m} S_i^1$ with $\xi(z) = 0$ we have $\xi'(z) \neq 0$.

The following three statements hold:
(1) The image of $\bigsqcup_{i=1}^{m} S_i^1$ under the map $\bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \xi(z))$ (together with the obvious ordering of the components) is an m-component link.
(2) Any two choices of ξ give rise to smoothly isotopic links,
(3) The resulting links are smoothly isotopic to the links obtained from the original Diagram-to-Link Lemma 112.5.

If we equip $\bigsqcup_{i=1}^{m} S_i^1$ with the standard orientation, then in the above statements we can also replace "link" by "oriented link".

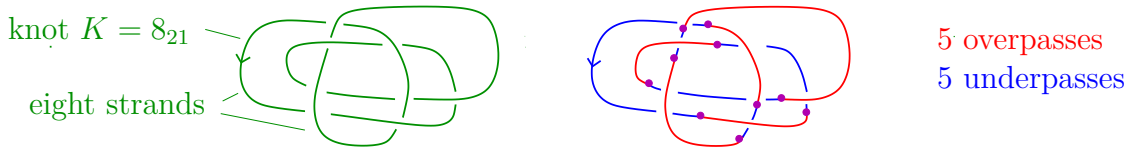


Proof. The proof of the lemma is very similar to the proof of the Diagram-to-Link Lemmas 112.5 and 113.8. We leave it to the reader to make the necessary modifications. ■

Definition 118.2. Let $L \subseteq \mathbb{R}^2 \times [-1,1] \subseteq \mathbb{R}^3$ be the oriented m-component link that is associated to the link diagram $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ via the Alternative Diagram-to-Link Lemma 118.1.

(1) We refer to each component of $L \cap (\mathbb{R}^2 \times [0,1])$ as an overpass.
(2) We refer to each component of $L \cap (\mathbb{R}^2 \times [-1,0])$ as an underpass.

The definition of an overpass and underpass is similar, but nonetheless different, from the definition of a strand which we gave in Definition 113.2. The following figure hopefully helps in distinguishing the different notions:



Remark. We will often use the following fact: If each component of a link diagram has at least one crossing, then the same argument as in the Crossing–Strand Lemma 113.3 shows that the number of overpasses equals the number of underpasses. □

The following proposition is an analog of the Wirtinger Presentation Proposition 113.6:

Proposition 118.3. (Over-Presentation Proposition) Let $\big(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\big)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 118.1 to the link diagram. We enumerate the overpasses by $x_1, \dots, x_n$ and we enumerate the underpasses by $1, \dots, n$.

- We pick **orientation-preserving** smooth embeddings $R_i\colon \overline{B}^2 \to \mathbb{R}^2$, $i = 1, \dots, n$ with disjoint images such that the projection of the i-th underpass to $\mathbb{R}^2$ is contained in $R_i(B^2)$ and such that $R_i(S^1)$ intersects $\gamma\big(\bigsqcup_{i=1}^{m} S_i^1\big)$ transversally.
- For the i-th underpass we define a relation $r_i \in \langle x_1, \dots, x_n \rangle$ as follows: We consider the path $[0,1] \to \mathbb{R}^2$ that is given by $t \mapsto R_i(\exp(2\pi i t))$. We start at $t = 0$ and going along $[0,1]$ we record a generator x_j (or its inverse x_j^{-1}) each time the path intersects the image of the j-th overpass with a positive (negative) sign.

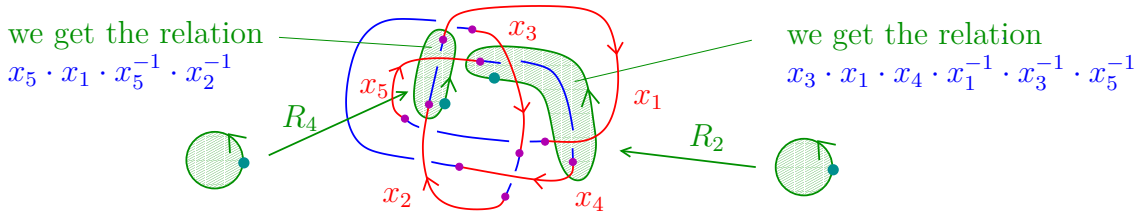
With this notation the following statements hold:

(1) We have an explicit isomorphism

$$\big\langle \underbrace{x_1, \dots, x_n}_{\substack{\text{correspond to}\\ \text{overpasses}}} \;\big|\; \underbrace{r_1, \dots, r_n}_{\substack{\text{correspond to}\\ \text{underpasses}}} \big\rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L)$$

where each x_i is given by a meridian corresponding to the i-th overpass.

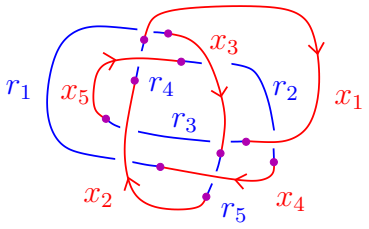
(2) In the presentation (1) we can drop any one of the relations and we still obtain an isomorphism.



Definition 118.4. Let $\big(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\big)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 118.1 to the link diagram. We refer to the presentation of $\pi_1(S^3 \setminus L)$ from the Over-Presentation Proposition 118.3 as the over-presentation of $\pi_1(S^3 \setminus L)$.

Sketch of proof. First recall that in the Alternative Diagram-to-Link Lemma 118.1 we picked a smooth function $\xi\colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1,1]$ with the following properties:

(a) $\xi^{-1}(1)$ is the set of overcrossings.
(b) $\xi^{-1}(-1)$ is the set of undercrossings.
(c) Each component of $\xi^{-1}([0,1])$ contains an overcrossing.
(d) Each component of $\xi^{-1}([-1,0])$ contains an undercrossing.



we get generators and relations x_1, x_2, x_3, x_4, x_5

$$\begin{aligned} r_1 &= x_3 \cdot x_1 \cdot x_2^{-1} \cdot x_4^{-1} \cdot x_2 \cdot x_1^{-1} \\ r_2 &= x_3 \cdot x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1} \cdot x_5^{-1} \\ r_3 &= x_2 \cdot x_5 \cdot x_2^{-1} \cdot x_3 \cdot x_1^{-1} : x_3^{-1} \\ r_4 &= x_5 \cdot x_1 \cdot x_5^{-1} \cdot x_2^{-1} \\ r_5 &= x_4^{-1} \cdot x_3^{-1} \cdot x_4 \cdot x_2 \end{aligned}$$

over-presentation for $\pi_1(S^3 \setminus 8_{21})$

(e) For each $z \in \bigsqcup_{i=1}^{m} S_i^1$ with $\xi(z) = 0$ we have $\xi'(z) \neq 0$.

Let L be the oriented m-component link that is associated to the link diagram via this function ξ. As before we refer to each component of $L \cap (\mathbb{R}^2 \times [-1, 0])$ as an underpass.

It is elementary to see that we can modify the above map ξ to obtain a smooth function $\widetilde{\xi} \colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1, 1]$ with the following modified properties

(a) $\widetilde{\xi}^{-1}(1) = \xi^{-1}(1)$ is the set of overcrossings.
(b) $\widetilde{\xi}^{-1}(-1)$ is the set of underpasses.
(c) Each component of $\widetilde{\xi}^{-1}([0, 1])$ contains an overcrossing.
(d) Each component of $\widetilde{\xi}^{-1}([-1, 0])$ contains an undercrossing.
(e) For each $z \in \bigsqcup_{i=1}^{m} S_i^1$ with $\xi(z) = 0$ we have $\xi'(z) \neq 0$.

It is easy to verify that

$$\begin{aligned} \left(\bigsqcup_{i=1}^{m} S_i^1 \right) \times [0, 1] &\to \mathbb{R}^3 \\ (z, t) &\mapsto (z, \xi(z) \cdot (1 - t) + \widetilde{\xi}(z) \cdot t) \end{aligned}$$

is a smooth isotopy from L to $\widetilde{L}$.

At that point the proof is almost identical to the proof of the Wirtinger Presentation Proposition 113.6 which we gave in Section 113.3. We introduce the following familiar notation:

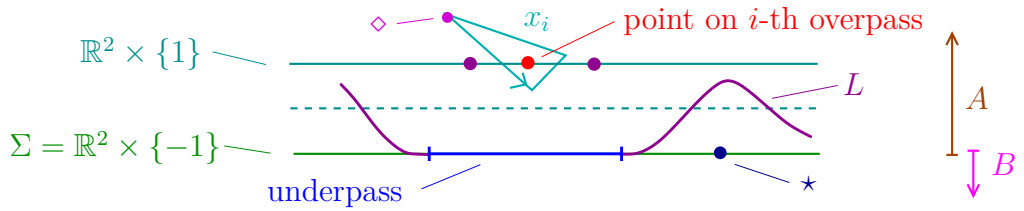
- We consider the following subspaces of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$:
$$\begin{aligned} A &:= \{(x, y, z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}, \\ B &:= \{(x, y, z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}, \\ S &:= \{(x, y, z) \in \mathbb{R}^3 \,|\, z = -1\} \cup \{\infty\} = A \cap B \end{aligned}$$
and we set $A_L := A \setminus L$, $B_L := B \setminus L$ and $S_L := S \setminus L$.
- We work with the base point $\diamond = (0, 0, 2) \in \mathbb{R}^3$ which lies "above" the link L.
- For the i-th overpass we denote by x_i an oriented triangle that starts at $\diamond$ and "circles once around the i-th overpass at any overcrossing" according to the "right-hand rule".
- We pick a base point $\star \in S_L$.

Similar to the arguments in Section 113.3 the following claims hold:

Claim 0. The inclusion maps induce an isomorphism

$$\pi_1(A_L, \star) *_{\pi_1(S_L, \star)} \pi_1(B_L, \star) \xrightarrow{\cong} \pi_1(S^3 \setminus L, \star).$$

Claim 1. The group $\pi_1(B_L, \star)$ is trivial.



Claim 2. The obvious map $\langle x_1, \dots, x_n \rangle \to \pi_1(A_L, \diamond)$ is an isomorphism.

Claim 3. The loops $R_1(S^1), \dots, R_n(S^1)$ form a normal generating set of $\pi_1(S_L, \star)$.

As on page 2204 the proposition is an easy consequence of the above claims. We leave it to the reader to fill in all the details. ■

118.2. **The symmetry of the Alexander polynomial.** In this section we give the long overdue proof of the following theorem:

Theorem 117.7. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link $L \subseteq S^3$ we have

$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$

The logic of the proof is as follows:

(1) In Subsection 118.2.1 we will introduce the notion of two presentations being "dual". We will see that if a group admits a pair of dual presentations, then the Alexander function is symmetric.
(2) In Subsection 118.2.2 we will introduce the under-presentation of $\pi_1(S^3 \setminus L)$.
(3) In Subsection 118.2.3 we will prove the Dual Presentation Theorem 118.12 which says that, for a given link diagram, the corresponding over-presentation and under-presentation are dual.
(4) In Subsection 118.2.4 we will wrap up the proof of the Alexander Polynomial–Symmetry Theorem 117.7.

118.2.1. **Dual presentations.** We start out with the following standard notation:

Definition 118.5.

(1) Let G a group. We equip the corresponding group ring $\mathbb{Z}[G]$ with the involution that is induced by $\overline{g} := g^{-1}$ for $g \in G$. More precisely, we write

$$\overline{\sum_{i=1}^{n} a_i \cdot g_i} := \sum_{i=1}^{n} a_i \cdot g_i^{-1}.$$

This involution is natural in the sense that for any group homomorphism $\varphi\colon G \to H$ and any $p \in \mathbb{Z}[G]$ we have $\varphi_*(\overline{p}) = \overline{\varphi_*(p)} \in \mathbb{Z}[H]$.
(2) If G is a free abelian group, then the involution on $\mathbb{Z}[H]$ defines also an involution on the quotient field $\mathbb{Q}(H)$ in an obvious way.

The following is the key definition in the proof of the Alexander Polynomial–Symmetry Theorem 117.7.

Definition 118.6. Let π be a group. We denote by $\Phi\colon \pi \to \pi_{\mathrm{ab}}$ the natural epimorphism. In the following we say that two presentations[1242]

$$\begin{aligned} P &= \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle, \text{ and} \\ P^\dagger &= \langle y_1, \dots, y_n \,|\, s_1, \dots, s_n \rangle \end{aligned}$$

for π are dual to one another if the following two conditions are satisfied:

(D1) For $i = 1, \dots, n$ we have $\Phi(x_i) = \Phi(\overline{y_i})$.

(D2) There exist $g_1, \dots, g_n, h_1, \dots, h_n \in \pi$ such that for every $i, j = 1, \dots, n$ we have[1243]

$$\Phi\big(g_i \cdot \frac{\partial r_i}{\partial x_j} \cdot (x_j - 1)\big) \;=\; \Phi\big(\overline{h_j \cdot \frac{\partial s_j}{\partial y_i}} \cdot (y_i - 1)\big).$$

The following proposition makes it clear why we are interested in dual presentations:

Proposition 118.7. (Dual Presentation–Symmetry Proposition) Let π be a group and let $\Phi\colon \pi \to H$ be a homomorphism a free abelian group H. We assume that π admits a pair of dual presentations $P = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle$ and $P^\dagger = \langle y_1, \dots, y_n \,|\, s_1, \dots, s_n \rangle$ with the following two properties:

- In each presentation we can drop any one of the relations.
- For all $i \in \{1, \dots, n\}$ the images $\Phi(x_i) \in H$ and $\Phi(y_i) \in H$ are non-trivial.

Then the following equality holds:

$$\Delta_{\pi,\Phi} \;\doteq\; \overline{\Delta_{\pi,\Phi}} \;\in\; \mathbb{Q}(H).$$

Proof. Let π be a group and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. We assume that there exist two presentations $P = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle$ and $P^\dagger = \langle y_1, \dots, y_n \,|\, s_1, \dots, s_n \rangle$ for π with the following three properties:

- The presentations are dual to one another.
- In both presentations we can drop any one of the relations.
- For all $i \in \{1, \dots, n\}$ the images $\Phi(x_i) \in H$ and $\Phi(y_i) \in H$ are non-trivial.

Evidently we now want to use these presentations to calculate the Alexander function.

By our hypothesis we can drop in both presentations the last relation. We denote the resulting presentations by $\widetilde{P}$ and $\widetilde{P}^\dagger$.

Claim. We have

$$\det \underbrace{\big(\Phi\big(\frac{\partial r_i}{\partial x_j}\big)_{i,j=1,\dots,n-1}\big)}_{=J^\Phi(\widetilde{P})_n} \;\doteq\; \overline{\det \underbrace{\big(\Phi\big(\frac{\partial s_i}{\partial y_j}\big)_{i,j=1,\dots,n-1}\big)}_{=J^\Phi(\widetilde{P}^\dagger)_n}}.$$

[1242] To simplify the notation we drop the isomorphism between π and its presentations from the notation.

[1243] The notion of dual presentations was introduced by Ralph Fox and Guillermo Torres [**FoT1954**], see also [**CF1977**]. Our definition differs slightly from their definitions since we allow for the $g_1, \dots, g_n$ and $h_1, \dots, h_n \in \pi$.

Proof. We have the following equalities in $\mathbb{Z}[H]$:

$$\begin{array}{ccc}
\det\left(\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{i,j=1,\dots,n-1}\right)\cdot\prod_{j=1}^{n-1}(\Phi(x_j)-1) & = & \det\left(\Phi\left(\frac{\partial r_i}{\partial x_j}\cdot(x_j-1)\right)_{i,j=1,\dots,n-1}\right) \\
\doteq & & = \\
\det\left(\Phi\left(\overline{\frac{\partial s_j}{\partial y_i}\cdot(y_i-1)}\right)_{i,j=1,\dots,n-1}\right) & & \det\left(\Phi\left(\overline{\frac{\partial s_i}{\partial y_j}\cdot(y_j-1)}\right)_{i,j=1,\dots,n-1}\right) \\
 & & = \\
 & & \overline{\det\left(\Phi\left(\frac{\partial s_i}{\partial y_j}\right)_{i,j=1,\dots,n-1}\right)}\cdot\prod_{j=1}^{n-1}(\Phi(x_j)-1).
\end{array}$$

since Φ is a ring homomorphism, since determinants commute with ring homomorphisms and since determinants are multilinear

by (D2), note that the $g_1,\dots,g_n$ and $h_1,\dots,h_n$ get swallowed by the "$\doteq$"

since $\det(A)=\det(A^{\mathrm{T}})$

by the same argument as above, we also use that $f\mapsto\overline{f}$ is a ring homomorphism and we use (D1)

To obtain the desired equality we need to dispose of the common factor $\prod_{j=1}^{n-1}(\Phi(x_j)-1)$. To achieve this we make two observations:

- By hypothesis we know that for all $j\in\{1,\dots,n\}$ the image $\Phi(x_j)\in H$ is non-trivial. This implies that each $\Phi(x_j)-1\in\mathbb{Z}[H]$ is non-zero.
- Since H is a free abelian group we know that $\mathbb{Z}[H]$ is a domain.

The desired equality follows from the above calculation and the above two observations. ⊞

We now see that

$$\begin{array}{ccccc}
\Delta_{\pi,\Phi} & \doteq & \frac{\det(J^{\Phi}(\widetilde{P})_n)}{\Phi(x_n)-1} & \doteq & \frac{\det\left(\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{i,j=1,\dots,n-1}\right)}{\Phi(x_n)-1} \\
 & & & & \\
 & \doteq & \frac{\overline{\det(J^{\Phi}(\widetilde{P}^{\dagger})_n)}}{\overline{\Phi(y_n)-1}} & \doteq & \frac{\overline{\det\left(\Phi\left(\frac{\partial s_i}{\partial y_j}\right)_{i,j=1,\dots,n-1}\right)}}{\overline{\Phi(y_n)-1}} \\
 & & & & \\
 & & & \doteq & \overline{\Delta_{\pi,\Phi}}.
\end{array}$$

by Definition 116.14

by Definition 116.9

by the claim and by (D2)

as above, this follows from the various definitions

■

118.2.2. **The under presentation.** The Dual Presentation–Symmetry Proposition 118.7 shows that one approach to proving the Alexander Polynomial–Symmetry Theorem 117.7 is to prove that $\pi_1(S^3\setminus L)$ admits a pair of dual presentations.

In this subsection we introduce, given a diagram of a link L, the under-presentation of $\pi_1(S^3\setminus L)$, which will later see is dual to the corresponding over-presentation of $\pi_1(S^3\setminus L)$.

Proposition 118.8. (Under-Presentation Proposition) Let $\left(\gamma\colon\bigsqcup_{i=1}^{m}S_i^1\to\mathbb{R}^2,c\right)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 118.1 to the link diagram. We enumerate the underpasses by $y_1,\dots,y_n$ and we enumerate the overpasses by $1,\dots,n$.

- We pick **orientation-reversing**[1244] smooth embeddings $S_i\colon\overline{B}^2\to\mathbb{R}^2$, $i=1,\dots,n$ with disjoint images and such that the projection of the i-th overpass to $\mathbb{R}^2$ is contained in $S_i(B^2)$ and such that $S_i(S^1)$ intersects $\gamma\left(\bigsqcup_{i=1}^{m}S_i^1\right)$ transversally.
- For the i-th overpass we define a relation $s_i\in\langle y_1,\dots,y_n\rangle$ as follows: We consider the path $[0,1]\to\mathbb{R}^2$ that is given by $t\mapsto S_i(\exp(2\pi\,\mathrm{i}t))$. We start at $t=0$ and going along

$[0,1]$ we record a generator y_j (or its inverse y_j^{-1}) each time the path intersects the image of the j-th overpass with a negative (positive) sign.

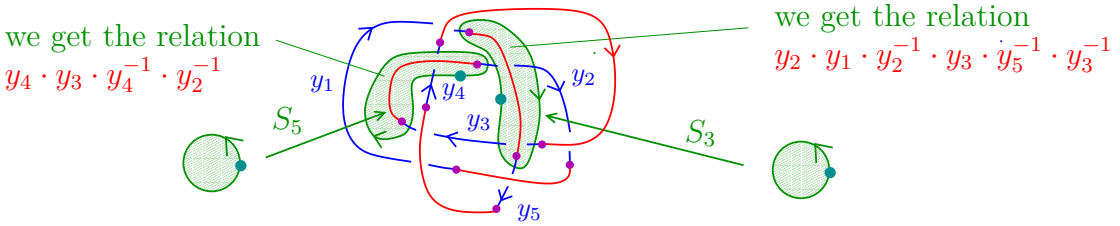
With this notation the following statements hold:

(1) We have an explicit isomorphism

$$\langle \underbrace{y_1,\dots,y_n}_{\text{correspond to underpasses}} \mid \underbrace{s_1,\dots,s_n}_{\text{correspond to overpasses}} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L)$$

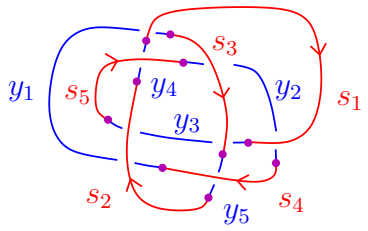
where each y_i is given by the inverse of a meridian corresponding to the i-th underpass.

(2) In the presentation (1) we can drop any one of the relations and we still obtain an isomorphism.



Proof. The proof of this proposition is almost identical to the proof of the Over-Presentation Proposition 118.3 which in turn is almost the same as the proof of the Wirtinger Presentation Proposition 113.6. We leave it to the reader to make the necessary modifications. ■

Definition 118.9. Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 118.1 to the link diagram. We refer to the presentation of $\pi_1(S^3 \setminus L)$ from the Under-Presentation Proposition 118.8 as the under-presentation of $\pi_1(S^3 \setminus L)$.



under-presentation for $\pi_1(S^3 \setminus 8_{21})$

we get generators and relations y_1, y_2, y_3, y_4, y_5

$$\begin{aligned} s_1 &= y_4 \cdot y_1 \cdot y_2^{-1} \cdot y_3^{-1} \cdot y_2 \cdot y_1^{-1} \\ s_2 &= y_3^{-1} \cdot y_1^{-1} \cdot y_5^{-1} \cdot y_1 \cdot y_2 \cdot y_4 \\ s_3 &= y_2 \cdot y_1 \cdot y_2^{-1} \cdot y_3 \cdot y_5^{-1} \cdot y_3^{-1} \\ s_4 &= y_5 \cdot y_2 \cdot y_5^{-1} \cdot y_1^{-1} \\ s_5 &= y_4 \cdot y_3 \cdot y_4^{-1} \cdot y_2^{-1} \end{aligned}$$

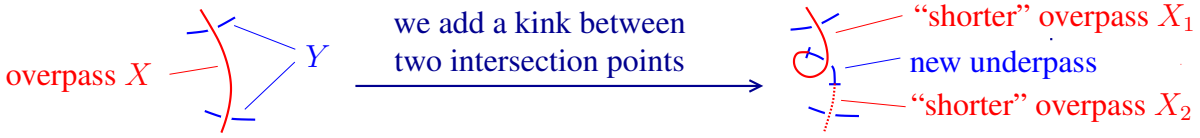
118.2.3. **The Dual Presentation Theorem.** In this subsection we want to show that over- and under-presentations are duals. To simplify our life we restrict ourselves to presentations arising from "simple" diagrams:

Definition 118.10. A link diagram is called simple if for each overpass X and each underpass Y intersect in at most one point, i.e. we have $\#(X \cap Y) \leq 1$.

[1244] Note that throughout we always use the opposite convention to the one used in the Over-Presentation Proposition 118.3. This is a little bit confusing, but it is necessary to obtain a presentation which is dual to the over-presentation.

Lemma 118.11. (Simple Diagram Lemma) Every link admits a simple diagram such that each component has at least one crossing.

Sketch of proof. Let $L \subseteq S^3$ be a link. By the Link Diagram Existence Proposition 112.7 and the One Crossing Exists Lemma 113.4 we know that L admits a diagram such that each component has at least one crossing. For each pair of overpass X and underpass Y with more than one intersection point we add a kink between the intersection points, as shown in the figure below:



This move splits the overpass into two overpasses X_1, X_2, each of which has at least one intersection point with the underpass Y less than the initial overpass. Furthermore it creates an underpass with a single crossing. It is clear that iterating this procedure (and doing the same for underpasses) creates a simple link diagram for L. ■

Theorem 118.12. (Dual Presentation Theorem) Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a simple[1245] link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 118.1 to the link diagram. We enumerate the underpasses by $1, \dots, n$ and we enumerate the overpasses by $1, \dots, n$. Since each component has at least one crossing we can and will do this in such a way that the i-th underpass and the i-th overpass correspond to the same component of L. The corresponding over-presentation and under-presentation of $\pi_1(S^3 \setminus L)$ are dual to one another.

Example. We consider again $K = 8_{21}$ and in the figure below we consider again our favorite over- and under-presentations:

over-presentation for $\pi_1(S^3 \setminus K)$

x_1, x_2, x_3, x_4, x_5

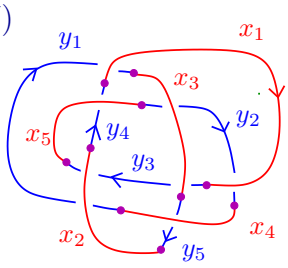
$r_1 = x_3 \cdot x_1 \cdot x_2^{-1} \cdot x_4^{-1} \cdot x_2 \cdot x_1^{-1}$

$r_2 = x_3 \cdot x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1} \cdot x_5^{-1}$

$r_3 = x_2 \cdot x_5 \cdot x_2^{-1} \cdot x_3 \cdot x_1^{-1} \cdot x_3^{-1}$

$r_4 = x_5 \cdot x_1 \cdot x_5^{-1} \cdot x_2^{-1}$

$r_5 = x_4^{-1} \cdot x_3^{-1} \cdot x_4 \cdot x_2$



under-presentation for $\pi_1(S^3 \setminus K)$

y_1, y_2, y_3, y_4, y_5

$s_1 = y_4 \cdot y_1 \cdot y_2^{-1} \cdot y_3^{-1} \cdot y_2 \cdot y_1^{-1}$

$s_2 = y_3^{-1} \cdot y_1^{-1} \cdot y_5^{-1} \cdot y_1 \cdot y_2 \cdot y_4$

$s_3 = y_2 \cdot y_1 \cdot y_2^{-1} \cdot y_3 \cdot y_5^{-1} \cdot y_3^{-1}$

$s_4 = y_5 \cdot y_2 \cdot y_5^{-1} \cdot y_1^{-1}$

$s_5 = y_4 \cdot y_3 \cdot y_4^{-1} \cdot y_2^{-1}$

For $i = 1$ and $j = 2$ we see that

$$\Phi_K\left(\tfrac{\partial r_1}{\partial x_2} \cdot (x_2 - 1)\right) = \Phi_K\left((-x_3 \cdot x_1 \cdot x_2^{-1} + x_3 \cdot x_1 \cdot x_2^{-1} \cdot x_4^{-1}) \cdot (x_2 - 1)\right) \overset{\text{since } \Phi_K(x_i) = t}{=} (-t+1) \cdot (t-1)$$

$$\Phi_K\left(\overline{\tfrac{\partial s_2}{\partial y_1} \cdot (y_1 - 1)}\right) = \Phi_K\left(\overline{(-y_3^{-1} \cdot y_1^{-1} + y_3^{-1} \cdot y_1^{-1} \cdot y_5^{-1}) \cdot (y_1 - 1)}\right) = (-t^{-2} + t^{-3}) \cdot (t-1).$$

since $\Phi_K(y_i) = t^{-1}$

[1245] The hypothesis that the diagram is simple simplifies the proof, but it is not really necessary. We leave it to the reader to modify the proof so that it can handle any diagram such that each component has at least one crossing.

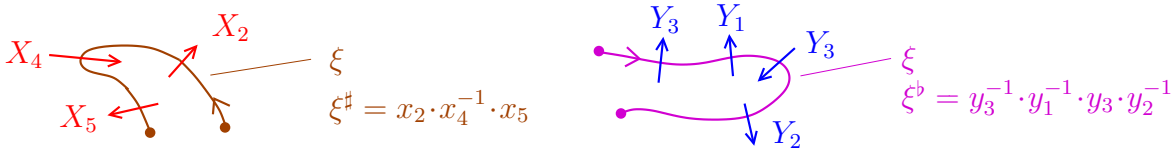
Thus we see that the two terms do indeed agree up to a power of t. □

Proof. [1246] Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 118.1 to the link diagram. We adopt the following notation:

(1) Since each component of the diagram has at least one crossing we see, as in the Crossing–Strand Lemma 113.3, that each component has the same number of under- and over-passes. This implies that we can enumerate the underpasses by $1, \dots, n$ and we can enumerate the overpasses by $1, \dots, n$ in such a way that the i-th underpass and the i-th overpass correspond to the same component of L.
(2) We denote by $\Phi_L\colon \pi_1(S^3 \setminus L) \to \langle t_1, \dots, t_m\rangle_{\mathrm{ab}}$ the epimorphism that sends the i-th meridian to t_i.
(3) (a) We denote by $X_1, \dots, X_n$ the projections of the overpasses to $\mathbb{R}^2$.
    (b) We denote by $Y_1, \dots, Y_n$ the projections of the underpasses to $\mathbb{R}^2$.
(4) (a) As in the Over-Presentation Proposition 118.3 we pick orientation-preserving smooth embeddings $R_1, \dots, R_n\colon \overline{B}^2 \to \mathbb{R}^2$ "around the Y_i". We obtain the corresponding over-presentation $\langle x_1, \dots, x_n \,|\, r_1, \dots, r_n\rangle$.
    (b) As in the Under-Presentation Proposition 118.8 we pick orientation-reversing smooth embeddings $S_1, \dots, S_n\colon \overline{B}^2 \to \mathbb{R}^2$ "around the X_i". We obtain the corresponding under-presentation $\langle y_1, \dots, y_n \,|\, s_1, \dots, s_n\rangle$.

Next we introduce two new objects:

(a) Given a smooth embedding $\xi\colon [0,1] \to \mathbb{R}^2$ that intersects the overpasses $X_1, \dots, X_n$ transversally we denote by $\xi^\sharp \in \langle x_1, \dots, x_n\rangle$ the following word: We start out at 0, going along $[0,1]$ we record a generator x_j (or its inverse x_j^{-1}) whenever the image of ξ intersects the image of the j-th underpass with a positive (negative) sign.
(b) Given a smooth embedding $\xi\colon [0,1] \to \mathbb{R}^2$ that intersects the underpasses $Y_1, \dots, Y_n$ transversally we denote by $\xi^\flat \in \langle y_1, \dots, y_n\rangle$ the following word: We start out at 0, going along $[0,1]$ we record a generator y_j (or its inverse y_j^{-1}) whenever the image of ξ intersects the image of the j-th underpass with a negative (positive) sign.



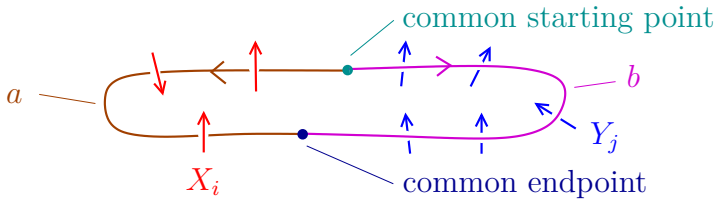
We make the following observations regarding these definitions:

(a) For two suitable smooth embeddings $\xi, \zeta\colon [0,1] \to \mathbb{R}^2$ with $\xi(1) = \zeta(0)$ we have, basically by definition, $(\xi * \zeta)^\sharp = \xi^\sharp \cdot \zeta^\sharp$. Furthermore, we have $R_i(t \mapsto \exp(2\pi i t))^\sharp = r_i$.
(b) The analogs for "$\flat$" also hold.

Claim 1. Suppose we are given the following:
- A smooth embedding $[0,1] \xrightarrow{a} \mathbb{R}^2 \setminus (Y_1 \cup \ldots \cup Y_n)$ that intersects $X_1, \ldots, X_n$ transversally.

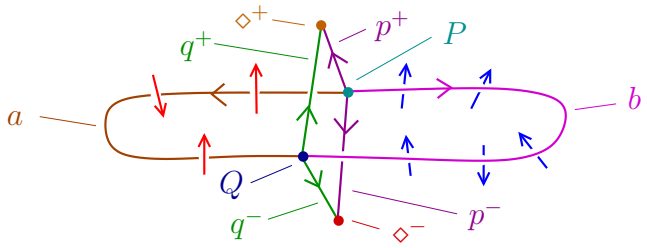
[1246] We will take a few minor liberties and omit a few details to keep the proof readable. A completely rigorous proof would certainly require more details.

• A smooth embedding $[0,1] \xrightarrow{b} \mathbb{R}^2 \setminus (X_1 \cup \ldots \cup X_n)$ that intersects $Y_1, \ldots, Y_n$ transversally. If $a(0) = b(0)$ and $a(1) = b(1)$, then $\Phi_L(a^\sharp) = \Phi_L(b^\flat)^{-1}$.



Proof. We introduce some extra notation:

- We set $P := a(0) = b(0)$ and $Q := a(1) = b(1)$.
- We write $\diamond^\pm := (0, 0, \pm 2)$.
- We denote by $p^\pm$ a path from P to $\diamond^\pm$. Similarly we define $q^\pm$.



We make the following observations:

(a) $a^\sharp \in \langle x_1, \ldots, x_n \rangle$ describes the loop $\overline{p^+} * a * q^+$ in $\pi_1((\mathbb{R}^2 \times [0,\infty)) \setminus L, \diamond^+) = \langle x_1, \ldots, x_n \rangle$.
(b) $b^\flat \in \langle y_1, \ldots, y_n \rangle$ describes the loop $\overline{p^-} * b * q^-$ in $\pi_1((\mathbb{R}^2 \times (-\infty, 0]) \setminus L, \diamond^-]) = \langle y_1, \ldots, y_n \rangle$.
(c) Since the image of a lies in the complement of $Y_1, \ldots, Y_n$ we see that a is path-homotopic in $\mathbb{R}^2 \times [-2, 0]$ to the path $p^- * \overline{q^-}$.
(d) Since the image of b lies in the complement of $X_1, \ldots, X_n$ we see that b is path-homotopic in $\mathbb{R}^2 \times [0, 2]$ to the path $p^+ * \overline{q^+}$.

We now consider the following diagram:

$$
\begin{array}{ccccc}
\langle x_1, \ldots, x_n \rangle & & \twoheadrightarrow & & \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle \\
\overset{a^\sharp}{} & & & & \\
\downarrow \mathrm{id} & & & & \downarrow \mathrm{id} \\
\pi_1((\mathbb{R}^2 \times [0,\infty)) \setminus L, \diamond^+) & \xrightarrow[\text{induced}]{\text{inclusion}} & \twoheadrightarrow & \pi_1(S^3 \setminus L, \diamond^+) & \xrightarrow{\Phi_L} \langle t_1, \ldots, t_m \rangle_{\mathrm{ab}} \\
{\scriptstyle [\overline{p^+} * a * q^+]} & & & {\scriptstyle [\overline{p^+} * p^- * \overline{q^-} * q^+]} & \\
 & & & \downarrow {\scriptstyle (\overline{p^-} * p^+)_*} & \\
{\scriptstyle [\overline{q^-} * \overline{b} * p^-]} & & & {\scriptstyle [\overline{q^-} * q^+ * \overline{p^+} * p^-]} & \\
\pi_1((\mathbb{R}^2 \times (-\infty, 0]) \setminus L, \diamond^-) & \xrightarrow[\text{induced}]{\text{inclusion}} & \twoheadrightarrow & \pi_1(S^3 \setminus L, \diamond^-) & \xrightarrow{\Phi_L} \langle t_1, \ldots, t_m \rangle_{\mathrm{ab}} \\
\uparrow \mathrm{id} & & & \uparrow \mathrm{id} & \\
\langle y_1, \ldots, y_n \rangle & & \twoheadrightarrow & & \langle y_1, \ldots, y_n \,|\, s_1, \ldots, s_{n-1} \rangle. \\
{\scriptstyle (b^\flat)^{-1}} & & & &
\end{array}
$$

Using (a)–(d) one sees that the maps send the given elements to the given elements. It follows that $\Phi_L(a^\sharp) = \Phi_L((b^\flat)^{-1})$, which corresponds to the desired result. ⊞

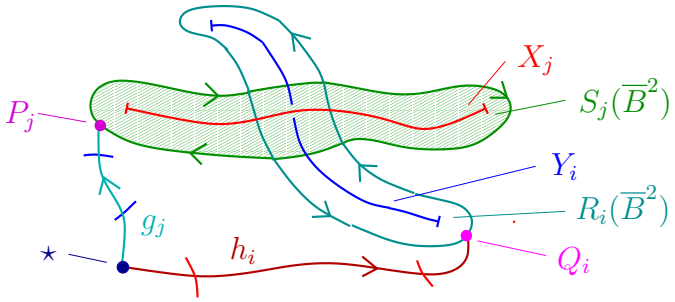
Now we turn to the actual proof that the over- and under-presentation are dual. We have to show that these two presentations satisfy the conditions (D1) and (D2) from Definition 118.6. First we consider (D1). Let $i \in \{1, \dots, n\}$. Recall that we arranged that there exists a $k \in \{1, \dots, m\}$ such that the i-th underpass and the i-th overpass both correspond to the k-th component of L. We now see that

$$\Phi_L(x_i) \underset{\uparrow}{=} \Phi_L(\mu_{L_m}) \underset{\uparrow}{=} \Phi_L(y_i^{-1}) = \Phi_L(\overline{y_i}).$$

by construction of the over- and under-presentation

So it remains to prove (D2). This will require a few more pieces of notation:

- We pick a base point $\star \in \mathbb{R}^2 \times \{0\}$.
- For $i = 1, \dots, n$ we pick a path g_i in $\mathbb{R}^2 \setminus (X_1 \cup \dots \cup X_n)$ from $\star$ to $S_i(1)$.
- For $i = 1, \dots, n$ we pick a path h_i in $\mathbb{R}^2 \setminus (Y_1 \cup \dots \cup Y_n)$ from $\star$ to $R_i(1)$.



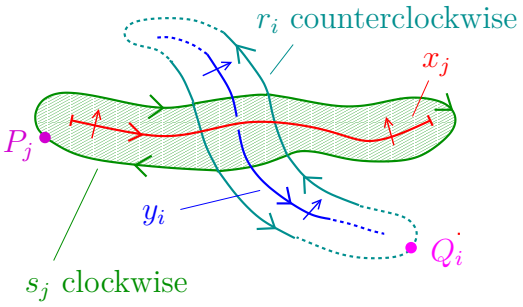
It remains to prove the following claim:

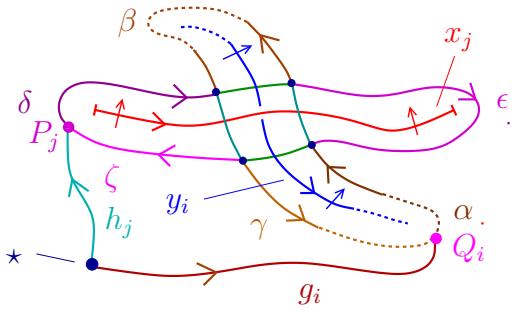
Claim 2. For any $i, j \in \{1, \dots, n\}$ we have

$$\Phi_L\big(g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1)\big) = \Phi_L\big(\overline{h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1)}\big).$$

Proof. Since we assume that the diagram is simple there exists at most one crossing between the i-th underpass and the j-th overpass. If there is no crossing, then one can easily see that x_j does not appear in the word r_i and y_i does not appear in the word s_j. This implies that both Fox derivatives $\frac{\partial r_i}{\partial x_j}$ and $\frac{\partial s_j}{\partial y_i}$ are zero. Thus the promised equality holds trivially.

Now we consider the much more interesting case, that there exists a crossing. We use the figure below to introduce some further notation.





In the setting of the figure we have

$$\begin{aligned} r_i &= \alpha^\sharp \cdot x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu} \cdot \gamma^\sharp \in \langle x_1, \dots, x_n \rangle \\ s_j &= \delta^\flat \cdot y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu} \cdot \zeta^\flat \in \langle y_1, \dots, y_n \rangle. \end{aligned}$$

(To be more precise, in the figure we have $\mu = 1$ and $\nu = 1$. But we want to take care of all possible orientations.) We calculate

$$\frac{\partial r_i}{\partial x_j} \underset{\text{by the discussion on page 2293}}{=} \frac{\partial}{\partial x_j}(\alpha^\sharp \cdot x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu} \cdot \gamma^\sharp) \underset{\text{since } X_i \text{ and } Y_j \text{ have only one intersection point we see that } \gamma \text{ and } \zeta \text{ play no role}}{=} \frac{\partial}{\partial x_j}(\alpha^\sharp \cdot x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu}) = (\alpha^\sharp - \alpha^\sharp \cdot x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu}) \cdot \frac{x_j^\mu - 1}{x_j - 1}$$

$$\frac{\partial s_j}{\partial y_i} = \frac{\partial}{\partial y_i}(\delta^\flat \cdot y_i^\nu \cdot \epsilon^\flat \cdot y_i^\nu \cdot \zeta^\flat) = \frac{\partial}{\partial y_i}(\delta^\flat \cdot y_i^\nu \cdot \epsilon^\flat \cdot y_i^\nu) = (\delta^\flat - \delta^\flat \cdot y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu}) \cdot \frac{y_i^\nu - 1}{y_i - 1}.$$

It follows that

$$\begin{aligned} g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1) &= (g_i \cdot \alpha)^\sharp \cdot (1 - x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu}) \cdot (x_j^\mu - 1) \\ h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1) &= (h_j \cdot \delta)^\flat \cdot (1 - y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu}) \cdot (y_i^\nu - 1). \end{aligned}$$

This implies that

$$\begin{aligned} g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1) &= (g_i * \alpha * x_j^\mu)^\sharp \cdot x_j^{-\mu} \cdot (1 - x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu}) \cdot x_j^\mu \cdot \overbrace{x_j^{-\mu} \cdot (x_j^\mu - 1)}^{=1 - x_j^{-\mu}} \\ h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1) &= (h_j * \delta * y_i^\nu)^\flat \cdot y_i^{-\nu} \cdot (1 - y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu}) \cdot y_i^\nu \cdot \underbrace{y_i^{-\nu} \cdot (y_i^\nu - 1)}_{=1 - y_i^{-\nu}}. \end{aligned}$$

Since Φ_L is a homomorphism and since the target of Φ_L is abelian we obtain that

$$\begin{aligned} \Phi_L(g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1)) &= \Phi_L((g_i * \alpha * x_j^\mu)^\sharp) \cdot \Phi_L(1 - \beta^\sharp) \cdot \Phi_L(1 - x_j^{-\mu}) \\ \Phi_L(h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1)) &= \Phi_L((h_j * \delta * y_i^\nu)^\flat) \cdot \Phi_L(1 - \epsilon^\flat) \cdot \Phi_L(1 - y_i^{-\nu}). \end{aligned}$$

Claim 2 follows from the above calculations and the observation that Claim 1 give us the following equalities:

$$\begin{aligned} \Phi_L((g_i * \alpha * x_j^\mu)^\sharp) &= \Phi_L((h_j * \delta * y_i^\nu)^\flat)^{-1}, \\ \Phi_L(\beta^\sharp) &= \Phi_L(y_i^{-\nu})^{-1}, \\ \Phi_L(\epsilon^\flat) &= \Phi_L(x_j^\nu)^{-1}. \end{aligned}$$

■

118.2.4. **Proof of the Alexander Polynomial–Symmetry Theorem 117.7.** For the reader's convenience we recall the theorem we need to prove:

Theorem 117.7. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link $L \subseteq S^3$ we have

$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$

Proof. Let L be an oriented m-component link. We adopt the following standard notation:

(1) We set $\pi := \pi_1(S^3 \setminus L)$ and we denote by $\Phi_L \colon \pi \to \langle t_1, \dots, t_m \rangle_{\text{ab}}$ the epimorphism given by $\mu_{L_i} \mapsto t_i$.
(2) As usual we identify the group ring of $\langle t_1, \dots, t_m \rangle_{\text{ab}}$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.
(3) Note that the above involution on group rings corresponds to the obvious involution on $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and $\mathbb{Q}(t_1, \dots, t_m)$ that is given by $t_i \mapsto t_i^{-1}$.

With the above notation our task can now be reformulated: We need to show that

$$\Delta_L \;=\; \overline{\Delta_L} \;\in\; \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}].$$

Next, recall that by the Simple Diagram Lemma 118.11 we know that L admits a simple diagram such that each component has a crossing. We enumerate the underpasses by $1, \ldots, n$ and we enumerate the overpasses by $1, \ldots, n$. Since each component has at least one crossing we can and will do this in such a way that the i-th underpass and the i-th overpass correspond to the same component of L. Next we apply the Over-Presentation Proposition 118.3 and the Under-Presentation Proposition 118.3 and we introduce the following notation:

- Let $P := \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle$ be the corresponding over-presentation.
- Let $P^\dagger := \langle y_1, \ldots, y_n \,|\, s_1, \ldots, s_n \rangle$ be the corresponding under-presentation.

By the Dual Presentation Theorem 118.12 we know that these two presentations are duals of one another. We make two more observations:

- By the last sentence of the Over-Presentation Proposition 118.3 and of the Under-Presentation Proposition 118.8 we can drop in both cases the n-th relation. We denote the resulting presentations by $\widetilde{P}$ and $\widetilde{P}^\dagger$.
- In both cases we argued that all $\Phi_L(x_i)$ and all $\Phi_L(y_i)$ are non-trivial.

The above observation implies that we can apply the Dual Presentation–Symmetry Proposition 118.7, and we see that

$$\Delta_{\pi,\Phi_L} \;\doteq\; \overline{\Delta_{\pi,\Phi_L}}.$$

For links this gives, by definition, immediately the desired statement that $\Delta_L \doteq \overline{\Delta_L}$. For knots basically the same argument works, the only extra ingredient we need is the little observation that $t - 1 \doteq t^{-1} - 1$. ∎

118.3. **The Alexander polynomial evaluated at $t = 1$.** We just showed that Alexander polynomials of links are symmetric. For knots we will now see that Alexander polynomials have one more property:

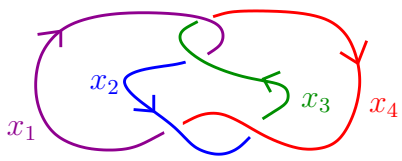
Proposition 118.13. (Alexander Polynomial-at-1 Proposition) For any given knot $K \subseteq S^3$ we have $\Delta_K(1) = \pm 1$.

Proof. Let $K \subseteq S^3$ be a knot. By the Link Diagram Existence Proposition 112.7 and the One Crossing Exists Lemma 113.4 we know that K admits a diagram with at least one crossing. We order the strands cyclically. By the Wirtinger Presentation Proposition 113.6 we obtain an $n \in \mathbb{N}$, a map $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ and a Wirtinger presentation

$$P := \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus K)$$

where each r_j is of the form

(a) $r_j = x_{\sigma(j)} \cdot x_{j+1} \cdot x_{\sigma(j)}^{-1} \cdot x_j^{-1}$ (positive crossing)
(b) $r_j = x_{\sigma(j)}^{-1} \cdot x_{j+1} \cdot x_{\sigma(j)} \cdot x_j^{-1}$ (negative crossing).



the generators are x_1, x_2, x_3, x_4
the relations are
$r_1 = x_3 \cdot x_2 \cdot x_3^{-1} \cdot x_1^{-1}$
$r_2 = x_4^{-1} \cdot x_3 \cdot x_4 \cdot x_2^{-1}$
$r_3 = x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1}$

Let $\Phi\colon P \to \langle t\rangle$ be the unique homomorphism given by $\Phi(x_i) = t$. As before we denote by $\epsilon\colon \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ the augmentation, i.e. the unique ring homomorphism with $\epsilon(t^k) = 1$.

We see that

$$\begin{array}{rcl} \Delta_K(1) = \epsilon(\Delta_K(t)) & \overset{\text{by Definition 117.2 of } \Delta_K(t)}{=} & \epsilon\Big((t-1)\cdot \dfrac{\det(\Phi_*(J(P))_n)}{\Phi_*(x_n - 1)}\Big) \\ & = & \epsilon(\det(\Phi_*(J(P))_n)) \end{array} \quad \begin{array}{rcl} & \overset{\text{since } x_n \text{ is a meridian}}{=} & \epsilon\Big((t-1)\cdot \dfrac{\det(\Phi_*(J(P))_n)}{t-1}\Big) \\ & = & \det((\epsilon\circ\Phi_*)(J(P))_n). \end{array}$$

since determinants commute with ring homomorphisms

It follows from this discussion that it remains to prove the following claim:

Claim. We have

$$(\epsilon\circ\Phi_*)(J(P)) = \underbrace{\begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}}_{(n-1)\times n\text{-matrix}} \quad \text{and thus} \quad (\epsilon\circ\Phi_*)(J(P))_n = \underbrace{\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & \ddots & \ddots \\ 0 & \dots & 0 & -1 \end{pmatrix}}_{\text{determinant } = (-1)^{n-1}}.$$

Proof. We just need to show that the j-th row of $(\epsilon\circ\Phi_*)(J(P))$ has an entry -1 in column j, an entry 1 in column $j+1$ and zeros otherwise. We look at the two types of relations:

(a) First we suppose that $r_j = x_{\sigma(j)}\cdot x_{j+1}\cdot x_{\sigma(j)}^{-1}\cdot x_j^{-1}$. If $\sigma(j) \neq \{j, j+1\}$, then

$$\begin{array}{rcll} (\epsilon\circ\Phi_*)\big(\frac{\partial r_j}{\partial x_j}\big) & = & (\epsilon\circ\Phi_*)(-x_{\sigma(j)}\cdot x_{j+1}\cdot x_{\sigma(j)}^{-1}\cdot x_j^{-1}) & = -1, \\ (\epsilon\circ\Phi_*)\big(\frac{\partial r_j}{\partial x_{j+1}}\big) & = & (\epsilon\circ\Phi_*)(x_{\sigma(j)}) & = 1, \\ (\epsilon\circ\Phi_*)\big(\frac{\partial r_j}{\partial x_{\sigma(j)}}\big) & = & (\epsilon\circ\Phi_*)(1 - x_{\sigma(j)}\cdot x_{j+1}\cdot x_{\sigma(j)}^{-1}) & = 0. \end{array}$$

apply the axioms from the Fox Derivative Proposition 116.3

All other Fox derivatives are evidently zero. The calculation for the special case that $\sigma(j) \in \{j, j+1\}$ is almost the same.

(b) We suppose that $r_j = x_{\sigma(j)}^{-1}\cdot x_{j+1}\cdot x_{\sigma(j)}\cdot x_j^{-1}$. This case is treated almost entirely the same way as the previous case. ∎

Let $K \subseteq S^3$ be a knot. By the Alexander Polynomial–Symmetry Theorem 117.7 and the Alexander Polynomial-at-1 Proposition 118.13 we now know that $\Delta_K(t) \doteq \Delta_K(t^{-1})$ and that $\Delta_K(1) = \pm 1$. The following proposition shows that these are the only two general properties of Alexander polynomials of knots:

Proposition 118.14. (Alexander Polynomial–Realization Proposition) Given any Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ with $p(t) \doteq p(t^{-1})$ and with $p(1) = \pm 1$ there exists a knot $K \subseteq S^3$ with $\Delta_K(t) \doteq p(t)$.

Proof. The proposition was first proved by Herbert Seifert [**Sei1934**, Satz 6]. An alternative proof was given by Jerome Levine [**Levine1965c**, p. 136]. ∎

118.4. **Appendix I: The Torres condition.** It is natural to ask whether the Alexander Polynomial-at-1 Proposition 118.13 generalizes to links. To formulate this extension we

need to recall the notion of the linking number of two oriented knots which we introduced in Definition 114.1

Definition 118.15. Let K and $\widetilde{K}$ be two disjoint oriented knots. By the Link Group-Abelianization Proposition 113.9 we have $\pi_1(S^3 \setminus K)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K$. Note that it follows from the Loop-π_1-Lemma 111.15 that $\widetilde{K}$ gives rise to a well-defined element of $\pi_1(S^3 \setminus K)_{\mathrm{ab}}$ which we denote by $[\widetilde{K}]$. We now define the linking number $\mathrm{lk}(K, \widetilde{K})$ to be the unique integer such that the following equality holds in $\pi_1(S^3 \setminus K)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K$:

$$[\widetilde{K}] \;=\; \mathrm{lk}(K, \widetilde{K}) \cdot \mu_K \;\in\; \pi_1(S^3 \setminus K)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K.$$

Using the notion of a linking number we can now formulate the following theorem:

Theorem 118.16. (Torres Condition Theorem) Let $K = K_1 \cup \cdots \cup K_m$ be an oriented m-component link with $m \geq 2$. For $i = 1, \ldots, m-1$ we set $\lambda_i := \mathrm{lk}(K_i, K_m)$. Then[1247]

$$\Delta_K(t_1, \ldots, t_{m-1}, 1) \doteq \begin{cases} (t_1^{\lambda_1} - 1) \cdot \frac{\Delta_{K_1}(t_1)}{t_1 - 1}, & \text{if } m = 2, \\ (t_1^{\lambda_1} \cdot \ldots \cdot t_{m-1}^{\lambda_{m-1}} - 1) \cdot \Delta_{K_1 \sqcup \cdots \sqcup K_{m-1}}(t_1, \ldots, t_{m-1}), & \text{if } m \geq 3. \end{cases}$$

Proof. The theorem was first proved by Guillermo Torres [**Tor1953**]. Alternative proofs can be found in [**Kaw1996**, Theorem 7.4.1], [**Hill2012**, Chapter 5] and [**Tur2002**, p. 86]. For the time being we do not attempt to prove this theorem. ■

In the Alexander Polynomial–Realization Proposition 118.14 we saw that we know which Laurent polynomials are realized as the Alexander polynomials of knots. For links this characterization problem is still open:

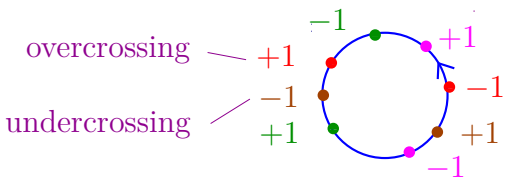
Problem 118.17. Which multivariable Laurent polynomials in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ are realized as the Alexander polynomial of an m-component link?

118.5. **Appendix II: Alexander polynomials of alternating knots.** We now introduce a particularly interesting class of knots:

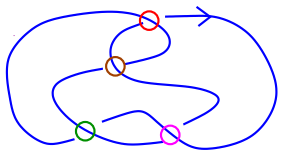
Definition 118.18.

(1) A knot diagram $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ is called alternating if on S^1 the overcrossings[1248] and undercrossings alternate.

(2) A knot is called alternating if it admits an alternating diagram.



alternating knot diagram



corresponding knot

Example.

(1) All diagrams shown in the list on page 2183 are alternating.

[1247] As we will see in Exercise 118.7 the formulation of this theorem would be neater if we had stuck to the Alexander function of fundamental groups.

[1248] We refer to Definition 112.4 for the definition of an overcrossing and an undercrossing.

(2) One can easily verify that the $(2, q)$-torus knots are alternating.
(3) The "standard diagram" for the knot 8_{19} which is shown below is not alternating. But in principle there could be a different diagram for this knot which is alternating.



trivial knot | trefoil | figure 8 knot | $(7,2)$-torus knot | 8_{19}

alternating | alternating?

Finally we point out that in general it is quite hard to decide whether or not a knot admits an alternating diagram. □

Theorem 118.19. (Alexander Polynomial–Alternating Theorem) Let $K \subseteq S^3$ be a knot. We write

$$\Delta_K(t) = \sum_{i=m}^{n} a_i \cdot t^i$$

with $a_m \neq 0$ and $a_n \neq 0$. If K is alternating, then the coefficients are alternating in sign, more precisely, there exists an $\epsilon \in \{-1, 1\}$ such that for all $i = m, \ldots, n$ we have $a_i \cdot \epsilon \cdot (-1)^i \geq 0$.

Proof. This theorem was first proved by Kunio Murasugi [**Muras1958**, Theorem 4.4]. See also [**Muras1960**, Theorem 3.12] for a related result. ■

Example. In the figure below we consider the same knots as above, but this time with their Alexander polynomials.



trivial knot | trefoil | figure 8 knot | $(7,2)$-torus knot | 8_{19}

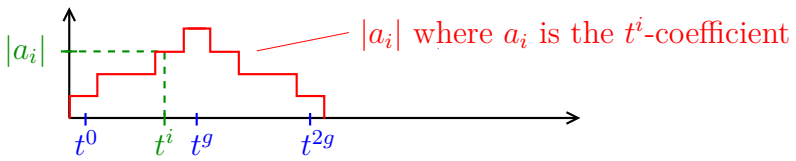
1 | $1-t+t^2$ | $1-3t+t^2$ | $1-t+t^2-t^3+t^4-t^5+t^6$ | $1-t+t^3-t^5+t^6$

For the first four knots we see that the signs of the coefficients of the Alexander polynomials are indeed alternating. For the knot $K = 8_{19}$ we have $\Delta_K(t) = 1-t+0t^2+t^3+0t^4-t^5+t^6$. We observe that the coefficients are *not* alternating. It follows from the Alexander Polynomial–Alternating Theorem 118.19 that $K = 8_{19}$ does not admit an alternating diagram. □

The Alexander polynomials of knots still holds some mysteries. For example the following conjecture, which goes back to a question of Ralph Fox from 1961 [**Fox1961**, p. 170], is still open:

Conjecture 118.20. (Trapezoidal Conjecture) If K is an alternating knot, then we can write $\Delta_K(t) = a_0 + a_1 \cdot t + \cdots + a_{2g} \cdot t^{2g}$ such that for $i = 0, \ldots, g-1$ we have $|a_i| \leq |a_{i+1}|$. (The figure below explains perhaps the name "trapezoidal".)

This conjecture was proved by Kunio Murasugi [**Muras1985**] and Peter Ozsváth and Zoltan Szabó [**OS2003a**, p. 233] for large classes of alternating knots.



Remark. We defined knots as submanifolds of S^3. The definition of an alternating knot is in terms of diagrams, which is arguably somewhat awkward. A diagram-free characterization of alternating knots was given by Josh Greene [**Greene2017**] and independently by Josh Howie [**HowB2017**]. □

118.6. Exercises for Chapter 118.

Exercise 118.1. For which diagrams does the over presentation have the same number of generators as the Wirtinger presentation?

Exercise 118.2. Let $m, n \in \mathbb{Z}$. As in the Baumslag-Solitar Group Proposition 130.25 we consider the Baumslag-Solitar group

$$\mathrm{BS}(m,n) \;=\; \langle x, y \,|\, x^{-1} \cdot y^m \cdot x \;=\; y^n \rangle.$$

For which $m, n \in \mathbb{Z}$ does there exist an oriented link $L \subseteq S^3$ such that $\pi_1(S^3 \setminus L)$ is isomorphic to $\mathrm{BS}(m,n)$?

Exercise 118.3. Let $K \subseteq S^3$ be a knot.

(a) Show that the number $\Delta_K(-1) \in \mathbb{Z}$ is odd.
(b) Show that $\deg(\Delta_K(t))$ is even.

Exercise 118.4. Let K be a knot let $\xi \in S^1$ such that there exists a prime p and a $k \in \mathbb{N}$ such that $\xi^{p^k} = 1$. Show that $\Delta_K(\xi) \neq 0$.
Hint. Consider the minimal polynomial of ξ and use the Alexander Polynomial-at-1 Proposition 118.13.

Exercise 118.5.

(a) Let $L = L_1 \sqcup L_2$ be an oriented 2-component link. Show that $\Delta_L(1,1) = \mathrm{lk}(L_1, L_2)$.
(b) Let L be an oriented link with ≥ 3 components. Show that $\Delta_L(1, \ldots, 1) = 0$.

Hint. In both cases make use of the Torres Condition Theorem 118.16.

Exercise 118.6. Let $K \subseteq S^3$ be a knot. By the Link Diagram Existence Proposition 112.7 and the One Crossing Exists Lemma 113.4 we know that K admits a diagram with at least one crossing. We order the strands cyclically. By the Wirtinger Presentation Proposition 113.6 we obtain an $n \in \mathbb{N}$, a map $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ and a Wirtinger presentation

$$P \;\coloneqq\; \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus K)$$

where each r_j is of the form $r_j = x_{\sigma(j)}^{\epsilon_j} \cdot x_{j+1} \cdot x_{\sigma(j)}^{-\epsilon_j} \cdot x_j^{-1}$ for some $\epsilon_j \in \{-1, 1\}$. Show that the diagram is alternating if and only if $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection.

Exercise 118.7. Let $K = K_1 \cup \cdots \cup K_m$ be an oriented m-component link with $m \geq 2$. We set $\widehat{K} \coloneqq K_1 \cup \cdots \cup K_{m-1}$.

(a) Reformulate the above Torres Condition Theorem 118.16 in terms of the Alexander functions of the two groups $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus \widehat{K})$ together with the homomorphisms $\pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus \widehat{K}) \to \pi_1(S^3 \setminus \widehat{K})_{\mathrm{ab}}$. and $\pi_1(S^3 \setminus \widehat{K}) \to \pi_1(S^3 \setminus \widehat{K})_{\mathrm{ab}}$.
Remark. If done correctly, this should lead to a more unified formulation of the Torres Condition Theorem 118.16.

(b) Prove the Torres Condition Theorem 118.16.
Hint. Start out with a diagram of K and use the Wirtinger presentation of K and $\widehat{K}$.

Exercise 118.8. For which coprime $p, q \in \mathbb{N}$ with $p \in \{1, 2, 3\}$ does the torus knot $T(p, q)$ admit an alternating diagram?

CHAPTER 119

The Alexander polynomial from diagrams

In this chapter we will get to know a way how to calculate the Alexander polynomial of an oriented link L fairly directly from a diagram of L. The key to doing so is the Dehn presentation of the link group $\pi_1(S^3 \setminus L)$, which can be viewed as a variation on the Wirtinger presentation. We will not make use of the results of this chapter in the later discussions. We use this fact as an excuse to take a few liberties throughout the chapter to keep the chapter readable.

This chapter is organized as follows:

- In Section 119.1 we will state the Alexander Polynomial-from-Diagram Theorem 119.5 which gives us a very pretty formula for computing the Alexander polynomial of an oriented link from a diagram.
- In Section 119.2 we will present the Dehn presentation of a link group. This gives another way to determine a presentation for a link group, starting from a diagram.
- In Section 119.3 we will use the Dehn presentation to prove the Alexander Polynomial-from-Diagram Theorem 119.5.
- In Section 119.4 we will provide exercises for Chapter 119.

In the next Chapter 120 we will study the relationship between braids, links and fundamental groups. This will allow us to show that the two definitions of the Alexander polynomial, via Fox calculus and the Burau representation, actually agree.

119.1. **The Alexander polynomial from diagrams.** Throughout this chapter we will make use of the notion of a link diagram which we first introduced in Definition 112.4. Recall that a link diagram for an oriented m-component link consists of two pieces of data:

- A diagrammatic map $\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2$.
- A map $c\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$.

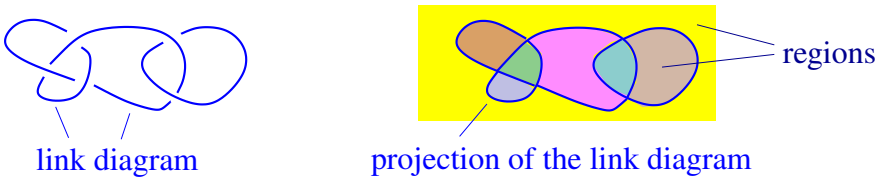
In Definition 112.4 we introduced the notion of crossing of a diagram. We now introduce the notion of a region of a diagram:

Definition 119.1. Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram.

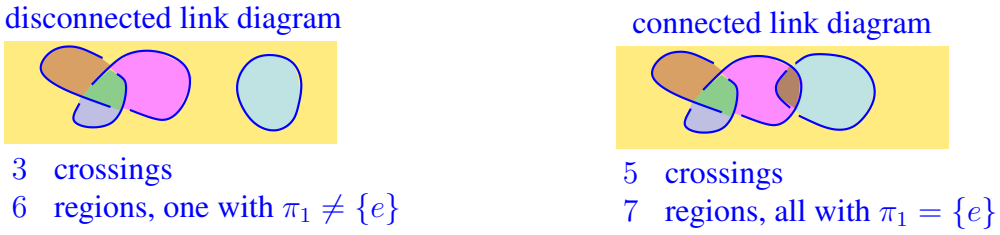
- The image of γ is called the projection of the diagram.
- Let $X \subseteq \mathbb{R}^2$ be the projection of the diagram. The components of $\mathbb{R}^2 \setminus X$ are called the regions of the link diagram.

At times we will think of $\mathbb{R}^2$ as a subspace of $S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ under the usual embedding that is given by $x \mapsto (x, 0)$.

Our next goal is to state a relationship between the number of regions and the number of crossings of a given link diagram. To state the result we will need the following definition:



Definition 119.2. We say that a link diagram is connected if the projection is a connected subspace of $\mathbb{R}^2$.



Lemma 119.3. (Diagram Region Lemma) Given any connected non-empty link diagram D we have the following equality:

$$\text{number of regions of } D \;=\; 2 + \text{number of crossings of } D,$$

furthermore each region is simply connected.

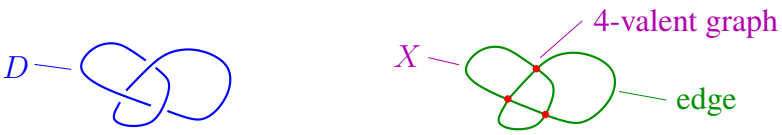
Sketch of proof. In the following we will sketch an argument that makes use of the notion of the homology groups $\mathrm{H}_k(W)$ and the cohomology groups $\mathrm{H}^k(W;\mathbb{Z})$ of a topological space W. We will also use several facts about these (co-) homology groups. Unfortunately we will introduce these (co-) homology groups only later on. We offer our most sincere apologies for using material that we will cover only much later.[1249]

Let D be a connected link diagram with c crossings. We make the following preparations and observations:

(a) We denote by $X \subseteq \mathbb{R}^2$ the projection of the link diagram.
(b) It is essentially clear that X is a topological graph where the vertices are given by the crossings. Note that the valence (in the sense of Definition 11.29) of each vertex is clearly 4.
(c) We denote by s the number of edges of the topological graph.
(d) Since the valence of each vertex is 4 we see that $4c = 2s$.
(e) We can think of X as a subspace of $S^2 =_{\mathrm{i}} \mathbb{R}^2 \cup \{\infty\}$. Since X is compact we obtain from an elementary argument, say using the elementary Union Connected Lemma 4.39, that $\#\pi_0(\mathbb{R}^2 \setminus X) = \#\pi_0(S^2 \setminus X)$.

We now see that

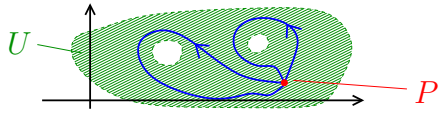
[1249] Our approach using (co-) homology groups might look like overkill, but it is actually quite tricky to give a rigorous elementary argument. The reader could try to prove the lemma using the Euler characteristic from Definition 156.1 and using the Schoenflies Theorem 170.6. But this might also become technically dicey.



$$\begin{array}{lll}
\#\text{regions of } D & & \\
\downarrow \text{ by (d)} & \downarrow \text{ see page 2758} & \downarrow \text{ by the General Alexander Duality Theorem 218.2 (2a) together with the Reduced Homology Lemma 147.3} \\
= \#\pi_0(S^2 \setminus X) & = \operatorname{rank}(\mathrm{H}_0(S^2 \setminus X)) & = \operatorname{rank}(\mathrm{H}^1(X;\mathbb{Z})) + 1 \\
= \operatorname{rank}(\mathrm{H}_1(X)) + 1 & = -\chi(X) + \operatorname{rank}(\mathrm{H}_0(X)) + 1 & = -\chi(X) + 1 + 1 \\
\uparrow \text{ by the Universal Coefficient H}^*\text{-Theorem 201.23} & \uparrow \text{ by the Homology } \chi\text{-Proposition 156.2} & \uparrow \text{ by the H}_0\text{-Proposition 143.25, since } X \text{ is connected} \\
= -c + s + 2 & = -c + 2c + 2 & = c + 2. \\
\uparrow \text{ by Definition 156.1} & \uparrow \text{ by (c)} &
\end{array}$$

It remains to prove that each region is simply connected. To do so we will need the following claim:

Claim. If $U \subseteq S^2$ is an open connected non-empty subset, then $\pi_1(U)$ is a free group.



$\pi_1(U, P) \cong \langle x_1, x_2 \rangle$

Proof. This claim can be proved using the Compact Submanifold Exhaustion Proposition 38.2 together with the Exhaustion-π_1-Proposition 105.2 and the Surface-with-Boundary–π_1-Lemma 107.8. The details were hopefully carried out in Exercise 107.7. ⊞

Now let $R_1, \ldots, R_{n+2}$ the regions of D. Note that

$$\begin{array}{lllllll}
\bigoplus_{i=1}^{n+2} \mathrm{H}_1(R_i) & \cong & \mathrm{H}_1\Big(\bigsqcup_{i=1}^{n+2} R_i\Big) & = & \mathrm{H}_1(S^2 \setminus X) & \cong & \widetilde{\mathrm{H}}^0(X;\mathbb{Z}) \quad = \quad 0. \\
& \uparrow & & & & \uparrow & \uparrow \\
& \text{by the Path-Component H}_*\text{-Lemma 143.21} & & & & \text{by the General Alexander Duality Theorem 218.2 (2b)} & \text{by the H}^0\text{-Proposition 198.9 and the Reduced Cohomology Lemma 198.17, since } X \text{ is connected and non-empty}
\end{array}$$

It follows that each $\mathrm{H}_1(R_i) = 0$. Finally note that it follows from this calculation, the Hurewicz Theorem 166.10, the claim and the observation that the abelianization of a free group $\langle S \rangle$ is the free abelian group $\mathbb{Z}^{(S)}$ (see page 2145) that each R_i is simply connected. ■

To state the main result of this chapter we need to introduce some notation:

Definition 119.4. Let $m \in \mathbb{N}$ and let D be the connected diagram of an oriented m-component link.

(1) By the Diagram Region Lemma 119.3 we can number the crossings of the diagram by $c_1, \ldots, c_n$ and we can number the regions by $r_1, \ldots, r_{n+2}$.
(2) To each crossing of the diagram we assign, as in figure below, values to the four regions that are adjacent to the crossing.

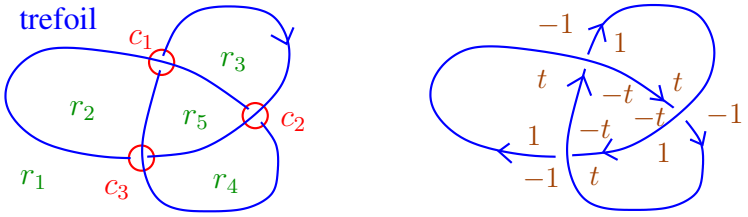


Definition 119.4.

(3) For each region r and each crossing c that is adjacent to r we define $a_{r,c} \in \mathbb{Z}[t_1, \ldots, t_m]$ to be the sum of the entries in (2).

(4) The Dehn matrix $\Theta(D)$ of D is the $n \times (n+2)$-matrix over $\mathbb{Z}[t_1, \ldots, t_m]$ where the entries are the polynomials in (3).

Example. Below we show a diagram D of the trefoil with crossings c_1, c_2, c_3 and regions r_1, r_2, r_3, r_4, r_5.



	1	2	3	4	5
1	−1	t	1	0	$-t$
2	−1	0	t	1	$-t$
3	−1	1	0	t	$-t$

To the right we also show the corresponding Dehn matrix $\Theta(D)$. □

Now we can formulate the main result of this chapter:

Theorem 119.5. (Alexander Polynomial-from-Diagram Theorem) Let L be an oriented non-empty link that is smoothly isotopic to a link that is associated to a connected diagram D. We number the crossings of the diagram by $c_1, \ldots, c_n$ and we number the regions by[1250] $r_1, \ldots, r_{n+2}$. Let $\Theta(D)$ be the corresponding $n \times (n+2)$-Dehn matrix. We pick two regions r_i, r_j that are adjacent, i.e. that are separated by a segment of the link.

(1) If L is a knot, then

$$\Delta_L(t) \doteq \det(\text{result of deleting the columns } i \text{ and } j \text{ from } \Theta(D)).$$

(2) Suppose that L has at least two components. If r_i and r_j are separated by the k-th component of L, then

$$\Delta_L(t_1, \ldots, t_m) \doteq \frac{1}{1-t_k} \cdot \det(\text{result of deleting the columns } i \text{ and } j \text{ from } \Theta(D)).$$

The remainder of this chapter is organized as follows:

- In this section we will discuss some examples for the Alexander Polynomial-from-Diagram Theorem 119.5.
- In Section 119.2 we will introduce the Dehn presentation of a link group.
- In Section 119.3 we will use the Dehn presentation to prove the Alexander Polynomial-from-Diagram Theorem 119.5.

Remark. One could also use the right hand side of the Alexander Polynomial-from-Diagram Theorem 119.5 as a definition of the Alexander polynomial. In fact this is the

[1250] Recall that by the Diagram Region Lemma 119.3 we know that the connected diagram D has $n+2$ regions.

original definition of the Alexander polynomial as given by James Alexander [**Ale1927, Ale1928**] and it is also used in [**Liv1993**, p. 49].

To make this idea work, one has to show that the right hand side does not depend on the choice of the link diagram and the choice of the two regions. The independence of the link diagram can be deduced from the Reidemeister Moves Theorem 112.10 (2). More details are provided in [**BAr2021**] and [**Lon2005**]. □

Example.

(1) Let D again be the above diagram of the trefoil K. Note that the regions r_4 and r_5 are adjacent. We see that

$$\Delta_K(t) \doteq \det \begin{pmatrix} -1 & t & 1 & \cancel{0} & \cancel{-t} \\ -1 & 0 & t & \cancel{1} & \cancel{-t} \\ -1 & 1 & 0 & \cancel{t} & \cancel{-t} \end{pmatrix} = t - 1 - t^2.$$

by the Alexander Polynomial-from-Diagram Theorem 119.5

Fortunately this is consistent with the calculation on page 2268.

(2) Next we consider two diagrams of the unknot:



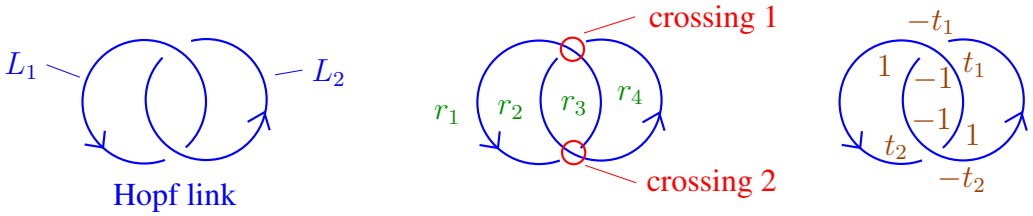
(a) The diagram to the left has zero crossings. So $\Theta(D)$ is a 0×2-matrix. After deleting the two columns we obtain a 0×0-matrix. But as we saw in the Determinant–Characterization Proposition 2.18 (ii), the determinant of a (0×0)-matrix equals by definition 1.

(b) The diagram to the right has one crossing. We obtain the (1×3)-Dehn matrix[1251] $\Theta(D) = (1{+}t \;\; -1 \;\; -t)$. Next note that here we have three regions:

- r_1 and r_2 are adjacent, thus $\Delta_K(t) \doteq \det(\cancel{1+t} \;\; \cancel{-1} \;\; -t) = -t$.
- r_1 and r_3 are adjacent, thus $\Delta_K(t) \doteq \det(\cancel{1+t} \;\; -1 \;\; \cancel{-t}) = -1$.
- r_2 and r_3 are *not* adjacent. In fact $\det(1+t \;\; \cancel{-1} \;\; \cancel{-t}) = 1+t$ would give us the wrong result.

Note that these calculations are consistent with the calculation on page 2268.

(3) Finally we consider the Hopf link with the diagram that is shown below.



We obtain the Dehn matrix

$$\Theta(D) = \begin{pmatrix} -t_1 & 1 & -1 & t_1 \\ -t_2 & t_2 & -1 & 1 \end{pmatrix}$$

[1251] Note that the region r_1 touches the crossing twice, so the entry is the sum of 1 and t.

Note that the two regions r_1 and r_2 are adjacent. Thus we see that

$$\Delta_L(t) \stackrel{\uparrow}{\doteq} \frac{1}{1-t_1} \cdot \det \begin{pmatrix} \cancel{-t_1} & \cancel{1} & -1 & t_1 \\ \cancel{-t_2} & \cancel{t_2} & -1 & 1 \end{pmatrix} = \frac{-1+t_1}{1-t_1} = -1.$$

by the Alexander Polynomial-from-Diagram Theorem 119.5, note that r_1 and r_2 are separated by L_1

Once again we gladly observe that this is consistent with the calculation on page 2271. □

119.2. **The Dehn presentation of a link group.** In the Wirtinger Presentation Proposition 113.6 we saw that a link diagram for a given L gives rise to a presentation of the link group $\pi_1(S^3 \setminus L)$ where

- the generators correspond to the strands,
- the relations correspond to the crossings.

In this section we will present the Dehn presentation, which gives us an alternative presentation of $\pi_1(S^3 \setminus L)$ where

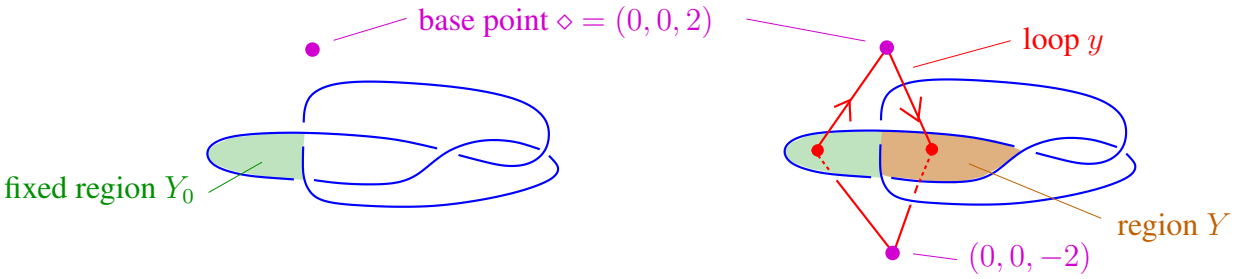
- the generators correspond to the regions,
- the relations correspond again to the crossings.

Before we present the Dehn presentation let us introduce some notation:

Notation 119.6. Let L be a link that is associated to a given link diagram.

(1) We work with the base point $\diamond := (0,0,2) \in \mathbb{R}^3$ which lies "above" the link L.
(2) We fix a region Y_0.
(3) Given a region Y we denote, using a hopefully suggestive notation, by y the quadrilateral that is given as follows:
   - We start at $\diamond$ and we go to a point in the region Y such that the path lies above L.
   - We go to $(0,0,-2)$ via a path that lies below L.
   - We go back to $\diamond$, but this time via Y_0 instead of Y.

   Note that y defines an element of $\pi_1(S^3 \setminus L, \diamond)$.

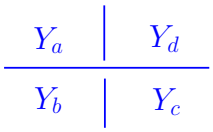


Now we can introduce the Dehn presentation of a link:

Proposition 119.7. (Dehn Presentation Proposition) Let $L \subseteq S^3$ be a non-empty link that is associated to a given connected link diagram. We fix a region Y_0, we enumerate the remaining regions by $Y_1, \ldots, Y_n$ and we enumerate the crossings by $1, \ldots, n-1$.[1252] For the i-th crossing we define a relation $r_i \in \langle y_0, y_1, \ldots, y_n \rangle$ as shown in the figure below.[1253]With these definitions we have an isomorphism[1254]

$$\langle y_0, y_1, \ldots, y_n \,|\, y_0, r_1, \ldots, r_{n-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond)$$

where the generator y_i gets sent to the loop y_i that we introduced in Notation 119.6.



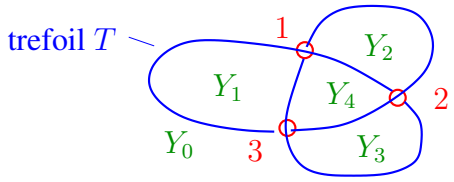
relation $y_a \cdot y_b^{-1} \cdot y_c \cdot y_d^{-1}$

Definition 119.8. Given a link L, any presentation as in the Dehn Presentation Proposition 119.7 is called a Dehn presentation of $\pi_1(S^3 \setminus L)$.

Remark. The Dehn Presentation Proposition 119.7 was first formulated by Max Dehn [**Deh1910**, p. 157], without much of a proof. The presentation is well known, see e.g. [**Neuw1965**, p. 15], [**PBIMGW2023**, p. 45] and [**GPor1994**, p. 154]. But it is hard to find a complete proof of the Dehn Presentation Proposition 119.7 in the literature. □

Example.

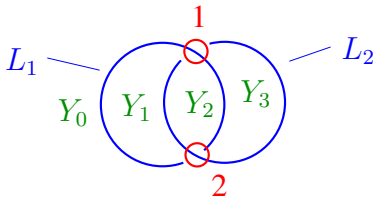
(1) Below we show the usual diagram of the trefoil T with three crossings and five regions.



$$\pi_1(S^3 \setminus T) \cong \langle y_0, y_1, y_2, y_3, y_4 \mid \begin{array}{l} y_0, \\ y_0 \cdot y_1^{-1} \cdot y_4 \cdot y_2^{-1}, \\ y_0 \cdot y_2^{-1} \cdot y_4 \cdot y_3^{-1}, \\ y_0 \cdot y_3^{-1} \cdot y_4 \cdot y_1^{-1} \rangle \end{array}$$

We also show the corresponding Dehn presentation. Note that using the Generator Elimination Lemma 100.30 one can easily turn the above presentation into a presentation with just two generators.[1255]

(2) Next we turn to the Hopf link H.



$$\pi_1(S^3 \setminus H) \cong \langle y_0, y_1, y_2, y_3 \mid \begin{array}{l} y_0, \\ y_2 \cdot y_3^{-1} \cdot y_0 \cdot y_1^{-1}, \\ y_2 \cdot y_1^{-1} \cdot y_0 \cdot y_3^{-1} \rangle \end{array}$$

[1252]Recall that by the Diagram Region Lemma 119.3 we know that the number of regions equals the number of crossings plus 2.

[1253]The only condition is that after an orientation-preserving local diffeomorphism a neighborhood of the crossing is of the given form where the overstrand is horizontal and the understrand is vertical. Note that there are essentially two ways for doing so, namely we can always rotate the picture by π. The resulting relators are not the same, but they are the same up to conjugation. So the different choices lead to essentially the same presentation.

[1254]Of course we could use the Generator Elimination Lemma 100.30 to immediately cancel the generator y_0 and the relator y_0. The reason why we introduce y_0 is to get a clean formulation of the relations, since we certainly allow that one of the regions Y_a, Y_b, Y_c, Y_d is our fixed region Y_0.

[1255]Recall that on page 2159 and 2199 we already gave two alternative presentations for $\pi_1(S^3 \setminus T)$. We leave it to the reader to give purely algebraic proofs why the three presentations correspond to isomorphic groups.

We now see that

$$\begin{aligned}\pi_1(S^3 \setminus H) &\cong \langle y_0, y_1, y_2, y_3 \,|\, y_0,\ y_2 \cdot y_3^{-1} \cdot y_0 \cdot y_1^{-1},\ y_2 \cdot y_1^{-1} \cdot y_0 \cdot y_3^{-1} \rangle \\ &\underset{\text{by the Generator Elimination Lemma 100.30 applied to } y_0}{\cong} \langle y_1, y_2, y_3 \,|\, y_2 \cdot y_3^{-1} \cdot y_1^{-1},\ y_2 \cdot y_1^{-1} \cdot y_3^{-1} \rangle \underset{\text{by the Generator Elimination Lemma 100.30 applied to } y_3}{\cong} \langle y_1, y_2 \,|\, y_2 \cdot y_1 \cdot y_2^{-1} \cdot y_1^{-1} \rangle \underset{\text{by the Free Abelian Group-Presentation Lemma 100.14}}{\cong} \mathbb{Z}^2.\end{aligned}$$

Thus we have recovered the calculation from the Hopf Link–π_1-Proposition 111.24 (1). □

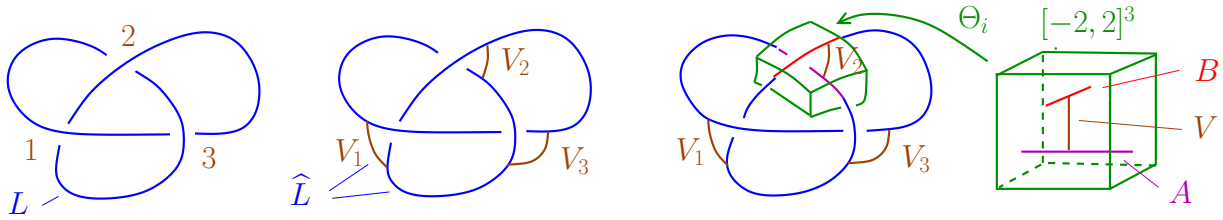
Sketch of proof. We give an extended sketch of a proof. But in contrast to the proof of the Wirtinger Presentation Proposition 113.6 we will not attempt to provide full details.

Let $L \subseteq S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ be a non-empty link that is associated to a given connected link diagram D. We fix a region Y_0. We introduce the following notation:

(a) We enumerate the remaining regions by $Y_1, \ldots, Y_n$.
(b) We enumerate the crossings by $1, \ldots, n-1$.
(c) We denote by $\widehat{L}$ the union of L together with the $n-1$ vertical arcs $V_1, \ldots, V_{n-1}$ that for each crossing connect the upper and lower point on L.
(d) For $i = 1, \ldots, n-1$ we pick smooth embeddings $\Theta_i \colon [-2,2]^3 \to \mathbb{R}^3$ with disjoint images such that for each $i \in \{1, \ldots, n-1\}$ the following holds:
- the preimage $\Theta_i^{-1}(L)$ consists of the two arcs

$$\begin{aligned} A &:= [-2,2] \times \{0\} \times \{-1\} \quad \text{(corresponding to the understrand)} \\ B &:= \{0\} \times [-2,2] \times \{1\} \quad \text{(corresponding to the overstrand),}\end{aligned}$$

- we have $\Theta_i^{-1}(V_i) = V$ where $V := \{0\} \times \{0\} \times [-1,1]$.



Note that

$$S^3 \setminus L \;=\; (S^3 \setminus \widehat{L}) \cup \bigcup_{i=1}^{n-1} \Theta_i([-2,2]^3 \setminus (A \cup B))$$

and note that $\Theta_i^{-1}(S^3 \setminus \widehat{L}) = [-2,2]^3 \setminus (A \cup B \cup V)$. Evidently we want to apply the Seifert-van Kampen Theorem 110.8 to this decomposition to calculate $\pi_1(S^3 \setminus L, \diamond)$.

To do so we first need to calculate the various fundamental groups involved and the inclusion induced maps. We will do so in the following claims.

Claim 1. The loops $y_0, y_1, \ldots, y_n$ from Notation 119.6 define an isomorphism

$$\langle y_0, y_1, \ldots, y_n \,|\, y_0 \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus \widehat{L}, \diamond).$$

Proof. Let $\gamma \colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2$ be the diagrammatic map that corresponds to the diagram D. Recall that by the definition in the Diagram-to-Link Lemma 112.5 we have a smooth map $\eta \colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1,0]$ such that

$$L \;=\; \left\{ (\gamma(P), \eta(P)) \;\middle|\; P \in \bigsqcup_{i=1}^{m} S_i^1 \right\}.$$

We use this notation to introduce two other subspaces of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$:

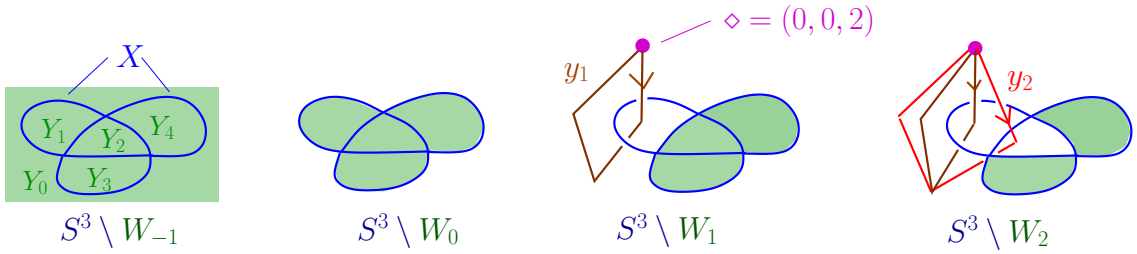
- Let X be the projection of D, viewed as a subspace of $\mathbb{R}^3$ via the standard embedding $\mathbb{R}^2 \xrightarrow{x\mapsto(x,0)} \mathbb{R}^3$.
- We set
$$Z \;:=\; \Big\{(\gamma(P),t)\;\Big|\; P \in \bigsqcup_{i=1}^{m} S^1_i \text{ and } t \in [\eta(P),0]\Big\}.$$

Note that we have the inclusions $\widehat{L} \subseteq Z \supseteq X$. Thus we obtain the following commutative diagram

$$\begin{array}{ccccc} & & \langle y_0, y_1, \dots, y_n \,|\, y_0\rangle & & \\ & \swarrow & \downarrow & \searrow & \\ \pi_1(S^3 \setminus \widehat{L}, \diamond) & \xrightarrow{\cong} & \pi_1(S^3 \setminus Z, \diamond) & \xleftarrow{\cong} & \pi_1(S^3 \setminus X, \diamond) \end{array}$$

where the bottom horizontal maps are induced by inclusions. We leave it to the reader to show that the bottom horizontal maps are isomorphisms.[1256] We need to show that the left diagonal map is an isomorphism. By the above discussion it now suffices to show that the right diagonal map $\langle y_0, y_1, \dots, y_n \,|\, y_0\rangle \to \pi_1(S^3 \setminus X, \diamond)$ is an isomorphism.

For $i = -1, \dots, n$ we set $W_i = \widehat{L} \cup Y_i \cup \dots \cup Y_n$. Note that $W_{-1} = S^2$ and $W_n = X$.



The claim that the right diagonal map is an isomorphism follows from the following three observations:

(1) The topological space $S^3 \setminus W_{-1} = S^3 \setminus S^2$ consists of two contractible components.

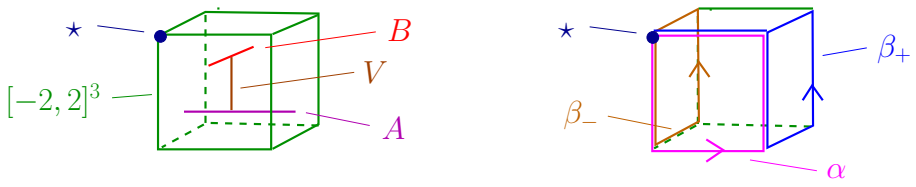
(2) Note that it follows immediately from (1) and the Seifert-van Kampen Theorem 110.8 that the topological space $S^3 \setminus W_0$ is simply connected. In particular we have an isomorphism $\langle y_0 \,|\, y_0\rangle \xrightarrow{\cong} \pi_1(S^3 \setminus W_0, \diamond)$.

(3) Recall that by the Diagram Region Lemma 119.3 we know that each region Y_i is simply connected. It thus follows from the HNN–Gluing Theorem 113.23 that passing from $S^3 \setminus W_i$ to $S^3 \setminus W_{i+1}$ changes the fundamental group by adding the generator y_i. ⊞

Next we have to understand the fundamental group of $[-2,2]^3 \setminus (A \cup B \cup V)$ and also of $[-2,2]^3 \setminus (A \cup B)$. To do so we introduce the following notation:

- We set $\star := [-2,-2,2]$.
- We denote by α, β_-, β_+ the loops in $[-2,2]^3 \setminus (A \cup B \cup V)$ that are shown in the figure below.
- We denote by $\iota\colon [-2,2]^3 \setminus (A \cup B \cup V) \to [-2,2]^3 \setminus (A \cup B)$ the inclusion.

[1256] One approach would be to show that $S^3 \setminus \widehat{L}$, $S^3 \setminus Z$ and $S^3 \setminus X$ have a common deformation retract.



Claim 2. These loops induce the following commutative diagram

$$\begin{array}{ccc} \langle \alpha, \beta_-, \beta_+ \rangle & \xrightarrow{\cong} & \pi_1([-2,2]^3 \setminus (A \cup B \cup V), \star) \\ \downarrow{\scriptstyle \alpha \mapsto \alpha, \beta_\pm \mapsto \beta} & & \downarrow{\scriptstyle \iota_*} \\ \langle \alpha, \beta \rangle & \xrightarrow{\cong} & \pi_1([-2,2]^3 \setminus (A \cup B), \star) \end{array}$$

where the horizontal maps are isomorphisms.

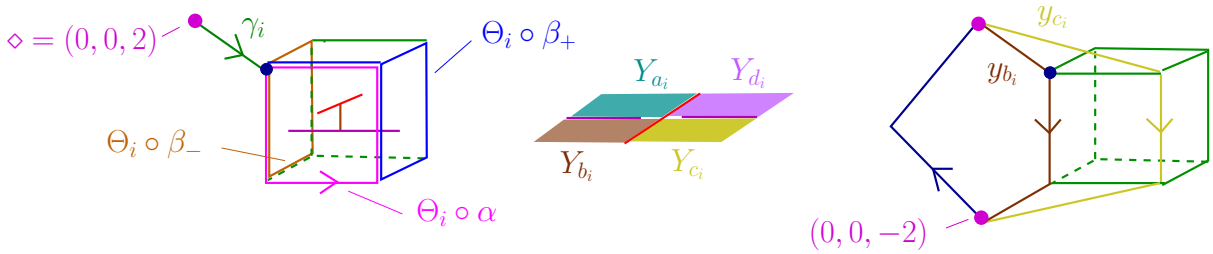
Proof. This claim can be proved using the Homotopy Equivalence-π_1-Proposition 90.9 (1b) together with the Graph-π_1-Proposition 99.13. We leave it to the reader to fill in the details. ⊞

Next, for each $i \in \{1, \dots, n-1\}$ we introduce the following notation:

- We denote by γ_i a path from $\diamond$ to $\Theta_i(\star)$ that "lies above" L.
- We write $\alpha_i := \gamma_i * (\Theta_i \circ \alpha) * \overline{\gamma}_i$ and $\beta_{\pm,i} := \gamma_i * (\Theta_i \circ \beta_\pm) * \overline{\gamma}_i$.
- We denote by a_i, b_i, c_i, d_i the indices of the four regions at the i-th crossing.

Claim 3. Let $i \in \{1, \dots, n-1\}$. We have the following equalities in $\pi_1(S^3 \setminus \widehat{L}, \diamond)$:

$$\begin{array}{rcl} [\alpha_i] & = & y_{b_i} \cdot y_{c_i}^{-1} \\ [\beta_{i,-}] & = & y_{b_i} \cdot y_{a_i}^{-1} \\ [\beta_{i,+}] & = & y_{c_i} \cdot y_{d_i}^{-1}. \end{array}$$



Proof. It is elementary to see that we have the following equalities in $\pi_1(S^3 \setminus \widehat{L}, \diamond)$:[1257]

$$\begin{array}{rcl} [\alpha_i] & = & [y_{b_i} * \overline{y}_{c_i}] \\ [\beta_{i,-}] & = & [y_{b_i} * \overline{y}_{a_i}] \\ [\beta_{i,+}] & = & [y_{c_i} * \overline{y}_{d_i}]. \end{array}$$

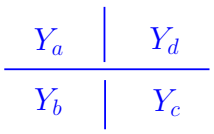
The claim follows immediately from these observation together with the π_1–Group Proposition 88.12. ⊞

[1257]Evidently we have no intention of writing down explicit path-homotopies in $S^3 \setminus \widehat{L}$.

In summary we now see that

$$\begin{array}{rl} & \text{by Claims 1,2 and 3 together with the Seifert-van Kampen Theorem 110.8 (iii), which we apply altogether } n-1 \text{ times} \\ & \downarrow \\ \pi_1(S^3 \setminus L, \diamond) & \cong \langle y_0, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1} \,|\, \begin{array}{l} y_0, \\ y_{b_i} \cdot y_{c_i}^{-1} = \alpha_i, \\ y_{b_i} \cdot y_{a_i}^{-1} = \beta_i, \\ y_{c_i} \cdot y_{d_i}^{-1} = \beta_i \rangle \end{array} \\ & \cong \langle y_0, y_1, \ldots, y_n \,|\, y_0, y_{b_i} \cdot y_{a_i}^{-1} = y_{c_i} \cdot y_{d_i}^{-1}, i = 1, \ldots, n-1 \rangle. \\ & \uparrow \\ & \text{follows from the Generator Elimination Lemma 100.30, first applied to } \alpha_1, \ldots, \alpha_{n-1} \text{ and then to } \beta_1, \ldots, \beta_{n-1} \\ & = \langle y_0, y_1, \ldots, y_n \,|\, y_0, y_{a_i} \cdot y_{b_i}^{-1} \cdot y_{c_i} \cdot y_{d_i}^{-1}, i = 1, \ldots, n-1 \rangle. \end{array}$$

For the reader's convenience we now recall the figure from page 2309.



relation $y_a \cdot y_b^{-1} \cdot y_c \cdot y_d^{-1}$

We see that our relations are exactly the ones that we promised. So we are done. ■

119.3. **Proof of the Alexander Polynomial-from-Diagram Theorem 119.5.** This section is totally dedicated to the long overdue proof of the following theorem:

Theorem 119.5. (Alexander Polynomial-from-Diagram Theorem) Let $m \in \mathbb{N}$ and let L be an oriented m-component link that is smoothly isotopic to a link that is associated to a connected diagram D. We number the crossings of the diagram by $c_1, \ldots, c_n$ and we number the regions by $Y_1, \ldots, Y_{n+2}$. Let $\Theta(D)$ be the corresponding Dehn matrix. We pick two regions Y_i, Y_j that are adjacent, i.e. that are separated by a strand.

(1) If L is a knot, then

$$\Delta_L(t) \doteq \det(\text{result of deleting columns } i \text{ and } j \text{ from } \Theta(D)).$$

(2) Suppose that L has at least two components. If r_i and r_j are separated by the k-th component of L, then

$$\Delta_L(t_1, \ldots, t_m) \doteq \frac{1}{1-t_k} \cdot \det(\text{result of deleting columns } i \text{ and } j \text{ from } \Theta(D)).$$

Proof. By the Isotopic Link-π_1-Lemma 111.4 we can assume that L is an oriented m-component link that is associated to a diagram D. We number the crossings of the diagram by $c_1, \ldots, c_n$. Clearly we can number the regions as $Y_1, \ldots, Y_{n+2}$ such that the adjacent regions are given by Y_{n+1} and Y_{n+2}. We let Y_{n+2} be our fixed region. It follows from the Dehn Presentation Proposition 119.7 that we have an isomorphism

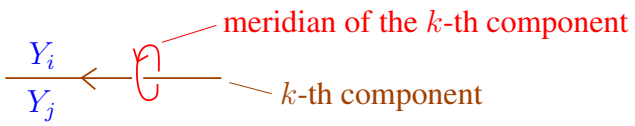
$$\langle y_1, \ldots, y_{n+1}, y_{n+2} \,|\, y_{n+2}, r_1, \ldots, r_n \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond)$$

where the generator y_i gets sent to the loop that we introduced in Notation 119.6 and which corresponds to the "base region" Y_{n+2}.

We see that these entries are exactly the same as in the figure on page 2309.



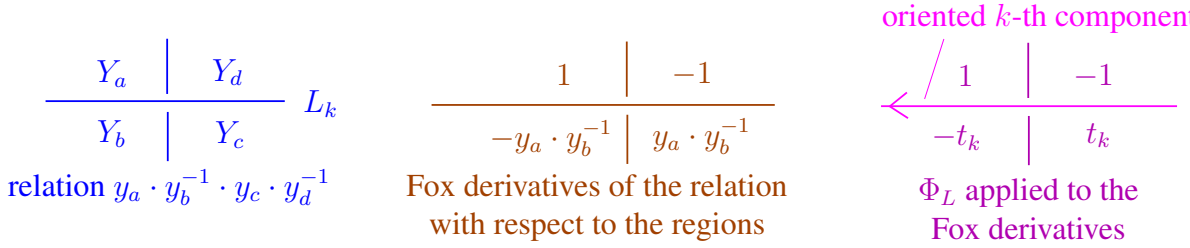
Now let $\Phi_L \colon \pi_1(S^3 \setminus L) \to \langle t_1, \dots, t_m \rangle_{\text{ab}}$ be the usual epimorphism from Definition 117.4. Note that for adjacent regions Y_i and Y_j that are separated by the k-th component, as in the figure below, then $y_i \cdot y_j^{-1}$ is a meridian for the k-th strand. Thus we see that $\Phi_L(y_i) = t_k \cdot \Phi_L(y_j)$.



here $\mu_k = y_i \cdot y_j^{-1}$, thus
$\Phi_L(y_i) = t_k \cdot \Phi_L(y_j)$

In the following figure we summarize what is happening at a crossing:

- To the left we see that a crossing gives rise to a relation.
- In the middle we see the Fox derivatives of the relation with respect to the generators given by the four regions. (For clarity we assume that the four regions are distinct.)
- To the right we show the result of applying Φ_L to the Fox derivatives. Note that the resulting four terms are exactly the same as in the figure on page 2306.



Now the arguments splits into the cases of a knot and of a link with ≥ 2 components:

(1) First we assume that we are dealing with an oriented knot K. Recall that by hypothesis the two regions Y_{n+1} and Y_{n+2} are adjacent. It follows from the above discussion that $\Phi_K(y_{n+1}) = t^\epsilon \cdot \Phi_K(y_{n+2}) = t^\epsilon$ for some $\epsilon \in \{-1, 1\}$. Now we see that

$$\Delta_K(t) \underset{\uparrow}{=} (t-1) \cdot \frac{\det \begin{pmatrix} & & 0 \\ & \Phi_{K*}\left(\frac{\partial r_i}{\partial y_j}\right)_{i,j=1,\dots,n} & \vdots \\ & & 0 \end{pmatrix}}{\Phi_K(y_{n+1}) - 1} \underset{\uparrow}{\doteq} \det\left(\Phi_{K*}\left(\frac{\partial r_i}{\partial y_j}\right)_{i,j=1,\dots,n}\right)$$

by Definition 117.2 and Definition 116.14 applied to the deficiency-one presentation $\pi_1(S^3 \setminus K) = \langle y_1, \dots, y_{n+1}, y_{n+2} \,|\, y_{n+2}, r_1, \dots, r_n \rangle$ and the generator y_{n+1}

follows from $\Phi_K(y_{n+1}) = t^\epsilon \cdot \Phi_K(y_{n+2}) = t^\epsilon$ and elementary algebra

$$\underset{\uparrow}{\doteq} \det(\text{Dehn matrix } \Theta(D), \text{ where we erased columns } n+1, n+2)$$

follows from comparing Definition 119.4 with the above calculation of Φ_K of the Fox derivatives

(2) Now we assume that we are dealing with an oriented m-component link $L \subseteq S^3$ with $m \geq 2$ components. Recall that by hypothesis the regions Y_{n+1} and Y_{n+2} are adjacent.

We assume that they are separated by the k-th component. It follows from the above discussion that $\Phi_L(y_{n+1}) = t_k^\epsilon \cdot \Phi_L(y_{n+2}) = t_k^\epsilon$ for some $\epsilon \in \{-1, 1\}$. Now we see that

$$\Delta_L(t_1, \dots, t_m)) \underset{\uparrow}{=} \frac{\det \begin{pmatrix} 0 & 1 \\ \Phi_{L*}\left(\frac{\partial r_i}{\partial y_j}\right)_{i,j=1,\dots,n} & 0 \end{pmatrix}}{\Phi_L(y_{n+1}) - 1} \underset{\uparrow}{\doteq} \frac{\det\left(\Phi_{L*}\left(\frac{\partial r_i}{\partial y_j}\right)_{i,j=1,\dots,n}\right)}{t_k - 1}$$

by Definition 117.4 and Definition 116.14

follows from $\Phi_L(y_{n+1}) = t_k^\epsilon \cdot \Phi_L(y_{n+2}) = t_k^\epsilon$

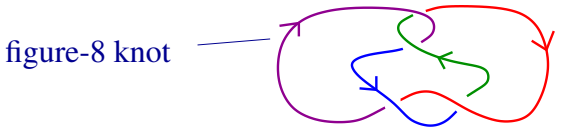
$$\underset{\uparrow}{\doteq} \frac{1}{t_k - 1} \cdot \det \begin{pmatrix} \text{Dehn matrix } \Theta(D), \text{ where we} \\ \text{erased columns } n+1, n+2 \end{pmatrix}$$

follows from comparing Definition 119.4 with the above calculation of Φ_L of the Fox derivatives

■

119.4. **Exercises for Chapter 119.** In the following exercises we will recover many properties of the Alexander polynomial from the description given in the Alexander Polynomial-from-Diagram Theorem 119.5. In other words, we will see, that if we use the expression in the Alexander Polynomial-from-Diagram Theorem 119.5 as the definition of the Alexander polynomial, we can recover many familiar statements.

Exercise 119.1. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to calculate the Alexander polynomial of the figure-8 knot.

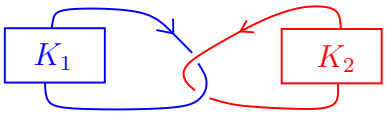


Exercise 119.2. Let K and J be two oriented knots. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to show that

$$\Delta_{K\#J}(t) \dodoteq \Delta_K(t) \cdot \Delta_J(t).$$

Exercise 119.3. We consider the oriented 2-component link L which is built out of oriented knots K_1 and K_2 as shown in the figure below. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to determine $\Delta_L(t_1, t_2)$ in terms of the Alexander polynomials of K_1 and K_2.

Remark. This exercise is closely related to Exercise 117.7.



Exercise 119.4.

(a) Let $n \in \mathbb{Z}$ be odd. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to calculate the Alexander polynomial of the $(2, n)$-torus knot $T(2, n)$.

(b) For the intrepid reader: Use the same approach to calculate the Alexander polynomial of any torus knot.

Exercise 119.5. Let L be a link that is splittable in the sense of Definition 109.5. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to show that $\Delta_L = 0$.
Remark. This exercise is closely related to Exercise 117.15.

Exercise 119.6. Let $L \subseteq S^3$ be an oriented link. We denote by L^{mir} the mirror of L, as introduced in Definition 108.23. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to show that

$$\Delta_{L^{\mathrm{mir}}}(t_1, \ldots, t_m) \doteq \Delta_L(t_1^{-1}, \ldots, t_m^{-1}).$$

Remark. This gives a new proof of the Alexander Polynomial–Mirror Lemma 117.15.

Exercise 119.7. In the Alexander Polynomial-from-Diagram Theorem 119.5 we showed that the Dehn Presentation Proposition 119.7 gives rise to an elementary way to calculate the Alexander polynomial of an oriented link from a diagram. Turn the Wirtinger Presentation Proposition 113.6 into another elementary way to calculate the Alexander polynomial from a diagram. In other words, calculate the Fox derivatives of the Wirtinger presentation and give a hands-on description of the resulting Jacobi matrix.

Exercise 119.8. Let L be an oriented m-component link and let $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$. We denote by $L^{(\epsilon_1, \ldots, \epsilon_m)}$ the link L but where we flipped the orientation of the i-th component precisely if $\epsilon_i = -1$. Use the Alexander Polynomial-from-Diagram Theorem 119.5 to show that

$$\Delta_{L^{(\epsilon_1, \ldots, \epsilon_m)}}(t_1, \ldots, t_m) \doteq \Delta_L(t_1^{\epsilon_1}, \ldots, t_m^{\epsilon_m}).$$

Remark. This gives a new proof of the Alexander Polynomial–Orientation Lemma 117.6.

CHAPTER 120

Braids, fundamental groups and Alexander polynomials

We start out by recalling two aspects from much earlier chapters:

- In Definition 64.1 we introduced the notion of an m-strand braid X and in Definition 66.3 we introduced the closure $\widetilde{X}$ of X, which is an oriented link. In this chapter we will calculate $\pi_1(S^3 \setminus \widehat{X})$ in terms of the braid X, more precisely, we will see how one can read off from a braid X an explicit deficiency one representation of $\pi_1(S^3 \setminus \widehat{X})$.
- In the Braid Group Lemma 64.4 we saw that m-strand braids give rise to the braid group B_m. Furthermore, in the Burau Representation Proposition 65.1 we introduced the Burau representation
$$\beta_m \colon \mathrm{B}_m \ \to \ \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$$
and in the Reduced Burau Representation Proposition 65.4 we introduced the reduced Burau representation
$$\beta_m^{\mathrm{red}} \colon \mathrm{B}_m \ \to \ \mathrm{GL}(m-1, \mathbb{Z}[t^{\pm 1}]).$$
Also recall that in the Reduced Burau Representation–Link Invariant Theorem 67.1 we used the reduced Burau representation to introduce the single-variable Alexander polynomial of an oriented link. In this chapter we will use the aforementioned deficiency-one representation to show that the various definitions of the Alexander polynomial (in terms of the Burau representation and Fox calculus) are consistent.

This chapter is organized as follows:

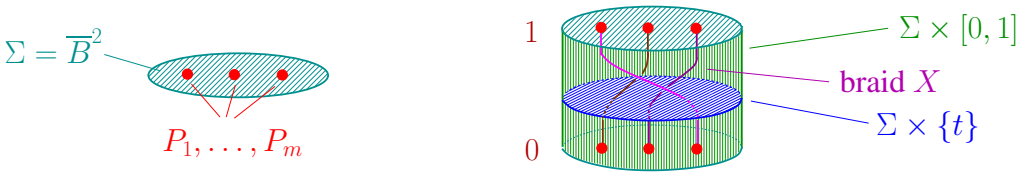
- In Section 120.1 we will recall the notion of a braid and of braid groups.
- In Section 120.2 we will recall the notion of the closure of a braid.
- In Section 120.3 we will state the Braid Closure–π_1–Proposition 120.10 which gives us a deficiency-one presentation of $\pi_1(S^3 \setminus \widehat{X})$ in terms of the braid X.
- In Section 120.4 we will provide a proof of the Braid Closure–π_1–Proposition 120.10.
- In Section 120.5 we will express the Burau representation in terms of Fox calculus.
- In Section 120.6 we will use Fox calculus to introduce the *single-variable* Alexander polynomial $\Delta_L(t)$ of an oriented link.
- In Section 120.7 we will see that the single-variable Alexander polynomial from Definition 67.2 (in terms of the Burau representation) agrees with the single-variable Alexander polynomial from Section 120.7 (in terms of Fox calculus).
- In Section 120.8 we will provide exercises for Chapter 120.

In the next Chapter 121 we will introduce Seifert surfaces of knots.

120.1. **Braids and braid groups.** In this section we will recall some basic notions from braid theory. We start out by recalling some of main aspects of Definition 64.1:

Definition 120.1. Let $\Sigma \subseteq \mathbb{R}^2$ be an open Euclidean disk and let $m_0 \in \mathbb{N}$. Furthermore let $P_1, \dots, P_m \in \Sigma$ be pairwise distinct points. An m-strand braid in the pair $(\Sigma, \{P_1, \dots, P_m\})$ is defined as a proper 1-dimensional submanifold $X \subseteq \Sigma \times [0,1]$ with the following properties:

(1) X has precisely m components, each of which is diffeomorphic to an interval.
(2) For each $t \in (0,1)$ the submanifold X is transverse to the submanifold $\Sigma \times \{t\}$.
(3) There exists an $\epsilon > 0$ such that the following conditions are satisfied:
- $X \cap (\Sigma \times [0, \epsilon]) = \{P_1, \dots, P_m\} \times [0, \epsilon]$.
- $X \cap (\Sigma \times [1-\epsilon, 1]) = \{P_1, \dots, P_m\} \times [1-\epsilon, 1]$.

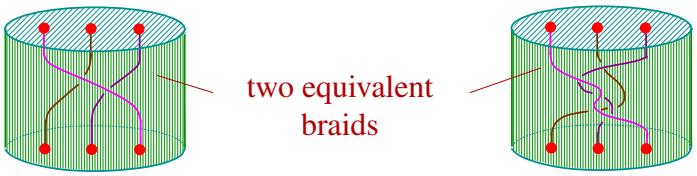


As we already pointed out on page 1394, similar to the setting for knots and links, we do not care about braids "on the nose", but only up to a suitable form of deformation. This thought leads us to the following slightly simplified version of Definition 64.2:

Definition 120.2. Let $\Sigma \subseteq \mathbb{R}^2$ be an open Euclidean disk and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma$ be pairwise distinct points.

(1) Two braids X_0, X_1 in $(\Sigma, \{P_1, \dots, P_m\})$ are called equivalent if there exists a proper smooth isotopy $\Phi\colon (\{1, \dots, m\} \times [0,1]) \times [0,1] \to \Sigma \times [0,1]$ with the following properties:
- For $t \in \{0,1\}$ we have $\Phi_t(\{1, \dots, m\} \times [0,1]) = X_t$.
- Each $\Phi_t(\{1, \dots, m\} \times [0,1])$ is a braid in $(\Sigma, \{P_1, \dots, P_m\})$.

(2) We denote by $\mathrm{B}^{P_1,\dots,P_m}(\Sigma)$ the set of equivalence classes of braids in $(\Sigma, \{P_1, \dots, P_m\})$.

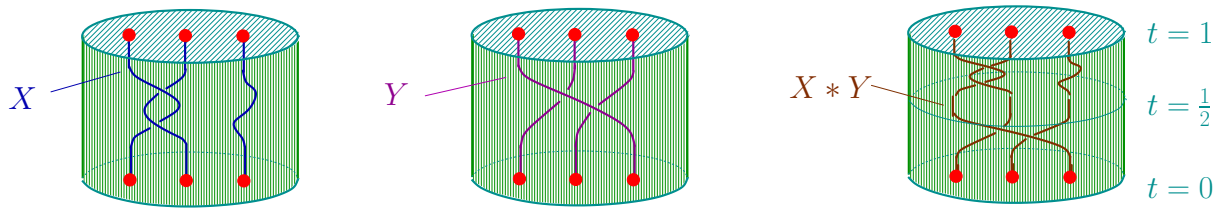


In Definition 120.3 we saw we can compose two braids $X * Y$ by "stacking X on top of Y". For the reader's convenience we recall the precise definition:

Definition 120.3. Let $\Sigma \subseteq \mathbb{R}^2$ be an open Euclidean disk and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \dots, P_m \in \Sigma$ be pairwise distinct points. Given two braids X, Y in $(\Sigma, \{P_1, \dots, P_m\})$ we define the composition

$$X * Y \;:=\; \{(P, \tfrac{t+1}{2}) \,|\, (P,t) \in X\} \cup \{(Q, \tfrac{t}{2}) \,|\, (Q,t) \in Y\}.$$

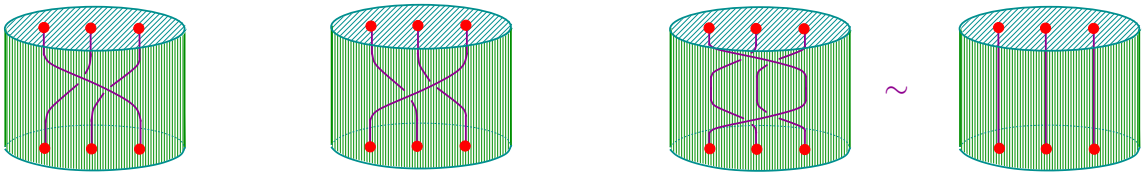
Note that using Condition (3) from Definition 120.1 one can easily verify that $X * Y$ is again a braid in $(\Sigma, \{P_1, \dots, P_m\})$.



Lemma 64.4. (Braid Group Lemma) Let $\Sigma \subseteq \mathbb{R}^2$ be an open Euclidean disk and let $m \in \mathbb{N}_0$. Furthermore let $P_1, \ldots, P_m \in \Sigma$ be pairwise distinct points. The map

$$\begin{array}{rcl} \mathrm{B}^{P_1,\ldots,P_m}(\Sigma) \times \mathrm{B}^{P_1,\ldots,P_m}(\Sigma) & \to & \mathrm{B}^{P_1,\ldots,P_m}(\Sigma) \\ ([X],[Y]) & \mapsto & [X * Y] \end{array}$$

is well-defined and it defines a group structure on $\mathrm{B}^{P_1,\ldots,P_m}(\Sigma)$.



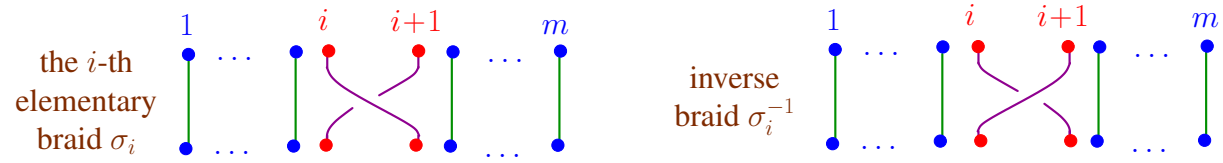
braid X | inverse braid X^{-1} given by reflecting in $z = \frac{1}{2}$ | $X * X^{-1}$ | trivial braid

For the remainder of this chapter we specialize to the following open Euclidean disk:

Notation 120.4. Let $m \in \mathbb{N}_0$.
(1) We consider the open disk $D := B^2_{\frac{m}{2}}(\frac{m+1}{2}, 0)$.
(2) We set $\mathrm{B}_m =_{\mathrm{i}} \mathrm{B}^{(1,0),\ldots,(m,0)}(D)$.[1258]

Next we want to describe a presentation for the braid groups B_m. To do so we introduce the following slight variation on Notation 64.17:

Notation 120.5. Let $m \in \mathbb{N}$ and let $i \in \{1, \ldots, m-1\}$. In the following we denote by $\sigma_i \in \mathrm{B}_m = \mathrm{B}^{(1,0),(2,0),\ldots,(m,0)}(D)$ the braid that is shown in the figure below to the left. We refer to it as the i-th elementary braid.

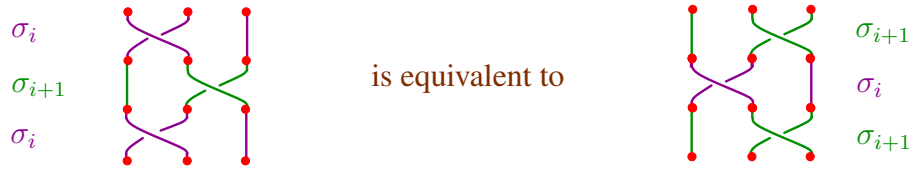


Next recall that in the Braid Group–Generating Set Proposition 64.18 we showed that the elementary braids $\sigma_1, \ldots, \sigma_{m-1}$ form a generating set for the braid group B_m. Furthermore

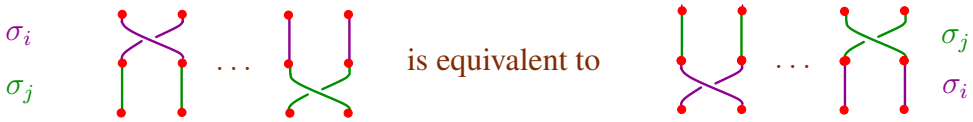
[1258] Recall that originally in Notation 64.6 we set $\mathrm{B}_m := \mathrm{B}^{(1,0),(2,0),\ldots,(m,0)}(\mathbb{R}^2)$. But similar to the Braid Group–Finite Radius Lemma 64.7 one can show quite easily that the inclusion induced homomorphism $\mathrm{B}^{(1,0),\ldots,(m,0)}(D) \to \mathrm{B}^{(1,0),(2,0),\ldots,(m,0)}(\mathbb{R}^2)$ is an isomorphism of groups. Thus the two slightly different definitions of B_m are essentially the same.

recall that on page 1402 we already mentioned the following two types of relations among the elementary braids $\sigma_1, \dots, \sigma_{m-1}$:

- Braid relation: For any $i \in \{1, \dots, m-2\}$ we have $[\sigma_i * \sigma_{i+1} * \sigma_i] = [\sigma_{i+1} * \sigma_i * \sigma_{i+1}]$.



- Far commutativity: For any $i, j \in \{1, \dots, m-1\}$ with $|i-j| \geq 2$ we have $[\sigma_i * \sigma_j] = [\sigma_j * \sigma_i]$.



The following proposition gives a completely algebraic description of the braid group B_m:

Proposition 64.19. (Braid Group–Presentation) Let $m \in \mathbb{N}$. The map

$$\left\langle \sigma_1, \dots, \sigma_{m-1} \;\middle|\; \begin{array}{ll} \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} & \text{for } i = 1, \dots, m-2 \\ \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i & \text{for } |i-j| \geq 2 \end{array} \right\rangle \to \mathrm{B}_m, \quad \sigma_i \mapsto [\sigma_i]$$

is well-defined and it is an isomorphism.

We conclude this section by recalling the following convention:

Convention 64.20. We will use the Braid Group–Presentation 64.19 to identify the braid group B_m with the above presentation. In particular, we will denote by σ_i indiscriminately a generator of the presentation, the corresponding group element and the i-th elementary braid.

120.2. **Closures of braids.** In this short section we want to recall the notion of the closure of a braid:

Notation 120.6. Let $m \in \mathbb{N}_0$.

(1) Similar to Notation 66.2 we consider the map

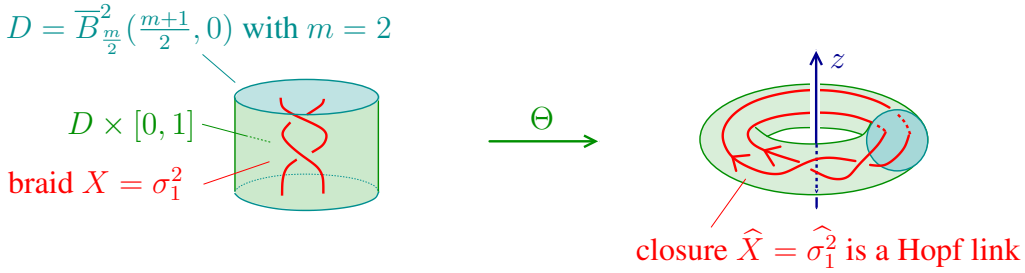
$$\begin{array}{rcl} \Theta\colon D \times [0,1] & \to & \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3 \\ ((x,z),t) & \mapsto & \underbrace{\begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) & 0 \\ \sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around the } z\text{-axis}} \cdot \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}. \end{array}$$

(2) Let X be an m-strand braid in $(D, \{(1,0), (2,0), \dots, (m,0)\})$. Similar to Definition 66.3 we define the closure of the braid X as the link[1259]

$$\widehat{X} := \Theta(X).$$

We give $\widehat{X} = \Theta(X)$ the orientation that is induced by the convention that the strands of X go "downward". As we mentioned in Definition 66.3, we have a well-defined map

$$\begin{array}{rcl} \mathrm{B}_m & \to & \mathtt{Link}^{\mathrm{or}}(S^3) \\ [X] & \mapsto & [\widehat{X}]. \end{array}$$



For the record we recall that it follows from the Alexander–Braid Closure Proposition 66.4 together with the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17 that the map

$$\begin{array}{rcl} \bigsqcup_{m\in\mathbb{N}_0} \mathrm{B}^{(1,0),(2,0),\dots,(m,0)}(D) =_{\mathrm{i}} \mathrm{B}_m & \to & \mathtt{Link}^{\mathrm{or}}(S^3) \\ [X] & \mapsto & [\widehat{X}] \end{array}$$

is surjective. Furthermore, the Markov Theorem 66.9 shows that two braids lead to smoothly isotopic links if and only if they are related by the two "Markov moves", i.e. by conjugation and (de-) stabilization. In this chapter we will not make use of these two facts.

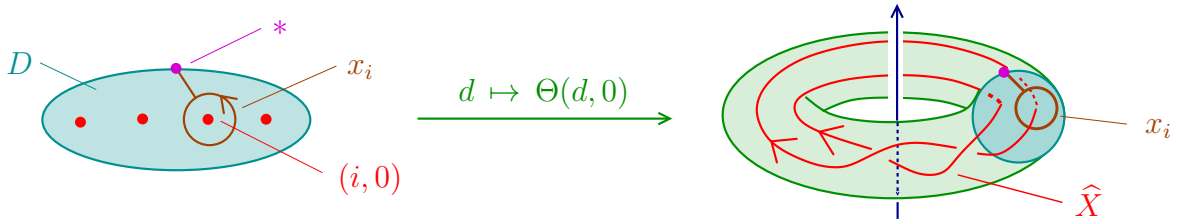
120.3. **Fundamental group from a braid presentation.** In this section we will see how one can obtain a presentation for the fundamental groups $\pi_1(S^3 \setminus \widehat{X})$ in terms of the braid X itself. To formulate this result we first need to specify some elements of $\pi_1(S^3 \setminus \widehat{X})$:

Notation 120.7. Let $m \in \mathbb{N}_0$.

(1) We fix the base point $* := (\frac{m}{2}, 1)$.

(2) We denote by $x_1, \dots, x_m$ the loops in $(D, *)$ that are shown in the figure below to the left.

(3) By a slight abuse of notation we denote by $x_1, \dots, x_m$ also the loops that are given by the images of the loops in D under the map $D \setminus \{(1,0), \dots, (m,0)\} \xrightarrow{d \mapsto \Theta(d,0)} S^3 \setminus \widehat{X}$. Note that each $x_i \in S^3 \setminus \widehat{X}$ is essentially a meridian.



We will soon see that these $x_1, \dots, x_m \in \pi_1(S^3 \setminus \widehat{X})$ form a generating set. To describe the relations among $x_1, \dots, x_m \in \pi_1(S^3 \setminus \widehat{X})$ we will need the following homomorphism:

[1259] It follows easily from Condition (3) in Definition 120.1 that $\Theta(X)$ is indeed a smooth submanifold.

Lemma 120.8. (Twist–π_1–Automorphism Lemma) Let $m \in \mathbb{N}$. There exists a unique homomorphism

$$\tau\colon \mathrm{B}_m \;\to\; \mathrm{Aut}\Big(\underbrace{\langle x_1,\ldots,x_m\rangle}_{\substack{\text{free group on}\\ \text{the set } \{x_1, dots, x_m\}}}\Big)$$

such that

$$\tau(\sigma_i)(x_j) \;=\; \begin{cases} x_j, & \text{if } j \neq i, i+1,\\ x_i \cdot x_{i+1} \cdot x_i^{-1}, & \text{if } j = i,\\ x_i, & \text{if } j = i+1. \end{cases}$$

Proof. First note that for any $i \in \{1,\ldots,m\}$ we have

$$\tau(\sigma_i)(x_j)^{-1} \;=\; \begin{cases} x_j, & \text{if } j \neq i, i+1,\\ x_{i+1}, & \text{if } j = i,\\ x_{i+1}^{-1} \cdot x_i \cdot x_{i+1}, & \text{if } j = i+1. \end{cases}$$

This shows that τ does indeed take values of the automorphisms of $\langle x_1,\ldots,x_m\rangle$.

At this point it remains to show that the given homomorphism

$$\left\langle \sigma_1,\ldots,\sigma_{m-1} \;\middle|\; \begin{array}{ll} \sigma_i\cdot\sigma_{i+1}\cdot\sigma_i = \sigma_{i+1}\cdot\sigma_i\cdot\sigma_{i+1} & \text{for } i = 1,\ldots,m-2\\ \sigma_i\cdot\sigma_j = \sigma_j\cdot\sigma_i & \text{for } |i-j| \geq 2 \end{array} \right\rangle \;\to\; \mathrm{Aut}(\langle x_1,\ldots,x_m\rangle)$$
$$\sigma_i \;\mapsto\; \tau(\sigma_i)$$

is well-defined. By the Presentation–Homomorphism Lemma 100.13 this means that we need to show that the $\tau(\sigma_i) \in \mathrm{Aut}(\langle x_1,\ldots,x_m\rangle)$ satisfy the braid relation and far commutativity. For the braid relation we observe that we have the following two little tables:

	x_i	x_{i+1}	x_{i+2}
$\tau(\sigma_i)$	$x_i \cdot x_{i+1} \cdot x_i^{-1}$	x_i	x_{i+2}
$\tau(\sigma_{i+1}\cdot\sigma_i)$	$x_i \cdot x_{i+1} \cdot x_{i+2} \cdot x_{i+1}^{-1} \cdot x_i^{-1}$	x_i	x_{i+1}
$\tau(\sigma_i\cdot\sigma_{i+1}\cdot\sigma_i)$	$\underbrace{x_i \cdot x_{i+1} \cdot x_i^{-1} \cdot x_i \cdot x_{i+2} \cdot x_i^{-1} \cdot (x_i \cdot x_{i+1} \cdot x_i^{-1})^{-1}}_{= x_i\cdot x_{i+1}\cdot x_{i+2}\cdot x_{i+1}^{-1}\cdot x_i^{-1}}$	$x_i \cdot x_{i+1} \cdot x_i^{-1}$	x_i

and

	x_i	x_{i+1}	x_{i+2}
$\tau(\sigma_{i+1})$	x_i	$x_{i+1} \cdot x_{i+2} \cdot x_{i+1}^{-1}$	x_{i+1}
$\tau(\sigma_i\cdot\sigma_{i+1})$	$x_i \cdot x_{i+1} \cdot x_i^{-1}$	$x_i \cdot x_{i+2} \cdot x_i^{-1}$	x_i
$\tau(\sigma_{i+1}\cdot\sigma_i\cdot\sigma_{i+1})$	$x_i \cdot x_{i+1} \cdot x_{i+2} \cdot x_{i+1}^{-1} \cdot x_i^{-1}$	$x_i \cdot x_{i+1} \cdot x_i^{-1}$	x_i

With a slight sigh of relief we notice that $\tau(\sigma_i\cdot\sigma_{i+1}\cdot\sigma_i) = \tau(\sigma_{i+1}\cdot\sigma_i\cdot\sigma_{i+1})$. This shows that the $\tau(\sigma_i)$ do indeed satisfy the braid relations.

Finally we point out that it is essentially clear that the $\tau(\sigma_i)$ satisfy far commutativity. ∎

To practice the definition of the homomorphism $\tau\colon \mathrm{B}_m \to \mathrm{Aut}(\langle x_1,\ldots,x_m\rangle)$ from the Twist–π_1–Automorphism Lemma 120.8 we now prove the following little lemma, which later on will play an essential role:

Lemma 120.9. (τ-Circle Lemma) Let $m \in \mathbb{N}_0$ and let $X \in \mathrm{B}_m$. In the group $\langle x_1,\ldots,x_m\rangle$ we have the equality

$$\tau(X)(x_1\cdot\ldots\cdot x_m) \;=\; x_1\cdot\ldots\cdot x_m.$$

Proof of the τ-Circle Lemma 120.9. Since $\tau\colon \mathrm{B}_m \to \mathrm{Aut}(\langle x_1,\ldots,x_m\rangle)$ is a homomorphism it suffices to verify the desired conclusion for the generators $X = \sigma_i$, $i = 1,\ldots,m$.

Thus let $i \in \{1, \ldots, m\}$. We see that

$$\begin{aligned} \tau(\sigma_i)(x_1 \cdot \ldots \cdot x_m) &= \tau(\sigma_i)(x_1) \cdot \ldots \cdot \tau(\sigma_i)(x_i) \cdot \tau(\sigma_i)(x_{i+1}) \cdot \ldots \cdot \tau(\sigma_i)(x_m) \\ &= x_1 \cdot \ldots \cdot x_i \cdot x_{i+1} \cdot x_i^{-1} \cdot x_i \cdot \ldots \cdot x_m \\ &\quad \uparrow \text{ by the Twist–}\pi_1\text{–Automorphism Lemma 120.8} \\ &= x_1 \cdot \ldots \cdot x_m. \end{aligned}$$

■

Now we can formulate the main result of this section:

Proposition 120.10. (Braid Closure–π_1–Proposition) Let $m \in \mathbb{N}_0$ and let $X \in \mathrm{B}_m$. We set $* := \Theta((\frac{m}{2}, 1, 0))$.

(1) We have a natural isomorphism

$$\begin{array}{ccc} \langle x_1, \ldots, x_m \,|\, x_j = \tau(X)(x_j), j = 1, \ldots, m \rangle & \xrightarrow{\cong} & \pi_1(S^3 \setminus \widehat{X}, *) \\ [x_j] & \mapsto & [x_j]. \\ & & \uparrow \text{ see Notation 120.7} \end{array}$$

(2) From the presentation in (1) we can drop any one of the m relations, i.e. for any $i \in \{1, \ldots, m\}$ we have a natural isomorphism

$$\begin{array}{ccc} \langle x_1, \ldots, x_m \,|\, x_j = \tau(X)(x_j), j \in \{1, \ldots, \widehat{i}, \ldots, m\} \rangle & \xrightarrow{\cong} & \pi_1(S^3 \setminus \widehat{X}) \\ [x_j] & \mapsto & [x_j]. \end{array}$$

Proof. The next Section 120.4 is completely dedicated to the proof of the Braid Closure–π_1–Proposition 120.10. Afterwards, in the subsequent Section 120.5 we will relate the Burau representation to Fox calculus. ■

Example.

(1) Let L be the trivial m-component link. It is the closure of the trivial m-strand braid T. Clearly $\tau(T) = \mathrm{id}_{\langle x_1, \ldots, x_m \rangle}$. Thus we see that

$$\pi_1(S^3 \setminus L) \underset{\uparrow}{\cong} \langle x_1, \ldots, x_m \,|\, x_i = x_i \rangle = \langle x_1, \ldots, x_m \rangle.$$

by the Braid Closure–π_1–Proposition 120.10

(2) Let U be the unknot. It is in particular the closure of the 2-strand braid σ_1. By the Twist–π_1–Automorphism Lemma 120.8 we have $\sigma(x_1) = x_1 \cdot x_2 \cdot x_1^{-1}$ and $\sigma(x_2) = x_1$. Thus we see that

$$\pi_1(S^3 \setminus U) \underset{\uparrow}{\cong} \langle x_1, x_2 \,|\, x_1 = x_1 \cdot x_2 \cdot x_1^{-1}, x_2 = x_1 \rangle.$$

by the Braid Closure–π_1–Proposition 120.10

One can easily verify that this group is indeed isomorphic to $\langle t \rangle$.

(3) Let H be the Hopf link. It is the closure of the 2-strand braid σ_1^2. One can easily verify that $\Theta(\sigma_1^2)(x_1) = x_1 \cdot x_2 \cdot x_1 \cdot x_2^{-1} \cdot x_1^{-1}$ and $\Theta(\sigma_1^2)(x_2) = x_1 \cdot x_2 \cdot x_1^{-1}$. Thus we see that

$$\pi_1(S^3 \setminus L) \underset{\uparrow}{\cong} \langle x_1, x_2 \,|\, x_1 = x_1 \cdot x_2 \cdot x_1 \cdot x_2^{-1} \cdot x_1^{-1}, x_2 = x_1 \cdot x_2 \cdot x_1^{-1} \rangle = \langle x_1, x_2 \,|\, [x_1, x_2] \rangle \underset{\uparrow}{\cong} \mathbb{Z}^2.$$

by the Braid Closure–π_1 Proposition 120.10

by the Free Abelian Group Presentation Lemma 100.14

□

120.4. **Proof of the Braid Closure–π_1–Proposition 120.10.** This section is organized as follows:

- In Section 120.4.1 we will see that each braid gives rise to a diffeotopy of the disk.
- In Section 120.4.2 we will give a reinterpretation of the homomorphism from the Twist–π_1–Automorphism Lemma 120.8
- In Section 120.4.3 we will conclude the proof of the Braid Closure–π_1–Proposition 120.10.

120.4.1. **Braids and diffeotopies of the disk.** In this subsection we will see that braids give rise to diffeotopies of the disk. We start out with the following definition:

Definition 120.11. Let $m \in \mathbb{N}$ and let $i \in \{1, \ldots, m-1\}$. First note that it follows from the Smooth Transition Function Lemma 24.9 that there exists a monotonously decreasing smooth map

$$\varphi\colon [0,1] \to [0,\pi]$$

with the following properties:

- For $t \in [0, \frac{5}{8}]$ we have $\varphi(t) = \pi$.
- For $t \in [\frac{3}{4}, 1]$ we have $\varphi(t) = 0$

and that there exists a monotonously decreasing smooth map

$$\eta\colon [0,1] \to [0,1]$$

with the following properties:

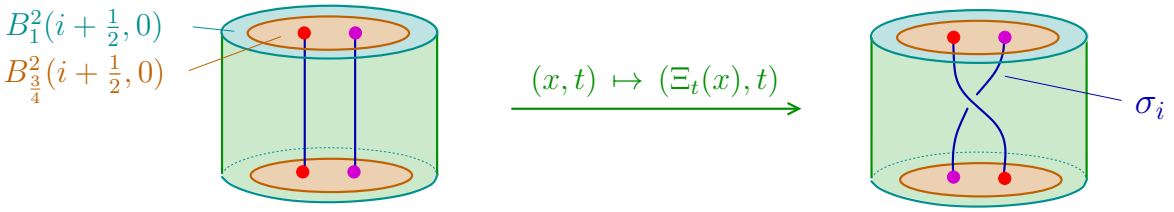
- For $t \in [0, \frac{1}{4}]$ we have $\eta(t) = 0$.
- For $t \in [\frac{3}{4}, 1]$ we have $\eta(t) = 1$.

For $Q \in \mathbb{R}^2$ we consider the diffeotopy[1260]

$$\begin{aligned} \rho_Q\colon \overline{B}_1^2(Q) \times [0,1] &\to \overline{B}_1^2(Q) \\ (P,t) &\mapsto \exp(\eta(t) \cdot \varphi(\|P-Q\|) \cdot \mathrm{i}) \cdot (P-Q) + Q. \end{aligned}$$

Finally we consider the diffeotopy[1261]

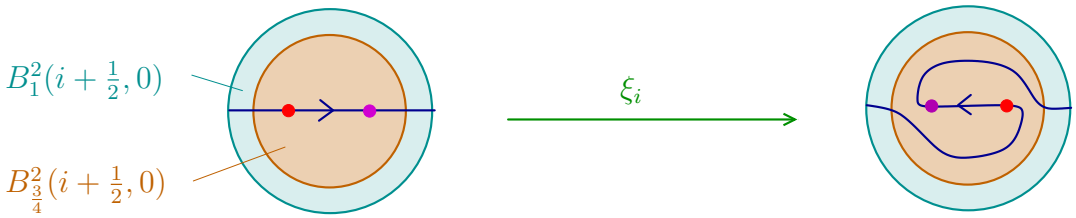
$$\begin{aligned} \Xi_i\colon D \times [0,1] &\to D \\ (P,t) &\mapsto \begin{cases} P, & \text{if } P \notin B_1^2(i+\frac{1}{2},0) \\ \rho_{(i+\frac{1}{2},0)}(P,t), & \text{otherwise.} \end{cases} \end{aligned}$$



Definition 120.11.
(4) We write $\xi_i := \Xi_{i1}\colon D \to D$.

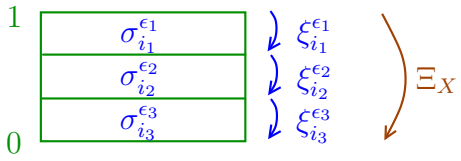
[1260] In other words, ρ_t is counter clockwise rotation around Q by the angle $\varphi(r)$ where r is the distance to the center Q. Note that $\rho_{Q,t}$ is the identity outside of $\overline{B}_{\frac{3}{4}}^2(Q)$ and that on the disk $\overline{B}_{\frac{5}{8}}^2(Q)$ the map $\rho_{Q,1}$ is rotation by π the center Q.

[1261] Note that the diffeotopy is stationary (in the sense of page 510) outside of $\overline{B}_{\frac{3}{4}}^2(i+\frac{1}{2},0)$.



Lemma 120.12. (Braid Closure–Complement Lemma) Let $X := \sigma_{i_1}^{\epsilon_1} \cdot \ldots \cdot \sigma_{i_l}^{\epsilon_l}$ be a word in $\langle \sigma_1, \ldots, \sigma_k \rangle$. Let $\xi_X := \xi_{i_l}^{\epsilon_l} \circ \ldots \circ \xi_{i_1}^{\epsilon_1} \colon D \to D$ be the diffeomorphism that we obtain from composing the diffeomorphisms from Definition 120.11 (and their inverses). Then we have a diffeomorphism

$$\underbrace{\mathrm{Tor}(D \setminus \{(1,0), \ldots, (m,0)\}, \xi_X)}_{\text{mapping torus, see Definition 21.19}} \;\to\; \Theta(D \times [0,1]) \setminus \widehat{X}.$$



Sketch of proof. Let $X := \sigma_{i_1}^{\epsilon_1} \cdot \ldots \cdot \sigma_{i_l}^{\epsilon_l}$ be a word in $\langle \sigma_1, \ldots, \sigma_k \rangle$. We define $\Xi_X \colon D \times [0,1] \to D^2$ as the smooth isotopy that we get by combining[1262] the diffeotopies Ξ_i from Definition 120.11. More precisely, we consider the map

$$\begin{array}{rcl} \Xi_X \colon D \times [0,1] & \to & D \\ (x,t) & \mapsto & \begin{cases} (\Xi_{i_1})_{l\cdot t}^{\epsilon_1}(x), & \text{if } t \in [0, \frac{1}{l}], \\ (\Xi_{i_2})_{l\cdot(t-\frac{1}{l})}^{\epsilon_2}(\xi_{i_1}^{\epsilon_1}(x)), & \text{if } t \in [\frac{1}{l}, \frac{2}{l}], \\ (\Xi_{i_3})_{l\cdot(t-\frac{2}{l})}^{\epsilon_3}((\xi_{i_2}^{\epsilon_2} \circ \xi_{i_1}^{\epsilon_1})(x)), & \text{if } t \in [\frac{2}{l}, \frac{3}{l}], \\ \quad \vdots & \end{cases} \end{array}$$

Note that $\Xi_{X,1} \colon D \to D$ is precisely the map $\xi_X := \xi_{i_l}^{\epsilon_l} \circ \ldots \circ \xi_{i_1}^{\epsilon_1}$. Next we consider the map

$$\begin{array}{rcl} f \colon \mathrm{Tor}(D, \xi_X) & \to & \Theta(D \times [0,1]) \\ [(d,t)] & \mapsto & \Theta(\Xi_X(d,t), t). \end{array}$$

We make a few observations:

- Since $\Xi_{X,1} = \xi_X \colon D \to D$ we see that this map f is well-defined. Furthermore it is easily seen to be a diffeomorphism.
- It follows fairly easily from the definitions that the closure $\widehat{X}$ is smoothly isotopic to $f(\mathrm{Tor}(\{(1,0), \ldots, (m,0)\}, \xi_X)$.

The desired statement now follows either from the Isotopy Extension Theorem 44.2 or from fine tuning the above argument where one defines σ_i to exactly match the ξ_i. We leave it to the reader to fill in all the details. ■

[1261] We refer to the Homotopy Combination Lemma 18.8 for the definition of what we mean by combining diffeotopies.

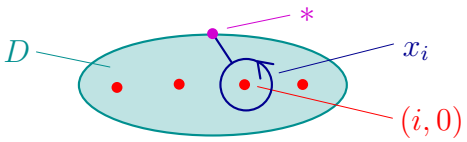
120.4.2. **Diffeomorphisms of the disks and automorphisms of free groups.** In this subsection we will use the diffeomorphisms from the previous subsection to give a reinterpretation of the Twist–π_1–Automorphism Lemma 120.8. To do this we should recall some aspects from Notation 120.7:

Notation 120.13. Let $m \in \mathbb{N}_0$.

(1) We fix the base point $* := (\frac{m}{2}, 1)$.

(2) We denote by $x_1, \dots, x_m$ the loops in $(D, *)$ that are shown in the figure below.

(3) It follows easily from the Seifert–van Kampen-Theorem 99.1 (or more directly from the isomorphism from the Punctured Sphere-π_1-Lemma 106.11) that the natural map $\langle x_1, \dots, x_m \rangle \to \pi_1(D \setminus \{(1,0), \dots, (m,0)\})$ is an isomorphism. In the following we will use this isomorphism to identify these two groups.



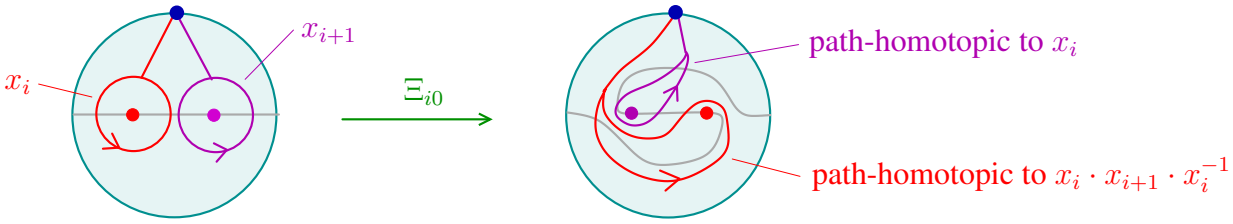
Proposition 120.14. (Twist–π_1–Proposition) Let $m \in \mathbb{N}_0$. The homomorphism

$$\begin{array}{ccc} \mathrm{B}_m & \to & \mathrm{Aut}(\langle x_1, \dots, x_m \rangle) \\ \sigma_i & \mapsto & \xi_{i*} \end{array}$$

ξ_{i*}: the map $\xi_i \colon D \to D$ from Definition 120.11

is precisely the homomorphism from the Twist–π_1–Automorphism Lemma 120.8 from the Twist–π_1–Automorphism Lemma 120.8.

Proof. In the figure below we show the effect of applying the map $\Xi_{i0} \colon D \to D$ to the loops x_i and x_{i+1}.



It should now be clear that we get the desired expression for $\tau(\sigma_i)(x_j)$ from the Twist–π_1–Automorphism Lemma 120.8. ∎

120.4.3. **Conclusion of the proof of the Braid Closure–π_1–Proposition 120.10.** For the reader's convenience we recall the proposition that we need to prove:

Proposition 120.10. (Braid Closure–π_1–Proposition) Let L be the closure of an m-strand braid X.

(1) We have a natural isomorphism

$$\begin{array}{ccc} \langle x_1, \dots, x_m \,|\, x_j = \tau(X)(x_j), j = 1, \dots, m \rangle & \xrightarrow{\cong} & \pi_1(S^3 \setminus L) \\ [x_j] & \mapsto & [x_j]. \end{array}$$

(2) From the presentation in (1) we can drop any one of the m relations, i.e. for any $i \in \{1, \ldots, m\}$ we have a natural isomorphism

$$\begin{array}{rcl} \langle x_1, \ldots, x_m \,|\, x_j = \tau(X)(x_j), j \in \{1, \ldots, \widehat{i}, \ldots, m\}\rangle & \xrightarrow{\cong} & \pi_1(S^3 \setminus L) \\ [x_j] & \mapsto & [x_j]. \end{array}$$

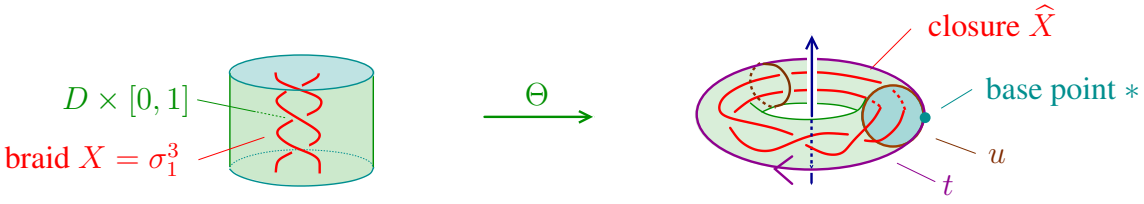
Proof of Braid Closure–π_1–Proposition 120.10 (1). Let $X \in \mathrm{B}_m$ be a braid. Similar to the Sphere–Solid Tori-Decomposition Lemma 111.6 we can write

$$S^3 \setminus \widehat{X} \;=\; \Theta((D \times [0,1]) \setminus \widehat{X}) \cup_T B$$

where $B = S^3 \setminus \Theta(\mathring{D} \times [0,1])$ is a solid torus and where $T = \Theta(\partial D \times [0,1]) = \partial B$ is a 2-dimensional torus. Evidently the idea is to apply the Manifold–Seifert-van Kampen Theorem 102.1 to the above decomposition of $S^3 \setminus \widehat{X}$. We make the following preparations:

(a) We pick the base point $* = \Theta(1,0) \in T = \Theta(\partial D \times [0,1])$.
(b) We consider the loops $t := \widetilde{t} := \Theta(1 \times [0,1])$ and $u := \widetilde{u} := \widehat{u} := \Theta(\partial D \times \{0\})$.
(c) It follows from the discussion on page 1806 together with the Free Abelian Group-Presentation Lemma 100.14 that $\pi_1(T, *) = \langle \widetilde{t}, \widetilde{u} \,|\, [\widetilde{t}, \widetilde{u}]\rangle$.
(d) It follows from the discussion on page 1806 together with the Homotopy Equivalence-π_1-Proposition 90.9 that $\pi_1(B, *) = \langle \widehat{u}\rangle$.
(e) It follows easily from the Braid Closure–Complement Lemma 120.12 together with the Mapping Torus-π_1-Proposition 93.13 and the Semidirect Product-Presentation Lemma 100.21 that we have an isomorphism

$$\langle x_1, \ldots, x_m, t \,|\, t \cdot x_i \cdot t^{-1} = \tau(X)(x_i), i = 1, \ldots, m\rangle \;\to\; \pi_1(\Theta(D \times [0,1]) \setminus \widehat{X}, *).$$



We now see that we have the following isomorphisms:

$$\begin{array}{c} \overbrace{\pi_1(\Theta(D \times [0,1]) \setminus \widehat{X}) \cup_T B, *)}^{= S^3 \setminus \widehat{X}} \\ \uparrow \cong \text{ (1)} \\ \pi_1(\Theta(D \times [0,1]) \setminus \widehat{X}, *) *_{\pi_1(T,*)} \pi_1(B, *) \\ \uparrow \cong \text{ (2)} \\ \langle x_1, \ldots, x_m, t \,|\, t \cdot x_i \cdot t^{-1} = \tau(X)(x_i), i = 1, \ldots, m\rangle *_{\langle \widetilde{t}, \widetilde{u} \,|\, [\widetilde{t}, \widetilde{u}]\rangle} \langle \widehat{u}\rangle \\ \uparrow \cong \text{ (3)} \\ \langle x_1, \ldots, x_m, t, \widehat{u} \,|\, t \cdot x_i \cdot t^{-1} = \tau(X)(x_i), t = e, x_1 \cdot \ldots \cdot x_m = \widehat{u}\rangle \\ \uparrow \cong \text{ (4)} \\ \langle x_1, \ldots, x_m \,|\, x_i = \tau(X)(x_i), i = 1, \ldots, m\rangle. \end{array}$$

Here are the justifications for the isomorphisms:

(1) This isomorphism is given by the Manifold–Seifert-van Kampen Theorem 102.1.

(2) This isomorphism is given by the discussion at the beginning of the proof. We make two observations:
- For the map $\pi_1(\Theta(D \times [0,1]) \setminus X) \leftarrow \pi_1(T)$ we clearly have $t \leftarrow\!\shortmid \widetilde{t}$, $x_1 \cdot \ldots \cdot x_m \leftarrow\!\shortmid \widetilde{u}$.
- For the map $\pi_1(T, *) \to \pi_1(B, *)$ we clearly have $\widetilde{t} \mapsto e$ and $\widetilde{u} \mapsto \widehat{u}$.

(3) This isomorphism is given by the Amalgamated Product–Presentation Lemma 100.20.
(4) This isomorphism is given by the Generator Elimination Lemma 100.30. ■

Proof of Braid Closure–π_1–Proposition 120.10 (2). Let $X \in \mathrm{B}_m$. In the following we write $\tau := \tau(X)$. Furthermore let $i \in \{1, \ldots, m\}$. By (1) it remains to show that the map

$$\begin{array}{rccc} \Phi\colon & \langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,\widehat{i},\ldots,m} \rangle & \to & \langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,m} \rangle \\ & [x_j] & \mapsto & [x_j] \end{array}$$

is an isomorphism. We will do so by writing down an inverse map:

Claim.
(1) The map

$$\begin{array}{rccc} \Psi\colon & \langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,m} \rangle & \to & \langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,\widehat{i},\ldots,m} \rangle \\ & [x_j] & \mapsto & [x_j] \end{array}$$

is well-defined.
(2) The maps Φ and Ψ are inverses of one another.

Proof.
(1) Let $i \in \{1, \ldots, m\}$. It follows easily from the Presentation–Homomorphism Lemma 100.13 that it suffices to show that in the group $\langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,\widehat{i},\ldots,m} \rangle$ we have $\tau(x_i) = x_i$. Indeed it turns out that we have the following equalities in the group $\langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,\widehat{i},\ldots,m} \rangle$:

$$\begin{array}{rcl} x_1 \cdot \ldots \cdot x_{i-1} \cdot \tau(x_i) \cdot x_{i+1} \cdot \ldots \cdot x_m & \overset{\text{since } \tau(x_j) = x_j \text{ for } j \neq i}{\overset{\downarrow}{=}} & \tau(x_1) \cdot \ldots \cdot \tau(x_m) \quad = \quad \tau(x_1 \cdot \ldots \cdot x_m) \\ & = & x_1 \cdot \ldots \cdot x_m \\ & \overset{\text{by the } \tau\text{-Circle Lemma 120.9}}{\overset{\uparrow}{=}} & \\ & = & x_1 \cdot \ldots \cdot x_{i-1} \cdot x_i \cdot x_{i+1} \cdot \ldots \cdot x_m. \end{array}$$

It follows that $\tau(x_i) = x_i \in \langle x_1, \ldots, x_m \,|\, \{x_j = \tau(x_j)\}_{j=1,\ldots,\widehat{i},\ldots,m} \rangle$.
(2) It is clear that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are both the identity on the obvious generating sets. So both are the identity. Thus we have shown that Φ and Ψ are inverses of one another. ■

120.5. **The Burau representation and Fox calculus.** In this section we will relate the Burau representation to Fox derivatives. For the reader's convenience we first recall Definition 65.2 of the Burau representation:

Definition 120.15. Let $m \in \mathbb{N}_{\geq 2}$. By the Burau Representation Proposition 65.1 we know that the map

$$\begin{array}{rcl} \beta_m \colon \mathrm{B}_m & \to & \mathrm{GL}(n, \mathbb{Z}[t^{\pm 1}]) \\ \sigma_i & \mapsto & \begin{pmatrix} \mathrm{id}_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{m-i-2} \end{pmatrix} \end{array}$$

is a well-defined homomorphism. It is called the Burau representation of B_m.

The following proposition gives us an alternative description of the Burau representation which is much less ad hoc:

Proposition 120.16. (Burau Representation–Fox Calculus Proposition) In the following let $m \in \mathbb{N}$. We denote by $\Phi \colon \langle x_1, \ldots, x_m \rangle \to \langle t \rangle$ the epimorphism that is given by $\Phi(x_i) = t$ for $i = 1, \ldots, m$. For every m-strand braid X we have the following equality in $\mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$:

$$\beta_m(X) = \Big(\underbrace{\Phi\Big(\frac{\partial \tau(X)(x_j)}{\partial x_i} \Big)}_{ji\text{-entry}} \Big)_{i,j=1,\ldots,m}.$$

In the proof of the Burau Representation–Fox Calculus Proposition 120.16 we will need the following lemma, which is also of independent interest:

Lemma 120.17. (Fox Derivative–Chain Rule) Let F be a finitely generated free group with basis $x_1, \ldots, x_m$ and let $\lambda \colon F \to F$ be any isomorphism. Given any $g \in F$ and given any $i \in \{1, \ldots, m\}$ we have the following equality in $\mathbb{Z}[F]$:

$$\frac{\partial \lambda(g)}{\partial x_i} = \sum_{r=1}^{m} \lambda\Big(\frac{\partial g}{\partial x_r} \Big) \cdot \frac{\partial \lambda(x_r)}{\partial x_i}.$$

Proof of the Fox Derivative–Chain Rule 125.13. We need to prove the desired equality for any $g \in F$. Since $x_1, \ldots, x_m$ is a basis for X it suffices to prove the following claim:

Claim.

(a) The desired equality holds for any $g = x_i$.
(b) The desired equality holds for any $g = x_i^{-1}$.
(c) If the desired equality holds for $u, v \in X$, then it also holds for $g = u \cdot v$.

Proof.

(a) It follows easily from $\frac{\partial x_j}{\partial x_k} = \delta_{ik}$ that the statement holds for any $g = x_i$.
(b) For $g = x_i^{-1}$ we have

$$\begin{aligned} \frac{\partial \lambda(x_i^{-1})}{\partial x_j} &= \frac{\partial (\lambda(x_i)^{-1})}{\partial x_j} \underset{\uparrow}{=} -\lambda(x_i)^{-1} \cdot \frac{\partial \lambda(x_i)}{\partial x_j} \underset{\uparrow}{=} -\lambda(x_i)^{-1} \cdot \sum_{r=1}^{m} \lambda\Big(\frac{\partial x_i}{\partial x_r} \Big) \cdot \frac{\partial \lambda(x_r)}{\partial y_j} \\ &\quad \text{by the Fox Derivative Properties Lemma 116.5 (5)} \qquad \text{by (a)} \\ &\underset{\uparrow}{=} \sum_{r=1}^{m} \lambda\Big(-x_i^{-1} \cdot \frac{\partial x_i}{\partial x_r} \Big) \cdot \frac{\partial \lambda(x_r)}{\partial y_j} \qquad = \sum_{r=1}^{m} \lambda\Big(\frac{\partial x_i^{-1}}{\partial x_r} \Big) \cdot \frac{\partial \lambda(x_r)}{\partial y_j}. \\ &\quad \text{since } \lambda \text{ is a homomorphism} \qquad \text{by the Fox Derivative Lemma 116.5 (5)} \end{aligned}$$

(c) Suppose the desired equality holds for $u, v \in X$. Then the following calculation shows that the desired equality also holds for $u \cdot v$:

$$\begin{aligned}
\frac{\partial\lambda(u\cdot v)}{\partial x_j} &\overset{\text{by the Leibniz Rule 116.3 (2) and since } \lambda(u\cdot v)=\lambda(u)\cdot\lambda(v)}{=} \frac{\partial\lambda(u)}{\partial x_j} + \lambda(u)\cdot\frac{\partial\lambda(v)}{\partial y_j} \\
&\overset{\text{since } u \text{ and } v \text{ satisfy the desired equality}}{=} \sum_{r=1}^{m} \lambda\Big(\frac{\partial u}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} + \lambda(u)\cdot\sum_{r=1}^{m}\lambda\Big(\frac{\partial v}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} \\
&\overset{\text{since } \lambda \text{ is a homomorphism}}{=} \sum_{r=1}^{m}\lambda\Big(\frac{\partial u}{\partial x_r} + u\cdot\frac{\partial v}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} \\
&\overset{\text{by the Leibniz Rule 116.3 (2)}}{=} \sum_{r=1}^{m}\lambda\Big(\frac{\partial(u\cdot v)}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}.
\end{aligned}$$

■

Proof of the Burau Representation–Fox Calculus Proposition 120.16. We start out with the most basic case:

Claim 1. The desired statement holds whenever $X = \sigma_i$ or $X = \sigma_i^{-1}$.

Proof. First we deal with $X = \sigma_i$. Here we see that

$$\begin{aligned}
\Phi\begin{pmatrix} \frac{\partial\tau(\sigma_i)(x_i)}{\partial x_i} & \frac{\partial\tau(\sigma_i)(x_i)}{\partial x_{i+1}} \\ \frac{\partial\tau(\sigma_i)(x_{i+1})}{\partial x_i} & \frac{\partial\tau(\sigma_i)(x_{i+1})}{\partial x_{i+1}} \end{pmatrix} &\overset{\text{by the Twist–}\pi_1\text{–Automorphism Lemma 120.8}}{=} \Phi\begin{pmatrix} \frac{\partial(x_i\cdot x_{i+1}\cdot x_i^{-1})}{\partial x_i} & \frac{\partial(x_i\cdot x_{i+1}\cdot x_i^{-1})}{\partial x_{i+1}} \\ \frac{\partial x_i}{\partial x_i} & \frac{\partial x_i}{\partial x_{i+1}} \end{pmatrix} \\
&\overset{\text{by the Fox Derivative Proposition 116.3}}{=} \Phi\begin{pmatrix} 1 - x_i\cdot x_{i+1}\cdot x_i^{-1} & x_i \\ 1 & 0 \end{pmatrix} \\
&\overset{\text{definition of } \Phi}{=} \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

But the right-hand matrix is exactly the interesting part of $\beta_m(\sigma_i)$, as defined in the Burau Representation Proposition 65.1. Now we turn to $X = \sigma_i^{-1}$. Similar to the above we see that

$$\begin{aligned}
\Phi\begin{pmatrix} \frac{\partial\tau(\sigma_i^{-1})(x_i)}{\partial x_i} & \frac{\partial\tau(\sigma_i^{-1})(x_i)}{\partial x_{i+1}} \\ \frac{\partial\tau(\sigma_i^{-1})(x_{i+1})}{\partial x_i} & \frac{\partial\tau(\sigma_i^{-1})(x_{i+1})}{\partial x_{i+1}} \end{pmatrix} &= \Phi\begin{pmatrix} \frac{\partial x_{i+1}}{\partial x_i} & \frac{\partial x_{i+1}}{\partial x_{i+1}} \\ \frac{\partial(x_{i+1}^{-1}\cdot x_i\cdot x_{i+1})}{\partial x_i} & \frac{\partial(x_{i+1}^{-1}\cdot x_i\cdot x_{i+1})}{\partial x_{i+1}} \end{pmatrix} \\
&= \Phi\begin{pmatrix} 0 & 1 \\ x_{i+1}^{-1} & -x_{i+1}^{-1} + x_{i+1}^{-1}\cdot x_i \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 0 & 1 \\ t^{-1} & -t^{-1}+1 \end{pmatrix}}_{=\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}^{-1}}.
\end{aligned}$$

⊞

Claim 2. Let $X, Y \in \mathrm{B}_m$. If the desired equality holds for X and Y, then it also holds for $X \cdot Y$.

Proof. Since the σ_i are a generating set for B_m and since $\beta_m\colon \mathrm{B}_m \to \mathrm{GL}(m, \mathbb{Z}[t^{\pm 1}])$ is a homomorphism it suffices to show that the right-hand side of the proposition is also a

homomorphism. In other words we have to show that

$$\begin{array}{rcl} \mathrm{B}_m & \to & \mathrm{GL}(n, \mathbb{Z}[t^{\pm 1}]) \\ X & \mapsto & \Big(\underbrace{\Phi\Big(\frac{\partial \tau(X)(x_j)}{\partial x_i} \Big)}_{ji\text{-entry}} \Big)_{i,j=1,\dots,m} \end{array}$$

is a group homomorphism. This follows from the following equalities:

$$\Phi\Big(\frac{\partial \tau(X \cdot Y)(x_j)}{\partial x_i} \Big) = \Phi\Big(\frac{\partial \tau(X)(\tau(Y)(x_j))}{\partial x_i} \Big) = \Phi\Big(\sum_{r=1}^{m} \tau(X)\Big(\frac{\partial \tau(Y)(x_j)}{\partial x_r} \Big) \cdot \frac{\partial \tau(X)(x_r)}{\partial x_i} \Big)$$

$\uparrow$ the Fox Derivative–Chain Rule 120.17 applied to $\lambda = \tau(X)$ and $g = \tau(Y)(x_j)$

$$= \sum_{r=1}^{m} \Phi\Big(\frac{\partial \tau(Y)(x_j)}{\partial x_r} \Big) \cdot \Phi\Big(\frac{\partial \tau(X)(x_r)}{\partial x_i} \Big)$$

$\uparrow$ since clearly $\Phi \circ \tau(X) = \Phi$

■

120.6. **The single-variable Alexander polynomial via Fox calculus.** Note that now we gave two quite different definitions of the Alexander polynomial of an oriented m-component link $L \subseteq S^3$:

- In Definition 67.2 we used the Burau representation to define the (single-variable) Alexander polynomial $\Delta_L(t)$.
- In Definition 117.2 and 117.4 we used Fox calculus to introduce the (multivariable) Alexander polynomial $\Delta_L(t_1, \dots, t_m)$.

To explain the relationship between these two notions we will now use Fox calculus to introduce a single-variable Alexander polynomial of links:

Definition 120.18. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link. As usual we denote by $\mu_{L_1}, \dots, \mu_{L_m}$ the meridians of the m components of L. By the Link Group-Abelianization Proposition 113.9 there exists a unique epimorphism

$$\widehat{\Phi}_L \colon \pi_1(S^3 \setminus L) \to \langle t \rangle_{\mathrm{ab}}$$

such that for $i = 1, \dots, m$ we have $\widehat{\Phi}_L(\mu_{L_i}) = t$. By the Wirtinger Presentation Proposition 113.6 we know that $\pi_1(S^3 \setminus L)$ admits a presentation of deficiency one. Thus it makes sense to define

$$\Delta_L = \Delta_L(t_1, \dots, t_m) \coloneqq (t-1) \cdot \underbrace{\Delta_{\pi_1(S^3 \setminus L), \widehat{\Phi}_L}}_{\text{see Definition 116.14}} \in \mathbb{Q}(t)$$

as the single-variable Alexander polynomial of L. The argument from the Knot–Alexander Polynomial Proposition 117.3 shows that $\Delta_L(t)$ is well-defined up to multiplication by $\pm t^k$, that $\Delta_L(t)$ lies in $\mathbb{Z}[t^{\pm 1}]$ and that $\Delta_L(t)$ is a smooth isotopy invariant of oriented links.

The following proposition explains the relationship between the single-variable and the multivariable Alexander polynomials (both defined in terms of Fox calculus) of links:

Proposition 120.19. (Multi-Single Variable-Alexander Polynomial Proposition) Let $m \in \mathbb{N}_{\geq 2}$ and let $L \subseteq S^3$ be an oriented m-component link. Then

$$\Delta_L(t, \ldots, t) \;=\; \tfrac{\Delta_L(t)}{t-1}.$$

Proof. The statement follows almost immediately from the Alexander Function–Functorial Lemma 116.15 (2). ■

120.7. **Equivalent definitions of the single-variable Alexander polynomial.** To a non-empty oriented link L we can now associate two single-variable Alexander polynomials: The one from Definition 67.2 (where we used the Burau representation) and the one from Definition 120.18 (where we used Fox calculus). In the following we want to show that these two concepts are the same:

Proposition 120.20. (Single Variable-Alexander Polynomials Agree Proposition) Let $m \in \mathbb{N}$ and let L be an oriented m-component link. By the Alexander–Braid Closure Proposition 66.4 we can pick an m-strand braid $X \in \mathrm{B}_m$ such that $\widehat{X}$ is smoothly isotopic to L. The following objects agree:[1263]

(1) The single-variable Alexander polynomial from Definition 67.2, i.e.

$$\Delta_L(t) \;=\; (-1)^{m-1} \cdot \tfrac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1}).$$

(2) The single-variable Alexander polynomial from Definition 120.18.

Remark.

(1) The proposition is stated in [**Bir1974**, Proposition 3.1], full details were given in [**Gol1990**, Theorem 1].
(2) Later in the Alexander Polynomials Agree Proposition **??** we will see that there exist many other alternative approaches to defining the Alexander polynomial of an oriented link. □

Proof. Let $m \in \mathbb{N}$ and let $X \in \mathrm{B}_m$. We introduce some objects and some notation:

(a) By the Braid Closure–π_1–Proposition 120.10 we have the deficiency one presentation

$$\begin{array}{ccc} \langle x_1, \ldots, x_m \,|\, x_j = \tau(X)(x_j), j \in \{1, \ldots, m-1\}\rangle & \xrightarrow{\cong} & \pi_1(S^3 \setminus \widehat{X}) \\ [x_j] & \mapsto & [x_j]. \end{array}$$

(b) Recall that each x_i is a meridian, so we have the natural epimorphism $\Phi_{\widehat{X}}$ to $\langle t\rangle$ that is given by $\Phi_{\widehat{X}}(x-i) = t$ for $i = 1, \ldots, m$.
(c) As in the Burau Representation–Fox Calculus Proposition 120.16 we consider the epimorphism $\Phi\colon \langle x_1, \ldots, x_m\rangle \to \langle t\rangle$ that is given by $\Phi(x_i) = t$ for $i = 1, \ldots, m$.
(d) Given a ring R, a matrix $A \in \mathrm{M}(m \times m, R)$ and $i, j \in \{1, \ldots, m\}$ we denote, as in Definition 2.20 and Notation 67.4, by $A_{\widehat{i},\widehat{j}}$ the matrix that is obtained from A by removing the i-th row and the j-th column.

[1263] By the discussion in Definition 120.18 we know that the Alexander polynomial defined via Fox calculus is well-defined only up to multiplication by a factor of the form $\pm t^k$ with $k \in \mathbb{Z}$.

Next note that up to multiplication by a factor of the form $\pm t^k$ we have the following equalities:

$$\Delta_{\widehat{X}}(t) \text{ as in (1)} \underset{\uparrow}{=} \frac{1-t}{1-t^m} \cdot \det(\beta_m^{\mathrm{red}}(X) - \mathrm{id}_{m-1})$$

Definition 67.2

$$\underset{\uparrow}{=} \det\left(\Phi\left(\frac{\partial \tau(X)(x_j)}{\partial x_i}\right)_{i,j=1,\dots,m-1} - \mathrm{id}_{m-1}\right)$$

by the Burau Representation Fox Calculus Proposition 120.16

$$\underset{\uparrow}{=} \Delta_{\widehat{X}}(t) \text{ as in (2)}.$$

see Definition 120.18

$$\underset{\uparrow}{=} \det((\beta_m(X) - \mathrm{id}_m)_{\widehat{m},\widehat{m}})$$

by the Alexander Polynomial-via-Burau Representation 67.5

$$\underset{\uparrow}{=} (t-1) \cdot \Delta_{\pi_1(S^3 \setminus \widehat{X}),\widehat{\Phi}_{\widehat{X}}}$$

Definition 116.14 applied to the presentation from (a)

■

120.8. **Exercises for Chapter 120.**

Exercise 120.1.

(a) Use the Braid Closure–π_1–Proposition 120.10 to find a presentation for the fundamental group of the complement of a trefoil knot.

(b) Give a purely algebraic proof that the presentation from (a) is isomorphic to the group given by the presentation $\langle x, y \mid x^2 = y^3 \rangle$ from page 2159.

Exercise 120.2.

(a) Find a braid such that its closure is smoothly isotopic to the figure-8 knot.

(b) Use the Braid Closure–π_1–Proposition 120.10 to find a presentation for the fundamental group of the complement of a figure-8 knot.

CHAPTER 121

Seifert surfaces

A Seifert surface for a knot K is defined as a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. In this chapter we will give many examples of Seifert surfaces and we will see that every knot admits a Seifert surface. We will also introduce the "genus of a knot" and we will see that it is additive under the connected sum operation.

In the next Chapter 122 we will introduce the related notion of a "fibered knot". Afterwards, in Chapter 123 we will see that the Alexander polynomial of a knot contains information about the "genus" and the "fiberedness" of a knot.

This chapter is organized as follows:

- In Section 121.1 we will introduce Seifert surfaces and we will show that every knot bounds a Seifert surface.
- In Section 121.2 we will introduce the genus of a surface.
- In Section 121.3 we will introduce the genus of a knot.
- In Section 121.4 we will show in the Genus–Connected Sum Proposition 121.12 that the genus of a knot is additive with respect to the connected sum operation.
- In Section 121.5 we will state the Knot–Prime Decomposition Theorem 121.18.
- In Appendix 121.6 we will prove the Interval–Isotopy Lemma 121.15.
- In Section 121.7 we will provide exercises for Chapter 121.

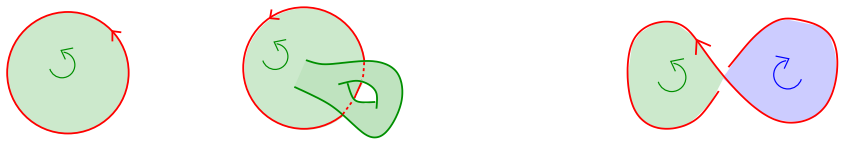
In the next Chapter 122 we will introduce the notion of a fibered knot.

121.1. **Seifert surfaces.** The following definition introduces the key new concept of this chapter:

Definition 121.1. Let $K \subseteq S^3$ be a knot. A Seifert surface[1264] for K is a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented smooth manifolds.

Examples.

(1) In the figure below we show Seifert surfaces for the trivial knot.



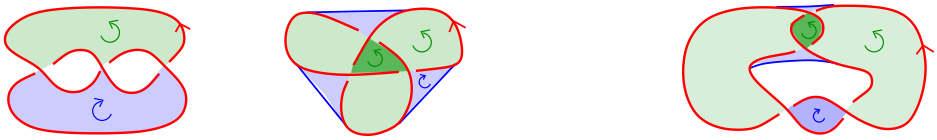
"standard" trivial knot

"8-shaped" trivial knot

[1264]Seifert surfaces are named after Herbert Seifert (1907-1996) who we had already encountered at the Seifert-van Kampen-Theorem 110.8.

To the left we see two different Seifert surfaces for the "standard" trivial knot and to the right we see a "'twisted disk" that is a Seifert surface for the "8-shaped" trivial knot.

(2) In the figure below we show Seifert surfaces for the trefoil (in two guises) and the figure-8 knot.

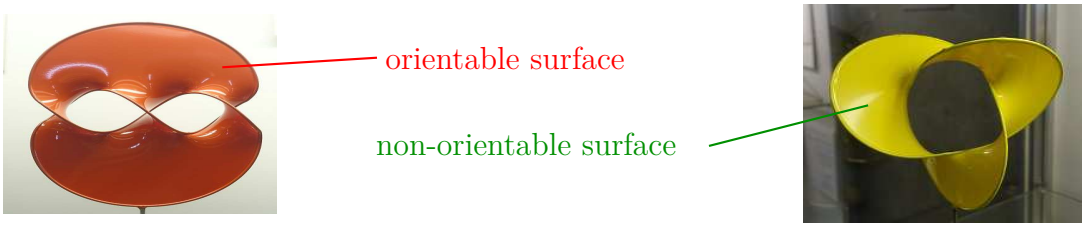


trefoil in two different guises

figure-8 knot

In the figure we illustrate the surface as a "two-sided" object with a green side and a blue side. This illustrates that the surface is orientable. The areas which are shaded in darker green are "above" the areas that are shaded in light green.

(3) In the figure below we see two surfaces of minimal surface area that bound the trefoil, both were manufactured by Jürgen Neukirch.



The one on the left is orientable, thus it is a Seifert surface. The one on the right is diffeomorphic to the Möbius band, thus it is non-orientable, which means that it does not count as a Seifert surface. □

The following theorem is arguably one of the most consequential results in knot theory:

Proposition 121.2. (Seifert Surface Existence Theorem) Every knot $K \subseteq S^3$ admits a Seifert surface.

Remark. The above theorem was first proved in 1930 by F. Frankl and Lev Pontryagin [**FraP1930**]. Shortly afterwards Herbert Seifert [**Sei1934**] gave a different proof which is the one we present below. The proof below is very visual, explicit and practical. It is arguably though not the most rigorous proof. We will make up for this deficiency by giving a much more formal, but much less explicit, proof later on page 5447. □

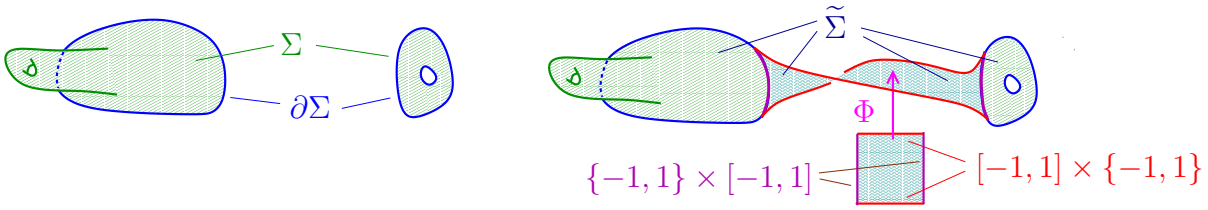
The proof of the Seifert Surface Existence Theorem 121.2 requires some preparations. First of all we need to introduce the following definition:

Definition 121.3. Let Σ be a 2-dimensional smooth submanifold of some smooth manifold M. (In most of our applications we work with $M = \mathbb{R}^3$ or $M = S^3$.) We say that a smooth submanifold $\widetilde{\Sigma}$ of M is obtained from Σ by a band attachment if there exists an embedding $\Phi\colon [-1,1] \times [-1,1] \to \widetilde{\Sigma}$ such that the following statements hold:[1265]

(1) $\widetilde{\Sigma} = \Sigma \cup \Phi([-1,1] \times [-1,1])$.

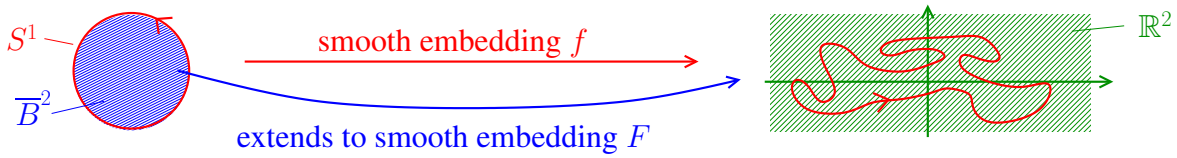
(2) $\Phi([-1,1] \times [-1,1]) \cap \Sigma = \Phi([-1,1] \times [-1,1]) \cap \partial\Sigma = \Phi(\{-1,1\} \times [-1,1])$.

(3) $\partial\widetilde{\Sigma} = (\partial\Sigma \setminus \Phi(\{-1,1\} \times (-1,1)) \cup \Phi([-1,1] \times \{-1,1\})$.



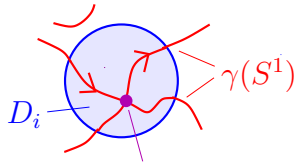
We will also need the following proposition.

Proposition 107.15. (Fill-in-Disk Proposition) Given any smooth embedding $f\colon S^1 \to \mathbb{R}^2$ there exists a smooth embedding $F\colon \overline{B}^2 \to \mathbb{R}^2$ with $F|_{S^1} = f$.

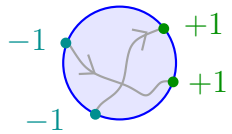


Sketch of proof of the Seifert Surface Existence Theorem 121.2. Let $K \subseteq S^3$ be a knot. It follows from the Knot Diagram Existence Proposition 112.7 and the Isotopy Extension Theorem 44.2 that we only have to deal with the case that K lies in $\mathbb{R}^3$ and that it arises from a knot diagram $(\gamma\colon S^1 \to \mathbb{R}^2, c)$. We equip K with an orientation and we perform the following three steps:

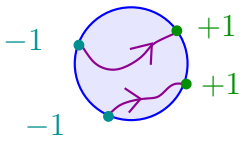
(1) First step we "resolve the crossings" in the diagram. More precisely we do the following:
   (a) We enumerate the crossings by $c_1, \ldots, c_n$.
   (b) We pick disjointly smoothly embedded closed disks $D_1, \ldots, D_n \subseteq \mathbb{R}^2$ with the following properties:
      - Each disk D_i contains precisely one crossing, namely c_i.
      - Each ∂D_i intersects $\gamma\colon S^1 \to \mathbb{R}^2$ transversally.
      - Each $D_i \cap \gamma(S^1)$ is the union of two oriented properly smoothly embedded intervals.
   (c) For each disk D_i we do the following: We consider the four *oriented* boundary points of the two oriented intervals. One can easily see that in the disk we can pick two *disjoint* oriented properly smoothly embedded intervals such that the boundary is precisely given by the four oriented points. We can arrange that in a small neighborhood of ∂D_i the new intervals agree with the original intervals.



crossing c_i



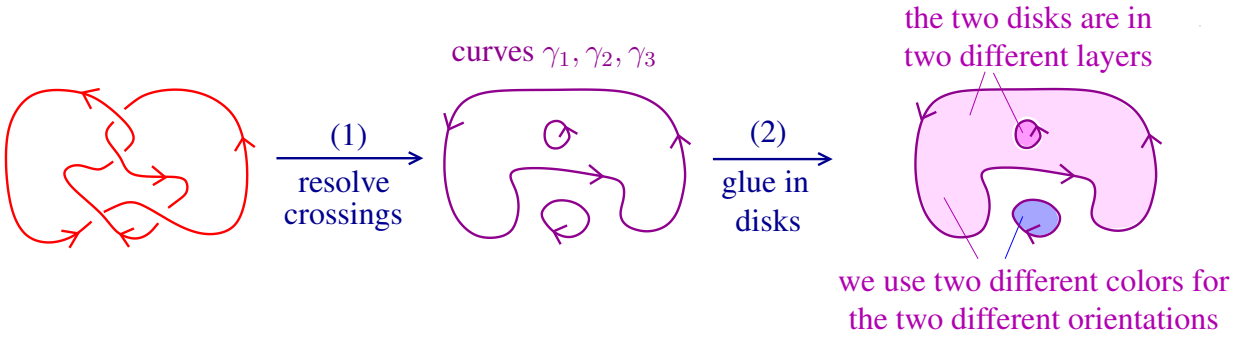
we consider the oriented boundary of the four intervals



we connect the endpoints by two intervals without crossing

[1265]The notion of a "band attachment" is closely related to the notion of "1-handle attachment" from Definition 246.4. The main difference is that the former is an operation on submanifolds whereas the latter is an operation on "abstract" smooth manifolds.

(2) (a) From the operation in (1) we obtain a collection of disjoint oriented smoothly embedded closed curves $\gamma_1, \dots, \gamma_k$ in $\mathbb{R}^2$. It follows from the Fill-in-Disk Proposition 107.15 that each of these smoothly embedded closed curves bounds a smoothly embedded closed disk $B_1, \dots, B_k \subseteq \mathbb{R}^2$. We equip these disks with the orientation coming from the oriented smoothly embedded closed curves $\gamma_1, \dots, \gamma_k$.



(b) We pick $t_1, \dots, t_k \in \mathbb{R}$ which satisfy the inequality $t_i < t_j$ whenever $B_j \subseteq B_i$.
(c) We consider the "wedding cake" $W \coloneqq (B_1 \times \{t_1\}) \cup \dots \cup (B_k \times \{t_k\}) \subseteq \mathbb{R}^3$.

(3) For each crossing we attach a band to W according to the type of crossing we started out with. More precisely: In the figure below we show this procedure for a positive crossing, it should then be clear, how the procedure works for a negative crossing.

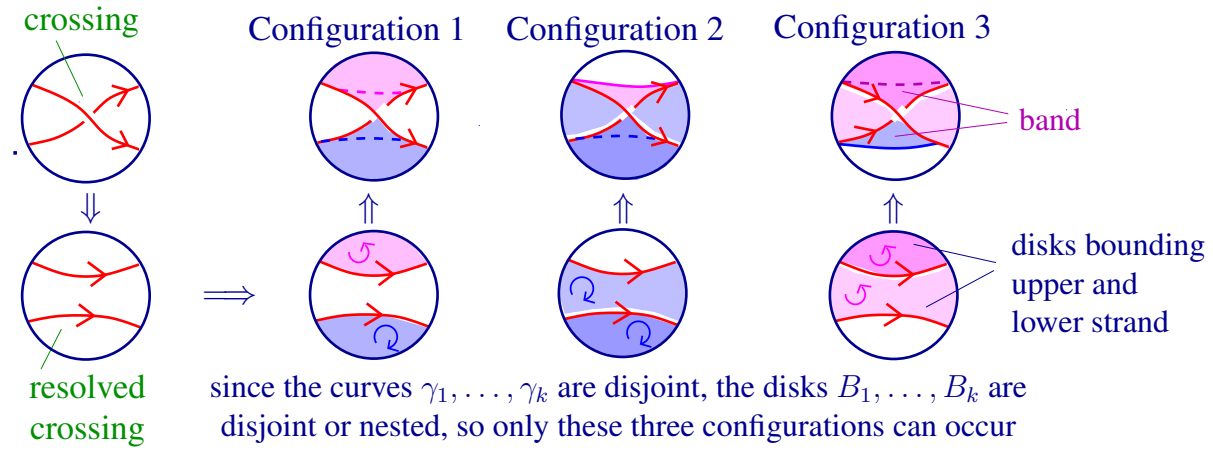
For the figure we make the following clarifications and observations:

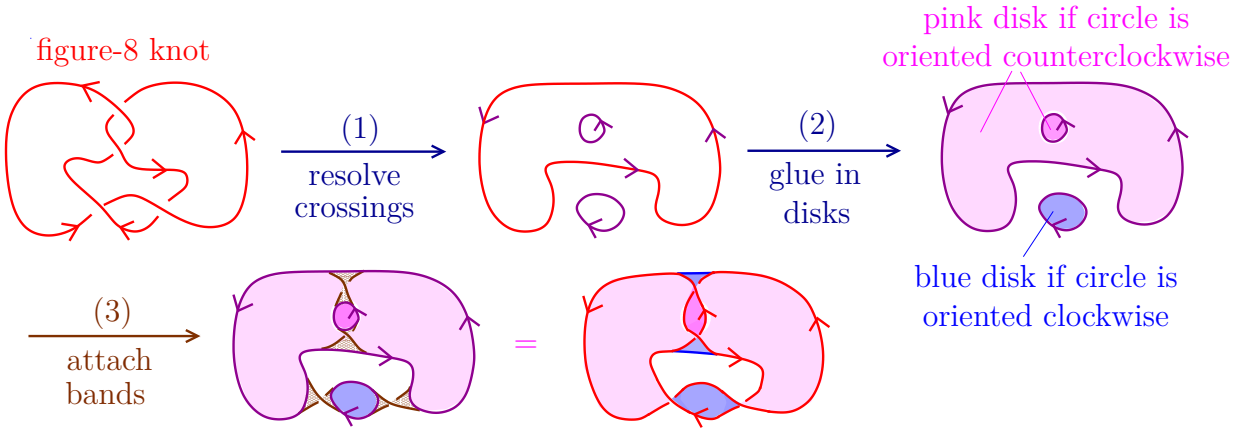
(a) After a smooth isotopy we can assume that we are in the situation on the left.
(b) The upper and lower interval each sit on the boundary of one of the oriented disks $B_1, \dots, B_k$. With respect to the y-coordinates these disks can be "above" or "below" the intervals of the resolved crossing.
(c) Since the smoothly embedded closed curves are disjoint the disks are disjoint or nested (here "nested" means that one is contained in the other).
(d) By (b) there are a priori four combinatorial possibilities for the disks. But it follows from (c) that it is not possible that for the lower strand the disk is above and that at the same time for the upper strand the disk is below. Thus there are only three possible configurations. These are shown in the figure below.
(e) For each of the three possible cases we show in the figure below how to attach a band to the collection of disks. This band, if we attach it to disks B_i and B_j, can be attached in the cylinder where the z-coordinate lies between t_i and t_j. It follows from (2b) that this cylinder is disjoint from all the other $B_k \times \{t_k\}$.

By construction the resulting submanifold of $\mathbb{R}^3$ is compact and orientable. We leave it to the zealous reader to show that the boundary of the resulting submanifold is smoothly isotopic to K. It follows easily from the explicit construction that the submanifold we constructed has no closed component. Thus we see that the submanifold is (after a smooth isotopy) a Seifert surface for K. ■

Example. In the figure below we apply the Seifert Surface Existence Theorem 121.2 to the usual diagram of the figure-8 knot.
We obtain a Seifert disk that is built out of three disks and four bands. □

121.2. **The genus of a surface.** Next we recall Notation 101.12 and 101.22:





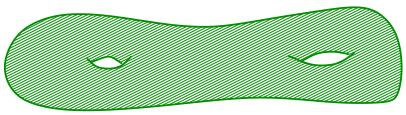
Definition 121.4.

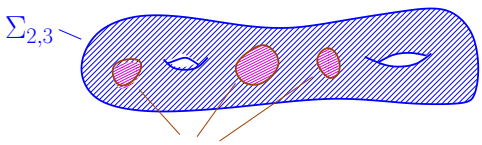
(1) Given $g, n \in \mathbb{N}_0$ we write[1266]

$$\Sigma_{g,n} \;:=\; \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Note that n is precisely the number of boundary components of $\Sigma_{g,n}$.

(2) Let M be a compact orientable connected non-empty 2-dimensional smooth manifold with n boundary components. By the Surface Classification Theorem 101.16 there exists a unique $g \in \mathbb{N}_0$ such that M is diffeomorphic to some $\Sigma_{g,n}$. We refer to g as the genus(M) of M.


surface Σ_2 of genus 2



three boundary components

[1266] Recall that according to Definition 101.10 by "minus n open disks" we mean the following: We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma_g$ with disjoint images. We refer to $\Sigma_g \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks. As we discussed in the Surface-Minus-Open Disks Lemma 101.11 the diffeomorphism type is independent of the choice of the smooth embeddings.

The question is, given a surface, how can we read off its genus in practice? As a first step towards answering this question, we will prove the following lemma, which gives us an easy way to read off the genus from the fundamental group:

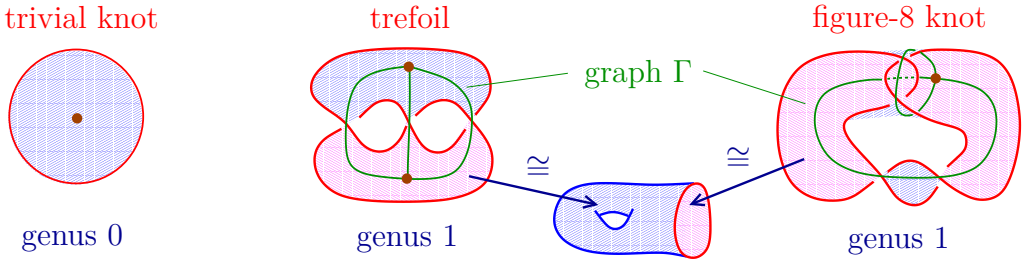
Lemma 121.5. (Surface-π_1-Genus Lemma) Let F be a compact orientable connected non-empty 2-dimensional smooth manifold with $n \in \mathbb{N}$ boundary components. The genus of F is the unique $g \in \mathbb{N}_0$ such that $\pi_1(F)$ is a free group on $2g + n - 1$ generators.

Before we provide the proof of the lemma, let us first formulate our next lemma:

Lemma 121.6. (Genus-via-Graph Lemma) Let F be a compact orientable connected 2-dimensional submanifold with $n \in \mathbb{N}$ boundary components. If $\Gamma \subseteq F$ is a topological graph that is a deformation retract of F, then

$$\text{genus of } F \;=\; \tfrac{1}{2} \cdot (2 - n - \chi(\Gamma)) \;=\; \tfrac{1}{2} \cdot (2 - n - \#\text{vertices of } \Gamma + \#\text{edges of } \Gamma).$$

Example. In the figure below we show some of the Seifert surfaces from page 2335.



With the Genus-via-Graph Lemma 121.6 it is straightforward to determine the genus in each case. □

Proof of Lemmas 121.5 and 121.6. We start out with the following claim:

Claim 1. Let F be a compact orientable connected 2-dimensional submanifold with $n \in \mathbb{N}$ boundary components. If $\Gamma \subseteq F$ is a topological graph that is a deformation retract of F, then $\pi_1(F)$ is a free group with rank

$$\text{rank}(\pi_1(F)) \;=\; 1 - \chi(\Gamma) \;=\; 1 - \#\text{vertices of } \Gamma + \#\text{edges of } \Gamma.$$

Proof. We denote by k the number of vertices and we denote by l the number of vertices of Γ. We calculate that

$$\pi_1(F) \;\underset{\text{by the Homotopy Equivalence } \pi_1\text{-Proposition 90.9 (2b)}}{\overset{\uparrow}{\cong}}\; \pi_1(|\Gamma|) \;\overset{\text{by the Graph-}\pi_1\text{-Proposition 99.13}}{\overset{\downarrow}{\cong}}\; \text{free group on } \underbrace{l - k}_{=\chi(\Gamma)} + 1 \text{ generators.}$$

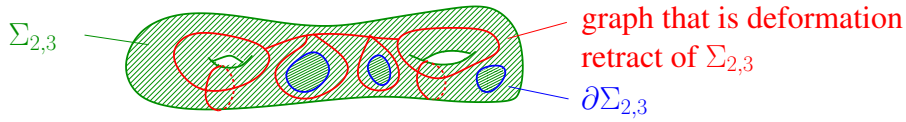
⊞

Now we turn to the proofs of the two lemmas. First we deal with the Surface-π_1-Genus Lemma 121.5. We need to show that the genus of F is the unique $g \in \mathbb{N}_0$ such that $\pi_1(F)$ is a free group on $2g + n - 1$ generators. First of all note that uniqueness is an immediate consequence of the Basis Cardinality Proposition 98.19. It follows from Claim 1 that it remains to prove the following claim:

Claim 2. The group $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

Proof. We proved this statement in the Surface–Full Invariants Proposition 101.20. For the reader's convenience we sketch the argument again. As we show below in the figure,

one can easily find a topological graph $\Gamma \subseteq F$ that is a deformation retract of F such that $\chi(\Gamma) = 2 - 2g - n$.



Thus the promised statement follows from Claim 1. ⊞

Finally note that the Genus-via-Graph Lemma 121.6 is an immediate consequence of Surface-π_1-Genus Lemma 121.5 and Claim 1. ■

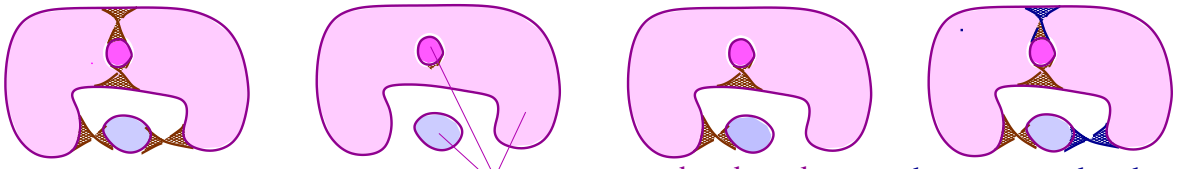
The next lemma is particularly suited for determining the genus of surfaces obtained through the Seifert Surface Existence Theorem 121.2.

Lemma 121.7. (Genus-via-Bands Lemma) Let Σ be a Seifert surface. If Σ is obtained from k smoothly disjointly embedded disks and the attachment of l bands, then

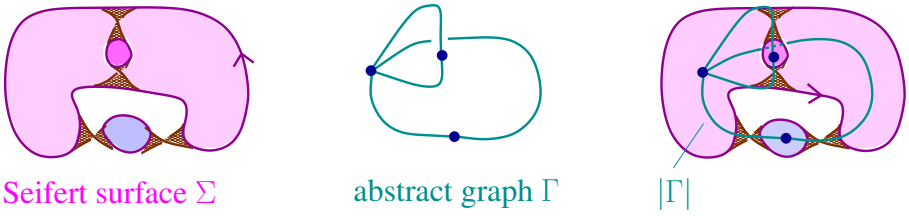
$$\text{genus}(\Sigma) \;=\; \tfrac{1}{2} \cdot (l - k + 1).$$

Sketch of proof. We present two different ways to prove the lemma:

(1) Note that a Seifert surface is connected. Since we start out out with k smoothly disjointly embedded disks it takes $k-1$ bands to obtain a connected smooth submanifold M. First note that it follows from a slight generalization of the Manifold–Seifert-van Kampen Theorem 110.8 that M is simply connected. Next note that it follows from a slight variation on the HNN–Gluing Theorem 113.23 that each of the remaining $l - (k-1)$ bands increases the rank of the fundamental group by 1, so that at the end we see that $\pi_1(\Sigma)$ is a free group on $l - k + 1$ generators. Finally note that it follows from the Surface-π_1-Genus Lemma 121.5 that $\text{genus}(\Sigma) = \frac{1}{2} \cdot (l - k + 1)$.



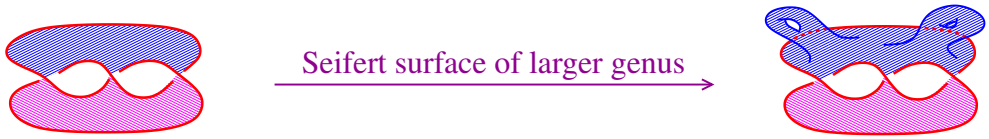
(2) Let $\Gamma = (V, E, \varphi)$ be the abstract graph where V is given by the set of disks, E is given by the set of bands and $\varphi\colon E \to \mathcal{P}(V)$ is the map which assigns to a band $b \in E$ the disks in V to which the band gets attached. It is not particularly difficult to show that the topological realization $|\Gamma|$ admits an embedding $|\Gamma| \to \Sigma$ such that $|\Gamma|$ is a deformation retract of Σ.



Since a Seifert surface is compact, orientable, connected and non-empty we obtain the promised result from the above discussion together with the Genus-via-Graph Lemma 121.6.

In both approaches we leave it to the reader to fill in the technical details. ■

121.3. **The genus of a knot.** If a knot K bounds a Seifert surface of genus g, then the construction sketched in the figure below shows pretty convincingly that K also bounds Seifert surfaces of any genus $h \geq g$.



So the interesting question is, what is the minimal genus of a Seifert surface for a given knot? This leads us to the following definition:

Definition 121.8. Given a knot $K \subseteq S^3$ we define

$$\text{genus}(K) := \text{minimal genus of a Seifert surface of } K.$$

The following lemma gets used subconsciously all the time:

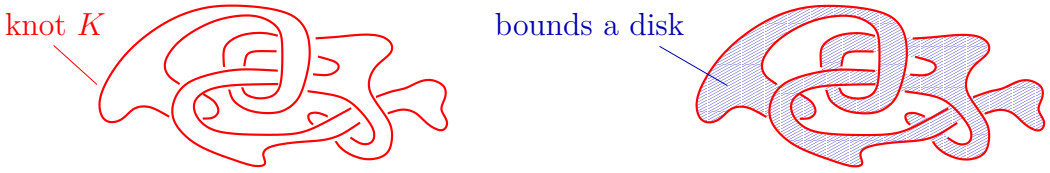
Lemma 121.9. (Genus–Smooth Isotopy Invariant Lemma) Let K and $\widetilde{K}$ be two knots. If K and $\widetilde{K}$ are smoothly isotopic, then $\text{genus}(K) = \text{genus}(\widetilde{K})$.

Proof. This statement is an almost immediate consequence of the Isotopy Extension Theorem 44.2. ■

Note that, almost by definition, a Seifert surface has genus zero if and only if it is diffeomorphic to a disk. The following proposition now shows that trivial knots are precisely the knots that admit a Seifert surface of genus zero:

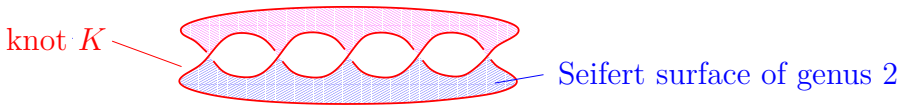
Proposition 121.10. (Genus-Zero Knots Proposition) Let K be a knot in S^3. The following two statements are equivalent:

(i) The knot K has genus zero, i.e. K bounds a smoothly embedded disk in S^3.
(ii) The knot K is smoothly isotopic to the trivial knot.



Example.

(1) We consider again the trefoil. On page 2339 we showed that the trefoil admits a Seifert surface of genus 1. Furthermore in the Trefoil–Non-Trivial Proposition 111.19 (2) we saw that the trefoil is not smoothly isotopic to the trivial knot. It follows from the Genus-Zero Knots Proposition 121.10 that genus(trefoil) = 1.
(2) We consider the knot K in the figure below. It admits a Seifert surface of genus two.



But does K also admit a Seifert surface of genus one? This does not seem to be the case, but right now we lack the tools to exclude the possibility that such a Seifert surface exists. □

Proof. Let $K \subseteq S^3$ be a knot. We need to prove the equivalence of (i) and (ii).

First we prove the almost trivial "(ii)⇒(i)"-direction. It is clear that the trivial knot $\{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1\} \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ bounds a disk. It follows from the Genus–Smooth Isotopy Invariant Lemma 121.9 that any knot that is smoothly isotopic to the trivial knot also bounds a smoothly embedded disk.

Now we turn to the proof of the much more interesting "(1)⇒(2)"-direction. In fact this is a special case of the Lower-Dimensional Smooth Ball Embedding Theorem 45.13. But to make this part of the lecture notes slightly more self-contained we show how one can deduce this statement directly from the Smooth Ball Embedding Theorem 45.10.

Thus let $\varphi\colon \overline{B}^2 \to S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ be a smooth embedding with $\varphi(S^1) = K$. It suffices to show that there exists a smooth isotopy from φ to a smooth embedding $\psi\colon \overline{B}^2 \to \mathbb{R}^3$ such that $\psi(S^1) = S^1 \times \{0\} \subseteq \mathbb{R}^2 \times \{0\}$. We perform the following steps:[1267] [1268]

(0) Using almost the same argument as in the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17 one can arrange that, after a smooth isotopy, we have $K \subseteq \mathbb{R}^3$ and $\varphi(\overline{B}^2) \subseteq \mathbb{R}^3$.

(1) Note that it follows from elementary linear algebra that there exists an $A \in \mathrm{SL}(3,\mathbb{R})$ such that $A \cdot (\mathrm{D}\varphi_0)(\mathrm{T}_0\overline{B}^2) = \mathbb{R}^2 \times \{0\}$. By the Matrix Group-$\pi_0$-Proposition 4.54 (2) we know that $\mathrm{SL}(3,\mathbb{R})$ is path-connected. Using the Smooth Path-Connectivity Proposition 25.32 we can find a smooth path γ in $\mathrm{SL}(3,\mathbb{R})$ from id to A. This path gives rise to the smooth isotopy

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{R}^3 \\ (x,t) & \mapsto & \gamma(t) \cdot \varphi(x). \end{array}$$

This shows that we might as well assume that $(\mathrm{D}\varphi_0)(\mathrm{T}_0\overline{B}^2) = \mathbb{R}^2 \times \{0\}$.

(2) (a) Let $p\colon \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the projection. It follows from (1) and the Chain Rule that the map $\mathrm{D}(p \circ \varphi)_0\colon \mathrm{T}_0\overline{B}^2 \to \mathrm{T}_0\mathbb{R}^2$ is an epimorphism. Since both vector spaces are 2-dimensional we see that the map is in fact an isomorphism. It follows from the Inverse Mapping Theorem 30.1 that there exists an open neighborhood U of $0 \in \overline{B}^2$ such that the restriction of $p \circ \varphi\colon \overline{B}^2 \to \mathbb{R}^2$ to $p \circ \varphi\colon U \to \mathbb{R}^2$ is a smooth embedding.

[1267] The following steps are similar to the steps in the proof of the Meridian Proposition 109.2.

[1268] Implicitly we also use the Smooth Isotopy-Equivalence Relation Proposition 41.5, which says that being smoothly isotopic is in fact an equivalence relation.

(b) We pick $\epsilon > 0$ such that $\overline{B}^2_\epsilon(0) \subseteq U$. We consider the smooth isotopy

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{R}^3 \\ (x,t) & \mapsto & \underbrace{\varphi(((1-t)+t\cdot\epsilon)\cdot x)}_{\substack{\text{for } t=0 \text{ this is the original map } \varphi, \\ \text{for } t=1, \text{ the composition with } p \text{ is} \\ \text{still a smooth embedding}}}. \end{array}$$

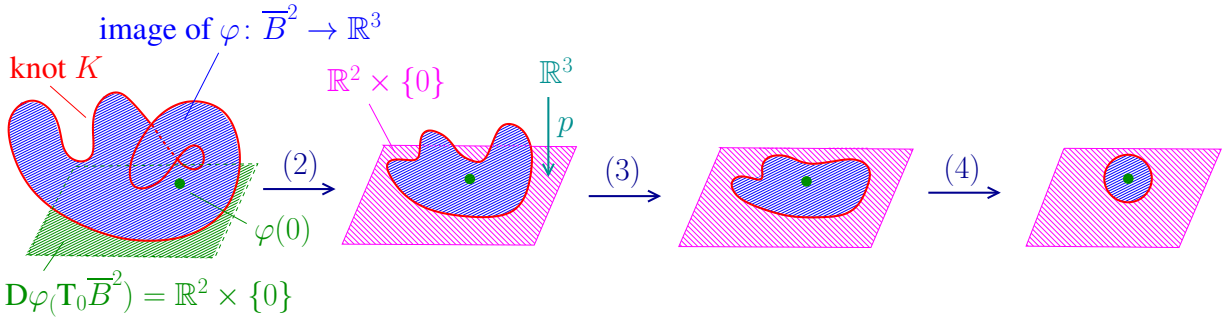
This discussion shows that we can now assume that we have a smooth embedding $\varphi\colon \overline{B}^2 \to \mathbb{R}^3$ such that the map $p \circ \varphi\colon \overline{B}^2 \to \mathbb{R}^2$ is a smooth embedding.

(3) We write $\varphi = \varphi^2_{\mathbb{R}} \times \varphi_{\mathbb{R}}\colon \overline{B}^2 \to \mathbb{R}^2 \times \mathbb{R}$. It follows from (2) that the map

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{R}^2 \times \mathbb{R} \\ (x,t) & \mapsto & (\varphi_{\mathbb{R}^2}(x), \varphi_{\mathbb{R}}(x)\cdot t) \end{array}$$

is a smooth isotopy. This shows that we only need to deal with a smooth embedding $\varphi\colon \overline{B}^2 \to \mathbb{R}^2 \times \{0\}$.

(4) Now we can finally apply the Smooth Ball Embedding Theorem 45.10 to see that our smooth embedding $\varphi\colon \overline{B}^2 \to \mathbb{R}^2 \times \{0\}$ is smoothly isotopic to the standard smooth embedding $\overline{B}^2 \to \mathbb{R}^2 \times \{0\}$.



We leave it to the reader to fill in all the technical details. ■

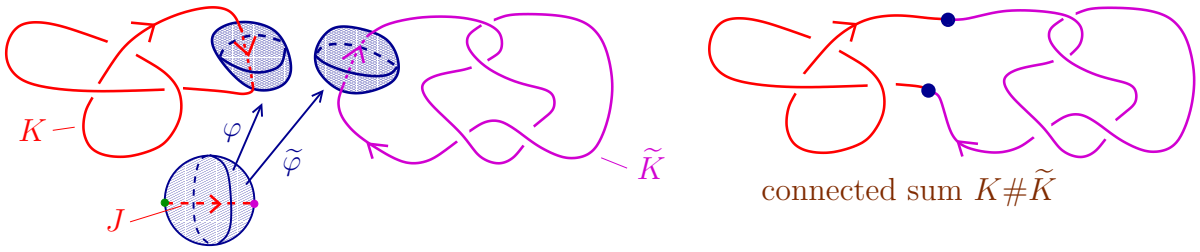
121.4. **Genus and the connected sum operation.** In this section we will see that the genus of a knot is additive under the connected sum operation. It is worthwhile recalling the definition of the connected sum of knots from Definition 109.12:

Definition 121.11.

(1)
- We set $J := \{(x,0) \in \overline{B}^3_2 \,|\, x \in [-2,2]\}$.
- Let $K \subseteq S^3$ be an oriented knot. We say that a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ (anti-) respects K if $\varphi(J) = \varphi(\overline{B}^3_2) \cap K$, if $\varphi\colon \overline{B}^3_2 \to S^3$ is orientation-preserving (orientation-reversing) and if $\varphi|_J\colon J \to \varphi(\overline{B}^3_2) \cap K$ is orientation-preserving (orientation-reversing).

(2) Let K and $\widetilde{K}$ be two oriented knots in S^3. We pick a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3_2 \to S^3$ that anti-respects $\widetilde{K}$. We define

$$\text{connected sum } K\#\widetilde{K} \;:=\; \underbrace{(K \setminus \varphi(\mathring{J})) \cup (\widetilde{K} \setminus \widetilde{\varphi}(\mathring{J}))}_{\subseteq (S^3 \setminus \varphi(B^3)) \cup (S^3 \setminus \widetilde{\varphi}(B^3)) \cong S^3}.$$



Proposition 121.12. (Genus–Connected Sum Proposition) Given any two oriented knots K and $\widetilde{K}$ we have

$$\operatorname{genus}(K\#\widetilde{K}) \;=\; \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

The remainder of this section is concerned with a (sketch of) the proof of the Genus–Connected Sum Proposition 121.12. More precisely, in the following two subsections we prove the "$\leq$"-inequality and the "$\geq$"-inequality. Together these two inequalities gives us the desired equality. Afterwards, in Section 121.5 we will use the Genus–Connected Sum Proposition 121.12 to study "prime decompositions" of knot.

In the following two subsections we will take a few liberties. The key ideas of the proofs are very pretty. Carrying out the technical details rigorously leads to a technical mess, involving smooth (sub-) manifolds with corner, which we do not want to get into.

121.4.1. **Proof of the "$\leq$"-inequality of Proposition 121.12.** The proof of the Genus–Connected Sum Proposition 121.12 rests on the shoulders of the following lemma:

Lemma 121.13. (Seifert Surface–Connected Sum Lemma) Let K and $\widetilde{K}$ be two oriented knots and let F and $\widetilde{F}$ be Seifert surfaces for K and $\widetilde{K}$. We introduce some objects and some notation:

- For $r \in \mathbb{R}_{\geq 0}$ we write $\overline{H}_r^{\geq 0} := \{(x,y,z) \in \overline{B}_r^3 \,|\, y \geq 0 \text{ and } z = 0\}$ and $H_r^{\geq 0} := \overline{H}_r^{\geq 0} \cap B_r^3$.
- Since F is a submanifold of S^3 we can pick a smooth embedding $\varphi\colon \overline{B}_2^3 \to S^3$ that respects K, such that φ restricts to a smooth embedding $\overline{H}_2^{\geq 0} \to F$ and such that $\varphi(\overline{B}_2^3) \cap F = \varphi(\overline{H}_2^{\geq 0}) \cap F$.
- Since $\widetilde{F}$ is a submanifold of S^3 we can pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}_2^3 \to S^3$ that anti-respects $\widetilde{K}$, such that $\widetilde{\varphi}$ restricts to a smooth embedding $\overline{H}_2^{\geq 0} \to \widetilde{F}$ and such that $\widetilde{\varphi}(\overline{B}_2^3) \cap \widetilde{F} = \widetilde{\varphi}(\overline{H}_2^{\geq 0}) \cap \widetilde{F}$.

We now consider the boundary connected sum [1269]

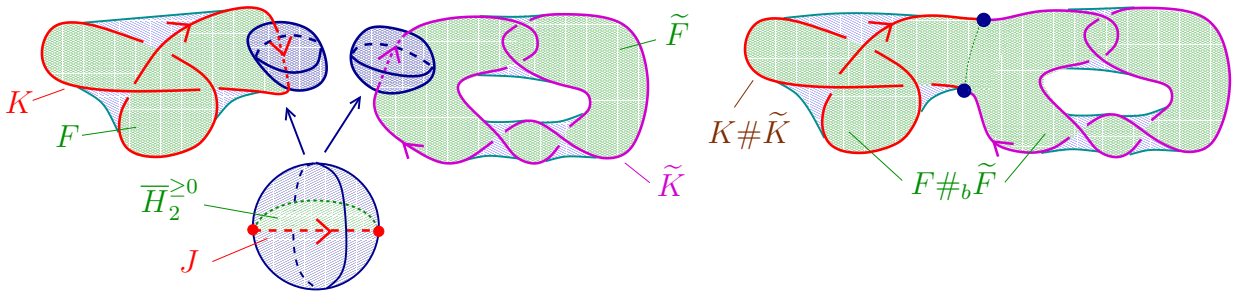
$$F \#_{\mathrm{b}} \widetilde{F} \;:=\; \underbrace{(F \setminus \varphi(H_1^{\geq 0})) \cup (\widetilde{F} \setminus \widetilde{\varphi}(H_1^{\geq 0}))}_{\subseteq (S^3 \setminus \varphi(B^3)) \cup (S^3 \setminus \widetilde{\varphi}(B^3)) \cong S^3}.$$

The following statements hold:

(1) $F \#_{\mathrm{b}} \widetilde{F}$ is a Seifert surface for $K\#\widetilde{K}$.
(2) We have $\operatorname{genus}(F \#_{\mathrm{b}} \widetilde{F}) = \operatorname{genus}(F) + \operatorname{genus}(\widetilde{F})$.

Sketch of proof of Lemma 121.13.

[1269] The name and the notation $F \#_{\mathrm{b}} \widetilde{F}$ is supposed to indicate that this construction is very similar to the notion of the boundary connected sum of two smooth manifolds that we introduced in Definition 61.10.



(1) This statement follows easily from the definitions.[1270] We leave it to the reader to fill in the details.

(2) We set $g := \operatorname{genus}(K)$ and $\widetilde{g} := \operatorname{genus}(\widetilde{K})$. We have

$$\begin{array}{rcl} \pi_1(F \#_b \widetilde{F}) & \overset{\cong}{\longleftarrow} & \pi_1(F \setminus \varphi(H_1^{\geq 0})) * \pi_1(\widetilde{F} \setminus \widetilde{\varphi}(H_1^{\geq 0})) \overset{\cong}{\longrightarrow} \pi_1(F) * \pi_1(\widetilde{F}) \\ & \cong & \{\text{free group of rank } 2g\} * \{\text{free group of rank } 2\widetilde{g}\} \\ & = & \text{free group of rank } 2 \cdot (g + \widetilde{g}). \end{array}$$

both are isomorphisms by a mild generalization of the Manifold–Seifert-van Kampen Theorem 110.8, see Exercise 102.10

It follows from the Surface-π_1-Genus Lemma 121.5 that $\operatorname{genus}(F \#_b \widetilde{F}) = g + \widetilde{g}$. ■

We can now already prove the "$\leq$"-inequality of the Genus–Connected Sum Proposition 121.12:

Proof of the "$\leq$"-inequality of Proposition 121.12. Let K and $\widetilde{K}$ be oriented knots. We want to prove that

$$\operatorname{genus}(K\#\widetilde{K}) \leq \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

Let F be a Seifert surface for K of minimal genus and let $\widetilde{F}$ be a Seifert surface for $\widetilde{K}$ of minimal genus. As in the Seifert Surface–Connected Sum Lemma 121.13 we form $F \#_b \widetilde{F}$. We now see that

$$\begin{array}{ccccccc} \operatorname{genus}(K\#\widetilde{K}) & \underset{\uparrow}{\leq} & \operatorname{genus}(F \#_b \widetilde{F}) & \underset{\uparrow}{=} & \operatorname{genus}(F) + \operatorname{genus}(\widetilde{F}) & \underset{\uparrow}{=} & \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}). \\ & \text{since } F \#_b \widetilde{F} \text{ is a Seifert surface for } K\#\widetilde{K} & & \text{by the Seifert Surface–Connected Sum Lemma 121.13 (2)} & & \text{by choice of } F \text{ and } \widetilde{F} & \end{array}$$

■

121.4.2. **Proof of the "$\geq$"-inequality of Proposition 121.12.** Our proof of the "$\geq$"-inequality of Proposition 121.12 requires two preparations. First of all we need the following proposition, which is also the content of Exercise 107.5 (a).

Proposition 121.14. (Fill-in-Disks Proposition) Let $\gamma\colon S^1 \to S^2$ be a smooth embedding. There exist two smooth embeddings $\varphi_1, \varphi_2\colon \overline{B}^2 \to S^2$ with $\varphi_1(\overline{B}^2) \cup \varphi_2(\overline{B}^2) = S^2$, with $\varphi_1(\overline{B}^2) \cap \varphi_2(\overline{B}^2) = \gamma(S^1)$ and such that for $i = 1, 2$ we have $\varphi_i|_{S^1} = \gamma$.



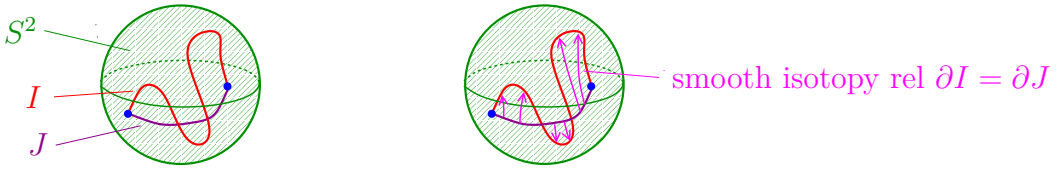
[1270] Note that here our careful choice of radii plays an essential role.

Sketch of proof. We recall the statement of the Fill-in-Disk Proposition 107.15: If Σ is a compact orientable 2-dimensional smooth manifold and if $f\colon S^1 \to \Sigma$ is a smooth embedding which is null-homotopic, then there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.

Now let $\gamma\colon S^1 \to S^2$ be a smooth embedding. By the Sphere-π_1-Proposition 88.24 we know that $\pi_1(S^2) = 0$. We can thus apply the Fill-in-Disk Proposition 107.15 and we obtain a smooth embedding $\varphi_1\colon \overline{B}^2 \to S^2$ with $\varphi_1|_{S^1} = \gamma$. Next we consider $\Sigma := S^2 \setminus \varphi_1(0)$. By the Stereographic Projection Lemma 4.24 we know that Σ is diffeomorphic to $\mathbb{R}^2$. This implies that Σ is simply connected. We can thus apply the Fill-in-Disk Proposition 107.15 to $\gamma\colon S^1 \to \Sigma$ and we obtain a smooth embedding $\varphi_2\colon \overline{B}^2 \to \Sigma$ with $\varphi_2|_{S^1} = \gamma$. ■

We will also need the following lemma, which might be of independent interest.

Lemma 121.15. (Interval–Isotopy Lemma) Let I and J be two connected 1-dimensional smooth submanifolds of S^2 that are diffeomorphic to the interval $[0,1]$. If I and J have the same boundary, then I and J are smoothly isotopic rel $\partial I = \partial J$.



Proof of Interval–Isotopy Lemma 121.15. The lemma can be deduced, with some effort, from the Fill-in-Disks Proposition 121.14. To keep the discussion flowing we outsource the proof of the lemma to Appendix 121.6. (Also note that the lemma is basically the content of Exercise 107.4.) ■

Now we are finally ready to prove the "$\geq$"-inequality of Proposition 121.12.

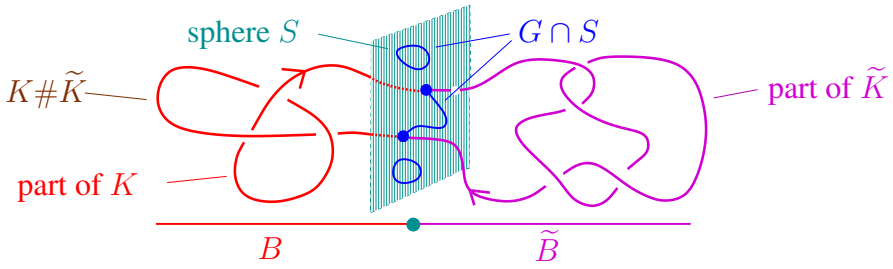
Sketch of proof of the "$\geq$"-inequality of Proposition 121.12. Let K and $\widetilde{K}$ be two oriented knots. We need to prove that

$$\operatorname{genus}(K\#\widetilde{K}) \;\geq\; \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

In other words, we need to show that if G is a Seifert surface of the connected sum $K\#\widetilde{K}$, then $\operatorname{genus}(G) \geq \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K})$. We introduce the following notation:

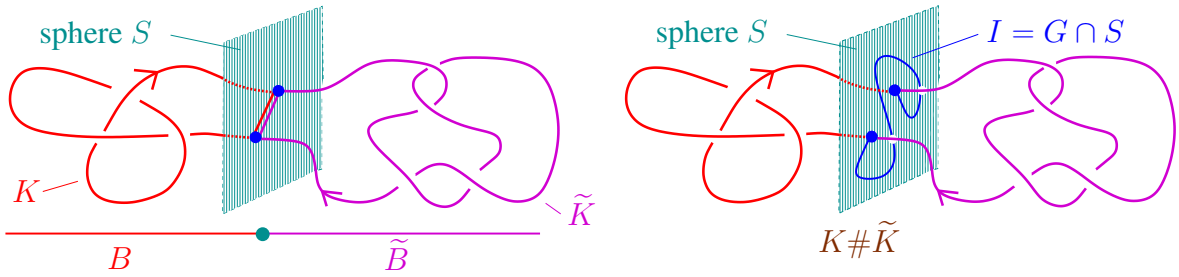
- Let $\varphi\colon \overline{B}^3 \to S^3$ and $\widetilde{\varphi}\colon \overline{B}^3 \to S^3$ be the two smooth embeddings in the definition of the connected sum of K and $\widetilde{K}$. We also consider the closed 3-balls $B := S^3 \setminus \varphi(B^3)$ and $\widetilde{B} := S^3 \setminus \widetilde{\varphi}(B^3)$.
- Let S be the sphere that comes with the definition of the connected sum of K and $\widetilde{K}$, i.e. S is the sphere that we obtain from gluing ∂B to $\partial\widetilde{B}$.

Now let G be a Seifert surface of $K\#\widetilde{K}$. Note that it follows from a suitable generalization of the Transversality Theorem 50.9 that, after a smooth isotopy, we can assume that G intersects S transversally. It is now a consequence of a suitable generalization of the Submanifold Transversal Intersection Theorem 50.4 that $G \cap S$ is a compact 1-dimensional submanifold of S with boundary $\partial G \cap S = (K\#\widetilde{K}) \cap S$.
Next note that it follows from the Smooth 1-Manifold Classification Theorem 28.8 that $G \cap S$ has precisely one component that is diffeomorphic to an interval and that all other



components are diffeomorphic to S^1. We now prove the desired conclusion by induction on the number of closed components of $G \cap S$.

If $G \cap S$ has no closed component, then $G \cap S$ consists of a single component I which is diffeomorphic to an interval.



We set $F := G \cap B$ and $\widetilde{F} := G \cap \widetilde{B}$. Note that $G = F \#_{\mathrm{b}} \widetilde{F}$. Also note that using the Interval–Isotopy Lemma 121.15 one can now show reasonably easily that (up to an isotopy) $F = G \cap B$ is a Seifert surface for K and that (up to an isotopy) $\widetilde{F} = G \cap \widetilde{B}$ is a Seifert surface for $\widetilde{K}$. It follows that

$$\mathrm{genus}(G) \;=\; \mathrm{genus}(F \#_{\mathrm{b}} \widetilde{F}) \;\underset{\text{by the Seifert Surface Connected Sum Lemma 121.13}}{=}\; \mathrm{genus}(F) + \mathrm{genus}(\widetilde{F}) \;\geq\; \mathrm{genus}(K) + \mathrm{genus}(\widetilde{K}).$$

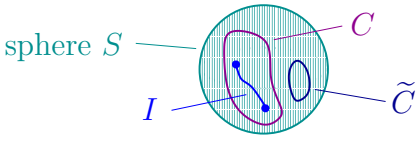
Now suppose that $G \cap S$ has at least one closed component. We start out with the following claim:

Claim 1. There exists a closed component C of $G \cap S$ and a closed disk $D \subseteq S$ such that $\partial D = C$ and such that $G \cap D = C$.

Proof. This can be seen by an "innermost curve" argument:

(a) We start with a closed component C of $G \cap S$. Note that it follows easily from the Fill-in-Disk Proposition 107.15 that there exist two closed disks $D, D' \subseteq S$ such that $D \cap D' = \partial D = \partial D' = C$. Let D be the disk that does not contain I.

(b) If $G \cap D = C$, then we are done. If not, then there exists a component $\widetilde{C}$ of $G \cap S$ that is contained in D. We now replace C by $\widetilde{C}$ and start again with (a).

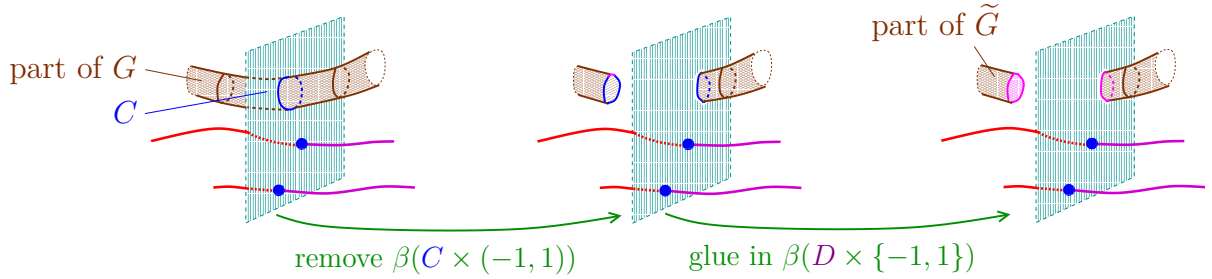


Since $G \cap S$ has only finitely many components the above process will come to an end, and we have found a closed component C of $G \cap S$ that bounds a closed disk $D \subseteq S$ such that $\partial D = C$ and such that $G \cap D = C$. ⊞

We now cut G along $G \cap D$ and glue in two copies of D. More precisely, by the Bicollar Neighborhood Theorem 55.2 there exists a bicollar for S, i.e. there exists a smooth embedding $\beta\colon S \times [-1,1] \to S^3$ such that $\beta(s,0) = s$ for all $s \in S$. Since G is compact we see, as in the Collar-for-Submanifold Proposition 56.3, that we can arrange that the restriction of β to $C \times [-1,1] \to S^3$ is in fact a bicollar $\beta\colon C \times [-1,1] \to G$ and that $\beta(S \times [-1,1]) \cap G = \beta(C \times [-1,1])$. We now consider

$$\widetilde{G} \;:=\; \underbrace{(G \setminus \beta(C \times (-1,1))) \cup \beta(D \times \{-1,1\})}_{\text{to be precise, } \widetilde{G} \text{ is this submanifold with the corner at } \beta(\partial D \times \{-1,1\} \text{ rounded}}.$$

One can now easily verify that $\widetilde{G}$ is a compact oriented smooth submanifold of S^3 with $\partial \widetilde{G} := K \# K_b$ and with $\widetilde{G} \cap S = (G \cap S) \setminus C$.



The idea now is to replace G by $\widetilde{G}$. Note though that there are a few issues: The surface $\widetilde{G}$ might be disconnected and since it is not entirely clear what the genus of $\widetilde{G}$ is. This leads us to the following claim:

Claim 2. The component $\widetilde{G}_0$ of $\widetilde{G}$ that contains $K \# \widetilde{K}$ is a Seifert surface of $K \# \widetilde{K}$ with $\operatorname{genus}(\widetilde{G}_0) \leq \operatorname{genus}(G)$.

Proof. It is basically clear that the component $\widetilde{G}_0$ of $\widetilde{G}$ that contains $K \# \widetilde{K}$ is a Seifert surface of $K \# \widetilde{K}$. It remains to show that $\operatorname{genus}(\widetilde{G}_0) \leq \operatorname{genus}(G)$. We start out with two preparations:

(i) For any compact orientable connected non-empty 2-dimensional smooth manifold M with $n \in \mathbb{N}_0$ boundary components we know by the discussion on page 2993[1271] that the Euler characteristic $\chi(M)$ is given by $\chi(M) = 2 - \operatorname{genus}(M) - n$.

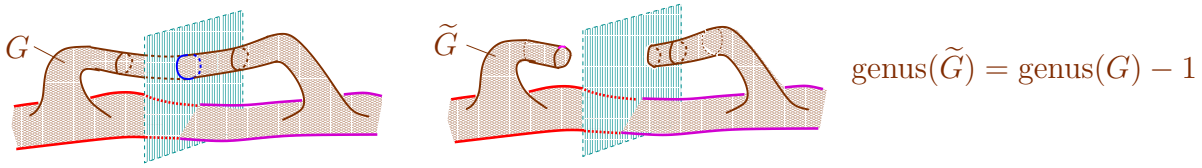
(ii) We consider the Euler characteristic of $\widetilde{G}$. We see that

$$\chi(\widetilde{G}) \underset{\text{follows from the Union-}\chi\text{-Proposition 156.10 and the fact that } \chi(C \times \{-1,1\}) = 0 \text{ and } \chi(\partial D \times \{-1,1\}) = 0}{=} \chi(G) - \chi(C \times [-1,1]) + \chi(D \times \{-1,1\}) \underset{\text{since } \chi(C \times [-1,1]) = 0 \text{ and since } \chi(D) = 1}{=} \chi(G) + 2.$$

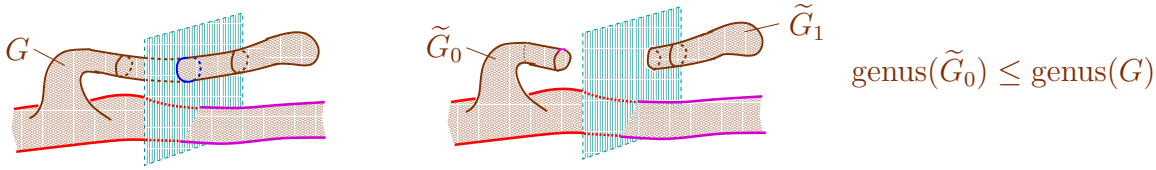
Now we consider two separate cases:

(1) If $\widetilde{G}$ is already connected, then it follows from (i) and (ii) that $\operatorname{genus}(\widetilde{G}) = \operatorname{genus}(G) - 1$.

[1271] We apologize for using a few basic facts about Euler characteristics which we only treat properly much later.



(2) If $\widetilde{G}$ is disconnected, then $\widetilde{G} = \widetilde{G}_0 \sqcup \widetilde{G}_1$. Note that $\widetilde{G}_1$ is closed, thus it follows from (i) that $\chi(\widetilde{G}_1) \leq 2$. It follows that $\chi(\widetilde{G}_0) = \chi(\widetilde{G}) - \chi(\widetilde{G}_1) = \chi(G) + 2 - \underbrace{\chi(\widetilde{G}_1)}_{\leq 2} \geq \chi(G)$.



By (i) this implies that $\mathrm{genus}(\widetilde{G}) \leq \mathrm{genus}(G)$. ⊞

Iterating this procedure we can dispose of the closed components of $G \cap S$ one by one, till we end up with the initial case, that $G \cap S$ has no closed component. ■

121.5. **The prime decomposition theorem.** We start out by recalling Definition 109.12:

Definition 121.16. We say that a knot K is prime if it is not the trivial knot and if it is not smoothly isotopic to the connected sum of two non-trivial knots.

In principle it would be more suitable to call a knot, which does not admit a non-trivial connected sum decomposition, "irreducible". The name "prime" is justified by the Knot–Prime Decomposition Theorem 121.18 which we will formulate in a minute. But first let us state the following proposition:

Proposition 121.17. (Prime Knot–Genus Proposition) For every knot K that is non-trivial and non-prime we have $\mathrm{genus}(K) \geq 2$.

Proof. Let K be a knot which is non-trivial and that is not prime. By definition this means that K is smoothly isotopic to the connected sum of two oriented non-trivial knots J and $\widetilde{J}$. We see that

$$\mathrm{genus}(K) \underset{\uparrow}{=} \mathrm{genus}(J\#\widetilde{J}) \underset{\uparrow}{=} \mathrm{genus}(J) + \mathrm{genus}(\widetilde{J}) \underset{\uparrow}{\geq} 1+1 = 2.$$

by the Genus–Smooth Isotopy Invariant Lemma 121.9 — by the Genus–Connected Sum Proposition 121.12 — by the Genus-Zero Knots Proposition 121.10, since the knots J and $\widetilde{J}$ are non-trivial ■

The Prime Knot–Genus Proposition 121.17 allows us to prove the first part of the following theorem:

Theorem 121.18. (Knot–Prime Decomposition Theorem) Let $K \subseteq S^3$ be an oriented knot.

(1) There exist oriented prime knots $K_1, \ldots, K_m$ such that K is smoothly isotopic to the connected sum $K_1\# \ldots \# K_m$.[1272]
(2) The oriented knots $K_1, \ldots, K_m$ in (1) are unique (as oriented knots) up to smooth isotopy and reordering.

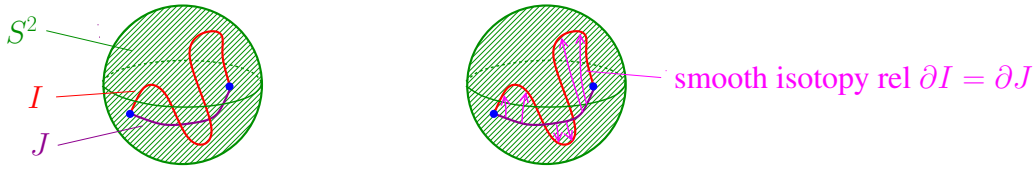
Proof.

(1) This statement can be proved easily using induction on the genus of K. Indeed, the base case $\text{genus}(K) \leq 1$ is an immediate consequence of the Prime Knot–Genus Proposition 121.17. The induction step follows immediately from the Genus–Connected Sum Proposition 121.12 and the associativity of the connected sum operation (which we proved in the Oriented Knot–Monoid Proposition 109.13).

(2) This statement was first proved by Horst Schubert in [**SchubH1949**]. Alternatively see [**BZH2014**, Chapter 7] or [**Lic1997a**, Theorem 2.12] or [**Crom2004**, Theorem 4.5.4] for textbook proofs of this statement. These proofs go well beyond what we can hope to achieve in these notes. ■

121.6. **Appendix: Proof of the Interval–Isotopy Lemma 121.15 .** In this appendix we will sketch the long overdue proof of the following lemma:

Lemma 121.15. (Interval–Isotopy Lemma) Let I and J be two connected 1-dimensional smooth submanifolds of S^2 that are diffeomorphic to the interval $[0,1]$. If I and J have the same boundary, then I and J are smoothly isotopic rel $\partial I = \partial J$.



Sketch of proof. Let P and Q be the two boundary points of I and J.

Claim. There exists a smooth isotopy rel ∂I from I to a smooth submanifold I' such that I' and J only intersect at the boundary.

Proof. We prove the claim with the following steps:

(0) We equip I and J with the orientation "which goes from P to Q". More formally, we equip them with the orientation such that $\partial I = \partial J = -\{P\} \cup \{Q\}$.

(1) We use a slight variation on the Transversality Theorem 50.9 to make I and J transverse through a smooth isotopy of I[1273], while keeping J fixed. Note that it follows from the Transversal Preimage Theorem 50.8 that $I \cap J$ is now finite.

(2) We keep J fixed. By "spiraling around P and Q" we can easily modify I such that $\widetilde{\mathrm{T}}_P I \neq \widetilde{\mathrm{T}}_P J$ and $\widetilde{\mathrm{T}}_Q I \neq \widetilde{\mathrm{T}}_Q J$, such that I and J are transverse and such that the algebraic intersection number of I and J (ignoring the two endpoints) is zero.[1274]

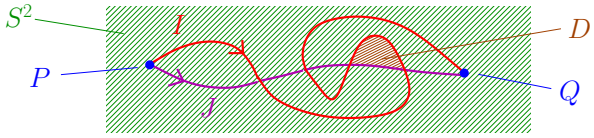
(3) We now want to arrange that I and J intersect only at P and Q. If $I \cap J = \{P, Q\}$, then there is nothing to show. Now assume that there are more intersection points. For this situation we need the following subclaim:

[1272] Recall that on page 2121 we defined the connected sum of the empty family of oriented knots to be the trivial knot.

[1273] Instead of the Transversality Theorem 50.9 one could also use a naive hands-on argument by breaking I into small enough intervals. We leave it to the reader to make this approach work.

[1274] We refer to Definition 51.3 for the definition of the algebraic intersection number.

Subclaim. If $I \cap J$ contains more intersection points than P and Q, then there exists a smoothly embedded disk $D \subseteq S^2 \setminus \{P, Q\}$ such that ∂D is the union of two intervals $\alpha \subseteq I$ and $\beta \subseteq J$.



Proof of subclaim.

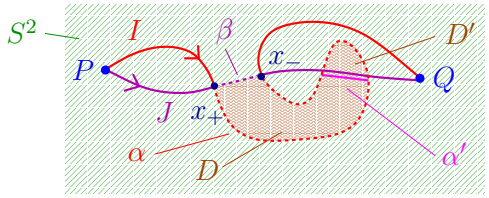
(a) If there are more intersection points than P and Q, then it follows from the fact that the algebraic intersection number is zero that there exist two adjacent intersection points x_- and x_+ on J with opposite signs.

(b) Let α and β be the intervals on I and J with endpoints x_-, x_+.

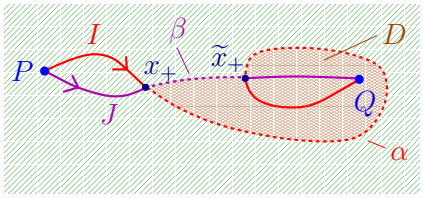
(c) By the Fill-in-Disk Proposition 107.15 we know that $\alpha \cup \beta$ is the boundary of two smoothly embedded disks. Since the signs of the intersection points are different we see that one of the two disks, we call it D, has the property that in a neighborhood of x_- and x_+ the intersection of D with I and J comes from α and β.

(d) It follows from (c) that $I \cap \partial D = \alpha$. (Here we use that x_- and x_+ are adjacent on J.) It follows easily that P, Q do not lie in D.

(e) It follows from (c) that $\beta \subseteq J \cap \partial D$. If $J \cap \partial D \neq \beta$, then let α' be another component of $J \cap D$ and let $D' \subseteq D$ be the disk that cobounds α' and that does not contain β in the boundary. We then restart the process at (d).



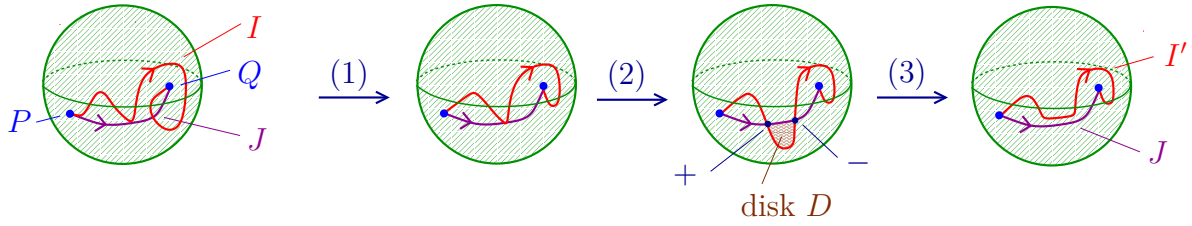
intersection points with opposite sign



intersection points with same sign, we do not need to consider this case, but the figure shows why the opposite signs are essential for the argument

This concludes the proof of the subclaim. □

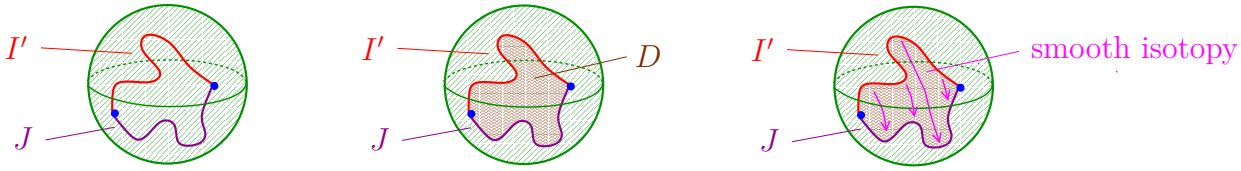
Let D be a disk as provided by the subclaim. Since the signs of the intersection points are opposite we can use the disk to push I across the disk to remove two intersection points with J.



We iterate this process till I and J only meet at the endpoints. We leave the challenging task of turning the above sketch into a proper proof to the reader. ⊞

In the claim we just showed that there exists a smooth isotopy rel ∂I from I to a smooth submanifold I' such that I' and J only intersect at the boundary. We apply once again

the Fill-in-Disk Proposition 107.15, this time to $I' \cup J$, and we see that there exists a disk $D \subseteq S^2$ with $\partial D = I' \cup J$.



We use this disk to define an isotopy from I' to J. ■

121.7. Exercises for Chapter 121.

Exercise 121.1. In the figure below we show to the left the trefoil knot K together with a 2-dimensional submanifold Σ with $\partial\Sigma = K$. (The colors are supposed to show that Σ is orientable and has "two sides".) To the right we show another knot $\widetilde{K}$.

(a) Show that K and $\widetilde{K}$ are smoothly isotopic.

(b) It follows from (a) and the Isotopy Extension Theorem 44.2 that there exists an orientation-preserving diffeomorphism $F\colon S^3 \to S^3$ with $F(K) = \widetilde{K}$. What does $F(\Sigma)$ look like?



Exercise 121.2.

(a) Let D and D' be two disks in $\mathbb{R}^2$ such that $\partial D \cap \partial D' = \varnothing$. Show that $D' \subseteq \mathring{D}$ or $D \subseteq \mathring{D}'$ or $D \cap D' = \varnothing$.

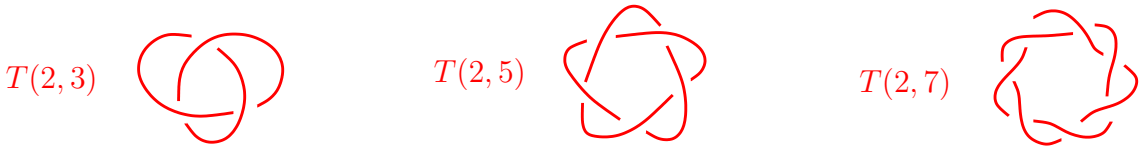
(b) Can we replace $\mathbb{R}^2$ by any 2-dimensional smooth manifold?

(c) Can we replace $\mathbb{R}^2$ by any topological space?

Exercise 121.3.

(a) Apply Seifert's Algorithm 121.2 to the diagram of the torus knot $T(2,n)$ that is illustrated below. What is the genus of the resulting Seifert surface?

(b) Let $p, q \in \mathbb{N}$ be coprime. Find a suitable diagram for the torus knot $T(p,q)$. Then apply Seifert's Algorithm 121.2. What is the smallest genus that you can achieve?

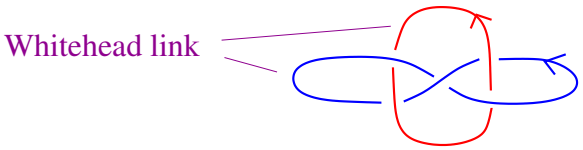


Exercise 121.4. Let D be a knot diagram with c crossings. Show that if we apply Seifert's Algorithm 121.2 to this diagram, then we obtain a surface F with $\text{genus}(F) \leq \frac{c}{2}$.

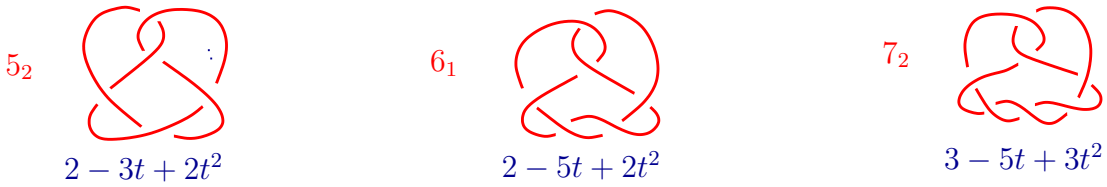
Exercise 121.5. Give an example of a diagram of a knot K where Seifert's Algorithm 121.2 does not give a Seifert surface of minimal genus.
Hint. You could even take K to be the trivial knot.

Exercise 121.6. Does the Seifert surface we obtain from Seifert's Algorithm 121.2 depend on the choice of the orientation of the knot?

Exercise 121.7. Apply Seifert's algorithm to the link shown below. Test question: The link has a rotational symmetry. Does the resulting Seifert surface also have a rotational symmetry?



Exercise 121.8. We consider the knots 5_2, 6_1 and 7_2 which are shown below (together with their Alexander polynomials.) Show that in each case the genus equals 1.



Exercise 121.9. Show that there exists infinitely many pairwise non-smoothly isotopic prime knots.
Remark. There are many approaches to showing this statement. One approach is to provide a non-constructive proof using torus knots and by arguing that their Alexander polynomials have infinitely many different irreducible factors.

Exercise 121.10.

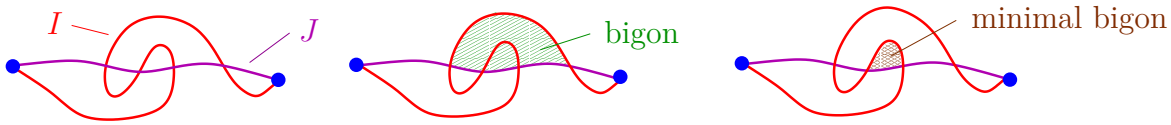
(a) Find a compact orientable 2-dimensional smooth submanifold of S^3 that cobounds the Hopf link.
(b) Can you find such a surface $F = F_1 \cup F_2$ that consists of two components such that each F_i cobounds one component of the Hopf link?
Remark. It might be hard to give a definite answer, but at least you should be able to make an educated guess.

Exercise 121.11. We say that an m-component link $L = L_1 \sqcup \cdots \sqcup L_m$ is a *boundary link* if there exists a collection of m disjoint Seifert surfaces $F_1, \dots, F_m$ such that for $i = 1, \dots, m$ we have $\partial F_i = L_i$.

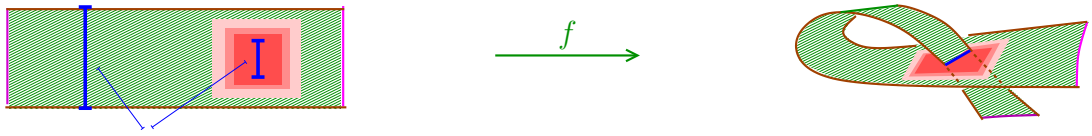
(a) Show that every 2-component *split* link is a boundary link.
(b) Give an example of a boundary link that is not split.
Hint. You can use the higher order Alexander polynomials of Exercise 117.17.
(c) Show that if L is an m-component boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group on m generators.
Hint. Use the Seifert surfaces and bicollars to construct an interesting continuous map $S^3 \setminus L \to \bigvee_{i=1}^{m} S^1$.
(d) Show that if $m \geq 2$ and if L is a boundary link, then $\Delta_L = 0$.

Exercise 121.12. Let I and J be two connected 1-dimensional smooth submanifolds of S^2 that are diffeomorphic to the interval $[0,1]$. We assume that I and J have the same boundary and that I and J intersect transversally. A *bigon for I and J* is a disk $D \subseteq S^2$ such that ∂D is the union of an interval on I and an interval on J. We order bigons by inclusion.

(a) Does a minimal bigon always exist?
(b) Let D be a minimal bigon. Is it true that $\mathring{D}$ is disjoint from I and J?
(c) Can we use a minimal bigon to reduce the number of intersection points of I and J?



Exercise 121.13. Using a specific color, e.g. "red" we can draw 4-dimensional pictures, where "redness" indicates the fourth dimension. For example in the figure below we show a map $f\colon [0,4]\times[0,1]\to\mathbb{R}^3$ which is not injective, but if we use the "redness" function $r\colon [0,4]\times[0,1]\to\mathbb{R}_{\geq 0}$, then the map $(f,r)\colon [0,4]\times[0,1]\to\mathbb{R}^3\times\mathbb{R}_{\geq 0}$ is injective.



images in $\mathbb{R}^3$ agree, but they differ in "redness" (fourth dimension)

A knot K is called *smoothly slice* if there exists a proper smooth embedding $\varphi\colon \overline{B}^2\to\overline{B}^4$ such that $K=\varphi(S^1)$.

(a) In the figure below we show two knots. Precisely two of these knots are smoothly slice. Which ones?



(b) Show that for every oriented knot K the connected sum $K\#K^{\mathrm{inv}} = K\#(K^{\mathrm{mir}})^{\mathrm{rev}}$ is smoothly slice.

CHAPTER 122

Fibered knots

In the last Chapter 121 we introduced the notion of a Seifert surface of a knot. In this chapter we will introduce the notion of a "fibered knot". These two notions are closely related, since fibered knot have, basically by definition, particularly pleasing Seifert surfaces. As our main example we will see that all torus knots are "fibered".

In the next chapter we will then see that the Alexander polynomial of a fibered knot is of a specific form. This will then allow us, among many other things, to show that there do exist knots that are not "fibered".

This chapter is organized as follows:

- In Section 122.1 we will introduce smooth bundle maps.
- In Section 122.2 we will state and prove the Ehresmann Fibration Theorem 122.6.
- In Section 122.3 we will introduce fibered knots and we will show in the Torus Knot–Fibered Proposition 122.9 that all torus knots are fibered.
- In Appendix 122.4 we will give more examples of fibered knots.
- In Section 122.5 we will provide exercises for Chapter 122.

In the next Chapter 123 we will relate the Alexander polynomials to the genus and to fiberedness.

122.1. **Smooth bundle maps.** Before we discussion the specific notion of a fibered knot, let us first discuss a few aspects of the more general theory of fiber bundles. In particular we want to recall Definition 57.1:

Definition 122.1.

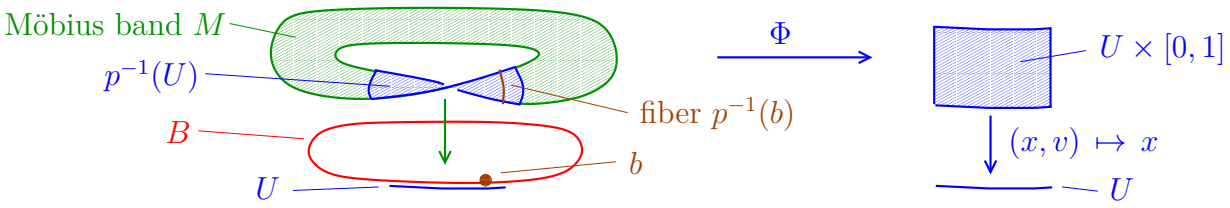
(1) Let M and B be smooth manifolds. We say that a smooth map $p\colon M \to B$ is a smooth bundle map if there exists a smooth manifold F such that for every $b \in B$ there exists a smooth trivialization of p around b, i.e. there exists an open neighborhood U of $b \in B$ and a diffeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\Phi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,v)\mapsto x} & \\ & U. & \end{array}$$

For each $b \in B$ we refer to $p^{-1}(b)$ as a fiber of p. Note that each fiber is diffeomorphic to F.

(2) Let M and B be smooth manifolds. We say that M fibers smoothly over B if there exists a smooth bundle map $p\colon M \to B$.

Example. In the following figure we show a map $p\colon M \to B$ from the Möbius band to a circle B.



We also illustrate that this map is a smooth bundle map and that each fiber is diffeomorphic to the interval $[0,1]$. □

In the following we will mostly be interested in fiber bundles over S^1. For such bundles there is a useful alternative description which makes use of the following concept, which we first introduced in Definition 21.19:

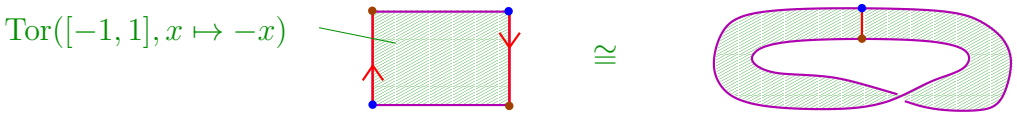
Definition 122.2. Let F be a topological space and let $\mu\colon F \to F$ be a continuous map.

(1) We call
$$\operatorname{Tor}(F,\mu) \;:=\; (F \times [0,1]) \,/\sim \qquad \text{where } (f,0) \sim (\mu(f),1) \text{ for all } f \in F,$$
the mapping torus of (F,μ).

(2) We refer to the map
$$\begin{array}{rcl} q\colon \operatorname{Tor}(F,\mu) = (F \times [0,1])/(f,0) \sim (\mu(f),1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}\,t) \end{array}$$
as the natural projection onto S^1

In the Mapping Torus-of-Manifold Lemma 25.36 we showed that if F is a k-dimensional smooth manifold F and if $\mu\colon F \to F$ is a diffeomorphism, then the mapping torus is naturally a $(k+1)$-dimensional smooth manifold with $\partial(\operatorname{Tor}(F,\mu)) = \operatorname{Tor}(\partial F, \mu|_{\partial F})$.

Example. Let $F = [-1,1]$ and let $\mu\colon [-1,1] \to [-1,1]$ be the diffeomorphism that is given by $x \mapsto -x$.



The corresponding mapping torus $\operatorname{Tor}(F,\mu)$ is, more or less by definition, diffeomorphic to the Möbius band. □

The following lemma says that smooth fiber bundles over S^1 are essentially the same as mapping tori of diffeomorphisms.

Lemma 122.3. (Smooth Mapping Torus–Bundle Lemma)

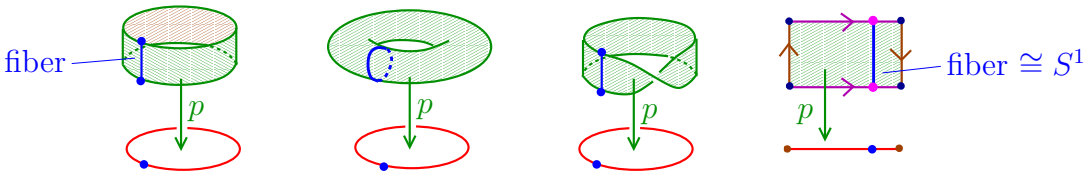
(1) Let F be a smooth manifold F and let $\mu\colon F \to F$ be a self-diffeomorphism. The natural projection
$$\begin{array}{rcl} q\colon \operatorname{Tor}(F,\mu) = (F \times [0,1])/(f,0) \sim (\mu(f),1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}\,t) \end{array}$$
is a smooth bundle map with fiber F.

(2) Let M be a smooth manifold. A smooth map $p\colon M \to S^1$ is a smooth bundle map if and only if there exists a smooth manifold F, a diffeomorphism $\mu\colon F \to F$ and a

diffeomorphism $\Theta\colon \operatorname{Tor}(F,\mu) \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Tor}(F,\mu) & \xrightarrow{\Theta} & M \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & S^1 & \end{array} \qquad q\colon [x,t] \mapsto \exp(2\pi i t)$$

Example. In the figure below we illustrate that the annulus, the Möbius band, the torus and the Klein bottle fiber smoothly over S^1.[1275]



In Exercise 122.1 we will use the Smooth Mapping Torus–Bundle Lemma 122.3 to show that all the remaining compact connected 2-dimensional smooth manifolds do *not* fiber smoothly over S^1. □

Proof. Let $\theta\colon [0,1] \to S^1$ be the map that is given by $\theta(t) := \exp(2\pi i t)$.

(1) By the usual diffeomorphism $[0,1]/0 \sim 1 \xrightarrow{[t]\mapsto\theta(t)} S^1$ we see that it suffices to show that the natural map $p\colon \operatorname{Tor}(F,\mu) \to [0,1]/0 \sim 1$ is a smooth bundle map.

We denote by $\pi\colon [0,1] \to [0,1]/0 \sim 1$ the natural projection. We consider the two open subsets $U_1 := \pi((\frac{1}{4},\frac{3}{4}))$ and $U_2 := \pi([0,\frac{1}{3}) \cup (\frac{2}{3},1])$ in $[0,1]/0 \sim 1$. Furthermore we consider the maps

$$\begin{array}{ccc} p^{-1}(U_1) & \xrightarrow{\Phi_1} & U_1 \times F \\ [(f,t)] & \mapsto & (t,f) \end{array} \quad \text{and} \quad \begin{array}{ccc} p^{-1}(U_2) & \xrightarrow{\Phi_2} & U_2 \times F \\ [(f,t)] & \mapsto & \begin{cases} (t,\mu(f)), & \text{if } t \in [0,\frac{1}{3}), \\ (t,f) & \text{if } t \in (\frac{2}{3},1]. \end{cases} \end{array}$$

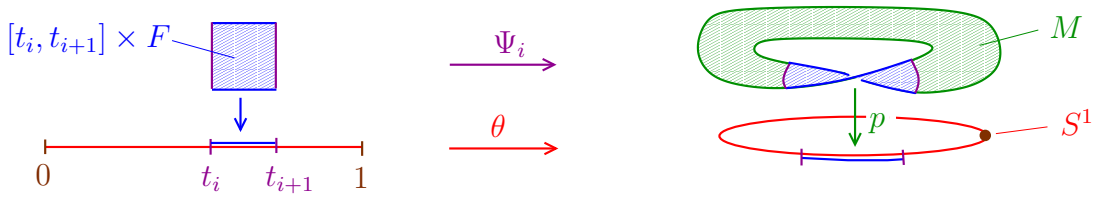
Note that the map on the right is indeed well-defined. Using the Topological-Quotient Proposition 8.2 (1b), the Homotopy Combination Lemma 18.8 (1) and the Topological-Product Proposition 7.2 (1) one can show quite easily that both maps are homeomorphisms. It is also straightforward to verify that these maps are even diffeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0 \sim 1$. It follows easily from this discussion that the natural map $p\colon \operatorname{Tor}(F,\mu) \to [0,1]/0 \sim 1$ is indeed a smooth bundle map.

(2) Let M be a smooth manifold and let $p\colon M \to S^1$ be a smooth bundle map with fiber F. It follows easily from the definition of a smooth bundle map and the Lebesgue Lemma 10.9 that there exists an $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for all $i = 0, \ldots, n-1$ there exists a diffeomorphism

$$\Psi_i\colon [t_i, t_{i+1}] \times F \;\to\; p^{-1}(\theta([t_i,t_{i+1}]))$$

that has the property that for all $(s,f) \in [t_i,t_{i+1}] \times F$ we have $p(\Psi_i(s,f)) = \theta(s)$. Given $s \in [t_i,t_{i+1}]$ we denote by $\Psi_i^s\colon F \to p^{-1}(\theta(s))$ the diffeomorphism that is given by the composition of the obvious diffeomorphism $F \to \{s\} \times F$ and the restriction

[1275] In fact we proved this statement already on page 575 and 575 and in the Mapping Torus-of-id-Lemma 21.22.



of Ψ_i to $\{s\} \times F$. Furthermore given $i = 0, \dots, n-1$ we consider the diffeomorphism $\nu_i := (\Psi_{i+1}^{t_i})^{-1} \circ \Psi_i^{t_i} \colon F \to F$ where we set $\Psi_n = \Psi_0$. We set $\mu := \nu_{n-1} \circ \cdots \circ \nu_0 \colon F \to F$. Finally we consider the map

$$\begin{array}{rccl} \Theta \colon & \operatorname{Tor}(F, \mu) & \to & M \\ & [(f, s)] & \mapsto & \Psi_i(s, (\nu_{i-1} \circ \cdots \circ \nu_0)(f)) \end{array}$$

where $s \in [t_i, t_{i+1}]$

One can easily verify that this map is well-defined, a bijection and that it is a local diffeomorphism. It follows that this map is a diffeomorphism. ■

122.2. **The Ehresmann Fibration Theorem.** In this section we want to recall one earlier theorem which are instrumental in showing that a given smooth map is actually a smooth bundle map. First we need to recall t Definition 32.1:

Definition 122.4. Let M and B be smooth manifolds and let $f \colon M \to B$ be a smooth map.

(1) Let $P \in M$. We say that P is a regular point if $\mathrm{D}f_P \colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} B$ is an epimorphism.
(2) We say that $Q \in B$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular.

After this remainder we can now recall the following theorem:

Theorem 56.9 (Neighborhood-Regular Value Theorem) Let $f \colon M \to B$ be a smooth map between smooth manifolds M and B with $\partial B = \varnothing$. Let $Q \in B$ be a regular value of $f \colon M \to B$ and also of $f|_{\partial M} \colon \partial M \to B$.

(1) $F := f^{-1}(Q)$ is a proper submanifold of M of dimension $\dim(M) - \dim(B)$.
(2) If M is orientable, then F is also orientable.
(3) If M is compact, then F is also compact.

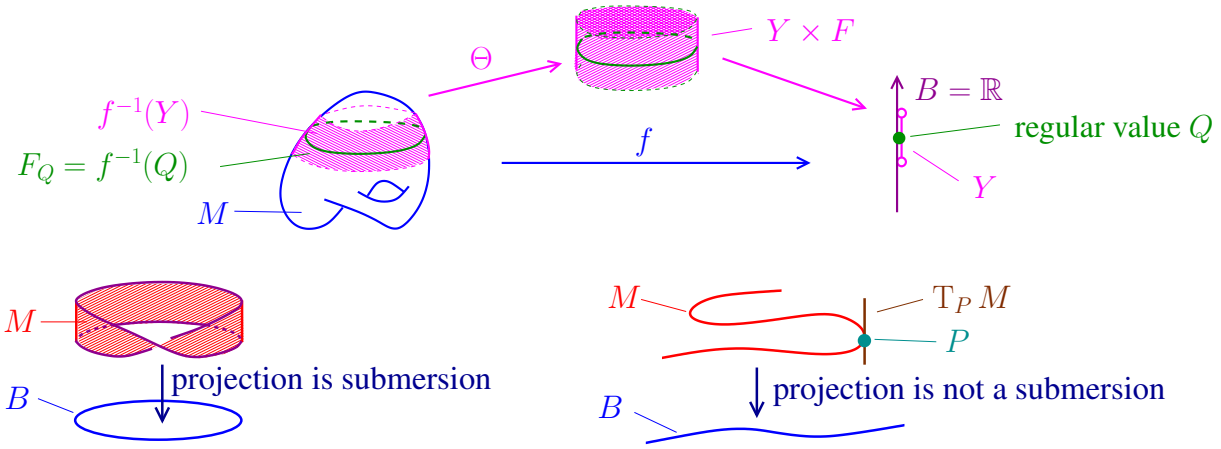
For the last statement we assume that M is compact.

(4) There exists an open neighborhood Y of $Q \in B$ together with a diffeomorphism $\Theta \colon f^{-1}(Y) \to Y \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(Y) & \xrightarrow{\Theta} & Y \times F \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle (y,k) \mapsto y} & \\ & Y. & \end{array}$$

The following definition arises naturally in this context:

Definition 122.5. Let M and B be smooth manifold where $\partial B = \varnothing$. A smooth map $\varphi \colon M \to B$ is called a submersion if every $Q \in B$ is a regular value of $\varphi \colon M \to B$ and also of $\varphi|_{\partial M} \colon \partial M \to B$.



One can easily verify that any smooth bundle map $p\colon M \to B$ to a smooth manifold with $\partial B = \varnothing$ is a submersion.[1276] The following theorem, which was first proved by Charles Ehresmann [**Ehr1951**] in 1951, says in particular that under some mild hypotheses the converse also holds.

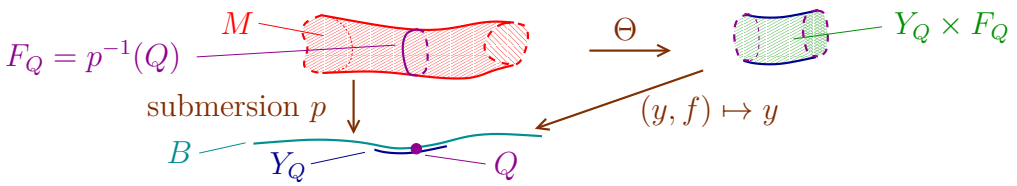
Theorem 122.6. (Ehresmann Fibration Theorem) Let M and B be smooth manifolds where $\partial B = \varnothing$. Let $p\colon M \to B$ be a submersion. If M is compact and if B is connected, then the map $p\colon M \to B$ is a smooth bundle map.

Proof. Let M and B be smooth manifolds where $\partial B = \varnothing$. Let $p\colon M \to B$ be a submersion. We assume that M is compact and that B is connected. Given $Q \in B$ we set $F_Q := p^{-1}(Q)$. The Neighborhood-Regular Value Theorem 56.9[1277] gives us the following two facts:

(a) Each F_Q is naturally a compact smooth manifold of dimension $\dim(M) - \dim(B)$.

(b) For each $Q \in B$ there exists an open neighborhood Y_Q of $Q \in B$ and a diffeomorphism $\Theta\colon p^{-1}(Y_Q) \to Y_Q \times F_Q$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(Y_Q) & \xrightarrow{\ \Theta\ } & Y_Q \times F_Q \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (y,f)\mapsto y} & \\ & Y_Q. & \end{array}$$



It follows easily from these statements and the hypothesis that B is connected that all F_Q are diffeomorphic. It is now clear that $p\colon M \to B$ is a smooth bundle map. ∎

122.3. **Fibered knots.** With the following definition we return to the setting of knots:

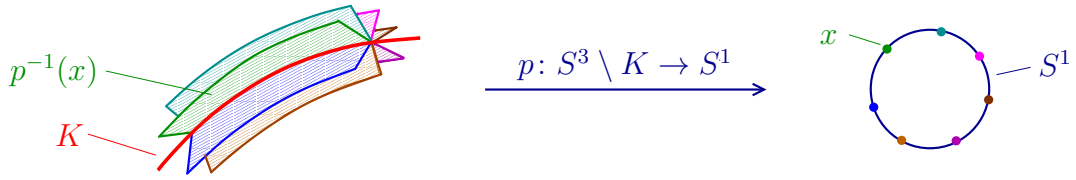
[1276] We carry out the details on page 4584.

[1277] Note that to apply this theorem we use that M is compact.

Definition 122.7. Let $K \subseteq S^3$ be a knot.

(1) We say that K is fibered if there exists a smooth bundle map $p\colon S^3 \setminus K \to S^1$ such that each fiber $p^{-1}(x)$ together with K forms a Seifert surface for K.

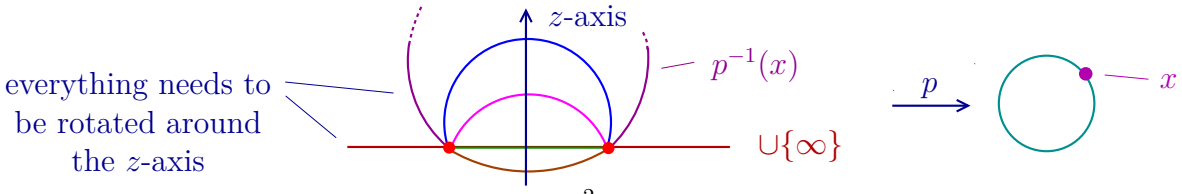
(2) We say that a Seifert surface F is a fiber Seifert surface if $F \setminus K = F \setminus \partial F$ is the fiber of a smooth bundle map $S^3 \setminus K \to S^1$.



Example. We consider the trivial knot $K = \{(z,0) \in S^3 \subseteq \mathbb{C}^2 \,|\, z \in S^1\}$. It is straightforward to verify that the maps

$$\begin{array}{rcl} f\colon S^3 \setminus K & \to & \mathbb{C} \times S^1 \\ (z,w) & \mapsto & \frac{1}{|w|} \cdot (z,w) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g\colon \mathbb{C} \times S^1 & \to & S^3 \setminus K \\ (z,w) & \mapsto & \frac{(z,w)}{\|(z,w)\|} \end{array}$$

are diffeomorphisms that are inverses of one another. It follows easily from this observation that the map $p\colon S^3 \setminus K \to S^1$ that is given by $p(z,w) = \frac{w}{|w|}$ is a smooth bundle map.



Next note that for any $x \in S^1$ the map $\varphi\colon \overline{B}^2 \to S^3$ that is given by $a \mapsto (a, x \cdot (1 - |a|^2))$ is a smooth embedding such that $\varphi(\overline{B}^2) = g(\mathbb{C} \times \{x\}) \cup K = p^{-1}(x) \cup K$. This shows that each fiber $p^{-1}(x)$ together with K does indeed form a Seifert surface for K. □

Our main example of fibered knots will be torus knots. It is worth recalling Definition 111.13:

Definition 122.8. Let $p, q \in \mathbb{Z}$ be coprime. We refer to

$$T(p,q) \;:=\; \left\{ \left(\tfrac{1}{\sqrt{2}} \exp(p \cdot \mathrm{i}t), \tfrac{1}{\sqrt{2}} \exp(q \cdot \mathrm{i}t) \right) \,\middle|\, t \in \mathbb{R} \right\}$$

as the (p,q)-torus knot $T(p,q)$.

Now we are ready to state the most interesting proposition of this chapter:

Proposition 122.9. (Torus Knot–Fibered Proposition) Let $p, q \in \mathbb{N}$ be coprime.[1278]

(1) The corresponding torus knot $T(p,q)$ is fibered.

(2) The fiber has genus $\frac{1}{2}(pq + 1 - p - q)$.

Remark. Alternative expositions and illustrations of parts of the argument and further discussions and generalizations can also be found in [**Miln1968a**, Lemmas 4.8 and 6.1], [**Sav2012**, Chapter 8], [**EN1985**, p. 125] and [**Rolf1990**, Chapter 10.I]. □

[1278] It follows from the Torus Knot Lemma 111.14 (2c) that any torus knot is smoothly isotopic to a (mirror image of a) torus knot $T(p,q)$ with $p, q \in \mathbb{N}$. Since the mirror of a fibered knot is evidently again fibered we see that every torus knot is fibered.

Example. It follows from the Torus Knot–Fibered Proposition 122.9 that the trefoil $T(2,3)$ is fibered. The fibers of a smooth bundle map $p\colon S^3 \setminus T(2,3) \to S^1$ are illustrated in the following video:

`https://www.youtube.com/watch?v=T1So80CDQ3g` □

In the proof of the Torus Knot–Fibered Proposition 122.9 (1) we will use at some point the following little lemma:

Lemma 122.10. (Connected Fiber Lemma) Let M be a connected smooth manifold and let $p\colon M \to B$ be a smooth bundle map. If for some $x \in B$ the fiber $p^{-1}(x)$ has a single boundary component, then $p^{-1}(x)$ is connected.

Sketch of proof of the Connected Fiber Lemma 122.10. Let M be a connected smooth manifold and let $p\colon M \to B$ be a smooth bundle map. Let $x \in B$ such that the fiber $p^{-1}(x)$ has a single boundary component. We set

$$M' \coloneqq \left\{ y \in M \,\middle|\, \begin{array}{l} p(y) \text{ is contained in the unique component} \\ \text{of } p^{-1}(p(y)) \text{ which contains } \partial(p^{-1}(p(y))) \end{array} \right\}.$$

Using the local condition of a smooth bundle map one can easily verify that M' is an open and closed subset of M. Since M is connected and M' is non-empty we see that $M = M'$. It follows that $p^{-1}(x)$ is connected. We leave it to the reader to fill in the details. ■

Proof of the Torus Knot–Fibered Proposition 122.9 (1). Let $p, q \in \mathbb{N}$ be coprime. Note that it follows from the Torus Knot Lemma 111.14 (1) that it suffices to deal with the case that $p \neq q$. Since the factors $\frac{1}{\sqrt{2}}$ in the definition of the (p,q)-torus knot are an eyesore we work throughout this proof with

$$\widetilde{T}(p,q) \coloneqq \{(\exp(p \cdot \mathrm{i}t), \exp(q \cdot \mathrm{i}t)) \,|\, t \in \mathbb{R}\} \subseteq \widetilde{S}^3 \coloneqq \{((z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 2\}$$

instead of $T(p,q)$ and S^3.

Claim 1. We consider the map

$$\begin{array}{rcl} f\colon \widetilde{S}^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 2\} & \to & \mathbb{C} \\ (z,w) & \mapsto & z^q - w^p. \end{array}$$

(1) We have $f^{-1}(0) = \widetilde{T}(p,q)$.
(2) 0 is a regular value of f.

Proof.

(1) This statement is elementary and left to the reader.
(2) We start out with two observations:

- It follows easily from the Smooth-Complex Differentiability Proposition 39.5 that for any $(z,w) \in \mathbb{C}^2$ we have
$$\mathrm{D}f_{(z,w)} = (q \cdot z^{q-1} \quad -p \cdot w^{p-1}) \in \mathrm{M}(1 \times 2, \mathbb{C}) \subseteq \mathrm{M}(2 \times 4, \mathbb{R}).$$
- Given $(z,w) \in \widetilde{S}^3$ we have
$$\mathrm{T}_{(z,w)}\widetilde{S}^3 \underset{\text{by the Physical Tangent Spaces–Lemma 29.2 (3)}}{\uparrow} = \{(c,d) \in \mathbb{C}^2 \,|\, \underbrace{\langle (z,w),(c,d)\rangle}_{\text{standard inner product in } \mathbb{R}^4 = \mathbb{C}^2} = 0\} \overset{\text{since for } a,b \in \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \text{ we have the equality } \langle a,b\rangle = \mathrm{Re}(a \cdot_{\mathbb{C}} \overline{b})}{\downarrow} = \left\{(c,d) \in \mathbb{C}^2 \,\middle|\, \mathrm{Re}\left(\underbrace{z \cdot_{\mathbb{C}} \overline{c} + w \cdot_{\mathbb{C}} \overline{d}}_{\text{calculation in } \mathbb{C}}\right) = 0\right\}.$$

It now follows easily that $\begin{pmatrix} \overline{w} \\ -\overline{z} \end{pmatrix}$ and $\mathrm{i} \cdot \begin{pmatrix} \overline{w} \\ -\overline{z} \end{pmatrix}$ are vectors in $\mathrm{T}_{(z,w)}\widetilde{S}^3$.

Now let $(z,w) \in f^{-1}(0) = \widetilde{T}(p,q)$. We need to show that (z,w) is a regular point of f, i.e. we need to show that the differential $\mathrm{D}f_{(z,w)} \colon \mathrm{T}_{(z,w)}\widetilde{S}^3 \to \mathrm{T}_{f(z,w)}\mathbb{C} = \mathbb{C}$ has rank two. Note that

$$\mathrm{D}f_{(z,w)} \cdot \begin{pmatrix} \overline{w} \\ -\overline{z} \end{pmatrix} \underset{\substack{\uparrow \\ \text{by the above calculation of } \mathrm{D}f_{(z,w)}}}{=} (q \cdot z^{q-1} \quad -p \cdot w^{p-1}) \cdot \begin{pmatrix} \overline{w} \\ -\overline{z} \end{pmatrix} = \underbrace{q \cdot \overline{w} \cdot z^{q-1} + p \cdot w^{p-1} \cdot \overline{z}}_{\substack{\text{non-zero, since } p \neq q \text{ and} \\ \text{since } |w| = |z| = 1}}$$

$$\mathrm{D}f_{(z,w)} \cdot \mathrm{i} \cdot \begin{pmatrix} \overline{w} \\ -\overline{z} \end{pmatrix} \underset{\substack{\downarrow \\ \text{by the above calculation of } \mathrm{D}f_{(z,w)}}}{=} (q \cdot z^{q-1} \quad -p \cdot w^{p-1}) \cdot \mathrm{i} \cdot \begin{pmatrix} \overline{w} \\ -\overline{z} \end{pmatrix} = \mathrm{i} \cdot \underbrace{(q \cdot \overline{w} \cdot z^{q-1} + p \cdot w^{p-1} \cdot \overline{z})}_{\text{same expression as above}}.$$

It follows from these calculations that the differential $\mathrm{D}f_{(z,w)}$ has rank two. This concludes the proof that (z,w) is a regular point. ⊞

Claim 2. We consider the map

$$\begin{aligned} \widetilde{f} \colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) = \widetilde{S}^3 \setminus f^{-1}(0) &\to S^1 \\ (z,w) &\mapsto \frac{z^q - w^p}{|z^q - w^p|} = \frac{f(z,w)}{|f(z,w)|}. \end{aligned}$$

Every $(z,w) \in \widetilde{S}^3 \setminus \widetilde{T}(p,q)$ is a regular point.

Proof. We need to show that for every $(z,w) \in \widetilde{S}^3 \setminus \widetilde{T}(p,q)$ the differential $\mathrm{D}\widetilde{f}_{(z,w)}$ is non-zero. We make a few preparations:

(a) We denote by $\pi \colon \mathbb{C} \setminus \{0\} \to S^1$ the map that is given by $z \mapsto \frac{z}{|z|}$.
(b) By definition we have $\widetilde{f} = \pi \circ f \colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) \to S^1$.
(c) A straightforward calculation shows that for every $z \in \mathbb{C} \setminus \{0\}$ and every $v \in \mathbb{C}$ we have $\mathrm{D}\pi_z \cdot v = \mathrm{Im}(v \cdot z^{-1}) \cdot \mathrm{i} \cdot \pi(z) \in \mathrm{T}_{\pi(z)}S^1 = \mathbb{R} \cdot \mathrm{i} \cdot \pi(z)$.
(d) It follows from (c) that for $z \in \mathbb{C} \setminus \{0\}$ and $v \in \mathbb{C}$ we have

$$\mathrm{D}\pi_z \cdot v = 0 \iff \mathrm{im}(v \cdot z^{-1}) = 0 \iff v \in \mathbb{R} \cdot z.$$

(e) It follows from the description of $\mathrm{T}_{(z,w)}\widetilde{S}^3$ in Claim 1 that $\begin{pmatrix} \mathrm{i} \cdot z \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \mathrm{i} \cdot w \end{pmatrix}$ are tangent vectors. In particular any real linear combination of these two vectors is a tangent vector.

Now let $(z,w) \in \widetilde{S}^3 \setminus \widetilde{T}(p,q)$. We calculate that

$$\mathrm{D}\widetilde{f}_{(z,w)} \cdot \overbrace{\begin{pmatrix} \mathrm{i} \cdot p \cdot z \\ \mathrm{i} \cdot q \cdot w \end{pmatrix}}^{\in \mathrm{T}_{(z,w)}\widetilde{S}^3 \text{ by (e)}} \underset{\substack{\uparrow \\ \text{by (b) and the Chain Rule}}}{=} \mathrm{D}\pi_{f(z,w)} \cdot \mathrm{D}f_{(z,w)} \cdot \begin{pmatrix} \mathrm{i} \cdot p \cdot z \\ \mathrm{i} \cdot q \cdot w \end{pmatrix} \underset{\substack{\uparrow \\ \text{calculation of } \mathrm{D}f_{(z,w)} \text{ in Claim 1}}}{=} \mathrm{D}\pi_{f(z,w)} \cdot (q \cdot z^{q-1} \quad -p \cdot w^{p-1}) \cdot \begin{pmatrix} \mathrm{i} \cdot p \cdot z \\ \mathrm{i} \cdot q \cdot w \end{pmatrix}$$

$$= \underbrace{\mathrm{D}\pi_{z^q - w^p} \cdot (\mathrm{i} \cdot p \cdot q \cdot (z^q - w^p))}_{\neq 0 \text{ by the discussion in (d)}}.$$

We have thus shown that the differential $\mathrm{D}\widetilde{f}_{(z,w)}$ is non-zero. ⊞

After these preliminary claims we turn to the proof of the actual statement:

Claim 3. The map

$$\begin{array}{rcl} \widetilde{f}\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) & \to & S^1 \\ (z,w) & \mapsto & \frac{z^q - w^p}{|z^q - w^p|} = \frac{f(z,w)}{|f(z,w)|} \end{array}$$

is a smooth bundle map.

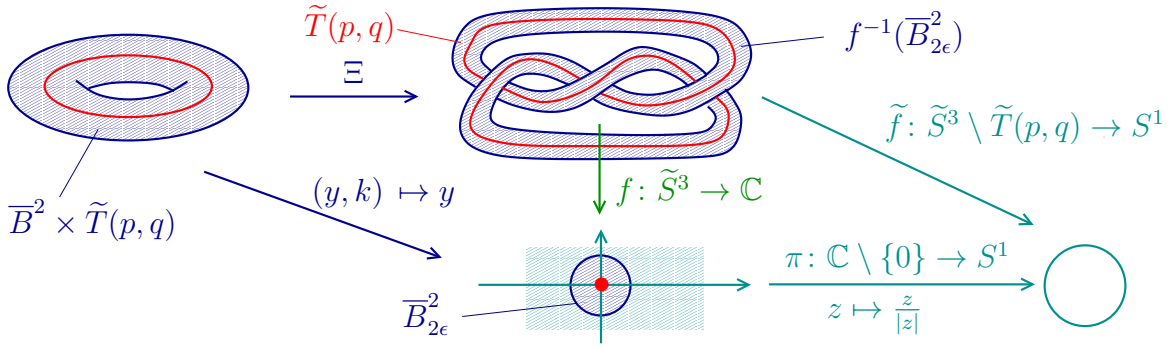
Proof. We first collect what we obtain from Claim 1 and Claim 2:

(1) It follows from Claim 1 and the Neighborhood-Regular Value Theorem 56.9 (4) that there exists an $\epsilon > 0$ and a diffeomorphism $\Xi\colon \overline{B}^2_{2\epsilon} \times \widetilde{T}(p,q) \to f^{-1}(\overline{B}^2_{2\epsilon})$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{B}^2_{2\epsilon} \times \widetilde{T}(p,q) & \xrightarrow[\cong]{\Xi} & f^{-1}(\overline{B}^2_{2\epsilon}) \\ & {\scriptstyle (y,k)\mapsto y} \searrow \quad \swarrow {\scriptstyle f} & \\ & \overline{B}^2_{2\epsilon}. & \end{array}$$

We set $X \coloneqq \widetilde{S}^3 \setminus f^{-1}(B^2_\epsilon)$.

(2) It follows from Claim 2 that the map $\widetilde{f}\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) \to S^1$ is a submersion.



Recall that $\pi\colon \mathbb{C} \setminus \{0\} \to S^1$ is the map that is given by $z \mapsto \frac{z}{|z|}$. It follows easily from (1) and (2) that the map $\widetilde{f}\colon X \to S^1$ is a submersion. (Note that the control on ∂X is provided by the commutative diagram in (1) since by the Codimension-Zero Submanifold Proposition 25.49 (3) we know that Ξ restricts to a diffeomorphism $S^1_\epsilon \times \widetilde{T}(p,q) \to \partial X$.) This observation implies that we can apply the Ehresmann Fibration Theorem 122.6 and we deduce that $\widetilde{f}\colon X \to S^1$ is a smooth bundle map. Using (1) one can easily verify that $\pi \circ f = \widetilde{f}\colon f^{-1}(\overline{B}^2_{2\epsilon}) \setminus \widetilde{T}(p,q) \to S^1$ is also a smooth bundle map. It is now pretty straightforward to combine these two observations and to deduce that $\widetilde{f}\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) \to S^1$ is a smooth bundle map. ⊞

Claim 4. For each $x \in S^1$ the fiber $\widetilde{f}^{-1}(x)$ together with $\widetilde{T}(p,q)$ forms a Seifert surface for $\widetilde{T}(p,q)$.

Proof. Let $x \in S^1$. We need to show that the union $\Sigma \coloneqq \widetilde{f}^{-1}(x) \cup \widetilde{T}(p,q)$ is a Seifert surface for $\widetilde{T}(p,q)$. Note that $\widetilde{S}^3$ is the union of the two open subsets $\widetilde{S}^3 \setminus \widetilde{T}(p,q))$ and $\Xi(B^2_\epsilon \times \widetilde{T}(p,q))$. We first consider the intersection of Σ with these two open subsets separately:

(1) Note that $\Sigma \cap (\widetilde{S}^3 \setminus \widetilde{T}(p,q)) = \widetilde{f}^{-1}(x)$. It follows from the Neighborhood-Regular Value Theorem 56.9 that $\Sigma \cap (\widetilde{S}^3 \setminus \widetilde{T}(p,q)) = \widetilde{f}^{-1}(x)$ is an orientable submanifold with empty boundary of the open subset $\widetilde{S}^3 \setminus \widetilde{T}(p,q)$.

(2) Note that $\Sigma \cap \Xi(B^2_\epsilon \times \widetilde{T}(p,q)) = \Xi(\{r \cdot x \,|\, r \in [0,\epsilon)\} \times \widetilde{T}(p,q))$. Since the above map $\Xi\colon \overline{B}^2_\epsilon \times \widetilde{T}(p,q) \to S^3$ is a smooth embedding we see that $\Sigma \cap \Xi(B^2_\epsilon \times \widetilde{T}(p,q))$ is a smooth submanifold with boundary $\Xi(\{0\} \times \widetilde{T}(p,q)) = \widetilde{T}(p,q)$.

It follows easily from these two observations that Σ is a compact oriented submanifold of S^3 with boundary given by $\widetilde{T}(p,q)$.

Finally, eagle eyed readers will have noticed that we still need to argue why Σ is connected. First note that it follows easily from the Connected Fiber Lemma 122.10 that $\Sigma \cap X$ is connected. Furthermore note that $\Sigma \cap \Xi(\overline{B}^2_{2\epsilon} \times \widetilde{T}(p,q)) \cong [0,2\epsilon] \times \widetilde{T}(p,q)$ is clearly connected. It thus follows from the Union Connected Lemma 4.39 that Σ is connected. ■

Proof of the Torus Knot–Fibered Proposition 122.9 (2). This statement follows from [**Miln1968a**, Theorem 7.2]. We will give an alternative proof a little later on page 2378. ■

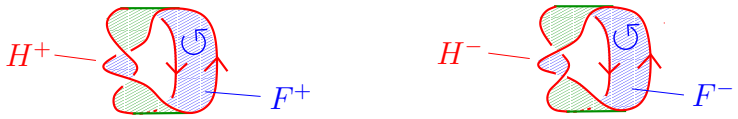
122.4. **Appendix: More fibered knots.** In this appendix we formulate a result which can be used frequently to show that knots are fibered. To state the result it is convenient to generalize some concepts from knots to links.

First of all we have the following generalizations of the concepts that we first introduced on Definition 121.1 and Definition 122.7:

Definition 122.11. Let $L \subseteq S^3$ be an oriented link.

(1) A Seifert surface for L is a compact oriented connected submanifold F of S^3 with $\partial F = L$.[1279]
(2) We say that L is fibered if there exists a smooth bundle map $p\colon S^3 \setminus L \to S^1$ such that each fiber $p^{-1}(x)$ together with L forms a Seifert surface for L.
(3) We say that a Seifert surface F is a fiber Seifert surface if $F \setminus L = F \setminus \partial F$ is the fiber of a smooth bundle map $S^3 \setminus L \to S^1$.

Example. In the figure below we show the positive and negative Hopf link $H^\pm$ with Seifert surfaces $F^\pm$.



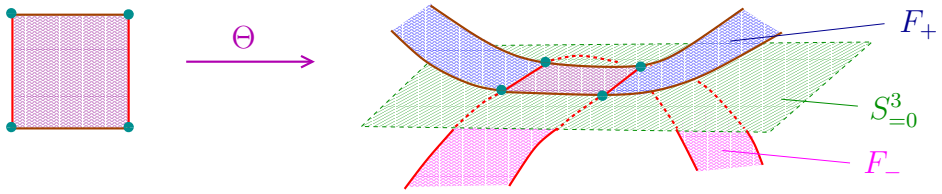
By the Hopf Link-Complement-Proposition 111.23 we already know that the complements $S^3 \setminus H^\pm$ are diffeomorphic to $S^1 \times S^1 \times (-1,1)$. In Exercise 122.6 we will strengthen this result to conclude that $F^\pm$ are fiber Seifert surfaces for the Hopf links $H^\pm$. We refer to $F^\pm$ as Hopf bands. □

Definition 122.12. Let $F_- \subseteq S^3_{\leq 0}$ and $F_+ \subseteq S^3_{\geq 0}$ be two Seifert surfaces such that there exists a diffeomorphism

[1279] For the record we explain why such Seifert surfaces always exist. The Seifert Surface Existence Theorem 121.2 can also be applied to the diagram of an oriented link L. The output is a compact oriented smooth submanifold F such that the boundary (with the induced orientation) equals L. The slight catch is that F does not need to be connected. One obtains an oriented connected submanifold with the same boundary by "internal connected sum". We explain this procedure, perhaps in excessive detail, in the Internal Connected Sum Construction 187.28. Since we will not make use of Seifert surfaces for links we will not elaborate further.

$$\Theta\colon [-1,1]\times[-1,1] \;\to\; F_-\cap F_+$$

with the following properties:
(1) $\Theta(\{\pm1\}\times[-1,1]) = \partial F_-\cap S^3_{=0}$.
(2) $\Theta([-1,1]\times\{\pm1\}) = \partial F_+\cap S^3_{=0}$.
We call $F_-\cup F_+$ the plumbing of F_- and F_+. After rounding corners this is a smooth submanifold of S^3.

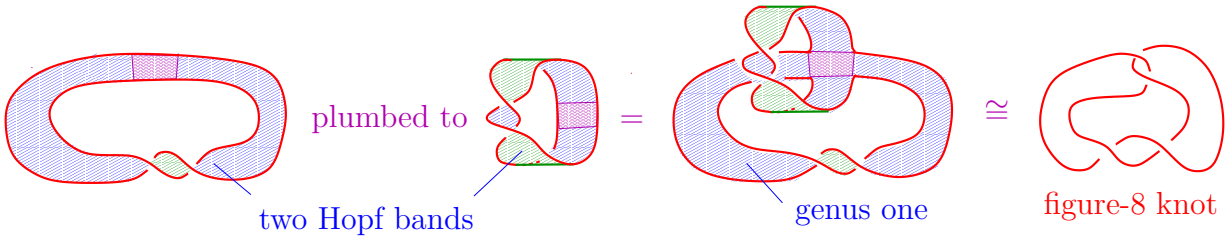


With this definition we can formulate the following surprising theorem.

Theorem 122.13. (Plumbing Fiber Theorem) Let $F\cup F'$ be the plumbing of two Seifert surfaces $F\subseteq S^3_{\leq 0}$ and $F'\subseteq S^3_{\geq 0}$. If F and F' are fiber Seifert surfaces, then $F\cup F'$ is also a fiber Seifert surface. In particular $\partial(F\cup F')$ is a fibered link.

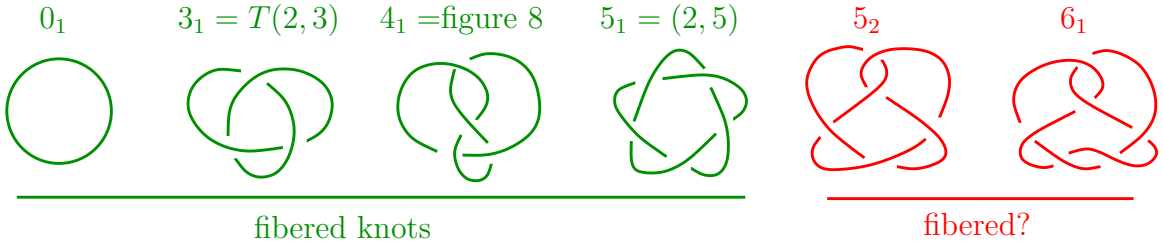
Proof. This theorem is first stated in [**Stal1978**, p. 56]. A proof is given in [**Gab1983a**, Theorem 3]. The proof goes well beyond our current technical abilities. ■

Example. In the figure below we show the plumbing of two Hopf bands. Note that the result is a surface of genus one with connected boundary.



By the Plumbing Fiber Theorem 122.13 the resulting knot is fibered. Since the knot is clearly isotopic to the figure-8 knot this implies that the figure-8 knot is fibered and that the figure-8 knot admits a fiber Seifert surface of genus one. □

We consider again some of the knots that were illustrated on page 2183.



It follows from the Torus Knot–Fibered Proposition 122.9 and the above discussion that the first four knots are fibered. But even with the powerful Plumbing Fiber Theorem 122.13 it seems hard to show that 5_2 and 6_1 are fibered. This raises the following question:

Question 122.14. Are the knots 5_2 and 6_1 fibered?

We will return to this question on page 2379.

We conclude this chapter with two theorems which we will not prove, but which are worth knowing. First of all we have the following classification of fibered knots that admit a fiber Seifert surface of genus one:

Theorem 122.15. (Fibered Genus-One Knots Theorem)

(1) (a) The trefoil is fibered such that the fiber Seifert surface has genus one.
(b) The figure-8 knot is fibered such that the fiber Seifert surface has genus one.
(2) If a knot K is fibered such that the fiber Seifert surface has genus one, then K is smoothly isotopic to a trefoil or to the figure-8 knot.

Proof.

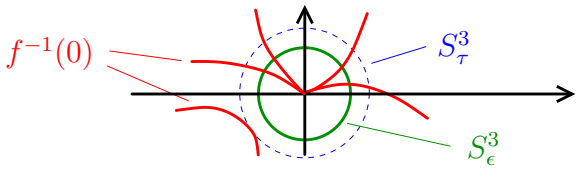
(1) (a) This statement follows from the Torus Knot–Fibered Proposition 122.9 applied to the trefoil $T(2,3)$. A more hands-on proof is given in [**Zee1965**, p. 475], [**Rolf1990**, p. 327-333] and [**BZH2014**, p. 77].
(b) As we just discussed this statement is a consequence of the Plumbing Fiber Theorem 122.13. An alternative proof is are given in [**BZH2014**, p. 77].
(2) This statement is proved in [**BZH2014**, Theorem 5.15, 6.1 and 15.7]. ■

Finally let us state the following theorem which relates knot theory to algebraic geometry. The theorem goes back to the work of John Milnor [**Miln1968a**].

Theorem 122.16. (Milnor Fibering Theorem) Let $f(z,w) \in \mathbb{C}[z,w]$ be a complex polynomial in two variables. We assume that $f(z,w)$ is square-free, i.e. we assume that $f(z,w)$ is the product of distinct irreducible polynomials. We assume that $f(0,0) = 0$. Given $\epsilon > 0$ we consider the sphere $S^3_\epsilon = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = \epsilon^2\}$ and we set $L_\epsilon := f^{-1}(0) \cap S^3_\epsilon$. There exists a $\tau \in \mathbb{R}_{>0} \cup \{\infty\}$ such that for every $\epsilon \in (0,\tau)$ the subset $L_\epsilon \subseteq S^3_\epsilon$ is a closed 1-dimensional smooth submanifold (in other words, it is a link) and such that the smooth function

$$\begin{array}{rccc} p\colon & S^3_\epsilon \setminus L_\epsilon & \to & S^1 \\ & (z,w) & \mapsto & \frac{f(z,w)}{|f(z,w)|} \end{array}$$

is a smooth bundle map which has the property that for each $\xi \in S^1$ the fiber $p^{-1}(\xi)$ together with L_ϵ forms a Seifert surface for L_ϵ.



Example. Let $p, q \in \mathbb{N}$ be coprime. We consider the polynomial $f(z,w) = z^p - w^q$. In this case one can in fact take $\tau = \infty$ and one can easily see, as on page 2361, that the resulting link $L_\epsilon \subseteq S^3_\epsilon$ is precisely the (p,q)-torus knot. The links corresponding to more complicated polynomials are studied in detail in [**Wall2004**, Chapters 5.3, 5.4] and [**EN1985**, Appendix to Chapter 1]. □

Proof. The theorem follows from [**Miln1968a**, Theorem 4.8] (with the hypothesis written in [**Miln1968a**, p. 35]), [**Miln1968a**, Lemma 6.1] and [**Miln1968a**, p. 81]. ■

122.5. Exercises for Chapter 122 .

Exercise 122.1. Recall that in the Diffeotopies-of-S^1-Proposition 42.8 we showed that every self-diffeomorphism of S^1 is diffeotopic to the identity or to the reflection in a hyperplane.

(a) Show that every self-diffeomorphism of $[-1,1]$ is diffeotopic to the identity or to the map that is given by $x \mapsto -x$.
(b) Let N be a compact 1-dimensional smooth manifold and let $\mu\colon N \to N$ be a diffeomorphism. Show that the mapping torus $\mathrm{Tor}(N,\mu)$ is diffeomorphic to the annulus, or the torus, or the Möbius band or the Klein bottle.

Exercise 122.2. Let W be a compact 1-dimensional smooth manifold and let $\mu\colon W \to W$ be a diffeomorphism. We consider the mapping torus $\mathrm{Tor}(W,\mu)$.

(a) Show that $\pi_1(\mathrm{Tor}(W,\mu))$ cannot be isomorphic to the fundamental group of a surface Σ_g of genus $g \geq 2$.
(b) Show that $\pi_1(\mathrm{Tor}(W,\mu))$ cannot be isomorphic to the fundamental group of a non-orientable surface N_k of genus $k \geq 3$.

Hint. Consider abelianizations.
Remark. The goal of this exercise is to give a proof of Exercise 122.1 using the fundamental group instead of using the classification of self-diffeomorphisms up to diffeotopy.

Exercise 122.3. Let M be a compact path-connected smooth manifold and let $p\colon M \to S^1$ be a smooth bundle map.

(a) Show that $p_*\colon \pi_1(M) \to \pi_1(S^1)$ has non-trivial image.
(b) Show that if $p_*\colon \pi_1(M) \to \pi_1(S^1)$ is an epimorphism, then each fiber $p^{-1}(x)$ is connected.
(c) Show that if $p_*\colon \pi_1(M) \to \pi_1(S^1)$ is not an epimorphism, then there exists a smooth bundle map $q\colon M \to S^1$ such that $q_*\colon \pi_1(M) \to \pi_1(S^1)$ is an epimorphism.

Exercise 122.4. Let $K \subseteq S^3$ be a knot. Show that there exists a continuous map $f\colon S^3 \setminus K \to S^1$ such that for every $x_0 \in M$ the induced map $f_*\colon \pi_1(S^3 \setminus K, x_0) \to \pi_1(S^1, p(x_0))$ is an epimorphism.
Hint. Use the fact that K admits a Seifert surface and use the Bicollar Neighborhood Theorem 55.2.

Exercise 122.5. Let $p\colon M \to S^1$ be a smooth bundle map. We suppose that for some $x \in S^1$ the fiber F has a non-compact component and a compact component. Show that M is disconnected.

Exercise 122.6. Let $H \subseteq S^3$ be the Hopf link from Definition 111.22.

(a) Show that there exists a smooth map

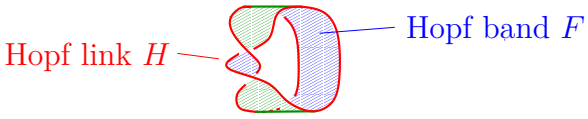
$$\Theta\colon S^1 \times S^1 \times [-1,1] \;\to\; S^3$$

with the following two properties:

(i) The map Θ restricts to a diffeomorphism $S^1 \times S^1 \times (-1,1) \to S^3 \setminus H$.

(ii) For each $z \in S^1$ the restriction of Φ to $\{z\} \times S^1 \times [-1,1]$ is a smooth embedding with $\Phi(\{z\} \times S^1 \times \{-1,1\}) = H$.

(b) Convince yourself that the Hopf band that is shown below is (after a smooth isotopy) given by $\Phi(\{z\} \times S^1 \times [-1,1])$.



Exercise 122.7. Let K and $\widetilde{K}$ be fibered knots. Is the connected sum $K \# \widetilde{K}$ also fibered?

Exercise 122.8. Show that the trefoil can be written as the boundary of a surface that is obtained from plumbing several Hopf bands.

CHAPTER 123

Alexander polynomials, genus and fiberedness

In this chapter we will explore the geometric content of the Alexander polynomial. More precisely, we will see that the Alexander polynomial gives us a lower bound on the genus of a knot. Furthermore we will see that the Alexander polynomial of a fibered knot is bimonic, which implies of course, that knots with non-bimonic Alexander polynomial are not fibered.

This chapter is organized as follows:

- In Section 123.1 we will state and prove the Genus–Alexander Polynomial Proposition 123.2 which says that the degree of the Alexander polynomial gives a lower bound on the genus of a knot.
- In Section 123.2 we will state and prove the Fibered Knot–Alexander Polynomial Proposition 123.12 which says in particular that the Alexander polynomials of fibered knots are bimonic.
- In Appendix 123.3 we will discuss that for alternating knots the Alexander polynomials detects the genus and fiberedness.
- In Section 123.4 we will provide exercises for Chapter 123.

In the next Chapter 124 we will introduce the twisted Alexander function of groups.

123.1. **Genus and the Alexander polynomial.** In this section, given a knot K, we will relate the degree of the Alexander polynomial to the genus of the knot K. First we have to define what we mean by the "degree" of a Laurent polynomial:

Definition 123.1. Given a Laurent polynomial $p(t) = \sum_{i=r}^{s} a_i \cdot t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its degree as[1280]

$$\deg(p(t)) \;=\; \deg\left(\sum_{i=r}^{s} a_i \cdot t^i\right) \;=\; s - r.$$

We extend this definition to $\deg(0) = -\infty$. Note that if we have $p(t), q(t) \in \mathbb{Z}[t^{\pm 1}]$ with $p(t) \doteq q(t)$, then $\deg(p(t)) = \deg(q(t))$.

Example. We have $\det(-5t^{-2} + 3t^{-1} + 7t^4 - t^5) = 7$ and $\deg(1) = 0$. □

The following is the first of the two main results of this chapter:

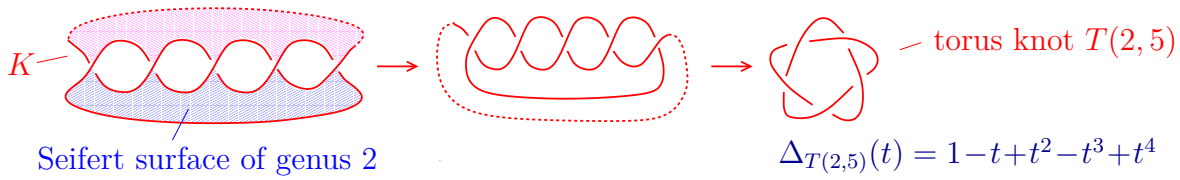
Proposition 123.2. (Genus–Alexander Polynomial Proposition) For every knot K in S^3 we have the following inequality:

$$\deg(\Delta_K(t)) \;\leq\; 2 \cdot \text{genus}(K).$$

[1280] The name "degree" for this concept is perhaps a little unfortunate, a name like "width" or "breadth" might be more suitable. But since "degree" usually gets used in our context, we stick to this name.

Example.

(1) We consider the knot K which is shown in the figure below.



We see a Seifert surface of genus two and on page 2342 we were wondering whether K also admits a Seifert surface of genus one. In the figure we show a smooth isotopy from K to the torus knot $T(2,5)$. Thus we see that

$$2 \underset{\text{by the above example}}{\geq} \text{genus}(K) \underset{\text{by the Genus–Smooth Isotopy Invariant Lemma 121.9}}{=} \text{genus}(T(2,5)) \underset{\text{by the Genus–Alexander Polynomial Proposition 123.2}}{\geq} \tfrac{1}{2}\cdot\deg(\Delta_{T(2,5)}(t)) \underset{\text{by the Torus Knot–Alexander Polynomial Proposition 117.11}}{=} \tfrac{1}{2}\cdot\deg\left(\overbrace{\frac{(t^1-1)\cdot(t^{10}-1)}{(t^2-1)\cdot(t^5-1)}}^{=1-t+t^2-t^3+t^4}\right) = 2.$$

(2) Unfortunately in general the inequality of the Genus–Alexander Polynomial Proposition 123.2 is not an equality. For example on page 2240 we introduced the Kinoshita-Terasaka knot, which we show again in the figure below with a different diagram.



We already mentioned that K is a non-trivial knot with $\Delta_K(t) \doteq 1$. Thus it follows that $\deg(\Delta_K(t)) = 0$, but since K is non-trivial we know by the Genus-Zero Knots Proposition 121.10 that $\text{genus}(K) \geq 1$. In fact we will see on page 2406 that the genus of K is two.[1281] □

To prepare ourselves for the proof of the Genus–Alexander Polynomial Proposition 123.2 we need to digest two lemmas:

Lemma 123.3. (Complement–Surface-π_1-Lemma) If $F \subseteq S^3$ is a Seifert surface, then $\pi_1(S^3 \setminus F)$ is finitely presented.

Proof of Lemma 123.3. This lemma is a special case of the Complement-of-Submanifold Invariants Proposition 179.12 which says that if M is a compact smooth manifold and F is a compact smooth submanifold of M, then $\pi_1(M \setminus F)$ is finitely presented. ■

Lemma 123.4. (Laurent Polynomial-Determinant-Degree Lemma) We suppose that we are given matrices $A_{(s-r)\times s} \in \text{M}((s-r)\times s, \mathbb{Z})$ and $B_{r\times s}, C_{r\times s} \in \text{M}(r\times s, \mathbb{Z})$. Then

$$\deg\left(\det\begin{pmatrix} A_{(s-r)\times s} \\ B_{r\times s}\cdot t + C_{r\times s}\end{pmatrix}\right) \leq r.$$

[1281] The first proof that the genus is two was given in [**Gab1986b**, Theorem 5.7].

Proof of Lemma 123.4. The statement follows almost immediately from the Leibniz expansion of the determinant, since any summand that appears is the product of $(s-r)$ integers and r linear terms of the form $b + t \cdot c$ with $b, c \in \mathbb{Z}$. ■

Proof of Proposition 123.2. Let $K \subseteq S^3$ be a knot. We need to prove that for every Seifert surface F of K we have

$$\deg(\Delta_K(t)) \;\leq\; 2 \cdot \mathrm{genus}(F).$$

So let F be a Seifert surface for K. We make a few preparations:

- We set $g \coloneqq \mathrm{genus}(F)$.
- We pick $b_0 \in S^3 \setminus F$.
- We pick an orientation for K and we equip F with the corresponding orientation.
- We denote as usual by $\Phi_K \colon \pi_1(S^3 \setminus K, b_0) \to \langle t \rangle$ the unique epimorphism which sends a meridian to t.

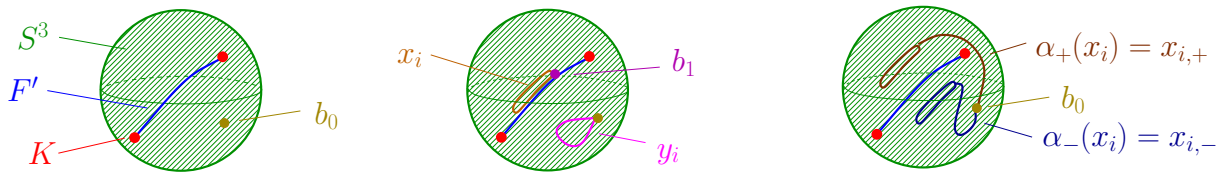
Claim 1. There exists a presentation P of $\pi_1(S^3 \setminus K, b_0)$ of the form

$$\pi_1(S^3 \setminus K, b_0) \;=\; \left\langle y_1, \dots, y_k, t \;\middle|\; \begin{matrix} r_1, \dots, r_l \\ t \cdot x_{i,+} \cdot t^{-1} \cdot x_{i,-}^{-1}, i = 1, \dots, 2g \end{matrix} \right\rangle$$

where $r_1, \dots, r_l, x_{1,\pm}, \dots, x_{2g,\pm} \in \langle y_1, \dots, y_k \rangle$ and where $\Phi_K(y_i) = e = t^0$ and $\Phi_K(t) = t$.

Proof. We make a few more preparations:

- We write $F' \coloneqq F \setminus K$. One can easily verify that F' is a proper submanifold (in the sense of Definition 23.18) of $S^3 \setminus K$. We pick $b_1 \in F'$.
- It follows from the Bicollar Neighborhood Theorem 55.2 that there exists an orientation-preserving smooth embedding $\beta \colon [-1, 1] \times F' \to S^3 \setminus K$ such that $\beta(0, f) = f$ for all $f \in F'$.



Next note that it follows easily from the above discussion and the HNN–Gluing Theorem 113.23 that there exist two homomorphisms $\alpha_\pm \colon \pi_1(F', b_1) \to \pi_1(S^3 \setminus F, b_0)$ and an isomorphism

$$\langle \pi_1(S^3 \setminus F, b_0), t \,|\, \alpha_-(\pi_1(F', b_1)) = t \cdot \alpha_+(\pi_1(F', b_1)) \cdot t^{-1} \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus K, b_0)$$

such that t corresponds to a meridian. We now need to describe the various groups and homomorphisms involved:

- By the Complement–Surface-π_1-Lemma 123.3 we know that $\pi_1(S^3 \setminus F, b_0)$ is finitely presented. Thus we can pick a finite presentation

$$\pi_1(S^3 \setminus F, b_0) \;=\; \langle y_1, \dots, y_k \,|\, r_1, \dots, r_l \rangle.$$

- Note that

$$\pi_1(F') \underset{\text{by the Push-Away-from-Boundary Corollary 35.8}}{\overset{\cong}{\to}} \pi_1(F) = \pi_1\left(\begin{array}{c}\text{surface of genus } g \text{ with one}\\ \text{boundary component}\end{array}\right) \underset{\text{by the Surface-with-Boundary } \pi_1\text{-Lemma 107.8}}{\cong} \begin{array}{c}\text{free group on } 2g\\ \text{generators } x_1,\dots,x_{2g}\end{array}.$$

It follows from the above together with the Amalgamated Product–Presentation Lemma 100.20 that we get a presentation of $\pi_1(S^3 \setminus K, b_0)$ of the form

$$P = \left\langle y_1,\dots,y_k,t \;\middle|\; \begin{array}{l} r_1,\dots,r_l,\\ t\cdot\alpha_+(x_i)\cdot t^{-1}\cdot\alpha_-(x_i)^{-1}, i=1,\dots,2g\end{array}\right\rangle$$

where $r_1,\dots,r_l$ and $\alpha_\pm(x_i)$ lie in $\langle y_1,\dots,y_k\rangle$. Since t corresponds to a meridian we know that $\Phi_K(t) = t$. We still need to show that for $i = 1,\dots,k$ we have $\Phi_K(y_i) = e$. We do this as follows: First note that it follows easily from the Presentation–Homomorphism Lemma 100.13 that there exists a homomorphism $\gamma\colon P \to \langle t\rangle$ with $\gamma(y_i) = e$ and $\gamma(t) = t$. Next note that by the Link Group-Abelianization Proposition 113.9 we know that $P_{\text{ab}} \cong \langle t\rangle$. In particular, up to a sign, there exists a unique epimorphism $P \to \langle t\rangle$. It follows from this observation and the fact that $\Phi_K(t) = t = \gamma(t)$ that $\Phi_K = \gamma$. Thus for $i = 1,\dots,k$ we have $\Phi_K(y_i) = \gamma(y_i) = e$.

Finally for $i = 1,\dots,2g$ we set $x_{i,\pm} \coloneqq \alpha_\pm(x_i)$. We have now obtained the promised presentation. ⊞

Recall that in Definition 116.9 we introduced the Jacobi matrix $J(P)$ of a finite presentation. Note that in our case the Jacobi matrix is an $(l+2g)\times(k+1)$-matrix. The following claim now describes $J^{\Phi_K}(P) \coloneqq \Phi_{K*}(J(P)) \in \mathrm{M}((l+2g)\times(k+1), \mathbb{Z}[t^{\pm1}])$.

Claim 2. We have

$$J^{\Phi_K}(P) = \begin{array}{c} \\ l \\ 2g\end{array}\begin{array}{c}\begin{array}{cc} k & 1\end{array}\\ \begin{pmatrix} A_{l\times k} & 0\\ B_{2g\times k}\cdot t + C_{2g\times k} & *\end{pmatrix}\end{array}$$

where $A_{l\times k}, B_{2g\times k}, C_{2g\times k}$ are matrices with entries in $\mathbb{Z}$.

Proof. It follows easily from the defining properties of Fox derivatives and the Fox Derivative Properties Lemma 116.5 that

$$J(P) = \begin{pmatrix} \left(\frac{\partial r_i}{\partial y_j}\right)_{\substack{i=1,\dots,l,\\ j=1,\dots,k}} & 0\\ \left(t\cdot\frac{\partial x_{i,+}}{\partial y_j} - t\cdot x_{i,+}\cdot t^{-1}\cdot\frac{\partial x_{i,-}}{\partial y_j}\right)_{\substack{i=1,\dots,2g,\\ j=1,\dots,k}} & *\end{pmatrix}$$

Since $x_1^\pm,\dots,x_{2g}^\pm \in \langle y_1,\dots,y_k\rangle$, since $\Phi_K(y_j) = e$ and since $\Phi_K(t) = t$ we see that the entries of $J^{\Phi_K}(P)$ are precisely of the form we promised. ⊞

We now see that

$$\Delta_K(t) \overset{\text{by Definition 117.2}}{\downarrow} = (t-1)\cdot\Delta_{\pi_1(S^3\setminus K),\Phi_K}(t) = \frac{t-1}{\Phi_{K*}(t-1)}\cdot \gcd_{\mathbb{Z}[t^{\pm 1}]}\{\det(J^{\Phi_K}(\pi)^Z_{k+1})\}_{\substack{Z\subseteq\{1,\dots,l+2g\}\\ \#Z=k}} = \uparrow$$

by the Independence-of-Presentation Lemma 116.22, note that it follows from $\Phi_K(t)=t$ that we can use the t-generator to calculate the Alexander function

follows from $\Phi_K(t)=t$ and the claim

$$= \gcd_{\mathbb{Z}[t^{\pm 1}]}\underbrace{\left\{\det\begin{pmatrix} A_{l\times k} & \cancel{0} \\ B_{2g\times k}\cdot t + C_{2g\times k} & \cancel{*}\end{pmatrix}^Z\right\}}_{\text{it follows from the Laurent Polynomial-Determinant-Degree Lemma 123.4 that degree } \le 2g}{}_{\substack{Z\subseteq\{1,\dots,l+2g\}\\ \#Z=k}}.$$

Thus we see that $\Delta_K(t)$ is the greatest divisor of a set of Laurent polynomials of degree $\le 2g$. It follows that $\deg(\Delta_K(t)) \le 2g$. But that is exactly what we had set out to show. ■

123.2. **Alexander polynomials and fibered knots.** Our next goal is to study Alexander polynomials of fibered knots. We will do so in the following three subsections:

(1) We will show that if K is fibered, then $\pi_1(S^3\setminus K)$ is a semidirect product $F\rtimes_\varphi\langle t\rangle$ where F is a free group.
(2) We will study Alexander functions of semidirect products $F\rtimes_\varphi\langle t\rangle$ where F is a free group.
(3) We will combine (1) and (2) to show that Alexander polynomials of fibered knots are "bimonic". This result will allow us to show that certain knots are not fibered.

123.2.1. **Fundamental groups and fibered knots.** By the Smooth Mapping Torus–Bundle Lemma 122.3 we know that if $K\subseteq S^3$ is fibered, then $S^3\setminus K$ is diffeomorphic to a mapping torus $\mathrm{Tor}(F,\mu)$. Since the Alexander polynomial is determined by the fundamental group it is a good idea to remind ourselves of our earlier results on fundamental groups of mapping tori.

To state the relevant result we need to recall t Definition 93.11:

Definition 123.5. Let N be a group and let $\varphi\colon N\to N$ be an isomorphism.

(1) We define the semidirect product of N and $\langle t\rangle$ with respect to φ as the group $N\rtimes_\varphi\langle t\rangle$ where the underlying set is given by the direct product $N\times\langle t\rangle$, but where the group multiplication is given by
$$(h,t^m)\cdot(\widetilde{h},t^{\widetilde{m}}) := (h\cdot\varphi^m(\widetilde{h}),t^{m+\widetilde{m}}).$$
We view N and $\langle t\rangle$ as subgroups of $N\rtimes_\varphi\langle t\rangle$ via the two natural monomorphisms $N\to N\rtimes_\varphi\langle t\rangle$ and $\langle t\rangle\to N\rtimes_\varphi\langle t\rangle$.
(2) We refer to the epimorphism $N\rtimes_\varphi\langle t\rangle\to\langle t\rangle$ that is given by $g\mapsto t^0$ for $g\in N$ and $t\mapsto t$ as the natural epimorphism.

Example. Note that, with the above notation, we have for any $x\in N$ that
$$t\cdot x\cdot t^{-1} = ((e,t)\cdot(x,t^0))\cdot(e,t^{-1}) = (\varphi(x),t)\cdot(e,t^{-1}) = (\varphi(x),e) = \varphi(x).$$

Note that this shows that in particular that N is a normal subgroup of $N \rtimes_\varphi \langle t \rangle$. □

The definition of the semidirect product allows us to formulate (a slightly simplified version of) the Mapping Torus-π_1-Proposition 93.13:

Proposition 93.13. (Mapping Torus-π_1-Proposition) Let (X, x_0) be a pointed path-connected topological space and let $\mu\colon X \to X$ be a homeomorphism. There exists an isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ and an isomorphism

$$\Theta\colon \pi_1(\mathrm{Tor}(X, \mu), [(x_0, 0)]) \xrightarrow{\cong} \pi_1(X, x_0) \rtimes_\varphi \langle t \rangle.$$

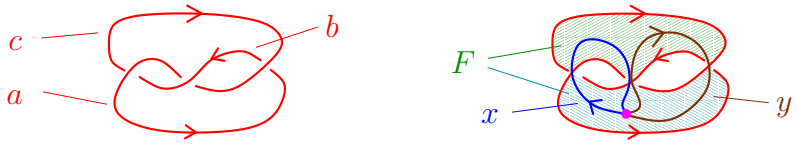
Using the Mapping Torus-π_1-Proposition 93.13 we can easily prove the following proposition:

Proposition 123.6. (Fibered Knot-π_1-Proposition) Let $K \subseteq S^3$ be a fibered knot and let F be a corresponding fiber Seifert surface.[1282] We denote by g the genus of F. There exists an isomorphism $\varphi\colon \langle y_1, \dots, y_{2g} \rangle \to \langle y_1, \dots, y_{2g} \rangle$ and an isomorphism

$$\pi_1(S^3 \setminus K) \cong \langle y_1, \dots, y_{2g} \rangle \rtimes_\varphi \langle t \rangle.$$

If K is oriented, then we can arrange that under this isomorphism the epimorphism $\Phi_K\colon \pi_1(S^3 \setminus K) \to \langle t \rangle$ agrees with the natural epimorphism from the semidirect product to $\langle t \rangle$.

Example. We consider the trefoil K with the diagram and the Seifert surface F that are shown below.



Similar to the discussion on page 2199 we see that a presentation for $\pi_1(S^3 \setminus K)$ is given by

$$\pi_1(S^3 \setminus K) = \langle a, b, c \,|\, \underbrace{c \cdot a \cdot c^{-1} \cdot b^{-1}}_{\text{i.e. } c \cdot a = b \cdot c}, a \cdot b \cdot a^{-1} \cdot c^{-1}, \underbrace{b \cdot c \cdot b^{-1} \cdot a^{-1}}_{\text{i.e. } a^{-1} \cdot b = b \cdot c^{-1}} \rangle.$$

With $x = a^{-1} \cdot b$ and $y = b^{-1} \cdot c$ and $t = a$ we see that in $\pi_1(S^3 \setminus K)$ we have

$$\begin{aligned} t^{-1} \cdot x \cdot t &= a^{-1} \cdot a^{-1} \cdot b \cdot a = a^{-1} \cdot b \cdot c^{-1} \cdot a = a^{-1} \cdot b \cdot c^{-1} \cdot b \cdot b^{-1} \cdot a = x \cdot y^{-1} \cdot x^{-1} \\ t^{-1} \cdot y \cdot t &= a^{-1} \cdot b^{-1} \cdot c \cdot a = a^{-1} \cdot b^{-1} \cdot b \cdot c = a^{-1} \cdot b \cdot b^{-1} \cdot c = x \cdot y. \end{aligned}$$

Let $\varphi\colon \langle x, y \rangle \to \langle x, y \rangle$ be the homomorphism[1283] that is given by $\varphi(x) := x \cdot y^{-1} \cdot x^{-1}$ and by $\varphi(y) := x \cdot y$. One can now easily verify, using the above considerations, that the assignment $x \mapsto a^{-1} \cdot b$, $y \mapsto b^{-1} \cdot c$ and $t \mapsto a$ defines an isomorphism $\langle x, y \rangle \rtimes_\varphi \langle t \rangle \to \pi_1(S^3 \setminus K)$. □

Proof. Let $K \subseteq S^3$ be a fibered knot and let F be a fiber Seifert surface of K. By definition this means that there exists a smooth bundle map $p\colon S^3 \setminus K \to S^1$ such that each fiber $p^{-1}(x)$ together with K forms a Seifert surface for K. We set $F' := F \setminus K$. We make a few preparations:

(a) We pick a base point $x_0 \in F'$.

[1282] We refer to Definition 122.7 for the definition of a fiber Seifert surface. Note that such a surface exists precisely if K is fibered.

[1283] In Exercise 123.5 we will see that φ is actually an isomorphism.

(b) It follows immediately from the Smooth Mapping Torus–Bundle Lemma 122.3 that there exists a self-diffeomorphism $\mu\colon F' \to F'$ such that $S^3 \setminus K$ is diffeomorphic to the mapping torus $\mathrm{Tor}(F', \mu) \coloneqq (F' \times [0,1])/(x,0) \sim (\mu(x), 1)$.

The proposition now follows from the following isomorphisms:

$$\begin{array}{ccccc} \pi_1(S^3 \setminus K) & \cong & \pi_1(\mathrm{Tor}(F', \mu)) & \cong & \pi_1(F', x_0) \rtimes_\varphi \langle t \rangle \\ & \uparrow & & \uparrow & \\ & \text{by (b)} & & \text{by the Mapping Torus-}\pi_1\text{-Proposition 93.13 there exists such an automorphism } \varphi \text{ of } \pi_1(F', x_0) \text{ and such that an isomorphism} & \\ & \cong & \pi_1(F, x_0) \rtimes_\varphi \langle t \rangle & \cong & \langle y_1, \dots, y_{2g} \rangle \rtimes_\varphi \langle t \rangle. \\ & \uparrow & & \uparrow & \\ & \text{it follows from the Basics-of-Collars Lemma 35.7, the Deformation Retract–Homotopy Equivalence Lemma 19.10 (2b) and the Homotopy Equivalence } \pi_1\text{-Proposition 90.9 that } \pi_1(F', x_0) \cong \pi_1(F, x_0) & & \text{it follows from the Surface-with-Boundary–}\pi_1\text{-Lemma 101.13 that } \pi_1(F, x_0) \text{ is a free group on } 2g \text{ generators} & \end{array}$$

Finally we assume that K is oriented. If the two epimorphisms to $\langle t \rangle$ match, then we are done. Otherwise we use the isomorphism

$$\begin{array}{ccc} \langle y_1, \dots, y_{2g} \rangle \rtimes_\varphi \langle t \rangle & \to & , \langle y_1, \dots, y_{2g} \rangle \rtimes_{\varphi^{-1}} \langle \widetilde{t} \rangle \\ (g, t^m) & \mapsto & (g, \widetilde{t}^m) \end{array}$$

to replace the semidirect product by a semidirect product with the "opposite" natural epimorphism to $\langle t \rangle$. ■

For the record we also state the following theorem which gives us the converse to the Fibered Knot-π_1-Proposition 123.6:

Theorem 123.7. (Stallings–Fibered Knot Characterization Theorem) Let $K \subseteq S^3$ be a knot and let $\Phi\colon \pi_1(S^3 \setminus K) \to \langle t \rangle$ be one of the two epimorphisms. The following statements are equivalent:

(1) K is fibered.
(2) $\pi_1(S^3 \setminus K)$ is isomorphic to a semidirect product $\langle y_1, \dots, y_k \rangle \rtimes \langle t \rangle$.
(3) $\ker(\Phi)$ is a finitely generated free group.
(4) $\ker(\Phi)$ is finitely generated.

Proof.
(1)⇒(2). This statement is the content of the Fibered Knot-π_1-Proposition 123.6.
(2)⇒(3). It follows basically immediately from the definitions that the kernel of the natural homomorphism $\langle y_1, \dots, y_k \rangle \rtimes \langle t \rangle \to \langle t \rangle$ is the free group $\langle y_1, \dots, y_k \rangle$.
(3)⇒(4). This statement is trivial.
(4)⇒(1). This direction was first proved by John Stallings [**Stal1962**, Theorem 2]. A textbook proof is given in [**BZH2014**, Theorem 5.1]. ■

123.2.2. **Alexander polynomials of semidirect products.** The Fibered Knot-π_1-Proposition 123.6 raises the following question: What can we say about Alexander functions of semidirect products $F \rtimes \langle \varphi \rangle$ where F is a free group? To state the main result of this subsection we need to introduce the following definition:

Definition 123.8. We say that a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic if the highest coefficient is ± 1. We say a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is bimonic if the lowest and the highest coefficient are ± 1.

Example. The Laurent polynomial $5t - 2t^2 - t^4$ is monic but it is not bimonic. □

Proposition 123.9. (Alexander Polynomial-of-Semidirect Product Proposition) Let $\pi = \langle y_1, \dots, y_k \rangle$ be a finitely generated free group and let $\varphi\colon \pi \to \pi$ be an isomorphism. We consider the corresponding semidirect product $\pi \rtimes_\varphi \langle t \rangle$ and the natural epimorphism $\Phi\colon \pi \rtimes_\varphi \langle t \rangle \to \langle t \rangle$. The following statements hold:

(0) $(t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}$ is a Laurent polynomial, i.e. it is contained in $\mathbb{Z}[t^{\pm 1}] \subseteq \mathbb{Q}(t)$.
(1) $(t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}$ is bimonic.
(2) We have $\deg((t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}) = k$.

The proof of the Alexander Polynomial-of-Semidirect Product Proposition 123.9 will require the remainder of this subsection. In the next Subsection 123.2.3 we will then return to the study of Alexander polynomials of (fibered) knots.

Lemma 123.10. (Abelianization-of-Jacobi Matrix Lemma) Let π be a finitely generated free group, let $\varphi\colon \pi \to \pi$ be a homomorphism and let $x_1, \dots, x_k$ be a basis for π. We introduce the following notation:

- We denote by $\alpha\colon \pi \to \pi_{\mathrm{ab}}$ the natural projection onto the abelianization.
- For $i = 1, \dots, k$ we set $\overline{x}_i := \alpha(x_i)$. By the Abelianization Coproduct Proposition 98.18 (2) we know that π_{ab} is a free abelian group with basis $\overline{x}_1, \dots, \overline{x}_k$.
- Let $\epsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ be the augmentation homomorphism from Definition 116.6, i.e. ϵ is the ring homomorphism which is given by $g \mapsto 1$ for all $g \in \pi$.

Then the following two statements hold:

(1) We have the following equality in $\mathrm{M}(k \times k, \mathbb{Z})$:

$$\epsilon_*\Big(\underbrace{\Big(\frac{\partial \varphi(x_i)}{\partial x_j}\Big)_{i,j=1,\dots,k}}_{(k \times k)\text{-matrix over } \mathbb{Z}[\pi]} \Big) \;=\; \begin{array}{c}\text{matrix that represents } \varphi_*\colon \pi_{\mathrm{ab}} \to \pi_{\mathrm{ab}} \text{ with} \\ \text{respect to the basis } \overline{x}_1, \dots, \overline{x}_k\end{array}$$

(2) If φ is an isomorphism, then $\epsilon_*\big(\big(\frac{\partial \varphi(x_i)}{\partial x_j}\big)_{i,j=1,\dots,k}\big) \in \mathrm{M}(k \times k, \mathbb{Z})$ is invertible.

Proof of Lemma 123.10.

(1) The statement follows easily from the definitions and from applying the following claim to $r = \varphi(x_i)$ for $i = 1, \dots, k$:

Claim. For any $r \in \langle x_1, \dots, x_k \rangle$ and any $i \in \{1, \dots, k\}$ we have

$$\sum_{i=1}^{k} \epsilon\Big(\frac{\partial r}{\partial x_i}\Big) \cdot \overline{x}_i \;=\; \alpha(r) \;\in\; \pi_{\mathrm{ab}}.$$

Proof. The claim follows from the following three observations:

- Basically by definition the claim holds for $r = x_j$ for any $j \in \{1, \dots, k\}$.
- Let $j \in \{1, \dots, k\}$. Recall that it follows from the Fox Derivative Properties Lemma 116.5 (5) that $\frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{ij} \cdot x_i^{-1}$. It follows easily from this observation and from the observation that $\epsilon(x_j^{-1}) = 1$ that the claim holds for $r = x_j^{-1}$.

- We suppose that the equality holds for $u, v \in \langle x_1, \dots, x_k \rangle$. We claim that it then also holds for $u \cdot v$. Indeed, this follows from the following little calculation in π_{ab}:

$$\begin{aligned} \sum_{i=1}^{k} \epsilon\Big(\frac{\partial(u \cdot v)}{\partial x_i}\Big) \cdot \overline{x}_i &\overset{\text{by the Leibniz rule}}{=} \sum_{i=1}^{k} \epsilon\Big(\frac{\partial u}{\partial x_i} + u \cdot \frac{\partial v}{\partial x_i}\Big) \cdot \overline{x}_i \\ &\overset{\text{since } \epsilon \text{ is a ring homomorphism and since } \epsilon(u) = 1}{=} \sum_{i=1}^{k} \epsilon\Big(\frac{\partial u}{\partial x_i}\Big) \cdot \overline{x}_i + \sum_{i=1}^{k} \epsilon\Big(\frac{\partial v}{\partial x_i}\Big) \cdot \overline{x}_i \\ &\overset{\text{since we assume that the equality holds for } u \text{ and } v}{=} \alpha(u) \cdot \alpha(v) = \alpha(u \cdot v). \end{aligned}$$

⊞

(2) If $\varphi\colon \pi \to \pi$ is an isomorphism, then it follows from the Abelianization Functor Proposition 98.16 together with the Isomorphism Preservation Lemma 5.26 that the induced homomorphism $\varphi_*\colon \pi_{\mathrm{ab}} \to \pi_{\mathrm{ab}}$ is also an isomorphism. This implies that the matrix on the right hand of (1) is an isomorphism. By (1) this also implies that $\epsilon_*\big(\big(\frac{\partial \varphi(x_i)}{\partial x_j}\big)_{i,j=1,\dots,k}\big)$ is an invertible matrix. ■

Lemma 123.11. (Polynomial Matrix–Determinant Lemma) Let $k \in \mathbb{N}_0$ and let R be a commutative ring. For any two matrices $A, B \in \mathrm{M}(k \times k, R)$ we have

$$\det(A \cdot t + B) = \det(A) \cdot t^n + \text{intermediate terms} + \det(B) \cdot t^0.$$

Proof. This statement follows immediately from the Leibniz Formula 2.18 for determinants. ■

Now we can finally provide the proof of the Alexander Polynomial-of-Semidirect Product Proposition 123.9:

Proof of Proposition 123.9. Let $\pi = \langle y_1, \dots, y_k \rangle$ be a finitely generated free group and let $\varphi\colon \pi \to \pi$ be an isomorphism. We consider the corresponding semidirect product $\pi \rtimes_\varphi \langle t \rangle$ and the natural epimorphism $\Phi\colon \pi \rtimes_\varphi \langle t \rangle \to \langle t \rangle$. Recall that we need to prove the following statements:

(0) $(t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}$ is a Laurent polynomial.
(1) $(t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}$ is bimonic.
(2) We have $\deg((t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}) = k$.

First note that we have the following isomorphisms:

$$\begin{aligned} \langle y_1, \dots, y_k \rangle \rtimes_\varphi \mathbb{Z} &\overset{\text{by the HNN–Semidirect Product Lemma 106.3, see also the example on page 2373}}{=} \langle y_1, \dots, y_k, t \,|\, t \cdot y_1 \cdot t^{-1} \cdot \varphi(y_1)^{-1}, \dots, t \cdot y_k \cdot t^{-1} \cdot \varphi(y_k)^{-1} \rangle \\ &= \langle y_1, \dots, y_k, t \,|\, t \cdot y_1 \cdot t^{-1} \cdot \varphi(y_1^{-1}), \dots, t \cdot y_k \cdot t^{-1} \cdot \varphi(y_k^{-1}) \rangle \\ &\overset{\text{substitution } y_i \mapsto x_i^{-1}}{=} \langle x_1, \dots, x_k, t \,|\, t \cdot x_1^{-1} \cdot t^{-1} \cdot \varphi(x_1), \dots, t \cdot x_k^{-1} \cdot t^{-1} \cdot \varphi(x_k) \rangle. \end{aligned}$$

Note that $\Phi(t) = t$ and $\Phi(x_1) = \cdots = \Phi(x_k) = 0$. We now see that

$$\begin{aligned}(t-1)\cdot\Delta_{\pi\rtimes_\varphi\mathbb{Z},\Phi} &\overset{\substack{\text{we calculate the}\\ \text{Alexander function}\\ \text{using the generator “}t\text{”}\\ \downarrow}}{=} (t-1)\cdot\frac{\det\left(\Phi\left(\frac{\partial(t\cdot x_i^{-1}\cdot t^{-1}\cdot\varphi(x_i))}{\partial x_j}\right)_{i,j}\right)}{\Phi(t-1)} = \det\left(\Phi\left(\frac{\partial(t\cdot x_i^{-1}\cdot t^{-1}\cdot\varphi(x_i))}{\partial x_j}\right)_{i,j}\right)\\ &\underset{\substack{\uparrow\\ \text{by the properties of Fox derivatives}\\ \text{and by definition of }\Phi}}{=} \det\left(-t\cdot\mathrm{id}_k + \Phi\left(\frac{\partial\varphi(x_i)}{\partial x_j}\right)_{i,j}\right) \underset{\substack{\uparrow\\ \text{since }\Phi\text{ is trivial}\\ \text{on }\langle x_1,\ldots,x_k\rangle}}{=} \det\left(-t\cdot\mathrm{id}_k + \underbrace{\epsilon\left(\frac{\partial\varphi(x_i)}{\partial x_j}\right)_{i,j}}_{\substack{\text{by the Abelianization-of}\\ \text{Jacobi Matrix Lemma 123.10}\\ \text{this matrix is invertible over }\mathbb{Z}}}\right)\end{aligned}$$

The desired statement follows from this calculation together with the claim in the beginning of the proof. ■

123.2.3. **Alexander polynomials of fibered knots.** As the reader will surely have noticed, the Fibered Knot-π_1-Proposition 123.6 together with the Alexander Polynomial-of-Semidirect Product Proposition 123.9 gives us an interesting proposition on Alexander polynomials of fibered knots:

Proposition 123.12. (Fibered Knot–Alexander Polynomial Proposition) In the following let $K \subseteq S^3$ be a fibered knot. The following two statements hold:

(1) The Alexander polynomial is bimonic.
(2) We have $\deg(\Delta_K(t)) = 2\cdot$ genus of a fiber Seifert surface.

Example. Let $p, q \in \mathbb{N}$ be coprime. By the Torus Knot–Fibered Proposition 122.9 we know that the torus knot $T(p,q)$ is fibered. Let F be the corresponding fiber Seifert surface. We now calculate that

$$\mathrm{genus}(F) \underset{\substack{\uparrow\\ \text{by the Fibered Knot–Alexander}\\ \text{Polynomial Proposition 123.12}}}{=} \tfrac{1}{2}\cdot\deg(\Delta_K(t)) \underset{\substack{\uparrow\\ \text{by the Torus Knot–Alexander}\\ \text{Polynomial Proposition 117.11}}}{=} \tfrac{1}{2}\cdot\deg\left(\frac{(t-1)\cdot(t^{p\cdot q}-1)}{(t^p-1)\cdot(t^q-1)}\right) = \tfrac{1}{2}\cdot(pq+1-p-q).$$

Note that this discussion also provides a proof for Statement (2) of the Torus Knot–Fibered Proposition 122.9. □

Proof. Let $K \subseteq S^3$ be a fibered knot with corresponding fiber Seifert surface F. We denote by g the genus of F. By the Fibered Knot-π_1-Proposition 123.6 there exists an isomorphism $\varphi\colon \langle y_1,\ldots,y_{2g}\rangle \to \langle y_1,\ldots,y_{2g}\rangle$ and an isomorphism

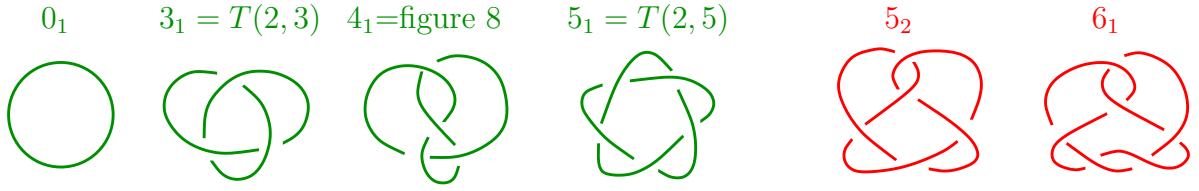
$$\Theta\colon \pi_1(S^3\setminus K) \xrightarrow{\cong} \langle y_1,\ldots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle.$$

We denote by $\Phi\colon \langle y_1,\ldots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle \to \langle t\rangle$ the natural epimorphism. As we discussed in Definition 117.10, we can calculate the Alexander polynomial of K using any epimorphism $\pi_1(S^3\setminus K) \to \langle t\rangle$. Note that

$$\Delta_K(t) \underset{\substack{\uparrow\\ \text{by the discussion in Definition 117.10, we can}\\ \text{calculate the Alexander polynomial of }K\\ \text{using any epimorphism }\pi_1(S^3\setminus K)\to\langle t\rangle}}{=} (t-1)\cdot\Delta_{\pi_K,\Phi\circ\Theta} \underset{\substack{\uparrow\\ \text{by the above isomorphism and the}\\ \text{Alexander Function Lemma 116.15}}}{=} (t-1)\cdot\Delta_{\langle y_1,\ldots,y_{2g}\rangle\rtimes_\varphi\langle t\rangle,\Phi}.$$

The proposition now follows from the Alexander Polynomial-of-Semidirect Product Proposition 123.9. ■

Example. We return to the list of knots that were shown on page 2365.



On page 2365 we discussed that the first four knots are fibered. In Question 122.14 we were wondering out loud whether the knots 5_2 and 6_1 are fibered. If we now consider the list of Alexander polynomials from page 2275 we see that the Alexander polynomials of 5_2 and 6_1 are not binomic. It thus follows from the Fibered Knot–Alexander Polynomial Proposition 123.12 that these two knots are not fibered. □

We conclude this section by picking some low-hanging fruit:

Corollary 123.13. (Fiber–Minimal Corollary) Let $K \subseteq S^3$ be a knot. If K is fibered, then for any corresponding fiber Seifert surface F we have the following equalities:

$$\text{genus}(K) \;=\; \text{genus}(F) \;=\; \tfrac{1}{2}\cdot \deg(\Delta_K(t)),$$

in particular F is a Seifert surface of minimal genus for K.

Proof. Let K be a fibered knot with corresponding fiber Seifert surface F. We see that

$$\begin{array}{ccccccc} \text{genus}(K) & \underset{\uparrow}{\leq} & \text{genus}(F) & \underset{\uparrow}{=} & \tfrac{1}{2}\cdot\deg(\Delta_K(t)) & \underset{\uparrow}{\leq} & \text{genus}(K). \\ \text{by Definition 121.8} & & & \text{by the Fibered Knot–Alexander Polynomial Proposition 123.12} & & \text{by the Genus–Alexander Polynomial Proposition 123.2} & \end{array}$$

We see that all inequalities have to be equalities and we get the promised result. ■

Remark. Let K be a fibered knot with fiber Seifert surface F. By the Fiber–Minimal Corollary 123.13 we know that F is a Seifert surface of minimal genus. In fact, as is pointed out in [**EL1983**, Lemma 5.1], using the Low-Dimensional s-Cobordism Theorem **??** one can show that every Seifert surface of minimal genus is smoothly isotopic, rel K, to F (we will also provide a proof in Exercise 126.3). In general though, for non fibered knots, minimal genus Seifert surfaces are not unique, examples are for instance provided in [**Alf1970**] and [**Alt2012**]. □

123.3. **Appendix: Alternating knots.** We have now shown that the Alexander polynomial gives lower bounds on the genus and that the Alexander polynomial of a fibered knot is bimonic. The Alexander polynomial is an imperfect invariant though, for example we saw on page 2370 that in general it does not determine the genus of a knot.

In this appendix we formulate two theorems which show that in contrast the Alexander polynomial of an *alternating* knot is a rather powerful invariant:

Theorem 123.14. (Alternating-Δ-Genus Theorem) Let K be an alternating knot. The following two statements hold:

(1) $\deg(\Delta_K(t)) = 2 \cdot \text{genus}(K)$.
(2) If we apply Seifert's algorithm to an alternating diagram, then we obtain a Seifert surface of minimal genus.

Proof. This theorem was first proved independently by Kunio Murasugi, see [**Muras1958**, p. 235 and Theorem 4.1], and Richard Crowell [**Crow1959**, Theorem 3.5]. Alternative proofs are given in [**Gab1986a**, Theorem 4] and [**Greene2017**]. ■

Theorem 123.15. (Alternating-Δ-Fibered Theorem) For any alternating knot K we have

$$\Delta_K(t) \text{ is bimonic} \iff K \text{ is fibered.}$$

Proof. This theorem was first proved by Kunio Murasugi [**Muras1963**, Theorem 2]. The theorem is also a consequence of [**Ni2007**, Theorem 1.1]. ■

123.4. Exercises for Chapter 123 .

Exercise 123.1. We consider the knots 6_2, 6_3 which are shown below (together with their Alexander polynomials.) Show that in each case the genus equals 2.
Remark. This exercise, together with the discussion on page 2337 and 2378 (and the observation that 5_1 and 7_1 are torus knots), and Exercise 121.8 gives us the genera for all knots that are shown in the figure on page 2275.

6_2



$1 - 3t + 3t^2 - 3t^3 + t^4$

6_3



$1 - 3t + 5t^2 - 3t^3 + t^4$

Exercise 123.2. Given a finitely generated group π we denote, for the purpose of this exercise, by $d(\pi)$ the minimal number of generators of π. Given a knot K we define

$$\widetilde{g}(K) \;:=\; \min\Big\{ d(\pi_1(\Sigma)) \;\Big|\; \begin{array}{l}\Sigma \text{ is a compact (possibly non-orientable)} \\ \text{connected submanifold of } S^3 \text{ with } \partial\Sigma = K\end{array} \Big\}.$$

(a) Determine $\widetilde{g}$ of the trefoil.
(b) Show that given any $m \in \mathbb{N}$ there exists a knot K with $\text{genus}(K) \geq m$ and $\widetilde{g}(K) = 1$.

Exercise 123.3. Let π and Γ be two groups and let $\alpha, \beta \colon \Gamma \to \pi$ be two homomorphisms. Recall that in Definition 106.2 we defined the corresponding HNN-extension:

$$\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g) \cdot t \cdot \beta(g)^{-1} \cdot t^{-1}\}_{g \in \Gamma} \rangle\rangle.$$

We introduce the following language:

- We say that an HNN-extension is *ascending* if α is an isomorphism.
- We refer to the homomorphism $\Phi \colon \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \to \langle t \rangle$ that is given by $g \mapsto t^0$ for $g \in \pi$ and $t \mapsto t$ as the *natural epimorphism*.

Show that if the HNN-extension is ascending, then the Alexander function corresponding to the natural homomorphism is monic and show that in general it is not bimonic.

Exercise 123.4. We say that an m-component link $L = L_1 \sqcup \cdots \sqcup L_m$ is a *boundary link* if there exists a collection of m disjoint Seifert surfaces $F_1, \ldots, F_m$ such that for $i = 1, \ldots, m$ we have $\partial F_i = L_i$.

(a) Give an example of a boundary link that is not split.
Remark. You could use the higher order Alexander polynomials of Exercise 117.16 to show that the boundary link is not split.

(b) Show that if L is an m-component boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group on m generators.
Hint. Use the Seifert surfaces and bicollars to construct an interesting continuous map $S^3 \setminus L \to \bigvee_{i=1}^{m} S^1$.

(c) Show that if $m \geq 2$ and if L is a boundary link, then $\Delta_L = 0$.

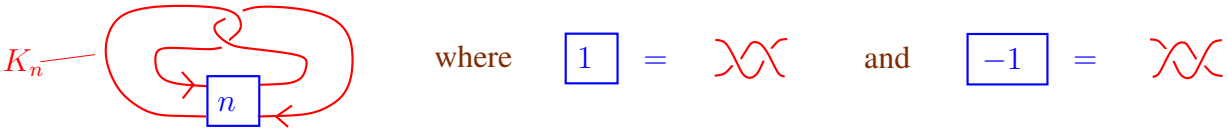
Exercise 123.5. Let $\varphi\colon \langle x, y\rangle \to \langle x, y\rangle$ be the homomorphism that is uniquely determined by $\varphi(x) = x \cdot y^{-1} \cdot x^{-1}$ and $\varphi(y) = x \cdot y$. Show that φ is an isomorphism.

Exercise 123.6. Let K be a non-trivial knot such that $\deg(\Delta_K(t)) = 2\text{genus}(K)$. Show that if $\Delta_K(t)$ is irreducible, then K is prime.
Remark. This exercise is identical to Exercise 123.6.

Exercise 123.7. Let K and $\widetilde{K}$ be two disjoint oriented links. Show that if there exists a Seifert surface for K that is disjoint from $\widetilde{K}$, then $\text{lk}(K, \widetilde{K}) = 0$.
Hint. Consider the description of $\pi_1(S^3 \setminus K)$ in the proof of the Genus–Alexander Polynomial Proposition 123.2.

Exercise 123.8. Given $n \in \mathbb{Z}$ we consider the n-twist knot K_n that is shown in the figure below.

(a) Determine the Alexander polynomial $\Delta_{K_n}(t)$.
(b) What is the genus of K_n?
(c) Which n-twist knots are fibered?



CHAPTER 124

The twisted Alexander function of groups

Let π be a group that admits a presentation of deficiency one and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group. In Definition 116.14 we introduced the corresponding Alexander function $\Delta_{\pi,\Phi} \in \mathbb{Q}(H)$, where $\mathbb{Q}(H)$ denotes the quotient field of $\mathbb{Z}[H]$.

In this chapter we will generalize this concept. More precisely if on top of π and Φ we are also given a unique factorization domain (UFD) R together with a homomorphism $\alpha\colon \pi \to \mathrm{GL}(n, R)$, then we will see in this chapter that we can define the twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi} \in Q_R(H)$, where $Q_R(H)$ denotes the quotient field of the group ring $R[H]$. The distribution of labor between this and the following chapter is as follows:

(1) In this chapter we will outline the definition and main properties of the twisted Alexander function.
(2) In the following chapter we will use the twisted Alexander function to introduce the twisted Alexander polynomial of oriented knots and links. Furthermore we will study many applications to the theory of knots and links.

Many of the proofs in this and the following chapter are slight generalizations of the earlier "untwisted" statements. To keep this and the next chapter at reasonable lengths, we will often just give outlines of the proofs. We leave it to the reader to fill in the details.

This chapter is organized as follows:

- In Section 124.1 we will define the twisted Alexander function $\Delta^{\alpha}_{G,\Phi}$ of a group G, a homomorphism $\Phi\colon G \to H$ to a free abelian group together with a homomorphism $\alpha\colon G \to \mathrm{GL}(n, R)$.
- In Section 124.2 we will provide examples of twisted Alexander functions.
- In Section 124.3 we will prove the twisted Alexander Function Theorem 124.10, which shows that the twisted Alexander function is well-defined.
- In Section 124.4 we will extend the notion of twisted Alexander functions to representations $\alpha\colon G \to \mathrm{Aut}_R(V)$ where V is a free R-module.
- In Section 124.5 we will provide exercises for Chapter 124.

In the next Chapter 125 we will use the twisted Alexander functions of groups to introduce the twisted Alexander polynomials of knots and links.

124.1. **Definition of the twisted Alexander function.** In this section we want to define twisted Alexander functions corresponding to homomorphisms $\pi \to \mathrm{GL}(n, R)$. We start out with some notation and three really elementary lemmas:

Notation 124.1. Given a ring R and given $k, l \in \mathbb{N}_0$ we denote by $\mathrm{M}(k \times l, R)$ the set of $k \times l$-matrices with entries in R. Furthermore for $k, l, m \in \mathbb{N}_0$ we denote by

$$\mu \colon \mathrm{M}(k \times l, R) \times \mathrm{M}(l \times m, R) \ \to \ \mathrm{M}(k \times m, R)$$

the map that is given by the usual matrix multiplication. Note this map turns $\mathrm{M}(k \times k, R)$ into a ring.

Lemma 124.2. (Group Ring-to-Matrix Ring Lemma) Let π be a group, let S be a ring and let $\gamma \colon \pi \to \mathrm{GL}(n, S)$ be a homomorphism. The map

$$\begin{array}{rcl} \gamma_* \colon \mathbb{Z}[\pi] & \to & \mathrm{M}(n \times n, S) \\ \sum\limits_{i=1}^{k} \underbrace{a_i \cdot g_i}_{\substack{\uparrow \\ \text{where } a_i \in \mathbb{Z} \text{ and } g_i \in \pi}} & \mapsto & \sum\limits_{i=1}^{k} a_i \cdot \gamma(g_i) \end{array}$$

is a ring homomorphism.

Proof. The lemma follows almost immediately from the definitions. ■

Lemma 124.3. (Matrix Multiplication–Change-of-Ring Lemma) Let $\varphi \colon R \to S$ be a ring homomorphism. The maps

$$\begin{array}{rcl} \varphi_* \colon \mathrm{M}(k \times l, R) & \to & \mathrm{M}(k \times l, S) \\ (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

respect matrix multiplication in the sense that for any $k, l, m \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{M}(k \times l, R) \times \mathrm{M}(l \times m, R) & \xrightarrow{\mu} & \mathrm{M}(k \times m, R) \\ \downarrow{\scriptstyle \varphi_*} \quad \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{M}(k \times l, S) \times \mathrm{M}(l \times m, S) & \xrightarrow{\mu} & \mathrm{M}(k \times m, S) \end{array}$$

commutes.

Proof. The lemma follows again almost immediately from the definitions. ■

Lemma 124.4. (Matrix-of-Matrices Lemma) Let S be a ring and let $n \in \mathbb{N}_0$. The maps[1284]

$$\begin{array}{rcl} \Xi \colon \mathrm{M}(k \times l, \mathrm{M}(n \times n, S)) & \to & \mathrm{M}((k \cdot n) \times (l \cdot n), S) \\ \underset{\substack{\uparrow \\ \text{each } A_{ij} \text{ is an} \\ (n \times n)\text{-matrix over } S}}{(\mathrm{A}_{ij})} & \mapsto & \begin{array}{l} \text{matrix where the } (i \cdot n + i') \times (j \cdot n + j') \text{ is} \\ \text{given by the } (i', j')\text{-entry of the matrix } A_{ij} \end{array} \end{array}$$

are isomorphisms of abelian groups that respect matrix multiplication, in the sense that for any $k, l, m \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{M}(k \times l, \mathrm{M}(n, \times n, S)) \times \mathrm{M}(l \times m, \mathrm{M}(n, \times n, S)) & \xrightarrow{\mu} & \mathrm{M}(k \times m, \mathrm{M}(n, \times n, S)) \\ \downarrow{\scriptstyle \Xi} \quad \downarrow{\scriptstyle \Xi} & & \downarrow{\scriptstyle \Xi} \\ \mathrm{M}((k \cdot n) \times (l \cdot n), S) \times \mathrm{M}((l \cdot n) \times (m \cdot n), S) & \xrightarrow{\mu} & \mathrm{M}((k \cdot n) \times (m \cdot n), S) \end{array}$$

commutes.

$$\Xi\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \Xi\begin{pmatrix} \boxed{\begin{matrix}(A_{11})_{11} & (A_{11})_{12} \\ (A_{11})_{21} & (A_{11})_{22}\end{matrix}} & \boxed{\begin{matrix}(A_{12})_{11} & (A_{12})_{12} \\ (A_{12})_{21} & (A_{12})_{22}\end{matrix}} \\ \boxed{\begin{matrix}(A_{21})_{11} & (A_{21})_{12} \\ (A_{21})_{21} & (A_{21})_{22}\end{matrix}} & \boxed{\begin{matrix}(A_{22})_{11} & (A_{22})_{12} \\ (A_{22})_{21} & (A_{22})_{22}\end{matrix}} \end{pmatrix} = \begin{pmatrix} (A_{11})_{11} & (A_{11})_{12} & (A_{12})_{11} & (A_{12})_{12} \\ (A_{11})_{21} & (A_{11})_{22} & (A_{12})_{21} & (A_{12})_{22} \\ (A_{21})_{11} & (A_{21})_{12} & (A_{22})_{11} & (A_{22})_{12} \\ (A_{21})_{21} & (A_{21})_{22} & (A_{22})_{21} & (A_{22})_{22} \end{pmatrix}$$

Proof. The lemma is elementary and follows quite easily from the definitions. We leave it to the reader to fill in the details. ■

Shortly we will also need the following notation:

Notation 124.5. Let π be a group, let S be a ring and let $\gamma\colon \pi \to \mathrm{GL}(n,S)$ be a homomorphism. Given $k,l \in \mathbb{N}$ we denote by γ_* also the map

$$\begin{array}{rcl} \gamma_*\colon \mathrm{M}(k\times l, \mathbb{Z}[\pi]) & \to & \mathrm{M}((k\cdot n)\times(l\cdot n), S) \\ (a_{ij}) & \mapsto & \Xi(\ \underbrace{\gamma_*(a_{ij})}_{\in \mathrm{M}(k\times l, \mathrm{M}(n\times n, S))}\ ). \end{array}$$

We point out that it follows from the Group Ring-to-Matrix Ring Lemma 124.2, the Matrix Multiplication–Change-of-Ring Lemma 124.3 and the Matrix-of-Matrices Lemma 124.4 that the above maps $\gamma_*\colon \mathrm{M}(k\times l,\mathbb{Z}[\pi]) \to \mathrm{M}((k\cdot n)\times(l\cdot n),S)$ again respect matrix multiplications.

After this tiring list of trivialities, let us continue with a definition, which recalls some concepts introduced in Definition 116.9:

Definition 124.6. Let $P = \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle$ be a finite presentation.
(1) As in Definition 116.8 we denote by $\mu_P\colon \langle x_1,\dots,x_k\rangle \to P$ the natural projection.
(2) As in Definition 116.9 we define the Jacobi matrix $J(P)$ to be the $(l\times k)$-matrix

$$J(P) := \left\{\mu_{P*}\Big(\ \underbrace{\frac{\partial r_i}{\partial x_j}}_{\in\mathbb{Z}[\langle x_1,\dots,x_k\rangle]}\ \Big)\right\}_{\substack{i=1,\dots,l\\ j=1,\dots,k}} \in \mathrm{M}(l\times k, \mathbb{Z}[P]).$$

We move on to a new concept:

Definition 124.7. Let π be a group, let $\Phi\colon \pi\to H$ be a homomorphism to an abelian group H, let R be a commutative ring and let $\alpha\colon \pi\to \mathrm{GL}(n,R)$ be a homomorphism. We consider the map

$$\begin{array}{rcl} \Phi\otimes\alpha\colon \pi & \to & \mathrm{GL}(n, R[H]) \\ g & \mapsto & \alpha(g)\cdot\Phi(g). \end{array}$$

It follows easily from the definitions and the hypothesis that R is commutative and that H is abelian that the map $\Phi\otimes\alpha\colon \pi\to\mathrm{GL}(n,R[H])$ is in fact a homomorphism.

Example. Let π be a group, let $\Phi\colon \pi\to H$ be a homomorphism to a free abelian group H, let R be a commutative ring and let $\alpha\colon \pi\to\mathrm{GL}(n,R)$ be a homomorphism. Finally let $x\in\pi$ such that $y := \Phi(x)$ is non-trivial. We can perform the following calculation in the Laurent polynomial ring $R[y^{\pm1}] \subseteq R[H]$:

[1284] In plain English the map Ξ does the following: An element of $\mathrm{M}(k\times l, \mathrm{M}(n\times n,S))$ is a $(k\times l)$-matrix of $(n\times n)$-matrices. We now view such a matrix as an $(k\cdot n)\times(l\cdot n)$-matrix in the obvious way. In practice we will often just apply Ξ without specifically pointing out, that we are applying Ξ.

$$\det((\Phi\otimes\alpha)_*(x-1)) \underset{\text{by the above Definition 124.7}}{=} \det(\alpha(x)\cdot y-\mathrm{id}_n) \underset{\text{by the Polynomial Matrix Determinant Lemma 123.11}}{=} \overbrace{\det(\alpha(x))\cdot y^n+\cdots+\det(-\mathrm{id}_n)}^{\text{since } H \text{ is free abelian and } y \text{ is non-trivial this is a polynomial in } y \text{ of degree } n} \neq 0$$

□

The following notation is a slight variation on the notation from Notation 116.13:

Notation 124.8. Let R be a commutative domain.

(1) We denote by Q_R the quotient ring of R.
(2) As in Definition 2.6 we denote by $R^\times$ the group of multiplicative units of R.

Let H be a free abelian group. (Note that this implies that $R[H]$ is a domain.)

(3) We denote by $Q_R(H)$ the quotient ring of $R[H]$.
(4) Given $p,q\in Q_R(H)$ we write[1285] $p \doteq_R q$ if there exists a unit $\epsilon\in R^\times$ and an $h\in H$ with $p=\epsilon\cdot h\cdot q$.

Finally we recall the following notation from Notation 116.10:

Notation 124.9. Let R be a ring and let $M\in \mathrm{M}(l\times k,R)$ be a matrix. Given $i\in\{1,\dots,k\}$ we denote by M_i the $l\times(k-1)$-matrix which we obtain by deleting the i-th column.

We can now formulate the following analog of the Presentation Function Quotient Proposition 116.11 and the Alexander Function Theorem 116.12:

Theorem 124.10. (Twisted Alexander Function Theorem) Let π be a group, let H be a free abelian group, let $\Phi\colon \pi\to H$ be a non-trivial homomorphism, let R be a domain and let $\alpha\colon \pi\to \mathrm{GL}(n,R)$ be a homomorphism. Finally assume that we are given a presentation[1286]

$$P=\langle x_1,\dots,x_k \,|\, r_1,\dots,r_{k-1}\rangle \xrightarrow{\cong} \pi$$

of deficiency one.

(1) There exists an $i\in\{1,\dots,k\}$ such that $\Phi(x_i)$ is non-trivial.
(2) If $\Phi(x_i)$ is non-trivial, then $\det((\Phi\otimes\alpha)_*(x_i-1))\neq 0$.

We pick an $i\in\{1,\dots,k\}$ such that $\Phi(x_i)$ is non-trivial. We consider

$$\Delta^\alpha_{\pi,\Phi} := \frac{\det((\Phi\otimes\alpha)_*(J(P)_i))}{\det((\Phi\otimes\alpha)_*(x_i-1))} \in Q_R(H).$$

The following statements hold:

(3) Up to a sign $\Delta^\alpha_{\pi,\Phi}$ does not depend on the choice of i.
(4) If R is a unique factorization domain (UFD), then up to "$\doteq_R$" $\Delta^\alpha_{\pi,\Phi}$ does not depend on the choice of the deficiency one presentation.

Proof. As we will see, the proof of the theorem is very similar to the earlier proofs of the Presentation Function Quotient Proposition 116.11 and the Alexander Function Theorem 116.12. We will provide the details in Section 124.3. ■

[1285] Since $-1,1$ are the only units of $\mathbb{Z}$ we see that "$\doteq_{\mathbb{Z}}$" is equivalent to the symbol "$\doteq$" that we introduced in Notation 116.13.

[1286] To simplify the notation, which is already heavy enough, we suppress the isomorphism from the presentation to the group from the notation.

The remainder of this chapter and the next chapter are organized as follows:

- In the remainder of this section we will discuss consequences of the Alexander Function Theorem 124.10, in particular we will state two very elementary lemmas about twisted Alexander functions.
- In Section 124.2 we will do an explicit calculation of a twisted Alexander function, just to get a sense of what is happening.
- In Section 124.3 we will finally provide a proof of the Twisted Alexander Function Theorem 124.10.
- In Section 124.4 we will slightly generalize our setup, namely we will define twisted Alexander functions for any pair $(\pi, \Phi\colon \pi \to H)$ together with any representation $\alpha\colon \pi \to \mathrm{Aut}_R(M)$, where R is a commutative ring and M is an R-module.
- In Chapter 125 we will apply the theory of twisted Alexander functions to knots and links.

The Alexander Function Theorem 124.10 allows us to make the following definition:

Definition 124.11. Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. If π admits a presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1}\rangle$ of deficiency one, then we refer to

$$\Delta^{\alpha}_{\pi,\Phi} \coloneqq \underbrace{\frac{\det((\Phi\otimes\alpha)_*(J(P)_i))}{\det((\Phi\otimes\alpha)_*(x_i - 1))}}_{\text{where } \Phi(x_i) \text{ is non-trivial}} \in Q_R(H)$$

as the twisted Alexander function of (π, Φ, α). By the Twisted Alexander Function Theorem 124.10 the twisted Alexander function is well-defined up to "$\doteq_R$".

Example. Suppose that in the previous definition we have $R = \mathbb{Z}$, $n = 1$ and that we are considering the trivial representation $\tau\colon \pi \to \mathrm{GL}(1, \mathbb{Z})$ that is given by $g \mapsto (1)$. It then follows immediately from the definition that the twisted Alexander function $\Delta^{\tau}_{\pi,\Phi}$ equals the Alexander function $\Delta_{\pi,\Phi}$ from Definition 116.14. □

Remark. The twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi}$ was first introduced by Masaaki Wada [**Wad1994**] and it is thus sometimes referred to as *Wada's invariant*. There is a large literature on this topic. In particular the twisted Alexander function equals or is closely related to at least two other "twisted invariants", namely the *twisted Reidemeister torsion* and the *twisted Alexander function* of (π, Φ, α). We refer to [**FV2011a**] and [**Kit2015**] for surveys on these invariants and a discussion how these various invariants are related. In particular in [**FV2011a**] it is shown that in suitable settings one can drop the hypothesis that R is a UFD and, more importantly, it is shown (using Chapman's Theorem **??**, the Whitehead Group-3D Theorem 344.31 and the Torsion–Whitehead Torsion Theorem 349.11) that in suitable settings the indeterminacy of the twisted Alexander function can be reduced to multiplication by $\{\pm\det(\alpha(g)) \cdot h \,|\, g \in \pi \text{ and } h \in H\}$. □

Remark. It is clear from the definition of the twisted Alexander function that the computational effort grows dramatically with the number of generators. Fortunately in Section 115.5 we learned about a practical way to reduce the number of generators of a presentation. □

We conclude this section with two elementary and formal lemmas about twisted Alexander functions. First of all we have the following lemma, which is a generalization of the Alexander Function–Functorial Lemma 116.15:

Lemma 124.12. (Twisted Alexander Function–Functorial Lemma)

(1) Let π and $\widetilde{\pi}$ be two groups which admit presentations of deficiency one, let $f\colon \widetilde{\pi} \to \pi$ be an isomorphism, let $\Phi\colon \pi \to H$ be a non-trivial homomorphisms to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n,R)$ be a homomorphism. Then

$$\Delta^{\alpha\circ f}_{\widetilde{\pi},\Phi\circ f} \doteq_R \Delta^{\alpha}_{\pi,\Phi} \quad \in Q_R(H).$$

(2) Let π be a group which admits a presentation of deficiency one. Let $\Phi\colon \pi \to G$ be a non-trivial homomorphism to a free abelian group, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n,R)$ be a homomorphism.

(a) Let H be a free abelian group and let $f\colon G \to H$ be a homomorphism such that $f\circ\Phi\colon \pi \to H$ is also non-trivial. Then[1287]

$$f_*(\Delta^{\alpha}_{\pi,\Phi}) \doteq_R \Delta^{\alpha}_{\pi,f\circ\Phi} \quad \in Q_R(H).$$

(b) Let S be a UFD and let $\varphi\colon R \to S$ be a ring homomorphism. As in the Matrix Multiplication–Change-of-Ring Lemma 124.3 let $\varphi_*\colon \mathrm{GL}(n,R) \to \mathrm{GL}(n,S)$ be the induced homomorphism. Then[1288]

$$\varphi_*(\Delta^{\alpha}_{\pi,\Phi}) \doteq_R \Delta^{\varphi_*\circ\alpha}_{\pi,\Phi} \quad \in Q_S(G).$$

Proof.

(1) This statement is an immediate consequence of the Twisted Alexander Function Theorem 124.10.

(2) By (1) we can assume that $\pi = \langle x_1,\dots,x_k \,|\, r_1,\dots,r_{k-1}\rangle$. We pick $i \in \{1,\dots,k\}$ such that $(f\circ\Phi)(x_i) \neq 0$.

(a) We see that

$$\begin{aligned} f_*(\Delta^{\alpha}_{\pi,\Phi}) &= f_*\Big(\frac{\det((\Phi\otimes\alpha)_*(J(\pi)_i))}{\det((\Phi\otimes\alpha)_*(x_i-1))}\Big) = \frac{f_*(\det((\Phi\otimes\alpha)_*(J(\pi)_i)))}{f_*(\det((\Phi\otimes\alpha)_*(x_i-1)))} \\ &\underset{\uparrow}{=} \frac{\det(f_*((\Phi\otimes\alpha)_*(J(\pi)_i)))}{\det(f_*((\Phi\otimes\alpha)_*(x_i-1)))} \underset{\uparrow}{=} \frac{\det((f\circ\Phi)\otimes\alpha)_*(J(\pi)_i)}{\det((f\circ\Phi)\otimes\alpha)_*(x_i-1))} = \Delta^{\alpha}_{\pi,f\circ\Phi}. \end{aligned}$$

since determinants commute with ring homomorphisms

it follows easily from the definitions that for any matrix A over $\mathbb{Z}[\pi]$ we $f_*((\Phi\otimes\alpha)_*(A)) = ((f\circ\Phi)\otimes\alpha)(A)$

(b) This statement follows, similar to (a), easily from the definitions. We leave it to the reader to fill in the details. ■

Before we can formulate the next lemma, we need to introduce a harmless piece of notation:

[1287]The group homomorphism $f\colon G \to H$ defines a ring homomorphism $f_*\colon R[G] \to R[H]$ and a ring homomorphism $f_*\colon \{\frac{p}{q} \,|\, p,q \in R[G] \text{ with } f_*(q) \neq 0\} \to Q_R(H)$.

[1288]The ring homomorphism $\varphi\colon R \to S$ defines a ring homomorphism $f_*\colon R[G] \to S[G]$ and a ring homomorphism $\varphi_*\colon \{\frac{p}{q} \,|\, p,q \in R[G] \text{ with } \varphi_*(q) \neq 0\} \to Q_S(G)$.

Notation 124.13. Let π be a group, let R be a commutative ring and let $\alpha\colon \pi \to \mathrm{GL}(n,R)$ be a homomorphism. Finally let $A \in \mathrm{GL}(n,R)$. We introduce the conjugated representation

$$\begin{array}{rccc} \alpha^A\colon & \pi & \to & \mathrm{GL}(n,R) \\ & g & \mapsto & A^{-1}\circ\alpha(g)\circ A. \end{array}$$

Lemma 124.14. (Conjugation Invariance Lemma) Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n,R)$ be a homomorphism. Finally let $A \in \mathrm{GL}(n,R)$. For the conjugated representation $\alpha^A\colon \pi \to \mathrm{GL}(n,R)$ we have

$$\Delta^{\alpha}_{\pi,\Phi} \;=\; \Delta^{\alpha_A}_{\pi,\Phi} \;\in\; Q_R(H).$$

Proof. One can easily verify that for any $(r\times s)$-matrix M over $\mathbb{Z}[\pi]$ we have

$$(\Phi\otimes\alpha^A)_*(M) \;=\; \mathrm{diag}\underbrace{(A^{-1},\dots,A^{-1})}_{r\text{ copies}}\cdot(\Phi\otimes\alpha)_*(M)\cdot\mathrm{diag}\underbrace{(A,\dots,A)}_{s\text{ copies}}.$$

In particular, if $r=s$, then we see that $\det((\Phi\otimes\alpha^A)_*(M)) = \det((\Phi\otimes\alpha)_*(M))$. The desired equality of twisted Alexander functions follows immediately from this observation and the definition of the twisted Alexander function. ∎

124.2. **Example of a twisted Alexander function.** In this section we want to calculate the twisted Alexander function in a particular case, just to get a sense for the definition and for what is happening. The group π under consideration will be $\pi := \pi_1(S^3\setminus T(2,3))$, where $T(2,3)$ is the trefoil. We make two observations:

(i) We already know, e.g. by considering one of the presentations below, that there exists (up to a sign) a unique epimorphism $\Phi\colon \pi \to \langle t\rangle$. It follows from the Twisted Alexander Function–Functorial Lemma 124.12 (1) that this indeterminacy has only a negligible effect on any twisted Alexander function.

(ii) We consider the permutation group S_3.

- Using the Presentation–Homomorphism Lemma 100.13 and say the presentation provided by the Torus Knot-π_1-Proposition 111.18 one can easily verify that, up to conjugation, there exists a unique epimorphism $\varphi\colon \pi_1(S^3\setminus K)\to S_3$.
- We consider the following two special elements of S_3:

$$\sigma \;=\; \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \qquad \text{(cyclic permutation)}$$

$$\tau \;=\; \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \qquad \text{(swapping of second and third number).}$$

One can easily show, say using Exercise 100.7 and using the Presentation–Homomorphism Lemma 100.13, that there exists a unique homomorphism $\theta\colon S_3\to\mathrm{GL}(2,\mathbb{Z})$ with

$$\theta(\sigma) = \begin{pmatrix}0&1\\-1&-1\end{pmatrix} \qquad \text{and} \qquad \theta(\tau) = \begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

In combination we get a homomorphism $\alpha := \theta\circ\varphi\colon \pi\to\mathrm{GL}(2,\mathbb{Z})$.

In the following, for practice, we will perform the calculation of the twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi}\in\mathbb{Q}(t)$ using two different presentations:

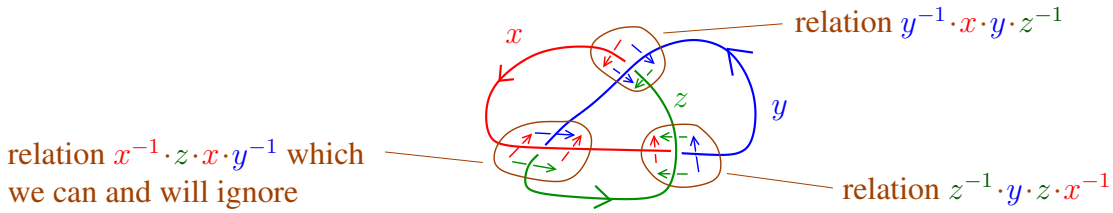
(1) We will use the presentation from the Wirtinger Presentation Proposition 113.6 (2).

(2) We will use the presentation from the Torus Knot-π_1-Proposition 111.18.

These calculations will happen in the following two subsections. The calculations show that calculating twisted Alexander functions is doable, but unfortunately also a pain, which is best left to computers.[1289] In this particular example it is also not entirely clear, what the resulting twisted Alexander function tries to tell us.

124.2.1. **Calculation using the Wirtinger presentation.** Recall that on page 2199 we used the Wirtinger Presentation Proposition 113.6 (2) to show that there exists an isomorphism

$$P := \langle x, y, z \,|\, y^{-1}\cdot x\cdot y\cdot z^{-1}, z^{-1}\cdot y\cdot z\cdot x^{-1}, \underbrace{x^{-1}\cdot z\cdot x\cdot y^{-1}}_{\text{not needed}}\rangle \xrightarrow{\cong} \pi_1(S^3\setminus K).$$



Using the above presentation P and the Presentation–Homomorphism Lemma 100.13 we obtain the following two homomorphisms:

(i) We obtain an epimorphism $\Phi\colon \pi \to \langle t\rangle$ that is given by $\Phi(x) = \Phi(y) = \Phi(z) = t$.
(ii) We consider again the above permutations $\sigma, \tau \in S_3$. Note that $\sigma^3 = \mathrm{id}$, $\tau^2 = \mathrm{id}$ and $\sigma\circ\tau\circ\sigma = \tau$. We can now easily verify that we have a well-defined epimorphism $\varphi\colon \pi \to S_3$ that is given by

$$\varphi(x) = \tau = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\quad \varphi(y) = \sigma\circ\tau\circ\sigma^{-1} = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix},\quad \varphi(z) = \sigma^2\circ\tau\circ\sigma^{-2} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}.$$

Note that with $\alpha = \theta\circ\varphi\colon \pi \to \mathrm{GL}(2,\mathbb{Z})$ we now have

$$\begin{aligned}
(\Phi\otimes\alpha)(x) &= \Phi(x)\cdot\alpha(x) = t\cdot\theta(\tau) &&= t\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} &&= \begin{pmatrix}0&t\\t&0\end{pmatrix},\\
(\Phi\otimes\alpha)(y) &= \Phi(y)\cdot\alpha(y) = t\cdot\theta(\sigma\circ\tau\circ\sigma^{-1}) &&= t\cdot\begin{pmatrix}-1&-1\\0&1\end{pmatrix} &&= \begin{pmatrix}-t&-t\\0&t\end{pmatrix},\\
(\Phi\otimes\alpha)(z) &= \Phi(z)\cdot\alpha(z) = t\cdot\theta(\sigma^2\circ\tau\circ\sigma^{-2}) &&= t\cdot\begin{pmatrix}1&0\\-1&-1\end{pmatrix} &&= \begin{pmatrix}t&0\\-t&-t\end{pmatrix}.
\end{aligned}$$

Next recall that on page 2253 we saw that the Jacobi matrix $J(P) \in \mathrm{M}(2\times 3, \mathbb{Z}[P])$ is given by

$$J(P) = \begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & -1 \\ -1 & z^{-1} & z^{-1}\cdot y - z^{-1}\end{pmatrix}.$$

[1289] Or for older people: Better left to students with programming skills.

We obtain the following equalities in $\mathbb{Q}(t)$:

$$\Delta^{\alpha}_{\pi,\Phi} = \frac{\det((\Phi\otimes\alpha)_*(J(P)_1)}{\det((\Phi\otimes\alpha)_*(x-1))} = \frac{\det\left((\Phi\otimes\alpha)_*\begin{pmatrix} \cancel{y^{-1}} & y^{-1}\cdot x - y^{-1} & -1 \\ \cancel{-1} & z^{-1} & z^{-1}\cdot y - z^{-1}\end{pmatrix}\right)}{\det((\Phi\otimes\alpha)_*(x-1))}$$

$$= \frac{\det\left(\begin{array}{cc|c} \begin{pmatrix}-t^{-1} & -t^{-1}\\ 0 & t^{-1}\end{pmatrix}\cdot\begin{pmatrix}0 & t\\ t & 0\end{pmatrix} - \begin{pmatrix}-t^{-1} & -t^{-1}\\ 0 & t^{-1}\end{pmatrix} & & \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \\ \hline & \begin{pmatrix}t^{-1} & 0\\ -t^{-1} & -t^{-1}\end{pmatrix} & \begin{pmatrix}t^{-1} & 0\\ -t^{-1} & -t^{-1}\end{pmatrix}\cdot\begin{pmatrix}-t & -t\\ 0 & t\end{pmatrix} - \begin{pmatrix}t^{-1} & 0\\ -t^{-1} & -t^{-1}\end{pmatrix}\end{array}\right)}{\det\left(\begin{pmatrix}0 & t\\ t & 0\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)}$$

$$= \frac{\det\left(\begin{array}{cc|c} \begin{pmatrix}-1+t^{-1} & -1+t^{-1}\\ 1 & -t^{-1}\end{pmatrix} & & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\ \hline & \begin{pmatrix}t^{-1} & 0\\ -t^{-1} & -t^{-1}\end{pmatrix} & \begin{pmatrix}-1-t^{-1} & -1\\ 1+t^{-1} & t^{-1}\end{pmatrix}\end{array}\right)}{\det\begin{pmatrix}-1 & t\\ t & -1\end{pmatrix}} = \frac{\det\left(\begin{array}{cc|cc} -1+t^{-1} & -1+t^{-1} & -1 & 0\\ 1 & -t^{-1} & 0 & -1 \\ \hline t^{-1} & 0 & -1-t^{-1} & -1\\ -t^{-1} & -t^{-1} & 1+t^{-1} & t^{-1}\end{array}\right)}{\det\begin{pmatrix}-1 & t\\ t & -1\end{pmatrix}}$$

$$= \frac{(t^{-2}-1)^2}{1-t^2} = t^{-2}-1.$$

124.2.2. **Calculation using the torus knot presentation.** For practice, and as a sanity check, we also calculate the twisted Alexander function of $\pi_1(S^3\setminus T(p,q))$ using a different presentation. Namely, for coprime $p,q\in\mathbb{N}$ we consider the presentation

$$Q := \langle x, y \,|\, x^p\cdot y^{-q}\rangle \xrightarrow{\cong} \pi_1(S^3\setminus T(p,q))$$

from the Torus Knot-π_1-Proposition 111.18, where for the trefoil we have $p=2$ and $q=3$. Using the above presentation Q and the Presentation–Homomorphism Lemma 100.13 we obtain the following two homomorphisms:

(i) We obtain an epimorphism $\Phi\colon Q\to\langle t\rangle$ that is given by $\Phi(x)=t^q$ and $\Phi(y)=t^p$.
(ii) We obtain an epimorphism $\varphi\colon \langle x,y\,|\,x^2\cdot y^{-3}\rangle \to S_3$ that is given by $\varphi(x)=\tau$ and $\varphi(y)=\sigma$.

Note that with $\alpha=\theta\circ\varphi\colon \pi\to \mathrm{GL}(2,\mathbb{Z})$ we now have

$$(\Phi\otimes\alpha)(x) = \Phi(x)\cdot\alpha(x) = t\cdot\theta(\tau) = t^3\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&t^3\\t^3&0\end{pmatrix},$$

$$(\Phi\otimes\alpha)(y) = \Phi(y)\cdot\alpha(y) = t^2\cdot\theta(\sigma) = t^2\cdot\begin{pmatrix}0&1\\-1&-1\end{pmatrix} = \begin{pmatrix}0&t^2\\-t^2&-t^2\end{pmatrix}.$$

Next recall that in the proof of the Torus Knot–Alexander Polynomial Proposition 117.11 we saw that

$$J(Q) = \begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix}\sum_{i=0}^{p-1} x^i & -x^p\cdot\sum_{i=1}^{q} y^{-i}\end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix}1+x & -x^2\cdot(y^{-1}+y^{-2}+y^{-3})\end{pmatrix}.$$

by the Fox Derivative Properties Lemma 116.5 (5) applied to $r=x^p\cdot y^{-q}$

specialized to $p=2$ and $q=3$

We obtain the following equalities in $\mathbb{Q}(t)$:

$$\begin{aligned}
\Delta^{\alpha}_{T(2,3)}(t) &= \frac{\det((\Phi\otimes\alpha)_*(J(Q)_2)}{\det((\Phi\otimes\alpha)_*(y-1))} &&= \frac{\det((\Phi\otimes\alpha)_*((1+x\ -x^2\cdot(y^{-1}+y^{-2}+y^{-3}))}{\det((\Phi\otimes\alpha)_*(y-1))}\\
&= \frac{\det\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t^3\\t^3&0\end{pmatrix}\right)}{\det\left(\begin{pmatrix}0&t^2\\-t^2&-t^2\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)} &&= \frac{\det\begin{pmatrix}1&t^3\\t^3&1\end{pmatrix}}{\det\begin{pmatrix}-1&t^2\\-t^2&-t^2-1\end{pmatrix}}\\
&= \frac{1-t^6}{1+t^2+t^4} &&= 1-t^2.
\end{aligned}$$

With a big sigh of relief the author can now proudly point to the fact that the two calculations of $\Delta^{\alpha}_{T(2,3)}(t)$, namely $t^{-2}-1$ and $1-t^2$, do indeed just differ by multiplication by a power of t.[1290]

124.3. **Proof of the Twisted Alexander Function Theorem 124.10 .** In this section we will prove the Twisted Alexander Function Theorem 124.10. As we will see, the proof is an adaptation of the arguments of Sections 116.5 and 116.6, where we dealt with the "untwisted" Alexander function.

In the following arguments we will on several occasions make use of the following elementary lemma:

Lemma 124.15. (Determinant-after-Homomorphism Lemma) Let π be a group, let S be a commutative ring and let $\varphi\colon \pi\to \mathrm{GL}(n,S)$ be a group homomorphism. Furthermore let $A,B\in \mathrm{M}(k\times k,\mathbb{Z}[\pi])$ be two square matrices and let $\lambda\in\mathbb{Z}[\pi]$. The following statements hold:

(1) (a) If we obtain B from A by right multiplication of a column by λ, then
$$\det(\varphi_*(B)) = \det(\varphi_*(A))\cdot\det(\varphi_*(\lambda)).$$
(b) If we obtain B from A by adding a right-multiple of column to another column, then
$$\det(\varphi_*(B)) = \det(\varphi_*(A)).$$
(2) (a) If we obtain B from A by left multiplication of a row by λ, then
$$\det(\varphi_*(B)) = \det(\varphi_*(\lambda))\cdot\det(\varphi_*(A)).$$
(b) If we obtain B from A by adding a left-multiple of a row to another row, then
$$\det(\varphi_*(B)) = \det(\varphi_*(A)).$$

Proof. In the following we provide the proof of (1a). The proofs of all other statements are basically the same. So suppose that we obtain B from A by right multiplication of a column by λ, then

$$\begin{aligned}
\det(\varphi_*(B)) &\overset{\text{by hypothesis}}{=} \det\left(\varphi_*\left(A\cdot\begin{pmatrix}\mathrm{id}_r&0&0\\0&\lambda&0\\0&0&\mathrm{id}_s\end{pmatrix}\right)\right) &&\overset{\text{follows from the Matrix Multiplication–Change-of-Ring Lemma 124.3 together with the Matrix-of-Matrices Lemma 124.4}}{=} \det\left(\varphi_*(A)\cdot\varphi_*\begin{pmatrix}\mathrm{id}_r&0&0\\0&\lambda&0\\0&0&\mathrm{id}_s\end{pmatrix}\right)\\
&= \det(\varphi_*(A))\cdot\det\left(\varphi_*\begin{pmatrix}\mathrm{id}_r&0&0\\0&\lambda&0\\0&0&\mathrm{id}_s\end{pmatrix}\right) &&= \det(\varphi_*(A))\cdot\det\begin{pmatrix}\mathrm{id}_{r\cdot n}&0&0\\0&\varphi_*(\lambda)&0\\0&0&\mathrm{id}_{s\cdot n}\end{pmatrix}\\
&= \det(\varphi_*(A))\cdot\det(\varphi_*(\lambda)).
\end{aligned}$$

[1290] Here we actually do not need a sign.

■

Next we recall Notation 116.19:

Notation 124.16. Let S be a ring and let $M \in \mathrm{M}(l \times k, S)$ be a matrix.

(1) Given $i \in \{1, \dots, k\}$ we denote by M_i the $l \times (k-1)$-matrix which is the result of deleting the i-th column.
(2) Given $Z \subseteq \{1, \dots, l\}$ with $\#Z = k-1$ we denote by M^Z the $(k-1) \times k$-matrix that is given by the rows in Z.
(3) Let $i \in \{1, \dots, k\}$ and let $Z \subseteq \{1, \dots, l\}$ with $\#Z = k-1$. We consider the corresponding $(k-1) \times (k-1)$-matrix $M_i^Z := (M^Z)_i$.

Using the Determinant-after-Homomorphism Lemma 124.15 we can easily prove the following generalization of the Matrix–Quotient Lemma 116.17:

Lemma 124.17. (Generalized Matrix–Quotient Lemma) Let π be a group, let $k \in \mathbb{N}$, let $M \in \mathrm{M}((k-1) \times k, \mathbb{Z}[\pi])$ and let $w = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} \in \mathbb{Z}[\pi]^k = \mathrm{M}(k \times 1, \mathbb{Z}[\pi])$ be such that $M \cdot w = 0$. Furthermore let S be a commutative ring and let $\varphi\colon \pi \to \mathrm{GL}(n, S)$ be a homomorphism. For any $i, j \in \{1, \dots, k\}$ with $\det(\varphi_*(w_i)) \neq 0$ and $\det(\varphi_*(w_j)) \neq 0$ we have the equality

$$\frac{\det(\varphi_*(M_i))}{\det(\varphi_*(w_i))} = \pm \frac{\det(\varphi_*(M_j))}{\det(\varphi_*(w_j))}.$$

Proof of Lemma 124.17. To simplify the notation we now assume that $i = 1$ and $j = 2$. (Note though that here we pick up a sign indeterminacy.) We denote the k columns of M by $c_1, \dots, c_k$. Note that with this notation our hypothesis becomes

$$(*) \qquad c_1 \cdot w_1 + c_2 \cdot w_2 + \cdots + c_k \cdot w_k = 0.$$

We now perform the following calculation:

$$\begin{aligned} \det(\varphi_*(M_1)) \cdot \det(\varphi_*(w_2)) &\overset{\text{by the Determinant-after-Homomorphism Lemma 124.15 (1)}}{\downarrow} \\ &= \det(\varphi_*(c_2 \cdot w_2 \; c_3 \; \dots \; c_k)) \\ &= \det(\varphi_*((-c_1 \cdot w_1 - c_3 \cdot w_3 - \cdots - c_k \cdot w_k) \; c_3 \; \dots \; c_k)) \\ &\overset{\text{follows from } (*)}{\uparrow} \\ &= \det(\varphi_*(-c_1 \cdot w_1 \, c_3 \; \dots \; c_k)) \overset{\text{this follows from the Determinant-after-Homomorphism Lemma 124.15 (1)}}{=\!\!\uparrow} (-1)^n \cdot \det(\varphi_*(M_2)) \cdot \det(\varphi_*(w_1)). \end{aligned}$$

This of course gives us the desired result. ■

Proof of Theorem 124.10. Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a domain and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. Finally assume that we are given a presentation

$$P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle \overset{\cong}{\to} \pi$$

of deficiency one.

(1) Since $x_1, \dots, x_k$ is a generating set of P and since $\Phi\colon P \to H$ is a non-trivial homomorphism we see that at least one $\Phi(x_i)$ is non-zero.

(2) Let $i \in \{1,\dots,k\}$ such that $\Phi(x_i)$ is non-trivial. It follows immediately from the example on page 2384 that $\det((\Phi\otimes\alpha)_*(x_i-1)) \neq 0$.

(3) We need to prove the following claim:

Claim. Let $i,j \in \{1,\dots,n\}$ such that $\Phi(x_i)$ and $\Phi(x_j)$ are non-trivial. Then

$$\frac{\det((\Phi\otimes\alpha)_*(J(P)_i))}{\det((\Phi\otimes\alpha)_*(x_i-1))} \;=\; \pm\frac{\det((\Phi\otimes\alpha)_*(J(P)_j))}{\det((\Phi\otimes\alpha)_*(x_j-1))} \;\in\; Q_R(H)$$

Proof. First recall that by the Jacobi Matrix–Vector Lemma 116.16 we know that

$$J(P)\cdot\begin{pmatrix}1-\mu_P(x_1)\\ \vdots \\ 1-\mu_P(x_k)\end{pmatrix} \;=\; 0.$$

The claim now follows from the Generalized Matrix–Quotient Lemma 124.17.

(4) We now assume that R is a UFD. We need to show that the definition of $\Delta^{\alpha}_{\pi,\Phi}$ (up to "$\doteq_R$") does not depend on the choice of the presentation. As we will see, the proof of this statement is very similar to the proof of the Alexander Function Theorem 116.12.

First we need to extend our definition of a twisted Alexander function to presentations that are not necessarily of deficiency one. Thus let $P=\langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle$ be any finite presentation for π and let $i\in\{1,\dots,k\}$ such that $\Phi(x_i)$ is non-trivial. Similar to the Independence-of-Generator Lemma 116.21 we consider[1291]

$$\Delta^{\alpha}_{P,\Phi} \;:=\; \frac{\gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(P)_i^Z))\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}}}{\det((\Phi\otimes\alpha)_*(x_i-1))} \;\in\; Q_R(H)$$

and we see that it is well-defined (i.e. independent of the choice of i) up to multiplication by a sign.

Now let P and Q be two deficiency-one presentations for π. Recall that by the Tietze Theorem 100.25 we know that the two presentations $\gamma\colon\pi\to P$ and $\delta\colon\pi\to Q$ are related by a finite sequence of Tietze transformations, as defined in Definition 116.18. So it suffices to deal with the case that the presentations P and Q are related by a single Tietze transformation.

(T1) We consider the following situation: We have

$$\underbrace{\langle x_1,\dots,x_k \,|\, r_1,\dots,r_l,s\rangle}_{=Q} \;=\; \underbrace{\langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle}_{=P} \;\xrightarrow{\gamma}\; \pi$$

where $s\in\langle\langle r_1,\dots,r_l\rangle\rangle\subseteq\langle X\rangle$. We see that the Jacobi matrix $J(Q)$ equals the Jacobi matrix $J(P)$, except that we are adding the row $\frac{\partial s}{\partial x_i}$, $i=1,\dots,k$. But since $s\in\langle\langle r_1,\dots,r_l\rangle\rangle$ one obtains easily from the Fox Derivative Properties Lemma 116.5 (5) that there exist $\lambda_1,\dots,\lambda_l\in\mathbb{Z}[P]$ such that for every $i\in\{1,\dots,k\}$ we have

$$\frac{\partial s}{\partial x_i} \;=\; \sum_{j=1}^{l}\lambda_j\cdot\frac{\partial r_j}{\partial x_i}.$$

[1291] Here we use that R is a UFD. More precisely, since R is a UFD and since H is a free abelian group it follows from the Laurent Polynomial Ring–UFD Proposition 287.12 that $R[H]$ is a unique factorization domain. Thus it makes sense to talk of the greatest common divisor in this context.

In other words the extra row of $J(Q)$ is just a "left-linear" combination of the previous rows.

Claim. Let $Z \subseteq \{1,\dots,l\}$ with $\#Z = k-2$. For any $i \in \{1,\dots,k\}$ we have

$$\det\left((\Phi\otimes\alpha)_*\left(J(Q)_i^{Z\cup\{l+1\}}\right)\right) \;=\; \sum_{\substack{j=1\\ j\notin Z}}^{l} \det(\Phi_*(\lambda_j))\cdot\det\left((\Phi\otimes\alpha)_*\left(J(Q)_i^{\{Z\cup\{j\}}\right)\right).$$

Proof. First note that by the above discussion we have

$$(l+1)\text{-st row of } J(Q) \;=\; \sum_{j=1}^{l} \lambda_j \cdot j\text{-th row of } J(Q).$$

The claim follows easily from this observation and the Determinant-after-Homomorphism Lemma 124.15 (2). ⊞

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\begin{aligned}
\gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(Q_i^Z)))\} &\underset{\substack{Z\subseteq\{1,\dots,l+1\}\\ \#Z=k-1}}{\doteq_R} \\
&\doteq_R \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(Q_i^Z)))\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}} \\
&\quad\uparrow \\
&\quad\text{follows easily from the above claim} \\
&\doteq_R \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(P_i^Z)))\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}}\cdot \\
&\quad\uparrow \\
&\quad\text{since the first } l \text{ rows of } J(Q) \text{ and } J(P) \text{ agree}
\end{aligned}$$

This calculation implies that $\Delta^{\alpha}_{Q,\Phi\circ\gamma} \doteq_R \Delta^{\alpha}_{P,\Phi\circ\gamma} \in Q_R(H)$.

(T2) We consider the following situation: We have $x \notin \{x_1,\dots,x_k\}$, $s,t \in \langle x_1,\dots,x_k\rangle$ and the presentations

$$\overbrace{\langle x_1,\dots,x_k,x \mid r_1,\dots,r_l, s\cdot x\cdot t\rangle}^{=Q} \xrightarrow{\Theta} \overbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}^{=P} \xrightarrow{\gamma} \pi.$$

$$\begin{aligned} x_i &\mapsto x_i \\ x &\mapsto s^{-1}\cdot t^{-1} \end{aligned}$$

We see that $$J(Q) \;=\; \begin{pmatrix} J(P) & 0 \\ * & s \end{pmatrix}.$$

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\begin{aligned}
\gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(Q)_i^Z))\} &\underset{\substack{Z\subseteq\{1,\dots,l+1\}\\ \#Z=k}}{\doteq_R} \\
&\doteq_R \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(Q)_i^Z))\}_{\substack{Z\subseteq\{1,\dots,l+1\}\\ \#Z=k,\ l+1\in Z}} \\
&\quad\uparrow \\
&\quad\text{by the above description of } J(Q) \text{ the determinant is zero, unless } l+1\in Z \\
&\doteq_R \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(P)_i^Z))\}_{\substack{Z\subseteq\{1,\dots,l\}\\ \#Z=k-1}}\cdot \\
&\quad\uparrow \\
&\quad\text{since } \det((\Phi\otimes\alpha)(s)) = \Phi(s)^n\cdot\det(\alpha(s)) \text{ and since } \det(\alpha(s)) \text{ is a unit in } R
\end{aligned}$$

This calculation implies that $\Delta^{\alpha}_{Q,\Phi\circ\gamma\circ\Theta} \doteq_R \Delta^{\alpha}_{P,\Phi\circ\gamma} \in Q_R(H)$. ■

124.4. **Twisted Alexander functions corresponding to representations.** In this final section of this chapter we will generalize our framework a little bit. We start out with a very general definition, which appears in many guises throughout large parts of mathematics:

Definition 124.18. Let π be a group and let R be a commutative ring.

(1) An R-representation of π consists of a finitely generated free R-module M and a homomorphism $\pi \to \mathrm{Aut}_R(M)$.
(2) We say that two representations $\varphi\colon \pi \to \mathrm{Aut}_R(M)$ and $\psi\colon \pi \to \mathrm{Aut}_R(N)$ are isomorphic if there exists an isomorphism $\Theta\colon M \to N$ of R-modules such that for each $g \in \pi$ we have $\varphi(g) = \Theta^{-1} \circ \psi(g) \circ \Theta\colon M \to M$.

Example. Let R be a commutative ring.

(1) Let G be any group. The map
$$\begin{array}{rcl} \tau_n^R\colon G & \to & \mathrm{Aut}_R(R^n) \\ g & \mapsto & \mathrm{id}_{R^n} \end{array}$$
is called the trivial n-dimensional R-representation of G. For $R = \mathbb{Z}$ we write $\tau_n \coloneqq \tau_n^{\mathbb{Z}}$ and often we write $\tau \coloneqq \tau_1$.
(2) Let G be a finite group. We consider the group ring $R[G]$. It follows easily from the definitions that the map
$$\begin{array}{rcl} \lambda_G^R\colon G & \to & \mathrm{Aut}_R(R[G]) \\ g & \mapsto & \begin{pmatrix} R[G] & \to & R[G] \\ p & \mapsto & g \cdot p \end{pmatrix} \end{array}$$
defines an R-representation of π. We call it the left regular R-representation of G. For $R = \mathbb{Z}$ we write $\lambda_G \coloneqq \lambda_G^{\mathbb{Z}}$.
(3) Let $n \in \mathbb{N}$.
- The map
$$\begin{array}{rcl} \sigma_n^R\colon S_n & \to & \mathrm{Aut}_R(R^n) \\ \sigma & \mapsto & \begin{pmatrix} R^n & \to & R^n \\ (v_1, \dots, v_n) & \mapsto & (v_{\sigma(1)}, \dots, v_{\sigma(n)}) \end{pmatrix} \end{array}$$
is clearly an R-representation of the permutation group S_n. We refer to it as the standard R-representation of the permutation group S_n. For $R = \mathbb{Z}$ we write $\sigma_n \coloneqq \sigma_n^{\mathbb{Z}}$.
- We set $(R^n)_0 \coloneqq \{(v_1, \dots, v_n) \in R^n \,|\, v_1 + \dots + v_n = 0\}$. It is clear that $(R^n)_0$ is preserved under permutation of the coordinates. Thus we obtain a well-defined representation
$$\begin{array}{rcl} \sigma_{n,0}^R\colon S_n & \to & \mathrm{Aut}_R((R^n)_0) \\ \sigma & \mapsto & \begin{pmatrix} (R^n)_0 & \to & (R^n)_0 \\ (v_1, \dots, v_n) & \mapsto & (v_{\sigma(1)}, \dots, v_{\sigma(n)}) \end{pmatrix}. \end{array}$$
We refer to this representation as the reduced standard R-representation of the permutation group. For $R = \mathbb{Z}$ we write $\sigma_{n,0} \coloneqq \sigma_{n,0}^{\mathbb{Z}}$.

We also view the above representations also as representations of the alternating group A_n in the obvious manner. □

The following lemma shows that representations are essentially the same as homomorphisms to matrix groups:

Lemma 124.19. (Representation Classification Lemma) Let π be a group, let R be a commutative ring and let $n \in \mathbb{N}_0$.

(1) Let M be a free R-module of rank n and let $v_1, \dots, v_n$ be a basis of M. The map

$$\begin{array}{rcl} \mathrm{Aut}_R(M) & \to & \mathrm{GL}(n,R) \\ \varphi & \mapsto & \text{matrix } [\varpi]_{(v_1,\dots,v_n)}^{(v_1,\dots,v_n)} \text{ representing}^{1292} \ \varphi\colon M \to M \text{ with} \\ & & \text{respect to the ordered basis } (v_1,\dots,v_n) \end{array}$$

is an isomorphism.

(2) The isomorphism from (1) defines a natural bijection between the set of isomorphism classes of representations $\pi \to \mathrm{Aut}_R(M)$, where M is any free R-module of rank n, and conjugacy classes of homomorphisms $\pi \to \mathrm{GL}(n,R)$.

Proof. Both statements follow easily from the definitions. We refer to the reader's preferred linear algebra book for details. ■

The Conjugation Invariance Lemma 124.14 allows us to extend our concept of the twisted Alexander function:

Definition 124.20. Let π be a group which admits a presentation of deficiency one, furthermore let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ be an R-representation. We pick a basis $v_1, \dots, v_n$ of M and we consider the corresponding homomorphism $\widetilde{\alpha}\colon \pi \to \mathrm{GL}(R,n)$. Note that it follows from the Representation Classification Lemma 124.19 and the Conjugation Invariance Lemma 124.14 that

$$\Delta^{\alpha}_{\pi,\Phi} \ := \ \Delta^{\widetilde{\alpha}}_{\pi,\Phi} \in \ Q_R(H)$$

does not depend on the choice of the basis $v_1, \dots, v_n$ and that in fact it does not even depend on the isomorphism type of the representation. We refer to $\Delta^{\alpha}_{\pi,\Phi}$ as the twisted Alexander function of (π, Φ, α). By the Twisted Alexander Function Theorem 124.10 the twisted Alexander function is well-defined up to "$\doteq_R$".

To formulate our next lemma we will need the following definition:

Definition 124.21. Let π be a group, let R be a commutative ring and let $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ and $\beta\colon \pi \to \mathrm{Aut}_R(N)$ be two R-representations. We define the direct sum representation

$$\begin{array}{rcl} \alpha \oplus \beta\colon \pi & \to & \mathrm{Aut}_R(M \oplus N) \\ g & \mapsto & \begin{pmatrix} M \oplus N & \to & M \oplus N \\ (m,n) & \mapsto & (\alpha(g)(m), \beta(g)(n)) \end{pmatrix}. \end{array}$$

Example. Let $n \in \mathbb{N}_0$. In Exercise 124.2 we will see that the direct sum of the reduced standard $\mathbb{Q}$-representation $\sigma^{\mathbb{Q}}_{n,0}$ of S_n and the trivial one-dimensional $\mathbb{Q}$-representation $\tau^{\mathbb{Q}}_1$ of S_n is isomorphic to the standard representation $\sigma^{\mathbb{Q}}_n$ of S_n. □

Lemma 124.22. (Direct Sum–Twisted Alexander Function Lemma) Let π be a group which admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ and

[1292] We refer to Definition 2.11 for the definition of this matrix.

$\beta\colon \pi \to \mathrm{Aut}_R(N)$ be two R-representations. Then

$$\Delta^{\alpha\oplus\beta}_{\pi,\Phi} \doteq_R \Delta^{\alpha}_{\pi,\Phi} \cdot \Delta^{\beta}_{\pi,\Phi}.$$

Sketch of proof. First note that clearly it suffices to deal with the case of two homomorphisms $\alpha\colon \pi \to \mathrm{GL}(m,R)$ and $\beta\colon \pi \to \mathrm{GL}(n,R)$. Next note that for any square matrix A over $\mathbb{Z}[\pi]$ we have

$$\det((\Phi\otimes(\alpha\oplus\beta))(A)) = \pm\det\begin{pmatrix}(\Phi\otimes\alpha)(A) & 0\\ 0 & (\Phi\otimes\beta)(A)\end{pmatrix} = \det((\Phi\otimes\alpha)(A))\cdot\det((\Phi\otimes\beta)(A)).$$

The lemma now follows easily from the definitions and these observations. We leave it to the reader to fill in the details. ■

The formulation of our next lemma makes use of the following notation:

Notation 124.23. Let R be a commutative ring.

(1) As in Definition 2.6 we denote by $R^\times$ the group of multiplicative units of R.

Let H be an abelian group and let $\rho\colon H \to R^\times$ be a homomorphism.

(2) We consider the map

$$\begin{array}{rcl}\lambda_*\colon R[H] & \to & R[H]\\ \sum_{i=1}^{k} r_i\cdot h_i & \mapsto & \sum_{i=1}^{k}(\rho(h_i)\cdot r_i)\cdot h_i.\end{array}$$

One can easily verify that this map is a ring homomorphism.

(3) Given a group π, given a homomorphism $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ and given a homomorphism $\rho\colon \pi \to R^\times$ we consider the map

$$\begin{array}{rcl}\rho\cdot\alpha\colon \pi & \to & \mathrm{Aut}_R(M)\\ g & \mapsto & (m \mapsto \rho(g)\cdot\alpha(g)(m)).\end{array}$$

One can easily verify that this map is a group homomorphism.

Lemma 124.24. (Modified Twisted Alexander Polynomial Lemma) Let π be a group which admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ be an R-representation. Furthermore let $\rho\colon H \to R^\times$ be a homomorphism. With the above notation we have

$$\Delta^{(\rho\circ\Phi)\cdot\alpha}_{\pi,\Phi} \doteq \lambda_*(\Delta^{\alpha}_{\pi,\Phi}).$$

Proof. Note that for any square matrix A over $\mathbb{Z}[\pi]$ we have

$$\det((\Phi\otimes(\rho\circ\Phi)\cdot\alpha)(A)) = \det(\lambda_*((\Phi\otimes\alpha)(A))) = \lambda_*(\det((\Phi\otimes\alpha)(A))).$$

The lemma now follows easily from the definitions and this observation. We leave it to the reader to fill in the details. ■

We conclude this chapter with a short discussion when the twisted Alexander function is actually a polynomial. To formulate the corresponding result we will need the following definition from representation theory.

Definition 124.25. Let π be a group, let $\mathbb{F}$ be a field and let $\alpha\colon \pi \to \mathrm{Aut}_{\mathbb{F}}(M)$ be a representation. We say that the representation α is reducible if there exists a subspace $V \subseteq M$ with $0 \subsetneq V \subsetneq M$ such that for every $g \in \pi$ we have $\alpha(g)(V) = V$. Otherwise we say that the representation is irreducible.

Now we can formulate the last result of this chapter:

Proposition 124.26. (Twisted Alexander Function–Polynomial Proposition) Let π be a group which admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ be a representation. We assume that one of the following conditions is satisfied:

(i) The image of $\Phi\colon \pi \to H$ is a free abelian group of rank ≥ 2.
(ii) The ring R is a field, the representation α is irreducible and the restriction of α to $\ker(\Phi)$ is non-trivial.

Then $\Delta^{\alpha}_{\pi,\Phi} \in R[H]$.

Sketch of proof. If (i) holds, then it follows by the same argument as in proof of the Link–Alexander Polynomial Proposition 117.5 (3) that $\Delta^{\alpha}_{\pi,\Phi} \in R[H]$. We will not really make use of the case (ii). We will thus not provide a proof. The statement that under the hypothesis (ii) we have $\Delta^{\alpha}_{\pi,\Phi} \in R[H]$ follows from [**FKK2012**, Proposition 9.3] and the discussion in [**FV2011a**, Chapter 3.3]. ■

124.5. Exercises for Chapter 124 .

Exercise 124.1. As in the Baumslag-Solitar Group Proposition 130.25 we consider the Baumslag-Solitar group

$$\mathrm{BS}(1,2) \;=\; \langle x, y \,|\, x^{-1} \cdot y \cdot x \;=\; y^2 \rangle.$$

(a) Show that $\mathrm{BS}(1,2)$ admits (up to conjugation) a unique epimorphism $\varphi\colon \mathrm{BS}(1,2) \to S_3$.
(b) Let $\Phi\colon \mathrm{BS}(1,2) \to \langle t \rangle$ be the epimorphism that is given by $\Phi(x) = t$ and $\Phi(y) = 1$. Furthermore let $\theta\colon S_3 \to \mathrm{GL}(2,\mathbb{Z})$ be the representation from page 2388. Compute the twisted Alexander function $\Delta^{\theta\circ\varphi}_{\mathrm{BS}(1,2),\Phi}$.

Exercise 124.2. Let $n \in \mathbb{N}_0$.

(a) Show that the direct sum of the reduced standard $\mathbb{Q}$-representation $\sigma^{\mathbb{Q}}_{n,0}$ of S_n and the trivial one-dimensional $\mathbb{Q}$-representation $\tau^{\mathbb{Q}}_1$ of S_n is isomorphic to the standard $\mathbb{Q}$-representation $\sigma^{\mathbb{Q}}_n$ of S_n.
(b) Does the conclusion of (a) also hold if we replace $\mathbb{Q}$ by $\mathbb{Z}$?
(c) Does the conclusion of (a) also hold if we replace $\mathbb{Q}$ by $\mathbb{F}_p$ for some prime p?

Exercise 124.3. Let G be a finite group.

(a) We denote by $\lambda^{\mathbb{Q}}_G\colon G \to \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}[G])$ the left regular $\mathbb{Q}$-representation from page 2395. Show that there exists a representation $\alpha\colon G \to \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}[G])$ such that $\sigma_{G,\mathbb{Q}} \cong \tau_{G,\mathbb{Q}} \oplus \alpha$.
(b) Does the conclusion of (a) also hold if we replace $\mathbb{Q}$ by $\mathbb{Z}$?
(c) Does the conclusion of (a) also hold if we replace $\mathbb{Q}$ by $\mathbb{F}_p$ for some prime p?

Exercise 124.4. Let π be a group that admits a presentation of deficiency one. Furthermore let $\Phi\colon \pi \to H$ be an epimorphism onto a free abelian group H with $\mathrm{rank}(H) \geq 2$, let $\alpha\colon \pi \to G$ be a homomorphism to a finite group G and let $\lambda^{\mathbb{Q}}_G\colon \pi \to \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}[G])$ be the corresponding left regular representation. Show that there exists an $f \in R[H]$ such that

$$\Delta^{\lambda^{\mathbb{Q}}_G}_{\pi,\Phi} \;=\; \Delta_{\pi,\Phi} \cdot f.$$

Hint. Make use of Exercise 124.3.

Exercise 124.5. Let π be a group, let $\mathbb{F}$ be a field and let $\alpha\colon \pi \to \mathrm{GL}(n,\mathbb{F})$ be a representation. Show that the representation is reducible if and only if there exist $r, s \in \{1,\dots,n-1\}$ with $r+s=n$, maps $A\colon \pi \to \mathrm{GL}(r,\mathbb{F})$, $B\colon \pi \to \mathrm{M}(r\times s,\mathbb{F})$ and $C\colon \pi \to \mathrm{GL}(s,\mathbb{F})$ and a matrix $P \in \mathrm{GL}(n,\mathbb{F})$ such that for every $g \in \pi$ we have

$$P \cdot \alpha(g) \cdot P^{-1} \;=\; \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}.$$

CHAPTER 125

The twisted Alexander polynomial of knots and links

Let R be a UFD. In Chapter 124 we associated to a group π (assuming it admits a presentation of deficiency one) together with a non-trivial homomorphism $\pi \to H$ to a free abelian group and a representation $\pi \to \mathrm{Aut}_R(M)$ the twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi} \in Q_R(H)$. Our favorite groups of deficiency one are the fundamental groups $\pi_1(S^3 \setminus L)$ of link complements. In this section we will thus use the general theory of twisted Alexander functions to introduce the twisted Alexander polynomials $\Delta^{\alpha}_L \in Q_R(t_1, \dots, t_m)$ of any oriented m-component link together with a representation $\pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$.

This chapter is organized as follows:

- In Section 125.1 we will introduce twisted Alexander polynomials of oriented links and we will study basic properties, e.g. we will study the effect of reversing orientations and of taking the mirror.
- In Section 125.2 we will see that twisted Alexander polynomials can be used to distinguish knots, in particular they can be used to distinguish the Kinoshita-Terasaka knot from the Conway knot.
- In Section 125.3 we will see that twisted Alexander polynomials can be used to give lower bounds on the knot genus. We will use this method to determine the rather elusive genera of the Kinoshita-Terasaka knot and of the Conway knot.
- In Section 125.4 we will see that twisted Alexander polynomials can be used to give new obstructions to a knot being fibered.
- In Appendix 125.5 we will discuss the question, for which representations twisted Alexander polynomials of links are symmetric. We will only state a symmetry result from the literature, without proving it.
- In Section 125.6 we will provide exercises for Chapter 125.

In the next Chapter 126 we will give a survey on different notions of equivalence of links.

125.1. **Twisted Alexander polynomials of oriented knots and links.** In Definition 117.4 we used Alexander functions to introduce the Alexander polynomial of an oriented link. In a very similar fashion we now use twisted Alexander functions to define the twisted Alexander polynomial of an oriented link:

Definition 125.1. Let $m \in \mathbb{N}$, let $L \subseteq S^3$ be an oriented m-component link, let R be a UFD, let M be a free R-module and let $\pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ be a representation. Recall that by the Link Group-Abelianization Proposition 113.9 there exists a unique epimorphism

$$\Phi_L \colon \pi_1(S^3 \setminus L) \ \to \ \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$$

such that for $i = 1, \dots, m$ we have $\Phi_L(\mu_{L_i}) = t_i$. By the Wirtinger Presentation Proposition 113.6 we know that $\pi_1(S^3 \setminus L)$ admits a presentation of deficiency one. Thus it makes sense to define[1293]

$$\Delta_L^\alpha \;=\; \Delta_L^\alpha(t_1, \dots, t_m) \;\coloneqq\; \underbrace{\Delta^\alpha_{\pi_1(S^3\setminus L),\Phi_L}}_{\substack{\text{as defined on}\\ \text{Definition 124.20}}} \;\in\; Q_R(t_1, \dots, t_m)$$

as the twisted Alexander polynomial of L.

Example. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link. We denote by $\tau\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(1, \mathbb{Z})$ the trivial one-dimensional representation. It follows from the discussion on page 2386 and Definition 117.2 of the Alexander polynomial of a knot that

$$\Delta_L^\tau(t_1, \dots, t_m) \;=\; \begin{cases} \frac{1}{t-1} \cdot \Delta_K(t), & \text{if } m = 1,\\ \Delta_L(t_1, \dots, t_m), & \text{if } m \geq 2. \end{cases}$$

In particular we see that the twisted Alexander polynomial of a knot is - alas - not necessarily a polynomial. Fortunately the Link–Twisted Alexander Polynomial Proposition 125.2 (2) shows that in almost all other cases the twisted Alexander polynomial of a link is in fact a polynomial. □

The following proposition can be viewed as an analog of the Knot–Alexander Polynomial Proposition 117.3 and the Link–Alexander Polynomial Proposition 117.5:

Proposition 125.2. (Link–Twisted Alexander Polynomial Proposition) Let $m \in \mathbb{N}$, let L be oriented m-component link $L \subseteq S^3$, let R be a UFD and let $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ be a representation.

(1) (a) The twisted Alexander polynomial $\Delta_L^\alpha(t_1, \dots, t_m)$ is well-defined up to "$\doteq_R$", i.e. up to multiplication by a unit in R and up to multiplication by a monomial $t_1^{d_1} \cdot \ldots \cdot t_m^{d_m}$.
(b) If $\beta\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(N)$ is a representation that is isomorphic to α, in the sense of the definition in Definition 124.18, then $\Delta_L^\alpha(t_1, \dots, t_m) \doteq_R \Delta_L^\beta(t_1, \dots, t_m)$.

(2) We assume that we are in one of the following settings:
(i) We have $m \in \mathbb{N}_{\geq 2}$.
(ii) The ring R is a field, the representation is irreducible and the image of α is a non-abelian subgroup of $\mathrm{Aut}_R(M)$.
Then the twisted Alexander polynomial Δ_L^α is an element of $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

(3) If $\widetilde{L}$ is an oriented m-component link that is smoothly isotopic to L, then there exists a representation $\widetilde{\alpha}\colon \pi_1(S^3 \setminus \widetilde{L}) \to \mathrm{Aut}_R(M)$ with $\Delta_L^\alpha \doteq_R \Delta_{\widetilde{L}}^{\widetilde{\alpha}}$.

Sketch of proof.

(1) (a) This statement is just a special case of the Twisted Alexander Function Theorem 124.10 (4).
(b) This statement follows immediately from the Representation Classification Lemma 124.19 together with the Conjugation Invariance Lemma 124.14.

(2) This statement follows immediately from the Twisted Alexander Function–Polynomial Proposition 124.26.

[1293] It follows from the Conjugation Invariance Lemma 124.14 that we can ignore base points.

(3) Let $\widetilde{L}$ be an oriented m-component link that is smoothly isotopic to L. As in the Isotopic Link-π_1-Lemma 111.4 we note that it follows from the Smooth Isotopy–Link Complement Proposition 108.12 that there exists an orientation-preserving diffeomorphism $f\colon S^3 \to S^3$ that restricts to a diffeomorphism $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$. It follows easily from the Meridian Proposition 109.2 that for each $i \in \{1, \dots, m\}$ the image of a meridian of L_i is a meridian of $\widetilde{L}_i$. This shows that there exists an isomorphism $f_*\colon \pi_1(S^3 \setminus L) \to \pi_1(S^3 \setminus \widetilde{L})$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(S^3 \setminus L) & \xrightarrow[\cong]{f_*} & \pi_1(S^3 \setminus \widetilde{L}) \\ & {\scriptstyle \Phi_L} \searrow \quad \swarrow {\scriptstyle \Phi_{\widetilde{L}}} & \\ & \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}. & \end{array}$$

Given a homomorphism $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ we set $\widetilde{\alpha} := \alpha \circ (f_*)^{-1}$. The desired statement now follows from the Twisted Alexander Function–Functorial Lemma 124.12 (1). ■

The following elementary lemma is a generalization of the Alexander Polynomial–Orientation Lemma 117.6:

Lemma 125.3. (Alexander Polynomial–Orientation Lemma) Let $m \in \mathbb{N}$ and let L be an oriented m-component link $L \subseteq S^3$. Given $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$ we denote by $L^{(\epsilon_1, \dots, \epsilon_m)}$ the link L but where we flipped the orientation of the i-th component precisely if $\epsilon_i = -1$. For any UFD R and any representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ we have

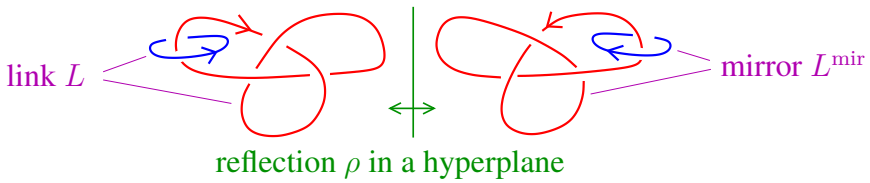
$$\Delta^{\alpha}_{L^{(\epsilon_1, \dots, \epsilon_m)}}(t_1, \dots, t_m) \doteq_R \Delta^{\alpha}_L(t_1^{\epsilon_1}, \dots, t_m^{\epsilon_m}).$$

Sketch of proof. The proof of the lemma is basically the same as the proof of the Alexander Polynomial–Orientation Lemma 117.6. The only real difference is that we need to replace the Alexander Function–Functorial Lemma 116.15 (2) by the Twisted Alexander Function–Functorial Lemma 124.12 (2). We leave it to the reader to fill in the details. ■

Now let L be an oriented m-component link. As in Definition 108.23 we consider the mirror L^{mir}. For the untwisted Alexander polynomial we know that

$$\Delta_{L^{\mathrm{mir}}}(t_1, \dots, t_m) \underset{\uparrow}{\doteq} \Delta_L(t_1^{-1}, \dots, t_m^{-1}) \underset{\uparrow}{\doteq} \Delta_L(t_1, \dots, t_m).$$

by the Alexander Polynomial Mirror Lemma 117.15 — by the Alexander Polynomial Symmetry Theorem 117.7



We have the following analog of the Alexander Polynomial–Mirror Lemma 117.15:

Lemma 125.4. (Twisted Alexander Polynomial–Mirror Lemma) Let L be an oriented m-component link, let R be a UFD and let $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ be a representation. Let $\rho\colon S^3 \to S^3$ be a reflection in a hyperplane. We consider the corresponding

mirror $L^{\mathrm{mir}} = \rho(L)$. Then

$$\Delta^{\alpha\circ(\rho^{-1})_*}_{\rho(L)}(t_1,\dots,t_m) \doteq \Delta^{\alpha}_{L}(t_1^{-1},\dots,t_m^{-1}).$$

Sketch of proof. The proof of the lemma is basically the same as the proof of the Alexander Polynomial–Mirror Lemma 117.15. The only real difference is that we need to replace the Alexander Function–Functorial Lemma 116.15 (1) by the Twisted Alexander Function–Functorial Lemma 124.12 (1). We leave it to the reader to fill in the details. ■

125.2. **Distinguishing knots and links.** Since the twisted Alexander polynomial is an invariant of an m-component link *together* with a representation, it is a priori not clear, how one could use twisted Alexander polynomials to distinguish smooth isotopy classes of oriented links.

In this section we will see that there are workarounds for this issue. In particular the following slight refinement of the Link–Twisted Alexander Polynomial Proposition 125.2 (3) will turn out to be helpful:

Proposition 125.2. (Link–Twisted Alexander Polynomial Proposition) In the following let $m \in \mathbb{N}$, let L be an oriented m-component link, let G be a group and let $\varphi\colon \pi_1(S^3 \setminus L) \to G$ be a homomorphism.

(3′) If $\widetilde{L}$ is an oriented m-component link that is smoothly isotopic to L, then there exists a homomorphism $\widetilde{\varphi}\colon \pi_1(S^3 \setminus \widetilde{L}) \to G$ with $\varphi(\pi_1(S^3 \setminus L)) = \widetilde{\varphi}(\pi_1(S^3 \setminus \widetilde{L}))$ such that for every UFD R and any representation $\alpha\colon G \to \mathrm{Aut}_R(M)$ we have

$$\Delta^{\alpha\circ\varphi}_{L} \doteq_R \Delta^{\alpha\circ\widetilde{\varphi}}_{\widetilde{L}}.$$

Proof. The proof of this statement is almost identical to the proof of the Link–Twisted Alexander Polynomial Proposition 125.2 (3). As so often we leave the details to the reader. ■

Example. Let C be the Conway knot and let K be the Kinoshita-Terasaka knot, that we introduced on page 2240.



On page 2240 we managed to distinguish these two knots using the somewhat naive but powerful knot coloring polynomial. On the other hand we saw on page 2278 that the Alexander polynomial is in both cases $= 1$. Thus the Alexander polynomial cannot distinguish these knots, and it cannot say anything interesting about these knots.

In both cases one can verify that, up to conjugation, there exists a unique epimorphism $\varphi_C\colon \pi_1(S^3 \setminus C) \to A_5 \subseteq S_5$ and a unique epimorphism $\varphi_K\colon \pi_1(S^3 \setminus K) \to A_5 \subseteq S_5$. If we work standard representation $\sigma_5\colon A_5 \to \mathrm{GL}(5,\mathbb{Z})$ which we introduced on page 2395, then we obtain the following result [**Pes2025**]:

$$\Delta^{\sigma_5^{\mathbb{Z}}\circ\varphi_C}_{C}(t) = \frac{5t^{18}-14t^{17}+15t^{16}-16t^{15}+19t^{14}-10t^{13}-5t^{12}+24t^{11}-34t^{10}+32t^{9}-34t^{8}+24t^{7}-5t^{6}-10t^{5}+19t^{4}-16t^{3}+15t^{2}-14t+5}{(t-1)^2\cdot(t^3-1)}$$

$$\Delta^{\sigma_5^{\mathbb{F}_7}\circ\varphi_K}_{K}(t) = \frac{-5t^{16}+15t^{15}-5t^{14}-11t^{13}-16t^{12}+14t^{11}+51t^{10}-34t^{9}-18t^{8}-34t^{7}+51t^{6}+14t^{5}-16t^{4}-11t^{3}-5t^{2}+15t-5}{(t-1)^2\cdot(t^3-1)}.$$

In general, calculations over the integers can be rather slow. So at times it is more efficient to work with a finite field instead of $\mathbb{Z}$. For example, if we replace $\mathbb{Z}$ by $\mathbb{F}_7$, i.e. if we use the representation $\sigma_5^{\mathbb{F}_7}\colon A_5 \to \mathrm{GL}(5,\mathbb{F}_7)$, then by [**FriK2006**] we obtain the following rational functions $\mathbb{F}_7(t)$:

$$\begin{aligned}\Delta_C^{\sigma_5^{\mathbb{F}_7}\circ\varphi_C}(t) &= \frac{1+t+4t^2+6t^3+t^5+2t^6+5t^7+2t^8+t^9+6t^{11}+4t^{12}+t^{13}+t^{14}}{1-t},\\ \Delta_K^{\sigma_5^{\mathbb{F}_7}\circ\varphi_K}(t) &= \frac{1+5t+6t^2+5t^3+t^4+2t^5+2t^6+2t^7+t^8+5t^9+6t^{10}+5t^{11}+t^{12}}{1-t}.\end{aligned}$$

It follows from either of these calculations, the Link–Twisted Alexander Polynomial Proposition 125.2 (3′) and the Link–Twisted Alexander Polynomial Proposition 125.2 (1b) that the two knots are *not* smoothly isotopic. □

125.3. **Twisted Alexander polynomials and the knot genus.** In the Genus–Alexander Polynomial Proposition 123.2 we showed that the degree of the Alexander polynomial of a knot gives a lower bound on $2\cdot\mathrm{genus}(K)$. In this section we will generalize this statement to twisted Alexander polynomials.

To do so we first of all need to slightly generalize the notion of the degree of a Laurent polynomial:

Definition 125.5. Let R be a domain.

(1) Given $p(t) = \sum_{i=r}^{s} a_i\cdot t^i \in R[t^{\pm1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define, as in Definition 123.1, its degree as
$$\deg(p(t)) \;=\; \deg\Big(\sum_{i=r}^{s} a_i\cdot t^i\Big) \;=\; s-r.$$
We extend this definition to $\deg(0) = -\infty$.

(2) Given $f(t)\in Q_R(t)$ we write $f(t) = \frac{p(t)}{q(t)}$ with $p(t), q(t)\in R[t^{\pm1}]$ and we set
$$\deg(f(t)) \;=\; \deg(p(t)) - \deg(q(t)).$$
One can easily verify that this degree is well-defined. It is also elementary to see that if we are given $f(t), g(t)\in Q_R(t)$ with $f(t) \doteq_R g(t)$, then $\deg(f(t)) = \deg(g(t))$.

Example. Let $K\subseteq S^3$ be an oriented knot and let $\tau\colon \pi_1(S^3\setminus K)\to \mathrm{GL}(1,\mathbb{Z})$ be the trivial 1-dimensional representation. Then

$$\deg(\Delta_K^{\tau}(t)) \underset{\substack{\uparrow\\ \text{by the discussion}\\ \text{on page 2401}}}{=} \deg\Big(\frac{\Delta_K(t)}{t-1}\Big) \;=\; \deg(\Delta_K(t)) - \deg(t-1) \;=\; \deg(\Delta_K(t)) - 1.$$

□

In light of the above example the following proposition can be viewed as a generalization of the Genus–Alexander Polynomial Proposition 123.2:

Proposition 125.6. (Genus–Twisted Alexander Polynomial Proposition) For every oriented knot K, ever UFD R and every representation $\alpha\colon \pi_1(S^3\setminus K)\to \mathrm{Aut}_R(M)$ we have[1294]
$$\deg(\Delta_K^{\alpha}(t)) \;\leq\; \mathrm{rank}(M)\cdot(2\cdot\mathrm{genus}(K)-1).$$
In other words, we have
$$\mathrm{genus}(K) \;\geq\; \tfrac{1}{2\cdot\mathrm{rank}(M)}\cdot\deg(\Delta_K^{\alpha}(t)) + \tfrac{1}{2}.$$

Example. Let $K = T(2,3)$ be the trefoil. As we pointed out on page 2341, the trefoil has genus one. Recall that in Section 124.2 we gave an explicit example of a representation $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(2,\mathbb{Z})$ such that $\Delta_K^\alpha(t) \doteq 1 - t^2$. Note that in this case the inequality of the Genus–Twisted Alexander Polynomial Proposition 125.6 is in fact an equality. □

Proof. Not surprisingly the proof of this proposition is very similar to the proof of the Genus–Alexander Polynomial Proposition 123.2. In particular we will import some results from the earlier proof.

Let $K \subseteq S^3$ be an oriented knot and let R be a UFD. It follows immediately from the definition of the twisted Alexander polynomial in Definition 124.20 that it suffices to consider the case that we are given a homomorphism $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(n,R)$. We need to prove that for every Seifert surface F of K we have

$$\deg(\Delta_K^\alpha(t)) \;\leq\; n \cdot (2 \cdot \mathrm{genus}(F) - 1).$$

So let F be a Seifert surface for K. We make a few preparations:

- We set $g := \mathrm{genus}(F)$.
- We pick $b_0 \in S^3 \setminus F$.
- We denote as usual by $\Phi_K\colon \pi_1(S^3 \setminus K, b_0) \to \langle t \rangle$ the unique epimorphism which sends a meridian to t.

Claim 1. There exists a presentation P of $\pi_1(S^3 \setminus K, b_0)$ of the form

$$\pi_1(S^3 \setminus K, b_0) \;=\; \left\langle y_1, \dots, y_k, t \;\middle|\; \begin{matrix} r_1, \dots, r_l \\ t \cdot x_{i,+} \cdot t^{-1} \cdot x_{i,-}^{-1}, i = 1, \dots, 2g \end{matrix} \right\rangle$$

where $r_1, \dots, r_l, x_{1,\pm}, \dots, x_{2g,\pm} \in \langle y_1, \dots, y_k \rangle$ and where $\Phi_K(y_i) = 0$ and $\Phi_K(t) = t$.

Proof. We proved this claim on page 2371. ⊞

Claim 2. We set $\Gamma := \ker(\Phi_K)$. There exist matrices $A_{l\times k}, B_{2g\times k}, C_{2g\times k}$ with entries in $\mathbb{Z}[\Gamma]$ such that

$$J(P) \;=\; \begin{matrix} \\ l \\ 2g \end{matrix} \begin{matrix} \begin{matrix} k & 1 \end{matrix} \\ \begin{pmatrix} A_{l\times k} & 0 \\ B_{2g\times k} \cdot t + C_{2g\times k} & * \end{pmatrix} \end{matrix}$$

Proof. The proof of this claim is very similar to the proof of the corresponding "untwisted" claim on page 2371. We leave it to the reader to fill in the details. ⊞

Evidently we now want to use the presentation of Claim 1 and the observation of Claim 2 to prove the desired inequality. First note that it follows from the proof of the Twisted Alexander Function Theorem 124.10 that we can use the presentation P from Claim 1 (even though it does not have deficiency one) to calculate the twisted Alexander polynomial. More precisely, it follows from that proof, applied to the $(k+1)$-st generator (i.e. applied to the generator t), that

$$\Delta_K^\alpha \;=\; \Delta^\alpha_{\pi_1(S^3\setminus K),\Phi_K} \;=\; \frac{\gcd_{R[t^{\pm1}]}\{\det((\Phi_K \otimes \alpha)_*(J(P)^Z_{k+1}))\}_{\substack{Z \subseteq \{1,\dots,l+2g\} \\ \#Z = k}}}{\det((\Phi_K \otimes \alpha)_*(t-1))} \;\in\; Q_R(t).$$

Next note that it follows from the discussion on page 2384 that the degree of the denominator is n. Thus it remains to prove the following claim:

[1294]The expression "$2 \cdot \mathrm{genus}(K) - 1$" is not as random as it looks. For a Seifert surface F of minimal genus we have $2 \cdot \mathrm{genus}(K) - 1 = 2 \cdot \mathrm{genus}(F) - 1 = -\chi(F)$.

Claim 3. For any $Z \subseteq \{1, \dots, l+2g\}$ with $\#Z = k$ we have
$$\deg(\det((\Phi_K \otimes \alpha)_*(J(P)^Z_{k+1}))) \;\leq\; n \cdot 2g.$$

Proof. Let $Z \subseteq \{1, \dots, l+2g\}$ with $\#Z = k$. Furthermore we set $Z' := Z \cap \{1, \dots, l+2g\}$ and we set $Z'' \cap \{k+1, \dots, l+2g\}$. We note that

$$\det((\Phi_K \otimes \alpha)_*(J(P)^Z_{k+1})) \overset{\text{by Claim 2}}{=} \det\left((\Phi_K \otimes \alpha)_*\begin{pmatrix} A_{l\times k} & \cancel{0} \\ B_{2g\times k} \cdot t + C_{2g\times k} & \cancel{*} \end{pmatrix}^Z\right)$$
$$= \det\left((\Phi_K \otimes \alpha)_*\begin{pmatrix} A^{Z'}_{l\times k} \\ B^{Z''}_{2g\times k} \cdot t + C^{Z''}_{2g\times k} \end{pmatrix}\right) \overset{\text{since } A,B \text{ and } C \text{ are matrices over } \mathbb{Z}[\ker(\Phi_K)] \text{ and since } \Phi_K(t) = t}{=} \det\begin{pmatrix} \alpha_*(A^{Z'}_{l\times k}) \\ \alpha_*(B^{Z''}_{2g\times k}) \cdot t + \alpha_*(C^{Z''}_{2g\times k}) \end{pmatrix}.$$

Note that $\alpha_*(B^{Z''}_{2g\times k})$ has $n \cdot 2g$ columns. Thus it follows from the Laurent Polynomial-Determinant-Degree Lemma 123.4 that the degree of the above determinant is indeed at most $n \cdot 2g$. ■

In the following two subsections we consider two of our favorite challenge examples, namely the Kinoshita-Terasaka knot K and the Conway knot C.

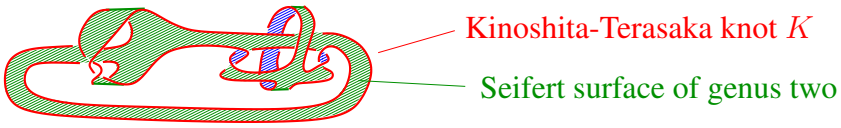
But before we do so, let us, for completeness' sake, state the following theorem, which complements the Genus–Twisted Alexander Polynomial Proposition 125.6:

Theorem 125.7. (Twisted Alexander Polynomial–Genus Detection Theorem) Given any oriented knot $K \subseteq S^3$ there exists a unitary representation $\pi_1(S^3 \setminus K) \to \mathrm{U}(n)$ such that
$$\deg(\Delta^\alpha_K(t)) \;=\; n \cdot (2 \cdot \mathrm{genus}(K) - 1).$$

Remark. Unfortunately the Twisted Alexander Polynomial–Genus Detection Theorem 125.7 is not as useful as one might hope, since it does not give us any indications how one could find these elusive unitary representations. □

Proof. This theorem is proved in [**FV2015**, Theorem 1.2]. The key input in the proof is provided by the Virtual Fibering Theorem **??**. ■

125.3.1. **The genus of the Kinoshita-Terasaka knot.** In the figure below we show the Kinoshita-Terasaka knot[1295] together with a Seifert surface of genus 2.



Recall that on page 2404 we saw that (up to conjugation) there exists a unique epimorphism $\varphi_K \colon \pi_1(S^3 \setminus K) \to A_5$. As on page 2395 we denote by $\sigma^{\mathbb{F}_7}_5 \colon A_5 \to \mathrm{Aut}_{\mathbb{F}_7}(\mathbb{F}^5_7)$ the standard representation. On page 2404 we saw that
$$\Delta^{\sigma^{\mathbb{F}_7}_5 \circ \varphi_K}_K(t) \;=\; \frac{1 + 5t + 6t^2 + 5t^3 + t^4 + 2t^5 + 2t^6 + 2t^7 + t^8 + 5t^9 + 6t^{10} + 5t^{11} + t^{12}}{1 - t} \;\in\; \mathbb{F}_7(t).$$

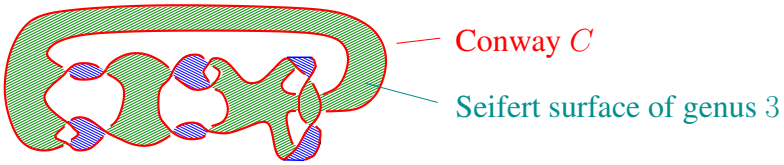
[1295] We leave it to the reader to verify that this diagram does indeed represent the Kinoshita-Terasaka knot as defined on page 2240.

If we plug this twisted Alexander polynomial into the Genus–Twisted Alexander Polynomial Proposition 125.6 we obtain that

$$\text{genus}(K) \underset{\substack{\uparrow\\ \text{by the Genus–Twisted Alexander}\\ \text{Polynomial Proposition 125.6}}}{\geq} \frac{1}{2\cdot\dim_{\mathbb{F}_7}(\mathbb{F}_7^5)} \cdot \deg(\Delta_K^{\sigma_5^{\mathbb{F}_7}\circ\varphi_K}(t)) + \frac{1}{2} \underset{\substack{\uparrow\\ \text{by the above}\\ \text{calculation}}}{=} \frac{1}{2\cdot 5}\cdot(12-1)+\frac{1}{2} = \frac{16}{10}.$$

Since the genus is a natural number, we see that $\text{genus}(K) \geq 2$. Thus the upper and lower bounds match, and we have shown that the genus of the Kinoshita-Terasaka knot K equals 2.

125.3.2. **The genus of the Conway knot.** In the figure below we show the Conway knot C together with a Seifert surface of genus 3.



Evidently we want to repeat the trick from the previous subsection. On page 2404 we saw that (up to conjugation) there exists a unique epimorphism $\varphi_C\colon \pi_1(S^3\setminus C)\to A_5$. As before we denote by $\sigma_5^{\mathbb{F}_7}\colon A_5\to \text{Aut}_{\mathbb{F}_7}(\mathbb{F}_7^5)$ the standard representation. On page 2404 we saw that

$$\Delta_C^{\sigma_5^{\mathbb{F}_7}\circ\varphi_C}(t) = \frac{1+t+4t^2+6t^3+t^5+2t^6+5t^7+2t^8+t^9+6t^{11}+4t^{12}+t^{13}+t^{14}}{1+t} \in \mathbb{F}_7(t).$$

This time we obtain that

$$\text{genus}(C) \underset{\substack{\uparrow\\ \text{by the Genus–Twisted Alexander}\\ \text{Polynomial Proposition 125.6}}}{\geq} \frac{1}{2\cdot\dim_{\mathbb{F}_7}(\mathbb{F}_7^5)} \cdot \deg(\Delta_C^{\sigma_5^{\mathbb{F}_7}\circ\varphi_C}(t)) + \frac{1}{2} \underset{\substack{\uparrow\\ \text{by the above}\\ \text{calculation}}}{=} \frac{1}{2\cdot 5}\cdot(14-1)+\frac{1}{2} = \frac{18}{10}.$$

Since the genus is a natural number we see that $\text{genus}(C)\geq 2$. Thus the upper and lower bounds do not match. As we will see shortly, we will get around this nuisance by replacing the standard representation of S_n by the reduced standard representation of S_n.

We will make use of the following lemma, which explains the relationship between the standard representation of A_5 by the reduced standard representation of A_5:

Lemma 125.8. (Standard Representation–Decomposition Lemma) Let $n\in\mathbb{N}$ and let $\mathbb{F}$ be some field such that the characteristic of $\mathbb{F}$ is coprime to n. As on page 2395 we write $(\mathbb{F}^n)_0 := \{(v_1,\dots,v_n)\in\mathbb{F}^n \,|\, v_1+\dots+v_n=0\}$ and as before we denote by $\sigma_{n,0}^{\mathbb{F}}\colon S_n\to\text{Aut}((\mathbb{F}^n)_0)$ the reduced standard representation. Then we have the following isomorphism of representations of S_n:

$$\sigma_{n,0}^{\mathbb{F}}\oplus\tau_1^{\mathbb{F}} \cong \sigma_n^{\mathbb{F}}.$$

The same statement holds if we replace S_n by A_n.

Proof. By hypothesis n is coprime to the characteristic of $\mathbb{F}$. This implies that there exists an $r \in \mathbb{F}$ with $n \cdot r = 1$. Now we consider the following homomorphisms of $\mathbb{F}$-vector spaces:

$$\begin{array}{rcl} \Phi\colon (\mathbb{F}^n)_0 \oplus \mathbb{F} & \to & \mathbb{F}^n \\ ((v_1,\dots,v_n), f) & \mapsto & (v_1,\dots,v_n) + f \cdot (1,\dots,1) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon \mathbb{F}^n & \to & (\mathbb{F}^n)_0 \oplus \mathbb{F} \\ (v_1,\dots,v_n) & \mapsto & (\underbrace{(v_1,\dots,v_n) - r\cdot(v_1+\dots+v_n)\cdot(1,\dots,1)}_{\text{sum of the coefficients is zero}}, r\cdot(v_1+\dots+v_n)). \end{array}$$

It follows easily from the definitions that the two maps are inverses of one another. In particular both are isomorphisms of $\mathbb{F}$-vector spaces. Finally note that it follows easily from the definitions that Φ preserves the S_n-action, thus it defines an isomorphism of S_n-representations. ∎

We return to the example of the Conway knot. The idea now is to replace the standard representation $\sigma_5^{\mathbb{F}_7}$ by the reduced standard representation $\sigma_{5,0}^{\mathbb{F}_7}$. We could calculate $\Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)$ again by a computer. But it is much more insightful to relate it to the twisted Alexander polynomial $\Delta_C^{\sigma_5^{\mathbb{F}_7}}(t)$ which we already calculated. In the following calculation we will use the following notation:

- We denote by $\rho\colon \mathbb{Z} \to \mathbb{F}_7$ the unique ring homomorphism.
- Given a representation α of A_5 we write $\Delta_C^{\alpha}(t) \coloneqq \Delta_C^{\alpha\circ\varphi_C}(t)$.

Now we see that

$$\begin{array}{lll} \Delta_C^{\sigma_5^{\mathbb{F}_7}}(t) \doteq \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}\oplus\tau_1^{\mathbb{F}_7}}(t) & \doteq \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\Delta_C^{\tau_1^{\mathbb{F}_7}}(t) & \doteq \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\Delta_C^{\lambda_*\circ\tau_1^{\mathbb{Z}}}(t) \\ \uparrow & \uparrow & \uparrow \\ \text{by the Standard Representation Decomposition Lemma 125.8} & \text{by the Direct Sum Twisted Alexander Function Lemma 124.22} & \text{since } \tau_1^{\mathbb{F}_7} = \lambda_*\circ\tau_1^{\mathbb{Z}} \text{ as maps } \pi_1(S^3\setminus C)\to \mathrm{GL}(1,\mathbb{F}_7) \end{array}$$

$$\begin{array}{lll} \doteq \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\lambda_*(\Delta_C^{\tau_1^{\mathbb{Z}}}(t)) & \doteq \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\lambda_*\left(\frac{\Delta_C(t)}{t-1}\right) & \doteq \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\frac{1}{t-1}. \\ \uparrow & \uparrow & \uparrow \\ \text{by the Twisted Alexander Function Functorial Lemma 124.12} & \text{by the discussion on page 2401} & \text{on page 2278 we saw that } \Delta_C(t)=1 \end{array}$$

Thus we see that the two twisted Alexander polynomials corresponding to the standard representation and corresponding to the reduced standard representation essentially agree. But since the rank of the reduced representation is lower, we get a better bound on the genus. More precisely, we now obtain that

$$\begin{array}{lll} \mathrm{genus}(C) \geq \frac{1}{2\cdot\dim_{\mathbb{F}_7}((\mathbb{F}_7^5)_0)}\cdot\deg(\Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}\circ\varphi_C}(t)) + \frac{1}{2} & = \frac{1}{2\cdot 4}\cdot(13+1)+\frac{1}{2} & = \frac{18}{8}. \\ \uparrow & \uparrow & \\ \text{by the Genus–Twisted Alexander Polynomial Proposition 125.6} & \text{by the above calculation} & \end{array}$$

Since the genus is a natural number we see that $\mathrm{genus}(C) \geq 3$. Thus the upper and lower bounds now do match. Thus we have finally shown that the genus of the Conway knot equals 3.

125.4. **Twisted Alexander polynomials and fiberedness.** In this section we will generalize the Fibered Knot–Alexander Polynomial Proposition 123.12 to the twisted context.

To formulate the main result of this section we need the following generalization of Definition 123.8:

Definition 125.9. Let R be a domain.

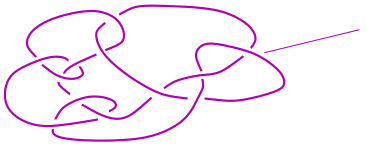
(1) We say that a Laurent polynomial $p(t) \in R[t^{\pm 1}]$ is bimonic if the highest and lowest coefficients both are a unit in R, i.e. they both lie in $R^\times$.
(2) We say $f(t) \in Q_R(t)$ is bimonic if it is the quotient of two bimonic Laurent polynomials.

In light of the discussion on page 2401, the following proposition can be viewed as a generalization of the Fibered Knot–Alexander Polynomial Proposition 123.12:

Proposition 125.10. (Fibered Knot–Twisted Alexander Polynomial Proposition) Let K be an oriented knot, let R be a UFD and let $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{Aut}_R(M)$ be a representation. If K is fibered, then the following two statements hold:

(1) The twisted Alexander polynomial $\Delta_K^\alpha(t)$ is bimonic.
(2) We have[1296]
$$\deg(\Delta_K^\alpha(t)) \;=\; \mathrm{rank}(M) \cdot (2 \cdot \mathrm{genus}(K) - 1).$$

Example. We consider the knot $K = 12_{1345} = 12_{n57}$.[1297] One can show that its genus equals two and that its Alexander polynomial equals $1 - 2t + 3t^2 - 2t^3 + t^4$.



So the Fibered Knot–Alexander Polynomial Proposition 123.12 has no objections to K being fibered. But one can show that there exists a homomorphism $\varphi\colon \pi_1(S^3 \setminus K) \to S_5$ such that
$$\Delta_K^{\sigma_5^{\mathbb{F}_2} \circ \varphi}(t) \;=\; 1 + t + t^3 + t^5 + t^7 + t^8 \;\in\; \mathbb{F}_2[t^{\pm 1}].$$
We see that
$$\deg(\Delta_K^{\sigma_5^{\mathbb{F}_2} \circ \varphi}(t)) \;=\; 8 \;\neq\; 12 \;=\; 4 \cdot (2 \cdot 2 - 1) \;=\; \dim_{\mathbb{F}_2}((\mathbb{F}_2^5)_0) \cdot (2 \cdot \mathrm{genus}(K) - 1).$$
Thus it follows from the Fibered Knot–Twisted Alexander Polynomial Proposition 125.10 that K is not fibered. □

Before we head to the proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 125.10, we state for completeness' sake the following theorem, which can be viewed as a converse to the Fibered Knot–Twisted Alexander Polynomial Proposition 125.10:

Theorem 125.11. (Twisted Alexander Polynomial–Fiberedness Detection Theorem) For any non-fibered knot $K \subseteq S^3$ there exists a representation $\pi_1(S^3 \setminus K) \to \mathrm{U}(n)$ such that
$$\Delta_K^\alpha(t) \;=\; 0.$$

Proof. This theorem is proved in [**FV2013**, Theorem 1.2]. (A somewhat weaker fiberedness detection theorem was first proved in [**FV2011b**, Theorem 1.2].) ■

The proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 125.10 will occupy the remainder of this chapter. We start out with a few preparations:

[1296] Recall that by the Fiber–Minimal Corollary 123.13 we know that if a knot K is fibered, then $\mathrm{genus}(K)$ equals the genus of any fiber Seifert surface.
[1297] Here we use the naming convention of [**LiM**]

Definition 125.12. Let $\varphi\colon F \to G$ be a homomorphism between two finitely generated free groups. Given a basis $x_1, \dots, x_k$ of F and given a basis $y_1, \dots, y_l$ of G we refer to

$$J^{y_1,\dots,y_l}_{x_1,\dots,x_k}(\varphi) \;:=\; \left(\frac{\partial \varphi(x_i)}{\partial y_j}\right)_{\substack{i=1,\dots,k,\\ j=1,\dots,l}} \;\in\; \mathrm{M}(k \times l, \mathbb{Z}[G])$$

as the Jacobi matrix of φ with respect to the bases $x_1, \dots, x_k$ and $y_1, \dots, y_l$. If we are given an endomorphism $\varphi\colon F \to F$ of a free group, then given any basis $x_1, \dots, x_k$ of F we write $J^{x_1,\dots,x_k}(\varphi) := J^{x_1,\dots,x_k}_{x_1,\dots,x_k}(\varphi)$.

Example.

(1) Let F be a free group with basis $x_1, \dots, x_k$. It follows immediately from the Fox Derivative Proposition 116.3 (1) that $J^{x_1,\dots,x_k}(\mathrm{id}) = \mathrm{id}_F \in \mathrm{M}(k \times k, \mathbb{Z}[F])$.

(2) Let $\pi = \langle y_1, \dots, y_l \,|\, r_1, \dots, r_k\rangle$ be a finite presentation. Let $\varphi\colon \langle x_1, \dots, x_k\rangle \to \langle y_1, \dots, y_l\rangle$ be the homomorphism given by $\varphi(x_i) = r_i$ for $i = 1, \dots, k$. It follows immediately from the definitions that

$$J^{y_1,\dots,y_l}_{x_1,\dots,x_k}(\varphi) \;=\; \underbrace{J(\text{finite presentation } \langle y_1, \dots, y_l \,|\, r_1, \dots, r_k\rangle)}_{\text{see Definition 116.9}}.$$

□

In a second we will need the following "Chain Rule":

Lemma 125.13. (Fox Derivative–Chain Rule) Let X be a finitely generated free group with basis $x_1, \dots, x_m$ and let Y be a finitely generated free group with basis $y_1, \dots, y_n$. Furthermore let $\lambda\colon X \to Y$ be any homomorphism. Given any $g \in X$ and given any $j \in \{1, \dots, m\}$ we have the following equality in $\mathbb{Z}[Y]$:

$$\frac{\partial \lambda(g)}{\partial y_j} \;=\; \sum_{r=1}^{m} \lambda\left(\frac{\partial g}{\partial x_r}\right) \cdot \frac{\partial \lambda(x_r)}{\partial y_j}.$$

Proof of the Fox Derivative–Chain Rule 125.13. We need to prove the desired equality for any $g \in X$. Since $x_1, \dots, x_m$ is a basis for X it suffices to prove the following claim:

Claim.

(a) The desired equality holds for any $g = x_i$.

(b) The desired equality holds for any $g = x_i^{-1}$.

(c) If the desired equality holds for $u, v \in X$, then it also holds for $g = u \cdot v$.

Proof.

(a) It follows easily from $\frac{\partial x_j}{\partial x_k} = \delta_{ik}$ that the statement holds for any x_i.

(b) For $g = x_i^{-1}$ we have

$$\begin{aligned}
\frac{\partial \lambda(x_i^{-1})}{\partial x_j} \;&=\; \frac{\partial (\lambda(x_i)^{-1})}{\partial x_j} \;\underset{\uparrow}{=}\; -\lambda(x_i)^{-1} \cdot \frac{\partial \lambda(x_i)}{\partial x_j} \;\underset{\uparrow}{=}\; -\lambda(x_i)^{-1} \cdot \sum_{r=1}^{m} \lambda\left(\frac{\partial x_i}{\partial x_r}\right) \cdot \frac{\partial \lambda(x_r)}{\partial y_j} \\
&\qquad\qquad \text{by the Fox Derivative Properties Lemma 116.5 (5)} \qquad\qquad \text{by (a)} \\
&=\; \sum_{r=1}^{m} \lambda\left(-x_i^{-1} \cdot \frac{\partial x_i}{\partial x_r}\right) \cdot \frac{\partial \lambda(x_r)}{\partial y_j} \qquad\qquad \underset{\uparrow}{=}\; \sum_{r=1}^{m} \lambda\left(\frac{\partial x_i^{-1}}{\partial x_r}\right) \cdot \frac{\partial \lambda(x_r)}{\partial y_j}. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by the Fox Derivative Lemma 116.5 (5)}
\end{aligned}$$

(c) Suppose the desired equality holds for $u, v \in X$. Then the following calculation shows that the desired equality also holds for $u \cdot v$:

$$\begin{aligned}\frac{\partial\lambda(u\cdot v)}{\partial x_j} &\overset{\text{by the Leibniz Rule 116.3 (2)}}{=} \frac{\partial\lambda(u)}{\partial x_j} + \lambda(u)\cdot\frac{\partial\lambda(v)}{\partial y_j} \overset{\text{since } u \text{ and } v \text{ satisfy the desired equality}}{=} \sum_{r=1}^{m}\lambda\Big(\frac{\partial u}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} + \lambda(u)\cdot\sum_{r=1}^{m}\lambda\Big(\frac{\partial v}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}\\ &= \sum_{r=1}^{m}\lambda\Big(\frac{\partial u}{\partial x_r} + u\cdot\frac{\partial v}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} \overset{\text{by the Leibniz Rule 116.3 (2)}}{=} \sum_{r=1}^{m}\lambda\Big(\frac{\partial(u\cdot v)}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}.\end{aligned}$$

■

The Fox Derivative–Chain Rule 125.13 allows us to prove the following lemma, which is going to be a key ingredient in the proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 125.10:

Lemma 125.14. (Jacobi Matrix-Isomorphism Lemma) Let π be a finitely generated free group. If $\varphi\colon \pi \to \pi$ is an isomorphism, then for any basis $x_1, \dots, x_n$ of π the Jacobi matrix $J^{x_1,\dots,x_n}(\varphi) = \big(\frac{\partial\varphi(x_i)}{\partial x_j}\big)_{i,j=1,\dots,n}$ has a right inverse in $\mathrm{M}(n\times n, \mathbb{Z}[\pi])$.[1298]

Proof of the Jacobi Matrix-Isomorphism Lemma 125.14. We apply the Fox Derivative–Chain Rule 125.13 in the setting where $X = Y = \pi$ and $\lambda = \mathrm{id}$. Since φ is an isomorphism we know that $y_i = \varphi(x_i)$, $i = 1, \dots, n$ also form a basis for π. Given any $i, j \in \{1, \dots, n\}$ we set $g := \varphi(x_i)$ and we obtain from the Fox Derivative–Chain Rule 125.13 that

$$\frac{\partial\varphi(x_i)}{\partial y_j} = \sum_{r=1}^{n}\frac{\partial\varphi(x_i)}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_j}.$$

But this implies that

$$\mathrm{id}_n \overset{\text{since } \varphi(x_i)=y_i \text{ and since } \mathrm{id}_n=(\delta_{ij})_{i,j=1,\dots,n}}{=} \Big(\frac{\partial\varphi(x_i)}{\partial y_j}\Big)_{i,j=1,\dots,n} \overset{\text{by the above}}{=} \Big(\sum_{r=1}^{n}\frac{\partial\varphi(x_i)}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_j}\Big)_{i,j=1,\dots,n} \overset{\text{definition of matrix multiplication}}{=} \overbrace{\Big(\frac{\partial\varphi(x_i)}{\partial x_r}\Big)_{i,r=1,\dots,n}}^{=J^{x_1,\dots,x_n}(\varphi)}\cdot\overbrace{\Big(\frac{\partial x_r}{\partial y_j}\Big)_{j,j=1,\dots,n}}^{\text{matrix over } \mathbb{Z}[\pi]}.$$

We have thus shown that $J^{x_1,\dots,x_n}(\varphi)$ has a right inverse in $\mathrm{M}(n\times n, \mathbb{Z}[\pi])$. ■

Now we can finally provide the proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 125.10.

Proof of Proposition 125.10. Let $K \subseteq S^3$ be a fibered oriented knot with corresponding fiber Seifert surface F. We denote by g the genus of F. Furthermore let R be a UFD. It follows immediately from the definition of the twisted Alexander polynomial in Definition 124.20 and the Fiber–Minimal Corollary 123.13 that it suffices to show that for any homomorphism $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(n, R)$ the following two statements hold:

(1) The twisted Alexander polynomial $\Delta_K^\alpha(t)$ is bimonic.
(2) We have $\deg(\Delta_K^\alpha(t)) = n\cdot(2\cdot g - 1)$.

[1298] In fact Joan Birman [**Bir1973**, p. 635] proved the following converse: If the matrix is invertible, then φ is an isomorphism.

First recall that by the Fibered Knot-π_1-Proposition 123.6 we know that there exists an isomorphism $\varphi\colon \langle y_1,\dots,y_{2g}\rangle \to \langle y_1,\dots,y_{2g}\rangle$ and an isomorphism

$$\Theta\colon \pi_1(S^3\setminus K) \xrightarrow{\cong} \langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle$$

such that under this isomorphism the epimorphism $\Phi_K\colon \pi_1(S^3\setminus K)\to\langle t\rangle$ agrees with the natural epimorphism $\Phi\colon \langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle \to \langle t\rangle$ that is given by $\Phi(y_i)=1$ and $\Phi(t)=t$.

Next note that, as in the proof of the Alexander Polynomial-of-Semidirect Product Proposition 123.9, we have the following isomorphisms:

$$\begin{aligned}
\langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \mathbb{Z} &\overset{\text{by the HNN–Semidirect Product Lemma 106.3, see also the example on page 2373}}{=} \langle y_1,\dots,y_{2g},t \,|\, t\cdot y_1\cdot t^{-1}\cdot\varphi(y_1)^{-1},\dots,t\cdot y_{2g}\cdot t^{-1}\cdot\varphi(y_{2g})^{-1}\rangle\\
&= \langle y_1,\dots,y_{2g},t \,|\, t\cdot y_1\cdot t^{-1}\cdot\varphi(y_1^{-1}),\dots,t\cdot y_{2g}\cdot t^{-1}\cdot\varphi(y_{2g}^{-1})\rangle\\
&\overset{\text{substitution } y_i\mapsto x_i^{-1}}{=} \langle x_1,\dots,x_{2g},t \,|\, t\cdot x_1^{-1}\cdot t^{-1}\cdot\varphi(x_1),\dots,t\cdot x_{2g}^{-1}\cdot t^{-1}\cdot\varphi(x_{2g})\rangle \;=:\; P.
\end{aligned}$$

We will now work with the presentation P that we just found. Using the $(2g+1)$-st generator, namely the generator t, we see that

$$\Delta_K^\alpha(t) \;=\; \frac{\det((\Phi\otimes\alpha)_*(J(P)_{2g+1}))}{\det((\Phi\otimes\alpha)_*(t-1))}.$$

By the discussion on page 2384 we know that the denominator of the above fraction is bimonic of degree n. Thus it suffices to prove the following claim about the numerator:

Claim. The determinant $\det((\Phi\otimes\alpha)_*(J(P)_{2g+1}))$ is bimonic of degree $n\cdot 2g$.

Proof. We perform the following calculation:

$$\begin{aligned}
\det((\Phi\otimes\alpha)_*(J(P)_{2g+1})) &= \det\Big((\Phi\otimes\alpha)\Big(\frac{\partial(t\cdot x_i^{-1}\cdot t^{-1}\cdot\varphi(x_i))}{\partial x_j}\Big)_{i,j=1,\dots,2g}\Big)\\
&\overset{\text{follows from the Fox Derivative Proposition 116.3 and the Fox Derivative Properties Lemma 116.5}}{=} \det\Big((\Phi\otimes\alpha)_*\Big(t\cdot\operatorname{diag}(-x_1,\dots,-x_{2g}) + \operatorname{diag}(tx_1^{-1}t^{-1},\dots,tx_{2g}^{-1}t^{-1})\cdot\underbrace{\Big(\frac{\partial\varphi(x_i)}{\partial x_j}\Big)_{i,j}}_{=J^{x_1,\dots,x_{2g}}(\varphi)}\Big)\Big)\\
&\overset{\text{by the definition of }\Phi}{=} \det(t\cdot\operatorname{diag}(\alpha(-x_1),\dots,\alpha(-x_{2g})) + \operatorname{diag}(\alpha(tx_1^{-1}t^{-1}),\dots,\alpha(tx_{2g}^{-1}t^{-1}))\cdot\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi)))\\
&\overset{\text{by the Polynomial Matrix Determinant Lemma 123.11}}{=} t^{2g\cdot n}\cdot\prod_{i=1}^{2g}\underbrace{\det(-\alpha(x_i))}_{\in R^\times} + \dots + t^0\cdot\Big(\prod_{i=1}^{2g}\underbrace{\det(\alpha(tx_i^{-1}t^{-1}))}_{\in R^\times}\Big)\cdot\underbrace{\det(\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi)))}_{\in R^\times \text{ by the discussion below}}.
\end{aligned}$$

By the Jacobi Matrix-Isomorphism Lemma 125.14 we know that $J^{x_1,\dots,x_{2g}}(\varphi)$ has a right inverse in $\mathrm{M}(2g\times 2g,\mathbb{Z}[\pi])$.[1299] It follows from the Group Ring-to-Matrix Ring Lemma 124.2, the Matrix Multiplication–Change-of-Ring Lemma 124.3 and the Matrix-of-Matrices Lemma 124.4 that $\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi))$ has a right inverse in $\mathrm{M}(2g\cdot n\times 2g\cdot n,R)$. But this implies that $\det(\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi)))$ is a unit in R. ∎

[1299] Note that at this point the proof starts to differ quite drastically from the original argument in the untwisted case.

125.5. **Appendix: Symmetries of twisted Alexander polynomials.** For ordinary Alexander polynomials we proved the Alexander Polynomial Symmetry Theorem 117.7 which says that for any oriented m-component link L we have

$$\Delta_L(t_1,\dots,t_m) \doteq \Delta_L(t_1^{-1},\dots,t_m^{-1}).$$

The reader might also remember that the proof was highly non-trivial. It is natural to ask whether a similar type of symmetry statement also holds for twisted Alexander polynomials. It turns out that this is a tricky question and, as we will see, even just to formulate the corresponding symmetry theorem requires some thought and preparations. Since we will not really make use of the symmetry statement, and since we will not prove it anyway, we are hiding this discussion in an appendix.

First of all we need the following definitions:

Definition 125.15. Let R be a commutative ring.

(1) An *involutionon*R is a map $R \to R$, usually written $z \mapsto \overline{z}$, with the following properties:
   (a) $\overline{z+w} = \overline{z} + \overline{w}$ for all $z, w \in R$,
   (b) $\overline{w \cdot z} = \overline{w} \cdot \overline{z}$ for all $z, w \in R$,
   (c) $\overline{\overline{z}} = z$ for all $z \in R$.

(2) Given an R-module M we denote by $\overline{M}$ the R-module where for $m \in \overline{M}$ we have $r \cdot_{\overline{M}} m := \overline{r} \cdot_M m$.

Example.

(1) For any commutative ring the identity map is evidently an involution.
(2) Complex conjugation turns $\mathbb{C}$ into a ring with involution. □

The following definition generalizes some concepts from Definition 2.29:

Definition 125.16. Let R be a commutative ring with (possibly trivial) involution and let M be a free R-module.

(1) A R-sesquilinear pairing on M is a map $\langle\ ,\ \rangle\colon M \times M \to R$ such that the following holds:
   - For all $v, v' \in M$ and all $w, w' \in M$ we have $\langle v+v', w\rangle = \langle v, w\rangle + \langle v', w\rangle$ and $\langle v, w+w'\rangle = \langle v, w\rangle + \langle v, w'\rangle$.
   - For all $v, w \in M$ and $r \in R$ we have $\langle r \cdot v, w\rangle = r \cdot \langle v, w\rangle$ and $\langle v, r \cdot w\rangle = \overline{r} \cdot \langle v, w\rangle$.

(2) We say that an R-sesquilinear pairing $\langle\ ,\ \rangle\colon M \times M \to R$ is non-singular if the induced maps

$$\begin{array}{ccc} M & \to & \mathrm{Hom}_R(\overline{M}, R) \\ v & \mapsto & \begin{pmatrix} \overline{M} & \to & R \\ w & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \to & \overline{\mathrm{Hom}_R(M, R)} \\ w & \mapsto & \begin{pmatrix} M & \to & R \\ v & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array}$$

are isomorphisms of R-modules.[1300]

Lemma 125.17. (Dual Representation Lemma) Let π be a group, let R be a commutative ring with (possibly trivial) involution, let $\alpha\colon \pi \to \mathrm{Aut}_R(M)$ be a representation

[1300] It is worth verifying that these maps are homomorphisms of R-modules.

and let

$$\langle -,-\rangle \colon M \times M \ \to \ R$$

be a non-singular R-sesquilinear pairing. In this setting there exists a unique representation $\alpha^\dagger \colon \pi \to \mathrm{Aut}_R(M)$ such that for all $m, m' \in M$ and all $g \in \pi$ we have[1301]

$$\langle \alpha(g)(m), m' \rangle \ = \ \langle m, \alpha^\dagger(g^{-1})(m') \rangle.$$

We refer to this representation as the Hermitian adjoint of α.

Example.

(1) Let R be a commutative ring with involution. We consider $M = R^n$ and the standard non-singular R-sesquilinear pairing

$$\begin{array}{rcl} \langle\ ,\ \rangle \colon R^n \times R^n & \to & R \\ ((v_1,\dots,v_n),(w_1,\dots,w_n)) & \mapsto & \sum_{i=1}^{n} v_i \cdot \overline{w}_i. \end{array}$$

It follows easily from the definitions that given any representation $\alpha \colon \pi \to \mathrm{GL}(n,R)$ and given any $g \in \pi$ we have $\alpha^\dagger(g) = (\overline{\alpha(g)^{\mathrm{T}}})^{-1}$. For $R = \mathbb{R}$ it follows from this discussion that for any orthogonal representation $\alpha \colon \pi \to \mathrm{O}(n)$ we have $\alpha = \alpha^\dagger$ and for $R = \mathbb{C}$ it follows from this discussion that for any unitary representation $\alpha \colon \pi \to \mathrm{U}(n)$ we have $\alpha = \alpha^\dagger$.

(2) Let R be a commutative ring. We consider the pairing

$$\begin{array}{rcl} \langle -,-\rangle \colon R^2 \times R^2 & \to & R \\ \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) & \mapsto & \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{array}$$

In Exercise 125.3 we will show that this R-bilinear pairing is non-singular and we will show that for any representation $\alpha \colon \pi \to \mathrm{GL}(2,R)$ we have $\alpha^\dagger(g) = \det(\alpha(g))^{-1} \cdot \alpha(g)$. In particular, if we are given a representation $\alpha \colon \pi \to \mathrm{SL}(2,R)$, then $\alpha^\dagger = \alpha$. □

Proof. Let $g \in \pi$ and let $m' \in M$. Since the pairing is sesquilinear we see that the map

$$\begin{array}{rcl} M & \to & R \\ m & \mapsto & \langle \alpha(g)(m), m' \rangle \end{array}$$

defines an R-linear homomorphism $M \to R$. Since the pairing is non-singular we see that there exists a unique $\widetilde{m} \in M$ such that $\langle m, \widetilde{m} \rangle = \langle \alpha(g)(m), m' \rangle$ for all $m' \in M$. We now set $(\alpha^\dagger(g^{-1}))(m') := \widetilde{m}$. We leave it to the reader, to verify that this way we defined a homomorphism $\alpha^\dagger(g) \colon M \to M$, that this defines a homomorphism $\alpha^\dagger \colon \pi \to \mathrm{Aut}_R(M)$ and that this homomorphism has the desired properties. ■

After all these preparations we can now finally formulate the following theorem which, by the discussion on page 2401, can be viewed as a generalization of the Alexander Polynomial–Symmetry Theorem 117.7:

Theorem 125.18. (Twisted Alexander Polynomial–Symmetry Theorem) Let $m \in \mathbb{N}$, let L be an oriented m-component link. Furthermore let R be a UFD with involution

[1301] In other words: For any $m, m' \in M$ and any $g \in \pi$ we have $\langle \alpha(g)(m), \alpha^\dagger(g)(m) \rangle = \langle m, m' \rangle$.

and let $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ be a representation. For any Hermitian adjoint representation $\alpha^\dagger$ as above we have

$$\Delta_L^\alpha(t_1, \dots, t_m) \doteq_R \Delta_L^{\alpha^\dagger}(t_1^{-1}, \dots, t_m^{-1}).$$

Proof. This theorem is proved in [**HSW2010**, Theorem 3.2], building on [**Miln1962c**, Lemma 2]. We also refer to [**Kit1996**, Theorem B], [**Hill2012**, Theorem 6.7], and [**FKK2012**, Theorem 1.1] for related results.

The courageous reader can also try to prove the theorem, using the approach taken in our proof of the Alexander Polynomial–Symmetry Theorem 117.7. ■

The following corollary summarizes the special cases one mostly cares about:

Corollary 125.19. (Twisted Alexander Polynomial–Symmetry Corollary) Let $m \in \mathbb{N}$ and let L be an oriented m-component link. We suppose that we are given one of the following types of presentations:

(1) (a) An orthogonal representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{O}(n)$, where $R = \mathbb{R}$.
(b) A unitary representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{U}(n)$, where $R = \mathbb{C}$.
(2) A representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{SL}(2, R)$ for some UFD R.

Then

$$\Delta_L^\alpha(t_1, \dots, t_m) \doteq_R \Delta_L^\alpha(t_1^{-1}, \dots, t_m^{-1}).$$

Proof. The corollary follows immediately from the Twisted Alexander Polynomial–Symmetry Theorem 125.18 together with the examples on page 2414. ■

Example. We consider again the trefoil T and as in Definition 108.21 we denote by T^{rev} the trefoil with the opposite orientation. Next recall that in Section 124.2 we introduced a representation $\alpha\colon \pi_1(S^3 \setminus T) \to \mathrm{GL}(2, \mathbb{Z})$ and we saw that $\Delta_T^\alpha(t) = 1 - t^2$. We now see that

$$\begin{array}{ccccccccc}
 & & \text{by the Twisted Alexander Polynomial Orientation Lemma 125.3} & & \text{by the Twisted Alexander Polynomial Symmetry Theorem 125.18} & & \text{by the example on page 2414 where } \rho\colon \langle t\rangle \to \{\pm 1\} \text{ is given by } \rho(t) = -1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\Delta_{T^{\mathrm{rev}}}^\alpha(t) & \doteq & \Delta_T^\alpha(t^{-1}) & \doteq & \Delta_T^{\alpha^\dagger}(t) & \doteq & \Delta_T^{(\rho\circ\Phi_K)\cdot\alpha}(t) & & \\
 & \doteq & \lambda_*(\Delta_T^\alpha(t)) & = & \Delta_T^\alpha(-t) & = & 1 - (-t)^2 & = & 1 - t^2. \\
 & & \uparrow & & \uparrow & & & & \\
 & & \text{by the Modified Twisted Alexander Polynomial Lemma 124.24} & & \text{follows easily from the definitions and } \rho(t) = -1 & & & &
\end{array}$$

We thus see that $\Delta_{T^{\mathrm{rev}}}^\alpha(t) \doteq \Delta_T^\alpha(t)$. But that is perfectly reasonable since in Exercise 108.2 we showed that the trefoil is reversible, i.e. it is smoothly isotopic to its reverse. □

This concludes our first discussion of invariants of knots and links that can be obtained from the fundamental group. In the next parts of the lecture notes we will introduce (co-) homology groups. After we have fully digested these concepts we will return to invariants of knots and links in Part XXXII.

125.6. Exercises for Chapter 125 .

Exercise 125.1. Let H be the Hopf link. Show that every twisted Alexander polynomial equals, up to the indeterminacy, 1.

Exercise 125.2. Let C be the Conway knot. Show that $\pi_1(S^3 \setminus C)$ admits a presentation with four generators.
Hint. Start out with a Wirtinger presentation and explain why you can apply the Generator Elimination Lemma 100.30 seven times. Note that this exercise does not require you to calculate such a presentation.

Exercise 125.3. Let R be a commutative ring. We consider the form

$$\begin{array}{rcl} \langle -,-\rangle \colon R^2 \times R^2 & \to & R \\ \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) & \mapsto & \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{array}$$

(a) Show that this pairing is non-singular.
(b) Show that for any representation $\alpha \colon \pi \to \mathrm{GL}(2,R)$ and any $g \in \pi$ we have the equality $\alpha^\dagger(g) = \det(\alpha(g))^{-1} \cdot \alpha(g)$.

Exercise 125.4. Let L be an oriented m-component link and let R be a UFD. Show that if the link L is splittable (in the sense of Definition 109.5), then for any representation $\alpha \colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ we have $\Delta_L^\alpha = 0$.
Remark. In [**FV2015**] the following converse is shown: If L is not splittable then there exists a representation $\pi_1(S^3 \setminus L) \to \mathrm{GL}(k,\mathbb{Q})$ such that $\Delta_L^\alpha \neq 0$.

Exercise 125.5. Let $p,q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p,q)$. Recall that by the Torus Knot-π_1-Proposition 111.18 we have an isomorphism

$$\pi_1(S^3 \setminus T(p,q)) \;\cong\; \langle x,y \mid x^p \cdot y^{-q} \rangle.$$

(a) Show that there exists a unique representation

$$\alpha \colon \langle x,y \mid x^p \cdot y^{-q} \rangle \;\to\; \mathrm{GL}(n,\mathbb{C})$$

such that

$$\alpha(x) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & \dots & & 0 \end{pmatrix} \qquad \text{and} \qquad \alpha(y) = \begin{pmatrix} \exp(2\pi \mathrm{i}/q) & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \exp(2\pi \mathrm{i}/q) \end{pmatrix}.$$

(b) Compute $\Delta^\alpha_{T(p,q)}$.
Hint. Modify the proof of the Torus Knot–Alexander Polynomial Proposition 117.11.
(c) Show that for $p=2$ and $q=3$ the representation from (a) is conjugate (over $\mathbb{C}$) to the representation $\pi_1(S^3 \setminus K) \to \mathrm{GL}(2,\mathbb{Z})$ that is given in Section 124.2.

Exercise 125.6. Let K_1 and K_2 be two oriented knots, let R be a commutative ring and let $\pi_1(S^3 \setminus (K_1 \# K_2)) \to \mathrm{GL}(n,R)$ be a representation. In the proof of the Oriented Knot–Monoid-π_1-Proposition 111.26 we gave an explicit isomorphism

$$\pi_1(S^3 \setminus K_1) *_{\mu_{K_1} = \mu_{K_2}} \pi_1(S^3 \setminus K_2) \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus (K_1 \# K_2)).$$

We can use this isomorphism to obtain homomorphisms $\lambda_i \colon \pi_1(S^3 \setminus K_i) \to \pi_1(S^3 \setminus (K_1 \# K_2))$. Determine the relationship between the twisted Alexander polynomials $\Delta_{K_1}^{\alpha \circ \lambda_1}(t)$, $\Delta_{K_2}^{\alpha \circ \lambda_2}(t)$ and $\Delta_{K_1 \# K_2}^{\alpha}(t)$.
Remark. This exercise is supposed to generalize the Connected Sum–Alexander Polynomial

Proposition 117.13. Note though that we do not just have a completely straightforward product formula.

CHAPTER 126

Different notions of equivalence of links

Let L and $\widetilde{L}$ be two links. If L and $\widetilde{L}$ are smoothly isotopic, then we know by the Smooth Isotopy–Link Complement Proposition 108.12, the Link Exterior Lemma 108.31 and the Isotopic Link-π_1-Lemma 111.4 that the following statements hold:

(1) There exists an orientation-preserving diffeomorphism $S^3 \setminus L \to S^3 \setminus \widetilde{L}$ between the link complements.
(2) There exists an orientation-preserving diffeomorphism $X_L \to X_{\widetilde{L}}$ between the link exteriors.
(3) There exists isomorphisms $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L})$ and $\pi_1(X_L) \cong \pi_1(X_{\widetilde{L}})$.

It is natural to ask to what degree the converses of these three conclusions results hold. We will discuss this question in great detail in this chapter.

This chapter is organized as follows:

- In Section 126.1 we will compare different notions of equivalence of links.
- In Section 126.2 we will prove the Link Equivalence Theorem 126.5.
- In Appendix 126.3 we list the definitions of the notions of "knot", "knot equivalence", "meridian" and "longitude" in various text books. We leave it to the reader to determine to what degree they are equivalent.
- In Section 126.4 we will provide exercises for Chapter 126.

126.1. **Comparing different notions of equivalence of links.** Before we can state our main result we need to introduce a few rather dry notions. We start out with the following definition which extends the notion of a group-pair system from Definition 115.9:

Definition 126.1.

(1) A group-pair system is a tuple $(G, P_1, \dots, P_m)$ where G is a group and $P_1, \dots, P_m$ are ordered pairs of elements of G.
(2) We say that two group-pair systems $(G, P_1, \dots, P_m)$ and $(H, Q_1, \dots, Q_m)$ are isomorphic if there exists an isomorphism $\varphi\colon G \to H$ such that for each $i \in \{1, \dots, m\}$ there exists an $h_i \in H$ with $h_i \cdot \varphi(P_i) \cdot h_i^{-1} = Q_i$ as an ordered pair of elements of H. In this case we write $(G, P_1, \dots, P_m) \cong (H, Q_1, \dots, Q_m)$.

Next we introduce a closely related concept:

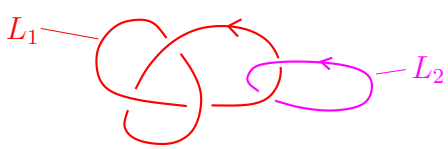
Definition 126.2.

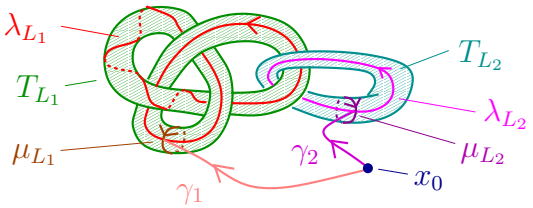
(1) A subgroup system is a tuple $(G, S_1, \dots, S_m)$ where G is a group and $S_1, \dots, S_m$ are subgroups of G.
(2) We say that two subgroup systems $(G, S_1, \dots, S_m)$ and $(H, T_1, \dots, T_m)$ are isomorphic if there exists an isomorphism $\varphi\colon G \to H$ such that for each $i \in \{1, \dots, m\}$ the group $\varphi(S_i)$ is conjugate to T_i. In this case we write $(G, S_1, \dots, S_m) \cong (H, T_1, \dots, T_m)$.

We now return to the study of links:

Definition 126.3. Let L be an oriented m-component link. We pick an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$.

(1) We define the exterior $X_L := S^3 \setminus \tau(B^2 \times L)$.
(2) For $i = 1, \dots, m$ we do the following:
  (a) We denote by T_{L_i} the boundary torus of X_L corresponding to the i-th component of L, i.e. $T_{L_i} = \tau(S^1 \times L_i)$. We denote by $\iota_i\colon T_{L_i} \to X_L$ the inclusion map.
  (b) We pick $y_i \in L_i$ and we define the meridian $\mu_{L_i} := \tau(S^1 \times \{y_i\}) \subseteq T_{L_i}$.
  (c) We define the longitude $\lambda_{L_i} \subseteq \partial T_{L_i}$ as in Definition 115.1.
(3) Let $x_0 \in X_L$.
  (a) For each boundary component we pick a path γ_i from x_0 to $\tau(1, y_i)$.
  (b) We use this common path to view μ_{L_i} and λ_{L_i} as elements of $\pi_1(X_L, x_0)$. Note that as a pair $(\mu_{L_i}, \lambda_{L_i}) \subseteq \pi_1(X_L, x_0)$ is well-defined up to a *common* conjugation. The group-pair system $(\pi_1(X_L), \mu_L, \lambda_L)$ of the link L is defined as the group-pair system $(\pi_1(X_L, x_0), (\mu_{L_1}, \lambda_{L_1}), \dots, (\mu_{L_m}, \lambda_{L_m}))$.[1302] [1303]
  (c) The peripheral subgroup system $(\pi_1(X_L), \pi_1(T_L))$ of the link L is defined as the fundamental group $\pi_1(X_L, x_0)$ together with the (ordered) family of subgroups $\gamma_{i*}(\iota_{i*}(\pi_1(T_{L_i}), \tau(1, y_i)))$ with $i = 1, \dots, m$.[1304]





Throughout this chapter we adopt the following convention:

Convention 126.4. If we talk about the exterior, meridians and longitudes of an oriented link it is understood that we used the same orientation-preserving tubular map to define them.

Now we can formulate the main result of this chapter:

Theorem 126.5. (Link Equivalence Theorem) Let L and $\widetilde{L}$ be two oriented m-component links. We pick orientation-preserving tubular maps for L and $\widetilde{L}$ to define the exteriors X_L and $X_{\widetilde{L}}$ and for $i = 1, \dots, m$ to define the boundary tori T_{L_i}, $T_{\widetilde{L}_i}$, the meridians μ_{L_i}, $\mu_{\widetilde{L}_i}$ and the longitudes λ_{L_i}, $\lambda_{\widetilde{L}_i}$. We consider the following statements:

[1302] Note that this definition generalizes the notion of a group-pair system from Definition 115.10 from oriented knots to links.
[1303] It follows easily from the Link Tubular Map Theorem 108.29 (2) that the isomorphism type of the group-pair system of an oriented link L is well-defined up to isomorphism.
[1304] It follows easily from the Link Tubular Map Theorem 108.29 (2) that the isomorphism type of the peripheral subgroup system of an oriented link L is well-defined up to isomorphism.

$L \cong \widetilde{L}$	L and $\widetilde{L}$ are smoothly isotopic.
$(S^3, L) \cong (S^3, \widetilde{L})$	There exists a diffeomorphism $f\colon S^3 \to S^3$ such that for every $i \in \{1, \dots, m\}$ we have $f(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds.
$X_L \cong X_{\widetilde{L}}$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$.
$(X_L, \mu_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \dots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as oriented smooth submanifolds.
$(X_L, \mu_L) \cong (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \dots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as *unoriented* smooth submanifolds.
$(X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \dots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and we have $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds.
$(X_L, T_{L_i}) \cong (X_{\widetilde{L}}, T_{\widetilde{L}_i})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \dots, m\}$ we have $f(T_{L_i}) = T_{\widetilde{L}_i}$.
$X_L \simeq X_{\widetilde{L}}$	X_L and $X_{\widetilde{L}}$ are homotopy equivalent.
$\pi_1(X_L) \cong \pi_1(X_{\widetilde{L}})$	The groups $\pi_1(X_L)$ and $\pi_1(X_{\widetilde{L}})$ are isomorphic.
$(\pi_1(X_L), \mu_L, \lambda_L) \cong (\pi_1(X_{\widetilde{L}}), \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$	The group-pair systems are isomorphic.
$(\pi_1(X_L), \pi_1(T_L)) \cong (\pi_1(X_{\widetilde{L}}), \pi_1(T_{\widetilde{L}}))$	The subgroup systems are isomorphic.

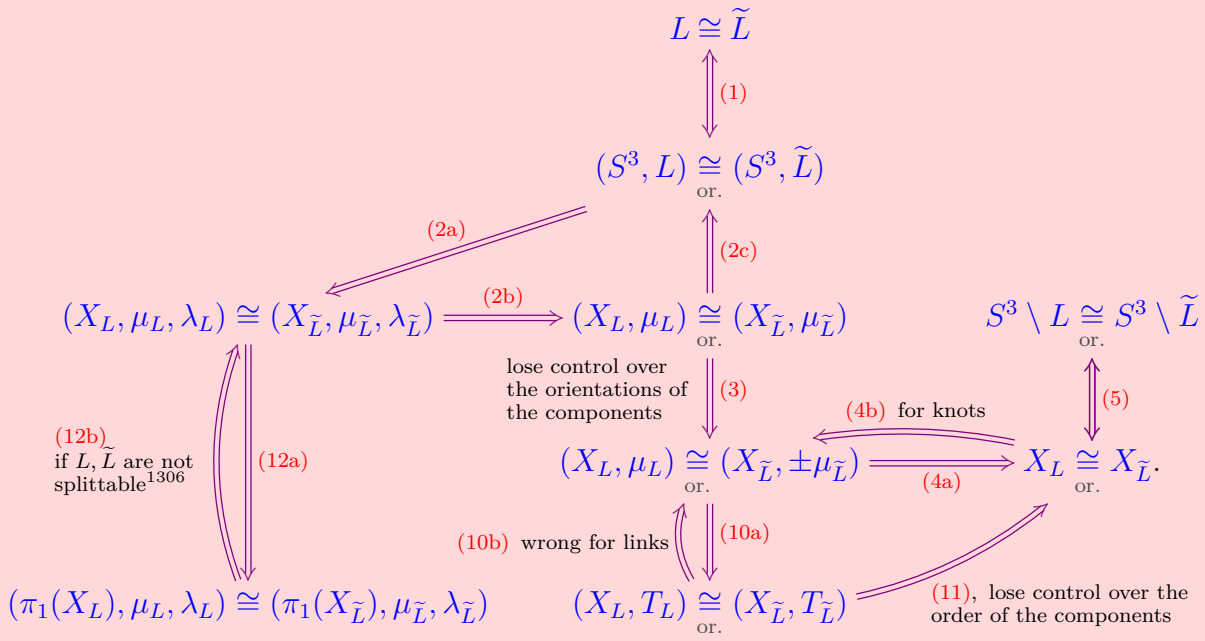
We will also use the following variations on the above:

(1) If we write $\underset{\text{or.}}{\cong}$, then we mean an orientation-preserving diffeomorphism.

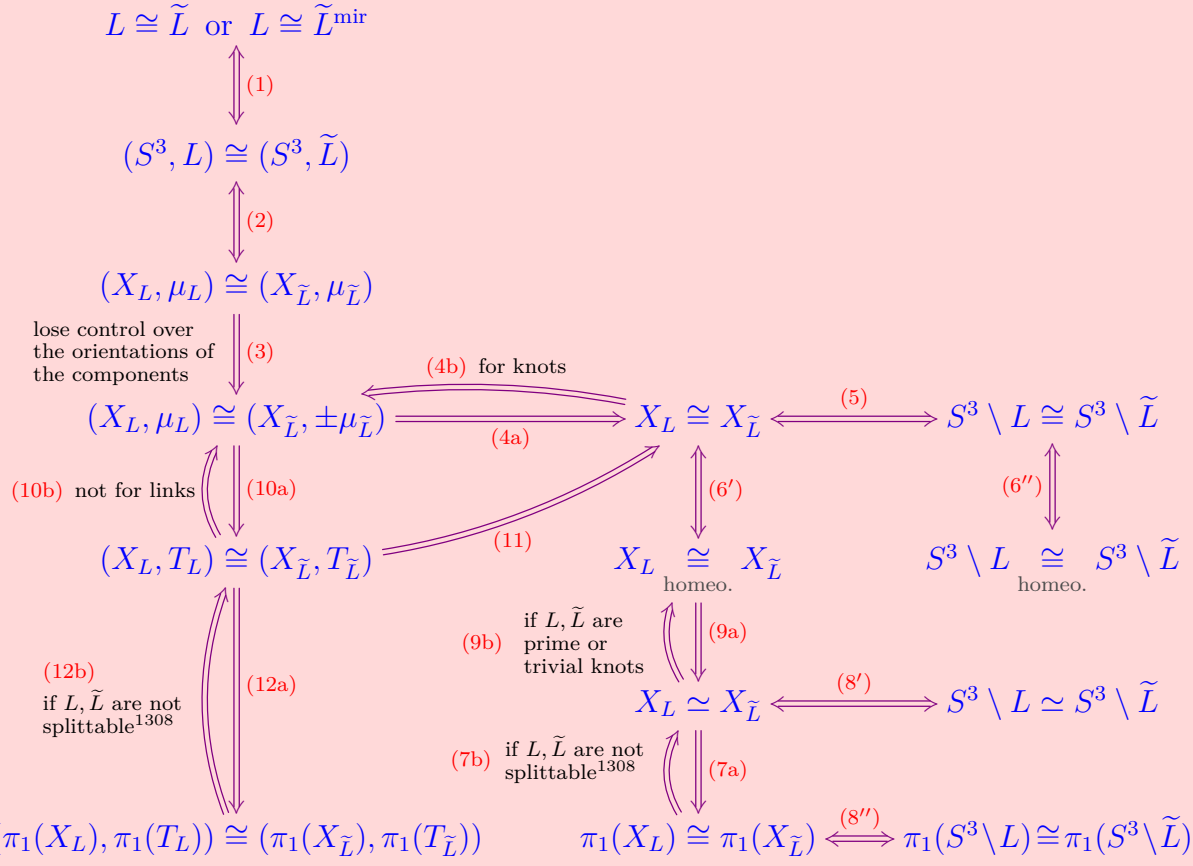
(2) If we write $\underset{\text{homeo.}}{\cong}$, then we mean homeomorphism instead of diffeomorphism.

(3) In some of the above statements we replace the link exteriors by the link complements.

With this notation we can formulate the following implications:



If we do not care about the difference between links and their mirrors, then we also have the following implications:



The proof of the Link Equivalence Theorem 126.5 will require the remainder of this chapter. It is unfortunately unavoidable that we will have to refer to many results which we will prove and discuss in later chapters. Furthermore we will also have to cite lots of results which we will not prove at all.

Even when in the following we provide proofs of statements our arguments will be more sketchy than in the other chapters.

126.2. **Proof of the Link Equivalence Theorem 126.5 .** In the following twelve subsections we consider all the implications of the Link Equivalence Theorem 126.5.

126.2.1. **Proof of (1).** The reader will surely have spotted that the Link Equivalence Theorem 126.5 (1) is essentially the same as the Smooth Isotopy–Diffeomorphism Proposition 108.11. This means, that we will have no troubles proving the Link Equivalence Theorem 126.5 (1):

Proof of the Link Equivalence Theorem 126.5 (1). Let L and $\widetilde{L}$ be two oriented m-component links. We need to prove the following two statements:

$$L \cong \widetilde{L} \iff (S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L}) \quad \text{and} \quad L \underset{\text{or}}{\cong} \widetilde{L} \text{ or } L \cong L^{\text{mir}} \iff (S^3, L) \cong (S^3, \widetilde{L}).$$

"$\Rightarrow$". We first suppose that $L \cong \widetilde{L}$, i.e. we assume that L and $\widetilde{L}$ are smoothly isotopic. As we discussed in the proof of the Smooth Isotopy–Diffeomorphism Proposition 108.11 (1)$\Rightarrow$(2) it follows from the Isotopy Extension Theorem 44.2 that there exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i$ as oriented smooth manifolds. We have thus shown that $(S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L})$.

If L and $\widetilde{L}^{\text{mir}}$ are smoothly isotopic, then we obtain from the above discussion that there exists an orientation-preserving diffeomorphism $(S^3, L) \to (S^3, \widetilde{L}^{\text{mir}})$. If we compose this diffeomorphism with the reflection in a suitable hyperplane we obtain an an orientation-reversing diffeomorphism $(S^3, L) \to (S^3, \widetilde{L})$.

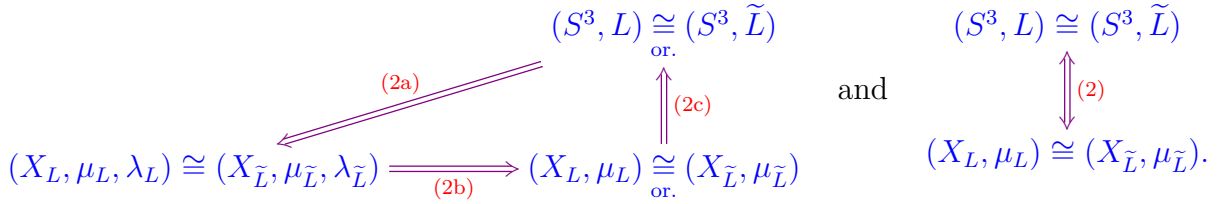
"$\Leftarrow$". First we assume that $(S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L})$. It follows immediately from the Smooth Isotopy–Diffeomorphism Proposition 108.11 (2)$\Rightarrow$(1) that there exists a smooth isotopy from L to $\widetilde{L}$.[1309]

Finally let us suppose that there exists an orientation-reversing self-diffeomorphism $\Phi\colon S^3 \to S^3$ such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds. If we compose this diffeomorphism with a reflection in a hyperplane, then we obtain an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i^{\text{mir}}$ as oriented smooth submanifolds. By the above this implies that $L \cong \widetilde{L}^{\text{mir}}$. ■

[1308] It follows from the Link Kneser Theorem 111.21 that if L and $\widetilde{L}$ are two links with $\pi_1(X_L) \cong \pi_1(\widetilde{X}_L)$, then L is splittable if and only if $\widetilde{L}$ is splittable.

[1309] Here is an alternative argument, which requires a much deeper input: Let $\Phi\colon S^3 \to S^3$ be an *orientation-preserving* diffeomorphism such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds. By the Cerf Theorem **??** there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \text{id}$ and $F_1 = \Phi$. The restriction of F to $L \times [0,1]$ is the desired smooth isotopy from L to $\widetilde{L}$.

126.2.2. **Proof of (2).** In this subsection we need to show the following implications:



In the proof we will need the following theorem on self-diffeomorphisms of the 2-dimensional torus:

Theorem 126.6. (Torus–Mapping Class Group Theorem)

(1) Given any automorphism $\varphi \in \mathrm{Aut}(\pi_1(S^1 \times S^1, (1,1)))$ there exists a diffeomorphism $f\colon S^1 \times S^1 \to S^1 \times S^1$ with $f_* = \varphi$.
(2) If $\varphi\colon S^1 \times S^1 \to S^1 \times S^1$ is a diffeomorphism such that $\varphi_* = \mathrm{id}$ on $\pi_1(S^1 \times S^1, (1,1))$, then φ is diffeotopic to the identity.

Proof.

(1) For the proof it is best to replace $S^1 \times S^1$ with $T = \mathbb{R}^2/\mathbb{Z}^2$. We use the generators from the Torus-π_1-Proposition 90.14 to make the usual identification $\pi_1(T, [(0,0)]) = \mathbb{Z}^2$. Now let $A \in \mathrm{Aut}(\pi_1(T, [(0,0)])) = \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2, \mathbb{Z})$. The map
$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \mapsto & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [A \cdot v] \end{array}$$
is clearly a diffeomorphism. It follows easily from the Torus-π_1-Proposition 90.14 that f_* is given by multiplication by A.
(2) In the Torus–Mapping Class Group Theorem **??** we prove this statement with the first homology group $\mathrm{H}_1(S^1 \times S^1)$ instead of $\pi_1(S^1 \times S^1, (1,1))$. The desired statement now follows from the natural isomorphism $\pi_1(S^1 \times S^1, (1,1)) \to \mathrm{H}_1(S^1 \times S^1)$ that is provided for us by the Hurewicz Theorem 166.10. ■

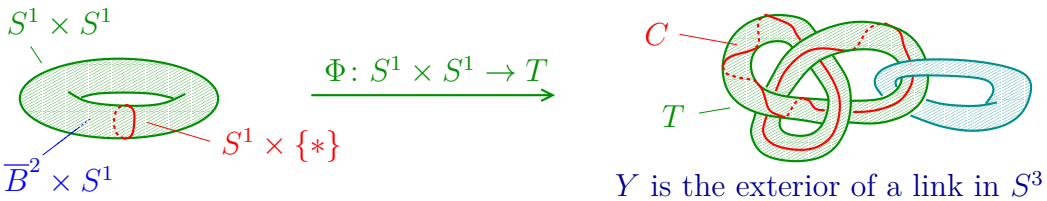
We use the Torus–Self Diffeomorphism Theorem 126.6 to introduce a construction which is very popular in low-dimensional topology:

Proposition 126.7. (Dehn Filling Proposition) Let Y be a 3-dimensional smooth manifold and let T be a component of the boundary ∂Y such that T is diffeomorphic to the 2-dimensional torus. Furthermore let $C \subseteq T$ be an oriented submanifold that is diffeomorphic to S^1 such that $[C] \in \pi_1(T)$ can be extended to a basis of $\pi_1(T) \cong \mathbb{Z}^2$.

(1) There exists a diffeomorphism $\Phi\colon S^1 \times S^1 \to T$ with $\Phi(S^1 \times \{*\}) = C$.
(2) The diffeomorphism type of
$$Y(C) \;:=\; Y \cup_{\partial Y \leftarrow S^1 \times S^1 : \Phi} (\overline{B}^2 \times S^1) \;=\; (Y \sqcup (\overline{B}^2 \times S^1))/\!\sim$$
where $\Phi(z,w) \sim (z,w)$ for $(z,w) \in S^1 \times S^1$
does not depend on the choice of Φ. We refer to the 3-dimensional smooth manifold $Y(C)$ as the Dehn filling of Y along C.

Remark. We will prove a closely related result in the Dehn Surgery Proposition **??**. □

Sketch of proof.



(1) By hypothesis we can extend $[C] \in \pi_1(T)$ to a basis of $\pi_1(T)$. The statement now follows quite easily from the Torus–Self Diffeomorphism Theorem 126.6 (1) and (2).

(2) Let $\Phi, \Psi\colon S^1 \times S^1 \to \partial Y$ be two diffeomorphisms with $\Phi(S^1 \times \{*\}) = C$ and with $\Psi(S^1 \times \{*\}) = C$. We make the following two observations:

(a) By the Diffeotopic-Gluings Lemma 54.7 we know that diffeotopic diffeomorphisms $S^1 \times S^1 \to T$ lead to diffeomorphic results.

(b) It follows easily from the Torus–Self Diffeomorphism Theorem 126.6 (2) that, up to smooth isotopy, $\Psi^{-1} \circ \Phi\colon S^1 \times S^1 \to S^1 \times S^1$ is of the form

$$\begin{array}{rcl} \Theta\colon S^1 \times S^1 & \to & S^1 \times S^1 \\ (z, w) & \mapsto & (z^\delta \cdot w^k, w^\epsilon) \end{array}$$

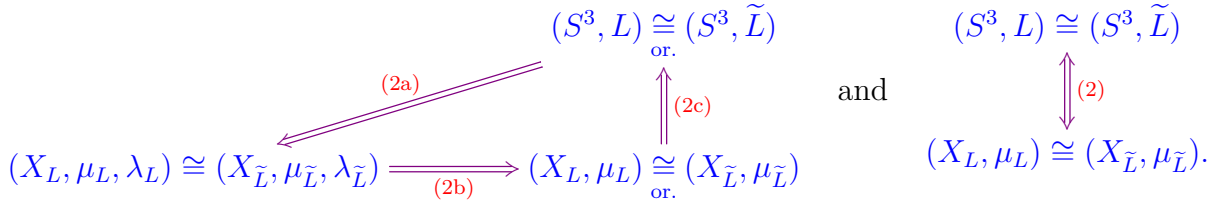
for some $k \in \mathbb{Z}$ and some $\delta, \epsilon \in \{-1, 1\}$. Note that this diffeomorphism extends to a diffeomorphism

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (z, w) & \mapsto & (z^\delta \cdot w^k, w^\epsilon) \end{array}$$

Using these two observations one can easily show that Φ and Ψ lead to diffeomorphic results. ■

Example. Let $L \subseteq S^3$ be an m-component link with link exterior X_L, let L_i be a component of L and let $\mu_i \subseteq \partial X_L$ be a meridian of L_i. We consider the Dehn filling $\varphi(\mu_i)$ as defined in the Dehn Filling Proposition 126.7. In Exercise 126.1 we will see that there exists an orientation-preserving diffeomorphism $X_L(\mu_i) \to X_{L \setminus L_i}$ which is the identity on X_L and which sends $\{0\} \times S^1$ to L_i as oriented smooth submanifolds. □

Proof of the Link Equivalence Theorem 126.5 (2). Let L and $\widetilde{L}$ be two oriented m-component links. We need to show the following implications:



In the following we will prove the oriented statements to the left. We leave it to the reader to modify the argument to deal with the unoriented statement to the right.

(2a). In the following we suppose that $(S^3, L) \cong (S^3, \widetilde{L})$, i.e. we suppose that there exists an orientation-preserving diffeomorphism $f\colon S^3 \xrightarrow{\text{gr.}} S^3$ such that for every $i \in \{1, \dots, m\}$ we have $f(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds. We pick an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L. It is clear that $(f \circ \tau) \circ (\operatorname{diag}(\mathrm{id}, (f|_L)^{-1}))\colon \overline{B}^2 \times \widetilde{L} \to S^3$ is an orientation-preserving tubular map for $\widetilde{L}$. If we use these tubular maps to define

the exteriors, the meridians and the longitudes we see that $f\colon S^3 \to S^3$ restricts to an orientation-preserving diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1,\dots,m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and we have $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds. Thus we have shown that $(X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$.

(2b). We now suppose that $(X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$, i.e. we suppose that there exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for $i = 1,\dots,m$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds. It follows easily that the restriction of f to $\partial X_L \to \partial X_{\widetilde{L}}$ is orientation-preserving. By the Boundary Orientation Lemma 31.38 this implies that the original map $f\colon X_L \to X_{\widetilde{L}}$ is also orientation-preserving. Thus we have shown that $(X_L, \mu_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}})$.

(2c). We suppose that $(X_L, \mu_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}})$, i.e. we suppose that there exists an orientation-preserving diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1,\dots,m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as oriented smooth manifolds.

Iterating the example preceding the proof we see that $f\colon X_L \to X_{\widetilde{L}}$ extends to an orientation-preserving diffeomorphism $f\colon S^3 \to S^3$ such that for every $i \in \{1,\dots,m\}$ the map f restricts to an orientation-preserving diffeomorphism $f\colon L_i \to \widetilde{L}_i$. Thus we have shown that $(S^3, L) \cong (S^3, \widetilde{L})$. ∎

126.2.3. **Proof of (3).** We need to show that

$$(X_L, \mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}}, \mu_{\widetilde{L}}) \implies (X_L, \mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}}).$$

In fact, this is of course trivial. It is perhaps more important to say what this step does: In the process we lose control over the orientation of the components of the link. In general the step is not reversible since we saw that on page 2241 there exist non-reversible knots, e.g. the Conway knot. For this knot K and its reverse K^{rev} we know, by the Meridian–Symmetries Lemma 109.3, that $(X_K, \mu_K) = (X_K, -(\mu_K)^{\text{rev}}) \cong (X_{K^{\text{rev}}}, -\mu_{K^{\text{rev}}})$, but we have $K \ncong K^{\text{rev}}$, thus it follows from (1) and (2) that we do *not* have $(X_K, \mu_K) \cong (X_{K^{\text{rev}}}, \mu_{K^{\text{rev}}})$.

126.2.4. **Proof of (4).** In this subsection we want to show

$$(X_L, \mu_L) \cong (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}}) \quad \underset{\text{(4a)}}{\overset{\text{(4b for knots}}{\rightleftarrows}} \quad X_L \cong X_{\widetilde{L}}$$

and we want to prove the corresponding oriented version. The statement (4a) is of course trivial. So it remains to deal with (4b). The proof of the Link Equivalence Theorem 126.5 (4b) rests squarely on the following theorem:

Theorem 126.8. (Gordon-Luecke Theorem) Let K be a knot and let $C \subseteq \partial X_K$ be an oriented submanifold that is diffeomorphic to S^1 such that $[C] \in \pi_1(\partial X_K)$ can be extended to a basis of $\pi_1(\partial X_K) \cong \mathbb{Z}^2$. If the Dehn filling $Y(C)$, which we introduced in the Dehn Filling Proposition 126.7, is diffeomorphic to S^3, then C is smoothly isotopic in ∂X_K to a meridian of K (as unoriented smooth submanifolds).

Proof. This theorem was proved in the 1980s by Cameron Gordon and John Luecke [**GLu1989**, Theorem 1]. The proof builds in particular on the work of William Thurston [**Thu1982**] for which William Thurston was awarded a Fields medal in 1982. ∎

Proof of the Link Equivalence Theorem 126.5 (4b). We need to show the following claim:

Claim. Let K and $\widetilde{K}$ be two oriented knots. If there exists an (orientation-preserving) diffeomorphism $f\colon X_K \xrightarrow{\cong} X_{\widetilde{K}}$, then there exists also an (orientation-preserving) diffeomorphism $g\colon X_K \xrightarrow{\cong} X_{\widetilde{K}}$ such that $g(\mu_K) = \mu_{\widetilde{K}}$ as *unoriented* smooth manifolds.[1310]

Proof. First note that we have the following (orientation-preserving) diffeomorphisms:

$$\begin{array}{ccccccc} S^3 & \cong & X_K(\mu_K) & \cong & f(X_K)(f(\mu_K)) & = & X_{\widetilde{K}}(f(\mu_K)). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the example} & & \text{this is clear} & & \text{since } f(X_K) = X_{\widetilde{K}} & \\ & \text{on page 2424} & & & & & \end{array}$$

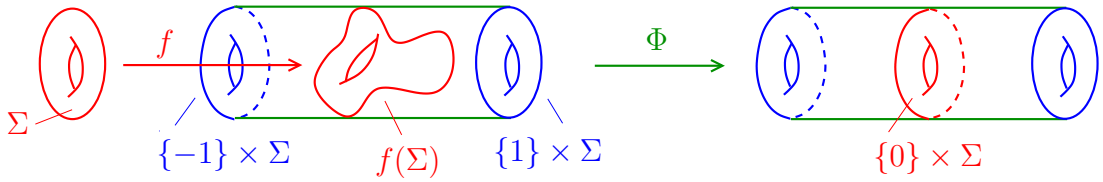
It follows from this discussion and the Gordon-Luecke Theorem 126.8 that $f(\mu_K)$ is smoothly isotopic in $\partial X_{\widetilde{K}}$ to $\mu_{\widetilde{K}}$. Using the Isotopy Extension Theorem 44.2 and the Smooth Collar Neighborhood Theorem 35.6 one can now easily find a diffeotopy from f to an (orientation-preserving) diffeomorphism $g\colon X_K \to X_{\widetilde{K}}$ such that $g(\mu_K) = \mu_{\widetilde{K}}$ (as unoriented smooth submanifolds). ■

126.2.5. **Proof of (5).** We need to show that

$$X_L \cong X_{\widetilde{L}} \iff S^3 \setminus L \cong S^3 \setminus \widetilde{L}$$

and we need to show the corresponding oriented analog. In the proof of the "⇐"-direction we will need the following non-trivial proposition:

Proposition 126.9. (3D-Product Splitting Proposition) Let Σ be a closed orientable connected 2-dimensional smooth manifold. If $f\colon \Sigma \to (-1,1) \times \Sigma$ is a smooth embedding such that $([-1,1] \times \Sigma) \setminus f(\Sigma)$ is disconnected, then there exists an orientation-preserving diffeomorphism $\Phi\colon [-1,1] \times \Sigma \to [-1,1] \times \Sigma$ which is the identity on a neighborhood of $\{\pm 1\} \times \Sigma$ and with $\Phi(f(\Sigma)) = \{0\} \times \Sigma$.



Proof. The proposition is a special case of the more general 3D-Product Splitting Proposition **??** which we will discuss (and partially prove) later. ■

Proof of the Link Equivalence Theorem 126.5 (5). Let L and $\widetilde{L}$ be two oriented m-component links. We need to show

$$X_L \cong X_{\widetilde{L}} \iff S^3 \setminus L \cong S^3 \setminus \widetilde{L}$$

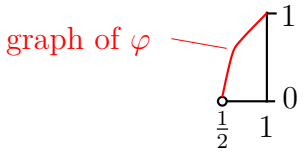
and we need to show the corresponding oriented analog.
"⇒". In the following we assume that there exists an (orientation-preserving) diffeomorphism $f\colon X_L \xrightarrow{\cong} X_{\widetilde{L}}$. In this case we now need to show that there exists an (orientation-preserving) diffeomorphism $S^3 \setminus L \xrightarrow{\cong} S^3 \setminus \widetilde{L}$. Note that f restricts to an (orientation-preserving) diffeomorphism $f\colon X_L \setminus \partial X_L \xrightarrow{\cong} X_{\widetilde{L}} \setminus \partial X_{\widetilde{L}}$. Thus it remains to prove the following claim:

[1310] It is clear that we cannot hope to recover the orientation of μ_K.

Claim 1. Let $J \subseteq S^3$ be an oriented link. There exists an orientation-preserving diffeomorphism between the smooth manifolds $X_J \setminus \partial X_J$ and $S^3 \setminus J$.

Proof. Let $\tau\colon \overline{B}^2 \times J \to S^3$ be an orientation-preserving tubular map for the oriented link J. It is clear that the map $\overline{B}^2 \times J \xrightarrow{(x,t)\mapsto(\frac{1}{2}x,t)} \overline{B}^2 \times J \xrightarrow{\tau} S^3$ is also an orientation-preserving tubular map for J. It follows from the Link Exterior Lemma 108.31 that there exists an orientation-preserving diffeomorphism $S^3 \setminus \tau(B^2 \times J) \to S^3 \setminus \tau(B^2_{\frac{1}{2}} \times J)$. Since the boundary of $S^3 \setminus \tau(B^2_{\frac{1}{2}} \times J)$ equals $\tau(S^1_{\frac{1}{2}} \times J)$ we now see that it suffices to provide an orientation-preserving diffeomorphism $S^3 \setminus \tau(\overline{B}^2_{\frac{1}{2}} \times J) \to S^3 \setminus J$.

Using the Smooth Transition Function Lemma 24.9 one can easily find a diffeomorphism $\varphi\colon (\frac{1}{2}, 1] \to (0, 1]$ such that $\varphi(t) = t$ for all $t \in [\frac{3}{4}, 1]$.



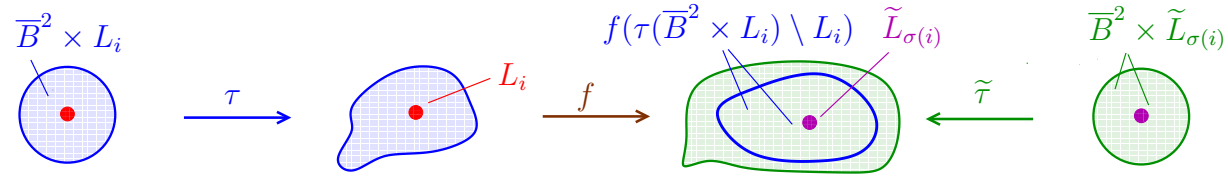
Using the Smooth Pasting Proposition 25.19 one can now verify that the map

$$\begin{array}{rcl} S^3 \setminus \tau(B^2_{\frac{1}{2}} \times J) & \to & S^3 \setminus J \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin \tau(\overline{B}^2_{\frac{7}{8}} \times J), \\ \tau(\varphi(\|y\|) \cdot y, z) & \text{if } x = \tau(y,z) \text{ with } y \in \overline{B}^2 \setminus \overline{B}^2_{\frac{1}{2}}, z \in J \end{cases} \end{array}$$

is an orientation-preserving diffeomorphism. ⊞

"⇐". In the following we assume that there exists an (orientation-preserving) diffeomorphism $f\colon S^3 \setminus L \xrightarrow{\cong} S^3 \setminus \widetilde{L}$. We need to show that there exists an (orientation-preserving) diffeomorphism $X_L \xrightarrow{\cong} X_{\widetilde{L}}$. We pick an orientation-preserving tubular map $\widetilde{\tau}\colon \overline{B}^2 \times \widetilde{L} \to S^3$ for $\widetilde{L}$.

Claim 2. There exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L and a bijection $\sigma\colon \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that for every $i \in \{1, \ldots, m\}$ we have the inclusion $f(\tau(\overline{B}^2 \times L_i) \setminus L_i) \subseteq \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_{\sigma(i)}) \setminus \widetilde{L}_{\sigma(i)}$.



Proof. We pick an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L. First note that it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 (1) that $\widetilde{\tau}(B^2 \times \widetilde{L}) \subseteq S^3$ is open. By the Compact-Closed Lemma 3.39 (1) this implies that the complement $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$ is compact. We note that it follows from the fact that $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$ is a diffeomorphism, together with the Codimension-Zero Smooth Embedding Proposition 30.21 (1), that the sets $\{f(S^3 \setminus \tau(B^2_t \times L))\}_{t \in (0,1)}$ form an open cover of $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$. It follows from compactness of $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$ that there exists an $s \in (0, 1)$ such that $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L}) \subseteq f(S^3 \setminus \tau(B^2_s \times L))$. From this inclusion and the

equalities

$$f(S^3 \setminus \tau(B^2_s \times L)) \;=\; f((S^3 \setminus L) \setminus \tau((B^2_s \times L) \setminus L)) \;=\; (S^3 \setminus \widetilde{L}) \setminus f(\tau(B^2_s \times L) \setminus L)$$

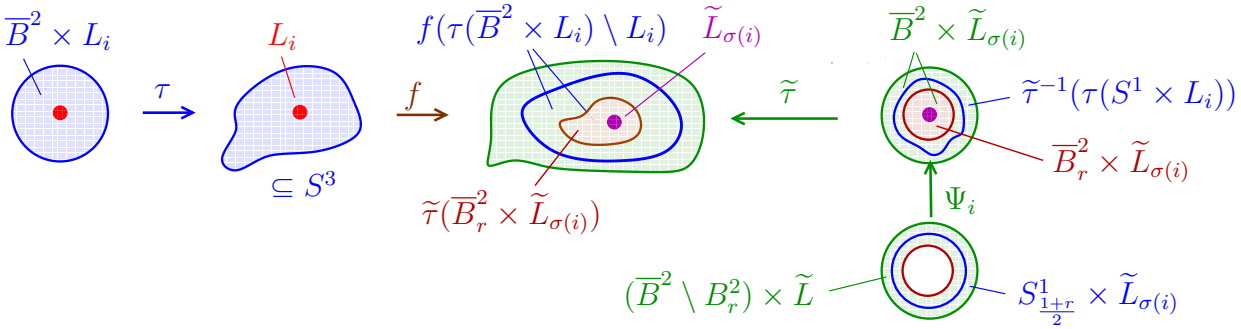
it follows that $f(\tau(B^2_s \times L) \setminus L) \subseteq \widetilde{\tau}(B^2 \times \widetilde{L}) \setminus \widetilde{L}$. We can now "radially rescale" τ to obtain an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L such that we have the desired inclusion $f(\tau(\overline{B}^2 \times L) \setminus L) \subseteq \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}) \setminus \widetilde{L}$.

We still need to show the existence of the bijection σ. Let $i \in \{1, \dots, m\}$. Since $f(\tau(\overline{B}^2 \times L_i) \setminus L_i)$ is connected and since $\widetilde{\tau}(\overline{B}^2 \times \widetilde{L}) \setminus \widetilde{L}$ is the disjoint union (as a topological space) of the subspaces $\widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_j) \setminus \widetilde{L}_j$ there exists a unique $\sigma(i) \in \{1, \dots, m\}$ such that $f(\tau(\overline{B}^2 \times L_i) \setminus L_i) \subseteq \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_{\sigma(i)}) \setminus \widetilde{L}_{\sigma(i)}$. It remains to argue that the resulting map $\sigma\colon \{1, \dots, m\} \to \{1, \dots, m\}$ is a bijection. Since we are dealing with a finite set it suffices to show that the map σ is surjective. Thus let $j \in \{1, \dots, m\}$. Since $f(S^3 \setminus \tau(B^2 \times L))$ is compact we obtain, by considering the open cover $\{S^3 \setminus \widetilde{\tau}(\overline{B}^2_t \times \widetilde{L}_j)\}_{t \in (0,1)}$ of $S^3 \setminus \widetilde{L}$ that there exists an $s \in (0,1)$ such that $f(S^3 \setminus \tau(B^2 \times L)) \subseteq S^3 \setminus \widetilde{\tau}(\overline{B}^2_s \times \widetilde{L}_j)$. This of course implies that $f(S^3 \setminus \tau(B^2 \times L)) \cap \widetilde{\tau}(\overline{B}^2_s \times \widetilde{L}_j) = \varnothing$. Since $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$ is a diffeomorphism (in particular a bijection) we see that there needs to exists an $i \in \{1, \dots, m\}$ with $f(\tau(\overline{B}^2 \times L_i)) \subseteq \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_j)$. ⊞

We continue with the notation from Claim 2. It remains to prove the following claim:

Claim 3. There exists an orientation-preserving tubular map $\widetilde{\tau}'\colon \overline{B}^2 \times \widetilde{L} \to S^3$ for $\widetilde{L}$ such that f restricts to a diffeomorphism $S^3 \setminus \tau(B^2 \times L) \to S^3 \setminus \widetilde{\tau}'(B^2 \times \widetilde{L})$.

Proof. Note that $f(S^3 \setminus \tau(B^2 \times L))$ is a closed subset of S^3 that is disjoint from $\widetilde{L}$. Since $f(S^3 \setminus L) = S^3 \setminus \widetilde{L}$ we now see that $f(\tau(B^2 \times L) \setminus L) \cup \widetilde{L}$ is an open neighborhood of $\widetilde{L}$. Using the Tube Lemma 7.12 one can easily show that there exists an $r > 0$ such that $\widetilde{\tau}(\overline{B}^2_r \times \widetilde{L}) \subseteq f(\tau(B^2 \times L) \setminus L) \cup \widetilde{L}$.



Next note that it follows easily from the 3D-Product Splitting Proposition 126.9 that for each $i \in \{1, \dots, m\}$ there exists an orientation-preserving diffeomorphism

$$\Psi_i\colon (\overline{B}^2 \setminus B^2_r) \times \widetilde{L}_{\sigma(i)} \;\to\; (\overline{B}^2 \setminus B^2_r) \times \widetilde{L}_{\sigma(i)},$$

with $\Psi_i(S^1_{\frac{1+r}{2}} \times \widetilde{L}_{\sigma(i)}) = (\widetilde{\tau}^{-1}(\tau(S^1 \times L_i)))$ and which is the identity in a neighborhood of the boundary. It is now straightforward to verify that the desired orientation-preserving tubular map $\widetilde{\tau}'\colon \overline{B}^2 \times \widetilde{L} \to S^3$ for $\widetilde{L}$ is given by the radial rescaling of the following map:

$$\begin{array}{rcl} \overline{B}^2_{\frac{1+r}{2}} \times \widetilde{L} & \to & S^3 \\ (x,y) & \mapsto & \begin{cases} \widetilde{\tau}(x,y), & \text{if } \|x\| \le r, \\ \widetilde{\tau}(\Psi_i^{-1}(x,y)), & \text{if } \|x\| \in [r, \frac{1+r}{2}] \text{ and } y \in \widetilde{L}_{\sigma(i)}. \end{cases} \end{array}$$

■

126.2.6. **Proof of (6).** We need to show that

$$X_L \cong X_{\widetilde{L}} \overset{(6')}{\Longleftrightarrow} X_L \underset{\text{homeo.}}{\cong} X_{\widetilde{L}} \quad \text{and} \quad S^3 \setminus L \cong S^3 \setminus \widetilde{L} \overset{(6'')}{\Longleftrightarrow} S^3 \setminus L \underset{\text{homeo.}}{\cong} S^3 \setminus \widetilde{L}.$$

The "$\Rightarrow$"-direction is of course in both cases trivial. The "$\Leftarrow$"-direction is in both cases a special case of the following deep theorem:

Theorem 126.10. (3D-Homeo-implies-Diffeo) If two 3-dimensional smooth manifolds are homeomorphic, then they are also diffeomorphic.

Proof. This theorem follows from work of Edwin Moise [**Moi1952**] [**Moi1977**, p. 252 and 253], William Thurston [**Thu1997**, Theorem 3.10.8] and James Munkres [**Mun1959a**, p. 333][**Mun1966a**][**Mun1960a**, Theorems 6.2 and 6.3]. We refer to **??** and the 3D-Unique Smooth Structure-Theorem **??** for a more detailed discussion. ■

Remark.

(1) The Smooth-Topological Orientation Proposition 191.25 implies that the "oriented analog" of the 3D-Homeo-implies-Diffeo 126.10 also holds.
(2) The statement of the 3D-Homeo-implies-Diffeo 126.10 is rather specific to dimension three. For example in the Exotic K3-Theorem 228.24 we will see that there exist closed 4-dimensional topological manifolds that are homeomorphic but not diffeomorphic. □

126.2.7. **Proof of (7).** We need to show that

$$X_L \simeq X_{\widetilde{L}} \quad \underset{\text{(7b) if } L, \widetilde{L} \text{ are not splittable}}{\overset{\text{(7a)}}{\rightleftarrows}} \quad \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}).$$

The "$\Rightarrow$"-statement (7a) is just a special case of the Homotopy Equivalence-π_1-Proposition 90.9. The other direction is much more interesting and much harder. It will build on several results which go well beyond the other material of this part of the lecture notes, but which we will cover in later parts of the lecture notes.

In the following discussion it is unavoidable to make use of the notion of higher homotopy groups $\pi_n(X, x_0)$, $n \geq 2$ of a pointed topological space, which we will only introduce a little later in Chapter 134. For the purpose of this section it is enough to know that $\pi_n(X, x_0)$ is defined as the set of homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$. Furthermore we will mention the notion of a CW-complex that we will introduce later in Section 136.1. But even if the reader is completely unaware of these concepts, it should not be a problem to follow the flow of the conversation.

Definition 126.11. Let X be a non-empty topological space. We say that X is aspherical if X is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

The following theorem explains why we are interested in these new notions:

Theorem 307.15 (Eilenberg-Maclane Uniqueness Theorem) Let (X, x_0) and (Y, y_0) be two pointed aspherical CW-complexes. For any isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ of the fundamental groups there exists a homotopy equivalence $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$.

The Eilenberg-Maclane Uniqueness Theorem 307.15 sounds like a promising approach to dealing with the "$\Leftarrow$"-statement formulated above. But to apply this theorem we need to

figure out which link exteriors are CW-complexes and which link exteriors are aspherical. Fortunately the next theorem says in particular that all link exteriors are CW-complexes:

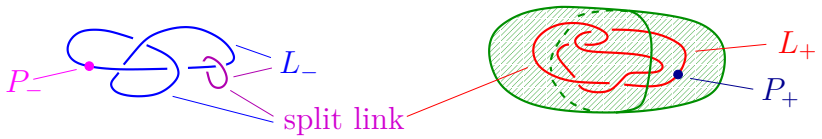
Theorem 179.9. (Smooth Manifold-CW-Theorem) Every compact smooth manifold M admits a CW-structure.

The question, which link exteriors are aspherical, is more subtle:

Theorem 126.12. (Aspherical Link Exterior Theorem) Let L be a non-empty link. Then

$$X_L \text{ is aspherical} \iff L \text{ is not splittable.}$$

Sketch of proof. The"$\Rightarrow$"-direction is quite straightforward: So let L be a link that is splittable in the sense of Definition 109.5. We need to show that X_L is not aspherical. Basically by definition this implies that there exist smooth embeddings $F_\pm\colon \overline{B}^3 \to S^3$ with $F_-(\overline{B}^3) \cap F_+(\overline{B}^3) = F_-(S^2) = F_+(S^2)$ such that $F_\pm(S^2) \cap L = \varnothing$ and such that $L_- = F_-(\overline{B}^3) \cap L$ and $L_+ = F_+(\overline{B}^3) \cap L$ are both non-empty. This allows us to pick points $P_- \in L_-$ and $P_+ \in L_+$.



It is elementary to see that there exists a retraction from $F_\pm(\overline{B}^3)\backslash\{P_\pm\}$ to $F_\pm(S^2)$. Together these two retractions give us a retraction r from $S^3 \setminus (L_- \cup L_+)$ to the sphere $F_-(S^2)$.[1311] It follows from the $\pi_n(S^n)$–Proposition 167.10 that $\pi_2(F_-(S^2)) \cong \mathbb{Z}$. By the functoriality of π_2 we see that $r_*\colon \pi_2(X_L) \to \pi_2(F_-(S^2))$ is an epimorphism. This implies that $\pi_2(X_L)$ is non-trivial. Thus we have shown that X_L is not aspherical.
The"$\Leftarrow$"-direction is much harder. We will prove it in greater detail in the later Aspherical Link Exterior Theorem **??**. Here we just mention the three main ingredients:

(1) The Sphere Theorem **??** which says that if N is an orientable 3-dimensional smooth manifold N with $\pi_2(N) \neq 0$, then there a smooth embedding $f\colon S^2 \to N$ such that $[f]$ is non-trivial in $\pi_2(N)$. This theorem was proved by Christos Papakyriakopoulos [**Pap1957a**] and John Whitehead [**WhdJ1958a**].
(2) The Generalized Smooth Schoenflies Theorem 170.7 which says that for any smooth embedding $f\colon S^2 \to S^3$ there exist two smooth embeddings $F_\pm\colon \overline{B}^3 \to S^3$ such that $F_-(\overline{B}^3) \cap F_+(\overline{B}^3) = f(S^2)$.
(3) The Hurewicz Theorem 167.9 and the Manifold-Top-H_*-Theorem 193.1 which imply that if $L \neq \varnothing$ and if $\pi_n(X_L) = 0$ for some $n \geq 2$, then $\pi_{n+1}(X_L) = 0$.

Now let L be a link that is not splittable. Using (1) and (2) one can easily show that $\pi_2(X_L) = 0$. Iteratively applying (3) we see that $\pi_n(X_L) = 0$ for all $n \geq 2$. Thus we have shown that X_L is aspherical. ■

[1311] This argument is basically the content of Exercise 110.9.

Proof of the Link Equivalence Theorem 126.5 (7). Let L and $\widetilde{L}$ be two oriented m-component links. We need to show that

$$X_L \simeq X_{\widetilde{L}} \underset{\text{(7b) if } L, \widetilde{L} \text{ are not splittable}}{\overset{\text{(7a)}}{\rightleftarrows}} \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}).$$

As we already mentioned in the beginning of this subsection, the "$\Rightarrow$"-direction is just a special case of the Homotopy Equivalence-π_1-Proposition 90.9.

We turn to the proof of the "$\Leftarrow$"-direction. We assume that L and $\widetilde{L}$ are not splittable. It follows from the Aspherical Link Exterior Theorem 126.12 that X_L and $X_{\widetilde{L}}$ are both aspherical. The promised statement now follows from the Eilenberg-Maclane Uniqueness Theorem 307.15 together with the Smooth Manifold-CW-Theorem 179.9. ■

126.2.8. **Proof of (8).** We need to prove

$$X_L \simeq X_{\widetilde{L}} \overset{(8')}{\Longleftrightarrow} S^3 \setminus L \simeq S^3 \setminus \widetilde{L} \quad \text{and} \quad \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}) \overset{(8'')}{\Longleftrightarrow} \pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L}).$$

Both statements follow from the Link Exterior Lemma 108.31, where we showed that for any link J the inclusion $X_J \to S^3 \setminus J$ is a homotopy equivalence, together with the Homotopy Equivalence-π_1-Proposition 90.9.

126.2.9. **Proof of (9).** We want to prove the following statement:

$$X_L \underset{\text{homeo.}}{\cong} X_{\widetilde{L}} \underset{\text{(9b) yes, if } L \text{ and } \widetilde{L} \text{ are prime or trivial knots,}}{\overset{\text{(9a)}}{\rightleftarrows}} X_L \simeq X_{\widetilde{L}}.$$
$$\text{no, in general}$$

The proof of (9b) for prime knots rests on the following theorem:

Theorem 126.13. (Whitten Theorem) Let K and $\widetilde{K}$ be two knots that are prime or trivial. If $\pi_1(X_K)$ is isomorphic to $\pi_1(X_{\widetilde{K}})$, then X_K is homeomorphic to $X_{\widetilde{K}}$.

Proof. Let K and $\widetilde{K}$ be two knots that are prime or trivial. We suppose that the fundamental groups $\pi_1(X_K)$ and $\pi_1(X_{\widetilde{K}})$ are isomorphic. Wilbur Whitten [**Whn1987**, Rigidity Theorem][1312] showed, using the highly non-trivial Property P Theorem **??**, that X_K is homeomorphic to $X_{\widetilde{K}}$. (In fact, if one reads the proof and the references one sees that the argument actually implies that X_K is diffeomorphic to $X_{\widetilde{K}}$.) ■

Proof of the Link Equivalence Theorem 126.5 (9). Let L and $\widetilde{L}$ be two oriented m-component links. The (9a)-statement is of course clear.

Now we turn to the (9b)-statement. First let L and $\widetilde{L}$ be two knots which are prime or trivial. We suppose that $X_L \simeq X_{\widetilde{L}}$. By the elementary (7a)-statement this implies that $\pi_1(X_L) \cong \pi_1(X_{\widetilde{L}})$. It thus follows from the Whitten Theorem 126.13 that X_L is homeomorphic to $X_{\widetilde{L}}$.

Now it remains to prove the following claim:

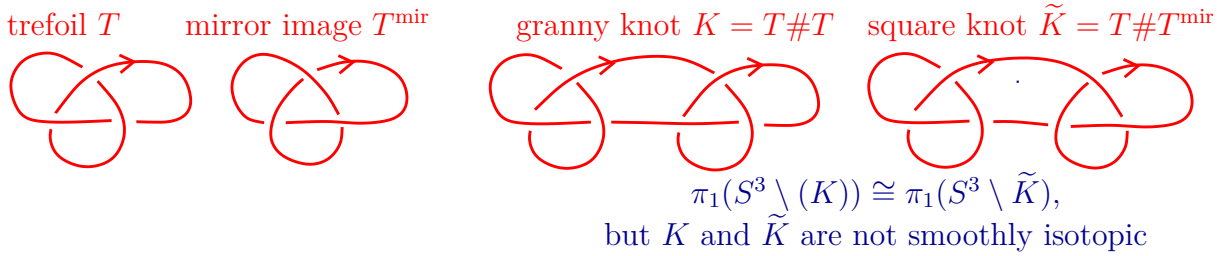
[1312] Note that [**Whn1987**, Rigidity Theorem] is formulated in terms of knot complements, but the proof shows that the statement is really about knot exteriors.

Claim. There exist knots K and $\widetilde{K}$ such that $X_K \simeq X_{\widetilde{K}}$, but such that X_K is not homeomorphic to $X_{\widetilde{K}}$.

Proof. Let T be the trefoil. We equip T with an orientation. As always we denote by T^{rev} the reverse of T, i.e. the trefoil with the opposite orientation. Let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane of $\mathbb{R}^4$. We set $T^{\mathrm{mir}} \coloneqq \rho(T)$. Let μ_T be a meridian for T.

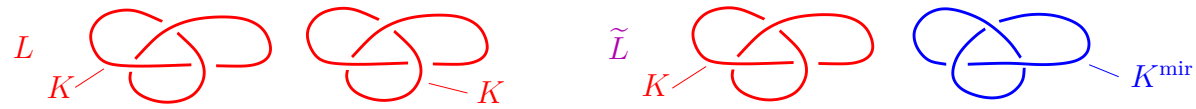
We consider the granny knot $K = T\#T$ and the square knot $\widetilde{K} = T\#T^{\mathrm{mir}}$. Note that

$$\begin{array}{lll}
 & \text{by the Oriented Knot–Monoid-}\pi_1\text{-Proposition 111.26 applied to } K = T\#T & \text{algebraic isomorphism} \\
 & \downarrow & \downarrow \\
\pi_1(S^3 \setminus K) & \cong \pi_1(S^3 \setminus T) *_{\mu_T = \mu_T} \pi_1(S^3 \setminus T) & \cong \pi_1(S^3 \setminus T) *_{\mu_T = \rho_*(\mu_T)} \rho_*(\pi_1(S^3 \setminus T)) \\
 & \cong \pi_1(S^3 \setminus T) *_{\mu_T = \rho_*(\mu_T)} \pi_1(S^3 \setminus \rho(T)) & \cong \pi_1(S^3 \setminus T) *_{\mu_T = \mu_{(T^{\mathrm{rev}})^{\mathrm{mir}}}} \pi_1(S^3 \setminus T^{\mathrm{mir}}) \\
 & \uparrow & \uparrow \\
 & \text{since } \rho(S^3 \setminus T) = S^3 \setminus \rho(T) & \text{by Definition 108.23 of } T^{\mathrm{mir}} = \rho(T) \text{ and since by the Meridian–Symmetries Lemma 109.3 we have } \rho_*(\mu_T) = \mu_{(T^{\mathrm{rev}})^{\mathrm{mir}}} \text{ as oriented smooth submanifolds} \\
 & \cong \pi_1(S^3 \setminus T) *_{\mu_T = \mu_{T^{\mathrm{mir}}}} \pi_1(S^3 \setminus T^{\mathrm{mir}}) & \cong \pi_1(S^3 \setminus \widetilde{K}). \\
 & \uparrow & \uparrow \\
 & \text{by Exercise 108.2 we have } T = T^{\mathrm{rev}} & \text{by the Oriented Knot–Monoid-}\pi_1\text{-Proposition 111.26 applied to } \widetilde{K} = T\#T^{\mathrm{mir}}
\end{array}$$



We have thus shown that $\pi_1(S^3 \setminus K) \cong \pi_1(S^3 \setminus \widetilde{K})$. From (8″), (7b) we obtain that $X_K \simeq X_{\widetilde{K}}$. On the other hand we saw in Exercise 115.8 that the square knot is neither smoothly isotopic to the granny knot nor to its mirror. By (1), (2), (3), (4a), (6) this implies that X_K is not homeomorphic to $X_{\widetilde{K}}$. ■

Remark. Let K be the trefoil. We consider the link $L \coloneqq K \sqcup K$ and the link $\widetilde{L} \coloneqq K \sqcup K^{\mathrm{mir}}$.



It follows from the Mirror Link-π_1-Lemma 111.5 together with the Split Union-π_1-Proposition 111.20 that $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L})$. On the other hand one can show, say using the 3D-Prime Decomposition Theorem **??**, that X_L is not diffeomorphic to $X_{\widetilde{L}}$. So one of the two statements (7b) and (9b) fails. But it is not entirely clear, which one. Put differently, the following question arises: Are $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$ homotopy equivalent?[1313] □

126.2.10. **Proof of (10).**

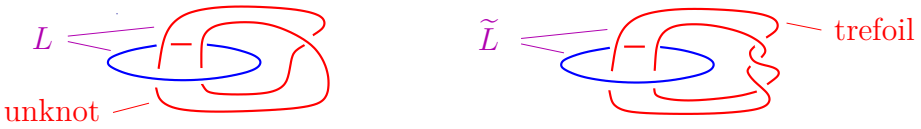
[1313] It seems to the author that one might be able to show that they are not homotopy equivalent by considering π_2 as a module over the group ring $\mathbb{Z}[\pi_1]$, but it seems like a daunting task to turn this into a proper proof.

Proof of the Link Equivalence Theorem 126.5 (10). We need to show that

$$(X_L, \mu_L) \cong (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}}) \quad \underset{\text{(10b) no for links}}{\overset{\text{(10a)}}{\rightleftarrows}} \quad (X_L, T_L) \cong (X_{\widetilde{L}}, T_{\widetilde{L}})$$

and we need to show the oriented analog. First we deal with the (10a)-statement, i.e. we deal with the "⇒"-direction. We assume that there exists an (orientation-preserving) diffeomorphism $f\colon (X_L, \mu_L) \xrightarrow{\cong} (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}})$. Recall that this means that there exists an (orientation-preserving) diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for $i = 1, \dots, m$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as (unoriented) smooth submanifolds. Since f is an diffeomorphism we see that f restricts to a diffeomorphism $\partial X_L \to \partial X_{\widetilde{L}}$. Since for each $i \in \{1, \dots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ we see that boundary components of X_L and $X_{\widetilde{L}}$ get matched in the way they are ordered. In other words, for each $i \in \{1, \dots, m\}$ we have $f(T_{L_i}) = T_{\widetilde{L}_i}$.

We turn to the proof of (10b). We need to show that there exist oriented links L and $\widetilde{L}$ with $(X_L, T_L) \cong (X_{\widetilde{L}}, T_{\widetilde{L}})$ but such that $(X_L, \mu_L) \ncong (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}})$. We consider the two 2-component links L and $\widetilde{L}$ that are shown in the figure below (and which we already considered in Exercise 111.20):



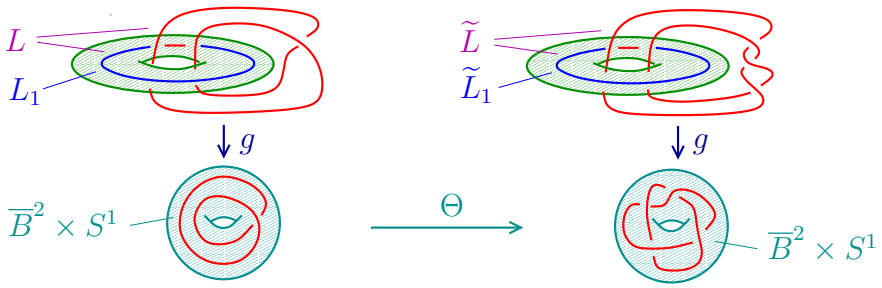
Note that all components of L are the trivial knot, whereas one component of $\widetilde{L}$ is a trefoil. This shows that in no reasonable sense are L and $\widetilde{L}$ smoothly isotopic. It follows from (1), (2) and (3) that there is no reasonable sense in which (X_L, μ_L) and $(X_{\widetilde{L}}, \pm\mu_{\widetilde{L}})$ are diffeomorphic. In contrast to this we have the following claim:

Claim. There exists an orientation-preserving diffeomorphism $(X_L, T_L) \to (X_{\widetilde{L}}, T_{\widetilde{L}})$.

Proof. Let $L_1 = \widetilde{L}_1$ be the blue component of L and $\widetilde{L}$. Since it is the trivial knot we know, e.g. by the Sphere–Solid Tori-Decomposition Lemma 111.6, that there exists a diffeomorphism $g\colon X_{L_1} \to \overline{B}^2 \times S^1$. We now consider the diffeomorphism

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (z, w) & \mapsto & (z \cdot w, w). \end{array}$$

(Pictorially speaking the diffeomorphism Θ gives the disk $\overline{B}^2$ one full twist as we walk along S^1.)



The figure above should convince the reader that the diffeomorphism $g^{-1}\circ\Theta\circ g\colon X_{L_1}\to X_{\widetilde{L}_1}$ sends L_2 to $\widetilde{L}_2$. It follows that for suitable tubular neighborhoods the map $g^{-1}\circ\Theta\circ g$ restricts to an orientation-preserving diffeomorphism $(X_L,T_L)\to(X_{\widetilde{L}},T_{\widetilde{L}})$. ■

126.2.11. **Proof of (11).** We need to show that

$$(X_L,T_L)\cong(X_{\widetilde{L}},T_{\widetilde{L}}) \underset{(11)}{\Longrightarrow} X_L\cong X_{\widetilde{L}}$$

and we need to show the oriented analog. In fact this statement is trivial. It is perhaps more important to point out what is happening: Going from left to right we lose control over the ordering of the components of the link. Of course in general this step is not reversible.

126.2.12. **Proof of (12).**

Proof of the Link Equivalence Theorem 126.5 (12). We need to show that

$$\begin{array}{ccc}(X_L,T_L)\cong(X_{\widetilde{L}},T_{\widetilde{L}}) & & (X_L,\mu_L,\lambda_L)\cong(X_{\widetilde{L}},\mu_{\widetilde{L}},\lambda_{\widetilde{L}})\\ \text{(12b}')\ \text{if } L,\widetilde{L} \text{ are not splittable}\ \Uparrow\ \Downarrow\ \text{(12a}') & \text{and} & \text{(12b}'')\ \text{if } L,\widetilde{L} \text{ are not splittable}\ \Uparrow\ \Downarrow\ \text{(12a}'')\\ (\pi_1(X_L),\pi_1(T_L))\cong(\pi_1(X_{\widetilde{L}}),\pi_1(T_{\widetilde{L}})) & & (\pi_1(X_L),\mu_L,\lambda_L)\cong(\pi_1(X_{\widetilde{L}}),\mu_{\widetilde{L}},\lambda_{\widetilde{L}}).\end{array}$$

The downward directions are trivial in both cases. We now turn to the proof of the upward directions.

First we consider the (12b′)-statement. In the following we assume that there exists an isomorphism $\varphi\colon(\pi_1(X_L),\pi_1(T_L))\xrightarrow{\cong}(\pi_1(X_{\widetilde{L}}),\pi_1(T_{\widetilde{L}}))$ of subgroup systems. It follows fairly easily from the Waldhausen Theorem [**Wald1968b**, Corollary 6.5] that there exists a diffeomorphism $f\colon(X_L,T_L)\to(X_{\widetilde{L}},T_{\widetilde{L}})$ such that $f_*=\varphi$. We will provide more details on page **??**.

Finally we consider the (12b″)-statement. In the following we assume that there exists an isomorphism $\varphi\colon(\pi_1(X_L),\mu_L,\lambda_L)\xrightarrow{\cong}(\pi_1(X_{\widetilde{L}}),\mu_{\widetilde{L}},\lambda_{\widetilde{L}})$ of group-pair systems. Since for each $i\in\{1,\dots,m\}$ the group $\pi_1(T_{L_i})$ is generated by μ_{L_i} and λ_{L_i} we see that φ also defines an isomorphism $\varphi\colon(\pi_1(X_L),\pi_1(T_L))\xrightarrow{\cong}(\pi_1(X_{\widetilde{L}}),\pi_1(T_{\widetilde{L}}))$. By the above discussion of the (12b′)-statement we see that there exists a diffeomorphism $f\colon(X_L,T_L)\to(X_{\widetilde{L}},T_{\widetilde{L}})$ such that $f_*=\varphi$. It follows that for each $i\in\{1,\dots,m\}$ we have $f_*(\mu_{L_i})=\mu_{\widetilde{L}_i}\in\pi_1(T_{L_i})$ and $f_*(\lambda_{L_i})=\lambda_{\widetilde{L}_i}\in\pi_1(T_{L_i})$. It follows easily from the Torus–Mapping Class Group Theorem 126.6 that f is diffeotopic to a diffeomorphism $f'\colon(X_L,T_L)\to(X_{\widetilde{L}},T_{\widetilde{L}})$ such that for each $i\in\{1,\dots,m\}$ we have $f(\mu_{L_i})=\mu_{\widetilde{L}_i}$ and $f(\lambda_{L_i})=\lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds of T_{L_i}. ■

126.3. **Appendix: Definitions in other textbooks.** Knot theory is peculiar in mathematics in the sense that one starts out with the notion of a "knot", where it is intuitively clear what is meant. Similarly it intuitively rather clear when two "physical" knots are equivalent. Also the notion of a meridian and a longitude are usually "clear".

It is a mathematical and stylistic challenge to turn the "obvious" into precise mathematical language. In the previous chapters the author attempted to do just that.

In this appendix we collect the definitions of knots, "knot equivalence", meridians and longitudes in the most common text books on knot theory. We mostly quote verbatim the definition, with only minor edits for clarity.

We leave it to the reader to decide whether these notions are mathematically precise and to determine to which degree the definitions below are equivalent to one another and to our definitions.

(1) Lickorish: An introduction to knot theory [**Lic1997a**].
   (a) *Knot.*
      - A link L of m components is a subset of S^3, or of $\mathbb{R}^3$, that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a knot ([**Lic1997a**, p. 1]).
      - Links L_1 and L_2 in S^3 are *equivalent* if there is an orientation-preserving piecewise linear homeomorphism $h\colon S^3 \to S^3$ such that $h(L_1) = L_2$ ([**Lic1997a**, p. 2]).
   (b) *Meridian and Longitude.* (page 13) Let K be an oriented knot in (oriented) S^3 with solid torus neighbourhood N. A meridian μ of K is a non-separating simple closed curve in ∂N that bounds a disc in N. A longitude λ of K is a simple closed curve in ∂N that is homologous to K in N and null-homologous in the exterior of K.
   (c) A fibred knot K is a knot with the property that its exterior X is a bundle over S with fibre an orientable surface F ([**Lic1997a**, p. 65]).

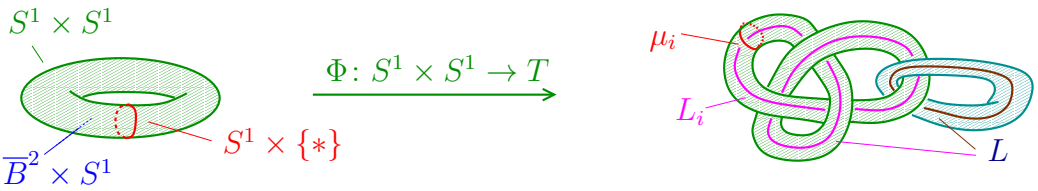
(2) Burde, Zieschang, Heusener: Knots [**BZH2014**].
   (a) *Knot.* (page 1 and 3) A knot is an embedding of a circle S^1 into Euclidean 3-space, $\mathbb{R}^3$, or the 3-sphere, S^3. A knot is called tame if it is ambient isotopic to a piecewise linear embedding in $\mathbb{R}^3$ resp. S^3. Meridian + Longitude (p. 21, Definition 2.6)
   p. 3: Two (p.l.)-knots are equivalent if they are (p.l.)-ambient isotopic.
   (b) A tubular neighborhood $V(K)$ of a knot $k \subseteq S^3$ is homeomorphic to a solid torus. There is a simple closed curve μ on $\partial V(k)$ which is null-homologous in $V(k)$ but not on $\partial V(k)$; we call μ *meridian* of k. It is easy to see that any two meridians (if suitably oriented) in $\partial V(k)$ are isotopic. A Seifert surface S will meet $\partial V(k)$ in a simple closed curve λ, if $V(k)$ is suitably chosen: λ is called a *longitude* of k.
   (c) The knots of the first class whose commutator subgroups are finitely generated [...] are called *fibered knots* by virtue of Theorem 5.1. (See page 74).

(3) Rolfsen: knots and links [**Rolf1990**].
   (a) *Knot.* A subset K of a space X is a knot if K is homeomorphic with a sphere S^p. Links K, K' are equivalent if there is a homeomorphism $h\colon X \to X$ such that $h(K) = K'$ (see page 2). A knot is *tame* if it is equivalent to a polygonal knot (see page 48).
   (b) *Meridian and longitude.* A simple closed curve in ∂V satisfying the conditions of the exercise [e.g. being null-homotopic] is called a meridian of V. A longitude of V is any simple closed curve in ∂V of the form $h(S^1 \times \{1\})$, for some framing $h\colon S^1 \times D^2 \to V$ [i.e. homeomorphism] of V.
   (c) A knot or link L^n in S^{n+2} is *fibred* if there exists a fibration map $f\colon S^{n+2} \setminus L \to S^1$. We require further that the fibration be well-behaved near L. That is, each component L_i is to have a neighbourhood framed as $S^n \times D^2$, with $L = S^n \times \{0\}$ in such a way that the restriction of f to $S^n \times (D^2 \setminus \{0\})$ is the map into S^1 given by $(x, y) \mapsto y/|y|$. (See page 323).

(4) Kawauchi: A survey of knot theory [**Kaw1996**].
   (a) *Knot.* We denote S^{n+2} or $\mathbb{R}^{n+2}$ by M. An n-dimensional PL link [...] is a locally flat compact PL submanifold L of M each component of which is PL homeomorphic to S^n.
   Two PL n-links L and L' are *equivalent* if there is a PL auto-homeomorphism h of M with $h(L) = L'$.
   (b) *Meridian and longitude.* p. 5. We say that $f(p_i \times D^2)$ [...] for a point $p_i i \in L_i$ is a meridian disk of L^i. There is an oriented PL simple loop ℓ_i in $\partial N(L)$ such that ℓ_i is homologous to L_i in $N(L)$ and null-homologous in $M \setminus L$. Such an ℓ_i is called a longitude of the component L_i.
   (c) *Fibered knot.* Let $(E', (\partial E)')$ be the manifold pair obtained from $(E, \partial E)$ by cutting E along S_E. A link L in S^3 is a *fibered link* if there is a Seifert surface S for L such that $(E', (\partial E)')$ is homeomorphic to $(S_E, \partial S_e) \times [0, 1]$. p.49
(5) Juhász: Differential and low-dimensional topology [**Juh2023**].
   (a) [**Juh2023**, Definition 4.1] A *smooth k-component n-link in an $(n+2)$-manifold M* is a smooth embedding $L\colon \bigcup_{i=1}^{k}(S^n \times \{i\}) \to M$. The links L_0 and L_1 are *equivalent* if they are ambient isotopic; that is, there is an isotopy $\Phi\colon M \times I \to M$ such that $\Phi_0 = \mathrm{id}_M$ and $\Phi_1(L_0) = L_1$.
   (b) *Meridian and longitude.* Undefined.
   (c) *Fibered knot.* [**Juh2023**, Definition 4.64] A knot K in a three-manifold Y is said to be fibred if its complement $Y \setminus K$ is a fibre bundle over S^1 such that the closure of each fibre is a Seifert surface of K.

126.4. Exercises for Chapter 126 .

Exercise 126.1. Let $L \subseteq S^3$ be an m-component link with link exterior X_L, let L_i be a component of L and let $\mu_i \subseteq \partial X_L$ be a meridian of L_i. We consider the Dehn filling $\varphi(\mu_i)$ as defined in the Dehn Filling Proposition 126.7. Show that there exists an orientation-preserving diffeomorphism $X_L(\mu_i) \to X_{L \setminus L_i}$ which is the identity on X_L and which sends $\{0\} \times S^1$ to L_i as oriented smooth manifolds.



Exercise 126.2. Let $L, \widetilde{L}$ be two m-component links. We consider the link exteriors X_L and $X_{\widetilde{L}}$. We denote by $T_1, \dots, T_m$ respectively $\widetilde{T}_1, \dots, \widetilde{T}_m$ the boundary components of X_L and $X_{\widetilde{L}}$ that correspond to the components of L and $\widetilde{L}$. We consider the following two statements:

(1) There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \dots, m\}$ we have $f(T_i) = \widetilde{T}_i$.
(2) There exists a diffeomorphism $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$ such that for every $i \in \{1, \dots, m\}$ we have ¿...?

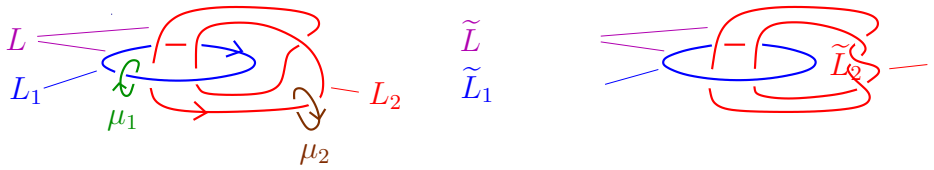
Formulate Statement (2) in such a way that the two statements are equivalent and prove the equivalence.
Hint. You might want to look up the notion of ends of a topological space, as introduced in Exercise 229.8.

Exercise 126.3. Let K be a fibered knot with fiber Seifert surface F. By the Fiber–Minimal Corollary 123.13 we know that F is a Seifert surface of minimal genus. Show that every Seifert surface G of minimal genus is smoothly isotopic, rel K, to F.
Hint. Consider the covering of $p\colon \widetilde{X}_K \to X_K$ corresponding to the kernel of the abelianization $\pi_1(X_K) \to \pi_1(X_K)_{\mathrm{ab}} \cong \mathbb{Z}$ and make use of the 3D-Product Splitting Proposition 126.9.

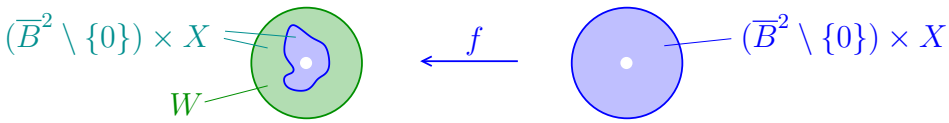
Exercise 126.4.

(a) Show that in general the statement of the Link Equivalence Theorem 126.5 (12b′) does not hold for links that are splittable.
(b) Show that in general the statement of the Link Equivalence Theorem 126.5 (12b″) does not hold for links that are splittable.

Exercise 126.5. We consider the links $L = L_1 \sqcup L_2$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \widetilde{L}_2$ that are shown in the figure below. We also show meridians μ_1, μ_2 for L_1, L_2. On page 2433 we gave a fairly explicit diffeomorphism of $S^3 \setminus L \to S^3 \setminus \widetilde{L}$. What are the images of the meridians μ_1, μ_2 under this diffeomorphism?



Exercise 126.6. Let X be a connected non-empty smooth manifold. Suppose we are given a smooth embedding $f\colon (\overline{B}^2 \setminus \{0\}) \times X \to (\overline{B}^2 \setminus \{0\}) \times X$ such that we have an inclusion $f((\overline{B}^2 \setminus \{0\}) \times X) \subseteq (B^2 \setminus \{0\}) \times X$. We set $W := ((\overline{B}^2 \setminus \{0\}) \times X) \setminus f((\overline{B}^2 \setminus \{0\}) \times X)$. Show that the inclusion induced map $\pi_1(S^1 \times X) \to \pi_1(W)$ is an isomorphism.
Remark. This statement can be used to give an alternative proof of the Link Equivalence Theorem 126.5 (5).



Part XIII

The universal covering

In this part of the lecture notes we introduce the universal covering of any (reasonable) topological space and consider examples and applications.

This part of the lecture notes is organized as follows:

- In Chapter 127 we will introduce universal coverings and we will show, under mild hypothesis, their existence and uniqueness.
- In Chapter 128 we will introduce the deck transformation group of any covering $\widetilde{X} \to X$.
- In Chapter 129 we will consider several topics in covering theory. For example we will give a new proof of the Diffeotopies-of-S^1-Proposition 42.8, in the Complex Tori–Classification Proposition 129.2 we will determine which lattices in $\mathbb{C}$ lead to biholomorphic tori and in the Complex Logarithm-Existence Proposition 129.5 we will determine which open subsets of $\mathbb{C}$ admit complex logarithms.
- In Chapter 130 we will explicitly construct the (universal) covering of topological graphs and we will prove a few applications of covering theory to group theory. For example in the Free Groups–Residually Finite Proposition 130.19 we will show that every free group is residually finite and in the Surface-π_1-Not Free Proposition 130.21 we will show that the fundamental group of the surface of genus 2 is not a free group.
- In Chapter 131 we will introduce basic concepts of hyperbolic geometry in dimension two. This theory will play an essential role in the next chapter.
- In Chapter 132 we will explicitly construct the universal covering of surfaces. We will see that in many cases the universal covering should be thought of as the hyperbolic half-space. As an application we prove Picard's Theorem 132.27 which says that if $f\colon \mathbb{C} \to \mathbb{C}$ is a non-constant holomorphic function, then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f.
- In the short and chatty Chapter 133 we will give a vague discussion of analogies between covering spaces and relayed concepts in algebraic geometry and Galois theory.

CHAPTER 127

The existence of universal coverings

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of coverings. For the reader's convenience we recall Definition 91.8.

Definition 127.1. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subseteq B$ is called uniformly covered, if $p^{-1}(U) \subseteq X$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of X with the property that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.

(2) We say that the map $p\colon X \to B$ is a covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of b that is uniformly covered.

(3) A covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces is a covering $p\colon X \to B$ with $p(x_0) = b_0$.

(4) We say a covering $p\colon X \to B$ is path-connected, if X is path-connected. We adopt obvious variations on this language.

One key goal of this section is to address the following question.

Question 127.2. Does every path-connected non-empty topological space B admit a path-connected covering $p\colon X \to B$ such that X is simply connected?

We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? Or does it exist for the complement $S^3 \setminus K$ of a knot $K \subseteq S^3$? How about the wedge $S^1 \vee S^1$ of two circles?

This chapter is organized as follows:

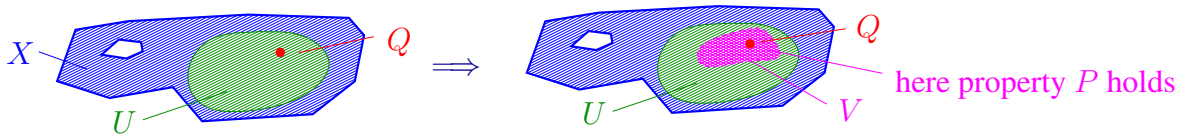
- In Section 127.1 we will study local properties of topological spaces.
- In Section 127.2 we will discuss lifting maps to coverings.
- In Section 127.3 we will show that every simply connected subset is uniformly covered.
- In Section 127.4 we will formulate the Covering Existence Theorem 127.10.
- In Section 127.5 we will start with a proof of the Covering Existence Theorem 127.10.
- In Section 127.6 we will complete the proof of the Covering Existence Theorem 127.10 using the "whisker topology".
- In Section 127.7 we will provide an alternative conclusion of the proof of the Covering Existence Theorem 127.10 using the familiar compact-open topology.
- In Section 127.8 we will show that coverings are essentially determined by the image in the fundamental group of the base space.
- In Section 127.9 we will introduce the universal covering of a topological space.
- In Section 127.10 we will give an explicit descriptions of the universal covering of a topological space.
- In Appendix 127.11 we will prove a few theoretical statements on universal coverings.

- In Section 127.12 we will provide exercises for Chapter 127.

In the next Chapter 128 we will introduce the deck transformation group of a covering.

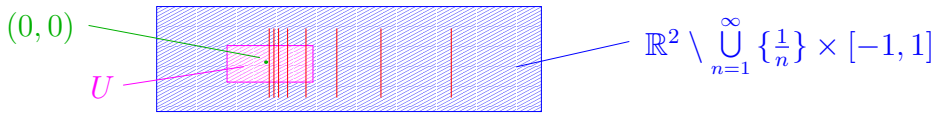
127.1. **Local properties of topological spaces.** Before we can address Question 127.2 we need to recall the definition from Section 4.9 of a local property of topological spaces.

Definition 127.3. Let P be a property of topological spaces. We say that a topological space X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that has the property P.



Examples.

(1) By the Manifold–Local Properties Lemma 23.8 (1) every topological manifold (in particular every open subset of $\mathbb{R}^n$) is locally contractible.
(2) In Exercise 96.9 we showed that every topological graph is locally contractible.
(3) In Exercise 127.1 we will show that any wedge of topological manifolds (e.g. any wedge of spheres) is locally contractible.
(4) By the Homotopy-Equivalence π_0-Lemma 19.17 and the Homotopy Equivalence-π_1-Proposition 90.9 we know that every contractible topological space is path-connected and simply connected. Thus every locally contractible topological space is also locally path-connected and locally simply connected.
(5) We consider
$$X \coloneqq \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \{\tfrac{1}{n}\} \times [-1,1],$$
with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. Indeed, we consider the point $(0,0) \in X$ and the neighborhood $U = (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0,0)$ which is contained in U.



(6) On numerous occasions we will use the following elementary fact: If $p\colon X \to B$ is a covering and if B is locally P, then so is X. □

127.2. **Lifting maps to coverings.** The following definition is a mild variation on the definition on Definition 92.1.

Definition 127.4. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z,z_0) \to (B,b_0)$ be a continuous map. A lift of f is a continuous map

$\widetilde{f}\colon (Z,z_0)\to(X,x_0)$ that makes the following diagram commute:

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0). \end{array}$$

It is worth recalling some earlier results on lifts of maps. First of all we have the following proposition that deals with any type of covering.

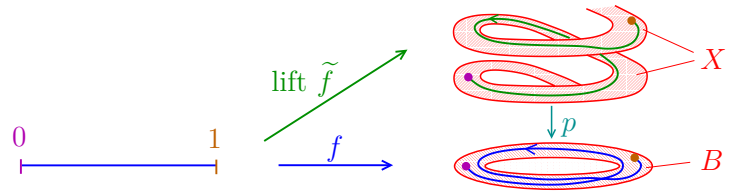
Proposition 92.2. (Lift Uniqueness Proposition) Let $p\colon X\to B$ be a covering and let Z be a connected topological space. Furthermore let $f\colon Z\to B$ be a continuous map and let $\widetilde{f},\widehat{f}\colon Z\to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $z_0\in Z$, then they agree everywhere.

We are already experts on lifts of paths, i.e. of continuous maps $[0,1]\to B$. We now recall two of the main results.

Proposition 92.6. (Path-Lifting Proposition) Let $p\colon X\to B$ be a covering.

(1) Let $f\colon [0,1]\to B$ be a path. Given any $\widetilde{b}\in X$ with $p(\widetilde{b})=f(0)$ there exists a unique lift $\widetilde{f}\colon[0,1]\to X$ of f to the starting point $\widetilde{b}$, i.e. with $\widetilde{f}(0)=\widetilde{b}$.

(2) Let $f,g\colon[0,1]\to B$ be two paths and let $\widetilde{f},\widetilde{g}\colon[0,1]\to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.



Proposition 92.8 (Loop–Lifting Proposition) Let $p\colon(X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. Let $f\colon[0,1]\to B$ be a loop in (B,b_0). We denote by $\widetilde{f}\colon[0,1]\to\widetilde{B}$ the lift of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X,x_0) \iff [f] \text{ lies in } p_*(\pi_1(X,x_0))\subseteq\pi_1(B,b_0).$$

Given a covering $p\colon(X,x_0)\to(B,b_0)$ we now want to study the existence of lifts of arbitrary continuous maps $(Z,z_0)\to(B,b_0)$. We start out with a very instructive example.

Example.

(1) We consider the maps:

$$\begin{array}{ccc} & & (S^1,1) \\ & \overset{\text{¿¿¿ lift } \widetilde{f}\,??}{\nearrow} & \big\downarrow p\,|_{z\mapsto z^3} \\ (S^1,1) & \xrightarrow[f]{z\mapsto z^2} & (S^1,1). \end{array}$$

Here the map p is a 3-fold covering. Does the map f lift to a continuous map $\widetilde{f}$?

(2) To answer the question we just posed in (1) it is helpful to consider a more general setup. Thus let $p\colon(X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces and let $f\colon(Z,z_0)\to(B,b_0)$ be a continuous map. Let us suppose that there exists a lift

$\widetilde{f}\colon (Z,z_0)\to(X,x_0)$. We obtain the commutative diagram

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p\circ\widetilde{f}\big)_* = p_*\circ\widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{array}{ccc} & & \pi_1(X,x_0) \\ & \overset{\widetilde{f}_*}{\nearrow} & \big\downarrow p_* \\ \pi_1(Z,z_0) & \xrightarrow[f_*]{} & \pi_1(B,b_0). \end{array}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) \;=\; \operatorname{im}((p\circ\widetilde{f})_*) \;=\; \operatorname{im}(p_*\circ\widetilde{f}_*) \subseteq \operatorname{im}(p_*).$$

(3) In the above example (1) we have $\pi_1(S^1,1) =_{\mathrm{i}} \mathbb{Z}$. By the discussion on page 1760 we have $\operatorname{im}(f_*) = 2\mathbb{Z}$ and we have $\operatorname{im}(p_*) = 3\mathbb{Z}$. Since $2\mathbb{Z}\not\subseteq 3\mathbb{Z}$ we see that the map f does *not* lift. □

The following proposition says, that the above conclusion has a converse for "reasonably" topological spaces.

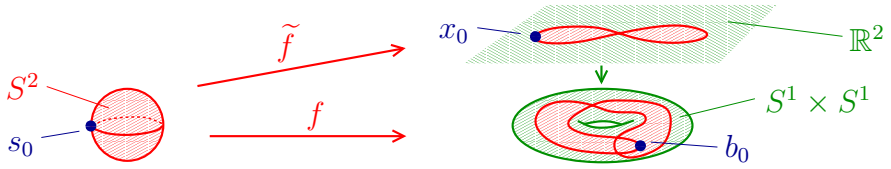
Proposition 127.5. (Map Lifting Criterion) Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z,z_0)\to(B,b_0)$ be a continuous map. If Z is locally path-connected, then we have the following equivalence of statements:[1314]

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\text{there exists a lift } \widetilde{f}}{\nearrow} & \big\downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array} \qquad\Longleftrightarrow\qquad \operatorname{im}(f_*)\subseteq\operatorname{im}(p_*).$$

Example. Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces.

(1) The condition on the fundamental groups in the Map Lifting Criterion 127.5 is automatically satisfied if Z is simply connected.

(2) Let $f\colon (S^n,s_0)\to(B,b_0)$ be a continuous map. The topological manifold S^n is path-connected and locally path-connected. By the Sphere-π_1-Proposition 88.24 the sphere S^n is simply connected if $n\geq 2$.



Therefore it follows from the observation in (1) that there exists a uniquely determined lift $\widetilde{f}\colon (S^n,s_0)\to(X,x_0)$ of the map $f\colon (S^n,s_0)\to(B,b_0)$. □

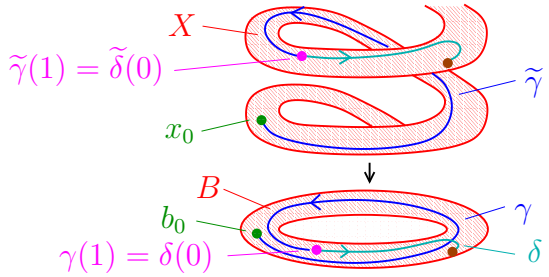
Remark. In Exercise 127.5 we will see that we cannot drop the hypothesis that Z is locally path-connected. □

In the proof of the Map Lifting Criterion 127.5 we will need the following lemma:

[1314] It is worth recalling that since Z is in particular connected, it follows from the Lift Uniqueness Proposition 92.2 that a lift $(Z,z_0)\to(X,x_0)$, if it exists, is unique.

Lemma 127.6. (Concatenation-of-Paths-Lifting Lemma) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0. Then

$$\begin{array}{c}\text{endpoint of the lift of } \gamma * \delta \\ \text{to the starting point } x_0\end{array} = \begin{array}{c}\text{endpoint of the lift of } \delta \\ \text{to the starting point } \widetilde{\gamma}(1).\end{array}$$



Proof of the Concatenation-of-Paths-Lifting Lemma 127.6. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0 and we denote by $\widetilde{\delta}$ the lift of δ to the starting point $\widetilde{\gamma}(1)$. Finally we denote by $\widetilde{\gamma * \delta}$ the lift of $\gamma * \delta$ to the starting point x_0. For $t \in [0,1]$ we write $\beta(t) = (\gamma * \delta)(\frac{1}{2}(t+1))$ and $\widetilde{\beta}(t) = (\widetilde{\gamma * \delta})(\frac{1}{2}(t+1))$. Note that by definition we have $\beta(t) = \delta(t)$ for $t \in [0,1]$ and that $\widetilde{\beta}$ and $\widetilde{\delta}$ are both lifts of $\beta = \delta$ to the starting point $\beta(0) = \gamma(1) = \delta(0)$. Thus it follows that

$$\text{endpoint of } \widetilde{\gamma * \delta} \;=\; (\widetilde{\gamma * \delta})(1) \;=\; \widetilde{\beta}(1) \underset{\substack{\uparrow \\ \text{by the Path-Lifting} \\ \text{Proposition 92.6 (1)}}}{=} \widetilde{\delta}(1) \;=\; \text{endpoint of } \widetilde{\delta}.$$

■

Proof of the Map Lifting Criterion 127.5. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map.

By the discussion on page 2444 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

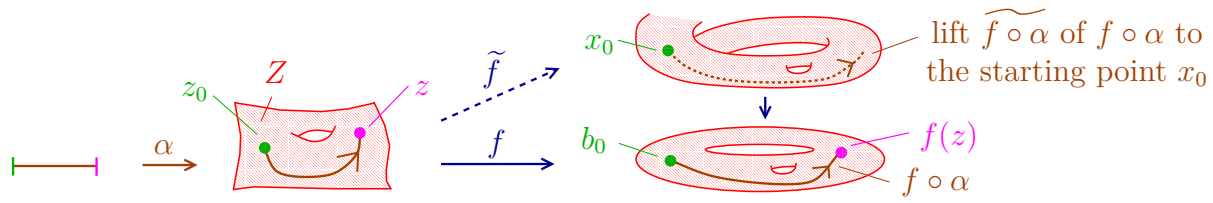
$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subseteq\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0,1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to the Path-Lifting Proposition 92.6 (1) we can lift the path $f \circ \alpha\colon [0,1] \to B$ to a path $(f \circ \alpha)\colon [0,1] \to X$ with starting point x_0. We set

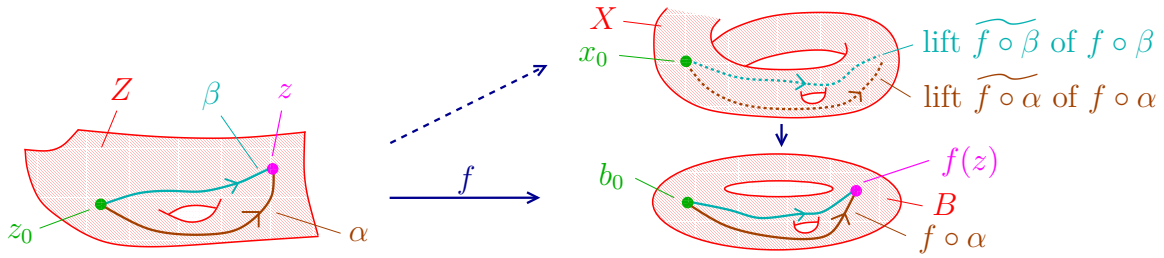
$$\widetilde{f}(z) \;:=\; (\widetilde{f \circ \alpha})(1).$$

It remains to prove the following claim:

Claim.
(1) The map $\widetilde{f}\colon Z \to X$ is well-defined.



(2) The map $\widetilde{f}$ is continuous.
(3) The map $\widetilde{f}$ is a lift.



Proof.

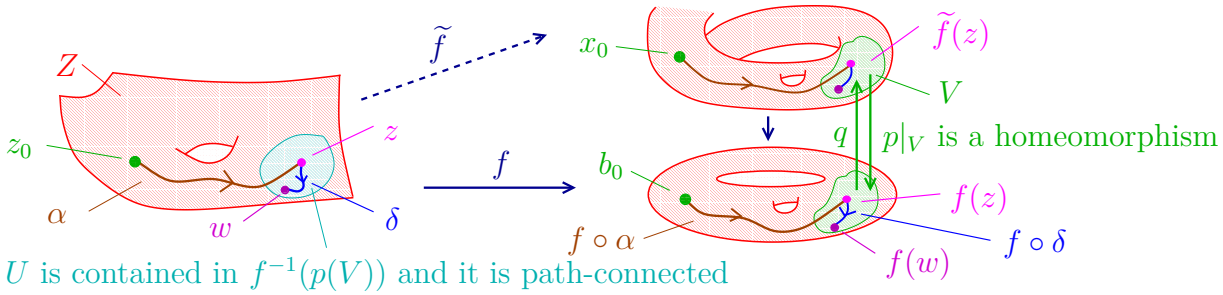
(1) We have to show that for $z \in Z$ the point $\widetilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α. Thus let $\beta\colon [0,1] \to Z$ be another path with $\beta(0) = z_0$ and $\beta(1) = z$. It follows that

$$\begin{aligned}
\widetilde{(f \circ \alpha)}(t=1) &= (\text{lift of } f \circ \alpha \text{ to the starting point } x_0)(t=1) \\
&= (\text{lift of } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&\quad \uparrow \\
&\quad \text{by the Path-Lifting Proposition 92.6 (2) since} \\
&\quad \text{the paths } f \circ \alpha \text{ and } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ are path-homotopic} \\
&= (\text{lift of } (f \circ (\underbrace{\alpha * \overline{\beta}}_{=:\gamma,\ \text{a loop in } (Z, z_0)})) * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&= (\text{lift of } f \circ \beta \text{ to the starting point } (\widetilde{f \circ \gamma})(1))(t=1) \\
&\quad \uparrow \\
&\quad \text{by the Concatenation-of-Paths-Lifting Lemma 127.6, here } \widetilde{f \circ \gamma} \text{ denotes} \\
&\quad \text{the lift of } f \circ \gamma \text{ to the starting point } x_0 \\
&= (\text{lift of } f \circ \beta \text{ to starting point } x_0)(t=1) \;=\; \widetilde{(f \circ \beta)}(t=1). \\
&\quad \uparrow \\
&\quad \text{by our hypothesis we have } [f \circ \gamma] = f_*([\gamma]) \in \operatorname{im}(f_*) \subseteq p_*(\pi_1(X, x_0)), \\
&\quad \text{hence } \widetilde{f \circ \gamma} \text{ is a loop in } x_0 \text{ by the Loop–Lifting Proposition 92.8}
\end{aligned}$$

Thus we have shown that the map $\widetilde{f}$ is well-defined.

(2) We need to show that $\widetilde{f}$ is continuous. By the Pasting Proposition 4.7 it suffices to show that $\widetilde{f}$ is locally continuous. So let $z \in Z$. We pick a path α from z_0 to z. Since p is a covering there exists an open neighborhood V of $\widetilde{f}(z)$ such that the restriction of p to V is an embedding. We denote by $q\colon p(V) \to V$ the inverse map. Since Z is locally path-connected there exists an open neighborhood U of z that is contained in $f^{-1}(p(V))$ and that is path-connected. Since f and q are continuous it suffices to prove the following subclaim:

Subclaim. On the open neighborhood U of z the map $\widetilde{f}$ agrees with $q \circ f$.



Proof. So let $w \in U$. Since U is path-connected there exists a path δ from z to w that lies in U. Then

$$\begin{aligned}\widetilde{f}(w) &= \text{endpoint of the lift of } f\circ(\alpha * \delta) = (f\circ\alpha)*(f\circ\delta) \text{ to the starting point } x_0\\ &= \text{endpoint of the lift of } f\circ\delta \text{ to the starting point } \widetilde{f}(z)\end{aligned}$$

by the Concatenation-of-Paths-Lifting Lemma 127.6 and the definition of $\widetilde{f}(z)$ as the endpoint of $\widetilde{f \circ \alpha}$

$$= \text{endpoint of } q\circ(f\circ\delta) \;=\; (q\circ f)(\delta(1)) \;=\; (q\circ f)(w).$$

since $q = p^{-1}$ we know that $q \circ (f \circ \delta)$ is the lift of $f \circ \delta$ to the starting point $f(z)$

(3) We need to show that the map $\widetilde{f}\colon Z \to X$ is a lift of $f\colon Z \to B$, i.e. we need to show that $p \circ \widetilde{f} = f$. Given any $z \in Z$ we have indeed the following equalities:

$$p\big(\widetilde{f}(z)\big) \;=\; p\big(\widetilde{(f\circ\alpha)}(1)\big) \;=\; (f\circ\alpha)(1) \;=\; f(\alpha(1)) \;=\; f(z).$$

we pick a path α from z_0 to z and we denote by $\widetilde{f \circ \alpha}$ the lift of $f \circ \alpha$

■

127.3. **More on lifts of maps.** In this section we will use the Map Lifting Criterion 127.5 to prove the following two statements:

Proposition 127.7. (Uniform Covering Criterion) Let $p\colon X \to B$ be a covering. We assume that B is locally path-connected.

(1) Let U be an open subset of B. We denote by $i\colon U \to B$ the inclusion. For every lift $\widetilde{i}\colon U \to X$ of $i\colon U \to B$ the image of $\widetilde{i}$ is open in X.

(2) Every open simply connected subset $U \subseteq B$ is uniformly covered.

Lemma 127.8. (Iterated Covering Lemma) Let $q\colon Y \to X$ and $p\colon X \to B$ be two coverings. If B is locally simply connected, then $p \circ q\colon Y \to B$ is also a covering.

Remark. In Exercise 127.4 we will see that in the Iterated Covering Lemma 127.8 we cannot drop the hypothesis that B is locally simply connected. □

The remainder of this section is only concerned with the rather dull proofs of the Uniform Covering Criterion 127.7 and of the Iterated Covering Lemma 127.8.

Proof of the Uniform Covering Criterion 127.7.

(1) Let U be an open subset of B. We denote by $i\colon U \to B$ the inclusion. Let $\widetilde{i}\colon U \to X$ be a lift of i. We will use the JH-Openness Criterion 3.3 to show that $\widetilde{i}(U)$ is an open

subset of X. Thus let $\widetilde{u} \coloneqq \widetilde{i}(u) \in \widetilde{i}(U)$. Since $p\colon X \to B$ is a covering and since B is locally path-connected we know that there exists a uniformly covered open path-connected neighborhood V of $u \in B$ with $V \subseteq U$. Thus we can write $p^{-1}(V) = \bigsqcup_{x \in p^{-1}(u)} W_x$ where each $W_x \subseteq X$ is open and such that each $p|_{W_x}\colon W_x \to V$ is a homeomorphism. Note that $(p|_{W_{\widetilde{u}}})^{-1}\colon W \to X$ and $\widetilde{i}|_W\colon W \to X$ are both lifts of $i|_W\colon W \to B$ which send u to $\widetilde{u}$. Since W is path-connected we obtain from the Lift Uniqueness Proposition 92.2 that these two lifts agree. Thus we have shown that the open neighborhood $W_{\widetilde{u}}$ of $\widetilde{u}$ is contained in $\widetilde{i}(U)$.

(2) Let $U \subseteq B$ an open simply connected subset. Note that the fact that U is open in B and the fact that B is locally path-connected implies that U itself is also locally path-connected.

Now we want to show that U is uniformly covered. We pick $b \in U$. Since U is simply connected and locally path-connected we obtain from the Map Lifting Criterion 127.5 that for each $x \in p^{-1}(b)$ there exists a lift $\widetilde{i}_x\colon U \to X$ of the inclusion $i\colon U \to B$ such that $\widetilde{i}_x(b) = x$.

Claim. We have $p^{-1}(U) = \bigsqcup_{x \in p^{-1}(b)} \widetilde{i}_x(U)$.

Proof. It follows easily from the Lift Uniqueness Proposition 92.2 and the fact that U is path-connected that the $\widetilde{i}_x(U)$ are disjoint. It is clear that we have the "$\supseteq$" inclusion. It remains to prove the "$\subseteq$" inclusion. Thus let $y \in p^{-1}(U)$. By the Map Lifting Criterion 127.5 there exists a lift $\widetilde{j}\colon U \to X$ with $\widetilde{j}(p(y)) = y$. For $x \coloneqq \widetilde{j}(b)$ we have $\widetilde{i}_x(b) = x = \widetilde{j}(b)$. It follows from the Lift Uniqueness Proposition 92.2 that $\widetilde{i}_x = \widetilde{j}$. In particular we see that $y \in \widetilde{i}_x(U)$. ⊞

By (1) we know that each $\widetilde{i}_x(U)$ is open. Since $\widetilde{i}_x\colon U \to X$ is a lift of $i\colon U \to B$ we have $p \circ \widetilde{i}_x = i$. It follows easily that $p\colon \widetilde{i}_x(U) \to U$ is a bijection. Since $p\colon X \to B$ is a local homeomorphism and since $\widetilde{i}_x(U) \subseteq X$ and $U \subseteq B$ are both open subsets we see that $p\colon \widetilde{i}_x(U) \to U$ is a local homeomorphism. Thus we obtain from the Bijective-Local Homeomorphism Lemma 4.62 that $p\colon \widetilde{i}_x(U) \to U$ is actually a homeomorphism. This concludes the proof that U is uniformly covered. ■

Proof of the Iterated Covering Lemma 127.8. It is clear that $p \circ q\colon Y \to B$ is also surjective. Now let $b \in B$. Since B is locally simply connected there exists an open simply connected neighborhood U of b. It follows from the Uniform Covering Criterion 127.7 (2) that U is uniformly covered. Thus we can write $p^{-1}(U) = \bigsqcup_{x \in p^{-1}(b)} V_x$ where each $V_x \subseteq X$ is open and such that each $p\colon V_x \to U$ is a homeomorphism.

Since $p\colon X \to B$ is a covering we see that X is also locally simply connected, in particular it is locally path-connected. Since each V_x is simply connected we obtain again from the Uniform Covering Criterion 127.7 (2) that each V_x is uniformly covered. This means that for each $x \in p^{-1}(b)$ we can write $q^{-1}(V_x) = \bigsqcup_{y \in q^{-1}(x)} W_{x,y}$ where each $W_{x,y} \subseteq Y$ is open and such that each $q\colon W_{x,y} \to V_x$ is a homeomorphism. Since

$$(p \circ q)^{-1}(U) = q^{-1}(p^{-1}(U)) = q^{-1}\Big(\bigsqcup_{x \in p^{-1}(b)} V_x\Big) = \bigsqcup_{y \in q^{-1}(x)} \bigsqcup_{y \in p^{-1}(b)} W_{x,y} = \bigsqcup_{y \in (p \circ q)^{-1}(b)} W_{x,y}$$

we see that U is uniformly covered with respect to the map $p \circ q\colon Y \to B$. ■

127.4. **Existence of covering spaces.** Our next goal will be to show that every "reasonable" path-connected non-empty topological space admits a covering that is simply connected. To make precise what we mean by "reasonable" we need to introduce the following definition:

Definition 127.9. A topological space X is called semi-locally simply connected[1315] if for every point $x \in X$ there exists an open[1316] neighborhood U of x such that the inclusion-induced map $\pi_1(U,x) \to \pi_1(X,x)$ is the trivial map.

Examples.

(1) Evidently every locally simply connected topological space and every simply connected topological space is semi-locally simply connected.
(2) Let H be the Hawaiian earring that we introduced in Exercise 105.1. In Exercise 127.2 we will see that H is locally path-connected but not semi-locally simply connected. □

Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of path-connected pointed topological spaces. In the Covering-π_1-Monomorphism Proposition 92.7 we saw that $p_*\colon \pi_1(X,x_0) \to \pi_1(B,b_0)$ is an injective homomorphism. Thus we can view $\pi_1(X,x_0)$ as a subgroup of $\pi_1(B,b_0)$. Generalizing Question 127.2 one can now ask, whether to each subgroup Γ of $\pi_1(B,b_0)$ there exists a corresponding covering with $p_*(\pi_1(X,x_0)) = \Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is locally path-connected and semi-locally simply connected.

Theorem 127.10. (Covering Existence Theorem) Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $y_0 \in Y$ and let $\Gamma \subseteq \pi = \pi_1(Y,y_0)$ be a subgroup.

(1) There exists a path-connected covering $p\colon (X,x_0) \to (Y,y_0)$ of pointed topological spaces such that $p_*(\pi_1(X,x_0)) = \Gamma$.[1317]
(2) For the covering p from (1) we have

$$\underbrace{[X:Y]}_{\substack{\text{degree from}\\ \text{Definition 91.10}}} = [\pi_1(Y,y_0):\Gamma].$$

We will provide the proof of the Covering Existence Theorem 127.10 in the next sections. In the following we discuss a few examples.

Example.

(1) We consider $(Y,y_0) = (S^1,1)$. We pick an identification[1318] $\pi_1(S^1,1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subseteq \mathbb{Z} =_{\mathrm{i}} \pi_1(S^1,1)$. It follows from the discussion on page 1760 that the covering

[1315] Note that in contrast to "locally simply connected" there is no general notion of what "semi-locally" means. In other words, we define "semi-locally simply connected" as a single object.
[1316] If there exists a neighborhood W of x such that the inclusion-induced map $\pi_1(W,x) \to \pi_1(X,x)$ is the trivial map, then the interior U of W is an open neighborhood of x and the inclusion-induced map $\pi_1(U,x) \to \pi_1(X,x)$ is trivial, since it factors through the trivial map $\pi_1(W,x) \to \pi_1(X,x)$. Thus in the definition it makes no real difference whether we demand a "neighborhood" or an "open neighborhood".
[1317] Note that it follows from the Covering-π_1-Monomorphism Proposition 92.7 that $\pi_1(X,x_0) \cong \Gamma$.
[1318] This means that we pick an isomorphism $\pi_1(S^1,1) \xrightarrow{\cong} \mathbb{Z}$ and then ignore it in the notation.

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

has the property that $p_*(\pi_1(S^1,1)) = n\cdot\mathbb{Z}$. Another example is given by the covering

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t+n\mathbb{Z} & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

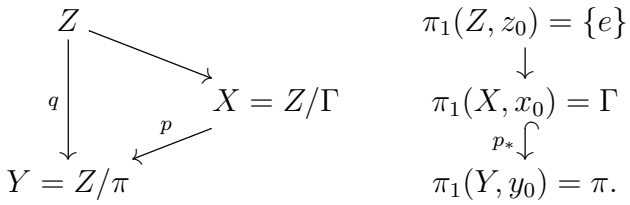
Here we also have $q_*(\pi_1(\mathbb{R}/n\mathbb{Z},[0])) = n\mathbb{Z}$.

(2) Let Z be a simply connected topological space and let $z_0 \in Z$. Furthermore let π be a group that acts on Z continuously and discretely. We consider $Y = Z/\pi$, we denote by $q\colon Z \to Y = Z/\pi$ the projection and we write $y_0 = q(z_0)$. Using the isomorphism of the Action–π_1-Theorem 93.1 we can identify π with $\pi_1(Y,y_0)$.

Now let $\Gamma \subseteq \pi = \pi_1(Y,y_0)$ be a subgroup. The group Γ also acts on Z and we can consider the projection

$$p\colon X \coloneqq Z/\Gamma \ \to\ Y \coloneqq Z/\pi$$

We denote by x_0 the image of z_0 under the projection $Z \to X = Z/\Gamma$. One can show easily that $p\colon X = Z/\Gamma \to Y = Z/\pi$ is a covering map with $p_*(\pi_1(X,x_0)) = \Gamma$. We summarize the situation in the following diagram:



□

127.5. **Proof of the Covering Existence Theorem 127.10 I.** We turn to the proof of the Covering Existence Theorem 127.10. Note that we only need to show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of the Covering-Degree Proposition 92.10.

We start out with a general discussion of the construction of the covering. Only afterwards we will turn to the nitty-gritty of the proof. In particular for the time being let Y be any path-connected topological space, without any other adjectives. Furthermore let $y_0 \in Y$ and let $\Gamma \subseteq \pi_1(Y,y_0)$ be a subgroup.

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p\colon X \to Y$ such that X is simply connected. Let us assume for a second that we have such an X. We pick $x_0 \in X$ with $p(x_0) = y_0$. Then, somewhat similar to the statement of the Action–π_1-Theorem 93.1, we have a map

$$\begin{array}{rcl} \Phi\colon X & \to & \{\text{path-homotopy classes of paths } [0,1] \to Y \text{ starting in } y_0\} \\ x & \mapsto & p \circ (\text{any path from } x_0 \text{ to } x). \end{array}$$

In fact the proof of the Action–π_1-Theorem 93.1 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right-hand side with an appropriate topology.

Now that we are not given X, the idea is, that given (Y,y_0) we can define the right-hand side of the above bijection, and with a suitable topology we will have found our X.

This discussion leads us to the following definition:

Definition 127.11. Let Y be a topological space, let $y_0 \in Y$ and let $\Gamma \subseteq \pi_1(Y, y_0)$ be a subgroup.

(1) As in Definition 16.2 we denote by $Y^{[0,1]}$ the set of all continuous maps $[0,1] \to Y$.

(2) We consider the path space[1319]

$$P_{y_0} \coloneqq \{\gamma \in Y^{[0,1]} \mid \gamma(0) = y_0\} = \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

(3) For $u, v \in P_{y_0}$ we write

$$u \sim_\Gamma v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [\underbrace{u * \overline{v}}_{\text{loop in } (Y, y_0)}] \in \Gamma.$$

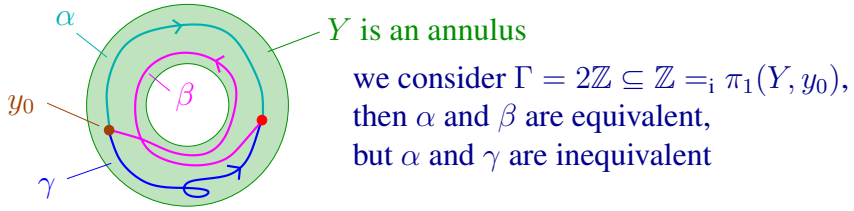
Using the fact that $\Gamma \subseteq \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim_\Gamma$ is indeed an equivalence relation on P_{y_0}.

(4) We write $X_{y_0,\Gamma} \coloneqq P_{y_0}/\sim_\Gamma$ and $x_0 \coloneqq [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.

(5) We consider the evaluation map

$$\begin{aligned} p\colon X_{y_0,\Gamma} = P_{y_0}/\sim_\Gamma &\to Y \\ [v] &\mapsto v(1). \end{aligned}$$

Note that this map is well-defined, i.e. it is independent of the choice of the representative v.



It remains to equip $X_{y_0,\Gamma} = P_{y_0}/\sim_\Gamma$ with a sensible topology and to show that the map $p\colon X_{y_0,\Gamma} = P_{y_0}/\sim_\Gamma \to Y$ has all the desired properties.

In fact in the following two sections we provide two approaches to defining such a topology:

(1) The first one is "by hand", namely in the Whisker Topology Lemma 127.12 we construct a topology on $X_{y_0,\Gamma} = P_{y_0}/\sim_\Gamma$ from scratch.

(2) The second approach is "off-the-shelves", meaning that in the CO-Topology Lemma 127.14 we make use of the compact-open topology that we introduced in Definition 16.2.

Put differently, both the Whisker Topology Lemma 127.12 and the CO-Topology Lemma 127.14 can be used to complete the proof of the Covering Existence Theorem 127.10. We will formulate and prove these two lemmas in the following two sections. Of course it is enough to digest one of the two lemmas for the proof of the Covering Existence Theorem 127.10.

[1319] In Definition 297.17 the path space will be called $P(Y, y_0)$. Later on it will play an important role, but for the purpose of this discussion it is easier to work with a shorter notation.

127.6. **Proof of the Covering Existence Theorem 127.10 II.** We start out with the "by hand" approach.

Lemma 127.12. (Whisker Topology Lemma) Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subseteq \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

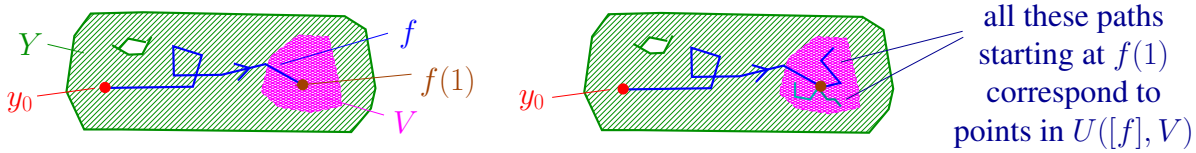
$$U(x, V) := \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subseteq X = P/\sim .$$

With this notation the following statement holds:

(0) The family of all such subsets $U(x, V)$ of X satisfy the "basis property" that we introduced in Definition 3.52.

The topology that is generated on $X = X_{y_0,\Gamma}$ by the sets $U(x, V)$ is called the whisker topology. If Y is locally path-connected and semi-locally simply connected, and if we equip $X = X_{y_0,\Gamma}$ with the whisker topology, then the following statements hold:

(1) The above evaluation map $p\colon X \to Y$ is continuous and open.
(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected.
(4) We have $p_*(\pi_1(X, x_0)) = \Gamma$.

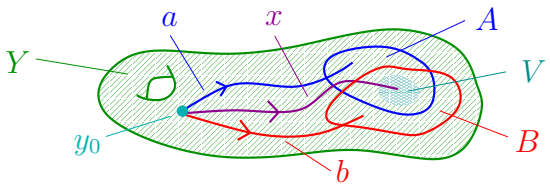


Proof. Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subseteq \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

$$U(x, V) := \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subseteq X = P/\sim .$$

We denote by $\mathcal{B}$ the family of all such subsets of X.

(0) We need to show that $\mathcal{B}$ has the basis property.
  (B1) For each $x \in X$ we have $x \in U(x, X)$. This shows that $\mathcal{B}$ satisfies (B1).
  (B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Let V be any open neighborhood V of $f(1)$ with $V \subseteq A \cap B$, e.g. we could take $V = A \cap B$. It is straightforward to see that $U(x, V) \subseteq U(a, A) \cap U(b, B)$.



In the following we assume that Y is locally path-connected and that Y is semi-locally simply connected. We equip $X = X_{y_0,\Gamma}$ with the *whisker topology*, i.e. we equip it with the topology generated by $\mathcal{B}$. Since Y is locally path-connected it follows easily that the

whisker topology is also generated by

$$\mathcal{C} \coloneqq \{U(x,V) \in \mathcal{B} \,|\, x \in X \text{ and } V \text{ is a } \textit{path-connected} \text{ open neighborhood of } p(x)\}.$$

(1) We need to show that the map $p\colon X = P/\sim \to Y$ is continuous and open. We start with the following observation: For every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x,V)) = V$.[1320]

Now we turn to the actual proof of continuity. So let $A \subseteq Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x,B) \subseteq p^{-1}(A)$. Since Y is locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x,B)) = B \subseteq A$, i.e. $U(x,B) \subseteq p^{-1}(A)$.

Finally we show that p is also open. By the Subbasis-Open Map Lemma 4.14 it suffices to verify that images of the basis $\mathcal{C}$ of the topology of X are open. But this statement we had just verified in the above observation.

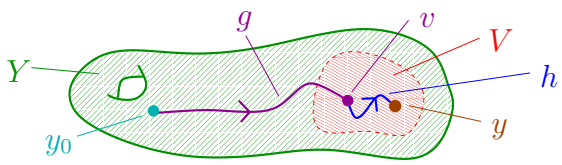
(2) Now we want to prove that $p\colon X = P/\sim \to Y$ is a covering map. First note that it follows from the hypothesis that Y is path-connected that p is surjective. Next let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

It follows easily from the hypothesis that Y is locally path-connected and that Y is semi-locally simply connected that there exists an open path-connected neighborhood V of y such that the inclusion-induced map $\pi_1(V,x) \to \pi_1(Y,y)$ is the trivial map. We claim that V already has the desired property.

Claim 1.
$$p^{-1}(V) = \bigsqcup_{\widetilde{y} \in p^{-1}(y)} U(\widetilde{y}, V).$$

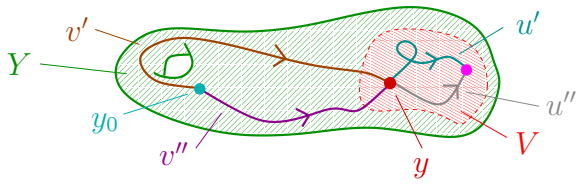
Proof. It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. Thus let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] = [g * h * \overline{h}] \in U(\underbrace{[g * h]}_{\in p^{-1}(y)}, V).$$



It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'],V) \cap U([v''],V)$. Then there exist paths u' and u'' in V with $[v' * u'] = [v'' * u'']$.

[1320] By definition we have $p(U(x,V)) \subseteq V$. But since V is path-connected we also have $V \subseteq p(U(x,V))$.



It follows that

$$[v'] \;=\; [v' * u' * \overline{u}'] \;=\; [v'' * u'' * \overline{u}'] \;\underset{\uparrow}{=}\; [v''].$$

$u'' * \overline{u}'$ is null-homotopic since the map $\pi_1(V, y) \to \pi_1(Y, y)$ is the trivial map

Thus $U([v'], V) \cap U([v''], V)$ intersect only if $[v'] = [v'']$. ⊞
The sets $U(\widetilde{y}, V)$ are by definition open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim:

Claim 2. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection $p\colon X = P/\!\sim \to Y$ to the map $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

Proof. So let $\widetilde{y} \in p^{-1}(y)$. As we had just pointed out in (1), the map $p\colon U(\widetilde{y}, V) \to V$ is surjective. From the fact that the inclusion-induced map $\pi_1(V, y) \to \pi_1(Y, y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism. ⊞

(3) Now we want to show that $X = P/\!\sim$ is path-connected. We start out with the following claim:

Claim 3. For any path $\alpha\colon [0,1] \to Y$ with $\alpha(0) = y_0$ the following map is continuous:

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\!\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(t \cdot s) \end{array}\right]. \end{array}$$

Proof. By the Subbasis-Continuity Proposition 4.4 it suffices to show that for each $U(x, V)$ in the basis $\mathcal{C}$ of the topology on X the preimage $\widetilde{\alpha}^{-1}(U(x, V))$ is open in $[0, 1]$. So suppose that we are given $U(x, V)$. We denote by $\{I_j\}_{j \in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0, 1]$. Note that for each $j \in J$ we either have[1321] $\widetilde{\alpha}(I_j) \subseteq U(x, V)$ or $\widetilde{\alpha}(I_j) \cap U(x, V) = \varnothing$. It follows that

$$\widetilde{\alpha}^{-1}(U(x, V)) \;=\; \text{union of all the } I_j \text{ with the property that } \widetilde{\alpha}(I_j) \subseteq U(x, V).$$

Thus we see that $\widetilde{\alpha}^{-1}(U(x, V))$ is a union of open subsets of $[0, 1]$, hence the preimage itself is open. ⊞

To show that X is path-connected it suffices to show that for every point $x \in X$ there exists a path in X from $x_0 = [e_{y_0}]$ to x. So let $x = [\alpha] \in X$. Then such a path is given by $\widetilde{\alpha}\colon [0, 1] \to X$, where $\widetilde{\alpha}$ is defined as in the claim. By the claim $\widetilde{\alpha}$ is indeed continuous.

[1321] This might require a few minutes of thought.

(4) It remains to show that $p_*(\pi_1(X,x_0)) = \Gamma$. Let $g \in \pi_1(Y,y_0)$. We choose a representatives $\alpha\colon [0,1] \to Y$ of g. We consider the following path[1322] in $X = P/\sim$

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\sim \\ t & \mapsto & \left[\begin{array}{ccc} [0,1] & \to & Y \\ s & \mapsto & \alpha(t\cdot s) \end{array}\right]. \end{array}$$

Note that $\widetilde{\alpha}$ is evidently the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \overset{\uparrow}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X = P/\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \overset{\uparrow}{\Leftrightarrow} \alpha \in \Gamma.$$

by the Loop–Lifting Proposition 92.8; definition of $\sim$ on P ■

127.7. **Proof of the Covering Existence Theorem 127.10 II′.** In this section we use the "off-the-shelves"-topology, namely we will make use of the compact-open topology. For the reader's convenience we recall Definition 16.2.

Definition 127.13. Let Y and A be topological spaces. We write

$$Y^A \;:=\; \text{set of all continuous maps } A \to Y.$$

The compact-open topology on the set Y^A is the topology generated, in the sense of Definition 3.56, by all the sets of the form

$$M(K,U) \;:=\; \{f \in Y^A \mid f(K) \subseteq U\}$$

where $K \subseteq A$ is compact and where $U \subseteq Y$ is open.

Lemma 127.14. (CO-Topology Lemma) Let Y be a path-connected topological space and let $y_0 \in Y$. Furthermore let $\Gamma \subseteq \pi_1(Y,y_0)$ be a subgroup.

(i) We equip $Y^{[0,1]} = \text{set of all continuous maps } [0,1] \to Y$ with the compact-open topology.
(ii) We equip $P_{y_0} = \{\gamma \in Y^{[0,1]} \mid \gamma(0) = y_0\}$ with the subspace topology.
(iii) We equip $X_{y_0,\Gamma} = P_{y_0}/\sim$ with the quotient topology.

We refer to this topology on $X_{y_0,\Gamma} = P_{y_0}/\sim$ as the CO-topology on $X_{y_0,\Gamma}$. We write $X := X_{y_0,\Gamma}$. If Y is locally path-connected and semi-locally simply connected, then the following statements hold:

(1) The above evaluation map $p\colon X \to Y$ is continuous and open.
(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected.
(4) We have $p_*(\pi_1(X,x_0)) = \Gamma$.

Proof. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $y_0 \in Y$ and let $\Gamma \subseteq \pi_1(Y,y_0)$ be a subgroup. We write $P := P_{y_0}$, $X := X_{y_0,\Gamma}$ and we equip $X := P/\sim$ with the CO-topology. We make a few preparations:

(i) We consider the maps

[1322] Note that this map is continuous by the claim in (3).

$$\begin{array}{ccc} P & \hookrightarrow & Y^{[0,1]} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}\ f\mapsto f(1)} \\ X = P/\!\sim & \xrightarrow{\quad p \quad} & Y \end{array}$$

where q is the projection and $\mathrm{ev}\colon Y^{[0,1]} \to Y$ is given by the evaluation at 1. By the Topological-Quotient Proposition 8.2 (1a) we know that q is continuous. Furthermore by the Evaluation Map Proposition 16.9 we know that the map $\mathrm{ev}\colon Y^{[0,1]} \to Y$ is continuous. Thus it follows from the Subspace Continuity Lemma 4.3 (2) and the Topological-Quotient Proposition 8.2 (1b) that the induced map $p\colon X = P/\!\sim \to Y$ is continuous.

(ii) Since Y is locally path-connected it follows from the Basis-with-Property Lemma 4.56 that the open path-connected subsets of Y form a basis for the topology of Y. Since $[0,1]$ is Hausdorff we obtain from the Compact-Open–Subbasis Lemma 16.13 that the sets $M(K,U)$, with $K \subseteq [0,1]$ compact and U open and path-connected form a basis for the topology of the compact-open topology on $Y^{[0,1]}$.

(iii) Let $x = [f] \in X = P/\!\sim$ and let V be an open path-connected neighborhood of the point $p(x) = f(1) \in Y$. As in the Whisker Topology Lemma 127.12 we consider

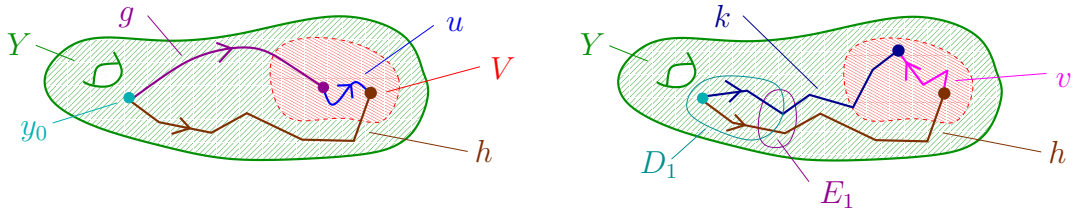
$$U(x,V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \;\subseteq\; X = P/\!\sim .$$

We want to show that $U(x,V)$ is an open subset of X. By Definition 8.3 of the quotient topology we need to show that $q^{-1}(U(x,V))$ is an open subset of P. This statement is easily seen to be a consequence of the following claim:

Claim 1. Let $g\colon [0,1] \to Y$ be a path with $g(0) = y_0$ and $g(1) \in V$. The set

$$\left\{ h \in P \;\middle|\; \begin{array}{c} \text{there exists a path } u\colon [0,1] \to V \text{ with } u(0) = g(1) \\ \text{such that } h \text{ is path-homotopic to } g * u \end{array} \right\}$$

is an open subset of P.



Proof. Let $h \in P$ be a path that is path-homotopic to $g * u$ for some path $u\colon [0,1] \to V$. Note that it follows from the fact that Y is semi-locally simply connected, together with the Lebesgue Lemma 10.9, that there exists an $n \in \mathbb{N}$ and open subsets $D_1, \ldots, D_n$ of Y with the following properties:

(a) For each $i \in \{0, \ldots, n-1\}$ we have $h([\frac{i}{n}, \frac{i+1}{n}]) \subseteq D_i$.

(b) For each $i \in \{0, \ldots, n-1\}$ the inclusion-induced map $\pi_1(D_i, h(\frac{i}{n})) \to \pi_1(Y, h(\frac{i}{n}))$ is the trivial map.

Since Y is locally path-connected we can furthermore find open path-connected neighborhoods $E_1, \dots, E_{n-1}$ of $h(\frac{1}{n}), \dots, h(\frac{n-1}{n})$. Now let

$$Z := \bigcap_{i=0}^{n-1} M([\tfrac{i}{n}, \tfrac{i+1}{n}], D_i) \cap \bigcap_{i=1}^{n-1} M(\{\tfrac{i}{n}\}, E_i) \cap M(\{1\}, V).$$

We make the following observations:

(α) By definition of the compact-open topology we know that Z is an open subset of $Y^{[0,1]}$. Furthermore it follows from (a) and the choice of $E_1, \dots, E_{n-1}$ that $h \in Z$.

(β) By the JH-Openness Criterion 3.3 it remains to show that $Z \cap P$ is contained in the set that we described in the statement of the claim.

Now let $k \in Z \cap P$.

(γ) Since $k(1) \in V$ and since V is path-connected there exists a path $v\colon [0,1] \to V$ from $h(1)$ to $k(1)$.

(δ) Since the $E_1, \dots, E_{n-1}$ are path-connected we can arrange, after a path-homotopy, that $k(\frac{i}{n}) = (h * v)(\frac{i}{n})$ for $i = 0, \dots, n-1$.

(η) It follows from (b) together with a slight variation on the Paths-in-Simply Connected Space Lemma 88.17 that we can show that k is path-homotopic to $h * v$.

Since k is path-homotopic to $h * v$ and since h is path-homotopic to $g * u$ we have now shown that k is path-homotopic to $g * (u * v)$. But this means that k lies again in the set defined in the statement of the claim. ⊞

(iv) Let $\alpha\colon [0,1] \to Y$ be a path with $\alpha(0) = y_0$. It follows easily from the Compact-Open Exponential Proposition 16.10 and the Subspace Continuity Lemma 4.3 (3) that the map

$$\begin{array}{ccc} [0,1] & \to & P \subseteq Y^{[0,1]} \\ t & \mapsto & \left(\begin{array}{ccc} [0,1] & \to & Y \\ s & \mapsto & \alpha(t \cdot s) \end{array} \right). \end{array}$$

is continuous.

Now we turn to the actual proofs of the four statements.

(1) In (i) we already showed that $p\colon X = P/\!\sim \to Y$ is continuous. We still need to show that p is open. It follows from an elementary argument, see Exercise 8.11, that it suffices to show that the evaluation map $\mathrm{ev} = p \circ q\colon P \to Y$ is open. In fact it follows from (ii), together with the Subspace-Basis Lemma 3.58 and the Subbasis-Open Map Lemma 4.14 that it suffices to prove the following claim:

Claim 2. For each compact subset $K \subseteq [0,1]$ and each path-connected open subset $V \subseteq Y$ we have $\mathrm{ev}(M(K,V) \cap P) = V$.

Proof. A straightforward argument, that uses that Y and V are path-connected, shows that

$$\mathrm{ev}(M(K,V) \cap P) = \begin{cases} \varnothing, & \text{if } M(K,V) = \varnothing, \\ Y, & \text{if } 1 \notin K \text{ and } M(K,V) \neq \varnothing, \\ V, & \text{if } 1 \in K. \end{cases}$$

Thus we see that in any case the image $\mathrm{ev}(M(K,V))$ is an open subset of Y. ⊞

(2) Now we want to prove that $p\colon X = P/\!\sim \to Y$ is a covering map. So let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is locally path-connected and semi-locally simply connected, there exists an open path-connected neighborhood V of y such that the inclusion-induced map $\pi_1(V,y) \to \pi_1(Y,y)$ is the trivial map. We claim that V already has the desired property.

Claim 3. $$p^{-1}(V) \;=\; \bigsqcup_{\widetilde{y}\in p^{-1}(y)} U(\widetilde{y},V).$$

Proof. For the reader's convenience we repeat word-by-word the proof from page 2454. First note that it is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] \;=\; [g*h*\overline{h}] \;\in\; U(\underbrace{[g*h]}_{\in p^{-1}(y)},V).$$

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'],[v''] \in p^{-1}(y)$ and let $[g] \in U([v'],V)\cap U([v''],V)$. Then there exist paths u' and u'' in V with $[v'*u'] = [v''*u'']$. It follows that

$$[v'] \;=\; [v'*u'*\overline{u}'] \;=\; [v''*u''*\overline{u}'] \underset{\uparrow}{=} [v''].$$

$u''*\overline{u}'$ is null-homotopic since the map $\pi_1(V,y)\to\pi_1(Y,y)$ is the trivial map

Thus $U([v'],V)\cap U([v''],V)$ intersect only if $[v'] = [v'']$. ⊞

By (iii) we know that the sets $U(\widetilde{y},V)$ are open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim:

Claim 4. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection $p\colon X = P/\!\sim \to Y$ to the map $p\colon U(\widetilde{y},V) \to V$ is a homeomorphism.

Proof. The proof of this claim is again almost verbatim the same as on page 2454. So let $\widetilde{y} \in p^{-1}(y)$. It follows easily from the fact that V is path-connected that the map $p\colon U(\widetilde{y},V) \to V$ is surjective. From the fact that the inclusion-induced map $\pi_1(V,y)\to\pi_1(Y,y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y},V)\to V$ is a homeomorphism. ⊞

(3) It follows from the fact that Y is path-connected together with (iv) that P is path-connected. By the Quotient Topology-Properties Proposition 8.5 (2) this implies that $X = P/\!\sim$ is path-connected.

(4) It remains to show that $p_*(\pi_1(X,x_0)) = \Gamma$. The proof we now give is basically identical to the proof on page 2455. Thus let $g \in \pi_1(Y,y_0)$. We choose a representatives $\alpha\colon [0,1]\to Y$ of g. We consider the following map:

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\!\sim \\ t & \mapsto & \left[\begin{array}{ccc} [0,1] & \to & Y \\ s & \mapsto & \alpha(t\cdot s) \end{array}\right]. \end{array}$$

By (iv) we know that $\widetilde{\alpha}$ is in fact continuous. It is now clear that $\widetilde{\alpha}$ is the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \underset{\text{by the Loop–Lifting Proposition 92.8}}{\Longleftrightarrow} \widetilde{\alpha} \text{ is a loop} \Longleftrightarrow \text{in } X = P/\!\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \underset{\text{definition of } \sim \text{ on } P}{\Longleftrightarrow} \alpha \in \Gamma.$$

■

Remark.

(1) The above construction of coverings of a topological space has its origin in the work of Hermann Weyl from 1913, see [**Wey1913**][**Wey1964**, p. 58].

(2) Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subseteq \pi = \pi_1(Y, y_0)$ be a subgroup. Recall that in the Whisker Topology Lemma 127.12 we introduced the whisker topology and that in the CO-Topology Lemma 127.14 we introduced the CO-topology on $X_{y_0,\Gamma} \coloneqq P_{y_0}/\!\sim$.

(a) In [**FZ2007**, Lemma 2.1] it is shown that the CO-topology is contained in the whisker topology. Furthermore in [**FZ2007**, Example 2.2] it is shown that the two topologies are in general not equal, for example it is shown that they disagree for the Hawaiian earring that we introduced in Exercise 105.1.

(b) Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. We consider the following commutative diagram:

$$\begin{array}{ccc} X^{\text{Whisker}}_{y_0,\Gamma} & \xrightarrow{\text{id}} & X^{\text{CO}}_{y_0,\Gamma} \\ & \searrow \quad \swarrow & \\ & Y. & \end{array}$$

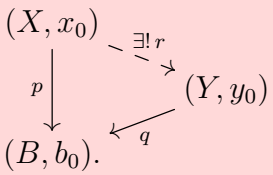
It follows from the Whisker Topology Lemma 127.12 and the CO-Topology Lemma 127.14 that the two diagonal maps are local homeomorphisms. This implies that the identity map is a local homeomorphism. It follows easily from the Pasting Proposition 4.7 (1) that the identity map is a homeomorphism, i.e. the whisker and the CO-topology agree. (An alternative, more direct, proof of this statement is given in [**FZ2007**, Lemma 2.1]).

(c) The various choices of topologies on $X_{y_0,\Gamma} = P_{y_0}/\!\sim$ are discussed in greater detail in [**FZ2007, VZ2014**] and [**Pav2020**]. The history of the construction of covering spaces is also outlined [**Pav2020**]. □

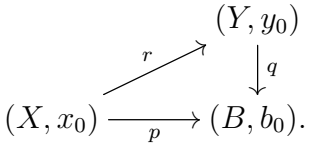
127.8. **Uniqueness of coverings.** Let (B, b_0) be a path connected topological space. We just saw, that under a few mild technical hypothesis, any subgroup $\Gamma \subseteq \pi_1(B, b_0)$ gives rise to a covering. The following question arises: to what degree is the covering corresponding to the subgroup Γ unique? To answer this question it is helpful to first prove the following universal property of coverings corresponding to a subgroup.

Theorem 127.15. (Covering–Covering Theorem) Let B be a locally path-connected topological space, let $b_0 \in B$ and let $p\colon (X, x_0) \to (B, b_0)$ be a path-connected covering of pointed topological spaces. For every other path-connected covering $q\colon (Y, y_0) \to (B, b_0)$ with $p_*(\pi_1(X, x_0)) \subseteq q_*(\pi_1(Y, y_0))$ there exists a *unique* covering $r\colon (X, x_0) \to (Y, y_0)$ of

pointed topological spaces such that the following diagram commutes:



Proof. Let B be a locally path-connected topological space and let $b_0 \in B$ and let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings such that $p_*(\pi_1(X, x_0)) \subseteq q_*(\pi_1(Y, y_0))$. Note that X is also locally path-connected. Thus it follows from the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 that there exists a unique continuous map $r\colon (X, x_0) \to (Y, y_0)$ that makes the following diagram commute:



It remains to prove that $r\colon X \to Y$ is a covering. Recall that this means that we need to prove the following two statements:

(a) Each $y_1 \in Y$ admits a uniformly covered open neighborhood.
(b) The map $r\colon X \to Y$ is surjective.

We turn to the proofs of these two statements:

(a) Let $y_1 \in Y$. We set $b_1 \coloneqq q(y_1)$. Since B is locally path-connected and since $p\colon X \to B$ and $q\colon Y \to B$ are coverings there exists, by the remark on page 1781, a path-connected open neighborhood W of b that is uniformly covered. Since $p\colon X \to B$ and $q\colon Y \to B$ are coverings we can write $p^{-1}(W) = \bigsqcup_{x \in p^{-1}(b_1)} U_x$ where each $U_x \subseteq X$ is open and each $p\colon U_x \to W$ is a homeomorphism and we can write $q^{-1}(W) = \bigsqcup_{y \in q^{-1}(b_1)} V_y$ where each $V_y \subseteq Y$ is open and each $q\colon V_y \to W$ is a homeomorphism.

Claim. Let $x \in p^{-1}(b_1)$ and let $y \in q^{-1}(b_1)$. If $r(U_x) \cap V_y \neq \varnothing$, then $r(x) = y$ and $r(U_x) = V_y$.

Proof. We make the following observations:
(i) Since W is path-connected it is in particular connected. Since U_x is homeomorphic to W we see that U_x is connected.
(ii) Since $p = q \circ r$ we see that $r(U_x) \subseteq q^{-1}(q(r(U_x)) = q^{-1}(p(U_x)) = q^{-1}(W)$.
(iii) It follows from the Component Lemma 4.48 together with the Image-Preimage-Component Lemma 4.50 together with (i), (ii), the fact that $q^{-1}(W) = \bigsqcup_{y \in q^{-1}(b_1)} V_y$ and the fact that the V_y are open that there exists a $y \in q^{-1}(b_1)$ with $r(U_x) \subseteq V_y$.
(iv) It follows from (iii) together with the observation that $p(x) = b_1 = q(y)$, $p^{-1}(b_1) \cap U_x = \{x\}$, $q^{-1}(b_1) \cap V_y = \{y\}$ and $p = q \circ r$ that $r(x) = y$.
(v) Since $p\colon U_x \to W$ and $q\colon V_y \to W$ are homeomorphisms and since $p = q \circ r$ we obtain from (iii) that $r\colon U_x \to V_y$ is a homeomorphism. ⊞

Next note that

$$\bigsqcup_{x\in r^{-1}(y)}\bigsqcup_{y\in q^{-1}(b)} U_x = \bigsqcup_{x\in(q\circ r)^{-1}(b)} U_x = \bigsqcup_{x\in p^{-1}(b)} U_x = p^{-1}(W) = (q\circ r)^{-1}(W)$$
$$= r^{-1}(q^{-1}(W)) = r^{-1}\Big(\bigsqcup_{y\in q^{-1}(b)} V_y\Big) \underset{\uparrow}{=} \bigsqcup_{y\in q^{-1}(b)} r^{-1}(V_y).$$

by the Image-Preimage Lemma 1.9 (6)

It follows from this equality and the above claim that

$$\bigsqcup_{x\in r^{-1}(y_1)} U_x = r^{-1}(V_{y_1}).$$

Next let $x \in r^{-1}(y_1)$. We already know that U_x is an open subset of X. Note that the restriction of r to U_x is given by $(q|_{V_y\to W})^{-1} \circ (p\colon U_x \to W)$. This shows that $r\colon U_x \to V_{y_1}$ is a homeomorphism. In summary, we have shown that V_{y_1} is uniformly covered.

(b) We want to show that $r\colon X \to Y$ is surjective. Let $y_1 \in Y$. We just showed in (a) that there exists an open neighborhood V_{y_1} that is uniformly covered with respect to the map $r\colon X \to Y$. But this means that either $V_{y_1} \subseteq r(X)$ or $V_{y_1} \cap r(X) = \varnothing$. It follows from the JH-Openness Criterion 3.3 that $r(X)$ and $Y \setminus r(X)$ are both open. Since Y is by hypothesis path-connected (and thus connected) and since $X \neq \varnothing$ we see that $r(X) = Y$. ■

Now we want to formulate the promised uniqueness statement for coverings. This statement builds on the following natural definition.

Definition 127.16. Let $p\colon X \to B$ and $q\colon Y \to B$ be two coverings of a topological space B. We say that p and q are isomorphic[1323], if there exists a homeomorphism $\Phi\colon X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow[\cong]{\Phi} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B. & \end{array}$$

Similarly we define an isomorphism of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t+n\mathbb{Z} & \mapsto & \exp(2\pi\,\mathrm{i}\,t) \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

are isomorphic. In fact the desired map $\Phi\colon \mathbb{R}/n\mathbb{Z} \to S^1$ is given by $[t] \mapsto \exp(2\pi\,\mathrm{i}\,\frac{t}{n})$. □

Now we can formulate the promised uniqueness theorem:

Theorem 127.17. (Covering Unique Existence Theorem) Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected.

[1323]The name "isomorphic" is appropriate since coverings over a fixed topological space form a category, where the morphisms are coverings between coverings, that lead to commutative diagrams. In this category an isomorphism is exactly what we just called an isomorphism.

Furthermore let $b_0 \in B$ and let $\Gamma \subseteq \pi_1(B, b_0)$ be a subgroup. There exists, up to isomorphism, a *unique* path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.

Proof. Let $b_0 \in B$ and let $\Gamma \subseteq \pi_1(B, b_0)$ be a subgroup. The existence of such a covering is just the statement of the Covering Existence Theorem 127.10.

Now suppose that we are given two path-connected coverings $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ with $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0)) = \Gamma$. There are two approaches to showing that the coverings p and q are isomorphic:

(1) The uniqueness follows immediately from the Covering–Covering Theorem 127.15 and the usual proof of uniqueness for an object satisfying a universal property. We leave it to the reader to fill in the details.

(2) By the Covering–Covering Theorem 127.15 there exists a covering $f\colon (X, x_0) \to (Y, y_0)$ with $p = q \circ f$. Next note that it follows from our hypotheses and the Covering-π_1-Monomorphism Proposition 92.7 that $p_*\colon \pi_1(X, x_0) \to \Gamma$ and $q_*\colon \pi_1(Y, y_0) \to \Gamma$ are isomorphisms. Since $p_* = q_* \circ f_*$ we see that f_* is an isomorphism. It follows from the Covering-Degree Proposition 92.10 that $f\colon X \to Y$ is a covering of degree 1. As we discussed on page 1783, it is an immediate consequence of the Bijective-Local Homeomorphism Lemma 4.62, that coverings of degree 1 are actually homeomorphisms. ■

The Covering Existence Theorem 127.10 and the Covering Unique Existence Theorem 127.17 lead us to the following definition:

Definition 127.18. Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $b_0 \in B$ and let $\Gamma \subseteq \pi_1(B, b_0)$ be a subgroup.

(1) As in Definition 127.11 we consider the set

$$\widetilde{B}_{b_0,\Gamma} \;\coloneqq\; \{\text{all paths in } B \text{ with starting point } b_0\}/\sim$$

where $u \sim v$ if and only if u and v have the same endpoint and if $[u * \overline{v}] \in \Gamma$. We equip the set $\widetilde{B}_{b_0,\Gamma}$ with the unique[1324] topology that turns $p\colon \widetilde{B}_{b_0,\Gamma} \to B$ into a covering. We refer to $\widetilde{B}_{b_0,\Gamma}$ together with map $p\colon \widetilde{B}_{b_0,\Gamma} \to B$ given by $[u] \mapsto u(1)$ as the natural covering corresponding to (B, b_0, Γ). Furthermore we refer to the point in $\widetilde{B}_{b_0,\Gamma}$ that is represented by the constant path as the natural base point of $\widetilde{B}_{b_0,\Gamma}$.

(2) We refer to any path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces with $p_*(\pi_1(X, x_0)) = \Gamma$ as a covering of B corresponding to Γ. By the Covering Unique Existence Theorem 127.17 any two such coverings are isomorphic, in particular any such covering is isomorphic to the above natural covering. In practice usually we treat all coverings corresponding to a subgroup as equal.

Example. Let M be a connected smooth manifold and let $x_0 \in M$. In Definition 95.15 we introduced the orientation character $w_1\colon \pi_1(M, x_0) \to \mathbb{Z}/2\mathbb{Z}$. It follows from the Orientation Covering–Properties-Proposition 95.10 (4) that the covering of M corresponding to the subgroup $\ker(w_1) \subseteq \pi_1(M, x_0)$ is isomorphic to the orientation covering that we introduced in Definition 95.9. □

[1324] The uniqueness follows easily from the Bijective-Local Homeomorphism Lemma 4.62.

Our definition of the natural coverings relies on a choice of a base point. We will now explore what happens if we move from one base point to another. More precisely, let B be a topological space and let $\rho\colon [0,1] \to B$ be a path from a point $b_0 = \rho(0)$ to a point $b_1 = \rho(1)$. In the Change-of-Base Point Proposition 88.14 (1) we saw that ρ induces an isomorphism

$$\begin{array}{rcl} \rho^*\colon \pi_1(B,b_1) & \to & \pi_1(B,b_0) \\ [\alpha] & \mapsto & [\rho * \alpha * \overline{\rho}] \end{array}$$

The following lemma, which is closely related to the Change-of-Base Point Proposition 88.14, shows that ρ also induce a homeomorphism of natural coverings.

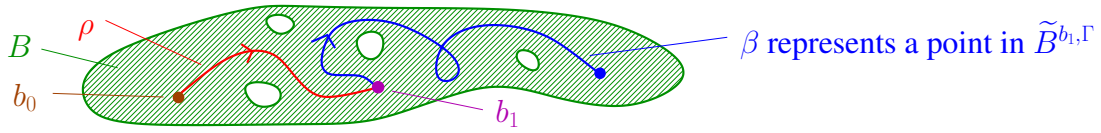
Lemma 127.19. (Natural Covering-Base Point Lemma) Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\rho\colon [0,1] \to B$ be a path from a point $b_0 = \rho(0)$ to a point $b_1 = \rho(1)$. Finally let $\Gamma \subseteq \pi_1(B,b_1)$ be a subgroup. The map[1325]

$$\begin{array}{rcl} \rho^*\colon \widetilde{B}_{b_1,\Gamma} & \to & \widetilde{B}_{b_0,\rho^*(\Gamma)} \\ [\beta] & \mapsto & [\rho * \beta] \end{array}$$

is a homeomorphism. This map has furthermore the following properties:

(1) The induced map γ^* only depends on the path-homotopy class of γ.
(2) (a) If p is a constant path, then $p^* = \mathrm{id}$.
    (b) If $q\colon [0,1] \to X$ is a path from x_1 to another point x_2, then

$$(p * q)^* = p^* \circ q^*\colon \pi_1(X,x_2) \to \pi_1(X,x_0).$$

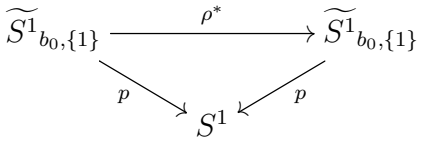


Proof. One can easily verify that the map $\rho^*\colon \widetilde{B}_{b_1,\Gamma} \to \widetilde{B}_{b_0,\rho^*(\Gamma)}$ is well-defined. Furthermore using the Subbasis-Continuity Proposition 4.4 one can easily show that the map is continuous. Finally note that an inverse is given by the map $\overline{\rho}^*$. This shows that the map ρ^* is indeed a homeomorphism. ∎

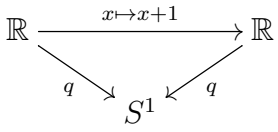
Example. The map defined in the Natural Covering-Base Point Lemma 127.19 is also interesting if $b_0 = b_1$ and if ρ is a loop in (B,b_0). Let us consider $B = S^1 \subseteq \mathbb{C}$ with base point $b_0 = 1 = \exp(0)$ and the trivial subgroup $\Gamma = \{1\} \subseteq \pi_1(S^1,1)$. It follows from the Covering Unique Existence Theorem 127.17 that the natural covering $p\colon \widetilde{S^1}_{b_0,\{1\}} \to S^1$ is isomorphic (in the sense of Definition 127.16) to the usual covering $q\colon \mathbb{R} \to S^1$ that is given by $q(t) = \exp(2\pi\,\mathrm{i}t)$. In fact they can become isomorphic via the homeomorphism $\mathbb{R} \to \widetilde{S^1}_{b_0,\{1\}}$ that sends $x \in \mathbb{R}$ to the path $[0,1] \to S^1$ that is given by $t \mapsto \exp(2\pi\,\mathrm{i}tx)$. Now let $\rho\colon [0,1] \to S^1$ be the loop that is given by $t \mapsto \exp(2\pi\,\mathrm{i}t)$. One can easily show

[1325] The notation ρ^* is motivated by Exercise 127.31 where we will see that the natural universal coverings are in a suitable sense contravariant in the base point.

that under the homeomorphism ϕ^{-1} the diagram



becomes



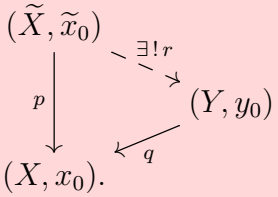
In summary we see that ρ^* acts non-trivially on $\widetilde{S^1}_{b_0,\{1\}}$. We will explore this action of loops on natural coverings in much greater detail in the subsequent Chapter 128 on the deck transformation group of coverings. □

127.9. **The universal covering.** In the Covering Unique Existence Theorem 127.17 we saw that subgroups of fundamental groups give rise to coverings which are essentially unique. An obvious choice of a subgroup is the trivial group and we immediately obtain the following corollary to the Covering Unique Existence Theorem 127.17.

Proposition 127.20. (Universal Covering Proposition) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. The following statements hold:

(1) There exists, up to isomorphism (in the sense of Definition 127.16), a unique path-connected covering
$$p\colon (\widetilde{X}, \widetilde{x}_0) \ \to \ (X, x_0)$$
such that $\widetilde{X}$ is simply connected.

(2) Let $q\colon (Y, y_0) \to (X, x_0)$ be another path-connected covering. There exists a unique covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:



Proof. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$.

(1) Note that if we apply the Covering Unique Existence Theorem 127.17 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to isomorphism, a unique path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the trivial group. It follows from the Covering-π_1-Monomorphism Proposition 92.7 that $\widetilde{X}$ is simply connected.

(2) Statement (2) is an immediate consequence of the Covering–Covering Theorem 127.15. ■

The Universal Covering Proposition 127.20 leads us to the following definition:

Definition 127.21. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected[1326] and let $x_0 \in X$.

(1) We refer to the natural covering $p\colon \widetilde{X}_{x_0} := \widetilde{X}_{x_0}^{\{e\}} \to X$ from Definition 127.18 as the natural universal covering of (X, x_0).

(2) We refer to any covering $p\colon (\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering of (X,x_0).

Example.

(1) For any $n\in\mathbb{N}$ the map
$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & (S^1)^n \\ (t_1,\dots,t_n) & \mapsto & \big(\exp(2\pi\,\mathrm{i}t_1),\dots,\exp(2\pi\,\mathrm{i}t_n)\big)\end{array}$$
is a universal covering of the n-dimensional torus $(S^1)^n$.

(2) Let $p,q\in\mathbb{N}$ be coprime. The natural projection $S^3\to S^3/(\mathbb{Z}/p\mathbb{Z})=L(p,q)$ is a universal covering of the lens space $L(p,q)$. □

Convention 127.22. In practice one often suppresses the actual map p from the language. For example, often one says that "$\mathbb{R}^n$ is a universal covering of $(S^1)^n$".

Example.

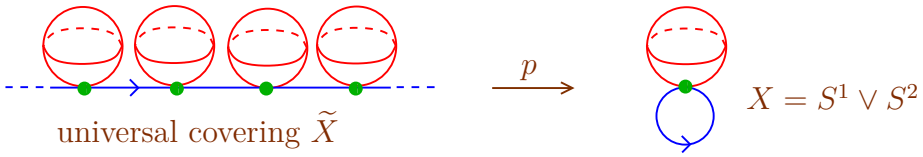
(1) Let $n\geq 2$. It follows from the Projective Space-Sphere Quotient Lemma 9.13 together with the Sphere-π_1-Proposition 88.24 that the universal covering of $\mathbb{R}\mathrm{P}^n$ is given by S^n.

(2) It follows from the Projective Space-Avatar Lemma 9.16 or the SU(2)-SO(3)-Covering Theorem 94.3 that a universal covering of SO(3) is given by SU(2).

(3) By the Klein Bottle-as-Quotient Lemma 91.2 we know that a universal covering of the Klein bottle is given by $\mathbb{R}^2$. □

127.10. **Explicit descriptions of the universal covering.** We have seen in the Universal Covering Proposition 127.20 that every "reasonable" topological space X admits a universal covering $p\colon\widetilde{X}\to X$. In the proof we gave an abstract construction. In the previous section we also gave an explicit description of the universal covering of some popular topological spaces, namely the n-dimensional torus $(S^1)^n$, the lens spaces, the projective spaces $\mathbb{R}\mathrm{P}^n$ with $n\geq 2$, SO(3) and the Klein bottle. In this section we will give explicit descriptions of the universal coverings of more complicated examples, namely of $X=S^1\vee S^2$ and of $X=S^1\vee S^1$.

Example. We consider $X=S^1\vee S^2$ where we identify $\mathrm{i}\in S^1$ with $(0,0,-1)\in S^2$. We also consider the topological space
$$\widetilde{X} \;=\; \big(\mathbb{R}\cup(\mathbb{Z}\times S^2)\big)\,/\,n\sim(n,(0,0,-1)),\quad n\in\mathbb{Z}.$$



[1325] Recall that most "reasonable" topological spaces (e.g. topological manifolds) are locally path-connected and semi-locally simply connected.

[1326] The name "universal covering" stems from the fact, proved in the Universal Covering Proposition 127.20 (2), that it has a universal property, namely the universal covering of a topological space X "covers" every other path-connected covering of X.

It is pretty straightforward to see that the map

$$\begin{array}{rcl} p\colon \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)) & \to & S^1 \vee S^2 \\ [P] & \mapsto & \begin{cases} \exp(\mathrm{i}(2\pi P + \frac{\pi}{2})), & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \in \mathbb{Z} \times S^2 \end{cases} \end{array}$$

is well-defined and in fact a covering. On page 1945 we showed that $\widetilde{X}$ is simply connected. Therefore $p\colon \widetilde{X} \to X$ is a universal covering of X. □

Now we turn to the universal covering of the wedge $X = S^1 \vee S^1$. By the discussion on page 1865 we know that X is homeomorphic to the topological realization of the unique abstract graph G that is given by the single vertex $*$ and the edge set $\{x, y\}$.[1327] It is convenient to work in the following in the setup of abstract graphs.

Convention 127.23. Let V be a set and let $E \subseteq V \times V$ be a subset. The pair (V, E) defines an abstract graph (V, E, i, t) by setting $i(g, h) = g$ and $t(g, h) = h$. We use this observation to view such (V, E) as an abstract graph.

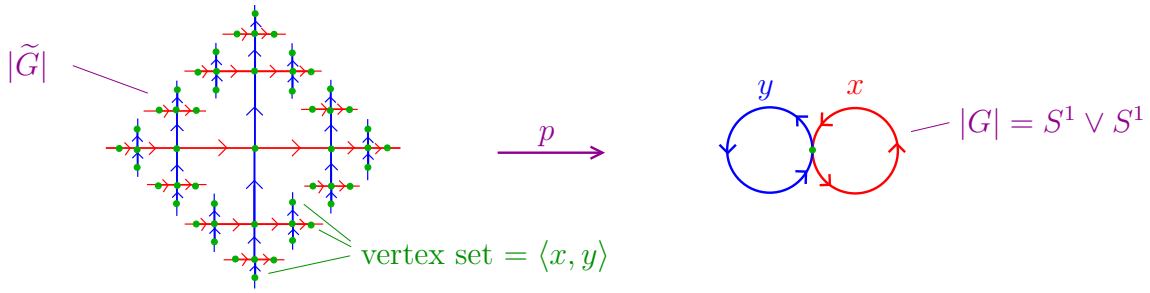
Proposition 127.24. (Wedge-of-Circles–Universal Covering Proposition) We consider the abstract graph $G = (\{*\}, \{x, y\})$. We also consider the abstract graph

$$\widetilde{G} \;=\; (\underbrace{\widetilde{V} = \text{ free group } \langle x, y\rangle}_{\text{vertex set}}, \underbrace{\widetilde{E} = \{(g,h) \in \widetilde{V} \times \widetilde{V} \,|\, h = g \cdot x \text{ or } h = g \cdot y\}}_{\text{edge set}}).$$

The maps

$$\begin{array}{rcl} \text{vertex set of } \widetilde{G} & \to & \text{vertex set of } G \\ g & \mapsto & * \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{edge set of } \widetilde{G} & \to & \text{edge set of } G \\ (g,h) & \mapsto & g^{-1} \cdot h \end{array}$$

define a morphism of graphs in the sense of Definition 11.3. The corresponding map $p\colon |\widetilde{G}| \to |G|$ is a universal covering of $|G| \cong S^1 \vee S^1$.



Proof. We leave it to the reader to verify, say using Exercise 91.14, that the given map $p\colon |\widetilde{G}| \to |G|$ is a covering. It remains to show that $|\widetilde{G}|$ is simply connected. This requires some preparations.

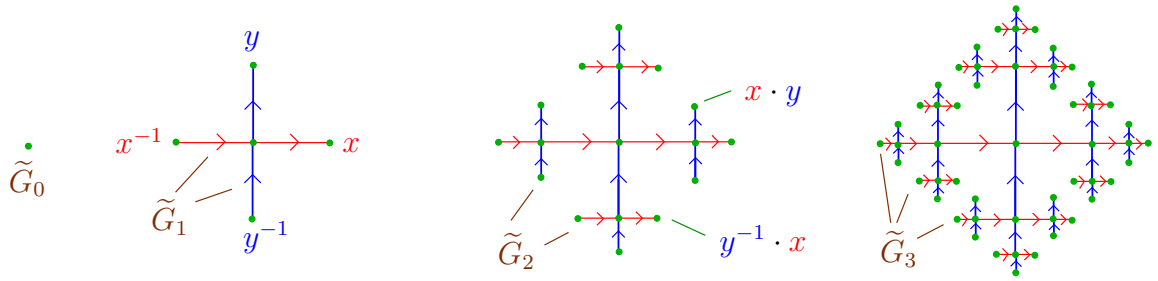
Given $g \in \widetilde{V} =_{\mathrm{i}} \langle x, y\rangle$ we define its length to be

$$l(g) \;:=\; \min\{k \in \mathbb{N}_0 \,|\, g = a_1 \cdot \ldots \cdot a_k \text{ for some } a_1, \ldots, a_k \in \{x^{\pm 1}, y^{\pm 1}\}\}.$$

Given $k \in \mathbb{N}_0$ we consider the subgraph $\widetilde{G}_k = (\widetilde{V}_k, \widetilde{E}_k)$ of $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ that is given by $\widetilde{V}_k := \{g \in \widetilde{V} =_{\mathrm{i}} \langle x, y\rangle \,|\, l(g) \leq k\}$ and $\widetilde{E}_k := \widetilde{E} \cap \mathcal{P}(\widetilde{V}_k \times \widetilde{V}_k)$.

[1327] Since we have a unique vertex we do not need to specify the maps from the edge set to the vertex set.

Claim. For each $k \in \mathbb{N}_0$ we have $\chi(\widetilde{G}_k) = 1$.[1328]



Proof. We prove the claim by induction on $k \in \mathbb{N}_0$. First note that $\widetilde{G}_0$ has a single vertex and no edge. Thus we see that $\chi(\widetilde{G}_0) = 1$. Now suppose that we know that $\chi(\widetilde{G}_k) = 1$. We make the following two observations:

(1) It follows easily from the definition of the free product $\langle x, y\rangle = \langle x\rangle * \langle y\rangle$ as a set of reduced words, see Definition 98.1, that each $g \in \widetilde{V}_{k+1} \setminus \widetilde{V}_k$ is uniquely of the form $a_1 \cdot \ldots \cdot a_{k+1}$ with $a_1, \ldots, a_{k+1} \in \{x^{\pm 1}, y^{\pm 1}\}$ and with $g := a_1 \cdot \ldots \cdot a_k \in \widetilde{V}_k$. Note that *precisely one* of $(g, g \cdot a_{k+1})$ or $(g \cdot a_{k+1}, g)$ lies in $\widetilde{E}_{k+1}$.

(2) Let $(g, h) \in \widetilde{E}_{k+1}$. It follows again easily from the definition of $\langle x, y\rangle = \langle x\rangle * \langle y\rangle$ as a set of reduced words that precisely one of g and h lies in $\widetilde{V}_k$ and $\widetilde{V}_{k+1}$.

Thus we see that

$$\chi(\widetilde{G}_{k+1}) \;=\; \chi(\widetilde{V}_k) + \underbrace{\#(\widetilde{V}_{k+1} \setminus \widetilde{V}_k) - \#(\widetilde{E}_{k+1} \setminus \widetilde{E}_k)}_{\substack{= 0, \text{ since by the above two observations} \\ \text{we have a bijection between the} \\ \text{sets } \widetilde{V}_{k+1} \setminus \widetilde{V}_k \text{ and } \widetilde{E}_{k+1} \setminus \widetilde{E}_k}} \;=\; \chi(\widetilde{G}_k) \;=\; 1.$$

⊞

It follows from the claim together with the Tree Characterization Proposition 11.28 that each $\widetilde{G}_k$ is a tree. The Finite Tree–Contractible Proposition 96.10 together with the Homotopy Equivalence-π_1-Proposition 90.9 now implies that each $|\widetilde{G}_k|$ is simply connected. It follows easily from the Exhaustion–Simply Connected Proposition 99.16 (or alternatively the Exhaustion-π_1-Proposition 105.2) that $|\widetilde{G}| = \bigcup_{k\in\mathbb{N}} |\widetilde{G}_k|$ itself is simply connected. ■

The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Therefore it is natural to ask the following question.

Question 127.25. What does the universal covering of the surface of genus $g \geq 2$ look like?

We will provide an interesting and hopefully surprising answer in Hadamard's Theorem 132.14.

Finally we note that in the Cyclic-Covering Proposition 234.21 we will get to know a method for constructing explicit coverings of smooth manifolds corresponding to kernels of homomorphisms of the form $\pi_1(M) \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}_0$.

127.11. **Appendix: More on universal coverings.** In this appendix we state and prove two technical propositions on universal coverings which will be useful on several occasions. These get hidden in the appendix since as standalone results they are arguably not particularly interesting.

After this warm paragraph, let us formulate the first of the two propositions.

[1328] Here χ denotes the Euler characteristic, i.e. the number of vertices minus number of edges.

Proposition 127.26. (Lift-to-Universal Coverings Proposition) Let X and Y be topological spaces that are path-connected, locally path-connected and semi-locally simply connected. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be universal coverings. Finally suppose that we are given a continuous map $f\colon X \to Y$ and points $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{y}_0 \in \widetilde{Y}$ with $p(\widetilde{y}_0) = f(x_0)$.

(1) There exists a unique continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ with $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

(2) If f is a homeomorphism, then $\widetilde{f}$ is also a homeomorphism.

(3) Suppose that X and Y are smooth manifolds and that p and q are coverings of smooth manifolds.[1329]
   (a) If f is smooth (respectively a diffeomorphism), then $\widetilde{f}$ is also smooth (respectively a diffeomorphism). The analogous statement also holds for complex manifolds.
   (b) If X and Y are oriented, and if p and q are orientation-preserving, then f is orientation-preserving if and only if $\widetilde{f}$ is orientation-preserving.

Remark.

(1) The Lift-to-Universal Coverings Proposition 127.26 is a generalization of Exercise 95.2 (a).

(2) In the Lift–Action Proposition 128.18 we will prove that the map $\widetilde{f}$ has an interesting extra property. □

Proof.

(1) We consider the map $g := f \circ p\colon \widetilde{X} \to Y$. Since $\widetilde{Y}$ is a universal covering, since $\pi_1(\widetilde{X}, \widetilde{x}_0) = 0$ and since X is locally path-connected and semi-locally simply connected we obtain from the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 that there exists a unique lift $\widetilde{g}\colon \widetilde{X} \to \widetilde{Y}$ of g with $\widetilde{g}(\widetilde{x}_0) = \widetilde{y}_0$. This map is precisely the map we were trying to find.

(2) This statement follows easily from the uniqueness statement of (1). We will fill in the details in Exercise 127.19.

(3) Note that coverings of smooth manifolds are in particular local diffeomorphisms. The statement now follows easily from (1) and (2), together with the Smooth Pasting Proposition 25.19. The case of complex manifolds is treated in almost entirely the same way. ■

We conclude this section with a more general statement. Recall that in the Action–π_1-Theorem 93.1 we saw that if X is a simply connected topological space and if G is a group that acts continuously and discretely on X, then the fundamental group of the quotient space X/G is isomorphic to G. Now we will show that, under modest extra hypotheses, the converse holds. More precisely, we have the following proposition.

[1329] We refer to Definition 95.3 for the definition of a covering between smooth manifolds.

Proposition 127.27. (Topological Space-as-Group Quotient Proposition) Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. We write $\pi = \pi_1(Y, y_0)$. Then there exists a simply connected topological space X, namely the universal covering of Y, and a continuous and discrete action of π on X such that Y is homeomorphic to X/π.

Proof. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$ be a base point. In the Covering Existence Theorem 127.10 we already showed that Y admits a simply connected covering. We recall the explicit construction given in the proof of the Covering Existence Theorem 127.10. Thus we consider

$$P \coloneqq \{f \in Y^{[0,1]} \mid f(0) = y_0\} = \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in P$ we write

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] = e \in \pi_1(Y, y_0)$$

and we put $X = P/\sim$. As in the Whisker Topology Lemma 127.12 given a point $x = [f] \in X = P/\sim$ and an open path-connected neighborhood V of $f(1)$ we define

$$U(x, V) \coloneqq \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subseteq X = P/\sim .$$

Furthermore, as in the Whisker Topology Lemma 127.12 we endow X with the whisker topology, i.e. with the topology generated by these sets $U(x, V)$.

Claim. The map

$$\begin{array}{rcl} \pi_1(Y, y_0) \times X & \to & X \\ ([g], [u]) & \mapsto & [g * u] \end{array}$$

is a well-defined action.

Proof. We first show that the map is independent of the choice of u. So suppose that $u, v \in P$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition in the definition of $\sim$. We have

$$[(g * u) * (\overline{g * v})] = [g * u * \overline{v} * \overline{g}] = [g] * \underbrace{[u * \overline{v}]}_{= e \in \pi_1(Y, y_0)} * [g]^{-1} = e \in \pi_1(Y, y_0).$$

It is straightforward to see that the map does not depend on the choice of representative g. Finally we want to show that it is indeed an action. Thus let $[g], [h] \in \pi_1(Y, y_0)$. Then

$$[g] \cdot ([h] \cdot [u]) = [g] \cdot [h * u] = [g * (h * u)] = [(g * h) * u] = [g * h] \cdot [u] = ([g] \cdot [h]) \cdot [u].$$

⊞

Now we need to show the following:

(1) the group $\pi_1(Y, y_0)$ acts continuously and discretely on $X = P/\sim$,
(2) there exists a homeomorphism $f \colon X/\pi_1(Y, y_0) \to Y$.

We do this in the following two steps:

(1) It follows immediately from the definition of the whisker topology on $X = P/\sim$ and from the Subbasis-Continuity Proposition 4.4 that the action is continuous.

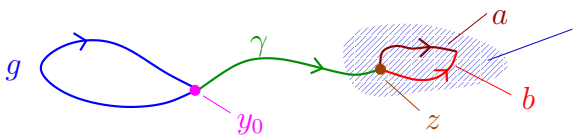
Now we need to show that the action is discrete. So let $x = [\gamma] \in P/\sim$. We set $z \coloneqq \gamma(1) \in Y$. Since Y is semi-locally simply connected there exists an open neighborhood V of $z = \gamma(1)$ such that the inclusion-induced map $\pi_1(V, z) \to \pi_1(Y, z)$

is trivial. It suffices to show that for $[g] \in \pi_1(Y, y_0)$ we have

$$[g] \cdot U(x,V) \cap U(x,V) \neq \varnothing \quad \Longrightarrow \quad [g] = e \in \pi_1(Y, y_0).$$

So suppose that $[g] \cdot U(x,V) \cap U(x,V) \neq \varnothing$. This means that there exist paths a and b in V, starting from z such that $g * \gamma * a$ and $\gamma * b$ have the same endpoint and they are path-homotopic in Y. But then

$$[g] \;=\; [g] \cdot \underbrace{[\gamma] \cdot \overbrace{[a * \bar{b}]}^{=e \in \pi_1(Y,y_0) = \{e\}} * [\gamma]^{-1}}_{=e \in \pi_1(Y,y_0)} \;=\; [(g * \gamma * a) * \overline{(\gamma * b)}] \;=\; e \;\in\; \pi_1(Y, y_0).$$



open neighborhood V of z such that $\pi_1(V, z) \to \pi_1(Y, z)$ is trivial

(2) We consider the map

$$\begin{array}{rcl} f\colon X/\pi_1(Y, y_0) & \to & Y \\ [\gamma] & \mapsto & \gamma(1). \end{array}$$

It is straightforward to see that this map is a bijection. It follows immediately from the Topological-Quotient Proposition 8.2 (1b) that the map is continuous and it follows from the Subbasis-Open Map Lemma 4.14 that the map is open. Thus it is a homeomorphism. ■

127.12. Exercises for Chapter 127.

Exercise 127.1.

(a) Let $\{X_i\}_{i \in I}$ be a family of topological manifolds. Show that any wedge $\bigvee_{i \in I} (X_i, x_i)$ is locally contractible.

Remark. Use the Wedge–Subspace Lemma 96.3 and the Wedge–Homotopy Lemma 96.8.

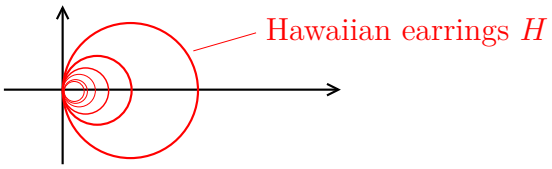
(b) Let $\{X_i\}_{i \in I}$ be a family of locally contractible topological spaces. Is the wedge $\bigvee_{i \in I} (X_i, x_i)$ necessarily locally contractible?

Exercise 127.2. Let H be the Hawaiian earring that we introduced in Exercise 105.1 and that we show in the figure below to the left.

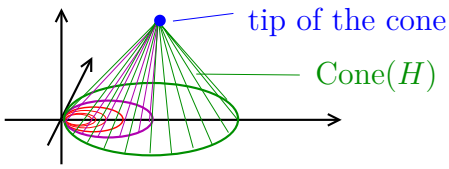
(a) Show that H is locally path-connected.
(b) Show that H is not locally simply connected.
(c) Show that $\mathrm{Cone}(H)$ is semi-locally simply connected.
(d) Show that $\mathrm{Cone}(H)$ is not locally simply connected.

Exercise 127.3.

(a) Show that every topological graph is locally contractible.
(b) Let $G = (V, E, i, t)$ be a connected abstract graph, let $v \in V$ and let $\Gamma \subseteq \pi_1(|G|, v)$ be a subgroup. Show that there exists an abstract graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$, a vertex $\widetilde{v} \in \widetilde{V}$ and a morphism $f\colon \widetilde{G} \to G$ of abstract graphs with $f(\widetilde{v}) = v$ such that $|f|\colon |\widetilde{G}| \to |G|$ is a covering with $|f|_*(\pi_1(|\widetilde{G}|, \widetilde{v})) = \Gamma$.

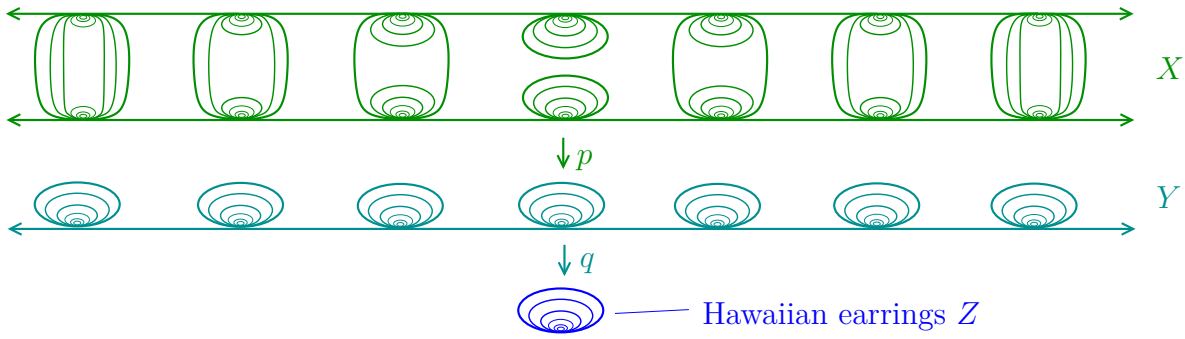


H is locally path-connected but not semi-locally simply connected



Cone(H) is semi-locally simply connected but not locally simply connected

Exercise 127.4. Let $p\colon X \to Y$ and $q\colon Y \to Z$ be the maps shown in the figure below. Now that p is a 2-fold covering and that q is an infinite covering. Show that the composition $q \circ p\colon X \to Z$ is *not* a covering.



Exercise 127.5. On page 195 we introduced the Topologist's Sine Curve

$$X \;:=\; \{(0,y)\,|\,y\in[-1,1]\} \cup \{(x,\sin(\tfrac{1}{x}))\,|\,x\in(0,\tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2.$$

As in Exercise 8.4 we refer to quotient space $Z := X/(0,0)\sim(\frac{1}{\pi},0)$ as the *Warsaw Circle*. We consider the map

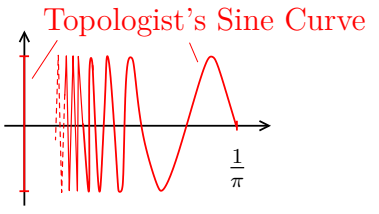
$$\begin{array}{rcl} f\colon Z & \to & S^1 \\ [(x,y)] & \mapsto & \exp(2\pi^2\cdot x\cdot \mathrm{i}) \end{array}$$

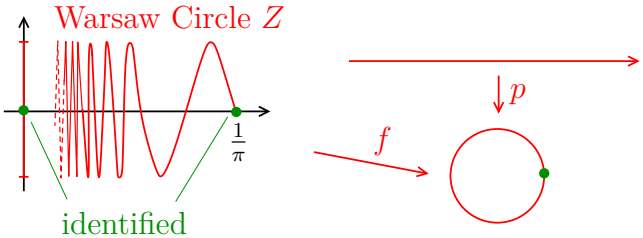
and the covering map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(t\cdot \mathrm{i}). \end{array}$$

(a) Show that $f\colon Z\to S^1$ is continuous.
(b) Show that there is no continuous map $\widetilde{f}\colon Z\to\mathbb{R}$ with $p\circ\widetilde{f}=f\colon Z\to S^1$.

Remark. This Exercise shows that in the Map Lifting Criterion 127.5 we cannot drop the hypothesis that Z is locally path-connected.





Exercise 127.6. Let $p\colon X\to B$ be a covering of topological spaces and let Z be a topological space that is path-connected and locally path-connected. Furthermore let $F\colon Z\times[0,1]\to B$ be a continuous map. Finally let $\widetilde{f}\colon Z\to X$ be a lift of $f:=F_1\colon Z\to B$. Show that there exists a lift $\widetilde{F}\colon Z\times[0,1]\to X$ with $\widetilde{F}_1:=\widetilde{f}$.

Exercise 127.7. We consider the figure below.

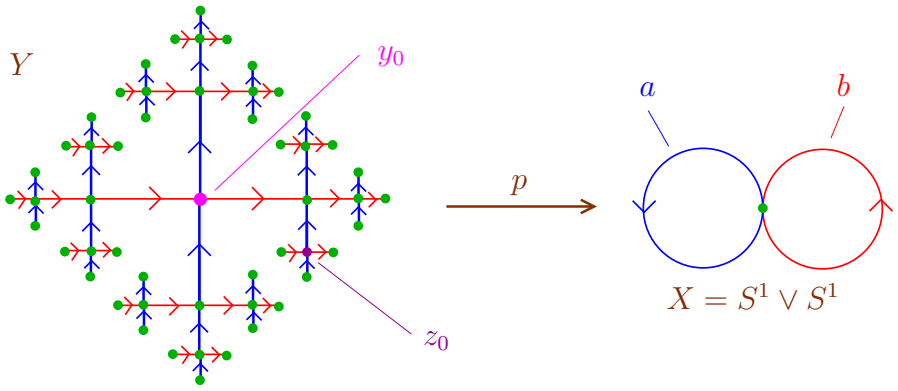
(a) Let X be the wedge of two circles and let $p\colon Y \to X$ be the universal covering that is illustrated in the figure. Denote the usual generators of $\pi_1(X)$ by a and b. By an abuse of notation we denote by a and b also representing loops. What does the lift of the loop $a * b * \overline{a} * \overline{b}$ to Y to the starting point y_0 look like?

(b) We continue with the notation from (1). Now consider the loop

$$\gamma := a * a * b * a * a * \overline{b} * \overline{a} * b * b * \overline{a} * \overline{a} * \overline{b} * \overline{a} * \overline{b}.$$

We denote by $\widetilde{\gamma}$ the lift of γ to the starting point y_0. Is y_0 the endpoint of $\widetilde{\gamma}$?

(c) Does there exist a homeomorphism $f\colon Y \to Y$ with $f(y_0) = z_0$?



Exercise 127.8. Let $X = S^1 \vee S^1$ be the wedge of two circles. Let $*$ be the wedge point and let $\theta\colon \langle x, y\rangle \to \pi_1(S^1 \vee S^1, *)$ be the obvious isomorphism. Next let $\varphi\colon \langle x, y\rangle \to \mathbb{Z}$ be the epimorphism that is given by $\varphi(x) = \varphi(y) = t$. Describe the covering of $X = S^1 \vee S^1$ corresponding to $\theta(\ker(\varphi))$.

Exercise 127.9. Let $X = S^1 \vee S^1$ be the wedge of two circles. Let $*$ be the wedge point and let $\theta\colon \langle x, y\rangle \to \pi_1(S^1 \vee S^1, *)$ be the obvious isomorphism. Describe the covering of $X = S^1 \vee S^1$ corresponding to the subgroup $\theta(\langle x\rangle)$.

Exercise 127.10.

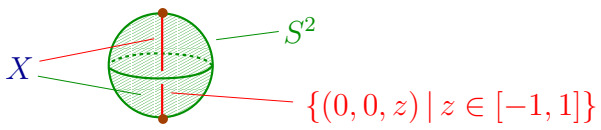
(a) Let $X = \mathbb{R} \cup \{*\}$ be the line with two zeros that we introduced on page 152. Show that X is locally simply connected.

(b) By (a) and the Universal Covering Proposition 127.20 we know that X admits a covering $p\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is simply connected. Give an explicit description of $\widetilde{X}$.

(c) Is $\widetilde{X}$ Hausdorff?

(d) What is the isomorphism type of $\pi_1(X)$?

Exercise 127.11. We consider the topological space $X = S^2 \cup \{(0, 0, z) \mid z \in [-1, 1]\}$. Give an explicit description of the universal covering of X as a subspace of $\mathbb{R}^3$.



Exercise 127.12.

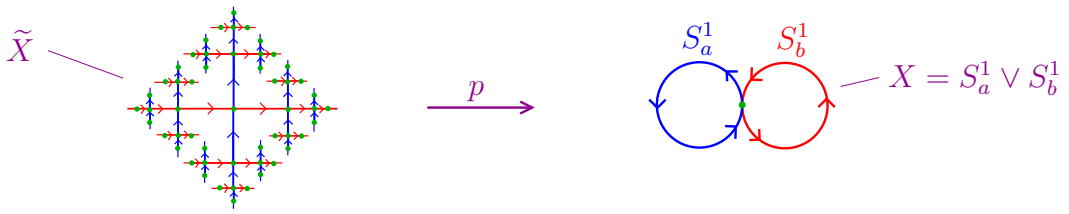
(a) Let $T = S^1 \times S^1$ be the torus. Classify all 2-fold coverings of T up to equivalence.
(b) Let $K = S^1 \times S^1$ be the Klein bottle. Classify all 2-fold coverings of K up to equivalence.

Exercise 127.13. We consider $X = \mathbb{R}\mathrm{P}^2 \vee S^1$ with the base point $*$ given by the wedge point. Note that we have an isomorphism $\pi_1(X,*) \cong \pi_1(\mathbb{R}\mathrm{P}^2) * \pi_1(S^1) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$.

(a) Let $\Gamma \subseteq \pi_1(X,*)$ be the subgroup corresponding to the kernel of the natural epimorphism $\pi_1(X,*) \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$. Give an explicit description of the covering corresponding to Γ.
(b) Let $\Gamma \subseteq \pi_1(X,*)$ be the subgroup corresponding to the kernel of the natural epimorphism $\pi_1(X,*) \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ and let $p\colon \widetilde{X} \to X$ be the covering corresponding to Γ. Show that $\widetilde{X}$ is homeomorphic to a subspace of $\mathbb{R}^3$.

Exercise 127.14. Let $X = S^1_a \vee S^1_b$ be the wedge of two circles. Let $\Gamma \subseteq \pi_1(X,*) = \langle a, b\rangle$ be the commutator subgroup. What does the covering $p\colon \widetilde{X} \to X$ corresponding to Γ look like? You can draw it as a subspace of $\mathbb{R}^2$.
Hint. You have seen $\widetilde{X}$ many times in your life.

Exercise 127.15. Let $X = S^1_a \vee S^1_b$ be the wedge of two circles. In the figure below we show the universal covering of X. How do you need to modify this figure to obtain the covering corresponding to the subgroup $\mathbb{Z} \cong \pi_1(S^1_a) \subseteq \pi_1(X,*) \cong \pi_1(S^1_a) * \pi_1(S^1_b)$?



Exercise 127.16. Let M be the Möbius band. For which $k \in \mathbb{N}$ does there exist a k-fold covering $p\colon N \to M$ such that N is an oriented connected smooth manifold?

Exercise 127.17. By the Matrix Group-Smooth Manifold Proposition 32.8 we know that $\mathrm{SL}(2,\mathbb{R})$ is a 3-dimensional smooth manifold.

(a) Show that $\pi_1(\mathrm{SL}(2,\mathbb{R})) \cong \mathbb{Z}$.
(b) Show that the universal covering $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is diffeomorphic to $\mathbb{R}^3$.
Suggestion. You could make use of the Polar Decomposition Theorem 239.18.

Exercise 127.18. Let X be a non-empty topological space that is path-connected and locally simply connected. Furthermore let $f\colon X \to X$ be a homeomorphism. We denote by $M \coloneqq \mathrm{Tor}(X,f) \coloneqq (X \times [0,1])/(x,0) \sim (f(x),1)$ the corresponding mapping torus from Definition 21.19.

(a) Let $x_0 \in X$. We equip M with the base point $y_0 = [(x_0,0)]$. Given $n \in \mathbb{N}$ we consider the homomorphism $\varphi\colon \pi_1(M,y_0) \to \pi_1(S^1,1) =_{\mathrm{i}} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.
  (i) Give an explicit description of a covering $p\colon (\widetilde{M},\widetilde{y}_0) \to (M,y_0)$ which satisfies $p_*(\pi_1(\widetilde{M},\widetilde{y}_0)) = \ker(\varphi_0)$.
  (ii) Let $n \in \mathbb{N}$. Let $p\colon \widetilde{M} \to M$ be a covering with $p_*(\pi_1(\widetilde{M}),\widetilde{y}_0) = \ker(\varphi_n)$. Show that $\widetilde{M}$ is homeomorphic to a mapping torus.

(b) Let $p\colon \widetilde{M} \to M$ be any finite covering. Show that $\widetilde{M}$ is homeomorphic to a mapping torus.

Exercise 127.19. Let X be a topological space and let $p\colon \widetilde{X} \to X$ be a covering. Let $f\colon X \to X$ be a homeomorphism and let $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ be a continuous map such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Show that $\widetilde{f}$ is also a homeomorphism.
Remark. This is of course the content of the Lift-to-Universal Coverings Proposition 127.26 (2). Note that this statement is also a generalization of Exercise 95.2 (b).

Exercise 127.20. Give an example of a topological space X, a finite covering $p\colon \widetilde{X} \to X$ and a homeomorphism $f\colon X \to X$ which does *not* lift to a continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$, i.e. such that there is no continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Exercise 127.21. Let X be a path-connected topological space that is locally path-connected and let $p\colon \widetilde{X} \to X$ be a finite covering of degree d.

(a) Let $f\colon X \to X$ be a self-homeomorphism of X. Show that there exists a continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that $p \circ \widetilde{f} = f^d \circ p$.
(b) Does the conclusion of (a) also hold, if we do not assume that f is a homeomorphism?

Exercise 127.22. Let X be a path-connected topological space that is locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. It follows from the the Covering Unique Existence Theorem 127.17 that there exists an, essentially unique, covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the commutator subgroup of $\pi_1(X, x_0)$. We refer this covering as the *universal abelian covering of* (X, x_0). Give an explicit description of the universal abelian cover of the following topological spaces:

(a) The Klein bottle K
(b) The wedge $X = S^1 \vee S^1$ of two circles. How does this generalize to more circles?
(c) The surface Y, which is given be the torus minus one open disk.
(d) The surface Z which is given by the sphere minus three open disks.

Evidently the expression "explicit" is a little vague, you could interpret it as an explicit description as a submanifold of say $\mathbb{R}^3$.

Exercise 127.23. Let X and Y be topological spaces that are path-connected, locally path-connected and semi-locally simply connected. Let $f\colon X \to Y$ be a continuous map, let $x_0 \in X$ and let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup. We consider the covering $p\colon \widetilde{X}_{x_0} \to X$

corresponding to the subgroup $\Gamma \subseteq \pi_1(X, x_0)$ and we consider the covering $q\colon \widetilde{Y}_{y_0} \to Y$ corresponding to the subgroup $f_*(\Gamma) \subseteq \pi_1(Y, y_0)$.

(a) Show that there exists a continuous map $\widetilde{f}\colon \widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow{\widetilde{f}} & \widetilde{Y}_{y_0} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

(b) Show that if $f\colon X \to Y$ is a homotopy equivalence, then any map $\widetilde{f}\colon \widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$ as in (a) is also a homotopy equivalence.

Remark. This exercise shows in particular that if X and Y are homotopy equivalent, then so are their universal coverings.

Exercise 127.24. Let X be a path-connected topological space that is locally path-connected and that is semi-locally simply connected. Furthermore let $p\colon \widetilde{X} \to X$ be the universal covering. Let $x_0 \in X$ be a base point and let $\varphi\colon \pi_1(X, x_0) \to \Gamma$ be a group homomorphism. We consider

$$\begin{array}{rcl} q\colon \widetilde{X}_\varphi \coloneqq (\widetilde{X} \times \Gamma)/\overbrace{(x,g) \sim (h \cdot x, \varphi(h) \cdot g)}^{x \in X, g \in \Gamma \text{ and } h \in \pi_1(X,x_0)} & \to & X \\ [(x,g)] & \mapsto & p(x) \end{array}$$

where we equip Γ with the discrete topology, $\widetilde{X} \times \Gamma$ with the product topology and $\widetilde{X}_\varphi$ with the quotient topology.

(a) Suppose that φ is an epimorphism. Show that $q\colon \widetilde{X}_\varphi \to X$ is the covering corresponding to $\ker(\varphi)$.

(b) In general, show that the number of components of $\widetilde{X}_\varphi$ has the same cardinality as $\operatorname{coker}(\varphi)$ and show that the restriction of $q\colon \widetilde{X}_\varphi \to X$ to each component of $\widetilde{X}_\varphi$ is the covering corresponding to $\ker(\varphi)$.

Exercise 127.25. Let X be a topological space and let $x_0 \in X$. We say that a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is *universal* if given any other covering $q\colon (Y, y_0) \to (X, x_0)$ (we do not demand that Y is path-connected) there exists a continuous map $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{x}_0) & & \\ {\scriptstyle p}\downarrow & \overset{r}{\searrow} & \\ & & (Y, y_0) \\ & \underset{q}{\swarrow} & \\ (X, x_0). & & \end{array}$$

Show that if X is path-connected, locally path-connected and semi-locally simply connected, then a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is universal in the above sense if and only if $\widetilde{X}$ is simply connected.

Remark 1. This exercise shows that the above definition of a universal covering is compatible with Definition 127.21.

Remark 2. In Exercise 127.26 we will see that in general a simply connected covering is not a universal covering in the above sense.

Exercise 127.26. On page 195 we introduced the Topologist's Sine Curve

$$S \;:=\; \{(0,y)\,|\,y\in[-1,1]\} \cup \{(x,\sin(\tfrac{1}{x}))\,|\,x\in(0,\tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2.$$

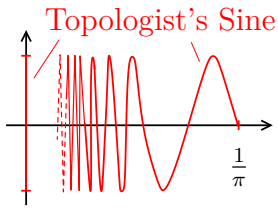
As in Exercise 8.4 we refer to quotient space $W := S/(0,0)\sim(\frac{1}{\pi},0)$ as the *Warsaw Circle*.

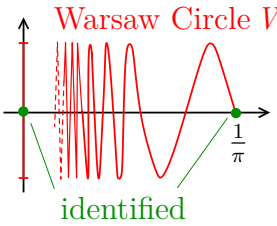
(a) Show that the Warsaw Circle is path-connected and simply connected.
Remark. This exercise is similar to Exercise 4.56 where we showed that the Topologist's Sine Curve S is connected but not path-connected.

(b) Show that the Warsaw Circle is not locally path-connected.

(c) We set $w_0 = [(0,0)]$. Provide an explicit example of a path-connected 2-fold covering $q\colon (Y,y_0)\to(W,w_0)$.

Remark. By (a) we know that (W,w_0) is simply connected. The example in (c) shows that the identity map $(W,w_0)\to(W,w_0)$ does not have the universal property stated in Exercise 127.25 respectively the Universal Covering Proposition 127.20, even though (W,w_0) is simply connected.





Exercise 127.27. We define the *Modified Warsaw Circle* to be the topological space

$$C \;:=\; X/(0,0)\sim(\tfrac{1}{\pi},0) \qquad \text{where } S \;:=\; \{(0,0)\}\cup\big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\tfrac{1}{\pi}]\big\} \;\subseteq\; \mathbb{R}^2.$$

As in Exercise 8.4 and 127.26 one can show that C is compact and simply connected.

(a) We define the *Topologist's Electrocardiogram* to be the topological space

$$E \;:=\; (S\times\mathbb{Z})/((\tfrac{1}{\pi},0),k)\sim((0,0),k+1) \quad \text{for } k\in\mathbb{Z}.$$

Show that the map

$$\begin{array}{rcl} p_E\colon E & \to & C \\ [(P,k)] & \mapsto & [P] \end{array}$$

is a covering.

(b) We consider the following subspace of $\mathbb{R}^2$:

$$L \;:=\; \Big(0\times\mathbb{Z} \;\sqcup\; \bigcup_{k\in\mathbb{Z}}\{(x,\tfrac{1}{x}+k)\,|\,x\in(0,\tfrac{1}{\pi}]\}\Big)\setminus(0,k)\sim(\tfrac{1}{\pi},k) \text{ for } k\in\mathbb{Z}.$$

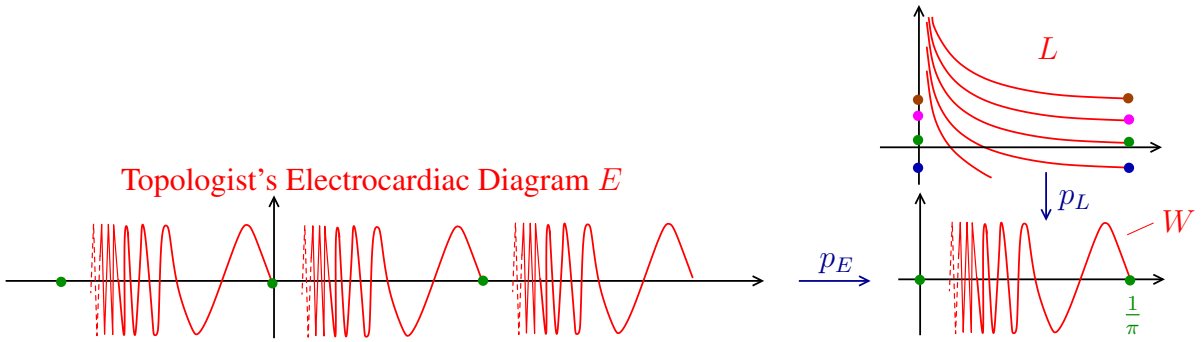
Show that the map

$$\begin{array}{rcl} p_L\colon L & \to & C \\ [(x,y)] & \mapsto & \begin{cases} [(0,0)], & \text{if } (x,y)\in\{0\}\times\mathbb{Z}, \\ [(x,\sin(\frac{1}{x}))], & \text{if } x>0. \end{cases} \end{array}$$

is a covering. This covering is called the *Löhlein covering*.

(c) Show that there is no continuous map $f\colon E\to L$ such that $p_L\circ f = p_E$.
Remark. This shows that the covering $p\colon E\to C$ is not a universal covering in any sense.

Exercise 127.28. Let G be a topological group that is connected and locally path-connected. Let $p\colon\widetilde{G}\to G$ be a covering of G and let $\widetilde{e}\in\widetilde{G}$ with $p(\widetilde{e})=e$. Show that there



exists a unique topological group structure on $\widetilde{G}$ with trivial element $\widetilde{e}$ that turns p into a group homomorphism.

Exercise 127.29.

(a) Show that every connected topological space X that admits a covering $X \to S^1 \times [0,1]$ is homeomorphic to $S^1 \times [0,1]$ or $\mathbb{R} \times [0,1]$.
(b) Show that every connected topological space X that admits a covering $X \to S^1 \times S^1$ is homeomorphic to $S^1 \times S^1$, $\mathbb{R} \times S^1$ or $\mathbb{R}^2$.

Exercise 127.30. Let X and Y be two topological spaces that are path-connected, locally path-connected and semi-locally simply connected. Show that if X and Y are homotopy equivalent, then so are their universal coverings.

Exercise 127.31. Given a topological space X the *fundamental groupoid* is defined as the category $\Pi(X)$ where

- The objects are points of X, i.e. $\mathrm{Ob}(\Pi(X)) = X$.
- Given $x, y \in \mathrm{Ob}(\Pi(X)) = X$ we define $\mathrm{Mor}_{\Pi(X)}(x,y)$ as the set of path-homotopy classes of paths from x to y. The composition of a morphism $[\alpha]$ from x to y and a morphism $[\beta]$ from y to z is hereby given by $[\beta] \circ [\alpha] := [\alpha * \beta]$.

Now we can formulate the following exercises:

(a) Show that $\Pi(X)$ is indeed a category.
(b) We suppose that X is path-connected, locally path-connected and semi-locally simply connected. Show that we have a *contravariant* functor $\Pi(X) \to \mathcal{T}op$ which on the level objects is given by $x \mapsto \widetilde{X}_{x_0}$.

Remark. This exercise has a significant overlap with Exercise 127.31.

CHAPTER 128

The deck transformation group

In this chapter we will introduce the deck transformation of a covering and we will discuss several features. In particular we will see that coverings fall naturally into two types: regular coverings and irregular coverings.

This chapter is organized as follows:

- In Section 128.1 we will introduce the deck transformation group of a covering.
- In Section 128.2 we will give a characterization of regular coverings.
- In Section 128.3 we will see that a suitable map between coverings induces a homomorphism of deck transformation groups.
- In Section 128.4 we will provide exercises for Chapter 128.

In the next Chapter 129 we will discuss several slightly random topics in covering theory.

128.1. **Definition of the deck transformation group.** The following definition introduces the main actor of this chapter.

Definition 128.1. Let $p\colon X \to B$ be a covering.

(1) A deck transformation is a homeomorphism $d\colon X \to X$ such that $p(d(x)) = p(x)$ for all $x \in X$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow[\cong]{d} & X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B. & \end{array}$$

The group of deck transformations[1330] is called the deck transformation group of the covering $p\colon X \to B$ and we denote it by $\mathrm{Deck}(p\colon X \to B)$.

(2) We say that the covering is regular if for every $b \in B$ the deck transformation group $\mathrm{Deck}(p\colon X \to B)$ acts transitively on $p^{-1}(b)$, i.e. if given any two points x, x' with $p(x) = p(x') = b$ there exists a deck transformation d with $d(x) = x'$. Otherwise we say that the covering is irregular.

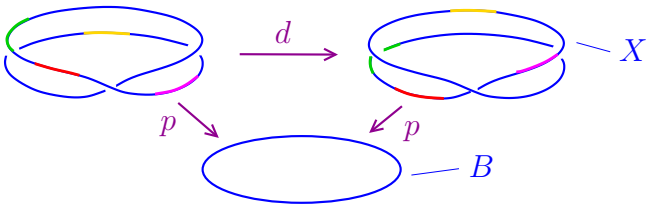
For completeness' sake we also mention the following definition even though we will barely make use of it:

(3) Let P be a property of isomorphism types of groups, e.g. cyclic, free, nilpotent, solvable etc. We say that the covering $p\colon X \to B$ is a P-covering if the covering is regular and if the deck transformation group has the property P.

Example.

(1) We consider the covering

[1330] Why do deck transformations form a group?



$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^2. \end{array}$$

The map $d\colon S^1 \to S^1$ given by $d(z) := -z$ is a deck transformation. It is now clear that the covering is regular.

(2) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$

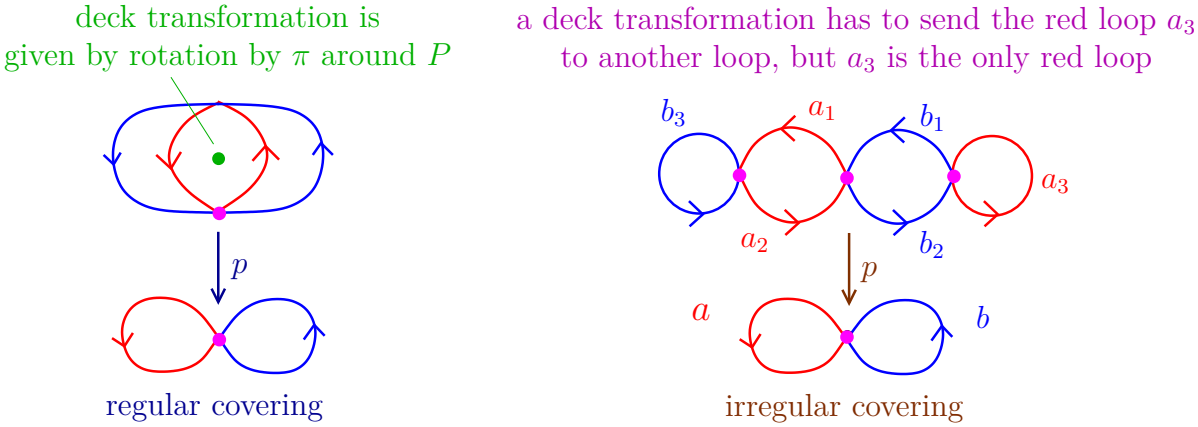
In the Preimage–Swapping Lemma 95.11 we saw that f is a deck transformation. It follows immediately that any 2-fold covering is regular.

(3) As an important special case of (2) consider a non-orientable path-connected smooth manifold M. We consider the orientation covering

$$\begin{array}{rcl} \widetilde{M} := \{(Q,O) \mid Q \in M \text{ and } O \text{ an orientation of } \mathrm{T}_Q M\} & \to & M \\ (Q,O) & \mapsto & Q \end{array}$$

from Definition 95.9. It is straightforward to show that the map $d\colon \widetilde{M} \to \widetilde{M}$ given by $d((Q,O)) := (Q,-O)$ is a deck transformation of p.

(4) We consider again the first two coverings of topological graphs from the figure on page 1784 which are also shown again in the figure below.



The covering maps are the ones suggested by the colors and the directions. We first make the following observations. A deck transformation d does the following:

(a) it sends a vertex again to a vertex,

(b) it sends a blue edge to a blue edge, preserving the orientation,

(c) it sends a red edge to a red edge, preserving the orientation,

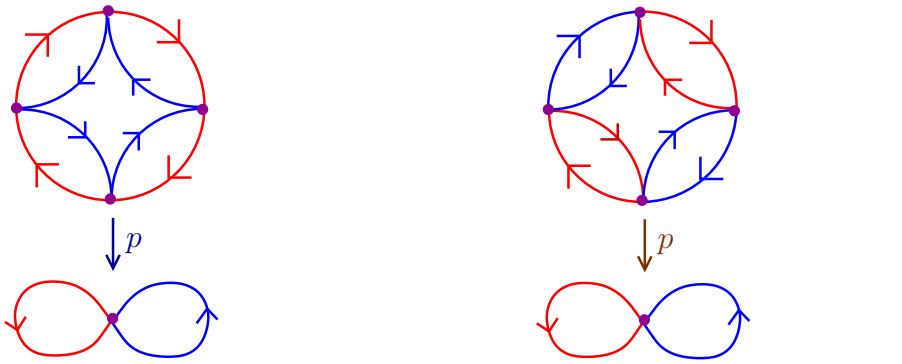
(d) if e is an edge with endpoints P and Q, then $d(e)$ is an edge with endpoints $d(P)$ and $d(Q)$, in particular if e is a loop, then $d(e)$ is also a loop.

Now we consider the two examples.

(a) The example on the left admits one non-trivial deck transformation, namely the map that is given by "rotation by π" around the center of the topological graph. The covering is now easily seen to be regular.

(b) The loop a_3 in the topological graph on the right is the only red loop, so any deck transformation has to send it to itself. It follows that there can be no deck transformation that sends the vertex to the right to any of the other vertices.[1331] Thus the covering on the right is irregular.

(5) Finally we consider the 4-fold coverings illustrated in the figure below.



deck transformation group $\mathbb{Z}_4$ deck transformation group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

They are both regular coverings. For the topological graph on the left the deck transformation group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, a generator is given by "rotation by $\frac{\pi}{2}$". The deck transformation group on the right is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, here one $\mathbb{Z}/2\mathbb{Z}$ summand is given by "rotation by π" and the other one corresponds to "swapping the inner and the outer circle". □

Lemma 128.2. (Deck Group–Discrete Action Lemma) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering between two path-connected topological spaces. The corresponding deck group $\mathrm{Deck}(p\colon X \to B)$ acts discretely (in the sense of Definition 91.1) and continuously on X.

Proof. By definition of a deck transformation the action is continuous. We will show in Exercise 128.1 that the action is discrete. ■

Lemma 128.3. (Deck Group-Free Action Lemma) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of path-connected topological spaces and let d_1 and d_2 be two deck transformations of p. Then the following holds

$$d_1 = d_2 \iff d_1(x_0) = d_2(x_0).$$

In other words, given any $b_0 \in B$ the deck transformation group $\mathrm{Deck}(p\colon X \to B)$ acts freely on $p^{-1}(b_0)$.

Proof. Evidently we only need to show the"$\Leftarrow$"-direction. Thus let d be a deck transformation of p. One way of stating that d is a deck transformation is to state that the

[1331] Does this covering admit any deck transformation?

following diagram commutes:

$$\begin{array}{ccc} & & (X, d(x_0)) \\ & \overset{d}{\nearrow} & \downarrow p \\ (X, x_0) & \xrightarrow[p]{} & (B, b_0). \end{array}$$

Put differently, the map d can be viewed as a lift of the map $(X, x_0) \to (B, b_0)$ to the covering $(X, d(x_0)) \to (B, b_0)$. From the Lift Uniqueness Proposition 92.2[1332] it follows that two deck transformations d_1 and d_2 of p with $d_1(x_0) = d_2(x_0)$ agree. ■

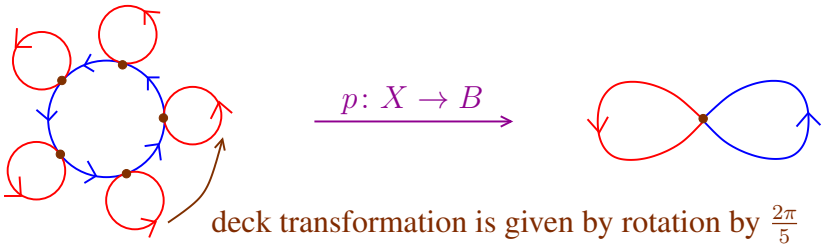
Example. We let $B = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1, a) \sim (1, b)$, i.e. B is the wedge of two circles and we consider

$$X \;=\; (S^1 \sqcup (S^1 \times \mathbb{Z}/5\mathbb{Z}))/(\exp(2\pi\,\mathrm{i}n/5) \sim (1, n)) \text{ for } n \in \mathbb{Z}/5\mathbb{Z},$$

i.e. X is a circle with five circles attached at the points $\exp(2\pi\,\mathrm{i}k/5)$, $k = 0, \ldots, 4$. Then the map

$$p\colon (S^1 \sqcup (S^1 \times \mathbb{Z}/5\mathbb{Z}))/\sim \;\to\; G = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1, a) \sim (1, b)$$

that is given by $p(z) = (z^5, a)$ for $z \in S^1$ and $p((z, n)) = (z, b)$ for $(z, n) \in S^1 \times \mathbb{Z}/5\mathbb{Z}$, is a covering map of degree five.



We consider the map

$$\begin{array}{ccccc} \mathbb{Z}/5\mathbb{Z} & \to & \mathrm{Deck}(p\colon X \to B) & & \\ & & X & \to & X \\ m & \mapsto & z & \mapsto & \exp(2\pi\,\mathrm{i}m/5) \cdot z, \quad \text{for } z \in S^1, \\ & & (z, n) & \mapsto & (z, m + n) \quad \text{for } (z, n) \in S^1 \times \mathbb{Z}/5\mathbb{Z} \end{array}$$

In the figure above this corresponds to "rotation of X by the angle $\frac{2\pi m}{5}$". It is straightforward to show that this map is well-defined, i.e. the maps on the right are deck transformations, and using the Deck Group-Free Action Lemma 128.3 one can easily see that this map is an isomorphism. Also one can now convince oneself easily that the covering $p\colon X \to B$ is regular. □

Lemma 128.4. (Group Action–Deck Transformation Lemma) Let X be a path-connected topological space on which a group G acts continuously and discretely. We consider the corresponding covering $p\colon X \to X/G$. Then $p\colon X \to X/G$ is regular and the

[1332] Here we use that X is path-connected.

map

$$\begin{array}{rcl} \Phi\colon G & \to & \mathrm{Deck}(p\colon X \to X/G) \\ g & \mapsto & \begin{pmatrix} X & \to & X \\ x & \mapsto & g\cdot x \end{pmatrix} \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to see that the maps $x \mapsto gx$ are indeed deck transformations. It follows almost immediately that the covering $p\colon X \to X/G$ is regular. Also it follows easily from the axioms of a group action that Φ is a homomorphism of groups. Furthermore it is clear that Φ is a monomorphism.

It remains to show that Φ is an epimorphism. So let $d \in \mathrm{Deck}(p\colon X \to X/G)$ be a deck transformation. We pick $x \in X$. Since $p(d(x)) = p(x)$ we have $d(x) = gx$ for some $g \in G$. Since X is path-connected it follows from the Deck Group-Free Action Lemma 128.3 that $d(y) = gy$ for all $y \in X$. This shows that Φ is an epimorphism. ■

Lemma 128.5. (2-Fold Covering–Regular Lemma) Any 2-fold covering of path-connected topological spaces is regular and the deck transformation group contains precisely two elements.

Proof. Let $p\colon X \to B$ be a 2-fold covering of path-connected topological spaces. By the Preimage–Swapping Lemma 95.11 we know that

$$\begin{array}{rcl} d\colon X & \to & X \\ x & \mapsto & \text{the unique other element of } p^{-1}(p(x)) \end{array}$$

is a deck transformation. Evidently it is regular. It follows easily from the Deck Group-Free Action Lemma 128.3 and our hypothesis that X, B are path-connected that any deck transformation of p is either the identity or it equals the above deck transformation d. ■

128.2. **Characterization of regular coverings.** We recall the following important, albeit somewhat technical, definition.

Definition 128.6. Let G be a group and let $H \subseteq G$ be a subgroup. The normalizer $N(H)$ of the subgroup $H \subseteq G$ is defined as

$$N(H) \;:=\; \{g \in G \mid g^{-1}Hg = H\}.$$

The following lemma summarizes a few elementary properties of the normalizer of a subgroup.

Lemma 128.7. (Normalizer Lemma) Let G be a group and let $H \subseteq G$ be a subgroup.

(1) The subgroup H of G is normal if and only if $N(H) = G$.
(2) The normalizer $N(H)$ of $H \subseteq G$ is a subgroup of G.
(3) The normalizer $N(H)$ of $H \subseteq G$ is the largest subgroup of G which contains H as a normal subgroup, in the sense that if A is a subgroup of G which contains H as a normal subgroup, then $A \subseteq N(H)$.

Proof. All three statements are easily verified. Alternatively see [**Isaa1994**, p. 26] for the elementary proof. ■

We return to topology and we consider one more example in detail.

Example. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. Furthermore let $\Gamma \subseteq \pi = \pi_1(Y, y_0)$ be a subgroup. By the Covering Existence Theorem 127.10 there exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$. We recall the construction of X. We first considered

$$P \coloneqq \{\text{all paths in } Y \text{ with starting point } y_0\}$$

and for $u, v \in W$ we wrote

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

We defined $X = P/\sim$ and the projection $p\colon P/\sim \to Y$ that is given by $[w] \mapsto w(1)$. We then showed that, with an appropriate topology on $P/\sim$, the map $p\colon P/\sim \to Y$ is a path-connected covering with $p_*(\pi_1(P/\sim, x_0)) = \Gamma$.

As in the proof of the Topological Space-as-Group Quotient Proposition 127.27 we now would like to construct deck transformations by precomposing elements in P by a loop in y_0. More precisely, let g be a loop in (Y, y_0). We consider the map

$$\begin{array}{ccc} P & \to & P \\ w & \mapsto & g * w \end{array}$$

We need to check that the map respects our equivalence relation $\sim$ on P. So suppose that $u, v \in P$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition. We have

$$[(g * u) * (\overline{g * v})] \;=\; [g * u * \overline{v} * \overline{g}] \;=\; [g] * \underbrace{[u * \overline{v}]}_{\in \Gamma} * [g]^{-1}.$$

In general it is not the case that the right-hand side lies again in Γ. But it is true if $[g]$ lies in the normalizer of Γ. As in the proof of the Topological Space-as-Group Quotient Proposition 127.27 it is now straightforward to see that the map

$$\begin{array}{rcl} \Psi\colon N(\Gamma)/\Gamma & \to & \mathrm{Deck}(p\colon X \to Y) \\ [g] & \mapsto & \begin{pmatrix} P/\sim & \to & P/\sim \\ [u] & \mapsto & [g * u] \end{pmatrix} \end{array}$$

is a well-defined homomorphism. We will see in the Deck Transformation–Normalizer Proposition 128.9 that this in fact an isomorphism of groups. □

Now we want to consider the general case of a covering. So let $p\colon (X, x_0) \to (B, b_0)$ be a covering of path-connected topological spaces. We write $\pi \coloneqq \pi_1(B, b_0)$ and we write $\Gamma \coloneqq p_*(\pi_1(X, x_0))$.[1333] Now the question arises whether there is a connection between the groups π, Γ and $\mathrm{Deck}(p\colon X \to B)$.

Definition 128.8. Similar to the proof of the Action–π_1-Theorem 93.1 we can consider the following map[1334]

$$\begin{array}{rcl} \Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) & \to & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{array}$$

[1333] Recall that by the Covering-π_1-Monomorphism Proposition 92.7 the induced map $p_*\colon \Gamma \to \pi$ is injective, thus $\Gamma \cong \pi_1(X, x_0)$.

It is straightforward to see that this map is well-defined, i.e. it does not depend on the choice of the path.

The question now arises, whether this map Φ is a group isomorphism. But in general this is not the case, after all, if Γ is not normal in π, then π/Γ is not even a group. But we have the following proposition:

Proposition 128.9. (Deck Transformation–Normalizer Proposition) Let X and B be two path-connected and locally path-connected topological spaces. Furthermore let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. We write $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$. The above map

$$\begin{aligned} \Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) &\to \pi/\Gamma \\ d &\mapsto \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{aligned}$$

restricts to an isomorphism of groups

$$\Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X \to B) \xrightarrow{\cong} N(\Gamma)/\Gamma$$

where the inverse is given by

$$\begin{aligned} N(\Gamma)/\Gamma &\to \mathrm{Deck}(p\colon X \to B) \\ [f\colon [0,1] \to B] &\mapsto \text{unique lift}^{1335} \text{ of } p\colon X \to B \text{ to } \widetilde{p}\colon X \to X \\ &\quad \text{with } \widetilde{p}(x_0) = \widetilde{f}(1), \text{ where } \widetilde{f}\colon [0,1] \to X \\ &\quad \text{is the unique lift of } f \text{ with } \widetilde{f}(0) = x_0 \end{aligned}$$

Proof. Throughout the proof let $p\colon (X, x_0) \to (B, b_0)$ be a covering between two path-connected topological spaces. In the following we write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$, $D := \mathrm{Deck}(p\colon X \to B)$ and $\Phi := \Xi_{(X,x_0)}$.

(A) First we have to show that $\Phi(D)$ lies indeed in $N(\Gamma)/\Gamma$. Thus let $d \in D$. We need to show that $\Phi(d)$ lies in $N(\Gamma)/\Gamma$. Thus let f be a loop in (X, x_0) and and let g be a path in X from x_0 to $d(x_0)$.



We have to verify that $[\overline{p \circ g}] \cdot [p \circ f] \cdot [p \circ g] \in \Gamma$. Indeed the following holds

$$\begin{aligned} [\overline{p \circ g}] \cdot [p \circ f] \cdot [p \circ g] &= \big[\overline{p \circ g} * p \circ f * p \circ g\big] = \big[p \circ \big(\overbrace{\overline{g} * f * g}^{\text{loop in } (X, d(x_0))}\big)\big] \\ &= \big[p \circ \underbrace{\big(d^{-1} \circ (\overline{g} * f * g)\big)}_{\text{loop in } (X, x_0)}\big] \in \underbrace{p_*(\pi_1(X, x_0))}_{=\Gamma}. \end{aligned}$$

$\uparrow$ we have $p \circ d = p$, thus we also have $p = p \circ d^{-1}$

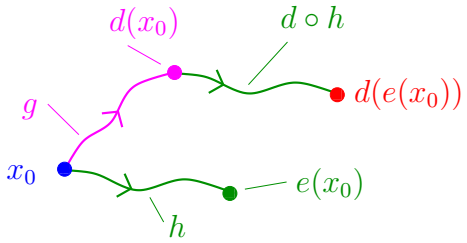
(B) Now we have to show that Φ is a group homomorphism. So let d and e be two deck transformations. We pick a path g from x_0 to $d(x_0)$ and we pick a path h from x_0

[1334] We denote hereby with $[p \circ f]$ not only the equivalence class of the loop $p \circ f$ in π, but also its equivalence class π/Γ, since the notation $[p \circ f]\Gamma$ is too cumbersome.

[1335] This unique lift is provided by the Covering–Covering Theorem 127.15.

to $e(x_0)$. Then

$$\Phi(d)\cdot\Phi(e) \underset{\text{by definition of } \Phi(e) \text{ and } \Phi(d)}{=} [p\circ g]\cdot[p\circ h] \underset{\text{since } p\circ d = p}{=} [p\circ g]\cdot[p\circ(d\circ h)] \overset{\text{the endpoint of } g \text{ equals the starting point of } d\circ h}{=} [g*(d\circ h)] \underset{\text{since } g*(d\circ h) \text{ is a path from } x_0 \text{ to } d(e(x_0)), \text{ see the figure below}}{=} \Phi(e\circ d).$$



(C) Now we will construct a map $\Psi\colon N(\Gamma)/\Gamma \to D$ in the inverse direction. Thus let $[f]\in N(\Gamma)\subseteq\pi=\pi_1(B,b_0)$ where f is a loop in (B,b_0). Let x_1 be the endpoint of the lift $\widetilde{f}$ of f to the starting point x_0.[1336] We have $p(x_1)=p\big(\widetilde{f}(1)\big)=f(1)=b_0$. Thus we obtain the following diagram of continuous maps of pointed topological spaces:

$$\begin{array}{ccc} & & (X,x_1) \\ & & \big\downarrow p \\ (X,x_0) & \xrightarrow{\ p\ } & (B,b_0). \end{array}$$

Then

$$p_*(\pi_1(X,x_1)) \underset{\text{by the Change-of-Base Point Proposition 88.14 since } f \text{ is a path from } x_0 \text{ to } x_1}{=} \big\{\big[p\circ\big(\overline{\widetilde{f}}*g*\widetilde{f}\big)\big]\ \big|\ [g]\in\pi_1(X,x_0)\big\} \underset{\text{since } [f]\in N(\Gamma)}{\begin{array}{l}=\ [f]\cdot p_*(\pi_1(X,x_0))\cdot[f]^{-1}\\ =\ p_*(\pi_1(X,x_0)).\end{array}}$$

Next note that it follows from the Map Lifting Criterion 127.5 that there exists a lift $\widetilde{p}\colon(X,x_0)\to(X,x_1)$ of p, i.e. there exists a continuous map $\widetilde{p}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & (X,x_1) \\ & \overset{\widetilde{p}}{\nearrow} & \big\downarrow p \\ (X,x_0) & \xrightarrow{\ p\ } & (B,b_0). \end{array}$$

We claim that $\widetilde{p}$ is a deck transformation. By definition it remains to prove the following claim:

Claim. The map $\widetilde{p}$ is a homeomorphism.

Proof. As above it follows from the Map Lifting Criterion 127.5 that there exists also a lift $\widehat{p}\colon(X,x_1)\to(X,x_0)$ of p, i.e. there exists a continuous map $\widehat{p}$ such that the following diagram commutes:

[1336] According to the Path-Lifting Proposition 92.6 (2) the definition of x_1 does not depend on the choice of the representative of $[f]\in N(\Gamma)$.

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\widehat{p}}{\nearrow} & \big\downarrow p \\ (X,x_1) & \xrightarrow{p} & (B,b_0). \end{array}$$

Note that $\widetilde{p}\circ\widehat{p}$ is a lift which sends x_1 to x_1. Another lift is given by the identity. It thus follows from the Lift Uniqueness Proposition 92.2 that $\widetilde{p}\circ\widehat{p}=\mathrm{id}_X$. The same argument shows that $\widehat{p}\circ\widetilde{p}=\mathrm{id}_X$. ⊞

We have thus constructed a map $N(\Gamma)\to D$. We still need to show that this map descends to a map $N(\Gamma)/\Gamma\to D$. So let $[f]\in N(\Gamma)$ and let $[g]\in\Gamma$. We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Similarly we define $g*f$. By the above we only need to show that the endpoints of $\widetilde{f}$ and $\widetilde{g*f}$ agree. In fact we have

$$\widetilde{g*f}(1) \underset{\substack{\uparrow\\ \text{by the Concatenation-of-Paths Lifting Lemma 127.6}}}{=} \begin{array}{c}\text{endpoint of the lift of } f\\ \text{to the starting point } \widetilde{g}(1)\end{array} \underset{\substack{\uparrow\\ \text{by the Loop–Lifting Proposition 92.8 since } [g]\in p_*(\pi_1(X,x_0))}}{=} \begin{array}{c}\text{endpoint of the lift of } f\\ \text{to the starting point } x_0\end{array} = \widetilde{f}(1)$$

(D) Now we want to show that Φ is a bijection. It suffices to show that $\Psi\circ\Phi=\mathrm{id}$ and $\Phi\circ\Psi=\mathrm{id}$.

First let $d\in D$. Then

$$\begin{aligned}\Psi(\Phi(d)) &= \Psi(\text{loop } f \text{ in } (B,b_0) \text{ that lifts to a path from } x_0 \text{ to } d(x_0))\\ &= \text{deck transformation that sends } x_0 \text{ to } d(x_0) \underset{\substack{\uparrow\\ \text{by the Deck Group-Free Action Lemma 128.3}}}{=} d\end{aligned}$$

This shows that $\Psi\circ\Phi$ is the identity.

Now let $g\in N(\Gamma)/\Gamma$. We pick a representative loop s in b_0 and we denote by $\widetilde{s}$ the lift of s to the starting point x_0. We denote by x_1 the endpoint of $\widetilde{s}$. Then

$$\Phi(\Psi(g)) = \big[p\circ\big(\text{path in } X \text{ from } x_0 \text{ to } \overbrace{\Psi(g)(x_0)}^{=x_1}\big)\big] \underset{\substack{\uparrow\\ \widetilde{s}\text{ is a path from } x_0 \text{ to } x_1}}{=} \big[p\circ\widetilde{s}\big] = [s] = g.$$

This shows that $\Phi\circ\Psi$ is the identity. ■

The following lemma shows that for group actions several of our isomorphisms are consistent:

Lemma 128.10. (Group Action–Deck Transformation-π_1-Lemma) Let (X,x_0) be a path-connected topological space on which a group G acts continuously and discretely. We consider the corresponding covering $p\colon X\to X/G$. The following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow[\cong]{\Psi} & \mathrm{Deck}(p\colon X\to X/G) \\ & {\scriptstyle\Theta_{x_0}}\searrow \quad \swarrow{\scriptstyle\Xi_{(X,x_0)}} & \\ & \pi_1(X/G,[x_0])/p_*(\pi_1(X,x_0)) & \end{array}$$

Here we consider the following isomorphisms:

$$\begin{array}{rcl}\Theta_{x_0}\colon G & \to & \pi_1(X/G,[x_0])/p_*(\pi_1(X,x_0)) \\ g & \mapsto & [p\circ(\text{path in } X \text{ from } x_0 \text{ to } g\cdot x_0)]\end{array} \quad \text{and} \quad \begin{array}{rcl}\Psi\colon G & \to & \mathrm{Deck}(p\colon X\to X/G) \\ g & \mapsto & \begin{pmatrix} X & \to & X \\ x & \mapsto & g\cdot x\end{pmatrix}\end{array}$$

from the Action–π_1-Theorem 93.1 and the Group Action–Deck Transformation Lemma 128.4 and we consider the isomorphism

$$\begin{array}{rcl}\Xi_{(X,x_0)}\colon \mathrm{Deck}(p\colon X\to X/G) & \to & \pi_1(X/G,[x_0])/p_*(\pi_1(X,x_0)) \\ d & \mapsto & \big[p\circ(\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]\end{array}$$

from the Deck Transformation–Normalizer Proposition 128.9.

Proof. The lemma follows immediately from the explicit definitions of the various maps. ∎

In most applications we will use the following special case of the Deck Transformation–Normalizer Proposition 128.9.

Proposition 128.11. (Deck Transformation–Normal Subgroup Proposition) In the following let (X,x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\gamma\colon \pi_1(X,x_0)\to G$ be an epimorphism onto a group G. We denote by $p\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ the covering corresponding to $\ker(\gamma)$ that is given by the Covering Existence Theorem 127.10 and that is unique up to isomorphism by the Covering Unique Existence Theorem 127.17. Then the map

$$\begin{array}{rcl}\Phi\colon \mathrm{Deck}(p\colon \widetilde{X}\to X) & \to & \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \xrightarrow{\gamma} G. \\ d & \mapsto & \big[p\circ(\text{path from } x_0 \text{ to } d(x_0))\big]\end{array}$$

is an isomorphism, where the inverse is given by

$$\begin{array}{rcl} G & \to & \mathrm{Deck}(p\colon \widetilde{X}\to X) \\ \gamma([f\colon[0,1]\to X]) & \mapsto & \text{unique lift of } p\colon\widetilde{X}\to X \text{ to } \widetilde{p}\colon\widetilde{X}\to\widetilde{X} \text{ with } \widetilde{p}(x_0)=\widetilde{f}(1), \\ & & \text{where } \widetilde{f}\colon[0,1]\to\widetilde{X} \text{ is the unique lift of } f \text{ with } \widetilde{f}(0)=\widetilde{x}_0\end{array}$$

Proof. First note that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))=\ker(\gamma)$ is a normal subgroup of $\pi_1(X,x_0)$. Therefore the normalizer of $\Gamma\coloneqq p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ equals $\pi_1(X,x_0)$. It now follows immediately from the Deck Transformation–Normalizer Proposition 128.9 that the homomorphism on the left is an isomorphism. Finally note that the homomorphism on the right is an isomorphism since $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))=\ker(\gamma)$. ∎

Convention 128.12. Let (X,x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected.

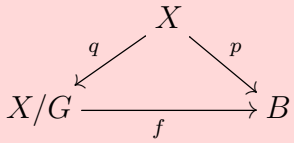
(1) Let $\gamma\colon\pi_1(X,x_0)\to G$ be an epimorphism onto some group G. Furthermore let $p\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ be a covering corresponding to $\ker(\gamma)$. We use the inverse of the isomorphism $\mathrm{Deck}(p\colon\widetilde{X}\to X)\xrightarrow{\cong} G$ from the Deck Transformation–Normal Subgroup Proposition 128.11 to equip $\widetilde{X}$ with a left G-action.

(2) Let $p\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ be a universal covering. We use the inverse of the isomorphism $\mathrm{Deck}(p\colon\widetilde{X}\to X)\xrightarrow{\cong}\pi_1(X,x_0)$ from the Deck Transformation–Normalizer Proposition 128.9 to equip $\widetilde{X}$ with a left $\pi_1(X,x_0)$-action.

The following corollary to the Deck Transformation–Normalizer Proposition 128.9 gives a complete characterization of regular coverings.

Corollary 128.13. (Regular Covering-Characterization Proposition) In the following let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces that are path-connected and locally path-connected. Then the following statements are equivalent:[1337]

(1) The covering is regular.
(2) The deck transformation group $\mathrm{Deck}(p\colon X \to B)$ acts transitively on $p^{-1}(b_0) \subseteq X$.[1338]
(3) The subgroup $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(B, b_0)$.
(4) There exists a group G that acts discretely on X (in the sense of Definition 91.1) and there exists a homeomorphism $f\colon X/G \to B$ such that the following diagram commutes:



where $q\colon X \to X/G$ is the natural projection.

If p is a finite covering then the above statements are furthermore equivalent to
(5) $\#\,\mathrm{Deck}(p\colon X \to B) = [X : B]$.[1339]

Remark. Let $p\colon (X, x_0) \to (B, b_0)$ be a 2-fold covering of path-connected pointed topological spaces. In the 2-Fold Covering–Regular Lemma 128.5 we saw that p is regular, which by the Regular Covering-Characterization Proposition 128.13 means that the index two subgroup $p_*(\pi_1(X, x_0))$ of $\pi_1(B, b_0)$ is normal. In fact we already showed in the Index 2–Subgroup Lemma 95.12 that any index 2 subgroup Γ of a group π is normal. □

Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$ and $D := \mathrm{Deck}(p\colon X \to B)$. The following four steps prove the equivalence of (1), (2) (3) and (4).

(1)⇔(2). Recall that a covering is regular if for every $b \in B$ the group D acts transitively on $p^{-1}(b)$. Thus (1)⇒(2) is trivial. Now suppose that d acts transitively on $p^{-1}(b_0)$. Let b be some other point in B. Since B is path-connected there exists a path s from b to b_0.

Now let x, x' be two points in $p^{-1}(b)$. We denote by $\widetilde{s}$ the lift of s to the starting point x and similarly we define $\widetilde{s}'$. We denote by x_0 and x_0' the endpoints of $\widetilde{s}$ and $\widetilde{s}'$. Since D acts transitively on $p^{-1}(b_0)$ there exists a deck transformation d with $d(x_0) = x_0'$. It follows from the Lift Uniqueness Proposition 92.2 that we have $d \circ \widetilde{s} = \widetilde{s}'$. But then we also have that d turns the starting point of $\widetilde{s}$ into the starting point of $\widetilde{s}'$, i.e. we have $d(x) = x'$.

(2)⇔(3). We consider again the map

$$\begin{aligned} \Phi\colon \mathrm{Deck}(p\colon X \to B) &\to \pi/\Gamma \\ d &\mapsto \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{aligned}$$

[1337]Note that being regular does not depend on the choice of $b_0 \in B$. Thus, by the equivalence of the statements, we see that if Statements (2) and (3) hold for a single b_0, they hold for any point in B.
[1338]Note that by the Deck Group-Free Action Lemma 128.3 we already know that the deck transformation group acts freely on $p^{-1}(b_0)$.
[1339]It follows from the Deck Transformation–Normalizer Proposition 128.9 together with the Covering-Degree Proposition 92.10 that we always have $\#\,\mathrm{Deck}(p\colon X \to B) \leq [X : B]$.

We first prove the following claim:

Claim. We have the following equivalence of statements:

$$D \text{ acts transitively on } p^{-1}(b_0) \subseteq X \quad \Leftrightarrow \quad \Phi\colon D \to \pi/\Gamma \text{ is surjective.}$$

Proof. First suppose that D acts transitively on $p^{-1}(b_0) \subseteq X$. Let $g \in \pi/\Gamma$ and pick a loop f that represents g. We denote by $\widetilde{f}$ the unique lift of f to the starting point x_0 and we denote by $\widetilde{x}_0$ the endpoint of $\widetilde{f}$. Since D acts transitively there exists a deck transformation d with $d(x_0) = \widetilde{x}_0$. Now we see that

$$\Phi(d) \;=\; [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))] \underset{\substack{\uparrow \\ \text{since } \widetilde{f} \text{ is a path from } x_0 \text{ to } d(x_0)}}{=} [p \circ \widetilde{f}] \underset{\substack{\uparrow \\ \text{since } \widetilde{f} \text{ is a lift of } f}}{=} [f] \;=\; g.$$

Now suppose that $\Phi\colon D \to \pi/\Gamma$ is surjective. Let $\widetilde{x}_0 \in p^{-1}(b_0)$. We pick a path $\widetilde{f}$ from x_0 to $\widetilde{x}_0$ and we denote by f the loop $p \circ \widetilde{f}$. Since Φ is surjective there exists a deck transformation d with $\Phi(d) = [f]$. But then $d(x_0) = \widetilde{x}_0$. ⊞

Now we have the following equivalences

$$\begin{aligned} D \text{ acts transitively on } p^{-1}(b_0) \subseteq X &\underset{\substack{\uparrow \\ \text{by the Deck Transformation} \\ \text{Normalizer Proposition 128.9}}}{\Longleftrightarrow} \Phi\colon D \to \pi/\Gamma \text{ is surjective} \\ &\Longleftrightarrow \pi = N(\Gamma) \underset{\substack{\uparrow \\ \text{by the definitions}}}{\Longleftrightarrow} \Gamma \text{ is normal in } \pi. \end{aligned}$$

(1)⇒(4). We will show that the deck transformation group D has the desired properties.

We first show that the deck transformation group acts discretely on X in the sense of Definition 91.1. So let $x \in X$. We write $b = p(x)$. Since $p\colon X \to B$ is a covering there exists a connected open neighborhood U of b that is uniformly covered. This means that the restriction of p to each component of $p^{-1}(U)$ is a homeomorphism. Now let V be the component of $p^{-1}(U)$ that contains x. Then $gV \cap V = \varnothing$ for each non-trivial $g \in D$. This shows that the action of D is discrete.

The map $p\colon X \to B$ has the property that $p(gx) = p(x)$ for all $g \in D$. Therefore the map p factors through a map $f\colon X/D \to B$. The map f is clearly surjective. Furthermore it is injective since we suppose that D acts transitively on each $p^{-1}(b)$. Finally the map f is not only continuous but, as follows easily from the Covering–Basics Lemma 91.12 (1), it is also open. Thus f is a homeomorphism.

(4)⇒(1). Now suppose that such a homeomorphism $f\colon X/G \to B$ exists. First note that for each $g \in G$ the map $x \mapsto gx$ not only defines a deck transformation of the covering $q\colon X \to X/G$ but also of the covering $p\colon X \to B$. Now let $b \in B$ and let $x, x' \in X$ with $p(x) = p(x')$. Since f is a homeomorphism we also have $q(x) = q(x')$. Hence there exists a $g \in G$ with $gx = x'$.

Now we suppose that p is a finite covering. We show the following equivalence:

(5)⇔(3). We have the following equivalences:[1340]

$$\begin{aligned} \#D = [X : B] &\underset{\substack{\downarrow \\ \text{by the Covering-Degree} \\ \text{Proposition 92.10}}}{\Longleftrightarrow} \#D = [\pi : \Gamma] \underset{\substack{\downarrow \\ \text{by the Deck Transformation–Normalizer} \\ \text{Proposition 128.9}}}{\Longleftrightarrow} [N(\Gamma) : \Gamma] = [\pi : \Gamma] \\ &\Longleftrightarrow N(\Gamma) = \pi \quad \Longleftrightarrow \Gamma \subseteq \pi \text{ is a normal subgroup.} \end{aligned}$$

■

[1340] Where do we actually use in the argument that the covering is finite?

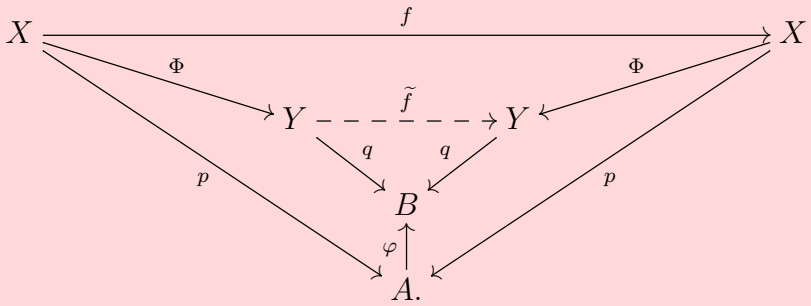
128.3. **Induced deck transformations.** The main goal of this section is to prove the following proposition:

Proposition 128.14. (Induced-Deck Transformation Proposition) Let X, Y, A and B be topological spaces that are path-connected and locally path-connected. Furthermore let $p\colon X \to A$ be a covering and let $q\colon Y \to B$ be a *regular* covering. Finally let $\varphi\colon A \to B$ and let $\Phi\colon X \to Y$ be continuous maps such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\varphi} & B. \end{array}$$

In this setting the following two statements hold:

(1) Given any $f \in \mathrm{Deck}(p\colon X \to A)$ there exists a unique $\widetilde{f} \in \mathrm{Deck}(q\colon Y \to B)$ such that the following diagram commutes:



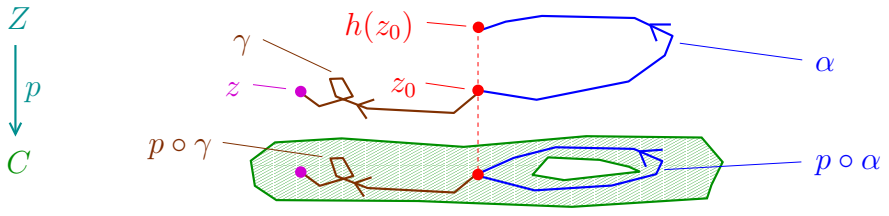
(2) The map $f \mapsto \widetilde{f}$ from (1) defines a group homomorphism

$$\mathrm{Deck}(p\colon X \to A) \;\to\; \mathrm{Deck}(q\colon Y \to B).$$

In the proof of the Induced-Deck Transformation Proposition 128.14 we will use the following lemma which gives a sometimes useful description of a deck transformation.

Lemma 128.15. (Explicit Deck Transformation Lemma) Let $r\colon (Z, z_0) \to (C, c_0)$ be a covering and let $h\colon Z \to Z$ be a deck transformation. Let $z \in Z$. We pick a path $\alpha\colon [0,1] \to Z$ from z_0 to $h(z_0)$ and we pick a path $\gamma\colon [0,1] \to Z$ from z_0 to z. Then

$h(z)$ = endpoint of the unique lift of $(p \circ \alpha) * (p \circ \gamma)\colon [0,1] \to C$ to the starting point z_0.



Proof of Lemma 128.15. Note that

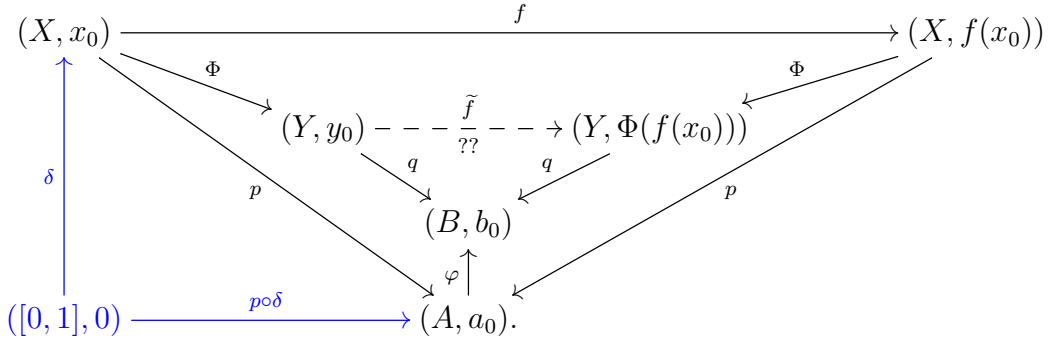
endpoint of the unique lift of $(p \circ \alpha) * (p \circ \gamma)\colon [0,1] \to C$ to the starting point z_0
$= \underset{\uparrow}{}$ endpoint of $\alpha * (h \circ \gamma) \;=\; (h \circ \gamma)(1) \;=\; h(\gamma(1)) \;=\; h(z)$.

since h is a deck transformation we see that $h \circ \gamma$ is the lift of $p \circ \gamma$ to starting point $\alpha(1) = h(z_0)$

■

Proof of the Induced-Deck Transformation Proposition 128.14.

(1) We pick $x_0 \in X$. We set $a_0 = p(x_0)$, $y_0 \coloneqq \Phi(x_0)$ and $b_0 \coloneqq q(y_0)$. We consider the following diagram of continuous maps between pointed topological spaces:



(The maps in blue will only be explained a little later on.)

Claim 1. We have $q_*(\pi_1(Y, y_0)) = q_*(\pi_1(Y, \Phi(f(x_0))))$.

Proof. Since Y is path-connected we can pick a path $\gamma\colon [0,1] \to Y$ with $\gamma(0) = y_0$ and $\gamma(1) = \Phi(f(x_0))$. Then

$$q_*(\pi_1(Y, y_0)) \underset{\uparrow}{=} \overbrace{[q \circ \gamma]}^{\in \pi_1(B, b_0)} \cdot q_*(\pi_1(Y, \Phi(f(x_0)))) \cdot [q \circ \gamma]^{-1} \underset{\uparrow}{=} q_*(\pi_1(Y, \Phi(f(x_0)))).$$

by the Change-of-Base Point Proposition 88.14

follows from the Regular Covering-Characterization Proposition 128.13, since $q\colon Y \to B$ is a regular covering ⊞

Since $q_*(\pi_1(Y, y_0)) = q_*(\pi_1(Y, \Phi(f(x_0))))$ and since Y is locally path-connected we obtain from the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 a unique continuous map $\widetilde{f}\colon (Y, y_0) \to (Y, \Phi(f(x_0)))$ such that $q = q \circ \widetilde{f}$, i.e. such that the small triangle commutes. It remains to prove the following claim:

Claim 2. The quadrilateral on top also commutes.

Proof. Let $x \in X$. We need to show that $(\widetilde{f} \circ \Phi)(x) = (\Phi \circ f)(x)$.

(a) We pick a path $\alpha\colon [0,1] \to X$ from x_0 to $f(x_0)$ and we pick a path $\gamma\colon [0,1] \to X$ from x_0 to x.

(b) Evidently $\Phi \circ \alpha\colon [0,1] \to Y$ is a path from $y_0 = \Phi(x_0)$ to $\Phi(f(x_0))$ and evidently $\Phi \circ \gamma\colon [0,1] \to Y$ is a path from $y_0 = \Phi(x_0)$ to $\Phi(x)$.

(c) Let $\delta\colon [0,1] \to X$ be the unique lift of

$$(p \circ \alpha) * (p \circ \gamma)\colon [0,1] \to A$$

to the starting point x_0.

(d) Note that

$$q \circ \Phi \circ \delta \underset{\uparrow}{=} \varphi \circ p \circ \delta \underset{\uparrow}{=} \varphi \circ ((p \circ \alpha) * (p \circ \gamma)) = (\varphi \circ p \circ \alpha) * (\varphi \circ p \circ \gamma) \underset{\uparrow}{=} (q \circ \Phi \circ \alpha) * (q \circ \Phi \circ \gamma).$$

since $q \circ \Phi = \varphi \circ p$

since δ is a lift of $(p \circ \alpha) * (p \circ \gamma)$

since $q \circ \Phi = \varphi \circ p$

(e) It follows from (d) that $\Phi \circ \delta$ is the (necessarily unique) lift of $(q \circ \Phi \circ \alpha) * (q \circ \Phi \circ \gamma)$ to the starting point y_0.

In summary we see that

$$\widetilde{f}(\Phi(x)) \underset{\uparrow}{=} (\Phi \circ \delta)(1) = \Phi(\delta(1)) \underset{\uparrow}{=} \Phi(f(x)) = (\Phi \circ f)(x).$$

by the Explicit Deck Transformation Lemma 128.15 with (b) and (e)

by the Explicit Deck Transformation Lemma 128.15 and by the choice of δ

⊞

(2) This statement follows easily from Statement (1). We leave it to the reader to fill in all details. ■

We continue with a more technical lemma.

Lemma 128.16. (Deck-to-π_1-Functoriality Lemma) Let X, Y, A and B be topological spaces that are path-connected and locally path-connected. Furthermore let $p\colon X \to A$ be a covering and let $q\colon Y \to B$ be a *regular* covering. Finally let $\varphi\colon A \to B$ and let $\Phi\colon X \to Y$ be continuous maps such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\varphi} & B. \end{array}$$

Let $x_0 \in X$. We write $a_0 = p(x_0)$, $y_0 = \Phi(x_0)$ and $b_0 = q(\Phi(x_0))$. In this setting the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Deck}(p\colon X \to A) & \xrightarrow{\Xi_{(X,x_0)}} & \pi_1(A,a_0)/p_*(\pi_1(X,x_0)) \\ \downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{Deck}(q\colon Y \to B) & \xrightarrow{\Xi_{(Y,y_0)}} & \pi_1(B,b_0)/q_*(\pi_1(Y,y_0)), \end{array}$$

where the horizontal maps are the maps we introduced in Definition 128.8, where the left vertical map is the map that is given by the Induced-Deck Transformation Proposition 128.14 and the right vertical map is induced by $\varphi_*\colon \pi_1(A,a_0) \to \pi_1(B,b_0)$.[1341]

Proof. The lemma follows immediately from the definitions. ■

The following proposition tells us how one can obtain in a relatively explicit way "all" coverings from the universal covering. It can be viewed as a refinement of the Topological Space-as-Group Quotient Proposition 127.27.

Proposition 128.17. (Universal Covering–Quotient Proposition) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $x_0 \in X$ and let $p\colon \widetilde{X} \to X$ be a universal covering. We pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. We endow $\widetilde{X}$ with the $\pi_1(X,x_0)$-action that we introduced on page 2487. Let $\Gamma \subseteq \pi_1(X,x_0)$ be a subgroup. The map

$$\begin{array}{rcl} q\colon \widetilde{X}/\Gamma & \to & X \\ [\widetilde{x}] & \mapsto & p(x) \end{array}$$

is a covering with $q_*(\pi_1(\widetilde{X}/\Gamma, [\widetilde{x}_0])) = \Gamma$.

[1341] It follows from $q \circ \Phi = \varphi \circ p$ that $\varphi_*(p_*(\pi_1(X,x_0))) \subseteq q_*(\pi_1(Y,y_0))$.

Proof. By the Deck Group–Discrete Action Lemma 128.2 we know that the action of Γ on $\widetilde{X}$ is discrete and continuous. Thus it follows from the Action–Covering Proposition 91.11 (2) that the map $q\colon \widetilde{X}/\Gamma \to X$ is indeed a covering. In the Action–π_1-Theorem 93.1 we gave an explicit isomorphism $\pi_1(\widetilde{X}/\Gamma, [\widetilde{x}_0]) \cong \Gamma$. One can easily verify that this map is precisely q_*. ■

Note that the following proposition is a refinement of the Lift-to-Universal Coverings Proposition 127.26.

Proposition 128.18. (Lift–Action Proposition) Let X and Y be topological spaces that are path-connected, locally path-connected and semi-locally simply connected. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be universal coverings. Furthermore suppose that we are given a continuous map $f\colon X \to Y$ and points $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{y}_0 \in \widetilde{Y}$ with $p(\widetilde{x}_0) = x_0$ and with $q(\widetilde{y}_0) = f(x_0)$. Note that by the Lift-to-Universal Coverings Proposition 127.26 there exists a unique continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ with $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

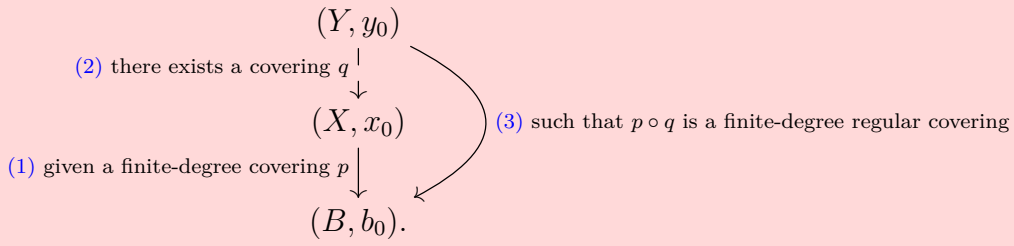
We equip $\widetilde{X}$ with the action by $\pi_1(X, x_0)$ and we equip $\widetilde{Y}$ with the action by $\pi_1(Y, y_0)$ that we introduced on page 2487. The map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ has the following property: Given any $\widetilde{x} \in \widetilde{X}$ and given any $g \in \pi_1(X, x_0)$ we have

$$\widetilde{f}(g \cdot \widetilde{x}) \;=\; \underbrace{f_*(g)}_{\in \pi_1(Y, y_0)} \cdot \widetilde{f}(\widetilde{x}).$$

Proof. Once one has digested the definition this statement follows immediately from the definitions. We leave it to the reader to fill in the details. ■

We conclude this chapter with the following proposition which often, rather conveniently, allows us to replace an irregular covering by a regular covering.

Proposition 128.19. (Replace-by-Regular Covering Proposition) In the following let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces that are path-connected, locally path-connected and semi-locally simply connected. If p is a finite-degree covering, then there exists a covering $q\colon (Y, y_0) \to (X, x_0)$ such that $p \circ q\colon (Y, y_0) \to (B, b_0)$ is a finite-degree *regular* covering. For convenience this statement is summarized in the following diagram:



Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a finite-degree covering of pointed topological spaces that are path-connected, that are locally path-connected and that are semi-locally simply connected. In the following we write $G := \pi_1(B, b_0)$ and $H := p_*(\pi_1(X, x_0))$. By the

Covering-Degree Proposition 92.10 we know that H is a finite-index subgroup of G. It follows from the Normal Core Lemma 130.17 that the normal core H^c of H in G is a finite-index normal subgroup of G that is contained in H. Since B is locally path-connected and semi-locally simply connected we obtain from the Covering Existence Theorem 127.10 a finite-degree path-connected covering $r\colon (Y, y_0) \to (B, b_0)$ of pointed topological spaces such that $r_*(\pi_1(Y, y_0)) = H^c$.

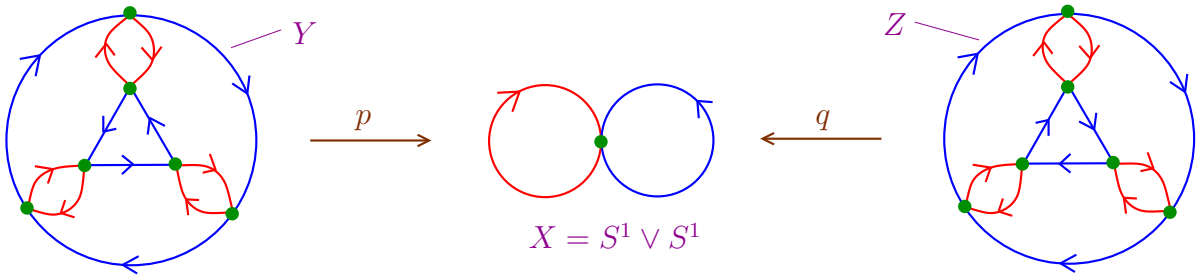
The existence of the desired covering map $q\colon (Y, y_0) \to (X, x_0)$ with $r = p \circ q$ is an immediate consequence of the Covering Unique Existence Theorem 127.17 and the fact that H^c is contained in H. ■

128.4. **Exercises for Chapter 128.**

Exercise 128.1. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of path-connected topological spaces. Show that the action of the deck group $\mathrm{Deck}(p\colon X \to B)$ on X is discrete.

Exercise 128.2. In the figure below we show two coverings $p\colon Y \to X = S^1 \vee S^1$ and $q\colon Z \to X = S^1 \vee S^1$.

(a) Determine which coverings are regular.
(b) For the regular covering(s) in (a), determine the isomorphism type of the deck transformation group.



Exercise 128.3. Let Σ be the surface of genus 2. Does there exist a regular covering $p\colon \widetilde{\Sigma} \to \Sigma$ such that the deck transformation group is isomorphic to the permutation group S_3?

Exercise 128.4. Let G be a finite group. Show that there exists a closed orientable 2-dimensional smooth manifold that admits a free and transitive action by G.

Exercise 128.5. Let $p\colon \widetilde{X} \to X$ be a covering with deck transformation group G and let $K \subseteq \widetilde{X}$ be a compact subset. We suppose that X is Hausdorff and regionally compact. Show that the set $\{g \in G \,|\, g \cdot K \cap K \neq \varnothing\}$ is finite.
Remark. In [**Scot1978**, Lemma 1.2] claims that the same conclusion holds without the hypothesis that X is regionally compact. Try to prove this more general statement.

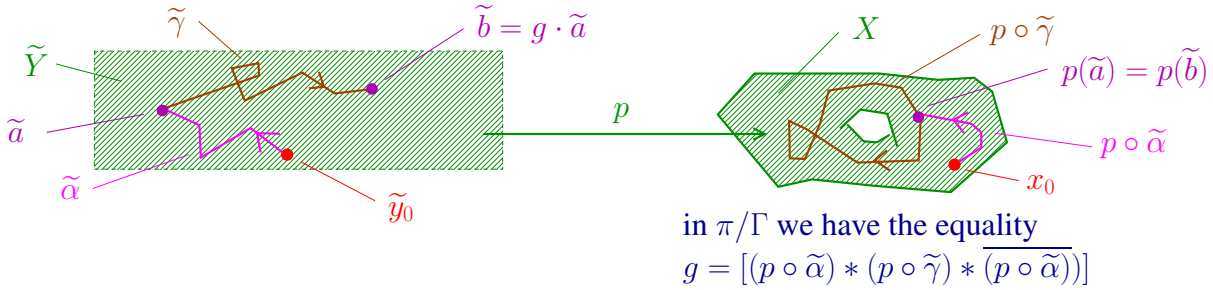
Exercise 128.6. Verify the Lift–Action Proposition 128.18.

Exercise 128.7. Let $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ be a regular covering between two topological spaces that are path-connected and locally contractible. We write $\pi := \pi_1(X, x_0)$ and

$\Gamma := p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$. We equip $\widetilde{Y}$ with the π/Γ-left action from the Deck Transformation–Normalizer Proposition 128.9. Now let $\widetilde{a}, \widetilde{b}$ be two points in $\widetilde{Y}$ with $p(\widetilde{a}) = p(\widetilde{b})$. We pick a path $\widetilde{\alpha}\colon [0,1] \to \widetilde{Y}$ from $\widetilde{y}_0$ to $\widetilde{a}$ and we pick a path $\widetilde{\gamma}\colon [0,1] \to \widetilde{Y}$ from $\widetilde{a}$ to $\widetilde{b}$. Show that with

$$g \;:=\; [\underbrace{(p\circ\widetilde{\alpha}) * (p\circ\widetilde{\gamma}) * \overline{(p\circ\widetilde{\alpha})}}_{\text{loop in } (X,x_0)}] \;\in\; \pi/\Gamma$$

we have $g \cdot [\widetilde{a}] = [\widetilde{b}]$.

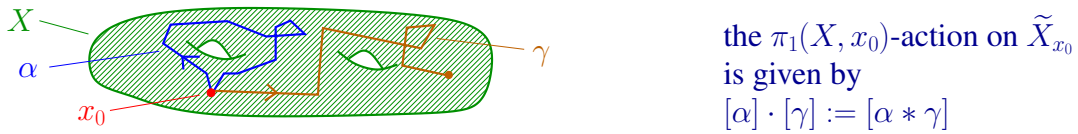


Exercise 128.8. Let X be a topological space that is locally contractible and connected and let $x_0 \in X$. As in the proof of the Covering Existence Theorem 127.10 we consider the topological space

$$\widetilde{X}_{x_0} \;=\; \{\gamma\colon [0,1] \to X \,|\, \gamma(0) = x_0\}/\!\sim \qquad \text{where } \alpha \sim \beta \text{ if } \alpha(1) = \beta(1) \text{ and } \alpha * \beta \text{ is null-homotopic.}$$

We denote by $p\colon \widetilde{X}_{x_0} \to X$ the covering map that is given by the evaluation map $[\gamma] \mapsto \gamma(1)$ In the Deck Transformation–Normalizer Proposition 128.9 we can an explicit isomorphism $\mathrm{Deck}(p\colon \widetilde{X}_{x_0} \to X) \cong \pi_1(X, x_0)$. Show that the corresponding action of the fundamental group $\pi_1(X, x_0)$ on $\widetilde{X}_{x_0}$ is given by

$$\begin{array}{rcl} \pi_1(X,x_0) \times \widetilde{X}_{x_0} & \to & \widetilde{X}_{x_0} \\ ([\alpha],[\gamma]) & \mapsto & [\alpha * \gamma]. \end{array}$$



Exercise 128.9. Let Y be a topological space and let $f\colon Y \to Y$ be a homeomorphism. We consider the action of $\langle t \rangle$ on $Y \times \mathbb{R}$ that is given by

$$t^n \cdot (y,s) \;:=\; (f^n(y), y+s).$$

Show that the quotient $(Y \times \mathbb{R})/\langle t \rangle$ is naturally homeomorphic to the mapping torus $\mathrm{Tor}(Y,f) := (Y \times [0,1])/(y,0) \sim (f(y),1)$.
Remark. This exercise is identical to Exercise 21.16.

Exercise 128.10. Let (X, x_0) be a path-connected pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let

$f\colon (X, x_0) \to (S^1, 1)$ be a continuous map such that $f_*\colon \pi_1(X, x_0) \to \pi_1(S^1, 1)$ is an epimorphism. Let $p\colon \widetilde{X} \to X$ be the covering corresponding to the subgroup $\ker(f_*) \subseteq \pi_1(X, x_0)$ that is provided by the Covering Existence Theorem 127.10.

(a) Show that there exists a continuous map $\widetilde{f}\colon \widetilde{X} \to \mathbb{R}$ that makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s \mapsto \exp(2\pi \mathrm{i} s)} \\ X & \xrightarrow{f} & S^1. \end{array}$$

(b) By the Deck Transformation–Normal Subgroup Proposition 128.11 we know that the deck transformation group of $p\colon \widetilde{X} \to X$ is isomorphic to the group $\pi_1(S^1, 1) \cong \mathbb{Z}$. Let $\varphi\colon \widetilde{X} \to \widetilde{X}$ be a deck transformation that generates the deck transformation group. Show that there exists a homeomorphism

$$\mathrm{Tor}(\widetilde{X}, \varphi) \;\to\; X \times \mathbb{R}.$$

Hint. The reformulation of the mapping torus given by Exercise 128.9 might come in handy.

(c) Let $X = S^1 \vee S^1$ be the wedge of two circles and let $f\colon X \to S^1$ be the map that is the identity on the first copy of S^1 and the constant map on the second copy of S^1. Illustrate $\widetilde{X}$, $\widetilde{f}$ and φ in this context.

CHAPTER 129

Topics in covering theory

In this chapter we will discuss several applications and examples of the existence of (universal) coverings of suitable topological spaces.

This chapter is organized as follows:

- In Section 129.1 we will classify self-diffeomorphisms of S^1.
- In Section 129.2 we will classify complex one-dimensional tori.
- In Section 129.3 we will introduce complex logarithms.
- In Section 129.4 we will study preimages of subsets under coverings.
- In Section 129.5 we will consider explicit examples of covering spaces of surfaces.
- In Section 129.6 we will provide exercises for Chapter 129.

In the next Chapter 130 we will study the coverings of topological graphs and we will obtain some applications of covering theory to group theory.

129.1. **Self-diffeomorphisms of S^1.** Using covering theorem we can now provide a new proof of the following proposition:

Proposition 42.8. (Diffeotopies-of-S^1-Proposition) Let $f\colon S^1 \to S^1$ be a diffeomorphism. If f is orientation-preserving, then f is diffeotopic to the identity, otherwise f is diffeotopic to the reflection $r\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, -y)$.

Remark. The proof of the Diffeotopies-of-S^1-Proposition 42.8 that we now provide is very similar to the original proof. The key difference is that we replace the ad hoc Lift-from-S^1-to-$\mathbb{R}$ Lemma 42.14 by the more general Map Lifting Criterion 127.5. □

Proof. In Torus-as-Quotient Lemma 25.42 we showed that the smooth manifolds S^1 and $\mathbb{R}/\mathbb{Z}$ are diffeomorphic. Thus we might as well work with $\mathbb{R}/\mathbb{Z}$. First let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving diffeomorphism. We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the usual covering map. By the Lift-to-Universal Coverings Proposition 127.26 there exists an orientation-preserving diffeomorphism $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z}. \end{array}$$

We make the following observations:

(a) The map $\widetilde{f}$ is in particular smooth.
(b) Since $\widetilde{f}$ is orientation-preserving we know, by Exercise 31.18, that $\widetilde{f}$ is a strictly monotonously increasing function.

(c) Since $\widetilde{f}$ is a diffeomorphism the restriction of $\widetilde{f}$ to the interval $[x, x+1)$ defines a bijection with $[f(x), f(x+1))$.
(d) Given $x \in \mathbb{R}$ we obtain from the commutative diagram that $\widetilde{f}(x+1) - \widetilde{f}(x) \in \mathbb{Z}$.
(e) It follows from (a), (b), (c) and (d) that for any $x \in \mathbb{R}$ we have $\widetilde{f}(x+1) = \widetilde{f}(x) + 1$.

Now we consider the map

$$\begin{aligned} \widetilde{F}\colon \mathbb{R} \times [0,1] &\to \mathbb{R} \\ (x,t) &\mapsto \widetilde{f}(x) \cdot (1-t) + x \cdot t. \end{aligned}$$

This map is evidently smooth. Since $x \mapsto x$ is a strictly monotonously increasing diffeomorphism of $\mathbb{R}$ one can see easily that for any $t \in [0,1]$ the map $\widetilde{F}_t\colon \mathbb{R} \to \mathbb{R}$ has the above Properties (a) to (e). This implies in particular that the map $\widetilde{F}$ descends to a map

$$\begin{aligned} F\colon (\mathbb{R}/\mathbb{Z}) \times [0,1] &\to \mathbb{R}/\mathbb{Z} \\ ([x],t) &\mapsto \widetilde{f}(x) \cdot (1-t) + x \cdot t \end{aligned}$$

where each $F_t\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a homeomorphism. Since each $\widetilde{F}_t$ is a local diffeomorphism and since p is a local diffeomorphism we obtain from the Smooth Pasting Proposition 25.19 (2) that each F_t is in fact a diffeomorphism. We have thus shown that F is indeed a diffeotopy from f to the identity.

Finally we consider the case that $f\colon S^1 \to S^1$ is orientation-reversing. In this case $r \circ f\colon S^1 \to S^1$ is orientation-preserving. We apply the above result to $r \circ g$ and we easily obtain the desired diffeotopy from $f\colon S^1 \to S^1$ to the reflection $r\colon S^1 \to S^1$. ∎

129.2. **Complex one-dimensional tori.** The following definition is inspired by the Complex Torus Lemma 40.9 and Exercise 25.32.

Notation 129.1.
(1) A lattice in $\mathbb{C}$ is a subgroup of $\mathbb{C}$ that is of the form $\mathbb{Z} \cdot a + \mathbb{Z} \cdot b$ where $a, b \in \mathbb{C}$ are two complex numbers that are linearly independent over $\mathbb{R}$.
(2) Given a lattice Λ we write $T(\Lambda) = \mathbb{C}/\Lambda$. The holomorphic analog of the Manifold–Quotient Proposition 25.39 shows that $T(\Lambda)$ is naturally a 1-dimensional complex manifold such that the projection $p\colon \mathbb{C} \to T(\Lambda)$ is a covering map of complex manifolds, in particular it is a local biholomorphism.
(3) We say that two lattices Λ and $\widetilde{\Lambda}$ in $\mathbb{C}$ are homothetic if there exists an $r \in \mathbb{C} \setminus \{0\}$ such that $r\cdot, \Lambda = \widetilde{\Lambda}$.

The following proposition gives a complete and satisfactory characterization for when two lattices define biholomorphic complex manifolds.

Proposition 129.2. (Complex Tori–Classification Proposition) Let Λ and $\widetilde{\Lambda}$ be two lattices in $\mathbb{C}$. The complex manifolds $T(\Lambda)$ and $T(\widetilde{\Lambda})$ are biholomorphic if and only if Λ and $\widetilde{\Lambda}$ are homothetic.

Proof. Let Λ and $\widetilde{\Lambda}$ be two lattices in $\mathbb{C}$. We denote by $p\colon \mathbb{C} \to T(\Lambda)$ and $\widetilde{p}\colon \mathbb{C} \to T(\widetilde{\Lambda})$ the natural projections. Note that they are local holomorphisms.

First assume that Λ and $\widetilde{\Lambda}$ are homothetic, i.e. we assume that there exists an $r \in \mathbb{C}\backslash\{0\}$ such that $r \cdot \Lambda = \widetilde{\Lambda}$. In this case we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z\mapsto r\cdot z} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ T(\Lambda) & \xrightarrow{[z]\mapsto[r\cdot z]} & T(\widetilde{\Lambda}). \end{array}$$

Note that our hypothesis shows that bottom horizontal map is defined and a bijection. The top horizontal map is of course a biholomorphism. Since the vertical maps are local biholomorphisms we see that the bottom horizontal map is a bijection and a local biholomorphism. In other words, it is a biholomorphism.

Now we assume that there exists a biholomorphism $f\colon T(\Lambda) \to T(\widetilde{\Lambda})$. By the Lift-to-Universal Coverings Proposition 127.26 there exists a biholomorphism $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widetilde{f}} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ T(\Lambda) & \xrightarrow{f} & T(\widetilde{\Lambda}). \end{array}$$

Next we make the following observations and preparations:

(a) We equip the universal covering $\mathbb{C}$ with the actions of $\pi_1(T(\Lambda), [0])$ and $\pi_1(T(\widetilde{\Lambda}), [s])$ that we introduced on page 2487.

(b) Note that it follows from the Lift–Action Proposition 128.18 that for all $z \in \mathbb{C}$ and all $g \in \pi_1(T(\Lambda), x_0)$ we have

$$\widetilde{f}(g \cdot z) \;=\; \underbrace{f_*(g) \cdot \widetilde{f}(z)}_{\in \pi_1(T(\widetilde{\Lambda}),[s])}.$$

(c) Note that the Action–π_1-Theorem 93.1 provides us with quite explicit isomorphisms $\Theta\colon \Lambda \to \pi_1(T(\Lambda), x_0)$ and $\widetilde{\Theta}\colon \widetilde{\Lambda} \to \pi_1(T(\widetilde{\Lambda}), x_0)$.

(d) It follows from the Group Action–Deck Transformation-π_1-Lemma 128.10 that (c) translates into the statement that for all $z \in \mathbb{C}$ and all $\lambda \in \Lambda$ we have

$$\widetilde{f}(\lambda + z) \;=\; (\widetilde{\Theta} \circ f_* \circ \Theta^{-1})(\lambda) + \widetilde{f}(z).$$

(e) Since $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ is a biholomorphism we know by the Biholomorphism-of-$\mathbb{C}$-Theorem 39.16 that there exist $r, s \in \mathbb{C}$ such that $\widetilde{f}(z) = r \cdot z + s$ for all $z \in \mathbb{C}$.

(f) The combination of (d), applied to $z = 0$, and (e) shows that for all $\lambda \in \Lambda$ we have

$$r \cdot \lambda + s \;=\; (\widetilde{\Theta} \circ f_* \circ \Theta^{-1})(\lambda) + s.$$

(g) The map $f_*\colon \pi_1(T(\Lambda), [0]) \to \pi_1(T(\widetilde{\Lambda}), [s])$ is an isomorphism since f is in particular a homeomorphism. In particular $\widetilde{\Theta} \circ f_* \circ \Theta^{-1}\colon \Lambda \to \widetilde{\Lambda}$ is an isomorphism.

The combination of (f) and (g) shows that $r \cdot \Lambda = \widetilde{\Lambda}$. We have thus shown that the lattices Λ and $\widetilde{\Lambda}$ are homothetic. ∎

Corollary 129.3. (Not–Biholomorphic Corollary) There are closed complex manifolds that are diffeomorphic but that are not biholomorphic.

Proof. Let Λ and $\widetilde{\Lambda}$ be two lattices in $\mathbb{C}$. It follows from Exercise 25.32 that the tori $T(\Lambda)$ and $T(\widetilde{\Lambda})$, viewed as smooth manifolds, are diffeomorphic. In Exercise 129.1 we will see that there exist lattices Λ and $\widetilde{\Lambda}$ in $\mathbb{C}$ that are not homothetic. It follows from the Complex Tori–Classification Proposition 129.2 that $T(\Lambda)$ and $T(\widetilde{\Lambda})$ are not biholomorphic. ■

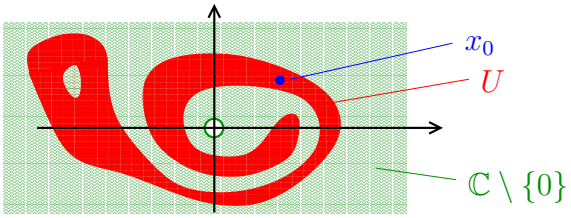
129.3. **Complex logarithms.** In this short section we will discuss complex logarithms. This discussion does not play a role in the remainder of these notes, but it is a fun application of covering theory.

Definition 129.4. Let $U \subseteq \mathbb{C} \setminus \{0\}$ be a subset. We say the complex logarithm exists on U if there exists a continuous map $\ln\colon U \to \mathbb{C}$ such that $\exp(\ln(z)) = z$ for all $z \in U$.

The following proposition gives a very pleasing sufficient and necessary criterion for complex logarithms to exist.

Proposition 129.5. (Complex Logarithm–Existence Proposition) Let $U \subseteq \mathbb{C} \setminus \{0\}$ be an open path-connected subset. The following statement holds:[1342]

the complex logarithm exists on U	$\iff$	for some $x_0 \in U$ the inclusion-induced map $\pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)$ is the trivial map.



the map $\pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)$ is the trivial map, hence the complex logarithm exists on U

Proof. Let $U \subseteq \mathbb{C} \setminus \{0\}$ be an open path-connected subset and let $x_0 \in U$. We consider the following rather simple minded diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "z \mapsto \exp(z)"] \\ U \arrow[ur, dashed] \arrow[r, hook, "i"'] & \mathbb{C} \setminus \{0\}. \end{tikzcd}$$

We make the following observations:

(1) As on page 1781 one can show that the vertical map exp is actually a covering map.
(2) The question whether the complex logarithm exists on U is equivalent to the question whether the inclusion map $i\colon U \to \mathbb{C} \setminus \{0\}$ lifts to a continuous map $U \to \mathbb{C}$.
(3) Since $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is surjective we can pick a $y_0 \in \mathbb{C}$ with $\exp(y_0) = x_0$.
(4) Note that $\mathbb{C}$ is simply connected. The image $\operatorname{im}(\exp_*\colon \pi_1(\mathbb{C}, y_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0))$ is therefore the trivial group.
(5) Since U is an open subset of $\mathbb{C} \setminus \{0\}$ it is in particular locally path-connected. Thus we can apply the Map Lifting Criterion 127.5. Together with (4) we obtain from the

[1342] It follows from the Change-of-Base Point Proposition 90.4 that the condition on the right-hand side does not depend on the choice of the base point.

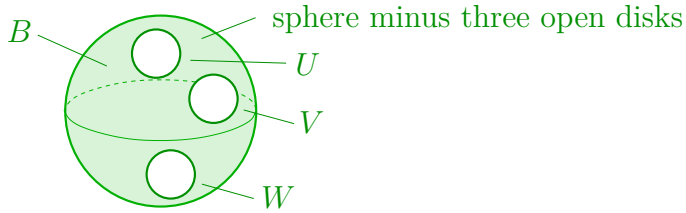
Map Lifting Criterion 127.5 the statement

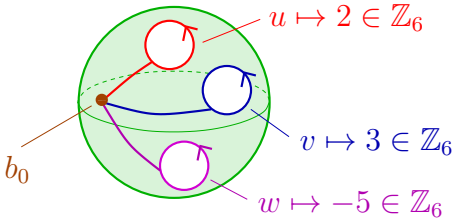
$$\begin{array}{c}\text{there exists a lift of}\\ \text{the map } i\colon U \to \mathbb{C}\setminus\{0\}\end{array} \iff \begin{array}{c}\text{the map } i_*\colon \pi_1(U,x_0) \to \pi_1(\mathbb{C}\setminus\{0\},x_0)\\ \text{is the trivial map.}\end{array}$$

(6) The combination of (2) and (5) is precisely the statement we had set out to prove. ■

129.4. **Preimages of subsets under coverings.** Let (B,b_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. In the Covering Existence Theorem 127.10 (2) we saw that subgroups of $\pi_1(B,b_0)$ give rise to coverings of B. Arguably the most common source of subgroups is given by considering kernels of group homomorphisms $\pi_1(B,b_0) \to G$.

Example. Let B be the 2-dimensional sphere S^2 minus three open disks. We pick a base point b_0 for B and we pick loops u,v,w as shown in the figure below.





By the Surface-with-Boundary–π_1-Lemma 107.8 we know that $\pi_1(B,b_0) = \langle u,v,w \,|\, uvw\rangle$. By the Normal Closure–Factorization Lemma 98.21 there exists a unique homomorphism $\varphi\colon \pi_1(B,b_0) \to \mathbb{Z}/6\mathbb{Z}$ with $\varphi(u) = 2$, $\varphi(v) = 3$ and $\varphi(w) = -5$. Let $p\colon X \to B$ be the covering corresponding to $\ker(\varphi)$. By the Manifold Covering Proposition 95.4 we know that X is again a compact orientable 2-dimensional smooth manifold, where the boundary is given by $\partial X = p^{-1}(\partial B)$. Thus we know from the Surface Classification Theorem 101.16 that there exist unique $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that X is homeomorphic to $\Sigma_{g,n}$, i.e. such that X is homeomorphic to the surface of genus g minus n open disks. □

Therefore the following question arises:

Question 129.6. In the previous example, what are g and n?

The final proposition of this chapter will allow us to answer one half of Question 129.6, namely we can determine n. We will determine g in Exercise 130.3.

Proposition 129.7. (Preimage-of-Covering Proposition) Let (B,b_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Let $\varphi\colon \pi_1(B,b_0) \to G$ be a group epimorphism and let $p\colon (X,x_0) \to (B,b_0)$ be the covering with $p_*(\pi_1(X,x_0)) = \ker(\varphi)$. Finally let A be a path-connected subset of B and let $a_0 \in A$. We denote by $i\colon A \to B$ the inclusion map and we pick a path γ from b_0 to a_0. We consider the homomorphism

$$\psi := \varphi \circ \gamma_* \circ i_*\colon \pi_1(A,a_0) \xrightarrow{i_*} \pi_1(B,a_0) \xrightarrow{\gamma_*} \pi_1(B,b_0) \xrightarrow{\varphi} G.$$

With these preparations the following two statements hold:

(1) Each path-component of $p^{-1}(A)$ is a covering of A corresponding to the subgroup $\ker(\psi\colon \pi_1(A,a_0) \to G)$.

(2) If G is a finite group, then

$$\text{number of path-components of } p^{-1}(A) \;=\; [G : \operatorname{im}(\psi\colon \pi_1(A, a_0) \to G)].$$

(3) Let $\widetilde{A}$ be a path-component of $p^{-1}(A)$. The following diagram commutes:

$$\begin{array}{ccc} \operatorname{Deck}(p\colon \widetilde{A} \to A) & \xrightarrow{\Xi_{(A,a_0)}} & \pi_1(A, a_0)/\ker(\psi) \\ \downarrow & & \downarrow{\scriptstyle \gamma_* \circ i_*} \\ \operatorname{Deck}(p\colon X \to B) & \xrightarrow{\Xi_{(B,a_0)}} & \pi_1(B, a_0)/\ker(\varphi \circ \gamma_*). \end{array}$$

Here the horizontal maps are the isomorphisms that we introduced in Definition 128.8 and the left vertical map is the map that we introduced in the Induced-Deck Transformation Proposition 128.14.

Example. As promised we can now determine the number n of boundary components of the covering X in the previous example. More precisely, denote by U, V, W the three boundary components of B. We make the following three observations:

(1) The image of $\pi_1(V)$ in $\mathbb{Z}/6\mathbb{Z}$ is the subgroup generated by $\varphi(u) = 2$, thus the subgroup has index two, which by the Preimage-of-Covering Proposition 129.7 implies that $p^{-1}(U)$ consists of two components.
(2) The image of $\pi_1(W)$ in $\mathbb{Z}/6\mathbb{Z}$ is the subgroup generated by $\varphi(v) = 3$, thus $p^{-1}(V)$ consists of three components.
(3) The image of $\pi_1(U)$ in $\mathbb{Z}/6\mathbb{Z}$ is the subgroup generated by $\varphi(w) = -5$. Since -5 is coprime to 6 we see that this subgroup is all of $\mathbb{Z}/6\mathbb{Z}$, thus $p^{-1}(W)$ consists of precisely one component.

Altogether we see that X has $2 + 3 + 1 = 6$ boundary components. □

Remark. The reader who is in need of a brain teaser might want to come up with an appropriate generalization of the Preimage-of-Covering Proposition 129.7 to the case where we consider coverings of B coming from a non-normal subgroup. □

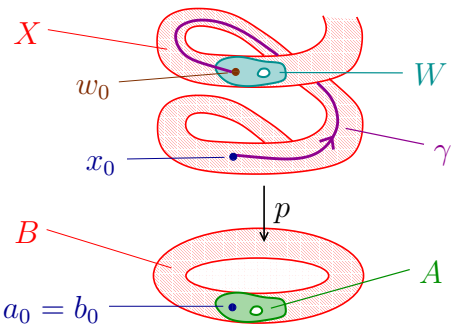
Proof. To simplify the notation a little bit we assume that $a_0 = b_0$, in particular we ignore the role of γ. Throughout the proof we refer to the figure below for an illustration.

(1) Let W be a path-component of $p^{-1}(A)$. Note that it follows from our hypothesis that B is locally path-connected and path-connected, together with the Locally Path-Connected Lemma 4.57 (1) and the Covering–Basics Lemma 91.12 (5), that the restriction of p to $p|_W\colon W \to A$ is also a covering. Therefore it now remains to show that $p_*(\pi_1(W, w_0)) = \ker(\psi)$.

Since we already established that $p\colon W \to A$ is a covering we know in particular this map is surjective. Thus we can pick a point $w_0 \in W$ with $p(w_0) = a_0$. We start out with the following slightly subtle claim.

Claim. We have the following equality:

$$p_*(\pi_1(X, w_0)) \;=\; \ker(\varphi\colon \pi_1(B, a_0) \to G).$$



Proof. We pick a path γ in X from x_0 to w_0. Since $p \circ \gamma$ is a loop in a_0 we can write $h \coloneqq [p \circ \gamma] \in \pi_1(B, a_0)$. We consider the following diagram:

$$\begin{array}{ccc} \pi_1(X, w_0) & \xrightarrow[\cong]{\gamma_*} & \pi_1(X, x_0) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \pi_1(B, a_0) & \xrightarrow[\cong]{g \mapsto hgh^{-1}} & \pi_1(B, a_0) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ G & \xrightarrow[\cong]{g \mapsto \varphi(h) g \varphi(h)^{-1}} & G. \end{array}$$

The top square commutes by the Change-of-Base Point Proposition 88.14 (1b) and by the Change-of-Base Point Proposition 90.4 (5). Evidently the bottom square also commutes. By definition of (X, x_0) we have

$$p_*(\pi_1(X, x_0)) \;=\; \ker(\varphi \colon \pi_1(B, a_0) \to G).$$

The claim now follows from the fact that the diagram commutes and that the horizontal maps are isomorphisms. ⊞

In the following we denote by $j \colon W \to X$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccc} \pi_1(W, w_0) & \xrightarrow{j_*} & \pi_1(X, w_0) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \pi_1(A, a_0) & \xrightarrow{i_*} & \pi_1(B, a_0) \\ & \searrow^{\psi} & \downarrow{\scriptstyle \varphi} \\ & & G. \end{array}$$

Recall that we need to show the equality $p_*(\pi_1(W, w_0)) = \ker(\psi)$. We show the two inclusions separately.

"$\subseteq$" This inclusion follows immediately from the definitions, from the commutativity of the above diagram and from the claim.

"$\supseteq$" Thus let $g \in \ker(\psi)$. We represent it by a loop γ in (A, a_0). By the above claim and the Loop–Lifting Proposition 92.8 the loop $i \circ \gamma$ in (B, a_0) lifts to a loop $\widetilde{\gamma}$ in (X, w_0). By definition the image of $\widetilde{\gamma}$ lies in $p^{-1}(A)$ and it contains w_0. In particular the image of $\widetilde{\gamma}$ lies in the path-component of $p^{-1}(A)$ that contains w_0, in other words, $\widetilde{\gamma}$ is actually a loop in (W, w_0). Since $g = p_*([\widetilde{\gamma}])$ we obtain that $g \in p_*(\pi_1(W, w_0))$.

(2) We write $n = |G|$ and $k = |\operatorname{im}(\psi\colon \pi_1(A, a_0) \to G)|$. It follows from the Covering-Degree Proposition 92.10 and (1) that each path-component of $p^{-1}(A)$ is a k-fold covering of A. By the Covering-Degree Proposition 92.10 the map $p\colon X \to B$ is an n-fold covering. Evidently $p\colon p^{-1}(A) \to A$ is also an n-fold covering. It follows almost immediately that $p^{-1}(A)$ has necessarily n/k path-components. But $n/k = |G|/|\operatorname{im}(\psi)|$ is by definition the degree of $\operatorname{im}(\psi)$ in G.

(3) This statement follows easily from the definition of the horizontal maps. We leave the details to the reader. ■

129.5. **Covering spaces of surfaces.** Suppose we start out with a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, then by the Covering Existence Theorem 127.10 we obtain a corresponding finite covering $p\colon \widetilde{F} \to F$. By the Manifold Covering Proposition 95.4 and the Covering–Basics Lemma 91.12 (4) the topological space $\widetilde{F}$ is again a closed orientable connected 2-dimensional smooth manifold. From the Surface Classification Theorem 101.16 we know that $\widetilde{F}$ is again a surface of some genus h. How can we determine h from g and Γ?

For future reference we record the question.

Question 129.8. Given a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, what is the genus of the corresponding finite cover?

It will take a while till we can give a satisfactory answer. In fact we will obtain a complete answer only once we have introduced the Euler characteristic.

But we want to discuss one example in detail. We consider the surface F of genus two that we view as the connected sum of two tori $S = S^1 \times S^1$ and $T = S^1 \times S^1$. We denote by u, v the standard generators for $\pi_1(S)$ given by the two copies of S^1 in each torus. Similarly we denote by x, y the standard generators for $\pi_1(T)$. We pick an embedded closed disk in S and an embedded closed disk in T which both lie in the complement of the standard generators. We denote the corresponding open disks by D and E and we write $S^* = S \setminus D$ and $T^* \setminus E$. We build the connected sum $S\#T$ by gluing S^* to T^* along the boundary.[1343] Furthermore, by a slight abuse of notation, we denote by u, v and x, y also the generators of $\pi_1(S^*) = \langle u, v\rangle$ and $\pi_1(T^*) =_{\mathrm{i}} \langle x, y\rangle$.

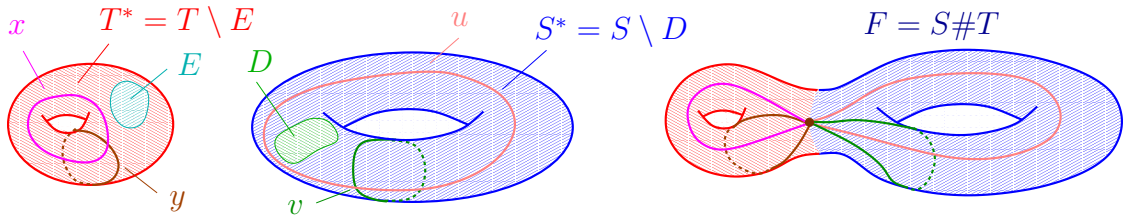
By the discussion on page 2083 the abelianization of $\pi_1(S\#T) = \pi_1(S^* \cup_{\partial S^* = \partial T^*} T^*)$ is the free abelian group on u, v, x, y. Now we consider the epimorphism

$$\varphi\colon \pi_1(S\#T) \;\to\; \pi_1(S\#T)/[\pi_1(S\#T), \pi_1(S\#T)] \;\to\; \mathbb{Z}/3\mathbb{Z}$$

given by $u \mapsto 1, v \mapsto 0, x \mapsto 0$ and $y \mapsto 0$. Our goal in the following discussion is to visualize the covering $p\colon \widetilde{F} \to F = S\#T$ corresponding to $\ker(\varphi)$. Since φ is an epimorphism onto a group of order 3 we see that $\ker(\varphi)$ is a subgroup of index 3. By the Covering-Degree Proposition 92.10 this implies that p is a 3-fold covering.

We first consider the torus $S = S^1 \times S^1$ and furthermore we consider the epimorphism $\psi\colon \pi_1(S) = \langle u, v \mid [u, v]\rangle \to \mathbb{Z}/3\mathbb{Z}$ given by $\psi(u) = 1$ and $\psi(v) = 0$. The covering

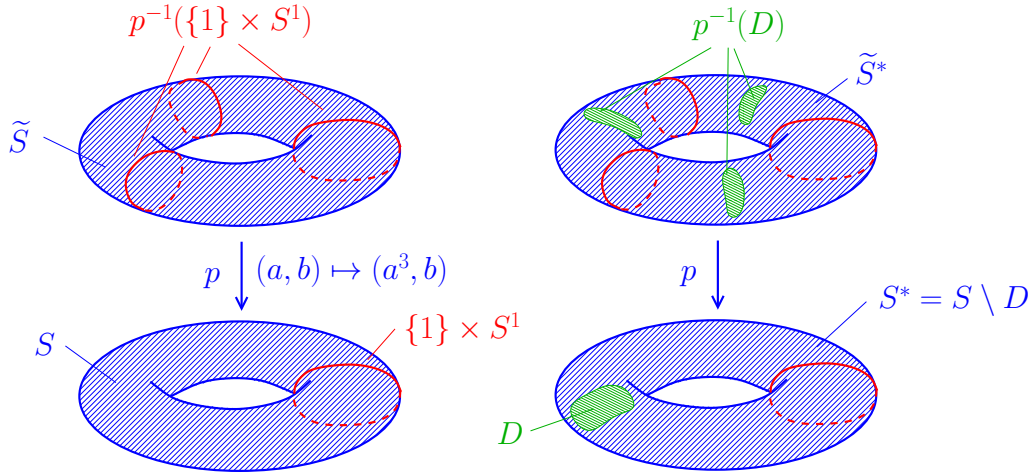
[1343] It follows from the discussion on page 1233 that we can be a bit casual in our language, i.e. we can talk about "gluing surfaces" without specifying explicit diffeomorphisms of boundary components, as long as we take into account orientations.



corresponding to $\ker(\psi) \subseteq \pi_1(S)$ is equivalent to the covering[1344]

$$\begin{array}{ccc} \widetilde{S} = S^1 \times S^1 & \to & S = S^1 \times S^1 \\ (a,b) & \mapsto & (a^3, b). \end{array}$$

Now we consider the epimorphism $\psi^*\colon \pi_1(S^*) \to \mathbb{Z}/3\mathbb{Z}$ given by $\psi^*(u) = 1$ and $\psi^*(v) = 0$. By the argument of Footnote 1344 the covering corresponding to $\ker(\psi^*)$ is then given by $p\colon \widetilde{S}^* := \widetilde{S} \setminus p^{-1}(D) \to S^* = S \setminus D$.

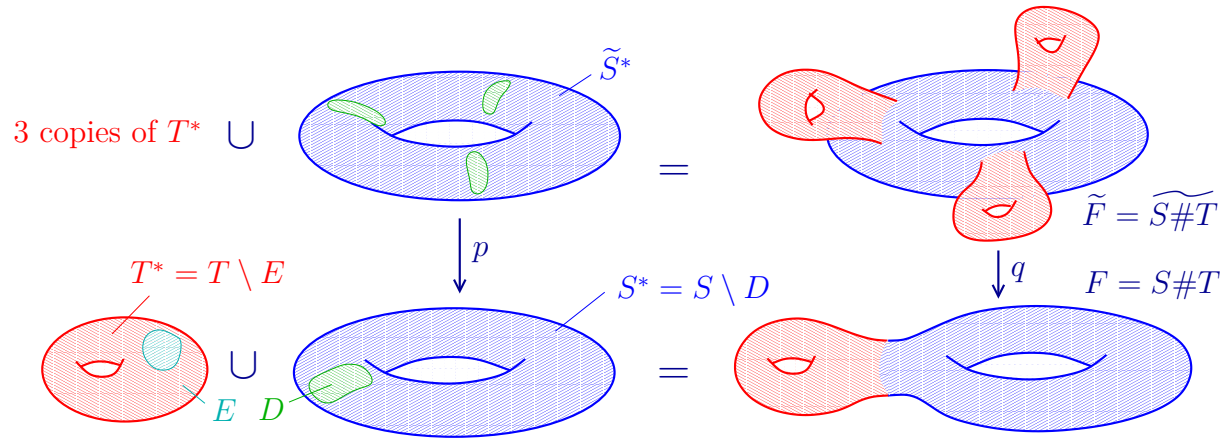


Recall that we perform the connected sum operation $S\#T$ by gluing $T^* := T \setminus E$ to $S^* = S \setminus D$ to obtain $F = S\#T$. The same way we can also glue three copies of T^*, i.e. $T^* \times \{1,2,3\}$, to the three boundary components of $\widetilde{S}^*$ to obtain a closed smooth manifold $\widetilde{F}$. It follows from the Pasting Proposition 4.7 (2′) that the map

$$\begin{array}{ccc} q\colon \widetilde{F} & \to & F \\ P & \mapsto & \begin{cases} p(P), & \text{if } P \in \widetilde{S}^*, \\ Q, & \text{if } P = Q \times \{n\} \text{ with } Q \in T^* \text{ and } n \in \{1,2,3\} \end{cases} \end{array}$$

is continuous. Similar to Footnote 1344 one can now see that $q\colon \widetilde{F} \to F$ is indeed the covering of F corresponding to $\ker(\varphi)$. This construction is illustrated in the figure below which also shows that $\widetilde{F}$ is a surface of genus 4.

[1344]Indeed, since $\widetilde{S} \to S$ is a connected 3-fold covering we know from the Covering-Degree Proposition 92.10 that $p_*(\pi_1(\widetilde{S}))$ is a subgroup of $\pi_1(S)$ of index three. Furthermore it is contained in the subgroup $\ker\left(\pi_1(S) \xrightarrow{\psi} \mathbb{Z}/3\mathbb{Z}\right)$ which also has index three, thus $p_*(\pi_1(\widetilde{S})) = \ker(\psi\colon \pi_1(S) \to \mathbb{Z}/3\mathbb{Z})$. But by the Covering Unique Existence Theorem 127.17 all path-connected coverings that correspond to the same subgroup are isomorphic, in the sense of Definition 127.16.



129.6. **Exercise for Chapter 129.**

Exercise 129.1. Show that there exist lattices Λ and $\widetilde{\Lambda}$ in $\mathbb{C}$ that are not homothetic, i.e. such that there exists no $r \in \mathbb{C} \setminus \{0\}$ such that $r \cdot \Lambda = \widetilde{\Lambda}$.

Exercise 129.2. We write $\mathbb{C}^* \coloneqq \mathbb{C} \setminus \{0\}$.

(a) Let $m \in \mathbb{N}$. Show that the map

$$\begin{array}{ccc} \mathbb{C}^* & \to & \mathbb{C}^* \\ z & \mapsto & z^m \end{array}$$

is a holomorphic map between complex manifolds and show that this map is a covering map with degree $= m$.

(b) Show that the exponential map

$$\begin{array}{rccl} \exp\colon \mathbb{C} & \to & \mathbb{C}^* \coloneqq \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$

is a holomorphic map between complex manifolds and show that this map is a covering map of infinite degree.

(c) A *complex logarithm* on an open subset $U \subseteq \mathbb{C}^*$ is a continuous map $f\colon U \to \mathbb{C}$ such that for all $z \in U$ we have $f(\exp(z)) = z$.

(i) Show that every complex logarithm is holomorphic.

(ii) Show that on any simply connected open subset $U \subseteq \mathbb{C}^*$ one can define a complex logarithm.

(iii) Show that it is not possible to define a complex logarithm on all of $\mathbb{C}^*$.

Exercise 129.3. Let Σ be a compact oriented connected 2-dimensional smooth manifold and let $f\colon \partial\Sigma \to \partial\Sigma$ be a diffeomorphism. Show that there exists a diffeomorphism $\Sigma \to \Sigma$ with $F|_{\partial\Sigma} = f$ if and only if f is either orientation-preserving or orientation-reversing.
Hint. Use the Boundary Component-Permutation Lemma 107.3, the Diffeotopies-of-S^1-Proposition 42.8 and the Smooth Collar Neighborhood Theorem 35.6.

Exercise 129.4. Let $n \in \mathbb{N}_{\geq 3}$.

(a) We consider the connected sum $M = \mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n$. We consider the following homomorphism

$$\begin{array}{ccccccccc} \varphi\colon \pi_1(\mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n) & \xrightarrow{\cong} & \pi_1(\mathbb{R}\mathrm{P}^n) * \pi_1(\mathbb{R}\mathrm{P}^n) & \xrightarrow{\cong} & \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} & = & \langle a \,|\, a^2 \rangle * \langle b \,|\, b^2 \rangle & \to & \mathbb{Z}/2\mathbb{Z}. \\ & & \uparrow & & \uparrow & & & & a \mapsto 1 \\ & & \text{by the Connected} & & \text{by the Real Projective} & & & & b \mapsto 1 \\ & & \text{Sum-}\pi_1\text{-Proposition 102.4} & & \text{Space–}\pi_1\text{-Proposition 93.3} & & & & \end{array}$$

We denote by $\widetilde{M}$ the cover corresponding to $\ker(\varphi\colon \pi_1(M) \to \mathbb{Z}/2\mathbb{Z})$. Show that $\widetilde{M}$ is diffeomorphic to $S^1 \times S^{n-1}$.

Hint. Use the Preimage-of-Covering Proposition 129.7.

(b) Let $k \in \mathbb{N}$. We consider the connected sum M of k copies of $\mathbb{R}\mathrm{P}^n$. As in (a) we have an obvious epimorphism $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$. It turns out that the corresponding covering space $\widetilde{M}$ is diffeomorphic to the connected sum of m copies of $S^1 \times S^{n-1}$. Determine m in terms of k.

CHAPTER 130

Coverings of topological graphs and applications to group theory

In this chapter we will apply the results obtained in Chapter 127 to prove several results in group theory.

This chapter is organized as follows:

- In Section 130.1 we will consider explicit examples of coverings of topological graphs.
- In Section 130.2 we will discuss fundamental group of countable topological graphs.
- In Section 130.3 we will show that subgroups of free groups are again free.
- In Section 130.4 we will study subgroups of surface groups.
- In Section 130.5 we will introduce residually finite groups and Hopfian groups.
- In Section 130.6 we will provide exercises for Chapter 130.

In the next Chapter 131 we will introduce some basic concepts from hyperbolic geometry.

130.1. **Coverings of topological graphs.** In our proofs of the promised group-theoretic results we will need the following proposition which is also of interest in its own right.

Proposition 130.1. (Covering-of-Graph Proposition) Let $G = (V, E, i, t)$ be a connected abstract graph and let $p\colon \widetilde{X} \to |G|$ be a covering of the topological realization $X := |G|$ of G. Then there exists a natural[1345] abstract graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, i, t)$, a natural homeomorphism $f\colon |\widetilde{G}| \to \widetilde{X}$ and a natural map $q\colon \widetilde{G} \to G$ of abstract graphs with the following properties:

(1) We have $\#\widetilde{V} = [\widetilde{X} : X] \cdot \#V$ and $\#\widetilde{E} = [\widetilde{X} : X] \cdot \#E$.

(2) The following diagram commutes:

$$\begin{array}{ccc} |\widetilde{G}| & \xrightarrow{f} & \widetilde{X} \\ {\scriptstyle |q|}\downarrow & & \downarrow{\scriptstyle p} \\ |G| & \xrightarrow{\mathrm{id}} & X. \end{array}$$

Here $|q|$ denotes the topological realization of the map q, as defined in the Topological Realization–Functor Lemma 11.7.

Remark. In the CW-Complex Covering Proposition 138.1 we will generalize the above proposition to the setting of "CW-complexes". □

Proof. Let $G = (V, E, i, t)$ be a connected abstract graph. Recall that in Definition 11.5 we defined the topological realization of G as follows:

$$|G| \ := \ \big(V \ \sqcup \ (E \times [0,1])\big)\big/ \sim \quad \text{where for } e \in E \text{ we have } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).$$

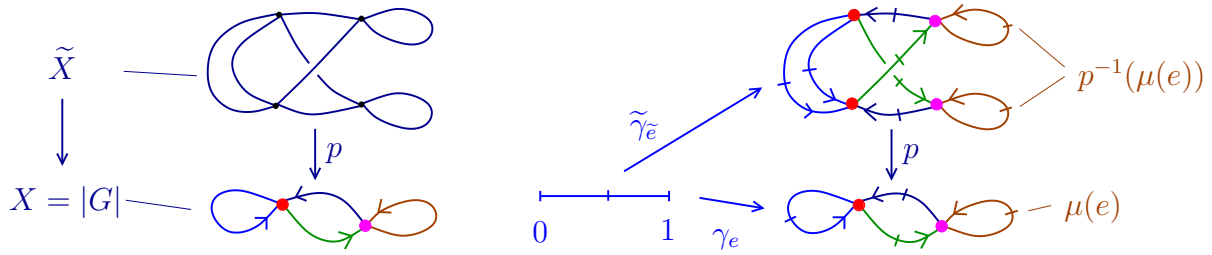
[1345] We leave the pleasure of figuring out what "natural" means to the reader.

Note that by the Topological Realization–Properties Lemma 11.8 (2) we know that $|G|$ is connected. Given $e \in E$ we set $\mu(e) \coloneqq [(e, \frac{1}{2})]$, i.e. $\mu(e)$ is the "midpoint" of e. Furthermore, given $e \in E$ we denote by $\gamma_e \colon [0,1] \to X$ the path given by $t \mapsto [(e,t)]$.

Note that in Exercise 96.12 we showed that $X = |G|$ is locally contractible, in particular it is locally path-connected. Thus in the following discussion we will have no troubles with applying the results from Chapter 127.

Now let $p \colon \widetilde{X} \to |G|$ be a covering of the topological realization $X = |G|$ of G. We set $\widetilde{V} \coloneqq p^{-1}(V) \subseteq \widetilde{X}$ and we set $\widetilde{E} \coloneqq p^{-1}(\{\mu(e) \,|\, e \in E\}) \subseteq \widetilde{X}$. Since $X = |G|$ is connected we know that the degree $[\widetilde{X} : X]$ is defined. By definition, or if you want, by the Covering–Preimage Lemma 91.9, we have $\#\widetilde{V} = [\widetilde{X} : X] \cdot \#V$ and $\#\widetilde{E} = [\widetilde{X} : X] \cdot \#E$.

Now let $\widetilde{e} \in \widetilde{E}$. We denote by $\varphi(\widetilde{e})$ the unique edge in E with $p(\widetilde{e}) = \mu(\varphi(\widetilde{e}))$. By the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 there exists a unique lift $\widetilde{\gamma}_{\widetilde{e}} \colon [0,1] \to \widetilde{X}$ of $\gamma_{\varphi(\widetilde{e})} \colon [0,1] \to X$ with $\widetilde{\gamma}(\frac{1}{2}) = \widetilde{e}$. We set $i(\widetilde{e}) = \widetilde{\gamma}(0)$ and $t(\widetilde{e}) = \widetilde{\gamma}(1)$. Since $\widetilde{\gamma}_{\widetilde{e}}$ is a lift of γ_e we see that $i(\widetilde{e})$ and $t(\widetilde{e})$ actually lie in $\widetilde{V}$. In other words, $(\widetilde{V}, \widetilde{E}, i, t)$ is actually an abstract graph. Furthermore let us point out that the two maps $p \colon \widetilde{V} \to V$ and $\varphi \colon \widetilde{E} \to E$ define by, basically by construction, a map $q \colon \widetilde{G} = (\widetilde{V}, \widetilde{E}, i, t) \to (V, E, i, t) = G$ of abstract graphs.



Next we define $f \colon |\widetilde{G}| \to \widetilde{X}$ via the inclusion $\widetilde{V} \to \widetilde{X}$ and by sending $(\widetilde{e}, t)$ to $\widetilde{\gamma}_{\widetilde{e}}(t)$. It follows immediately from the Topological-Quotient Proposition 8.2 (1b) that f is continuous. Furthermore, using the Map Lifting Criterion 127.5 and the Lift Uniqueness Proposition 92.2, it is not difficult to see that the map f is actually a bijection. Next note that, with a little bit of an effort, one can also show that f is an open map. Thus we obtain from the Open-Closed–Injective Lemma 4.22 that f is actually a homeomorphism.

Finally note that it is straightforward to see that the given diagram in the proposition does indeed commute. ■

130.2. **Fundamental group of countable topological graphs.** In the Graph-π_1-Proposition 99.13 we showed that the fundamental group of any finite topological graph is a free group. We generalize this result to countable topological graphs.

Proposition 130.2. (Countable Graph-π_1-Proposition) If X is a countable connected non-empty topological graph, then $\pi_1(X)$ is a free group.

Remark. In the 1-Complex-π_1-Theorem 137.6 we will show that the fundamental of any (not necessarily countable) connected topological graph is a free group. Since the proof of this more general statement is technically rather demanding we feel that it is worth to prove this intermediate statement. □

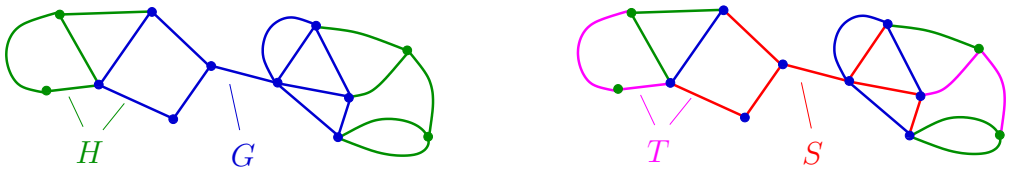
The proof of the Countable Graph-π_1-Proposition 130.2 builds on the following reasonably straightforward lemma.

Lemma 130.3. (Subgraph-π_1-Lemma) Let H be a finite connected graph and let G be a connected non-empty subgraph of H. Let v be a vertex of G. We denote by $i\colon |G| \to |H|$ the inclusion. We set $s := \chi(G) - \chi(H)$. Then there exists a homomorphism $\varphi\colon \langle x_1, \dots, x_s\rangle \to \pi_1(|H|, v)$ such that the map

$$i_* * \varphi\colon \pi_1(|G|, v) * \langle x_1, \dots, x_s\rangle \;\to\; \pi_1(|H|, v)$$

is an isomorphism.

Proof of Lemma 130.3. Let $H = (W, F, i, t)$ be a finite connected graph. Furthermore let $G = (V, E, i, t)$ be a connected non-empty subgraph of H. By the Spanning Tree–Existence Proposition 11.26 there exists a spanning tree S for G. By a rather elementary argument, which we outsourced to Exercise 130.1, there exists a spanning tree T of H such that $T \cap G = S$.



Let $e_1, \dots, e_m$ be the edges of G not contained in S and let $e_{m+1}, \dots, e_n$ be the remaining edges of H not contained in T. For $j = 1, \dots, m$ we pick a path α_j in $|S|$ from v to $i(e_j)$, we denote by β_j the obvious path form $i(e_j)$ to $t(e_j)$ and we pick a path γ_j in $|S|$ from $t(e_j)$ to v. We proceed analogously for $j = m+1, \dots, n$, except that now can no longer demand that the paths α_j and γ_j lie in $|S|$, but we will demand that they lie in $|T|$. Finally we consider the following diagram:

$$\begin{array}{ccc} & & \pi_1(|G|, v) * \langle y_{m+1}, \dots, y_{m+n}\rangle \\ & \overset{\Theta_G * \mathrm{id}}{\nearrow} & \searrow \\ \langle y_1, \dots, y_m\rangle * \langle y_{m+1}, \dots, y_n\rangle & \xrightarrow{\qquad\qquad\Theta_H\qquad\qquad} & \pi_1(|H|, v). \end{array}$$

Here the homomorphisms Θ_G and Θ_H are the maps induced by $y_j \mapsto [\alpha_j * \beta_j * \gamma_j]$. By the Graph-$\pi_1$-Proposition 99.13 we know that both Θ_G and Θ_H are isomorphisms. This shows that the homomorphism $\langle x_1, \dots, x_{n-m}\rangle \to \pi_1(|H|, v)$ given by $x_j \mapsto \Theta_H(y_{m+j})$ has the desired properties. Finally one can easily verify that $\chi(G) = 1 - m$ and $\chi(H) = 1 - n$. Thus we see that $n - m = \chi(G) - \chi(H)$. ∎

Proof of Proposition 130.2. Let $G = (V, E, i, t)$ be a countable connected non-empty abstract graph. We pick a vertex $v \in V$. We need to show that $\pi_1(G, v)$ is a free group. By Exercise 11.12 we know that there exists a sequence $\{G_j\}_{j\in\mathbb{N}}$ of subgraphs with the following properties:

(1) Each G_j is finite connected.
(2) Each G_j contains v.
(3) We have $G = \bigcup_{i\in\mathbb{N}} G_j$ in the sense that $V = \bigcup_{j\in\mathbb{N}} V_j$ and $E = \bigcup_{j\in\mathbb{N}} E_j$.

We see that

$$\pi_1(|G|, v) \;=\; \pi_1\Big(\bigcup_{j\in\mathbb{N}} |G_j|, v\Big) \;\underset{\uparrow}{=}\; \varinjlim \pi_1(|G_j|, v) \;\underset{\uparrow}{=}\; \text{free group.}$$

by the Topological Realization–Properties Lemma 11.8 (1a) and the Exhaustion-π_1-Proposition 105.2

follows from the Subgraph-π_1-Lemma 130.3 and the discussion on page 2026

■

130.3. **Subgroups of free groups.** In this section we will apply our topological results to the study of free groups. First we introduce the following definition:

Definition 130.4. Given a free group F, i.e. given a group F that is isomorphic to the free group $\langle S\rangle$ generated by some set S, we refer to the cardinality of S as the rank of F.[1346]

The following proposition is arguably the most interesting result of this chapter. The statement is purely group-theoretical, but as we will see, a topological point of view leads to a very neat proof.

Proposition 130.5. (Countable Free Subgroup Proposition) Let F be a free group of countable rank.

(1) Every subgroup of F is again a free group.
(2) If F is finitely generated and if G is a finite-index subgroup of F, then[1347]

$$\operatorname{rank}(G) - 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Examples.

(1) If $F = \mathbb{Z}$ then the formula from the Countable Free Subgroup Proposition 130.5 does indeed show that every finite-index subgroup of $\mathbb{Z}$ is again a free group of rank one, i.e. every finite-index subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.
(2) Let $\alpha\colon F := \langle a, b\rangle \to \mathbb{Z}/3\mathbb{Z}$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. Note that it follows from the Countable Free Subgroup Proposition 130.5 that the kernel $G := \ker(\alpha)$ is a free group of rank $[F : G] \cdot (\operatorname{rank}(F) - 1) + 1 = 3 \cdot 1 + 1 = 4$. In Exercise 130.2 we will determine four elements of $G = \ker(\alpha)$ that are generators for this free group. □

Remark.

(1) In the Free Subgroup Proposition 138.6 we will extend the statement to all free groups, i.e. we will drop the condition that the rank should be countable. The idea of the proof is fairly similar to the one used below, but the proof of the more general statement is overall "technologically" somewhat more fancy.
(2) The Countable Free Subgroup Proposition 130.5 and the Free Subgroup Proposition 138.6 are special cases of the *Kurosh Subgroup Theorem*. This theorem says that if G is the product of groups $A_i, i \in I$ and if H is a subgroup of G, then H is a free product of a free group and groups $B_i, i \in I$ where each B_i is a (potentially trivial) group that is conjugate to a subgroup of A_i. We refer to [**LyS1977**, Theorem IV.1.10] for details and a proof. □

Proof. Let $F = \langle E\rangle$ be a free group on a countable generating set E and let G be a subgroup of F. Note that it follows easily from the Countability Lemma 1.37 that G is a countable set. Let $\Gamma = (\{x_0\}, E, i, t)$ be the abstract graph with one vertex and one edge for

[1346] In principle this rank is not uniquely defined. We will later see, using the notion of an abelianization, that the rank is indeed well-defined.
[1347] Does the equality hold if G is infinite-index and if we interpret $[F : G] \cdot (\operatorname{rank}(F) - 1)$, as usual, as $\infty \cdot (\operatorname{rank}(F) - 1) = \infty$ for $F \ncong \mathbb{Z}$?

each element of E. By the discussion on page 1865 we know that the topological realization $X := |G|$ is homeomorphic to the wedge $\bigvee_{e \in E} S^1_e$. It follows from the Wedge-of-Circles–π_1-Lemma 99.11 that there exists an isomorphism $\varphi\colon F = \langle E \rangle \to \pi_1(X, x_0)$. It is rather elementary to show that X is locally contractible (see e.g. Exercise 19.33). Thus it follows from the Covering Existence Theorem 127.10 that there exists a path-connected covering $p\colon (Y, y_0) \to (X, x_0)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, y_0)) = G$. By the Covering-π_1-Monomorphism Proposition 92.7 the map p_* is injective. Thus we see that $\pi_1(Y, y_0)$ itself is isomorphic to G. By the Covering-of-Graph Proposition 130.1 we can equip Y with the structure of a topological graph. Since E and G are countable we see that Y is a countable topological graph. We deduce from the Countable Graph-π_1-Proposition 130.2 that $G \cong \pi_1(Y, y_0)$ is a free group. This completes the proof of (1).

Now suppose that F is finitely generated and that G is a finite-index subgroup of F. We see that we have the following equalities:

$$\underbrace{\operatorname{rank}(\pi_1(Y, y_0))}_{=\operatorname{rank}(G)} - 1 \overset{\text{both equalities follow from the Graph-}\pi_1\text{-Proposition 99.13 (2)}}{\overset{\downarrow}{=}} \chi(Y) \overset{\text{by the Covering-of-Graph Proposition 130.1}}{=} [Y : X] \cdot \chi(X) \overset{\downarrow}{=} [F : G] \cdot (\underbrace{\operatorname{rank}(\pi_1(X, x_0))}_{=\operatorname{rank}(F)} - 1).$$

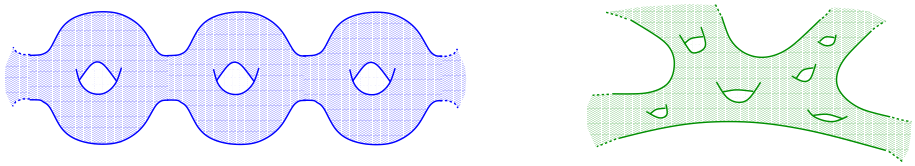
This completes the proof of (2). ■

130.4. **Subgroups of surface groups.** We start out with the following proposition that is similar in spirit to the Countable Graph-π_1-Proposition 130.2, but potentially it is more surprising.

Proposition 130.6. (Non-Compact Surface-π_1-Proposition) If M is a connected 2-dimensional smooth manifold that is non-compact, then $\pi_1(M)$ is a free group.

Remark. A somewhat different proof of the Non-Compact Surface-π_1-Proposition 130.6 is given in [**Stil1993**, p. 142]. □

Example. The 2-dimensional smooth manifolds shown in the figure below are non-compact and thus their fundamental groups are free.



For the example on the left we saw this already in the Surface-of-Infinite Genus–π_1-Lemma 105.5. □

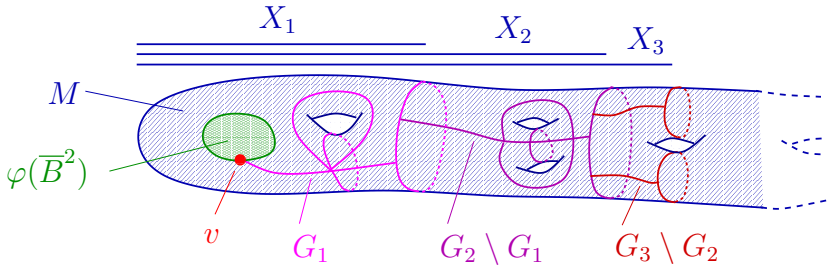
Proof. Let M be a connected 2-dimensional smooth manifold that is non-compact. In the Push-Away-from-Boundary Corollary 35.8 we showed that M is homotopy equivalent to a smooth manifold with empty boundary. So we might as well assume that M has no boundary.

By the Compact Submanifold Exhaustion Proposition 38.2 we know that there exists a sequence $X_1, X_2, \ldots$ of n-dimensional smooth submanifolds of M with the following four properties:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subseteq X_{i+1}$.
(2) Each X_i is compact and connected.
(3) The X_i are submanifolds in the sense of Definition 23.18.
(4) We have $\bigcup_{i\in\mathbb{N}} \mathring{X}_i = M$.

We pick an embedding $\varphi\colon \overline{B}^2 \to X_1 \setminus \partial X_1$. We set $C := \varphi(S^1)$. Furthermore, given any $i \in \mathbb{N}$ we set $X_i' := X_i \setminus \varphi(B^2)$. With a moderate effort we see, by iteratively applying the Surface Classification Theorem 101.16 and the Surface-with-Boundary–Retraction Lemma 107.9, that there exists a sequence of finite topological graphs $\{G_i\}_{i\in I}$ with the following properties:

(5) G_1 contains a point $v \in \varphi(S^1)$.
(6) For each $i \in \mathbb{N}$ we have $G_i \subseteq G_{i+1}$.
(7) Each G_i is a deformation retract of X_i'.



We see that

$$\begin{array}{ccccc} \pi_1(M \setminus \varphi(B^2), v) & = & \pi_1\Big(\bigcup_{j\in\mathbb{N}} X_j', v\Big) & = & \operatorname{colim} \pi_1(X_j', v) \\ & = & \operatorname{colim} \pi_1(|G_j|, v) & = & \pi_1(G_1, v) * \text{free group}. \end{array}$$

(follows from (4) and the Exhaustion-π_1-Proposition 105.2; follows from the Subgraph-π_1-Lemma 130.3 and the discussion on page 2026; by (7) and the Homotopy Equivalence-π_1-Proposition 90.9)

Now we can perform the following calculation:

$$\begin{array}{ccccc} \pi_1(M, v) & = & \pi_1((M \setminus \varphi(B^2)) \cup_C \varphi(\overline{B}^2), v) & \cong & \pi_1(M \setminus \varphi(B^2), v) *_{\pi_1(C,v)} \overbrace{\pi_1(\varphi(\overline{B}^2), v)}^{=\{e\}} \\ & \cong & (\pi_1(G_1, v) * \text{free group})/\langle\langle \pi_1(C, v) \rangle\rangle & = & \text{free group}. \end{array}$$

(by the Seifert–van Kampen Theorem 99.1; by the above calculation of $\pi_1(M \setminus \varphi(B^2), v)$ together with the Amalgamated Product–Quotient Lemma 98.25; by the Surface-with-Boundary–Retraction Lemma 107.9 we know that we can write $\pi_1(G_1, v) = \pi_1(C, v) * \text{free group}$)

■

Next we address the following question:

Question 101.4. Let π be the fundamental group of the surface of genus $g \geq 2$. What isomorphism types of subgroups can appear? Is every subgroup again a free group or the fundamental group of a surface?

The following proposition gives an affirmative answer.

Proposition 130.7. (Subgroups-of-Surface Groups Proposition) Let M be the surface of genus g and let $\Gamma \subseteq \pi_1(M)$ be a subgroup.

(1) If the index $[\pi : \Gamma]$ is finite, then Γ is the fundamental group of the surface of genus $[\pi : \Gamma] \cdot (g-1) + 1$.
(2) If the index $[\pi : \Gamma]$ is infinite, then Γ is a free group.

Remark. In Exercise 130.6 we will consider the case of subgroups of the fundamental group of a *non-orientable* surface of some genus $g \in \mathbb{N}$. □

Proof. Let $g \in \mathbb{N}_{\geq 2}$ and let Σ be the surface of genus g. We pick a base point $x_0 \in \Sigma$. Let $\Gamma \subseteq \pi_1(\Sigma, x_0)$ be a subgroup. We obtain from the CW-Complex–Local Properties Proposition 137.8 (2) and the Covering Existence Theorem 127.10 that there exists a path-connected covering $p\colon (\widetilde{\Sigma}, \widetilde{x}_0) \to (\Sigma, x_0)$ of index $[\pi_1(M, x_0) : \Gamma]$ with $p_*(\pi_1(\Sigma, \widetilde{x}_0)) = \Gamma$. By the Manifold Covering Proposition 95.4 we know that $\widetilde{\Sigma}$ is an orientable 2-dimensional smooth manifold with empty boundary.

(1) If the index $[\pi : \Gamma]$ is finite, then we know by the Covering–Basics Lemma 91.12 (4) that $\widetilde{\Sigma}$ is again compact. This implies that we now know that $\widetilde{\Sigma}$ is a closed 2-dimensional smooth manifold. In Exercise 130.6 we will show that the genus is given by the slightly weird formula
(2) By the Covering–Basics Lemma 91.12 (4) we know that $\widetilde{\Sigma}$ is non-compact. Thus it follows from the Non-Compact Surface-π_1-Proposition 130.6 that $\Gamma \cong \pi_1(\widetilde{\Sigma})$ is a free group. ■

Note that the following lemma gives an affirmative answer to Question 101.6 (1).

Corollary 130.8. (Surface Group–Torsion Free Corollary) For every $g \in \mathbb{N}$ the fundamental group of the surface Σ_g of genus g is torsion-free.

Proof. We will prove the corollary to the Subgroups-of-Surface Groups Proposition 130.7 in Exercise 130.4. ■

We conclude this section with the following proposition.[1348] We will not prove the proposition, but we will it use it on one occasion later on.

Proposition 130.9. (Surface Group–Free Subgroup Proposition) Let M be a surface of genus ≥ 2. If $\Gamma \subseteq \pi_1(M)$ is a *normal* non-trivial subgroup of infinite-index, then $\pi_1(M)$ is a free group of *infinite* rank.

Remark. If π is a finitely generated free group or the fundamental group of a surface of genus ≥ 2, then we know by the Countable Free Subgroup Proposition 130.5 and the Surface Group–Free Subgroup Proposition 130.9 that every infinite index subgroup of π is a free group. It is an open question, see e.g. [**Wilt2024**], whether these two classes of

[1348]The proposition is closely related to the fact that every non-trivial, normal, infinite-index subgroup of a free group is a free group on an infinite set. This fact is proved in [**LyS1977**, Proposition I.3.12] and [**dlH2000**, p. 45].

groups are the only infinite, finitely presented groups with the property that every infinite index subgroup of π is a free group. □

Proof. Let M be a surface of genus ≥ 2 and let $\Gamma \subseteq \pi_1(M)$ be a *normal* non-trivial subgroup of infinite-index. By the Subgroups-of-Surface Groups Proposition 130.7 it remains to show that Γ is not finitely generated. A proof for this statement is given in [**Cat2003**, Lemma 3.4], alternatively one can prove Statement (2) using [**Lü1997**, Theorem 0.7]. A significantly stronger statement is proved in [**BrHo2007**, Theorem 1]. ■

130.5. **Residually finite groups and Hopfian groups.** Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. In the Covering Unique Existence Theorem 127.17 we saw that there exists a bijection between the set of subgroups of $\pi_1(X, x_0)$ and the set of isomorphism classes (in the sense of Definition 127.16) of path-connected coverings of (X, x_0).

Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected that is compact. It follows from the above, that given a (finite-index) subgroup of $\pi_1(X, x_0)$ we can associate to (X, x_0) a (finite) covering $p\colon \widetilde{X} \to X$. For the most part we like to stick to the world of compact spaces X. By the Covering–Basics Lemma 91.12 (4) this means that we are interested in finite-index subgroups of $\pi_1(X, x_0)$. This leads us to the following, purposefully vague question.

Question 130.10. Which (finitely generated) groups have lots of finite-index subgroups?

In this section we will try to address Question 130.10. The following definition, which we already stated in Exercise 98.12, gives one popular way to turn Question 130.10 into a proper mathematical question.

Definition 130.11. A group π is called residually finite if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

Examples.

(1) The group $\mathbb{Z}$ is residually finite, indeed given $g \in \mathbb{Z}$ we can consider the epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}/(|g|+1)\cdot\mathbb{Z}$ onto a finite group, which evidently satisfies $\alpha(g) \neq 0$.
(2) It follows easily from the Abelian Group-Classification Theorem 97.24, a slight generalization of (1), that every finitely generated abelian group is residually finite.
(3) In Exercise 130.12 we will see that the group $(\mathbb{Q}, +)$ is not residually finite. □

The following is now a meaningful interpretation of Question 130.10.

Question 130.12. Which (finitely generated) groups are residually finite?

Before we address Question 130.12 let us heighten the interest in Question 130.12 by stating a surprising property of residually finite groups.

Definition 130.13. A group π is called Hopfian[1349] if every epimorphism $\varphi\colon \pi \to \pi$ is in fact a monomorphism.

Example.

[1349] This notion is once again named after Heinz Hopf.

(1) It follows immediately from the Epi-Automorphism Lemma 97.27 that every finitely generated abelian group is Hopfian.
(2) Note that in contrast to (1), the free abelian group $\mathbb{Z}^\infty$ of infinite rank is actually *not* Hopfian. This can be seen a s follows: The homomorphism $\varphi\colon \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ which is given by $\varphi(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ is an epimorphism, but evidently it is not a monomorphism. □

It is self-evident that being Hopfian is a convenient property. As we will see later in the Hopf Conjecture 186.15, the notion of a Hopfian group also naturally arises in topology. The following proposition relates the two concepts that we just introduced.

Proposition 130.14. (Residually Finite-Hopfian Proposition) Let π be a finitely generated group. If π is residually finite, then it is Hopfian.

Remark. Shortly we will see that the converse to the Residually Finite-Hopfian Proposition 130.14 does not hold. □

Proof. Let π be a finitely generated group that is residually finite. Let $\varphi\colon \pi \to \pi$ be an epimorphism. Now we assume that φ is not a monomorphism. We pick a non-trivial element $g \neq e \in \ker(\varphi)$. Since π is residually finite there exists a homomorphism $\alpha\colon \pi \to G$ onto a finite group G such that $\alpha(g) \neq e$.

Since π is finitely generated we know by the rather elementary Exercise 100.11 that the set $\operatorname{Hom}(\pi, G)$ of homomorphisms from π to G is finite. We set $n := \#\operatorname{Hom}(\pi, G)$ and we denote by $\beta_1, \dots, \beta_n\colon \pi \to G$ the distinct homomorphisms.

Next we consider the homomorphisms $\beta_1 \circ \varphi, \dots, \beta_n \circ \varphi\colon \pi \to G$. Since φ is an epimorphism we see that these homomorphisms are also distinct. But note they all satisfy $(\beta_i \circ \varphi)(g) = \beta_i(\varphi(g)) = \beta_i(e) = e$. So these n homomorphisms are different from our initial homomorphism $\alpha\colon \pi \to G$. This means that we have found $n+1$ distinct homomorphisms from $\pi \to G$. This is a contradiction to the definition of n. ■

Now that we are fully convinced of the value of residually finite groups, let us explore this class of groups in greater detail. The following proposition gives a convenient characterization of residually finite groups.

Proposition 130.15. (Residually Finite–Characterization Proposition) Let π be a group. The following two statements are equivalent.

(1) The group is residually finite.
(2) Given any non-trivial element $g \in \pi$ there exists a finite-index subgroup Γ with $g \notin \Gamma$.

If π is countable, then the above are also equivalent to the following statement:

(3) There exists a sequence $\pi \supseteq \pi_0 \supseteq \pi_1 \supseteq \pi_2 \supseteq \dots$ of finite index normal subgroups of π such that $\bigcap\limits_{i\in\mathbb{N}} \pi_i = \{e\}$.

The proof of the Residually Finite–Characterization Proposition 130.15 rests on the following group-theoretic concept.

Definition 130.16. Let G be a group and let H be a subgroup. We refer to

$$H^c := \bigcap_{g\in G} gHg^{-1}$$

as the normal core of G.

The following lemma is the key ingredient for the proof of the Residually Finite–Characterization Proposition 130.15. But note that the lemma is also interesting in its own right.

Lemma 130.17. (Normal Core Lemma) Let G be a group and let H be a subgroup.

(1) The normal core of H is a normal subgroup of G.
(2) If X is a subgroup of G that is contained in H and that is normal in G, then $X \subseteq H^c$.
(3) If H is of finite-index d in G, then H^c is a subgroup of G of index at most $d!$, in particular H^c is also of finite-index in G.

Proof of the Normal Core Lemma 130.17.

(1) Let $k \in G$. We have

$$kH^ck^{-1} = \bigcap_{g\in G} k \cdot gHg^{-1} \cdot k^{-1} = \bigcap_{g\in G} (kg)H(kg)^{-1} = \bigcap_{g\in G} gHg^{-1} = H^c.$$

(2) Let X be a subgroup of G that is contained in H and that is normal in G. Then

$$X \underset{\substack{\uparrow\\ \text{since } X \text{ normal in } G}}{=} \bigcap_{g\in G} gXg^{-1} \underset{\substack{\uparrow\\ \text{since } X\subseteq H}}{\subseteq} \bigcap_{g\in G} gHg^{-1} = H^c.$$

(3) The proof of the last statement is arguably more interesting. Thus let H be a subgroup of index $d \in \mathbb{N}$ of G. We consider the set G/H of left cosets of G with respect to H. We denote by $\mathrm{Perm}(G/H)$ the group of permutations of G/H, i.e. it is the group of all bijections of the set G/H. Note that G/H and thus also $\mathrm{Perm}(G/H)$ are finite. We consider the map

$$\begin{array}{rcl} \varphi\colon G & \to & \mathrm{Perm}(G/H) \\ g & \mapsto & \begin{pmatrix} G/H & \to & G/H \\ xH & \mapsto & gxH \end{pmatrix}. \end{array}$$

This map is easily seen to be a homomorphism. Since $\mathrm{Perm}(G/H)$ is finite we see that $\ker(\varphi)$ is a finite-index normal subgroup of G. We claim that $\ker(\varphi) \subseteq H$. Thus let $g \in \ker(\varphi)$. This implies in particular that $\varphi(g)(H) = H$, but that means that $gH = H$, i.e. that $g \in H$. We have thus shown that indeed $\ker(\varphi) \subseteq H$. From (2) and the above observation that $\ker(\varphi)$ is normal in G we obtain that $\ker(\varphi) \subseteq H^c$. But since the index of $\ker(\varphi)$ in G is at most $d!$ we see that the index of the normal H^c in G is also at most $d!$. ■

Proof of Proposition 130.15. Let π be a group.

(1)⇒(2) Let $g \in \pi$ be a non-trivial element. Since π is residually finite there exists a homomorphism $\alpha\colon \pi \to G$ to some finite group G such that $\alpha(g)$ is non-trivial. The kernel $\Gamma := \ker(\alpha)$ is a finite-index subgroup of π with $g \notin \Gamma$.

(2)⇒(1) Let $g \in \pi$ be a non-trivial element and let Γ be a finite-index subgroup of π with $g \notin \Gamma$. By the Normal Core Lemma 130.17 we know that the normal core Γ^c is a finite-index normal subgroup of π. Thus we see that the homomorphism $\alpha\colon \pi \to \pi/\Gamma^c$ is an epimorphism onto a finite group with $\gamma(g) \neq e$.

(3)⇒(2) This statement is trivial. Note that here we do not need that π is countable.

(1)⇒(3) We outsource the straightforward proof to Exercise 130.9. ■

The following lemma gives a topological criterion for a (fundamental) group to be residually finite.

Lemma 130.18. (Residually Finite–Loop Non-Lifting Lemma) Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. The following two statements are equivalent:

(1) The group $\pi_1(X, x_0)$ is residually finite.
(2) Given any loop $\gamma\colon [0,1] \to X$ in (X, x_0) that represents a non-trivial element in $\pi_1(X, x_0)$ there exists a finite covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop.

Proof. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. First we prove the (1)⇒(2)-implication. Thus we assume that the group $\pi_1(X, x_0)$ is residually finite. Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0) that represents a non-trivial element in $\pi_1(X, x_0)$. By the Residually Finite–Characterization Proposition 130.15 there exists a finite-index subgroup Γ of $\pi_1(X, x_0)$ such that $[\gamma] \notin \Gamma$. Since X is locally path-connected and semi-locally simply connected we know by the Covering Existence Theorem 127.10 that there exists a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of X of degree $[\pi : \Gamma]$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$. By the Loop–Lifting Proposition 92.8 we know that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop.

Now we prove the (2)⇒(1)-implication. Let $g \in \pi_1(X, x_0)$ be a non-trivial element. We pick a representative loop $\gamma\colon [0,1] \to (X, x_0)$. By hypothesis there exists a finite covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop. By the Loop–Lifting Proposition 92.8 we know that $g = [\gamma]$ does not lie in $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ and by the Covering-Degree Proposition 92.10 we know that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is a finite-index subgroup of $\pi_1(X, x_0)$. It follows from the Residually Finite–Characterization Proposition 130.15 that $\pi_1(X, x_0)$ is residually finite. ■

Proposition 130.19. (Free Groups–Residually Finite Proposition) Every free group is residually finite.

In the following we will provide two proofs of the Free Groups–Residually Finite Proposition 130.19. The first is algebraic, very short and elegant. The other proof is topological in nature and great fun. The eagle-eyed reader will notice that the basic ideas of the two proofs are actually not that dissimilar.

Remark. In Exercises 98.12 and 98.13 we gave two other proofs for the Free Groups–Residually Finite Proposition 130.19. Furthermore, a completely different proof of the Free Groups–Residually Finite Proposition 130.19, based again on topology, is given in [**Hem1972**]. □

First proof of Proposition 130.19. Let X be a set and let $g \in \langle X \rangle$ be a non-trivial element in the free group generated by the set X. We can and will write g as a product $g = y_1^{\epsilon_1} \cdot \ldots \cdot y_k^{\epsilon_k}$ with $y_1, \ldots, y_k \in X$ and $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$ such that the following condition is satisfied:

(∗) For every $i \in \{1, \ldots, k-1\}$ we have $(y_i, \epsilon_i) \neq (y_{i+1}, -\epsilon_{i+1})$.

It follows easily from (∗) that for every $x \in X$ we can pick a permutation $\sigma_x \in S_{k+1}$ with the following two properties:

(1) For every $i \in \{1, \dots, k\}$ with $x = y_i$ and $\epsilon_i = 1$ we have $\sigma_x(i+1) = i$.
(2) For every $i \in \{1, \dots, k\}$ with $x = y_i$ and $\epsilon_i = -1$ we have $\sigma_x(i) = i+1$. In other words, we have $\sigma_x^{-1}(i+1) = i$.

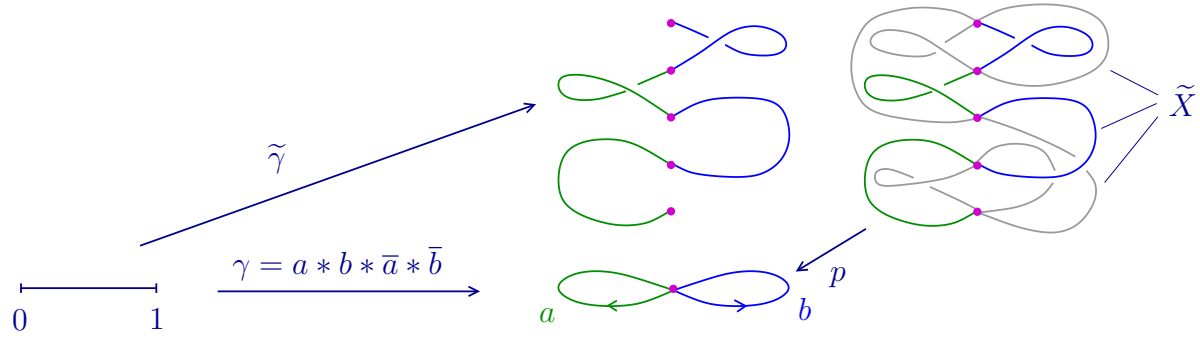
Note that by the Free Group–Existence Lemma 98.9 there exists a unique homomorphism $\varphi\colon \langle X\rangle \to S_{k+1}$ such that $\varphi(x) = \sigma_x$ for every $x \in X$. We claim that $\varphi(g) \in S_{k+1}$ is a non-trivial permutation. In fact we have

$$\varphi(g)(k+1) = \varphi(y_1^{\epsilon_1} \cdot \ldots \cdot y_k^{\epsilon_k})(k+1) = \varphi(y_1)^{\epsilon_1}(\ldots(\underbrace{\varphi(y_k)^{\epsilon_k}(k+1)}_{=k})\ldots) = \ldots = \varphi(y_1)^{\epsilon_1}(2) = 1.$$

This shows that the permutation $\varphi(g) \in S_{k+1}$ sends $k+1$ to 1, in other words, it is a non-trivial permutation. ■

Sketch of second proof of Proposition 130.19. For simplicity we only provide the proof that the free group on two generators is residually finite. Recall that by the Wedge-of-Circles–π_1-Lemma 99.11 we can make an identification $\langle a, b\rangle = \pi_1(S_a^1 \vee S_b^1, *)$ where a and b correspond to loops around S_a^1 and S_b^1. Next we point out that it follows from the Residually Finite–Loop Non-Lifting Lemma 130.18 that it remains to show that for every loop $\gamma = a_1^{m_1} * b_1^{n_1} * \cdots * a_k^{m_k} * b_k^{n_k}\colon [0,1] \to X = S_a^1 \vee S_b^1$ there exists a finite-covering $p\colon \widetilde{X} \to X$ such that the lift of γ is not a loop. In the figure below we show the following two-step construction of such a covering for the loop $\gamma = a * b * \overline{a} * \overline{b}$.

(1) First we build an "upward turning lift $\widetilde{\gamma}$ of γ".
(2) Afterward we complete the "image of $\widetilde{\gamma}$" to a finite covering $\widetilde{X}$.



It is an extraordinary charming exercise to turn the above idea into a rigorous argument for the general case. We would not dream of robbing the reader from this experience. ■

Basically for free we now get the following result:

Corollary 130.20. (Free Group–Hopfian Corollary) Every finitely generated free group is Hopfian.

Proof. This statement follows immediately from the Free Groups–Residually Finite Proposition 130.19 together with the Residually Finite-Hopfian Proposition 130.14. An alternative proof of the corollary is given in [**MKS1976**, p. 109]. ■

In Question 101.3 we asked whether the fundamental group of the surface of genus g is isomorphic to the free group F_{2g} on $2g$ generators. Now we can answer this question in the negative.

Proposition 130.21. (Surface-π_1–Not Free Proposition) If $g \geq 1$, then the group

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

is not a free group.

Remark. We will give alternative proofs of the Surface-π_1–Not Free Proposition 130.21 on page 2999 and in the Surface-π_1–Not Free Proposition 130.21 and in Exercises 116.11 and 116.12. □

In fact, as we will see, the Surface-π_1–Not Free Proposition 130.21 is an almost immediate consequence of the following more general statement.

Proposition 130.22. (One-Relator Group–Not Free Proposition) Let X be a finite set with k elements $\{x_1, \ldots, x_k\}$. If r is a non-trivial element in the free group $\langle x_1, \ldots, x_k \rangle$, then the group $\langle x_1, \ldots, x_k \,|\, r \rangle$ is not isomorphic to $\langle x_1, \ldots, x_k \rangle$.

Proof of the One-Relator Group–Not Free Proposition 130.22. Let X be a finite set and let r be an element in $\langle X \rangle$. We now suppose that there exists an isomorphism $\varphi\colon \langle X \,|\, r \rangle \to \langle X \rangle$. We denote by $\psi\colon \langle X \rangle \to \langle X \,|\, r \,\rangle$ the obvious epimorphism. Then $\varphi \circ \psi\colon \langle X \rangle \to \langle X \rangle$ is an epimorphism. It follows from the Free Group–Hopfian Corollary 130.20 that $\varphi \circ \psi$ is actually an isomorphism. This implies in particular that ψ is a monomorphism. But r lies in the kernel of ψ, this is only possible if $r = e \in \langle X \rangle$. ■

Proof of the Surface-π_1–Not Free Proposition 130.21. Let $g \in \mathbb{N}_0$. We consider the group

$$\pi \;:=\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle.$$

Note that it follows from the Abelianization-of-Presentation Proposition 100.17 that the abelianization of π is isomorphic to $\mathbb{Z}^{2g}$. It follows from the Abelianization Coproduct Proposition 98.18 (2) together with the Basis Cardinality Theorem 97.12 that if π is a free group, then π is a free group of rank $2g$. By the One-Relator Group–Not Free Proposition 130.22 this is only possible if $g = 0$.[1349] ■

Our next goal is to give a useful criterion for a group to be residually finite. To do so we need to introduce the following definition:

Definition 130.23. Let R be a ring. We say that a group G is linear over R if there exists an $n \in \mathbb{N}$ and a monomorphism $G \to \mathrm{GL}(n, R)$.

Examples.

(1) In Exercise 98.11 we showed that the free group on two generators is linear over $\mathbb{Z}$.
(2) Let G be a group and let R be a ring. In Exercise 130.15 we will show that if G admits a finite-index subgroup which is linear over R, then G itself is linear over R. □

Now we can formulate the promised proposition.

[1349] It follows from considering the abelianizations, see the Surface-π_1-Proposition 101.1, that $\pi_1(\text{surface of genus } g)$ cannot be isomorphic to any other free group either.

Proposition 130.24. (Linear–Residually Finite Proposition) Let G be a finitely generated group. If G is linear over $\mathbb{C}$, then G is residually finite.

Sketch of proof. We only provide a full proof of the following easier claim which is the content of Exercise 98.12 (a).

Claim. The group $\mathrm{GL}(n,\mathbb{Z})$ is residually finite.

Proof. Let $A = (a_{ij}) \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix with $A \neq \mathrm{id}$. We pick some $m \in \mathbb{N}$ such that

$$\begin{aligned} m &> \max\{\text{absolute value of the } ij\text{-entry of } A - \mathrm{id} \,|\, i,j = 1,\ldots,n\} \\ &= \max\{|a_{ij} - \delta_{ij}| \,|\, i,j = 1,\ldots,n\}. \end{aligned}$$

Next not that the obvious ring homomorphism $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ induces a group homomorphism $\alpha\colon \mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{Z}/m\mathbb{Z})$. Basically by construction we have $\alpha(A) \neq \mathrm{id}$. Since $\mathrm{GL}(n,\mathbb{Z}/m\mathbb{Z})$ is a finite group we are done. ⊞

Now let G be a finitely generated subgroup of $\mathrm{GL}(n,\mathbb{C})$. The proof that G is residually finite consists of two steps:

(1) First, since G is finitely generated one sees, almost immediately, that there exists a finitely generated subring S of $\mathbb{C}$ such that $G \subseteq \mathrm{GL}(n,S)$.
(2) Next one needs to show that S is a "residually finite ring", i.e. one needs to show that given any non-zero $s \in S$ there exists a ring homomorphism $\alpha\colon S \to T$ to some finite ring T with $\alpha(s) \neq 0$. This part requires some input from ring theory.

Once one has proved (1) and (2) it is straightforward to modify the proof of the claim to conclude that G is residually finite.

The full details for the above proof are given in [**Mal1940, Mal1965**] and also in [**Weh1973**, Theorems 4.1, 4.3] and in [**Hem87**]. ■

We already mentioned that $(\mathbb{Q},+)$ is not residually finite and that $\mathbb{Z}^\infty$ is not Hopfian. These are amusing observations, but most group theorists and geometric topologists restrict their interest to finitely generated groups. Thus the question arises, whether perhaps every finitely generated group is residually finite or Hopfian.

Proposition 130.25. (Baumslag-Solitar Group Proposition) Given $m,n \in \mathbb{N}$ we consider the Baumslag-Solitar group

$$\mathrm{BS}(m,n) \;=\; \langle x,y \,|\, x^{-1}y^m x \,=\, y^n \rangle.$$

The following statements hold:

(1) The group $\mathrm{BS}(m,n)$ is Hopfian if and only if m and n are "meshed", i.e. if and only if m or n divides the other or if m and n have precisely the same prime divisors.
(2) The group $\mathrm{BS}(m,n)$ is residually finite if and only if $|m| = 1$, or if $|n| = 1$ or if $|m| = |n|$.

Example. By the Baumslag-Solitar Group Proposition 130.25 the group $\mathrm{BS}(2,6)$ is Hopfian but not residually finite. This shows, as promised, that the converse to the Residually Finite-Hopfian Proposition 130.14 does not hold. □

Proof. Statement (1) was proved in 1962 by Gilbert Baumslag and Solitar [**BaS1962**]. Statement (2) is proved in [**Mes1972**, Theorem C] and also [**Camp1990**, p. 61]. ■

By the Baumslag-Solitar Group Proposition 130.25 we now know that there exist finitely generated groups which do not have "many" finite quotients. The question arises, does every non-trivial group admit at least some finite index subgroups? It is now time to recall the following standard definition from group theory.

Definition 130.26. A group G is called simple if the only normal subgroups are the trivial group $\{e\}$ and the whole group G.

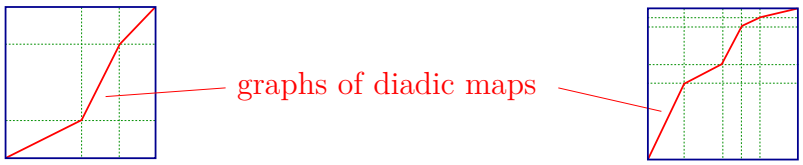
Example. One of the most basic statements in the theory of finite groups is the fact that for $n \geq 5$ the alternating group A_n is actually simple, we refer to [**Bog2008**, Chapter 1.12] for a proof. □

It is now natural to ask whether there exist infinite simple groups. The answer is yes, but one needs to be rather creative when it comes to producing such groups. This discussion leads us to the following groups which were introduced by Richard Thompson in 1965.

Definition 130.27.

(1) A continuous self-homeomorphism $f\colon [0,1] \to [0,1]$ is called dyadic if there exists an $m \in \mathbb{N}$ such that the restriction of f to each interval $[\frac{i}{2^m}, \frac{i+1}{2^m}]$ is linear and the slope on each such interval is a (possibly negative) power of 2.

(2) We say that a map $g\colon S^1 \to S^1$ is dyadic if there exists a dyadic map $f\colon [0,1] \to [0,1]$ such that $g(\exp(2\pi\,\mathrm{i}t)) = \exp(2\pi\,\mathrm{i}f(t))$.



We leave it to the reader to verify that the set of dyadic self-homeomorphisms of $[0,1]$ and of S^1 actually form groups. This observation leads us to the key definition.

Definition 130.28.

(1) Thompson's group F is defined as the group of dyadic self-homeomorphisms of $[0,1]$.

(2) Thompson's group T is defined as the subgroup of the self-homeomorphisms of S^1 that is generated by the dyadic self-homeomorphisms together with the homeomorphism given by $z \mapsto -z$.

The following theorem summarizes some of the many interesting properties of Thompson's groups.

Theorem 130.29. (Thompson's Group T–Properties Theorem)

(1) Thompson's group F and Thompson's group T are both finitely presented.

(2) The commutator subgroup of Thompson's group F is simple, in particular Thompson's group F is not residually finite.

(3) Thompson's group T is simple, in particular it is not residually finite.

Proof.

(1) The statements are proved in [**CFP1996**] and [**Geo2008**, Chapters 9.2].

(2) By [**CFP1996**] and [**Geo2008**, Chapters 9.2] the commutator subgroup of Thompson's group F is simple. One can easily verify that Thompson's group F is non-abelian and torsion-free. It now follows from an elementary argument that it is not residually finite.
(3) It is shown in [**CFP1996**] and [**Geo2008**, Theorem 9.4.3] that Thompson's group T is simple. Clearly T is infinite, hence it cannot be residually finite. ■

In this section we have shown, among many other statements, that free groups are residually finite. It is natural to ask which of the other friendly groups which we have encountered are actually residually finite. Let us pose two questions:

Question 130.30.
(1) Are the fundamental groups of closed 2-dimensional smooth manifolds residually finite?
(2) Is the fundamental group of every knot complement residually finite?

We will return to these questions later on in the Surface Group–Linear Proposition 132.25.

130.6. Exercises for Chapter 130.

Exercise 130.1. Let H be a finite connected graph and let G be a connected non-empty subgraph of H. Let S be a spanning tree of G. Show that there exists a spanning tree T of H such that $T \cap G = S$.
Remark. This exercise is a rather straightforward generalization of the Spanning Tree–Existence Proposition 11.26.

Exercise 130.2. Let $\alpha\colon \langle a, b\rangle \to \mathbb{Z}/3\mathbb{Z}$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. Furthermore let X be the wedge of two circles A and B. As usual we equip X with the CW-structure with one 0-cell P and two 1-cells a and b. Furthermore we make the identification $\pi_1(X) = \langle a, b\rangle$.
(a) "Construct" the covering $p\colon \widetilde{X} \to X$ corresponding to $\ker(\alpha)$, i.e. construct the covering p such that $p_*(\pi_1(\widetilde{X})) = \ker(\alpha)$. More precisely, give the CW-structure on $\widetilde{X}$ and indicate, e.g. using colors, the cellular map p.
(b) We know from the Countable Free Subgroup Proposition 130.5 (2) that $\ker(\alpha)$ is a free group on four generators. Determine four elements of $\ker(\alpha)$ that generate $\ker(\alpha)$ as a free group.

Exercise 130.3. In Question 129.6 we considered an explicit 6-fold covering $X \to B$ where B is the 2-dimensional sphere minus three open disks. On page 2502 we saw that X has six boundary components. Determine the genus of X.

Exercise 130.4. Let $g \in \mathbb{N}_{\geq 2}$. Use the Non-Compact Surface-π_1-Proposition 130.6 to show that the fundamental group of the surface of genus g is torsion-free.

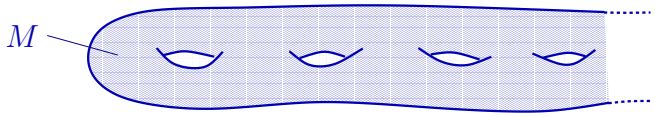
Exercise 130.5. Let P be a property of groups. We say that a group G is *virtually* P if G admits a finite-index subgroup that has property P. Let G be a finitely generated group. Show that if G is virtually abelian, then there exists a short exact sequence of the form $0 \to \mathbb{Z}^d \to G \to A \to 1$ where $d \in \mathbb{N}_0$ and A is a finite group.

Exercise 130.6. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be a covering map of degree d between two closed *non-orientable* connected 2-dimensional smooth manifolds. Show that the following equality holds:

$$\operatorname{genus}(\widetilde{\Sigma}) \;=\; d \cdot (\operatorname{genus}(\Sigma) - 1) + 1.$$

Hint. Let $\varphi\colon \overline{B}^2 \to \Sigma$ be an embedding. Consider the compact connected 2-dimensional smooth manifolds $M := \Sigma \setminus \varphi(B^2)$ and $\widetilde{M} := p^{-1}(M)$. What can you say about the fundamental group and the number of boundary components of $\widetilde{M}$?
Remark. This exercise is meant as the non-orientable analog of the Subgroups-of-Surface Groups Proposition 130.7. In the Genus-of-Covering Lemma 156.18 we will use "Euler characteristics" to give an alternative proof of this equality.

Exercise 130.7. We consider the non-compact 2-dimensional smooth manifold M shown in the figure below. Does there exist a closed 2-dimensional smooth manifold N such that M is a covering space of N?



Exercise 130.8. In Exercise 98.12 we introduced the notion of a residually finite group and we showed that the free group on two generators is residually finite.

(a) Show that every finitely generated free group is residually finite.
Remark. Use the Countable Free Subgroup Proposition 130.5 (2).
(b) Show that every free group is residually finite.

Exercise 130.9. Let π be a countable residually finite group. Show that there exists a descending sequence $\pi \supseteq \pi_0 \supseteq \pi_1 \supseteq \pi_2 \supseteq \dots$ of finite index normal subgroups of π such that we have $\bigcap\limits_{i\in\mathbb{N}} \pi_i = \{e\}$.

Exercise 130.10. Let G be a finitely generated group. Show that given any $n \in \mathbb{N}$ there exist only finitely many subgroups of G of index n.

Exercise 130.11. Let G be a group. A subgroup Γ of G is called *characteristic* if for every automorphism $\varphi\colon G \to G$ we have $\varphi(\Gamma) = \Gamma$.

(a) Show that if subgroup is characteristic, then it is also normal.
(b) Let H be a finite-index subgroup of G. Suppose that G is finitely generated. Show that there exists a characteristic subgroup $K \subseteq G$ with $K \subseteq H$ and such that K has finite index in G.
(c) Show that in (b) we cannot drop the hypothesis that G is finitely generated.

Exercise 130.12. Show that the group $(\mathbb{Q}, +)$ is not residually finite.

Exercise 130.13. Let $\varphi\colon A \to B$ be a homomorphism between free groups of the same finite rank. Suppose that $\varphi_*\colon A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is an isomorphism. Show that φ is a monomorphism.
Hint. Use the Countable Free Subgroup Proposition 130.5 and the Free Group–Hopfian Corollary 130.20.

Exercise 130.14. Let G be a group and let $\varphi\colon G \to G$ be an automorphism. We suppose that G is residually finite and finitely generated. Show that the semidirect product $G \rtimes_\varphi \mathbb{Z}$, that we introduced in Definition 93.11, is also residually finite.
Hint. Make good use of Exercise 130.11.

Exercise 130.15. Let R be a ring, let G be a group and let H be a subgroup of finite index $d \in \mathbb{N}$. Suppose that there exists a monomorphism $\varphi\colon H \to \mathrm{GL}(n, R)$. Show that there exists a monomorphism $\varphi\colon G \to \mathrm{GL}(n \cdot d, R)$.

Exercise 130.16.

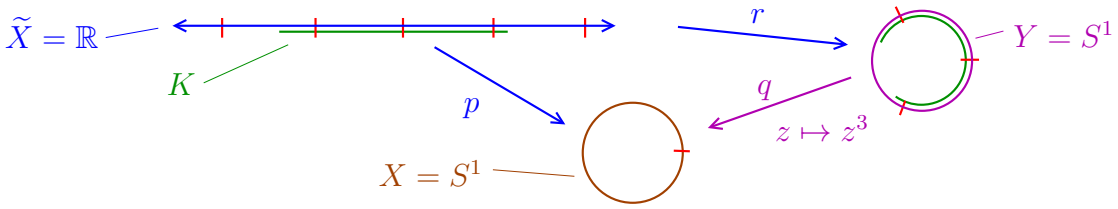
(a) Provide an example of a group that is not finitely generated but that is nonetheless Hopfian.

(b) Show that every infinite simple group is Hopfian.

Exercise 130.17. Let $k \in \mathbb{N}_{\geq 2}$. We consider the direct product $\langle x_1, \ldots, x_k \rangle \times \mathbb{Z}$. Show that given any $N \in \mathbb{N}$ there exists a finitely generated free group F of rank $\geq N$ and an automorphism φ of F such that $\langle x_1, \ldots, x_k \rangle \times \mathbb{Z}$ is isomorphic to the semidirect product $F \rtimes_\varphi \mathbb{Z}$.

Remark. Most of the hard work was already done in Exercise 100.10.

Exercise 130.18. Let X be a path-connected non-empty topological space. We denote by $p\colon \widetilde{X} \to X$ the universal covering. Let $K \subseteq \widetilde{X}$ be a compact subset. Show that if $\pi_1(X)$ is residually finite, then there exists a finite covering $q\colon Y \to X$ such that for the induced covering $r\colon \widetilde{X} \to Y$, that is given by the Universal Covering Proposition 127.20, the restriction of r to $r|_K\colon K \to Y$ is an embedding.



CHAPTER 131

Hyperbolic geometry

In this chapter we will switch gears and we will introduce some basics on hyperbolic geometry. The reason for doing so is that in Chapter 132 we will see that every surface of genus ≥ 2 admits a hyperbolic structure, and this fact will be extremely helpful in getting a better understanding of surfaces.

This chapter is organized as follows:

- In Section 131.1 we will introduce hyperbolic space.
- In Section 131.2 we will discuss angles in Riemannian manifolds.
- In Section 131.3 we will study the distance metric of a Riemannian manifold.
- In Section 131.4 we will discuss the hyperbolic distance function.
- In Section 131.5 we will provide exercises for Chapter 131.

In the next Chapter 132 we will use hyperbolic geometry to discuss the universal coverings of surfaces.

131.1. **Hyperbolic space.** We start out with the following definition, which is modelled on Definitions 2.25 and 2.27.

Definition 131.1. Let V be a real vector space. A symmetric form on V is a map[1350]

$$g\colon V \times V \to \mathbb{R}$$

that is $\mathbb{R}$-bilinear, i.e. for all $u, v, w \in V$ and $\mu, \lambda \in \mathbb{R}$ we have

$$\begin{aligned} g(\lambda \cdot u + \mu v, w) &= \lambda \cdot g(u, w) + \mu g(v, w) \\ g(u, \lambda \cdot v + \mu \cdot w) &= \lambda \cdot g(u, v) + \mu \cdot g(u, w) \end{aligned}$$

and that is symmetric, i.e. for all $v, w \in V$ we have

$$g(v, w) = g(w, v).$$

We say that g is positive-definite if for all $v \in V \setminus \{0\}$ we have

$$g(v, v) > 0.$$

Now we can recall the following key definition from the theory of smooth manifolds:

Definition 131.2.

(1) A Riemannian structure on a smooth manifold M is a smooth[1351] map g which assigns to each point P a positive-definite symmetric form g_P on the tangent space $\mathrm{T}_P M$.

(2) A Riemannian manifold is a pair (M, g) consisting of a smooth manifold M together with a Riemannian structure g.

[1350] Sometimes such a structure is called a *bilinear symmetric form*, but we abbreviate this to *symmetric form*.

As so often in mathematics, given some structure one is very much interested in maps that preserve the given structure. This leads us to the following definition:

Definition 131.3.

(1) Given a smooth map $f\colon M \to N$ between smooth manifolds and given $P \in M$ we denote by $\mathrm{D}f_P = f_*\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}N$ the differential, i.e. the induced homomorphism from the tangent space $\mathrm{T}_P M$ to the tangent space $\mathrm{T}_{f(P)}N$.

(2) Let $f\colon M \to N$ be a local diffeomorphism. Given a Riemannian structure h on N we denote by f^*h the Riemannian structure on M that at each point $P \in M$ is given by $(f^*h)(v,w) = h(f_*v, f_*w)$.[1352]

(3) We say that a map $f\colon (M,g) \to (N,h)$ between Riemannian manifolds is a (local) isometry if f is a (local) diffeomorphism and if $g = f^*h$.

(4) Given a Riemannian manifold (M,g) we denote by $\mathrm{Isom}(M,g)$ the group of isometries of (M,g). If M is oriented, then we denote by $\mathrm{Isom}^+(M,g)$ the group of orientation-preserving isometries of (M,g).

(5) We say that Riemannian manifolds (M,g) and (N,h) are isometric if there exists an isometry from (M,g) to (N,h).

In this chapter and the following chapter we are mostly interested in Riemannian structures on surfaces, and more specifically, in hyperbolic structures. Now we give the definition of hyperbolic space.

Definition 131.4. The hyperbolic half-plane is the smooth manifold

$$\mathbb{H} \;=\; \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \;=\; \{z = x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$$

together with the Riemannian structure that assigns to the tangent space $\mathrm{T}_z\mathbb{H} = \mathbb{R}^2$ at a point $z = x + \mathrm{i}y \in \mathbb{H}$ the positive-definite symmetric form

$$\begin{array}{rcl} g\colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (v,w) & \mapsto & \frac{1}{y^2} \cdot \langle v, w\rangle \end{array}$$

where $\langle v, w\rangle$ denotes the usual standard inner product on $\mathbb{R}^2$.

Example. In the figure below we see a decomposition of $\mathbb{H}$ into angels and devils.



[1351]The notion of "smooth" in this context can be made precise in various equivalent ways. We will give a formal definition in 269.8. For the purpose of the current discussion one can also say that g is smooth if for any two smooth vector fields v, w on M the map $M \to \mathbb{R}$ that is given by $P \mapsto g(v,w)$ is smooth.

[1352]It follows from [**Lee2002**, Proposition 13.9] that f^*h is indeed a Riemannian structure on M. Note that in the proof, to show that f^*h is indeed non-degenerate, one needs that f is a local diffeomorphism.

With respect to the hyperbolic metric all the angels have the same size and all the devils have the same size, even though they differ in the usual Euclidean sense. □

As we mentioned before, we are very much interested in the group $\mathrm{Isom}^+(\mathbb{H})$ of orientation-preserving isometries of the Riemannian manifold $\mathbb{H}$. This leads us to the following definition:

Definition 131.5. A Möbius transformation of $\mathbb{H}$ is a map of the form[1353]

$$\begin{array}{ccc} \mathbb{H} & \to & \mathbb{H} \\ z & \mapsto & \dfrac{az+b}{cz+d} \end{array}$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

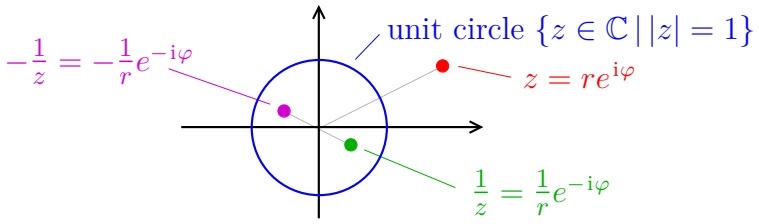
Examples. We give three special types of Möbius transformations of $\mathbb{H}$:

(1) Given $r > 0$ the scalar multiplication $z \mapsto rz = \frac{\sqrt{r}\cdot z+0}{0\cdot z+\frac{1}{\sqrt{r}}}$ is a Möbius transformation.

(2) Given $d \in \mathbb{R}$ the horizontal translation $z \mapsto z + d = \frac{1\cdot z+d}{0\cdot z+1}$ is a Möbius transformation.

(3) The map $z \mapsto -\frac{1}{z} = \frac{0\cdot z+1}{(-1)\cdot z+0}$ is a Möbius transformation.

The third Möbius transformation is the composition of taking the inverse $z \mapsto \frac{1}{z}$ and the reflection $z \mapsto -z$ in the origin. We illustrate this map in the figure below.



For simplicity we refer to the map $z \mapsto -\frac{1}{z}$ as an *inversion*. □

Rather amazingly we are already done with listing all elements in $\mathrm{Isom}^+(\mathbb{H})$. More precisely, we have the following proposition:

Proposition 131.6. (Hyperbolic Isometries Proposition) The map

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \to & \mathrm{Isom}^+(\mathbb{H}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} \mathbb{H} & \to & \mathbb{H} \\ z & \mapsto & \frac{az+b}{cz+d} \end{pmatrix} \end{array}$$

is a group isomorphism.

Sketch of proof. One can easily calculate by hand that each Möbius transformation is an orientation-preserving isometry of $\mathbb{H}$. Furthermore, it is an amusing calculation to verify that the given map is indeed a group homomorphism. It is elementary to see that the map is a monomorphism. The only bit of the proposition which requires some thought is the statement that every orientation-preserving isometry is already a Möbius transformation. This statement is for example shown in [**And2005**, Theorem 3.19]. ■

[1353] An elementary calculation shows that the image of a Möbius transformation $\mathbb{H} \to \mathbb{C}$ does indeed lie again in $\mathbb{H}$.

Now that we have seen that every orientation-preserving isometry of $\mathbb{H}$ is a Möbius transformation it makes sense to study Möbius transformations in greater detail.

Proposition 131.7. (Möbius Transformations–Characterization Proposition) Every Möbius transformation of $\mathbb{H}$ is the composition of scalar multiplications, horizontal translations and inversions.

Proof. Let $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and denote by

$$\Phi(z) \;=\; \frac{az+b}{cz+d}$$

the corresponding Möbius transformation. If $c = 0$, then

$$\Phi(z) \;=\; \frac{a}{d}z + \frac{b}{d}$$

is the composition of the scalar multiplication[1354] $z \mapsto \frac{a}{d}z$ with the horizontal translation $z \mapsto \frac{b}{d} + z$. If $c \neq 0$, then

$$\Phi(z) \;=\; \frac{az+b}{cz+d} \;=\; \frac{acz+bc}{c^2z+dc} \underset{\text{since } ad-bc=1}{\overset{\uparrow}{=}} \frac{acz+ad-1}{c^2z+cd} \;=\; \frac{-1+a(cz+d)}{c^2z+cd} \;=\; \frac{-1}{c^2z+dc} + \frac{a}{c}.$$

Thus we see that Φ is the composition of the following maps:

$$z \mapsto c^2z, \quad z \mapsto z+dc, \quad z \mapsto \frac{-1}{z} \quad \text{and } z \mapsto z + \frac{a}{c}. \qquad \blacksquare$$

Definition 131.8.

(1) The hyperbolic disk[1355] is the smooth manifold

$$\mathbb{D} \;=\; \{z \in \mathbb{C} \mid |z|^2 < 1\}$$

together with the Riemannian structure which assigns to the tangent space $\mathrm{T}_z\mathbb{D} = \mathbb{R}^2$ at a point $z \in \mathbb{D}$ the positive-definite symmetric form

$$\begin{array}{rcl} g_z \colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (v,w) & \mapsto & \dfrac{2}{(1-|z|^2)^2}\langle v, w\rangle \end{array}$$

where, as above, $\langle v, w\rangle$ denotes the usual standard inner product on $\mathbb{R}^2$.

(2) A Möbius transformation of $\mathbb{D}$ is a map of the form[1356]

$$\begin{array}{rcl} \mathbb{D} & \to & \mathbb{D} \\ z & \mapsto & \exp(\mathrm{i}\theta) \cdot \frac{z-a}{1-\bar{a}z} \end{array}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

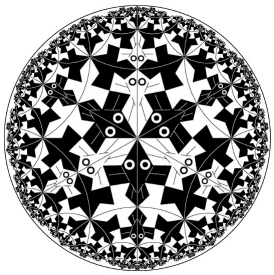
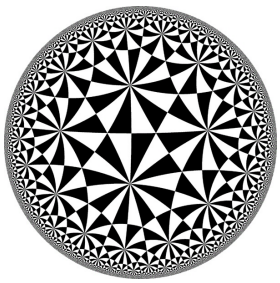
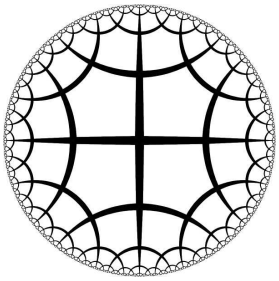
Examples.

(1) In the figure below we see three different decompositions of $\mathbb{D}$ into subsets. In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual Euclidean sense.

[1354] Note that $c = 0$ implies that $1 = ad - bc = ad$, in particular $ad > 0$, hence $\frac{a}{d} > 0$.

[1355] The hyperbolic disk is often also called the *Poincaré model for hyperbolic space*.

[1356] An elementary calculation shows that the image of a Möbius transformation $\mathbb{D} \to \mathbb{C}$ does indeed lie again in $\mathbb{D}$.

(2) If we set $a = 0$ in the definition of a Möbius transformation of $\mathbb{D}$, then we obtain precisely the rotation by the angle θ. Put differently, the Möbius transformations preserving 0 are precisely given by rotations around the origin. □

The following proposition will allow us to go back and forth between the hyperbolic half-plane and the hyperbolic disk.

Proposition 131.9. (Hyperbolic Models Proposition) The maps

$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{\mathrm{i}+\mathrm{i}w}{1-w} \end{array}$$

have the following properties:

(1) Φ and Ψ are inverses to one another, in particular they are biholomorphisms.
(2) They extend to homeomorphisms between $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$[1357] and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$.
(3) They are orientation-preserving.[1358]
(4) They are isometries.
(5) They give a bijection between the Möbius transformations of $\mathbb{D}$ and $\mathbb{H}$.[1359]

Proof. All five statements follow from a straightforward calculation. The statements can also be deduced, more or less directly, from any book that deals with Möbius transformations, see e.g. [**Schwe1979**, Chapter 6] and [**And2005**, Chapter 2]. ■

The following proposition summarizes some of the key properties of Möbius transformations.

Proposition 131.10. (Möbius Transformations–Proposition)[1360]

(1) Compositions of Möbius transformations are again Möbius transformations and the inverse of a Möbius transformation is again a Möbius transformation.
(2) Möbius transformations are biholomorphisms, they are isometries and they are orientation-preserving.
(3) Möbius transformations act transitively, i.e. given any two points P and Q there exists a Möbius transformation which sends P to Q.

[1357] Here $\overline{\mathbb{H}}$ has the same topology as defined on page 152.
[1358] The hyperbolic half-plane $\mathbb{H}$ and the hyperbolic disk $\mathbb{D}$ are both subsets of $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ and inherit from $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ the natural orientation.
[1359] More precisely, if α is a Möbius transformation of $\mathbb{H}$, then $\Phi \circ \alpha \circ \Psi$ is a Möbius transformation of $\mathbb{D}$ and conversely, if α is a Möbius transformation of $\mathbb{D}$, then $\Psi \circ \alpha \circ \Phi$ is a Möbius transformation of $\mathbb{H}$.

(4) Given $P, Q \in \mathbb{D}$ and non-zero vectors $v \in \mathrm{T}_P\mathbb{D}$ and $w \in \mathrm{T}_Q\mathbb{D}$ with $\|v\|_g = \|w\|_g$ there exists a Möbius transformation ϕ with $\phi(P) = Q$ and $d\phi_P(v) = w$.
(5) Let ϕ and ψ be two Möbius transformations. Suppose that there exists a $P \in \mathbb{D}$ such that $\phi(P) = \psi(P)$ and such that $\mathrm{D}\phi_P = \mathrm{D}\psi_P$. Then $\phi = \psi$.

Proof. By the Hyperbolic Models Proposition 131.9 it suffices to prove the statements for $\mathbb{D}$ or for $\mathbb{H}$, whichever is more convenient.

(1),(2) The first two statements follow from a straightforward calculation, see e.g. [**Schwe1979**, p. 43].

(3) We prove this statement for $\mathbb{D}$. Note that by (1) it suffices to show that given any $P \in \mathbb{D}$ there exists a Möbius transformation ψ with $\psi(P) = 0$. But such a Möbius transformation is given by

$$\psi(z) \;=\; \frac{z-P}{1-\overline{P}z}.$$

(4) Using (3) we see that we only have to deal with the case that $P = Q = 0$. But then the desired Möbius transformation is given by a rotation.

(5) By (3) there exists a Möbius transformation θ such that $\theta(0) = P$ and there exists a Möbius transformation ω such that $\omega(\phi(P)) = \omega(\psi(P)) = 0$. We consider the Möbius transformations $\phi' = \omega \circ \phi \circ \theta$ and $\psi' = \omega \circ \psi \circ \theta$. It follows from our hypothesis that both ϕ' and ψ' fix 0. As we pointed out above, the only Möbius transformations of $\mathbb{D}$ that fix 0 are rotations.

By our hypothesis we have $\mathrm{D}\phi'_0 = \mathrm{D}\psi'_0$, which means that the rotations ϕ' and ψ' agree. But this implies that ϕ and ψ also agree. ■

Note that Möbius transformations extend in an obvious way uniquely to homeomorphisms of $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. We will denote the extension of a Möbius transformation with the same symbol.

Lemma 131.11. (Boundary Triples Transitivity Lemma) Let $\{P, Q, R\}$ and let $\{S, T, U\}$ be two sets of three distinct points on the boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. Then there exists a Möbius transformation ψ of $\mathbb{H}$ with $\psi(P) = S$ and $\psi(\{Q, R\}) = \{T, U\}$.

Proof. Let P, Q, R be three distinct points on $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. It suffices to show that there exists a Möbius transformation ψ with $\psi(P) = \infty$ and with $\psi(\{Q, R\}) = \{0, 1\}$. We proceed as follows:

(1) We first claim that there exists a Möbius transformation α with $\alpha(P) = \infty$. Indeed, if $P = \infty$ then we just take $\alpha = \mathrm{id}$. Otherwise we can combine a horizontal translation with the inversion to obtain a Möbius transformation α with $\alpha(P) = \infty$. Given such α we write $Q' = \alpha(Q)$ and $R' = \alpha(R)$. Note that the points Q', R' lie on $\mathbb{R} = \overline{\mathbb{H}} \setminus \{\infty\}$. Without loss of generality we can assume that $Q' < R'$.

(2) Next we pick a horizontal translation such that $\beta(Q') = 0$. Note that we still have $(\beta \circ \alpha)(P) = \beta(\infty) = \infty$. Also note that it follows from $Q' < R'$ that $\beta(R') > 0$.

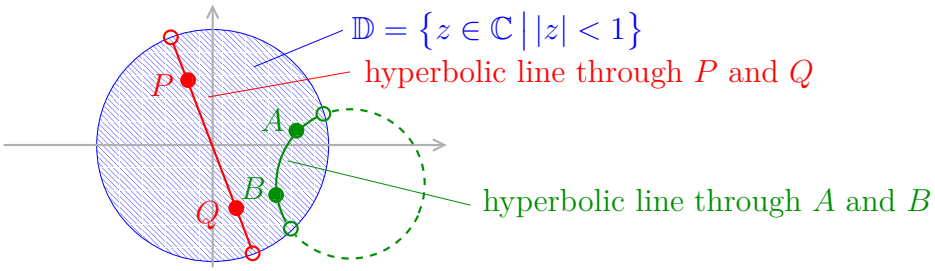
(3) Finally we pick a scalar multiplication γ such that $(\gamma \circ \beta)(R') = 1$. Note that γ fixes 0 and ∞. Thus we see that $\psi = \gamma \circ \beta \circ \alpha$ has the desired property. ■

[1360]The statement of the proposition applies to Möbius transformations of $\mathbb{D}$ and to Möbius transformations of $\mathbb{H}$.

Definition 131.12. A hyperbolic line is a subset of $\mathbb{D}$ of one of the following two types:

(1) it is the intersection of $\mathbb{D}$ with a Euclidean line through the origin,
(2) it is the intersection of $\mathbb{D}$ with a Euclidean circle that intersects the circle $S^1 = \partial\overline{\mathbb{D}}$ orthogonally.

In each of the two cases we refer to the intersection of the Euclidean object with $S^1 = \partial\overline{\mathbb{D}}$ as the endpoints of the hyperbolic line.



Lemma 131.13. (Hyperbolic Lines Lemma)

(1) For any two distinct points P and Q in $\mathbb{D}$ there exists a unique hyperbolic line that contains P and Q.
(2) For any two distinct points P and Q in $\partial\overline{\mathbb{D}}$ there exists a unique hyperbolic line with P and Q as endpoints.
(3) Let P and Q be two distinct points in $\partial\overline{\mathbb{D}}$. Every Möbius transformation ϕ sends the hyperbolic line with endpoints P and Q to the hyperbolic line with endpoints $\phi(P)$ and $\phi(Q)$.
(4) Möbius transformations act transitively on hyperbolic lines, i.e. given any two hyperbolic lines g and h there exists a Möbius transformation ϕ with $\phi(g) = h$.

Proof. In principle one can prove the lemma directly in $\mathbb{D}$. But it is easier to prove it using $\mathbb{H}$. In $\mathbb{H}$ the hyperbolic lines are defined to be vertical lines $\{x + \mathrm{i}y \mid y > 0\}$ and Euclidean half-circles such that the origin lies on $\mathbb{R}$. It is straightforward to see that the diffeomorphisms of the Hyperbolic Models Proposition 131.9 define a bijection between the hyperbolic lines in $\mathbb{D}$ and in $\mathbb{H}$.

The corresponding statements for hyperbolic lines in $\mathbb{H}$ are straightforward to prove. They can also be obtained from any textbook on hyperbolic geometry, see e.g. [**And2005**] and [**Bona2009**, Chapter 2]. ■

131.2. **Angles in Riemannian manifolds.** We continue our quick introduction to Riemannian manifolds and hyperbolic space with a short discussion of angles.

Definition 131.14.

(1) Let g be a positive-definite symmetric form on a real vector space V. Given two non-zero vectors $v, w \in V \setminus \{0\}$ we define[1361]

$$\text{angle between } v \text{ and } w \;:=\; \text{unique } \alpha \in [0, \pi] \text{ with } g(v, w) = \cos(\alpha) \cdot \|v\| \cdot \|w\|.$$

(2) Given a Riemannian manifold (M, g) and non-zero vectors $v, w \in \mathrm{T}_P M$ we use the form g_P to define the angle between v and w.

[1361] We have $|g(v, w)| \leq \big|\|v\| \cdot \|w\|\big|$ by the Cauchy-Schwarz Inequality 2.28.

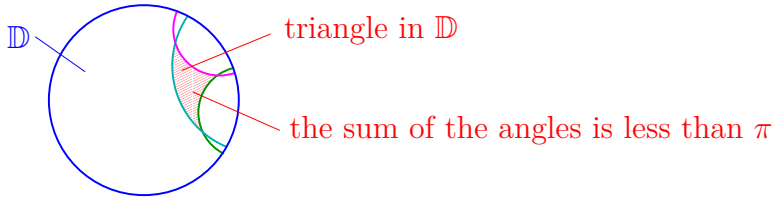
Examples.

(1) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}_{>0}$ be a smooth function. For any $P \in U$ we can define a positive-definite symmetric form g_P on $\mathrm{T}_P U = \mathbb{R}^n$ via

$$g_P(v,w) \;=\; f(P)\cdot\langle v,w\rangle,$$

where $\langle\,,\,\rangle$ denotes the usual standard inner product. A straightforward calculation shows that in this Riemannian structure the angle at any point is the usual Euclidean angle in $\mathbb{R}^n$. This argument implies in particular to the Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$, in both cases the hyperbolic angle, defined by the hyperbolic Riemannian structure, is the same as the Euclidean angle.

(2) A hyperbolic triangle is defined to be a compact subset of $\mathbb{H}$ whose boundary is given by hyperbolic segments of three hyperbolic lines. One of the many elegant theorems in hyperbolic geometry, see [**And2005**, Theorem 5.16], says that given such a hyperbolic triangle its angle sum equals π minus its "area".



Even without the definition of "area" it is pretty clear that this result implies that the angle sum of a hyperbolic triangle is always less than π. □

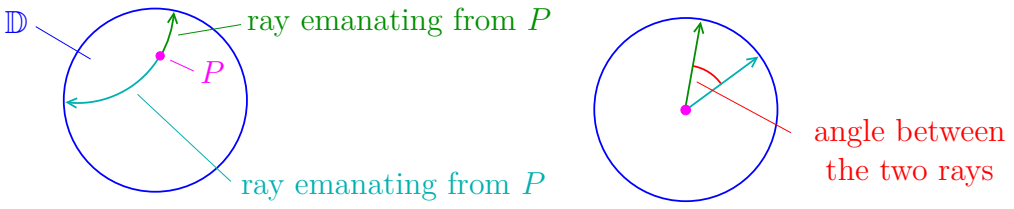
Lemma 131.15. (Möbius Transformations–Angles Lemma) Local isometries, in particular Möbius transformations, preserve angles.

Proof. The lemma follows easily from the definitions and from the Möbius Transformations–Proposition 131.10 (2). ■

Definition 131.16.

(1) Let $P \in \mathbb{D}$ and let g be a hyperbolic line through P. Let Q be one of the two endpoints of g. We refer to the set of points between P and Q on g as a ray emanating from P.

(2) Let $P \in \mathbb{D}$ and let r, s be two rays emanating from P. We refer to the angle between the tangent vectors of r and s at P as the angle between the rays r and s.



Examples.

(1) Given $\varphi \in [0, 2\pi)$ the set $S_\varphi := \{r \cdot \exp(\mathrm{i}\varphi) \mid r \in [0,1)\}$ is a ray emanating from 0.

(2) It follows from the above discussion of angles in $\mathbb{D}$, that given $\varphi, \theta \in [0, 2\pi)$ the angle between the two rays S_φ and S_θ in $\mathbb{D}$ is the usual Euclidean angle. □

Lemma 131.17. (Ray–Angle Lemma) Let r, s be two rays emanating from $P \in \mathbb{D}$ and let r', s' be two rays emanating from $P' \in \mathbb{D}$. Then the following holds:

$$\text{there is a Möbius transformation } \phi \text{ with } \phi(r \cup s) = r' \cup s' \iff \text{angle between } r, s \;=\; \text{angle between } r', s'.$$

Furthermore, if such a Möbius transformation exists, then it is unique.

Proof. By the Möbius Transformations–Angles Lemma 131.15 Möbius transformations preserve angles and by the Möbius Transformations–Proposition 131.10 Möbius transformations act transitively on $\mathbb{D}$. Therefore we can without loss of generality assume that $P = P' = 0$. The lemma follows from the observation, made on page 2530, that Möbius transformations preserving 0 are precisely given by rotations around the origin. ∎

131.3. The distance metric of a Riemannian manifold.

Definition 131.18. Let (M, g) be a connected Riemannian manifold.

(1) Given $v \in \mathrm{T}_P M$ we write $\|v\|_g := \sqrt{g_P(v, v)}$. If g is understood from the context, then we just write $\|v\|$.

(2) Let $\gamma\colon [a, b] \to M$ be a piecewise smooth path. By definition we can pick a subdivision $a = s_0 < s_1 < \cdots < s_m = b$ such that for $i = 0, \ldots, m-1$ the map $\gamma|_{[s_i, s_{i+1}]}$ is smooth. We define the length of γ as[1362]

$$\ell_{(M,g)}(\gamma) := \sum_{i=0}^{m-1} \int_{t=s_i}^{t=s_{i+1}} \|\gamma'(t)\|_g \, dt.$$

If (M, g) is understood from the context, then we write $\ell(\gamma)$ instead of $\ell_{(M,g)}(\gamma)$.

(3) Let P and Q be two points on M.

$$d_{(M,g)}(P, Q) := \inf\{\ell(\gamma) \mid \underbrace{\gamma \text{ is a smooth path in } M \text{ from } P \text{ to } Q}_{\substack{\text{exists by the Smooth Path-Connectivity} \\ \text{Proposition 25.32}}}\}.$$

If (M, g) is understood from the context, then we just write $d(P, Q)$.

We start our discussion of the distance function d on Riemannian manifolds with the following somewhat technical proposition.

Proposition 131.19. (Riemannian Metric Proposition) Let (M, g) be a connected Riemannian manifold. The following statements hold:

(1) The pair $(M, d_{(M,g)})$ is a metric space.

(2) The topology defined by the metric $d_{(M,g)}$ agrees with the topology of M.

(3) Given any $P \in M$ and given any open neighborhood U of P there exists an $r > 0$ such that $B_r^{(M,g)}(P) := \{Q \in M \mid d_{(M,g)}(P, Q) < r\} \subseteq U$.

Proof. Let (M, g) be a connected n-dimensional Riemannian manifold. We first show that $(M, d_{(M,g)})$ is a metric space.

[1362] Recall that given a smooth path $\delta\colon [a, b] \to M$ and given any $t \in [a, b]$ the path defines, basically by definition of the tangent space $\mathrm{T}_{\delta(t)} M$, an element in $\mathrm{T}_{\delta(t)} M$. We denote this vector by $\delta'(t)$.

It is clear that $d = d_{(M,g)}$ is symmetric. It follows easily from the definitions and the fact that the concatenation of two piecewise smooth paths is again piecewise smooth that d satisfies the triangle inequality.

Now let $P \neq Q$ be two distinct points in M. We need to show that $d(P,Q) > 0$. Since M is Hausdorff and since it is locally diffeomorphic to open subsets of $\mathbb{R}^n$ we can find a chart $\Phi\colon U \to B_{2r}(0)$ for P with $\Phi(P) = 0$ and such that $Q \notin U$. We write $\Psi = \Phi^{-1}$. Before we continue we introduce the following definitions:

(1) We write $U_r = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| = r\})$ and given a subset $I \subseteq [0,2r)$ we write $U_I = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| \in I\})$.

(2) For a positive-definite bilinear form h on $\mathbb{R}^n$ we set

$$\|h\| \coloneqq \min\left\{\sqrt{h(v,v)} \,\middle|\, v \in S^{n-1}\right\}.$$

Note that h is positive-definite, that $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that S^{n-1} is compact. It follows from the Compact Image Lemma 4.15 that the minimum exists and that $\|h\| > 0$. It is straightforward to see that for any $v \in \mathbb{R}^n$ we have

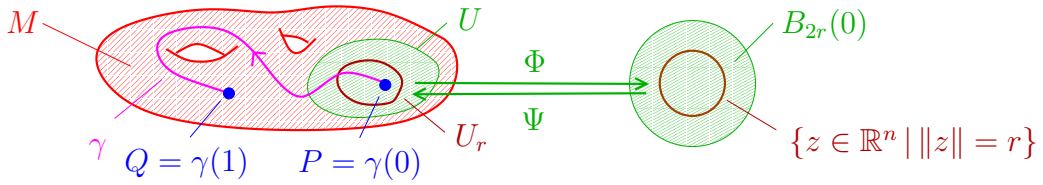
$$h(v,v) \geq \|h\| \cdot \|v\|_{\text{eucl}}.$$

(3) We set

$$C \coloneqq \min\left\{\|(\Psi^* g)_P\| \,\middle|\, P \in \overline{B_r(0)}\right\}.$$

Since $P \mapsto \|(\Psi^* g)_P\|$ is continuous and since $\overline{B_r(0)}$ is compact it follows from the Compact Image Lemma 4.15 that $C > 0$.

Claim 1. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. There exists a $t \in [0,1]$ with $\gamma(t) \in U_r$.



Proof. Suppose that there is no $t \in (0,1)$ with $\gamma(t) \in U_r$. We consider the sets $A = U_{[0,r)}$ and $B = M \setminus U_{[0,r]}$. The set A is clearly open. Furthermore it follows from the Compactness-Chart Lemma 23.11 that $U_{[0,r]}$ is closed, hence B is open. Since $\gamma([0,1]) \cap U_r = \varnothing$ we have $[0,1] = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$. But both sets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are open by the continuity of γ and both are non-empty since $0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$. But this contradicts the fact, established with some effort in the Interval-Connected Proposition 4.32, that $[0,1]$ is connected. ⊞

Now it suffices to prove the following claim:

Claim 2. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. Then $\ell_g(\gamma) \geq C \cdot r$.

Proof. Let $T \coloneqq \inf\{t \in (0,1) \mid \gamma(t) \in U_r\}$. By Claim 1 this definition makes sense since we take the infimum of a non-empty set. By the Compactness-Chart Lemma 23.11 the set U_r

is a closed subset of M. Since γ is continuous we also have $\gamma(T) \in U_r$. Now we have

$$\begin{array}{rcll} \ell_g(\gamma) & \geq & \ell_g(\gamma|_{[0,T]}) & = \ell_{\Psi^* g}(\Phi \circ \gamma|_{[0,T]}) \\ & & & = \int_{t=0}^{t=T} \|(\Phi \circ \gamma)'(t)\|_{\Psi^* g}\, dt \\ & & \geq \int_{t=0}^{t=T} C \cdot \|(\Phi \circ \gamma)'(t)\|_{\text{eucl}}\, dt & \geq C \cdot \left\| \int_{t=0}^{t=T} (\Phi \circ \gamma)'(t)\, dt \right\|_{\text{eucl}} \\ & & \uparrow & \uparrow \\ & & \text{for any } v \in \mathrm{T}_P \overline{B_r(0)} = \mathbb{R}^n \text{ we have} & \text{by the Integral Inequality Lemma 2.45} \end{array}$$

$$\begin{aligned} \|v\|_{(\Psi^* g)_P} &\geq \|(\Psi^* g)_P\| \cdot \|v\|_{\text{eucl}} \\ &= C \cdot \|\underbrace{(\Phi \circ \gamma)(T)}_{\|-\|_{\text{eucl}} = r} - \underbrace{(\Phi \circ \gamma(0))}_{=0}\|_{\text{eucl}} = C \cdot r. \end{aligned}$$

This concludes the proof of the claim and therefore the proof of Statement (1) of the proposition. ⊞

Next we turn to the proof of Statement (2). This means that we need to show that the topology defined by d is the same as the original topology of M. The proof is again somewhat technical and uses ideas similar to the proof of (1). We refer to [**Lee1997**, Lemma 6.2] for the proof.

Finally Statement (3) is just an immediate consequence of the definitions and of Statement (2). ■

Lemma 131.20. (Local Isometry–Distance Lemma) Let (M, g) and (N, h) be two connected Riemannian manifolds. Furthermore let $f\colon (M, g) \to (N, h)$ be a local isometry. Then the following hold:

(1) For any piecewise smooth path $\gamma\colon [a, b] \to M$ we have
$$\ell_{(N,h)}(f \circ \gamma) = \ell_{(M,g)}(\gamma).$$
(2) For any $P, Q \in M$ we have
$$d_{(N,h)}(f(P), f(Q)) \leq d_{(M,g)}(P, Q).$$

Proof. Let $f\colon (M, g) \to (N, h)$ be a local isometry. We start out with the following claim:

Claim.
(1) For a smooth path $\gamma\colon [c, d] \to M$ and $t \in (c, d)$ we have $\|(f \circ \gamma)'(t)\|_{(N,h)} = \|\gamma'(t)\|_{(M,g)}$.
(2) For any piecewise smooth path $\gamma\colon [a, b] \to M$ we have $\ell_{(N,h)}(f \circ \gamma) = \ell_{(M,g)}(\gamma)$.

Proof.

(1) Statement (1) is an immediate consequence of the definitions and the hypothesis that f is a local isometry.
(2) By the additivity of the lengths of piecewise smooth paths it suffices to consider the case that $\gamma\colon [a, b] \to M$ is a smooth path. But in that case the equality follows immediately from the definitions and from (1). ⊞

Now let P and Q be two points on M. For any piecewise smooth path p in M from P to Q we have

$$\begin{array}{ccccc} d_{(N,h)}(f(P), f(Q)) & \leq & \ell_{(N,h)}(f \circ p) & = & \ell_{(M,g)}(p). \\ & \uparrow & & \uparrow & \\ & \text{by definition of } d_{(N,h)} & & \text{by the claim} & \end{array}$$

It follows that the left-hand side is also less or equal than the infimum over all paths on the right-hand side. Thus $d_{(N,h)}(f(P), f(Q)) \leq d_{(M,g)}(P, Q)$. ■

131.4. **The hyperbolic distance function.** We start out with the following definition:

Definition 131.21. Let (M, g) be a Riemannian manifold. A path $\gamma\colon [0, a] \to M$ is called a geodesic if the following conditions are satisfied:
(1) γ is smooth,
(2) $\|\gamma'(t)\|_g = 1$ for every $t \in [0, a]$ and
(3) $\ell(\gamma) = d_{(M,g)}(\gamma(0), \gamma(a))$.

The following proposition summarizes the key facts about the distance metric on $\mathbb{D}$.

Proposition 131.22. (Geodesic Proposition)
(1) Let $r \in (0, 1)$ and $\varphi \in \mathbb{R}$. In the Riemannian manifold $\mathbb{D}$ the path

$$\begin{array}{rcl} \gamma\colon [0, r] & \to & \mathbb{D} \\ t & \mapsto & t \cdot \exp(\mathrm{i}\varphi) \end{array}$$

is, up to reparametrization, the unique shortest path from 0 to $r \cdot \exp(\mathrm{i}\varphi)$.
(2)
$$d_{(\mathbb{D},g)}\big(0, r \cdot \exp(\mathrm{i}\varphi)\big) \;=\; \ell(\gamma) \;=\; \int_{t=0}^{t=r} \frac{\sqrt{2}}{1 - |\gamma(t)|^2} \cdot \|\gamma'(t)\|_{\mathrm{eucl}}\, dt \;=\; \int_{t=0}^{t=r} \frac{\sqrt{2}}{1 - t^2}\, dt.$$
(3) Given $\varphi \in \mathbb{R}$ we have
$$\lim_{r \to 1} d_{(\mathbb{D},g)}\big(0, r \cdot \exp(\mathrm{i}\varphi)\big) \;=\; \infty.$$
(4) Given any P and Q in $\mathbb{D}$ there is a unique geodesic from P to Q.

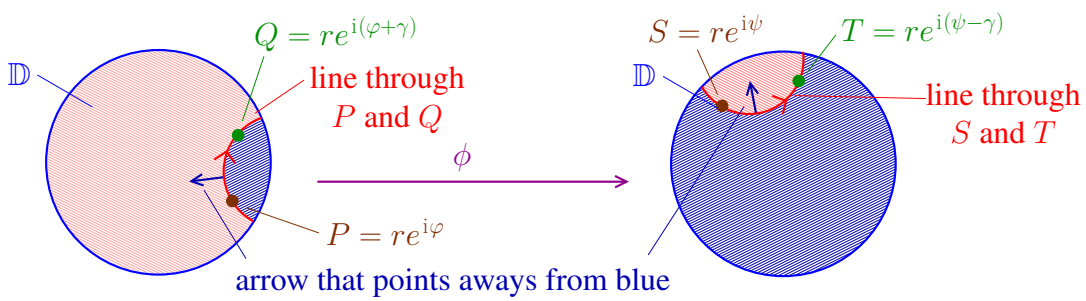
Proof. Statement (1) can be deduced easily from [**Bona2009**, Lemma 2.4 and Proposition 2.22]. Statement (2) follows immediately from Statement (1), and Statement (3) is a straightforward consequence of Statement (2). For Statement (4), note that it suffices to prove it for $P = 0$. In that case it is a straightforward consequence of (1). ■

Lemma 131.23. (Möbius-Transitive Lemma) Let (P, Q) and (S, T) be two pairs of distinct points on $\mathbb{D}$ such that $d(P, Q) = d(S, T)$. Then there exists a unique Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$.

Example. Let $r \in (0, 1)$, $\varphi \in [0, 2\pi]$, $\psi \in [0, 2\pi]$ and $\gamma \in (0, \pi)$. We consider the pairs of points $P = r \cdot \exp(\mathrm{i}\varphi), Q = r \cdot \exp(\mathrm{i}(\varphi + \gamma))$ and $S = r \cdot \exp(\mathrm{i}\psi), T = r \cdot \exp(\mathrm{i}(\psi - \gamma))$. We then have

$$d(S, T) \;=\; d(T, S) \;=\; d(r \cdot \exp(\mathrm{i}(\psi - \gamma)), r \cdot \exp(\mathrm{i}\psi)) \underset{\substack{\uparrow \\ \text{since rotation by } \varphi + \gamma - \psi \text{ is} \\ \text{an isometry}}}{=} d(r \cdot \exp(\mathrm{i}\varphi), r \cdot \exp(\mathrm{i}(\varphi + \gamma))).$$

By the Möbius-Transitive Lemma 131.23 there exists a Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$.
Recall that from the Möbius Transformations–Proposition 131.10 and the Hyperbolic Lines Lemma 131.13 we know that Möbius transformations preserve hyperbolic lines and that they are orientation-preserving. In particular ϕ sends the hyperbolic line through P and Q to the hyperbolic line through S and T. Since ϕ is orientation-preserving it "swaps" the two sides of the hyperbolic line. We refer to the figure below for an illustration. □



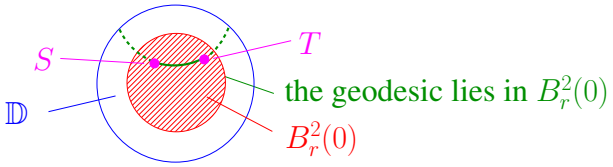
Proof of Lemma 131.23. It follows from the Möbius Transformations–Proposition 131.10 (3) that, without loss of generality, we can assume that $P = 0$ and $S = 0$. We write $Q = r \cdot \exp(i\varphi)$ and $T = s \cdot \exp(i\psi)$. It is a direct consequence of the Geodesic Proposition 131.22 (1) and our hypothesis that

$$d(0, r \cdot \exp(i\varphi)) \;=\; d(P, Q) \;=\; d(S, T) \;=\; d(0, s \cdot \exp(i\psi))$$

that we have $r = s$. But this means that we can apply the rotation by the angle $\varphi - \psi$, which is a Möbius transformation, to turn (P, Q) into (S, T). This proves the existence of ϕ. The uniqueness follows from the observation, that we had made already on page 2530, that the only Möbius transformations that fix 0 are precisely rotations. ■

Lemma 131.24. (Geodesic-stays-in-Ball Lemma) Let $P \in \mathbb{D}$ and $r > 0$ such that[1363] $B_r^2(P) \subseteq \mathbb{D}$. Let $S, T \in B_r^2(P)$. Then the image of the unique geodesic in $\mathbb{D}$ from S to T lies in $B_r^2(P)$.

Proof. As usual, this time by the Möbius Transformations–Proposition 131.10, the Hyperbolic Lines Lemma 131.13 and the Geodesic Proposition 131.22, it suffices to prove the lemma for $P = 0$.



In this special case the reader will have no troubles providing the elementary argument. We also refer to the figure above for an illustration. ■

Next we recall Definition 10.15:

Definition 131.25. We say that a metric space (X, d) is complete if every Cauchy sequence in (X, d) converges.

The present setting motivates the following definition:

Definition 131.26. We say that a connected Riemannian manifold (M, g) is complete if the corresponding metric space $(M, d_{(M,g)})$ is complete.

In the Compact-Complete Proposition 10.17 we showed that every compact metric space is complete. The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are of course not compact, but nonetheless,

[1363] Here by $B_r^2(P)$ we still mean the good old Euclidean disk.

as we will now see, we can use the Compact-Complete Proposition 10.17 to show that $\mathbb{D}$ and $\mathbb{H}$ are complete.

Lemma 131.27. (Hyperbolic Completeness Lemma) The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are complete.

Proof. By the Hyperbolic Models Proposition 131.9 the two Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are isometric. Therefore it suffices to show that $\mathbb{D}$ is complete. We denote by $d = d_{(\mathbb{D},g)}$ the metric on $\mathbb{D}$ corresponding to the Riemannian structure. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d. An elementary argument using the triangle inequality and the definition of a Cauchy sequence shows that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

It follows from the Geodesic Proposition 131.22 that there exists an $r \in (0, 1)$ such that all a_n are contained in the closed Euclidean ball $\overline{B}_r^{eucl}(0)$. By the Heine-Borel Theorem 3.44 this ball is a compact subset of $\mathbb{R}^2$, viewed with the usual topology. It follows from the Riemannian Metric Proposition 131.19 that $\overline{B}_r^{eucl}(0)$ is also compact in the topology defined by d. Now we see that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the compact metric space $(\overline{B}_r^{eucl}(0), d)$. It follows from the Compact-Complete Proposition 10.17 that the sequence converges. ■

131.5. Exercises for Chapter 131.

Exercise 131.1. Does the smooth manifold $B^2\backslash\{0\}$ admit a complete Riemannian metric?

CHAPTER 132

The universal covering of surfaces

Throughout this chapter we mean by a *surface* a connected 2-dimensional smooth manifold. For example the surface of genus g, $g \geq 0$, but also $\mathbb{C}$, $\mathbb{C} \setminus \{0, 1\}$ and the result of removing finitely many points from Σ_g are surfaces.

Definition 132.1. Let M be a surface. We say that a Riemannian structure g on M is hyperbolic if the following two conditions are satisfied:

(1) (M, g) is locally isometric to $\mathbb{D}$, i.e. if given any $P \in M$ there exists an isometry $\Phi \colon U \to V$ from an open neighborhood U of P to an open subset V of $\mathbb{D}$,

(2) (M, g) is complete.

If g is hyperbolic, then we refer to (M, g) as a hyperbolic surface.

Remark. By the Compact-Complete Proposition 10.17 the condition (2) of the definition is automatically satisfied if M is a closed surface. □

In this chapter we will pursue the following three goals:

(1) We want to show that for every $g \geq 2$ the surface of genus g is a complex manifold.
(2) We intend to prove that for any $g \geq 2$ the surface of genus g is hyperbolic.
(3) Given $g \geq 2$ we want to determine the universal covering of the surface of genus g.

This chapter is organized as follows:

- In Section 132.1 we will show that surfaces of genus ≥ 2 are complex manifolds and that they are hyperbolic.
- In Section 132.2 we will construct more hyperbolic structures on the surfaces of genus $g \geq 2$.
- In Section 132.3 we will see that some non-compact surfaces also admit hyperbolic structures.
- In Section 132.4 we will state Hadamard's Theorem 132.14 which gives us in particular an explicit description of the universal covering of surfaces of genus 2.
- In Section 132.5 we will provide the first part of the proof of Hadamard's Theorem 132.14.
- In Section 132.6 we will conclude the proof of Hadamard's Theorem 132.14.
- In Section 132.7 we will use Hadamard's Theorem 132.14 to draw several conclusions on fundamental groups of surfaces.
- In Section 132.8 we will state and prove Picard's Theorem 132.27 from complex analysis.
- In Section 132.9 we will discuss triangle groups.
- In Section 132.10 we will provide exercises for Chapter 132.

In the next Chapter 133 we will discuss similarities between covering theory and topics in other branches of mathematics.

132.1. **Hyperbolic surfaces.** In this section we will show that the surfaces of genus ≥ 2 are complex manifolds and that they are hyperbolic. In the subsequent subsection we will then consider more general types of surfaces.

Definition 132.2. Let M be a 2-dimensional topological manifold with empty boundary. A Möbius structure for M is a family of homeomorphisms $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ from open subsets of M to open subsets of $\mathbb{D}$ such that $\bigcup_{i\in I} U_i = M$ and such that for every $i, j \in I$ the transition map

$$\underbrace{\Phi_i(U_i \cap U_j)}_{\subseteq \mathbb{D}} \xrightarrow{(\Phi_i|_{U_i\cap U_j})^{-1}} U_i \cap U_j \xrightarrow{\Phi_j|_{U_i\cap U_j}} \underbrace{\Phi_j(U_i \cap U_j)}_{\subseteq \mathbb{D}}$$

is given by a Möbius transformation. Sometimes we refer to a smooth manifold together with a Möbius structure as a Möbius manifold.

Example.

(1) We will always view $\mathbb{D}$ as equipped with the Möbius structure that is given by the chart $\mathrm{id} \colon \mathbb{D} \to \mathbb{D}$.
(2) Any open subset of a Möbius manifold is also a Möbius manifold. □

The following lemma says that Möbius structures are useful for solving two problems at once: we can use them to show that a smooth manifold is a complex manifold and we can use them to show that a smooth manifold has a hyperbolic structure.

Lemma 132.3. (Möbius Structure Lemma) Let M be a surface. Furthermore let $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M. Then the following hold:

(1) The charts form a holomorphic atlas for M, in particular M is a complex 1-dimensional smooth manifold.
(2) (a) The smooth manifold M admits a unique Riemannian structure g such that all the charts in the atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ are isometries.
    (b) If the Riemannian structure from (a) is complete, then (M, g) is hyperbolic.

In the following we will always view a Möbius manifold as a 1-dimensional complex manifold and as a Riemannian manifold with the structures coming from the Möbius Structure Lemma 132.3.

Proof. Let M be a surface and let $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M.

(1) Statement (1) is an immediate consequence of the definitions and of the fact, proved in the Möbius Transformations–Proposition 131.10 (2), that Möbius transformations are biholomorphisms.
(2) (a) We denote by g the Riemannian structure on $\mathbb{D}$. We define a Riemannian structure on M as follows. Let $P \in M$. We pick a chart $\Phi_i \colon U_i \to V_i$ from our Möbius structure and we define $h_P := \Phi_i^*(g_{\Phi_i(P)})$. Now suppose that we had picked a different chart Φ_j from the Möbius structure. The transition map from Φ_i to Φ_j is by definition a Möbius transformation, which is an isometry by the Möbius Transformations–Proposition 131.10 (2). It follows easily that Φ_j gives rise to the same definition of h_P. It is now straightforward to verify that this defines a Riemannian structure on M and that it has the desired properties.

(b) This statement is an immediate consequence of (a) and of the definition of a hyperbolic surface. ■

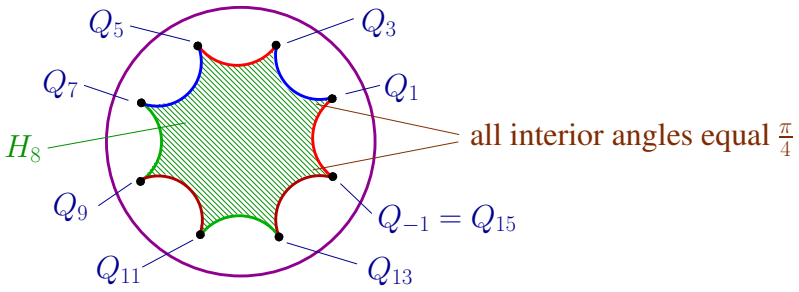
The following proposition gives in particular an affirmative answer to Question 40.21.

Proposition 132.4. (Surface–Möbius Structure Proposition) Let $g \geq 2$. Then the following hold:

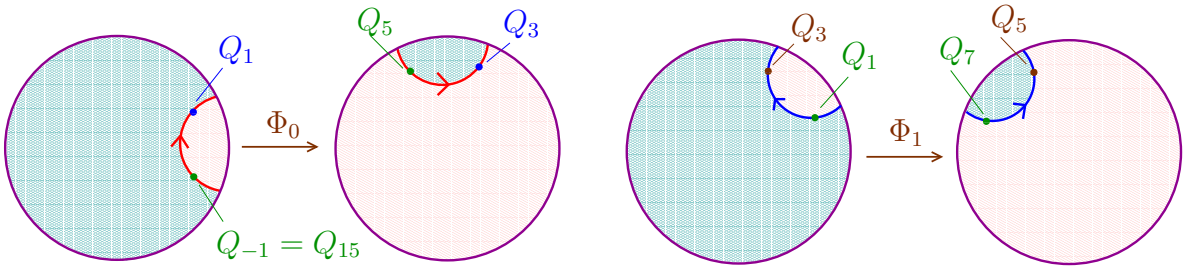
(1) The surface of genus g admits a Möbius structure.
(2) The surface of genus g admits the structure of a 1-dimensional complex manifold.
(3) The surface of genus g admits a hyperbolic Riemannian structure.

Proof. We prove the proposition for the case that $g = 2$. The general case is proved by almost the same argument.

We first show that the surface of genus 2 admits a Möbius structure. By [**Bona2009**, Lemma 5.3] (see also [**Frie2016a**, Lemma 21.5]) there exists an $r > 0$ such that the interior angle at any vertex of the hyperbolic octagon H_8 in $\mathbb{D}$, defined by $Q_k = r \cdot \exp(2\pi \mathrm{i} k/16)$, $k = 1, 3, \ldots, 15$, equals $\frac{\pi}{4}$.[1364]

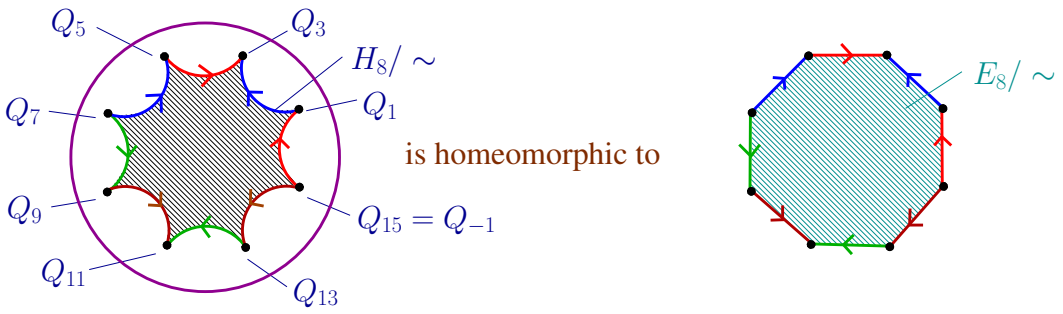


Let $k \in \{0, 1, 4, 5\}$. Note that it follows from the Möbius-Transitive Lemma 131.23 and the example on page 2537 that there exists a unique Möbius transformation Φ_k with $\Phi_k(Q_{2k-1}) = Q_{2k+5}$ and with $\Phi_k(Q_{2k+1}) = Q_{2k+3}$.



Given $k \in \{0, 1, 4, 5\}$ we declare any point P on the edge connecting Q_{2k-1} and Q_{2k+1} to be equivalent to $\Phi_k(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences. It is straightforward to see that $H_8/\sim$ is homeomorphic to the surface of genus 2 that we defined page 342 as $E_8/\sim$. In the following we denote by $p\colon H_8 \to H_8/\sim$ the natural projection.

[1364] Here is a sketch for the proof: For $r \to 0$ the interior angle converges to the interior angle of a Euclidean regular octagon, namely $\frac{3\pi}{4}$, whereas for $r \to \infty$ the interior angle converges to 0. The interior angle changes continuously with r (that is sort of clear, except that it is somewhat painful to provide a rigorous proof), so by the Intermediate Value Theorem there exists an r such that the interior angle equals $\frac{\pi}{4}$.



Now we will give an explicit Möbius structure for $H_8/\sim$. More precisely, given any $P \in H_8/\sim$ we will give a chart ψ_P such that the transitions maps are given by Möbius transformations. So let $P \in H_8/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_8$, then the map

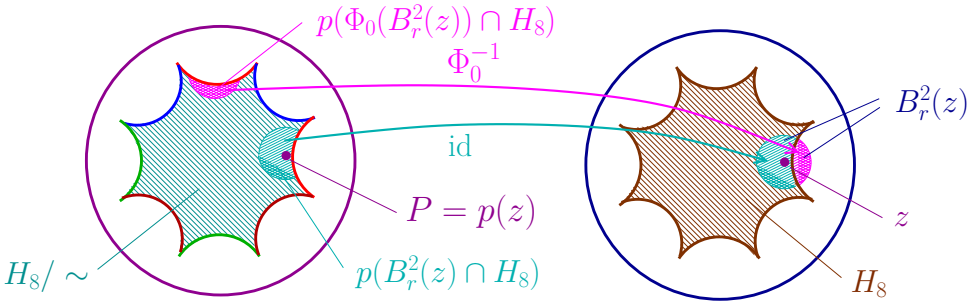
$$\begin{array}{rcl} \psi_P \colon p\big(\mathring{H}_8\big) & \to & \mathring{H}_8 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies in the interior of one of the eight edges of ∂H_8. In the following we deal with the case that z lies on the edge from Q_{-1} to Q_1. We pick an $r > 0$ such that[1365] $B^2_{2r}(z)$ does not contain any of the vertices $Q_1, Q_3, \ldots, Q_{15}$ and does not hit any of the other edges. Then we consider the map

$$\begin{array}{rcl} \psi_P \colon p(B^2_r(z) \cap H_8) \cup p(\Phi_0(B^2_r(z)) \cap H_8) & \to & B^2_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B^2_r(z) \cap H_8 \\ \Phi_0^{-1}(w), & \text{if } w \in \Phi_0(B^2_r(z)) \cap H_8. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$. We refer to the figure below for an illustration of the definition of the map ψ_P.



(3) Finally suppose that $P = p(Q_1) = \cdots = p(Q_{15})$. We choose $r \in (0, \frac{1}{2}\|Q_{-1} - Q_1\|)$. Given $k \in \{1, 3, \ldots, 15\}$ we define $\gamma(k)$ via the following table:

k	1	3	5	7	9	11	13	15
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$.

[1365] Here we consider Euclidean balls.

Recall that we had chosen the octagon H_8 such that the angle at any vertex equals $\frac{\pi}{4}$. For $k = 1, 3, \dots, 15$ we can therefore apply the Ray–Angle Lemma 131.17 to find a unique Möbius transformation θ_k with the following two properties:

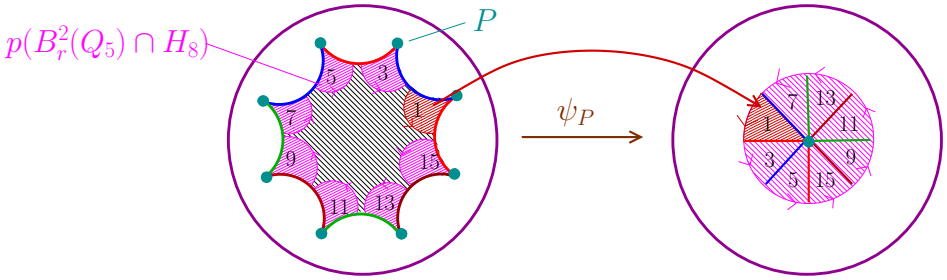
(a) $\theta_k(Q_k) = 0$,

(b) The map θ_k applied to the two rays emanating from Q_k equals the two rays $\{s \cdot \exp(\mathrm{i}\gamma(k)) \mid s \geq 0\}$ and $\{s \cdot \exp(\mathrm{i}(\gamma(k) + \frac{\pi}{4})) \mid s \geq 0\}$ emanating from 0.

We consider the map

$$\begin{aligned} \psi_P \colon p(B_r^2(Q_1) \cap H_8) \cup \dots \cup p(B_r^2(Q_{15}) \cap H_8) &\to B_r^2(0) \\ p(z) &\mapsto \theta_k(z), \quad \text{if } z \in B_r^2(Q_k) \cap H_8. \end{aligned}$$

We leave it to the reader to verify that ψ_P is well-defined and that it is indeed a homeomorphism. This map is a chart around $P = p(z)$. We refer to the figure below for an illustration of the definition of the map ψ_P.



It is straightforward to verify that all the transition maps between the above maps are given by Möbius transformations. We have thus shown that the maps $\{\psi_P\}_{P \in H_8}$ form a Möbius structure for $H_8/\sim$. This concludes the proof of (1).

Now we turn to the proof of (2). The existence of a Möbius structure together with the Möbius Structure Lemma 132.3 (2) shows immediately that the surface of genus 2 is a 1-dimensional complex manifold.

Finally we provide the proof of (3). In light of (1) and the Möbius Structure Lemma 132.3 (2) it suffices to show that the Riemannian manifold $(H_8/\sim, g)$ is complete. By the Riemannian Metric Proposition 131.19 the topology on M defined by the metric $d_{(M,g)}$ agrees with the given topology of M. Since M is compact it now follows from the Compact-Complete Proposition 10.17 that (M, g) is complete. ∎

132.2. **More hyperbolic structures on the surfaces of genus $g \geq 2$.** Now we want to study the question, how many hyperbolic metrics does a surface of genus g admit. To make this question more precise, we introduce the following definition:

Definition 132.5. Let M be a surface of genus $g \geq 2$ and let h, h' be two hyperbolic metrics on M. We say that h and h' are equivalent if there exists an isometry from (M, h) to (M, h').

This raises the question, whether a surface of genus $g \geq 2$ admits non-equivalent hyperbolic metrics. We return to the construction of a hyperbolic metric in the Surface–Möbius Structure Proposition 132.4. In our construction we started out with a regular $4g$-gon with interior angle $\frac{2\pi}{4g}$ and we identified pairs of sides. Two facts were crucial to obtain a hyperbolic metric:

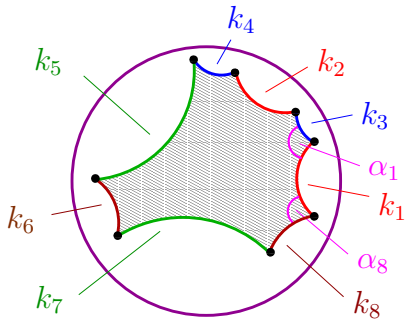
(1) the sides were identified via an isometry,

(2) the sum of the interior angles equals 2π.

Or put differently, any choice of a $4g$-gon with side lengths $k_1, \dots, k_{4g}$ and interior angles $\alpha_1, \dots, \alpha_{4g}$ that satisfy the equations

$$k_1 = k_3,\ k_2 = k_4,\ \dots,\ k_{4g-3} = k_{4g-1},\ k_{4g-2} = k_{4g} \quad \text{and} \quad \alpha_1 + \cdots + \alpha_{4g} = 2\pi$$

gives rise to a hyperbolic metric on a topological space that is homeomorphic to the surface of genus g. It is of course not clear whether different parameters give rise to non-equivalent hyperbolic structures. One way to distinguish them, in principle at least, is to look at the shortest length of a closed curve that is not null-homotopic. We will not elaborate on this procedure.



How many such $4g$-gons are there? On the one hand we have parameters $k_1, k_3, \dots, k_{4g-1}$ and $\alpha_1, \dots, \alpha_{4g}$. On the other hand we have the condition that $\alpha_1 + \cdots + \alpha_{4g} = 2\pi$. Furthermore we want a $4g$-gon, so the last angle and the length of the last edge are determined by the others. Thus we obtain two other conditions. Summarizing we have $2g + 4g = 6g$ parameters that need to satisfy 3 conditions. Of course it is not clear which of these Riemannian structures are isometric. This naive discussion suggests that the space of hyperbolic metrics is $(6g-3)$-dimensional. Surprisingly this is not too far off the correct answer, in fact in [**BP1992**, Chapter B.4] the following proposition is proved.

Proposition 132.6. (Teichmüller Proposition) The topological space of hyperbolic metrics on a surface of genus $g \geq 2$ can be viewed as a smooth manifold of dimension $6g - 6$.

We will not make use of this proposition. The topological space of hyperbolic metric is called the *Teichmüller space*, more information can be found at

`https://en.wikipedia.org/wiki/Teichmueller_space`.

The theory is named after Oswald Teichmüller (1913-1943) who was a brilliant mathematician and an ardent Nazi, see

`https://en.wikipedia.org/wiki/Oswald_Teichmueller`

`http://www-history.mcs.st-andrews.ac.uk/Biographies/Teichmuller.html`

and

`https://de.wikipedia.org/wiki/Oswald_Teichmueller`.

132.3. **More examples of hyperbolic surfaces.** In this section we will generalize the discussion from the previous section to surfaces that are non-compact.

In the following let $Q_1 = 1, Q_2 = \mathrm{i}, Q_3 = -1$ and $Q_0 = Q_4 = -\mathrm{i}$. We denote by H_4 the closed, non-compact subset of $\mathbb{D}$ that is bounded by the four hyperbolic lines with endpoints Q_k, Q_{k+1} where $k = 0, 1, 2, 3$. We refer to the figure on page 2547 for an illustration.

We consider the Möbius transformations[1366]

$$\Phi_1(z) := (-\mathrm{i}) \cdot \frac{z - \frac{-1+\mathrm{i}}{2}}{1 - \frac{-1-\mathrm{i}}{2} z} \quad \text{and} \quad \Phi_2(z) := (-\mathrm{i}) \cdot \frac{z + \frac{-1-\mathrm{i}}{2}}{1 + \frac{-1+\mathrm{i}}{2} z}.$$

We record a few key properties of Φ_1 and Φ_2 in a lemma.

Lemma 132.7. (Explicit Möbius Transformation Lemma)

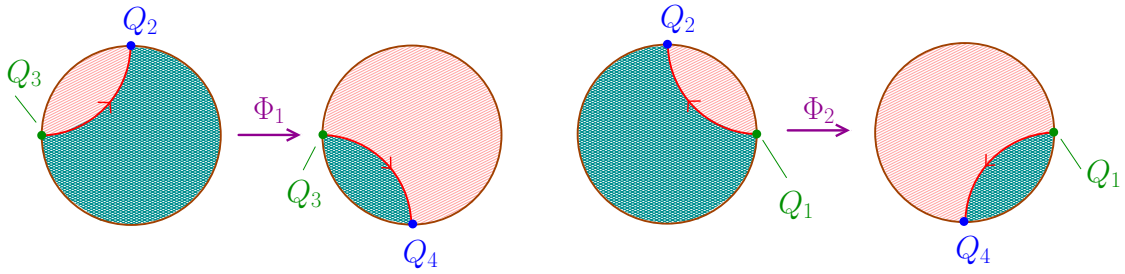
(1) We have $\Phi_1(Q_2) = Q_4$ and $\Phi_1(Q_3) = Q_3$ and we have $\Phi_2(Q_2) = Q_4$ and $\Phi_2(Q_1) = Q_1$.
(2) The restriction of Φ_1 to the hyperbolic line with endpoints Q_3 and Q_2 is given by reflection in the x-axis.[1367]
(3) The restriction of Φ_2 to the hyperbolic line with endpoints Q_1 and Q_2 is given by reflection in the x-axis.

Proof. Statement (1) follows immediately from plugging in the points. A straightforward, albeit slightly painful calculation shows that for any $\alpha \in \mathbb{R}$ we have

$$\Phi_1\big(-1 + \mathrm{i} + e^{-\mathrm{i}\alpha}\big) = -1 - \mathrm{i} + \exp(\mathrm{i}\alpha).$$

This proves Statement (2). Statement (3) is proved by a similarly tedious calculation. ■

Remark. We consider the Möbius transformation Φ_1 in more detail. It sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. In particular it sends the two halves determined by the hyperbolic lines to one another. Which half gets sent to which half is determined by the fact that Φ_1 is orientation-preserving.



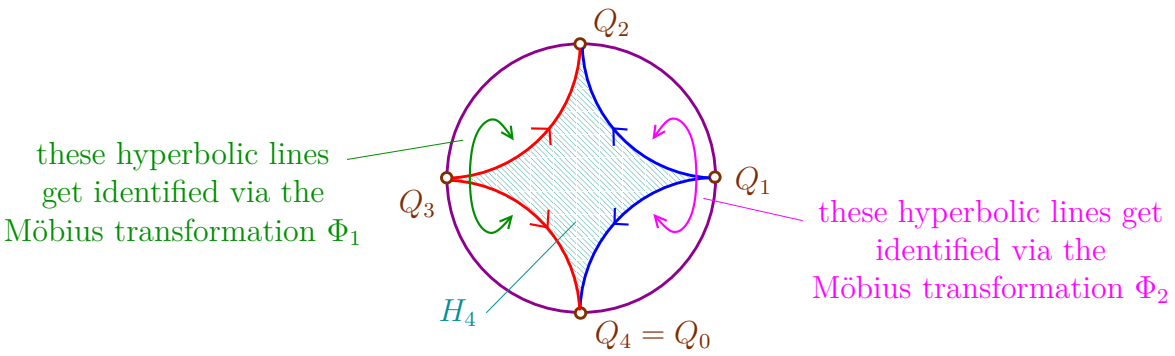
We refer to the figure above for an illustration of the two Möbius transformations Φ_1 and Φ_2. □

Given $k \in \{1, 2\}$ we declare any point P on the line with the endpoints Q_k and Q_{k+1} to be equivalent to $\Phi_{3-k}(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences.

Proposition 132.8. (Punctured Sphere–Möbius Structure Proposition)

[1366]Recall that a Möbius transformation of $\mathbb{D}$ is a map of the form $\psi(z) = e^{\mathrm{i}\theta} \cdot \frac{z-a}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

[1367]The Möbius transformation Φ_1 sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. There are many such Möbius transformations, but the Möbius transformation we picked has the extra property, that on the hyperbolic line it is the "obvious map".



(1) The topological space $H_4/\sim$ admits a Möbius structure. In particular it is a 1-dimensional complex manifold.
(2) The Riemannian structure on $H_4/\sim$ coming from (1) and the Möbius Structure Lemma 132.3 is complete.
(3) The topological space $H_4/\sim$ admits a hyperbolic metric.

Proof. We start out with the proof of (1). We have to show that $H_4/\sim$ admits a Möbius structure. The following argument is almost identical to the proof of the Surface–Möbius Structure Proposition 132.4 (1).

We denote by $p\colon H_4 \to H_4/\sim$ the projection. In the following let $P \in H_4/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_4$, then

$$\begin{array}{rcl} \psi_P\colon p\big(\mathring{H}_4\big) & \to & \mathring{H}_4 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies on ∂H_4. In the following we deal with the case that z lies on the edge from Q_1 to Q_2. All other cases are dealt with almost the same way. We pick an $r > 0$ such that $B^2_{2r}(z)$ intersects no other component of ∂H_4. We consider the map

$$\begin{array}{rcl} \psi_P\colon p(B^2_r(z) \cap H_4) \cup p(\Phi_2(B^2_r(z)) \cap H_4) & \to & B^2_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B^2_r(z) \cap H_4 \\ \Phi_2^{-1}(w), & \text{if } w \in \Phi_2^{-1}(B^2_r(z)) \cap H_4. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$.

The charts that we just constructed form an atlas for $H_4/\sim$. It follows immediately from the definitions that all transition maps are given by Möbius transformations. It follows that the maps ψ_P form a Möbius structure for $H_4/\sim$. As in the proof of the Surface–Möbius Structure Proposition 132.4 it follows that $H_4/\sim$ is a 1-dimensional complex manifold and that these charts satisfy the conditions of the Möbius Structure Lemma 132.3. This concludes the proof of (1).

We continue with the proof of (2). We have to show that the Riemannian structure on $H_4/\sim$ coming from (1) and the Möbius Structure Lemma 132.3 is complete. The subsequent argument is a variation on the proof of the Compact-Complete Proposition 10.17.

We denote by d the metric corresponding to the Riemannian structure on $H_4/\sim$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_4/\sim$ with respect to d. As in the proof of the Compact-Complete Proposition 10.17 we see that there exists a $C\in\mathbb{R}$ such that $d(0,a_n)\leq C$ for all $n\in\mathbb{N}$.

Claim. There exists an $r\in[0,1)$ such that for every $z\in H_4$ with $d(0,p(z))\leq C$ we have $|z|\leq r$.

Proof. Given $s\in[0,1)$ we set

$$g(s) \;:=\; \int_{t=0}^{t=s} \frac{\sqrt{2}}{1-t^2}\,dt.$$

Clearly we have $\lim_{s\to 1} g(s)=\infty$. In particular there exists an $r\in[0,1)$ such that $g(s)>C+1$ for all $s\geq \frac{1}{2}r$.

We claim that r has the desired property. So let $z=x+\mathrm{i}y\in H_4$ with $|z|>r$. Since H_4 is a subset of the square spanned by ± 1 and $\pm i$ we see that we have $|z|\leq |x|+|y|$. Put differently, we have $|x|\geq\frac{1}{2}r$ or $|y|\geq\frac{1}{2}r$. Without loss of generality we can assume that $|x|\geq\frac{1}{2}r$.

We need to show that $d(0,p(z))\geq C+1$. It suffices to show that for every piecewise smooth path $\gamma\colon[0,a]\to H_4/\sim$ with $\gamma(0)=0$ and $\gamma(a)=p(z)$ we have $\ell(\gamma)\geq C+1$.

We denote by

$$\begin{array}{rcl} q\colon H_4/\sim & \to & (-1,1) \\ [x+\mathrm{i}y] & \mapsto & x \end{array}$$

the projection onto the real part. It follows from the Explicit Möbius Transformation Lemma 132.7 that this map is indeed well-defined. A straightforward calculation shows that for all $t\in[0,a]$ we have

$$\big\|\gamma'(t)\big\|_{H_4/\sim} \;\geq\; \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}.$$

It follows that

$$\ell(\gamma) \;=\; \int_0^a \big\|\gamma'(t)\big\|_{H_4/\sim}\,dt \;\underset{\text{above inequality}}{\geq} \int_0^a \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}\,dt \;\underset{\text{definition of } d_{(\mathbb{D},g)}}{\geq} d_{(\mathbb{D},g)}(0,\overbrace{q(\gamma(a))}^{=x}) \;\underset{\text{by the Geodesic Proposition 131.22}}{=} g(|x|) \underset{\text{since } |x|\geq\frac{1}{2}r}{\geq} C+1.$$

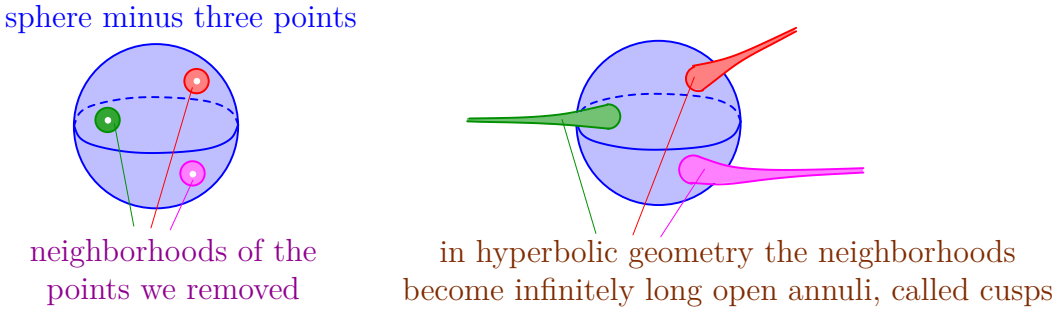
By the above claim the sequence $\{a_n\}_{n\in\mathbb{N}}$ lies in the compact set $p(\{z\in H_4\mid |z|\leq r\})$. But by the Compact-Complete Proposition 10.17 the metric space $p(\{z\in H_4\mid |z|\leq r\})$ is complete, so the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges in $p(\{z\in H_4\mid |z|\leq r\})$, in particular it converges in $p(H_4)=H_4/\sim$. Thus we have proved that $H_4/\sim$ is complete. This concludes the proof of (2).

Finally Statement (3) is an immediate consequence of the definition of a hyperbolic metric and of Statement (2). ∎

We had just seen in the Punctured Sphere–Möbius Structure Proposition 132.8 that $H_4/\sim$ admits a Möbius structure, in particular it is a 1-dimensional complex manifold. It is relatively straightforward to show that $H_4/\sim$ is homeomorphic to the sphere minus three points, or equivalently, to $\mathbb{C}$ minus two points. Now we show the much stronger statement that $H_4/\sim$ is actually biholomorphic to $\mathbb{C}\setminus\{0,1\}$.

Proposition 132.9. (Punctured Sphere–Biholomorphism Proposition) There exists a biholomorphism $H_4/\sim\to\mathbb{C}\setminus\{0,1\}$.

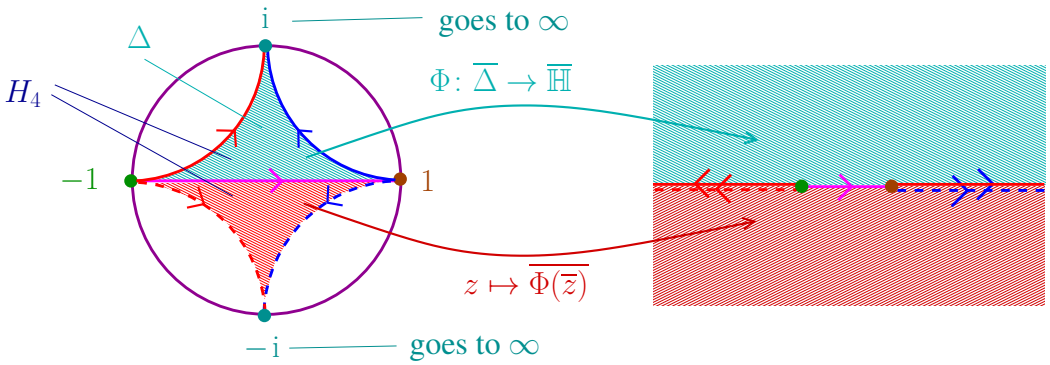
Remark. We equip $H_4/\sim$ with the Riemannian metric g coming from the Möbius structure and we equip $\mathbb{C} \setminus \{0,1\}$ with the usual Euclidean Riemannian metric that we denote by h.



Note that it follows from the Punctured Sphere–Möbius Structure Proposition 132.8 that the metric space $(H_4/\sim, g)$ is complete. On the other hand $(\mathbb{C} \setminus \{0,1\}, h)$ is evidently not complete. Thus we see that the biholomorphism $H_4/\sim \to \mathbb{C} \setminus \{0,1\}$ from the Punctured Sphere–Biholomorphism Proposition 132.9 is *not* an isometry. □

Proof of the Punctured Sphere–Biholomorphism Proposition 132.9. We denote by Δ the open subset of $\mathbb{D}$ bounded by the three hyperbolic lines with endpoints $-1, 1$ and i. (We refer to the figure below for an illustration of Δ.) It follows from applying the Riemann Mapping Theorem 39.18 together with Carathéodory's Theorem 39.20 that there exists a biholomorphism $\Psi\colon \Delta \to \mathbb{D}$ that extends to a homeomorphism $\Psi\colon \overline{\Delta} \to \overline{\mathbb{D}}$. We combine Ψ with the biholomorphism from $\mathbb{D}$ to $\mathbb{H}$ coming from the Hyperbolic Models Proposition 131.9. This way we obtain a biholomorphism $\Phi\colon \Delta \to \mathbb{H}$ that extends to a homeomorphism $\Phi\colon \overline{\Delta} \to \overline{\mathbb{H}}$.

Using the Boundary Triples Transitivity Lemma 131.11 we can, if necessary, compose Φ with a Möbius transformation of $\mathbb{H}$, to arrange that $\Phi(i) = \infty$ and that $\Phi(\{-1,1\}) = \{0,1\}$.



Note that Φ restricts to a homeomorphism $\Phi\colon \partial\overline{\Delta} \to \partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. By the above we have $\Phi(\partial\overline{\Delta} \setminus \{\mathrm{i}, 1, -1\}) = \mathbb{R} \setminus \{0,1\}$. Now we consider the map

$$\begin{aligned} \Psi\colon H_4 &\to \mathbb{C} \setminus \{0,1\} \\ z &\mapsto \begin{cases} \Phi(z), & \text{if } z \in \overline{\Delta}, \\ \overline{\Phi(\overline{z})}, & \text{if } \overline{z} \in \overline{\Delta}, \end{cases} \end{aligned}$$

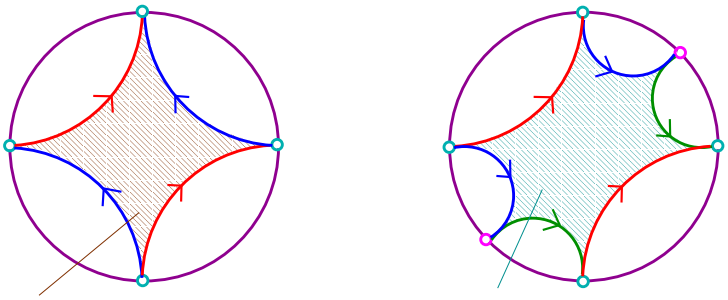
Note that by the above we have $\Psi((-1,1)) = (0,1)$. Thus it follows from the Schwarz Reflection Principle, see Proposition 39.11, that the restriction of Ψ to the interior of H_4 is holomorphic. It follows from the Explicit Möbius Transformation Lemma 132.7 that if P and Q are two equivalent points on ∂H_4, then $\Psi(P) = \Psi(Q)$. Therefore the map Ψ factors through a map $H_4/\sim \to \mathbb{C} \setminus \{0,1\}$. Using once again the Schwarz Reflection Principle one can show that this map is in fact a biholomorphism. We leave the verification of this step to the reader. ■

Definition 132.10. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we now denote by $F_{g,n}$ the surface that is obtained from the surface of genus g by removing n points. We refer to $F_{g,n}$ as the n-punctured surface of genus g.

For example the three-punctured sphere $F_{0,3}$ is the sphere with three points removed. As we pointed out above, $F_{0,3}$ is diffeomorphic to $\mathbb{C} \setminus \{0,1\}$. In particular, as we have just seen, the three-punctured sphere admits a hyperbolic structure. The following theorem says that "most" surfaces admit a hyperbolic structure.

Theorem 132.11. (Hyperbolic Structure–Existence Theorem) Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$, then $F_{g,n}$ admits a hyperbolic structure.

Sketch of a proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$ then variations on the argument of the Surface–Möbius Structure Proposition 132.4 and the Punctured Sphere–Möbius Structure Proposition 132.8 show that $F_{g,n}$ admits a hyperbolic metric. In the figure below we sketch the constructions needed to show that the once-punctured and the twice-punctured torus have a Möbius structure.



once punctured torus twice punctured torus

The proof that the corresponding Riemannian manifolds are complete is similar to the proof of the Punctured Sphere–Möbius Structure Proposition 132.8. ■

132.4. The universal covering of surfaces.

Definition 132.12. Let $f\colon M \to N$ be a continuous map between two Möbius manifolds.

(1) We say that f is a Möbius map if for every two chart $\Phi\colon U \to V$ in the Möbius structure of M and any chart $\Psi\colon X \to Y$ in the Möbius structure of N the restriction of the map

$$\underbrace{\Phi(U \cap f^{-1}(X))}_{\subseteq \mathbb{D}} \xrightarrow{\Psi \circ f \circ \Phi^{-1}} \underbrace{\Psi(X \cap f(U))}_{\subseteq \mathbb{D}}$$

to any component of $\Phi(U \cap f^{-1}(X))$ is given by a Möbius transformation.

(2) We say that f is a Möbius isomorphism, if f is a Möbius map, if f is a bijection, and if f^{-1} is also a Möbius map.

The following lemma is an immediate consequence of the definitions and the fact, proved in the Möbius Transformations–Proposition 131.10, that Möbius transformations are biholomorphisms and isometries.

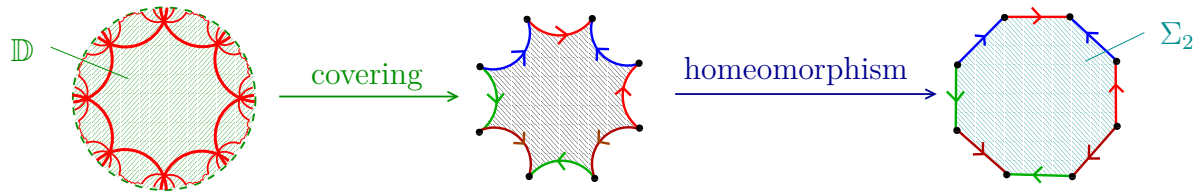
Lemma 132.13. (Möbius Map Lemma) Let M and N be two Möbius manifolds and let $f\colon M \to N$ be a Möbius map. Then the following two statements hold:

(1) the map f is a local biholomorphism,
(2) the map f is a local isometry.

The following theorem is the main result of this chapter.

Theorem 132.14. (Hadamard's Theorem) Let $g \geq 2$, and let $M = H_{4g}/\sim$ be the surface of genus g with the Möbius structure constructed in the Surface–Möbius Structure Proposition 132.4. Then there exists a covering map $p\colon \mathbb{D} \to M$ with the following properties:

(1) p is a Möbius map,
(2) p is a local isometry,
(3) p is a local biholomorphism.



The hyperbolic disk $\mathbb{D}$ is of course simply connected. The theorem thus says that for any $g \geq 2$ the hyperbolic disk $\mathbb{D}$ is a universal covering of the surface of genus g. Thus we have now answered Question 127.25.

The proof of Hadamard's Theorem 132.14 will require the following two sections.

132.5. Proof of Hadamard's Theorem 132.14 I.

In this section we will lay the technical foundations for the proof of Hadamard's Theorem 132.14. In the following lemma we first summarize a few properties of Möbius maps.

Auxiliary Lemma 132.15.

(1) If $U \subseteq \mathbb{D}$ is an open connected subset and if $f\colon U \to \mathbb{D}$ is a Möbius map, then f is given by a Möbius transformation.
(2) Let $f\colon M \to N$ be a continuous map between two Möbius manifolds. If f is locally a Möbius map, then f is a Möbius map.
(3) The composition of two Möbius maps is again a Möbius map.
(4) A bijective Möbius map is a Möbius isomorphism.

Proof.

(1) Let $U \subseteq \mathbb{D}$ be an open connected subset and let $f\colon U \to \mathbb{D}$ be a Möbius map. It follows from the Möbius Transformations–Proposition 131.10 (4) and (5) that given $P \in U$ there exists a unique Möbius transformation ϕ_P such that $f = \phi_P$ in an open

neighborhood of P. We consider the map

$$\begin{array}{rcl} \phi\colon U & \to & \text{set of all Möbius transformations } \mathbb{D} \to \mathbb{D} \\ P & \mapsto & \phi_P. \end{array}$$

We equip the right-hand side of the map ϕ with the discrete topology. It follows from the definitions that this map is continuous. But then it follows from the Connected-to-Discrete Lemma 4.37 that this map is constant. But that is exactly what we had tried to show.

(2) This statement follows immediately from (1) and the definitions.

(3) This statement is an immediate consequence of definitions and the fact, proved in the Möbius Transformations–Proposition 131.10, that the compositions of two Möbius transformations is again a Möbius transformation.

(4) Let $f\colon M \to N$ be a bijective Möbius map. In the Möbius Transformations–Proposition 131.10 we saw that the inverse of a Möbius transformation is again a Möbius transformation. It follows that f^{-1} is locally a Möbius map. But then it follows from (2), that f^{-1} is in fact a Möbius map. ■

Auxiliary Lemma 132.16. Let U be an open connected subset of $\mathbb{D}$ and let M be a Möbius manifold. Furthermore let $\Phi, \Psi\colon U \to M$ be two Möbius maps.[1368] Suppose one of the following two conditions is satisfied:

(1) There exists an open non-empty subset of U on which Φ and Ψ agree, or

(2) there exists a point $Q \in U$ with $\Phi(Q) = \Psi(Q)$ and a non-zero vector $v \in \mathrm{T}_Q\mathbb{D}$ such that $\mathrm{D}\Phi_Q(v) = \mathrm{D}\Psi_Q(v)$,

then the Möbius maps Φ and Ψ agree on all of U.

Proof. Let U be an open connected subset of $\mathbb{D}$, let M be a Möbius manifold and let $\Phi, \Psi\colon U \to M$ be two Möbius maps. We consider one more statement:

(3) There exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $\mathrm{D}\Phi_Q = \mathrm{D}\Psi_Q$.

Clearly we have (1)⇒(3). We also have (2)⇒(3). Indeed, the differentials $\mathrm{D}\Phi_Q$ and $\mathrm{D}\Psi_Q$ are homomorphisms between two 1-dimensional complex vector spaces. So if they are agree on one non-zero vector, then they agree on all of the vector space $\mathrm{T}_Q\mathbb{D}$.

So it suffices to show that if (3) is satisfied, then the maps Φ and Ψ agree on all of U. So suppose that there exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $\mathrm{D}\Phi_Q = \mathrm{D}\Psi_Q$. We consider

$$Y \;:=\; \{P \in U \mid \Phi(P) = \Psi(P) \text{ and } \mathrm{D}\Phi_P = \mathrm{D}\Psi_P\}.$$

By our hypothesis we know that Y contains Q, in particular Y is non-empty. Since Φ and Ψ are smooth it follows easily from the fact that U is Hausdorff, Exercise 4.41 and the Differential-equals-Differential Lemma 29.6 that Y is closed. It remains to show that Y is open.

So let $P \in Y$. We pick a map $\Theta\colon W \to \mathbb{D}$ from the Möbius structure of M such that W is an open neighborhood of $\Phi(P) = \Psi(P)$. We denote by V the component of $\Phi^{-1}(W) \cap \Psi^{-1}(W)$ that contains P. Since Φ and Ψ are Möbius maps it follows from Lemma 132.15 (3) that the maps $\Theta \circ \Phi\colon V \to \mathbb{D}$ and $\Theta \circ \Psi\colon V \to \mathbb{D}$ are also Möbius maps.

[1368] Here we view $U \subseteq \mathbb{D}$ as equipped with the obvious Möbius structure.

Since V is connected it follows from Lemma 132.15 (1) that both maps are given by Möbius transformations. These two Möbius transformations agree at P and they have the same differential at P. It follows from the Möbius Transformations–Proposition 131.10 (5) that Φ and Ψ agree on an open neighborhood of P. This shows that Y is also open. ∎

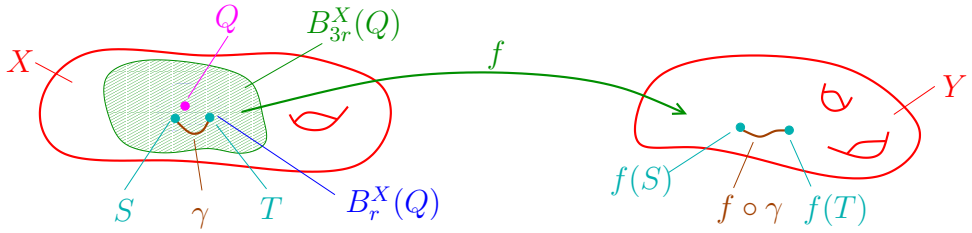
Definition 132.17. Let M be a Möbius manifold and let $Q \in M$ and $r > 0$. We say that $B_r^M(Q)$ is small if there exists a Möbius isomorphism $\Omega\colon B_r^{\mathbb{D}}(0) \to B_r^M(Q)$ with $\Omega(0) = Q$.

Proposition 132.18. (Small Disk Proposition) Let M be a Möbius manifold and let $Q \in M$. Then there exists an $r > 0$ such that $B_r^M(Q)$ is small.

In the proof of the Small Disk Proposition 132.18 we will make use of the following lemma:

Lemma 132.19. Let (X, g) and (Y, h) be Riemannian manifolds. Let $Q \in X$ and $r > 0$. Let $f\colon B_{3r}^X(Q) \to (Y, h)$ be a local isometry and let $S, T \in B_r^X(Q)$. Then we have

$$d_Y(f(S), f(T)) \;\leq\; d_X(S, T).$$



Proof. Let $S, T \in B_r^X(Q)$. Furthermore let $\epsilon > 0$. By the triangle inequality we have $d_X(S, T) < 2r$. In particular there exists a piecewise smooth path γ in X from S to T such that $\ell_X(\gamma) < 2r$ and with $\ell_X(\gamma) < d_X(S, T) + \epsilon$. Given any t we have

$$d_X(Q, \gamma(t)) \;\leq\; d_X(Q, S) + d_X(S, \gamma(t)) \;<\; r + \ell_X(\gamma) \;<\; 3r.$$

Thus we have shown that the image of γ lies in $B_{3r}^X(Q)$. In particular the image of γ lies in the domain of f. Thus we see that

$$d_Y(f(S), f(T)) \;\underset{\uparrow}{\leq}\; \ell_Y(f \circ \gamma) \;\underset{\uparrow}{=}\; \ell_X(\gamma) \;<\; d_X(S, T) + \epsilon.$$

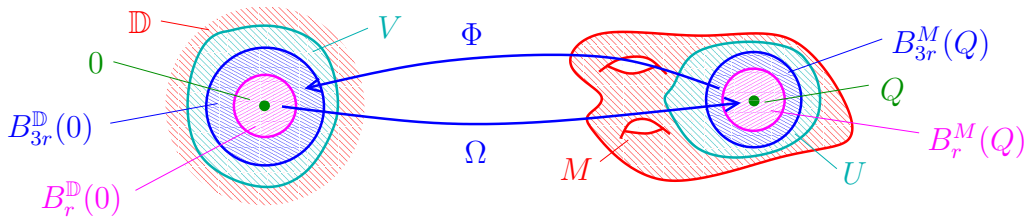
since $f \circ \gamma$ is a path from $f(S)$ to $f(T)$ — by the Local Isometry–Distance Lemma 131.20, since f is a local isometry

Since this inequality holds for all $\epsilon > 0$ we see that $d_Y(f(S), f(T)) \leq d_X(S, T)$. ∎

Proof of the Small Disk Proposition 132.18. Let M be a Möbius manifold and let $Q \in M$. We pick a map $\Phi\colon U \to V$ from the Möbius structure of M such that U is an open neighborhood of Q. By the Möbius Transformations–Proposition 131.10 (3), after possibly composing Φ with a Möbius transformation, we can furthermore assume that $\Phi(Q) = 0$. We write $\Omega := \Phi^{-1}$. By the Riemannian Metric Proposition 131.19 (3) and by the openness of V there exists an $r > 0$ such that $B_{3r}^M(Q) \subseteq U$ and such that $B_{3r}^{\mathbb{D}}(0) \subseteq V$.[1369]
It follows immediately from the Möbius Map Lemma 132.13 and Lemma 132.19 and from our choice of r that the following holds:

(1) For any $S, T \in B_r^M(Q)$ we have $d_{\mathbb{D}}(\Phi(S), \Phi(T)) \leq d_M(S, T)$.
(2) For any $S, T \in B_r^{\mathbb{D}}(0)$ we have $d_M(\Omega(S), \Omega(T)) \leq d_{\mathbb{D}}(S, T)$.

[1369] Note the discrete factor 3 in the radius of the ball.

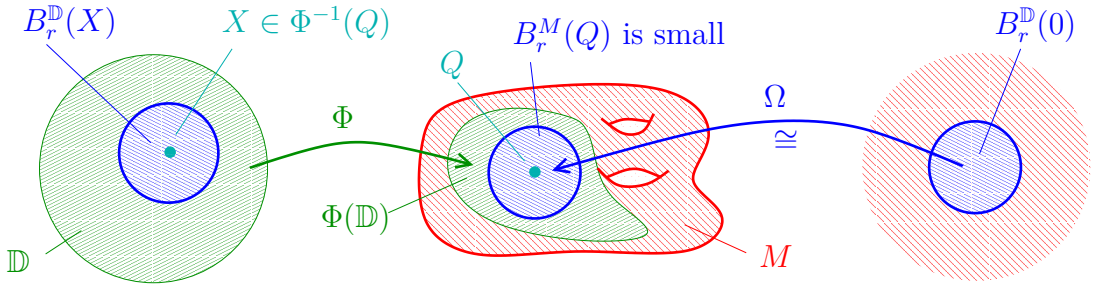


It follows immediately from these two inequalities, from $\Phi(Q) = 0$ and from the fact that $\Phi = \Omega^{-1}$ that Ω restricts to an isometry $\Omega\colon (B^{\mathbb{D}}_r(0), d_{\mathbb{D}}) \to (B^M_r(Q), d_M)$ of metric spaces which is furthermore a bijection. Furthermore, since Φ is a Möbius isomorphism it follows that the inverse map $\Omega = \Phi^{-1}\colon (B^{\mathbb{D}}_r(0), d_{\mathbb{D}}) \to (B^M_r(Q), d_M)$ is also a Möbius isomorphism. ■

Auxiliary Lemma 132.20. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B^M_r(Q)$ is small. Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B^M_r(Q) \subseteq \Phi(\mathbb{D})$. Then the following hold:

(1) For any $X \in \Phi^{-1}(Q)$ the map Φ restricts to a Möbius isomorphism $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$.
(2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B^M_s(Q)$. Then there exists a point $X \in \Phi^{-1}(Q)$ with $Y \in B^{\mathbb{D}}_s(X)$.

Proof. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B^M_r(Q)$ is small. We pick a Möbius isomorphism $\Omega\colon B^{\mathbb{D}}_r(0) \to B^M_r(Q)$ as in the definition of "small". Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B^M_r(Q) \subseteq \Phi(\mathbb{D})$.

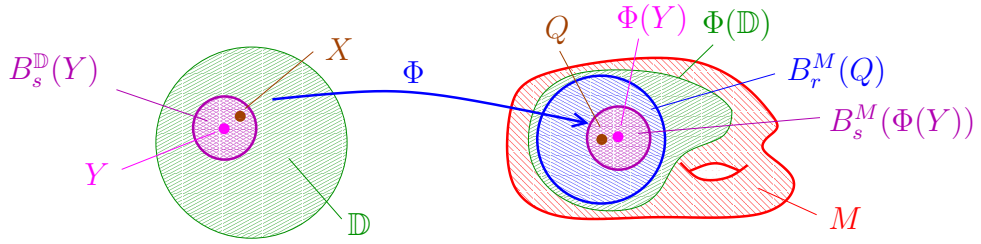


(1) Let $X \in \Phi^{-1}(Q)$. It follows immediately from the fact that Möbius maps are local isometries and from Lemma 132.19 that $\Phi(B^{\mathbb{D}}_r(X)) \subseteq B^M_r(Q)$. Therefore we can consider the Möbius map $\Omega^{-1} \circ \Phi\colon B^{\mathbb{D}}_r(X) \to B^{\mathbb{D}}_r(0)$. By Lemma 132.15 this Möbius map is the restriction of a Möbius transformation. In particular it is a Möbius isomorphism. But since Ω is a Möbius isomorphism as well we obtain that $\Phi\colon B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is also a Möbius isomorphism.

(2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B^M_s(Q)$, i.e. such that $d_M(Q, \Phi(Y)) < s$. By the triangle inequality we have

$$B^M_s(\Phi(Y)) \subseteq B^M_{2s}(Q) \subseteq B^M_r(Q).$$

The same argument as in (1) shows that the map Φ restricts to a Möbius isomorphism $B^{\mathbb{D}}_s(Y) \to B^M_s(\Phi(Y))$.
We have $d_M(Q, \Phi(Y)) < s$, i.e. we have $Q \in B^M_s(\Phi(Y))$. Hence there exists a point $X \in B^{\mathbb{D}}_s(Y)$ with $\Phi(X) = Q$. ■



132.6. **Proof of Hadamard's Theorem 132.14 II.** In this section we will now finally provide the proof of Hadamard's Theorem 132.14. We will only cover the case $g = 2$, the proof of the general case is almost identical.

Throughout this section let $M = H_8/\sim$ be the surface of genus 2 with the Möbius structure constructed in the Surface–Möbius Structure Proposition 132.4. In the following we will show that there exists a Möbius map $p\colon \mathbb{D} \to M$ that is in fact a covering map. It then follows from the Möbius Map Lemma 132.13 that p is also a local isometry and a local biholomorphism.

For the remainder of the proof we adopt the following notation and conventions.

(1) We view $\mathring{H}_8$ as a subset of $\mathbb{D}$ and also of $H_8/\sim$.

(2) Given $Z \in \mathbb{D}$ we denote by
$$\begin{aligned} \gamma_Z\colon [0,1] &\to \mathbb{D} \\ t &\mapsto t \cdot Z \end{aligned}$$
the radial path from the origin $0 \in \mathbb{D}$ to Z.

(3) We say that $U \subseteq \mathbb{D}$ is *convex* if it is convex in the usual Euclidean sense, i.e. if given any $P, Q \in U$ the points $t \cdot P + (1-t) \cdot Q$, $t \in [0,1]$ also lie in U. On several occasions throughout the proof we will make use of the following elementary observations:
   (a) convex sets are path-connected,
   (b) the intersection of two convex sets is also convex,
   in particular, combining (a) and (b) we get
   (c) the intersection of two convex sets is path-connected.

Our first goal is to construct a suitable map $p\colon \mathbb{D} \to M = H_8/\sim$. So let $Z \in \mathbb{D}$. We write $\gamma = \gamma_Z$. We say that $t \in [0,1]$ is *good* if there exists an open convex neighborhood U of $\gamma([0,t])$ and a continuous map $\Psi\colon U \to M$ which has the following properties:

(i) Ψ is a Möbius map,

(ii) it restricts to the identity on $\mathring{H}_8 \cap U \subseteq M$.

We start out with the following lemma:

Auxiliary Lemma 132.21. The point $t = 1$ is good.

Proof. We set
$$T \coloneqq \{t \in [0,1] \mid t \text{ is good}\}.$$
We start out with the following observations:

(1) We have that $0 \in T$ since the identity on the open neighborhood $\mathring{H}_8$ of 0 has the required properties.[1370]

(2) Being good is clearly an open condition, hence $T \subseteq [0,1]$ is open.

[1370] Here we use that for each point $Z \in \mathring{H}_8$ the identity map $\mathrm{id}\colon \mathring{H}_8 \to \mathring{H}_8$ is part of the Möbius structure of M.

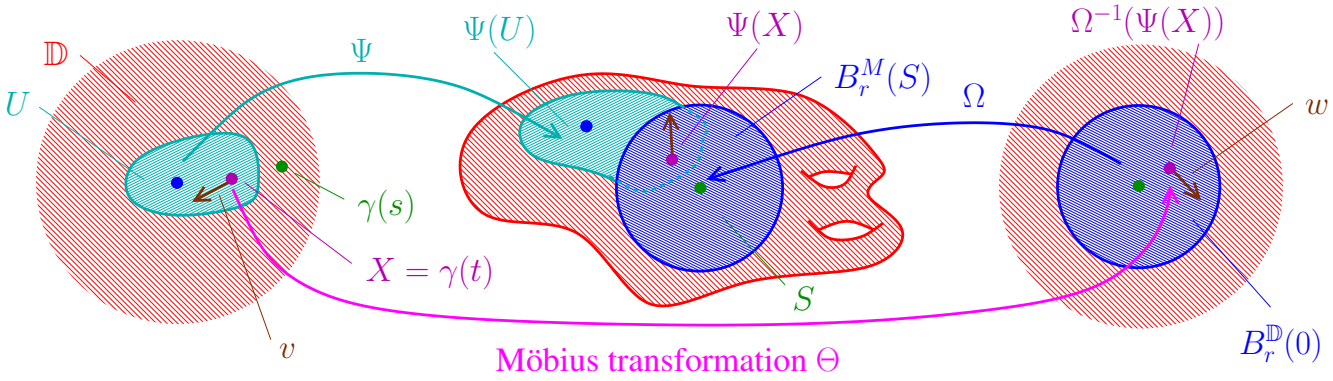
We want to show that $T = [0,1]$. By the above, and since $[0,1]$ is connected, it suffices to show that $s := \sup(T)$ is good.

Since $0 \in S$ and since T is open we see that $s > 0$. Therefore there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of good numbers $t_n \in [0,t)$ that converges to s. Since each t_n is good we can pick for each n an open convex neighborhood U_n of $\gamma([0,t_n])$ and a continuous map $\Psi_n \colon U_n \to M$ that has the desired properties (i) and (ii).

Let $m,n \in \mathbb{N}$. The Möbius maps Ψ_m and Ψ_n agree on the open non-empty connected subset $\mathring{H}_8 \cap U_n \cap U_m$. It follows from Lemma 132.16 that the maps Ψ_m and Ψ_n agree on $U_n \cap U_m$.

By the continuity of γ the sequence $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ converges to $\gamma(s)$, in particular it is a Cauchy sequence with respect to $d_{\mathbb{D}}$. It follows from the Local Isometry–Distance Lemma 131.20 that $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d_M.[1371] It follows from the Surface–Möbius Structure Proposition 132.4 that the metric space $(M = H_8/\sim, d_M)$ is complete. Therefore the Cauchy sequence $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ converges to a point $S \in M$.

It follows from the Small Disk Proposition 132.18 that there exists an $r > 0$ and a Möbius isomorphism $\Omega \colon B_r^{\mathbb{D}}(0) \to B_r^M(S)$ such that $\Omega(0) = S$. We pick an n such that $\Psi_n(\gamma(t_n)) \in B_r^M(S)$ and such that $d_{\mathbb{D}}(\gamma(t),\gamma(s)) < r$. We write $t = t_n, \Psi = \Psi_n, U = U_n$ and we write $X = \gamma(t)$. Let $v \in \mathrm{T}_X\mathbb{D}$ be a non-zero vector. We let $w := D(\Omega^{-1} \circ \Psi)_X(v)$. The Möbius maps Ω and Ψ are in particular local isometries, therefore we have $\|w\| = \|v\|$. It follows from the Möbius Transformations–Proposition 131.10 (4) that there exists a Möbius transformation Θ with $\Theta(X) = \Omega^{-1}(\Psi(X))$ and such that $d\Theta_X(v) = w$.



The Möbius maps $\Omega \circ \Theta \colon B_r^{\mathbb{D}}(X) \to M$ and $\Psi \colon U \to M$ agree at $X = \gamma(t)$ and we have $\mathrm{D}(\Omega \circ \Theta)_X(v) = \mathrm{D}\Psi_X(v)$. It follows from Lemma 132.16 that $\Omega \circ \Theta$ and Ψ agree on the path-connected set $B_r^{\mathbb{D}}(X) \cap U$.

[1371] Indeed, suppose we have $\epsilon > 0$. Since $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an N such that $d_{\mathbb{D}}(\gamma(t_m),\gamma(t_n)) < \epsilon$ for all $m,n \geq N$. Now let $m,n \geq N$. Without loss of generality we have $m > n$. Then

$$d_M(\Psi_m(\gamma(t_m)),\Psi_n(\gamma(t_n))) \underset{\uparrow}{=} d_M(\Psi_m(\gamma(t_m)),\Psi_m(\gamma(t_n))) \underset{\uparrow}{\leq} d_{\mathbb{D}}(\gamma(t_m),\gamma(t_n)) < \epsilon.$$

since Ψ_m and Ψ_n agree on $\gamma|_{[0,t_n]}$ — by the Local Isometry–Distance Lemma 131.20

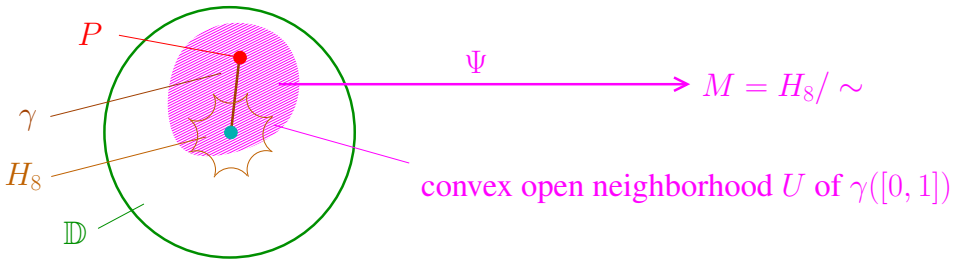
Now we consider the map

$$\begin{aligned} \Phi\colon U \cup B_r^{\mathbb{D}}(\gamma(t)) &\to M \\ Q &\mapsto \begin{cases} \Psi(Q), & \text{if } Q \in U, \\ \Omega^{-1}(\Theta(Q)), & \text{if } Q \in B_r^{\mathbb{D}}(\gamma(t)). \end{cases} \end{aligned}$$

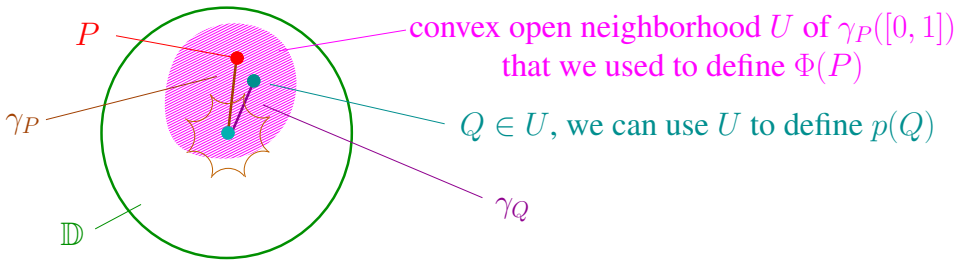
By the above discussion this map is well-defined and it is locally a Möbius map. It follows from Lemma 132.15 (3) that this map is a Möbius map. It follows from $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$ and the fact that γ is a geodesic in $\mathbb{D}$ that $\gamma([t,s]) \subseteq B_r^{\mathbb{D}}(\gamma(t))$, in particular $U \cup B_r^{\mathbb{D}}(\gamma(t))$ is an open subset of $\mathbb{D}$ that contains $\gamma([0,t]) \cup \gamma([t,s]) = \gamma([0,s])$. It is now straightforward to see that we can find a *convex* open subset W of $U \cup B_r^{\mathbb{D}}(\gamma(t))$ that contains $\gamma([0,s])$. The Möbius map $\Phi\colon W \to M$ now certifies that s is also good. ∎

Thus we have now shown that there exists an open convex neighborhood U of $\gamma([0,1])$ and a local isometry $\Psi\colon U \to H_8/\sim$ which agrees with the identity map on $U \cap H_8$. We define $p(Z) \coloneqq \Psi(Z)$. By Lemma 132.16 this definition does not depend on the choice of U and Ψ.

Lemma 132.22. The map $p\colon \mathbb{D} \to M$ is a Möbius map.



Proof. By Lemma 132.15 (2) it suffices to show that $p\colon \mathbb{D} \to M$ is locally a Möbius map. This means that it suffices to show that every point $P \in \mathbb{D}$ admits an open neighborhood U such that the restriction of p to W is a Möbius map. Thus let $P \in \mathbb{D}$. By the above discussion we know that there exists an open convex neighborhood U of $\gamma_P([0,1])$ and a Möbius map $\Psi\colon U \to M = H_8/\sim$ which restricts to the identity on $U \cap H_8$. Since U is open we know that it is an open neighborhood of P. Since U is convex and since $0 \in U$ we know that for all $Q \in U$ the image of the radial path γ_Q from 0 to Q also lies in U.



This implies that we can use the Möbius map $\Psi\colon U \to M$ to define p for any $Q \in U$. Put differently, $p = \Psi$ on U. We have thus shown that the restriction of p to the open neighborhood U of P is a Möbius map. ∎

The following lemma now concludes the proof of Hadamard's Theorem 132.14

Lemma 132.23. (Hadamard Lemma) The map $p\colon \mathbb{D} \to M = H_8/\sim$ is a covering map.

Proof. First we show that the map $p\colon \mathbb{D} \to M = H_8/\sim$ is surjective. Recall that by construction p is the identity on $\mathring{H}_8$. Since p is continuous[1372] it follows that the restriction of p to $H_8 \to M = H_8/\sim$ is surjective.[1373]

It remains to show that every $Q \in M$ admits a uniformly covered neighborhood. Let $Q \in M$. By the Small Disk Proposition 132.18 there exists an $r > 0$ such that $B^M_{2r}(Q)$ is small.[1374] We claim that $B^M_r(Q)$ is uniformly covered. More precisely, we make the following claims:

(1) $p^{-1}(B^M_r(Q)) = \bigcup_{X \in p^{-1}(Q)} B^{\mathbb{D}}_r(X)$,
(2) for $X \neq X' \in p^{-1}(Q)$ we have $B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X') = \varnothing$,
(3) given any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is a homeomorphism.

We apply Lemma 132.20 to the map $p\colon \mathbb{D} \to M$. It gives us the following three facts:

(a) If $Y \in \mathbb{D}$ satisfies $p(Y) \in B^M_r(Q)$, then there exists an $X \in p^{-1}(Q)$ with $Y \in B^{\mathbb{D}}_r(X)$.
(b) For any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is a Möbius isomorphism, in particular it is a homeomorphism.
(c) For any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_{2r}(X)$ is injective.

Now the desired properties (1) and (3) follow immediately from (a) and (b). It remains to prove (2). Suppose that we have $X \neq X' \in p^{-1}(Q)$ with $B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X') \neq \varnothing$. We need to show that $X = X'$. We pick $Y \in B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X')$. The triangle inequality implies that $X' \in B^{\mathbb{D}}_{2r}(X)$. But p restricted to $B^{\mathbb{D}}_{2r}(X)$ is injective by (d). Hence $X = X'$. ∎

132.7. **Fundamental groups of surfaces.** In this section we will combine Hadamard's Theorem 132.14 and the discussion of the deck transformation group in Chapter 128 to draw some interesting consequences on the structure of fundamental groups of surfaces. First we have the following proposition.

Proposition 132.24. (Surface Group–$\mathrm{SL}(2,\mathbb{R})$-Proposition) Let $g \in \mathbb{N}_{\geq 2}$. We denote by Σ_g the surface of genus g. We pick a base point $x_0 \in \Sigma_g$. There exists a monomorphism $\pi_1(\Sigma_g, x_0) \to \mathrm{SL}(2,\mathbb{R})$.

Proof. Throughout the proof we will make use of the notion of a Möbius structure, of a Möbius manifold and of a Möbius map between Möbius manifolds. We refer to Definition 132.2 and Definition 132.12 for the definitions, in case the reader needs a refresher.

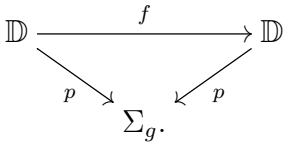
Now let $g \geq 2$. As in the proof of the Surface–Möbius Structure Proposition 132.4 we can and will make the identification $\Sigma_g = H_{4g}/\sim$ where H_{4g} is a regular hyperbolic $4g$-gon in $\mathbb{D}$ with interior angle $\frac{\pi}{2g}$. Furthermore, as in the proof of the Surface–Möbius Structure Proposition 132.4 we equip $\Sigma_g = H_{4g}/\sim$ with the canonical Möbius structure.

[1372] The map p is a Möbius map, in particular it is a local biholomorphism which implies that it is a local homeomorphism. Thus it follows from the Pasting Proposition 4.7 that p itself is continuous.
[1373] This is the only part of the proof where we actually use that we work with $H_8/\sim$, the remainder of the proof works for any smooth manifold M with a Möbius structure such that the corresponding Riemannian manifold is complete.
[1374] Note the factor 2.

By Hadamard's Theorem 132.14 there exists a covering map $p\colon \mathbb{D} \to \Sigma_g$ such that p is a Möbius map.

Claim. Every deck transformation of the covering map $p\colon \mathbb{D} \to \Sigma_g$ is a Möbius transformation.

Proof. Let $f\colon \mathbb{D} \to \mathbb{D}$ be a deck transformation. Recall that this means that the following diagram commutes:



Since p is a Möbius map and since p is a local diffeomorphism we see that f is locally a Möbius map. By Lemma 132.16 (2) this implies that f is in fact a Möbius map. It now follows from Lemma 132.16 (1) that f is in fact a Möbius transformation. ⊞

Now we can easily conclude the proof of the proposition, indeed we have the following isomorphisms:

$$\pi_1(\Sigma_g, x_0) \overset{\text{by the Deck Transformation–Normalizer Proposition 128.9}}{\cong} \text{Deck}(p\colon \mathbb{D} \to \Sigma_g) \underset{\text{by the claim}}{\subseteq} \text{group of Möbius transformations of } \mathbb{D} \underset{\text{by the Hyperbolic Isometries Proposition 131.6}}{\cong} \text{SL}(2,\mathbb{R}).$$

■

The following proposition gives in particular a positive answer to Question 130.30.

Proposition 132.25. (Surface Group–Linear Proposition) Let Σ be a closed connected 2-dimensional smooth manifold. We pick a base point $x_0 \in \Sigma$. The following two statements hold:

(1) The group $\pi_1(\Sigma, x_0)$ is linear over $\mathbb{R}$.
(2) The group $\pi_1(\Sigma, x_0)$ is residually finite.

Remark.

(1) In [**NewM1985**, Lemma 1], see also [**Scot1978**, Section 3] together with [**Bou1981**, Chapitre V, §4, Section 4], it shown that $\pi_1(\Sigma, x_0)$ is in fact linear over $\mathbb{Z}$.
(2) There are many other proofs that $\pi_1(\Sigma, x_0)$ is residually finite, see e.g. [**Chis1999**, Corollary 4.9] and [**Hem1972**] for alternative arguments. □

Proof. Let Σ be a closed connected 2-dimensional smooth manifold. By the Orientation Covering–Existence Proposition 95.8 we know that there exists a finite covering $p\colon (\widetilde{\Sigma}, \widetilde{x}_0) \to (\Sigma, x_0)$ such that $\widetilde{\Sigma}$ is a closed orientable connected 2-dimensional smooth manifold. By the Surface Group–SL$(2,\mathbb{R})$-Proposition 132.24 we know that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is isomorphic to a subgroup of SL$(2,\mathbb{R})$. In particular we see that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is linear over $\mathbb{R}$.

By the Covering-π_1-Monomorphism Proposition 92.7 and the Covering-Degree Proposition 92.10 we know that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is isomorphic to a finite-index subgroup of $\pi_1(\Sigma, x_0)$. Thus it follows from the above together with Exercise 130.15 that $\pi_1(\Sigma, x_0)$ is also linear over $\mathbb{R}$. By the Linear–Residually Finite Proposition 130.24 this implies that $\pi_1(\Sigma, x_0)$ is residually finite. ■

132.8. **Picard's Theorem.** We also have the following variation on Hadamard's Theorem 132.14.

Proposition 132.26. (Punctured Sphere–Covering Proposition) Let $H_4/\sim$ be the three-punctured sphere with the Möbius structure constructed in the Punctured Sphere–Möbius Structure Proposition 132.8. Then there exists a covering map $p\colon \mathbb{D} \to H_4/\sim$ with the following properties:

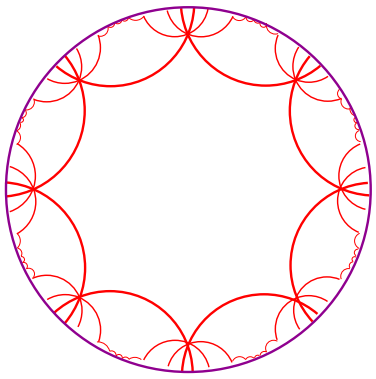
(1) p is a Möbius map,
(2) p is a local isometry,
(3) p is a local biholomorphism.

Proof. The proof of the Punctured Sphere–Covering Proposition 132.26 is almost the same as the proof of Hadamard's Theorem 132.14. In the proof of Hadamard's Theorem 132.14 we used that $H_8/\sim$ is complete. In the present case it follows from the Punctured Sphere–Möbius Structure Proposition 132.8 (3) that $H_4/\sim$ is complete. The construction of the Möbius map $\Phi\colon \mathbb{D} \to H_4/\sim$ is verbatim the same as the construction of the Möbius map $\Phi\colon \mathbb{D} \to H_8/\sim$. The same argument as before shows that $\Phi\colon \mathbb{D} \to H_4/\sim$ is surjective and that it is a covering map. ■

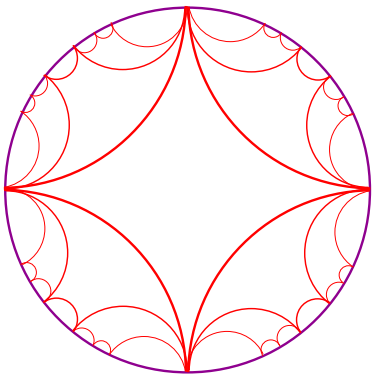
Remark. We consider again the covering $p\colon \mathbb{D} \to H_8/\sim$ that we had just constructed. It follows from the Uniform Covering Criterion 127.7 (2) that the interior of H_8 is uniformly covered. In particular we see that

$$\mathbb{D} \setminus \underbrace{p^{-1}(\partial H_8)}_{\text{“1-dimensional”}} \quad = \quad \bigsqcup \text{ copies of } \mathring{H}_8.$$

Put differently, up to the "one-dimensional" subset $p^{-1}(\partial H_8)$ we can cover $\mathbb{D}$ by infinitely many disjoint copies of the open hyperbolic octagon. Such a decomposition is often called a *tessellation*. This tessellation is shown in the figure below on the left.



preimages of $\mathring{H}_8$ under the universal covering $p\colon \mathbb{D} \to H_8/\sim$



preimages of $\mathring{H}_4$ under the universal covering $p\colon \mathbb{D} \to H_4/\sim$

The same argument also applies to the covering $\mathbb{D} \to H_4/\sim$ that we also had just constructed. The corresponding tessellation of the disk $\mathbb{D}$ is shown in the figure above on the right. □

At the end of this chapter we return to complex analysis. There are different types of holomorphic functions defined on $\mathbb{C}$.

(1) There are the (rather dull) constant functions.

(2) There are holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ that are surjective. For example it is an easy consequence of the fundamental theorem of algebra that all maps defined by non-constant polynomials are surjective.
(3) There are also non-constant maps that are not surjective, for example the exponential function $z \mapsto \exp(z)$ is never zero[1375], in fact its image is $\mathbb{C} \setminus \{0\}$.

On the other hand it is fairly easy prove, using Liouville's Theorem 39.10, that the image of a non-constant holomorphic function on $\mathbb{C}$ is dense.

Now we can prove Picard's Theorem[1376] which is a significantly stronger statement than the above statement that the image is dense.

Theorem 132.27. (Picard's Theorem) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f.

Proof. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there exist two different complex numbers a, b that do not lie in the image of f. We need to show that f is constant. We consider the biholomorphism

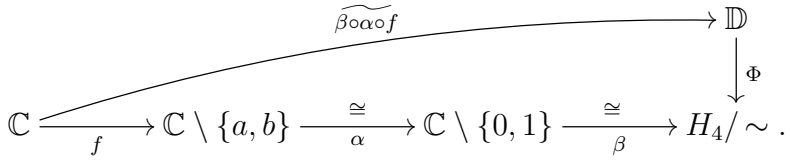
$$\begin{array}{rcl} \alpha\colon \mathbb{C} \setminus \{a,b\} & \to & \mathbb{C} \setminus \{0,1\} \\ z & \mapsto & \dfrac{z-a}{b-a}. \end{array}$$

Furthermore we denote by $\beta\colon \mathbb{C} \setminus \{0,1\} \to H_4/\!\sim$ the biholomorphism from the Punctured Sphere–Biholomorphism Proposition 132.9. Now we consider the following diagram of maps

$$\begin{array}{ccccccc} & & & & & & \mathbb{D} \\ & & & & & & \downarrow \Phi \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{a,b\} & \xrightarrow[\cong]{\alpha} & \mathbb{C} \setminus \{0,1\} & \xrightarrow[\cong]{\beta} & H_4/\!\sim \end{array}$$

where $\Phi\colon \mathbb{D} \to H_4/\!\sim$ is the covering map from the Punctured Sphere–Covering Proposition 132.26. Recall that Φ is a Möbius map which implies by the Möbius Map Lemma 132.13 that Φ is a local biholomorphism.

Since $\mathbb{C}$ is simply connected we can appeal to the Map Lifting Criterion 127.5 to obtain a lift of the map $\beta \circ \alpha \circ f\colon \mathbb{C} \to H_4/\!\sim$ to the universal covering. More precisely, there exists a continuous map $\beta \circ \alpha \circ f\colon \mathbb{C} \to \mathbb{D}$ such that the following diagram commutes:



Note that it follows from the fact that $\beta \circ \alpha \circ f$ is holomorphic and the fact that Φ is a local biholomorphism, that the map $\beta \circ \alpha \circ f\colon \mathbb{C} \to \mathbb{D}$ is also holomorphic. But $\mathbb{D}$ is of course bounded. Therefore it follows from Liouville's Theorem 39.10 that $\beta \circ \alpha \circ f$ is constant. But then $\beta \circ \alpha \circ f$ is also constant. Since α and β are biholomorphisms this implies that f itself is already constant. ∎

[1375] Why not?

[1376] Emile Picard (1856-1941) was a French mathematician. Many readers will have encountered him already in real analysis, as one of the two namesakes of the Picard-Lindelöf Theorem.

Remark. In the literature Theorem 132.27 is sometimes referred to as Picard's Little Theorem, to distinguish it from Picard's Great Theorem which says that if z_0 is an essential singularity of a holomorphic function, then in any neighborhood of z_0 the function f assumes all, but possibly one, value *infinitely* many times. We refer to [**Conw78**, Section 12] for details. □

132.9. **Triangle groups.** In this short bonus section we will discuss the triangle groups. But before we can do so we need to introduce the notion of the reflection in a hyperbolic plane.

Definition 132.28. The reflection in the hyperbolic line $l := \{x + 0\mathrm{i} \,|\, x \in (-1, 1)\} \subseteq \mathbb{D}$ is defined to be the orientation-reversing isometry

$$\begin{array}{rcl} \rho_l \colon \mathbb{D} & \to & \mathbb{D} \\ z = x + \mathrm{i}y & \mapsto & \overline{z} = x - \mathrm{i}y. \end{array}$$

Now let k be any hyperbolic line in $\mathbb{D}$. By the Hyperbolic Lines Lemma 131.13 there exists a Möbius transformation Φ with $\Phi(k) = l$. We define the reflection in k to be the map

$$\begin{array}{rcl} \rho_k \colon \mathbb{D} & \to & \mathbb{D} \\ x & \mapsto & (\Phi^{-1} \circ \rho_l \circ \Phi)(x) \end{array}$$

In Exercise 132.1 we will see that the definition of ρ_k does not depend on the choice of Φ.

Next recall that given any two distinct points $P, Q \in \mathbb{D}$ there exists by the Hyperbolic Lines Lemma 131.13 a unique hyperbolic line $g(P, Q)$ through P and Q. Now we can formulate the key proposition of this section.

Proposition 132.29. (Hyperbolic Triangle Proposition) Let $l, m, n \in \mathbb{N}$ such that the inequality $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ holds.

(1) There exists a hyperbolic triangle Δ_{ABC} with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ at the vertices A, B and C.

Let G be the subgroup of $\mathrm{Isom}(\mathbb{D})$ that is generated by the reflections in the hyperbolic lines $g(C, A)$, $g(A, B)$ and $g(B, C)$. The following statements hold:

(2) (a) Given any $g \neq h \in G$ the intersection $g \cdot \Delta_{ABC} \cap h \cdot \Delta_{ABC}$ is either empty or a vertex or a side.

(b) $\bigcup_{g \in G} g \cdot \Delta_{ABC} = \mathbb{D}$.

(3) The map

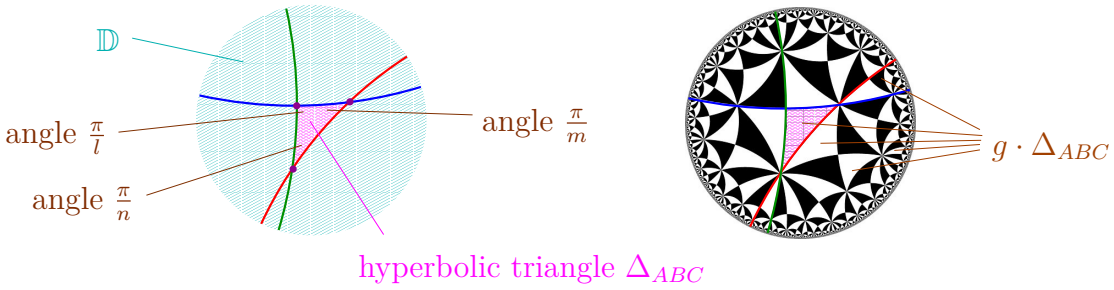
$$\begin{array}{rcl} \langle x, y, z \,|\, x^2, y^2, z^2, (xy)^l = (yz)^m = (zx)^n \rangle & \to & G \\ x & \mapsto & \text{reflection in } g(C, A) \\ y & \mapsto & \text{reflection in } g(A, B) \\ z & \mapsto & \text{reflection in } g(B, C). \end{array}$$

is an isomorphism.

Proof.

(1) This statement is shown in [**Rat2019**, Theorem 3.5.6]. In fact in that reference it is also shown that the triangle is unique up to an isometry.

(2) The statements (2a) and (2b) are both shown in [**Cara1954**, p. 177-182] or alternatively in [**Mask1971**].

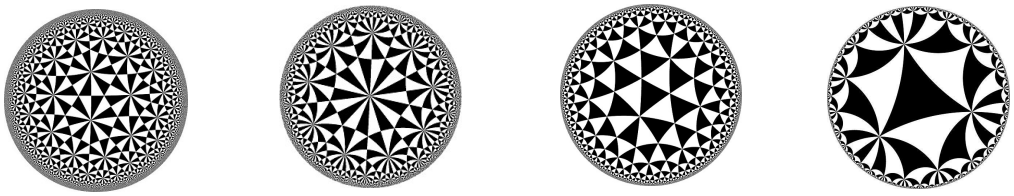


(3) (a) First note that the map is actually well-defined. In fact this follows from the observation that every reflection has order 2 and that xy, yz and zx get sent to a rotation around a vertex with rotation angles $\frac{2\pi}{l}$, $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$.

(b) The fact that the map is in fact an isomorphism is shown in [**Mag1974**, Theorem 2.8]. ■

Remark.

(1) A decomposition of the Euclidean plane $\mathbb{R}^2$ or the hyperbolic disk $\mathbb{D}$ into isometric geometric shapes that only overlap on the boundary is called a *tessellation*. The Hyperbolic Triangle Proposition 132.29 thus shows that the hyperbolic disk $\mathbb{D}$ can be tessellated by any hyperbolic triangle with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ for $l, m, n \in \mathbb{N}$ with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Tessellations for various values of (l, m, n) are shown in the figure below.



$(2, 3, 7)$ $(2, 3, 9)$ $(3, 3, 4)$ $(6, 6, 6)$

(2) Note that, as illustrated in the figure on page 2560, Hadamard's Theorem 132.14 and the Punctured Sphere–Covering Proposition 132.26 also provide tessellations of $\mathbb{D}^2$. □

The Hyperbolic Triangle Proposition 132.29 partly motivates the following definition:

Definition 132.30. Let $l, m, n \in \mathbb{N}$.

(1) The group
$$T^*(l,m,n) := \langle x, y, z \,|\, x^2, y^2, z^2, (xy)^l = (yz)^m = (zx)^n \rangle$$
is called the (l, m, n)-triangle group.

(2) The group
$$T(l,m,n) := \ker \left\{ \begin{array}{rcl} T^*(l,m,n) & \to & \mathbb{Z}/2\mathbb{Z} \\ x, y, z & \mapsto & 1 \end{array} \right\}$$
is called the (l, m, n)-von Dyck group.

Remark. Let $l, m, n \in \mathbb{N}$.

(1) (a) If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then we saw in the Hyperbolic Triangle Proposition 132.29 that $T^*(l, m, n)$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{D})$ generated by reflections in the hyperbolic lines defined by a hyperbolic triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(b) If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=1$, then it is shown in [**Mag1974**, Section II.4] that $T^*(l,m,n)$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ generated by reflections in the Euclidean lines defined by a *euclidean* triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(c) Finally, if $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}>1$, then it is shown in [**Mag1974**, Section II.4] that $T^*(l,m,n)$ is isomorphic to the subgroup of $\mathrm{Isom}(S^2)$ generated by reflections in the great circles defined by a *spherical* triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(2) In each of the three cases above the (l,m,n)-von Dyck group is the intersection of the (l,m,n)-triangle group with the subgroup of orientation-preserving isometries. □

The following lemma gives a particularly simple presentation of the von Dyck groups.

Lemma 132.31. (Van Dyck Group-Presentation Lemma) Given any $l,m,n\in\mathbb{N}$ the map

$$\begin{array}{rcl}\langle a,b\,|\,a^l,b^m,(ab)^n\rangle & \to & T(l,m,n)\\ a & \mapsto & xy\\ b & \mapsto & yz\end{array}$$

is an isomorphism.

Proof. We will prove the Van Dyck Group-Presentation Lemma 132.31 in Exercise 132.3. ■

We conclude this chapter with the following corollary to the Hyperbolic Triangle Proposition 132.29.

Corollary 132.32. (Van Dyck Group-Cardinality Corollary) Let $l,m,n\in\mathbb{N}$. If the inequality $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$ holds, then the triangle group $T^*(l,m,n)$ and the von Dyck group $T(l,m,n)$ are infinite.

Proof. Let $l,m,n\in\mathbb{N}$ with $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. We denote by G the group introduced in the Hyperbolic Triangle Proposition 132.29. Note that it follows from the Hyperbolic Triangle Proposition 132.29 (2), the fact that Δ_{ABC} is compact and the fact that $\mathbb{D}$ is non-compact that G is infinite. By the Hyperbolic Triangle Proposition 132.29 (3) this implies that the triangle group $T^*(l,m,n)$ is infinite. The index two subgroup $T(l,m,n)$ is evidently also infinite. ■

132.10. Exercises for Chapter 132.

Exercise 132.1. We define reflection in the hyperbolic line $l:=\{x\,|\,x\in(-1,1)\}\subseteq\mathbb{D}$ to be the map

$$\begin{array}{rcl}\rho_l\colon\mathbb{D} & \to & \mathbb{D}\\ z=x+\mathrm{i}y & \mapsto & \overline{z}=x-\mathrm{i}y.\end{array}$$

Now let k be any hyperbolic line in $\mathbb{D}$. By the Hyperbolic Lines Lemma 131.13 there exists a Möbius transformation Φ with $\Phi(k)=l$. We define the *reflection in* k to be the map

$$\begin{array}{rcl}\rho_k\colon\mathbb{D} & \to & \mathbb{D}\\ x & \mapsto & (\Phi^{-1}\circ\rho_l\circ\Phi)(x)\end{array}$$

Show that the definition of ρ_k does not depend on the choice of Φ.

Exercise 132.2. Show that the map

$$\begin{array}{rcl} \mathrm{SL}(2,\mathbb{R}) \times \mathbb{H} & \to & \mathbb{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) & \mapsto & \dfrac{az+b}{cz+d} \end{array}$$

defines an action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$ and show that this action is transitive.

Exercise 132.3. Let $l, m, n \in \mathbb{N}$. Show that the map

$$\begin{array}{rcl} \langle a, b \,|\, a^l, b^m, (ab)^n \rangle & \to & T(l,m,n) \\ a & \mapsto & xy \\ b & \mapsto & yz \end{array}$$

is an isomorphism.
Hint. First use Proposition 113.19 to find an explicit presentation for $T(l,m,n)$.

CHAPTER 133

Related constructions in algebraic geometry and Galois theory

In this short chapter we indicate some connections between the topics that we have covered so far to algebraic geometry and Galois theory. Since the author of these notes does not know much about either field the following discussion should be taken with a grain of salt. In particular for an expert the subsequent discussion will certainly look too simplistic.

This chapter is organized as follows:

- In Section 133.1 we will discuss the fundamental group of an algebraic variety.
- In Section 133.2 we will discuss similarities between Galois theory and coverings.

133.1. **The fundamental group of an algebraic variety.** The following definition might be familiar from an elementary course on algebraic geometry:

Definition 133.1. Let $\mathbb{K}$ be a field.[1377]

(1) Given a set of multivariable polynomials S, i.e. given a subset $S \subseteq \mathbb{K}[t_1, \ldots, t_n]$ we denote by

$$V(S) = \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in S\}$$

the vanishing set of S, i.e. the set of all points in $\mathbb{K}^n$ where all the polynomials in S, viewed as maps $\mathbb{K}^n \to \mathbb{K}$, vanish.

(2) We say that $U \subseteq \mathbb{K}^n$ is Zariski open,[1378] if there exists a subset S of $\mathbb{K}[t_1, \ldots, t_n]$ such that $U = \mathbb{K}^n \setminus V(S)$.

Remark. Let S be a subset of $\mathbb{K}[t_1, \ldots, t_n]$. Since the ring $\mathbb{K}[t_1, \ldots, t_n]$ is Noetherian there exists a finite set $\{f_1, \ldots, f_k\} \subseteq \mathbb{K}[t_1, \ldots, t_n]$ such that $V(S) = V(f_1, \ldots, f_k)$. □

Lemma 133.2. (Zariski Topology Lemma) Let $\mathbb{K}$ be a field. The Zariski open subsets form a topology on $\mathbb{K}^n$.

Remark.

(1) The topology defined by the Zariski open subsets is of course called the *Zariski topology on* $\mathbb{K}^n$.

(2) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then every Zariski open subset is also open with respect to the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n = \mathbb{R}^{2n}$. But for $n \geq 1$ the converse does not hold, in fact "most" subsets that are open with respect to the usual topology are not open with respect to the Zariski topology. □

[1377] For example we could take $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ but $\mathbb{K}$ could also be a field of non-zero characteristic, e.g. for a prime p we could take $\mathbb{K} = \mathbb{F}_p$ or $\mathbb{K}$ could be the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.

[1378] Oscar Zariski (1899-1986) was a Russian-American mathematician working in algebraic geometry.

Proof. Let $\mathbb{K}$ be a field.

(1) The vanishing set of the empty set, i.e. $V(\varnothing)$ is all of $\mathbb{K}^n$. It follows that the empty set $\varnothing = \mathbb{K}^n \setminus V(\varnothing)$ is open.
(2) The vanishing set of the constant polynomial $f = 1$ is the empty set. It follows that $\mathbb{K}^n = \mathbb{K}^n \setminus V(1)$ is open.
(3) We need to show that the intersection of finitely many Zariski open subsets is again Zariski open. By induction it suffices to consider the case of two Zariski open subsets. So suppose that we are given Zariski open subsets $X = \mathbb{K}^n \setminus V(S)$ and $Y = \mathbb{K}^n \setminus V(T)$. Then
$$\mathbb{K}^n \setminus (X \cap Y) \;=\; (\mathbb{K}^n \setminus X) \cup (\mathbb{K}^n \setminus Y) \;=\; V(S) \cup V(T) \;=\; V\big(\{s \cdot t\}_{s \in S, t \in T}\big).$$
(4) We need to show that the union of Zariski open subsets is again Zariski open. So suppose that we are given a family of Zariski open subsets $X_i = \mathbb{K}^n \setminus V(S_i)$, $i \in I$. Then
$$\mathbb{K}^n \setminus \bigcup_{i \in I} X_i \;=\; \bigcap_{i \in I} (\mathbb{K}^n \setminus X_i) \;=\; \bigcap_{i \in I} V(S_i) \;=\; V\Big(\bigcup_{i \in I} S_i\Big).$$

Thus we have verified that the Zariski open subsets have all the properties of a topology. ■

Definition 133.3. Let $\mathbb{K}$ be a field.

(1) A variety over $\mathbb{K}$ is defined as the vanishing set of a set of multivariable polynomials.
(2) The Zariski topology on a variety $V \subseteq \mathbb{K}^n$ is defined as the subspace topology on V induced from the Zariski topology on $\mathbb{K}^n$.

Example. Let $\mathbb{K}$ be a field and let and $f(s,t) = s^2 + t^2 - 1$. We define
$$S^1_{\mathbb{K}} \;:=\; V(f) \;=\; \{(x,y) \in \mathbb{K}^2 \mid f(x,y) = x^2 + y^2 - 1 = 0\} \;=\; \{(x,y) \in \mathbb{K}^2 \mid x^2 + y^2 = 1\}.$$
For $\mathbb{K} = \mathbb{R}$ we obtain the usual circle S^1. □

Any variety V is again a topological space, so in principle one could study the usual fundamental group. But in many cases, e.g. if the characteristic of $\mathbb{K}$ is non-zero, one will see that all continuous maps $S^1 \to V$ are constant, hence the usual fundamental group is trivial.

The correct analog of a finite covering $p\colon X \to B$ between topological spaces for varieties is a finite étale morphism. The precise definition of a finite étale morphism is irrelevant for our purpose.[1379] A heuristic fact is that varieties tend to have many finite étale morphisms, but there is no well-behaved analog of an infinite covering in the context of varieties. In particular usually there is no concept of a "universal" étale morphism. The question is now, whether one can recover the notion of a fundamental group from the knowledge of finite covers.

Given a variety V we can consider all finite regular étale morphisms $p_i\colon W_i \to V$, $i \in I$. The corresponding deck transformations groups $\mathrm{Deck}(p_i\colon W_i \to V)$, $i \in I$ are finite groups that form an inverse system. Now we can define the étale fundamental group of V as
$$\pi_1^{\text{ét}}(V) \;:=\; \varprojlim_i \mathrm{Deck}(p_i\colon W_i \to V).$$

[1379] Furthermore it is unknown to the author of these lecture notes.

By the Finite Groups–Inverse Limit Proposition 229.9 the étale fundamental group $\pi_1^{\text{ét}}(V)$ is naturally a compact topological group. In algebraic geometry the étale fundamental group of a variety plays a role similar to the role of the fundamental group of a topological space in topology.

If V is a complex variety without singularities, then the étale fundamental group is isomorphic to the profinite completion[1380] of the usual topological fundamental group $\pi_1(V)$, i.e. we have

$$\pi_1^{\text{ét}}(V) \;\cong\; \widehat{\pi_1(V)}.$$

We refer to [**Groth1971**] for a proof of this statement.

More information on this topic can be found at

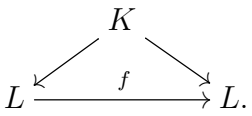
`https://en.wikipedia.org/wiki/etale_fundamental_group`

133.2. **Galois theory.** The theory of coverings of topological spaces has some similarities to the Galois theory of field extensions.

Throughout this section let K be a field of characteristic zero[1381] and let L/K be a field extension of K, i.e. L is a field which contains K as a subfield. The dimension of L as a K vector space is denoted by $[L : K]$. We consider the group

$$\mathrm{Gal}(L/K) = \text{ all field automorphisms of } L \text{ which are the identity on } K.$$

Put differently, a field automorphism $f\colon L \to L$ lies in $\mathrm{Gal}(L/K)$ if and only if the following diagram commutes:



We consider two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt{2})\colon \mathbb{Q}] = 2$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ consists of the identity and the homomorphism $a + b\sqrt{2} \mapsto a - b\sqrt{2}$.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt[3]{2})\colon \mathbb{Q}] = 3$ and the group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ consists only of the identity.

Given a field K we denote by $\overline{K}$ the algebraic closure of K. A field extension L/K is called *Galois*, if for each $f \in \mathrm{Gal}(\overline{K}/K)$ we have $f(L) = L$.[1382] In the usual course on Galois theory one shows that L/K is Galois if and only if each polynomial in $K[x]$ which admits a zero in L splits over L, i.e. all other zeros of f lies also in L. For a Galois extension L/K we refer to $\mathrm{Gal}(L/K)$ as the *Galois group* of the field extension.

We return to the above two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois field extension.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois field extension, since $\sqrt[3]{2}$ is a zero of $x^3 - 2$, but the complex zero $\sqrt[3]{2} \cdot \exp(2\pi\,\mathrm{i}/3)$ of f does not lie in $\mathbb{Q}(\sqrt[3]{2})$.[1383]

[1380] Recall that in Definition 229.10 we introduced the profinite-completion $\widehat{\pi}$ of a group π.

[1381] This restriction to characteristic zero is not necessary, but it simplifies the discussion.

[1382] Strictly speaking this is the definition of a normal extension, but since K is a field of characteristic zero, a normal extension is already Galois.

[1383] We put $z = \sqrt[3]{2}$ and $w = \exp(2\pi\,\mathrm{i}/3)$. It is a good exercise to show that $\mathbb{Q}(z, w)/\mathbb{Q}$ is a normal field extension with $[\mathbb{Q}(z, w)\colon \mathbb{Q}] = 6$. The group $\mathrm{Gal}(\mathbb{Q}(z, w)/\mathbb{Q})$ is isomorphic to the permutation group S_3.

If L/K is a finite extension, then the main theorem of Galois theory says that the following two statements are equivalent:

(1) The field extension L/K is Galois.
(2) $\#\operatorname{Gal}(L/K) = [L : K]$.

The statement is formally very similar to the statement of the Regular Covering-Characterization Proposition 128.13 which says that for some finite covering $p\colon X \to B$ of topological spaces the following statements are equivalent:

(1) The covering $p\colon X \to B$ is regular.
(2) $\#\operatorname{Deck}(p\colon X \to B) = [X : B]$.

The analogy between Galois theory and covering theory goes considerably further. Many statements about the Galois group of a Galois extension have a corresponding statement for the deck transformation group of a normal covering. For example, given a finite extension L/K the (normal) subgroups of Galois group $\operatorname{Gal}(L/K)$ correspond to (normal) intermediate fields and (normal) subgroups of the deck transformation group correspond to (regular) intermediate coverings.

Now let L/K be a Galois extension of infinite degree. All finite intermediate Galois extensions of K, i.e. all finite Galois extensions F/K with $F \subseteq L$, form a directed set via the inclusion. The corresponding Galois groups form an inverse set. Indeed, given two finite Galois extensions F/K and G/K with $F \subseteq G$ the restriction of an automorphism of G to an automorphism of F defines a group homomorphism $\operatorname{Gal}(G/K) \to \operatorname{Gal}(F/K)$. Therefore we can form the inverse limit of these Galois groups. By [**FJ2008**, Section 1.3] there exists a canonical isomorphism of groups

$$\operatorname{Gal}(L/K) \;=\; \varprojlim_{\substack{F/K \text{ finite} \\ \text{Galois extension}}} \operatorname{Gal}(F/K).$$

On the right-hand side we consider the inverse limit of an inverse system of finite groups. It follows from the Finite Groups–Inverse Limit Proposition 229.9 that the right-hand side, and thus also the Galois group $\operatorname{Gal}(L/K)$, is naturally a compact topological group. This topology on $\operatorname{Gal}(L/K)$ is called the *Krull topology*.

Example. Let p be a prime. We denote by $\mathbb{F}_p$ the finite field with p elements and we denote by $\overline{\mathbb{F}}_p$ its algebraic closure. It is shown in [**FJ2008**, Section 1.5] that there exists a canonical isomorphism

$$\operatorname{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right) \;\cong\; \widehat{\mathbb{Z}}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$, or equivalently the profinite integers that we will introduced on page 4437. □

Part XIV

Higher homotopy groups

In the previous chapters we introduced and studied in great detail the fundamental groups of pointed topological spaces. Using the fundamental group we managed to prove several non-trivial results, for example we proved that

(1) S^1 and S^k with $k \geq 2$ are not homeomorphic,
(2) $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n \geq 3$ are not homeomorphic,
(3) surfaces of different genus are not homeomorphic,
(4) we showed that S^1 is not a deformation retract of $\overline{B}^2$,
(5) the trefoil is not isotopic to the trivial knot.

But there are still many even rather elementary questions for which the answer eludes. For example at the moment we cannot show that for $k \neq l$ with $k, l \geq 2$ the spheres S^k and S^l are not homeomorphic and we cannot show that for $k \neq l$ with $k, l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. Also, the somewhat mysterious lens spaces $L(p, q)$ still evade our attempts at distinguishing them for a given p.

The problem with fundamental groups is that we only probe a topological space X with loops. The natural idea is now to go from using maps $S^1 \to X$ to studying maps $S^n \to X$ for $n \geq 2$. As we will see, this idea leads us naturally to the definition of the "higher homotopy groups" of a topological space.

Thus, in this part of the lecture notes we will introduce the higher homotopy groups of a pointed topological space. In these notes a "part" is usually a cluster of chapters that deals with one particular aspect of topology. For example in Part X we introduced the fundamental groups of pointed topological spaces and then we got to know many different techniques for calculating fundamental groups. Similarly in Part XVI will introduce the singular homology groups of a topological space and we will get to know many different techniques for calculating singular homology groups.

In this part we will introduce the higher homotopy groups of a pointed topological space. But when it comes to calculations we almost immediately run out of steam. In fact there is not really a dedicated machine to computing higher homotopy groups (in contrast to fundamental groups and singular homology groups). In fact the first really non-trivial calculation of a higher homotopy group builds on the theory of homology groups.

Throughout these notes, as we expand our knowledge of topology, we will get to know many tricks for computing higher homotopy groups (or at least for obtaining partial results), and over time we will see many applications. In this sense higher homotopy groups are absolutely essential to topology, even though our initial attempt to study higher homotopy groups will be a somewhat frustrating exercise.

CHAPTER 134

Higher homotopy groups: Definition and basics

In this chapter we will define higher homotopy groups and we collect several basic properties.
This chapter is organized as follows:

- In Section 134.1 we will define the higher homotopy groups $\pi_n(X, x_0)$.
- In Section 134.2 we will consider several alternative descriptions of the higher homotopy groups.
- In Section 134.3 we will collect several essential statements about higher homotopy groups. In particular we will see that for $n \geq 2$ these groups are in fact abelian.
- In Section 134.4 we will see that higher homotopy groups define covariant functors.
- In Section 134.5 we will see that for coverings $p\colon \widetilde{X} \to X$ and any $n \geq 2$ the higher homotopy groups of $\widetilde{X}$ and X are isomorphic.
- In Section 134.6 we will introduce the Hopf map $S^3 \to S^2$ as a candidate for map which is a non-trivial element in $\pi_3(S^2)$.
- In Appendix 134.7 we will use de Rham cohomology to obtain some results on $\pi_n(S^n)$.
- In Appendix 134.8 we will use the theory of degrees to determine $\pi_n(S^n)$.
- In Section 134.9 we will provide exercises for Chapter 134.

In the next Chapter 135 we will discuss cell attachments.

134.1. **Definition of the higher homotopy groups.** We start out by recalling Definition 19.24.

Definition 134.1. Let W and X be topological spaces and let $A \subseteq W$ and $B \subseteq X$ be subsets.

(1) A continuous map $(W, A) \to (X, B)$ of pairs of topological spaces is a continuous map $f\colon W \to X$ with $f(A) \subseteq B$.
(2) Let $f, g\colon (W, A) \to (X, B)$ be two continuous maps of pairs of topological spaces.
   (a) A homotopy between f and g is a continuous map
   $$H\colon W \times [0,1] \to X,$$
   such that $H_0 = f$ and $H_1 = g$ and such that each H_t is a map $(W, A) \to (X, B)$.
   (b) If there exists a homotopy, then we say that f and g are homotopic and we write $f \simeq g$.
(3) We set
   $$[(W, A), (X, B)] := \{\text{continuous maps } (W, A) \to (X, B)\}/\simeq .$$

Notation 134.2. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}_0$ we denote by $I^n \subseteq \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote by ∂I^n its boundary as a subspace of $\mathbb{R}^n$.



$\alpha, \beta\colon (W, A) \to (X, B)$ are homotopic but
$\alpha, \gamma\colon (W, A) \to (X, B)$ are not homotopic

Let (X, x_0) be a pointed topological space. With the above definitions we can write

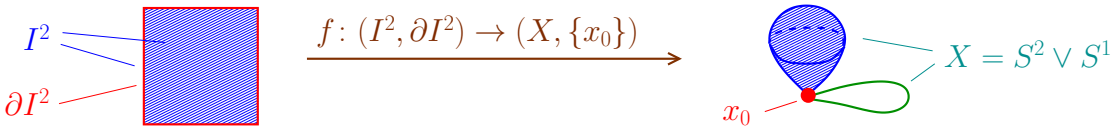
$$\pi_1(X, x_0) \;=\; [(I, \partial I), (X, \{x_0\})].$$

This definition can be easily generalized to higher dimensions:

Definition 134.3. Given a pointed topological space (X, x_0) and given $n \in \mathbb{N}_0$ we define

$$\pi_n(X, x_0) \;\coloneqq\; \underbrace{[(I^n, \partial I^n), (X, \{x_0\})]}_{\substack{\text{homotopy classes of continuous} \\ \text{maps } (I^n, \partial I^n) \to (X, \{x_0\})}}.$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n-th homotopy group. We refer to any $\pi_n(X, x_0)$ with $n \geq 2$ as a higher homotopy group.



Remark. This definition makes in particular sense for $n = 0$, we can thus consider $\pi_0(X, x_0)$. In Section 4.6 we had already defined $\pi_0(X)$ as the set of path-components of X. In fact it follows almost immediately from the fact that $I^0 = \{0\}$ and $\partial I^0 = \varnothing$ that the map

$$\begin{array}{rcl} \pi_0(X, x_0) & \to & \pi_0(X) \\ [\gamma\colon I^0 = \{0\} \to X] & \mapsto & \text{path-component of } X \text{ that contains } \gamma(0) \end{array}$$

is a bijection. □

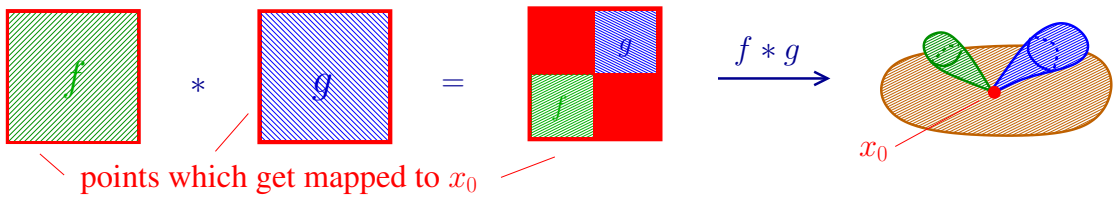
Note that $\pi_n(X, x_0)$ is a priori only a set. So to justify the name "homotopy group" we need to define a group structure on $\pi_n(X, x_0)$ which for $n = 1$ agrees with our previous definition of the group structure. Here is what we do:

Definition 134.4. Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f, g\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be two continuous maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, \{x_0\}) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, \ldots, 2x_n), & \text{if } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \ldots, 2x_n - 1), & \text{if } (x_1, \ldots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

Using the Pasting Proposition 4.7 (2) one can easily show that this map is indeed continuous.

It is straightforward to see that the homotopy class of $f * g$ only depends on the homotopy classes of f and g. This leads us to the following definition:



Definition 134.5. Let (X, x_0) be a pointed topological space and let $n \geq 1$. We define

$$\begin{array}{rcl} \pi_n(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([f],[g]) & \mapsto & [f] \cdot [g] \coloneqq [f * g]. \end{array}$$

Note that for $n = 1$ this is precisely the same definition of the product structure that we already introduced in Section 88.2.

Now we can formulate the following proposition:

Proposition 134.6. (π_n-Group Proposition) Let (X, x_0) be a pointed topological space and let $n \geq 1$. The following two statements hold:

(1) The map

$$\begin{array}{rcl} \pi_n(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([f],[g]) & \mapsto & [f] \cdot [g] \coloneqq [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

(2) Given a continuous map $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$ the inverse of $[f] \in \pi_n(X, x_0)$ is represented by the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, x_2, \ldots, x_n) & \mapsto & f(1 - x_1, x_2, \ldots, x_n). \end{array}$$

Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P. In this case the only map $I^n \to \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}, P) = 0$ for any $n \geq 1$. □

The proof of the π_n-Group Proposition 134.6 requires the remainder of this section. As we will see, the proof of the π_n-Group Proposition 134.6 is quite similar to the proof of the Path-Homotopy Product Proposition 88.10 and the proof of the π_1–Group Proposition 88.12.

The technicalities are evidently more difficult than before. We will provide (most) of the gory details. But most readers will be best served by just glimpsing at the pictures in the proof and by ignoring the technical details.

After this preamble let us turn to the actual proof of the π_n-Group Proposition 134.6. To deal with the technicalities we need a few more definitions and one lemma.

Definition 134.7.

(1) A hyperrectangle is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.

(2) A hyperrectangle $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is called non-degenerate if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

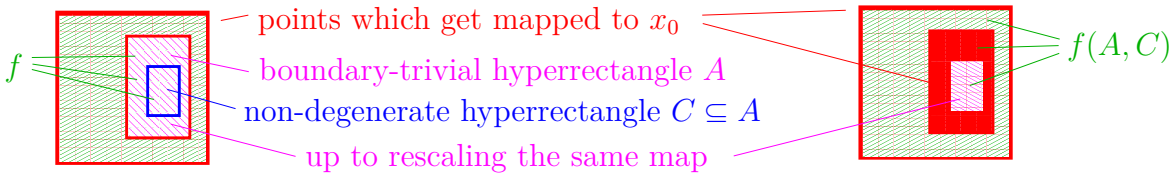
Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a continuous map.

(3) We say that a hyperrectangle $A \subseteq I^n$ is boundary-trivial if f sends all points on the boundary ∂A of A to x_0.

(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq I^n$ be a boundary-trivial hyperrectangle. Furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. We define

$$f(A,C)\colon I^n \to X$$
$$x = (x_1, \ldots, x_n) \mapsto \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\left(a_1 + (x_1 - c_1) \cdot \frac{b_1 - a_1}{d_1 - c_1}, \cdots, a_n + (x_n - c_n) \cdot \frac{b_n - a_n}{d_n - c_n}\right) & \text{if } x \in C. \end{cases}$$

Using the Pasting Proposition 4.7 (2) one can verify that this map is continuous.



Lemma 134.8. (Shrinking Lemma) Let (X, x_0) be a pointed topological space. Furthermore let $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be a continuous map. Furthermore let $A \subseteq I^n$ be a boundary-trivial hyperrectangle and let C be a non-degenerate hyperrectangle that is contained in A. Then the maps f and $f(A,C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.

Proof. Let $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be a continuous map between pairs of topological spaces. Furthermore let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq I^n$ be a boundary-trivial hyperrectangle and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. For $t \in [0,1]$ we write

$$C_t = [a_1 \cdot (1-t) + c_1 \cdot t, b_1(1-t) + d_1 \cdot t] \times \cdots \times [a_n \cdot (1-t) + c_n \cdot t, b_n \cdot (1-t) + d_n \cdot t].$$

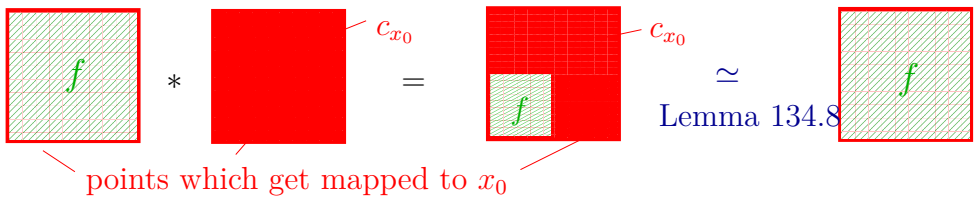
Note that $C_0 = A$ and $C_1 = C$. Using the Pasting Proposition 4.7 (2) one can show fairly easily that the map

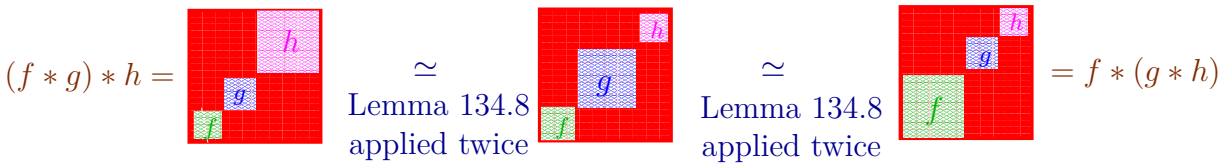
$$F\colon I^n \times [0,1] \to X, \quad (x,t) \mapsto f(A, C_t)(x)$$

is continuous. By construction we have $F_0 = F$ and $F_1 = f(A,C)$, thus F is a homotopy between f and $f(A,C)$. ∎

Proof of the π_n-Group Proposition 134.6.

- The neutral element of the product structure is given by the equivalence class of the constant map
$$c_{x_0}\colon I^n \to X, \quad x \mapsto x_0.$$
Indeed, it follows immediately from the Shrinking Lemma 134.8, applied to $A = [0,1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any continuous map $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.
- We consider three maps $f, g, h\colon (I^n, \partial I^n) \to (X, \{x_0\})$. By applying the Shrinking Lemma 134.8 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. the figure below. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.





- Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$: In the following we suppose that we are given a continuous map $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \ldots, x_n) & \mapsto & f(1 - x_1, x_2, \ldots, x_n). \end{array}$$

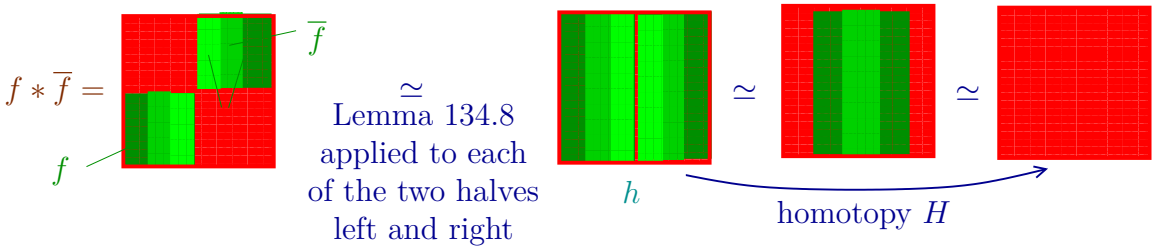
We claim that $[f] \cdot [\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from the Shrinking Lemma 134.8 that $f * \overline{f}$ is homotopic to the map

$$\begin{array}{rcl} h\colon (I^n, \partial I^n) & \to & (X, \{x_0\}) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, x_2, \ldots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \underbrace{\overline{f}(2x_1 - 1, x_2, \ldots, x_n)}_{= f(2 - 2x_1, x_2, \ldots, x_n)}, & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map[1384] [1385]

$$\begin{array}{rcl} H\colon I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \ldots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \ldots, x_n), & \text{if } x_1 \in [\frac{1}{2}, 1 - \frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{array}$$

is a homotopy from h to c_{x_0}. We refer to the figure below for an illustration of the argument.



Basically the same argument also shows that $[\overline{f}] \cdot [f] = [\overline{f} * f] = [c_{x_0}]$. ∎

[1384] It follows from the Pasting Proposition 4.7 (2) that this map is continuous.

[1385] Even for $n = 1$ this homotopy is different from the homotopy used in the Path-Homotopy Product Proposition 88.10 (3). Why is that?

134.2. **Alternative descriptions of homotopy groups.** In Section 88.6 we showed that there are two different but equivalent ways to define fundamental groups, namely we can either consider the set $[(I,\partial I),(X,\{x_0\})]$ or we can consider the set $[(S^1,1),(X,x_0)]$. Similarly, in this section we will see that there are several alternative points of view regarding elements in homotopy groups.

Identification 134.9. Let $n\in\mathbb{N}$.

(1) Let
$$\begin{aligned}\Phi\colon (I^n,\partial I^n) &\to (\overline{B}^n,S^{n-1})\\ x &\mapsto \begin{cases}\frac{x}{\|x\|}\cdot\max\{|x_1|,\dots,|x_n|\}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0\end{cases}\end{aligned}$$
be the homeomorphism from page 189.

(2) We equip S^n throughout with the base point $*:=(0,\dots,0,1)\in S^n$.[1386]

(3) We denote by $\Psi\colon(\overline{B}^n/S^{n-1},[S^{n-1}])\to(S^n,*)$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 8.10.

(4) Let X be a topological space and let $x_0\in X$. It follows easily from the Topological-Quotient Proposition 8.2 (1) together with the Homotopy Quotient Theorem 18.11 that the above homeomorphisms Φ and Ψ define bijections
$$\begin{aligned}&\text{(homotopy classes) of continuous maps } (I^n,\partial I^n)\to(X,\{x_0\})\\ \iff\ &\text{(homotopy classes) of continuous maps } (\overline{B}^n,S^{n-1})\to(X,\{x_0\})\\ \iff\ &\text{(homotopy classes) of continuous maps } (\overline{B}^n/S^{n-1},[S^{n-1}])\to(X,x_0)\\ \iff\ &\text{(homotopy classes) of continuous maps } (S^n,*)\to(X,x_0).\end{aligned}$$
The three equivalences combined say that we can view elements in $\pi_n(X,x_0)$ as homotopy classes of continuous maps $(S^n,*)\to(X,x_0)$ of pointed topological spaces. This point of view is perhaps closest to our visual approach to thinking about topological spaces. But as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n,\partial I^n)\to(X,\{x_0\})$.

Remark. It is visually most natural to work with the last point of view, namely to think of elements in $\pi_n(X,x_0)$ as homotopy classes of continuous maps $(S^n,*)\to(X,x_0)$. Unfortunately with this definition it is a trifle less clear how the group structure on $\pi_n(X,x_0)$ is defined. To visualize the group structure we need to work a little:

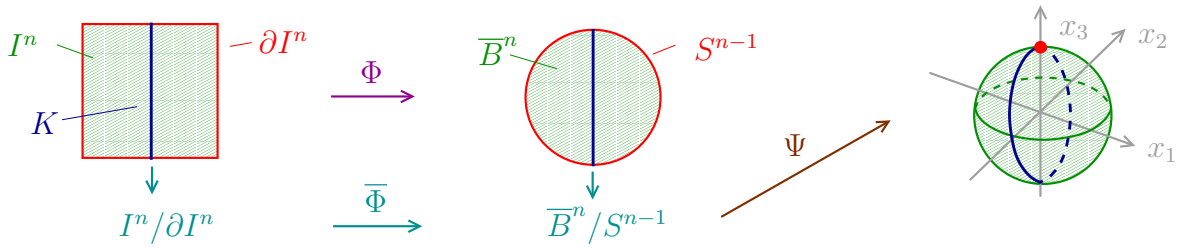
(1) Note that the homeomorphism $\Phi\colon(I^n,\partial I^n)\to(\overline{B}^n,S^{n-1})$ descends to a homeomorphism $\overline{\Phi}\colon I/\partial I^n\to\overline{B}^n/S^{n-1}$.

(2) We consider the homeomorphism $\Theta:=\overline{\Phi}^{-1}\circ\Psi^{-1}\colon S^n\to I^n/\partial I^n$. Note that it identifies the North Pole $(0,\dots,0,1)$ with the point $[\partial I^n]$.

(3) We write $K:=\{(x_1,\dots,x_n)\in I^n\,|\,x_1=\frac{1}{2}\}$.

(4) We denote by $S^n_1\vee S^n_2$ the wedge of two copies of S^n along the point $*\in S^n$.

[1386] Here is a slight nuisance: For S^1, viewed as a subspace of $\mathbb{C}$, it seems most reasonable to pick the base point $1=(1,0)\in\mathbb{C}=_{\mathrm{i}}\mathbb{R}^2$. On the other hand for $S^n\subseteq\mathbb{R}^{n+1}$ it seems most reasonable to pick the North Pole $(0,\dots,0,1)$ as a base point. Of course mathematically there is no difference and we are fully confident that the reader can handle the fact that for S^1 we use at times two different base points.



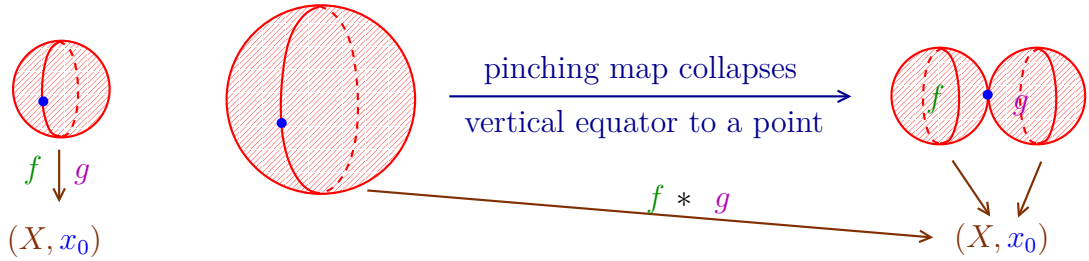
(5) We refer to the map[1387]

$$S^n \xrightarrow{\Theta} I^n/\partial I^n \to I^n/(\partial I^n \cup K) \to S_1^n \vee S_2^n$$
$$[x_1, \ldots, x_n] \mapsto \begin{cases} \Theta^{-1}([2x_1, x_2, \ldots, x_n]) \in S_1^n, & \text{if } x_1 \in [0, \frac{1}{2}], \\ \Theta^{-1}([2x_1 - 1, x_2, \ldots, x_n]) \in S_2^n, & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases}$$

as the *pinching map*.

(6) It follows basically immediately from the definitions that the group structure on $\pi_n(X, x_0)$ is given by

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$
$$([f\colon S^n \to X], [g\colon S^n \to X]) \mapsto \left[S^n \to S_1^n \vee S_2^n \xrightarrow{f \vee g} X\right].$$



In the figure above we sketch the pinching map and we sketch the corresponding definition of the group structure on $\pi_n(X, x_0)$.[1388] □

We conclude this section with the following generalization of the Trivial-in-π_1-Lemma 88.28.

Lemma 134.10. (Trivial-in-π_n-Lemma) Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. Furthermore let $f\colon (S^n, *) \to (X, x_0)$ be a continuous map. The following statements are equivalent:

(1) The map $f\colon S^n \to X$ is homotopic to a constant map.
(2) There exists a continuous map $F\colon \overline{B}^{n+1} \to X$ so that $F|_{S^n} = f$.
(3) The map f represents the trivial element in $\pi_n(X, x_0)$.

Proof. The proof of the lemma is basically identical to the proof of the Trivial-in-π_1-Lemma 88.28. For the reader's convenience we recall the argument.

(1)⇒(2). We assume that $f\colon S^n \to X$ is homotopic to a constant map c_{x_0}. In other words,

[1387] It follows easily from the Pasting Proposition 4.7 (2′), the Topological-Quotient Proposition 8.2 (1b) and the Wedge Proposition 96.2 (1a) that this map is continuous.

[1388] The meticulous reader will notice that it is perhaps not completely clear how the maps f and g are defined on the two smaller spheres.

we assume that there exists a homotopy $H\colon S^n \times [0,1] \to X$ with $H_0 = f$ and $H_1 = c_{x_0}$. We make a few preparations:

(a) Let $\sim$ be the equivalence relation on $S^n \times [0,1]$ given by $(P,1) \sim (Q,1)$ for all $P, Q \in S^n$.

(b) We consider the map

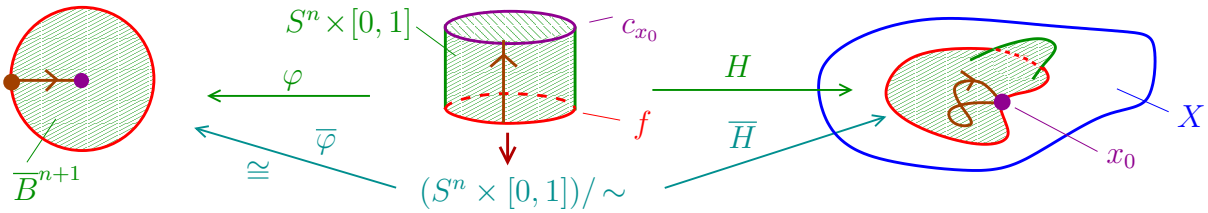
$$\begin{array}{rcl} \varphi\colon S^n \times [0,1] & \to & \overline{B}^{n+1} \\ (x,t) & \mapsto & x \cdot (1-t). \end{array}$$

(c) Note that $\varphi(P) = \varphi(Q)$ and $H(P) = H(Q)$ whenever $P \sim Q$.

(d) From the Topological-Quotient Proposition 8.2 (1b) we obtain induced continuous maps $\overline{\varphi}\colon (S^n \times [0,1])/\sim \to \overline{B}^{n+1}$ and $\overline{H}\colon (S^n \times [0,1])/\sim \to X$.

(e) Using the Compact-Hausdorff Proposition 4.23 (3) one can easily verify that $\overline{\varphi}\colon (S^n \times [0,1])/\sim \to \overline{B}^{n+1}$ is actually a homeomorphism.

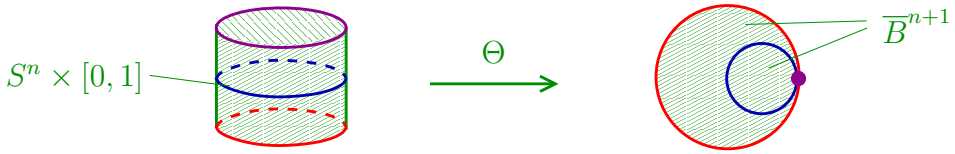
The promised map $F\colon \overline{B}^{n+1} \to X$ is now given by $F := \overline{H} \circ \overline{\varphi}^{-1}$.



(2)⇒(3). Let $F\colon \overline{B}^{n+1} \to X$ be a continuous map with $F|_{S^n} = f$. We consider the map

$$\begin{array}{rcl} \Theta\colon S^n \times [0,1] & \to & \overline{B}^{n+1} \\ \underbrace{((x_1,\dots,x_{n+1}),t)}_{=(x,t)} & \mapsto & \underbrace{(t\cdot x_1,\dots,t\cdot x_n, t\cdot x_{n+1} + (1-t))}_{=x\cdot t + (1-t)\cdot *}. \end{array}$$

One can now easily verify that the desired homotopy of maps $(S^n, *) \to (X, x_0)$ is given by $H = F \circ \Theta\colon S^n \times [0,1] \to X$.



(3)⇒(1). This direction is basically obvious. Indeed, the fact that f represents the trivial element in $\pi_n(X, x_0)$ means that there exists a homotopy of maps of pairs $(S^n, *) \to (X, x_0)$ from f to the constant map. But this means in particular that f is homotopic to the constant map c_{x_0}. ∎

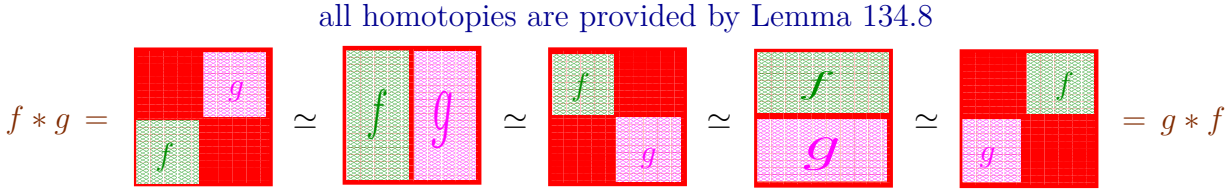
134.3. **Properties and basic calculations of higher homotopy groups.** Now we want to understand some of the basic properties of higher homotopy groups. The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler namely, with a great sight of relief, we will now see that higher homotopy groups are abelian.

Proposition 134.11. (π_n-Abelian Proposition) For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.

Remark. The higher homotopy groups were initially introduced by Eduard Čech in 1932 [**Čec1932**]. Shortly afterwards it was shown by Witold Hurewicz [**Hur1935a**, Satz II] that

they are abelian. Therefore it was initially assumed that higher homotopy groups are of little interest. This perception changed soon afterwards. □

Proof. Suppose we are given two continuous maps $f, g\colon (I^n, \partial I^n) \to (X, \{x_0\})$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in the figure below for $n = 2$.



It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by the Shrinking Lemma 134.8. We leave the details to the reader. (An alternative, more formal proof of the proposition will be given in Exercise 134.10.) ■

The following proposition can be viewed as a generalization of the key statements of the Change-of-Base Point Proposition 88.14.

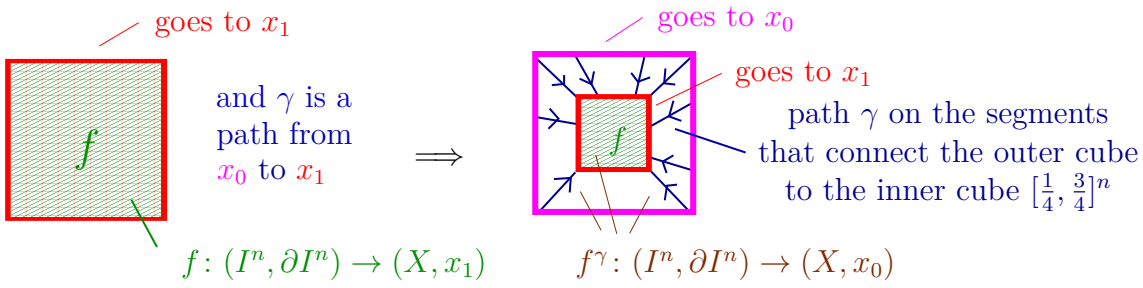
Proposition 134.12. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. Given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we consider the map[1389]

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, \{x_0\}) \\ (t_1, \dots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(2-4t), & \text{if } t := \max\{|t_1 - \frac{1}{2}|, \dots, |t_n - \frac{1}{2}|\} \in [\frac{1}{4}, \frac{1}{2}]. \end{cases} \end{array}$$

The map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is well-defined and it is a group isomorphism.

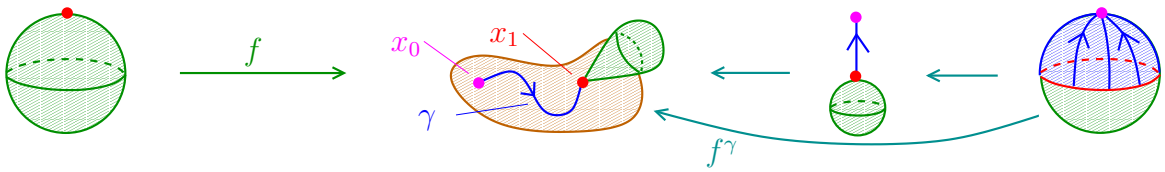


Remark.

(1) In the figure below we show the induced map $\gamma_*\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ that is given by $f \mapsto f^\gamma$, if in the definition of homotopy groups we work with $(S^n, *)$ instead of $(I^n, \partial I^n)$.

(2) In the Change-of-Base Point Proposition 296.12 we will prove a refined version of the above proposition. □

[1389] Using the Pasting Proposition 4.7 (2) one can easily verify that the map is continuous.



The Change-of-Base Point Proposition 134.12 implies that for a path-connected topological space the isomorphism type of the homotopy groups does not depend on the choice of the base points. This allows us to introduce the following definition:

Definition 134.13. Let X be a path-connected non-empty topological space. Given $n \in \mathbb{N}$ we denote by $\pi_n(X)$ the isomorphism type of the n-th homotopy groups.

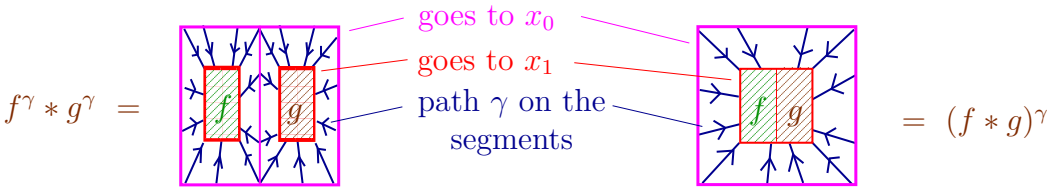
Proof of the Change-of-Base Point Proposition 134.12. Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. We need to prove the following claim:

Claim.

(1) The map $\Phi\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $[f] \mapsto [f^\gamma]$ is well-defined.
(2) The map Φ is a bijection.
(3) The map Φ is a group homomorphism.

Proof.

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t) = \gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$ given by $[f] \mapsto [f^{\overline{\gamma}}]$ is an inverse to Φ.
(3) Let $f, g\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be continuous maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f * g)^\gamma$.



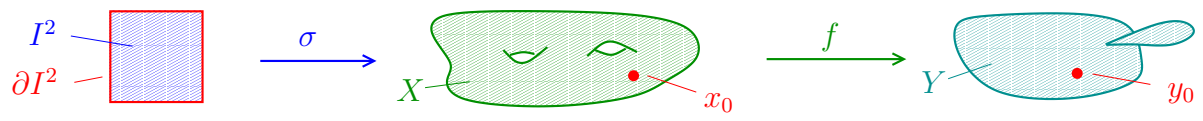
It should be clear that verifying this statement is elementary, in the sense that one can write down an explicit homotopy. On the other hand it should also be clear that writing down an explicit homotopy is supremely painful and highly non-instructive. Thus we will skip this step and leave it to the eager reader to fill in the details. ■

134.4. **Functoriality of homotopy groups.** Next we will see that higher homotopy groups are covariantly functorial.

Definition 134.14. Let $f\colon X \to Y$ be a continuous map between two topological spaces, let $n \in \mathbb{N}_0$ and let $x_0 \in X$. We write $y_0 = f(x_0)$. One can easily verify that the map

$$\begin{array}{rcl} f_*\colon \pi_n(X, x_0) & \to & \pi_n(Y, y_0) \\ [\sigma\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f \circ \sigma\colon (I^n, \partial I^n) \xrightarrow{\sigma} (X, \{x_0\}) \xrightarrow{f} (Y, \{y_0\})] \end{array}$$

is well-defined. As usual we say that f_* is the map induced by f.



Example. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We denote by Y the path-component of X that contains x_0 and we denote by $i\colon Y \to X$ the inclusion map. It follows from the fact that I^n is path-connected and the Image-Preimage-Component Lemma 4.50 that the map

$$i_*\colon \pi_n(Y, x_0) \to \pi_n(X, x_0)$$

is an isomorphism. □

The next proposition is a straightforward generalization of the π_1-Functor Proposition 90.3.

Proposition 134.15. (π_n-Functor Proposition) Let $n \geq 1$. As usual we denote by $\mathcal{PTop}$ the category of pointed topological spaces and that we denote by $\mathcal{Group}$ the category $\mathcal{Group}$ of groups. The map[1390]

$$\begin{aligned} \mathrm{Ob}(\mathcal{PTop}) &\to \mathrm{Ob}(\mathcal{Group}) \\ (X, x_0) &\mapsto \pi_n(X, x_0) \end{aligned}$$

and the maps

$$\begin{aligned} \mathrm{Mor}(\mathcal{PTop}) &\to \mathrm{Mor}(\mathcal{Group}) \\ (f\colon (X, x_0) \to (Y, y_0)) &\mapsto (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{aligned}$$

form a covariant functor.

Proof. First note that it follows immediately from the definitions that given a continuous map $f\colon (X, x_0) \to (Y, y_0)$ the induced map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is actually a group homomorphism. Furthermore note that it follows immediately from the definitions that

$$\begin{aligned} \mathrm{id}_* &= \mathrm{id} && \text{for all pointed pairs } (X, x_0) \\ (g \circ f)_* &= g_* \circ f_* && \text{for all continuous maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0). \end{aligned}$$

But this is exactly what we needed to show. ■

We continue with the following proposition, which generalizes the essentials of the Homotopy-π_1-Proposition 90.8.

Proposition 134.16. (Homotopy-π_n-Proposition) Let $n \geq 1$. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then

$$\begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

Proof. The proof is quite similar to the proof of the Homotopy-π_1-Proposition 90.8. The only problem is that the notation gets a little more messy. We leave it to the reader to fill in the details. ■

[1390] It follows from the π_n-Abelian Proposition 134.11 that for $n \geq 2$ this is also a covariant functor to the category $\mathcal{AbGroup}$ of abelian groups.

Proposition 134.17. (Homotopy Equivalence-π_n-Proposition) Let $n \geq 1$.

(1) (1) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then
$$f_*\colon \pi_n(X, x_0) \ \to \ \pi_n(Y, f(x_0))$$
is an isomorphism.

(2) Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map
$$i_*\colon \pi_n(A, a_0) \ \to \ \pi_n(X, a_0)$$
is an isomorphism.

(2) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof.

(1) As in the proof of the Homotopy Homotopy-π_1-Proposition 90.9 one can easily deduce this statement from the Homotopy-π_n-Proposition 134.16.

(2) If $X = \{x_0\}$ is a singleton, then there exists only a unique map $(I^n, \partial I^n) \to (X, \{x_0\})$, thus we see that in this case all homotopy groups are trivial. The case that X is contractible, i.e. the case that X is homotopy equivalent to a singleton, now follows from this calculation together with Statement (1). ■

Example. The topological space $\mathbb{R}^k$ is homotopy equivalent to a point $\{P\}$. Thus it follows from the Homotopy Equivalence-π_n-Proposition 134.17 and the discussion on page 2575 that $\pi_n(\mathbb{R}^k) \cong \pi_n(\{P\}) \cong 0$. Exactly the same way, using the discussion on page 531, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^k$ vanish, e.g. we have $\pi_n\big(\overline{B}^k\big) = \pi_n(B^k) = 0$. □

The next proposition is the high-dimensional analog of the Product-π_1-Proposition 90.13.

Proposition 134.18. (Product-π_n-Proposition) Let $n \in \mathbb{N}$.

(1) Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j\colon \prod_{i\in I} X_i \to X_j$ the natural projection. The induced homomorphism[1391]
$$\begin{array}{rcl} \Phi\colon \pi_n\Big(\prod_{i\in I} X_i, (x_i)_{i\in I}\Big) & \to & \prod_{i\in I} \pi_n(X_i, x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i\in I} \end{array}$$
is an isomorphism.

(2) If we are given a finite family $(X_1, x_1), \ldots, (X_k, x_k)$ of pointed topological spaces, then the inverse to the map from (1) is given by the map
$$\begin{array}{rcl} \Psi\colon \pi_n(X_1, x_1) \times \cdots \times \pi_n(X_k, x_k) & \to & \pi_n(X_1 \times \cdots \times X_k, (x_1, \ldots, x_k)) \\ (g_1, \ldots, g_k) & \mapsto & \iota_{1*}(g_1) \cdot \ldots \cdot \iota_{k*}(g_k) \end{array}$$
where $\iota_i\colon X_i \to X_1 \times \cdots \times X_k$ is given by $y \mapsto (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$.

Proof. The proof of the above proposition is basically identical to the proof of the Product-π_1-Proposition 90.13. For the reader's convenience we carry out the proof again.

[1391] The projections p_i induce homomorphisms on fundamental groups which in turn, by the Product-of-Sets-and-Groups Lemma 6.4, induce the given homomorphism.

(1) Given pairs of topological spaces (W,A) and (X,B) we denote by $C((W,A),(X,B))$ the set of all continuous maps $(W,A)\to(X,B)$. As in Definition 134.1 we denote by $[(W,A),(X,B)]$ the set of all homotopy classes of continuous maps $(W,A)\to(X,B)$. We set $x:=(x_i)_{i\in I}$. It follows from the Topological-Product Proposition 7.2 (1) that the map

$$\begin{array}{rcl} C\Big((S^n,*),\Big(\prod\limits_{i\in I}X_i,x\Big)\Big) & \to & \prod\limits_{i\in I}C((S^n,*),(X_i,x_i)) \\ f & \mapsto & (p_i\circ f)_{i\in I}\end{array}$$

is a bijection. The analogous statement holds if we replace the pair $(S^n,*)$ by the pair $(S^n\times[0,1],*\times[0,1])$. This shows that the map

$$\begin{array}{rcccl} \pi_n\Big(\prod\limits_{i\in I}X_i,x\Big) & = & \Big[(S^n,*),\Big(\prod\limits_{i\in I}X_i,x\Big)\Big] & \to & \prod\limits_{i\in I}[(S^n,*),(X_i,x_i)] \;=\; \prod\limits_{i\in I}\pi_n(X_i,x_i) \\ & & [f] & \mapsto & \underbrace{[(p_i\circ f)_{i\in I}]}_{=p_{i*}(f)}\end{array}$$

is a bijection. But this is precisely the homomorphism that we are considering.

(2) To simplify the notation we just work with two pointed topological spaces (X,x_0) and (Y,y_0) and we denote the two inclusions by $i\colon X\to X\times Y$ and $j\colon Y\to X\times Y$. Since by (1) we already know that Φ is an isomorphism it remains to prove the following claim:

Claim. We have $\Phi\circ\Psi=\mathrm{id}$ as self-maps of $\pi_n(X,x_0)\times\pi_n(Y,y_0)$.

Proof. Given $(\alpha,\beta)\in\pi_n(X,x_0)\times\pi_n(Y,y_0)$ we obtain the following equalities in the group $\pi_n(X,x_0)\times\pi_n(Y,y_0)$:

$$\begin{array}{rcccccl} & & \overset{\text{definition of }\Psi}{\downarrow} & & \overset{\text{definition of }\Phi}{\downarrow} & & \overset{\text{since } p_* \text{ and } q_* \text{ are homomorphisms}}{\downarrow} \\ (\Phi\circ\Psi)(\alpha,\beta) & = & \Phi(i_*(\alpha)\cdot j_*(\beta)) & = & (p_*(i_*(\alpha)\cdot j_*(\beta)),q_*(i_*(\alpha)\cdot j_*(\beta))) & = & \\ & = & (\underbrace{p_*(i_*(\alpha))}_{=(p\circ i)_*=\mathrm{id}_*}\cdot\underbrace{p_*(j_*(\beta))}_{=(p\circ j)_*=0},\underbrace{q_*(i_*(\alpha))}_{=(q\circ i)_*=0}\cdot\underbrace{q_*(j_*(\beta))}_{=(q\circ j)_*=\mathrm{id}_*}) & & & & = & (\alpha,\beta).\end{array}$$

■

In the Sphere-π_1-Proposition 88.24 we saw that $\pi_1(S^n)=0$ for $n\geq 2$. The following proposition generalizes this result to suitable higher dimensions.

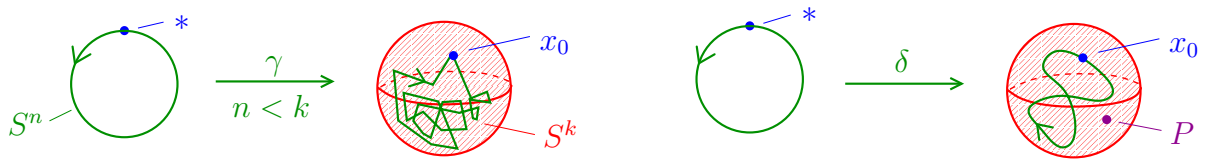
Proposition 134.19. (Sphere–Low-π_n-Proposition) Let $k\in\mathbb{N}$. For any $n<k$ we have

$$\pi_n(S^k) \;=\; 0.$$

Proof. Let $k\geq 2$, let $x_0\in S^k$ and let $\gamma\colon(S^n,*)\to(S^k,x_0)$ be a continuous map. It follows immediately from the Whitney Approximation Theorem 36.1 (or the more elementary Exercise 27.4), applied to the closed subset $\{*\}\subseteq S^n$, that there exists a homotopy rel $\{*\}$ from $\gamma\colon S^n\to S^k$ to a smooth map $\delta\colon S^n\to S^k$ with $\delta(*)=x_0$. It follows from the Measure-Zero Properties Proposition 37.3 (4) and the Full Measure-Density Proposition 37.4 (1) and the hypothesis that $k>n$ that the smooth map $\delta\colon S^n\to S^k$ is not surjective. We pick $P\in S^n\setminus\delta(S^n)$.

In the proof of the Sphere-Minus-Point Lemma 88.25 we saw that $S^k\setminus\{P\}$ is homeomorphic to $\mathbb{R}^k$, in particular contractible, which implies that $[\delta]=0\in\pi_n(S^k\setminus\{P\},*)$, which implies that $[\delta]=0\in\pi_n(S^k,*)$. [1392] ■

[1392] On page 2723 we will give an alternative proof of the proposition using the theory of CW-complexes.



The following lemma is the obvious extension of the Exhaustion–Simply Connected Proposition 99.16.

Proposition 134.20. (Exhaustion-π_n-Zero-Proposition) Let $n \in \mathbb{N}$, let X be a path-connected topological space and let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$

be a sequence of subspaces of X such that the following two conditions are satisfied:
(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subseteq \mathring{X}_{i+1}$.
(3) For each $i \in \mathbb{N}_0$ the following holds: X_i is path-connected and we have $\pi_n(X_i) = 0$.
Then for each $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof. Let $x_0 \in X$ and let $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be a continuous map. By the Compact Image Lemma 4.15 we know that $f([0,1]^n)$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 99.17 we know that $f([0,1]^n)$ is contained in a single X_i. But then it follows from Hypothesis (1) that $[f] = 0 \in \pi_n(X_i, x_0)$, which implies that we have $[f] = 0 \in \pi_n(X, x_0)$. ∎

For the reader who is familiar with the concept of direct colimits, which we introduced in Chapter 104, we also have the following proposition, which is a generalization of the above Exhaustion-π_n-Zero-Proposition 134.20.

Proposition 134.21. (Exhaustion-π_n-Proposition) Let $n \in \mathbb{N}$, let X be a topological space and let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$

be a sequence of subspaces of X. We assume that the following conditions are satisfied
(1) We have $\bigcup_{i \in I} X_i = X$.
(2) For each $i \in \mathbb{N}$ we have $X_i \subseteq \mathring{X}_{i+1}$.
Then for any $w \in X_0$ we have a natural isomorphism

$$\operatorname*{colim}_{\longrightarrow} \pi_n(X_i, w) \xrightarrow{\cong} \pi_n(X, w).$$

Proof. The proof is almost verbatim the same as the proof of the Exhaustion-π_1-Proposition 105.2. We will not bore the reader with a repetition of the argument. ∎

134.5. **Covering spaces and higher homotopy groups.** Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. In the Covering-π_1-Monomorphism Proposition 92.7 we saw that the induced map $p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism. We saw in many examples that the map is in general not an isomorphism. Perhaps somewhat surprisingly, for the homotopy groups π_n with $n \geq 2$ the following proposition, which was

first proved by Witold Hurewicz [**Hur1935a**, Satz II], shows that the induced map is in fact always an isomorphism:

Proposition 134.22. (Covering-π_n-Isomorphism Proposition) Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. Given any $n\geq 2$ the following two statements hold:

(1) The map $p_*\colon \pi_n(X,x_0)\to\pi_n(B,b_0)$ is an isomorphism.
(2) The map[1393] [1394]
$$\begin{array}{rcl}\pi_n(B,b_0) & \to & \pi_n(X,x_0)\\ [f\colon (S^n,*)\to(B,b_0)] & \mapsto & [\text{unique lift of } f \text{ to a map } (S^n,*)\to(X,x_0)]\end{array}$$
is well-defined and it is the inverse to p_*.

Before we provide the proof of the Covering-π_n-Isomorphism Proposition 134.22 let us first give a few examples. We will consider a few more examples after the proof of the Covering-π_n-Isomorphism Proposition 134.22.

Examples.

(1) For any $k\in\mathbb{N}_0$ and any $n\geq 2$ we have
$$\pi_n((S^1)^k) \quad\cong\quad \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \quad\underset{\text{by the Covering } \pi_n\text{-Isomorphism Proposition 134.22}}{\overset{\cong}{\leftarrow}}\quad \pi_n(\mathbb{R}^k) \quad\underset{\text{see page 2584}}{=}\quad 0.$$
In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1\times S^1$ vanish.

(2) Let Σ be a surface of genus ≥ 2. For any $n\geq 2$ we see that
$$\pi_n(\Sigma) \quad\underset{\text{by the Covering } \pi_n\text{-Isomorphism Proposition 134.22}}{\cong}\quad \pi_n(\text{universal covering of } \Sigma) \quad\underset{\text{by Hadamard's Theorem 132.14}}{\cong}\quad \pi_n(B^2) \quad\underset{\text{see page 2584}}{=}\quad 0.$$
Together with the example on page 2587 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish.

(3) One of our goals is to complete the classification of lens spaces, in particular we would like to answer Question 91.6. So let $L(p,q)$ and $L(p,r)$ be two lens spaces. For any $n\geq 2$ we have
$$\pi_n(L(p,q)) \quad\underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\overset{\cong}{\leftarrow}}\quad \pi_n(S^3) \quad\underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\overset{\cong}{\rightarrow}}\quad \pi_n(L(p,r)).$$
We do not know the higher homotopy groups of S^3, but this argument shows that higher homotopy groups are useless for distinguishing lens spaces. □

Proof of the Covering-π_n-Isomorphism Proposition 134.22. Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces and let $n\geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X,x_0)\to\pi_n(B,b_0)$.

[1393] Here we take the point of view from Identification 134.9 where we saw that we view the elements of $\pi_n(B,b_0)$ as homotopy classes of continuous maps $(S^n,*)\to(B,b_0)$.
[1394] We will discuss this map $\pi_n(B,b_0)\to\pi_n(X,x_0)$ in greater detail in the proof.

Recall that we saw in Identification 134.9 that we can think of elements in $\pi_n(B, b_0)$ as homotopy classes of continuous maps $(S^n, *) \to (B, b_0)$ of pointed topological spaces. Thus let $f\colon (S^n, *) \to (B, b_0)$ be a continuous map. We are in the following setting[1395] [1396]

$$\begin{array}{ccc} & & (X, x_0) \\ & & \downarrow p \\ (S^n, *) & \xrightarrow[f]{} & (B, b_0). \end{array} \qquad \begin{array}{c} \text{by the Map Lifting Criterion 127.5} \\ \text{since by } n \geq 2 \text{ we have } \pi_1(S^n) = 0 \end{array} \qquad \begin{array}{ccc} & \exists!\, \widetilde{f} \nearrow & (X, x_0) \\ & & \downarrow p \\ (S^n, *) & \xrightarrow[f]{} & (B, b_0). \end{array}$$

Claim. The map

$$\begin{array}{rccc} \lambda\colon & \pi_n(B, b_0) & \to & \pi_n(X, x_0) \\ & [f] & \mapsto & [\widetilde{f}] \end{array}$$

is well-defined.

Proof. Let $f, g\colon (S^n, *) \to (B, b_0)$ be two continuous maps which represent the same element in $\pi_n(B, b_0)$. We denote by $\widetilde{f}, \widetilde{g}\colon (S^n, *) \to (X, x_0)$ the corresponding lifts. Since f, g represent the same element in $\pi_n(B, b_0)$ there exists a homotopy $F\colon S^n \times [0,1] \to B$ between these two maps $f, g\colon (S^n, *) \to (B, b_0)$. It follows from the Map Lifting Criterion 127.5, applied to the pointed topological space $(Z, z_0) = (S^n \times [0,1], (*, 0))$, that we can lift F to a continuous map $\widetilde{F}\colon S^n \times [0,1] \to X$ such that $\widetilde{F}(*, 0) = x_0$. We still need to show that $\widetilde{F}(*, t) = x_0$ for all $t \in [0,1]$. We know that for every $t \in [0,1]$ we have $\widetilde{F}(*, t) \in p^{-1}(b_0)$. By the Covering–Basics Lemma 91.12 (2) the preimage $p^{-1}(b_0)$ is discrete. Since $[0,1]$ is connected it follows that the map $p\colon [0,1] \to p^{-1}(b_0)$ is constant by the Connected-to-Discrete Lemma 4.37. But since it takes the value x_0 for $t = 0$ it takes the value x_0 for all $t \in [0,1]$. ⊞

We have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B, b_0)$ we have

$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; [p \circ \widetilde{f}] \;\underset{\substack{\uparrow \\ \text{by Definition 92.1} \\ \text{of a lift}}}{=}\; [f].$$

Now let $[g] \in \pi_n(X, x_0)$. We then have

$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \;\underset{\substack{\uparrow \\ \text{since } g \text{ is the unique} \\ \text{lift of } p \circ g \text{ to } x_0.}}{=}\; [g].$$

■

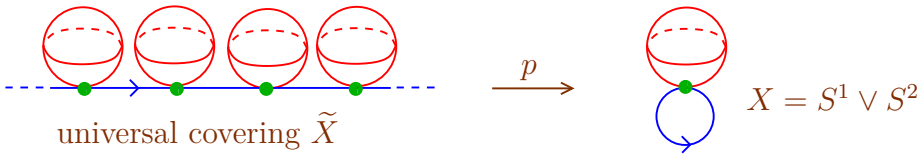
Now we give a few more applications of the Covering-π_n-Isomorphism Proposition 134.22.

Example.

(1) We consider $X = S^1 \vee S^2$. On page 2466 we determined its universal covering $\widetilde{X}$, which is given by the wedge of $\mathbb{R}$ with infinitely many spheres. It follows from the Covering-π_n-Isomorphism Proposition 134.22 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big. Much later, in the $S^1 \vee S^2$–π_2–Proposition 169.9, we will give a complete calculation of $\pi_2(S^1 \vee S^2)$.

[1395] In the Map Lifting Criterion 127.5 we need that the sphere S^n is path-connected and locally path-connected, but there are no hypotheses on B or X.

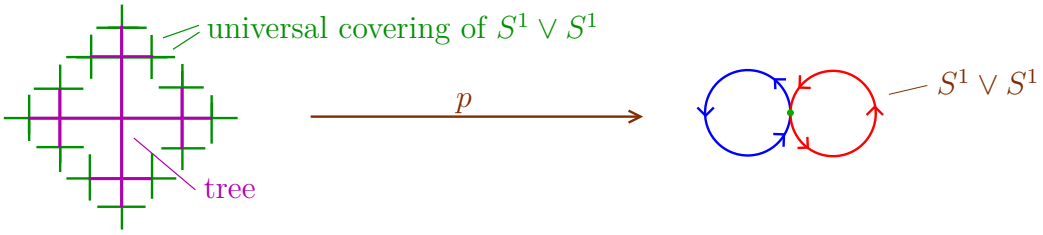
[1396] We also use throughout the proof that the Lift Uniqueness Proposition 92.2 guarantees that the lifts, if they exist, are unique.



(2) We have

$$\pi_n(S^1 \vee S^1) \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_n(\text{universal covering of } S^1 \vee S^1) \underset{\text{by the Wedge-of-Circles–Universal Covering Proposition 127.24}}{\cong} \pi_n\Big(\bigcup_{k\in\mathbb{N}} \text{trees}\Big) \underset{\text{by the Exhaustion-}\pi_n\text{-Zero Proposition 134.20, since trees are contractible}}{=} 0.$$

We will generalize this argument in the 1-Complex-π_n-Lemma 138.2.



(3) Let Σ be a surface of genus ≥ 1 and let $f\colon \Sigma \to \Sigma$ be a self-homeomorphism. We denote by $\mathrm{Tor}(\Sigma, f) = (\Sigma \times [0,1])/(x,0) \sim (f(x),1)$ the mapping torus that we introduced in Definition 21.19. For any $n \geq 2$ we have the following isomorphisms:

$$\pi_n(\mathrm{Tor}(\Sigma, f)) \underset{\text{by the Mapping Torus Covering Lemma 93.10 and the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_n(\Sigma \times \mathbb{R}) \underset{\text{by the Homotopy Equivalence }\pi_n\text{-Proposition 134.16 (1b)}}{\cong} \pi_n(\Sigma) \underset{\text{see the example on page 2587}}{=} 0.$$

□

134.6. **Are there any higher homotopy groups that are non-trivial?** We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.

So are there are any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 134.23. Let $n \geq 2$.
(1) Does the identity map $\mathrm{id}\colon S^n \to S^n$ define a non-trivial element in the homotopy group $\pi_n(S^n, *)$?
(2) Do we have an isomorphism $\pi_n(S^n, *) \cong \mathbb{Z}$?

Throughout these notes we will give many (partial) answers to the above question:

- For readers who are familiar with differential forms and de Rham cohomology we will answer Question 134.23 (1) in the De Rham-π_n-Proposition 134.29.
- For readers familiar with the degree of a smooth map between smooth manifolds, that we introduced on page 1069, we will provide an answer to Question 134.23 (2) in the $\pi_n(S^n)$–Proposition 134.31.

- In the Hurewicz Sphere Epimorphism Lemma 167.5 we will use homology groups to answer Question 134.23 (1).
- In the $\pi_n(S^n)$–Proposition 167.10 we will also use homology groups to answer Question 134.23 (2)
- In the $\pi_n(S^n)$-via-Thom-Pontryagin Corollary 300.20 we will use methods from differential topology to answer Question 134.23 (2).

We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k > n$? On page 2587 we saw that all these groups are trivial for $n = 1$. But what happens for $n > 1$? Can the higher homotopy groups of spheres be non-trivial?

To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that on page 340 we gave an identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$. which identifies $[0:1]$ with the North Pole $N = (0,0,1)$.

Definition 134.24. We refer to

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

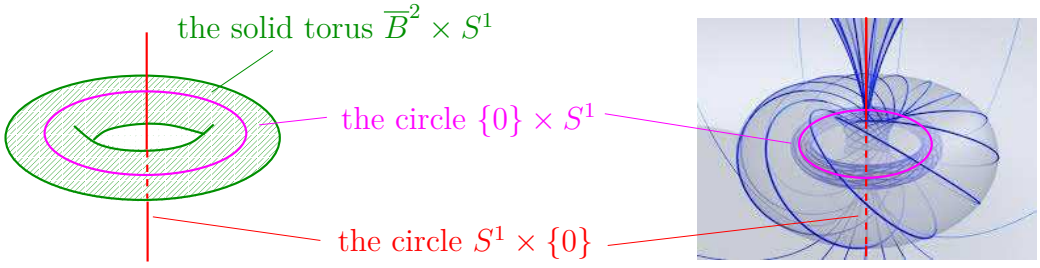
as the Hopf map[1397]. The homotopy class of H defines an element in $\pi_3(S^2, N)$.

Question 134.25. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2, N)$?

We will answer Question 134.25 in the Hopf Map–Homotopy Theorem 216.6 using the cup product on singular cohomology and will also provide an answer in the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12 using fiber bundles. Finally in the Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25 we will use methods from differential topology to provide an answer.

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. So how are supposed to think about the Hopf map? A straightforward calculation shows that for each $P \in S^2$ the preimage $H^{-1}(P)$ is a circle. Furthermore, if we use the explicit homeomorphism from the Stereographic Projection Lemma 4.24 to make the identification of S^3 with $\mathbb{R}^3 \cup \{\infty\}$ then we can make the following observations:

(1) The circle $S^1 \times \{0\}$ corresponds to the points $(z,0) \in S^3$, in particular the circle gets sent to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.
(2) The circle that is given by the z-axis together with the point ∞ corresponds to the points $(0,z) \in S^3$, in particular the circle gets sent to the point $[0:1] \in \mathbb{C}\mathrm{P}^1$.



[1397]Heinz Hopf (1894-1971) was a German mathematician.

Next recall that the Sphere–Solid Tori-Decomposition Lemma 111.6 we gave an explicit homeomorphism[1398] $(S^1 \times \overline{B}^2) \cup_{S^1\times S^1} (\overline{B}^2 \times S^1) \xrightarrow{\cong} S^3$.

It follows from the definition of this identification that on the torus $S^1 \times S^1$ the "diagonal circle" $\{(z,z) \,|\, z \in S^1\}$ is the preimage of the point $[1:1] \in \mathbb{C}\mathrm{P}^1$. In the figure above we show the preimages of H for some points in $\mathbb{C}\mathrm{P}^2 = S^2$. □

We introduce one more construction of continuous maps that represent potentially non-trivial elements in homotopy groups of spheres.

Definition 134.26.

(1) Let X be a non-empty topological space. Recall that in Definition 20.21 we defined the suspension of X to be the topological space

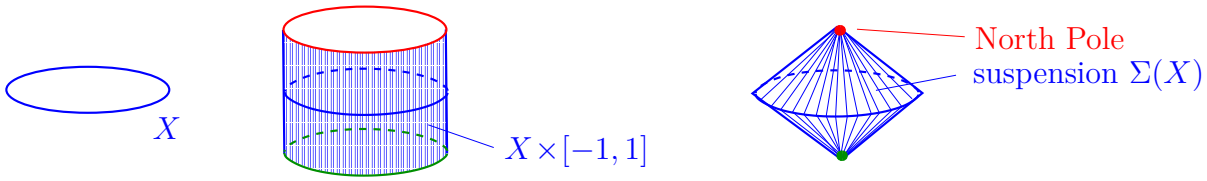
$$\Sigma(X) \;:=\; (X \times [-1,1])/(X \times \{-1\})/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the North Pole of $\Sigma(X)$.

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the suspension of f.



Remark. Let $n \in \mathbb{N}_0$. Recall that on page 556 we wrote down an explicit homeomorphism $\Sigma(S^n) \to S^{n+1}$ that sends the North Pole of $\Sigma(S^n)$ to the North Pole $(0,\dots,0,1)$ of S^{n+1}. We will use this homeomorphism to identify $\Sigma(S^n)$ with S^{n+1}. □

Now we have the following elementary lemma:

Lemma 134.27. (Suspension Homomorphism Lemma) Let $k \in \mathbb{N}_0$.

(1) Let (X, x_0) be a pointed topological space. We denote by $*$ the North Pole of $\Sigma(X)$. The map

$$\begin{array}{rcl} \pi_k(X,x_0) & \to & \pi_{k+1}(\Sigma(X),*) \\ [f\colon S^k \to X] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(X)\big] \end{array}$$

is a well-defined group homomorphism. We refer to it as the suspension homomorphism.

(2) Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map

$$\begin{array}{rcl} \pi_k(S^n,*) & \to & \pi_{k+1}(S^{n+1},*) \\ [f\colon S^k \to S^n] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) =_{\mathrm{i}} S^{k+1} \to \Sigma(S^n) =_{\mathrm{i}} S^{n+1}\big] \end{array}$$

[1398]Note that in the Torus Embeddings–Isotopic Lemma 111.8 we saw that the corresponding smooth embedding $S^1 \times \overline{B}^2 \to S^3$ is essentially given by the standard smooth embedding.

is a well-defined group homomorphism. We refer to it also as the suspension homomorphism.

Proof.

(1) It follows from the Suspension Functor Lemma 20.23 and the Suspension–Homotopy Lemma 20.28 (1) that this map is well-defined. We leave the task of verifying that the map is in fact a group homomorphism as a somewhat challenging exercise to the reader. Alternatively we refer to [**Hilt1953**, Theorem 2.1] for a proof.

(2) This statement follows immediately from (1). ■

We conclude this chapter with the following question:

Question 134.28. Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?

We answer will answer Question 134.28 in the Steenrod Operations-π_n-Theorem 236.19 using "Steenrod operations". Alternatively we will answer Question 134.28 in the $\pi_{n+1}(S^n)$–Theorem 302.1 using methods from differential topology.

Unfortunately at the moment we do not have many tools to address the above questions. (In the following appendices we will use results from earlier chapters on differential forms and degrees to get a few partial answers.) In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analog to the Seifert–van Kampen-Theorem 99.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in much later chapters, since the calculations will require the use of singular homology groups. But even then we will only be able to determine some homotopy groups of spheres. The results we obtain are summarized in Chapter 303. But note that the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the part of the lecture notes.

134.7. **Appendix I: $\pi_n(S^n)$ and differential forms.** For readers who are familiar with differential forms and de Rham cohomology, which we introduced and studied in Part IX, we can state and prove two propositions in this appendix.

Proposition 134.29. (De Rham-π_n-Proposition) Given any dimension $n \in \mathbb{N}$ we have $[\mathrm{id}_{S^n}] \neq 0 \in \pi_n(S^n, *)$.

Proof. If id_{S^n} did represent the trivial element of $\pi_n(S^n, *)$, then it would follow from the Replace-by-Smooth Homotopy Proposition 36.6 that id_{S^n} is smoothly homotopic to a constant map $\mathrm{const}\colon S^n \to S^n$. It follows that

$$0 = \mathrm{const}^* \underset{\uparrow}{=} (\mathrm{id}_{S^n})^* = \mathrm{id}_{\mathrm{H}^n_{\mathrm{dR}}(S^n)} \underset{\uparrow}{=} \mathrm{id}_{\text{non-trivial vector space}}$$

by the above discussion together with the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2

by the Sphere–$\mathrm{H}^*_{\mathrm{dR}}$ Proposition 80.4

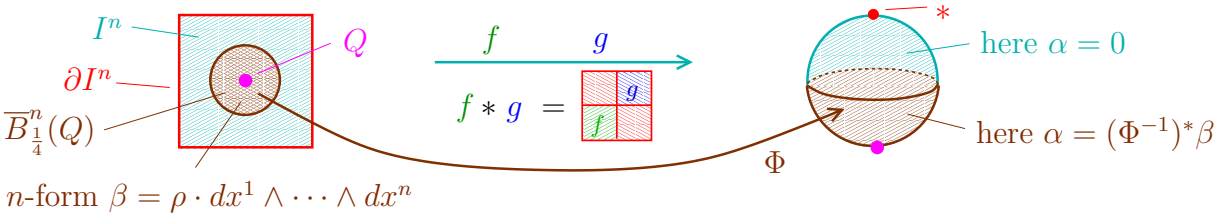
We have thus obtained a contradiction. ■

We also have the following more substantial result.

Proposition 134.30. (Integration-π_n-Epimorphism Lemma) Given any $n \in \mathbb{N}$ there exists a non-trivial homomorphism $\pi_n(S^n, *) \to \mathbb{R}$.

Sketch of proof. We start out with a few preparations:

(1) Let $Q := (\frac{1}{2}, \ldots, \frac{1}{2}) \in I^n$ be the "center" of I^n.
(2) Let $\Phi\colon \overline{B}^n_{\frac{1}{4}}(Q) \to S^n_{\leq 0}$ be a diffeomorphism.
(3) We pick a smooth map $\rho\colon \overline{B}^n_{\frac{1}{4}}(Q) \to \mathbb{R}_{\geq 0}$ with $\rho(Q) > 0$ and which vanishes outside of $\overline{B}^n_{\frac{1}{8}}(Q)$. We consider the smooth n-form $\beta := \rho \cdot dx^1 \wedge \cdots \wedge dx^n$ on $\overline{B}^n_{\frac{1}{4}}(Q)$. We obtain the smooth n-form $(\Phi^{-1})^*(\beta)$ on $\Phi(\overline{B}^n_{\frac{1}{4}}(Q)) = S^n_{\geq 0}$. This n-form vanishes on a neighborhood of $S^n_{=0}$. Thus we can extend it by the zero n-form to a smooth n-form α on all of S^n.



We will also use the following two observations and facts:

(a) In the following we can and will view I^n as a subspace of the closed smooth manifold[1399] $S^n =_{\text{i}} \mathbb{R}^n \cup \{\infty\}$. Any continuous map on I^n which is constant on ∂I^n can be extended continuously to a map on $\mathbb{R}^n \cup \{\infty\}$. This trick allows us to apply various results on maps on closed smooth manifolds to the setting of maps on I^n. We leave it to the reader to work out the details.
(b) It follows easily from the Shrinking Lemma 134.8 together with the Whitney Approximation Theorem 36.1 that any continuous map $f\colon (I^n, \partial I^n) \to (S^n, *)$ is homotopic rel ∂I^n to a map $\widetilde{f}\colon (I^n, \partial I^n) \to (S^n, *)$ that is constant map in a neighborhood of ∂I^n and that is smooth in the sense of Definition 24.3.

Claim 1. Let $f\colon (I^n, \partial I^n) \to (S^n, *)$ be a continuous map. By (b) we know that f is homotopic rel ∂I^n to a smooth map $\widetilde{f}\colon (I^n, \partial I^n) \to (S^n, *)$. The integral[1400]

$$\int_{I^n} \widetilde{f}^*\alpha \ \in \ \mathbb{R}$$

does not depend on the choice of $\widetilde{f}$.

Proof. Suppose that $\widehat{f}\colon (I^n, \partial I^n) \to (S^n, *)$ is also smooth and homotopic to f. Note that $\widetilde{f}, \widehat{f}$ are thus homotopic rel ∂I^n. It follows from the Replace-by-Smooth Homotopy Proposition 36.6 (together with (a)) that there exists in fact a smooth homotopy $F\colon I^n \times [0,1] \to I^n$

[1399] We equip $\mathbb{R}^n \cup \{\infty\}$ with the smooth structure defined on page 660.

[1400] Since $\widetilde{f}$ is smooth we obtain, using the Regular Closed–∂_*-Lemma 24.5, the pullback form $\widetilde{f}^*\alpha$ on I^n. As in the Integral-of-Form-in-Chart Proposition 74.12 the integral of an n-form ω on I^n is defined as the integral of the real valued function $x \mapsto \omega_x(e_1, \ldots, e_n)$.

rel ∂I^n between $\widetilde{f}$ and $\widehat{f}$. Using basically the same calculation as in the Hyperrectangle–Stokes Lemma 75.15 (see also Exercise 75.15) one sees that the integrals of the smooth n-forms $\widetilde{f}^*\alpha$ and $\widehat{f}^*\alpha$ over I^n agree. ⊞

Claim 2. The map

$$\begin{array}{rcl} \omega\colon \pi_n(S^n,*) & \to & \mathbb{R} \\ [f\colon (I^n,\partial I^n)\to(S^n,*)] & \mapsto & \omega([f]) \coloneqq \int\limits_{I^n} \underset{\text{where } \widetilde{f} \text{ is chosen as in (b)}}{\uparrow \widetilde{f}^*\alpha} \end{array}$$

is a homomorphism.

Proof. Let $f\colon (I^n,\partial I^n)\to(S^n,*)$ and $g\colon (I^n,\partial I^n)\to(S^n,*)$ be two continuous maps. We apply (b) to get corresponding smooth maps $\widetilde{f}$ and $\widetilde{g}$. Since $\widetilde{f}$ and $\widetilde{g}$ are constant in a neighborhood of ∂I^n we obtain easily from the Smooth Pasting Proposition 25.19 that the "product" $\widetilde{f}*\widetilde{g}\colon (I^n,\partial I^n)\to(S^n,*)$ is also smooth. We now see that

$$\omega([f])+\omega([g]) \;=\; \int\limits_{I^n}\widetilde{f}^*\alpha+\int\limits_{I^n}\widetilde{g}^*\alpha \;\underset{\text{follows easily from the explicit definitions of the integrals}}{\uparrow =}\; \int\limits_{I^n}(\widetilde{f}^**\widetilde{g}^*)\alpha \;=\; [f*g] \;=\; [f]\cdot[g].$$

⊞

Claim 3. The map[1401]

$$\begin{array}{rcl} \omega\colon \pi_n(S^n,*) & \to & \mathbb{R} \\ [f\colon (I^n,\partial I^n)\to(S^n,*)] & \mapsto & \int\limits_{I^n}\widetilde{f}^*\alpha \end{array}$$

is non-trivial.

Proof. Using the fact that $S^n = S^n_{\leq 0}\cup S^n_{\geq 0}$ is diffeomorphic to $\overline{B}^n_-\cup_{S^{n-1}}\overline{B}^n_+$ it is not hard to write down an explicit smooth map $h\colon (I^n,\partial I^n)\to(S^n,*)$ which agrees with Φ on $\overline{B}^n_{\frac{1}{4}}(Q)$ and which satisfies $h^{-1}(S^n_{\leq 0}) = \overline{B}^n_{\frac{1}{4}}(Q)$. It follows easily from the definitions that

$$\begin{aligned} \omega(h) \;=\; \int\limits_{I^n} h^*\alpha \;&=\; \int\limits_{\overline{B}^n_{\frac{1}{4}}(Q)} h^*\alpha \;=\; \int\limits_{\overline{B}^n_{\frac{1}{4}}(Q)} \Phi^*((\Phi^{-1})^*(\beta)) \;=\; \int\limits_{\overline{B}^n_{\frac{1}{4}}(Q)} \beta \\ &=\; \int\limits_{\overline{B}^n_{\frac{1}{4}}(Q)} \rho\cdot dx^1\wedge\cdots\wedge dx^n \;=\; \int\limits_{x\in\overline{B}^n_{\frac{1}{4}}(Q)} \rho(x)\,dx \;>\; 0. \end{aligned}$$

■

134.8. **Appendix II: $\pi_n(S^n)$ with degrees.** In Definition 46.5 we introduced the degree $\deg(f)$ of a continuous map $f\colon M\to N$ between two closed oriented connected non-empty n-dimensional smooth manifolds. For the reader who is familiar with the Hopf Degree Theorem 46.22 we can now easily give the first non-trivial calculation of a higher homotopy group:

Proposition 134.31. ($\pi_n(S^n)$–Proposition) For every $n\in\mathbb{N}$ the map

$$\begin{array}{rcl} \pi_n(S^n,*) & \to & \mathbb{Z} \\ [f] & \mapsto & \deg(f) \end{array}$$

is an isomorphism which sends $[\mathrm{id}_{S^n}]$ to 1.

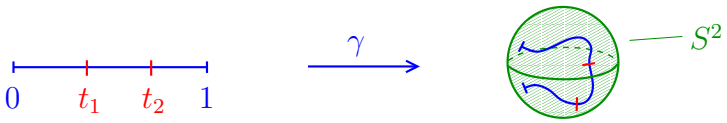
In the $\pi_n(S^n)$–Proposition 134.31 we will need the following lemma which might be of independent interest.

[1401] It is possible, but technically somewhat painful, to show that $[\mathrm{id}_{S^n}]$, viewed as an element of $\pi_n(S^n,*)$ gets sent to a non-zero real number.

Lemma 134.32. (Matrix Path–Lemma)[1402] Let $\gamma\colon [0,1] \to S^n$ be a continuous map.
(1) There exists a continuous map $A\colon [0,1] \to \mathrm{SO}(n+1)$ such that $A(t) \cdot e_1 = \gamma(t)$ for every $t \in [0,1]$.
(2) If $\gamma(0) = \gamma(1)$, then we can arrange that $A(0) = A(1)$.

Proof of the Matrix Path–Lemma 134.32. [1403] Let $\gamma\colon [0,1] \to S^n$ be a continuous map.

(1) We need to show that there are continuous maps $\alpha_1, \dots, \alpha_n\colon [0,1] \to \mathbb{R}^{n+1}$ such that for every $t \in [0,1]$ we have $(\gamma(t)\ \alpha_1(t)\ \dots\ \alpha_n(t)) \in \mathrm{SO}(n+1)$.
   (a) Suppose that on an interval $[a,b]$ the i-th coordinate of $\gamma(t)$ is also ways non-zero. This means that for each $t \in [a,b]$ the vectors $\gamma(t), e_1(t), \dots, \widehat{e_i(t)}, \dots, e_{n+1}(t)$ are linearly independent. We apply the Gram-Schmidt Orthonormalization Process 2.32 to this set of $n+1$ vectors and obtain a continuous map $B\colon [a,b] \to \mathrm{SO}(n+1)$ with the desired properties.
   (b) Using the Lebesgue Lemma 10.8 we can find a subdivision $0 = t_0 < t_1 < \dots < t_m = 1$ such that on each subinterval $[t_j, t_{j+1}]$ the situation is as described in (a).



   (c) From (a) we get get for $j = 0, \dots, m-1$ a continuous map $B_j\colon [t_j, t_{j+1}] \to \mathrm{SO}(n+1)$ such that $A(t) \cdot e_1 = \gamma(t)$ for all $t \in [0,1]$. Finally we consider

$$\begin{array}{rcl} A\colon [0,1] & \to & \mathrm{SO}(n+1) \\ t & \mapsto & \begin{cases} B_0(t), & \text{if } t \in [0,t_1], \\ B_1(t) \cdot B_1(t_1)^{-1} \cdot B_0(t_1), & \text{if } t \in [t_1,t_2], \\ \vdots \end{cases} \end{array}$$

   It follows from the Pasting Proposition 4.7 that this map is continuous. One can now easily verify that this map has all the right properties.

(2) Now we suppose that $\gamma(0) = \gamma(1)$. This implies that $A(1)^{-1} \cdot A(0) \cdot e_1 = e_1$ which implies, by some elementary linear algebra, that $A(1)^{-1} \cdot A(0) = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$ for some $B \in \mathrm{SO}(n)$. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n)$ is path-connected. This implies that there exists a continuous map $\beta\colon [0,1] \to \mathrm{SO}(n)$ with $\beta(0) = \mathrm{id}$ and $\beta(1) = B$. One can now easily verify that the map

$$\begin{array}{rcl} [0,1] & \to & \mathrm{SO}(n+1) \\ t & \mapsto & A(t) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \beta(t) \end{pmatrix} \end{array}$$

   has all the promised qualities. ∎

Proof of the $\pi_n(S^n)$–Proposition 134.31. Let $n \in \mathbb{N}$. We need to prove the following claim:

[1402] This lemma is precisely the content of Exercise 41.14.
[1403] A more conceptual proof can be given using the Matrix Group-Fiber Proposition 299.8.

Claim. The map
$$\begin{array}{rcl} \pi_n(S^n,*) & \to & \mathbb{Z} \\ [f] & \mapsto & \deg(f) \end{array}$$
has the following properties:

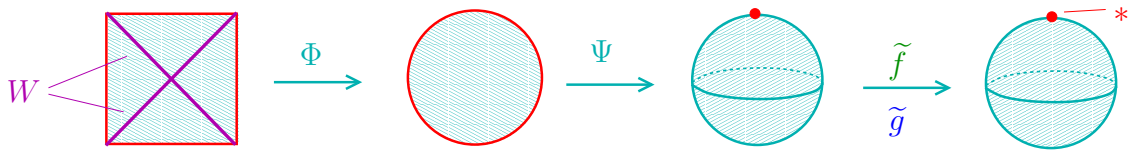
(1) it is well-defined,
(2) it sends $[\mathrm{id}_{S^n}]$ to 1,
(3) it is surjective,
(4) it is injective,
(5) it is a homomorphism.

Proof.

(1) This statement follows immediately from the Degree Homotopy Invariance Theorem 46.6.
(2) This statement follows immediately from the definition of the degree.
(3) Let $k \in \mathbb{Z}$. The Degree Realization Proposition 46.19 says that there exists a continuous map $f\colon S^n \to S^n$ of degree k. By the Orthogonal Action Lemma 2.37 there exists an $A \in \mathrm{SO}(n+1)$ with $A \cdot f(*) = *$. It follows from the Basics-of-Degree Lemma 46.12 together with the Degree Multiplicativity Theorem 46.7 that the composition of f with multiplication by A is a map $(S^n,*) \to (S^n,*)$ of degree k.
(4) Let $f,g\colon (S^n,*) \to (S^n,*)$ be two continuous maps with $\deg(f) = \deg(g)$. It follows from the Hopf Degree Theorem 46.22 that there exists a homotopy $F\colon S^n \times [0,1] \to S^n$ with $F_0 = f$ and $F_1 = g$. We set $\gamma(t) \coloneqq F(*,t)$. By the Matrix Path–Lemma 134.32 there exists a continuous map $A\colon [0,1] \to \mathrm{SO}(n+1)$ with $A(0) = A(1)$ such that for every $t \in [0,1]$ we have $A(t) \cdot e_1 = \gamma(t)$. One can now easily verify the map
$$\begin{array}{rcl} S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & (A(0)^{-1} \cdot A(t))^{-1} \cdot F(x,t) \end{array}$$
is a homotopy between f and g as maps $(S^n,*) \to (S^n,*)$.
(5) Let $f,g\colon (S^n,*) \to (S^n,*)$ be two continuous maps. Recall that by the Whitney Approximation Theorem 36.1 we know that the two maps f,g are homotopic rel $*$ to smooth maps $\widetilde{f},\widetilde{g}\colon (S^n,*) \to (S^n,*)$.

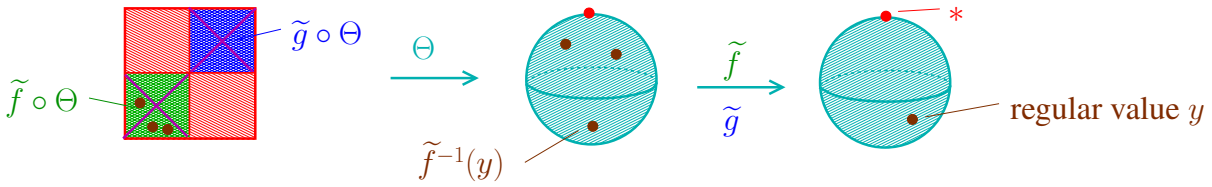
As on page 2578 we denote by $\Phi\colon (I^n,\partial I^n) \to (\overline{B}^n, S^{n-1})$ the explicit homeomorphism from page 2578 and we denote by $\Psi\colon (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (S^n,*)$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 8.10. We denote by $p\colon \overline{B}^n \to \overline{B}^n/S^{n-1}$ the projection and we set $\Theta \coloneqq \Psi \circ p \circ \Phi\colon I^n \to S^n$.

The map Ψ is evidently smooth. It follows easily from the explicit description of Φ on page 2578 that there exist finitely many subsets $W_1,\dots,W_m \subseteq I^n$ (which are given by cones on open lower-dimensional faces of ∂I^n) such that each W_i is a smooth manifold of dimension $< n$, such that each $\Phi|_{W_i}$ is smooth and such that Φ is smooth on the complement of $W \coloneqq W_1 \cup \dots \cup W_m$.



It follows from the Measure-Zero Properties Proposition 37.3 (4), the Full Measure-Density Proposition 37.4 and Sard's Theorem 38.1 that we can find a $y \in S^n$ which is

a regular value for $\widetilde{f}$ and $\widetilde{g}$, which does not lie in $(\widetilde{f}\circ\Theta)(W)$ and which does not lie in $(\widetilde{g}\circ\Theta)(W)$.



Now we see that

$$\begin{aligned}
\deg([\widetilde{f}])+\deg([\widetilde{g}]) &\overset{\text{by Definition 46.5 of degrees}}{=} \sum_{x\in\widetilde{f}^{-1}(y)}\deg(\widetilde{f},y)+\sum_{x\in\widetilde{g}^{-1}(y)}\deg(\widetilde{g},y) \overset{\text{since }\Theta\text{ is an orientation-preserving diffeomorphism outside of }W}{=}\\
&= \sum_{x\in(\Theta\circ\widetilde{f})^{-1}(y)}\deg(\widetilde{f}\circ\Theta,y)+\sum_{x\in(\Theta\circ\widetilde{g})^{-1}(y)}\deg(\widetilde{g}\circ\Theta,y)\\
&= \sum_{x\in((\Theta\circ\widetilde{f})*(\Theta\circ\widetilde{g}))^{-1}(y)}\deg((\widetilde{f}\circ\Theta)*(\widetilde{g}\circ\Theta),y) = \deg(\widetilde{f}*\widetilde{g}) = \deg([\widetilde{f}]\cdot[\widetilde{g}]).
\end{aligned}$$

follows easily from the definition of $*$

by definition of $\widetilde{f}*\widetilde{g}$ and by the same arguments as above ■

134.9. Exercises for Chapter 134.

Exercise 134.1. Let X be a topological space, let $x_0\in X$ and let $f\colon (I^n,\partial I^n)\to(X,\{x_0\})$ be a continuous map with $n\geq 1$. We consider the map

$$\begin{array}{rcl}\overline{f}\colon I^n & \to & X\\ (x_1,\ldots,x_n) & \mapsto & f(1-x_1,1-x_2,\ldots,1-x_n).\end{array}$$

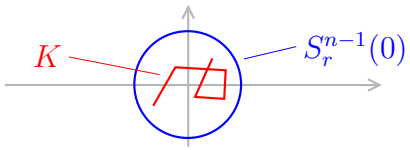
Is $[\overline{f}]\in\pi_n(X,x_0)$ the inverse of $[f]\in\pi_n(X,x_0)$?

Exercise 134.2. Let $k\leq l$ be natural numbers, let $f\colon S^k\to S^l$ be a continuous map and let $x_0\in S^k$. Show that for any $i\in\mathbb{N}$ the map $f_*\colon \pi_i(S^k,x_0)\to\pi_i(S^l,f(x_0))$ is the trivial map.
Hint. Use the Sphere–Low-π_n-Proposition 134.19.

Exercise 134.3. Show that there exists an epimorphism $\pi_n(S^1\vee S^n)\to\bigoplus_{k\in\mathbb{Z}}\pi_n(S^n)$.

Exercise 134.4. In this exercise you can assume that we already know that $\pi_3(S^2)\cong\mathbb{Z}$. Show that there exists an epimorphism $\pi_3(S^1\vee S^2)\to\mathbb{Z}^3\oplus\mathbb{Z}/4\mathbb{Z}$.

Exercise 134.5. Let $n\in\mathbb{N}$ and let K be a compact non-empty subset of $\mathbb{R}^n$. Let $r\in\mathbb{R}$ such that $r>\sup\{\|P\|\mid P\in K\}$. Show that for any $k\in\mathbb{N}$ the inclusion-induced map $\pi_k(S_r^{n-1}(0))\to\pi_k(\mathbb{R}^n\setminus K)$ is a monomorphism.



Exercise 134.6. We consider the pseudocircle, i.e. we consider the slightly weird topological space $X = \{A, B, C, D\}$ from page 172. Show that $\pi_n(X, A) = 0$ for any $n \geq 2$.
Hint. Find a convenient covering of X.

Exercise 134.7. Let (X, x_0) be a topological space. We denote by $\Sigma(X)$ the suspension of X and we denote by $*$ the North Pole of $\Sigma(X)$. Is the suspension homomorphism

$$\pi_1(X_0, x_0) \ \to \ \pi_2(\Sigma(X), *),$$

that we introduced in the Suspension Homomorphism Lemma 134.27, necessarily a monomorphism?

Exercise 134.8. Let M be a smooth manifold, let $Y \subseteq M$ be a compact submanifold and let $P \in M \setminus Y$.

(a) Show that for every $i < \dim(M) - \dim(Y) - 1$ the map $\pi_i(M \setminus Y, P) \to \pi_1(M, P)$ is an isomorphism.
(b) Show that for $i = \dim(M) - \dim(Y) - 1$ the map $\pi_i(M \setminus Y, P) \to \pi_1(M, P)$ is an epimorphism.

Hint. Use the Transversality Theorem 50.9.

Exercise 134.9. On page 195 we introduced the Topologist's Sine Curve

$$S \ := \ \{(0, y) \,|\, y \in [-1, 1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \ \subseteq \ \mathbb{R}^2.$$

As in Exercise 8.4 we refer to quotient space $W := S/(0,0) \sim (\frac{1}{\pi}, 0)$ as the *Warsaw Circle*.

(a) Show that the image of every continuous map $S^n \to W$ lies in a contractible subset of W.
Hint. Use Exercise 4.56 and use the fact that each S^n is path-connected.
(b) Show that for every $n \in \mathbb{N}$ we have $\pi_n(W, [(0,0)]) = 0$.

Exercise 134.10.

(a) Let X be a set.
- A *binary operation on* X is a map $X \times X \to X$.
- A binary operation $\mu\colon X \times X \to X$ is called *unital* if there exists an element $1 \in X$ such that $\mu(1, x) = \mu(x, 1) = x$ for all $x \in X$.

Let "$\circ$" and "$\otimes$" be two unital binary operations on X. We suppose that for every choice of $a, b, c, d \in X$ we have the equality

$$(a \otimes b) \circ (c \otimes d) \ = \ (a \circ c) \otimes (b \circ d).$$

Show that "$\circ$" and "$\otimes$" are the same and that they are commutative and associative.
Remark. This statement is known as the Eckmann-Hilton Theorem.
(b) Use (a) to give a new proof that higher homotopy groups are abelian.
(c) Use (a) to give a new proof that fundamental groups of topological groups are abelian.

Exercise 134.11. Let $X = S^2$ with the *trivial* topology $\mathcal{T}_{\text{tr}}$, i.e. the only open sets are $\varnothing$ and S^1. Which higher homotopy groups can you determine?

Exercise 134.12. Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. Show that the map

$$\begin{array}{rcl} \Theta\colon \pi_n(A,a_0)\times\pi_n(B,b_0) & \to & \pi_n(A\times B,(a_0,b_0)) \\ ([f\colon (I^n,\partial I^n)\to(A,a_0)],[g\colon (I^n,\partial I^n)\to(B,b_0)]) & \mapsto & \left[\begin{array}{c}(I^n,\partial I^n)\to(A\times B,(a_0,b_0))\\ z\mapsto (f(z),g(z))\end{array}\right]. \end{array}$$

agrees with the isomorphism Φ from the Product-π_n-Proposition 90.13.

Exercise 134.13. Determine $\pi_2(\mathrm{SO}(3))$.

Exercise 134.14. Let $X\subsetneq S^n$ be a connected subspace and let $x_0\in X$. Show that for each $i\in\mathbb{N}$ the inclusion-induced map $\pi_i(X,x_0)\to\pi_i(S^n,x_0)$ is the zero map.

Exercise 134.15. Can homotopy groups be used to distinguish the homeomorphism types of $\mathbb{R}\mathrm{P}^2\times S^3$ and $S^2\times\mathbb{R}\mathrm{P}^3$?

Part XV

CW-complexes

In this part of the lecture notes we will give an example-driven but hopefully thorough introduction to CW-complexes. We will see that the notion of a CW-complex strikes the right balance:

(1) CW-complexes are topological space that have many desirable properties,
(2) the notion is flexible enough so that most of the topological spaces that we are interested in are actually CW-complexes.

We will study many properties of CW-complexes, but for more specialized results we refer the reader to the two classical textbooks [**LW1969**] and [**FrPi1990a**] on CW-complexes.

This part of the lecture notes is organized as follows:

- In Chapter 135 we will explain what it means to attach a cell to a topological space.
- In Chapter 136 we will introduce the notion of a CW-complex and we will discuss many examples and basic properties. For example we will see that all projective spaces are CW-complexes and we will see that CW-complexes are Hausdorff.
- In the somewhat technical Chapter 137 we will study homotopies on CW-complexes. The difficulty is to justify why certain maps are actually continuous. This discussion will allow us to show that CW-complexes are locally contractible.
- In Chapter 138 we will show that every covering of a CW-complex is naturally again a CW-complex.
- In Chapter 139 we will give several methods for constructing CW-complexes. In particular we will consider quotients and produces of CW-complexes.
- In Chapter 140 we will prove the Homotopy Extension Theorem 140.2 which says that a homotopy on a subcomplex always extends to a homotopy on the total CW-complex. As a typical application we will prove the Quotient-by-Contractible Corollary 140.4 which says that the quotient of a CW-complex X by a contractible subcomplex is actually homotopy equivalent to the original CW-complex.
- In Chapter 141 we will prove the Cellular Approximation Theorem 141.2 which says that any continuous map between two CW-complexes is homotopic to a cellular map.

CHAPTER 135

Cell attachments

In this chapter we introduce the concept of attaching a cell to a topological space. We will see that many topological spaces arise that way and we will see that attaching cells changes the fundamental group in a controlled way. This allows us to give the overdue calculation of the fundamental groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. More importantly though the study of cell attachments will be essential in the follow-up chapter where will graduate to the more general concept of a CW-complex.

This chapter is organized as follows:

- In Section 135.1 we will define cell attachments.
- In Section 135.2 we will see the effect of a cell attachment on fundamental groups.
- In Section 135.3 we will see that the real and complex projective spaces can be obtained by iterative cell attachments.
- In Section 135.4 we will discuss what it means to attach arbitrarily many cells.
- In Section 135.5 we will show that the homotopy type of a cell attachment does not change if we change the attaching map by a homotopy or if we replace the original topological space by a homotopy equivalent topological space.
- In Appendix 135.6 we will see that if we attach a cell to a normal topological space, then the result is also normal.
- In Section 135.7 we will provide exercises for Chapter 135.

In the next Chapter 136 we will introduce CW-complexes.

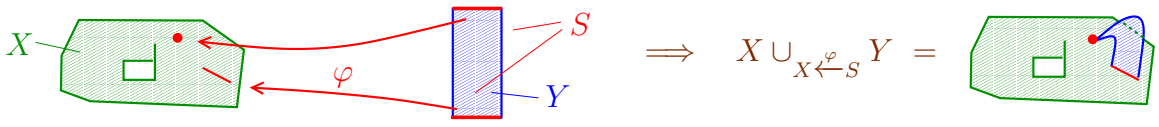
135.1. **Attaching a cell to a topological space.** We start out with a very general definition.

Definition 135.1. Let X and Y be topological spaces, let $S \subseteq Y$ be a subspace and let $\varphi\colon S \to X$ be a continuous map. (Note that φ does not need to be injective.) We write

$$X \cup_\varphi Y \;:=\; X \cup_{X \xleftarrow{\varphi} S} Y \;:=\; (X \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation generated by $\varphi(s) \sim s$ for $s \in S$. We call $X \cup_\varphi Y$ the topological space obtained by gluing Y to X via the attaching map φ. Note that according to the Topological Pushout Proposition 8.12 we obtain a pushout diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup_\varphi Y. \end{array}$$



In this chapter we will mostly be interested in the following special case:

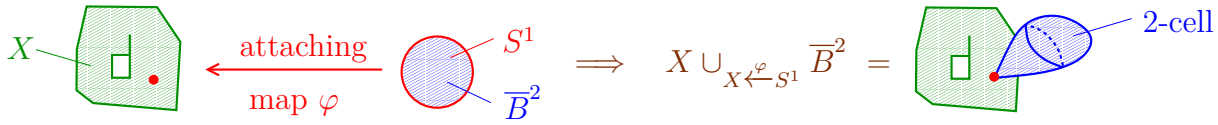
Definition 135.2. Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We consider the topological space

$$X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n.$$

We introduce the following language:

(1) We refer to the image of B^n as an open n-cell.
(2) We refer to the image of $\overline{B}^n$ as an n-cell.
(3) We refer to φ as the attaching map of the n-cell.

We say that $X \cup_\varphi \overline{B}^n$ is obtained from X by attaching an n-cell along the attaching map φ. It follows from the fact that the inclusion $S^{n-1} \to \overline{B}^n$ is a closed embedding together with the Topological Pushout-Maps Lemma 8.16 (4) that the natural map $X \to X \cup_\varphi \overline{B}^n$ is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of $X \cup_\varphi \overline{B}^n$.



Remark. On page 552 we gave an explicit homeomorphism $\mathrm{Cone}(S^{n-1}) \xrightarrow{\cong} \overline{B}^n$. Using this homeomorphism one sees almost immediately that, using the notation from the above definition, we have a homeomorphism $\mathrm{Cone}(\varphi\colon S^{n-1} \to X) \xrightarrow{\cong} X \cup_\varphi \overline{B}^n$ which is the identity on X. In other words, attaching a cell is just a special case of taking a mapping cone. □

In the following we provide a very long list of examples of cell attachments. These examples train our geometric intuition and they will come in super handy in the next chapter, when we study CW-complexes.

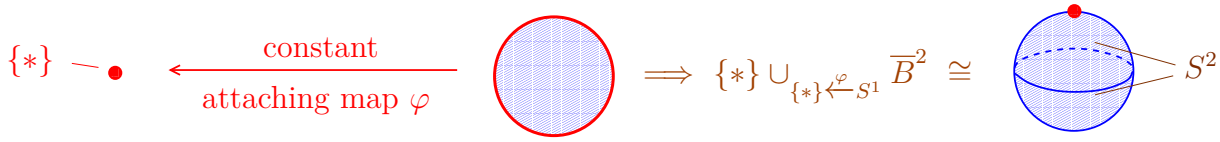
Examples.

(1) We consider the effect of adding a 0-cell to a topological space X. Since $S^{-1} = \varnothing$ and since $\overline{B}^0 = \{0\}$ we see that attaching a 0-cell is the same as taking the disjoint union of X with a point.
(2) Let $X = \{*\}$ and let $\varphi\colon S^{n-1} \to \{*\}$ be the constant map. We have homeomorphisms

$$\{*\} \cup_{\{*\} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \quad \underset{\uparrow}{\xrightarrow{\cong}} \quad \overline{B}^n / S^{n-1} \quad \underset{\uparrow}{\xrightarrow{\cong}} \quad S^n.$$

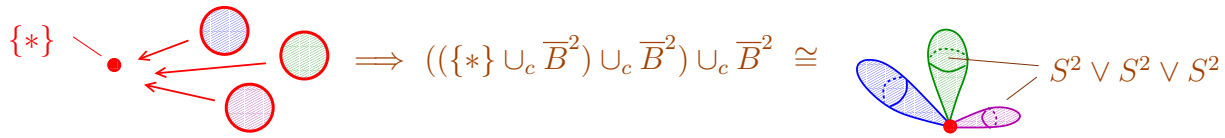
see the homeomorphism from page 318 — identification from the Ball Quotient-Sphere Lemma 8.10

In fact, looking at these explicit homeomorphisms we see that these homeomorphisms send $\{*\}$ to the North Pole $(0, \ldots, 0, 1)$.

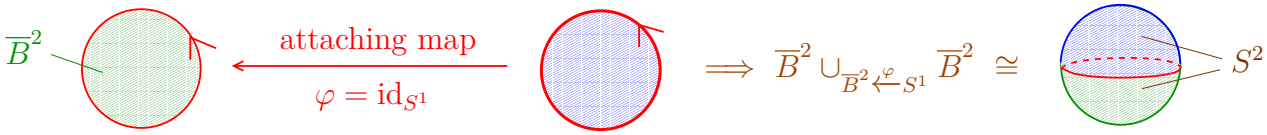


(3) Again let $X = \{*\}$. Iteratively we attach k n-cells according to the constant attaching map $c\colon S^{n-1} \to \{*\}$. It follows easily from the Topological Pushout Proposition 8.12

(1∗), the Compact-Hausdorff Proposition 4.23 (3), and the Wedge-Properties Lemma 96.4 (3) that the resulting topological space is homeomorphic to the wedge of k copies of S^n.

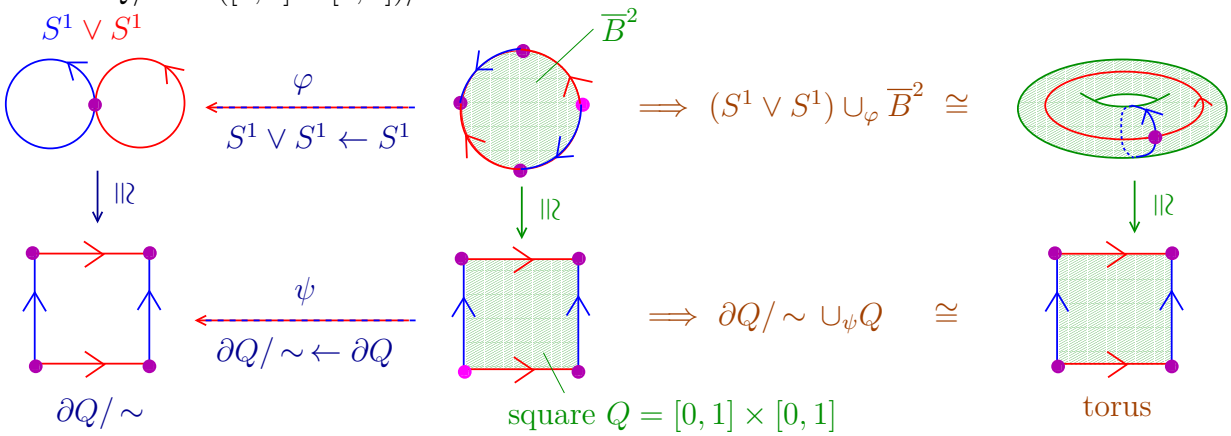


(4) Let $n \in \mathbb{N}_0$. We consider the n-ball $\overline{B}^n$ and we attach an n-cell to it along the attaching map $\varphi\colon S^{n-1} \to \overline{B}^n$ that is given by the inclusion. On page 319 we already saw that $\overline{B}^n \cup_{\overline{B}^n \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ is homeomorphic to the n-sphere S^n.



(5) We consider $X = S^1 \vee S^1$.

(a) We attach a 2-cell along the attaching map $\varphi\colon S^1 \to S^1 \vee S^1$ that is illustrated below. The Convex-to-Ball Proposition 4.28 gives us an explicit homeomorphism $\overline{B}^2 \to Q := [0,1] \times [0,1]$. This homeomorphism, together with the Topological Pushout Proposition 8.12 (1∗) and the Compact-Hausdorff Proposition 4.23 (3) can be used to show that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \xleftarrow{\varphi} S^1} \overline{B}^2$ is homeomorphic to the 2-torus $Q/\!\sim \cong ([0,1] \times [0,1])/\!\sim$.

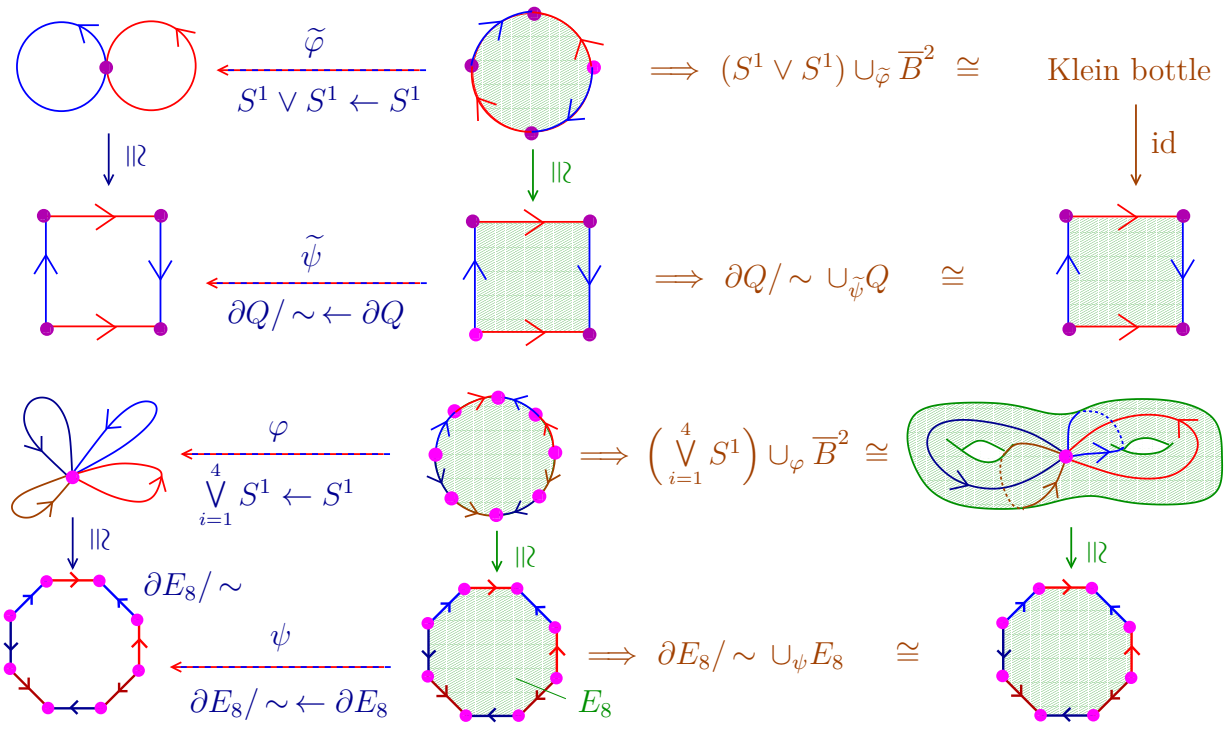


(b) If instead we consider the attaching map $\widetilde{\varphi}\colon S^1 \to S^1 \vee S^1$ that is shown in the figure below, then one easily sees that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \xleftarrow{\widetilde{\varphi}} S^1} \overline{B}^2$ is homeomorphic to the Klein bottle.

(c) We turn to our last example for the time being:
In the figure above we show that if we attach a 2-cell in a suitable way to the wedge of four circles, then we obtain a topological space that is homeomorphic to the surface of genus 2. □

Later in this chapter we will give more examples of (iterated) cell attachments.

We conclude this section with the following more theoretical lemma which shows that attaching cells is basically just a special case of taking mapping cones. This little observation allows us at times to import results on mapping cones to the study of cell attachments.



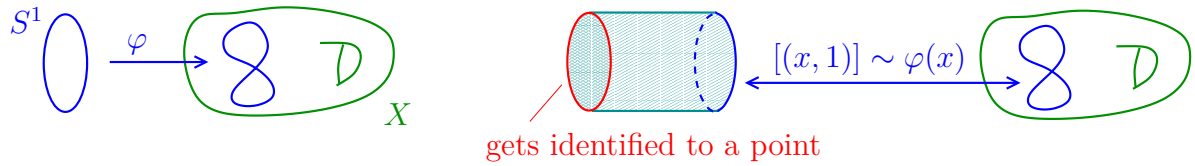
Lemma 135.3. (Mapping Cone–Cell Attachment Lemma) Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a continuous map. As in Definition 21.7 we consider

$$\begin{array}{rcll} \mathrm{Cone}(S^{n-1}) & := & (S^{n-1} \times [0,1])/(S^{n-1} \times \{0\}) & \text{(cone of } S^{n-1}\text{)} \\ \mathrm{Cone}(f\colon S^{n-1} \to X) & := & (\mathrm{Cone}(S^{n-1}) \sqcup X)/[(x,1)] \sim f(x) & \text{(mapping cone of } f\text{).} \end{array}$$

The map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon S^{n-1} \to X) & \to & X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \\ [P] & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [x \cdot r], & \text{if } P = [(x,r)] \text{ with } (x,r) \in S^{n-1} \times [0,1] \end{cases} \end{array}$$

is a homeomorphism.



Proof. This lemma follows almost immediately from the fact that by the popular Compact-Hausdorff Proposition 4.23 the map

$$\begin{array}{rcl} (S^{n-1} \times [0,1])/(a,0) \sim (b,0) & \to & \overline{B}^n \\ [(x,r)] & \mapsto & x \cdot r \end{array}$$

is a homeomorphism. We leave it to the reader to fill in the remaining details. ∎

135.2. **Attaching cells, connectedness and fundamental groups.** The following lemma tells us how attaching a cell affects the set of path-components of a given topological space:

Lemma 135.4. (Cell Attachment-π_0-Lemma) Let X be a topological space, let $n \in \mathbb{N}_0$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. The following statements hold:

- ($n \geq 2$) The inclusion-induced map $\pi_0(X) \to \pi_0(X \cup_\varphi \overline{B}^n)$ is a bijection.
- ($n = 1$) (a) If $\varphi(S^0)$ lies in a single path-component of X, then the inclusion-induced map $\pi_0(X) \to \pi_0(X \cup_\varphi \overline{B}^n)$ is a bijection.
  (b) If $\varphi(S^0)$ takes values in two path-components X' and X'' of X, then the inclusion-induced map $\pi_0(X) \to \pi_0(X \cup_\varphi \overline{B}^n)$ is a surjection, the two path-components X' and X'' get sent to a single path-component and all other path-components are unaffected.
- ($n = 0$) $X \cup_\varphi \overline{B}^0$ has an extra path-component.

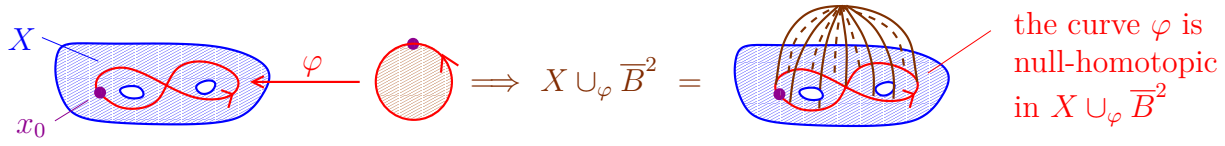
Sketch of proof. Morally the lemma follows from the Disjoint Union Topology-Properties Lemma 7.28 (2b), the Union Connected Lemma 4.39, the Double–Quotient Lemma 8.7 and the Path–Component Lemma 4.44. A full lean-ready argument might be a little longer than one might hope. ■

The following much more interesting proposition tells us how attaching a cell affects the fundamental group of a given topological space:

Proposition 135.5. (Cell Attachment-π_1-Proposition) Let X be a topological space, let $n \geq 1$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 \coloneqq \varphi(\underbrace{(0,\ldots,0,1)}_{\in S^{n-1}})$. The following statements hold:

- ($n \geq 3$) The inclusion-induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.
- ($n = 2$) The map $[0,1] \to X$ given by $t \mapsto \varphi(\exp(2\pi i t))$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$. The inclusion-induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an isomorphism

$$\pi_1(X, x_0)/\langle\langle [\varphi] \rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^2, x_0\big).$$



Proposition 135.5. (Cell Attachment-π_1-Proposition) We continue with the above notation.

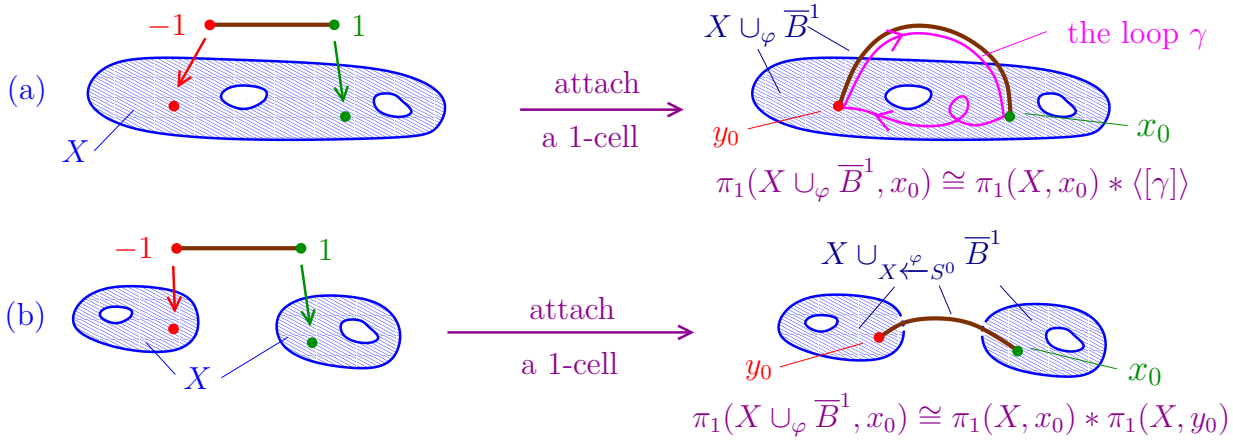
- ($n = 1$) We set $x_0 \coloneqq \varphi(1)$ and $y_0 \coloneqq \varphi(-1)$.
  (a) Suppose that x_0 and y_0 lie in the same path-component of X. Let γ be a loop in $X \cup_\varphi \overline{B}^1$ that is given by the concatenation of the obvious path in the 1-cell from $x_0 = \varphi(1)$ to $y_0 = \varphi(-1)$ and some path in X from $y_0 = \varphi(-1)$ to $x_0 = \varphi(1)$. With this notation the map

$$\pi_1(X, x_0) * \langle t \rangle \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0)$$

  that is given by the inclusion-induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^1, x_0)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that x_0 and y_0 lie in two different path components of X. Then we get an isomorphism

$$\pi_1(X, x_0) * \pi_1(X, y_0) \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0).$$

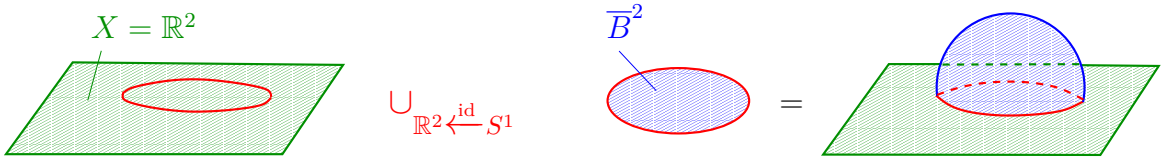


Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then

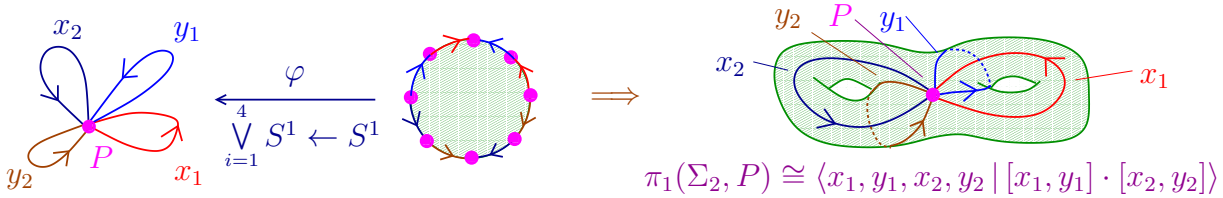
$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\text{by the Cell Attachment-}\pi_1\text{ Proposition 135.5}}{\overset{\uparrow}{=}} \pi_1(S^1, 1)/\langle\langle[\varphi]\rangle\rangle \underset{\text{see the Circle-}\pi_1\text{ Theorem 88.22}}{\overset{\uparrow}{=}} \langle x\rangle/\langle\langle x^n\rangle\rangle \;\cong\; \mathbb{Z}/n\mathbb{Z}.$$

(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\varphi := \mathrm{id}_{S^1}\colon S^1 \to S^1$ that is given by the identity. By the Cell Attachment-π_1-Proposition 135.5 the new topological space $\mathbb{R}^2 \cup_\varphi \overline{B}^2$ is again simply connected



(3) In the previous section we already saw that many interesting topological spaces can be built out of easier topological spaces by attaching cells. In many cases we can use the Cell Attachment-π_1-Proposition 135.5 to calculate the fundamental group. For example, on page 2605 we showed that we can obtain the surface of genus 2 from the wedge of four circles by attaching a 2-cell. With the notation from the figure below we now see that

$$\pi_1(\Sigma_2, P) \;\cong \pi_1\Big(\Big(\bigvee_{i=1}^4 S^1\Big) \cup_\varphi \overline{B}^2, P\Big) \underset{\text{by the Cell Attachment-}\pi_1\text{ Proposition 135.5}}{\overset{\downarrow}{\cong}} \pi_1\Big(\bigvee_{i=1}^4 S^1, P\Big)/\langle\langle[\varphi]\rangle\rangle \underset{\text{by the Wedge-}\pi_1\text{-Proposition 99.10 and since } [\varphi] = [x_1, y_1] \cdot [x_2, y_2]}{\overset{\uparrow}{\cong}} \langle x_1, y_1, x_2, y_2\rangle/\langle\langle[x_1, y_1] \cdot [x_2, y_2]\rangle\rangle.$$



Of course we obtained the same result much earlier. The advantage of the current approach is that the Cell Attachment-π_1-Proposition 135.5 absorbs many of the technical arguments we had to deal with in the proof of the Surface-π_1-Proposition 101.1, i.e. in our initial calculation of $\pi_1(\Sigma_2, P)$. □

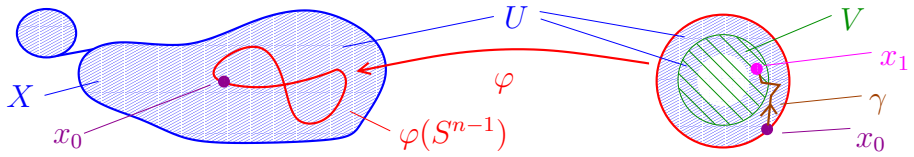
Proof of the Cell Attachment-π_1-Proposition 135.5 for $n \geq 2$. This statement follows immediately from the Mapping Cone–Cell Attachment Lemma 135.3 together with the Mapping Cone-π_1-Lemma 99.15. For the reader's convenience we give an independent proof here, which is basically the same as the proof of the Mapping Cone-π_1-Lemma 99.15, just with slightly different notation and a few simplifications.

Let X be a topological space, let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 := \varphi((0, \dots, 0, 1))$. The key idea of the proof is to apply the Seifert–van Kampen-Theorem 99.1 to the following two subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$

$$U \;=\; X \cup_{X \xleftarrow{\varphi} S^{n-1}} \left\{ z \in \overline{B}^n \;\middle|\; \|z\| > \tfrac{1}{4} \right\} \quad \text{and} \quad V \;=\; \left\{ z \in \overline{B}^n \;\middle|\; \|z\| < \tfrac{3}{4} \right\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path γ in U from x_0 to x_1.



Claim. The following diagram of maps

$$\begin{array}{ccc}
\pi_1(X, x_0) & \longrightarrow & \pi_1(X \cup_\varphi \overline{B}^n, x_0) \\
\cong\downarrow & & \downarrow\mathrm{id} \\
\pi_1(U, x_0) & \longrightarrow & \pi_1(X \cup_\varphi \overline{B}^n, x_0) \\
\cong\uparrow\gamma_* & & \cong\uparrow\gamma_* \\
\pi_1(U, x_1) & \longrightarrow & \pi_1(X \cup_\varphi \overline{B}^n, x_1) \\
\downarrow & & \downarrow\mathrm{id} \\
\pi_1(U, x_1)/\langle\langle \pi_1(U \cap V, x_1) \rangle\rangle & \longrightarrow & \pi_1(X \cup_\varphi \overline{B}^n, x_1) \\
\cong\uparrow \text{Amalgamated Product–Quotient Lemma 98.25} & & \downarrow\mathrm{id} \\
\pi_1(U, x_1) *_{\pi_1(U \cap V, x_1)} \underbrace{\pi_1(V, x_1)}_{=\{e\}} & \xrightarrow[\text{Seifert–van Kampen Theorem 99.1}]{\cong} & \pi_1(X \cup_\varphi \overline{B}^n, x_1)
\end{array}$$

has the following three properties:

(i) The diagram commutes.
(ii) All vertical maps except possibly the third vertical map on the left are isomorphisms.
(iii) The bottom horizontal map is an isomorphism.

Proof.

(i) The two vertical maps from the third to the second level are the homomorphisms induced by the choice of base point coming from the path γ from x_0 to x_1. It follows from the Change-of-Base Point Proposition 90.4 that the second square commutes. All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily from functoriality that the diagram commutes.

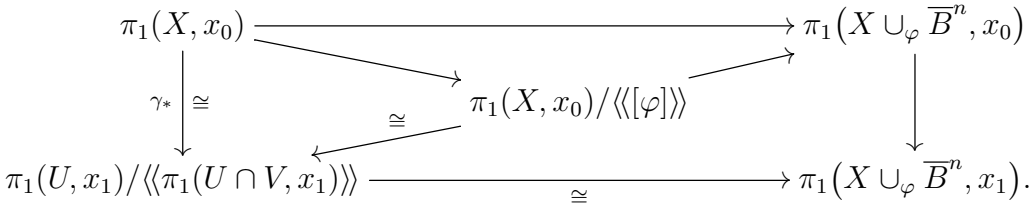
(ii) First note that it follows from the Change-of-Base Point Proposition 88.14 (1) that the vertical maps from the third to the second level are isomorphisms. Using the Homotopy Pushout Lemma 18.13 one can easily show that X is a deformation retract of U. Thus it follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1b) that the top left vertical map is an isomorphism.

The natural map $\{z \in B^n \,|\, \|z\| < \frac{3}{4}\} \to X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ is easily seen to be an embedding. It follows that V is homeomorphic to an open ball. Hence $\pi_1(V, x_1)$ is the trivial group. It follows from this observation and the Amalgamated Product–Quotient Lemma 98.25 that the bottom vertical map on the left is an isomorphism.

(iii) It follows easily from the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0) that U and V are indeed open subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$. We can therefore apply the Seifert–van Kampen-Theorem 99.1 and we obtain that the bottom horizontal map is an isomorphism. ⊞

Now we consider the case $n \geq 3$. In this case $\pi_1(U \cap V, x_1)$ is trivial since $U \cap V$ is homotopy equivalent to S^{n-1}. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 and the Sphere-π_1-Proposition 88.24 that $\pi_1(U \cap V, x_1) \cong \pi_1(S^{n-1}) = 0$. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are isomorphisms, since the diagram commutes and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U \cap V$ is homotopy equivalent to S^1 which implies, by the explicit isomorphism from the Circle-π_1-Theorem 88.22 and the Homotopy Equivalence-π_1-Proposition 90.9, that $\pi_1(U \cap V, x_1) \cong \mathbb{Z}$. Note that under the explicit isomorphism $\Phi\colon \pi_1(X, x_0) \to \pi_1(U, x_1)$ this infinite cyclic subgroup corresponds to $\langle [\varphi] \rangle$. It is straightforward to see that we get a commutative diagram



This concludes the proof of the proposition. ■

Proof of the Cell Attachment-π_1-Proposition 135.5 for $n = 1$.

(a) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in the same path-component of X. The proof of the case $n = 1$ is similar in spirit to the proof of the case $n \geq 2$. The only serious difference is that we need to replace the Seifert–van Kampen-Theorem 99.1 by the HNN-Seifert–van Kampen Theorem 106.7. We leave it to the reader to fill in the details.

(b) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in different path-components of X. This case follows once again fairly easily from the Seifert–van Kampen-Theorem 99.1. Again we leave it to the reader to fill in the details. ■

135.3. **Projective spaces.** We just saw that the Cell Attachment-π_1-Proposition 135.5 can be very useful for calculating fundamental groups. Before we had already been quite successful with calculating fundamental groups, but it turns out that there is one interesting example of topological spaces that we could not handle yet, namely the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

For the reader's convenience we recall the definition of real and complex projective spaces which we introduced in Definition 9.11.

Definition 135.6. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.

(1) For $n \in \mathbb{N}_0$ we set
$$\mathbb{A}\mathrm{P}^n := (\mathbb{A}^{n+1} \setminus \{0\})/\sim \quad \text{where } v \sim \lambda \cdot w \text{ for } \lambda \in \mathbb{A} \setminus \{0\}.$$
Given a point $(v_0, \ldots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \ldots : v_n]$ its equivalence class.

(2) For $k \leq n$ we refer to the map
$$\begin{array}{rcl} \iota\colon \mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$
as the standard embedding. We saw on page 338 that this map is indeed an embedding.

The following lemma shows that we can obtain $\mathbb{R}\mathrm{P}^n$ from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching a single n-cell and that we can obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a single $2n$-cell.

Lemma 135.7. (Projective Space–Cell Attachment Lemma) Let $n \in \mathbb{N}_0$.

(1) We consider the continuous map
$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x_1, \ldots, x_n) & \mapsto & [x_1 : \ldots : x_n]. \end{array}$$
There exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1} \cup_{\mathbb{R}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ such that the restriction to $\mathbb{R}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^n$.

(2) We think of $\overline{B}^{2n}$ and S^{2n-1} as subspaces of $\mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n}$. We consider the attaching map
$$\begin{array}{rcl} \varphi\colon S^{2n-1} & \to & \mathbb{C}\mathrm{P}^{n-1} \\ (z_1, \ldots, z_n) & \mapsto & [z_1 : \ldots : z_n]. \end{array}$$
There exists a homeomorphism $\mathbb{C}\mathrm{P}^{n-1} \cup_{\mathbb{C}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{2n-1}} \overline{B}^{2n} \to \mathbb{C}\mathrm{P}^n$ such that the restriction to $\mathbb{C}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n & \to & \mathbb{R}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [x_1 : \cdots : x_n : 0], & \text{if } P = [x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^{n-1}, \\ [P : \sqrt{1 - \|P\|^2}], & \text{if } P \in \overline{B}^n. \end{cases} \end{array}$$



It follows immediately from the definition of φ that the map Θ is well-defined. Furthermore it follows easily from the Topological Pushout Proposition 8.12 (1∗) that the map Θ is continuous. It is basically clear that the map Θ is a surjection and it is straightforward to verify that the map Θ is an injection. Using the Topological-Quotient Proposition 8.2 (1b), the Projective Spaces-Compact+Hausdorff Proposition 9.15 and the Compact-Hausdorff Proposition 4.23 (3) one can easily verify that the map is a homeomorphism. Finally it is clear that the restriction of the map Θ to $\mathbb{R}\mathrm{P}^{n-1}$ is the standard embedding.

(2) Similar to (1) we see that the following map is a homeomorphism:

$$\begin{array}{rcl} \Theta\colon \mathbb{C}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^{2n} & \to & \mathbb{C}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [z_1 : \cdots : z_n : 0], & \text{if } P = [z_1 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^{n-1}, \\ [P : \underbrace{\sqrt{1 - \|P\|^2}}_{\in \mathbb{R} \subseteq \mathbb{C}}], & \text{if } P \in \overline{B}^{2n}. \end{cases} \end{array}$$

■

The combination of the Cell Attachment-π_1-Proposition 135.5 together with the Projective Space–Cell Attachment Lemma 135.7 allows us to (re-) calculate the fundamental groups of projective spaces.

Proposition 135.8. (Projective Space–π_1-Proposition)
(1) We have $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$ and for $n \geq 2$ we have $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$.
(2) For any $n \in \mathbb{N}_0$ we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$.

Proof.
(1) Since we already calculated the fundamental groups of $\mathbb{R}\mathrm{P}^n$ in the Real Projective Space–π_1-Proposition 93.3 we outsource this part to Exercise 135.3.
(2) Note that $\mathbb{C}\mathrm{P}^0$ consists of a single point. It follows from the Projective Space–Cell Attachment Lemma 135.7 that we obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^0$ by iteratively attaching cells of dimension $2, 4, \ldots, 2n$. It follows from the Cell Attachment-π_1-Proposition 135.5 that $\pi_1(\mathbb{C}\mathrm{P}^n) \cong \pi_1(\mathbb{C}\mathrm{P}^0) = 0$. ■

135.4. **Attaching arbitrarily many cells.** In the following we slightly generalize the concepts from Definition 135.1 to the following notationally less pleasant definition.

Definition 135.9. Let X be a topological space and let $\{\varphi_i \colon S^{n_i-1} \to X\}_{i\in I}$ be a family of continuous maps.

(1) We consider the topological space

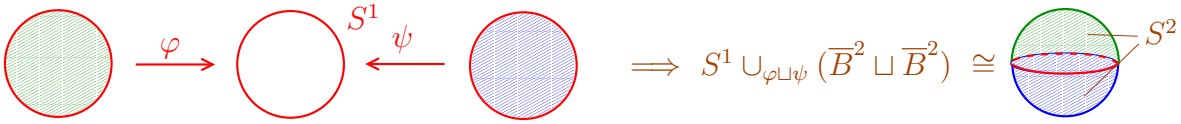
$$X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i}.$$

As before we introduce the notion of an (open) cell and of an attaching map of a cell. We say that the above topological space is obtained from X by attaching cells along the attaching maps φ_i.

(2) It follows from the fact that the inclusions $S^{n-1} \to \overline{B}^n$ are closed embeddings together with the Topological Pushout-Maps Lemma 8.16 (4) that the natural map of X into the above topological space is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of the above topological space.

Example.

(1) Let $n \in \mathbb{N}_0$. We attach two n-cells to the sphere S^{n-1} where in both cases the attaching map is given by the identity map $\varphi := \psi := \mathrm{id} \colon S^{n-1} \to S^{n-1}$. Using the Compact-Hausdorff Proposition 4.23 (3) one can easily show that there exists a homeomorphism $S^{n-1} \cup_{\varphi\sqcup\psi} (\overline{B}^n \sqcup \overline{B}^n) \to S^n$ which on S^{n-1} is given by $x \mapsto (x, 0)$.



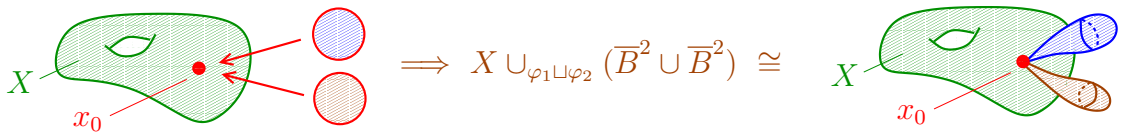
(2) Let (X, x_0) be a pointed topological space. Furthermore let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}_0$. Given $i \in I$ we denote by $\varphi_i \colon S^{n_i-1} \to X$ the constant map that sends all points to x_0. Using the Ball Quotient-Sphere Lemma 8.10 one can show that there exists a homeomorphism

$$X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i} \xrightarrow{\cong} \underbrace{(X, x_0) \vee \bigvee_{i\in I} (S^{n_i}, N)}_{\substack{\text{wedge of pointed topological} \\ \text{spaces where } N \in S^{n_i} \text{ is the} \\ \text{North Pole}}}$$

that is the identity on X. [1404]

This homeomorphism is illustrated in the figure above. □

[1404] For fans of checking continuity let us elaborate a little bit. Let $p_n \colon \overline{B}^n \to \overline{B}^n/S^{n-1}$ be the projection and let $\Theta_n \colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 8.10. The maps $\Theta_{n_i} \circ p_{n_i} \colon \overline{B}^{n_i} \to S^{n_i}$ together with the identity on X define, by the Topological-Coproduct Proposition 7.26 (1b), Topological Pushout Proposition 8.12 (1b) and the Wedge Proposition 96.2 (1a), a continuous map $\alpha \colon X \cup \bigsqcup_{i\in I} \overline{B}^{n_i} \to X \vee \bigvee_{i\in I} S^{n_i}$. Conversely note that the natural map $\overline{B}^{n_i} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$ descends to a continuous map $f_i \colon \overline{B}^{n_i}/S^{n_i-1} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$. The maps $f_i \circ \Theta_{n_i}^{-1} \colon S^n \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$ and the identity on X define, by the Wedge Proposition 96.2 (1a), a continuous map $\beta \colon X \vee \bigvee_{i\in I} S^{n_i} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$. One can easily verify that these two maps α and β are inverses of one another.



Lemma 135.10. (Iterative Cell-Attachment Lemma) Let X be a topological space and let $\{\varphi_i \colon S^{n_i} \to X\}_{i \in I}$ be a family of continuous maps. Let $J \subseteq I$ be a subset. We denote by X_J the result of attaching the cells in J and we denote by X_I the result of attaching the cells in I.

(1) The natural map $X_J \to X_I$ is a closed embedding.
(2) If we attach the cells corresponding to $J \setminus I$ to X_J, then the resulting topological space is naturally homeomorphic to X_I.

Proof.

(1) It follows from the Topological-Pushout Proposition 8.12 (1∗) and the Topological-Coproduct Proposition 7.26 (1∗) that the natural map $X_J \to X_I$ is continuous. Furthermore, using the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0) one can easily verify that this map is closed. It follows from the Open-Closed–Injective Lemma 4.22 (2) that the given map is a closed embedding.
(2) This statement is just a special case of the Pushout Composition Lemma 6.25. ∎

The following proposition shows that the Hausdorff property is preserved under cell attachments.

Proposition 135.11. (Cell Attachment–Hausdorff Proposition) Let X be a topological space and let $\{\varphi_i \colon S^{n_i - 1} \to X\}_{i \in I}$ be a family of continuous maps. We consider the topological space

$$\widehat{X} \coloneqq X \cup_{X \leftarrow \bigsqcup_{i \in I} S^{n_i - 1} \,:\, \bigsqcup_{i \in I} \varphi_i} \bigsqcup_{i \in I} \overline{B}^{n_i}$$

that we obtain from X by attaching cells along the attaching maps φ_i.

(1) If U, V are disjoint open subset of X, then there exist disjoint open subsets $\widehat{U}$ and $\widehat{V}$ of $\widehat{X}$ with $X \cap \widehat{U} = U$ and $X \cap \widehat{V} = V$.
(2) If X is Hausdorff, then $\widehat{X}$ is also Hausdorff.

Proof. For each $i \in I$ we consider the natural map $\Phi_i \colon \overline{B}^{n_i} \to \widehat{X}$. It follows immediately from the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0) that given any subset $U \subseteq \widehat{X}$ the following statement holds:

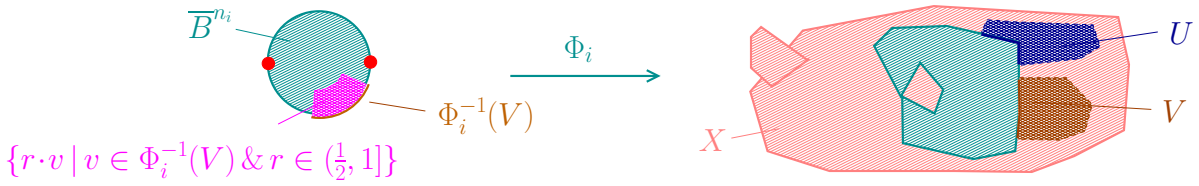
$$U \text{ is open in } \widehat{X} \iff \begin{array}{ll} \text{(a)} & \text{the intersection } U \cap X \text{ is open in } X \text{ and} \\ \text{(b)} & \text{for all } i \in I \text{ the preimage } \Phi_i^{-1}(U) \text{ is open in } \overline{B}^{n_i}. \end{array}$$

After this preparation we turn to the actual proof of the two statements:

(1) Let U and V be two disjoint open subsets of X. We set

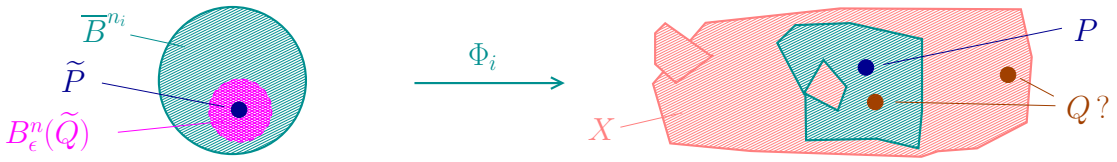
$$\begin{aligned} \widehat{U} &\coloneqq U \cup \bigcup_{i \in I} \Phi_i(\{r \cdot u \mid u \in \Phi_i^{-1}(U) \text{ and } r \in (\tfrac{1}{2}, 1]\}). \\ \widehat{V} &\coloneqq V \cup \bigcup_{i \in I} \Phi_i(\{r \cdot v \mid v \in \Phi_i^{-1}(V) \text{ and } r \in (\tfrac{1}{2}, 1]\}). \end{aligned}$$

It follows immediately from the above discussion that $\widehat{U}$ and $\widehat{V}$ are open subsets of $\widehat{X}$. Furthermore it is clear that they are disjoint.



(2) Let $P, Q \in \widehat{X}$ be two distinct points. If P and Q lie in X, then it follows from the hypothesis that X is Hausdorff and (1) that we can find disjoint open neighborhoods $\widehat{U}$ of P and $\widehat{V}$ of Q in $\widehat{X}$.

It remains to consider the case that at least one of P and Q lies in $\widehat{X} \setminus X$. Let us assume that $P \in \widehat{X} \setminus X$. This means that there exists an $i \in I$ and a $\widetilde{P} \in B^{n_i}$ such that $P = \Phi_i(\widetilde{P})$. Since $Q \neq P$ there exists an $\epsilon > 0$ such that $B^{n_i}_\epsilon(\widetilde{P}) \subseteq B^{n_i}$ and such that $Q \notin \Phi_i(B^{n_i}_\epsilon(\widetilde{P}))$.

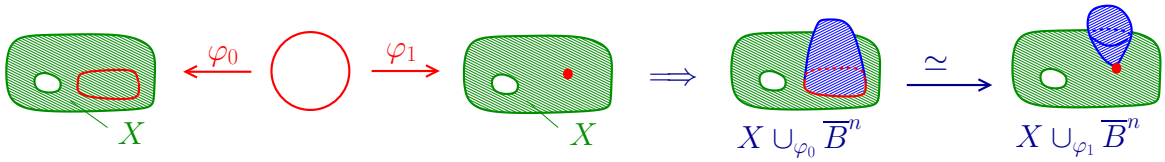


We set $U := \Phi_i(B^{n_i}_{\frac{\epsilon}{2}}(\widetilde{Q}))$ and $V := \widehat{X} \setminus \Phi_i(\overline{B}^{n_i}_{\frac{\epsilon}{2}}(\widetilde{Q}))$. It is clear that U and V are disjoint. It follows from the discussion at the beginning of the proof that U and V are open subsets of $\widehat{X}$. ■

135.5. **Cell attachments and homotopy equivalences.** In this section we will prove two results which show that the homotopy type of a cell attachment $X \cup_\varphi \overline{B}^n$ does not change if we replace φ by a homotopic map or if we replace X by a homotopy equivalent topological space.

We start out with the easier of these two results:

Lemma 135.12. (Homotopic Attaching Map Lemma) Let X be a topological space and let $\varphi_0, \varphi_1 \colon S^{n-1} \to X$ be two continuous maps. If φ_0 and φ_1 are homotopic, then there exists a homotopy equivalence $X \cup_{X \leftarrow S^{n-1}:\varphi_0} \overline{B}^n \to X \cup_{X \leftarrow S^{n-1}:\varphi_1} \overline{B}^n$ which is the identity on X.

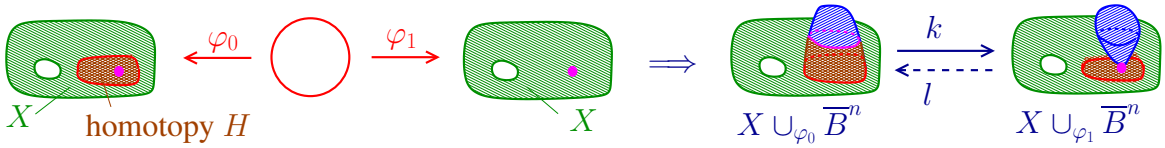


Remark. On page 2604 we saw that a cell attachment is a special case of a mapping cone. Thus the Homotopic Attaching Map Lemma 135.12 can be viewed as a special case of the Homotopic Map–Mapping Cone Proposition 21.15. For the reader's convenience we will in the following provide an independent proof of the Homotopic Attaching Map Lemma 135.12. □

Proof. Let $H\colon S^{n-1}\times[0,1]\to X$ be a homotopy between $H_0=\varphi_0$ and $H_1=\varphi_1$. We consider the map

$$\begin{array}{rcl} f\colon X\cup_{\varphi_0}\overline{B}^n & \to & X\cup_{\varphi_1}\overline{B}^n \\ [P] & \mapsto & \begin{cases} [P], & \text{if } P\in X, \\ [H(Q,2t-2)], & \text{if } P=Q\cdot t, \text{ with } Q\in S^{n-1} \text{ and } t\in[\frac{1}{2},1], \\ [Q\cdot(2t)], & \text{if } P=Q\cdot t, \text{ with } Q\in S^{n-1} \text{ and } t\in[0,\frac{1}{2}]. \end{cases} \end{array}$$

Note that it follows from the Topological-Pushout Proposition 8.12 that this map is continuous.



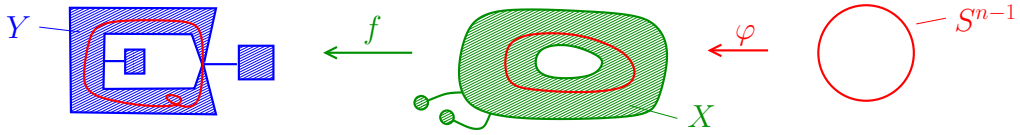
By taking the "inverse homotopy" $(z,t)\mapsto H(z,1-t)$ we also obtain a continuous map $g\colon X\cup_{\varphi_1}\overline{B}^n\to X\cup_{\varphi_0}\overline{B}^n$. It is elementary, but notationally slightly messy, to write down explicit homotopies which show that $f\circ g\sim\mathrm{id}$ and which show that $g\circ f\sim\mathrm{id}$. We leave it to the reader to fill in the details. ■

We turn to our second, and more sophisticated result of this section:

Proposition 135.13. (Homotopy Equivalent Cell Attachment Proposition) In the following let $f\colon X\to Y$ be a continuous map between topological spaces and let $\varphi\colon S^{n-1}\to X$ be an attaching map. If f is a homotopy equivalence, then the map

$$X\cup_{X\leftarrow S^{n-1}:\varphi}\overline{B}^n\ \xrightarrow{\simeq}\ Y\cup_{Y\leftarrow S^{n-1}:f\circ\varphi}\overline{B}^n$$

which is given by f on X and by id on $\overline{B}^n$ is a homotopy equivalence.



Remark. As we mentioned above, on page 2604 we saw that a cell attachment is a special case of a mapping cone. Thus the Homotopy Equivalent Cell Attachment Proposition 135.13 can be viewed as a special case of the Homotopy Equivalence-Mapping Cone Proposition 22.25, which in turn is a consequence of the daunting Homotopy Pushout Theorem 22.24. □

Proof. In the proof we will work in the homotopy category $\mathcal{HomTop}$. Recall that the objects in the homotopy category are topological spaces, but the morphism set between two topological spaces X and Y is given by the set $[X,Y]$ of homotopy equivalence classes of continuous maps $X\to Y$. We recall two essential observations:

Observation.

(1) A map $f\colon X\to Y$ is a homotopy equivalence if and only if $[f]\in[X,Y]$ is an isomorphism in the homotopy category.

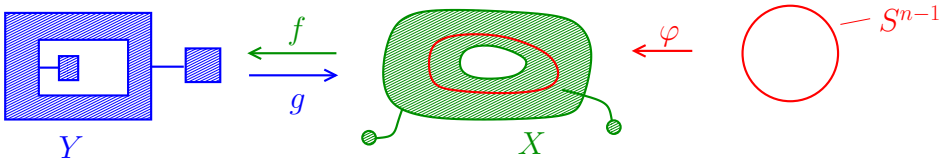
(2) If $\mathcal{C}$ is a category and if $f\colon X\to Y$ is a morphism which admits a left-inverse $a\colon Y\to X$ and a right-inverse $b\colon Y\to X$, then $a=b$. In particular f is an isomorphism.

Proof.

(1) This is just the content of the Homotopy Equivalence-Reformulation Lemma 19.13.
(2) This statement follows from (1) and the Inverse Uniqueness Lemma 5.10 (2). ⊞

After these preparations we start out with the actual proof. Thus let $f\colon X \to Y$ be a homotopy equivalence and let $\varphi\colon S^{n-1} \to X$ be an attaching map. We introduce many extra maps:

- We consider the map
$$F\colon X \cup_{X \leftarrow S^{n-1}:\varphi} \overline{B}^n \xrightarrow{\simeq} Y \cup_{Y \leftarrow S^{n-1}:f\circ\varphi} \overline{B}^n$$
which is given by f on X and by id on $\overline{B}^n$.
- Since $f\colon X \to Y$ is a homotopy equivalence there exists a homotopy inverse $g\colon Y \to X$.



- We consider the map
$$G\colon Y \cup_{Y \leftarrow S^{n-1}:f\circ\varphi} \overline{B}^n \to X \cup_{X \leftarrow S^{n-1}:g\circ f\circ\varphi} \overline{B}^n$$
which is given by g on Y and by id on $\overline{B}^n$. Note that this map is indeed well-defined.
- Since g is a homotopy inverse to f we can pick a homotopy $H\colon X \times [0,1] \to X$ between $g \circ f$ and id_X.
- Note that $H \circ \mathrm{diag}(\varphi, \mathrm{id}_{[0,1]})\colon S^{n-1} \times [0,1] \to X$ is a homotopy between $g \circ f \circ \varphi$ and φ. As in the proof of the Homotopic Attaching Map Lemma 135.12 we consider the homotopy equivalence
$$\begin{aligned} k\colon X \cup_{g\circ f\circ\varphi} \overline{B}^n &\to X \cup_\varphi \overline{B}^n \\ [P] &\mapsto \begin{cases} [P], & \text{if } P \in X, \\ [H(\varphi(Q), 2t-2)], & \text{if } P = Q \cdot t, \text{ with } Q \in S^{n-1} \text{ and } t \in [\frac{1}{2}, 1], \\ [Q \cdot (2t)], & \text{if } P = Q \cdot t, \text{ with } Q \in S^{n-1} \text{ and } t \in [0, \frac{1}{2}]. \end{cases} \end{aligned}$$

Claim. The map $k \circ G \circ F\colon X \cup_\varphi \overline{B}^n \to X \cup_\varphi \overline{B}^n$ is homotopic to the identity.

Proof. We consider the following map:
$$\begin{aligned} \Theta\colon (X \cup_\varphi \overline{B}^n) \times [0,1] &\to X \cup_\varphi \overline{B}^n \\ ([P], t) &\mapsto \begin{cases} [H(P,t)], & \text{if } P \in X, \\ \frac{2r}{1+t} \cdot v, & \text{if } P = v \cdot r \text{ with } v \in S^{n-1} \text{ and } r \in [0, \frac{1+t}{2}], \\ H(\varphi(v), 2-2r+t), & \text{if } P = v \cdot r \text{ with } v \in S^{n-1} \text{ and } r \in [\frac{1+t}{2}, 1]. \end{cases} \end{aligned}$$
The following discussion shows that this map is the desired homotopy:

- By plugging $r = \frac{1+t}{2}$ into the two expression at the bottom one sees that the bottom two definitions agree on the overlap.
- By plugging in $r = 1$ we see that the first and last definition agree on the overlap.

- It follows from the explicit definitions of F, G and k that the following equalities hold:

$$\begin{array}{lrcll} \text{(a)} & (k\circ G\circ F)(x) & = & (g\circ f)(x), & \text{for all } x\in X,\\ \text{(b)} & (k\circ G\circ F)(r\cdot v) & = & 2\cdot r\cdot v, & \text{for all } v\in S^{n-1} \text{ and } r\in[0,\tfrac{1}{2}],\\ \text{(c)} & (k\circ G\circ F)(r\cdot v) & = & H(\varphi(v),2-2r), & \text{for all } v\in S^{n-1} \text{ and } r\in[\tfrac{1}{2},1]. \end{array}$$

  One can easily verify that these equalities imply that $k\circ G\circ F=\Theta_0$.
- Clearly we have $\Theta_1=\mathrm{id}$.
- It follows easily from the Pasting Proposition 4.7 (2) and the Homotopy Pushout Lemma 18.13 that the map Θ is continuous. ⊞

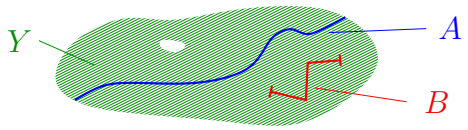
We continue our proof of the proposition with four fairly formal observations:

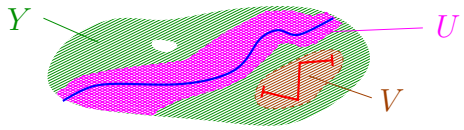
(i) It follows from the claim and Observation (1) that $[k\circ G]\circ[F]=[\mathrm{id}]$. In other words, $[F]$ has a left inverse in the homotopy category.
(ii) The same argument as in (i) shows that $[G]$ has a left inverse in the homotopy category.
(iii) By (i) we have $[k]\circ[G\circ F]=[k\circ G]\circ[F]=[\mathrm{id}]$. Since $[k]$ is invertible we obtain from Observation (2) that $[G\circ F]\circ[k]=[\mathrm{id}]$.
(iv) From (iii) we obtain that $[G]\circ[F\circ k]=[\mathrm{id}]$ and since by (ii) we know that $[G]$ has a left inverse we obtain from Observation (2) that $[F]\circ[k\circ G]=[F\circ k]\circ[G]=[\mathrm{id}]$.

By (i) and (iv) we now know that $[F]$ has a left-inverse and a right-inverse. It follows from this discussion and Observations (1) and (2) that F is indeed a homotopy equivalence. ■

135.6. **Appendix: Cell attachments and normality.** In this appendix we hide a technical proposition which, in light of Exercise 135.6, can be viewed as a generalization of the Cell Attachment–Hausdorff Proposition 135.11. Before we get to this generalization, we need to remind the reader of Definition 3.48:

Definition 135.14. A topological space Y is called normal if any two disjoint closed subsets A and B of X can be separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.





Now we can formulate the one result of this appendix.

Proposition 135.15. (Cell Attachment–Normal Proposition) Let X be a topological space and let $\{\varphi_i\colon S^{n_i-1}\to X\}_{i\in I}$ be a family of continuous maps. We consider the topological space

$$\widehat{X} \;:=\; X\cup_{X\leftarrow \bigsqcup_{i\in I} S^{n_i-1}\,:\,\bigsqcup_{i\in I}\varphi_i} \bigsqcup_{i\in I}\overline{B}^{n_i}$$

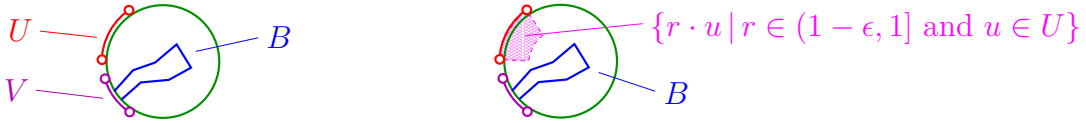
that we obtain from X by attaching cells along the attaching maps φ_i.

(1) Let $\widehat{A},\widehat{B}\subseteq\widehat{X}$ be two disjoint closed subsets. We set $A:=X\cap\widehat{A}$ and $B:=X\cap\widehat{B}$. If there exist disjoint open neighborhoods $U\subseteq X$ and $V\subseteq X$ of A and B then there exist disjoint open neighborhoods $\widehat{U}\subseteq\widehat{X}$ and $\widehat{V}\subseteq\widehat{X}$ of $\widehat{A}$ and $\widehat{B}$ with $U=X\cap\widehat{U}$ and $V=X\cap\widehat{V}$.
(2) If X is normal, then $\widehat{X}$ is also normal.

The proof of the Cell Attachment–Normal Proposition 135.15 rests on the following little lemma.

Auxiliary Lemma 135.16. Let $U, V \subseteq S^{n-1}$ be disjoint open subsets and let $B \subseteq \overline{B}^n$ be a closed subset with $B \cap S^{n-1} \subseteq V$. There exists an $\epsilon > 0$ such that

$$\{r \cdot u \,|\, r \in (1-\epsilon, 1] \text{ and } u \in U\} \cap B = \varnothing.$$



Proof of Lemma 135.16. It follows easily from the fact that $U \cap V = \varnothing$ and that V is an open neighborhood of $B \cap S^{n-1}$ that $\overline{U}$, the closure of U in S^{n-1}, is disjoint from B. By the Compact-Closed Lemma 3.39 we know that $\overline{U}$ is compact. Next let $f\colon [\frac{1}{2}, 1] \times S^{n-1} \to \overline{B}^n$ be the obvious embedding given by $f(r, v) = r \cdot v$. It follows from the Tube Lemma 7.12, applied to $\{1\} \times \overline{U}$ and the open set $f^{-1}(\overline{B}^n \setminus B)$ that there exists an $\epsilon \in (0, \frac{1}{2})$ such that $[1-\epsilon, 1] \times \overline{U} \subseteq f^{-1}(\overline{B}^n \setminus B)$. This ϵ has the desired properties. ■

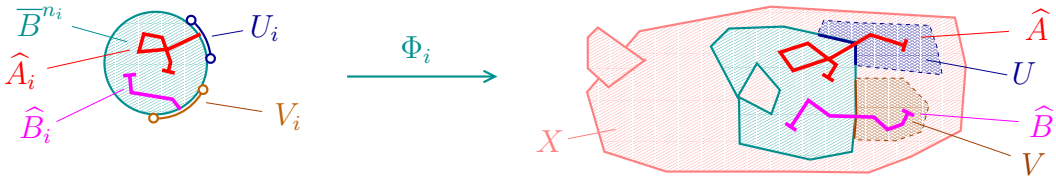
Proof of the Cell Attachment–Normal Proposition 135.15. Note that Statement (2) is an immediate consequence of Statement (1). Thus it suffices to prove Statement (1). Before we get to the actual argument let us make a few preparations. For each $i \in I$ we consider the natural map $\Phi_i\colon \overline{B}^n_i \to \widehat{X}$. It follows immediately from the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0) that given any subset $W \subseteq \widehat{X}$ the following statement holds:

$$W \text{ is open in } \widehat{X} \iff \begin{array}{ll} \text{(a)} & \text{the intersection } W \cap X \text{ is open in } X \text{ and} \\ \text{(b)} & \text{for all } i \in I \text{ the preimage } \Phi_i^{-1}(W) \text{ is open in } \overline{B}^n. \end{array}$$

Furthermore, the same references also shows that we can replace "open" by "closed".

Let $\widehat{A}, \widehat{B} \subseteq \widehat{X}$ be two disjoint closed subsets. We set $A \coloneqq X \cap \widehat{A}$ and $B \coloneqq X \cap \widehat{B}$. Furthermore let $U \subseteq X$ and $V \subseteq X$ be disjoint open neighborhoods of A and B.

Claim. Let $i \in I$. We set $U_i \coloneqq \Phi_i^{-1}(U)$, $V_i \coloneqq \Phi_i^{-1}(V)$, $\widehat{A}_i \coloneqq \Phi_i^{-1}(\widehat{A})$ and $\widehat{B}_i \coloneqq \Phi_i^{-1}(\widehat{B})$. There exist disjoint open neighborhoods $\widehat{U}_i$ and $\widehat{V}_i$ of $\widehat{A}_i \subseteq \overline{B}^{n_i}$ and $\widehat{B}_i \subseteq \overline{B}^{n_i}$ with $\widehat{U}_i \cap S^{n_i-1} = U_i$ and $\widehat{V}_i \cap S^{n_i-1} = V_i$.



Proof. We make two observations:

(a) It follows from the above Lemma 135.16 that there exists an $\epsilon > 0$ such that

$$\begin{aligned} \{r \cdot u \,|\, r \in (1-\epsilon, 1] \text{ and } u \in U_i\} \cap \widehat{B}_i &= \varnothing \\ \{r \cdot v \,|\, r \in (1-\epsilon, 1] \text{ and } v \in V_i\} \cap \widehat{A}_i &= \varnothing. \end{aligned}$$

(b) Since $\overline{B}^{n_i}$ is a metric space we know by the Metric Spaces-Normal Proposition 12.2 that $\overline{B}^{n_i}$ is normal. This implies that there exist disjoint open neighborhoods $\widetilde{U}_i$ and $\widetilde{V}_i$ of $\widehat{A}_i$ and $\widehat{B}_i$.

We set

$$\begin{aligned}\widehat{U}_i &:= (\widetilde{U}_i \cap B^{n_i}) \cup \{r \cdot u \,|\, r \in (1-\epsilon, 1] \text{ and } u \in U_i\}\\ \widehat{V}_i &:= (\widetilde{V}_i \cap B^{n_i}) \cup \{r \cdot v \,|\, r \in (1-\epsilon, 1] \text{ and } v \in V_i\}.\end{aligned}$$

One can now easily verify that $\widehat{U}_i$ and $\widehat{V}_i$ have the desired properties. ⊞

Finally we set

$$\begin{aligned}\widehat{U} &:= U \cup \bigcup_{i\in I} \Phi_i(\widehat{U}_i)\\ \widehat{V} &:= V \cup \bigcup_{i\in I} \Phi_i(\widehat{V}_i).\end{aligned}$$

It follows immediately from the discussion at the beginning of the proof that $\widehat{U}$ and $\widehat{V}$ are open subsets of $\widehat{X}$. Furthermore it is clear that they are disjoint. It is elementary to verify that $\widehat{U}$ and $\widehat{V}$ have all the other desired properties. ■

135.7. Exercises for Chapter 135.

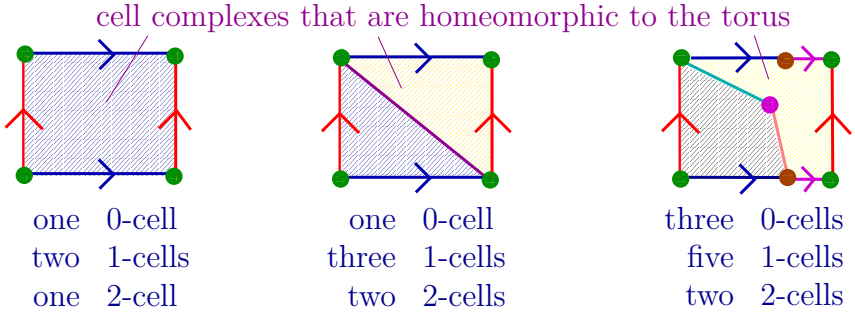
Exercise 135.1. Given $n \in \mathbb{Z}$ we consider the map $\varphi_n \colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. We use φ_n to attach a 2-cell to S^1. For which $n \in \mathbb{Z}$ is the resulting topological space $S^1 \cup_{S^1 \xleftarrow{\varphi_n} S^1} \overline{B}^2$ a topological manifold?

Exercise 135.2. We consider the attaching map $\varphi \colon S^{n-1} \to \overline{B}^n$ that is given by $x \mapsto -x$ and we use it to attach an n-cell to the n-ball $\overline{B}^n$. Is $\overline{B}^n \cup_{\overline{B}^n \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ homeomorphic to a known topological space?

Exercise 135.3. Use the Projective Space–Cell Attachment Lemma 135.7 and the Cell Attachment-π_1-Proposition 135.5 to show that $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$ and to show that for every $n \geq 2$ we have $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

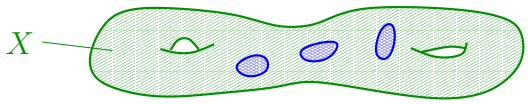
Exercise 135.4. A *cell complex* is a topological space that can be obtained from the empty topological spaces by attaching finitely many cells.

(a) In the figure below we show that we can view the torus as cell complexes in different ways with different numbers of cells. Find a non-trivial linear combination of the number of cells which is the same for each of the three cell complexes.



(b) Show that if X and Y are cell complexes, then so is $X \times Y$.

Exercise 135.5. A *cell complex* is a topological space that can be obtained from the empty topological spaces by attaching finitely many cells. Let X be the surface of genus 2 minus three open disks. Show that X is homeomorphic to a cell complex.



Exercise 135.6. Let X be a topological space and let $\{\varphi_i \colon S^{n_i} \to X\}_{i \in I}$ be a family of continuous maps. Show that if X has the property that each point defines a closed subset, then the topological space

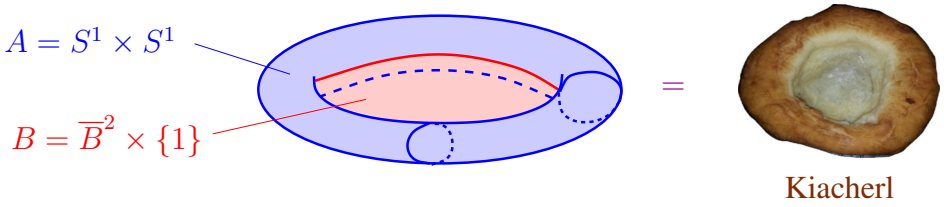
$$\widehat{X} \coloneqq X \cup_{X \leftarrow \bigsqcup_{i \in I} S^{n_i - 1} \colon \bigsqcup_{i \in I} \varphi_i} \bigsqcup_{i \in I} \overline{B}^{n_i}$$

has the same property.

Exercise 135.7. We consider

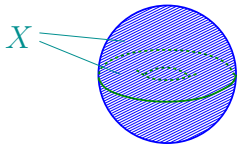
$$K \coloneqq \underbrace{(S^1 \times S^1)}_{=:A} \cup \underbrace{(\overline{B}^2 \times \{1\})}_{=:B},$$

i.e. K is given by a torus where the "central hole" is filled in with a disk. Determine the fundamental group of K.



Exercise 135.8. We consider the Klein bottle K. What is the minimal number of 2-cells one needs to attach to K to obtain a simply connected topological space?

Exercise 135.9. We consider the topological space X that is given by the union of S^2 with the torus $S^1 \times S^1$ such that they touch exactly on the equatorial circle. Determine $\pi_1(X)$.



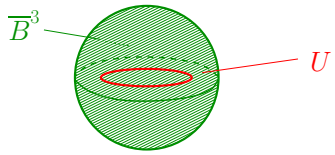
Exercise 135.10. Let X be a compact 2-dimensional topological manifold and let $c \in \mathbb{N}_0$. We say that X is *c-colorable* if for every CW-structure one can find a map $\varphi \colon \mathrm{Cell}_n(X) \to \{1, \ldots, c\}$ such that for every two n-cells A and B with $A \cap B \cap X^1 \neq \varnothing$ we have $\varphi(A) \neq \varphi(B)$. The chromatic number $c(X)$ of X is defined as the minimal c such that every CW-structure for X is c-colorable. The famous Four Color Theorem of Kenneth Appel and Wolfgang Haken [**ApH1976**] says that $c(S^2) = 4$.

(a) Can you determine the chromatic number of the annulus?
(b) Show that $c(\mathbb{R}\mathrm{P}^2) \geq 5$.
(c) Show that $c(S^1 \times S^1) \geq 7$.

Exercise 135.11. We consider the trivial knot $U := \{(x, y, 0) \,|\, x +^2 +y^2 = \frac{1}{2}\} \subseteq \overline{B}^3$.

(a) Show that $\overline{B}^3 \setminus U$ contains a 2-dimensional cell complex X such that X is a deformation retract of $\overline{B}^3 \setminus U$.

(b) Show that $\overline{B}^3 \setminus U$ is homotopy equivalent to $S^1 \vee S^2$.



CHAPTER 136

CW-complexes: Definitions and basic properties

In this chapter we introduce CW-complexes, we will see that many popular topological spaces have the structure of a CW-complex and we discuss basic properties of CW-complexes. The notion of a CW-complex will play an essential role throughout these lecture notes.

This chapter is organized as follows:

- In Section 136.1 we will define CW-complexes and we will provide some examples.
- In Section 136.2 we will introduce attaching maps and characteristic maps of cells.
- In Section 136.3 we will introduce subcomplexes of CW-complexes.
- In Section 136.4 we will prove the Finiteness Theorem 136.16 for CW-complexes.
- In Section 136.5 we will discuss fundamental groups of CW-complexes.
- In Section 136.6 we will discuss CW-complexes and direct colimits
- In Appendix 136.7 we will prove that CW-Complexes are normal and paracompact.
- In Section 136.8 we will provide exercises for Chapter 136.

In the next Chapter 137 we will discuss homotopies on CW-complexes.

136.1. **Definition of CW-complexes and examples.** After the long discussion of cell attachments in the last chapter we are now mentally ready for the all-important notion of a CW-complex:

Definition 136.1. A CW-complex X[1405 1406] is a topological space X together with a filtration

$$\varnothing = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \ldots \subseteq X$$

by subspaces which has the following properties:

(CW1) We have $X = \operatorname{colim} X^n$, as described in the Topological-Direct Colimit Proposition 104.10. In other words, as a set we have $X = \bigcup_{n\in\mathbb{N}_0} X^n$ and $U \subseteq X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is open in X^n.

(CW2) For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism[1407 1408]

$$\Xi_n \colon X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

A CW-structure on a topological space X is a filtration as above.

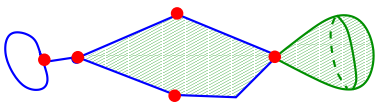
Before we discuss examples of CW-complexes it is useful to already introduce some extra language:

Definition 136.2. Using the notation from above we introduce the following language:

(1) Given $n \in \mathbb{Z}_{\geq -1}$ we refer to X^n as the n-skeleton of X.
(2) The components of $X^n \setminus X^{n-1}$ are called the open n-cells of X.[1409 1410]
(3) For each open n-cell σ of X we refer to its closure $\overline{\sigma}$ in X as an n-cell of X.
(4) We set
$$\dim(X) \;:=\; \sup\{n \in \mathbb{Z}_{\geq -1} \,|\, X^{n-1} \neq X^n\} \;\in\; \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$
(5) We say that the CW-complex is finite if it has only finitely many cells and we say that it is countable if it has only countably many cells.

Illustration. In the figure below we show a 2-dimensional CW-complex X with its filtration $X^0 \subseteq X^1 \subseteq X^2$.

CW-complex X



X^0 five 0-cells
$X^1 \setminus X^0$ six 1-cells
$X^2 \setminus X^1$ two 2-cells

In the next figure we indicate how we obtain the 2-skeleton from the 1-skeleton by gluing on 2-disks.
These examples should already convince the reader that CW-complexes cover many types of topological spaces. □

Our long list of examples of cell attachments from page 2605 provides us, almost for free, with a long list of CW-complexes.

Examples.

(1) The one type of CW-complexes that is easy to understand are 0-dimensional CW-complexes. Similar to the discussion on page 2604 it follows from the observation that $\overline{B}^0$

[1405]CW-complexes were introduced by John H. C. Whitehead[1406]. The name comes from "C" for "closure-finite", and the "W" for "weak topology":

(1) "Closure finite" means that the closure of each open cell is contained in the union of finitely many open cells. It follows from the Contained-in-Finite Subcomplex Lemma 136.18 (3) and the Cell-Compact Proposition 136.8 (2) that this is in fact the case.
(2) "Weak topology" means that a subset is open if and only if each intersection $U \cap X^n$ is open.

Some people also say that the name CW goes back to the initials C. W. of Whitehead.

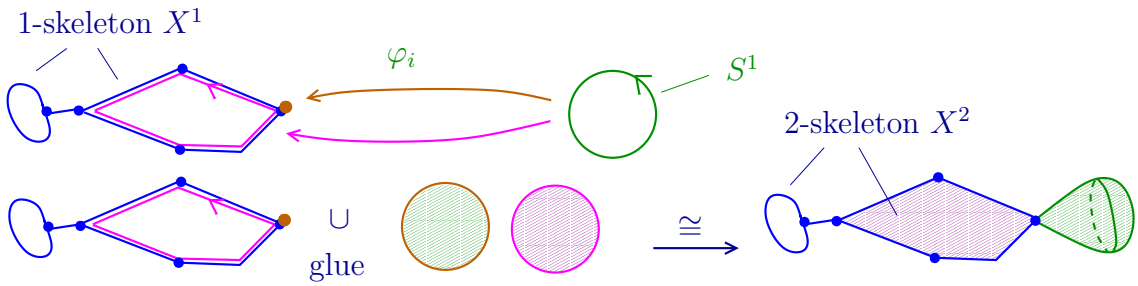
[1406]At first glance the definitions of CW-complexes that appear in the literature look widely different. Some differences just stem from different notation. Some authors, see e.g. [**Lee2000**, p. 132] and [**Mun1984**, p. 214] require "closure-finiteness" as part of the definition, but as we will see in the Contained-in-Finite Subcomplex Lemma 136.18 (3) and the Cell-Compact Proposition 136.8 (2), this property follows from the definition as given. One could also make the case that the definitions of CW-complexes in some references are a little vague.

[1407]In other words, X^n is obtained from attaching n-cells to X^{n-1} along the maps φ_i.

[1408]Note that the φ_c and the homeomorphism are *not* part of the data of a CW-complex.

[1409]It follows from the Disjoint Union Topology-Properties Lemma 7.28 (2b) and the Topological Pushout Proposition 8.12 (0) that, in the notation of (CW2), the open cells are precisely the images of the $B^n_i = B^n$ and that each open n-cell is an open subset of X^n.

[1410]It follows almost immediately from the definitions that, *as a set*, any CW-complex is the disjoint union of all the open cells.

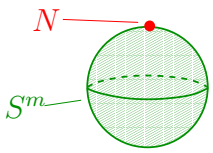


consists of a point and that $S^{-1} = \varnothing$ that a CW-complex of dimension ≤ 0 is essentially the same as a set of points with the discrete topology.

(2) Let $m \in \mathbb{N}_0$. As usual we denote by $N := (0, \ldots, 0, 1)$ the North Pole of S^m. We consider the filtration $\varnothing \subseteq \{N\} \subseteq S^m$. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since the m-skeleton equals S^m.

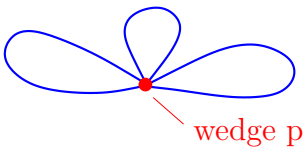
(CW2) The case $n = 0$ is clear since $\{N\}$ has the discrete topology. For $n = 1, \ldots, m-1$ there is nothing to show. Finally note that on page 2605 we showed that we obtain S^m from $\{N\}$ by attaching an m-cell.



the filtration $\varnothing \subseteq \{N\} \subseteq S^m$ is a CW-structure with
one 0-cell
one m-cell

(3) Let $n \in \mathbb{N}_0$ and let $X = \bigvee_{i=1}^{n} S^1$ be the wedge of n circles with wedge point $*$. The discussion on page 2605 implies that the filtration $\varnothing \subseteq \{*\} \subseteq \bigvee_{i=1}^{n} S^1$ is a CW-complex with one 0-cell and n 1-cells.

$X = S^1 \vee S^1 \vee S^1$



the filtration $\varnothing \subseteq \{*\} \subseteq S^1 \vee S^1 \vee S^1$
is a CW-complex with
one 0-cell
three 1-cells

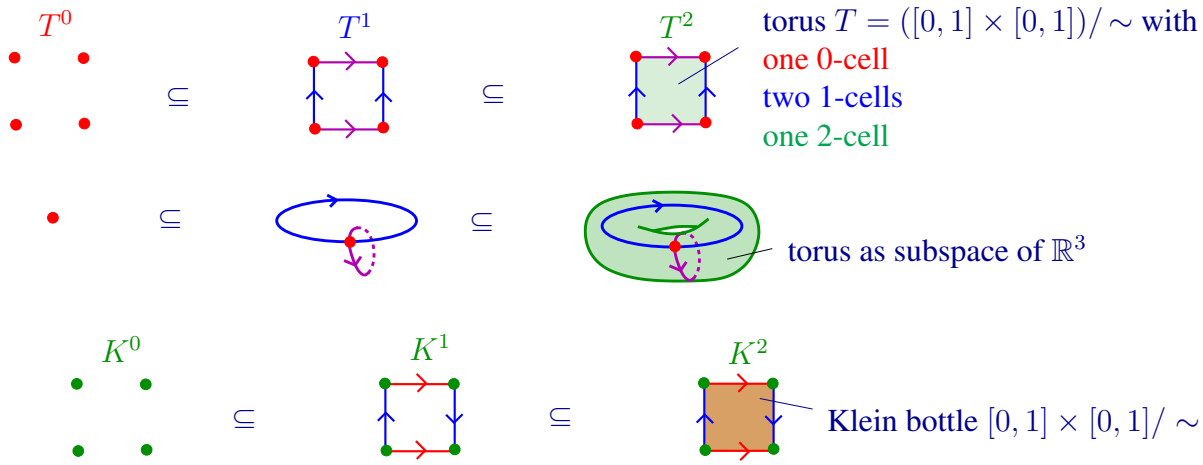
(4) The figure below shows a filtration of the torus $T = ([0,1] \times [0,1])/\sim$ that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since $T^2 = T$.

(CW2) The case $n = 0$ is clear since T^0 has the discrete topology. For $n = 1$ note that T^1 is clearly homeomorphic to the wedge of two circles along the unique point of T^0. Thus we obtain from the previous example that (CW2) is satisfied for $n = 1$. Finally note that on page 2605 we showed that we obtain T from attaching a 2-cell to $T^1 \cong S^1 \vee S^1$.

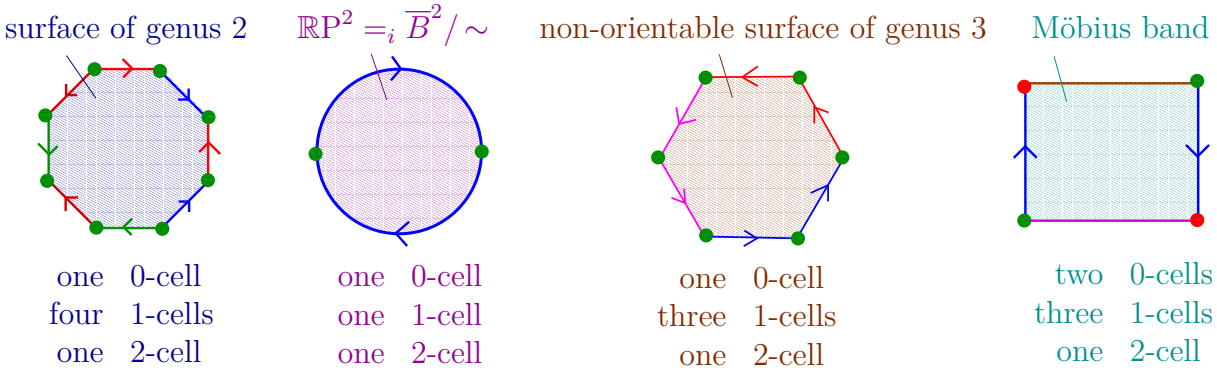
For fun we also show the filtration if we view the torus as a subspace of $\mathbb{R}^3$:

(5) The figure below shows a filtration of the Klein bottle that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. We see that the two CW-structures of the torus



and the Klein bottle have the same 1-skeleton, but the 2-cells have different attaching maps.

(6) In the figure below we sketch CW-structures for the surface of genus 2, the projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$, the non-orientable surface of genus 3 and the Möbius band as CW-complexes.



Generalizing these examples one can easily show, using the Surface Classification Theorem 101.16, that every compact 2-dimensional topological manifold admits a CW-structure. □

So far we only considered CW-complexes with finitely many cells and for the most part we just recycled material from the previous chapter. We now move on to CW-complexes with infinitely many cells:

Examples.

(1) (a) Let us consider the topological space $\mathbb{R}$. We equip it with the filtration $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$. We consider the family $\{\varphi_i \colon S^0 \to \mathbb{Z}\}_{i\in\mathbb{Z}}$ of maps given by $\varphi_i(\epsilon) = i + \frac{1+\epsilon}{2}$. Using the Cover-Open Subset Lemma 3.20 (2) one can verify that the map

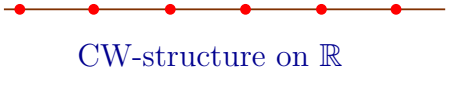
$$\begin{aligned} \mathbb{Z} \cup_{\mathbb{Z} \leftarrow \bigsqcup_{i\in\mathbb{Z}} S_i^0 \,:\, \bigsqcup_{i\in\mathbb{Z}} \varphi_i} \bigsqcup_{i\in\mathbb{Z}} \overline{B}_i^1 \;&\to\; \mathbb{R} \\ P \;&\mapsto\; \begin{cases} n, & \text{if } P = [n] \text{ for some } n \in \mathbb{Z} \\ i + \frac{1+t}{2}, & \text{if } P = [(t,i)] \text{ for some } t \in \overline{B}^1 = [-1,1] \text{ and } i \in \mathbb{Z} \end{cases} \end{aligned}$$

is a homeomorphism. Thus the filtration $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$ is a CW-structure on $\mathbb{R}$.

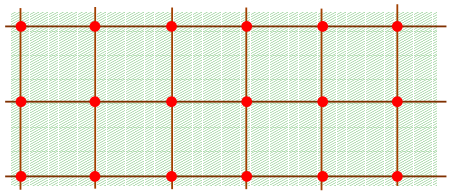
(b) Using the Convex-to-Ball Proposition 4.28 one can easily generalize the above example and one can show that the subspaces

$$X^i := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \text{at least } n-i \text{ coordinates lie in } \mathbb{Z}\} \text{ with } i = 0, \ldots, n$$

form a CW-structure on $\mathbb{R}^n$.



CW-structure on $\mathbb{R}$

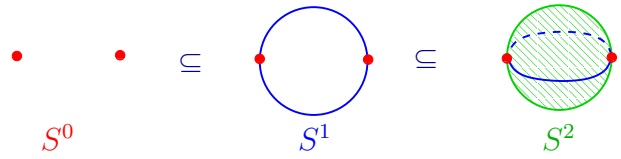


CW-structure on $\mathbb{R}^2$

(2) Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. We consider the wedge $\bigvee_{i \in I} S_i^{n_i}$. Given $n \in \mathbb{N}_0$ we denote by X_n the wedge of all spheres of dimension $\leq n$. It follows from the discussion on page 2613, the Iterative Cell-Attachment Lemma 135.10 and the Wedge–Subspace Lemma 96.3 that the filtration $\varnothing \subseteq X_0 \subseteq X_1 \subseteq \ldots$ is a CW-structure on $\bigvee_{i \in I} S_i^{n_i}$. □

Next we want to consider the infinite families S^n, $\mathbb{R}\mathrm{P}^n := (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ and $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$. To study these family we introduce, with great reluctance, the following convention.

Convention 136.3. For $k < n$ we use in the following discussion the embedding $S^k \to S^n$ given by $x \mapsto (x, 0)$ to think of S^k as a subspace of S^n. Similarly we view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$ and we view $\mathbb{C}\mathrm{P}^k$ as a subspace of $\mathbb{C}\mathrm{P}^n$.



It is natural to try to find CW-structures which are "compatible" with these inclusions. For example we could try to find CW-complexes such that the "smaller family members" are "subcomplexes". This leads us to the following definition:

Definition 136.4. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subseteq X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.

(2) Each open n-cell of A is also an open n-cell of X.[1411]

(For example, it follows immediately from the definitions that any skeleton is a subcomplex.)

Examples.

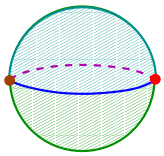
[1411] In the CW-Subcomplex–Definition Lemma 170.13 we will show, using the Topological Invariance-of-Domain Theorem 170.9, that Condition (2) follows from Condition (1). But the proof of the Topological Invariance-of-Domain Theorem 170.9 requires deep results on singular homology, that we have not proved yet.

(1) We consider the spheres S^n.

(a) On page 2625 we found a "tiny" CW-structure for S^n with just one 0-cell and one n-cell. But this CW-structure does not have any interesting subcomplexes. To fix this, recall that on page 2613 we saw that we obtain S^{k+1} from S^k by attaching two $(k+1)$-cells. It follows that a CW-structure for S^n is given by

$$\varnothing \subseteq S^0 \subseteq S^1 \subseteq S^2 \subseteq \ldots \subseteq S^n.$$

With this CW-structure we can view each S^k with $k < n$ as a subcomplex of S^n.



the filtration $\varnothing \subseteq S^0 \subseteq S^1 \subseteq S^2$ is a CW-structure with
two 0-cells
two 1-cells
two 2-cells

(b) The new CW-structure on spheres gives us an immediate bonus: As in Definition 104.16 we consider the topological space

$$S^\infty := \left\{(x_1, x_2, \ldots) \in \mathbb{R}^\infty \,\middle|\, \sum_{i=1}^{\infty} x_i^2 = 1\right\} = \operatorname*{colim} S^{i-1}.$$

It follows immediately from (a) and the definitions that a CW-structure for S^∞ is given by

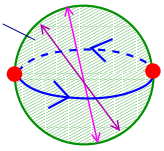
$$\varnothing \subseteq S^0 \subseteq S^1 \subseteq S^2 \subseteq \ldots \subseteq S^\infty.$$

(2) We consider the real projective spaces $\mathbb{R}\mathrm{P}^n$.

(a) In the Projective Space–Cell Attachment Lemma 135.7 (1) we showed that we can obtain $\mathbb{R}\mathrm{P}^{k+1}$ from $\mathbb{R}\mathrm{P}^k$ by attaching a single $(k+1)$-cell. It follows that a CW-structure for $\mathbb{R}\mathrm{P}^n$ is given by

$$\varnothing \subseteq \mathbb{R}\mathrm{P}^0 \subseteq \mathbb{R}\mathrm{P}^1 \subseteq \mathbb{R}\mathrm{P}^2 \subseteq \ldots \subseteq \mathbb{R}\mathrm{P}^n$$

and that this CW-structure has precisely one cell in dimensions $0, 1, 2, \ldots, n$.

$\mathbb{R}\mathrm{P}^2 =_i S^2/\sim$ 

the filtration $\varnothing \subseteq \mathbb{R}\mathrm{P}^0 \subseteq \mathbb{R}\mathrm{P}^1 \subseteq \mathbb{R}\mathrm{P}^2$ is a CW-structure with
one 0-cell
one 1-cell
one 2-cell

(b) As in Definition 104.21 we consider $\mathbb{R}\mathrm{P}^\infty = \operatorname*{colim} \mathbb{R}\mathrm{P}^i$. As in (1b) we see that

$$\varnothing \subseteq \mathbb{R}\mathrm{P}^0 \subseteq \mathbb{R}\mathrm{P}^1 \subseteq \mathbb{R}\mathrm{P}^2 \subseteq \ldots \subseteq \mathbb{R}\mathrm{P}^\infty$$

is a CW-structure which has precisely one cell in each dimension.

(3) Finally we consider the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

(a) As in (2a) we see, this time using the Projective Space–Cell Attachment Lemma 135.7 (2), that a CW-structure for $\mathbb{C}\mathrm{P}^n$ is given by[1412]

$$\varnothing \subseteq \underset{\substack{\uparrow\\ \text{0-skeleton}}}{\mathbb{C}\mathrm{P}^0} \subseteq \underset{\substack{\uparrow\\ \text{2-skeleton}}}{\mathbb{C}\mathrm{P}^1} \subseteq \underset{\substack{\uparrow\\ \text{4-skeleton}}}{\mathbb{C}\mathrm{P}^2} \subseteq \ldots \subseteq \underset{\substack{\uparrow\\ 2n\text{-skeleton}}}{\mathbb{C}\mathrm{P}^n}$$

and that this CW-structure has precisely one cell in dimensions $0, 2, 4, \ldots, 2n$.

(b) As in Definition 104.21 we consider $\mathbb{C}\mathrm{P}^\infty = \operatorname*{colim} \mathbb{C}\mathrm{P}^i$ and we see that

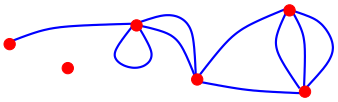
$$\varnothing \subseteq \mathbb{C}\mathrm{P}^0 \subseteq \mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2 \subseteq \ldots \subseteq \mathbb{C}\mathrm{P}^\infty$$

is a CW-structure which has precisely one cell in each even dimension. □

[1412] For better readability we suppress the odd skeleta from the notation.

The following lemma shows that topological graphs and CW-complexes of dimension ≤ 1 are essentially the same.

Lemma 136.5. (Graph–CW-Complex Lemma)
(1) If $(X, G = (V, E, \theta), \Theta\colon |G| \to X)$ is a topological graph, in the sense of Definition 11.18, then $\varnothing \subseteq V \subseteq X$ is a CW-structure.
(2) Let $\varnothing \subseteq X^0 \subseteq X^1 = X$ be a CW-complex of dimension ≤ 1. There exists an abstract graph (V, E, θ) and a homeomorphism $f\colon |G| \to X$ such that $f(V) = X^0$.



vertices form the 0-skeleton

edges correspond to 1-cells

Sketch of proof.

(1) This statement follows basically immediately from the definitions.
(2) By definition of a CW-complex there exists a set C, a family $\{\varphi_c\colon S^0 \to X^0\}_{c\in C}$ of continuous maps and a homeomorphism

$$\Xi\colon X^0 \cup_{X^0 \leftarrow \bigsqcup_{c\in C} S^0_c \,:\, \bigsqcup_{c\in C} \varphi_c} \bigsqcup_{c\in C} \overline{B}^1_c \;\to\; X^1 = X$$

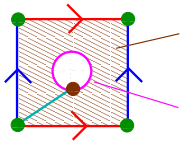
which is the inclusion on X^0. Given $c \in C$ we set $\theta(c) \coloneqq \{\varphi_c(-1), \varphi_c(1)\} \subseteq \mathcal{P}(X^0)$. It is clear that $G \coloneqq (X^0, C, \theta)$ is an abstract graph. The map Ξ gives rise to a homeomorphism $f\colon |G| \to X$ with $f(V) = X^0$. ■

We conclude our long list of examples of CW-complexes with a sneak preview of a later result.

Theorem 179.9. (Smooth Manifold-CW-Theorem) Every compact connected n-dimensional smooth manifold M admits a finite CW-structure with the following properties:
(1) The maximal dimension of a cell is n.
(2) The boundary ∂M is a subcomplex.

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW-structure on M which has all the above properties and such that A and B are subcomplexes.

Example. In the figure below we consider the smooth manifold M that is given by removing an open disk from a torus.



M is the torus $[0,1] \times [0,1]/\sim$ with an open disk removed

∂M is a subcomplex with one 0-cell and one 1-cell

We see that M has a CW-structure such that the boundary ∂M is a subcomplex. □

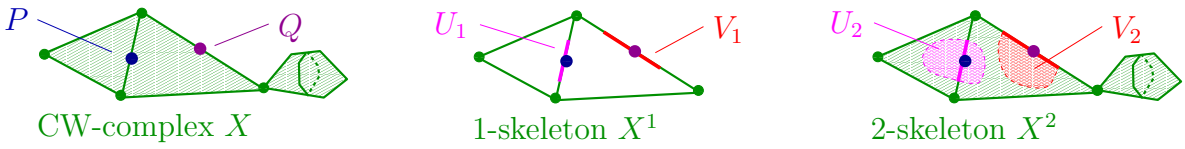
136.2. **Attaching maps and characteristic maps.** Before we continue in developing our language for CW-complexes it is essential to get the following proposition out of the way.

Proposition 136.6. (CW-Complex–Hausdorff Proposition) Every CW-complex is Hausdorff.

Proof. Let X be a CW-complex and let P, Q be two distinct points in X. Recall that $X = \bigcup_{n\in\mathbb{N}_0} X^n$. Thus there exists an $n \in \mathbb{N}_0$ with $P, Q \in X^n$.

Since the empty topological space is Hausdorff we obtain from iteratively applying the Cell Attachment–Hausdorff Proposition 135.11 (2) that $X^0, \ldots, X^n$ are Hausdorff. In particular we see that there exist disjoint open neighborhoods U_n of $P \in X^n$ and V_n of $Q \in X^n$.

It follows from iteratively applying the Attachment–Hausdorff Proposition 135.11 (1) that for each $d \geq n$ we can find disjoint open neighborhoods U_d of $P \in X^d$ and V_d of $Q \in X^d$ such that for $d \geq n$ we have $U_d \subseteq U_{d+1}$ and $V_d \subseteq V_{d+1}$.



It remains to prove the following claim:

Claim. The sets $U := \bigcup_{d\geq n} U_d$ and $V := \bigcup_{d\geq n} V_d$ are disjoint open neighborhoods of P and Q in X.

Proof. Note that for each $d \geq n$ we have $(U \cap X^d) \cap (V \cap X^d) = U^d \cap V^d = \varnothing$. This shows that U and V are disjoint. It follows immediately from the Direct Colimit–Open Subset Lemma 104.14 that U and V are open subsets of X. ∎

Before we can continue with the theory of CW-complexes we need to introduce the following definition:

Definition 136.7. Let X be a CW-complex

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X. Furthermore we denote by $\mathrm{Cell}(X) := \bigcup_{n\in\mathbb{N}_0} \mathrm{Cell}_n(X)$ the set of all cells of X.

(2) Let $c \in \mathrm{Cell}_n(X)$ be an n-cell of X.

(a) An attaching map for c is a continuous map $\varphi\colon S^{n-1} \to X^{n-1}$ such that there exists a homeomorphism

$$X^{n-1} \cup_{X^{n-1} \leftarrow S^{n-1} :\varphi} \overline{B}^n \quad \to \quad \underbrace{X^{n-1} \cup c}_{\text{subspace of } X^n}$$

which is the identity on X^{n-1}.

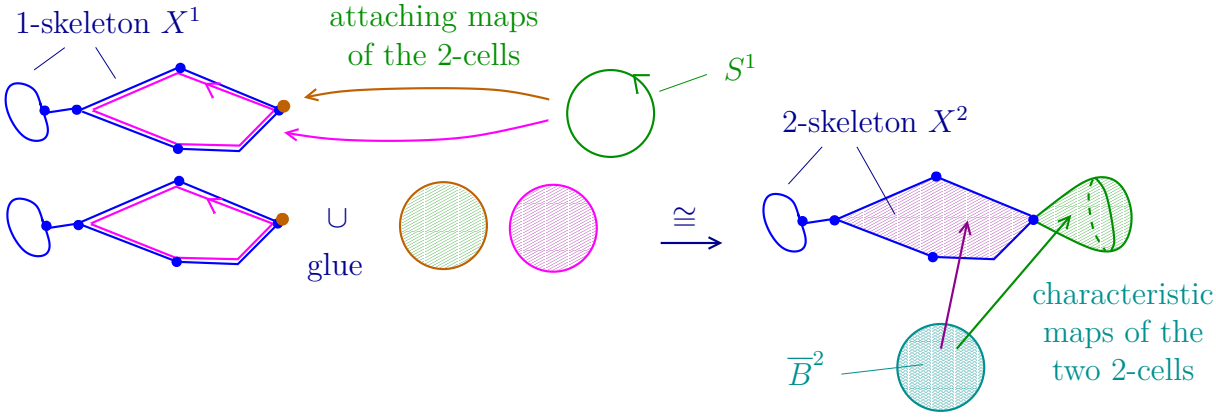
(b) A characteristic map for c is a continuous map $\Phi\colon \overline{B}^n \to X$ such that the map

$$X^{n-1} \cup_{X^{n-1} \leftarrow S^{n-1} :\Phi|_{S^{n-1}}} \overline{B}^n \quad \to \quad X^{n-1} \cup c$$

induced by the identity on X^{n-1} and by the map Φ on $\overline{B}^n$ is well-defined and it is a homeomorphism.

It follows from the Iterative Cell-Attachment Lemma 135.10 (1), applied to $\{c\} \subseteq \mathrm{Cell}_n(X)$ that a characteristic map, and thus also an attaching map, always exists.[1413]

(3) A full set of attaching maps for the cells of the CW-complex X is defined as a family of continuous maps $\{\varphi_c \colon S^{\dim(c)-1} \to X^{\dim(c)-1}\}_{c\in\mathrm{Cell}(X)}$ such that each φ_c is an attaching map for the cell c. Similarly we define a full set of characteristic maps for the n-cells of X.



Using the CW-Complex–Hausdorff Proposition 136.6 we can prove the following essential proposition.

Proposition 136.8. (Cell-Compact Proposition) Let X be a CW-complex.
(1) Let c be an n-cell of X. If $\Phi \colon \overline{B}^n \to X$ is a characteristic map for c, then $\Phi(\overline{B}^n) = c$.
(2) Every cell is compact.

Proof. Let c be an n-cell of X.

(1) Let $\Phi \colon \overline{B}^n \to X$ be a characteristic map for c. We start out with the following claim:

Claim. The image $\Phi(\overline{B}^n)$ is a closed subset of X.

[1413] Let us elaborate a bit. Let X be a CW-complex. By definition we know that for each $n \in \mathbb{N}_0$ there exists a family $\{\varphi_d \colon S^{n-1} \to X^{n-1}\}_{c\in\mathrm{Cell}_n(X)}$ of continuous maps and a homeomorphism

$$\Xi_n \colon X^{n-1} \cup_{X \leftarrow \bigsqcup_d S^{n-1}_d \,:\bigsqcup_d \varphi_d} \bigsqcup_{d\in\mathrm{Cell}_n(X)} \overline{B}^n_d \;\to\; X^n$$

which is the inclusion on X^{n-1}. For the n-cell $c \in \mathrm{Cell}_n(X)$ we consider the map

$$\begin{array}{rcll} \Phi_c \colon \overline{B}^n & \to & X^{n-1} \cup c & \\ x & \mapsto & \Xi_n([(x,c)]) & \text{where } (x,c) \in \overline{B}^n_c. \end{array}$$

It follows from the Topological-Pushout Proposition 8.12 (1a) and the Topological-Coproduct Proposition 7.26 (1a) that this map is continuous. It follows from the Iterative Cell-Attachment Lemma 135.10 (1) that the map

$$X^{n-1} \cup_{X^{n-1} \leftarrow S^{n-1} \,:\varphi_c} \overline{B}^n \;\to\; \underbrace{X^{n-1} \cup c}_{\text{subspace of } X^n}$$

which is induced by $\mathrm{id}_{X^{n-1}}$ and the map Φ_c on $\overline{B}^n$ is a homeomorphism. It follows immediately from this observation that Φ_c is a characteristic map for the cell c and that φ_c is an attaching map for the cell c.

Proof. Since characteristic maps are continuous and since $\overline{B}^n$ is compact we obtain from the Compact Image Lemma 4.15 that $\Phi(\overline{B}^n)$ is compact. By the CW-Complex–Hausdorff Proposition 136.6 we know that X is Hausdorff. Thus it follows from the Compact-Closed Lemma 3.39 (2) that $\Phi(\overline{B}^n)$ is a closed subset of X. ⊞

Note that we have the following inclusions

$$\Phi(\overline{B}^n) \underset{\text{by the Closure-Interior Mapping Lemma 4.6}}{\subseteq} \underbrace{\overline{\Phi(B^n)}}_{=c=\overline{f}} \underset{\text{by Definition 3.21 the closure}}{=} \bigcap \text{closed subsets containing } \Phi(B^n) \underset{\text{by the claim}}{\subseteq} \Phi(\overline{B}^n).$$

It follows that $\overline{\Phi(B^n)} = \Phi(\overline{B}^n)$. Next recall that by definition of an n-cell there exists an open n-cell $f \subseteq X$ with $\overline{f} = c$. It follows easily from the definition of a characteristic map that the restriction of Φ to B^n defines a homeomorphism $\Phi\colon B^n \to f$. It follows from the above that we have $c = \overline{f} = \overline{\Phi(B^n)} = \Phi(\overline{B}^n)$.

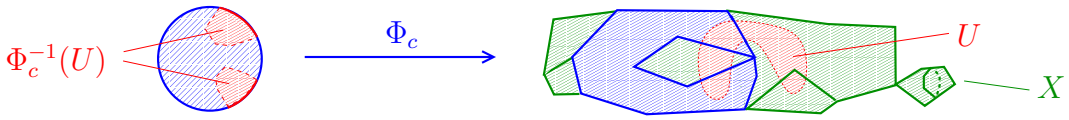
(2) This statement follows from (1) and the Compact Image Lemma 4.15. ■

Proposition 136.9. (Characteristic Maps Proposition) Let X be a CW-complex. Furthermore let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the cells of X.

(1) Given any subset $U \subseteq X$ the following statements are equivalent:
   (a) U is open in X.
   (b) For every $c \in \mathrm{Cell}(X)$ the intersection $c \cap U$ is open in c.
   (c) For every $c \in \mathrm{Cell}(X)$ the preimage $\Phi_c^{-1}(U)$ is open in $\overline{B}^{\dim(c)}$.

   The same statements also hold if we replace "open" by "closed".

(2) Let $f\colon X \to Y$ be a map to a topological space Y. The following statements are equivalent:
   (a) f is continuous.
   (b) For every $c \in \mathrm{Cell}(X)$ the restriction of f to c is continuous.
   (c) For every $c \in \mathrm{Cell}(X)$ the map $f \circ \Phi_c\colon \overline{B}^{\dim(c)} \to Y$ is continuous.



Proof. Let X be a CW-complex. We write $C := \mathrm{Cell}(X)$ and given $n \in \mathbb{N}_0$ we write $C_n := \mathrm{Cell}_n(X)$. By definition of a CW-complex we know that for each $n \in \mathbb{N}_0$ there exists a family $\{\varphi_c\colon S^{n-1} \to X^{n-1}\}_{c\in C_n}$ of continuous maps and a homeomorphism

$$\Xi_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_c S_c^{n-1} \,:\, \bigsqcup_c \varphi_c} \bigsqcup_{c\in C_n} \overline{B}_c^n \;\to\; X^n$$

which is the inclusion on X^{n-1}. We start out with the following technical claim:

Claim. For each $n \in \mathbb{N}_0$ the map

$$\Omega_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C} S_c^{n-1} \,:\, \bigsqcup_{c\in C} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C} \overline{B}_c^n \;\to\; X^n$$

induced by the identity on X^{n-1} and the maps $\Phi_c\colon \overline{B}_c^n \to X^n$ is well-defined and it is a homeomorphism.

Proof. We consider the map

$$\underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S^{n-1}_c \,:\, \bigsqcup_{c\in C_n} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C_n} \overline{B}^n_c}_{=:\widetilde{S}} \xrightarrow{\Omega_n} X^n \xrightarrow[\cong]{\Xi_n^{-1}} \underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S^{n-1}_c \,:\, \bigsqcup_{c\in C_n} \varphi_c} \bigsqcup_{c\in C_n} \overline{B}^n_c}_{=:S}.$$

Since Ξ_n is a homeomorphism it suffices to show that $\Xi_n^{-1} \circ \Omega_n \colon \widetilde{S} \to S$ is a homeomorphism.

First note that one can easily verify that $\Xi_n^{-1} \circ \Omega_n \colon \widetilde{S} \to S$ is a bijection. Since each Φ_c is continuous we obtain from the Topological-Pushout Proposition 8.12 (1b) and the Topological-Coproduct Proposition 7.26 (1b) that $\Xi_n^{-1} \circ \Omega_n \colon \widetilde{S} \to S$ is continuous.

Given $c \in C_n$ we denote by $\widetilde{c} \subseteq \widetilde{S}$ the image of $\overline{B}^n_c$ on the left-hand side. Note that $\Xi_n^{-1} \circ \Omega_n \colon \widetilde{c} \to c$ is a continuous bijection. By the Compact Image Lemma 4.15 we know that $\widetilde{c}$ is compact and by the CW-Complex–Hausdorff Proposition 136.6 we know that c is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 4.23 (3) that $\Xi_n^{-1} \circ \Omega_n \colon \widetilde{c} \to c$ is a homeomorphism. We denote by $g_c \colon c \to \widetilde{c}$ its inverse. Furthermore we denote by $p_c \colon \overline{B}^n_c \to S$ the natural map. Since each $g_c \circ p_c$ is continuous we obtain from the Topological-Pushout Proposition 8.12 (1b) and the Topological-Coproduct Proposition 7.26 (1b) that $\Omega_n^{-1} \circ \Xi_n \colon S \to \widetilde{S}$ is continuous. ⊞

Now we turn to the actual proof of the statements of the proposition.

(1) Let $U \subseteq X$ be a subset. The (a)⇒(b)-direction follows from the definition of the subspace topology and the (b)⇒(c)-direction follows from the continuity of the attaching maps. It remains to deal with the (c)⇒(a)-direction. Thus we assume that U satisfies (c). It follows from the claim and the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0), applied iteratively for $n = 0, 1, 2, \ldots$, that each $X^n \cap U$ is open. But by definition of a CW-complex this means that U is open in X.

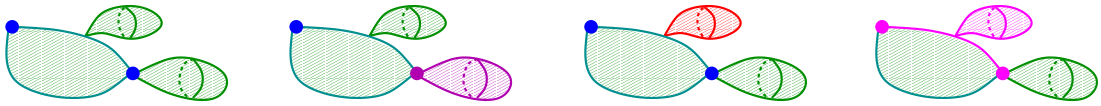
(2) This statement follows immediately from (1). ■

136.3. **Subcomplexes.** We start out by recalling Definition 136.4:

Definition 136.10. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subseteq X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.

(2) Each open n-cell of A is also an open n-cell of X.



Example. If X is a CW-complex, then any skeleton X^n, with the obvious filtration, is basically by definition a subcomplex of X. □

The above definition of a subcomplex is short and succinct. But in practice it is not always that easy to work with it. In the following lemma we first show that subcomplexes are always closed subsets. Afterwards we will give a convenient characterization of subcomplexes, which in practice is quite helpful.

Lemma 136.11. (Subcomplex–Closed Lemma) Each subcomplex of a CW-complex X is a closed subset of X.

Proof. Let X be a CW-complex and let A be a subcomplex of X. We start out with the following claim:

Claim 1. Let a be an open n-cell of A. The closure of a in A equals the closure of a in X.

Proof. Let a be an open n-cell of A. By definition of a subcomplex we know that a is also an open n-cell of X. Let $\Phi_a : \overline{B}^n \to A^n$ be a characteristic map of the n-cell $\overline{a}$ of A. Since the subspace topology on $A^{n-1} \cup a$ agrees with the subspace topology coming from $X^{n-1} \cup a$ we see that Φ_a is also a characteristic map of the n-cell $\overline{a}$ of X. The claim now follows from the Cell-Compact Proposition 136.8 (1). ⊞

Claim 2. If A intersects an open n-cell x of X, then $\overline{x}$ is an n-cell of A.

Proof. Let x be an open n-cell of X with $x \cap A \neq \varnothing$. Let $P \in x \cap A$. Since x is a component of $X^n \setminus X^{n-1}$ and since $A \cap X^i = A^i$ we see that $P \in A^n \setminus A^{n-1}$. Thus there exists an open n-cell a of A with $P \in a$. By definition of a subcomplex we know that a is an open n-cell of X. Since $x \cap a \neq \varnothing$ and since X is the disjoint union of its open cells we see that $a = x$. The claim now follows from Claim 1. ⊞

Now we turn to the actual proof of the statement that $A \subseteq X$ is a closed subset. Thus we need to show that $X \setminus A$ is an open subset of X. By definition of a CW-complex it suffices to show that for each $n \in \mathbb{N}_0$ the intersection $(X \setminus A) \cap X^n$ is an open subset of X^n. Since $(X \setminus A) \cap X^n = X^n \setminus (A \cap X^n)$ we see that it suffices to prove the following claim:

Claim 3. For each $n \in \mathbb{N}_0$ the intersection $A \cap X^n$ is a closed subset of X^n.

Proof. We prove the claim by induction on $n \in \mathbb{N}_0$. For $n = 0$ the statement is clear since, as we pointed out on page 2625, we know that X^0 has the discrete topology. Now assume that the statement holds for $n - 1$. Let $\{\Phi_c : \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X. By the Characteristic Maps Proposition 136.9 (1), applied to the CW-complex X^n, it suffices to show that for each cell $c \in \mathrm{Cell}_n(X)$ the preimage $\Phi_c^{-1}(A) \subseteq \overline{B}^n$ is closed.

Let $c \in \mathrm{Cell}_n(X)$. We denote by $\mathring{c}$ the corresponding open cell.

(a) If $\mathring{c}$ is an open n-cell of A, then we see that

$$\Phi_c^{-1}(A) \quad = \quad \Phi_c^{-1}(c \cap A \cap X^n) \quad \underset{\uparrow}{=} \quad \Phi_c^{-1}(c) \quad \underset{\uparrow}{=} \quad \Phi_c^{-1}(\Phi_c(\overline{B}^n)) \quad = \quad \overline{B}^n.$$

by Claim 2 we know that c is an n-cell of A

by the Cell-Compact Proposition 136.8 we know that $c = \Phi_c(\overline{B}^n)$

(b) If $\mathring{c}$ is not an open n-cell of A, then it follows from the definition of a subcomplex (or alternatively from Claim 2) that $\Phi_c^{-1}(A \cap X^n) = \Phi_c^{-1}(A \cap X^{n-1})$. By induction $A \cap X^{n-1}$ is a closed subset of X^{n-1}. Thus we see that $\Phi_c^{-1}(A \cap X^{n-1})$ is a closed subset of $\overline{B}^n$. ■

The following proposition gives a very convenient characterization of subcomplexes.

Proposition 136.12. (Subcomplex–Characterization Proposition) Let X be a CW-complex.

(1) Every subcomplex of X is the union of open cells of X.

(2) Let $\{\sigma_i\}_{i\in I}$ be a family of open cells of X. We set $A := \bigcup_{i\in I} \sigma_i$. The following statements are equivalent:
   (a) A is a subcomplex.
   (b) A is a closed subset of X.
   (c) For every $i \in I$ the closure $\overline{\sigma}_i$ of $\sigma_i \subseteq X$ is contained in A.

Proof. Let X be a CW-complex.

(1) Let A be a subcomplex of X. Recall that X is the union of its open cells. Thus we need to show that if c is an open cell of X with $c \cap A \neq \varnothing$, then $c \subseteq A$. Since A is a CW-complex we see that A itself is the union of its open cells. Thus we see that $c \cap A \subseteq a_1 \cup \cdots \cup a_k$ where $a_1, \ldots, a_k$ are open cells of A. But by definition of a CW-complex $a_1, \ldots, a_k$ are also open cells of X. Since the open cells of X are disjoint and since $c \cap A \neq \varnothing$ we see that $c = a_i$ for some $i \in \{1, \ldots, k\}$. Thus we have shown that $c \subseteq A$.

(2) Let $\{\sigma_i\}_{i\in I}$ be a family of open cells of X. We set $A := \bigcup_{i\in I} \sigma_i$.

(a)⇒(b). If A is a subcomplex, then we know by the Subcomplex–Closed Lemma 136.11 that A is a closed subset of X.

(b)⇒(c). This statement follows immediately from the Definition 3.21 of the closureof a subset.

(c)⇒(a). We set $A^n := A \cap X^n$. We need to show that this defines a CW-structure on A and that each open n-cell of A is also an open n-cell of X. It is in fact elementary to see that each open n-cell of A is also an open n-cell of X. So let us move on to the following more interesting claim.

Claim 1. For each $n \in \mathbb{N}_0$ the topological space A^n is obtained from A^{n-1} by attaching n-cells and A^n is a closed subset of X^n.

Proof. For $n = 0$ the claim follows from the observation that X^0 is equipped with the discrete topology. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. In particular we assume that $A^n \subseteq X^n$ is closed. We denote by $I_{n+1} \subseteq I$ the family of open $(n+1)$-cells. For each $c \in I_{n+1}$ we pick a characteristic map $\Phi_c \colon \overline{B}^{n+1} \to X^n$. By the Cell-Compact Proposition 136.8 we have $\Phi_c(\overline{B}^{n+1}) = \overline{c}$. By hypothesis $\overline{c} \subseteq A$. In particular we see that $\Phi_c(S^n) \subseteq A^n$. This shows that we can use the restrictions of the maps Φ_c to S^n to attach $(n+1)$-cells to A. We consider the natural map

$$A^n \cup \bigsqcup_{c\in I_{n+1}} \overline{B}^{n+1} \;\to\; X^{n+1} = X^n \cup \bigsqcup_{c\in \mathrm{Cell}_{n+1}(X)} \overline{B}^n.$$

This map is continuous and injective. By the Open-Closed–Injective Lemma 4.22 (1) it remains to show that the map is closed. Let B be a closed subset of the left-hand side. It follows easily from the Topological-Pushout Proposition 8.12 (0), the Topological-Coproduct Proposition 7.26 and the induction hypothesis that $A^n \cap X^n$ is closed in X^n that the image of B is indeed a closed subset. ⊞

It remains to prove the following claim:

Claim 2. We have $A = \operatorname{colim} A^i$.

Proof. First note that it follows from Claim 1 that $A \subseteq X$ is closed. Also note that it follows either from Claim 1 or much more directly, from the Topological Pushout-Maps Lemma 8.16 (4), that each i-skeleton $X^i \subseteq X$ is closed. It follows from these observations that we can apply the Direct Colimit–Subspace Lemma 104.13 which tells us that $A = \varinjlim (A \cap X^i)$, which is exactly what we set out to show. ∎

The Subcomplex–Characterization Proposition 136.12 allows us to prove the following two frequently used lemmas.

Lemma 136.13. (Subcomplex–Intersection-Union Lemma) Let X be a CW-complex. The union and also the intersection of arbitrarily many subcomplexes of X is again a subcomplex of X.

Proof. Let X be a CW-complex and let $\{A_k\}_{k\in K}$ be a family of subcomplexes of X. For each $k \in K$ let $\{\sigma_i\}_{i\in I_K}$ be the corresponding family of open cells of X.

(1) It follows easily from the Subcomplex–Characterization Proposition 136.12 (1), together with the(a)⇔(c)-direction of the Subcomplex–Characterization Proposition 136.12 (2) that $\bigcup_{k\in K} A_k = \bigcup_{k\in K} \bigcup_{i\in I_K} \sigma_i$ is a subcomplex of X.

(2) It follows from the Subcomplex–Characterization Proposition 136.12 (1), together with the(a)⇔(b)-direction of the Subcomplex–Characterization Proposition 136.12 (2) and the fact that the intersection of closed sets is always closed that $\bigcap_{k\in K} A_k = \bigcup_{i\in \bigcap_{k\in K} I_K} \sigma_i$ is a subcomplex of X. ∎

Lemma 136.14. (Component–Subcomplex Lemma) Let X be a CW-complex. Every (path-) component of X is a subcomplex of X.[1414]

Proof. Let A be a (path-) component of X. By the Subcomplex–Characterization Proposition 136.12 it suffices to prove the following claim:

Claim. If c is an open n-cell of X with $c \subseteq A$, then $\overline{c} \subseteq A$.

Proof. Let c be an open n-cell of X. We pick a characteristic map $\Phi_c\colon \overline{B}^n \to X$ for c. It follows from the Cell-Compact Proposition 136.8 that $\overline{c} = \Phi(\overline{B}^n)$. The claim now follows from the Image-Preimage-Component Lemma 4.50 which implies that $\Phi(\overline{B}^n)$ is contained in a (path-) component of X. ∎

Lemma 136.15. (CW-Complex Pasting Lemma) Let X be a CW-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup_{i\in I} X_i = X$. The following three statements hold:

(1) (a) A subset $U \subseteq X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.
(b) A subset $A \subseteq X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.

(2) A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof.

[1414] In the CW-Complex–Local Properties Proposition 137.8 (3) we will see that every CW-complex is the coproduct (i.e. the disjoint union as a topological space) of its components.

(1) (a) Given any subset $U \subseteq X$ we have

$$U \subseteq X \text{ is open} \overset{\text{by the Characteristic Maps Proposition 136.9 (1)}}{\Leftrightarrow} \forall_{c \in \mathrm{Cell}(X)} \, c \cap U \text{ is open in } c \overset{\substack{\text{follows from } \cup X_i = X \text{ and the observation that it is a consequence of the Subcomplex–Characterization Proposition 136.12 (a)⇒(c) that every cell of some } X_i \text{ is a cell of } X}}{\Leftrightarrow} \forall_{i \in I} \forall_{c \in \mathrm{Cell}(X_i)} \, c \cap U \text{ is open in } c$$
$$\overset{\text{follows from the Characteristic Maps Proposition 136.9 (1) applied to the CW-complex } X_i}{\Leftarrow} \forall_{i \in I} \, X_i \cap U \text{ is open in } X_i.$$

(b) The same logic as in (a) also applies for "closed" instead of "open". This statement is an easy consequence of the Subcomplex–Characterization Proposition 136.12 (a)⇒(c) and the Characteristic Maps Proposition 136.9 (1).

(2) The "only if"-direction is an immediate consequence of the Subspace Continuity Lemma 4.3 (2). The "if"-direction follows from (1a) and the Topology-Defined-by-Family Lemma 4.8 (1)⇒(3). ■

136.4. **The Finiteness Theorem for CW-complexes.** The following proposition is one of the key technical results about infinite CW-complexes.

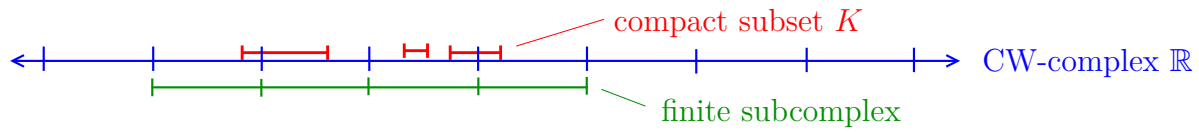
Theorem 136.16. (CW-Complex-Finiteness Theorem) Every compact (connected) subset of a CW-complex X is contained in a finite (connected) subcomplex.

Remark. The statement of the proposition and also its proof have more than a passing resemblance with Exercise 104.18. □

Example.

(1) We consider the wedge $X := \bigvee_{i \in \mathbb{N}_0} S^1_i$. As discussed on page 2627, we can view X as a CW-complex with one 0-cell, given by the wedge point, and 1-cells indexed by $c \in \mathbb{N}_0$. It follows from the CW-Complex-Finiteness Theorem 136.16 that each compact subset of X is contained in the union of finitely many S^1_i. In fact this statement already follows from Exercise 96.6.

(2) We consider the real line $\mathbb{R}$ with the CW-structure $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$.



In this context the CW-Complex-Finiteness Theorem 136.16 gives us the familiar statement that any compact subset of $\mathbb{R}$ is bounded. □

Before we start out with the proof of the CW-Complex-Finiteness Theorem 136.16 we formulate the following pretty corollary.

Corollary 136.17. (CW-Complex Compactness Corollary) A CW-complex is compact if and only if it is finite.

Proof. A finite CW-complex is by definition the union of finitely many cells. Thus it follows from the Cell-Compact Proposition 136.8 (2) together with the Compact–Union-Intersection Lemma 3.41 that a finite CW-complex is compact.

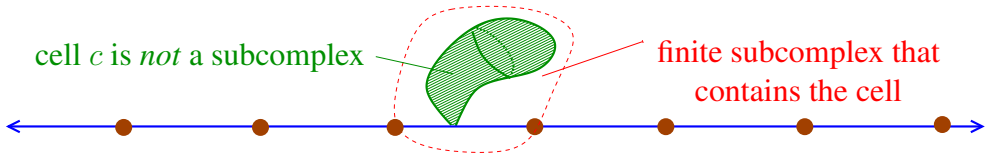
Conversely let X be a compact CW-complex. It follows from the CW-Complex-Finiteness Theorem 136.16 that X is contained in a finite subcomplex Y of X. But from the inclusions $X \subseteq Y \subseteq X$ it follows that $X = Y$, i.e. X is a finite CW-complex. ■

The proof of the CW-Complex-Finiteness Theorem 136.16 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 136.18. (Contained-in-Finite Subcomplex Lemma) Let X be a CW-complex.

(1) Let $A \subseteq X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.
(2) If $C \subseteq X$ is a compact subset, then C is contained in a finite union of open cells.
(3) Every cell of X is contained in a finite subcomplex of X.

Example. We consider the CW-complex X that is shown in the figure below together with a cell c.

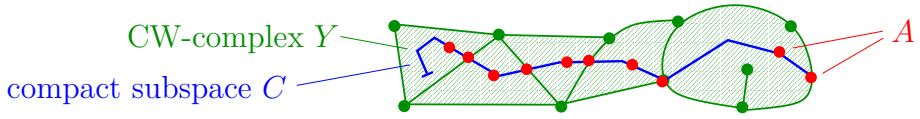


The example below shows that a cell itself is not necessarily a subcomplex, but, as predicted by the Contained-in-Finite Subcomplex Lemma 136.18, we see in our example that every cell is contained in a finite subcomplex. □

Proof of Lemma 136.18. We start out with the following claim:

Claim. For any CW-complex Y the statements (1), (2) and (3) of the lemma are equivalent.

Proof. Let Y be a CW-complex.
(1)⇒(2). Let $C \subseteq Y$ be a compact subset. For each open cell σ with $C \cap \sigma \neq \varnothing$ we pick a single point in $C \cap \sigma$. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of the compact set C we obtain from the Compact-Closed Lemma 3.39 (1) that A is compact. Since A is compact and discrete we obtain from the Compact-Discrete Lemma 3.40 (1) that A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells. This is precisely what we had to show.

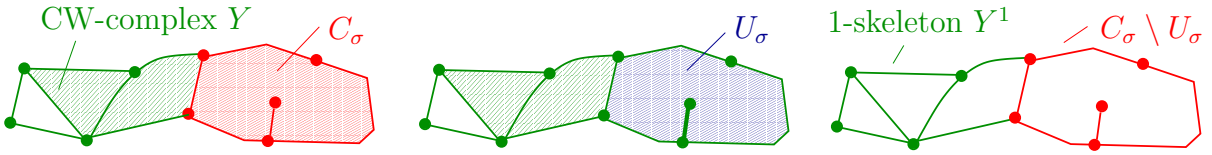


(2)⇒(3). This direction follows from the following stronger subclaim.

Subclaim. For every $n \in \mathbb{N}_0$ the following holds: If Y^{n-1} satisfies (2), then Y^n satisfies (3).

Proof of subclaim. We prove the claim by induction on $n \in \mathbb{N}_0$. For $n = 0$ there is nothing really to show. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $C_\sigma \subseteq Y$ be an $(n+1)$-cell. We denote by U_σ the corresponding open cell. Note that by the Cell-Compact Proposition 136.8 (2) we know that C_σ is compact. Furthermore note that it follows from

the discussion on page 2624 that U_σ is an open subset of Y^{n+1}. In particular we see that U_σ is an open subset of C_σ.



Finally note that it follows from these observations and the Compact-Closed Lemma 3.39 that $C_\sigma \setminus U_\sigma$ is a compact subset of the n-skeleton Y^n.

Since we assume that (2) holds for n it follows that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n$. By induction each of these open cells is contained in a finite subcomplex of Y. By the Subcomplex–Intersection-Union Lemma 136.13 the union of these finitely many finite subcomplexes forms a subcomplex of Y. This subcomplex, together with U_σ, forms a finite subcomplex which contains C_σ. □

(3)⇒(1). We start out with the following subclaim.

Subclaim. If $B \subseteq Y$ is a subset such that no two points lie in the same open cell, then B is a closed subset.

Proof of subclaim. Let $B \subseteq Y$ be a subset such that no two points lie in the same open cell. We need to show that B is a closed subset of X. By the Characteristic Maps Proposition 136.9 it suffices to show that the intersection of B with each cell c is closed. Thus let c be a cell of X. By (3) we know that c is contained in a finite subcomplex Y'. Since B has at most one point in common with any open cell we see that the intersection $B \cap Y'$ must be finite. Thus the intersection $B \cap c$ is finite. It follows from the CW-Complex–Hausdorff Proposition 136.6 and the Points-in-Hausdorff Lemma 3.32 that $B \cap c$ is a closed subset of c. □

Now let $A \subseteq Y$ be a subset such that no two points lie in the same open cell. By the subclaim we know that A is a closed subset of Y. We still need to show that A is discrete. Note that for any point $x \in A$ the subset $A \setminus \{x\}$ also satisfies the hypothesis of the subclaim. Thus $A \setminus \{x\}$ is a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. ⊞

Finally we turn to the actual proof of the lemma. Let X be a CW-complex. We need to show that for X all three statements hold. First note that Statement (1) holds for X^0 by definition of the topology on X^0. Now suppose that for some $n \in \mathbb{N}_0$ we have (1) for the n-skeleton X^n By the claim we know that (2) and (3) also hold for X^n. By the subclaim in the proof of (2)⇒(3) we know that (3) actually holds for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all $n \in \mathbb{N}_0$. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. ■

Proof of the CW-Complex-Finiteness Theorem 136.16. Let X be a CW-complex and let $C \subseteq X$ be a compact subset. By the Contained-in-Finite Subcomplex Lemma 136.18 (2) the subset C is contained in a union of finitely many open cells. By Lemma 136.18 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.

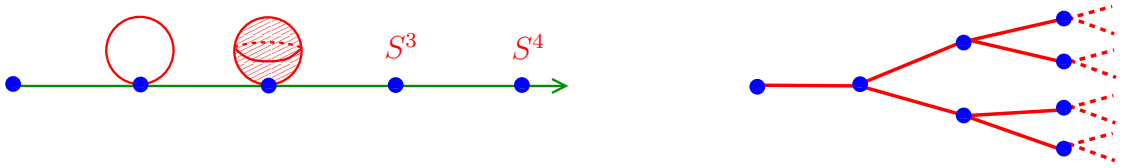
If C is connected, then we take the component of Z that contains C. By the Component–Subcomplex Lemma 136.14 we know that each path-component of a CW-complex is a subcomplex. In particular we see in our case that C is again a subcomplex. ■

For completeness we want to cite a characterization of a CW-complex to be *regionally compact*. To do so we need to introduce the following definition:

Definition 136.19. We say that a CW-complex is locally finite if every cell is disjoint from all but finitely many cells of X.

Examples.

(1) The CW-complexes that are illustrated in the figure below are locally finite.



(2) One can easily show that the infinite-dimensional CW-complexes $\mathbb{C}\mathrm{P}^\infty$ and $\bigvee_{i\in\mathbb{N}_0} S^1$ are not locally finite. □

We can now formulate the following proposition:

Proposition 136.20. (CW-Complex–Regionally Compact Proposition) A CW-complex is regionally compact if and only if it is locally finite.

Proof. We will not really make use of the CW-Complex–Regionally Compact Proposition 136.20, except that in the negative sense, that now we know that many CW-complexes we are interested in are unfortunately not regionally compact. The proposition is proved in [**Geo2008**, Proposition 10.1.8] or alternatively in [**FrPi1990a**, Proposition 1.5.10]. ■

136.5. **Fundamental groups of CW-complexes.** Before we head to the fundamental groups of CW-complexes let us quickly study the path-connectedness of CW-complexes. The following lemma says that the path-connectedness depends only on the 1-skeleton.

Lemma 136.21. (CW-Skeleton-π_0-Lemma) Let X be a CW-complex.

(1) The inclusion-induced map $\pi_0(X^0) \to \pi_0(X)$ is a surjection.
(2) For every $k \geq 1$ the inclusion-induced map $\pi_0(X^k) \to \pi_0(X)$ is a bijection.

Remark. In the CW-Complex–Local Properties Proposition 137.8 (2) we will see that CW-complexes are locally path-connected. Thus it follows from the Locally Path-Connected Lemma 4.57 that the components of a CW-complex are precisely the path-components. □

Proof. We will prove this lemma in Exercise 136.13. ■

Example. We equip S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ with the usual CW-structures. Their 1-skeleta are $S^1, \mathbb{R}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^0$. Since these are path-connected we obtain from the CW-Skeleton-π_0-Lemma 136.21 that S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ are path-connected. □

In a similar fashion the following proposition says that for CW-complexes the fundamental group is determined by the fundamental group of its 2-skeleton.

Proposition 136.22. (CW-Skeleton-π_1-Proposition) Let X be a connected CW-complex and let x_0 be a point in the 0-skeleton of X.

(1) The inclusion-induced map $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ is an epimorphism.
(2) For any $k \geq 2$ the inclusion-induced map $\pi_1(X^k, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

Remark. In the CW-Skeleton-π_n-Proposition 141.3 we will prove an analog of the CW-Skeleton-π_1-Proposition 136.22 for higher homotopy groups. □

Example. It follows from the CW-Skeleton-π_1-Proposition 136.22 together with some of our earlier calculations that

$$\begin{array}{lllll} \pi_1(S^\infty) & \cong \pi_1(\text{2-skeleton of } S^\infty) & \cong \pi_1(S^2) & = 0, \\ \pi_1(\mathbb{R}\mathrm{P}^\infty) & \cong \pi_1(\text{2-skeleton of } \mathbb{R}\mathrm{P}^\infty) & \cong \pi_1(\mathbb{R}\mathrm{P}^2) & = \mathbb{Z}/2\mathbb{Z}, \\ \pi_1(\mathbb{C}\mathrm{P}^\infty) & \cong \pi_1(\text{2-skeleton of } \mathbb{C}\mathrm{P}^\infty) & \cong \pi_1(\mathbb{C}\mathrm{P}^1) & = 0. \end{array}$$

□

Proof.

(1) Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). It follows from the Compact Image Lemma 4.15 and the CW-Complex-Finiteness Theorem 136.16 that there exists a finite connected subcomplex A of X with $\gamma([0,1]) \subseteq A$. We consider the following commutative diagram:

$$\begin{array}{ccc} \pi_1(A^1, x_0) & \longrightarrow & \pi_1(X^1, x_0) \\ \downarrow\!\!\downarrow & & \downarrow \\ \pi_1(A, x_0) & \longrightarrow & \pi_1(X, x_0). \end{array}$$

Since A is a finite CW-complex we see that A is obtained from its 1-skeleton A^1 by attaching finitely many cells of dimension ≥ 2. We obtain from iteratively applying the Cell Attachment-π_1-Proposition 135.5 that the left vertical map is an epimorphism. By choice of A we see that $[\gamma]$ lies in the image of the bottom horizontal map. Since the diagram commutes we see that $[\gamma]$ lies in the image of the right vertical map. We have thus shown that the right vertical map is indeed an epimorphism.

(2) The proof of this statement is a variation on the argument of (1) and it follows from the same logic as in the proof of the Direct Colimit-π_1-Proposition 105.1. We will fill in the details in Exercise 136.18. ■

Proposition 136.23. (CW-π_1-Finite Presentation–Proposition) Let X be a connected CW-complex and let $x_0 \in X$.

(1) If there are only finitely many cells in dimensions 0 and 1, then $\pi_1(X, x_0)$ is finitely generated.
(2) If there are only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X, x_0)$ is finitely presented.

Example. Recall that the Smooth Manifold-CW-Theorem 179.9 says that every compact smooth manifold admits the structure of a finite CW-complex. Together with the CW-π_1-Finite Presentation–Proposition 136.23 we now obtain that the fundamental group of any compact connected smooth manifold M is finitely presented. □

Proof.

(1) By the CW-Skeleton-π_1-Proposition 136.22 it suffices to show that the fundamental group of the 1-skeleton X^1 is finitely generated. Note that by hypothesis and by

the CW-Skeleton-π_0-Lemma 136.21 we know that the 1-skeleton is a finite connected 1-dimensional CW-complex. It follows now easily from the Cell Attachment-π_1-Proposition 135.5, or alternatively from the Graph–CW-Complex Lemma 136.5 together with the Graph-π_1-Proposition 99.13, that $\pi_1(X^1)$ is a finitely generated free group.

(2) As in (1) we see that by the CW-Skeleton-π_1-Proposition 136.22 it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. We pick $x_0 \in X$. Let $\varphi_1, \dots, \varphi_k \colon S^1 \to X$ be attaching maps of all of the 2-cells. For $i = 1, \dots, k$ we consider the corresponding path $\widetilde{\varphi}_i \colon [0,1] \to X$ that is given by $\widetilde{\varphi}_i(t) \coloneqq \varphi(\exp(2\pi \mathrm{i} \cdot t))$. Furthermore, for $i = 1, \dots, k$ we pick a path p_i from x_0 to the point $\widetilde{\varphi}_i(0) = \widetilde{\varphi}_i(1)$ and we write $\psi_i = p_i * \widetilde{\varphi}_i * \overline{p}_i$. Now we see that

$$\begin{aligned} \pi_1(X) &\underset{\uparrow}{\cong} \pi_1(X^1)/\langle\langle [\psi_1] \rangle\rangle \dots /\langle\langle [\psi_k] \rangle\rangle \underset{\uparrow}{=} \pi_1(X^1)/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle \\ &\underset{\uparrow}{=} \{\text{finitely generated free group}\}/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle. \end{aligned}$$

by the Change-of-Base Point Proposition 88.14 and by the Cell Attachment-π_1-Proposition 135.5

follows from the Quotient Presentation–Lemma 100.18

see the argument in (1)

We have thus shown that $\pi_1(X)$ is a finitely presented group. ■

In the CW-π_1-Finite Presentation–Proposition 136.23 we just showed that the fundamental group of a finite CW-complex is finitely generated. Basically the same argument, using the CW-Skeleton-π_1-Proposition 136.22 and the Graph-π_1-Proposition 99.13, shows that the fundamental group of a countable CW-complex admits a countable generating set. On the other hand, the discussion of $\pi_2(S^1 \vee S^2)$ on page 2588 strongly suggests that even simple topological spaces can have surprisingly large higher homotopy groups. This raises the following question.

Question 136.24. Are the homotopy groups of a countable CW-complex countable?

We will answer this question in the CW-Complex–π_n-Countable Proposition 190.8.

136.6. **CW-complexes and direct colimits.** The following propositions give a convenient variation on the Exhaustion-π_n-Proposition 134.21.

Proposition 136.25. (CW-Direct Colimit-π_n-Proposition) Let X be a CW-complex and let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$.

(1) We have $X = \varinjlim X_i$.

(2) For any $n \in \mathbb{N}_0$ and any $w \in X_1$ we have a natural isomorphism[1415]

$$\varinjlim \pi_n(X_i, w) \xrightarrow{\cong} \pi_n\big(\underbrace{\varinjlim X_i}_{=X}, w\big).$$

Sketch of proof.

(1) This statement follows immediately from the characterization of the direct colimit $\varinjlim X_i$ in the Topological-Direct Colimit Proposition 104.10 together with the CW-Complex Pasting Lemma 136.15.

[1415] For $n = 0$ this is an isomorphism of sets, i.e. a bijection. For $n \geq 1$ this is an isomorphism of groups.

(2) The proof of this proposition is basically identical to the proof of the Exhaustion-π_1-Proposition 105.2. We just need to replace the Compact Subspace–Exhaustion Lemma 99.17 by the CW-Complex-Finiteness Theorem 136.16. ■

Examples.

(1) For every $n \in \mathbb{N}_0$ we have

$$\begin{array}{ccccc} \pi_n(S^\infty) & \underset{\uparrow}{\cong} & \varinjlim \pi_n(S^i) & \underset{\uparrow}{=} & \varinjlim (\,? \to \ldots \to ? \to \overset{\pi_n(S^{n+1})}{\overset{\downarrow}{0}} \to 0 \to \ldots) \\ & \text{by the Direct Colimit-}\pi_n & & \text{since by the Sphere-Low-}\pi_n\text{ Proposition 134.19} & \\ & \text{Proposition 136.25} & & \text{we have } \pi_n(S^i) = 0 \text{ for } i > n & \\ & & & \underset{\uparrow}{\cong}\ 0. & \\ & & & \text{by the CW-Direct Colimit} & \\ & & & \text{Stabilization Lemma 104.5} & \end{array}$$

(2) Recall that by the Infinite Projective Space-Sphere Quotient Lemma 104.22 we know that the group $\{\pm 1\}$ acts continuously and discretely on S^∞ and that $S^\infty/\{\pm 1\}$ is homeomorphic to $\mathbb{R}\mathrm{P}^\infty$. Therefore it follows from (1) and the Action-π_1–Theorem 93.1 that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \cong \pi_1(S^\infty/\{\pm 1\}) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore it follows from (1), the Action–Covering Proposition 91.11 and the Covering-π_n-Isomorphism Proposition 134.22 that for every $n \in \mathbb{N}_{\geq 2}$ we have $\pi_n(\mathbb{R}\mathrm{P}^\infty) \cong \pi_n(S^\infty) = 0$.

(3) For fans of the CW-Direct Colimit-π_n-Proposition 136.25 we can also calculate $\pi_1(\mathbb{R}\mathrm{P}^\infty)$ directly:

$$\begin{array}{ccccccccccccc} & & & & & & & & & & & & \text{by the CW-Direct Colimit} \\ & & & & & & & & & & & & \pi_n\text{-Proposition 136.25} \\ & & & & & & & & & & & & \overset{\cong}{\downarrow} \\ \pi_1(\mathbb{R}\mathrm{P}^\infty) & \overset{\cong}{\to} & \varinjlim & (& \pi_1(\mathbb{R}\mathrm{P}^1) & \to & \pi_1(\mathbb{R}\mathrm{P}^2) & \to & \pi_1(\mathbb{R}\mathrm{P}^3) & \to & \ldots &) & \\ & \underset{\uparrow}{\cong} & \varinjlim & (& \mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \overset{\cong}{\to} & \mathbb{Z}/2\mathbb{Z} & \overset{\cong}{\to} & \ldots &) & \overset{\cong}{\to}\ \mathbb{Z}/2\mathbb{Z}. \\ & \text{the Real Projective Space–}\pi_1\text{-Proposition 93.3 says that for all } i \geq 2 & & & & & & & & & & & \text{by the Direct Colimit} \\ & \text{we have } \pi_1(\mathbb{R}\mathrm{P}^i) \cong \mathbb{Z}/2\mathbb{Z}\text{, in fact the explicit isomorphism implies that the} & & & & & & & & & & & \text{Stabilization} \\ & \text{inclusion-induced maps } \pi_1(\mathbb{R}\mathrm{P}^i) \to \pi_1(\mathbb{R}\mathrm{P}^{i+1}) \text{ are isomorphisms for all } i \geq 2 & & & & & & & & & & & \text{Lemma 104.5} \end{array}$$

Furthermore for any $n \geq 2$ we have

$$\begin{array}{ccccc} & & & & \text{by the Covering-}\pi_n\text{-Isomorphism} \\ & & & & \text{Proposition 134.22} \\ & & & & \overset{\cong}{\downarrow} \\ \pi_n(\mathbb{R}\mathrm{P}^\infty) & \overset{\cong}{\downarrow} & \varinjlim \pi_n(\mathbb{R}\mathrm{P}^i) & \overset{\cong}{\downarrow} & \varinjlim \pi_n(S^i) \\ & \text{by the CW-Direct Colimit-}\pi_n & & \underset{\uparrow}{=} & \varinjlim (\,? \to ? \to ? \to 0 \to 0 \to \ldots) \\ & \text{Proposition 136.25} & & \text{since by the Sphere-Low-}\pi_n\text{ Proposition 134.19} & \\ & & & \text{we have } \pi_n(S^i) = 0 \text{ for } i > n & \\ & & & \underset{\uparrow}{\cong}\ 0. & \\ & & & \text{by the Direct Colimit} & \\ & & & \text{Stabilization Lemma 104.5} & \end{array}$$

□

136.7. **Appendix: CW-Complexes are normal and paracompact.** As the title of this appendix strongly suggests, in this appendix we will show (among many other things) that CW-complexes are normal and paracompact. It is nice to know that this is true, but

we will use these statements only rarely and this is why we "hide" the statements, let alone the proofs, in this appendix.

We start out with the first promised property of CW-complexes.

Proposition 136.26. (CW-Complex–Normal Proposition) Every CW-complex is normal.

Proof. Similar to the argument in the proof of the CW-Complex–Hausdorff Proposition 136.6, this proposition can be deduced easily from the Cell Attachment–Normal Proposition 135.15. ■

We continue with another mildly interesting property of CW-complexes, which we will rarely use.

Criterion 136.27. (CW-Complex Second-Countability Criterion) If X is a connected CW-complex, then the following statements are equivalent:

(1) X is second-countable.
(2) X is locally finite.

Proof. We will not really make use of this lemma, thus we outsource the proof to Exercise 136.12. ■

The last two results lead us to the following pleasant lemma.

Lemma 136.28. (CW-Complex Metrizability Lemma) Every connected locally finite CW-complex is metrizable.

Proof. Let X be a connected locally finite CW-complex. It follows from the CW-Complex Second-Countability Criterion 136.27 that X is second-countable. By the CW-Complex–Normal Proposition 136.26 we know that X is normal and by the CW-Complex–Hausdorff Proposition 136.6 we know that X is Hausdorff. It follows from the Urysohn Metrization Theorem 13.8 that X is metrizable. ■

Next we recall Definition 14.14:

Definition 136.29. Let X be a topological space.

(1) The support of a map $f\colon X \to \mathbb{R}$ is defined as
$$\operatorname{supp}(f) \coloneqq \overline{\{x \in X \mid f(x) \neq 0\}}.$$
(2) A partition of unity on X is a family of continuous maps $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
   (a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i.
   (b) For every $x \in X$ we have
   $$\sum_{i\in I} f_i(x) = 1.$$
(3) Let $\{U_i\}_{i\in I}$ be an open cover. A partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$ is a partition of unity $\{f_j\colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subseteq U_i$.

Proposition 136.30. (CW-Complex–Partition-of-Unity Proposition) Let X be a CW-complex. Given any open cover $\{U_i\}_{i\in I}$ of X there exists a partition of unity subordinate to U_i.

The proof of the CW-Complex–Partition-of-Unity Proposition 136.30 rests heavily on the shoulders of the following lemma:

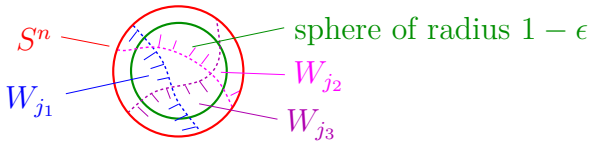
Lemma 136.31. (Partition-of-Unity–Extension Lemma) Let $n \in \mathbb{Z}_{\geq -1}$ and let $\{W_j\}_{j\in J}$ be an open cover of $\overline{B}^{n+1}$. Furthermore let $\{\varphi_k \colon S^n \to [0,1]\}_{k\in K}$ be a partition of unity on S^n and let $f \colon K \to J$ be a map such that for each $k \in K$ we have $\operatorname{supp}(\varphi_k) \subseteq W_{f(k)}$. There exist functions $\{\widetilde{\varphi}_k \colon \overline{B}^n \to [0,1]\}_{k\in K}$, $\{\widetilde{\psi}_j \colon \overline{B}^n \to [0,1]\}_{j\in J}$ on $\overline{B}^{n+1}$ and a map $\widetilde{f} \colon K \sqcup J \to J$ with the following properties:

(0) For each $k \in K$ we have $\widetilde{\varphi}_k|_{S^n} = \varphi_k$ and we have $\widetilde{f}|_K = f \colon K \to J$.
(1) The functions $\{\widetilde{\varphi}_k \colon \overline{B}^n \to [0,1]\}_{k\in K}$, $\{\widetilde{\psi}_j \colon \overline{B}^n \to [0,1]\}_{j\in J}$ form a partition of unity on $\overline{B}^n$.
(2) For each $k \in K$ we have $\operatorname{supp}(\widetilde{\varphi}_k) \subseteq W_{\widetilde{f}(k)}$ and for each $j \in J$ we have $\operatorname{supp}(\widetilde{\psi}_j) \subseteq W_{\widetilde{f}(j)}$.
(3) For each open subset U of S^n there exists an open neighborhood $\Theta(U)$ of $U \subseteq \overline{B}^{n+1}$ with the following property: if φ_k vanishes on U, then $\widetilde{\varphi}_k$ vanishes on $\Theta(U)$.
(4) There exists an open neighborhood M of $S^n \subseteq \overline{B}^{n+1}$ such that for all $j \in J$ the function $\widetilde{\psi}_j$ vanishes on M.

Proof of Lemma 136.31. It follows easily from the fact that S^n is compact and the fact that the sets $\{\varphi_k^{-1}((0,\infty))\}_{k\in K}$ form a locally finite open cover of S^n that there exist finitely many $k_1, \ldots, k_m \in K$ such that for any $k \in K \setminus \{k_1, \ldots, k_m\}$ the map $\varphi_k \colon S^n \to [0,1]$ is identically zero. For $i = 1, \ldots, m$ we set $j_i = f(k_i)$.

Note that $S^n \subseteq W_{j_1} \cup \cdots \cup W_{j_m}$. Also note that $W_{j_1}, \ldots, W_{j_m}$ are open subsets of $\overline{B}^{n+1}$. It follows easily from the Tube Lemma 7.12 that there exists an $\epsilon \in (0,1)$ with the following two properties:

- We have $\{r \cdot x \,|\, r \in [1-\epsilon, 1] \text{ and } x \in S^n\} \subseteq W_{j_1} \cup \cdots \cup W_{j_m}$.
- For each $k \in K$ we have $\{r \cdot x \,|\, r \in [1-\epsilon, 1] \text{ and } x \in \operatorname{supp}(\varphi_k)\} \subseteq W_{f(k)}$.



Next we pick a continuous map $\rho \colon [0,1] \to [0,1]$ with $\rho(x) = 0$ for $x \in [0, 1-\epsilon]$ and with $\rho(x) = 1$ for $x \in [1-\frac{\epsilon}{2}, 1]$. Given $k \in K$ we set

$$\begin{array}{rccl} \widetilde{\varphi}_k \colon \overline{B}^{n+1} & \to & [0,1] & \\ x & \mapsto & \begin{cases} \rho(\|x\|) \cdot \varphi_k(\frac{x}{\|x\|}), & \text{if } \|x\| \geq 1-\epsilon \\ 0, & \text{if } \|x\| \leq 1-\epsilon \end{cases} \end{array}$$

It follows easily from the Pasting Proposition 4.7 that each $\widetilde{\varphi}_k$ is continuous. Also note that it follows from the second property of ϵ that for each $k \in K$ we have $\operatorname{supp}(\widetilde{\varphi}_k) \subseteq W_{f(k)}$. Next we set $\widetilde{\varphi} := \widetilde{\varphi}_1 + \cdots + \widetilde{\varphi}_k$. Note that $\widetilde{\varphi}(x) = 0$ if $\|x\| \in [0, 1-\epsilon]$ and that $\widetilde{\varphi}(x) = 1$ if $\|x\| \geq 1 - \frac{\epsilon}{2}$.

We know, for example by the Paracompact-Criterion Lemma 14.5, that $\overline{B}^{n+1}$ is paracompact. It thus follows from the Paracompact-Partition of Unity Theorem 14.15 that there exists a partition of unity $\{\psi_i \colon \overline{B}^{n+1} \to [0,1]\}_{j\in J}$ such that for each $j \in J$ we have

$\operatorname{supp}(\widetilde{\psi}_j) \subseteq W_j$. Given $j \in J$ we consider the continuous map

$$\begin{array}{rcl} \widetilde{\psi}_j \colon \overline{B}^{n+1} & \to & [0,1] \\ x & \mapsto & (1 - \widetilde{\varphi}(x)) \cdot \psi_j(x). \end{array}$$

One can easily verify that the functions $\{\widetilde{\varphi}_k \colon \overline{B}^{n+1} \to [0,1]\}_{k \in K}$, $\{\widetilde{\psi}_j \colon \overline{B}^{n+1} \to [0,1]\}_{j \in J}$, together with the map $\widetilde{f} \colon K \cup J \to J$ that is given by $\widetilde{f}|_K = f$ and $\widetilde{f}|_J = \mathrm{id}_J$, the map $U \mapsto \Theta(U) := \{r \cdot u \,|\, r \in (0,1] \text{ and } u \in U\}$ and together with the open neighborhood $M := \{x \in \overline{B}^n \,|\, \|x\| > 1 - \frac{\epsilon}{2}\}$ of S^n have all the desired properties. ■

Proof of the CW-Complex–Partition-of-Unity Proposition 136.30. Let $\{U_j\}_{j \in J}$ be an open cover of X.

Claim 1. Let $n \in \mathbb{Z}_{\geq -1}$, let $\{\varphi_k^n\}_{k \in K_n}$ be a partition of unity on X^n and let $f^n \colon K_n \to J$ be a map such that for each $k \in K_n$ we have $\operatorname{supp}(\varphi_k^n) \subseteq U_{f(k)}$. There exists a set K_{n+1} which contains K_n, there exists a family of functions $\{\varphi_k^{n+1} \colon X^{n+1} \to [0,1]\}_{k \in K_{n+1}}$ and there exists a map $f^{n+1} \colon K_{n+1} \to J$ with the following properties:

(0) For each $k \in K_n$ we have $\varphi_k^{n+1}|_{X^n} = \varphi_k^n$ and we have $f^{n+1}|_{K_n} = f^n \colon K_n \to J$.
(1) The functions $\{\varphi_k^{n+1} \colon X^{n+1} \to [0,1]\}_{k \in K_{n+1}}$ form a partition of unity on X^{n+1}.
(2) For each $k \in K_{n+1}$ we have $\operatorname{supp}(\varphi_k^{n+1}) \subseteq U_{f(k)}$.
(3) For each open subset U of X^n there exists an open neighborhood $\Theta_{n+1}(U)$ of $U \subseteq X^{n+1}$ with the following property: if we are given $k \in K_n$ such that φ_k^n vanishes on U, then φ_k^{n+1} vanishes on $\Theta_{n+1}(U)$.
(4) There exists an open neighborhood M_{n+1} of $X^n \subseteq X^{n+1}$ such that for all $k \in K_{n+1} \setminus K_n$ the function φ_k^{n+1} vanishes on M_n.

Proof. Let $\{\Phi_c \colon \overline{B}^{n+1} \to X^n\}_{c \in \mathrm{Cell}_{n+1}(X)}$ be a full set of characteristic maps for the $(n+1)$-cells of X. For each $c \in \mathrm{Cell}_{n+1}(X)$ we apply the Partition-of-Unity–Extension Lemma 136.31 to the partition of unity $\{\varphi_j \circ \Phi_c \colon S^n \to [0,1]\}_{j \in K_n}$ on $\overline{B}^{n+1}$ and the open cover $\{\Phi_c^{-1}(U_j)\}_{j \in J}$ of $\overline{B}^{n+1}$. It follows from the Characteristic Maps Proposition 136.9 (1) and (2) that the output of the Partition-of-Unity–Extension Lemma 136.31 gives a partition of unity on X^{n+1}, a map f, a map $U \mapsto \Theta_{n+1}(U)$ and an open neighborhood M_{n+1} that satisfy all the desired conditions. ⊞

Now we turn to the actual proof of the proposition. For $n = -1$ we pick the empty partition of unity. Applying the claim iteratively we get the following:

(i) We get sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$.
(ii) For each $n \in \mathbb{Z}_{\geq -1}$ we get a map Θ_{n+1} that associates to an open subset $U \subseteq X^n$ an open subset $\Theta_n(U)$ of X^{n+1} with $U \subseteq \Theta_n(U)$.
(iii) For each $n \in \mathbb{Z}_{\geq -1}$ we get an open neighborhood M_{n+1} of $X^n \subseteq X^{n+1}$.
(iv) For each $k \in K := \bigcup_{n \in \mathbb{N}_0} K_n$ we get a map $\varphi_k \colon X \to [0,1]$.
(v) We get a map $f \colon K \to J$.

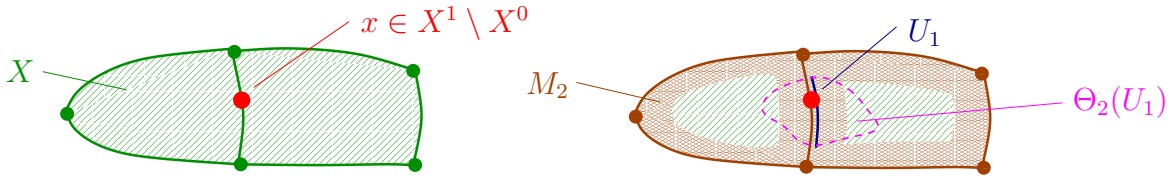
For each $n \in \mathbb{Z}_{\geq -1}$ these objects have the following properties:

(1) The maps $\{\varphi_k|_{X^n}\}_{k \in K_n}$ are a partition of unity on X^n.
(2) For each $k \in K_n$ we have $\operatorname{supp}(\varphi_k|_{X^n}) \subseteq U_{f(k)}$.
(3) If we are given $k \in K_n$ such that φ_k vanishes on $U \subseteq X^n$, then φ_k also vanishes on $\Theta_{n+1}(U) \subseteq X^{n+1}$.
(4) For all $k \in K_{n+1} \setminus K_n$ the function φ_k vanishes on M_{n+1}.

First note that it follows from (1) and the Characteristic Maps Proposition 136.9 (2) that the maps $\varphi_k \colon X \to [0,1]$ are continuous. Next note that it follows from (1) and $X = \bigcup_{n\in\mathbb{N}_0} X^n$ that the maps φ_k add up to one. Furthermore note that it follows from (2), and the fact that each $X^n \subseteq X$ is closed, that for each $k \in K$ we have $\operatorname{supp}(\varphi_k) \subseteq U_{f(k)}$.
It remains to prove the following claim:

Claim 2. For each $x \in X$ there exists an open neighborhood U of x such that φ_k vanishes on U for all but finitely many $k \in K$.

Proof. Let $x \in X$. Let $n \in \mathbb{N}_0$ such that $x \in X^n \setminus X^{n-1}$. Since $\{\varphi_k|_{X^n} \colon X^n \to [0,1]\}_{k\in K_n}$ form a partition of unity on X^n we know that there exists an open neighborhood U_n of $x \in X^n$ such that there exists a finite subset $F \subseteq K_n$ such that for all $k \notin F$ the map φ_k vanishes on U_n.



We set
$$U := \bigcup_{k>n} \bigcup_{m>k} (\Theta_m \circ \cdots \circ \Theta_{k+1})(M_k) \cup \bigcup_{m>n} (\Theta_m \circ \cdots \circ \Theta_{n+1})(U_n).$$
It follows from the Direct Colimit–Open Subset Lemma 104.14 that U is an open neighborhood of x. One can easily verify, using (3) and (4), that by construction for any $k \in K \setminus F$ the map φ_k vanishes on U. In particular, all but finitely many φ_k vanish on this open neighborhood of x. ■

Finally we recall the following definitions from Definition 3.19 and Definition 14.3.

Definition 136.32. Let X be a topological space.
(1) We say that a family $\{U_i\}_{i\in I}$ of subsets of X is locally finite if each $x \in X$ admits an open neighborhood W such that $\{i \in I \mid U_i \cap W \neq \varnothing\}$ is finite.
(2) A topological space X is called paracompact if every open cover of X admits a locally finite open refinement. More precisely, we saw X is paracompact if for any open cover $\{U_i\}_{i\in I}$ of X there exists a locally finite open cover $\{V_j\}_{j\in J}$ of X such that each V_j is contained in some U_i.

Now we can easily prove the other statement about CW-complexes that we promised in the title of the appendix.

Proposition 136.33. (CW-Complex–Paracompact Proposition) Every CW-complex is paracompact.

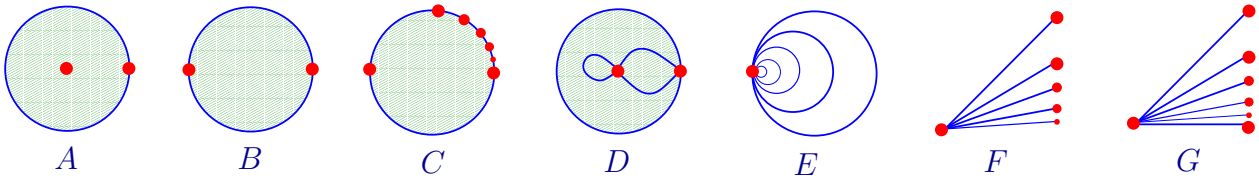
Remark. By the CW-Complex–Hausdorff Proposition 136.6 we know that every CW-complex is Hausdorff and by the CW-Complex–Paracompact Proposition 136.33 we know that every CW-complex is paracompact. These two facts together with the Dieudonné Theorem 14.9 give an alternative proof of the fact, proved above in the CW-Complex–Normal Proposition 136.26, that every CW-complex is normal. □

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X. By the CW-Complex–Partition-of-Unity Proposition 136.30 there exists a partition of unity $\{f_j \colon X \to [0,1]\}_{j\in J}$ subordinate to the

open cover $\{U_i\}_{i\in I}$. It follows immediately from the definitions that $\{f_j^{-1}((0,\infty))\}_{j\in J}$ is a locally finite open cover that is a refinement of $\{U_i\}_{i\in I}$. ■

136.8. Exercises for Chapter 136.

Exercise 136.1. In the figure below we consider various subspaces of $\mathbb{R}^2$ with filtrations. Which of these filtrations is a CW-structure?



Exercise 136.2. Let $v, e, f \in \mathbb{N}_0$ with $v - e + f = 2$ and $v, f \geq 1$. Show that there exists a CW-structure for S^2 with v 0-cells, e 1-cells and f 2-cells.

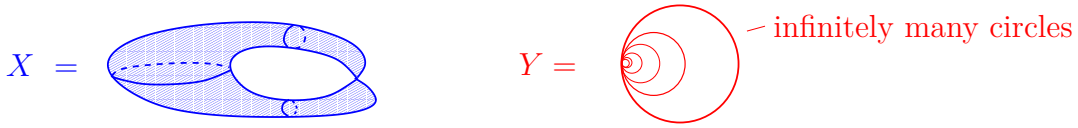
Exercise 136.3. Classify all CW-structures on S^1.

Exercise 136.4. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ let $\Sigma_{g,n}$ be the surface of genus g minus n open disks, as introduced in Notation 101.12. Provide an explicit CW-structure for $\Sigma_{g,n}$.

Exercise 136.5.

(a) Provide a CW-structure for $\overline{B}^n$.
(b) Does every compact subset of $\mathbb{R}^2$ admit a CW-structure?
(c) Let $n \in \mathbb{N}$. Does the open ball $B_1^n(0)$ admit a CW-structure?
(d) Let C be the Cantor set from Definition 3.45. Does C admit a CW-structure?

Exercise 136.6. Which (if any) of the two subspaces of $\mathbb{R}^3$ that are shown in the figure admit a CW-structure?

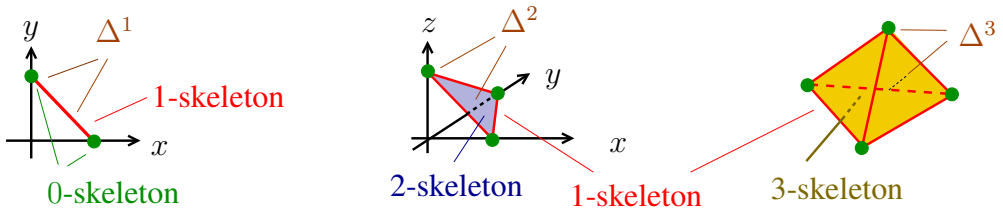


Exercise 136.7. Let $k \in \mathbb{N}_0$.

(a) We consider the filtration of the standard k-simplex Δ^k that is given by

$$(\Delta^k)^i \;=\; \{(t_0,\ldots,t_k) \in \Delta^k \,|\, \text{at least } k-i \text{ coordinates are zero}\}$$

for $i = 0,\ldots,k$. Show that this defines a CW-structure on Δ^k.
(b) How many i-cells does this CW-structure have?



Exercise 136.8. We consider the sphere S^2 with the CW-structure given by $\varnothing \subseteq \{N\} \subseteq S^2$. Show that there exist two characteristic maps $\Phi\colon \overline{B}^2 \to S^2$ and $\Psi\colon \overline{B}^2 \to S^2$ for the unique 2-cell, such that Φ and Ψ do *not* differ by a homeomorphism of $\overline{B}^2$.

Exercise 136.9. Given a compact topological space X that admits a CW-structure we define

$$\Theta(X) \coloneqq \text{minimal number of cells in any CW-structure for } X.$$

(a) What is $\Theta(S^n)$?
(b) What is $\Theta(S^1 \times S^1)$?
(c) What is $\Theta(\mathbb{R}\mathrm{P}^2)$?
(d) What is $\Theta(\text{Klein bottle})$?
(e) Let $n \in \mathbb{N}_0$. What is $\Theta(\overline{B}^n)$?

Exercise 136.10. Let X be a connected CW-complex and let $A \subseteq X$ be a subspace. We suppose that A has the following property: For every open cell c of X with $c \cap A \neq \varnothing$ we have $\overline{c} \subseteq A$. Show that A, together with the filtration given by $A \cap X^i$, is a subcomplex of A.

Exercise 136.11.

(a) Let X be a CW-complex that is a 1-dimensional topological manifold. Let $v \in X^0$.
  (i) Show that there exists at most two 1-cells that contain v.
  (ii) Show that v is contained in a single 1-cell if and only if v is a boundary point.
  Hint. What happens with open neighborhoods of v if you remove v?
(b) By the Smooth Manifold-CW-Theorem 179.9 we know that every compact connected n-dimensional smooth manifold M admits a finite CW-structure. Use this theorem to show that every compact connected non-empty 1-dimensional smooth manifold is homeomorphic to $[0, 1]$ or to S^1.

Exercise 136.12. Let X be a connected CW-complex. Show that the following statements are equivalent:

(a) X is second-countable.
(b) X is locally finite.

Exercise 136.13. Let X be a CW-complex.

(a) Show that the inclusion-induced map $\pi_0(X^0) \to \pi_0(X)$ is a surjection.
(b) Show that for every $k \geq 1$ the inclusion-induced map $\pi_0(X^k) \to \pi_0(X)$ is a bijection.

Hint. First consider the case that X is finite.

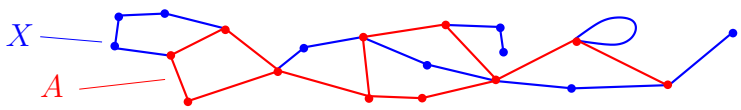
Exercise 136.14. Let X be a CW-complex consisting of precisely two cells. Is X necessarily homeomorphic to a sphere?

Exercise 136.15. Let X be a finite CW-complex. Show that there exists an $n \in \mathbb{N}_0$ and a map $\varphi\colon |X| \to \mathbb{R}^n$ that is an embedding.

Exercise 136.16. Let X be a CW-complex. Is the union of cells necessarily a closed subset of X?

Exercise 136.17.

(a) Let X be a 1-dimensional CW-complex and let $A \subseteq X$ be a subcomplex. Show that if A is connected and non-empty, then A is a retract of X.



(b) Let X be any CW-complex. Is every connected non-empty subcomplex a retract of X?

Exercise 136.18. Let X be a connected CW-complex and let x_0 be a point in the 0-skeleton of X. Show that for any $k \geq 2$ the inclusion-induced map $\pi_1(X^k, x_0) \to \pi_1(X, x_0)$ is an isomorphism.
Remark. Make use of the Cell Attachment-π_1-Proposition 135.5 and apply the logic of the proof of the Direct Colimit-π_1-Proposition 105.1.

Exercise 136.19. Let X be a CW-complex with countably many cells.

(a) Show that there exists a sequence $A_1 \subseteq A_2 \subseteq \dots$ of *finite* subcomplexes A_i of X such that $X = \bigcup_{i \in \mathbb{N}_0} A_i$.
(b) If X is connected, then show that we can arrange in (a) that every A_i is connected.

Exercise 136.20. Show that given any finite connected CW-complex Y there exists a sequence of CW-complexes $Y_0 = \{*\} \subseteq Y_1 \subseteq \dots \subseteq Y_m = Y$ where each Y_{i+1} is obtained from Y_i by attaching a single cell and where each Y_{i+1} is connected.
Hint. Make use of the Spanning Tree–Existence Proposition 11.26.

Exercise 136.21. What is the minimal $n \in \mathbb{N}_0$ that not every n-cell is a subcomplex.

Exercise 136.22. In the following we consider the wedge $X := \bigvee_{i \in \mathbb{N}_0} [0,1]_i$ given by "wedging" the points $0 \in [0,1]_i$, $i \in \mathbb{N}_0$.

(a) Show that the sequences of points $x_i := \frac{1}{2^i} \in [0,1]_i \subseteq X$, $i \in \mathbb{N}_0$ does not converge to the wedge point.
(b) Show that X is not metrizable, i.e. show that it does not admit a metric such that the topology given by the metric agrees with the topology of X.

Remark. This shows that CW-complexes are in general not metrizable.

Exercise 136.23. Use the results of Chapter 15 to show that every finite CW-complex X of dimension n admits an embedding $X \to \mathbb{R}^{2n+1}$.

CHAPTER 137

Homotopies on CW-complexes

In this chapter we study homotopies on CW-complexes. The technical tool for doing so is the all important CW×Interval Proposition 137.1. We will use this result to show that the infinite-dimensional sphere S^∞ is contractible, we will use this result to show that CW-complexes are locally contractible and we will use it to prove a convenient variation on the Seifert–van Kampen Theorem 99.1.

This chapter is organized as follows:

- In Section 137.1 we will show that the product of a CW-complex with an interval has an obvious CW-structure.
- In Section 137.2 we will study trees.
- In Section 137.3 we will consider the fundamental group of 1-dimensional CW-complexes.
- In Section 137.4 we will study local properties of CW-complexes
- In Section 137.5 we will state and prove the CW–Seifert–van Kampen Theorem 137.10.
- In Section 137.6 we will provide exercises for Chapter 137.

In the next Chapter 138 we will discuss coverings of CW-complexes.

137.1. **The topology of the product of a CW-complex with an interval.** We want to study homotopies of continuous maps $X \to Y$ between topological spaces. In other words, we want to study *continuous* maps $X \times [0,1] \to Y$. In general it is quite difficult to show that a given map out of a product is continuous. Fortunately the following important proposition simplifies life quite a bit if X is a CW-complex.

Proposition 137.1. (CW×Interval Proposition) Let X be a CW-complex.

(1) The following three topologies on $X \times [0,1]$ agree:
   (a) The product topology on the product $X \times [0,1]$.
   (b) The topology on $X \times [0,1]$ viewed as the direct colimit $\operatorname{colim}_{\longrightarrow} (\underbrace{X^k \times [0,1]}_{\text{product topology}})$.
   (c) The topology which is defined on the set $X \times [0,1]$ by the condition that a given subset $U \subseteq X \times [0,1]$ is open if and only if for each characteristic map $\Phi\colon \overline{B}^n \to X$ the preimage $(\Phi \times \mathrm{id}_{[0,1]})^{-1}(U)$ is an open subset of $\overline{B}^n \times [0,1]$.[1416]

(2) Let $f\colon X \times [0,1] \to Y$ be a map to a topological space Y. The following statements are equivalent:
   (a) The map f is continuous if $X \times [0,1]$ is equipped with the product topology.
   (b) For every $k \in \mathbb{N}_0$ the restriction of f to $X^k \times [0,1]$ is continuous.

(c) For every characteristic map $\Phi\colon \overline{B}^n \to X$ of a cell of X the map
$$\overline{B}^n \times [0,1] \xrightarrow[\Phi\times \mathrm{id}]{(v,t)\mapsto(\Phi(v),t)} X \times [0,1] \xrightarrow{f} Y$$
is continuous.

Remark. Since $[0,1]$ is regionally compact we see that the equivalence of (a) and (b) in the CW×Interval Proposition 137.1 (1) is a special case of the Direct Colimit-Product Topology Proposition 104.24. □

Proof. Let X be a CW-complex. Note that Statement (2) of the proposition is an immediate consequence of Statement (1) together with the Topological-Direct Colimit Proposition 104.10 (1∗). Thus it remains to show that the three topologies (a), (b) and (c) on $X \times [0,1]$ agree.

First we show that any subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b). Thus let $U \subseteq X \times [0,1]$ be a subset that is open with respect to the product topology on $X \times [0,1]$. We need to show that U is open in $\operatorname{colim}(X^k \times [0,1])$. By definition of the direct colimit topology this means that we need to show that for each $k \in \mathbb{N}_0$ the intersection $U \cap (X^k \times [0,1])$ is open in $X^k \times [0,1]$ where we equip $X^k \times [0,1]$ with the product topology. By the Product Topology-Basics Lemma 7.6 (1a) we know that the subspace topology is compatible with taking products. In our context this implies that the product topology on $X^k \times [0,1]$ conveniently enough agrees with the subspace topology coming from $X \times [0,1]$. It follows from this discussion that $U \cap (X^k \times [0,1])$ is indeed open in $X^k \times [0,1]$.

Next we show that any subset that is open with respect to (b) is also open with respect to (c). Thus let $\Phi\colon \overline{B}^n \to X$ be a characteristic map. It follows from the Topological-Product Proposition 7.2 (1) that the map $\Phi \times \mathrm{id}_{[0,1]}\colon \overline{B}^n \times [0,1] \to X \times [0,1]$ is continuous. This implies that for any open subset $U \subseteq X \times [0,1]$ the preimage $(\Phi \times \mathrm{id}_{[0,1]})^{-1}(U)$ is an open subset of $\overline{B}^n \times [0,1]$.

At this point it suffices to show that any set that is open with respect to (c) is also open with respect to (a). This is by far the most interesting direction of the proposition. Thus let $W \subseteq X \times [0,1]$ be a subset that is open with respect to (c). We need to show that W is also open with respect to the product topology on $X \times [0,1]$.

By the JH-Openness Criterion 3.3 and the Definition 7.3 of the product topology on $X \times [0,1]$ it suffices to prove the following statement:

(∗) For each $(x_0, t_0) \in W$ there exists a neighborhood V of $x_0 \in X$ and a neighborhood K of $t_0 \in [0,1]$ such that $V \times K$ is contained in W.

For the remainder of the proof we fix a point $(x_0, t_0) \in W$. To prove (∗) for (x_0, t_0) we start out with the following claim:

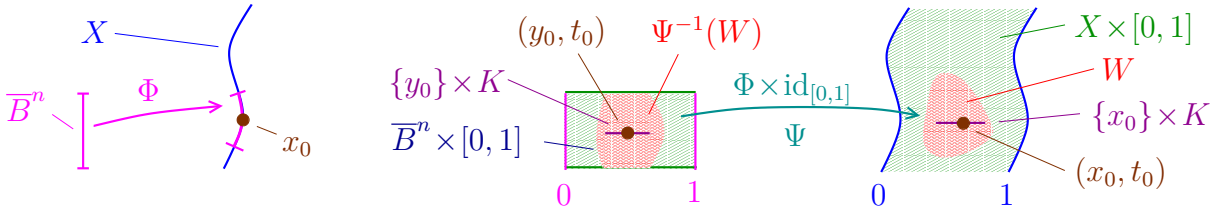
[1414]Note that it follows from the Product Topology-Basics Lemma 7.6 (1a) that all the natural maps $X^k \times [0,1] \to X^{k+1} \times [0,1]$ are embeddings. Thus the direct colimit exists by the Topological-Direct Colimit Proposition 104.10.

[1415]Recall that we showed in the Topological-Direct Colimit Proposition 104.10 that this is the topology where a given subset W of $X \times [0,1] = \operatorname{colim}(X^k \times [0,1])$ is open if and only each intersection $W \cap (X^k \times [0,1])$ is an open subset of $X^k \times [0,1]$.

[1416]It follows from the Image-Preimage Lemma 1.9 that this recipe defines a topology on $X \times [0,1]$.

Claim 1. There exists a compact neighborhood K of $t_0 \in [0,1]$ such that $\{x_0\} \times K \subseteq W$.

Proof. Let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of the open cell that contains $x_0 \in X$. Note that $\Phi\colon B^n \to X$ is injective. Thus there exists a unique $y_0 \in B^n$ with $\Phi(y_0) = x_0$. We consider the map $\Psi := \Phi \times \mathrm{id}_{[0,1]}\colon \overline{B}^n \times [0,1] \to X \times [0,1]$.



It follows from the Topological-Product Proposition 7.2 (1) that Ψ is continuous. Since W is open with respect to (c) we know that $\Psi^{-1}(W)$ is an open subset of $\overline{B}^n \times [0,1]$. Since $(y_0, t_0) \in \Psi^{-1}(W)$ we obtain from the Definition 7.3 of the product topology on $\overline{B}^n \times [0,1]$ that there exists in particular an open neighborhood J of $t_0 \in [0,1]$ such that $\{y_0\} \times J \subseteq \Psi^{-1}(W)$. Since $[0,1]$ is regionally compact we see that there exists a compact neighborhood K of $t_0 \in [0,1]$ that is contained in J. Now we see that

$$\{x_0\} \times K \;=\; \{\Phi(y_0)\} \times K \underset{\text{definition of } \Psi}{=} \Psi(\{y_0\} \times K) \subseteq \Psi(\{y_0\} \times J) \underset{\text{since } \{y_0\} \times J \subseteq \Psi^{-1}(W)}{\subseteq} \Psi(\Psi^{-1}(W)) \underset{\text{Image-Preimage Lemma 1.9 (2)}}{\subseteq} W.$$

Thus K has the desired property. ⊞

Next we consider $V := \{x \in X \mid \{x\} \times K \subseteq W\}$. By the choice of K we have $x_0 \in V$. Thus to prove (∗) it remains to prove the following claim:

Claim 2. The subset V is open in X.

Proof. We use the Characteristic Maps Proposition 136.9 (1) to show that V is open in X. Thus let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of an n-cell of X. We need to show that the preimage $\Phi^{-1}(V)$ is open in $\overline{B}^n$. We write $U := (\Phi \times \mathrm{id}_{[0,1]})^{-1}(W) \subseteq \overline{B}^n \times [0,1]$. Since W is open with respect to (c) we see that U is an open subset of $\overline{B}^n \times [0,1]$. Basically by definition we have the equality

$$\Phi^{-1}(V) \;=\; \{y \in \overline{B}^n \mid \{y\} \times K \subseteq U\}.$$

It follows easily from the Tube Lemma 7.12 that the set on the right-hand side is indeed open. This concludes the proof of the second claim and thus of the lemma. ■

We now consider the infinite-dimensional sphere S^∞

$$S^\infty \;:=\; \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \;\middle|\; \sum_{j=1}^{\infty} x_j^2 = 1\right\} \;=\; \operatorname*{colim}_{\longrightarrow} S^i$$

from Definition 104.16. On page 2643 we showed that $\pi_n(S^\infty) = 0$ for every $n \in \mathbb{N}$. Thus none of the homotopy groups prevent S^∞ from being contractible. Indeed the next mind-bending lemma shows that S^∞ is, in contrast to the finite-dimensional spheres, contractible.

Lemma 104.25. **(S^∞–Contractibility Lemma)** The infinite-dimensional sphere S^∞ is contractible.

Remark. Much later, in the Whitehead Deformation Retract Proposition 305.2 (2) we will show that a non-empty connected CW-complex X with $\pi_n(X) = 0$ for all $n \in \mathbb{N}$ is in fact

contractible. On page 2628 we showed that S^∞ admits a CW-structure and on page 2643 we showed that $\pi_n(S^\infty) = 0$ for all $\in \mathbb{N}$. Thus we have shown that this general proposition applies to S^∞. In other words, the S^∞–Contractibility Lemma 104.25 can be viewed as a special case of the Whitehead Deformation Retract Proposition 305.2 (2). □

Proof. We want to show that S^∞ is contractible. Note that by the Contractability Criterion Lemma 19.20 it suffices to show that there exists a homotopy H from id_{S^∞} to a constant map. Our goal is to find such H. We set

$$X \coloneqq \{(0, x_2, x_3, \dots) \in \mathbb{R}^\infty \mid x_2, x_3, \dots \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{aligned} F\colon \mathbb{R}^\infty \times [0,1] &\to \mathbb{R}^\infty \\ (P,t) &\mapsto \underbrace{P\cdot(1-t) + (0,P)\cdot t}_{\neq 0 \text{ if } P \neq 0} \end{aligned} \quad \text{and} \quad \begin{aligned} G\colon X \times [0,1] &\to \mathbb{R}^\infty \\ (Q,t) &\mapsto \underbrace{Q\cdot(1-t) + (1,0,\dots)\cdot t}_{\neq 0 \text{ if } Q \neq 0}. \end{aligned}$$

Note that F_1 takes values in X and note that G_1 is a constant map. For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t) \in S^\infty$ and $\widetilde{G}(P,t) \in S^\infty$ to be the normalizations of $F(P,t)$ and $G(P,t)$ to length one. Next we consider the map[1417]

$$\begin{aligned} H\colon S^\infty \times [0,1] &\to S^\infty \\ (P,t) &\mapsto \begin{cases} \widetilde{F}(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ \widetilde{G}(\underbrace{\widetilde{F}(P,1)}_{\in X}, 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

Clearly $H_0 = \mathrm{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show that H is continuous. We will do so in two steps:

Claim 1. For each $n \in \mathbb{N}$ the map

$$\begin{aligned} H\colon S^n \times [0,1] &\to S^\infty \\ (x,t) &\mapsto H((x,0),t) \end{aligned}$$

is continuous.

Proof. Let $n \in \mathbb{N}_0$. Note that $H(S^n \times [0,1]) \subseteq S^{n+1}$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} S^n \times [0,1] & \xrightarrow{H} & S^\infty \\ & \searrow^{H} & \uparrow \\ & & S^{n+1} \end{array}.$$

By the Topological-Direct Colimit Proposition 104.10 (1a) we know that the vertical map is a continuous. It follows from this discussion that it suffices to show that for each $n \in \mathbb{N}_0$ the diagonal map is a continuous.

By the Homotopy Combination Lemma 18.8 (1) it suffices to show that for each $n \in \mathbb{N}_0$ the restriction of $H\colon S^n \times [0,1] \to S^{n+1}$ to the subspaces $S^n \times [0, \frac{1}{2}]$ and $S^n \times [\frac{1}{2}, 1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ⊞

Claim 2. The map H is continuous.

[1417] One can easily verify that the map is well-defined for $t = \frac{1}{2}$.

Proof. There are two approaches to proving the claim:

(1) First recall that on page 2628 we saw that we can equip S^∞ with a CW-structure where each i-skeleton is given by $S^i \subseteq S^\infty$. The claim now follows from Claim 1 together with the CW×Interval Proposition 137.1 (2).
(2) One can use Claim 1 together with the Direct Colimit–Product Topology Proposition 104.24 and the Topological-Direct Colimit Proposition 104.10 (1b). ∎

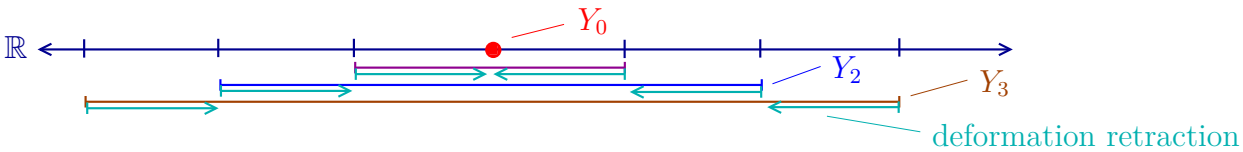
137.2. **Trees.** In this section we want to return to the study of "trees", which we first encountered in Definition 11.21. Our main result on trees builds on the following proposition, which is of interest in its own right, and which is an application of the CW×Interval Proposition 137.1 (2).

Proposition 137.2. (Iterated Deformation Retract Proposition) Let X be a CW-complex and let $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ be a sequence of subcomplexes of X such that the following two conditions are satisfied:

(1) we have $X = \bigcup_{k\in\mathbb{N}_0} Y_k$,
(2) each Y_k is a deformation retract of Y_{k+1}.

Then Y_0 is a deformation retract of X.

Example. We consider the CW-complex $\mathbb{R}$ with the usual CW-structure that is given by $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$. For $k \in \mathbb{N}_0$ we consider the subcomplex $Y_k := [-k, k]$. Evidently each Y_k is a deformation retract of Y_{k+1}. Thus we obtain from the Iterated Deformation Retract Proposition 137.2 that $\{0\}$ is a deformation retraction of $\mathbb{R}$.



This statement is not very surprising. But it is perhaps somewhat puzzling how one can get out of this infinite family of deformation retractions from Y_{k+1} to Y_k a deformation retraction from $\mathbb{R}$ to $\{0\}$. □

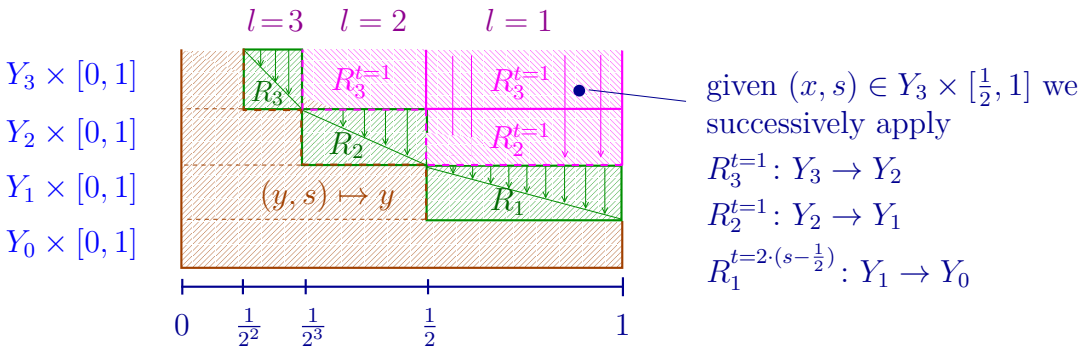
Proof. By hypothesis there exists for each $k \in \mathbb{N}_0$ a deformation retraction

$$R_k \colon Y_k \times [0,1] \;\to\; Y_k$$

from Y_k to Y_{k-1}. We consider the map

$$F \colon X \times [0,1] \;\to\; X$$
$$(x,s) \;\mapsto\; \begin{cases} x, & \text{if } x \in Y_k \text{ and } s \in [0, \frac{1}{2^k}], \\ \underbrace{(R_l^{t=2^l\cdot(s-\frac{1}{2^l})} \circ R_{l+1}^{t=1} \circ \dots \circ R_k^{t=1})(x)}_{\in Y_{l-1}}, & \text{if } x \in Y_k \text{ and } s \in [\frac{1}{2^l}, \frac{1}{2^{l-1}}] \text{ with } 1 \le l \le k. \end{cases}$$

One can easily verify that this map is well-defined. It follows from the Homotopy Combination Lemma 18.8 (1) that the restriction of F to each $Y_k \times [0,1]$ is continuous. Thus it follows from the CW×Interval Proposition 137.1 (2) that $F \colon X \times [0,1] \to X$ is continuous. This map is the desired deformation retraction from X to Y_0. ∎



Next we turn to the study of trees. In Definition 11.21 we defined what it means for an abstract graph to be a tree. We now define what it means for a CW-complex to be a tree:

Definition 137.3. A simply connected CW-complex of dimension ≤ 1 is called a tree.

Remark. As we just mentioned, in Definition 11.21 we defined what it means for an abstract graph to be a tree. In the Graph–CW-Complex Lemma 136.5 we saw that abstract graphs and CW-complexes of dimension ≤ 1 are essentially the same. It follows from the Tree–π_1-Proposition 99.18 that the two definitions of a tree are compatible. □

In the Finite Tree–Contractible Proposition 96.10 we showed that finite trees are contractible. Now we can generalize this statement to arbitrary trees:

Proposition 137.4. (Tree–Contractible Proposition) Let T be a non-empty connected CW-complex of dimension ≤ 1. The following statements are equivalent:

(1) T is a tree.
(2) Each 0-cell is a deformation retract of T.
(3) T is contractible.

Proof. Let T be a non-empty connected CW-complex of dimension ≤ 1. The (2)⇒(3)-direction follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 and the (3)⇒(1)-direction follows from the Homotopy Equivalence-π_1-Proposition 90.9.

Thus at this stage it remains to prove the (1)⇒(2)-direction. Let $v_0 \in T$ be a 0-cell. By the Iterated Deformation Retract Proposition 137.2 it suffices to prove the following claim:

Claim. There exists a sequence $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$ of subcomplexes of T with the following properties:

(1) We have $\bigcup_{i \in \mathbb{N}_0} Y_i = T$.
(2) Each Y_i is a deformation retract of Y_{i+1}.

Proof. We construct a sequence $\{v_0\} = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$ of subcomplexes of T as follows:[1418]

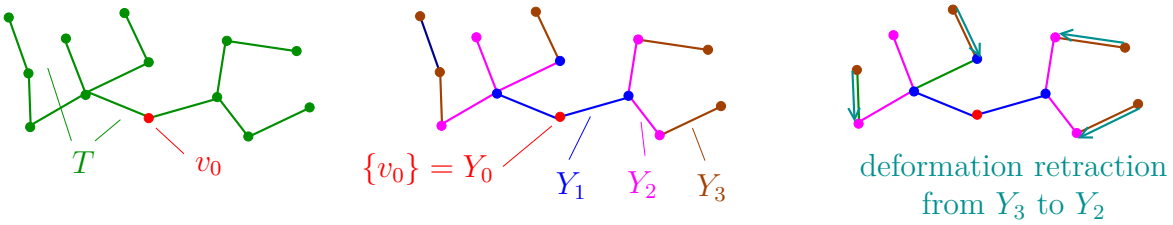
(a) We set $Y_0 := \{v_0\}$.
(b) Given Y_i we define Y_{i+1} to be the union of Y_i with all 1-cells which contain at least one 0-cell of Y_i.

[1418] The construction is very similar to a construction that we gave in the proof of the Spanning Tree–Existence Proposition 11.26.

We make the following observations:

(i) Note that every 1-cell of any CW-complex defines a subcomplex of T.[1419] Thus it follows from the Subcomplex–Intersection-Union Lemma 136.13 that Y_{i+1} is a subcomplex of T.

(ii) We set $Y := \bigcup_{i\in\mathbb{N}} Y_i$. We claim that $T = Y$. This can be seen as follows. First note that, basically by construction of Y, we have for each characteristic map[1420] $\Phi\colon \overline{B}^n \to T$, $n = 0, 1$ that $\Phi^{-1}(Y) = \varnothing$ or $\Phi^{-1}(Y) = \overline{B}^n$. Since T is 1-dimensional it follows from the Characteristic Maps Proposition 136.9 (1) that Y is a subset of T that is closed and that is open. Since T is connected and since Y is non-empty we see that $Y = T$.



It remains to show that each Y_i is a deformation retract of Y_{i+1}. As we will see, this is almost obvious, except that we need to worry a little bit about continuity issues. Let V be the set of 0-cells in Y_{i+1} that are not contained in Y_i. Note that for each $v \in V$ the corresponding 1-cell contains two 0-cells, namely v and one 0-cell w in Y_i. Finally we consider the map

$$\begin{aligned} F\colon \Big(\overbrace{\Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)}^{=Y_{i+1}}\Big) \times [0,1] &\to Y_{i+1} = \Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v) \\ ([P],t) &\mapsto \begin{cases} P, & \text{if } P \in Y_i, \\ [(x\cdot(1-t),v)], & \text{if } P = (x,v) \in [0,1]_v. \end{cases} \end{aligned}$$

It follows almost immediately from the Homotopy Pushout Lemma 18.13 that the map F is continuous. It is now basically clear that F defines a deformation retraction from Y_{i+1} to Y_i. ∎

137.3. **The fundamental group of 1-dimensional CW-complexes.** Our next goal is to use the results from the previous section to calculate the fundamental groups of 1-dimensional CW-complexes. To formulate our theorem we need to introduce the following definition:

Definition 137.5. Let X be a 1-dimensional CW-complex. A spanning tree is a subcomplex T of X that is a tree and that contains all 0-cells of X.[1421]

[1419] This follows from the observation that any non-empty subset of a 0-cell is all of the 0-cell.

[1420] For $n = 0$ the statement is a tautology. For $n = 1$ note that if $\Phi^{-1}(Y) \neq \varnothing$, then there exists an $i \in \mathbb{N}_0$ with $\Phi^{-1}(Y_i) \neq \varnothing$. Since Y_i is a subcomplex we see that $\Phi^{-1}(Y_i)$ contains at least one 0-cell. But by construction this means that $\Phi^{-1}(Y_{i+1}) = \overline{B}^1$.

[1421] In Definition 11.23 we introduced the notion of a spanning tree for an abstract graph. It follows immediately from the definitions together with the Tree–π_1-Proposition 99.18 that the two notions of spanning tree match under the correspondence given by the Graph–CW-Complex Lemma 136.5.



The following theorem can be viewed as a generalization of the Graph-π_1-Proposition 99.13 and the Graph-Homotopy Type Corollary 96.12.

Theorem 137.6. (1-Complex-π_1-Theorem) Let X be a connected non-empty 1-dimensional CW-complex X.

(1) X admits a spanning tree.

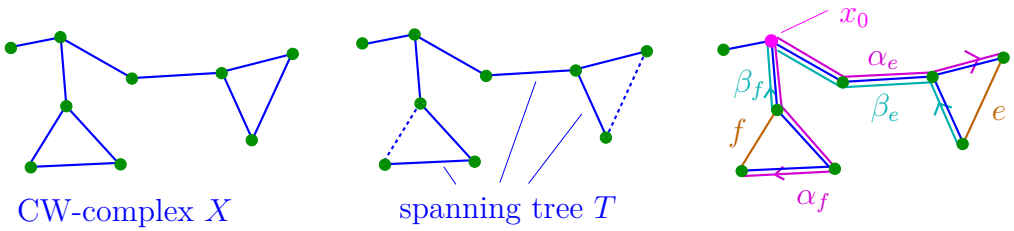
Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) X is homotopy equivalent to $\bigvee_{i \in I} S^1$.
(b) Given any $x_0 \in X$ the group $\pi_1(X, x_0)$ is isomorphic to the free group on the set I.
(c) An explicit description of the isomorphism in (b) is given as follows. Pick a base point x_0 in the spanning tree and for each 1-cell i that is not contained in T we pick a path α_i from x_0 to one of the endpoints and pick a path β_i from the other endpoint back to x_0[1422] and let t_i be the loop given by first going along α_i, then going through the 1-cell i and then going back to the base point via β_i. Then the map

$$\begin{array}{ccc} \langle \{x_i\}_{i \in I} \rangle & \to & \pi_1(X, x_0) \\ x_i & \mapsto & [t_i] \end{array}$$

is an isomorphism.

(3) If the CW-complex X is finite, then we have $\#I = 1 - \chi(X)$, where the Euler characteristic $\chi(X)$ of X is defined as $\chi(X) := \#\{\text{0-cells}\} - \#\{\text{1-cells}\}$.



The proof of the 1-Complex-π_1-Theorem 137.6 rests on the following proposition, which is perhaps of independent interest.

Proposition 137.7. (1-Complex-mod-Tree Proposition) Let X be a 1-dimensional CW-complex. For every subtree T of X the projection $X \to X/T$ is a homotopy equivalence.

Proof of the 1-Complex-mod-Tree Proposition 137.7. The proof of the proposition is basically identical to the proof of the Graph-mod-Tree Proposition 96.11. We just need to replace the Finite Tree–Contractible Proposition 96.10 by the Tree–Contractible Proposition 137.4. Alternatively in the Quotient-by-Contractible Corollary 140.4 we will prove

[1422] The two paths exist since a tree is in particular connected.

a more general result, which also shows how to deduce the 1-Complex-mod-Tree Proposition 137.7 from the Tree–Contractible Proposition 137.4. ■

Proof of the 1-Complex-π_1-Theorem 137.6. Let X be a connected non-empty 1-dimensional CW-complex X.

(1) It follows almost immediately from the Spanning Tree–Existence Proposition 11.26 together with the correspondence given by the Graph–CW-Complex Lemma 136.5 that X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) By the 1-Complex-mod-Tree Proposition 137.7 we know that the projection $X \to X/T$ is a homotopy equivalence. It follows from the Abstract Graph–Quotient Lemma 11.32 (or alternatively the more general CW-Quotient Lemma 139.10 that X/T admits the structure of a 1-dimensional CW-complex with a single 0-cell and one 1-cell for each $i \in I$. It follows from the discussion on page 1865 (or from the more general Skeleton-Quotient Lemma 139.12) that X/T is homeomorphic to the wedge $\bigvee_{i\in I} S^1$.

(b) This statement follows from (a) together with the Homotopy Equivalence-π_1-Proposition 90.9 and the Wedge-of-Circles–π_1-Lemma 99.11.

(c) Let x_0 be a point in T. We denote by $p\colon X \to X/T$ the projection. We write $* = p(x_0)$. In (b) we saw that we have the following explicit isomorphisms:

$$\pi_1(X, x_0) \underset{\cong}{\xrightarrow{p_*}} \pi_1(X/T, *) \xrightarrow[\uparrow]{\cong} \pi_1\Big(\bigvee_{e\in E} S^1_e, *\Big) \xleftarrow[\uparrow]{\cong} \langle\{x_e\}_{e\in E}\rangle.$$

follows from the Abstract Graph–Quotient Lemma 11.32

isomorphism given by the Wedge-of-Circles–π_1-Lemma 99.11

If one goes through the explicit isomorphism, then one sees that the map given in (c) is indeed an isomorphism.

(3) This statement follows from the Graph-Homotopy Type Corollary 96.12 together with the Graph–CW-Complex Lemma 136.5. ■

137.4. **Local properties of CW-complexes.** The following, rather technical proposition, says that CW-complexes are locally in many ways very well-behaved.

Proposition 137.8. (CW-Complex–Local Properties Proposition)

(1) Let X be a CW-complex, let $x \in X$ and let U be an open neighborhood of x. There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

(2) CW-complexes are locally contractible, locally path-connected, locally simply-connected and semi-locally simply connected.

(3) Every CW-complex is the coproduct (i.e. the disjoint union as a topological space) of its components.



Remark. It follows from the CW-Complex–Local Properties Proposition 137.8 (2) and the Locally Path-Connected Lemma 4.57 that a CW-complex is path-connected if and only if it is connected. Subconsciously we will use this fact frequently. □

Proof. Note that Statement (2) follows immediately from (1) since every topological space that admits a deformation retraction to a point is contractible, since every contractible topological space is path-connected and simply connected and since every topological space that is locally simply connected is in particular semi-locally simply connected. Also note that Statement (3) follows immediately from Statement (2), together with the Locally Path-Connected Lemma 4.57 and the Disjoint Union Embedding Lemma 7.29.

So let us turn to the proof of Statement (1). We start out with some notation. Given a subset A of $\overline{B}^n$ and $\epsilon > 0$ we denote by

$$N_\epsilon(A) \coloneqq \big\{x \in \overline{B}^n \,\big|\, \text{there exists an } y \in A \text{ with } \|x-y\| < \epsilon\big\} = \bigcup_{y\in A} (B^n_\epsilon(y) \cap \overline{B}^n)$$

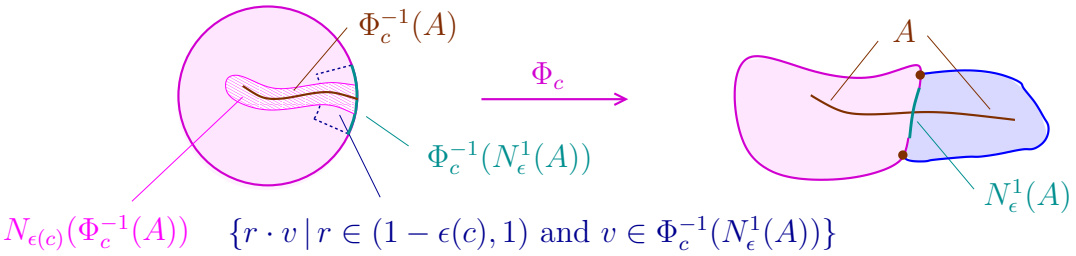
the ϵ-neighborhood around A. It follows from the second description that $N_\epsilon(A)$ is an open subset of $\overline{B}^n$.

Now let X be a CW-complex. We pick a full set $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)}$ of characteristic maps for the cells of X.

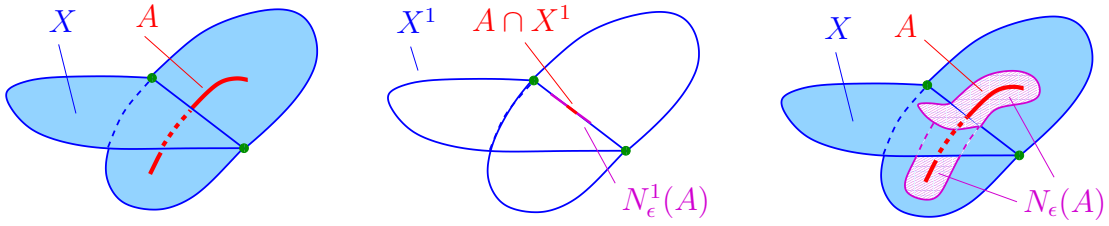
Let $A \subseteq X$ be a subset and let $\epsilon \colon \mathrm{Cell}(X) \to (0,1)$ be a map. Given $n \in \mathbb{N}_0$ we define a subset $N^n_\epsilon(A) \subseteq X$ inductively as follows:

- For $n = 0$ we set $N^0_\epsilon(A) \coloneqq X^0 \cap A$.
- If $N^n_\epsilon(A)$ is already been defined, then we introduce the following subset of X^{n+1}:

$$N^{n+1}_\epsilon(A) \coloneqq N^n_\epsilon(A) \cup \bigcup_{c\in \mathrm{Cell}_{n+1}(X)} \Phi_c\big(N_{\epsilon(c)}(\underbrace{\Phi_c^{-1}(A)}_{\subseteq \overline{B}^{n+1}}) \cup \underbrace{\{r\cdot v \,|\, r \in (1-\epsilon(c),1) \;\&\; v \in \overbrace{\Phi_c^{-1}(N^n_\epsilon(A))}^{\subseteq S^n}\}}_{\subseteq B^{n+1}}\big).$$



Finally we set $N_\epsilon(A) \coloneqq \bigcup_{n\in\mathbb{N}_0} N^n_\epsilon(A)$. Using the Characteristic Maps Proposition 136.9 (1) one can easily verify that each $N^n_\epsilon(A)$ is an open neighborhood of $A \cap X^n$ in X^n. It follows from the Direct Colimit–Open Subset Lemma 104.14 that $N_\epsilon(A)$ is an open neighborhood of A in X.

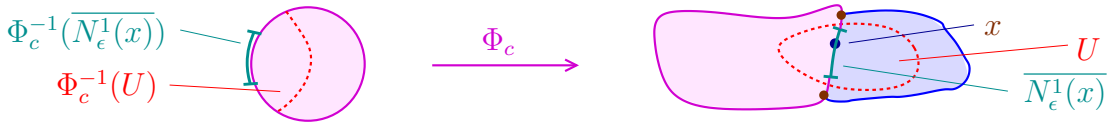


Now we turn to the actual proof of Statement (1). Let $x \in X$ and let U be an open neighborhood of x. We need to prove that there exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$. We denote by m the unique natural number with $x \in X^m \setminus X^{m-1}$.

Claim 1. There exists a map $\epsilon\colon \mathrm{Cell}(X) \to (0,1)$ such that $N_\epsilon(x)$ is contained in U.

Proof. We show a slightly stronger statement: There exists a map $\epsilon\colon \mathrm{Cell}(X) \to (0,1)$ such that each $\overline{N_\epsilon^n(x)}$ is contained in U. For the cells of dimension $< m$ we take arbitrary ϵ-values. We also assign to all m-cells arbitrary ϵ-values, except for the one m-cell that contains x for which we can pick an appropriate ϵ-value such that $\overline{N_\epsilon^m}(x) \subseteq V$. Suppose we constructed $\epsilon\colon \mathrm{Cell}_n(X) \to (0,1)$ for some $n \geq m$ such that $\overline{N_\epsilon^n(x)}$ is contained in $X^n \cap U$.

Let $c \in \mathrm{Cell}_{n+1}(X)$ be an $(n+1)$-cell. By induction $\Phi_c^{-1}(\overline{N_\epsilon^n(x)})$ and $\overline{B}^{n+1} \setminus \Phi_c^{-1}(U)$ are disjoint closed subsets of $\overline{B}^{n+1}$.



It follows from the Distance-to-Subspace Lemma 10.4 that there exists an $\epsilon(c) > 0$ such that the $\epsilon(c)$-neighborhood of $\Phi_c^{-1}(\overline{N_\epsilon^n(X)})$ is still disjoint from $\overline{B}^{n+1} \setminus \Phi_c^{-1}(U)$. This way we can define a suitable map $\epsilon\colon \mathrm{Cell}_{n+1}(X) \to (0,1)$. ⊞

It remains to show that $N_\epsilon(x)$ admits a deformation retraction to $\{x\}$. We start out with the following claim:

Claim 2. There exists a deformation retraction from $N_\epsilon^m(x)$ to $\{x\}$.

Proof. Since $x \in X^m \setminus X^{m-1}$ we see that $N_\epsilon^m(x)$ is contained in an open cell and that $N_\epsilon^m(x)$ is homeomorphic to an open ball. It is thus clear that there exists a deformation retraction from $N_\epsilon^m(x)$ to $\{x\}$. ⊞

It follows from Claim 2 together with the Iterated Deformation Retract Lemma 19.5 that it suffices to find a deformation retraction from $N_\epsilon(x)$ to $N_\epsilon^m(x)$. We will construct this deformation retraction iteratively using the following claim:

Claim 3. For each $n > m$ there exists a homotopy

$$R_n\colon X^n \times [0,1] \;\to\; X^n$$

rel X^{n-1} such that the restriction of R_n to $N_\epsilon^n(x) \times [0,1]$ defines a deformation retraction

$$R_n\colon N_\epsilon^n(x) \times [0,1] \;\to\; N_\epsilon^n(x)$$

from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$.

Proof. Given $\mu \in (0,1)$ we consider the map

$$\begin{aligned} S_\mu\colon \overline{B}^n \times [0,1] &\to \overline{B}^n \\ (x,t) &\mapsto \begin{cases} x \cdot ((1-t) + t \cdot \frac{x}{\|x\|}), & \text{if } \|x\| \geq 1-\mu, \\ x \cdot ((1-t) + t \cdot \frac{1}{1-\mu}), & \text{if } \|x\| \leq 1-\mu. \end{cases} \end{aligned}$$

Using the Pasting Proposition 4.7 one can verify that the map S_μ is continuous. Note that the restriction of the map S_μ to $(\overline{B}^n \setminus \overline{B}^n_{1-\mu}) \times [0,1]$ is a deformation retraction from

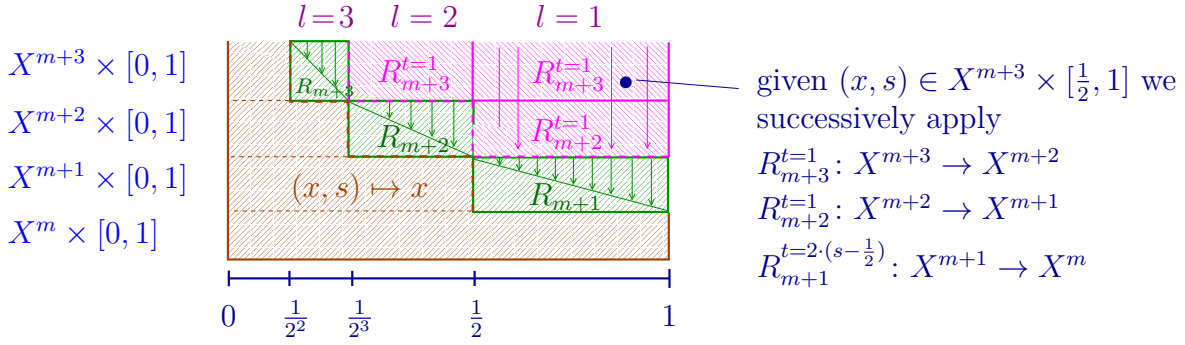
$(\overline{B}^n \setminus \overline{B}^n_{1-\mu})$ to S^{n-1}. We now consider the following map

$$R_n\colon \Big(X^{n-1} \cup \bigsqcup_{c\in \mathrm{Cell}_n(X)} \overline{B}^n_c\Big) \times [0,1] \xleftarrow{\cong} \Big((X^{n-1}\times[0,1]) \sqcup \bigsqcup_{c\in\mathrm{Cell}_n(X)} (\overline{B}^n_c \times [0,1])\Big)/\sim \to X^n.$$

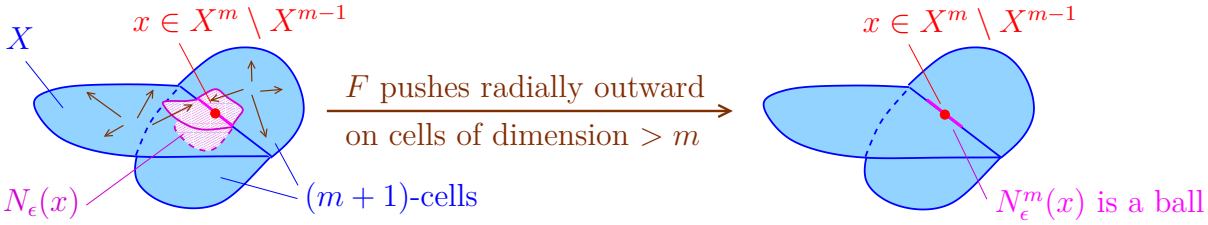
Here the map to the left is the natural homeomorphism from the Homotopy Pushout Lemma 18.13 and the map to the right is induced by the projection $X^{n-1}\times[0,1]\to X^{n-1}$ and by the maps $\Phi_c \circ S_{\epsilon(c)}\colon \overline{B}^n_c \times [0,1] \to X^n$. It follows from the Topological-Coproduct Proposition 7.26 (1∗) and the Topological-Quotient Proposition 8.2 (1∗) that this map R_n is continuous. ⊞

Finally, similar to the discussion in Section 137.2, we consider the map

$$F\colon X\times[0,1] \to X$$
$$(x,s) \mapsto \begin{cases} x, & \text{if } x\in X^{m+k} \text{ and } s\in[0,\frac{1}{2^k}],\\ \underbrace{\big(R^{t=2^l\cdot(s-\frac{1}{2^l})}_{m+l} \circ R^{t=1}_{m+l+1}\circ\cdots\circ R^{t=1}_{m+k}\big)(x)}_{\in X^{m+l-1}}, & \text{if } x\in X^{m+k} \text{ and } s\in[\tfrac{1}{2^l},\tfrac{1}{2^{l-1}}] \\ & \text{with } 1\le l\le k. \end{cases}$$



One can easily verify that this map is well-defined. It follows from the Homotopy Combination Lemma 18.8 (1) that the restriction of F to each $X^k\times[0,1]$ is continuous. Thus it follows from the CW×Interval Proposition 137.1 (2) that $F\colon X\times[0,1]\to X$ is continuous.



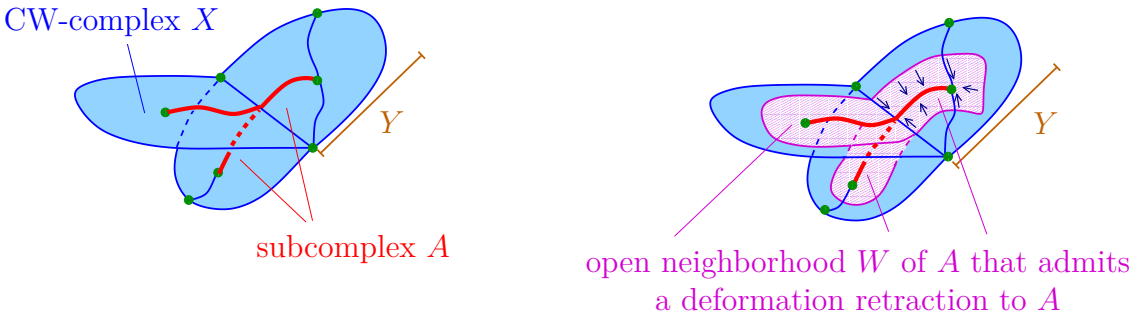
Finally note that it follows from the fact that $F\colon X\times[0,1]\to X$ is continuous, the Product Topology-Basics Lemma 7.6 (1a) and the Subspace Continuity Lemma 4.3 (2) and (3) that $F\colon N_\epsilon(x)\times[0,1]\to N_\epsilon(x)$ is also continuous.[1423] This map is the desired deformation retraction. ■

On several occasions we will need the following variation on the CW-Complex–Local Properties Proposition 137.8.

[1423]Of course one could have just written down the maps $N_\epsilon(x)\times[0,1]\to N_\epsilon(x)$. But without the existence of a continuous extension to all $X\times[0,1]$ it is unclear how one can show that a map $N_\epsilon(x)\times[0,1]\to N_\epsilon(x)$ is continuous.

Proposition 137.9. (Subcomplex–Neighborhood Proposition) Let X be a CW-complex.

(1) For each subcomplex A of X there exists an open neighborhood W of A and a deformation retraction $F\colon W \times [0,1] \to W$ from W to A.
(2) One can choose the deformation retraction such that for each subcomplex Y of X the map F restricts to a deformation retraction $F\colon (Y \cap W) \times [0,1] \to Y \cap W$ from $Y \cap W$ to $Y \cap A$.



Sketch of proof. In the following we will use, without much comment, the notation and some of the results of the proof of the CW-Complex–Local Properties Proposition 137.8.

Let X be a CW-complex. We pick a full set $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)}$ of characteristic maps for the cells of X. For the remainder of the proof we pick an $\epsilon \in (0,1)$ and we denote by $\epsilon\colon \mathrm{Cell}(X) \to (0,1)$ the map that assigns to each cell the value ϵ.

Given a subcomplex A of X we set, as in Section 137.4, $W := N_\epsilon(A)$. In Section 137.4, we already saw that $W = N_\epsilon(A)$ is an open neighborhood of A. Thus it remains to show that A is a deformation retract of $N_\epsilon(A)$. This leads us to the following analog of Claim 3 on page 2661.

Claim 3. For each $n \in \mathbb{N}_0$ there exists a homotopy

$$R_n\colon X^n \times [0,1] \;\to\; X^n$$

rel X^{n-1} such that the restriction of R_n to $N^n_\epsilon(A) \times [0,1]$ defines a deformation retraction

$$R_n\colon N^n_\epsilon(x) \times [0,1] \;\to\; N^n_\epsilon(A)$$

from $N^n_\epsilon(A)$ to $N^{n-1}_\epsilon(A)$.

Proof. As on page 2661 we consider the map

$$\begin{array}{rcl} S_\epsilon\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x\cdot((1-t)+t\cdot\frac{x}{\|x\|}), & \text{if } \|x\| \geq 1-\epsilon, \\ x\cdot((1-t)+t\cdot\frac{1}{1-\epsilon}), & \text{if } \|x\| \leq 1-\epsilon. \end{cases} \end{array}$$

We now consider a map

$$R_n\colon \Big(X^{n-1} \cup \coprod_{c\in \mathrm{Cell}_n(X)} \overline{B}^n_c\Big) \times [0,1] \;\xleftarrow{\cong}\; \Big((X^{n-1}\times[0,1]) \sqcup \coprod_{c\in \mathrm{Cell}_n(X)} (\overline{B}^n_c \times [0,1]\Big)\Big/\sim \;\to\; X^n$$

which is given as follows:

(1) The map to the left is the natural homeomorphism from the Homotopy Pushout Lemma 18.13.

(2) The map to the right is given as follows:
   (a) We use the projection $X^{n-1} \times [0,1] \to X^{n-1}$.
   (b) If $c \subseteq A$, then we use the map $\overline{B}^n_c \times [0,1] \to X^n$ given by $(x,t) \mapsto \Phi_c(x)$. If $c \not\subseteq a$, then we use the map $\Phi_c \circ S_\epsilon \colon \overline{B}^n_c \times [0,1] \to X^n$.

It follows from the Topological-Coproduct Proposition 7.26 (1∗) and the Topological-Quotient Proposition 8.2 (1∗) that this map R_n is continuous. Finally recall that by the Subcomplex–Characterization Proposition 136.12 we know that if an n-cell c is not contained in A, then $c \cap (A^n \setminus A^{n-1}) = \varnothing$. It follows easily from this observation that $R_n \colon N^n_\epsilon(x) \times [0,1] \to N^n_\epsilon(A)$ is indeed a deformation retraction from $N^n_\epsilon(A)$ to $N^{n-1}_\epsilon(A)$. ⊞

As on page 2662 we combine these homotopies, more precisely we consider the map

$$\begin{array}{rcl} F \colon X \times [0,1] & \to & X \\ (x,s) & \mapsto & \begin{cases} x, & \text{if } x \in X^k \text{ and } s \in [0, \frac{1}{2^k}], \\ \underbrace{\big(R_l^{t=2^l\cdot(s-\frac{1}{2^l})} \circ R_{l+1}^{t=1} \circ \ldots \circ R_k^{t=1}\big)(x)}_{\in X^{l-1}}, & \text{if } x \in X^k \text{ and } s \in [\frac{1}{2^l}, \frac{1}{2^{l-1}}] \text{ with } 1 \le l \le k. \end{cases} \end{array}$$

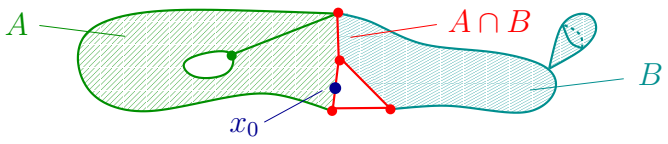
One can now easily verify that the restriction of this map to $N_\epsilon(A) \times [0,1] \to N_\epsilon(A)$ has all the desired properties. ■

137.5. **Seifert–van Kampen Theorem for CW-complexes.** Using the Subcomplex–Neighborhood Proposition 137.9 we can prove the following quite useful variation of the Seifert–van Kampen-Theorem 99.1.

Theorem 137.10. (CW-Seifert–van Kampen Theorem) Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Given any $x_0 \in A \cap B$ the natural map

$$\pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

is an isomorphism.

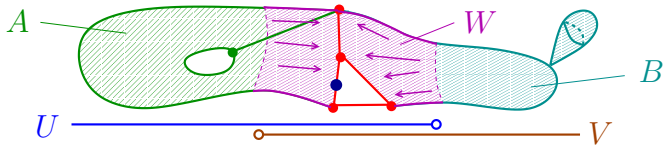


Remark. In view of the Smooth Manifold-CW-Theorem 179.9, which we mentioned already on page 2629, the CW-Seifert–van Kampen Theorem 137.10 can be viewed as a generalization of the Manifold–Seifert-van Kampen Theorem 99.1. □

Sketch of proof of Theorem 137.10. [1424] Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. It follows from the Subcomplex–Intersection-Union Lemma 136.13 that $A \cap B$ is also a subcomplex of X. Let $x_0 \in A \cap B$.

[1424] The proof is similar to the proof of the Manifold–Seifert-van Kampen Theorem 102.1. The key difference is that we need to replace the Smooth Collar Neighborhood Theorem 35.6 by the Subcomplex–Neighborhood Proposition 137.9.

By the Subcomplex–Neighborhood Proposition 137.9 there exists an open neighborhood W of $A \cap B \subseteq X$ and a deformation retraction $F\colon W \times [0,1] \to W$ from W to $A \cap B$ such that $F\colon (A \cap W) \times [0,1] \to A \cap W$ is a deformation retraction from $A \cap W$ to $A \cap B$ and such that $F\colon (B \cap W) \times [0,1] \to B \cap W$ is a deformation retraction from $B \cap W$ to $A \cap B$. We write $U = A \cup (B \cap W)$ and $V = B \cup (A \cap W)$. Note that $U \cap V = W$.



We consider the following diagram:

$$\begin{array}{ccc} \pi_1(A,x_0) *_{\pi_1(A\cap B,x_0)} \pi_1(B,x_0) & \longrightarrow & \pi_1(X,x_0) \\ \downarrow & & \downarrow \mathrm{id} \\ \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) & \xrightarrow{\cong} & \pi_1(X,x_0), \end{array}$$

where the left vertical maps are induced by the inclusions $A \to U$ and $B \to V$. We make the following observations:

(1) Since all maps are induced by inclusions we see that the diagram commutes.

(2) By construction the map F is a deformation retraction from W to $A \cap B$. Recall that F also defines a deformation restriction from $B \cap W$ to $A \cap B$. Together with the stationary homotopy on A the map F thus gives rise to deformation retractions from $U = A \cup (B \cap W)$ to $A \cup (B \cap A) = A$. Similarly we see that F gives rise to a homotopy from $V = B \cup (A \cap W)$ to $B \cup (B \cap A) = B$. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 (2) that the three inclusion-induced maps $\pi_1(A,x_0) \to \pi_1(U,x_0)$, $\pi_1(B,x_0) \to \pi_1(V,x_0)$ and $\pi_1(A \cap B,x_0) \to \pi_1(U \cap V,x_0)$ are isomorphisms.

(3) Since $A \cap B$ is path-connected and since $A \cap B$ is a deformation retract of W we see that $W = U \cap V$ is also path-connected. One can easily verify that U and V are open subsets of X. Therefore we can apply the Seifert–van Kampen-Theorem 99.1 to the decomposition $X = U \cup V$ and we obtain that the bottom horizontal map is an isomorphism.

(4) It follows from (1), (2) and (3) that the top horizontal map is also an isomorphism. ■

137.6. Exercises for Chapter 137.

Exercise 137.1. We consider the infinite-dimensional sphere S^∞. We equip S^∞ with the Euclidean metric

$$d((x_1,x_2,\dots),(y_1,y_2,\dots)) \coloneqq \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

By the Metric-Topology Lemma 3.7 the metric d defines a topology $\mathcal{M}$ on S^∞. Show that $(S^\infty, \mathcal{M})$ is also contractible

Exercise 137.2. Let X be a CW-complex and let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ be a sequence of subcomplexes. We suppose that each inclusion $A_i \to A_{i+1}$ is homotopic to a constant map. Show that $\bigcup_{n\in\mathbb{N}_0} A_n$ is contractible.

Exercise 137.3. Does there exist a 2-dimensional CW-complex that contains both an annulus $S^1 \times [0,1]$ and a Möbius band as deformation retracts?

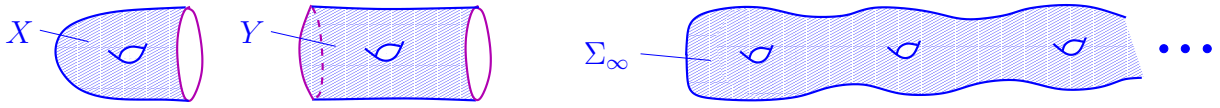
Exercise 137.4. We consider the CW-complex $\mathbb{R}$ where the CW-structure is as usual given by $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$. For each $k \in \mathbb{N}_0$ we consider the subcomplex $Y_k = [-k, k]$. for each $k \in \mathbb{N}_0$ there exists an obvious deformation retraction from Y_{k+1} to Y_k. The proof of the Iterated Deformation Retract Proposition 137.2 gives an explicit deformation retraction from $\mathbb{R}$ to $\{0\}$. Visualize this deformation retraction.

Exercise 137.5.

(a) We consider the surfaces X and Y shown in the figure below. Show that, with a suitable CW-structure, both X and Y admit a deformation retraction to a 1-dimensional subcomplex.

(b) We consider the surface Σ_∞ shown in the figure below. Show that, with a suitable CW-structure, Σ_∞ admits a deformation retraction to a 1-dimensional subcomplex.

Remark. This exercise gives a new proof that the fundamental group of the surface of infinite genus is a free group.



Exercise 137.6. Does every closed subset A of a CW-complex X admit an open neighborhood $U \subseteq X$ such that A is a deformation retract of X?

Exercise 137.7. Let $A = S^1 \times [0, 1]$ be an annulus.

(a) (i) We consider

$$X \coloneqq \Big(\bigcup_{k \in \mathbb{N}_0} A \times \{k\} \Big) / (z, 0, k) \sim (z^2, 1, k-1) \quad \text{for } z \in S^1, k \in \mathbb{N}.$$

Determine the isomorphism type of $\pi_1(X)$.

(ii) We consider

$$Y \coloneqq \Big(\bigcup_{k \in \mathbb{N}_0} A \times \{-k\} \Big) / (z, 0, -k) \sim (z^2, 1, -k-1) \quad \text{for } z \in S^1, k \in \mathbb{N}.$$

Determine the isomorphism type of $\pi_1(Y)$.

In both cases the group you obtain is an abelian group that is isomorphic to a subgroup of $(\mathbb{R}, +)$.

(b) Give an example of topological space Z such that $\pi_1(Z) \cong (\mathbb{Q}, +)$.

Exercise 137.8. Show that any point of S^∞ is a deformation retract of S^∞.

Remark. Note that in the S^∞–Contractibility Lemma 104.25 we proved a related, but weaker statement.

CHAPTER 138

Coverings of CW-complexes

Let X be a connected CW-complex, let $x_0 \in X$ and let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup. By the CW-Complex–Local Properties Proposition 137.8 we know that X is locally simply connected. It follows from the Covering Existence Theorem 127.10 that there exists an (essentially unique) covering $p\colon \widetilde{X} \to X$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = \Gamma$. This discussion shows that CW-complexes tend to have many coverings. In this short chapter we will see that any covering of a CW-complex inherits a natural CW-structure. We will use this topological result to prove several "topology free" statements in group theory.

This chapter is organized as follows:

- In Section 138.1 we will show that coverings of CW-complexes admit a natural CW-structure.
- In Section 138.2 we will show that finite index subgroups of finitely presented groups are again finitely present.
- In Section 138.3 we will show that arbitrary subgroups of free groups are again free.
- In Section 138.4 we will provide exercises for Chapter 138.

In the next Chapter 139 we will discuss various constructions of CW-complexes.

138.1. **Coverings of CW-complexes.** In this section we will see that, as promised above, a covering of a CW-complex inherits a natural CW-structure.

Proposition 138.1. (CW-Complex Covering Proposition) Let $p\colon \widetilde{X} \to X$ be a covering of a CW-complex X.

(1) The filtration
$$\varnothing = p^{-1}(X^{-1}) \subseteq p^{-1}(X^0) \subseteq p^{-1}(X^1) \subseteq \ldots \subseteq \widetilde{X}$$
defines a CW-structure for $\widetilde{X}$ which has the property that for any $k \in \mathbb{N}_0$ we have[14251426]
$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

(2) (a) Each characteristic map $\Phi\colon \overline{B}^n \to X^n$ admits precisely $[\widetilde{X} : X]$ lifts to $\widetilde{X}$.
(b) Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. The set of all lifts of all these characteristic maps is a full set of characteristic maps for the CW-complex $\widetilde{X}$.

Remark. The CW-Complex Covering Proposition 138.1 is a generalization of the Covering-of-Graph Proposition 130.1 where we showed that any covering of a topological graph is

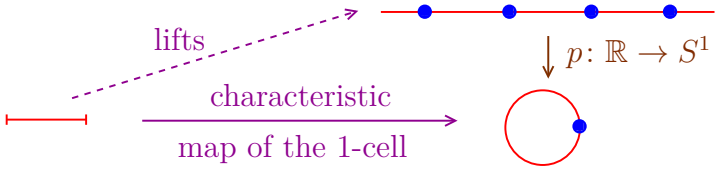
[1425]Recall that in Definition 91.10 we defined $[\widetilde{X} : X] := \#p^{-1}(x)$ where x is any point in X. Since X is non-empty and connected this definition makes sense.

[1426]Note that this is an equality of cardinalities. For example, if $[\widetilde{X} : X]$ is infinite countable and if X has finitely many k-cells, then $\widetilde{X}$ has infinitely many, but countably many, k-cells.

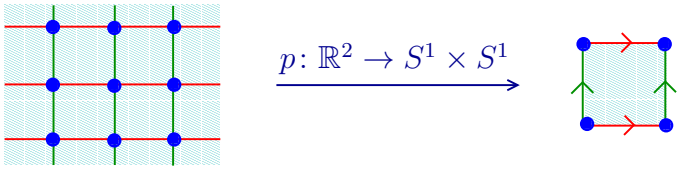
again a topological graph. In fact, as we will see, the two proofs are structurally very similar. □

Examples.

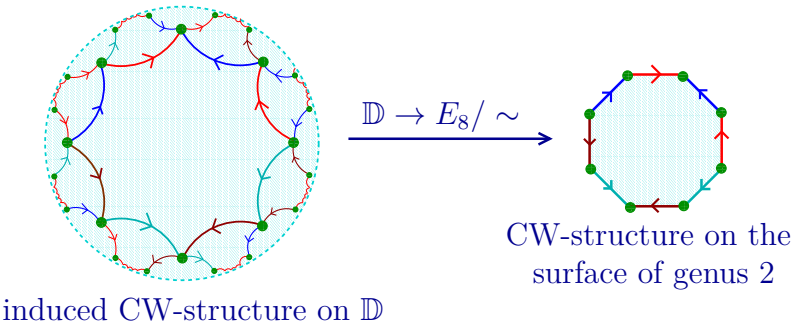
(1) We equip S^1 with the CW-structure $\varnothing \subseteq \{1\} \subseteq S^1$ consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi\,\mathrm{i}t)$ the universal covering of S^1. It follows from the CW-Complex Covering Proposition 138.1, that a CW-structure on $\mathbb{R}$ is given by $\varnothing \subseteq p^{-1}(\{1\}) \subseteq S^1$, i.e. by $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$.



(2) We equip $S^1 \times S^1 =_{\mathrm{i}} ([0,1]\times[0,1])/\sim$ with the usual CW-structure with one 0-cell, two 1-cells and one 2-cell. We consider the universal covering $\mathbb{R}^2 \to S^1 \times S^1$. The resulting CW-structure on $\mathbb{R}^2$ is illustrated in the figure below.



(3) We consider $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. In the proof of Hadamard's Theorem 132.14 we constructed an explicit covering map $\mathbb{D} \to H_8/\sim \cong E_8/\sim$.



If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from the CW-Complex Covering Proposition 138.1 the CW-structure on $\mathbb{D}$ that is shown in the figure below.□

Proof. We start out with an observation. Let $\Phi\colon \overline{B}^n \to X$ be any continuous map. Since each $\overline{B}^n$ is simply connected and locally path-connected we obtain from the Map Lifting Criterion 127.5 and the Lift Uniqueness Proposition 92.2 that for each $y \in p^{-1}(\Phi(0))$ there exists a unique lift $\widetilde{\Phi}_y\colon \overline{B}^n \to \widetilde{X}$ of $\Phi\colon \overline{B}^n \to X$ with $\widetilde{\Phi}_y(0) = y$. Note that this observation already implies (2a).

We turn to the proof of (1) and (2b). We introduce the following notation:

- For $n \in \mathbb{N}_0$ we set $\widetilde{X}^n := p^{-1}(X^n)$.
- Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in\mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. By the above observation we obtain for each $c \in \mathrm{Cell}(X)$ the family of continuous maps $\{\widetilde{\Phi}_{c,y}\colon \overline{B}^{\dim(c)} \to \widetilde{X}\}_{y\in p^{-1}(\Phi_c(0))}$.

Statements (1) and (2b) follow from the following two claims.

Claim 1. A set $U \subseteq \widetilde{X}$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $\widetilde{X}^n \cap U$ is open in $\widetilde{X}^n$.

Claim 2. Let $n \in \mathbb{N}_0$. We consider the map

$$\Theta\colon \widehat{X}^n := \widetilde{X}^{n-1} \cup \bigcup_{c\in \mathrm{Cell}_n(X)} \bigcup_{y\in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; \widetilde{X}^n$$

that is induced by the identity on $\widetilde{X}^{n-1}$ and by the maps $\widetilde{\Phi}_{c,y}\colon \overline{B}^n \to \widetilde{X}^n$. This map is a homeomorphism.

Proof of Claim 1. Since $p\colon \widetilde{X} \to X$ is a covering there exists an open cover $\{\widetilde{W}_i\}_{i\in I}$ of $\widetilde{X}$ such that for each $i \in I$ the image $p(\widetilde{W}_i) =: W_i$ is an open subset of X and that the map $p_i := p|_{\widetilde{W}_i}\colon \widetilde{W}_i \to W_i$ is a homeomorphism. Now we see that

$$\begin{aligned}
&\forall_{n\in\mathbb{N}_0}\ \widetilde{X}^n \cap U \text{ is open in } \widetilde{X}^n \\
\iff\ &\forall_{n\in\mathbb{N}_0}\ \forall_{i\in I}\ \widetilde{W}_i \cap (\widetilde{X}^n \cap U) \text{ is open in } \widetilde{W}_i \cap \widetilde{X}^n && \text{(Cover-Open Subset Lemma 3.20)} \\
\iff\ &\forall_{n\in\mathbb{N}_0}\ \forall_{i\in I}\ p_i(\widetilde{W}_i \cap (\widetilde{X}^n \cap U)) \text{ is open in } p_i(\widetilde{W}_i \cap \widetilde{X}^n) && \text{(since each } p_i \text{ is a homeomorphism)} \\
\iff\ &\forall_{i\in I}\ \forall_{n\in\mathbb{N}_0}\ (W_i \cap p_i(U)) \cap X^n \text{ is open in } W_i \cap X^n && \text{(since } \widetilde{X}^n = p^{-1}(X^n) \text{ and } p(\widetilde{W}_i) = W_i) \\
\iff\ &\forall_{i\in I}\ \forall_{n\in\mathbb{N}_0}\ (W_i \cap p_i(U)) \cap X^n \text{ is open in } X^n && \text{(since } W_i \text{ is open in } X) \\
\iff\ &\forall_{i\in I}\ W_i \cap p_i(U) \text{ is open in } W_i && \text{(since } X \text{ is a CW-complex)} \\
\iff\ &\forall_{i\in I}\ \widetilde{W}_i \cap U \text{ is open in } \widetilde{W}_i && \text{(since } p_i \text{ is a homeomorphism)} \\
\iff\ &U \text{ is open in } \widetilde{X} && \text{(Cover-Open Subset Lemma 3.20).}
\end{aligned}$$

⊞

Proof of Claim 2. First note that one can easily verify that Θ is a bijection. Also note that it follows from the Topological-Pushout Proposition 8.12 (0) that the map Θ is continuous. To show that it is actually a homeomorphism we have to work a little harder.

To do so we consider the map

$$\widehat{p}\colon \widehat{X}^n = \widetilde{X}^{n-1} \cup \bigcup_{c\in \mathrm{Cell}_n(X)} \bigcup_{y\in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; X^n$$

that is induced by the projection $p\colon \widetilde{X}^{n-1} \to X^{n-1}$ and the maps $\overline{B}^n_{c,y} \to \overline{B}^n_c \xrightarrow{\Phi_c} X^n$. It follows again from the Topological-Pushout Proposition 8.12 (0) that $\widehat{p}$ is continuous. Also note that p satisfies $p \circ \Theta = \widehat{p}\colon \widehat{X}^n \to X^n$, i.e. we have the following little commutative diagram:

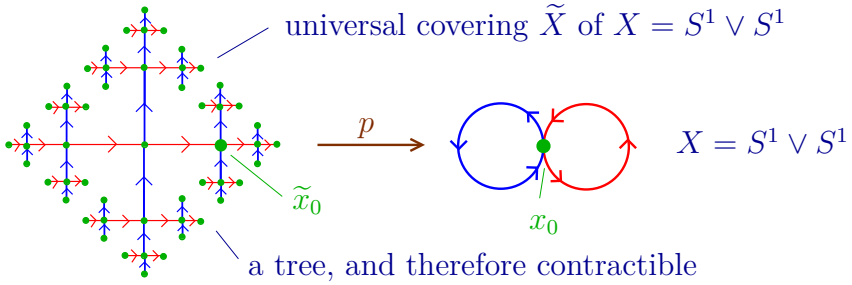
$$\begin{array}{ccc} \widehat{X}^n & \xrightarrow{\ \Theta\ } & \widetilde{X}^n \\ & {\scriptstyle\widehat{p}}\searrow \quad \swarrow{\scriptstyle p} & \\ & X^n. & \end{array}$$

It is not hard to show that $\widehat{p}\colon \widehat{X}^n \to X^n$ is a covering. (We leave it to the reader to write down the details.) We now see that p and $\widehat{p}$ are both local homeomorphisms. It follows from $p \circ \Theta = \widehat{p}$ that Θ is a local homeomorphism. Since Θ is a bijection it follows from the Bijective-Local Homeomorphism Lemma 4.62 that Θ is a homeomorphism. ■

The CW-Complex Covering Proposition 138.1, together with the Tree–Contractible Proposition 137.4, allows us to calculate the higher homotopy groups of 1-dimensional CW-complexes.

Lemma 138.2. (1-Complex-π_n-Lemma) Let X be a connected 1-dimensional CW-complex and let $x_0 \in X$ be a base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.

Proof. [1427] Let X be a connected 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. By the CW-Complex–Local Properties Proposition 137.8 we know that X is locally simply connected. It follows from the Covering Existence Theorem 127.10 that there exists a universal covering $p\colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ is simply connected. We pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$.



We perform the following calculation:

$$\begin{array}{ccccc} \pi_n(X, x_0) & \xleftarrow{\cong} & \pi_n(\widetilde{X}, \widetilde{x}_0) & = & \pi_n(\text{simply connected 1-dimensional CW-complex}) \\ & & & = & \pi_n(\text{contractible topological space}) \quad = \quad 0. \end{array}$$

by the Covering-π_n-Isomorphism Proposition 134.22, since $n \geq 2$

by the CW-Complex Covering Proposition 138.1

by the Tree–Contractible Proposition 137.4

by the Homotopy Equivalence-π_n Proposition 134.16 (2)

■

138.2. **Subgroups of finitely presented groups.** We conclude this chapter with two purely group theoretic results, which we will prove using topological methods. In particular the CW-Complex Covering Proposition 138.1 will play a prominent role in both proofs. We start out with the following result:

Proposition 138.3. (Finite-Index Subgroup–Presentation Proposition) Every finite-index subgroup of a finitely presented group is finitely presented.

Remark.

(1) We will provide a proof of the Finite-Index Subgroup–Presentation Proposition 138.3 that makes heavy use of topological methods. Not surprisingly one can also give a purely group theoretic proof. In fact such a proof can be given by the Reidemeister-Schreier process that we alluded to in Section 113.5. We refer to [**LyS1977**, p. 103] and [**MKS1976**, p. 90] for details.

(2) In Exercise 100.9 we saw that every finite-index subgroup of a finitely generated group is finitely generated. In Exercise 138.3 we will give a new proof of this result by modifying the proof of the Finite-Index Subgroup–Presentation Proposition 138.3. □

[1427] The proof can be viewed as a generalization of the argument provided on page 2589.

We will prove the Finite-Index Subgroup–Presentation Proposition 138.3 by switching from the group-theoretic setting to the topological setting. The key to doing so is the following construction, which associates to any finite presentation a 2-dimensional CW-complex.

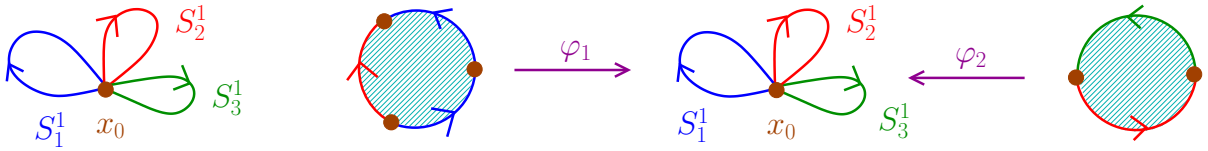
Construction 138.4. Let $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$ be a finite presentation. For $i = 1, \dots, k$ let S^1_i be a copy of S^1. For $j = 1, \dots, m$ we do the following:

(1) We write the relation r_j as a reduced word $g_{s_1}^{\epsilon_1} \cdot \ldots \cdot g_{s_n}^{\epsilon_n}$ with $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$.

(2) We denote by $\varphi_j \colon S^1 =_{\mathrm{i}} [0,1]/0 \sim 1 \to \bigvee_{i=1}^{k} S^1_i$ the (hopefully obvious) path which on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ travels through $S^1_{s_i}$ at constant speed in the direction determined by ϵ_i.

We denote by Y the CW-complex that is given by attaching m 2-cells to $\bigvee_{i=1}^{k} S^1_i$ along the attaching maps $\varphi_1, \dots, \varphi_m$.[1428] We refer to Y as the CW-complex associated to the presentation. Sometimes we also refer to Y as the presentation complex of the presentation.

2-dimensional CW-complex for the presentation
$\langle g_1, g_2, g_3 \,|\, g_1 \cdot g_2^{-1} \cdot g_1, g_3 \cdot g_2 \rangle$



CW-complex is given by attaching two 2-cells to $S^1_1 \vee S^1_2 \vee S^1_3$

Now we can state the following unsurprising lemma.

Lemma 138.5. (Presentation–CW Lemma) Let $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$ be a finite presentation for a group π. If Y is the associated CW-complex, then $\pi_1(Y) \cong \pi$.

Proof of the Presentation–CW Lemma 138.5. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$ be a finitely presented group. We use the notation introduced in the above construction of the CW-complex associated to π. We denote by X the wedge of the k circles and we denote by x_0 the wedge point. By the Wedge-of-Circles–π_1-Lemma 99.11 we can make the identification $\pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. This identification has the property that for $j = 1, \dots, m$ the corresponding loop $\varphi_j \colon S^1 \to X$ represents $r_j \in \pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. We obtain that

$$\begin{array}{ccccc} & \text{by the Cell Attachment-}\pi_1 & & \text{since } [\varphi_j] = r_j & \\ & \text{Proposition 135.5,} & & & \\ & \text{applied } m \text{ times} & & & \\ & \downarrow & & \downarrow & \\ \pi_1(Y, x_0) & \cong & \langle g_1, \dots, g_k \rangle / \langle\langle [\varphi_1] \rangle\rangle \dots / \langle\langle [\varphi_m] \rangle\rangle & = & \langle g_1, \dots, g_k \rangle / \langle\langle r_1 \rangle\rangle \dots / \langle\langle r_m \rangle\rangle \\ & = & \langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_m \rangle\rangle & = & \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle. \\ & \uparrow & & \uparrow & \\ & \text{follows from the Quotient} & & \text{by Definition 100.10} & \\ & \text{Presentation Lemma 100.18} & & & \end{array}$$

■

Proof of the Finite-Index Subgroup–Presentation Proposition 138.3. Suppose we are given a finitely presented group $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$. By the Presentation–CW

[1428] Strictly speaking we need the Iterative Cell-Attachment Lemma 135.10 to argue that we just constructed a CW-complex.

Lemma 138.5 there exists a finite connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$. Now let Γ be a finite-index subgroup of π. By the CW-Complex–Local Properties Proposition 137.8 (2) together with the Covering Existence Theorem 127.10 there exists a finite covering $p\colon \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y}) \cong \Gamma$. By the CW-Complex Covering Proposition 138.1 (1) the 2-complex $\widetilde{Y}$ has finitely many 1-cells and 2-cells. Thus it follows from the CW-π_1-Finite Presentation–Proposition 136.23 that $\Gamma \cong \pi_1(\widetilde{Y})$ is a finitely presented group. ■

138.3. **Subgroups of free groups.** We conclude this chapter with our second application of our topological methods and results to group theory:

Proposition 138.6. (Free Subgroup Proposition)
(1) Every subgroup of a free group (of any rank) is again a free group.
(2) If F is a finitely generated free group and if G is a finite-index subgroup of F, then

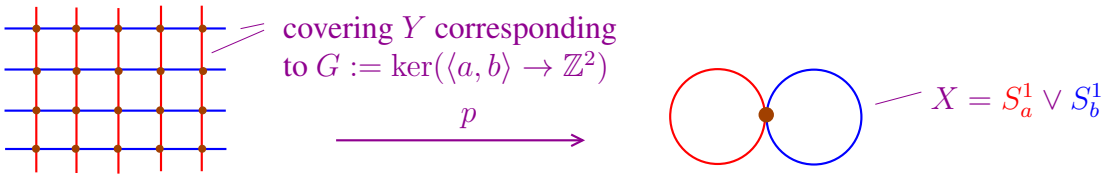
$$\operatorname{rank}(G) - 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Remark.

(1) In the Free Abelian Subgroup Theorem 97.14 we showed that any subgroup of a free *abelian* group is again a free *abelian* group. The Free Subgroup Proposition 138.6 (1) can be viewed as the "non-abelian analog" of the previous result on free abelian groups.
(2) In the Countable Free Subgroup Proposition 130.5 we proved the same result as in the Free Subgroup Proposition 138.6, except that we demanded that the free group is countable.
(3) Of course there are purely group theoretic proofs of the Free Subgroup Proposition 138.6, see e.g. [**Schre1927**], [**HallM1976**, Theorem 7.2.1] or [**Baum1993**, Chapter III.3]. But the topological proof that we will provide below is arguably conceptually the clearest proof. □

Proof. Let $F = \langle S \rangle$ be a free group on a set S and let G be a subgroup of F.

(1) We consider the 1-dimensional CW-complex $X = \bigvee_{s \in S} S^1_s$. By the 1-Complex-$\pi_1$-Theorem 137.6 we have $\pi_1(X, P) \cong F$. By the CW-Complex–Local Properties Proposition 137.8 (2) and the Covering Existence Theorem 127.10 there exists a covering $p\colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$.



Recall that it follows from the Covering-π_1-Monomorphism Proposition 92.7 that the induced map $p_*\colon \pi_1(Y, Q) \to \pi_1(X, P)$ is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to G. By the CW-Complex Covering Proposition 138.1 we can equip Y with the structure of a 1-dimensional CW-complex. We deduce from the 1-Complex-π_1-Theorem 137.6 that $G \cong \pi_1(Y, Q)$ is a free group.

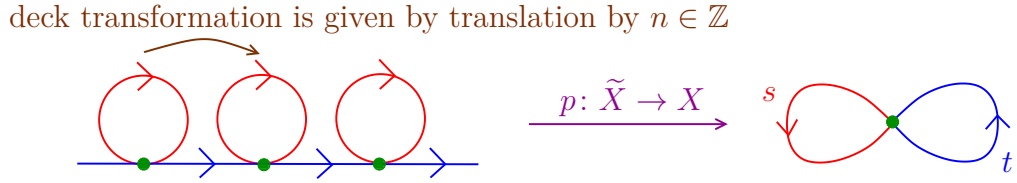
(2) Now we suppose that F is a free group of finite rank and we suppose that G is a finite-index subgroup of F. Then

$$\begin{array}{rcl} & & \text{by the 1-Complex-}\pi_1\text{-Theorem 137.6} \\ & & \downarrow \\ \operatorname{rank}(G) & = & \#\{\text{1-cells of } Y\} - \#\{\text{0-cells of } Y\} + 1 \\ & = & [F:G] \cdot (\underbrace{\#\{\text{1-cells of } X\}}_{=\operatorname{rank}(F)} - \underbrace{\#\{\text{0-cells of } X\}}_{=1}) + 1 \\ & & \uparrow \\ & & \text{by the CW-Complex Covering Proposition 138.1} \\ & = & [F:G] \cdot (\operatorname{rank}(F) - 1) + 1. \end{array}$$

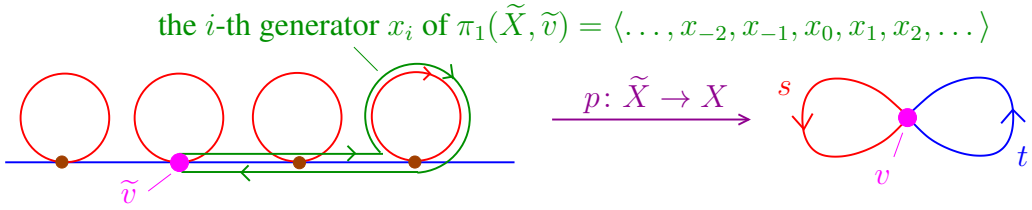
■

Example. We consider the free group on two generators s and t and we consider the homomorphism $\phi\colon \langle s,t\rangle \to \mathbb{Z}$ given by $\phi(s) = 0$ and $\phi(t) = 1$. We want to get a good understanding of $\ker(\phi)$. As on other occasions we turn to topology. Thus let X be the CW-complex consisting of one 0-cell v and two 1-cells which we denote by s and t. By the 1-Complex-π_1-Theorem 137.6 we can identify $\pi_1(X,v)$ with the free group $\langle s,t\rangle$ on the generators s and t.

(1) It follows from the Covering Unique Existence Theorem 127.17 that there exists an essentially unique covering $q\colon (\widehat{X},\widehat{v}) \to (X,v)$ with $q_*(\pi_1(\widehat{X},\widehat{v})) = \ker(\phi)$.
(2) We consider the infinite covering $p\colon \widetilde{X} \to X$ that is illustrated in the figure below. It is straightforward to verify that the deck transformation group is isomorphic to $\mathbb{Z}$.



For $i \in \mathbb{Z}$ we denote by $x_i \in \pi_1(\widetilde{X},\widetilde{v})$ the element that is illustrated in the figure below. It follows from the 1-Complex-π_1-Theorem 137.6 that $\pi_1(\widetilde{X},\widetilde{v})$ equals the free group $\langle \dots x_{-2}, x_{-1}, x_0, x_1, x_2 \dots \rangle$. It is straightforward to see that $p_*(x_i) = t^i \cdot s \cdot t^{-i}$. This implies in particular that $p_*(\pi_1(\widetilde{X},\widetilde{v})) \subseteq \ker(\phi\colon \pi_1(X,v) \to \mathbb{Z})$.



Next we claim that the coverings $p\colon \widetilde{X} \to X$ and $q\colon \widehat{X} \to X$ are isomorphic, in the sense of Definition 127.16. By the Covering Unique Existence Theorem 127.17 it suffices to show that $p_*(\pi_1(\widetilde{X},\widetilde{v})) = q_*(\pi_1(\widehat{X},\widehat{v}))$. We consider the following sequence of maps:

$$\begin{array}{ccccccc} & & & & \text{isomorphism of the Deck Transformation Normal Subgroup Proposition 128.11} & & \\ & & & & \downarrow & & \\ \mathbb{Z} & \cong & \operatorname{Deck}(p\colon \widetilde{X} \to X) & \cong & \pi_1(X,v)/p_*(\pi_1(\widetilde{X},\widetilde{v})) & \twoheadrightarrow & \pi_1(X,v)/\ker(\phi\colon \pi_1(X,v) \to \mathbb{Z}) \xrightarrow[\cong]{\phi} \mathbb{Z}. \\ \uparrow & & & & & & \uparrow \\ \text{see (2)} & & & & & & \text{since } p_*(\pi_1(\widetilde{X},\widetilde{v})) \subseteq \ker(\phi\colon \pi_1(X,v) \to \mathbb{Z}) \end{array}$$

Thus, reading from left to right, we get an epimorphism from $\mathbb{Z}$ onto itself. But any such epimorphism has to be an isomorphism. It follows that the third map is also an isomorphism, i.e. $p_*(\pi_1(\widetilde{X}, \widetilde{v})) = \ker(\phi\colon \pi_1(X, v) \to \mathbb{Z})$, but the latter group is precisely $q_*(\pi_1(\widehat{X}, \widehat{v}))$. So we get the desired equality of subgroups.

In particular we now see that

$$\ker(\phi) \;=\; q_*(\pi_1(\widehat{X}, \widehat{v})) \underset{\text{by the above}}{\uparrow}{=} p_*(\pi_1(\widetilde{X}, \widetilde{v})) \underset{\text{by (2)}}{\uparrow}{=} p_*(\langle\{x_i\}_{i\in\mathbb{Z}}\rangle) \underset{\substack{\text{by the Covering}\\ \pi_1\text{-Monomorphism}\\ \text{Proposition 92.7}}}{\uparrow}{=} \langle\{t^i \cdot s \cdot t^{-i}\}_{i\in\mathbb{Z}}\rangle.$$

□

138.4. Exercises for Chapter 138.

Exercise 138.1. We equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one cell in dimensions $0, 1, 2$. Let $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\pm 1$ be the usual 2-fold covering. The CW-structure on $\mathbb{R}\mathrm{P}^2$ induces a CW-structure on S^2. What does this CW-structure on S^2 look like?

Exercise 138.2. We start out with two definitions:

- The *deficiency* of a finite presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle$ is defined to be $k - l$.
- Let π be a finitely presented group. We define the *deficiency* $\mathrm{def}(\pi)$ *of* π as the maximal deficiency of a finite presentation for π.

Let G be a finitely presented group and let $H \subseteq G$ be a subgroup of finite index. Show that

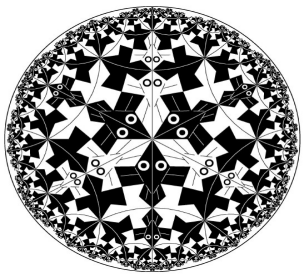
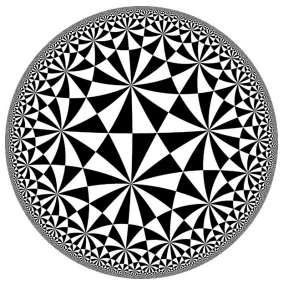
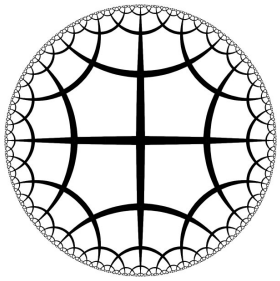
$$\mathrm{def}(H) \geq [G : H] \cdot (\mathrm{def}(G) - 1).$$

Exercise 138.3.

(a) Show that given any finitely generated group π there exists a connected non-empty CW-complex X with finitely many 1-cells such that $\pi_1(X) \cong \pi$.

(b) Show that every finite-index subgroup of a finitely generated group is finitely generated.

Exercise 138.4. In the figure below we show three CW-structures on the open disk B^2. For which of these does there exist a covering $p\colon B^2 \to X$ where X is a finite CW-complex, such that the CW-structure on B^2 is given by the preimage of the CW-structure on X?

(a)  (b)  (c) 

Exercise 138.5. We consider the free group $\langle s, t\rangle$.

(a) We consider the epimorphism $\phi\colon \langle s, t\rangle \to \mathbb{Z}/4\mathbb{Z}$ that is given by $\phi(s) = 1$ and $\phi(t) = 0$. By the Free Subgroup Proposition 138.6 (2) we know that $\ker(\phi)$ is a free group of rank 5. Provide an explicit basis for this free group.

(b) We consider the epimorphism $\phi\colon \langle s,t\rangle \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ that is given by $\phi(s) = (1,0)$ and $\phi(t) = (0,1)$. By the Free Subgroup Proposition 138.6 (2) we know that $\ker(\phi)$ is a free group of rank 7. Provide an explicit basis for this free group.

CHAPTER 139

Constructions of CW-complexes

In this chapter we will introduce and discuss several ways to construct CW-complexes out of given CW-complexes. In particular we will consider the disjoint union, the pushout, the quotient and the wedge of CW-complexes. We will also consider the somewhat tricky topic of products of CW-complexes.

This chapter is organized as follows:

- In Section 139.1 we will introduce the category of CW-complexes where the objects are CW-complexes and the morphisms are cellular maps.
- In Section 139.2 we will discuss pushouts and direct colimits in the category of CW-complexes.
- In Section 139.3 we will discuss products of CW-complexes.
- In Section 139.4 we will discuss more constructions of CW-complexes, namely cones, suspensions and mapping cones.
- In Section 139.5 we will state and prove the CW-HNN-Seifert-van Kampen Theorem 139.28.
- In Appendix 139.6 we will discuss the category of CGWH (compactly generated, weakly Hausdorff) topological spaces.
- In Section 139.7 we will provide exercises for Chapter 139.

In the next Chapter 140 we will state and prove the Homotopy Extension Theorem 140.2.

139.1. **The two categories of CW-complexes.** As fans of category theory it is natural to view CW-complexes as objects of a category. We start out with one obvious way of doing so.

Definition 139.1. The full subcategory $\mathcal{FullCW}$ of CW-complexes is the full subcategory[1429] $\mathcal{FullCW}$ of the category $\mathcal{Top}$ of topological spaces that is given by

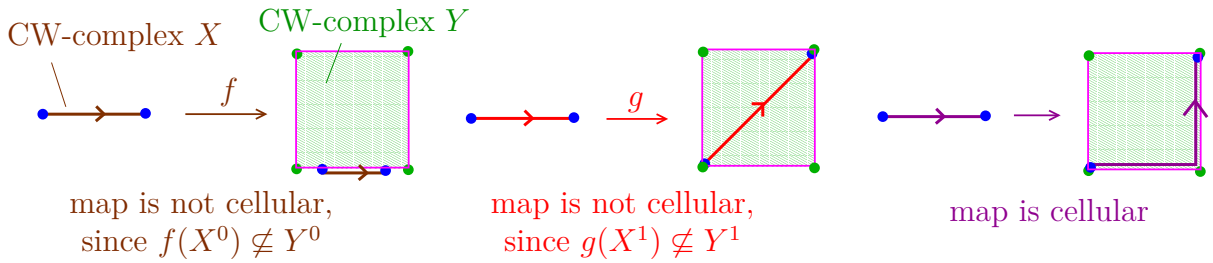
$$\begin{aligned} \mathrm{Ob}(\mathcal{FullCW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{continuous} \text{ maps from } X \text{ to } Y. \end{aligned}$$

As we will see, in many settings it is reasonable to consider maps between CW-complexes that respect the CW-structure to a certain degree. This leads us to the following definition:

Definition 139.2. A map $f\colon X \to Y$ between two CW-complexes is called cellular if f is continuous and if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subseteq Y^n$.

Example. In the figure below we consider maps from the CW-complex $X = [0,1]$ to the CW-complex $Y = [0,1] \times [0,1]$, where each is equipped with the "obvious" CW-structure. The first two maps are not cellular, the one to the right is cellular. □

[1429] Recall that in Definition 5.8 we introduced the notion of a full subcategory of a category.



It is clear that the composition of two cellular maps is again cellular. This leads us to the following definition:

Definition 139.3. The category $\mathcal{CW}$ of CW-complexes is the category $\mathcal{CW}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{CW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y. \end{aligned}$$

Remark. Let X and Y be two CW-complexes and let $f\colon X \to Y$ be a map. It follows easily from the definitions that f is an isomorphism in the category $\mathcal{CW}$ from Definition 139.3 if and only if f is a homeomorphism which for each $k \in \mathbb{N}_0$ restricts to a homeomorphism $X^k \to Y^k$ between the k-skeleta. □

Now is as good as any other moment to introduce a little more categorical language.

Definition 139.4. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories is called faithful if for every two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(F(X),F(Y))$ is an injection. If the map is in fact a bijection, then we call the functor fully faithful.

To practice the definitions we formulate the following lemma. The lemma also explains the name "full category $\mathcal{CW}$ of CW-complexes".

Lemma 139.5. (CW-to-Topological Spaces–Functor Lemma) The maps

$$\begin{aligned} \text{CW-complex } (X, \{X^n\}_{n\in\mathbb{N}_0}) &\mapsto X \\ (f\colon X \to Y) &\mapsto (f\colon X \to Y) \end{aligned}$$

define a fully faithful covariant functor from the full category $\mathcal{FullCW}$ of CW-complexes to the category of topological spaces. The maps also define a faithful covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category of topological spaces, this time the functor is not fully faithful.

Proof. The statements are basically obvious. The fact that we just found a continuous map which is not cellular shows that the functor $\mathcal{CW} \to \mathcal{Top}$ is not fully faithful. ■

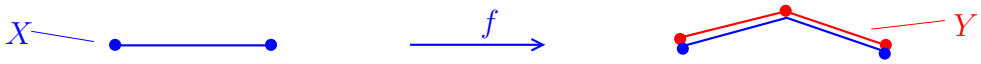
The above definition of a cellular map and of the category $\mathcal{CW}$ of CW-complexes leads us straight to our next definition.

Definition 139.6. Let X and Y be CW-complex.

(1) A map $f\colon X \to Y$ between two CW-complexes is called a cellular isomorphism if it is an isomorphism in the category $\mathcal{CW}$ of CW-complexes. In other words: f is a cellular isomorphism, if f is cellular, if f is a bijection and if f^{-1} is also cellular.

(2) A continuous map $f\colon X \to Y$ is called a cellular embedding if f is injective and if the image of an open n-cell of X is an open n-cell of Y.

Example. In the figure below we see a map $f\colon X \to Y$ between two CW-complexes.



This map is cellular and it is a homeomorphism, but $f^{-1}\colon Y \to X$ is not cellular. Therefore f is not a cellular isomorphism. □

One of the curses of the working topologist is the unfortunate fact that an injective continuous map between topological spaces does not need to be an embedding. The following lemma gives a nice criterion for a map between CW-complexes to be an embedding.

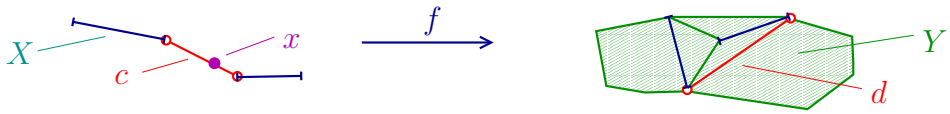
Lemma 139.7. (Cellular Embedding Lemma) Let X and Y be CW-complexes and let $f\colon X \to Y$ be a cellular embedding. The following statements hold:

(1) The image $f(X)$ is a subcomplex of Y.
(2) The map $f\colon X \to f(X)$ is a cellular isomorphism.[1430]

Proof.

(1) We need to show that $f(X)$ is a subcomplex of Y. By hypothesis $f(X)$ is a union of open cells of Y. By the Subcomplex–Characterization Proposition 136.12 it remains to show that for each open cell d of Y with $d \subseteq f(X)$ we also have $\overline{d} \subseteq f(X)$.

Let $x \in X$ with $f(x) \in d$. Let c be the unique open k-cell of X that contains x. By hypothesis we know that $f(c)$ is a k-cell of Y. Since Y is the disjoint union of open cells we see that $f(c) = d$. Let $\overline{c}$ be the closure of c in X.



By the Cell-Compact Proposition 136.8 we know that $\overline{c}$ is compact and by the Compact Image Lemma 4.15 we know that $f(\overline{c})$ is compact. It follows from the Compact-Closed Lemma 3.39 together with the CW-Complex–Hausdorff Proposition 136.6 that $f(\overline{c})$ is a closed subset of Y. By the Definition 3.21 of the closurewe thus have $\overline{f(c)} \subseteq f(\overline{c})$. It follows that $\overline{d} = \overline{f(c)} \subseteq f(\overline{c}) \subseteq f(X)$. ⊞

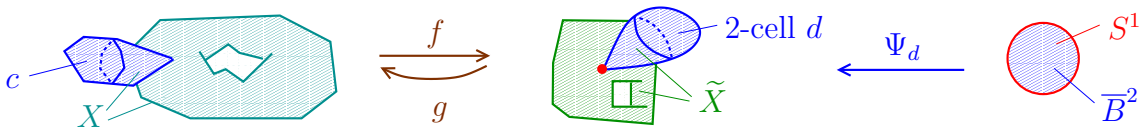
(2) We set $\widetilde{X} \coloneqq f(X)$. Since f is continuous and since $f\colon X \to \widetilde{X}$ is a bijection it remains to show that $g \coloneqq f^{-1}\colon \widetilde{X} \to X$ is continuous. By (1) we can view $\widetilde{X}$ as a CW-complex. Let $\{\Psi_d\colon \overline{B}^{n_d} \to \widetilde{X}\}_{d \in \mathrm{Cell}(\widetilde{X})}$ be a full set of characteristic maps of the CW-complex $\widetilde{X}$. By the Characteristic Maps Proposition 136.9 (2) it suffices to prove the following claim:

Claim. For each $d \in \mathrm{Cell}_n(\widetilde{X})$ the map $g \circ \Psi_d\colon \overline{B}^n \to X$ is continuous.

Proof. Let d be an n-cell of $\widetilde{X}$. We denote by[1431] $\mathring{d}$ the corresponding open n-cell of $\widetilde{X}$. By hypothesis there exists an n-cell c of X such that for the corresponding open n-cell $\mathring{c}$ we have $f(\mathring{c}) = \mathring{d}$.

[1430] In particular $f\colon X \to f(X)$ is a homeomorphism, which implies that $f\colon X \to Y$ is an embedding.

[1431] This notation is somewhat risque, since given an n-cell d, the corresponding open n-cell is usually not the interior of d.



Next we want to show that $f(c) = d$. In fact in (1) we already saw that $d \subseteq f(c)$. Conversely by the Closure-Interior Mapping Lemma 4.6 (2) we know that $\overline{f(c)} = \overline{d} = d$, which gives us the reverse inclusion $f(c) \subseteq d$. Thus we have shown that $f(c) = d$.

By the Cell-Compact Proposition 136.8 we know that the cell c is compact. Since by the CW-Complex–Hausdorff Proposition 136.6 we know that $d = \Psi_d(\overline{B}^{n_d}) \subseteq \widetilde{X}$ is Hausdorff it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map $f\colon c \to d = \Psi_d(\overline{B}^{n_d})$ is a homeomorphism. But this implies that $f^{-1} = g$ is continuous on $\Psi_d(\overline{B}^{n_d})$, which implies that the map $g \circ \Psi_d\colon \overline{B}^{n_d} \to X$ is continuous. ■

139.2. **Constructions of CW-structures: Pushouts and direct colimits .** In this section we finally turn to constructions of CW-complexes. We start out with the following elementary lemma, which is modelled somewhat on the Topological-Coproduct Proposition 7.26.

Lemma 139.8. (CW-Disjoint Union Lemma) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes.

(0) The disjoint union $\bigsqcup_{i\in I} X_i$ admits a natural CW-structure where each n-skeleton is given by $\bigsqcup_{i\in I} X_i^n$.

(1) (a) The natural maps $X_j \to \bigsqcup_{i\in I} X_i$ are cellular.

(b) If Z is a CW-complex and if $f_i\colon X_i \to Z$, $i \in I$ are continuous (cellular) maps, then the corresponding map $\bigsqcup_{i\in I} f_i\colon \bigsqcup_{i\in I} X_i \to Z$ is also continuous (cellular).

(2) The CW-complex $\bigsqcup_{i\in I} X_i$ is a coproduct in the full category $\mathcal{FullCW}$ of CW-complexes and in the category $\mathcal{CW}$ of CW-complexes.

Example. Let X be a CW-complex with components $\{X_i\}_{i\in I}$. By the CW-Complex–Local Properties Proposition 137.8 (2) we know that X is locally path-connected. It follows from the Locally Path-Connected Lemma 4.57 that the components of X are the path-components of X. By the Component–Subcomplex Lemma 136.14 we know that each X_i is a CW-complex in its own right. Furthermore, since X is locally connected, it follows from the discussion on page 298 that X is (as a topological space) the disjoint union of the X_i. It follows from the CW-Disjoint Union Lemma 139.8 that X is (as a CW-complex) the disjoint union of the (path-) components X_i. □

Proof. The proof is elementary, and the more one says, the more confusing it gets. ■

We continue with a much more interesting result which is modelled on the Topological-Pushout Proposition 8.12.

Proposition 139.9. (CW-Gluing Proposition) Let X and Y be CW-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. As in Definition 8.13 we

consider the following topological space:

$$Y \cup_{Y \xleftarrow{f} A} X \;:=\; Y \cup_f X \;:=\; (Y \sqcup X)/\!\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A$$

By the Topological-Pushout Proposition 8.12 the following diagram is a pushout diagram in the category *Top* of topological spaces:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p \;\; x \mapsto [x]} \\ Y & \xrightarrow[q]{y \mapsto [y]} & Y \cup_f X. \end{array}$$

The following statements hold:

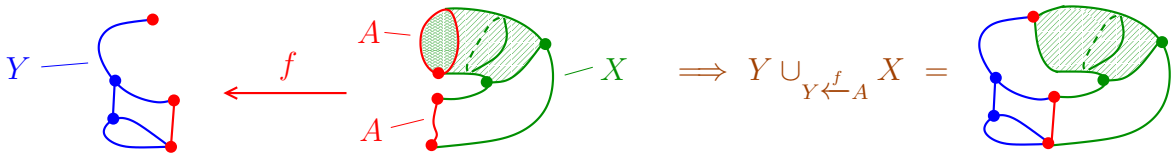
(0) (a) The subspaces $(Y \cup_f X)^n := q(Y^n) \cup p(X^n)$ define a CW-structure on $Y \cup_f X$.

(b) If $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ and $\{\Phi_d \colon \overline{B}^{\dim(d)} \to Y\}_{d \in \mathrm{Cell}(Y)}$ are full sets of characteristic maps of X and Y, then $\{p \circ \Phi_c \colon \overline{B}^{\dim(c)} \to Y \cup_f X\}_{c \in \mathrm{Cell}(X) \setminus \mathrm{Cell}(A)}$, $\{q \circ \Phi_d \colon \overline{B}^{\dim(d)} \to Y \cup_f X\}_{d \in \mathrm{Cell}(Y)}$ is a full set of characteristic maps of $Y \cup_f X$.

(1) (a)
- The natural map $Y \to Y \cup_f X$ is a cellular embedding.
- If $f \colon A \to Y$ is a cellular embedding, then the "opposite" map $X \to Y \cup_f X$ is also a cellular embedding.[1432]

(b) If Z is a CW-complex and if $\alpha \colon X \to Z$ and $\beta \colon Y \to Z$ are continuous (cellular) maps with $\alpha|_A = \beta \circ f \colon A \to Z$, then the induced map $Y \cup_f X \to Z$ is also continuous (cellular).

(2) The CW-complex $Y \cup_f X$ is a pushout in the full category *FullCW* of CW-complexes and in the category $\mathcal{CW}$ of CW-complexes.



Example. A typical example is given as follows: Let X be a CW-complex and let $n \in \mathbb{N}_0$. We equip S^{n-1} and $\overline{B}^n$ with the standard CW-structure given by $\varnothing \subseteq \{N\} \subseteq S^{n-1} \subseteq \overline{B}^n$. Let $f \colon S^{n-1} \to X$ be a cellular map. It follows from the CW-Gluing Proposition 139.9 that $X \cup_f \overline{B}^n$ is a CW-complex, such that X is a subcomplex, and with one extra n-cell. □

Proof.

(0) We start out with the following preliminary claim.

Claim 0. Given any $n \in \mathbb{N}_0$ the natural map[1433] $X^n \cup_{f|_{A^n}} Y^n \to (Y \cup_f X)^n$ is a homeomorphism.

Proof. This statement is a special case of a general fact about pushouts, namely that pushouts commute with taking subspaces. We proved this elementary statement in Exercise 8.25. ⊞

[1432] This statement should be viewed as an extension of the discussion of the properties of the pushout that we gave in the Topological Pushout-Maps Lemma 8.16.

[1433] Note that the fact that f is cellular implies that $f(A^n) \subseteq X^n$.

We now turn to the actual proof that this filtration defines a CW-structure. In the following two claims we will verify that the conditions (CW1) and (CW2) from Definition 136.1 are satisfied. Along the way we will also prove Statement (b).

Claim 1. For any subset $U \subseteq Y \cup_f X$ we have

$$U \subseteq Y \cup_f X \text{ is open} \iff \text{for all } n \in \mathbb{N}_0 \text{ the intersection } (Y \cup_f X)^n \cap U \text{ is open.}$$

Proof. Note that for any subset $U \subseteq Y \cup_f X$ we have

$$\begin{aligned}
& U \subseteq Y \cup_f X \text{ is open} \\
\iff\ & p^{-1}(U) \subseteq X \text{ and } q^{-1}(U) \subseteq Y \text{ are open} \\
\iff\ & \text{for all } n \in \mathbb{N}_0 \text{ the intersections } X^n \cap p^{-1}(U) \subseteq X^n \text{ and } Y^n \cap q^{-1}(U) \subseteq Y^n \text{ are open} \\
\iff\ & \text{for all } n \in \mathbb{N}_0 \text{ the intersection } (X^n \cup_{f|_{A^n}} Y^n) \cap U \text{ is open} \\
\iff\ & \text{for all } n \in \mathbb{N}_0 \text{ the intersection } (Y \cup_f X)^n \cap U \text{ is open.}
\end{aligned}$$

The equivalences follow from the Topological-Pushout Proposition 8.12 (0), the fact that X and Y are CW-complexes, again the Topological-Pushout Proposition 8.12 (0) and Claim 0. ⊞

Claim 2. Let $n \in \mathbb{N}_0$. Let $\{\Phi_c \colon \overline{B}^n \to X\}_{c \in \mathrm{Cell}_n(X^n)}$ and $\{\Phi_d \colon \overline{B}^n \to Y\}_{d \in \mathrm{Cell}_n(Y^n)}$ be full sets of characteristic maps of the n-cells of X and Y. The map[1434]

$$\underbrace{q(Y^{n-1}) \cup p(X^{n-1})}_{=(Y \cup_f X)^{n-1}} \cup_\sim \bigsqcup_{d \in \mathrm{Cell}(Y)} \overline{B}^n_d \cup \bigsqcup_{c \in \mathrm{Cell}(X) \setminus \mathrm{Cell}(A)} \overline{B}^n_c \ \to\ \underbrace{q(Y^n) \cup p(X^n)}_{=(Y \cup_f X)^n},$$

which is induced by the two inclusions $q(Y^{n-1}) \to q(Y^n)$, $p(X^{n-1}) \to p(X^n)$ and by the maps $q \circ \Phi_d$ and $p \circ \Phi_c$ is a homeomorphism.

Proof. One can easily verify, using the Topological-Pushout Proposition 8.12 (1∗), that the given map is continuous. It remains to show that the map has a continuous inverse. We consider the map

$$\begin{aligned}
\underbrace{q(Y^n) \cup p(X^n)}_{=(Y \cup_f X)^n} \overset{\cong}{\leftarrow}\ Y^n \cup_{f|_{A^n}} X^n \ &= \ \left(Y^{n-1} \cup_\sim \bigcup_{d \in \mathrm{Cell}_n(Y)} \overline{B}^n_d\right) \cup_\sim \left(X^{n-1} \cup_\sim \bigcup_{c \in \mathrm{Cell}_n(X)} \overline{B}^n_c\right) \\
&\to\ \underbrace{q(Y^{n-1}) \cup p(X^{n-1})}_{=(Y \cup_f X)^{n-1}} \cup_\sim \bigsqcup_{d \in \mathrm{Cell}(Y)} \overline{B}^n_d \cup \bigsqcup_{c \in \mathrm{Cell}(X) \setminus \mathrm{Cell}(A)} \overline{B}^n_c
\end{aligned}$$

which is given as follows:

- The first map is the homeomorphism from Claim 0.
- The last map is given by $q|_{Y^{n-1}}$, $p|_{X^{n-1}}$, by the characteristic maps of the n-cells of Y, the characteristic maps of the n-cells of X that are not contained in A and for each n-cell in A the map is given by the composition of $f \colon A^n \to Y^n$ (here we use that f is cellular) and the above map on Y^n. It follows from the Topological-Coproduct Proposition 7.26 (1∗) and the Topological-Pushout Proposition 8.12 (1∗) that this map is continuous.

One can easily verify that the above map is the inverse to the map of the claim. ⊞

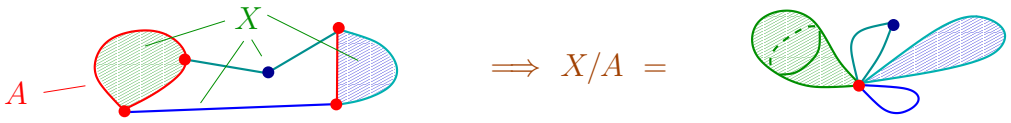
[1434]Here "$\cup_\sim$" is short for gluing the n-balls to $p(X^{n-1}) \cup q(Y^{n-1})$ via the restrictions of the characteristic maps to the spheres S^{n-1}.

(1) (a) These statements follow easily from the Cellular Embedding Lemma 139.7 and arguments as in the proof of the Topological Pushout-Maps Lemma 8.16. We leave the details to the reader.
(b) This statement also follows easily from the definitions.
(2) This statement follows, as in the case of the Topological-Pushout Proposition 8.12 (2), fairly easily from the above discussion and some straightforward extra arguments. ■

In the following lemmas we will spell out some important special cases of the CW-Gluing Proposition 139.9.

Lemma 139.10. (CW-Quotient Lemma) Let X be a CW-complex and let A be a subcomplex.

(0) (a) We denote by $p\colon X \to X/A$ the natural projection. If $A \neq \varnothing$, then the subsets $p(X^n)$, $n \in \mathbb{N}_0$, define a CW-structure on X/A. Furthermore, for $A = \varnothing$ the subsets $X^n \cup \{\varnothing\}$ define a CW-structure on $X/\varnothing := X \cup \{Xt\}$.[1435]
(b) The cells of X/A are in one-to-one correspondence to the cells of X that do not entirely lie in A together with one 0-cell given by the point $[A]$.
(1) The natural map $p\colon X \to X/A$ is cellular.



Proof. On page 318 we pointed out that we have the following pushout diagram:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle x\mapsto[x]} \\ \{*\} & \longrightarrow & X/A. \end{array}$$

The lemma is thus an almost immediate consequence of the CW-Gluing Proposition 139.9. ■

Lemma 139.11. (CW-Wedge Lemma) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes and suppose that for each $i \in I$ we are given a point $x_i \in X_i^0$. The corresponding wedge $\bigvee_{i\in I}(X_i, x_i)$, admits a natural CW-complex structure.

Proof. It follows from the CW-Disjoint Union Lemma 139.8 that $X := \bigsqcup_{i\in I} X_i$ has a natural CW-structure such that $A := \bigsqcup_{i\in I}\{x_i\}$ is a subcomplex of X. It follows from the CW-Quotient Lemma 139.10 that $X/A = \bigvee_{i\in I}(X_i, x_i)$ inherits a natural CW-structure. ■

Later on we will make good use of the following lemma:

Lemma 139.12. (Skeleton-Quotient Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$. Let $\{\Phi_c\colon \overline{B}^{n+1} \to X^n\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps of the $(n+1)$-cells of X. The map

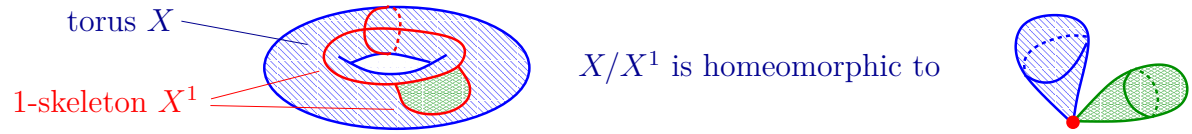
[1435] In Definition 8.8 we defined $X/\varnothing$ as the disjoint union of X with the topological space $\{X\}$ consisting of a single point.

by the Ball Quotient-Sphere Lemma 8.10 the pointed topological space $(\overline{B}^{n+1}/S^n, [S^n])$ is homeomorphic to (S^{n+1}, N)

$$\bigvee_{c\in\mathrm{Cell}(X)} \overbrace{(\overline{B}_c^{n+1}/S_c^n, [S_c^n])}^{} \quad \to \quad X^{n+1}/X^n$$
$$[x] \quad \mapsto \quad [\Phi_c(x)] \quad \text{if } x \in \overline{B}_c^{n+1}$$

is a homeomorphism.

Example. In the figure below we see the torus together with a CW-structure X that has two 2-cells.



It follows from the Skeleton-Quotient Lemma 139.12 that $X/X^1 = X^2/X^1$ is homeomorphic to the wedge of two spheres. □

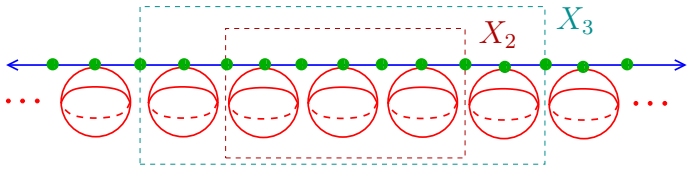
Proof. It is basically clear that the given map $f\colon \bigvee_{c\in\mathrm{Cell}(X)} (\overline{B}_c^{n+1}/S_c^n, [S_c^n]) \to X^{n+1}/X^n$ is a bijection. Next note that it follows immediately from the Quotient-to-Quotient Lemma 8.6, the Wedge Proposition 96.2 (1b) and the Topological-Quotient Proposition 8.2 (1) that f is continuous. To upgrade this bijective continuous map to a homeomorphism we have to work a little harder.

To do so note that by the Ball Quotient-Sphere Lemma 8.10 the pointed topological space $(\overline{B}^{n+1}/S^n, [S^n])$ is homeomorphic to (S^{n+1}, N). It follows from the discussion on page 2625 that we can view $\overline{B}^{n+1}/S^n$ as a CW-complex where the point $[S^n]$ is the unique 0-cell. Therefore we can use the CW-Wedge Lemma 139.11 and the CW-Quotient Lemma 139.10 to view both sides of the Skeleton-Quotient Lemma 139.12 as CW-complexes. The map f gives a bijection between the open cells of the two sides. Thus it follows from the Cellular Embedding Lemma 139.7 (2) that f is a homeomorphism. ■

We conclude this section with the following lemma:

Lemma 139.13. (Direct Colimit–CW Lemma) Let $\{f_i\colon X_i \to X_{i+1}\}_{i\in\mathbb{N}_0}$ be a sequence of cellular embeddings between CW-complexes. The direct colimits $\varinjlim X_i^n$ of the skeleta of the X_i define a CW-structure on $\varinjlim X_i$.

Example. We consider the line with infinitely many spheres attached.



the topological space
$X = \bigcup_{i\in\mathbb{N}_0} X_i = \varinjlim X_i$

The Direct Colimit–CW Lemma 139.13 allows us to view the line with infinitely many spheres attached as a CW-complex.[1436] □

[1436] It follows from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 that the topology on the "line with infinitely many spheres" coming from $\mathbb{R}^3$ agrees with the topology coming from the CW-structure.

Proof. It follows from the Topological-Direct Colimit Proposition 104.10 (1) that we can assume that each $X_i \to X_{i+1}$ is an inclusion and that $X \coloneqq \operatorname{colim}\limits_{\longrightarrow} X_i = \bigcup\limits_{i\in\mathbb{N}_0} X_i$ where a subset $U \subseteq X = \bigcup\limits_{i\in\mathbb{N}_0} X_i$ is open if and only if each $X_i \cap U$ is open in X_i. Given $n \in \mathbb{N}_0$ we set $X^n \coloneqq \bigcup\limits_{i\in\mathbb{N}_0} X_i^n$. We start out with the following claim:

Claim 0. Let $n \in \mathbb{N}_0$. The direct colimit topology $\operatorname{colim}\limits_{\longrightarrow} X_i^n$ on $\bigcup\limits_{i\in\mathbb{N}_0} X_i^n$ agrees with the subspace topology coming from $\operatorname{colim}\limits_{\longrightarrow} X_i = \bigcup\limits_{i\in\mathbb{N}_0} X_i$.

Proof. First note that it follows immediately from the Subcomplex–Closed Lemma 136.11 that each $X_i^n \subseteq X_i$ is closed. Also note that for $i, j \in \mathbb{N}_0$ we have $X_j \cap X_i^n = X^n_{\min\{i,j\}}$. It follows from these two observations together with the Topological-Direct Colimit Proposition 104.10 that each $\bigcup\limits_{i\in\mathbb{N}_0} X_i^n$ is closed in $\operatorname{colim}\limits_{\longrightarrow} X_i = \bigcup\limits_{i\in\mathbb{N}_0} X_i$. The claim now follows from the Direct Colimit–Subspace Lemma 104.13. ⊞

The following two claims show that the two conditions (CW1) and (CW2) from Definition 136.1 are satisfied.

Claim 1. A subset $U \subseteq X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $X^n \cap U$ is open in X^n.

Proof. Given $U \subseteq X$ we see that

$$\begin{array}{ccccc} U \subseteq X \text{ is open} & \overset{\text{since } X = \operatorname{colim} X_i}{\iff} & \forall\limits_{i\in\mathbb{N}_0} X_i \cap U \text{ is open in } X_i & \overset{\text{since } X_i^n \text{ is a CW-complex}}{\iff} & \forall\limits_{i\in\mathbb{N}_0}\forall\limits_{n\in\mathbb{N}_0} X_i^n \cap U \text{ is open in } X_i^n \\ & \iff & \forall\limits_{n\in\mathbb{N}_0}\forall\limits_{i\in\mathbb{N}_0} X_i^n \cap U \text{ is open in } X_i^n & \underset{\text{since by Claim 0 we have } X^n = \operatorname{colim} X_i^n}{\iff} & \forall\limits_{n\in\mathbb{N}_0} X^n \cap U \text{ is open in } X^n. \end{array}$$

⊞

Claim 2. For each $n \in \mathbb{N}_0$ there exists a set J, a family $\{\varphi_j \colon S^{n-1} \to X^{n-1}\}_{j\in J}$ of continuous maps and a homeomorphism

$$X^{n-1} \cup_{X \leftarrow \bigsqcup_{j\in J} S_j^{n-1} \,:\, \bigsqcup_{j\in J} \varphi_j} \bigsqcup_{j\in J} \overline{B}_j^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

Proof. Let $n \in \mathbb{N}_0$. For each $i \in \mathbb{N}_0$ let $\{\Phi_j \colon \overline{B}_j^n \to X_i^n\}_{j\in J_i}$ be a full set of characteristic maps for the n-cells of X_i that are not contained in X_{i-1}. We set $J \coloneqq \bigsqcup\limits_{i\in\mathbb{N}_0} J_i$ and given $j \in J$ we write $\varphi_j \coloneqq \Phi_j|_{S^{n-1}} \colon S^{n-1} \to X^{n-1}$. Basically by construction we know that for each $i \in \mathbb{N}_0$ the map

$$X_i^{n-1} \cup_\sim \bigcup_{k=0}^{i} \bigsqcup_{j\in J_k} \overline{B}_j^n \;\to\; X_i^n$$

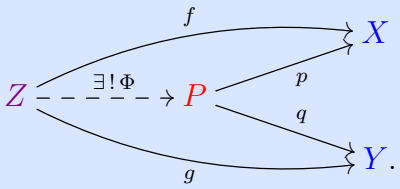
that is induced by the identity on X_i^{n-1} and the characteristic maps is a homeomorphism. It follows that the map

$$\underbrace{\operatorname*{colim}_i \left(X^{n-1} \cup_\sim \bigsqcup_{k=0}^{i} \bigsqcup_{j\in J_k} \overline{B}_j^n \right)}_{=X^{n-1}\cup_\sim \bigsqcup_{k\in\mathbb{N}_0} \bigsqcup_{j\in J_i} \overline{B}_j^n} \to \underbrace{\operatorname*{colim}_i X_i^n}_{=X^n \text{ by Claim } 0}$$

is also a homeomorphism. But that is exactly what we wanted to show. ■

139.3. **Products of CW-complexes.** The reader will have noticed that a discussion of products is long overdue. As we will see, there was a reason why we were procrastinating. In fact it is perhaps helpful to reflect about what "product" is supposed to mean in the first place. Thus let us recall the following from Definition 6.1.

Definition 139.14. Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following property is satisfied: Whenever we are given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram:



By the Product Uniqueness Lemma 6.2 we know that if the product exists, then it is unique in a suitable sense.

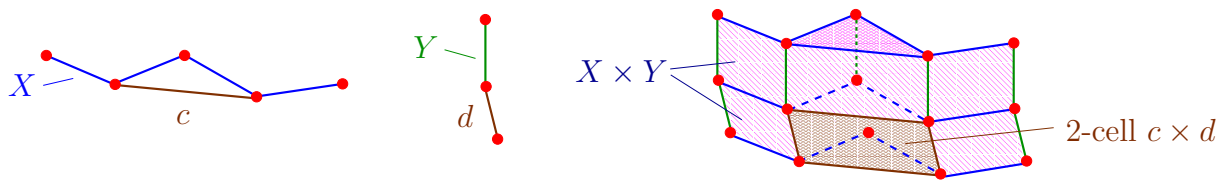
We remind the reader of the following simplified version of the Topological-Product Proposition 7.2.

Proposition 7.2. (Topological-Product Proposition) Let X and Y be two topological spaces. We consider the topology given by

$$\mathcal{TP} := \left\{ U \subseteq X \times Y \;\middle|\; \begin{array}{l} \text{for all } (x,y) \in U \text{ there exist open neighborhoods} \\ A \subseteq X \text{ of } x \text{ and } B \subseteq Y \text{ of } y \text{ with } A \times B \subseteq U \end{array} \right\}.$$

(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.
(b) Given a topological space Z and given continuous maps $f\colon Z \to X$ and $g\colon Z \to Y$ the map $(f,g)\colon Z \to (X \times Y, \mathcal{TP})$ that is given by $z \mapsto (f(z), g(z))$ is continuous.
(2) The topological space $X \times Y$ with the natural projections is a product of X and Y in the category $\mathcal{Top}$ of topological spaces.

Now let X and Y be CW-complexes. We want to equip $X \times Y$ with a CW-structure. Perhaps initially it is easier to think in terms of cells instead of filtrations. If c is a p-cell of X and if d is a q-cell of Y, then it seems reasonable that $c \times d$ should be a $(p+q)$-cell of $X \times Y$.
If that is the idea, then it follows from the Characteristic Maps Proposition 136.9 that $X \times Y$ should have the topology that is described in the following definition:



Definition 139.15. Let X and Y be two CW-complexes.

(1) Given a p-cell c for X and given a q-cell d for Y we equip c and d with the subspace topology and we equip $c \times d$ with the product topology.

(2) We denote by $\mathcal{CWP}$ the topology on $X \times Y$ that is defined by the condition that $U \subseteq X \times Y$ is open if and only if each intersection $(c \times d) \cap U \subseteq c \times d$ is open. Sometimes we refer to $\mathcal{CWP}$ as the CW-product topology.

Proposition 139.16. (CW-Category–Product Proposition) Let X and Y be two CW-complexes.

(0) (a) A CW-structure for the topological space $(X \times Y, \mathcal{CWP})$ is given by the filtration

$$(X \times Y)^n := \bigcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} (X^p \times Y^q) \qquad \text{with } n \in \mathbb{N}_0.$$

(b) The n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y.

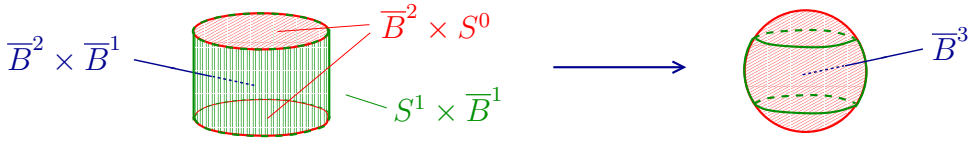
(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are cellular, in particular they are continuous with respect to $\mathcal{CWP}$.

(b) If Z is a CW-complex and if $f\colon Z \to X$ and $g\colon Z \to Y$ are continuous maps, then the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.[1437]

(2) The CW-complex $(X \times Y, \mathcal{CWP})$ with the natural projections is a product of the CW-complexes X and Y in the full category $\mathcal{FullCW}$ of CW-complexes.

In the proof of the CW-Category–Product Proposition 139.16 we will need the following elementary lemma:

Lemma 139.17. (Ball-times-Ball Lemma) Let $p, q \in \mathbb{N}_0$. There exists a homeomorphism $\overline{B}^p \times \overline{B}^q \to \overline{B}^{p+q}$ that restricts to a homeomorphism $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q) \to S^{p+q-1}$.



Proof of the Ball-times-Ball Lemma 139.17. This statement follows from the Product Metric Lemma 7.4 together with the Convex-to-Ball Proposition 4.28 (2). ■

Proof of the CW–Product Proposition 139.16. Let X and Y be two CW-complexes.

(0) We start out with the following claim:

[1436] It follows easily from the Image-Preimage Lemma 1.9 that this defines a topology on $X \times Y$.

[1437] Note that if f and g are cellular, this does *not* mean that the map $(f, g)\colon Z \to X \times Y$ is cellular.

Claim 0. Let $p, q \in \mathbb{N}_0$. We view the skeleta X^p and Y^q as CW-complexes in their own right. The $\mathcal{CWP}$-topology on $X^p \times Y^q$ agrees with the subspace topology coming from $(X \times Y, \mathcal{CWP})$

Proof. We consider the inclusion map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$. Note that it follows from the Subcomplex–Characterization Proposition 136.12 that every cell of X^p is a cell of X and that every cell of Y^q is a cell of Y. It follows basically immediately from this observation and the definition of the CW-product topology that the map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$ is continuous. By the Open-Closed–Injective Lemma 4.22 it remains to show that f is a closed map. Let A be a subset that is closed in $(X^p \times Y^q, \mathcal{CWP})$. We need to show that A is closed in $(X \times Y, \mathcal{CWP})$. By definition of $(X \times Y, \mathcal{CWP})$ and by the elementary Topology-Defined-by-Family Lemma 4.8 it suffices to show that for each cell c of X and each cell d of Y the intersection $(c \times d) \cap A$ is closed in $c \times d$.

Note that it follows from the Cell-Compact Proposition 136.8, the Subcomplex–Closed Lemma 136.11 and the CW-Complex-Finiteness Theorem 136.16 that there exist finitely many cells $x_1, \dots, x_r$ of X^p such that $c \cap X^p \subseteq x_1 \cup \dots \cup x_r$ and that there exist finitely many cells $y_1, \dots, y_s$ of X^p such that $d \cap Y^q \subseteq y_1 \cup \dots \cup y_s$. We now see that

$$\begin{aligned}
(c \times d) \cap A &\overset{\text{since } A \subseteq X^p \times Y^q}{=} (X^p \times Y^q) \cap ((c \times d) \cap A) = ((c \cap X^p) \times (d \cap Y^q)) \cap A \\
&= ((c \cap X^p) \times (d \cap Y^q)) \cap ((x_1 \cup \dots \cup x_r) \times (y_1 \cup \dots \cup y_s)) \cap A \\
&= \underbrace{((c \cap X^p) \times (d \cap Y^q))}_{\substack{\text{closed in } c \times d \text{ since it follows from} \\ \text{the Subcomplex–Closed Lemma 136.11} \\ \text{that } X^p \subseteq X \text{ and } Y^q \subseteq Y \text{ are closed}}} \cap \Big(\bigcup_{i=1}^{r} \bigcup_{j=1}^{s} \underbrace{((x_i \times y_j) \cap A)}_{\substack{\text{by hypothesis closed} \\ \text{in } x_i \times y_j, \text{ thus compact,} \\ \text{thus closed in } c \times d}} \Big).
\end{aligned}$$

We have thus shown that $(c \times d) \cap A$ is indeed a closed subset of $c \times d$. ⊞

Now we turn to the actual proof that the filtration $(X \times Y)^n := \bigcup_{p+q=n} (X^p \times Y^q)$ defines a CW-structure on $(X \times Y, \mathcal{CWP})$. The following two claims show that this filtration satisfies the two conditions (CW1) and (CW2) from Definition 136.1.

Claim 1. A subset $U \subseteq (X \times Y, \mathcal{CWP})$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $(X \times Y)^n \cap U$ is open in $(X \times Y)^n$.

Proof. Let $U \subseteq X \times Y$. Then:

$$\begin{aligned}
U \text{ is open in } \mathcal{CWP} &\iff \forall_{p \in \mathbb{N}_0} \forall_{q \in \mathbb{N}_0} \forall_{c \in \mathrm{Cell}_p(X)} \forall_{d \in \mathrm{Cell}_q(Y)} (c \times d) \cap U \text{ is open in } c \times d \\
&\iff \forall_{p \in \mathbb{N}_0} \forall_{q \in \mathbb{N}_0} (X^p \times Y^q) \cap U \text{ is open in } X^p \times Y^q \\
&\quad \uparrow \text{ follows from Claim 0} \\
&\iff \forall_{n \in \mathbb{N}_0} (X \times Y)^n \cap U \text{ is open in } (X \times Y)^n \\
&\quad \uparrow \text{ follows from the Cover-Open Subset Lemma 3.20 (2), since } (X \times Y)^n \text{ is the} \\
&\quad \text{union of the finitely many closed subsets } X^p \times Y^q, \text{ where } p + q = n
\end{aligned}$$

⊞

Claim 2. For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to (X \times Y)^{n-1}\}_{i \in I}$ of continuous maps and a homeomorphism

$$(X \times Y)^{n-1} \cup_{X \times Y \leftarrow \bigsqcup_{i \in I} S_i^{n-1} \,:\, \bigsqcup_{i \in I} \varphi_i} \bigsqcup_{i \in I} \overline{B}_i^n \;\to\; (X \times Y)^n$$

which is the inclusion on $(X \times Y)^{n-1}$.

Proof. Let $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ and $\{\Psi_d \colon \overline{B}^{\dim(d)} \to Y\}_{d \in \mathrm{Cell}(Y)}$ be full sets of characteristic maps of the CW-complexes X and Y. Note that for each $p, q \in \mathbb{N}_0$ the restriction of the map $\Phi_c \times \Psi_d \colon \overline{B}^p \times \overline{B}^q \to X \times Y$ to $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q)$ takes values in $(X^p \times Y^{q-1}) \cup (X^{p-1} \cup Y^q) \subseteq (X \times Y)^{n-1}$. Thus we can consider the topological space

$$(X \times Y)^{n-1} \;\cup_\sim \bigsqcup_{\substack{p,\, q \,\in\, \mathbb{N}_0 \text{ with} \\ p+q=n}} \; \bigsqcup_{c \in \mathrm{Cell}_p(X)} \; \bigsqcup_{d \in \mathrm{Cell}_q(Y)} (\overline{B}^p \times \overline{B}^q)$$

where given c, d we use the restrictions of $\Phi_p \times \Psi_q$ to $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q)$ to glue $\overline{B}^p \times \overline{B}^q$ to $(X \times Y)^{n-1}$. Next we consider the map

$$(X \times Y)^{n-1} \;\cup_\sim \bigsqcup_{\substack{p,\, q \,\in\, \mathbb{N}_0 \text{ with} \\ p+q=n}} \; \bigsqcup_{c \in \mathrm{Cell}_p(X)} \; \bigsqcup_{d \in \mathrm{Cell}_q(Y)} (\overline{B}^p \times \overline{B}^q) \;\to\; (X \times Y)^n$$

that is given by the inclusion on $(X \times Y)^{n-1}$ and that is given by all the product maps $\Phi_c \times \Psi_d \colon \overline{B}^p \times \overline{B}^q \to X \times Y$. It follows from the Topological Pushout Proposition 8.12 (0∗) that this map is continuous. It is clearly a bijection. Using the $\mathcal{CWP}$ topology on $(X \times Y)^n$ it is also not difficult to show that the map is closed. It follows from the Open-Closed–Injective Lemma 4.22 that the above map is a homeomorphism.

The claim now follows from the above discussion and the observation that the Ball-times-Ball Lemma 139.17 allows us to replace $\overline{B}^p \times \overline{B}^q$ by $\overline{B}^{p+q} = \overline{B}^n$ and it allows us to replace $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q)$ by $S^{p+q-1} = S^{n-1}$. ⊞

It remains to show that the n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y. But this follows easily from Claim 2 together with the Product-of-Subsets Lemma 7.8.

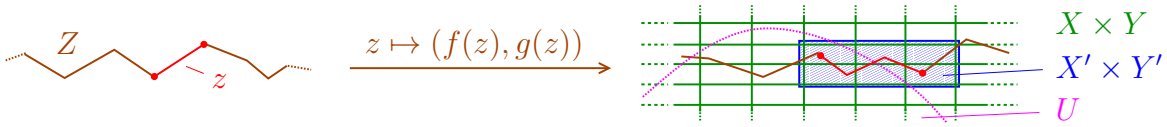
(1) (a) We consider the natural projection $p \colon (X \times Y, \mathcal{CWP}) \to X$. We need to show that this map is continuous. Note that by the Topology-Defined-by-Family Lemma 4.8 (3) it suffices to show that for each cell $c \subseteq X$ and each cell $d \subseteq Y$ the map $c \times d \to X \times Y \to X$ is continuous. But by the Product Topology-Basics Lemma 7.6 (1a) and the Topological-Product Proposition 7.2 (1a) we know that the maps $c \times d \to (X \times Y, \mathcal{TP})$ and $(X \times Y, \mathcal{TP}) \to X$ are continuous. Thus the composition is also continuous. The same argument of course also shows that the natural projection $p \colon X \times Y \to Y$ is continuous.

(b) Let Z be a CW-complex and let $f \colon Z \to X$ and $g \colon Z \to Y$ be two continuous maps. We need to show that the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.[1438] Let $U \subseteq (X \times Y, \mathcal{CWP})$ be an open subset. We need to show that $(f, g)^{-1}(U)$ is an open subset of the CW-complex Z. By the Characteristic

[1438] In the CW-Complex Product Proposition 139.18 we will see that $\mathcal{TP} \subseteq \mathcal{CWP}$. This implies that it is easier to show that maps *out* of $(X \times Y, \mathcal{CWP})$ are continuous than to show that maps *out* of $(X \times Y, \mathcal{TP})$ are continuous. But this fact also implies that it harder to show that maps *to* $(X \times Y, \mathcal{CWP})$ are continuous

Maps Proposition 136.9 it remains to show that for each cell z of Z the intersection $z \cap (f,g)^{-1}(U)$ is open in z.

It follows from the Cell-Compact Proposition 136.8, the Compact Image Lemma 4.15 and the CW-Complex-Finiteness Theorem 136.16 that there exist finite subcomplexes X' of X and Y' of Y with $f(z) \subseteq X'$ and $g(z) \subseteq Y'$.



Claim. The intersection $(X' \times Y') \cap U$ is an open subset of $(X' \times Y', \mathcal{TP})$.

Proof. First note that we have the equality

$$(X' \times Y') \cap U \;=\; \bigcup_{c \in \mathrm{Cell}(X')} \bigcup_{d \in \mathrm{Cell}(Y')} (c \times d) \cap U.$$

Since U is open in $\mathcal{CWP}$ we see that each $(c \times d) \cap U$ is open in $c \times d$. Note that each $c \times d$ is a closed subset of $X' \times Y'$. It follows from these observations, and the fact that the finite CW-complexes X' and Y' have only finitely many cells, together with the Cover-Open Subset Lemma 3.20 (2) that $(X' \times Y') \cap U$ is indeed an open subset of $X' \times Y'$. ⊞

Next note that $$z \cap (f,g)^{-1}(U) \;=\; z \cap (f,g)^{-1}((X' \times Y') \cap U).$$

By the elementary Subspace Continuity Lemma 4.3 the map $(f,g)\colon Z \to X' \times Y'$ is also continuous. Thus it follows from the above equality and the claim, that $z \cap (f,g)^{-1}(U)$ is indeed an open subset of z.

(2) As in the case of the Topological-Product Proposition 7.2 this statement follows almost immediately from (1). ■

Let X and Y be two CW-complexes. The obvious question that now arises is, under what circumstances does the CW-product topology $\mathcal{CWP}$ that we used in the CW-Category–Product Proposition 139.16 agree with the usual product topology $\mathcal{TP}$. The following proposition shows that in two settings, which are of relevance to us, this is indeed the case.

Proposition 139.18. (CW-Complex Product Proposition) Let X and Y be two CW-complexes.

(1) The identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous, in other words, we have the inclusion $\mathcal{TP} \subseteq \mathcal{CWP}$.

(2) Suppose we are in one of the following two settings:[1439]
   (a) *X and Y* are both finite CW-complexes, or
   (b) $Y = [0,1]$ where Y is equipped with the usual CW-structure with two 0-cells and one 1-cell.[1440]

   then the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is a homeomorphism, in other words, we have $\mathcal{CWP} = \mathcal{TP}$. Thus the CW-product structure from the CW-Category–Product Proposition 139.16 is a CW-structure on $(X \times Y, \mathcal{TP})$.

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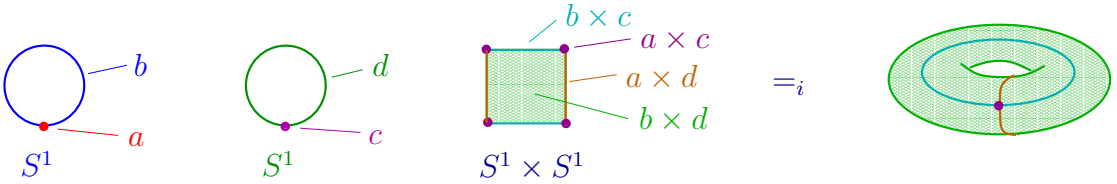
than to show that maps *to* $(X \times Y, \mathcal{TP})$ are continuous. This observation explains why the proof of (1b) is trickier than the proof of (1a).

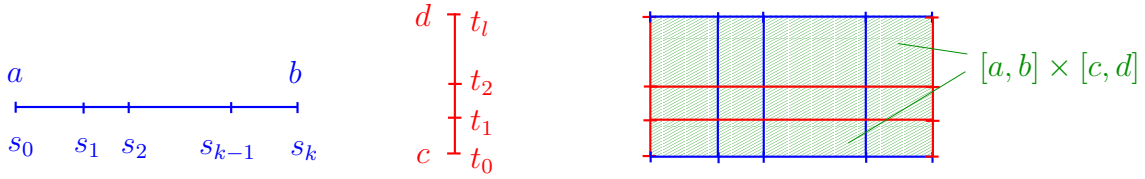
Example.

(1) If we equip $S^1 =_{\mathrm{i}} [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$ is precisely the CW-structure that we had already encountered on page 2626. We refer to the figure below for an illustration.



Iterating this example we see that for each $m \in \mathbb{N}_0$ the m-dimensional torus $(S^1)^m$ admits a CW-structure with $\binom{m}{k}$ cells of dimension k.

(2) Given $a = s_0 < s_1 < \cdots < s_k = b$ and $c = t_0 < t_1 < \cdots < t_l = d$ we can view the intervals $[a,b]$ and $[c,d]$ as CW-complexes where the 1-cells are given precisely by the intervals $[s_i, s_{i+1}]$, $i = 0, \ldots, k-1$ and $[t_j, t_{j+1}]$, $j = 0, \ldots, l-1$.



The product CW-complex is the topological space $[a,b] \times [c,d]$ where the 2-cells are given by the rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ with $i \in \{0, \ldots, k-1\}$ and $j \in \{0, \ldots, l-1\}$.

□

Proof. Let X and Y be two CW-complexes.

(1) In the Product Topology-Basics Lemma 7.6 (1a) we showed that the product topology is compatible with taking the subspace topology. In our context this means that for cells $c \subseteq X$ and $d \subseteq Y$ the product topology on $c \times d$ agrees with the subspace topology on $c \times d \subseteq (X \times Y, \mathcal{TP})$. It follows immediately from this discussion that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous.

(2) (a) Suppose that X and Y are both finite CW-complexes. It follows from the CW-Category–Product Proposition 139.16 that the CW-complex $X \times Y$ has only finitely many cells. It thus follows from the CW-Complex Compactness Corollary 136.17 that $(X \times Y, \mathcal{CWP})$ is compact. Furthermore note that it follows from the CW-Complex–Hausdorff Proposition 136.6 and the Product Topology-Properties Proposition 7.9 that $(X \times Y, \mathcal{TP})$ is Hausdorff. Finally note that it follows from these facts, the Compact-Hausdorff Proposition 4.23 (3) and Statement (1) that $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is indeed a homeomorphism.

[1439] We will add more cases in the Countable CW-Complex Product Proposition 139.23.

[1440] This special case might appear somewhat random, but as we will see in Section 139.4, this case will actually prove important for showing that several constructions, like mapping cylinders and mapping cones, give rise to CW-complexes.

(b) Suppose we have $Y = [0,1]$ with the obvious CW-structure. By (1) we know that $\mathcal{TP} \subseteq \mathcal{CWP}$. It remains to show that $\mathcal{CWP} \subseteq \mathcal{TP}$. Thus let $U \in \mathcal{CWP}$. Let $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. Since $U \in \mathcal{CWP}$ we know that for each $c \in \mathrm{Cell}(X)$ the intersection $(c \times [0,1]) \cap U$ is open in $c \times [0,1]$. Next note that it follows from the Topological-Product Proposition 7.2 (1) that the map $\Phi_c \times \mathrm{id}_{[0,1]} \colon \overline{B}^{\dim(c)} \times [0,1] \to X \times [0,1]$ is continuous. Thus we see that for each $c \in \mathrm{Cell}(X)$ the preimage $(\Phi_c \times \mathrm{id}_{[0,1]})^{-1}(U)$ is open in $\overline{B}^{\dim(c)} \times [0,1]$. It follows from this discussion together with the (c)⇒(b)-direction of the CW×Interval Proposition 137.1 (1) that $U \in \mathcal{TP}$. ■

The formulation of the CW-Complex Product Proposition 139.18 already suggests that the conclusion does not hold without some assumptions on X and Y. In fact, the following lemma, which goes back to [**Dow1952**], shows that there are indeed examples of CW-complexes X and Y such that the map $\Theta \colon Z \to X \times Y$ is *not* a homeomorphism. In other words, in general the "naive" attempt to endow a product of CW-complexes with a CW-structure fails.

Lemma 139.19. (Product-not-Product Lemma) Let $S := \{(s_1, s_2, \dots) \mid s_i \in \mathbb{N}\}$ be the set of all infinite sequences of positive integers. Let $X = \bigvee_{s \in S} ([0,1]_s, 0)$ be the wedge of (uncountably) many copies of $[0,1]$, obtained by wedging all the $0 \in [0,1]_s$. Similarly we form the wedge $Y = \bigvee_{j \in \mathbb{N}} ([0,1]_j, 0)$ of countably many copies of $[0,1]$. We equip X and Y with the CW-structure given by the CW-Wedge Lemma 139.11. In this setting the identity map $\mathrm{id} \colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is *not* a homeomorphism.

Proof. For $s \in S$ and $j \in \mathbb{N}$ we consider $(\frac{1}{s_j}, \frac{1}{s_j}) \in [0,1]_s \times [0,1]_j \in X \times Y$. We denote by $P \subseteq X \times Y$ the subset defined by all these points. Since each 2-cell of the CW-complex $X \times Y$ contains a single point of P we obtain from the Characteristic Maps Proposition 136.9 (1) that P is a closed subset of $(X \times Y, \mathcal{CWP})$. To show that the identity map $\mathrm{id} \colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is not a homeomorphism it remains to prove the following claim:

Claim. The set P is not closed with respect to the product topology $\mathcal{TP}$ on $X \times Y$.

Proof. We denote by $x_0 \in X$ and $y_0 \in Y$ the wedge points. Note that $(x_0, y_0) \notin P$. Thus it suffices to show that every neighborhood of $(x_0, y_0) \in X \times Y$ contains a point of P. In fact, by Definition 7.3 of the product topology, see the Topological-Product Proposition 7.2, it suffices to show that any subset $U \times V$ of $X \times Y$, where U and V are of the form[1441] $U = \bigvee_{s \in S} ([0, a_s), 0) \subseteq X$ and $V = \bigvee_{j \in \mathbb{N}} ([0, b_j), 0) \subseteq Y$ contains a point of P. We pick a sequence $t = (t_1, t_2, \dots) \in S$ such that for every $j \in \mathbb{N}$ we have $t_j > j$ and such that $t_j > \frac{1}{b_j}$. Furthermore we pick a $k \in \mathbb{N}$ with $k > \frac{1}{a_t}$. It follows that $t_k > k > \frac{1}{a_t}$, which implies that $\frac{1}{t_k} < a_t$. Recall that by construction we also have $\frac{1}{t_k} < b_k$. Thus we see that $(\frac{1}{t_k}, \frac{1}{t_k})$ is a point of P that lies in $[0, a_t) \times [0, b_k)$. Hence it also lies in $U \times V$. ■

The Product-not-Product Lemma 139.19 is a rather unwelcome nasty little surprise. Fortunately the following proposition shows that in some respects the two topologies $\mathcal{TP}$ and $\mathcal{CWP}$ on a product of CW-complexes are nonetheless quite similar and in fact hard to distinguish with our tools.

[1441] It follows from the Wedge Proposition 96.2 (0) that U and V are neighborhoods of the wedge point.

Proposition 139.20. (CW-Products–Invariants Proposition) Let X and Y be two CW-complexes.

(1) A subset $K \subseteq X \times Y$ is compact with respect to $\mathcal{TP}$ if and only if it is compact with respect to $\mathcal{CWP}$.

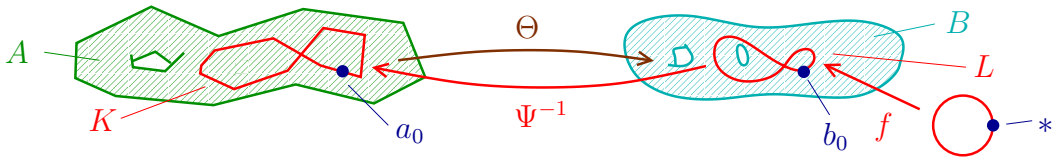
(2) By the CW-Complex Product Proposition 139.18 (1) we know that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous. Given any $n \in \mathbb{N}_0$ and given any $(x, y) \in X \times Y$ the induced map $\pi_n((X \times Y, \mathcal{CWP}), (x, y)) \to \pi_n((X \times Y, \mathcal{TP}), (x, y))$ is an isomorphism.

The proof of the CW-Product–Invariants Proposition 139.20 (2) is mostly outsourced to the following lemma:

Lemma 139.21. (Same Compacts–Same-π_n-Lemma) Let $\Theta\colon A \to B$ be a continuous bijection between two topological spaces which has the property that a subset $K \subseteq A$ is compact if and only if $\Theta(K) \subseteq B$ is compact. If B is Hausdorff, then for any $a_0 \in A$ and any $n \in \mathbb{N}_0$ the induced map $\Theta_*\colon \pi_n(A, a_0) \to \pi_n(B, f(a_0))$ is an isomorphism.

Proof. We write $b_0 \coloneqq f(a_0)$. The proof of the lemma naturally breaks up into showing that $\Theta_*\colon \pi_n(A, a_0) \to \pi_n(B, b_0)$ is an epimorphism and that it is a monomorphism.

Let us first show that the map Θ_* is an epimorphism. Suppose we are given an element[1442] $[f\colon (S^n, *) \to (B, b_0)] \in \pi_n(B, b_0)$. By the Compact Image Lemma 4.15 we know that $L \coloneqq f(S^n)$ is a compact subset of B. We set $K \coloneqq \Theta^{-1}(L)$. Since Θ is a bijection we have $\Theta(K) = L$. It follows from the fact that L is compact and our hypothesis on the map $\Theta\colon A \to B$ that K is a compact subset of A. Since Θ is a bijection, since K is compact and since L is Hausdorff we obtain from the Compact-Hausdorff Proposition 4.23 (3) that $\Psi \coloneqq \Theta|_K\colon K \to L$ is a homeomorphism. In particular the inverse $\Psi^{-1}\colon L \to K$ is continuous. Next we set $g \coloneqq \Psi^{-1} \circ f\colon (S^n, *) \to (A, a_0)$. We have $\Theta_*([g]) = [\Theta \circ g] = [\Theta|_K \circ \Psi^{-1} \circ f] = [f]$. Thus we have shown that Θ_* is an epimorphism.



The proof that Θ_* is a monomorphism is quite similar. The key observation is that the image of a homotopy $F\colon S^n \times [0, 1] \to B$ in B is again a compact subset. We leave it to the reader to fill in the few remaining details. ∎

Proof of the CW-Product–Invariants Proposition 139.20. Let X and Y be two CW-complexes.

(1) Let $K \subseteq X \times Y$. First we suppose that K is compact with respect to $\mathcal{CWP}$. Recall that in the CW-Complex Product Proposition 139.18 (1) we showed that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous. It follows from the Compact Image Lemma 4.15 that K is also compact in $\mathcal{TP}$.

[1442] As explained on page 2578, we can and will think of elements of $\pi_n(X, x_0)$ as equivalence classes of continuous maps $(S^n, *) \to (X, x_0)$.

Now suppose conversely that K is compact with respect to $\mathcal{TP}$.

(a) Note that by the CW-Complex–Hausdorff Proposition 136.6 and the Product Topology-Properties Proposition 7.9 we know that $(X \times Y, \mathcal{TP})$ is Hausdorff. Since K is compact in $\mathcal{TP}$ we obtain from the Compact-Closed Lemma 3.39 (2) that K is a closed subset of $(X \times Y, \mathcal{TP})$. By the CW-Complex Product Proposition 139.18 (1) we know that $\mathcal{TP} \subseteq \mathcal{CWP}$. Thus we have shown that K is closed in $(X \times Y, \mathcal{CWP})$.

(b) We denote by $p\colon (X \times Y, \mathcal{TP}) \to X$ and $q\colon (X \times Y, \mathcal{TP}) \to Y$ the natural projections. It follows from the Compact Image Lemma 4.15 that $p(K)$ and $q(K)$ are compact. By the CW-Complex-Finiteness Theorem 136.16 we know that there exist finite subcomplexes $X' \subseteq X$ and $Y' \subseteq Y$ such that $p(K) \subseteq X'$ and $q(K) \subseteq Y'$. But this implies that K is contained in the finite subcomplex $(X' \times Y', \mathcal{CWP})$ of the CW-complex $(X \times Y, \mathcal{CWP})$. By the CW-Complex-Finiteness Theorem 136.16 we know that $(X' \times Y', \mathcal{CWP})$ is compact. Together with (a) we see that K is a closed subset of $(X \times Y, \mathcal{CWP})$ that is contained in a compact subset $X' \times Y'$. It follows from the Compact-Closed Lemma 3.39 (1) that K itself is compact in $\mathcal{CWP}$.

(2) Again we consider the identity map $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$, which is continuous by the CW-Complex Product Proposition 139.18 (1). It follows from the CW-Complex–Hausdorff Proposition 136.6 together with the Product Topology-Properties Proposition 7.9 (3) that $(X \times Y, \mathcal{TP})$ is Hausdorff. It follows from these two observations, together with Statement (1), that we can apply the above Same Compacts–Same-π_n-Lemma 139.21 and we get the desired statement. ■

For completeness' sake we want to mention a generalization of the CW-Complex Product Proposition 139.18, even though we will not really make use of it. To formulate the generalization we need to recall and extend Definition 136.19.

Definition 139.22. A CW-complex is called locally finite (countable) if each cell meets only finitely (countably) many cells.

Proposition 139.23. (Countable CW-Complex Product Proposition) Let X and Y be two CW-complexes. We assume that one of the following two statements holds:

(1) *At least one* of X and Y is locally finite (e.g. X or Y is finite).

(2) *Both* X and Y are locally countable (e.g. both are countable)

Then the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is a homeomorphism, i.e. $\mathcal{CWP} = \mathcal{TP}$.

Proof. This statement is proved in [**LW1969**, Theorem II.5.2]. If one of X or Y is a finite CW-complex, then the statement is also proved in [**Hat2002**, Theorem A.6] or alternatively in Exercise 139.4. If both CW-complexes have countably many cells, then the statement is also proved in [**FrPi1990a**, Proposition 2.2.3] or [**Miln1956b**, Lemma 2.1]. ■

Remark. Note that [**Brooke2017**, Theorem 1] gives a necessary and sufficient condition for the equality $\mathcal{CWP} = \mathcal{TP}$ to hold. □

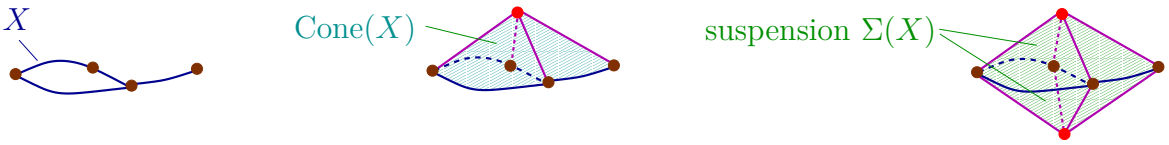
139.4. **More constructions of CW-complexes: cones, suspensions etc.** In the CW-Complex Product Proposition 139.18 we showed in particular that the product of a CW-complex with the interval $[0, 1]$ has a natural CW-structure. This product is at the heart of

many constructions of topological spaces, e.g. the cone, the suspension, the mapping cone and so on. This observation leads us straight to the following two propositions that show that several constructions that we are familiar with can also be performed with CW-complexes.

Proposition 139.24. (CW-Complex–Cone+Suspension Proposition) Let X be a CW-complex.

(1) The cone
$$\mathrm{Cone}(X) \;:=\; (X \times [0,1])/(X \times \{0\})$$
admits a natural CW-structure such that the inclusion map $X \to \mathrm{Cone}(X)$ given by $x \mapsto [(x,1)]$ is a cellular embedding.

(2) The suspension
$$\Sigma(X) \;:=\; ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\})$$
admits a natural CW-structure such that the map $X \to \Sigma(X)$ given by $x \mapsto [(x,0)]$ is a cellular embedding.

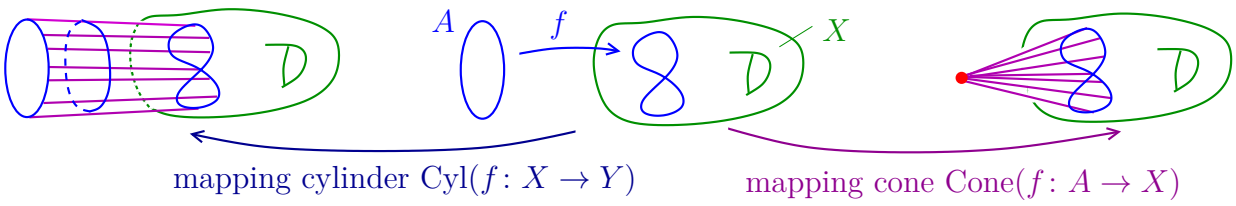


Proof. We view $[0,1]$ as a CW-complex with two 0-cells and we view $[-1,1]$ as CW-complex with the three 0-cells $-1, 0, 1$. Note that it follows from the CW-Complex Product Proposition 139.18 (2b) that we can equip $X \times [0,1]$ and $X \times [-1,1]$ with natural CW-structures. Using the Subcomplex–Characterization Proposition 136.12 one can easily verify that $X \times \{0\}$ is a subcomplex of $X \times [0,1]$ and that $X \times \{-1,1\}$ is a subcomplex of $X \times [-1,1]$. By the Cone–Pushout Lemma 20.16 and the Suspension–Pushout Lemma 20.25 we can view $\mathrm{Cone}(X)$ and $\Sigma(X)$ as pushouts as follows:

$$\begin{array}{ccc} X \times \{0\} & \hookrightarrow & X \times [0,1] \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{Cone}(X) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X \times \{-1,1\} & \hookrightarrow & X \times [-1,1] \\ \downarrow{\scriptstyle (x,\epsilon)\mapsto\epsilon} & & \downarrow \\ \{-1,1\} & \longrightarrow & \Sigma(X). \end{array}$$

Since the top horizontal are inclusions of subcomplexes and since the left vertical maps are cellular we obtain the desired results from the CW-Gluing Proposition 139.9. ∎

Proposition 139.25. (CW-Complex Mapping Cylinder+Cone Proposition) Let X be a CW-complex, let A be a CW-complex and let $f\colon A \to X$ be a cellular map. The mapping cylinder
$$\mathrm{Cyl}(f\colon A \to X) \;:=\; ((A \times [0,1]) \sqcup X)/\sim \;\text{ where } (a,1) \sim f(a) \text{ for all } a \in A$$
and the mapping cone
$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \sqcup X)/\sim \;\text{ where } (a,1) \sim f(a) \text{ for all } a \in A$$
admit natural CW-structures. Furthermore the natural maps $X \to \mathrm{Cone}(f\colon A \to X)$ and $\mathrm{Cyl}(f\colon A \to X)$ are cellular embeddings and the map $A \mapsto \mathrm{Cyl}(f\colon A \to X)$ given by $a \mapsto [(a,0)]$ is a cellular embedding.



Proof. By the Mapping Cylinder–Pushout Lemma 21.3 we have the following pushout diagram:

$$\begin{array}{ccc} A\times\{1\} & \hookrightarrow & A\times[0,1] \\ \downarrow{\scriptstyle f} & & \downarrow \\ X & \longrightarrow & \mathrm{Cyl}(f\colon A\to X). \end{array}$$

Furthermore, by the Mapping Cone–Pushout Lemma 21.9 we also have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & \mathrm{Cone}(A) \\ \downarrow{\scriptstyle f} & & \downarrow \\ X & \longrightarrow & \mathrm{Cone}(f\colon A\to X). \end{array}$$

Since in both cases the top horizontal are cellular embeddings of subcomplexes and since in both cases the left vertical maps are cellular we obtain the desired results from the CW-Gluing Proposition 139.9. ∎

We conclude our discussion with one last standard construction:

Proposition 139.26. (CW-Complex Mapping Torus Proposition) Let X be a CW-complex, let A be a CW-complex and let $f\colon A\to X$ be a cellular map. The (partial) mapping torus[1443]

$$\mathrm{Tor}(f) \;:=\; \mathrm{Tor}(f\colon A\to X) \;:=\; (X\sqcup(A\times[0,1]))/\!\sim \quad \begin{array}{l}\text{where } (a,0)\sim a \text{ and } (a,1)\sim f(a)\\ \text{for every } a\in A\end{array}$$

admits a natural CW-structure. Furthermore the natural map $X\to\mathrm{Tor}(f\colon A\to X)$ is a cellular embedding.

Proof. Similar to the Mapping Torus–Pushout Lemma 21.21 we see that we have the following pushout diagram:

$$\begin{array}{ccc} (A\times\{0\})\sqcup(A\times\{1\}) & \hookrightarrow & A\times[0,1] \\ \downarrow \begin{array}{l}(a,0)\mapsto a\\(a,1)\mapsto f(a)\end{array} & & \downarrow \\ X & \longrightarrow & \mathrm{Tor}(f). \end{array}$$

The claim now follows from the CW-Gluing Proposition 139.9. ∎

139.5. **Application: CW-HNN-Seifert-van Kampen Theorem.** Next we want to compute the fundamental group of a CW-complex that is obtained by gluing two subcomplexes of a connected CW-complex. We recall and introduce the following two definitions.

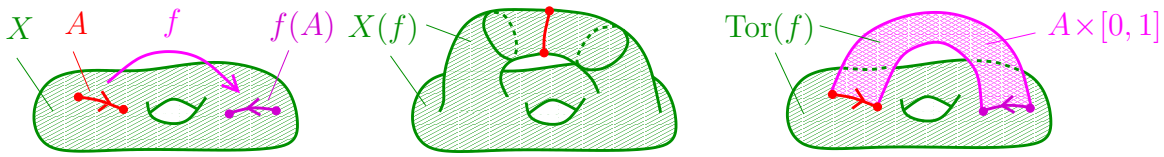
[1443] If $A = X$, then one can easily show that the partial mapping torus is naturally homeomorphic to the mapping torus that we introduced in Definition 21.19.

Definition 139.27. Let X be a topological space, let A be a subset of X and let $f\colon A \to X$ be an embedding. Following the discussion in Notation 106.1 we define

$$X(f) \coloneqq X(f\colon A \to X) \coloneqq X/\sim \qquad \text{where } f(a) \sim a \text{ for every } a \in A.$$

Furthermore we define the partial mapping torus

$$\mathrm{Tor}(f) \coloneqq \mathrm{Tor}(f\colon A \to X) \coloneqq (X \sqcup (A \times [0,1]))/\sim \qquad \text{where } (a,0) \sim a \text{ and } (a,1) \sim f(a) \text{ for every } a \in A.$$



Remark. Continuing with the above notation, note that the projection $A \times [0,1] \to A$ induces a continuous map $p\colon \mathrm{Tor}(f) \to X(f)$. In the Collapse-Cylinder Proposition 140.7 we will see that under reasonably mild hypotheses the map $p\colon \mathrm{Tor}(f) \to X(f)$ is a homotopy equivalence. In particular by the Homotopy Equivalence-π_1-Proposition 90.9 we see that p induces an isomorphism of fundamental groups. We will use this fact as an excuse to work in the following with the technically somewhat easier $\mathrm{Tor}(f)$. □

Now we can formulate, prove and discuss the following variation on the HNN-Seifert–van Kampen Theorem 106.7.

Theorem 139.28. (CW-HNN-Seifert–van Kampen Theorem) Suppose we are given a path-connected CW-complex X and two path-connected disjoint subcomplexes A and B. Let $f\colon A \to B$ be a cellular isomorphism. We pick a base point $x_0 \in A$. Let $\gamma\colon [0,1] \to X$ be an embedding with $\gamma(0) = x_0$ and with $\gamma(1) = f(x_0)$ such that $\gamma([0,1])$ is a subcomplex of the 1-skeleton X^1.[1444] We write $\Gamma = \pi_1(A, x_0)$. Finally we denote by

$$\alpha\colon \Gamma = \pi_1(A, x_0) \to \pi_1(X, x_0)$$

the inclusion-induced map and we denote by

$$\beta\colon \Gamma = \pi_1(A, x_0) \xrightarrow{f_*} \pi_1(B, f(x_0)) \to \pi_1(X, f(x_0)) \xrightarrow{\gamma_*} \pi_1(X, x_0)$$

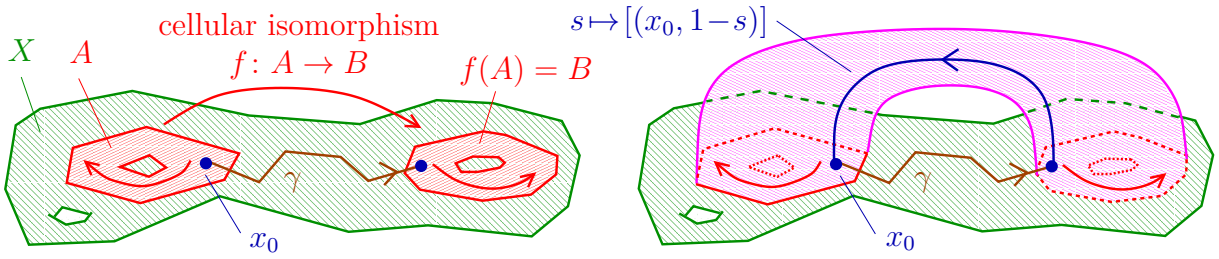
the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from the Change-of-Base Point Proposition 88.14 (1), using the path γ. The natural map

$$\Phi\colon \underbrace{\langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle}_{\text{HNN-extension as in Definition 106.2}} \xrightarrow{\cong} \pi_1(\mathrm{Tor}(f), x_0)$$

which is induced by the natural map $X \to \mathrm{Tor}(f)$ and which is given by sending t to the composition of γ with the path $s \mapsto [(x_0, 1-s)]$, is an isomorphism.

Remark.

[1444] Note that it follows from our hypothesis that X is path-connected together with the Graph-CW-Complex Lemma 136.5 and the Abstract Graph-Topology Lemma 11.6 that such γ exists.

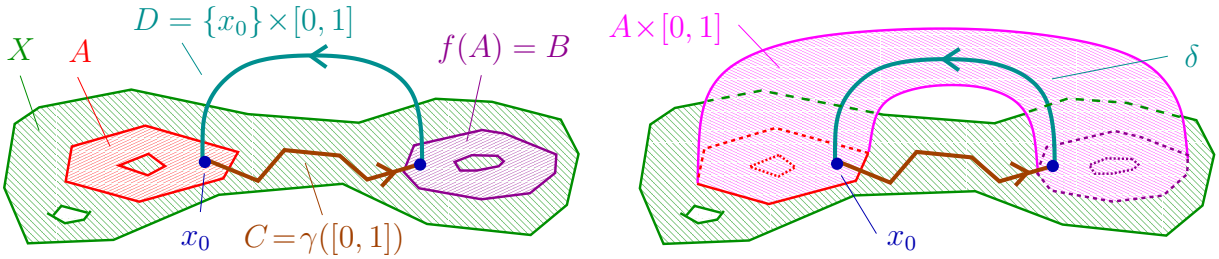


(1) Basically the same way that we obtained the HNN-Seifert–van Kampen Theorem 106.7 from the Seifert–van Kampen-Theorem 99.1 one can also obtain the above CW-HNN-Seifert–van Kampen Theorem 139.28 from the CW-Seifert–van Kampen Theorem 137.10. Below we will give a different proof of the CW-HNN-Seifert–van Kampen Theorem 139.28 which is arguably easier than the argument provided in the proof of the HNN-Seifert–van Kampen Theorem 106.7.

(2) Using the Collapse-Cylinder Proposition 140.7 one can also deal with certain situations where the map $f\colon A \to B$ is not a cellular isomorphism. We leave it to the reader to figure out what these generalized statements might be. □

Proof. Let X be a path-connected CW-complex and let $f\colon A \to B$ be a cellular isomorphism between two path-connected disjoint subcomplexes of X. Let $x_0 \in A$ and let $\gamma\colon [0,1] \to X$ be an embedding with $\gamma(0) = x_0$ and with $\gamma(1) = f(x_0)$ such that $C := \gamma([0,1])$ is a subcomplex. We equip the mapping torus $\mathrm{Tor}(f)$ with the natural CW-structure from the CW-Complex Mapping Torus Proposition 139.26.

We consider $Y := (X \sqcup (\{x_0\} \times [0,1])/\sim$ where $(x_0, 0) \sim x_0$ and $(x_0, 1) \sim f(x_0)$. By the CW-Gluing Proposition 139.9 we can view Y as a CW-complex given by attaching a single 1-cell to X. We denote by D the image of $\{x_0\} \times [0,1]$ in Y. Furthermore we denote by $\delta\colon [0,1] \to D$ the path that is given by $s \mapsto (x_0, 1-s)$.



Now note that we have the following two isomorphisms:

$$\pi_1(Y, x_0) \underset{\uparrow}{\cong} \pi_1(X, x_0) *_{\pi_1(C, x_0)} \pi_1(C \cup D, x_0) \underset{\uparrow}{\cong} \pi_1(X, x_0) * \langle t = [\gamma * \delta] \rangle$$

by the CW-Seifert–van Kampen Theorem 137.10

we have $\pi_1(C, x_0) = \{e\}$ since C is contractible and we have $\pi_1(C \cup D, x_0) = \langle [\gamma * \delta] \rangle$ by the Circle-π_1-Theorem 88.22

and

$$\pi_1(A, x_0) * \pi_1(B, f(x_0)) \underset{\uparrow}{\cong} \pi_1(A, x_0) * \pi_1(D \cup B, x_0) \underset{\uparrow}{\cong} \pi_1(A \cup D \cup B, x_0).$$

by the CW-Seifert–van Kampen Theorem 137.10 and the base point change $f(x_0) \mapsto x_0$ induced by δ

Finally we also have the isomorphism:

$$\pi_1(\mathrm{Tor}(f), x_0) \underset{\uparrow}{\cong} \pi_1(Y, x_0) *_{\pi_1(A\cup D\cup B, x_0)} \underbrace{*\pi_1(A \times [0,1], x_0)}_{\cong \pi_1(A, x_0)}.$$

by the CW-Seifert–van Kampen Theorem 137.10, here we use that $Y \cap (A \times [0,1]) = A \cup D \cup B$ is connected

Note that in the above discussion basically all maps are induced by the obvious inclusion maps. We leave it to the intrepid reader to go through the explicit descriptions of the various homomorphisms and the description of the amalgamated product in the Group Pushout Proposition 98.22 to show that we do indeed end up with the promised description of $\pi_1(\mathrm{Tor}(f), x_0)$. ■

139.6. **Appendix: A convenient category of topological spaces.** Let X and Y be two CW-complexes. Recall that we just saw in the previous sections that we can equip the set $X \times Y := \{(x, y) \mid x \in X \text{ and } y \in Y\}$ with two different topologies, namely the familiar product topology $\mathcal{TP}$ from the Topological-Product Proposition 7.2 and the less familiar CW-product topology $\mathcal{CWP}$ from Definition 139.15. As we saw in the Product-not-Product Lemma 139.19, in general these two topologies are different.

This is quite a nuisance. In this section we will discuss one approach to addressing this issue. We will not make much use of the ideas discussed in this appendix, thus the reader can safely move on to the next section.

The following theorem shows that if we move from considering all topological spaces to a slightly more constrained setup, then we can combine the two concepts of products that we introduced for CW-complexes.

Theorem 139.29. (Convenient Category Theorem) There exists a full subcategory $\mathcal{C}$ of the category of topological spaces with the following properties:

(1) Every regionally compact Hausdorff space is an object of $\mathcal{C}$.
(2) Every CW-complex is an object of $\mathcal{C}$.
(3) Every diagram in $\mathcal{C}$ has a limit and a colimit.
(4) If X and Y are two CW-complexes, then the topological space $(X \times Y, \mathcal{CWP})$ is in fact the product in $\mathcal{C}$.

Remark. A category as in the Convenient Category Theorem 139.29 is often called a *convenient category*. Note though that the notion of a convenient category might be used somewhat differently by different authors. In fact there are many "convenient categories", possibly with somewhat different properties from the ones we stated. We refer to [**Stee1967, Miln1959a, Spa1963, VogR1971, McCor1967, Stri, Pre2002, BrownR1963**] and [**Stro2011**, Chapter 3.4] for more examples and for the discussion of various notions of a "convenient category". □

We will outline the proof of the Convenient Category Theorem 139.29 by giving an explicit example of a convenient category. We will need the following definitions.

Definition 139.30.

(1) A subset A of a topological space X is called compactly closed if for every continuous map $\varphi\colon K \to X$, where K is a compact Hausdorff space, the preimage $\varphi^{-1}(A)$ is a closed subset of K.[1445]

(2) A topological space X is called compactly generated if every compactly closed subset of X is actually closed.[1446]
(3) A topological space X is called weakly Hausdorff if for every compact Hausdorff space K and every continuous map $\varphi\colon K \to X$, the image $\varphi(K)$ is closed.[1447]
(4) A topological space X is called a CGWH-space[1448] if it is compactly generated and if it is weakly Hausdorff.

First, for the readers who like weird topological spaces we give some non-examples.

Non-Examples.

(1) We equip the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{*\}$ with the topology where $U \subseteq X$ is open if one of the following holds:
   (a) the point $*$ is contained in U, or
   (b) for all but finitely many $n \in \mathbb{N}$ the set U contains all but finitely many points in $\{n\} \times \mathbb{N}$.

   In Exercise 139.6 we will see that X is not compactly generated.
(2) We consider $X = \mathbb{R} \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ and we consider $Y = \mathbb{R}/Z$ where $Z \subseteq \mathbb{R}$ is the subset given by the positive integers.[1449] In [**Enge1989**, Example 3.3.29], see also Exercise 139.9, it is shown that the product topological space $X \times Y$ is not compactly generated.
(3) In Exercise 139.10 we will give an example of a topological space that is weakly Hausdorff but that is not Hausdorff. □

We proceed with an actual example of a CGWH-space.

Proposition 139.31. (CW-CGWH Proposition) Every CW-complex is a CGWH-space.

Proof. Let X be a CW-complex.

(1) Let A be a compactly closed subset of X. It follows easily from the Characteristic Maps Proposition 136.9 (1) and the fact that the closed balls $\overline{B}^n$ are compact and Hausdorff, that A is in fact a closed subset of X.
(2) By the CW-Complex–Hausdorff Proposition 136.6 we know that X is Hausdorff. It follows from the Compact-Closed Lemma 3.39 (2) that X is in fact weakly Hausdorff. ■

We continue with the following simple-minded definition.

Definition 139.32. The category $\mathcal{CGWH}$ of CGWH-spaces is the full subcategory of the category of topological spaces whose objects are precisely the CGWH-spaces.

[1445] It follows from the Continuity-via-Closed Subsets Lemma 4.5 that every closed subset of X is actually compactly closed.
[1446] Sometimes in the literature a compactly generated topological space is also called a *k-space*.
[1447] By the Compact-Closed Lemma 3.39 (2) we know that a topological space that is Hausdorff is also, as it should be, weakly Hausdorff.
[1448] Evidently the acronym CGWH stands for Compactly Generated and Weakly Hausdorff.
[1449] We use this cumbersome notation to make sure that we do not confuse the quotient space $\mathbb{R}/Z$ with the quotient of $\mathbb{R}$ by the obvious action of the group $(\mathbb{Z}, +)$ on the topological space $\mathbb{R}$.

Remark. The category of CGWH-spaces was introduced by Michael McCord [**McCor1969**], building on work by Norman Steenrod [**Stee1967**]. □

The following theorem shows that the category of CGWH-spaces has many charming properties. The theorem provides us in particular with a proof of the Convenient Category Theorem 139.29.

Theorem 139.33. (CGWH-Category Theorem) The category $\mathcal{CGWH}$ of CGWH-spaces has the following properties:

(0) The category $\mathcal{CGWH}$ is a full subcategory of the category of topological spaces.
(1) Every regionally compact Hausdorff space is an object of $\mathcal{CGWH}$.
(2) Every CW-complex is an object of $\mathcal{CGWH}$.
(3) Every diagram in $\mathcal{CGWH}$ has a limit and a colimit.
(4) Given any family $\{X_i\}_{i\in I}$ of objects in $\mathcal{CGWH}$ there exists a product in the category $\mathcal{CGWH}$.
(5) Let X and Y be two objects in $\mathcal{CGWH}$. The following statements hold:
  (a) The product of X and Y in $\mathcal{CGWH}$ is given by the set $X \times Y$ together with a topology $\mathcal{CGWHP}$ that contains the usual product topology $\mathcal{TP}$. In particular the identity map
$$(X \times Y, \mathcal{CGWHP}) \;\to\; (X \times Y, \mathcal{TP})$$
  is a bijection and continuous.
  (b) If X and Y are compact and Hausdorff, then $\mathcal{CGWHP} = \mathcal{TP}$, i.e. the map in (a) is a homeomorphism.
  (c) If X and Y are two CW-complexes, then $\mathcal{CGWHP} = \mathcal{CWP}$.

Example. It follows from the Manifold–Local Properties Lemma 23.8 (4) and the CGWH-Category Theorem 139.33 (2) that every topological manifold is a CGWH-space. □

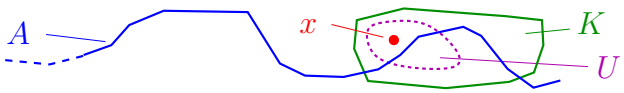
Sketch of proof. Most of the statements below were first proved by Michael McCord [**McCor1969**], building on work by Norman Steenrod [**Stee1967**]. Arguably the best account of the proof is given in [**Stri**]. In the following we show that our theorem, as formulated, can be deduced from the statements in [**Stri**].

(0) This statement holds by definition.
(1) Let X be a topological space that is regionally compact and Hausdorff. It follows from the Compact-Closed Lemma 3.39 (2) that X is also weakly Hausdorff.

Now let $A \subseteq X$ be a subset that is compactly closed. We need to show that A is actually closed. It follows from the Cover-Open Subset Lemma 3.20 that it suffices to prove the following claim:

Claim. For every $x \in X$ there exists an open neighborhood U of x such that $U \cap A$ is closed in U.

Proof. Let $x \in X$. Since X is regionally compact we know that there exists a neighborhood K of $x \in X$ that is compact. Since X is Hausdorff and since $A \subseteq X$ is compactly closed we see that $K \cap A$ is closed in K.



Since K is a neighborhood of x there exists an open subset $U \subseteq X$ with $x \in U \subseteq K$. Since $K \cap A$ is closed in K we obtain from the elementary Iterated Subspace Lemma 3.18 that $U \cap A$ is closed in U. ⊞

(2) In the CW-CGWH Proposition 139.31 we just saw that every CW-complex is a CGWH-space.

(3) This statement follows from [**Stri**, Propositions 2.23 and 2.30].

(4) As we discussed on page 258, taking the product is a special case of a limit. Thus this statement is just a special case of (3).

(5) Let X and Y be two topological spaces. Let $\mathcal{TP}$ be the usual product topology on $X \times Y$. Furthermore let $\mathcal{CGWHP}$ be the topology on $X \times Y$ where the closed sets are precisely the compactly closed subsets of $(X \times Y, \mathcal{TP})$.[1450] By the Continuity-via-Closed Subsets Lemma 4.5 we know that every closed subset is compactly closed, thus we see that $\mathcal{TP} \subseteq \mathcal{CGWHP}$. It follows from [**Stri**, Proposition 2.4 and Corollary 2.16] that $(X \times Y, \mathcal{CGWHP})$ is the product in the category of CGWH-spaces.

   (a) This statement follows immediately from the above discussion.

   (b) Now suppose that X and Y are compact and Hausdorff. By the Product Topology-Properties Proposition 7.9 we know that the topological space $(X \times Y, \mathcal{TP})$ is also compact and Hausdorff. But it follows almost immediately from the definitions that in a compact Hausdorff space a subset is compactly closed if and only if it is closed. Thus we see that $\mathcal{TP} = \mathcal{CGWHP}$.

   (c) Now let X and Y be CW-complexes. We denote by $\mathcal{CWP}$ the topology on the set $X \times Y$ that we introduced in Definition 139.15. Recall that we need to show that $\mathcal{CGWHP} = \mathcal{CWP}$. We make the following two observations:

      (a) By definition a subset $A \subseteq X \times Y$ is closed in $\mathcal{CGWHP}$ if and only if the intersection of A with each compact subset of $(X \times Y, \mathcal{TP})$ is closed.

      (b) By definition a subset $A \subseteq X \times Y$ is closed in $\mathcal{CWP}$ if and only if the intersection of A with each cell of the CW-complex $(X \times Y, \mathcal{CWP})$ is closed.

   Using the Characteristic Maps Proposition 136.9 (1), the CW-Complex-Finiteness Theorem 136.16 and Exercise 7.16 one can show that $\mathcal{CGWHP}$ and $\mathcal{CWP}$ have the same closed subsets. We leave it to the reader to fill in the details. ■

Remark. Given CGWH-spaces X and Y it is also convenient to equip X^Y, i.e. the set of continuous maps from Y to X, with a slight modification of the usual compact-open topology. We refer to [**Stri**, Proposition 2.24] for details. □

We conclude this section with the following proposition that gives us three other convenient properties of the category of CGWH-spaces. The reader who has struggled with the intricacies of the Whitehead Quotient Theorem 17.6 and the Product-with-Quotient Map Proposition 17.14 will surely appreciate these statements.

Proposition 139.34. (CGWH Properties Proposition)

(1) Let $f\colon X \to Y$ be a bijection between two topological spaces. If f is proper in the sense of Definition 4.63, i.e. if the preimage of every compact subset of Y is a compact subset of X, and if X and Y are CGWH-spaces, then f is actually a homeomorphism.

[1450] One can easily verify that this defines a topology.

(2) Let $f\colon W \to X$ and $g\colon Y \to Z$ be quotient maps, in the sense of Definition 17.3. If W, X, Y and Z are CGWH-spaces, then $f \times g\colon (W \times Y, \mathcal{CGWH}) \to (X \times Z, \mathcal{CGWH})$ is also a quotient map.
(3) Let X and Y be topological spaces and let $B \subseteq Y$ be a closed subset. If X and Y are CGWH-spaces, then the obvious map $(X \times Y)/(X \times B) \to X \times (Y/B)$, where each product is equipped with the $\mathcal{CGWH}$ topology, is a homeomorphism.

Proof. This proposition is proved in [**Stri**, Propositions 2.20, 2.40 and 3.17]. ■

Remark. The category of CGWH-spaces admits a fully faithful embedding into the category of "quasiseparated condensed sets". The category of "quasiseparated condensed sets" is particularly well-suited for dealing with technical problems that arise in topology. We refer to [**Schol2019**, Theorem 2.16] and [**Schol2019b**, Proposition 1.2] for details. □

139.7. Exercises for Chapter 139.

Exercise 139.1. Let $f\colon X \to Y$ be a map between two CW-complexes. Show that if f is a cellular isomorphism, then for each cell c of X the image $f(c)$ is a cell of Y.

Exercise 139.2. Let $\mathcal{C}$ be the category where the objects are abstract graphs and the morphisms are isomorphisms of abstract graphs. Furthermore let $\mathcal{D}$ be the category where the objects are CW-complexes of dimension ≤ 1 and the morphisms are cellular maps. Show that there exists a covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ that is faithful and essentially surjective in the sense of the definition on page 245.

Exercise 139.3. Let $n \in \mathbb{N}_0$. We equip $\mathbb{R}\mathrm{P}^n$ with the CW-structure from page 2628. Show that $\mathbb{C}\mathrm{P}^n$ has a CW-structure such that the obvious map $\mathbb{R}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ is cellular.

Exercise 139.4. Let X and Y be two CW-complexes. Show that if X is finite, then the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{P})$ is a homeomorphism.
Hint. Formulate and prove a suitable generalization of the CW×Interval Proposition 137.1 and use the Pasting Proposition 4.7.

Exercise 139.5. Given a finite CW-complex X we define its *Euler characteristic* $\chi(X)$ as follows:

$$\chi(X) \coloneqq \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

(a) Let X be a finite CW-complex and let A be a subcomplex. What is the relationship between $\chi(X)$, $\chi(A)$ and $\chi(X/A)$?
(b) Let $p\colon \widetilde{X} \to X$ be a finite covering of CW-complexes such that p is cellular. What is the relationship between $\chi(\widetilde{X})$ and $\chi(X)$?
(c) Let X and Y be two finite CW-complexes. In the CW-Complex Product Proposition 139.18 together with the CW-Category–Product Proposition 139.16 we introduced the product CW-structure on $X \times Y$. Is $\chi(X \times Y)$ determined by $\chi(X)$ and $\chi(Y)$?
(d) Let X be a finite CW-complex. is there a relationship between $\chi(X)$ and $\chi(\mathrm{Cone}(X))$?
(e) Let X be a finite CW-complex. Is there a relationship between $\chi(X)$ and $\chi(\Sigma(X))$?

Exercise 139.6. We consider the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{*\}$. We say that $U \subseteq X$ is open if $* \notin U$ or if for all but finitely many $n \in \mathbb{N}$ the set U contains all but finitely many points in $\{n\} \times \mathbb{N}$.

(a) Show that these open subsets do indeed define a topology on X.
(b) Show that X is Hausdorff.
(c) Show that every infinite subset of X contains an infinite closed and discrete subset.
(d) Show that every compact subset of X is finite.
(e) Show that the compactly generated topology on X is the discrete topology.

These steps show that X is not compactly generated.

Exercise 139.7. Let $\mathcal{C}$ be a convenient category of topological spaces, i.e. let $\mathcal{C}$ be a full subcategory $\mathcal{C}$ of the category of topological spaces that has the properties stated in the Convenient Category Theorem 139.29. Let X and Y be CW-complexes. Show that if the usual product $(X \times Y, \mathcal{TP})$ lies in $\mathcal{C}$, then $(X \times Y, \mathcal{TP})$ is homeomorphic to the product CW-complex $(X \times Y, \mathcal{CWP})$.

Exercise 139.8. Let X and Y be CGWH-spaces. Show that for each $n \in \mathbb{N}$, each $x_0 \in X$ and $y_0 \in Y$ the natural maps $p\colon (X \times Y, \mathcal{CGWHP}) \to X$ and $q\colon (X \times Y, \mathcal{CGWHP}) \to Y$ induce an isomorphism

$$p_* \times q_*\colon \pi_n((X \times Y, \mathcal{CGWHP}), (x_0, y_0)) \xrightarrow{\cong} \pi_n(X, x_0) \times \pi_n(Y, y_0).$$

Remark. The exercise is to verify that the proofs of the Product-π_1-Proposition 90.13 and the Product-π_n-Proposition 134.18 also apply to $(X \times Y, \mathcal{CGWHP})$ instead of the usual product topological space $(X \times Y, \mathcal{TP})$.

Exercise 139.9. We consider $X = \mathbb{R} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and we consider $Y = \mathbb{R}/Z$ where $Z \subseteq \mathbb{R}$ is the subset given by the positive integers. We consider the subset

$$A \;=\; \big\{\big(\tfrac{1}{i} + \tfrac{a_i}{j}, i + \tfrac{1}{2j}\big) \in X \times Y \,\big|\, i, j \in \mathbb{N}\big\} \quad \text{where } a_i = \tfrac{1}{2}\big(\tfrac{1}{i} - \tfrac{1}{i+1}\big).$$

(a) Show that the closure of A contains $(0, [0])$.
(b) Show that for any compact subset $K \subseteq X \times Y$, the set $K \cap A$ has only finitely many points.
(c) Show that X is not compactly generated.

Exercise 139.10.

(a) We consider $$X \;:=\; \{0\} \cup \bigcup_{n,m \in \mathbb{N}} \{\tfrac{1}{n} + \tfrac{1}{m+n}\} \;\subseteq\; \mathbb{R}.$$
We equip X with the usual metric coming from $\mathbb{R}$. We set $x := 0$.
(i) Show that each point in $X \setminus \{x\}$ is open.
(ii) Show that X is not regionally compact at x.
(b) Let X be a countable metric space which admits a point $x \in X$ such that the following conditions are satisfied:
(i) Each point in $X \setminus \{x\}$ is open.
(ii) The topological space X is not regionally compact at x.
Let $Y := X \cup \{\infty\}$ be the one-point compactification as defined in Exercise 4.67.
(α) Show that compact sets of Y are closed. (This implies that Y is weakly Hausdorff.)
(β) Show, by considering the points x and ∞, that Y is not Hausdorff.

Remark. The combination of (a) and (b) shows that there exist topological spaces that are weakly Hausdorff but not Hausdorff.

Exercise 139.11. Let X be a CW-complex and let $f\colon X \to X$ be a cellular map. We denote by $\mathrm{Tor}(X,f) \coloneqq (X \times [0,1])/(x,0) \sim (f(x),1)$ the corresponding mapping torus from Definition 21.19. Show that $\mathrm{Tor}(X,f)$ admits a natural CW-structure.

CHAPTER 140

The Homotopy Extension Theorem

In this chapter and the following chapter we will prove several key technical theorems on CW-complexes, in particular we will prove the Homotopy Extension Theorem 140.2 and the Cellular Approximation Theorem 141.2. In all likelihood for most readers it is best to skip these two chapters and to come back to them, when later on these results actually get used.

This chapter is organized as follows:

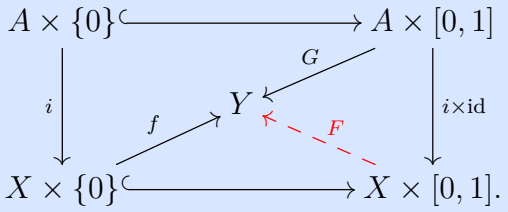
- In Section 140.1 we will state the Homotopy Extension Theorem 140.2.
- In Section 140.2 we will prove the Homotopy Extension Theorem 140.2.
- In Section 140.3 we will use the Homotopy Extension Theorem 140.2 to show that, for maps between CW-complexes, we can often ignore base points
- In Section 140.4 we will see under what circumstances one can combine infinitely many homotopies between CW-complexes.
- In Section 140.5 we will provide exercises for Chapter 140.

In the next Chapter 141 we will state and prove the Cellular Approximation Theorem 141.2.

140.1. **The Homotopy Extension Theorem.** We start out by recalling Definition 22.3:

Definition 140.1.

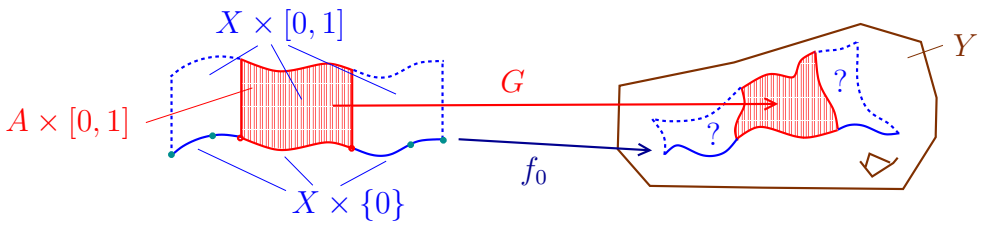
(1) A continuous map $i\colon A \to X$ between topological spaces is called a cofibration if it has the homotopy extension property, i.e. if given any topological space Y and any two continuous maps $f\colon X \times \{0\} \to Y$ and $G\colon A \times [0,1] \to Y$ with $f(i(a),0) = G(a,0)$ for all $a \in A$ there exists a continuous map $F\colon X \times [0,1] \to Y$ such that the following diagram commutes:



We illustrate the definition in the figure below.

(2) If $i\colon A \to X$ is a cofibration such that $i(A)$ is a closed subset of X, then we refer to i as a closed cofibration.

Now we can already formulate the first main theorem of this chapter.



Theorem 140.2. (Homotopy Extension Theorem) Let X be a CW-complex and let A be a subcomplex of X.

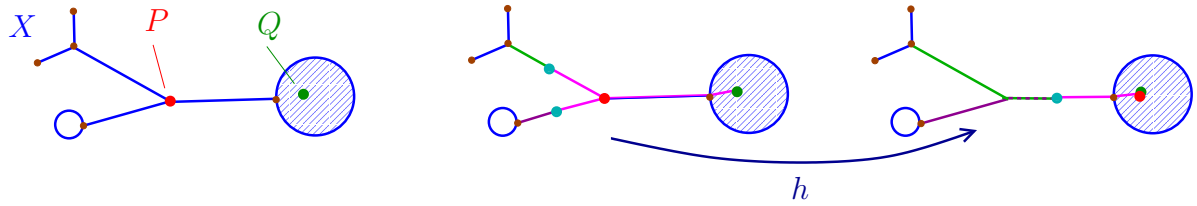
(1) The inclusion map $A \to X$ is a closed cofibration.
(2) In slightly different words: let Y be a topological space and let $f_0 \colon X \to Y$ be a continuous map. Furthermore, suppose that $A \subseteq X$ is a subcomplex and suppose that we are given a homotopy $G \colon A \times [0,1] \to Y$ from $g_0 := f_0|_A$ to some continuous map $g_1 \colon A \to Y$. Then there exists a homotopy $F \colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A \times [0,1]}$ equals G.

We postpone the long and strenuous proof of the Homotopy Extension Theorem 140.2 to the following section. Let us first revel in some exciting corollaries of the Homotopy Extension Theorem 140.2.

Corollary 140.3. (CW-Complex–Vertices–Homotopy Corollary) Let X be a connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y.

(1) Given any path $\gamma \colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$ there exists a homotopy $F \colon X \times [0,1] \to X$ with $F(P,t) = \gamma(t)$ for all $t \in [0,1]$ and such that $F_0 = \mathrm{id}$.
(2) There exists a continuous map $h \colon X \to X$ with $h(P) = Q$ that is homotopic to the identity.

Example. We consider the CW-complex X and the two points P and Q shown in the figure below.



It should be pretty clear that there is no homeomorphism f of X with $f(P) = Q$. But the CW-Complex–Vertices–Homotopy Corollary 140.3 gives us a decent replacement, namely a map $h \colon X \to X$ with $h(P) = Q$ which is homotopic to the identity. □

Proof. Let X be a connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y.

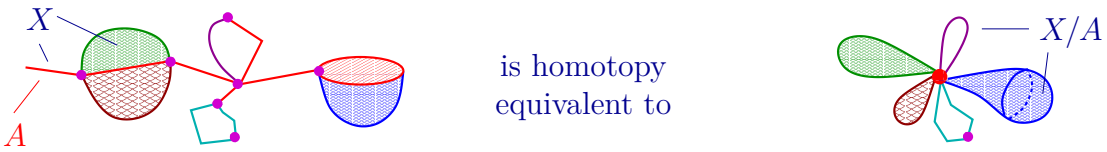
(1) Let $\gamma \colon [0,1] \to X$ be a path with $\gamma(0) = P$ and $\gamma(1) = Q$. We write $A := \{P\}$. Since P is a point in the 0-skeleton of X we see that A is a subcomplex of A. We apply the Homotopy Extension Theorem 140.2 to $f_0 = \mathrm{id} \colon X \to X$, $g_0(P) := P$, $g_1(P) = Q$ and $G(P,t) = \gamma(t)$. The resulting homotopy $F \colon X \times [0,1] \to X$ has all the desired properties.

(2) Since X is connected we obtain from the CW-Complex–Local Properties Proposition 137.8 (2) together with the Locally Path-Connected Lemma 4.57 that there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$. From (1) we obtain a homotopy $F\colon X \times [0,1] \to X$ with $F_0 = \mathrm{id}$ and $F_1(P) = Q$. The map $h \coloneqq F_1$ evidently has all the promised properties. ■

The Homotopy Extension Theorem 140.2 allows us to dip into the theory of cofibrations that we developed in Chapter 22. In particular we obtain the following useful corollary.

Corollary 140.4. (Quotient-by-Contractible Corollary) Let X be a CW-complex and let $A \subseteq X$ be a subcomplex.

(1) If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.
(2) If $a_0 \in A$ is a point such that $\{a_0\}$ is a deformation retract of A, then the projection $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.



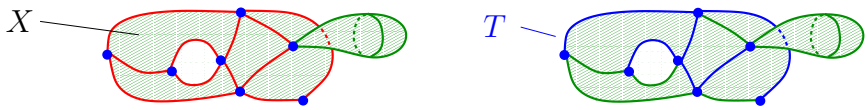
Remark. In the Tree–Contractible Proposition 137.4 we showed that trees are contractible. The Quotient-by-Contractible Corollary 140.4 (1) can thus be viewed as a generalization of the 1-Complex-mod-Tree Proposition 137.7. □

Proof. By the Subcomplex Cofibration Theorem 22.12 we know that the inclusion $A \to X$ is a closed cofibration. The two statements now follow immediately from the corresponding statements in the Cofibration-Quotient Lemma 22.10. ■

The following proposition is a neat corollary to the Quotient-by-Contractible Corollary 140.4.

Proposition 140.5. (CW-Complex–Spanning Tree Proposition) Let X be a 0-connected CW-complex.

(1) There exists a contractible 1-dimensional subcomplex T which contains all 0-cells.[1451]
(2) Given any T as in (1) the quotient X/T is a CW-complex with precisely one 0-cell. Furthermore the projection $X \to X/T$ is a homotopy equivalence.



Remark. Using the Cell Attachment-π_1-Proposition 135.5 it is very easy to calculate the fundamental group of a finite non-empty connected CW-complex with a single 0-cell. The CW-Complex–Spanning Tree Proposition 140.5 says in particular that given a finite 0-connected CW-complex we can always reduce the calculation of the fundamental group to the case of a finite CW-complex with a single 0-cell. □

[1451] Such a subcomplex is basically what we called a spanning tree in Definition 11.23.

Proof. Let X be a 0-connected CW-complex.

(1) It follows from the Graph–CW-Complex Lemma 136.5 together with the Spanning Tree–Existence Proposition 11.26 and the Tree–Contractible Proposition 137.4 that the 1-skeleton X^1 admits a subcomplex T that contains all 0-cells of X and that admits a deformation retraction to x_0.

(2) By the CW-Quotient Lemma 139.10 the quotient X/T admits a CW-structure with one 0-cell. Furthermore note that it follows from the Quotient-by-Contractible Corollary 140.4 (2) that the projection $X \to X/T$ is a homotopy equivalence between. ■

The following proposition gives us a very appealing calculation of fundamental groups of quotients.

Proposition 140.6. (Quotient-π_1-Proposition) Let X be a connected CW-complex and let $A \subseteq X$ be a connected non-empty subcomplex. Given any $a \in A$ the natural map

$$\pi_1(X,a)/\langle\langle i_*(\pi_1(A,a))\rangle\rangle \;\to\; \pi_1(X/A,[x]),$$

which is induced by the natural projection $X \to X/A$, is an isomorphism.

Proof. We denote by $i\colon A \to X$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccc} \pi_1(X,a)/\langle\langle i_*(\pi_1(A,a))\rangle\rangle & \longrightarrow & \pi_1(\mathrm{Cone}(i\colon A\to X),a) \\ & \searrow \quad \swarrow & \\ & \pi_1(X/A,[x]) & \end{array}$$

We make the following clarifications and observations:

(1) The top horizontal map is induced by the natural map $A \to \mathrm{Cone}(i\colon A \to X)$. It follows from the Mapping Cone-π_1-Lemma 99.15 that this map is an isomorphism.

(2) The right diagonal map is induced by the natural map $\mathrm{Cone}(i\colon A \to X) \to X/A$. It follows from the Cofibration-Quotient Lemma 22.10 together with the Homotopy Extension Theorem 140.2 that the right diagonal map is an isomorphism.

(3) One can easily verify that the diagram commutes.

It follows from the above discussion that the left diagonal map is an isomorphism. ■

The following proposition gives another typical application of the Homotopy Extension Theorem 140.2.

Proposition 140.7. (Collapse-Cylinder Proposition) Let $f\colon A \to B$ be a cellular map between CW-complexes that has the following property:

(∗) For each n-cell σ of A we have either $f(\sigma) \subseteq B^{n-1}$ or $f|_\sigma$ is a homeomorphism.

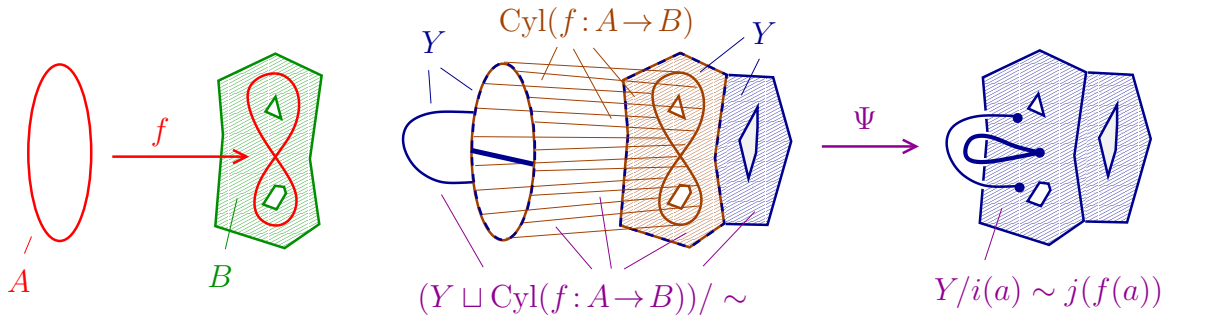
We consider the corresponding mapping cylinder

$$\mathrm{Cyl}(f) \coloneqq \mathrm{Cyl}(f\colon A\to B) \coloneqq ((A\times[0,1]) \sqcup B)/\sim \text{ where } (a,1)\sim f(a) \text{ for all } a\in A.$$

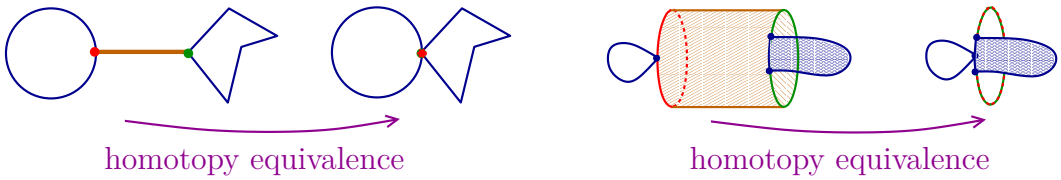
that we introduced in Definition 21.1. Finally let Y be a CW-complex together with a cellular embedding $i \sqcup j\colon A \sqcup B \to Y$. The map

$$\begin{array}{rcl} \Phi\colon (Y \sqcup \mathrm{Cyl}(f\colon A\to B))/\sim & \to & Y/i(a)\sim j(f(a)) \\ [P] & \mapsto & \begin{cases} [P], & \text{if } P\in Y, \\ [i(a)], & \text{if } P=[(a,t)] \text{ with } a\in A \text{ and } t\in[0,1], \\ [j(b)], & \text{if } P=[b] \text{ with } b\in B \end{cases} \end{array}$$

is a homotopy equivalence.



Example. In many applications the map f will actually be a cellular isomorphism. In this special case the Collapse-Cylinder Proposition 140.7 basically says that one can compress a product $A \times [0,1]$ to a single copy of A without changing the homotopy type.



Two such examples are shown in the figure above. □

Remark. Note that the statement of the Collapse-Cylinder Proposition 140.7 is related in spirit to the statement of the Cofibration-Quotient Lemma 22.10 (2). □

The proof of the Collapse-Cylinder Proposition 140.7 rests on the following lemma.

Lemma 140.8. (CW-Quotient Lemma) Let X be a CW-complex, let A be a complex of X and let $f\colon A \to X$ be a cellular map with the following properties:

(1) For each n-cell σ of A we have either $f(\sigma) \subseteq X^{n-1}$ or $f|_\sigma$ is a homeomorphism.
(2) The image of f is contained in a subcomplex of X that is disjoint from A.

Then $X/\sim$, where $a \sim f(a)$ for all $a \in A$, admits a natural CW-structure such that for every subcomplex B that is disjoint from $f(A)$ the natural map $B \to X/\sim$ is a cellular embedding.

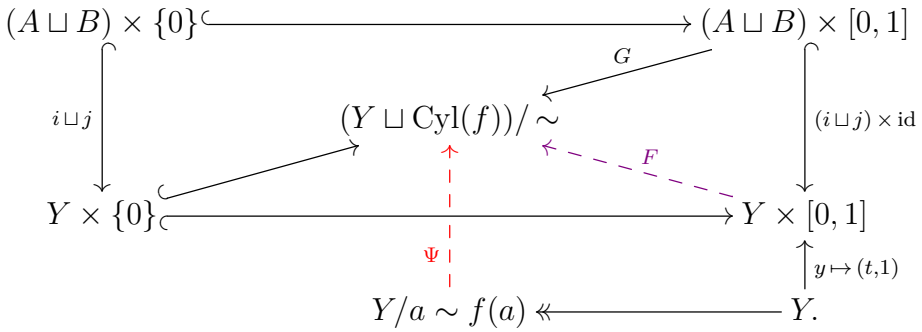
Proof of the CW-Quotient Lemma 140.8. This lemma follows from [**LW1969**, Proposition 5.8]. Note that we can apply [**LW1969**, Proposition 5.8] since our hypothesis on f ensures that f defines a "cellular equivalence relation" in the sense of [**LW1969**, p. 32]. ■

Proof of the Collapse-Cylinder Proposition 140.7. First note that by the Cellular Embedding Lemma 139.7 we can and will view $A \sqcup B$ as a subcomplex of Y. In the following let $R\colon \operatorname{Cyl}(f\colon A \to B) \times [0,1] \to B$ be the natural deformation retraction given by the Mapping Cylinder–Homotopy Lemma 21.5. Furthermore note that it follows quite easily from the CW-Complex Mapping Cylinder+Cone Proposition 139.25 and the CW-Gluing Proposition 139.9 that we can and will equip $\operatorname{Cyl}(f)$ and $(Y \sqcup \operatorname{Cyl}(f))/\sim$ with natural CW-structures. Finally, it follows from our slightly technical hypothesis on f together with the CW-Quotient Lemma 140.8 that $Y/\sim$ also admits a natural CW-structure. Next

we consider the homotopy

$$\begin{array}{rcl} G\colon (A\sqcup B)\times[0,1] & \to & \mathrm{Cyl}(f)\subseteq (Y\sqcup \mathrm{Cyl}(f))/\sim \\ [(Q,t)] & \mapsto & \begin{cases} R([(Q,0)],t), & \text{if } Q\in A,\\ Q, & \text{if } Q\in B. \end{cases} \end{array}$$

We obtain the following diagram:



We make the following observations and clarifications:

(1) The map $i\sqcup j\colon A\sqcup B\to Y$ is the inclusion map.
(2) It follows from the Homotopy Extension Theorem 140.2 and the fact that $A\sqcup B$ is a subcomplex of Y that there exists a continuous map $F\colon Y\times[0,1]\to (Y\sqcup \mathrm{Cyl}(f))/\sim$ that makes the two triangles involving F commute.
(3) For any $a\in A$ we have

$$F_1(a)=F(a,1)=G(a,1)=\underbrace{f(a)}_{\in B}=G(f(a),1)=F(f(a),1)=F_1(f(a)).$$

It follows from this calculation that the map $F_1\colon Y\to (Y\sqcup \mathrm{Cyl}(f))/\sim$ descends to a map $\Psi\colon Y/a\sim f(a)\to (Y\sqcup \mathrm{Cyl}(f))/\sim$.

It remains to show the following claim:

Claim. The maps $\Phi\colon (Y\sqcup \mathrm{Cyl}(f))/\sim\to Y/a\sim f(a)$ and $\Psi\colon Y/a\sim f(a)\to (Y\sqcup \mathrm{Cyl}(f))/\sim$ are homotopy inverses of one another.

Proof. First we consider the map

$$\begin{array}{rcl} H\colon (Y/a\sim f(a))\times[0,1] & \to & Y/a\sim f(a)\\ (\underbrace{[P]}_{\in Y},t) & \mapsto & \Phi(F(P,t)). \end{array}$$

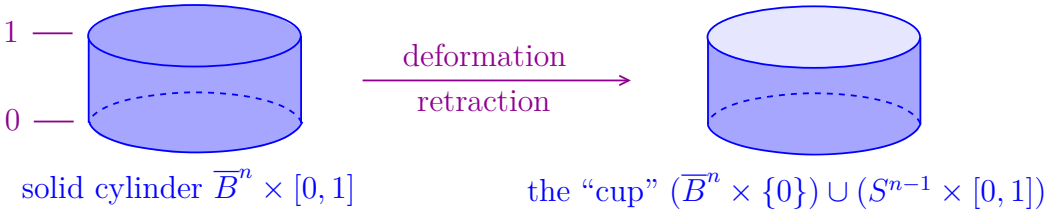
One can easily verify that the map is, despite first appearances, actually well-defined. Using the fact that $Y/a\sim f(a)$ is a CW-complex and using the CW×Interval Proposition 137.1 (2) one can show quite easily that H is continuous. Basically by construction we have $H_0=\mathrm{id}$ and $H_1=\Phi\circ\Psi$. Thus we have found our first homotopy. Finally we consider the map

$$\begin{array}{rcl} H\colon (Y\sqcup \mathrm{Cyl}(f))/\sim & \to & (Y\sqcup \mathrm{Cyl}(f))/\sim\\ ([P],t) & \mapsto & \begin{cases} R(P,t), & \text{if } P\in \mathrm{Cyl}(f),\\ F(P,t), & \text{if } P\in Y. \end{cases} \end{array}$$

Similar to the above one sees that H is continuous, that $H_0=\mathrm{id}$ and that $H_1=\Psi\circ\Phi$. ∎

140.2. **Proof of the Homotopy Extension Theorem 140.2 .** Before we can provide the proof of Theorem 140.2 we will need to formulate and prove several lemmas.

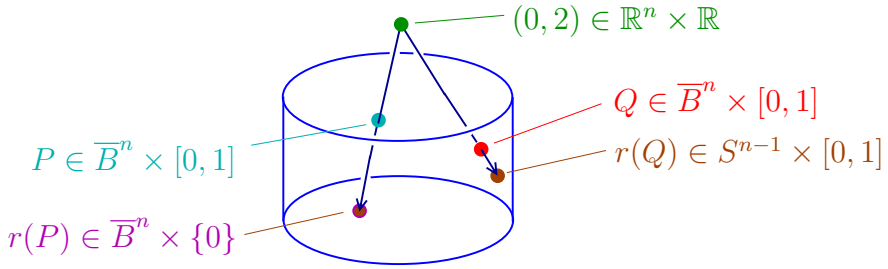
Lemma 140.9. (Cup–Retraction Lemma) For any $n \in \mathbb{N}_0$ there exists a deformation retraction from the solid cylinder $\overline{B}^n \times [0,1]$ to the "cup" $(\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1])$.



Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} r\colon \overline{B}^n \times [0,1] & \to & (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \\ P & \mapsto & \text{the unique point on } \overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1] \text{ that} \\ & & \text{gets hit by the ray from } (0,2) \in \mathbb{R}^n \times \mathbb{R} \text{ through } P. \end{array}$$

is a retraction. We refer to the figure below for an illustration.

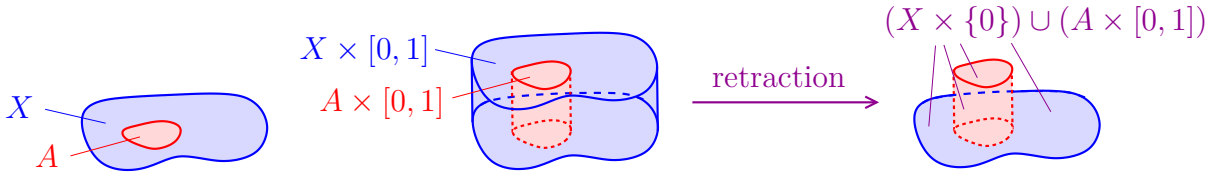


Since $\overline{B}^n \times [0,1]$ is convex we can define the corresponding deformation retraction as follows:

$$\begin{array}{rcl} (\overline{B}^n \times [0,1]) \times [0,1] & \to & \overline{B}^n \times [0,1] \\ (P,t) & \mapsto & P \cdot (1-t) + r(P) \cdot t. \end{array}$$

■

Lemma 140.10. (Subcomplex–Cylinder Retraction Lemma) If X is a CW-complex and if A is a subcomplex of X, then there exists a deformation retraction from $X \times [0,1]$ to the subcomplex $(X \times \{0\}) \cup (A \times [0,1])$.



Proof. Let X be a CW-complex and let A be a subcomplex. For each $n \in \mathbb{N}_0$ we set

$$Z_n \;\coloneqq\; X \times \{0\} \;\cup\; (X^{n-1} \cup A) \times [0,1].$$

Note that each Z_n is a subcomplex of the product CW-complex $X \times [0,1]$. Evidently we have a sequence

$$(X \times \{0\}) \cup (A \times [0,1]) = Z_0 \;\subseteq\; Z_1 \;\subseteq\; Z_2 \;\subseteq\; \ldots \qquad \text{with } X \times [0,1] \;=\; \bigcup_{n \in \mathbb{N}_0} Z_n.$$

Thus it follows from the Iterated Deformation Retract Proposition 137.2 that it suffices to prove the following claim:

Claim. For every $n \in \mathbb{N}_0$ there exists a deformation retraction from Z_{n+1} to Z_n.

Proof. Let $n \in \mathbb{N}_0$. We denote by $\Phi_i \colon \overline{B}^n \to X$, $i \in I$, the characteristic maps of all the n-cells of X that do not lie in A. By the Cup–Retraction Lemma 140.9 there exists a deformation retraction $r \colon \overline{B}^n \times [0,1] \times [0,1] \to \overline{B}^n \times [0,1]$ from $\overline{B}^n \times [0,1]$ to $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$. The idea is to apply this retraction to all k-cells of X to obtain the desired deformation retraction. More precisely we consider the map

$$r_n \colon \overbrace{((X \times \{0\}) \cup ((X^n \cup A) \times [0,1]))}^{=Z_{n+1}} \times [0,1] \to \overbrace{(X \times \{0\}) \cup ((X^n \cup A) \times [0,1])}^{=Z_{n+1}}$$
$$((P,t),s) \mapsto \begin{cases} \Phi_i(r((Q,t),s)), & \text{if } P = \Phi_i(Q) \text{ with } Q \in \overline{B}^n \text{ and } i \in I, \\ (P,t), & \text{otherwise.} \end{cases}$$

This map is easily seen to be a deformation retraction from Z_{n+1} to Z_n.[1452] ∎

Proof of the Homotopy Extension Theorem 140.2. Let X be a CW-complex and let $A \subseteq X$ be a subcomplex. We need to show that the inclusion map $i \colon A \to X$ is a closed cofibration. Note that by the Subcomplex–Closed Lemma 136.11 we know that A is a closed subset of X. It remains to show that $i \colon A \to X$ is a cofibration.

Thus let Y be a topological space and let $f_0 \colon X \to Y$ be a continuous map. Furthermore let $G \colon A \times [0,1] \to Y$ be a homotopy from $g_0 \coloneqq f_0|_A$ to some continuous map $g_1 \colon A \to Y$. We need to show that we can extend G to a homotopy on all of $X \times [0,1]$.

Note that by the Subcomplex–Cylinder Retraction Lemma 140.10 there exists a retraction

$$H \colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1]).$$

Therefore the theorem follows from the Cofibration Characterization Lemma 22.5 or, equivalently, from the following claim:

Claim. The map

$$\begin{aligned} F \colon X \times [0,1] \xrightarrow{H} (X \times \{0\}) \cup (A \times [0,1]) &\to Y \\ (x,0) &\mapsto f_0(x), \quad \text{for } x \in X, \\ (a,t) &\mapsto G(a,t), \quad \text{for } (a,t) \in A \times [0,1]. \end{aligned}$$

has all the desired properties.

Proof. First we have to verify that the map is continuous. In fact, for continuity we do have to worry a little bit about the right-hand map. But note that the Subcomplex–Closed Lemma 136.11 says that the subcomplex A is a closed subset of X. Thus we see that the map on the right-hand side is continuous when restricted to the *closed* subsets $X \times \{0\}$ and $A \times [0,1]$. Thus it is continuous by the Pasting Proposition 4.7 (2). Note that it is clear that F has the desired properties, i.e. that it is a homotopy $F \colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A \times [0,1]}$ equals G. ∎

[1452] It follows from the CW-Complex Product Proposition 139.18 and the Characteristic Maps Proposition 136.9 (2) that the map is in fact continuous.

140.3. **Dropping base points.** Even though base points are necessary and valuable, in practice they are frequently a major nuisance. Fortunately, using the Homotopy Extension Theorem 140.2, we can prove a proposition that often allows us to stop worrying about base points.

We start out with a few definitions. Some of them we have encountered before.

Definition 140.11.

(1) A pointed topological space is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a point. We refer to x_0 as the base point of the pointed topological space.
(2) A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces is a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$.
(3) We say that two maps $f, g\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces are homotopic if there exists a homotopy $F\colon X \times [0,1] \to Y$ rel $\{x_0\}$ from f to g.
(4) Given two pointed topological spaces (X, x_0) and (Y, y_0) we write[1453]

$$\langle (X, x_0), (Y, y_0) \rangle := \{\text{maps } (X, x_0) \to (Y, y_0)\}/\sim$$

where $f \sim g$ if there exists a homotopy in the above sense between f and g.

Example. Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$ and let $*$ be the usual base point of S^n. By definition we have $\langle (S^n, *), (X, x_0) \rangle = \pi_n(X, x_0)$. □

We have a natural map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$. It is worth spending a minute on trying to find examples that show that in general this map is neither an injection nor a surjection. In this section we want to give some criteria that give us some control over this map. We start out with the following easy, albeit fairly restrictive, example.

Proposition 140.12. (Circle Pointed-Homotopy Proposition) Let (Y, y_0) be a path-connected topological space. We consider the natural map

$$\begin{array}{ccc} \langle (S^1, *), (Y, y_0) \rangle & \to & [S^1, Y] \\ [f] & \mapsto & [f] \end{array}$$

The following two statements hold:[1454]

(1) This map is a surjection.
(2) If $\pi_1(Y, y_0)$ is abelian, then the map is a bijection.

Proof.

(1) It follows from the fact that Y is path-connected together with the Homotopies-of-Loops Proposition 88.29 (1) that the map is a surjection.
(2) It follows from the extra hypothesis that $\pi_1(Y, y_0)$ is abelian together with the Homotopies-of-Loops Proposition 88.29 (2) that the map is also an injection. ■

Now we want to move away from the fairly simple topological space S^1. In the proof of the Homotopies-of-Loops Proposition 88.29 we had to construct fairly explicitly maps on S^1 and $S^1 \times [0,1]$. In the following we will now replace S^1 by CW-complexes, we have

[1453] The notation $\langle X, Y \rangle$ goes back to [**Hat2002**, p. 357], where it says "the notation $\langle \dots \rangle$ is intended to suggest pointed homotopy classes".
[1454] In the Sphere Pointed-Homotopy Corollary 140.15 below we will deal with S^n instead of S^1.

now developed several tools for building maps *out* of a CW-complex. This leads us to the following definition:

Definition 140.13. A pointed CW-complex is a pair (X, x_0) where X is a CW-complex and x_0 is a point in the 0-skeleton X^0 of X.

Proposition 140.14. (CW–Complex Pointed Homotopy Proposition) Let (X, x_0) be pointed a CW-complex and let (Y, y_0) be a pointed topological space. We consider the map

$$\begin{array}{rcl} \langle (X, x_0), (Y, y_0) \rangle & \to & [X, Y] \\ [f] & \mapsto & [f]. \end{array}$$

The following two statements hold:

(1) If Y is path-connected, then the map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$ is a surjection.
(2) If Y is simply connected, then the map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$ is a bijection.

Remark. In the Abelian-π_1–Drop Base Point Proposition 308.42 we will formulate a variation on the CW–Complex Pointed Homotopy Proposition 140.14 for CW-complexes Y with abelian fundamental groups. □

For the record we state two corollaries to our results.

Corollary 140.15. (Sphere Pointed-Homotopy Corollary) Let (Y, y_0) be a path-connected pointed topological space and let $n \in \mathbb{N}$. We consider the following map:

$$\underbrace{\langle (S^n, x_0), (Y, y_0) \rangle}_{= \pi_n(Y, y_0)} \xrightarrow{[f] \mapsto [f]} [S^n, Y].$$

The following two statements hold:

(1) The map is a surjection.
(2) If $n = 1$, then we now assume that $\pi_1(Y, y_0)$ is abelian and if $n \geq 2$ we now assume that $\pi_1(Y, y_0)$ is trivial. Under these stronger hypothesis the above map is a bijection.

Proof of the Sphere Pointed-Homotopy Corollary 140.15.

- The case $n = 1$ is precisely the content of the Circle Pointed-Homotopy Proposition 140.12.
- Now let $n \in \mathbb{N}_{\geq 2}$. We can and will equip S^n with a CW-structure such that x_0 is a 0-cell. The desired statement now follows from the CW–Complex Pointed Homotopy Proposition 140.14 (2). ■

Corollary 140.16. (Sphere-to-Sphere Pointed-Homotopy Corollary) For any $n \in \mathbb{N}$ and given any base point $x_0 \in S^n$ the map

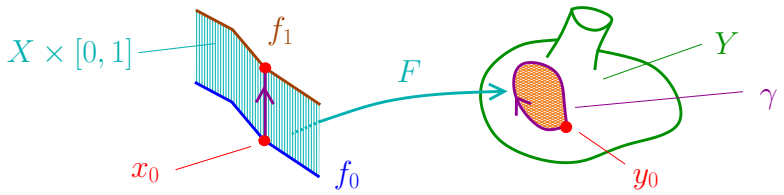
$$\langle (S^n, x_0), (S^n, x_0) \rangle \xrightarrow{[f] \mapsto [f]} [S^n, S^n]$$

is a bijection.

Proof of the Sphere-to-Sphere Pointed-Homotopy Corollary 140.16. This statement follows immediately from the Sphere Pointed-Homotopy Corollary 140.15 (2) and the fact, established in the Sphere-π_1-Proposition 88.24, that for $n \geq 2$ the fundamental group $\pi_1(S^n)$ is trivial. ■

Now we turn to the proof of the CW–Complex Pointed Homotopy Proposition 140.14. This proof relies very much on the following lemma.

Lemma 140.17. (CW–Ignore Base Point Lemma) Let (X, x_0) and (Y, y_0) be pointed CW-complexes and let $F\colon X \times [0,1] \to Y$ be a homotopy with $F_0(x_0) = F_1(x_0) = y_0$. We consider the loop $\gamma\colon [0,1] \to Y$ that is given by $\gamma(t) := F(x_0, t)$. If the loop γ is null-homotopic, then there exists a homotopy $J\colon X \times [0,1] \to Y$ with $J_0 = F_0$ and $J_1 = F_1$ and such that $J(x_0, t) = y_0$ for all $t \in [0,1]$.



Proof of Lemma 140.17. We prove the lemma in three easily digestible steps:

(1) We consider the loop $\gamma\colon [0,1] \to Y$ that is given by $\gamma(t) := F(x_0, t)$. By our hypothesis the loop γ is null-homotopic. By definition this means that there exists a continuous map $\varphi\colon [0,1]\times[0,1] \to Y$ with $\varphi(t,0) = \gamma(t)$ for all $t \in [0,1]$ and such that $\varphi(x,y) = y_0$ for all other points in $\partial([0,1] \times [0,1])$.

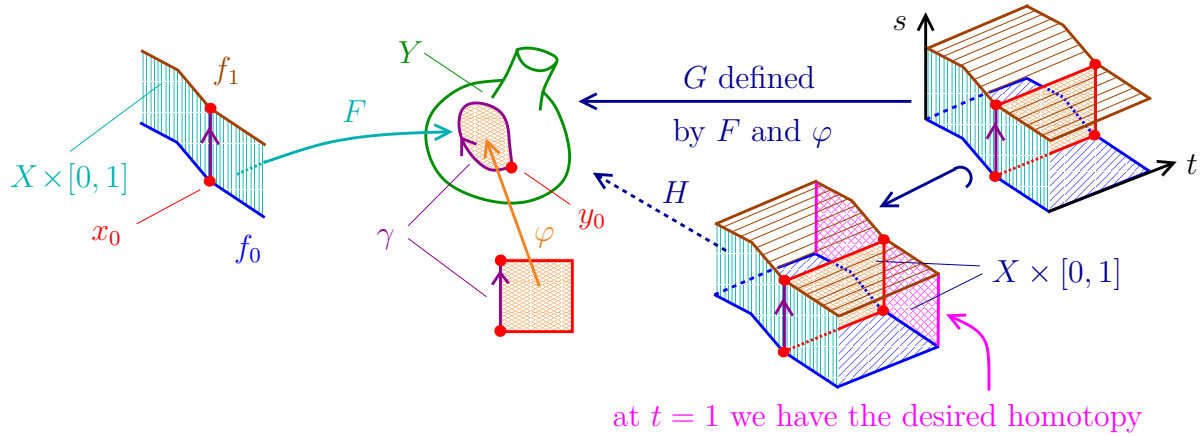
(2) We consider the map[1455]

$$G\colon \overbrace{((X\times[0,1])\times\{0\}) \cup (\underbrace{(X\times\{0,1\} \cup \{x_0\}\times[0,1])}_{\text{subcomplex of } X\times[0,1]}\times[0,1])}^{\text{subcomplex of } (X\times[0,1])\times[0,1]} \to Y$$

$$(x,s,t) \mapsto \begin{cases} F(x,s), & \text{if } t = 0, \\ F(x,s), & \text{if } s = 0,1, \\ \varphi(s,t), & \text{if } x = x_0. \end{cases}$$

It follows from the CW-Complex Pasting Lemma 136.15 that this map G is continuous.

(3) By the Homotopy Extension Theorem 140.2 the map G from (2) extends to a map $H\colon (X \times [0,1])\times[0,1] \to Y$.



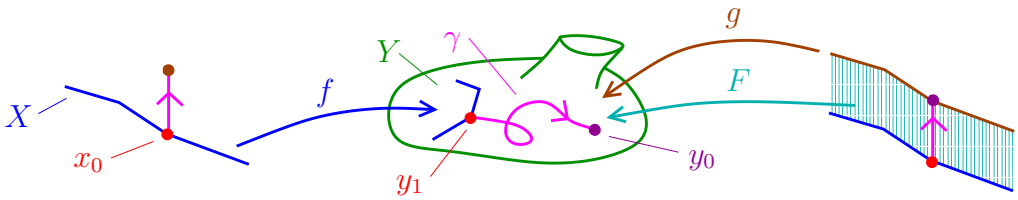
It follows basically immediately from the definitions that $J := H_1\colon X \times [0,1] \to Y$ has the desired properties. ∎

[1455] Secretly we use the CW-Complex Product Proposition 139.18 which tells us in this instance that the product CW-structure $X \times [0,1] \times [0,1]$ is indeed a CW-structure for the topological space $X \times [0,1] \times [0,1]$.

Now we turn to the actual proof of the CW–Complex Pointed Homotopy Proposition 140.14.

Proof of the CW–Complex Pointed Homotopy Proposition 140.14. Let (X, x_0) and (Y, y_0) be pointed CW-complexes.

(1) We suppose that Y is path-connected. Let $f\colon X \to Y$ be a continuous map. We set $y_1 = f(x_0)$. Since Y is path-connected there exists a path $\gamma\colon [0,1] \to Y$ with $\gamma(0) = y_1$ and $\gamma(1) = y_0$. Since $\{x_0\}$ is a subcomplex of X we obtain from the CW-Complex–Vertices–Homotopy Corollary 140.3 that we can extend γ, viewed as a homotopy $\{x_0\} \times [0,1] \to Y$, to a homotopy $F\colon X \times [0,1] \to Y$. This shows that $f = F_0$ is homotopic to a map $g \coloneqq F_1$ with $g(x_0) = y_0$. This implies that the map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$ is a surjection.



(2) By (1) it remains to show that the map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$ is injective. Thus let $f_0, f_1\colon (X, x_0) \to (Y, y_0)$ be two continuous maps which represent the same element in $[X, Y]$. This means that there exists a homotopy $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. We need to show that there exists a homotopy $J\colon X \times [0,1] \to Y$ with $J_0 = f_0$ and $J_1 = f_1$ and such that $J(x_0, t) = y_0$ for all $t \in [0,1]$. We consider the loop $\gamma\colon [0,1] \to Y$ that is given by $\gamma(t) \coloneqq F(x_0, t)$. Since Y is simply connected we know that γ is null-homotopic. Thus we can apply the CW–Ignore Base Point Lemma 140.17 and we obtain the desired homotopy J. ■

We conclude this section with the following proposition:

Proposition 140.18. (CW–Push Points Proposition) Let X be a CW-complex and let x_0 and x_1 be two points in the 0-skeleton of X. If X is connected, then there exists a continuous map $f\colon X \to X$ with the following properties:

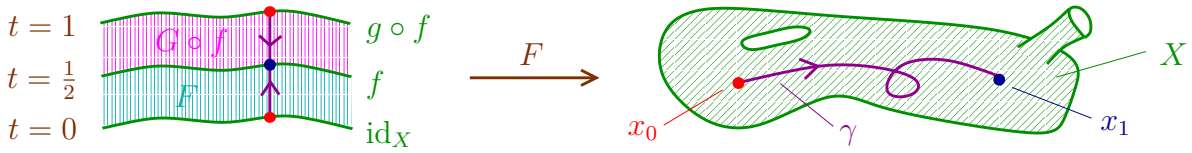
(1) The map f is homotopic to the identity.
(2) We have $f(x_0) = x_1$.
(3) The map f defines a homotopy equivalence $(X, x_0) \to (X, x_1)$ of pointed topological spaces.

Proof. Since X is connected we obtain from the CW-Complex–Local Properties Proposition 137.8 (2) together with the Locally Path-Connected Lemma 4.57 that there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Since x_0, x_1 lie in the 0-skeleton of X we can apply the CW-Complex–Vertices–Homotopy Corollary 140.3 and we obtain a homotopy $F\colon X \times [0,1] \to X$ with $F(x_0, t) = \gamma(t)$ for all $t \in [0,1]$ and such that $F_0 = \mathrm{id}$. We set $f \coloneqq F_1$. Evidently f is homotopic to the identity and $f(x_0) = x_1$. It remains to show that $f\colon (X, x_0) \to (X, x_1)$ is a homotopy equivalence of pointed topological spaces.

We need to find a homotopy inverse to f. Once again we appeal to the CW-Complex–Vertices–Homotopy Corollary 140.3 to obtain a homotopy $G\colon X \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $G(x_1, t) = \overline{\gamma}(t) \coloneqq \gamma(1-t)$ and such that $G_0 = \mathrm{id}$. We set $g \coloneqq G_1$. We

need to show that $g \circ f \colon (X, x_0) \to (X, x_0)$ and $f \circ g \colon (X, x_1) \to (X, x_1)$ are homotopic[1456] to the identity. We consider the homotopy[1457]

$$\begin{array}{rcl} F \colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t \in [0, \frac{1}{2}], \\ G(f(x), 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$



Note that this is a homotopy from id_X to $g \circ f \colon X \to X$. Furthermore note that the loop given by $\gamma(t) = F(x_0, t)$ is precisely the loop $\gamma * \overline{\gamma}$. By the Path-Homotopy Product Proposition 88.10 (3) we know that this loop is null-homotopic. Thus we can apply the CW–Ignore Base Point Lemma 140.17. We obtain a homotopy $J \colon X \times [0,1] \to X$ with $J_0 = F_0 = \mathrm{id}_X$, $J_1 = F_1 = g \circ f$ and $J(x_0, t) = x_0$ for all $t \in [0,1]$. But that is precisely what we needed. ∎

140.4. **Combining infinitely many homotopies and the telescope construction.** In the Homotopy Combination Lemma 18.8 (2) we saw that one can combine finitely many homotopies. Later on we will want to combine infinitely many homotopies of CW-complexes. This slightly delicate procedure can be performed using the following lemma:

Lemma 140.19. (Infinite Homotopies–CW–Lemma) Let X be a CW-complex, let A be a subset of X and let Y be a topological space. Finally let $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots$ be a sequence of subcomplexes of X such that for each $k \in \mathbb{N}_0$ there exists an $l \in \mathbb{N}_0$ such that the k-skeleton X^k is contained in W_l. (In many applications we have $W_k = X^k$.) Let $f_0, f_1, \ldots$ be continuous maps from X to Y. Suppose that for each $k \in \mathbb{N}_0$ we are given a homotopy $H(k) \colon X \times [0,1] \to Y$ rel $A \cup W_k$ from f_k to f_{k+1}. (Since it is easy to overlook this bit of the hypothesis we stress that $H(k)$ is in particular a homotopy rel W_k). We define[1458]

$$\begin{array}{rcl} g \colon X & \to & Y \\ x & \mapsto & f_k(x) \quad \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{array}$$

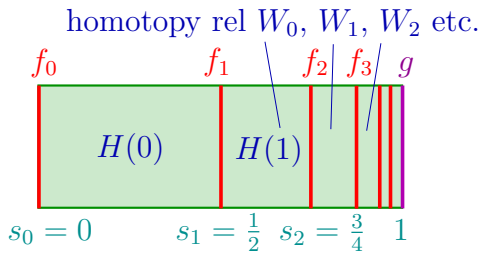
For $k = 0, 1, 2, \ldots$ we write $s_k = \sum_{j=1}^{k} \frac{1}{2^j}$. Then the map

$$\begin{array}{rcl} H \colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} H(k)(x, 2^{k+1} \cdot (t - s_k)), & \text{if } t \in [s_k, s_{k+1}), \\ g(x), & \text{if } t = 1 \end{cases} \end{array}$$

that is illustrated in the figure below is a homotopy rel A from f_0 to g. We refer to the homotopy H as the combination of $\{H(k)\}_{k \in \mathbb{N}_0}$.

Proof. We only need to show that H is continuous. Note that for each $l \in \mathbb{N}_0$ it follows from our choice of maps f_l and the Homotopy Combination Lemma 18.8 (1) that the

[1456] Here, as a reminder, "homotopic" means "homotopic as maps of pointed topological spaces".
[1457] It follows from the Homotopy Combination Lemma 18.8 (1) that F is indeed continuous.
[1458] It follows easily from the setup that the map g is well-defined, i.e. given $x \in X$ the point $g(x)$ does not depend on the choice of k.



restriction of H to $W_l \times [0,1]$ is continuous. Since each X^k is contained in some W_l we see that the restriction of H to each $X^k \times [0,1]$ is continuous. But now it follows from the CW×Interval Proposition 137.1 (2) that $H\colon X \times [0,1] \to Y$ is also continuous. ■

We conclude this technical section with an application of the Infinite Homotopies–CW–Lemma 140.19 that we will need once, namely in the proof of the Skeleton-H*-Theory Proposition **??**. But the application doubles also as a nice illustration for how to apply the previous lemma.

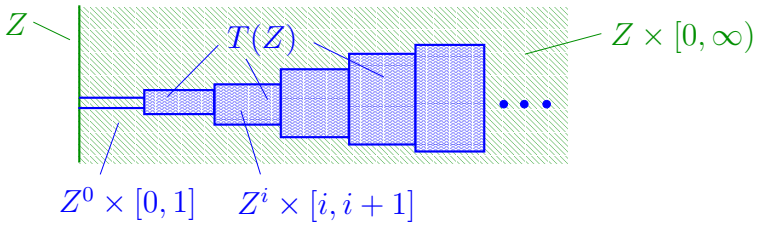
Definition 140.20. Let Z be a CW-complex. We equip $[0,\infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$. We equip

$$Z \times [0,\infty) \overset{\text{holds by the Union-equals-Direct Colimit Criterion 104.12}}{=} \varinjlim (Z \times [0,i]) = \underbrace{\varinjlim ((Z \times [0,1]) \cup_{Z\times\{1\}} \cdots \cup (Z \times [i-1,i]))}_{\text{CW-complex by the CW-Complex Product Proposition 139.18 together with the CW-Gluing Proposition 139.9}}$$

with the CW-structure coming from Direct Colimit–CW Lemma 139.13.[1459] We refer to the subcomplex[1460]

$$T(Z) \coloneqq \bigcup_{i\in\mathbb{N}_0} (Z^i \times [i,i+1]) \quad \subseteq \quad Z \times [0,\infty)$$

as the telescope of Z.



The following lemma shows that the telescope $T(Z)$ is homotopy equivalent to Z.

Lemma 140.21. (Telescope Lemma) Let Z be a CW-complex. The following statements hold:

(1) $T(Z)$ is a deformation retract of $Z \times [0,\infty)$.

[1459] We could also use the Countable CW-Complex Product Proposition 139.23 to show that $Z \times [0,1)$ has a natural CW-structure.

[1460] It follows pretty easily from the explicit construction of the CW-structure on $Z \times [0,\infty)$ and the Subcomplex–Characterization Proposition 136.12 that $T(Z)$ is indeed a subcomplex.

(2) The telescope $T(Z)$ is homotopy equivalent to Z.

Remark. One can also show that the Telescope Lemma 140.21 is a consequence of the much more general Whitehead Deformation Retract Proposition 305.2. □

Proof. First note that $Z \times \{0\}$ is clearly a deformation retract of $Z \times [0,\infty)$. It follows from the fact that Z is homeomorphic to $Z \times \{0\}$, the Deformation Retract–Homotopy Equivalence Lemma 19.10 and the Homotopy-Equivalence Basics Lemma 19.14 that (2) is a consequence of (1). Therefore it remains to prove (1).

As in the definition of the telescope $T(Z)$ we view $[0,\infty)$, $X := Y := Z \times [0,\infty)$ and $T(Z)$ as CW-complexes. It follows from the Subcomplex–Cylinder Retraction Lemma 140.10 that for each $k \in \mathbb{N}_0$ there exists a deformation retraction

$$r_k \colon (Z \times [0,1]) \times [0,1] \;\to\; Z \times [0,1]$$

from $Z \times [0,1]$ to $(Z^k \times [0,1]) \cup (Z \times \{1\})$. We denote by

$$\begin{array}{rcl} \rho_k \colon Z \times [k,k+1] & \to & (Z^{k-1} \times [k,k+1]) \cup (Z^k \times \{k+1\}) \\ (z,t) & \mapsto & (r_k(z,t-k) + (0,k),1) \end{array}$$

the corresponding retraction, with the intervals shifted by k. We make the following preparations:

(1) Given $k \in \mathbb{N}_0$ we define $W_k := Z^k \times [0,\infty)$. Note that it follows from the definition of the product CW-structure on $Z \times [0,\infty)$ that the k-skeleton of $X = Z \times [0,\infty)$ is contained in W_{k+1}.

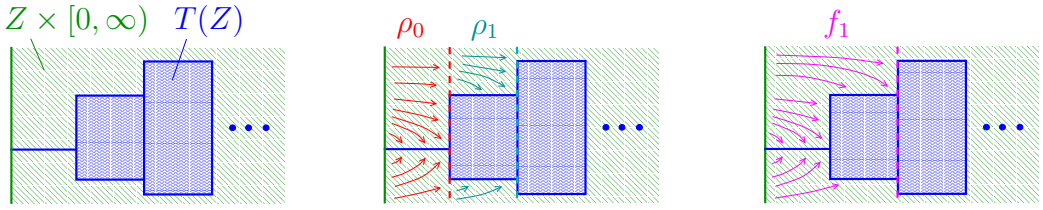
(2) For each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} f_k \colon X = Z \times [0,\infty) & \to & Y = Z \times [0,\infty) \\ (z,t) & \mapsto & \begin{cases} (z,t), & \text{if } x \in Z^k \text{ and } t \geq k+1, \\ (\rho_k \circ \cdots \circ \rho_1 \circ \rho_0)(x), & \text{if } z \in Z^k \text{ and } t < k+1. \end{cases} \end{array}$$

(3) We consider the map

$$\begin{array}{rcl} H(k) \colon X \times [0,1] & \to & Y = Z \times [0,\infty) \\ ((z,t),s) & \to & \begin{cases} (z,t), & \text{if } t \geq k+1, \\ r_{k+1}(\rho_k(z,t),s), & \text{if } t < k+1. \end{cases} \end{array}$$

This defines a homotopy from f_k to f_{k+1} rel W_k. Note that $H(0)_1 = \mathrm{id}_X$.



As in the statement of the Infinite Homotopies–CW–Lemma 140.19 we consider the map

$$\begin{array}{rcll} g \colon X & \to & Y & \\ x & \mapsto & f_k(x) & \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{array}$$

One can easily verify that this is a retraction of $X = Z \times [0,\infty)$ to the telescope $T(Z)$. It follows from the Infinite Homotopies–CW–Lemma 140.19 that there exists a homotopy from $H(0)_1 = \mathrm{id}$ to g, i.e. $T(Z)$ is in fact a deformation retract of $X = Z \times [0,\infty)$. ■

140.5. Exercises for Chapter 140.

Exercise 140.1. Let X be a contractible CW-complex and let $x_0 \in X$. Show that X admits a deformation retraction to the point x_0.
Hint. Use the Homotopy Extension Theorem 140.2.
Remark. In the S^∞–Contractibility Lemma 104.25 we showed that S^∞ is contractible. The exercise now shows that S^∞ admits a deformation retraction to a point.

Exercise 140.2. We consider the map

$$\begin{array}{rcl} \varphi\colon S^1 & \to & S^1 \\ \exp(\mathrm{i}t) & \mapsto & \begin{cases} \exp(3\mathrm{i}t), & \text{if } t \in [0, \frac{2\pi}{3}], \\ \exp(-3\mathrm{i}t), & \text{if } t \in [\frac{2\pi}{3}, \frac{4\pi}{3}], \\ \exp(3\mathrm{i}t), & \text{if } t \in [\frac{4\pi}{3}, 2\pi]. \end{cases} \end{array}$$

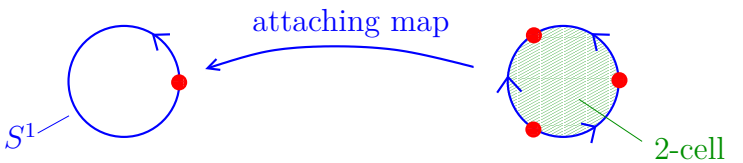
Put differently, φ wraps S^1 first around S^1 clockwise, then counterclockwise and then once again clockwise. We write $Y = S^1$ and we consider

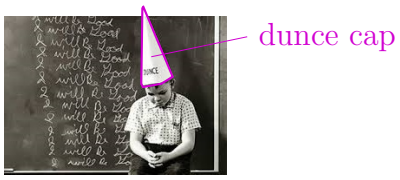
$$D \coloneqq Y \cup_{S^1 \xleftarrow{\varphi} S^1} \overline{B^2},$$

where we identify $z \in \partial\overline{B}^2 = S^1$ with $\varphi(z) \in S^1 = Y$. (This topological space is sometimes called the *dunce cap*).

(a) Show that D is homotopy equivalent to $\overline{B}^2$.
Hint. Make use of the Homotopy Equivalent Cell Attachment Proposition 135.13 or the Homotopy Equivalence-Mapping Cone Proposition 22.25.

(b) Is D homeomorphic to $\overline{B}^2$?





Exercise 140.3. Show that for any path-connected topological group G and any $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \pi_n(G, e) & \to & [S^n, G] \\ [f\colon (S^n, *) \to (G, e)] & \mapsto & [f] \end{array}$$

is a bijection.

CHAPTER 141

The Cellular Approximation Theorem

In this chapter we prove and discuss the Cellular Approximation Theorem 141.2 which, in a nutshell, says that every continuous map between CW-complexes is homotopic to a cellular map.

This chapter is organized as follows:

- In Section 141.1 we will state and discuss the Cellular Approximation Theorem 141.2.
- In Section 141.2 we will prove the Cellular Approximation Theorem 141.2.
- In Appendix 141.3 we will give an alternative (more self-contained) proof for an important ingredient in the proof of the Cellular Approximation Theorem 141.2.
- In Section 141.4 we will provide exercises for Chapter 141.

141.1. The Cellular Approximation Theorem.

We start out by recalling Definition 139.2:

Definition 141.1. A continuous map $g\colon X \to Y$ between CW-complexes is called cellular if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subseteq Y^n$

The following theorem is the main result of this chapter.

Theorem 141.2. (Cellular Approximation Theorem) Let $g\colon X \to Y$ be a continuous map between CW-complexes and let A be a (possibly empty) subcomplex of X such that the restriction of g to A is a cellular map. The map g is homotopic rel A to a cellular map $\widetilde{g}\colon X \to Y$.

We will provide the proof of the Cellular Approximation Theorem 141.2 in Section 141.2. First we want to discuss some examples and applications.

Example.

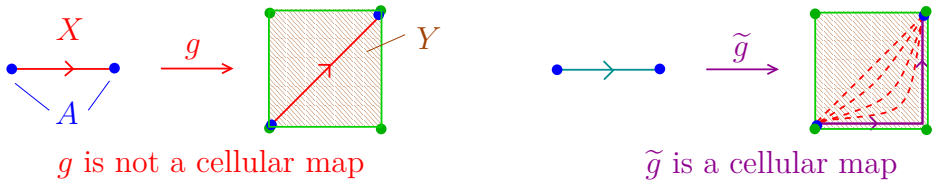
(1) We consider the CW-complexes $X = [0,1]$ and $Y = [0,1] \times [0,1]$ with the obvious CW-structures. Furthermore we consider the continuous map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t). \end{array}$$

Note that the restriction of g to the 0-skeleton $X^0 = \{0,1\}$ is cellular, but g is not cellular, since $g(X^1) \not\subseteq Y^1$. As shown in the figure below, there exists a homotopy rel $X^0 = \{0,1\}$ from g to the cellular map $\widetilde{g}$ that is given by

$$\begin{array}{rcl} \widetilde{g}\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & \begin{cases} (2t,0), & \text{if } t \in [0,\frac{1}{2}] \\ (1,2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

One way of stating this example is that we turn g into a cellular map $\widetilde{g}$ by "pushing g off the high-dimensional cell". The proof of Cellular Approximation Theorem 141.2 is

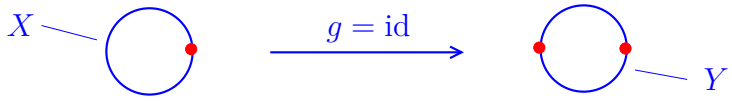


g is not a cellular map $\widetilde{g}$ is a cellular map

built on the same concept. But somewhat similar to the proof of the Sphere-π_1-Proposition 88.24, we have to worry about space-filling paths that exist by the Space-Filling Curve Theorem 10.21.

(2) Let $g\colon X \to Y$ be a *homeomorphism* between CW-complexes. By the Cellular Approximation Theorem 141.2 we know that g is homotopic to a cellular map h. But in general we cannot arrange that h is again a *homeomorphism*.

(a) The simplest example is arguably the case where $X = S^1$ with one 0-cell and $Y = S^1$ with two 0-cells. Then the identity map is a homeomorphism, but clearly there is no cellular homeomorphism.



(b) As a more fancy example consider the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ with the standard CW-structure from page 2625 which has precisely one 0-cell $*$ that corresponds to the origin. As in the Action–π_1-Theorem 93.1 we make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^2, *) = \mathbb{Z}^2$. Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [P \cdot v]. \end{array}$$

In the Maps-on-Torus Lemma 25.46 we saw that this map is a homeomorphism. Furthermore, in Exercise 93.8 we saw that the induced map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *) = \mathbb{Z}^2$ is given by multiplication by P. On the other hand in Exercise 141.2 we will see that the map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *) = \mathbb{Z}^2$ induced by any *cellular homeomorphism* $\widetilde{g}$ on $\mathbb{R}^2/\mathbb{Z}^2$ is represented by a matrix of the form $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. Since the type of maps are different and since $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, *)$ is abelian it follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1a) that g cannot be homotopic, rel $*$, to any cellular homeomorphism from X to Y.[1461]

In fact one can show, see Exercise 141.3, that there is no CW-structure on $T = \mathbb{R}^2/\mathbb{Z}^2$ such that every homeomorphism is homotopic to a cellular homeomorphism. □

Next we will use the Cellular Approximation Theorem 141.2 to prove the following generalization of the CW-Skeleton-π_1-Proposition 136.22 to higher homotopy groups.

Proposition 141.3. (CW-Skeleton-π_n-Proposition) Let X be a connected CW-complex, let x_0 be a point in the 0-skeleton of X and let $j \in \mathbb{N}_0$.

(1) The inclusion-induced map $\pi_j(X^j, x_0) \to \pi_j(X, x_0)$ is an epimorphism.

[1461] In Exercise 179.3 we will see one approach how at times one can turn a homeomorphism into a cellular homeomorphism.

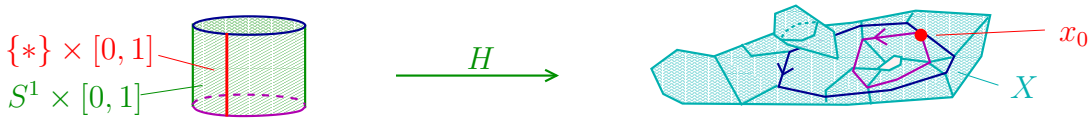
(2) For any $k \geq j+1$ the inclusion-induced map $\pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is an isomorphism.

Proof. Let X be a CW-complex and let $x_0 \in X^0$. We take the point of view from Identification 134.9, i.e. we view elements of $\pi_j(X, x_0)$ as homotopy classes of continuous maps $(S^j, *) \to (X, x_0)$. As on page 2625 we consider the j-sphere S^j as a CW-complex with one 0-cell $*$ and one j-cell.

(1) We denote by $i\colon X^j \to X$ the inclusion. We need to show that the induced map $i_*\colon \pi_j(X^j, x_0) \to \pi_j(X, x_0)$ is an epimorphism. Thus let $[g] \in \pi_j(X, x_0)$ be some element. In other words, we are given a continuous map $g\colon (S^j, *) \to (X, x_0)$. Since $x_0 \in X^0$ and since $g(*) = x_0$ we can apply the Cellular Approximation Theorem 141.2 to the continuous map $g\colon S^j \to X$ and the subcomplex $\{*\} \subseteq S^j$ and we obtain a homotopy F rel $\{*\}$ from the map g to a cellular map $\widetilde{g}$. Since F is a homotopy rel $\{*\}$ we see that F also defines a homotopy of maps $(S^j, *) \to (X, x_0)$ of pairs of topological spaces. Thus we see that $[g] = [\widetilde{g}] \in \pi_j(X, x_0)$.

Since $\widetilde{g}$ is cellular we know that $\widetilde{g}(S^j)$ lies in the j-skeleton of X. In other words, $\widetilde{g}$ is a continuous map $(S^j, *) \to (X^j, x_0)$. This shows that $[g] = [\widetilde{g}] = [i \circ \widetilde{g}] = i_*([\widetilde{g}])$ lies in the image of $\pi_j(X^j, x_0)$.

(2) Let $k \in \mathbb{N}$. By (1) it suffices to show that for $j = 1, \ldots, k-1$ the induced map $i_*\colon \pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is a monomorphism. Thus we now suppose that we are given $[f_0], [f_1] \in \pi_j(X^k, x_0)$ such that $i_*([f_0]) = i_*([f_1])$. By the Cellular Approximation Theorem 141.2 we can assume that the maps $f_0, f_1\colon (S^j, *) \to (X^k, x_0)$ are cellular. The fact that $i_*([f_0]) = i_*([f_1])$ means that there exists a homotopy $H\colon S^j \times [0,1] \to X$ with $H_0 = f_0$, with $H_1 = f_1$ and such that $H(*, t) = x_0$ for all $t \in [0,1]$. This time we apply the Cellular Approximation Theorem 141.2 to the continuous map $H\colon S^j \times [0,1] \to X$ and the subcomplex $(S^j \times \{0\}) \cup (S^j \times \{1\}) \cup (\{*\} \times [0,1])$ of $S^j \times [0,1]$ and we obtain a cellular map $\widetilde{H}\colon S^j \times [0,1] \to X$ such that $\widetilde{H}_0 = H_0 = f_0$ and $\widetilde{H}_1 = H_1 = f_1$.



Since the map $\widetilde{H}$ is cellular it takes values in the $(j+1)$-skeleton of $X \times [0,1]$. Since $j < k$ we see in particular that $\widetilde{H}$ takes values in $X^k \times [0,1]$. But this means that we can use $\widetilde{H}$ to show that $[f_0] = [f_1] \in \pi_j(X^k, x_0)$. ∎

As an example we can now give a new proof of the Sphere–Low-π_n-Proposition 134.19.

Proposition 134.19. (Sphere–Low-π_n-Proposition) Let $n \in \mathbb{N}$. For any choice of $k \in \{1, \ldots, n-1\}$ we have $\pi_k(S^n) = 0$.

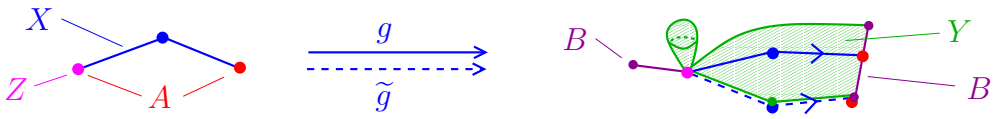
Proof. Again, as on page 2625 we consider the n-sphere S^n as a CW-complex with one 0-cell $\{N\}$ and one n-cell. With this CW-structure it follows immediately from the CW-Skeleton-π_n-Proposition 141.3 that $\pi_k(S^n) \cong \pi_k(\{N\}) = 0$ for $k = 1, \ldots, n-1$. ∎

Before we attack the proof of the Cellular Approximation Theorem 141.2 we want to formulate a relative version thereof, which will come in handy later on. To formulate this relative version we need the following semi-obvious definition.

Definition 141.4. A pair of CW-complexes is a pair (X, A) where X is a CW-complex and A is a subcomplex of X.

Now we can formulate the promised theorem.

Theorem 141.5. (Cellular Approximation Theorem for Pairs) Suppose we are given a continuous map $g\colon (X, A) \to (Y, B)$ between pairs of CW-complexes. Let Z be a subcomplex of A such that the restriction of g to Z is a cellular map. Then g is homotopic rel Z to a cellular map $\widetilde{g}\colon (X, A) \to (Y, B)$.[1462]

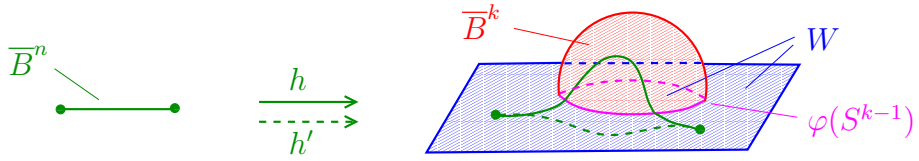


Proof of Theorem 141.5 assuming Theorem 141.2. The desired homotopy is constructed as follows:

(1) By the Cellular Approximation Theorem 141.2 applied to $g|_A\colon A \to B$ there exists a homotopy F rel Z from $g|_A\colon A \to B$ to a cellular map.
(2) By the Homotopy Extension Theorem 140.2 we can extend the homotopy F from (1) to a homotopy $G\colon X \times [0, 1] \to Y$. Note that G is still a homotopy rel Z. We write $h = g_1\colon X \to Y$. Note that $h|_A$ is cellular and that $h(A) \subseteq B$.
(3) We apply the Cellular Approximation Theorem 141.2 to h and we obtain a homotopy H rel A from h to a cellular map.
(4) As in the Homotopy Combination Lemma 18.8 we can combine the two homotopies G and H to obtain a homotopy that has all the desired properties. ■

141.2. **Proof of the Cellular Approximation Theorem 141.2 .** In this section we finally turn to the proof of the Cellular Approximation Theorem 141.2. The proof happens in several stages. The key step is performed in the following lemma:

Lemma 141.6. (Ball-avoid-Cell Lemma) Let W be a topological space, let $k \in \mathbb{N}_0$, let $\varphi\colon S^{k-1} \to W$ be a continuous map and let $h\colon \overline{B}^n \to W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ be a continuous map with $h(S^{n-1}) \subseteq W$. If $n < k$, then h is homotopic rel S^{n-1} to a map h' that misses B^k, i.e. that takes values in W.



If there exists a point $P \in B^k$ that does not get hit by $h(B^n)$, then the statement of the Ball-avoid-Cell Lemma 141.6 is fairly obvious, Indeed, if such a point P exists, then we can use the deformation retraction $\overline{B}^k \setminus \{p\} \to S^{k-1}$ to push h on B^n off the ball B^k. Put in general such a point P does not exist, since even for $n < k$ we know by the Space-Filling Curve Theorem 10.21 that there exist continuous surjective maps $\overline{B}^n \to \overline{B}^k$. As we will see, getting around this problem requires quite some technical work.

[1462]Here we refer to page 535 for the discussion of the notion of homotopies of pairs of topological spaces rel a subset.

Proof. In the following we provide a proof of the lemma using the Smooth-to-Cell Proposition 36.9. In Appendix 141.3 we provide an alternative, self-contained, proof of the lemma which does not make use of our results on smooth manifolds,

Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be a continuous. Furthermore let $h\colon \overline{B}^n \to X := W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ be a continuous map with $h(S^{n-1}) \subseteq W$. We suppose that $n < k$. We have to show that h is homotopic rel S^{n-1} to a map h' that misses B^k.

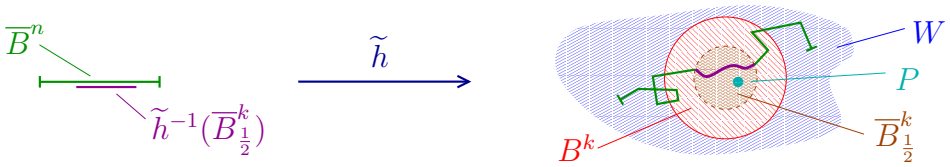
First note that it follows from the Smooth-to-Cell Proposition 36.9, applied to the smooth manifold $M := \overline{B}^n$, the topological space $X := W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ and the embedded disk $\overline{B}^k_{\frac{3}{4}} \subseteq B^k \subseteq X$, that there exists a continuous map $\widetilde{h}\colon \overline{B}^n \to W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ with the following properties:

(1) The map $\widetilde{h}$ is homotopic to h rel $\overline{B}^n \setminus h^{-1}(B^k_{\frac{1}{2}})$.
(2) The map

$$\begin{aligned} \widetilde{h}\colon h^{-1}(B^k_{\frac{1}{2}}) &\to B^k_{\frac{1}{2}} \\ x &\mapsto \widetilde{h}(x) \end{aligned}$$

is smooth.

Since $h^{-1}(B^k_{\frac{1}{2}})$ is an open subset of B^n, since $\widetilde{h}|_{h^{-1}(B^k_{\frac{1}{2}})}\colon h^{-1}(B^k_{\frac{1}{2}}) \to B^k$ is smooth and since $n < k$ we obtain from the Measure-Zero Properties Proposition 37.3 that the image $\widetilde{h}(h^{-1}(B^k_{\frac{1}{2}}))$ is a subset of measure zero in B^k. (We refer to Definition 37.1 of a subset of measure zero.) On the other hand it follows from the Full Measure-Density Proposition 37.4 that any open non-empty subset does not have measure zero. This means that we can pick a point $P \in B^k_{\frac{1}{2}} \setminus \widetilde{h}(h^{-1}(B^k_{\frac{1}{2}}))$. Note that P does not lie in the image of $\widetilde{h}(\overline{B}^n)$.



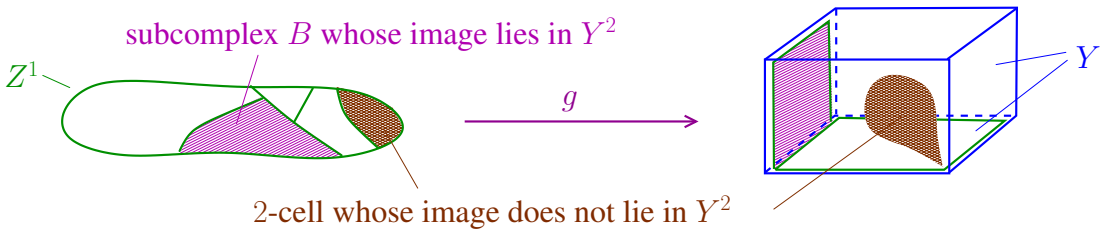
As we pointed out on page 523, there exists a deformation retraction from $\overline{B}^k \setminus \{P\}$ to S^{k-1}. Using the Homotopy Pushout Lemma 18.13 we can extend this deformation retraction via the identity map to a deformation retraction from $X = W \cup_\varphi (\overline{B}^k \setminus \{P\})$ to W. If we compose $\widetilde{h}$ with this deformation retraction we obtain a homotopy from h to a map h' that misses B^k. Finally note that it follows from $h(S^{n-1}) \subseteq W$ that this homotopy is rel S^{n-1}, ∎

In the following lemma we work our way up towards a proof of the Cellular Approximation Theorem 141.2.

Lemma 141.7. (Avoid-higher-Skeleton Lemma) Let Z be an n-dimensional CW-complex, let $g\colon Z \to Y$ be a continuous map to a CW-complex Y such that $g(Z^{n-1}) \subseteq Y^{n-1}$ and let B be a subcomplex of Z with $g(B) \subseteq Y^n$. The map g is homotopic rel $Z^{n-1} \cup B$ to a continuous map g' with $g'(Z) \subseteq Y^n$.

Proof. Let Y be CW-complex.

Claim 1. Let $n \in \mathbb{N}_0$. Given any $k > n$ and given any continuous map $h\colon \overline{B}^n \to Y^k$ with $h(S^{n-1}) \subseteq Y^{k-1}$ there exists a homotopy $G\colon \overline{B}^n \times [0,1] \to Y$ rel S^{n-1} with $G_0 = h$ and such that $G_1(\overline{B}^n) \subseteq Y^{k-1}$.



Proof. It follows from the Compact Image Lemma 4.15 together with the CW-Complex-Finiteness Theorem 136.16 that $h(\overline{B}^n)$ is contained in a finite subcomplex of Y^k. In other words, we might as well assume that Y is in fact a finite k-dimensional CW-complex. We denote by $c_1, \ldots, c_s$ the k-cells of Y.

In the following we apply the Ball-avoid-Cell Lemma 141.6 altogether s times to get the desired homotopy. More precisely, we do the following: Since we assume that $k > n$ and since we assume that $h(S^{n-1}) \subseteq Y^{k-1}$ we can apply the Ball-avoid-Cell Lemma 141.6 to the continuous map

$$g\colon \overline{B}^n \;\to\; (Y^{k-1} \cup c_1 \cup \cdots \cup c_{s-1}) \cup c_s.$$

This way we obtain a homotopy rel S^{n-1} from h to a continuous map h_{s-1} such that $h_{s-1}(\overline{B}^n) \subseteq Y \cup c_1 \cup \cdots \cup c_{s-1}$. Next we apply the Ball-avoid-Cell Lemma 141.6 to the continuous map

$$h_{s-1}\colon \overline{B}^n \;\to\; Y^{k-1} \cup c_1 \cup \cdots \cup c_{s-1}.$$

Iterating this process, and using the Homotopy Combination Lemma 18.8 (2), we obtain a homotopy rel S^{n-1} to a continuous map h' with $h'(\overline{B}^n) \subseteq Y^{k-1}$. ⊞

Now we turn to the actual proof of the lemma. Let Z be an n-dimensional CW-complex, let $g\colon Z \to Y$ be a continuous map such that $g(Z^{n-1}) \subseteq Y^{n-1}$ and let B be a subcomplex of Z with $g(B) \subseteq Y^n$. For each cell d of Z that is not contained in B we pick a characteristic map $\Phi^d\colon \overline{B}^{\dim(d)} \to Z$.

Claim 2. Let d be an n-cell of Z that is not contained in the subcomplex B. There exists a homotopy $\Theta^d\colon \overline{B}^n \times [0,1] \to Y$ rel S^{n-1} with $\Theta^d_0 = g \circ \Phi^d$ and such that $\Theta^d_1(\overline{B}^n) \subseteq Y^n$.

Proof. Let d be an n-cell of Z that is not contained in the subcomplex B. It follows from the Compact Image Lemma 4.15 together with the CW-Complex-Finiteness Theorem 136.16 that $(g \circ \Phi_d)(\overline{B}^n)$ is contained in a finite subcomplex of Y. In particular there exists a $k \in \mathbb{N}_0$ with $(g \circ \Phi_d)(\overline{B}^n) \subseteq Y^k$. If $k \leq n$ there is nothing to show. Now suppose that $k > n$. We apply Claim 1 to $h = g \circ \Phi_d\colon \overline{B}^n \to Y$ and we obtain a homotopy rel S^{n-1} to a map $h_{k-1}\colon \overline{B}^n \to Y^{k-1}$. Next we apply Claim 1 to h_{k-n} and we iterate this procedure altogether $k - n$ times. At the end we obtain the desired homotopy from the Homotopy Combination Lemma 18.8 (2). ⊞

Finally we consider the following map:

$$G\colon Z \times [0,1] \;\to\; Y$$
$$(P,t) \;\mapsto\; \begin{cases} \Theta^d(x,t), & \text{if } P = \Phi^d(x) \text{ for some } x \in \overline{B}^n, \text{ where } d \text{ is an } n\text{-cell} \\ & \qquad\qquad \text{that is not contained in } B \\ g(P), & \text{otherwise.} \end{cases}$$

It follows from the CW×Interval Proposition 137.1 that this map is continuous. It is now clear that it has all the desired properties. ■

Now we turn to the proof of the Cellular Approximation Theorem 141.2:

Proof of Theorem 141.2. Let X be a CW-complex, let A be a (possibly empty) subcomplex of X, let Y be a CW-complex and let $g\colon X \to Y$ be a continuous map such that the restriction of g to A is cellular.



We iteratively construct homotopies as follows:

- (F^0) We apply the Avoid-higher-Skeleton Lemma 141.7 to $Z = X^0$, $g = g|_{X^0}$ and to $B = X^0 \cap A$. We obtain a homotopy $F^0\colon X^0 \times [0,1] \to Y$ rel $X^0 \cap A$ with $F^0_0 = g|_{X^0}$ and such that $F^0_1(X^0) \subseteq Y^0$.
- (G^0) It follows from the Homotopy Extension Theorem 140.2 that there exists a homotopy $G^0\colon X \times [0,1] \to Y$ rel A with $G^0_0 = g$ and such that $G^0_1 = F^0_1|_{X^0}$. In particular this implies that $G^0_1(X^0) \subseteq Y^0$.
- (F^1) We apply the Avoid-higher-Skeleton Lemma 141.7 to $Z = X^1$, $g = G^0_1$ and with $B = X^0 \cup (X^1 \cap A)$. We obtain a homotopy $F^1\colon X^1 \times [0,1] \to Y$ rel $(X^0 \cup (X^1 \cap A))$ with $F^1_0 = G^0_1$ and such that $F^1_1(X^1) \subseteq Y^1$.
- (G^1) It follows from the Homotopy Extension Theorem 140.2 that there exists a homotopy $G^1\colon X \times [0,1] \to Y$ rel $X^0 \cup A$ with $G^1_0 = G^0_1$ and such that $G^1_1 = F^1_1|_{X^1}$. In particular this implies that $G^1_1(X^1) \subseteq Y^1$.

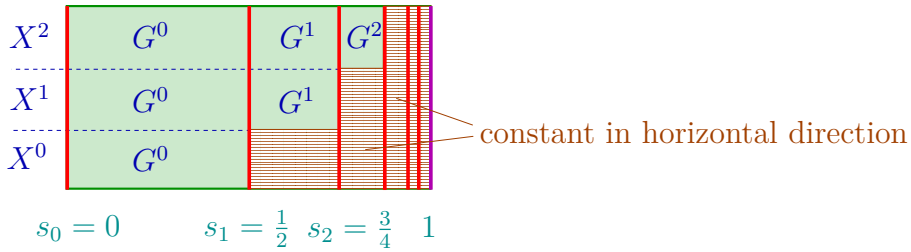
We proceed indefinitely and we obtain a sequence of homotopies $F^n\colon X^n \times [0,1] \to Y$ rel $X^{n-1} \cup (X^n \cap A)$ and a sequence of homotopies $G^n\colon X \times [0,1] \to Y$ rel A. As in the Infinite Homotopies–CW–Lemma 140.19 we can combine these infinitely many homotopies to get the desired homotopy. More precisely, we define

$$\begin{array}{rcl} \widetilde{g}\colon X & \to & Y \\ x & \mapsto & G^k_1(x) \quad \text{where } k \in \mathbb{N}_0 \text{ such that } x \in X^k. \end{array}$$

For $k = 0, 1, 2, \ldots$ we write $s_k = \sum_{j=1}^{k} \frac{1}{2^j}$. We consider the map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} G^k(x, 2^{k+1} \cdot (t - s_k)), & \text{if } t \in [s_k, s_{k+1}], \\ \widetilde{g}(x), & \text{if } t = 1 \end{cases} \end{array}$$

that is illustrated in the figure below.



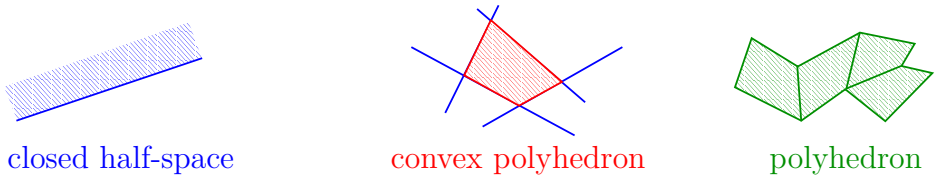
It follows from the elementary Homotopy Combination Lemma 18.8 (1) that the restriction of H to each $X^k \times [0,1]$ is continuous. Furthermore note that it follows from the CW×Interval Proposition 137.1 (2) that $H\colon X \times [0,1] \to Y$ is also continuous. Now one can easily verify that this map has the desired properties. ■

141.3. **Appendix: Alternative proof of the Ball-avoid-Cell Lemma 141.6 .** A key ingredient in the proof of the Cellular Approximation Theorem 141.2 was the Ball-avoid-Cell Lemma 141.6. On page 2725 we gave a proof using the Smooth-to-Cell Proposition 36.9. In this appendix we will provide an alternative, self-contained proof, of the Ball-avoid-Cell Lemma 141.6 which does not make use of our results on smooth manifolds.

Before we can give our new proof of the Ball-avoid-Cell Lemma 141.6 we need to introduce a few more definitions and we need to prove a rather technical lemma.

Definition 141.8.

(1) A closed affine half-space in $\mathbb{R}^n$ is a set of the form $\{v \in \mathbb{R}^n \mid \langle v, w\rangle \geq a\}$, for some fixed non-zero vector $w \in \mathbb{R}^n$ and some $a \in \mathbb{R}$.
(2) A convex polyhedron in $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ that is the intersection of finitely many closed affine half-spaces.
(3) A polyhedron in $\mathbb{R}^n$ is the union of finitely many convex polyhedra.[1463]
(4) Given a polyhedron P a map $f\colon P \to \mathbb{R}^k$ is called piecewise linear if there exists a subdivision of P into finitely many convex polyhedra $Q_1, \ldots, Q_m$ such that the restriction of f to each Q_i is affine linear.[1464]



closed half-space | convex polyhedron | polyhedron

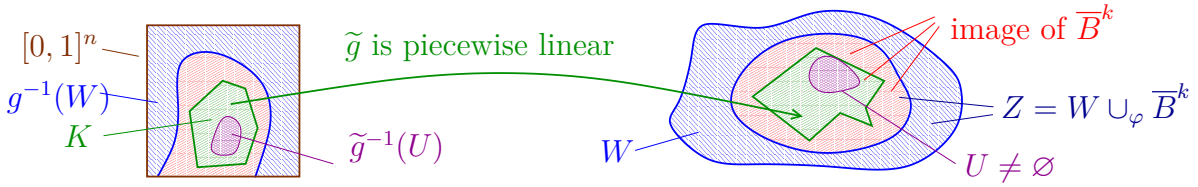
The following, very technical lemma, will get us around the problem of space-filling paths, i.e. it will get us around the problem that we saw in Space-Filling Curve Theorem 10.21 that there exist surjective continuous maps $\overline{B}^n \to \overline{B}^k$ even if $n < k$. The lemma can be viewed as an analog of the Smooth-to-Cell Proposition 36.9.

Lemma 141.9. (Homotopic-to-Piecewise Linear Lemma) Let W be a topological space, let $\varphi\colon S^{k-1} \to W$ be a continuous map and let $g\colon [0,1]^n \to Z := W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ be a continuous map. There exists a (possibly empty) polyhedron $K \subseteq [0,1]^n$ and a continuous map $\widetilde{g}\colon [0,1]^n \to Z$ with the following properties:

(1) The map $\widetilde{g}$ is homotopic to g rel $g^{-1}(W)$.
(2) We have $\widetilde{g}(K) \subseteq B^k$.
(3) The restriction $\widetilde{g}|_K\colon K \to B^k$ is piecewise linear.
(4) There exists an open non-empty subset $U \subseteq B^k$ such that $\widetilde{g}^{-1}(U) \subseteq K$.

[1463] Note that the union of zero convex polyhedra is the empty set.

[1464] A map $f\colon V \to \mathbb{R}^k$ from a subset V of $\mathbb{R}^n$ is called *affine linear* if there exists a $P \in \mathbb{R}^k$ and a $k \times n$–matrix A such that $f(v) = P + Av$ for all $v \in V$.

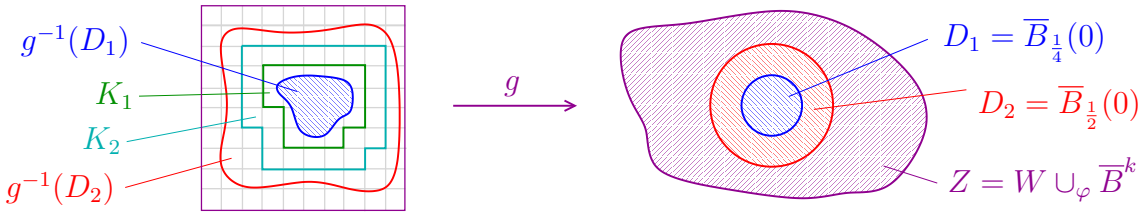


Example. Suppose that in the setting of the Homotopic-to-Piecewise Linear Lemma 141.9 there exists an open subset $U \subseteq B^k$ such that $g([0,1]^n) \cap U = \varnothing$. Then the empty polyhedron $K = \varnothing$ and the map $\widetilde{g} = g$ already have the desired property. □

Proof. Let W be a topological space, let $\varphi\colon S^{k-1} \to W$ be a continuous map and let $g\colon [0,1]^n \to Z := W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ be a continuous map. We write $D_1 := \overline{B}_{\frac{1}{4}}(0) \subseteq B^k$ and $D_2 := \overline{B}_{\frac{1}{2}}(0) \subseteq B^k$. The set $g^{-1}(D_2) \subseteq [0,1]^n$ is closed and therefore compact by the Heine-Borel Theorem 3.44. We obtain from the Compact–Uniformly Continuous Proposition 10.11 that g is uniformly continuous on $g^{-1}(D_2)$.[1465] This implies that we can find an $\epsilon > 0$ with the following two properties:

(i) For any $x, y \in g^{-1}(D_2)$ with $\|x - y\| < \epsilon$ we have $\|g(x) - g(y)\| < \frac{1}{8}$.
(ii) We have[1466]
$$\epsilon \ < \ \text{distance between } g^{-1}(D_1) \text{ and } [0,1]^n \setminus g^{-1}\big(\mathring{D}_2\big).$$

We divide $[0,1]^n$ into cubes such that the diameter of each cube is less than ϵ. We denote by K_1 the union of all the cubes that meet $g^{-1}(D_1)$ and we denote by K_2 the union of all the cubes meeting K_1. By the second property of ϵ we have $K_2 \subseteq g^{-1}(D_2)$.



We need one more definition: Suppose we are given points $P_1, \ldots, P_{n+1} \in \mathbb{R}^n$ such that the vectors $P_2 - P_1, \ldots, P_{n+1} - P_1$ are linearly independent. We refer to
$$\text{convex hull of } P_1, \ldots, P_{n+1} \ = \ \left\{ \sum_{i=1}^{n+1} t_i \cdot P_i \,\middle|\, \sum_{i=1}^{n+1} t_i = 1 \right\}$$
as a *simplex* in $\mathbb{R}^n$.[1467] We will make use of the following trivial observation: If we are given $w_1, \ldots, w_{n+1} \in \mathbb{R}^k$, then there exists a unique affine linear map Φ with $\Phi(P_i) = w_i$,

[1465] Why can't we conclude that g is uniformly continuous on all of $[0,1]^n$?

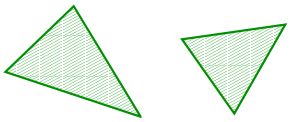
[1466] Here recall that, according to Definition 10.3, for two non-empty subsets A and B of $\mathbb{R}^n$ we refer to
$$d(A,B) \ := \ \inf\{\|P - Q\| \mid P \in A \text{ and } Q \in B\}$$
as the distance between A and B. If A and B are compact and disjoint, then it follows from the Disjoint Compact-Closed–Distance Lemma 10.5 that $d(A,B) > 0$.

[1467] Given a subset S of $\mathbb{R}^k$ we introduced in Definition 4.27 the *convex hull* of S as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. As we saw in Exercise 4.46, if S is a finite subset of $\mathbb{R}^k$, then the *convex hull* of S is given by $\left\{ \sum_{s \in S} a_s \cdot s \,\middle|\, a_s \in [0,1] \text{ and } \sum_{s \in S} a_s = 1 \right\}$.

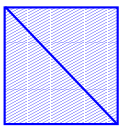
namely the map that is given by

$$\Phi\Big(\sum_{i=1}^{n+1} t_i \cdot P_i\Big) \;:=\; \sum_{i=1}^{n+1} t_i \cdot w_i.$$

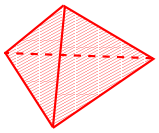
We leave it to the reader to show that each cube can be subdivided[1468] into simplices. We also refer to the figure below for an illustration.



simplices in $\mathbb{R}^2$



a square can be decomposed into two simplices



simplex in $\mathbb{R}^3$

Now we subdivide all the cubes in K_2 into simplices. By the above discussion there exists a continuous map $h\colon K_2 \to \mathbb{R}^k$ that equals g on all the vertices of the simplices in the subdivision of the cubes.[1469] Furthermore we choose a continuous map $\psi\colon K_2 \to [0,1]$ with $\psi|_{\partial K_2} \equiv 0$ and $\varphi|_{K_1} \equiv 1$.[1470] Now we consider the homotopy[1471]

$$\begin{array}{rcl} F\colon [0,1]^n \times [0,1] & \to & Z = W \cup_\varphi \overline{B}^k \\ (P,t) & \mapsto & \begin{cases} (1 - t\cdot\psi(P))\cdot g(P) + t\cdot\psi(P)\cdot h(P), & \text{if } P \in K_2 \\ g(P), & \text{otherwise.} \end{cases} \end{array}$$

This is a homotopy from g to a map $\widetilde{g}\colon [0,1] \to Z = W \cup_\varphi \overline{B}^k$ that restricts to the piecewise linear map h on K_1.

Claim. The point $0 \in D_1$ does not lie in $\widetilde{g}(\overline{[0,1]^n \setminus K_1})$.

Proof. The claim is proved in two steps:

(1) On $[0,1]^n \setminus K_2$ we have $\widetilde{g} = g$. By Property (ii) of ϵ we have $g^{-1}(D_1) \subseteq K_2$. It follows that the image of $[0,1]^n \setminus K_2$ lies outside of D_1.

(2) Now let σ be a simplex of K_2 that does not lie in K_1. Note that $K_2 \subseteq g^{-1}(D_2)$. It follows from Property (i) of ϵ that $g(\sigma)$ is contained in some ball B_σ of radius $\frac{1}{8}$. Since $g(\sigma) \subseteq B_\sigma$ and since B_σ is convex we also have $h(\sigma) \subseteq B_\sigma$. Again by the convexity of B_σ we have $\widetilde{g}(\sigma) \subseteq B_\sigma$. We know that B_σ is not contained in D_1 since σ contains points outside K_1, hence outside $g^{-1}(D_1)$. The radius of B_σ is half that of D_1, so it follows that 0 is not in B_σ, and hence 0 does not lie in $\widetilde{g}(\sigma)$. ⊞

Since $\widetilde{g}\big(\overline{[0,1]^n \setminus K_1}\big)$ is compact it follows from the claim that there exists in fact an open neighborhood U of 0 that does not intersect the image of $[0,1]^n \setminus K_1$. Put differently, we have $\widetilde{g}^{-1}(U) \subseteq K_1$. It is now straightforward to verify that $K = K_1$, $\widetilde{g}$ and U have the desired properties. ■

[1468] Here, by "subdivide" we mean that given any cube K there exist simplices $S_1, \ldots, S_k$ such that the interiors of the S_i are disjoint and such that $K = S_1 \cup \cdots \cup S_k$.

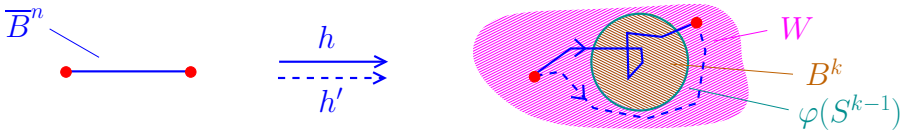
[1469] Note that it follows from the uniqueness statement above that the map g is in fact continuous.

[1470] Such a map can be constructed as follows: we assign the value 1 to all vertices in K_1 and we assign the value 0 to all other vertices in K_2. By the above "trivial observation" this defines a map on K_2 with the desired property.

[1471] Since $\varphi|_{\partial K_2} \equiv 0$ we see that F is indeed continuous.

Now we can give our alternative proof of the Ball-avoid-Cell Lemma 141.6:

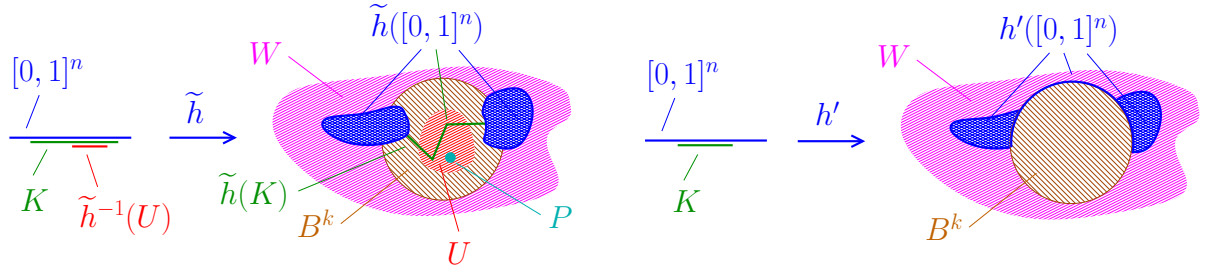
Lemma 141.6. (Ball-avoid-Cell Lemma) Let W be a topological space, let $k \in \mathbb{N}_0$, let $\varphi\colon S^{k-1} \to W$ be a continuous map and let $h\colon \overline{B}^n \to W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ be a continuous map with $h(S^{n-1}) \subseteq W$. If $n < k$, then h is homotopic rel S^{n-1} to a map h' that misses B^k, i.e. that takes values in W.



Proof. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ be a continuous map with $h(S^{n-1}) \subseteq W$. We suppose that $n < k$. We have to show that h is homotopic rel S^{n-1} to a map h' that misses B^k.

By the Convex-to-Ball Proposition 4.28 there exists a homeomorphism $\Phi\colon [0,1]^n \to \overline{B}^n$ with $\Phi(\partial([0,1]^n)) = S^{n-1}$. So after precomposing h with Φ we might as well assume that the domain of h is $[0,1]^n$, i.e. that h is a continuous map $h\colon [0,1]^n \to Z := W \cup_{W \xleftarrow{\varphi} S^{k-1}} \overline{B}^k$ with $h(\partial([0,1]^n)) \subseteq W$.

We pick a polyhedron K, a map $\widetilde{h}\colon [0,1]^n \to Z$ and an open non-empty subset $U \subseteq B^n$ with $\widetilde{h}^{-1}(U) \subseteq K$ as in the statement of the Homotopic-to-Piecewise Linear Lemma 141.9. Since $n < k$ and since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear we deduce that $\widetilde{h}(K)$ cannot contain the open subset U of B^n.[1472] We pick a point $P \in U \setminus \widetilde{h}(K)$. By Statement (4) of the Homotopic-to-Piecewise Linear Lemma 141.9 we know that P does not lie in the image of $\widetilde{h}\colon [0,1]^n \to Z$.



As we pointed out on page 523, there exists a deformation retraction from $\overline{B}^k \setminus \{P\}$ to S^{k-1}. Using the Homotopy Pushout Lemma 18.13 we can extend this deformation retraction via the identity map to a deformation retraction from $Z = W \cup_\varphi (\overline{B}^k \setminus \{P\})$ to W. If we compose $\widetilde{h}$ with this deformation retraction we obtain a homotopy, rel S^{n-1}, from $\widetilde{h}$ to a map h' that misses B^k. ∎

141.4. Exercises for Chapter 141.

[1472] This can be seen as follows. Since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear and since $n < k$ we obtain from the Measure-Zero Properties Proposition 37.3 that the image $\widetilde{h}(K)$ is a subset of measure zero in B^n. On the other hand it follows from the Full Measure-Density Proposition 37.4 that any open non-empty subset does not have measure zero.

Exercise 141.1. Provide an example of a 1-dimensional CW-complex X and a self-homeomorphism $f\colon X \to X$ that is not homotopic to a cellular homeomorphism.

Exercise 141.2. We consider the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ with the standard CW-structure from page 2625 which has precisely one 0-cell $*$ that corresponds to the origin. As in the Action–π_1-Theorem 93.1 we make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^2, *) = \mathbb{Z}^2$. Let $f\colon T \to T$ be a cellular isomorphism with $f(*) = *$. Show that $f_*\colon \pi_1(T,*) \to \pi_1(T,*)$ is represented by a matrix of the form $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$.

Exercise 141.3. Show that there is no CW-structure on the 2-dimensional torus $S^1 \times S^1$ such that every homeomorphism is homotopic to a cellular homeomorphism.

Exercise 141.4. We equip the torus $\mathbb{R}^2/\mathbb{Z}^2$ with the standard CW-structure from page 2625. Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [P \cdot v]. \end{array}$$

Give an example of an explicit cellular map $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ that is homotopic to f.

Exercise 141.5. Let X be a CW-complex which admits a single 0-cell. Show that every subcomplex Y of X is path-connected.

Exercise 141.6. Let X be a CW-complex. Use the Cellular Approximation Theorem 141.2 to give a new proof (first proved in the CW-Skeleton-π_0-Lemma 136.21) that if X is path-connected, then its 1-skeleton X^1 is also path-connected.

Exercise 141.7. Show that given any finitely presented group π there exists a connected non-empty CW-complex X with $\pi_1(X) \cong \pi$ and such that for all $n \geq 2$ we have $\pi_n(X) = 0$.

Part XVI

Singular Homology – the essentials

In Part X we introduced the fundamental group $\pi_1(X, x_0)$ of a pointed topological space (X, x_0). We observed and proved several statements regarding fundamental groups:

(1) In the π_1-Functor Proposition 90.3 we learned that fundamental groups are covariantly functorial.
(2) In the Homotopy Equivalence-π_1-Proposition 90.9 we showed that homotopy equivalent topological spaces have isomorphic fundamental groups.
(3) We got to know several techniques for calculating fundamental groups. In particular we formulated and proved the Seifert–van Kampen Theorem 99.1 which allows us to calculate the fundamental groups of a pointed topological space (X, x_0) with a suitable decomposition $X = U \cup V$ in terms of the fundamental groups of $U, V, U \cap V$ and the induced maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

We also saw that fundamental groups have many applications. For example we used them for the proof of one direction of the Surface Classification Theorem 101.16, namely we used fundamental groups to show that surfaces of different genus are non-homeomorphic.

Unfortunately fundamental groups have serious drawbacks. Perhaps the biggest issue is that fundamental groups are by construction impervious to higher-dimensional phenomena. For example, we would like to show that for $m \neq n$ the spheres S^m and S^n are not homeomorphic. We have seen that for most cases fundamental groups are of no use. In Part IX we introduced the higher homotopy groups $\pi_n(X, x_0)$, $n \geq 1$ of a pointed topological space. In principle higher homotopy groups might be able to distinguish such spheres, but since currently we cannot compute higher homotopy groups, those are as of now of no use either. So we need another approach to finding invariants that can distinguish spheres.

In Part IX we saw that de Rham cohomology groups can in fact be used to distinguish spheres. But we really would like to have invariants of topological spaces, and not just of smooth manifolds. For example de Rham cohomology is already useless for showing that the wedges $S^m \vee S^m$ and $S^n \vee S^n$ for $m \neq n$ are not homeomorphic.

In this part of the lecture notes we will introduce the singular homology groups $\mathrm{H}_k(X)$, $k \in \mathbb{N}_0$ of a topological space X to address these issues. These have in particular the following properties:

(1) For each $k \in \mathbb{N}_0$ we will see that the map $X \mapsto \mathrm{H}_k(X)$ define a covariant functor from the category of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups.
(2) We will see that if X and Y are homotopy equivalent topological spaces, then for any $k \in \mathbb{N}_0$ the corresponding singular homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(Y)$ are isomorphic.
(3) Suppose we have a decomposition $X = U \cup V$ of a topological space X into open subset U and V. In the Seifert–van Kampen Theorem we will see that the singular homology groups of X are for the most part determined by the singular homology groups of U, V and $U \cap V$ and the induced maps $\mathrm{H}_k(U \cap V) \to \mathrm{H}_k(U)$ and $\mathrm{H}_k(U \cap V) \to \mathrm{H}_k(V)$.

Unfortunately it will take quite some time to perform a first non-trivial calculation of these singular homology groups. But once we have managed to calculate the singular homology groups of spheres and once we have proved the above three properties, we will have a long string of applications and we will see that in many situations singular homology groups are quite computable and very powerful.

This part of the lecture notes is organized as follows:

- In Chapter 142 we will explain the basic geometric idea behind singular homology.
- In Chapter 143 we will introduce the singular homology groups $\mathrm{H}_k(X)$ of a topological space X, we will show that homology defines a covariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups. We will also compute homology groups in a few basic settings.
- In Chapter 144 we will show in the Homotopy Equivalence-H_*-Proposition 144.14 that the homology groups of homotopy equivalent topological spaces are isomorphic.
- In the purely algebraic Chapter 145 we will introduce the notion of an exact sequence and we will show in the LES-Proposition 145.10 that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.
- In Chapter 146 we introduce, given a pair (X, A) of topological spaces, the relative homology $\mathrm{H}_k(X, A)$. These will play an essential role in the development of the theory, even if one only cares about absolute homology groups.
- In the short Chapter 147 we will introduce the reduced homology groups $\widetilde{\mathrm{H}}_k(X)$ of a topological space. These are a minute, but surprisingly convenient, variation on the ordinary homology groups.
- In Chapter 148 we will prove the Excision Theorem 148.1 which is the key to all non-trivial calculations of homology groups.
- In Chapter 149 we will study applications of the Excision Theorem 148.1. In particular in the Sphere-H_*-Proposition 149.4 we will use it to calculate the homology groups of spheres. This calculation allows us to give a new proof of the Brouwer Fixed Point Theorem 38.7.
- In Chapter 150 we will study the local homology groups of a topological space X at a point $x \in X$ and we will use them to provide explicit generators of homology groups.
- In Chapter 151 we will introduce the degree of a continuous self-map of a sphere and we will calculate it in several instances.
- In Chapter 152 we will introduce local degrees of self-maps of spheres and we will show how to use them to calculate even more degrees of self-maps of spheres.
- In Chapter 153 we will prove the Mayer–Vietoris Theorem 153.2 which can be viewed as the homological analog of the Seifert-van Kampen Theorem 99.1.
- In Chapter 154 we will discuss the homology groups of a direct colimit of topological spaces.
- In Chapter 155 we will use the results from Chapter 154 to introduce the cellular homology of a CW-complex. We will see that the calculation of cellular homology boils down to calculating degrees of self-maps of spheres. Since we already developed many tools for calculating these degrees we see that cellular homology is imminently computable. Furthermore in the Singular-Cellular-H_*-Proposition 155.8 we will that cellular homology is naturally isomorphic to ordinary homology. This fact opens the door to many calculations of homology groups. For example we will be able to compute homology groups of the real and complex projective spaces and of compact surfaces.
- In Chapter 156 we will introduce the Euler characteristic $\chi(X)$ of a CW-complex X. In the Homology-χ-Proposition 156.2 we will show, using cellular homology, that the Euler characteristic only depends on the homotopy type of X.

- In Chapter 157 we will discuss several fun applications of the Euler characteristic. In particular we will study regular convex polyhedra in $\mathbb{R}^3$ and we will see that not every finite topological graph can be embedded into $\mathbb{R}^2$.

CHAPTER 142

The idea behind singular homology

The definition and the theory of singular homology was developed over several decades in the first half of the 20th century. In this chapter we will try to outline how one might have come up with the idea behind singular homology. The stress in this chapter is on explaining ideas and examples, without getting bogged down with technical discussions. We will not make use of any of the results and definitions of this chapter. We will use this feature as an excuse for taking a few liberties here and there.

This chapter is organized as follows:

- In Section 142.1 we will give an introduction to "naive" simplicial complexes.
- In Section 142.2 we will define "naive" simplicial homology.
- In Section 142.3 we will provide exercises for Chapter 142.

In the next Chapter 143 we will introduce the singular homology groups of a topological space.

142.1. **Simplicial complexes.** So let's start with our story. Historically mathematicians found, and still find it, convenient to work with simplicial complexes. Loosely speaking a simplicial complex is a subset of $\mathbb{R}^n$ that is built out of simplices. Thus we first should tell the reader what we mean by a simplex:

Definition 142.1.

(1) We say that $k+1$ points in $\mathbb{R}^n$ are in general position if they are not contained in a $(k-1)$-dimensional affine hyperplane of $\mathbb{R}^n$.[1473]

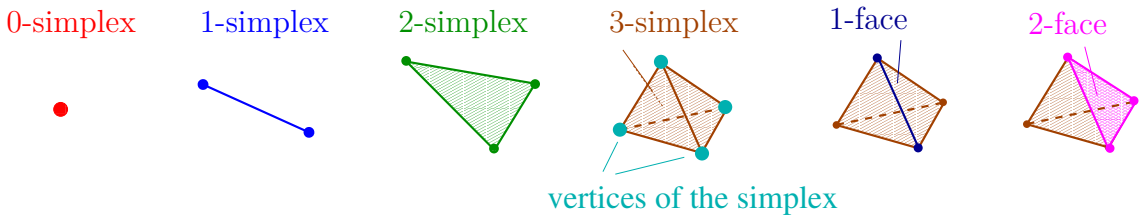
(2) Given any finite subset $S \subseteq \mathbb{R}^n$ we denote by[1474]
$$\text{Hull}(S) \ := \ \left\{ \sum_{x \in S} \lambda_x \cdot x \ \middle| \ \text{each } \lambda_x \in \mathbb{R}_{\geq 0} \text{ and } \sum_{x \in S} \lambda_x = 1 \right\}$$
the convex hull of S.

(3) A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ points of $\mathbb{R}^n$ that are in general position.

(4) Let $\sigma \subseteq \mathbb{R}^n$ be a k-simplex.
   (a) There exists a unique[1475] subset $V \subseteq \mathbb{R}^n$ with $k+1$ elements and with $\sigma = \text{Hull}(V)$. We refer to V as the vertex set of σ.
   (b) Let V be the vertex set of σ. For any subset $T \subseteq V$ of with $(l+1)$ elements we refer to $\text{Hull}(T)$ as an l-face of $\sigma = \text{Hull}(V)$.

Simplices can be viewed as convenient building blocks to describe topological spaces. This observation leads us straight to the following definition:

[1473] In other words, $P_0, \ldots, P_k \in \mathbb{R}^n$ are in general position if $P_1 - P_0, \ldots, P_k - P_0$ are linearly independent.
[1474] Strictly speaking we are using Exercise 4.46 to get to this description of the convex hull.
[1475] It is a nice little exercise to figure out what S is unique.



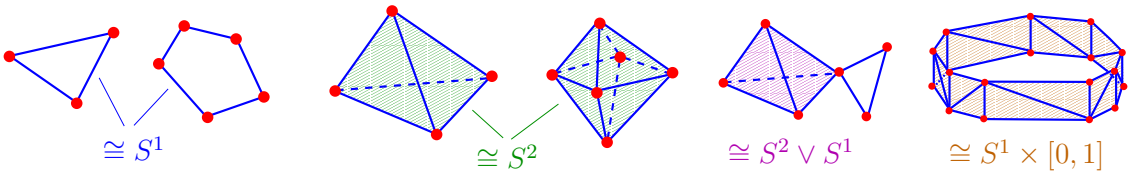
Definition 142.2. A simplicial complex in $\mathbb{R}^n$ is a finite family $X = \{\sigma_i\}_{i=1,\dots,k}$ of simplices in $\mathbb{R}^n$ with the following properties:

(1) Any face of a simplex is also contained in the family.
(2) The intersection of any two simplices is either empty or it is a single simplex, which is a face of both simplices.

Given a simplicial complex $X = \{\sigma_i\}_{i=1,\dots,k}$ as above we denote by X also the subspace $\sigma_1 \cup \cdots \cup \sigma_k \subseteq \mathbb{R}^n$.[1476]

Example.

(1) In the figure below we see many examples of simplicial complexes. In particular we see that many topological spaces that we like, are actually homeomorphic to a simplicial complex.



(2) In the Smooth Manifold-Simplicial Structure Theorem 179.5 we will show that every compact smooth manifold is homeomorphic to a simplicial complex and in the CW–Simplicial Homotopy Type Proposition 175.21 we will see that every finite CW-complex is homotopy equivalent to a simplicial complex. □

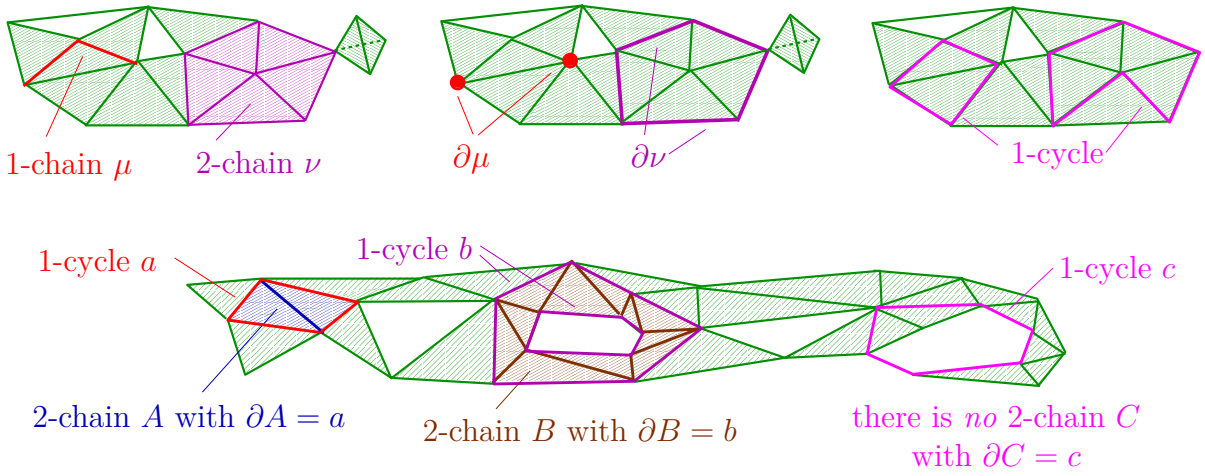
142.2. **Simplicial homology.** The next definition is a key step towards introducing simplicial homology:

Definition 142.3. Let X be a simplicial complex.

(1) An n-chain in X is a collection of n-simplices of X, in other words, it is a subset of the set of n-simplices of X.
(2) The boundary $\partial_n\sigma$ of an n-chain σ is the union of all $(n-1)$-simplices that are the faces of an *odd* number of n-simplices of the n-chain.
(3) We say that an n-chain is an n-cycle if its boundary is the empty set of $(n-1)$-simplices.

In the next figure we see that cycles come in two different flavors. Some n-cycles are the boundary of an $(n+1)$-chain and are thus not very interesting. On the other hand there are n-cycles which are not the boundary of an $(n+1)$-chain, and these non-bounding cycles seem to detect something interesting, like the presence of a "hole".

[1476] We will discuss simplicial complexes in much greater detail later in Part XIX.



The idea now is to "measure" how many n-cycles are not the boundary of an $(n+1)$-cycle by introducing some algebra. This leads us to the following definition:

Definition 142.4. Let X be a simplicial complex.

(1) We define the n-th simplicial chain group of X as

$$\mathrm{C}_n^{\mathrm{simp}}(X) \coloneqq \text{set of all } n\text{-chains} \underset{\uparrow}{=} (\mathbb{Z}/2\mathbb{Z})^{\text{set of } n\text{-simplices of } X}.$$

for any set W we have $\mathcal{P}(W) = (\mathbb{Z}/2\mathbb{Z})^W$, we apply this logic to the set of all n-simplices of X

In other words the n-th chain group $\mathrm{C}_n^{\mathrm{simp}}(X)$ of X is defined as the free $\mathbb{Z}/2\mathbb{Z}$-vector space on the set of all n-simplices of X. More informally speaking, $\mathrm{C}_n^{\mathrm{simp}}(X)$ is the set of all finite $\mathbb{Z}/2\mathbb{Z}$-linear combinations of n-simplices of X.

(2) We define the boundary map

$$\begin{aligned} \partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) &\to \mathrm{C}_{n-1}^{\mathrm{simp}}(X) \\ \sum_{i=1}^{r} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{r} a_i \cdot \underbrace{\partial\sigma_i}_{\uparrow} \end{aligned}$$

we view $\partial\sigma_i$ as a linear combination of $(n-1)$-simplices

(3) We refer to the sequence

$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2^{\mathrm{simp}}(X) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{simp}}(X) \xrightarrow{\partial_1} \mathrm{C}_0^{\mathrm{simp}}(X) \to 0$$

as the simplicial chain complex of X.

Remark. Note that it follows easily from the definitions that an n-chain μ is an n-cycle if and only if $\partial_n(\mu) = 0$.[1477] □

The following lemma is the crucial observation, which makes the whole theory come alive.

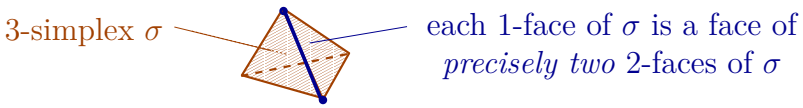
Lemma 142.5. (Simplicial–Double Boundary Lemma) Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X)$ of two consecutive boundary maps is the zero map.

[1477] Here the odd "odd" condition in the definition of the boundary of an n-chain in Definition 142.3 is absolutely crucial.

Sketch of proof. Since $\mathrm{C}_{n+1}^{\mathrm{simp}}(X)$ is a vector space with basis given by the $(n+1)$-simplices of X it suffices to prove that for each $(n+1)$-simplex of X we have $\partial_n(\partial_{n+1}(\sigma)) = 0$. Thus let σ be an $(n+1)$-simplex with vertex set $V = \{x_0, \ldots, x_{n+1}\}$. We compute[1478]

$$\begin{aligned}\partial_n(\partial_{n+1}(\sigma)) &= \partial_n\Big(\sum_{i=0}^{n+1} \operatorname{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, x_{n+1}\})\Big) \\ &= \sum_{i=0}^{n+1} \sum_{\substack{j=0 \\ i \neq j}}^{n+1} \operatorname{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, x_{n+1}\}) \underset{\uparrow}{=} 0.\end{aligned}$$

since each $(n-1)$-face appears twice and since we work with $\mathbb{Z}/2\mathbb{Z}$-coefficients



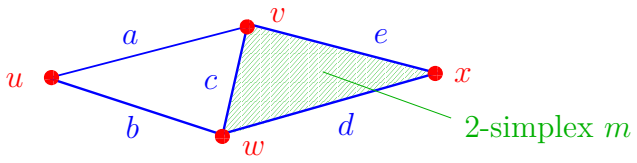
Another way of stating this argument is to say that each $(n-1)$-face of σ is the face of precisely two n-faces of σ. ■

The Simplicial–Double Boundary Lemma 142.5 allows us to introduce the following definition:

Definition 142.6. Let X be a simplicial complex. Furthermore let $n \in \mathbb{N}_0$. It follows immediately from the above Simplicial–Double Boundary Lemma 142.5 that we have the inclusion $\operatorname{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X)) \subseteq \ker(\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))$. Thus it makes sense to define the n-th simplicial homology group[1479] as follows:

$$\mathrm{H}_n^{\mathrm{simp}}(X) \coloneqq \frac{\ker(\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))}{\operatorname{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X))}.$$

Example. We consider the simplicial complex X that is shown below. It has four 0-simplices u, v, w, x, five 1-simplices a, b, c, d, e and one 2-simplex m.



One can easily verify that the corresponding simplicial chain complex is given as follows:

$$0 \to \underbrace{\mathbb{Z}/2\mathbb{Z} \cdot m}_{=\mathrm{C}_2^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \underbrace{\begin{matrix} \mathbb{Z}/2\mathbb{Z} \cdot a \\ \mathbb{Z}/2\mathbb{Z} \cdot b \\ \mathbb{Z}/2\mathbb{Z} \cdot c \\ \mathbb{Z}/2\mathbb{Z} \cdot d \\ \mathbb{Z}/2\mathbb{Z} \cdot e \end{matrix}}_{=\mathrm{C}_1^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}} \underbrace{\begin{matrix} \mathbb{Z}/2\mathbb{Z} \cdot u \\ \mathbb{Z}/2\mathbb{Z} \cdot v \\ \mathbb{Z}/2\mathbb{Z} \cdot w \\ \mathbb{Z}/2\mathbb{Z} \cdot x \end{matrix}}_{=\mathrm{C}_0^{\mathrm{simp}}(X)} \to 0.$$

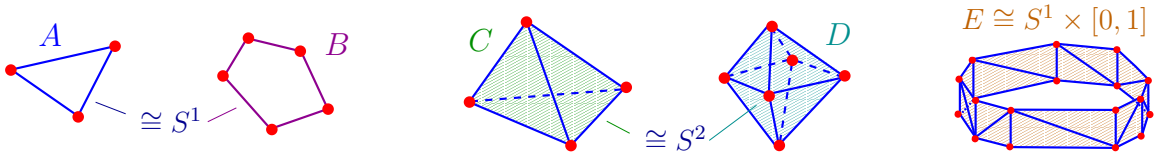
[1478] Here $\widehat{x_i}$ and $\widehat{x_j}$ means that we leave out this point.

[1479] We will introduce and study the simplicial homology groups in much greater detail in Chapter 176. In particular we will then see what needs to be done to work with $\mathbb{Z}$-coefficients instead of $\mathbb{Z}/2\mathbb{Z}$-coefficients.

An elementary, albeit rather painful calculation shows that

$$\mathrm{H}_k^{\mathrm{simp}}(X) \;\cong\; \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } k=0,\\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k=1,\\ 0, & \text{if } k\geq 2. \end{cases}$$ □

The previous example should convince the reader that simplicial homology groups can be computed, with enough patience, for any given simplicial complex. This is a good moment to return to some our previous examples. With sufficient energy one obtains the following simplicial homology groups:



$\mathrm{H}_0^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(A) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(B) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(C) = 0$
$\mathrm{H}_2^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$
$\mathrm{H}_3^{\mathrm{simp}}(C) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(D) = 0$
$\mathrm{H}_2^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$
$\mathrm{H}_3^{\mathrm{simp}}(D) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(E) = 0$

Using these examples we can already make a couple of educated guesses:

- The simplicial complexes A and B are homeomorphic and the corresponding simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(A)$ and $\mathrm{H}_k^{\mathrm{simp}}(B)$ are isomorphic. We see the same behavior for the simplicial complexes C and D which are also homeomorphic. This suggests that the isomorphism type of simplicial homology might only depend on the homeomorphism type of the simplicial complex.
- If we look at the simplicial homology groups of the simplicial complexes that are homeomorphic to S^1 and that are homeomorphic to S^2 we might spot a pattern, namely one could make the guess that for any[1480] $n \geq 1$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k^{\mathrm{simp}}(\text{simplicial complex homeomorphic to } S^n) \;\cong\; \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } k=0,n,\\ 0, & \text{otherwise.} \end{cases}$$

- The simplicial complexes $A \cong S^1$ and $E \cong S^1 \times [0,1]$ are homotopy equivalent. We notice that the simplicial homology groups are isomorphic. This suggests, on admittedly rather tenuous evidence, that the isomorphism type of simplicial homology might actually only depend on the homotopy type of the simplicial complex.

All this sounds very promising. But there are serious issues with the concept of simplicial homology:

- First of all it seems rather restrictive to work just with simplicial complexes. Even though many topological spaces are homeomorphic to simplicial complexes, already for simple topological spaces, like the torus $S^1 \times S^1$, it is quite painful and unnatural to represent them as a simplicial complex.
- For simplicial homology to be really useful as a topological tool one would need to show that simplicial homology groups depend only on the homeomorphism type (or better

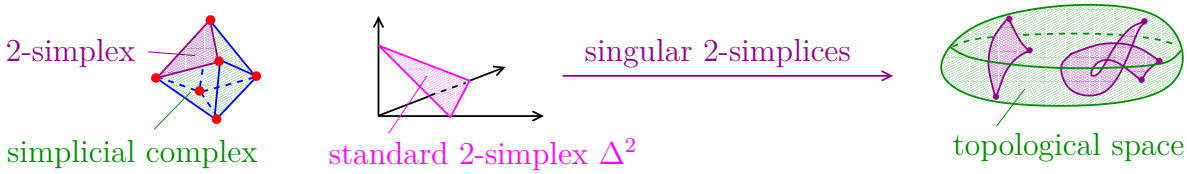
[1480] What about $n = 0$?

homotopy type) of the simplicial complex. The longer one thinks about it, the harder it seems to prove such a statement.

- When we studied fundamental groups we saw that in applications it was essential that fundamental groups are covariantly functorial, i.e. that continuous maps induce maps on fundamental groups. Given a continuous map between two simplicial complexes there is no reasonable way how such a map will induce a map on the corresponding simplicial homology groups.

Because of all these issues we have to go back to the drawing board and come up with a better concept. The key idea is as follows:

- In *simplicial* homology $\mathrm{H}_k^{\mathrm{simp}}(X)$ we consider only simplicial complexes X and we study the simplices contained in them.
- In *singular* homology $\mathrm{H}_k(X)$ we allow any topological space X and this time we take the drastic step of considering "singular simplices", namely we study all continuous maps of the standard simplices Δ^n to X, regardless of whether or not the maps are injective.



This new concept will eventually solve all of the technical problems. For example we will see that, almost by definition, singular homology groups are covariantly functorial under continuous maps and thus homeomorphism invariants. Furthermore it will also not be particularly difficult to show that singular homology groups depend only on the homotopy type of the topological space. The problem with singular homology is that in our attempt to solve the technical issues of simplicial homology we threw computability out of the window. We will have to work quite hard to calculate singular homology groups. But eventually we will see that singular homology encompasses simplicial homology and that it is the best of two worlds: it has very nice theoretical properties and highly computable.

142.3. Exercises for Chapter 142 .

Exercise 142.1.

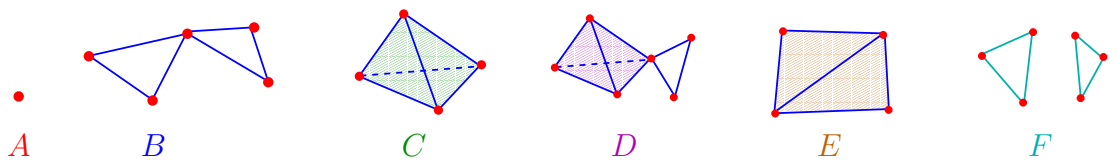
(a) What is a smallest (in terms of number of simplices) simplicial complex $A \subseteq \mathbb{R}^2$ that is homeomorphic to S^1 ?

(b) What is a smallest (in terms of number of simplices) simplicial complex $B \subseteq \mathbb{R}^3$ that is homeomorphic to S^2 ?

(c) Give an example of a simplicial complex C in $\mathbb{R}^4$ that is homeomorphic to S^3 and compute the simplicial homology groups of C.

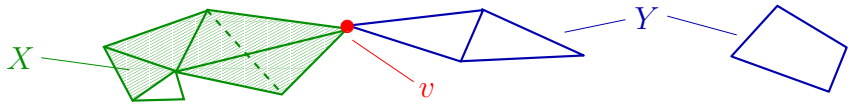
Exercise 142.2. We consider the simplicial complexes that are shown in the figure below. Determine their simplicial homology groups.

Exercise 142.3. Let X be a simplicial complex such that all simplices have dimension ≤ 1. Can you determine the simplicial homology groups of X in terms of known invariants,



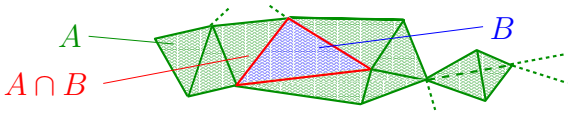
like the number of 0-simplices, number of 1-simplices, connectedness etc.?
Remark. Note that such a simplicial complex has a natural structure of a topological graph.

Exercise 142.4. Let Z be a simplicial complex and let $X, Y \subseteq Z$ be subcomplexes with $X \cup Y = Z$ and such that $X \cap Y$ is a vertex v. Try to determine the relationship between the simplicial homology groups of X, Y and Z.
Remark. Note that Z is homeomorphic to the wedge $(X, v) \vee (Y, v)$.



Exercise 142.5.
(a) How many $(n-1)$-faces does an n-simplex have?
(b) How many faces does an n-simplex have?

Exercise 142.6. Let X be a simplicial complex and suppose that we have a decomposition $X = A \cup B$ where A is a simplicial complex and where B is an n-simplex such that $A \cap B = \partial B$ and such that $A \cap B$ is a union of simplices of A. What is the relationship between the simplicial homology groups of A and B?
Hint. Note that $A \cap B = \partial B$ defines an $(n-1)$-chain in A. Distinguish the two cases that $A \cap B = \partial B$ represents the zero or a non-zero element of $\mathrm{H}_{n-1}(A)$.

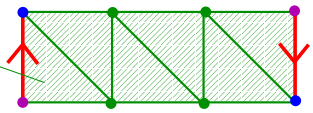


Exercise 142.7. Which finite sequences $A_0, \ldots, A_k$ of finite $\mathbb{Z}/2\mathbb{Z}$-vector spaces can be realized as the sequence of homology groups of a simplicial complex? More precisely, for which finite sequences $A_0, \ldots, A_k$ of finite $\mathbb{Z}/2\mathbb{Z}$-vector spaces does there exist a simplicial complex X such that for $i = 0, \ldots, k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X) \cong A_i$ and such that $\mathrm{H}_i^{\mathrm{simp}}(X) = 0$ for $i > k$?

Exercise 142.8. Let X be a simplicial complex. Try to determine $\mathrm{H}_0^{\mathrm{simp}}(X)$ in terms of other topological invariants.

Exercise 142.9. One can generalize the notion of a simplicial complex by gluing simplices along along faces via linear maps. This is not the right moment to worry about the very precise definitions, instead, in the figure we show a generalized simplicial complex that is homeomorphic to the Möbius band.
The notion of simplicial homology makes sense in this more general context.
(a) Find a generalized simplicial complex that is homeomorphic to the torus and compute its simplicial homology groups.

generalized simplicial complex that is homeomorphic to the Möbius band



(b) Find a generalized simplicial complex that is homeomorphic to the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ and compute its simplicial homology groups.

CHAPTER 143

The singular homology groups of a topological space

In the previous chatty chapter we gave a very short introduction to simplicial homology, we calculated a few examples and we discovered serious issues with the concept of simplicial homology. As indicated towards the end of the previous chapter the solution to these problems is to introduce a "homology theory" where we study "singular simplices". In this chapter we will follow up on this idea, we will introduce singular homology, we will see it is covariantly functorial and we will calculate singular homology in a few very basic examples. In the coming chapters we will then develop the theory of singular homology groups in its full glory.

This chapter is organized as follows:

- In Section 143.1 we will introduce algebraic chain complexes, homology groups of chain complex and chain maps.
- In Section 143.2 we will introduce the singular chain complex $(\mathrm{C}_*(X), \partial_*)$ of a topological space.
- In Section 143.3 we will introduce the singular homology groups $\mathrm{H}_k(X)$ of a topological space X and we will state a few basic properties. For example we will see that the homology groups of a topological space are the direct sums of the homology groups of its path-components.
- In Section 143.4 we will perform a few first basic calculations of homology groups. In particular we will calculate the 0-th homology groups of any topological space and we will calculate the homology groups of a singleton.
- In Section 143.5 we will study singular 1-chains in greater detail.
- In Section 143.6 we will provide exercises for Chapter 143.

In the next Chapter 144 we will see that homotopic maps define the same map on homology.

143.1. **Algebraic chain complexes.** In the previous chapter, in the discussion of simplicial homology, we implicitly briefly encountered the purely algebraic notion of a chain complex. This notion will play a prominent role in the discussion of homology and it thus useful to introduce this algebraic notion before we return to topology and before we get to the actual definition of singular homology.

Definition 143.1.

(1) An chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

of homomorphisms between abelian groups such that the composition of any two consecutive boundary maps is the zero map, i.e. for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. For $n \in \mathbb{Z}_{<0}$ we always set $C_n := 0$.

(2) Let (C_*, ∂_*) be a chain complex and let $n \in \mathbb{N}_0$. By definition of a chain complex we have $\operatorname{im}(\partial_{n+1}\colon C_{n+1} \to C_n) \subseteq \ker(\partial_n\colon C_n \to C_{n-1})$. Thus it makes sense to define the n-th homology group of the chain complex as follows:

$$\mathrm{H}_n(C_*) \coloneqq \frac{\ker(\partial_n\colon C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1}\colon C_{n+1} \to C_n)}.$$

(3) (a) We refer to elements in the kernel of $\partial_n\colon C_n \to C_{n-1}$ as cycles.
(b) We say that two elements $c, d \in C_n$ are homologous if there exists an $e \in C_{n+1}$ with $c - d = \partial_{n+1}e$.
(c) We say that a cycle $z \in C_n$ is null-homologous if z is homologous to the zero element of C_n, i.e. if $[z] = 0 \in \mathrm{H}_n(C_*)$.

(4) A chain complex (C_*, ∂_*) is called acyclic if all homology groups are zero.

On some occasions we will need the following generalization:

(5) A generalized chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \ldots$$

of homomorphisms between abelian groups such that for every $i \in \mathbb{Z}$ we have the equality $\partial_{i-1} \circ \partial_i = 0$. Basically all present and future definitions and results for chain complexes also make sense for generalized chain complexes. In most cases we will not explicitly state the generalization to generalized chain complexes.

As always we not only care about objects, but we also want to allow reasonable morphisms between the objects. This quest leads us to the following definition:

Definition 143.2. A chain map $f\colon C_* \to D_*$ between two chain complexes[1481] consists of a family $\{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ the following equality holds:[1482]

$$f_{n-1} \circ \partial_n = \partial_n \circ f_n.$$

Put differently the homomorphisms f_n have the property that the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \xrightarrow{\partial_{n+2}} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots
\end{array}$$

The above definitions naturally lead to the following definition:

Definition 143.3. We refer to category $\mathcal{ChCplx}$ that is given by

$$\operatorname{Ob}(\mathcal{ChCplx}) \coloneqq \text{all chain complexes,}$$

and with morphisms

$$\operatorname{Mor}(C_*, D_*) \coloneqq \text{all chain maps from } C_* \text{ to } D_*.$$

with the obvious composition of morphisms[1483] as the category of chain complexes.

[1481] As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying "let (C_*, ∂_*) be a chain complex" we say simply "let C_* be a chain complex".

[1482] More precisely, for any $c \in C_n$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $C_n \to C_{n-1}$, and on the right-hand side we consider the boundary map $D_n \to D_{n-1}$ which we also denote by ∂_n. Usually it is obvious from the context which boundary map we consider.

Now we can formulate and prove the following lemma:

Lemma 143.4. (Chain Complex-H_*-Functor Lemma)

(1) If $f\colon C_* \to D_*$ is a chain map between chain complexes C_* and D_*, then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined homomorphism.

(2) Given $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon C_* \to D_*) & \mapsto & (f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)) \end{array}$$

define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Definition 143.5. In the setting of the Chain Complex-H_*-Functor Lemma 143.4 (1) we refer to the corresponding map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ as the induced map on homology.

Proof.

(1) Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\substack{\uparrow \\ \text{by Definition 143.2} \\ \text{of a chain map}}}{=} f_{n-1}(\partial_n(c)) \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} f_n(0) \underset{\substack{\uparrow \\ \text{since } f_n \text{ is a} \\ \text{homomorphism}}}{=} 0.$$

Now let $c, d \in C_n$ be two cycles which represent the same element in $\mathrm{H}_n(C_*)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] = [f_n(c-d)] \underset{\substack{\uparrow \\ \text{since } c,d \text{ are homologous} \\ \text{there exists an } e \in C_{n+1} \\ \text{with } \partial_{n+1}(e) = c-d}}{=} [f_n(\partial_{n+1}(e))] \underset{\substack{\uparrow \\ \text{by Definition 143.2} \\ \text{of a chain map}}}{=} [\partial_{n+1}(f_{n+1}(e))] \underset{\substack{\uparrow \\ \text{by Definition 143.1} \\ \text{of homology}}}{=} 0 \in \mathrm{H}_n(D).$$

This shows that the map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is well-defined. It follows easily from the definitions that this map is actually a homomorphism.

(2) The statement follows easily from (1) and the definitions. ∎

At this stage there is only one little lemma on chain complexes that we want to state. Before we can do so, we recall Definition 97.1:

Definition 143.6. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) We define the product of the X_s to be the abelian group

$$\prod_{s\in S} X_s := \left\{ f\colon S \to \bigcup_{s\in S} X_s \;\middle|\; \text{for every } s \in S \text{ we have } f(s) \in X_s \right\},$$

where the group structure is given by the obvious "componentwise addition".

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s := \left\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \;\middle|\; \text{all but finitely many } g_s \text{ are trivial} \right\}.$$

[1483] Here we use the rather elementary observation that the composition of two chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ defines a chain map $g \circ f\colon C_* \to E_*$.

(3) Given $t \in S$ we refer to

$$\begin{array}{rcl} \iota_t \colon X_t & \to & \bigoplus_{s\in S} X_s \\ x & \mapsto & \left(\begin{array}{ccl} S & \to & \bigcup_{s\in S} X_s \\ s & \mapsto & \begin{cases} x, & \text{if } s = t, \\ 0_{X_t}, & \text{if } s \neq t. \end{cases} \end{array} \right) \end{array}$$

as the natural homomorphism.

In the Coproduct-of-Abelian Groups Lemma 97.2 we showed that the direct sum with the above natural homomorphisms is the coproduct of $\{X_s\}_{s\in S}$ in the category $\mathcal{AbGroup}$ of abelian groups.

Basically the same way as we defined the direct sum of a family of abelian groups we can also define the direct sum of a family of chain complexes. Now we can formulate the following lemma:

Lemma 143.7. (Direct Sum-H$_*$-Lemma) Let $\{C_a\}_{a\in A}$ be a family of chain complexes. For every $n \in \mathbb{N}_0$ the natural maps $C_a \to \bigoplus_{a\in A} C_a$ induce an isomorphism

$$\begin{array}{rcl} \bigoplus_{a\in A} \mathrm{H}_n(C_a) & \xrightarrow{\cong} & \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a \Big) \\ ([c_a])_{a\in A} & \mapsto & [(c_a)_{a\in A}]. \end{array}$$

Proof. One can easily verify that the map

$$\begin{array}{rcl} \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a \Big) & \xrightarrow{\cong} & \bigoplus_{a\in A} \mathrm{H}_n(C_a) \\ [(c_a)_{a\in A}] & \mapsto & ([c_a])_{a\in A} \end{array}$$

is an inverse to the given map. ∎

We could easily fill the remainder of these notes with the general theory of chain complexes. We will not do so, but instead we will develop the theory of chain complexes along the way, as we develop the theory of the singular chain complexes and singular homology.

143.2. **The singular chain complex.** With this section we return to topology. As a first step towards the definition of singular homology we need to introduce the notion of a (standard) n-simplex:

Definition 143.8. Let $n \in \mathbb{N}_0$.

(1) Given $i \in \{0, \dots, n\}$ we write

$$\mathfrak{v}_i \;\coloneqq\; (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th coordinate}}}{1}, 0, \dots, 0).$$

(2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of $\mathfrak{v}_1, \dots, \mathfrak{v}_{n+1} \in \mathbb{R}^{n+1}$, i.e. to

$$\Delta^n \;\coloneqq\; \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \cdots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

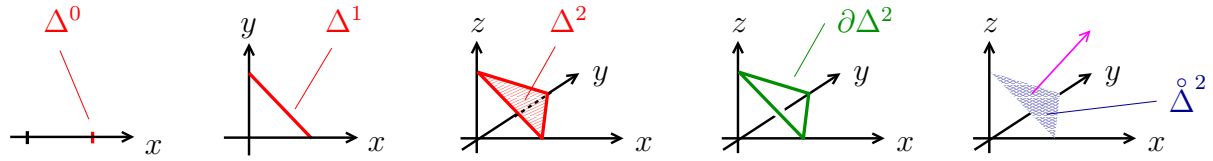
as the standard n-simplex.[1484]

(3) Given the standard n-simplex Δ^n we define its boundary as[1485]

$$\partial\Delta^n \;\coloneqq\; \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

(4) We write $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$.

Sometimes it will be helpful to view $\mathring{\Delta}^n$ as a smooth manifold where a smooth atlas for $\mathring{\Delta}^n$ is given by projection to the first n coordinates. We equip the smooth manifold $\mathring{\Delta}^n$ with the orientation where for each $P \in \mathring{\Delta}^n$ a basis $v_1, \ldots, v_n$ of $\widetilde{\mathrm{T}}_P\mathring{\Delta}^n$ is positive if the vectors $\{(1, \ldots, 1), v_1, \ldots, v_n\}$ are a positive basis for $\mathbb{R}^{n+1}$.



For the record we state the following elementary lemma:

Lemma 143.9. (Δ^n-Homeomorphism Lemma) Let $n \in \mathbb{N}_0$.

(1) We set

$$\triangle_n := \left\{(x_1, \ldots, x_n) \in [0,1]^n \,\middle|\, \sum_{i=1}^{n} x_i \in [0,1]\right\}$$

and refer to it as the planar n-simplex. The maps[1486]

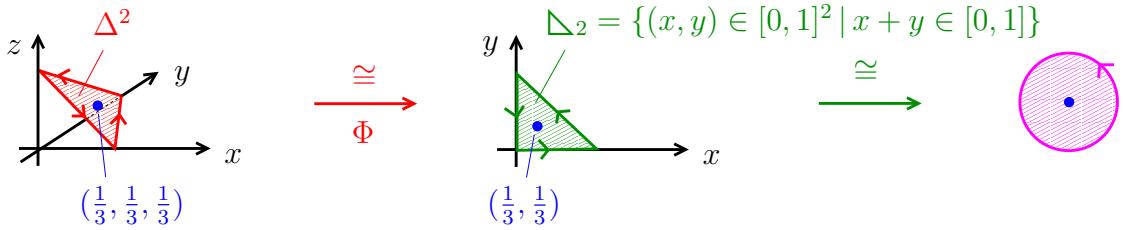
$$\begin{array}{rcl} \Phi\colon \Delta^n & \to & \triangle_n \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \triangle_n & \to & \Delta^n \\ (x_1, \ldots, x_n) & \mapsto & \left(x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i\right) \end{array}$$

are homeomorphisms that are inverses of one another.

(2) The map

$$\Delta^n \xrightarrow{\Phi} \triangle_n \xrightarrow{\text{the homeomorphism } \Xi \text{ obtained from the Convex-to-Ball Proposition 4.28 (2) applied to } \triangle_n \text{ and } Q = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})} \overline{B}^n$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.



Proof.

(1) Note that Φ extends "by the same formula" to a map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ which is evidently continuous. Hence the restriction to Δ^n is also continuous. Now that this is settled we see that the remaining statements are basically trivial.

[1484]Recall that in Exercise 4.46 (a) we showed that the standard n-simplex Δ^n is the convex hull of $e_i \in \mathbb{R}^{n+1}$, $i = 1, \ldots, n+1$. In particular the standard n-simplex is an n-simplex in the sense of Definition 142.1.

[1485]The definition of the "boundary" of the standard n-simplex Δ^n is the "obvious one". Nonetheless note that if we had taken the original definition of the boundary of a subset A in some subspace X as introduced in Definition 3.21 and applied it to $\Delta^n \subseteq \mathbb{R}^{n+1}$, then the boundary of Δ^n in $\mathbb{R}^{n+1}$ would be all of Δ^n.

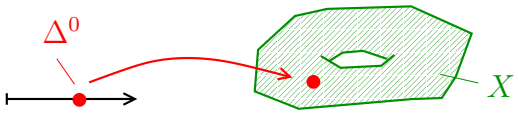
[1486]Note that Φ and Ψ restrict to diffeomorphisms of the "interiors". If one goes carefully through all orientation conventions, then one sees that Φ and Ψ are orientation-preserving if and only if n is even.

(2) The statement follows easily from (1), the observation that $\triangle_n$ is compact (which can be shown easily using the Heine-Borel Theorem 3.44) and the Convex-to-Ball Proposition 4.28 (2). ■

Definition 143.10. Let X be a topological space and let $n \in \mathbb{N}_0$. A singular n-simplex is a continuous map[1487] $\sigma\colon \Delta^n \to X$.

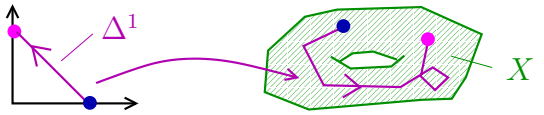
Example. Let X be a topological space.

(1) A singular 0-simplex is a continuous map from Δ^0 to X. Note that $\Delta^0 = \{1\} \subseteq \mathbb{R}$. Thus a map $\Delta^0 = \{1\} \to X$ is uniquely determined by a point in X. Given a point $P \in X$ we frequently denote by P also the singular 0-simplex $\Delta^0 \to X$ that is given by sending the single point in Δ^0 to P.



a singular 0-simplex in X is essentially the same as a point in X

(2) A singular 1-simplex is a continuous map $\Delta^1 \to X$.



a singular 1-simplex in X is essentially the same as a path in X

Since the map $[0,1] \to \Delta^1$ given by $t \mapsto (1-t, t)$ is a homeomorphism we see that a singular 1-simplex is essentially the same as a path in X. □

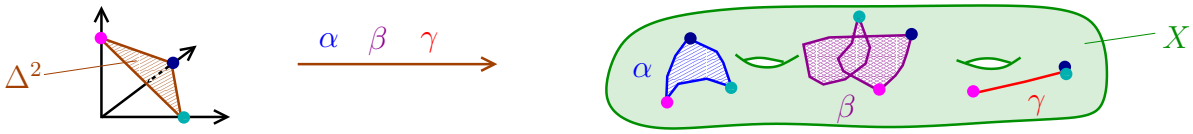
Definition 143.11. Let X be a topological space and let $n \in \mathbb{N}_0$.

(1) We define the n-th singular chain group as follows:[1488]

$$\mathrm{C}_n(X) \coloneqq \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $\mathrm{C}_n(X) \coloneqq 0$.

(2) We refer to an element in $\mathrm{C}_n(X)$ as a singular n-chain in X.



$-2\alpha + \beta + 3\gamma$ is a singular 2-chain

Motivated by the discussion of the last chapter we now want to find suitable boundary maps between the singular chain groups $\mathrm{C}_n(X)$. In the simplicial chain complex our generators were simplices and in Definition 142.4 we defined the boundary of a simplex to be a linear combination of codimension-one faces of the simplex. In our current setting generators

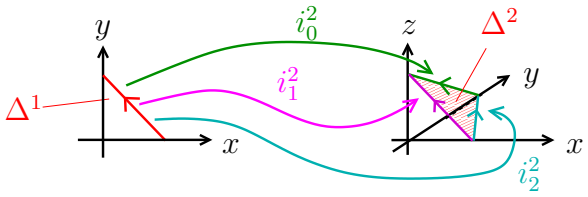
[1487] In principle, given a map $\sigma\colon \Delta^n \to X$ to some topological space X we need to show that this map is actually continuous. If one really feels compelled to give an argument, then often it is more convenient to show the equivalent statement that the map $\sigma \circ \Psi\colon \triangle_n \to X$ is continuous, where $\Psi\colon \triangle_n \to \Delta^n$ is the homeomorphism given in the Δ^n-Homeomorphism Lemma 143.9 (1).

[1488] Let X be a topological space. By the discussion in Definition 97.7 we can view every element in $\mathrm{C}_n(X)$, i.e. every singular n-chain, as a linear combination $a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k$ where $a_1, \ldots, a_k \in \mathbb{Z}$ and where $\sigma_1, \ldots, \sigma_k$ are singular n-simplices.

are singular simplices, i.e. continuous maps $\Delta^n \to X$. To such a singular simplex we now have to associate a singular $(n-1)$-simplex, i.e. a linear combination of continuous maps $\Delta^{n-1} \to X$. To do so it is convenient to introduce the following definition:

Definition 143.12. Let $n \in \mathbb{N}_0$. For $j \in \{0, \dots, n\}$ we define the j-th face map as follows:[14891490]

$$\begin{array}{rcl} i_j^n \colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{array}$$



Now we can define the promised boundary maps $\mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$.

Definition 143.13. Let X be a topological space and let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. The boundary $\partial_n \sigma \in \mathrm{C}_{n-1}(X)$ of σ is defined as[1491]

$$\partial_n(\sigma) \coloneqq \sum_{j=0}^{n} (-1)^j \cdot \underbrace{\sigma \circ i_j^n}_{\text{map } \Delta^{n-1} \to X}.$$

By the Free Abelian Group–Existence Lemma 97.8 the map $\sigma \mapsto \partial_n \sigma$ extends uniquely to a linear map

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n-1}(X) \\ \sum_{r=1}^{k} a_r \cdot \sigma_r & \mapsto & \sum_{r=1}^{k} a_r \cdot \partial_n(\sigma_r), \end{array}$$

which we refer to as the n-th boundary map. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

Examples.

(1) For any singular 1-simplex $\sigma \colon \Delta^1 \to X$ we have[1492]

$$\partial_1(\sigma) = (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 = \sigma(0,1) - \sigma(1,0) \in \mathrm{C}_0(X).$$

Recall that a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$.

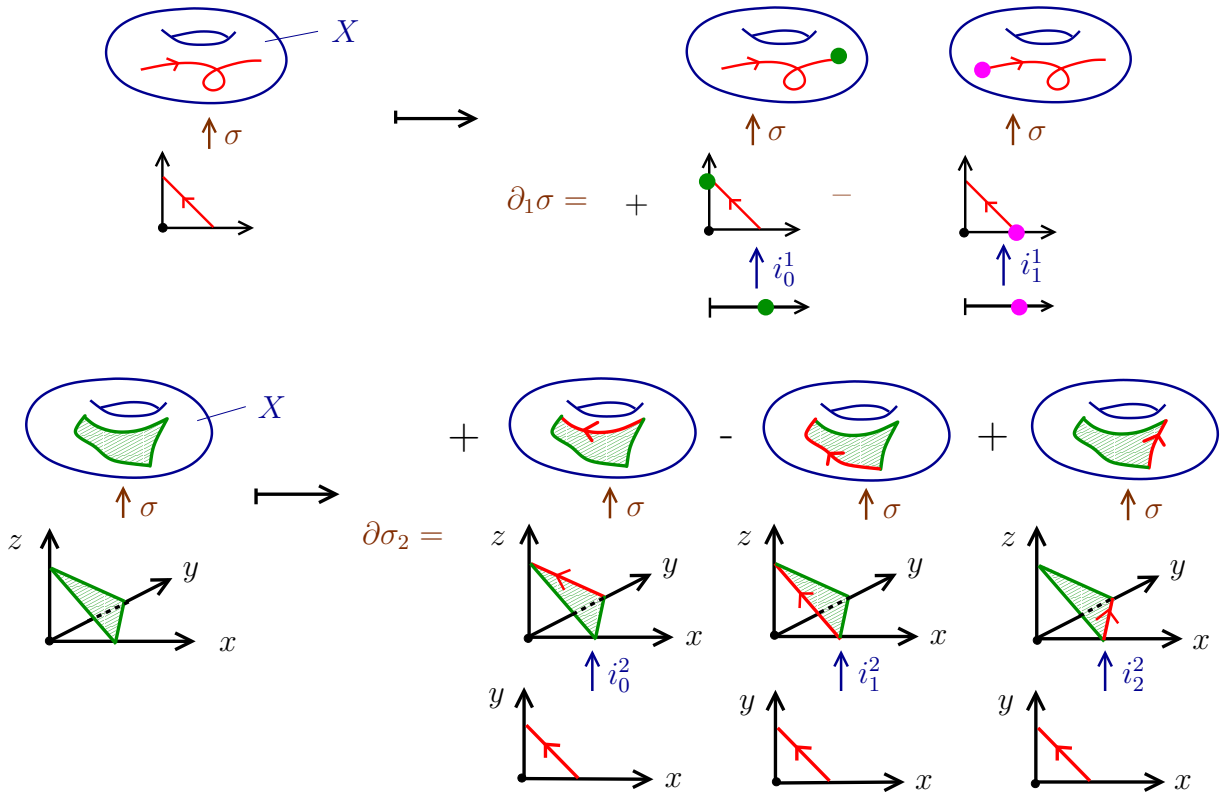
(2) In the figure below we sketch the boundary of a singular 2-simplex.

[1490] One can of course easily verify that this map is continuous.

[1490] The image of a face map is often called an $(n-1)$-*dimensional face* of the standard simplex. The boundary $\partial\Delta^n$ of the standard n-simplex is the union of all $(n-1)$-dimensional faces of Δ^n.

[1491] The sign $(-1)^j$ is surely somewhat unexpected. As we will see shortly, it is necessary to ensure that the composition of two consecutive boundary maps is indeed zero.

[1492] Here we again use the convention that given a point $x \in X$ we denote by x also the corresponding singular 0-simplex.



The following proposition, which is an analog of the Simplicial–Double Boundary Lemma 142.5, shows that these boundary maps have the fundamental property that the composition of two consecutive boundary maps is the zero map.

Proposition 143.14. (Double Boundary Proposition) Let X be a topological space. For each $n \in \mathbb{N}_0$ the composition

$$\partial_{n-1} \circ \partial_n \colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n-2}(X)$$

is the zero map.

Proof. Let $n \in \mathbb{N}_0$. It suffices to show that for every singular n-simplex $\sigma \colon \Delta^n \to X$ we have $\partial_{n-1}(\partial_n(\sigma)) = 0$. Thus let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &\overset{\text{Definition 143.8 of } \partial_n}{=} \partial_{n-1}\left(\sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n\right) \overset{\text{Definition 143.8 of } \partial_{n-1}}{=} \sum_{k=0}^{n-1}(-1)^k \cdot \sum_{j=0}^{n}(-1)^j \cdot (\sigma \circ i_j^n) \circ i_k^{n-1} \\ &= \sum_{0=k<j\leq n}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\leq k\leq n-1}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) =: (*).\end{aligned}$$

Now we want to show that each expression $\sigma \circ (i_j^n \circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j \leq k$ we have[1493]

$$i_j^n \circ i_k^{n-1} \;=\; i_{k+1}^n \circ i_j^{n-1}.$$

[1493]Indeed, if $j \leq k$, then the zero-entries of $(i_j^n \circ i_k^{n-1})(t_0, \ldots, t_{n-1})$ are the entries j and $k+1$.

Now we continue with the above calculation:

$$(*) \underset{\substack{\uparrow \\ \text{above calculation} \\ \text{of } i^n_j \circ i^{n-1}_k}}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i^n_j \circ i^{n-1}_k) + \sum_{0=j\leq k\leq n-1} (-1)^{k+j} \cdot \sigma \circ (i^n_{k+1} \circ i^{n-1}_j)$$

$$\underset{\substack{\uparrow \\ \text{substitution } l=k+1}}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i^n_j \circ i^{n-1}_k) + \sum_{0=j<l\leq n} (-1)^{l+j-1} \cdot \sigma \circ (i^n_l \circ i^{n-1}_j) \underset{\substack{\uparrow \\ \text{all summands left} \\ \text{and right cancel}}}{=} 0. \quad \blacksquare$$

The Double Boundary Proposition 143.14 says that the map $X \mapsto (\mathrm{C}_*(X), \partial_*)$ assigns to each topological space a chain complex. Of course, as experienced algebraic topologists, we want this assignment to be functorial. This goal leads us to the following definition:

Definition 143.15. Let $f\colon X \to Y$ be a continuous map between topological spaces. For each $n \in \mathbb{N}_0$ we refer to

$$\begin{array}{rcl} f_*\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(Y) \\ \sum\limits_{i=1}^{k} a_i \cdot \underbrace{\sigma_i}_{\substack{\uparrow \\ \text{map } \Delta^n \to X}} & \mapsto & \sum\limits_{i=1}^{k} a_i \cdot \underbrace{(f \circ \sigma_i)}_{\substack{\uparrow \\ \text{map } \Delta^n \to Y}}. \end{array}$$

as the induced chain map.



The above Definition 143.15 leads to the following lemma:

Lemma 143.16. (C_*–Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. The induced maps $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ form a chain map, i.e. given any $c \in \mathrm{C}_n(X)$ we have

$$f_*(\partial_n(c)) \;=\; \partial_n(f_*(c)) \;\in\; \mathrm{C}_n(Y).$$

(2) The maps

$$\text{topological space } X \;\mapsto\; (\mathrm{C}_*(X), \partial_*)$$

and

$$(f\colon X \to Y) \;\mapsto\; (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y))$$

define a covariant functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{C}h\mathcal{C}pl\mathcal{X}$ of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have the following equalities

$$\begin{aligned} f_*(\partial_n(\sigma)) &\overset{\text{Definition 143.13 of } \partial_n}{=} f_*\Big(\sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n\Big) \overset{\text{Definition 143.15 of } f_*}{=} \sum_{j=0}^{n} (-1)^j \cdot f \circ (\sigma \circ i_j^n) \\ &= \sum_{j=0}^{n} (-1)^j \cdot \underbrace{(f \circ \sigma)}_{=f_*\sigma} \circ i_j^n \overset{\text{Definition 143.13 of } \partial_n}{=} \partial_n(f_*\sigma). \end{aligned}$$

Since the singular n-simplices generate $\mathrm{C}_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) It follows immediately from the definitions that for two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Z)$. With this observation Statement (2) is an almost immediate consequence of Statement (1). ■

For any functor it is interesting to know whether or not products, coproducts and more generally limits and colimits, etc. are preserved. The following basic lemma (together with the Component Lemma 4.48 (5)) shows that the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves coproducts.

Lemma 143.17. (Path-Component–C_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. The map

$$\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_*(X_a) \to \mathrm{C}_*(X)$$

is an isomorphism of chain complexes.

Proof. Evidently the standard simplex Δ^n is path-connected. Thus it follows from the Image-Preimage-Component Lemma 4.50 that the image of a continuous map $\sigma\colon \Delta^n \to X$ is contained in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{set of singular } n\text{-simplices in } X_a = \text{set of singular } n\text{-simplices in } X.$$

It follows that the map

$$\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_n(X_a) \to \mathrm{C}_n(X)$$

is an isomorphism of free abelian groups. It follows immediately from the definitions that this a chain map. ■

143.3. **Definition of the singular homology groups.** It follows from the Double Boundary Proposition 143.14 that given a topological space X the sequence $(\mathrm{C}_*(X), \partial_*)$ is a chain complex. This allows us to make the following definition, which is arguably one of the most influential and important definitions of algebraic topology.

Definition 143.18. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th singular homology group of X as the n-th homology group of the singular chain complex. In other words, we set

$$\mathrm{H}_n(X) := \mathrm{H}_n(\mathrm{C}_*(X)) = \frac{\ker(\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X))}{\mathrm{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}(X) \to \mathrm{C}_n(X))}.$$

For $n \in \mathbb{Z}_{<0}$ we extend this definition to $\mathrm{H}_n(X) := 0$. Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".[1494]

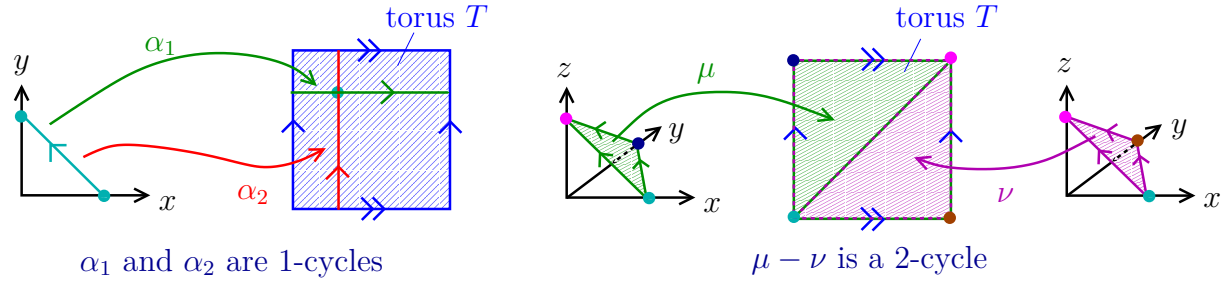
Example. There is only one example that we can do immediately: For the empty topological space $\varnothing$ we have for all $n \in \mathbb{N}_0$ that $\mathrm{C}_n(\varnothing) = 0$. It follows that for all $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(\varnothing) = 0$. □

The following definitions are just adaptations of Definition 143.1.

Definition 143.19. Let X be a topological space.

(1) We say that $c \in \mathrm{C}_n(X)$ is a singular n-cycle, or short n-cycle, if $\partial_n c = 0$.
(2) We say that two singular n-chains c and d are homologous if there exists a singular $(n+1)$-chain $e \in \mathrm{C}_{n+1}(X)$ with $\partial_{n+1} e = c - d$.
(3) An n-cycle c is called null-homologous, if $[c] = 0 \in \mathrm{H}_n(X)$.

Example. In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$ together with two singular 1-simplices α_1 and α_2 and together with two singular 2-simplices μ and ν.



It follows easily from the definitions that α_1 and α_2 are 1-cycles. It is a good exercise to verify that $\mu - \nu$ is a 2-cycle. With a little bit of practice it is fairly easy to write down cycles, but usually it is very hard to show that a cycle represents a non-trivial element in homology. For example, with our present knowledge it is impossible though to determine whether the cycles $\alpha_1, \alpha_2, \mu - \nu$ represent non-trivial homology classes. □

The following lemma is so fundamental that we will use it constantly without referring to it.

Lemma 143.20. (H_n–Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. The map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array}$$

is a well-defined homomorphism.

[1494] In the previous chapter we already saw a basic version of simplicial homology. Later on, in Chapters 155 and 176, we will also introduce "cellular homology" and we will introduce "simplicial homology" in much greater generality. At that point it will be useful to remember that our present homology groups have a more precise name, namely singular homology groups.

(2) For each $n \in \mathbb{N}_0$ the map

$$X \mapsto \mathrm{H}_n(X)$$

and the map

$$(f\colon X \to Y) \mapsto \begin{pmatrix} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{pmatrix}$$

define a covariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups.

Example. It follows from the H_n–Functor Lemma 143.20 together with the elementary Isomorphism Preservation Lemma 5.26 that every homeomorphism induces an isomorphism of homology groups. □

Proof.

(1) This statement follows from the C_*–Functor Lemma 143.16 (1) together with the Chain Complex-H_*-Functor Lemma 143.4 (1).

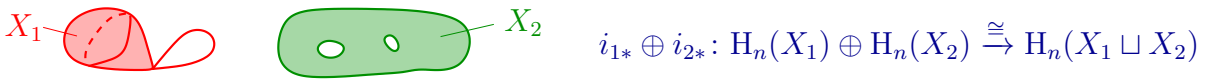
(2) This statement follows from the combination of (1), the C_*–Functor Lemma 143.16 (2) and the Chain Complex-H_*-Functor Lemma 143.4 (2). ■

Again the question is how singular homology groups behave under coproducts, products, etc. At this stage we have the following very long hanging fruit, which, together with the Component Lemma 4.48 (5), implies in particular that homology groups preserve coproducts.

Lemma 143.21. (Path-Component–H_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. For each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A} \mathrm{H}_n(X_a) \to \mathrm{H}_n(X)$$

is an isomorphism.



Proof. This lemma is an immediate consequence of the Path-Component–C_*–Lemma 143.17 and the Direct Sum-H_*-Lemma 143.7. ■

143.4. **First calculations of homology groups.** Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will not be finitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the most basic calculation, which often gets used implicitly without referring to it:

Lemma 143.22. (Empty Space-H_*-Lemma) For the empty topological space $\varnothing$ we have $\mathrm{H}_k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$.

Proof. This lemma follows immediately from the observation that in this case we have $\mathrm{C}_k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$. ■

We continue with the somewhat more interesting calculation of the 0-th homology of a path-connected topological space. The key to the calculation is the following definition:

Definition 143.23. Given a topological space X we refer to the homomorphism

$$\begin{array}{rcl} \epsilon_X \colon \mathrm{C}_0(X) & \to & \mathbb{Z} \\ \sum_{i=1}^{k} a_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} a_i \end{array}$$

as the augmentation homomorphism.

The following lemma shows in particular that the augmentation map is a neat addition to the singular chain complex of a topological space.

Lemma 143.24. (Augmentation Lemma) Let X be a topological space.

(1) The map $\epsilon_X \circ \partial_1 \colon \mathrm{C}_1(X) \to \mathbb{Z}$ is the zero map.

(2) The map

$$\begin{array}{rcl} \epsilon_X \colon \mathrm{H}_0(X) & \to & \mathbb{Z} \\ [\sigma] & \mapsto & \epsilon_X(\sigma) \end{array}$$

is well-defined. In the following we refer to this map also as the augmentation map.

(3) The augmentation map is natural in the sense that for any continuous map $f \colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{f_*} & \mathrm{H}_0(Y). \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

Proof. Let X be a topological space.

(1) Since the abelian group $\mathrm{C}_1(X)$ is generated by singular 1-simplices it suffices to prove that for each singular 1-simplex $\sigma \colon \Delta^1 \to X$ we have $(\epsilon_X \circ \partial_1)(\sigma) = 0$. But this is the case since

$$\epsilon_X(\partial_1(\sigma)) \underset{\substack{\uparrow \\ \text{see the discussion} \\ \text{on page 2751}}}{=} \epsilon_X(\sigma(0,1) - \sigma(1,0)) \underset{\substack{\uparrow \\ \text{by Definition 143.23 of } \epsilon_X}}{=} 1 - 1 = 0.$$

(2) If we have $\sigma, \tau \in \mathrm{C}_0(X)$ with $[\sigma] = [\tau] \in \mathrm{H}_0(X)$, then by definition of $\mathrm{H}_0(X)$ there exists a $\mu \in \mathrm{C}_1(X)$ with $\partial_1(\mu) = \sigma - \tau$. We now see that

$$\epsilon_X(\sigma) = \epsilon_X(\tau + \partial_1(\mu)) = \epsilon_X(\tau) + \underbrace{\epsilon_X(\partial_1(\mu))}_{= 0 \text{ by (1)}} = \epsilon_X(\tau).$$

(3) This statement follows easily from the definitions. ∎

Using the augmentation map we can formulate the following proposition:

Proposition 143.25. (H_0-Proposition) For any path-connected non-empty topological space X the augmentation map

$$\epsilon_X \colon \mathrm{H}_0(X) \to \mathbb{Z}$$

is a natural isomorphism.[1495] In fact given any $P \in X$ the inverse is given by the map

$$\begin{array}{rcl} \iota \colon \mathbb{Z} & \to & \mathrm{H}_0(X) \\ n & \mapsto & n \cdot [P]. \end{array}$$

Example. Let X be a topological space. As in Definition 4.43 we denote by $\pi_0(X)$ the set of path-components of X. It follows from the H_0-Proposition 143.25 together with the Path-Component–H_*–Lemma 143.21 that we have a natural isomorphism

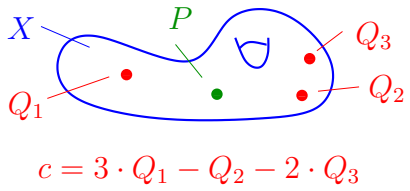
$$\mathrm{H}_0(X) \xrightarrow{\cong} \mathbb{Z}^{(\pi_0(X))}.$$

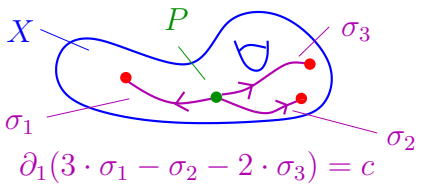
In particular, if X consists of n path-components with $n \in \mathbb{N}_0$, then we have an isomorphism

$$\mathrm{H}_0(X) \xrightarrow{\cong} \mathbb{Z}^n. \qquad \square$$

Proof. Let X be a path-connected non-empty topological space. The naturality statement is taken care of by the Augmentation Lemma 143.24 (3). Now let P be a point in X (which exists, since X is non-empty). It is clear that $\epsilon_X \circ \iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ_X is surjective. It remains to show that ϵ_X is injective.

Thus let $c \in \mathrm{C}_0(X)$ with $\epsilon_X(c) = 0$. By definition of $\mathrm{C}_0(X)$ we can find points $Q_1, \dots, Q_k$ in X and $a_1, \dots, a_k \in \mathbb{Z}$ such that $c = \sum_{i=1}^{k} a_i \cdot Q_i$. Since X is path-connected we can find, for $i = 1, \dots, k$, a singular 1-simplex σ_i with $\sigma_i((0,1)) = Q_i$ and $\sigma_i((1,0)) = P$.





Then we have

$$\partial_1\Big(\sum_{i=1}^{k} a_i \cdot \sigma_i\Big) = \sum_{i=1}^{k} a_i \cdot \partial_1\sigma_i \underset{\uparrow}{=} \sum_{i=1}^{k} a_i \cdot (Q_i - P) = \sum_{i=1}^{k} a_i \cdot Q_i - \underbrace{\Big(\sum_{i=1}^{k} a_i\Big)}_{=\epsilon_X(c)=0} \cdot P = \underbrace{\sum_{i=1}^{k} a_i \cdot Q_i}_{=c}.$$

see the discussion on page 2751

This implies that $[c] = 0 \in \mathrm{H}_0(X)$. Thus we have shown that ϵ_X is injective. ■

We continue with another elementary but essential calculation.

Lemma 143.26. (Singleton-H_*-Lemma) If X is a singleton, i.e. if X is a topological space which consists of a single point, then

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. Let $X = \{\star\}$ be a topological space consisting of a single point. Given $n \in \mathbb{N}_0$ let $\sigma_n \colon \Delta^n \to \{\star\}$ be the unique map. We calculate that

$$\partial_n(\sigma_n) \underset{\uparrow}{=} \sum_{j=0}^{n} (-1)^j \cdot \underbrace{\sigma_n \circ i_j^n}_{=\sigma_{n-1}} = \sigma_{n-1} \cdot \sum_{j=0}^{n} (-1)^j = \begin{cases} \sigma_{n-1}, & \text{if } n \text{ even}, \\ 0, & \text{if } n \text{ odd}. \end{cases}$$

see Definition 143.13 of the boundary map ∂_n

[1495] Here "natural means that given any continuous map $f \colon X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{f_*} & \mathrm{H}_0(Y). \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

We see that the singular chain complex is of the form

$$\dots \mathbb{Z}\cdot\sigma_4 \xrightarrow{\cong} \mathbb{Z}\cdot\sigma_3 \xrightarrow{=0} \mathbb{Z}\cdot\sigma_2 \xrightarrow{\cong} \mathbb{Z}\cdot\sigma_1 \xrightarrow{=0} \mathbb{Z}\cdot\sigma_0 \to 0.$$

The lemma follows almost immediately from this observation. ■

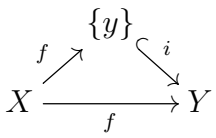
Example.

(1) One of our goals is to calculate the homology groups of the spheres. Now we can at least calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, it follows from the Singleton-H_*-Lemma 143.26 and the Path-Component–H_*–Lemma 143.21 that

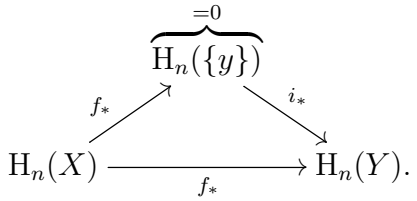
$$\mathrm{H}_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Later in the proof of the Sphere-H_*-Proposition 149.4 we will cleverly reduce the computation of the homology groups of all other spheres to this calculation.

(2) Let X and Y be non-empty topological spaces. Let $f\colon X \to Y$ be a continuous map. We suppose that f is constant, i.e. we suppose that there exists a $y \in Y$ such that $f(X) = \{y\}$. We denote by $i\colon \{y\} \to Y$ the inclusion. Let $n \in \mathbb{N}$. We consider the commutative diagram



which, by the functoriality of homology, gives rise to the commutative diagram



By the Singleton-H_*-Lemma 143.26 we know that $\mathrm{H}_n(\{y\}) = 0$. Thus we see that $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ factors through the trivial group, i.e. the map is the trivial map. On many occasions we will make use of this observation without referring to this discussion. □

The calculation of $\mathrm{H}_n(\varnothing)$ on page 2755, of $\mathrm{H}_0(X)$ in the H_0-Proposition 143.25 and the calculation of the homology of a point in the Singleton-H_*-Lemma 143.26 are the only calculations of homology groups of path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder, and this task will keep us busy for many chapters.

143.5. **Singular 1-chains.** It follows from the H_0-Proposition 143.25 together with the Direct Sum-H_*-Lemma 143.7 that we have no troubles calculating 0-th homology groups. Unfortunately, going up by one dimension, things become already much harder. At the moment we do not have the tools to calculate the homology groups in dimension 1. But in this last section we will prove one result on singular 1-chains which will come in handy on numerous occasions. To formulate our lemma we need the following notation:

Notation 143.27. Let X be a topological space. Given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) = \sigma(1-t, t)$. Informally speaking $\overline{\sigma}$ is given by reversing the orientation of σ.

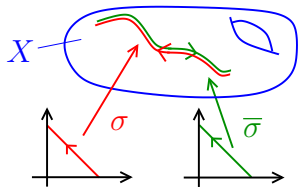
Lemma 143.28. (Singular 1-Simplex Lemma) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.

(1) σ is a cycle if and only if $\sigma((0,1)) = \sigma((1,0))$.
(2) The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.
(3) Let $s \in (0,1)$. We consider the singular 1-simplices[1496]

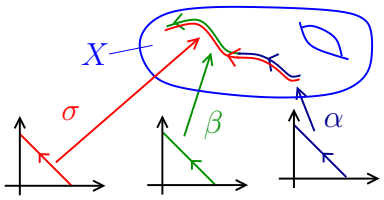
$$\begin{array}{rcl} \alpha\colon \Delta^1 & \to & X \\ (t,1-t) & \mapsto & \sigma(st,1-st) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \Delta^1 & \to & X \\ (t,1-t) & \mapsto & \sigma(s+t(1-s),1-(s+t(1-s))). \end{array}$$

The singular 1-chains σ and $\alpha+\beta$ are homologous.

The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.



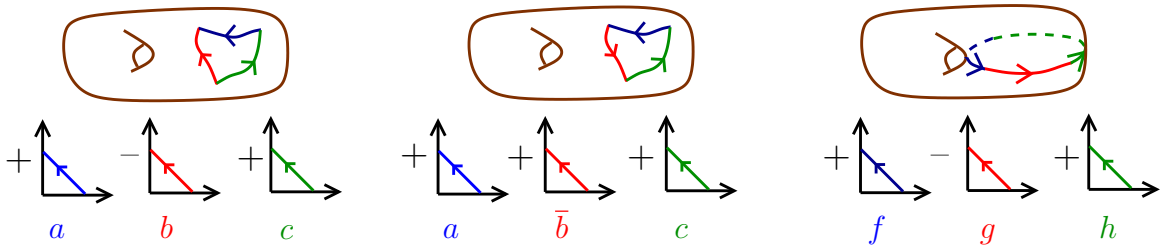
σ and $-\overline{\sigma}$ are homologous



σ and $\alpha+\beta$ are homologous

Proof. Statement (1) follows from the discussion of the boundary of a singular 1-simplex on page 2751. The other two statements will be proved in Exercise 143.9. ■

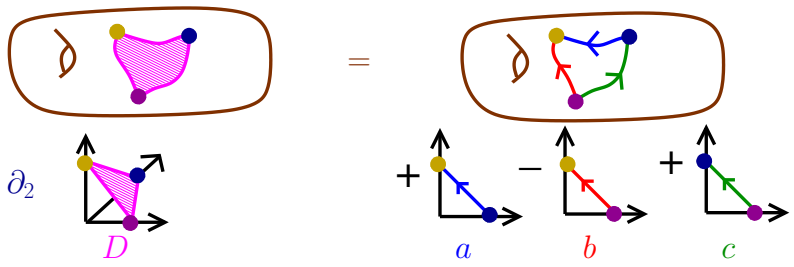
Example. In the figure below we see three singular chains on the torus. It is straightforward to show that all three singular chains are cycles.



Now the question arises, which of these cycles are in fact null-homologous and which represent non-zero classes in homology? In the figure below we see that the first cycle $a-b+c$ is the boundary of a singular 2-simplex D.[1497]
One can easily see that the second cycle, $a+\overline{b}+c$ in the above figure is *not* the boundary of a singular 2-simplex. On the other hand by the Singular 1-Simplex Lemma 143.28 we know that $b+\overline{b}$ is a null-homologous cycle, i.e. there exists an $E \in \mathrm{C}_2(X)$ such that $b+\overline{b} = \partial_2 E \in \mathrm{C}_1(X)$. Thus it follows that

$$a+\overline{b}+c \;=\; a+\partial_2(E)-b+c \;=\; a-b+c+\partial_2(E) \;=\; \partial_2(D)+\partial_2(E) \;=\; \partial_2(D+E),$$

[1496] Loosely speaking we break the singular 1-simplex σ into two singular 1-simplices α and β.
[1497] At this point it might be helpful to consider the figure on page 2752.



i.e. $a + \overline{b} + c$ is null-homologous.

It is not clear whether the third cycle in the figure below is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement. □

143.6. Exercises for Chapter 143 .

Exercise 143.1. Provide an example of a chain complex (C_*, ∂_*) with the following properties:

(1) All C_* are finitely generated free abelian groups.
(2) For all even $i \in \mathbb{N}_0$ we have $\mathrm{H}_i(C_*) = 0$.
(3) For all odd $i \in \mathbb{N}_0$ we have $\mathrm{H}_i(C_*) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 143.2.

(a) We consider the chain complex

$$0 \to C_1 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix}} C_0 = \mathbb{Z}^2 \to 0.$$

(i) Determine the isomorphism types of the homology groups of this chain complex.
(ii) Does the vector $\begin{pmatrix} 3 \\ 7 \end{pmatrix} \in \mathbb{Z}^2 = C_0$ represent a non-trivial element of $\mathrm{H}_0(C_*)$?

(b) Determine the isomorphism types of the homology groups of the chain complex

$$0 \to C_2 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix}} C_1 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -2 & 1 \end{pmatrix}} C_0 = \mathbb{Z} \to 0.$$

(Why is this a chain complex?)

Exercise 143.3. Let $f\colon C \to B$ be a homomorphism between abelian groups and let A be a subgroup of B with the following two properties:

(1) We have $f(C) \cap A = 0$.
(2) The subgroups $f(C)$ and A generate B.

Show that the map

$$\begin{array}{ccc} A & \to & \operatorname{coker}(f) = B/\operatorname{im}(f) \\ a & \mapsto & [a] \end{array}$$

is an isomorphism.

Exercise 143.4. We define the direct sum of two chain complexes (C_*, ∂_*) and (D_*, ∂_*) in the obvious way. A chain complex is called *indecomposable* if it cannot be written as

the direct sum of two non-zero chain complexes. Show that every chain complex (C_*, ∂_*), where the groups C_i are all finitely generated free abelian groups, can be written as a direct sum of chain complexes of the form

$$0 \to \underset{\substack{\uparrow\\ \text{degree } i}}{\mathbb{Z}} \to 0 \qquad \text{and} \qquad 0 \to \underset{\substack{\uparrow\\ \text{degree } i}}{\mathbb{Z}} \xrightarrow{\cdot n} \underset{\substack{\uparrow\\ \text{degree } i-1}}{\mathbb{Z}} \to 0.$$

Remark. Use the Smith Normal Form Theorem 97.22.

Exercise 143.5. Let (C_*, ∂_*) be a chain complex of abelian groups. Show that there exists a chain complex (F_*, ∂_*) of *free abelian groups* together with a chain map $\varphi\colon F_* \to C_*$ such that the induced maps $f_*\colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ are isomorphisms for all $k \in \mathbb{N}_0$.
Hint. Construct the groups $F_0, F_1, \ldots$ and the desired homomorphisms iteratively.

Exercise 143.6. Let (C_*, c_*) and (D_*, d_*) be a chain complex and let $\varphi\colon C_* \to D_*$ be a chain map. Show that the groups $K_n \coloneqq \ker(\varphi_n\colon C_n \to D_n)$ form a subcomplex of (C_*, c_*), i.e. show that for each $n \in \mathbb{N}_0$ the boundary map c_n restricts to a map $c_n|_{K_n}\colon K_n \to K_{n-1}$.

Exercise 143.7. Let $n \in \mathbb{N}$. As in the Δ^n-Homeomorphism Lemma 143.9 (1) we consider

$$\triangle_n \coloneqq \left\{(x_1, \ldots, x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \in [0,1]\right\} \subseteq \mathbb{R}^n.$$

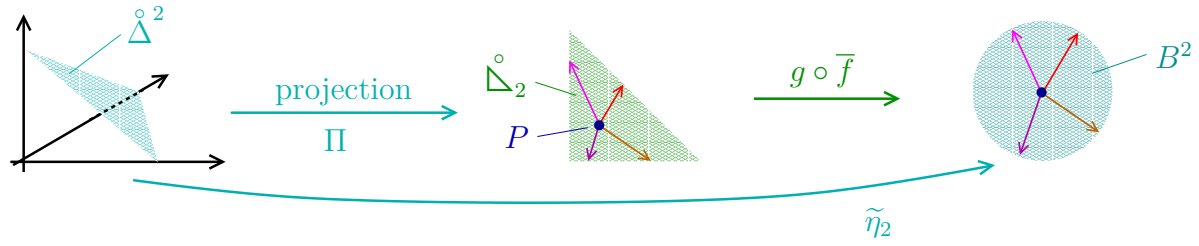
We denote by $\Phi\colon \Delta^n \to \triangle_n$ the natural projection. Let $P \in \overset{\circ}{\triangle}_n$. We consider the maps

$$\begin{array}{rcl} f_P\colon \overset{\circ}{\triangle}_n & \to & \mathbb{R}^n \\ (t_1, \ldots, t_n) & \mapsto & \dfrac{((t_1, \ldots, t_n) - P)}{t_1 \cdot \ldots \cdot t_n \cdot (1 - t_1 - \cdots - t_n)} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g\colon \mathbb{R}^n & \to & B^n \\ x & \mapsto & \dfrac{x}{\sqrt{1+\|x\|^2}}. \end{array}$$

Note that f_P is defined in such a way that it sends rays emanating from P to rays emanating from the origin.

(a) Show that $f_P\colon \overset{\circ}{\triangle}_n \to \mathbb{R}^n$ is an orientation-preserving diffeomorphism.
(b) Show that $g\colon \mathbb{R}^n \to B^n$ is an orientation-preserving diffeomorphism.
Remark. Note that for the most part we had solved this problem on page 772.
(c) Show that $g \circ f_P\colon \overset{\circ}{\triangle}_n \to B^n$ extends to a homeomorphism $\Theta\colon \triangle_n \to \overline{B}^n$.
(d) Let $\Xi\colon \triangle_n \to \overline{B}^n$ be the homeomorphism obtained from the Convex-to-Ball Proposition 4.28 (2) applied to P. Show that Θ and Ξ are homotopic rel P.

Remark. The composition $\Psi \circ \Phi\colon \Delta^n \to \overline{B}^n$ is a homeomorphism such that the restriction $\overset{\circ}{\Delta}{}^n \to B^n$ is a diffeomorphism. Furthermore this diffeomorphism is orientation-preserving if and only if n is even. Here we use Definition 143.8 to view $\overset{\circ}{\Delta}{}^n$ as an oriented smooth manifold.



Exercise 143.8. Let X be a topological space and let $A, B \subseteq X$ be subspaces.
(a) Let $n \in \mathbb{N}_0$. Is it true that $\mathrm{C}_n(A \cap B) = \mathrm{C}_n(A) \cap \mathrm{C}_n(B)$?

(b) Let $n \in \mathbb{N}_0$. Is it true that $C_n(A \cup B) = C_n(A) \cup C_n(B)$?

Exercise 143.9. Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.

(a) We consider the singular 1-simplex

$$\begin{array}{rcl} \overline{\sigma}\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(1-t, t). \end{array}$$

Show that $\sigma + \overline{\sigma}$ is a 1-cycle and that this 1-cycle is null-homologous.

(b) We consider the singular 1-simplices

$$\begin{array}{rcl} \sigma_1\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(\frac{1}{2}t, 1-\frac{1}{2}t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \sigma_2\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & (\frac{1}{2}+\frac{1}{2}t, \frac{1}{2}-\frac{1}{2}t). \end{array}$$

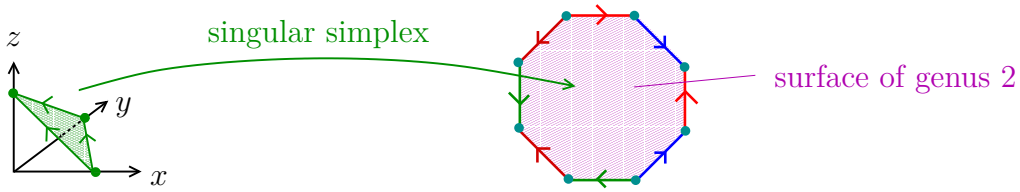
Show that σ and $\sigma_1 + \sigma_2$ are homologous.

Remark. This exercise is basically the content of the Singular 1-Simplex Lemma 143.28 (2) and (3).

Exercise 143.10. Let $\Sigma = E_8/\sim$ be the surface of genus 2. Provide an example of a 2-cycle $\sum_{i=1}^{m} a_i \cdot \sigma_i \in C_2(\Sigma)$ such that there exists an $i \in \{1, \dots, m\}$ with the following properties:

(1) $a_i \neq 0$.
(2) The map $\sigma_i\colon \mathring{\Delta}^2 \to \Sigma$ is an embedding and the image is an open subset of Σ.
(3) For any $j \neq i$ we have $\sigma_j(\Delta^2) \cap \sigma_i(\mathring{\Delta}^2) = \varnothing$.

Remark. Note that on page 2755 we provided an example of such a 2-cycle for the torus $\Sigma = ([0,1] \times [0,1])/\sim$.
Remark. In the Detect-Cycle Proposition 150.5 we will see that such cycles represent non-trivial homology classes.



Exercise 143.11.

(a) Try to find "interesting" elements in the following homology groups:
  (i) $H_2(S^2)$,
  (ii) H_2(non-orientable surface of genus 2),
  (iii) H_2(Klein bottle K) and
  (iv) $H_2(\mathbb{R}P^2)$.
  (v) H_2(annulus $S^1 \times [-1, 1]$).
  More precisely, try to find 2-cycles which have the properties (1), (2) and (3) from Exercise 143.10. (Mental health advice: In some cases this might be impossible.)

(b) Make a guess for which surfaces you expect H_2 to be zero and for which it is non-zero.

Exercise 143.12. Let $n \in \mathbb{N}$.

(a) Provide an example of an n-cycle $\sum_{i=1}^{m} a_i \cdot \sigma_i \in \mathrm{C}_n(S^n)$ which has the properties (1), (2) and (3) from Exercise 143.10 (where we of course need to replace the dimension 2 by the dimension n).

(b) Make an educated guess what the homology groups of S^n might be.

Exercise 143.13. Given a topological space X and $n \in \mathbb{N}_0$ we consider the map

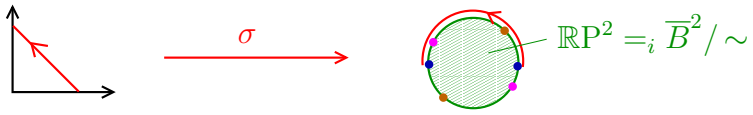
$$\begin{array}{rcl} \epsilon_X \colon \mathrm{C}_n(X) & \to & \mathbb{Z} \\ \sum_{i=1}^{k} a_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} a_i. \end{array}$$

(a) Show that the following diagram commutes:

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \mathrm{C}_3(X) & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_2(X) & \xrightarrow{\partial_0} & 0 & & \\ & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_X} & & & & \\ \ldots & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0. & & \end{array}$$

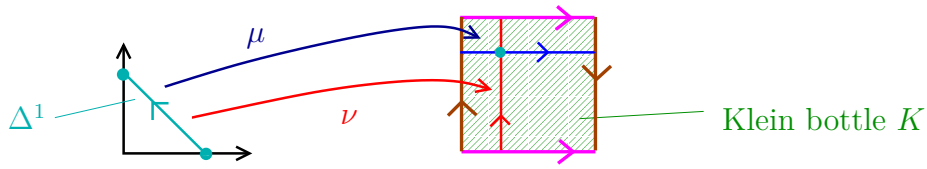
(b) Show that if n is even, then for every cycle $\sigma \in \mathrm{C}_n(X)$ we have $\epsilon_X(\sigma) = 0$.

Exercise 143.14. We consider the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ where $z \sim -z$ for $z \in S^1$. We consider the singular 1-simplex $\sigma \colon \Delta^1 \to \overline{B}^2/\sim$ that is given by the map $(1-t,t) \mapsto [\exp(\pi \mathrm{i} t)]$. Find as many $n \in \mathbb{Z}$ as possible for which the 1-cycle $n \cdot \sigma$ in $\mathrm{C}_1(\overline{B}^2/\sim)$ is null-homologous.



Exercise 143.15. Let X and Y be two topological spaces. What is the relationship between $\mathrm{H}_0(X)$, $\mathrm{H}_0(Y)$ and $\mathrm{H}_0(X \times Y)$? You should strive for the most elegant way to formulate this result.

Exercise 143.16. We consider the Klein bottle K and we consider the two singular 1-cycles μ and ν that are shown in the figure below. Precisely one has order 2 in $\mathrm{H}_1(K)$ and precisely one has infinite order in $\mathrm{H}_1(K)$. Which one is which?



Exercise 143.17. Let Σ be the torus minus two open disks as illustrated below. Furthermore let μ and ν be the two singular 1-chains illustrated in the figure below. Note that both are cycles. Find the largest number of values $m, n \in \mathbb{Z}$ such that $m \cdot \mu$ and $n \cdot \nu$ are homologous.



Exercise 143.18. Let X be a topological space and let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$$

be a sequence of subspaces such that $X = \operatorname{colim}_{\longrightarrow} X_i$ and such that for each $i \in \mathbb{N}_0$ we have $X_i \subseteq \mathring{X}_{i+1}$. By the Direct Colimit-$\pi_1$-Proposition 105.1 we know that for any $x_0 \in X_0$ we have

$$\operatorname{colim}_{\longrightarrow} \pi_1(X_i, x_0) \xrightarrow{\cong} \pi_1\big(\operatorname{colim}_{\longrightarrow} X_i, x_0\big).$$

Does the corresponding statement also hold for homology groups? More precisely, is it true that for every $k \in \mathbb{N}_0$ we have

$$\operatorname{colim}_{\longrightarrow} \mathrm{H}_k(X_i) \xrightarrow{\cong} \mathrm{H}_k\big(\operatorname{colim}_{\longrightarrow} X_i\big).$$

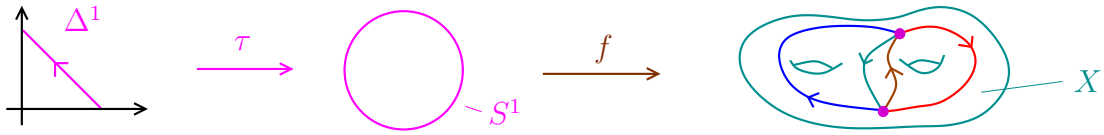
Exercise 143.19. Is it true that homology groups preserve products? More precisely, is it true that for any $k \in \mathbb{N}_0$ and any topological space A and B there exists an isomorphism $\mathrm{H}_k(A \times B) \cong \mathrm{H}_k(A) \times \mathrm{H}_k(B)$?

Exercise 143.20. Let X be a path-connected topological space and let $\theta \in \mathrm{H}_1(X)$.

(a) Show that there exists a singular simplex $\sigma\colon \Delta^1 \to X$ with $\partial_1(\sigma) = 0$ and such that $[\sigma] = \theta$.

Hint. Use the Singular 1-Simplex Lemma 143.28 excessively.

(b) Show that there exists a $\tau \in \mathrm{H}_1(S^1)$ and a continuous map $f\colon S^1 \to X$ with $f_*(\tau) = \theta$.



CHAPTER 144

Homology and homotopies

In this chapter we will see that homotopy maps between two topological spaces induce chain homotopic maps on the chain complexes, and thus they induce the same map in homology. It follows in particular that the homology groups of homotopy equivalent topological spaces are isomorphic.

This chapter is organized as follows:

- In Section 144.1 we will recall the definition of the homotopy category.
- In Section 144.2 we will introduce the algebraic notion of a chain homotopy.
- In Section 144.3 we will state the Homotopy-H_*-Proposition 144.12 which says that homotopic maps induce the same map on homology. We will discuss several consequences of this proposition.
- In Section 144.4 we will prove the Homotopy-H_*-Proposition 144.12.
- In Section 144.5 we will provide exercises for Chapter 144.

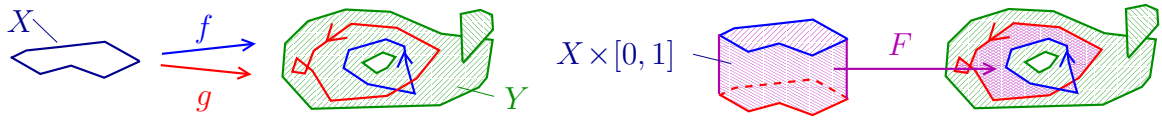
In the next purely algebraic Chapter 145 we will study exact sequences and we will see that a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.

144.1. **Homotopies and the homotopy category.** We start out with recalling Definition 18.2:

Definition 144.1. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. A homotopy between the maps f and g is a continuous map

$$F\colon X \times [0,1] \to Y,$$

such that for all $x \in X$ we have $F(x,0) = f(x)$ and $F(x,1) = g(x)$. If there exists a homotopy between f and g, then we say that f and g are homotopic and we write $f \simeq g$.



We also recall the following related two definitions.

Definition 144.2. Let X and Y be topological spaces.

(1) A continuous map $f\colon X \to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent and we write $X \simeq Y$.

Example.

(1) In the Deformation Retract–Homotopy Equivalence Lemma 19.10 we showed that if A is a deformation retract of a topological space X, then the inclusion $A \to X$ is a homotopy equivalence.
(2) It follows easily from (1) that each non-empty convex subset of $\mathbb{R}^n$ is homotopy equivalent to a singleton, i.e. to a topological space consisting of a single point. □

Next we recall Definition 19.12:

Definition 144.3. We define the homotopy category $\mathcal{HomTop}$ to be the category given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomTop}) &:= \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) &:= \text{set of homotopy equivalence classes of continuous maps } X \to Y \end{aligned}$$

with the composition[1498]

$$\begin{array}{ccc} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

It is worth recalling that in the Homotopy Equivalence-Reformulation Lemma 19.13 we proved the elementary fact that a continuous map $f\colon X \to Y$ is a homotopy equivalence if and only if f represents an isomorphism in the homotopy category.

In the Homotopy-π_1–Proposition 90.8 and in the Homotopy-π_n-Proposition 134.16 we showed that homotopic maps induce essentially the same map on homotopy groups. We used this fact to show that homotopy equivalent path-connected topological spaces have isomorphic homotopy groups.
In this chapter we will see that homotopic maps induce the same map on homology. We will use this result to show that homotopy equivalent topological spaces have isomorphic homology groups. This result, together with the results from the previous chapter, allows us for example to compute the homology groups of closed and open balls.

In the next section we first make some algebraic preparations. In particular we will introduce algebraic analogs of homotopies, of being "homotopy equivalent" and of the homotopy category.

144.2. **Chain homotopies.** In this purely algebraic section we introduce a criterion that ensures that two chain maps $f, g\colon C_* \to D_*$ induce the same maps on the corresponding homology groups. In the next section we will use this to study the effect of homotopic maps on homology groups.

Definition 144.4. [1499] Let $f, g\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*) and (D_*, ∂_*). A chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g is defined as a family $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

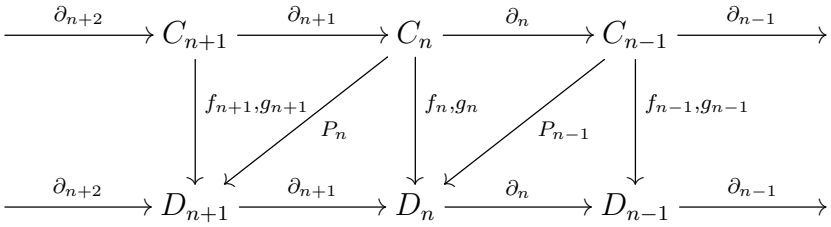
$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n\colon C_n \;\to\; D_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects

[1498] It follows from the Homotopy Composition Lemma 18.10 (2) that this composition map is well-defined.
[1499] We apologize for the fact that we present this definition without any motivation. One could spend some time trying to motivate this definition, but we think that this would not really improve matters.

involved in the definition:



The following lemma explains perhaps why we are interested in chain homotopies.

Lemma 144.5. (Chain Homotopy–H_*-Lemma) Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ for all $n \in \mathbb{N}_0$.

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$\begin{aligned} g(z) - f(z) \;=\; (g-f)(z) \;&=\; (\partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n)(z) \\ &=\; (\partial_{n+1}\circ P_n)(z) + P_{n-1}(\underbrace{\partial_n(z)}_{=0}) \;=\; \partial_{n+1}(P_n(z)), \end{aligned}$$

i.e. $f(z)$ and $g(z)$ are homologous. ∎

Now that we know that chain homotopies are useful it seems reasonable to develop the theory a bit further. The next lemma can be viewed as an algebraic analog of the Homotopy-Equivalence Relation Lemma 18.9 and the Homotopy Composition Lemma 18.10.

Lemma 144.6. (Chain Homotopy-Lemma) Let C_* and D_* be two chain complexes.

(1) Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*. In particular, if $f, g, h\colon C_* \to D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.

(2) Let $f, f'\colon C_* \to D_*$ and $g, g'\colon D_* \to E_*$ be chain maps. Then

$$f \simeq f' \;\text{ and }\; g \simeq g' \implies g\circ f \simeq g'\circ f'.$$

Proof. The lemma will be proved in Exercise 144.1. ∎

The following definition is modelled on the definition of the homotopy category $\mathcal{HomTop}$ of topological spaces which we recalled in the beginning of the chapter.

Definition 144.7. We define the chain homotopy category $\mathcal{HomCplx}$

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomCplx}) &:= \text{all chain complexes,} \\ \mathrm{Mor}(C_*, D_*) &:= \text{set of chain homotopy equivalence classes of chain maps } C_* \to D_* \end{aligned}$$

with the composition[1500]

$$\begin{array}{ccc} \mathrm{Mor}(C_*, D_*) \times \mathrm{Mor}(D_*, E_*) & \to & \mathrm{Mor}(C_*, ED_*) \\ ([f], [g]) & \mapsto & [g\circ f]. \end{array}$$

The following definition is also modelled on a definition which we recalled in the beginning of the chapter.

[1500]It follows from the Chain Homotopy-Lemma 144.6 (2) that this composition map is well-defined.

Definition 144.8. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon C_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to C_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.

(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

The following lemma is the algebraic analog of the Homotopy Equivalence-Reformulation Lemma 19.13.

Lemma 144.9. (Chain Homotopy Equivalence-Reformulation Lemma) Let C_* and D_* be two chain complexes and let $f\colon C_* \to D_*$ be a chain map. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomCplx}}(C_*, D_*)$ is an isomorphism in the chain homotopy category $\mathcal{HomCplx}$.

(2) f is a chain homotopy equivalence.

In particular the following statements are equivalent:

(1′) C_* and D_* are isomorphic in the chain homotopy category $\mathcal{HomCplx}$.

(2′) C_* and D_* are chain homotopy equivalent.

Example. It follows from the Chain Homotopy Equivalence-Reformulation Lemma 144.9 and the Isomorphism Lemma 5.12 that the relation "chain homotopy equivalence" of chain complexes is indeed an equivalence relation on the class of all chain complexes. □

Proof. As in the case of the proof of the Homotopy Equivalence-Reformulation Lemma 19.13 the statements follow almost immediately from the definitions. ■

For later we record the following straightforward corollary to the Chain Homotopy–H_*-Lemma 144.5.

Corollary 144.10. (Chain Homotopy Equivalence-H_*-Corollary) If $f\colon C_* \to D_*$ is a chain homotopy equivalence between two chain complexes, then for each $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is an isomorphism.

Proof. In the following we give the same proof twice, but with different language. First the language is a little more high-brow, the second time it is more down-to-earth:

(a) First note that it follows from the Chain Homotopy–H_*-Lemma 144.5 that the homology functor $\mathrm{H}_n\colon \mathcal{ChCplx} \to \mathcal{AbGroup}$ from the category of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups factors through a functor $\mathcal{HomCplx} \to \mathcal{AbGroup}$ from the homotopy category of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups. It follows from this observation, the Chain Homotopy Equivalence-Reformulation Lemma 144.9 and the Isomorphism Preservation Lemma 5.26 that chain homotopies give rise to isomorphisms of abelian groups.

(b) Let $g\colon D_* \to C_*$ be a chain homotopy inverse to f. Then we see that

$$g_* \circ f_* \quad = \quad (g \circ f)_* \quad \underset{\uparrow}{=} \quad (\mathrm{id}_{C_*})_* \quad = \quad \mathrm{id}_{\mathrm{H}_n(C_*)}\,.$$

by the Chain Homotopy–H_*-Lemma 144.5
since $g \circ f$ is chain homotopic to the identity

The same way we see that $f_* \circ g_* = \mathrm{id}_{\mathrm{H}_n(D)}$. ■

We conclude this section with the following little lemma, which we will use much later on two occasions.

Lemma 144.11. (Chain Null-Homotopy Lemma) Let (C_*, ∂_*) be a chain complex. If there exist "diagonal" maps $\{P_n \colon C_n \to C_{n+1}\}_{n\mathbb{Z}_{\geq -1}}$, to obtain a diagram[1501]

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\
 & \swarrow_{P_3} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow_{P_2} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow_{P_1} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow_{P_0} & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & C_3 & \xrightarrow[\partial_3]{} & C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 & \xrightarrow[\partial_0]{} & 0
\end{array}$$

such that for every $n \in \mathbb{N}_0$ we have the equality

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \ = \ \mathrm{id}_{C_n},$$

then all homology groups of (C_*, ∂_*) vanish, i.e. the chain complex (C_*, ∂_*) is acyclic.

Proof. Let D_* be the zero chain complex and for each $n \in \mathbb{N}_0$ let $f_n \colon C_n \to D_n = 0$ be the trivial map. Since the homology groups of D_* are evidently zero it suffices, by the Chain Homotopy Equivalence-H_*-Corollary 144.10, to show that f_* is a chain homotopy equivalence. For each $n \in \mathbb{N}_0$ let $g_n \colon D_n \to C_n$ be the trivial map. Evidently $f_n \circ g_n = \mathrm{id}_{D_*}$. Thus it remains to show that $g_* \circ f_* \simeq \mathrm{id}_{C_*}$. In other words we need to show that $0 \simeq \mathrm{id}_{C_*}$. But that is exactly what the maps P_n do. ■

144.3. **Homology and homotopic maps.** The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. The following proposition shows that this is indeed the case.

Proposition 144.12. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g \colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_* \colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g \colon X \to Y \implies f_* \simeq g_* \colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y) \implies f_* = g_* \colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y). \qquad \square$$

Example. Let $f \colon X \to Y$ be a continuous map between two topological spaces. If f is null-homotopic, i.e. if f is homotopic to a constant map $c \colon x \mapsto P$, then it follows from the Homotopy-H_*-Proposition 144.12 that $f_* = c_*$. If we denote by c also the map $X \to \{P\}$ and if we denote by $i \colon \{P\} \to Y$ the inclusion, then it follows from the functoriality of homology that we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_n(X) & \xrightarrow{f_* = c_*} & \mathrm{H}_n(Y). \\
 & {\scriptstyle c_*} \searrow \quad \nearrow {\scriptstyle i_*} & \\
 & \underbrace{\mathrm{H}_n(\{P\})}_{= 0 \text{ for } n \geq 1} &
\end{array}$$

[1501]The diagram has no particular role, except that it shows the domain and the target of the maps P_n and that it is convenient for keeping track of the various maps.

By the Singleton-H_*-Lemma 143.26 we know that $\mathrm{H}_n(\{P\}) = 0$ for $n \geq 1$. In summary we have shown that $f_* = 0$ for $n \geq 1$. This observation often gets used subconsciously.

We postpone the lengthy proof of the Homotopy-H_*-Proposition 144.12 to Section 144.4. Let us first turn to the fun part, namely let us jot down a few consequences. We start out with the following straightforward consequence.

Corollary 144.13. (Homotopy Category-H_*-Functor Corollary) Each of our homology functors $\mathrm{H}_n\colon \mathcal{T}op \to \mathcal{A}b\mathcal{G}roup$ defines a functor $\mathrm{H}_n\colon \mathcal{H}om\mathcal{T}op \to \mathcal{A}b\mathcal{G}roup$ from the homotopy category of topological spaces to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups.

Proof. This statement follows immediately from the Homotopy-H_*-Proposition 144.12 and the definitions. ∎

The following corollary to the above results mirrors the fact, proved in the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8, the Homotopy Equivalence-π_1-Proposition 90.9 (1) and the Homotopy Equivalence-π_n-Proposition 134.17 (1a), that homotopy equivalences induce isomorphisms of de Rham cohomology groups fundamental groups and higher homotopy groups.

Proposition 144.14. (Homotopy Equivalence-H_*-Proposition) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.
(b) If X is a deformation retract of Y, then the inclusion-induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.

(2) If X and Y are homotopy equivalent, then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.

Proof.

(1) (a) This statement follows immediately from the Homotopy Category-H_*-Functor Corollary 144.13 and the Isomorphism Preservation Lemma 5.26.
(b) This statement follows immediately from (1a) and the Deformation Retract–Homotopy Equivalence Lemma 19.10.

(2) This statement follows immediately from (1a). ∎

We can now easily prove the following analog of the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary 79.9.

Corollary 144.15. (Contractible-H_*-Corollary) Let X be a topological space. If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. This statement follows from the Homotopy Equivalence-H_*-Proposition 144.14 (2) and the Singleton-H_*-Lemma 143.26. ∎

Example. On page 531 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from the Contractible-H_*-Corollary 144.15, that for any $i \in \mathbb{N}_0$ we have

$$\mathrm{H}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n}) \ \cong\ \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

□

144.4. **Proof of the Homotopy-H_*-Proposition 144.12.** For the reader's convenience we recall the proposition that we need to prove.

Proposition 144.12. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic, i.e. there exists a family $\{P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \ = \ g_n - f_n\colon \mathrm{C}_n(X) \ \to\ \mathrm{C}_{n+1}(Y).$$

(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

We start out with a discussion of the key idea behind the proof of the Homotopy-H_*-Proposition 144.12. Afterwards we will fill in the details. So let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. Our goal is to find a chain homotopy between f_* and g_*. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We need to find $P(\sigma) \in \mathrm{C}_{n+1}(Y)$ and $P(\partial_n\sigma) \in \mathrm{C}_n(Y)$ such that

$$\partial_{n+1}P(\sigma) \ = \ -P(\partial_n\sigma) \ + \ f_*(\sigma) \ - \ g_*(\sigma).$$

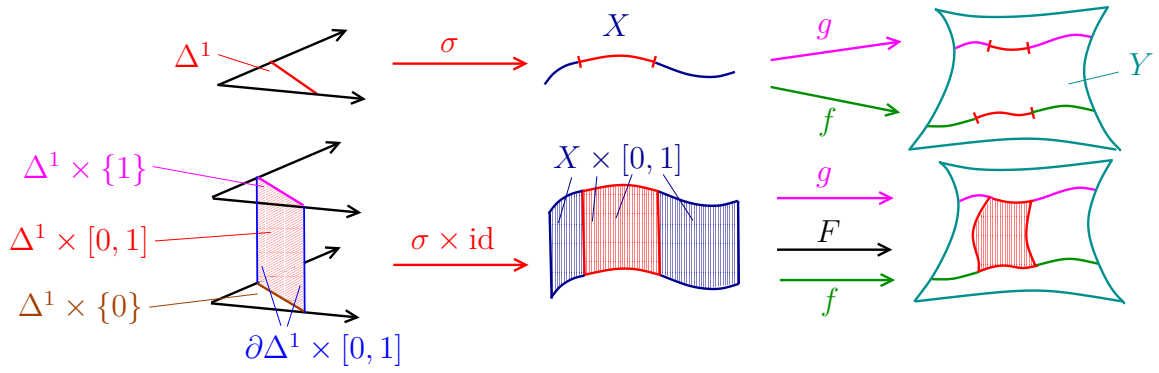
Note that $F \circ (\sigma \times \mathrm{id}_{[0,1]})$ defines a map $\Delta^n \times [0,1] \to Y$. Now we can rewrite the above desired equality as follows:

$$\partial_{n+1}P(\sigma) \ = \ -P(\partial_n\sigma) + \underbrace{\begin{matrix}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \Delta^n \times \{0\}\end{matrix}}_{=_{\mathrm{i}}\Delta^n} - \underbrace{\begin{matrix}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \Delta^n \times \{1\}\end{matrix}}_{=_{\mathrm{i}}\Delta^n}.$$

To get an idea we consider the figure below. Looking at the figure we note that

$$\partial(\Delta^n \times [0,1]) \ = \ (\partial\Delta^n \times [0,1]) \ \cup \ (\Delta^n \times \{0\}) \ \cup \ (\Delta^n \times \{1\}).$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(Y)$ should take the role of replacing a singular k-simplex $\mu\colon \Delta^k \to X$ by the map $F \circ (\mu \times \mathrm{id}_{[0,1]})\colon \Delta^k \times [0,1] \to Y$. This is a good idea, but the problem is that $\Delta^k \times [0,1]$ is not a $(k+1)$-simplex. The idea now is to write $\Delta^k \times [0,1]$ as the union of $(k+1)$-simplices. In the figure below we see for example that we can view $\Delta^1 \times [0,1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.
To generalize this observation we have to introduce some more notation.



Notation 144.16. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \ldots, n$ we write

$$v_i \coloneqq ((0, \ldots, 1, 0, \ldots, 0), 0) \in \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i \coloneqq ((0, \ldots, 1, 0, \ldots, 0), 1) \in \mathbb{R}^{n+1} \times \{1\}.$$

The points $v_0, \ldots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $w_0, \ldots, w_n$ span the simplex $\Delta^n \times 1$. Furthermore for each i the points $v_0, \ldots, v_i, w_i, \ldots, w_n$ span an $(n+1)$-simplex that is a subset of $\Delta^n \times [0,1]$.[1502]

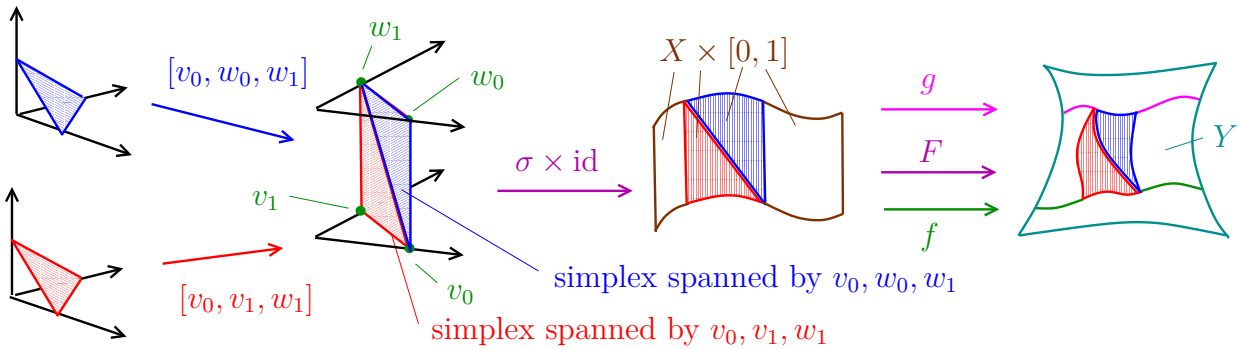
(2) For $a_0, \ldots, a_k \in \Delta^n \times [0,1]$ we consider the map

$$\begin{aligned} [a_0, \ldots, a_k] \colon \Delta^k &\to \Delta^n \times [0,1] \\ (\lambda_0, \ldots, \lambda_k) &\mapsto \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

The map $[a_0, \ldots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \ldots, a_k$ in $\Delta^n \times [0,1]$.

(3) We set

$$\Omega_n \coloneqq \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \ldots, v_j, w_j, \ldots, w_n]}_{\text{map } \Delta^{n+1} \to \Delta^n \times [0,1]} \in \mathrm{C}_{n+1}(\Delta^n \times [0,1]).$$



In the following lemma we compute the boundary of $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$. The statement might not be particularly visually appealing, but it will turn out to be very useful.

[1502] It is a straightforward exercise to show that the union of these $(n+1)$-dimensional simplices is precisely $\Delta^n \times [0,1]$. We will not make use of this fact.

Lemma 144.17. (Prism Boundary Lemma) Let $n \in \mathbb{N}_0$. Given $t \in \{0,1\}$ we write

$$\begin{array}{rcl} \eta_t \colon \Delta^n & \to & \Delta^n \times [0,1] \\ x & \mapsto & (x,t). \end{array}$$

With this notation we have[1503]

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*\Big(\underbrace{\Omega_{n-1}}_{\in \mathrm{C}_n(\Delta^{n-1}\times[0,1])} \Big) \;=\; \eta_1 - \eta_0 \;\in\; \mathrm{C}_n(\Delta^n \times [0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{aligned} \partial_{n+1}\Omega_n &= \partial_{n+1}\Big(\sum_{j=0}^{n} (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, w_n] \Big) \\ &= \sum_{k=0}^{n+1} (-1)^k \cdot \sum_{j=0}^{n} (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, w_n] \circ i_k^{n+1}. \end{aligned}$$

It is straightforward to see that for any $k \in \{0, \ldots, n+1\}$ we have[15041505]

$$[v_0, \ldots, v_j, w_j, \ldots, w_n] \circ i_k^{n+1} \;=\; \begin{cases} [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n], & \text{if } k \le j, \\ [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_{k-1}}, \ldots, w_n], & \text{if } k > j. \end{cases}$$

Thus it follows that

$$\begin{aligned} \partial_{n+1}\Omega_n \;=\; & \sum_{0 \le k \le j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\ & + \sum_{n+1 \ge k > j \ge 0} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_{k-1}}, \ldots, w_n]. \end{aligned}$$

Now we split the first sum into the cases $k = j$ and $k < j$ and we split the second sum into the cases $k = j+1$ and $k > j+1$. This leads us to the following equality:

$$\begin{aligned} \partial_{n+1}\Omega_n \;=\; & \sum_{k=0}^{n} (-1)^{2k} \cdot [v_0, \ldots, v_{k-1}, w_k, \ldots, w_n] \\ & + \sum_{k=1}^{n+1} (-1)^{2k+1} \cdot [v_0, \ldots, v_{k-1}, w_k, \ldots, w_n] \\ & + \sum_{0 \le k < j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\ & + \sum_{n+1 \ge k > j+1 \ge 1} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_{k-1}}, \ldots, w_n]. \end{aligned}$$

[1503] Here $i_j^n \colon \Delta^{n-1} \to \Delta^n$ denotes the j-th face map as introduced in Definition 143.12.

[1504] The first case, $k \le j$ is slightly easier: one has to figure out, where the h-th vertex e_h of the standard n-simplex gets sent to. If $h < k$, then i_k^{n+1} sends the h-th vertex again to the h-th vertex, and $[v_0, \ldots, v_j, w_j, \ldots, w_n]$ sends this vertex to v_h. If $h \ge k$, then i_k^{n+1} sends the h-th vertex again to the $(h+1)$-th vertex, and $[v_0, \ldots, v_j, w_j, \ldots, w_n]$ sends this vertex to v_{h+1}. If $k \le j$ this corresponds just to the map $[v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n]$.
The second case, $k > j$ is proved precisely the same way.

[1505] Here we use the usual notation that $\widehat{v_k}$ means that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

Next we consider the first two sums. The summands cancel each other, except for the summands for $k = 0$ and $k = n + 1$. Thus we obtain that

$$
\begin{aligned}
\partial_{n+1}\Omega_n \;=\;& [w_0, \dots, w_n] - [v_0, \dots, v_n] \\
(*) \qquad & + \sum_{0 \le k < j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\
& + \sum_{n+1 \ge k > j+1 \ge 1} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_k}, \dots, w_n].
\end{aligned}
$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof with the discussion of the one term in the statement of the lemma that we have not studied yet. We have[1506]

$$
\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;=\; \sum_{r=0}^{n} \sum_{s=0}^{n-1} (-1)^r \cdot (-1)^s \cdot (i_r^n \times \mathrm{id}_{[0,1]}) \circ [v_0, \dots, v_s, w_s, \dots, w_{n-1}].
$$

$\uparrow$ Notation 144.16 of Ω_{n-1} and substitution $r = j$

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n \to \Delta^n \times [0,1]$:

$$
(i_r^n \times \mathrm{id}_{[0,1]}) \circ [v_0, \dots, v_s, w_s, \dots, w_{n-1}] \;=\; \begin{cases} [v_0, \dots, \widehat{v_r}, \dots, v_{s+1}, w_{s+1}, \dots, w_n], & \text{if } r \le s, \\ [v_0, \dots, v_s, w_s, \dots, \widehat{w_{r+1}}, \dots, w_n], & \text{if } r > s. \end{cases}
$$

Thus we see that

$$
\begin{aligned}
\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;=\;& \sum_{0 \le r \le s \le n-1} (-1)^s \cdot (-1)^r \cdot [v_0, \dots, \widehat{v_r}, \dots, v_{s+1}, w_{s+1}, \dots, w_n] \\
& + \sum_{n \ge r > s \ge 0} (-1)^s \cdot (-1)^r \cdot [v_0, \dots, v_s, w_s, \dots, \widehat{w_{r+1}}, \dots, w_n].
\end{aligned}
$$

In the first term we perform the substitutions $r = k$ and $s = j - 1$ and in the second term we perform the substitutions $r = k - 1$, $s = j$. We obtain the following equality:

$$
\begin{aligned}
\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;=\;& \sum_{0 \le k \le j-1 \le n-1} (-1)^{j-1} \cdot (-1)^k \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\
& + \sum_{n \ge k-1 > j \ge 0} (-1)^j \cdot (-1)^{k-1} \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_k}, \dots, w_n].
\end{aligned}
$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation $(*)$. ∎

Now we can finally provide the proof of the Homotopy-H_*-Proposition 144.12.

Proof of the Homotopy-H_*-Proposition 144.12. Let X be a topological space and let $F\colon X \times [0,1] \to Y$ be a homotopy from a map f to a map g.

(1) This statement follows immediately from the following claim:

[1506] Here $v_0, \dots, v_{n-1}, w_0, \dots, w_{n-1}$ are now, unfortunately somewhat confusingly, the corresponding vertices of $\Delta^{n-1} \times [0,1]$ instead of $\Delta^n \times [0,1]$.

Claim. Given any $n \in \mathbb{N}_0$ we define

$$P_n \colon \mathrm{C}_n(X) \ \to \ \mathrm{C}_{n+1}(Y)$$

via

$$(\sigma \colon \Delta^n \to X) \ \mapsto \ (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n).$$

↑ we defined $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$ in Notation 144.16

For any $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \ = \ g_* - f_*$$

as maps from $\mathrm{C}_n(X)$ to $\mathrm{C}_n(Y)$.

Proof. As always it suffices to verify the equality for each singular n-simplex. Thus let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}
(\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n)(\sigma) \ &= \ \partial_{n+1}(P_n(\sigma)) + P_{n-1}\Big(\sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n\Big) \\
&= \ \partial_{n+1}(P_n(\sigma)) + \sum_{j=0}^{n} (-1)^j \cdot P_{n-1}(\sigma \circ i_j^n) \\
&= \ \partial_{n+1}((F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n)) + \sum_{j=0}^{n} (-1)^j \cdot (F \circ (\sigma \circ i_j^n \times \mathrm{id}_{[0,1]}))_*(\Omega_{n-1}) \\
&= \ (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*\Big(\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big) \\
&\quad \uparrow \text{ by the } \mathrm{C}_*\text{–Functor Lemma 143.16 (1) and since we have } (\sigma \times \mathrm{id}) \circ (i_j^n \times \mathrm{id}) = (\sigma \circ i_j^n) \times \mathrm{id} \text{ as maps } \Delta^n \times [0,1] \times Y \times [0,1] \\
&= \ (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\eta_1 - \eta_0) \ = \ f \circ \sigma - g \circ \sigma \ = \ g_*(\sigma) - f_*(\sigma).
\end{aligned}$$

↑ by the Prism Boundary Lemma 144.17 ↑ since F is a homotopy from f to g ⊞

(2) This statement follows immediately from Statement (1) together with the Chain Homotopy–H_*-Lemma 144.5. ■

144.5. Exercises for Chapter 144 .

Exercise 144.1. Let C_* and D_* be two chain complexes.

(a) Show that "chain homotopy" defines an equivalence relation on the set of all chain maps from C_* to D_*.

(b) Let $f, f' \colon C_* \to D_*$ be chain maps, and let $e \colon B_* \to \mathrm{C}_*$ and $h \colon D_* \to E_*$ be two other chain maps. Show that if $f \simeq f'$, then $f \circ e \simeq f \circ e$ and $h \circ f \simeq h \circ f'$.

(c) Let $f, f' \colon C_* \to D_*$ and $g, g' \colon D_* \to E_*$ be chain maps. Show that

$$f \simeq f' \ \text{ and } \ g \simeq g' \ \Longrightarrow \ g \circ f \simeq g' \circ f'.$$

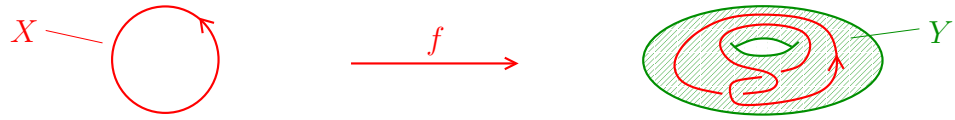
Exercise 144.2. Let $\varphi \colon A \to B$ be a homomorphism of free abelian groups and let $C \subseteq A$ and $D \subseteq B$ be subgroups such that the natural map $C/D \to A/B$ is an isomorphism. Show that the chain complex $0 \to A \xrightarrow{\varphi} B \to 0$ is chain homotopy equivalent to the chain complex $0 \to C \xrightarrow{\varphi} D \to 0$.

Exercise 144.3. Let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in A$ be a point.

(a) Let A be a convex subset of $\mathbb{R}^n$ which contains the origin 0. Let $r > 0$ such that $A \subseteq B_r^n(0)$. Show that $\mathbb{R}^n \setminus B_r^n(0)$ is a deformation retract of $\mathbb{R}^n \setminus A$.

(b) Show that the inclusion $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus \{x\}$ induces for each $i \in \mathbb{N}_0$ an isomorphism $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{x\})$ for all i.

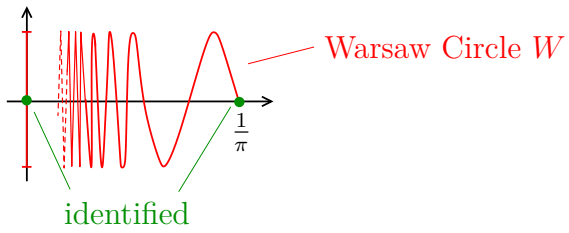
Exercise 144.4. Let $X = S^1$ and let $Y = \overline{B}^2 \times S^1$. We consider the map $f\colon X \to Y$ shown in the figure below. Show that for every $i \in \mathbb{N}$ the induced map $f_*\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is the zero map.



Exercise 144.5. Determine the isomorphism types of the homology groups of the Topologist's Sine Curve that we introduced on page 195.
Hint. Use Exercise 4.56.



Exercise 144.6. Let $S := \{(0,y)\,|\,y \in [-1,1]\} \cup \{(x,\sin(\frac{1}{x}))\,|\,x \in (0,\frac{1}{\pi}]\} \subseteq \mathbb{R}^2$.
be the "Topologist's Sine Curve" from page 195. As in Exercise 8.4 we refer to the quotient space $W := S/(0,0) \sim (\frac{1}{\pi},0)$ as the *Warsaw Circle*. Determine as many homology groups of the Warsaw circle as possible.



Exercise 144.7. Let $X = S^1$ with the *trivial* topology $\mathcal{T}_{\mathrm{tr}}$, i.e. the only open sets are $\varnothing$ and S^1. Which homology groups can you determine?

Exercise 144.8. Let (X,x_0) be a pointed topological space. We consider the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & \left[\begin{array}{rcl} \Delta^1 & \to & X \\ (1-t,t) & \mapsto & \gamma(t). \end{array}\right] \end{array}$$

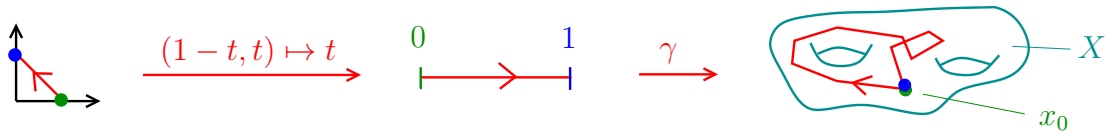
(a) Show that the map $\Phi\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is actually well-defined.

(b) Show that the map $\Phi\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is a homomorphism.

(c) Suppose that X is path-connected.

(i) Show that the map $\Phi\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is surjective.

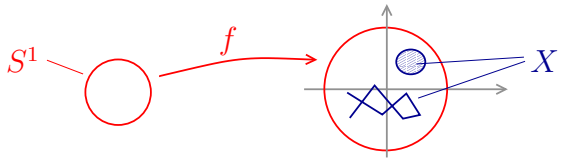
(ii) Show that the map $\Phi\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is in general not injective.



Exercise 144.9. Let X be a non-empty bounded subset of $\mathbb{R}^n$. We pick an $r \in \mathbb{R}_{>0}$ such that X is contained in the bounded n-ball $\overline{B}_r^n$ of radius r. We consider the map

$$\begin{array}{rcl} f\colon S^{n-1} & \to & \mathbb{R}^n \setminus X \\ x & \mapsto & 2r \cdot x. \end{array}$$

Show that the induced map $f_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus X)$ is a monomorphism.



CHAPTER 145

Exact sequences and long exact sequences

In the next Chapter 146, given a topological space X and a subspace $A \subseteq X$, we will introduce the relative chain complex given by the groups $\mathrm{C}_n(X, A) \coloneqq \mathrm{C}_n(X)/\mathrm{C}_n(A)$. To understand the relationship between the homology groups of the chain complexes $\mathrm{C}_*(A)$, $\mathrm{C}_*(X)$ and $\mathrm{C}_*(X, A)$ we will need to develop our algebraic skills.

In this purely algebraic chapter we will therefore introduce the concept of a "(short) exact sequence" and will see that a "short exact sequence" of chain complexes gives rise to a "long exact sequence" of homology groups. These concepts will play an essential role in the theory of homology.

This chapter is organized as follows:

- In Section 145.1 we will introduce exact sequences of abelian groups.
- In Section 145.2 we will study split exact sequences.
- In Section 145.3 we will introduce long exact sequence of homology groups.
- In Section 145.4 we will provide exercises for Chapter 145.

In the next Chapter 146 we will introduce the relative homology groups of a pair of topological spaces.

145.1. **Exact sequences.** We start out with the following definition which appears in many parts of mathematics.

Definition 145.1. A sequence of group[1507] homomorphisms

$$\ldots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

is called exact, if for each n we have

$$\operatorname{im}(f_{n+1}\colon A_{n+1} \to A_n) \;=\; \ker(f_n\colon A_n \to A_{n-1}).$$

In the following lemma we collect a few basic statements about exact sequences which we will use frequently without referring to the lemma.

Lemma 145.2. (Exactness Lemma)

(1) In any exact sequence the composition of two consecutive maps is the zero map.

(2) We have the following equivalences of statements:

$$\begin{aligned}
&\text{the sequence } 0 \to A \xrightarrow{f} B \text{ is exact} &&\iff f \text{ is injective}\\
&\text{the sequence } A \xrightarrow{f} B \to 0 \text{ is exact} &&\iff f \text{ is surjective}\\
&\text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \text{ is exact} &&\iff f \text{ is an isomorphism.}
\end{aligned}$$

[1507] Note that we do not assume that the groups are abelian. In most cases the groups will be abelian though, that is why in this section we denote the trivial group by "0".

(3) Given a monomorphism $i\colon A \to B$ and an epimorphisms $p\colon B \to C$ of groups we have

$$0 \to A \xrightarrow{i} B \xrightarrow{f} C \text{ is exact} \iff i \text{ induces an isomorphism } A \xrightarrow{\cong} \ker(f)$$
$$A \xrightarrow{f} B \xrightarrow{p} C \to 0 \text{ is exact} \iff p \text{ descends to an isomorphism } \operatorname{coker}(f) \xrightarrow{\cong} C.$$

Proof. The lemma follows easily from the definitions. We leave it to the reader to provide the details. ■

Example. Let $f\colon A \to B$ be a homomorphism of groups. It follows from the Exactness Lemma 145.2 that the following sequence is exact

$$0 \to \ker(f) \to A \xrightarrow{f} B \to \operatorname{coker}(f) \to 0.$$ □

145.2. **Split exact sequences.** In many future settings we will be given an exact sequence where we know some of the groups and we will wonder, whether we can determine the missing groups. In this section we will study the simplest instant of such a problem, namely the case of a short exact sequence:

Definition 145.3. A short exact sequence is an exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

which consists of at most three non-trivial groups.

Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence. If we know $\varphi\colon A \to B$ or if we know $\psi\colon B \to C$ then using the Exactness Lemma 145.2 we can read off the third group. Now suppose we know the two "outer" groups A and C. Can we read "read off" B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

could be of the form

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{x\mapsto(x,0)} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{(x,y)\mapsto y} \mathbb{Z}/2\mathbb{Z} \to 0$$

but it could also be of the form[1508]

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{[x]\mapsto[2x]} \mathbb{Z}/4\mathbb{Z} \xrightarrow{[x]\mapsto[x]} \mathbb{Z}/2\mathbb{Z} \to 0.$$

In the remainder of this section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition 145.4. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of groups splits, if there exists a splitting $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Example.

(1) The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto(k,0)} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{(k,l)\mapsto l} \mathbb{Z}/n\mathbb{Z} \to 0$$

[1508] Are there any other possibilities for the isomorphism type of B?

splits. In fact a splitting is given by the homomorphism $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

does not split.

(2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. It follows from the Free Abelian Group–Splitting Lemma 97.11, that if the last group C is a free abelian group, then the short exact sequence splits. □

The following lemma will be useful on numerous occasions.

Lemma 145.5. (Splitting Lemma) Let

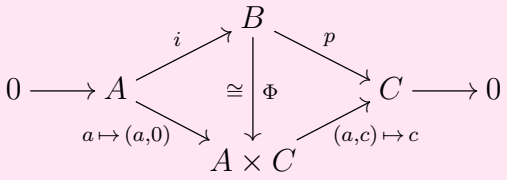
$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. The following three statements are equivalent:

(1) The short exact sequence splits, i.e. there exists a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

(2) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.

(3) There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:



where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.

Remark. The implication (3)⇒(1) and the equivalence (2)⇔(3) of the Splitting Lemma 145.5 also hold if the groups are non-abelian, with basically the same proof. But if the middle group B is not abelian, then the implication (1)⇒(2) of the Splitting Lemma 145.5 does in general not hold. For example, let us consider the exact sequence $0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{i} S_3 \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0$ where S_3 is the permutation group on the three elements, where $i\colon \mathbb{Z}/3\mathbb{Z} \to S_3$ is the homomorphism given by $i(1) = (1\,2\,3)$ and where $p\colon S_3 \to \mathbb{Z}/2\mathbb{Z}$ is the unique epimorphism from the permutation group S_3 onto $\mathbb{Z}/2\mathbb{Z}$. A splitting of p is given by the homomorphism $s\colon \mathbb{Z}/2\mathbb{Z} \to S_3$ that is given by setting $s(1) := (1\,2)$. On the other hand, since p is (essentially) the only epimorphism from S_3 onto a non-trivial group we see there exists no homomorphism $t\colon S_3 \to \mathbb{Z}/3\mathbb{Z}$ with $t \circ i = \mathrm{id}_{\mathbb{Z}/3\mathbb{Z}}$. □

Proof. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups.

(1)⇒(3). Suppose that $s\colon C \to B$ is a splitting, i.e. suppose that s is a homomorphism such that $p \circ s = \mathrm{id}_C$. We consider the map

$$\begin{aligned} \Psi\colon A \times C &\to B \\ (a, c) &\mapsto i(a) + s(c). \end{aligned}$$

Using the fact that B is abelian one can easily verify that the map is a homomorphism.

Claim. The above homomorphism $\Psi\colon A \times C \to B$ is an isomorphism and the inverse $\Phi := \Psi^{-1}\colon B \to A \times C$ has all desired properties.

Proof. First we show that Ψ is an epimorphism. Let $b \in B$. Note that

$$p(b - s(p(b))) \underset{\substack{\uparrow \\ \text{since } p \text{ is a} \\ \text{homomorphism}}}{=} p(b) - \underbrace{(p \circ s)}_{=\mathrm{id}}(p(b)) = p(b) - p(b) = 0.$$

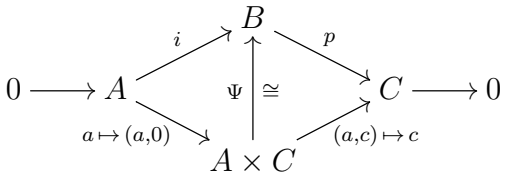
It follows from this calculation and the exactness of the short exact sequence that there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. Then $\Psi(a, p(b)) = i(a) + s(p(b)) = b$.

Next we show that Ψ is a monomorphism. Let $(a, c) \in A \times B$ with $\Psi(a, c) = 0$. Then

$$0 = p(0) = p(\Psi(a,c)) = p(i(a) + s(c)) = \underbrace{(p \circ i)}_{=0}(a) + \underbrace{(p \circ s)}_{=\mathrm{id}_C}(c) = c.$$

Thus we have shown that $c = 0$. It follows that $0 = \Psi(a, c) = i(a) + s(c) = i(a)$. Since i is a monomorphism we see that $a = 0$. In summary we have shown that $(a, c) = 0$.

It remains to show that the following diagram commutes:



It is clear that the left hand triangle commutes. Furthermore it follows from $p \circ i = 0$ and $p \circ s = \mathrm{id}_C$ that the triangle to the right commutes.

(2)⇒(3). Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{aligned} \Phi\colon B &\to A \times C \\ b &\mapsto (t(b), p(b)). \end{aligned}$$

As in the proof of (1)⇒(3) it follows easily from the exactness of the initial sequence that this map is in fact an isomorphism. It is straightforward to verify that Φ has all the other desired properties.

(3)⇒(1), (2). Now suppose that we are given an isomorphism $\Phi\colon B \to A \times C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the maps

$$B \xrightarrow{\Phi} A \times C \xrightarrow{(a,c)\mapsto a} A \quad \text{and} \quad C \xrightarrow{c \mapsto (0,c)} A \times C \xrightarrow{\Phi^{-1}} B$$

have the required properties. ■

Example. Let B be an abelian group and let $0 \to \ker(p) \to B \to \mathbb{Z} \to 0$ be a short exact sequence. Let $b \in B$ with $p(b) = 1$. It follows from the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 (1)⇒(3), that we can write $B = \ker(p) \oplus \mathbb{Z} \cdot b$. □

The above discussion gives us tools for dealing with some short exact sequence. The following lemma allows us to break a long exact sequence into shorter exact sequences.

Lemma 145.6. (Break-into-SES Lemma) Let

$$\ldots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

be an exact sequence of abelian groups. For any $n \in \mathbb{N}_0$ the sequences

$$0 \to \operatorname{coker}\left(A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right) \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

and

$$\ldots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker\left(A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right) \to 0$$

are also exact. In particular

$$0 \to \operatorname{coker}\left(A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right) \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker\left(A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right) \to 0$$

is a short exact sequence.

Proof. The statements follow immediately from the definitions. We leave it to the reader to provide the details. ■

Example. A typical application is that if we are given an exact sequence

$$\ldots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n=0} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \ldots$$

of abelian groups such that the map f_n is the zero map, then the exact sequence can be split "into the two halves" along that zero map, i.e.

$$0 \to A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \ldots$$

and

$$\ldots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \to 0$$

are also exact sequences. In the remainder of this course we will frequently use this fact without explicitly citing the Break-into-SES Lemma 145.6. □

145.3. **Long exact sequence of homology groups.** In the previous two sections we studied (short) exact sequences of abelian groups. In this section we will study short exact sequences of chain complexes and we will see that they give rise to an exact sequence of homology groups.

Definition 145.7. A short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to C_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$$

form a short exact sequence.

We move straight to another, much more clever definition:

Definition 145.8. Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of chain complexes. Let $n \in \mathbb{N}_0$. We define a homomorphism

$$\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$$

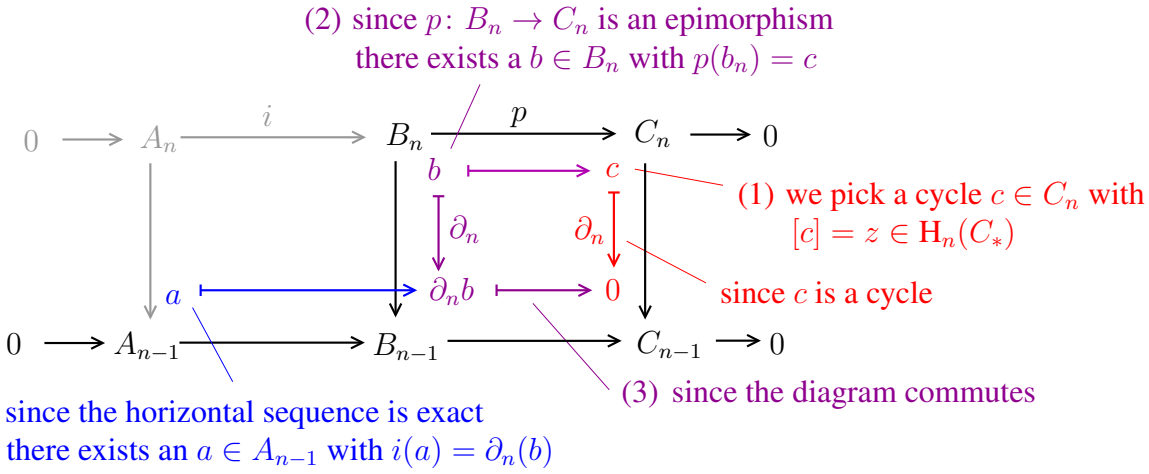
as follows:

(0) Let $z \in \mathrm{H}_n(C_*)$.
(1) We pick an n-cycle $c \in C_n$ with $[c] = z$.
(2) Since $0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$ is exact we know by the Exactness Lemma 145.2 that $p \colon B_n \to C_n$ is an epimorphism. Thus there exists a $b \in B_n$ with $p(b) = c$.
(3) We have

$$p(\partial_n(b)) \underset{\text{since } p \text{ is a chain map}}{=} \partial_n(p(b)) = \partial_n(c) \underset{\text{since } c \text{ is a cycle}}{=} 0.$$

(4) Since $0 \to A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \to 0$ is exact it follows from (3) together with the Exactness Lemma 145.2 that there exists a unique $a \in A_{n-1}$ with $i(a) = \partial_n(b)$.
(5) We set $\partial_n(z) := [a] \in \mathrm{H}_{n-1}(A_*)$.

We refer to this map $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ as the connecting homomorphism.[1509]



The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 145.9. (Connecting Homomorphism Lemma)

(1) Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. For each $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is well-defined and it is indeed a homomorphism.
(2) The connecting homomorphism is natural, i.e. if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i'} & B'_* & \xrightarrow{p'} & C'_* & \longrightarrow & 0 \end{array}$$

[1509] The fact that here the symbol "∂_n" denotes the boundary map of the various chain complexes and also the connecting homomorphism $\mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is somewhat unfortunate.

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes: For every $n \in \mathbb{N}$:

$$\begin{array}{ccc} \mathrm{H}_n(C_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A_*) \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle a_*} \\ \mathrm{H}_n(C'_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A'_*) \end{array}$$

Remark. Let us explain why we say that the connecting homomorphism is "natural". Let $\mathcal{S}$ be the category of short exact sequences of chain complexes where the morphisms are chain maps. The Connecting Homomorphism Lemma 145.9 (2) says that given any $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is a natural transformation from the functor

$$\begin{array}{rcl} \mathcal{S} & \to & \mathcal{A}\mathit{bGroup} \\ (A_* \to B_* \to C_*) & \mapsto & \mathrm{H}_n(C_*) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{S} & \to & \mathcal{A}\mathit{bGroup} \\ (A_* \to B_* \to C_*) & \mapsto & \mathrm{H}_{n-1}(A_*). \end{array} \quad \square$$

Proof. In the proof of the lemma it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{i} & B_{n+1} & \xrightarrow{p} & C_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \\ 0 & \longrightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{p} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{p} & C_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \\ 0 & \longrightarrow & A_{n-2} & \xrightarrow{i} & B_{n-2} & \xrightarrow{p} & C_{n-2} & \longrightarrow & 0. \end{array}$$

The proof of this lemma and the following proposition are done by a "diagram chase". This means that one needs to keep the above commutative diagram in mind and on several occasions one has to use that the vertical maps are boundary maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary and utterly unreadable. In both cases it is thus highly recommended to prove the statements on your own and to ignore the proofs provided below.

For the undeterred reader we now give the fully gory details.

(1) We start out with the following claim:

Claim 1. The map $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is well-defined.

Proof. Let $z \in \mathrm{H}_n(C_*)$. We denote by $c \in C_n$, $b \in B_n$ and $a \in A_{n-1}$ the elements that arise in the definition of $\partial_n(z)$. We need to show that the element a is indeed a cycle and that the definition of $\partial_n(z)$ does not depend on the choices of c and b.

[1509] In the proof we will write the differentials ∂_n of the chain complexes in black and we will write the connecting homomorphisms ∂_n in purple. Hopefully this makes the exposition less confusing.

The proofs of these two statements are not particularly difficult and they are done via "diagram chasing". First we note that

$$i(\partial_{n-1}(a)) \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial_{n-1}(i(a)) = \partial_{n-1}(\partial_n(b)) = 0.$$

Since i is injective we obtain that $\partial_{n-1}(a) = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a] \in \mathrm{H}_{n-1}(A)$ depends only on the homology class $z \in \mathrm{H}_n(C_*)$.

(a) If we chose a different element $\widetilde{b}$ in B_n with $p(\widetilde{b}) = p(b) = c$, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $\widetilde{b} = b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality in B_{n-1}:

$$\partial_n(\widetilde{b}) = \partial_n(b + i(a')) = \partial_n(b) + \partial_n(i(a')) \overset{\substack{\text{since } i \text{ is a chain map} \\ \downarrow}}{=} \underbrace{\partial_n(b)}_{=i(a)} + i(\partial_n(a')) = i(a + \partial_n(a')).$$

Since $[a] = [a + \partial_n(a')] \in \mathrm{H}_{n-1}(A)$ this does not change the homology class.

(b) If we picked a different representative $\widetilde{c}$ for $z \in \mathrm{H}_n(C_*)$, then this representative would be of the form $\widetilde{c} = c + \partial_{n+1}(c')$ for some $c' \in C_{n+1}$. Since the homomorphism $p\colon B_{n+1} \to C_{n+1}$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b + \partial_n(b')) = p(b) + p(\partial_{n+1}(b')) \underset{\substack{\uparrow \\ \text{since } p \text{ is a chain map}}}{=} c + \partial_{n+1}(p(b')) = c + \partial_{n+1}(c') = \widetilde{c} \in C_n.$$

Since $\partial_n(b + \partial_{n+1}(b')) = \partial_n(b)$ we see that we have ended up with the same element in B_{n-1} as before. ⊞

Claim 2. The map $\partial_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is a homomorphism.

Proof. Let $[c_1], [c_2] \in \mathrm{H}_n(C_*)$. For $j = 1, 2$ we choose $b_j \in B_n$ as in the definition of the connecting homomorphism $\partial_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ and we denote by $a_j \in A_{n-1}$ the uniquely determined element with $i(a_j) = \partial_n(b_j)$. Since i and p are homomorphisms we also have $p(b_1 + b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial_n(b_1) + \partial_n(b_2) = \partial_n(b_1 + b_2)$. It follows that

$$\partial_n([c_1 + c_2]) = [a_1 + a_2] = [a_1] + [a_2] = \partial_n([c_1]) + \partial_n([c_2]).$$ ⊞

(2) It follows easily from the definitions that the connecting homomorphisms are natural. We leave the verification of the details to the reader. ■

Now we can formulate the following proposition which explains the name "connecting homomorphism".

Proposition 145.10. (LES-Proposition)

(1) Let

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

be a short exact sequence of chain complexes. The sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B_*) \xrightarrow{p_*} \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A_*) \xrightarrow{i_*} \mathrm{H}_{n-1}(B_*) \xrightarrow{p_*} \ldots$$

is exact. This sequence is called the long exact sequence in homology corresponding to the short exact sequence of chain complexes.

(2) If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i'} & B'_* & \xrightarrow{p'} & C'_* & \longrightarrow & 0 \end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(A_*) & \xrightarrow{i_*} & \mathrm{H}_n(B_*) & \xrightarrow{p_*} & \mathrm{H}_n(C_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A_*) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle a_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_n(A'_*) & \xrightarrow{i'_*} & \mathrm{H}_n(B'_*) & \xrightarrow{p'_*} & \mathrm{H}_n(C'_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A'_*) & \longrightarrow & \ldots \end{array}$$

Proof of the LES-Proposition 145.10. First note that Statement (2) is an immediate consequence of the Chain Complex-H_*-Functor Lemma 143.4 (2) together with the Connecting Homomorphism Lemma 145.9. Thus let us turn to the actual meat, namely the proof of Statement (1).

Let

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

be a short exact sequence of chain complexes. We need to show that the sequence

$$\ldots \to \mathrm{H}_{n+1}(C_*) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(A_*) \xrightarrow{i_*} \mathrm{H}_n(B_*) \xrightarrow{p_*} \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A_*) \xrightarrow{i_*} \mathrm{H}_{n-1}(B_*) \xrightarrow{p_*} \ldots$$

is exact. We examine the exactness at the three groups $\mathrm{H}_n(A_*), \mathrm{H}_n(B_*)$ and $\mathrm{H}_n(C_*)$. Put differently, at each of the three groups we want to show the following two statements:

(i) The image of the map to the left is contained in the kernel of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,

(ii) The kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) (i) Let $[c] \in \mathrm{H}_{n+1}(C_*)$. We want to show that $i_*(\partial_{n+1}([c])) = 0$. We choose $b \in B_{n+1}$ with $p(b) = c$ and we denote by $a \in A_n$ the element with $i(a) = \partial_{n+1}(b)$. Then we have

$$i_*(\partial_{n+1}([c])) \underset{\substack{\uparrow \\ \text{by Definition 145.8 of} \\ \partial_{n+1}\colon \mathrm{H}_{n+1}(C_*) \to \mathrm{H}_n(A_*)}}{=} i_*([a]) = [\partial_{n+1}(b)] = 0 \in \mathrm{H}_n(B_*).$$

(ii) Let $[a] \in \ker(i_*)$. Then $i(a) = \partial_{n+1}(b)$ for some $b \in B_{n+1}$. It follows immediately from the definitions that $\partial_{n+1}([p(b)]) = [a]$.

(B) (i) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\mathrm{im}(i_*) \subseteq \ker(p_*)$.

(ii) Let $b \in \ker(p_*)$. Then $p(b) = \partial_{n+1}(c')$ for some $c' \in C_{n+1}$. Since p is an epimorphism, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. Since p is a chain map we see that $p(\partial_{n+1}(b')) = \partial_{n+1}(p(b')) = \partial_{n+1}(c') = p(b)$. Thus it follows that $p(b - \partial_{n+1}(b')) = 0$, i.e. $b - \partial_{n+1}(b') = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial_{n+1}(b')] = [i(a)] \in \mathrm{im}(i_*)$.

[1509] In the proof we will write the differentials ∂_n of the chain complexes in black and we will write the connecting homomorphisms ∂_n in purple.

(C) (i) Let $[b] \in \mathrm{H}_n(B)$. We have to show that $\partial_n(p_*([b])) = 0$. In the definition of $\partial_n(p_*([b]))$ we need to choose a $\widetilde{b} \in B_n$ with $[p(\widetilde{b})] = p_*([b])$. Of course we take $\widetilde{b} = b$. Since $\partial_n(b) = 0$ it follows immediately that $\partial_n(p_*([b])) = 0$.

(ii) Let $[c] \in \ker(\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*))$. We choose $b \in B_n$ with $p(b) = c$. Recall that there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial_n(b)$. Since we have $[a] = \partial_n([c]) = 0 \in \mathrm{H}_{n-1}(A_*)$ there exists an $a' \in A_n$ with $\partial_n(a') = a$. It follows that

$$\partial_n(b - i(a')) \;=\; \partial_n(b) - \partial_n(i(a')) \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial_n(b) - i(\partial_n(a')) \;=\; \partial_n(b - i(a)) \;=\; 0,$$

i.e. $b - i(a') \in B_n$ is a cycle. On the other hand note that we have the equalities $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial_n) \subseteq \mathrm{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. ∎

We continue with the following lemma, which will often be useful when we compare two long exact sequences of homology groups.

Lemma 145.11. (Five Lemma) Let

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle a} & & \cong\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \cong\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

be a diagram of groups (here for once we do not suppose that the groups are abelian) such that the horizontal sequences are exact and such that every square commutes (if all the groups are abelian, then it suffices that the squares commute up to a sign). If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.[1510]

Proof. One can prove the lemma via a "diagram chase". Filling in the details is precisely the task of Exercise 145.23. ∎

We conclude this chapter with the following lemma, which gets used frequently to "simplify" a given exact sequence:

Lemma 145.12. (Exact Sequence–Cancellation Lemma) Suppose we are given a chain complex of the following form

$$\dots \longrightarrow C_{n+2} \xrightarrow[\partial_{n+2}]{} \begin{matrix} A_{n+1} \\ B_{n+1} \end{matrix} \xrightarrow[\partial_{n+1}]{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} \begin{matrix} A_n \\ B_n \end{matrix} \xrightarrow[\partial_n]{} C_{n-1} \longrightarrow \dots$$

We suppose that the following two hypothesis hold:

(1) The map $e \colon A_{n+1} \to A_n$ is an isomorphism.

(2) (a) The map $g \colon A_{n+1} \to B_n$ is the zero map, or

(b) the map $f \colon B_{n+1} \to A_n$ is the zero map.

[1510] In many applications it suffices to remember the following slightly weaker statement: If the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

Next we consider the chain complex

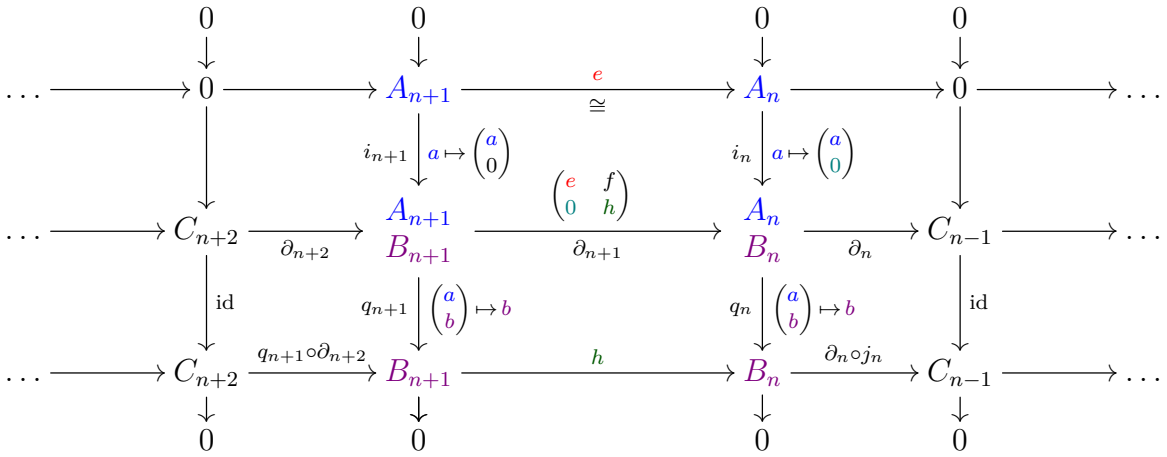
$$\widetilde{D}_* = \ldots \to C_{n+2} \xrightarrow{\text{(projection to } B_{n+1}) \circ \partial_{n+2}} B_{n+1} \xrightarrow{h} B_n \xrightarrow{b \mapsto \partial_n(0,b)} C_{n-1} \to \ldots$$

where we "canceled" A_{n+1} and A_n from the chain complex. There exists a chain map $D_* \to \widetilde{D}_*$ which induces for each $i \in \mathbb{N}_0$ an isomorphism $\mathrm{H}_i(D_*) \to \mathrm{H}_i(\widetilde{D}_*)$.[1511]

Proof under hypothesis (1) and (2a). We now suppose that the following two hypothesis hold:

(1) The map $e\colon A_{n+1} \to A_n$ is an isomorphism, and
(2) the map $g\colon A_{n+1} \to B_n$ is the zero map.

We consider the following vertically expanded diagram:



We make the following initial observations:

- The middle sequence is precisely our original chain complex D_*, where we use a highly suggestive notation to describe the boundary map ∂_{n+1}. Note that with the notation we have $e = p_n \circ \partial_{n+1} \circ i_{n+1}$, in particular we know by hypothesis (a) that e is an isomorphism. Furthermore note that by hypothesis (b) we know that bottom left entry of the matrix is zero.
- The bottom sequence is precisely the modified chain complex $\widetilde{D}_*$.
- Clearly the vertical sequences are exact.

Claim. That the diagram commutes.

Proof. For the most part it is clear that the squares commute. But there are three places where we need to think a little longer:

(i) For the top middle square to commute it is essential that the bottom left corner of the matrix representing ∂_{n+1} is the zero map.
(ii) Next we consider the square with A_n in the upper left corner. So let $a \in A_n$. Since e is an isomorphism we can consider $e^{-1}(a) \in A_{n+1}$. Since the middle sequence is a chain complex we see that $e^{-1}(a) \in A_{n+1}$ gets sent to $0 \in C_{n-1}$ by going right-down-right. Finally, since in (i) we just showed that the middle rectangle commutes we see that e also gets sent to zero by going down-right.

[1511] In practice we mostly care about the case that D_* is exact, then the conclusion is, that $\widetilde{D}_*$ is also exact.

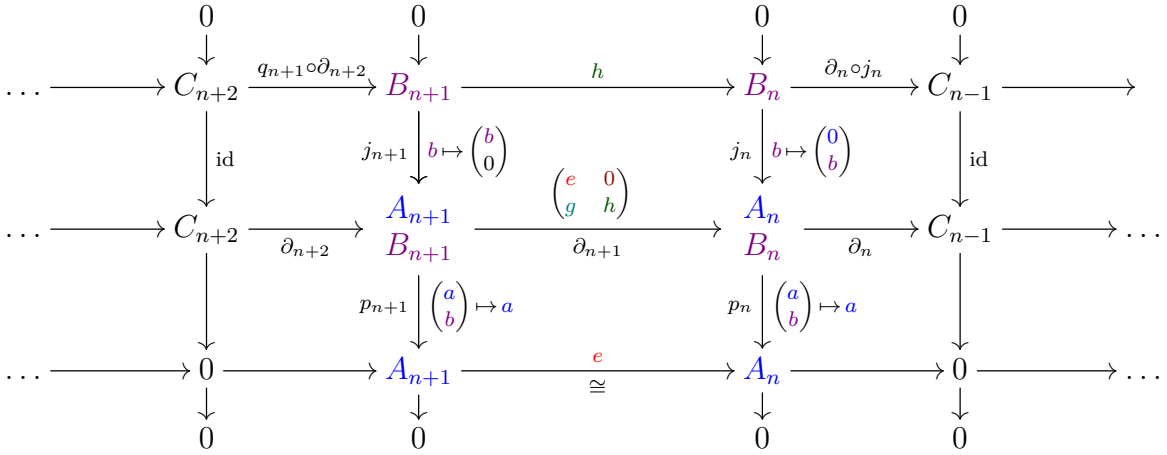
(iii) Next we consider the square with $A_n \oplus B_n$ in the upper left corner. For this square to commute we need to show that the right-down map is zero on A_n. But this follows from (ii). ⊞

It follows from the above discussion that all these maps form a "vertical" short exact sequence of "horizontal" chain complexes. Since e is an isomorphism we see that the homology groups of the top chain complex are zero. It follows easily from the LES-Proposition 145.10 that the bottom vertical maps induce isomorphisms of homology. Thus we see that the bottom vertical maps form a chain map $D_* \to \widetilde{D}_*$ which induces for each $i \in \mathbb{N}_0$ an isomorphism $\mathrm{H}_i(D_*) \to \mathrm{H}_i(\widetilde{D}_*)$. ■

Proof under hypothesis (1) and (2b). We now suppose that the following two hypothesis hold:

(1) The map $e\colon A_{n+1} \to A_n$ is an isomorphism, and
(2) the map $f\colon B_{n+1} \to A_n$ is the zero map.

This time we consider the following diagram:



We now leave it to the reader to modify the previous proof to obtain the desired result. ■

145.4. Exercises for Chapter 145 .

Exercise 145.1. Suppose we are given abelian groups A and B that fit into an exact sequence as follows:

$$0 \to A \to \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 2 & 8 & -4 \\ 0 & 12 & 2 \end{pmatrix}} \mathbb{Z}^2 \to B \to 0.$$

What is the isomorphism type of A and B?

Exercise 145.2. Does there exist a short exact sequence of abelian groups which is of the form $0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0$?

Exercise 145.3. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. Inspired by the Splitting Lemma 145.5 we consider the following two statements:

(1) The short exact sequence splits.

(3′) The group B is isomorphic to $B \to A \times C$.

With these notions we can formulate the exercise:

(a) Show that if A, B and C are finitely generated abelian groups, then (1) and (3′) are equivalent.
(b) Show that in (a) we cannot drop the finite generation hypothesis.

Exercise 145.4. Let
$$\mathbb{Z} \xrightarrow{f} B \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \to 0$$
be an exact sequence of abelian groups, i.e. we know that B is abelian, we know that $g \circ f = 0$, we know that $\operatorname{im}(f) = \ker(g)$ and we know that g is an epimorphism. What are the possible isomorphism types of the abelian group B?

Exercise 145.5. Recall that in Definition 6.23 we defined the notion of a pushout. Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be homomorphisms of abelian groups. Complete the right hand side so that the following two statements are equivalent:

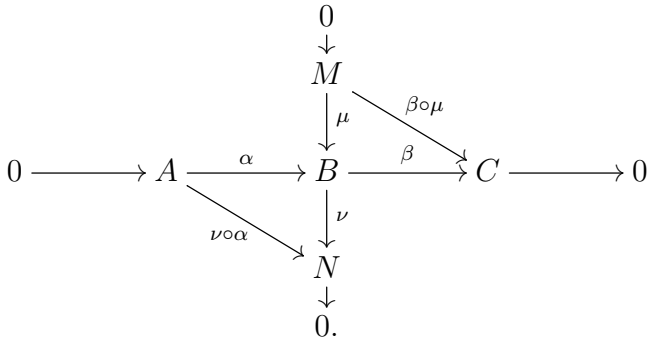
$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \text{ is exact} \quad \Longleftrightarrow \quad \begin{array}{ccc} ?? & \longrightarrow & ?? \\ \downarrow & & \downarrow \\ ?? & \longrightarrow & ??. \end{array}$$
is a pushout diagram in the category *AbGroup* of abelian groups.

Exercise 145.6. Recall that in Definition 6.13 we defined the notion of a pullback. Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be homomorphisms of abelian groups. Complete the right hand side so that the following two statements are equivalent:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \text{ is exact} \quad \Longleftrightarrow \quad \begin{array}{ccc} ?? & \longrightarrow & ?? \\ \downarrow & & \downarrow \\ ?? & \longrightarrow & ??. \end{array}$$
is a pullback diagram in the category *AbGroup* of abelian groups.

Exercise 145.7. Complete the following sentence in the most elegant way possible: a chain complex (C_*, ∂_*) is an exact sequence if and only if ....

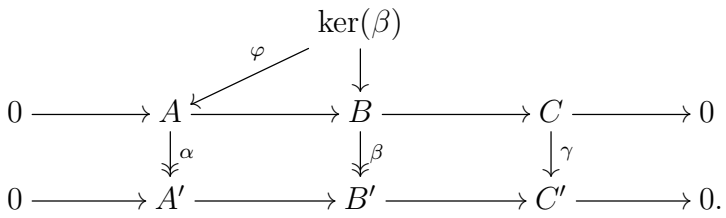
Exercise 145.8. We consider the following diagram of homomorphisms between abelian groups:



We suppose that the horizontal and the vertical sequences are exact.

(a) Show that $\beta \circ \mu$ is a monomorphism if and only if $\nu \circ \alpha$ is a monomorphism.
(b) Show that $\beta \circ \mu$ is an epimorphism if and only if $\nu \circ \alpha$ is an epimorphism.

Exercise 145.9. We consider the following commutative diagram of homomorphisms between abelian groups:



Evidently we assume that the horizontal sequences are exact. Furthermore we assume that $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ are epimorphisms. Show that $\gamma\colon C \to C'$ is an isomorphism.

Exercise 145.10. Let $0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between abelian groups. Show that the following two statements are equivalent:

(1) The sequence is exact and the sequence splits.
(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma\colon G \to F$ and $\tau\colon H \to G$ such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_B$.

Exercise 145.11. Let $\mathbb{F}$ be a field.

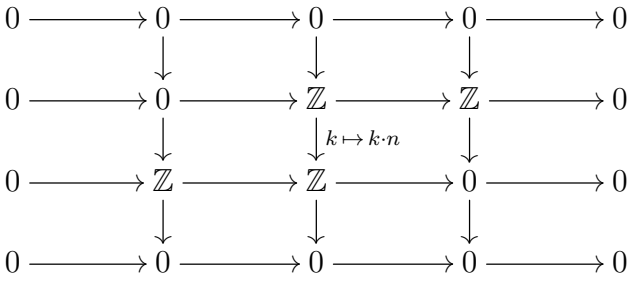
(a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of homomorphisms between finite-dimensional $\mathbb{F}$-vector spaces. Show that $\dim(B) = \dim(A) + \dim(C)$.
(b) Let $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$ be an exact sequence of homomorphisms between finite-dimensional $\mathbb{F}$-vector spaces. Formulate and prove a non-trivial formula that involves the dimensions of all of the C_i.

Remark. This exercise is identical to Exercise 77.1.

Exercise 145.12. Given a finite set S we denote by $|S| \in \mathbb{N}_0$ the number of elements in S.

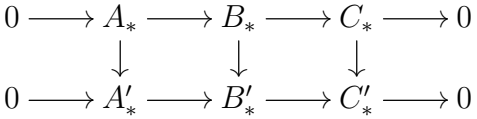
(a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finite abelian groups. How can you relate $|A|$, $|B|$ and $|C|$?
(b) Let $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$ be an exact sequence of homomorphisms between finite abelian groups. Formulate and prove a non-trivial formula that involves the numbers of elements of all of the C_i.

Exercise 145.13. Let $n \in \mathbb{N}$. We consider the following short exact sequence of chain complexes:



Determine the corresponding long exact sequence of homology groups.

Exercise 145.14. Let



be a commutative diagram of short exact sequences of chain complexes. Show that if two of the vertical maps induce isomorphisms on homology, then so does the third.

Exercise 145.15. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0. \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

We assume that the three horizontal sequences are exact and that the two left vertical sequences are exact. Show that the right vertical sequence is also exact.

Exercise 145.16. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

(a) Show that if all columns and the bottom two rows are exact, then the top row is exact.
(b) Show that if all columns and the top two rows are exact, then the bottom row is exact.
(c) Show that if all columns are exact, if the top and bottom row are exact and if the map $B_1 \to B_3$ is the zero map, then the middle row is also exact.

Remark. This statement is sometimes referred to as the *Nine Lemma*.

Exercise 145.17. Let A, B, C and D be abelian groups. We denote by $p_B \colon B \oplus C \to B$ the projection. Suppose that we are given a sequence

$$A \xrightarrow{\alpha} B \oplus C \xrightarrow{\varphi} D$$

of homomorphisms such that the following conditions are satisfied:

(1) We have $\varphi \circ \alpha = 0$.
(2) The map $p_B \circ \alpha \colon A \to B$ is an epimorphism.

We denote by $i_C \colon C \to B \oplus C$ the natural inclusion. Show that the map

$$\begin{array}{rcl} \ker(\varphi \circ i_C \colon C \to D) & \to & \dfrac{\ker(\varphi \colon B \oplus C \to D)}{\operatorname{im}(\alpha \colon A \to B \oplus C)} \\ c & \mapsto & [i_C(c)] \end{array}$$

is well-defined and that it is an isomorphism.

Exercise 145.18. For readers who really really enjoy diagram chases we have the following problem on offer. Suppose we are given the following commutative diagram of chain maps

between chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_* & \longrightarrow & B_* & \xrightarrow{h_B} & C_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle v_B} & & \downarrow & & \\
0 & \longrightarrow & D_* & \xrightarrow{h_D} & E_* & \longrightarrow & F_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_D} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_* & \longrightarrow & H_* & \longrightarrow & J_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the vertical and horizontal sequences are exact. We suppose that the chain complex H_* is acyclic, i.e. we suppose that all homology groups of H_* vanish. Show that for every $k \in \mathbb{N}_0$ the following diagram commutes up to a minus sign:

$$\begin{array}{ccccc}
\mathrm{H}_k(D_*) & \xrightarrow{v_{D*}} & \mathrm{H}_k(G_*) & \xleftarrow[\cong]{\partial_{n+1}} & \mathrm{H}_{k+1}(K_*) \\
\downarrow & & & & \cong\downarrow{\scriptstyle \partial_{n+1}} \\
\mathrm{H}_k(E) & \xrightarrow[\cong]{v_{B*}^{-1}} & \mathrm{H}_k(B) & \xrightarrow{h_{B*}} & \mathrm{H}_k(C_*).
\end{array}$$

Exercise 145.19. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow \\
0 & \longrightarrow & \widetilde{A} & \longrightarrow & \widetilde{B} & \longrightarrow & \widetilde{C} & \longrightarrow & 0.
\end{array}$$

We suppose that the horizontal sequences are exact and that the maps a and c to the left and right are the zero map. Does it follow that the middle map b is also the zero map?

Exercise 145.20. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of abelian groups. Furthermore let Z be an abelian group and let $g\colon Z \to C$ be a group homomorphism. Show that there exists a commutative diagram of group homomorphisms of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\widetilde{\alpha}} & Y & \xrightarrow{\widetilde{\beta}} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0
\end{array}$$

such that the top horizontal sequence is also exact.

Exercise 145.21. Let $\mathbb{F}$ be a field. Given a $(k \times l)$-matrix A over $\mathbb{F}$ we define

$$\operatorname{rank}(A) \;:=\; \dim_{\mathbb{F}} \big(\operatorname{im} \big(\mathbb{F}^l \xrightarrow{v \mapsto A \cdot v} \mathbb{F}^k \big) \big).$$

Now suppose we have the following type of matrices:

- A is an $(k \times l)$-matrix over $\mathbb{F}$.
- B is an $(k \times m)$-matrix over $\mathbb{F}$.
- C is an $(l \times n)$-matrix over $\mathbb{F}$.
- D is an $(m \times n)$-matrix over $\mathbb{F}$.

Show that if

$$\begin{pmatrix} A & B \end{pmatrix} \cdot \begin{pmatrix} C \\ D \end{pmatrix} = 0,$$

and if A has rank l, then $\operatorname{rank}(D) = \operatorname{rank} \begin{pmatrix} C \\ D \end{pmatrix}$.
Hint. Apply the LES-Proposition 145.10.

Exercise 145.22. For the following conditions (a)–(d) provide a short exact sequence of chain complexes $0 \to A_* \to B_* \to C_* \to 0$ such that the connecting homomorphism $\mathrm{H}_1(C_*) \to \mathrm{H}_0(A_*)$ is

(a) ... the zero homomorphism,
(b) ... an isomorphism,
(c) ... a monomorphism but not an epimorphism,
(d) ... an epimorphism but not a monomorphism.

Exercise 145.23.

(a) Let

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

be a commutative diagram of homomorphisms between groups (we do not assume that the groups are abelian) such that the horizontal sequences are exact. Show that if a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.

(b) Does the statement in (a) also hold, if a injective and if e is surjective ?

Remark. The task of this exercise is to provide a proof of the Five Lemma 145.11.

Exercise 145.24. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of homomorphisms between abelian groups. Show that there exists an exact sequence

$$\begin{array}{l} 0 \longrightarrow \ker(a) \xrightarrow{\varphi} \ker(b) \xrightarrow{\psi} \ker(c) \\ \qquad\qquad \longrightarrow \operatorname{coker}(a) \xrightarrow{\varphi'} \operatorname{coker}(b) \xrightarrow{\psi'} \operatorname{coker}(c) \longrightarrow 0. \end{array}$$

This statement is sometimes referred to as the "Snake Lemma".
Hint. Try to prove the statement with as little effort as possible.

Exercise 145.25. Let $m \in \mathbb{Z}$. Suppose we are given a commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle \cdot m} & & \downarrow{\scriptstyle \cdot m} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \cdot m} & & \downarrow{\scriptstyle \cdot m} \\ A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \end{array}$$

of homomorphisms between abelian groups such that the horizontal sequences are exact. Note that the two outermost vertical maps to the left and right are given by multiplication

by m. For which $m \in \mathbb{Z}$ does it follow that the middle vertical map is also given by multiplication by m?

Exercise 145.26. We suppose that we have the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \widetilde{A} & \longrightarrow & \widetilde{B} & \longrightarrow & \widetilde{C} & \longrightarrow & 0. \end{array}$$

Furthermore we assume that the horizontal sequences are exact. Which of the following conclusions hold?

(a)	if α and γ are monomorphisms	then β is a monomorphism ?
(b)	if α and γ are epimorphisms	then β is an epimorphism ?
(c)	if α and γ are isomorphisms	then β is an isomorphism ?
(d)	if α and γ are the identity	then β is the identity ?
(e)	if α and γ are the zero map	then β is the zero map ?

Exercise 145.27. We suppose that we are given an exact sequence of abelian groups of the following form:

$$\cdots \longrightarrow C_{n+2} \xrightarrow[\partial_{n+2}]{} \begin{matrix} A_{n+1} \\ B_{n+1} \end{matrix} \xrightarrow[\partial_{n+1}]{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} \begin{matrix} A_n \\ B_n \end{matrix} \xrightarrow[\partial_n]{} C_{n-1} \longrightarrow \cdots$$

We suppose that $e\colon A_{n+1} \to A_n$ is an isomorphism. Show that if $f\colon B_{n+1} \to A_n$ and $h\colon A_{n+1} \to B_n$ are both non-zero, then we cannot necessarily cancel A_n and A_{n+1} from the sequence without affecting the homology groups.

Remark. This exercise shows that in the Exact Sequence–Cancellation Lemma 145.12 we cannot just drop the hypothesis that f or g is the zero map.

CHAPTER 146

Relative homology

Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. Does there exist a relationship between the homology groups of X and A? The first, very optimistic, guess might be that perhaps the projection $X \to X/A$ induces an isomorphism $\mathrm{H}_k(X)/i_*(\mathrm{H}_k(A)) \to \mathrm{H}_k(X/A)$. But this idea breaks down almost immediately, for example if we have $X = A$, then for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X)/i_*(\mathrm{H}_k(A)) = 0$ but by the H_0-Proposition 143.25 we have $\mathrm{H}_0(X/A) \cong \mathrm{H}_0(\{*\}) \cong \mathbb{Z}$.

This simple example shows that the covariant functor $X \to \mathrm{H}_k(X)$ does not preserve quotients. But recall that this functor is really the composition of the two covariant functors $X \mapsto (\mathrm{C}_*(X), \partial_*)$ and $(C_*, \partial_*) \mapsto \mathrm{H}_k((C_*, \partial_*))$. In this and the following chapters we will study the question to which degree the covariant functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves quotients (at least say up to chain homotopy).

This chapter is organized as follows:

- In Section 146.1 we will define the relative homology groups $\mathrm{H}_k(X, A)$ of a pair of topological spaces.
- In Section 146.2 we will introduce the long exact sequence in homology of a triple (X, B, A)
- In Section 146.3 we will study the relationship between relative homology groups and homotopies.
- In Section 146.4 we will provide exercises for Chapter 146.

In the next Chapter 147 we will introduce the reduced homology groups of a topological space.

146.1. **Definition of the relative homology groups.** As we just discussed it is worth exploring whether the chain complex functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves taking quotients. This question motivates the following definition, which will play a major role for the remainder of these notes.

Definition 146.1. Let (X, A) be a pair of topological spaces.[1512] We view each chain group $\mathrm{C}_n(A)$ as a subgroup of $\mathrm{C}_n(X)$. Thus we can define

$$\mathrm{C}_n(X, A) \coloneqq \mathrm{C}_n(X)/\mathrm{C}_n(A)$$

and we consider the boundary map[1513]

$$\begin{aligned} \partial_n\colon \mathrm{C}_n(X, A) &\to \mathrm{C}_{n-1}(X, A) \\ [c] &\mapsto [\partial_n(c)]. \end{aligned}$$

We refer to

$$\ldots \xrightarrow{\partial_{n+2}} \mathrm{C}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X, A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \ldots$$

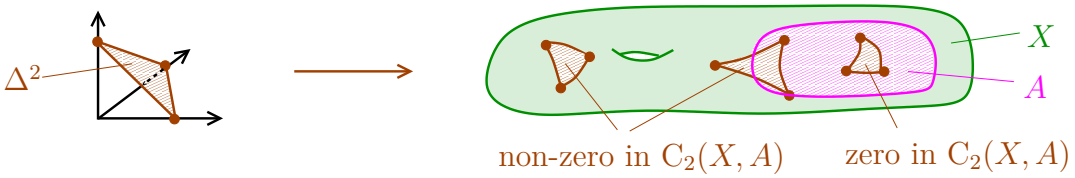
as the chain complex of the pair (X, A).[1514] Given any $n \in \mathbb{N}_0$ we define the n-th relative homology group of (X, A) as follows:

$$\mathrm{H}_n(X, A) \coloneqq \mathrm{H}_n(\mathrm{C}_*(X, A)).$$

Remark. Let (X, A) be a pair of topological spaces and let $n \in \mathbb{N}_0$. We write

$$S \coloneqq \{\sigma\colon \Delta^n \to X \mid \sigma \text{ is continuous and } \operatorname{im}(\sigma) \cap (X \setminus A) \neq \varnothing\},$$

i.e. S is the set of all singular n-simplices of X whose image is not completely contained in A.



Then it is straightforward to show that the map

$$\begin{aligned} \mathbb{Z}^{(S)} &\to \mathrm{C}_n(X, A) \\ \sum_{i=1}^{k} r_i \cdot \sigma_i &\mapsto \Big[\sum_{i=1}^{k} r_i \cdot \sigma_i\Big] \end{aligned}$$

is an isomorphism. In particular we see that $\mathrm{C}_n(X, A)$ is a free abelian group. □

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $\mathrm{C}_n(\varnothing) = 0$ and thus we have a natural isomorphism $\mathrm{C}_n(X, \varnothing) = \mathrm{C}_n(X)/\{0\} \xrightarrow{\cong} \mathrm{C}_n(X)$, i.e. we have a natural isomorphism $\mathrm{H}_n(X, \varnothing) \xrightarrow{\cong} \mathrm{H}_n(X)$. In the following we will use these natural isomorphisms to freely go back and forth between the notations $\mathrm{H}_n(X, \varnothing)$ and $\mathrm{H}_n(X)$.

(2) Whereas it is often difficult to write down singular cycles in absolute chain complexes $\mathrm{C}_*(X)$, it can be surprisingly easy to write down cycles in a relative chain complex $\mathrm{C}_*(X, A)$: Let $\sigma\colon \Delta^n \to X$ be a continuous map with $\sigma(\partial\Delta^n) \subseteq A$. Note that

$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n \underset{\uparrow}{=} 0 \in \mathrm{C}_{n-1}(X, A) = \mathrm{C}_{n-1}(X)/\mathrm{C}_{n-1}(A).$$

since $\sigma \circ i_j^n\colon \Delta^{n-1}$ takes values in A
we see that $\sigma \circ i_j^n \in \mathrm{C}_{n-1}(A)$

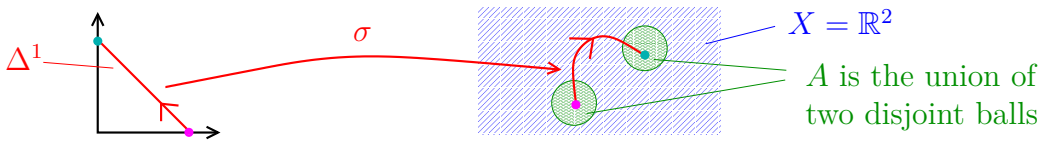
Thus we have shown that *any continuous map* $\sigma\colon \Delta^n \to X$ *with* $\sigma(\partial\Delta^n) \subseteq A$ *defines a cycle in the chain complex* $\mathrm{C}_n(X, A)$.

As an example we consider $X = \mathbb{R}^2$ with the subspace $A = B_1 \sqcup B_2$ that is given by the union of two disjoint closed disks in $\mathbb{R}^2$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0, 1) \in B_1$ and $\sigma(1, 0) \in B_2$. It follows from the above discussion that σ is a cycle in $\mathrm{C}_*(X, A)$. Later, on page 2803, we will see whether or not $[\sigma] \in \mathrm{H}_1(X, A)$ is non-trivial.

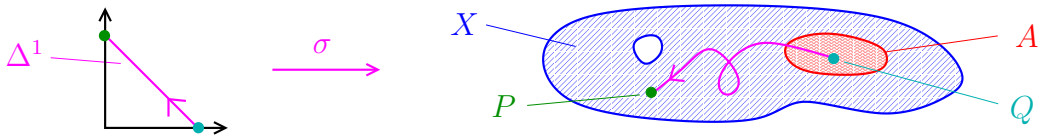
[1512] Recall that according to Definition 19.23 this means that X is a topological space and $A \subseteq X$ is a subspace.

[1513] The map ∂_n is well-defined, since for any $c \in \mathrm{C}_n(A) \subseteq \mathrm{C}_n(X)$ the boundary $\partial_n(c)$ obviously lies in $\mathrm{C}_{n-1}(A) \subseteq \mathrm{C}_{n-1}(X)$.

[1514] Since $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-2}(X)$ is the zero homomorphism it follows immediately that the composition $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n-2}(X, A)$ is also the zero map.



(3) If X is a path-connected topological space and if A is non-empty, then $\mathrm{H}_0(X, A) = 0$. This can be seen as follows: First note that $\mathrm{H}_0(X, A)$ is a quotient of $\mathrm{C}_0(X)$, which is generated by singular 0-simplices, i.e. by points. Thus it suffices to show that for each $P \in X$ we have $[P] = 0 \in \mathrm{H}_0(X, A)$. Thus let $P \in X$ be a point. Since A is non-empty we can pick a point $Q \in A$. Since X is path-connected we can pick a continuous map $\sigma\colon \Delta^1 \to X$ such that $\sigma(0, 1) = P$ and $\sigma(1, 0) = Q$.



We now see that

$$\partial_1([\sigma]) \;=\; [\partial_1(\sigma)] \underset{\uparrow}{=} [P - Q] \;=\; [P] - [Q] \underset{\uparrow}{=} [P].$$

see page 2751

since $Q \in A$ we have $[Q] = 0 \in \mathrm{C}_0(X, A)$

We have thus shown that $\partial_1([\sigma]) = [P]$, which implies that $[P] = 0 \in \mathrm{H}_0(X, A)$. □

As always we expect that relative homology groups are functorial in a suitable way. This thought leads us to the following definition:

Definition 146.2.

(1) As in Definition 19.23 we refer to the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X, A), (Y, B)) &:= \text{all continuous maps } f\colon X \to A \text{ with } f(A) \subseteq B \end{aligned}$$

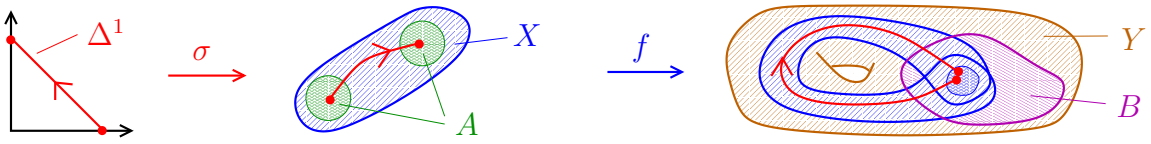
with the usual composition of maps as the category of pairs of topological spaces.

(2) Let $f\colon (X, A) \to (Y, B)$ be a continuous map of pairs of topological spaces, i.e. a continuous map $f\colon X \to Y$ with $f(A) \subseteq B$. If follows from $f(A) \subseteq B$ that the induced chain maps $f_*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ from the H_n–Functor Lemma 143.20 descends to a chain map

$$f_*\colon \mathrm{C}_n(X, A) \;\to\; \mathrm{C}_n(Y, B).$$

In particular we also get induced homomorphisms

$$f_*\colon \mathrm{H}_n(X, A) \;\to\; \mathrm{H}_n(Y, B).$$



The following lemma is a variation on the C_*–Functor Lemma 143.16 and the H_n–Functor Lemma 143.20. Often we will use the lemma without actually referring to it.

Lemma 146.3. (Relative-H_*–Functor Lemma) The maps

$$\begin{array}{rcl}(X,A) & \mapsto & (\mathrm{C}_*(X,A),\partial_*)\\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f_*\colon \mathrm{C}_*(X,A)\to \mathrm{C}_*(Y,B))\end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes and for each $n\in\mathbb{N}_0$ the maps

$$\begin{array}{rcl}(X,A) & \to & \mathrm{H}_n(X,A)\\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f_*\colon \mathrm{H}_n(X,A)\to \mathrm{H}_n(Y,B))\end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups.

Proof. The lemma follows almost immediately from the definitions. ∎

146.2. **The long exact sequence of a triple.** Let (X,A) be a pair of topological spaces. In the following we want to consider the relationship between the homology groups of A, X and (X,A). Cleverly we prepared the ground in Chapter 145. Using the tools from the previous chapter we can easily show the existence of interesting long exact sequences of homology groups.

In fact it is convenient to study a slightly more general setup, which leads us to the harmless notion of a triple of topological spaces:

Definition 146.4.

(1) A triple of topological spaces is a triple (X,B,A) where X is a topological space and $A\subseteq B\subseteq X$ are subspaces.
(2) A map $f\colon (X,B,A)\to(X',B',A')$ between triples of topological spaces is a continuous map $f\colon X\to X'$ such that $f(B)\subseteq B'$ and $f(A)\subseteq A'$.

We can now formulate the main result of this section:

Proposition 146.5. (LES-of-Triple Proposition) Let X be a topological space and let $A\subseteq B\subseteq X$ be two subspaces. Let $i\colon (B,A)\to(X,A)$ and by $p\colon (X,A)\to(X,B)$ be the natural maps.

(1) For each $n\in\mathbb{N}_0$ the following map is a well-defined homomorphism:[1515]

$$\begin{array}{rcl}\partial_n\colon \mathrm{H}_n(X,B) & \to & \mathrm{H}_{n-1}(B,A)\\ \Big[\sum_{j=1}^k s_j\cdot(\sigma_j\colon\Delta^n\to X)\Big] & \mapsto & \Big[\sum_{j=1}^k s_j\cdot \underbrace{\partial_n(\sigma_j)}_{\text{singular } (n-1)\text{-chain in } B}\Big].\end{array}$$

We refer to ∂_n as a connecting homomorphism of the triple (X,B,A).

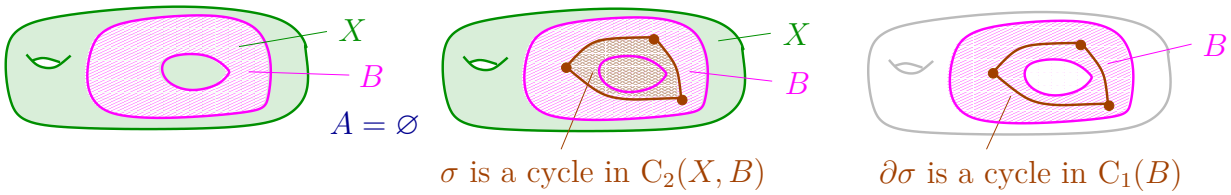
(2) The connecting homomorphisms are natural. In particular for any continuous map $f\colon (X,A,B)\to(\widetilde{X},\widetilde{A},\widetilde{B})$ of triples of topological spaces the following diagram commutes:[1516]

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(B,A) & \xrightarrow{i_*} & \mathrm{H}_n(X,A) & \xrightarrow{p_*} & \mathrm{H}_n(X,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B,A) & \longrightarrow & \ldots\\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\ldots & \longrightarrow & \mathrm{H}_n(\widetilde{B},\widetilde{A}) & \xrightarrow{\widetilde{i}_*} & \mathrm{H}_n(\widetilde{X},\widetilde{A}) & \xrightarrow{\widetilde{p}_*} & \mathrm{H}_n(\widetilde{X},\widetilde{B}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\widetilde{B},\widetilde{A}) & \longrightarrow & \ldots.
\end{array}$$

(3) The maps

$$\ldots \to \mathrm{H}_n(B,A) \xrightarrow{i_*} \mathrm{H}_n(X,A) \xrightarrow{p_*} \mathrm{H}_n(X,B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B,A) \to \ldots$$

form an exact sequence. It is called the called the long exact sequence (LES) of the triple (X,B,A).



Proof.

(1) First note that it follows immediately from the definitions and the third isomorphism theorem in group theory[1517] that

$$0 \to \mathrm{C}_*(B,A) \xrightarrow{i_*} \mathrm{C}_*(X,A) \xrightarrow{p_*} \mathrm{C}_*(X,B) \to 0$$

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as introduced in Definition 145.8, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined homomorphisms.

Proof. The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_n(B,A) & \longrightarrow & \mathrm{C}_n(X,A) & \longrightarrow & \mathrm{C}_n(X,B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \\
0 & \longrightarrow & \mathrm{C}_{n-1}(B,A) & \longrightarrow & \mathrm{C}_{n-1}(X,A) & \longrightarrow & \mathrm{C}_{n-1}(X,B) & \longrightarrow & 0.
\end{array}$$

Let $z \in \mathrm{H}_n(X,B)$. Now we will determine the image of z under the connecting homomorphism $\mathrm{H}_n(X,B) \to \mathrm{H}_{n-1}(B,A)$ which during the proof of the claim we denote by ∂_n^c.

We pick a representative $\sum_{j=1}^{k} s_j \cdot (\sigma_j \colon \Delta^n \to X) \in \mathrm{C}_n(X)$ of $z \in \mathrm{H}_n(X,B)$. Note that $\sum_{j=1}^{k} s_j \cdot (\sigma_j \colon \Delta^n \to X)$ also defines an element in $\mathrm{C}_n(X,A)$. We take the boundary of the above singular n-chain under the boundary map $\partial_n \colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n-1}(X,A)$ and we end up with the singular n-chain $\sum_{j=1}^{k} s_j \cdot \partial_n(\sigma_j) \in \mathrm{C}_{n-1}(X,A)$. As we have seen in the proof of the Connecting Homomorphism Lemma 145.9 this singular n-chain lies in fact in $\mathrm{C}_{n-1}(B,A) \subseteq \mathrm{C}_{n-1}(X,A)$. The homology class thereof is by definition precisely

[1514] Here we use that every homology class in $\mathrm{H}_n(X,B)$ admits by definition a representative in the quotient $\mathrm{C}_n(X,B) = \mathrm{C}_n(X)/\mathrm{C}_n(B)$, so it can be represented by a singular n-chain in X.

[1515] We consider the category of triples of topological spaces. The connecting homomorphisms ∂_n define a natural transformation from the functor $(X,B,A) \to \mathrm{H}_n(X,B)$ to the functor $(X,B,A) \to \mathrm{H}_{n-1}(B,A)$.

[1516] The fact that ∂_n is a natural transformation means precisely that the third square commutes.

[1517] The third isomorphism theorem in group theory implies that we have a natural isomorphism $(\mathrm{C}_*(X)/\mathrm{C}_*(A))/(\mathrm{C}_*(B)/\mathrm{C}_*(A)) \cong \mathrm{C}_*(X)/\mathrm{C}_*(B)$.

$\partial_n^c(z) \in \mathrm{H}_{n-1}(B, A)$. Summarizing we have shown that

$$\partial_n^c(z) \;=\; \Big[\sum_{j=1}^{k} s_j \cdot \partial_n(\sigma_j)\Big].$$

This shows that the map defined in (1) agrees with ∂_n^c. ⊞

(2) It follows from the above claim together with the Connecting Homomorphism Lemma 145.9 that the connecting homomorphisms are natural.

(3) It follows from the above claim together with the LES-Proposition 145.10 that the given sequence is exact. ■

In a second we will state an important special case of the LES-of-Triple Proposition 146.5. This will require the following notation:

Definition 146.6. Let X be a topological space, let $A \subseteq X$ be a subspace and let $n \in \mathbb{N}_0$. We introduce the following homomorphisms[1518]

$$\mathrm{H}_n(X) \xrightarrow{\mathrm{id}} \mathrm{H}_n(X, \varnothing) \longrightarrow \mathrm{H}_n(X, A)$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{=:\text{я}=:\text{Я}_{X,A}}$$

Now we can formulate the promised proposition.

Proposition 146.7. (LES-of-Pair Proposition)

(1) Let X be a topological space and let $B \subseteq X$ be a subspace. The connecting homomorphism of the LES-of-Triple Proposition 146.5 applied to $A = \varnothing$ gives rise to an exact sequence

$$\begin{array}{ccccccccccc} \ldots & \to & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \to & \ldots \\ & & & & & & \Big[\sum_{j=1}^{k} s_j \cdot \sigma_j\Big] & \mapsto & \Big[\sum_{j=1}^{k} s_j \cdot \partial_n(\sigma_j)\Big] & & \end{array}.$$

This exact sequence is called the long exact sequence of the pair (X, B).

(2) The connecting homomorphisms in (1) are natural. In particular, given a continuous map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \longrightarrow & \ldots \end{array}$$

Proof. This proposition follows immediately from the LES-of-Triple Proposition 146.5 applied to $A = \varnothing$. ■

Example. As on page 2798 we consider the topological space $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By the LES-of-Pair Proposition 146.7 we have a long exact sequence

$$\mathrm{H}_2(X) \to \mathrm{H}_2(X, A) \to \mathrm{H}_1(A) \xrightarrow{i_*} \mathrm{H}_1(X) \to \mathrm{H}_1(X, A) \to \mathrm{H}_0(A) \to \mathrm{H}_0(X) \to \mathrm{H}_0(X, A) \to 0.$$

[1518] We apologize to whoever wants to teach out of these notes for using Cyrillic symbols.

By the Contractible-H_*-Corollary 144.15 together with the Path-Component–H_*–Lemma 143.21 we have $\mathrm{H}_i(X) = \mathrm{H}_i(\mathbb{R}^2) = 0$ and also $\mathrm{H}_i(A) = \mathrm{H}_i(B_1) \oplus \mathrm{H}_i(B_2) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X, A) = 0$ for $i \geq 2$.

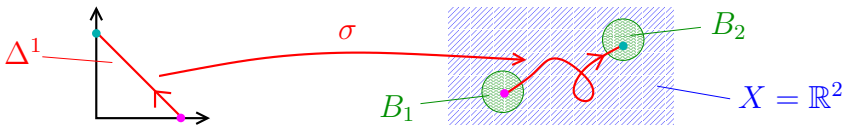
Next let $\epsilon\colon \mathrm{H}_0(\mathbb{R}^2) \to \mathbb{Z}$ be the augmentation map. Since $\mathbb{R}^2$ is path-connected we know by the H_0-Proposition 143.25 that this map is an isomorphism. Similarly we denote by $\mu_1\colon \mathrm{H}_0(B_1) \to \mathbb{Z}$ and $\mu_2\colon \mathrm{H}_0(B_2) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_1(X, A) & \xrightarrow{\partial_1} & \mathrm{H}_0(A) & \xrightarrow{i_*} & \mathrm{H}_0(X) & \longrightarrow & \mathrm{H}_0(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mu_1 \oplus \mu_2}{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong}{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{H}_1(X, A) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{H}_0(X, A) & \longrightarrow & 0.
\end{array}$$

It follows easily from the Augmentation Lemma 143.24 that the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$. It follows that

$$\mathrm{H}_1(X, A) \underset{\text{by the Exactness Lemma 145.2 (3)}}{\cong} \ker\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \cong \mathbb{Z} \qquad \text{and} \qquad \mathrm{H}_0(X, A) \underset{\text{by the Exactness Lemma 145.2 (3)}}{\cong} \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \cong 0.$$

Finally we want to find an explicit generator of the relative homology group $\mathrm{H}_1(X, A) \cong \mathbb{Z}$. As on page 2798 let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0, 1) \in B_2$ and $\sigma(1, 0) \in B_1$.



Note that

$$((\mu_1 \oplus \mu_2) \circ \partial_1)([\sigma]) \underset{\text{see the description of the connecting homomorphism } \partial_1\colon \mathrm{H}_1(X, A) \to \mathrm{H}_0(A) \text{ in the LES-of-Triple Proposition 146.5}}{=} (\mu_1 \oplus \mu_2)([\partial_1 \sigma]) = (\mu_1 \oplus \mu_2)([\underbrace{\sigma(0, 1)}_{\in B_2}] - [\underbrace{\sigma(1, 0)}_{\in B_1}]) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $\mathrm{H}_1(X, A) \cong \mathbb{Z}$. In particular it is non-zero. □

We conclude this section with two more theoretical applications of the LES-of-Pair Proposition 146.7.

Corollary 146.8. (Homotopy Equivalence-Relative-H_*-Corollary) In the following let $f\colon (X, A) \to (Y, B)$ be a continuous map of pairs of topological spaces. If $f\colon X \to Y$ is a homotopy equivalence and if $f\colon A \to B$ is a homotopy equivalence[1519], then for every $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$ is an isomorphism.

[1519] In many applications $f\colon (X, A) \to (Y, B)$ will be an inclusion, where X is a deformation retract of Y and A is a deformation retract of B. It then follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $f\colon X \to Y$ and $f\colon A \to B$ are homotopy equivalences.

Example. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that for each $k \in \mathbb{N}_0$ the inclusion-induced homomorphisms $\mathrm{H}_n(\overline{B}^k, S^{k-1}) \to \mathrm{H}_n(\overline{B}^k, \overline{B}^k \setminus \{0\})$ are isomorphisms. □

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \xrightarrow{i_*} & \mathrm{H}_{n-1}(X) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\ldots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \xrightarrow{i_*} & \mathrm{H}_{n-1}(Y) & \longrightarrow & \ldots
\end{array}$$

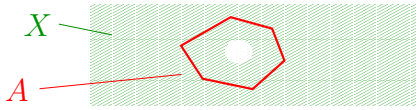
By the LES-of-Pair Proposition 146.7 (2) we know that the diagram commutes and that the horizontal sequences are exact. Furthermore note that it follows from our hypothesis on f and the Homotopy Equivalence-H_*-Proposition 144.14 that the two vertical maps to the left and the two vertical maps to the right are isomorphisms. It follows from the Five Lemma 145.11 that the middle vertical map is, as promised, an isomorphism. ■

Corollary 146.9. (Deformation Retract-Relative H_*-Corollary) Let X be a topological space. If $A \subseteq X$ is a deformation retract of X, then $\mathrm{H}_n(X,A) = 0$ for all $n \in \mathbb{N}_0$.

Proof. Let $i\colon A \to X$ be the inclusion map. By the LES-of-Pair Proposition 146.7 we have a long exact sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_n(X) \to \mathrm{H}_n(X,A) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_{n-1}(X) \to \ldots$$

it follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $i\colon A \to X$ is a homotopy equivalence, thus it follows from the Homotopy Equivalence-H_*-Proposition 144.14 that i_* is an isomorphism

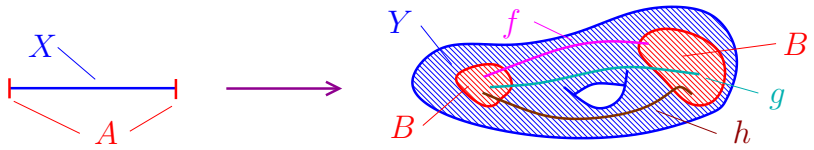


A is a deformation retract of X therefore each $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_k(X)$ is an isomorphism

Since the maps $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_n(X)$ are isomorphisms and since the sequence is exact we see that the relative homology groups $\mathrm{H}_n(X,A)$ are all zero. ■

146.3. **Relative homology groups and homotopies.** Next we recall Definition 19.24.

Definition 146.10. We say that two continuous maps $f, g\colon (X,A) \to (Y,B)$ between pairs of topological spaces are homotopic, if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$, and such that $F(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.



$f, g\colon (X,A) \to (Y,B)$ are homotopic but $f, h\colon (X,A) \to (Y,B)$ are not homotopic

The following proposition extends the Homotopy-H_*-Proposition 144.12 to the relative case.

Proposition 146.11. (Homotopy-H_*-Proposition) Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces. If f and g are homotopic, then the chain maps $f_*, g_*\colon \mathrm{C}_*(X, A) \to \mathrm{C}_*(Y, B)$ are chain homotopic, in particular for every $n \in \mathbb{N}_0$ we have

$$f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B).$$

Proof. Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces and let $F\colon X \times [0, 1] \to Y$ be a homotopy between f and g as maps of pairs of topological spaces.[1520] Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0, 1])$ be the singular chain that we introduced in Notation 144.16. On page 2776 we showed that the homomorphisms

$$\begin{aligned} P_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(Y) \\ \sum_{i=1}^{k} s_i \cdot (\sigma\colon \Delta^n \to X) &\mapsto \sum_{i=1}^{k} s_i \cdot (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n) \end{aligned}$$

define a chain homotopy between the maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$, i.e. for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = f_* - g_*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y).$$

Since F restricts to a map $A \times [0, 1] \to B$ we see that these homomorphisms P_n have the property that $\mathrm{C}_n(A)$ gets sent to $\mathrm{C}_{n+1}(B)$. Thus the maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ descend to homomorphisms $\overline{P}_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n+1}(Y, B)$. These satisfy

$$\partial_{n+1} \circ \overline{P}_n + \overline{P}_{n-1} \circ \partial_n = f_* - g_*\colon \mathrm{C}_n(X, A) \to \mathrm{C}_n(Y, B),$$

i.e. the maps $\overline{P}_n$ form a chain homotopy between the maps f_* and g_* from $\mathrm{C}_*(X, A)$ to $\mathrm{C}_*(Y, B)$. It follows from this observation together with the Chain Homotopy–H_*-Lemma 144.5 that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. ∎

146.4. Exercises for Chapter 146 .

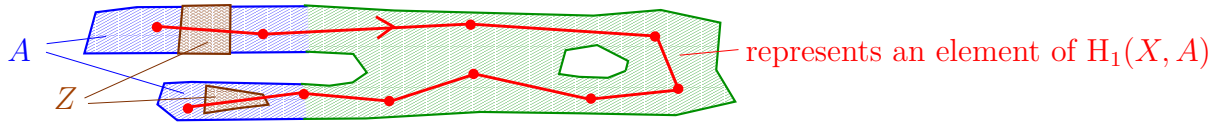
Exercise 146.1. Let $X = \mathbb{R}$ and let $A = \bigcup_{i \in \mathbb{Z}} [2i, 2i + 1]$. Compute the relative homology groups $\mathrm{H}_*(X, A)$ and for each homology group provide a family of cycles in $\mathrm{C}_*(X, A)$ which correspond to a basis of the homology groups.

Exercise 146.2.

(a) Let X be a topological space, let $A \subseteq X$ be an open subset and let $Z \subseteq X$ be a closed subset with $Z \subseteq A$. Show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an epimorphism

$$\mathrm{H}_1(X \setminus Z, A \setminus Z) \to \mathrm{H}_1(X, A).$$

Hint. Use the Lebesgue Lemma 10.8 and the Singular 1-Simplex Lemma 143.28.
Remark. In the Excision Theorem 148.1 we will prove the much stronger result that the above map is actually an isomorphism.



[1520] In most cases we just want the statement that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. This statement follows easily from the LES-of-Pair Proposition 146.7 (2), the Homotopy-H_*-Proposition 144.12 and the Five Lemma 145.11.

(b) Show that there exists an epimorphism $\mathbb{Z} \to \mathrm{H}_1(S^1)$.
Hint. Use (a) with $X = S^1$ and appropriate choices of A and Z.

Exercise 146.3. Let $A, B, C \in S^1$ be three distinct points. Show that $\mathrm{H}_1(S^1, \{A, B, C\})$ admits an epimorphism onto $\mathbb{Z}^2$.

Exercise 146.4. Let (X, A) be a pair of topological spaces. Suppose that the long exact sequence of the pair is as follows:

$$0 \to \mathrm{H}_2(X, A) \to \underbrace{\mathrm{H}_1(A)}_{\cong \mathbb{Z}/5\mathbb{Z}} \to \underbrace{\mathrm{H}_1(X)}_{\cong \mathbb{Z}/3\mathbb{Z}} \to \mathrm{H}_1(X, A) \xrightarrow{\partial_1} \mathrm{H}_0(A) \to \underbrace{\mathrm{H}_0(X)}_{\cong \mathbb{Z}^5} \to \underbrace{\mathrm{H}_0(X, A)}_{\cong \mathbb{Z}^2} \to 0.$$

Determine values for the missing homology groups. Are the isomorphism types of the missing groups uniquely determined?

Exercise 146.5.

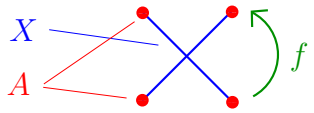
(a) Let X be a non-empty path-connected topological space and let $A \subseteq X$ be a subspace. Show that

$$\mathrm{H}_0(X, A) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } A = \varnothing, \\ 0, & \text{if } A \neq \varnothing. \end{cases}$$

Hint. Look at the example on page 2803.
(b) Let (X, A) be a pair of (possibly disconnected) topological spaces. Give a completely topological criterion for $\mathrm{H}_0(X, A)$ to be zero.

Exercise 146.6. We consider the pair (X, A) of topological spaces that is shown in the figure below.

(a) Determine the relative homology groups $\mathrm{H}_n(X, A)$.
(b) We consider the homeomorphism $f\colon X \to X$ that is given by rotation by $\frac{\pi}{2}$. For each $n \in \mathbb{N}_0$ determine $f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, A)$.



Exercise 146.7. Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_m$ be a nested sequence of topological spaces.

(a) Suppose that for each $k = 0, \ldots, m-1$ we have $\mathrm{H}_n(X_{k+1}, X_k) = 0$ for all $n \in \mathbb{N}_0$. Show that $\mathrm{H}_n(X_m, X_0) = 0$ for all $n \in \mathbb{N}_0$.
(b) Let $n \in \mathbb{N}_0$. Suppose that for each $k = 0, \ldots, m-1$ we have $\mathrm{H}_n(X_{k+1}, X_k) = 0$. Does it also follow that $\mathrm{H}_n(X_m, X_0) = 0$?

Exercise 146.8. Let (X, A) be a pair of topological spaces. Show that $\mathrm{H}_1(X, A) = 0$ if and only if $\mathrm{H}_1(A) \to \mathrm{H}_1(X)$ is surjective and if each path-component of X contains at most one path-component of A.

Exercise 146.9. Let (X, A) be a pair of topological spaces. Show that if A is a retract of X, then for each $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_n(X) \;\cong\; \mathrm{H}_n(A) \oplus \mathrm{H}_n(X, A).$$

Exercise 146.10. Given a topological space X and given $x \in X$ we refer to the homology groups of the pair $(X, X \setminus \{x\})$ as the *local homology of X at x*.

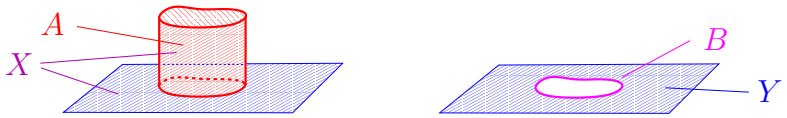
(a) For each point $x \in (0,1)$ compute the local homology of $(0,1)$ at x.
(b) For each point $x \in [0,1)$ compute the local homology of $[0,1)$ at x.
(c) Show that $(0,1)$ and $[0,1)$ are not homeomorphic.

Exercise 146.11. Let X be a topological space and let $A \subseteq X$ be a retract of X. Recall that this means that there exists a continuous map $r\colon X \to A$ with $r(a) = a$ for all $a \in A$.

(a) Show that each connecting homomorphism $\partial_n\colon \mathrm{H}_n(X,A) \to \mathrm{H}_{n-1}(A)$ is the zero homomorphism.
(b) Show that for each $k \in \mathbb{N}_0$ we have an isomorphism $\mathrm{H}_k(X) \cong \mathrm{H}_k(A) \oplus \mathrm{H}_k(X,A)$.

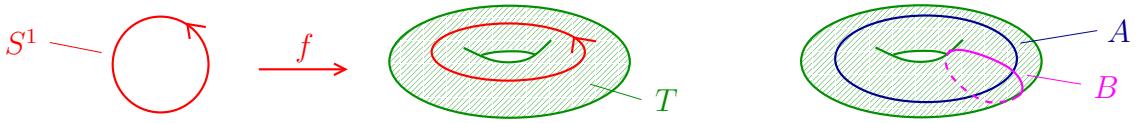
Exercise 146.12. Find interesting cycles in disk rel boundary etc.

Exercise 146.13. Let (X,A) be a pair of topological spaces. Suppose that there exists a deformation retraction $r\colon X \times [0,1] \to X$ from X to some subspace $Y = r_1(X)$. We write $B = r_1(A)$. Show that the inclusion-induced maps $\mathrm{H}_k(Y,B) \to \mathrm{H}_k(X,A)$ are isomorphisms.



Exercise 146.14. We consider the map $f\colon S^1 \to T := S^1 \times S^1$ that is shown in the figure below and we consider the subspaces $A := S^1 \times \{1\}$ and $B := \{1\} \times S^1$ of $T = S^1 \times S^1$ that are shown in the figure below.

(a) Show that the map $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(T)$ is a monomorphism.
(b) Is the map $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(T,A)$ also a monomorphism?
(c) Is the map $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(T,B)$ also a monomorphism?



Exercise 146.15. We consider the following pairs of topological spaces:

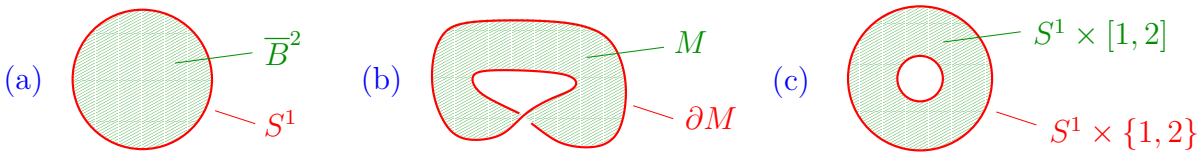
(a) $(X,A) = (\overline{B}^2, S^1)$.
(b) $(X,A) = (M, \partial M)$, where M is the Möbius band.
(c) $(X,A) = (S^1 \times [1,2], S^1 \times \{1,2\})$.

In each case try to provide an example of a 2-cycle $\sum_{i=1}^{m} s_i \cdot \sigma_i \in \mathrm{C}_2(X,A)$ with the following three properties:

(1) At least one s_i is non-zero.
(2) The images of the singular 2-simplices cover the whole surface.
(3) The images of the singular 2-simplices overlap only in subsets of dimension "≤ 1".

Remark. One can do so for precisely two of the three settings.

Exercise 146.16. Give an example of two pairs of topological spaces (X,A) and (Y,B) such that X is homotopy equivalent to Y, such that A is homotopy equivalent to B but such that the relative homology groups of (X,A) and (Y,B) are not isomorphic.


(a)
$\overline{B}^2$
S^1
(b)
M
∂M
(c)
$S^1 \times [1,2]$
$S^1 \times \{1,2\}$

CHAPTER 147

Reduced homology

In this very short chapter we will introduce the *reduced* homology groups of topological spaces. These reduced homology groups are almost the same as the by now familiar homology groups. But they will come in handy in the calculations of homology groups in the next chapters, since reduced homology groups are often a little easier to deal with than ordinary homology groups, even though they contain the same information.

This chapter is organized as follows:

- In Section 147.1 we will introduce the reduced homology groups $\widetilde{\mathrm{H}}_*(X)$ of a topological space X and we establish some basic properties of reduced homology
- In Section 147.2 we will introduce the long exact sequence of a pair in reduced homology.
- In Section 147.3 we will provide exercises for Chapter 147.

In the next Chapter 148 we will formulate and prove the all important Excision Theorem 148.1.

147.1. **Definition and basic properties of reduced homology.** The Augmentation Lemma 143.24 (1) motivates the following definition of the reduced homology groups of a topological space.

Definition 147.1. Let X be a topological space. By the Augmentation Lemma 143.24 (1) we know that the following is a generalized[1521] chain complex

$$\begin{array}{ccccccccccc} & & & & & & \text{degree } 0 & & \text{degree } -1 & & \\ & & & & & & \downarrow & & \downarrow & & \\ \ldots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^k r_i \cdot \sigma_i & \mapsto & \sum_{i=1}^k r_i & & \\ & & & & & & \uparrow & & & & \\ & & & & & & \text{augmentation map} & & & & \\ & & & & & & \text{from Definition 143.23} & & & & \end{array}$$

We introduce the following two definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.
(2) Given $n \in \mathbb{Z}_{\geq -1}$ we refer to $\widetilde{\mathrm{H}}_n(X) := \mathrm{H}_n(\widetilde{\mathrm{C}}_*(X))$ as the n-th reduced homology group of X.

These reduced homology groups are a minute but at times surprisingly useful variation on the usual homology groups. In the following we will explore the properties of reduced homology groups and we will explore their relationship to the traditional homology groups.

The following lemma is a variation on the by now familiar H_n–Functor Lemma 143.20.

[1521] Recall that a generalized chain complex is like a chain complex, but we also allow non-zero groups in negative degrees.

Lemma 147.2. ($\widetilde{\mathrm{H}}_n$–Functor Lemma)

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term. This map is a chain map and thus induces homomorphisms $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ on the reduced homology groups.

(2) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)) \end{array}$$

define a covariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups.

(3) The maps $\mathrm{id}\colon \widetilde{\mathrm{C}}_n(X) \to \mathrm{C}_n(X)$ for $n \geq 0$ and the zero map $\widetilde{\mathrm{C}}_{-1}(X) \to \mathrm{C}_{-1}(X) = 0$ define a natural chain map and for any $n \geq -1$ they induce a natural transformation

$$\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$$

from the functor $X \mapsto \widetilde{\mathrm{H}}_n(X)$ to the functor $X \mapsto \mathrm{H}_n(X)$.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. As in the Augmentation Lemma 143.24 (3) it is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_0(X) & \xrightarrow{f_*} & \mathrm{C}_0(Y) \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

It follows from this observation together with the C_*–Functor Lemma 143.16 that the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. Furthermore we obtain from the Chain Complex-H_*-Functor Lemma 143.4 (1) that we get induced maps on reduced homology.

(2) This statement is basically obvious.

(3) This statement follows easily from the definitions. ∎

The following elementary lemma explains the relationship between ordinary and reduced homology: Basically they are the same, except that for non-empty topological spaces they "differ by $\mathbb{Z}$" in dimension 0.

Lemma 147.3. (Reduced Homology Lemma) For every topological space X the following statements hold:

(-1) We have[1522]

$$\widetilde{\mathrm{H}}_{-1}(X) = \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathbb{Z}, & \text{if } X = \varnothing. \end{cases}$$

(0) (a) If X is non-empty, then the sequence

$$0 \to \widetilde{\mathrm{H}}_0(X) \xrightarrow{\text{ж}} \mathrm{H}_0(X) \xrightarrow[\text{Augmentation Lemma 143.24}]{\text{augmentation map } \epsilon_X \text{ from the}} \mathbb{Z} \to 0$$

is exact.

(b) Given any point $P \in X$ the map

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$

is an isomorphism.

(c) If X has $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In particular X is path-connected if and only if $\widetilde{\mathrm{H}}_0(X) = 0$.

(1) For every $n \geq 1$ the natural map $\varkappa\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the identity.

The lemma shows in particular that reduced homology groups are almost the same as the usual homology groups. So it might sound slightly silly to introduce them. But we will see that on several occasions reduced homology groups make it easier to formulate some statements. Throughout these notes we will often use the Reduced Homology Lemma 147.3 without explicitly mentioning it. For example we will refer to earlier results about homology groups and state the corresponding results for reduced homology groups.

Proof. Let X be a topological space.

(-1) First note that clearly we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Now we assume that X is non-empty. It follows immediately from the definitions that the augmentation map ϵ_X is an epimorphism which entails that $\widetilde{\mathrm{H}}_{-1}(X) = 0$.

(0) (a) We consider the following short exact sequence of generalized chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{C}_1(X) & \xrightarrow{\mathrm{id}} & \mathrm{C}_1(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{C}_0(X) & \xrightarrow{\mathrm{id}} & \mathrm{C}_0(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \partial_0} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

The desired statement follows easily from the corresponding long exact sequence of homology groups that is given by the LES-Proposition 145.10.

(b) Let $P \in X$ be a point. It is straightforward to show that the maps

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(X) & \to & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ [\sigma] & \mapsto & [\sigma - \epsilon_X(\sigma) \cdot P] + \epsilon_X(\sigma) \end{array}$$

are inverses of one another. In particular both maps are isomorphisms.

(c) This statement follows immediately from (0a), the H_0-Proposition 143.25 together with the Direct Sum-H_*-Lemma 143.7, and the Cancellation Proposition 97.26.

(1) This statement is clear. ∎

Many statements for ordinary homology groups carry over to the setting of reduced homology groups. For completeness' sake we spell out the analogs of the Homotopy-H_*-Proposition 144.12 and the Homotopy Equivalence-H_*-Proposition 144.14:

[1522] This calculation shows that in almost all cases we can ignore $\widetilde{\mathrm{H}}_{-1}(X)$. Nonetheless, some formulas later on become cleaner if we take into account that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

Proposition 147.4. (Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\widetilde{\mathrm{C}}_*(X)$ to $\widetilde{\mathrm{C}}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof.

(1) Recall that by the Homotopy-H_*-Proposition 144.12 (2) we know that there exists a chain homotopy $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$. We extend this chain homotopy by the zero-map $P_{-1}\colon \widetilde{\mathrm{C}}_{-1}(X) = \mathbb{Z} \to \widetilde{C}_0(Y)$. It is straightforward to verify that this is a chain homotopy. For convenience and reassurance we show the relevant part in a diagram:

$$\begin{array}{ccccccccc}
\mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_*-f_*} & \swarrow{\scriptstyle P_0} & \downarrow{\scriptstyle g_*-f_*} & \swarrow{\scriptstyle 0} & \downarrow{\scriptstyle \mathrm{id}-\mathrm{id}} & & \\
\mathrm{C}_1(Y) & \xrightarrow[\partial_1]{} & \mathrm{C}_0(Y) & \xrightarrow[\epsilon_Y]{} & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

(2) This statement follows immediately from (1). ■

Corollary 147.5. (Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.
(b) If X is a deformation retract of Y, then the inclusion-induced map $\widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.
(2) If X and Y are homotopy equivalent, then $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof. This corollary can be obtained easily from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 147.4. ■

The statement of the following corollary differs subtly from the corresponding "unreduced" version.

Corollary 147.6. (Contractible-$\widetilde{\mathrm{H}}_*$-Corollary) If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have $\widetilde{\mathrm{H}}_n(X) = 0$ for all $n \in \mathbb{Z}_{\geq -1}$.

Example. On page 531 we saw that every convex subset of $\mathbb{R}^n$ is contractible. Thus it follows from the Contractible-H_*-Corollary 144.15, that for any $i \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n \text{ or simplex}}) \ = \ 0.$$

□

Proof. This corollary follows immediately from the Contractible-H_*-Corollary 144.15 together with the Reduced Homology Lemma 147.3. ■

147.2. **The long exact sequence of a pair in reduced homology.** Shortly we will see that a pair of topological spaces gives rise to a long exact sequence that involves reduced homology groups. Before we can formulate this result we need to introduce the following notation:

Notation 147.7. Let X be a topological space, let $A \subseteq X$ and let $n \in \mathbb{N}_0$. We introduce the following map:

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}} \mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X,A), \qquad \widetilde{\mathrm{H}}_n(X) \xrightarrow{=:\text{Ж}_{X,A}} \mathrm{H}_n(X,A),$$

where $\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the natural homomorphism introduced in the $\widetilde{\mathrm{H}}_n$–Functor Lemma 147.2 and $\text{я}\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X,A)$ is the natural homomorphism that we introduced in Definition 146.6.

The following slight variation on the LES-of-Pair Proposition 146.7 will turn out to be surprisingly useful.

Proposition 147.8. (Reduced LES-of-Pair Proposition)

(1) Let X be a topological space and let $A \subseteq X$ be a subspace. The following sequence is exact[1523]

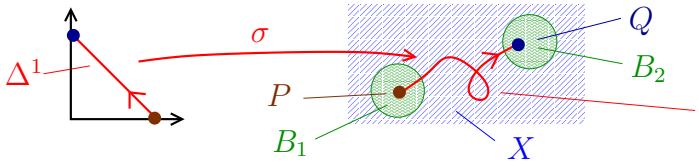
$$\begin{array}{ccccccccccc} \ldots & \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{Ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \to & \ldots \\ & & & & & & \left[\sum_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum_{j=1}^{k} s_j \cdot \partial_n(\sigma_j)\right] & & \end{array}$$

This sequence is called the long exact sequence in reduced homology of the pair (X,A).

(2) The connecting homomorphisms in (1) are natural. In particular, given a continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{Ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(B) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(Y) & \xrightarrow{\text{Ж}_{Y,B}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(B) & \longrightarrow & \ldots \end{array}$$

Example. As on page 2798 and 2803 we consider the topological space $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X,A).



We denote by $i\colon A \to X$ the inclusion map. By the Reduced LES-of-Pair Proposition 147.8 we have a long exact sequence

$$\widetilde{\mathrm{H}}_2(X) \to \mathrm{H}_2(X,A) \to \widetilde{\mathrm{H}}_1(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_1(X) \to \mathrm{H}_1(X,A) \to \widetilde{\mathrm{H}}_0(A) \to \widetilde{\mathrm{H}}_0(X) \to \mathrm{H}_0(X,A) \to 0.$$

By the Contractible-H_*-Corollary 144.15, the Path-Component–H_*–Lemma 143.21 and the Reduced Homology Lemma 147.3 we have $\widetilde{\mathrm{H}}_i(X) = \widetilde{\mathrm{H}}_i(\mathbb{R}^2) = 0$ for all $i \geq 0$, we have $\widetilde{\mathrm{H}}_i(A) = 0$ for $i \geq 1$ and we have $\widetilde{\mathrm{H}}_0(A) \cong \mathbb{Z}$. Thus we see from the above long exact

[1523] Note that if the subspace B is non-empty, then it follows from the Reduced Homology Lemma 147.3 (-1) that $\widetilde{\mathrm{H}}_{-1}(B) = 0$. Thus in this case the last possibly non-zero term is $\mathrm{H}_0(X,B)$. On the other hand, if $B = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(B) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$, and if $X = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(X) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Therefore, if $B = \varnothing$ the sequence is understood to continue with $\mathrm{H}_0(X,\varnothing) \xrightarrow{\epsilon} \mathrm{H}_{-1}(\varnothing) \to \mathrm{H}_{-1}(X) \to 0$.

sequence that $\mathrm{H}_i(X, A) = 0$ for $i \neq 1$ and that $\mathrm{H}_1(X, A) \cong \mathbb{Z}$. We get the same result as on page 2803, but with much less pain. □

Proof.

(1) We consider the following short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_1(A) & \longrightarrow & \mathrm{C}_1(X) & \longrightarrow & \mathrm{C}_1(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \\
0 & \longrightarrow & \mathrm{C}_0(A) & \longrightarrow & \mathrm{C}_0(X) & \longrightarrow & \mathrm{C}_0(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \partial_0} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

From the LES-Proposition 145.10 we obtain a long exact sequence of homology groups, which in this case involve the reduced homology groups of A and X, and the relative homology groups of (X, A). Furthermore, as in the proof of the LES-of-Triple Proposition 146.5 we see that the connecting homomorphism is given by the map

$$\begin{array}{rcl}
\mathrm{H}_n(X, A) & \to & \widetilde{\mathrm{H}}_{n-1}(A) \\
\left[\sum_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum_{j=1}^{k} s_j \cdot \partial_n(\sigma_j)\right]
\end{array}$$

Finally note that one can easily verify that the maps $\widetilde{\mathrm{H}}_k(X) \to \mathrm{H}_k(X, A)$ are precisely the above maps $\text{ж}_{X,A}$.

(2) This statement follows easily from the naturality of the connecting homomorphism proved in the LES-Proposition 145.10 ■

We conclude this chapter with the following lemma that elucidates the relationship between reduced and relative homology.

Lemma 147.9. (Reduced-to-Relative-H_*-Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. For each $x_0 \in X$ the map

$$\text{ж}_{X,\{x_0\}} \colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X, \{x_0\})$$

is a natural isomorphism.

Proof. Let X be a topological space and let $x_0 \in X$. We apply the Reduced LES-of-Pair Proposition 147.8 to the pair $(X, \{x_0\})$ and we obtain a long exact sequence

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_n(\{x_0\})}_{=0} \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}_{X,\{x_0\}}} \mathrm{H}_n(X, \{x_0\}) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(\{x_0\})}_{=0} \to \ldots$$

The lemma follows from the exactness of this sequence and the observation that it follows from the Singleton-H_*-Lemma 143.26 and the Reduced Homology Lemma 147.3 that all reduced homology groups of $\{x_0\}$ vanish. ■

147.3. Exercises for Chapter 147 .

Exercise 147.1. Let $f\colon S^0 \to S^0$ be the map that is given by $f(\epsilon) = -\epsilon$. What is the induced map on $\widetilde{\mathrm{H}}_0(S^0) \cong \mathbb{Z}$?

Exercise 147.2. Let X be a non-empty topological space. Recall that by the Reduced Homology Lemma 147.3 (0b) we know that there exists an isomorphism $\mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}$. Show that there is no *natural* isomorphism $\mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}$. More precisely, show that it is not possible to assign to each non-empty topological space X an isomorphism $\Phi_X\colon \mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}$ such that for every continuous map $f\colon X \to Y$ between non-empty topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{\Phi_X} & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_* \oplus \mathrm{id}_{\mathbb{Z}}} \\ \mathrm{H}_0(Y) & \xrightarrow{\Phi_Y} & \widetilde{\mathrm{H}}_0(Y) \oplus \mathbb{Z}. \end{array}$$

Hint. Let X be the discrete space consisting of two points. If such a natural isomorphism Φ_X did exist, what should the map Φ_X do?

Exercise 147.3. Let X be a topological space with $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}$.

(a) Show that given any $\epsilon \in \{-1, 1\}$ there exists a continuous map $f\colon X \to X$ such that the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by ϵ.

(b) Show that given any continuous map $f\colon X \to X$ the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 1\}$.

Exercise 147.4. We fix a singleton $\{\star\}$, i.e. we fix a topological space with a single element called $\star$. Given any topological space X we denote by $f^X\colon X \to \{\star\}$ the unique map.

(a) Show that for each $n \in \mathbb{N}_0$ the map $X \mapsto \ker(f^X_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(\{\star\}))$ can be used to define a covariant functor from the category of topological spaces to the category *AbGroup* of abelian groups.

(b) Show that for non-empty topological spaces the functor in (a) is naturally isomorphic to reduced homology.

Exercise 147.5. Given a topological space X the *cone on* X is defined to be the topological space

$$\mathrm{Cone}(X) \;\coloneqq\; (X \times [0,1])/(X \times \{0\}).$$

The map $X \to \mathrm{Cone}(X)$ given by $x \mapsto [(x, 1)]$ is an embedding and we use it to view X as a subspace of $\mathrm{Cone}(X)$. Determine $\mathrm{H}_1(\mathrm{Cone}(X), X)$ in terms of the topology of X and describe a generating set for this homology group.



CHAPTER 148

The Excision Theorem

We introduced singular homology in Definition 143.18, many chapters ago. Since then we have developed the theory of singular homology groups and related topics. But the reader will have noticed that up to now we have not really calculated any interesting homology groups.

This is about to change. In this chapter we will state and prove the Excision Theorem 148.1, which is easily one of the most essential theorems in algebraic topology. We will use it, and the techniques from the previous chapters, to finally provide the calculation of the homology groups of spheres. In the next chapter we will use the Excision Theorem 148.1 to find, given a suitable pair of topological spaces (X, A), a connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$.

This chapter is organized as follows:

- In Section 148.1 we will state the Excision Theorem 148.1
- In Section 148.2 we will discuss variations on the Excision Theorem 148.1.
- In Section 148.3 we will present the idea for the proof of the Excision Theorem 148.1.
- In Section 148.4 we will provide the full details for the proof of the Excision Theorem 148.1.
- In Section 148.5 we will provide exercises for Chapter 148.

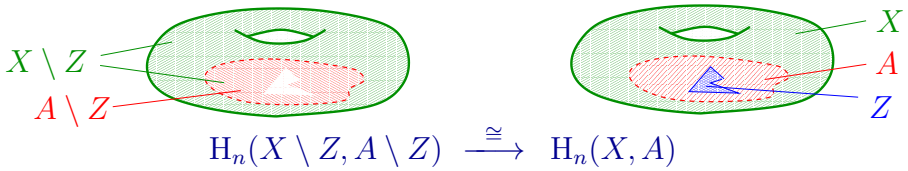
In the next Chapter 149 we will study many applications of the Excision Theorem 148.1.

148.1. The Excision Theorem.

Here is the main theorem of this chapter.

Theorem 148.1. (Excision Theorem) Let X be a topological space and let $Z \subseteq A \subseteq X$ be subsets. If the closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

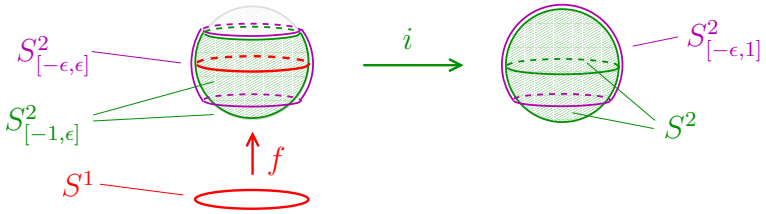
$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$



The Excision Theorem 148.1 thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise a closed subset Z from the interior of A, then this does not affect the relative homology groups. In the next chapter we will use the Excision Theorem 148.1 to perform many calculations. As an "amuse bouche" we will now compute the homology groups of the spheres S^n:

Example. We want to determine the reduced homology groups of S^n. We pick some $\epsilon \in (0,1)$ and for each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{tikzcd}
\arrow[r] & \widetilde{\mathrm{H}}_m(S^n_{[-\epsilon,1]}) \arrow[r] & \widetilde{\mathrm{H}}_m(S^n) \arrow[r, "\cong"] & \mathrm{H}_m(S^n, S^n_{[-\epsilon,1]}) \arrow[r, "\partial_m"] & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,1]}) \arrow[r] & \dots \\
\dots \arrow[r] & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]}) \arrow[r] & \widetilde{\mathrm{H}}_m(S^n_{[-1,\epsilon]}) \arrow[r] & \mathrm{H}_m(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]}) \arrow[r, "\partial_m", "\cong"'] \arrow[u, "i_*", "\cong"'] & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]}) \arrow[r] & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-1,\epsilon]}) \arrow[r] & {} \\
& & & & \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \arrow[u, "f_*", "\cong"'] & &
\end{tikzcd}$$



We make the following clarifications and explanations:

(1) For an interval $I \subseteq [-1,1]$ we write $S^n_I := \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \in I\}$.
(2) The top horizontal sequence is the long exact sequence in reduced homology of the pair $(S^n, S^n_{[-\epsilon,1]})$ that is given by the Reduced LES-of-Pair Proposition 147.8.
(3) The bottom horizontal sequence is the long exact sequence in reduced homology of the pair $(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]})$ that is given by the Reduced LES-of-Pair Proposition 147.8.
(4) It follows from the Stereographic Projection Lemma 4.24 that $S^n_{[-\epsilon,1]}$ and $S^n_{[-1,\epsilon]}$ are homeomorphic to the closed n-ball $\overline{B}^n$. It follows from this observation and the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that for all $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_k(S^n_{[-\epsilon,1]}) = 0$ and that we have $\widetilde{\mathrm{H}}_k(S^n_{[-1,\epsilon]}) = 0$.
(5) It follows from (4) and the exactness of the horizontal sequences that the horizontal maps shown in red are isomorphisms.
(6) It follows from the Excision Theorem 148.1, applied to $X = S^n$ and the two subspaces $A = S^n_{[-\epsilon,1]}$ and $Z = S^n_{(\epsilon,1]}$, that the vertical map $i_*\colon \mathrm{H}_m(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]}) \to \mathrm{H}_m(S^n, S^n_{[-\epsilon,1]})$ is an isomorphism.
(7) We denote by $f\colon S^{n-1} \to S^n_{[-\epsilon,\epsilon]}$ the map that is given by $x \mapsto (x,0)$. This map is clearly an embedding. It is not hard to show that $f(S^{n-1})$ is a deformation retract of $S^n_{[-\epsilon,\epsilon]}$. It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $f\colon S^{n-1} \to S^n_{[-\epsilon,\epsilon]}$ is a homotopy equivalence.
(8) It follows from (7) together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 for reduced homology that the induced map $f_*\colon \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]})$ is an isomorphism.

We have now shown that all red homomorphisms are isomorphisms. We now see that for each $m \in \mathbb{Z}_{\geq -1}$ we have[1524]

$$\widetilde{\mathrm{H}}_m(S^n) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \underset{\uparrow}{\cong} \dots \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{m-n}(S^0) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

since all the red homomorphisms are isomorphisms

iterating the argument

follows from the Singleton-H_*-Lemma 143.26, the Path-Component–H_*–Lemma 143.21 and the Reduced Homology Lemma 147.3

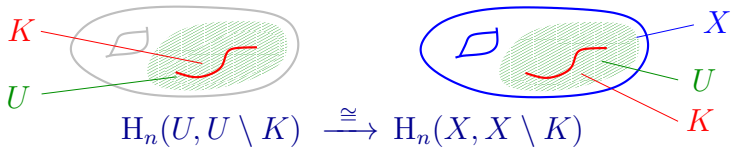
We have thus finally calculated the homology groups of spheres. We will redo the calculation in the Sphere-H_*-Proposition 149.4 using a slightly different approach. Furthermore in Section 149.2 we will discuss several applications of this calculation. □

The plan for the remainder of this chapter is as follows: In the next section we will formulate several variations on the Excision Theorem 148.1. We give a short outline of the proof of the Excision Theorem 148.1 in Section 148.3 and we will provide the full daunting details of the proof of the Excision Theorem 148.1 in Section 148.4. Readers who want to see the Excision Theorem 148.1 in action are advised to move to the next chapter.

148.2. **Variations on the Excision Theorem 148.1 .** In this section we will state and prove several variations on the Excision Theorem 148.1. The following theorem is in fact just a convenient reformulation of the Excision Theorem 148.1.

Theorem 148.2. (Excision Theorem) Let X be a topological space, let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism[1525]

$$\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K).$$



Proof of Theorem 148.2 assuming Theorem 148.1. We have

$$\mathrm{H}_n(U, U \setminus K) \underset{\uparrow}{=} \mathrm{H}_n(X \setminus Z, A \setminus Z) \underset{\uparrow}{\xrightarrow{\cong}} \mathrm{H}_n(X, A) = \mathrm{H}_n(X, X \setminus K).$$

we set $A := X \setminus K$ and we set $Z := X \setminus U$

since U is a neighborhood of the closure of K we see that the closure of Z is contained in the interior of A, thus we can apply the Excision Theorem 148.1

■

In many occasions we will deal with manifolds and CW-complexes. In these settings the following two theorems will be useful.

[1524] We could even have gone one step further and ended with $S^{-1} = \varnothing$ and the final calculation would have been provided by the Reduced Homology Lemma 147.3 (-1).

[1525] It follows from the Compact-Closed Lemma 3.39 (2) that the hypotheses are for example satisfied if X is Hausdorff (e.g. if X is a topological manifold) and if K is a compact subset that is contained in an open subset U.

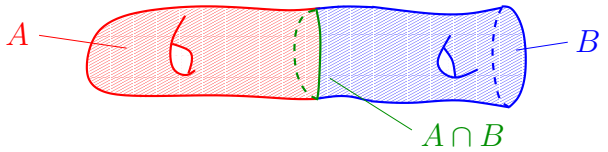
Theorem 148.3. (Manifold–Excision Theorem) Let X be an m-dimensional topological manifold and let $A, B \subseteq X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.[1526]

For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(B, A \cap B) \;\to\; \mathrm{H}_n(X, A)$$

is an isomorphism.



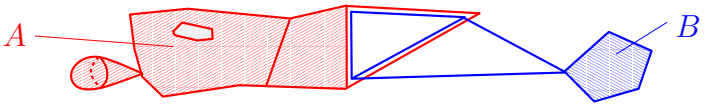
the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

Remark. In the Manifold–Excision Theorem 212.4 we will generalize the statement of the Manifold–Excision Theorem 148.3 to the setting that B is a submanifold "with corner". □

Theorem 148.4. (CW-Excision Theorem) Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(B, A \cap B) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A)$$

is an isomorphism.



the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

The proofs of the Manifold–Excision Theorem 148.3 and the CW-Excision Theorem 148.4 both rely on the following somewhat technical theorem.

Theorem 148.5. (Excision-via-Deformation Theorem) Let X be a topological space and let $Z \subseteq A \subseteq X$ be two subsets. We suppose that there exists a subset $U \subseteq Z$ such that the closure of U is contained in the interior of A and such that the following statements hold:

(1) $A \setminus Z$ is a deformation retract of $A \setminus U$.
(2) $X \setminus Z$ is a deformation retract of $X \setminus U$

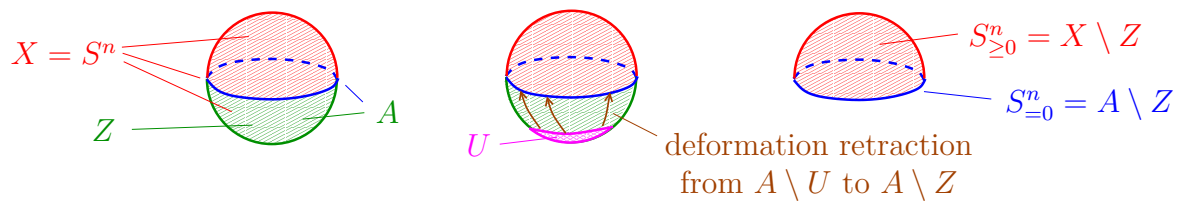
Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A).$$

Example. Let $n \in \mathbb{N}$. As usual we denote by $S^n_{\geq 0}$ and $S^n_{\leq 0}$ the upper and lower hemisphere of S^n and we write $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$.
It follows from the Excision-via-Deformation Theorem 148.5 and the choices of X, A, Z and U illustrated in the figure above that for each $k \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_k(S^n_{\geq 0}, S^n_{=0}) \;\to\; \mathrm{H}_k(S^n, S^n_{\leq 0})$$

[1526] Note that by the Compact-Closed Lemma 3.39 (2) this condition is satisfied if A and B are compact.



is an isomorphism. □

Proof of Theorem 148.5 assuming the Excision Theorem 148.1. We consider the following commutative diagram of inclusion-induced maps:

$$\begin{array}{ccc} \mathrm{H}_n(X \setminus Z, A \setminus Z) & \longrightarrow & \mathrm{H}_n(X, A) \\ \downarrow & \nearrow_{\cong} & \\ \mathrm{H}_n(X \setminus U, A \setminus U). & & \end{array}$$

Since by hypothesis we know that $\overline{U} \subseteq \mathring{A}$ we obtain from the Excision Theorem 148.1 that the diagonal map is an isomorphism. Furthermore it follows from our hypothesis (1) and (2) together with the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the vertical map is an isomorphism. It follows that the top horizontal map is an isomorphism. ■

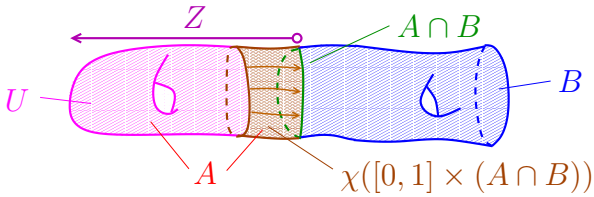
Proof of Theorem 148.3 assuming Theorem 148.5. Let X be an m-dimensional topological manifold and let $A, B \subseteq X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.

It follows from (b) together with the Topological Collar Neighborhood Theorem 35.14 that there exists a continuous map $\chi\colon [0,1] \times (A \cap B) \to A$ with the following three properties:

(d) The map χ is an embedding.
(e) The image of $[0,1) \times (A \cap B)$ is an open subset of A.
(f) For every $P \in A \cap B$ we have $\chi(0, P) = P$.

We set $Z := X \setminus B$ and we set $U := X \setminus (B \cup \chi([0,1) \times (A \cap B))) = A \setminus \chi([0,1) \times (A \cap B))$.



Claim.
(0) The closure of U is contained in the interior of A.
(1) $A \setminus Z = A \cap B$ is a deformation retract of $A \setminus U = \chi([0,1] \times (A \cap B))$.
(2) $X \setminus Z = B$ is a deformation retract of $X \setminus U = (\chi([0,1] \times (A \cap B)) \cup B$.

Proof.

(0) It follows from (b) that $A \setminus (A \cap B) = X \setminus B$ is an open subset of X. Furthermore it follows from (c), (e), (f) and the Cover-Open Subset Lemma 3.20 (2b) that U is a closed subset of X. It follows from this discussion with the fact that $U \subseteq X \setminus B$ that the closure of U is contained in the interior of A.
(1) We have an obvious deformation retraction from $[0,1] \times (A \cap B)$ to $\{0\} \times (A \cap B)$. It follows from (d) and (f) that this deformation retraction gives rise to a deformation retraction from $\chi([0,1] \times (A \cap B))$ to $A \cap B$.
(2) It follows from (c) and the Homotopy Combination Lemma 18.8 that we can extend the deformation retraction from (2) in an obvious way to the desired deformation retraction from $X \setminus Z$ to $X \setminus U$. ⊞

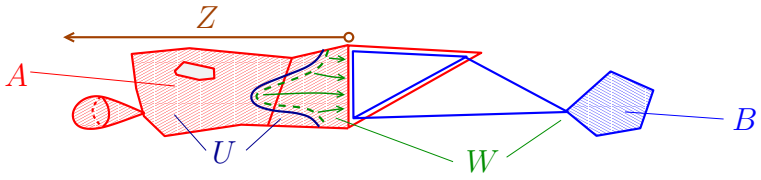
It follows from the claim that we can apply the Excision-via-Deformation Theorem 148.5 which tells us that the inclusion-induced map

$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A\cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

Proof of the CW-Excision Theorem 148.4 assuming Theorem 148.5. Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. Note that by the Subcomplex-Neighborhood Proposition 137.9, applied to the subcomplex B, there exists an open neighborhood W of B and a deformation retraction $F\colon W \times [0,1] \to W$ from W to B with the extra property that the map F restricts to a deformation retraction $F\colon (A \cap W) \times [0,1] \to A \cap W$ from $A \cap W$ to $A \cap B$.

We set $Z := X \setminus B$ and we set $U := X \setminus W$. Since W is an open neighborhood of B we see that the closure of U is contained in the interior of A.



Recall that F is a deformation retraction from $W = X \setminus U$ to $B = X \setminus Z$ and that F restricts to a deformation retraction from $A \cap W = A \setminus U$ to $A \cap B = A \setminus Z$. Therefore it follows from the Excision-via-Deformation Theorem 148.5 that the inclusion-induced map

$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A\cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

148.3. **The proof of the Excision Theorem 148.1: The idea.** In this section we want to come up with the idea for the proof of the Excision Theorem 148.1. The full details of the proof will then be given in the following section.

Thus let X be a topological space and let $Z \subseteq A \subseteq X$ be two subsets such that the closure of Z is contained in the interior of A. Our goal is to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A).$$

Let us attempt to prove that the map $\mathrm{H}_n(X\setminus Z, A\setminus Z) \to \mathrm{H}_n(X, A)$ is surjective. We take an element in $\mathrm{H}_n(X, A) = \mathrm{H}_n(\mathrm{C}_*(X, A)) = \mathrm{H}_n(\mathrm{C}_*(X)/\mathrm{C}_*(A))$ and we choose a representative $a_1\cdot\sigma_1+\cdots+a_r\cdot\sigma_r \in \mathrm{C}_n(X)$ where $\sigma_i\colon \Delta^n \to X$, $i = 1, \ldots, r$ are singular n-simplices in X.

We make the following two basic observations:

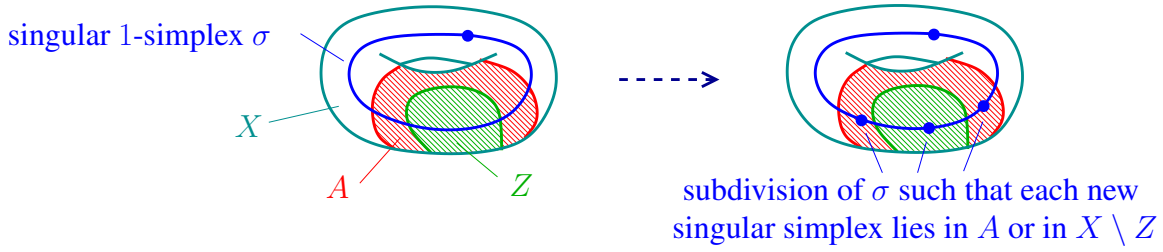
(1) If the singular n-simplices $\sigma_1, \ldots, \sigma_r$ lie in $X \setminus Z$,[1527] then $[a_1\cdot\sigma_1+\cdots+a_r\cdot\sigma_r]$ evidently lies in the image of the inclusion-induced map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.

(2) If a singular n-simplex lies entirely in A, then it represents the zero element in the quotient group $\mathrm{C}_n(X, A) = \mathrm{C}_n(X)/\mathrm{C}_n(A)$.

The combination of these two observations proves the following lemma:

Lemma 148.6. Let X be a topological space and let $Z \subseteq A$ be two subsets of X. Let $c = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X \setminus Z$. Then $[c]$ lies in the image of $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.

Unfortunately not every cycle in $\mathrm{C}_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular simplices that lie entirely in A or lie entirely in $X \setminus Z$. This then implies that the inclusion-induced map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$ is an epimorphism. A very similar argument then also shows that the map is a monomorphism.



148.4. **The proof of the Excision Theorem 148.1: The full details.** In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 148.1. This will take a surprisingly long time.

First we introduce a few definitions.

Definition 148.7. Let again X be a topological space.

(1) In Definition 3.19 we defined a cover of X to be a family $\{U_i\}_{i\in I}$ of subsets of X such that $X = \bigcup_{i\in I} U_i$.

(2) We say that a cover $\{U_i\}_{i\in I}$ is comfortable if $X = \bigcup_{i\in I} \mathring{U}_i$, i.e. if the interiors of the U_i also form a cover of X.

We move straight to the next definition.

Definition 148.8. Let again X be a topological space. Given a comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^{k} a_j\cdot\sigma_j \;\middle|\; \text{for each } j \text{ there exists a } U_i\in\mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

[1527] Given a topological space X and a subset A we say that *a singular simplex* $\sigma\colon \Delta^n \to X$ *lies in* A if the image of σ lies in A.

It follows quite easily from the definitions that the restriction of the usual boundary map $\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ to $\mathrm{C}_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n\colon \mathrm{C}_n^{\mathcal{U}}(X) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ is the zero map. Thus we can define

$$\begin{aligned}\mathrm{H}_n^{\mathcal{U}}(X) &:= \mathrm{H}_n\big(\mathrm{C}_*^{\mathcal{U}}(X), \partial_*\big)\\ &= \ker\big(\partial_n\colon \mathrm{C}_n^{\mathcal{U}}(X) \to \mathrm{C}_{n-1}^{\mathcal{U}}(X)\big) \,/\, \mathrm{im}\,\big(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\big).\end{aligned}$$

The homology groups $\mathrm{H}_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are "small" in the sense that the image of each singular simplex lies in some U_i.

The following proposition says that we can determine the homology of a topological space through such "small" singular chains. As we will see later on many occasions, this proposition is at the heart of many methods for computing homology groups.

Proposition 148.9. (Comfortable Cover Proposition) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. The inclusion map

$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

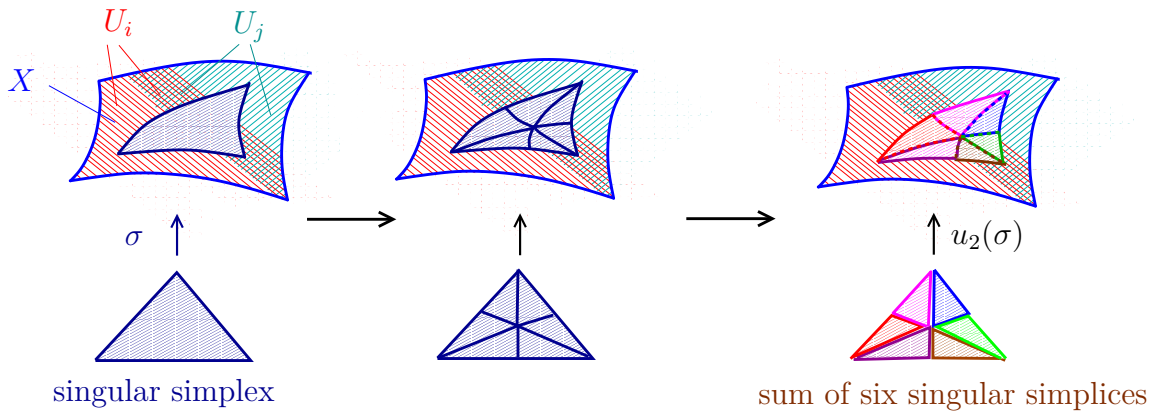
$$\mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

Remark. In fact a stronger statement holds: the inclusion map $\mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ is a chain homotopy equivalence. Later, in Section 158.2, we will show that this stronger statement follows in fact from the Comfortable Cover Proposition 148.9 and a purely algebraic argument. Alternatively one can fight one's way through the proof of the Comfortable Cover Proposition 148.9 and upgrade the proof to this stronger statement. This approach is for example pursued in [**Hat2002**, Proposition 2.21]. The disadvantage is that it makes the proof even more complicated than it already is. □

The idea of the proof of the Comfortable Cover Proposition 148.9 is the one we already described in the previous section. We want to replace systematically each singular n-simplex $\sigma\colon \Delta^n \to X$ by a singular n-chain of the form $\sum_{j=1}^{r} a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in the Singular 1-Simplex Lemma 143.28 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use for the general case is given by "barycentric subdivision". The key idea is sketched in the figure below for the case $n = 2$.
Before we can execute this idea we have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \ldots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter*[1528] of the standard n-simplex Δ^n.

[1528]The word "barycenter" comes from the Greek word "barys" which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n-simplex is the center of mass if we view Δ^k as a physical object of uniform density.

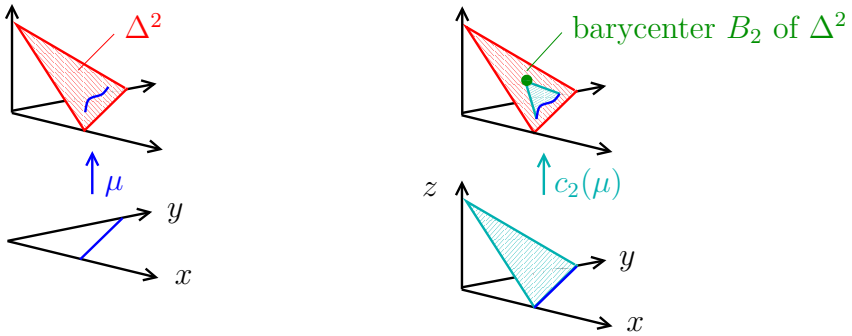


(2) Given a singular l-simplex $\mu\colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu)\colon \Delta^{l+1} \to \Delta^n$ which is defined as the composition of the (inverses) of the following two maps:

homeomorphism by the Compact-Hausdorff Proposition 4.23 (3) — continuous by the Topological-Quotient Proposition 8.2 (1b)

$$\begin{array}{rcccl} c_{l+1}(\mu)\colon \Delta^{l+1} & \xleftarrow{\cong} & (\Delta^l \times [0,1])/\Delta^l \times \{1\} & \to & \Delta^n \\ ((t_0,\dots,t_l)\cdot(1-s),s) & \mapsto & [((t_0,\dots,t_l),s)] & & \\ & & [((t_0,\dots,t_l),s)] & \mapsto & \underbrace{\mu(t_0,\dots,t_l)\cdot(1-s)+B\cdot s.}_{\text{contained in } \Delta^n \text{ since } \Delta^n \text{ is convex}} \end{array}$$

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. Note that by the Free Abelian Group–Existence Lemma 97.8 this map $\mu \mapsto c_{l+1}(\mu)$ defines a linear map $\mathrm{C}_l(\Delta^n) \to \mathrm{C}_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-st cone map.



(3) For $n = 0, 1, 2, 3, \dots$ we inductively do the following:
- (a) we define an element $\chi_n \in \mathrm{C}_n(\Delta^n)$, and
- (b) for each topological space X we define a homomorphism $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

We do this inductive procedure as follows:
- (i) We set $\chi_0 = \mathrm{id}\colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $\mathrm{C}_0(X)$.

(ii) Now we suppose that $\chi_{n-1} \in \mathrm{C}_{n-1}(\Delta^{n-1})$ is already defined and we suppose that the homomorphisms $u_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-1}(X)$ are already defined. Then we set

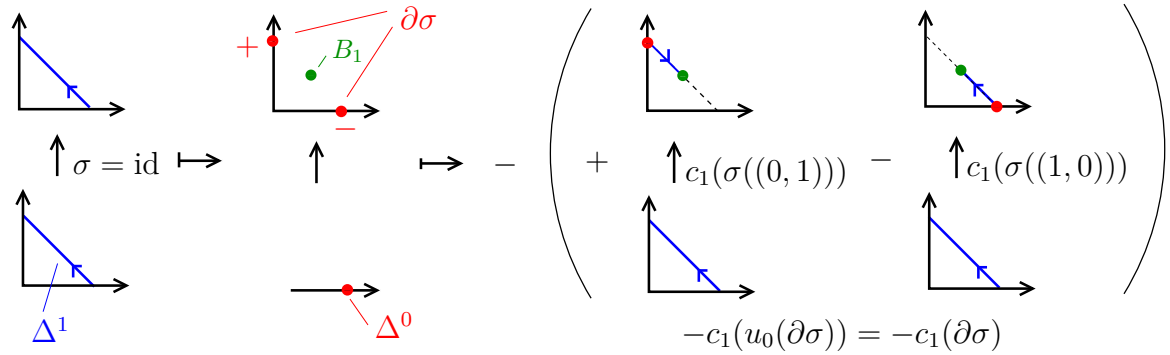
$$\chi_n \;:=\; (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial_n(\mathrm{id}_{\Delta^n})}_{\in \mathrm{C}_{n-1}(\Delta^n)})) \;\in\; \mathrm{C}_n(\Delta^n).$$

Furthermore, for each topological space X we define

$$\begin{array}{rcl} u_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) & \mapsto & \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{\chi_n}_{\in \mathrm{C}_n(\Delta^n)}). \end{array}$$

the continuous map $\sigma_l\colon \Delta^l \to X$ induces a homomorphism $(\sigma_l)_*\colon \mathrm{C}_n(\Delta^l) \to \mathrm{C}_n(X)$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$. The map u_n is illustrated for $n = 1$ in the figure below. For $n = 2$ we already gave a more schematic sketch in the figure above without considering signs and orientations.



Before we can continue we need to prove several technical lemmas about the subdivision maps. We start out with a straightforward lemma.

Lemma 148.10. (Subdivision–Natural Lemma) The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ are natural,[1529] in particular for any continuous map $f\colon X \to Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have

$$f_* \circ u_n \;=\; u_n \circ f_*\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(Y).$$

[1529]Let us explain in what sense these maps deserve to be called "natural". Given $n \in \mathbb{N}_0$ we consider the functor C_* from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes which is given by

$$\begin{array}{rcl} \text{topological space } X & \mapsto & \mathrm{C}_*(X) \\ \text{and} \qquad (f\colon X \to Y) & \mapsto & (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)). \end{array}$$

Given a topological space X we denote by $\Phi_X\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)$ the map that is given by the n-th subdivision map that we defined on page 2825. Lemma 148.11 and the Subdivision–Natural Lemma 148.10 say that these maps define a natural transformation from the functor C_* to itself.

Proof. This lemma follows immediately from the definitions. Indeed, let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then

$$f_*(u_n(\sigma)) \underset{\substack{\uparrow\\ \text{by definition}\\ \text{of } u_n(\sigma)}}{=} f_*(\sigma_*(\chi_n)) \underset{\substack{\uparrow\\ \text{since } (f\circ\sigma)_* = f_*\circ\sigma_*\\ \text{as maps } \mathrm{C}_n(\Delta^n)\to\mathrm{C}_n(Y)}}{=} (f\circ\sigma)_*(\chi_n) \underset{\substack{\uparrow\\ \text{by definition}\\ \text{of } u_n(f_*(\sigma))}}{=} u_n(f_*(\sigma)).$$

Since the desired equality holds for a generating set of $\mathrm{C}_n(X)$ it also holds for all of $\mathrm{C}_n(X)$. ■

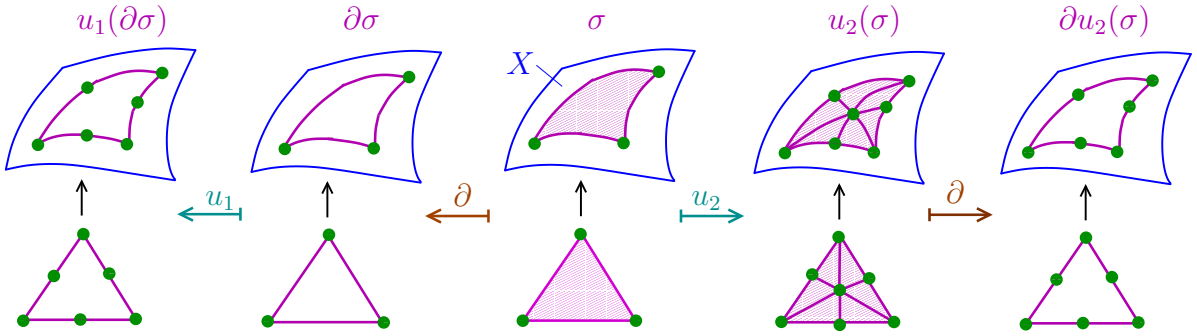
We continue with a lemma which is much harder to prove.

Lemma 148.11. (Chain Subdivision Lemma) Let X be a topological space. The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(\mathrm{C}_*(X), \partial_*)$ to itself.

The statement of the Chain Subdivision Lemma 148.11 is equivalent to the statement that for every singular n-simplex σ in X we have

$$u_{n-1}(\partial_n(\sigma)) \;=\; \partial_n(u_n(\sigma)).$$

Put differently, the Chain Subdivision Lemma 148.11 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in the figure below.



Proof of the Chain Subdivision Lemma 148.11. [1530] We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.

(2) Given points $x_0, \ldots, x_k \in \Delta^n$ we denote by $[x_0, \ldots, x_k]$ the singular k-chain $\Delta^k \to \Delta^n$ that is given by[1531]

$$\begin{aligned} \Delta^k &\to \Delta^n \\ (t_0, \ldots, t_k) &\mapsto \sum_{i=0}^{k} t_i \cdot x_i, \end{aligned}$$

we refer to such a singular k-simplex as a *linear* singular simplex.

(3) We denote by $\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $\mathrm{C}_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial_k\colon \mathrm{C}_k(\Delta^n) \to \mathrm{C}_{k-1}(\Delta^n)$ restricts to a boundary map $\partial_k\colon \mathrm{C}_k^{\mathrm{lin}}(\Delta^n) \to \mathrm{C}_{k-1}^{\mathrm{lin}}(\Delta^n)$.

[1530] We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

[1531] It is straightforward to show that this map does indeed take values in Δ^n.

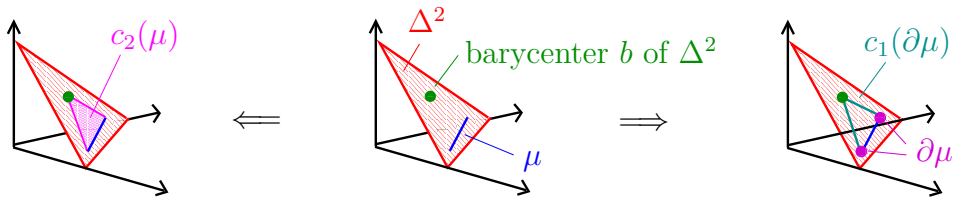
(4) For $x_0, \dots, x_k \in \Delta^n$ we write

$$[x_0, \dots, \widehat{x_i}, \dots, x_k] \;:=\; [x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k].$$

The proof of the lemma will be broken into three claims.

Claim 1. For any $\mu \in \mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial_{k+1}(c_{k+1}(\mu)) \;=\; c_k(\partial_k \mu) + (-1)^{k+1} \cdot \mu \;\in\; \mathrm{C}_k(\Delta^n).$$



Proof. To prove the claim we first note that for any choice of points $x_0, \dots, x_{l-1} \in \Delta^n$ we have by definition of c_l that

$$c_l([x_0, \dots, x_{l-1}]) \;=\; [x_0, \dots, x_{l-1}, b].$$

It follows that for any $w_0, \dots, w_k \in \Delta^n$ we have the following equality in $\mathrm{C}_k(\Delta^n)$:

$$\begin{aligned}
\partial_{k+1}(c_{k+1}([w_0, \dots, w_k])) &\overset{\text{by the above equality}}{\downarrow} \\
&= \partial_{k+1}([w_0, \dots, w_k, b]) \\
&\underset{\text{by Definition 143.13 of } \partial_{k+1} \text{ and by the above equality}}{\uparrow} \\
&= \sum_{j=0}^{k} (-1)^j \cdot [w_0, \dots, \widehat{w_j}, \dots, w_k, b] + (-1)^{k+1} \cdot [w_0, \dots, w_k] \\
&\overset{\downarrow}{=} c_k(\partial_k([w_0, \dots, w_k])) + (-1)^{k+1} \cdot [w_0, \dots, w_k].
\end{aligned}$$

The claim immediately follows from this equality. ⊞

Claim 2. For any $n \in \mathbb{N}_0$ and any $\nu \in \mathrm{C}_n^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial_n u_n(\nu) \;=\; u_{n-1}(\partial_n \nu).$$

Proof. We will prove this claim by induction on n. The case $n = 0$ is trivial. So now suppose we have already shown the case $n - 1$. It suffices to show the claim for all linear

singular n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We then have

$$\begin{aligned}
\partial_n u_n(\nu) &\overset{\text{definition of } u_n \text{ and } \chi_n, \text{ see page 2825}}{\downarrow} \partial_n(\nu_*(\chi_n)) \overset{\downarrow}{=} (-1)^n \cdot \partial_n(\nu_*(c_n(u_{n-1}(\partial_n \mathrm{id}_{\Delta^n})))) \overset{\text{since } \partial_n \circ \nu_* = \nu_* \circ \partial_n}{\downarrow} (-1)^n \cdot \nu_*\big(\partial_n(c_n(\underbrace{u_{n-1}(\partial_n \mathrm{id}_{\Delta^n})}_{=:\mu}))\big) \\
&\overset{\uparrow}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(\partial_{n-1}(u_{n-1}(\partial_n \mathrm{id}_{\Delta^n})) + (-1)^n \cdot u_{n-1}(\partial_n \mathrm{id}_{\Delta^n})\big) \\
&\quad \text{we can apply the previous claim since } \mu := u_{n-1}(\partial_n \mathrm{id}_{\Delta^n}) \in \mathrm{C}_{n-1}^{\mathrm{lin}}(\Delta^n) \\
&= (-1)^n \cdot \nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial_{n-1}\partial_n \mathrm{id}_{\Delta^n}}_{=0}))\big) + \nu_*(u_{n-1}(\partial_n \mathrm{id}_{\Delta^n})) \\
&\quad \text{by induction hypothesis we have } \partial_{n-1}u_{n-1}(\rho) = u_{n-2}(\partial_{n-1}\rho) \text{ for } \rho \in \mathrm{C}_{n-1}^{\mathrm{lin}}(\Delta^n), \text{ applied here to } \rho = \partial_n \mathrm{id}_{\Delta^n} \\
&\overset{\uparrow}{=} u_{n-1}(\nu_*(\partial_n \mathrm{id}_{\Delta^n})) \overset{\text{by definition}}{=} u_{n-1}(\partial_n \nu).
\end{aligned}$$

Subdivision–Natural Lemma 148.10 applied to $f = \nu$ ⊞

The following claim is just a reformulation of the statement of the lemma.

Claim 3. For any singular n-simplex σ in X we have $u_{n-1}(\partial_n \sigma) = \partial_n(u_n(\sigma))$.

Proof. So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{aligned}
\partial_n(u_n(\sigma)) &= \partial_n(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) \overset{\text{by the Subdivision–Natural Lemma 148.10 applied to } f=\sigma}{\downarrow} \partial_n(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) \overset{\text{since } \sigma_* \text{ is a chain map}}{\downarrow} \sigma_*(\partial_n(u_n(\mathrm{id}_{\Delta^n}))) \\
&\overset{\uparrow}{=} \sigma_*(u_{n-1}(\partial_n \mathrm{id}_{\Delta^n})) \overset{\uparrow}{=} u_{n-1}(\sigma_*(\partial_n \mathrm{id}_{\Delta^n})) \overset{\uparrow}{=} u_{n-1}(\partial_n \sigma_*(\mathrm{id}_{\Delta^n})) = u_{n-1}(\partial_n \sigma).
\end{aligned}$$

by Claim 2; by the Subdivision–Natural Lemma 148.10 applied to $f=\sigma$; since σ_* is a chain map ■

The proof of the next lemma is again rather involved.

Lemma 148.12. (Subdivision–Chain Homotopy-Lemma) Let X be a topological space. The chain maps

$$u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1} \circ P_n = -P_{n-1} \circ \partial_n + u_n - \mathrm{id}\,.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $\mathrm{C}_{-1}(X) = 0$, accordingly we define $P_{-1}\colon \mathrm{C}_{-1}(X) \to \mathrm{C}_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1}, \dots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) = -(P_{n-1} \circ \partial_n)(\sigma) + (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in \mathrm{C}_{n+1}(X)$ with

$$\partial_{n+1}(E_\sigma) = -(P_{n-1} \circ \partial_n)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

so that we can then define $P_n(\sigma) \coloneqq E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have
$$\big(\underbrace{P_{-1}}_{=0} \circ \partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) \;=\; 0 \in \mathrm{C}_0(X),$$
and we put $E_\sigma \coloneqq 0 \in \mathrm{C}_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 2772 that $\mathrm{H}_n(X) = 0$. Therefore to show that an element $c \in \mathrm{C}_n(X)$ is the boundary of some element in $\mathrm{C}_{n+1}(X)$ it suffices to show that $\partial_n c = 0$. In our specific situation we calculate

by the Chain Subdivision Lemma 148.11 the subdivision maps form a chain map, i.e. $\partial_n \circ u_n = u_{n-1} \circ \partial_n$

$$\begin{aligned}\partial_n\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) &\overset{\downarrow}{=} -(\partial_n \circ P_{n-1} \circ \partial_n)(\sigma) + (u_{n-1} \circ \partial_n)(\sigma) - \partial_n(\sigma)\\ &= \underbrace{(-\partial_n P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\circ\partial_{n-1}}(\partial_n\sigma) \;=\; P_{n-2}(\underbrace{\partial_{n-1}\partial_n\sigma}_{=0}) \;=\; 0.\end{aligned}$$

As we had just remarked, this implies that there exists an $E_\sigma \in \mathrm{C}_{n+1}(X)$ which satisfies $\partial_{n+1}(E_\sigma) = (P_{n-1} \circ \partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$ a map which assigns to each singular n-simplex σ an element $E_\sigma \in \mathrm{C}_{n+1}(X)$ as above. It follows from the definition of E_σ that
$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; u_n - \mathrm{id}$$
as maps $\mathrm{C}_n(X) \to \mathrm{C}_n(X)$. This concludes the proof of the Subdivision–Chain Homotopy-Lemma 148.12 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider
$$\begin{aligned} P_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(X)\\ \sum_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) &\mapsto \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\uparrow\text{ defined since } \Delta^n \text{ is convex}}).\end{aligned}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. ■

Remark. Later on, in Section 235.2, we will generalize the idea behind the proof of the Subdivision–Chain Homotopy-Lemma 148.12, to prove the Acyclic Model Theorems. □

Definition 148.13. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

Lemma 148.14. (Iterated Subdivision Lemma) For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$ that is chain homotopic to the identity.

Proof. Recall that in the Chain Subdivision Lemma 148.11 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$. As we remarked on page 2747, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By the Subdivision–Chain Homotopy-Lemma 148.12 we have $u_* \simeq \mathrm{id}$. By the Chain Homotopy-Lemma 144.6 (2) this implies $u_*^2 \simeq u_*$, and by the Chain Homotopy-Lemma 144.6 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. ■

Definition 148.15. Given a singular chain c we can write it uniquely (up to reordering) as $c = a_1 \cdot \sigma_1 + \cdots + a_s \cdot \sigma_s$ with $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We call $\sigma_1, \ldots, \sigma_s$ the pieces of c.

If $\sigma\colon \Delta^n \to X$ is a singular n-simplex, then the pieces of $u_n(\sigma)$ are evidently "smaller" than the pieces of σ, and by iteratively applying u_n one can hope to make the resulting pieces "arbitrary small". In the following two lemmas we will make it precise what we mean by "arbitrary small". Before we formulate the first lemma we need to recall and introduce several definitions.

Definition 148.16.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the diameter is defined as

$$\mathrm{diam}(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a,b \in A\big\} \in \mathbb{R}_{\geq 0}.$$

(2) Given a singular simplex $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

Definition 148.17. For points $z_0, \ldots, z_k \in \mathbb{R}^n$ we consider the map

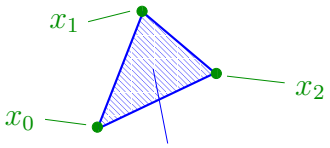
$$\begin{aligned}[z_0, \ldots, z_k]\colon \Delta^k &\to \mathbb{R}^n \\ (t_0, \ldots, t_k) &\mapsto \sum_{j=0}^{k} t_j \cdot z_j.\end{aligned}$$

The map $[z_0, \ldots, z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0, \ldots, z_k$ in $\mathbb{R}^n$. Note that $[z_0, \ldots, z_k]$ is a singular k-simplex in $\mathbb{R}^n$.

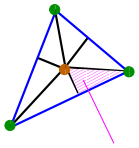
Now we can formulate the following lemma:

Lemma 148.18. (Subdivision–Diameter Lemma) For any $x_0, \ldots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \ldots, x_k])$ we have

$$\mathrm{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1} \cdot \mathrm{diam}(|[x_0, \ldots, x_k]|).$$



the image of $[x_0, x_1, x_2]$



the image of a piece τ of $u_2([x_0, x_1, x_2])$

In the proof of the Subdivision–Diameter Lemma 148.18 we will need the following elementary[1532] geometric lemma.

Lemma 148.19. (Simplex–Diameter Lemma) Let $y_0, \ldots, y_k \in \mathbb{R}^n$.

(1) We have the equality

$$\mathrm{diam}(|[y_0, \ldots, y_k]|) \;=\; \max\big\{\|y_i - y_j\| \,\big|\, i,j = 1, \ldots, k\big\}.$$

[1532]Here "elementary" means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

(2) We denote by $b = \frac{1}{k+1} \cdot \sum_{i=0}^{k} y_i$ the barycenter of $y_0, \ldots, y_k$. For any $i = 0, \ldots, k$ we have
$$\|y_i - b\| \ \leq \ \tfrac{k}{k+1} \cdot \operatorname{diam}(|[y_0, \ldots, y_k]|).$$

Proof of the Simplex–Diameter Lemma 148.19.

(1) The inequality "$\geq$" follows immediately from the definitions and the observation that each y_i lies in
$$|[y_0, \ldots, y_k]| \ = \ \Big\{ \sum_{i=0}^{k} t_i \cdot y_i \,\Big|\, t_i \in [0,1] \text{ and } \sum_{i=0}^{k} t_i = 1 \Big\}.$$
We turn to the proof of the reverse inequality "$\leq$". So we pick $a, b \in |[y_0, \ldots, y_k]|$. By definition we can write $a = \sum_{i=0}^{k} s_i \cdot y_i$ and $b = \sum_{i=0}^{k} t_i \cdot y_i$ with $s_i, t_i \in [0,1]$ and such that $\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then see that
$$\begin{aligned} \|a-b\| \ &= \ \Big\| a - \sum_{i=0}^{k} t_i \cdot y_i \Big\| \qquad \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{\downarrow} \\ &= \ \Big\| \sum_{i=0}^{k} t_i \cdot (a - y_i) \Big\| \ \leq \ \sum_{i=0}^{k} \|t_i \cdot (a - y_i)\| \\ &\leq \ \sum_{i=0}^{k} t_i \cdot \max_{i=0,\ldots,k} \{\|a - y_i\|\} \ = \ \max_{i=0,\ldots,k} \{\|a - y_i\|\} \ \leq \ \max_{i,j=0,\ldots,k} \{\|y_j - y_i\|\} \\ &\qquad \uparrow \text{ since } \sum_{i=0}^{k} t_i = 1 \qquad \uparrow \text{ same argument applied to } a = \sum_{j=0}^{k} s_j \cdot y_j \end{aligned}$$

(2) We denote by b_i the barycenter of $y_0, \ldots, \widehat{y_i}, \ldots, y_k$, i.e.
$$b_i \ = \ \tfrac{1}{k} \cdot \sum_{j \neq i} y_j.$$
We then see that
$$b \ = \ \tfrac{1}{k+1} \cdot y_i + \tfrac{k}{k+1} \cdot b_i.$$
It follows that
$$\begin{aligned} \|b - y_i\| \ &= \ \Big\| \tfrac{1}{k+1} \cdot y_i + \tfrac{k}{k+1} \cdot b_i - y_i \Big\| \ = \ \Big\| \tfrac{k}{k+1} \cdot b_i - \tfrac{k}{k+1} \cdot y_i \Big\| \\ &= \ \tfrac{k}{k+1} \cdot \|b_i - y_i\| \ \leq \ \tfrac{k}{k+1} \cdot \operatorname{diam}(|[y_0, \ldots, y_k]|). \end{aligned}$$
Thus we have obtained the desired inequality. ∎

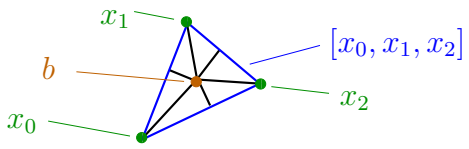
Now we are ready to provide the proof of the Subdivision–Diameter Lemma 148.18.

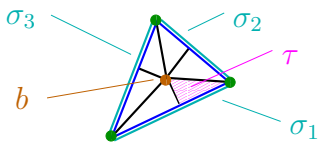
Proof of the Subdivision–Diameter Lemma 148.18. Recall that we need to show that for every $x_0, \ldots, x_k \in \mathbb{R}^n$ and every piece τ of $u_k([x_0, \ldots, x_k])$ we have
$$\operatorname{diam}(|\tau|) \ \leq \ \tfrac{n}{n+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|).$$
We prove the claim by induction on k. For $k = 0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0, \ldots, x_k \in \Delta^n$. We denote by $b = \frac{1}{k} \cdot \sum_{i=0}^{k} x_i$ the barycenter of $x_0, \ldots, x_k$. We denote by $\sigma_1, \ldots, \sigma_r$ the pieces of $\partial_k[x_0, \ldots, x_k]$.[1533]

[1533] Why does it is not say that the number of pieces is $k+1$?

Now let τ be a piece of $u_k([x_0,\dots,x_k])$. By the definition of the subdivision map given on page 2825 we see that $|\tau| = |[w_0,\dots,w_{k-1},b]|$ where $[w_0,\dots,w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1,\dots,r\}$. By the Simplex–Diameter Lemma 148.19 (1) it now suffices to prove the following claim:

Claim.
(1) For any $i,j = 0,\dots,k-1$ we have $\|w_i - w_j\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0,\dots,x_k]|)$.
(2) For any $i = 0,\dots,k-1$ we have $\|w_i - b\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0,\dots,x_k]|)$.





Proof. First let $i,j \in \{0,\dots,k-1\}$. Then

$$\|w_i - w_j\| \leq \operatorname{diam}([w_0,\dots,w_{k-1}]) \underset{\substack{\uparrow \\ \text{induction hypothesis} \\ \text{applied to } \sigma_s}}{\leq} \frac{k-1}{k} \cdot \operatorname{diam}(\sigma_s) \underset{\substack{\uparrow \\ \text{since } |\sigma_s| \subseteq |[x_0,\dots,x_k]| \text{ and } \frac{k-1}{k} \leq \frac{k}{k+1}}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}([x_0,\dots,x_k]).$$

Now let $i \in \{0,\dots,k-1\}$. We then see that

$$\|w_i - b\| \underset{\substack{\uparrow \\ \text{by the Simplex–Diameter} \\ \text{Lemma 148.19 (2)}}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[w_0,\dots,w_{k-1}]|) \underset{\substack{\uparrow \\ \text{since } |[w_0,\dots,w_{k-1}]| \subseteq |[x_0,\dots,x_k]|}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0,\dots,x_k]|)$$

■

Lemma 148.20. (Simplex–Reduction Lemma) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. For every singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an $m \in \mathbb{N}_0$ such that $u_n^m(\sigma)$ lies in $\mathrm{C}_n^{\mathcal{U}}(X)$.

Proof. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. By the Lebesgue Lemma 10.8 there exists a $\delta > 0$ such that for every subset A of Δ^n with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ such that $A \subseteq \sigma^{-1}(U_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We calculate that

$$u_n^m(\sigma) \underset{\substack{\uparrow \\ \text{by the Subdivision} \\ \text{Natural Lemma 148.10}}}{=} u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) = \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow \\ \text{by the Subdivision–Diameter Lemma 148.18} \\ \text{and since } \left(\frac{n}{n+1}\right)^m < \delta}}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image} \\ \text{of each simplicial simplex} \\ \text{has diameter less than } \delta\end{pmatrix}$$
$$= \underset{\substack{\uparrow \\ \text{by definition of } \delta}}{\sigma_*}\begin{pmatrix}\text{simplicial chain such that the image} \\ \text{of each piece lies in some } \sigma^{-1}(U_i)\end{pmatrix}.$$

But this means precisely that $u_n^m(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$. ■

Now we can prove the Comfortable Cover Proposition 148.9.

Proof of the Comfortable Cover Proposition 148.9. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We want to show that the inclusion map

$\iota\colon \mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X).$$

We first show that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective. Let $z \in \mathrm{H}_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j \cdot \sigma_j \in \mathrm{C}_n(X)$ that represents $z \in \mathrm{H}_n(X)$. It follows from the Simplex–Reduction Lemma 148.20, applied to the singular n-simplices $\sigma_1, \ldots, \sigma_r$, that there exists an $m \in \mathbb{N}_0$ such that $u_n^m(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. By the Iterated Subdivision Lemma 148.14 the maps $\{u_n^m\}_{n\in\mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$z = [c] \underset{\substack{\uparrow\\ \text{since we have the equality}\\ \partial_{n+1}\circ P_n + P_{n-1}\circ \partial_n = u_n^m - \mathrm{id}}}{=} \big[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\big] = \big[u_n^m(c) - \partial_{n+1}P_n(c)\big] \underset{\substack{\uparrow\\ \text{equality in } \mathrm{H}_n(X)}}{=} \overbrace{[u_n^m(c)]}^{\in \iota_*(\mathrm{H}_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective.

The argument that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective is very similar. Indeed, let $[c] \in \mathrm{H}_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in \mathrm{H}_n(X)$. This means that there exists an $e \in \mathrm{C}_{n+1}(X)$ with $\partial_{n+1}e = c$. From the Simplex–Reduction Lemma 148.20 it follows, as above, that there exists an $m \in \mathbb{N}_0$ with $u_{n+1}^m(e) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$. Again we choose a chain homotopy $\{P_k\}_{k\in\mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k\in\mathbb{N}_0}$ and id. Then we have

$$c = \partial_{n+1}e \underset{\substack{\uparrow\\ \text{since we have the equality}\\ \partial_{n+2}\circ P_{n+1} + P_n \circ \partial_{n+1} = u_{n+1}^m - \mathrm{id}}}{=} \partial_{n+1}\big(u_{n+1}^m(e) - \partial_{n+2}P_{n+1}(e) - P_n\partial_{n+1}e\big) = \partial_{n+1}\underbrace{u_{n+1}^m(e)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)} - \partial_{n+1}\underbrace{P_n(c)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in \mathrm{H}_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective. ∎

Using the Comfortable Cover Proposition 148.9 we can now prove the Excision Theorem 148.1.

Proof of the Excision Theorem 148.1. Let X be a topological space. Furthermore let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

We put $B \coloneqq X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B. Put differently, the two sets $\{A, B\}$ form a comfortable cover of X.



For each $n \in \mathbb{N}_0$ we consider the following commutative diagram of abelian groups:[1534]

$$\begin{array}{ccccccccc} & & & & & & \mathrm{C}_n(B, A\cap B) & & \\ & & & & & & \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma]} & & \\ 0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X,A) & \longrightarrow & 0. \end{array}$$

We make the following two observations:

(1) By definition the horizontal sequences are short exact sequences of chain complexes.
(2) It follows immediately from the definition of the group $\mathrm{C}_n^{\{A,B\}}(X)$ that the inclusion $\mathrm{C}_n(B) \to \mathrm{C}_n^{\{A,B\}}(X)$ induces a chain map $\mathrm{C}_n(B, A\cap B) \to \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A)$ which is in fact an isomorphism.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. It follows from the LES-Proposition 145.10, together with the above vertical isomorphism, that we obtain the following commutative diagram where the horizontal sequences are exact:

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n^{\{A,B\}}(X) & \longrightarrow & \overbrace{\mathrm{H}_n(B, A\cap B)}^{=\mathrm{H}_n(X\setminus Z, A\setminus Z)} & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}^{\{A,B\}}(X) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}(X) & \longrightarrow & \ldots \end{array}$$

From the Comfortable Cover Proposition 148.9 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 145.11 that the middle map is also an isomorphism. ■

148.5. Exercises for Chapter 148 .

Exercise 148.1. Provide an example of a topological space X and subspaces $Z \subseteq A \subseteq X$ such that the inclusion-induced map

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is not an isomorphism for some $n \in \mathbb{N}_0$.
Remark. The Excision Theorem 148.1 says that if the closure of Z is contained in the interior of A, then the above map is an isomorphism.

Exercise 148.2. Let $A \subseteq \mathbb{R}^n$ be a compact subset and let $r \in \mathbb{R}$ such that $A \subseteq B_r^n$. Show that for each $k \in \mathbb{N}_0$ the inclusion-induced map $\mathrm{H}_k(\overline{B}_r^n, \overline{B}_r^n \setminus A) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ is an isomorphism.

[1534] Here we use Definition 148.8 which in this context says that

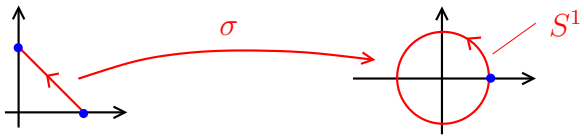
$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(A) \text{ and } \tau \in \mathrm{C}_n(B)\}.$$

Exercise 148.3. Determine the homology groups of $S^2 \vee S^1$.
Hint. Use long exact sequences for appropriate choices of pairs of topological spaces.

Exercise 148.4. Show that the singular 1-simplex

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & S^1 \\ (1-t,t) & \mapsto & \exp(2\pi\,\mathrm{i}\,t) \end{array}$$

is a cycle that represents a generator of $\mathrm{H}_1(S^1) \cong \mathbb{Z}$.
Hint. Work with the fairly explicit isomorphism $\mathrm{H}_1(S^1) \cong \widetilde{\mathrm{H}}_0(S^0)$ from page 2818 and make use of the Singular 1-Simplex Lemma 143.28.



Exercise 148.5. Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the following two maps.
(1) Let $p\colon X \times S^1 \to X$ be the projection.
(2) We consider the homomorphism $\psi\colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$ that is given by

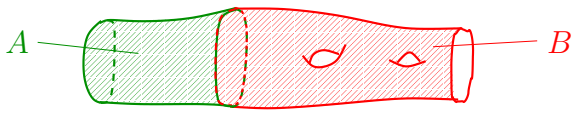
$$\begin{array}{ccccc} \mathrm{H}_n(X\times S^1) & \to & \mathrm{H}_n(X\times S^1, X\times S^1_{\leq 0}) & \overset{\cong}{\leftarrow} & \mathrm{H}_n(X\times S^1_{\geq 0}, X\times\{\pm 1\}) \\ & & & & \uparrow \\ & & & & \text{isomorphism by the Excision-via-Deformation Theorem 148.5} \\ & \to & \mathrm{H}_{n-1}(X\times\{\pm 1\}) & \to & \mathrm{H}_{n-1}(X\times\{1\}) \;=\; \mathrm{H}_{n-1}(X). \\ & & \uparrow & & \uparrow \\ & & \text{the connecting homomorphism} & & \text{projection induced homomorphism} \end{array}$$

Show that the map

$$p_* \oplus \psi\colon \mathrm{H}_n(X\times S^1) \;\to\; \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X)$$

is an isomorphism.
Hint. Consider the long exact sequence of the pair $(X \times S^1, X \times S^1_{\leq 0})$.

Exercise 148.6. Let X be an m-dimensional topological manifold and let $A, B \subseteq X$ be two codimension-zero submanifolds such that the following statements hold:
(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.
Show that if each inclusion-induced map $\mathrm{H}_k(A \cap B) \to \mathrm{H}_k(A)$ is an isomorphism, then each inclusion-induced map $\mathrm{H}_k(B) \to \mathrm{H}_k(M)$ is an isomorphism.



Exercise 148.7. Let M be a smooth manifold and let $W \subseteq M$ be a compact codimension-zero submanifold with corner. As in the Smooth Submanifold-with-Corner Lemma 26.16

and the Boundary-of-Submanifold Proposition 25.48 (0) we set $\partial_0 W \coloneqq W \cap (\overline{M \setminus W})$ and $\mathring{W} \coloneqq W \setminus \partial_0 W$. Furthermore let A be a union of components of ∂M. Show that each inclusion-induced map

$$\mathrm{H}_k(M \setminus \mathring{W}, A \setminus \mathring{W}) \;\to\; \mathrm{H}_k(M, A)$$

is an isomorphism.

Hint. In the proof of the Manifold–Excision Theorem 148.3 you should replace the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7.

CHAPTER 149

Applications of the Excision Theorem

In the previous Chapter 148 we formulated and proved the Excision Theorem 148.1. The proof turned out to be long and difficult. But in this chapter we will see that this effort really pays off. For example we will use the Excision Theorem 148.1 to compute homology groups of suspensions and wedges of topological spaces.

We will also return to the challenge, posed in Chapter 146, to calculate homology groups of quotient spaces X/A. Using the Excision Theorem 148.1 we will be able, in reasonable settings, to relate the homology groups of A, X and X/A via a long exact sequence.

This chapter is organized as follows:

- In Section 149.1 we will determine the homology groups of suspensions.
- In Section 149.2 we will determine the homology groups of spheres.
- In Section 149.3 we will restate and reprove the Brouwer Fixed Point Theorem 38.7.
- In Section 149.4 we will introduce good subsets and the relationship between relative homology and homology groups of quotients.
- In Section 149.5 we will determine the homology groups of the wedge of topological spaces.
- In Section 149.6 we will formulate and prove the long exact sequence for quotients.
- In Appendix 149.7 we will discuss the effect of a reflection on the homology groups of a suspension.
- In Section 149.8 we will provide exercises for Chapter 149.

In the next Chapter 150 we will study local homology and we will provide some explicit generators of homology groups.

149.1. **Suspensions and homology groups.** Let us recall the following definitions from Definition 20.21.

Definition 149.1. Let X be a non-empty topological space.

(1) We define the suspension of X to be the topological space

$$\Sigma(X) \;:=\; ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\}).$$

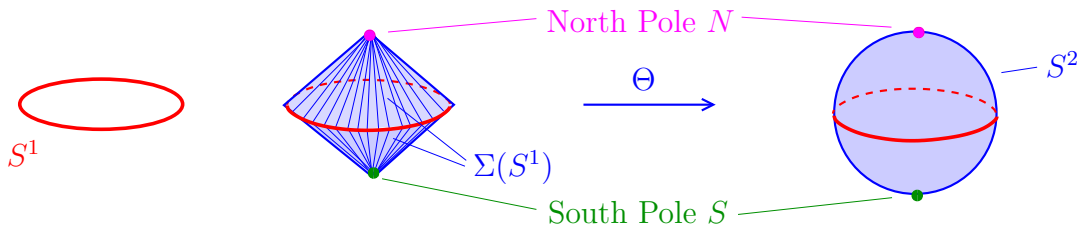
(2) We call $S := [X \times \{-1\}]$ the South Pole and we call $N := [X \times \{1\}]$ the North Pole of $\Sigma(X)$.[1535]

Example. One can easily verify that the map

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subseteq \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \underbrace{\left(P \cdot \cos\left(\frac{\pi}{2}t\right)\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}} \end{array}$$

[1535] If $X = \varnothing$, then $\Sigma(X)$ consists of precisely two points, which we also call North and South Pole.

is a homeomorphism.



This homeomorphism $\Sigma(S^n) \to S^{n+1}$ is illustrated in the figure above. □

In the Suspension-π_1-Lemma 99.3 and Exercise 106.5 we calculated the fundamental group of the suspension of a non-empty topological space with finitely many path components. In our next proposition we will determine the homology groups of a suspension. To formulate the proposition we need to introduce the following definition:

Definition 149.2.

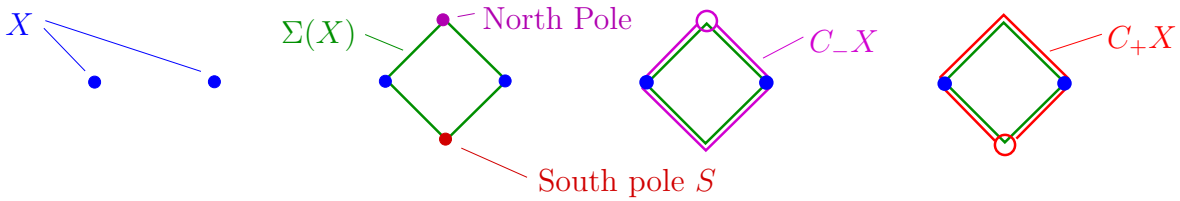
(1) We set

$$C_-X \coloneqq \Sigma(X) \setminus \{N\} \quad \text{and} \quad C_+X \coloneqq \Sigma(X) \setminus \{S\}.$$

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we refer, as in Definition 20.22, to

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

as the suspension of f.



Proposition 149.3. (Suspension-H_*-Proposition) Let X be a topological space and let $n \in \mathbb{Z}$.

(1) We write $I \coloneqq (-1,1)$. The following maps are isomorphisms:[1536]

$$\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow[\uparrow]{\text{ж}} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xleftarrow[\downarrow]{} \mathrm{H}_{n+1}(C_+X, X\times I) \xrightarrow[\uparrow]{\partial_{n+1}} \widetilde{\mathrm{H}}_n(X\times I) \xleftarrow[\downarrow]{} \widetilde{\mathrm{H}}_n(X).$$

ж: the natural map defined in Notation 147.7

∂_{n+1}: connecting homomorphism of the pair $(C_+X, X\times I)$ from the Reduced LES-of-Pair Proposition 147.8

left $\downarrow$: induced by the inclusion

right $\downarrow$: induced by the embedding $x \mapsto (x,0)$

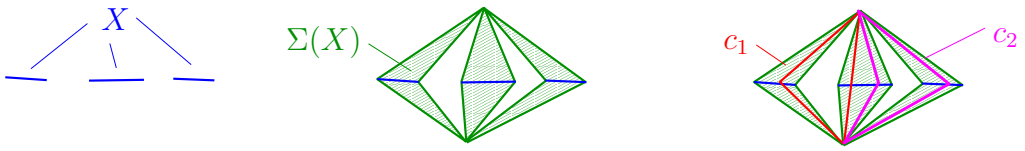
(2) The resulting isomorphism

$$\Sigma_X\colon \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_n(X)$$

is natural. In other words, given any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) & \xrightarrow{\Sigma_X} & \widetilde{\mathrm{H}}_n(X) \\ \downarrow{\scriptstyle \Sigma(f)_*} & & {\scriptstyle f_*}\downarrow \\ \widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) & \xrightarrow{\Sigma_Y} & \widetilde{\mathrm{H}}_n(Y). \end{array}$$

Example. Let $n \in \mathbb{N}$ and let X be the disjoint union of $n \in \mathbb{N}$ intervals. It follows from the Suspension-H_*-Proposition 149.3 together with the Reduced Homology Lemma 147.3 that we have isomorphisms $\mathrm{H}_1(\Sigma(X)) \cong \widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In the figure below we show a generating set for $\mathrm{H}_1(\Sigma(X))$ for $n = 3$.



$\{c_1, c_2\}$ is a basis for $\mathrm{H}_1(\Sigma(X))$

We leave it to the reader to deduce from the proof of the Suspension-H_*-Proposition 149.3 that the given cycles do indeed form a generating set for $\mathrm{H}_1(\Sigma(X))$. □

Remark. In the Reflection-on-Suspension-H_*-Proposition 149.18 we will calculate the effect of the obvious reflection on $\Sigma(X)$ on the homology groups of $\Sigma(X)$. □

Proof. [1537]

(1) Let X be a topological space. For each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{array}{l} \longrightarrow \widetilde{\mathrm{H}}_{n+1}(C_-X) \to \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{H}}_n(C_-X) \longrightarrow \dots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow{\scriptstyle \cong} \\ \dots \to \widetilde{\mathrm{H}}_{n+1}(X \times I) \to \widetilde{\mathrm{H}}_{n+1}(C_+X) \to \mathrm{H}_{n+1}(C_+X, X \times I) \xrightarrow[\cong]{\partial_{n+1}} \widetilde{\mathrm{H}}_n(X \times I) \to \widetilde{\mathrm{H}}_n(C_+X) \to \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad {\scriptstyle i_*}\uparrow{\scriptstyle \cong} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \widetilde{\mathrm{H}}_n(X) \end{array}$$

We make the following clarifications and explanations:

(a) The top horizontal sequence is the long exact sequence in reduced homology of the pair $(\Sigma(X), C_-X)$ that is given by the Reduced LES-of-Pair Proposition 147.8.

(b) The bottom horizontal sequence is the long exact sequence in reduced homology of the pair $(C_+X, X \times I)$ that is given by the Reduced LES-of-Pair Proposition 147.8.

[1536] Using the Open-Closed–Injective Lemma 4.22 one can easily show that the maps $X \times (-1,1) \to C_\pm$ and $X \times (-1,1) \to \Sigma(X)$ given by $(x,t) \mapsto [(x,t)]$ are embeddings. We use this embedding to view $X \times (-1,1)$ as a subspace of $C_\pm X$ and of $\Sigma(X)$.

[1537] The proof of the Suspension-H_*-Proposition 149.3 is very similar to the calculation of the homology groups of spheres on page 2818.

(c) By the same argument as in the proof of the Cone Contractability Lemma 20.20 we see that both C_-X and C_+X admit a deformation retraction to a point. It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 together with the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that for all $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_k(C_-X) = 0$ and that we have $\widetilde{\mathrm{H}}_k(C_+X) = 0$.

(d) It follows from (c) and the exactness of the horizontal sequences that the horizontal maps shown in red are isomorphisms.

(e) As before let $S = [X \times \{-1\}]$ be the South Pole of the suspension. Note that $\Sigma(X) \setminus \{S\} = C_+X$ and $C_-X \setminus \{S\} = C_-X \cap C_+X = X \times I$. Using this observation we obtain immediately from the Excision Theorem 148.1[1538] that the vertical map $\mathrm{H}_{n+1}(C_+X, X \times I) \to \mathrm{H}_{n+1}(\Sigma(X), C_-X)$ is an isomorphism.

(f) We denote by $i\colon X \to X \times I$ the natural embedding given by $x \mapsto (x, 0)$. Since I is an interval one can easily show, using the Deformation Retract–Homotopy Equivalence Lemma 19.10, that i is a homotopy equivalence.

(g) It follows from (f) together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 that the induced map $i_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(X \times I)$ is an isomorphism.

We have now shown that all red homomorphisms are isomorphisms. But that is exactly what we had to do.

(2) Let $f\colon X \to Y$ be a continuous map between two topological spaces. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & \overset{\Sigma_X}{\frown} & & & & & \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) & \xrightarrow{\text{ж}} & \mathrm{H}_{n+1}(\Sigma(X), C_-X) & \longleftarrow & \mathrm{H}_{n+1}(C_+X, X\times I) & \xrightarrow{\partial_{n+1}} & \widetilde{\mathrm{H}}_n(X\times I) & \xleftarrow{i_*} & \widetilde{\mathrm{H}}_n(X) \\
\downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (f\times\mathrm{id})_*} & & \downarrow{\scriptstyle (f\times\mathrm{id})_*} & & \downarrow{\scriptstyle f_*} \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) & \xrightarrow{\text{ж}} & \mathrm{H}_{n+1}(\Sigma(Y), C_-Y) & \longleftarrow & \mathrm{H}_{n+1}(C_+Y, Y\times I) & \xrightarrow{\partial_{n+1}} & \widetilde{\mathrm{H}}_n(Y\times I) & \xleftarrow{i_*} & \widetilde{\mathrm{H}}_n(Y). \\
 & & & \underset{\Sigma_Y}{\smile} & & & & &
\end{array}$$

The first, second and fourth square commutes by functoriality of (reduced) homology. The third square commutes since in the LES-of-Triple Proposition 146.5 (2) we showed that the connecting homomorphism is natural. We have thus shown that the diagram commutes, which implies that Σ_X itself is natural. ■

149.2. **The homology groups of spheres.** Using the Suspension-H_*-Proposition 149.3 we can now redo the calculation of the homology groups of spheres which we first did on page 2818.

Proposition 149.4. (Sphere-H_*-Proposition) For any $n \in \mathbb{Z}_{\geq -1}$ we have[1539]

$$\widetilde{\mathrm{H}}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

[1538] One can easily verify that the hypotheses for the Excision Theorem 148.1 are satisfied.

[1539] For ordinary homology recall that by the Reduced Homology Lemma 147.3 we know that for $n \geq 0$ we have $\mathrm{H}_0(S^n) \cong \widetilde{\mathrm{H}}_0(S^n) \oplus \mathbb{Z}$ and $\mathrm{H}_k(S^n) \cong \widetilde{\mathrm{H}}_k(S^n)$ for $k > 0$.

Proof. For each $m \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_m(S^n) \underset{\text{since } S^n \text{ is homeomorphic to the suspension } \Sigma(S^{n-1})}{\cong} \widetilde{\mathrm{H}}_m(\Sigma(S^{n-1})) \overset{\text{by the Suspension-H}_* \text{ Proposition 149.3}}{\cong} \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \overset{\text{iterating the argument}}{\cong} \ldots \cong \widetilde{\mathrm{H}}_{m-n}(S^0) \underset{\text{by the Singleton-H}_*\text{-Lemma 143.26, the Path-Component–H}_*\text{–Lemma 143.21 and the Reduced Homology Lemma 147.3}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } m-n=0, \\ 0, & \text{otherwise.} \end{cases}$$

■

Remark.

(1) In the Sphere-Singular Cycle Proposition 150.7 we will improve on the above result, namely we will describe an explicit n-cycle that represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$.
(2) Note that the statement of the Sphere-H$_*$-Proposition 149.4 is in marked contrast to the situation for higher homotopy groups. As formulated in Question 134.25 we have the strong suspicion that $\pi_3(S^2) \neq 0$, whereas we just showed that $\mathrm{H}_3(S^2) = 0$. □

By the Smooth Manifold-Dimension Proposition 25.31 we know that for $k \neq l$ the spheres S^k and S^l are not diffeomorphic. The following corollary is a serious upgrade of this fact.

Corollary 149.5. (Sphere–Homeomorphism Corollary) For $k \neq l$ the spheres S^k and S^l are not homotopy equivalence, let alone homeomorphic.

So we have finally managed to distinguish spheres of different dimensions. Our initial idea had been to use higher homotopy groups to prove that result. Even though the definition of higher homotopy groups is arguably much more natural than the definition of homology groups, it turns out that the latter groups are significantly easier to compute.

Proof. This corollary follows immediately from the calculation of the (reduced) homology groups of spheres in the Sphere-H$_*$-Proposition 149.4 together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5. ■

In the $\mathbb{R}$-$\mathbb{R}^2$-Non Homeomorphism–Lemma 4.36 we used elementary connectivity arguments to show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Furthermore, in the $\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma 90.10 we used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$. Now we can finally show that for any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 149.6. ($\mathbb{R}^k$-$\mathbb{R}^l$-Non Homeomorphism–Proposition) For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proof. Let $k, l \in \mathbb{N}_0$ and suppose that there exists a homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Then we obtain that

$$\mathbb{Z} \underset{\text{by the Sphere-H}_* \text{ Proposition 149.4}}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{k-1}) \overset{\text{this follows from the Homotopy Equivalence-}\widetilde{\mathrm{H}}_*\text{-Corollary 147.5 (1a), since for any } P \in \mathbb{R}^m \text{ the topological space } \mathbb{R}^m \setminus \{P\} \text{ is homotopy equivalent to } S^{m-1}}{\cong} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^k \setminus \{0\}) \underset{\text{isomorphism since } f \text{ is a homeomorphism}}{\xrightarrow{f_*}} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^l \setminus \{f(0)\}) \overset{}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{l-1}).$$

So we have $\widetilde{\mathrm{H}}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by the Sphere-H$_*$-Proposition 149.4 that is only possible if $k = l$. ■

We conclude this section with a calculation of a relative homology group, which we will frequently use.

Lemma 149.7. (Ball-rel-Sphere-H$_*$-Lemma) Given any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\overline{B}^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$ which is provided for us by the Reduced LES-of-Pair Proposition 147.8:

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^n)}_{=0, \text{ see page 2812}} \to \mathrm{H}_k(\overline{B}^n, S^{n-1}) \xrightarrow{\partial_k} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{n-1})}_{\substack{\text{calculated in the} \\ \text{Sphere-H}_*\text{-Proposition 149.4}}} \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(\overline{B}^n)}_{=0, \text{ see page 2812}} \to \ldots$$

The statement of the lemma is now an immediate consequence of the Sphere-H$_*$-Proposition 149.4. ∎

149.3. **The Brouwer Fixed Point Theorem II .** In this section we discuss a major application of the calculation of the homology groups of spheres, namely the Brouwer Fixed Point Theorem.

For the reader's convenience we recall Definition 19.1:

Definition 149.8. Let X be a topological space. We say that a subset $A \subseteq X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

In the Circle–Non Retract Proposition 90.6 we used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with the Sphere-H$_*$-Proposition 149.4, now gives us the following high-dimensional generalization.

Proposition 149.9. (Sphere-Non Retract Proposition) For every $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial\overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.

Remark.

(1) In the Boundary Non-Retraction Proposition 75.13 we gave, using de Rham cohomology groups, an alternative proof of the Sphere-Non Retract Proposition. As we just mentioned, the case $n = 2$ was also proved in the Circle–Non Retract Proposition 90.6.

(2) I found the following amusing description of the Sphere-Non Retract Proposition 149.9 online:
"Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn't know where to go, and stays there for a moment, much to the shark's satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep." □

Proof. Suppose that there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccc} & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^n) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & \xrightarrow{(r\circ i)_*} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \end{array}$$

This diagram commutes since it follows from the functoriality of the reduced homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ which, by the Sphere-H_*-Proposition 149.4, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism. But the upper map factorizes through the group $\widetilde{\mathrm{H}}_{n-1}(\overline{B}^n)$, which is the trivial group by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6. This implies that the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. ∎

Now we can again provide a proof of the Brouwer Fixed Point Theorem 38.7.

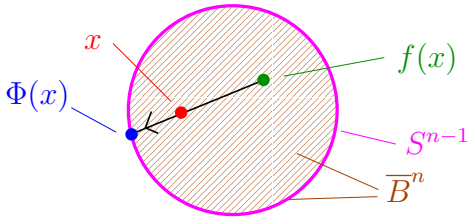
Theorem 38.7. (Brouwer Fixed Point Theorem) Every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.

Remark. On page 939 we already provided a proof of the theorem, and we will provide another proof on page 3443. On page 941 we also mentioned a few non-mathematical applications of the Brouwer Fixed Point Theorem 38.7. □

Proof. The proof of the Brouwer Fixed Point Theorem 38.7 is basically identical to the one provided on page 2843. We just need to replace the Boundary Non-Retraction Proposition 38.6 by the Sphere-Non Retract Proposition 149.9. For the reader's convenience we provide the argument, minus the technical details, again.

Suppose that there exists a continuous map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. On page 939 we saw that it follows from the hypothesis that f admits no fixed points that for each $x \in \overline{B}^n$ the open ray $\{f(x)+t\cdot(x-f(x))\,|\,t>0\}$ which starts at $f(x)$ and which goes through x contains precisely one point in S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rccl} \Phi\colon & \overline{B}^n & \to & \overline{B}^n \\ & x & \mapsto & \text{the intersection point of } S^{n-1} \text{ with the} \\ & & & \text{uniquely determined open ray from } f(x) \text{ to } x. \end{array}$$



As we saw on page 940, it is elementary to show that Φ is continuous. It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in \overline{B}^n$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to the Sphere-Non Retract Proposition 149.9 this not possible. Thus we have obtained a contradiction. ∎

As we mentioned already in Section 38.2, the Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 38.8. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $(n \times n)$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Proof. For the reader's convenience we recall the proof from page 941. We need to introduce two players:

(1) Given $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum_{i=1}^{n} |v_i|$.

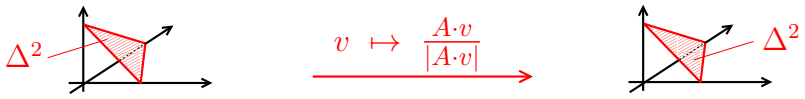
(2) We consider again the standard $(n-1)$-simplex

$$\Delta^{n-1} \;:=\; \{v \in [0,1]^n \,\big|\, |v| = 1\}.$$

We saw in the Δ^n-Homeomorphism Lemma 143.9 (2) that Δ^{n-1} is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 38.7 that any continuous self-map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point.

Now let $A = (a_{ij})$ be an $(n \times n)$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon \Delta^{n-1} & \to & \Delta^{n-1} \\ v = (v_1, \dots, v_n) & \mapsto & \frac{A \cdot v}{|A \cdot v|}. \end{array}$$



Note that this map is defined since our hypothesis that all a_{ij} are positive, the fact that all entries of v are non-negative and the fact that at least one entry of v is positive imply that $A \cdot v \neq 0$. It is clear that f is continuous. As we had just discussed above, the map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point $v \in \Delta^{n-1}$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

149.4. **Good subsets and relative homology.** A very long time ago, namely in Chapter 146, we started out with our attempt to calculate homology groups of quotients X/A. We developed the concept of relative chain complexes $\mathrm{C}_n(X/A) := \mathrm{C}_n(X)/\mathrm{C}_n(A)$ which, as we saw, turned out to be very interesting in its own right.

In this section we return to the original question and we will try to determine homology groups of quotients. There is now an obvious idea: we suspect that in many cases the relative homology groups $\mathrm{H}_n(X, A)$ should be isomorphic to the homology groups of the quotient X/A. In this section we will see that this hunch is basically correct, if A is a "good" subset of X. In the next section we will use this result to determine the homology groups of wedges of topological spaces.

Before we can state the main result of this section we need to introduce the following definition:

Definition 149.10. Let X be a topological space and let $A \subseteq X$ be a[1540] subset. We denote by $p\colon (X,A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ we define

$$\mathrm{H}_n(X,A) \xrightarrow{p_*} \mathrm{H}_n(X/A, A/A) \xleftarrow[\cong]{\text{isomorphism ж from the Reduced-to-Relative-H}_*\text{-Lemma 147.9}} \widetilde{\mathrm{H}}_n(X/A)\ .$$
$$\text{=: ю}$$

The following lemma summarizes two properties of the maps ю.

Lemma 149.11. (Relative-to-Quotient-H$_*$-Lemma)

(1) Let X be a topological space and let $A \subseteq X$ be a subset. We denote by $p\colon X \to X/A$ the natural projection. The following diagram commutes:[1541]

$$\begin{array}{ccc} & \mathrm{H}_n(X,A) & \\ {\scriptstyle \text{ж}}\nearrow & & \searrow{\scriptstyle \text{ю}} \\ \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A). \end{array}$$

(2) We denote by *PairTop* the category of pairs of topological spaces that we introduced in Definition 19.23. Furthermore as usual we denote by *AbGroup* the category *AbGroup* of abelian groups. The above maps ю$\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ define a natural transformation from the functor

$$\begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X,A) & \mapsto & \mathrm{H}_n(X,A) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X,A) & \mapsto & \widetilde{\mathrm{H}}_n(X/A). \end{array}$$

Proof.

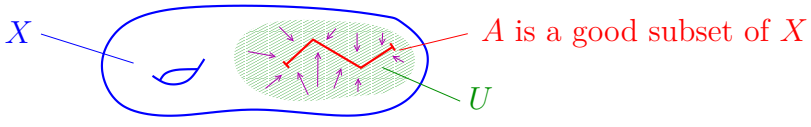
(1) We consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_n(X,A) & & \\ & {\scriptstyle \text{ж}}\nearrow & \downarrow{\scriptstyle p_*} & \searrow{\scriptstyle \text{ю}} & \\ & & \mathrm{H}_n(X/A, A/A) & & \\ \widetilde{\mathrm{H}}_n(X) & \xrightarrow{[\sigma]\mapsto[\sigma]} & & \xleftarrow[\cong]{\text{Lemma 147.9}} & \widetilde{\mathrm{H}}_n(X/A). \\ & & \xrightarrow[p_*]{} & & \end{array}$$

It follows immediately from the definitions that all three regions of the diagram commute. This observation gives us the desired statement.

(2) This statement follows basically immediately from the definitions. ∎

Definition 149.12. Let X be a topological space and let $A \subseteq X$ be a subset. We say that A is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.



[1540] Note that the definition also makes sense if A is the empty set.

[1541] The left diagonal map is the composition of the natural map $\widetilde{\mathrm{H}}_k(X) \to \mathrm{H}_n(X)$ from the Reduced Homology Lemma 147.3 with the natural map $\mathrm{H}_n(X) \to \mathrm{H}_n(X,A)$.

Examples.

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subseteq X$ is good in the above sense if and only the point is good in the sense of Definition 99.8.

(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. Indeed, it is clear that S^{n-1} is a closed subset of $\overline{B}^n$. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$, then a deformation retraction from U to S^{n-1} is given by the map

$$\begin{aligned} (\overline{B}^n \setminus \{0\}) \times [0,1] &\to \overline{B}^n \setminus \{0\} \\ (x,t) &\mapsto \left(1 - t \cdot \tfrac{1-\|x\|}{\|x\|}\right) \cdot x. \end{aligned}$$

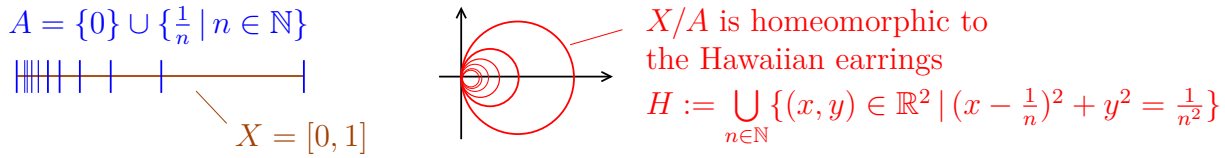
(3) We can generalize the second example: It follows easily from the Topological Manifold–Boundary Proposition 23.15 (2a) and the Topological Collar Neighborhood Theorem 35.14 that every union of boundary components of a topological manifold is good.

(4) Note that it follows from the Subcomplex–Closed Lemma 136.11 and the Subcomplex–Neighborhood Proposition 137.9 (1) that every subcomplex A of a CW-complex X is good. □

Proposition 149.13. (Relative-to-Quotient-H$_*$-Proposition) Let X be a topological space, let $A \subseteq X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset, then the natural homomorphism $\text{ю}\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism.[1542]

Example. Let X be a topological space, let $A \subseteq X$ be a subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $\mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is in general not an isomorphism. In fact in general the two groups $\mathrm{H}_n(X,A)$ and $\widetilde{\mathrm{H}}_n(X/A)$ are not even abstractly isomorphic.

For example let us consider the topological space $X = [0,1]$ together with the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. In Exercise 105.1 we showed that the quotient space X/A is homeomorphic to the Hawaiian earring $H := \bigcup_{n\in\mathbb{N}} \{(x,y) \in \mathbb{R}^2 \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$.



In Exercise 149.9 we will prove the following two statements:

(1) The group $\mathrm{H}_1(X,A)$ is countable.

(2) The group $\mathrm{H}_1(X/A)$ is uncountable. □

The proof of the Relative-to-Quotient-H$_*$-Proposition 149.13 relies on the following elementary little lemma, which is precisely the content of Exercise 8.20.

Auxiliary Lemma 149.14. Let X be a topological space and let A be a closed subset of X.

(1) For every open subset $U \subseteq X \setminus A$ the obvious map $U \to X/A$ is an open embedding.

[1542] In the Cofibration–Relative-to-Quotient-H$_*$-Corollary 153.16 we will show the analog of the Relative-to-Quotient-H$_*$-Proposition 149.13 under the slightly different hypothesis that the inclusion map $i\colon A \to X$ is a closed cofibration.

(2) For every open neighborhood W of A the obvious map $W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism.



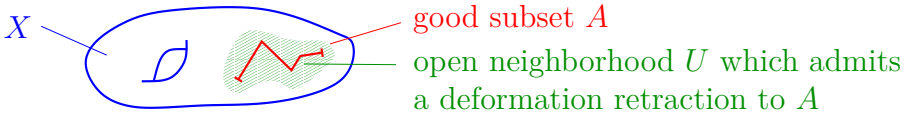
Proof of Lemma 149.14. Let X be a topological space and let A be a closed subset of X. We denote by $p\colon X \to X/A$ the natural projection.

(1) Let $U \subseteq X \setminus A$ be an open subset. We need to show that the map $q\colon U \to X \to X/A$ is an open embedding. It follows from $U \subseteq X \setminus A$ that q is an injection. Furthermore note that by the Topological-Quotient Proposition 8.2 (1a) we know that q is continuous. By the Open-Closed–Injective Lemma 4.22 (2) it remains to show that $q\colon U \to X/A$ is an open map. Now let $V \subseteq U$ be an open subset. We need to show that $q(V) \subseteq X/A$ is open. We break the argument into three easy steps:
   (a) It follows from the Open-Closed Inclusion Lemma 4.13 (1) and the fact that A is by hypothesis closed, that V is also an open subset of X.
   (b) Since $A \cap V = \varnothing$ we see that $p^{-1}(p(V)) = V$, which by (a) is in fact an open subset of X.
   (c) It follows from (b) and the Definition 8.3 of the quotient topology that $p(V)$ is an open subset of X/A.

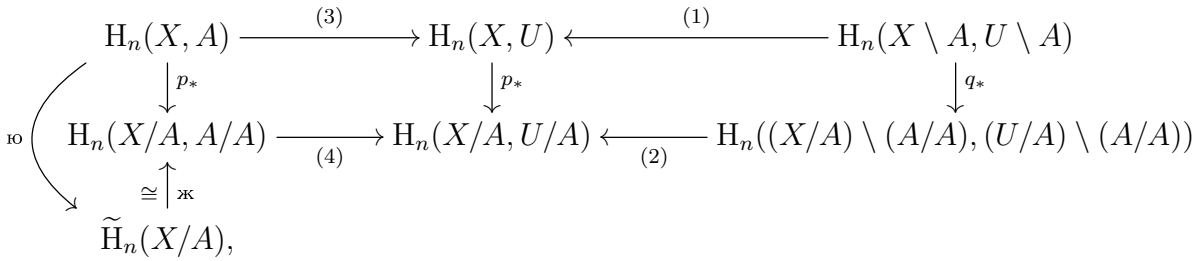
(2) Let W be an open neighborhood of A. We set $U \coloneqq W \setminus A$. Since A is closed we see that U is an open subset of $W \setminus A$. Recall that we want to show that the obvious map $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism. It is clear that q is a bijection and it follows from the Quotient-to-Quotient Lemma 8.6 that q is continuous. By the Open-Closed–Injective Lemma 4.22 (2) it remains to show that $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is an open map. Thus let V be an open subset of $W \setminus A$. We have $q(V) = p(V) \cap (W/A \setminus A/A)$. It follows from (1c) and the definition of the subspace topology that $q(V)$ is an open subset of $(W/A) \setminus (A/A)$. ∎

Proof of the Relative-to-Quotient-H_*-Proposition 149.13. Let X be a topological space, let $A \subseteq X$ be a good subset and let $n \in \mathbb{N}_0$. We make the following simple preparations:

(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.
(2) We denote by $p\colon X \to X/A$ the projection.
(3) We denote by $q\colon X \setminus A \to (X/A) \setminus (A/A)$ the restriction of p to $X \setminus A$. For completeness' sake we point out that it follows from the Subspace Continuity Lemma 4.3 (2) that q is continuous.



Now we consider the following diagram:



where all the horizontal maps are induced by inclusions of pairs of topological spaces. Recall that we need to show that ю: $\mathrm{H}_n(X, A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism. By the Reduced-to-Relative-H_*-Lemma 147.9 we know that ж: $\widetilde{\mathrm{H}}_n(X/A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism. Thus it suffices to show that the top left vertical map p_*: $\mathrm{H}_n(X, A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism.

First note that it follows easily from the functoriality of relative homology groups that the two squares of the diagram commute. Next note that it follows from our hypothesis that the subset A is closed, together with the above auxiliary Lemma 149.14 (2), that the map q: $(X \setminus A, U \setminus A) \to ((X/A) \setminus (A/A), (U/A) \setminus (A/A))$ is a homeomorphism of pairs of topological spaces. We obtain from this observation that the vertical map on the right is an isomorphism. At this stage it remains to prove the following claim:

Claim. All horizontal maps are isomorphisms.

Proof.

(1) Since $A \subseteq X$ is closed and since $U \subseteq X$ is open it follows from the Excision Theorem 148.1 that both the top right horizontal map is an isomorphism.
(2) The hypothesis that $A \subseteq X$ is closed and that $U \subseteq X$ is an open neighborhood of $A \subseteq X$ implies, by Definition 8.3 of the quotient topology, that A/A is a closed subset of X/A and that U/A is an open neighborhood of $A/A \subseteq X/A$. Thus we can also apply the Excision Theorem 148.1 to the horizontal map on the bottom right, and we see that this map is also an isomorphism.
(3) Since A is deformation retract of U we obtain from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the top left horizontal map is an isomorphism.
(4) Recall that A is a deformation retract of U. It follows from the Deformation Retraction-Quotient Theorem 19.7 (2) that A/A is a deformation retract of U/A.[1543] As in (3) we now see that the bottom-left horizontal map is an isomorphism. ■

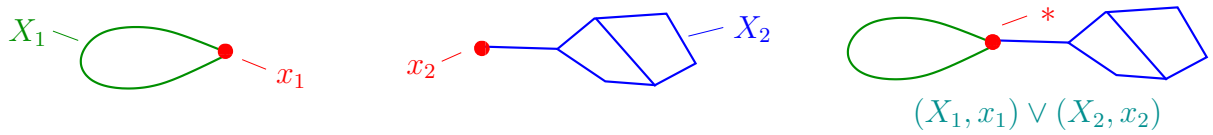
149.5. **The homology groups of the wedge of topological spaces.** We start out by recalling Definition 96.1:

Definition 149.15. Let $\{(X_k, x_k)\}_{k \in K}$ be a family of pointed topological spaces. We refer to

$$\bigvee_{k \in K} (X_k, x_k) \coloneqq \Big(\bigsqcup_{k \in K} X_k \Big) \Big/ \bigsqcup_{k \in K} \{x_k\}$$

as the wedge of the pointed topological spaces (X_k, x_k).

[1543] The diligent reader who checks the statements we refer to will not fail to notice, that the Deformation Retraction-Quotient Theorem 19.7 (2) rests eventually on the Whitehead Quotient Theorem 17.6.



Recall that in the Wedge Proposition 96.2 we showed that the wedge $\bigvee_{k\in K}(X_k, x_k)$ is the coproduct in the category $\mathcal{PTop}$ of *pointed* topological spaces. Also recall that we showed in the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 that in the category $\mathcal{AbGroup}$ of abelian groups the coproduct is given by the direct sum. It is again natural to ask which functors from the category $\mathcal{PTop}$ of pointed topological groups to the category $\mathcal{AbGroup}$ of abelian groups preserve coproducts. One can see in very easy examples that ordinary homology groups do not quite preserve coproducts. But we have alternative functors, namely for each $n \in \mathbb{N}_0$ we can consider the functor

$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{AbGroup} \\ (X, x_0) & \mapsto & \mathrm{H}_n(X, \{x_0\}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \mathrm{H}_n(X, \{x_0\}) \to \mathrm{H}_n(Y, \{y_0\})). \end{array}$$

The next shows that these relative homology functors preserve coproducts (at least under rather mild hypotheses on the points $x_k \in X_k$).

Proposition 149.16. (Wedge-H_*-Proposition) Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces such that each $x_k \in X_k$ is good[1544]. We denote the wedge point of $\bigvee_{k\in K}(X_k, x_k)$ by $*$ and given $k \in K$ we denote by $i_k\colon X_k \to \bigvee_{k\in K}(X_k, x_k)$ the natural inclusion that we introduced in Definition 96.1. Let $n \in \mathbb{N}_0$.

(1) The natural map

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \to \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big)$$

is an isomorphism.

(2) For $k \in K$ let $p_j\colon \bigvee_{k\in K} x_k \to X_j$ be the natural map that we introduced in Definition 96.1. If K is finite, then the map

$$\prod_{k\in K} p_k^*\colon \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big) \to \prod_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) = \underset{\text{since } K \text{ is finite}}{\uparrow} \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\})$$

is defined and it is the inverse to the isomorphism from (1).

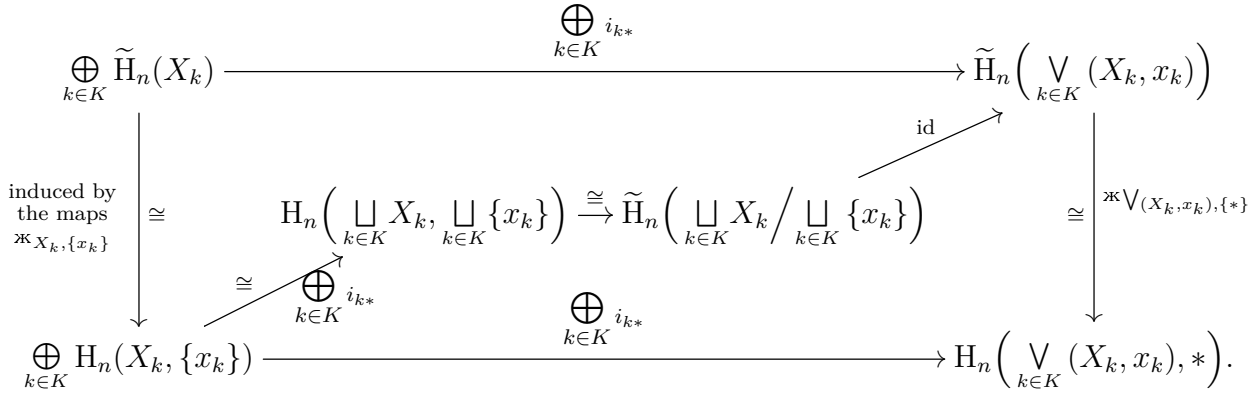
Both statements also hold for reduced homology instead of relative homology.

Remark.

(1) In Exercise 149.19 we generalize the Wedge-H_*-Proposition 149.16 (2) to infinite K.
(2) The Wedge-H_*-Proposition 149.16 is an analog of the Wedge-π_1-Proposition 99.10. □

[1544]Recall that by the Good Point-Lemma 99.9 (1) we know that any point on a topological manifold is good and that it follows from the CW-Complex-Hausdorff Proposition 136.6 and the CW-Complex–Local Properties Proposition 137.8 (1) that any point on a CW-complex is good.

Proof of the Wedge-H$_*$-Proposition 149.16 (1). We consider the following diagram:



The proposition follows from the following clarifications and observations:

(a) It follows from the Reduced-to-Relative-H$_*$-Lemma 147.9 that the vertical maps to the left and right are isomorphisms.

(b) It follows from the obvious generalization of the Path-Component–H$_*$–Lemma 143.21 that the bottom left diagonal map is an isomorphism.

(c) By hypothesis each $x_k \in X_k$ is good. It follows easily from the definitions that $\bigsqcup_{k\in K}\{x_k\}$ is a good subset of $\bigsqcup_{k\in K} X_k$. Thus we can apply the Relative-to-Quotient-H$_*$-Proposition 149.13 to see that the middle horizontal map is an isomorphism.

(d) The top right diagonal map is indeed by definition the identity.

(e) One can easily verify that the lower triangle commutes. This implies the desired statement for relative homology groups.

(f) One can also easily verify that the upper triangle commutes. Thus we also get the desired statement on reduced homology groups. ∎

Proof of the Wedge-H$_*$-Proposition 149.16 (2). By (1) it suffices to prove the following claim:

Claim. The composition

$$\bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \xrightarrow{\bigoplus_{k\in K} i_{k*}} \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big) \xrightarrow{\prod_{k\in K} p_k^*} \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\})$$

is the identity.

Proof. This claim follows immediately from the observation that $p_k \circ i_k = \mathrm{id}_{X_k}$ and that for $k \neq l$ the map $p_k \circ i_l$ is the constant map $x \mapsto x_k$ which induces the zero map on relative homology. ∎

149.6. **The long exact sequence for quotients.** The following theorem gives us, at least in reasonable settings, the promised connection between the homology groups of a topological space X, a subspace $A \subseteq X$ and the quotient space X/A.

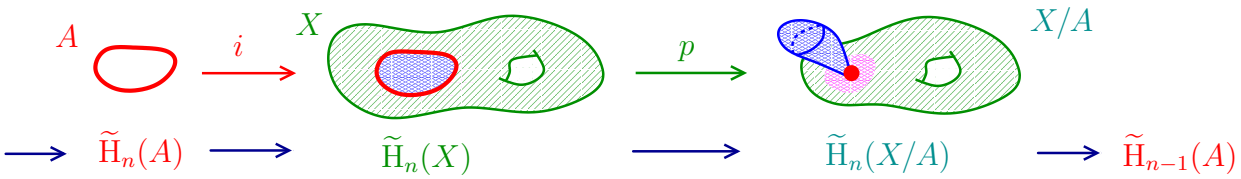
Theorem 149.17. (Quotient–LES Theorem) Let X be a topological space. Furthermore let $A \subseteq X$ be a good subset. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/A$ the projection. For each $n \in \mathbb{N}_0$ there exists a natural[1545]

homomorphism $\partial_n\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$ such that the sequence

$$\begin{array}{rccccccc} & & & & \dots & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{n+1}(X/A) & \xrightarrow{\partial_{n+1}} \\ \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} & \\ \to & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \dots & & & & \end{array}$$

is exact.

Example. In the figure below we show a topological space X together with a good subset A and we show the inclusion map $i\colon A \to X$ and the projection $p\colon X \to X/A$.



The Quotient–LES Theorem 149.17 relates the (reduced) homology groups of A, X and X/A in a mysterious way. □

Remark. In Exercise 149.10 we will see that in general the conclusion of the Quotient–LES Theorem 149.17 does not hold, if we do not assume that A is a good subset. □

Proof. Let X be a topological space and let $A \subseteq X$ be a good subset. We denote by $i\colon A \to X$ the inclusion and we denote by $p\colon X \to X/A$ the projection. We consider the following diagram:

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_{n-1}(X) & \longrightarrow & \dots \\ & & & & & \searrow^{p_*} & \downarrow{\scriptstyle \text{ю}} & & & & & & \\ & & & & & & \widetilde{\mathrm{H}}_n(X/A). & & & & & & \end{array}$$

We make the following comments and observations:

(1) The horizontal sequence is the long exact sequence coming from the Reduced LES-of-Pair Proposition 147.8.
(2) The vertical map is the natural homomorphism that we defined on Definition 149.10. By the Relative-to-Quotient-H_*-Lemma 149.11 (1) we know that the triangle commutes.
(3) Since A is good we obtain from the Relative-to-Quotient-H_*-Proposition 149.13 that the vertical map ю is an isomorphism.

Now we consider the homomorphisms $\widetilde{\partial}_n := \partial_n \circ \text{ю}^{-1}\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$. Since ю and ∂_n are natural we see that the maps $\widetilde{\partial}_n = \partial_n \circ \text{ю}^{-1}$ are also natural. It follows from the above discussion that the sequence

$$\dots \longrightarrow \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \xrightarrow{\widetilde{\partial}_n} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \dots$$

[1545]Here "natural" means the following: If $f\colon (X,A) \to (Y,B)$ is a continuous map between pairs of topological spaces and if A and B are good subsets, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{\mathrm{H}}_n(Y/B) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(B). \end{array}$$

is exact. ∎

Using the Quotient–LES Theorem 149.17 we can now give a somewhat shorter proof of the Sphere-H_*-Proposition 149.4.

Proposition 149.4. (Sphere-H_*-Proposition) For each $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. We prove the proposition by induction on n. First note that for $n = -1$ we have $S^n = S^{-1} = \varnothing$, thus the statement follows immediately from the Reduced Homology Lemma 147.3 (-1).[1546]

Now we suppose that the statement of the proposition holds for $n-1$. We want to determine the reduced homology of S^n. The key idea is to use that by the Ball Quotient-Sphere Lemma 8.10 we know that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. On page 2846 we saw that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply the Quotient–LES Theorem 149.17 to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence:

$$\begin{array}{rcccccc} \dots & \longrightarrow & \widetilde{H}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{H}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_{k+1}} & \\ \to \quad \widetilde{H}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{H}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_k} & \\ \to \quad \widetilde{H}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & & \end{array}$$

By the Contractible-$\widetilde{H}_*$-Corollary 147.6 we know that $\widetilde{H}_i(\overline{B}^n) = 0$ for all i. Since the sequence is exact we see that all the connecting homomorphisms are isomorphisms. Thus we see that

$$\widetilde{H}_k(S^n) \underset{\uparrow}{\cong} \widetilde{H}_k(\overline{B}^n/S^{n-1}) \underset{\uparrow}{\cong} \widetilde{H}_{k-1}(S^{n-1}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k-1 = n-1 \\ 0, & \text{if } k \neq n. \end{cases}$$

in the Ball Quotient-Sphere Lemma 8.10 we showed that $S^n \cong \overline{B}^n/S^{n-1}$ — by the above exact sequence — by induction hypothesis

∎

149.7. **Appendix: Reflections of suspensions.** In the Suspension-H_*-Proposition 149.3 we calculated the homology groups of the suspension $\Sigma(X)$ of a topological space. There is an obvious reflection map on the suspension. In this appendix we calculate the induced map on homology:

Proposition 149.18. (Reflection-on-Suspension-H_*-Proposition) Let X be a topological space. Let

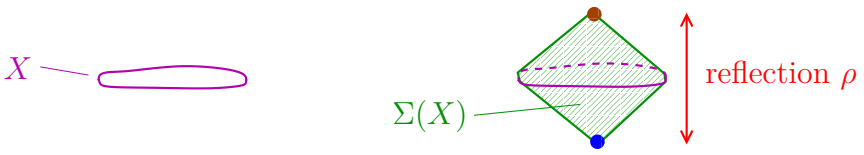
$$\begin{array}{rcl} \rho\colon \Sigma(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,-t)] \end{array}$$

be the natural reflection. For any $k \in \mathbb{N}_0$ the induced map

$$\rho_*\colon \widetilde{H}_k(\Sigma(X)) \to \widetilde{H}_k(\Sigma(X))$$

is given by multiplication by -1.

[1546] To readers who prefer S^0 as a starting point we point out that S^0 consists of precisely two points. Thus the promised isomorphisms follow from the Singleton-H_*-Lemma 143.26, the Path-Component–H_*–Lemma 143.21 and the Reduced Homology Lemma 147.3.



Proof. We denote by ρ also the reflection of $X \times [-1,1]$ in the last coordinate. The following claim contains the crucial calculation required to prove the proposition.

Claim. For any $k \in \mathbb{N}_0$ the induced map[1547]
$$\rho_* \colon \mathrm{H}_k(X \times [-1,1], X \times \{\pm 1\}) \;\to\; \mathrm{H}_k(X \times [-1,1], X \times \{\pm 1\})$$
is given by multiplication by -1.

Proof. We consider the following diagram:
$$\begin{array}{ccccccc}
\to \mathrm{H}_{i+1}(X\times[-1,1], X\times\{\pm1\}) & \xrightarrow{\partial_{i+1}} & \mathrm{H}_i(X\times\{-1\})\oplus\mathrm{H}_i(X\times\{1\}) & \xrightarrow{\iota_*^-\oplus\iota_*^+} & \mathrm{H}_i(X\times[-1,1]) \to \\
\downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\
\to \mathrm{H}_{i+1}(X\times[-1,1], X\times\{\pm1\}) & \xrightarrow{\partial_{i+1}} & \mathrm{H}_i(X\times\{-1\})\oplus\mathrm{H}_i(X\times\{1\}) & \xrightarrow{\iota_*^-\oplus\iota_*^+} & \mathrm{H}_i(X\times[-1,1]) \to
\end{array}$$
We make the following clarifications and observations regarding this diagram:

(1) We denote by $\iota^{\pm} \colon X \times \{\pm 1\} \to X \times [-1,1]$ the inclusion maps.
(2) The two horizontal sequences in the above diagram are obtained from the long exact sequence of the pair $(X \times [-1,1], X \times \{\pm 1\})$ together with the natural isomorphism $\mathrm{H}_i(X \times \{-1\}) \oplus \mathrm{H}_i(X \times \{1\}) \to \mathrm{H}_i(X \times \{\pm 1\})$ that is given by the Path-Component–H_*–Lemma 143.21.
(3) It follows from the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 146.7 that the diagram commutes.
(4) It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the maps $\iota^{\pm} \colon X \times \{\pm 1\} \to X \times [-1,1]$ are homotopy equivalences. It follows from the Homotopy Equivalence-H_*-Proposition 144.14 that the induced maps on homology are isomorphisms.
(5) It follows from (4) that the maps $\iota_*^- \oplus \iota_*^+$ are epimorphisms.
(6) It follows from (5) and the exactness of the horizontal sequences that the connecting homomorphisms ∂_{i+1} are monomorphisms.

Now we turn to the actual proof of the claim. Thus let $a \in \mathrm{H}_i(X \times [-1,1], X \times \{\pm 1\})$. We denote by $\varphi^{\pm} \colon X \to X \times \{\pm 1\}$ the obvious homeomorphisms. Since the induced maps $\varphi_*^{\pm} \colon \mathrm{H}_i(X) \to \mathrm{H}_i(X \times \{\pm 1\})$ are isomorphisms there exist $b^-, b^+ \in \mathrm{H}_i(X)$ with $\partial_{i+1}(a) = (\varphi_*^-(b^-), \varphi_*^+(b^+)) \in \mathrm{H}_i(X \times \{-1\}) \oplus \mathrm{H}_i(X \times \{1\})$. Next note that we have the following equalities in $\mathrm{H}_i(X \times [-1,1])$:

[1547] Later in Exercise 164.1 we will use the Künneth Theorem 164.15 to give an alternative proof of the claim.

$$
\begin{aligned}
(\iota_*^- \circ \rho_* \circ \varphi_*^+)(b^+) + (\iota_*^- \circ \varphi_*^-)(b^-) &\overset{\substack{\text{since } \iota^- \circ \rho = \rho \circ \iota^+ \text{ as} \\ \text{maps } X \times \{1\} \to X \times [-1,1]}}{=} (\rho_* \circ \iota_*^+ \circ \varphi_*^+)(b^+) + (\iota_*^- \circ \varphi_*^-)(b^-) \overset{\substack{\text{follows from the Homotopy-H}_*\text{-Proposition 144.12} \\ \text{since the two maps } \rho \circ \iota^+ \circ \varphi^+ \text{ and } \iota^+ \circ \varphi^+ \\ \text{from } X \text{ to } X \times [-1,1] \text{ are evidently homotopic}}}{=} \\
&= (\iota_*^+ \circ \varphi_*^+)(b^+) + (\iota_*^- \circ \varphi_*^-)(b^-) \\
&\underset{\substack{\uparrow \\ \text{by definition of } \iota_*^+ \oplus \iota_*^-}}{=} (\iota_*^+ \oplus \iota_*^-)(\varphi_*^+(b^+), \varphi_*^-(b^-)) \underset{\substack{\uparrow \\ \text{by definition of } b^\pm}}{=} (\iota_*^+ \oplus \iota_*^-)(\partial_{i+1}(a)) \underset{\substack{\uparrow \\ \text{since the horizontal} \\ \text{sequence is exact}}}{=} 0.
\end{aligned}
$$

Since ι_*^- is an isomorphism we see that $\rho_*(\varphi_*^+(b^+)) + \varphi_*^-(b^-) = 0$. We now see that

$$
\begin{aligned}
\partial_{i+1}(\rho_*(a) + a) &\overset{\substack{\text{since the above diagram commutes} \\ \downarrow}}{=} \rho_*(\partial_{i+1}(a)) + \partial_{i+1}(a) \overset{\substack{\text{by definition of } b^\pm \\ \downarrow}}{=} \rho_*(\varphi_*^-(b^-), \varphi_*^+(b^+)) + (\varphi_*^-(b^-), \varphi_*^+(b^+)) \\
&= (\rho_*(\varphi_*^+(b^+)), \rho_*(\varphi_*^-(b^-))) + (\varphi_*^-(b^-), \varphi_*^+(b^+)) \\
&= (\underbrace{\rho_*(\varphi_*^+(b^+)) + \varphi_*^-(b^-)}_{=0, \text{ by the above}}, \underbrace{\rho_*(\varphi_*^-(b^-)) + \varphi_*^+(b^+)}_{\substack{=0, \text{ by the same} \\ \text{argument as above}}}) = 0.
\end{aligned}
$$

By (6) we know that ∂_{i+1} is a monomorphism. Thus it follows from the above calculation that $\rho_*(a) + a = 0 \in \mathrm{H}_{i+1}(X \times [-1,1], X \times \{\pm 1\})$. In other words, $\rho_*(a) = -a$. ⊞

With the next claim we are getting very close to the desired statement.

Claim 2. We denote by $N_\pm$ the North and South Pole of $\Sigma(X)$. The induced map

$$\rho_* \colon \mathrm{H}_i(\Sigma(X), \{N_-\} \cup \{N_+\}) \to \mathrm{H}_i(\Sigma(X), \{N_-\} \cup \{N_+\})$$

is given by multiplication by -1.

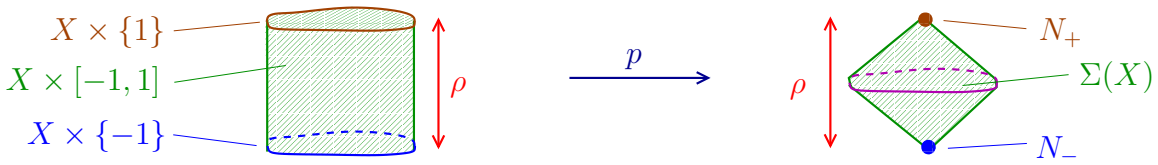
Proof. We consider the following diagram:

$$
\begin{array}{ccc}
\mathrm{H}_i(X \times [-1,1], (X \times \{-1\}) \cup (X \times \{1\})) & \xrightarrow{\rho_*} & \mathrm{H}_i(X \times [-1,1], (X \times \{-1\}) \cup (X \times \{1\})) \\
\cong \downarrow & & \downarrow \cong \\
\mathrm{H}_i(X \times [-1,1], (X \times [-1,-\frac{1}{2}]) \cup (X \times [\frac{1}{2},1])) & \xrightarrow{\rho_*} & \mathrm{H}_i(X \times [-1,1], (X \times [-1,-\frac{1}{2}]) \cup (X \times [\frac{1}{2},1])) \\
\cong \uparrow & & \uparrow \cong \\
\mathrm{H}_i(X \times (-1,1), (X \times (-1,-\frac{1}{2}]) \cup (X \times [\frac{1}{2},1])) & \xrightarrow{\rho_*} & \mathrm{H}_i(X \times (-1,1), (X \times (-1,-\frac{1}{2}]) \cup (X \times [\frac{1}{2},1])) \\
\cong \downarrow p_* & & \cong \downarrow p_* \\
\mathrm{H}_i(\Sigma(X), p((X \times (-1,-\frac{1}{2}]) \cup (X \times [\frac{1}{2},1]))) & \xrightarrow{\rho_*} & \mathrm{H}_i(\Sigma(X), p((X \times (-1,-\frac{1}{2}]) \cup (X \times [\frac{1}{2},1]))) \\
\cong \uparrow & & \uparrow \cong \\
\mathrm{H}_i(\Sigma(X), \{N_-\} \cup \{N_+\}) & \xrightarrow{\rho_*} & \mathrm{H}_i(\Sigma(X), \{N_-\} \cup \{N_+\}).
\end{array}
$$

We make the following observations and clarifications:

(1) The map $p \colon X \times [-1,1] \to \Sigma(X)$ is the natural projection. All other vertical maps are induced by the natural maps between pairs of topological spaces.
(2) The diagram commutes by the functoriality of relative homology.
(3) It follows fairly easily from the Excision Theorem 148.1 and the Homotopy Equivalence-Relative-H$_*$-Corollary 146.8 that all of the vertical maps are isomorphisms.
(4) By Claim 1 we know that the top horizontal map is given by multiplication by -1.

(5) It follows from (1)–(4) that the bottom horizontal map is indeed given by multiplication by -1. ⊞



Now we can easily complete the argument. We consider the long exact sequence in reduced homology of the pair $(\Sigma(X), \{N_-\} \cup \{N_+\})$:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\widetilde{\mathrm{H}}_i(\{N_-\} \cup \{N_+\}) & \xrightarrow{\rho_*} & \widetilde{\mathrm{H}}_i(\{N_-\} \cup \{N_+\}) \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{H}}_i(\Sigma(X)) & \xrightarrow{\rho_*} & \widetilde{\mathrm{H}}_i(\Sigma(X)) \\
\downarrow & & \downarrow \\
\mathrm{H}_i(\Sigma(X), \{N_-\} \cup \{N_+\}) & \xrightarrow{\rho_*} & \mathrm{H}_i(\Sigma(X), \{N_-\} \cup \{N_+\}) \\
\downarrow \partial_i & & \downarrow \partial_i
\end{array}$$

(1) It follows from the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 146.7 that the diagram commutes.
(2) It follows from an elementary argument, see Exercise 147.1, that the top horizontal map is given by multiplication by -1.[1548]
(3) By Claim 2 we know that the bottom horizontal map is given by multiplication by -1.
(4) It follows from (1)–(3) together with the elementary Exercise 145.25 that the middle horizontal map is also given by multiplication by -1. ■

149.8. Exercises for Chapter 149 .

Exercise 149.1. Do there exist non-empty topological spaces X and Y such that for every $x \in X$ and every $y \in Y$ the topological spaces $X \setminus \{x\}$ and $Y \setminus \{y\}$ are homotopy equivalent, but such that X and Y are not homeomorphic?

Exercise 149.2. We say that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has *bounded displacement* if there exists a $C \in \mathbb{R}$ such that for every $x \in \mathbb{R}^n$ we have $\|x - f(x)\| \leq C$. Show that every continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded displacement is onto.
Hint. Use the Brouwer Fixed Point Theorem 38.7.

Exercise 149.3. We consider

$$\mathbb{R}^\infty \coloneqq \{(x_1, x_2, \dots) \,|\, x_i \in \mathbb{R} \text{ and there exists a } k \in \mathbb{N}_0 \text{ such that } x_i = 0 \text{ for } i \geq k\}.$$

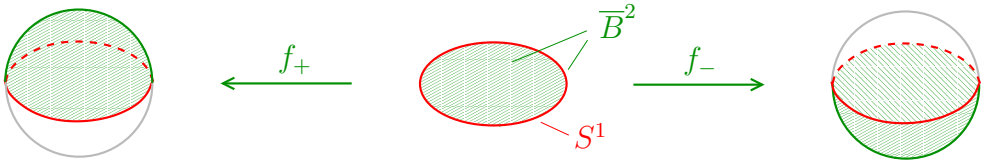
We equip $\mathbb{R}^\infty$ with the usual Euclidean metric. This allows us to define the infinite-dimensional closed ball $\overline{B}^\infty$. Does every continuous self-map of $\overline{B}^\infty$ admit a fixed point?

[1548] Evidently except for $i = 0$ there is nothing to show anyway.

Exercise 149.4. We consider the two maps

$$\begin{array}{rcl} f_+\colon \overline{B}^2 & \to & S^2 \\ (x,y) & \mapsto & \big(x,y,\sqrt{1-x^2-y^2}\,\big) \end{array} \quad \text{and} \quad \begin{array}{rcl} f_-\colon \overline{B}^2 & \to & S^2 \\ (x,y) & \mapsto & \big(x,y,-\sqrt{1-x^2-y^2}\,\big) \end{array}$$

Show that these two maps $f_-, f_+\colon \overline{B}^2 \to S^2$ are not homotopic rel S^1.



Exercise 149.5.

(a) Show that the first homology of the torus $S^1 \times S^1$ has rank at least 2.

(b) Let Σ be the surface of genus 2. What is the largest $k \in \mathbb{N}_0$ for which you can find an epimorphism $\mathrm{H}_1(\Sigma) \to \mathbb{Z}^k$?

Exercise 149.6. Let $n \in \mathbb{N}$. Show that $\mathrm{H}_1(\mathrm{GL}(n,\mathbb{C}))$ admits an epimorphism onto $\mathbb{Z}$.

Exercise 149.7. Let $n \in \mathbb{N}_0$ and let $\rho_n\colon S^n \to S^n$ be the reflection in the $(x_1{=}0)$-hyperplane. Show that the induced map $\rho_{n*}\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$ is given by multiplication by -1.

Hint. First consider the very explicit case $n = 0$. For the higher dimensional case use the Suspension-H_*-Proposition 149.3 (2).

Exercise 149.8. Let X and Y be topological spaces.

(a) Let $f\colon X \to Y$ be an injective continuous map.
  (i) Is it necessarily true that $f_*\colon \mathrm{C}_k(X) \to \mathrm{C}_k(Y)$ is injective?
  (ii) Is it necessarily true that $f_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ is injective?

(b) Let $f\colon X \to Y$ be a surjective continuous map.
  (i) Is it necessarily true that $f_*\colon \mathrm{C}_k(X) \to \mathrm{C}_k(Y)$ is surjective?
  (ii) Is it necessarily true that $f_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ is surjective?

Exercise 149.9. We consider the topological space $X = [0,1]$ and together with the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Prove the following two statements:

(a) The group $\mathrm{H}_1(X,A)$ is countable.

(b) The group $\mathrm{H}_1(X/A)$ is uncountable. For psychological reasons it might be convenient to recall that in Exercise 105.1 we showed that the quotient space X/A is homeomorphic to the Hawaiian earring $H = \bigcup_{n\in\mathbb{N}} \{(x,y) \in \mathbb{R}^2 \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$.

Remark. The homology groups of the Hawaiian earring are computed in [**EKa2000**]. In particular it is shown that the higher homology groups $\mathrm{H}_i(X/A)$ are zero for $i \geq 2$, which is not clear at all, see e.g. [**BaM1962**].

Exercise 149.10. Give an example of a topological space X and some non-empty subset A for which there is no exact sequence of the form

$$\ldots \to \widetilde{\mathrm{H}}_{n+1}(X/A) \to \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \to \ldots$$

where $i\colon A \to X$ is the inclusion and $p\colon X \to X/A$ is the projection.

Remark. You might want to consult your calculations from Exercise 149.9.

Exercise 149.11. Compute the homology groups of the quotient of S^2 obtained by identifying the North Pole N and the South Pole S to a point.



Exercise 149.12. Let $T = S^1 \times S^1$ be the torus. We consider the subspace $A \coloneqq S^1 \times \{1\}$.

(a) Show that T/A is homeomorphic to the topological space $S^2/(S \cup N)$ considered in Exercise 149.11.

(b) Compute the homology groups of the torus.

(c) Bonus question: How far can you push the above approach if we start out with the Klein bottle instead of the torus?

Remark. We will a more conceptual, less ad hoc, approach to calculating the homology groups of the torus.

Exercise 149.13. Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{AbGroup}$ be the category of abelian groups. Does the functor

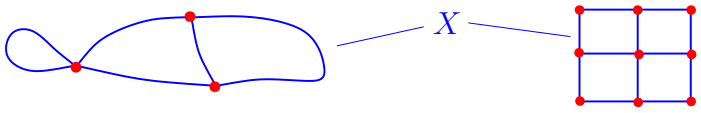
$$\begin{array}{ccc} \mathcal{PTop} & \to & \mathcal{AbGroup} \\ (X, x_0) & \mapsto & \mathrm{C}_n(X, \{x_0\}) \end{array}$$

preserve coproducts?

Exercise 149.14. We consider the two pairs (X, A) of topological spaces shown in the figure below. In each case determine the homology groups of the quotient X/A.



Exercise 149.15. We consider the topological graph X shown in the figure below. For each $n \in \mathbb{N}_0$ determine the homology groups $\mathrm{H}_n(X)$ and determine a set of n-cycles that represent a basis for $\mathrm{H}_n(X)$.



Exercise 149.16. Given $m < n$ we view S^m as a subspace of S^n in the obvious way and we view $\mathbb{R}\mathrm{P}^m$ as a subspace of $\mathbb{R}\mathrm{P}^n$ in the obvious way. We want to show that $\mathbb{R}\mathrm{P}^m$ is not a retract of $\mathbb{R}\mathrm{P}^n$. Suppose that there exists a retraction $r\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$.

(a) Given $k \in \mathbb{N}$ let $p_k\colon S^k \to \mathbb{R}\mathrm{P}^k =_{\mathrm{i}} S^k/x \sim -x$ be the natural projection. Show that there exists a retraction $\widetilde{r}\colon S^n \to S^m$ such that the following diagram commutes:

$$\begin{array}{ccc} S^n & \xrightarrow{\widetilde{r}} & S^m \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_n} \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{r} & \mathbb{R}\mathrm{P}^m. \end{array}$$

Hint. Use the Map Lifting Criterion 127.5.

(b) Show that S^m is not a retract of S^n.

Exercise 149.17. Let $X := \{(x,y,z) \in \mathbb{R}^3 \mid xyz = 0\}$ be the union of the three coordinate axis in $\mathbb{R}^3$. Compute the homology groups of $\mathbb{R}^3 \setminus X$.
Hint. Find a nice deformation retract.

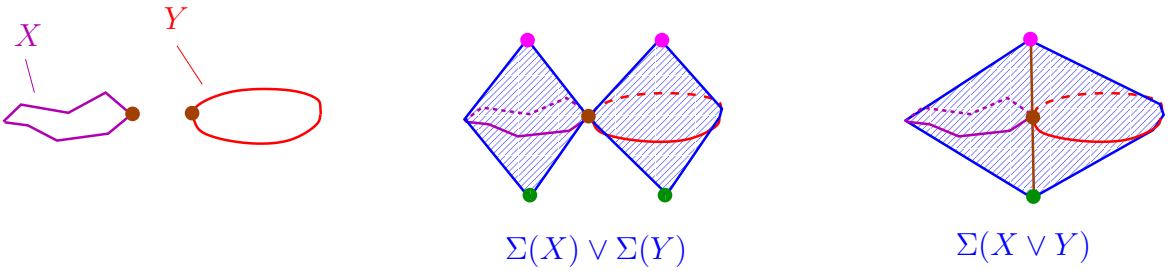
Exercise 149.18. Let X and Y be topological spaces and let $x_0 \in X$ and $y_0 \in Y$ be good points. We write $\widetilde{x}_0 = [(x_0, 0)] \in \Sigma(X)$ and $\widetilde{y}_0 = [(y_0, 0)] \in \Sigma(Y)$.

(a) Show that $\widetilde{x}_0 \in \Sigma(X)$ and $\widetilde{y}_0 \in \Sigma(Y)$ are good points. We use these points to define the wedge $\Sigma(X) \vee \Sigma(Y)$.

(b) Let $i\colon X \to X \vee Y$ and $j\colon Y \to X \vee Y$ be the natural maps. Show that the map

$$(\Sigma(i) \vee \Sigma(j))_*\colon \mathrm{H}_k(\Sigma(X) \vee \Sigma(Y)) \;\to\; \mathrm{H}_k(\Sigma(X \vee Y))$$

is an isomorphism for each k.
Remark. You could make use of the *natural* isomorphism of the Suspension-H_*-Proposition 149.3.



Exercise 149.19. Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces.

- We denote by $*$ the wedge point of $\bigvee_{k\in K} (X_k, x_k)$.
- Given $k \in K$ we denote by $i_k\colon X_k \to \bigvee_{l\in K} (X_l, x_l)$ the natural inclusion.
- Given $k \in K$ we denote by $p_k\colon \bigvee_{l\in K} (X_l, x_l) \to X_k$ the continuous map that is given by the identity on X_k and the constant map $x \mapsto x_k$ on X_l for $l \neq k$.

Let $n \in \mathbb{N}_0$. We consider the homomorphism

$$\prod_{k\in K} p_{k*}\colon \mathrm{H}_n\Big(\bigvee_{l\in K} (X_l, x_l), *\Big) \;\to\; \prod_{k\in K} \mathrm{H}_n(X_k, x_k).$$

(a) Show that in general the above homomorphism does not take values in $\bigoplus_{k\in K} \mathrm{H}_n(X_k, x_k)$.

(b) Now suppose that each $x_k \in X_k$ is good. Show that the above homomorphism takes values in $\bigoplus_{l\in K} \mathrm{H}_n(X_l, x_l)$. Furthermore show that the map

$$\prod_{k\in K} p_{k*}\colon \mathrm{H}_n\Big(\bigvee_{l\in K} (X_l, x_l), *\Big) \;\to\; \prod_{k\in K} \mathrm{H}_n(X_k, x_k)$$

is an isomorphism and that it is an inverse to the isomorphism

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \;\to\; \mathrm{H}_n\Big(\bigvee_{l\in K} (X_l, x_l), \{*\}\Big)$$

from the Wedge-H_*-Proposition 149.16.

CHAPTER 150

Local homology and explicit generators of homology groups

By the Sphere-H_*-Proposition 149.4 we know that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the isomorphism types of the homology groups of all finite-dimensional spheres. In this chapter we will determine an explicit singular n-cycle in $\mathrm{C}_n(S^n)$ that represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. Along the way we introduce the concept of the local homology of a topological space X at a point x. This concept will turn out to be quite interesting in its own right.

This chapter is organized as follows:

- In Section 150.1 we will determine an explicit generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$.
- In Section 150.2 we will introduce the local homology groups of a topological space.
- In Section 150.3 we will introduce the standard generator of $\mathrm{H}_n(S^n)$.
- In Section 150.4 we will study the local homology groups of topological manifolds.
- In Appendix 150.5 we will use homology groups to discuss the topology of topological manifolds.
- In Section 150.6 we will provide exercises for Chapter 150.

In the next Chapter 151 we will study the effect of self-maps of spheres on their homology groups.

150.1. **An explicit generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$.** As we mentioned above, in this chapter we eventually want to find explicit generators of $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ for $n \in \mathbb{N}_0$. We start out with the following lemma, which gives us a closely related statement. In fact the following lemma is the first instance where we can nail down a generator of an interesting homology group.

Lemma 150.1. (Identity-is-Generator Lemma) For every $n \in \mathbb{N}_0$ the following two statements hold:

(1) For $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Proof. Statement (1) follows immediately from the Ball-rel-Sphere-H_*-Lemma 149.7 and the Δ^n-Homeomorphism Lemma 143.9.

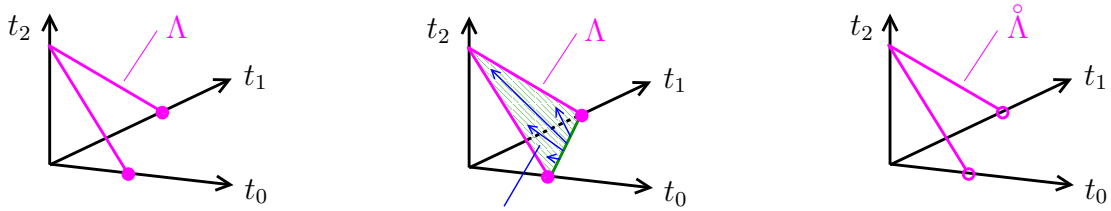
Thus it remains to prove Statement (2). We will prove this statement by induction on $n \in \mathbb{N}_0$. It follows easily from the H_0-Proposition 143.25 that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n - 1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ and $\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. To do so we introduce the following notation:

(a) We consider
$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\},$$
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. A generalization of the argument of Exercise 19.24 (b) shows that Λ is a deformation retract of Δ^n.

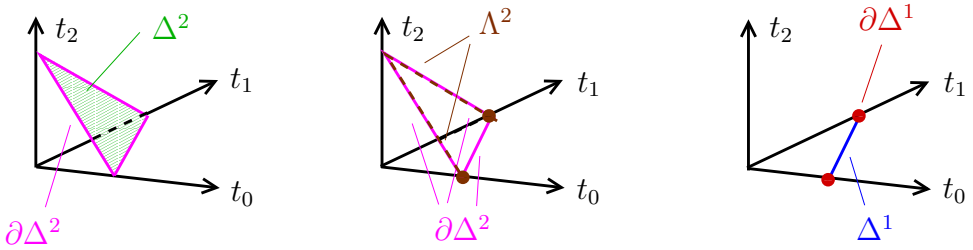
(b) We write
$$\mathring{\Lambda} \;:=\; \{(t_0, \ldots, t_n) \in \Lambda \,|\, t_n > 0\}.$$
The set $\mathring{\Lambda}$ can be viewed as the "interior" of Λ.

In the following we view $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subspace of Δ^n. Note that with this convention we have $\partial\Delta^{n-1} \subseteq \Lambda$.



deformation retraction from Δ^2 to Λ

Claim. We consider the maps
$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$
Here the map ∂_n on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ and the right-hand map is induced by the inclusion. Both maps are isomorphisms and the following equality holds:
$$\partial_n([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]).$$



It is clear that a proof of the claim together with the induction hypothesis implies that $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ and that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. In particular a proof of the claim implies the Identity-is-Generator Lemma 150.1.

Proof. We consider the two maps separately.

(a) Of the two maps in the claim we first consider the map ∂_n on the left. It follows from the LES-of-Triple Proposition 146.5 applied to the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ of topological spaces that the following sequence is exact:
$$\mathrm{H}_n(\Delta^n, \Lambda) \xrightarrow{p_*} \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^n, \Lambda)$$

where $p\colon (\Delta^n, \Lambda) \to (\Delta^n, \partial\Delta^n)$ is the obvious inclusion. As we remarked above, one can easily show that Λ is a deformation retract of Δ^n. It follows from the Deformation Retract-Relative H_*-Corollary 146.9 that $\mathrm{H}_n(\Delta^n, \Lambda) = 0$, which implies that the connecting homomorphism $\partial_n\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ is an isomorphism. Furthermore note that we have the following equalities in $\mathrm{H}_n(\partial\Delta^n, \Lambda)$:

$$\partial_n([\mathrm{id}_{\Delta^n}]) \underset{\uparrow}{=} [\partial_n(\mathrm{id}_{\Delta^n})] \overset{\downarrow}{=} \Big[\underbrace{\sum_{j=0}^{n-1} (-1)^j \cdot \mathrm{id} \circ i_j^n}_{\text{lies in } \mathrm{C}_{n-1}(\Lambda),\ \text{thus} = 0 \text{ in } \mathrm{C}_{n-1}(\partial\Delta^n, \Lambda)} + (-1)^n \cdot \underbrace{\mathrm{id} \circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}} \Big] \underset{\uparrow}{=} (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}].$$

by definition of the connecting homomorphism in the LES-of-Triple Proposition 146.5

see Definition 143.13 of $\partial_n\colon \mathrm{C}_n(\Delta^n) \to \mathrm{C}_{n-1}(\Delta^n)$

since we work in $\mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$

(b) We turn our attention to the right-hand map $\iota_*\colon \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ in the statement of the claim. It follows from the Excision-via-Deformation Theorem 148.5, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = (0, \ldots, 0, 1)$[1549] that the inclusion-induced map

$$\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) = \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n \setminus \mathring{\Lambda}}_{=\Delta^{n-1}}, \underbrace{\Lambda \setminus \mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$$

is an isomorphism. Finally note that evidently we have $\iota_*([\mathrm{id}_{\Delta^{n-1}}]) = [\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial_n([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}] = (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]). \qquad \blacksquare$$

150.2. **The local homology groups of a topological space.** We start out this section with the following definition which at first glance might look somewhat artificial.

Definition 150.2. Given a topological space X and a point $x \in X$ the relative homology group $\mathrm{H}_n(X, X \setminus \{x\})$ is called the n-th local homology group of X at the point x.

As we will see shortly, the concept of local homology is surprisingly useful. In many settings the following proposition allows us to calculate the local homology of a topological space at a point.

Proposition 150.3. (Local-H_*-Proposition) Let X be a topological space and let $x \in X$.

(1) If $\{x\} \subseteq X$ is a closed subset[1550], then for any neighborhood U of $x \in X$ the inclusion-induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.[1551]

(2) For any homeomorphism $f\colon X \to Y$ the maps $f_*\colon \mathrm{H}_n(X, X \setminus \{x\}) \to \mathrm{H}_n(Y, Y \setminus \{f(x)\})$ are isomorphisms.

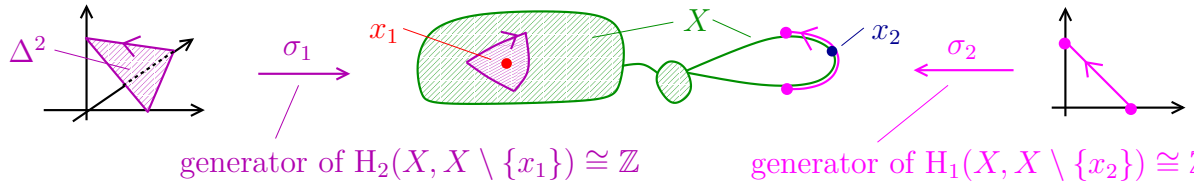
(3) (a) If X is an open subset of $\mathbb{R}^n$, then for any $x \in X$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

[1549] One can easily show that U has the property required in the Excision-via-Deformation Theorem 148.5.

(b) If X is an open subset of $\mathbb{H}^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_n \geq 0\}$ and if $x \in \mathbb{R}^{n-1} \times \{0\}$, then $\mathrm{H}_k(X, X \setminus \{x\}) = 0$ for all $k \in \mathbb{N}_0$.

(4) If $\sigma\colon \Delta^n \to X$ is a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding, then for any $x \in \sigma(\mathring{\Delta}^n)$ we have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and the singular n-simplex $\sigma\colon \Delta^n \to X$ represents a generator of $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$.



Proof. Let X be a topological space and let $x \in X$.

(1) Let U be a neighborhood of $x \in X$. It follows immediately from the Excision Theorem 148.2 that the inclusion-induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.

(2) This statement follows from the functoriality of relative homology groups.

(3) (a) It follows easily from (1), applied to $x \in X \subseteq \mathbb{R}^n$, that it suffices to consider the case that $X = \mathbb{R}^n$. For any $k \in \mathbb{N}_0$ we see that

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^n \setminus \{x\}) \xleftarrow{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}(x)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

this is the connecting homomorphism of the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ in reduced homology, it is an isomorphism since on page 2812 we showed that $\widetilde{\mathrm{H}}_*(\mathbb{R}^n) = 0$

isomorphism since the inclusion $S^{n-1}(x) \to \mathbb{R}^n \setminus \{x\}$ is a homotopy equivalence

by the Sphere-H_* Proposition 149.4

(b) It follows easily from (1) that it suffices to consider the case that $X = \mathbb{H}^n$ and that $x \in \mathbb{R}^{n-1} \times \{0\}$. By the Reduced LES-of-Pair Proposition 147.8 we have a long exact sequence of the form

$$\dots \to \widetilde{\mathrm{H}}_k(\mathbb{H}^n) \to \mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(\mathbb{H}^n \setminus \{x\}) \to \dots$$

It follows easily from the fact that $x \in \mathbb{R}^{n-1} \times \{0\}$ that both $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ are contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that the reduced homology groups of $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ vanish. But this implies, by exactness, that also the "middle" groups $\mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\})$ have to vanish.

(4) Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding. Furthermore let $x \in \sigma(\mathring{\Delta}^n)$. Since $x \in \sigma(\mathring{\Delta}^n)$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding there exists a unique $a \in \mathring{\Delta}^n$ with $\sigma(a) = x$.

Claim. There exists a continuous map $\tau\colon \Delta^n \to X$ with the following three properties:
(a) $\tau\colon \Delta^n \to X$ is an embedding.
(b) $\tau(\Delta^n)$ is a neighborhood of $x \in X$.
(c) We have $[\sigma] = [\tau] \in \mathrm{H}_n(X, X \setminus \{x\})$.

[1550] Recall that by the Points-in-Hausdorff Lemma 3.32 we know that this condition is satisfied if X is Hausdorff, e.g. if X is a topological manifold.

[1551] This statement hopefully justifies the name "local" homology.

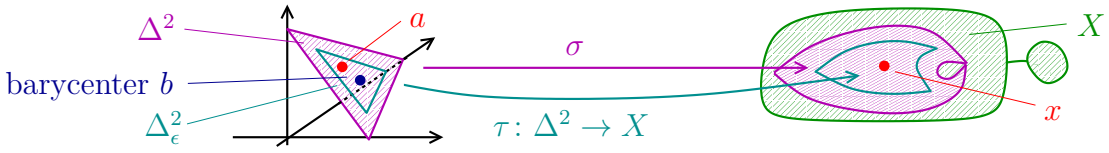
Proof. We introduce the following notation: We denote by $b := (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \Delta^n$ the barycenter of Δ^n. Furthermore given $\epsilon > 0$ we write

$$\Delta^n_\epsilon \;=\; \{(x-b)\cdot(1-\epsilon)+b \,|\, x \in \Delta^n\}.$$

Note that $\Delta^n_\epsilon \subseteq \Delta^n$ and note that $\Delta^n_0 = \Delta^n$. It follows from an elementary argument and the hypothesis that $a \in \Delta^n \setminus \partial\Delta^n$ that there exists an $\epsilon > 0$ such that $a \in \Delta^n_\epsilon$. Next we consider the homotopy

$$\begin{array}{rcl} F\colon \Delta^n \times [0,1] & \to & \Delta^n \\ (x,t) & \mapsto & (x-b)\cdot(1-t+(1-\epsilon)\cdot t)+b. \end{array}$$

Note that $F_0 = \mathrm{id}_{\Delta^n}$ and note that F_1 defines a homeomorphism $\Delta^n \to \Delta^n_\epsilon$. We set $\tau := \sigma \circ F_1\colon \Delta^n \to X$.



We have to verify that τ has the desired properties:

(a) Since $F_1\colon \Delta^n \to \Delta^n$ is an embedding with image Δ^n_ϵ, since $\Delta^n_\epsilon \subseteq \mathring{\Delta}^n$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding we see that $\tau = \sigma \circ F_1\colon \Delta^n \to X$ is an embedding.

(b) It is straightforward to verify that Δ^n_ϵ is a neighborhood of $a \in \Delta^n$. By hypothesis $\sigma(\mathring{\Delta}^n)$ is an open subset of X. It follows from the Open-Closed Inclusion Lemma 4.13 that $\tau(\Delta^n) = \sigma(\Delta^n_\epsilon)$ is a neighborhood of $x = \sigma(a) \in X$.

(c) We have the following equalities in $\mathrm{H}_n(X, X\setminus\{x\})$:

$$[\sigma] \;=\; (\sigma\circ F_0)_*([\mathrm{id}_{\Delta^n}]) \;\underset{\uparrow}{=}\; (\sigma\circ F_1)_*([\mathrm{id}_{\Delta^n}]) \;=\; [\sigma\circ F_1\circ \mathrm{id}_{\Delta^n}] \;=\; [\tau].$$

follows from the Homotopy-H_*-Proposition 146.11 since $\sigma\circ F$ defines a homotopy of maps of pairs $(\Delta^n, \partial\Delta^n) \to (X, X\setminus\{x\})$

⊞

Finally we consider the following sequence of homomorphisms:

since $\tau(\Delta^n)$ is a neighborhood of x it follows from the Excision Theorem 148.2 that this inclusion induced map is an isomorphism

isomorphism since τ is an embedding

$\partial\Delta^n$ is a deformation retract of $\Delta^n \setminus \{a\}$, thus it follows from the Homotopy Equivalence–Relative-H_*-Corollary 146.8 that the map is an isomorphism

$$\begin{array}{ccccccc} \mathrm{H}_n(X, X\setminus\{x\}) & \leftarrow & \mathrm{H}_n(\tau(\Delta^n), \tau(\Delta^n)\setminus\{x\}) & \xleftarrow{\tau_*} & \mathrm{H}_n(\Delta^n, \Delta^n\setminus\{a\}) & \leftarrow & \mathrm{H}_n(\Delta^n, \partial\Delta^n). \\ [\sigma] = [\tau] & \mapsto & [\tau] & \mapsto & [\mathrm{id}_{\Delta^n}] & \mapsto & [\mathrm{id}_{\Delta^n}]. \end{array}$$

By the Identity-is-Generator Lemma 150.1 we know that id_{Δ^n} represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. It follows from the above explicit isomorphisms that σ represents a generator of $\mathrm{H}_k(X, X\setminus\{x\})$. ■

We will now see that the Local-H_*-Proposition 150.3 can also be used to show that given cycles represent non-trivial homology classes. To state this result it is convenient to introduce the following definition:

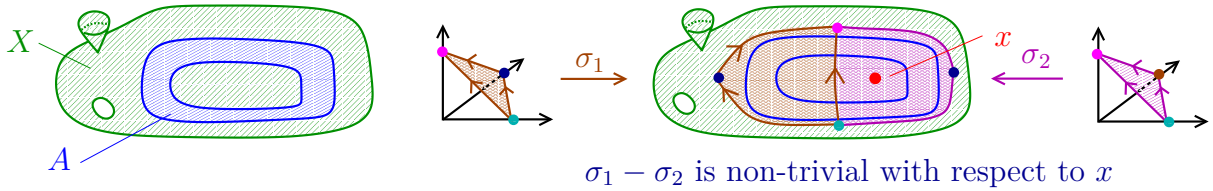
Definition 150.4. Let (X, A) be a pair of topological spaces. In the following we say that a singular n-chain $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ is non-trivial with respect to $x \in X \setminus A$ if the following statements hold:[1552]

(a) We have $\partial_n(c) \in \mathrm{C}_{n-1}(A)$, in other words, c represents an n-cycle in $\mathrm{C}_n(X, A)$.
(b) There exists a $j \in \{1, \ldots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, such that $x \in \sigma_j(\mathring{\Delta}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

The following proposition gives a convenient criterion for showing that a given cycle represents a non-trivial element in homology.

Proposition 150.5. (Detect-Cycle Proposition) Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain that is non-trivial with respect to some $x \in X \setminus A$. The following statements hold:

(1) We have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and under the map[1553] $\mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$ the singular n-cycle $c \in \mathrm{C}_n(X, A)$ gets sent to a generator of $\mathrm{H}_n(X, X \setminus \{x\})$.
(2) If $\mathrm{H}_n(X, A) \cong \mathbb{Z}$, then c represents a generator of $\mathrm{H}_n(X, A)$.



$\sigma_1 - \sigma_2$ is non-trivial with respect to x
thus $[\sigma_1 - \sigma_2] \neq 0 \in \mathrm{H}_2(X, A)$

Proof. Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain with the following two properties:

(a) c represents an n-cycle in $\mathrm{C}_n(X, A)$.
(b) There exists an $x \in X \setminus A$ and a $j \in \{1, \ldots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, such that $x \in \sigma_j(\mathring{\Delta}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

We turn to the actual proofs of the statements:

(1) First note that it follows from the hypothesis that $x \in \sigma_j(\mathring{\Delta}^n)$ and the hypothesis that $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, together with the Local-H_*-Proposition 150.3 that $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and that $[\sigma_j] \in \mathrm{H}_n(X, X \setminus \{x\})$ is a generator. Next we consider the induced homomorphism $\varphi \colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$. We see that

$$\varphi([c]) = \varphi\Big(\Big[\sum_{i=1}^{m} t_i \cdot \sigma_i\Big]\Big) = \underbrace{t_j}_{=\pm 1} \cdot [\sigma_j] + \varphi\Big(\Big[\sum_{i \neq j} \underbrace{t_j \cdot \sigma_j}_{\substack{= 0 \in \mathrm{C}_n(X, X \setminus \{x\}), \\ \text{since } \sigma_j(\Delta^n) \subseteq X \setminus \{x\}}}\Big]\Big) = \pm \cdot \underbrace{[\sigma_j]}_{\substack{\uparrow \\ \text{generator of} \\ \mathrm{H}_n(X, X \setminus \{x\})}}.$$

(2) It follows from the hypothesis that $\mathrm{H}_n(X, A) \cong \mathbb{Z}$ and the Epi-Automorphism Lemma 97.27 that any epimorphism $\mathrm{H}_n(X, A) \to \mathbb{Z}$ is actually an isomorphism. Statement (2) is an immediate consequence of this observation together with Statement (1). ∎

[1552] For $A = \varnothing$ we encountered this concept (with minor modifications) already in Exercise 143.10 and Exercise 143.11.
[1553] Note that it follows from $x \in X \setminus A$ that $A \subseteq X \setminus \{x\}$, in particular this map is indeed defined.

150.3. **The standard generator of $\mathrm{H}_n(S^n)$.** In this section we will use the Local-H_*-Proposition 150.3 to finally exhibit explicit generators of $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ for any $n \in \mathbb{N}_0$.

We start out by introducing two singular n-simplices on S^n:

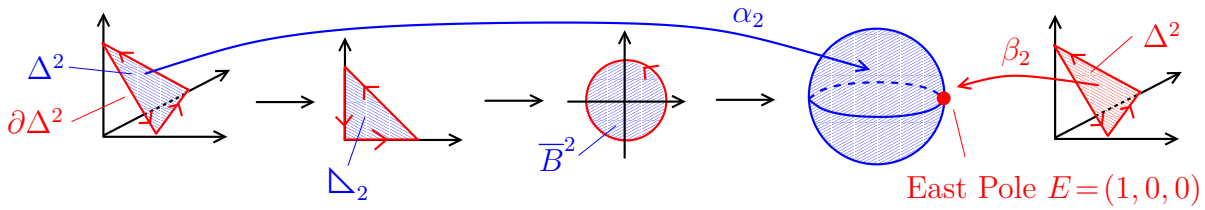
Notation 150.6. Let $n \in \mathbb{N}_0$.

(1) We consider the singular n-simplex $\alpha_n \colon \Delta^n \to S^n$ that is given by the concatenation of the following two continuous maps[1554]

$$\begin{array}{ccccccc} \Delta^n & \xrightarrow[\Delta^n\text{-Homeomorphism Lemma 143.9 (2)}]{\text{homeomorphism given by the}} & \overline{B}^n & \to & S^n & \subseteq & \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n \\ & & r \cdot v & \mapsto & \underbrace{\begin{pmatrix} -\cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}}. & & \end{array}$$

Similar to the Ball Quotient-Sphere Lemma 8.10 we see that this map $\Delta^n \to S^n$ sends $\partial\Delta^n$ to the "East Pole" $E \coloneqq (1, 0, \ldots, 0)$ and that α_n descends to a homeomorphism $\Delta^n/\partial\Delta^n \to S^n$.

(2) We denote by $\beta_n \colon \Delta^n \to S^n$ the constant map that sends all points to the "East Pole" $(1, 0, \ldots, 0)$.



Proposition 150.7. (Standard Generator-$\mathrm{H}_n(S^n)$-Proposition) Let $n \in \mathbb{N}_0$.

(1) The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle and it represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. (In particular for $n \geq 1$ it represents a generator of $\mathrm{H}_n(S^n)$.)

(2) If n is odd, then $\alpha_n \in \mathrm{C}_n(S^n)$ is also a cycle with $[\alpha_n] = [\alpha_n - \beta_n] \in \widetilde{\mathrm{H}}_n(S^n)$.

Definition 150.8. Let $n \in \mathbb{N}_0$. We refer to[1555]

$$[S^n] \coloneqq (-1)^{n-1} \cdot [\alpha_n - \beta_n] \quad \in \widetilde{\mathrm{H}}_n(S^n)$$

as the standard generator of $\widetilde{\mathrm{H}}_n(S^n)$.

Example.

(1) For $n = 0$ we see that $[S^0] = -[\alpha_0 - \beta_0] = -[(-1) - (1)] = [(1) - (-1)] \in \widetilde{\mathrm{H}}_0(S^0)$.

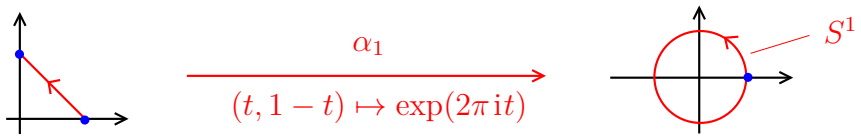
[1554] The given map $\overline{B}^n \to S^n$ is a variation on the map given in the Ball Quotient-Sphere Lemma 8.10.

[1555] The slightly arbitrary looking factor $(-1)^{n-1}$ will be helpful later to stay consistent with various orientation conventions. In a nutshell it comes from the fact, that if we orient Δ^n with the "outward pointing" normal vector and if we give S^n the standard orientation, then α_n is orientation-preserving if and only if n is odd.

(2) Next we consider the case $n = 1$. If one goes carefully through the definition of the map $\alpha_1 \colon \Delta^1 \to S^1$ one sees that

$$\alpha_1(1-t,t) \underset{\uparrow}{=} (-\cos(\pi(1-2t)), \sin(\pi(1-2t))) = (\cos(2t\pi), \sin(2t\pi)) = \exp(2\pi\,\mathrm{i}t).$$

since the explicit homeomorphism $\Delta^1 \to \overline{B}^1$ is given by $(t, 1-t) \mapsto 1-2t$



Thus we see that the singular 1-simplex $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents the standard generator of $\mathrm{H}_1(S^1)$. □

Proof.

(1) We start out with the following claim:

Claim. The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle.

Proof. Let $\gamma \colon \Delta^{n-1} \to S^n$ be the singular $(n-1)$-simplex that sends all points to the "East Pole" $E = (1, 0, \ldots, 0)$. For $n \neq 0$ we calculate that

$$\partial_n(\alpha_n - \beta_n) = \partial_n(\alpha_n) - \partial_n(\beta_n) \underset{\uparrow}{=} \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\alpha_n \circ i_j^n}_{\substack{=\gamma, \text{ since} \\ \alpha_n(\partial\Delta^n) = \{E\}}} - \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\beta_n \circ i_j^n}_{\substack{=\gamma, \text{ since} \\ \beta_n(\partial\Delta^n) = \{E\}}} = 0.$$

see Definition 143.13 of ∂_n

For $n = 0$ the same argument also shows that $\epsilon_{S^0}(\alpha_0 - \beta_0) = 0$ where $\epsilon_{S^0} \colon \mathrm{C}_0(S^0) \to \mathbb{Z}$ denotes the augmentation map. ⊞

We set $x \coloneqq (-1, 0, \ldots, 0)$. We consider the long exact sequence in reduced homology of the pair $(S^n, S^n \setminus \{x\})$:

$$\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) \to \widetilde{\mathrm{H}}_n(S^n) \xrightarrow{\text{я}} \mathrm{H}_n(S^n, S^n \setminus \{x\}) \xrightarrow{\partial_n} \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}).$$

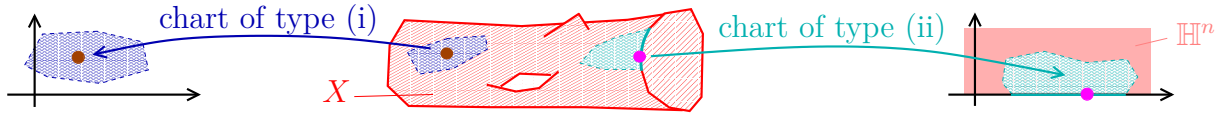
It follows from the Orthogonal Action Lemma 2.37 (3) together with the Stereographic Projection Lemma 4.24 that $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$. This implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) = \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}) = 0$. It follows that $\text{я} \colon \widetilde{\mathrm{H}}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{x\})$ is an isomorphism.

Next note that it follows easily from the definitions that $\alpha_n - \beta_n$ is non-trivial with respect to the given $x \in S^n$. It follows from the Detect-Cycle Proposition 150.5 (1) that $\alpha_n - \beta_n$ represents a generator of $\mathrm{H}_n(S^n, S^n \setminus \{x\})$. Thus by the above it also represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$.[1556]

(2) If n is odd, then we know by the simple calculation in the proof of the Singleton-H_*-Lemma 143.26 that β_n is a cycle which is the boundary of the singular $(n+1)$-simplex that is given by the constant map $\gamma \colon \Delta^{n+1} \to (1, 0, \ldots, 0)$. Thus we see that $[\alpha_n - \beta_n] = [\alpha_n - \partial_{n+1}\gamma] = [\alpha_n] \in \widetilde{\mathrm{H}}_n(S^n)$. ■

[1556] We could also have argued using the Detect-Cycle Proposition 150.5 (2). But it seems to us that it is more elegant to give a more direct proof.

150.4. **Local homology groups of topological manifolds.** Our next goal is to calculate the local homology groups of topological manifolds and to draw several consequences from this calculation. At this point it is helpful to recall Definition 23.2 of charts of type (i) and (ii) of a topological manifold. For the reader's convenience we illustrate the definitions in the figure below. We also recall that in Definition 23.2 we said that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i).



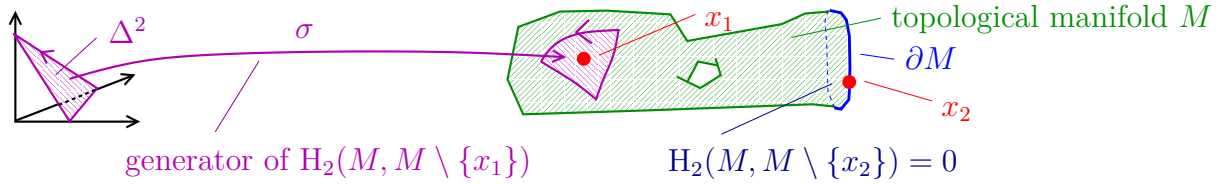
The following lemma calculates in particular the local homology groups of topological manifolds.

Lemma 150.9. (Manifold-Local-H$_*$-Lemma) Let M be an n-dimensional topological manifold.

(1) Every point on X admits either a chart of type (i) or it admits a chart of type (ii).
(2) Given any point $x \in M$ and given any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } x \in M \setminus \partial M \text{ and if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

If $x \in M \setminus \partial M$, then there exists an embedding $\sigma\colon \Delta^n \to X$ such that $\sigma(\mathring{\Delta}^n)$ is an open neighborhood of x. Furthermore any such singular n-simplex σ represents a generator of $\mathrm{H}_n(M, M \setminus \{x\})$.



Proof. Let x be a point on an n-dimensional topological manifold M. We start out with the following claim:

Claim. Given any $k \in \mathbb{N}_0$ we have the following isomorphisms:

$$\mathrm{H}_k(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } x \text{ admits a chart of type (i) and if } k = n, \\ 0, & \text{if } x \text{ admits a chart of type (ii) or if } k \neq n. \end{cases}$$

Proof. If x admits a chart of type (i), then by definition x admits an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. Thus the promised statement follows immediately from the Local-H$_*$-Proposition 150.3 (1), (2) and (3a).

If x admits a chart of type (ii), then by definition there exists an open neighborhood U of x and a homeomorphism $\Phi\colon U \to V$ where V is an open subset of $\mathbb{H}^n$ and $\Phi(x) \in \mathbb{R}^{n-1} \times \{0\}$. Thus the promised statement follows immediately from the Local-H$_*$-Proposition 150.3 (1), (2) and (3b). ⊞

Now it is straightforward to prove the statements of the lemma.

(1) It follows from the above claim that $\mathrm{H}_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ if and only if x admits a chart of type (i). This observation immediately implies the promised statement.

(2) First note that the claim gives us in particular the desired calculation of the isomorphism type of the local homology groups $\mathrm{H}_n(M, M \setminus \{x\})$.

Finally let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to B^n$ around x with $\Phi(x) = 0$ and we pick an embedding $\tau\colon \Delta^n \to B^n$ such that $\tau(\mathring{\Delta}^n)$ is an open neighborhood of 0.[1557] It is clear that $\sigma := \Phi^{-1} \circ \tau\colon \Delta^n \to M$ is an embedding such that $\sigma(\mathring{\Delta}^n)$ is an open neighborhood of x. It follows from the Local-H_*-Proposition 150.3 (1), (2) and (4) that any such σ represents a generator of $\mathrm{H}_n(M, M \setminus \{x\})$. ■

We conclude this section with the following proposition which is the generalization of the Smooth Manifold-Dimension Proposition 25.31 from smooth manifolds to topological manifolds.

Proposition 150.10. (Topological Manifold-Dimension Proposition) Let M be a non-empty m-dimensional topological manifold and let N be an n-dimensional topological manifold. If M is homeomorphic to an open subset of N (e.g. if M is homeomorphic to N), then $m = n$.

It follows in particular from the Topological Manifold-Dimension Proposition 150.10 that the dimension of a non-empty topological manifold is well-defined.

Proof. By the remark on page 639 it suffices to consider the case that we are given a homeomorphism $f\colon M \to N$ between a non-empty m-dimensional topological manifold M and a topological manifold N. Since M is non-empty we obtain from the Interior-Non Compact Lemma 23.16 that $M \setminus \partial M$ is non-empty. In particular we can find a point $P \in M \setminus \partial M$. Then

$$\begin{array}{ccccc} \mathbb{Z} & \underset{\uparrow}{\cong} & \mathrm{H}_m(M, M \setminus \{P\}) & \underset{\uparrow}{\xrightarrow{\cong}} & \mathrm{H}_m(N, N \setminus \{f(P)\}). \\ & \text{by the Manifold-Local-H}_* & & \text{since } f \text{ is a} & \\ & \text{Lemma 150.9} & & \text{homeomorphism} & \end{array}$$

It now follows from the Manifold-Local-H_*-Lemma 150.9 that N is also m-dimensional. ■

150.5. **Appendix: The topology of topological manifolds.** As the title of this appendix suggests, in this appendix we will consider some applications of homology groups to the study of topological manifolds. In particular we will prove several statements on the topology of topological manifolds which extend the results obtained in Section 25.6 from the setting of smooth manifolds to the more general context of topological manifolds.

First of all we have the following proposition which is a refinement of the Unique Type-of-Chart Proposition 23.13 and the topological analog of the Unique Type-of-Chart Proposition 25.26:

Proposition 150.11. (Unique Type-of-Chart Proposition) Let M be a topological manifold.

(i) Let $P \in M$. Precisely one of the following two statements holds:[1558]

- The point P admits a chart of type (i) as in Definition 23.2.
- The point P admits a chart of type (ii) as in Definition 23.2.

[1557] It is elementary to see that such an embedding exists, for example we could consider the projection $\Delta^n \to \triangle^n$ followed by a translation.

(ii) If M is equipped with a smooth atlas, i.e. if we view M as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of M viewed as a topological manifold.

Example. It follows from the Unique Type-of-Chart Proposition 150.11 and the examples of charts for $\overline{B}^n$ given on page 623 that the boundary of the n-dimensional topological manifold is indeed S^{n-1}. We had suspected this for a long time, but only recently did we acquire the tools for providing a proof. □

Proof.

(i) This statement is precisely the statement of the Manifold-Local-H_*-Lemma 150.9 (1). The only reason why we wrote down the statement again is to preserve the analogy with the Unique Type-of-Chart Proposition 25.26.

(ii) Suppose that X is equipped with a smooth atlas. For the purpose of this proof we write $\partial^{\text{top}}X$ and $\partial^{\text{sm}}X$. By the definitions on Definition 23.2 and Definition 25.10 we have

$$\begin{aligned}\partial^{\text{top}}X &= \text{all points that do not admit a chart of type (i)}\\ \partial^{\text{sm}}X &= \text{all points that do not admit a chart of type (i) from the smooth atlas.}\end{aligned}$$

By definition we have $\partial^{\text{top}}X \subseteq \partial^{\text{sm}}X$. Now let $P \in \partial^{\text{sm}}X$. This means that P does not admit a chart of type (i) from the smooth atlas. By definition of a smooth manifold it admits a chart of type (ii). But by (1) this means that it cannot admit any chart of type (i). Thus we see that $P \in \partial^{\text{sm}}X$. ■

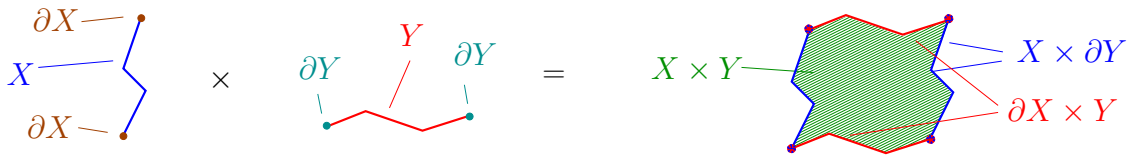
We now now remove the slight ambiguity in the Topological Manifold Product Proposition 23.6 and we obtain the following topological analog of the Smooth Manifold Product Proposition 26.12:

Proposition 150.12. (Topological Manifold Product Proposition) Let X be an m-dimensional topological manifold and let Y be an n-dimensional topological manifold. Then the following hold:

(1) The product $X \times Y$ is an $(m+n)$-dimensional topological manifold.

(2) (a) We have $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$.

(b) The subspaces $\partial X \times Y$ and $X \times \partial Y$ are codimension-zero submanifolds of $\partial(X \times Y)$ with $\partial(\partial X \times Y) = \partial(X \times \partial Y) = \partial X \times \partial Y$.



Example. Let M and N be two connected n-dimensional topological manifolds with empty boundary. If $M \times [0,1]$ is homeomorphic to $N \times [0,1]$, then evidently the corresponding boundaries are also homeomorphic. It follows from the Topological Manifold Product Proposition 150.12 that $\partial(M \times [0,1]) = M \times \{0,1\}$ and $\partial(N \times [0,1]) = N \times \{0,1\}$. Since M and

[1558]For 2-dimensional topological manifolds we proved this statement in the Unique 2D-Type-of-Chart Proposition Proposition 101.8.

N are connected we obtain easily from the above that M and N are homeomorphic. Even though this sounds quite reasonable, very similar sounding statements can easily be wrong. For example there exists a 3-dimensional topological manifold W (called the *Whitehead manifold*) which is non-compact and has no boundary, which is not homeomorphic to $\mathbb{R}^3$ but such that $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. We refer to [**Kir1989**, p. 87] and [**AnR1965**] for details. □

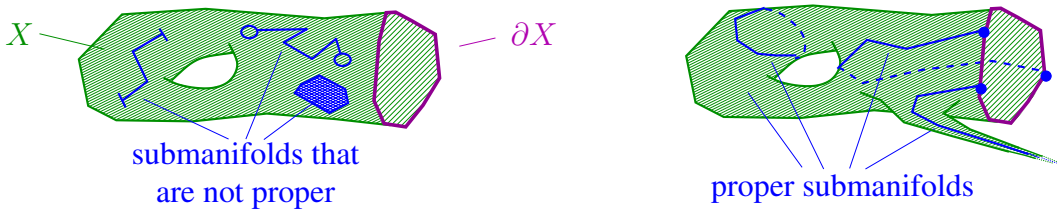
Proof. First we recall that in the Topological Manifold Product Proposition 23.6 we already showed that $X \times Y$ is an $(m+n)$-dimensional topological manifold. Furthermore recall that in the Topological Manifold Product Proposition 23.6 we already showed that each point in $(X \times Y) \setminus ((\partial X \times Y) \cup (X \times \partial Y))$ admits a chart of type (i) and we showed that each point in $(\partial X \times Y) \cup (X \times \partial Y)$ admits a chart of type (ii). It now follows from the Unique Type-of-Chart Proposition 150.11 that we have indeed $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$. The last statement can again be proved by considering product charts and restricting them to the boundary. ■

We move on to submanifolds. More precisely, we continue with the following topological analog of the Proper Submanifold Proposition 25.30:

Proposition 150.13. (Proper Submanifold Proposition) Let X be an n-dimensional topological manifold. The following statements hold:

(1) Let Y be a submanifold of X. The following two statements are equivalent:
  (a) Y is a proper submanifold in the sense of Definition 23.18.
  (b) The following two conditions are satisfied:
    (i) Y is a closed subset of X.
    (ii) We have the equality $\partial Y = Y \cap \partial X$.

(2) If Y is a proper submanifold of X, then $\partial Y = Y \cap \partial X$ is a proper submanifold of ∂X.



Proof.

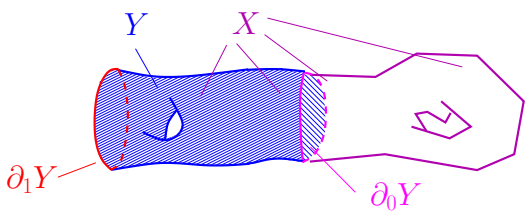
(1) Similar to what we saw in the proof of the Proper Submanifold Proposition 25.30 (1), this statement is a fairly straightforward consequence of the Unique Type-of-Chart Proposition 150.11.

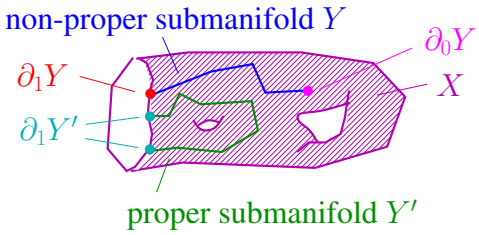
(2) As we already saw in the proof of the Proper Submanifold Proposition 25.30 (2), this statement is a straightforward consequence of (1) and the Unique Type-of-Chart Proposition 150.11. ■

Inspired by Definition 23.2 (4), Definition 25.10 and the Unique Type-of-Chart Proposition 150.11 we introduce the following analog of Notation 25.47

Notation 150.14. Let X be a topological manifold and let Y be a submanifold. We set:

$$\begin{aligned} \partial_0 X &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\gamma), \\ \partial_1 X &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\beta). \end{aligned}$$





The following proposition is the topological analog of the Boundary-of-Submanifold Proposition 25.48:

Proposition 150.15. (Boundary-of-Submanifold Proposition) Let X be an n-dimensional topological manifold and let Y be a k-dimensional submanifold. The following statement holds:

(0) (a) $\partial_0 Y = \partial Y \setminus \partial X$,[1559]
    (b) $\partial_1 Y = \partial Y \cap \partial X$.
(1) $\partial_0 Y$ and $\partial_1 Y$ are unions of components of ∂Y.
(2) $\partial_0 Y$ is an $(k-1)$-dimensional submanifold of X. If Y is a closed subset of X, then $\partial_0 Y$ is a proper submanifold of X.

Proof. The proof is almost identical to the proof of the Boundary-of-Submanifold Proposition 25.48, we just need to replace the Unique Type-of-Chart Proposition 25.26 by the Unique Type-of-Chart Proposition 150.11. ■

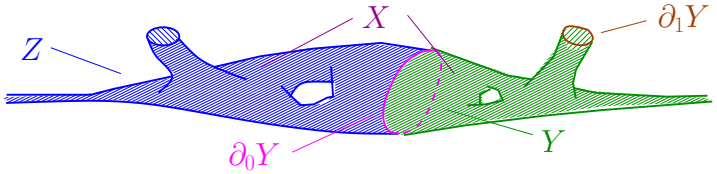
We continue with a topological analog of the Codimension-Zero Submanifold Proposition 25.49:

Proposition 150.16. (Codimension-Zero Submanifold Proposition) Let X be an n-dimensional topological manifold and let Y be a codimension-zero submanifold that is a closed subset of X. Then the following statements hold:

(1) The boundary of Y as a subset of X agrees with $\partial_0 Y$.
(2) The interior $\overset{\circ}{Y}$ of Y as a subset of X agrees with $Y \setminus \partial_0 Y$.

We can furthermore make some statements about the complement $Z := X \setminus \overset{\circ}{Y}$ of the interior:

(3) $Z := X \setminus \overset{\circ}{Y}$ is a codimension-zero submanifold with $\partial_0 Z = \partial_0 Y$ and $\partial_1 Z = \partial X \setminus \partial_1 Y$.
(4) Z is a closed subset of X and if X is compact, then $Z = X \setminus \overset{\circ}{Y}$ is also compact.
(5) If X is connected and if $\partial_0 Y$ is connected, then $Z = X \setminus \overset{\circ}{Y}$ is also connected.



Proof. The proof is almost identical to the proof of the Codimension-Zero Submanifold Proposition 25.49. Again we just need to replace the Unique Type-of-Chart Proposition 25.26 by the Unique Type-of-Chart Proposition 150.11. ■

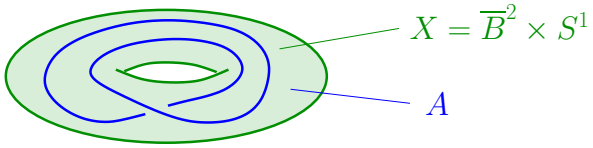
150.6. Exercises for Chapter 150 .

Exercise 150.1. Let $n \in \mathbb{N}$. Let

$$\begin{array}{rcl} \rho\colon \Delta^n & \to & \Delta^n \\ (t_0, t_1, t_2, \ldots, t_n) & \mapsto & (t_1, t_0, t_2, \ldots, t_n) \end{array}$$

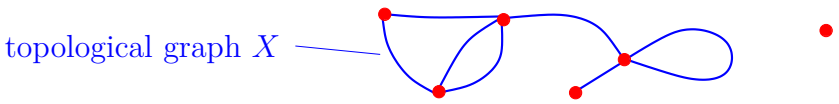
be the map that swaps the first two coordinates of Δ^n. Show that the induced map $\rho_*\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is given by multiplication by -1.
Hint. First prove the statement for $n = 1$. Then use an induction argument similar to the proof of the Identity-is-Generator Lemma 150.1 (2). Hereby you need to use the naturality of the connecting homomorphism of the long exact sequence in homology of a pair of topological spaces.

Exercise 150.2. We consider the solid torus $X = \overline{B}^2 \times S^1$ together with the subspace $A \subseteq X$ that are shown in the figure below. Compute the relative homology groups $\mathrm{H}_k(X, A)$.



Exercise 150.3. Let $f\colon \Delta^n \to \Delta^n$ be a continuous map that sends each $(n-1)$-dimensional face to itself. Show that $f_*\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is an isomorphism.
Hint. You could prove the statement by induction on n.

Exercise 150.4. Let X be the topological graph that is shown in the figure below. Compute the local homology of X at each point $x \in X$.



Exercise 150.5. We consider the topological space $X := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = z^2\}$. Compute the local homology groups of X.

Exercise 150.6. Let $n_1, \ldots, n_k \in \mathbb{N}$. We form the wedge $X := S^{n_1} \vee \cdots \vee S^{n_k}$. We denote by P the wedge point. Compute the local homology groups $\mathrm{H}_*(X, X \setminus \{P\})$.

Exercise 150.7. Show that for $k \geq 1$ the wedge $S^k \vee S^k$ is not a topological manifold.

Exercise 150.8. Does there exist a continuous map $f\colon S^1 \times S^1 \to S^2$ which induces an epimorphism $\mathrm{H}_2(S^1 \times S^1) \to \mathrm{H}_2(S^2)$?

Exercise 150.9. Let M be a compact non-empty n-dimensional topological manifold. Show that if M admits a CW-structure, then the CW-complex also has dimension n.

Exercise 150.10. We consider the topological space $X = \mathbb{Q}$. What is the local homology of $X = \mathbb{Q}$ at $x = 0$?

Exercise 150.11. Let M be a topological manifold. Furthermore let $x_1, \ldots, x_k$ be distinct points in M. Let $n \in \mathbb{N}_0$. For $i \in \{1, \ldots, k\}$ we denote by

$$\nu_i \colon \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_k\}) \to \mathrm{H}_n(M, M \setminus \{x_i\})$$

the map that is induced by the inclusion $M \setminus \{x_1, \ldots, x_k\} \to M \setminus \{x_i\}$. Show that the map

$$\bigoplus_{i=1}^{k} \nu_i \colon \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_k\}) \to \bigoplus_{i=1}^{k} \mathrm{H}_n(M, M \setminus \{x_i\})$$

is an isomorphism.
Hint. Use the Excision Theorem 148.2.

Exercise 150.12. We recall the following definitions:

(1) The suspension of a non-empty topological space X is defined to be the topological space
$$\Sigma(X) := (X \times [-1, 1])/\sim$$
where we identify all points in $X \times \{-1\}$ and we identify all points in $X \times \{1\}$.
(2) A topological space X that has the property that for each $x \in X$ we have
$$\mathrm{H}_k(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$
is called an *n-dimensional homology manifold*.
(3) We say that an n-dimensional topological manifold X is a *topological homology n-sphere* if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$.

Let Y be a homology n-sphere.

(a) Show that $Y \times \mathbb{R}$ is an $(n+1)$-dimensional homology manifold.
(b) Show that the suspension $\Sigma(Y)$ of Y is an $(n+1)$-dimensional homology manifold.
(c) Show that if $n \geq 2$ and if $\pi_1(Y)$ is non-trivial, then the suspension $\Sigma(Y)$ is *not* a topological manifold.

Remark. In the Poincaré Homology Sphere Proposition 183.24 we will see that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group. Together with the above this shows that there exist homology manifolds that are not topological manifolds.

Exercise 150.13. Let X be a compact topological space such that for each $x \in X$ we have

$$\mathrm{H}_n(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Is X necessarily a 1-dimensional topological manifold?

Exercise 150.14. Let X be a topological space. Suppose that the suspension $\Sigma(X)$ is a closed n-dimensional topological manifold. Show that X is a homology $(n-1)$-sphere, i.e. show that $\mathrm{H}_i(X) \cong \mathrm{H}_i(S^{n-1})$ for all $i \in \mathbb{N}_0$.
Remark. Also have a look at Exercise 150.15.

Exercise 150.15. Let $n \in \mathbb{N}_{\geq 3}$ and let M be a connected non-empty topological space with non-trivial fundamental group.

(a) Show that the suspension $\Sigma(M)$ is *not* an n-dimensional topological manifold.
Hint. There are many different approaches one can take. One possibility is to study the fundamental group of $\Sigma(M)$ and to study the fundamental group of $\Sigma(M)$ with some points removed, and to use Exercise 99.3.

(b) Show that the suspension $\Sigma(M)$ is not a topological manifold of any dimension.

Remark. In Exercise 150.14 we just showed that if $\Sigma(M)$ is a closed n-dimensional topological manifold, then M is a topological homology $(n-1)$-sphere, i.e. $\mathrm{H}_i(M) \cong \mathrm{H}_i(S^{n-1})$ for all $i \in \mathbb{N}_0$. Together with (a) and the proof of the Poincaré Conjecture in all dimensions, see **??**, this shows that if $n \in \mathbb{N}_{\geq 2}$ and if M is a connected non-empty n-dimensional topological manifold such that $\Sigma(M)$ is a topological manifold, then M is necessarily homeomorphic to S^n.

Exercise 150.16. Let X be a topological space and let $x \in X$. In Definition 11.33 we defined the *topological valence of* x as

$$\mathrm{topval}(x, X) \coloneqq \min \left\{ n \in \mathbb{N} \cup \{\infty\} \;\middle|\; \begin{array}{c} \text{for every open neighborhood } U \text{ of } x \text{ there} \\ \text{exists an open neighborhood } V \text{ of } x \text{ with } V \subseteq U \\ \text{such that } V \setminus \{x\} \text{ has at least } n \text{ components} \end{array} \right\}.$$

(a) Show that if X is locally contractible and if x is not an isolated point, then

$$\mathrm{topval}(x, X) \;=\; \mathrm{rank}(\mathrm{H}_1(X, X \setminus \{x\})).$$

(b) Show that the conclusion of (a) does not necessarily hold if X is not locally contractible.

CHAPTER 151

Homology and self-maps of spheres

To understand the concept of homology we need to be able to compute homology groups of interesting topological spaces *and* we should acquire techniques for calculating induced maps $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$. In the last chapters we got to know the homology groups of spheres quite well. In this and the following chapter we will get to know several techniques for determining induced maps $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$.

These results will allow us to prove various interesting results, e.g. the mysterious sounding Hairy Ball Theorem. The results of this chapter will also be essential later on in various other calculations of homology groups, in particular we will need them in Chapter 153 when we study Mayer–Vietoris sequence and in Chapter 155 when we will calculate cellular homology groups of CW-complexes.

This chapter is organized as follows:

- In Section 151.1 we will introduce the degree of a self-map of a sphere.
- In Section 151.2 we will use this degree to give yet another proof of the Hairy Ball Theorem 46.18.
- In Section 151.3 we will provide exercises for Chapter 151.

In the next Chapter 152 we will learn about local degrees of self-maps of spheres and we will see that they can be used to calculate the degrees of self-maps of spheres.

151.1. **The degree of a self-map of a sphere.** If we want to understand the concept of homology properly, then not only do we need to determine the isomorphism types of homology groups of interesting topological spaces, but we also need to determine induced maps on homology. In this section we will study the maps on homology induced by continuous self-maps of spheres. This topic is interesting in its own right, but the results will also play a major role later on in Chapters 153 and 155 when we will study Mayer-Vietoris sequences and when we try to determine the homology groups of CW-complexes.

Throughout this chapter we will work a lot with the following simple-minded definition which is analogous to Definition 80.5:

Definition 151.1. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$ and let $\Phi\colon G \to G$ be a homomorphism. Note that Φ is given by multiplication by a unique integer which we refer to as the degree $\deg(\Phi)$.

For completeness' sake we record two basically obvious statements regarding the degree.

Lemma 151.2. (Algebraic Degree Lemma) Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$.

(1) Let $g \in G$ be any non-trivial element. Furthermore let $\Phi\colon G \to G$ be a homomorphism. The degree $\deg(\Phi)$ is uniquely determined by the equality $\Phi(g) = \deg(\Phi) \cdot g$.

(2) Let $\varphi\colon G \to H$ be an isomorphism and let $f\colon H \to H$ be a homomorphism. Then $\deg(\varphi^{-1} \circ f \circ \varphi\colon G \to G) = \deg(f\colon H \to H)$.

Proof. The two statements follow easily from the definitions. ■

Next we relate the above Definition 151.1 to topology.

Definition 151.3. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be a continuous map. By the Sphere-$\widetilde{\mathrm{H}}_*$-Proposition 149.4 we know that $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$. Thus we can define the degree of f:

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

Remark. Let $n \in \mathbb{N}$. Recall that in Definition 46.5 we used smooth techniques to define the degree of a continuous map $f\colon S^n \to S^n$. On page 2889 we will see that the two definitions of the degree agree. □

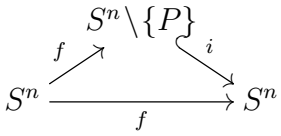
The following lemma, which can be viewed as an analog of the Sphere–De Rham-Degree Lemma 80.8, summarizes several basic properties of the degree of continuous self-maps of spheres.

Lemma 151.4. (Sphere–Degree Lemma) Let $n \in \mathbb{N}_0$ and let $f, g\colon S^n \to S^n$ be two continuous maps.

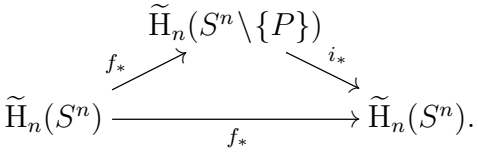
(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$.[1560]
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) If f is a self-homeomorphism of S^n, then $\deg(f^{-1}) = \deg(f)^{-1}$.

Proof.

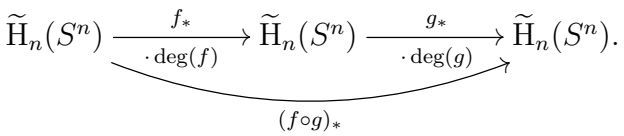
(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\widetilde{\mathrm{H}}_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) Suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram



which gives rise to the commutative diagram



As we already mentioned on page 2866, $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$, which implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{P\}) = 0$. But this means that $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.
(3) If f is homotopic to g, then it follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 147.4 that $\deg(f) = \deg(g)$.
(4) We consider the following diagram:



This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

(5) Let $f\colon S^n \to S^n$ be a self-homeomorphism. The desired statement follows from the following equalities:

$$1 \underset{\text{by (1)}}{=} \deg(\mathrm{id}) = \deg(f \circ f^{-1}) \underset{\text{by (4)}}{=} \deg(f) \cdot \deg(f^{-1}).$$

Thus $\deg(f^{-1}) = \deg(f)^{-1}$. ■

In the following we want to determine the degrees of interesting self-maps of spheres. We start out with self-maps of circles:

Lemma 151.5. (Degree-of-Power Lemma) Given $k \in \mathbb{Z}$ we have[1561]

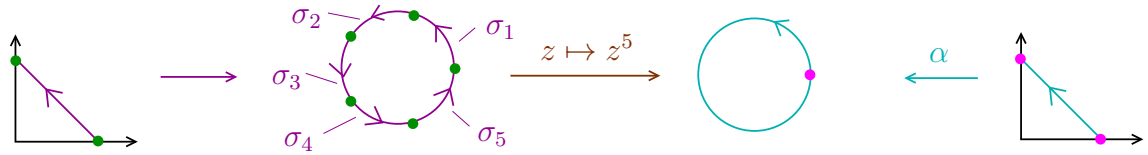
$$\deg \begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{pmatrix} = k.$$

Proof. Let $k \in \mathbb{Z}$. We denote by $f\colon S^1 \to S^1$ the map that is given by $z \mapsto z^k$. We need to show that $\deg(f) = k$. Let $\alpha\colon \Delta^1 \to S^1$ be the singular 1-simplex that is given by $\alpha(1-t,t) = \exp(2\pi \mathrm{i} t)$. By the discussion on page 2866 we know that $\alpha \in \mathrm{C}_1(S^1)$ is a cycle and $[\alpha] \in \mathrm{H}_1(S^1)$ is a generator.

First we first consider the case $k \geq 1$. For $j = 0, \ldots, k-1$ we denote by $\sigma_j\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(2\pi \mathrm{i}(\frac{t+j}{k}))$. Then

$$f_*([\alpha]) \underset{\text{by the Singular 1-Simplex Lemma 143.28 (3)}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = [\underbrace{f \circ \sigma_0}_{=\alpha} + \cdots + \underbrace{f \circ \sigma_{k-1}}_{=\alpha}] = k \cdot [\alpha].$$

It follows from this calculation together with the Algebraic Degree Lemma 151.2 (1) that $\deg(f) = k$.



The case $k < 0$ is proved is proved in a similar fashion using the Singular 1-Simplex Lemma 143.28 (2). Finally, if $k = 0$, then $f\colon S^1 \to S^1$ is not surjective, thus it follows from the Sphere–Degree Lemma 151.4 (2) that $\deg(f) = 0$. ■

The combination of the Degree-of-Power Lemma 151.5 and the following lemma shows that every $k \in \mathbb{Z}$ is realized as the degree of any sphere S^n of dimension $n \geq 1$.

Lemma 151.6. (Degree-Suspension Lemma) Let $n \in \mathbb{N}_0$.

(1) The map

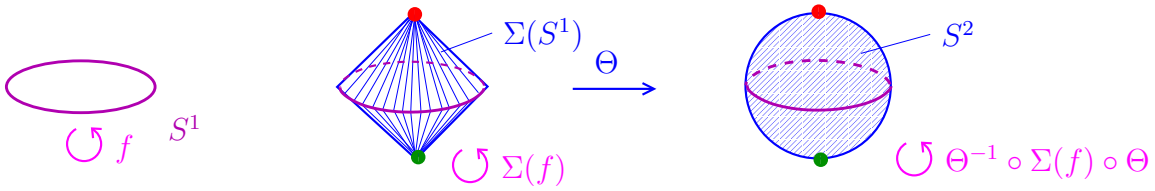
$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \to & S^{n+1} \subseteq \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & (\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}) \end{array}$$

is a homeomorphism.

[1561] This result is analogous to the fact, observed on page 1760, that for fundamental groups the induced map $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$ is given by multiplication by k.

(2) Let $f\colon S^n \to S^n$ be a continuous map. Let $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ be the suspension of f that we defined on Definition 20.22. Furthermore let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be any homeomorphism. Then

$$\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}\colon S^{n+1} \to S^{n+1}) \;=\; \deg(f\colon S^n \to S^n).$$



Proof. Let $n \in \mathbb{N}_0$.

(1) On page 556 we showed that the given map $\Theta\colon \Sigma(S^n) \to S^{n+1}$ is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map and let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be a homeomorphism. We consider the following diagram:

$$\begin{array}{ccccc}
\widetilde{\mathrm{H}}_n(S^n) & \xleftarrow[\cong]{\Sigma_{S^{n+1}}} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) & \xrightarrow[\cong]{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}) \\
{\scriptstyle \cdot \deg(f)}\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (\Theta \circ \Sigma(f) \circ \Theta^{-1})_*} \\
\widetilde{\mathrm{H}}_n(S^n) & \xleftarrow[\cong]{\Sigma_{S^{n+1}}} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) & \xrightarrow[\cong]{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}).
\end{array}$$

First note that it follows from the Suspension-H_*-Proposition 149.3 (2) that the square to the left commutes and that the horizontal maps to the left are isomorphisms. It is clear that the square to the right commutes. Furthermore, since Θ is a homeomorphism we see that the horizontal maps to right are isomorphisms. It follows immediately from the above discussion together with the Algebraic Degree Lemma 151.2 (2) that $\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}) = \deg(f)$. ∎

We move on to our final calculation of a degree:

Lemma 151.7. (Sphere–Degree–Matrix Lemma) Given any $A \in \mathrm{O}(n+1)$ we have

$$\deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & A \cdot x \end{pmatrix} \;=\; \underbrace{\det(A)}_{\in\{\pm 1\}}.$$

Example.

(1) If $f\colon S^n \to S^n$ is the reflection in any hyperplane of $\mathbb{R}^{n+1}$, then it follows from elementary linear algebra and the Sphere–Degree–Matrix Lemma 151.7 that $\deg(f) = -1$,

(2) If $f\colon S^n \to S^n$ is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then it follows from the Sphere–Degree–Matrix Lemma 151.7 that we have the equalities $\deg(f) = \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}$.

(3) Let $A \in \mathrm{O}(n+1)$. Let $f\colon S^n \to S^n$ be the map that is given by $f(P) = A \cdot P$. If $\det(A) = 1$, then we saw on page 510 that f is actually homotopic to the identity. On the other hand, if $\det(A) = -1$, then it now follows from the Sphere–Degree Lemma 151.4 (1), (3) together with the Sphere–Degree–Matrix Lemma 151.7 that f is not homotopic to the identity. □

[1561] In the Degree Isomorphism Proposition 169.2 we will see that the converse to this statement holds.

Proof. We start out with the following claim:

Claim. [1562] Given $n \in \mathbb{N}_0$ we denote by $\rho_n \colon S^n \to S^n$ the reflection in the $(x_1{=}0)$-hyperplane. For any $n \in \mathbb{N}_0$ we have $\deg(\rho_n) = -1$.

Proof. [1563] We start out with the case $n = 0$. We denote by $\sigma_+ \colon \Delta^0 \to S^0$ and $\sigma_- \colon \Delta^0 \to S^0$ the singular 0-simplices that are given by $\{1\} \mapsto \{1\}$ and $\{1\} \mapsto \{-1\}$. With this notation let us consider the reduced chain complex of S^0:

$$\ldots \to \mathrm{C}_1(S^0) \xrightarrow[\substack{=0,\text{ see}\\ \text{page 2751}}]{\partial_1} \underbrace{\mathrm{C}_0(S^0)}_{=\mathbb{Z}\cdot\sigma_+\oplus\mathbb{Z}\cdot\sigma_-} \xrightarrow[=\epsilon_{S^0}]{\substack{\sigma_+ \mapsto 1\\ \sigma_- \mapsto 1}} \mathbb{Z} \to 0.$$

We see that $\widetilde{\mathrm{H}}_0(S^0) = \ker(\epsilon_{S^0} \colon \mathbb{Z}\cdot\sigma_+ \oplus \mathbb{Z}\cdot\sigma_- \to \mathbb{Z}) = \mathbb{Z}\cdot(\sigma_+ - \sigma_-)$. Thus we see that

$$\rho_{0*}(\sigma_+ - \sigma_-) = \underbrace{\rho_0 \circ \sigma_+}_{=\sigma_-} - \underbrace{\rho_0 \circ \sigma_-}_{=\sigma_+} = \sigma_- - \sigma_+ = (-1)\cdot(\sigma_+ - \sigma_-).$$

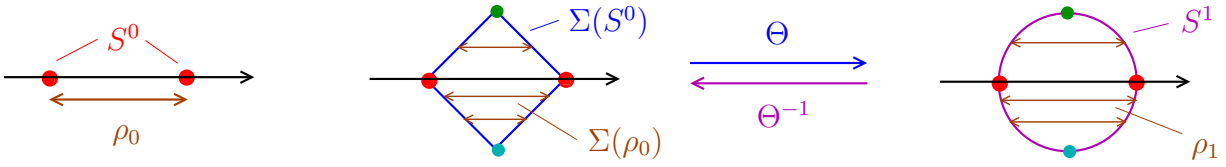
It follows from the Algebraic Degree Lemma 151.2 (1) that $\deg(\rho_0) = -1$. We have thus proved the desired statement for $n = 0$.

Now we turn to the proof of the induction step. Thus we assume that we know that the statement holds for some $n \in \mathbb{N}_0$. Let

$$\begin{array}{rcl} \Theta \colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subseteq \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & (P \cdot \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)) \end{array}$$

be the homeomorphism from page 2838. It follows immediately from the explicit definitions of the maps that for each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \Sigma(S^n) & \xrightarrow{\Theta} & S^{n+1} \\ {\scriptstyle \Sigma(\rho_n)}\downarrow & & \downarrow{\scriptstyle \rho_{n+1}} \\ \Sigma(S^n) & \xrightarrow{\Theta} & S^{n+1}. \end{array}$$



Thus it follows from the Degree-Suspension Lemma 151.6 and the induction hypothesis that $\deg(\rho_n) = \deg(\rho_{n-1}) = -1$. ⊞

Before we get to the denouement of the proof, it is helpful to introduce the following notation: given $B \in \mathrm{O}(n+1)$ we denote by $\mu(B) \colon S^n \to S^n$ the map that is given by $x \mapsto B \cdot x$.

Now let $A \in \mathrm{O}(n+1)$. First we consider the case $\det(A) = 1$, i.e. we assume that $A \in \mathrm{SO}(n+1)$. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n+1)$ is path-connected, so there exists a continuous map $h \colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and

[1562] This claim is the solution to Exercises 147.1 and 149.7.

[1563] This claim can also be proved easily using the Reflection-on-Suspension-H_*-Proposition 149.18.

$h(1) = A$. But then $F(x,t) = h(t) \cdot x$ defines a homotopy from $\mathrm{id}_{S^n} = \mu(\mathrm{id})$ to $\mu(A)$. It follows from the Sphere–Degree Lemma 151.4 (1) and (3) that $\deg(\mu(A)) = \deg(\mathrm{id}_{S^n}) = 1$.

Finally we consider the case that $\det(A) \neq 1$. Let $R \in \mathrm{O}(n+1)$ be the matrix such that $\mu(R) = \rho_n$. Now we see that

$$\deg(\mu(A)) = \deg(\mu(A \cdot R \cdot R^{-1})) \underset{\uparrow}{=} \deg(\mu(A \cdot R)) \cdot \deg(\mu(R)) \underset{\uparrow}{=} 1 \cdot (-1) = -1.$$

by the Sphere–Degree Lemma 151.4 (4); since $\det(A \cdot R) = 1$ we have $\deg(\mu(A \cdot R)) = 1$ and by the claim we have $\deg(\mu(R)) = -1$ ■

151.2. **Application: The Hairy Ball Theorem.** After the above discussion of degrees of continuous self-maps of spheres it makes sense to recall the following theorem:

Theorem 46.15. (Sphere-Degree Fixed Point Theorem) Let $f\colon S^n \to S^n$ be a continuous map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Example. The Sphere-Degree Fixed Point Theorem Theorem 46.15 says in particular that every continuous map $f\colon S^2 \to S^2$ of degree one has a fixed point. Together with the Sphere–Degree Lemma 151.4 this implies that every continuous map $f\colon S^2 \to S^2$ that is homotopic to the identity has a fixed point. This means that we cannot "deform" the identity to a continuous map that is fixed point free. This is in contrast to the case $n = 1$ where we can "deform" the identity into the map that is given by rotation by a small angle, which is of course a fixed point free map. □

Proof. We proved the Sphere-Degree Fixed Point Theorem 46.15 on page 1080 using the notion of the degree of a smooth map $f\colon S^n \to S^n$. In a rather blatant case of self-plagiarism the proof we now provide is almost identical to the proof on page 1080. Let $f\colon S^n \to S^n$ be a continuous map that does not admit a fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\mathrm{id}$.

Our hypothesis implies that $(1-t) \cdot f(x) \neq t \cdot x$ for all $t \in [0,1]$ and all $x \in S^n$.[1564] Therefore we can consider the map

$$\begin{aligned} F\colon S^n \times [0,1] &\to S^n \\ (x,t) &\mapsto \frac{(1-t) \cdot f(x) - t \cdot x}{\|(1-t) \cdot f(x) - t \cdot x\|}. \end{aligned}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \underset{\uparrow}{=} \deg(-\mathrm{id}) \underset{\uparrow}{=} \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$

by the Sphere–Degree Lemma 151.4 (3); by the Sphere–Degree Matrix Lemma 151.7 ■

We use this opportunity to reprove the following result:

Proposition 46.17. (Free Action-on-Sphere Proposition) If a finite group G acts freely and continuously on an even-dimensional sphere, then G is either trivial or $G \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Let G be a group that acts freely and continuously on a sphere S^n. We consider the map

[1564] This can be seen as follows: If $t \neq \frac{1}{2}$ we have $\|(1-t) \cdot f(x)\| = 1 - t \neq t = \|t \cdot x\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t) \cdot f(x) = \frac{1}{2} f(x) \neq \frac{1}{2} x = t \cdot x$.

$$\begin{array}{ccc} G & \to & \mathbb{Z} \\ g & \mapsto & \deg \begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & g \cdot x \end{pmatrix}. \end{array}$$

It follows from the Sphere–Degree Lemma 151.4 (5) that this map takes values in $\{-1, 1\}$ and it follows from the Sphere–Degree Lemma 151.4 (4) that the map $G \to \{-1, 1\}$ is actually a group homomorphism.

Now if n is even, then it follows from the Sphere-Degree Fixed Point Theorem 46.15 and our hypothesis that G acts freely that the above map $G \to \{-1, 1\}$ is actually a monomorphism. ■

Next we recall Definition 29.27 of a vector field on a submanifold of $\mathbb{R}^n$.

Definition 151.8. Let $M \subseteq \mathbb{R}^n$ be a submanifold. Given $P \in M \setminus \partial M$ we denote by $\widetilde{\mathrm{T}}_P M \subseteq \mathbb{R}^n$ the physical tangent vector space as introduced in Definition 29.1. A vector field on M is a continuous map $v \colon M \to \mathbb{R}^n$ such that for every $P \in M$ we have $v(P) \in \widetilde{\mathrm{T}}_P M$.

Examples.

(1) Let P be a point on S^n. By the Physical Tangent Spaces–Lemma 29.2 we have

$$\widetilde{\mathrm{T}}_P S^n = \{v \in \mathbb{R}^{n+1} \mid \langle P, v \rangle = 0\}.$$

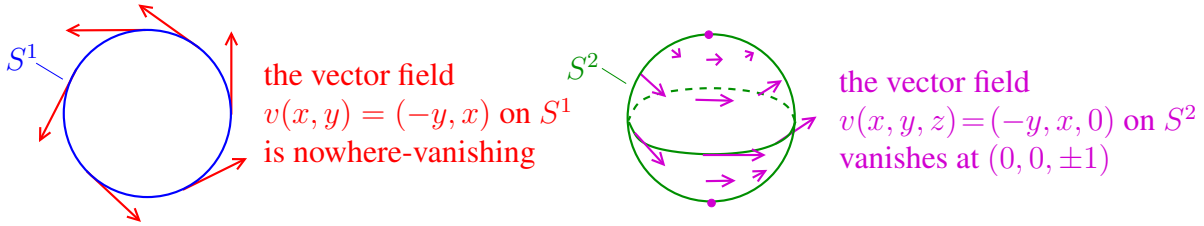
(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by[1565]

$$\begin{array}{ccc} v \colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{array}$$

This vector field has the property that it is nowhere-vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{ccc} v \colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$



This vector field vanishes at the "North Pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "South Pole" $(0, \ldots, 0, -1)$. □

In Question 29.28 we asked whether it is possible to find a nowhere-vanishing vector field on even-dimensional spheres. The following theorem says that this is not possible.

[1565] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P) \rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\widetilde{\mathrm{T}}_P S^{2k-1}$.

Theorem 46.18. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.[1566]

Remark.

(1) For $n = 2$ we get the following "real life application": on the surface of earth there always exists at least one point where the wind does not blow at all. This fact is illustrated at the following website

`https://earth.nullschool.net/`

(2) The "Hairy Ball Theorem" gets its name from the observation that for $n = 2$ the theorem says that "you can't comb a hairy ball flat without creating a cowlick".

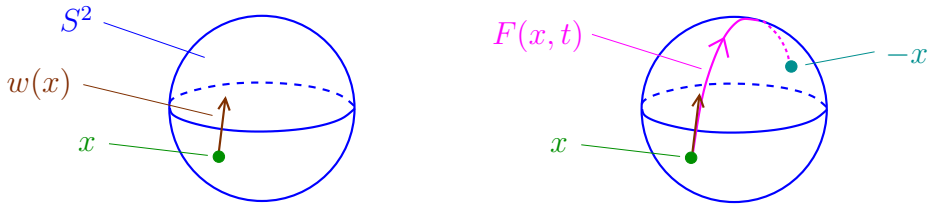
(3) Evidently the fun never stops. Now that we know that odd-dimensional spheres admit one nowhere-vanishing vector field, the question arises, how many linearly independent vector fields can we find on a given odd-dimensional sphere. It turns out that there is a complete and highly non-trivial answer, see [**Adam1962**], [**MillH1989**] and [**VogI**] for details. □

Proof. We proved the Hairy Ball Theorem 46.18 on page 1082 using the notion of the degree of a smooth map $f\colon S^n \to S^n$ and on page 1632 using de Rham cohomology. The proof we now provide is again almost identical to our earlier proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. Since v is nowhere vanishing we can consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) \coloneqq \frac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

Furthermore we consider the map[1567]

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$



The map $F_0 = F(-,0)$ is the identity and the map $F_1 = F(-,1)$ equals $F_1 = -\mathrm{id}$. Put differently, the map F is a homotopy between the two maps id and $-\mathrm{id}$. It follows that

$$1 \underset{\uparrow}{=} \deg(\mathrm{id}) \underset{\uparrow}{=} \deg(-\mathrm{id}) \underset{\uparrow}{=} \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$

by the Sphere–Degree Lemma 151.4 (1) — Lemma 151.4 (3) — by the Sphere–Degree–Matrix Lemma 151.7

[1566] On S^{2k} we just found a vector field which vanishes at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in Exercise 151.8.

[1567] The map does indeed take values in S^n, since

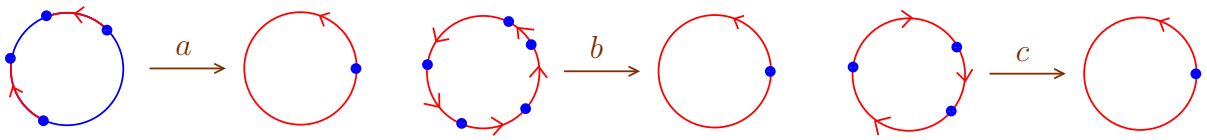
$$\langle \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x), \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x) \rangle \underset{\uparrow}{=} \cos(\pi t)^2 \cdot \|x\|^2 + \sin(\pi t)^2 \cdot \|w(x)\|^2 \underset{\uparrow}{=} 1.$$

since $\langle x, w(x) \rangle = 0$ — since $\|x\| = \|w(x)\| = 1$

But this means that n is odd. ■

151.3. Exercises for Chapter 151 .

Exercise 151.1. Determine the degrees of the maps $a, b, c\colon S^1 \to S^1$ that are shown in the figure below.



Exercise 151.2. Let $A \in \mathrm{O}(n)$. We consider the map

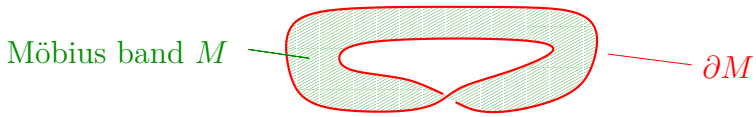
$$\begin{array}{rcl} \rho(A)\colon \overline{B}^n & \to & \overline{B}^n \\ v & \mapsto & A \cdot v. \end{array}$$

Determine $\rho(A)_*\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \mathrm{H}_n(\overline{B}^n, S^{n-1})$.

Exercise 151.3. Give an example of a topological space X and a subspace A such that $\mathrm{H}_1(A) \cong \mathbb{Z}$ and $\mathrm{H}_1(X) \cong \mathbb{Z}$ and such that the inclusion induced map $\mathrm{H}_1(A) \to \mathrm{H}_1(X)$ corresponds to multiplication by ± 5.

Exercise 151.4. Let M be the Möbius band.

(a) Compute the homology groups of M.
(b) Determine the relative homology groups of the pair $(M, \partial M)$.
(c) Is ∂M a retract of M?



Exercise 151.5.

(a) Let $n \in \mathbb{N}_{\geq 2}$. Show that given any $k \in \mathbb{N}$ there exists a continuous map $f\colon \overline{B}^n \to \overline{B}^n$ with $f(S^{n-1}) \subseteq S^{n-1}$ such that

$$\deg\left(f_*\colon \underbrace{\mathrm{H}_n(\overline{B}^n, S^{n-1})}_{\cong \mathbb{Z}} \to \mathrm{H}_n(\overline{B}^n, S^{n-1})\right) \;=\; k.$$

(b) Show that the conclusion of (a) does not hold for $n = 1$.

Exercise 151.6. We consider the continuous map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^1 \\ [z_0 : z_1] & \mapsto & [\overline{z}_0 : \overline{z}_1]. \end{array}$$

What is the induced map on $\mathrm{H}_2(\mathbb{C}\mathrm{P}^1)$?
Remark. Recall that by the Projective Space-Avatar Lemma 9.16 we know that $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2.

Exercise 151.7. Let $n \in \mathbb{N}_0$ odd.

(a) Give an example of a continuous map $f\colon S^n \to S^n$ that has no fixed points.
(b) Can you also find a continuous map $f\colon S^n \to S^n$ that has no fixed points *and* that is homotopic to the identity?

Exercise 151.8. On S^{2k} we had found a vector field which vanishes at *precisely two* points. Does there exist a vector field on S^2 which vanishes on only *one* point?

Exercise 151.9. Let $n \in \mathbb{N}$ be even. Show that every continuous map $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ has a fixed point.
Hint. Apply the Lift-to-Universal Coverings Proposition 127.26 to get a suitable map $\widetilde{f}\colon S^n \to S^n$. If $\deg(\widetilde{f}) \neq -1$, then we are basically done by the Sphere-Degree Fixed Point Theorem Theorem 46.15. What can we do if $\deg(\widetilde{f}) = -1$?

Exercise 151.10. Let $n \in \mathbb{N}_{\geq 2}$. In this exercise we identify S^{n-1} with the equatorial sphere $\{(x,0) \in S^n \mid x \in S^{n-1}\}$.

(a) Let $f\colon S^n \to S^n$ be a continuous map with $f(S^n_{\geq 0}) \subseteq S^n_{\geq 0}$ and with $f(S^n_{\leq 0}) \subseteq S^n_{\leq 0}$. Show that $\deg(f\colon S^n \to S^n) = \deg(f\colon S^{n-1} \to S^{n-1})$.
(b) Let $f\colon S^n \to S^n$ be a continuous map with $f(S^n_{\geq 0}) \subseteq S^n_{\leq 0}$ and with $f(S^n_{\leq 0}) \subseteq S^n_{\geq 0}$. Show that $\deg(f\colon S^n \to S^n) = -\deg(f\colon S^{n-1} \to S^{n-1})$.

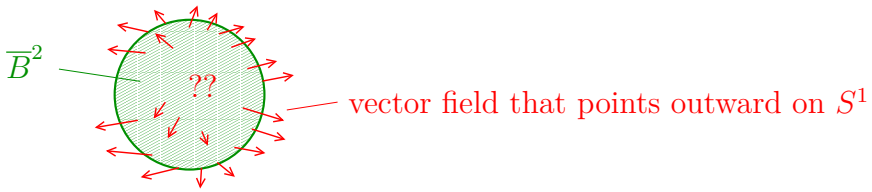
Exercise 151.11. Let X be a topological space. Given a homology class $\varphi \in \mathrm{H}_k(X)$ we define

$$\|\varphi\|_{\mathbb{Z}} \coloneqq \inf\Big\{ \sum_{i=1}^n |a_i| \,\Big|\, \sigma = \underbrace{\sum_{i=1}^n a_i \cdot \sigma_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i \neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X) \Big\}.$$

(a) Show that $\| - \|_{\mathbb{Z}}$ defines a norm on $\mathrm{H}_k(X)$.
(b) Let X be a path-connected topological space. Show that for any non-zero $\varphi \in \mathrm{H}_1(X)$ we have $\|\varphi\|_{\mathbb{Z}} = 1$.
(c) Let $n \in \mathbb{N}$. Determine $\|[S^n]\|_{\mathbb{Z}}$, where $[S^n] \in \mathrm{H}_n(S^n)$ is the standard generator.

Remark. The notation with the subscript "$\mathbb{Z}$" is used to distinguish this definition from the concept that we will introduce in Exercise 186.9.

Exercise 151.12. Let $n \in \mathbb{N}$. Show that every vector field on $\overline{B}^n$ that points outward on S^{n-1} admits at least one zero in $\overline{B}^n$.



Exercise 151.13.

(a) Let $f, g\colon S^n \to S^n$ be two continuous maps. We assume that for every $x \in S^n$ we have $f(x) \neq g(x)$. Show that the maps $x \mapsto f(x)$ and $x \mapsto -g(x)$ are homotopic.
(b) Let $f\colon S^n \to S^n$ be a continuous map which is homotopic to a constant map. Show that f admits a fixed point and show that there exists a $y \in S^n$ with $f(y) = -y$.

Exercise 151.14. Let $f\colon S^{2n} \to S^{2n}$ be a continuous map. Show that there exists an $x \in S^{2n}$ with $f(x) = x$ or with $f(x) = -x$.

Exercise 151.15. Let $f, g\colon S^n \to S^n$ be two continuous maps. Show that if $\deg(f) \neq \deg(g)$, then there exists an $x \in S^n$ where $f(x)$ and $g(x)$ are orthogonal, i.e. with $\langle f(x), g(x)\rangle = 0$.

Exercise 151.16. Let X be a topological space with subspaces A, B, U, V and suppose that we have the following diagram of inclusion maps

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & U \\ \downarrow{\scriptstyle j} & & \uparrow{\scriptstyle k} \\ V & \xleftarrow{\quad l \quad} & B. \end{array}$$

Suppose that the inclusion-induced maps i_*, j_*, k_*, l_* on the first homology groups are isomorphisms. Does it follow that $i_* \circ j_*^{-1} = k_* \circ l_*^{-1}\colon \mathrm{H}_1(V) \to \mathrm{H}_1(U)$?

Exercise 151.17. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a continuous map. We pick $x_0, y_0 \in S^n$. We define the *bidegree* of f as

$$\mathrm{bideg}(f) \coloneqq (p, q) \text{ where } p \coloneqq \deg\begin{pmatrix} S^n & \xrightarrow{i} & S^n \\ x & \mapsto & f(x, y_0) \end{pmatrix} \text{ and } q \coloneqq \deg\begin{pmatrix} S^n & \xrightarrow{j} & S^n \\ y & \mapsto & f(x_0, y) \end{pmatrix}.$$

(a) Show that the definition not depend on the choice of x_0 and y_0.

(b) Let $f\colon S^1 \times S^1 \to S^1$ be the map that is given by complex multiplication. Show that the bidegree of f equals $(1, 1)$.
Remark. If you already know quaternions, perform the same exercise for the map $f\colon S^3 \times S^3 \to S^3$ that is given by quaternionic multiplication.

(c) Show that the map

$$\mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(S^n \times S^n) \xrightarrow{f_*} \mathrm{H}_n(S^n)$$

is of the form $\varphi \oplus \psi \mapsto p \cdot \varphi + q \cdot \psi$.

Exercise 151.18. Let $n \in \mathbb{N}_0$. We consider

$$\Lambda \coloneqq \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0\},$$

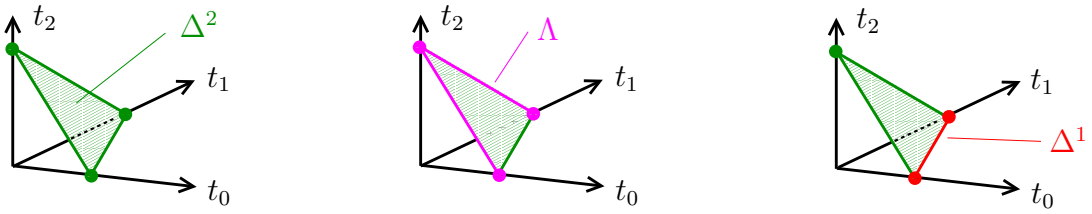
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. We make the identification

$$\Delta^{n-1} \coloneqq \{(t_0, \ldots, t_n) \in \Delta^n \mid t_n = 0\}.$$

Let $f\colon \Delta^n \to \Delta^n$ be a continuous map with $f(\Lambda) \subseteq \Lambda$, with $f(\Delta^{n-1}) \subseteq \Delta^{n-1}$ and which also satisfies $f(\partial\Delta^{n-1}) \subseteq \partial\Delta^{n-1}$. Show that

$$\deg\left(\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{f_*} \mathrm{H}_n(\Delta^n, \partial\Delta^n)\right) = \deg\left(\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \xrightarrow{f_*} \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})\right).$$

Exercise 151.19. Let $\mathcal{C}$ be a category and let $f\colon X \to Y$ be a morphism. A *right-inverse* of f is a morphism $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_X$. Let $\mathcal{HomTop}$ be the homotopy category of topological spaces that we introduced in Definition 19.12. For which $n \in \mathbb{Z}$ does the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$ admit a right-inverse in the homotopy category?



Exercise 151.20. We consider the two topological manifolds X and Y that are shown in the figure below. Use (relative?) homology groups to show that X and Y are not homeomorphic.



CHAPTER 152

Local degrees of self-maps of spheres

In the previousChapter 151 we introduced the notion of the degree of a continuous self-map of a sphere. We managed to calculate the degree in simple situations, but so far we have no general scheme for calculating degrees. For example in the proof of the Degree-of-Power Lemma 151.5 we had to use a rather ad hoc argument to get the desired result.
In this chapter we will formulate and prove the S^n-Local-Global Degree Proposition 152.5 and the Degree-via-Local Orientations Proposition 152.11. These propositions show how one can determine the degree of a continuous map $f\colon S^n \to S^n$ by considering "local degrees" and "local information" of f.

This chapter is organized as follows:

- In Section 152.1 we will introduce the notion of a local degree of a map between oriented smooth manifolds of the same dimension.
- In Section 152.2 we will generalize the notion of a local degree.
- In Section 152.3 we will see that for local diffeomorphisms the local degree is determined by the orientation behavior.
- In Section 152.4 we will provide yet another proof of the Fundamental Theorem 32.4 of Algebra.
- In Section 152.5 we will provide exercises for Chapter 152.

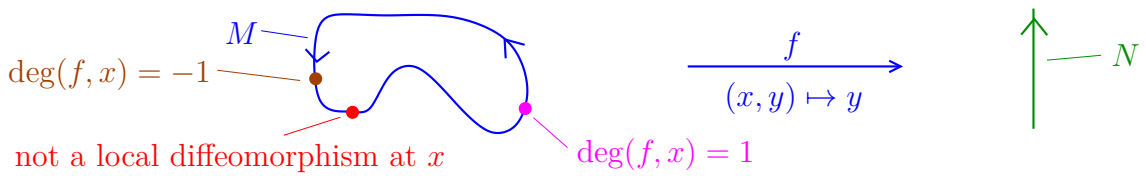
In the next Chapter 153 we will formulate and prove the Mayer–Vietoris Theorem 153.2 which is a key tool for calculating homology groups.

152.1. Degrees and local orientations.

First we recall Definition 46.1:

Definition 152.1. Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional smooth manifolds and let $x \in M \setminus \partial M$. If f is a local diffeomorphism[1568] at x, then we define the local degree of f at the point x as[1569]

$$\deg(f,x) := \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



[1570]By the Inverse Mapping Theorem 30.1 the map f is a local diffeomorphism at $x \in M$ if there exists an open neighborhood U around x such that $f\colon U \to N$ is smooth and such that $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_xN$ is an isomorphism.

For most computational purposes the following proposition is the only result from this chapter one really needs:

Proposition 152.11. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:[1571]

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then[1572] [1573]
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

As the number of the proposition suggests, it will take a while to get to the proof. Meanwhile let us consider some examples:

Examples.

(1) Let $m \in \mathbb{Z}$. We consider the map
$$\begin{array}{rcl} f\colon S^1 & \to & S^1 \\ z & \mapsto & z^m \end{array}$$
In the Degree-of-Power Lemma 151.5 we already proved that $\deg(f) = m$. Now we will show, just for fun, that the degree of f can also easily be determined using the Degree-via-Local Orientations Proposition 152.11. First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = \exp(2\pi \mathrm{i} j/m)$ where $j = 1, \ldots, m$. It follows easily from the definitions that for each $j \in \{1, \ldots, m\}$ we have
$$\deg(f, x_j) \;=\; \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$
It follows from this calculation and the Degree-via-Local Orientations Proposition 152.11 that $\deg(f) = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then we take $y = -1$. Note that for $m = 0$ we have $f^{-1}(y) = \varnothing$, thus we see that $\deg(f) = 0$.

(2) If $f\colon S^n \to S^n$ is a diffeomorphism, then we can apply the Degree-via-Local Orientations Proposition 152.11 with any $y \in S^n$ and we obtain that
$$\deg(f) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving}, \\ -1, & \text{if } f \text{ is orientation-reversing}. \end{cases}$$
This fact can be viewed as a generalization of the Sphere–Degree–Matrix Lemma 151.7.

(3) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a continuous map. We have two definitions of "degree", namely we have Definition 151.3 and also from Definition 46.5. Mercifully these two notions agree:
- For smooth maps this equality follows from the Degree-via-Local Orientations Proposition 152.11.

[1571] If $f\colon S^n \to S^n$ is in fact smooth, then it follows from Sard's Theorem 38.1 and the Inverse Mapping Theorem 30.1 that such y always exists.
[1572] In the Standard Smooth Structures Proposition 25.6 we equipped S^n with a standard smooth structure, which we will use throughout this chapter.
[1573] Recall that on page 817 we equipped each sphere S^n with the following standard orientation: Given $P \in S^n$ we say that a basis $(v_1, \ldots, v_n)$ of the tangent space $\widetilde{\mathrm{T}}_P S^n \subseteq \mathbb{R}^{n+1}$ is positive if $\det(P\, v_1 \ \ldots \ v_n) > 0$.

- For continuous maps this equality follows from the smooth case, together with the Whitney Approximation Theorem 36.1, the Sphere–Degree Lemma 151.4 and the Degree Homotopy Invariance Theorem 46.6.

We now see that we have some overlap of results, for example the Sphere–Degree Lemma 151.4 overlaps with the Degree Homotopy Invariance Theorem 46.6, the Degree Multiplicativity Theorem 46.7 and the Basics-of-Degree Lemma 46.12. □

152.2. **The local degree.** In this section we define, for continuous self-maps $f\colon S^n \to S^n$ and suitable points in S^n a "local degree". We will then prove that the degree of a continuous map $S^n \to S^n$ is determined by the local degrees. In the next section we will relate the new notion of local degree to the notion introduced in the beginning of the chapter. This will then allow us to finally prove the Degree-via-Local Orientations Proposition 152.11 which we formulated in the beginning of the chapter.

Definition 152.2. We say that a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

Examples.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $x \in X$. If U is a neighborhood such that $f|_U$ is injective, then U is evidently a nice neighborhood of x.

(2) We consider the maps

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t = 0, \\ t \cdot \sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the map f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the map g the point $x = 0$ is not nice. □

The following is the key definition of this section.

Definition 152.3. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x. The local degree of f at the point x is defined as the degree[1574] of the homomorphism

$$\mathrm{H}_n(S^n) \underset{\cong}{\xrightarrow{\text{я}}} \mathrm{H}_n(S^n, S^n \setminus \{x\}) \overset{\substack{\text{isomorphism by the} \\ \text{Excision Theorem 148.2} \\ \downarrow \\ \cong}}{\xleftarrow{}} \mathrm{H}_n(U, U \setminus \{x\}) \underset{1575}{\xrightarrow{f_*}} \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}) \underset{\cong}{\xleftarrow{\text{я}}} \mathrm{H}_n(S^n).$$

the long exact sequence of the pair $(S^n, S^n \setminus \{y\})$ shows that these maps are isomorphisms

We denote this local degree by $\deg(f, x)$.

[1574] By the Sphere-H_*-Proposition 149.4 we know that $\mathrm{H}_n(S^n) \cong \mathbb{Z}$. As in Definition 151.1 we note that any homomorphism $\mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is multiplication by an integer, which we call the degree of this homomorphism.

[1575] Note that f induces a map of pairs $(U, U \setminus \{x\}) \to (S^n, S^n \setminus \{f(x)\})$ since U is nice.

Remark. We will generalize the notion of a local degree in Definition 195.4 to maps between oriented topological manifolds. □

Lemma 152.4. (Local Degree–Well-Defined Lemma) The local degree $\deg(f,x)$ of a map $f\colon S^n \to S^n$ at a nice point $x \in S^n$ is well-defined.

Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x. After replacing U' by $U \cap U'$ we can without loss of generality assume that $U' \subseteq U$. We consider the following diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{H}_n(U', U'\setminus\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\setminus\{f(x)\}) & & \\
 & \overset{\cong}{\swarrow} & \downarrow \cong & & \downarrow \mathrm{id} & \overset{\cong}{\nwarrow} & \\
\mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n, S^n\setminus\{x\}) & & & & & & \mathrm{H}_n(S^n). \\
 & \underset{\cong}{\nwarrow} & & & & \underset{\cong}{\swarrow} & \\
 & & \mathrm{H}_n(U, U\setminus\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\setminus\{f(x)\}). & &
\end{array}$$

All the non-decorated maps are induced by inclusions. It follows from the functoriality of relative homology that the diagram commutes. Also note that the left vertical map is an isomorphism by the Excision Theorem 148.2. It follows immediately from these observations that the local degree defined using U' agrees with the local degree defined using U. ■

Example. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a homeomorphism and let $x \in S^n$. Since f is a homeomorphism we see that $U = S^n$ is a nice neighborhood for x. By considering the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_n(S^n) & \xrightarrow{\qquad f_* \qquad} & \mathrm{H}_n(S^n) \\
{\scriptstyle\cong}\big\downarrow{\scriptstyle \text{я}} & & {\scriptstyle \text{я}}\big\downarrow{\scriptstyle\cong} \\
\mathrm{H}_n(S^n, S^n\setminus\{x\}) & \xrightarrow{\qquad f_* \qquad} & \mathrm{H}_n(S^n, S^n\setminus\{f(x)\})
\end{array}$$

we see that $\deg(f,x) = \deg(f)$. □

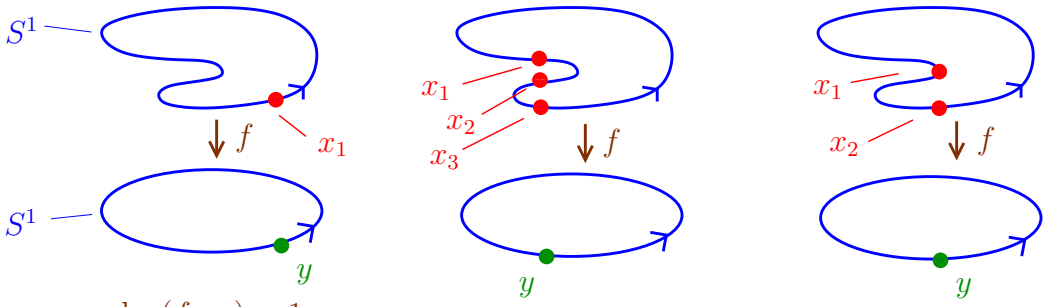
Using the following proposition we can determine the degree of a continuous map from local degrees.

Proposition 152.5. (S^n-Local-Global Degree Proposition) Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Then all the points $x_1, \ldots, x_m$ are nice and we obtain the equality

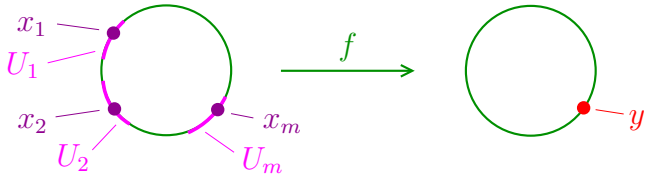
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Remark. In the Local-Global Degree Proposition 195.6 we will generalize the S^n-Local-Global Degree Proposition 152.5 to a more general setting. □

Example. In the figure below we sketch a map $f\colon S^1 \to S^1$ and we choose several points $y \in S^1$.
We see that the number of preimages and the local degrees at the points in the preimages vary, but in all three cases the sum of the local degrees at the points in the preimage is the same. □



Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$. As we saw in Exercise 3.12, it follows easily from the fact that S^n is Hausdorff that we can choose pairwise disjoint open neighborhoods $U_1, \dots, U_m$ for $x_1, \dots, x_m$. Note that this observation implies already that all the x_i are nice.



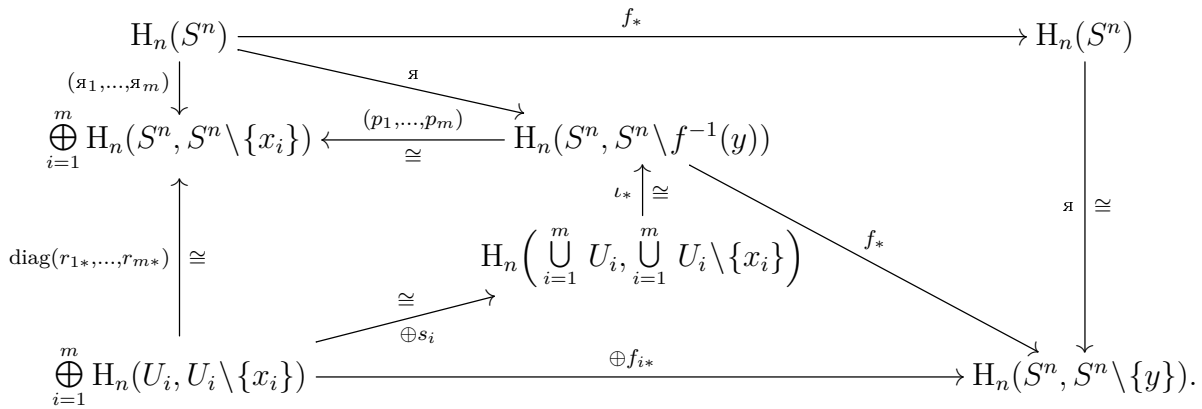
In the following we adopt the following notation:

(1) Given homomorphisms $\varphi_i\colon A_i \to B$ we write $\bigoplus_{i=1}^m \varphi_i\colon \bigoplus_{i=1}^m A_i \to B$.

(2) Given homomorphisms $\varphi_i\colon A \to B_i$ we write $(\varphi_1, \dots, \varphi_m)\colon A \to \bigoplus_{i=1}^m B_i$.

(3) Given homomorphisms $\varphi_i\colon A_i \to B_i$ we write $\operatorname{diag}(\varphi_1, \dots, \varphi_m)\colon \bigoplus_{i=1}^m A_i \to \bigoplus_{i=1}^m B_i$.

We consider the following diagram:



We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i\colon (U_i, U_i \setminus \{x_i\}) \to (S^n, S^n \setminus \{y\})$ are given by the restriction of f to U_i.

(2) It follows pretty easily from the functoriality of relative homology groups and elementary considerations that the diagram commutes.

(3) It follows easily from the fact that the U_i are disjoint open subsets of S^n, the Disjoint Union Embedding Lemma 7.29, the Path-Component–H_*–Lemma 143.21 and the Component Lemma 4.48 (4) and (5) that the diagonal map $\bigoplus_{i=1}^m s_i$ to the bottom left is an isomorphism.

(4) It follows from the Excision Theorem 148.2 that the vertical map ι_* and the maps r_{i*} are isomorphisms.

(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $(p_1, \dots, p_m)$ is also an isomorphism.

(6) As we pointed out on page 2866, it follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that the reduced homology groups of $S^n \setminus \{y\}$ vanish. Since $n \geq 1$ it follows from the LES-of-Pair Proposition 146.7 that $\mathit{я}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{y\})$ is an isomorphism.

By (6) it suffices to prove the following claim:

Claim. We have $$\mathit{я}(f_*([S^n])) \;=\; \sum_{i=1}^m \deg(f, x_i) \cdot \mathit{я}([S^n]) \;\in\; \mathrm{H}_n(S^n, S^n \setminus \{y\}).$$

Proof. We verify the claim by performing the following calculation:

$$\begin{aligned}
\mathit{я}(f_*([S^n])) &\overset{\text{since the diagram commutes and since the map } \mathrm{diag}(r_{1*}, \dots, r_{m*}) \text{ is an isomorphism}}{=} \left(\bigoplus_{i=1}^m f_{i*}\right) \circ \mathrm{diag}(r_{1*}, \dots, r_{m*})^{-1} \circ (\mathit{я}_1, \dots, \mathit{я}_m)([S^n]) \overset{\text{since each } r_{i*} \text{ is an isomorphism}}{=} \\
&= \left(\bigoplus_{i=1}^m f_{i*}\right) \circ (r_{1*}^{-1} \circ \mathit{я}_1, \dots, r_{m*}^{-1} \circ \mathit{я}_m)([S^n]) \\
&\overset{\text{since we always have the equality } \left(\left(\bigoplus_{i=1}^m \varphi_i\right) \circ (\psi_1, \dots, \psi_m)\right)(x) = \sum_{i=1}^m (\varphi_i \circ \psi_i)(x)}{=} \sum_{i=1}^m (f_{i*} \circ r_{i*}^{-1} \circ \mathit{я}_i)([S^n]) \overset{\text{by Definition 152.3 of the local degree}}{=} \sum_{i=1}^m \deg(f, x_i) \cdot \mathit{я}([S^n]).
\end{aligned}$$

■

152.3. **The local degree and local diffeomorphisms.** In the previous section we introduced the notion of a local degree of a continuous self-map of a sphere. In this section we will see that for local diffeomorphisms the above notion of a local degree matches the definition that we gave in the beginning of this chapter.

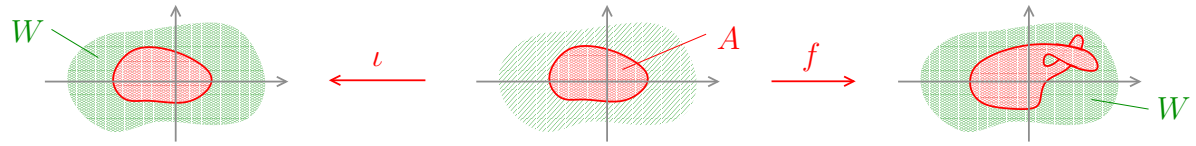
Our proof of the equality of the two local degrees relies on the following proposition, which is interesting in its own right.

Proposition 152.6. (Local Diffeomorphism-Degree Proposition) Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior of A. We denote by $\iota\colon A \to W$ the inclusion. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Then the degree of the

homomorphism[1576]

$$\mathrm{H}_n(W, W\setminus\{0\}) \xleftarrow[\cong]{\iota_*} \mathrm{H}_n(A, A\setminus\{0\}) \xrightarrow{f_*} \mathrm{H}_n(W, W\setminus\{0\})$$

equals $+1$ if f is orientation-preserving at 0 and -1 if f is orientation-reversing at 0.



In the proof of the Local Diffeomorphism-Degree Proposition 152.6 we will make use of the following two lemmas which we will also use on several other occasions later on.

Lemma 152.7. (Linearization–Homotopy Lemma) Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by

$$\begin{aligned} \lambda\colon \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto \mathrm{D}f_P \cdot x + f(P) \end{aligned}$$

($\mathrm{D}f_P$: the differential $\mathrm{D}f_P \in \mathrm{M}(n\times n, \mathbb{R})$ from Definition 24.15)

the linearization of f at P. There exists an open neighborhood U' of P such that the restrictions of f and λ to maps $(U', U'\setminus\{P\}) \to (\mathbb{R}^n, \mathbb{R}^n\setminus\{f(P)\})$ of pairs of topological spaces are homotopic.

Remark. The Linearization–Homotopy Lemma 152.7 is the key to translating information from smooth methods into topology and later on into (relative) homology groups. □

Proof. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. To simplify the notation we assume that $P = 0$ and that $f(P) = 0$. Furthermore, by restriction to an open ball we might as well assume that U is convex. We write $D := \mathrm{D}f_0$. Thus the linearization of f at $P = 0$ is given by the map $\lambda(x) = D\cdot x$. Since f is in particular a local diffeomorphism a $P = 0$ we know by the Chain Rule 24.17 together with the Differential-of-Diffeomorphism Proposition 24.19, that D is an invertible matrix.

Since $U \subseteq \mathbb{R}^n$ is convex we obtain from the Map-as-Matrix Multiplication Proposition 24.23 that there exists a smooth map $A\colon U \to \mathrm{M}(n\times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that $f(x) = A(x)\cdot x$ for all $x \in U$. Since $\det(A(0)) \neq 0$ we see that there exists an $r > 0$ with $B_r^n(0) \subseteq U$ such that $\det(A(x)) \neq 0$ for all $x \in B_r^n(0)$. This implies that the map

$$\begin{aligned} B_r^n(0)\times[0,1] &\to \mathbb{R}^n \\ (x,t) &\mapsto A(x\cdot(1-t))\cdot x \end{aligned}$$

is a homotopy of the maps f and λ as maps $(B_r^n(0), B_r^n(0)\setminus\{0\}) \to (\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$ of pairs of topological spaces. We now see that $U' = B_r^n(0)$ has the desired properties. ■

Before we can formulate our next lemma we need to introduce the following notation:

Notation 152.8.

(1) Given a matrix $P \in \mathrm{GL}(n,\mathbb{R})$ we denote by $\rho(P)\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by $x \mapsto P\cdot x$.

[1576] By the Local-H_*-Proposition 150.3 we know that $\mathrm{H}_n(W, W\setminus\{0\}) \cong \mathbb{Z}$, thus it makes sense to talk of the degree of an automorphism of $\mathrm{H}_n(W, W\setminus\{0\})$.

(2) Given a non-zero real number r we denote its sign by $\operatorname{sign}(r) \in \{-1, +1\}$.

Lemma 152.9. (Matrix Multiplication–Degree Lemma) Let $n \in \mathbb{N}$. Given any invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$ we have[1577]

$$\deg\left(\rho(A)_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) = \operatorname{sign}(\det(A)).$$

Proof of the Matrix Multiplication–Degree Lemma 152.9. Let $A \in \mathrm{GL}(n, \mathbb{R})$. We start out by picking an orthogonal matrix $B \in \mathrm{O}(n)$ such that $\operatorname{sign}(\det(B)) = \operatorname{sign}(\det(A))$. Next we consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow[\cong]{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \\
\downarrow{\scriptstyle \rho(A)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow[\cong]{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}).
\end{array}$$

We make the following observations:

(1) Since $\operatorname{sign}(\det(B)) = \operatorname{sign}(\det(A))$ we obtain from the Matrix Group-π_0-Proposition 4.54 (1) that there exists a path $[0, 1] \to \mathrm{GL}(n, \mathbb{R})$ from A to B. Similar to the discussion on page 510 this path gives rise to a homotopy between $\rho(A)$ and $\rho(B)$ as maps of pairs of topological spaces $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. We obtain from the Homotopy-H_*-Proposition 146.11 that $\rho(A)_* = \rho(B)_*$, in other words, the square to the left commutes.

(2) The square in the middle commutes by the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 146.7.

(3) The horizontal maps to the right are induced by the inclusion $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. Since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$ we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 that these horizontal maps are isomorphisms.

(4) Since $B \in \mathrm{O}(n)$ we see that $\rho(B)\colon \mathbb{R}^n \to \mathbb{R}^n$ restricts to a self-map of S^{n-1}. It follows from the functoriality of reduced homology groups that the square to the right commutes.

Now we see that

$$\begin{array}{ccccc}
\deg\left(\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow{\rho(A)_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) & \overset{\text{by the above commutative diagram together with the Algebraic Degree Lemma 151.2 (2)}}{=} & \deg\left(S^{n-1} \xrightarrow{\rho(B)} S^{n-1}\right) & \overset{\text{by the Sphere–Degree–Matrix Lemma 151.7}}{=} & \det(B) \\
& = & \operatorname{sign}(\det(B)) & = & \operatorname{sign}(\det(A)). \\
& \uparrow & & \uparrow & \\
& \text{since } B \in \mathrm{O}(n) \text{ we have } \det(B) \in \{\pm 1\} & & \text{choice of } B &
\end{array}$$

■

Now we are finally in a position to prove the Local Diffeomorphism-Degree Proposition 152.6.

Proof of the Local Diffeomorphism-Degree Proposition 152.6. Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Recall that this means that there exists an open neighborhood U of

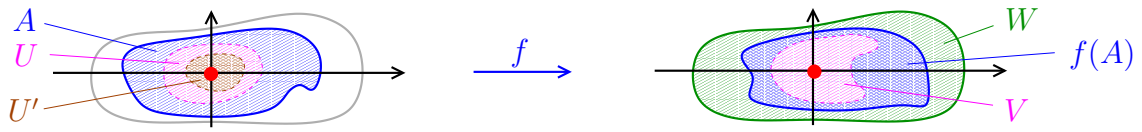
[1577] It follows from the Reduced LES-of-Pair Proposition 147.8, the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6, the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 and the Sphere-H_*-Proposition 149.4 that we have isomorphisms $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Thus the degree of $\rho(A)_*$ is indeed defined.

0 that is contained in A and an open neighborhood V of 0 such that the map $f\colon U \to V$ is a diffeomorphism.

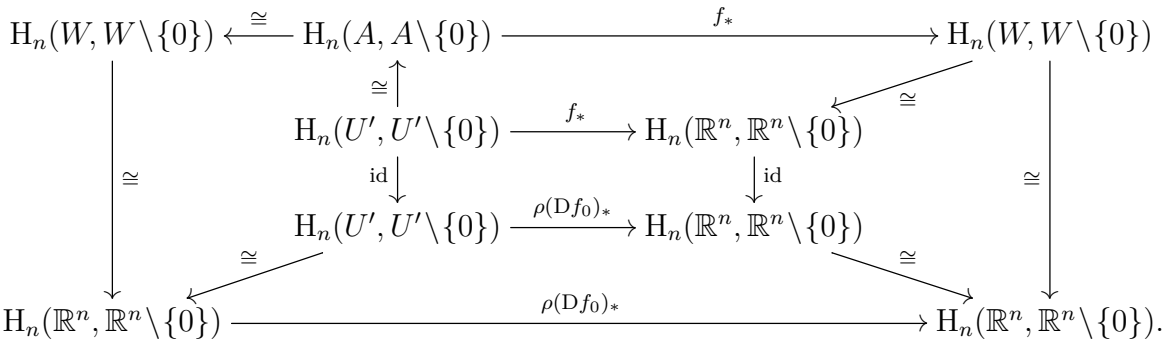
We start out with the following preparations.

(1) We set
$$\epsilon \coloneqq \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } 0, \\ -1, & \text{if } f \text{ is orientation-reversing at } 0. \end{cases}$$
Note that it follows from the Orientation Checking-via-Differentials Lemma 31.30 that we have $\epsilon = \operatorname{sign}(\det(\mathrm{D}f_0))$.

(2) Note that by the Linearization–Homotopy Lemma 152.7 there exists an open neighborhood U' of 0 that is contained in U such that the restrictions of f and $\rho(\mathrm{D}f_0)$ to maps $(U', U' \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopic.



Next we consider the following diagram:



We continue with the following clarifications and observations:

(3) The undecorated maps are induced by inclusions. All of them are isomorphisms by the Excision Theorem 148.2.

(4) It follows from (2) and the Homotopy-H_*-Proposition 146.11 that the central square commutes. All other regions of the diagram commute by the functoriality of relative homology groups.

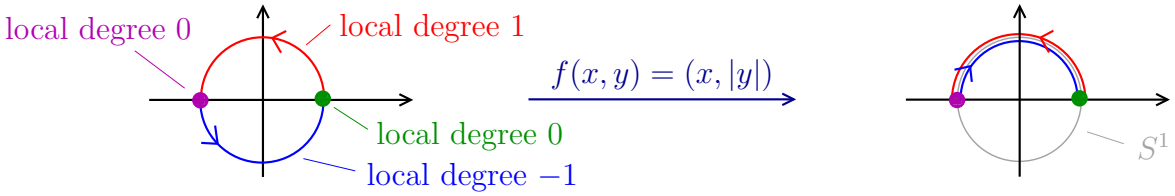
(5) By the Matrix Multiplication–Degree Lemma 152.9 the bottom horizontal map equals multiplication by $\operatorname{sign}(\det(\mathrm{D}f_0))$.

(6) It follows from all of the above that the top map is given by multiplication by ϵ. But that is precisely what we wanted to show. ∎

Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$ be a point such that f is a local diffeomorphism at x. In this setting we now have two different notions of a local degree, namely the notion from Definition 152.1 and the notion from Definition 152.3. The following proposition shows that the two notions of a local degree agree.

Proposition 152.10. (Local Degree-Calculation Proposition) Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. If f is a local diffeomorphism at x, then x is nice and
$$\deg(f, x) = \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f\colon S^1 \to S^1$ that is given by $(x, y) \mapsto (x, |y|)$. It follows from the Local Degree-Calculation Proposition 152.10 that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1.



In Exercise 152.6 we will prove that the local degree at the two points ± 1 is in both cases zero. □

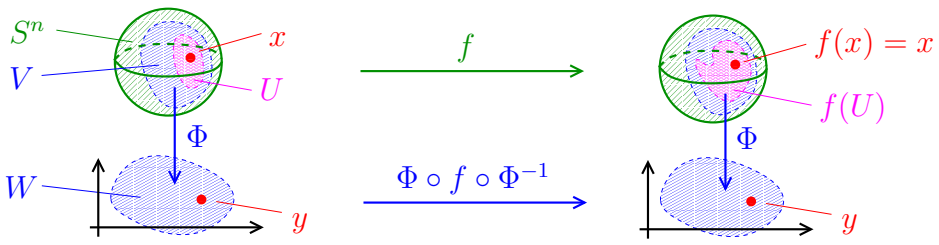
Remark. The Local Degree-Calculation Proposition 152.10 shows that for a smooth map $f\colon S^n \to S^n$ and a point $x \in S^n$ at which f is a local diffeomorphism the definition of the local degree in Definition 46.1 agrees with the above Definition 152.3. □

Proof. Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. We assume that f is a local diffeomorphism at x. By the Orthogonal Action Lemma 2.37 we know that there exists an $A \in \mathrm{SO}(n+1)$ with $A \cdot f(x) = x$. By Exercise 31.5 we know that the self-map of S^n given by $z \mapsto A \cdot z$ is orientation-preserving. Furthermore by the Sphere–Degree–Matrix Lemma 151.7 we know that $\deg(z \mapsto A \cdot z) = 1$. It follows from these observations, together with the Sphere–Degree Lemma 151.4 (4), that after replacing f by the map $z \mapsto A \cdot f(z)$, we might as well assume that $f(x) = x$.

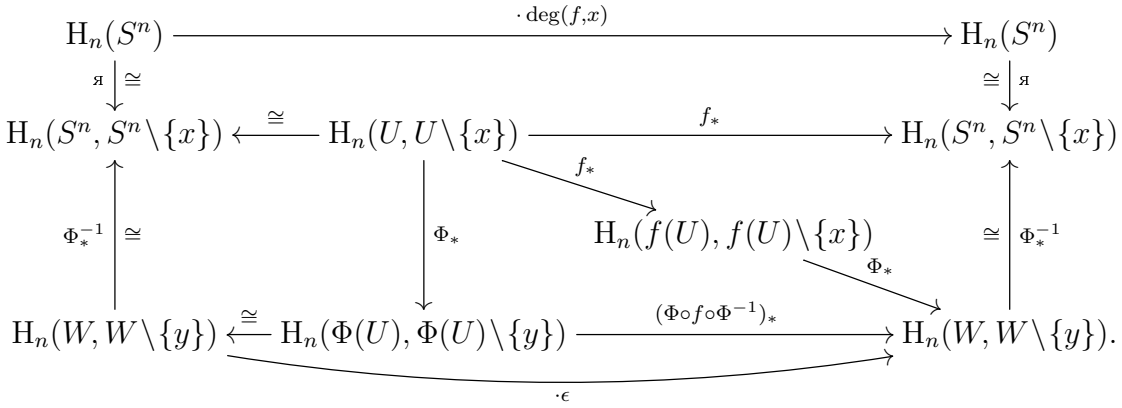
We make a few further preparations.

(1) We pick a chart $\Phi\colon V \to W$, from the smooth atlas, around $x \in S^n$.
(2) Since f is a local diffeomorphism at x there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. Note that after possibly replacing U by the subset $U \cap V \cap f^{-1}(V)$ we can assume that $U \subseteq V$ and $f(U) \subseteq V$.
(3) We set $y := \Phi(x)$.
(4) We set

$$\epsilon := \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



Next we consider the following diagram:



We make the following observations and clarifications.

(1) As usual the undecorated maps are induced by inclusions.
(2) The top rectangle commutes by the definition of $\deg(f,x)$. The middle regions of the diagram commute by the functoriality of relative homology. Finally, note that it follows from the Manifold–Orientation Basics Lemma 31.31 (5) that $\Phi\circ f\circ\Phi^{-1}\colon \Phi^{-1}(U)\to W$ is orientation-preserving at y if and only if f is orientation-preserving at x. Thus we see that the bottom part of the diagram commutes by the Local Diffeomorphism-Degree Proposition 152.6.
(3) By the Excision Theorem 148.2 the map $\Phi_*\colon \mathrm{H}_n(W, W\setminus\{y\})\to \mathrm{H}_n(S^n, S^n\setminus\{x\})$ is an isomorphism.
(4) It follows from the above together with the Algebraic Degree Lemma 151.2 (2) that $\deg(f,x)=\epsilon$. ■

Now we can finally prove the following proposition, which we already mentioned on page 2888, at the very beginning of this chapter.

Proposition 152.11. (Degree-via-Local Orientations Proposition) Let $n\in\mathbb{N}_0$. Let $f\colon S^n\to S^n$ be a continuous map and let $y\in S^n$ such that the following conditions are satisfied:

(1) $f^{-1}(y)$ consists of finitely many points $x_1,\dots,x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then
$$\deg(f) \;=\; \sum_{i=1}^{m}\deg(f,x_i) \;=\; \sum_{i=1}^{m}\begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i,\\ -1, & \text{if } f \text{ is orientation-reversing at } x_i.\end{cases}$$

Proof. The proposition is an immediate consequence of the S^n-Local-Global Degree Proposition 152.5 together with the Local Degree-Calculation Proposition 152.10. ■

152.4. **Application: Another proof of the Fundamental Theorem of Algebra.** We conclude this chapter with yet another proof of the Fundamental Theorem of Algebra. First we introduce the following notation:

Definition 152.12. Let f be a complex polynomial of degree ≥ 1. We consider the following map:

$$\begin{array}{rcl} \Theta(f)\colon S^2 =_{\mathrm{i}} \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} =_{\mathrm{i}} S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty. \end{cases} \end{array}$$

By Exercise 4.43 we know that $\Theta(f)$ is continuous.

The following proposition determines the degree of the above maps.

Proposition 152.13. (Polynomial–Degree Proposition) Let f be a complex polynomial of degree ≥ 1. The degree of the map

$$\begin{array}{rcl} \Theta(f)\colon S^2 =_{\mathrm{i}} \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} =_{\mathrm{i}} S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases} \end{array}$$

equals the degree of the polynomial f.

Sketch of proof. Let f be a complex polynomial of degree $n \geq 1$. We also consider the polynomial $g(z) = z^n$. We calculate that

$$\deg(\Theta(f)) \overset{\text{by the Sphere–Degree Lemma 151.4 (3), since by Exercise 18.11 we know that } \Theta(f) \text{ and } \Theta(g) \text{ are homotopic}}{\downarrow =} \deg(\Theta(g(z)=z^n)) \underset{\text{follows from the } S^n\text{-Local-Global Degree Proposition 152.5 and the Local Degree-Calculation Proposition 152.10, see Exercise 152.5}}{\uparrow =} n.$$

■

We can now provide the promised new proof of the Fundamental Theorem of Algebra.

Theorem 32.4. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Remark. On page 842 and page 842. we already gave two alternative topological proofs of the Fundamental Theorem of Algebra 32.4. □

Proof. Let f be a nonconstant polynomial with complex coefficients. In other words, f is a complex polynomial of some degree $n \geq 1$. By the Polynomial–Degree Proposition 152.13 we know that the degree of the self-map $\Theta(f)$ of $S^2 =_{\mathrm{i}} \mathbb{C}\cup\{\infty\}$ equals $n \geq 1$. By the Sphere–Degree Lemma 151.4 (2) this implies that $\Theta(f)\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is surjective, in particular 0 lies in the image of $\Theta(f)$. Since $\Theta(f)(\infty) = \infty$ we see that 0 lies in the image of $f\colon \mathbb{C} \to \mathbb{C}\cup\{\infty\}$. In other words, there exists a $z \in \mathbb{C}$ with $\underbrace{\Theta(f)(z)}_{=f(z)} = 0$. ■

152.5. Exercises for Chapter 152 .

Exercise 152.1. Let $n \in \mathbb{N}_0$. Let $f\colon S^n \to S^n$ be the composition of the following three maps:

$$\begin{array}{ccccccc} S^n & \to & \overline{B}^n & \to & \overline{B}^n/S^{n-1} & \to & S^n \\ (x_1,\dots,x_{n+1}) & \mapsto & (x_1,\dots,x_n) & & [r\cdot v] & \mapsto & \begin{pmatrix} \cos(\pi\cdot r) \\ \sin(\pi\cdot r)\cdot v \end{pmatrix} \end{array}$$

Determine $\deg(f)$.

Exercise 152.2. We consider the following maps:

$$\begin{array}{rcl} p\colon S^1 & \to & S^1/z\sim -z \\ z & \mapsto & [z] \end{array} \quad \text{and} \quad \begin{array}{rcl} f\colon S^1/z\sim -z & \to & \mathbb{R}\mathrm{P}^1 \\ [z] & \mapsto & [z] \end{array}$$

$$\begin{array}{rcl} g\colon \mathbb{R}\mathrm{P}^1 & \to & \mathbb{R}\cup\{\infty\} \\ [x:y] & \mapsto & \begin{cases} \frac{x}{y}, & \text{if } y\neq 0, \\ \infty, & \text{if } y=0. \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} h\colon S^1 & \to & \mathbb{R}\cup\infty \\ (x,y) & \mapsto & \begin{cases} \frac{x}{1-y}, & \text{if } y<1, \\ \infty, & \text{if } y=1. \end{cases} \end{array}$$

By the Projective Space-Sphere Quotient Lemma 9.13, the Projective Space-Avatar Lemma 9.16 and by the Stereographic Projection Lemma 4.24 we know that f, g and h are homeomorphisms. Determine the degree of the map $h^{-1}\circ g\circ f\circ p\colon S^1\to S^1$.

Exercise 152.3. Let $f\colon S^1\to S^1$ be a continuous map. Does there necessarily exist a $y\in S^1$ such that the preimage $f^{-1}(y)$ consists of finitely many points?

Exercise 152.4. Let $n\in\mathbb{N}$. For which $k\in\mathbb{Z}$ does there exist a local diffeomorphism $f\colon S^n\to S^n$ with $\deg(f)=k$?

Exercise 152.5. Let $f=f(z)$ be a polynomial with complex coefficients of degree ≥ 1. We consider the map

$$\begin{array}{rcl} \Theta(f)\colon S^2=_{\mathrm{i}}\mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\}=_{\mathrm{i}}S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases} \end{array}$$

which we first introduced in Definition 152.12.

(a) Show that the local degree of $\Theta(f)$ at a root of $f(z)$ equals the multiplicity of the root.
(b) Let $n\geq 1$. We consider the complex polynomial $g(z)=z^n$. Show that the degree of the map $\Theta(g)\colon S^2\to S^2$ equals n.

Exercise 152.6.

(a) We consider the map $f\colon S^1\to S^1$ that is given by $(x,y)\mapsto(x,|y|)$. Show that the local degree at the point -1 is zero.
(b) Let $f\colon S^1\to S^1$ be a continuous map and let $z\in S^1$ be a nice point. Suppose that $\deg(f,z)=1$. Does it follow that f is a local homeomorphism at z?

Exercise 152.7. Let $k,l\in\mathbb{Z}$. We consider the sphere $S^3=\{(z,w)\in\mathbb{C}^2\,|\,|z|^2+|w^2|=1\}$ and the self-map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (z^k,w^l). \end{array}$$

Determine the degree of f.

Exercise 152.8. Let $f\colon S^n\to S^n$ be an orientation-preserving diffeomorphism. Show that f is homotopic to the identity.
Hint. You can easily arrange that f fixes say the North Pole $N=(0,\dots,0,1)$. Now use the Linearization–Homotopy Lemma 152.7 and push everything that is "awkward" into the South Pole.

CHAPTER 153

The Mayer–Vietoris sequence and its applications

Recall that in the Seifert–van Kampen-Theorem 99.1 we saw how, given a suitable decomposition $X = U \cup V$ of a topological space, one can determine the fundamental group $\pi_1(X, x_0)$ in terms of the two inclusion-induced homomorphisms $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

In this chapter we will state and prove the Mayer–Vietoris Theorem 153.2 which gives us a similar result for the homology groups of $X = U \cup V$. As we will see shortly, the Mayer–Vietoris Theorem 153.2 is just a user friendly variation on the Excision Theorem 148.1. But we will also see that it is a very useful tool for computing homology groups. In particular, together with results of the previous chapters, we will be able to determine the homology groups of the torus, the Klein bottle and the real projective plane $\mathbb{R}\mathrm{P}^2$.

This chapter is organized as follows:

- In Section 153.1 we will state and prove the Mayer–Vietoris Theorem 153.2.
- In Section 153.2 we will apply the Mayer–Vietoris sequence to calculate the homology groups of suspensions.
- In Section 153.3 we will determine the homology groups of a mapping torus.
- In Section 153.4 we will discuss the effect of a cell attachment on homology.
- In Section 153.5 we will determine the homology groups of a mapping cone.
- In Section 153.6 we will provide exercises for Chapter 153.

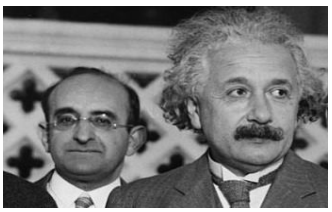
In the next Chapter 154 we will discuss the relationship between homology groups and direct colimits.

153.1. **The Mayer–Vietoris sequence.** In the following theorem and throughout the subsequent discussion we will use the following notation:

Notation 153.1. Given a topological space D and a subspace $C \subseteq D$ we denote the corresponding inclusion map $C \to D$ by i_C^D. To avoid pandemonium with subscripts we often drop the "$*$" when we deal with inclusion-induced maps on homology, e.g. we will write $i_C^D \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ instead of the more accurate $(i_C^D)_* \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$. We use similar notation for inclusions of pairs of topological spaces.

The following theorem, which can be viewed as an analog of the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2, has its origin in the work of Walther Mayer [**MayeW1929**] and Leopold Vietoris [**Vie1930**].

Theorem 153.2. (Mayer–Vietoris Theorem) Let X be a topological space and let $A, B \subseteq X$ be subspaces such that one of the following three conditions holds:
(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.



Walther Mayer (1887-1948) with some other person



Leopold Vietoris (1891-2002) at 110

(3) X is a topological manifold[1578] and $A, B \subseteq X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
- $X = A \cup B$.
- $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
- A and B are closed subsets of X.[1579]

By (1) the Excision Theorem 148.1, (2) the CW-Excision Theorem 148.4 or (3) the Manifold–Excision Theorem 148.3 each inclusion-induced map $i^{(X,B)}_{(A,A\cap B)} \colon \mathrm{H}_n(A, A\cap B) \to \mathrm{H}_n(X,B)$ is an isomorphism. We refer to the homomorphism

$$\mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X,B) \xrightarrow{\left(i^{(X,B)}_{(A,A\cap B)}\right)^{-1}} \mathrm{H}_n(A, A\cap B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A\cap B)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A\cap B)$, provided by the LES-of-Pair Proposition 146.7

as the connecting homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$ corresponding to (X, A, B). The connecting homomorphism is natural in a suitable sense and with this notation the following sequence is exact:[1580]

$$\begin{array}{ccccccccc} \ldots \to \mathrm{H}_n(A\cap B) & \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) & \xrightarrow{i^X_A \oplus i^X_B} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B) \to \ldots \\ s & \longmapsto & (i^A_{A\cap B}(s), -i^B_{A\cap B}(s)) & & & & \\ & & s & \longmapsto & i^X_A(s) + i^X_B(s) & & \end{array}$$

Totally analogous statements also hold if we replace the homology groups of $A\cap B, A, B$ and X by the corresponding reduced homology groups.[15811582]

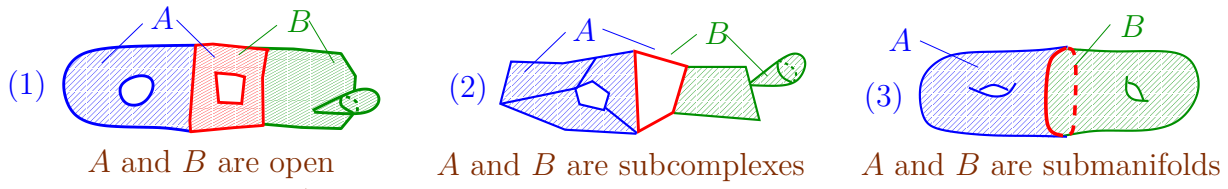
[1578]For smooth manifolds there is also a slight and useful variation: If X is a smooth manifold and if A, B are codimension-zero submanifolds with corner (in the sense of Definition 26.14) and if $A\cap B$ is a smooth boundary of A and B (in the sense of Definition 59.4), then the desired statements also hold. The only difference is that in the proof we need to replace the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7.

[1579]Note that by the Compact-Closed Lemma 3.39 (2) this condition is satisfied if A and B are compact.

[1580]Note that in the sequence precisely one of the inclusion induced maps has a minus sign. It does not really matter where it is, as long as there is precisely one. Our convention of $(i, -i)$ and $i \oplus i$ is also used in [**Hat2002**, p. 150] and [**Mun1984**, p. 187]. On the other hand [**Bre1993**, p. 229] uses (i, i) and $i \oplus -i$ and [**GrH1981**, p. 99] uses (i, i) and $-i \oplus i$.

[1581]Note though that for reduced homology groups we might need to use the fact, obtained in the Reduced Homology Lemma 147.3 (-1), that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

[1582]In many occasions some of the topological spaces $A, B, A\cap B$ or X will not be path-connected. In ordinary homology the homology is the direct sum of the homology groups of the components. But this is



$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$

Definition 153.3. The long exact sequence of the Mayer–Vietoris Theorem 153.2 is called the Mayer–Vietoris sequence of the decomposition $X = A \cup B$.

Remark. In general it is quite tricky to determine the connecting homomorphism in the Mayer–Vietoris sequence. One elementary calculation is given in Exercise 153.3 and one convenient calculation is provided by the Manifold–MV-Connecting Homomorphism Corollary 194.16. □

Proof. First note that it follows immediately from the definitions and the LES-of-Pair Proposition 146.7 (2) that the connecting homomorphism $\mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ is natural in a suitable sense. It remains to show that the given sequence is exact. There are two approaches to proving this statement:

(1) One can deduce the Mayer–Vietoris Theorem 153.2 directly from the Excision Theorem 148.1 using an algebraic manipulation. We will carry out this approach in Subsection 153.1.1.
(2) One can prove the Mayer–Vietoris Theorem 153.2 using the Comfortable Cover Proposition 148.9. We will carry out this approach in Subsection 153.1.2.

The first proof is slicker and it has the advantage that it applies to other functors, besides singular homology, that also satisfy a suitable version of the Excision Theorem 148.1. On the other hand the second proof makes it arguably clearer how the long exact sequence arises. ■

Example. Suppose we are in the setting of the Mayer–Vietoris Theorem 153.2 (1) with $X = \mathring{A} \cup \mathring{B}$. It follows easily from the definitions that in this setting the connecting homomorphism

$$\partial_n \colon \mathrm{H}_n(X) \;\to\; \mathrm{H}_{n-1}(A \cap B)$$

is characterized by the following property: For every $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ such that $a + b \in \mathrm{C}_n(X)$ is a cycle we have[1583]

$$\partial_n([a+b]) \;=\; [\,\partial_n(a)\,] \;=\; -\,[\,\partial_n(b)\,] \;\in\; \mathrm{H}_{n-1}(A \cap B).$$

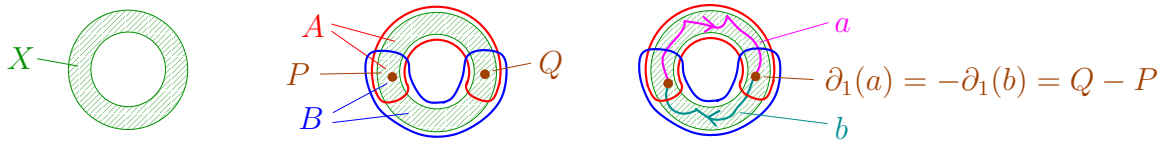
In the figure below we consider the annulus X together with two closed subsets A and B with $X = \mathring{A} \cup \mathring{B}$. Furthermore we show singular chains $a \in \mathrm{C}_1(A)$ and $b \in \mathrm{C}_1(B)$ such that

---

not the case for reduced homology. Thus, if one of the topological spaces involved is not path-connected it is much safer to stick to ordinary homology.

[1583] In the expression "$\partial_n(a)$" we mean by ∂_n the boundary map $\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$. Since $\partial_n(a+b) = 0$ we have $\partial_n(a) = -\partial_n(b)$. Since $\partial_n(a) \in \mathrm{C}_{n-1}(A) \subseteq \mathrm{C}_{n-1}(X)$ and since $\partial_n(b) \in \mathrm{C}_{n-1}(B) \subseteq \mathrm{C}_{n-1}(X)$ we see that $\partial_n(a) \in \mathrm{C}_{n-1}(A) \cap \mathrm{C}_{n-1}(B) = \mathrm{C}_{n-1}(A \cap B) \subseteq \mathrm{C}_{n-1}(A)$. A similar statement holds for $\partial_n(b)$.

$a+b$ form a cycle in $\mathrm{C}_1(X)$. By the above, the connecting homomorphism of the Mayer–Vietoris sequence sends $[a+b] \in \mathrm{H}_1(X)$ to $[\partial_1(a)] = -[\partial_1(b)] = [Q-P] \in \mathrm{H}_0(A \cap B)$.



We see that the Mayer–Vietoris sequence in reduced homology looks as follows:

$$\overbrace{\widetilde{\mathrm{H}}_1(A\cap B)}^{=0} \to \overbrace{\widetilde{\mathrm{H}}_1(A)}^{=0} \oplus \overbrace{\widetilde{\mathrm{H}}_1(B)}^{=0} \to \widetilde{\mathrm{H}}_1(X) \xrightarrow{\partial_1} \overbrace{\widetilde{\mathrm{H}}_0(A\cap B)}^{=\mathbb{Z}\cdot[Q-P]} \to \overbrace{\widetilde{\mathrm{H}}_0(A)}^{=0} \oplus \overbrace{\widetilde{\mathrm{H}}_0(B)}^{=0} \to \widetilde{\mathrm{H}}_0(X) \to 0.$$
$$[a+b] \mapsto [Q-P]$$

Note that A and B are contractible and note that the two path-components of $A \cap B$ are contractible. Thus we can use the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 and the Path-Component–H_*–Lemma 143.21 to determine the reduced homology groups of A and B as indicated in the exact sequence above. We have thus shown that $\mathrm{H}_1(X) \cong \mathbb{Z}$ and that $a+b$ represents a generator of $\mathrm{H}_1(X)$. □

Remark. Suppose that we are given a topological space X together with an open cover $\{U_i\}_{i\in I}$. If the cover consists of precisely two open sets U_1 and U_2, then the Mayer–Vietoris Theorem 153.2 (1) relates the homology groups of X, U_1, U_2 and the intersection $U_1 \cap U_2$. If the open cover consists of more than two sets, then the situation is much more tricky. In this more general setting the homology groups of X are related to the homology groups of the U_i and the homology groups of all finite intersections $U_{i_1} \cap \cdots \cap U_{i_m}$ via the "Mayer–Vietoris spectral sequence". We refer to [**LSVJ2011, Staf2015**], [**Dim2004**, Remark 2.3.9], [**TC2019**, Chapter 3] and [**BoT1982**, Chapter 15] for details. It is also worth having a look at Exercise 153.29. □

153.1.1. **First proof of the Mayer–Vietoris Theorem 153.2.** Our first proof of the Mayer–Vietoris Theorem 153.2 rests on the shoulders of the following purely algebraic lemma.

Lemma 153.4. (Barratt-Whitehead Lemma) We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{i_n} & C_n & \xrightarrow{p_n} & C''_n & \xrightarrow{\partial_n} & C'_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n-1}} & & \\
\dots & \xrightarrow{\partial_{n+1}} & D'_n & \xrightarrow{j_n} & D_n & \xrightarrow{q_n} & D''_n & \xrightarrow{\partial_n} & D'_{n-1} & \longrightarrow & \dots
\end{array}$$

We suppose that the horizontal sequences are exact and we suppose that each $f''_n \colon C''_n \to D''_n$ is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n \colon D_n \to C'_{n-1}$. The following sequence is exact:

$$\dots \longrightarrow C'_n \xrightarrow{(i_n,-f'_n)} C_n \oplus D'_n \xrightarrow{f_n \oplus j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \dots$$

Proof of the Barratt-Whitehead Lemma 153.4. Let us prove exactness at D_n. First we show that $\Delta_n \circ (f_n \oplus j_n) = 0$. Thus let $(c,d) \in C_n \oplus D'_n$. Then

$$\begin{array}{rcl} (\Delta_n \circ (f_n \oplus j_n))((c,d)) & \overset{\text{definition of } \Delta_n}{=} & (\partial_n \circ (f''_n)^{-1} \circ q_n)(f_n(c) + j_n(d)) \\ & = & (\partial_n \circ (f''_n)^{-1} \circ f''_n \circ p_n)(c) \\ & \uparrow & \\ & \text{since the diagram commutes} & \\ & \text{we have } q_n \circ f_n = f''_n \circ p_n & \end{array} \quad \begin{array}{rcl} \overset{\text{since } q_n \circ j_n = 0}{=} & (\partial_n \circ (f''_n)^{-1} \circ q_n \circ f_n)(c) \\ = & (\partial_n \circ p_n)(c) \overset{\uparrow}{=} 0. \\ & \text{since } \partial_n \circ p_n = 0 \end{array}$$

Now suppose we have $d \in D_n$ with $\Delta_n(d) = 0$. Since $0 = \Delta_n(d) = \partial_n((f''_n)^{-1} \circ q_n)(d))$ and since the top horizontal sequence is exact we see that there exists a $c \in C_n$ with $p_n(c) = ((f''_n)^{-1} \circ q_n)(d)$. Next we see that

$$q_n(\underbrace{f_n(c) - d}_{\in D_n}) = (q_n \circ f_n)(c) - q_n(d) \overset{\text{since the diagram commutes}}{=} (f''_n \circ p_n)(c) - q_n(d) \overset{\text{choice of } c}{=} 0.$$

It follows from this calculation and the hypothesis that the bottom horizontal sequence is exact that there exists a $d' \in D'_n$ with $j_n(d') = f_n(c) - d$. Finally we calculate that

$$(f_n \oplus j_n)((c, -d')) = f_n(c) - j_n(d') = f_n(c) - (f_n(c) - d) = d.$$

We have thus shown that the given sequence is exact at D_n. The proof of the exactness at C'_n and $C_n \oplus D'_n$ is done in a similar fashion. We outsource the proofs to Exercise 153.1. ■

With the Barratt-Whitehead Lemma 153.4 at our disposal we can easily prove the Mayer–Vietoris Theorem 153.2:

Proof of the Mayer–Vietoris Theorem 153.2. Let X be a topological space. Furthermore let $A, B \subseteq X$ be two subspaces. We assume that one of the following three conditions holds:

(1) $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subseteq X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.

We consider the following commutative diagram of continuous maps between pairs of topological spaces:

$$\begin{array}{ccccc} (A \cap B, \varnothing) & \xrightarrow{i^A_{A\cap B}} & (A, \varnothing) & \longrightarrow & (A, A \cap B) \\ \downarrow{\scriptstyle i^B_{A\cap B}} & & \downarrow{\scriptstyle i^X_A} & & \downarrow{\scriptstyle i^{(X,B)}_{(A,A\cap B)}} \\ (B, \varnothing) & \xrightarrow{i^X_B} & (X, \varnothing) & \longrightarrow & (X, B). \end{array}$$

From the LES-of-Pair Proposition 146.7 we get two long exact sequences which, together with the vertical maps, form the following commutative diagram:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}_n(A\cap B) & \xrightarrow{i^A_{A\cap B}} & \mathrm{H}_n(A) & \xrightarrow{\text{я}} & \mathrm{H}_n(A, A\cap B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle i^B_{A\cap B}} & & \downarrow{\scriptstyle i^X_A} & & \downarrow{\scriptstyle i^{(X,B)}_{(A,A\cap B)}} & & \downarrow{\scriptstyle i^B_{A\cap B}} & & \\
\dots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i^X_B} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \longrightarrow & \dots.
\end{array}$$

Depending on the setting we are in we obtain from

(1) the Excision Theorem 148.1 (applied with $Z = X \setminus A$), or
(2) the Manifold–Excision Theorem 148.3, or
(3) the CW-Excision Theorem 148.4

that the inclusion-induced map $i^{(X,B)}_{(A,A\cap B)} \colon \mathrm{H}_n(A, A\cap B) \to \mathrm{H}_n(X, B)$ is an isomorphism. We apply the Barratt-Whitehead Lemma 153.4 to the above setup and we obtain that the following sequence is exact:

$$\dots \to \mathrm{H}_n(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n \circ (i^{(X,B)}_{(A,A\cap B)})^{-1} \circ \text{я}} \mathrm{H}_{n-1}(A\cap B) \to \dots$$

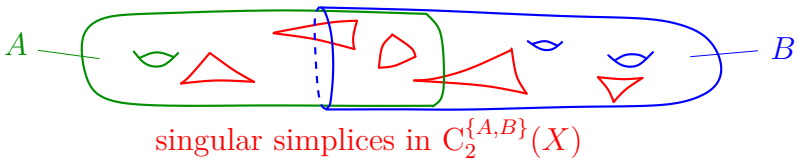
But that was exactly our goal. Finally note that by replacing the LES-of-Pair Proposition 146.7 by the Reduced LES-of-Pair Proposition 147.8 we get the analogous result for reduced homology groups. ∎

153.1.2. **Second proof of the Mayer–Vietoris Theorem 153.2.** In this section we give our second proof of the Mayer–Vietoris Theorem 153.2. Here it is:

Proof of the Mayer–Vietoris Theorem 153.2. Let X be a topological space.

(1) First we suppose that we are given two subsets $A, B \subseteq X$ such that $X = \mathring{A} \cup \mathring{B}$. We recall Definition 148.8:

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{a + b \,|\, a \in \mathrm{C}_n(A) \text{ and } b \in \mathrm{C}_n(B)\}.$$



The inclusion maps induce the following sequence of chain maps:

$$0 \to \mathrm{C}_*(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0.$$

Claim. The above sequence of chain maps is exact.

Proof. We prove the claim in the following four easy steps:
(a) It is clear that each of the maps $\mathrm{C}_*(A\cap B) \to \mathrm{C}_*(A)$ and $\mathrm{C}_*(A\cap B) \to \mathrm{C}_*(B)$ is already injective, in particular the map $(i^A_{A\cap B}, -i^B_{A\cap B})$ is injective.
(b) It is an immediate consequence of the definition of $\mathrm{C}_*^{\{A,B\}}(X)$ that the map $i^X_A \oplus i^X_B$ is surjective.

(c) It follows immediately from the definitions that the composition of both maps is the zero map,[1584] i.e. $\operatorname{im}((i^A_{A\cap B}, -i^B_{A\cap B})) \subseteq \ker(i^X_A \oplus i^X_B)$.

(d) It remains to show that $\ker(i^X_A \oplus i^X_B) \subseteq \operatorname{im}((i^A_{A\cap B}, -i^B_{A\cap B}))$. Thus let $x \in \mathrm{C}_n(A)$ and $y \in \mathrm{C}_n(B)$ such that $(i^X_A \oplus i^X_B)((x,y)) = 0$. By definition this means that x and y, viewed as elements in $\mathrm{C}_n(X)$, satisfy $x = -y$. This means that x lies in $\mathrm{C}_n(A)$ and also in $\mathrm{C}_n(B)$. Put differently, $x \in \mathrm{C}_n(A) \cap \mathrm{C}_n(B) = \mathrm{C}_n(A \cap B)$. But then we have $((i^A_{A\cap B}, -i^B_{A\cap B}))(x) = x + y$. ⊞

Next we consider the following maps:

$$\begin{array}{ccccccccc} \dots \longrightarrow \mathrm{H}_n(A\cap B) & \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} & \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) & \xrightarrow{i^X_A \oplus i^X_B} & \mathrm{H}^{\{A,B\}}_n(X) & \xrightarrow{p_n} & \mathrm{H}_{n-1}(A\cap B) \longrightarrow \dots \\ & \searrow^{i^X_A \oplus i^X_B} & & & \Phi_n \downarrow \cong & \nearrow_{:=\partial_n} & \\ & & & & \mathrm{H}_n(X) & & \end{array}$$

We make the following clarifications and observations:

(a) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{H}^{\{A,B\}}_n(X)$ the n-th homology group of the chain complex $\mathrm{C}^{\{A,B\}}_*(X)$ and we denote by $p_n \colon \mathrm{H}^{\{A,B\}}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ the connecting homomorphism of the above short exact sequence of chain complexes, see Definition 145.8.

(b) By the LES-Proposition 145.10 the top horizontal sequence is exact.

(c) It follows from the Comfortable Cover Proposition 148.9, together with our hypothesis that $X = \mathring{A} \cup \mathring{B}$, that for every $n \in \mathbb{N}_0$ the inclusion map

$$\mathrm{C}^{\{A,B\}}_*(X) \quad \to \quad \mathrm{C}_*(X)$$

induces an isomorphism

$$\Phi_n \colon \mathrm{H}^{\{A,B\}}_n(X) \quad \xrightarrow{\cong} \quad \mathrm{H}_n(X).$$

(d) For each $n \in \mathbb{N}_0$ we write $\partial_n := p_n \circ \Phi_n^{-1} \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$. Using the explicit description of the connecting homomorphisms in Definition 145.8 and in the LES-of-Pair Proposition 146.7 one can show easily that the map $\partial_n := p_n \circ \Phi_n^{-1} \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ equals the connecting homomorphism given in the theorem.

(2) Now we suppose that we are dealing with a CW-complex X and two subcomplexes A and B with $X = A \cup B$. It is worth recalling how we dealt with the related situation for fundamental groups: First we proved the Seifert–van Kampen-Theorem 99.1 for the decomposition of a topological space X into two *open* subsets. Afterwards in the proof of the CW-Seifert–van Kampen Theorem 137.10 we used the Subcomplex–Neighborhood Proposition 137.9 together with the Homotopy Equivalence-π_1-Proposition 90.9 to reduce the case of CW-complexes and subcomplexes to the case of an open cover. A similar approach, now using the Homotopy Equivalence-H_*-Proposition 144.14, can be used to prove the existence of the long exact sequence. We will not carry out the technical details for CW-complexes, since we already provided a complete proof of the Mayer–Vietoris Theorem 153.2 (2) in Section 153.1.1.

[1584] Note that here we use the minus sign that appears in the first map.

(3) Similarly to (2), as in the proof of the Submanifold–Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 85.12 and the Manifold–Seifert-van Kampen-Theorem 102.1, we can use the Topological Collar Neighborhood Theorem 35.14 to reduce the manifold case to the case of an open cover. As in (2) we leave the details to the reader.

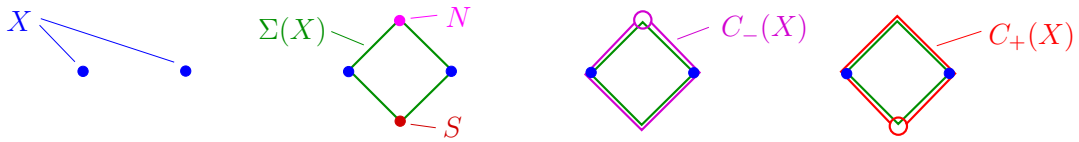
Finally note that the statements for reduced homology can be proved as above, we just need to replace $\mathrm{C}_*^{\{A,B\}}(X)$ by its reduced analog. Alternatively one can deduce the reduced long exact sequence from the above unreduced long exact sequence and the Reduced Homology Lemma 147.3. Again there is no point in providing the details. ■

153.2. **The Mayer–Vietoris sequence applied to suspensions.** To get used to dealing with the Mayer–Vietoris sequence we will give an alternative proof of a slightly simplified version of the Suspension-H_*-Proposition 149.3:

Proposition 149.3. (Suspension-H_*-Proposition) Given any topological space X and given any $k \in \mathbb{Z}$ we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X).$$

Proof. Let X be a topological space and let $k \in \mathbb{Z}$. As in Definition 149.2 we consider the subspaces $C_+ \coloneqq C_+(X) \coloneqq \Sigma(X) \setminus \{S\}$ and $C_- \coloneqq C_-(X) \coloneqq \Sigma(X) \setminus \{N\}$, where S and N denote the South and North Pole of the suspension $\Sigma(X)$. Note that $C_+(X)$ and $C_-(X)$ are both open subsets of $\Sigma(X)$ and that $\Sigma(X) = C_+(X) \cup C_-(X)$.



We consider the corresponding long exact sequence in reduced homology from the Mayer–Vietoris Theorem 153.2 (1):

$$\underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0} \longrightarrow \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\partial_{k+1}} \widetilde{\mathrm{H}}_k(C_+ \cap C_-) \longrightarrow \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0}$$

$$\widetilde{\mathrm{H}}_k(X), \quad \Sigma_X \colon \widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)), \quad \cong \uparrow i_* \colon \widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_k(C_+ \cap C_-)$$

We make the following clarifications and observations:

(1) As in the proof of the Cone Contractability Lemma 20.20 we see that both $C_+ = C_+(X)$ and $C_- = C_-(X)$ are contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that the reduced homology groups of $C_+(X)$ and $C_-(X)$ vanish.

(2) Next note that it follows immediately from (1) that the connecting homomorphism $\partial_{k+1} \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \to \widetilde{\mathrm{H}}_k(C_+(X) \cap C_-(X))$ is in fact an isomorphism.

(3) We consider the following little diagram:

$$X \xrightarrow{x \mapsto (x,0)} X \times (-1,1) \xrightarrow{(x,t) \mapsto [(x,t)]} C_+(X) \cap C_-(X), \qquad X \xrightarrow{=:i} C_+(X) \cap C_-(X).$$

The map to the left is evidently a homotopy equivalence. Furthermore one can easily show, either by hand, or using Lemma 149.14 (1), that the map to the right is a

homeomorphism. We obtain that the map $i\colon X \to C_+(X) \cap C_-(X)$ is a homotopy equivalence. Thus we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 (2) that the induced map $i_*\colon \widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_k(C_+(X) \cap C_-(X))$ is an isomorphism.

It follows from the above that we get an isomorphism $\Sigma_X\colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X)$. It follows easily from the definition of Σ_X and the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence (which is a consequence of the LES-of-Pair Proposition 146.7 (2)), that the isomorphism Σ_X is natural. ■

153.3. **The homology groups of a mapping torus.** For convenience we recall from Definition 21.19 the definition of the mapping torus:

Definition 153.5. Let X be a topological space and let $f\colon X \to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X, f) \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1)$$

as the mapping torus of (X, f).

Examples.

(1) Let X be a topological space. In the Mapping Torus-of-id-Lemma 21.22 we showed that the map

$$\begin{array}{rccc} \Phi\colon & \mathrm{Tor}(X, \mathrm{id}) & \to & X \times S^1 \\ & [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a homeomorphism.

(2) Let $X = [-1,1]$ and let $f\colon [-1,1] \to [-1,1]$ be the map that is given by $x \mapsto -x$.

The corresponding mapping torus $\mathrm{Tor}([-1,1], x \mapsto -x)$ is clearly homeomorphic to the Möbius band. □

In the Mapping Torus-π_1-Proposition 93.13 we determined the fundamental group of a given mapping torus. In many cases the following proposition allows us to determine the homology groups of a given mapping torus.

Proposition 153.6. (Mapping Torus-H_*-Proposition) Let X be a topological space and let $f\colon X \to X$ be a continuous map. We denote by $i\colon X \to \mathrm{Tor}(X, f)$ the natural map that is given by $x \mapsto [(x,0)]$. For every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n\colon \mathrm{H}_n(\mathrm{Tor}(X, f)) \;\to\; \mathrm{H}_{n-1}(X)$$

(which is described in greater detail in the proof of the proposition) such that the following sequence is exact[1585]

$$\cdots \to \mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{Tor}(X,f)) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X) \to \dots$$

In the proof of the Mapping Torus-H_*-Proposition 153.6 we will need the following purely algebraic lemma which can be viewed as a slight variation on the Exact Sequence–Cancellation Lemma 145.12:

[1585] It follows from the Break-into-SES Lemma 145.6 that we get the following short exact sequence:

$$0 \to \mathrm{coker}\left(\mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X)\right) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{Tor}(X,f)) \xrightarrow{\partial_n} \ker\left(\mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X)\right) \to 0.$$

Lemma 153.7. (Canceling Exact Sequence Lemma) Suppose we are given the following commutative diagram of homomorphisms of abelian groups:

$$\begin{array}{ccccccccccc}
\longrightarrow & \begin{matrix}A_n\\A_n\end{matrix} & \xrightarrow{\begin{pmatrix}\varphi_n & \psi_n\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}B_n\\A_n\end{matrix} & \xrightarrow{(\beta_n\ \alpha_n)} & C_n & \xrightarrow{\partial_n} & \begin{matrix}A_{n-1}\\A_{n-1}\end{matrix} & \xrightarrow{\begin{pmatrix}\varphi_{n-1} & \psi_{n-1}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}B_{n-1}\\A_{n-1}\end{matrix} & \longrightarrow\\
 & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}} & \\
\longrightarrow & A_n & \xrightarrow{\psi_n-\varphi_n} & B_n & \xrightarrow{\beta_n} & C_n & \xrightarrow{\widetilde{\partial}_n} & A_{n-1} & \xrightarrow{\psi_{n-1}-\varphi_{n-1}} & B_{n-1} & \longrightarrow
\end{array}$$

If the top sequence is exact, then there exist unique homomorphisms $\widetilde{\partial}_n\colon C_n \to A_{n-1}$ that make the diagram commute. With these homomorphisms the bottom sequence is also exact.

Proof of the Canceling Exact Sequence Lemma 153.7. First note that we know that the composition of the two top right homomorphisms is zero. It follows that for any $c \in C_n$ there exists a unique $a \in A_{n-1}$ such that $\partial_n(c) = \begin{pmatrix}-a\\ a\end{pmatrix}$. We now set $\widetilde{\partial}_n(c) := a$. One can easily see that this is the unique homomorphism that makes the diagram commute.

It remains to show that the bottom sequence is exact. One can do so directly, using a diagram chase. In the following we perform a little trick to outsource the diagram chase to a case we already dealt with: We consider the following vertically expanded diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & A_n & \xrightarrow{-\mathrm{id}} & A_n & \longrightarrow & 0 & \longrightarrow & A_{n-1} & \xrightarrow{-\mathrm{id}} & A_{n-1} & \longrightarrow\\
 & \uparrow{\scriptstyle (\mathrm{id}\ \ \mathrm{id})} & & \uparrow{\scriptstyle (0\ \ \mathrm{id})} & & \uparrow & & \uparrow{\scriptstyle (\mathrm{id}\ \ \mathrm{id})} & & \uparrow{\scriptstyle (0\ \ \mathrm{id})} & \\
\longrightarrow & \begin{matrix}A_n\\A_n\end{matrix} & \xrightarrow{\begin{pmatrix}\varphi_n & \psi_n\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}B_n\\A_n\end{matrix} & \xrightarrow{(\beta_n\ \alpha_n)} & C_n & \xrightarrow{\partial_n} & \begin{matrix}A_{n-1}\\A_{n-1}\end{matrix} & \xrightarrow{\begin{pmatrix}\varphi_{n-1} & \psi_{n-1}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}B_{n-1}\\A_{n-1}\end{matrix} & \longrightarrow\\
 & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}} & \\
\longrightarrow & A_n & \xrightarrow{\psi_n-\varphi_n} & B_n & \xrightarrow{\beta_n} & C_n & \xrightarrow{\widetilde{\partial}_n} & A_{n-1} & \xrightarrow{\psi_{n-1}-\varphi_{n-1}} & B_{n-1} & \longrightarrow
\end{array}$$

By construction the bottom part of the diagram commutes. It follows easily from the definitions that the top part also commutes. Furthermore note that the vertical sequences, with 0 added on top and bottom, are exact.

Next note that evidently the top sequence is exact. Furthermore by hypothesis the middle sequence is exact. Since the bottom vertical maps are monomorphisms we see that the bottom sequence is a chain complex.

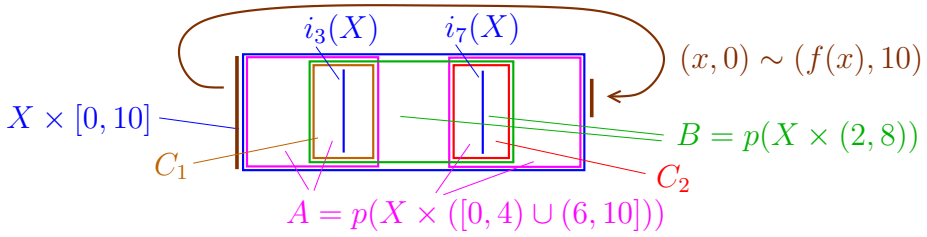
It follows from the above discussion that all these maps form a "vertical" short exact sequence of "horizontal" chain complexes. Clearly the homology groups of the top chain complex are zero. By hypothesis the homology groups of the middle chain complex are also zero. It follows from the LES-Proposition 145.10 that the homology groups of the bottom chain complex are also zero. But this means that the bottom sequence is exact. ∎

Proof of the Mapping Torus-H_*-Proposition 153.6. Let X be a topological space and let $f\colon X \to X$ be a continuous map. For notational convenience we work in this proof with the following topological space which is clearly naturally homeomorphic to the mapping torus:

$$M \ := \ (X \times [0,10])/(x,0) \sim (f(x),10).$$

We introduce the following notation:

(1) We denote by $p\colon X \times [0,10] \to M = (X \times [0,1])/\sim$ the natural projection.
(2) We consider the subsets $A \coloneqq p(X \times ([0,4) \cup (6,10])) \subseteq M$ and $B \coloneqq p(X \times (2,8)) \subseteq M$.
(3) We consider the two subsets $C_1 = p(X \times (2,4))$ and $C_2 = p(X \times (6,8))$. Note that $A \cap B = C_1 \cup C_2$.
(4) Given $t \in [0,10]$ we consider the map
$$\begin{array}{rccc} i_t\colon & X & \to & p(X \times \{t\}) \\ & x & \mapsto & p(x,t). \end{array}$$
(5) We write $i_3^A\colon X \to A$ for the composition of $i_3\colon X \to p(X \times \{3\})$ with the inclusion map $p(X \times \{3\}) \to A$. We denote analogous compositions by the obvious variations in the notation.
(6) To simplify the notation we drop the "$*$" when we write down the corresponding induced maps on homology.



We can now consider the following diagram:[1586]

$$\begin{array}{ccccccccccc}
\to & \begin{matrix}\mathrm{H}_n(C_1)\\ \mathrm{H}_n(C_2)\end{matrix} & \xrightarrow{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2}\end{pmatrix}} & \begin{matrix}\mathrm{H}_n(A)\\ \mathrm{H}_n(B)\end{matrix} & \xrightarrow{(i^M_A \;\; i^M_B)} & \mathrm{H}_n(M) & \xrightarrow{\partial_n} & \begin{matrix}\mathrm{H}_{n-1}(C_1)\\ \mathrm{H}_{n-1}(C_2)\end{matrix} & \xrightarrow{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2}\end{pmatrix}} & \begin{matrix}\mathrm{H}_{n-1}(A)\\ \mathrm{H}_{n-1}(B)\end{matrix} & \to \\
 & \uparrow{\scriptstyle \begin{pmatrix} i_3 & 0 \\ 0 & i_7\end{pmatrix}}_{\cong} & & \uparrow{\scriptstyle \cong \begin{pmatrix} i_7 & 0 \\ 0 & i_7\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} i_3 & 0 \\ 0 & i_7\end{pmatrix}}_{\cong} & & \uparrow{\scriptstyle \cong \begin{pmatrix} i_7 & 0 \\ 0 & i_7\end{pmatrix}} & \\
\to & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} & \xrightarrow{\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} & \xrightarrow{(i_7 \;\; i_7)} & \mathrm{H}_n(M) & \xrightarrow{=:\partial_n} & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} & \xrightarrow{\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} & \to \\
 & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0\end{pmatrix}} & \\
\to & \mathrm{H}_n(X) & \xrightarrow{\mathrm{id} - f_*} & \mathrm{H}_n(X) & \xrightarrow{i_7} & \mathrm{H}_n(M) & \xrightarrow{\widetilde{\partial}_n} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id} - f_*} & \mathrm{H}_{n-1}(X) & \to
\end{array}$$

[1586]Here we write
$$\begin{matrix}\mathrm{H}_i(\mathrm{C}_1)\\ \mathrm{H}_i(\mathrm{C}_2)\end{matrix} \text{ instead of } \mathrm{H}_i(\mathrm{C}_1) \oplus \mathrm{H}_i(\mathrm{C}_2) \quad \text{and} \quad \begin{matrix}\mathrm{H}_i(A)\\ \mathrm{H}_i(B)\end{matrix} \text{ instead of } \mathrm{H}_i(A) \oplus \mathrm{H}_i(B).$$
The reason is that we think of direct sums as columns vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

We make the following clarifications and observations:

(1) Note that $A = p(X \times ([0,4) \cup (6,10])) \subseteq M$ and $B = p(X \times (2,8)) \subseteq M$ are open subsets of M that cover all of M. Furthermore note that we have $A \cap B = C_1 \sqcup C_2$.[1587] On the top of the diagram we wrote down the corresponding exact Mayer–Vietoris sequence from the Mayer–Vietoris Theorem 153.2 (1).

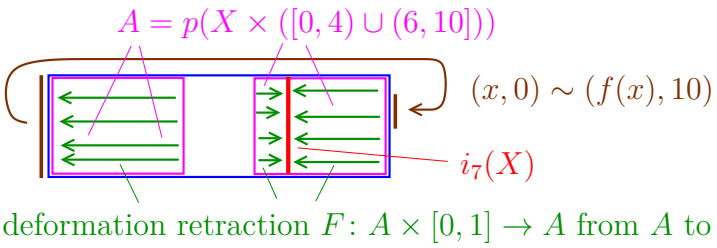
(2) We want to show that the vertical maps from the second to the first row are isomorphisms. In fact this statement follows immediately from the following claim together with the Homotopy Equivalence-H_*-Proposition 144.14 (1a).

Claim. The maps $i_3 \colon X \to C_1$, $i_7 \colon X \to C_2$, $i_7 \colon X \to B$ and $i_7 \colon X \to A$ are homotopy equivalences.

Proof. One can easily verify that the maps $i_t \colon X \to p(X \times \{t\})$, $p \colon X \times (2,4) \to C_1$, $p \colon X \times (6,8) \to C_2$ and $p \colon X \times (2,8) \to B$ are homeomorphisms. Using this observation one can easily verify that the first three maps are homotopy equivalences.

If f is not a homeomorphism, then A is not necessarily homeomorphic to a product. Thus we need to work somewhat harder. Let us consider the following map:

$$\begin{aligned} F \colon A \times [0,1] &\to A \\ ([(x,t)],s) &\mapsto \begin{cases} [(x,t)], & \text{if } s \in [0,\tfrac{1}{2}] \text{ and } t \in (6,10], \\ [(x, t\cdot(1-2s))], & \text{if } s \in [0,\tfrac{1}{2}] \text{ and } t \in [0,4), \\ [(x, t\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\tfrac{1}{2},1] \text{ and } t \in (6,10], \\ [(f(x), 10\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\tfrac{1}{2},1] \text{ and } t \in [0,4). \end{cases} \end{aligned}$$



First note that it follows from $(x,0) \sim (f(x),10)$ that this map F is well-defined. One can now easily verify, say using the Pasting Proposition 4.7, that the map F is continuous. It is now clear that the above map F is a deformation retraction from A to $i_7(X)$. This implies, by the Deformation Retract–Homotopy Equivalence Lemma 19.10, that $i_7 \colon X \to A$ is a homotopy equivalence. ⊞

(3) Next we want to show that the top part of the diagram commutes.

(a) We start with the square to the left. We need to show that the following equality holds:

$$\underbrace{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}\begin{pmatrix} i_3^{C_1} & 0 \\ 0 & i_7^{C_2} \end{pmatrix}}_{=\begin{pmatrix} i_3^A & i_7^A \\ -i_3^B & -i_7^B \end{pmatrix}} = \underbrace{\begin{pmatrix} i_7^A & 0 \\ 0 & i_7^B \end{pmatrix}\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}}_{=\begin{pmatrix} i_7^A \circ f_* & i_7^A \circ \mathrm{id} \\ i_7^B \circ (-\mathrm{id}) & i_7^B \circ (-\mathrm{id}) \end{pmatrix}}.$$

[1587] It follows easily from the Disjoint Union Embedding Lemma 7.29 that the subspace topology on $C_1 \cup C_2$ coming from M agrees with the disjoint union topology. In particular it follows from the Path-Component-H_*-Lemma 143.21 that we have a natural isomorphism $\mathrm{H}_n(A \cap B) = \mathrm{H}_n(C_1 \sqcup C_2) \xleftarrow{\cong} \mathrm{H}_n(C_1) \oplus \mathrm{H}_n(C_2)$.

If we compare the two matrices we see that it remains to prove the following claim:

Claim. For each $n \in \mathbb{N}_0$ we have $i_3^A = i_7^A \circ f_* \colon \mathrm{H}_n(X) \to \mathrm{H}_n(A)$ and we also have $i_3^B = i_7^B \colon \mathrm{H}_n(X) \to \mathrm{H}_n(B)$.

Proof. We consider the maps[1588]

$$\begin{aligned} X \times [0,1] &\to A \\ (x,t) &\mapsto \begin{cases} p(x, 3-6t), & \text{if } t \in [0, \frac{1}{2}], \\ p(f(x), 13-6t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{aligned} \quad \& \quad \begin{aligned} X \times [0,1] &\to B \\ (x,t) &\mapsto p(x, 3\cdot(1-t) + 7\cdot t). \end{aligned}$$

These two maps are homotopies between the maps $i_3^A, i_7^A \circ f \colon X \to A$ and the maps $i_3^B, i_7^B \colon X \to B$. The claim now follows from the Homotopy-H_*-Proposition 144.12. ⊞

(b) The second square commutes by the functoriality of homology groups.

(c) Recall that in (1) we just showed that the vertical maps from the second to the first row are isomorphisms. Therefore we can define the connecting homomorphism $\partial_n \colon \mathrm{H}_n(M) \to \begin{smallmatrix} \mathrm{H}_{n-1}(X) \\ \mathrm{H}_{n-1}(X) \end{smallmatrix}$ in the second row in such a way that the third square commutes.

(4) It follows from the above that the middle horizontal sequence is exact.

(5) It follows from (4) and the Canceling Exact Sequence Lemma 153.7 that there exists a unique homomorphism $\widetilde{\partial}_n \colon \mathrm{H}_n(M) \to \mathrm{H}_{n-1}(X)$ that makes the bottom part of the diagram commute and that with this homomorphism the bottom horizontal sequence is exact.

We have thus completed the proof that the bottom horizontal sequence of the diagram on page 2911 is exact. If one goes carefully through the definitions one sees that the connecting homomorphism $\partial_n \colon \mathrm{H}_n(\mathrm{Tor}(X,f)) \to \mathrm{H}_{n-1}(X)$ is given by the composition of the homomorphisms in the following unreadable sequence of homomorphisms:[1589]

$$\begin{aligned} \mathrm{H}_n(\mathrm{Tor}(X,f)) &\xrightarrow{\text{я}} \mathrm{H}_n(\mathrm{Tor}(X,f), A) \xleftarrow{\cong} \mathrm{H}_n(B, C_1 \cup C_2) \\ &\xrightarrow{\partial_n} \mathrm{H}_{n-1}(C_1 \cup C_2) \xleftarrow{\cong} \mathrm{H}_{n-1}(C_1) \oplus \mathrm{H}_{n-1}(C_2) \to \mathrm{H}_{n-1}(C_2) \underset{i_7^{C_2}}{\overset{\cong}{\leftarrow}} \mathrm{H}_{n-1}(X). \end{aligned}$$ ■

Now we can easily prove the following proposition, which can be viewed as an analog of the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 153.8.

Proposition 153.8. (Product-with-S^1-H_*-Proposition) Let X be a topological space. We denote by $p \colon X \times S^1 \to X$ the natural projection. For every $n \in \mathbb{N}_0$ there exists a natural[1590] homomorphism $\psi_X \colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$ (which will be described in greater detail in the proof) such that the map

$$\Xi_X \coloneqq p_* \oplus \psi_X \colon \mathrm{H}_n(X \times S^1) \xrightarrow{\cong} \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X)$$

is an isomorphism.[1591]

[1588]Note that it follows from $(x,0) \sim (f(x),10)$ that the map to the left is well-defined. It now follows from the Homotopy Combination Lemma 18.8 that the map to the left is indeed continuous.

[1589]The second homomorphism is an isomorphism by the Excision Theorem 148.1, the third homomorphism is the connecting homomorphism of the given pair of topological spaces, the fourth homomorphism is the isomorphism given by the Path-Component-H_*-Lemma 143.21, the fifth homomorphism is the natural projection and the last isomorphism is induced by the map $i_7^{C_2} \colon X \to C_2 = p(X \times (\frac{6}{10}, \frac{8}{10}))$, which is easily seen to be a homotopy equivalence.

Proof. Let $n \in \mathbb{N}_0$. Recall that by the Mapping Torus-of-id Lemma 21.22 the map

$$\begin{array}{rccc} \Theta\colon & \mathrm{Tor}(X,\mathrm{id}) & \to & X \times S^1 \\ & [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}\,t)) \end{array}$$

is a natural homeomorphism. We denote by $i\colon X \to \mathrm{Tor}(X,\mathrm{id})$ the natural map that is given by $x \mapsto [(x,0)]$ and we denote by $j\colon X \to X \times S^1$ the natural embedding that is given by $x \mapsto (x,1)$. Next we consider the following diagram:

$$\begin{array}{ccccccccc} \mathrm{H}_n(X) & \xrightarrow{\mathrm{id}_* - \mathrm{id}_*} & \mathrm{H}_n(X) & \xrightarrow{i_*} & \mathrm{H}_n(\mathrm{Tor}(X,\mathrm{id})) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id}_* - \mathrm{id}_*} & \mathrm{H}_{n-1}(X) \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \cong\downarrow{\scriptstyle \Theta_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{H}_n(X) & \underset{p_*}{\overset{j_*}{\rightleftarrows}} & \mathrm{H}_n(X \times S^1) & \xrightarrow{\partial_n \circ \Theta_*^{-1}} & \mathrm{H}_{n-1}(X) & \longrightarrow & 0. \end{array}$$

We make the following clarifications and observations:

(1) The top sequence is the short exact sequence that we obtain from the Mapping Torus-H_*-Proposition 153.6.

(2) Since $\mathrm{id}_* - \mathrm{id}_* = 0$ we see that the homomorphisms to the top left and top right are the zero map.

(3) Since Θ is a homeomorphism and since $\Theta \circ i = j$ we obtain from (1) and (2) that the bottom sequence is also exact.

(4) The map $p\colon X \times S^1 \to X$ is the natural map that is given by projection onto the first factor. Note that $p_* \circ j_* = (p \circ j)_* = \mathrm{id}$.

(5) We set $\psi_X := \partial_n \circ \Theta_*^{-1}\colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$.

Finally note that as in the proof of the Splitting Lemma 145.5 (2)⇒(3) one can show that the map

$$\begin{array}{rccc} \Xi_X\colon & \mathrm{H}_n(X \times S^1) & \to & \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X) \\ & \sigma & \mapsto & p_*(\sigma) \oplus \underbrace{(\partial_n \circ \Theta_*^{-1})(\sigma)}_{=\psi_X(\sigma)} \end{array}$$

is an isomorphism. ∎

Using the Product-with-S^1-H_*-Proposition 153.8 we can now finally compute the homology groups of tori $(S^1)^n$:

Proposition 153.9. (Torus-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For any $k \in \mathbb{N}_0$ we have[1592]

$$\mathrm{H}_k(n\text{-dimensional torus } (S^1)^n) \;=\; \text{free abelian group of rank } \binom{n}{k}.$$

[1590]Here natural means that for any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X \times S^1) & \xrightarrow[\cong]{\psi_X} & \mathrm{H}_{n-1}(X) \\ \downarrow{\scriptstyle (f \times \mathrm{id})_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}_n(Y \times S^1) & \xrightarrow[\cong]{\psi_Y} & \mathrm{H}_{n-1}(Y). \end{array}$$

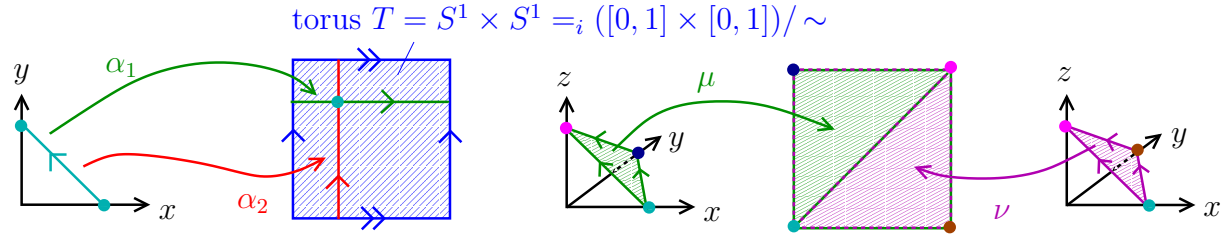
[1591]The direct sum of natural homomorphisms is of course also natural.

(2) For $i = 1, \dots, n$ we consider the singular 1-cycles

$$\begin{array}{rcl} \alpha_i \colon \Delta^1 & \to & (S^1)^n \\ (1-t,t) & \mapsto & (1, \dots, 1, \underbrace{\exp(2\pi i t)}_{\substack{\uparrow \\ i\text{-th coordinate}}}, 1, \dots, 1). \end{array}$$

These singular 1-cycles represent a basis for $\mathrm{H}_1((S^1)^n) \cong \mathbb{Z}^n$.

Example. Let $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$ be the 2-dimensional torus.



It follows from the Torus-H_*-Proposition 153.9 and the Detect-Cycle Proposition 150.5 that

$$\mathrm{H}_n(T) \;=\; \begin{cases} \mathbb{Z} \cdot [P], & \text{if } n = 0, \\ \mathbb{Z} \cdot [\alpha_1] \oplus \mathbb{Z} \cdot [\alpha_2], & \text{if } n = 1, \\ \mathbb{Z} \cdot [\mu - \nu], & \text{if } n = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $P \in T$ is any point and where $\alpha_1, \alpha_2 \colon \Delta^1 \to T$ and $\mu, \nu \colon \Delta^2 \to T$ are indicated in the figure above.[1593] □

Proof.

(1) We prove this statement by induction on n. For $n = 0$ this statement follows from the Singleton-H_*-Lemma 143.26. The induction step is a straightforward consequence of the Product-with-S^1-H_*-Proposition 153.8 together with the recurrence relation

$$\binom{n}{k} \;=\; \binom{n-1}{k} + \binom{n-1}{k-1}$$

from page 111. We leave it to the reader to fill in the details.

(2) For $i = 1, \dots, n$ let $p_i \colon (S^1)^n \to S^1$ be the projection onto the i-th factor. We consider the homomorphism

$$\begin{array}{rcl} \Psi \colon \mathrm{H}_1((S^1)^n) & \to & \mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1) \\ \alpha & \mapsto & (p_{1*}(\alpha), \dots, p_{n*}(\alpha)) \end{array}$$

We make the following observations:

(a) By Statement (1) we know that Ψ is a homomorphism between abelian groups that are both isomorphic to $\mathbb{Z}^n$.

(b) On page 2866 we showed that the singular 1-cycle $\alpha \colon \Delta^1 \to S^1$ that is given by $(t, 1-t) \mapsto \exp(2\pi i t)$ represents a generator of $\mathrm{H}_1(S^1)$.

[1592] We refer to Definition 1.43 for the definition of $\binom{n}{k}$.

[1593] In particular this answers our question from page 2755, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $\mathrm{H}_2(T)$.

(c) It follows easily from (b) and the discussion on page 2759 that Ψ is an epimorphism that sends $[\alpha_1], \dots, [\alpha_n]$ to a basis of $\mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1)$.
(d) It follows from (a) and (c), together with the Epi-Automorphism Lemma 97.27, that Ψ is an isomorphism.
(e) It follows from (c) and (d) that $[\alpha_1], \dots, [\alpha_n]$ is a basis of $\mathrm{H}_1((S^1)^n)$. ■

The Mapping Torus-H_*-Proposition 153.6 also provides us with the tools for computing the homology groups of the Klein bottle:

Lemma 153.10. (Klein Bottle-H_*-Lemma) Let

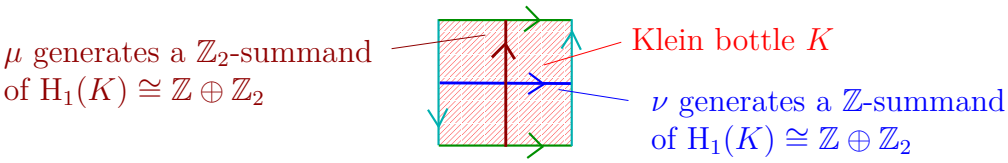
$$K \;=\; ([0,1]\times[0,1])/\!\sim \qquad \text{where } (x,0)\sim(x,1) \text{ and } (0,y)\sim(1,1-y)$$

be the Klein bottle as defined on page 312. We consider the two singular 1-cycles

$$\begin{array}{rcl} \mu\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \nu\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(t,\frac{1}{2})] \end{array}$$

For any $P \in K$ we have[1594]

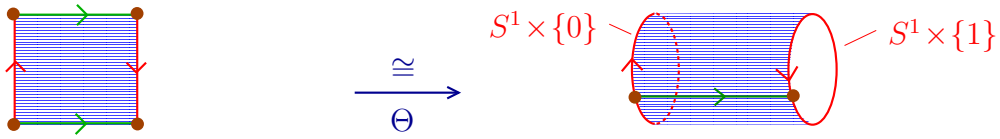
$$\mathrm{H}_n(K) \;=\; \begin{cases} \mathbb{Z}\cdot[P], & \text{if } n=0, \\ \mathbb{Z}/2\mathbb{Z}\cdot[\mu]\oplus\mathbb{Z}\cdot[\nu], & \text{if } n=1, \\ 0, & \text{otherwise.} \end{cases}$$



Proof. We consider $X = S^1$ and we consider the self-homeomorphism $f\colon S^1 \to S^1$ that is given by $f(z) = \overline{z} = z^{-1}$. Recall that on page 575 we showed that the map

$$\begin{array}{rcl} \Theta\colon ([0,1]\times[0,1])/\!\sim & \to & \mathrm{Tor}(S^1, z\mapsto\overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi\,\mathrm{i}x), y)] \end{array}$$

is a homeomorphism. We will use this homeomorphism as an excuse to study the homology groups of the mapping torus $\mathrm{Tor}(S^1, z\mapsto\overline{z})$.



Klein bottle $([0,1]\times[0,1])/\!\sim$ mapping torus $\mathrm{Tor}(S^1, z\mapsto\overline{z})$

First note that it follows from the Mapping Torus-H_*-Proposition 153.6 that we get the long exact sequence

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & \mathrm{H}_2(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_2} & \\ \to & \mathrm{H}_1(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_1(S^1) & \xrightarrow{i_*} & \mathrm{H}_1(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_1} & & \\ \to & \mathrm{H}_0(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_0(S^1) & \xrightarrow{i_*} & \mathrm{H}_0(\mathrm{Tor}(S^1,f)) & \longrightarrow & 0. & \end{array}$$

[1594]This lemma gives in particular a solution to Exercise 143.16.

To get our hands on the homology groups of $K \cong \operatorname{Tor}(S^1, f)$ we make the following observations:

(1) By the Sphere-H_*-Proposition 149.4 we know that $\mathrm{H}_0(S^1) \cong \mathbb{Z}$, $\mathrm{H}_1(S^1) \cong \mathbb{Z} \cdot [\alpha]$ where $\alpha(t, 1-t) = \exp(2\pi\,\mathrm{i}t)$, and that $\mathrm{H}_k(S^1) = 0$ for $k \geq 2$.
(2) It follows from the H_0-Proposition 143.25 that the induced map $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity map.
(3) It follows from the Sphere–Degree–Matrix Lemma 151.7 that the induced homomorphism $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by -1.

It follows easily from these observations that $\mathrm{H}_2(K) \cong \mathrm{H}_2(\operatorname{Tor}(S^1, f)) = 0$ and that $\mathrm{H}_0(K) \cong \mathrm{H}_0(\operatorname{Tor}(S^1, f)) \cong \mathbb{Z}$. Furthermore it follows that we have a short exact sequence

$$0 \;\to\; \underbrace{\operatorname{coker}(\mathrm{H}_1(S^1) \xrightarrow{\cdot(-2)} \mathrm{H}_1(S^1))}_{\cong \mathbb{Z}/2\mathbb{Z}} \;\xrightarrow{i_*}\; \mathrm{H}_1(\operatorname{Tor}(S^1, f)) \;\xrightarrow{\partial_1}\; \underbrace{\mathrm{H}_0(S^1)}_{\cong \mathbb{Z}} \;\to\; 0.$$

Note that $i_*([\alpha]) = [\Theta \circ \mu]$. Furthermore, using the explicit description of the connecting homomorphisms ∂_1 on page 2912 and using the Singular 1-Simplex Lemma 143.28 one can show that $\partial_1([\Theta \circ \nu])$ is a generator of $\mathrm{H}_0(S^1)$. Using the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 (1)⇒(3) we now deduce that

$$\mathrm{H}_1(\operatorname{Tor}(S^1, f)) \;=\; \mathbb{Z}/2\mathbb{Z} \cdot \underbrace{[\Theta \circ \mu]}_{=\Theta_*([\mu])} \oplus \mathbb{Z} \cdot \underbrace{[\Theta \circ \nu]}_{=\Theta_*([\nu])}.$$

Finally note that by the isomorphism $\Theta_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(\operatorname{Tor}(S^1, f))$ this implies that $\mathrm{H}_1(K) = \mathbb{Z}/2\mathbb{Z} \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu]$. ∎

The result of the Mapping Torus-H_*-Proposition 153.6 is a little unsatisfactory since it does not give us a complete calculation of the homology groups of a mapping torus, it only puts them into a long exact sequence. For the first homology groups we have a refinement, namely we have the following lemma, which can be viewed as the analog of the Mapping Torus-π_1-Proposition 93.13.

Lemma 153.11. (Mapping Torus-H_1-Lemma) Let X be a path-connected topological space and let $f\colon X \to X$ be a continuous map. Let $i\colon X \xrightarrow{x \mapsto [(x,0)]} \operatorname{Tor}(X, f)$ be the natural embedding and let $q\colon \operatorname{Tor}(X, f) \to S^1$ be the natural projection that we introduced in Definition 21.19. With this notation the following sequence is exact:

$$0 \;\to\; \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) \;\xrightarrow{i_*}\; \mathrm{H}_1(\operatorname{Tor}(X, f)) \;\xrightarrow{q_*}\; \mathrm{H}_1(S^1) \;\to\; 0.$$

In particular we have an isomorphism

$$\mathrm{H}_1(\operatorname{Tor}(X, f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big).$$

Proof. Since X is path-connected we obtain immediately from the H_0-Proposition 143.25 that the map $f_* - \mathrm{id}\colon \mathrm{H}_0(X) \to \mathrm{H}_0(X)$ is the zero map. We consider the following maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) & \xrightarrow{i} & \mathrm{H}_1(\operatorname{Tor}(X, f)) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \\ & & & & \downarrow{\scriptstyle p_*} & \swarrow_{\Psi} & & & \\ & & & & \underbrace{\mathrm{H}_1(S^1)}_{\cong \mathbb{Z}}. & & & & \end{array}$$

We make the following observations:

(1) The horizontal map is the short exact sequence from the Mapping Torus-H_*-Proposition 153.6.
(2) Since $p \circ i$ is a constant map we see that $p_* \circ i_*$ is the trivial map.
(3) It follows from (1) and (2) that there exists a continuous map $\Psi\colon \mathbb{Z} \to \mathrm{H}_1(S^1)$ that makes the diagram commute.
(4) Furthermore, since X is path-connected one can easily construct a left inverse to $p\colon \mathrm{Tor}(X,f) \to S^1$. This implies that $p_*\colon \mathrm{H}_1(\mathrm{Tor}(X,f)) \to \mathrm{H}_1(S^1)$ is an epimorphism.
(5) Since $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ and since by the Epi-Automorphism Lemma 97.27 we know that any epimorphism from $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism we obtain from the above that Ψ is an isomorphism.

This concludes the proof that the sequence in the statement of the lemma is indeed exact.

Finally note that since $\mathbb{Z}$ is free abelian it follows from the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 (1)⇒(3) that we have an isomorphism

$$\mathrm{H}_1(\mathrm{Tor}(X,f)) \;\cong\; \mathbb{Z} \oplus \mathrm{coker}\left(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\right). \qquad \blacksquare$$

153.4. **Homology and cell attachments.** In Chapter 135 we saw that many interesting topological spaces can be obtained from the empty topological space via iterated cell attachments. Thus, to calculate invariants of topological space, it is convenient to know how attaching a cell attachment alters the chosen invariants.

In the Cell Attachment-π_1-Proposition 135.5 we saw how attaching an n-cell to a topological space affects its fundamental group. The following proposition gives us an analogous result for homology groups:

Proposition 153.12. (Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition) Let X be a topological space, let $n \in \mathbb{N}_0$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We denote by $i\colon X \to X \cup_\varphi \overline{B}^n$ the natural map. There exists a natural homomorphism $\partial_n\colon \widetilde{\mathrm{H}}_n(X \cup_\varphi \overline{B}^n) \to \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ such that the following sequence is exact:

$$0 \to \widetilde{\mathrm{H}}_n(X) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n\big(X \cup_\varphi \overline{B}^n\big) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(S^{n-1})}_{\cong \mathbb{Z}} \xrightarrow{\varphi_*} \widetilde{\mathrm{H}}_{n-1}(X) \xrightarrow{i_*} \widetilde{\mathrm{H}}_{n-1}\big(X \cup_\varphi \overline{B}^n\big) \to 0.$$

Furthermore, for any $k \neq n-1, n$ the inclusion-induced map

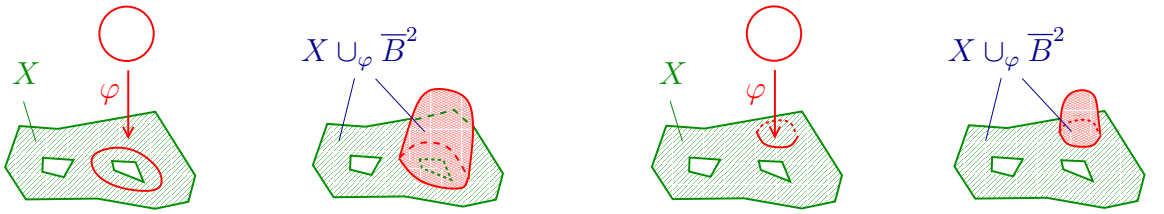
$$i_*\colon \widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X \cup_\varphi \overline{B}^n)$$

is an isomorphism.

Example. Let X be a topological space with finitely generated homology groups. Let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map.

Loosely speaking the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 says that gluing on a copy of $\overline{B}^n$ along φ has the following effect on homology:

	$\widetilde{\mathrm{H}}_n$	$\widetilde{\mathrm{H}}_{n-1}$
$\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \xrightarrow{\varphi_*} \widetilde{\mathrm{H}}_{n-1}(X)$ has infinite image	unchanged	rank goes down by 1
$\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \xrightarrow{\varphi_*} \widetilde{\mathrm{H}}_{n-1}(X)$ has finite image	rank goes up by 1	unchanged. $\square$

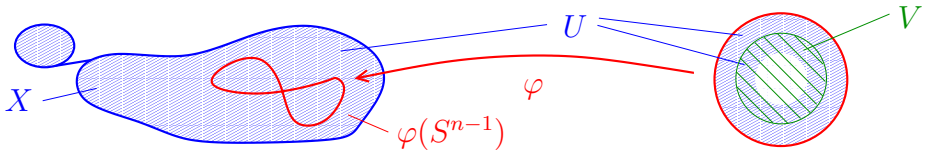


$\varphi_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(X)$ has infinite image
rank of H_2 is unchanged
rank of H_1 goes down by one

$\varphi_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(X)$ has trivial image
rank of H_2 goes up by one
rank of H_1 is unchanged

Proof. [1595] Let X be a topological space, let $n \in \mathbb{N}_0$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We consider the following two subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$

$$U \;=\; X \cup_{X \xleftarrow{\varphi} S^{n-1}} \left\{ z \in \overline{B}^n \,\middle|\, \|z\| > \tfrac{1}{4} \right\} \quad \text{and} \quad V \;=\; \left\{ z \in \overline{B}^n \,\middle|\, \|z\| < \tfrac{3}{4} \right\}.$$



We consider the following diagram:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{\varphi_*} & \widetilde{\mathrm{H}}_k(X) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k\big(X \cup_\varphi \overline{B}^n\big) & \xrightarrow{=:\partial_k} & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \longrightarrow \dots \\
 & & \downarrow{\scriptstyle \mu_* \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mu_* \cong} \\
\dots & \longrightarrow & \widetilde{\mathrm{H}}_k(U \cap V) & \xrightarrow{j_*} & \widetilde{\mathrm{H}}_k(U) \oplus \underbrace{\widetilde{\mathrm{H}}_k(V)}_{=0} & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k\big(X \cup_\varphi \overline{B}^n\big) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_k(U \cap V) \longrightarrow \dots
\end{array}$$

We make the following clarifications and observations:

(1) It follows easily from the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0) that U and V are open subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$. We can therefore apply the Mayer–Vietoris Theorem 153.2 (1) and we obtain the connecting homomorphism $\partial_k\colon \widetilde{\mathrm{H}}_k(X \cup_\varphi \overline{B}^n) \to \widetilde{\mathrm{H}}_k(U \cap V)$ which makes the bottom sequence exact.

(2) We denote by $\mu\colon S^{n-1} \to U \cap V$ the map that is given by $x \mapsto \frac{1}{2} \cdot x$. Using the Homotopy Pushout Lemma 18.13 one can easily show that μ is a homotopy equivalence. This implies, by the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5, that the left and right vertical maps are isomorphisms.

(3) We denote by $j\colon U \cap V \to U$ the inclusion map. One can easily see that $j \circ f\colon S^{n-1} \to U$ is homotopic to $\varphi\colon S^{n-1} \to X \subseteq U$. It follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 147.4 that the left square commutes.

(4) The natural map $\{z \in B^n \,|\, \|z\| < \frac{3}{4}\} \to X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ is easily seen to be an embedding. It follows that V is homeomorphic to an open ball. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that the reduced homology groups of V vanish.

[1595] The proof of the proposition we will give follows the same logic as the proof of the Cell Attachment-π_1-Proposition 135.5. For the most part we just need to replace the Seifert–van Kampen-Theorem 99.1 by the Mayer–Vietoris Theorem 153.2 (1).

(5) Using the Homotopy Pushout Lemma 18.13 one can easily show that X is a deformation retract of U. It follows from this observation, the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 and (4) that the second vertical map is an isomorphism.
(6) It follows easily from the functoriality of homology groups that second square commutes.
(7) Since all vertical maps are isomorphisms and since the bottom sequence is exact we can define $\partial_k\colon \widetilde{\mathrm{H}}_k(X \cup_\varphi \overline{B}^n) \to \widetilde{\mathrm{H}}_{k-1}(S^{n-1})$ such that the third square commutes.
(8) It follows from the above that the top sequence is exact.
(9) Since the connecting homomorphism at the bottom is natural the connecting homomorphism at the top is also natural.

It follows from the discussion that the top sequence is exact. The promised statement now follows from the Sphere-H_*-Proposition 149.4. ■

Using the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 we can perform many interesting calculations of homology groups. We give one sample calculation:

Lemma 153.13. (Projective Plane-$\widetilde{\mathrm{H}}_*$-Lemma) Given $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_n(\mathbb{R}\mathrm{P}^2) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Remark.

(1) In Exercise 153.18 we will determine an explicit cycle that represents the unique non-trivial element of $\widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2)$.
(2) (a) In Exercise 153.43 and in Exercise 153.44 we will use the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 to compute the homology groups of the surface Σ_g of genus $g \geq 2$ and of the complex projective space $\mathbb{C}\mathrm{P}^n$.
(b) It is a non-trivial, but certainly very amusing exercise to generalize the calculation of the Projective Plane-$\widetilde{\mathrm{H}}_*$-Lemma 153.13 to compute the homology groups of any real projective space $\mathbb{R}\mathrm{P}^n$. We cordially invite the reader to try to make an educated guess of what the homology groups of $\mathbb{R}\mathrm{P}^n$ might be.

We will calculate the above homology groups in Chapter 155 using cellular homology, which is relatively easy to calculate. But in principle we have the tools right now to perform the above calculations. □

Proof. We consider the continuous map

$$\begin{array}{rcl} \varphi\colon S^1 & \to & \mathbb{R}\mathrm{P}^1 \\ (x,y) & \mapsto & [x:y]. \end{array}$$

In the Projective Space–Cell Attachment Lemma 135.7 we showed that there exists a homeomorphism $\mathbb{R}\mathrm{P}^1 \cup_{\mathbb{R}\mathrm{P}^1 \xleftarrow{\varphi} S^1} \overline{B}^2 \to \mathbb{R}\mathrm{P}^2$. Next we consider the following maps:

$$0 \longrightarrow \overbrace{\widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^1)}^{=0} \xrightarrow{i_*} \widetilde{\mathrm{H}}_2\big(\overbrace{\mathbb{R}\mathrm{P}^1 \cup_\varphi \overline{B}^2}^{\cong \mathbb{R}\mathrm{P}^2}\big) \longrightarrow \overbrace{\widetilde{\mathrm{H}}_1(S^1)}^{\cong \mathbb{Z}} \xrightarrow{\varphi_*} \overbrace{\widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^1)}^{\cong \mathbb{Z}} \xrightarrow{i_*} \widetilde{\mathrm{H}}_1\big(\overbrace{\mathbb{R}\mathrm{P}^1 \cup_\varphi \overline{B}^2}^{\cong \mathbb{R}\mathrm{P}^2}\big) \longrightarrow 0.$$

$$\widetilde{\mathrm{H}}_1(S^1) \xrightarrow{(g\circ\varphi)_*} \mathrm{H}_1(S^1), \qquad \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^1) \xrightarrow{\cong} \mathrm{H}_1(S^1) \text{ via } g_*$$

We make the following clarifications and observations:

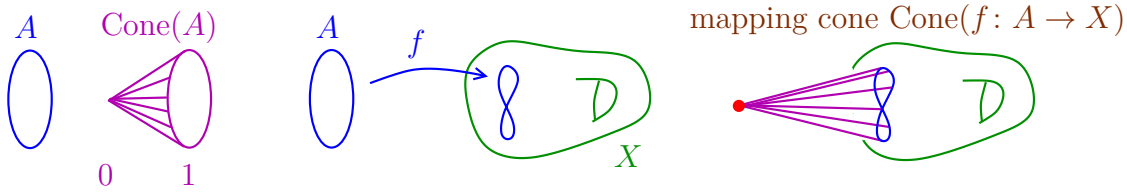
(1) By the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 we know that the horizontal sequence is exact.
(2) In the Projective Space-Avatar Lemma 9.16 (1) we gave an explicit homeomorphism $g\colon \mathbb{R}\mathrm{P}^1 \to S^1$.
(3) Using the Degree-via-Local Orientations Proposition 152.11 one can show fairly easily, see also Exercise 152.2, that $\deg(g \circ \varphi\colon S^1 \to S^1) = \pm 2$.[1596]

It follows immediately from the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 and these calculations that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^2) \cong \mathrm{H}_n(S^1) = 0$ for $n \geq 2$, $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{H}_0(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}$. ■

153.5. **The homology groups of a mapping cone.** In this section we will discuss the homology groups of mapping cones. Before we start the discussion we recall the relevant definitions.

Definition 153.14.

(1) In Definition 20.12, given a topological space A we defined the cone on A as
$$\mathrm{Cone}(A) \coloneqq (A \times [0,1])/(A \times \{0\}).$$
(2) Given a continuous map $f\colon A \to X$ between topological spaces we introduced in Definition 21.7 the corresponding mapping cone as
$$\mathrm{Cone}(f\colon A \to X) \coloneqq (\mathrm{Cone}(A) \sqcup X)/\sim \text{ where } (a,1) \sim f(a) \text{ for all } a \in A.$$



The following lemma gives an elegant calculation of the reduced homology groups of a mapping cone of an inclusion.

Lemma 153.15. (Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma)

(1) Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion map. For any $n \in \mathbb{N}_0$ the two maps
$$\mathrm{H}_n(X,A) \underset{\substack{\uparrow \\ \text{induced by the embedding} \\ X \to \mathrm{Cone}(i\colon A \to X)}}{\to} \mathrm{H}_n(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A)) \xleftarrow{\text{ЖК}} \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X))$$
are isomorphisms.

[1596] Alternatively note that it is pretty elementary to show that $g \circ \varphi\colon S^1 \to S^1$ is homotopic to the map $S^1 \to S^1$ given by $z \mapsto z^2$. Thus the calculation of the degree follows from the Degree-of-Power Lemma 151.5 together with the Sphere–Degree Lemma 151.4 (3).

(2) We denote by *PairTop* the category of pairs of topological spaces and we denote by *AbGroup* the category *AbGroup* of abelian groups. The composition of the two isomorphisms in (1) defines a natural isomorphism from the functor[1597]

$$\begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X,A) & \mapsto & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X,A) & \mapsto & \mathrm{H}_n(X,A). \end{array}$$

Remark.

(1) It is worth comparing the Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15 to the Reduced-to-Relative-H_*-Lemma 147.9 which says that given a pointed topological space (X, x_0) the reduced homology groups $\widetilde{\mathrm{H}}_i(X)$ are naturally isomorphic to the relative homology groups $\mathrm{H}_i(X, \{x_0\})$.

(2) In the Relative-to-Quotient-H_*-Proposition 149.13 we showed that, under favorable circumstances, we can replace the relative homology group $\mathrm{H}_n(X, A)$ by the homology group $\widetilde{\mathrm{H}}_n(X/A)$. The Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15 says that we can replace any relative homology group by the reduced homology group of the mapping cone of the inclusion. □

Proof. We denote by $P \coloneqq [A \times \{0\}]$ the tip of $\mathrm{Cone}(A)$ and of $\mathrm{Cone}(i\colon A \to X)$.

(1) Just for kicks we throw in one more map:

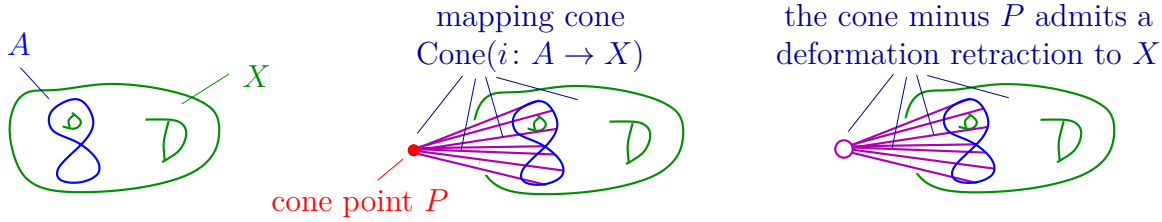
$$\begin{array}{ccccc} & & \mathrm{H}_n\big(\mathrm{Cone}(A \xrightarrow{i} X) \setminus \{P\}, \mathrm{Cone}(A) \setminus \{P\}\big) & & \\ & \overset{\text{(a)}}{\nearrow} & & \overset{\text{(b)}}{\downarrow} & \\ \mathrm{H}_k(X,A) & \longrightarrow & \mathrm{H}_k(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A)) & \underset{\text{(c)}}{\xleftarrow{\cong}} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i\colon A \to X)). \end{array}$$

The diagram commutes by the functoriality of relative homology. Thus it suffices to show that the maps (a), (b) and (c) are isomorphisms.

(a) From a slight variation on the Mapping Cylinder–Homotopy Lemma 21.5 (1) we obtain a deformation retraction from $\mathrm{Cone}(i\colon A \to X) \setminus \{P\}$ to X which restricts to a deformation retraction from $\mathrm{Cone}(A)$ to A. Thus it follows from Exercise 146.13, which is based on the Homotopy-H_*-Proposition 146.11, that the map (a) is an isomorphism.

(b) It follows from the Excision Theorem 148.1 that the vertical map (b) is an isomorphism.

(c) By the Cone Contractability Lemma 20.20 we know that the cone $\mathrm{Cone}(A)$ is contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 together with the LES-of-Pair Proposition 146.7 that the map (c) is an isomorphism.



[1597]By the Mapping Cone–Functor Lemma 21.13 we know that the left-hand side does indeed define a covariant functor.

(2) It is clear that the two maps in (1) are natural. Thus it follows that the two maps together define a natural isomorphism. ■

We obtain the following convenient corollary which is a very close relative of the Relative-to-Quotient-H_*-Proposition 149.13.

Corollary 153.16. (Cofibration–Relative-to-Quotient-H_*-Corollary) Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion map. If i is a closed cofibration, then for any $n \in \mathbb{N}_0$ the natural map $\imath\mathrm{o}\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ from Definition 149.10 is an isomorphism.

Proof. The case that $A = \varnothing$ is basically trivial and is left to the reader. In the following we assume that $A \subseteq X$ is a non-empty subspace. Let $n \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(X,A) & \xrightarrow{\quad\imath\mathrm{o}\quad} & \widetilde{\mathrm{H}}_n(X/A) \\ & {\scriptstyle\cong}\searrow \quad \nearrow{\scriptstyle\varphi_*} & \\ & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)). & \end{array}$$

We make the following clarifications and observations:

(1) The top horizontal map is the natural homomorphism that we introduced in Definition 149.10.

(2) The left diagonal map is the natural *isomorphism* of the Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15.

(3) Since $A \neq \varnothing$ the right diagonal map is induced by the continuous map

$$\begin{array}{rcl} \varphi\colon \mathrm{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases} \end{array}$$

that we introduced in the Cofibration-Quotient Lemma 22.10 (1).

(4) It follows easily from the definitions that the diagram commutes.

(5) Since $i\colon A \to X$ is by hypothesis a closed cofibration we obtain from the Cofibration-Quotient Lemma 22.10 (1) that φ is a homotopy equivalence. Therefore we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 (1a) that the right diagonal map is also an isomorphism.

(6) Putting everything together we see that the top map is indeed an isomorphism. ■

Let A be a subspace of a topological space X. We denote by $i\colon A \to X$ the corresponding inclusion map. It follows from the LES-of-Pair Proposition 146.7 together with the Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15 that the inclusion-induced maps $i_*\colon \widetilde{\mathrm{H}}_k(A) \to \widetilde{\mathrm{H}}_k(X)$ fit into the following exact sequence

$$\ldots \to \widetilde{\mathrm{H}}_k(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_k(X) \to \underbrace{\mathrm{H}_k(X,A)}_{\cong \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i\colon A \to X))} \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_{k-1}(X) \to \ldots$$

The next lemma, which can be viewed as a homological analog of the Mapping Cone-π_1-Lemma 99.15, generalizes this neat fact to the setting of any continuous map $f\colon A \to X$ between topological spaces.

Lemma 153.17. (Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma)[1598]

(1) Let $f\colon A \to X$ be a continuous map between topological spaces.[1599]
   (a) We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\sim$ the natural embedding.
   (b) We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also the figure below.) We consider the following diagram:[1600]

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_k(U) \oplus \widetilde{\mathrm{H}}_k(V) & \longrightarrow & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_{k-1}(U \cap V) \longrightarrow \dots \\ & & & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \cong} \\ & & & & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{=:\partial_k(f)} & \widetilde{\mathrm{H}}_{k-1}(A). \end{array}$$

   Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 153.2 (1)[1601] and the vertical map is induced by the inclusion $a \mapsto (a, \frac{1}{2})$ which is a homotopy equivalence. From the above diagram we obtain a homomorphism
$$\partial_k(f)\colon \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \to \widetilde{\mathrm{H}}_{k-1}(A).$$

With these definitions the following sequence is exact

$$\dots \longrightarrow \widetilde{\mathrm{H}}_k(A) \xrightarrow{f_*} \widetilde{\mathrm{H}}_k(X) \xrightarrow{j(f)_*} \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \xrightarrow{\partial_k(f)} \widetilde{\mathrm{H}}_{k-1}(A) \longrightarrow \dots$$

(2) The connecting homomorphism in (1) is natural. In particular given any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \widetilde{A} & \xrightarrow{\tilde{f}} & \widetilde{X} \end{array}$$

of continuous maps between topological spaces the following diagram commutes:

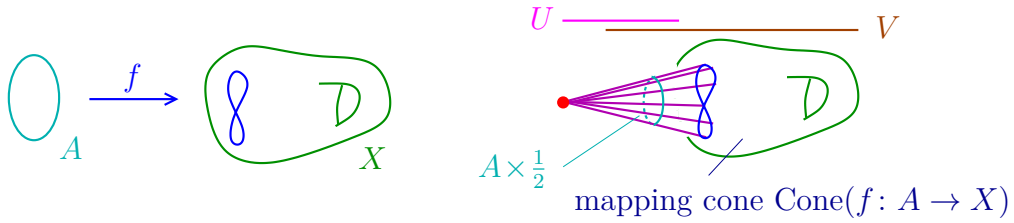
$$\begin{array}{ccccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_k(A) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_k(X) & \xrightarrow{j(f)_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k(f)} & \widetilde{\mathrm{H}}_{k-1}(A) \longrightarrow \dots \\ & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \psi_*} & & \downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \dots & \longrightarrow & \widetilde{\mathrm{H}}_k(\widetilde{A}) & \xrightarrow{\tilde{f}_*} & \widetilde{\mathrm{H}}_k(\widetilde{X}) & \xrightarrow{j(\tilde{f})_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(\tilde{f})) & \xrightarrow{\partial_k(\tilde{f})} & \widetilde{\mathrm{H}}_{k-1}(\widetilde{A}) \longrightarrow \dots . \end{array}$$

Here the vertical maps between the homology groups of the mapping cones are induced by the continuous maps from the Mapping Cone–Functor Lemma 21.13.

(3) Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i\colon A \to X$ the inclusion map. The following diagram commutes:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_k(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(X) & \xrightarrow{я} & \mathrm{H}_k(X, A) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_{k-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{k-1}(X) \longrightarrow \dots \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \dots & \longrightarrow & \widetilde{\mathrm{H}}_k(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(X) & \xrightarrow{j(f)_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i)) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_{k-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{k-1}(X) \longrightarrow \dots \end{array}$$

Here the middle vertical map is the natural isomorphism from the Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.15, the top sequence is the long exact sequence arising from the Reduced LES-of-Pair Proposition 147.8 and the bottom sequence is the exact sequence from (1).



Example. In the Mapping Cone–Cell Attachment Lemma 135.3 we saw that attaching an n-cell to a topological space is essentially the same as considering the mapping cone of a continuous map $S^{n-1} \to X$. We see that the earlier Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 is essentially just a special case of the above Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.17. □

Proof.

(1) Similar to the proof of the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 we consider the following diagram:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \widetilde{\mathrm{H}}_k(A) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_k(X) & \xrightarrow{j(f)_*} & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k(f)} & \widetilde{\mathrm{H}}_{k-1}(A) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \mu_*}\cong & & \downarrow\cong & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mu_*}\cong & & \\
\dots & \longrightarrow & \widetilde{\mathrm{H}}_k(U\cap V) & \longrightarrow & \underbrace{\widetilde{\mathrm{H}}_k(U)}_{=0} \oplus \widetilde{\mathrm{H}}_k(V) & \longrightarrow & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_{k-1}(U\cap V) & \longrightarrow & \dots
\end{array}$$

We make the following clarifications and observations:

(a) The bottom sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 153.2 (1) applied to the open cover $\mathrm{Cone}(f) = U \cup V$.

(b) The map $\mu\colon A \to U \cap V$ is given by $a \mapsto (a, \frac{1}{2})$. All the undecorated maps are induced by inclusions. Using the Homotopy Pushout Lemma 18.13 one can easily show that $\mu\colon A \to U \cap V$ is a homotopy equivalence. It follows from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 that the left and right vertical maps are isomorphisms.

(c) It follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 144.12 that the left square commutes.

(d) As in the Cone Contractability Lemma 20.20 we see that U is contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that $\widetilde{\mathrm{H}}_k(U) = 0$.

(e) Using the Homotopy Pushout Lemma 18.13 one can easily show that the inclusion $X \to V$ is a homotopy equivalence. It follows from this observation, the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 and (d) that the second vertical map is an isomorphism.

(f) The homomorphism $\partial_k(f)$ is the unique homomorphism that makes the third diagram commute.

(g) It follows from the above that the top horizontal sequence is exact.

[1598] If $A = \varnothing$, then some of the statements need to be reinterpreted in a pretty obvious way.

[1599] In the statement and the proof of the lemma we implicitly use Mapping Cone–Embedding Lemma 21.11 (2) which says that the map $j(f)$ given in (1a) is actually a closed embedding.

[1600] We leave it to the reader to give a reasonable interpretation of what we say here in the special case that $A = \varnothing$.

[1601] Note that U and V are open subsets of $\mathrm{Cone}(f)$.

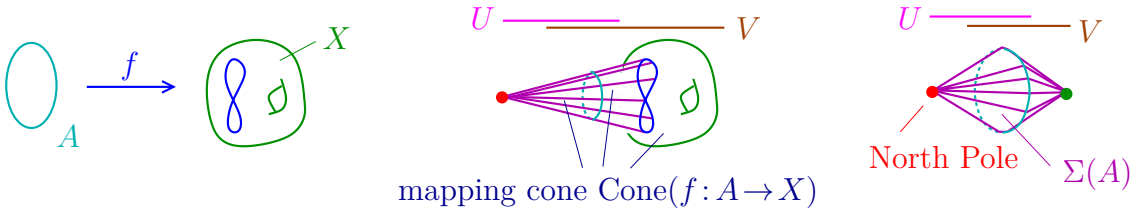
(2) This statement follows from the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence (which in turn is a consequence of the LES-of-Pair Proposition 146.7 (2)).
(3) We will prove this statement in Exercise 153.48. ■

The last lemma of this chapter gives a neat reinterpretation of the connecting homomorphism in the long exact sequence of the Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.17.

Lemma 153.18. (Cone–Connecting Homomorphism Lemma) Let $f\colon A \to X$ be a continuous map between topological spaces. For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f\colon A \to X)) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_{k-1}(A) \\ \downarrow & & \cong\downarrow \Sigma_A \\ \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f\colon A \to X)/X) & \xrightarrow[\cong]{\varphi_*} & \widetilde{\mathrm{H}}_k(\Sigma(A)). \end{array}$$

Here we used the following notation:
(1) The top horizontal map is the connecting homomorphism of the long exact sequence given in the Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.17.
(2) The left vertical map is induced by the natural projection.
(3) The map $\varphi\colon \mathrm{Cone}(f\colon A \to X)/X \to \Sigma(A)$ is the natural homeomorphism from the Mapping Cone–Quotient Lemma 21.14.
(4) The map $\Sigma_A\colon \widetilde{\mathrm{H}}_{k-1}(A) \to \widetilde{\mathrm{H}}_k(\Sigma(A))$ is the natural isomorphism from the Suspension-$\widetilde{\mathrm{H}}_*$-Proposition 149.3.



Proof. The statement follows immediately from the explicit definition of the homomorphisms ∂_k and Σ_A via Mayer–Vietoris sequences, from the explicit definition of the map φ, and from the naturality of the connecting homomorphism of the Mayer–Vietoris sequence (which is a consequence of the LES-of-Pair Proposition 146.7 (2)). We leave it to the reader to fill in the details. ■

153.6. Exercises for Chapter 153 .

Exercise 153.1. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{i_n} & C_n & \xrightarrow{p_n} & C''_n & \xrightarrow{\partial_n} & C'_{n-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n-1}} & & \\ \dots & \xrightarrow{\partial_{n+1}} & D'_n & \xrightarrow{j_n} & D_n & \xrightarrow{q_n} & D''_n & \xrightarrow{\partial_n} & D'_{n-1} & \longrightarrow & \dots \end{array}$$

We suppose that the horizontal sequences are exact and we suppose that each $f''_n\colon C''_n \to D''_n$ is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n\colon D_n \to C'_{n-1}$. Show that

the following sequence is exact:

$$\ldots \longrightarrow C'_n \xrightarrow{(i_n,-f'_n)} C_n \oplus D'_n \xrightarrow{f_n \oplus j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \ldots$$

Remark. In the proof of the Barratt-Whitehead Lemma 153.4 we showed that the sequence is exact at D_n.

Exercise 153.2. Let X be a topological space.

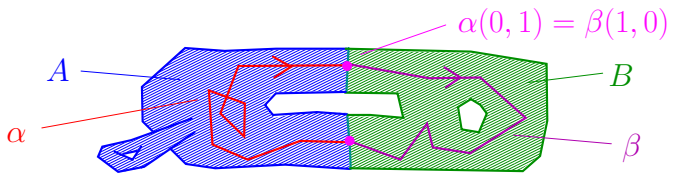
(a) Let U and V be open subsets of X. Show that for all $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n \colon \mathrm{H}_n(X, U \cup V) \to \mathrm{H}_{n-1}(X, U \cap V)$ such that
$\to \mathrm{H}_n(X, U \cap V) \xrightarrow{(i,-i)} \mathrm{H}_n(X,U) \oplus \mathrm{H}_n(X,V) \xrightarrow{i \oplus i} \mathrm{H}_n(X, U \cup V) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X, U \cap V) \to$
is an exact sequence.

(b) Let X be a topological space, let $A, B \subseteq X$ be open subsets with $X = A \cup B$ and let $U \subseteq A$ and $V \subseteq B$ be open subsets. Formulate a generalization of (a) involving the relative homology groups of (A, U) and (B, V).

Exercise 153.3. Let X be a topological space and let $X = A \cup B$ be a decomposition as in the Mayer–Vietoris Theorem 153.2. Let $\alpha \colon \Delta^1 \to A$ and $\beta \colon \Delta^1 \to B$ be two singular simplices with $\alpha(1,0) = \beta(0,1) \in A \cap B$ and with $\alpha(0,1) = \beta(1,0) \in A \cap B$. (Note that by the discussion on page 2751 this implies that $\alpha - \beta$ is a cycle in X.) We consider the long exact sequence from the Mayer–Vietoris Theorem 153.2. Show that under the connecting homomorphism

$$\begin{array}{rcl} \partial_1 \colon \mathrm{H}_1(A \cap B) & \to & \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) \\ [\alpha - \beta] & \mapsto & ([\alpha(0,1)], -[\alpha(1,0)]). \end{array}$$

Hint. This exercise can be solved by looking carefully at the various definitions.

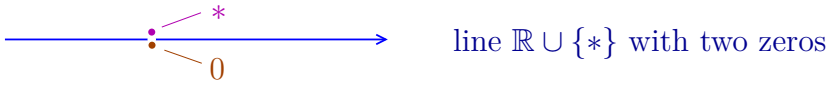


Exercise 153.4. Give examples of two contractible subspaces A and B of $\mathbb{R}^2$ such that $\mathrm{H}_1(A \cup B)$ is not finitely generated.

Exercise 153.5. Let X be a topological space and let U and V be two open subset of X such that $X = U \cup V$ and such that X and $U \cap V$ are path-connected. Show that U and V are path-connected.

Remark. This is of course just a special case of Exercise 10.12. But it is great fun to prove this statement using the Mayer–Vietoris Theorem 153.2 (1).

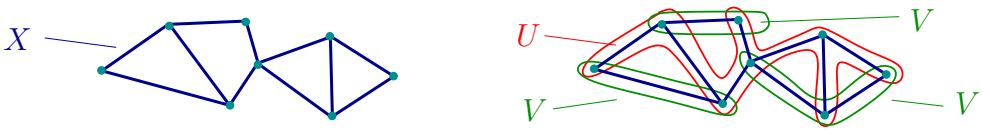
Exercise 153.6. Compute the isomorphism types of the homology groups of the real line $\mathbb{R} \cup \{*\}$ with two zeros, as defined on page 152.



Exercise 153.7. Given $k \in \mathbb{Z}$ let $f\colon S^1 \to S^1$ be the map that is given by $z \mapsto z^k$. Determine the homology groups of the mapping torus $\mathrm{Tor}(S^1, f)$.

Exercise 153.8. Let $g\colon S^1 \to S^1$ be the map that is given by $(x, y) \mapsto (-x, -y)$. Determine the homology groups of the mapping torus $\mathrm{Tor}(S^1, g)$.

Exercise 153.9. Let X be the topological space shown in the figure below and let $U \cup V$ be the open cover of X that is shown in the figure below. Use this open cover and the Mayer–Vietoris Theorem 153.2 (1) to determine the homology groups of X.



Exercise 153.10. Let $K = (V, E, i, t)$ be a connected abstract graph. Determine the homology groups of the topological realization $|K|$.
Hint. Use a spanning tree, which exists by the Spanning Tree–Existence Proposition 11.26, and the Mayer–Vietoris Theorem 153.2 (2).

Exercise 153.11. We consider the following three subspaces of $\mathbb{R}^2$:

$$\begin{aligned} S &:= \{(\tfrac{1}{n}, 0) \,|\, n \in \mathbb{N}\} \cup \{(0,0)\}, \\ A &:= S \cup ([0,1] \times (0,1], \\ B &:= S \cup ([0,1] \times [-1,0). \end{aligned}$$

Now we can formulate the corresponding problems:

(a) Show that $X := A \cup B$ is the pushout of A and B.
Hint. Use the Regluing Lemma 8.17.

(b) Show that there can be now exact sequence of the form

$$\mathrm{H}_1(A) \oplus \mathrm{H}_1(B) \;\to\; \mathrm{H}_1(X) \;\to\; \mathrm{H}_0(\underbrace{A \cap B}_{=S})$$

Remark. This shows that in the Mayer–Vietoris Theorem 153.2 one cannot replace the stated hypotheses by the natural sounding hypothesis that A and B should form a push-out. In particular this exercise can be viewed as the "homological analog" of Exercise 102.11.



Exercise 153.12. Let M be a compact smooth manifold. Use a doubling construction, the Smoothly Good Cover Theorem 81.2 and the Mayer–Vietoris Theorem 153.2 (1) to show that each homology group $\mathrm{H}_k(M)$ is finitely generated.
Remark. This exercise is related to Exercise 106.7, where we used a similar approach to show that the fundamental group of M is finitely presented.

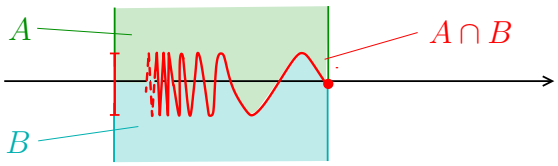
Exercise 153.13. We consider the following two subsets of $\mathbb{R}^2$:

$$\begin{aligned} A &:= \{(0,y) \in \mathbb{R}^2 \,|\, y \geq -1\} \;\cup\; \{(x,y) \in \mathbb{R}^2 \,|\, x \in (0, \tfrac{1}{\pi}] \text{ and } y \geq \sin(\tfrac{1}{x})\}, \\ B &:= \{(0,y) \in \mathbb{R}^2 \,|\, y \leq 1\} \;\cup\; \{(x,y) \in \mathbb{R}^2 \,|\, x \in (0, \tfrac{1}{\pi}] \text{ and } y \leq \sin(\tfrac{1}{x})\}. \end{aligned}$$

Note that $A \cup B = [0, \frac{1}{\pi}] \times \mathbb{R}$.

(a) Show that A and B are closed subsets of $X := A \cup B$.

(b) Show that the conclusion of the Mayer–Vietoris Theorem 153.2 does not hold for this decomposition $X = A \cup B$.



Exercise 153.14. Let $n \in \mathbb{N}$. Provide explicit singular 2-cycles that represent a basis for $\mathrm{H}_2((S^1)^n)$.

Exercise 153.15. Let $T = S^1 \times S^1$ be the torus and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2 \times 2, \mathbb{Z})$ be a matrix. We consider the map

$$\begin{array}{rccc} f_A \colon & S^1 \times S^1 & \to & S^1 \times S^1 \\ & (x, y) & \mapsto & (x^a \cdot y^b, x^c \cdot y^d). \end{array}$$

(a) Determine the induced map $f_{A*} \colon \mathrm{H}_1(S^1 \times S^1) \to \mathrm{H}_1(S^1 \times S^1)$.

(b) For which $A \in \mathrm{M}(2 \times 2, \mathbb{Z})$ is the map $f_A \colon S^1 \times S^1 \to S^1 \times S^1$ a homeomorphism?

Remark. If we work with $\mathbb{R}^2/\mathbb{Z}^2$ instead of $S^1 \times S^1$, then the map f_A corresponds to the map $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ that is given by $[v] \mapsto [A \cdot v]$.

Exercise 153.16. Does there exist a self-homeomorphism $f \colon S^1 \times S^1 \to S^1 \times S^1$ such that we get the following homology groups?

(a) $$\mathrm{H}_k(\mathrm{Tor}(S^1 \times S^1, f)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 1, 2, 3, \\ 0, & \text{otherwise} \end{cases}$$

(b) $$\mathrm{H}_k(\mathrm{Tor}(S^1 \times S^1, f)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 3, \\ \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, & \text{if } k = 1, \\ 0, & \text{otherwise} \end{cases}$$

(c) $$\mathrm{H}_k(\mathrm{Tor}(S^1 \times S^1, f)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 1, 3, \\ \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise} \end{cases}$$

Remark. Make use of Exercise 153.15.

Exercise 153.17. Let $T = S^1 \times S^1$ be the 2-dimensional torus and let $f \colon T \to T$ be a homeomorphism.

(a) Is $\mathrm{H}_1(\mathrm{Tor}(T, f))$ determined by $\mathrm{H}_2(\mathrm{Tor}(T, f))$?

(b) Is $\mathrm{H}_2(\mathrm{Tor}(T, f))$ determined by $\mathrm{H}_1(\mathrm{Tor}(T, f))$?

Exercise 153.18. We consider the real projective plane

$$\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim \quad \text{where } x \sim y \text{ if } x, y \in \partial\overline{B}^2 \text{ and } x = -y.$$

Determine an explicit cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$ that represents the unique non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$.

Exercise 153.19. We recall a few definitions from Definition 108.3 and Definition 109.1.

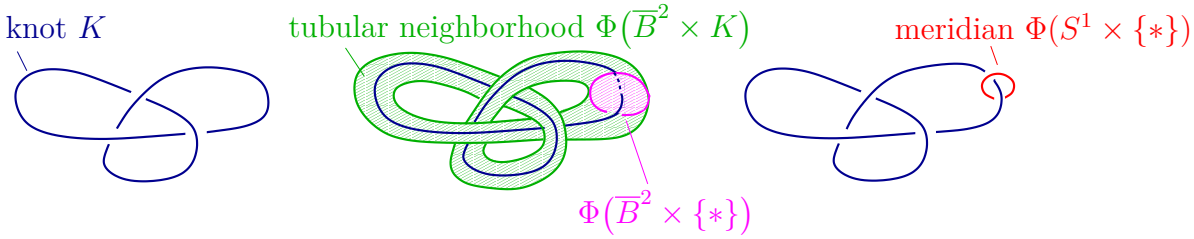
- A *knot* is a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1.
- Let K be a knot in S^3. By the Product Tubular Neighborhood Theorem 58.4 (2) there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0,P) = P$ for all $P \in K$. Let $* \in S^1$. By the Meridian Proposition 109.2 (2) we know that the submanifold $\mu := \Phi(S^1 \times \{*\})$ is a *meridian* μ of the knot K.
- Let $[\mu] \in \mathrm{H}_1(\mu) \cong \mathrm{H}_1(S^1) \cong \mathbb{Z}$ be a generator.

Now we turn to the actual exercise:

(a) Show that for any knot $K \subseteq S^3$ we have

$$\mathrm{H}_0(S^3 \setminus K) \;\cong\; \mathbb{Z} \quad \text{and} \quad \mathrm{H}_1(S^3 \setminus K) \;=\; \mathbb{Z} \cdot [\mu].$$

(b) Can you calculate any of the other homology groups of $S^3 \setminus K$? Can you make any educated guesses?



Exercise 153.20.

(a) Let $P_1, \dots, P_k \in S^n$ be distinct points. Determine the isomorphism types of the homology groups of $S^n \setminus \{P_1, \dots, P_k\}$.

(b) Let $P_1, \dots, P_k \in S^1 \times S^1$ be distinct points. Determine the isomorphism types of the homology groups of $(S^1 \times S^1) \setminus \{P_1, \dots, P_k\}$.

Exercise 153.21. Let $X \subseteq \mathbb{R}^n$ be a non-empty compact subset. As in the Stereographic Projection Lemma 4.24 we make the identification $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ of the sphere with the space $\mathbb{R}^n$ together with a "point at infinity".

(a) Show that for every $k \neq n-1$ the inclusion-induced map

$$\mathrm{H}_k(\mathbb{R}^n \setminus X) \;\to\; \mathrm{H}_k(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=S^n} \setminus X)$$

is an isomorphism.

(b) Show that there exists a short exact sequence

$$0 \;\to\; \mathbb{Z} \;\to\; \mathrm{H}_{n-1}(\mathbb{R}^n \setminus X) \;\to\; \mathrm{H}_{n-1}((\mathbb{R}^n \cup \{\infty\}) \setminus X) \;\to\; 0.$$

(c) Show that the short exact sequence in (b) splits.

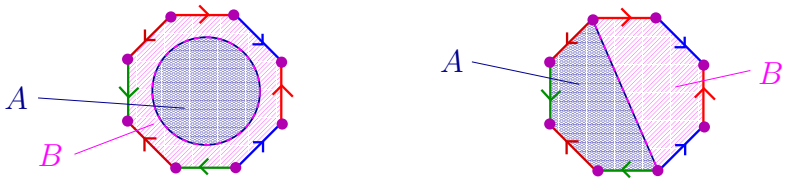
Exercise 153.22. Let X be a non-empty topological space and let $f\colon X \to X$ be a continuous map.

(a) Show that if X is furthermore path-connected, then there exists an epimorphism $\mathrm{H}_1(\mathrm{Tor}(X,f)) \to \mathbb{Z}$.

(b) Is the hypothesis in (a) that X is path-connected necessary?

Exercise 153.23.

(a) Let $g \in \mathbb{N}_{\geq 2}$. Use the Mayer–Vietoris Theorem 153.2 (3) to determine the isomorphism types of the homology groups of the surface of genus g.
Hint. In the figure below we show two possible approaches to decomposing the surface of genus two.



(b) Calculate the isomorphism types of the homology groups of the non-orientable surface of genus g.

Exercise 153.24. Let X be a topological space, let $k \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. What is the isomorphism type of $\mathrm{H}_n(X \times S^k)$ in terms of the homology groups of X?
Remark. This exercise can be viewed as an analog of the Product-with-S^k-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 80.10.

Exercise 153.25. Let $k \in \mathbb{N}$. We consider the topological space

$$X_k \coloneqq ([-k,k] \cup (S^2 \times \{-k,\dots,k\}))/\sim$$

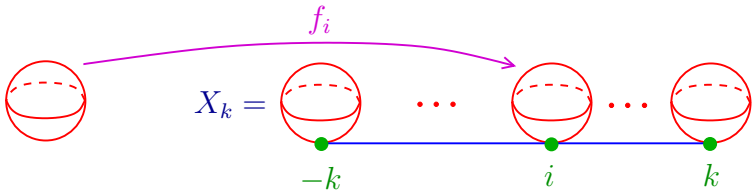
where given $i \in \{-k,\dots,k\}$ we identify $i \in [-k,k]$ with $((0,0,-1),i) \in S^2 \times \{-k,\dots,k\}$. For $i = -k,\dots,k$ let

$$\begin{array}{rcl} f_i\colon S^2 & \to & X_k \\ P & \mapsto & (P,i) \end{array}$$

be the "obvious" inclusion map.

(a) Show that $\mathrm{H}_0(X_k) \cong \mathbb{Z}$.
(b) Show that $\mathrm{H}_i(X_k) = 0$ for $i \neq 0, 2$.
(c) Let $[S^2] \in \mathrm{H}_2(S^2)$ be the standard generator. Show that the following map is an isomorphism:

$$\begin{array}{rcl} \mathbb{Z}^{2k+1} & \to & \mathrm{H}_2(X_k) \\ (a_{-k},\dots,a_k) & \mapsto & \sum\limits_{i=-k}^{k} a_i \cdot f_{i*}([S^2]) \end{array}$$



Exercise 153.26. Let X be a topological space. We consider the corresponding suspension $\Sigma(X) = (X \times [-1,1])/\sim$ which is given by identifying all points in $X \times \{-1\}$ and by identifying all points in $X \times \{1\}$.

(a) Given a singular k-simplex $\sigma\colon \Delta^k \to X$ we consider the two maps

$$\begin{array}{rcl} c_\pm(\sigma)\colon \Delta^{k+1} & \to & \Sigma(X) \\ (t_0,t_1,\dots,t_{k+1}) & \mapsto & \begin{cases} \big[\big(\sigma\big(\frac{1}{1-t_0}\cdot(t_1,\dots,t_{k+1})\big),\pm t_0\big)\big], & \text{if } t_0 < 1, \\ [X \times \{\pm 1\}], & \text{if } t_0 = 1. \end{cases} \end{array}$$

Show that the maps $c_-(\sigma)$ and $c_+(\sigma)$ are continuous.
Hint. Consider $\Delta^k \times [0,1]$ and $(\Delta^k \times [0,1])/(\Delta^k \times \{1\})$.

(b) Note that the maps $\sigma \mapsto c_\pm(\sigma)$ define homomorphisms $c_\pm \colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(\Sigma(X))$. We set $s = (-1)^{k+1} \cdot (c_+ - c_-) \colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(\Sigma(X))$. Show that for any $\sigma \colon \Delta^k \to X$ we have
$$\partial_{k+1}(s(\sigma)) \;=\; \begin{cases} s(\partial_k(\sigma)), & \text{if } k > 0, \\ [X \times \{1\}] - [X \times \{-1\}], & \text{if } k = 0. \end{cases}$$

(c) Let $k \in \mathbb{N}_0$. Show that the map
$$\begin{array}{rcl} \widetilde{\mathrm{H}}_k(X) & \to & \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \\ \sigma & \mapsto & s(\sigma) \end{array}$$
is well-defined and that it is an isomorphism.

(d) Let $\tau \colon \Sigma(X) \to \Sigma(X)$ be the map that is given by $[(x,t)] \mapsto [(x,-t)]$. What is the induced map $\tau_* \colon \mathrm{H}_k(\Sigma(X)) \to \mathrm{H}_k(\Sigma(X))$?

Remark. We refer to c_+, c_- as *cone operators* and we refer to s as the *suspension operator*.

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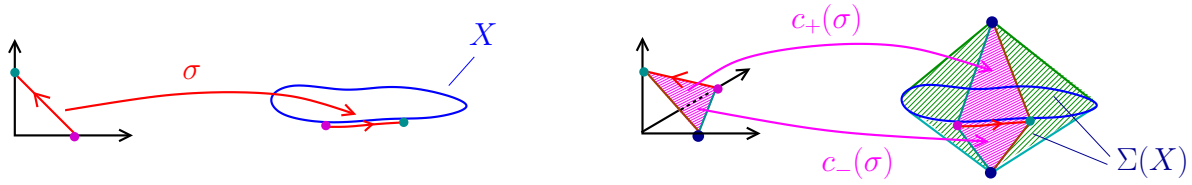
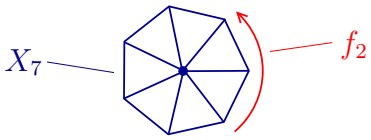


Exercise 153.27. Let $k \in \mathbb{N}_{\geq 3}$ and let $m \in \mathbb{N}$. We consider the topological space
$$X_k \;:=\; S^1 \cup \{r \cdot \exp(2\pi \mathrm{i} j/k) \,|\, r \in [0,1] \text{ and } j \in \{0, \ldots, k-1\}\}$$
and we consider the homeomorphism
$$\begin{array}{rcl} f_m \colon X_k & \to & X_k \\ z & \mapsto & \exp(2\pi \mathrm{i} m/k) \cdot z. \end{array}$$
Determine the homology groups of the mapping torus $\mathrm{Tor}(X_k, f_m)$.



Exercise 153.28. Let X be a topological space, let $x_0 \in X$ and let $f \colon X \to X$ be the constant map that is given by $x \mapsto x_0$. Determine the homology groups of the mapping torus $\mathrm{Tor}(X, f)$.

Exercise 153.29. Let X be a topological space and let U_1, U_2 and U_3 be open subsets of X with $X = U_1 \cup U_2 \cup U_3$.

(a) We suppose that $U_1 \cap U_2 \cap U_3 = \varnothing$. Show that there exists a long exact sequence of the form
$$\ldots \to \begin{array}{c} \mathrm{H}_n(U_1 \cap U_2) \\ \mathrm{H}_n(U_2 \cap U_3) \\ \mathrm{H}_n(U_1 \cap U_3) \end{array} \xrightarrow{\begin{pmatrix} i_{U_1 \cap U_2} & 0 & i_{U_1 \cap U_3} \\ -i_{U_1 \cap U_2} & i_{U_2 \cap U_3} & 0 \\ 0 & -i_{U_2 \cap U_3} & -i_{U_1 \cap U_3} \end{pmatrix}} \begin{array}{c} \mathrm{H}_n(U_1) \\ \mathrm{H}_n(U_2) \\ \mathrm{H}_n(U_3) \end{array} \xrightarrow{(i_{U_1}\ i_{U_2}\ i_{U_3})} \mathrm{H}_n(X) \xrightarrow{\partial_n} \ldots$$

(b) Show that in general the conclusion of (a) does not hold if $U_1 \cap U_2 \cap U_3 \neq \varnothing$.

Remark. This exercise is related to the discussion on page 2903.

Exercise 153.30. Let $X = \mathbb{Z}/n\mathbb{Z}$ be the topological space whose underlying set is $\mathbb{Z}/n\mathbb{Z}$ and whose topology is given by the discrete topology.

(a) Let $f\colon X = \mathbb{Z}/n\mathbb{Z} \to X = \mathbb{Z}/n\mathbb{Z}$ be the homeomorphism given by $f(x) = x+1 \in \mathbb{Z}/n\mathbb{Z}$. Compute the homology groups of the mapping torus $\operatorname{Tor}(X, f)$.

(b) More generally, let $f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be any bijection (i.e. let f be a permutation). Compute the homology groups of the mapping torus $\operatorname{Tor}(X, f)$ in terms of standard data about permutations.

(c) What can you say about the homeomorphism types of the mapping tori in (a) and (b)?

Exercise 153.31. Let X and Y be connected CW-complexes such that for all $k \in \mathbb{N}_0$ the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(Y)$ are isomorphic. Does it follow that X and Y are homotopy equivalent?

Exercise 153.32. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T_{p,q}$ as defined on Definition 111.13. Compute the homology groups of $S^3 \setminus T_{p,q}$.

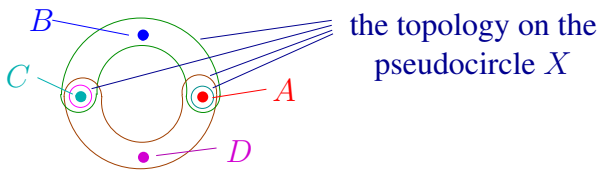
Hint. Use the decomposition of $S^3 \setminus T_{p,q}$ given in the proof of the Torus Knot-π_1-Proposition 111.18.

Exercise 153.33. Let K be the Klein bottle minus one open disk. Show that ∂K is not a retract of K.

Exercise 153.34. Let $n \in \mathbb{N}$ and let M be S^2 minus n open disks. Show that the inclusion-induced map $\mathrm{H}_1(\partial M) \to \mathrm{H}_1(M)$ is an epimorphism.

Exercise 153.35. Let $X = \{A, B, C, D\}$ be the pseudocircle from page 172. Calculate all homology groups of X.

Remark. This exercise is the homological version of Exercise 93.1.



Exercise 153.36. Show that given any finitely generated abelian group A there exists a topological space X with $\mathrm{H}_0(X) = \mathbb{Z}$ and $\mathrm{H}_1(X) \cong A$ and with $\mathrm{H}_i(X) = 0$ for $i \geq 2$.

Exercise 153.37. Given an n-dimensional smooth manifold M we define the

$$\text{embedding covering number } N(M) \;:=\; \begin{array}{c}\text{minimal cardinality of an open cover } \{U_i\}_{i\in I} \\ \text{of } M \text{ such that each } U_i \text{ admits a} \\ \text{smooth embedding into } \mathbb{R}^n.\end{array}$$

In other words, the embedding covering number is precisely the minimal cardinality of a smooth atlas of M. Let M be a non-orientable smooth manifold. Show that if $N(M) = 2$, then $\mathrm{H}_1(M)$ is infinite.

Hint. Assume that there exists a smooth atlas for M consisting of two charts $\Phi_1\colon U_1 \to V_1$ and $\Phi_2\colon U_2 \to V_2$. Consider the Mayer–Vietoris sequence corresponding to the open cover

$M = U_1 \cup U_2$ and use the Orientations-via-Transition Maps Lemma 31.35. It helps to first consider the case that U_1 and U_2 are connected.

Remark. This exercise, together with the calculation of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$ in the Projective Plane-H_*-Lemma 153.13, shows that the embedding covering number of the real projective plane $\mathbb{R}\mathrm{P}^2$ equals three.

Exercise 153.38. For a topological space X and $n \in \mathbb{N}_0$ we denote by $b_n(X) \in \mathbb{N}_0 \cup \{\infty\}$ the rank of $\mathrm{H}_n(X)$. Does there exist a topological space X such that $b_1(X) < \infty$ but such that for a *finite* cover $\widetilde{X}$ of X we have $b_1(\widetilde{X}) = \infty$?

Exercise 153.39.

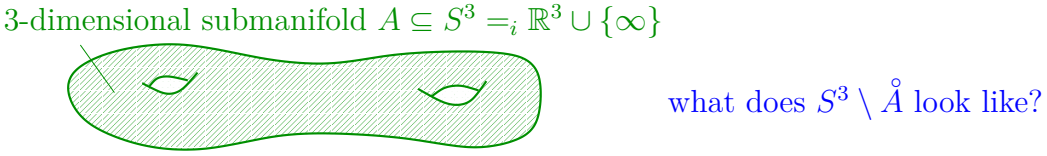
(a) We set $A \coloneqq \overline{B}^2 \times S^1$ and $B = S^1 \times \overline{B}^2$. Let $f\colon \partial A \to \partial B$ be the identity map. If you have never done so, show that

$$A \cup_f B \;\coloneqq\; (A \sqcup B)/(z,w) \sim f(z,w) \qquad \text{for all } (z,w) \in \partial A$$

is homeomorphic to S^3.

(b) Use (a) and the Mayer–Vietoris Theorem 153.2 to give another calculation of the homology groups of the torus $S^1 \times S^1 = \partial A = \partial B$.

(c) Can you use a similar approach to calculate the isomorphism types of the homology groups of higher genus surfaces?



Exercise 153.40.

(a) Compute the relative homology groups $\mathrm{H}_*(S^1 \times S^1, \{1\} \times S^1)$.

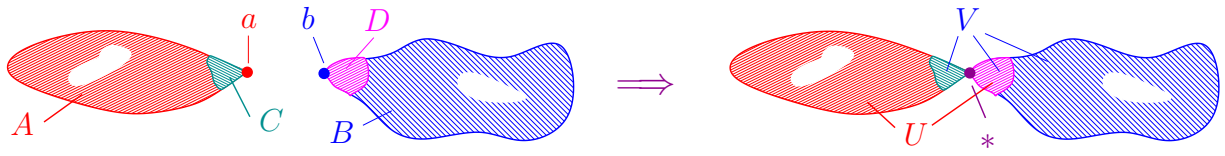
(b) Let A, B, C be three distinct points on $S^1 \times S^1$. Compute the relative homology groups $\mathrm{H}_*(S^1 \times S^1, \{A, B, C\})$.

Exercise 153.41. Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be points that are good in the sense of Definition 99.8. We denote by $(A,a) \vee (B,b)$ the wedge defined using the points $a \in A$ and $b \in B$ and we denote by $i\colon A \to (A,a) \vee (B,b)$ and $j\colon B \to (A,a) \vee (B,b)$ the natural inclusion maps that we introduced in Definition 96.1. Use the Mayer–Vietoris Theorem 153.2 to give a new proof for the fact that for every $n \in \mathbb{N}_0$ the map

$$i_* \oplus j_*\colon \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) \;\to\; \widetilde{\mathrm{H}}_n(A \vee B)$$

is an isomorphism.

Remark. This exercise gives a new proof of the Wedge-H_*-Proposition 149.16 for finite wedges.



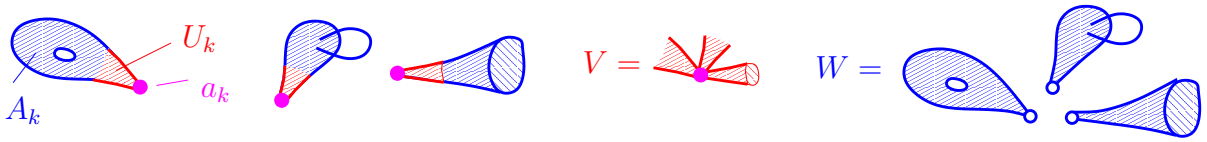
Exercise 153.42. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. By definition, see Definition 99.8, this means that given any $k \in K$ the following statements hold:

(1) $\{a_k\}$ is a closed subset of A_k and
(2) there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

Provide another proof of the Wedge-H_*-Proposition 149.16 using the following approach. First we write $W := \bigcup_{k\in K}(A_k \setminus \{a_k\})$. Now apply the (natural) reduced Mayer–Vietoris sequences from the Mayer–Vietoris Theorem 153.2 (1) to the decompositions

$$\bigvee_{k\in K} A_k \;=\; V \cup W \quad \text{and} \quad \bigsqcup_{k\in K} A_k \;=\; \Big(\bigsqcup_{k\in K}(U_k \setminus \{a_k\})\Big) \cup \Big(\bigsqcup_{k\in K}(A_k \setminus \{a_k\})\Big).$$

Remark. Do not forget to make use of the Wedge–Subspace Lemma 96.3 and the Wedge–Homotopy Lemma 96.8.



Exercise 153.43. Let $g \geq 2$. Use the description of the surface of genus g given on page 2605 and the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 to compute the homology groups of Σ_g.

Exercise 153.44. Let $k \in \mathbb{N}$. Use the description of the complex projective space given in the Projective Space–Cell Attachment Lemma 135.7 and the Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 to compute the homology groups of $\mathbb{C}\mathrm{P}^n$.
Remark. The undaunted reader will see that this calculation is surprisingly easy.

Exercise 153.45. Let $n \in \mathbb{N}_0$. We consider the real projective space $\mathbb{C}\mathrm{P}^n$ and for $i = 0, \ldots, n$ we consider the chart

$$\begin{aligned} \Phi_i \colon \overbrace{\{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\}}^{=:U_i} &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\tfrac{z_0}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_n}{z_i}\right) \end{aligned}$$

from the Projective Space–Manifold Proposition 25.13.

(a) Show that for each $k \in \{0, \ldots, n\}$ the union $U_0 \cup \cdots \cup U_k$ is homotopy equivalent to the complex projective space $\mathbb{C}\mathrm{P}^k$.
(b) Show that for each $k \in \{1, \ldots, n\}$ the intersection $(U_0 \cup \cdots \cup U_{k-1}) \cap U_k$ is homotopy equivalent to S^{2k-1}.
(c) Use (a) to compute the homology groups of $\mathbb{C}\mathrm{P}^n$.

Remark. Parts (a) and (b) are essentially the same as Exercise 25.12.

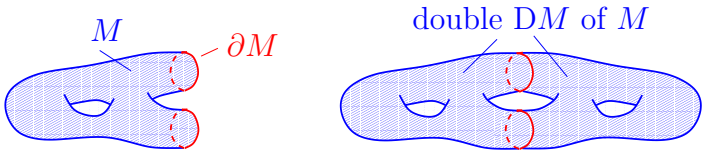
Exercise 153.46. Let X be a finite CW-complex and let $n \in \mathbb{N}_0$.

(a) Show that for $k < n$ the inclusion-induced map $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism.
(b) Show that the inclusion-induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism.

Exercise 153.47. Let M be a compact connected n-dimensional smooth manifold such that ∂M has $k \in \mathbb{N}$ components. As in Definition 54.8 we refer to

$$\mathrm{D}M \;:=\; (M \times \{1\}) \cup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$$

as the double of M. Show that $\mathrm{rank}(\mathrm{H}_1(\mathrm{D}M)) \geq k - 1$.



Exercise 153.48. Let X be a topological space and let $A \subseteq X$ be a subspace. We denote by $i \colon A \to X$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i} & \widetilde{\mathrm{H}}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}(X) & \longrightarrow & \ldots \\
 & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \cong & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\
\ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i} & \widetilde{\mathrm{H}}_n(X) & \longrightarrow & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i)) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}(X) & \longrightarrow & \ldots
\end{array}$$

where the middle vertical map is the natural isomorphism from the Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15, where the top sequence is the long exact sequence arising from the Reduced LES-of-Pair Proposition 147.8 and where the bottom sequence is the exact sequence from the Mapping Cone-H_*-Lemma 153.17. Show that the diagram commutes.

CHAPTER 154

Homology groups and direct colimits

On several occasions we were already wondering whether various functors, e.g. π_1, π_n and H_n, preserve (co-) limits. In particular in the Exhaustion-π_1-Proposition 105.2, in the Exhaustion-π_n-Proposition 134.21 and in the CW-Direct Colimit-π_n-Proposition 136.25 we studied the question whether (higher) homotopy groups preserve direct colimits.

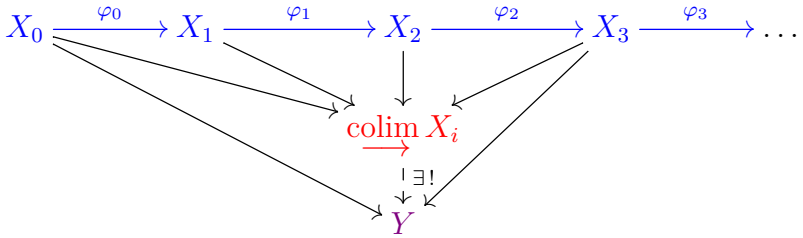
In this chapter we will discuss the question to what degree homology groups preserve direct colimits. The results of this chapter will allow us in particular to determine the homology groups of suitable non-compact topological spaces. On a more theoretical level, the results of this chapter will be important in the next chapter when we study cellular homology of CW-complexes.

This chapter is organized as follows:

- In Section 154.1 we will state the purely algebraic Direct Colimit-H_*-Lemma 154.8 which says that direct colimits and homology of chain complexes commute. In the Exhaustions-H_*-Proposition 154.2 we will use this lemma to calculate homology groups of suitable ascending unions of topological spaces.
- In Section 154.2 we will prove the Direct Colimit-H_*-Lemma 154.8.
- In Section 154.3 we will provide exercises for Chapter 154.

In the next Chapter 155 we will use the results of this chapter to introduce cellular homology of a CW-complex.

154.1. **Direct colimits of topological spaces and homology.** First let us remind the reader of the notion of a direct colimit of a sequence of morphisms, as introduced in Definition 104.4: Given a sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct colimit can be summarized in the following diagram:



More precisely, a direct colimit $\operatorname{colim} X_i$, if it exists, comes with morphisms $X_i \to \operatorname{colim} X_i$ which form a commutative diagram; furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\operatorname{colim} X_i \to Y$ which makes everything commute.

Next let us look at two of our favorite categories and types of sequences:

- First we consider the category *Top* of topological spaces. Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be a sequence of subspaces of X with $X = \bigcup_{i\in\mathbb{N}_0} X_i$. We suppose that one of the following two conditions is satisfied:
  (1) Each X_i is contained in the interior of X_{i+1}.
  (2) X is a CW-complex and each X_i is a subcomplex of X.
  It follows from (1) the Union-equals-Direct Colimit Criterion 104.12 and (2) the CW-Complex Pasting Lemma 136.15 that in these two settings we have $X = \varinjlim X_i$.
- Next we consider the category *AbGroup* of abelian groups. If we are given a sequence $A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \dots$ of homomorphisms between abelian groups, then it follows easily from the Algebraic–Direct Colimit Existence Proposition 104.7 (1b), that[1602] [1603]
$$\varinjlim A_i \;=\; \Big(\bigoplus_{i\in\mathbb{N}_0} A_i\Big) / \langle \varphi_i(a) - a \mid \; i \in \mathbb{N}_0 \text{ and } a \in A_i\rangle.$$

The following lemma, together with the above discussion, shows that in reasonable circumstances for each $n \in \mathbb{N}_0$ the singular chain group functor $X \mapsto \mathrm{C}_n(X)$ from the category *Top* of topological spaces to the category *AbGroup* of abelian groups preserves direct colimits.

Lemma 154.1. (Exhaustion-C$_*$-Lemma) Let X be a topological space. Furthermore let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subspaces of X with $\bigcup_{i\in\mathbb{N}_0} X_i = X$. Let $n \in \mathbb{N}_0$.

By the universal property of direct colimits the inclusion-induced maps $\mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ induce a natural homomorphism
$$\varinjlim_i \mathrm{C}_n(X_i) \;\to\; \mathrm{C}_n(X).$$
If one of the following conditions is satisfied
(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,
then the above map $\varinjlim_i \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism.

Proof. Let X be a topological space and let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subspaces with $\bigcup_{i\in\mathbb{N}_0} X_i = X$ such that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

We need to show that for each $n \in \mathbb{N}_0$ the natural map $\varinjlim \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism. We start out with the following claim:

Claim. For each compact subset $K \subseteq X$ there exists an $i \in \mathbb{N}_0$ with $K \subseteq X_i$.

Proof. Under the hypothesis (1) the claim follows immediately from the elementary Compact Subspace–Exhaustion Lemma 99.17, whereas under the hypothesis (2) the claim follows immediately from the CW-Complex-Finiteness Theorem 136.16. ⊞

[1602] Given a group G and a subset $S \subseteq G$ we denote by $\langle S\rangle \subseteq G$ the subgroup generated by S, see Definition 98.20 for details.

[1603] We use the natural monomorphism $A_j \to \bigoplus_{i\in\mathbb{N}_0} A_i$ to view A_j as a subgroup of $\bigoplus_{i\in\mathbb{N}_0} A_i$.

In the following, given a topological space Y we denote by $\mathrm{S}_n(Y)$ the set of singular n-simplices in Y. We consider the following diagram:

$$\begin{array}{ccccc}
\operatorname*{colim}_i \mathrm{C}_n(X_i) & & \longrightarrow & & \mathrm{C}_n(X) \\
\downarrow \mathrm{id} & & & & \downarrow \mathrm{id} \\
\operatorname*{colim}_i \mathbb{Z}^{(\mathrm{S}_n(X_i))} & \xrightarrow{\cong} & \mathbb{Z}^{\left(\bigcup_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)\right)} & \xrightarrow{\mathrm{id}} & \mathbb{Z}^{(\mathrm{S}_n(X))}.
\end{array}$$

The lemma follows from the following observations:

(a) The top map and the bottom left map are induced by the inclusions $X_i \to X$.
(b) It follows easily from (a) and the definitions that the diagram commutes.
(c) On page 2026 we saw that the bottom left homomorphism is an isomorphism.
(d) It follows immediately from the fact that Δ^n is compact, the Compact Image Lemma 4.15 and the above claim that the image of any continuous map $\Delta^n \to X$ is contained in some X_i. It follows from this observation and the hypothesis that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ that $\mathrm{S}_n(X) = \bigcup_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)$. In other words, the bottom right map is indeed the identity.
(e) It follows from (b), (c) and (d) that the top homomorphism is also an isomorphism. ■

Next we would like to understand the relationship between direct colimits and homology groups. The key to doing so is the following purely algebraic lemma.

Lemma 154.8. (Direct Colimit-H_*-Lemma) Suppose that for each $i \in \mathbb{N}_0$ we are given a chain complex $(C_{i*}, \partial_*) = (\{C_{in}\}_{n\in\mathbb{N}_0}, \{\partial_n\}_{n\in\mathbb{N}_0})$. Furthermore suppose that we are given a sequence $C_{0*} \to C_{1*} \to C_{2*} \to \dots$ of chain maps.

(1) The groups $\operatorname*{colim}_i C_{in}$ with $n \in \mathbb{N}_0$ naturally form a chain complex such that the natural maps $C_{j*} \to \operatorname*{colim}_i C_{i*}$ are chain maps.

(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\operatorname*{colim}_i C_{i*}\big)$ from (1) induce an isomorphism

$$\operatorname*{colim}_j \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname*{colim}_i C_{i*}\big).$$

Proof. To keep the conversation going we outsource the proof of this algebraic lemma to Section 154.2. In fact in that section we will formulate and prove a slightly more general statement which we will need later on in a different setting. ■

By the discussion on page 2937 the following proposition shows in particular that homology groups preserve many direct colimits.

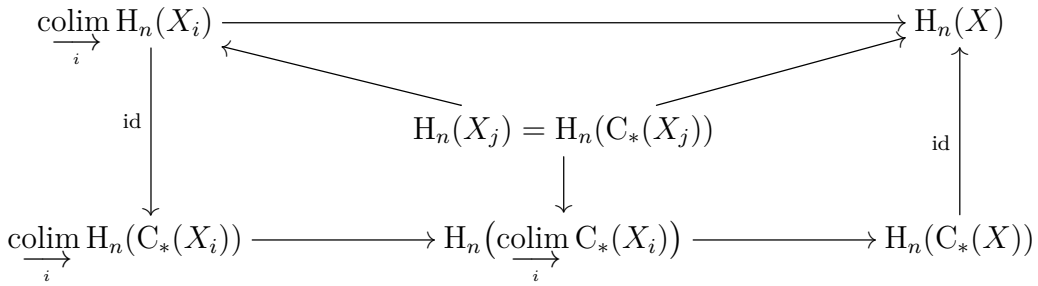
Proposition 154.2. (Exhaustions-H_*-Proposition) Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be a sequence of subspaces such that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ and such that one of the following conditions holds:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,

then for any $n \in \mathbb{N}_0$ the inclusion-induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism[1604]

$$\operatorname*{colim}_{\longrightarrow} \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

Proof. For each $j \in \mathbb{N}_0$ we consider the following diagram:



One can easily verify that each segment of the diagram involving $\mathrm{H}_n(X_j)$ commutes. It follows from the universal property of direct colimits that the total diagram commutes. Next note that the bottom left map is an isomorphism by the purely algebraic Direct Colimit-H_*-Lemma 154.8 and that the bottom right map is an isomorphism by the Exhaustion-C_*-Lemma 154.1 (here we use our hypothesis on the sequence of subspaces). It follows that the top map is also an isomorphism. ■

We conclude this section with a few examples:

Example. For any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(S^\infty) \overset{\uparrow}{=} \operatorname*{colim}_{\longrightarrow k} \mathrm{H}_n(S^k) \overset{\uparrow}{=} \operatorname*{colim}_{\longrightarrow k} \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} \overset{\uparrow}{=} \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

follows from the Exhaustion-H_*-Proposition 154.2, where we view S^∞ as a CW-complex as on page 2628

by the Sphere-H_* Proposition 149.4

by the Direct Colimit Stabilization Lemma 104.5

□

The next lemma is basically also just an example. But since we will need it later on we formulate it as a lemma.

Lemma 154.3. (Infinite String-of-Spheres-H_*-Lemma) We consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X := (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim \qquad \text{where } n \sim ((0,0,-1),n) \text{ for } n \in \mathbb{Z}.$$

Then[1605]

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^\infty, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

[1604]This statement is very similar to the statements of the Exhaustion-π_1-Proposition 105.2, the Exhaustion-π_n-Proposition 134.21 and the CW-Direct Colimit-π_n-Proposition 136.25. In fact, as we will see, the proof of our current proposition is at least philosophically also very similar to the earlier proofs.

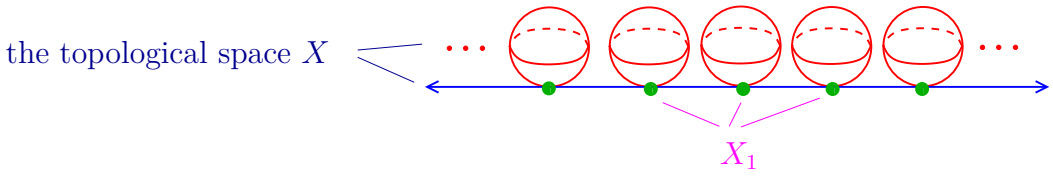
In fact we can specify an isomorphism on $\mathrm{H}_2(X)$: Given $i \in \mathbb{Z}$ we denote by $f_i \colon S^2 \to X$ the "obvious" i-th inclusion map. Then an isomorphism is given by

$$\begin{aligned} \mathbb{Z}^\infty \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} &\to \mathrm{H}_2(X) \\ (\dots, 0, a_k, \dots, a_l, 0, \dots) &\mapsto \sum_{j=k}^{l} a_j \cdot f_{j*}(\underbrace{[S^2]}_{\substack{\uparrow \\ \text{standard generator of } \mathrm{H}_2(S^2) \\ \text{from Definition 150.8}}}). \end{aligned}$$

Proof. Using the methods from Chapter 139 we can view X as a CW-complex in a pretty obvious way. Furthermore, given $k \in \mathbb{N}$ we consider the subcomplex

$$X_k \coloneqq ([-k,k] \cup (S^2 \times \{-k, \dots, k\}))/\sim .$$

Note that X_k is the interval $[-k,k]$ with $2k+1$ spheres attached.



We see that

$$\mathrm{H}_n(X) \underset{\substack{\uparrow \\ \text{by the Exhaustion-H}_* \\ \text{Proposition 154.2}}}{=} \varinjlim \mathrm{H}_n(X_k) \underset{\substack{\uparrow \\ \text{see Exercise 153.25}}}{=} \varinjlim \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{2k+1}, & \text{if } n = 2, \\ 0, & \text{else} \end{cases} \underset{\substack{\uparrow \\ \text{see page 2026 and the Direct Colimit} \\ \text{Stabilization Lemma 104.5}}}{=} \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^\infty, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

Finally note that the statement about the precise isomorphism of $\mathrm{H}_2(X)$ follows from the explicit description of the isomorphism in Exercise 153.25 and the discussion on page 2026. We leave it to the reader to fill in the details. ■

154.2. **Direct colimits of chain complexes.** In this section we will provide a proof of the purely algebraic Direct Colimit-H$_*$-Lemma 154.8. In fact we will consider a slightly more general statement than before, namely we will now consider general direct systems instead of just considering sequences.

First we recall the following segments from Definition 104.1 and Definition 104.2:

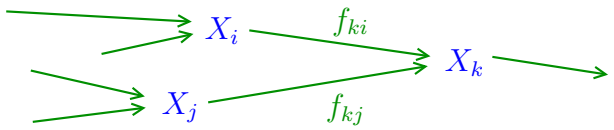
Definition 154.4.

(1) A directed set is a set I together with a partial order $\leq$ that has the extra property that for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$. (In most cases the directed set will just be $(\mathbb{N}_0, \leq)$.)

(2) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}_{i\leq j}$ such that the following two conditions are satisfied:

(a)	f_{ii}	$= \mathrm{id}_{X_i}$	for all $i \in I$,
(b)	f_{ki}	$= f_{kj} \circ f_{ji}$	for all $i, j, k \in I$ with $i \leq j \leq k$.

[1605] Recall that $\mathbb{Z}^\infty = \{(x_1, x_2, \dots) \,|\, x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\}$.



Example. Let $\mathcal{C}$ be a category. Any sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms in $\mathcal{C}$ gives rise to a direct system over the directed set $(\mathbb{N}_0, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} \coloneqq \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i \colon X_i \to X_j$. □

Next we recall Definition 104.9:

Definition 154.5. Let $(I, \leq)$ be a directed set. Furthermore let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems in some category $\mathcal{C}$.

(1) A morphism between the above two direct systems is a family $\{f_i \colon K_i \to L_i\}_{i\in I}$ of morphisms such that for every $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle \kappa_{ji}}\downarrow & & \downarrow{\scriptstyle \lambda_{ji}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$

(2) By the universal property of direct colimits the morphisms $K_i \to L_i \to \varinjlim L_i$ give rise to a natural morphism

$$\varinjlim f_i \colon \varinjlim K_i \to \varinjlim L_i.$$

Now we can formulate our first result of this section.

Proposition 154.6. (Direct Colimit-Exactness Proposition) Let $(I, \leq)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$, $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{\mu_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Finally suppose that we are given morphisms $f_i \colon K_i \to L_i$ and $g_i \colon L_i \to M_i$ of the direct systems.

(1) If for all $i \in I$ we have $g_i \circ f_i = 0$, then $\varinjlim g_i \circ \varinjlim f_i = 0$.

(2) If for all $i \in I$ the sequence

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. Let $(I, \leq)$ be a directed set. We start out with two general observations regarding direct colimits:

(a) Let $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ be a direct system of abelian groups. By the Algebraic–Direct Colimit Existence Proposition 104.7 (1b) we know that

$$\varinjlim A_i = \Big(\bigoplus_{i\in I} A_i\Big) / \langle \alpha_{ji}(x) - x \mid i \leq j \text{ and } x \in A_i \rangle.$$

We adopt the following notation: Given $a_j \in A_j \subseteq \bigoplus_{i\in I} A_i$ we denote by $[a_j] \in \varinjlim A_j$ the element represented by a_j.

Claim.
(i) Any element in $\varinjlim A_i$ is of the form $[a_i]$ for some $a_i \in A_i$.
(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then there exists a $k \geq j$ such that $\alpha_{kj}(a) = 0 \in A_k$.

Proof.
(i) By the above construction any element in $\varinjlim A_i$ is of the form $[a_{i_1}] + \cdots + [a_{i_m}]$ where $a_{i_j} \in A_{i_j}$. Since $(I, \leq)$ is a directed set we see easily, by considering an $i \in I$ with $i \geq i_1, \ldots, i_m$, that any element of $\varinjlim A_i$ is in fact contained in the image of a *single* A_i, i.e. is of the form $[a_i]$ for some $a_i \in A_i$.
(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then it follows from the above description of $\varinjlim A_i$ that in $\bigoplus_{i\in I} A_i$ we can write $a_j = \sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r) \in \bigoplus_{i\in I} A_i$ where $c_r \in \mathbb{Z}$, $x_r \in A_{s_r}$ and $t_r \geq s_r$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq j, t_{r_1}, \ldots, t_{r_m}$. Then we obtain the following equality in A_k:

$$\begin{aligned}\alpha_{kj}(a_j) \;=\; \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(a_j) \;&=\; \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)\Big(\sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r)\Big) \\ &=\; \sum_{r=1}^{m} c_r \cdot \Big(\underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(\alpha_{t_r,s_r}(x_r))}_{=\alpha_{k,t_r}(\alpha_{t_r,s_r}(x_r))=\alpha_{k,s_r}(x_r)} - \underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(x_r)}_{=\alpha_{k,s_r}(x_r)} \Big) \;=\; 0.\end{aligned}$$

⊞

(b) If $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ and $(\{B_i\}_{i\in I}, \{\beta_{ji}\}_{i\leq j})$ are two direct systems of abelian groups and if $\{f_i \colon A_i \to B_i\}_{i\in I}$ is a morphism between these two direct systems, then it follows easily from the definitions and (a) that the induced map of direct systems is given by

$$\begin{array}{ccc} \varinjlim A_i & \to & \varinjlim B_i \\ [a_i] & \mapsto & [f_i(a_i)] \\ \uparrow & & \\ \text{lies in } A_i & & \end{array}$$

Now we turn to the actual proof of the two statements of the proposition. We write $f \coloneqq \varinjlim f_i$ and $g \coloneqq \varinjlim g_i$.

(1) We assume that for all $i \in I$ we have $g_i \circ f_i = 0$. We need to show that $g \circ f = 0$. Suppose we are given an element of $\varinjlim K_i$. By (a-i) we know that it is of the form $[k_i]$ for some $i \in I$ and some $k_i \in K_i$. We now see that

$$(g \circ f)([k_i]) \;=\; g(f([k_i])) \underset{\text{by (b)}}{=} g([f_i(k_i)]) \underset{\text{by (b)}}{=} [g_i(f_i(k_i))] \;=\; [(g_i \circ f_i)(k_i)] \underset{\text{since } g_i \circ f_i = 0}{=} 0.$$

(2) Now we assume that each $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact. Note that by (1) it remains to prove that $\ker(g) \subseteq \operatorname{im}(f)$. Thus suppose that we are given an element of $\ker(g) \in \varinjlim L_i$. By (a-i) we know that it is of the form $[l_i]$ for some $i \in I$ and some $l_i \in L_i$. Since $g([l_i]) = 0$ it follows from (a-ii) and (b) that there exists a $j \geq i$ such that $\mu_{ji}(g_i(l_i)) = 0$.

Next we consider the following diagram:

$$\begin{array}{ccccc} K_i & \xrightarrow{f_i} & L_i & \xrightarrow{g_i} & M_i \\ \downarrow{\scriptstyle \kappa_{ji}} & & \downarrow{\scriptstyle \lambda_{ji}} & & \downarrow{\scriptstyle \mu_{ji}} \\ K_j & \xrightarrow{f_j} & L_j & \xrightarrow{g_j} & M_j. \end{array}$$

By hypothesis the diagram commutes, thus we see that $g_j(\lambda_{ji}(l_i)) = \mu_{ji}(g_i(l_i)) = 0$. Since $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact we know that there exists a $k_j \in K_j$ with $f_j(k_j) = \lambda_{ji}(l_i)$. It follows from (b) that $f([k_j]) = [f_j(k_j)] = [\lambda_{ji}(l_j)] = [l_i] \in \operatorname{colim} L_i$. We have thus shown that $[l_i] \in \operatorname{im}(f)$. ∎

The following corollary says that direct colimits commute with kernels and cokernels.

Corollary 154.7. (Direct Colimit-(Co-)-Kernel Corollary) Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a morphism $f_i\colon K_i \to L_i$ between the direct systems. There exist natural isomorphisms

$$\operatorname{colim} \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\big(\operatorname{colim} f_i\colon \operatorname{colim} K_i \to \operatorname{colim} L_i\big)$$

and

$$\operatorname{colim} \operatorname{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \operatorname{coker}\big(\operatorname{colim} f_i\colon \operatorname{colim} K_i \to \operatorname{colim} L_i\big).$$

Proof. By the elementary Exactness Lemma 145.2 we know that for each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \operatorname{coker}(f_i) \to 0.$$

is exact. One can easily verify that these homomorphisms form morphisms between the various direct systems. It follows from the Direct Colimit-Exactness Proposition 154.6 (2) (applied four times!) that the sequence

$$0 \to \operatorname{colim} \ker(f_i) \to \operatorname{colim} K_i \xrightarrow{\operatorname{colim} f_i} \operatorname{colim} L_i \to \operatorname{colim} \operatorname{coker}(f_i) \to 0$$

is also exact. By the Exactness Lemma 145.2 this observation implies the desired statements. ∎

The following lemma is a slightly more general version of the lemma stated on page 2938 in the first section of this chapter.

Lemma 154.8. (Direct Colimit-H_*-Lemma) Let $(I, \leq)$ be a directed set. Furthermore let $(\{C_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system of chain complexes.

(1) The groups $\operatorname{colim}_i C_{i*}$ naturally form a chain complex such that the natural maps $C_{j*} \to \operatorname{colim}_i C_{i*}$ are chain maps.

(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\operatorname{colim}_i C_{j*}\big)$ from (1) induce an isomorphism

$$\operatorname{colim}_j \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname{colim}_i C_{i*}\big).$$

Proof.

(1) It follows immediately from the Direct Colimit-Exactness Proposition 154.6 (1), applied to the maps $C_{i,n+1} \xrightarrow{\partial_{n+1}} C_{i,n} \xrightarrow{\partial_n} C_{i,n-1}$, that the direct colimits $\operatorname{colim}_i C_{i*}$ form a chain complex. Furthermore note that it follows immediately from the universal properties of direct colimits that the natural maps $C_{j*} \to \operatorname{colim}_i C_{i*}$ are chain maps.

(2) Given any $n \in \mathbb{N}_0$ we see that

$$\begin{aligned}
\operatorname*{colim}_i \mathrm{H}_n(C_{i*}) &= \operatorname*{colim}_i \overbrace{\operatorname{coker}\left(C_{i,n+1} \xrightarrow{\partial_{n+1}} \ker(C_{i,n} \xrightarrow{\partial_n} C_{i,n-1})\right)}^{=\mathrm{H}_n(C_{i*})} \\
&\overset{\cong}{\uparrow} \operatorname{coker}\left(\operatorname*{colim}_i C_{i,n+1} \xrightarrow{\operatorname{colim}\partial_{n+1}} \ker\left(\operatorname*{colim}_i C_{i,n} \xrightarrow{\operatorname{colim}\partial_n} \operatorname*{colim}_i C_{i,n-1}\right)\right) \\
&\qquad \text{follows from applying the Direct Colimit-(Co-)-Kernel Corollary 154.7 twice} \\
&= \mathrm{H}_n\left(\operatorname*{colim}_i C_{i*}\right). \\
&\qquad \text{by definition of the } n\text{-th homology group}
\end{aligned}$$

The fact that this isomorphism is induced by the maps $\mathrm{H}_n(C_{i*}) \to \mathrm{H}_n(\operatorname{colim} C_{i*})$ follows easily from the various definitions. We leave the details to the reader. ■

154.3. Exercises for Chapter 154 .

Exercise 154.1. We consider the topological spaces $\mathbb{R}^2$ and $\mathbb{Z}$.

(a) Show that there exist two injective maps $\varphi, \psi\colon \mathbb{Z} \to \mathbb{R}^2$ such that the first homology groups of $\mathbb{R}^2 \setminus \varphi(\mathbb{Z})$ and of $\mathbb{R}^2 \setminus \psi(\mathbb{Z})$ are not isomorphic.
Hint. Two groups are not isomorphic if one is countable and the other one is uncountable.

(b) What happens if we demand that φ and ψ are embeddings of topological spaces?

Exercise 154.2. Let $A \subseteq \mathbb{R}^3$ be a countable subset. What can you say about $\mathrm{H}_2(\mathbb{R}^3 \setminus A)$? Can you determine it completely? Does the isomorphism type depend on A?

Exercise 154.3. We consider the two topological spaces $X = S^1 \times S^2$ and $Y = S^1 \vee S^2 \vee S^3$.

(a) Show that the homology groups and the fundamental groups of X and Y are isomorphic.

(b) Show that the universal coverings of X and Y are not homotopy equivalent.
Remark. It follows from Exercise 127.30 that X and Y themselves are not homotopy equivalent.

Exercise 154.4.

(a) Let
$$\mathbb{Z}/3\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z}/3\mathbb{Z} \xrightarrow{\varphi_2} \mathbb{Z}/3\mathbb{Z} \xrightarrow{\varphi_3} \dots$$
be a sequence of homomorphisms, with unknown homomorphisms φ_i. What are the possible isomorphism types of the direct colimit?

(b) Let
$$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z} \xrightarrow{\varphi_2} \mathbb{Z} \xrightarrow{\varphi_3} \dots$$
be a sequence of homomorphisms, with unknown homomorphisms φ_i. What are the possible isomorphism types of the direct colimit?

(c) Let
$$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z}^2 \xrightarrow{\varphi_2} \mathbb{Z}^3 \xrightarrow{\varphi_3} \dots$$
be a sequence of homomorphisms, with unknown homomorphisms φ_i. What are the possible isomorphism types of the direct colimit?

Remark. In some case it might be hard to give a complete answer, but one can at least try to come up with as many examples as possible.

Exercise 154.5. We consider the direct colimit $\overline{B}^\infty = \operatorname{colim}\overline{B}^n$. Let $X \subseteq \mathbb{R}^\infty$ be a discrete subspace.

(a) Show that X is countable.
(b) Determine the homology groups of $\mathbb{R}^\infty \setminus X$.

Exercise 154.6. Does there exist a CW-complex X together with a sequence of subcomplexes $X_1 \subseteq X_2 \subseteq \dots$ with $X = \operatorname{colim} X_i = \bigcup_{i\in\mathbb{N}} X_i$ such that each inclusion-induced map $\mathrm{H}_1(X_i) \to \mathrm{H}_1(X_{i+1})$ is neither injective nor surjective?

Exercise 154.7.

(a) Show that if $X \subseteq \mathbb{R}^n$ is the union of m open convex subsets of $\mathbb{R}^n$, then all homology groups of X are finitely generated.
Hint. Do induction on m.
(b) Show that for any open subset $U \subseteq \mathbb{R}^n$ all homology groups are countable.
(c) Let X be a topological manifold. Show that all homology groups are countable.
Hint. First consider the case that the topological manifold has a finite atlas.

Exercise 154.8.

(a) Provide an example of a topological space X together with a sequence of subspaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ with $X = \bigcup_{i\in\mathbb{N}_0} X_i$ but such that for some $k \in \mathbb{N}_0$ the natural map $\operatorname{colim} \mathrm{H}_k(X_i) \to \mathrm{H}_k(X)$ is not an epimorphism.
(b) Provide an example of a topological space X together with a sequence of subspaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ with $X = \bigcup_{i\in\mathbb{N}_0} X_i$ but such that for some $k \in \mathbb{N}_0$ the natural map $\operatorname{colim} \mathrm{H}_k(X_i) \to \mathrm{H}_k(X)$ is not a monomorphism.

CHAPTER 155

Cellular homology

It follows from the 1-Complex-π_1-Theorem 137.6 and the Cell Attachment-π_1-Proposition 135.5 that it is relatively easy to compute the fundamental group of a finite CW-complex. The Cell Attachment-$\widetilde{\mathrm{H}}_*$-Proposition 153.12 also gives us a tool for calculating homology groups of finite CW-complexes, even though in practice this proposition can be tricky to apply.

In this chapter we will develop an algorithm which reduces the calculation of homology groups of CW-complexes to determining the degrees of suitable self-maps of spheres. We will use this method to finally determine the homology groups of the (non-orientable) surfaces of genus ≥ 2, of all the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ and even of the infinite-dimensional projective spaces $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$.[1606]

This chapter is organized as follows:

- In Section 155.1 we will study certain relative homology groups of CW-complexes.
- In Section 155.2 we will introduce cellular homology.
- In Section 155.3 we will prove that cellular and singular homology are naturally isomorphic.
- In Section 155.4 we will study the cellular boundary maps.
- In Section 155.5 we will calculate the homology groups of surfaces.
- In Section 155.6 we will calculate the homology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$.
- In Appendix 155.7 we will collect a few more results on homology groups of surfaces
- In Appendix 155.8 we will discuss the relationship between standard generators of the homology groups of spheres and balls.
- In Section 155.9 we will provide exercises for Chapter 155.

In the next Chapter 156 we will introduce the Euler characteristic of a finite CW-complex.

155.1. **Relative homology groups of CW-complexes.** In the following section we will introduce and study the cellular homology groups of a CW-complex X. Throughout this chapter we will use the following familiar notation:

Notation 155.1. Let X be a CW-complex

(1) As usual, we denote the filtration of X by $\varnothing = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$ and we refer to X^n as the n-skeleton of X.

(2) We denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X.

In preparation for the next section we now prove several basic statements about homology groups of CW-complexes:

[1606] As we will make our way through the lectures we will see that the topological spaces $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ are much prettier and much more important than they initially appear to be.

Lemma 155.2. (CW-Skeleton-H_*-Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$.

(1) Let $\{\Phi_c \colon \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.[1607] For every $k \in \mathbb{N}_0$ the following map is an isomorphism:

$$\bigoplus_{c \in \mathrm{Cell}_n(X)} \Phi_{c*} \colon \bigoplus_{c \in \mathrm{Cell}_n(X)} \mathrm{H}_k(\overline{B}^n, S^{n-1}) \to \mathrm{H}_k(X^n, X^{n-1}).$$

In particular the following statements hold:

(a) For $k \neq n$ we have $\mathrm{H}_k(X^n, X^{n-1}) = 0$.

(b) The group $\mathrm{H}_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set $\mathrm{Cell}_n(X)$ of n-cells in the CW-structure.

(2) For $k > n$ we have $\mathrm{H}_k(X^n) = 0$.

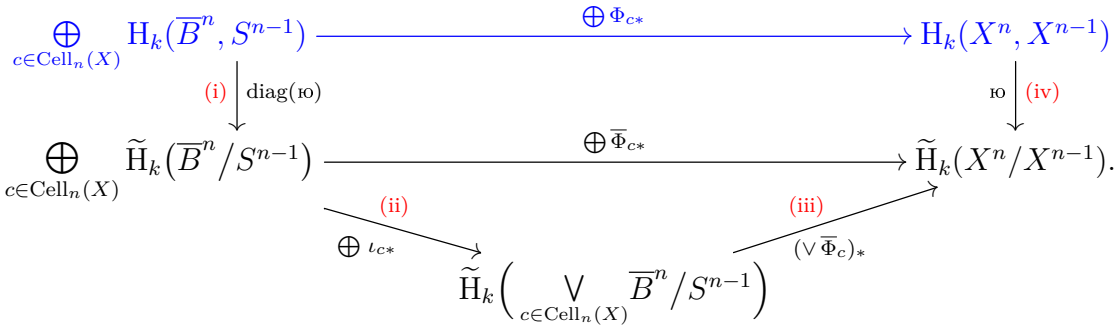
(3) For $k < n$ the inclusion $i \colon X^n \to X$ induces an isomorphism $i_* \colon \mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X)$.[1608]

(4) The inclusion $i \colon X^n \to X$ induces an epimorphism $i_* \colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Remark. The statements (3) and (4) of the CW-Skeleton-H_*-Lemma 155.2 can be viewed as "homological analogs" of the CW-Skeleton-π_1-Proposition 136.22 and the CW-Skeleton-π_n-Proposition 141.3. □

Proof. Let X be a CW-complex and let $\{\Phi_c \colon \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.

(1) We consider the following diagram:



We make the following clarifications and observations:

- Given a pair (Y, A) of topological space we consider the natural homomorphism $\imath\jmath \colon \mathrm{H}_n(Y, A) \to \widetilde{\mathrm{H}}_n(Y/A)$ from Definition 149.10.
- We use (obvious generalizations of) the Section 152.2 regarding maps out of direct sums of abelian groups.
- Given $c \in \mathrm{Cell}_n(X)$ we denote by $\overline{\Phi}_c \colon \overline{B}^n / S^{n-1} \to X^n / X^{n-1}$ the map that is induced by the characteristic map $\Phi_c \colon \overline{B}^n \to X^n$.
- It follows almost immediately from the naturality of the maps $\imath\jmath$ that the rectangle commutes.
- We use the points $[S^{n-1}] \in \overline{B}^n / S^{n-1}$ to form the wedge $\bigvee_{c \in \mathrm{Cell}_n(X)} \overline{B}^n / S^{n-1}$.
- For $j \in \mathrm{Cell}_n(X)$ let $\iota_j \colon \overline{B}^n / S^{n-1} \to \bigvee_{c \in \mathrm{Cell}_n(X)} \overline{B}^n / S^{n-1}$ be the natural inclusion.

[1607] We refer to Definition 136.7 for the definition of a full set of characteristic maps and a justification why such a set always exists.

[1608] For finite CW-complexes we proved Statements (3) and (4) in Exercise 153.46.

- For each $j \in \mathrm{Cell}_n(X)$ we have $\overline{\Phi}_j = \big(\bigvee_{c\in \mathrm{Cell}_n(X)} \overline{\Phi}_c\big) \circ \iota_j$. It follows from this observation and the definition of $\bigoplus_{c\in \mathrm{Cell}_n(X)} \overline{\Phi}_{c*}$ that the triangle commutes.

We want to show that the top horizontal map is an isomorphism. By the above it remains to show that the maps (i)–(iv) are isomorphisms.

(i) This map is an isomorphism by the Relative-to-Quotient-H_*-Proposition 149.13.[1609]

(ii) This map is an isomorphism by the Wedge-H_*-Proposition 149.16.

(iii) This map is an isomorphism by the Skeleton-Quotient Lemma 139.12.

(iv) It follows from the Relative-to-Quotient-H_*-Proposition 149.13 together with the Subcomplex–Neighborhood Proposition 137.9 that this map is an isomorphism.

The remaining statements of (1) are now an immediate consequence of the Ball-rel-Sphere-H_*-Lemma 149.7.

(2) We fix $k \in \mathbb{N}$. We need to prove the following claim:

Claim. For $n = -1, \dots, k-1$ we have $\mathrm{H}_k(X^n) = 0$.

Proof. We prove the claim by induction on $n \in \mathbb{Z}_{\geq -1}$. First we consider the case $n = -1$. In this case $X^n = X^{-1} = \varnothing$ and the statement follows from the Empty Space-H_*-Lemma 143.22.

Now assume that the statement holds for some $n \in \{-1, \dots, k-2\}$. We consider the long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to \mathrm{H}_{k+1}(X^{n+1}, X^n) \to \underbrace{\mathrm{H}_k(X^n)}_{=0 \text{ by induction}} \to \mathrm{H}_k(X^{n+1}) \to \underbrace{\mathrm{H}_k(X^{n+1}, X^n)}_{\substack{=0 \text{ by (1) since } n < k-1 \\ \text{which implies that } k \neq n+1}} \to \dots$$

It follows that we have an epimorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X^{n+1})$. Since $\mathrm{H}_k(X^n) = 0$ we conclude that $\mathrm{H}_k(X^{n+1}) = 0$. ⊞

(3) Let $k \in \mathbb{N}_0$ and $n > k$. Recall that we need to show that the inclusion-induced map $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism. We consider the following diagram of inclusion-induced homomorphisms:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{H}_k(X^{n-1}) & \longrightarrow & \mathrm{H}_k(X^n) & \xrightarrow{\cong} & \mathrm{H}_k(X^{n+1}) & \xrightarrow{\cong} & \mathrm{H}_k(X^{n+2}) & \xrightarrow{\cong} & \dots \\ & & & \searrow & \downarrow & \swarrow & & \swarrow & & & \\ & & & & \mathrm{H}_k(X). & & & & & & \end{array}$$

We make the following observations:

(a) The diagram commutes by the functoriality of homology.

(b) We just want to prove the following claim:

Claim. For any $m \geq n$ the inclusion-induced homomorphism $\mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1})$ is an isomorphism.

Proof. This claim follows by considering the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{k+1}(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < n \leq m} \to \mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1}) \to \underbrace{\mathrm{H}_k(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < n \leq m} \to \dots$$

⊞

[1609] Here we use that $S^{n-1} \subseteq \overline{B}^n$ is clearly a good subset.

(c) By the Exhaustions-H_*-Proposition 154.2 (2) we know that $\mathrm{H}_k(X) = \operatorname{colim}\limits_{\longrightarrow} \mathrm{H}_k(X^n)$.

(d) It follows from (a)–(c) and the Direct Colimit–Stabilization Lemma 104.5 that the vertical inclusion-induced homomorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism.

(4) We consider the following excerpt from the long exact sequence of the pair (X^{n+1}, X^n):

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[r] & \mathrm{H}_n(X^n) \arrow[r] \arrow[dr] & \mathrm{H}_n(X^{n+1}) \arrow[r] \arrow[d, "\cong \text{ by } (3)"] & \underbrace{\mathrm{H}_n(X^{n+1}, X^n)}_{= 0 \text{ by } (1)} \arrow[r] & \dots \\
& & & \mathrm{H}_n(X) & &
\end{tikzcd}$$

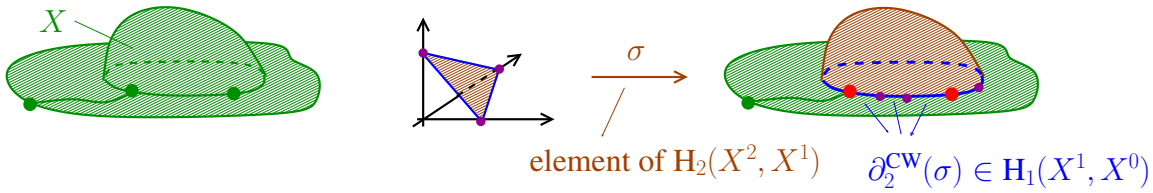
We see that the inclusion-induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism. ■

155.2. **The definition of cellular homology.** In this section we will introduce the main protagonists of this chapter, namely the cellular homology groups of a CW-complex. We start out with the definition of the cellular boundary map.

Definition 155.3. Given a CW-complex X and given $n \in \mathbb{N}_0$ we denote by ∂_n^{CW} the homomorphism

$$\begin{tikzcd}
\mathrm{H}_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "=:\partial_n^{\mathrm{CW}}"'] & \mathrm{H}_{n-1}(X^{n-1}) \arrow[r, "я_{n-1}"] & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})
\end{tikzcd}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where $я_{n-1} \colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map introduced in Definition 146.6. We refer to ∂_n^{CW} as the cellular boundary map.[1610]



The name "cellular boundary map" already suggests that the maps ∂_n^{CW} have the following important property:

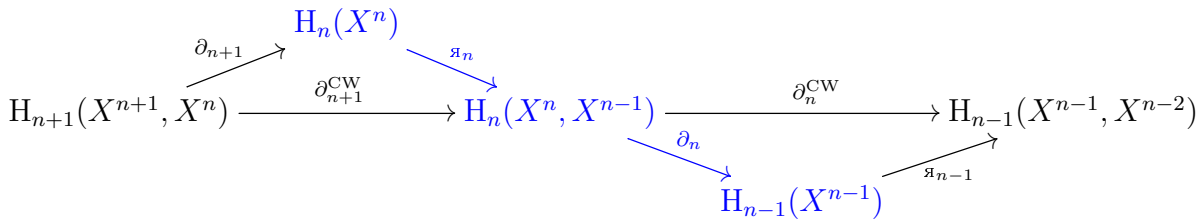
[1610] We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[d, "\mathrm{id}"] \arrow[dr, "=:\partial_n^{\mathrm{CW}}"] & \mathrm{H}_{n-1}(X^{n-1}) \arrow[d, "я_{n-1}"] \\
\mathrm{H}_n(X^n, X^{n-1}) \arrow[r, "\partial_n"'] & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).
\end{tikzcd}$$

Here the horizontal maps are the connecting homomorphisms of the triples $(X^n, X^{n-1}, \varnothing)$ and (X^n, X^{n-1}, X^{n-2}) from the LES-of-Triple Proposition 146.5. By the LES-of-Triple Proposition 146.5 (2) we know that this connecting homomorphism is natural. Thus we see that the diagram commutes. This little discussion shows that the cellular boundary map $\partial_n^{\mathrm{CW}} \colon \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is precisely the connecting homomorphism of the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2}). But for the purpose of proofs of the Cellular Boundary Map Lemma 155.4 and of the Singular-Cellular-H_*-Proposition 155.8 it is better to adopt our two-step definition.

Lemma 155.4. (Cellular Boundary Map Lemma) For all $n \in \mathbb{N}_0$ we have $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}} = 0$.

Proof. We consider the following diagram of homomorphisms:



Note that the diagram commutes by the definition of the cellular boundary maps. The map $\partial_n \circ \mathrm{я}_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}). This implies that the map $\partial_n \circ \mathrm{я}_n$ is the zero map. Since the diagram commutes we obtain that $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}} = 0$. ■

By the Cellular Boundary Map Lemma 155.4 it now makes sense to introduce the following definition:

Definition 155.5. Let X be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we write $\mathrm{C}_n^{\mathrm{CW}}(X) \coloneqq \mathrm{H}_n(X^n, X^{n-1})$.[1611]
(2) We refer to $(\mathrm{C}_*^{\mathrm{CW}}(X), \partial_*^{\mathrm{CW}})$ as the cellular chain complex of the CW-complex X.[1612]
(3) We denote the homology groups of the cellular chain complex by $\mathrm{H}_n^{\mathrm{CW}}(X)$ and we refer to these groups as the cellular homology groups of the CW-complex X.

Using the approach of Definition 147.1 we can also introduce the augmented cellular chain complex $\widetilde{\mathrm{C}}_n^{\mathrm{CW}}(X)$ and the reduced cellular homology groups $\widetilde{\mathrm{H}}_n^{\mathrm{CW}}(X)$.

Example. For $n \geq 2$ we consider the n-dimensional sphere S^n. On page 2625 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. It follows from the CW-Skeleton-H_*-Lemma 155.2 (1) that the cellular chain complex $(\mathrm{C}_*^{\mathrm{CW}}(S^n), \partial_*^{\mathrm{CW}})$ is of the form

$$0 \to \underbrace{\mathrm{C}_n^{\mathrm{CW}}(S^n)}_{\cong \mathbb{Z}} \to 0 \to \ldots \to 0 \to \underbrace{\mathrm{C}_0^{\mathrm{CW}}(S^n)}_{\cong \mathbb{Z}} \to 0.$$

Since $n \geq 2$ the two $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps. This implies that[1613]

$$\mathrm{H}_i^{\mathrm{CW}}(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These cellular homology groups are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence. □

[1611] Note that the CW-Skeleton-H_*-Lemma 155.2 (1) says that the n-th cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group where the rank is given by the cardinality of the set of n-cells. In the Cellular Chain Complex–Basis Lemma 155.13 we will discuss in greater detail how one get a basis for the cellular chain groups.

[1612] It follows from the Cellular Boundary Map Lemma 155.4 that $(\mathrm{C}_*^{\mathrm{CW}}(X), \partial_*^{\mathrm{CW}})$ is indeed a chain complex.

[1613] It is a fun exercise to calculate the cellular boundary maps of S^1.

The "usual" homology groups define covariant functors from the category of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups. We will now see that in a more restricted setting cellular homology groups also define covariant functors.

Definition 155.6.

(1) Recall that according to Definition 139.2 a map $f\colon X \to Y$ between CW-complexes is called cellular if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subseteq Y^n$.

(2) As in Definition 139.3 we denote by $\mathcal{CW}$ the category $\mathcal{CW}$ of CW-complexes where the objects are CW-complexes and where the morphisms are cellular maps.

We conclude this section with the following lemma:

Lemma 155.7. (Cellular Homology-Functor Lemma)

(1) Every cellular map $f\colon X \to Y$ between two CW-complexes induces a chain map

$$f_*\colon \mathrm{C}_*^{\mathrm{CW}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(Y)$$

and hence also homomorphisms

$$f_*\colon \mathrm{H}_*^{\mathrm{CW}}(X) \to \mathrm{H}_*^{\mathrm{CW}}(Y)$$

on cellular homology.

(2) For any $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} X &\mapsto \mathrm{H}_n^{\mathrm{CW}}(X) \\ (f\colon X \to Y) &\mapsto (f_*\colon \mathrm{H}_n^{\mathrm{CW}}(X) \to \mathrm{H}_n^{\mathrm{CW}}(Y)) \end{aligned}$$

define a covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof.

(1) Let $f\colon X \to Y$ be a cellular map. Since the map f is cellular we get for every $n \in \mathbb{N}_0$ a continuous map $(X^n, X^{n-1}) \to (Y^n, Y^{n-1})$ of pairs of topological spaces and hence an induced homomorphism $\mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_n(Y^n, Y^{n-1})$, i.e. we get an induced homomorphism $f_*\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(Y)$. It remains to show that these maps form a chain map. To do so we consider for each $n \in \mathbb{N}_0$ the following diagram:

$$\begin{array}{ccccc} & & =:\partial_n^{\mathrm{CW}} & & \\ \mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow[\partial_n]{} & \mathrm{H}_{n-1}(X^{n-1}) & \xrightarrow[\text{я}_{n-1}]{} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(Y^n, Y^{n-1}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(Y^{n-1}) & \xrightarrow{\text{я}_{n-1}} & \mathrm{H}_{n-1}(Y^{n-1}, Y^{n-2}). \\ & & =:\partial_n^{\mathrm{CW}} & & \end{array}$$

It follows from the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 146.7 (2), that the square to the left commutes. Furthermore it follows from the functoriality of relative homology that the square to the right commutes. In summary, the diagram commutes, which implies that the cellular homology groups are chain maps.

(2) This statement follows easily from the fact that the usual relative homology groups are covariantly functorial. ∎

155.3. **The relationship between cellular and singular homology.** Given a CW-complex X we now have two "competing" homology groups, namely the singular homology groups $\mathrm{H}_n(X)$ and the cellular homology group $\mathrm{H}_n^{\mathrm{CW}}(X)$. The following proposition explains the relationship between these two homology groups.

Proposition 155.8. (Singular-Cellular-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For each CW-complex X there exists a uniquely determined *isomorphism*

$$\Theta_X \colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[1614]

$$\begin{array}{ccc} \mathrm{H}_n(X^n) & \twoheadrightarrow & \mathrm{H}_n(X) \\ {\scriptstyle я_n}\downarrow & & \downarrow{\scriptstyle \Theta_X} \\ \ker(\partial_n^{\mathrm{CW}}) & \xrightarrow{a \mapsto [a]} & \mathrm{H}_n^{\mathrm{CW}}(X). \end{array}$$

(2) The homomorphisms Θ_X from (1) define a natural isomorphism between the functor $X \mapsto \mathrm{H}_n(X)$ and the functor $X \mapsto \mathrm{H}_n^{\mathrm{CW}}(X)$. In other words, for every cellular map $f \colon X \to Y$ between CW-complexes the following diagram commutes:

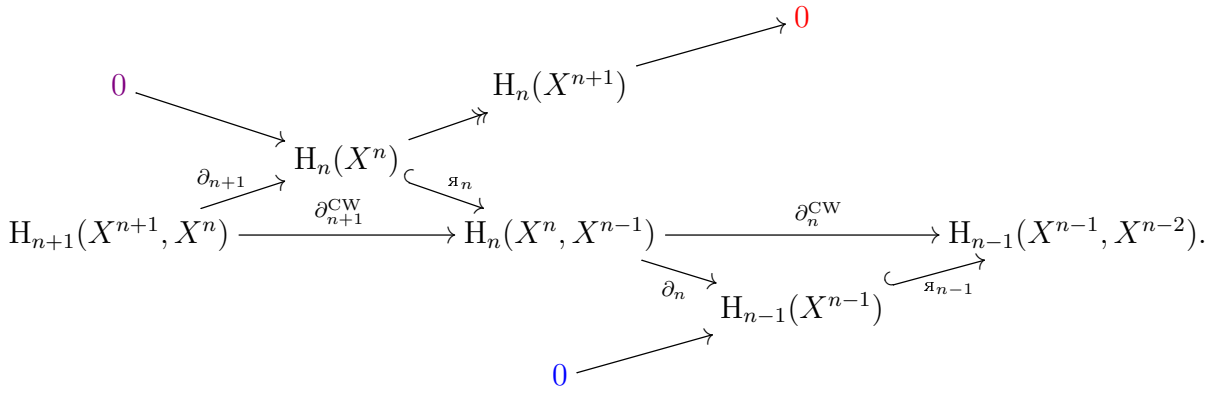
$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y) \\ {\scriptstyle \Theta_X}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Theta_Y} \\ \mathrm{H}_n^{\mathrm{CW}}(X) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{CW}}(Y). \end{array}$$

The analogous statements also hold for reduced cellular and singular homology.

Remark. Later on, in the Singular-Cellular Chain Complex Proposition 158.16, we will give an alternative proof that singular and cellular homology groups are naturally isomorphic. The latter proof has the advantage that it shows a stronger statement, namely it shows in fact that the singular and cellular chain complexes are chain homotopy equivalent. The latter proof also the nice feature that it generalizes well to other contexts, like cohomology and twisted (co-) homology. □

[1614] Here we again denote by $я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. Since $\partial_n^{\mathrm{CW}} = я_n \circ \partial_n$ this implies that the map $я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(\partial_n^{\mathrm{CW}})$.

Proof of Proposition 155.8 (1). We add a couple of maps to the commutative diagram that we had already encountered in the proof of the Cellular Boundary Map Lemma 155.4:



Claim 1. All the diagonal sequences in the above commutative diagram are exact.

Proof. By the CW-Skeleton-H_*-Lemma 155.2 (1) and (2) we know that

$$\mathrm{H}_n(X^{n+1}, X^n) = 0, \quad \mathrm{H}_n(X^{n-1}) = 0 \quad \text{and} \quad \mathrm{H}_{n-1}(X^{n-2}) = 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just excerpts from the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. ⊞

Claim 2. The map $\mathrm{я}_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ induces an isomorphism

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \;\to\; \underbrace{\ker(\partial_n^{\mathrm{CW}})/\operatorname{im}(\partial_{n+1}^{\mathrm{CW}})}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}.$$

Proof.

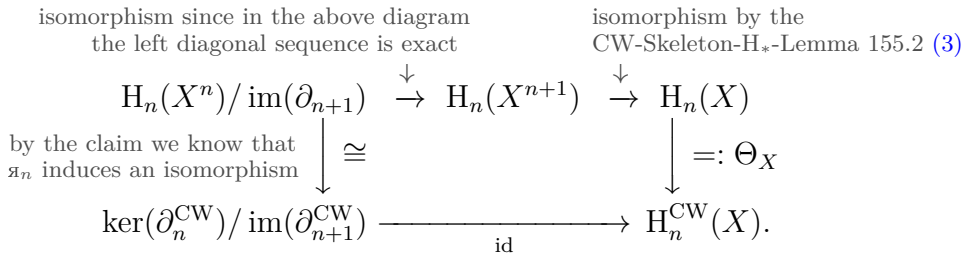
(1) The given map is well-defined since it follows immediately from $\partial_n \circ \mathrm{я}_n = 0$ that $\mathrm{я}_n(\mathrm{H}_n(X^n)) \subseteq \ker(\partial_n^{\mathrm{CW}})$.

(2) [1615] The given map is an epimorphism. This can be seen as follows:
   (a) It follows from Claim 1 that the map $\mathrm{я}_{n-1}$ is injective.
   (b) It follows from (a) and Claim 1 that $\ker(\partial_n^{\mathrm{CW}}) = \ker(\partial_n) = \operatorname{im}(\mathrm{я}_n)$.
   (c) It follows from (b) that the map $\mathrm{я}_n$ induces an epimorphism $\mathrm{H}_n(X^n) \to \ker(\partial_n^{\mathrm{CW}})$.
   (d) Since the left triangle commutes we obtain from (c) that $\mathrm{я}_n$ also induces an epimorphism $\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(\partial_n^{\mathrm{CW}})/\operatorname{im}(\partial_{n+1}^{\mathrm{CW}})$.

(3) The given map is a monomorphism: We need to show that if we are given $c \in \mathrm{H}_n(X^n)$ such that $\mathrm{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of $\partial_{n+1}^{\mathrm{CW}}$, then $c \in \operatorname{im}(\partial_{n+1})$. Thus suppose that $\mathrm{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of $\partial_{n+1}^{\mathrm{CW}}$. This means that there exists an $e \in \mathrm{H}_{n+1}(X^{n+1}, X^n)$ with $\mathrm{я}_n(c) = \partial_{n+1}^{\mathrm{CW}}(e) = \mathrm{я}_n(\partial_{n+1}(e))$. Since by Claim 1 we know that $\mathrm{я}_n$ is injective we obtain that $c = \partial_{n+1}(e)$. ⊞

[1615] It remains to show that the given map is an epimorphism and a monomorphism. This is shown by a diagram chase. It is arguably best not to read the remainder of the proof of the claim but to go on the chase on one's own.

Next we consider the following diagram:



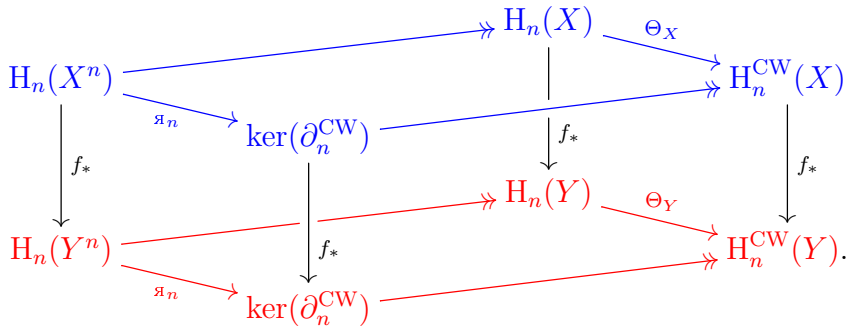
Since all the given maps are isomorphisms we can now define the right vertical map Θ_X as the unique map which makes the diagram commute. Evidently Θ_X is also an isomorphism.

Finally we note that by construction the isomorphism Θ_X has the property that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X^n) & \twoheadrightarrow & \mathrm{H}_n(X) \\ {\scriptstyle \text{я}_n}\downarrow & & \downarrow{\scriptstyle \Theta_X} \\ \ker(\partial_n^{\mathrm{CW}}) & \twoheadrightarrow & \mathrm{H}_n^{\mathrm{CW}}(X). \end{array}$$

Note that by the CW-Skeleton-H$_*$-Lemma 155.2 (4) the top horizontal map is an epimorphism. This implies immediately that Θ_X is the unique homomorphism that makes the diagram commute. ■

Proof of Proposition 155.8 (2). We need to show that the isomorphisms Θ_X that we just constructed are natural. Thus suppose that we are given a cellular map $f\colon X \to Y$ and let $n \in \mathbb{N}_0$. We consider the following diagram:



We make the following observations:

(a) The sides of the hyperrectangle that do not involve Θ_X and Θ_Y commute basically by definition.
(b) The top and bottom parallelogram commute by definition of Θ_X and Θ_Y.
(c) By the CW-Skeleton-H$_*$-Lemma 155.2 (4) the map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism.

It follows from these observation that also the parallelogram involving Θ_X and Θ_Y commutes. But that is exactly, what we had to show. ■

Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious, we can already use the above proposition to carry out an interesting calculation,

namely we can finally calculate the homology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ and of $\mathbb{C}\mathrm{P}^\infty = \operatorname{colim}_{\longrightarrow} \mathbb{C}\mathrm{P}^n$:

Proposition 155.9. ($\mathbb{C}\mathrm{P}^n$-H_*-Proposition)

(1) Given $n \in \mathbb{N}_0$ we have[1616]

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is } k \text{ is odd.} \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 2628 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with exactly one cell in dimensions $0, 2, \ldots, 2n$ and no other cells. It follows from the CW-Skeleton-H_*-Lemma 155.2 (1) that the corresponding cellular chain complex is of the form

$$0 \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2n}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2n-1}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \ldots \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{1}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{0}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$

Since every other group is trivial it follows that all boundary maps are zero. In particular the cellular homology groups equal the cellular chain groups. It follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \underset{\text{by the Singular-Cellular Homology Proposition 155.8}}{\overset{\uparrow}{\cong}} \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n) \underset{\text{follows from the above discussion}}{\overset{\uparrow}{\cong}} \mathrm{C}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n) \overset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Just for fun we provide two different ways to calculate $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$:

(a) On page 2628 we saw that we can equip the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ with a CW-structure that has precisely one cell in each even dimension and no cells in odd-dimensions. From the same arguments as in (1) it follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{C}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(b) We start out with the following claim:

Claim. Let $j \geq i$. Let $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ be the usual embedding that is given by $[x] \mapsto [(x, 0)]$. The induced map $\iota_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^j)$ on singular homology is an isomorphism for $k = 0, \ldots, 2i$.

Proof. Note that $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ is clearly cellular. Thus we obtain from the Singular-Cellular-H_*-Proposition 155.8 (2) the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_k(\mathbb{C}\mathrm{P}^i) & \xrightarrow{\iota_*} & \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j) \\ {\scriptstyle\Theta_{\mathbb{C}\mathrm{P}^i}}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Theta_{\mathbb{C}\mathrm{P}^j}} \\ \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^i) & \xrightarrow{\iota_*} & \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j) \end{array}.$$

Next note that it follows from inspecting the cellular chain complexes that for any $k \leq 2i$ the bottom horizontal map is an isomorphism. The claim follows immediately from these two observations.[1617] ⊞

It follows from the claim, the Exhaustions-H_*-Proposition 154.2 and the Direct Colimit–Stabilization Lemma 104.5 that for any $k \leq 2i$ the inclusion-induced map $\mathrm{H}_k(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$ is an isomorphism. Together with (1) we thus get the promised result. ■

We move on to another, more theoretical application, of the Singular-Cellular-H_*-Proposition 155.8. The Singular-Cellular-H_*-Proposition 155.8 lets us compute the singular homology groups through the cellular chain complex. Whereas the singular chain groups are in most cases uncountable, the ranks of the cellular chain groups are given by the number of cells. Using this observation we can now easily prove the following proposition which can be viewed as an analog to the CW-Skeleton-π_1-Proposition 136.22.

Proposition 155.10. (CW-Complex-H_*-Proposition) Let X be a CW-complex. The following statements hold:

(1) If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then $\mathrm{H}_n(X)$ is generated by at most d elements. In particular, if X has no cells of dimension n, then $\mathrm{H}_n(X) = 0$. Similarly, if X contains only countable many cells of dimension n, then $\mathrm{H}_n(X)$ is countable.
(2) If X is finite-dimensional, then for any $k > \dim(X)$ we have $\mathrm{H}_k(X) = 0$.
(3) If X is an n-dimensional CW-complex, then $\mathrm{H}_n(X)$ is a free abelian group.

Example. Let G be a topological graph. It follows from the Graph–CW-Complex Lemma 136.5 and the CW-Complex-H_*-Proposition 155.10 (3) that we have $\mathrm{H}_i(G) = 0$ for every $i \geq 2$ and that $\mathrm{H}_1(G)$ is a free abelian group. □

Remark. In Chapter 252, in particular in the Algebraic Morse Inequalities Proposition 252.10, we will formulate and prove a pretty elementary algebraic statement that allows us in the Morse Inequalities Proposition 252.13 (2) to prove that the "sizes" of homology groups give lower bounds on the number of cells in a CW-structure. This result can be viewed as a refinement of the CW-Complex-H_*-Proposition 155.10 (1). □

Proof. Let X be a CW-complex.

(1) If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then it follows from the CW-Skeleton-H_*-Lemma 155.2 (1) that $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group of rank d. It follows from the Free Abelian Subgroup Theorem 97.14 that any subgroup of $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group of rank $\leq d$. It follows that

$$\mathrm{H}_n(X) \underset{\substack{\uparrow \\ \text{by the Singular-Cellular-H}_* \\ \text{Proposition 155.8}}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(X) = \overbrace{\ker\left(\mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{\partial_n^{\mathrm{CW}}} \mathrm{C}_{n-1}^{\mathrm{CW}}(X)\right) / \operatorname{im}\left(\mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{\partial_{n+1}^{\mathrm{CW}}} \mathrm{C}_n^{\mathrm{CW}}(X)\right)}^{\text{free abelian group of rank} \leq d}$$

is generated by at most d elements. The argument in the case that X has countably many cells of dimension n is very similar, we just need to use the Countability Lemma 1.37 several times.

[1617] Alternatively this claim follows from the CW-Skeleton-H_*-Lemma 155.2 (3).

(2) This statement follows immediately from (1). Alternatively note that it is in fact already a consequence of the CW-Skeleton-H_*-Lemma 155.2 (2).
(3) Suppose that X is an n-dimensional CW-complex. We have

$$\mathrm{H}_n(X) \underset{\substack{\uparrow \\ \text{by the Singular-Cellular-H}_*\text{-} \\ \text{-Proposition 155.8}}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(X) \underset{\substack{\uparrow \\ \text{we have } \mathrm{C}_{n+1}^{\mathrm{CW}}(X)=0 \text{ since} \\ X \text{ is } n\text{-dimensional}}}{=} \ker\left(\partial_n^{\mathrm{CW}} \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)\right),$$

i.e. $\mathrm{H}_n(X)$ is a subgroup of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X)$. It follows from the Free Abelian Subgroup Theorem 97.14 that the group $\mathrm{H}_n(X)$ itself is a free abelian group. ■

We conclude this section with the following application to smooth manifolds.

Proposition 179.10. (Smooth Manifolds-Invariants Proposition) Let M be a compact n-dimensional smooth manifold. The following holds:
(2) All homology groups are finitely generated abelian groups.
(3) For any $k > n$ we have $\mathrm{H}_k(M) = 0$.

Proof. By the Smooth Manifold-CW-Theorem 179.9, which we already mentioned in a sneak preview on page 2629, we know that every compact n-dimensional smooth manifold admits a CW-structure with finitely many cells and such that the maximal dimension of a cell is n. If we combine this result with the CW-Complex-H_*-Proposition 155.10 (1) and (2) we obtain immediately the desired result. ■

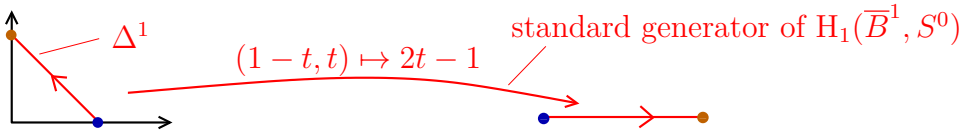
155.4. **The cellular boundary maps.** In this section we want to describe an approach to determine the cellular boundary homomorphisms for a given CW-complex. Note that in general an explicit description of a homomorphism $\varphi \colon C \to D$ between two free abelian groups requires the following:

(1) We need bases $\mathcal{C}$ and $\mathcal{D}$ for the free abelian groups.
(2) With respect to these bases we need to express φ by a matrix, i.e. we need to determine the integral matrix $\{a_{dc}\}_{d\in\mathcal{D},c\in\mathcal{C}}$ such that for every $c \in \mathcal{C}$ we have $\varphi(c) = \sum_{d\in\mathcal{D}} a_{dc} \cdot d$.

Our next goal is to apply this scheme to the cellular boundary maps $\partial_n^{\mathrm{CW}} \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$. Let us start with Step (1): By the Singular-Cellular-H_*-Proposition 155.8 (1) the key to describing such a basis is to fix a basis for $\mathrm{H}_n(\overline{B}^n, S^{n-1}) \cong \mathbb{Z}$, i.e. we need to fix a preferred generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$. We do so in the following definition:

Definition 155.11. Let $n \in \mathbb{N}$.
(1) We denote by $[S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ the standard generator of $\widetilde{\mathrm{H}}_n(S^{n-1})$ from Definition 150.8.
(2) The standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element with $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ where $\partial_n \colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ is the connecting homomorphism of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$.

Example. Recall that on page 2865 we saw that the standard generator of $\widetilde{\mathrm{H}}_0(S^0)$ is given by $[S^0] = [(+1) - (-1)] \in \widetilde{\mathrm{H}}_0(S^0)$.



It follows easily from this observation, the explicit description of the connecting homomorphism in the LES-of-Triple Proposition 146.5, and the explicit description of the boundary of a singular 1-simplex on page 2751, that the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ is represented by the following cycle in $\mathrm{C}_1(\overline{B}^1, S^0)$:

$$\begin{array}{rcl} \Delta^1 & \to & \overline{B}^1 \\ (1-t,t) & \mapsto & 2t-1. \end{array}$$ □

Now we can provide a scheme for equipping the cellular chain groups with bases.

Definition 155.12.

(1) As in Definition 164.17 we define a charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c \in \mathrm{Cell}(X)}$ of characteristic maps of the cells of X.

(2) Let X be a charted CW-complex as in (1). For an n-cell α we write[1618] [1619]

$$\Xi_\alpha \coloneqq \Phi_{\alpha *}([\overline{B}^n]) \in \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X).$$

With the above notation we can now formulate the following essential lemma.

Lemma 155.13. (Cellular Chain Complex–Basis Lemma) Suppose that we are given a charted CW-complex $(X, \{\Phi_c \colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c \in \mathrm{Cell}(X)})$. For every $n \in \mathbb{N}_0$ the set $\{\Xi_c\}_{c \in \mathrm{Cell}_n(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.

Proof. This lemma follows immediately from the CW-Skeleton-H_*-Lemma 155.2 (1). ■

Our next goal is, given a charted CW-complex X, to determine the matrix of the cellular boundary map

$$\partial_n^{\mathrm{CW}} \colon \underbrace{\mathrm{C}_n^{\mathrm{CW}}(X)}_{\substack{\text{free abelian group with} \\ \text{basis } \{\Xi_c\}_{c \in \mathrm{Cell}_n(X)}}} \to \underbrace{\mathrm{C}_{n-1}^{\mathrm{CW}}(X)}_{\substack{\text{free abelian group with} \\ \text{basis } \{\Xi_c\}_{c \in \mathrm{Cell}_{n-1}(X)}}}$$

As a warm-up we start out with the case $n = 1$. The following lemma shows that it is straightforward to determine the first cellular boundary map:

Lemma 155.14. (Cellular-First Boundary Map Lemma) Let X be a charted CW-complex. For a 1-cell α with characteristic map $\Phi_\alpha \colon \overline{B}^1 \to X^1$ we have

$$\partial_1^{\mathrm{CW}}(\underbrace{\Xi_\alpha}_{\substack{\uparrow \\ \text{element of } \mathrm{C}_1^{\mathrm{CW}}(X)}}) = \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)} \in \mathrm{C}_0^{\mathrm{CW}}(X).$$

[1618] Note that Φ_α defines a map $(\overline{B}^n, S^{n-1}) \to (X^n, X^{n-1})$ of pairs of topological spaces. Thus we get an induced map $\Phi_{\alpha *} \colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \mathrm{H}_n(X^n, X^{n-1})$, which we can apply to the standard generator $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.

[1619] It is quite common to just write α instead of Ξ_α, i.e. it is common to view the cells themselves as elements of the cellular chain complex. With this risque language one can then view $\mathrm{Cell}_n(X)$ as a basis of $\mathrm{C}_n^{\mathrm{CW}}(X)$. For clarity's sake we will not adopt this language.

In particular, if X has precisely one 0-cell, then the cellular boundary map ∂_1^{CW} is the zero map.

Proof. Recall that the cellular boundary map ∂_n^{CW} is defined as follows:

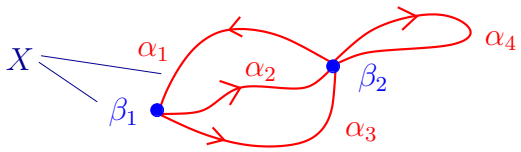
$$\begin{array}{ccc} \mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow{=:\partial_n^{\mathrm{CW}}} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \\ & {\scriptstyle \partial_n} \searrow \quad \nearrow {\scriptstyle \text{я}} & \\ & \mathrm{H}_{n-1}(X^{n-1}) & \end{array}$$

In our special case $n = 1$ we see that $X^{n-2} = \varnothing$, thus ∂_1^{CW} equals the connecting homomorphism ∂_1 of the pair (X^1, X^0) as defined in the LES-of-Pair Proposition 146.7. Now we see that

$$\begin{array}{ccccccc} & \text{by definition of } \Xi_\alpha \in \mathrm{C}_1^{\mathrm{CW}}(X) & & \text{by the above discussion} & & \text{see the description of } [\overline{B}^1] \text{ on page 2958} & \\ & \downarrow & & \downarrow & & \downarrow & \\ \partial_1^{\mathrm{CW}}(\Xi_\alpha) & = & \partial_1^{\mathrm{CW}}(\Phi_{\alpha *}([\overline{B}^1])) & = & \partial_1(\Phi_{\alpha *}([\overline{B}^1])) & = & \partial_1([(1-t,t) \mapsto \Phi_\alpha(2t-1)]) \\ & & & & & = & \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)}. \\ & & & & & \uparrow & \end{array}$$

see the definition of the connecting homomorphism ∂_1 in the LES-of-Pair Proposition 146.7 and the discussion of ∂_1 on page 275 ■

Example. We consider the charted 1-dimensional CW-complex X with two 0-cells β_1, β_2 and four 1-cells $\alpha_1, \ldots, \alpha_4$. that is given in the figure below:



By the Cellular-First Boundary Map Lemma 155.14 the cellular chain complex of X is of the form

$$0 \to \underbrace{\begin{array}{l} \mathbb{Z} \cdot \Xi_{\alpha_1} \\ \mathbb{Z} \cdot \Xi_{\alpha_2} \\ \mathbb{Z} \cdot \Xi_{\alpha_3} \\ \mathbb{Z} \cdot \Xi_{\alpha_4} \end{array}}_{=\mathrm{C}_1^{\mathrm{CW}}(X)} \xrightarrow{\begin{array}{c} \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\ \beta_1 \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \\ \beta_2 \end{array}} \underbrace{\begin{array}{l} \mathbb{Z} \cdot \Xi_{\beta_1} \\ \mathbb{Z} \cdot \Xi_{\beta_2} \end{array}}_{=\mathrm{C}_0^{\mathrm{CW}}(X)} \to 0.$$

It follows that

$$\begin{array}{ccccc} \mathrm{H}_n(X) & \cong & \mathrm{H}_n^{\mathrm{CW}}(X) & = & \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^3, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\ \uparrow & & \uparrow & & \\ \text{by the Singular-Cellular-H}_*\text{-Proposition 155.8} & & \text{by the above calculation and elementary linear algebra} & & \end{array}$$

□

The next proposition shows that we can calculate the cellular boundary maps ∂_n^{CW} in terms of degrees of suitable continuous self-maps of S^{n-1}.

Proposition 155.15. (Cellular Boundary–Degree Proposition) Let X be a charted CW-complex and let $n \in \mathbb{N}$.[1620] Given an n-cell α and an $(n-1)$-cell β we consider the

following continuous self-map of S^{n-1}:

$$S^{n-1} \xrightarrow{\Phi_\alpha|_{S^{n-1}}} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \uparrow \bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j \bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \xrightarrow{p_\beta} \overline{B}^{n-1}/S^{n-2} \xrightarrow[\cong]{g} S^{n-1}$$

where we adopt the following notation:

- For $c \in \mathrm{Cell}(X)$ we denote by Φ_c the corresponding characteristic map.
- $q\colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the natural projection.
- Let $j \in \mathrm{Cell}_{n-1}(X)$. Since $\Phi_j(S^{n-2}) \subseteq X^{n-2}$ we see that the characteristic map $\Phi_j\colon \overline{B}^{n-1} \to X^{n-1}$ descends to a continuous map $\overline{\Phi}_j\colon \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$.
- $\bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\colon \bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$ is the natural map provided by the Wedge Proposition 96.2. Note that it follows from the Skeleton-Quotient Lemma 139.12 that this map is a homeomorphism.
- $p_\beta\colon \bigvee_{j\in \mathrm{Cell}_n(X)} \overline{B}^{n-1}/S^{n-2} \to \overline{B}^{n-1}/S^{n-2}$ is the natural map that is induced by the identity on the β-copy of $\overline{B}^{n-1}/S^{n-2}$ and the constant map on all other copies of $\overline{B}^{n-1}/S^{n-2}$.
- $g\colon \overline{B}^{n-1}/S^{n-2} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from the Ball Quotient-Sphere Lemma 8.10.

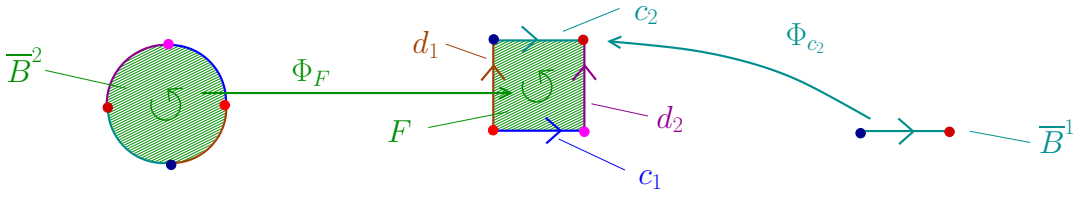
For any n-cell α we have[1621]

$$\partial_n^{\mathrm{CW}}(\Xi_\alpha) = \sum_{\beta\in \mathrm{Cell}_{n-1}(X)} \deg(\Omega_{\beta,\alpha}) \cdot \Xi_\beta \in \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$$

where[1622]

$$\Omega_{\beta,\alpha} := g \circ p_\beta \circ \Big(\bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\Big)^{-1} \circ q \circ \Phi_\alpha|_{S^{n-1}}\colon S^{n-1} \to S^{n-1}.$$

Example. We consider the CW-complex $W = [0,1] \times [0,1]$ that is shown below.

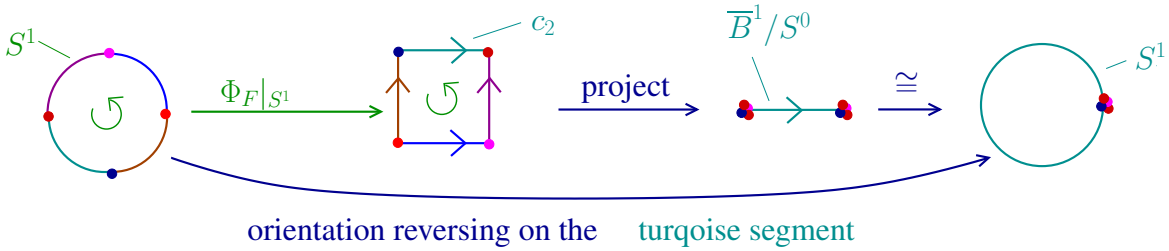


We also show a characteristic map for the unique 2-cell F and for the four 1-cells. We now want to calculate $\partial_2^{\mathrm{CW}}(\Xi_F)$. It follows from the Cellular Boundary–Degree Proposition 155.15. that the Ξ_{c_2}-coefficient of $\partial_2^{\mathrm{CW}}(\Xi_F)$ is given by the degree of the map that is shown in the figure below:

[1620] It takes a while to see that the statement for $n = 1$ is equivalent to the statement of the Cellular-First Boundary Map Lemma 155.14.

[1621] We refer to Definition 151.3 for the definition of the degree of a continuous self-map of a sphere.

[1622] It follows easily from the CW-Complex-Finiteness Theorem 136.16 that for all but finitely many β the given map $\Omega_{\beta,\alpha}\colon S^{n-1} \to S^{n-1}$ is the constant map, which implies that for all but finitely many β we have $\deg(\alpha_{\beta,\alpha}) = 0$.



Using the Degree-via-Local Orientations Proposition 152.11 one can easily verify that the degree is -1. If one proceeds like this for all 1-cells we see that

$$\partial_2^{\mathrm{CW}}(\Xi_F) \;=\; \Xi_{c_1} + \Xi_{\partial_2^{\mathrm{CW}}} - \Xi_{c_2} - \Xi_{\partial_1^{\mathrm{CW}}}.$$ □

The proof of the Cellular Boundary–Degree Proposition 155.15 requires the following lemma:

Lemma 155.29. (Standard Generator-of-$\overline{B}^n$-Lemma) Let $n \in \mathbb{N}_0$. If we denote by $\mathrm{ю}\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1})$ the natural map and if we denote by $g\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from the Ball Quotient-Sphere Lemma 8.10, then

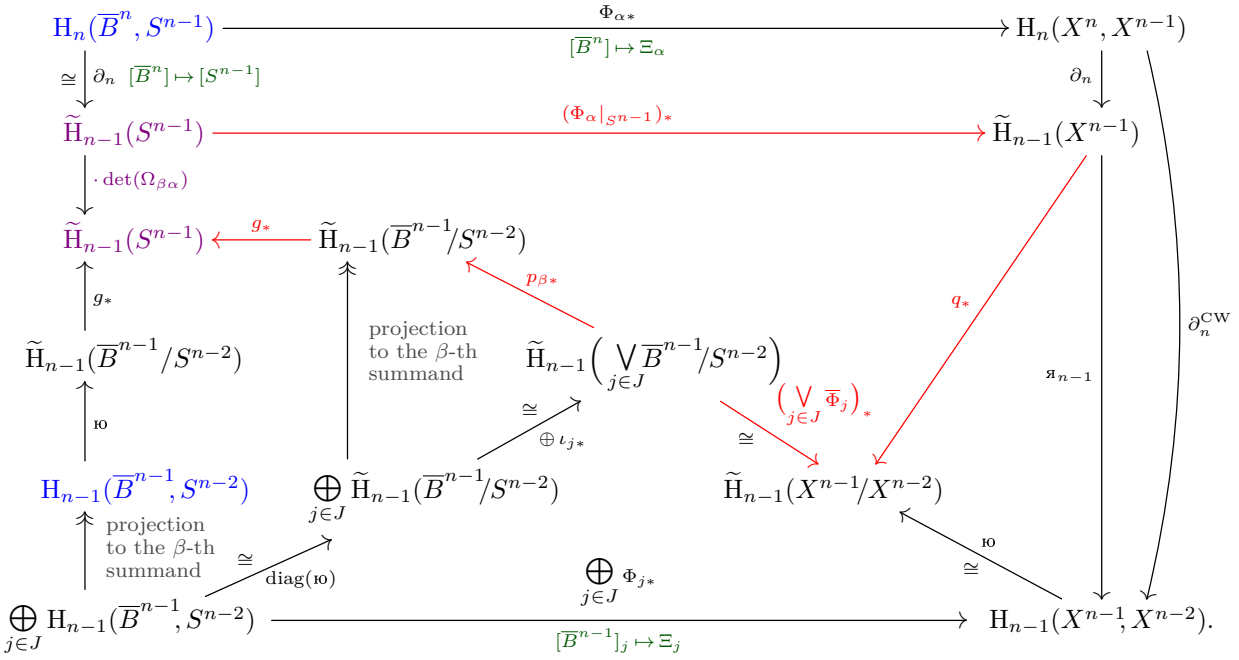
$$(g_* \circ \mathrm{ю})([\overline{B}^n]) \;=\; [S^n] \quad \in \widetilde{\mathrm{H}}_n(S^n).$$

Proof of the Standard Generator-of-$\overline{B}^n$-Lemma 155.29. First note that it follows immediately from the Relative-to-Quotient-H_*-Lemma 149.11 that the homomorphism $g_* \circ \mathrm{ю}\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(S^n)$ is an isomorphism. Since $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ and $[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$ are generators we thus see that $(g_* \circ \mathrm{ю})([\overline{B}^n]) = \pm[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$. Therefore it remains to determine the precise sign. This is a little tricky and we outsource the proof to Appendix 155.8. For most purposes the precise sign is not really relevant though. For example in the statement of the Cellular Boundary–Degree Proposition 155.15 a different sign in the equality $(g_* \circ \mathrm{ю})([\overline{B}^n]) = \pm[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$ changes at worst all $\deg(\Omega_{\beta,\alpha})$ by the *same* sign, which has no effect later on when we need to determine (co-) kernels. ■

Proof of the Cellular Boundary–Degree Proposition 155.15. We adopt the following notation:

(1) We write $I := \mathrm{Cell}_n(X)$ and $J := \mathrm{Cell}_{n-1}(X)$.
(2) Given $k \in J$ we denote by $\iota_k\colon \overline{B}^{n-1}/S^{n-2} \to \bigvee_{j\in J} \overline{B}^{n-1}/S^{n-2}$ the natural injection.

For any n-cell α and any $(n-1)$-cell β we consider the following diagram:



We make the following claim:

Claim. The diagram commutes and all the maps decorated with "$\cong$" are indeed isomorphisms.

Proof. We consider the various parts of the diagram.

(1) It follows from the naturality of the connecting homomorphism, see the Reduced LES-of-Pair Proposition 147.8 (2), that the upper rectangle in the diagram commutes.

(2) The red arrows trace out the map $\Omega_{\beta,\alpha}$ of which we take the degree. In other words, the oddly shaped hexagon in the center of the diagram commutes by definition.

(3) We consider the hexagon to the lower left of the diagram. It follows easily from the definition of the diagonal map diag(ю) that the hexagon commutes on each summand, hence it commutes for the direct sum.

(4) (a) Note that the map $p_\beta \circ \iota_j$ equals $\mathrm{id}\,\overline{B}^{n-1}/S^{n-2}$ if $j = \beta$ and that it is constant otherwise. It follows easily from this observation that the triangle towards the center left, involving the map $p_{\beta *}$ commutes.

(b) It follows from the Wedge-H_*-Proposition 149.16 that the map decorated with "$\cong$" in the triangle is indeed an isomorphism.

(5) In the proof of the CW-Skeleton-H_*-Lemma 155.2 (1) we showed that the large triangle at the bottom commutes and we showed that all the maps involved in the large triangle are indeed isomorphisms.

(6) (a) It follows from the Relative-to-Quotient-H_*-Lemma 149.11 (1) that the triangle to the bottom right commutes.

(b) It follows from the Relative-to-Quotient-H_*-Proposition 149.13 that the diagonal map at the very bottom right is an isomorphism. ⊞

Next note that by the definition of the standard generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ in Definition 155.11 we have $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ and note that the Standard Generator-of-$\overline{B}^n$-Lemma 155.29 says that $g_*(\mathrm{ю}([\overline{B}^{n-1}])) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$. The reader who is still awake will notice that the proposition follows immediately from the claim and this observation. ■

155.5. **The homology groups of surfaces.** In this section we will use the Cellular Boundary–Degree Proposition 155.15 to calculate the isomorphism types of the homology groups of compact 2-dimensional topological and smooth manifolds. First let us recall the following standard notation.

Notation 155.16. Given $g \in \mathbb{N}_0$ and given $k \in \mathbb{N}$ we write as usual

$$\begin{aligned} \Sigma_g &:= \text{surface of genus } g \\ N_k &:= \text{non-orientable surface of genus } k. \end{aligned}$$

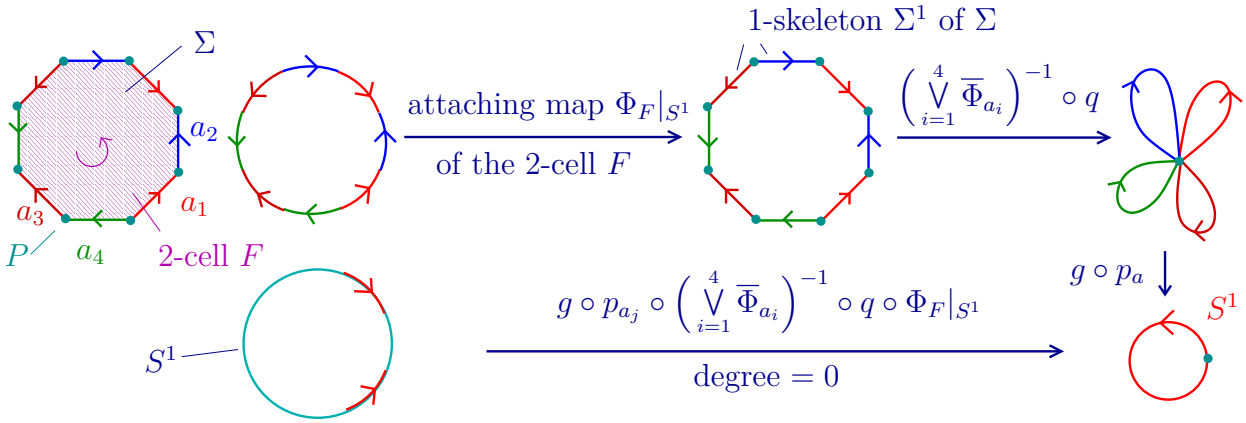
Furthermore given $m \in \mathbb{N}_0$ we write [1623] [1624]

$$\begin{aligned} \Sigma_{g,m} &:= \text{the surface of genus } g \text{ minus } m \text{ open disks} \\ N_{k,m} &:= \text{the non-orientable surface of genus } k \text{ minus } m \text{ open disks.} \end{aligned}$$

In the following we will first consider some explicit examples before stating a proposition that gives us the calculation of the isomorphism types of the homology groups for all the above surfaces.

Example.

(1) We consider the surface Σ_2 of genus 2. As usual we view Σ_2 as a charted CW-complex with one 0-cell P, four 1-cells a_1, a_2, a_3, a_4 and one 2-cell F.



Note that it follows easily from the definitions, or alternatively from the Degree-via-Local Orientations Proposition 152.11, that for each $j \in \{1, \dots, 4\}$ the degree of the map $\Omega_{F,a_j} = g \circ p_a \circ \left(\bigvee_{i=1}^4 \overline{\Phi}_{a_i}\right)^{-1} \circ q \circ \Phi_F|_{S^1} \colon S^1 \to S^1$ is zero. It follows from the Cellular Boundary–Degree Proposition 155.15 that the cellular boundary map ∂_2^{CW} in the cellular chain complex of Σ_2 is zero. Since Σ_2 has only one 0-cell we know from

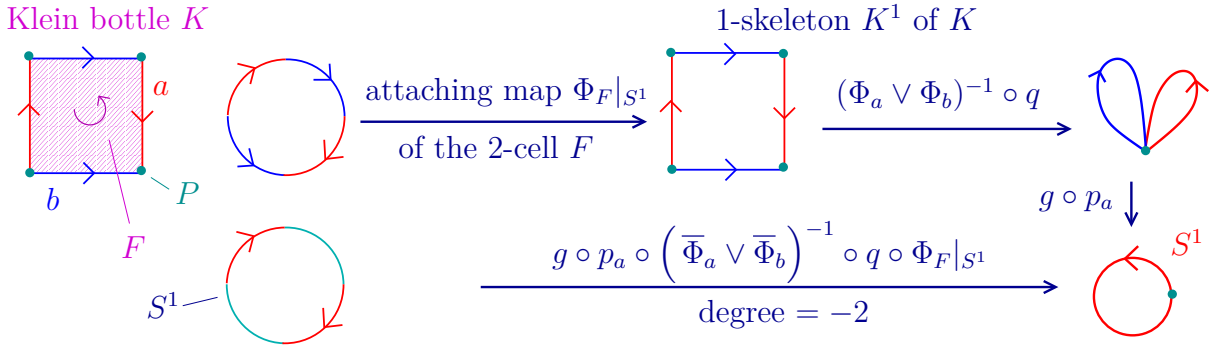
[1623] Recall Definition 101.10 of what we mean by a "surface minus m open disks".

[1624] Recall that in the Surface Classification Theorem 101.16 we saw that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $N_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

the Cellular-First Boundary Map Lemma 155.14 that the cellular boundary map ∂_1^{CW} is also zero. Thus we see that

$$\begin{array}{ccccccc} \mathrm{H}_n(\Sigma_2) & \cong & \mathrm{H}_n^{\mathrm{CW}}(\Sigma_2) & = & \mathrm{C}_n^{\mathrm{CW}}(\Sigma_2) & \cong & \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\ & \uparrow & & \uparrow & & & \\ & \text{by the Singular-Cellular-H}_*\text{-Proposition 155.8} & & \text{since all the cellular boundary maps are zero} & & & \end{array}$$

(2) As a reality check we consider once again the Klein bottle $K = N_2$. We view K as a charted CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F, as shown in the figure below.



It follows from the Cellular Boundary–Degree Proposition 155.15, the Degree-via-Local Orientations Proposition 152.11 and the Cellular-First Boundary Map Lemma 155.14 that the cellular chain complex is given by

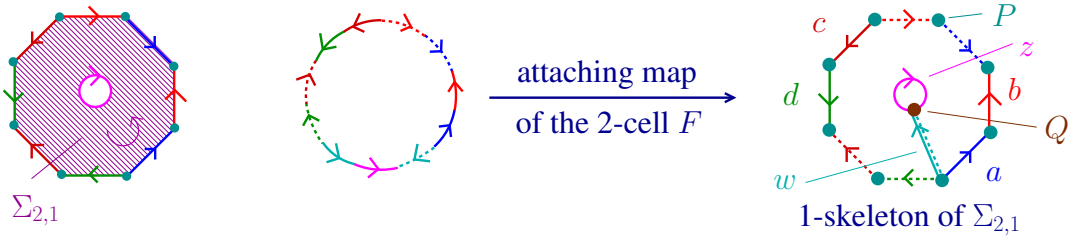
$$0 \;\to\; \mathbb{Z} \cdot \Xi_F \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \begin{matrix} \mathbb{Z} \cdot \Xi_a \\ \mathbb{Z} \cdot \Xi_b \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \cdot \Xi_P \;\to\; 0.$$

Therefore we see that

$$\begin{array}{ccccc} \mathrm{H}_n(K) & \cong & \mathrm{H}_n^{\mathrm{CW}}(K) & = & \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\ & \uparrow & & \uparrow & \\ & \text{by the Singular-Cellular-H}_*\text{-Proposition 155.8} & & \text{by the above calculation and elementary linear algebra} & \end{array}$$

This is of course the same result that we had already obtained in the Klein Bottle-H$_*$-Lemma 153.10.

(3) We consider $\Sigma_{2,1}$ and we equip it with the charted CW-structure that is illustrated in the figure below.



This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and one 2-cell F. Using the Cellular Boundary–Degree Proposition 155.15 and using the Cellular-First Boundary Lemma 155.14 one can now easily show that the cellular chain complex is of the following form:

$$0 \to \mathbb{Z}\cdot\Xi_F \xrightarrow{\begin{matrix} F \\ a \\ b \\ c \\ d \\ w \\ z \end{matrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot\Xi_a \\ \mathbb{Z}\cdot\Xi_b \\ \mathbb{Z}\cdot\Xi_c \\ \mathbb{Z}\cdot\Xi_d \\ \mathbb{Z}\cdot\Xi_w \\ \mathbb{Z}\cdot\Xi_z \end{matrix} \xrightarrow{\begin{matrix} \quad a & b & c & d & w & z \\ P \\ Q \end{matrix}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot\Xi_P \\ \mathbb{Z}\cdot\Xi_Q \end{matrix} \to 0.$$

Now we see that

$$\mathrm{H}_n(\Sigma_{2,1}) \cong \mathrm{H}_n^{\mathrm{CW}}(\Sigma_{2,1}) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

by the Singular-Cellular-H_*-Proposition 155.8 — by the above chain complex and some elementary linear algebra □

In the following proposition we give the calculation for the isomorphism types of the homology groups of all surfaces $\Sigma_{g,n}$ and $N_{k,m}$:

Proposition 155.17. (Surface-H_*-Proposition) For any $g \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \text{Furthermore for } m \geq 1 \text{ we have} \quad \mathrm{H}_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$\mathrm{H}_n(N_k) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \text{Furthermore for } m \geq 1 \text{ we have} \quad \mathrm{H}_n(N_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. The calculation of the homology groups Σ_g is a slight generalization of the arguments that we provided on page 2963. Furthermore the case N_k is dealt with in the same way as we dealt with the Klein bottle $K = N_2$ on page 2964. Finally note that for $m \in \mathbb{N}$ the calculation of the homology groups of $\Sigma_{g,m}$ and $N_{k,m}$ is a slight generalization of the argument preceding the proposition. We leave the details to the reader. ■

In Appendix 155.7 we will study homology groups of surfaces in greater detail, in particular we will deal with the following questions:

(1) What are explicit bases for the first homology groups of surfaces?.
(2) Given a surface $\Sigma_{g,n}$ with $n \geq 1$, what can we say about the inclusion-induced map $\mathrm{H}_1(\partial\Sigma_{g,n}) \to \mathrm{H}_1(\Sigma_{g,n})$?

155.6. **The homology groups of real projective spaces.** Recall that in the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 we calculated the homology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In this section we will finally compute the homology groups of the real projective spaces

$\mathbb{R}\mathrm{P}^n$. This calculation is significantly trickier than the calculation for $\mathbb{C}\mathrm{P}^n$, since this time we will need the theory of local degrees, which we developed in Section 152.2, to determine cellular boundary maps.

Proposition 155.18. ($\mathbb{R}\mathrm{P}^n$-H_*-Proposition)

(1) For any $n \in \mathbb{N}_0$ we have[1625][1626]

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 338 we showed that for any $k \leq n$ the map $\mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ that is given by $[x] \mapsto [(x, 0)]$ is an embedding. For the duration of the proof we use this embedding as an excuse to view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$.

Next recall we recall the CW-structure on $\mathbb{R}\mathrm{P}^n$ which we introduced on page 2628: For each $k = 0, \ldots, n$ we define

$$\begin{aligned} \Phi_k \colon \overline{B}^k &\to \mathbb{R}\mathrm{P}^k \\ x &\mapsto [(x, \sqrt{1 - \|x\|^2})]. \end{aligned}$$

We saw in the proof of the Projective Space–Cell Attachment Lemma 135.7 that the map

$$\begin{aligned} \mathbb{R}\mathrm{P}^{k-1} \cup_{\mathbb{R}\mathrm{P}^{k-1} \xleftarrow{\Phi_k|_{S^{k-1}}} S^{k-1}} \overline{B}^k &\xrightarrow{\cong} \mathbb{R}\mathrm{P}^k \\ [x] &\mapsto \begin{cases} [(x, 0)], & \text{if } x \in \mathbb{R}\mathrm{P}^{k-1} \\ \Phi_k(x), & \text{if } x \in \overline{B}^k \end{cases} \end{aligned}$$

is a homeomorphism. Thus we see inductively that we can view $\mathbb{R}\mathrm{P}^n$ as a charted CW-complex with a single cell in the dimensions $0, \ldots, n$ where the characteristic maps are given precisely by the maps $\Phi_0, \ldots, \Phi_n$.

Claim. For each $k = 1, \ldots, n$ the cellular boundary map

$$\partial_k^{\mathrm{CW}} \colon \mathbb{Z} \cong \mathrm{C}_k^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$$

is given by multiplication by $\pm(1 + (-1)^k)$.

Proof. Let $k \in \{1, \ldots, n\}$. We denote by $q \colon \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$ the projection and we denote by $g \colon \overline{B}^{k-1}/S^{k-2} \to S^{k-1}$ the homeomorphism from the Ball Quotient-Sphere Lemma 8.10. By the Cellular Boundary–Degree Proposition 155.15 it suffices

[1625] The calculation looks more complicated than it is, in fact the most efficient way to remember the calculation is to remember the description of the cellular chain complex given in the proof.

[1626] For $n = 2$ we thus obtain, as we should, the same result as in Section 153.2.

to show that the degree of the map

$$S^{k-1} \xrightarrow{\Phi_k|_{S^{k-1}}} \mathbb{R}\mathrm{P}^{k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \xrightarrow{(\overline{\Phi}_{k-1})^{-1}} \overline{B}^{k-1}/S^{k-2} \xrightarrow{g} S^{k-1}$$

$$=:\Omega$$

equals $\pm(1+(-1)^k)$. We will do so using the Degree-via-Local Orientations Proposition 152.11. To apply this proposition we recall a few definitions and we make a few preparations:

(a) If $\Omega\colon S^{k-1} \to S^{k-1}$ is a local diffeomorphism at a point x, then as in Definition 152.1 we write

$$\deg(\Omega, x) \coloneqq \begin{cases} +1, & \text{if } \Omega \text{ is orientation-preserving at } x \\ -1, & \text{if } \Omega \text{ is orientation-reversing at } x. \end{cases}$$

(b) If $f\colon S^{k-1} \to S^{k-1}$ is a diffeomorphism, then we write

$$\deg(f) \coloneqq \begin{cases} +1, & \text{if } f \text{ is orientation-preserving} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

(c) If $\Omega\colon S^{k-1} \to S^{k-1}$ is a local diffeomorphism at a point x and if $f\colon S^{k-1} \to S^{k-1}$ is a diffeomorphism, then it follows immediately from $\mathrm{D}(\Omega\circ f)_{f^{-1}(x)} = \mathrm{D}\Omega_x \circ \mathrm{D}\Omega_{f^{-1}(x)}$ that $\deg(\Omega\circ f, f^{-1}(x)) = \deg(\Omega, x)\cdot\deg(f)$.

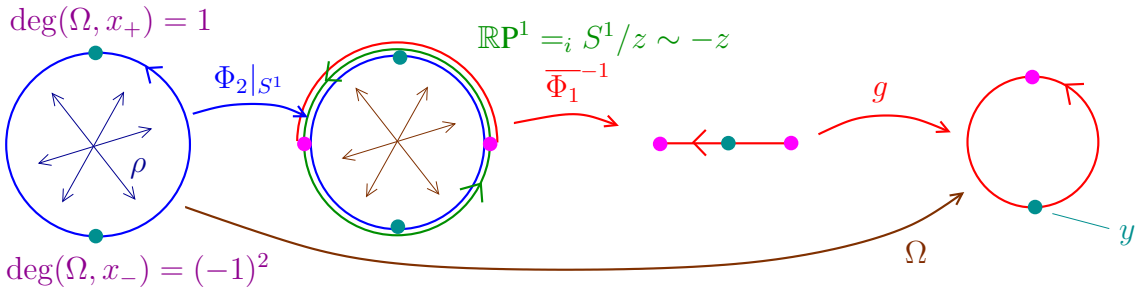
Now we turn to our specific situation:

(d) We pick the point $y = (0,\dots,0,-1) \in S^{k-1}$. The preimage of y under the map g is $[(0)]$ and the preimages of y under the map Ω are $x_+ = (0,\dots,0,1)$ and $x_- = (0,\dots,0,-1)$. One can easily verify that the restrictions of Ω to the open hemispheres $S^n_{>0}$ and $S^n_{<0}$ are smooth embeddings. Since $x_+ \in S^n_{>0}$ and $x_- \in S^n_{<0}$ we see that Ω is a local diffeomorphism around x_+ and x_-.[1627]

(e) Let $\rho\colon S^{k-1} \to S^{k-1}$ be the reflection in the origin, i.e. ρ is given by $z \mapsto -z$.

(f) By the Sphere-Degree-Matrix Lemma 151.7 we have $\deg(\rho) = \det(-\mathrm{id}_k) = (-1)^k$.

(g) Note that $(\Phi_k|_{S^{k-1}})\circ\rho = \Phi_k|_{S^{k-1}}\colon S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$. This implies that $\Omega\circ\rho = \Omega$.



We now see that

$$\begin{array}{rcll} \deg(\Omega) & \overset{\text{by the Degree-via-Local Orientations Proposition 152.11 and by (d)}}{=} & \deg(\Omega, x_+) + \deg(\Omega, x_-) & \overset{\text{since by (g) we know that } \Omega\circ\rho = \Omega}{=} \deg(\Omega, x_+) + \deg(\Omega\circ\rho, x_-) \\ & = & \deg(\Omega, x_+) + \deg(\Omega, x_+)\cdot\deg(\rho) & = \underbrace{\deg(\Omega, x_+)}_{=\pm 1}\cdot(1 + \underbrace{\deg(\rho)}_{=(-1)^k \text{ by (f)}}). \\ & & \uparrow \text{ follows from (c), since } \rho(x_+) = x_- & \end{array}$$

[1627] If one struggles through all the orientation conventions and all the maps, then one can show that $\deg(\Omega, x_+) = 1$ and that $\deg(\Omega, x_-) = (-1)^k$. We will not go down that route, instead we will use a more indirect, but hopefully more robust and more convincing argument.

⊞

It follows from the claim that the cellular chain complex of $\mathbb{R}\mathrm{P}^n$ is of the form

$$0 \to \underset{\substack{\uparrow\\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\pm(1+(-1)^n)} \mathbb{Z} \to \dots \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \underset{\substack{\uparrow\\ \text{degree } 0}}{\mathbb{Z}} \to 0.$$

It is now straightforward to determine the cellular homology groups which, by the Singular-Cellular-H_*-Proposition 155.8, are isomorphic to the singular homology groups of $\mathbb{R}\mathrm{P}^n$.

(2) On page 2628 we equipped $\mathbb{R}\mathrm{P}^\infty = \operatorname{colim}\mathbb{R}\mathrm{P}^n$ with a CW-structure with precisely one cell in each dimension. Basically the same calculation as in (1) gives the desired result. Alternatively, as in the proof of the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 (2) one can use the calculation of (1) and the Exhaustions-H_*-Proposition 154.2 to obtain the desired result. We leave it to the reader to work out the details for this second approach. ■

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes? There are two related answers:

(1) We also want to study topological spaces, e.g. topological manifolds, that are not (known to be) CW-complexes.

(2) It is much harder to prove some statements using cellular chain complexes, for example it is very difficult to show (without making recourse to singular homology) that homeomorphic (let alone homotopy equivalent) CW-complexes have isomorphic homology groups.

A general slogan in topology (and many other fields of mathematics) is that for calculations it is usually best to work with "small objects" (in our case the cellular chain complex), whereas for proving general properties, it is often best to work with "big objects" (in our case the singular chain complex). □

155.7. **Appendix: More on homology groups of surfaces.** In Section 155.5 we determined the *isomorphism types* of the homology groups of compact surfaces. But the calculation is somewhat unsatisfactory, since we did not specify generators of the homology groups.

The problem is that in Section 155.5 we computed cellular homology groups of surfaces and then we appealed to the Singular-Cellular Homology Isomorphism Proposition 155.8. For the cellular homology groups we know explicitly which cycles in the cellular chain complex represent generators of the cellular homology groups. Unfortunately the Singular-Cellular Homology Isomorphism Proposition 155.8 only gives us the isomorphism type of the singular homology groups, it does not give us a way to obtain cycles in the singular chain complex from cycles in the cellular chain complex.

The next lemma will give us a tool to deal with this issue in some basic cases.

Lemma 155.19. (Singular-to-Cellular Homology Lemma) Let X be a charted CW-complex, let $n \in \mathbb{N}$ and let $\Phi_\alpha \colon \overline{B}^n \to X$ be the characteristic map of an n-cell α such that $\Phi_\alpha(S^{n-1})$ consists of a single point. Note that this implies that Φ_α descends to a continuous

map $\overline{\Phi}_\alpha\colon \overline{B}^n/S^{n-1} \to X$. Let $g\colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 8.10. Under the natural isomorphism $\Theta_X\colon \mathrm{H}_n(X) \to \mathrm{H}_n^{\mathrm{CW}}(X)$ of the Singular-Cellular-H_*-Proposition 155.8 we have

$$\Theta_X(\underbrace{(\overline{\Phi}_\alpha \circ g^{-1})_*([S^n])}_{\in \mathrm{H}_n(X)}) \;=\; [\Xi_\alpha] \;\in\; \mathrm{H}_n^{\mathrm{CW}}(X).$$

($[\Xi_\alpha]$: see Definition 155.12)

Proof. As usual we can view the sphere $\overline{B}^n/S^{n-1}$ as a CW-complex with a single 0-cell given by the point $[S^{n-1}]$ and with a single n-cell. This allows us to formulate the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccccc}
\widetilde{\mathrm{H}}_n(S^n) & \xleftarrow{g_*} & \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1}) & \xleftarrow{\text{ю}} & \mathrm{H}_n(\overline{B}^n, S^{n-1}) \\
\downarrow{\scriptstyle \text{ж}}\cong & & \downarrow{\scriptstyle \text{ж}}\cong & & \\
\mathrm{H}_n(S^n,\{N\}) & \xleftarrow[\cong]{g_*} & \mathrm{H}_n(\overline{B}^n/S^{n-1},[S^{n-1}]) & \searrow^{a\mapsto[a]} & \\
\uparrow{\scriptstyle \text{я}}\cong & & \uparrow{\scriptstyle \text{я}}\cong & & \\
\mathrm{H}_n(S^n) & \xleftarrow[\cong]{g_*} & \mathrm{H}_n(\overline{B}^n/S^{n-1}) & \xrightarrow[\Theta_{\overline{B}^n/S^{n-1}}]{} & \mathrm{H}_n^{\mathrm{CW}}(\overline{B}^n/S^{n-1}) \\
 & & \downarrow{\scriptstyle \overline{\Phi}_{\alpha*}} & & \downarrow{\scriptstyle \overline{\Phi}_{\alpha*}} \\
 & & \mathrm{H}_n(X) & \xrightarrow{\Theta_X} & \mathrm{H}_n^{\mathrm{CW}}(X)
\end{array}$$

and all the maps decorated with "$\cong$" are isomorphisms

Proof. We make the following observations:

- The maps ж: $\widetilde{\mathrm{H}}_n(S^n) \to \mathrm{H}_n(S^n,\{N\})$ and ж: $\widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1}) \to \mathrm{H}_n(\overline{B}^n/S^{n-1},[S^{n-1}])$ are isomorphisms by the Reduced-to-Relative-H_*-Lemma 147.9. It follows from the naturality of ж that the top rectangle commutes.
- Since $n \geq 1$ we see that the vertical maps я are isomorphisms. Since these maps are natural the rectangle towards the center left also commutes.
- Note that by our above choice of the CW-structure on $X := \overline{B}^n/S^{n-1}$ we have that $X = X^n = \overline{B}^n/S^{n-1}$ and $X^{n-1} = \{[S^{n-1}]\}$. Note that this implies that we have the equalities $\mathrm{H}_n^{\mathrm{CW}}(X^n) = \mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{H}_n(\overline{B}^n/S^{n-1},[S^{n-1}])$. It follows from this discussion that the diagonal map exists.
- It follows from the Singular-Cellular-H_*-Proposition 155.8 (1) that the triangle commutes.
- Note that with our CW-structure on $\overline{B}^n/S^{n-1}$ the map $\overline{\Phi}_\alpha\colon \overline{B}^n/S^{n-1} \to X$ is cellular. It follows that the right vertical map exists.
- It follows from the Singular-Cellular-H_*-Proposition 155.8 (2) that the bottom rectangle commutes. ⊞

The lemma follows from all of the above and the following three further observations:

- By the Standard Generator-of-$\overline{B}^n$-Lemma 155.29 we know that $(g_* \circ \text{ю})([\overline{B}^n]) = [S^n]$.
- Since $n \geq 1$ the composition of the two vertical maps to the top left and top center is the identity.
- It follows almost immediately from the definition of Ξ_α in Definition 155.12 that we have the equality $\Phi_{\alpha*}([\text{ж} \circ \text{ю})([\overline{B}^n]) = [\Phi_{\alpha*}([\overline{B}^n])] = [\Xi_\alpha]$. ■

Before we return to the study of homology groups of surfaces let us introduce the following notation:

Notation 155.20.

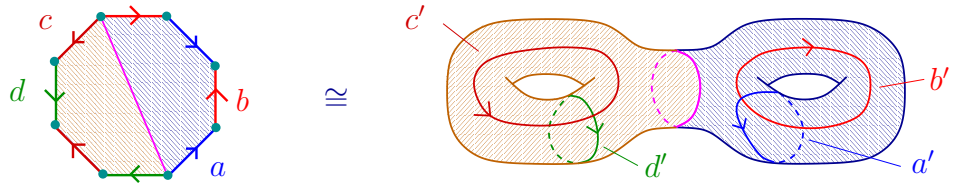
(1) Let X be a topological space. A closed curve is a continuous map $f\colon S^1 \to X$. We call $f_*([S^1]) \in \mathrm{H}_1(X)$ the homology class represented by the closed curve.

(2) Let M be a smooth manifold. Let C be an oriented submanifold of M (or an oriented boundary component of M) that is diffeomorphic[1628] to S^1. We pick an orientation-preserving diffeomorphism $\gamma\colon S^1 \to C$ and we set

$$[C] \coloneqq \gamma_*([S^1]) \quad \in \mathrm{H}_1(M).$$

It follows from the Diffeotopies-of-S^1-Proposition 42.8 and the Homotopy-H_*-Proposition 144.12 that $[C] \in \mathrm{H}_1(M)$ is well-defined. We refer to $[C]$ as the homology class represented by C.

Example. In the figure below on the left-hand side we show the surface Σ_2 of genus 2 as a charted CW-complex with four 1-cells a, b, c, d.



We make the following two observations:

(1) On page 2963 we saw that the cellular boundary maps are all zero. This implies that $[\Xi_a], [\Xi_b], [\Xi_c], [\Xi_d]$ form a basis for $\mathrm{H}_1^{\mathrm{CW}}(\Sigma_2) \cong \mathbb{Z}^4$.

(2) The closed curves a, b, c, d are isotopic to the closed curves a', b', c', d' on the right-hand side.

It follows from (a) and (b), the Singular-to-Cellular Homology Lemma 155.19 and the Homotopy-H_*-Proposition 144.12 that a', b', c', d' represent a basis for $\mathrm{H}_1(\Sigma_2) \cong \mathbb{Z}^4$. □

Next we want to study the homology groups of surfaces with boundary in greater detail:

Lemma 155.21. (Surface-with-Boundary-H_*-Lemma) Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. As in Notation 101.12 we consider $\Sigma_{g,n}$, i.e. we consider the surface of genus g minus n open disks. We denote by $C_1, \dots, C_n$ the components of $\partial\Sigma_{g,n}$ and we equip them with the orientation given by the Boundary Orientation Lemma 31.38.

(1) There exists an isomorphism $\mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}$. Furthermore there exists a unique generator $[\Sigma_{g,n}] \in \mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n})$ with

$$\underset{\substack{\uparrow \\ \text{connecting homomorphism of} \\ \text{the pair } (\Sigma_{g,n}, \partial\Sigma_{g,n})}}{\partial_2}([\Sigma_{g,n}]) \quad = \quad [C_1] + \cdots + [C_n] \quad \in \mathrm{H}_1(\partial\Sigma_{g,n}).$$

[1628] Note that it follows from the Submanifold Smooth Structure Lemma 25.23 or the Smooth Manifold–Boundary Proposition 25.27 (1) that we can view C as a smooth manifold in its own right.

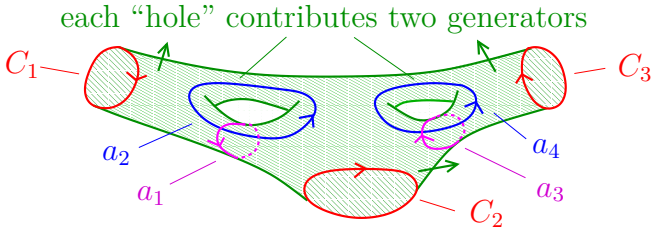
(2) The following sequence is exact:

$$0 \to \underbrace{\mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n})}_{=\mathbb{Z}\cdot[\Sigma_{g,n}]} \xrightarrow[[\Sigma_{g,n}]\mapsto \sum_{i=1}^{n}[C_i]]{\partial_2} \underbrace{\mathrm{H}_1(\partial\Sigma_{g,n})}_{=\bigoplus_{i=1}^{n}\mathbb{Z}\cdot[C_i]} \to \mathrm{H}_1(\Sigma_{g,n}) \to \underbrace{\mathrm{H}_1(\Sigma_g)}_{\cong\mathbb{Z}^{2g}} \to 0.$$

Here the first map is the connecting homomorphism ∂_2 of the pair $(\Sigma_{g,n}, \partial\Sigma_{g,n})$ and the other two maps are induced by the inclusions $\partial\Sigma_{g,n} \to \Sigma_{g,n}$ and $\Sigma_{g,n} \to \Sigma_g$.

(3) A basis for $\mathrm{H}_1(\Sigma_{g,n}) \cong \mathbb{Z}^{2g+n-1}$ can be obtained as follows:
   (a) We can and will pick $a_1, \dots, a_{2g} \in \mathrm{H}_1(\Sigma_{g,n})$ which represent a basis of $\mathrm{H}_1(\Sigma_g)$.
   (b) We pick any $i \in \{1, \dots, n\}$.
   Then $a_1, \dots, a_{2g}, [C_1], \dots, \widehat{[C_i]}, \dots, [C_n] \in \mathrm{H}_1(\Sigma_{g,n})$ form a basis of $\mathrm{H}_1(\Sigma_{g,n})$.

Example. As we saw in the figure above, a basis for $\mathrm{H}_1(\Sigma_g)$ is given by a pair of curves for each "hole".



two out of the three boundary components are needed to obtain, together with a_1, a_2, a_3, a_4, a basis for $\mathrm{H}_1(\Sigma_{2,3})$

It follows from the Surface-with-Boundary-H_*-Lemma 155.21 (2) that a basis for the homology group $\mathrm{H}_1(\Sigma_{g,n}) \cong \mathbb{Z}^{2g+n-1}$ is obtained by a pair of curves for each "hole" together with all *but one* boundary curve. □

Remark.

(1) It follows from the Surface Classification Theorem 101.16 that the statement of the Surface-with-Boundary-H_*-Lemma 155.21 also holds for any compact orientable connected 2-dimensional smooth manifold with non-empty boundary.

(2) In the Kernel-of-Boundary Inclusion Corollary 185.20 and in the Topological Fundamental Class-of-Boundary 194.14 we will generalize the statement of the Surface-with-Boundary-H_*-Lemma 155.21 to compact oriented smooth and topological manifolds in a suitable way. □

Sketch of proof. We consider the surface $\Sigma_g = E_{2g}/\sim$ of genus g and we consider n disjoint closed Euclidean disks $D_1, \dots, D_n \subseteq E_{2g} \subseteq \Sigma_g$. We set $\Sigma_{g,n} = \Sigma_g \setminus (\mathring{D}_1 \cup \dots \cup \mathring{D}_n)$ and we set $C_i = \partial D_i$. We start out with the following claim:

Claim. The following sequence is exact:

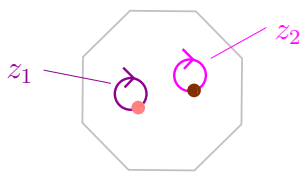
$$0 \to \mathbb{Z}\cdot([C_1] + \dots + [C_n]) \to \mathrm{H}_1(\partial\Sigma_{g,n}) \underset{\uparrow}{\to} \mathrm{H}_1(\Sigma_{g,n}) \underset{\uparrow}{\to} \mathrm{H}_1(\Sigma_g) \to 0.$$

both maps are induced by the inclusions

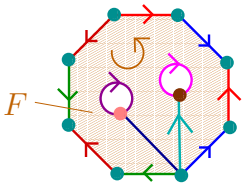
Proof. As in the figure below we can pick a CW-structure for Σ_g with the following properties:

- Each of the n embedded disks $D_1, \dots, D_n$ is a 2-cell.
- Each $C_i = \partial D_i$ has precisely one 1-cell z_i, which we equip with a characteristic map that respects the given orientation of $\partial\Sigma_{g,n}$.

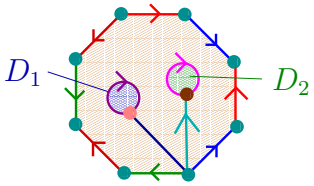
- We have only one extra 2-cell F.



CW-structure for $\partial\Sigma_{g,n}$



CW-structure for $\Sigma_{g,n}$



CW-structure for Σ_g

Using this CW-structure one can easily show that the following sequence is exact:

$$0 \to \mathbb{Z}\cdot(\Xi_{z_1}+\cdots+\Xi_{z_n}) \to \mathrm{H}_1^{\mathrm{CW}}(\partial\Sigma_{g,n}) \underset{\uparrow}{\to} \mathrm{H}_1^{\mathrm{CW}}(\Sigma_{g,n}) \underset{\uparrow}{\to} \mathrm{H}_1^{\mathrm{CW}}(\Sigma_g) \to 0.$$

both maps are induced by the inclusions

Indeed, for the inclusion-induced map $\mathrm{H}_1^{\mathrm{CW}}(\partial\Sigma_{g,n}) \to \mathrm{H}_1^{\mathrm{CW}}(\Sigma_{g,n})$ we get the statement about the kernel from the observation that $\partial_2(\Xi_F) = \Xi_{z_1}+\cdots+\Xi_{z_n}$. Furthermore for the inclusion-induced map $\mathrm{H}_1^{\mathrm{CW}}(\Sigma_{g,n}) \to \mathrm{H}_1^{\mathrm{CW}}(\Sigma_g)$ we just need to determine the effect of adding the 2-cells $D_1,\dots,D_n$. This can be done easily by hand.

The claim follows from the above "cellular" exact sequence together with the Singular-Cellular-H_*-Proposition 155.8 and the Singular-to-Cellular Homology Lemma 155.19. ⊞

Now we turn to the actual proof of the statements:

(1) We consider the long exact sequence in homology of the pair $(\Sigma_{g,n}, \partial\Sigma_{g,n})$. Since $n \geq 1$ we obtain from the Surface-H_*-Proposition 155.17 that $\mathrm{H}_2(\Sigma_{g,n}) = 0$. The desired statements follow immediately from comparing this exact sequence with the exact sequence from the claim.

(2) This statement follows from (1) and the claim.

(3) By the Surface-H_*-Proposition 155.17 we know that $\mathrm{H}_1(\Sigma_g) \cong \mathbb{Z}^{2g}$, in particular this homology group is a free abelian group. The desired statements follow from this observation, Statement (2), together with the Free Abelian Group–Splitting Lemma 97.11 and the Splitting Lemma 145.5. ■

It is worth considering the case of a single boundary component in greater detail. To do so it is convenient to introduce the following definition and notation.

Definition 155.22. Let X be a topological space. A closed embedded curve C is a subspace of X that is homeomorphic to S^1. The image of either generator of $\mathrm{H}_1(C) \cong \mathbb{Z}$ in $\mathrm{H}_1(X)$ is denoted by $[C]$.[1629]

Lemma 155.23. (Surface-with-One Boundary Component-H_*-Lemma)

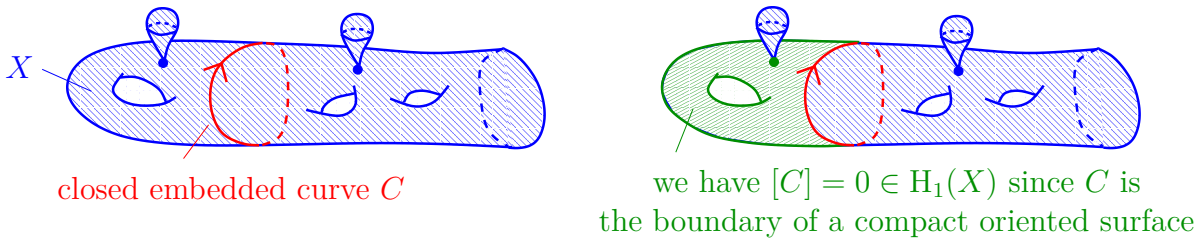
(1) For any $g \in \mathbb{N}_0$ the inclusion-induced map $\mathrm{H}_1(\partial\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1})$ is the zero map.

(2) Let X be a topological space and let C be a closed embedded curve in X. If there exists a $g \in \mathbb{N}_0$ and a continuous map $\psi\colon \Sigma_{g,1} \to X$ which restricts to a homeomorphism $\partial\Sigma_{g,1} \to C$, then $[C] = 0 \in \mathrm{H}_1(X)$.

Proof.

(1) This statement follows from the Surface-with-Boundary-H_*-Lemma 155.21 (1) which tells us that $\ker(\mathrm{H}_1(\partial\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1})) = \mathbb{Z}\cdot[\partial\Sigma_{g,1}] = \mathrm{H}_1(\partial\Sigma_{g,1})$.

[1629] In other words, $[C]$ is determined up to sign. In practice this indeterminacy usually causes no problems.



(2) Let X be a topological space and let C be a closed embedded curve in X. We denote by $i\colon C \to M$ the inclusion map. We suppose that there exists a $g \in \mathbb{N}_0$ and a continuous map $\psi\colon \Sigma_{g,1} \to M$ which restricts to a homeomorphism $\partial\Sigma_{g,1} \to C$. We denote by $j\colon \partial\Sigma_{g,1} \to \Sigma_{g,1}$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_1(\partial\Sigma_{g,1}) & \xrightarrow{\quad j_* \quad} & \mathrm{H}_1(\Sigma_{g,1}) \\ {\scriptstyle \psi_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi_*}\downarrow \\ \mathrm{H}_1(C) & \xrightarrow{\quad i_* \quad} & \mathrm{H}_1(M). \end{array}$$

We make the following observations:

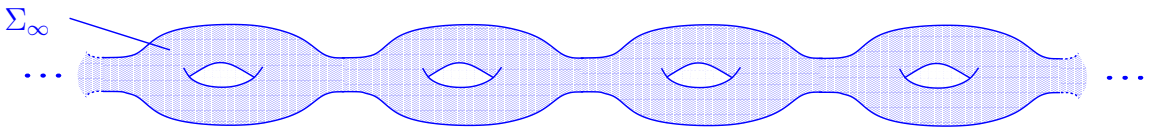
- It follows from the functoriality of homology groups that the diagram commutes.
- By Statement (1) the top horizontal map is the zero map.
- Since $\partial\Sigma_{g,1} \to C$ is a homeomorphism we see that the left vertical map is an isomorphism.

It follows that the bottom horizontal map $i_*\colon \mathrm{H}_1(C) \to \mathrm{H}_1(X)$ is the zero map. This implies that the homology class represented by C is zero. ■

We conclude this appendix with the surface Σ_∞ of infinite genus which we had already encountered on page 2054.

Lemma 155.24. (Surface-of-Infinite Genus–H_*-Lemma) If Σ_∞ is the surface of infinite genus that is shown below, then

$$\mathrm{H}_n(\Sigma_\infty) \;\cong\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^\infty, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$



Proof. The lemma can be proved using the Surface-with-Boundary-H_*-Lemma 155.21 and the Exhaustions-H_*-Proposition 154.2. We will fill in the details in Exercise 155.28. ■

155.8. **Appendix: Relationship between standard generators.** For the tenacious reader we provide in this appendix the long-delayed proof of the Standard Generator-of-$\overline{B}^n$-Lemma 155.29. We start out with recalling the definitions of the two types of standard generators we are dealing with.

Notation 155.25. Let $n \in \mathbb{N}_0$.

(1) As on page 2865 we consider the singular n-simplex $\alpha_n \colon \Delta^n \to S^n$ that is given by the concatenation of the following two continuous maps

$$\begin{array}{ccccccc} \Delta^n & \xrightarrow[\Delta^n\text{-Homeomorphism Lemma 143.9 (2)}]{\text{homeomorphism given by the}} & \overline{B}^n & \to & S^n & \subseteq & \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ & & r \cdot v & \mapsto & \underbrace{\begin{pmatrix} -\cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}}. & & \end{array}$$

(2) We denote by $\beta_n \colon \Delta^n \to S^n$ the constant map that sends all points to the "East Pole" $(1, 0, \ldots, 0)$.

Recall that in the Standard Generator-$\mathrm{H}_n(S^n)$-Proposition 150.7 we saw that the singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle and that it represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. This lead us in Definition 150.8 to the following definition:

Definition 155.26. Let $n \in \mathbb{N}_0$. We refer to $[S^n] \coloneqq (-1)^{n-1} \cdot [\alpha_n - \beta_n] \in \widetilde{\mathrm{H}}_n(S^n)$ as the standard generator of $\widetilde{\mathrm{H}}_n(S^n)$.

We also recall Definition 155.11:

Definition 155.27. Given $n \in \mathbb{N}_0$ we denote by $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ the unique element with $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$.

Now we can formulate the two technical lemmas on $[\overline{B}^n]$ that we are going to prove in this appendix:

Lemma 155.28. (Standard Generator-$\overline{B}^n$-Cycle Lemma) Let $n \in \mathbb{N}_0$. Furthermore let $f \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from the Δ^n-Homeomorphism Lemma 143.9 (2). Then

$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \text{standard generator } [\overline{B}^n] \text{ of } \mathrm{H}_n(\overline{B}^n, S^{n-1}).$$

Our second lemma was already stated on page 2961.

Lemma 155.29. (Standard Generator-of-$\overline{B}^n$-Lemma) Let $n \in \mathbb{N}_0$. If we denote by $ю \colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(\overline{B}^n / S^{n-1})$ the natural map and if we denote by $g \colon \overline{B}^n / S^{n-1} \to S^n$ the homeomorphism from the Ball Quotient-Sphere Lemma 8.10, then

$$(g_* \circ ю)([\overline{B}^n]) \;=\; [S^n] \quad \in \widetilde{\mathrm{H}}_n(S^n).$$

The proof of the Standard Generator-$\overline{B}^n$-Cycle Lemma 155.28 and the Standard Generator-of-$\overline{B}^n$-Lemma 155.29 require one technical lemma, which in turn requires the following notation:

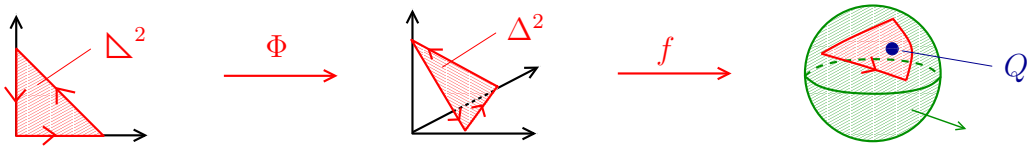
Notation 155.30. Let $n \in \mathbb{N}_0$.

(1) Given $P \in S^n$ we denote by $[S^n]_P$ the image of the standard generator $[S^n]$ under the natural map $ж \colon \widetilde{\mathrm{H}}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{P\})$.

(2) Let $\Psi \colon \Delta^n \to \triangle_n$ be the obvious projection. In the Δ^n-Homeomorphism Lemma 143.9 (1) we saw that Ψ is a homeomorphism. We write $\Phi \coloneqq \Psi^{-1} \colon \triangle_n \to \Delta^n$.

Auxiliary Lemma 155.31. (Standard Generator-at-Point Lemma) Let $n \in \mathbb{N}_0$. If $f\colon \Delta^n \to S^n$ is a continuous map such that $f \circ \Phi\colon \triangle_n \to S^n$ is a smooth embedding, [1630] then for every $Q \in f(\mathring{\Delta}^n)$ we have the following equality in $\mathrm{H}_n(S^n, S^n \setminus \{Q\})$:

$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; \epsilon \cdot [S^n]_Q \qquad \text{where } \epsilon = \begin{cases} +1, & \text{if } f \circ \Phi\colon \mathring{\triangle}^n \to S^n \text{ is orientation-preserving,} \\ -1, & \text{if } f \circ \Phi\colon \mathring{\triangle}^n \to S^n \text{ is orientation-reversing.} \end{cases}$$



Proof of Lemma 155.31. First we deal with the special case that $Q = W := (-1, 0, \ldots, 0)$ is the West Pole and that we have the inclusion $f(\Delta^n) \subseteq S^n \setminus \{-W\}$. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) & & \xrightarrow{\cdot\epsilon} & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
\uparrow{\scriptstyle i_*}_{\cong} & & & & \uparrow{\scriptstyle j_*}_{\cong} \\
\mathrm{H}_n(t_P(\triangle_n), t_P(\triangle_n)\setminus\{0\}) & & \xrightarrow{(t_X\circ\Psi\circ\alpha_n^{-1}\circ f\circ\Psi^{-1}\circ t_P^{-1})_*} & & \mathrm{H}_n(t_X(\triangle_n), t_X(\triangle_n)\setminus\{0\}) \\
\uparrow{\scriptstyle t_{P*}}_{\cong} & & & & \uparrow{\scriptstyle t_{X*}}_{\cong} \\
\mathrm{H}_n(\triangle_n, \triangle_n\setminus\{P\}) & & \xrightarrow{(\Psi\circ\alpha_n^{-1}\circ f\circ\Psi^{-1})_*} & & \mathrm{H}_n(\triangle_n, \triangle_n\setminus\{X\}) \\
\uparrow{\scriptstyle \mu_*}_{\cong} & \searrow^{\cong}_{(f\circ\Psi^{-1})_*} & & \swarrow^{\cong}_{(\alpha_n\circ\Psi^{-1})_*} & \uparrow{\scriptstyle \mu_*}_{\cong} \\
\mathrm{H}_n(\triangle_n, \partial\triangle_n) & & \mathrm{H}_n(S^n, S^n\setminus\{W\}) & & \mathrm{H}_n(\triangle_n, \partial\triangle_n) \\
\uparrow{\scriptstyle \Psi_*}_{\cong} & \nearrow_{f_*} & & \nwarrow_{\alpha_{n*}} & \uparrow{\scriptstyle \Psi_*}_{\cong} \\
\mathrm{H}_n(\Delta^n, \partial\Delta^n) & & \xrightarrow{\cdot\epsilon} & & \mathrm{H}_n(\Delta^n, \partial\Delta^n).
\end{array}$$

$(i\circ t_P\circ\Psi)_*$ on the left; $(j\circ t_X\circ\Psi)_*$ on the right.

We make the following clarifications and observations:

(1) We set $P := \Psi(f^{-1}(W)) \in \mathring{\triangle}_n$ and $X := \Psi(\alpha_n^{-1}(W)) \in \mathring{\triangle}_n$. Evidently we have the equality $(f \circ \Psi^{-1})(P) = W = (\alpha_n \circ \Psi^{-1})(X)$.

(2) Given $v \in \mathbb{R}^n$ we denote by $t_v\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by $x \mapsto x - v$.

(3) The maps $i\colon t_P(\triangle_n) \to \mathbb{R}^n$ and $j\colon t_X(\triangle_n) \to \mathbb{R}^n$ are the inclusion maps.

(4) The maps $\mu\colon (\triangle_n, \partial\triangle_n) \to (\triangle_n, \triangle_n \setminus \{P\})$ and $\mu\colon (\triangle_n, \partial\triangle_n) \to (\triangle_n, \triangle_n \setminus \{X\})$ are the inclusion maps of pairs of topological spaces.

(5) Since $f(\Delta^n) \subseteq S^n \setminus \{-W\}$ we have in particular that $f(\Delta^n) \subseteq \alpha_n(\mathring{\Delta}^n)$ which means that the map $\alpha_n^{-1} \circ f$ is actually defined.

(6) One can easily verify that $\alpha_n \circ \Psi^{-1}$ is orientation-preserving. As mentioned on page 818, the translation maps t_v are also orientation-preserving. Therefore we see that the map $t_X^{-1} \circ \Psi \circ \alpha_n^{-1} \circ f \circ \Psi^{-1} \circ t_P$ is orientation-preserving if and only if the map $f \circ \Psi^{-1} = f \circ \Phi$ is orientation-preserving. Thus we obtain from the Local Diffeomorphism-Degree Proposition 152.6 that the top square commutes.

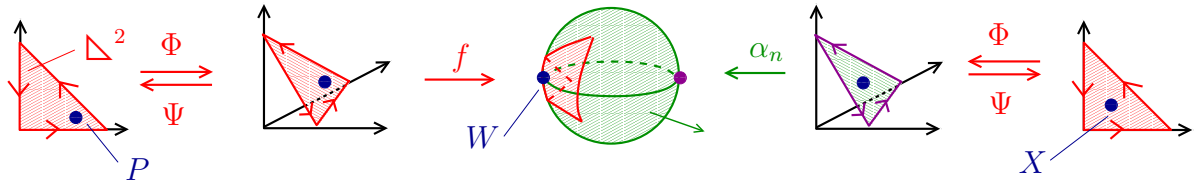
(7) The second square from the top, the top triangle and the two triangles to the left and right commute basically by the functoriality of relative homology.

[1630] Note that $\mathring{\triangle}_n$ is an open subset of $\mathbb{R}^n$, in particular, following the convention established on page 812, we will view it as an oriented n-dimensional smooth manifold.

(8) It is straightforward to see that the maps $i \circ t_P \circ \Psi$ and $j \circ t_X \circ \Psi$ are homotopic as maps of pairs $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. In particular we obtain from this observation together with the Homotopy-H_*-Proposition 146.11 that the two sequences of vertical maps on the left and on the right are actually the same map.

(9) Using the Homotopy Equivalence-Relative-H_*-Corollary 146.8 and using the Excision Theorem 148.2 one can show fairly easily that all the vertical maps are isomorphisms and we see that the upper diagonal maps are isomorphisms. Since the triangles to the left and right commute we see that all diagonal maps are isomorphisms.

(10) It follows from (6), (7), (8) and (9) that the bottom triangle also commutes.



Now we are basically done with the proof for our special case. Indeed we see that

$$f_*([\mathrm{id}_{\Delta^n}]) \underset{\text{by (10)}}{=} \epsilon \cdot \alpha_{n*}([\mathrm{id}_{\Delta^n}]) = \epsilon \cdot [\alpha_n] \underset{\text{since } \beta_n = 0 \in \mathrm{C}_n(S^n, S^n \setminus \{W\})}{=} \epsilon \cdot [\alpha_n - \beta_n]_Q \underset{\text{Definition 150.8 of } [S^n]}{=} \epsilon \cdot [S^n]_Q.$$

Finally it remains to deal with the general case. Thus let $f\colon \Delta^n \to S^n$ be a continuous map such that $f \circ \Phi\colon \triangle_n \to S^n$ is a smooth embedding and let $Q \in f(\mathring{\Delta}^n)$.

(1) As in the proof of the Local-H_*-Proposition 150.3 (4), see in particular the proof of the claim on page 2862, we see that there exists a homotopy $F\colon \Delta^n \times [0,1] \to S^n$ from f to a map $\widetilde{f}$ such that the following holds:
   (a) For each $t \in [0,1]$ we have $Q \in F_t(\mathring{\Delta}^n)$.
   (b) The corresponding map $\mathring{\triangle}^n \times [0,1] \to S^n$, given by $(x,t) \mapsto F_t(\Phi(x))$, is a smooth isotopy.
   (c) We have $\widetilde{f}(\Delta^n) \subseteq S^n \setminus \{-Q\}$.

(2) By the Orthogonal Action Lemma 2.37 (4) there exists an $A \in \mathrm{SO}(n+1)$ such that $A \cdot Q = W$. We make the following observations:
   (a) By the discussion on page 510 the map $\varphi\colon S^n \to S^n$ that is given by $\varphi(x) = A \cdot x$ is smoothly isotopic to the identity.
   (b) We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(S^n) & \xrightarrow{\varphi_* = \mathrm{id}} & \mathrm{H}_n(S^n) \\ \downarrow{\scriptstyle я_Q} & & \downarrow{\scriptstyle я_W} \\ \mathrm{H}_n(S^n, S^n \setminus \{Q\}) & \xrightarrow{\varphi_*} & \mathrm{H}_n(S^n, S^n \setminus \{W\}). \end{array}$$

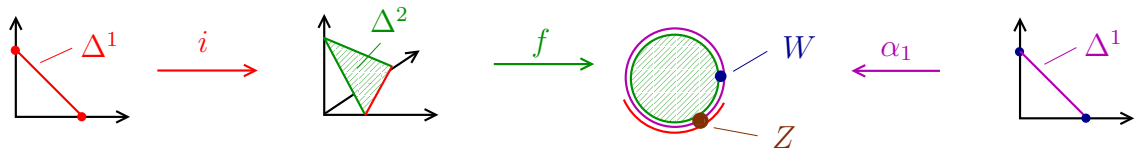
   The diagram commutes by functoriality of relative homology. Furthermore by (a) together with the Homotopy-H_*-Proposition 144.12 we have $\varphi_* = (\mathrm{id}_{S^n})* = \mathrm{id}_{\mathrm{H}_n(S^n)}$.

(3) Since φ is orientation-preserving, since by (1a) we have $W \in (\varphi \circ \widetilde{f})(\mathring{\Delta}^n)$ and since by (1c) we have $(\varphi \circ \widetilde{f})(\Delta^n) \subseteq S^n \setminus \{-W\}$ we obtain from the above special case and (1b) that $(\varphi \circ \widetilde{f})_*([\mathrm{id}_{\Delta^n}]) = \epsilon \cdot я_W([S^n])$.

(4) From (2b) and (3) we obtain the equality $\widetilde{f}_*([\mathrm{id}_{\Delta^n}]) = \epsilon \cdot я_Q([S^n])$.
(5) From (1a) and (4) together with the Homotopy-H$_*$-Proposition 144.12 we obtain the desired equality $f_*([\mathrm{id}_{\Delta^n}]) = \epsilon \cdot я_Q([S^n])$. ∎

Proof of the Standard Generator-$\overline{B}^n$-Cycle Lemma 155.28. Let $n \in \mathbb{N}_0$. We denote by $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from the Δ^n-Homeomorphism Lemma 143.9 (2). We need to show that $f_*([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\overline{B}^n]$. To do so we recall the following maps:

(1) As in Definition 143.12 we denote by $i = i_n^n\colon \Delta^{n-1} \to \Delta^n$ the n-th face map that is given by $i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{n-1}, 0)$.
(2) Since $f\colon \partial\Delta^n \xrightarrow{\cong} S^{n-1}$ is a homeomorphism one sees easily that we can pick a point $Z \in (f \circ i)(\mathring{\Delta}^{n-1}) \cap (S^{n-1} \setminus \{-W\})$.[1631]
(3) Below we will denote by ж the natural map from reduced homology to relative homology that we introduced in Notation 147.7.



We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\partial\Delta^n) & \xrightarrow{ж} & \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) & \xleftarrow[\cong]{i_*} & \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \\
\overset{[\mathrm{id}_{\Delta^n}]}{\downarrow} & & \overset{\partial_n[\mathrm{id}_{\Delta^n}]}{\downarrow} & & \overset{\partial_n[\mathrm{id}_{\Delta^n}]=(-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}])}{\downarrow} & & \overset{(-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}]}{\downarrow} \\
\mathrm{H}_n(\overline{B}^n, S^{n-1}) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & \xrightarrow{ж} & \mathrm{H}_{n-1}(S^{n-1}, S^{n-1}\setminus\{Z\}) & \xleftarrow[\alpha_{n*}]{} & \mathrm{H}_i(\Delta^{n-1}, \partial\Delta^{n-1}). \\
[\overline{B}^n] & & [S^{n-1}]=[\alpha_{n-1}-\beta_{n-1}] & & [\alpha_{n-1}-\beta_{n-1}]=[\alpha_{n-1}] & & [\mathrm{id}_{\Delta^{n-1}}]
\end{array}$$

Note that the diagram commutes by the LES-of-Pair Proposition 146.7 (2) and the discussion in Notation 147.7. Now we see that we have the following equalities in the group $\mathrm{H}_{n-1}(S^{n-1}, S^{n-1} \setminus \{Z\})$:

$$\begin{array}{lll}
\text{since the diagram commutes} & & \text{we have } ж(\partial_n(\mathrm{id}_{\Delta^n})) = (-1)^n \cdot i_*(\mathrm{id}_{\Delta^{n-1}}) \in \mathrm{C}_{n-1}(\partial\Delta^n, \Lambda) \text{ since all other summands of } \partial_n(\mathrm{id}_{\Delta^n}) \text{ are zero in } \mathrm{C}_{n-1}(\partial\Delta^n, \Lambda) \\
ж(\partial_n(f_*([\mathrm{id}_{\Delta^n}]))) \overset{\downarrow}{=} f_*(ж(\partial_n([\mathrm{id}_{\Delta^n}]))) & & \overset{\downarrow}{=} f_*((-1)^n \cdot i_*([\mathrm{id}_{\Delta^{n-1}}])) \\
= (-1)^n \cdot (f \circ i)_*([\mathrm{id}_{\Delta^{n-1}}]) & & \overset{\uparrow}{=} (-1)^n \cdot \alpha_{n-1,*}([\mathrm{id}_{\Delta^{n-1}}]) = (-1)^n \cdot [\alpha_{n-1}]
\end{array}$$

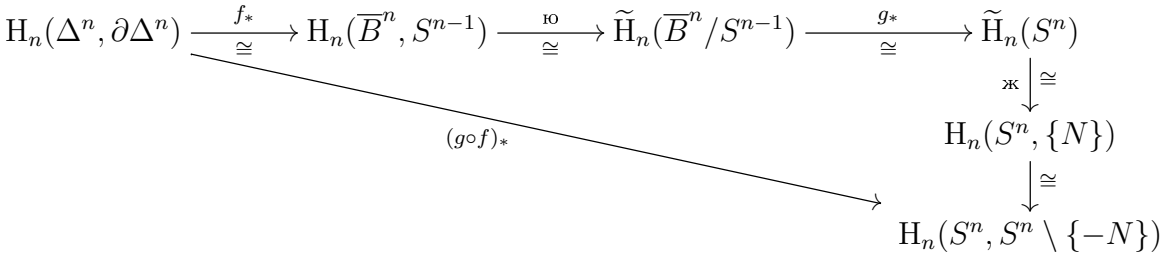
we leave it to the reader to verify that $f \circ i\colon \Delta^{n-1} \to S^{n-1}$ and $\alpha_{n-1}\colon \Delta^{n-1} \to S^{n-1}$ are both orientation-preserving, thus the equality follows from the Standard Generator-at-Point Lemma 155.31

$$\begin{array}{lll}
= (-1)^n \cdot ([\alpha_{n-1} - \beta_{n-1}]) & \overset{\uparrow}{=} (-1)^n \cdot ж([S^{n-1}]) & \overset{\uparrow}{=} (-1)^n \cdot ж(\partial_n([\overline{B}^n])). \\
\text{since } \beta_n = 0 \in \mathrm{C}_{n-1}(S^n, S^{n-1} \setminus \{Z\}) & \text{Definition 150.8 of } [S^{n-1}] & \text{Definition 155.11 of } [\overline{B}^n]
\end{array}$$

As discussed earlier, by the various long exact sequences we know that the maps ∂_n and ж are both isomorphisms. Thus we obtain from the above discussion the desired equality $f_*([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\overline{B}^n]$. ∎

[1631] Note that this also works for $n = 1$.

Proof of the Standard Generator-of-$\overline{B}^n$-Lemma 155.29. Let $n \in \mathbb{N}_0$. We denote by $g\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from the Ball Quotient-Sphere Lemma 8.10. Furthermore let $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from the Δ^n-Homeomorphism Lemma 143.9 (2). We consider the following diagram:



We make the following observations:

(1) Note that $g([S^{n-1}]) = \{N\}$. It follows that the diagonal map is defined.
(2) It follows from the functoriality of relative homology and the naturality of ж that the diagram commutes.
(3) Next note that it follows from the Standard Generator-$\overline{B}^n$-Cycle Lemma 155.28 that $f_*([\mathrm{id}_{\Delta^n}] = (-1)^n \cdot [\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.
(4) Let $\Phi\colon \bigtriangledown_n \to \Delta^n$ be the map that we introduced in Notation 155.30. One can easily verify that the map $(g \circ f) \circ \Phi\colon \bigtriangledown_n \to S^n$ is orientation-preserving if and only if n is even. This observation, together with the Standard Generator-at-Point Lemma 155.31 implies that the image of $[\mathrm{id}_{\Delta^n}]$ in $\mathrm{H}_n(S^n, S^n \setminus \{-N\})$ under the diagonal map $(g \circ f)_*$ agrees with $(-1)^n \cdot [S^n]_{-N}$.

Since all maps are easily seen to be isomorphisms we obtain that $g_*([\overline{B}^n] = [S^n]$. ■

We conclude this section with the following lemma which gives us another slightly subtle sign calculation.

Lemma 155.32. (Restriction-to-Equator-H$_*$-Lemma) Let $n \in \mathbb{N}_0$. We consider the following sequence of maps:

$$\mathrm{H}_n(S^n) \xrightarrow{\text{я}} \mathrm{H}_n(S^n, S^n_{\geq 0}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \xrightarrow{\partial_n} \widetilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

inclusion-induced map is an isomorphism
by the discussion on page 2820

The image of the standard generator $[S^n] \in \mathrm{H}_n(S^n)$ under this map is $(-1)^n \cdot [S^{n-1}]$.

Example. Let $n \in \mathbb{N}_0$. Let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 556. We point out that it follows quite easily from the explicit definition of the suspension isomorphism $\Sigma_{S^n}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_{n+1}(S^{n+1})$ and the Restriction-to-Equator-H$_*$-Lemma 155.32 that the isomorphism from the Suspension-H$_*$-Proposition 149.3 satisfies

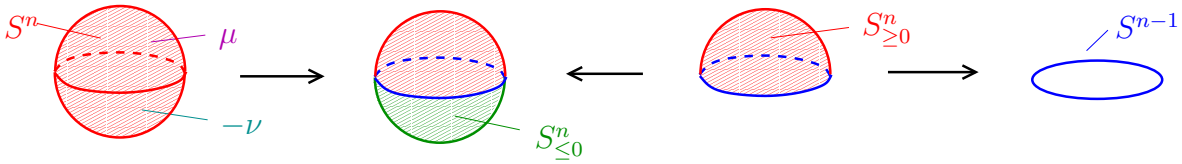
$$\Theta_*(\Sigma_{S^n}([S^n])) \;=\; (-1)^{n+1} \cdot [S^{n+1}] \quad \in \mathrm{H}_{n+1}(S^{n+1}). \qquad \square$$

Sketch of proof. We recall and introduce the following notation:

(1) Let $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from the Δ^n-Homeomorphism Lemma 143.9 (2).

(2) Let $p_+ \colon S^n_{\geq 0} \to \overline{B}^n$ be the natural projection and let $q_+ \colon \overline{B}^n \to S^n_{\geq 0}$ be the map that is given by $x \mapsto (x, \sqrt{1 - \|x\|^2})$. Evidently p and q are inverses of one another. We define $p_- \colon S^n_{\leq 0} \to \overline{B}^n$ and $q_- \colon \overline{B}^n \to S^n_{\leq 0}$ in an analogous fashion.

(3) We consider the singular n-simplices $\mu := q_+ \circ f \colon \Delta^n \to S^n$ and $\nu := q_- \circ f \colon \Delta^n$. One can easily verify that $\mu - \nu \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle. In fact, using the Standard Generator-at-Point Lemma 155.31 and using the observation that the homomorphism $\widetilde{\mathrm{H}}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{W\})$ is an isomorphism one can easily verify that $[\mu - \nu] = [S^n] \in \widetilde{\mathrm{H}}_n(S^n)$.



Next we consider the following diagram:

$$\begin{array}{ccccccc} \mathrm{H}_n(S^n) & \xrightarrow{\text{я}} & \mathrm{H}_n(S^n, S^n_{\geq 0}) & \xleftarrow{i} & \mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) & \xrightarrow[n]{\partial} & \mathrm{H}_{n-1}(S^{n-1}). \\ & & & & \uparrow{\scriptstyle q_+} \; \downarrow{\scriptstyle p_+} & \nearrow_{\vartheta} & \\ & & & & \mathrm{H}_n(\overline{B}^n, S^{n-1}) & & \end{array}$$

Note that the triangle to the right commutes by the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 146.7. Now we see that

$$\begin{array}{cccccc} & \text{see the above discussion} & & \text{Definition 146.6 of я} & & \text{since } \nu = 0 \in \mathrm{C}_n(S^n, S^n_{\leq 0}) \\ \partial_n(i_*^{-1}(\text{я}([S^n]))) & \overset{\downarrow}{=} \partial_n(i_*^{-1}(\text{я}([\mu - \nu]))) & \overset{\downarrow}{=} \partial_n(i_*^{-1}([\mu - \nu])) & \overset{\downarrow}{=} \partial_n(i_*^{-1}([\mu])) & = \partial_n([\mu]) \\ & = \partial_n(q_{+*}([p_+ \circ \mu])) & = \partial_n(q_{+*}([f])) & = \partial_n([f]) & = (-1)^n \cdot [S^{n-1}]. \\ & \uparrow & \uparrow & \uparrow & \uparrow \\ & \text{since } p_+ = q_+^{-1} & \text{since } \mu = q_+ \circ f & \text{since the triangle commutes} & \text{Standard Generator Relationship Lemma 155.28} \end{array}$$ ■

155.9. Exercises for Chapter 155 .

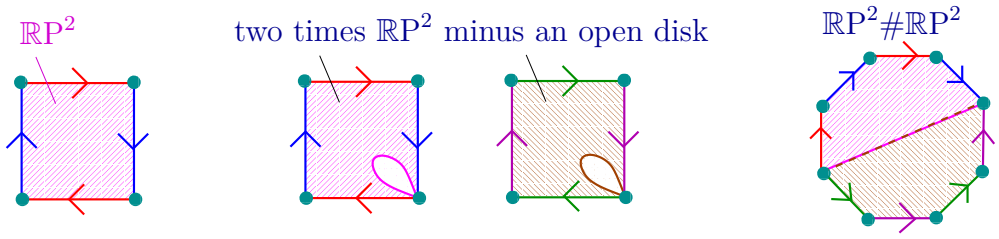
Exercise 155.1.

(a) Determine the homology groups of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$.

(b) Determine the homology groups of $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ minus an open disk.

Note: In both parts you can make use of the CW-structure for $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ illustrated in the figure below.

Remark. The homology groups are stated as the Surface-H_*-Proposition 155.17. But it is a very good exercise to try to determine the homology groups without looking at the proposition.



Exercise 155.2. Let A be a finitely generated abelian group. Does there exist a topological space X with $\mathrm{H}_0(X) = \mathbb{Z}$, $\mathrm{H}_3(X) \cong A$ and $\mathrm{H}_i(X) = 0$ for $i \neq 0, 3$?

Exercise 155.3. On S^2 we consider the equivalence relation that is given by identifying opposite points on the equator, i.e. we have $(x, y, 0) \sim (-x, -y, 0)$ for all $(x, y, 0) \in S^2$. Compute the homology groups of $S^2/\sim$.

Exercise 155.4. We consider the infinite-dimensional sphere $S^\infty = \operatorname{colim} S^n$. On page 2628 we equipped S^∞ with a CW-structure. Calculate the corresponding cellular chain complex and use it to determine the homology groups of S^∞.

Exercise 155.5. Use cellular homology to determine the homology groups of the product $S^m \times S^n$ of two spheres.

Exercise 155.6. Let $g \geq k$. Show that there exists a continuous map $\Sigma_g \to \Sigma_k$ from the surface of genus g to the surface of genus k which induces epimorphisms on all homology groups.

Exercise 155.7.

(a) Determine $\min\{\#\text{cells of } X \mid X \text{ is a CW-complex with } \mathrm{H}_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\}$.
(b) Determine $\min\{\#\text{cells of } X \mid X \text{ is a CW-complex with } \mathrm{H}_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\}$.
(c) Determine $\min\{\#\text{cells of } X \mid X \text{ is a connected CW-complex with } \pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$.

Exercise 155.8.

(a) Let K be the Klein bottle.
  (i) What is the minimal number of cells of any CW-structure for K?
  (ii) What is the minimal number of cells of any CW-complex that is homotopy equivalent to K?
(b) Let $n \in \mathbb{N}_0$.
  (i) What is the minimal number of cells of any CW-structure for $\overline{B}^n$?
  (ii) What is the minimal number of cells of any CW-complex that is homotopy equivalent to $\overline{B}^n$?

Exercise 155.9. We consider the CW-complex $\mathbb{R}\mathrm{P}^2$ with the subcomplex $\mathbb{R}\mathrm{P}^1$. We denote by $i\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^2$ the usual embedding that is given by $[(x, y)] \mapsto [(x, y, 0)]$.

(a) Compute the homology groups of $A \coloneqq \mathrm{Cone}(i\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^2)$.
(b) We set $B \coloneqq \mathrm{Cone}(\mathbb{R}\mathrm{P}^2)$ and we set $X \coloneqq A \cup_{\mathbb{R}\mathrm{P}^2} B$. We equip X with a CW-complex structure such that A and B are subcomplexes. Show that the Mayer–Vietoris sequence in reduced homology corresponding to $X = A \cup B$ is a short exact sequence that does not split.
(c) Compute the homology groups of X.

Exercise 155.10. Let $n \in \mathbb{N}_0$. On page 2628 we showed that $\mathbb{R}\mathrm{P}^n$ admits a CW-structure which has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells. Show that this is the "smallest" CW-structure on $\mathbb{R}\mathrm{P}^n$. More precisely, show that every CW-structure for $\mathbb{R}\mathrm{P}^n$ has at least one cell in the dimensions $0, 1, \ldots, n$.
Hint. Consider the (cellular) homology groups of $\mathbb{R}\mathrm{P}^n$.

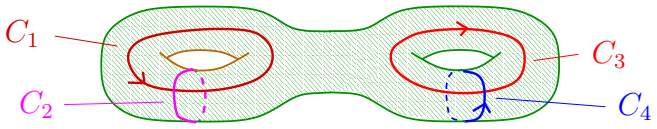
Exercise 155.11. Let n be odd. Recall that by the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we know that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Show that there exists a singular simplex $\sigma\colon \Delta^n \to \mathbb{R}\mathrm{P}^n$ such that $[\sigma]$ is a generator of $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n)$.
Hint.
(a) First you need to find an "interesting" cycle $\sigma \in \mathrm{C}_n(\mathbb{R}\mathrm{P}^n)$. You could use the identification $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} \overline{B}^n/\sim$ from page 340. Or perhaps better, write $\mathbb{R}\mathrm{P}^n = \Delta^n/\sim$ for a suitable equivalence relation $\sim$.
(b) To show that the "interesting" cycle σ you could use the Detect-Cycle Proposition 150.5.

Exercise 155.12. Let $n \in \mathbb{N}$ be odd. By the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we know that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ be the projection. Determine the induced map $p_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(\mathbb{R}\mathrm{P}^n)$.
Hint. You could use the fact that the isomorphism between cellular and singular homology is natural.

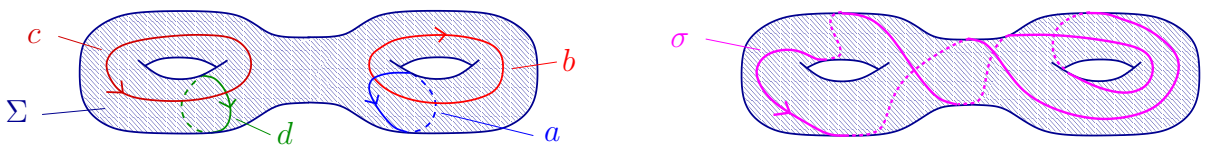
Exercise 155.13. Let $k \leq n$ and let $i\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ be the embedding that is given by $i([x_0 : \cdots : x_k]) = [x_0 : \cdots : x_k : 0 : \cdots : 0]$. Use the relative cellular chain complex of Exercise 155.26 to compute the relative singular homology groups $\mathrm{H}_*(\mathbb{R}\mathrm{P}^n, i(\mathbb{R}\mathrm{P}^k))$.

Exercise 155.14. Let $n \in \mathbb{N}$ be odd. By the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we know that $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$. Let $A \in \mathrm{GL}(n+1, \mathbb{R})$. We denote by $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ the map that is given by $f(x) := A \cdot x$. Determine the degree of the map $f_*\colon \mathrm{H}_n(\mathbb{R}\mathrm{P}^n) \to \mathrm{H}_n(\mathbb{R}\mathrm{P}^n)$.
Remark. You could make use of the homotopies considered on page 516.

Exercise 155.15. Let Σ be the surface of genus 2. We continue the four closed oriented curves C_1, C_2, C_3, C_4 shown in the figure below. Show that there exist continuous maps $f_i\colon \Sigma \to S^1$, $i = 1, 2, 3, 4$, such that for each $i = 1, 2, 3, 4$ the following two statements hold:
(1) $f_{i*}([C_i])$ is a generator of $\mathrm{H}_1(S^1) \cong \mathbb{Z}$.
(2) For $j \neq i$ we have $f_{i*}([C_j]) = 0 \in \mathrm{H}_1(S^1)$.
Remark. By the Surface-H_*-Proposition 155.17 we know that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$. It follows from this calculation, the Epi-Automorphism Lemma 97.27 and the exercise that the homology classes $[C_1], [C_2], [C_3], [C_4]$ form a basis for $\mathrm{H}_1(\Sigma)$. This gives an alternative proof for the statement on page 2970.



Exercise 155.16. In the figure below we see to the left the surface Σ of genus two together with four closed oriented curves a, b, c, d and to the right we see a closed oriented curve σ. On page 2970 we showed that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ and we showed that the closed oriented curves a, b, c and d represent a basis for $\mathrm{H}_1(\Sigma)$. In particular we can write $[\sigma] \in \mathrm{H}_1(\Sigma)$ uniquely as a linear combination of $[a]$, $[b]$, $[c]$ and $[d]$. Determine the coefficients of this linear combination.
Hint. You could use Exercise 155.15.
Remark. In Exercise 220.18 we will get to know a charming approach to dealing with problems of this form.



Exercise 155.17. We consider the surface Σ of genus 2. By the Surface-H_*-Proposition 155.17 we know that $\mathrm{H}_2(\Sigma) \cong \mathbb{Z}$. Provide an explicit singular cycle which represents a generator of $\mathrm{H}_2(\Sigma)$.

Exercise 155.18.

(a) Let (X, A) be a pair of topological spaces. Show that if A is a retract of X, then for each $n \in \mathbb{N}_0$ there exists an isomorphism

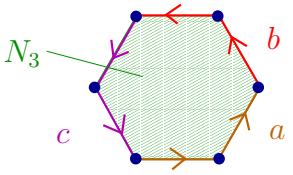
$$\mathrm{H}_n(X) \;\cong\; \mathrm{H}_n(A) \oplus \mathrm{H}_n(X, A).$$

Remark. This statement is identical to Exercise 146.9.

(b) Show that the isomorphism from (a) cannot be chosen to be natural. More precisely, provide a pair (X, A) of topological space and a continuous map $f\colon X \to Y$ with $f(A) \subseteq B$ such that there is no commutative diagram of the form

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow[\cong]{\Phi} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(X, A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \quad \downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(X) & \xrightarrow[\cong]{\Phi} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(X, A). \end{array}$$

Exercise 155.19. We consider the non-orientable surface N_3 of genus 3 with the CW-structure that is shown in the figure below. Recall that by the Surface-H_*-Proposition 155.17 we know that $\mathrm{H}_1(N_3) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. Use the 1-cells a, b, c to give an explicit isomorphism $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathrm{H}_1(N_3)$.



Exercise 155.20.

(a) Let Σ be a compact orientable connected 2-dimensional smooth manifold and let $F \subseteq \Sigma$ be a compact codimension-zero submanifold. We denote by $i\colon F \to \Sigma$ the inclusion map. Show that $i_*(\mathrm{H}_1(F)) \subseteq \mathrm{H}_1(\Sigma)$ is a summand.
Hint. Consider the long exact sequence of the pair (Σ, F).

(b) Does the conclusion of (a) also hold if we do not assume that Σ is orientable?

Exercise 155.21. Let $N_{k,1}$ be the non-orientable surface minus one open disk. What can you say about the inclusion-induced map $\mathrm{H}_1(\partial N_{k,1}) \to \mathrm{H}_1(N_{k,1})$. Is it injective? Is it the zero map? Does the answer depend on k?

Exercise 155.22. Let $g \in \mathbb{N}_0$, let $k \in \mathbb{N}$ and let $m, n \in \mathbb{N}_0$. As in Notation 101.12 we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks and we denote by $N_{k,n}$ the non-orientable surface of genus k minus n open disks.

(a) Determine the relative homology groups $\mathrm{H}_*(\Sigma_{g,m}, \partial\Sigma_{g,m})$.
(b) Determine the relative homology groups $\mathrm{H}_*(N_{k,n}, \partial N_{k,n})$.
(c) Use (a) and (b) to show that the topological spaces $\Sigma_{g,m}$ and $N_{k,n}$ are pairwise non-homeomorphic.

Hint. Make use of the Surface-with-Boundary-H_*-Lemma 155.21. and the Surface-H_*-Proposition 155.17.
Remark. This gives a new, perhaps more natural proof for the Surfaces-Non Homeomorphic Theorem 101.18.

Exercise 155.23. We consider the knot $K \subseteq \mathbb{R}^3$ shown in the figure below and we show the closed oriented curve $C \subseteq \mathbb{R}^3 \setminus K$. Is $[C] = 0 \in \mathrm{H}_1(\mathbb{R}^3 \setminus K)$?



Exercise 155.24. Let $m \leq n$. We denote by $i\colon \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ the inclusion map. Determine for each $k \in \mathbb{N}_0$ the induced map $i_*\colon \mathrm{H}_k(\mathbb{R}\mathrm{P}^m) \to \mathrm{H}_k(\mathbb{R}\mathrm{P}^n)$.

Exercise 155.25. Let $m \leq n$. On page we showed that we can view the real projective space $\mathbb{R}\mathrm{P}^m$ as a subspace of the projective space $\mathbb{R}\mathrm{P}^n$. For which values can you now exclude that $\mathbb{R}\mathrm{P}^m$ is a retract of $\mathbb{R}\mathrm{P}^n$.

Exercise 155.26. Let X be a CW-complex and let A be a subcomplex of X. We denote by $i\colon A \to X$ the inclusion map. We define the relative cellular chain complex

$$\mathrm{C}_*^{\mathrm{CW}}(X, A) \;\coloneqq\; \mathrm{coker}\left(i_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)\right),$$

and we refer to its homology groups as the relative cellular homology groups $\mathrm{H}_*^{\mathrm{CW}}(X, A)$. Show that for each $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n\colon \mathrm{H}_n^{\mathrm{CW}}(X, A) \;\to\; \mathrm{H}_{n-1}^{\mathrm{CW}}(A)$$

and a natural isomorphisms

$$\Theta_{(X,A)}\colon \mathrm{H}_n(X, A) \;\to\; \mathrm{H}_n^{\mathrm{CW}}(X, A)$$

such that we obtain a commutative diagram of exact sequences as follows:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}^{\mathrm{CW}}(A) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \Theta_A} & & \downarrow{\scriptstyle \Theta_X} & & \downarrow{\scriptstyle \Theta_{(X,A)}} & & \downarrow{\scriptstyle \Theta_A} & & \\
\ldots & \longrightarrow & \mathrm{H}_n^{\mathrm{CW}}(A) & \xrightarrow{i_*} & \mathrm{H}_n^{\mathrm{CW}}(X) & \longrightarrow & \mathrm{H}_n^{\mathrm{CW}}(X, A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}^{\mathrm{CW}}(A) & \longrightarrow & \ldots
\end{array}$$

Exercise 155.27.

(a) What is the minimal number of cells you need to attach to the torus $S^1 \times S^1$ to get an acyclic topological space, i.e. a topological space whose reduced homology groups all vanish?

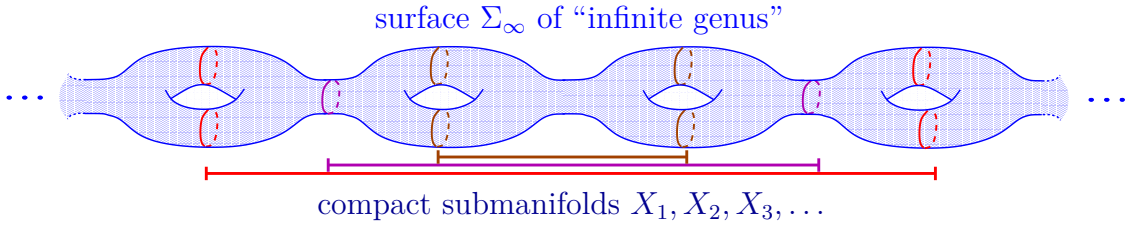
(b) What is the minimal number of cells you need to attach to the Klein bottle to get an acyclic topological space?

Exercise 155.28. We consider the surface $\Sigma_\infty = \operatorname{colim} X_i$ as shown in the figure below.

(a) For each $i \in \mathbb{N}_0$ determine the inclusion-induced maps $\mathrm{H}_*(X_i) \to \mathrm{H}_*(X_{i+1})$.
Hint. Use the Surface-with-Boundary-H_*-Lemma 155.21.

(b) Show that
$$\mathrm{H}_n(\Sigma_\infty) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^\infty, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$
Hint. Use the Exhaustions-H_*-Proposition 154.2.

(c) Determine a basis for $\mathrm{H}_1(\Sigma_\infty)$.



Exercise 155.29. Let F be a compact oriented connected 2-dimensional smooth manifold with n boundary components. For which $n \in \mathbb{N}_0$ is the natural map $\mathrm{H}_1(F) \to \mathrm{H}_1(F, \partial F)$ an isomorphism?

Exercise 155.30. Does $\mathbb{R}\mathrm{P}^3$ admit a CW-structure with two 3-cells and one 2-cell?

Exercise 155.31. Let M be a closed orientable connected non-empty smooth manifold. We define its *cut number* $\mathrm{cut}(M)$ as the maximal number of pairwise disjoint closed orientable codimension-one submanifolds $F_1, \ldots, F_c$ such that $M \setminus (F_1 \cup \cdots \cup F_c)$ is still connected.

(a) Show that if M has cut number $c = \mathrm{cut}(M)$, then $\pi_1(M)$ admits an epimorphism onto a free subgroup of rank c.
Hint. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can pick tubular neighborhoods $[-1, 1] \times F_i$ and use these to construct an epimorphism onto a graph.

(b) Let $g \in \mathbb{N}_0$. Determine the cut number of the surface of genus g.

CHAPTER 156

The Euler characteristic

In this chapter we will introduce the Euler characteristic $\chi(X) \in \mathbb{N}_0$ of a finite CW-complex X. This invariant has two interesting features:

(1) The Euler characteristic is easy to define and it is fun to calculate Euler characteristics.
(2) Using singular homology we will see that the Euler characteristic depends only on the homotopy type of X.

In this and the next chapter we will see that the Euler characteristic can be used to prove many statements and answer many classical problems (e.g. why does a football have 12 pentagons?).

This chapter is organized as follows:

- In Section 156.1 we will introduce the Euler characteristic of a CW-complex.
- In Section 156.2 we will prove the Homology-χ-Proposition 156.2.
- In Section 156.3 we will extend the definition of the Euler characteristic to more general topological spaces.
- In Section 156.4 we will use the Euler characteristic to determine the homology groups of topological graphs.
- In Section 156.5 we will study properties of the Euler characteristic.
- In Section 156.6 we will perform some calculations of Euler characteristics.
- In Section 156.7 we will consider applications of Euler characteristics.
- In Section 156.8 we will study groups acting on smooth manifolds.
- In Section 156.9 we will provide exercises for Chapter 156.

In the next Chapter 157 we will consider several fun applications of the Euler characteristic.

156.1. **The Euler characteristic of a CW-complex.** Let us start out with the straightforward definition of the Euler characteristic of a finite CW-complex.

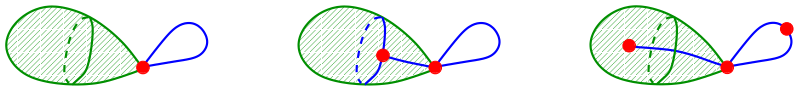
Definition 156.1. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) := \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite topological graph, then, according to the Graph–CW-Complex Lemma 136.5 we can view G as a 1-dimensional CW-complex where the number of 0-cells equals the number of vertices and the number of 1-cells equals the number of edges of G. Thus it follows immediately from the definitions that in this case the above Euler characteristic agrees with the definition of the Euler characteristic of a finite topological graph that we gave in Definition 11.27.

(2) In the figure below we show the topological space $S^2 \vee S^1$ with three different CW-structures.



number of 0-cells	1	2	3
number of 1-cells	1	3	3
number of 2-cells	1	2	1
Euler characteristic	1	1	1

We notice that even though the numbers of cells in the various dimensions differ, the Euler characteristic is always the same.

(3) Let M be a closed orientable smooth manifold. In Definition 53.4 we introduced the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of M. We will see in the Cup Product–Intersection Number Theorem 220.10 and the Lefschetz-Intersection Number Proposition 222.4, or alternatively in the Euler Characteristics–Equality Corollary 223.6 (which we prove twice, once on page 4309 and once on page **??**) that the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$ of M agrees with the above Euler characteristic $\chi(M)$. □

The above example suggests that the Euler characteristic depends perhaps only on the homeomorphism type of the CW-complex. Our next result shows that this is indeed the case.

Proposition 156.2. (Homology-χ-Proposition) Given any finite CW-complex X we have[1632]

$$\chi(X) = \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \mathrm{rank}(\mathrm{H}_n(X)).$$

Proof. We will provide the proof of Homology-χ-Proposition 156.2 in the next section. ■

Before we do anything else let us record the following pretty obvious corollary.

Lemma 156.3. (Homotopy Type-χ-Lemma) Let X and Y be finite CW-complexes. If X and Y are homotopy equivalent, then $\chi(X) = \chi(Y)$.[1633]

Proof of the Homotopy Type-χ-Lemma 156.3. The corollary is an immediate consequence of the Homology-χ-Proposition 156.2 together with the Homotopy Equivalence-H_*-Proposition 144.14 (2). ■

Now that we know that the Euler characteristic is a homotopy invariant (in particular a homeomorphism invariant) it is worth determining it for some of our favorite topological spaces.

Examples.

[1632] We introduced the rank of an abelian group in Definition 97.15. For our current setup the following characterization might be the most convenient way to determine the rank of a finitely generated abelian group: If G is a finitely generated abelian group, then it follows from the Abelian Group-Classification Theorem 97.24 that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. It follows from the Rank Additivity Proposition 97.18 and the Rank Comparison Lemma 97.16 that r is the rank of G.

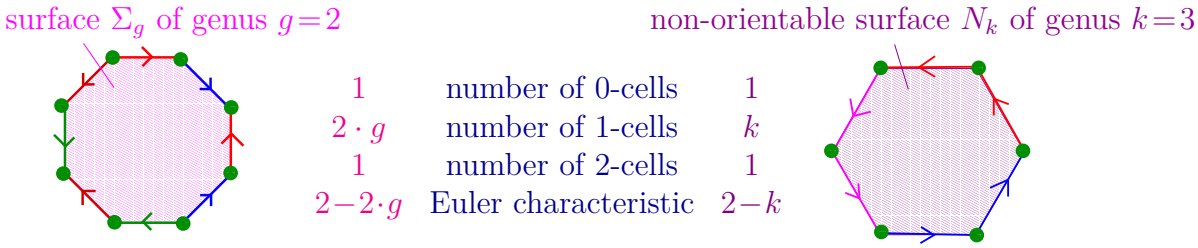
[1633] This corollary implies in particular, that the Euler characteristic of a CW-complex does not depend on the choice of CW-structure.

(1) Let $n \in \mathbb{N}_0$. On page 2625 we showed that the n-dimensional sphere S^n has a CW-structure of S^n with one 0-cell and one extra cell in dimension n. It follows that[1634]

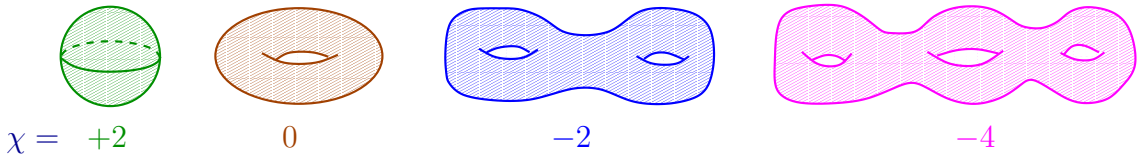
$$\chi(S^n) \;=\; 1+(-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Given $g \in \mathbb{N}_0$ we consider the surface Σ_g of genus g and given $k \in \mathbb{N}$ we consider the non-orientable surface N_k of genus k. On page 2987 we equipped Σ_g and N_k with CW-structures that are also shown (for $g=2$ and $k=3$) in the figure below. One can now easily calculate that[1635]

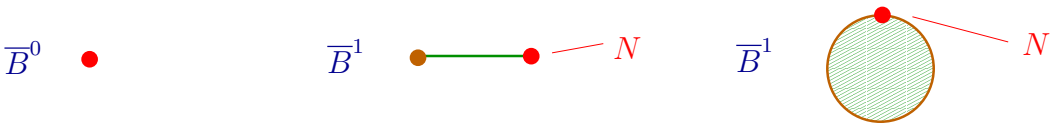
$$\chi(\Sigma_g) \;=\; 2-2g \qquad \text{and} \qquad \chi(N_k) \;=\; 2-k.$$



In the following figure we show the surfaces with increasing genus. The surface of genus 0, i.e. the sphere S^2, has Euler characteristic +2. When we increase the genus by one, the Euler characteristic drops by two.



(3) Let $n \in \mathbb{N}_0$. For $n=0$ we have $\overline{B}^0 = \{0\}$, thus $\chi(\overline{B}^0)=1$. For $n \geq 2$ a CW-structure for the ball $\overline{B}^n$ is given by the filtration $\varnothing \subseteq \{N\} \subseteq S^{n-1} \subseteq \overline{B}^n$, thus we have a CW-structure with one 0-cell $\{N\}$, one extra cell in dimension $n-1$ and one extra cell of dimension n. Thus again we obtain that $\chi(\overline{B}^n) = 1+(-1)^{n-1}+(-1)^n = 1$.[1636]



[1634] One can also calculate $\chi(S^n)$ using the Sphere-H_*-Proposition 149.4 and Homology-χ-Proposition 156.2.

[1635] One can also calculate the Euler characteristics using the Surface-H_*-Proposition 155.17 together with the Homology-χ-Proposition 156.2.

[1636] Here is another approach to calculating the Euler characteristic: Note that $\overline{B}^n$ is contractible, i.e. $\overline{B}^n$ is homotopy equivalent to a singleton $\{*\}$. Thus it follows from the Homotopy Type-χ-Lemma 156.3 that $\chi(\overline{B}^n) = \chi(\{*\}) = 1$.

(4) On page 2628 we showed that $\mathbb{R}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 1, \ldots, n$ and no other cell. It follows that[1637]

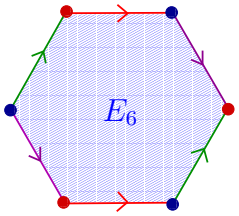
$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \sum_{i=0}^{n} (-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

(5) On page 2628 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 2, \ldots, 2n$ and no other cell. It follows that[1638] $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$. □

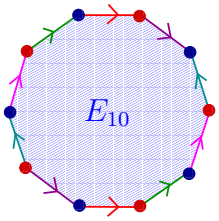
The fact that the Euler characteristic is readily computable and a homeomorphism invariant can be employed in many situations. For example it gives us an easy way to detect the genus of a surface:

Example. On page 1988 we introduced two examples of closed orientable 2-dimensional smooth manifolds for which we found it difficult to determine the genus. We show the same surfaces also in the figure below.

hexagon with opposite sides identified



dodecagon with opposite sides identified



in each case there are two equivalence classes of vertices: red and blue

Let us first consider the surface on the left: it is given by identifying opposite sides of the hexagon E_6. The obvious CW-structures on the hexagon E_6 descends to a CW-structure on $E_6/\!\sim$. We make the following observations:

(a) the single 2-cell on E_6 defines a 2-cell for $E_6/\!\sim$,
(b) we identify every 1-cell of E_6 with the opposite 1-cell, thus $E_6/\!\sim$ has three 1-cells,
(c) we identify all 0-cells that differ by an even number of vertices, thus $E_6/\!\sim$ has precisely two 0-cells.

Summarizing we see that $E_6/\!\sim$ has Euler characteristic $2 - 3 + 1 = 0$. We had just seen on page 2987 that the surface of genus g has Euler characteristic $2 - 2g$. Thus we obtain from the Surface Classification Theorem 101.16 that $E_6/\!\sim$ is diffeomorphic to the surface of genus zero, i.e. the torus.

The same argument shows that $E_{10}/\!\sim$ has Euler characteristic $2-5+1 = -2$ and that it is diffeomorphic to the surface of genus two. Evidently the same argument shows that if we take a regular $(4g+2)$-gon and if we identify opposite sides, then we obtain the surface of genus g. [1639] □

[1637] We can also calculate $\chi(\mathbb{R}\mathrm{P}^n)$ using the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 and the Homology-χ-Proposition 156.2.

[1638] We can also calculate $\chi(\mathbb{C}\mathrm{P}^n)$ using the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 and the Homology-χ-Proposition 156.2.

[1639] It is a great mental exercise to visualize that the surface on the left is a torus and that the surface on the right is a surface of genus two.

156.2. **Proof of the Homology-χ-Proposition 156.2 .** The following purely algebraic lemma is the key ingredient in the proof of the Homology-χ-Proposition 156.2.

Lemma 156.4. (Chain Complex–H_*–χ-Lemma) Given any chain complex

$$0 \;\to\; C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots\; C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

of abelian groups of finite rank the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(C_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(C_*)).$$

The same statement holds if the C_i are finite-dimensional vector spaces over a field and if we replace ranks by dimensions.

Proof. For the above chain complex (C_*, ∂_*) we write for each $n \in \mathbb{N}_0$[1640]

$$Z_n \coloneqq \ker(\partial_n), \quad B_n \coloneqq \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n \;=\; Z_n/B_n.$$

Note that these groups form the following two types of short exact sequences

$$\begin{array}{lccccccccc} \text{(a)} & & 0 & \hookrightarrow & Z_n & \to & C_n & \xrightarrow{\partial_n} & B_{n-1} & \to & 0, \\ \text{(b)} & & 0 & \hookrightarrow & B_n & \to & Z_n & \to & H_n & \to & 0. \end{array}$$

We see that

$$\begin{aligned} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} C_n \;&\overset{\text{by (a) and the Rank Additivity Proposition 97.18 we have } \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}}{\downarrow} \\ &= \sum_{n=0}^{k} (-1)^n \cdot (\operatorname{rank} Z_n + \operatorname{rank} B_{n-1}) \\ &\overset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot (\operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}) \\ &\qquad \text{by (b) and the Rank Additivity Proposition 97.18 we have } \operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n \\ &= \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} H_n \;+ \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} B_n + \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} B_{n-1} \\ &\overset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} H_n \;+ \sum_{n=0}^{k-1} (-1)^n \cdot \operatorname{rank} B_n - \sum_{m=0}^{k-1} (-1)^m \cdot \operatorname{rank} B_m \\ &\qquad \text{we do the substitution } m = n-1 \text{ and we use that } B_k = 0 \text{ and } B_{-1} = 0 \\ &= \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} H_n. \end{aligned}$$

Basically the same proof also applies if the C_i are finite-dimensional vector spaces over a field and if we replace ranks by dimensions. ∎

Now we can easily provide the proof of the Homology-χ-Proposition 156.2.

[1640] It is a common notation to call these groups Z_n and B_n, the names go back to the German words "Zykel" and "Bild".

Proof of the Homology-χ-Proposition 156.2. Let X be a finite k-dimensional CW-complex. We perform the following calculation:

$$\begin{array}{ccccc} & \text{by Definition 156.1 of } \chi(X) & & \text{by the CW-Skeleton-H}_* \text{ Lemma 155.2 (1)} & \\ \chi(X) & \overset{\downarrow}{=} & \sum_{n=0}^{k} (-1)^n \cdot \text{number of } n\text{-cells} & \overset{\downarrow}{=} & \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{C}_n^{\mathrm{CW}}(X)) \\ & \overset{\uparrow}{=} & \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n^{\mathrm{CW}}(X)) & \overset{\uparrow}{=} & \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)). \\ & \text{by the Chain Complex–H}_*\text{–}\chi \text{ Lemma 156.4} & & \text{by the Singular-Cellular-H}_* \text{ Proposition 155.8} & \end{array}$$

■

Finally note, that using the Chain Complex–H$_*$–χ-Lemma 156.4 we can also easily prove the following lemma, which be will useful on several occasions.

Lemma 156.5. (Exact Sequence-χ-Lemma) If

$$0 \to A_k \to A_{k-1} \to \ldots \to A_1 \to A_0 \to 0$$

is an exact sequence of finite-rank abelian groups, then

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(A_n) = 0.$$

Proof. We have

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(A_n) \underset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(A_*)) \underset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot 0 = 0.$$

by the Chain Complex–H$_*$–χ-Lemma 156.4, which we can apply since an exact sequence is also a chain complex

since the sequence is exact

■

156.3. **The Euler characteristic of topological spaces.** In this section we will see that the Homology-χ-Proposition 156.2 allows us to define the notion of a Euler characteristic in a somewhat more general setup:

Definition 156.6. Let X be a topological space and let $A \subseteq X$ be a subspace.

(1) We define the n-th Betti[1647] number of (X, A) to be

$$b_n(X, A) := \operatorname{rank}(\mathrm{H}_n(X, A)) \in \mathbb{N}_0 \cup \{\infty\}$$

(2) If $\sum_{n \in \mathbb{N}_0} b_n(X, A)$ is finite, then we define the Euler characteristic of (X, A) to be

$$\chi(X, A) := \sum_{n \in \mathbb{N}_0} (-1)^n \cdot b_n(X, A).$$

Evidently we write $b_n(X) := b_n(X, \varnothing)$ and $\chi(X) := \chi(X, \varnothing)$.

Example. It now makes sense to say that $\chi(\mathbb{R}^n) = 1$, even though $\mathbb{R}^n$ is non-compact. □

Remark.

(1) It follows from the Homology-χ-Proposition 156.2 that for finite CW-complexes this new definition of the Euler characteristic agrees with the earlier Definition 156.1.

[1647] These numbers are named after the Italian mathematician Enrico Betti (1823-1892) who studied in [**Bet1870**] an early version of homology groups.

(2) In the Smooth Manifold-CW-Theorem 179.9 we pointed out that compact smooth manifolds admit CW-structures. But it is actually unknown whether all compact *topological* manifolds admit CW-structures. Nonetheless we will see in the Topological Manifolds-Invariants Proposition 189.14 that the above Definition 156.6 of a Euler characteristic makes sense for compact topological manifolds. □

Now that we know that Definition 156.6 extends Definition 156.1 it is natural to wonder whether the "new" Euler characteristic shares the properties of the original Euler characteristics. Fortunately this is mostly the case, for example, we have the following slight generalization of the Homotopy Type-χ-Lemma 156.3:

Lemma 156.7. (Homotopy Type-χ-Lemma) Let X and Y be homotopy equivalent topological spaces. If the Euler characteristic is defined for one of the two topological spaces, then it is defined for the other topological space and we have

$$\chi(X) \;=\; \chi(Y).$$

Proof. This statement follows immediately from the definitions and the Homotopy Equivalence-H_*-Proposition 144.14 (2). ■

We conclude this section with the following elementary lemma:

Lemma 156.8. (Relative-χ-Lemma) Let (X, A) be a pair of topological spaces. If $\chi(X), \chi(A)$ and $\chi(X, A)$ are defined, then

$$\chi(X) \;=\; \chi(X, A) + \chi(A).$$

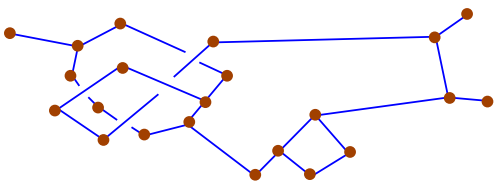
Proof. This lemma follows immediately from the LES-Proposition 145.10 and the Exact Sequence-χ-Lemma 156.5. ■

156.4. **Homology groups of topological graphs.** Using the Homology-χ-Proposition 156.2 we can now easily prove the following lemma:

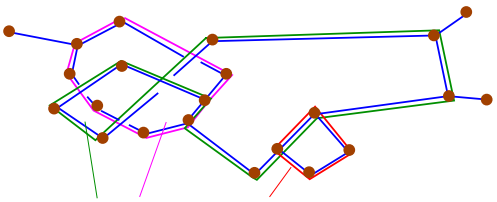
Lemma 156.9. (Graph-H_*-Lemma) Let G be a connected non-empty topological graph with v vertices and e edges. Then

$$\mathrm{H}_0(G) \;\cong\; \mathbb{Z}, \quad \mathrm{H}_1(G) \;\cong\; \mathbb{Z}^{e-v+1} \quad \text{and} \quad \mathrm{H}_i(G) \;=\; 0 \text{ for } i \geq 2.$$

Example. In the figure below we see a connected topological graph G with 22 vertices and 24 edges.



topological graph G with 22 vertices and 24 edges



basis for $\mathrm{H}_1(G) \cong \mathbb{Z}^3$

It follows from the Graph-H_*-Lemma 156.9 that its first homology group is isomorphic to $\mathbb{Z}^3$. Using the Epi-Automorphism Lemma 97.27 one can show fairly easily that the three cycles which are shown in the figure below represent a basis for $\mathrm{H}_1(G) \cong \mathbb{Z}^3$. □

Proof. Let G be a connected topological graph with v vertices and e edges. Recall that by the Graph–CW-Complex Lemma 136.5 we can view G as a 1-dimensional CW-complex

with v 0-cells and e 1-cells. Now that we have developed so many techniques for computing homology groups it is perhaps not surprising that we can provide many different proofs for the lemma. We already sketched one proof in Exercise 153.10. We now provide two more proofs.[1648]

(1) First of all, from the H_0-Proposition 143.25 together with the CW-Complex–Local Properties Proposition 137.8 (2) and the Locally Path-Connected Lemma 4.57 it follows immediately that $\mathrm{H}_0(G) \cong \mathbb{Z}$. Furthermore, from the CW-Complex-H_*-Proposition 155.10 (1) and (3) it follows that $\mathrm{H}_i(G) = 0$ for $i \geq 2$ and that $\mathrm{H}_1(G)$ is torsion-free. We now see that

$$1 - \operatorname{rank}(\mathrm{H}_1(G)) \underset{\substack{\uparrow \\ \text{by the above}}}{=} \sum_i (-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(X)) \underset{\substack{\uparrow \\ \text{by the Homology-}\chi \\ \text{Proposition 156.2}}}{=} \chi(X) = \#0\text{-cells} - \#1\text{-cells} = v - e.$$

Since $\mathrm{H}_1(G)$ is torsion-free we obtain that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-v+1}$.

(2) An alternative proof for the Graph-H_*-Lemma 156.9 is given as follows: we have

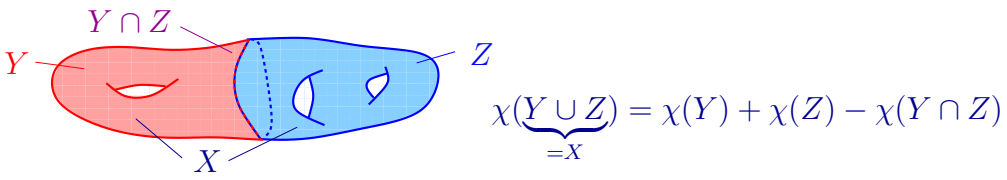
$$\widetilde{\mathrm{H}}_j(G) \underset{\substack{\uparrow \\ \text{by the 1-Complex-}\pi_1 \\ \text{Theorem 137.6 (2a) and} \\ \text{the Homotopy Equivalence-H}_* \\ \text{Proposition 144.14 (2)}}}{\cong} \widetilde{\mathrm{H}}_j(\text{wedge of } 1-\chi(G) \text{ circles}) \underset{\substack{\downarrow \\ \text{by the Wedge-H}_* \\ \text{Proposition 149.16}}}{\cong} \bigoplus_{i=1}^{1-\chi(G)} \widetilde{\mathrm{H}}_j(S^1) \underset{\substack{\uparrow \\ \text{by the Sphere-H}_* \\ \text{Proposition 149.4}}}{\cong} \begin{cases} \mathbb{Z}^{1-\chi(G)}, & \text{if } j=1, \\ 0, & \text{else.} \end{cases}$$

It is a matter of taste which of the two proofs is the nicest. ■

156.5. **Properties of the Euler characteristic.** The Euler characteristic is an invariant which "sees less" than the homology groups. On the other hand we will see in this section that it behaves better than the homology groups under several natural operations. This makes the Euler characteristic an eminently computable invariant.

Proposition 156.10. (Union-χ-Proposition) Let $X = Y \cup Z$ be a decomposition of a finite CW-complex X into two subcomplexes Y and Z. The following equality holds [1649]

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$



[1648] The reader is invited to come up with more proofs.

[1649] It follows from the Subcomplex–Intersection-Union Lemma 136.13 that $Y \cap Z$ is also a CW-complex.

Proof. We give two proofs for the proposition:

(1) Given $n \in \mathbb{N}_0$ and given a subcomplex W of X we denote by $n(W)$ the number of n-cells that are contained in W. In our setting we have

$$n(X) \underset{\uparrow}{=} n(Y) + n(Z) - n(Y \cap Z).$$

since $X = Y \cup Z$ and since Y and Z are subcomplexes we obtain from the Subcomplex–Characterization Proposition 136.12 together with the Subcomplex–Closed Lemma 136.11 that every n-cell of X lies in Y or Z, the n-cells that lie in the intersection $Y \cap Z$ get counted twice in $n(Y) + n(Z)$, so we have to subtract $n(Y \cap Z)$ to obtain $n(X)$

By performing the alternating sum over all $n \in \mathbb{N}_0$ we obtain immediately that

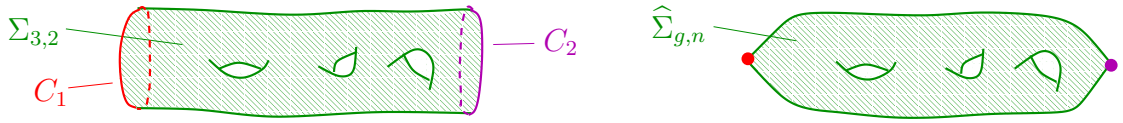
$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

(2) The Mayer–Vietoris Theorem 153.2 (2) applied to the decomposition $X = Y \cup Z$ gives us a long exact sequence. If we apply the Exact Sequence-χ-Lemma 156.5 and the Homology-χ-Proposition 156.2 to this long exact sequence, we obtain the desired result. We will fill in the details in Exercise 156.11.[1650] ■

Proposition 156.11. [1651] **(Quotient-χ-Proposition)** Let X be a finite CW-complex and let A be a subcomplex of X. We equip the quotient X/A with the CW-structure given by the CW-Quotient Lemma 139.10. Then

$$\chi(X/A) = \chi(X) - \chi(A) + 1.$$

Example. Let $\Sigma_{g,n}$ be the surface of genus g minus n open disks. We equip $\Sigma_{g,n}$ with a CW-structure. Let $C_1, \ldots, C_n$ be the boundary components of $\Sigma_{g,n}$. We denote by $\widehat{\Sigma}_{g,n}$ the topological space that is obtained from $\Sigma_{g,n}$ by collapsing each boundary component to a point.



By the Surface Recovery Lemma 101.15 we know that $\widehat{\Sigma}_{g,n}$ is homeomorphic to the surface of genus g. We calculate that

$$2-2g \underset{\uparrow}{=} \chi(\widehat{\Sigma}_{g,n}) = \chi(\overbrace{(\ldots(\Sigma_{g,n}/C_1)/\ldots/C_k)}^{=\widehat{\Sigma}_{g,n}}) \underset{\uparrow}{=} \chi(\Sigma_{g,n}) - \sum_{i=1}^{n} \underbrace{\chi(C_i)}_{=0} + n = \chi(\Sigma_{g,n}) + n.$$

see page 2987

by the Quotient-χ-Proposition 156.11, applied n times

Thus it follows that $\chi(\Sigma_{g,n}) = 2 - 2g - n$. □

[1650] This second proof can be viewed as overkill. But it does have the advantage that it works in the more general context of the Euler characterization from Definition 156.6.

[1651] This proposition can be viewed as a generalization of the Abstract Graph–Quotient Lemma 11.32.

Proof. Again we provide two proofs of the proposition:

(1) Recall that in the CW-Quotient Lemma 139.10 we equipped X/A with a CW-structure such that for each $k \in \mathbb{N}_0$ we have the following equalities:

$$\#k\text{-cells of } X/A \;=\; \begin{cases} \#k\text{-cells of } X \text{ not in } A, & \text{if } k > 0, \\ \#k\text{-cells of } X \text{ not in } A \;+\; 1, & \text{if } k = 0. \end{cases}$$

Evidently these equalities imply the equality $\chi(X/A) = \chi(X) - \chi(A) + 1$.

(2) The desired equality also follows from the Relative-χ-Lemma 156.8 together with the Relative-to-Quotient-H_*-Proposition 149.13 (which we can apply by the Subcomplex–Closed Lemma 136.11 and the Subcomplex–Neighborhood Proposition 137.9 (1)) and the Reduced Homology Lemma 147.3. We will fill in the details in Exercise 156.11. ■

Proposition 156.12. (Product-χ-Proposition) Let X and Y be two finite CW-complexes. If we equip $X \times Y$ with the CW-structure from the CW-Category–Product Proposition 139.16 and the CW-Complex Product Proposition 139.18, then[1652]

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Example. Let X be a finite CW-complex.

(1) It follows from the Product-χ-Proposition 156.12 that $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.

(2) We consider the suspension $\Sigma(X) = ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\})$. We see that

$$\begin{array}{rcl} \chi(\Sigma(X)) & \overset{\text{follows from applying the Quotient-}\chi\text{-Proposition 156.11 twice}}{\downarrow} & \\ & = & \chi((X \times [-1,1]) - \chi(X \times \{-1\}) + 1 - \chi(X \times \{1\}) + 1 \\ & = & 2 + \chi(X) - \chi(X) - \chi(X) \;=\; 2 - \chi(X). \\ & \underset{\text{follows from the Product-}\chi\text{-Proposition 156.12 and the observation that } \chi([-1,1]) = 1}{\uparrow} & \end{array}$$

□

Proof. [1653] Let X and Y be two finite CW-complexes. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y. We equip $X \times Y$ with the product CW-structure from the CW-Category–Product Proposition 139.16 and the CW-Complex Product Proposition 139.18. By construction we have

$$\#n\text{-cells of } X \times Y \;=\; \sum_{k+l=n} \#k\text{-cells of } X \cdot \#l\text{-cells of } Y \;=\; \sum_{k+l=n} c_k \cdot d_l.$$

It follows that

$$\begin{array}{rcl} \chi(X) \cdot \chi(Y) \;=\; \Big(\sum\limits_{k \in \mathbb{N}_0} (-1)^k \cdot c_k\Big) \cdot \Big(\sum\limits_{l \in \mathbb{N}_0} (-1)^l \cdot d_l\Big) & \overset{\text{by multiplying out}}{\downarrow} & \\ & = & \sum\limits_{n \in \mathbb{N}_0} \sum\limits_{k+l=n} (-1)^{k+l} \cdot c_k \cdot d_l \\ & = & \sum\limits_{n \in \mathbb{N}_0} (-1)^n \cdot \sum\limits_{k+l=n} c_k \cdot d_l \;=\; \chi(X \times Y). \\ & & \qquad\qquad \underset{\text{by the above discussion}}{\uparrow} \end{array}$$

We have thus proved the desired equality $\chi(X) \cdot \chi(Y) = \chi(X \times Y)$. ■

[1652] Note that the proposition gives in particular a solution to Exercise 139.5.

[1653] In the proofs of the Union-χ-Proposition 156.10 and the Quotient-χ-Proposition 156.11 we provided two arguments: one given by counting cells and one given by considering homology groups. For the current proposition we will have to wait till the Künneth Theorem 164.6 to get a proof using homology groups.

Next let $p\colon \widetilde{X} \to X$ be a covering of path-connected non-empty topological spaces. We know that the fundamental groups and the higher homotopy groups behave well under coverings:

(1) In the Covering-π_1-Monomorphism Proposition 92.7 we showed that the induced map $p_*\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism and we saw in the Covering-Degree Proposition 92.10 that $[\pi_1(X) : p_*(\pi_1(\widetilde{X}))] = \deg(p)$.
(2) For $n \geq 2$ we obtain from the Covering-π_n-Isomorphism Proposition 134.22 that the induced map $p_*\colon \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism.

There are no analogous statements for homology groups. For example we saw on page 1783 that there exists a 2-fold covering $p\colon S^1 \times S^1 \to K$ from the torus to the Klein bottle K. In this case we have by the Torus-H_*-Proposition 153.9 and the Klein Bottle-H_*-Lemma 153.10 that

$$\begin{array}{rcl} \mathrm{H}_0(S^1 \times S^1) &\cong& \mathbb{Z}, \\ \mathrm{H}_1(S^1 \times S^1) &\cong& \mathbb{Z}^2, \\ \mathrm{H}_2(S^1 \times S^1) &\cong& \mathbb{Z}, \\ \mathrm{H}_i(S^1 \times S^1) &\cong& 0, \text{ for } i \geq 3, \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(K) &\cong& \mathbb{Z}, \\ \mathrm{H}_1(K) &\cong& \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\ \mathrm{H}_2(K) &\cong& 0, \\ \mathrm{H}_i(K) &\cong& 0, \text{ for } i \geq 3. \end{array}$$

Thus we see that the homology groups are quite different and there is no clear connection between the n-th homology group of a covering and the n-th homology group of the base.

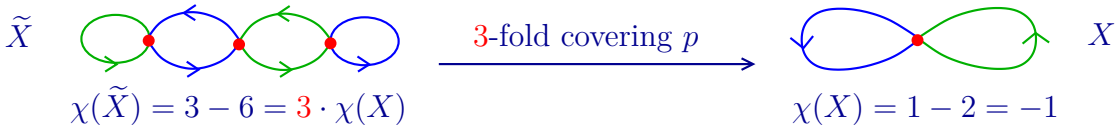
It is therefore perhaps surprising that we have the following proposition:

Proposition 156.13. (Covering-χ-Proposition) Let $p\colon \widetilde{X} \to X$ be a finite covering of a finite path-connected non-empty CW-complex X. We equip $\widetilde{X}$ with the structure of a CW-complex given by the CW-Complex Covering Proposition 138.1. Then

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Example.

(1) We consider again the 2-fold covering $p\colon S^1 \times S^1 \to K$. Recall that on page 2987 we saw that $\chi(S^1 \times S^1) = 0$ and $\chi(K) = 0$. This is consistent with the Covering-χ-Proposition 156.13.
(2) We consider the 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2$. In this case $\chi(S^2) = 2$ is indeed twice $\chi(\mathbb{R}\mathrm{P}^2) = 1$.
(3) In the Graph-π_1-Proposition 99.13 we showed that if X is a connected finite non-empty 1-dimensional CW-complex, then $\pi_1(X)$ is a free group of rank $1 - \chi(X)$. As we implicitly already argued in the Free Subgroup Proposition 138.6, it follows fairly easily from this observation and the Covering-χ-Proposition 156.13 that if F is a free group of finite rank and if G is a finite-index subgroup of F, then G is a free group with $\mathrm{rank}(G) - 1 = [F : G] \cdot (\mathrm{rank}(F) - 1)$.



Proof. [1654] As we mentioned in the CW-Complex Covering Proposition 138.1 (1), it follows immediately from the construction of the CW-structure on $\widetilde{X}$ that for each $k \in \mathbb{N}_0$ we have

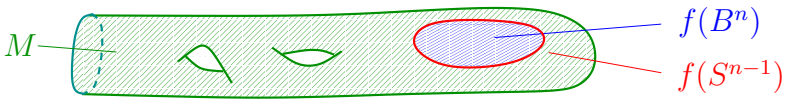
$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

The proposition is an immediate consequence of this equality. ■

156.6. **Calculations of Euler characteristics.** The results from the previous section, especially the Union-χ-Proposition 156.10 can be extremely useful for calculating Euler characteristics. We start out with the following simple lemma.

Lemma 156.14. (Manifold-minus-Ball–χ-Lemma) Let M be a compact n-dimensional smooth manifold and let $f\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding. Then[1655]

$$\chi(M \setminus f(B^n)) \;=\; \chi(M) + (-1)^{n-1}.$$



Proof. By the Smooth Manifold-CW Structure Theorem we can equip M with a CW-structure such that $M \setminus f(B^n)$ and $f(\overline{B}^n)$ are subcomplexes. We perform the following simple calculation:

$$\begin{array}{rcl} \chi(M) \;=\; \chi((M \setminus f(B^n)) \cup f(\overline{B}^n)) & \overset{\text{by the Union-}\chi\text{ Proposition 156.10}}{\downarrow} & \\ & = \chi(M \setminus f(B^n)) + \overbrace{\chi(f(\overline{B}^n))}^{=1} - \overbrace{\chi(f(S^{n-1}))}^{=1+(-1)^{n-1}} \\ & = \chi(M \setminus f(B^n)) - (-1)^{n-1}. \end{array}$$

It follows that $\chi(M \setminus f(B^n)) = \chi(M) + (-1)^{n-1}$. ■

The following lemma serves in particular as an example for the previous lemma.

Lemma 156.15. (Surface-χ-Lemma)

(1) Let $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. If we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks, then $\chi(\Sigma_{g,n}) = 2 - 2g - n$.

(2) Let Σ and Σ' be two compact orientable connected 2-dimensional smooth manifolds. If Σ and Σ' have the same number of boundary components and if $\chi(\Sigma) = \chi(\Sigma')$, then Σ and Σ' are diffeomorphic.

Proof.

(1) On page 2987 we showed that $\chi(\Sigma_{g,0}) = 2-2g$. The equality now follows from applying the Manifold-minus-Ball–χ-Lemma 156.14 altogether n times.[1656]

[1654] It is quite tricky to provide a proof of this proposition using homology groups. One approach is via the Serre spectral sequence, which we will not cover in this course.

[1655] It follows from the Codimension-Zero Submanifold Proposition 25.49 (3) that $M \setminus f(B^n)$ is a compact n-dimensional smooth manifold. Thus it follows from the Smooth Manifold-CW-Theorem 179.9 that the Euler characteristic of $M \setminus f(B^n)$ is defined.

[1656] Of course there are many other approaches to proving this equality. For example one can easily write down an explicit CW-structure for $\Sigma_{g,n}$. Alternatively we can use the Homology-χ-Proposition 156.2 together with the Surface-H_*-Proposition 155.17. Finally note that we already gave an argument on page 2993.

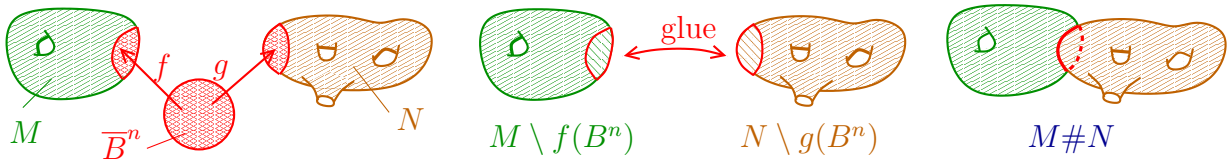
(2) This statement follows immediately from (1) and the Surface Classification Theorem 101.16 together with the elementary observation that the Euler characteristic $\chi(\Sigma_{g,n}) = 2-2g-n$ and the number of boundary components n determines the genus g. ■

In Definition 54.12 we introduced the connected sum of two smooth manifolds. We will not recall the definition, but we hope that the figure below jogs the reader's memory. In the following lemma we determine the Euler characteristic of a connected sum of two smooth manifolds

Lemma 156.16. (Connected Sum-χ-Lemma) If M and N are two compact connected non-empty n-dimensional smooth manifolds, then we have the following equality:[1657]

$$\chi(M\#N) \;=\; \chi(M)+\chi(N)+\begin{cases} -2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

Proof. [1658] Let M and N be two compact connected non-empty n-dimensional smooth manifolds and let $f\colon \overline{B}^n \to M \setminus \partial M$ and $g\colon \overline{B}^n \to N \setminus \partial N$ be two smooth embeddings.



Note that it follows from the Smooth Connected Sum-Proposition 54.13 together with the Smooth Manifold-CW Structure Theorem that we can equip $M\#N$ with a CW-structure such that $M \setminus f(B^n)$ and $N \setminus g(B^n)$ are subcomplexes. We see that

$$\begin{aligned}
\chi(M\#N) &\overset{\text{by Definition 54.12 of } M\#N}{=} \chi((M \setminus f(B^n)) \cup_{f(S^{n-1})=g(S^{n-1})} (N \setminus g(B^n))) \\
&\overset{\text{by the Euler Characteristic Addition Proposition 156.10}}{=} \chi(M \setminus f(B^n)) + \chi(N \setminus g(B^n)) - \chi(S^{n-1}) \\
&\overset{\text{follows from the Manifold-minus-Ball–}\chi\text{-Lemma 156.14 together with the calculation } \chi(S^n)=1+(-1)^{n-1} \text{ from page 2987}}{=} \chi(M)+(-1)^{n-1} \;+\; \chi(N)+(-1)^{n-1} \;-\; (1+(-1)^{n-1}) \\
&= \chi(M)+\chi(N)-1+(-1)^{n-1}.
\end{aligned}$$

■

Example. Let $g \in \mathbb{N}_0$. We see that

$$\chi(\text{surface of genus } g) \underset{\text{by the Surface–Connected Sum Lemma 54.15}}{=} \text{connected sum of } g \text{ tori} \underset{\text{by the Connected Sum-}\chi\text{-Lemma 156.16 applied } g-1 \text{ times}}{=} g\cdot\overbrace{\chi(S^1\times S^1)}^{=0} - 2\cdot(g-1) \;=\; 2-2g.$$

[1657] Strictly speaking, if M and N are both orientable, then we need to assume that M and N are oriented to get a well-defined connected sum.

[1658] In the Connected Sum-H_*-Proposition 185.23 we will calculate the homology groups of the connected sum of two closed oriented smooth manifolds. Together with the Homology-χ-Proposition 156.2 this gives an alternative proof of the lemma for closed smooth manifolds.

Similarly we see that for $k \in \mathbb{N}$ we have

$$\chi(\text{non-orientable surface of genus } k) \overset{\text{by the Surface–Connected Sum Lemma 54.15}}{=} \text{connected sum of } k \text{ copies of } \mathbb{R}\mathrm{P}^2 \overset{\text{by the Connected Sum } \chi\text{-Lemma 156.16 applied } k-1 \text{ times}}{=} \\ = k \cdot \underbrace{\chi(\mathbb{R}\mathrm{P}^2)}_{=1} - 2 \cdot (k-1) = 2-k.$$

We obtained the same results on page 2987 using a different method. □

We conclude this section with the following variation on the Manifold-minus-Ball–χ-Lemma 156.14.

Lemma 156.17. (Link Exterior–χ-Lemma) Let $L \subseteq S^3$ be a link. Furthermore let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map for L. For the link exterior $X_L := S^3 \setminus \tau(B^2 \times L)$ from Definition 108.30 we have

$$\chi(X_L) = 0.$$



Proof. By the Smooth Manifold-CW Structure Theorem we can equip S^3 with a CW-structure such that $X_L = S^3 \setminus \tau(B^2 \times L)$ and $\tau(\overline{B}^2 \times L)$ are subcomplexes. We perform the following simple calculation:

$$0 \overset{\text{see page 2987}}{=} \chi(S^3) = \chi((S^3 \setminus \tau(B^2 \times L)) \cup \tau(\overline{B}^2 \times L)) \\ = \chi(S^3 \setminus \tau(B^2 \times L)) + \underbrace{\chi(\tau(\overline{B}^2 \times L))}_{=0,\text{ see page 2994}} - \underbrace{\chi(\tau(S^1 \times S^1))}_{=0,\text{ see page 2987}} = \chi(S^3 \setminus \tau(B^2 \times L)).$$

(by the Union-χ Proposition 156.10)

It follows that $\chi(S^3 \setminus \tau(B^2 \times L)) = 0$. ■

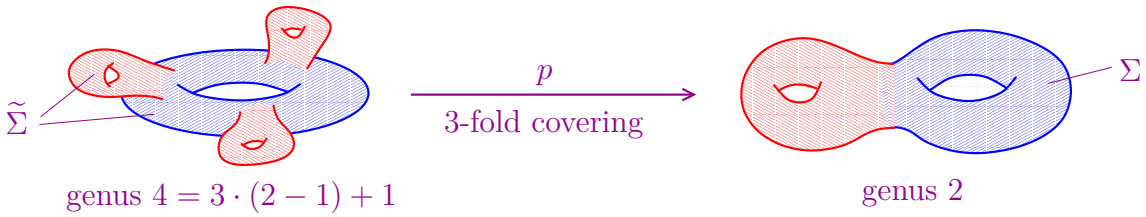
156.7. **Applications of Euler characteristics.** In Question 129.8 we asked how the genus of a surface changes if we take finite covers. We had answered this question already in the Subgroups-of-Surface Groups Proposition 130.7 (1) and Exercise 130.6. The proof of the following lemma is arguably more conceptual and easier to remember.

Lemma 156.18. (Genus-of-Covering Lemma) Let Σ be the surface of genus g. For any connected finite covering $p\colon \widetilde{\Sigma} \to \Sigma$ we have[1659]

$$\text{genus}(\widetilde{\Sigma}) = \deg(p) \cdot (g-1) + 1.$$

Example. In the figure below we illustrate a 3-fold covering of the surface of genus 2 by the surface of genus $4 = 3 \cdot (2-1) + 1$.
Fortunately this example is consistent with the Genus-of-Covering Lemma 156.18. □

[1659] Recall that by the Surface Classification Theorem 101.16 there exists precisely one $\widetilde{g}$ such that the closed, oriented 2-dimensional smooth manifold $\widetilde{\Sigma}$ is diffeomorphic to the surface of genus $\widetilde{g}$.



Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the discussion on page 2987 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.
(2) By the Covering-χ-Proposition 156.13 we have $\chi(\widetilde{\Sigma}) = \deg(p) \cdot \chi(\Sigma)$. ∎

Rather surprisingly we can use the Genus-of-Covering Lemma 156.18 to give a new proof of the following proposition.

Proposition 130.21. (Surface-π_1–Not Free Proposition) If $g \geq 1$, then the group

$$\pi_1(\text{surface of genus } g) \underset{\substack{\uparrow \\ \text{by the Surface-}\pi_1 \\ \text{Proposition 101.1}}}{\cong} \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

is not free.

Proof. We start out with two observations about abelianizations:

(1) Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g. It follows from the Surface-π_1-Proposition 101.1 that $\pi_1(\Sigma_g)_{\text{ab}} \cong \mathbb{Z}^{2g}$.
(2) Given $k \in \mathbb{N}_0$ we denote by $F_k = \langle x_1, \ldots, x_k \rangle$ the free group on k generators $x_1, \ldots, x_k$. It follows from the Abelianization Coproduct Proposition 98.18 (2) that $(F_k)_{\text{ab}} \cong \mathbb{Z}^k$.

Now we turn to the actual proof of the proposition. Let $g \geq 1$. We write $\pi = \pi_1(\Sigma_g)$. Let us assume that π is free. it follows from the above two observations and the Basis Cardinality Theorem 97.12 that π is free of rank $2g$. This implies that there exists an isomorphism $\varphi \colon \pi \to F_{2g}$.

By Observation (1) we know that $\pi_{\text{ab}} \cong \mathbb{Z}^{2g}$. Since we assume that $g \geq 1$ we see that there exists an epimorphism $\alpha \colon \pi \to \mathbb{Z}/k\mathbb{Z}$ for some $k \geq 2$. By the Covering Existence Theorem 127.10 there exists a connected covering $p \colon \widetilde{\Sigma} \to \Sigma$ with $p_*(\pi_1(\widetilde{\Sigma})) = \ker(\alpha)$. By the Covering-Degree Proposition 92.10 this is a k-fold covering. We get the following commutative diagram:

$$\begin{array}{ccccccc}
\pi & \longleftarrow & \ker(\alpha) & \xrightarrow[\substack{\text{by the Genus-of-Covering} \\ \text{Lemma 156.18}}]{\cong} & \pi_1(\Sigma_{k\cdot(g-1)+1}) & \xrightarrow{\text{abelianization}} & \mathbb{Z}^{2k\cdot(g-1)+2} \\
\varphi \downarrow \cong & & \varphi \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
F & \longleftarrow & \ker(\alpha \circ \varphi^{-1}) & \xrightarrow[\substack{\text{by the Free Subgroup} \\ \text{Proposition 138.6}}]{\cong} & F_{k\cdot(2g-1)+1} & \xrightarrow{\text{abelianization}} & \mathbb{Z}^{k\cdot(2g-1)+1}
\end{array}.$$

It follows from $k \geq 2$ that $2k \cdot (g-1) + 2 \neq k \cdot (2g-1) + 1$, hence the two groups on the right cannot by isomorphic. We have thus obtained a contradiction. ∎

156.8. **Groups acting on smooth manifolds.** We recall the following Definition 2.36 and Definition 25.38:

Definition 156.19. Let G be a group that acts on a topological space X.

(1) We say that the action is continuous if for every $g \neq e$ the map $X \to X$ given by $x \mapsto g \cdot x$ is continuous.
(2) If X is in fact a smooth manifold, then we say that the action is smooth if for every $g \neq e$ the map $X \to X$ given by $x \mapsto g \cdot x$ is smooth.
(3) We say that the action is free if for every $g \neq e$ and every $x \in X$ we have $g \cdot x \neq x$.

Example. Let $k \in \mathbb{N}$. It is straightforward to show that the map

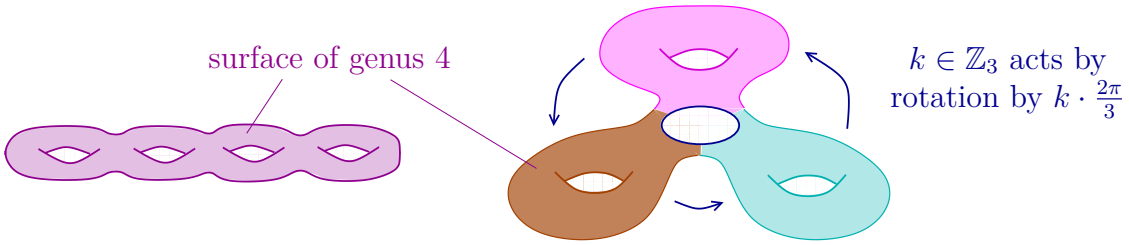
$$\begin{array}{rcl} \mathbb{Z}/k\mathbb{Z} \times S^{2n-1} & \to & S^{2n-1} \\ ([l], (z_1, \ldots, z_n)) & \mapsto & (\exp(2\pi \mathrm{i} l/k) \cdot z_1, \ldots, \exp(2\pi \mathrm{i} l/k) \cdot z_n) \end{array}$$

is a free and smooth action of $\mathbb{Z}/k\mathbb{Z}$ on the odd-dimensional sphere S^{2n-1}. □

Proposition 156.20. (Free Smooth Action-on-Manifold Proposition) If a finite group G acts freely and smoothly[1660] on a compact smooth manifold M, then $|G|$ divides the Euler characteristic $\chi(M)$.

Example.

(1) We consider the case of the even-dimensional sphere $M = S^{2n}$. On page 2987 we saw that $\chi(S^{2n}) = 2$. It follows from the Free Smooth Action-on-Manifold Proposition 156.20 that any group that acts freely and smoothly on M is necessarily trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In fact in the Free Action-on-Sphere Proposition 46.17 we proved a stronger statement, namely we saw that we can replace "smoothly" by "continuously".
(2) Let Σ be the surface of genus four. Recall that by the discussion on page 2987 we know that $\chi(\Sigma) = 2 - 2 \cdot 4 = -6$. By the Free Smooth Action-on-Manifold Proposition 156.20 we know that the order of any finite group acting freely and smoothly on Σ has to have an order that divides 6. In the figure below we sketch a free and smooth action of the group $\mathbb{Z}/3\mathbb{Z}$ on Σ.



In Exercise 156.12 we will see that given any $n \in \mathbb{N}$ with $n \,|\, \chi(\Sigma)$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ admits a free and smooth action on Σ. □

Proof. Let G be a finite group that acts freely and smoothly on a compact smooth manifold M. We make the following observation:

(1) Since M is Hausdorff and since G is finite we obtain from the Free-Proper Action Lemma 25.44 that the action is discrete.

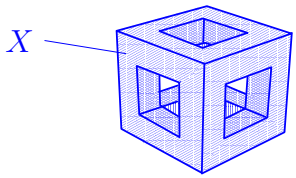
[1660] As we will see, we need the hypothesis that M is a smooth manifold and that G acts freely and smoothly to ensure that the quotient is again a smooth manifold, and thus admits a CW-structure. If one starts out with a topological manifold and an action that is free and continuous, then one can use the Manifold-χ-Proposition 190.24 (5) to prove that $|G|$ divides $\chi(M)$.

(2) It follows from (1) and the Action–Covering Proposition 91.11 that the natural projection $M \to M/G$ is a covering of degree $|G|$.
(3) Since G is finite and since the action is smooth we obtain from the Manifold–Quotient Proposition 25.39 that M/G inherits the structure of a compact smooth manifold. It follows from the Smooth Manifold-CW Structure Theorem 179.9 that we can equip M/G with a CW-structure.

It follows from (2), (3) and the Covering-χ-Proposition 156.13 that $\chi(M) = |G| \cdot \chi(M/G)$. ■

156.9. Exercises for Chapter 156.

Exercise 156.1. Determine the Euler characteristic of the 2-dimensional CW-complex shown in the figure below.



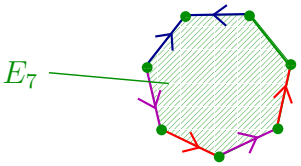
Exercise 156.2. Given a finite set S we denote by $|S| \in \mathbb{N}_0$ the number of elements in S. Let $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$ be a chain complex consisting of homomorphisms between finite abelian groups. Show that

$$\prod_{i=0}^{k} |C_i|^{(-1)^i} = \prod_{i=0}^{k} |\mathrm{H}_i(C)|^{(-1)^i}.$$

Exercise 156.3. Show that every CW-structure for the torus with one 0-cell has precisely one 2-cell.

Exercise 156.4. Let X and Y be two finite CW-complexes. Show that if X and Y are homotopy equivalent, then the total number of cells of X is even (odd) if and only if the total number of cells of Y is even (odd).

Exercise 156.5. What is the homeomorphism type of the surface $E_7/\sim$ shown in the figure below?



Exercise 156.6. Let $K \subseteq S^3$ be a knot. By the Product Tubular Neighborhood Theorem 58.4 (2) there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0, P) = P$ for all $P \in K$. We set $X_K := S^3 \setminus \Phi(B^2 \times K)$.

(a) By the Smooth Manifold-CW-Theorem 179.9 there exists a CW-structure for S^3 such that $X_K = S^3 \setminus \Phi(B^2 \times K)$ and $\Phi(\overline{B}^2 \times S^1)$ are subcomplexes. Determine the Euler characteristic of X_K.

(b) Use the Mayer-Vietoris Theorem 153.2 (3) to determine $\mathrm{H}_0(X_K)$ and $\mathrm{H}_1(X_K)$.
Remark. That is basically the content of Exercise 153.19.

(c) In the No Top-Cell Proposition 183.31 we will see that X_K is homotopy equivalent to a finite CW-complex of dimension two. Use this fact, together with (a) and (b) to calculate the homology groups of X_K.

Exercise 156.7.

(a) Let $k \in \mathbb{N}$. By the Orientation Covering–Existence Proposition 95.8 we know that the non-orientable surface N_k admits an essentially unique 2-fold covering $\Sigma \to N_k$ such that Σ is orientable. What is the genus of Σ?

(b) Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. By the Orientation Covering–Existence Proposition 95.8 we know that the non-orientable surface $N_{k,m}$ admits an essentially unique 2-fold covering $\Sigma \to N_{k,m}$ such that Σ that is orientable. What is the genus of Σ and what is the number of boundary components of Σ?
Remark. You could use Exercise 95.13.

Remark. These are precisely the same questions as in Exercise 101.19. But this time you can give a different, shorter proof using Euler characteristics.

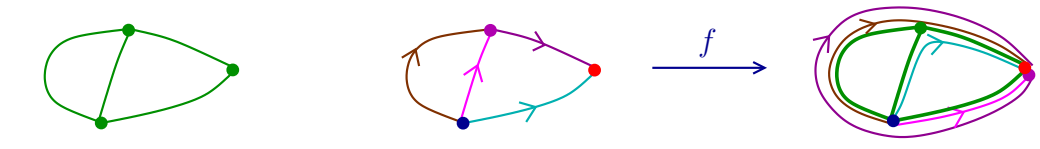
Exercise 156.8. Let X be a topological space that is homotopy equivalent to a finite CW-complex. Show that for any self-homeomorphism f of X we have

$$\chi(\mathrm{Tor}(X, f)) \;=\; 0.$$

Hint. Use the Mapping Torus-H_*-Proposition 153.6.

Exercise 156.9. Given $g, n \in \mathbb{N}_0$ we denote, as usual, by $\Sigma_{g,n}$ the surface of genus g minus n open disks. Let $g, k, m, n \in \mathbb{N}_0$. We now suppose that there exists a smooth embedding $\Sigma_{g,m} \to \Sigma_{k,n} \setminus \partial\Sigma_{k,n}$. Show that $g \leq k$.
Hint. Use the Codimension-Zero Submanifold Proposition 25.49.

Exercise 156.10. We consider the topological graph G together with the map $f\colon G \to G$ that are shown in the figure below. Determine the determinant and the trace of the induced map $f_*\colon \mathrm{H}_1(G) \to \mathrm{H}_1(G)$.
Remark. In the proof of the Graph-H_*-Lemma 156.9 we gave two different, more or less explicit, isomorphisms $\varphi\colon \mathrm{H}_1(G) \xrightarrow{\cong} \mathbb{Z}^2$. The challenge now is to find an isomorphism φ that is explicit enough so that one can calculate the map $\varphi \circ f_* \circ \varphi^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$.



topological graph G

Exercise 156.11. Given a topological space X we define its *total Betti number* as the sum $b(X) \coloneqq \sum\limits_{n\in\mathbb{N}_0} b_n(X) \in \mathbb{N}_0 \cup \{\infty\}$. Given a topological space X with finite total Betti number we define its Euler characteristic as

$$\chi(X) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X).$$

(a) Let $Y, Z \subseteq X$ be closed subsets of a topological space with $X = Y \cup Z$. Show that if $X, Y, Z, Y \cap Z$ have finite total Betti number and if $Y \cap Z$ is a good subset of X and Y, then
$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$
(b) Let X be a topological space and let $A \subseteq X$ be a subspace such that the total Betti numbers of X, A and X/A are finite. We suppose that A is a good subset of X. Show that
$$\chi(X/A) \;=\; \chi(X) - \chi(A) + 1.$$

Exercise 156.12.

(a) Let $N_3 = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ be the non-orientable surface of genus three. Show that given any even $n \in \mathbb{N}$ there exists a connected orientable covering $p\colon \Sigma \to N_3$ of degree n.
(b) Let Σ be the surface of genus g. Show that given any $k \in \mathbb{N}$ that divides $\chi(\Sigma)$ there exists a free and smooth action by the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on Σ.

Remark. This exercise gives in particular a more systematic solution to Exercise 9.7.

Exercise 156.13. Let G be a finite group. Does there exist a closed connected non-empty 2-dimensional smooth manifold on which G acts freely and smoothly?

Exercise 156.14. Let $k \in \mathbb{N}$. Show that the fundamental group of the non-orientable surface of genus k is not a free group.

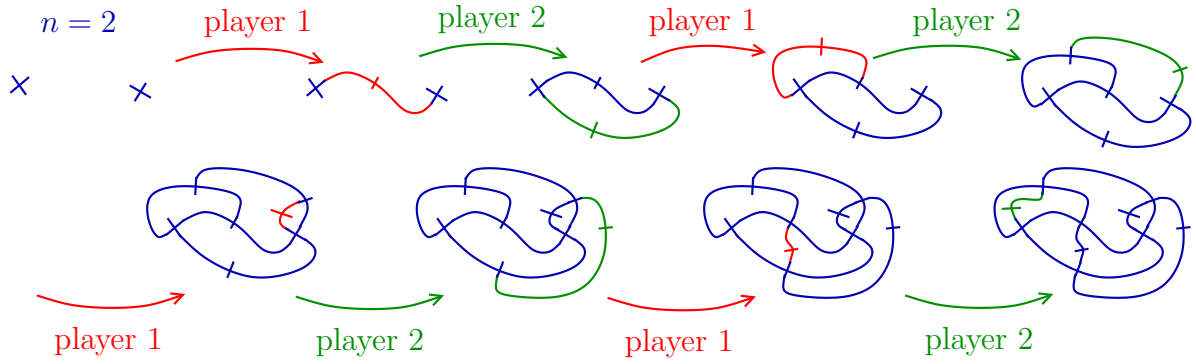
Exercise 156.15. Let Σ be the torus minus one open disk. Furthermore let $\alpha, \beta\colon S^1 \to \Sigma$ be two disjoint embeddings with $\alpha_*([S^1]) = \beta_*([S^1]) \in \mathrm{H}_1(\Sigma)$. We denote this homology class by x.

(a) Suppose that $x \neq 0$. Show that α is isotopic to β.
(b) Show that the conclusion of (a) does not necessarily hold, if we drop the hypothesis that $x \neq 0$.

Exercise 156.16. The two player game of "Brussels sprouts" works as follows:

(1) Initially one draws n disjoint crosses on a 2-dimensional smooth manifold.
(2) The two players take turns to do the following move: Join two free ends with a curve, without crossing any existing curve, and then put a short stroke across the new curve to create two new free ends.

Such a game with $n = 2$ is illustrated in the figure below.



(a) Suppose that the game takes place on the plane $\mathbb{R}^2$. Show that if n is odd, then the first player will always win and show that if n is even, then the second player invariably wins.
(b) Show that the conclusion of (a) also holds if the game takes place on a closed orientable 2-dimensional smooth manifold.

Remark. The game of "Brussels sprouts" is a variation on the more interesting game "sprouts" introduced by John Conway and Michal Paterson in the 1967. More information on "Sprouts" and "Brussels sprouts" can be found in [**FiG1991**, Chapter 6] and [**Gardn1989**, Chapter 1].

Exercise 156.17. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected it follows from the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 that there exist precisely k different lifts of σ to $\widetilde{X}$, i.e. there exist precisely k different singular simplices $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \dots, k$. We set

$$p^*(\sigma) \;:=\; \sum_{i=1}^{k} \widetilde{\sigma}_i.$$

We refer to the map

$$\begin{aligned} p^*\colon \mathrm{C}_*(X) &\to \mathrm{C}_*(\widetilde{X}) \\ \sum_{j=1}^{s} a_j \cdot \sigma_j &\mapsto \sum_{j=1}^{s} a_j \cdot p^*(\sigma_j) \end{aligned}$$

as the *transfer map* of the covering.

(a) Show that the transfer map is in fact a chain map.
(b) By (a) we get an induced homomorphism $p^*\colon \mathrm{H}_*(X) \to \mathrm{H}_*(\widetilde{X})$. What can you say about the maps $p_* \circ p^*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X)$?
(c) Show that for each $n \in \mathbb{N}_0$ we have $\operatorname{rank}(\mathrm{H}_n(\widetilde{X})) \geq \operatorname{rank}(\mathrm{H}_n(X))$.

Exercise 156.18. We say that two smooth manifolds M and N are *commensurable* if M and N admit finite covers $\widetilde{M}$ and $\widetilde{N}$ that are diffeomorphic.

(a) Show that the torus and the surface of genus two are not commensurable.
(b) Let $g, h \in \mathbb{N}_{n\geq 2}$. Let M be the surface of genus g and let N be the surface of genus h. Show that M and N are commensurable.
(c) Determine which compact connected 2-dimensional smooth manifolds are commensurable (Note that we do not make any orientability assumptions.)

Exercise 156.19.

(a) For which dimensions $n \in \mathbb{N}_0$ do there exist compact connected n-dimensional smooth manifolds of arbitrarily large Euler characteristic?
(b) For which dimensions $n \in \mathbb{N}_0$ do there exist compact connected n-dimensional smooth manifolds of arbitrarily low Euler characteristic?

In the following exercises we will see that the Euler characteristic together with the classification of compact smooth manifolds can be used to prove many statements about curves on surfaces:

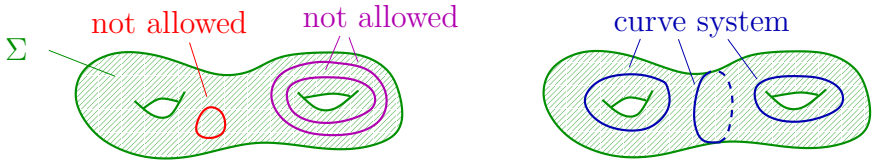
Exercise 156.20. Let $A = S^1 \times [-1,1]$ be the annulus. Let C and D be two closed oriented curves on A. Suppose that $[C] = [D] \neq 0$. Show that the following two statements hold:

(a) The two closed oriented curves are smoothly isotopic,
(b) If C and D are furthermore disjoint, then there exists an embedded annulus $X \subseteq A$ with $\partial A = C \cup -D$.

Hint. Using the fact that A is an annulus you can find a smooth isotopy that makes the closed curves disjoint. At that point it is perhaps better to first prove (b).

Exercise 156.21. Let F be an oriented connected 2-dimensional smooth manifold and let C and D be distinct boundary curves. Suppose that C and D are homotopic. Show that F is diffeomorphic to the annulus $S^1 \times [-1,1]$.

Exercise 156.22. Let Σ be a 2-dimensional smooth manifold. A *curve system* on Σ is a collection of disjointly smoothly embedded closed curves $C_1, \ldots, C_k$ such that no C_i bounds a disk and such that no C_i, C_j bound an annulus. Show that the maximal cardinality of a curve system on the surface of genus g has cardinality $2g-1$.



Exercise 156.23. Let M be a compact orientable connected 2-dimensional smooth manifold.

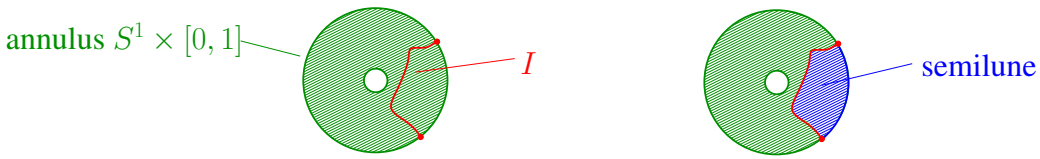
(a) Show that there exists a $g \in \mathbb{N}_0$ such that M admits g disjoint curves such that the complement is connected, but such that any $g+1$ disjoint curves disconnect M.
(b) Show that g equals the genus of M from Definition 101.22.

Remark. This exercise gives an intrinsic definition of the genus of a compact orientable connected 2-dimensional smooth manifold that does not require the language of fundamental groups or homology groups.

Exercise 156.24. Let M be a compact non-orientable connected 2-dimensional smooth manifold. We define $k(M) \in \mathbb{N}_0 \cup \{\infty\}$ to be the maximum $k \in \mathbb{N}_0 \cup \{\infty\}$ such that there exist k smooth embeddings of the Möbius band into M with the following two properties: the images are disjoint and the complement of images is connected.

(a) Let K be the Klein bottle. Show that $k(K) \geq 2$.
(b) Let M be a closed non-orientable connected 2-dimensional smooth manifold.
  (i) Show that $k(M)$ is finite.
  (ii) Show that $k(M)$ equals the genus of M, as introduced in Definition 101.22.
  Remark. This exercise gives an intrinsic definition of the genus of a compact non-orientable connected 2-dimensional smooth manifold that does not require the language of fundamental groups or homology groups.

Exercise 156.25. We consider the annulus $S^1 \times [0,1]$. Show that for any connected proper 1-dimensional smooth manifold $I \subseteq S^1 \times [0,1]$ with $\partial I \subseteq S^1 \times \{0\}$ there exists a "semilune", i.e. an embedding $\varphi \colon \overline{B}^2_{\geq 0} \to S^1 \times [0,1]$ such that $\varphi(\overline{B}^2_{=0}) \subseteq S^1 \times \{0\}$ and such that $\varphi(S^1_{\geq 0}) = I$.



Exercise 156.26. We consider the Klein bottle K and the two smooth submanifolds A and B that are shown in the figure below. Are they smoothly isotopic?



Exercise 156.27. We consider the torus $T = S^1 \times S^1$. Let $p, q \in \mathbb{Z}$ be coprime and let C be a closed oriented connected 1-dimensional smooth manifold that represents the homology class $p \cdot [S^1 \times \{1\}] + q \cdot [S^1 \times \{1\}] \in \mathrm{H}_1(S^1 \times S^1)$. Show that C is smoothly isotopic to the "standard curve"

$$T_{p,q} \;=\; \{(\exp(2\pi\,\mathrm{i}tp), \exp(2\pi\,\mathrm{i}tq) \,|\, t \in \mathbb{R}\}.$$

You can try to go through the following steps:

(a) Show that after applying a diffeomorphism we might as well assume that $p = 1$ and $q = 0$. Thus we need to show that C is smoothly isotopic to $S^1 \times \{*\}$.

(b) By the Transversality Theorem 50.9 we can assume that C is transverse to $S^1 \times \{*\}$. Consider the intersection points $P_1, \ldots, P_m$ with signs $\epsilon_1, \ldots, \epsilon_m$.
- Deal with the case $m = 0$.
- Show that if $m > 0$ then there exist two intersection points of opposite sign.
- Show that for any two points of opposite sign we can find a semilune as in Exercise 156.25 and use this to remove at least two intersection points.

CHAPTER 157

Applications of the Euler characteristic

In this chapter we will show that the fairly simple concept of the Euler characteristic, which we introduced in the previous chapter, can be used to address the following questions about 3-dimensional objects:

(1) Why is a classical football made using precisely 12 pentagons?
(2) What is Euler's famous formula?
(3) Why are there precisely five types of platonic solids?
(4) Can we embed all finite topological graphs into the plane?

The reader is warned that in this chapter at times we will trade notational and mathematical rigor for readability.

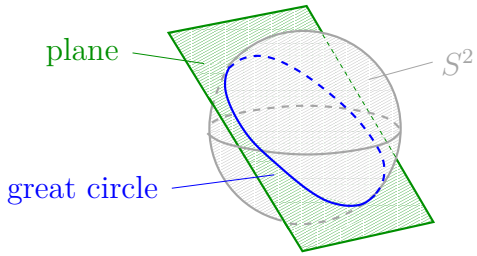
This chapter is organized as follows:

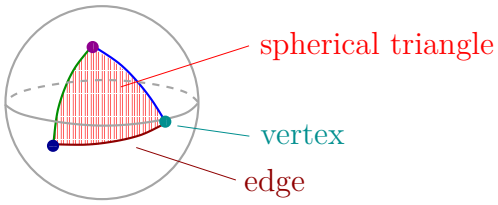
- In Section 157.1 we will use Euler characteristics to discuss the pattern of the classical leather football.
- In Section 157.2 we will prove Euler's Formula 157.8.
- In Section 157.3 we will classify the possible combinatorics of Platonic solids.
- In Section 157.4 we will show that not every finite topological graph is planar.
- In Section 157.5 we will provide exercises for Chapter 157.

157.1. **Building a leather football.** In this section we want to study the classic leather football from a mathematical point of view. This discussion will require the following definitions:

Definition 157.1.

(1) A great circle on the 2-dimensional sphere S^2 is the intersection of S^2 with a 2-dimensional vector subspace of $\mathbb{R}^3$.
(2) The angle between two great circles is defined as the smaller of the two angles given by the corresponding 2-dimensional vector subspaces.
(3) A spherical n-gon is a subset of S^2 such that the boundary is the union of n circle segments (in the sense of Definition 2.51) of n different great circles. These segments are called edges. The points where two edges intersect are called vertices.





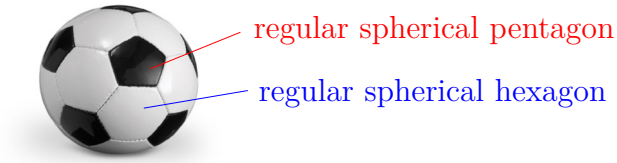
In the following we are mostly interested in highly symmetric spherical n-gons. To define what we mean by "highly symmetric" we need to introduce the following general concept.

Definition 157.2. Let P be a subset of $\mathbb{R}^k$.
(1) A symmetry of P in $\mathbb{R}^k$ is a matrix $A \in \mathrm{O}(k)$ such that $A \cdot P = P$.
(2) The symmetry group $\mathrm{Sym}(P) \subseteq \mathrm{O}(k)$ is defined as the group of all symmetries of P.
(3) We set $\mathrm{Sym}^+(P) := \mathrm{Sym}(P) \cap \mathrm{SO}(k)$.

Using this general definition we can now introduce the main players of this section:

Definition 157.3. A spherical polygon P is called regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of pairs (v, e) where v is a vertex of P and E is an edge of P with $v \in e$.

Example. The classical leather football is made out of 12 regular spherical pentagons and 20 regular spherical hexagons.



Unfortunately modern footballs often no longer exhibit this beautiful structure. □

We want to determine how many regular spherical pentagons and hexagons are required to make a football. So suppose a football is made out of p regular spherical pentagons and h regular spherical hexagons such that any edge of a polygon is the edge of another polygon. This endows S^2 with a CW-structure such that the following holds:

- The 0-cells correspond to the vertices.
- The 1-cells correspond to the edges.
- The 2-cells correspond to the polygons.

We make the following observations:

(1) We first note that each 1-cell bounds two 2-cells.
(2) The interior angles of a regular n-gon are all the same.
(3) One can show, using elementary spherical geometry (see e.g. [**Bona2009**, p. 53] or [**Fen2001**, p. 260]), that the interior angle of a regular spherical n-gon is larger than the angle of the corresponding Euclidean regular n-gon.
(4) It follows from (2) and (3) that in our setting all the interior angles are larger than[1661] $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. Since the total angle at a vertex is 2π we see that at any vertex there are not more than three regular spherical hexagons and pentagons that meet. Of course this implies that at any vertex precisely three regular spherical hexagons and pentagons meet.

It follows that

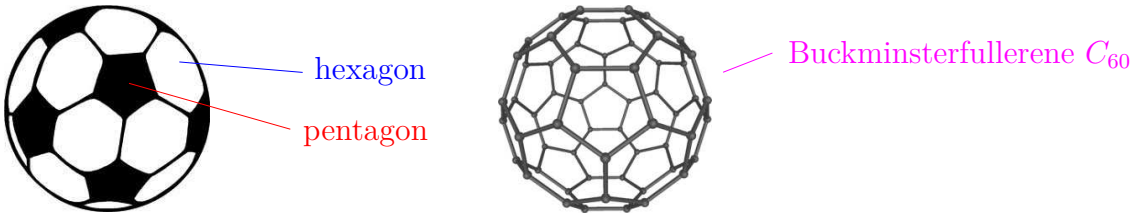
$$\begin{aligned} \#\text{2-cells} &= p + h \\ \#\text{1-cells} &= \tfrac{1}{2}(5p + 6h) \quad \text{here we use Observation (1)} \\ \#\text{0-cells} &= \tfrac{1}{3}(5p + 6h) \quad \text{here we use Observation (4).} \end{aligned}$$

[1661] Where does that formula come from?

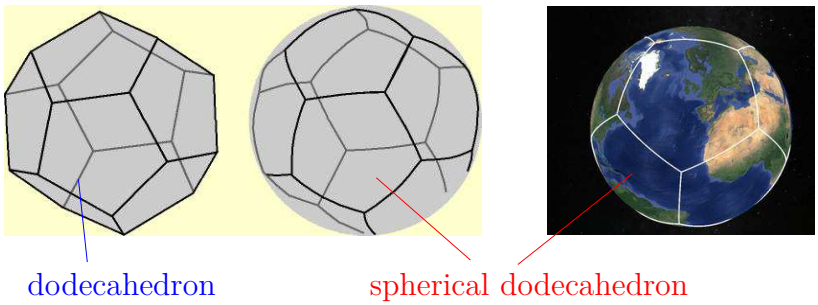
We now see that

$$2 \underset{\substack{\uparrow \\ \text{by the Homology-}\chi \\ \text{Proposition 156.2}}}{=} \chi(S^2) = \sum_{n=0}^{2}(-1)^n \cdot \#n\text{-cells} \underset{\substack{\uparrow \\ \text{by the above} \\ \text{three equalities}}}{=} \tfrac{1}{3}(5p+6h) - \tfrac{1}{2}(5p+6h) + (p+h) = \tfrac{1}{6}p.$$

Thus we see that a football has to have precisely 12 regular spherical pentagons. On the other hand, the number of regular spherical hexagons is not fixed. For example, as we mentioned above, the standard football has 20 hexagons.



But in principle one could also build a football with zero hexagons, see the figure below on the right. This way one obtains a spherical dodecahedron. One could also build footballs with more hexagons, but they would be much less round since they would have bigger flat parts.



Finally note that the 1-skeleton of this CW-structure was in fact already "invented by nature", more precisely, there exists a molecule, called the Buckminsterfullerene C_{60}, built out of 60 carbon atoms, which has exactly the same shape as a football, except that it is a little smaller.

157.2. **Euler's Formula.** We start out by recalling Definition 4.27:

Definition 157.4. Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.

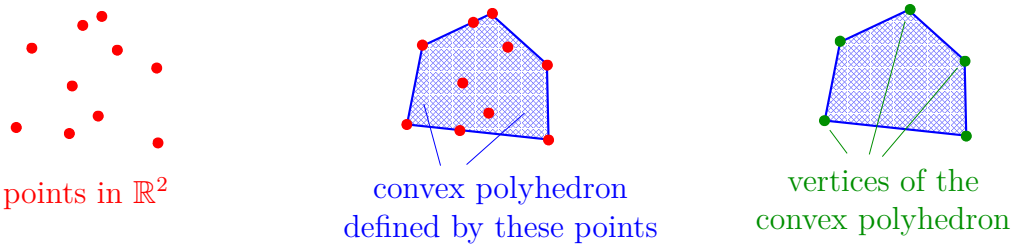
We move on to a new definition.

Definition 157.5.

(1) A convex polyhedron is the convex hull of finitely many points in $\mathbb{R}^k$.[1662 1663 1664]
(2) The dimension of a convex polyhedron is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

This definitions leads us to the following lemma:

Lemma 157.6. (Vertices-of-Polyhedron Lemma) If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set $V \subseteq \mathbb{R}^k$ such that P is the convex hull of V, but such that for every proper subset W of V, P is not the convex hull of W.



Proof. We outsource the proof of the lemma to Exercise 157.2. ■

The Vertices-of-Polyhedron Lemma 157.6 allows us to introduce the following notions:

Definition 157.7. Let P be a convex polyhedron in $\mathbb{R}^k$.

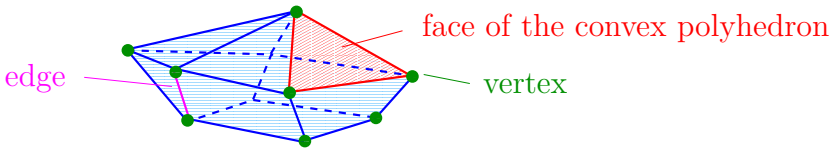
(1) We refer to the finite set of the Vertices-of-Polyhedron Lemma 157.6 as the vertices of the convex polyhedron P.

(2) (a) A closed affine half-space in $\mathbb{R}^k$ is a set of the form $\{v \in \mathbb{R}^k \,|\, \langle v, w\rangle \geq a\}$ for some fixed non-zero vector $w \in \mathbb{R}^k$ and some $a \in \mathbb{R}$.

(b) A subset $F \subseteq P$ is called a face of P if there exists a closed affine half-space H of $\mathbb{R}^k$ with $P \subseteq H$ such that $F = P \cap \partial H$.

Note that a face is a convex polyhedron in its own right, which implies in particular that it makes sense to talk of the dimension of a face.

(3) An edge is a 1-dimensional face. Sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.



Proposition 157.8. (Euler's Formula) Let P be a convex k-dimensional polyhedron in $\mathbb{R}^k$. The following statements hold:

(1) The interior $\mathring{P}$ of P is non-empty and all faces of P are of dimension $\leq k-1$.

(2) The boundary ∂P admits a CW-structure such that for each $j \in \mathbb{N}_0$ the j-cells are precisely the j-dimensional faces of P.

[1662] It is an amusing little exercise to show that this notion of a convex polyhedron agrees with the notion that we originally defined (using intersections of closed affine half-spaces) in Definition 141.8.

[1663] In the literature a convex polyhedron is often also called a polytope.

[1664] In Exercise 4.46 (a) we saw that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\left\{ \sum_{i=1}^{n} t_i \cdot P_i \,\middle|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}.$$

[1664] Here ∂P is the boundary of P viewed as a subset of $\mathbb{R}^k$.

(3) The following equality hold:

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \;=\; 1+(-1)^{k-1}.$$

Remark. The 3-dimensional version of Proposition 157.8 gives us that for the boundary of a convex 3-dimensional polyhedron in $\mathbb{R}^3$ we have

$$\text{number of vertices} - \text{number of edges} + \text{number of faces} \;=\; 2.$$

This equality is known as Euler's Formula and was formulated by Leonhard Euler [**Eul1758**, p. 119] in 1758. This equality was already noted before by Francesco Maurolico [**FriedmM2018**, p. 71] in 1537 and René Descartes [**Rabo2010**, p. 447] in 1628. Arguably Euler's Formula is historically the first result in topology. A famous book by Imre Lakatos [**Lak1976**] is dedicated to the challenge of finding the correct statement of Euler's formula and of providing a rigorous proof of Euler's Formula without employing the technical machinery, namely singular homology, that we are about to use: □

Sketch of proof. We leave the pleasure of proving the first two statements to the reader as Exercise 157.4. Now we calculate that

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \underset{\text{by (1) and (2)}}{=} \chi(\partial P) \underset{\text{by (1) together with the the Convex-to-Ball Proposition 4.28(2)}}{=} \chi(S^{k-1}) \underset{\text{see page 2987}}{=} 1+(-1)^{k-1}.$$

■

157.3. **Platonic solids.** In this section we will use Euler's Formula 157.8 to address the question: Why are there only five types of Platonic solids. Evidently we first need to introduce the notion of a Platonic solid. This requires some preparations:

Definition 157.9.

(1) Let P be a convex polyhedron in $\mathbb{R}^k$.
   (a) A flag of P is a sequence $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_{k-1}$ of faces of P such that each F_i is i-dimensional.
   (b) We say that P is regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of flags of P.

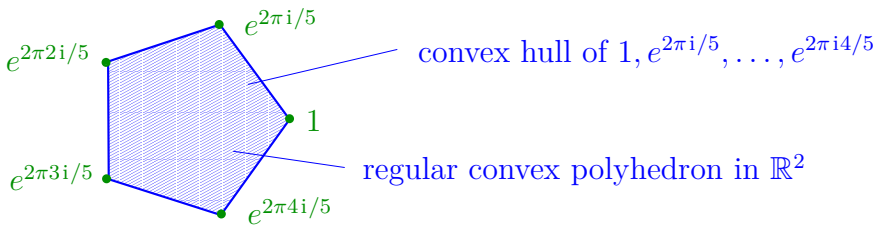
(2) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called equivalent if there exists an $r > 0$, a $v \in \mathbb{R}^k$ and an orthogonal matrix $A \in \mathrm{O}(k)$ with $P = A \cdot (r \cdot Q) + v$.

Examples.

(1) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the n points $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$ in $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$. We leave it to the reader to show that, up to equivalence, these are all regular convex 2-dimensional polyhedra in $\mathbb{R}^2$.

(2) The standard n-simplex Δ^n is a convex n-dimensional polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. It is straightforward to see that Δ^n is regular. □

We now specialize to the "classical" setting of 3-dimensional objects. The next definition introduces one of the oldest concepts in the history of mathematics.



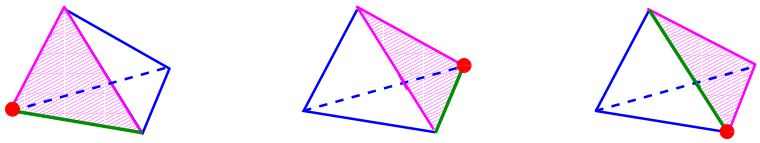
Definition 157.10. A Platonic solid is a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$.

Before we discuss the five classical examples of Platonic solids let us introduce the following definition:

Definition 157.11. The valence of a vertex of a convex polyhedron is defined as the number of edges that contain the vertex.

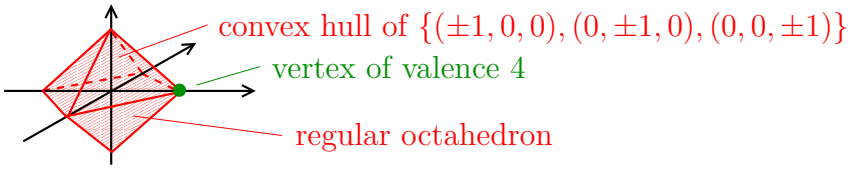
Examples.

(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. In the figure below we show T together with three flags. Some classical geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as *the regular tetrahedron*. Finally note that each vertex has valence 3.

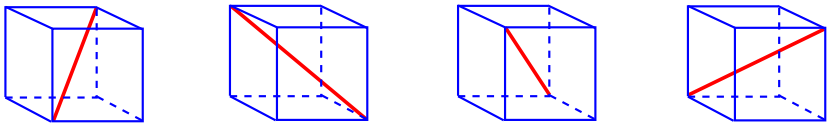


three flags in the regular tetrahedron T

(2) The convex polyhedron in $\mathbb{R}^3$ spanned by $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ is a regular convex 3-dimensional polyhedron where each vertex has valence 4. This convex polyhedron is usually called *the regular octahedron*.



(3) We consider the cube $[-1, 1]^3 \subseteq \mathbb{R}^3$. In [**Arm1988**, p. 38] it is shown that the group $\mathrm{Sym}^+([-1, 1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the diagonals in the cube, i.e. the Euclidean segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements.



In [**Arm1988**, p. 38] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}^+([-1,1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \begin{pmatrix} D & \to & D \\ d & \mapsto & A \cdot d \end{pmatrix} \end{array}$$

is an isomorphism. It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *the regular hexahedron*.

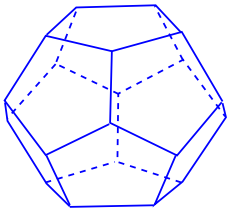
(4) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by D the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

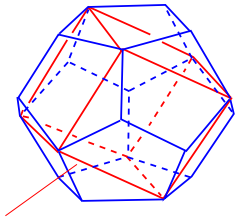
By [**Cox1948**, p. 53] this is a regular polyhedron. We refer to it as *the regular dodecahedron*. Note that it has 12 faces and 20 vertices. We can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. In [**Arm1988**, p. 40] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}^+(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

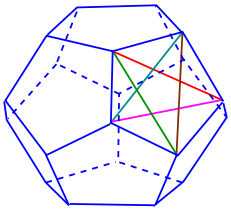
is an isomorphism.[1665]



regular dodecahedron



cube inscribed into the regular dodecahedron



the five diagonals of a face correspond precisely to one inscribed cube

(5) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by I the polyhedron that is spanned by the 12 vertices

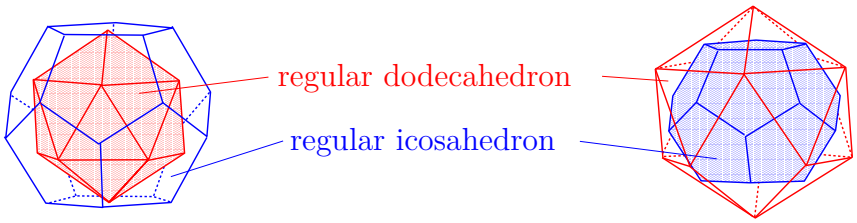
$$(0, \pm 1, \pm z), \quad (\pm z, 0, \pm 1) \quad \text{and} \quad (\pm 1, \pm z, 0).$$

By [**Cox1948**, p. 52] this is a regular polyhedron. We refer to it is as the *regular icosahedron*.

Note that the regular dodecahedron has 20 vertices and 12 faces whereas the regular icosahedron has 12 vertices and 20 faces. In fact the regular icosahedron is dual to the regular dodecahedron in the following sense: up to a scaling factor and a rotation, the regular icosahedron is spanned by the centers of the faces of the regular dodecahedron and vice-versa. We refer to the figure below for an illustration of this fact.

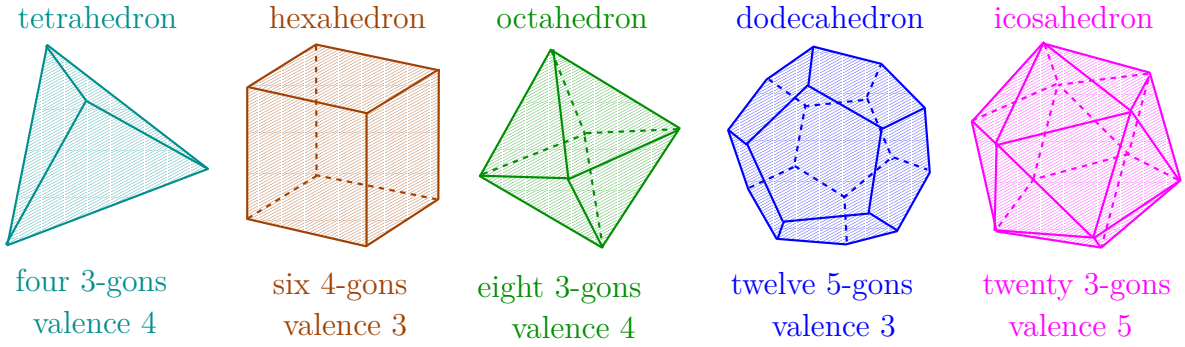
It follows that the symmetry group of the regular icosahedron I is in a suitable sense equal to the symmetry group of the regular dodecahedron D. In particular the group $\mathrm{Sym}^+(I)$ is isomorphic to A_5. □

[1665]The regular dodecahedron, and its symmetry group, will play a prominent role later on. More precisely in Section 183.4 we will use these to construct the Poincaré homology sphere which is a closed 3-dimensional smooth manifold M such that $\pi_1(M)$ is a group of order 120, but with trivial abelianization.



Theorem 157.12. (Platonic Solid Theorem) Up to equivalence the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids discussed above and illustrated in the figure below. They have the following types of faces and valences and they have the following symmetry groups:

	tetrahedron	hexahedron	octahedron	dodecahedron	icosahedron
number of faces	4	6	8	12	20
shape of face	3-gon	4-gon	3-gon	5-gon	3-gon
valence of vertex	4	3	4	3	5
$\mathrm{Sym}^+(P)$	A_4	S_4	S_4	A_5	A_5



Sketch of proof. We will not provide a full proof for the theorem. Instead we refer to [**Cox1948**, Section 6.7] for a full proof that the list of five platonic solids is indeed complete. Furthermore we refer to [**Arm1988**, p. 40] or alternatively to [**Aa2008**, Chapter 5.5] for a proof of the statement regarding the symmetry groups.[1666]

In the following we will show the weaker statement that any platonic solid fits into the table of the theorem. Put differently, we will show that given any platonic solid the number of faces, the shape of faces and the valence of a vertex corresponds to one of the five columns in the above table.

Thus let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

(1) It follows easily from the definition of a regular polyhedron that there exists an $m \in \mathbb{N}$ such that each face of P is a regular m-gon.
(2) The number of faces that meet at a vertex equals the valence of a vertex.
(3) The valences of the vertices are the same and it is at least three.

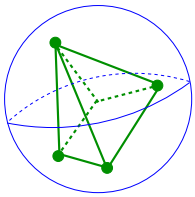
[1666] We sketched the proofs for the regular hexahedron, the regular dodecahedron and the regular icosahedron.

The following claim tells us the possible values for m and the number f of faces that meet at a vertex.

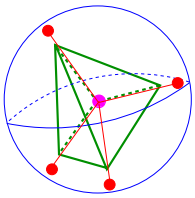
Claim.

possible values for m	3	4	5
f = number of faces that meet at a vertex	3, 4, 5	3	3

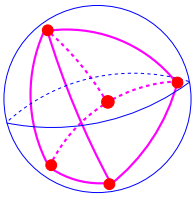
Proof. After a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere S^2.



embed polyhedron in the 3-ball



projection of vertices onto the sphere



projection of edges onto the sphere

Recall that on page 3008 we noted that the interior angle of a regular spherical m-gon is greater than the angle of a regular Euclidean m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $f \geq 3$ and that $f \cdot (\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. ⊞

Now we treat the cases $m = 3, 4, 5$ separately. We start out with the case $m = 3$, i.e. we assume that all the faces of the convex polyhedron are equilateral triangles. Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#\text{faces} =: k, \qquad \#\text{edges} \underset{\uparrow}{=} \tfrac{1}{2} \cdot 3k \qquad \text{and} \qquad \#\text{vertices} \underset{\uparrow}{=} \tfrac{1}{3} \cdot 3k.$$

since $m = 3$ and since each edge cobounds two faces

since $m = 3$ and since each vertex cobounds three faces

Now we see that

$$2 \underset{\uparrow}{=} \#\text{faces} - \#\text{edges} + \#\text{vertices} \underset{\uparrow}{=} k - \tfrac{3}{2}k + \tfrac{3}{3}k = \tfrac{1}{2}k.$$

Euler's Formula 157.8

by the above

It follows that $k = 4$. Note that this case is realized by the regular tetrahedron. Almost the same calculation shows that if four triangles meet at a vertex, then $2 = k - \frac{3}{2}k + \frac{3}{4}k$, i.e. $k = 8$. Note that this case is realized by the regular octahedron. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $2 = k - \frac{3}{2}k + \frac{3}{5}k$, i.e. $k = 20$. Note that this case is realized by the regular icosahedron.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Note that this case is realized by the regular hexahedron (aka cube).

Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. Note that this case is realized by the regular dodecahedron. ■

Remark. On page 3011 we saw that there exist infinitely many equivalence classes of regular convex polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in the Platonic Solid Theorem 157.12 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular convex polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that in $\mathbb{R}^4$ there are six equivalence classes of regular convex polyhedra, whereas in all higher dimensions there are only three equivalence classes of convex polyhedra. A nice exposition of these facts is given on the following website:

`http://math.ucr.edu/home/baez/platonic.html` □

157.4. **Planar graphs.** We move on to a completely different subject, namely the study of graphs and their embeddings into the plane. We start out by recalling Definition 11.12:

Definition 157.13.

(1) An abstract graph G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

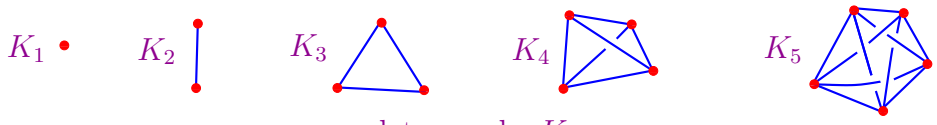
$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called vertices of G and the elements of E are called the edges of G.

(2) In Definition 11.17 we introduced the topological realization[1667] [1668] $|G|$ of a graph G.[1669] It follows from the Topological Realization–Homeomorphism Lemma 11.19 and the Graph–CW-Complex Lemma 136.5 that $|G|$ is naturally a 1-dimensional CW-complex where the 0-cells correspond to the vertices and the 1-cells correspond to the edges.

Examples.

(1) For any $n \in \mathbb{N}$ we define the complete graph K_n as the abstract graph with vertex set $V = \{v_1, \dots, v_n\}$, edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$ and where we define $\varphi\colon E \to \mathcal{P}(V)$ via $\varphi(e) = e$. Put differently, the complete graph K_n has n vertices and each vertex is connected to any other vertex by precisely one edge.



complete graphs K_n

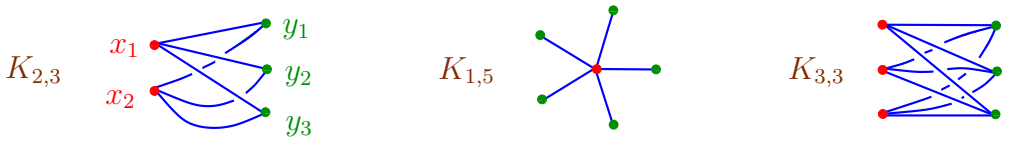
(2) For every choice of $m, n \in \mathbb{N}$ we define the complete bipartite graph $K_{m,n}$ as the abstract graph which is defined by the vertex set $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, the edge set

[1667] The definition of the topological realization $|G|$ is a little technical. For the purpose of this slightly informal section it suffices to know that $|G|$ is built out of the vertex set V by attaching an interval for each edge $e \in E$ to the vertices given by $\varphi(e) \subseteq V$.

[1668] In this section we will refer to any topological space that is homeomorphic to $|G|$ as a topological realization of G.

[1669] On Definition 11.1 we defined the notion of an abstract graph. By the Topological Realization–Homeomorphism Lemma 11.19 and the discussion on page 388 we know that the concepts of abstract graphs and abstract graphs lead to homeomorphic topological realizations. Therefore in this section we will mostly deal with abstract graphs. In fact, since this name is rather unwieldy, we will refer to them just as graphs.

$E = \{\{x_i, y_j\} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$ and the map $\varphi\colon E \to \mathcal{P}(V)$ that is given by $\varphi(e) = e$.



complete bipartite graphs $K_{m,n}$

Put differently, the complete bipartite graph $K_{m,n}$ consists of m "red-colored" vertices, n "green-colored" vertices and each "red-colored" vertex is connected to any "green-colored" vertex by precisely one edge. □

Let G be a finite graph. On page 386 we mentioned already that there exists an embedding of its topological realization $|G|$ into $\mathbb{R}^3$. In Question 11.11 we asked, without really expecting a positive answer, whether one can always find an embedding of $|G|$ into $\mathbb{R}^2$. This leads us to the following definition:

Definition 157.14. We say that a finite graph G is planar if its topological realization $|G|$ admits an embedding $|G| \to \mathbb{R}^2$.

Examples.

(1) The above figures show that the graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. On the other hand the above topological realizations of K_4 and $K_{2,3}$ are not subspaces of $\mathbb{R}^2$. But as we see in the figure below both graphs admit topological realizations that are subspaces of $\mathbb{R}^2$.



topological realizations of K_4 topological realizations of $K_{2,3}$

(2) One can easily verify that given an $m \in \mathbb{N}$ the bipartite graph $K_{2,m}$ is planar. □

Now the following question arises:

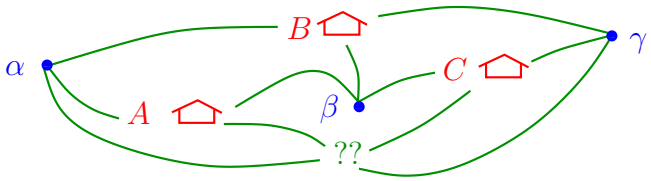
Question 157.15. Are the graphs K_5 and $K_{3,3}$ also planar?

If this question sounds to abstract and bipartite graphs sound too otherworldly let us reformulate the question regarding $K_{3,3}$:

Problem 157.16. (Utility Problem) Suppose that three new houses A, B and C are built and they need to be connected to three utilities, e.g. a water source α, an electricity provider β and an internet provider γ. Can we perform the connections so that no two lines/pipes intersect, except at the endpoints?[1670]

To answer that question we need a few more definitions.

[1670] It follows easily from the Homogeneous-via-Diffeotopies Proposition 44.6 that the answer to this question does not depend on the precise location of A, B, C and α, β, γ, as long as they are distinct.



Definition 157.17. Let G be a graph.

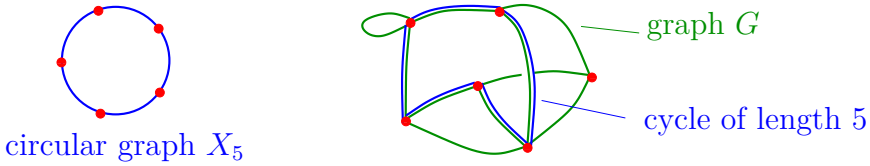
(1) Given $n \in \mathbb{N}$ we denote by X_n the "circular graph with n vertices and n edges", more precisely, X_n is the graph with

$$V(X_n) = \{1, \ldots, n\} \quad \text{and} \quad E(X_n) = \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\}$$

and where $\varphi\colon E(X_n) \to \mathcal{P}(V(X_n))$ is the obvious map.

(2) A cycle of length n in G is an injective morphism $\varphi\colon X_n \to G$ of graphs.

(3) The girth of G is the shortest length of a cycle in G. If G has no cycles, then we define its girth as ∞.



Examples.

(1) The girth of the graph shown in the figure above to the right is one, since it has an edge where the two endpoints agree.

(2) One can easily show that for any $n \geq 3$ the girth of the complete graphs K_n equals three.



(3) Similarly one can easily show that for any $n, m \geq 2$ the girth of the complete bipartite graphs $K_{n,m}$ equals four. □

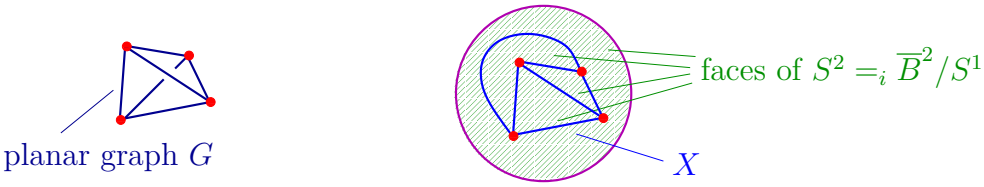
The following proposition gives an obstruction to a graph being planar.

Proposition 157.18. (Planar Graph–Girth Proposition) Let G be a graph with v vertices and e edges. We denote by g the girth of G. If G is planar and if $g < \infty$, then

$$e \leq \frac{g}{g-2} \cdot (v-2).$$

Sketch of proof. Let G be a graph with v vertices and e edges. We suppose that the girth g of G is finite. Furthermore we suppose that G is planar, i.e. we suppose that there exists an embedding $\varphi\colon |G| \to \mathbb{R}^2$. We write $X = \varphi(|G|)$. Without loss of generality we can assume that X lies in the open disk $B^2 \subseteq \mathbb{R}^2$.

Now we view X as a subspace of the sphere $S^2 =_{\text{i}} \overline{B}^2/S^1$. We refer to the closures of the components of $S^2 \setminus X$ as *faces* and we denote by f the number of faces.



We make the following observation:

(1) The vertices, edges and faces define a CW-structure for $S^2 =_{\mathrm{i}} \overline{B}^2/S^1$.[1671]

(2) We make the following two observations:

(a) The boundary of each face corresponds to a cycle in the graph. By the definition of the girth each face is bounded by at least g edges.

(b) Each edge lies in the boundary of at most two faces.[1672]

Now we deduce that

$$\begin{aligned} g \cdot f &\overset{\text{follows from (a)}}{\leq} \sum_{\text{faces } f} \text{number of edges in the boundary of } f \\ &= \text{number of pairs } (f,e) \text{ where } f \text{ is a face and } e \text{ an edge in the boundary of } f \\ &= \sum_{\text{edges } e} \text{number of faces that contain } e \text{ in the boundary} \overset{\text{follows from (b)}}{\leq} 2e. \end{aligned}$$

Summarizing we see that $g \cdot f \leq 2e$.

Now we see that

$$2 \overset{\text{see page 2987}}{=} \chi(S^2) \overset{\text{by (1)}}{=} v - e + f \overset{\text{since by (2) we see that } g \cdot f \leq 2e}{\leq} v - e + \frac{2}{g} \cdot e = v - \frac{g-2}{g} \cdot e.$$

Solving for e we get the desired inequality $e \leq \frac{g}{g-2} \cdot (v-2)$. ■

The following proposition gives in particular a negative answer to Question 157.15.

Proposition 157.19. (Non-Planar Graph Proposition) If a graph G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph, then G itself cannot be planar.

Proof.

(1) Let $m \in \mathbb{N}_{\geq 3}$. It is straightforward to see that the complete graph K_m has m vertices and $\frac{1}{2}m(m-1)$ edges. Furthermore, as we pointed out on page 3018, the girth is three. It is straightforward to verify that the inequality of the Planar Graph–Girth Proposition 157.18 is only satisfied for $m = 3$ and $m = 4$.

(2) Let $m, n \in \mathbb{N}_{\geq 2}$. It is easy to verify that the complete bipartite graph $K_{m,n}$ has $m+n$ vertices and $m \cdot n$ edges. Furthermore, as we pointed out above, the girth is four.

[1671] Here is the reason why it is the sketch of a proof: we do not prove that each face is in fact homeomorphic to a closed disk. This statement is "clear from the picture" and it can be proved using the Schoenflies Theorem 170.6. But we will not attempt to give a rigorous proof.

[1672] This statement also requires a little bit of thought.

It is straightforward to verify that the inequality of the Planar Graph–Girth Proposition 157.18 is only satisfied if $m = 2$ or $n = 2$.

(3) The proposition now follows from (1) and (2) and the observation that if a graph is planar, then evidently any subgraph is also planar. ■

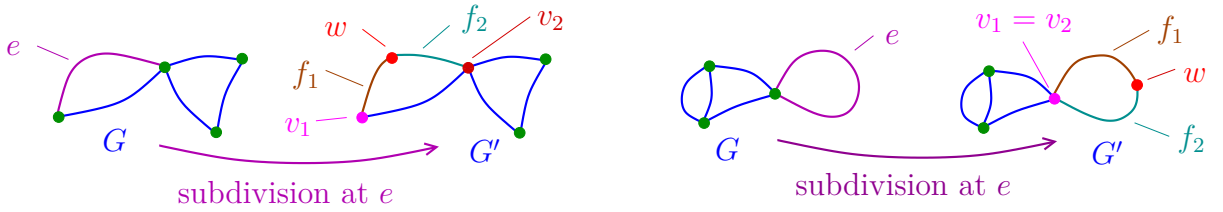
Remark. Let G be a graph that contains the complete graph K_7. By the Non-Planar Graph Proposition 157.19 the graph G cannot be planar. John Conway and Cameron Gordon proved in 1984 the following interesting variation on this fact: given any embedding $\varphi\colon |G| \to \mathbb{R}^3$ such that the image is a polygon (i.e. it is the union of finitely many Euclidean segments) the image $\varphi(|G|)$ contains a non-trivial polygonal knot. More precisely, there exists a closed polygon C in $\varphi(|G|)$ such that $\pi_1(\mathbb{R}^3 \setminus C)$ is not isomorphic to $\mathbb{Z}$. We refer to [**CoG1983**] for the precise formulation of the theorem and the proof. □

In the remainder of this section we will consider the question, to what degree the converse to the Non-Planar Graph Proposition 157.19 holds. To answer this question we need to introduce the notion of a subdivision of a graph. From the figure below it should become clear what is meant be a subdivision. As so often, the formal definition is a little cumbersome:

Definition 157.20. Let $G = (V, E, \varphi\colon E \to \mathcal{P}(V))$ be a graph.

(1) Let $e \in E$ be an edge. We say that a graph $G' = (V', E', \varphi'\colon E' \to \mathcal{P}(V'))$ is a subdivision of G at e if $V' = V \sqcup \{w\}$ and $E' = (E \setminus \{e\}) \sqcup \{f_1, f_2\}$ with $w \notin V$ and $f_1, f_2 \notin E$ such that $\varphi'(e') = e$ for $e' \in E \setminus \{e\}$, $\varphi'(f_1) = \{v_1, *\}$ and $\varphi'(f_2) = \{v_2, *\}$ where $\varphi(e) = \{v_1, v_2\}$.

(2) We say that a graph G' is a subdivision of G if G' is obtained from G by a finite sequence of subdivisions along edges.



The following lemma says in particular that "planarity" is unaffected by subdivisions.

Lemma 157.21. (Graph–Subdivision Lemma) Let G be a graph. If G' is a subdivision of G, then the topological realizations of G' and G are homeomorphic.

Proof. We leave it to the reader to provide the elementary proof of this lemma. ■

The following theorem, which was first proved by Kazimierz Kuratowski [**Kura1930**] in 1930, shows that the graphs K_5 and $K_{3,3}$ turn out to be the only obstruction to a graph being planar.

Theorem 157.22. (Kuratowski's Reduction Theorem) For any finite graph G the following statements are equivalent:

(1) The graph G is planar.

(2) G contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ as a subgraph.

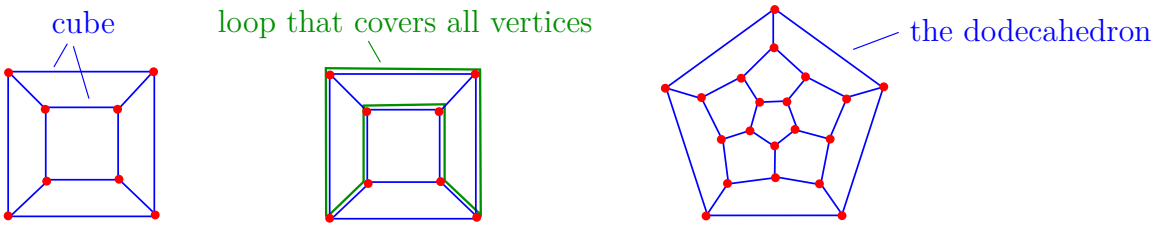
Proof. A textbook proof of the theorem can be found in [**BoM2008**, Chapter 10.5] or alternatively in [**BaR2021**, Chapter 8.7]. ■

157.5. Exercises for Chapter 157 .

Exercise 157.1. A *closed affine half-space* in $\mathbb{R}^k$ is a subset of the form $\{x \in \mathbb{R}^k \mid \langle x, v\rangle \geq r\}$ for some $v \in \mathbb{R}^k \setminus \{0\}$ and some $r \in \mathbb{R}$. Let P be a subset of $\mathbb{R}^k$. Show that P is a convex polyhedron (in the sense of Definition 157.5) if and only if P is the intersection of finitely many half-spaces.

Exercise 157.2. Let P be a convex polyhedron in $\mathbb{R}^k$. Show that there exists a unique finite set V such that P is the convex hull of V, but such that for every proper subset W of V, P is not the convex hull of W.
Remark. In other words, the task is to prove the Vertices-of-Polyhedron Lemma 157.6.

Exercise 157.3. Find an embedded loop on the edges of the dodecahedron that covers all vertices.
Remark. This is known as Hamilton's "icosian game". This puzzle was actually marketed in the 19th century as a board game.



Exercise 157.4.

(a) Let P be a convex k-dimensional polyhedron in $\mathbb{R}^k$. Show that the interior $\mathring{P}$ of P is non-empty.

(b) Let Q be a convex k-dimensional polyhedron in some $\mathbb{R}^n$. Show that Q is homeomorphic to $\overline{B}^k$.
Hint. Use (a) and the Convex-to-Ball Proposition 4.28 (2).

(c) Let P be a convex k-dimensional polyhedron in some $\mathbb{R}^n$. Show that P admits a CW-structure such that the boundary ∂P is a subcomplex and such that for each $j \in \mathbb{N}_0$ the j-cells in ∂P are precisely the j-dimensional faces of P.

Exercise 157.5.

(a) Show that there exist embeddings $K_5 \to S^1 \times S^1$ and $K_{3,3} \to S^1 \times S^1$.

(b) Let $n \in \mathbb{N}_{\geq 6}$. What is the smallest g that you can find such that there exists an embedding from K_n to the surface of genus g?

Exercise 157.6. In his last will, a farmer, who owns a rectangular plot, wants to distribute his land among his sons as follows:

(1) every son gets a compact connected field with non-empty area,
(2) every son should be the neighbor of every other son.

What is maximal number of sons for which the farmer's last will can be fulfilled?
Remark. On the margin of the last will the farmer insists that the new distribution should give a CW-structure for the rectangular plot.



Exercise 157.7. Does there exist an embedding of the topological realization of the graph $K_{3,3}$ into $\mathbb{R}\mathrm{P}^2$?

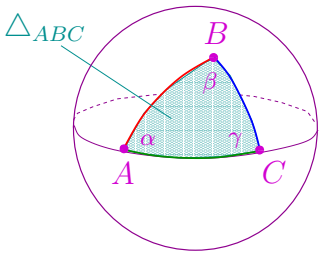
Exercise 157.8.

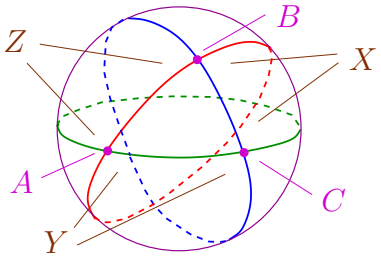
(a) Show that given any finite topological graph G there exists a closed orientable connected surface Σ that admits an embedding $G \to \Sigma$.
(b) Suppose that G is a graph with v vertices and e edges on a closed orientable connected surface. Show that $e \leq 6v - 2\chi(\Sigma)$.
(c) What is the minimal genus of a surface Σ such that there exists an embedding of $K_{3,3}$ into Σ?

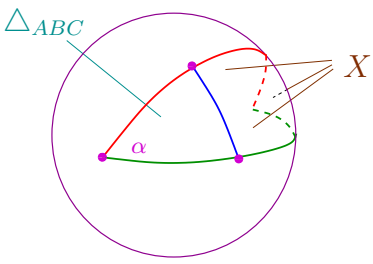
Remark. It is best not get burdened down by an exaggerated level of rigor in the arguments.

Exercise 157.9. In this exercise we want to sketch a down-to-earth proof of the classical Euler Formula using elementary spherical geometry.

(a) A *spherical bigon* is a subset of S^2 that is bounded by two great circles. Let A be a spherical bigon of angle α. Show that $\mathrm{area}(A) = 2\pi \cdot \alpha$.
Remark. Recall that on page 1573 we showed that $\mathrm{area}(S^2) = 4\pi$.
(b) Let Δ_{ABC} be a spherical triangle with interior angles α, β, γ at the vertices A, B, C. Show that $\mathrm{area}(\Delta_{ABC}) = \alpha + \beta + \gamma - \pi$.
Hint. The great circles formed by the edges of the triangle divide S^2 into eight spherical triangles:
- the original spherical triangle Δ_{ABC},
- spherical triangles X, Y, Z which together with Δ_{ABC} form spherical bigons with angles α, β, γ.
- the reflections of the spherical triangles Δ_{ABC}, X, Y, Z in the origin.







$\triangle_{ABC} \cup X$ is a spherical bigon with angle α

(c) For a spherical n-gon determine a relationship between the area and the sum of the interior angles.

(d) Suppose that S^2 is divided into spherical n-gons such that an edge of one n-gon is an edge of another n-gon. Show that with this decomposition of S^2 we have

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} \;=\; 2.$$

Hint. Cleverly make use (b) and (c) and use that at a vertex the sum of the angles equals 2π.

(e) Prove Euler's Formula 157.8 in dimension three. More precisely, let P be a convex 3-dimensional polyhedron in $\mathbb{R}^3$. Show that for the boundary ∂P we have

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} \;=\; 2.$$

Part XVII

Singular Homology – Algebra and coefficients

In this part of the lecture notes we will develop our algebraic skills and bring them to bear on topology. In particular we will learn about homology with coefficients and we will determine the homology groups of a product of topological spaces.
This part of the lecture notes is organizes as follows:

- In Chapter 158 we will prove the Homological Algebra–Fundamental Theorem 158.4 which says that every abelian group has a free resolution which is unique up to chain homotopy. We will also prove that a chain map between chain complexes of free abelian groups, that induces isomorphisms on all homology groups, is in fact a chain homotopy equivalence.
- In the purely algebraic Chapter 159 we will introduce and study the tensor product of abelian groups.
- In Chapter 160 we will introduce the homology groups $\mathrm{H}_k(X, A; G)$ of a pair (X, A) of topological spaces with coefficients in an abelian group G. We will explore the main properties and we will see that most results on ordinary homology generalize in a straightforward way to homology with G-coefficients.
- In the short Chapter 161 we will introduce exact functors, we will study their effect on homology and we will give a few basic examples.
- In Chapter 162 we will prove the Universal Coefficient Theorems 162.15 and 162.16 which let us calculate the homology groups $\mathrm{H}_k(X, A; G)$ in terms of the ordinary homology groups $\mathrm{H}_k(X, A)$ and $\mathrm{H}_{k-1}(X, A)$.
- In Chapter 163 we will introduce the tensor product $\mathcal{C}_* \otimes \mathcal{D}_*$ of two chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$. In the Algebraic–Künneth Theorem 163.6 we will compute the homology groups of a tensor product chain complex $\mathcal{C}_* \otimes \mathcal{D}_*$ in terms of homology groups of the chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$.
- In Chapter 164 we will state and prove the Eilenberg-Zilber Theorem 164.2 which says that for two topological spaces X and Y the singular chain complex $\mathrm{C}_*(X \times Y)$ is chain homotopic to the tensor chain complex $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$. We will use the Algebraic–Künneth Theorem 163.6 to prove the Künneth Theorem 164.15 which lets us compute the homology groups of a product $X \times Y$ of topological spaces in terms of the homology groups of X and Y.
- In Chapter 165 we will consider several applications of homology groups. In particular we will give a very very concise introduction to persistent homology and we will use homology with $\mathbb{F}_2$-coefficients to give one more proof of the Borsuk-Ulam Theorem 47.12.

CHAPTER 158

The Fundamental Theorem of Homological Algebra

We start out by recalling two results from earlier on:

(1) In the C_*–Functor Lemma 143.16 and the Relative-H_*–Functor Lemma 146.3 we showed that the maps

$$\begin{array}{rcl} \text{pair } (X,A) \text{ of topological spaces} & \mapsto & (\mathrm{C}_*(X,A),\partial_*) \\ (\text{continuous map } f\colon (X,A)\to(Y,B)) & \mapsto & (f_*\colon (\mathrm{C}_*(X,A),\partial_*)\to(\mathrm{C}_*(Y,B),\partial_*)) \end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes.

(2) In the Chain Complex-H_*-Functor Lemma 143.4 we showed that for every $n\in\mathbb{N}_0$ the maps

$$\begin{array}{rcl} (C_*,\partial_*) & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon (C_*,\partial_*)\to(D_*,\partial_*)) & \mapsto & (f_*\colon \mathrm{H}_n(C_*)\to\mathrm{H}_n(D_*)) \end{array}$$

define a covariant functor from the category *ChCplx* of chain complexes to the category *AbGroup* of abelian groups.

In the previous chapters we mostly studied the composition of the two functors, i.e. for $n\in\mathbb{N}_0$ we studied the covariant functor

$$(X,A) \;\mapsto\; \mathrm{H}_n(X,A)$$

from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups. In this and the following chapters we want to consider the two functors more in their own right and we will see that it can be interesting to "intersperse" a functor from the category *ChCplx* of chain complexes to itself, between these two functors.

A typical question we want to answer in this chapter is the following: Recall that in the Excision Theorem 148.1 we showed that if X is a topological space and $Z\subseteq A\subseteq X$ are subsets such that the closure of Z is contained in the interior of A, then the inclusion $(X\setminus Z, A\setminus Z)\to(X,A)$ induces for each $n\in\mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X\setminus Z, A\setminus Z) \;\to\; \mathrm{H}_n(X,A).$$

It is reasonable to ask whether the inclusion-induced map

$$(\mathrm{C}_*(X\setminus Z, A\setminus Z),\partial_*) \;\to\; (\mathrm{C}_*(X,A),\partial_*).$$

is in fact a chain homotopy equivalence.

In this chapter we will see that, for purely algebraic reasons, the above question can always be answered in the affirmative. More precisely, we will show the following completely algebraic statement: If a chain map $C_*\to D_*$ induces isomorphisms on all homology groups and if all chain groups are free abelian groups, then the chain map is in fact a chain homotopy equivalence. This algebraic result will be very useful when we study homology groups with coefficients and cohomology groups.

This chapter is organized as follows:

- In Section 158.1 we will state and prove the Homological Algebra–Fundamental Theorem 158.4.
- In Section 158.2 we will state the Quasi Iso–Homotopy Equivalence Proposition 158.8 which says that quasi-isomorphisms are chain homotopy equivalences.
- In Section 158.3 we will prove the Quasi Iso–Homotopy Equivalence Proposition 158.8
- In Section 158.4 we will state the CW-Intermediate-Singular Proposition 158.15 which implies that the singular and cellular chain complexes are chain homotopy equivalent.
- In Section 158.5 we will prove the CW-Intermediate-Singular Proposition 158.15
- In Section 158.6 we will introduce excisive triads.
- In Appendix 158.7 we will see that if for two free chain complexes the homology groups are abstractly isomorphic, then the chain complexes are actually chain homotopy equivalent.
- In Section 158.8 we will provide exercises for Chapter 158.

In the purely algebraic Chapter 159 we will introduce the tensor product of abelian groups.

158.1. **The Fundamental Theorem of Homological Algebra.** We start out by recalling the following definitions from Definition 97.5 and Definition 97.7.

Definition 158.1.

(1) We say that a group F is a free abelian group on a subset $S \subseteq F$, if the following universal property holds: For any *map* $f\colon S \to G$ of the set S to an abelian group G there exists a unique *homomorphism* $\psi\colon F \to G$ that makes the following diagram commute:

$$\begin{array}{ccc} S & \hookrightarrow & F \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\,\psi} \\ & & G. \end{array}$$

We call S a basis of the free abelian group F.

(2) Let S be a set. By the Free Abelian Group–Existence Lemma 97.8 we know that

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S$$

is a free abelian group generated by S.[1673]

We move on to a new definition, which will turn out to be surprisingly important for our future discussions.

Definition 158.2. A (free) resolution of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where the F_i are (free) abelian groups.

Example. The following is a free resolution for the abelian group $H = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$:

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 5 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\text{obvious projection}} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \to 0. \qquad \square$$

[1673] Here we identify $s \in S$ with the homomorphism $S \to \mathbb{Z}$ which is given by $s \mapsto 1$ and $t \mapsto 0$ for $t \neq s$. This way we can view S as a subset of $\mathbb{Z}^{(S)}$.

Definition 158.3. Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a resolution of H and let F'_* be a resolution of H'. An extension of α to the resolutions F_* and F'_* is a sequence of homomorphisms $\alpha_i\colon F_i \to F'_i$, $i \in \mathbb{N}_0$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0. \end{array}$$

Note that the horizontal exact sequences on top and bottom are in particular chain complexes and that the maps α_* define a chain map from one chain complex to the other.

The following fundamental theorem underlies many arguments in homological algebra.

Theorem 158.4. (Homological Algebra–Fundamental Theorem) Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F'_* be a resolution[1674] of H'.

(1) There exists an extension of α to the resolutions F_* and F'_*.
(2) Any two extensions α_* and α'_*, viewed as chain maps, are chain homotopic.

Proof. We start out with the following claim:

Claim 0. For every free abelian group F, every abelian group M, every homomorphism $\psi\colon F \to M$ and every epimorphism $\varphi\colon N \to M$ there exists a homomorphism $\sigma\colon F \to N$ such that the following diagram commutes:[1675]

$$\begin{array}{ccc} F & & \\ {\scriptstyle \exists\sigma}\downarrow & \searrow^{\psi} & \\ N & \xrightarrow[\varphi]{} & M. \end{array}$$

This diagram can also be drawn as follows:

$$\begin{array}{ccc} & & F \\ & {\scriptstyle \exists\sigma}\nearrow & \downarrow{\scriptstyle \psi} \\ N & \xrightarrow[\varphi]{} & M. \end{array}$$

Proof. We pick a basis $\mathcal{B}$ for F. Since φ is an epimorphism we see that for each $b \in \mathcal{B}$ we can pick an $n_b \in N$ with $\varphi(n_b) = \psi(b)$. Now let $\sigma\colon F \to N$ be the unique homomorphism with $\sigma(b) = n_b$ for all $b \in B$. Note that for all $b \in \mathcal{B}$ we have $(\varphi \circ \sigma)(b) = \varphi(n_b) = \psi(b)$. It thus follows from the uniqueness statement for free abelian groups that $\varphi \circ \sigma = \psi$. ⊞

Now let F_* be a free resolution of H and let F'_* be a resolution of H'.

(1) We have to show that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0. \end{array}$$

[1674]Note that we do not demand that F'_* is a *free* resolution. Also if one goes through the proof then one realizes that we do not need that F_* is a resolution, i.e. we do not need that $F_* \to H \to 0$ is exact, it suffices to know that $F_* \to H \to 0$ is a chain complex.
[1675]In the language of Definition 309.25 this claim says that free groups are "projective".

First we define the homomorphism $\alpha_0\colon F_0 \to F'_0$. Since $f'_0\colon F'_0 \to H'$ is an epimorphism and since F_0 is free abelian we obtain directly from Claim 0 that there exists a homomorphism $\alpha_0\colon F_0 \to F'_0$ that makes the right square commute.

Now we turn to the definition of the homomorphism $\alpha_1\colon F_1 \to F'_1$. Since F_1 is free abelian it suffices by Claim 0 to prove the following claim:

Claim 1. We have $\operatorname{im}(\alpha_0 \circ f_1) \subseteq \operatorname{im}(f'_1)$.

Proof. Since the diagram commutes and since the top sequence is a chain complex we see that $\operatorname{im}(\alpha_0 \circ f_1) \subseteq \ker(f'_0)$. Since the bottom sequence is exact we see that $\ker(f'_0) = \operatorname{im}(f'_1)$. ⊞

It is now clear how inductively we can construct the homomorphisms $\alpha_2, \alpha_3, \ldots$.

(2) Now let $\alpha_i, \alpha'_i\colon F_i \to F'_i, i \in \mathbb{N}_0$ be two extensions of α to the free resolutions F_* and the resolution F'_*. We have to find a chain homotopy from the chain map α_i to the chain map α'_i. Note that the maps $\beta_i := \alpha_i - \alpha'_i$ are an extension of the zero map $\beta = \alpha - \alpha = 0\colon H \to H'$. Therefore it suffices to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero chain map. As we will see, the construction of the chain homotopy is very similar to the construction of the homomorphisms α_i in the first part of the proof.

Recall that we have to find a sequence of homomorphisms λ_i, $i \in \mathbb{N}_0$ such that for every $i \in \mathbb{N}_0$ we have

$$f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i \;=\; \beta_i.$$

In the following discussion it is convenient to keep an eye on the following diagram that contains all the homomorphisms that we are given and that we need to construct:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & 0 & & \\
 & \swarrow^{\lambda_2} & \downarrow{\scriptstyle \beta_2} & \swarrow^{\lambda_1} & \downarrow{\scriptstyle \beta_1} & \swarrow^{\lambda_0} & \downarrow{\scriptstyle \beta_0} & \swarrow^{\lambda_{-1}} & & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow[f'_2]{} & F'_1 & \xrightarrow[f'_1]{} & F'_0 & \xrightarrow[f'_0]{} & 0. & &
\end{array}$$

We put $\lambda_{-1} = 0$. Next we need to find a homomorphism $\lambda_0\colon F_0 \to F'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. Since F_0 is free abelian it suffices by Claim 0 to prove the following claim:

Claim 2. We have $\operatorname{im}(\beta_0) \subseteq \operatorname{im}(f'_1)$.

Proof. Since the diagram commutes and since $\beta = 0$ we see that $\operatorname{im}(\beta_0) \subseteq \ker(f'_0)$. Since the bottom sequence is exact we know that $\ker(f'_0) = \operatorname{im}(f'_1)$. ⊞

Now we suppose that we have already found homomorphisms $\lambda_0, \ldots, \lambda_{i-1}$ which have the desired property that

$$(*) \qquad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} \;=\; \beta_{j-1} \quad \text{for } j = 0, \ldots, i.$$

Note that we need to find a homomorphism $\lambda_i\colon F_i \to F'_{i+1}$ such that we have the equality $f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i = \beta_i$. Since F_i is free abelian it suffices by Claim 0 to prove the following claim:

Claim 3. We have $\operatorname{im}(\beta_i - \lambda_{i-1} \circ f_i) \subseteq \operatorname{im}(f'_{i+1})$.

Proof. Since the bottom sequence is exact we know that $\operatorname{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $\operatorname{im}(\beta_i - \lambda_{i-1} \circ f_i) \subseteq \ker(f'_i)$. It turns out that indeed we have

$$\begin{array}{rcccl} f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) & = & f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i & \overset{\text{since the } \beta_i\text{'s are a chain map, i.e. since } f'_i \circ \beta_i = \beta_{i-1} \circ f_i}{=} & \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\ & = & (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i & = & \lambda_{i-2} \circ f_{i-1} \circ f_i \quad = \quad 0. \\ & & & \text{by } (*) & \text{since } f_{i-1} \circ f_i = 0 \end{array}$$

■

We conclude this section with two applications of the Homological Algebra–Fundamental Theorem 158.4. The following corollary says in particular that any two free resolutions of a given abelian group are chain homotopy equivalent.

Corollary 158.5. (Free Resolution–Chain Homotopic Corollary) Let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

and

$$\ldots \to F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} H \to 0$$

be two free resolutions of the *same* abelian group H. Let $\{\alpha_i \colon F_i \to F'_i\}_{i \in \mathbb{N}}$ be an extension of the identity map $\mathrm{id} \colon H \to H$ to the free resolutions F_* and F'_*. The maps α_* form a chain homotopy equivalence between the two chain complexes F_* and F'_*.

Proof. Let $\{\alpha_i \colon F_i \to F'_i\}_{i \in \mathbb{N}}$ be an extension of the identity map $\mathrm{id} \colon H \to H$. We apply the Homological Algebra–Fundamental Theorem 158.4 to the free resolutions F'_* and F_* (i.e. we swap the roles of the free resolutions) and we obtain homomorphisms $\{\beta_i \colon F'_i \to F_i\}_{i \in \mathbb{N}_0}$ that give us the lower half of the following commutative diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0. \end{array}$$

The maps $\beta_i \circ \alpha_i$ now provide us with the following commutative diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_2 \circ \alpha_2} & & \downarrow{\scriptstyle \beta_1 \circ \alpha_1} & & \downarrow{\scriptstyle \beta_0 \circ \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0. \end{array}$$

This shows that the maps $\beta_i \circ \alpha_i$ define an extension of $\mathrm{id} \colon H \to H$ to the free resolutions F_i and F_i. But the identity maps also define an extension. By the Homological Algebra–Fundamental Theorem 158.4 the extensions $\{\beta_i \circ \alpha_i\}_{i \in \mathbb{N}_0}$ and $\{\mathrm{id}\}_{i \in \mathbb{N}_0}$ are chain homotopic. The same way one shows that the extensions $\{\alpha_i \circ \beta_i\}_{i \in \mathbb{N}_0}$ and $\{\mathrm{id}\}_{i \in \mathbb{N}_0}$ are chain homotopic. This shows that the maps α_i form a chain homotopy equivalence. ■

Using the Homological Algebra–Fundamental Theorem 158.4 we can now prove the following criterion for a chain complex to be chain homotopy equivalent to the zero chain complex:

Proposition 158.6. (Null-Homotopic Criterion Proposition) Let (C_*, ∂_*) be a chain complex of free abelian groups. The following four statements are equivalent:

(a) The chain complex (C_*, ∂_*) is chain homotopy equivalent to the zero chain complex.
(b) The identity map and the zero map of C_* to itself are chain homotopic.
(c) There exists an automorphism of C_* that is chain homotopic to the zero map.
(d) The chain complex (C_*, ∂_*) is acyclic in the sense of Definition 143.1, i.e. $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$.

Proof. The (a)⇔(b)-equivalence follows easily from the definitions. The (b)⇒(c)-implication is a tautology. The (c)⇒(d)-implication is a straightforward consequence of the Chain Homotopy–H_*-Lemma 144.5.

We move on to the most interesting part of the proof, namely the proof of the (d)⇒(b)-implication. Thus we assume that the chain complex (C_*, ∂_*) is acyclic. Next we consider the following pretty diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0} \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0} \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0} & & \\
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0.
\end{array}$$

The hypothesis that $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$ means that C_* is a free resolution of the trivial group, i.e. the top and bottom are free resolutions. The zero map on the right now has two extensions:

(1) The identity maps $\mathrm{id}\colon C_i \to C_i$ define an extension.
(2) The zero maps $\mathrm{id}\colon C_i \to C_i$ define an extension.

By Statement (2) of the Homological Algebra–Fundamental Theorem 158.4 these two extensions are chain homotopic. But that is precisely Statement (b). ∎

158.2. **Quasi-isomorphisms are chain homotopy equivalences.** In this section we will state and discuss more consequences of the Homological Algebra–Fundamental Theorem 158.4. We start out with the following definition:

Definition 158.7. Let $f\colon C_* \to D_*$ be a chain map between chain complexes. We say that f is a quasi-isomorphism if for each $k \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.

In the Chain Homotopy–H_*-Lemma 144.5 we saw that chain homotopy equivalences are quasi-isomorphisms. If we are dealing with chain complexes of free abelian groups, the following proposition gives us now the converse statement.

Proposition 158.8. (Quasi Iso–Homotopy Equivalence Proposition) Let C_* and D_* be two chain complexes of *free* abelian groups and let $f\colon C_* \to D_*$ be a chain map. If f is a quasi-isomorphism, then f is in fact a chain homotopy equivalence.

Proof. We postpone the proof to the next section. ∎

Example. As on page 3027 we consider the following setting: Let X be a topological space and let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. By the Excision Theorem 148.1 we know that the inclusion-induced chain map

$$\mathrm{C}_*(X \setminus Z, A \setminus Z) \;\to\; \mathrm{C}_*(X, A)$$

is a quasi-isomorphism. By the discussion on page 2798 we know that the relative chain groups are free abelian groups. Thus it follows from the Quasi Iso–Homotopy Equivalence Proposition 158.8 that the above chain map is in fact a chain homotopy equivalence. □

Remark. The hypothesis that C_* and D_* are free abelian groups is essential. For example in Exercise 158.3 we will see that there exists a quasi-isomorphism between the chain complex

$$C_* = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and the zero chain complex D_*, but we will see that the two chain complexes are not chain homotopy equivalent. □

Before we turn to the proof of the Quasi Iso–Homotopy Equivalence Proposition 158.8, let us formulate the following corollary:

Corollary 158.9. (Two-Out-Of-Three-Homotopy Equivalence Corollary)[1676] Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes of free abelian groups. If two of the vertical maps are chain homotopy equivalences, then the third is also a chain homotopy equivalence.

Proof. It follows from the LES-Proposition 145.10 (1) and (2) that we get a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{\partial_{n+1}} & \mathrm{H}_n(A_*) & \longrightarrow & \mathrm{H}_n(B_*) & \longrightarrow & \mathrm{H}_n(C_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A_*) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \mathrm{H}_{n+1}(C'_*) & \xrightarrow{\partial_{n+1}} & \mathrm{H}_n(A'_*) & \longrightarrow & \mathrm{H}_n(B'_*) & \longrightarrow & \mathrm{H}_n(C'_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A'_*) & \longrightarrow & \dots \end{array}$$

By our hypothesis, together with the Chain Homotopy Equivalence-H_*-Corollary 144.10, we know that always two out of three vertical maps are isomorphisms. But then it follows from the Five Lemma 145.11 that the third vertical map is also an isomorphism. The Quasi Iso–Homotopy Equivalence Proposition 158.8 implies that the corresponding map of chain complexes is also a chain homotopy equivalence. ■

158.3. **Proof of the Quasi Iso–Homotopy Equivalence Proposition 158.8 .** This section is completely dedicated to the proof of the Quasi Iso–Homotopy Equivalence Proposition 158.8. We start out with the following definition which associates to a chain map a chain complex.

Definition 158.10. Let (C_*, ∂_*^C) and (D_*, ∂_*^D) be two chain complexes and let $\varphi\colon C_* \to D_*$ be a chain map. The algebraic mapping cone of φ is the chain complex $(\mathrm{M}(\varphi), \partial_*)$ that is given by the chain groups

$$\mathrm{M}(\varphi)_n := \begin{matrix} C_{n-1} \\ D_n \end{matrix} := C_{n-1} \oplus D_n$$

[1676] This corollary is very close to the content of Exercise 145.14.

and the boundary maps[1677]

$$\partial_n\colon \begin{array}{ccc} \overbrace{\begin{array}{c} C_{n-1} \\ D_n \end{array}}^{\mathrm{M}(\varphi)_n} & \to & \overbrace{\begin{array}{c} C_{n-2} \\ D_{n-1} \end{array}}^{\mathrm{M}(\varphi)_{n-1}} \\ x & \mapsto & -\partial^C_{n-1}(x) \\ y & & \varphi_{n-1}(x)+\partial^D_n(y) \end{array} \qquad \text{which we write as block matrices} \qquad \begin{array}{c} \begin{array}{cc} C_{n-1} & D_n \end{array} \\ \begin{pmatrix} -\partial^C_{n-1} & 0 \\ \varphi_{n-1} & \partial^D_n \end{pmatrix} \end{array} \begin{array}{c} C_{n-2} \\ D_{n-1} \end{array}$$

Remark. Recall that in Definition 21.7 we associated to any continuous map $f\colon A \to X$ between topological spaces a topological space, namely we defined the corresponding *mapping cone* as

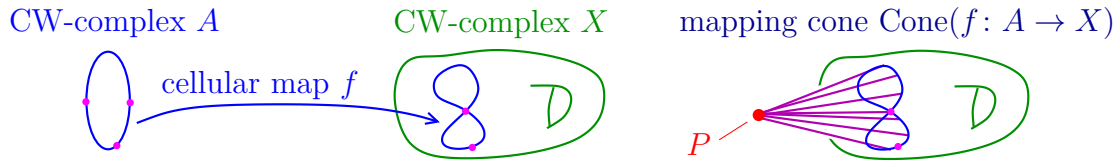
$$\mathrm{Cone}(f\colon A \to X) \coloneqq ((A \times [0,1]) \sqcup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a,b \in A$. We make the following two observations:

(1) In the Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.17 we showed that there exists a long exact sequence of the form

$$\ldots \to \widetilde{\mathrm{H}}_n(A) \xrightarrow{f_*} \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(\mathrm{Cone}(f)) \xrightarrow{\partial_n} \widetilde{\mathrm{H}}_{n-1}(A) \to \ldots$$

(2) Let us suppose that A and X are finite CW-complexes and that f is a cellular map. By the CW-Complex Mapping Cylinder+Cone Proposition 139.25 we can equip the mapping cone $\mathrm{Cone}(f\colon A \to X)$ with a natural CW-structure.



In Exercise 158.2 we will show that the corresponding cellular chain complex of the topological mapping cone is isomorphic to the algebraic mapping cone of the chain map $f_*\colon \mathrm{C}^{\mathrm{CW}}_*(A) \to \mathrm{C}^{\mathrm{CW}}_*(X)$. □

The following lemma can be viewed as an algebraic analog of the Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.17.

Lemma 158.11. (Mapping Cone-LES Lemma) Let $\varphi\colon C_* \to D_*$ be a chain map between chain complexes (C_*, ∂^C_*) and (D_*, ∂^D_*). We denote by $i_n\colon D_n \to \mathrm{M}(\varphi)_n = C_{n-1} \oplus D_n$ the obvious natural inclusion map and we denote by $p_n\colon \mathrm{M}(\varphi)_n = C_{n-1} \oplus D_n \to C_{n-1}$ the obvious natural projection. These maps induce maps on homology groups and the following sequence is exact

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\varphi_*} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(D_*) \to \ldots$$

[1677] Note that

$$\partial_{n-1}\circ\partial_n = \begin{pmatrix} -\partial^C_{n-2} & 0 \\ \varphi_{n-2} & \partial^D_{n-1} \end{pmatrix}\begin{pmatrix} -\partial^C_{n-1} & 0 \\ \varphi_{n-1} & \partial^D_n \end{pmatrix} = \begin{pmatrix} \partial^C_{n-2}\circ\partial^C_{n-1} & 0 \\ -\varphi_{n-2}\circ\partial^C_{n-1} + \partial^D_{n-1}\circ\varphi_{n-1} & \partial^D_{n-1}\circ\partial^D_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

since φ is a chain map, since $\partial^C_{n-2}\circ\partial^C_{n-1} = 0$ and since $\partial^D_{n-1}\circ\partial^D_n = 0$

This shows that we have indeed defined a chain complex. This calculation also shows the necessity of the $-$-sign in the definition of the boundary map of the algebraic mapping cone.

Proof. For each $n \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \overbrace{\begin{array}{c} C_{n-1} \\ D_n \end{array}}^{=\mathrm{M}(\varphi)_n} & & & & \\
0 & \longrightarrow & D_n & \xrightarrow{i_n} & & \xrightarrow{p_n} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n^D} & & \downarrow{\scriptstyle \partial_n} \begin{pmatrix} -\partial_{n-1}^C & 0 \\ \varphi_{n-1} & \partial_n^D \end{pmatrix} & & \downarrow{\scriptstyle \partial_{n-1}^C} & & \\
0 & \longrightarrow & D_{n-1} & \xrightarrow[i_{n-1}]{} & \underbrace{\begin{array}{c} C_{n-2} \\ D_{n-1} \end{array}}_{=\mathrm{M}(\varphi)_{n-1}} & \xrightarrow[p_{n-1}]{} & C_{n-2} & \longrightarrow & 0.
\end{array}$$

We make the following observations:

(1) The horizontal sequences are exact.
(2) The left-hand square commutes, i.e. the maps i_* define a chain map.
(3) The right-hand square commutes up to a minus sign, i.e. we have

$$\partial_{n-1}^C \circ p_n \;=\; -\,p_{n-1} \circ \partial_n.$$

Fortunately the minus sign in the above equality does not change the fact that the maps p_n induce a map between homology groups.

Despite the minus sign in (3) we can use verbatim the same construction as in Definition 145.8 to obtain connecting homomorphisms $\partial_n \colon \mathrm{H}_n(D_*) \to \mathrm{H}_n(C_*)$, $n \in \mathbb{N}_0$, and verbatim the same proof as in the LES-Proposition 145.10 shows that these connecting homomorphisms give rise to a long exact sequence of the form

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\partial_{n-1}} \mathrm{H}_{n-1}(D_*) \to \ldots$$

Keeping the above diagram in mind one sees that it follows easily from the definition of the connecting homomorphism on Definition 145.8 that for each $n \in \mathbb{N}_0$ we have the equality[1678]

$$\partial_n \;=\; \varphi_* \colon \mathrm{H}_n(C_*) \;\to\; \mathrm{H}_n(D_*).$$

We have thus shown that the sequence in the statement of the lemma is exact. ∎

Now we are finally fully prepared to provide the proof of the Quasi Iso–Homotopy Equivalence Proposition 158.8.

Proof of the Quasi Iso–Homotopy Equivalence Proposition 158.8. Let $\varphi \colon C_* \to D_*$ be a chain map between chain complexes (C_*, ∂_*^C) and (D_*, ∂_*^D) of free abelian groups which is a quasi-isomorphism, i.e. which induces an isomorphism on all homology groups. We need to show that φ is in fact a chain homotopy equivalence.

As above we denote by $(\mathrm{M}(\varphi)_*, \partial_*)$ the algebraic mapping cone of φ. We consider the long exact sequence from the Mapping Cone-LES Lemma 158.11:

$$\ldots \to \mathrm{H}_n(C_*) \underset{\substack{\uparrow \\ \text{isomorphism} \\ \text{by hypothesis}}}{\xrightarrow{\varphi_*}} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \underset{\substack{\uparrow \\ \text{isomorphism} \\ \text{by hypothesis}}}{\xrightarrow{\varphi_*}} \mathrm{H}_{n-1}(D_*) \to \ldots$$

[1678] It is worth verifying that $\partial_n = \varphi_*$.

By our hypothesis the maps φ induce isomorphisms of homology groups, thus we see that $\mathrm{H}_n(\mathrm{M}(\varphi)) = 0$ for all $n \geq 0$. By the Null-Homotopic Criterion Proposition 158.6 this means that we have a chain homotopy $\{P_n \colon \mathrm{M}(\varphi)_n \to \mathrm{M}(\varphi)_{n+1}\}_{n\geq 0}$ from the identity map of $\mathrm{M}(\varphi)_*$ to the zero map. (Note that here we use that by our hypothesis on C_* and D_* each $\mathrm{M}(\varphi)_n = C_{n-1} \oplus D_n$ is a free abelian group.)

Recall that this means that for each $n \in \mathbb{N}_0$ we have the equality

$$P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n \;=\; \mathrm{id}_{C_{n-1}\oplus D_n} - 0. \tag{$*$}$$

Using the obvious matrix notation we write

$$P_n =: \begin{matrix} C_n \\ D_{n+1} \end{matrix} \begin{matrix} \begin{matrix} C_{n-1} & D_n \end{matrix} \\ \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} \end{matrix} : \underbrace{\begin{matrix} C_{n-1} \\ D_n \end{matrix}}_{=\mathrm{M}(\varphi)_n} \to \underbrace{\begin{matrix} C_n \\ D_{n+1}. \end{matrix}}_{=\mathrm{M}(\varphi)_{n+1}}$$

Now we can rewrite the above equality $(*)$ and we obtain that

$$\begin{pmatrix} P_{n-1}^{11} & P_{n-1}^{12} \\ P_{n-1}^{21} & P_{n-1}^{22} \end{pmatrix} \circ \begin{pmatrix} -\partial_{n-1}^C & 0 \\ \varphi_{n-1} & \partial_n^D \end{pmatrix} + \begin{pmatrix} -\partial_n^C & 0 \\ \varphi_n & \partial_{n+1}^D \end{pmatrix} \circ \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{C_{n-1}} & 0 \\ 0 & \mathrm{id}_{D_n} \end{pmatrix}.$$

Looking at the four entries separately we obtain the following four equalities:

$$\text{(a)} \qquad -P_{n-1}^{11} \circ \partial_{n-1}^C + P_{n-1}^{12} \circ \varphi_{n-1} - \partial_n^C \circ P_n^{11} \;=\; \mathrm{id}_{C_{n-1}},$$

$$\text{(b)} \qquad P_{n-1}^{12} \circ \partial_n^D - \partial_n^C \circ P_n^{12} \;=\; 0,$$

$$\text{(c)} \qquad -P_{n-1}^{21} \circ \partial_{n-1}^C + P_{n-1}^{22} \circ \varphi_{n-1} + \varphi_n \circ P_n^{11} + \partial_{n+1}^D \circ P_n^{21} \;=\; 0,$$

$$\text{(d)} \qquad P_{n-1}^{22} \circ \partial_n^D + \varphi_n \circ P_n^{12} + \partial_{n+1}^D \circ P_n^{22} \;=\; \mathrm{id}_{D_n}.$$

We make the following observation:

(1) Equality (b) implies that the homomorphisms $\{P_n^{12} \colon D_n \to C_n\}_{n\in\mathbb{N}_0}$ define a chain map $D \to C$.

Now we claim that the chain map $\{P_n^{12}\}_{n\in\mathbb{N}_0}$ is a chain homotopy inverse to the chain map φ. We do so with the following two observations:

(2) Equality (a) implies that $P^{12} \circ \varphi \simeq \mathrm{id}_{C_*}$ where the chain homotopy is given by the maps $P_n^{11} \colon C_{n-1} \to C_n$.

(3) Equality (d) implies that $\varphi \circ P^{12} \simeq \mathrm{id}_{D_*}$, where the chain homotopy is given by the maps $P_n^{22} \colon D_n \to D_{n+1}$,

The last two statements together imply that φ is indeed a chain homotopy equivalence. ∎

158.4. **Singular and cellular chain complexes.** In Chapter 155 we introduced the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and the cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(X)$ of a CW-complex X. In the Singular-Cellular-H_*-Proposition 155.8 we showed that the cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(X)$ are naturally isomorphic to the singular homology groups $\mathrm{H}_k(X)$. At that point we did not say anything about the relationship between the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and the singular chain complex $\mathrm{C}_*(X)$. We will rectify this in this section. The extra information that we gain in this section will stand us in good stead in Section 160.3 when we explore the relationship between cellular and singular homology with coefficients, and again in Section 199.1 when we study the relationship between cellular and singular cohomology.

First we recall some definitions and some notation, mostly from Definition 155.5.

Definition 158.12. Let X be a CW-complex and let $n \in \mathbb{N}_0$.

(1) We denote by X^n the n-skeleton of X.

(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we denote by $\partial_n^{CW} \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from Definition 155.3, i.e. ∂_n^{CW} is the homomorphism that is given by the composition of the following two homomorphisms:

$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{\text{я}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).$$

(∂_n: connecting homomorphism of the long exact sequence of the pair (X^n, X^{n-1}))

In the Cellular Boundary Map Lemma 155.4 we showed that for every $n \in \mathbb{N}_0$ we have $\partial_n^{CW} \circ \partial_{n+1}^{CW} = 0$.

(3) We refer to

$$\ldots \xrightarrow{\partial_{n+2}^{CW}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{\partial_{n+1}^{CW}} \mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{\partial_n^{CW}} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{\partial_{n-1}^{CW}} \ldots$$

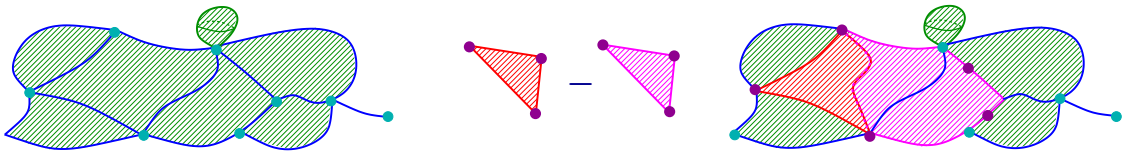
as the cellular chain complex. Its homology groups are denoted by $\mathrm{H}_n^{\mathrm{CW}}(X)$.

In the following we want to relate the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ to the singular chain complex $\mathrm{C}_*(X)$. There is no obvious "interesting" chain map between these two chain complexes. But as we will see in a minute, it turns out that there is a nice "intermediate" chain complex which maps to both of the above chain complexes:

Definition 158.13. Given a CW-complex X and given $n \in \mathbb{N}_0$ we define the intermediate cellular chain complex[1679]

$$\mathrm{C}_n^{\mathrm{CW,int}}(X) := \ker\left(\mathrm{C}_n(X^n) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X^n) \xrightarrow{\pi_{n-1}} \mathrm{C}_{n-1}(X^n, X^{n-1})\right)$$

where $\pi_{n-1} \colon \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})$ is the natural projection.



The following lemma summarizes a few basic properties of the intermediate cellular chain complexes.

Lemma 158.14. (CW-Intermediate-Singular-Lemma)

(1) Given any CW-complex X the intermediate cellular chain complex $\mathrm{C}_*^{\mathrm{CW,int}}(X)$ is a subcomplex of the singular chain complex $\mathrm{C}_*(X)$.

(2) If $f \colon X \to Y$ is a cellular map between two CW-complexes, then the induced map $f_* \colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ restricts to a chain map $\mathrm{C}_*^{\mathrm{CW,int}}(X) \to \mathrm{C}_*^{\mathrm{CW,int}}(Y)$.

(3) For each $k \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}_k(\mathrm{C}_*^{\mathrm{CW,int}}(X))$ and $f \mapsto f_*$ define a covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof.

[1679] Strictly speaking we only show in the CW-Intermediate-Singular-Lemma 158.14 that this is indeed a chain complex.

(1) First note that by definition we have for any $k \in \mathbb{N}_0$ the equality

$$(*) \qquad \mathrm{C}_k^{\mathrm{CW,int}}(X) \;=\; \{\sigma \in \mathrm{C}_k(X^k) \,|\, \partial_k(\sigma) \in \mathrm{C}_{k-1}(X^{k-1})\}.$$

Now let $\sigma \in \mathrm{C}_n^{\mathrm{CW,int}}(X)$. We need to show that $\partial_n(\sigma) \in \mathrm{C}_{n-1}^{\mathrm{CW,int}}(X)$.

(a) By $(*)$ we know that $\partial_n(\sigma) \in \mathrm{C}_{n-1}(X^{n-1})$.

(b) Note that $\partial_{n-1}(\partial_n(\sigma)) = (\partial_{n-1} \circ \partial_n)(\sigma) = 0$. It follows from this observation that $\partial_{n-1}(\partial_n(\sigma)) \in \mathrm{C}_{n-2}(X^{n-2})$.

(c) The combination of (a), (b) and $(*)$ shows that $\partial_n(\sigma) \in \mathrm{C}_{n-1}^{\mathrm{CW,int}}(X)$.

(2) Let $f\colon X \to Y$ be a cellular map between two CW-complexes. We obtain the following diagram:

$$\begin{array}{ccccc}
\mathrm{C}_n(X^n) & \xrightarrow{\partial_n} & \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(X^n, X^{n-1}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{C}_n(Y^n) & \xrightarrow{\partial_n} & \mathrm{C}_{n-1}(Y^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(Y^n, Y^{n-1}).
\end{array}$$

Note that it follows from the C_*–Functor Lemma 143.16 that this diagram commutes. It thus follows immediately that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ restricts to a chain map $\mathrm{C}_*^{\mathrm{CW,int}}(X) \to \mathrm{C}_*^{\mathrm{CW,int}}(Y)$.

(3) This statement follows easily from the definitions. ■

As a standalone invariant the intermediate chain complexes are not particular interesting. But as we will see in the following proposition, the intermediate chain complexes have interesting chain maps to the singular chain complex and to the cellular chain complex. The following proposition also gives us in particular a new proof of the main statements of the Singular-Cellular-H_*-Proposition 155.8.

Proposition 158.15. (CW-Intermediate-Singular Proposition) Let X be a CW-complex.

- We denote by $i\colon \mathrm{C}_*^{\mathrm{CW,int}}(X) \to \mathrm{C}_*(X)$ the natural inclusion map.
- We consider the natural map

$$\begin{array}{rcl} p\colon \mathrm{C}_*^{\mathrm{CW,int}}(X) & \to & \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1}) \\ \sigma & \mapsto & [\pi_n(\sigma)]. \end{array}$$

For each $n \in \mathbb{N}_0$ the natural maps

(1) $i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*(X))}_{=\mathrm{H}_n(X)}$ and (2) $p_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(X))}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}$

are isomorphisms.

Proof. We outsource the lengthy and dull proof of the proposition to the next section. ■

The following proposition gives us the promised relationship between the singular and cellular chain complex of a given CW-complex.

Proposition 158.16. (Singular-Cellular Chain Complex Proposition) Let X be a CW-complex.

(1) The two natural chain maps

$$i\colon \mathrm{C}_*^{\mathrm{CW,int}}(X) \hookrightarrow \mathrm{C}_*(X) \quad \text{and} \quad p\colon \mathrm{C}_*^{\mathrm{CW,int}}(X) \twoheadrightarrow \mathrm{C}_*^{\mathrm{CW}}(X)$$

from the CW-Intermediate-Singular Proposition 158.15, that are given by the natural inclusion and the natural projection, are chain homotopy equivalences.
(2) There exists a[1680] chain homotopy equivalence $C_*(X) \to C_*^{CW}(X)$.

Remark. In the proof of the Singular-Cellular Chain Complex Proposition 158.16 we will make use of the Free Abelian Subgroup Theorem 97.14 which says that the subgroup of a free abelian group is again a free abelian group. This purely algebraic fact lets us conclude that the intermediate chain groups $C_k^{CW,int}(X)$ are free abelian groups. In Section 328.1 we will use a somewhat different approach to relate the chain complexes $C_*(X)$ and $C_*^{CW}(X)$ that does not make use of this algebraic fact, and which thus generalizes to more general contexts. □

Proof. Let X be a CW-complex.

(1) By definition each $C_k^{CW,int}(X)$ is a subgroup of the free abelian group $C_k(X)$. It follows from the Free Abelian Subgroup Theorem 97.14 that each $C_k^{CW,int}(X)$ itself is a free abelian group. In particular $C_*^{CW,int}(X)$ is a chain complex of free abelian groups. By the CW-Intermediate-Singular Proposition 158.15 we know that the chain maps $i\colon C_*^{CW,int}(X) \hookrightarrow C_*(X)$ and $p\colon C_*^{CW,int}(X) \twoheadrightarrow C_*^{CW}(X)$ are quasi-isomorphisms. Since all chain complexes involved consist of free abelian groups we deduce from the Quasi Iso–Homotopy Equivalence Proposition 158.8 that the two chain maps are both chain homotopy equivalences.
(2) Recall that on page 2769 we pointed out that it follows from the Chain Homotopy Equivalence-Reformulation Lemma 144.9 and the Isomorphism Lemma 5.12 that the relation "chain homotopy equivalence" of chain complexes is an equivalence relation on the class of all chain complexes. The desired statement follows immediately from this observation and Statement (1). ■

158.5. **Proof of the CW-Intermediate-Singular Proposition 158.15 .** We prove the two parts of the CW-Intermediate-Singular Proposition 158.15 separately.

Proof of Proposition 158.15 (1). Let X be a CW-complex and let $n \in \mathbb{N}_0$. We need to show that the map

$$i_*\colon H_n(C_*^{CW,int}(X)) \to H_n(C_*(X))$$

is an isomorphism. Before we start, let us recall that in the CW-Skeleton-H_*-Lemma 155.2 (3) and (4) we proved the following two convenient facts:

(a) The inclusion $k''\colon X^{n+1} \to X$ induces an isomorphism $k''_*\colon H_n(X^{n+1}) \xrightarrow{\cong} H_n(X)$.
(b) The inclusion $k'\colon X^n \to X$ induces an epimorphism $k'_*\colon H_n(X^n) \twoheadrightarrow H_n(X)$.

Claim. Let $n \in \mathbb{N}_0$.
(i) The map $i_*\colon H_n(C_*^{CW,int}(X)) \to H_n(C_*(X)) = H_n(X)$ is a monomorphism.
(ii) The map $i_*\colon H_n(C_*^{CW,int}(X)) \to H_n(C_*(X)) = H_n(X)$ is an epimorphism.

[1680]Note that we do not claim that there exists a *natural* chain homotopy equivalence.

Proof. We consider the following two commutative diagrams

$$\begin{array}{ccccc} \mathrm{C}_{n+1}^{\mathrm{CW,int}}(X) & \xhookrightarrow{j} & \mathrm{C}_{n+1}(X^{n+1}) & \xhookrightarrow{k''_*} & \mathrm{C}_{n+1}(X) \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ \mathrm{C}_{n}^{\mathrm{CW,int}}(X) & \xhookrightarrow{j} & \mathrm{C}_{n}(X^{n+1}) & \xhookrightarrow{k''_*} & \mathrm{C}_{n}(X) \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathrm{C}_{n}^{\mathrm{CW,int}}(X) & & \xrightarrow{j} & & \mathrm{C}_{n}(X^{n}) & & \xrightarrow{k'_*} & & \mathrm{C}_{n}(X) \\ \downarrow{\scriptstyle \partial_{n}} & & & & \downarrow{\scriptstyle \partial_{n}} & & & & \downarrow{\scriptstyle \partial_{n}} \\ \mathrm{C}_{n-1}^{\mathrm{CW,int}}(X) & & \xhookrightarrow{j} & & \mathrm{C}_{n-1}(X^{n}) & & \xhookrightarrow{k'_*} & & \mathrm{C}_{n-1}(X). \end{array}$$

Here in each diagram the maps to the left are the obvious inclusion maps; for better or worse we refer to all of them as j. The maps to the right are induced by the inclusions $k'' \colon X^{n+1} \to X$ and $k' \colon X^n \to X$. As we will now see, both statements of the claim follow easily from the diagrams together with (a) and (b) above:

(i) We consider the diagram to the left. Let $\sigma \in \mathrm{C}_n^{\mathrm{CW,int}}(X)$ be a cycle with the property that $i(\sigma) \in \mathrm{C}_n(X)$ is null-homologous. It follows from (a) that $j(\sigma) \in \mathrm{C}_n(X^{n+1})$ is already null-homologous, i.e. there exists a $\tau \in \mathrm{C}_{n+1}(X^{n+1})$ with $\partial_{n+1}(\tau) = j(\sigma)$. By definition of $\mathrm{C}_n^{\mathrm{CW,int}}(X)$ we know that $\sigma \in \mathrm{C}_n(X^n) \subseteq \mathrm{C}_n(X^{n+1})$. Thus we see that τ actually lies in

$$\mathrm{C}_{n+1}^{\mathrm{CW,int}}(X) \;=\; \ker\big(\mathrm{C}_{n+1}(X^{n+1}) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X^{n+1}) \xrightarrow{\pi_n} \mathrm{C}_n(X^{n+1}, X^n)\big).$$

But this implies that σ is null-homologous in $\mathrm{C}_*^{\mathrm{CW,int}}(X)$.

(ii) We consider the diagram to the right. Let $[\sigma] \in \mathrm{H}_n(X)$. By (b) we know that there exists a cycle $\tau \in \mathrm{C}_n(X^n)$ with $[k'_*(\tau)] = [\sigma]$. Since $\partial_n \tau = 0$ we see that $\tau \in \mathrm{C}_n^{\mathrm{CW,int}}(X)$. Since the bottom-left horizontal map is a monomorphism we see that $\tau \in \mathrm{C}_n^{\mathrm{CW,int}}(X)$ is also a cycle. But this shows that $[\sigma] = [i(\tau)]$. ∎

Proof of Proposition 158.15 (2). Let X be a CW-complex. For orientation purposes it is convenient to keep the following simple commutative diagram in mind:

$$\begin{array}{ccc} \mathrm{C}_n(X^n) & \xrightarrow{\pi_n} & \mathrm{C}_n(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(X^n, X^{n-1}). \end{array}$$

Furthermore it is convenient to recall the following definitions:

$$\begin{aligned} \mathrm{C}_n^{\mathrm{CW,int}}(X) &= \ker(\pi_{n-1} \circ \partial_n \colon \mathrm{C}_n(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})) \\ \mathrm{C}_n^{\mathrm{CW}}(X) &= \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{H}_n(\mathrm{C}_*(X^n, X^{n-1})). \end{aligned}$$

Claim 0. The maps

$$\begin{aligned} p \colon \mathrm{C}_*^{\mathrm{CW,int}}(X) &\to \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1}) \\ \sigma &\mapsto [\pi(\sigma)] \end{aligned}$$

are well-defined, in the sense that $[\pi(\sigma)]$ is a cycle in $\mathrm{C}_*^{\mathrm{CW}}(X)$, and these maps form a chain map.

Proof. We consider the above commutative diagram. If $\sigma \in \mathrm{C}_n^{\mathrm{CW,int}}(X)$, then the bottom-right map is by definition zero. Since the diagram commutes we see that the right-bottom map is also zero. But this means precisely that $\pi_n(\sigma)$ is a cycle in $\mathrm{C}_n(X^n, X^{n-1})$, thus

$\pi_n(\sigma)$ defines an element in $\mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$. We leave it to the reader to verify that the maps $p\colon \mathrm{C}_*^{\mathrm{CW,int}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$ form a chain map. ⊞

Now let $n \in \mathbb{N}_0$. We need to show that the map

$$\begin{array}{rcl} p_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(X)) & \to & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(X)) \\ [\sigma] & \mapsto & [\pi_n(\sigma)] \end{array}$$

is an isomorphism.

Claim 1. For each $n \in \mathbb{N}_0$ the map $p\colon \mathrm{C}_n^{\mathrm{CW,int}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(X)$ is an epimorphism.

Proof. Let $[\sigma] \in \mathrm{C}_n^{\mathrm{CW}}(X)$, i.e. let $\sigma \in \mathrm{C}_n(X^n, X^{n-1})$ such that $\partial_n(\sigma) = 0$. Since the projection $\pi_n\colon \mathrm{C}_n(X^n) \to \mathrm{C}_n(X^n, X^{n-1})$ is an epimorphism there exists a $\nu \in \mathrm{C}_n(X^n)$ with $\pi_n(\nu) = \sigma$. Using the above commutative diagram one sees easily that $\nu \in \mathrm{C}_n^{\mathrm{CW,int}}(X)$. ⊞

Given $n \in \mathbb{N}_0$ we set

$$D_n \coloneqq \ker\left\{p\colon \mathrm{C}_n^{\mathrm{CW,int}}(X) \xrightarrow{p} \mathrm{C}_n^{\mathrm{CW}}(X)\right\}.$$

Note that $\nu \in \mathrm{C}_n(X^n)$ lies in D_n if and only if the following two conditions are satisfied:
(a) $(\pi_{n-1} \circ \partial_n)(\nu) = 0$,
(b) $\pi_n(\nu) \in \mathrm{C}_n(X^n, X^{n-1})$ represents the zero element of $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.
Furthermore note that the groups D_* form a subcomplex of $\mathrm{C}_*^{\mathrm{CW,int}}(X)$.

Claim 2. For each $n \in \mathbb{N}_0$ we have $\mathrm{C}_{n+1}(X^n) \subseteq D_{n+1}$.

Proof. For orientation purposes we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{C}_{n+1}(X^n) & \hookrightarrow & \mathrm{C}_{n+1}(X^{n+1}) & \xrightarrow{\pi_{n+1}} & \mathrm{C}_{n+1}(X^{n+1}, X^n) \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ \mathrm{C}_n(X^n) & \hookrightarrow & \mathrm{C}_n(X^{n+1}) & \xrightarrow{\pi_n} & \mathrm{C}_n(X^{n+1}, X^n). \end{array}$$

Let $\Theta \in \mathrm{C}_{n+1}(X^n)$. Since the diagram commutes and since the horizontal sequences are exact we see that Θ satisfies (a) and we see that $\pi_{n+1}(\Theta) = 0 \in \mathrm{C}_{n+1}(X^{n+1}, X^n)$, the latter statement implies that (b) is also satisfied. ⊞

Note that by Claim 1 we have a short exact sequence

$$0 \to D_* \to \mathrm{C}_n^{\mathrm{CW,int}}(X) \xrightarrow{p} \mathrm{C}_n^{\mathrm{CW}}(X) \to 0$$

of chain complexes. Considering the corresponding long exact sequence of homology groups, see the LES-Proposition 145.10, it suffices to prove the following claim:

Claim 3. The chain complex D_* is acyclic, i.e. all homology groups of D_* are zero.

Proof. For orientation purposes we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{C}_{n+1}(X^{n-1}) & \xrightarrow{i} & \mathrm{C}_{n+1}(X^n) & \xrightarrow{\pi_{n+1}} & \mathrm{C}_{n+1}(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ \mathrm{C}_n(X^{n-1}) & \xrightarrow{i} & \mathrm{C}_n(X^n) & \xrightarrow{\pi_n} & \mathrm{C}_n(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ \mathrm{C}_{n-1}(X^{n-1}) & \xrightarrow{i} & \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(X^n, X^{n-1}). \end{array}$$

So let $\mu \in \ker(\partial_n\colon D_n \to D_{n-1}) \subseteq D_n \subseteq \mathrm{C}_n^{\mathrm{CW,int}}(X) \subseteq \mathrm{C}_n(X)$. We need to show that there exists a $\Theta \in D_{n+1}$ with $\partial_{n+1}(\Theta) = \mu$. By (b) we know that $\pi_n(\mu) \in \mathrm{C}_n(X^n, X^{n-1})$ represents the zero element of $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$. It follows that there exists a

$\gamma \in \mathrm{C}_{n+1}(X^n, X^{n-1})$ with $\partial_{n+1}(\gamma) = \pi_n(\mu)$. Since the top horizontal sequence is exact we can pick $\widetilde{\gamma} \in \mathrm{C}_{n+1}(X^n)$ with $\pi_{n+1}(\widetilde{\gamma}) = \gamma$. Note that we have the following equality in $\mathrm{C}_n(X^n, X^{n-1})$:

$$\pi_n(\partial_{n+1}(\widetilde{\gamma}) - \mu) \;=\; (\pi_n \circ \partial_{n+1})(\widetilde{\gamma}) - \pi(\mu) \;\underset{\substack{\uparrow \\ \text{since we have} \\ \partial_{n+1} \circ \pi_{n+1} = \pi_n \circ \partial_{n+1}}}{=}\; \partial_{n+1}\overbrace{(\pi_{n+1}(\widetilde{\gamma}))}^{=\gamma} - \pi_n(\mu) \;\underset{\substack{\uparrow \\ \text{since we have} \\ \partial_{n+1}(\gamma) = \pi_n(\mu)}}{=}\; 0.$$

Since the middle horizontal sequence is exact there exists an $\alpha \in \mathrm{C}_n(X^{n-1})$ such that $i(\alpha) = \partial_{n+1}(\widetilde{\gamma}) - \mu$. Note that

$$i(\partial_n(\alpha)) \;=\; \partial_n(i(\alpha)) \;=\; \partial_n(\partial_{n+1}(\widetilde{\gamma}) - \mu) \;=\; \underbrace{(\partial_n \circ \partial_{n+1})}_{=0}(\widetilde{\gamma}) - \underbrace{\partial_n(\mu)}_{\substack{=0, \text{ since} \\ \mu \in \ker(\partial_n)}} \;=\; 0.$$

Since i is a monomorphism we see that $\partial_n(\alpha) = 0$. As we mentioned above, by the CW-Skeleton-H_*-Lemma 155.2 (2) we know that $\mathrm{H}_n(X^{n-1}) = 0$. This implies that there exists an $\widetilde{\alpha} \in \mathrm{C}_{n+1}(X^{n-1})$ with $\partial_{n+1}(\widetilde{\alpha}) = \alpha$. Then

$$\begin{aligned} \partial_{n+1}(\widetilde{\gamma} - i(\widetilde{\alpha})) \;&\underset{\substack{\downarrow \\ \text{since } \partial_{n+1} \circ i = i \circ \partial_{n+1}}}{=}\; \partial_{n+1}(\widetilde{\gamma}) - (i \circ \partial_{n+1})(\widetilde{\alpha}) \;=\; \partial_{n+1}(\widetilde{\gamma}) - \overbrace{i(\alpha)}^{=\partial_{n+1}(\widetilde{\gamma}) - \mu} \\ &=\; \partial_{n+1}(\widetilde{\gamma}) - \partial_{n+1}(\widetilde{\gamma}) + \mu \;=\; \mu. \end{aligned}$$

We set $\Theta := \widetilde{\gamma} - i(\widetilde{\alpha}) \in \mathrm{C}^{n+1}(X^n)$. We just showed that $\partial_{n+1}(\Theta) = \mu$. Since $\Theta \in \mathrm{C}^{n+1}(X^n)$ we obtain from Claim 2 that Θ actually lies in D_{n+1}. ∎

158.6. **Excisive triads.** In this section we introduce the notion of an "excisive triad". The name sounds mysterious. It turns out that the concept is a little technical, but it will appear on many occasions. The Quasi Iso–Homotopy Equivalence Proposition 158.8, which we just proved, will help us to prove a characterization of excisive triads, which we will use on many occasions in the future.

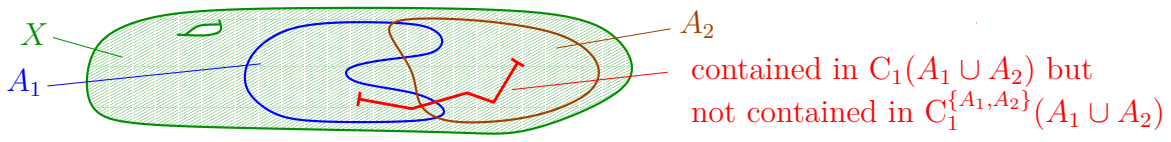
First let us recall Definition 148.8.

Definition 158.17. Let again X be a topological space and let $A_1, A_2 \subseteq X$ be subsets of X. We consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\{A_1,A_2\}}(X) \;:=\; \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \begin{array}{l} \text{for each } j \text{ there exists an } i \in \{1,2\} \text{ such} \\ \text{that the image of } \sigma_j \text{ lies in } A_i \end{array} \right\}.$$

In Definition 148.8 we saw that these groups define a subcomplex of $\mathrm{C}_*(X)$. Thus we can define

$$\mathrm{H}_n^{\{A_1,A_2\}}(X) \;:=\; \mathrm{H}_n\big(\mathrm{C}_*^{\{A_1,A_2\}}(X), \partial_*\big).$$



Now we can provide the definition of an (excisive) triad of topological spaces:

Definition 158.18.

(1) A triad of topological spaces is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subspaces of X.[1681]
(2) A continuous map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ between two triads is a continuous map $f\colon X \to Y$ with $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$.
(3) We say that a triad (X, A_1, A_2) is excisive if the inclusion map

$$\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{C}_*(A_1 \cup A_2)$$

is a chain homotopy equivalence.

We will see that many of the triads that a working topologist comes across are excisive. But before we get to this point let us first formulate the following characterization of excisive triads.

Lemma 158.19. (Excisive Characterization Lemma) Let (X, A_1, A_2) be a triad of topological spaces. The following statements are equivalent:

(1) The triad is excisive.
(2) For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2)$$

is an isomorphism.
(3) For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$

is an isomorphism.

Proof. Let (X, A_1, A_2) be a triad of topological spaces. The equivalence of (1) and (2) is an immediate consequence of the Chain Homotopy Equivalence-H_*-Corollary 144.10 and the Quasi Iso–Homotopy Equivalence Proposition 158.8.

Now we turn to the proof that (2) and (3) are equivalent. We write $Y = A_1 \cup A_2$. We consider the following sequences of chain complexes:

$$\text{(I)} \quad 0 \longrightarrow \mathrm{C}_*^{\{A_1,A_2\}}(Y) \longrightarrow \mathrm{C}_*(Y) \longrightarrow \mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y) \longrightarrow 0$$

$$\text{(II)} \quad 0 \longrightarrow \underbrace{\mathrm{C}_*(A_1)/\mathrm{C}_*(A_1 \cap A_2)}_{=\mathrm{C}_*(A_1, A_1\cap A_2)} \longrightarrow \underbrace{\mathrm{C}_*(Y)/\mathrm{C}_*(A_2)}_{=\mathrm{C}_*(Y, A_2)} \longrightarrow \mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y) \longrightarrow 0.$$

The top sequence is exact by definition. It is straightforward to see that the bottom sequence is also exact. We can thus apply the LES-Proposition 145.10 to get the corresponding long exact sequences of homology groups:

$$\text{(I)} \quad \to \mathrm{H}_n(\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \longrightarrow \mathrm{H}_n(Y) \longrightarrow \mathrm{H}_n(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \to$$

$$\text{(II)} \quad \to \mathrm{H}_n(A_1, A_1 \cap A_2) \to \mathrm{H}_n(Y, A_2) \to \mathrm{H}_n(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A_1, A_1 \cap A_2) \to$$

But using these long exact sequences we see that

$$\text{(2) holds} \underset{\text{long exact sequence (I)}}{\iff} \mathrm{H}_n(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)) = 0 \text{ for all } n \in \mathbb{N}_0 \underset{\text{long exact sequence (II)}}{\iff} \text{(3) holds.}$$

[1681] This definition should not be confused with the definition of a *triple* of topological spaces, which was defined as a triple (X, B, A) where X is a topological space and $A \subseteq B \subseteq X$ are subsets. In a triad we do not demand that the third set is contained in the second set.

■

The following proposition gives us a very long list of examples of excisive triads:

Proposition 158.20. (Excisive Triad-Proposition) Let (X, A_1, A_2) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:

(0) (a) $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.
(b) $A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$.
(c) A_1 and A_2 are both unions of path components of $A_1 \cup A_2$.

(1) (a) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$.
(b) A_1 and A_2 are both open subsets of $A_1 \cup A_2$.

(2) $A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2.[1682]

(3) X is a CW-complex and A_1 and A_2 are both subcomplexes.

(4) Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$ be good points.[1683] Then[1684]$((A, a_0) \vee (B, b_0), A, B)$ is an excisive triad.



Proof. Let (X, A_1, A_2) be a triad of topological spaces.

(0) (a) If $A_1 \subseteq A_2$, then $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) = \mathrm{C}_*(A_2) = \mathrm{C}_*(A_1 \cup A_2)$, so the triad is excisive. The same argument also applies if $A_2 \subseteq A_1$.
(b) This case is just a special case of (a).
(c) If A_1 and A_2 are both unions of path components of $A_1 \cup A_2$, then it follows from the Path-Component–C_*–Lemma 143.17 that $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) = \mathrm{C}_*(A_1 \cup A_2)$.

(1) (a) We suppose that $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$. By the Comfortable Cover Proposition 148.9 (which we proved in the context of the Excision Theorem 148.1) we know that the natural chain maps $\mathrm{C}_k^{\{A_1,A_2\}}(A_1 \cup A_2) \to \mathrm{C}_k(A_1 \cup A_2)$ induce isomorphisms of all homology groups. It thus follows from the Excisive Characterization Lemma 158.19 (2)⇒(1) that the triad (X, A_1, A_2) is excisive.
(b) The case that A_1 and A_2 are both open subsets of $A_1 \cup A_2$ is a special case of (a).

(2) Suppose that $A_1 \cup A_2$ is a compact topological manifold and that A_1 and A_2 are submanifolds such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2. The Manifold–Excision Theorem 148.3 implies that for every $n \in \mathbb{N}_0$ the inclusion-induced map

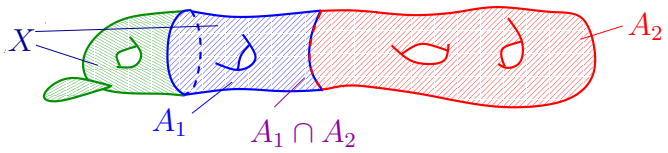
$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$

is an isomorphism. By the Excisive Characterization Lemma 158.19 (3)⇒(1) this implies that the triad (X, A_1, A_2) is excisive.

[1682] In Exercise 158.8 we will state a generalization of this statement.

[1683] Recall that according to Definition 99.8 we say that a point x in a topological space X is *good*, if $\{x\}$ is a closed subset and if there exists an open neighborhood U of x such that $\{x\}$ is a deformation retract of U.

[1683] As we discussed on page 1864, we can view X and Y as subspaces of the wedge.



(X, A_1, A_2) is excisive

(3) Suppose that X is a CW-complex and that A_1 and A_2 are both subcomplexes. The proof is identical to the proof of (2), except that we need to replace the Manifold–Excision Theorem 148.3 by the CW-Excision Theorem 148.4.

(4) We denote by z the point in $A \vee B$ that is given by identifying a_0 with b_0. For each $k \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_k(B, \{b_0\}) & \longrightarrow & \mathrm{H}_k(A, \{a_0\}) \oplus \mathrm{H}_k(B, \{b_0\}) & \longrightarrow & \mathrm{H}_k(A, \{a_0\}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_k(B, \{b_0\}) & \longrightarrow & \mathrm{H}_k(A \vee B, \{z\}) & \longrightarrow & \mathrm{H}_k(A \vee B, B) & \longrightarrow & 0 \end{array}$$

where all the maps are the natural maps. The top horizontal sequence is evidently exact. The bottom horizontal sequence is an excerpt from the long exact sequence of the triple $(\{b_0\} = \{z\}, B, A \vee B)$, see the LES-of-Triple Proposition 146.5. The excerpt shown is also exact since $(B, \{b_0\})$ is a retract of $(A \vee B, \{z\})$, which implies that for any $l \in \mathbb{N}_0$ the inclusion-induced map $\mathrm{H}_l(B, \{b_0\}) \to \mathrm{H}_l(A \vee B, \{z\})$ is a monomorphism.

The middle vertical map in the above diagram is an isomorphism by the Wedge-H_*-Proposition 149.16 and the left vertical map is the identity. It follows from the Five Lemma 145.11 that the right vertical map is an isomorphism. We obtain from the Excisive Characterization Lemma 158.19 (3)⇒(1) that $(A \vee B, A, B)$ is an excisive triad. ■

158.7. **Appendix: Abstractly isomorphic homology groups.** The following proposition shows that if we are given two chain complexes of free abelian groups, then any homomorphism on homology groups is induced by a chain map.

Proposition 158.21. (Lift-to-Chain Map-Proposition) Suppose that (C_*, ∂_*) and (C'_*, ∂'_*) are two chain complexes of free abelian groups.

(1) If $\gamma_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$, is a family of homomorphisms, then there exists a chain map $f \colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have

$$f_* = \gamma_n \colon \mathrm{H}_n(C_*) \ \to \ \mathrm{H}_n(C'_*).$$

(2) If the maps γ_n in (1) are isomorphisms, then the chain maps f in (1) are chain homotopy equivalences.

Remark.

(1) We consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle n \mapsto (0,n)} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(m,n) \mapsto m+2n]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

We view the top and bottom sequences as chain complexes C_* respectively D_*. Furthermore we view the vertical maps as a chain map f. We will see in Exercise 158.1 that f induces the zero map on homology. Evidently the zero maps from top to bottom are also a chain map that induce the zero map on homology. In Exercise 158.1 we will see that f that is not chain homotopic to the zero chain map. This shows that the chain map from the Lift-to-Chain Map-Proposition 158.21 is in general *not* unique up to chain homotopy.

(2) The combination of the Lift-to-Chain Map-Proposition 158.21 with the Singular-Cellular-H_*-Proposition 155.8 gives us a new proof of the Singular-Cellular Chain Complex Proposition 158.16 (2). □

Proof of the Lift-to-Chain Map-Proposition 158.21. Note that Statement (2) of the proposition is an immediate consequence of Statement (1), the fact that we are dealing with chain complexes of free abelian groups and the Quasi Iso–Homotopy Equivalence Proposition 158.8. Thus it remains to prove Statement (1).

Let (C_*, ∂_*) and (C'_*, ∂'_*) be two chain complexes of free abelian groups. Furthermore let $\gamma_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$ be a family of homomorphisms. We have to show that there exists a chain map $f \colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have

$$f_* = \gamma_n \colon \mathrm{H}_n(C_*) \ \to \ \mathrm{H}_n(C'_*).$$

For each $n \in \mathbb{N}_0$ we write

$$Z_n \coloneqq \ker(\partial_n \colon C_n \to C_{n-1}), \quad B_n \coloneqq \mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n) \quad \text{and} \quad H_n = Z_n/B_n.$$

Note that these groups form the following two types of short exact sequences

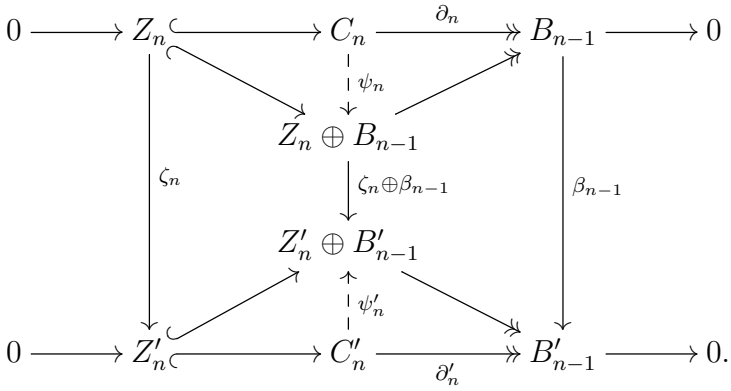
$$\text{(a)} \qquad 0 \ \to \ Z_n \ \hookrightarrow \ C_n \ \xrightarrow{\partial_n} \ B_{n-1} \ \to \ 0,$$
$$\text{(b)} \qquad 0 \ \to \ B_n \ \hookrightarrow \ Z_n \ \to \ H_n \ \to \ 0.$$

Similarly we define Z'_n, B'_n and H'_n for the chain complex C'_*. Evidently these groups also form short exact sequences. We denote these by (a′) and (b′).

Next note that it follows from the hypothesis that all the chain groups are free abelian and the Free Abelian Subgroup Theorem 97.14 that the subgroups Z_n, B_n, Z'_n and B'_n are also free abelian. In particular the sequences (b) and (b′) are free resolutions of H_n respectively H'_n. It follows from the Homological Algebra–Fundamental Theorem 158.4 that for each $n \in \mathbb{N}_0$ there exist homomorphisms β_n and ζ_n such that the following diagram commutes:

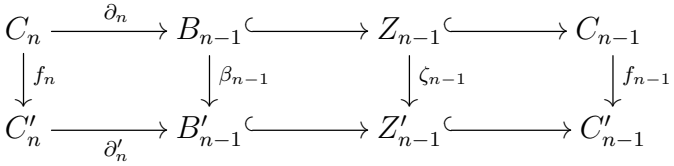
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \zeta_n} & & \downarrow{\scriptstyle \gamma_n} & & \\ 0 & \longrightarrow & B'_n & \longrightarrow & Z'_n & \longrightarrow & H'_n & \longrightarrow & 0. \end{array}$$

Our goal is to extend the maps $\zeta_n\colon Z_n \to Z'_n$ to a chain map $f\colon C_n \to C'_n$. To do so we consider the following diagram:



Here the top sequence is precisely the short exact sequence (a) and the bottom sequence is the short exact sequence (a′). Furthermore the unlabeled maps are the obvious inclusion and projections. Since B_{n-1} and B'_{n-1} are free abelian groups it follows immediately from the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 (1)⇒(3) that there exist isomorphisms $\psi_n\colon C_n \xrightarrow{\cong} Z_n \oplus B_{n-1}$ and $\psi'_n\colon C'_n \xrightarrow{\cong} Z'_n \oplus B'_{n-1}$ which make the diagram commute. Now we define $f_n\colon C_n \to C'_n$ to be the vertical homomorphism given by the diagram, i.e. $f_n = (\psi'_n)^{-1} \circ (\zeta_n \oplus \beta_{n-1}) \circ \psi_n$.

We need to show that the maps f_n form a chain map. We consider the diagram



where all unlabeled maps are the obvious inclusion maps. The middle square commutes by the choice of β_{n-1} and ζ_{n-1}. We had arranged the maps f_n, f_{n-1} in such a way that the squares to the left and right commute. This shows that the rectangle commutes. But that was exactly what we needed to do to show that the maps f_n, $n \in \mathbb{N}_0$ form a chain map.

Finally note that it follows easily from the choice of ζ_n that the chain map f_n induces the given maps γ_n on homology. ■

We conclude this chapter with the following lemma which we will use on several occasions.

Lemma 158.22. (Finite-Rank Chain Complex Lemma) Let (C_*, ∂_*) be a chain complex of free abelian groups such that for each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(C_*)$ is finitely generated. Then there exists a chain complex (D_*, ∂_*) with the following two properties:
(1) The chain complexes (C_*, ∂_*) and (D_*, ∂_*) are chain homotopy equivalent.
(2) Each chain group D_k is a free abelian group of *finite rank*.

Proof. We start out with the following basic claim.

Claim. Given any $k \in \mathbb{N}_0$ there exists a chain complex D^k, such that all chain groups are finitely generated free abelian groups, such that at most two chain groups are non-zero and such that

$$\mathrm{H}_i(D^k) \cong \begin{cases} \mathrm{H}_k(C_*), & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $k \in \mathbb{N}_0$. Recall that we assume that $\mathrm{H}_k(C_*)$ is finitely generated. By definition this means that there exists a finitely generated free abelian group F and an epimorphism $\varphi_k \colon F \to \mathrm{H}_k(C_*)$. By the Free Abelian Subgroup Theorem 97.14 we know that $\ker(\varphi_k)$ is a finitely generated free abelian group. Clearly the chain complex

$$D_*^k \coloneqq \quad \ldots \to 0 \to \underset{\substack{\uparrow \\ \text{degree } k+1}}{\ker(\varphi_k)} \hookrightarrow \underset{\substack{\uparrow \\ \text{degree } k}}{F} \to 0 \to \ldots$$

has the desired properties. ⊞

We now consider the direct sum of the chain complexes D^k, i.e. we set $D_* \coloneqq \bigoplus_{k \in \mathbb{N}_0} D_*^k$. Evidently each chain group is a finitely generated free abelian group.

Note that for each $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(D_*) \quad = \quad \mathrm{H}_k\Big(\bigoplus_{j \in \mathbb{N}_0} D_*^j\Big) \quad \underset{\substack{\uparrow \\ \text{by the Direct} \\ \text{Sum-H}_*\text{-Lemma 143.7}}}{=} \quad \bigoplus_{j \in \mathbb{N}_0} \mathrm{H}_k(D_*^j) \quad \underset{\substack{\uparrow \\ \text{by construction}}}{\cong} \quad \mathrm{H}_k(C_*).$$

Finally note that it follows immediately from the Lift-to-Chain Map-Proposition 158.21 (1) and (2) that the chain complex D_* is chain homotopy equivalent to (C_*, ∂_*). ■

158.8. Exercises for Chapter 158 .

Exercise 158.1.

(a) We consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} & & \\ & & {\scriptstyle n \mapsto (0,n)} & & {\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(m,n) \mapsto m+2n]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f.

(i) Convince yourself that $f \colon C_* \to D_*$ induces the zero map on homology.

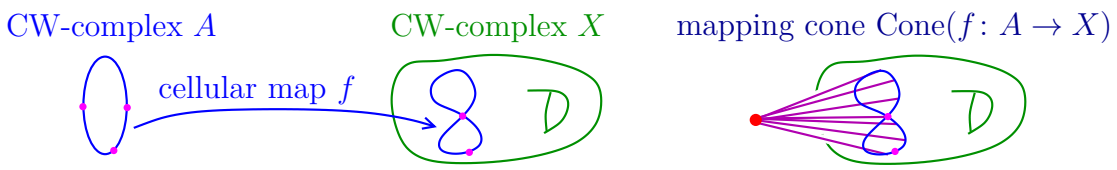
(ii) Show that the chain map $f \colon C_* \to D_*$ is not chain homotopic to the zero chain map.

(b) Give examples of chain complexes C_* and D_* of free abelian groups and quasi-isomorphisms $f_*, g_* \colon C_* \to D_*$ that induce the same isomorphism on homology groups, but that are not chain homotopic.

Exercise 158.2. Let $f \colon A \to X$ be a continuous map between topological spaces. We define the corresponding *mapping cone* as

$$\mathrm{Cone}(f \colon A \to X) \coloneqq ((A \times [0,1]) \sqcup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a, b \in A$. Now suppose that A and X are finite CW-complexes and that f is a cellular map.



We equip the mapping cone $\mathrm{Cone}(f\colon A \to X)$ with the natural CW-structure of the CW-Complex Mapping Cylinder+Cone Proposition 139.25. Show that there exists an isomorphism

$$\Theta\colon \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cone}(f\colon A \to X)) \;\to\; \underbrace{\mathrm{M}(f_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X))}_{\text{algebraic mapping cone}}$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n^{\mathrm{CW}}(X) & \longrightarrow & \mathrm{H}_*^{\mathrm{CW}}(\mathrm{Cone}(f\colon A \to X)) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}^{\mathrm{CW}}(A) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}_n^{\mathrm{CW}}(X) & \xrightarrow{i_*} & \mathrm{H}_n(\mathrm{M}(f_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X))) & \xrightarrow{p_*} & \mathrm{H}_{n-1}^{\mathrm{CW}}(A),
\end{array}$$

where the horizontal maps are given as in the Mapping Cone-$\widetilde{\mathrm{H}}_*$-Lemma 153.17 and the Mapping Cone-LES Lemma 158.11.

Exercise 158.3. We consider the chain complex

$$C_* = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and the zero chain complex D_*.

(a) Show that there exists a chain map $\varphi\colon C_* \to D_*$ which is a quasi-isomorphism.
(b) Show that the chain complexes C_* and D_* are not chain homotopy equivalent.

Remark. This exercise shows that in the Quasi Iso–Homotopy Equivalence Proposition 158.8 we cannot drop the hypothesis that we are dealing with chain complexes of free abelian groups.

Exercise 158.4. Let $0 \to A_* \xrightarrow{\varphi_*} B_* \xrightarrow{\psi_*} C_* \to 0$ be a short exact sequence of chain complexes. Show that the following exact sequences are isomorphic:

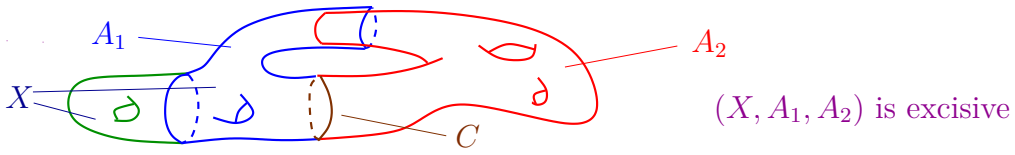
(1) The long exact sequence provided by the LES-Proposition 145.10.
(2) The long exact sequence of the Mapping Cone-LES Lemma 158.11 corresponding to the chain map $\varphi_*\colon A_* \to B_*$.
(3) The long exact sequence of the Mapping Cone-LES Lemma 158.11 corresponding to the chain map $\psi_*\colon B_* \to C_*$.

Exercise 158.5. Let C_* and D_* be two chain complexes and let $f\colon C_* \to D_*$ be a quasi-isomorphism. Is every element $d \in D_k$ homologous to an element of the form $f(c)$ with $c \in C_k$?

Exercise 158.6. Give an example of a chain map $\varphi\colon C_* \to D_*$ between chain complexes C_* and D_* consisting of finitely generated free abelian groups such that for some $n \in \mathbb{N}_0$ the natural maps $i_*\colon \mathrm{H}_n(D_*) \to \mathrm{H}_n(\mathrm{M}(\varphi))$ and $p_*\colon \mathrm{H}_n(\mathrm{M}(\varphi)) \to \mathrm{H}_{n-1}(C_*)$ from the Mapping Cone-LES Lemma 158.11 are both non-trivial.

Exercise 158.7. Show that in the Lift-to-Chain Map-Proposition 158.21 we cannot drop the hypothesis that the chain complexes are free abelian groups. More precisely, give an example of chain complexes (C_*, ∂_*) and (C'_*, ∂'_*) and a sequence of homomorphisms $\gamma_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$, that is not realized by a chain map.

Exercise 158.8. Let (X, A_1, A_2) be a triad of topological spaces. We suppose that $A_1 \cup A_2$ is a topological manifold and that A_1 and A_2 are codimension-zero submanifolds which are both closed subsets. We also suppose that $A_1 \cup A_2 = (A_1 \setminus \partial A_1) \cup (A_2 \setminus \partial A_2) \cup C$ where each component of C is a boundary component of A_1 and a boundary component of A_2. Show that (X, A_1, A_2) is an excisive triad.
Remark. This exercise is a generalization of the Excisive Triad-Proposition 158.20 (2).



Exercise 158.9. Show that given any excisive triad (X, A, B) with $X = A \cup B$ there exists a natural homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ such that the following sequence

$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$

is exact, where the various maps denoted by i are the obvious inclusion-induced maps.
Remark. By the Excisive Triad-Proposition 158.20 the above statement can be viewed as a mild generalization of the Mayer–Vietoris Theorem 153.2.

CHAPTER 159

Algebraic interlude: The tensor product of abelian groups

In this purely algebraic chapter we introduce and study the tensor product of two abelian groups. It follows from the fact that an abelian group is essentially the same as a $\mathbb{Z}$-module that the tensor product of two abelian groups is essentially the same as the tensor product of two $\mathbb{Z}$-modules. Thus if the reader is familiar with the tensor product of abelian groups or of modules over a commutative ring, then the reader might as well move straight to the next chapter.

The reader might also recall that in Section 81.3 we introduced the tensor product of two vector spaces. The theory of tensor products of abelian groups is similar, but more tricky, since abelian groups are more complicated than vector spaces.

This chapter is organized as follows:

- In Section 159.1 we will define the tensor product of two abelian groups.
- In Section 159.2 we will consider homomorphisms between tensor products.
- In Section 159.3 we will perform some basic calculations of tensor products.
- In Section 159.4 we will prove several formal properties of the tensor product.
- In Section 159.5 we will see that tensoring with a fixed group is a left adjoint functor, in particular it commutes with colimits.
- In Section 159.6 we will study the special case that we tensor with a commutative ring.
- In Section 159.7 we will provide exercises for Chapter 159.

In the next Chapter 160 we will use tensor products to introduce homology with coefficients.

159.1. **The definition of the tensor product of abelian groups.** We start out by recalling Definition 2.24:

Definition 159.1. Let A, B and C be abelian groups. We say that a map $f\colon A \times B \to C$ is bilinear, if if for all $a, a' \in A$, $b, b' \in B$ we have

$$f(a+a', b) \;=\; f(a,b) + f(a',b) \quad \text{and} \quad f(a, b+b') \;=\; f(a,b) + f(a,b').$$

Example. Let A, B and C be abelian groups and let $f\colon A \times B \to C$ be a bilinear map. As on page 3051 we see that for all $a \in A$, $b \in B$ and $m \in \mathbb{Z}$ we have the equality[1684]

$$f(m \cdot a, b) \;=\; f(a, m \cdot b) \;=\; m \cdot f(a,b).$$

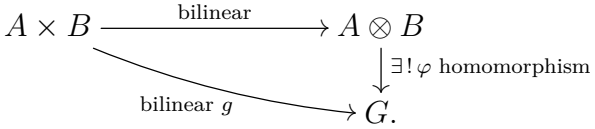
[1684]Here is a proof: We first consider the case that $m \geq 0$. We have the following equalities:

$$f((m \cdot a), b) \;=\; f\Big(\sum_{i=1}^{m} a, b\Big) \;\underset{\text{since } f \text{ is bilinear}}{\uparrow}\; \sum_{i=1}^{m} f(a,b) \;=\; m \cdot f(a,b) \;\underset{\text{same argument backwards}}{\uparrow}\; \ldots \;=\; f(a, (m \cdot b)).$$

Using the bilinearity one can also easily show that $(-a) \otimes b = a \otimes (-b) = -a \otimes b$. Using this observation we can easily reduce the case $m < 0$ to the case $m > 0$, which we just dealt with.

We will use this basic fact frequently. □

Our next goal is the following: Given abelian groups A and B we want to find "the universal bilinear map out of $A \times B$". More precisely, we want to find an abelian group, let's call it $A \otimes B$, together with a bilinear map $A \times B \to A \otimes B$ such that given any abelian group G and given any bilinear map $g\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ of abelian groups such that the following diagram commutes:



The following definition is the key step to providing such an object:

Definition 159.2. Let A and B be abelian groups.

(1) We consider the free abelian group $\mathbb{Z}^{(A\times B)}$ that is generated by the set $A \times B$. We define the tensor product $A \otimes B$ to be the following abelian group:

$$A \otimes B \;:=\; \mathbb{Z}^{(A\times B)}/N(A,B)$$

where $N(A,B)$ is the subgroup of $\mathbb{Z}^{(A\times B)}$ generated by the elements of the form

$$((a+a'),b) - ((a,b)+(a',b)) \qquad \text{and} \qquad (a,(b+b')) - ((a,b)+(a,b'))$$

with $a, a' \in A$ and $b, b' \in B$.

(2) For $a \in A$ and $b \in B$ we write

$$a \otimes b \;:=\; [(a,b)] \quad \in\; \mathbb{Z}^{(A\times B)}/N(A,B) =: A \otimes B.$$

Such an element of $A \otimes B$ is sometimes called an elementary tensor.

Lemma 159.3. (Tensor Product–Basics Lemma) Let A and B be abelian groups.

(1) The tensor product $A \otimes B$ is an abelian group.

(2) Any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$.[1685]

Proof.

(1) This statement follows from the fact that $A \otimes B$ is defined as the quotient of the free abelian group $\mathbb{Z}^{(A\times B)}$ by a (necessarily normal) subgroup $N(A,B)$.

(2) It follows immediately from the definition of the tensor product that any element in the tensor product $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i)$, with $r_1, \ldots, r_n \in \mathbb{Z}$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$. Thus it remains to prove the following claim:

Claim. For any $a \in A$, $b \in B$ and $m \in \mathbb{Z}$ we have $m \cdot (a \otimes b) = (m \cdot a) \otimes b \in A \otimes B$.

Proof. Note that, essentially by definition of $N(A,B)$, the map $A \times B \to A \otimes B$ that is given by $(a,b) \mapsto a \otimes b$ is bilinear. The claim now follows from the discussion on page 3051. ■

The following lemma shows that the above $A \otimes B$ is indeed the object of our wildest dreams:

[1685] Note that this description is in general by no means unique.

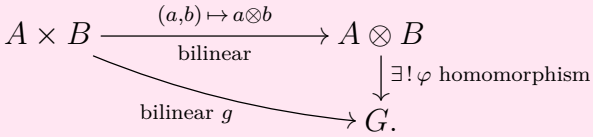
Lemma 159.4. (Tensor Product–Universal Property Lemma) Let A and B be abelian groups.

(1) The map
$$\begin{array}{rcl} f\colon A\times B & \to & A\otimes B \\ (a,b) & \mapsto & a\otimes b \end{array}$$
is bilinear. In particular the following statements hold:

(a) For any $a,a'\in A$ and any $b,b'\in B$ we have
$$\begin{array}{rcl} (a+a')\otimes b & = & a\otimes b+a'\otimes b \ \in \ A\otimes B \\ a\otimes(b+b') & = & a\otimes b+a\otimes b' \ \in \ A\otimes B. \end{array}$$

(b) For any $a\in A, b\in B$ and any $m\in\mathbb{Z}$ we have
$$(m\cdot a)\otimes b \ = \ a\otimes(m\cdot b) \ = \ m\cdot(a\otimes b).$$

(2) Given any abelian group G and given any bilinear map $g\colon A\times B\to G$ there exists a unique homomorphism $\varphi\colon A\otimes B\to G$ such that the following diagram commutes:



In fact this homomorphism is given by[1686]
$$\begin{array}{rcl} \varphi\colon A\otimes B & \to & G \\ \sum\limits_{i=1}^{n} a_i\otimes b_i & \mapsto & \sum\limits_{i=1}^{n} g(a_i,b_i). \end{array}$$

Remark.

(1) The tensor product of two abelian groups was first introduced by the topologist Hassler Whitney [**Why1934**, p. 497]. The characterization through a universal property was first given by Nicolas Bourbaki in 1942 [**Bou2007**, Chapitre II.3.1].

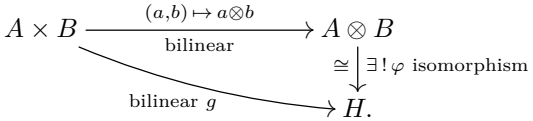
(2) For the most part we will not work with the actual definition of the tensor product $A\otimes B=\mathbb{Z}^{(A\times B)}/N(A,B)$. Instead we will mostly work with the universal property formulated in the Tensor Product–Universal Property Lemma 159.4 (2).

(3) In Section 312.2 we will introduce the more general notion of a tensor product of modules over rings. □

Proof. Let A and B be two abelian groups.

(1) We defined $N(A,B)$ exactly in such a way to ensure that the map
$$\begin{array}{rcl} f\colon A\times B & \to & A\otimes B:=\mathbb{Z}^{(A\times B)}/N(A,B) \\ (a,b) & \mapsto & a\otimes b \end{array}$$

[1686]This universal property gives us the usual uniqueness property: Let H be an abelian group and let $g\colon A\times B\to H$ be a bilinear map. If this bilinear map satisfies the universal property of (2), then the standard argument (as in the proof of the Product Uniqueness Lemma 6.2) shows that there exists a unique *isomorphism* $\varphi\colon A\otimes B\to H$ such that the following diagram commutes:



is bilinear. Statement (a) follows immediately from the definition of bilinear and Statement (b) follows from the discussion on page 3051.

(2) Let G be an abelian group and let $g\colon A \times B \to G$ be a bilinear map.

Claim. The map
$$\begin{array}{rcl} A \otimes B & \to & G \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i) \end{array}$$
is a well-defined homomorphism and it is the unique homomorphism $\varphi\colon A \otimes B \to G$ such that the following diagram commutes:
$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b)\mapsto a\otimes b} & A \otimes B \\ & \searrow^{g} & \downarrow \varphi \\ & & G. \end{array}$$

Proof. First note that it follows from the fact that $A \times B$ is a basis for the free abelian group $\mathbb{Z}^{(A\times B)}$ that there exists a *unique* homomorphism
$$\mathbb{Z}^{(A\times B)} \to G$$
which sends each $(a, b) \in A \times B$ to $g(a, b)$. Since g is bilinear we see that the subgroup $N(A, B)$ lies in the kernel of the above homomorphism. It follows from the Quotient Group Lemma 2.4 that the above homomorphism descends to a well-defined homomorphism
$$\begin{array}{rcl} \varphi\colon A \otimes B \coloneqq \mathbb{Z}^{(A\times B)}/N(A, B) & \to & G \\ \left[\sum_{i=1}^{n} (a_i, b_i)\right] = \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i). \end{array}$$
It is clear that the following diagram commutes:
$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b)\mapsto a\otimes b} & A \otimes B \\ & \searrow^{g} & \downarrow \varphi \\ & & G. \end{array}$$
Finally note that it follows easily from the fact that $\mathbb{Z}^{(A\times B)} \to A \otimes B$ is an epimorphism that φ is uniquely determined. ∎

To practice the definitions and to see the Tensor Product–Universal Property Lemma 159.4 in action we will now perform a first non-trivial calculation of a tensor product:

Lemma 159.5. (Cyclic-Tensor-Cyclic Lemma) Let $m, n \in \mathbb{N}$. The map
$$\begin{array}{rcl} \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Z}/\gcd(m, n)\mathbb{Z} \\ \sum_{i=1}^{d} [a_i] \otimes [b_i] & \mapsto & \left[\sum_{i=1}^{d} a_i \cdot b_i\right] \end{array}$$
is well-defined and it is an isomorphism of abelian groups.

Proof. Using the fact that $\gcd(m, n)$ divides m and n one can easily verify that the map
$$\begin{array}{rcl} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Z}/\gcd(m, n)\mathbb{Z} \\ ([a], [b]) & \mapsto & [a \cdot b] \end{array}$$

is a well-defined bilinear map. Thus it follows from the Tensor Product–Universal Property Lemma 159.4 (2) that the map given in the lemma is a well-defined homomorphism of abelian groups. By the Tensor Product–Universal Property Lemma 159.4 (2) we know that the tensor product is defined by a universal property. It thus follows from the discussion in Footnote 1686 that it now suffices to prove the following claim:

Claim. Given any abelian group G (which we write additively) and given any bilinear map $g\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to G$ there exists a unique homomorphism $\varphi\colon \mathbb{Z}/\gcd(m,n)\mathbb{Z} \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \xrightarrow[\text{bilinear}]{([a],[b]) \mapsto [a\cdot b]} & \mathbb{Z}/\gcd(m,n)\mathbb{Z} \\ & \searrow^{\text{bilinear } g} & \downarrow \exists!\, \varphi \text{ homomorphism} \\ & & G. \end{array}$$

Proof. Let $g\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to G$ be a bilinear map. We consider the map

$$\begin{array}{rcl} \varphi\colon \mathbb{Z}/\gcd(m,n)\mathbb{Z} & \to & G \\ [d] & \mapsto & d \cdot g([1],[1]). \end{array}$$

First we need to show that this map is well-defined. So let us consider $d + r \cdot \gcd(m,n)$ for some $r \in \mathbb{Z}$. By elementary number theory we know that there exist $x, y \in \mathbb{Z}$ with $x \cdot m + y \cdot n = \gcd(m,n)$. We now see that we have the following equalities in G:

$$\begin{array}{rcl} (d + r \cdot \gcd(m,n)) \cdot g([1],[1]) & = & (d + rx \cdot m + ry \cdot n) \cdot g([1],[1]) \\ & = & d \cdot g([1],[1]) + rx \cdot m \cdot g([1],[1]) + ry \cdot n \cdot g([1],[1]) \\ & = & d \cdot g([1],[1]) + rx \cdot g(\underbrace{[m \cdot 1]}_{=0 \in \mathbb{Z}/m\mathbb{Z}}, [1]) + ry \cdot g([1], \underbrace{[n \cdot 1]}_{=0 \in \mathbb{Z}/n\mathbb{Z}}) \\ & & \uparrow \text{ since } g \text{ is bilinear} \\ & = & d \cdot g([1],[1]). \end{array}$$

This shows that $\varphi\colon \mathbb{Z}/\gcd(m,n)\mathbb{Z} \to G$ is well-defined. Next note that it follows immediately from the definition of $\varphi([d]) := d \cdot g([1],[1])$ that φ is a homomorphism. Using the bilinearity of g one can also easily verify that φ makes the diagram commute. Finally, since the given map $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/\gcd(m,n)\mathbb{Z}$ is an epimorphism it is also clear that φ is uniquely determined. ∎

159.2. **Homomorphisms between tensor products.** In this section we will study homomorphisms between tensor products. Let us start out with the following definition:

Definition 159.6. Let $f\colon A \to A'$ and $g\colon B \to B'$ be homomorphisms between abelian groups. It follows from the Tensor Product–Universal Property Lemma 159.4 (1) that the map

$$\begin{array}{rcl} A \times B & \to & A' \otimes B' \\ (a,b) & \mapsto & f(a) \otimes g(b) \end{array}$$

is bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 159.4 (2) that the map

$$\begin{array}{rcl} f \otimes g\colon A \otimes B & \to & A' \otimes B' \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} f(a_i) \otimes g(b_i) \end{array}$$

is a well-defined homomorphism. We refer to $f \otimes g$ as the tensor product of the homomorphisms f and g.

The following lemma summarizes three key properties of the tensor product of two homomorphisms.

Lemma 159.7. (Tensor–Map Composition Lemma)
(1) Given any abelian groups A and B we have $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B}$.
(2) Let $f\colon A \to A'$, $f'\colon A' \to A''$, $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms between abelian groups. Then we have
$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon A \otimes B \;\to\; A'' \otimes B''.$$
(3) Let $f, g\colon A \to B$ be homomorphisms between abelian groups and let G be an abelian group. Then
$$(f+g) \otimes \mathrm{id}_G \;=\; f \otimes \mathrm{id}_G + g \otimes \mathrm{id}_G\colon A \otimes G \;\to\; B \otimes G.$$
The same statement also holds if we swap the roles of the two sides.

Proof. All the statements follow immediately from the above explicit description of the tensor product of homomorphisms. Alternatively, as we will see in Exercise 159.2, one can also deduce the statements from the universal property stated in the Tensor Product–Universal Property Lemma 159.4 (2). ∎

Subconsciously we will use the following lemma frequently:

Lemma 159.8. (Tensor–Functor Lemma) Given any abelian group G the maps
$$\begin{array}{rcl} A & \mapsto & A \otimes G \\ (f\colon A \to B) & \mapsto & (f \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \end{array}$$
define a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself. Of course an analogous statement holds if we tensor with G from the left.

Proof. This lemma follows immediately from the Tensor–Map Composition Lemma 159.7. ∎

159.3. **Calculations of tensor products.** The next lemma (which is an analog of the Tensor-with-$\mathbb{F}$–Lemma 81.18) gives us an important, albeit slightly dull, calculation of a tensor product.

Lemma 159.9. (Tensor-with-$\mathbb{Z}$–Lemma) Let A be an abelian group. The maps
$$\begin{array}{rcl} \Theta\colon \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^m n_i \otimes a_i & \mapsto & \sum_{i=1}^m n_i \cdot a_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Omega\colon A & \to & \mathbb{Z} \otimes A \\ a & \mapsto & 1 \otimes a \end{array}$$
are well-defined homomorphisms, they are natural and they are inverses to one another. In particular both are natural isomorphisms. Here "natural" means that for any homomorphism $\varphi\colon A \to B$ of abelian groups the following diagrams commute:
$$\begin{array}{ccc} \mathbb{Z} \otimes A & \xrightarrow[\cong]{\Theta} & A \\ {\scriptstyle \mathrm{id}_{\mathbb{Z}} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{Z} \otimes B & \xrightarrow[\cong]{\Theta} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow[\cong]{a \mapsto 1\otimes a} & \mathbb{Z} \otimes A \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}} \otimes \varphi} \\ B & \xrightarrow[\cong]{b \mapsto 1\otimes b} & \mathbb{Z} \otimes B. \end{array}$$
The analogous statements also hold if the $\mathbb{Z}$-term is on the right instead of the left.

Proof. To practice working with tensor products we provide two slightly different proofs:

(1) Note that the map

$$\begin{array}{rcl} \mathbb{Z} \times A & \to & A \\ (n, a) & \mapsto & n \cdot a \end{array}$$

is bilinear. It thus follows from the Tensor Product–Universal Property Lemma 159.4 (2) that the given map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i \cdot a_i \end{array}$$

is a well-defined homomorphism. It follows easily from the definitions that this map is natural.

It is clear that $\Omega\colon A \to \mathbb{Z} \otimes A$ is a well-defined natural homomorphism. Furthermore it is clear that $\Theta \circ \Omega\colon A \to A$ is the identity. It remains to prove that $\Omega \circ \Theta\colon \mathbb{Z} \otimes A \to \mathbb{Z} \otimes A$ is the identity. This follows easily from the definitions together with the following claim:

Claim. Any element in $\mathbb{Z} \otimes A$ is represented by an element of the form $1 \otimes a$.

Proof. By the Tensor Product–Basics Lemma 159.3 (2) any element of $\mathbb{Z} \otimes A$ is of the form $\sum_{i=1}^{m} n_i \otimes a_i \in \mathbb{Z} \otimes A$. We see that

$$\sum_{i=1}^{m} n_i \otimes a_i \;=\; \sum_{i=1}^{m} (n_i \cdot 1) \otimes a_i \;\underset{\uparrow}{=}\; \sum_{i=1}^{m} 1 \otimes (n_i \cdot a_i) \;\underset{\uparrow}{=}\; 1 \otimes \Big(\sum_{i=1}^{m} (n_i \cdot a_i) \Big).$$

by the Tensor Product Universal Lemma 159.4 (1b) — by the Tensor Product Universal Lemma 159.4 (1a)

The argument if the $\mathbb{Z}$-term is on the right is evidently almost identical.[1687] ⊞

(2) Our second proof works directly with the universal property of tensor products. We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} \mathbb{Z} \times A & \to & A \\ (r, b) & \mapsto & r \cdot b \end{array}$$

has the defining property from the Tensor Product–Universal Property Lemma 159.4 (2).

Proof. The map is clearly bilinear. Given any bilinear homomorphism $g\colon \mathbb{Z} \times A \to G$ we consider the homomorphism $\varphi\colon A \to G$ that is given by $\varphi(a) = g(1, a)$. In summary we have the following diagram:

$$\begin{array}{ccc} \mathbb{Z} \times A & \xrightarrow{(n,a) \mapsto n \cdot a} & A \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi(a) := g(1,a)} \\ & & G. \end{array}$$

We now see that the map g satisfies all conditions and that it is unique. ⊞

The desired statement now follows from the claim and the uniqueness statement for tensor products from Footnote 1686. ■

We move on to a slightly more interesting calculation of a tensor product.

[1687] Alternatively one could use the elementary Tensor Commutative-Associative Lemma 159.11 (1) below.

Lemma 159.10. (Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma) Let $k \in \mathbb{N}_0$ and let A be an abelian group. We write

$$k \cdot A \coloneqq \{k \cdot a \,|\, a \in A\}.$$

The maps

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}/k\mathbb{Z} \otimes A & \to & A/k \cdot A \\ \sum_{i=1}^{m} [n_i] \otimes a_i & \mapsto & \left[\sum_{i=1}^{m} n_i \cdot a_i\right] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Omega\colon A/k \cdot A & \to & \mathbb{Z}/k\mathbb{Z} \otimes A \\ [a] & \mapsto & [1] \otimes a \end{array}$$

are well-defined homomorphisms, they are natural and they are inverses to one another. In particular both are natural isomorphisms. Here "natural" means that for any homomorphism $\varphi\colon A \to B$ of abelian groups the following diagrams commute:

$$\begin{array}{ccc} \mathbb{Z}/k\mathbb{Z} \otimes A & \xrightarrow[\cong]{\Theta} & A/k \cdot A \\ {\scriptstyle \mathrm{id}_{\mathbb{Z}/k\mathbb{Z}} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{Z}/k\mathbb{Z} \otimes B & \xrightarrow[\cong]{\Theta} & B/k \cdot B \end{array} \quad \text{and} \quad \begin{array}{ccc} A/k \cdot A & \xrightarrow[\cong]{a \mapsto [1] \otimes a} & \mathbb{Z}/k\mathbb{Z} \otimes A \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}/k\mathbb{Z}} \otimes \varphi} \\ B/k \cdot B & \xrightarrow[\cong]{b \mapsto [1] \otimes b} & \mathbb{Z}/k\mathbb{Z} \otimes B. \end{array}$$

The analogous statements also hold if the $\mathbb{Z}$-term is on the right instead of the left.

Proof. [16881689] First note that one can easily show that the map

$$\begin{array}{rcl} \mathbb{Z}/k\mathbb{Z} \times A & \to & A/k \cdot A \\ ([n], a) & \mapsto & [n \cdot a] \end{array}$$

is well-defined and bilinear. It thus follows from the Tensor Product–Universal Property Lemma 159.4 (2) that the given map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}/k\mathbb{Z} \otimes A & \to & A/k \cdot A \\ \sum_{i=1}^{m} [n_i] \otimes a_i & \mapsto & \left[\sum_{i=1}^{m} n_i \cdot a_i\right] \end{array}$$

is a well-defined homomorphism. It follows easily from the definitions that this map is natural.

Using the Tensor Product–Universal Property Lemma 159.4 (1) one can easily verify that the map $\Omega\colon A/k \cdot A \to \mathbb{Z}/k\mathbb{Z} \otimes A$ is a well-defined natural homomorphism. Furthermore it is clear that the map $\Theta \circ \Omega\colon A/k \cdot A \to A/k \cdot A$ is the identity. It remains to prove that the map $\Omega \circ \Theta\colon \mathbb{Z}/k\mathbb{Z} \otimes A \to \mathbb{Z}/k\mathbb{Z} \otimes A$ is the identity. This follows easily from the definitions together with the following claim:

Claim. Any element in $\mathbb{Z}/k\mathbb{Z} \otimes A$ is represented by an element of the form $[1] \otimes a$.

Proof. By the Tensor Product–Basics Lemma 159.3 (2) any element of $\mathbb{Z}/k\mathbb{Z} \otimes A$ is of the form $\sum_{i=1}^{m} [n_i] \otimes a_i \in \mathbb{Z}/k\mathbb{Z} \otimes A$. We see that

$$\sum_{i=1}^{m} [n_i] \otimes a_i = \sum_{i=1}^{m} (n_i \cdot [1]) \otimes a_i \underset{\uparrow}{=} \sum_{i=1}^{m} [1] \otimes (n_i \cdot a_i) \underset{\uparrow}{=} [1] \otimes \left(\sum_{i=1}^{m} (n_i \cdot a_i)\right).$$

by the Tensor Product Universal Lemma 159.4 (1b) — by the Tensor Product Universal Lemma 159.4 (1a) ⊞

[1688] The proof of the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 is very similar to the proof of the Tensor-with-$\mathbb{Z}$–Lemma 159.9. In fact the case $k = 0$ of the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 is essentially the content of the Tensor-with-$\mathbb{Z}$–Lemma 159.9.

[1689] On page 3095 we will give a different, more conceptual proof, of the lemma.

The argument if the $\mathbb{Z}/k\mathbb{Z}$-term is on the right is evidently almost identical.[1690] ■

159.4. **Formal properties of the tensor product.** We continue with general results on tensor products.

Lemma 159.11. (Tensor Commutative-Associative Lemma)

(1) For any abelian group A we have $A \otimes 0 \cong 0 \cong 0 \otimes A$.

(2) Given any two abelian groups A and B the map

$$\begin{array}{rcl} A \otimes B & \to & B \otimes A \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} b_i \otimes a_i \end{array}$$

is well-defined and it is a natural isomorphism of abelian groups. Here "natural" means that if $f\colon A \to A'$ and $g\colon B \to B'$ are two homomorphisms between abelian groups, then the following diagram commutes:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{a\otimes b \mapsto b\otimes a} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ A' \otimes B' & \xrightarrow{a'\otimes b' \mapsto b'\otimes a'} & B' \otimes A'. \end{array}$$

(3) Let A, B, C be abelian groups. The map[1691]

$$\begin{array}{rcl} (A \otimes B) \otimes C & \to & A \otimes (B \otimes C) \\ \sum_{j=1}^{n} (a_j \otimes b_j) \otimes c_j & \mapsto & \sum_{j=1}^{n} a_j \otimes (b_j \otimes c_j) \end{array}$$

is well-defined and it is a natural isomorphism of abelian groups.

Proof.

(1) Let A be an abelian group. We need to show that $A \otimes \{0\} \cong \{0\}$. As we pointed out in Footnote 1686 it remains to show that the group $\{0\}$ has the universal property from the Tensor Product–Universal Property Lemma 159.4. Let G be an abelian group and let $\varphi\colon A \times \{0\} \to G$ be a bilinear map. It follows from the discussion on page 3051 that for any $a \in A$ we have $\varphi(a, 0) = \varphi(a, 0 \cdot 0) = 0 \cdot \varphi(a, 0) = 0$. Thus the unique homomorphism $\{0\} \to G$ is the unique homomorphism that gives us the commutative diagram:

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & \{0\} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \exists !} \\ & & G. \end{array}$$

(2) First note that the maps

$$\begin{array}{rcl} A \times B & \to & B \otimes A \\ (a, b) & \mapsto & b \otimes a \end{array} \qquad \text{and} \qquad \begin{array}{rcl} B \times A & \to & A \otimes B \\ (b, a) & \mapsto & a \otimes b \end{array}$$

[1690] Alternatively one could use the elementary Tensor Commutative-Associative Lemma 159.11 (1) below.

[1691] It follows easily from the Tensor Product–Universal Property Lemma 159.4 (1) and (2) that every element of $(A \otimes B) \otimes C$ is of that form.

are bilinear. It follows from the Tensor Product–Universal Property Lemma 159.4 (2) that we get induced homomorphisms

$$\Phi\colon A\otimes B \;\to\; B\otimes A \qquad \text{and} \qquad \Psi\colon B\otimes A \;\to\; A\otimes B.$$

Since $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are the identity on the generating sets $\{b\otimes a \,|\, b\in B, a\in A\}$ of $B\otimes A$ and $\{a\otimes b \,|\, a\in A, b\in B\}$ of $A\otimes B$ we see that $\Phi\circ\Psi = \mathrm{id}_{B\otimes A}$ and $\Psi\circ\Phi = \mathrm{id}_{A\otimes B}$. This shows that Φ and Ψ are isomorphisms. It follows easily from the definitions that Φ and Ψ are natural.

(3) Let $\widetilde{c}\in C$ and let $\widetilde{a}\in A$. One can easily verify that the maps

$$\begin{array}{rcl} A\times B & \to & A\otimes(B\otimes C) \\ (a,b) & \mapsto & a\otimes(b\otimes\widetilde{c}) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} B\times C & \to & (A\otimes B)\otimes C \\ (b,c) & \mapsto & (\widetilde{a}\otimes b)\otimes c \end{array}$$

are bilinear. It follows from the Tensor Product–Universal Property Lemma 159.4 (2) that we get induced homomorphisms $A\otimes B\to A\otimes(B\otimes C)$ and $B\otimes C\to(A\otimes B)\otimes C$. Thus we have defined maps

$$\begin{array}{rcl} (A\otimes B)\times C & \to & A\otimes(B\otimes C) \\ \left(\sum_{i=1}^n a_i\otimes b_i, c\right) & \mapsto & \sum_{i=1}^n a_i\otimes(b_i\otimes c) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} A\times(B\otimes C) & \to & (A\otimes B)\otimes C \\ \left(a, \sum_{i=1}^n (b_i\otimes c_i)\right) & \mapsto & \sum_{i=1}^n (a\otimes b_i)\otimes c_i. \end{array}$$

Again one can easily verify that these maps are bilinear. Thus it follows again from the Tensor Product–Universal Property Lemma 159.4 (2) that we get induced homomorphisms

$$\Phi\colon (A\otimes B)\otimes C \;\to\; A\otimes(B\otimes C) \qquad \text{and} \qquad \Psi\colon A\otimes(B\otimes C) \;\to\; (A\otimes B)\otimes C.$$

It is clear that the homomorphisms $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are the identity on the generating sets $\{a\otimes(b\otimes c) \,|\, a\in A, b\in B, c\in C\}$ and $\{(a\otimes b)\otimes c) \,|\, a\in A, b\in B, c\in C\}$. We see that $\Phi\circ\Psi = \mathrm{id}_{A\otimes(B\otimes C)}$ and $\Psi\circ\Phi = \mathrm{id}_{(A\otimes B)\otimes C}$. This shows that Φ and Ψ are isomorphisms. Finally note that it follows easily from the definitions that Φ and Ψ are natural. ■

The Tensor Commutative-Associative Lemma 159.11 leads us to the following convention:

Convention 159.12. Let $A_1,\dots,A_k$ be abelian groups. It follows from the Tensor Commutative-Associative Lemma 159.11, together with the discussion on page 115, that the isomorphism type of the tensor product of $A_1,\dots,A_k$ does not depend on the choice of the parentheses. We use this observation as an excuse to write $A_1\otimes A_2\otimes\cdots\otimes A_k$, even though strictly speaking as a set this object does depend on the parentheses.

159.5. **The Hom-Tensor adjunction and colimits.** For reader's who are familiar with adjoint functors in the sense of Definition 6.33 we will put tensor products and the Hom-functor in this context:

Lemma 159.13. (Hom–Tensor Adjunction Lemma) Let $\mathcal{A}\mathit{bGroup}$ be the category of abelian groups and let A be an abelian group. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{A}\mathit{bGroup} & \to & \mathcal{A}\mathit{bGroup} \\ X & \mapsto & G\otimes X \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{A}\mathit{bGroup} & \to & \mathcal{A}\mathit{bGroup} \\ Y & \mapsto & \mathrm{Hom}(G,Y) \end{array}$$

from the Tensor–Functor Lemma 159.8 and the Fixed Domain-Covariant Functor Lemma 5.20. For abelian groups X and Y the maps[1692]

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{A}b\mathcal{G}roup}(\overbrace{G \otimes X}^{=F(X)}, Y) & \to & \mathrm{Mor}_{\mathcal{A}b\mathcal{G}roup}(X, \overbrace{\mathrm{Hom}(G,Y)}^{=G(Y)}) \\ (f\colon G \otimes X \to Y) & \mapsto & \left(\begin{array}{ccc} X & \to & \mathrm{Hom}(G,Y) \\ x & \mapsto & \left(\begin{array}{ccc} G & \to & Y \\ a & \mapsto & f(a \otimes x) \end{array} \right) \end{array} \right) \end{array}$$

form a bijection that is natural in X and that it also natural in Y. In other words, the tensor functor is a left adjoint of the Hom-functor in the sense of Definition 6.33.

Proof. Using the Tensor Product–Universal Property Lemma 159.4 (2) one can easily show that this map is a natural bijection. This shows that F and G form an adjoint pair. ■

For completely formal reasons we now obtain the following result:

Proposition 159.14. (Tensor Product–Colimit Proposition) Let G be an abelian group. The tensor functor

$$\begin{array}{rcl} F\colon \mathcal{A}b\mathcal{G}roup & \to & \mathcal{A}b\mathcal{G}roup \\ X & \mapsto & G \otimes X \end{array}$$

preserves all colimits. The same statement holds of course also if we tensor with G from the right.

Proof. This proposition is an immediate consequence of the Hom–Tensor Adjunction Lemma 159.13 together with the Left Adjoint–Colimit Proposition 6.35 ■

In the following we will consider two explicit examples for the Tensor Product–Colimit Proposition 159.14:

Lemma 159.15. (Tensor–Direct Sum Lemma) Let $\{A_i\}_{i\in I}$ be a family of abelian groups and let B be an abelian group. Given $j \in I$ let $\iota_j\colon A_j \to \bigoplus_{i\in I} A_i$ be the natural homomorphism. Note that it follows from the Coproduct-of-Abelian Groups Lemma 97.2 that the homomorphisms $\iota_j \otimes \mathrm{id}_B\colon A_j \otimes B \to \left(\bigoplus_{i\in I} A_i\right) \otimes B$ from Definition 159.6 give rise to a natural homomorphism

$$\begin{array}{rcl} \bigoplus_{i\in I} (A_i \otimes B) & \to & \left(\bigoplus_{i\in I} A_i\right) \otimes B \\ \sum_i a_i \otimes b & \mapsto & \left(\sum_i a_i\right) \otimes b \end{array}$$

This homomorphism is a natural isomorphism of abelian groups.

Sketch of proof. This lemma is an immediate consequence of the Tensor Product–Colimit Proposition 159.14 and the observation from Definition 6.5 that the coproduct is just a colimit. For the reader's convenience we will also give a down-to-earth proof of the lemma.

We need to show that the above homomorphism

$$\Phi\colon \bigoplus_{i\in I} (A_i \otimes B) \to \left(\bigoplus_{i\in I} A_i\right) \otimes B$$

has an inverse. Let $b \in B$. For each $j \in I$ we consider the homomorphism $A_j \to A_j \otimes B$ that is given by $a \mapsto a \otimes b$. We compose this homomorphism with the natural homomorphism

[1692] Recall that for abelian groups C and D we have by definition that $\mathrm{Mor}_{\mathcal{A}b\mathcal{G}roup}(C,D) = \mathrm{Hom}(C,D)$.

$A_j \otimes B \to \bigoplus_{i\in I}(A_i \otimes B)$ to obtain a homomorphism $A_j \to \bigoplus_{i\in I}(A_i \otimes B)$. By the Coproduct-of-Abelian Groups Lemma 97.2 we get an induced homomorphism

$$\bigoplus_{j\in I} A_j \;\to\; \bigoplus_{i\in I}(A_i \otimes B).$$

In summary we have now constructed a map

$$\left(\bigoplus_{j\in I} A_j\right) \times B \;\to\; \bigoplus_{i\in I}(A_i \otimes B).$$

One can easily verify that this map is bilinear. It follows from the Tensor Product–Universal Property Lemma 159.4 (2) that we get an induced homomorphism

$$\Psi\colon \left(\bigoplus_{j\in I} A_j\right) \otimes B \;\to\; \bigoplus_{i\in I}(A_i \otimes B).$$

Note that the two homomorphisms $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are both the identity on the generating set $\left\{a \otimes b \;\middle|\; a \in \bigcup_{j\in I} A_j \text{ and } b \in B\right\}$. Thus we see that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. This implies that Φ is indeed an isomorphism. Finally note that one can easily verify that Φ is natural. ■

The combination of the Cyclic-Tensor-Cyclic Lemma 159.5, the Tensor-with-$\mathbb{Z}$–Lemma 159.9, the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10, the Tensor Commutative-Associative Lemma 159.11 and the Tensor–Direct Sum Lemma 159.15 allows us to calculate most tensor products that arise naturally.

Examples.

(1) Let A be a finitely generated abelian group of rank r. By the Abelian Group-Classification Theorem 97.24 we have $A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}/a_i\mathbb{Z}$ where $a_1, \ldots, a_n$ are non-zero natural numbers. For any abelian group S we have

$$A \otimes S \cong \left(\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}/a_i\mathbb{Z}\right) \otimes S \;\underset{\uparrow}{\cong}\; (\mathbb{Z} \otimes S)^r \oplus \left(\bigoplus_{i=1}^n (\mathbb{Z}/a_i\mathbb{Z} \otimes S)\right) \;\underset{\uparrow}{\cong}\; S^r \oplus \bigoplus_{i=1}^n S/a_i \cdot S.$$

by the Tensor Commutative Associative Lemma 159.11 (2); by the Tensor-with-$\mathbb{Z}$–Lemma 159.9 and the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$ Lemma 159.10

In particular for $S = \mathbb{Q}, S = \mathbb{R}$ or $S = \mathbb{C}$ we have

$$A \otimes S \cong \left(\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}/a_i\mathbb{Z}\right) \otimes S \quad\underset{\uparrow}{\cong}\quad S^r \oplus \bigoplus_{i=1}^n S/a_i \cdot S \quad\underset{\uparrow}{=}\quad S^r.$$

by the above; since a_i are invertible in $S = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

(2) Recall that given a set W we denote by $\mathbb{Z}^{(W)}$ the free abelian group generated by W. Let A and B be two sets. We then have

$$\mathbb{Z}^{(A)} \otimes \mathbb{Z}^{(B)} \;=\; \left(\bigoplus_{a\in A} \mathbb{Z}\cdot a\right) \otimes \left(\bigoplus_{b\in B} \mathbb{Z}\cdot b\right) \;\underset{\downarrow}{\cong}\; \bigoplus_{\substack{a\in A\\ b\in B}} \mathbb{Z}\cdot a \otimes \mathbb{Z}\cdot b \;\underset{\downarrow}{\cong}\; \bigoplus_{\substack{a\in A\\ b\in B}} \mathbb{Z}\cdot(a,b) \;\cong\; \mathbb{Z}^{(A\times B)}.$$

follows from applying the Tensor Commutative-Associative Lemma 159.11 (2) twice; by the Tensor-with-$\mathbb{Z}$ Lemma 159.9

In summary we have shown that the tensor product of two free abelian groups on sets A and B is the free abelian group on the set $A \times B$. □

We conclude this section with the following lemma, which we will use sparingly.

Lemma 159.16. (Tensor–Direct Colimit Lemma) Let $(I, \leq)$ be a directed set (in the sense of Definition 104.1) and let $(\{A_i\}_{i\in I}, \{\varphi_{ji}\}_{i\leq j})$ be a direct system of abelian groups (in the sense of Definition 104.2). Furthermore let B be an abelian group. Given $i \in I$ we consider the natural homomorphism $\iota_i \colon A_i \to \varinjlim A_i$ and the natural homomorphisms $\iota_i \otimes \mathrm{id}_B \colon A_i \otimes B \to \left(\varinjlim A_i\right) \otimes B$. These induce a natural isomorphism[1693]

$$\varinjlim (A_i \otimes B) \;\to\; \left(\varinjlim A_i\right) \otimes B.$$

Sketch of proof. Again this lemma is an immediate consequence of the Tensor Product–Colimit Proposition 159.14. For the reader's convenience we will again give a down-to-earth proof.

It follows from the definition of a direct colimit and the Tensor–Map Composition Lemma 159.7 that for any $j \leq k$ we get the following commutative diagram

$$\begin{array}{ccc} A_k \otimes B & & \\ \uparrow{\scriptstyle \varphi_{kj}\otimes \mathrm{id}_B} & \searrow^{\iota_k\otimes \mathrm{id}_B} & \\ & & (\varinjlim A_i) \otimes B. \\ A_j \otimes B & \nearrow_{\iota_j\otimes \mathrm{id}_B} & \end{array}$$

Thus we obtain a homomorphism

$$\Phi \colon \varinjlim (A_i \otimes B) \;\to\; \left(\varinjlim A_i\right) \otimes B.$$

One can easily verify that this homomorphism is natural. It remains to show that it has an inverse. Let $b \in B$. For each $j \in I$ we consider the homomorphism $A_j \to A_j \otimes B$ that is given by $a \mapsto a \otimes b$ and we consider the natural homomorphisms $A_j \otimes B \to \varinjlim (A_i \otimes B)$ which are part of the definition of a direct colimit (see Definition 104.4). Note that for any $j \leq k$ we get the following diagram

$$\begin{array}{ccccc} A_k & \xrightarrow{a\mapsto a\otimes b} & A_k \otimes B & & \\ \uparrow{\scriptstyle \varphi_{kj}} & & \uparrow{\scriptstyle \varphi_{kj}\otimes \mathrm{id}_B} & \searrow & \\ & & & & \varinjlim (A_i \otimes B). \\ A_j & \xrightarrow{a\mapsto a\otimes b} & A_j \otimes B & \nearrow & \end{array}$$

It is clear that the left part of the diagram commutes. The triangle to the right commutes by definition of a direct colimit. It follows from the observation that the above diagram commutes and the definition of a direct colimit that we get an induced homomorphism

$$\varinjlim A_i \;\to\; \varinjlim (A_i \otimes B).$$

In summary we have now constructed a map

$$\left(\varinjlim A_i\right) \times B \;\to\; \varinjlim (A_i \otimes B).$$

[1693] We refer to Definition 6.21 for the definition of a colimit of a direct system. Note that by the Algebraic–Direct Colimit Existence Proposition 104.7 we know that the direct colimit always exists in the category *AbGroup* of abelian groups.

One can easily verify that this map is bilinear. It follows from the Tensor Product–Universal Property Lemma 159.4 (2) that we obtain a homomorphism

$$\Psi\colon (\varinjlim A_i) \otimes B \;\to\; \varinjlim (A_i \otimes B).$$

Again one can show easily that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on generating sets. Thus Φ is indeed an isomorphism. ■

159.6. **Tensoring with commutative rings.** For this section it is convenient to introduce the following notation:

Notation 159.17. Let R be a commutative ring. We denote by $\mathcal{M}od_R$ the category of R-modules.

The following elementary lemma will be used frequently, without actually referring to it.

Lemma 159.18. (Tensor-with-Ring Lemma) Let R be a commutative ring.

(1) Given any abelian group A the map

$$\begin{array}{rcl} R \times (A \otimes R) & \to & A \otimes R \\ \left(\sum_{i=1}^n a_i \otimes r_i, s\right) & \mapsto & \sum_{i=1}^n a_i \otimes \underbrace{(s \cdot r_i)}_{\in R} \end{array}$$

is well-defined and it defines an R-module structure on $A \otimes R$.

(2) If $f\colon A \to B$ is a homomorphism of abelian groups, then the map

$$\begin{array}{rcl} f \otimes \mathrm{id}_R\colon A \otimes R & \to & B \otimes R \\ \sum_{i=1}^n a_i \otimes r_i & \mapsto & \sum_{i=1}^n f(a_i) \otimes r_i \end{array}$$

is a homomorphism of R-modules.

(3) The maps

$$\begin{array}{rcl} A & \mapsto & A \otimes R \\ (f\colon A \to B) & \mapsto & (f \otimes \mathrm{id}_R\colon A \otimes R \to B \otimes R) \end{array}$$

define a covariant functor from the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups to the category $\mathcal{M}od_R$ of R-modules.

Remark. The Tensor-with-Ring Lemma 159.18 shows that tensoring with a commutative ring "turns" an abelian group into a module. In particular if we tensor with a field, then we turn an abelian group into a vector space. This is very convenient since sometimes it allows us to reduce problems about abelian groups to problems about vector spaces that we can handle with techniques from linear algebra. For example in Appendix 161.4 we will use this idea to get a new perspective on the rank of abelian groups. □

Proof.

(1) Let $s \in R$. Using the fact that R is a *commutative* ring and using the Tensor Product–Universal Property Lemma 159.4 (1) one can easily verify that the map

$$\begin{array}{rcl} A \times R & \to & A \otimes R \\ (a, r) & \mapsto & a \otimes (s \cdot r) \end{array}$$

is bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 159.4 (2) that the map descends to a well-defined homomorphism $A \otimes R \to A \otimes R$. This

shows that the given map $R \times (A \otimes R) \to A \otimes R$ is well-defined. One can now easily verify that this defines an R-module structure on $A \otimes R$.

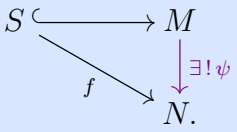
(2) This statement follows easily from the definitions.

(3) This statement also follows immediately from (2) together with the Tensor–Map Composition Lemma 159.7. ■

We continue with the following variation on the definition of a free abelian group (see Definition 97.5).

Definition 159.19. Let R be a commutative ring.

(1) We say that an R-module M is a free R-module on a subset $S \subseteq M$, if the following universal property holds: For any *map* $f\colon S \to N$ of the set S to an R-module N there exists a unique *homomorphism* $\psi\colon M \to N$ of R-modules that makes the following diagram commute:



We call S a basis of the R-module M.[1694] [1695]

(2) Given a set S we consider

$$R^{(S)} \;:=\; \text{all maps from } S \text{ to } R \text{ which are non-zero for only finitely many } s \in S$$

with the obvious R-module structure. The same argument as in the Free Abelian Group–Existence Lemma 97.8 shows that the R-module $R^{(S)}$ is a free R-module on the subset $S \subseteq R^{(S)}$

(3) We use Definition 97.7 to identify elements in $R^{(S)}$ with finite formal linear combinations $r_1 \cdot s_1 + \cdots + r_k \cdot s_k$ where $r_1, \ldots, r_k \in R$ and $s_1, \ldots, s_k \in S$.

Lemma 159.20. (Free Module–Tensor-with-Ring Lemma) Let R be a commutative ring.

(1) (a) Let F be a free abelian group with basis $\mathcal{B}$ in the sense of Definition 97.5. Then $F \otimes R$ is a free R-module with basis $\{b \otimes 1_R\}_{b \in \mathcal{B}}$ in the sense of Definition 159.19.

(b) Let I be a set. There exists a unique isomorphism

$$\begin{array}{rccc} \Theta_I\colon & \mathbb{Z}^{(I)} \otimes R & \xrightarrow{\cong} & R^{(I)} \\ & i \otimes 1_R & \mapsto & i \end{array}$$

of R-modules. This isomorphism is natural in R.

(2) Let A be an $(m \times n)$-matrix over $\mathbb{Z}$. We denote by $\varphi\colon \mathbb{Z} \to R$ the ring homomorphism that is given by $\varphi(k) = k \cdot 1_R$. With this notation the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}^n \otimes R & \xrightarrow[\cong]{\Theta_n} & R^n \\ \downarrow{\scriptstyle (v \mapsto A \cdot v) \otimes \mathrm{id}_R} & & \downarrow{\scriptstyle w \mapsto \varphi(A) \cdot w} \\ \mathbb{Z}^m \otimes R & \xrightarrow[\cong]{\Theta_m} & R^m. \end{array}$$

[1694] One can easily verify that this definition is equivalent to the low-tech Definition 2.9.

[1695] It follows easily from the definitions that a $\mathbb{Z}$-module is essentially the same as an abelian group.

[1695] Recall that in Footnote 976 we saw that the to the left and right we do indeed have the identity.

Example. Let $m, n \in \mathbb{N}_0$ such that $\mathbb{Z}^m \cong \mathbb{Z}^n$. We want to show that $m = n$. Indeed, we have

$$m = \dim_{\mathbb{Q}}(\mathbb{Q}^m) \underset{\text{follows from the Free Module–Tensor-with-Ring Lemma 159.20 (2)}}{=} \dim_{\mathbb{Q}}(\mathbb{Z}^m \otimes \mathbb{Q}) \overset{\text{by the Tensor-with-Ring Lemma 159.18 (3)}}{=} \dim_{\mathbb{Q}}(\mathbb{Z}^n \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathbb{Q}^n) = n.$$

□

Proof.

(1) (a) We need to verify the condition from Definition 159.19:

Claim. Let N be an R-module and let $f\colon \{b \otimes 1_R\}_{b\in\mathcal{B}} \to N$ be a map. There exists a unique homomorphism $\Psi\colon F \otimes R \to N$ of R-modules such that for all $b \in \mathcal{B}$ we have $\Psi(b \otimes 1_R) = f(b)$.

Proof. Since $\mathcal{B}$ is a basis of the free abelian group F we obtain from Definition 97.5 a unique homomorphism $\Phi\colon F \to N$ of abelian groups with $\Phi(b) = f(b \otimes 1_R)$ for all $b \in B$. Next we consider the following diagram:

$$\begin{array}{ccc} F \times R & \xrightarrow{(f,r)\mapsto f\otimes r} & F \otimes R \\ & \searrow^{(f,r)\mapsto \Phi(f)\cdot r} & \downarrow{\scriptstyle \exists! \Psi} \\ & & N. \end{array}$$

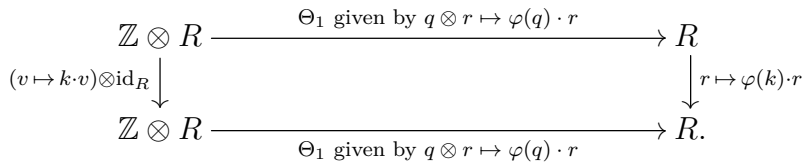
Clearly the diagonal map is bilinear. By the Tensor Product–Universal Property Lemma 159.4 we get a unique homomorphism $\Psi\colon F \otimes R \to N$ of abelian groups that makes the diagram commute. It follows easily from the uniqueness property that Ψ is actually R-linear. Finally note that for each $b \in \mathcal{B}$ we have by definition $\Psi(b \otimes 1_R) = \Phi(b) \cdot 1_R = \Phi(b)$. This gives us the existence statement.
It remains to show that Ψ is uniquely determined by the values on $\{b \otimes 1_R\}_{b\in\mathcal{B}}$. So suppose we have two homomorphisms $\Psi, \Psi'\colon F \otimes R \to N$ such that for all $b \in \mathcal{B}$ we have $\Psi(b \otimes 1_R) = \Psi'(b \otimes 1_R)$. We consider the homomorphisms $\Phi\colon F \to N$ and $\Phi'\colon F \to N$ that are given by the precomposition of Ψ and Ψ' by $f \mapsto f \otimes 1_R$. Note that Ψ and Ψ' are now the unique homomorphisms that make the above diagram commute for Φ and Φ'. Since $\mathcal{B}$ is a basis of F we see that $\Phi = \Phi'$. But then it follows from the above characterization of Ψ and Ψ' that $\Psi = \Psi'$. ⊞

(b) This statement follows easily from (a). Alternatively one can see that the given map is the composition of the following isomorphisms of R-modules:[1696]

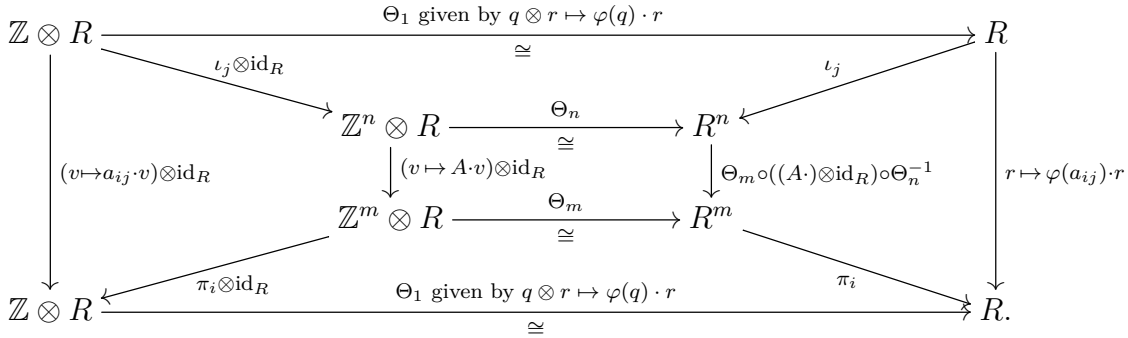
$$\mathbb{Z}^{(I)} \otimes R = \left(\bigoplus_{i\in I} \mathbb{Z}\right) \otimes R \underset{\text{isomorphism from the Tensor–Direct Sum Lemma 159.15}}{\overset{\cong}{\longleftarrow}} \bigoplus_{i\in I} (\mathbb{Z} \otimes R) \underset{\text{isomorphism from the Tensor-with-}\mathbb{Z}\text{ Lemma 159.9}}{\overset{\cong}{\longrightarrow}} \bigoplus_{i\in I} R = R^{(I)}.$$

(2) We start out with the following simple observation: It follows easily from the Tensor Product–Basics Lemma 159.3 (1b) that given any $k \in \mathbb{Z}$ the following diagram commutes:

[1696] Recall that in Footnote 976 we saw that to the left and right we do indeed have the identity.



Now let $A = (a_{ij}) \in \mathrm{M}(m \times n, \mathbb{Z})$. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. We denote by $\pi_i \colon \mathbb{Z}^m \to \mathbb{Z}$ the projection onto the i-th term and we denote by $\iota_j \colon \mathbb{Z} \to \mathbb{Z}^n$ the inclusion into the j-th coordinate. We consider the following diagram:



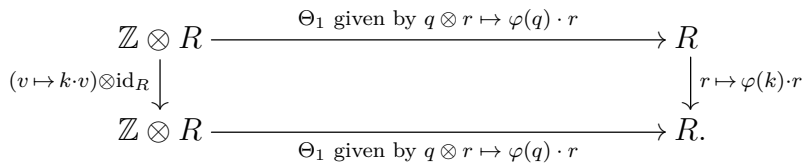
We make the following observations:

(a) The outer rectangle commutes by the discussion at the beginning of the proof.
(b) The small inner rectangle commutes by definition.
(c) The quadrilateral to the left commutes by the Tensor–Map Composition Lemma 159.7 (2).
(d) The quadrilaterals to the top and bottom commute by definition of Θ_m and Θ_n.

It follows from these observations and the fact that the horizontal maps are isomorphisms that the quadrilateral to the right commutes. But this implies precisely that the vertical map from R^n to R^m is given by multiplication by the matrix $\varphi(A)$. ■

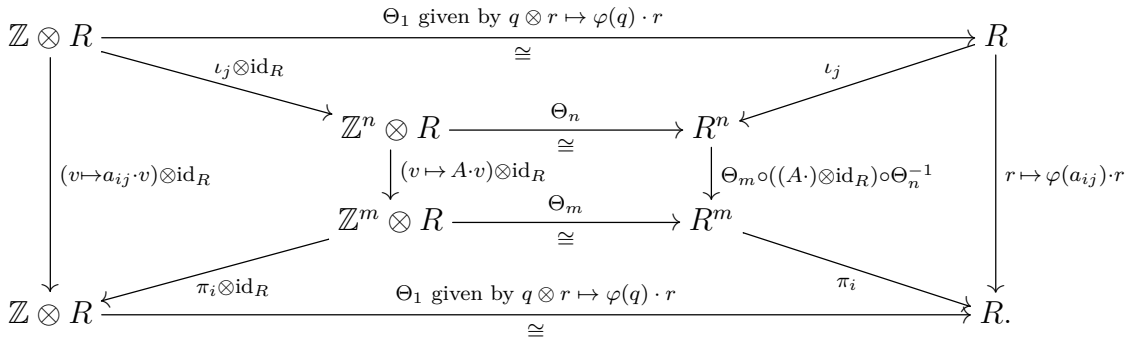
Proof.

(1) This statement follows almost immediately from the Tensor–Direct Sum Lemma 159.15 and the Tensor-with-$\mathbb{Z}$–Lemma 159.9.
(2) We start out with the following simple observation: It follows easily from the Tensor Product–Basics Lemma 159.3 (1b) that given any $k \in \mathbb{Z}$ the following diagram commutes:



Now let $A = (a_{ij}) \in \mathrm{M}(m \times n, \mathbb{Z})$. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. We denote by $\pi_i \colon \mathbb{Z}^m \to \mathbb{Z}$ the projection onto the i-th term and we denote by $\iota_j \colon \mathbb{Z} \to \mathbb{Z}^n$ the

inclusion into the j-th coordinate. We consider the following diagram:



We make the following observations:

(a) The outer rectangle commutes by the discussion at the beginning of the proof.

(b) The small inner rectangle commutes by definition.

(c) The quadrilateral to the left commutes by the Tensor–Map Composition Lemma 159.7 (2).

(d) The quadrilaterals to the top and bottom commute by definition of Θ_m and Θ_n.

It follows from these observations and the fact that the horizontal maps are isomorphisms that the quadrilateral to the right commutes. But this implies precisely that the vertical map from R^n to R^m is given by multiplication by the matrix $\varphi(A)$. ■

159.7. Exercises for Chapter 159 .

Exercise 159.1. Give an example of abelian groups A and B such that not every element of $A \otimes B$ can be written as $a \otimes b$ for some $a \in A$ and $b \in B$.

Exercise 159.2. Use the universal property formulated in the Tensor Product–Universal Property Lemma 159.4 (2) to prove the following statements:

(a) Given any abelian groups A and B we have $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B}$.

(b) Let $f\colon A \to A'$, $f'\colon A' \to A''$, $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms between abelian groups. Then we have

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon A \otimes B \;\to\; A'' \otimes B''.$$

(c) Let $f, g\colon A \to B$ be homomorphisms between abelian groups and let G be an abelian group. Then

$$(f + g) \otimes \mathrm{id}_G \;=\; f \otimes \mathrm{id}_G + g \otimes \mathrm{id}_G\colon A \otimes G \;\to\; B \otimes G.$$

Exercise 159.3. Show that if A is a finitely generated abelian group, then

$$\ker \begin{pmatrix} A & \to & A \otimes \mathbb{Q} \\ a & \mapsto & a \otimes 1 \end{pmatrix} \;=\; \text{all torsion elements of } A.$$

Hint. Make use of the classification of finitely generated abelian groups.

Remark. In Exercise 161.8 we will deal with the case that A is not necessarily finitely generated.

Exercise 159.4. Determine the isomorphism type of the tensor product of the two groups

$$\begin{aligned} A &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}, \\ B &= \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}. \end{aligned}$$

Exercise 159.5. We consider the abelian group $\mathbb{Q}/\mathbb{Z}$.

(a) Show that $\mathbb{Q}/\mathbb{Z}$ can be written as a direct colimit of finite abelian groups.
(b) For each $k \in \mathbb{N}$ determine $\mathbb{Z}/k\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}$.
(c) Is $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}$ non-trivial?

Exercise 159.6. Let G be an abelian group and let $\varphi\colon A \to B$ be a homomorphism between abelian groups. Which of the following two statements is correct:

(a) If $\varphi\colon A \to B$ is an epimorphism, then $\varphi \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G$ is also an epimorphism.
(b) If $\varphi\colon A \to B$ is a monomorphism, then $\varphi \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G$ is also a monomorphism.

Exercise 159.7. Show that for any three abelian groups A, B and G there exists a natural isomorphism

$$\mathrm{Hom}(A, \mathrm{Hom}(B, G)) \xrightarrow{\cong} \mathrm{Hom}(A \otimes B, G).$$

Exercise 159.8. Let R and S be two rings (not necessarily commutative).

(a) Show that there exists a unique bilinear map

$$\cdot\colon R \otimes S \times R \otimes S \to R \otimes S$$

such that for all $r, \widetilde{r} \in R$ and all $s, \widetilde{s} \in S$ we have the following equality in $R \otimes S$:

$$(r \otimes s) \cdot (\widetilde{r} \otimes \widetilde{s}) = r \cdot_R \widetilde{r} \otimes s \cdot_S \widetilde{s}.$$

(b) Show that this multiplication defines a ring structure on the abelian group $R \otimes S$.

Exercise 159.9. Let F be a free abelian group and let A be an abelian group. Show that there exists a natural monomorphism $\mathrm{Hom}(F, \mathbb{Z}) \otimes A \to \mathrm{Hom}(F, A)$.

Exercise 159.10. Classify all abelian groups G with $G \otimes \mathbb{Z}/9\mathbb{Z} \cong G$.

Exercise 159.11. Let A, A', B, B' be abelian groups. Is the map

$$\begin{aligned} \mathrm{Hom}(A, A') \times \mathrm{Hom}(B, B') &\to \mathrm{Hom}(A \otimes B, A' \otimes B') \\ (f, g) &\mapsto f \otimes g \end{aligned}$$

an isomorphism of abelian groups?

Exercise 159.12. We consider the abelian group $(\mathbb{Q}, +)$. By the Tensor-with-Ring Lemma 159.18 we know that $\mathbb{Q} \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space. What is $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes \mathbb{Q})$?

Exercise 159.13. Does there exist an abelian group G with $G \otimes G = 0$?

Exercise 159.14. Let G be an abelian group and let $n \in \mathbb{N}_0$. Show that if we have an inequality $\dim_{\mathbb{Q}}(G \otimes \mathbb{Q}) \geq n$, then there exists a monomorphism $\varphi\colon \mathbb{Z}^n \to G$.

Exercise 159.15. Let A be a finitely generated torsion-free abelian group. Show that the homomorphism $A \mapsto A \otimes \mathbb{Q}$ that is given by $a \mapsto a \otimes 1$ is injective.
Hint. Try to reduce the problem to the case that A is finitely generated.

Exercise 159.16. We denote by $\mu\colon \mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Z}$ the homomorphism that is induced by $\mu(a\otimes b)=a\cdot b$. Let A and B be abelian groups. We consider the homomorphism

$$\begin{array}{rccl}\Phi\colon & \operatorname{Hom}(A,\mathbb{Z})\otimes\operatorname{Hom}(B,\mathbb{Z}) & \to & \operatorname{Hom}(A\otimes B,\mathbb{Z})\\ & \alpha\otimes\beta & \mapsto & \mu\circ(\alpha\otimes\beta).\end{array}$$

(a) Show that in general this map is not an isomorphism.
(b) Show that the map is an isomorphism if A is a finitely generated free abelian group.

CHAPTER 160

Homology with coefficients

After the algebraic interlude from the last Chapter 159 we now return to topology. As we discussed in the beginning of Chapter 158, the functors we were so far mostly interested in are the covariant functor

$$(X, A) \ \mapsto \ (\mathrm{C}_*(X, A), \partial_*)$$

from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes and the covariant functors

$$(C_*, \partial_*) \ \to \ \mathrm{H}_k(C_*, \partial_*)$$

from the category *ChCplx* of chain complexes to the category *AbGroup* of abelian groups. The idea now is to "intersperse" a functor between these two functors.

In this chapter the functor of choice is the following: Given a fixed abelian group G we consider the functor $A \mapsto A \otimes G$ from the category *AbGroup* of abelian groups to itself. Combining this functor with the above two by now "classical" functors we will obtain the functor

$$(X, A) \ \mapsto \ \underbrace{\mathrm{H}_n(\mathrm{C}_*(X, A) \otimes G)}_{=:\mathrm{H}_n(X,A;G)}$$

from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups (or the category $\mathcal{M}od_G$ of G-modules.) We call $\mathrm{H}_n(X, A; G)$ the *n-th homology group of (X, A) with coefficients in an abelian group G* (or with coefficients in a commutative ring G). These homology groups will have many applications:

(1) If we take a "small" ring, like $G = \mathbb{F}_2$,[1697] then the homology groups with coefficients become much more computable.
(2) If we consider homology groups with coefficients in a field $\mathbb{F}$, then these homology groups are $\mathbb{F}$-vector spaces and thus easier to handle than abelian groups.
(3) Homology with coefficients will be essential in proving theoretical statements. For example we will need to work with $\mathbb{F}_2$-coefficients to prove the higher-dimensional analog of the 2D-Borsuk-Ulam Theorem 89.1.

This chapter is organized as follows:

- In Section 160.1 we will see what happens if we apply an additive functor to a chain complex.
- In Section 160.2 we will tensor chain complexes with a fixed abelian groups.
- In Section 160.3 we will introduce the homology groups $\mathrm{H}_k(X; G)$ of a topological space X with coefficients in an abelian group G.
- In Appendix 160.4 we will discuss the Eilenberg-Steenrod axioms for homology.
- In Section 160.5 we will provide exercises for Chapter 160.

[1697] Recall that according to the Notation 2.7 we denote for a prime p by $\mathbb{F}_p$ the field where the underlying abelian group is $\mathbb{Z}/p\mathbb{Z}$ and where the multiplication is the one coming from the usual multiplication on $\mathbb{Z}$.

In the next Chapter 161 we will introduce exact functors.

160.1. **Additive functors applied to chain complexes.** Our next goal is to figure out what properties on a self-functor of the category *AbGroup* of abelian groups we need to get a self-functor of the category *ChCplx* of chain complexes.

As topologists we also want our functors to preserve the property of being chain-homotopic. Some thought leads us to the following definition:

Definition 160.1. We say that a covariant functor $F\colon \mathit{AbGroup} \to \mathit{AbGroup}$ from the category *AbGroup* of abelian groups to itself is additive if for any two abelian groups A and B and any two homomorphisms $\varphi, \psi\colon A \to B$ we have $F(\varphi+\psi) = F(\varphi)+F(\psi)\colon F(A) \to F(B)$.

Lemma 160.2. (Additive Functor–Basics Lemma) Let $F\colon \mathit{AbGroup} \to \mathit{AbGroup}$ be a covariant functor from the category *AbGroup* of abelian groups to itself. If F is additive, then the following two statement holds:

(1) $F(\text{zero homomorphism}) = \text{zero homomorphism}$.
(2) $F(\text{zero group}) = \text{zero group}$.

Proof. Let $F\colon \mathit{AbGroup} \to \mathit{AbGroup}$ be an additive covariant functor.

(1) Let A, B be two abelian groups. Since F is an additive covariant functor we see that $F(0\colon A \to B) = F(0+0) = F(0)+F(0)\colon F(A) \to F(B)$. It follows that $F(0)$ is the zero homomorphism.
(2) Let G be a trivial abelian group. Then $\mathrm{id}_G = 0$. It follows from functoriality and (1) that $\mathrm{id}_{F(G)} = F(\mathrm{id}_G) = F(0) = 0$. This implies that $F(G)$ is a trivial group. ∎

We can now formulate the elementary but useful lemma:

Lemma 160.3. (Functor-applied-to-Chain Complex Lemma) Suppose we are given an additive covariant functor $F\colon \mathit{AbGroup} \to \mathit{AbGroup}$.

(1) (a) If (C_*, ∂_*) is a chain complex, then
$$\ldots \to F(C_2) \xrightarrow{F(\partial_2)} F(C_1) \xrightarrow{F(\partial_1)} F(C_0) \xrightarrow{F(\partial_0)} 0$$
is also a chain complex.
(b) If $\varphi_*\colon C_* \to D_*$ is a chain map, then $F(\varphi)_*\colon F(C)_* \to F(D)_*$ is also a chain map.
(2) The maps
$$\begin{aligned} (C_*, \partial_*) &\mapsto (F(C_*), F(\partial_*)) \\ (\varphi_*\colon C_* \to D_*) &\mapsto (F(\varphi)_*\colon F(C)_* \to F(D)_*) \end{aligned}$$
define a covariant functor from the category *ChCplx* of chain complexes to itself.
(3) If $\varphi, \psi\colon C_* \to D_*$ are chain homotopic, then $F(\varphi), F(\psi)\colon F(C_*) \to F(D_*)$ are also chain homotopic.

Proof. Let $F\colon \mathit{AbGroup} \to \mathit{AbGroup}$ be an additive covariant functor from the category *AbGroup* of abelian groups to itself.

(1) (a) Let (C_*, ∂_*) be a chain complex. Given any $n \in \mathbb{N}_0$ we see that
$$\begin{array}{ccccc} F(\partial_{n-1}) \circ F(\partial_n) & = & F(\partial_{n-1} \circ \partial_n) & = & F(0) & = & 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } F \text{ is a covariant functor} & & \text{since } (C_*, \partial_*) \text{ is a chain complex} & & \text{by the Additive Functor Basics Lemma 160.2 (1)} & \end{array}$$

(b) Let $\varphi\colon (C_*, \partial_*^C) \to (D_*, \partial_*^D)$ be a chain map. Given any $n \in \mathbb{N}_0$ we have

$$F(\varphi_{n-1}) \circ F(\partial_n^C) \underset{\text{since } F \text{ is a covariant functor}}{=} F(\varphi_{n-1} \circ \partial_n^C) \underset{\text{since } \varphi_* \text{ is a chain map}}{=} F(\partial_n^D \circ \varphi_n) \underset{\text{since } F \text{ is a covariant functor}}{=} F(\partial_n^D) \circ F(\varphi_n).$$

(2) This statement follows easily from the definitions.

(3) Let $\varphi, \psi\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*^C) and (D_*, ∂_*^D). Furthermore let $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy between the chain maps φ and ψ. Given any $n \in \mathbb{N}_0$ we have

$$\begin{aligned} F(\partial_{n+1}^D) \circ F(P_n) + F(P_{n-1}) \circ F(\partial_n^C) &\underset{\text{since } F \text{ is an additive functor}}{=} F(\partial_{n+1}^D \circ P_n + P_{n-1} \circ \partial_n^C) \\ &\underset{\text{since the } P_n \text{ form a chain homotopy between } \varphi \text{ and } \psi}{=} F(\varphi_n - \psi_n) \underset{\text{since } F \text{ is additive}}{=} F(\varphi_n) - F(\psi_n). \end{aligned}$$

We have thus shown that the homomorphisms $F(P_n)\colon F(C_n) \to F(D_{n+1})$ form a chain homotopy between the chain maps $F(\varphi)$ and $F(\psi)$. ■

Fortunately the tensor product from the last chapter gives us an interesting additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself:

Lemma 160.4. (Tensor–Functor Lemma) Let G be an abelian group. The maps

$$\begin{aligned} A &\mapsto A \otimes G \\ (f\colon A \to B) &\mapsto (f \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \end{aligned}$$

define an additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

Proof.

(1) We verify the two properties of a covariant functor from Definition 5.18:

(a) Let A be an abelian group. By the Tensor–Map Composition Lemma 159.7 (1) we have $\mathrm{id}_A \otimes \mathrm{id}_G = \mathrm{id}_{A\otimes G}$.

(b) Let $f\colon A \to B$ and $g\colon B \to C$ be two homomorphisms between two abelian groups. It follows immediately from the Tensor–Map Composition Lemma 159.7 (2) that we have $(g \otimes \mathrm{id}_G) \circ (f \otimes \mathrm{id}_G) = (g \circ f) \otimes \mathrm{id}_G$.

(2) It follows from the Tensor–Map Composition Lemma 159.7 (3) that the functor is additive. ■

160.2. **Tensoring chain complexes.** In this section we will apply the results from the previous section to tensor a chain complex with a fixed abelian group. In the following section we will apply this principle to the singular chain complexes of a topological space.

Definition 160.5. Let

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex in the sense of Definition 143.1.

(1) Let G be an abelian group. It follows from the Tensor–Functor Lemma 160.4 together with the Functor-applied-to-Chain Complex Lemma 160.3 (1a) that the following sequence of homomorphisms
$$\ldots \to C_n \otimes G \xrightarrow{\partial_n \otimes \mathrm{id}_G} C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \mathrm{id}_G} \ldots \xrightarrow{\partial_2 \otimes \mathrm{id}_G} C_1 \otimes G \xrightarrow{\partial_1 \otimes \mathrm{id}_G} C_0 \otimes G \to 0.$$
is again a chain complex. We refer to $\mathrm{H}_n(C_* \otimes G) = \mathrm{H}_n(C_* \otimes G, \partial_* \otimes \mathrm{id}_G)$ as the homology groups of C_* with G-coefficients.

(2) If we tensor the chain complex C_* with a commutative ring R, then it follows from the Tensor-with-Ring Lemma 159.18 that we obtain a chain complex of R-modules, i.e. all chain groups are R-modules and all chain maps are homomorphisms of R-modules. In particular also all homology groups are R-modules.

The following lemma collects some low-hanging mathematical fruit.

Lemma 160.6. (Homology-with-Coefficients Lemma) In the following let G be an abelian group.

(1) (a) If $f\colon C_* \to D_*$ is a chain map between two chain complexes C_* and D_*, then the maps $f \otimes \mathrm{id}_G\colon C_* \otimes G \to D_* \otimes G$ also form a chain map. Thus for each $n \in \mathbb{N}_0$ we get an induced homomorphism
$$(f \otimes \mathrm{id}_G)_*\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(D_* \otimes G).$$

(b) Given $n \in \mathbb{N}_0$ the maps
$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_* \otimes G) \\ (f\colon C_* \to D_*) & \mapsto & (\underbrace{(f \otimes \mathrm{id}_G)_*}_{=:f_*}\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(D_* \otimes G)) \end{array}$$
define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups. Furthermore, if G is a commutative ring, then we get a functor to the category $\mathcal{Mod}_G$ of G-modules.

(2) (a) Let $f, g\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*) and (D_*, ∂_*). If $\{P_n\colon C_n \to D_{n+1}\}_{n \in \mathbb{N}_0}$ is a chain homotopy between f and g, then the maps $\{P_n \otimes \mathrm{id}_G\colon C_n \otimes G \to D_{n+1} \otimes G\}_{n \in \mathbb{N}_0}$ form a chain homotopy between $f \otimes \mathrm{id}_G$ and $g \otimes \mathrm{id}_G$. It follows from the Chain Homotopy–H_*-Lemma 144.5 that we have
$$(f \otimes \mathrm{id}_G)_* = (g \otimes \mathrm{id}_G)_*\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(D_* \otimes G).$$

(b) Let C_* and D_* be chain complexes. If $f\colon C_* \to D_*$ is a chain homotopy equivalence, then $f \otimes \mathrm{id}_G\colon C_* \otimes G \to D_* \otimes G$ is also a chain homotopy equivalence. It follows immediately from the Chain Homotopy Equivalence-H_*-Corollary 144.10, that the induced map $(f \otimes \mathrm{id}_G)_*\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(D_* \otimes G)$ is an isomorphism.

Proof. All statements follow almost immediately from the Tensor–Functor Lemma 160.4 together with the Functor-applied-to-Chain Complex Lemma 160.3. ∎

One of the most important tools in homology is the long exact sequence of homology groups corresponding to a short exact sequence of chain complexes. In the setting of homology with coefficients this requires a little bit of care.

Proposition 160.7. (LES-with-Coefficients Proposition) Let G be an abelian group and let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of chain complexes such that all chain groups C_* are free abelian. We get natural connecting homomorphisms $\partial_k \colon \mathrm{H}_k(C_* \otimes G) \to \mathrm{H}_{k-1}(A_* \otimes G)$ and a long exact sequence

$$\ldots \to \mathrm{H}_k(A_* \otimes G) \xrightarrow{(i\otimes \mathrm{id}_G)_*} \mathrm{H}_k(B_* \otimes G) \xrightarrow{(p\otimes \mathrm{id}_G)_*} \mathrm{H}_k(C_* \otimes G) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(A_* \otimes G) \to \ldots$$

in homology with G-coefficients. Furthermore, if G is a commutative ring, then the connecting homomorphisms are in fact G-module homomorphisms.

The proof of the proposition requires some preparations. First let us recall Definition 145.4:

Definition 160.8. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of groups splits, if there exists a splitting $s \colon C \to B$, i.e. a homomorphism $s \colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Lemma 160.9. (Tensor-Split-SES Lemma) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups and let G be an abelian group. If the short exact sequence splits[1698], then

$$0 \to A \otimes G \xrightarrow{i\otimes \mathrm{id}} B \otimes G \xrightarrow{p\otimes \mathrm{id}} C \otimes G \to 0$$

is also a short exact sequence and it also splits.

Proof of the Tensor-Split-SES Lemma 160.9. We start out with the following claim, which is the content of Exercise 145.10:

Claim. Let $0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between abelian groups. The following two statements are equivalent:

(1) The sequence is exact and the sequence splits.
(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma \colon G \to F$ and $\tau \colon H \to G$ such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_G$.

Proof. The (1)⇒(2) Statement follows almost immediately from the Splitting Lemma 145.5. Conversely the (2)⇒(1)-Statement can easily be verified by hand. ⊞

Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups that splits and let G be an abelian group. We pick homomorphisms $\sigma \colon B \to A$ and $\tau \colon C \to B$ as in the claim. Note that it follows from the Tensor–Map Composition Lemma 159.7 (2) and (3) that we have the equalities $(p \otimes \mathrm{id}_G) \circ (\iota \otimes \mathrm{id}_G) = (p \circ \iota) \otimes \mathrm{id}_G = 0 \otimes \mathrm{id}_G = 0$ and that the homomorphisms $\sigma \otimes \mathrm{id}_G \colon B \otimes G \to A \otimes G$ and $\tau \otimes \mathrm{id}_G \colon C \otimes G \to B \otimes G$ have the desired properties for the sequence

$$0 \to A \otimes G \xrightarrow{i\otimes \mathrm{id}_G} B \otimes G \xrightarrow{p\otimes \mathrm{id}_G} C \otimes G \to 0.$$

It follows from the claim that this sequence is exact and that it splits. ■

[1698] Recall that by the Free Abelian Group–Splitting Lemma 97.11 this happens if C is free abelian.

Proof of the LES-with-Coefficients Proposition 160.7. In the following let G be an abelian group and let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes such that all chain groups C_* are free abelian. Since each chain group C_* is free abelian we obtain from the Free Abelian Group–Splitting Lemma 97.11 that each of the above short exact sequences splits. It follows from this observation together with the Tensor-Split-SES Lemma 160.9 that $0 \to A_* \otimes G \to B_* \otimes G \to C_* \otimes G \to 0$ is again a short exact sequence of chain complexes. Thus we see that the Connecting Homomorphism Lemma 145.9 and the LES Proposition 145.10 apply in the present context. Finally, if G is a commutative ring, then one can easily verify that the connecting homomorphisms are G-module homomorphisms. ■

The following lemma shows that, not surprisingly, tensoring with $\mathbb{Z}$ has no real effect on the homology groups.

Lemma 160.10. (Homology-$\mathbb{Z}$-Coefficients Lemma) Let (C_*, ∂_*) be a chain complex. The natural isomorphisms $\Theta\colon C_k \otimes \mathbb{Z} \to C_k$ from the Tensor-with-$\mathbb{Z}$–Lemma 159.9 define a natural isomorphism of chain complexes, in particular for each $k \in \mathbb{N}_0$ we obtain natural isomorphisms
$$\Theta\colon \mathrm{H}_k(C_* \otimes \mathbb{Z}, \partial_* \otimes \mathrm{id}_{\mathbb{Z}}) \ \to\ \mathrm{H}_k(C_*, \partial_*).$$

Proof. It follows from the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that for each $k \in \mathbb{N}_0$ we obtain the following *commutative* diagram with vertical isomorphisms:
$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{k+1} \otimes \mathbb{Z} & \xrightarrow{\partial_{k+1} \otimes \mathrm{id}_{\mathbb{Z}}} & C_k \otimes \mathbb{Z} & \xrightarrow{\partial_k \otimes \mathrm{id}_{\mathbb{Z}}} & C_{k-1} \otimes \mathbb{Z} & \longrightarrow & \ldots \\
 & & \Theta \downarrow \cong & & \Theta \downarrow \cong & & \Theta \downarrow \cong & & \\
\ldots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \longrightarrow & \ldots
\end{array}$$
But this observation gives us precisely the desired statement. ■

The following elementary lemma shows that homology groups are "covariantly functorial in the coefficients".

Lemma 160.11. (Homology-Coefficients-Functor Lemma) Let C_* be a chain complex and let $\varphi\colon G \to \widetilde{G}$ be a homomorphism of abelian groups. The maps
$$\mathrm{id}_{C_*} \otimes \varphi\colon C_* \otimes G \ \to\ C_* \otimes \widetilde{G}$$
form a natural chain map. In particular for each $n \in \mathbb{N}_0$ we obtain a natural induced homomorphism
$$(\mathrm{id}_{C_*} \otimes \varphi)_*\colon \mathrm{H}_n(C_* \otimes G) \ \to\ \mathrm{H}_n(C_* \otimes \widetilde{G}).$$
Here "natural" means that for any chain map $f\colon C_* \to D_*$ we get a commutative diagram
$$\begin{array}{ccc}
\mathrm{H}_n(C_* \otimes G) & \xrightarrow{(\mathrm{id}_{C_*} \otimes \varphi)_*} & \mathrm{H}_n(C_* \otimes \widetilde{G}) \\
{\scriptstyle (f \otimes \mathrm{id}_G)_*}\downarrow & & \downarrow{\scriptstyle (f \otimes \mathrm{id}_{\widetilde{G}})_*} \\
\mathrm{H}_n(D_* \otimes G) & \xrightarrow{(\mathrm{id}_{C_*} \otimes \varphi)_*} & \mathrm{H}_n(D_* \otimes \widetilde{G}).
\end{array}$$

Example. Let R be a commutative ring with neutral element 1_R. We have a natural homomorphism $\varphi\colon \mathbb{Z} \to R$ that is given by $n \mapsto n \cdot 1_R$. We get the following natural homomorphisms

$$\mathrm{H}_k(C_*) \xleftarrow{\cong} \mathrm{H}_k(C_* \otimes \mathbb{Z}) \xrightarrow{(\mathrm{id}\otimes\varphi)_*} \mathrm{H}_k(C_* \otimes R).$$

by the Homology-$\mathbb{Z}$-Coefficients Lemma 160.10 — from the Homology-Coefficients Functor Lemma 160.11 □

Proof. Given $k \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccc} C_k \otimes G & \xrightarrow{\mathrm{id}_{C_k}\otimes\varphi} & C_k \otimes \widetilde{G} \\ \downarrow{\scriptstyle \partial_k \otimes \mathrm{id}_G} & & \downarrow{\scriptstyle \partial_k \otimes \mathrm{id}_{\widetilde{G}}} \\ C_{k-1} \otimes G & \xrightarrow{\mathrm{id}_{C_{k-1}}\otimes\varphi} & C_{k-1} \otimes \widetilde{G}. \end{array}$$

It follows immediately from the Tensor–Map Composition Lemma 159.7 (2) that the diagram commutes. This means precisely that the horizontal maps are chain maps. The proof that the horizontal maps are natural follows by the same type of argument. ■

160.3. **The homology groups of a topological space with coefficients.** Now we are ready to define the homology groups of a (pair of) topological spaces with coefficients in an abelian group G.

Definition 160.12. Let (X, A) be a pair of topological spaces. For an abelian group G we define

$$\begin{aligned} \mathrm{C}_*(X, A; G) &:= \mathrm{C}_*(X, A) \otimes G \\ \mathrm{H}_k(X, A; G) &:= \mathrm{H}_k(\mathrm{C}_*(X, A; G), \partial_* \otimes \mathrm{id}_G). \end{aligned}$$

We call $\mathrm{H}_k(X, A; G)$ the k-th homology group of (X, A) with G-coefficients. If $A = \varnothing$, then we just drop it from the notation.

We start out with the following three observations. Let (X, A) be a pair of topological spaces.

(1) If the coefficients are given by a commutative ring R, e.g. if $R = \mathbb{Z}/n\mathbb{Z}$ or $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, then, as we pointed out above, each $\mathrm{C}_n(X, A; R) = \mathrm{C}_n(X, A) \otimes R$ is an R-module, each boundary map is an R-module homomorphism, and each homology group $\mathrm{H}_n(X, A; R)$ is also an R-module.

(2) It follows from the Homology-$\mathbb{Z}$-Coefficients Lemma 160.10 that we have natural isomorphisms

$$\mathrm{C}_*(X, A; \mathbb{Z}) \xrightarrow{\cong} \mathrm{C}_*(X, A)$$

of chain complexes and natural isomorphisms

$$\mathrm{H}_*(X, A; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_*(X, A)$$

of abelian groups. In the following we will identify these chain complexes and homology groups and we will go back and forth between these two points of view, depending on which one is more convenient.

(3) Let G be an abelian group. It follows easily from the Tensor–Direct Sum Lemma 159.15 together with the Tensor-with-$\mathbb{Z}$–Lemma 159.9 (1) and the remark on page 2798 that any element in $\mathrm{C}_k(X, A; G)$ can be written uniquely as $\sum_{i=1}^{n} \sigma_i \otimes g_i$ where the following holds:

(a) $\sigma_1, \ldots, \sigma_n \colon \Delta^k \to X$ are pairwise distinct singular k-simplices whose images are not entirely contained in A.

(b) $g_1, \ldots, g_n \in G \setminus \{0\}$.

Most results for homology groups of (pairs of) topological spaces carry over, with the obvious modifications, to the theory of homology groups with G-coefficients. For example, given an abelian group G we obtain from the Homology-with-Coefficients Lemma 160.6 the following statements:

(1) (a) Let $f\colon (X,A) \to (Y,B)$ be a continuous map between two pairs of topological spaces. It follows from the C_*–Functor Lemma 143.16 (and the Relative-H_*–Functor Lemma 146.3) together with the Homology-with-Coefficients Lemma 160.6 (1a) that

$$f_* \otimes \mathrm{id}\colon \mathrm{C}_*(X,A;G) \ \to \ \mathrm{C}_*(Y,B;G)$$

is a chain map which in turn, for each $k \in \mathbb{N}_0$, induces a homomorphism

$$(f_* \otimes \mathrm{id})_*\colon \mathrm{H}_k(X,A;G) \ \to \ \mathrm{H}_k(Y,B;G).$$

To simplify the notation we usually denote this induced map on homology groups also by f_*.

(b) It follows from the C_*–Functor Lemma 143.16 (and the Relative-H_*–Functor Lemma 146.3) together with the Homology-with-Coefficients Lemma 160.6 (1b) that for any $k \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} (X,A) & \mapsto & \mathrm{H}_k(X,A;G) \\ (f\colon (X,A) \to (Y,B)) & \mapsto & (f_* = (f_* \otimes \mathrm{id}_G)_*\colon \mathrm{H}_k(X,A;G) \to \mathrm{H}_k(Y,B;G)) \end{array}$$

define a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups.

As we mentioned in the Homology-with-Coefficients Lemma 160.6 (1b), if G is a commutative ring, then the above induced maps are G-module homomorphisms and we get a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{Mod}_G$ of G-modules.

(2) (a) In the Homotopy-H_*-Proposition 144.12 we showed that if $f, g\colon X \to Y$ are homotopic maps between topological spaces, then the induced maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ are chain homotopic. It follows from the Homology-with-Coefficients Lemma 160.6 (2a) that for any $n \in \mathbb{N}_0$ we have

$$f_* = g_*\colon \mathrm{H}_n(X;G) \ \to \ \mathrm{H}_n(Y;G).$$

(b) It follows from (a) that if X and Y are homotopy equivalent topological spaces, then for any $n \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_n(X;G) \ \xrightarrow{\cong} \ \mathrm{H}_n(Y;G).$$

(3) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. It follows from the Path-Component-C_*-Lemma 143.17, the Tensor–Direct Sum Lemma 159.15 and the Direct Sum-H_*-Lemma 143.7 that for each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A} \mathrm{H}_n(X_a;G) \ \to \ \mathrm{H}_n(X;G)$$

is an isomorphism. This result is the analog of the Path-Component-H_*-Lemma 143.21.



$i_{1*} \oplus i_{2*}\colon \mathrm{H}_n(X_1;G) \oplus \mathrm{H}_n(X_2;G) \xrightarrow{\cong} \mathrm{H}_n(X_1 \sqcup X_2;G)$

(4) (a) Precisely as in Definition 147.1 we can use the natural augmentation map

$$\begin{array}{rcl} \epsilon_X \colon \mathrm{C}_0(X;G) & \to & G \\ \sum\limits_{i=1}^{k} \sigma_i \otimes g_i & \mapsto & \sum\limits_{i=1}^{k} g_i. \end{array}$$

to define reduced homology groups $\widetilde{\mathrm{H}}_k(X;G)$ with G-coefficients.

(b) One can easily verify that the augmentation homomorphism from (a) descends to a natural augmentation homomorphism

$$\begin{array}{rcl} \epsilon_X \colon \mathrm{H}_0(X;G) & \to & G \\ \left[\sum\limits_{i=1}^{k} \sigma_i \otimes g_i\right] & \mapsto & \sum\limits_{i=1}^{k} g_i. \end{array}$$

If X is path-connected and non-empty, then the argument of the H_0-Proposition 143.25 implies immediately that the augmentation map $\epsilon \colon \mathrm{H}_0(X;G) \to G$ is an isomorphism.

(c) If $f \colon X \to Y$ is a continuous map between path-connected non-empty topological spaces, then one can easily show, as in the H_0-Proposition 143.25, that the induced map $f_* \colon \mathrm{H}_0(X;G) \to \mathrm{H}_0(Y;G)$ is an isomorphism.

All the other earlier results on reduced homology groups also have obvious analogs for reduced homology with G-coefficients. There is surely no need to spell out all of these results. The earlier results on reduced homology groups have obvious analogs for reduced homology with G-coefficients.

(5) Let (X, B, A) be a triple of topological spaces. As we mentioned in the proof of the LES-of-Triple Proposition 146.5, it follows from the third isomorphism theorem that we have the following short exact sequence of chain complexes:

$$0 \ \to \ \mathrm{C}_n(B,A) \ \xrightarrow{i} \ \mathrm{C}_n(X,A) \ \xrightarrow{p} \ \mathrm{C}_n(X,B) \ \to \ 0.$$

As we pointed out on page 2798, the relative chain groups $\mathrm{C}_n(X,B)$ are free abelian groups. Note that it follows immediately from these two observations and the LES-with-Coefficients Proposition 160.7 that we obtain natural connecting homomorphisms $\partial_n \colon \mathrm{H}_n(X,B;G) \to \mathrm{H}_{n-1}(B,A;G)$ which give rise to the following long exact sequence of homology groups with G-coefficients:

$$\ldots \ \to \ \mathrm{H}_n(B,A;G) \ \xrightarrow{i_*} \ \mathrm{H}_n(X,A;G) \ \xrightarrow{p_*} \ \mathrm{H}_n(X,B;G) \ \xrightarrow{\partial_n} \ \mathrm{H}_{n-1}(B,A;G) \ \to \ \ldots$$

If G is a commutative ring, then the connecting homomorphism is a G-module homomorphism.

(6) (a) Let X be a topological space and let $Z \subseteq A \subseteq X$ be subsets. We assume that the closure of Z is contained in the interior of A. It follows from the Excision Theorem 148.1, the fact that relative chain groups are free abelian groups, the Quasi Iso–Homotopy Equivalence Proposition 158.8 and the Homology-with-Coefficients Lemma 160.6 (2) that for each $n \in \mathbb{N}_0$ the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z; G) \ \xrightarrow{\cong} \ \mathrm{H}_n(X, A; G).$$



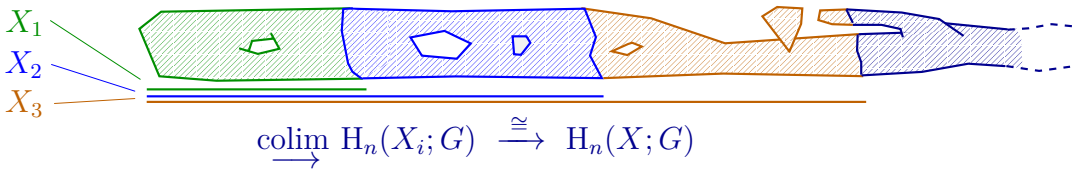
(b) We have an obvious version of the Mayer–Vietoris Theorem 153.2 for homology groups with G-coefficients. We can either deduce this version from (a) and the argument of Section 153.1.1. Alternatively we can argue as in Section 153.1.2. For this argument we only have to note that when we consider the exact sequence

$$0 \to \mathrm{C}_*(A \cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i^X_A \oplus i^X_B} \underbrace{\mathrm{C}_*^{\{A,B\}}(X)}_{\text{free abelian}} \to 0,$$

then it is a consequence of the Tensor-Split-SES Lemma 160.9 together with the Free Abelian Group–Splitting Lemma 97.11 that the corresponding long exact sequence of homology groups with G-coefficients is also exact.

(7) Let X be a topological space and let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following two conditions holds:
(a) each X_i is contained in the interior of X_{i+1}, or
(b) X is a CW-complex and each X_i is a subcomplex of X.
As in the proof of the Exhaustions-H_*-Proposition 154.2 it follows quite easily from the Exhaustion-C_*-Lemma 154.1, the Tensor–Direct Colimit Lemma 159.16 and the Direct Colimit-H_*-Lemma 154.8 that for every $n \in \mathbb{N}_0$ the inclusion-induced homomorphisms $\mathrm{H}_n(X_i; G) \to \mathrm{H}_n(X; G)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \mathrm{H}_n(X_i; G) \xrightarrow{\cong} \mathrm{H}_n(X; G).$$



(8) For a CW-complex X we can also consider the corresponding cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X; G) := \mathrm{C}_*^{\mathrm{CW}}(X) \otimes G$ and the corresponding cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(X; G) = \mathrm{H}_k(\mathrm{C}_*^{\mathrm{CW}}(X); G)$.

Recall that by the Singular-Cellular Chain Complex Proposition 158.16 we have natural chain homotopy equivalences

$$\mathrm{C}_*(X) \xleftarrow{i} \mathrm{C}_*^{\mathrm{CW,int}}(X) \xrightarrow{p} \mathrm{C}_*^{\mathrm{CW}}(X).$$

It follows from the Homology-with-Coefficients Lemma 160.6 (2) that we obtain natural isomorphisms[1699]

$$\mathrm{H}_k(X; G) \xleftarrow[\cong]{(i\otimes \mathrm{id}_G)_*} \mathrm{H}_k(\mathrm{C}_*^{\mathrm{CW,int}}(X) \otimes G) \xrightarrow[\cong]{(p\otimes \mathrm{id}_G)_*} \mathrm{H}_*^{\mathrm{CW}}(X; G).$$

[1699] Alternatively we could easily modify the proof of the Singular-Cellular-H_*-Proposition 155.8 to see that there exists a natural isomorphism $\mathrm{H}_*(X; G) \xrightarrow{\cong} \mathrm{H}_*^{\mathrm{CW}}(X; G)$.

We can draw the following conclusions:
(a) If X is an n-dimensional CW-complex, then $\mathrm{H}_i(X;G) = 0$ for $i \geq n+1$.
(b) If X has only finitely many n-cells, then given any field $\mathbb{F}$ the vector space $\mathrm{H}_n(X;\mathbb{F})$ is finite-dimensional.

Next we record two important calculations of homology groups with coefficients as lemmas:

Proposition 160.13. (Sphere-H_*-Proposition) For any $n \in \mathbb{Z}_{\geq -1}$ and any abelian group G we have

$$\widetilde{\mathrm{H}}_k(S^n;G) \;\cong\; \begin{cases} G, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. We can calculate these homology groups by exactly the same argument as on page 2818 or as in the proof of the original Sphere-H_*-Proposition 149.4. Alternatively we can equip S^n with the CW-structure with one 0-cell and one n-cell, which we introduced on page 2625. With this CW-structure we can easily calculate the cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(S^n;G)$. By the discussion on page 3081 we obtain this way also the singular homology groups $\mathrm{H}_k(S^n;G)$. Using the obvious analog of the Reduced Homology Lemma 147.3 we finally obtain the reduced homology groups $\widetilde{\mathrm{H}}_k(S^n;G)$. ■

The next calculation is more interesting and more surprising.

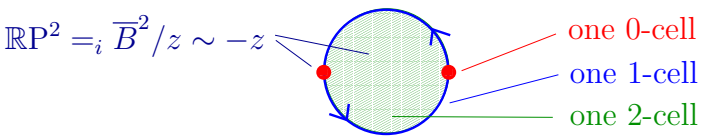
Proposition 160.14. ($\mathbb{R}\mathrm{P}^n$-H_*-Proposition)
(1) Let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\ldots,n \\ 0, & \text{if } k > n. \end{cases}$$

(2) For every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof.

(1) Let $n \in \mathbb{N}_0$. As usual we equip $\mathbb{R}\mathrm{P}^n$ with the CW-structure that we introduced on page 2628 and which has precisely one cell in dimensions $0,\ldots,n$.



In the proof of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we saw that the cellular chain complex is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

We tensor the above cellular chain complex with $\mathbb{F}_2$. Using the Free Module–Tensor-with-Ring Lemma 159.20 we obtain the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z}\otimes\mathbb{F}_2 & \xrightarrow{(\cdot(1+(-1)^n))\otimes\mathrm{id}_{\mathbb{F}_2}} & \mathbb{Z}\otimes\mathbb{F}_2 & \to \ldots \to & \mathbb{Z}\otimes\mathbb{F}_2 & \xrightarrow{(\cdot 2)\otimes\mathrm{id}_{\mathbb{F}_2}} & \mathbb{Z}\otimes\mathbb{F}_2 & \xrightarrow{0\otimes\mathrm{id}_{\mathbb{F}_2}} & \mathbb{Z}\otimes\mathbb{F}_2 \to 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & \mathbb{F}_2 & \xrightarrow{0} & \mathbb{F}_2 & \xrightarrow{\cdot 0} \ldots \longrightarrow & \mathbb{F}_2 & \xrightarrow{\cdot 0} & \mathbb{F}_2 & \xrightarrow{\cdot 0} & \mathbb{F}_2 \longrightarrow 0.
\end{array}$$

We now see that

$$\begin{array}{ccccc} \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \cong & \mathrm{H}_k^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \cong & \begin{cases} \mathbb{F}_2, & \text{if } k=0,\dots,n, \\ 0, & \text{otherwise.} \end{cases} \\ \uparrow & & \uparrow & & \end{array}$$

since by the discussion on page 3081 we know that singular and cellular homology groups are isomorphic

by the above commutative diagram

(2) For $\mathbb{R}\mathrm{P}^\infty$ we use the CW-structure from page 2628 which has precisely one n-cell in each dimension. The same argument as in (1) gives us the desired result. ■

We conclude this chapter with the following analog of the Homology-χ-Proposition 156.2:

Proposition 160.15. (Homology-χ-Proposition) Let X be a finite CW-complex. For any field $\mathbb{F}$ we have

$$\chi(X) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X;\mathbb{F})).$$

Remark. In the Twisted Homology-χ-Proposition 331.12 we will prove a generalization of the Homology-χ-Proposition 160.15 to homology with "twisted coefficients". □

Example. Let us consider the real projective plane $\mathbb{R}\mathrm{P}^2$. By the $\mathbb{R}\mathrm{P}^n$-H_*-Propositions 155.18 and 160.14 we have the following table:

	$k=3$	$k=2$	$k=1$	$k=0$
$\mathrm{H}_k(\mathbb{R}\mathrm{P}^2)$	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
$\mathrm{H}_k(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$

But even though the "classical homology groups" of $\mathbb{R}\mathrm{P}^2$ and the homology groups with $\mathbb{F}_2$-coefficients are quite different, in both cases the alternating sums of the ranks and dimensions give, as promised in the Homology-χ-Propositions 156.2 and 160.15, the Euler characteristic. Indeed, we have

$$\mathrm{rank}(\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)) - \mathrm{rank}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)) + \mathrm{rank}(\mathrm{H}_0(\mathbb{R}\mathrm{P}^2)) \;=\; 0-0+1 \;=\; 1,$$

and

$$\dim_{\mathbb{F}_2}(\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)) - \dim_{\mathbb{F}_2}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)) + \dim_{\mathbb{F}_2}(\mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)) \;=\; 1-1+1 \;=\; 1.$$

□

Proof. [1700] Our proof builds on the following claim, which is a variation on the Chain Complex–H_*–χ-Lemma 156.4.

Claim. Given any chain complex

$$0 \;\to\; V_k \xrightarrow{\partial_k} V_{k-1} \xrightarrow{\partial_{k-1}} \dots\; V_1 \xrightarrow{\partial_1} V_0 \;\to\; 0$$

of finite-dimensional $\mathbb{F}$-modules the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(V_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(V_*)).$$

Proof. The proof builds is almost identical to the proof of the Chain Complex–H_*–χ-Lemma 156.4. All we need to do is to replace the Rank Additivity Proposition 97.18 by the

[1700] The proof of this proposition is of course very similar to the proof of the Homology-χ-Proposition 156.2.

well-known fact that for any short exact sequence $0 \to A \to B \to C \to 0$ of finite-dimensional $\mathbb{F}$-vector spaces we have

$$\dim_{\mathbb{F}}(B) \;=\; \dim_{\mathbb{F}}(A) + \dim_{\mathbb{F}}(C). \qquad \boxplus$$

Now let X be a finite k-dimensional CW-complex. As on page 2990 we perform the following calculation:

$$\begin{array}{ccc}
 & \overset{\text{by Definition 156.1 of } \chi(X)}{\downarrow} & \overset{\text{by the CW-Skeleton-H}_*\text{-Lemma 155.2 (1)}}{\underset{\text{and the Free Module–Tensor-with-Ring Lemma 159.20 (1)}}{\downarrow}} \\
\chi(X) & = \sum_{n=0}^{k} (-1)^n \cdot \text{number of } n\text{-cells} & = \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{C}_n^{\mathrm{CW}}(X) \otimes \mathbb{F}) \\
 & = \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n^{\mathrm{CW}}(X;\mathbb{F})) & = \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(X;\mathbb{F})). \\
 & \underset{\text{by the claim}}{\uparrow} & \underset{\text{by the discussion on page 3081}}{\uparrow}
\end{array}$$

■

Remark. Why should we study the homology groups of a topological space with G-coefficients? As we mentioned in the beginning of the chapter, there are several answers one can give:

(1) If we work over the field $\mathbb{F}_2$ then many calculations become particularly simple, since we do not need to worry about signs. Even in the previous example we already saw that working with $\mathbb{F}_2$-coefficients can simplify calculations.
(2) The theory of (finite-dimensional) vector spaces is much easier than the theory of (finitely generated) abelian groups. Thus on many occasions it is easier to use homology with coefficients in a field, then to work with the "tradition" homology groups.
(3) We will use homology groups with coefficients for more theoretic purposes, e.g. in the proof of the Borsuk–Ulam Theorem 47.12 that we will give on page 3175. □

160.4. **Appendix: The Eilenberg-Steenrod axioms for homology.** We just learned that besides the usual homology groups $\mathrm{H}_n(X, A)$ there are also the homology groups $\mathrm{H}_n(X, A; G)$ with pretty much the same formal properties (homotopy invariance, long exact sequences, excision etc.).

This observation leads us the to the concept of (co-) homology theories which was introduced in the seminal book of Eilenberg–Steenrod [**EiS1952**]. In this appendix we will give a very short introduction to homology theories. We will not really make use of the results of this appendix.

We start out with the following definition:

Definition 160.16. Let R be a commutative ring and let S be a set.

(1) An S-graded R-module is defined as an R-module M together with a direct sum decomposition $M = \bigoplus_{s \in S} M_s$ into R-modules M_s, $s \in S$. If $x \in M$ lies in some M_s, then we call x homogeneous and we refer to $\deg(x) := s$ as the degree of x.
(2) A morphism $f\colon M \to N$ between S-graded R-modules M and N is an R-module homomorphism such that for each $s \in S$ we have $f(M_s) \subseteq N_s$. Equivalently, it is a family of R-module homomorphisms $\{f_s\colon M_s \to N_s\}_{s \in S}$.
(3) The category $S\text{-}\mathcal{GradMod}_R$ of S-graded R-modules is the category where the objects are S-graded R-modules and the morphisms are the ones described in (2).

(4) Since abelian groups are for all intents and purposes the same as $\mathbb{Z}$-modules we define the category *S-GradAb* of S-graded abelian groups to be the category of S-graded $\mathbb{Z}$-modules.

Example. Given a pair of topological spaces (X, Y) and an abelian group G we write

$$\mathrm{H}_*(X, Y; G) \coloneqq \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_n(X, Y; G)$$

and we view $\mathrm{H}_*(X, Y; G)$ as an $\mathbb{N}_0$-graded $\mathbb{Z}$-module. □

The following definition goes back to [**EiS1952**] and [**Miln1962b**].

Definition 160.17. A homology theory is a covariant functor

$$\mathcal{H}\colon \begin{array}{c}\text{category } \mathit{PairTop} \text{ of}\\ \text{pairs of topological spaces}\end{array} \to \begin{array}{c}\text{category } \mathbb{N}_0\text{-}\mathit{GradAb} \text{ of}\\ \mathbb{N}_0\text{-graded abelian groups}\end{array}$$

that satisfies the following axioms:[1701]

(1) (Homotopy invariant) If $f, g\colon (X, A) \to (Y, B)$ are homotopic maps between pairs of topological spaces, then $f_* = g_*$.

(2) (Connecting homomorphism) Given any pair (X, A) of topological spaces and any $n \in \mathbb{N}$ there exists a natural connecting homomorphism $\partial_n\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$[1702] such that the sequence

$$\ldots \to \mathcal{H}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathcal{H}_n(A) \to \mathcal{H}_n(X) \to \mathcal{H}_n(X, A) \xrightarrow{\partial_n} \mathcal{H}_{n-1}(A) \to \ldots$$

is exact.

(3) (Excision) If U and A are subsets of a topological space X with $\overline{U} \subseteq \mathring{A}$, then for each $n \in \mathbb{N}_0$ the map $\mathcal{H}_n(X \setminus U, A \setminus U) \to \mathcal{H}_n(X, A)$ is an isomorphism.

(4) (Additivity) Let $\{(X_j, A_j)\}_{j\in J}$ be a family of pairs of topological spaces. We write $X = \bigsqcup_{j\in J} X_j$ and $A = \bigsqcup_{j\in J} A_j$. If given $j \in J$ we denote by $i_j\colon (X_j, A_j) \to (X, A)$ the obvious inclusion map, then for any $k \in \mathbb{N}_0$ the induced map

$$\bigoplus_{j\in J} i_{j*}\colon \bigoplus_{j\in J} \mathcal{H}_k(X_j, A_j) \to \mathcal{H}_k(X, A)$$

is an isomorphism.

(5) (Dimension axiom) We denote by $\star$ the topological space that consists of a single point. For $n \neq 0$ we have $\mathrm{H}_n(\star) = 0$.

Example.

(1) It follows from the Homotopy-H_*-Proposition 146.11, the LES-of-Pair Proposition 146.7, the Excision Theorem 148.1, the obvious generalization of the Path-Component–H_*–Lemma 143.21 and the Singleton-H_*-Lemma 143.26 that singular homology $\mathrm{H}_*(X, A)$ is indeed a homology theory with $\mathrm{H}_n(\star) \cong \mathbb{Z}$.

(2) Let G be an abelian group. It follows from the obvious analogs of the references in (1) that singular homology $\mathrm{H}_*(X, A; G)$ with G-coefficients is a homology theory with $\mathrm{H}_n(\star; G) \cong G$. □

[1701] The axioms are usually called Eilenberg-Steenrod axioms.

[1702] Here we use the obvious notation that for a topological space Y we write $\mathcal{H}_n(Y) \coloneqq \mathcal{H}_n(Y, \varnothing)$.

We conclude this chapter with a uniqueness theorem of homology theories. To formulate the theorem we need the following definition:

Definition 160.18. We call the category $\mathcal{FullPairCW}$

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullPairCW}) &:= \text{all pairs of CW-complexes,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f\colon X \to Y \text{ with } f(A) \subseteq B, \end{aligned}$$

together with the usual composition of maps the full category of pairs of CW-complexes.[1703]

Now we can formulate the following uniqueness statement for homology theories.

Theorem 160.19. (Eilenberg-Steenrod Uniqueness Theorem) Let $\mathcal{H}$ and $\mathcal{H}'$ be two homology theories and let $\varphi\colon \mathcal{H}_0(\star) \to \mathcal{H}'_0(\star)$ be an isomorphism. Then there exists a natural isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ viewed as functors

$$\begin{array}{c} \text{full category } \mathcal{FullPairCW} \text{ of} \\ \text{pairs of CW-complexes} \end{array} \to \begin{array}{c} \text{category } \mathbb{N}_0\text{-}\mathcal{GradAb} \text{ of} \\ \mathbb{N}_0\text{-graded abelian groups.} \end{array}$$

Sketch of a proof. Let G be an abelian group. We start our proof with the following observation: Given a pair (X,A) of CW-complexes we saw in the CW-Complex Mapping Cylinder+Cone Proposition 139.25 that we can equip the mapping cone $\mathrm{Cone}(i\colon A \to X)$ of the inclusion map with a natural CW-structure. Given $n \in \mathbb{N}_0$ we define the n-th cellular homology $\mathrm{H}_n^{\mathrm{CW}}(X,A;G)$ as the reduced cellular homology with G-coefficients of the mapping cone.

The Singular-Cellular-H_*-Proposition 155.8, together with the Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15, the Singular-Cellular-H_*-Proposition 155.8 and the discussion on page 3081 shows that for pairs of CW-complexes there exists a natural isomorphism between singular homology with G-coefficients and cellular homology with G-coefficients. Closer inspection of the proof shows that we can prove this statement only using the above Eilenberg-Steenrod axioms and the fact that $\mathrm{H}_0(\star;G) \cong G$.

Now let $\alpha\colon \mathcal{H}_0(\star) \to G$ be an isomorphism. By the above one can show that there exists a natural isomorphism from $\mathcal{H}$ to $\mathrm{H}_n^{\mathrm{CW}}(-;G)$. The same applies to $\mathcal{H}'$ and we obtain the theorem as long as φ is an isomorphism. This approach to the proof is developed in more detail in [**tD2008**, Chapter 12.2]. ■

Remark.

(1) Closely related results are also proved in [**EiS1952**, p. 100] and [**Hu1966**, p. 52], see also [**GrH1981**, p. 138].
(2) The axioms of a cohomology theory are defined basically the same way with the obvious modifications and the analog of the Eilenberg-Steenrod Uniqueness Theorem 160.19 also holds in this context, see [**DaK2001**, Theorem 1.31].
(3) If one drops the dimension axiom, then there are other (co-) homology theories, for example "topological K-theory". We refer to [**Karo2008**] for details. □

[1703] Note that we consider all continuous maps, we do not restrict ourselves to cellular maps. Thus $\mathcal{FullPairCW}$ is a full subcategory of the category $\mathcal{PairTop}$ of pairs of topological spaces.

160.5. Exercises for Chapter 160 .

Exercise 160.1. Let

$$A_* \;=\; 0 \;\to\; A_2 \;\xrightarrow{i}\; A_1 \;\xrightarrow{p}\; A_0 \;\to\; 0$$

be a sequence of abelian groups and let G be an abelian group. In the Tensor-Split-SES Lemma 160.9 we showed that if the sequence A_* is exact and if it splits, then $A_* \otimes G$ is also exact and it also splits.

(a) If the sequence A_* is exact, does it follow that $A_* \otimes G$ is also exact?
(b) If the sequence A_* splits, does it follow that $A_* \otimes G$ also splits?

Exercise 160.2.

(a) Let C_* be a chain complex of free abelian groups. Show that if C_* is acyclic (i.e. if $\mathrm{H}_k(C_*) = 0$ for all $k \in \mathbb{N}_0$), then for every abelian group G the chain complex $C_* \otimes G$ is also acyclic.
Hint. Use the Tensor-Split-SES Lemma 160.9 and use the Free Abelian Subgroup Theorem 97.14.
(b) Let $f\colon C_* \to D_*$ be a chain map between complexes of free abelian groups. Show that if $f_*\colon C_* \to D_*$ is a quasi-isomorphism, then for every abelian group G the map $f_* \otimes \mathrm{id}_G\colon C_* \otimes G \to D_* \otimes G$ is also a quasi-isomorphism.
Hint. Make use of algebraic mapping cones.

Exercise 160.3. We consider an electric circuit as a finite abstract graph X where at the vertices we have resistors or capacitors. Kirchhoff's current rule, see e.g.

`https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws`

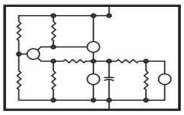
states that

> *at any node (vertex) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node,*

or equivalently

> *the algebraic sum of currents in a network meeting at a point is zero.*

Explain in what sense a current defines a 1-cycle in $\mathrm{C}_1(X;\mathbb{R})$.
Remark. This thought and the figure are from Robert Ghrist's fun book on applied topology [**Gh2014**].

electric circuit 

Exercise 160.4. Let X be a topological space and let G be an abelian group. We consider the suspension $\Sigma(X)$. Relate the homology groups $\mathrm{H}_*(X;G)$ to the homology groups $\mathrm{H}_*(\Sigma(X);G)$.

Exercise 160.5. Does there exist a topological space X such that $\mathrm{H}_2(X;\mathbb{F}_5) \neq 0$ but such that $\mathrm{H}_2(X;\mathbb{F}_3) = 0$?

Exercise 160.6. Let X be a topological space and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of abelian groups.

(a) Show that there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_k(X;A) \xrightarrow{\alpha_*} \mathrm{H}_k(X;B) \xrightarrow{\beta_*} \mathrm{H}_k(X;C) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(X;A) \to \ldots$$

where α_* and β_* are the maps from the Homology-Coefficients-Functor Lemma 160.11.

(b) Provide an example where the connecting homomorphism in (a) is a non-zero map.

Exercise 160.7. Let $n \in \mathbb{N}$.

(a) Let $p \in \mathbb{N}_{\geq 2}$ be a prime. What are the isomorphism types of the homology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_p$-coefficients?

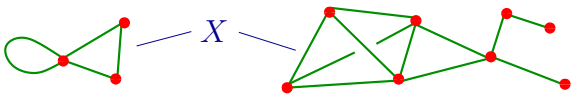
(b) What are the isomorphism types of the homology groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Q})$?

Exercise 160.8. Let $n \in \mathbb{N}$. Compute the homology groups of $\mathbb{R}\mathrm{P}^n$ with coefficients in the abelian group $\mathbb{Q}/\mathbb{Z}$, i.e. compute the groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Q}/\mathbb{Z})$.

Exercise 160.9. Does there exist a topological space X and a $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X)$ is not finitely generated but such that $\mathrm{H}_k(X;\mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$-vector space?

Exercise 160.10. Let G be an abelian group and let $n \in \mathbb{N}_0$. Compute the homology groups $\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;G)$.

Exercise 160.11. Let X be a finite topological graph and let G be an abelian group. Compute the homology groups $\mathrm{H}_k(X;G)$.



$\mathrm{H}_k(X;G) \ = \ ?$

Exercise 160.12. Let $k \in \mathbb{Z}$. We consider the continuous map $f_k \colon S^1 \to S^1$ that is given by $z \mapsto z^k$. For which $m \in \mathbb{N}$ is the induced map $f_{k*} \colon \mathrm{H}_1(S^1;\mathbb{Z}/m\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z}/m\mathbb{Z})$ an isomorphism?

Exercise 160.13. Show that given any prime $p \in \mathbb{N}$ there exists a topological space X, a continuous map $f \colon X \to X$ and a $k \in \mathbb{N}$ such that $f_* \colon \mathrm{H}_k(X) \to \mathrm{H}_k(X)$ is a non-zero map but such that $f_* \colon \mathrm{H}_k(X;\mathbb{F}_p) \to \mathrm{H}_k(X;\mathbb{F}_p)$ is the zero map.

Exercise 160.14. Let $g \in \mathbb{N}$. We denote by N_g the non-orientable surface of genus g from Definition 9.19.

(a) Show that

$$\mathrm{H}_i(N_g;\mathbb{F}_2) \ \cong \ \begin{cases} 0, & \text{if } i > 2, \\ \mathbb{F}_2, & \text{if } i = 0 \text{ or } i = 2, \\ \mathbb{F}_2^g, & \text{if } i = 1. \end{cases}$$

Remark. We calculated $\mathrm{H}_*(N_g)$ in the Surface-H_*-Proposition 155.17.

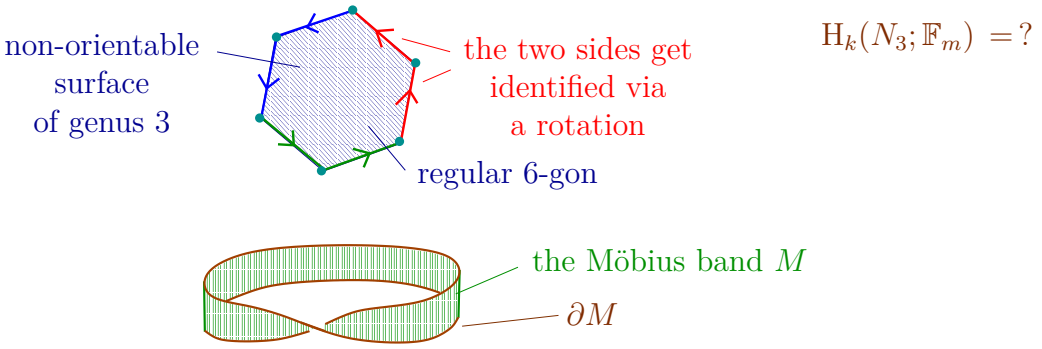
(b) Compute $\mathrm{H}_*(N_g;\mathbb{F}_p)$ for all primes $p \geq 3$.

Exercise 160.15. Let M be the Möbius band.

(a) Determine the map $\mathrm{H}_1(\partial M;\mathbb{F}_2) \to \mathrm{H}_1(M;\mathbb{F})$.

(b) Show that ∂M is not a retract of M.

(c) For $k \in \mathbb{N}$ compute the relative homology groups $\mathrm{H}_i(M,\partial M;\mathbb{Z}/k\mathbb{Z})$.



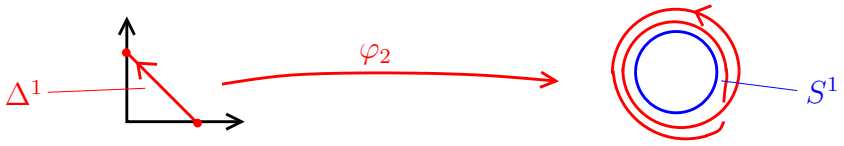
(d)

Exercise 160.16. Let X be an n-dimensional CW-complex. If $\mathrm{H}_n(X) \neq 0$, does it follow that $\mathrm{H}_n(X;\mathbb{F}_2) \neq 0$?

Exercise 160.17. Let $k \in \mathbb{Z}\backslash\{0\}$ and let $\varphi_k\colon \Delta^1 \to S^1$ be the continuous map that is given by $(1-t,t) \mapsto \exp(k\cdot 2\pi t\,\mathrm{i})$. Furthermore let $i\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1;\mathbb{R})$ be the homomorphism from the Homology-$\mathbb{Z}$-Coefficients Lemma 160.10 and the Homology-Coefficients-Functor Lemma 160.11 (1) that is induced by $\sigma \mapsto \sigma \otimes 1$. Show that

$$\left[\tfrac{1}{n} \otimes \varphi\right] \;=\; \tfrac{1}{n} \cdot i_*([S^1]) \;\in\; \mathrm{H}_1(S^1;\mathbb{R}).$$

We refer to the figure below for an illustration.



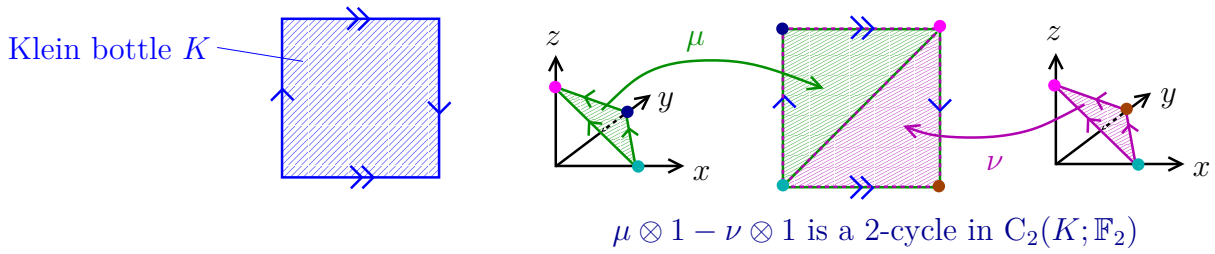
Exercise 160.18. Let $\varphi\colon \mathbb{Z} \to Z$ be the homomorphism that is given by multiplication by 3. Does there exist a topological space X and a $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X;\mathbb{Z})$ is non-zero and such that $\varphi_*\colon \mathrm{H}_k(X;\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z})$ is an isomorphism?

Exercise 160.19. We consider the Klein bottle $K = ([0,1]\times[0,1])/\sim$, where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$. We consider the singular simplices $\mu,\nu\colon \Delta^2 \to K$ that are shown in the figure below.

(a) Is $\mu - \nu$ a 2-cycle in $\mathrm{C}_*(K)$?
(b) Show that $\mu \otimes 1 - \nu \otimes 1$ a 2-cycle in $\mathrm{C}_*(K) \otimes \mathbb{Z}/2\mathbb{Z}$.

Remark. It follows from the obvious analog of the Local-H_*-Proposition 150.3 and the Detect-Cycle Proposition 150.5 for homology with $\mathbb{F}_2$-coefficients that $\mu\otimes 1-\nu\otimes 1$ represents a non-zero element of $\mathrm{H}_2(K;\mathbb{F}_2)$.

Exercise 160.20. We consider the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ where $z \sim -z$ for $z \in S^1$. By the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 160.14 we know that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$. Use the obvious analog of the Local-H_*-Proposition 150.3 and the Detect-Cycle Proposition 150.5 for homology with $\mathbb{F}_2$-coefficients to find an explicit cycle in $\mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ that represents a generator of $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$.



$\mu \otimes 1 - \nu \otimes 1$ is a 2-cycle in $\mathrm{C}_2(K; \mathbb{F}_2)$

Exercise 160.21. Let $k \leq l$ in $\mathbb{N}_0$. We consider the embedding $\iota\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$ that is given by $[x] \mapsto [(x, 0)]$.

(a) Show that for $i = 0, \ldots, k$ the induced map $\iota_*\colon \mathrm{H}_i(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2) \to \mathrm{H}_i(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2)$ is an isomorphism.

(b) Does the conclusion of (a) also hold if we replace $\mathbb{F}_2$ by $\mathbb{F}_3$?

Exercise 160.22. Let M and M' be two oriented connected n-dimensional smooth manifolds with non-empty boundary. We consider the boundary connected sum $M \#_{\mathrm{b}} M'$ that we introduced in Definition 61.10. Show that for any $i \geq 1$ and any commutative ring R the inclusion maps induce an isomorphism

$$\mathrm{H}_i(M; R) \oplus \mathrm{H}_i(M'; R) \xrightarrow{\cong} \mathrm{H}_i(M \#_{\mathrm{b}} M'; R).$$

Remark. Use the Boundary Connected Sum Proposition 61.11 (3) and collars or tubular neighborhoods.

Exercise 160.23. Let X be a connected CW-complex with the following properties:

- X has two 0-cells, one 1-cell, X has finitely many cells in dimension 2,3 and 5 and X no other cells.
- We have $\mathrm{H}_2(X; \mathbb{F}_2) \cong \mathbb{F}_2^3$, $\mathrm{H}_3(X; \mathbb{F}_2) \cong \mathbb{F}_2^2$ and $\mathrm{H}_5(X; \mathbb{F}_2) \cong \mathbb{F}_2^7$.

(a) Does this data determine $\mathrm{H}_*(X)$?

(b) Does this data determine $\mathrm{H}_*(X; \mathbb{Q})$?

Exercise 160.24. Let $f\colon (X, A) \to (Y, B)$ be a continuous map between pairs of topological spaces and let G be an abelian group. Show that if for all $n \in \mathbb{N}_0$ the induced map

$$f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$$

is an isomorphism, then it is also true that for any $n \in \mathbb{N}_0$ and any abelian group G the induced map

$$f_*\colon \mathrm{H}_n(X, A; G) \to \mathrm{H}_n(Y, B; G)$$

is an isomorphism.

Exercise 160.25. Show that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

Exercise 160.26. Given a pair of topological spaces (X, A) and given $n \in \mathbb{N}_0$ we consider

$$\mathrm{T}_n(X, A) := \{g \in \mathrm{H}_n(X, A) \,|\, \text{there exists } m \in \mathbb{N} \text{ with } m \cdot g = 0\}.$$

Furthermore we denote by $\partial_n\colon \mathrm{T}_n(X, A) \to \mathrm{T}_{n-1}(A)$ the restriction of the usual connecting homomorphism $\partial_n\colon \mathrm{H}_n(X, A) \to \mathrm{H}_{n-1}(A)$. Do these groups $\mathrm{T}_n(X, A)$ define a homology theory?

CHAPTER 161

Exact functors and homology

The following question naturally arises from the discussion in the previous chapter:

Question 161.1. Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?

We will discuss this question in great detail in this and in the following chapter.

This chapter is organized as follows:

- In Section 161.1 we will introduce exact functors.
- In Section 161.2 we will study whether taking tensor products is exact.
- In Section 161.3 we will see that taking tensor products with torsion-free abelian groups is exact.
- In Appendix 161.4 we will use tensor products to give a new definition of the rank of an abelian group
- In Section 161.5 we will provide exercises for Chapter 161.

In the next Chapter 162 we will prove the Universal Coefficient Theorem 162.16.

161.1. **Exact functors.** Before we can hope to answer Question 161.1 we have to introduce many more definitions and algebraic tools. We start out with the following definition from category theory.

Definition 161.2. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(1) We say that F is left-exact, if for every exact sequence[1704]

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C)$$

is also exact.

(2) We say that F is right-exact, if for every exact sequence

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of abelian groups the sequence

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0$$

is also exact.

(3) We say that F is exact if it is left-exact and right-exact.

Remark. Let $F\colon \mathcal{A}b\mathcal{G}roup \to \mathcal{A}b\mathcal{G}roup$ be an additive covariant functor and let $\varphi\colon A \to B$ be a homomorphism between abelian groups.

(1) If F is left-exact, then "taking F" commutes with "taking kernels". More precisely, if we apply the left-exact functor F to the exact sequence $0 \to \ker(\varphi) \xrightarrow{\iota} A \xrightarrow{\varphi} B$, then we obtain the *exact* sequence $0 \to F(\ker(\varphi)) \xrightarrow{F(\iota)} F(A) \xrightarrow{F(\varphi)} F(B)$. From the Exactness Lemma 145.2 (3) we obtain that
$$F(\ker(\varphi\colon A \to B)) \xrightarrow{F(\iota)} \ker(F(\varphi)\colon F(A) \to F(B))$$
is an isomorphism.

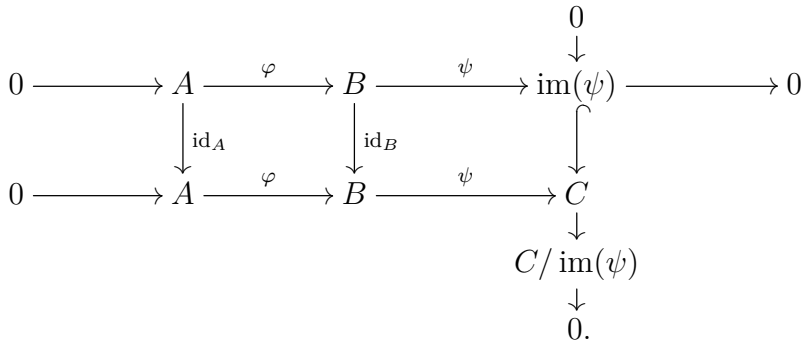
(2) If F is right-exact, then "taking F" commutes with "taking cokernels". More precisely, if we apply the right-exact functor F to the exact sequence $A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker}(\varphi) \to 0$, then we obtain the *exact* sequence $F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\pi)} F(\operatorname{coker}(\varphi)) \to 0$. From the Exactness Lemma 145.2 (3) we obtain that $F(\pi)$ induces an isomorphism
$$\operatorname{coker}(F(\varphi)\colon F(A) \to F(B)) \xrightarrow{\text{induced by } F(\pi)} F(\operatorname{coker}(\varphi)).$$ □

Lemma 161.3. (Exact Functor Lemma) Let $F\colon \mathcal{A}b\mathcal{G}roup \to \mathcal{A}b\mathcal{G}roup$ be an additive covariant functor. The following two statements are equivalent:

(1) F is exact.

(2) For every short exact sequence
$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$
of abelian groups the sequence
$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0$$
is also exact.

Proof. The (1)⇒(2)-direction is clear. Now suppose that F satisfies (2). We first show that F is left-exact. Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ be an exact sequence of abelian groups. We consider the following little diagram:



We apply the functor F to this diagram. Since we assume that F satisfies condition (2) we see that the resulting top horizontal sequence and the resulting right vertical sequence are

[1704]Note that we know by the Additive Functor–Basics Lemma 160.2 (2) that $F(0) \cong 0$.

exact. It is not hard to see that this implies that the bottom horizontal sequence is also exact. The same type of argument also shows that F is right-exact. ■

The following lemma shows why we are suddenly interested in *exact* additive covariant functors.

Lemma 161.4. (Exact Functor–Homology Lemma) Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor and let (C_*, ∂_*) be a chain complex. If F is exact, then for any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$F(\mathrm{H}_k(C_*, \partial_*)) \ \to \ \mathrm{H}_k(F(C_*), F(\partial_*)).$$

Proof. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an *exact* additive covariant functor and let (C_*, ∂_*) be a chain complex. Let $k \in \mathbb{N}_0$. We denote by $\iota\colon \ker(\partial_k\colon C_k \to C_{k-1}) \to C_k$ the inclusion. Since $\operatorname{im}(\partial_{k+1}\colon C_{k+1} \to C_k) \subseteq \ker(\partial_k\colon C_k \to C_{k-1})$ we see that there exists a homomorphism $\widetilde{\partial}_{k+1}\colon C_{k+1} \to \ker(\partial_k\colon C_k \to C_{k-1})$ with $\partial_{k+1} = \iota \circ \widetilde{\partial}_{k+1}\colon C_{k+1} \to C_k$.

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & C_{k+1} & & & & \\
 & & & & \downarrow{\scriptstyle \widetilde{\partial}_{k+1}} & \searrow^{\partial_{k+1}} & & & \\
0 & \longrightarrow & \ker(\partial_k\colon C_k \to C_{k-1}) & \xrightarrow[\iota]{} & C_k & \xrightarrow[\partial_k]{} & C_{k-1} & & \\
 & & \downarrow{\scriptstyle \pi} & & & & & & \\
 & & \mathrm{H}_k(C_*) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

Note that the vertical and the horizontal sequence are exact. Since F is an additive covariant functor we obtain from the Additive Functor–Basics Lemma 314.2 the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & F(C_{k+1}) & & & & \\
 & & & & \downarrow{\scriptstyle F(\widetilde{\partial}_{k+1})} & \searrow^{F(\partial_{k+1})} & & & \\
0 & \longrightarrow & F(\ker(\partial_k\colon C_k \to C_{k-1})) & \xrightarrow[F(\iota)]{} & F(C_k) & \xrightarrow[F(\partial_k)]{} & F(C_{k-1}) & & \\
 & & \downarrow{\scriptstyle F(\pi)} & & & & & & \\
 & & F(\mathrm{H}_k(C_*)) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

Since F is exact we see that the vertical and the horizontal sequence are still exact. It follows from the exactness that the following maps are isomorphisms:

$$F(\mathrm{H}_k(C_*)) \xleftarrow{F(\pi)} \frac{F\Big(\ker(C_k \xrightarrow{\partial_k} C_{k-1})\Big)}{\operatorname{im}(F(\widetilde{\partial}_{k+1}))} \xrightarrow{F(\iota)} \frac{\ker\Big(F(C_k) \xrightarrow{F(\partial_k)} F(C_{k-1})\Big)}{\operatorname{im}\Big(F(C_{k+1}) \xrightarrow{F(\partial_{k+1})} F(C_k)\Big)} = \mathrm{H}_k(F(C_*)).$$

Since all the isomorphisms involved are natural we see also that the isomorphism from left to right is natural. ■

161.2. **Tensor products and exactness.** It is now natural to ask whether taking tensor products is an exact functor. We start out with some bad news:

Example. Given a fixed abelian group G we have studied in great detail the covariant functor $A \mapsto A \otimes G$. For better or worse we will now see that in general this is not a *left-exact* functor. For example for any $k \in \mathbb{N}_{\geq 2}$ the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \xrightarrow{n \mapsto [n]} \mathbb{Z}/k\mathbb{Z}$$

is exact. But after tensoring with $G = \mathbb{Z}/k\mathbb{Z}$ we obtain, by the Free Module–Tensor-with-Ring Lemma 159.20 and the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10, the following diagram with vertical isomorphisms:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/k\mathbb{Z} & \xrightarrow[0]{\cdot k} & \mathbb{Z}/k\mathbb{Z} & \xrightarrow{\mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}} & \mathbb{Z}/k\mathbb{Z} \\
 & & \cong\downarrow [n]\mapsto 1\otimes[n] & & \cong\downarrow [n]\mapsto 1\otimes[n] & & \cong\downarrow [n]\mapsto [1]\otimes[n] \\
0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z}/k\mathbb{Z} & \xrightarrow[(\cdot k)\otimes\mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}]{} & \mathbb{Z}\otimes\mathbb{Z}/k\mathbb{Z} & \xrightarrow[(n\mapsto[n])\otimes\mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}]{} & \mathbb{Z}/k\mathbb{Z}\otimes\mathbb{Z}/k\mathbb{Z}
\end{array}$$

It follows from $(k \cdot a) \otimes b = a \otimes (k \cdot b)$ that the diagram commutes. Note that the top left horizontal homomorphism $\mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ is not a monomorphism, thus the top horizontal sequence is not exact. But this implies that the bottom horizontal sequence is not exact either. □

In contrast to the previous example we will now see that tensoring with an abelian group G is *right-exact*.

Lemma 161.5. (Tensor–Right-Exact Lemma) For every abelian group G the functor

$$\begin{array}{ccc} \mathcal{A}b\mathcal{G}roup & \to & \mathcal{A}b\mathcal{G}roup \\ A & \mapsto & A \otimes G \end{array}$$

is right-exact.

We give two different proofs of the Tensor–Right-Exact Lemma 161.5, the first one is short and formal. The second one is more elementary and self-contained:

First proof of the Tensor–Right-Exact Lemma 161.5. [1705] Let G be an abelian group. By the Tensor Product–Colimit Proposition 159.14 we know that the functor

$$\begin{array}{ccc} \mathcal{A}b\mathcal{G}roup & \to & \mathcal{A}b\mathcal{G}roup \\ A & \mapsto & A \otimes G \end{array}$$

preserves all colimits. Next recall that in Definition 6.23 we defined pushouts as colimits. Thus the desired statement thus follows from the following claim:

Claim. Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be homomorphisms of abelian groups. Then the following two statements are equivalent:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \text{ is exact} \iff \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & \ulcorner & \downarrow\beta \\ 0 & \longrightarrow & C. \end{array} \text{ is a pushout diagram}$$

[1705] This is the proof which convinced the author that it is worth introducing the notion of adjoint functors.

Proof. As we saw in Exercise 145.5, the claim follows from elementary algebra which we leave to the reader. ∎

Second proof of the Tensor–Right-Exact Lemma 161.5. Let

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be an exact sequence of abelian groups and let G be an abelian group. We have to show that the sequence

$$A \otimes G \xrightarrow{i\otimes \mathrm{id}_G} B \otimes G \xrightarrow{j\otimes \mathrm{id}_G} C \otimes G \to 0$$

is also exact.

(a) Since $j\colon B \to C$ is an epimorphism it follows immediately from the Tensor Product–Basics Lemma 159.3 (2) that the homomorphism

$$\begin{aligned} j \otimes \mathrm{id}_G\colon B \otimes G &\to C \otimes G \\ \sum_{i=1}^{n} b_i \otimes g_i &\mapsto \sum_{i=1}^{n} j(b_i) \otimes g_i \end{aligned}$$

is also an epimorphism.

(b) As on so many other occasions we obtain from the Tensor–Map Composition Lemma 159.7 that we have $(j \otimes \mathrm{id}_G) \circ (i \otimes \mathrm{id}_G) = (j \circ i) \otimes \mathrm{id}_G = 0$, i.e. we have

$$\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \subseteq \ker(j \otimes \mathrm{id}_G\colon B \otimes G \to C \otimes G).$$

Note that this implies that we get a well-defined homomorphism

$$\begin{aligned} \Psi\colon (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) &\to C \otimes G \\ \left[\sum_{i=1}^{k} b_i \otimes g_i\right] &\mapsto \sum_{i=1}^{k} j(b_i) \otimes g_i. \end{aligned}$$

(c) By (a) and (b) it remains to show that

$$\ker(j \otimes \mathrm{id}_G\colon B \otimes G \to C \otimes G) \subseteq \mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G).$$

Put differently it remains to show that the above homomorphism

$$\begin{aligned} \Psi\colon (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) &\to C \otimes G \\ \left[\sum_{i=1}^{k} b_i \otimes g_i\right] &\mapsto \sum_{i=1}^{k} j(b_i) \otimes g_i \end{aligned}$$

is a monomorphism. We will do so by providing a left inverse for Ψ, i.e. we will provide a map $\Phi\colon C \otimes G \to (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G)$ with $\Phi \circ \Psi = \mathrm{id}_{(B\otimes G)/\mathrm{im}(i\otimes \mathrm{id}_G)}$. As a first step we prove the following claim

Claim 1. The map

$$\begin{aligned} \varphi\colon C \times G &\to (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \\ (c, g) &\mapsto [b \otimes g] \end{aligned}$$

where $b \in B$ is an element with $j(b) = c$

is well-defined.

Proof. Let $(c, g) \in C \times G$. Since j is by hypothesis surjective we know that there exists a $b \in B$ with $j(b) = c$. We have to show that $\varphi(c, g)$ is independent of the choice of b. So let $b, b' \in B$ with $j(b) = j(b') = c$, i.e. with $j(b - b') = 0$. From the exactness of the original short exact sequence at B it follows that there exists an $a \in A$ with

$b - b' = i(a)$. Then we obtain that

$$b \otimes g - b' \otimes g \underset{\substack{\uparrow \\ \text{by the Tensor Product–Universal} \\ \text{Property Lemma 159.4 (1)}}}{=} (b - b') \otimes g \;=\; i(a) \otimes g \;\in\; \operatorname{im}(i \otimes \mathrm{id}_G \colon A \otimes G \to B \otimes G).$$

Thus we have proved that φ is well-defined. ⊞

The map φ is easily seen to bilinear. It follows from the Tensor Product–Universal Property Lemma 159.4 (2) that φ induces a well-defined homomorphism

$$\begin{aligned} \Phi \colon C \otimes G &\to (B \otimes G)/\operatorname{im}(i \otimes \mathrm{id}_G \colon A \otimes G \to B \otimes G) \\ \sum_{i=1}^{k} c_i \otimes g_i &\mapsto \sum_{i=1}^{k} \varphi(c_i, g_i). \end{aligned}$$

It remains to prove the following claim:

Claim 2. We have $\Phi \circ \Psi = \mathrm{id}_{(B \otimes G)/\operatorname{im}(i \otimes \mathrm{id}_G)}$.

Proof. We make two observations:

(a) It follows easily from the definitions that for all $b \in B$ and $g \in G$ we have

$$(\Phi \circ \Psi)([b \otimes g]) \;=\; \Phi(j(b) \otimes g) \;=\; [b \otimes g] \in (B \otimes G)/\operatorname{im}(i \otimes \mathrm{id}_G).$$

(b) By the Tensor Product–Basics Lemma 159.3 (2) we know that the elements $b \otimes g$ generate $B \otimes G$.

It follows from the two observations that $\Phi \circ \Psi = \mathrm{id}_{(B \otimes G)/\operatorname{im}(i \otimes \mathrm{id}_G)}$. ■

Using the Tensor–Right-Exact Lemma 161.5 we can now provide a new, arguably more conceptual, proof of the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10.

Lemma 159.10 (Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma) Let $k \in \mathbb{N}$ and let A be an abelian group. The maps

$$\begin{aligned} \Theta \colon \mathbb{Z}/k\mathbb{Z} \otimes A &\to A/k \cdot A \\ \sum_{i=1}^{m} [n_i] \otimes a_i &\mapsto \Big[\sum_{i=1}^{m} n_i \cdot a_i\Big] \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Omega \colon A/k \cdot A &\to \mathbb{Z}/k\mathbb{Z} \otimes A \\ [a] &\mapsto [1] \otimes a \end{aligned}$$

are well-defined, they are inverses of one another and they are natural isomorphisms.

Proof. We already discussed in our original proof of the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 that the two maps are well-defined natural homomorphisms. It is clear that $\Theta \circ \Omega = \mathrm{id}_{A/k \cdot A}$. So let us concentrate on the most interesting aspect of the proof, namely let us prove that $\Omega \colon A/k \cdot A \to \mathbb{Z}/k\mathbb{Z} \otimes A$ is an isomorphism. We consider the exact sequence

$$\mathbb{Z} \xrightarrow{a \mapsto k \cdot a} \mathbb{Z} \xrightarrow{n \mapsto [n]} \mathbb{Z}/k\mathbb{Z} \to 0.$$

Tensoring this exact sequence with A gives use the following diagram:

$$\begin{array}{ccccccc} \mathbb{Z} \otimes A & \xrightarrow{(a \mapsto a \cdot k) \otimes \mathrm{id}_A} & \mathbb{Z} \otimes A & \xrightarrow{n \otimes a \mapsto [n] \otimes a} & \mathbb{Z}/k\mathbb{Z} \otimes A & \longrightarrow & 0 \\ {\scriptstyle a \mapsto 1 \otimes a}\uparrow \cong & & {\scriptstyle a \mapsto 1 \otimes a}\uparrow \cong & & {\scriptstyle [a] \mapsto [1] \otimes a}\uparrow \Omega & & \\ A & \xrightarrow{a \mapsto k \cdot a} & A & \xrightarrow{a \mapsto [a]} & A/k \cdot A & \longrightarrow & 0. \end{array}$$

We make the following observations:

(1) It follows from the Tensor–Right-Exact Lemma 161.5 that the top sequence is exact.

(2) The two vertical maps to the left are the natural isomorphisms given by the Tensor-with-$\mathbb{Z}$–Lemma 159.9. It follows from the naturality that the square to the left commutes.

(3) It follows immediately from the definitions that the square to the right commutes.

It follows from the above and the Five Lemma 145.11 that the vertical map to the right is also an isomorphism. ■

161.3. **Tensor products with torsion-free abelian groups.** On page 3093 we saw that tensoring with $\mathbb{Z}/k\mathbb{Z}$ is not exact. The following proposition shows that in contrast tensoring with torsion-free abelian groups is in fact exact:

Proposition 161.6. (Tensor-with-Torsion Free Proposition) For any torsion-free abelian group G the functor $A \mapsto A \otimes G$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is exact.

Proof. Let G be a torsion-free abelian group. We want to show that tensoring with G is exact. By the Exact Functor Lemma 161.3 it suffices to show that for every short exact sequence of abelian groups

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0,$$

the "tensored" sequence is still exact. First we consider the special case that G is a finitely generated torsion-free abelian group:

Claim 1. If G is in fact a finitely generated torsion-free abelian group, then the sequence $0 \to A \otimes G \xrightarrow{\varphi\otimes\mathrm{id}_G} B \otimes G \xrightarrow{\psi\otimes\mathrm{id}_G} C \otimes G \to 0$ is exact.

Proof. Since G is a *finitely generated* torsion-free abelian group we obtain from the Abelian Group-Classification Theorem 97.24 that there exists an $m \in \mathbb{N}_0$ and an isomorphism $\Xi\colon \bigoplus_{i=1}^{m} \mathbb{Z} \to G$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{m} A & \xrightarrow{\operatorname{diag}_{i=1,\dots,m}(\varphi)} & \bigoplus_{i=1}^{m} B & \xrightarrow{\operatorname{diag}_{i=1,\dots,m}(\psi)} & \bigoplus_{i=1}^{m} C & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{m} (A\otimes\mathbb{Z}) & \xrightarrow{\operatorname{diag}_{i=1,\dots,m}(\varphi\otimes\mathrm{id}_{\mathbb{Z}})} & \bigoplus_{i=1}^{m} (B\otimes\mathbb{Z}) & \xrightarrow{\operatorname{diag}_{i=1,\dots,m}(\psi\otimes\mathrm{id}_{\mathbb{Z}})} & \bigoplus_{i=1}^{m} (C\otimes\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & A\otimes\left(\bigoplus_{i=1}^{m}\mathbb{Z}\right) & \xrightarrow{\varphi\otimes\mathrm{id}_{\mathbb{Z}^m}} & B\otimes\left(\bigoplus_{i=1}^{m}\mathbb{Z}\right) & \xrightarrow{\psi\otimes\mathrm{id}_{\mathbb{Z}^m}} & C\otimes\left(\bigoplus_{i=1}^{m}\mathbb{Z}\right) & \longrightarrow & 0 \\
 & & \mathrm{id}_A\otimes\Xi\downarrow\cong & & \mathrm{id}_B\otimes\Xi\downarrow\cong & & \mathrm{id}_C\otimes\Xi\downarrow\cong & & \\
0 & \longrightarrow & A\otimes G & \xrightarrow{\varphi\otimes\mathrm{id}_G} & B\otimes G & \xrightarrow{\psi\otimes\mathrm{id}_G} & C\otimes G & \longrightarrow & 0.
\end{array}$$

The top vertical maps are the natural isomorphisms from the Tensor-with-$\mathbb{Z}$–Lemma 159.9. Note that naturality implies that the top diagram commutes. The middle vertical maps are the natural isomorphisms from the Tensor–Direct Sum Lemma 159.15. Again naturality implies that the middle part of the diagram commutes. The bottom vertical maps are induced by the isomorphism Ξ, thus they are isomorphisms. It follows from the Tensor–Map Composition Lemma 159.7 that the bottom part of the diagram commutes.

Evidently the top sequence is exact. It follows from the above discussion that the bottom sequence is also exact. ⊞

Now we turn to the general case:

Claim 2. For every torsion-free abelian group G the corresponding "tensored" sequence $0 \to A \otimes G \xrightarrow{\varphi\otimes \mathrm{id}_G} B \otimes G \xrightarrow{\psi\otimes G} C \otimes G \to 0$ is exact.

Proof. By the Tensor–Right-Exact Lemma 161.5 we only need to worry about the left part of the sequence. In other words, it suffices to prove that for any monomorphism $\varphi\colon A \to B$ of abelian groups the induced homomorphism $\varphi \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G$ is also injective.

We introduce and recall the following notation:

- We denote by $R(B,G)$ the set of elements of $\mathbb{Z}^{(B\times G)}$ that are of the form
$$((b+b'),g) - ((b,g)+(b',g)) \qquad \text{or} \qquad (b,(g+g')) - ((b,g)+(b,g'))$$
with $b, b' \in B$ and $g, g' \in G$.
- Let $N(B,G)$ be the subgroup of $\mathbb{Z}^{(B\times G)}$ that is generated by $R(B,G)$.
- By definition $B \otimes G = \mathbb{Z}^{(B\times G)}/N(B,G)$.
- Given $b \in B$ and $g \in G$ we denote by $b\otimes g$ the image of (b,g) in $B\otimes G = \mathbb{Z}^{(B\times G)}/N(B,G)$.

Now we turn to the actual proof.

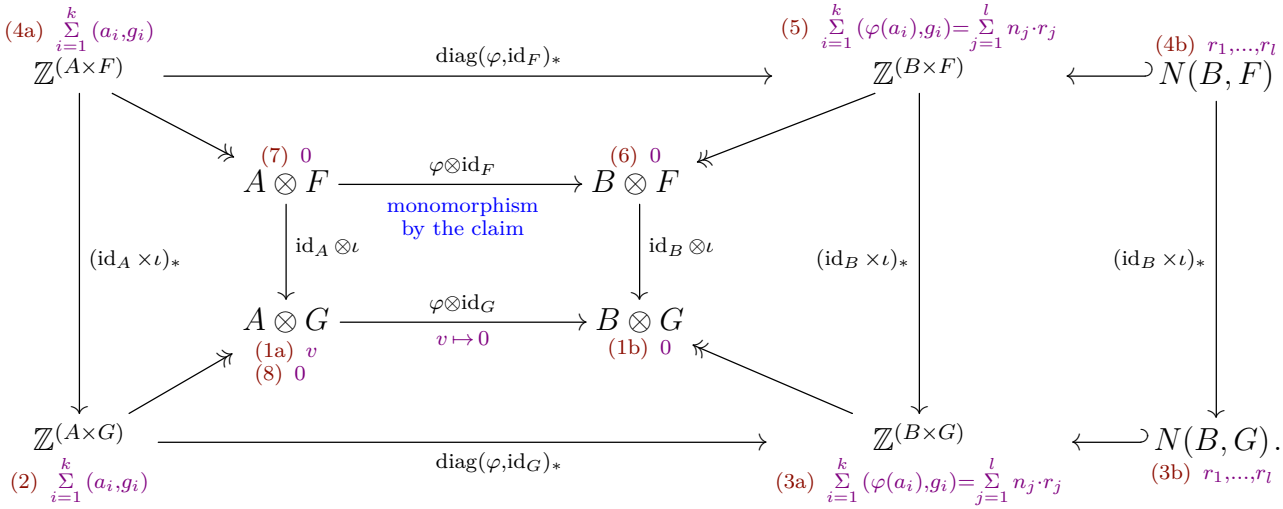
(1) Let $v \in \ker(\varphi \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes B)$. We want to show that $v = 0$.

(2) We write $v = \sum_{i=1}^{k} a_i \otimes g_i \in A \otimes G$ and we consider $\sum_{i=1}^{k} (a_i, g_i) \in \mathbb{Z}^{(A\times G)}$.

(3) We have the following equality in $B \otimes G = \mathbb{Z}^{(B\times G)}/N(B,G)$:
$$\Big[\sum_{i=1}^{k} (\varphi(a_i), g_i)\Big] \;=\; \sum_{i=1}^{k} \varphi(a_i) \otimes g_i \;=\; (\varphi \otimes \mathrm{id}_G)\Big(\sum_{i=1}^{k} a_i \otimes g_i\Big) \;=\; (\varphi \otimes \mathrm{id}_G)(v) \;=\; 0.$$
By definition of $N(B,G)$ we see that there exist $r_1, \dots, r_l \in R(B,G)$ and $n_1, \dots, n_l \in \mathbb{Z}$ such that
$$\sum_{i=1}^{k} (\varphi(a_i), g_i) \;=\; \sum_{j=1}^{l} n_j \cdot r_j \;\in\; \mathbb{Z}^{(B\times G)}.$$

Let F be the subgroup of G that is generated by $g_1, \dots, g_k$ and by all the second entries appearing in the summands of $r_1, \dots, r_l \in R(B,G)$. By construction F is finitely generated. Furthermore, as the subgroup of the torsion-free group G we see that F is also torsion-free. In summary, F is a finitely generated torsion-free group. We denote by $\iota\colon F \to G$ the inclusion homomorphism.

We consider the following diagram:



Note that it follows from the Tensor–Map Composition Lemma 159.7 that the small inner rectangle commutes. All other quadrilaterals commute basically by definition.

We continue with the following observations:

(4) (a) By definition of F we have $\sum_{i=1}^{k}(a_i, g_i) \in \mathbb{Z}^{(A\times F)} \subseteq \mathbb{Z}^{(A\times G)}$.

(b) By definition of F we have $r_1, \ldots, r_l \in R(B, F) \subseteq R(B, G)$.

(5) (a) It follows from (4a) that $\sum_{i=1}^{k}(\varphi(a_i), g_i) \in \mathbb{Z}^{(B\times F)} \subseteq \mathbb{Z}^{(B\times G)}$.

(b) It follows from (4b) that $\sum_{j=1}^{l} n_j \cdot r_j \in N(B, F) \subseteq N(B, G)$.

It follows from (3) that these two terms are equal in $\mathbb{Z}^{(B\times F)}$.

(6) By (5) we know that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $\mathbb{Z}^{(B\times F)}$ is a linear combination of $r_1, \ldots, r_l$. Since $r_1, \ldots, r_l$ lie in the image of $N(B, F) \to N(B, G)$ we see that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $\mathbb{Z}^{(B\times F)}$ lies in the image of $N(B, F)$. It follows that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $B \otimes F = \mathbb{Z}^{(B\times F)}/N(B, F)$ is zero.

(7) Since F is a free abelian group it follows from Claim 1 that the horizontal map $\varphi \otimes \mathrm{id}_F \colon A\otimes F \to B\otimes F$ is a monomorphism. It follows from (6) that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $A \otimes F$ is zero.

(8) It follows from (7) that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $A \otimes G$ is zero. But the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $A \otimes G$ is precisely v. Thus we have shown that $v = 0$. ∎

The combination of the Exact Functor–Homology Lemma 161.4 and the Tensor-with-Torsion Free Proposition 161.6 shows that for torsion-free abelian groups G we have isomorphisms $\mathrm{H}_n(C_*)\otimes G \to \mathrm{H}_n(C_*\otimes G)$. The following lemma allows us to specify an explicit isomorphism:

Lemma 161.7. (Homology-Multiplication Lemma) Let (C_*, ∂_*) be a chain complex, let G be an abelian group and let $n \in \mathbb{N}_0$. The map[1706]

$$\begin{array}{rcl} \mu\colon \mathrm{H}_n(C_*) \otimes G & \to & \mathrm{H}_n(C_* \otimes G) \\ \sum_{i=1}^{k} [\sigma_i] \otimes g_i & \mapsto & \sum_{i=1}^{k} [\sigma_i \otimes g_i] \end{array}$$

is a well-defined natural[1707] homomorphism. If G is a commutative ring, then this is a homomorphism of G-modules.

Proof. It is straightforward to show that the map

$$\begin{array}{rcl} \mathrm{H}_n(C_*) \times G & \to & \mathrm{H}_n(C_*; G) \\ ([\sigma], g) & \mapsto & [\sigma \otimes g] \end{array}$$

[1706] We will discuss in Exercise 161.13 whether the "obvious converse" is also well-defined.

[1707] Here "natural" means that for a chain map $f_*\colon C_* \to D_*$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(C_*) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(C_* \otimes G) \\ \downarrow{\scriptstyle f_*\otimes\varphi} & & \downarrow{\scriptstyle (f\otimes\varphi)_*} \\ \mathrm{H}_n(D_*) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(D_* \otimes G') \end{array}$$

is well-defined and bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 159.4 (1) that the given map is well-defined. It follows easily from the definitions that the resulting homomorphism μ is natural. If G is a commutative ring, then one can easily verify that the given map is an isomorphism of G-modules. ■

We can now formulate the following convenient proposition.

Proposition 161.8. (Torsion–Free Coefficients Proposition) Let (C_*, ∂_*) be a chain complex, let G be an abelian group and let $n \in \mathbb{N}_0$. If G is a torsion-free abelian group, then the map

$$\mu\colon \mathrm{H}_n(C_*) \otimes G \;\to\; \mathrm{H}_n(C_*; G)$$

from the Homology-Multiplication Lemma 161.7 is a natural isomorphism. In particular, if (X, A) is a pair of topological spaces, then the map

$$\mu\colon \mathrm{H}_n(X, A) \otimes G \;\to\; \mathrm{H}_n(X, A; G)$$

is a natural isomorphism.

Example. We have the following calculation:

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^m; \mathbb{R}) \overset{\cong}{\underset{\uparrow}{\leftarrow}} \mathrm{H}_k(\mathbb{R}\mathrm{P}^m) \otimes \mathbb{R} \underset{\uparrow}{\cong} \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ or if } k = m \text{ and } m \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

by the Torsion–Free Coefficients Proposition 161.8

follows from the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 together with the Tensor-with-$\mathbb{Z}$–Lemma 159.9 and the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 □

Proof. It follows from the Tensor-with-Torsion Free Proposition 161.6 that we can apply the Exact Functor–Homology Lemma 161.4 in our context. This implies that we have a natural isomorphism

$$\mathrm{H}_n(C_*) \otimes G \;\to\; \mathrm{H}_n(C_*; G).$$

In fact, in the Exact Functor–Homology Lemma 161.4 this natural isomorphism is given explicitly: For an exact additive covariant functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ and for the chain complex C_* we showed that we get the following two isomorphisms:

$$F(\mathrm{H}_k(C_*)) \underset{\cong}{\xleftarrow{F(\pi)}} \frac{F\Big(\ker(C_k \xrightarrow{\partial_k} C_{k-1})\Big)}{\mathrm{im}(F(\widetilde{\partial}_{k+1}))} \underset{\cong}{\xrightarrow{F(\iota)}} \frac{\ker\Big(F(C_k) \xrightarrow{F(\partial_k)} F(C_{k-1})\Big)}{\mathrm{im}\Big(F(C_{k+1}) \xrightarrow{F(\partial_{k+1})} F(C_k)\Big)} = \mathrm{H}_k(F(C_*)).$$

It follows immediately from the definitions that in our context this isomorphism is precisely the homomorphism μ from the Homology-Multiplication Lemma 161.7. ■

161.4. **Appendix: The rank of an abelian group.** We start out by recalling Definition 97.15:

Definition 161.9. Let G be an abelian group.

(1) Given a subset $S \subseteq G$ we denote by $\Xi_S\colon \mathbb{Z}^{(S)} \to G$ the natural homomorphism.
(2) We define the $\mathrm{rank}(G) \in \mathbb{N}_0 \cup \{\infty\}$ as the largest cardinality[1708] of a subset $S \subseteq G$ such that the natural homomorphism $\Xi_S\colon \mathbb{Z}^{(S)} \to G$ is a monomorphism.

The following proposition gives us an alternative, and arguably much more elegant, point of view on ranks of abelian groups:

[1708] For simplicity we view the cardinalities as elements of $\mathbb{N}_0 \cup \{\infty\}$, i.e. here we do not distinguish between the different infinite cardinalities.

Proposition 161.10. (Rank-via-Dimensions Proposition) For any abelian group G and any field $\mathbb{F}$ of characteristic zero we have

$$\operatorname{rank}(G) \;=\; \dim_{\mathbb{F}}(G \otimes \mathbb{F}).$$

Remark. Let A be an abelian group. We can view A as a $\mathbb{Z}$-module. It follows from the Rank-via-Dimensions Proposition 161.10 that the rank of the abelian group A, see Definition 97.15, agrees with the rank of the corresponding $\mathbb{Z}$-module as introduced in Definition 2.13. □

Proof of the Rank-via-Dimensions Proposition 161.10 for $\mathbb{F} = \mathbb{Q}$. In the following let G be an abelian group. We prove the two inequalities "$\leq$" and "$\geq$" separately:

"$\leq$" Let S be a subset such that the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to G$ is a monomorphism. Note that

$$0 \;\to\; \mathbb{Z}^{(S)} \;\xrightarrow{\Xi_S}\; G \;\to\; G/\Xi_S(\mathbb{Z}^{(S)}$$

is exact. Since $\mathbb{Q}$ has characteristic zero we obtain from the Tensor-with-Torsion Free Proposition 161.6 that

$$0 \;\to\; \mathbb{Z}^{(S)} \otimes \mathbb{Q} \;\xrightarrow{\Xi_S \otimes \mathrm{id}_{\mathbb{Q}}}\; G \otimes \mathbb{Q} \;\to\; (G/\Xi_S(\mathbb{Z}^{(S)}) \otimes \mathbb{Q}$$

is also exact. It follows that

$$\#S \;=\; \dim_{\mathbb{Q}}(\mathbb{Q}^{(S)}) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(\mathbb{Z}^{(S)} \otimes \mathbb{Q}) \underset{\uparrow}{\leq} \dim_{\mathbb{Q}}(G \otimes \mathbb{Q}).$$

by the Matrix–Tensor-with-Ring Lemma 159.20 — by the above

"$\geq$" It suffices to show that for every $n \in \mathbb{N}_0$ with $\dim_{\mathbb{Q}}(G \otimes \mathbb{Q}) \geq n$ there exists a monomorphism $\mathbb{Z}^n \to G$.[1709] So suppose we have $n \in \mathbb{N}_0$ with $\dim_{\mathbb{Q}}(G \otimes \mathbb{Q}) \geq n$. This means that we have a monomorphism $\varphi \colon \mathbb{Q}^n \to G \otimes \mathbb{Q}$. Before we continue let us prove the following claim:

Claim 1. Given any $v \in G \otimes \mathbb{Q}$ there exists a $g \in \mathbb{Q}$ and a $t \in \mathbb{N}$ with $t \cdot v = g \otimes 1$.

Proof. Let $v \in Q \otimes \mathbb{Q}$. By the Tensor Product–Basics Lemma 159.3 (2) there exist $g_1, \ldots, g_m \in G$, $s_1, \ldots, s_m \in \mathbb{Z}$ and $t_1, \ldots, t_m \in \mathbb{N}$ with $v = \sum_{i=1}^{m} g_i \otimes \frac{s_i}{t_i}$. We set $t \coloneqq t_1 \cdot \ldots \cdot t_m$. Note that in $G \otimes \mathbb{Q}$ we have the following equalities:

$$t \cdot v \;=\; \sum_{i=1}^{m} t \cdot (g_i \otimes \tfrac{s_i}{t_i}) \underset{\uparrow}{=} \sum_{i=1}^{m} g_i \otimes \Big(s_i \cdot \prod_{j \neq i} t_j \cdot 1 \Big) \underset{\uparrow}{=} \Big(\sum_{i=1}^{m} \Big(s_i \cdot \prod_{j \neq i} t_j \Big) \cdot g_i \Big) \otimes 1.$$

by the Tensor Product–Universal Property Lemma 159.4 (1) ⊞

We denote by $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ the usual basis vectors of $\mathbb{Q}^n$. It follows from Claim 1 that for each $i \in \{1, \ldots, n\}$ there exists a $g_i \in \mathbb{Z}$ and a $t_i \in \mathbb{N}$ such that $\varphi(t_i \cdot \mathfrak{e}_i) = g_i \otimes 1$. We consider the homomorphism $\Xi \colon \mathbb{Z}^n \to G$ that is given by $\Xi(\mathfrak{e}_i) = g_i$. It remains to prove the following claim:

Claim 2. The homomorphism $\Xi \colon \mathbb{Z}^n \to G$ is a monomorphism.

[1709] This is precisely the content of Exercise 159.14.

Proof. We consider the following diagram:

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\quad\Xi\quad} & G \\ \downarrow{\scriptstyle \mathfrak{e}_i \mapsto t_i \cdot \mathfrak{e}_i} & & \downarrow{\scriptstyle v \mapsto v\otimes 1} \\ \mathbb{Q}^n & \xrightarrow{\quad\varphi\quad} & G \otimes \mathbb{Q}. \end{array}$$

It follows immediately from the definitions that the diagram commutes. Since the t_i are all non-zero we see that the left vertical map is a monomorphism. By hypothesis the bottom horizontal map is a monomorphism. It follows that the top horizontal map is also a monomorphism. ∎

Proof of the Rank-via-Dimensions Proposition 161.10 for any $\mathbb{F}$. In the following let G be an abelian group and let $\mathbb{F}$ be a field of characteristic zero. Note that we can view $\mathbb{Q}$ as a subfield of $\mathbb{F}$. To deal with this situation we need the generalization of the notion of a tensor product over abelian groups to the tensor product of $\mathbb{Q}$-vector spaces. We then see that

$$\begin{array}{ccccccc} \operatorname{rank}(G) & = & \dim_{\mathbb{Q}}(G\otimes\mathbb{Q}) & = & \dim_{\mathbb{F}}((G\otimes\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{F}) & = & \dim_{\mathbb{F}}(G\otimes\mathbb{F}). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the above} & & \text{by the Tensor Product} & & \text{follows from the Tensor-with-Base} & \\ & & & \text{Cancel Ring-Lemma 312.12} & & \text{Ring Lemma 312.10} & \end{array}$$

∎

As an illustration of these techniques we now give a new proof of the following proposition:

Proposition 97.18. (Rank Additivity Proposition) Given any short exact sequence of abelian groups

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

we have the equality in $\mathbb{N}_0 \cup \{\infty\}$:

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Proof. Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be a short exact sequence of abelian groups. It follows from the Tensor-with-Torsion Free Proposition 161.6 and the Tensor-with-Ring Lemma 159.18 that

$$0 \to A\otimes\mathbb{Q} \xrightarrow{\alpha\otimes\mathrm{id}_{\mathbb{Q}}} B\otimes\mathbb{Q} \xrightarrow{\beta\otimes\mathrm{id}_{\mathbb{Q}}} C\otimes\mathbb{Q} \to 0$$

is a short exact sequence of $\mathbb{Q}$-vector spaces. We obtain the following equalities in $\mathbb{N}_0\cup\{\infty\}$:

$$\begin{array}{ccccccc} \operatorname{rank}(B) & = & \dim_{\mathbb{Q}}(B\otimes\mathbb{Q}) & = & \dim_{\mathbb{Q}}(A\otimes\mathbb{Q}) + \dim_{\mathbb{Q}}(C\otimes\mathbb{Q}) & = & \operatorname{rank}(A) + \operatorname{rank}(C). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the Rank-via-Dimensions} & & \text{since dimensions are additive under} & & \text{by the Rank-via-Dimensions} & \\ & \text{Proposition 161.10} & & \text{short exact sequences} & & \text{Proposition 161.10} & \end{array}$$

∎

We conclude this appendix with a short discussion of Betti numbers. To do so we recall and expand Definition 156.6:

Definition 161.11. Given a pair (X, A) of topological spaces and $n \in \mathbb{N}_0$ we refer to

$$b_n(X, A) := \operatorname{rank}(\mathrm{H}_n(X, A))$$

as the n-th Betti number of (X, A). For $A = \varnothing$ we write of course $b_n(X) := b_n(X, \varnothing)$.

Lemma 161.12. (Betti-via-Dimension Lemma) Let (X, A) be a pair of topological spaces. For any $n \in \mathbb{N}_0$ and any field $\mathbb{F}$ of characteristic zero we have the following equality in $\mathbb{N}_0 \cup \{\infty\}$:

$$\operatorname{rank}(\mathrm{H}_n(X, A)) = \dim_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F})).$$

Proof. Let (X, A) be a pair of topological spaces and let $\mathbb{F}$ be a field of characteristic zero. We have

$$\begin{array}{ccccc} \operatorname{rank}(\mathrm{H}_n(X,A)) & = & \dim_{\mathbb{F}}(\mathrm{H}_n(X,A)\otimes\mathbb{F}) & = & \dim_{\mathbb{F}}(\mathrm{H}_n(X,A;\mathbb{F}). \\ & \uparrow & & \uparrow & \\ & \text{by the Rank-via-Dimensions} & & \text{by the Torsion–Free Coefficients} & \\ & \text{Proposition 161.10} & & \text{Proposition 161.8} & \end{array}$$ ■

161.5. Exercises for Chapter 161 .

Exercise 161.1. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself. Show that if F is exact, then for every exact sequence

$$\ldots \to C_{k+1} \xrightarrow{\varphi_{k+1}} C_k \xrightarrow{\varphi_k} C_{k-1} \xrightarrow{\varphi_{k-1}} C_{k-2} \to \ldots$$

the sequence

$$\ldots \to F(C_{k+1}) \xrightarrow{F(\varphi_{k+1})} F(C_k) \xrightarrow{F(\varphi_k)} F(C_{k-1}) \xrightarrow{F(\varphi_{k-1})} F(C_{k-2}) \to \ldots$$

is also exact.

Exercise 161.2. Show that the abelianization functor (which we introduced in Definition 98.15) from the category $\mathcal{Group}$ of groups to the category $\mathcal{AbGroup}$ of abelian groups is right-exact and that it is not left-exact.
Remark. The notion of exactness generalizes in an obvious way to the category $\mathcal{Group}$ of groups.

Exercise 161.3. Let G be a fixed abelian group. We consider the covariant functor

$$\begin{array}{rcl} A & \mapsto & \operatorname{Hom}(G,A) \\ (\varphi\colon A\to B) & \mapsto & \begin{pmatrix} \operatorname{Hom}(G,A) & \to & \operatorname{Hom}(G,B) \\ (\alpha\colon G\to A) & \mapsto & (\varphi\circ\alpha\colon G\to B) \end{pmatrix} \end{array}$$

from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(a) Is this functor additive?
(b) Is this functor exact?

Exercise 161.4. Let $\mathcal{C}$ be the category of finitely generated abelian groups. Given an abelian group A we denote by $\operatorname{Tor}(A)$ as usual its torsion subgroup and furthermore we denote by $\mathrm{F}A \coloneqq A/\operatorname{Tor}(A)$ the maximal torsion-free quotient. We consider the following two covariant functors:

$$\begin{array}{rcl} \mathrm{F}\colon \mathcal{C} & \to & \mathcal{C} \\ A & \mapsto & \mathrm{F}A \end{array} \quad \text{and} \quad \begin{array}{rcl} \operatorname{Tor}\colon \mathcal{C} & \to & \mathcal{C} \\ A & \mapsto & \operatorname{Tor}(A). \end{array}$$

(a) Show that the functor F is left-exact.
(b) Show that the functor F is not right-exact.
(c) Is the functor Tor left-exact?
(d) Is the functor Tor is right-exact?

Remark. The notion of exactness generalizes in an obvious way to the category of finitely generated abelian groups.

Exercise 161.5. Let G be a fixed abelian group. If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence. Is the induced sequence $\operatorname{Hom}(C,G) \xrightarrow{\beta^*} \operatorname{Hom}(B,G) \xrightarrow{\alpha^*} \operatorname{Hom}(A,G) \to 0$ exact?

Exercise 161.6. Let G be an abelian group. Show that if the functor $A \mapsto A\otimes G$ is exact, then G is torsion free.

Remark. In other words, the exercise is to show that the converse to the Tensor-with-Torsion Free Proposition 161.6 holds.

Exercise 161.7. Does there exist a torsion-free abelian group that is not a free abelian group?

Exercise 161.8. Show that for any abelian group A we have

$$\ker \begin{pmatrix} A & \to & A \otimes \mathbb{Q} \\ a & \mapsto & a \otimes 1 \end{pmatrix} \;=\; \text{all torsion elements of } A.$$

Remark. In Exercise 159.3 we dealt with the case that A is finitely generated. You can use the Tensor–Direct Colimit Lemma 159.16 and the Tensor-with-Torsion Free Proposition 161.6 to deal with the case that A is not necessarily finitely generated.

Exercise 161.9. Show that if A is an abelian group, then

$$\ker \begin{pmatrix} A & \to & A \otimes \mathbb{Q} \\ a & \mapsto & a \otimes 1 \end{pmatrix} \;=\; \text{all torsion elements of } A.$$

Hint. The "$\supseteq$"-inclusion is clear. For the "$\subseteq$"-inclusion let $a \in A$. Let $f\colon \mathbb{Z} \to A$ be the map that is given by $f(n) = n \cdot a$. Consider the map $f \otimes \mathrm{id}\colon (\mathbb{Z}/\ker(f)) \otimes \mathbb{Q} \to A \otimes \mathbb{Q}$ and use the Tensor-with-Torsion Free Proposition 161.6.
Remark. This hint gives a different approach to Exercise 161.8.

Exercise 161.10.

(a) Let A and B be two finitely generated abelian groups. Show that we have an equality $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A) \cdot \mathrm{rank}(B) \in \mathbb{N}_0$.

(b) Let A and B be two arbitrary abelian groups. Show that in this case we also have the equality $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A) \cdot \mathrm{rank}(B) \in \mathbb{N}_0 \cup \{\infty\}$.

Exercise 161.11.

(a) Show that $\left(\prod_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \right) \otimes \mathbb{Q} \neq 0$.

(b) Show that $\prod_{n \in \mathbb{N}} (\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Q}) = 0$.

Remark. In the Tensor–Direct Sum Lemma 159.15 we saw that the tensor product commutes with forming direct sums. The above exercise shows that the analogous statement is not true if we replace direct sums by direct products. In fact, that is not really a surprise, since there are no natural maps relating products and tensor products.

Exercise 161.12. Given an abelian group G we define the $\mathbb{F}_2$-rank as

$$\mathrm{rank}_{\mathbb{F}_2}(G) \;:=\; \dim_{\mathbb{F}_2}(G \otimes \mathbb{F}_2).$$

Does the analog of the Rank Additivity Proposition 97.18 hold for the $\mathbb{F}_2$-rank? In other words, if

$$0 \;\to\; A \;\xrightarrow{\alpha}\; B \;\xrightarrow{\beta}\; C \;\to\; 0$$

is a short exact sequence of abelian groups, do we have the equality

$$\mathrm{rank}_{\mathbb{F}_2}(B) \;=\; \mathrm{rank}_{\mathbb{F}_2}(A) + \mathrm{rank}_{\mathbb{F}_2}(C)\,?$$

Exercise 161.13. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. We consider the following map:

$$\begin{array}{rcl} \mathrm{H}_n(C_* \otimes G) & \to & \mathrm{H}_n(C_*) \otimes G \\ \left[\sum_{i=1}^{n} c_i \otimes g_i \right] & \mapsto & \sum_{i=1}^{n} [c_i] \otimes g_i \end{array}$$

Is this map well-defined?

CHAPTER 162

The Universal Coefficient Theorem

We are still trying to answer the following question:

Question 161.1. Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?

In the Torsion–Free Coefficients Proposition 161.8 we showed that if G is a torsion-free abelian group, then the natural map $\mu\colon \mathrm{H}_k(X)\otimes G \to \mathrm{H}_k(X;G)$ is an isomorphism.

Nonetheless, in general the situation is more complex. For example, if we compare the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 with the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 160.14 we see that

$$\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{F}_2 \;=\; 0 \;\ncong\; \mathbb{F}_2 \;=\; \mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2).$$

This shows that, for better or worse, in general it is not true that $\mathrm{H}_k(X)\otimes G \cong \mathrm{H}_k(X;G)$. In this chapter will see that, despite this sobering example, one can always deduce the homology groups $\mathrm{H}_*(X;G)$ of a topological space X from the usual homology groups $\mathrm{H}_*(X)$. But the connection between these homology groups will turn out to be less naive than one might have thought initially.

This chapter is organized as follows:

- In Section 162.1 we will introduce the G-torsion $\mathrm{Tor}(H,G)$ of an abelian group H.
- In Section 162.2 we will state and prove the Algebraic Universal Coefficient Theorem 162.15
- In Section 162.3 we will discuss splittings of the short exact sequence that is provided by the Universal Coefficient Theorem
- In Section 162.4 we will provide exercises for Chapter 162.

In the next purely algebraic Chapter 163 we will introduce the tensor product of chain complexes.

162.1. The G-torsion of an abelian group.

Our goal in this chapter is to relate the homology groups $\mathrm{H}_*(C_*)$ of a chain complex C_* to the homology groups $\mathrm{H}_*(C\otimes G)$ of the same chain complex with G-coefficients. To state the theorem relating these groups we need to introduce the G-torsion of an abelian group.

We start our discussion with recalling the definition of a (free) resolution, which we had introduced in Definition 158.2.

Definition 162.1. A (free) resolution of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \;\to\; 0$$

where the F_i are (free) abelian groups. The length of the resolution is the maximal $n\in\mathbb{Z}_{\geq -1}\cup\{\infty\}$ such that $F_n\neq 0$.

Examples.

(1) Let $m \in \mathbb{N}$. A free resolution of length 1 for $H = \mathbb{Z}/m\mathbb{Z}$ is given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0.$$

(2) Let G be a *finitely generated* abelian group. By the Abelian Group-Classification Theorem 97.24 there exists an $r \in \mathbb{N}_0$, non-zero natural numbers $a_1, \dots, a_k$ and an isomorphism

$$\Phi\colon G \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/a_i\mathbb{Z}.$$

Thus we obtain a free resolution[1710]

$$0 \to \mathbb{Z}^k \xrightarrow{0 \oplus \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}} \mathbb{Z}^r \oplus \mathbb{Z}^k \to \underbrace{\mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/a_i\mathbb{Z}}_{\cong G} \to 0.$$

□

The final example is so important that it becomes a definition.

Definition 162.2.

(1) We recall the following basic notation: If M is an abelian group, then we know by the Free Abelian Group–Existence Lemma 97.8 that there exists a unique homomorphism $\alpha(M)\colon \mathbb{Z}^{(M)} \to M$ such that for each $m \in M$ we have $\alpha(M)(m) = m$. Evidently this map is an epimorphism.

(2) Let H be an abelian group. We consider the sequence

$$\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

The group $\mathbb{Z}^{(H)}$ is free abelian and $\alpha(H)$ is an epimorphism. In particular the sequence is exact. We set $H_1 := \ker(\alpha(H)\colon \mathbb{Z}^{(H)} \to H)$. Then

$$H_1 \hookrightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

is exact. Now we consider

$$\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H_1)} \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

This sequence is exact and both the groups $\mathbb{Z}^{(H_1)}$ and $\mathbb{Z}^{(H)}$ are free abelian. Now we set $H_2 := \ker(\alpha(H_1)\colon \mathbb{Z}^{(H_1)} \to H_1)$ and we iterate this procedure. This way we obtain a free resolution of H.[1711] Note that this free resolution is unambiguously defined, we did not make any choices along the way. Therefore we refer to this free resolution of H as the natural[1718] free resolution of H.

Lemma 162.3. (Natural Free Resolution Lemma) Every abelian group admits a natural free resolution.

[1710] Note that the homomorphism to the left is a monomorphism since $a_1, \dots, a_k$ are non-zero.

[1717] Note that this procedure does not stop, i.e. it produces an infinite free resolution. If H is non-trivial, then basically all groups involved are free groups of infinite rank.

[1718] This resolution is "natural" in the weakest sense, namely only under isomorphisms.

Proof. The lemma follows from the definition preceding the lemma. ■

The following definition is one of the key definitions in homological algebra.

Definition 162.4. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the natural free resolution from Definition 162.2. We define the n-th G-torsion group of H as

$$\mathrm{Tor}_n(H,G) := \mathrm{H}_n(\underbrace{F_* \otimes G}_{\text{chain complex}}) = \mathrm{H}_n(\ldots \to F_2 \otimes G \xrightarrow{f_2\otimes \mathrm{id}_G} F_1 \otimes G \xrightarrow{f_1\otimes \mathrm{id}_G} \underbrace{F_0 \otimes G}_{\text{degree } 0.} \xrightarrow{0} 0).$$

We will now see that the torsion groups $\mathrm{Tor}_n(H,G)$ are "covariantly functorial in both entries":

Definition 162.5.

(1) We fix an abelian group G. Let $\beta\colon H \to H'$ be a homomorphism between two abelian groups. It follows from the Homological Algebra–Fundamental Theorem 158.4 applied to the natural free resolutions F_* and F'_* of H and H', that we get an extension $F_* \to F'_*$ of β to the free resolutions, and this extension is unique up to chain homotopy. It follows from the Chain Homotopy–H_*-Lemma 144.5, that β induces for each $n \in \mathbb{N}_0$ a well-defined homomorphism

$$\beta_*\colon \underbrace{\mathrm{Tor}_n(H,G)}_{=\mathrm{H}_n(F_*\otimes G)} \to \underbrace{\mathrm{Tor}_n(H',G)}_{=\mathrm{H}_n(F'_*\otimes G)}.$$

One can now easily show that the maps given by

$$\begin{aligned} H &\mapsto \mathrm{Tor}_n(H,G) \\ (\beta\colon H \to H') &\mapsto (\beta_*\colon \mathrm{Tor}_n(H,G) \to \mathrm{Tor}_n(H',G)) \end{aligned}$$

define a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups $\mathcal{AbGroup}$ to itself.

(2) We fix an abelian group H. Let $\alpha\colon G \to G'$ be a homomorphism between two abelian groups. Let F_* be the natural free resolution of H. Then $\mathrm{id}\otimes\alpha$ defines a chain map from $F_* \otimes G$ to $F_* \otimes G'$ and, as in the Homology-Coefficients-Functor Lemma 160.11, we get an induced homomorphism

$$\alpha_*\colon \mathrm{Tor}_n(H,G) \to \mathrm{Tor}_n(H,G').$$

One can now easily show that the maps given by

$$\begin{aligned} G &\mapsto \mathrm{Tor}_n(H,G) \\ (\alpha\colon G \to G') &\mapsto (\alpha_*\colon \mathrm{Tor}_n(H,G) \to \mathrm{Tor}_n(H,G')) \end{aligned}$$

define a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

The following proposition shows that we can use any free resolution to determine the G-torsion groups. This fact is the key to calculating G-torsion groups.

Proposition 162.6. (Tor-Free Resolution Proposition) Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. For every $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{Tor}_n(H,G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G)$$

which is natural in G.

Proof. Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. We want to show that there exists a natural isomorphism

$$\mathrm{Tor}_n(H,G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G).$$

We denote by (F'_*, f'_*) the natural free resolution of H. Since F'_* is a free resolution we obtain from the Homological Algebra–Fundamental Theorem 158.4 that there exists an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}_H \colon H \to H$ from the free resolutions F'_* to the free resolution F_*. In other words, we obtain we obtain a commutative diagram of the following form:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}_H} & & \\
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
\end{array}$$

Since both resolutions are free we obtain from the Free Resolution–Chain Homotopic Corollary 158.5 that the maps α_* form a chain homotopy equivalence between the two chain complexes F'_* and F_*. It follows from the Homology-with-Coefficients Lemma 160.6 (2) that $\alpha_* \otimes \mathrm{id}_G \colon F'_* \otimes G \to F_* \otimes G$ is also a chain homotopy equivalence. It follows from this observation together with the Chain Homotopy Equivalence-H_*-Corollary 144.10 that for every $n \in \mathbb{N}_0$ the map

$$\begin{array}{rccc}
\alpha_n \otimes \mathrm{id}_G \colon & \overbrace{\mathrm{H}_n(F'_* \otimes G)}^{=:\mathrm{Tor}_n(H,G)} & \to & \mathrm{H}_n(F_* \otimes G) \\
 & \left[\sum_{i=1}^{k} c_i \otimes g_i\right] & \mapsto & \left[\sum_{i=1}^{k} \alpha_n(c_i) \otimes g_i\right]
\end{array}$$

is an isomorphism. Using Statement (2) of the Homological Algebra–Fundamental Theorem 158.4 one can easily show that this map does not depend on the choice of the extension α_*. Finally note that it follows easily from the Tensor–Map Composition Lemma 159.7 (2) that this isomorphism is natural in G. ■

We now turn to the calculation of the G-torsion groups. The following lemma shows that the 0-th G-torsion groups are objects that we are already familiar with.

Lemma 162.7. (Tor_0–Lemma) For any choice of abelian groups G and H there exists a natural isomorphism [1719] $\mathrm{Tor}_0(H,G) \xrightarrow{\cong} H \otimes G$.

Identification 162.8. In the remainder of this course we will use the natural isomorphisms from the Tor_0–Lemma 162.7 to make the identification $\mathrm{Tor}_0(H,G) =_{\mathrm{i}} H \otimes G$.

[1719] Note that "natural isomorphism" means in this context that given an abelian group G we have natural isomorphisms from the functor $H \mapsto \mathrm{Tor}_0(H,G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H,G)$ to the functor $G \mapsto H \otimes G$.

Proof. Let G and H be abelian groups. We consider the natural free resolution of H:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

By the Tensor–Right-Exact Lemma 161.5 we know that the sequence

$$(*) \qquad F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}_G} F_0 \otimes G \xrightarrow{f_0 \otimes \mathrm{id}_G} H \otimes G \to 0$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned} \mathrm{Tor}_0(H,G) &\overset{\text{by Definition 162.4}}{=} \mathrm{H}_0\big(\cdots \to F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}_G} F_0 \otimes G \to 0\big) \\ &= \mathrm{coker}\big(F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}_G} F_0 \otimes G\big) \xrightarrow[\text{since } (*) \text{ is exact we obtain from the Exactness Lemma 145.2 (3) that this map is an isomorphism}]{f_0 \otimes \mathrm{id}_G} H \otimes G. \end{aligned}$$

It follows easily from the Tensor–Map Composition Lemma 159.7 (2) that this isomorphism $\mathrm{Tor}_0(H,G) \to H \otimes G$ is natural in G. Furthermore, using the Homological Algebra–Fundamental Theorem 158.4 one can easily show that this isomorphism is natural in H. ■

The following two lemmas show that we do not need to worry about G-torsion groups $\mathrm{Tor}_n(H,G)$ of degree $n \geq 2$:

Lemma 162.9. (Length 1–Free Resolution Lemma) Every abelian group H admits a free resolution of length ≤ 1.

Proof. Let H be an abelian group. Recall that we have the exact sequence

$$0 \to \ker(\alpha(H) \colon \mathbb{Z}^{(H)} \to H) \to \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

where $\alpha(H)$ is defined as in the proof of the Natural Free Resolution Lemma 162.3. The lemma follows from the observation that it follows from the Free Abelian Subgroup Theorem 97.14 that the group $\ker(\alpha(H) \colon \mathbb{Z}^{(H)} \to H)$, as the subgroup of the free abelian group $\mathbb{Z}^{(H)}$, is itself free abelian. ■

Lemma 162.10. (Higher Tor Lemma) For any two abelian groups H and G we have $\mathrm{Tor}_n(H,G) = 0$ for $n \geq 2$.

Proof. Let G and H be abelian groups. By the Length 1–Free Resolution Lemma 162.9 there exists a free resolution of H of length 1, i.e. there exists a free resolution of the form

$$0 \to F_1 \to F_0 \to H \to 0.$$

It now follows that for every $n \geq 2$ we see that

$$\mathrm{Tor}_n(H,G) \underset{\text{by the Tor-Free Resolution Proposition 162.6}}{\cong} \mathrm{H}_n(F_* \otimes G) \underset{\text{since } n \geq 2 \text{ implies that } F_n = 0}{=} 0.$$

■

The Tor_0–Lemma 162.7 together with the Higher Tor Lemma 162.10 motivates the following simplification of our notation:

Definition 162.11. Given two abelian groups H and G we write $\mathrm{Tor}(H,G) \coloneqq \mathrm{Tor}_1(H,G)$ and we refer to this group as the G-torsion of H.

The following is one of the most important calculations of G-torsion groups.

Lemma 162.12. ($\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma) Let $k \in \mathbb{N}$.

(1) For any abelian group G we have a natural isomorphism

$$\ker\left(G \xrightarrow{g \mapsto g \cdot k} G\right) \xrightarrow{\cong} \operatorname{Tor}(\mathbb{Z}/k\mathbb{Z}, G).$$

(2) Given any $m \in \mathbb{N}$ we have $\operatorname{Tor}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}_{\gcd(k,m)}$.

Proof. [1720]

(1) We consider the free resolution $0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0$ of $\mathbb{Z}/k\mathbb{Z}$. Given any abelian group G this gives rise to the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overbrace{\ker\left(\mathbb{Z} \otimes G \xrightarrow{(n \mapsto n \cdot k) \otimes \mathrm{id}_G} \mathbb{Z} \otimes G\right)}^{\cong \operatorname{Tor}(\mathbb{Z}/k\mathbb{Z}, G)} & \longrightarrow & \mathbb{Z} \otimes G & \xrightarrow{(n \mapsto n \cdot k) \otimes \mathrm{id}_G} & \mathbb{Z} \otimes G \\
 & & \uparrow & & \uparrow{\scriptstyle g \mapsto 1 \otimes g}_{\cong} & & \uparrow{\scriptstyle g \mapsto 1 \otimes g}_{\cong} \\
0 & \longrightarrow & \ker(G \xrightarrow{g \mapsto g \cdot k} G) & \longrightarrow & G & \xrightarrow{g \mapsto g \cdot k} & G.
\end{array}$$

We make the following observations:

(a) Since $0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/k\mathbb{Z}$ we obtain from the Tor-Free Resolution Proposition 162.6 the isomorphism in the top left corner.

(b) The top horizontal sequence is exact by definition.

(c) The two vertical maps to the right are the natural isomorphisms given by the Tensor-with-$\mathbb{Z}$–Lemma 159.9. It follows immediately from the Tensor Product–Universal Property Lemma 159.4 (1) that the square to the right commutes.

It now follows from a mild diagram chase that from the above we get an induced isomorphism $\ker\left(G \xrightarrow{g \mapsto g \cdot k} G\right) \xrightarrow{\cong} \operatorname{Tor}(\mathbb{Z}/k\mathbb{Z}, G)$. One can easily verify that this isomorphism is natural.

(2) We write $r = \gcd(m, k)$ and $s = m/r$. We see that

$$\begin{array}{ccccccc}
\operatorname{Tor}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) & \cong & \ker(\mathbb{Z}/m\mathbb{Z} \xrightarrow{h \mapsto h \cdot k} \mathbb{Z}/m\mathbb{Z}) & = & s \cdot \mathbb{Z}/m\mathbb{Z} & \cong & \mathbb{Z}_r. \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{by (1)} & & & \text{elementary number theory} & &
\end{array}$$

■

The following lemma contains many useful properties of the torsion groups $\operatorname{Tor}(H, G)$:

Lemma 162.13. (Tor-Properties Lemma) Let G, H, $\{H_i\}_{i \in I}$ and $\{G_j\}_{j \in J}$ be abelian groups. Then the following holds:

(1) There exists a natural isomorphism $\operatorname{Tor}\left(\bigoplus_i H_i, G\right) \cong \bigoplus_i \operatorname{Tor}(H_i, G)$.

(2) There exists a natural isomorphism $\operatorname{Tor}\left(H, \bigoplus_j G_j\right) \cong \bigoplus_j \operatorname{Tor}(H, G_j)$.

(3) If G is a torsion-free abelian group, then $\operatorname{Tor}(H, G) = 0$.

(4) (a) If H is free abelian group, then $\operatorname{Tor}(H, G) = 0$.

For completeness sake we state a few more properties, which we will not need, but which might be useful for the reader in other settings:

(4) (b) If H is torsion-free abelian group, then $\operatorname{Tor}(H, G) = 0$.

[1720] The following proof shares some features with the proof of the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 on page 3095.

(5) For any short exact sequence $0 \to B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$ of abelian groups there exists a natural homomorphism $\partial\colon \operatorname{Tor}(A,D) \to A \otimes B$ such that the following sequence is exact:

$$0 \to \operatorname{Tor}(A,B) \xrightarrow{\beta_*} \operatorname{Tor}(A,C) \xrightarrow{\gamma_*} \operatorname{Tor}(A,D) \xrightarrow{\partial} A\otimes B \xrightarrow{\mathrm{id}_A \otimes \beta} A\otimes C \xrightarrow{\mathrm{id}_A \otimes \gamma} A\otimes D \to 0.$$

(6) There exists a natural isomorphism $\operatorname{Tor}(H,G) \cong \operatorname{Tor}(G,H)$.
(7) If T denotes the torsion subgroup[1721] of H, then $\operatorname{Tor}(H,G) = \operatorname{Tor}(T,G)$.

Example. Using this lemma one can usually determine the G-torsion of an abelian group H without any problems. For example the groups $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are torsion-free abelian groups. Thus it follows immediately from the Tor-Properties Lemma 162.13 (3) that for every abelian group H we have

$$\operatorname{Tor}(H,\mathbb{Q}) \;=\; \operatorname{Tor}(H,\mathbb{R}) \;=\; \operatorname{Tor}(H,\mathbb{C}) \;=\; 0. \qquad \square$$

The proof of the Tor-Properties Lemma 162.13 will require the remainder of this section. The reader who is not that interested in the proof can jump straight to Section 162.2.

Proof.

(1) For each $i \in I$ let F^i_* be the natural free resolution of H_i. One can easily verify that $\bigoplus_{i\in I} F^i_*$ is a free resolution of $\bigoplus_{i\in I} H_i$. We have the following natural isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Tor}\Big(\bigoplus_{i\in I} H_i, G\Big) & \overset{\text{by the Tor-Free Resolution Proposition 162.6}}{\cong} & \mathrm{H}_1\Big(\Big(\bigoplus_{i\in I} F^i_*\Big)\otimes G\Big) & \overset{\text{by the Tensor–Direct Sum Lemma 159.15}}{\cong} & \mathrm{H}_1\Big(\bigoplus_{i\in I}(F^i_*\otimes G)\Big) \\
 & & & \overset{\text{by the Direct Sum H}_*\text{-Lemma 143.7}}{\cong} & \bigoplus_{i\in I}\mathrm{H}_1(F^i_*\otimes G) \\
 & & & & \overset{\text{by Definition 162.4}}{\cong} \bigoplus_{i\in I}\operatorname{Tor}(H_i,G).
\end{array}$$

(2) Let F_* be the natural resolution for H. We have the following natural isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Tor}\Big(H, \bigoplus_{j\in J} G_j\Big) & \overset{\text{by Definition 162.4}}{=} & \mathrm{H}_1\Big(F_*\otimes \bigoplus_{j\in J} G_j\Big) & \overset{\text{by the Tensor–Direct Sum Lemma 159.15}}{\cong} & \mathrm{H}_1\Big(\bigoplus_{j\in J}(F_*\otimes G_j)\Big) \\
 & & & \overset{\text{by the Direct Sum H}_*\text{-Lemma 143.7}}{\cong} & \bigoplus_{j\in J}\mathrm{H}_1(F_*\otimes G_j) \\
 & & & & \overset{\text{by Definition 162.4}}{=} \bigoplus_{j\in J}\operatorname{Tor}(H,G_j).
\end{array}$$

(3) We need to show that if G is a torsion-free abelian group, then for any abelian group H we have $\operatorname{Tor}(H,G) = 0$. We consider the natural free resolution of H:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

[1721] The torsion subgroup of an abelian group is defined as the subgroup of all elements of finite order.

We denote by F'_* the above sequence *with* H, i.e. $F'_i = F_i$ for $i \in \mathbb{H}_0$ and $F_{-1} = H$. Note that for each $k \in \mathbb{H}$ we have the following isomorphisms:

$$\begin{array}{ccccccc} \mathrm{Tor}^R(H,G) & = & \mathrm{H}_1(F_* \otimes G) & = & \mathrm{H}_1(F'_* \otimes G) & & \\ & & \uparrow & & \uparrow & & \\ & & \text{by Definition 314.20} & & \text{since } k \geq 1 & & \\ & & & \cong & \mathrm{H}_1(F'_*) \otimes G & = & 0 \otimes G \quad = \quad 0. \\ & & & & \uparrow & & \uparrow \\ & & & & \text{by the Exact Functor–Homology Lemma 161.4} & & \text{since } F'_* \text{ is exact} \\ & & & & \text{and the Tensor-with-Torsion Free Proposition 161.6} & & \end{array}$$

(4) (a) Let H be a *free* abelian group. Note that in this case $0 \to F_0 := H \xrightarrow{\mathrm{id}} H \to 0$ is already a free resolution, and it follows immediately from the Tor-Free Resolution Proposition 162.6 that $\mathrm{Tor}(H,G) = 0$.

(5) Let $0 \to B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$ be a short exact sequence of abelian groups. We need to show that there exists a natural homomorphism $\partial\colon \mathrm{Tor}(A,D) \to A \otimes B$ such that the following sequence is exact:

$$0 \to \mathrm{Tor}(A,B) \xrightarrow{\beta_*} \mathrm{Tor}(A,C) \xrightarrow{\gamma_*} \mathrm{Tor}(A,D) \xrightarrow{\partial} A \otimes B \xrightarrow{\mathrm{id}_A \otimes \beta} A \otimes C \xrightarrow{\mathrm{id}_A \otimes \gamma} A \otimes D \to 0.$$

First note that by the Length 1–Free Resolution Lemma 162.9 we can pick a free resolution $0 \to F_1 \xrightarrow{\varphi} F_0 \to A$ of A. We get the following commutative diagram:

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & F_1 \otimes B & \xrightarrow{\mathrm{id}\otimes\beta} & F_1 \otimes C & \xrightarrow{\mathrm{id}\otimes\beta} & F_1 \otimes D & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle \varphi\otimes\mathrm{id}_B} & & \downarrow{\scriptstyle \varphi\otimes\mathrm{id}_B} & & \downarrow{\scriptstyle \varphi\otimes\mathrm{id}_B} & & & & \\ 0 & \longrightarrow & F_0 \otimes B & \xrightarrow{\mathrm{id}\otimes\beta} & F_0 \otimes C & \xrightarrow{\mathrm{id}\otimes\beta} & F_0 \otimes D & \longrightarrow & 0. & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

Since F_1 and F_0 are free abelian we obtain from the Tensor-with-Torsion Free Proposition 161.6 that the horizontal sequences are exact. In other words, we have a short exact sequence of chain complexes. From the LES-Proposition 145.10 we obtain a natural connecting homomorphism and a long exact sequence of homology groups. We now obtain the desired natural homomorphism and the promised exact sequence from the definition of Tor and the Tor_0–Lemma 162.7.

(6) We need to show that there exists a natural isomorphism $\mathrm{Tor}(G,H) \to \mathrm{Tor}(H,G)$. First note that by the Length 1–Free Resolution Lemma 162.9 we can pick a free resolution $0 \to F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\pi} H$ of H. We consider the following diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}(G,H) & \xrightarrow{\partial} & G \otimes F_1 & \xrightarrow{\mathrm{id}_G\otimes\varphi} & G \otimes F_0 & \xrightarrow{\mathrm{id}_G\otimes\pi} & G \otimes H & \longrightarrow & 0 \\ & & \vdots & & \cong\downarrow{\scriptstyle \tau} & & \cong\downarrow{\scriptstyle \tau} & & \cong\downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & \mathrm{Tor}(H,G) & \hookrightarrow & F_1 \otimes G & \xrightarrow{\varphi\otimes\mathrm{id}_G} & F_0 \otimes G & \xrightarrow{\pi\otimes\mathrm{id}_G} & H \otimes G & \longrightarrow & 0. \end{array}$$

We make the following observations:

- Since F_0 is *free* abelian we know by the above Statement (4a) that $\mathrm{Tor}(G,F_0) = 0$. The top horizontal sequence is now the exact sequence provided by Statement (5).

- The vertical maps are the natural isomorphisms from the Tensor Commutative-Associative Lemma 159.11 (2). Note that it follows from the naturality that the two squares to the right of the diagram commute.
- It follows from the definition of $\mathrm{Tor}(H, G)$ together with the Tor_0–Lemma 162.7 that the bottom sequence is exact.
- It follows from a very mild diagram chase that there exists a unique vertical homomorphism $\mathrm{Tor}(G, H) \to \mathrm{Tor}(H, G)$ to the left that makes the diagram commute. Furthermore it follows from the Five Lemma 145.11 that this vertical homomorphism is an isomorphism.

We leave it to the reader to verify that the isomorphism $\mathrm{Tor}(G, H) \to \mathrm{Tor}(H, G)$ is natural. ⊞

(4) (b) Let G be any abelian group and let H be a torsion-free abelian group. We now see that

$$\mathrm{Tor}(H, G) \underset{\text{by Statement (6)}}{\overset{\uparrow}{\cong}} \mathrm{Tor}(G, H) \underset{\substack{\text{by Statement (3),}\\ \text{since } H \text{ is torsion-free}}}{\overset{\uparrow}{=}} 0.$$

(7) If H is a finitely generated abelian group, then it follows from the Abelian Group-Classification Theorem 97.24 that we can write H as a direct sum $F \oplus T$, where is T the torsion subgroup and where F is a free abelian group. It follows from (1) and (2) that

$$\mathrm{Tor}(H, G) \cong \mathrm{Tor}(F \oplus T, G) \underset{\text{by (1)}}{\overset{\uparrow}{\cong}} \mathrm{Tor}(F, G) \oplus \mathrm{Tor}(T, G) \underset{\text{by (3)}}{\overset{\uparrow}{=}} \mathrm{Tor}(T, G).$$

For the general case we consider the short exact sequence $0 \to T \xrightarrow{\iota} G \xrightarrow{\pi} G/T \to 0$. We consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}(T, H) & \xrightarrow{\iota_*} & \mathrm{Tor}(G, H) & \xrightarrow{\pi_*} & \mathrm{Tor}(G/T, H) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Tor}(H, T) & \xrightarrow{\iota_*} & \mathrm{Tor}(H, G) & \xrightarrow{\pi_*} & \mathrm{Tor}(H, G/T) \end{array}$$

We make the following observations:

- The vertical isomorphisms come from Statement (6). Since these isomorphisms are natural we see that the diagram commutes.
- It follows from Statement (5) that the bottom sequence is exact.
- Note that G/T is torsion-free. It follows from (3) that $\mathrm{Tor}(H, G/T) = 0$.

It follows from the above discussion that $\mathrm{Tor}(T, H) \to \mathrm{Tor}(G, H)$ is an isomorphism. ■

162.2. **The Universal Coefficient Theorem.** Before we turn to the Algebraic Universal Coefficient Theorem let us recall the following concept that we first introduced in the Homology-Multiplication Lemma 161.7:

Definition 162.14. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. Throughout this chapter, given $n \in \mathbb{N}_0$, we denote by μ the natural homomorphism

$$\begin{array}{rcl} \mu\colon \mathrm{H}_n(C_*) \otimes G & \to & \mathrm{H}_n(C_* \otimes G) \\ \sum\limits_{i=1}^{m} [c_i] \otimes g_i & \mapsto & \Big[\sum\limits_{i=1}^{m} c_i \otimes g_i \Big]. \end{array}$$

If G is a commutative ring, then we saw that this is a homomorphism of G-modules.

Now we can formulate and prove the following theorem which is the key result of the whole chapter.

Theorem 162.15. (Algebraic Universal Coefficient Theorem) Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\nu\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G)$ such that the following sequence is exact:

$$0 \to \mathrm{H}_n(C_*) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C_* \otimes G) \xrightarrow{\nu} \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \to 0.$$

If G is a commutative ring, then ν is a G-module homomorphism.

Remark.

(1) The fact that the homomorphisms μ and ν are natural implies that given a chain map $f\colon C_* \to D_*$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_n(C_*) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(C_* \otimes G) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_n(D_*) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(D_* \otimes G') & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(D_*), G') & \longrightarrow & 0 \end{array}$$

commutes. Here the vertical maps are all induced by the chain map $f\colon C \to D$ and the homomorphism $\varphi\colon G \to G'$.

(2) In the Algebraic Universal Coefficient Theorem 314.33 we will prove a generalization of the Algebraic Universal Coefficient Theorem 162.15 to modules over PIDs. Furthermore in the Homological UCSS Theorem 316.5 we will state (but not prove) a generalization to modules over arbitrary rings. □

Proof. Let (C_*, ∂_*) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always

$$\begin{array}{rcl} Z_n & := & \ker(\partial_n\colon C_n \to C_{n-1}) \\ B_n & := & \mathrm{im}(\partial_{n+1}\colon C_{n+1} \to C_n). \end{array}$$

Since C_n is by hypothesis free abelian we obtain from the Free Abelian Subgroup Theorem 97.14 that B_n and Z_n are free abelian groups.

We consider the following diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \xhookrightarrow{\iota_n} & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \xhookrightarrow{\iota_{n-1}} & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. We can and will view the left and the right vertical columns as chain complexes. In other words, the above diagram is a short exact sequence of chain complexes.

Now let G be an abelian group. We tensor the above short exact sequence of chain complexes with G. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n \otimes G & \xrightarrow{\iota_n \otimes \mathrm{id}_G} & C_n \otimes G & \xrightarrow{\partial_n \otimes \mathrm{id}_G} & B_{n-1} \otimes G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n \otimes \mathrm{id}_G} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} \otimes G & \xrightarrow{\iota_{n-1} \otimes \mathrm{id}_G} & C_{n-1} \otimes G & \xrightarrow{\partial_{n-1} \otimes \mathrm{id}_G} & B_{n-2} \otimes G & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Tensor–Functor Lemma 160.4 together with the Functor-applied-to-Chain Complex Lemma 160.3 (1a) that the vertical sequences are again chain complexes. Thus the above is a sequence of "vertical" chain complexes.

(2) Recall that the groups B_n are free abelian groups. Thus it follows from the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 (1)⇒(3) that each of the horizontal sequences in the original diagram splits. It follows from the Tensor-Split-SES Lemma 160.9 that the horizontal sequences obtained from tensoring with G are still exact. ⊞

We apply the LES-Proposition 145.10 to the short exact sequence of chain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{B_n \otimes G}_{=\mathrm{H}_n(B_* \otimes G)} \to \underbrace{Z_n \otimes G}_{=\mathrm{H}_n(Z_* \otimes G)}.$$

In fact, it follows rather easily from Definition 145.8 that the connecting homomorphism $B_n \otimes G \to Z_n \otimes G$ is the map that is given by $i_n \otimes \mathrm{id}_G \colon B_n \otimes G \to Z_n \otimes G$ where $i_n \colon B_n \to Z_n$ denotes the obvious inclusion map. By the LES-Proposition 145.10 this connecting homomorphism gives us the following long exact sequence

$$\ldots \to B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G \to \mathrm{H}_n(C_* \otimes G) \xrightarrow{\partial_n} B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G \to \ldots$$

Using the Break-into-SES Lemma 145.6 the above discussion implies that we obtain the following short exact sequence

$$0 \to \underbrace{\operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G\right)}_{\text{need to show that } = \mathrm{H}_n(C_*) \otimes G} \to \mathrm{H}_n(C_* \otimes G) \to \underbrace{\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G\right)}_{\text{need to show that } = \operatorname{Tor}(\mathrm{H}_{n-1}(C_*), G)} \to 0.$$

We still have to show that the expressions left and right coincide with the desired expressions:

(1) We have natural isomorphisms

$$\operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G\right) \xleftarrow[\uparrow]{\cong} \operatorname{Tor}_0(\mathrm{H}_n(C_*), G) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n(C_*) \otimes G$$

by the Tor-Free Resolution Proposition 162.6 since $0 \to B_n \xrightarrow{i_n} Z_n \to \mathrm{H}_n(C_*)$ is a free resolution of $\mathrm{H}_n(C_*)$

by the Tor_0–Lemma 162.7

One can easily verify that the resulting homomorphism $\mathrm{H}_n(C_*) \otimes G \to \mathrm{H}_n(C_* \otimes G)$ is the given homomorphism μ.

(2) Similarly to (1) we have a natural isomorphism

$$\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G\right) \xrightarrow[\uparrow]{\cong} \operatorname{Tor}(\mathrm{H}_{n-1}(C_*), G).$$

by the Tor-Free Resolution Proposition 162.6 since $0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to \mathrm{H}_{n-1}(C_*)$ is a free resolution of $\mathrm{H}_{n-1}(C_*)$

In summary we have now finally shown the existence of the promised homomorphism $\nu\colon \mathrm{H}_n(C_* \otimes G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(C_*), G)$ and we have shown that μ and ν give rise to a short exact sequence. The statement that the homomorphism ν is natural follows easily from the definitions and Statement (2) of the Homological Algebra–Fundamental Theorem 158.4. We leave the verification of the details to the reader. ■

The following theorem is for us the main application of the Algebraic Universal Coefficient Theorem 162.15.

Theorem 162.16. (Universal Coefficient Theorem) Let (X, A) be a pair of topological spaces and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\nu\colon \mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G)$ such that the following sequence is exact:

$$0 \;\to\; \mathrm{H}_n(X, A) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(X, A; G) \;\xrightarrow{\nu}\; \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) \;\to\; 0.$$

If G is a commutative ring, then the natural homomorphism ν is in fact a G-module homomorphism.

Remark. Note that the naturality of the maps μ and ν implies that for every continuous map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces and any homomorphism $\varphi\colon G \to G'$ between abelian groups we obtain the following *commutative* diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_n(X, A) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(X, A; G) & \xrightarrow{\nu} & \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}_n(Y, B) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(Y, B; G') & \xrightarrow{\nu} & \operatorname{Tor}(\mathrm{H}_{n-1}(Y, B), G') & \longrightarrow & 0,
\end{array}$$

where the vertical maps are the obvious (?) ones induced by f and φ. □

Proof. In the following let (X, A) be a pair of topological spaces and let G be an abelian group. As we pointed out on page 2798, the chain groups $\mathrm{C}_n(X, A)$, $n \in \mathbb{N}_0$, are free abelian groups. Thus we can apply the Algebraic Universal Coefficient Theorem 162.15 to the chain complex $\mathrm{C}_*(X, A)$ and we immediately obtain the desired natural homomorphism $\nu\colon \mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G)$ and the desired short exact sequence. ■

Example.

(1) Let (X, A) be a pair of topological spaces and let G be a torsion-free abelian group. From the Universal Coefficient Theorem 162.16 we obtain the short exact sequence

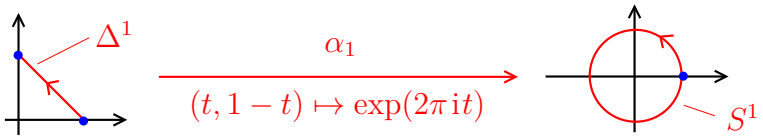
$$0 \to \mathrm{H}_n(X, A) \otimes G \xrightarrow{\mu} \mathrm{H}_n(X, A; G) \xrightarrow{\nu} \underbrace{\mathrm{Tor}(\mathrm{H}_{n-1}(X, A), G)}_{\substack{= 0 \text{ by the Tor-Properties} \\ \text{Lemma 162.13 (3)}}} \to 0.$$

Thus we see that the map to the left is an isomorphism. In other words, we have reproved the Torsion–Free Coefficients Proposition 161.8.

(2) Let $n \in \mathbb{N}$ and let G be an abelian group. We consider the sphere S^n. By the Universal Coefficient Theorem 162.16 we know that the following sequence is exact:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{H}_n(S^n) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(S^n; G) & \xrightarrow{\nu} & \underbrace{\mathrm{Tor}(\underbrace{\mathrm{H}_{n-1}(S^n)}_{= 0 \text{ or } \cong \mathbb{Z}}, G)}_{\substack{= 0 \text{ by the Tor-Properties} \\ \text{Lemma 162.13 (4)}}} \to 0. \\ & & \sum_{i=1}^{k} [\sigma_i] \otimes g_i & \mapsto & \sum_{i=1}^{k} [\sigma_i \otimes g_i] & & \end{array}$$

Since the term on the right is zero, we see that the map μ to the left is an isomorphism. Thus we see that $G \cong \mathbb{Z} \otimes G \cong \mathrm{H}_n(S^n) \otimes G \cong \mathrm{H}_n(S^n; G)$. This gives us a new proof of the Sphere-H_*-Proposition 160.13. But this approach gives us more: In the Standard Generator-$\mathrm{H}_n(S^n)$-Proposition 150.7 we gave an explicit singular cycle $\alpha_n - \beta_n$ in $\mathrm{C}_n(S^n)$ that represents a generator of $\mathrm{H}_n(S^n)$. If G is a commutative ring, then it follows easily from the above that $(\alpha_n - \beta_n) \otimes 1_G \in \mathrm{C}_n(S^n; G)$ is a cycle that represents a generator of the G-module $\mathrm{H}_n(S^n; G) \cong G$. Furthermore, if n is odd, then we know by the Standard Generator-$\mathrm{H}_n(S^n)$-Proposition 150.7 that we can drop β_n, i.e. $\alpha_n \otimes 1_G$ represents already a generator.



$[\alpha_1] \otimes 1_{\mathbb{F}}$ is a generator of $\mathrm{H}_1(S^1; \mathbb{F})$

(3) We consider again the real projective space $\mathbb{R}\mathrm{P}^n$. In the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we saw that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \leq n, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

It follows from the Universal Coefficient Theorem 162.16, the Cyclic-Tensor-Cyclic Lemma 159.5, the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12 and the Tor-Properties Lemma 162.13 (4) that

for any even number $2k \in \{2, \dots, n\}$ we have a short exact sequence

$$0 \;\to\; \underbrace{\mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{F}_2}_{\cong 0 \otimes \mathbb{F}_2 = 0} \;\xrightarrow{\mu}\; \mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\to\; \underbrace{\mathrm{Tor}(\mathrm{H}_{2k-1}(\mathbb{R}\mathrm{P}^n), \mathbb{F}_2)}_{\cong \mathrm{Tor}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2} \;\to\; 0$$

whereas for an odd number $2k+1 \in \{1, \dots, n\}$ we have a short exact sequence

$$0 \;\to\; \underbrace{\mathrm{H}_{2k+1}(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{F}_2}_{\substack{\cong \mathbb{F}_2 \otimes \mathbb{F}_2 \cong \mathbb{F}_2 \\ \text{or } \cong \mathbb{Z} \otimes \mathbb{F}_2 \cong \mathbb{F}_2}} \;\xrightarrow{\mu}\; \mathrm{H}_{2k+1}(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\to\; \underbrace{\mathrm{Tor}(\mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n), \mathbb{F}_2)}_{\substack{\cong \mathrm{Tor}(0, \mathbb{F}_2) = 0 \\ \text{or } \cong \mathrm{Tor}(\mathbb{Z}, \mathbb{F}_2) = 0}} \;\to\; 0.$$

Also it follows easily that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) = 0$ for $k > n$. Summarizing we obtain the following result:

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Fortunately this calculation agrees with the earlier calculation in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 160.14.

□

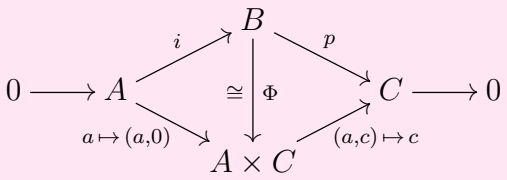
162.3. **Splittings of the Universal Coefficient Theorem.** Our goal, first formulated in Question 161.1, has been to determine the homology groups $\mathrm{H}_n(C_* \otimes G)$ only in terms of the original groups $\mathrm{H}_k(C_*)$ and the abelian group G. The Algebraic Universal Coefficient Theorem 162.15 gives us a short exact sequence

$$0 \;\to\; \mathrm{H}_n(C_*) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(C_* \otimes G) \;\xrightarrow{\nu}\; \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \;\to\; 0.$$

But in general, as we already saw on page 2780, given a short exact sequence, it is not possible to determine the middle group of a short exact sequence from the groups to the left and the right. At this stage it is worth recalling the following lemma:

Lemma 145.5. (Splitting Lemma) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. The following three statements are equivalent:

(1) The short exact sequence splits, i.e. there exists a homomorphism $s\colon C \to B$ with $p \circ s = \mathrm{id}_C$.
(2) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.
(3) There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:



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The above discussion raises the question whether the short exact sequence in the Algebraic Universal Coefficient Theorem 162.15 splits. This is fortunately the case:

Proposition 162.17. (Algebraic UCT-Splitting Proposition) Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. For any $n \in \mathbb{N}_0$ the short

exact sequence

$$0 \;\to\; \mathrm{H}_n(C_*) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(C_* \otimes G) \;\xrightarrow{\nu}\; \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \;\to\; 0$$

of the Algebraic Universal Coefficient Theorem 162.15 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(C_* \otimes G) \;\cong\; \mathrm{H}_n(C_*) \otimes G \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G).$$

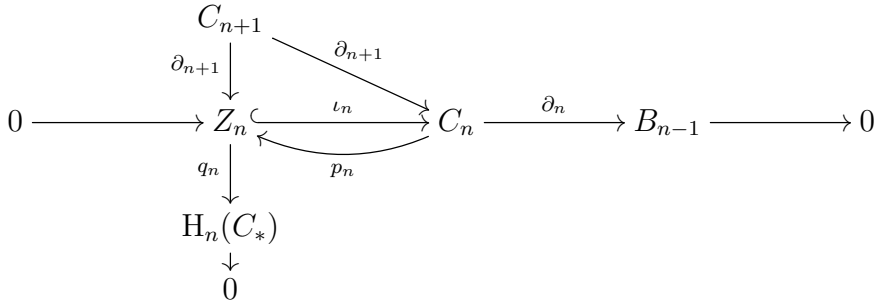
Proof. Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. According to the Splitting Lemma 145.5 (2)⇒(1) it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(C_*) \otimes G$ such that $\eta \circ \mu$ is the identity on $\mathrm{H}_n(C_*) \otimes G$. We summarize this problem as follows:

$$0 \longrightarrow \mathrm{H}_n(C_*) \otimes G \underset{\overset{\leftarrow}{?\,\eta\,?}}{\xrightarrow{\mu}} \mathrm{H}_n(C_* \otimes G) \xrightarrow{\nu} \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \longrightarrow 0.$$

For each $n \in \mathbb{N}_0$ we consider, as on page 3114, $Z_n \coloneqq \ker(\partial_n)$ and $B_n \coloneqq \mathrm{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \xhookrightarrow{\iota_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

The group $B_{n-1} \subseteq C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by the Free Abelian Subgroup Theorem 97.14. Therefore we can deduce from the Free Abelian Group–Splitting Lemma 97.11 that the above short exact sequence splits. In particular, according to the Splitting Lemma 145.5 (1)⇒(2), for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. Again we summarize the setting as follows:



where q_n is just the natural projection.

Claim 1. The following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle q_{n+1}\circ p_{n+1}} & & \downarrow{\scriptstyle q_n \circ p_n} & & \downarrow{\scriptstyle q_{n-1}\circ p_{n-1}} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{0} & \mathrm{H}_n(C_*) & \xrightarrow{0} & \mathrm{H}_{n-1}(C_*) & \longrightarrow & \ldots
\end{array}$$

Proof. The map down-right is evidently zero. On the other hand we also have

$$(q_n \circ p_n) \circ \partial_{n+1} \underset{\uparrow}{=} q_n \circ p_n \circ \iota_n \circ \partial_{n+1} \underset{\uparrow}{=} q_n \circ \partial_{n+1} \underset{\uparrow}{=} 0.$$

since $\iota_n \circ \partial_{n+1} = \partial_{n+1}$; since $p_n \circ \iota_n = \mathrm{id}_{Z_n}$; since $\mathrm{H}_n(C_*) = Z_n / \mathrm{im}(\partial_{n+1})$ ⊞

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps. Now

we tensor the top and the bottom horizontal sequences with G. From the Functor-applied-to-Chain Complex Lemma 160.3 together with the Tensor–Functor Lemma 160.4 we obtain the following diagram of chain complexes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{n+1}\otimes G & \xrightarrow{\partial_{n+1}\otimes \mathrm{id}_G} & C_n\otimes G & \xrightarrow{\partial_n\otimes \mathrm{id}_G} & C_{n-1}\otimes G & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle (q_{n+1}\circ p_{n+1})\otimes \mathrm{id}_G} & & \downarrow{\scriptstyle (q_n\circ p_n)\otimes \mathrm{id}_G} & & \downarrow{\scriptstyle (q_{n-1}\circ p_{n-1})\otimes \mathrm{id}_G} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*)\otimes G & \xrightarrow{0} & \mathrm{H}_n(C_*)\otimes G & \xrightarrow{0} & \mathrm{H}_{n-1}(C_*)\otimes G & \longrightarrow & \ldots
\end{array}$$

The induced maps on the homology of these two chain complexes give us a homomorphism

$$\eta\colon \mathrm{H}_n(C_*\otimes G) \;\to\; \mathrm{H}_n(\text{lower sequence}) = \mathrm{H}_n(C_*)\otimes G.$$

It remains to prove the following claim:

Claim 2. The composition $\eta\circ\mu$ is the identity on $\mathrm{H}_n(C_*)\otimes G$.

Proof. First let $[\sigma]\otimes g\in \mathrm{H}_n(C_*)\otimes G$ where $\sigma\in Z_n$. Then

$$(\eta\circ\mu)([\sigma]\otimes g) \underset{\substack{\uparrow\\ \text{definition of } \mu}}{=} \eta([\sigma\otimes g]) \underset{\substack{\uparrow\\ \text{definition of } \eta}}{=} (q_n(p_n(\sigma)))\otimes g \underset{\substack{\uparrow\\ \text{since } p_n=\mathrm{id} \text{ on } Z_n}}{=} q_n(\sigma)\otimes g \underset{\substack{\uparrow\\ \text{definition of } q_n}}{=} [\sigma]\otimes g.$$

The claim now follows from the observation that the Tensor Product–Basics Lemma 159.3 (2) tells us that the $[\sigma]\otimes g$ generate $\mathrm{H}_n(C_*)\otimes G$. ■

The following immediate corollary to the Algebraic UCT-Splitting Proposition 162.17 shows in particular that the isomorphism types of the homology groups with G-coefficients of a topological space X are completely determined by the usual homology groups of X and the abelian group G.

Theorem 162.18. (UCT-Splitting Theorem) Let (X,A) be a pair of topological spaces and let G be an abelian group. For each $n\in\mathbb{N}_0$ the short exact sequence

$$0 \;\to\; \mathrm{H}_n(X,A)\otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(X,A;G) \;\to\; \mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G) \;\to\; 0$$

from the Universal Coefficient Theorem 162.16 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(X,A;G) \;\cong\; \mathrm{H}_n(X,A)\otimes G \;\oplus\; \mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G).$$

In particular the isomorphism types of the G-homology groups $\mathrm{H}_n(X,A;G)$ of (X,A) are completely determined by the isomorphism types of the "classical" homology groups $\mathrm{H}_n(X,A)$.

Proof. This theorem follows basically immediately from the Algebraic UCT-Splitting Proposition 162.17 applied to the chain complex $\mathrm{C}_*(X,A)$. ■

It is worth recording the following straightforward corollary:

Corollary 162.19. (G-Homology Uselessness Corollary) Let (X,A) and (Y,B) be topological spaces and let G be an abelian group. Then

$$\begin{array}{c}\mathrm{H}_n(X,A) \text{ and } \mathrm{H}_n(Y,B) \text{ are}\\ \text{isomorphic for all } n\in\mathbb{N}_0\end{array} \quad\Longrightarrow\quad \begin{array}{c}\mathrm{H}_n(Y,B;G) \text{ and } \mathrm{H}_n(X,A;G) \text{ and}\\ \text{are isomorphic for all } n\in\mathbb{N}_0.\end{array}$$

Remark. The G-Homology Uselessness Corollary 162.19 is in some sense a disappointment, it shows that homology groups with coefficients are not better at distinguishing topological spaces than ordinary homology groups. □

Proof. The corollary is an immediate consequence of the UCT-Splitting Theorem 162.18. Alternatively the G-Homology Uselessness Corollary 162.19 is also an almost immediate consequence of the Lift-to-Chain Map-Proposition 158.21, the Quasi Iso–Homotopy Equivalence Proposition 158.8 and the Homology-with-Coefficients Lemma 160.6. ■

The careful reader might have noticed that the keyword in the Algebraic UCT-Splitting Proposition 162.17 and the UCT-Splitting Theorem 162.18 that did not bark is the adjective "natural". Put differently, the question arises, whether given an abelian group G there exist *natural* isomorphisms

$$\begin{array}{rcl} \mathrm{H}_n(C_* \otimes G) & \cong & \mathrm{H}_n(C_*) \otimes G \quad \oplus \quad \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \\ \mathrm{H}_n(X, A; G) & \cong & \mathrm{H}_n(X, A) \otimes G \quad \oplus \quad \mathrm{Tor}(\mathrm{H}_{n-1}(X, A), G). \end{array}$$

The following two propositions say that, given any cyclic group $G = \mathbb{Z}/k\mathbb{Z}$, for chain complexes of free abelian groups such a natural isomorphism does not exist:

Proposition 162.20. (Algebraic No-Natural Splitting Proposition) Let $k \in \mathbb{N}_{\geq 2}$. It is not possible to assign to each chain complex C_* of free abelian groups an isomorphism

$$\Phi_{C_*} \colon \mathrm{H}_n(C_* \otimes \mathbb{Z}/k\mathbb{Z}) \ \to \ \mathrm{H}_n(C_*) \otimes \mathbb{Z}/k\mathbb{Z} \ \oplus \ \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), \mathbb{Z}/k\mathbb{Z})$$

such that for every chain map $\psi \colon C_* \to D_*$ between chain complexes of free abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(C_* \otimes \mathbb{Z}/k\mathbb{Z}) & \xrightarrow[\cong]{\Phi_{C_*}} & \mathrm{H}_n(C_*) \otimes \mathbb{Z}/k\mathbb{Z} \ \oplus \ \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), \mathbb{Z}/k\mathbb{Z}) \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} \qquad \downarrow{\scriptstyle \psi_*} \\ \mathrm{H}_n(D_* \otimes \mathbb{Z}/k\mathbb{Z}) & \xrightarrow[\cong]{\Phi_{D_*}} & \mathrm{H}_n(D_*) \otimes \mathbb{Z}/k\mathbb{Z} \ \oplus \ \mathrm{Tor}(\mathrm{H}_{n-1}(D_*), \mathbb{Z}/k\mathbb{Z}). \end{array}$$

Proof. So suppose that such a natural isomorphism Φ_{C_*} exists for every chain complex C_* of free abelian groups. We consider the following "vertical" chain map ψ between "horizontal" chain complexes C_* and D_*:

$$\begin{array}{lccccccc} C_* := & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot k} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi} & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ D_* := & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

From the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 we obtain the following commutative diagram:

$$\begin{array}{lccccccc} C_* \otimes \mathbb{Z}/k\mathbb{Z} \ \cong & 0 & \longrightarrow & \mathbb{Z}/k\mathbb{Z} & \xrightarrow{(\cdot k)=0} & \mathbb{Z}/k\mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi \otimes \mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}} & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ D_* \otimes \mathbb{Z}/k\mathbb{Z} \ \cong & 0 & \longrightarrow & \mathbb{Z}/k\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Using the above information, using the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 and using the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12 we can calculate the various homology groups in the following

diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}_1(C_*;\mathbb{Z}/k\mathbb{Z})}^{\cong\mathbb{Z}/k\mathbb{Z}} & \xrightarrow[\cong]{\Phi_{C_*}} & \overbrace{\mathrm{H}_1(C_*)\otimes\mathbb{Z}/k\mathbb{Z}}^{=0} & \oplus & \overbrace{\mathrm{Tor}(\mathrm{H}_0(C_*),\mathbb{Z}/k\mathbb{Z})}^{\cong\mathbb{Z}/k\mathbb{Z}} \\
\downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle =0}\ {\scriptstyle \psi_*} & & \downarrow{\scriptstyle =0}\ {\scriptstyle \psi_*} \\
\underbrace{\mathrm{H}_1(D_*;\mathbb{Z}/k\mathbb{Z})}_{\cong\mathbb{Z}/k\mathbb{Z}} & \xrightarrow[\cong]{\Phi_{D_*}} & \underbrace{\mathrm{H}_1(D_*)\otimes\mathbb{Z}/k\mathbb{Z}}_{\cong\mathbb{Z}/k\mathbb{Z}} & \oplus & \underbrace{\mathrm{Tor}(\mathrm{H}_0(D_*),\mathbb{Z}/k\mathbb{Z})}_{=0}.
\end{array}$$

Since the left vertical map is an isomorphism of the non-trivial group $\mathbb{Z}/k\mathbb{Z}$ (here we use that $k \geq 2$) and since the right vertical map is the zero map we see that this diagram just cannot be commutative. We have thus obtained a contradiction to our assumption on the existence of natural isomorphisms. ■

Proposition 162.21. (No-Natural Splitting Proposition) Let $k \in \mathbb{N}_{\geq 2}$. It is not possible to assign to each topological space X an isomorphism

$$\Phi_X \colon \mathrm{H}_n(X;\mathbb{Z}/k\mathbb{Z}) \;\to\; \mathrm{H}_n(X)\otimes\mathbb{Z}/k\mathbb{Z} \;\oplus\; \mathrm{Tor}(\mathrm{H}_{n-1}(X),\mathbb{Z}/k\mathbb{Z})$$

such that for every continuous map $f \colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n(X;\mathbb{Z}/k\mathbb{Z}) & \xrightarrow[\cong]{\Phi_X} & \mathrm{H}_n(X)\otimes\mathbb{Z}/k\mathbb{Z} & \oplus & \mathrm{Tor}(\mathrm{H}_{n-1}(X),\mathbb{Z}/k\mathbb{Z}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{H}_n(Y;\mathbb{Z}/k\mathbb{Z}) & \xrightarrow[\cong]{\Phi_Y} & \mathrm{H}_n(Y)\otimes\mathbb{Z}/k\mathbb{Z} & \oplus & \mathrm{Tor}(\mathrm{H}_{n-1}(Y),\mathbb{Z}/k\mathbb{Z}).
\end{array}$$

Proof. In the following we deal with the case $k = 2$. In Exercise 162.19 we will deal with the other cyclic groups. So suppose that such a natural isomorphism Φ_X exists for every topological space X and for $\mathbb{Z}/2\mathbb{Z}$.

The idea of the proof is quite simple: We want to realize (a slight variation) on the chain complexes and the chain map of the Algebraic No-Natural Splitting Proposition 162.20 by the cellular chain complexes of CW-complexes and by a suitable cellular map.

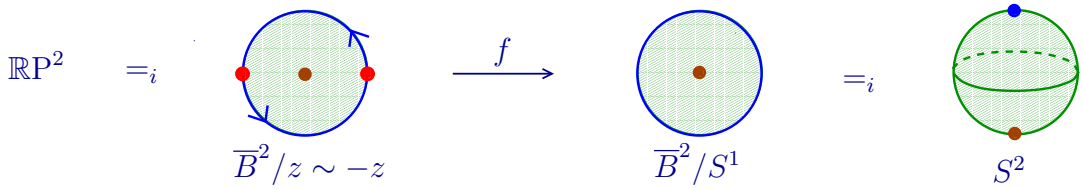
We make the following preparations:

(1) As in the Ball Quotient-Sphere Lemma 8.10 and as on page 340 we make the usual identifications $S^2 =_{\mathrm{i}} \overline{B}^2/S^1$ and $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z$ where $z \in S^1$.

(2) We consider the map

$$\begin{array}{rcl}
f \colon \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z & \to & \overline{B}^2/S^1 =_{\mathrm{i}} S^2 \\
[P] & \mapsto & [P]
\end{array}$$

which is continuous by the Quotient-to-Quotient Lemma 8.6.



(3) As in on page 2628 we equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell and as on page 2625 we equip S^2 with the CW-structure with one 0-cell and one 2-cell. Note that with this CW-structure the map f is cellular.

Claim. With suitable bases of the cellular chain complexes we get the following description of the cellular chain complexes and the chain map induced by the cellular map f:

$$\begin{array}{ccccccccccc}
\mathrm{C}_*^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2) & \cong & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_*} & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathrm{C}_*^{\mathrm{CW}}(S^2) & \cong & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Proof. It follows from the discussion on page 2966 that the horizontal cellular chain complexes are of the stated form. The fact that the right vertical map is an isomorphism is almost trivial and also irrelevant for what comes next. So let us concentrate on showing that the left vertical map is an isomorphism.

We consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}_2(\mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1) & \xleftarrow[\cong]{\text{ю}} & \mathrm{H}_2(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^1) & \xrightarrow{\mathrm{id}} & \mathrm{C}_2^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{H}_2(S^2/\{N\}) & \xleftarrow[\cong]{\text{ю}} & \mathrm{H}_2(S^2, \{N\}) & \xrightarrow{\mathrm{id}} & \mathrm{C}_2^{\mathrm{CW}}(S^2).
\end{array}$$

We make the following clarifications and observations:

(1) The left horizontal maps ю are the natural isomorphisms from the Relative-to-Quotient-H_*-Proposition 149.13.
(2) It follows from our choice of CW-structures and the definition of the cellular chain groups $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ that the second horizontal maps are the identity.
(3) It follows quite easily from the Compact-Hausdorff Proposition 4.23 (3) that the induced map $f\colon \mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1 \to S^2/\{N\}$ is a homeomorphism. In particular we see that the left vertical map is an isomorphism. But this implies that the right vertical map is also an isomorphism. But that is precisely what we needed to show. ⊞

Now we consider the following diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)}^{\cong \mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{\mathbb{R}\mathrm{P}^2}} & \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2}^{=0} & \oplus & \overbrace{\mathrm{Tor}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2), \mathbb{F}_2)}^{\cong \mathbb{F}_2} \\
\downarrow{\scriptstyle f_*} & & \overset{=0}{\downarrow}{\scriptstyle f_*} & & \overset{=0}{\downarrow}{\scriptstyle f_*} \\
\underbrace{\mathrm{H}_2(S^2; \mathbb{F}_2)}_{\cong \mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{S^2}} & \underbrace{\mathrm{H}_2(S^2) \otimes \mathbb{F}_2}_{\cong \mathbb{F}_2} & \oplus & \underbrace{\mathrm{Tor}(\mathrm{H}_1(S^2), \mathbb{F}_2)}_{=0}.
\end{array}$$

We make the following observations:

(1) By hypothesis the horizontal maps are isomorphisms and by hypothesis the diagram commutes.
(2) The various homology groups involved can be calculated easily using the claim, the Singular-Cellular-H_*-Proposition 155.8 and the discussion on page 3081. The resulting tensor products and Tor-groups can be calculated using the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 and using the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12.

(3) It follows from the claim and the discussion on page 3081 that the vertical map on the left is an isomorphism.
(4) The two vertical maps on the right are zero since either the domain or the target is zero. Thus this diagram cannot possibly commute.

We have thus found a contradiction to our assumption about the existence of the natural maps Φ_X. ■

162.4. Exercises for Chapter 162 .

Exercise 162.1. Let $A \to B \to C$ be a short exact sequence of abelian groups and let G be an abelian group. Show that there exists an exact sequence of the form

$$0 \to \mathrm{Tor}(A,G) \to \mathrm{Tor}(B,G) \to \mathrm{Tor}(C,G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0.$$

Hint. Use the Snake Lemma that we proved in Exercise 145.24.
Remark. This exact sequence is sometimes called the *six-term sequence* in homological algebra.

Exercise 162.2. Let $n \in \mathbb{N}$. Let G be an abelian group (which is not necessarily finitely generated). Suppose that G admits an element of order n, i.e. we suppose that there exists a $g \in G$ with $n \cdot g = 0$ and such that $m \cdot g \neq 0$ for $m = 1, \ldots, n-1$. Show that $\mathrm{Tor}(G, \mathbb{Z}/n\mathbb{Z}) \neq 0$.
Remark. You could make use of Exercise 162.1.

Exercise 162.3.

(a) Let G_* be a chain complex. Suppose that for every $k \in \mathbb{N}_0$ and every prime $p \in \mathbb{N}$ we know that $\mathrm{H}_k(G_*;\mathbb{Q}) = 0$ and that $\mathrm{H}_k(G_*;\mathbb{Z}/p\mathbb{Z}) = 0$.
  (i) First suppose that all chain groups G_k are finitely generated. Show that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(G_*) = 0$.

  (ii) Show without any hypothesis that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(G_*) = 0$.
  Remark. You might want to use Exercise 162.2.

(b) Let $f\colon C_* \to D_*$ be a chain map between two chain complexes. (We do not assume that the chain groups are finitely generated.) Suppose that for every $k \in \mathbb{N}_0$ and every prime $p \in \mathbb{N}$ the maps $f_*\colon \mathrm{H}_k(C_*;\mathbb{Q}) \to \mathrm{H}_k(D_*;\mathbb{Q})$ and $f_*\colon \mathrm{H}_k(C_*;\mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}_k(D_*;\mathbb{Z}/p\mathbb{Z})$ are isomorphisms. Show that for every $k \in \mathbb{N}_0$ the map $f_*\colon \mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.
  Hint. Use the algebraic mapping cone that we introduced in Definition 158.10.

Exercise 162.4. Show that for every finite abelian group A we have $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong A$.

Exercise 162.5. Let $T = S^1 \times S^1$ and let K be the Klein bottle.

(a) Does there exist a continuous map $f\colon K \to T$ such that $f_*\colon \mathrm{H}_2(K;\mathbb{F}_2) \to \mathrm{H}_2(T;\mathbb{F}_2)$ is an isomorphism?
(b) Does there exist a continuous map $g\colon T \to K$ such that $g_*\colon \mathrm{H}_2(T;\mathbb{F}_2) \to \mathrm{H}_2(K;\mathbb{F}_2)$ is an isomorphism?

Exercise 162.6. Let X be an n-dimensional CW-complex. Show that if $\mathrm{H}_n(X) \neq 0$, then for any abelian group G we have $\mathrm{H}_n(X;G) \neq 0$.

Exercise 162.7. We suppose that we have the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \widetilde{A} & \longrightarrow & \widetilde{B} & \longrightarrow & \widetilde{C} & \longrightarrow & 0. \end{array}$$

Furthermore we assume that the horizontal sequences are exact. Which of the following conclusions hold?

(a)	if α and γ are monomorphisms	then β is a monomorphism ?
(b)	if α and γ are epimorphisms	then β is an epimorphism ?
(c)	if α and γ are isomorphisms	then β is an isomorphism ?
(d)	if α and γ are the identity	then β is the identity ?
(e)	if α and γ are the zero map	then β is the zero map ?

Remark. This exercise is a copy of Exercise 145.26.

Exercise 162.8. We consider the wedge $S^2_a \vee S^2_b$ of two spheres. Let $\mathbb{F}$ be a field. Determine explicit cycles in $\mathrm{C}_*(S^2_q \vee S^2_b) \otimes \mathbb{F}$ that represent a basis for $\mathrm{H}_2(S^2_a \vee S^2_b; \mathbb{F})$.

Exercise 162.9. Let $f\colon (X, A) \to (Y, B)$ be a continuous map between pairs of topological spaces and let G be an abelian group. Show, just using the Universal Coefficient Theorem 162.16, that if for all $n \in \mathbb{N}_0$ the induced map

$$f_*\colon \mathrm{H}_n(X, A) \ \to \ \mathrm{H}_n(Y, B)$$

is an isomorphism, then it is also true that for any $n \in \mathbb{N}_0$ and any abelian group G the induced map

$$f_*\colon \mathrm{H}_n(X, A; G) \ \to \ \mathrm{H}_n(Y, B; G)$$

is an isomorphism.

Remark. In Exercise 162.9 we gave a different proof of this statement.

Exercise 162.10. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$ be the projection. Does there exist an abelian group G such that $p_*\colon \mathrm{H}_n(S^n; G) \to \mathrm{H}_n(\mathbb{R}\mathrm{P}^n; G)$ is a non-zero homomorphism?

Exercise 162.11. Give an example of a chain complex C_* such that $\mathrm{H}_k(C_*) = 0$ for every $k \geq 1$ but such that $\mathrm{H}_k(C_* \otimes \mathbb{Z}/2\mathbb{Z}) \neq 0$ for every $k \geq 1$.

Exercise 162.12. Let X be a topological space, let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_k(X) & \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1]} & \mathrm{H}_k(X; \mathbb{Z}/n\mathbb{Z}) \\ {\scriptstyle [\sigma] \mapsto [\sigma] \otimes 1}\downarrow & & \uparrow{\scriptstyle [\sigma] \otimes s \mapsto [\sigma \otimes s]} \\ \mathrm{H}_k(X) \otimes \mathbb{Z} & \xrightarrow{[\sigma] \otimes s \mapsto [\sigma] \otimes [s]} & \mathrm{H}_k(X) \otimes \mathbb{Z}/n\mathbb{Z}. \end{array}$$

It follows immediately from the definitions that this diagram commutes.

(a) Show that the top map is an epimorphism if $k = 1$.

(b) Show that for $k \geq 2$ the top map is in general not an epimorphism.

Exercise 162.13. Let $f, g\colon X \to Y$ be two continuous maps such that for every $k \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(Y)$. Does this imply that for every abelian group G and every $k \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_k(X; G) \to \mathrm{H}_k(Y; G)$?

Exercise 162.14. We denote by $\iota\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^2$ the usual embedding. We consider the continuous map $f\colon \mathbb{R}\mathrm{P}^2 \to \Sigma(\mathbb{R}\mathrm{P}^2)$ that is given by

$$\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z \xrightarrow{[P]\mapsto[P]} \overline{B}^2/S^1 =_{\mathrm{i}} S^2 \cong \Sigma(S^1) \cong \Sigma(\mathbb{R}\mathrm{P}^1) \xrightarrow{\Sigma(\iota)} \Sigma(\mathbb{R}\mathrm{P}^2).$$

Here we denote by $\Sigma(X)$ the suspension of a topological space X, as defined on Definition 20.21.

(a) Show that for each $k \in \mathbb{N}$ the map $f_*\colon \mathrm{H}_k(\mathbb{R}\mathrm{P}^2) \to \mathrm{H}_k(\Sigma(\mathbb{R}\mathrm{P}^2))$ is the zero map.
(b) Show that the map $f_*\colon \mathrm{H}_k(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \to \mathrm{H}_k(\Sigma(\mathbb{R}\mathrm{P}^2);\mathbb{F}_2)$ is an isomorphism.

Exercise 162.15. Let X be a topological space and let $\mathbb{F}$ be a field. We say that X is $\mathbb{F}$-*finite* if $\sum_{k=0}^{\infty} \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F}))$ is finite. Otherwise we say that X is $\mathbb{F}$-*infinite*. Does there exist a topological space that is $\mathbb{F}$-finite with respect to one field $\mathbb{F}$ and that is $\widetilde{\mathbb{F}}$-infinite with respect to another field $\widetilde{\mathbb{F}}$?

Exercise 162.16. Let X be a topological space such that all homology groups $\mathrm{H}_k(X)$ are finitely generated and such that only finitely many are non-zero.

(a) Show that for any field $\mathbb{F}$ of characteristic 0 we have

$$\sum_{k=0}^{\infty} (-1)^k \cdot \mathrm{rank}(\mathrm{H}_k(X)) = \sum_{k=0}^{\infty} (-1)^k \cdot \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F})).$$

(b) Let p be a prime. Show that for any field $\mathbb{F}$ of characteristic p we also have

$$\sum_{k=0}^{\infty} (-1)^k \cdot \mathrm{rank}(\mathrm{H}_k(X)) = \sum_{k=0}^{\infty} (-1)^k \cdot \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F})).$$

Remark. This exercise can be viewed as a generalization of the Homology-χ-Proposition 160.15.

Exercise 162.17. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a continuous map. Recall that in Definition 151.3 we defined the degree $\deg(f) \in \mathbb{Z}$ as the unique integer such that $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by $\deg(f)$. Show that for every $k \in \mathbb{N}$ the induced map

$$f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z})$$

is given by multiplication by $\deg(f)$.

Exercise 162.18. We consider a chain complex of the form $0 \to C_1 \xrightarrow{\varphi} C_0 \to 0$ where C_1 and C_0 are free abelian groups. We consider the following two special cases:

(a) φ is a monomorphism.
(b) φ is an epimorphism.

Furthermore we consider the field $G = \mathbb{F}_p$ for some prime p. In each of the two cases formulate the conclusion of the Algebraic Universal Coefficient Theorem 162.15 in the simplest terms possible. Are the statements you get elementary?

Exercise 162.19. Let $m \in \mathbb{N}_{\geq 2}$. Show that there is no *natural* isomorphism

$$\Phi_X\colon \mathrm{H}_n(X;\mathbb{Z}/m\mathbb{Z}) \to \mathrm{H}_n(X) \otimes \mathbb{Z}/m\mathbb{Z} \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X), \mathbb{Z}/m\mathbb{Z}).$$

In other words, you are supposed to modify the example in the proof of the No-Natural Splitting Proposition 162.21.

Exercise 162.20. We consider the Universal Coefficient Theorem 162.16 for a topological space X. Is it possible to replace, in the formulation of the Universal Coefficient Theorem 162.16, the absolute homology groups by reduced homology groups?

Exercise 162.21. Let X be a connected 3-dimensional CW-complex with no 1-cell, with $\chi(X) = 5$ and with $\mathrm{H}_2(X) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$.

(a) Determine the homology groups $\mathrm{H}_n(X; \mathbb{Z}/2\mathbb{Z})$.
(b) Determine the homology groups $\mathrm{H}_n(X; \mathbb{Z}/6\mathbb{Z})$.

Exercise 162.22. Let $T = S^1 \times S^1$ be the torus and let K be the Klein bottle.

(a) Show that $\mathrm{H}_2(T; \mathbb{F}_2) \cong \mathbb{F}_2$ and that $\mathrm{H}_2(K; \mathbb{F}_2) \cong \mathbb{F}_2$.
(b) Show that for every continuous map $f\colon T \to K$ the corresponding induced homomorphism $f_*\colon \mathrm{H}_2(T; \mathbb{F}_2) \to \mathrm{H}_2(K; \mathbb{F}_2)$ is the zero map.
(c) Show that for every continuous map $f\colon K \to T$ the corresponding induced homomorphism $f_*\colon \mathrm{H}_2(K; \mathbb{F}_2) \to \mathrm{H}_2(T; \mathbb{F}_2)$ is the zero map.

CHAPTER 163

The tensor product of chain complexes

In this and the following chapter we want to study the following question:

Question 163.1. Let X and Y be two topological spaces. What is the connection between the homology groups of X and Y on the one hand and the homology groups of the product topological space $X \times Y$ on the other hand?

The idea for answering Question 163.1 is as follows:

(1) In this chapter we will introduce the completely algebraic notion of the tensor product $\mathcal{C} \otimes \mathcal{D}$ of two chain complexes $\mathcal{C}$ and $\mathcal{D}$.
(2) In this chapter we will prove the Algebraic–Künneth Theorem 163.6 which relates the homology groups of the chain complex $\mathcal{C} \otimes \mathcal{D}$ to the homology groups of the original chain complexes $\mathcal{C}$ and $\mathcal{D}$.
(3) In the next chapter we will prove the Eilenberg-Zilber Theorem 164.2 which says that, given two topological spaces X and Y, the singular chain complex $\mathrm{C}_*(X \times Y)$ is chain homotopy equivalent to the tensor product of the singular chain complexes $\mathrm{C}_*(X)$ and $\mathrm{C}_*(Y)$.

The combination of (2) and (3) then gives us an answer to Question 163.1, see the Künneth Theorem 164.6 for details.

This chapter is organized as follows:

- In Section 163.1 we will introduce the tensor product of chain complexes.
- In Section 163.2 we will state and prove Algebraic–Künneth Theorem 163.6 for chain complexes
- In Section 163.3 we will provide exercises for Chapter 163.

In the next Chapter 164 we will use tensor products of chain complexes to prove the Künneth Theorem 164.6.

163.1. The tensor product of chain complexes.

In this section we will introduce the tensor product of two chain complexes. Initially this definition might come across as being rather ad hoc. Historically the concepts from the next, topological chapter, motivate the definition. But instead of going through a long-winded motivation we just provide the definition and hope that the reader enjoys the corresponding algebra.

Without further ado, here is the definition:

Definition 163.2. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we define

$$(\mathcal{C} \otimes \mathcal{C}')_n := \bigoplus_{p+q=n} C_p \otimes C'_q.$$

We consider the homomorphisms

$$\partial_n^{\otimes}\colon (\mathcal{C}\otimes\mathcal{C}')_n \;\to\; (\mathcal{C}\otimes\mathcal{C}')_{n-1}$$

which are uniquely determined by[1722]

$$c_p\otimes c'_q \;\mapsto\; \underbrace{\partial_p(c_p)\otimes c'_q + (-1)^p\cdot c_p\otimes\partial'_q(c'_q)}_{=(\partial_p\otimes\mathrm{id}\,+(-1)^p\cdot\mathrm{id}\,\otimes\partial'_q)(c_p\otimes c'_q)}$$

for $c_p\in C_p$ and $c'_q\in C'_q$. By the Tensor–Boundary Lemma 163.3 below these maps $\partial_*^{\otimes}$ turn $\mathcal{C}\otimes\mathcal{C}'$ into a chain complex, called the tensor product[1729] $\mathcal{C}\otimes\mathcal{C}'$ of the two chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Visualization. We illustrate the definition of the tensor product of chain complexes in the following diagram:

$$\begin{array}{ccccccc}
C_0\otimes C'_3 & \xleftarrow{\partial_1\otimes\mathrm{id}} & C_1\otimes C'_3 & \xleftarrow{\partial_2\otimes\mathrm{id}} & C_2\otimes C'_3 & \xleftarrow{\partial_3\otimes\mathrm{id}} & C_3\otimes C'_3\\
\downarrow{\scriptstyle \mathrm{id}\otimes\partial'_3} & & \downarrow{\scriptstyle -\mathrm{id}\otimes\partial'_3} & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial'_3} & & \downarrow{\scriptstyle -\mathrm{id}\otimes\partial'_3}\\
C_0\otimes C'_2 & \xleftarrow{\partial_1\otimes\mathrm{id}} & C_1\otimes C'_2 & \xleftarrow{\partial_2\otimes\mathrm{id}} & C_2\otimes C'_2 & \xleftarrow{\partial_3\otimes\mathrm{id}} & C_3\otimes C'_2\\
\downarrow{\scriptstyle \mathrm{id}\otimes\partial'_2} & & \downarrow{\scriptstyle -\mathrm{id}\otimes\partial'_2} & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial'_2} & & \downarrow{\scriptstyle -\mathrm{id}\otimes\partial'_2}\\
C_0\otimes C'_1 & \xleftarrow{\partial_1\otimes\mathrm{id}} & C_1\otimes C'_1 & \xleftarrow{\partial_2\otimes\mathrm{id}} & C_2\otimes C'_1 & \xleftarrow{\partial_3\otimes\mathrm{id}} & C_3\otimes C'_1\\
\downarrow{\scriptstyle \mathrm{id}\otimes\partial'_1} & & \downarrow{\scriptstyle -\mathrm{id}\otimes\partial'_1} & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial'_1} & & \downarrow{\scriptstyle -\mathrm{id}\otimes\partial'_1}\\
C_0\otimes C'_0 & \xleftarrow{\partial_1\otimes\mathrm{id}} & C_1\otimes C'_0 & \xleftarrow{\partial_2\otimes\mathrm{id}} & C_2\otimes C'_0 & \xleftarrow{\partial_3\otimes\mathrm{id}} & C_3\otimes C'_0.
\end{array}$$

Here the "antidiagonal" groups of the same color "add up" to the chain groups of $\mathcal{C}\otimes\mathcal{C}'$.□

In the above Definition 163.2 we already made use of the following little lemma:

Lemma 163.3. (Tensor–Boundary Lemma) Let $\mathcal{C}=(C_*,\partial_*)$ and $\mathcal{C}'=(C'_*,\partial'_*)$ be two chain complexes. The maps $\partial_*^{\otimes}$ introduced above turn $\mathcal{C}\otimes\mathcal{C}'$ into a chain complex.

Proof. Let $\mathcal{C}=(C_*,\partial_*)$ and $\mathcal{C}'=(C'_*,\partial'_*)$ be two chain complexes. We have to show that the maps $\partial_*^{\otimes}$ are indeed boundary maps of a chain complex, i.e. we have to verify that $\partial_n^{\otimes}\circ\partial_{n+1}^{\otimes}=0$. This is the case since for any $c_p\in C_p$ and $c'_q\in C'_q$ we calculate that

$$\begin{aligned}
\partial_n^{\otimes}(\partial_{n+1}^{\otimes}(c_p\otimes c'_q)) &= \partial_n^{\otimes}(\overbrace{\partial_p(c_p)}^{\in C_{p-1}}\otimes c'_q) + \partial_n^{\otimes}((-1)^p\cdot c_p\otimes \overbrace{\partial'_q(c'_q)}^{\in C'_{q-1}})\\
&= \underbrace{\partial_{p-1}(\partial_p(c_p))}_{=0}\otimes c'_q + (-1)^{p-1}\cdot\partial_p(c_p)\otimes\partial'_q(c'_q)\\
&\qquad +(-1)^p\cdot\partial_p(c_p)\otimes\partial'_q(c'_q) + (-1)^p\cdot c_p\otimes\underbrace{\partial'_{q-1}(\partial'_q(c'_q))}_{=0}\\
&= (-1)^{p-1}\cdot\partial_p(c_p)\otimes\partial'_q(c'_q) - (-1)^{p-1}\cdot\partial_p(c_p)\otimes\partial'_q(c'_q) \;=\; 0.
\end{aligned}$$

[1728] Secretly we are using here the Tensor Product–Universal Property Lemma 159.4 (2) and the Coproduct-of-Abelian Groups Lemma 97.2 (2).

[1729] The author is not aware of any universal properties of the tensor product of chain complexes.

Thus we have shown that $(\mathcal{C} \otimes \mathcal{C}', \partial^{\otimes})$ is indeed a chain complex.[1730] ■

Example. Let

$$\mathcal{C} \;=\; \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \ldots \qquad \text{and} \qquad \mathcal{C}' \;=\; 0 \;\to\; \underset{\text{in degree } k}{G} \;\to\; 0$$

be two chain complexes where the second consists of a single abelian group G. In this case we have for each $n \in \mathbb{N}_0$ the equality

$$(\mathcal{C}_* \otimes \mathcal{C}'_*)_n \;=\; \mathcal{C}_{n-k} \otimes G$$

and the boundary map is given by $\partial_{n-k} \otimes \mathrm{id}_G$. Thus we see that in this case the tensor product of chain complexes is just given by tensoring the chain groups of $\mathcal{C}$ by G and shifting the degrees "inward " by k. □

We conclude this section with the following lemma, which we will use subconsciously on several occasions:

Lemma 163.4. (Tensoring Chain Maps Lemma) Let $\mathcal{C}, \mathcal{D}, \mathcal{C}', \mathcal{D}'$ be chain complexes.

(1) Given two chain maps $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ the induced map

$$\begin{aligned} f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' &\to \mathcal{D} \otimes \mathcal{D}' \\ \sum_{i=1}^{n} c_i \otimes c'_i &\mapsto \sum_{i=1}^{n} f(c_i) \otimes f'(c'_i) \end{aligned}$$

is also a chain map.

(2) If $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ are chain homotopic maps, then the induced maps

$$f \otimes f', g \otimes g'\colon \mathcal{C} \otimes \mathcal{C}' \;\to\; \mathcal{D} \otimes \mathcal{D}'$$

are also chain homotopic.

(3) If $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain homotopy equivalences, then the induced map

$$f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' \;\to\; \mathcal{D} \otimes \mathcal{D}'$$

is also a chain homotopy equivalence.

Proof.

(1) Let $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ be chain maps. Given any $c \in C_p$ and $c' \in C'_q$ we calculate that

$$\begin{aligned} \partial^{\otimes}_n(f(c) \otimes f'(c')) &\overset{\text{since } \deg(f(c)) = \deg(c) = p}{=} \partial_p(f(c)) \otimes f'(c') + (-1)^p \cdot f(c) \otimes \partial_q(f'(c')) \\ &= f(\partial_p(c)) \otimes f'(c') + (-1)^p \cdot f(c) \otimes f'(\partial_q(c')) \\ &\overset{\text{since } f \text{ and } f' \text{ are chain maps}}{=} (f \otimes f')\big(\partial_p(c) \otimes c' + (-1)^p \cdot c \otimes \partial_q(c')\big) \;=\; (f \otimes f')(\partial^{\otimes}_n(c \otimes c')). \end{aligned}$$

(2) Let $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ be chain maps.

[1730] Note that in the calculation we made use of the extra-term $(-1)^p$ in the definition of the boundary map. If we had defined the boundary map "naively" as $\partial^{\otimes}_n(c_p \otimes c'_q) = \partial_p(c_p) \otimes c'_q + c_p \otimes \partial'_q(c'_q)$, then the result would in general not have been a chain complex.

Claim 1. If $P\colon \mathcal{C} \to \mathcal{D}$ is a chain homotopy between the chain maps f and g, then the maps $P_p \otimes f'_q\colon C_p \otimes \mathcal{C}'_q \to D_{p+1} \otimes \mathcal{D}'_q$ are a chain homotopy between $f \otimes f'$ and $g \otimes f'$.

Proof. We need to show that $\partial^{\otimes}_{p+q+1} \circ (P \otimes f') + (P \otimes f') \circ \partial^{\otimes}_{p+q} = f \otimes f' - g \otimes f'$. We only have to verify that this statement holds on each $C_p \otimes C'_q$. As maps $C_p \otimes C'_q \to D_p \otimes D'_q$ we have

$$\begin{aligned}
&(\partial^{\otimes}_{p+q+1}) \circ (P \otimes f') + (P \otimes f') \circ \partial^{\otimes}_{p+q} \\
&= \underbrace{(\partial_{p+1} \otimes \mathrm{id} + (-1)^{p+1} \cdot \mathrm{id} \otimes \partial'_q)}_{=\partial^{\otimes}_{p+q+1}\colon C_{p+1}\otimes D_q \to C_p \otimes D_q \oplus C_{p+1} \otimes D_{q-1}} \circ (P \otimes f') + (P \otimes f') \circ \underbrace{(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_q)}_{=\partial^{\otimes}_{p+q}\colon C_p \otimes D_q \to C_{p-1} \otimes D_q \oplus D_p \otimes D_{q-1}} \\
&= (\partial_{p+1} \circ P) \otimes f' + (-1)^{p+1} \cdot P \otimes (\partial'_q \circ f') + (P \circ \partial_p) \otimes f' + (-1)^p \cdot P \otimes \underbrace{(f' \circ \partial'_q)}_{=\partial'_q \circ f'} \\
&= (\partial_{p+1} \circ P) \otimes f' + (P \circ \partial_p) \otimes f' \;=\; (\partial_{p+1} \circ P + P \circ \partial_p) \otimes f' \;=\; (f_p - g_p) \otimes f' \\
&= f_p \otimes f' - g_p \otimes f'.
\end{aligned}$$

⊞

Since the situation is not entirely symmetric between the two factors let us also consider explicitly the situation with the two entries reversed:

Claim 2. Let $P'\colon \mathcal{C}' \to \mathcal{D}'$ be a chain homotopy between the chain maps f' and g'. Then the maps $(-1)^p \cdot f_p \otimes P'_q\colon C_p \otimes C'_q \to C_p \otimes D'_{q+1}$ define a chain homotopy between $f \otimes f'$ and $f \otimes g'$.

Proof. We need to show that $\partial^{\otimes}_{p+q+1} \circ (f \otimes P') + (f \otimes P') \circ \partial^{\otimes}_{p+q} = f \otimes f' - f \otimes g'$. We only have to verify that this statement holds on each $C_p \otimes C'_q$. We set $Q := (-1)^p \cdot f_p \otimes P'_q$. As maps $C_p \otimes C'_q \to D_p \otimes D'_q$ we have

$$\begin{aligned}
&\partial^{\otimes}_{p+q+1} \circ Q + Q \circ \partial^{\otimes}_{p+q} \\
&= \underbrace{(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_{q+1})}_{=\partial^{\otimes}_{p+q+1}\colon C_p \otimes D_{q+1} \to C_{p-1} \otimes D_{q+1} \oplus C_p \otimes D_q} \circ Q + Q \circ \underbrace{(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_q)}_{=\partial^{\otimes}_{p+q}\colon C_p \otimes D_q \to C_{p-1} \otimes D_q \oplus C_p \otimes D_{q-1}} \\
&= (-1)^p \cdot (\partial_p \circ f_p) \otimes P' + f_p \otimes (\partial'_{q+1} \circ P') + (-1)^{p-1} \cdot \underbrace{(f_{p-1} \circ \partial_p)}_{=\partial_p \circ f_p} \otimes P' + f_p \otimes (P' \circ \partial'_q) \\
&= f_p \otimes (\partial'_{q+1} \circ P' + P' \circ \partial'_q) \;=\; f_p \otimes (f'_q - g'_q) \;=\; f_p \otimes f'_q - f_p \otimes g'_q.
\end{aligned}$$

⊞

Now we finally turn to the actual proof of Statement (2) of the lemma. Thus let $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ be chain homotopic chain maps. It follows from Claim 1, Claim 2 and the Chain Homotopy-Lemma 144.6 that $f \otimes f', g \otimes g'\colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{D} \otimes \mathcal{D}'$ are chain homotopic.

(3) This statement follows easily from the definitions and (2). ■

163.2. **The Künneth Theorem for chain complexes.** Let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes of free abelian groups. In this section we want to determine the homology groups of $\mathcal{C} \otimes \mathcal{D}$ in terms of the homology groups of $\mathcal{C}$ and $\mathcal{D}$. We start out with the following elementary lemma:

Lemma 163.5. (Cycle Tensor Lemma) Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. The map

$$\begin{aligned}
\mu\colon \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') &\to \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\
\sum_{i=1}^{r} [c_i] \otimes [c'_i] &\mapsto \sum_{i=1}^{r} [c_i \otimes c'_i]
\end{aligned}$$

is a well-defined homomorphism and it is natural with respect to the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Proof. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. We start out with the following claim:

Claim. The map
$$\begin{array}{rccc} \mu\colon & \mathrm{H}_p(\mathcal{C}) \times \mathrm{H}_q(\mathcal{C}') & \to & \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\ & ([c],[c']) & \mapsto & [c \otimes c'] \end{array}$$
is well-defined and bilinear.

Proof. First note that if $c \in C_p$ and $c' \in C_q$ are cycles, then
$$\partial^{\otimes}_{p+q}(c \otimes c') \;=\; \underbrace{\partial_p(c)}_{=0} \otimes c' + (-1)^p \cdot c \otimes \underbrace{\partial_q(c')}_{=0} \;=\; 0$$
i.e. $c \otimes c' \in \mathcal{C} \otimes \mathcal{C}'$ is indeed a cycle. Second, if $c + \partial_{p+1}(e)$ is another representative of $[c]$ then we have the following equality in $\mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}')$:
$$[(c + \partial_{p+1}(e)) \otimes c'] \;=\; [c \otimes c' + \partial_{p+1}(e) \otimes c'] \underset{\substack{\uparrow \\ \text{since } \partial'_q(c') = 0}}{=} [c \otimes c' + \partial^{\otimes}_{p+q+1}(e \otimes c')] \;=\; [c \otimes c'].$$

The same way one can show that the value of μ does not depend on the choice of representative of the class in $\mathrm{H}_q(\mathcal{C}')$. We have thus shown that the map given in the claim is well-defined. It is clear that it is bilinear. ⊞

It follows from the claim and the Tensor Product–Universal Property Lemma 159.4 (2) that the map μ of the lemma is a well-defined homomorphism. Finally note that it is straightforward to verify that μ is in fact a natural map. ■

Example. In some instances the homomorphism μ from the Cycle Tensor Lemma 163.5 is an isomorphism. For example let $\mathcal{C} = (C_*, \partial_*)$ be a chain complex of free abelian groups such that all boundary maps $C_n \to C_{n-1}$ in $\mathcal{C}$ are the zero map. In this case we have for any chain complex $\mathcal{C}'$ and any $n \in \mathbb{N}_0$ the following natural isomorphisms
$$\bigoplus_{p \in \mathbb{N}_0} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_{n-p}(\mathcal{C}'_*) \underset{\substack{\uparrow \\ \text{since all the boundary maps in } \mathcal{C} \text{ are zero}}}{=} \bigoplus_{p \in \mathbb{N}_0} C_p \otimes \mathrm{H}_n(\mathcal{C}'_{*-p}) \underset{\substack{\uparrow \\ \text{by the Algebraic Universal Coefficient Theorem 162.15 and the Tor-Properties Lemma 162.13 (4) (here we use that all } C_p \text{ are free abelian)}}}{\overset{\cong}{\to}} \bigoplus_{p \in \mathbb{N}_0} \mathrm{H}_n(C_p \otimes \mathcal{C}'_{*-p}) \underset{\substack{\uparrow \\ \text{by the example on page 3130 since } \mathcal{C} \text{ is a direct sum of trivial chain complexes}}}{\overset{\cong}{\to}} \mathrm{H}_n(\mathcal{C}_* \otimes \mathcal{C}'_*).$$

It follows easily from the definitions that the maps from left to right are induced by the maps μ from the previous lemma. We will shortly make good use of this example. □

Now we can formulate the following promised theorem which is the key result of this chapter.

Theorem 163.6. (Algebraic–Künneth Theorem)[1731] Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. If all chain groups of the chain complex $\mathcal{C}$ are free abelian groups, then there exists a natural homomorphism
$$\nu\colon \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \;\to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}'))$$

such that the following sequence is exact:

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{\mu} \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \xrightarrow{\nu} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) \to 0.$$

If the chain groups of $\mathcal{C}'$ are also free abelian, then this short exact sequence splits.[1732]

Remark.

(1) The adjective "natural" means in this context that if $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain maps, then we obtain a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{\mu} & \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') & \xrightarrow{\nu} & \displaystyle\bigoplus_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*\otimes f'_*} & & \downarrow{\scriptstyle (f\otimes f')_*} & & \downarrow{\scriptstyle \mathrm{Tor}(f_*, f'_*)} & & \\
0 & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{D}) \otimes \mathrm{H}_q(\mathcal{D}') & \xrightarrow{\mu} & \mathrm{H}_n(\mathcal{D} \otimes \mathcal{D}') & \xrightarrow{\nu} & \displaystyle\bigoplus_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{D}), \mathrm{H}_q(\mathcal{D}')) & \longrightarrow & 0.
\end{array}$$

Here the middle map is given by the Tensoring Chain Maps Lemma 163.4 (1) and the map on the right is given by the functoriality of the torsion-groups that we had discussed on page 3107.

(2) If $\mathcal{C}'$ is a chain complex which consists of a single abelian group G, i.e. if it is of the form $0 \to G \to 0$ with G in degree 0, then $\mathrm{H}_0(\mathcal{C}') = G$ and all other homology groups are zero. As on page 3130 we see that $(\mathcal{C} \otimes \mathcal{C}')_n = C_n \otimes G$. Thus the Algebraic–Künneth Theorem 163.6 gives us the short exact sequence

$$0 \to \mathrm{H}_n(\mathcal{C}) \otimes G \to \mathrm{H}_n(\underbrace{\mathcal{C} \otimes G}_{=\mathcal{C}\otimes\mathcal{C}'}) \to \mathrm{Tor}(\mathrm{H}_{n-1}(\mathcal{C}), G) \to 0,$$

i.e. we recover the Algebraic Universal Coefficient Theorem 162.15 as a special case of the Algebraic–Künneth Theorem 163.6. In fact, as we will see, the proof of the Künneth Theorem is quite similar to the proof of the Algebraic Universal Coefficient Theorem 162.15. □

Now we turn to the proof of the Algebraic–Künneth Theorem 163.6. To improve readability we break the proof into two parts. First we define ν and we show that we obtain a short exact sequence. Afterwards we will show that the short exact sequence splits if all chain groups of $\mathcal{C}'$ are also free abelian.

Proof that the short exact sequence exists. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes such that all C_n are free abelian groups. First we introduce the following notation and objects:

(1) (a) For $n \in \mathbb{N}_0$ we set $Z_n \coloneqq \ker(\partial_n\colon C_n \to C_{n-1})$.
(b) We denote by $\mathcal{Z}$ the chain complex which is given by the groups Z_n where all the boundary maps are the zero map.

[1731] Hermann Künneth (1892-1975) was a German mathematician. In [**Kün1923a, Kün1924**] he proved early versions of the above theorem. In [**Kün1923b**] he also gave one of the first proofs of the (much easier) Product-π_1-Proposition 90.13.

[1732] If the Künneth Theorem splits, then it follows from the Splitting Lemma 145.5 that the middle group is isomorphic to the direct sum of the two groups to the left and right. In Exercise 163.2 we will prove, analogous to the Algebraic No-Natural Splitting Proposition 162.20, that there is no natural isomorphism.

(c) We denote by $\iota\colon Z_n \to C_n$ the inclusion map. Note that these maps form a chain map $\iota\colon \mathcal{Z} \to \mathcal{C}$.

(2) (a) For $n \in \mathbb{N}_0$ we set $B_n := \operatorname{im}(\partial_{n+1}\colon C_{n+1} \to C_n)$.

(b) We denote by $\mathcal{B}$ the chain complex which is given by the groups B_n where all the boundary maps are the zero map.

(c) We denote by $\mathcal{B}[1]$ the chain complex with $\mathcal{B}[1]_n = \mathcal{B}_{n-1}$, i.e. it is the chain complex $\mathcal{B}$ but all groups got shifted by 1 to the left.

(d) Note that the boundary maps $\partial_n\colon C_n \to B_{n-1}$ define a chain map $\partial\colon \mathcal{C} \to \mathcal{B}[1]$.

Claim. The maps

$$0 \;\to\; \mathcal{Z} \otimes \mathcal{C}' \;\xrightarrow{\iota\otimes\mathrm{id}}\; \mathcal{C} \otimes \mathcal{C}' \;\xrightarrow{\partial\otimes\mathrm{id}}\; \mathcal{B}[1] \otimes \mathcal{C}' \;\to\; 0$$

form a short exact sequence of chain complexes.

Proof. It follows from the Tensoring Chain Maps Lemma 163.4 (1) that the homomorphisms $\iota \otimes \mathrm{id}\colon \mathcal{Z} \otimes \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$ and $\partial \otimes \mathrm{id}\colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{B}[1] \otimes \mathcal{C}$ are chain maps. It remains to show that the sequence is exact at each degree. The argument is exactly the same as on page 3115. For the reader's convenience we recall the argument. First note that by definition the sequences

$$0 \;\to\; Z_p \;\xrightarrow{\iota}\; C_p \;\xrightarrow{\partial_p}\; \underbrace{B_{p-1}}_{=\mathcal{B}[1]_p} \;\to\; 0$$

are exact. Also recall that by hypothesis each C_n is a free abelian group. It follows from the Free Abelian Subgroup Theorem 97.14 that each B_n is also free abelian. By the Free Abelian Group–Splitting Lemma 97.11 this means that the above short exact sequence splits. But then it follows from the Tensor-Split-SES Lemma 160.9 that the sequence stays exact even after tensoring it with any C'_q. The claim is an immediate consequence of these observations. ⊞

From the LES-Proposition 145.10 and the claim we obtain a natural connecting homomorphism

$$\underbrace{\mathrm{H}_{n+1}(\mathcal{B}[1] \otimes \mathcal{C}')}_{=\mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}')} \;\to\; \mathrm{H}_n(\mathcal{Z} \otimes \mathcal{C}').$$

As in the proof of the Algebraic Universal Coefficient Theorem 162.15 it follows easily from the definition that the connecting homomorphism $\mathrm{H}_n(\mathcal{B} \otimes \mathcal{C}') \to \mathrm{H}_n(\mathcal{Z} \otimes \mathcal{C}')$ is induced by the map $i \otimes \mathrm{id}$ where i denotes the inclusion $\mathcal{B} \to \mathcal{Z}$. From the LES-Proposition 145.10 we obtain the following long exact sequence[1733]

$$\cdots \to \mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \mathrm{H}_n(\mathcal{Z}\otimes\mathcal{C}') \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \mathrm{H}_{n-1}(\mathcal{Z}\otimes\mathcal{C}') \to \dots$$

Since the boundary maps in the chain complexes $\mathcal{B}$ and $\mathcal{Z}$ are the zero maps it follows from the example on page 3132 that we obtain the following long exact sequence:

$$\dots \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n} Z_p \otimes \mathrm{H}_q(\mathcal{C}') \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \bigoplus_{p+q=n-1} B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n-1} Z_p \otimes \mathrm{H}_q(\mathcal{C}') \to \dots$$

[1733] The index goes down by one in the map $\mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B} \otimes \mathcal{C}')$, since we replace $\mathcal{B}[1]$ by $\mathcal{B}$.

From the Break-into-SES Lemma 145.6 we obtain a short exact sequence[1734] [1735]

$$0 \to \bigoplus_{p+q=n} \overbrace{\operatorname{coker}\left(B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes \mathrm{id}} Z_p \otimes \mathrm{H}_q(\mathcal{C}')\right)}^{\substack{\cong \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}'),\ \text{since} \\ 0 \to B_p \to Z_p \to \mathrm{H}_p(\mathcal{C}) \to 0 \text{ is a free resolution}}} \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to$$

$$\to \bigoplus_{p+q=n-1} \underbrace{\ker\left(B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes \mathrm{id}} Z_p \otimes \mathrm{H}_q(\mathcal{C}')\right)}_{\substack{\cong \operatorname{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')),\ \text{since} \\ 0 \to B_p \to Z_p \to \mathrm{H}_p(\mathcal{C}) \to 0 \text{ is a free resolution}}} \to 0.$$

This proves the existence of the homomorphism ν and the fact that μ and ν give us a short exact sequence. Using Statement (2) of the Homological Algebra–Fundamental Theorem 158.4 one can show that ν is natural. We leave it to the reader to verify the details. ■

Proof that the short exact sequence splits. [1736] Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes such that all C_n and all C'_n are free abelian groups. Recall that according to the Splitting Lemma 145.5 (3)⇒(1) it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C})\otimes\mathrm{H}_q(\mathcal{C}')$ such that $\eta\circ\mu$ is the identity on $\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C})\otimes\mathrm{H}_q(\mathcal{C}')$. We summarize this problem as follows:

$$0 \longrightarrow \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \underset{?\,\eta\,?}{\overset{\mu}{\rightleftarrows}} \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \xrightarrow{\ \nu\ } \bigoplus_{p+q=n-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) \longrightarrow 0.$$

For each $n \in \mathbb{N}_0$ we consider $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \xhookrightarrow{\ \iota_n\ } C_n \xrightarrow{\ \partial_n\ } B_{n-1} \longrightarrow 0.$$

Since the chain groups C_n are free abelian we see, as in the proof of the Algebraic UCT-Splitting Proposition 162.17, that for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. We denote by $q_n\colon Z_n \to \mathrm{H}_n(\mathcal{C})$ the natural projection and we set $f_n := q_n \circ p_n\colon C_n \to \mathrm{H}_n(\mathcal{C})$.

Next recall that in the Algebraic UCT-Splitting Proposition 162.17 we showed that the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{n+1} & \xrightarrow{\ \partial_{n+1}\ } & C_n & \xrightarrow{\ \partial_n\ } & C_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \longrightarrow & \mathrm{H}_{n+1}(\mathcal{C}) & \xrightarrow{\ 0\ } & \mathrm{H}_n(\mathcal{C}) & \xrightarrow{\ 0\ } & \mathrm{H}_{n-1}(\mathcal{C}) & \longrightarrow & \dots
\end{array}$$

[1734] We are using that for homomorphisms $\{\varphi_i\colon V_i \to W_i\}_{i\in I}$ we have $\ker\left(\bigoplus_{i\in I} \varphi_i\right) = \bigoplus_{i\in I} \ker(\varphi_i)$ and similarly for cokernels.

[1735] As before we also make use of the Tor-Free Resolution Proposition 162.6 and the Tor_0–Lemma 162.7.

[1736] As we will see, the proof of this statement is quite similar to the proof of the Algebraic UCT-Splitting Proposition 162.17.

Since each C'_n is also a free abelian group we can do all the definitions and constructions for the chain complex $\mathcal{C}'$. In particular we obtain homomorphisms $f'_n\colon C'_n \to \mathrm{H}_n(\mathcal{C}')$ which have the same properties as the homomorphisms f_n.

Using the above commutative diagrams involving f_n and f'_n one can easily show that the following diagram of two horizontal chain complexes commutes:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \overbrace{\bigoplus_{p+q=n+1} \mathcal{C}_p \otimes \mathcal{C}'_q}^{=(\mathcal{C}\otimes\mathcal{C}')_{n+1}} & \xrightarrow{\partial^{\otimes}_{n+1}} & \overbrace{\bigoplus_{p+q=n} \mathcal{C}_p \otimes \mathcal{C}'_q}^{=(\mathcal{C}\otimes\mathcal{C}')_{n}} & \xrightarrow{\partial^{\otimes}_{n}} & \overbrace{\bigoplus_{p+q=n-1} \mathcal{C}_p \otimes \mathcal{C}'_q}^{=(\mathcal{C}\otimes\mathcal{C}')_{n-1}} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \bigoplus_{p+q=n+1} f_p\otimes f'_q} & & \downarrow{\scriptstyle \bigoplus_{p+q=n} f_p\otimes f'_q} & & \downarrow{\scriptstyle \bigoplus_{p+q=n-1} f_p\otimes f'_q} & & \\
\dots & \to & \bigoplus_{p+q=n+1} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{0} & \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{0} & \bigoplus_{p+q=n-1} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \to & \dots
\end{array}$$

Since the diagram commutes we obtain induced maps on the homology of these two chain complexes. These give us a homomorphism

$$\eta\colon \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \;\to\; \mathrm{H}_n(\text{lower sequence}) = \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}').$$

As in the proof of the Algebraic UCT-Splitting Proposition 162.17 it follows easily from the definitions that the composition $\eta \circ \mu$ is the identity on $\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}')$. ■

163.3. Exercises for Chapter 163 .

Exercise 163.1.
(a) Let C_*, D_* be chain complexes. Is $C_* \otimes D_*$ is isomorphic to $D_* \otimes C_*$?
(b) Let C_*, D_*, E_* be chain complexes. Is $C_* \otimes (D_* \otimes E_*)$ isomorphic to $(C_* \otimes D_*) \otimes E_*$?
If yes, then write down isomorphisms.
Remark. Beware of signs.

Exercise 163.2. Show that the splitting in the Algebraic–Künneth Theorem 163.6, for chain complexes C_* and C'_* of free abelian groups, is not natural.

Exercise 163.3. Show that there is no *natural* isomorphism $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ of singular chain complexes.
Hint. Assume such a natural isomorphism exists. Consider chains in degree 0 and 1 and apply naturality cleverly to get to a contradiction.

Exercise 163.4. How can one deduce the Algebraic UCT-Splitting Proposition 162.17 from the Algebraic–Künneth Theorem 163.6?

CHAPTER 164

The Eilenberg-Zilber Theorem and the Künneth Theorem

As we already mentioned in the beginning of the last Chapter 163, the goal of this chapter is to formulate and prove the Eilenberg-Zilber Theorem 164.2 which says that for topological spaces X and Y there exists a natural chain homotopy equivalence between $\mathrm{C}_*(X \times Y)$ and the tensor product $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ of the chain complexes $\mathrm{C}_*(X)$ and $\mathrm{C}_*(Y)$. Together with the Algebraic–Künneth Theorem 163.6 this allows us to calculate the homology groups of products of topological spaces.

In fact the Eilenberg-Zilber Theorem 164.2 gives us much more, namely it allows us to define the cross product on homology and later it will be at the heart of the all important definition of the cup product on cohomology (see Chapter 203).

This chapter is organized as follows:

- In Section 164.1 we will state the Eilenberg-Zilber Theorem 164.2.
- In Section 164.2 we will use the Eilenberg-Zilber Theorem 164.2 to prove the Künneth Theorem 164.6 for topological spaces
- In Section 164.3 we will prove the Eilenberg-Zilber Theorem 164.2.
- In Section 164.4 we will introduce the relative Eilenberg–Zilber map Υ.
- In Appendix 164.5 we will discuss the cellular chain complex of a product.
- In Appendix 164.6 we will introduce the shuffle product.
- In Section 164.7 we will provide exercises for Chapter 164.

In the next Chapter 165 we will discuss several applications of homology groups.

164.1. **The Eilenberg-Zilber theorem.** We start out by recalling and introducing some definitions:

Definition 164.1.

(1) Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C} = (C'_*, \partial'_*)$ be two chain complexes. Recall that in Definition 163.2 we defined the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes where the n-th chain group is defined as

$$(\mathcal{C} \otimes \mathcal{C}')_n \quad := \quad \bigoplus_{p+q=n} C_p \otimes C'_q$$

and where the boundary homomorphisms are given by

$$\begin{array}{rccl} \partial^{\otimes}_n \colon & (\mathcal{C} \otimes \mathcal{C}')_n & \to & (\mathcal{C} \otimes \mathcal{C}')_{n-1} \\ & c_p \otimes c'_q & \mapsto & \partial_p(c_p) \otimes c'_q + (-1)^p \cdot c_p \otimes \partial'_q(c'_q). \end{array}$$

(2) Given a point x in a topological space X we denote by x also the singular 0-simplex given by sending the unique point of Δ^0 to x.

(3) Let X be a topological space and let $x \in X$. If $\sigma\colon \Delta^n \to Y$ is a singular n-simplex in a topological space Y, then we denote by $x \times \sigma\colon \Delta^n \to X \times Y$ the singular n-simplex given by $v \mapsto (x, \sigma(v))$. We use the analogous notation for the roles of X and Y swapped.

(4) Given two topological spaces X and Y we refer to the maps

$$\begin{array}{ccc} \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) & \to & \mathrm{C}_0(X \times Y) \\ x \otimes y & \mapsto & (x, y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{C}_0(X \times Y) & \to & \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) \\ (x, y) & \mapsto & x \otimes y \end{array}$$

as the canonical maps. We refer to the identity maps on $\mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$ and on $\mathrm{C}_0(X \times Y)$ also as the canonical maps.

The following theorem, which was first proved in 1953 by Samuel Eilenberg and Joseph Zilber [**EZ1953**], relates the singular chain complex $\mathrm{C}_*(X \times Y)$ of the product of two topological spaces X and Y to the tensor product $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ of the two individual chain complexes.

Theorem 164.2. (Eilenberg-Zilber Theorem)

(1) Given any topological spaces X and Y there exist natural[1737] chain maps

$$\begin{array}{ccc} \Upsilon = \Upsilon_{X,Y}\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \to & \mathrm{C}_*(X \times Y) \end{array}$$

and

$$\begin{array}{ccc} \Theta = \Theta_{X,Y}\colon \mathrm{C}_*(X \times Y) & \to & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y), \end{array}$$

which on the 0-level are the above canonical maps.

(2) The maps Υ and Θ are unique up to natural chain homotopies.[1738]

(3) The maps Υ and Θ are naturally[1739] chain homotopy inverses of one another.

In fact, we can find $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ with the following extra property:

(∗) For any $x \in X$ and any singular simplex $\sigma\colon \Delta^q \to Y$ we have $\Upsilon(x \otimes \sigma) = x \times \sigma$ and the same statement also holds with the roles of X and Y swapped.

Definition 164.3. We refer to the maps Υ and Θ from the Eilenberg-Zilber Theorem 164.2 as Eilenberg-Zilber maps.

Proof. We postpone the elegant, but technically slightly intricate, proof of the Eilenberg-Zilber Theorem 164.2 to Section 164.3.[1740] ∎

164.2. **The Künneth Theorem for topological spaces.** Before we turn to the proof of the Eilenberg-Zilber Theorem 164.2 let us first discuss a major application. To formulate the application it is convenient to introduce the following definition:

Definition 164.4. In the following let X and Y be two topological spaces. Furthermore let $\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$ be an Eilenberg-Zilber map. We refer to the map[1741]

$$\times\colon \mathrm{H}_p(X)\otimes\mathrm{H}_q(Y) \xrightarrow{\mu} \mathrm{H}_{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)) \xrightarrow{\Upsilon_*} \mathrm{H}_{p+q}(X\times Y).$$

(μ: map from the Cycle Tensor Lemma 163.5)

as the cross product.

In the following lemma we summarize some elementary properties of the cross product.

Lemma 164.5. (Cross Product Lemma)
(0) The cross product does not depend on the choice of the Eilenberg-Zilber map Υ.

[1737]Here "natural" requires some explanation. Let $\mathcal{TupTop}$ be the category of tuples of topological spaces with

$$\begin{aligned}\mathrm{Ob}(\mathcal{TupTop}) &= \text{all tuples } (X,Y) \text{ of topological spaces,}\\ \mathrm{Mor}((X,Y),(X',Y')) &= \text{all tuples } (f,g) \text{ of continuous maps } f\colon X\to X' \text{ and } g\colon Y\to Y'.\end{aligned}$$

Now "natural" means that the chain maps $\Upsilon_{X,Y}$ form a natural transformation between the functors

$$\begin{array}{ccl}\mathcal{TupTop} & \to & \text{category } \mathcal{ChCplx} \text{ of chain complexes}\\ (X,Y) & \mapsto & \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\\ (X,Y) & \mapsto & \mathrm{C}_*(X\times Y)\end{array}$$

and similarly for Θ. In other words, this means that for continuous maps $f\colon X\to X'$ and $g\colon Y\to Y'$ of topological spaces we obtain commutative diagrams

$$\begin{array}{ccc}\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) & \xrightarrow{\Upsilon_{X,Y}} & \mathrm{C}_*(X\times Y)\\ \downarrow{\scriptstyle f_*\otimes f'_*} & & \downarrow{\scriptstyle (f\times g)_*}\\ \mathrm{C}_*(X')\otimes\mathrm{C}_*(Y') & \xrightarrow{\Upsilon_{X',Y'}} & \mathrm{C}_*(X'\times Y')\end{array} \quad\text{and}\quad \begin{array}{ccc}\mathrm{C}_*(X\times Y) & \xrightarrow{\Theta_{X,Y}} & \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\\ \downarrow{\scriptstyle (f\times g)_*} & & \downarrow{\scriptstyle f_*\otimes f'_*}\\ \mathrm{C}_*(X'\times Y') & \xrightarrow{\Theta_{X',Y'}} & \mathrm{C}_*(X')\otimes\mathrm{C}_*(Y').\end{array}$$

[1738]Here "natural chain homotopy" means that given two such maps Υ and Υ' there exists a choice of chain homotopies $(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_k\to\mathrm{C}_{k+1}(X\times Y)$, which is natural in X and Y, between Υ and Υ'. An analogous statement holds for maps Θ and Θ'.
[1739]Here "naturally" means that $\Upsilon\circ\Theta$ and $\Theta\circ\Upsilon$ are naturally homotopic to the identity maps, i.e. there exist natural chain homotopies from $\Upsilon\circ\Theta$ to the identity map and from $\Theta\circ\Upsilon$ to the identity map.
[1740]The existence proof for the two maps Υ and Θ which we will provide in the following sections is non-explicit. A little later, in the Shuffle Proposition 164.24 we will give an explicit example of an Eilenberg-Zilber map Υ. For the most part the existence is enough, but on a few occasions it will be helpful to have an explicit example of an Eilenberg-Zilber map Υ at our disposal.
[1741]As we just mentioned in footnote 1740, in the Shuffle Proposition 164.24 we will give an explicit example of an Eilenberg-Zilber map Υ. This makes it possible to write down the cross product explicitly.

(1) Let X and Y be two topological spaces. The map

$$\times\colon \mathrm{H}_0(X) \otimes \mathrm{H}_n(Y) \;\to\; \mathrm{H}_n(X \times Y)$$

is given by

$$[x] \otimes \left[\sum_{i=1}^{k} r_i \cdot \sigma_i\right] \;\mapsto\; \left[\sum_{i=1}^{k} r_i \cdot (x \times \sigma_i)\right].$$

The statement also holds with the roles of X and Y reversed.

(2) The cross product is natural, i.e. if $f\colon X \to X'$ and $g\colon Y \to Y'$ are continuous maps between topological spaces, then for any $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) & \xrightarrow{\times} & \mathrm{H}_{p+q}(X \times Y) \\ \downarrow{\scriptstyle f_*} \quad \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ \mathrm{H}_p(X') \otimes \mathrm{H}_q(Y') & \xrightarrow{\times} & \mathrm{H}_{p+q}(X' \times Y'). \end{array}$$

Proof.

(0) This statement follows from the Eilenberg-Zilber Theorem 164.2 (2) together with the Chain Homotopy–H_*-Lemma 144.5.

(1) This statement follows easily from Statement $(*)$ of the Eilenberg-Zilber Theorem 164.2.

(2) This statement follows immediately from the naturality of μ and Υ. ■

With the Eilenberg-Zilber Theorem 164.2, the results from the previous chapter and the cross product we have now all the tools to deal with Question 163.1. More precisely, the following theorem tells us what is the relationship between the homology groups of the topological spaces X, Y and $X \times Y$.

Theorem 164.6. (Künneth Theorem) For all topological spaces X and Y there exists a natural homomorphism

$$\widetilde{\nu}\colon \mathrm{H}_n(X \times Y) \;\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y))$$

such that the following sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \xrightarrow{\times} \mathrm{H}_n(X \times Y) \xrightarrow{\widetilde{\nu}} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0$$

is exact and such that it splits.

Remark. The Künneth Theorem 164.6 can be viewed as a homological analog of the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21. □

Proof. Let X and Y be topological spaces. Let

$$\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X \times Y)$$

[1741] As in the case of the Algebraic Universal Coefficient Theorem 162.15, see the No-Natural Splitting Proposition 162.21, one can show that the splitting is not natural.

be an Eilenberg-Zilber map. We consider the following maps:

$$0 \longrightarrow \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \xrightarrow{\mu} \mathrm{H}_n(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \xrightarrow{\nu} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \longrightarrow 0$$

$$\cong \Big\downarrow \Upsilon_* \qquad \nearrow =:\widetilde{\nu}$$

$$\underbrace{\mathrm{H}_n(\mathrm{C}_*(X \times Y))}_{=\mathrm{H}_n(X \times Y)}.$$

We make the following observations and clarifications:

(a) The homomorphisms μ and ν are the natural homomorphisms from the Cycle Tensor Lemma 163.5 and the Algebraic–Künneth Theorem 163.6.

(b) By definition we have $\Upsilon_* \circ \mu = \times$. In other words, the left triangle commutes.

(c) Since the chain groups $\mathrm{C}_*(X)$ and $\mathrm{C}_*(Y)$ are free abelian groups it follows from the Algebraic–Künneth Theorem 163.6 that the top sequence is exact and that it splits.

(d) The vertical map is a natural isomorphism by the Eilenberg-Zilber Theorem 164.2 together with the Chain Homotopy Equivalence-H_*-Corollary 144.10.

It follows from the above that the homomorphism $\widetilde{\nu} := \nu \circ (\Upsilon_*)^{-1}$ has all the desired properties. ■

Examples.

(1) Let X be a topological space and let $k \in \mathbb{N}$. We want to determine the homology groups of $X \times S^k$. By the Künneth Theorem 164.6 we obtain a short exact sequence[1742]

$$0 \to \bigoplus_{p+q=n} \underbrace{\mathrm{H}_p(X) \otimes \mathrm{H}_q(S^k)}_{\substack{= \mathrm{H}_n(X), \quad \text{for } p=n, q=0, \\ = \mathrm{H}_{n-k}(X), \quad \text{for } p=n-k, q=k, \\ = 0, \quad \text{else}}} \xrightarrow{\times} \mathrm{H}_n(X \times S^k) \xrightarrow{\widetilde{\nu}} \underbrace{\bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(S^k))}_{=0} \to 0$$

where all maps are natural. Thus we see that we have a natural isomorphism

$$\mathrm{H}_n(X) \oplus \mathrm{H}_{n-k}(X) \xrightarrow{\cong} \mathrm{H}_n(X \times S^k).$$

For $k = 1$ this is basically the same result as in Exercise 148.5 and the Product-with-S^1-H_*-Proposition 153.8 and for $k \geq 1$ this basically recovers the result from Exercise 153.24.

(2) Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. Using (1) and using an induction argument one can easily show that for any $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i((S^m)^n) \cong \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k! \cdot (n-k)!}} & \text{if } i = k \cdot m \text{ with } k \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

We will fill in the details in Exercise 164.10.

(3) Let X and Y be path-connected topological spaces. Let $x \in X$ and let $y \in Y$. We denote by $i \colon X \to X \times Y$ the inclusion map that is given by $i(x) = (x, y)$ and similarly

[1742]Here we use the following:

- The homology groups of S^k are isomorphic to $\mathbb{Z}$ in the dimensions 0 and k and 0 otherwise. Furthermore the standard generator $[S^k]$ gives us a preferred isomorphism $\mathbb{Z} \to \mathrm{H}_k(S^k)$.
- For any abelian group A we have by the Tensor-with-$\mathbb{Z}$–Lemma 159.9 a natural isomorphism $A \otimes \mathbb{Z} \xrightarrow{\cong} A$ and by the Tor-Properties Lemma 162.13 (3) we have $\mathrm{Tor}(A, \mathbb{Z}) = 0$.

we denote by $j\colon Y \to X \times Y$ the inclusion map that is given by $j(y) = (x, y)$. It follows from the H_0-Proposition 143.25, the Cross Product Lemma 164.5 (1) the Künneth Theorem 164.6 and the Tor-Properties Lemma 162.13 that the map

$$i_* \oplus j_*\colon \mathrm{H}_1(X) \oplus \mathrm{H}_1(Y) \;\to\; \mathrm{H}_1(X \times Y)$$

is an isomorphism. □

In some cases we cannot only determine the isomorphism types of homology groups, but we can actually specify a basis:[1743]

Lemma 164.7. ($(S^k \times S^l)$-H_*-Lemma) Let $k, l \in \mathbb{N}$ and let $P_1 \in S^k$ and $P_2 \in S^l$. We consider the maps $i\colon S^k \to S^k \times S^l$ and $j\colon S^k \to S^k \times S^l$ that are given by $i(x) := (x, P_2)$ and $j(x) := (P_1, x)$.

(1) If $k = l$, then we have the following equalities:

$$\mathrm{H}_n(S^k \times S^k) \;=\; \begin{cases} \mathbb{Z} \cdot [(P_1, P_2)], & \text{if } n = 0, \\ \mathbb{Z} \cdot i_*([S^k]) \oplus \mathbb{Z} \cdot j_*([S^k]), & \text{if } n = k, \\ \mathbb{Z} \cdot (i_*([S^k]) \times j_*([S^k])), & \text{if } n = 2k, \\ 0, & \text{else.} \end{cases}$$

(2) If $k \neq l$, then we have the following equalities:

$$\mathrm{H}_n(S^k \times S^l) \;=\; \begin{cases} \mathbb{Z} \cdot [(P_1, P_2)], & \text{if } n = 0, \\ \mathbb{Z} \cdot i_*([S^k]), & \text{if } n = k, \\ \mathbb{Z} \cdot j_*([S^l]), & \text{if } n = l, \\ \mathbb{Z} \cdot (i_*([S^k]) \times j_*([S^l])), & \text{if } n = k + l, \\ 0, & \text{else.} \end{cases}$$

Example. Let $k \in \mathbb{N}$. We pick some $P \in S^k$. We introduce the following maps:
(1) The embedding $\iota_1\colon S^k \xrightarrow{x \mapsto (x,P)} S^k \times S^k$ and the projection $\mathrm{pr}_1\colon S^k \times S^k \xrightarrow{(x_1,x_2) \mapsto x_1} S^k$.
(2) The embedding $\iota_2\colon S^k \xrightarrow{x \mapsto (P,x)} S^k \times S^k$ and the projection $\mathrm{pr}_2\colon S^k \times S^k \xrightarrow{(x_1,x_2) \mapsto x_2} S^k$.
Next we consider the homomorphisms

$$\begin{array}{ccccc} \mathbb{Z} \cdot [S^k] \oplus \mathbb{Z}^k \cdot [S^k] & \to & \mathrm{H}_k(S^k \times S^k) & \to & \mathbb{Z} \cdot [S^k] \oplus \mathbb{Z} \cdot [S^k]. \\ (\varphi_1, \varphi_2) & \mapsto & \iota_{1*}(\varphi_1) + \iota_{2*}(\varphi_2) & & \\ & & \varphi & \mapsto & (\mathrm{pr}_{1*}(\varphi), \mathrm{pr}_{2*}(\varphi)). \end{array}$$

First note that it follows from the $(S^k \times S^l)$-H_*-Lemma 164.7 (1) that the map to the left is an isomorphism. Next note that it follows easily from the functoriality of homology that the composition of the two maps is the identity. Thus we see that right map is also an isomorphism.

Finally let $d\colon S^k \to S^k \times S^k$ be the diagonal map that is given by $x \mapsto (x, x)$. Note that it follows easily from the above discussion that the "diagonal sphere" $d_*([S^k])$ is homologous to $\iota_{1*}([S^k]) + \iota_{2*}([S^k])$. □

Proof. Both statements follow easily from the Künneth Theorem 164.6 together with the Sphere-H_*-Proposition 149.4 and the Cross Product Lemma 164.5 (1). We leave it to the reader to fill in the details. ■

[1743] As we saw in Exercise 155.5 one can also use cellular homology to prove the $(S^k \times S^l)$-H_*-Lemma 164.7.

Example. Let us discuss the product $S^2 \times S^4$ in slightly more detail. It follows from the $(S^k \times S^l)$-H_*-Lemma 164.7 that

$$\mathrm{H}_k(S^2 \times S^4) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6, \\ 0, & \text{otherwise.} \end{cases}$$

But by the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 these are precisely the homology groups of the complex projective space $\mathbb{C}\mathrm{P}^3$. Furthermore it follows from the Sphere-π_1-Proposition 88.24 and the Product-π_1-Proposition 90.13 that $S^2 \times S^4$ is simply connected and it follows from the Projective Space–π_1-Proposition 135.8 that $\mathbb{C}\mathrm{P}^3$ is also simply connected. So all the invariants we are aware of, and that we know how to compute[1744], agree for the two smooth manifolds $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$. □

The last example raises the following question which we will answer in the Doppelgänger Corollary 213.11.

Question 164.8. Are the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ homeomorphic?

164.3. **Proof of the Eilenberg-Zilber Theorem 164.2 .** The proof of the Eilenberg-Zilber Theorem 164.2 breaks up into three parts:

(1) In the EZ-Construction Proposition 164.10 we build $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$.
(2) In the EZ-Inverse Proposition 164.11 we construct $\Theta\colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$.
(3) Using the EZ–Natural Chain Homotopy Proposition 164.13 we show that these maps Υ and Θ are unique in a suitable sense and that they have the desired properties.

We start out with the following definition:

Definition 164.9. A topological space X is called acyclic if $\mathrm{H}_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and if all higher homology groups are zero.

Example. In the Contractible-H_*-Corollary 144.15 we showed that every contractible topological space is acyclic. In particular every standard simplex Δ^n is contractible and thus acyclic. Furthermore, all products $\Delta^p \times \Delta^q$ of standard simplices are contractible and thus acyclic.[1745] □

Now we can formulate the first of the three promised propositions.

Proposition 164.10. (EZ-Construction Proposition) For all topological spaces X and Y and all $p, q \in \mathbb{N}_0$ there exists a homomorphism

$$\Upsilon_{p,q}\colon \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y) \to \mathrm{C}_{p+q}(X \times Y)$$

such that the following conditions are satisfied:

(1) For any $x \in X$ and any singular simplex $\sigma\colon \Delta^q \to Y$ we have $\Upsilon_{0,q}(x \otimes \sigma) = x \times \sigma$. The same statement also holds with the roles of X and Y swapped.

[1744] As of right now we do not know the higher homotopy groups of these two smooth manifolds. Later in Exercise 297.4 we will see that we can use them to distinguish $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$.

[1745] This can be seen in many ways, for example pick points $x \in \Delta^p$ and $y \in \Delta^q$, it is straightforward to show that there exists a deformation retraction r from Δ^p to $\{x\}$ and that there exists a deformation retraction s from Δ^q to $\{y\}$. But these two deformation retractions also define a deformation retraction from $\Delta^p \times \Delta^q$ to (x, y).

(2) The maps $\Upsilon_{p,q}$ form a chain map, i.e. for any $a \in \mathrm{C}_p(X)$ and any $b \in \mathrm{C}_q(Y)$ we have the following equality in $\mathrm{C}_{p+q-1}(X \times Y)$:

$$\partial_{p+q}(\Upsilon_{p,q}(a \otimes b)) \;=\; \underbrace{\Upsilon_{p-1,q}(\partial_p(a) \otimes b) + (-1)^p \cdot \Upsilon_{p,q-1}(a \otimes \partial_q(b))}_{=\Upsilon(\partial^{\otimes}_{p+q}(a \otimes b))}.$$

(3) The map $\Upsilon_{p,q}$ is natural, i.e. if $f\colon X' \to X$ and $g\colon Y' \to Y$ are continuous maps between topological spaces, then for any $a \in \mathrm{C}_p(X')$ and any $b \in \mathrm{C}_q(Y')$ we have

$$\underbrace{\Upsilon_{p,q}(f_*(a) \otimes g_*(b))}_{\in \mathrm{C}_{p+q}(X \times Y)} \;=\; (f \times g)_*(\underbrace{\Upsilon_{p,q}(a \otimes b)}_{\in \mathrm{C}_{p+q}(X' \times Y')}).$$

Remark. In the proof of the EZ-Construction Proposition 164.10 we give a non-constructive proof of the existence of such maps $\Upsilon_{p,q}$. Later in the Shuffle Proposition 164.24 we will give an explicit construction of such maps $\Upsilon_{p,q}$. □

Proof. We start out with the outline of proof:

(a) We use (1) to define $\Upsilon_{p,q}$ for $p = 0$ and $q = 0$.
(b) We assume that $\Upsilon_{p',q'}$ is defined for $p' \cdot q' < p \cdot q$. We then use (2) to find a suitable value for $\Upsilon_{p,q}(\mathrm{id}_{\Delta^p}, \mathrm{id}_{\Delta^q})$.
(c) We then use the naturality statement of (3) to define $\Upsilon_{p,q}$ for all topological spaces X and Y and all singular simplices in $\mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)$.

Now we carry out the above program. We prove the existence of the maps $\Upsilon_{p,q}$ by induction on $p \cdot q$. It is quite elementary to show that the maps from (1) have the properties (2) and (3).[1746] So now suppose that we are given $p, q \in \mathbb{N}_0$ with $p \cdot q > 0$ such that the maps $\Upsilon_{p',q'}$ have been defined for all p', q' with $p' \cdot q' < p \cdot q$.

Given any $n \in \mathbb{N}_0$ we denote by $\mathrm{id}_n\colon \Delta^n \to \Delta^n$ the identity map. We first want to come up with a reasonable definition of $\Upsilon_{p,q}(\mathrm{id}_p \otimes \mathrm{id}_q) \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$. If we want (2) to be satisfied we have to define $\Upsilon_{p,q}(\mathrm{id}_p \otimes \mathrm{id}_q) =: z_{p,q}$ where $z_{p,q} \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$ satisfies

$$\partial_{p+q}(z_{p,q}) \;=\; \Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p) \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon_{p,q-1}(\mathrm{id}_p \otimes \partial_q(\mathrm{id}_q)) \;\in\; \mathrm{C}_{p+q-1}(\Delta^p \times \Delta^q).$$

In other words, we have to prove the following claim:

Claim 1. There exists a $z_{p,q} \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$ with

$$\partial_{p+q}(z_{p,q}) \;=\; \Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p) \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon_{p,q-1}(\mathrm{id}_p \otimes \partial_q(\mathrm{id}_q)) \;\in\; \mathrm{C}_{p+q-1}(\Delta^p \times \Delta^q).$$

Proof. As we pointed out on page 3143, the product $\Delta^p \times \Delta^q$ is acyclic. Since $p, q \geq 1$ this implies that $\mathrm{H}_{p+q-1}(\Delta^p \times \Delta^q) = 0$. This in turn implies that every singular $(p+q-1)$-chain *that is a cycle* is in fact the boundary of some $(p+q)$-chain $z_{p,q}$. So all we have to do is to

[1746] Note though that the proof of property (3) is not completely content-free.

show that the right-hand side of the claim is indeed a cycle. We compute that

$$\begin{aligned}
&\partial_{p+q-1}(\Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p)\otimes\mathrm{id}_q)+(-1)^p\cdot\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial_q(\mathrm{id}_q))) \overset{\text{by the linearity of }\partial_{p+q}}{=}\\
&=\partial_{p+q}(\Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p)\otimes\mathrm{id}_q))+(-1)^p\cdot\partial_{p+q}(\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial_q(\mathrm{id}_q)))\\
&=\Upsilon_{p-2,q}(\underbrace{\partial_{p-1}\partial_p\,\mathrm{id}_p}_{=0}\otimes\mathrm{id}_q)+(-1)^{p-1}\cdot\Upsilon_{p-1,q-1}(\partial_p(\mathrm{id}_p)\otimes\partial_q(\mathrm{id}_q))\\
&\quad+(-1)^p\cdot\Upsilon_{p-1,q-1}(\partial_p(\mathrm{id}_p)\otimes\partial_q(\mathrm{id}_q))+(-1)^{2p}\cdot\Upsilon_{p,q-2}(\mathrm{id}_p\otimes\underbrace{\partial_{q-1}\partial_q\,\mathrm{id}_q}_{=0})=0.
\end{aligned}$$

by our induction hypothesis we can apply the boundary formula (2) to $\Upsilon_{p-1,q}(\partial_{p+q}\,\mathrm{id}_p\otimes\mathrm{id}_q)$ and $\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial_q\,\mathrm{id}_q)$ ⊞

This discussion gives us the definition of $\Upsilon(\mathrm{id}_p\otimes\mathrm{id}_q):=z_{p,q}\in\mathrm{C}_{p+q}(\Delta^p\times\Delta^q)$. Now let X and Y be topological spaces and let $\varphi\colon\Delta^p\to X$ and $\psi\colon\Delta^q\to Y$ be singular simplices. If we want (2) to be satisfied then we have to define

$$\Upsilon_{p,q}(\underbrace{\varphi\otimes\psi}_{\in\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)}):=(\varphi,\psi)_*(\underbrace{\Upsilon_{p,q}(\mathrm{id}_p\otimes\mathrm{id}_q)}_{\in\mathrm{C}_{p+q}(\Delta^p\times\Delta^q)})=\mathrm{diag}(\varphi,\psi)_*(z_{p,q})\in\mathrm{C}_{p+q}(X\times Y).$$

It follows from the discussion on page 3062 that the $\varphi\otimes\psi$ form a basis of $\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)$. Thus we can extend this map Θ linearly to all of $\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)$.

Claim 2. The map $\Upsilon_{p,q}\colon\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)\to\mathrm{C}_{p+q}(X\times Y)$ has all the desired properties.

Proof. First note that it follows almost immediately from the definition of Υ that the map Υ is natural in X and Y.

Thus it remains to prove (2). This property holds basically by definition. But for completeness' sake we carry out the proof. So let $\varphi\colon\Delta^p\to X$ and $\psi\colon\Delta^q\to Y$ be singular simplices. We then have the following equality in $\mathrm{C}_{p+q-1}(X\times Y)$:

$$\begin{aligned}
\partial_{p+q}(\Upsilon_{p,q}(\varphi\otimes\psi))&\overset{\text{definition of }\Upsilon_{p,q}}{=}\partial_{p+q}(\mathrm{diag}(\varphi,\psi)_*(\Upsilon_{p,q}(\mathrm{id}_p\otimes\mathrm{id}_q)))\overset{\text{since }\mathrm{diag}(\varphi,\psi)_*\text{ is a chain map}}{=}\mathrm{diag}(\varphi,\psi)_*(\partial_{p+q}(\overbrace{\Upsilon_{p,q}(\mathrm{id}_p\otimes\mathrm{id}_q)}^{=z_{p,q}}))\\
&\overset{\text{by the choice of }z_{p,q}\text{ and the linearity of }\mathrm{diag}(\varphi,\psi)_*}{=}\mathrm{diag}(\varphi,\psi)_*(\Upsilon_{p-1,q}(\partial_p\,\mathrm{id}_p\otimes\mathrm{id}_q))+(-1)^p\cdot\mathrm{diag}(\varphi,\psi)_*(\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial_q\,\mathrm{id}_q))\\
&\overset{\text{by induction we can apply the naturality of }\Upsilon}{=}\Upsilon_{p-1,q}(\varphi_*(\partial_p\,\mathrm{id}_p)\otimes\psi_*(\mathrm{id}_q)))+(-1)^p\cdot\Upsilon_{p,q-1}(\varphi_*(\mathrm{id}_p)\otimes\psi_*(\partial_q\,\mathrm{id}_q))\\
&\overset{\text{since }\varphi_*\text{ and }\psi_*\text{ are chain maps and since }\varphi_*(\mathrm{id}_p)=\varphi\text{ and }\psi_*(\mathrm{id}_q)=\psi}{=}\Upsilon_{p-1,q}(\partial_p(\varphi)\otimes\psi)+(-1)^p\cdot\Upsilon_{p,q-1}(\varphi\otimes\partial_q(\psi))).
\end{aligned}$$

■

Now we turn to the construction of the second map in the Eilenberg-Zilber Theorem 164.2.

Proposition 164.11. (EZ-Inverse Proposition) For all topological spaces X and Y and all $k\in\mathbb{N}_0$ there exists a homomorphism

$$\Theta_k\colon\mathrm{C}_k(X\times Y)\to(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_k=\bigoplus_{p+q=k}\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)$$

such that the following conditions are satisfied:

(1) The map Θ_0 is the canonical map on the 0-level, i.e. for every $x \in X$ and $y \in Y$ we have $\Theta_0((x,y)) = x \otimes y$.

(2) The maps Θ_* form a chain map, i.e. for any $k \in \mathbb{N}_0$ we have

$$\partial_k^{\otimes} \circ \Theta_k \;=\; \Theta_{k-1} \circ \partial_k.$$

(3) The map Θ_k is natural in X and Y, i.e. for any two continuous maps $f\colon X' \to X$ and $g\colon Y' \to Y$ of topological spaces and any $c \in \mathrm{C}_k(X' \times Y')$ we have

$$\Theta_k((f \times g)_*(c)) \;=\; (f_* \otimes g_*)(\Theta_k(c)).$$

In the proof of the EZ-Inverse Proposition 164.11 we will need the following lemma:

Lemma 164.12. (Acyclic Tensor Product Lemma) Let X and Y be two acyclic topological spaces. For every $k \geq 1$ we have

$$\mathrm{H}_k(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \;=\; 0.$$

Furthermore we have an isomorphism

$$\begin{array}{rcl} \epsilon\colon \mathrm{H}_0(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) & \xrightarrow{\cong} & \mathbb{Z} \\ \Big[\sum\limits_{i=1}^{k} \underbrace{a_i}_{\in\mathbb{Z}} \cdot \underbrace{x_i}_{\in X} \otimes \underbrace{y_i}_{\in Y}\Big] & \mapsto & \sum\limits_{i=1}^{k} a_i. \end{array}$$

Proof of the Acyclic Tensor Product Lemma 164.12. The lemma is a straightforward consequence of our hypotheses, the H_0-Proposition 143.25 and the Algebraic–Künneth Theorem 163.6. ∎

Proof of the EZ-Inverse Proposition 164.11. [1747] We start out with an outline of the proof:

(a) We use (1) to define Θ_0.

(b) We suppose that the maps $\Theta_0, \ldots, \Theta_{k-1}$ are defined. We use (2) to find a suitable value for $\Theta_k(d_k)$ where we consider
  (i) the topological spaces $X = Y = \Delta^k$ and
  (ii) the singular k-simplex $d_k\colon \Delta^k \to \Delta^k \times \Delta^k$ that is given by the diagonal map $d_k(x) = (x,x)$.

(c) We then use the naturality statement of (3) to define Θ_k for all topological spaces X and Y and all singular simplices in $\mathrm{C}_*(X \times Y)$.

Now we carry out the above program. First of all, given a 0-simplex (x,y) in $X \times Y$ we define $\Theta_0((x,y)) = x \otimes y$. Then we extend this definition linearly to get the desired map $\Theta_0\colon \mathrm{C}_0(X \times Y) \to \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$.

Now suppose we have already constructed the maps $\Theta_0, \ldots, \Theta_{k-1}$. We want to define Θ_k and we start out by defining $\Theta_k(d_k\colon \Delta^k \to \Delta^k \times \Delta^k)$. Since we need (2) to be satisfied we need to define $\Theta_k(d_k) \coloneqq c$ where $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ satisfies $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial_k(d_k))$.

Claim 1. There exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial_k(d_k))$.

[1747] As we will see the idea behind the proof of the EZ-Inverse Proposition 164.11 is very similar to the idea of the proof of the EZ-Construction Proposition 164.10.

Proof. We first consider the case $k \geq 2$. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\partial^{\otimes}_{k-1}(\Theta_{k-1}(\partial_k(d_k))) \underset{\text{by induction we can apply (2)}}{=} (\Theta_{k-2} \circ \partial_{k-1})(\partial_k(d_k)) = \Theta_{k-2}(\underbrace{(\partial_{k-1} \circ \partial_k)}_{=0}(d_k)) = 0,$$

i.e. the chain $\Theta_{k-1}(\partial_k(d_k)) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k-1}$ is in fact a cycle. Since Δ^k is acyclic it follows immediately from the Acyclic Tensor Product Lemma 164.12 that this cycle is null-homologous. Thus there exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial^{\otimes}_k(c) = \Theta_{k-1}(\partial_k(d_k))$.

Now we consider the case $k = 1$. In this case one verifies easily, using the explicit description of the map $\Theta_0 \colon \mathrm{C}_0(\Delta^k \times \Delta^k) \to \mathrm{C}_0(\Delta^k) \otimes \mathrm{C}_0(\Delta^k)$, that $\Theta_0(\partial_1(d_1))$ represents an element in the kernel of the isomorphism $\epsilon \colon \mathrm{H}_0(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \to \mathbb{Z}$. Again this means that there exists a singular chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial^{\otimes}_k(c) = \Theta_{k-1}(\partial_k(d_k))$. ⊞

We set $\Theta_k(d_k) \coloneqq c$. Next we define Θ_k for arbitrary topological spaces X and Y and an arbitrary singular k-simplex $\sigma \colon \Delta^k \to X \times Y$. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the natural projections. Note that we have a commutative diagram

$$\begin{array}{ccc} & & \Delta^k \times \Delta^k \\ & \overset{d_k}{\nearrow} & \big\downarrow {\scriptstyle (p\circ\sigma)\times(q\circ\sigma)} \\ \Delta^k & & \\ & \underset{\sigma}{\searrow} & X \times Y, \end{array}$$

i.e. we have $\sigma = ((p \circ \sigma) \times (q \circ \sigma)) \circ d_k = ((p \circ \sigma) \times (q \circ \sigma))_*(d_k)$. For the naturality condition (2) to be satisfied we need to define

$$\Theta_k(\sigma) \coloneqq ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\underbrace{\Theta_k(d_k)}_{=c}) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k.$$

We extend Θ_k linearly to $\Theta_k \colon \mathrm{C}_k(X \times Y) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k$.

Claim 2. The map $\Theta_k \colon \mathrm{C}_k(X \times Y) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k$ has all the desired properties.

Proof. First note that it follows easily from the definitions that the map Θ_k is natural in X and Y. Thus it remains to show that Θ_k satisfies the boundary formula (2). So as before let $\sigma \colon \Delta^k \to X \times Y$ be a singular k-simplex. Then

$$\begin{array}{lll} & \text{by definition of } \Theta_k(\sigma) & \text{by the Tensoring Chain Maps Lemma 163.4 we know that } (p\circ\sigma)_* \otimes (q\circ\sigma)_* \colon \mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \text{ is in fact a chain map} \\ \partial^{\otimes}_k(\Theta_k(\sigma)) \overset{\downarrow}{=} & \partial^{\otimes}_k((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\Theta_k(d_k)) & \overset{\downarrow}{=} ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\partial^{\otimes}_k(\Theta_k(d_k))) \\ = & ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\Theta_{k-1}(\partial_k(d_k))) & = \Theta_{k-1}(((p \circ \sigma) \times (q \circ \sigma))_*(\partial_k(d_k))) \\ & \uparrow \text{ by definition of } \Theta_k(d_k) & \uparrow \text{ by naturality of } \Theta_{k-1} \\ = & \Theta_{k-1}(\partial_k(((p \circ \sigma) \times (q \circ \sigma))_*(d_k))) & = \Theta_{k-1}(\partial_k(\sigma)). \\ & \uparrow \text{ since } ((p \circ \sigma) \times (q \circ \sigma))_* \text{ is a chain map} & \uparrow \text{ since } \sigma = ((p \circ \sigma) \times (q \circ \sigma)) \circ d_k \end{array}$$

We have now verified that Θ_k has all the desired properties. ■

Next recall that in the proofs of the EZ-Construction Proposition 164.10 and of the EZ-Inverse Proposition 164.11 we saw that the constructions of the two natural chain maps $\Upsilon \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ and $\Theta \colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ depend on many

choices. The next theorem says that different choices in the proofs lead to natural chain maps that are naturally homotopy equivalent to one another.

Proposition 164.13. (EZ–Natural Chain Homotopy Proposition) Let X and Y be topological spaces.

(a) Any two natural chain maps

$$\mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y),$$

that are the canonical maps on the 0-level, are naturally chain homotopic.

The same conclusion also holds for any two natural chain maps that are the canonical maps on the 0-level in the following three setups:

(b) $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$.

(c) $\mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X \times Y)$.

(d) $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X \times Y)$.

Proof. [1748] In the following we prove Statement (1). The proofs of the other three statements are very similar and they are left to the reader. So let

$$F, G\colon \mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$$

be two natural chain maps that are the canonical maps on the 0-chains, i.e. which satisfy $F((x,y)) = x \otimes y = G((x,y))$ for all $x \in X, y \in Y$. For every $k \in \mathbb{Z}_{\geq -1}$ we have to define a homomorphism

$$P_k\colon \mathrm{C}_k(X \times Y) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+1}$$

that satisfies the following three properties:

(1) The map P_{-1} is the zero map.

(2) We have $\partial^{\otimes}_{k+1} \circ P_k + P_{k-1} \circ \partial_k = F_k - G_k$.

(3) The map P_k is natural in X and Y.

So suppose we have already defined the maps $P_{-1}, \ldots, P_{k-1}$ with $k \geq 0$. As before we denote by $d_k\colon \Delta^k \to \Delta^k \times \Delta^k$ the singular k-simplex that is given by the diagonal map $d_k(x) := (x,x)$. We first want to define $P_k(d_k) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$.

Claim 1. There exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ such that

$$\partial^{\otimes}_{k+1}(c) \;=\; (F_k - G_k - P_{k-1} \circ \partial_k)(d_k).$$

Proof. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\begin{aligned}
\partial^{\otimes}_k((F_k - G_k - P_{k-1} \circ \partial_k)(d_k)) \;&=\; \partial^{\otimes}_k(F_k(d_k)) - \partial^{\otimes}_k(G_k(d_k)) - (\partial^{\otimes}_k \circ P_{k-1})(\partial_k(d_k)) \\
&\overset{\uparrow}{=}\; F_{k-1}(\partial_k(d_k)) - G_{k-1}(\partial_k(d_k)) - (F_{k-1} - G_{k-1} - (P_{k-2} \circ \partial_{k-1}))(\partial_k(d_k))
\end{aligned}$$

since F and G are chain maps and since inductively we know by (2) that $\partial^{\otimes}_k \circ P_{k-1} + P_{k-2} \circ \partial_{k-1} = F_{k-1} - G_{k-1}$

$$= \; P_{k-2}(\underbrace{(\partial_{k-1} \circ \partial_k)}_{=0}(d_k)) \;=\; 0.$$

We first assume that $k \geq 1$. It follows in this case from the above calculation and the Acyclic Tensor Product Lemma 164.12 that there exists a $(k+1)$-chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}_{k+1}(c) = (F_k - G_k - P_{k-1} \circ \partial_k)(d_k)$. Now we consider the case $k = 0$. Using the

[1748] The proof of this theorem is in many ways quite similar to the proofs of the EZ-Construction Proposition 164.10 and the EZ-Inverse Proposition 164.11.

fact that F and G agree on 0-simplices one can easily verify that for the augmentation homomorphism $\epsilon\colon \mathrm{C}_0(X)\otimes\mathrm{C}_0(Y)\to\mathbb{Z}$ from the Acyclic Tensor Product Lemma 164.12 we have $\epsilon((F_0-G_0)(d_0))=0$. It follows again from the Acyclic Tensor Product Lemma 164.12 that there exists a 1-chain $c\in(\mathrm{C}_*(\Delta^0)\otimes\mathrm{C}_*(\Delta^0))_1$ with $\partial_1^{\otimes}(c)=(F_0-G_0)(d_0)$. ⊞

We set $P_k(d_k):=c$. Next let X and Y be two topological spaces and furthermore let $\sigma\in\Delta^k\to X\times Y$ be a continuous map. As in the proof of the EZ-Construction Proposition 164.10 we denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the natural projections and we set

$$P_k(\sigma) \;:=\; ((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\underbrace{P_k(d_k)}_{=c}) \;\in\; (\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_{k+1}.$$

Claim 2. The map $P_k\colon \mathrm{C}_k(X\times Y)\to(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_k$ has all the desired properties.

Proof. First note that basically by definition this map is natural in X and Y. Finally note that, as in the proof of the EZ-Construction Proposition 164.10, one easily verifies that condition (2) is satisfied. We leave the details to the reader. ■

Now we can easily provide the proof of the Eilenberg-Zilber Theorem 164.2.

Proof of the Eilenberg-Zilber Theorem 164.2. We pick natural chain maps

$$\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X\times Y)$$

and

$$\Theta\colon \mathrm{C}_*(X\times Y) \;\to\; \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$$

as provided by the EZ-Construction Proposition 164.10 and by the EZ-Inverse Proposition 164.11. These maps are the canonical maps on the 0-level. It follows from the EZ–Natural Chain Homotopy Proposition 164.13 (d) and (a) that the maps Υ and Θ are unique up to a natural chain homotopy.

It remains to show that the maps Υ and Θ are naturally chain homotopy inverses of one another. The map $\Theta\circ\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ is a natural chain map that is the canonical map on the 0-level. But this is also the case for the identity map. It follows from the EZ–Natural Chain Homotopy Proposition 164.13 (b) that $\Theta\circ\Upsilon$ is naturally chain homotopic to the identity. The same argument, using the EZ–Natural Chain Homotopy Proposition 164.13 (c), shows that $\Upsilon\circ\Theta$ is naturally chain homotopic to the identity. ■

Remark. The proofs of the EZ-Construction Proposition 164.10, the EZ-Inverse Proposition 164.11 and of the four parts of the EZ–Natural Chain Homotopy Proposition 164.13 share many features. In the Acyclic Model Theorem 235.1 and the Acyclic Model Theorem 235.7 we will formalize some of the arguments. But as of right now there is perhaps no point in building up too much of an overarching theory. □

164.4. **The relative Eilenberg–Zilber map Υ.** In this section we introduce a relative version of the Eilenberg–Zilber map Υ and we prove a relative version of the Künneth Theorem 164.6. Most of the discussion is straightforward, but at some point some care needs to be taken.

The following definition is a mild generalization of the Definition 164.4:

Definition 164.14. Let (X, A) and (Y, B) be two pairs of topological spaces. Furthermore let $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ be an Eilenberg-Zilber map. We refer to the homomorphism[1749] [1750]

$$\begin{array}{rcl} \times\colon \mathrm{H}_p(X, A) \otimes \mathrm{H}_q(Y, B) & \xrightarrow{\times} & \mathrm{H}_{p+q}(X \times Y, (A \times Y) \cup (X \times B)) \\ [c] \otimes [d] & \mapsto & [c] \times [d] \coloneqq [\Upsilon(c \otimes d)] \end{array}$$

as the cross product.

We have the following slightly subtle generalization of the earlier Künneth Theorem 164.6.

Theorem 164.15. (Künneth Theorem) Let (X, A) and (Y, B) be pairs of topological spaces. We assume that one of the following three statements holds:

(1) $A = \varnothing$ or $B = \varnothing$.
(2) $A \subseteq X$ and $B \subseteq Y$ are open subsets.
(3) (X, A) and (Y, B) are pairs of CW-complexes and $X \times Y$ has the topology from Definition 139.15 which turns $X \times Y$ into a CW-complex.[1751]

For each $n \in \mathbb{N}$ there exists a natural homomorphism

$$\widetilde{\nu}\colon \mathrm{H}_n(X \times Y, (A \times Y) \cup (X \times B)) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X, A), \mathrm{H}_q(Y, B))$$

such that the following sequence

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X, A) \otimes \mathrm{H}_q(Y, B) &\xrightarrow{\times} \mathrm{H}_n(X \times Y, (A \times Y) \cup (X \times B)) \\ &\xrightarrow{\widetilde{\nu}} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X, A), \mathrm{H}_q(Y, B)) \to 0. \end{aligned}$$

is exact and such that it splits.

Example. We consider the pairs $(\overline{B}^p, S^{p-1})$ and $(\overline{B}^q, S^{q-1})$.

[1749] We will now see that it follows from the naturality of Υ that this map on relative homology is well-defined. Indeed: As we discussed in Footnote 1737, naturality of Υ implies that the inclusions $i\colon A \to X$ and $j\colon B \to Y$ give rise to the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{C}_*(A) \otimes \mathrm{C}_*(Y) & \xrightarrow{\Upsilon_{A,Y}} & \mathrm{C}_*(A \times Y) \\ \downarrow{\scriptstyle i_* \otimes \mathrm{id}_{Y*}} & & \downarrow{\scriptstyle (i \times \mathrm{id}_Y)_*} \\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \xrightarrow{\Upsilon_{X,Y}} & \mathrm{C}_*(X \times Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{C}_*(X) \otimes \mathrm{C}_*(B) & \xrightarrow{\Upsilon_{X,B}} & \mathrm{C}_*(X \times B) \\ \downarrow{\scriptstyle \mathrm{id}_{X*} \otimes j_*} & & \downarrow{\scriptstyle (\mathrm{id}_X \times j)_*} \\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \xrightarrow{\Upsilon_{X,Y}} & \mathrm{C}_*(X \times Y) \end{array}$$

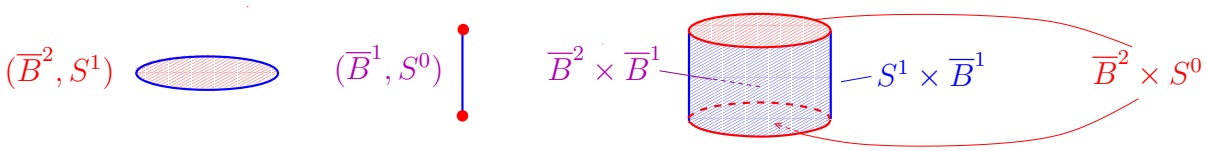
Thus we see that the map

$$\begin{array}{ccccc} \mathrm{C}_p(X, A) \otimes \mathrm{C}_q(Y, B) & \to & \mathrm{C}_{p+q}(X \times Y)/(\mathrm{C}_{p+q}(A \times Y) + \mathrm{C}_{p+q}(X \times B)) & \to & \mathrm{C}_{p+q}(X \times Y, (A \times Y) \cup (X \times B)) \\ [c] \otimes [d] & \mapsto & [\Upsilon(c \otimes d)] & & \\ & & [e] & & \mapsto [e] \end{array}$$

is well-defined.

[1750] As in the Cross Product Lemma 164.5 we see that this map is natural and that it is independent of the choice of the Eilenberg-Zilber map Υ.

[1751] If X and Y are finite CW-complexes, then it follows from the CW-Complex Product Proposition 139.18 that this topology is the usual product topology.



In this context the Künneth Theorem 164.15 (3) simplifies to the statement that the cross product[1752]

$$\underbrace{\mathrm{H}_p(\overline{B}^p, S^{p-1})}_{\cong\mathbb{Z}} \otimes \underbrace{\mathrm{H}_q(\overline{B}^q, S^{q-1})}_{\cong\mathbb{Z}} \xrightarrow{\times} \mathrm{H}_{p+q}\underbrace{(\overline{B}^p \times \overline{B}^q, (S^{p-1} \times \overline{B}^q) \cup (\overline{B}^p \times S^{q-1}))}_{\cong(\overline{B}^{p+q}, S^{p+q-1})}$$

is an isomorphism. □

The extra requirements in the Künneth Theorem 164.15 already indicate that the proof of the Künneth Theorem 164.15 is a little less straightforward than one might have hoped. As a first step to proving the Künneth Theorem 164.15 let us formulate and prove the following lemma:

Lemma 164.16. (Relative–Υ-Chain Equivalence Lemma) Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Any choice of an Eilenberg-Zilber map $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ induces a natural chain homotopy equivalence[1753]

$$\begin{array}{rcl} \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Y, B) & \to & \mathrm{C}_*(X \times Y)/(\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B)) \\ \sum_{i=1}^{n} [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^{n} [\Upsilon(c_i \otimes d_i)] \end{array}$$

which we also denote by Υ.

Proof of the Relative–Υ-Chain Equivalence Lemma 164.16. Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subspaces. We start out with the following claim:

Claim. For any topological space Z the Eilenberg-Zilber map

$$\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Z) \to \mathrm{C}_*(X \times Z)$$

induces a chain homotopy equivalence

$$\overline{\Upsilon}\colon \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Z) \to \mathrm{C}_*(X \times Z, A \times Z).$$

Proof. Let $i\colon A \to Z$ be the inclusion. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*(A) \otimes \mathrm{C}_*(Z) & \xrightarrow{i_* \otimes \mathrm{id}} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Z) & \longrightarrow & \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Z) & \longrightarrow & 0 \\ & & \simeq\downarrow \Upsilon & & \simeq\downarrow \Upsilon & & \overline{\Upsilon}\downarrow [\sigma]\otimes\tau \mapsto [\Upsilon(\sigma\otimes\tau)] & & \\ 0 & \longrightarrow & \mathrm{C}_*(A \times Z) & \xrightarrow{(i \times \mathrm{id})_*} & \mathrm{C}_*(X \times Z) & \longrightarrow & \mathrm{C}_*(X \times Z)/\mathrm{C}_*(A \times Z) & \longrightarrow & 0. \end{array}$$

We make the following observations:

(1) Since each chain group $\mathrm{C}_k(Z)$ is free abelian we obtain from the Tensor-with-Torsion Free Proposition 161.6 that the top horizontal sequence is exact.

[1752]Recall that it follows from the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$ that $\mathrm{H}_k(\overline{B}^n, S^{n-1}) \cong \widetilde{\mathrm{H}}_{k-1}(S^{n-1})$ which is $\cong \mathbb{Z}$ for $k = n$ and $= 0$ otherwise.

[1753]Note that $\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B)$ is in fact a subcomplex of the chain complex $\mathrm{C}_*(X \times Y)$, so the right-hand side is indeed a chain complex.

(2) The bottom sequence is exact by definition.
(3) By the Tensoring Chain Maps Lemma 163.4 we know that $i_* \otimes \mathrm{id}$ is a chain map.
(4) It follows from the Eilenberg-Zilber Theorem 164.2 that two vertical maps to the left are chain homotopy equivalences.
(5) It follows from the naturality of Υ that the square to the left commutes.
(6) It follows immediately from (5) that $\overline{\Upsilon}$ is well-defined. It is clear that the right square also commutes.

It follows from the above and the Two-Out-Of-Three-Homotopy Equivalence Corollary 158.9 that the right vertical map is also a chain homotopy equivalence. ⊞



Next we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(B) & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(Y) & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(Y,B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\Upsilon}} & & \downarrow{\scriptstyle \overline{\Upsilon}} & & \downarrow{\scriptstyle \overline{\Upsilon}} & & \\ 0 & \longrightarrow & \mathrm{C}_*(X\times B)/\mathrm{C}_*(A\times B) & \longrightarrow & \mathrm{C}_*(X\times Y)/\mathrm{C}_*(A\times Y) & \longrightarrow & \dfrac{\mathrm{C}_*(X\times Y)}{\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)} & \longrightarrow & 0. \end{array}$$

We make the following observations:

(1) It follows from the fact that each $\mathrm{C}_k(X,A)$ is a free abelian group and the Tensor-with-Free Abelian Group Proposition 161.6 that the top sequence is exact.
(2) It follows easily from

$$\mathrm{C}_*(X\times B)\cap\mathrm{C}_*(A\times Y) \;=\; \mathrm{C}_*((X\times B)\cap(A\times Y)) \;=\; \mathrm{C}_*(A\times B)$$

that the bottom sequence is also exact.
(3) It follows from the above claim that the two vertical maps to the left are chain homotopy equivalences.
(4) It follows easily from the naturality of Υ that the square to the left commutes. The square to the right commutes by definition of the third vertical map.

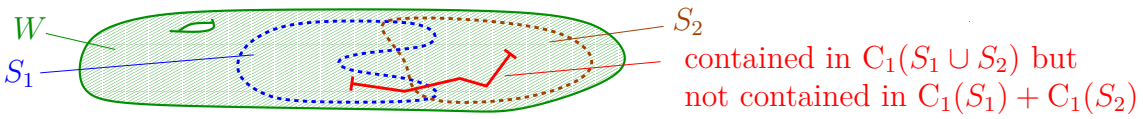
It follows again from the Two-Out-Of-Three-Homotopy Equivalence Corollary 158.9 that the right vertical map is also a homotopy equivalence. ■

For the reader's convenience we recall some statements from the Excisive Triad-Proposition 158.20:

Proposition 158.20. (Excisive Triad-Proposition) Let W be a topological space and let $S_1, S_2 \subseteq W$ be subspaces. We assume that one of the following conditions is satisfied:
(1) $S_1 = \varnothing$ or $S_2 = \varnothing$.
(2) S_1 and S_2 are open subsets of W.
(4) W is a CW-complex and S_1 and S_2 are both subcomplexes.
Then the inclusion maps

$$\underbrace{\mathrm{C}_k(S_1)+\mathrm{C}_k(S_2)}_{\text{subgroup of } \mathrm{C}_k(S_1\cup S_2)} \;\to\; \mathrm{C}_k(S_1\cup S_2)$$

form a chain homotopy equivalence.



Proof of the Künneth Theorem 164.15. We consider the following maps:

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes \mathrm{H}_q(Y,B) \xrightarrow[\text{(a)}]{\mu} \mathrm{H}_n(\mathrm{C}_*(X,A)\otimes \mathrm{C}_*(Y,B)) \xrightarrow[\text{(a)}]{\nu} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X,A),\mathrm{H}_q(Y,B)) \to 0.$$

$$\text{(b)}\ \downarrow \Upsilon_*$$

$$\mathrm{H}_n(\mathrm{C}_*(X\times Y)/(\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)))$$

$$\text{(c)}\ \downarrow$$

$$\mathrm{H}_n(\mathrm{C}_*(X\times Y)/\mathrm{C}_*((A\times Y)\cup(X\times B)))$$

$$\downarrow \mathrm{id}$$

$$\mathrm{H}_n(X\times Y,(A\times Y)\cup(X\times B))$$

The theorem is a consequence of the following observations and clarifications:

(a) The top horizontal maps μ and ν are the natural maps from the Cycle Tensor Lemma 163.5 and the Algebraic–Künneth Theorem 163.6. The Algebraic–Künneth Theorem 163.6 implies that the top horizontal sequence is exact and that it splits.[1754]

(b) It follows from the Relative–Υ-Chain Equivalence Lemma 164.16 that the top vertical map is an isomorphism.

(c) Now we consider the middle vertical map:

Claim. The middle vertical map is an isomorphism.

Proof. Recall that we assume that one of the following three statements holds:
(1) $A = \varnothing$ or $B = \varnothing$.
(2) $A \subseteq X$ and $B \subseteq Y$ are open subsets.
(3) (X, A) and (Y, B) are pairs of CW-complexes.
In the three cases we get the following three corresponding consequences:
(1) $A \times Y = \varnothing$ or $X \times B = \varnothing$.
(2) $A \times Y$ and $X \times B$ are open subsets of $(A \times Y) \cup (X \times B)$.
(3) It follows easily from the CW-Category–Product Proposition 139.16 that $A \times Y$ and $X \times B$ are subcomplexes of the CW-complex $X \times Y$.
It follows from these two observations and the Excisive Triad-Proposition 158.20 (0), (1) respectively (3) that the inclusion map

$$\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B) \ \to\ \mathrm{C}_*((A\times Y)\cup(X\times B))$$

is a chain homotopy equivalence. We now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B) & \to & \mathrm{C}_*(X\times Y) & \to & \mathrm{C}_*(X\times Y)/(\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)) & \to & 0 \\
 & & \downarrow & & \downarrow \mathrm{id} & & \downarrow & & \\
0 & \to & \mathrm{C}_*((A\times Y)\cup(X\times B)) & \to & \mathrm{C}_*(X\times Y) & \longrightarrow & \mathrm{C}_*(X\times Y,(A\times Y)\cup(X\times B)) & \longrightarrow & 0.
\end{array}$$

By the above discussion the vertical maps to the left induce isomorphisms on homology. If follows from the long exact sequences in homology and the Five Lemma 145.11 that

[1754] Here we use again the fact (shown on page 2798) that relative chain groups are free abelian groups.

the right vertical map also induces an isomorphism on homology. But that is exactly what we had to show.

(d) It follows immediately from the definition of the cross product that the induced "diagonal map"

$$\bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes \mathrm{H}_q(Y,B) \;\to\; \mathrm{H}_n(X\times Y,(A\times Y)\cup(X\times B))$$

is precisely the cross product. ∎

164.5. **Appendix: The cellular chain complex of products.** In this appendix we consider the cellular chain complex of the product of two finite CW-complexes X and Y. We will see that the cellular chain complex of $X\times Y$ is the tensor product of the cellular chain complexes of X and Y. We will not make use of this result. We will use this as an excuse to omit an important proof.

But first let us remind the readers of a few definitions and notions regarding CW-complexes. We refer to Definition 136.1 and Definition 155.12 for more precise and detailed formulations:

Definition 164.17.

(1) A CW-complex X is a topological space X together with a filtration

$$\varnothing = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \ldots \subseteq X$$

by subspaces which has the following properties:

(CW1) We have $X = \operatorname{colim} X^n$.

(CW2) For each $n\in\overrightarrow{\mathbb{N}_0}$ there exists a set I, a family $\{\varphi_i\colon S^{n-1}\to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism

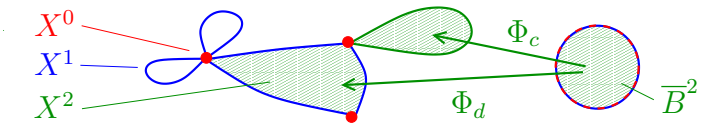
$$\Xi_n\colon X^{n-1}\cup_{X^{n-1}\leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I}\varphi_i} \bigsqcup_{i\in I}\overline{B}^n \;\to\; X^n.$$

The components of $X^n\setminus X^{n-1}$ are called the open n-cells of X. The closures of the open n-cells are call the n-cells of X.

(2) We denote by $\mathrm{Cell}(X)$ the set of all cells of X and given $n\in\mathbb{N}_0$ we denote by $\mathrm{Cell}_n(X)$ the set of all n-cells of X.

(3) A charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c\colon \overline{B}^{\dim(c)}\to X^{\dim(c)}\}_{c\in\mathrm{Cell}(X)}$ of characteristic maps[1755] of the cells of X.

(4) For $n\in\mathbb{N}_0$ we denote by $[\overline{B}^n]\in \mathrm{H}_n(\overline{B}^n,S^{n-1})$ the standard generator of $\mathrm{H}_n(\overline{B}^n,S^{n-1})$ from Definition 155.11.



the maps $\Phi_c,\Phi_d\colon \overline{B}^2\to X$ are a full set of characteristic maps for the 2-cells of X

Let us remind the reader of the following lemma:

Lemma 155.13 (Cellular Chain Complex–Basis Lemma) Suppose that we are given a charted CW-complex $(X,\{\Phi_c\colon \overline{B}^{\dim(c)}\to X^{\dim(c)}\}_{c\in\mathrm{Cell}(X)})$. For every $n\in\mathbb{N}_0$ the set $\{\Phi_{c*}([\overline{B}^n])\}_{c\in\mathrm{Cell}_n(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X)=\mathrm{H}_n(X^n,X^{n-1})$.

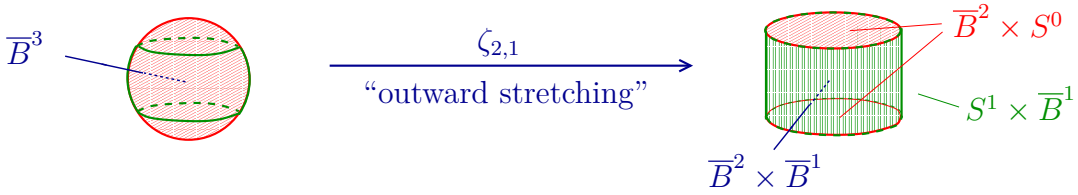
[1755] We refer to Definition 136.7 for the definition of a full set of characteristic maps.

Now let X and Y be finite CW-complexes. Eventually our goal will be to relate the cellular chain complex of $X \times Y$ to the cellular chain complexes of X and Y. To get to this point we first need to recall the definition of the CW-structure on $X \times Y$ and how to obtain a full set of characteristic maps for the cells of $X \times Y$. At this stage it is convenient to introduce the following notation:

Notation 164.18. Let $p, q \in \mathbb{N}_0$. Using the Compact-Hausdorff Proposition 4.23 (3) and some elementary arguments one can easily show that the map

$$\begin{array}{rcl} \zeta_{p,q}\colon \overline{B}^{p+q} & \to & \overline{B}^p \times \overline{B}^q \\ (x,y) & \mapsto & \begin{cases} \frac{(x,y)}{\max\{\|x\|,\|y\|\}} \cdot \|(x,y)\|, & \text{if } (x,y) \neq (0,0), \\ (0,0), & \text{if } (x,y) = (0,0) \end{cases} \end{array}$$

is a homeomorphism.



For the reader's convenience we recall a few facts on products of CW-complexes which we had established earlier:

Proposition 164.19. (Finite CW–Product Proposition) Let X and Y be two finite CW-complexes.

(1) A CW-structure for the topological space $X \times Y$ is given by the filtration

$$(X \times Y)^n := \bigcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} (X^p \times Y^q) \qquad \text{with } n \in \mathbb{N}_0.$$

(2) The n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y. In other words

$$\mathrm{Cell}_n(X \times Y) = \bigcup_{p+q=n} \bigcup_{\substack{c \in \mathrm{Cell}_p(X) \\ d \in \mathrm{Cell}_q(Y)}} c \times d.$$

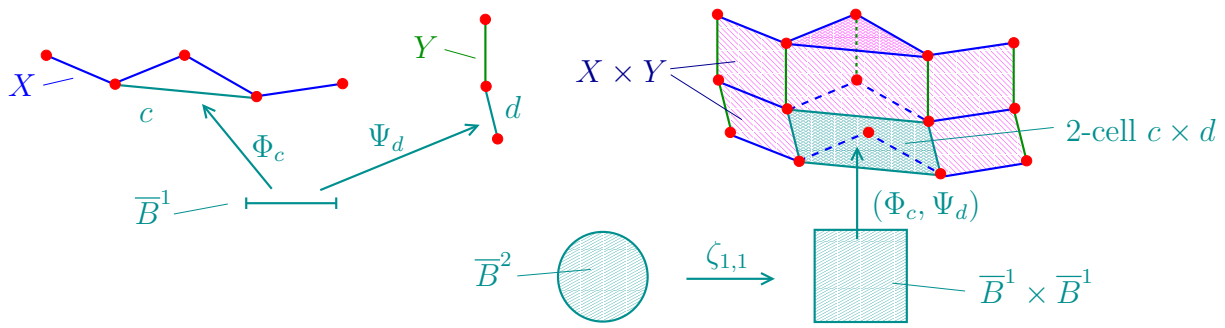
(3) If $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ and $\{\Psi_d\colon \overline{B}^{\dim(d)} \to Y\}_{d \in \mathrm{Cell}(Y)}$ are full sets of characteristic maps of the CW-complexes X and Y, then the maps

$$\overline{B}^{p+q} \xrightarrow{\zeta_{p,q}} \overline{B}^p \times \overline{B}^q \xrightarrow{(\Phi_c, \Psi_d)} X^p \times Y^q \subseteq (X \times Y)^{p+q}$$

form a full set of characteristic maps of the CW-complex $X \times Y$.

Proof. The first two statements of the proposition follow immediately from the combination of the CW-Category–Product Proposition 139.16 (1) and the CW-Complex Product Proposition 139.18 (2a). Statement (3) was proved on page 2688. ■

The following proposition gives us a precise relationship between the cellular chain complexes of X, Y and $X \times Y$.



Proposition 164.20. (Cellular Chain Complex–Product Proposition) Suppose we are given two charted finite CW-complexes $(X, \{\Phi_c\}_{c\in \mathrm{Cell}(X)})$ and $(Y, \{\Psi_d\}_{d\in \mathrm{Cell}(Y)})$. We adopt the following notation from Definition 155.12:

- Given $c \in \mathrm{Cell}_p(X)$ we write $\Xi_c \coloneqq \Phi_{c*}([\overline{B}^p]) \in \mathrm{C}_p^{\mathrm{CW}}(X)$.
- Given $d \in \mathrm{Cell}_q(Y)$ we write $\Xi_d \coloneqq \Psi_{d*}([\overline{B}^q]) \in \mathrm{C}_q^{\mathrm{CW}}(Y)$.
- Given c, d as above we write $\Xi_{c\times d} \coloneqq ((\Phi_c, \Psi_d) \circ \zeta_{p,q})_*([\overline{B}^{p+q}]) \in \mathrm{C}_{p+q}^{\mathrm{CW}}(X \times Y)$.

The following two statements hold:

(1) There exists a unique isomorphism

$$\Upsilon\colon \bigoplus_{p+q=n} F\mathrm{C}_p^{\mathrm{CW}}(X) \otimes \mathrm{C}_q^{\mathrm{CW}}(Y) \ \to\ \mathrm{C}_n^{\mathrm{CW}}(X \times Y)$$

which has the property that for every $c \in \mathrm{Cell}_p(X)$ and every $d \in \mathrm{Cell}_q(Y)$ with $p + q = n$ we have

$$\Xi_c \otimes \Xi_d \ \mapsto\ \Xi_{c\times d}.$$

(2) The isomorphisms from (1) define an isomorphism

$$\Upsilon\colon \mathrm{C}_*^{\mathrm{CW}}(X) \otimes \mathrm{C}_*^{\mathrm{CW}}(Y) \ \to\ \mathrm{C}_*^{\mathrm{CW}}(X \times Y)$$

of chain complexes.

Sketch of proof.

(1) The desired statement follows easily from the following three inputs:
  (a) The Cellular Chain Complex–Basis Lemma 155.13, which we had recalled above.
  (b) The fact, shown on page 3062, that if C is a free abelian group on a set S and if D is a free abelian group on a set T, then $C \otimes D$ is a free abelian group on the set $\{s \otimes t\}_{(s,t)\in S\times T}$.
  (c) The Finite CW–Product Proposition 164.19 (2) and (3).
  We leave it to the reader to fill in the details.

(2) To prove that the isomorphisms define an isomorphism of chain complexes we need to show that the maps Υ from (1) are chain maps. In other words, one needs to prove the following claim:

Claim. For any p-cell c of X and any q-cell d of Y we have the following equality in $\mathrm{C}_{p+q-1}^{\mathrm{CW}}(X \times Y)$:

$$\underbrace{\partial_{p+q}(\Xi_{c\times d})}_{\in \mathrm{C}_{p+q-1}^{\mathrm{CW}}(X\times Y)} = \Upsilon\Big(\underbrace{\partial_p(\Xi_c) \otimes \Xi_d}_{\in \mathrm{C}_{p-1}^{\mathrm{CW}}(X)\otimes \mathrm{C}_q^{\mathrm{CW}}(Y)}\Big) + (-1)^p \cdot \Upsilon\Big(\underbrace{\Xi_c \otimes \partial_q(\Xi_d)}_{\in \mathrm{C}_p^{\mathrm{CW}}(X)\otimes \mathrm{C}_{q-1}^{\mathrm{CW}}(Y)}\Big).$$

Proof. This statement is proved in [**Hat2002**, Proposition 3.B.1]. An alternative proof is also sketched in [**Bre1993**, Theorem IV.12.2]. ■

164.6. **Appendix: The shuffle product.** In the EZ-Construction Proposition 164.10 we showed that there exist homomorphisms

$$\Upsilon_{p,q}\colon \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y) \;\to\; \mathrm{C}_{p+q}(X \times Y)$$

which have the following properties:

(1) On the 0-level they equal the canonical maps that we introduced in Definition 164.1. In other words, for all $x \in X$ and $y \in Y$ we have $\nabla_{0,0}(x \otimes y) = (x, y) \in \mathrm{C}_0(X \times Y)$.
(2) They form a chain map $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$.
(3) They are natural.

We gave a non-constructive proof of the existence of the map Υ. For the most part the non-explicit version of the Eilenberg-Zilber map Υ is enough. But on a few occasions, e.g. if one wants to make the Künneth Theorem 164.6 explicit, or in the proof of Product Orientation Proposition 192.7 (2), it will be essential to have an explicit Eilenberg-Zilber map.

In this appendix we will give an *explicit* description of a map ∇ which has all the properties that we stated above. We start out with several definitions.

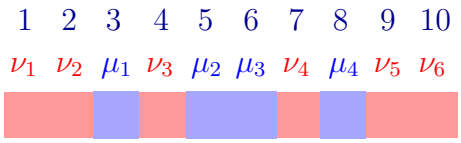
Definition 164.21. Let $m, n \in \mathbb{N}_0$.

(1) An (m,n)-shuffle is a pair (μ, ν) of disjoint sets of integers

$$1 \le \mu_1 < \mu_2 < \cdots < \mu_m \le m+n \quad \text{and} \quad 1 \le \nu_1 < \nu_2 < \cdots < \nu_n \le m+n.$$

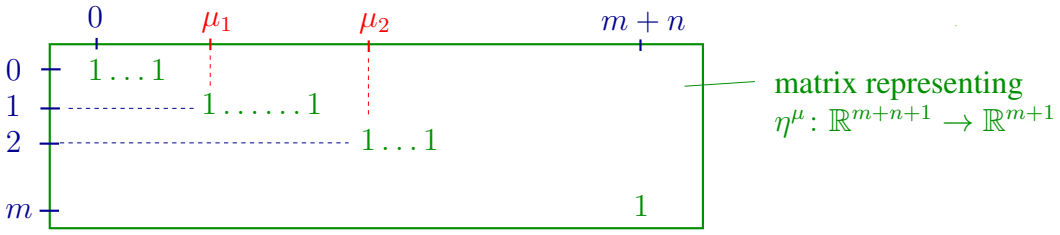
Note that an (m,n)-shuffle is the same data as picking m numbers out of $\{1, \ldots, m+n\}$.
(2) We denote by $S_{m,n}$ the set of all (m,n)-shuffles.
(3) We denote by $\operatorname{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$.

$(4,6)$-shuffle



Definition 164.22.

(4) Given a sequence $1 \le \mu_1 < \mu_2 < \cdots < \mu_m \le m+n$ we denote by $\eta^\mu\colon \mathbb{R}^{m+n+1} \to \mathbb{R}^{m+1}$ the unique linear map which satisfies $\eta^\mu(\mathfrak{v}_i) = \mathfrak{v}_j$ for $\mu_j \le i < \mu_{j+1}$.[1756] One can easily verify that η^μ restricts to a map $\Delta^{m+n} \to \Delta^m$ which we also denote by η^μ.



Definition 164.23.

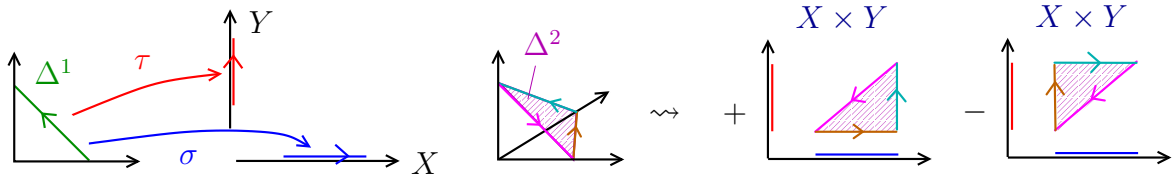
(5) Given two topological spaces X and Y we consider the homomorphism[1757]

$$\begin{aligned} \nabla := \nabla_{m,n} \colon \mathrm{C}_m(X) \otimes \mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X \times Y) \\ (\sigma \otimes \tau) &\mapsto \sum_{(\mu,\nu) \in S_{m,n}} \operatorname{sign}(\mu,\nu) \cdot \Big(\overbrace{(\sigma \times \tau)}^{\Delta^m \times \Delta^n \to X \times Y} \circ \overbrace{(\eta^\mu, \eta^\nu)}^{\Delta^{m+n} \to \Delta^m \times \Delta^n} \Big). \end{aligned}$$

map $\Delta^{m+n} \to X \times Y$

Example.

(1) We consider the case $X = Y = \mathbb{R}$ and $m = n = 1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1 = 1, \nu_1 = 2\}$ and $\{\mu_1 = 2, \nu_1 = 1\}$. Note that $\eta^1 \colon \Delta^2 \to \Delta^1$ equals the map[1758] $[\mathfrak{v}_0, \mathfrak{v}_1, \mathfrak{v}_1]$ and that $\eta^2 \colon \Delta^2 \to \Delta^1$ equals the map $[\mathfrak{v}_0, \mathfrak{v}_0, \mathfrak{v}_1]$. In the figure below we illustrate $\nabla_{1,1}(\sigma \otimes \tau)$ for two affine linear singular 1-simplices $\sigma \colon \Delta^1 \to X = \mathbb{R}$ and $\tau \colon \Delta^1 \to Y = \mathbb{R}$.



$$\underbrace{(\sigma \circ \eta^1, \tau \circ \eta^2) - (\sigma \circ \eta^2, \tau \circ \eta^1)}_{= \nabla(\sigma \otimes \tau)}$$

(2) We consider the case $X = Y = S^1$ and $m = n = 1$. Furthermore we denote by $\mu \colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t,t) = \exp(2\pi i t)$. On page 3536 we saw that μ represents the standard generator of S^1 with the standard orientation. Comparing the figure on page 2914 with the above figure we see that the shuffle $\nabla_{1,1}(\mu \otimes \mu)$ is precisely the usual representative for the standard generator of the homology groups $\mathrm{H}_2(S^1 \times S^1) = \mathrm{H}_2(([0,1] \times [0,1])/\sim)$. □

Now we can formulate the main result of this appendix:

Proposition 164.24. (Shuffle Proposition) The maps $\nabla_{m,n}$ that we had just defined have all the properties that we had mentioned in the EZ-Construction Proposition 164.10, in other words, the maps $\nabla_{m,n} \colon \mathrm{C}_m(X) \otimes \mathrm{C}_n(Y) \to \mathrm{C}_{m+n}(X \times Y)$ have the following properties:

(1) On the 0-level they equal the canonical maps that we introduced in Definition 164.1. In other words, for all $x \in X$ and $y \in Y$ we have $\nabla_{0,0}(x \otimes y) = (x,y) \in \mathrm{C}_0(X \times Y)$.
(2) They form a chain map $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$.
(3) They are natural.
(∗) For any $x \in X$ and any singular simplex $\sigma \colon \Delta^q \to Y$ we have $\nabla(x \otimes \sigma) = x \times \sigma$ and the same statement also holds with the roles of X and Y swapped.[1759]

[1756] Let $X = \Delta^m$, $Y = \Delta^n$ and $\sigma = \mathrm{id}_{\Delta^m}$ and $\tau = \mathrm{id}_{\Delta^n}$. If we compare the above definitions with Definition 174.37 of the product of two ordered simplicial complexes we see that the summands correspond precisely to the $(m+n)$-simplices of the product of the ordered simplicial complexes Δ^m and $Delta^n$.
[1757] Here we implicitly use that $\mu_0 = 0$ and $\mu_{m+1} = m+n+1$.
[1758] As in Definition 143.8, given points $a_0, \ldots, a_k \in \mathbb{R}^n$ we denote by $[a_0, \ldots, a_k] \colon \Delta^k \to \mathbb{R}^n$ the unique affine linear map that sends the i-th standard basis vector e_i to a_i.
[1759] This shows that ∇ has the extra property (∗) which we formulated in the Eilenberg-Zilber Theorem 164.2.

The key to the proof of the Shuffle Proposition 164.24 is the following lemma:

Lemma 164.25. (∇-Chain Map Lemma) For any choice of $m, n \in \mathbb{N}_0$ we have the following equality in $\mathrm{C}_{m+n-1}(\Delta^m \times \Delta^n)$:

$$\nabla(\partial^{\otimes}_{m+n}(\underbrace{\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_m(\Delta^m)\otimes \mathrm{C}_n(\Delta^n)})) \;=\; \partial_{m+n}(\nabla(\underbrace{\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_m(\Delta^m)\otimes \mathrm{C}_n(\Delta^n)})).$$

Proof of Proposition 164.24 assuming Lemma 164.25. It follows immediately from the definitions that the maps ∇ satisfy (1), (3) and (∗). Thus it remains to show that the maps form a chain map. In other words, we need to prove the following claim:

Claim. Let X and Y be topological spaces. For any $\sigma \in \mathrm{C}_m(X)$ and $\tau \in \mathrm{C}_n(Y)$ we have

$$\nabla(\partial^{\otimes}_{m+n}(\sigma \otimes \tau)) \;=\; \partial_{m+n}(\nabla(\sigma \otimes \tau)).$$

Proof. As we will see this claim follows easily from the naturality of ∇, the naturality of boundary maps and from the ∇-Chain Map Lemma 164.25. More precisely, we have the following equalities:

$$\begin{aligned}
\nabla(\partial^{\otimes}_{m+n}(\sigma \otimes \tau)) &= \nabla(\partial^{\otimes}_{m+n}(\sigma_*(\mathrm{id}_{\Delta^m}) \otimes \tau_*(\mathrm{id}_{\Delta^n}))) &&= \nabla(\partial^{\otimes}_{m+n}((\sigma \otimes \tau)_*(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})))\\
&= \nabla((\sigma \otimes \tau)_*(\partial^{\otimes}_{m+n}(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}))) &&= (\sigma \times \tau)_*(\nabla(\partial^{\otimes}_{m+n}(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})))\\
&\quad\uparrow \text{ since } \sigma_* \text{ and } \tau_* \text{ are chain maps} && \quad\uparrow \text{ by the naturality of } \nabla\\
&= (\sigma \times \tau)_*(\partial_{m+n}(\nabla(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}))) &&= \partial_{m+n}((\sigma \times \tau)_*(\nabla(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})))\\
&\quad\uparrow \text{ by the ∇-Chain Map Lemma 164.25} && \quad\uparrow \text{ since } (\sigma \times \tau)_* \text{ is a chain map}\\
&= \partial_{m+n}(\nabla((\sigma \otimes \tau)_*(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}))) &&= \partial_{m+n}(\nabla(\sigma \otimes \tau)).\\
&\quad\uparrow \text{ by the naturality of } \nabla
\end{aligned}$$

■

Proof of the ∇-Chain Map Lemma 164.25. Let $m, n \in \mathbb{N}_0$. We start out by spelling out the left and right-hand side of the desired equality:

(1) We have the following equalities in $\mathrm{C}_{m+n-1}(\Delta^m \times \Delta^n)$:

$$\begin{aligned}
\nabla(\partial^{\otimes}_{m+n}(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})) &\overset{\text{see Definition 163.2 of } \partial^{\otimes} \text{ in}}{=} \nabla(\partial_m(\mathrm{id}_{\Delta^m}) \otimes \mathrm{id}_{\Delta^n} + (-1)^m \cdot \mathrm{id}_{\Delta^m} \otimes \partial_n(\mathrm{id}_{\Delta^n})) \overset{\text{Definition 143.13 of } \partial_m \text{ and } \partial_n}{=}\\
&= \nabla\Big(\underbrace{\sum_{j=0}^{m} (-1)^j \cdot i^m_j \otimes \mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_{m-1}(\Delta^m)\otimes \mathrm{C}_n(\Delta^n)} + (-1)^m \cdot \underbrace{\sum_{j=0}^{n} (-1)^j \cdot \mathrm{id}_{\Delta^m} \otimes i^n_j}_{\in \mathrm{C}_m(\Delta^m)\otimes \mathrm{C}_{n-1}(\Delta^n)}\Big)\\
&= \sum_{(\mu,\nu)\in S_{m-1,n}} \sum_{j=0}^{m} (-1)^j \cdot \mathrm{sign}(\mu,\nu) \cdot (i^m_j \circ \eta^\mu, \eta^\nu)\\
&\quad + (-1)^m \cdot \sum_{(\mu,\nu)\in S_{m,n-1}} \sum_{j=0}^{n} (-1)^j \cdot \mathrm{sign}(\mu,\nu) \cdot (\eta^\mu, i^n_j \circ \eta^\nu).
\end{aligned}$$

(2) Conversely we obtain from the definitions of $\Delta_{m,n}$ and ∂_{m+n} the following equalities in $\mathrm{C}_{m+n-1}(\Delta^m \times \Delta^n)$:

$$\begin{aligned}
\partial_{m+n}(\nabla_{m,n}(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})) &= \partial_{m+n}\Big(\sum_{(\mu,\nu)\in S_{m,n}} \mathrm{sign}(\mu,\nu) \cdot (\eta^\nu, \eta^\nu)\Big)\\
&= \sum_{(\mu,\nu)\in S_{m,n}} \mathrm{sign}(\mu,\nu) \cdot \sum_{j=0}^{m+n} (-1)^j \cdot (\eta^\mu, \eta^\nu) \circ i^{m+n}_j.
\end{aligned}$$

The idea now is to show that the purple expression is the sum of the blue expression, the red expression and two other expressions that gracefully cancel each other out. To reach this advanced level of illumination we need to introduce lots of notation:

(3) Let $k, l \in \mathbb{N}_0$.

(a) We denote by $\widetilde{S}_{k,l}$ the set of all maps $\xi\colon \{1,\dots,k+l\} \to \{\mathrm{b},\mathrm{r}\}$ where the value b (i.e. the color blue) is chosen precisely k times and the value r (i.e. the color red) is chosen precisely l times.

(b) One can easily verify that the map

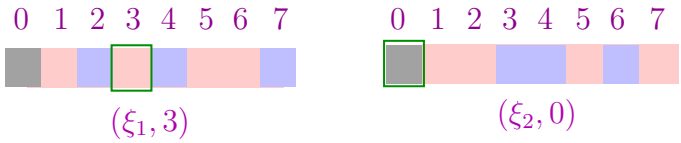
$$\begin{array}{rcl} S_{k,l} & \to & \widetilde{S}_{k,l} \\ (\mu,\nu) & \mapsto & \left(\begin{array}{rcl} \{1,\dots,k+l\} & \to & \{\mathrm{b},\mathrm{r}\} \\ i & \mapsto & \begin{cases} \mathrm{b}, & \text{if } i \in \{\mu_1,\dots,\mu_k\}, \\ \mathrm{r}, & \text{if } i \in \{\nu_1,\dots,\nu_l\} \end{cases} \end{array} \right) \end{array}$$

is a bijection. We will use this bijection to go back and forth between the two sets $S_{k,l}$ and $\widetilde{S}_{k,l}$.

(c) We visualize an element $(\xi, j) \in \widetilde{S}_{k,l} = S_{k,l}$ as a horizontal sequence of boxes in positions $0, 1, \dots, k+l$ where the zeroth box is gray and where the other boxes are colored by $\xi(1), \dots, \xi(k+l)$.

(d) We visualize an element $(\xi, j) \in \widetilde{S}_{k,l} \times \{0,\dots,k+l\} = S_{k,l} \times \{0,\dots,k+l\}$ by marking the j-th box in green.

elements of $\widetilde{S}_{3,4} \times \{0,\dots,7\}$



$(\xi_1, 3)$ $\qquad$ $(\xi_2, 0)$

(4) Given $(\xi = (\mu,\nu), j) \in \widetilde{S}_{m,n} \times \{0,\dots,m+n\}$ we write $\Omega_{(\xi,j)}\colon \Delta^{m+n-1} \to \Delta^m \times \Delta^n$ for the map $(\eta^\mu, \eta^\nu) \circ i_j^{m+n}$.

(5) (a) Let $\xi\colon \{1,\dots,m+n-1\} \to \{\mathrm{b},\mathrm{r}\} \in \widetilde{S}_{m-1,n}$. Given $i \in \{1,\dots,m-1\}$ we write

$$p_{\mathrm{b}}(i) \;\coloneqq\; \min\{j \in \{1,\dots,m+n-1\} \,|\, \#(\xi^{-1}(\mathrm{b}) \cap \{1,\dots,j\}) = i\}.$$

In other words, $p_{\mathrm{b}}(i)$ is the position where the value b appears for the i-th time. We also set $p_{\mathrm{b}}(m) = m+n$. Given $i \in \{1,\dots,m\}$ we consider the map

$$\begin{array}{rcl} \xi_i^{\mathrm{b}}\colon \{1,\dots,m+n\} & \to & \{\mathrm{b},\mathrm{r}\} \\ j & \mapsto & \begin{cases} \xi(j), & \text{if } j < p_{\mathrm{b}}(i), \\ \mathrm{b}, & \text{if } j = p_{\mathrm{b}}(i), \\ \xi(j-1), & \text{if } j > p_{\mathrm{b}}(i). \end{cases} \end{array}$$

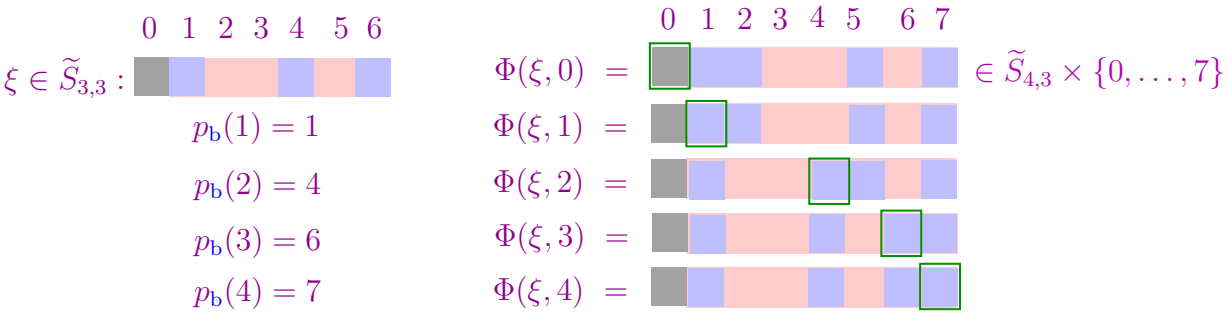
Note that ξ_i^{b} lies in $\widetilde{S}_{m,n}$. Pictorially ξ_i^{b} is obtained from ξ by inserting an extra b in front of the i-th occurrence of blue. We use this definition to consider the map

$$\begin{array}{rcl} \Phi\colon \widetilde{S}_{m-1,n} \times \{0,\dots,m\} & \to & \widetilde{S}_{m+n} \times \{0,\dots,m+n\} \\ (\xi, i) & \mapsto & \begin{cases} (\xi_1^{\mathrm{b}}, 0), & \text{if } i = 0, \\ (\xi_i^{\mathrm{b}}, p_{\mathrm{b}}(i)), & \text{if } i \in \{1,\dots,m\}. \end{cases} \end{array}$$

Given $(\xi, i) \in \widetilde{S}_{m-1,n} \times \{0,\dots,m\}$ we can describe the effect of Φ as follows:

- If $i = 0$, then we insert a blue box in the first spot and we mark the gray box.

- For $i = 1, \dots, m-1$ we insert a blue box right in front of the i-th occurrence of b in ξ and we mark this box.
- If $i = m$, then we add a blue box at the very end and we mark it.



(b) Let $(\mu, \nu) \in S_{m-1,n} = \widetilde{S}_{m-1,n}$ and let $j \in \{0, \dots, m\}$. It follows easily from the definitions that we have

$$(i_j^m \circ \eta^\mu, \eta^\nu) \;=\; \Omega_{\Phi((\mu,\nu),j)} \colon \Delta^{m+n} \;\to\; \Delta^m \times \Delta^n.$$

Furthermore, if we write $\Phi((\mu, \nu), j) =: (\zeta, k)$, then it follows again easily from the definitions that[1760]

$$\operatorname{sign}(\mu, \nu) \cdot (-1)^j \;=\; \operatorname{sign}(\zeta) \cdot (-1)^k.$$

(6) (a) As in (5a) we define the map $\Psi \colon \widetilde{S}_{m,n-1} \times \{0, \dots, n\} \to \widetilde{S}_{m,n} \times \{0, \dots, m+n\}$.

(b) As in (5b) it follows easily from the definitions that for $(\mu, \nu) \in S_{m,n-1} = \widetilde{S}_{m,n-1}$ and $j \in \{0, \dots, n\}$ we have

$$(\eta^\mu, i_j^n \circ \eta^\nu) \;=\; \Omega_{\Psi((\mu,\nu),j)} \colon \Delta^{m+n} \;\to\; \Delta^m \times \Delta^n.$$

Furthermore, if we write $\Psi((\mu, \nu), j) =: (\zeta, k)$, then it follows again easily from the definitions that[1761]

$$\operatorname{sign}(\mu, \nu) \cdot (-1)^{j+m} \;=\; \operatorname{sign}(\zeta) \cdot (-1)^k.$$

(7) We set

$$\begin{aligned} X_{m,n}^{\mathrm{b}} &:= \{(\xi, i) \in \widetilde{S}_{m,n} \times \{1, \dots, m+n-1\} \,|\, \xi(i) = \mathrm{b} \text{ and } \xi(i+1) = \mathrm{r}\} \\ X_{m,n}^{\mathrm{r}} &:= \{(\xi, i) \in \widetilde{S}_{m,n} \times \{1, \dots, m+n-1\} \,|\, \xi(i) = \mathrm{r} \text{ and } \xi(i+1) = \mathrm{b}\}. \end{aligned}$$

In other words, $X_{m,n}^{\mathrm{b}}$ correspond to the marked sequences where the marked square is blue and the square to the right of it is red. For $X_{m,n}^{\mathrm{r}}$ it is the opposite.

(8) (a) Note that we have a bijection

$$\begin{aligned} K \colon X_{m,n}^{\mathrm{b}} &\to X_{m,n}^{\mathrm{r}} \\ (\xi, i) &\mapsto \left(\begin{array}{rcl} \{1, \dots, m+n\} & \to & \{\mathrm{b}, \mathrm{r}\} \\ j & \mapsto & \begin{cases} \xi(j), & \text{if } j \neq i, i+1, \\ \text{opposite of } \xi(j), & \text{if } j = i, i+1. \end{cases} \end{array} \right., i \end{aligned}$$

Pictorially speaking this map just swaps the colors of the marked box and the square to the right of it.

[1760] This can for example be seen by considering the matrix that corresponds to the permutation ζ. This matrix has the (j, k)-entry given by 1. Thus if we first develop along the k-th column we obtain that $\operatorname{sign}(\zeta) = (-1)^{k+j} \cdot \operatorname{sign}(\mu, \nu)$.

[1761] As in (5b) we consider the matrix that corresponds to the permutation ζ. This matrix has the $(j+m, k)$-entry given by 1. Thus if we first develop the determinant along the k-th column we obtain the equality $\operatorname{sign}(\zeta) = (-1)^{k+j+m} \cdot \operatorname{sign}(\mu, \nu)$.

(b) It follows quite easily from the definitions that for every $(\xi,i)\in X^{\mathrm{b}}_{m,n}$ we have $\Omega_{\Phi(\xi,i)}=\Omega_{\Psi(K(\xi,i))}\colon \Delta^{m+n-1}\to\Delta^m\times\Delta^n$. Furthermore it is straightforward to show that the sign of the shuffle flips.

(9) It is elementary to verify that

$$\widetilde{S}_{m,n}\times\{0,\dots,m+n\}=\Phi(\widetilde{S}_{m-1,n}\times\{0,\dots,m\})\sqcup\Psi(\widetilde{S}_{m,n-1}\times\{0,\dots,n\})\sqcup X^{\mathrm{b}}_{m,n}\sqcup X^{\mathrm{r}}_{m,n}.$$

Finally note that we have the following equalities in $\mathrm{C}_{m+n-1}(\Delta^m\times\Delta^n)$:

$$\begin{aligned}
\partial_{m+n}(\nabla_{m,n}(\mathrm{id}_{\Delta^m}\otimes\mathrm{id}_{\Delta^n}))&\overset{(2)}{=}\sum_{(\mu,\nu)\in S_{m,n}}\mathrm{sign}(\mu,\nu)\cdot\sum_{k=0}^{m+n}(-1)^k\cdot(\eta^\mu,\eta^\nu)\circ i_k^{m+n}\\
&=\sum_{((\mu,\nu),k)\in S_{m,n}\times\{0,\dots,m+n\}}\mathrm{sign}(\mu,\nu)\cdot(-1)^k\cdot(\eta^\mu,\eta^\nu)\circ i_k^{m+n}\\
&\overset{(3,4)}{=}\sum_{(\zeta,k)\in\widetilde{S}_{m,n}\times\{0,\dots,m+n\}}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}\\
&\overset{(9)}{=}\sum_{(\zeta,k)\in\Phi(\widetilde{S}_{m-1,n}\times\{0,\dots,m\})}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}+\sum_{(\zeta,k)\in\Psi(\widetilde{S}_{m,n-1}\times\{0,\dots,n\})}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}\\
&\quad+\underbrace{\sum_{(\zeta,k)\in X^{\mathrm{b}}_{m,n}}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}+\sum_{(\zeta,k)\in X^{\mathrm{r}}_{m,n}}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}}_{=0,\text{ by (7b)}}\\
&\overset{(7b)}{=}\sum_{(\zeta,k)\in\Phi(\widetilde{S}_{m-1,n}\times\{0,\dots,m\})}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}+\sum_{(\zeta,k)\in\Psi(\widetilde{S}_{m,n-1}\times\{0,\dots,n\})}\mathrm{sign}(\zeta)\cdot(-1)^k\cdot\Omega_{(\zeta,k)}\\
&\overset{(5b,6b)}{=}\sum_{(\mu,\nu)\in S_{m-1,n}}\sum_{j=0}^{m}(-1)^j\cdot\mathrm{sign}(\mu,\nu)\cdot(i_j^m\circ\eta^\mu,\eta^\nu)\\
&\quad+(-1)^m\cdot\sum_{(\mu,\nu)\in S_{m,n-1}}\sum_{j=0}^{n}(-1)^j\cdot\mathrm{sign}(\mu,\nu)\cdot(\eta^\mu,i_j^n\circ\eta^\nu)\\
&\overset{(6)}{=}\nabla(\partial^{\otimes}_{m+n}(\mathrm{id}_{\Delta^m}\otimes\mathrm{id}_{\Delta^n})).
\end{aligned}$$

■

164.7. Exercises for Chapter 164 .

Exercise 164.1. Let X be a topological space. As usual we denote by $\Sigma(X)$ the suspension of X. Let

$$\begin{aligned}\rho\colon\Sigma(X)&\to\Sigma(X)\\ [(x,t)]&\mapsto[(x,-t)]\end{aligned}$$

be the natural reflection. Use the Künneth Theorem 164.15 to show that for every $k\in\mathbb{N}_0$ the induced map

$$\rho_*\colon\widetilde{\mathrm{H}}_k(\Sigma(X))\to\widetilde{\mathrm{H}}_k(\Sigma(X))$$

is given by multiplication by -1.

Remark. This gives a new proof of the Reflection-on-Suspension-H_*-Proposition 149.18.

Exercise 164.2. Let X be a topological space such that all homology groups are finitely generated abelian groups and such that $\sum_{k=0}^{\infty}\mathrm{rank}(\mathrm{H}_k(X))<\infty$. We consider the map $\rho\colon S^1\to S^1$ that is given by $\rho(x,y)=(x,-y)$. What is

$$\sum_{k=0}^{\infty}\mathrm{tr}((\mathrm{id}_X\times\rho)_*\colon\mathrm{H}_k(X\times S^1;\mathbb{Q})\to\mathrm{H}_k(X\times S^1;\mathbb{Q}))\,?$$

Exercise 164.3. Let $m,n\in\mathbb{N}_0$. Show that for each $k\in\mathbb{N}_0$ we have an isomorphism $\mathrm{H}_k(S^m\times S^n,S^m\vee S^n)\cong\widetilde{\mathrm{H}}_k(S^{m+n})$.

Exercise 164.4. Let $p, q \in \mathbb{N}$ and let $x \in S^p$ and $y \in S^q$. The map $S^p \to S^p \times S^q$ given by $z \mapsto (z, y)$ and the map $S^q \to S^p \times S^q$ given by $z \mapsto (x, z)$ give rise to an embedding $i\colon (S^p, x) \vee (S^q, y) \to S^p \times S^q$. Compute the relative homology groups $(S^p \times S^q, i(S^p, x) \vee (S^q, y))$.

Exercise 164.5. Given a topological space X we define its *total Betti number* as the sum $b(X) := \sum\limits_{n\in\mathbb{N}_0} b_n(X) \in \mathbb{N}_0 \cup \{\infty\}$. Given a topological space X with finite total Betti number we define its Euler characteristic as

$$\chi(X) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X).$$

Let X and Y be two topological spaces such that all homology groups are finitely generated abelian groups and such that $\sum\limits_{k=0}^{\infty} \operatorname{rank}(\mathrm{H}_k(X)) < \infty$ and $\sum\limits_{k=0}^{\infty} \operatorname{rank}(\mathrm{H}_k(X)) < \infty$ the Künneth Theorem 164.6 to show that

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Remark. This result generalizes the Product-χ-Proposition 156.12.

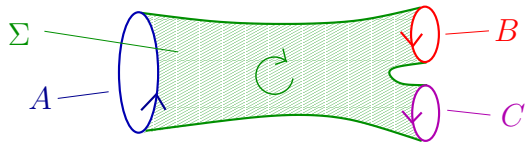
Exercise 164.6. We consider the oriented 2-dimensional smooth manifold Σ that is shown in the figure below. It has three oriented boundary components A, B, C. We pick orientation-preserving diffeomorphisms $\beta\colon S^1 \to B$ and $\gamma\colon S^1 \to C$. We consider

$$M \;:=\; (\Sigma \times S^1)/\!\sim \qquad \text{where } (\varphi(z), w) \sim (\psi(z), \overline{w}) \text{ for } (z, w) \in S^1 \times S^1.$$

By the Gluing-Smooth Manifolds-Proposition 54.5 we know that M is a compact orientable 3-dimensional smooth manifold with boundary $\partial M = A \times S^1$. We denote by $i\colon \partial M \to M$ the inclusion. Show that there exist isomorphisms $\Phi\colon \mathbb{Z}\oplus\mathbb{Z} \to \mathrm{H}_1(\partial M)$ and $\Psi\colon \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \to \mathrm{H}_1(M)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(a,b)\mapsto(2a,[b])} & \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \\ {\scriptstyle\Phi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Psi} \\ \mathrm{H}_1(\partial M) & \xrightarrow{\quad i_*\quad} & \mathrm{H}_1(M). \end{array}$$

Remark. The oriented boundary components A, B, C define elements $a, b, c \in \mathrm{H}_1(\Sigma)$. On page 2971 we saw that $\mathrm{H}_1(\Sigma) \cong (\mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c)/\mathbb{Z}\cdot(a+b+c)$.



Exercise 164.7. Let X and Y be two non-empty topological spaces. As in Definition 20.1 we define the join $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

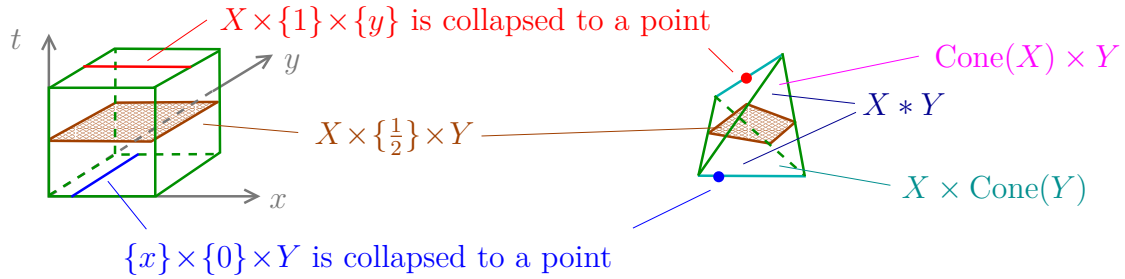
- For every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point.
- For every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Show that for each $k \in \mathbb{N}_0$ there exists a *split* short exact sequence of the following form:

$$0 \;\to\; \widetilde{\mathrm{H}}_k(X * Y) \;\to\; \widetilde{\mathrm{H}}_{k-1}(X \times Y) \;\to\; \widetilde{\mathrm{H}}_{k-1}(X) \oplus \widetilde{\mathrm{H}}_{k-1}(Y) \;\to\; 0.$$

Hint. There are two possible approaches:

- In Exercise 20.3 we gave an explicit homeomorphism
$$X * Y \;\cong\; (\mathrm{Cone}(X) \times Y) \cup_{(X\times\{1\})\times Y = X\times(Y\times\{1\})} (X \times \mathrm{Cone}(Y))$$
where $\mathrm{Cone}(X) = (X \times [0,1])/(x,0) \sim (x',0)$. Note though that you need to argue why you can use this decomposition.
- Let $\pi\colon X \times [0,1] \times Y \to X * Y$ be the natural projection. Consider the open subsets $U := \pi(X \times [0,1) \times Y)$ and $V := \pi(X \times (0,1] \times Y)$.



Exercise 164.8. Is it possible to replace in the Künneth Theorem 164.6 "homology" by "reduced homology"?

Exercise 164.9. Let X and Y be topological spaces. Show that for any $\sigma \in \mathrm{H}_0(X)$ and $\tau \in \mathrm{H}_0(X)$ we have
$$\epsilon_X(\sigma) \cdot \epsilon_Y(\tau) \;=\; \epsilon_{X\times Y}(\sigma \underset{\uparrow}{\times} \tau).$$
cross product of σ and τ

Here the maps ϵ are as always the various augmentation maps.

Exercise 164.10. Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. Show that for every $k \in \mathbb{N}_0$ there exists an isomorphism
$$\mathrm{H}_i((S^m)^n) \;\cong\; \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k!\cdot(n-k)!}} & \text{if } i = k\cdot m \text{ with } k \in \{0,\dots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 164.11. Let $n \in \mathbb{N}_0$. Show that
$$\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \;=\; 0.$$

Remark. There are many elementary ways to prove this equality (think of one!). One fun way of doing this is by computing the Euler characteristic of the n-dimensional torus in two different ways.

Exercise 164.12.

(a) Compute the homology groups $\mathrm{H}_*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2)$.
(b) Let $m, n \in \mathbb{N}_0$. Compute the homology groups $\mathrm{H}_*(\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$.

Exercise 164.13. Let p be a prime and let X and Y be two topological spaces such that all $\mathrm{H}_k(X)$ and $\mathrm{H}_k(Y)$ are of finite rank. Compute the homology groups $\mathrm{H}_*(X \times Y; \mathbb{F}_p)$ in terms of $\mathrm{H}_*(X; \mathbb{F}_p)$ and $\mathrm{H}_*(Y; \mathbb{F}_p)$.
Remark. This can be seen as a generalization of Exercise 164.12.

Exercise 164.14. Let X and Y be path-connected non-empty topological spaces. What is the most concise expression for $\mathrm{H}_2(X \times Y)$ in terms of the homology groups of X and Y that you can find?

Exercise 164.15. Let X and Y be path-connected topological spaces. Let $x \in X$ and let $y \in Y$. We denote by $i\colon X \to X \times Y$ the inclusion map that is given by $i(x) = (x, y)$ and similarly we denote by $j\colon Y \to X \times Y$ the inclusion map that is given by $j(y) = (x, y)$. Finally let G be any abelian group. Show that the map

$$i_* \oplus j_*\colon \mathrm{H}_1(X;G) \oplus \mathrm{H}_1(Y;G) \;\to\; \mathrm{H}_1(X \times Y;G)$$

is an isomorphism.

Exercise 164.16. Let Υ be an Eilenberg-Zilber map and let X be a topological space. We denote by $\pi\colon X \times X \to X$ the projection onto the first factor. Let $p, q \in \mathbb{N}_0$. We consider the map

$$Z_{p,q}\colon \mathrm{H}_p(X) \otimes \mathrm{H}_q(X) \xrightarrow{\times} \mathrm{H}_{p+q}(X \times X) \xrightarrow{\pi_*} \mathrm{H}_{p+q}(X).$$

(a) Show that for $q \neq 0$ the map $Z_{p,q}$ is the zero map.
(b) Let $x \in X$. Show that the map

$$\mathrm{H}_p(X) \xrightarrow{\sigma \mapsto \sigma \otimes [x]} \mathrm{H}_p(X) \otimes \mathrm{H}_0(X) \xrightarrow{\mathbb{Z}_{p,0}} \mathrm{H}_p(X)$$

is the identity.

Hint. You could either formulate and prove a variation on the EZ–Natural Chain Homotopy Proposition 164.13, or you could use the Shuffle Proposition 164.24.

Exercise 164.17. Let $n \in \mathbb{N}$ and let $\sigma \in S_n$ be a permutation. We consider the homeomorphism

$$\begin{aligned} f_\sigma\colon (S^1)^n &\to (S^1)^n \\ (z_1, \ldots, z_n) &\mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(n)}). \end{aligned}$$

What is the induced map $f_{\sigma *}\colon \mathrm{H}_n((S^1)^n) \to \mathrm{H}_1((S^1)^n)$?

Exercise 164.18.

(a) Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. Write down an isomorphism $\mathcal{C}_* \otimes \mathcal{D}_* \to \mathcal{C}_* \otimes \mathcal{D}_*$ which in degree zero is given by $v \otimes w \mapsto w \otimes v$.
(b) Let X and Y be two topological spaces. Given $\sigma \in \mathrm{H}_k(X)$ and $\tau \in \mathrm{H}_l(Y)$, what is the relationship between $\sigma \times \tau \in \mathrm{H}_{k+l}(X \times Y)$ and $\tau \times \sigma \in \mathrm{H}_{k+l}(Y \times X)$?

Exercise 164.19. We consider the two topological spaces $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$.

(a) Show that the homology groups of $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$ are isomorphic.
(b) Use suitable homotopy groups to show that $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$ are not homeomorphic.

Exercise 164.20. Let X and Y be two arbitrary CW-complexes. Let $\mathcal{TP}$ and $\mathcal{CWP}$ be the two topologies on the set $X \times Y$ that we introduced in the Topological-Product Proposition 7.2 and in Definition 139.15. By the CW-Complex Product Proposition 139.18 we know that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous. Show that the continuous map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ induces an isomorphism of chain complexes.
Hint. Make use of the statement and the proof of the CW-Products-Invariants Proposition 139.20.

Exercise 164.21. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n \times S^n$ be the map that is given by swapping the two factors, i.e. that is given by $f(x,y) = (y,x)$. Determine the trace and the determinant of the induced map $f\colon \mathrm{H}_n(S^n \times S^n) \to \mathrm{H}_n(S^n \times S^n)$.

Exercise 164.22. Let $n \in \mathbb{N}$. Let $\sim$ be the equivalence relation on $[0,1]^n$ which is generated by

$$(x_1, \ldots, 0, \ldots, x_n) \ \sim \ (x_1, \ldots, 1, \ldots, x_n).$$

With this notation we can pose the following problems:

(a) Show that $[0,1]^n/\sim$ is homeomorphic to $(S^1)^n$.

(b) Provide an explicit singular cycle σ such that $\mathrm{H}_n([0,1]^n/\sim) = \mathbb{Z} \cdot [\omega]$.

CHAPTER 165

Applications of homology groups

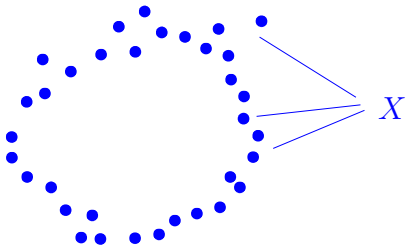
In this chapter we will present two very different applications of homology groups. First we will give a very quick, not entirely serious, introduction to persistent homology. Afterwards we will provide one more proof of the Borsuk-Ulam Theorem 47.12.

This chapter is organized as follows:

- In Section 165.1 we will give a very quick introduction to persistent homology.
- In Section 165.2 we will introduce the transfer map.
- In Section 165.3 we will use the transfer map to provide one more proof of the Borsuk-Ulam Theorem 47.12.
- In Section 165.4 we will provide exercises for Chapter 165.

165.1. **Persistent homology.** In this section we outline the idea behind "persistent homology". Persistent homology was introduced by Gunnar Carlsson [**Carl2009**] in 2009. More information and a much more careful treatment of persistent homology can for example be found in [**EdH2010**]. Some extra information can also be found in [**Gh2014**].

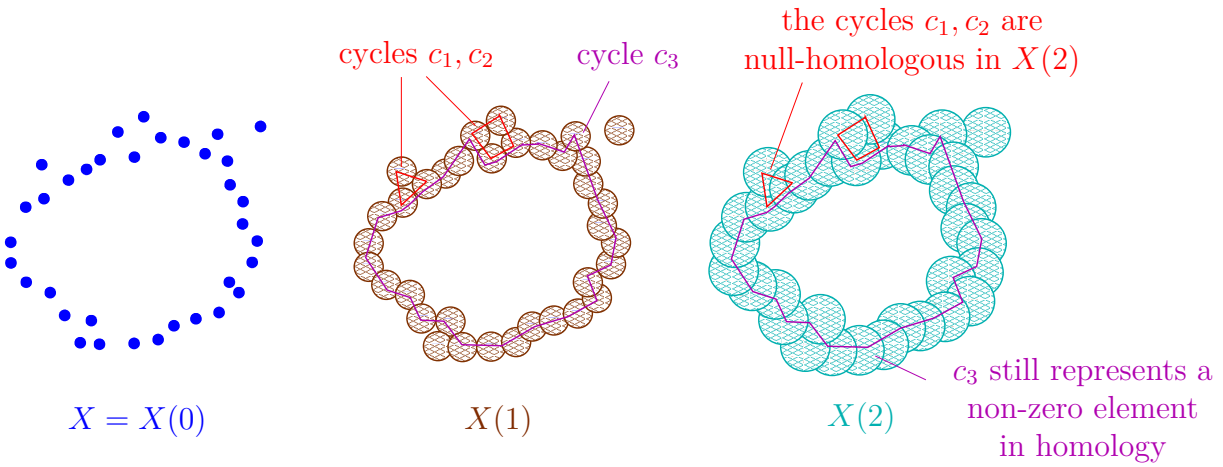
We consider the figure below. What do we see? If we forget about topology for a second, then we would say that we see a set X of dots that form a loop.



dots that appear to the eye to form a loop

If we view X as a topological space, then it is just a finite discrete subset of $\mathbb{R}^2$. In particular its homology groups are non-zero only in dimension 0. So how can we use the mathematical machine that we have developed over the last two semesters to "see the loop"?

Given $r \in [0, \infty)$ we denote by $X(r)$ the subset of $\mathbb{R}^2$ that is given by replacing each point $x \in X$ by the closed ball $\overline{B}_r(x)$ of radius r around x. The homeomorphism type of $X(r)$ changes with the radius r. For example in the figure below we see that at $r = 1$ all but one ball are joined up and we see that $\mathrm{H}_1(X(1)) \cong \mathbb{Z}^3$. We show three cycles c_1, c_2, c_3 in $X(1)$ which form a basis for $\mathrm{H}_1(X(1))$. Two of the generators, namely c_1, c_2 are formed "by accident", namely they arise from balls which are close by. This gets captured by the fact that they become null-homologous in $\mathrm{H}_1(X(2))$ whereas c_3 represents a non-trivial element for "much longer", i.e. for much larger values of r. Put differently, c_3 is a persistent element in homology, hence it captures global information.



Summarizing, given any subset X of $\mathbb{R}^2$ we can do the following: For any $r \in [0, \infty)$ we can study the topological spaces $X(r)$ and furthermore for any $r < s$ we can study the inclusion-induced map $\mathrm{H}_n(X(r)) \to \mathrm{H}_n(X(s))$. If there is a non-zero element in $\mathrm{H}_n(X(r))$ which stays non-zero for a "long time", then we say that it is persistent and we hope that it captures something interesting. For computational convenience it might also be interesting to consider homology with say $\mathbb{F}_2$-coefficients.

The same approach can be applied to any finite subset of some $\mathbb{R}^n$. Pretty much any data one has in practice, e.g. blue pixels of a picture, measurements in an experiment, statistical data and so on, can be viewed as a finite subset of some $\mathbb{R}^n$, so in principle one can study its persistent homology. The real question is, whether one captures something significant or not. For example persistent homology was applied in [**SHP2016**] to study the "The topological shape of Brexit". It is perhaps debatable whether one can really obtain anything meaningful about the politics of Brexit using persistent homology.

165.2. **The transfer map.** Before we can give the next application of homology theory, namely the Borsuk-Ulam Theorem in higher dimensions, we need to introduce and study the transfer map in homology corresponding to a finite covering.[1762]

Definition 165.1. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected[1763]it follows from the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 that there exist precisely k different lifts of σ to $\widetilde{X}$, i.e. there exist precisely k different singular simplices $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \ldots, k$. We write

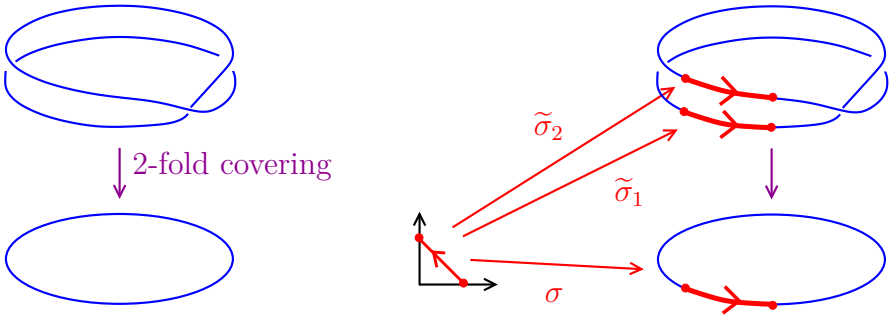
$$p^*(\sigma) \;:=\; \sum_{i=1}^{k} \widetilde{\sigma}_i \;\in\; \mathrm{C}_n(\widetilde{X}).$$

[1762]The following definition of the transfer map is the analog of the definition in Definition 85.4 of the transfer map for smooth forms on manifold.

Given any commutative ring R we refer to the R-module homomorphism[1764]

$$\begin{aligned} p^*\colon \mathrm{C}_n(X;R) &\to \mathrm{C}_n(\widetilde{X};R) \\ \sum_{i=1}^m \sigma_i \otimes r_i &\mapsto \sum_{i=1}^m p^*(\sigma_i)\otimes r_i \end{aligned}$$

as the transfer map of the covering.



The following lemma, which is the analog of the De Rham–Transfer Map Lemma 85.5, summarizes a few essential properties of transfer maps:

Lemma 165.2. (Transfer Map Lemma) Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X and let R be a commutative ring.

(1) Each transfer map $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_n(\widetilde{X};R)$ is a monomorphism.
(2) The transfer map $p^*\colon \mathrm{C}_*(X;R) \to \mathrm{C}_*(\widetilde{X};R)$ is a chain map.
(3) The transfer map is natural in the following sense: Suppose we are given a diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{p} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{q} & Y \end{array}$$

of continuous maps such that the following statements hold:
(a) The diagram commutes.
(b) $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ are k-fold coverings.
(c) For each $x \in X$ the map $\widetilde{f}|_{p^{-1}(x)}\colon p^{-1}(x) \to q^{-1}(f(x))$ is a bijection.
Then for every $n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_n(X;R) & \xrightarrow{p^*} & \mathrm{C}_n(\widetilde{X};R) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \widetilde{f}_*} \\ \mathrm{C}_n(Y;R) & \xrightarrow{q^*} & \mathrm{C}_n(\widetilde{Y};R). \end{array}$$

Proof. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X.

(1) Let $c = \sum_{i=1}^m \sigma_i \otimes r_i \in \mathrm{C}_n(X) \otimes R$ be a non-zero element. We can assume that the σ_i are pairwise distinct and that all the r_i are non-zero. Let $\widetilde{\sigma}_i\colon \Delta^n \to \widetilde{X}$ be a lift of σ_i. Let $\pi\colon \mathrm{C}_n(\widetilde{X}) \to \mathbb{Z}$ be the epimorphism with $\widetilde{\sigma}_i \mapsto 1$ and that sends all other singular

[1763]To be precise, we also use that Δ^n is locally path-connected.
[1764]Here we use that it follows easily from the Free Module–Tensor-with-Ring Lemma 159.20 that for any $k \in \mathbb{N}_0$ the set $\{\sigma \otimes 1_R\}_{\sigma\colon \Delta^k \to X}$ forms a basis for the free R-module $\mathrm{C}_k(X;R)$.

simplices to 0. Clearly $((\pi \otimes \mathrm{id}_R) \circ p^*)(c) = 1 \otimes r_i \in \mathbb{Z} \otimes R \cong R$ is non-zero. But this shows that p^* is a monomorphism.

For the proof of (2) and (3) let $\sigma\colon \Delta^n \to X$ be a singular simplex. Furthermore let $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ be the k different continuous maps with $p \circ \widetilde{\sigma}_j = \sigma$ for $j = 1, \dots, k$.

(2) This statement follows immediately from the observation that for any $r = 0, \dots, n$ the maps $\widetilde{\sigma}_j \circ i_r^n\colon \Delta^{n-1} \to \widetilde{X}$, $j = 1, \dots, k$ are k different lifts of $\sigma_j \circ i_r^n\colon \Delta^{n-1} \to X$.

(3) This statement follows immediately from the following observation: It is a consequence of (c) that for any $r = 0, \dots, n$ the maps $\widetilde{f} \circ \widetilde{\sigma}_j\colon \Delta^n \to \widetilde{Y}$, $j = 1, \dots, k$ are k different lifts of $f \circ \sigma\colon \Delta^n \to Y$. ■

The Transfer Map Lemma 165.2 allows us to introduce the following definition:

Definition 165.3. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Recall that it follows from the Transfer Map Lemma 165.2 (2) that the natural transfer map $p^*\colon \mathrm{C}_*(X; R) \to \mathrm{C}_*(\widetilde{X}; R)$ induces a natural R-module homomorphism

$$p^*\colon \mathrm{H}_*(X; R) \to \mathrm{H}_*(\widetilde{X}; R)$$

which we also call transfer map. Finally suppose that we are given a subset $A \subseteq X$. We write $\widetilde{A} \coloneqq p^{-1}(A)$. One easily verifies that the transfer map $p^*\colon \mathrm{C}_n(X; R) \to \mathrm{C}_*(\widetilde{X}; R)$ induces an R-module homomorphism

$$p^*\colon \mathrm{H}_*(X, A; R) \to \mathrm{H}_*(\widetilde{X}, \widetilde{A}; R).$$

The following proposition is an analog of the De Rham-Transfer–Projection Proposition 85.7.

Proposition 165.4. (Transfer–Projection Proposition) Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X and let $n \in \mathbb{N}_0$. The following statements hold:

(1) For any commutative ring R the map $p_* \circ p^*\colon \mathrm{H}_n(X; R) \to \mathrm{H}_n(X; R)$ is given by multiplication by k.

(2) The projection induced map $p_*\colon \mathrm{H}_n(\widetilde{X}; \mathbb{Q}) \to \mathrm{H}_n(X; \mathbb{Q})$ is an epimorphism.

The analogous statements also hold in the relative setting.

Remark. The conclusion of the Transfer–Projection Proposition 165.4 (2) does not hold if we drop the condition that the covering is finite. For example, we have an infinite covering $p\colon \mathbb{R} \to S^1$ but $0 = \mathrm{H}_1(\mathbb{R}; \mathbb{Q}) \xrightarrow{p_*} \mathrm{H}_1(S^1; \mathbb{Q}) = \mathbb{Q}$ is not an epimorphism. □

Proof. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X and let $n \in \mathbb{N}_0$.

(1) Let R be a commutative ring. We denote by $p^*\colon \mathrm{C}_n(X; R) \to \mathrm{C}_n(\widetilde{X}; R)$ the transfer map. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. We denote the corresponding lifts by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k$. Then

$$(p_* \circ p^*)(\sigma) = p_*(\widetilde{\sigma}_1 + \dots + \widetilde{\sigma}_k) = \underbrace{p \circ \widetilde{\sigma}_1 + \dots + p \circ \widetilde{\sigma}_k}_{k \text{ summands}} \overset{\text{since } \widetilde{\sigma}_i \text{ is a lift of } \sigma}{=} \underbrace{\sigma + \dots + \sigma}_{k \text{ summands}} = k \cdot \sigma.$$

We thus see that $p_* \circ p^*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ is multiplication by k. It follows easily that the map

$$\mathrm{H}_n(X; R) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X}; R) \xrightarrow{p_*} \mathrm{H}_n(X; R)$$

is also given by multiplication by k.

(2) Since multiplication by $k \neq 0$ is an isomorphism of any rational vector space we obtain from (1) that $p_* \circ p^* \colon \mathrm{H}_n(X;\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an isomorphism. In particular the second map, namely p_*, is an epimorphism.

The proofs in the relative setting are basically identical to the above. ■

For two-fold coverings we will now see that the transfer map fits into a long exact sequence:

Proposition 165.5. (Double Covering-LES-Proposition) Let $p \colon \widetilde{X} \to X$ be a 2-fold covering. There exist homomorphisms $\partial_n \colon \mathrm{H}_n(X;\mathbb{F}_2) \to \mathrm{H}_{n-1}(X;\mathbb{F}_2)$, which are natural in the sense of the Transfer Map Lemma 165.2, such that the following sequence is exact:

$$\ldots \longrightarrow \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};\mathbb{F}_2) \xrightarrow{p_*} \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X;\mathbb{F}_2) \longrightarrow \ldots .$$

Example. Let $n \in \mathbb{N}$. We will see that we can use the Double Covering-LES-Proposition 165.5 to calculate the homology groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ just using the following basic facts:

(a) By the Sphere-H_*-Proposition 160.13 we know that

$$\mathrm{H}_k(S^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0 \text{ or } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since $\mathbb{R}\mathrm{P}^n$ is an n-dimensional CW-complex we know from the discussion on page 3081 that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$.

In the following discussion we write $\mathbb{F} = \mathbb{F}_2$ for space reasons. Applying the Double Covering-LES-Proposition 165.5 to the obvious 2-fold covering $p \colon S^n \to \mathbb{R}\mathrm{P}^n$ we obtain the following long exact sequence:

$$\underbrace{\mathrm{H}_{n+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=0 \text{ by (b)}} \xrightarrow{\partial_{n+1}} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F} \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0 \text{ by (a)}} \to$$

$$\ldots \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_1(S^n;\mathbb{F})}_{=0 \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_1} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_0(S^n;\mathbb{F})}_{=\mathbb{F} \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \to 0.$$

Now we can make the following observations:

(1) Considering the bottom part of the long exact sequence we see that $\mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$.

(2) If we work our way upward through the long exact sequence we see that the connecting homomorphisms give us isomorphisms

$$\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial_{n-1}} \mathrm{H}_{n-2}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial_{n-2}} \ldots \xrightarrow[\cong]{\partial_2} \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial_1} \mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \;\cong\; \mathbb{F}.$$

(3) Now that we know that $\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$ let us look again at the upper end of the long exact sequence:

$$0 \longrightarrow \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_n} \underbrace{\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0} \longrightarrow$$

It follows easily from staring at this data that the maps $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \mathrm{H}_n(S^n;\mathbb{F})$ and $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})$ are also isomorphisms.

In particular we have shown that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \;\cong\; \begin{cases} \mathbb{F}, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

Fortunately this agrees with the result that we had obtained earlier in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 160.14. □

Proof of the Double Covering-LES-Proposition 165.5. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering.

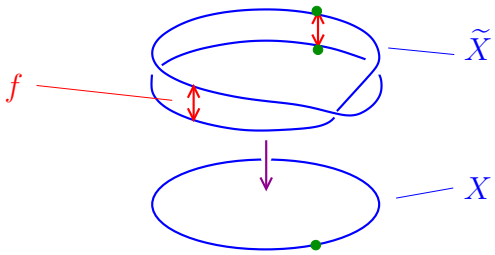
Claim. The following sequence of chain maps

$$0 \;\to\; \mathrm{C}_*(X;\mathbb{F}_2) \;\xrightarrow{p^*}\; \mathrm{C}_*(\widetilde{X};\mathbb{F}_2) \;\xrightarrow{p_*}\; \mathrm{C}_*(X;\mathbb{F}_2) \;\to\; 0$$

is in fact exact.

Proof. Since $p\colon \widetilde{X} \to X$ is a 2-fold cover we can consider, as in the Preimage–Swapping Lemma 95.11, the continuous map

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ x & \mapsto & \text{the unique other element of } p^{-1}(p(x)). \end{array}$$



Using the Map Lifting Criterion 127.5 we pick for each singular simplex $\sigma\colon \Delta^k \to X$ a lift $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$. We make the following observations which follow easily from the definitions and the Map Lifting Criterion 127.5:

(a) Given any singular n-simplex σ we have $p_*(\widetilde{\sigma}) = \sigma$.
(b) Given any singular n-simplex σ we have $p^*(\sigma) = \widetilde{\sigma} + f_*(\widetilde{\sigma})$.
(c) Given any singular simplex μ of $\widetilde{X}$ we have $p_*(f_*(\mu)) = p_*(\mu)$.
(d) Every singular n-simplex μ of $\widetilde{X}$ is of the form $\mu = \widetilde{\sigma}$ or $\mu = f_*(\widetilde{\sigma})$ for some singular n-simplex σ of X.

Now we turn to the proof of the exactness of the above sequence.

(1) We first show that the map $p_*\colon \mathrm{C}_n(\widetilde{X};\mathbb{F}_2) \to \mathrm{C}_n(X;\mathbb{F}_2) = \mathrm{C}_n(X) \otimes \mathbb{F}_2$ is surjective. So let $c = \sigma_1 \otimes b_1 + \cdots + \sigma_k \otimes b_k \in \mathrm{C}_n(X;\mathbb{F}_2)$. It follows from (a) that

$$p_*(\widetilde{\sigma}_1 \otimes b_1 + \cdots + \widetilde{\sigma}_k \otimes b_k) \;=\; \sigma_1 \otimes b_1 + \cdots + \sigma_k \otimes b_k \;=\; c.$$

(2) Note that it follows from the Transfer Map Lemma 165.2 (1) that the transfer map $p^*\colon \mathrm{C}_*(X;\mathbb{F}_2) \to \mathrm{C}_*(\widetilde{X};\mathbb{F}_2)$ is a monomorphism.

(3) Next we show that $\operatorname{im}(p^*) \subseteq \ker(p_*)$. For any singular n-simplex σ of $\widetilde{X}$ we have[1765]
$$(p_* \circ p^*)(\sigma \otimes 1) \;=\; p_*(\widetilde{\sigma} \otimes 1 + f_*(\widetilde{\sigma}) \otimes 1) \;\underset{\text{by (b) and (c)}}{=}\; 2\sigma \otimes 1 \;=\; \sigma \otimes 2 \;=\; 0 \;\in\; \mathrm{C}_*(X) \otimes \mathbb{F}_2.$$

(4) Finally we have to show that $\ker(p_*) \subseteq \operatorname{im}(p^*)$. So let $c \in \ker(p_*) \subseteq \mathrm{C}_*(\widetilde{X}; \mathbb{F}_2)$. By (d) we can write
$$c \;=\; (\widetilde{\sigma}_1 \otimes a_1 + f_*(\widetilde{\sigma}_1) \otimes b_1) + \cdots + (\widetilde{\sigma}_k \otimes a_k + f_*(\widetilde{\sigma}_k) \otimes b_k)$$
where $\sigma_1, \ldots, \sigma_k$ are pairwise different and where $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{F}_2$. By (c) we have
$$p_*(c) \;=\; \sigma_1 \otimes (a_1 + b_1) + \cdots + \sigma_k \otimes (a_k + b_k).$$
But since $c \in \ker(p_*)$ this implies that $a_i = -b_i$ for $i = 1, \ldots, k$. But since we work over $\mathbb{F}_2$ we actually have $a_i = b_i$ for $i = 1, \ldots, k$. It follows from (b) that
$$c \;=\; p^*(\sigma_1 \otimes a_1 + \cdots + \sigma_k \otimes a_k) \;\in\; \operatorname{im}(p^*)$$
as desired. ⊞

It follows from the claim and the LES-Proposition 145.10 that we get the desired connecting homomorphism $\partial_n \colon \mathrm{H}_n(X; \mathbb{F}_2) \to \mathrm{H}_{n-1}(X; \mathbb{F}_2)$. Finally note that the naturality of the connecting homomorphism follows from the naturality statement of the Transfer Map Lemma 165.2 and the Connecting Homomorphism Lemma 145.9 (2). ■

165.3. **The Borsuk-Ulam Theorem.** Our goal in this section is to give another proof of the following theorem:

Theorem 47.12. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every continuous map $f \colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.

Example. If our universe happens to be diffeomorphic to S^3, then there exists a pair of antipodal points where the temperature, distance to the Kneitinger and strength of the magnetic field agree. □

Remark.

(1) We already gave a proof of the Borsuk-Ulam Theorem 47.12 in Section 47.3. In Sections 182.6 and 213.3 we will provide two other proofs of the Borsuk-Ulam Theorem 47.12. Also recall that in the 2D-Borsuk Ulam Theorem 89.1 we proved the Borsuk-Ulam Theorem 47.12 in the special case that $n = 2$.

(2) The Borsuk-Ulam Theorem 47.12 has many applications, in particular the Lusternik-Schnirelmann Theorem 48.2, the Ham-Sandwich Theorem 48.4, Pfister's Level Theorem 48.15 and the Necklace Theorem 48.17. □

The key ingredient in the proof of the Borsuk-Ulam Theorem 47.12 is the following proposition:

Proposition 165.6. (Odd Map–Odd Degree Proposition) Let $f \colon S^n \to S^n$ be a continuous map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.

[1765] That is of course basically the same argument as in the proof of the Transfer–Projection Proposition 165.4.

Remark. In the Smooth Odd Map–Odd Degree Proposition 47.14 we used a very different argument to provide a proof of the Odd Degree Proposition 165.6 under the stronger hypothesis that f is actually smooth. Furthermore in the Odd Degree Proposition 89.6 we provided a proof of the Odd Map–Odd Degree Proposition 165.6 for the case $n = 1$. Finally an alternative proof will be given in the Odd Map–Even Lefschetz Number Proposition 182.24. □

Proof of the Odd Map–Odd Degree Proposition 165.6. We start out with a very general claim, which might be interesting in its own right.

Claim. Let $f\colon S^n \to S^n$ be a continuous map. For every $k \in \mathbb{N}$ the induced map

$$f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z})$$

is given by multiplication by $\deg(f)$.[1766]

Proof. Let $f\colon S^n \to S^n$ be a continuous map and let $k \in \mathbb{N}$. Since the two homomorphisms μ and ν of the Universal Coefficient Theorem 162.16 are natural we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_n(S^n)\otimes\mathbb{Z}/k\mathbb{Z} & \xrightarrow{\mu} & \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z}) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(S^n),\mathbb{Z}/k\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*\otimes \mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ 0 & \longrightarrow & \mathrm{H}_n(S^n)\otimes\mathbb{Z}/k\mathbb{Z} & \xrightarrow{\mu} & \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z}) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(S^n),\mathbb{Z}/k\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

of short exact sequences. Note that $\mathrm{H}_{n-1}(S^n) = 0$ or $\cong \mathbb{Z}$, hence it follows from the Tor-Properties Lemma 162.13 (4) that $\mathrm{Tor}(\mathrm{H}_{n-1}(S^n),\mathbb{Z}/k\mathbb{Z}) = 0$. It follows from this observation and the exactness of the horizontal sequences that the horizontal maps μ on the left are isomorphisms.

By definition the map $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by $\deg(f)$. It follows from the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10 that $f_* \otimes \mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}$ is also given by multiplication by $\deg(f)$. It now follows that the middle vertical map is also given by multiplication by $\deg(f)$. ⊞

Now we turn to the actual proof of the proposition. Thus let $f\colon S^n \to S^n$ be a continuous map such that $f(-x) = -f(x)$ for all $x \in S^n$. We need to show that $\deg(f)$ is odd. By the above claim it suffices to show that $f_*\colon \mathrm{H}_n(S^n;\mathbb{F}_2) \to \mathrm{H}_n(S^n;\mathbb{F}_2)$ is an isomorphism.

We start out with the observation that it follows from the Quotient-to-Quotient Lemma 8.6 that the map $f\colon S^n \to S^n$ descends to a well-defined continuous map

$$\begin{array}{rcl} \overline{f}\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [f(x)]. \end{array}$$

Note that this map gives rise to the following diagram:

$$\begin{array}{ccc} S^n & \xrightarrow{p} & \mathbb{R}\mathrm{P}^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ S^n & \xrightarrow{p} & \mathbb{R}\mathrm{P}^n \end{array}$$

of continuous maps. Note that the following statements hold:

(a) The diagram commutes.

[1766] This claim is the content of Exercise 162.17.

(b) $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ is a 2-fold covering.
(c) For each $x \in \mathbb{R}\mathrm{P}^n$ the map $f|_{p^{-1}(x)}\colon p^{-1}(x) \to p^{-1}(\overline{f}(x))$ is a bijection. (Note that here we use that $f(-x) = -f(x)$).

Now we consider the following diagram:

$$\begin{array}{ccccccccccc}
\mathrm{H}_n(S^n;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_2} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_1} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} \\
\mathrm{H}_n(S^n;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_2} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_1} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2).
\end{array}$$

We make the following clarifications and observations:

(1) The map p^* is of course the natural transfer map from the Transfer Map Lemma 165.2. It follows from (a)–(c) above that the naturality statement applies, i.e. the square to the left commutes.
(2) The maps $\partial_n, \dots, \partial_1$ are the natural connecting homomorphisms from the Double Covering-LES-Proposition 165.5. It follows from (a)–(c) above that the naturality statement applies, i.e. the squares to the right commutes.
(3) All the horizontal maps are isomorphisms by the discussion on page 3171.
(4) It follows from the obvious generalization of the H_0-Proposition 143.25, see also the discussion on page 3079, that the right vertical map is an isomorphism.

It follows from the above that the left vertical map is an isomorphism. But that is exactly what we had wanted to prove. ∎

Now we finally turn to the proof of the Borsuk-Ulam Theorem 47.12.

Proof of the Borsuk-Ulam Theorem 47.12. [1767] Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. We suppose that there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. Put differently, we suppose that for every $x \in S^n$ we have $f(x) \neq f(-x)$. We consider the composition of the following two maps

$$\begin{array}{rccc}
\Phi\colon S^{n-1} \times [0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\
(y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\
 & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}}_{\text{defined since } f(x) \neq f(-x)}
\end{array}$$

Now we see that

$$\text{odd number} \overset{\text{by definition we have } \Phi(-x,0) = -\Phi(x,0), \text{ thus it follows from the Odd Map–Odd Degree Proposition 165.6 that } \deg(\Phi_0) \text{ is odd}}{\downarrow =} \deg(\Phi_0) \overset{\text{by the Sphere–Degree Lemma 151.4 (3) since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{\uparrow =} \deg(\Phi_1) \overset{\text{by definition of } \Phi_1}{\uparrow =} \deg(\text{constant map}) \overset{\text{by the Sphere–Degree Lemma 151.4 (2)}}{\downarrow =} 0.$$

We have thus obtained a contradiction. ∎

[1767] The following proof is almost verbatim the same as the proof of the Borsuk-Ulam Theorem that we gave on page 1107 and on page 1755. Our only excuse for repeating the proof is that we hope that the reader finds this repetition convenient.

165.4. Exercises for Chapter 165 .

Exercise 165.1. Let $p\colon \widetilde{X} \to X$ be a covering odd degree. Show that for each $k \in \mathbb{N}_0$ the induced map $p_*\colon \mathrm{H}_k(\widetilde{X};\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z}/2\mathbb{Z})$ is an epimorphism.

Exercise 165.2. Let $p\colon \widetilde{X} \to X$ be a finite covering and let $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X)$ is finitely generated. Show that the cokernel of $p_*\colon \mathrm{H}_k(\widetilde{X}) \to \mathrm{H}_k(X)$ is finite.

Exercise 165.3. Let $p, q \in \mathbb{N}$ be coprime. We consider the lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$. Determine $\mathrm{H}_k(L(p,q);\mathbb{Q})$.

Exercise 165.4. Let $d \in \mathbb{Z}$ be an odd integer and let $n \in \mathbb{N}$. Give an example of a continuous map $f\colon S^n \to S^n$ with $\deg(f) = d$ and such that $f(-x) = -f(x)$ for all $x \in S^n$.

Exercise 165.5. Let $T = S^1 \times S^1$ be the torus and let K be the Klein bottle. Recall that on page 1783 we saw that there exists a 2-fold covering $p\colon T \to K$. Also note that we know that

$$\mathrm{H}_k(T;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,2 \\ \mathbb{F}_2^2, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Is the above information and the Double Covering-LES-Proposition 165.5 enough to determine the groups $\mathrm{H}_k(K;\mathbb{F}_2)$?

Exercise 165.6. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a finite CW-complex. Are the groups $\mathrm{H}_*(\widetilde{X};\mathbb{F}_2)$ uniquely determined by the groups $\mathrm{H}_*(X;\mathbb{F}_2)$?

Exercise 165.7. Let $p\colon \widetilde{X} \to X$ be a 3-fold covering. Do there exist natural homomorphisms $\partial_n\colon \mathrm{H}_n(X;\mathbb{F}_3) \to \mathrm{H}_{n-1}(X;\mathbb{F}_3)$ such that the following sequence is exact:

$$\ldots \longrightarrow \mathrm{H}_n(X;\mathbb{F}_3) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};\mathbb{F}_3) \xrightarrow{p_*} \mathrm{H}_n(X;\mathbb{F}_3) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X;\mathbb{F}_3) \longrightarrow \ldots ?$$

Exercise 165.8. Does there exist a CW-complex X with the following properties:

(a) $\mathrm{H}_2(X;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
(b) There exists a covering $p\colon S^4 \to X$.

Exercise 165.9. Let $n \in \mathbb{N}_0$. Show that the following are equivalent statements:

(a) For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = f(-x)$.
(b) For every odd continuous map $f\colon S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ with $f(x) = 0$.
(c) There is no odd continuous map $f\colon S^n \to S^{n-1}$.
(d) There is no continuous map $f\colon \overline{B}^n \to S^{n-1}$ such that the restriction of f to $S^{n-1} \to S^{n-1}$ is odd.

Exercise 165.10. Let $n \in \mathbb{N}_0$. Show that the Borsuk-Ulam Theorem 47.12 implies the Brouwer Fixed Point Theorem 38.7.

Remark. From Exercise 47.8 you can pick your favorite formulation of the Borsuk-Ulam Theorem 47.12.

Part XVIII

Singular Homology – extra topics

In this last part of the lecture notes, which is dedicated to homology groups, we prove a couple of loosely or even unconnected results. The main topic will be the proof and discussion of the Hurewicz Theorems 166.10 and 167.9.

This part of the lecture notes is organized as follows:

- In Chapter 166 we will prove the Hurewicz Theorem 166.10 which says that for a pointed connected topological space (X, x_0) there exists a natural isomorphism between the abelianization $\pi_1(X, x_0)_{\mathrm{ab}}$ of the fundamental group and the first homology group $\mathrm{H}_1(X)$.
- In Chapter 167 we will prove a high-dimensional analog of the Hurewicz Theorem 166.10. More precisely, the Hurewicz Theorem 167.9 says that if (X, x_0) is a pointed connected topological space such that $\pi_i(X, x_0) = 0$ for $i = 1, \ldots, n-1$ with $n \geq 2$, then $\pi_n(X, x_0)$ is naturally isomorphic to $\mathrm{H}_n(X)$. This result allows us in particular to finally show that for any $n \in \mathbb{N}$ we have an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$.
- In Chapter 168 we will use the calculation $\pi_n(S^n) \cong \mathbb{Z}$ to study the not entirely serious "social choice problem".
- In Chapter 169 we will introduce groups rings and we will see that for $n \geq 2$ we should view $\pi_n(X, x_0)$ as a module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Furthermore we will use the calculation $\pi_n(S^n) \cong \mathbb{Z}$ to calculate the $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$-module $\pi_2(S^1 \vee S^2)$.
- In Chapter 170 we will prove the Sphere Complement-H_*-Theorem 170.2 which gives us the calculation of the homology groups of $S^n \setminus h(S^k)$ for any embedding $S^k \to S^n$. We will use this calculation to prove the Jordan-Brouwer Separation Theorem 170.3 and to prove the Closed Manifold–Injection Corollary 170.10 which says that an injective continuous map between closed connected topological manifolds of the same dimension is already a homeomorphism.
- In Chapter 171 we will show in the Commutative Division Algebra Theorem 171.6, using the Closed Manifold–Injection Corollary 170.10 that any finite-dimensional division algebra over $\mathbb{R}$ which is *commutative* and which has a multiplicatively neutral element is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
- In Chapter 172 we will return to the study of the covering dimension of topological spaces, which we first introduced in Chapter 15. Now that we have homology groups at our disposal we can finally show that the covering dimension of any non-empty n-dimensional manifold equals n.
- In Chapter 173 we will introduce for each abelian group π and each $k \in \mathbb{N}$ the corresponding Moore space $\mathrm{M}(\pi, k)$.

CHAPTER 166

The Hurewicz Theorem in dimension one

Given a pointed path-connected topological space (X, x_0) we can consider the fundamental group $\pi_1(X, x_0)$ and the first homology group $\mathrm{H}_1(X)$. Both groups are defined by mapping "intervals" into X. Hence it is reasonable to expect that there is a relationship between these two groups. But in general these two groups cannot be isomorphic, since the first homology group is by definition abelian, whereas we saw on many occasions that fundamental groups can be non-abelian.
In this chapter we will prove and discuss the Hurewicz Theorem 166.10 which explains the relationship between the fundamental group $\pi_1(X, x_0)$ and the first homology group $\mathrm{H}_1(X)$. In the next chapter we will explore the relationship between higher homology groups and higher homotopy groups.

This chapter is organized as follows:

- In Section 166.1 we will state and prove the Hurewicz Theorem 166.10 in dimension one.
- In Section 166.2 we will discuss applications of the Hurewicz Theorem 166.10.
- In Section 166.3 we will provide exercises for Chapter 166.

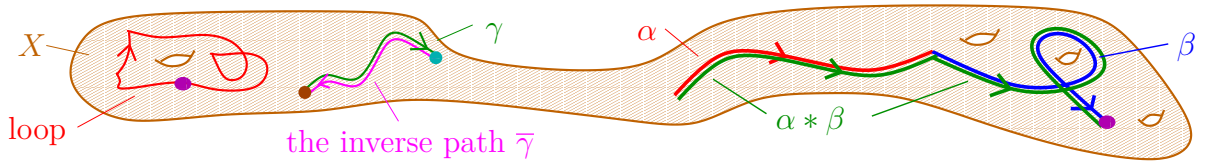
In the next Chapter 167 we will state and prove the higher-dimensional Hurewicz Theorem 167.9.

166.1. **The Hurewicz Theorem in dimension one.** We start out by reminding the reader of Definitions 88.1 and 88.7:

Definition 166.1. Let X be a topological space.

(1) A path in X is a continuous map $\gamma\colon [0,1] \to X$. We call $\gamma(0)$ the starting point of γ and we call $\gamma(1)$ the endpoint of γ. Often we say that γ is a path from $\gamma(0)$ to $\gamma(1)$.
(2) A loop is a path for which the starting point and the endpoint coincide.
(3) Given a path $\gamma\colon [0,1] \to X$ we refer to the path $\overline{\gamma}\colon [0,1] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(1-t)$ as the inverse path of γ.
(4) Let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in [0, \frac{1}{2}] \\ \beta(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$



Notation 166.2. Throughout this chapter we work with the homeomorphism

$$\begin{array}{rcl} \zeta\colon \Delta^1 & \to & [0,1] \\ (1-t,t) & \mapsto & t. \end{array}$$

The following proposition is the first step towards relating the fundamental group and the first homology group.

Proposition 166.3. (Hurewicz Homomorphism Proposition) Let (X, x_0) be a pointed topological space.

(0) The map

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & [\gamma \circ \zeta\colon \Delta^1 \to X] \end{array}$$

is well-defined.

(1) The map $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is a group homomorphism.

(2) Given any path $\gamma\colon [0,1] \to X$ from x_0 to some $x_1 \in X$ the following diagram commutes:[1768]

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \\ \Big\downarrow \text{homomorphism } \gamma_* \text{ from the Change-of-Base Point Proposition 88.14 (1)} & & \mathrm{H}_1(X). \\ \pi_1(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \end{array}$$

(3) Let *PTop* be the category of pointed topological spaces and let *Group* be the category of groups. The homomorphisms

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$$

define a natural transformation from the functor[1769]

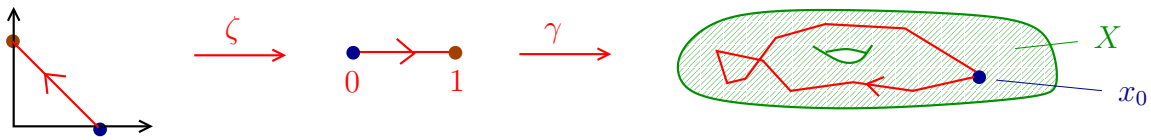
$$\begin{array}{rcl} \mathit{PTop} & \to & \mathit{Group} \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array}$$

to the functor

$$\begin{array}{rcl} \mathit{PTop} & \to & \mathit{Group} \\ (X,x_0) & \mapsto & \mathrm{H}_1(X). \end{array}$$

In plain English this means the following: Given any continuous map $f\colon X \to Y$ between topological spaces and given any $x_0 \in X$ the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,f(x_0)) \\ \Big\downarrow{\scriptstyle \Phi_{(X,x_0)}} & & \Big\downarrow{\scriptstyle \Phi_{(Y,f(x_0))}} \\ \mathrm{H}_1(X) & \xrightarrow{f_*} & \mathrm{H}_1(Y). \end{array}$$



The proof of the Hurewicz Homomorphism Proposition 166.3 relies on the following elementary lemma:

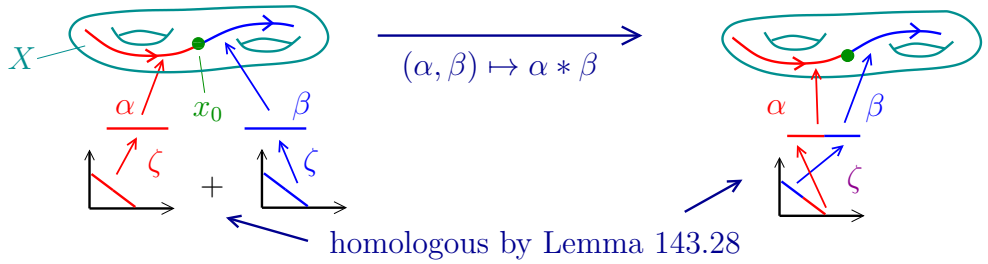
[1768] This statement shows in particular that the question of whether the Hurewicz homomorphism is a monomorphism, epimorphism, or isomorphism does not depend on the choice of a base point in a path-component.

[1769] Strictly speaking we viewed H_1 as a functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups. But of course it also defines a functor from the category *PTop* of pointed topological spaces to the category *Group* of groups.

Lemma 166.4. (Path-Chain–Homomorphism Lemma) Let X be a topological space.
(1) If γ is a loop in X, then $\gamma \circ \zeta \in \mathrm{C}_1(X)$ is a cycle.
(2) If γ is a path in X, then $-\gamma \circ \zeta$ and $\overline{\gamma} \circ \zeta$ are homologous.
(3) If α and β are two paths in X such that the endpoint of α agrees with the starting point of β, then the singular 1-chains $\alpha \circ \zeta + \beta \circ \zeta$ and $(\alpha * \beta) \circ \zeta$ are homologous.

Proof of the Path-Chain–Homomorphism Lemma 166.4.

(1) This statement is just the Singular 1-Simplex Lemma 143.28 (1).
(2) This statement follows easily from the Singular 1-Simplex Lemma 143.28 (2).
(3) Let α and β be two paths in X such that the endpoint of α agrees with the starting point of β.



It follows easily from the Singular 1-Simplex Lemma 143.28 (3) that the singular 1-chains $\alpha \circ \zeta + \beta \circ \zeta$ and $(\alpha * \beta) \circ \zeta$ are homologous. ■

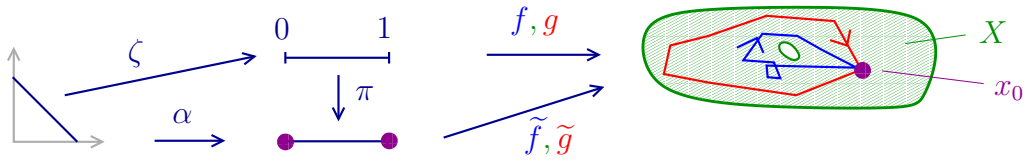
Proof of the Hurewicz Homomorphism Proposition 166.3.

(0) We need to show that the given map $\Phi_{(X,x_0)} \colon \pi_1(X, x_0) \to \mathrm{H}_1(X)$ is well-defined. First note that it follows from the Path-Chain–Homomorphism Lemma 166.4 (1) that if $\gamma \colon [0,1] \to X$ is a loop in (X, x_0), then $\gamma \circ \zeta \in \mathrm{C}_1(X)$ is a cycle. It remains to prove the following claim:

Claim. If $f, g \colon [0,1] \to X$ are two path-homotopic loops in x_0, then $f \circ \zeta$ and $g \circ \zeta$ are homologous.

Proof. We denote by $\sim$ the equivalence relation on $[0,1]$ that is generated by $0 \sim 1$. We denote by $\pi \colon [0,1] \to [0,1]/\!\sim$ the projection. We set $\alpha := \pi \circ \zeta$. Note that α defines an element in $\mathrm{H}_1([0,1]/\!\sim)$. Now suppose that $f, g \colon [0,1] \to X$ are two path-homotopic loops in x_0. Recall that this means that there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ with $F_0 = f$, $F_1 = g$ and such that for all $t \in [0,1]$ we have $F_t(0) = f(0) = g(0)$ and we have $F_t(1) = f(1) = g(1)$.

We denote by $\widetilde{f}, \widetilde{g} \colon [0,1]/\!\sim \to X$ and $\widetilde{F} \colon ([0,1]/\!\sim) \times [0,1] \to X$ the induced maps.



It follows from the Topological-Quotient Proposition 8.2 (1b) and the Homotopy Quotient Theorem 18.11 that these induced maps are continuous. In particular we see that

$\widetilde{f}$ and $\widetilde{g}$ are homotopic. We obtain the following equalities in $\mathrm{H}_1(X)$:

$$[f\circ\zeta] \underset{\substack{\uparrow\\ \text{since } f\circ\zeta \text{ and } \widetilde{f}\circ\alpha \text{ are}\\ \text{the same map } \Delta^1\to X}}{=} [\widetilde{f}\circ\alpha] = \widetilde{f}_*([\alpha]) \underset{\substack{\uparrow\\ \text{by the Homotopy-H}_*\text{-Proposition 144.12,}\\ \text{since } \widetilde{f} \text{ and } \widetilde{g} \text{ are homotopic maps } [0,1]/{\sim}\to X}}{=} \widetilde{g}_*([\alpha]) = [\widetilde{g}\circ\alpha] = [g\circ\zeta].$$

⊞

(1) This statement follows from the Path-Chain–Homomorphism Lemma 166.4 (3).
(2) This statement follows from the Path-Chain–Homomorphism Lemma 166.4 (2) and (3).
(3) The last statement follows immediately from the definitions. ■

The Hurewicz Homomorphism Proposition 166.3 allows us to introduce the following definition:

Definition 166.5. Given a pointed topological space (X,x_0) we refer to the homomorphism

$$\begin{array}{rcl}\Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X)\\ [\gamma\colon [0,1]\to X] & \mapsto & [\gamma\circ\zeta\colon \Delta^1\to X]\end{array}$$

as the Hurewicz homomorphism.

Example. We consider the pointed topological space $(X,x_0)=(S^1,1)$. It follows from the discussion on page 2866 that $\Phi_{(S^1,1)}([t\mapsto\exp(2\pi\,\mathrm{i}t)])=[S^1]\in\mathrm{H}_1(S^1)$. □

It is clear that the Hurewicz homomorphism is in general neither injective nor surjective:

(1) The Hurewicz homomorphism is in general not injective since $\pi_1(X,x_0)$ is in many cases a non-abelian group, whereas $\mathrm{H}_1(X)$ is an abelian group, and it is not possible that a non-abelian group is a subgroup of an abelian group.
(2) The Hurewicz homomorphism is in general not surjective, since $\pi_1(X,x_0)$ depends only on the path-component of x_0, while $\mathrm{H}_1(X)$ "sees" all path-components of X. But as we will see in the next proposition, this turns out to be the only obstacle to Φ being surjective.

Proposition 166.6. (Hurewicz Epimorphism Proposition) Let X be a topological space and let $x_0\in X$. If X is path-connected, then the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$$

is surjective.

In the proof of the Hurewicz Epimorphism Proposition 166.6 we will need the following elementary lemma:

Lemma 166.7. (Matching Lemma) Let X be a set and let $x_1,\dots,x_k$ and $y_1,\dots,y_l$ be elements in X. If

$$\sum_{i=1}^{k} x_i - \sum_{i=1}^{l} y_i = 0 \in \mathbb{Z}^{(X)},$$

then $k=l$ and there exists a permutation $s\in S_k$ such that $y_i=x_{s(i)}$ for $i=1,\dots,k$.

Proof. Let X be a set and let $x_1,\dots,x_k$ and $y_1,\dots,y_l$ be elements in X such that

$$\underbrace{\sum_{i=1}^{k} x_i}_{=:a} = \underbrace{\sum_{i=1}^{l} y_i}_{=:b} \in \mathbb{Z}^{(X)}.$$

It follows from the Free Abelian Group–Existence Lemma 97.8 that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x) = 1$ for all $x \in X$. By definition we have $\psi(a) = k$ and $\psi(b) = l$. So from $a = b$ we deduce that $k = l$.

Note that it follows from applying the Free Abelian Group–Existence Lemma 97.8 once again that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(y_k) = 1$ and $\psi(y) = 0$ for all $y \in X \setminus \{y_k\}$. By definition we have $\psi(b) > 0$. From $\psi(b) = \psi(a) > 0$ it follows that there exists an $n \in \{1, \ldots, k\}$ such that $y_k = x_n$. We define $s(k) = n$. We obtain that

$$\sum_{i=1}^{k-1} y_i \;=\; \sum_{i \in \{1,\ldots,k\} \setminus \{s(k)\}} x_i \;\in\; \mathbb{Z}^{(X)}$$

and we iterate this procedure to define $s(k-1), \ldots, s(1)$. By construction the map $s\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a bijection. ∎

Proof of Proposition 166.6. The homeomorphism $\zeta\colon \Delta^1 \to [0,1]$ from Notation 166.2 gives us a bijection

$$\begin{array}{rcl} \Phi\colon \{\text{paths } [0,1] \to X\} & \to & \{\text{singular 1-simplices in } X\} \\ (\gamma\colon [0,1] \to X) & \mapsto & (\gamma \circ \zeta\colon \Delta^1 \to X). \end{array}$$

We denote by Ψ the inverse of the above bijection.

We pick a homology class $z \in \mathrm{H}_1(X)$. Let $c = \sum_{i=1}^{k} a_i \cdot \sigma_i$ be a cycle in $\mathrm{C}_1(X)$ that represents z. It follows from the Singular 1-Simplex Lemma 143.28 (2) that, after possibly replacing a singular chain σ_i by $\overline{\sigma}_i$, we can assume that all the coefficients a_i are positive. Furthermore, since we do not require that the σ_i's are different, we can in fact suppose that $a_1 = \cdots = a_k = 1$. Next note that we have

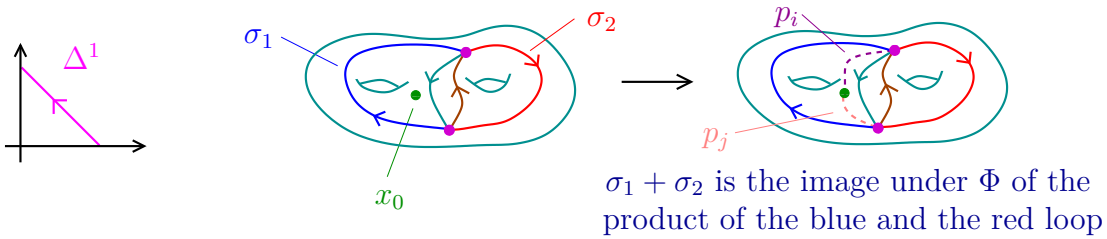
$$0 \underset{\uparrow}{=} \partial_1(c) = \partial_1\Big(\sum_{i=1}^{k} \sigma_i\Big) \underset{\uparrow}{=} \sum_{i=1}^{k} \sigma_i(0,1) - \sigma_i(1,0) = \sum_{i=1}^{k} \sigma_i(0,1) - \sum_{i=1}^{k} \sigma_i(1,0).$$

since c is a cycle

see the discussion on page 2751

On the right-hand side each "positive summand" needs to cancel with a "negative summand". This means, see also the Matching Lemma 166.7, that there exists a permutation $\theta \in S_k$ such that for each $i \in \{1, \ldots, k\}$ the "endpoint" $\sigma_i(0,1)$ of σ_i is just the "starting point" $\sigma_{\theta(i)}(1,0)$ of $\sigma_{\theta(i)}$.

Since X is path-connected we can choose for each $i \in \{1, \ldots, k\}$ a path p_i from the base point x_0 to the starting point $\sigma_i(1,0)$ of σ_i.



We obtain the following equality in $\mathrm{H}_1(X)$:

$$\begin{array}{lll} & \text{by the Singular 1-Simplex Lemma 143.28, the singular chain } \Phi(p_i)+\Phi(\overline{p_i}) \text{ is a null-homologous cycle} & \text{since } \Phi\circ\Psi=\mathrm{id} \text{ and since permutation of the summands does not change the sum} \\ c = \sum_{i=1}^{k}\sigma_i & \downarrow & \downarrow \\ = \sum_{i=1}^{k}\sigma_i + \sum_{i=1}^{k}\Phi(p_i) + \sum_{i=1}^{k}\Phi(\overline{p}_i) & & = \sum_{i=1}^{k}\big(\Phi(p_i)+\Phi(\Psi(\sigma_i))+\Phi(\overline{p}_{\theta(i)})\big) \\ = \sum_{i=1}^{k}\Phi\big(\underbrace{p_i * \Psi(\sigma_i) * \overline{p}_{\theta(i)}}_{\text{loop in } (X,x_0), \text{ by choice of } \theta}\big) & & = \Phi\Big(\underbrace{\prod_{i=1}^{k} p_i * \Psi(\sigma_i) * \overline{p}_{\theta(i)}}_{\text{loop in } (X,x_0)}\Big). \\ \uparrow & & \uparrow \\ \text{by the Path-Chain–Homomorphism Lemma 166.4, the two sides are homologous, so they represent the same element in } \mathrm{H}_1(X) & & \text{by the Path-Chain Homomorphism Lemma 166.4} \end{array}$$

Thus we have shown that $z = [c] \in \Phi_{(X,x_0)}(\pi_1(X,x_0))$. ■

In the following we want to study to what degree the Hurewicz homomorphism fails to be injective. Before we can do so, we need to recall some definitions and results from Section 98.3.

Definition 166.8. Let π be a group.

(1) For $x, y \in \pi$ we denote by $[x,y] \coloneqq xyx^{-1}y^{-1}$ the commutator of x and y.

(2) We refer to

$$[\pi,\pi] \coloneqq \left\{ \prod_{i=1}^{n} [x_i,y_i] \,\middle|\, x_1,y_1,\ldots,x_n,y_n \in \pi \right\}$$

as the commutator subgroup of π.[1770]

(3) We refer to $\pi_{\mathrm{ab}} \coloneqq \pi/[\pi,\pi]$ as the abelianization of π.

In Section 98.3 we saw that the abelianization is "the largest abelian quotient" of π. More precisely, if $\alpha\colon \pi \to H$ is a homomorphism from π to an abelian group H, then the Commutator Subgroup Lemma 98.14 (3) says that there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi & \longrightarrow & \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \\ & \searrow^{\alpha} & \downarrow \exists!\,\beta \\ & & H. \end{array}$$

This discussion leads us to the following definition:

Definition 166.9. Let (X,x_0) be a pointed topological space (X,x_0). It follows from the Commutator Subgroup Lemma 98.14 (3) that the corresponding Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ induces a homomorphism $\pi_1(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_1(X)$. We denote this homomorphism by $\Phi_{(X,x_0)}$ as well and we refer to it also as the Hurewicz homomorphism.

Now we can formulate the main result of this chapter:[1771]

[1770] By the Commutator Subgroup Lemma 98.14 (1) we know that $[\pi,\pi]$ is a normal subgroup of π.

[1771] This theorem is called Hurewicz Theorem since Witold Hurewicz [**Hur1935b**] proved the high-dimensional generalization, namely the Hurewicz Theorem 167.9. This 1-dimensional statement was known before the work of Witold Hurewicz, see e.g. [**ST1934**, Satz 48.3].

Theorem 166.10. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. If X is path-connected, then the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \;\to\; \mathrm{H}_1(X)$$

is a natural isomorphism[1772].

Example. Using our earlier calculations of fundamental groups and homology groups in the Torus-π_1-Proposition 90.14, the Real Projective Space–π_1-Proposition 93.3, the Surface-π_1-Proposition 101.1, the Torus-H_*-Proposition 153.9 (1), the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18, the Klein Bottle-H_*-Lemma 153.10 and the Surface-H_*-Proposition 155.17 we obtain the following table:

topological space	$(S^1)^n$	$\mathbb{R}\mathrm{P}^n$	Klein bottle	surface of genus g
π_1	$\mathbb{Z}^n$	$\mathbb{Z}/2\mathbb{Z}$	$\langle x, y \mid yxyx^{-1}\rangle$	$\langle x_1, y_1, \ldots, x_g, y_g \mid [x_1,y_1]\cdot\ldots\cdot[x_g,y_g]\rangle$
H_1	$\mathbb{Z}^n$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{2g}$

Using the discussion on page 1906 and the Abelianization-of-Presentation Proposition 100.17 one can verify that these calculations are indeed consistent with the Hurewicz Theorem 166.10. □

Example. Let G be a connected non-empty topological graph with v vertices and e edges. It follows from the discussion on page 2956, the Graph-π_1-Proposition 99.13 and the Hurewicz Theorem 166.10 that

$$\mathrm{H}_0(G) \cong \mathbb{Z}, \quad \mathrm{H}_1(G) \cong \mathbb{Z}^{e-v+1} \quad \text{and} \quad \mathrm{H}_i(G) = 0 \text{ for } i \geq 2.$$

This gives a new proof of the Graph-H_*-Lemma 156.9. □

The proof of the Hurewicz Theorem 166.10 requires the remainder of this section. We continue in Section 166.2 with applications.

Proof of the Hurewicz Theorem 166.10. Throughout this proof we consider the following two homeomorphisms:

$$\begin{array}{rcl} \zeta\colon \Delta^1 & \to & [0,1] \\ (1-t,t) & \mapsto & t \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \psi\colon [0,1] & \to & \Delta^1 \\ t & \mapsto & (1-t,t). \end{array}$$

Let (X, x_0) be a path-connected pointed topological space. By the Hurewicz Homomorphism Proposition 166.3 (3) we know that the Hurewicz homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} & \to & \mathrm{H}_1(X) \\ [f\colon [0,1]\to X] & \mapsto & [f\circ\zeta] \end{array}$$

is natural. Furthermore it follows from the Hurewicz Epimorphism Proposition 166.6 that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_1(X)$ is surjective. It remains to show that $\Phi_{(X,x_0)}$ is injective.

So let $f\colon [0,1]\to X$ be a loop in (X,x_0) such that $\Phi_{(X,x_0)}([f]) = [f\circ\zeta] = 0 \in \mathrm{H}_1(X)$. By definition of $\mathrm{H}_1(X)$ this means that there exists a singular 2-chain $\sum_{j=1}^{s} n_j\cdot\sigma_j$ in X whose boundary is the singular 1-simplex $f\circ\zeta$. Note that, since we do not demand that the σ_j

[1772]If $\pi_1(X,x_0)$ is abelian, then it follows from the elementary discussion on page 1906 that we obtain an isomorphism $\pi_1(X,x_0) \xrightarrow{\cong} \mathrm{H}_1(X)$.

are distinct, we can assume that all n_j are -1 or 1. Given $j \in \{1,\dots,s\}$ and $k \in \{0,1,2\}$ we write[1773] $\mu_{(j,k)} \coloneqq \sigma_j \circ i_k^2 \circ \psi \colon [0,1] \to X$ and $n_{(j,k)} \coloneqq (-1)^k \cdot n_j$. We have the following equality in $\mathrm{C}_1(X)$:

$$f \circ \zeta \;=\; \partial_2\Big(\sum_{j=1}^{s} n_j \cdot \sigma_j\Big) \;=\; \sum_{j=1}^{s}\sum_{k=0}^{2} n_{(j,k)} \cdot (\mu_{j,k} \circ \zeta) \;=\; \sum_{a \in \{1,\dots,s\}\times\{0,1,2\}} \underbrace{n_a}_{=\pm 1} \cdot (\mu_a \circ \zeta).$$

by Definition 143.13 of ∂_2 and since $\psi \circ \zeta = \mathrm{id}_{\Delta^1}$

We make the following choices and we introduce the following definitions:

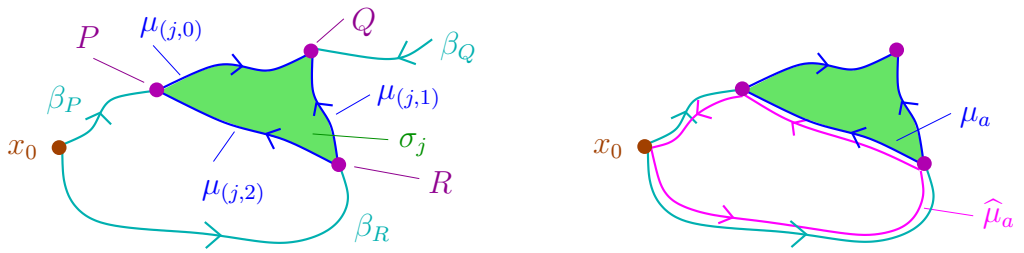
(1) We set $A \coloneqq \{1,\dots,s\} \times \{0,1,2\}$.
(2) By comparing the left and the right-hand side of the above equality we see that there exists a $b \in A$ with $1 \cdot (f \circ \zeta) = n_b \cdot (\mu_b \circ \zeta)$.
(3) We set $A_{\pm} \coloneqq \{a \in A \setminus \{b\} \,|\, n_a = \pm 1\}$.
(4) We obtain from the above equality that

$$\sum_{a \in A_+} (\mu_a \circ \zeta) \;-\; \sum_{a \in A_-} (\mu_a \circ \zeta) \;=\; 0.$$

It follows from the Matching Lemma 166.7 that there exists a bijection $\theta \colon A_+ \to A_-$ such that for each $a \in A_+$ we have $\mu_{\theta(a)} = \mu_a$.
(5) For each point P in X that is a starting point or an endpoint of some μ_a we pick a path β_P from x_0 to P. For $P = x_0$ we just pick the constant path const_{x_0}.
(6) Given $(i,j) \in A$ we set

$$\widehat{\mu}_{(i,j)} \;\coloneqq\; \beta_{\text{starting point of } \mu_{(i,j)}} * \mu_{(i,j)} * \overline{\beta}_{\text{endpoint of } \mu_{(i,j)}}.$$

Note that $\widehat{\mu}_{(i,j)}$ is a loop in (X, x_0), in particular it defines an element in the group $\pi_1(X, x_0)$.



Next note that it follows easily from the description of inverses in the π_1-Group Proposition 88.12 that for any $(i,j) \in \{1,\dots,s\} \times \{0,1,2\}$ we have the following equality in $\pi_1(X, x_0)$:

$$(*) \qquad [\widehat{\mu}_{(i,j)}]^{-1} \;\coloneqq\; [\,\beta_{\text{starting point of } \overline{\mu}_{(i,j)}} * \overline{\mu}_{(i,j)} * \overline{\beta}_{\text{endpoint of } \overline{\mu}_{(i,j)}}].$$

[1773] Recall that for $j \in \{0,\dots,n\}$ we denote by i_j^n the j-th face map

$$\begin{array}{rcl} i_j^n \colon \Delta^{n-1} & \to & \Delta^n \\ (t_0,\dots,t_{n-1}) & \mapsto & (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}). \end{array}$$

Finally note that in $\pi_1(X,x_0)_{\mathrm{ab}}$ we have the following equalities:

$$0 = [e] \underset{\uparrow}{=} \prod_{j=1}^{s} [\beta_{\text{starting point of } \mu_{(j,0)}} * (\overbrace{\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}}^{\text{loop in the starting point of } \mu_{(j,0)}})^{n_j} * \overline{\beta}_{\text{starting point of } \mu_{(j,0)}}]$$

since $\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}$ is null-homotopic

$$\underset{\uparrow}{=} \prod_{j=1}^{s} [\widehat{\mu}_{(j,0)}]^{n_{(j,0)}} \cdot [\widehat{\mu}_{(j,1)}]^{n_{(j,1)}} \cdot [\widehat{\mu}_{(j,2)}]^{n_{(j,2)}} \underset{\uparrow}{=} \prod_{a\in A=\{1,\dots,s\}\times\{0,1,2\}} [\widehat{\mu}_a]^{n_a}$$

follows from inserting null-homotopic paths $\overline{\beta}_P * \beta_P$ together with the above equality (∗) and the fact that $n_{(j,k)} = (-1)^k \cdot n_j$

since $\pi_1(X,x_0)_{\mathrm{ab}}$ is abelian

$$\underset{\uparrow}{=} [f] \cdot \prod_{a\in A_+} [\widehat{\mu}_a]^{n_a} \cdot [\widehat{\mu}_{\theta(a)}]^{n_{\theta(a)}} \underset{\uparrow}{=} [f] \cdot \prod_{a\in A_+} [\widehat{\mu}_a]^{n_a} \cdot [\widehat{\mu}_a]^{-n_a} = [f].$$

since $\{1,\dots,k\}\times\{0,1,2\} = \{b\} \sqcup A_+ \sqcup \theta(A_+)$, since $\mu_b = f$, since $n_b = 1$ and since $\beta_{x_0} = \mathrm{const}_{x_0}$

since for $a \in A_+$ we have $\mu_{\theta(a)} = \mu_a$ and we have $n_{\theta(a)} = -n_a$ ■

166.2. **Applications of the Hurewicz Theorem 166.10 .** For most topological spaces for which we computed the fundamental group we also determined the (first) homology groups. So we do not really gain much new information from the Hurewicz Theorem 166.10. But there is one rather interesting nugget, namely the lens spaces $L(p,q) = S^3/\sim$ which we introduced in Definition 91.4:

Corollary 166.11. (Lens Space-H_1-Corollary) For any coprime $p,q \in \mathbb{N}$ we have

$$\mathrm{H}_1(\text{lens space } L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}.$$

Remark. We will calculate the remaining homology group of lens spaces in the Lens Space-H_*-Lemma 183.21 or alternatively in the Generalized Lens Space-H_*-Proposition 306.7. □

Proof. We have $\mathrm{H}_1(L(p,q)) \underset{\uparrow}{\cong} \pi_1(L(p,q)) \underset{\uparrow}{\cong} \mathbb{Z}/p\mathbb{Z}.$

by the Hurewicz Theorem 166.10 — by the Lens Space-π_1 Proposition 93.2 ■

Remark. The Hurewicz Theorem 166.10 is perhaps a mixed blessing. On the one hand it allows us to use the techniques that we had developed in earlier chapters to determine first homology groups. On the other hand it tells us that all the information that is contained in the first homology group is already contained in the fundamental group. So we do not get any new information. In particular, as we saw in the Lens Space-H_1-Corollary 166.11, the first homology group cannot help us in our long-standing goal of distinguishing lens spaces with the same fundamental group. □

The task of calculating (isomorphism types) of homology groups can get quite involved. But what is often even harder is to determine the maps on homology groups induced by maps between topological spaces. Using the Hurewicz Homomorphism Proposition 166.3 (3) we can transfer calculations of induced maps on fundamental groups to obtain the corresponding induced maps on first homology groups. We will now see one instance where this observation turns out to be helpful.

In the Torus-H_*-Proposition 153.9 we calculated the homology groups of tori. The following lemma gives an alternative calculation of the first homology group, and more importantly, gives a nice calculation of induced maps on first homology groups.

Lemma 166.12. (Action-on-Torus Homology Lemma) Let $n \in \mathbb{N}$. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n =_{\mathrm{i}} (S^1)^n$ the n-dimensional torus.

(1) We denote by $[S^1] \in \mathrm{H}_1(S^1)$ the standard generator that we introduced in Definition 150.8. Given $i \in \{1, \ldots, n\}$ let

$$\begin{array}{rcl} \iota_i \colon S^1 & \to & (S^1)^n \\ z & \mapsto & (1, \ldots, 1, z, 1, \ldots, 1) \end{array}$$

be the obvious i-th inclusion map. If we set $v_i := \iota_{i*}([S^1])$, then $v_1, \ldots, v_n$ is a basis of $\mathrm{H}_1(T)$.

(2) Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. We consider the map[1774]

$$\begin{array}{rcl} f(A) \colon T = \mathbb{R}^n/\mathbb{Z}^n & \to & T = \mathbb{R}^n/\mathbb{Z}^n \\ [v] & \mapsto & [A \cdot v]. \end{array}$$

With respect to the basis $v_1, \ldots, v_n$ of $\mathrm{H}_1(T)$ which we introduced in (1), the induced map $f(A)_* \colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)$ is given by multiplication by the matrix A.

Remark.

(1) In Exercise 207.2 we will study what self-maps of $\mathrm{H}_2(S^2 \times S^2)$ can be realized by continuous self-maps of $S^2 \times S^2$.

(2) In Exercise 153.15 we gave an alternative proof of the Action-on-Torus Homology Lemma 166.12 (2). □

Proof.

(1) Recall that on page 2866 we saw that the singular 1-simplex $\sigma \colon \Delta^1 \to S^1$ that is given by $(1-t, t) \mapsto \exp(2\pi \mathrm{i} t)$ represents the standard generator $[S^1]$. The promised statement follows easily from this observation, the Action–π_1-Theorem 93.1 and the Hurewicz Theorem 166.10.

(2) Let $\Theta \colon \pi_1(T, [0]) \to \mathbb{Z}^n$ be the isomorphism from the Action–π_1-Theorem 93.1. By the Hurewicz Homomorphism Proposition 166.3 (3), the Action–π_1-Theorem 93.1 and Exercise 93.8 we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(T) & \xleftarrow{\Phi_{(T,[0])}} & \pi_1(T,[0]) & \xrightarrow[\cong]{\Theta} & \mathbb{Z}^n \\ \downarrow{\scriptstyle f(A)_*} & & \downarrow{\scriptstyle f(A)_*} & & \downarrow{\scriptstyle v \mapsto A \cdot v} \\ \mathrm{H}_1(T) & \xleftarrow{\Phi_{(T,[0])}} & \pi_1(T,[0]) & \xrightarrow[\cong]{\Theta} & \mathbb{Z}^n. \end{array}$$

The desired statement follows from the observation that the horizontal isomorphism $\Theta \circ \Phi_{(T,[0])} \colon \mathrm{H}_1(T) \to \mathbb{Z}^n$ is precisely the one corresponding to the basis $v_1, \ldots, v_n$. ■

166.3. Exercises for Chapter 166 .

Exercise 166.1. Show that the abelianization functor (which we introduced in Definition 98.15) from the category *Group* of groups to the category *AbGroup* of abelian groups is

[1774] If $A \in \mathrm{GL}(n, \mathbb{Z})$, then it follows from the Maps-on-Torus Lemma 25.46 that this map is a homeomorphism.

right-exact and that it is not left-exact.
Remark. This exercise is identical to Exercise 161.2.

Exercise 166.2. Let $p\colon S^1 \times S^1 \to S^1$ and $q\colon S^1 \times S^1 \to S^1$ be the projection onto the first respectively second factor and let $\mu\colon S^1 \times S^1 \to S^1$ be the multiplication map. Show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(S^1 \times S^1) & \xrightarrow{\quad\mu_*\quad} & \mathrm{H}_1(S^1). \\ & {\scriptstyle p_*\oplus q_*}\searrow \quad \nearrow{\scriptstyle (\varphi,\psi)\mapsto \varphi+\psi} & \\ & \mathrm{H}_1(S^1) \oplus \mathrm{H}_1(S^1) & \end{array}$$

Exercise 166.3. Let $T = S^1 \times S^1$ be the torus. Let $\varphi, \psi\colon S^1 \to T$ be two continuous maps such that $\varphi_*([S^1]) = \psi_*([S^1]) \in \mathrm{H}_1(T)$. Show that φ and ψ are homotopic maps.

Exercise 166.4. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. Prove the following statements:
(a) If $\mathrm{tr}(A) = 2$, then A is conjugate over $\mathbb{Z}$ to a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
(b) If $\mathrm{tr}(A) = -2$, then A is conjugate over $\mathbb{Z}$ to a matrix of the form $\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$.
(c) If $\mathrm{tr}(A) \in (-2,2)$, then A is a matrix of finite order in the group $\mathrm{SL}(2,\mathbb{Z})$.
(d) If $\mathrm{tr}(A) \notin [-2,2]$, then A has two real eigenvalues λ, λ^{-1} with $|\lambda| > 1$.

Exercise 166.5. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rccc} f(A)\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [A \cdot v] \end{array}$$

and following Definition 21.19 we consider the corresponding mapping torus

$$\mathrm{Tor}(\mathbb{R}^2/\mathbb{Z}^2, f(A)) \;=\; ((\mathbb{R}^2/\mathbb{Z}^2) \times [0,1])/(x,0) \sim (f(A)(x),1).$$

(a) Show that $\mathrm{Tor}(\mathbb{R}^2/\mathbb{Z}^2, f(A))$ is a closed oriented 3-dimensional smooth manifold.
(b) Determine $\mathrm{rank}(\mathrm{H}_1(\mathrm{Tor}(\mathbb{R}^2/\mathbb{Z}^2, f(A))$ and $\mathrm{rank}(\mathrm{H}_2(\mathrm{Tor}(\mathbb{R}^2/\mathbb{Z}^2, f(A)))$ in terms of the classification of Exercise 166.4.
Remark. You might want to use the Mapping Torus-H_*-Proposition 153.6 and the Action-on-Torus Homology Lemma 166.12.
(c) Show that $\mathrm{H}_2(\mathrm{Tor}(\mathbb{R}^2/\mathbb{Z}^2, f(A)))$ is torsion-free.

Exercise 166.6. Show that there exist continuous maps $f, g\colon S^1 \times S^1 \to S^1 \times S^1$ such that $f \circ g$ and $g \circ f$ are not homotopic.
Remark. This exercise is the content of Exercise 46.6 (b). But with the methods of this chapter it is much easier to solve this exercise.

Exercise 166.7. Let $p\colon Y \to X$ be a covering of degree d of path-connected topological spaces. Use the Hurewicz Theorem 166.10 and the Covering-Degree Proposition 92.10 to show that the cardinality of the cokernel of the map $p_*\colon \mathrm{H}_1(Y) \to \mathrm{H}_1(X)$ is at most d.
Remark. Note that for first homology groups this exercise is a slight refinement of Exercise 165.2.

Exercise 166.8. Let $p\colon \widetilde{X} \to X$ be a finite covering. Use the Hurewicz Theorem 166.10 to show that the induced map $p\colon \mathrm{H}_1(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_1(X;\mathbb{Q})$ is an epimorphism.

Remark. In the Transfer–Projection Proposition 165.4 we showed, using a totally different argument, that this conclusion holds for rational homology groups in all dimensions.

Exercise 166.9. To what degree does the combination of the Seifert–van Kampen-Theorem 99.1 and the Hurewicz Theorem 166.10 imply the "low degree" part of the Mayer–Vietoris Theorem 153.2?

Exercise 166.10. Let X and Y be two path-connected non-empty topological spaces. Use the Hurewicz Theorem 166.10 to relate the Product-π_1-Proposition 90.13 to the Künneth Theorem 164.6.

Exercise 166.11. In the Hurewicz Theorem 166.10 we have the hypothesis that "X is path-connected". Can we replace this hypothesis by "X is connected"?

Exercise 166.12. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. Show that there exists a natural epimorphism $\pi_1(X, x_0) \to \mathrm{H}_1(X; \mathbb{Z}/n\mathbb{Z})$.
Hint. You could consider Exercise 162.12.

Exercise 166.13. Let X be a path-connected 2-dimensional CW-complex with Euler characteristic $\chi(X) = -3$ and with $\mathrm{H}_2(X; \mathbb{F}_3) \cong \mathbb{F}_3$. Does X necessarily admit a path-connected 5-fold covering $p\colon \widetilde{X} \to X$?

Exercise 166.14. Let $p, q \in \mathbb{N}$ be coprime. We consider the lens space $L(p, q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ which we introduced in Definition 91.4. To what degree can you determine the homology groups of the lens space $L(p, q) = S^3/(\mathbb{Z}/p\mathbb{Z})$? Throw everything you have at this problem. For example we already calculated in the Lens Space-π_1-Proposition 93.2 that $\pi_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$. Furthermore we know by the Smooth Manifold-CW-Theorem 179.9 that $L(p, q)$ admits a finite 3-dimensional CW-structure.

Exercise 166.15. Let X be a connected CW-complex, let $A \subseteq X$ be a connected subcomplex and let $P \in A$. Let $i\colon A \to X$ be the inclusion and let $\pi\colon X \to X/A$ be the projection. We consider the following diagram:

$$\begin{array}{ccccccc}
\pi_1(A, P) & \xrightarrow{i_*} & \pi_1(X, P) & \xrightarrow{\pi_*} & \pi_1(X/A, [P]) & \longrightarrow & \{e\} \\
\downarrow{\scriptstyle \Phi_{(A,P)}} & & \downarrow{\scriptstyle \Phi_{(X,P)}} & & \downarrow{\scriptstyle \Phi_{(X/A,[P])}} & & \downarrow \\
\mathrm{H}_1(A) & \xrightarrow{i_*} & \mathrm{H}_1(X) & \xrightarrow{\pi_*} & \mathrm{H}_1(X/A) & \longrightarrow & 0.
\end{array}$$

(a) Does the diagram commute?
(b) Is the top horizontal sequence exact?
(c) Is the bottom horizontal sequence exact?

Exercise 166.16. Does there exist a topological space X with $\mathrm{H}_1(X) = 0$ which admits a finite covering $p\colon \widetilde{X} \to X$ such that $\mathrm{H}_1(\widetilde{X}) \neq 0$?

Exercise 166.17. Let $p, q \in \mathbb{N}$ be coprime. We consider the corresponding lens space $L(p, q) = S^3/\sim$ from Definition 91.4. For which $k \in \mathbb{Z}$ does there exist a self-homeomorphism $f\colon L(p, q) \to L(p, q)$ such that the map $f_*\colon \mathrm{H}_1(L(p, q)) \to \mathrm{H}_1(L(p, q))$ is given by multiplication by k?

CHAPTER 167

The Hurewicz Theorem in higher dimensions

The main goal of this chapter is to formulate and prove a suitable high-dimensional analog of the Hurewicz Theorem 166.10 from Chapter 166. Furthermore in this chapter and in Chapters 168 and 169 we will discuss several applications of the higher-dimensional Hurewicz Theorem.

This chapter is organized as follows:

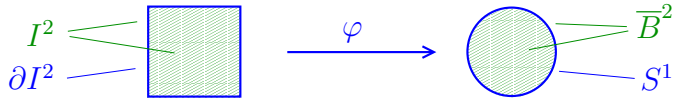
- In Section 167.1 we will introduce the Hurewicz homomorphism in higher dimensions.
- In Section 167.2 we will discuss first applications of the Hurewicz homomorphism.
- In Section 167.3 we will state the Hurewicz Theorem 167.9.
- In Section 167.4 we will provide a proof of the Hurewicz Theorem 167.9, which builds on the n-Connected–Chain Group Proposition 167.17 and the Homotopy-via-Homology Proposition 167.20.
- In Section 167.5 we will prove the n-Connected–Chain Group Proposition 167.17.
- In Section 167.6 we will provide a proof of the Homotopy-via-Homology Proposition 167.20, which builds on the technical Faces–Addition Lemma 167.23.
- In Appendix 167.7 we will start out with the proof of the Faces–Addition Lemma 167.23.
- In Appendix 167.8 we will conclude the proof of the Faces–Addition Lemma 167.23 II.
- In Section 167.9 we will provide exercises for Chapter 167.

In the next Chapter 168 we will use the Hurewicz Theorem 167.9 to discuss the "social choice problem".

167.1. **The Hurewicz homomorphism in higher dimensions.** The Hurewicz Theorem 166.10 evidently raises the question whether there is a connection between higher homotopy groups and higher homology groups. As in the 1-dimensional case we first try to find a promising map from homotopy groups to homology groups.

We start out with introducing the following notation:

Notation 167.1.
(1) We write $I := [0,1]$.
(2) Given $n \in \mathbb{N}_0$ we denote by $I^n \subseteq \mathbb{R}^n$ the cube in $\mathbb{R}^n$.
(3) We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
(4) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 2578.
(5) Let $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ be the standard generator from Definition 155.11.[1775] We refer to $[I^n] := (\varphi^{-1})_*([\overline{B}^n]) \in \mathrm{H}_n(I^n, \partial I^n)$ as the standard generator of $\mathrm{H}_n(I^n, \partial I^n)$.



Now comes the actual definition we are interested in.

Definition 167.2. Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. Note that it follows from the LES-of-Pair Proposition 146.7 together with the Singleton-H_*-Lemma 143.26 that the natural map $\text{я}\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X, \{x_0\})$ is an isomorphism. We refer to the map

$$\begin{array}{rcccc} \Phi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n(X,\{x_0\}) & \xrightarrow[\cong]{\text{я}^{-1}} & \mathrm{H}_n(X) \\ [f\colon (I^n,\partial I^n) \to (X,x_0)] & \mapsto & f_*([I^n]) & & \end{array}$$

as the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$.[1776]

Remark. It follows rather easily from the explicit description of the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ on page 2958 and the fact that the homeomorphism $\varphi\colon I^1 \to \overline{B}^1$ from page 2578 is the map $t \mapsto 2t-1$, that for $n=1$ the above Definition 167.2 of the Hurewicz homomorphism agrees with Definition 166.5. □

In the next proposition, which can be viewed as an analog of the Hurewicz Homomorphism Proposition 166.3, we show in particular that the Hurewicz homomorphism is in fact a homomorphism:

Proposition 167.3. (Hurewicz Homomorphism Proposition) Let $n \in \mathbb{N}$.

(1) For any pointed topological space (X, x_0) the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$$

is indeed a homomorphism.

(2) Let X be a topological space and let $\gamma\colon I \to X$ be a path from a point x_0 to some point x_1. The following diagram commutes:

$$\begin{array}{ccc} \pi_n(X,x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \mathrm{H}_n(X) \\ {\scriptstyle \gamma_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_n(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X). \end{array}$$

Here the vertical map to the left is defined in the Change-of-Base Point Propositions 88.14 (1) and 134.12.

(3) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category of groups. The Hurewicz homomorphisms define a natural transformation from the functor

$$\begin{array}{ccc} \mathcal{PTop} & \to & \mathcal{Group} \\ (X,x_0) & \mapsto & \pi_n(X,x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{PTop} & \to & \mathcal{Group} \\ (X,x_0) & \mapsto & \mathrm{H}_n(X). \end{array}$$

(4) Let (X, x_0) be a pointed topological space. If we take the point of view from page 2578, i.e. if we view elements in the n-th homotopy group $\pi_n(X,x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$, then the Hurewicz homomorphism is given by the map

$$\begin{array}{ccc} \pi_n(X,x_0) & \to & \mathrm{H}_n(X) \\ [f\colon (S^n,*) \to (X,x_0)] & \mapsto & f_*([S^n]) \end{array}$$

[1775]For the most part the precise choice of the generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is irrelevant.

[1776]It follows from the Homotopy-H_*-Proposition 144.12 that the Hurewicz homomorphism is well-defined.

where $[S^n] \in \mathrm{H}_n(S^n)$ is the standard generator from Definition 150.8.

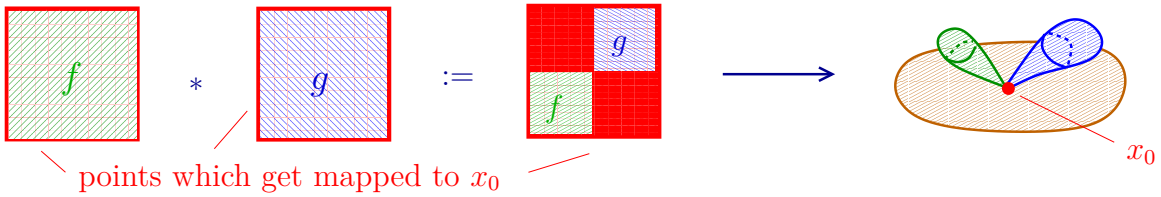
The proof of the Hurewicz Homomorphism Proposition 167.3 will require the remainder of this section. The reader who instead wants to see first applications can jump to Section 167.2.

The key ingredient in the proof of the Hurewicz Homomorphism Proposition 167.3 (1) is the following technical lemma:

Lemma 167.4. (Multiplication–Addition Lemma) Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$, let $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ and let $f, g\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be two continuous maps. Then

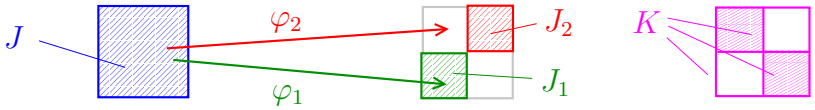
$$f_*(\omega) + g_*(\omega) \;=\; (f * g)_*(\omega) \;\in\; \mathrm{H}_n(X, x_0).$$

($f * g$: product of f and g from Definition 134.4)



Proof of the Multiplication–Addition Lemma 167.4. We introduce the following notation:

(1) We write $J \coloneqq I^n$, $J_1 \coloneqq [0, \frac{1}{2}]^n$ and $J_2 \coloneqq [\frac{1}{2}, 1]^n$.
(2) We write $K \coloneqq J \setminus (\mathring{J}_1 \cup \mathring{J}_2)$.
(3) For $k = 1, 2$ we denote by $i_k\colon (J_k, \partial J_k) \to (J, K)$ the inclusion map.
(4) We denote by $\varphi_k\colon (J, \partial J) \to (J_k, \partial J_k)$ the obvious homeomorphism given by reparametrizing the coordinates.



Claim. Given any $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ we have

$$i_*(\omega) \;=\; i_{1*}(\varphi_{1*}(\omega)) + i_{2*}(\varphi_{2*}(\omega)) \;\in\; \mathrm{H}_n(J, K).$$

Proof. Let $\omega \in \mathrm{H}_n(I^n, \partial I^n)$. For $s = 1, 2$ we write $\omega_s = \varphi_{s*}(\omega) \in \mathrm{H}_n(J_s, \partial J_s)$. To prove the desired equality we consider the following diagram:

$$\begin{array}{ccccc}
 & & \mathrm{H}_n(J_1, \partial J_1) & \xrightarrow{k_{1*}} & \mathrm{H}_n(J, J \setminus \mathring{J}_1) \\
 & \overset{\varphi_{1*}}{\nearrow} & \downarrow{\scriptstyle i_{1*}} & \overset{j_{1*}}{\nearrow} & \\
\mathrm{H}_n(J, \partial J) & \xrightarrow{i_*} & \mathrm{H}_n(J, K) & & \\
 & \underset{\varphi_{2*}}{\searrow} & \uparrow{\scriptstyle i_{2*}} & \underset{j_{2*}}{\searrow} & \\
 & & \mathrm{H}_n(J_2, \partial J_2) & \xrightarrow[k_{2*}]{} & \mathrm{H}_n(J, J \setminus \mathring{J}_2),
\end{array}$$

where the five maps i, j_1, j_2, k_1, k_2 are the obvious inclusion maps of pairs of topological spaces. Note that the two triangles to the right commute by the functoriality of relative homology groups. Furthermore note that one can easily verify that the two maps $j_s \circ i$ and

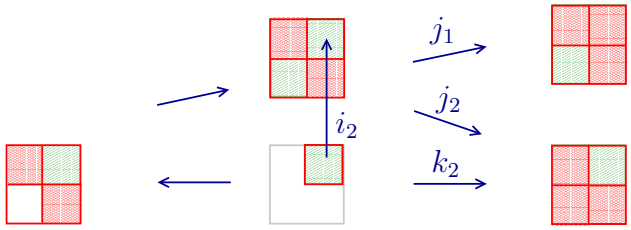
$k_s \circ \varphi_s \colon (J, \partial J) \to (J, J \setminus \mathring{J}_s)$ are homotopic as maps of pairs of topological spaces. Thus it follows from the Homotopy-H_*-Proposition 146.11 that the top and bottom parallelogram commute.[1777] Next note that

$$\begin{array}{ccccccc} & \text{since the top parallelogram commutes} & & \text{since the top right triangle commutes} & & \text{since } (j_1 \circ i_2)_* \colon \mathrm{H}_n(J_2, \partial J_2) \to \mathrm{H}_n(J, J \setminus \mathring{J}_1) \text{ is zero we have } j_{1*} \circ i_{2*} = (j_1 \circ i_2)_* = 0 & \\ j_{1*}(i_*(\omega)) & \overset{\downarrow}{=} & (k_{1*} \circ \varphi_{1*})(\omega) & = & k_{1*}(\omega_1) & \overset{\downarrow}{=} & j_{1*}(i_{1*}(\omega_1)) \overset{\downarrow}{=} j_{1*}(i_{1*}(\omega_1)) + j_{1*}(i_{2*}(\omega_2)) \\ & & & & & & = j_{1*}(i_{1*}(\omega_1) + i_{2*}(\omega_2)). \end{array}$$

and similarly we show that $j_{2*}(i_*(\omega)) = j_{2*}(i_{1*}(\omega_1)) + j_{2*}(i_{2*}(\omega_2)) = j_{2*}(i_{1*}(\omega_1) + i_{2*}(\omega_2))$.

To prove the desired equality $i_*(\omega) = i_{1*}(\varphi_{1*}(\omega)) + i_{2*}(\varphi_{2*}(\omega))$ it now remains to show that $\ker(j_{1*}) \cap \ker(j_{2*}) = 0$. We consider the following diagram:

$$\begin{array}{ccccc} & & & & \mathrm{H}_n(J, J \setminus \mathring{J}_1) \\ & & & \overset{j_1}{\nearrow} & \\ & & \mathrm{H}_n(J, K) & & \\ & \nearrow & \uparrow{\scriptstyle i_2} & \searrow{\scriptstyle j_2} & \\ \mathrm{H}_n(J \setminus \mathring{J}_1, K) & \xleftarrow[\cong]{} & \mathrm{H}_n(J_2, \partial J_2) & \xrightarrow[\cong]{k_2} & \mathrm{H}_n(J, J \setminus \mathring{J}_2). \end{array}$$



We make the following clarifications and observations:

(1) The unnamed maps are induced by the obvious inclusions of pairs of topological spaces.
(2) The diagram commutes by the functoriality of relative homology.
(3) The diagonal sequence is exact by the LES-of-Triple Proposition 146.5.
(4) It follows from the CW-Excision Theorem 148.4 that the two bottom horizontal maps are isomorphisms.
(5) It now follows from some mild diagram chasing that $\ker(j_{1*}) \cap \ker(j_{2*}) = 0$. ⊞

Now we can prove the actual statement of the lemma.

Claim 2. For any $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ we have the following equality:

$$f_*(\omega) + g_*(\omega) \;=\; (f * g)_*(\omega) \;\in\; \mathrm{H}_n(X, x_0).$$

Proof. The claim follows from the following equalities:

$$\begin{array}{ccccc} & \text{since } (f*g)(K) = \{x_0\} & & \text{by Claim 1} & \\ (f*g)_*(\omega) & \overset{\downarrow}{=} & (f*g)_*(i_*(\omega)) & \overset{\downarrow}{=} & (f*g)_*(i_{1*}(\varphi_{1*}(\omega)) + i_{2*}(\varphi_{2*}(\omega))) \\ & & & & = (f*g)_*(i_{1*}(\varphi_{1*}(\omega)) + (f*g)_*(i_{2*}(\varphi_{2*}(\omega))) \\ & & & & = f_*(\omega) + g_*(\omega). \\ & & & & \uparrow \\ & & & & \text{since } (f*g) \circ i_1 \circ \varphi_1 = f \text{ and} \\ & & & & \text{since } (f*g) \circ i_2 \circ \varphi_2 = g \end{array}$$

[1777] The triangles to the left do not commute, but this is irrelevant.

■

The proof of the Hurewicz Homomorphism Proposition 167.3 is now, except for one technicality, almost a breeze:

Proof of the Hurewicz Homomorphism Proposition 167.3.

(1) We need to show that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$ is indeed a homomorphism. Let $[I^n] \in \mathrm{H}_n(I^n,\partial I^n)$ be the standard generator. Note that it follows immediately from the Multiplication–Addition Lemma 167.4 that for all continuous maps $f,g\colon (I^n,\partial I^n) \to (X,\{x_0\})$ we have the desired equality $f_*([I^n]) + g_*([I^n]) = (f*g)_*([I^n]) \in \mathrm{H}_n(X,x_0)$.

(2) Let X be a topological space and let $\gamma\colon I \to X$ be a path from a point x_0 to some point x_1. We need to show that the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X,x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \mathrm{H}_n(X) \\ {\scriptstyle\gamma_*}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\ \pi_n(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X). \end{array}$$

We start out with the following technical claim, which later will allow us to describe the Hurewicz homomorphism in an alternative way:

Claim. Let $g\colon (I^n,\partial I^n) \to (X,x)$ be a continuous map. Note that g descends to a continuous map $\overline{g}\colon I^n/\partial I^n \to X$. We denote by $p\colon X \to X/\{x\}$ the natural projection. With this notation we obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_n(I^n,\partial I^n) & \xrightarrow{g_*} & \mathrm{H}_n(X,\{x\}) & \xleftarrow{\text{я}} & \mathrm{H}_n(X) \\ \downarrow{\scriptstyle\text{ю}} & & \downarrow{\scriptstyle\text{ю}} & \overset{\text{ж}_{(X,\{x\})}}{\swarrow} & \uparrow{\scriptstyle\text{ж}} \\ \widetilde{\mathrm{H}}_n(I^n/\partial I^n) & \xrightarrow{(p\circ\overline{g})_*} & \widetilde{\mathrm{H}}_n(X/\{x\}) & \xleftarrow{p_*} & \widetilde{\mathrm{H}}_n(X). \\ & \searrow^{\overline{g}_*} & & & \end{array}$$

We claim that this diagram commutes.

Proof. The square to the left commutes by the Relative-to-Quotient-H_*-Lemma 149.11 (2). The triangle to the top right commutes by definition, see Notation 147.7. The triangle to the bottom right commutes by the Relative-to-Quotient-H_*-Lemma 149.11 (1). The bottom part commutes by the functoriality of reduced homology. ⊞

Now let $f\colon (I^n,\partial I^n) \to (X,x_1)$ be a continuous map and let $\gamma\colon I \to X$ be a path from a point x_0 to some point x_1. As we mentioned above, f descends to a continuous map $\overline{f}\colon I^n/\partial I^n \to X$ and f^γ descends to a continuous map $\overline{f^\gamma}\colon I^n/\partial I^n \to X$.



One can easily write down an explicit homotopy $F\colon I^n \times [0,1] \to X$ from $F_0 = f$ to $F_1 = f^\gamma$ such that for each $t \in [0,1]$ the image $F_t(\partial I^n)$ consists of a single point. It follows from the Homotopy Quotient Theorem 18.11 that F descends to a homotopy $\overline{F}\colon (I^n/\partial I^n) \times [0,1] \to X$ from $\overline{f}$ to $\overline{f^\gamma}$.

We now obtain the following equalities in $\mathrm{H}_n(X,\{x_1\})$:[1778]

$$\begin{array}{ccccc} & \text{by Definition 167.2 of } \Phi_{(X,x_1)} & & \text{since the diagram commutes} & & \text{by the Homotopy-H}_*\text{-Proposition 144.12 since } \overline{f} \text{ and } \overline{f^\gamma} \text{ are homotopic} \\ & \downarrow & & \downarrow & & \downarrow \\ \text{я}(\Phi_{(X,x_1)}([f])) & = & f_*([I^n]) & = & \text{ж}_{(X,\{x\})}(\overline{f}_*(\text{ю}([I^n]))) & = & \text{ж}_{(X,\{x\})}(\overline{f}^\gamma_*(\text{ю}([I^n]))) \\ & = & f^\gamma_*([I^n]) & = & \text{я}(\Phi_{(X,x_0)}([f^\gamma])) & = & \text{я}(\Phi_{(X,x_0)}(\gamma_*([f]))). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{since the diagram commutes} & & \text{by Definition 167.2 of } \Phi_{(X,x_0)} & & \text{by definition of } \gamma_* & \end{array}$$

Since $n \geq 1$ we obtain from the LES-of-Pair Proposition 146.7 together with the Singleton-H_*-Lemma 143.26 that я is an isomorphism. It follows that the diagram given in the statement is commutative.

(3) This statement follows immediately from the functoriality of relative homology groups.

(4) This statement is a straightforward consequence of the Standard Generator-of-$\overline{B}^n$-Lemma 155.29. ∎

167.2. **First applications of the Hurewicz homomorphism.** The following lemma gives in particular an affirmative answer to Question 134.23 (1).

Lemma 167.5. (Hurewicz Sphere Epimorphism Lemma) For any $n \in \mathbb{N}$ the Hurewicz homomorphism

$$\Phi_{(S^n,*)} \colon \pi_n(S^n,*) \;\to\; \mathrm{H}_n(S^n) \cong \mathbb{Z}$$

sends $[\mathrm{id}_{S^n} \colon S^n \to S^n] \in \pi_n(S^n)$ to the standard generator of $\mathrm{H}_n(S^n)$, in particular this Hurewicz homomorphism is an epimorphism.

Proof. We denote by $[S^n] \in \mathrm{H}_n(S^n)$ the standard generator. Note that

$$\Phi_{(S^n,*)}(\underbrace{[\mathrm{id}_{S^n} \colon S^n \to S^n]}_{\in \pi_n(S^n,*)}) \;\underset{\text{by the Hurewicz Homomorphism Proposition 167.3 (4)}}{=}\; (\mathrm{id}_{S^n})_*([S^n]) \;=\; [S^n].$$

Thus we see that the standard generator of $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ lies in the image of the Hurewicz homomorphism. But this implies that the Hurewicz homomorphism $\Phi \colon \pi_n(S^n,*) \to \mathrm{H}_n(S^n)$ is an epimorphism. ∎

The fairly simple-minded Hurewicz Sphere Epimorphism Lemma 167.5 actually already turns out to be quite useful. As an example, recall that on page 338 we showed that we can view the real projective space $\mathbb{R}\mathrm{P}^m$ as a subspace of the projective space $\mathbb{R}\mathrm{P}^n$. Now we can formulate the following application of the Hurewicz Sphere Epimorphism Lemma 167.5.

Proposition 167.6. ($\mathbb{R}\mathrm{P}^n$-Non-Retraction Proposition) If $1 \leq m < n$, then the subspace $\mathbb{R}\mathrm{P}^m \subseteq \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

Proof. [1779] Note that

[1778] This argument might seem somewhat convoluted. But to the author it does not seem like there is an easy way out, since we cannot appeal to the Homotopy-H_*-Proposition 146.11 since we do not have homotopies of maps of pairs of topological spaces.

[1779] We gave a somewhat different proof of the proposition in Exercise 149.16.

$$\pi_m(\mathbb{R}\mathrm{P}^m) \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_m(S^m) \underset{\text{by the Hurewicz Epimorphism Sphere Lemma 167.5}}{\neq} 0 \qquad \text{and} \qquad \pi_m(\mathbb{R}\mathrm{P}^n) \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_m(S^n) \underset{\text{follows from } m<n \text{ and the Sphere–Low-}\pi_n\text{-Proposition 134.19}}{=} 0.$$

With this calculation, and the fact that π_m is functorial, the argument of the Circle–Non Retract Proposition 90.6 can be easily modified to show that $\mathbb{R}\mathrm{P}^m$ is not a retract of $\mathbb{R}\mathrm{P}^n$. ■

The $\mathbb{R}\mathrm{P}^n$-Non-Retraction Proposition 167.6 raises the following question:

Question 167.7. Let $m < n$. Is the complex projective space $\mathbb{C}\mathrm{P}^m$ a retract of $\mathbb{C}\mathrm{P}^n$?

We will answer Question 167.7 in the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 213.15.

167.3. **The Hurewicz Theorem.** It is natural to ask whether, given a path-connected topological space X and $n \geq 2$, the Hurewicz homomorphism

$$\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \to \mathrm{H}_n(X)$$

is necessarily an isomorphism. Since $n \geq 2$ we know by the π_n-Abelian Proposition 134.11 that the group $\pi_n(X, x_0)$ is abelian. So this time, in contrast to the situation for π_1 which we discussed on page 3182, there is no obvious obstruction to this map being a monomorphism.

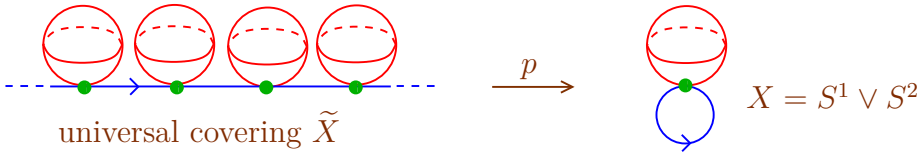
Nonetheless, for better or worse, browsing through our examples shows that the Hurewicz homomorphism is in general neither an epimorphism nor a monomorphism:

(1) In the Torus-H_*-Proposition 153.9 (1) we showed that $\mathrm{H}_2(S^1 \times S^1) \cong \mathbb{Z}$ is non-zero, whereas on page 2587 we showed that $\pi_2(S^1 \times S^1) = 0$. This shows that the Hurewicz homomorphism $\pi_2(S^1 \times S^1) \to \mathrm{H}_2(S^1 \times S^1)$ is not an epimorphism.

(2) Here are two examples that show that in general the Hurewicz homomorphism is not a monomorphism:

(a) We consider $X = \mathbb{R}\mathrm{P}^2$. By the Covering-$\pi_n$-Isomorphism Proposition 134.22 and the Hurewicz Sphere Epimorphism Lemma 167.5 we know that $\pi_2(\mathbb{R}\mathrm{P}^2) \cong \pi_2(S^2)$ is non-trivial. On the other hand we saw in the Projective Plane-H_*-Lemma 153.13 (or alternatively the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18) that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) = 0$.

(b) We consider $X = S^1 \vee S^2$. It follows immediately from the Wedge-H_*-Proposition 149.16 and the Sphere-H_*-Proposition 149.4 that $\mathrm{H}_2(S^1 \vee S^2) \cong \mathbb{Z}$. On the other hand we know by the Covering-π_n-Isomorphism Proposition 134.22 and the discussion on page 2466 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$, where $\widetilde{X}$ is the real line "with infinitely many 2-spheres attached".



Using the projections onto the various spheres and using the Hurewicz Sphere Epimorphism Lemma 167.5 one can now show that $\pi_2(\widetilde{X})$ admits an epimorphism onto $\mathbb{Z}^\infty$.[1780] It follows that there is no monomorphism from $\pi_2(X) = \pi_2(\widetilde{X})$ into

[1780] The proof of this statement is a slight variation on the proof of Exercise 134.4.

$\mathrm{H}_2(X) \cong \mathbb{Z}$. In particular the Hurewicz homomorphism $\pi_2(S^1 \vee S^2) \to \mathrm{H}_2(S^1 \vee S^2)$, whatever it is precisely, cannot be a monomorphism.

The above examples show that we need to put some hypotheses on X for a higher Hurewicz homomorphism to be an isomorphism. Some quick thinking leads us to the following definition:

Definition 167.8. Let $k \in \mathbb{N}_0$. We say that a topological space X is k-connected if $\pi_i(X) = 0$ for $i = 0, \dots, k$.[1781] [1782]

The main goal of this chapter is to prove the following theorem which was first proved by Witold Hurewicz [**Hur1935b**]. It can be viewed as the high-dimensional analog of the original Hurewicz Theorem 166.10.

Theorem 167.9. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. Furthermore let $n \in \mathbb{N}_{\geq 2}$. If X is $(n-1)$-connected, then the following two statements hold:

(1) We have $\widetilde{\mathrm{H}}_k(X) = 0$ for $k = 0, \dots, n-1$.
(2) The Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \to \mathrm{H}_n(X)$ is an isomorphism.

Remark.

(1) The above examples of $X = \mathbb{R}\mathrm{P}^2$ and $X = S^1 \vee S^2$ show that in general the hypothesis in the Hurewicz Theorem 167.9 that X is $(n-1)$-connected cannot be dropped.
(2) In the Relative Hurewicz Theorem 296.25 we will prove a relative version of the Hurewicz Theorem 167.9 that relates "relative homotopy groups" to relative homology groups.
(3) In the Hurewicz-plus-One Theorem 307.34 we will prove an addendum to the Hurewicz Theorem 167.9, which says in particular that for $n \geq 2$ and an $(n-1)$-connected topological space the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_{n+1}(X, x_0) \to \mathrm{H}_{n+1}(X)$ is an epimorphism. □

Using the Hurewicz Theorem 167.9 we can now determine some homotopy groups that had been eluding us for a long time:

Proposition 167.10. ($\pi_n(S^n)$–Proposition) For any $n \in \mathbb{N}$ we have[1783] [1784]

$$\pi_n(S^n) \;=\; \mathbb{Z} \cdot [\mathrm{id}_{S^n}].$$

Proof. Let $n \in \mathbb{N}$. By the Sphere–Low-π_n-Proposition 134.19 we know that S^n is $(n-1)$-connected. Thus the corollary follows immediately from the Hurewicz Theorems 166.10

[1781] Recall that according to our Definition 4.43 we write $\pi_0(X) = 0$ if and only if X is path-connected and *non-empty*. Furthermore note that by Definition 88.16 a topological space X is 1-connected if and only if it is simply connected.
[1782] Note that a k-connected space is in particular 0-connected and thus path-connected. It follows from the Change-of-Base Point Proposition 134.12 that we can ignore base points.
[1783] Here, following the discussion on page 2578, we view $\pi_n(X, x_0)$ as the set of homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$.
[1784] In the $\pi_n(S^n)$–Proposition 134.31 we gave an alternative proof of this result using the Hopf Degree Theorem 46.22.

and 167.9, the discussion of the Hurewicz homomorphism in the Hurewicz Sphere Epimorphism Lemma 167.5, and the fact that by the Sphere-H_*-Proposition 149.4 we know that $\mathrm{H}_n(S^n) = \mathbb{Z} \cdot [S^n]$. ■

Corollary 167.11. ($\pi_n(\mathbb{R}\mathrm{P}^n)$–Corollary) For any $n \in \mathbb{N}$ we have

$$\pi_n(\mathbb{R}\mathrm{P}^n, *) \;=\; \mathbb{Z} \cdot \big[p\colon S^n \xrightarrow{x \mapsto [x]} \mathbb{R}\mathrm{P}^n\big].$$

Proof. This corollary follows immediately from the fact, established on page 1784, that the projection $p\colon S^n \to \mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is a covering, the Covering-π_n-Isomorphism Proposition 134.22 and the $\pi_n(S^n)$–Proposition 167.10. ■

Corollary 167.12. ($\pi_2(\mathbb{C}\mathrm{P}^n)$–Corollary) Let $n \in \mathbb{N} \cup \{\infty\}$. Let $f\colon S^2 \to \mathbb{C}\mathrm{P}^1$ be the standard homeomorphism from page 340 and let $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ be the continuous map that is given by $[x] \mapsto [(x, 0)]$. With this notation we have

$$\pi_2(\mathbb{C}\mathrm{P}^n) \;=\; \mathbb{Z} \cdot [i \circ f\colon S^2 \to \mathbb{C}\mathrm{P}^n].$$

Proof. Let $n \in \mathbb{N} \cup \{\infty\}$. We consider the following diagram:

$$\begin{array}{ccccc}
\pi_2(S^2, *) & \xrightarrow[\cong]{f_*} & \pi_2(\mathbb{C}\mathrm{P}^1, *) & \xrightarrow{i_*} & \pi_2(\mathbb{C}\mathrm{P}^n, *) \\
\cong\big\downarrow{\scriptstyle (\Phi_{(S^2,*)}} & & \cong\big\downarrow{\scriptstyle \Phi_{(\mathbb{C}\mathrm{P}^1,*)}} & & \cong\big\downarrow{\scriptstyle \Phi_{(\mathbb{C}\mathrm{P}^n,*)}} \\
\mathrm{H}_2(S^2) & \xrightarrow[\cong]{f_*} & \mathrm{H}_2(\mathbb{C}\mathrm{P}^1) & \xrightarrow[\cong]{i_*} & \mathrm{H}_2(\mathbb{C}\mathrm{P}^n).
\end{array}$$

We make the following observations:

(1) It follows from the Hurewicz Homomorphism Proposition 167.3 that the diagram commutes.
(2) The left horizontal maps are isomorphisms since f is a homeomorphism.
(3) By the Projective Space-π_1-Proposition 135.8 and the discussion on page 2641 we know that $\pi_1(S^2) = \pi_1(\mathbb{C}\mathrm{P}^1) = \pi_1(\mathbb{C}\mathrm{P}^n) = 0$.
(4) It follows from (3) and the Hurewicz Theorem 167.9 that the vertical maps are isomorphisms.
(5) It follows from the discussion on page 2956 that the bottom right horizontal map is an isomorphism.
(6) It follows from (1), (4) and (5) that the top right horizontal map is an isomorphism.
(7) By the $\pi_n(S^n)$–Proposition 167.10 we know that $\pi_2(S^2, *) = \mathbb{Z} \cdot [\mathrm{id}_{S^n}]$.

It follows from the above discussion that $\pi_2(\mathbb{C}\mathrm{P}^n, *) = \mathbb{Z} \cdot [i \circ f\colon S^2 \to \mathbb{C}\mathrm{P}^n]$. ■

On several occasions we will also need the following straightforward corollary to the Hurewicz Theorem 167.9.

Corollary 167.13. (Hurewicz Corollary) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. We suppose that X is *simply connected*. If given every $k \in \{2, \ldots, n-1\}$ we have $\pi_k(X, x_0) = 0$ or $\mathrm{H}_k(X) = 0$, then the following two statements hold:

(1) For $k = 1, \ldots, n-1$ we have $\pi_k(X, x_0) = \mathrm{H}_k(X) = 0$.
(2) The Hurewicz homomorphism $\pi_n(X, x_0) \to \mathrm{H}_n(X)$ is an isomorphism.

Proof. We prove the corollary by induction on n. First consider the case $n = 2$. Since X is simply connected we obtain from the Hurewicz Theorem 166.10 that $\mathrm{H}_1(X) = 0$.

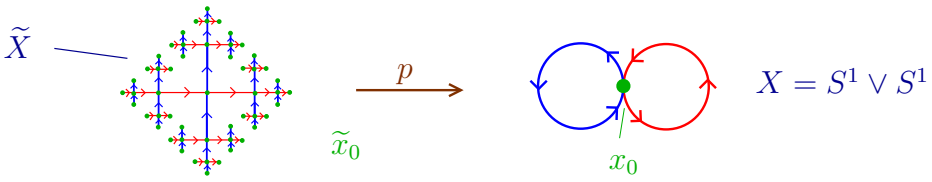
Furthermore it follows from the Hurewicz Theorem 167.9 that $\Phi_{(X,x_0)}\colon \pi_2(X,x_0) \to \mathrm{H}_2(X)$ is an isomorphism.

Now suppose the statement holds for some n with $n \geq 2$. We need to show that the statement holds for $n+1$. Thus we now suppose that for every $k \in \{2,\dots,n\}$ we have $\pi_k(X,x_0) = 0$ or $\mathrm{H}_k(X) = 0$. By induction we know that the homotopy and homology groups vanish up to degree $n-1$. Thus we can apply the Hurewicz Theorem 167.9 which says that $\pi_n(X)$ and $\mathrm{H}_n(X)$ are isomorphic. Since one is zero, both are zero. Hence we can apply the Hurewicz Theorem 167.9 once more to see that the Hurewicz homomorphism $\pi_{n+1}(X,x_0) \to \mathrm{H}_{n+1}(X)$ is an isomorphism. ■

As a nice application to the Hurewicz Corollary 167.13 we can prove the following lemma.[1785]

Lemma 167.14. (1-Complex-π_n-Lemma) Let X be a 1-dimensional CW-complex and let $x_0 \in X$ be base point. Then $\pi_n(X,x_0) = 0$ for every $n \geq 2$.

Proof. Let X be a 1-dimensional CW-complex, let $n \geq 2$ and let $x_0 \in X$ be a base point. Using the Component-Subcomplex Lemma 136.14 we can replace X by the path-component of x_0. In other words, without loss of generality we can assume that X is path-connected. We denote by $\widetilde{X}$ the universal covering of X and we pick $\widetilde{x}_0 \in p^{-1}(x_0)$.



Let $n \geq 2$. We calculate that

$$0 \underset{\uparrow}{=} \mathrm{H}_n(\widetilde{X}) \xleftarrow[\text{isomorphism by the Hurewicz Corollary 167.13, since } \widetilde{X} \text{ is simply connected}]{\Phi_{(\widetilde{X},\widetilde{x}_0)}} \pi_n(\widetilde{X},\widetilde{x}_0) \xleftarrow[\text{isomorphism by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}]{p_*} \pi_n(X,x_0).$$

by the CW-Complex Covering Proposition 138.1 we know that $\widetilde{X}$ is a 1-dimensional CW-complex, it follows from the CW-Complex-H_*-Proposition 155.10 (2) that $\mathrm{H}_n(\widetilde{X}) = 0$ for all $n \in \mathbb{N}_{\geq 2}$

■

Now that we are fully convinced of the value of the Hurewicz Theorem 167.9 we turn to the proof thereof. We will discuss more applications in Chapters 168 and 169.

167.4. **The proof of the Hurewicz Theorem 167.9 .** The proof of the Hurewicz Theorem 167.9 requires some more preparations.

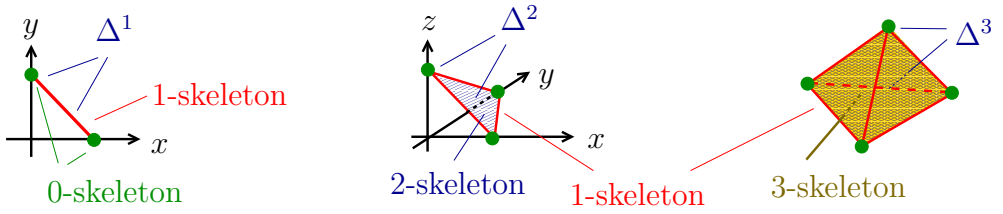
Convention 167.15. Let $k \in \mathbb{N}_0$. We view Δ^k as a CW-complex where the i-skeleton is given by[1786]

$$(\Delta^k)^i := \{(t_0,\dots,t_k) \in \Delta^k \mid \text{at least } k-i \text{ coordinates are zero}\}.$$

Note that the $(k-1)$-skeleton of Δ^k is given by $\partial\Delta^k$.

[1785] The content of this lemma is the same as for the 1-Complex-π_n-Lemma 138.2, but the proof is quite different.

[1786] In Exercise 136.7 we showed that this filtration does indeed define a CW-structure on Δ^k.

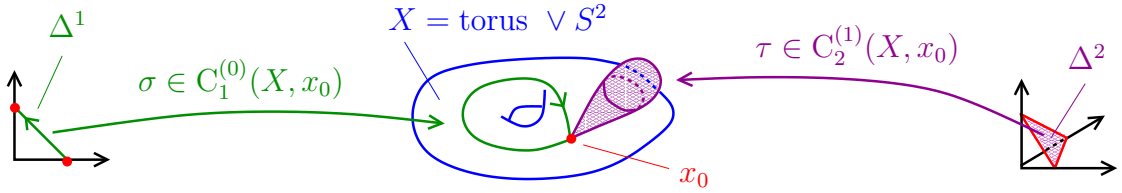


Definition 167.16. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we define[1787]

$$\mathrm{C}_k^{(n)}(X, x_0) \;:=\; \begin{array}{l}\text{the subgroup of the relative chain group } \mathrm{C}_k(X, \{x_0\}) \text{ generated by} \\ \text{all singular simplices } \sigma\colon \Delta^k \to X \text{ with } \sigma(n\text{-skeleton of } \Delta^k) \subseteq \{x_0\}.\end{array}$$

It follows almost immediately from the definitions that for fixed $n \in \mathbb{N}_0$ these groups define a subcomplex of $\mathrm{C}_*(X, \{x_0\})$. In particular we obtain corresponding homology groups $\mathrm{H}_k^{(n)}(X, x_0)$. The inclusions $\mathrm{C}_*^{(n)}(X, x_0) \to \mathrm{C}_*(X, \{x_0\})$ are evidently chain maps and thus they define natural maps $\mathrm{H}_k^{(n)}(X, x_0) \to \mathrm{H}_k(X, \{x_0\})$.

Example. In the figure below we show a pointed topological space (X, x_0) and we consider an element $\sigma \in \mathrm{C}_1^{(0)}(X, x_0)$ and an element $\tau \in \mathrm{C}_2^{(1)}(X, x_0)$.



From the pictures it becomes apparent that elements in $\mathrm{C}_n^{(n-1)}(X, x_0)$ are "close" to elements in $\pi_n(X, x_0)$. In particular it seems reasonable to expect that the groups $\mathrm{H}_n^{(k)}(X, x_0)$ with $k = 0, \ldots, n-1$ "sit between" $\mathrm{H}_n(X)$ and $\pi_n(X, x_0)$. □

The following proposition is one of the two key ingredients in the proof of the Hurewicz Theorem 167.9.

Proposition 167.17. (n-Connected–Chain Group Lemma) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. If X is n-connected, then for any $k \in \mathbb{N}_0$ the induced map

$$\mathrm{H}_k^{(n)}(X, x_0) \;\to\; \mathrm{H}_k(X, \{x_0\})$$

is an isomorphism.

Proof. We postpone the proof to Section 167.5. ■

Before we continue let us introduce some notations and conventions.

Notation 167.18.

(1) We write $* = (0, \ldots, 0, 1) \in S^n$.

(2) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 2578.

[1787] Note that the notation $\mathrm{C}_k^{(n)}(X, x_0)$ instead of $\mathrm{C}_k^{(n)}(X, \{x_0\})$ is on purpose.

(3) As on page 2578 we use φ to make the identifications

$$\pi_n(X,x_0) \;:=\; \begin{array}{c}\text{set of homotopy classes}\\ \text{of continuous maps}\\ (I^n,\partial I^n)\to(X,x_0)\end{array} \overset{\text{via }\varphi}{=} \begin{array}{c}\text{set of homotopy classes}\\ \text{of continuous maps}\\ (\overline{B}^n,S^{n-1})\to(X,x_0).\end{array}$$

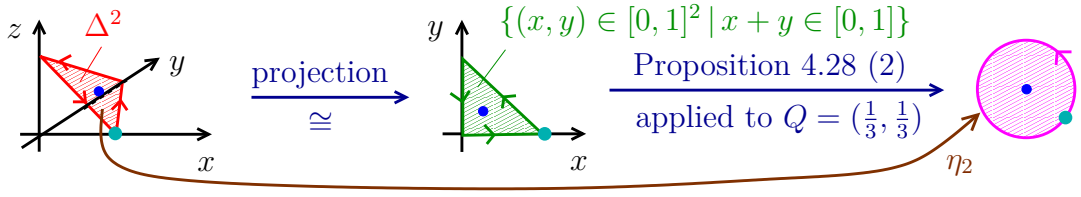
In the remainder of this chapter we will mostly work with the description on the right.

Next we need to relate the standard balls (used in the definition of homotopy groups) to the standard simplex (used in the above definition of singular homology). We do so via the following definition:

Notation 167.19. Let $n\in\mathbb{N}$. We consider the homeomorphism $\eta_n\colon(\Delta^n,\partial\Delta^n)\to(\overline{B}^n,S^{n-1})$ that is given by the composition of the following two maps:[1788]

$$\begin{array}{rcl}\eta_n\colon \Delta^n & \overset{\cong}{\to} & \left\{(x_1,\ldots,x_n)\in[0,1]^n \,\middle|\, \sum_{i=1}^n x_i\in[0,1]\right\} \xrightarrow[\text{with } Q=(\frac{1}{n+1},\ldots,\frac{1}{n+1})]{\text{Proposition 4.28 (2)}} \overline{B}^n.\\ (t_0,\ldots,t_n) & \mapsto & (t_0,\ldots,t_{n-1})\end{array}$$

Note that η_n restricts to a homeomorphism $\eta_n|_{\partial\Delta^n}\colon\partial\Delta^n\to S^{n-1}$.



The following proposition is the other key ingredient in the proof of the Hurewicz Theorem 167.9.

Proposition 167.20. (Homotopy-via-Homology Proposition) Let $k\in\mathbb{N}$ and let (X,x_0) be a pointed $(k-1)$-connected[1789] topological space.

(i) The map

$$\begin{array}{rcl}\widetilde{\Phi}_{(X,x_0)}\colon \pi_k(X,x_0) & \to & \mathrm{H}_k^{(k-1)}(X,x_0)\\ [f\colon(\overline{B}^k,S^{k-1})\to(X,x_0)] & \mapsto & \big[\underbrace{f\circ\eta_k\colon\Delta^k\to X}_{\in \mathrm{C}_k^{(k-1)}(X,x_0)}\big]\end{array}$$

is well-defined and it is an isomorphism.

(ii) The following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc}\pi_k(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_k(X)\\ {\scriptstyle\widetilde{\Phi}_{(X,x_0)}}\downarrow & & {\scriptstyle \text{я}}\downarrow{\scriptstyle\cong}\\ \mathrm{H}_k^{(k-1)}(X,x_0) & \xrightarrow{\iota} & \mathrm{H}_k(X,\{x_0\}),\end{array}$$

here ι is induced by the inclusion $\mathrm{C}_*^{(k-1)}(X,x_0)\to\mathrm{C}_k(X,\{x_0\})$.

(iii) The right vertical map of the diagram in (ii) is an isomorphism.

Proof. We postpone the proof of the proposition to Section 167.6. ∎

[1788] This is precisely the homeomorphism from the Δ^n-Homeomorphism Lemma 143.9 (2).

[1789] It is not clear to the author whether this hypothesis is strictly necessary.

After cleverly hiding two proofs in future sections we can now turn to the proof of the Hurewicz Theorem 167.9.

Proof of the Hurewicz Theorem 167.9. Let $n \in \mathbb{N}_{\geq 2}$. Furthermore let (X, x_0) be a pointed $(n-1)$-connected topological space. Note that it suffices to prove the following claim:

Claim. For $k = 1, \ldots, n$ the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_k(X, x_0) \to \mathrm{H}_k(X)$ is an isomorphism.

Proof. Let $k \in \{1, \ldots, n\}$. As in the Homotopy-via-Homology Proposition 167.20 we consider the following diagram:

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_k(X) \\ {\scriptstyle \widetilde{\Phi}_{(X,x_0)}}\downarrow & & {\scriptstyle \text{я}}\downarrow{\scriptstyle \cong} \\ \mathrm{H}_k^{(k-1)}(X, x_0) & \xrightarrow[\iota]{\cong} & \mathrm{H}_k(X, \{x_0\}) \end{array}$$

We make the following observations:

(a) By the Homotopy-via-Homology Proposition 167.20 we know that the diagram commutes up to the sign $(-1)^k$ and that the two vertical maps are isomorphisms.

(b) Since X is $(n-1)$-connected and since $1 \leq k \leq n$ we obtain from the n-Connected–Chain Group Proposition 167.17 that the bottom horizontal map is an isomorphism.

It follows from (a) and (b) that the top horizontal map is an isomorphism. ■

The remainder of this chapter is taken up by the proofs of the n-Connected–Chain Group Proposition 167.17 and the Homotopy-via-Homology Proposition 167.20.

167.5. **Proof of the n-Connected–Chain Group Proposition 167.17 .** For the reader's convenience we recall the statement we need to prove:

Proposition 167.17. (n-Connected–Chain Group Lemma) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. If X is n-connected, then the inclusion map

$$\mathrm{C}_*^{(n)}(X, x_0) \;\to\; \mathrm{C}_*(X, \{x_0\})$$

is a chain homotopy equivalence. In particular for any $k \in \mathbb{N}_0$ the induced map

$$\mathrm{H}_k^{(n)}(X, x_0) \;\to\; \mathrm{H}_k(X, \{x_0\})$$

is an isomorphism.

Proof. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is n-connected. We recall the following notation from the Prism Boundary Lemma 144.17: Given $t = 0$ and $t = 1$ and $k \in \mathbb{N}_0$ we denote by

$$\begin{array}{rccc} \eta_t \colon & \Delta^k & \to & \Delta^k \times [0,1] \\ & x & \mapsto & (x, t) \end{array}$$

the obvious embedding. We start out with the following claim which basically says that singular simplices $\Delta^k \to X$ can be homotoped consistently to singular simplices that lie in $\mathrm{C}_k^{(n)}(X, x_0)$.

Claim 1. Given any $k \in \mathbb{N}_0$ and given any singular simplex $\sigma\colon \Delta^k \to X$ we can choose a continuous map $P_k(\sigma)\colon \Delta^k \times [0,1] \to X$ such that the following four statements are satisfied:

(a) We have $P_k(\sigma) \circ \eta_0 = \sigma\colon \Delta^k \to X$.
(b) The singular k-simplex $P_k(\sigma) \circ \eta_1\colon \Delta^k \to X$ lies in $\mathrm{C}_k^{(n)}(X, x_0)$.
(c) If $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then $P_k(\sigma) = \sigma \circ \pi$ where $\pi\colon \Delta^k \times [0,1] \to \Delta^k$ is the projection.
(d) For any $j \in \{0, \dots, k\}$ we have

$$P_k(\sigma) \circ \underbrace{(i_j^k \times \mathrm{id}_{[0,1]})}_{\Delta^{k-1}\times[0,1] \to \Delta^k \times [0,1]} = P_{k-1}(\underbrace{\sigma \circ i_j^k}_{\Delta^{k-1} \to X})\colon \Delta^{k-1} \times [0,1] \to X.$$

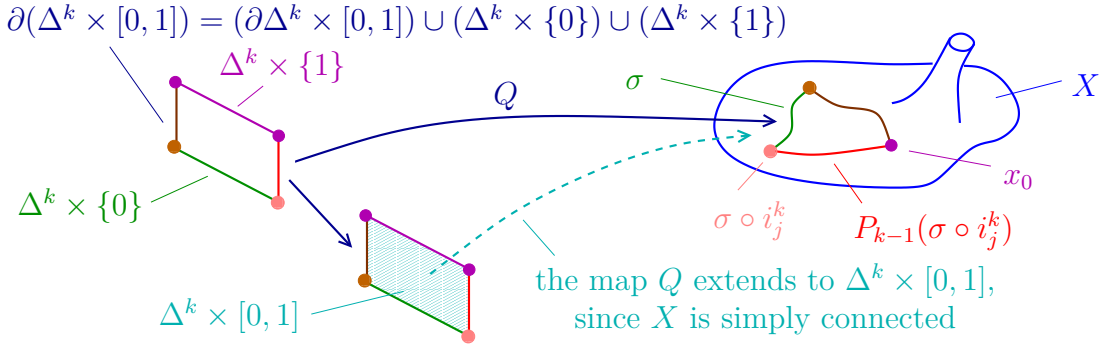
Proof. Property (d) already suggests that we should prove the claim by induction on k. It is convenient to view $k = -1$ as the (trivial) base case. Now let $k \in \mathbb{N}_0$ and suppose that we have proved the claim for all singular i-simplices with $i < k$. Now let $\sigma\colon \Delta^k \to X$ be a singular simplex. If $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then we set $P_k(\sigma) = \sigma \circ \pi$. In this case it is straightforward to verify that the conditions (a), (b) and (d) are satisfied.

Now consider the case that $\sigma \notin \mathrm{C}_k^{(n)}(X, x_0)$. We distinguish two cases.

(1) First we consider the case that $k \leq n$. By induction the maps $P_{k-1}(\sigma \circ i_j^k)$ have been defined for all j. We set $\partial(\Delta^k \times [0,1]) \coloneqq (\partial\Delta^k \times [0,1]) \cup (\Delta^k \times \{0,1\})$ and we consider the map

$$Q\colon \partial(\Delta^k \times [0,1]) \to X$$

that is defined via σ on $\Delta^k \times \{0\}$, via the maps $P_{k-1}(\sigma \circ i_j^k)$ on $\partial\Delta^k \times [0,1]$[1790] and which sends all points in $\Delta^k \times \{1\}$ to x_0.[1791] (We refer to the figure below for an illustration of this map.) It follows almost immediately from the Convex-to-Ball Proposition 4.28 (2) that the pair $(\Delta^k \times [0,1], \partial(\Delta^k \times [0,1]))$ is homeomorphic to $(\overline{B}^{k+1}, S^k)$. Since $k \leq n$ and since X is n-connected we obtain from the Trivial-in-π_n-Lemma 134.10 that there exists an extension of Q to a continuous map $\Delta^k \times [0,1] \to X$. We define $P_k(\sigma)$ to be any such extension. We obtain a map $P_k(\sigma)$ which is easily seen to have all the desired properties.



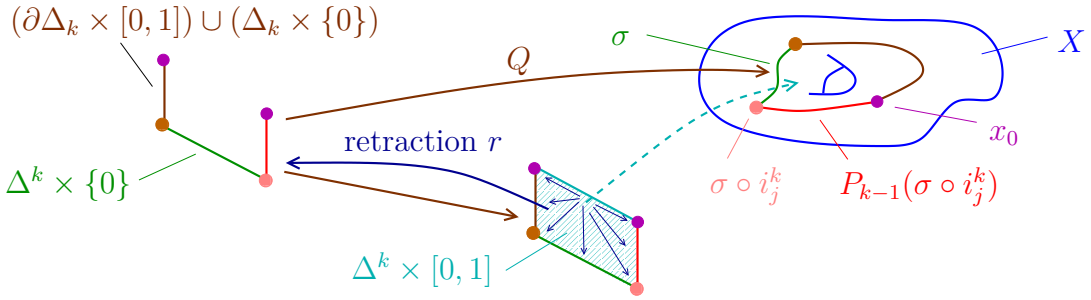
(2) Now we consider the case that $k > n$. Note that in this case condition (b) is satisfied "for free" by any continuous map $\Delta^k \times [0,1] \to X$ that satisfies (d). Thus we only have to find a map $P_k(\sigma)$ that satisfies (a), (c) and (d). This is much easier than in (1).

[1790] Here we use that we assume that the maps $P(\mu)$ are defined for simplices of dimension $< k$.
[1791] It follows from the Pasting Proposition 4.7 (2′) that this map is indeed continuous.

Indeed, similar to (1) we consider the map

$$Q\colon (\Delta^k \times \{0\}) \cup (\partial\Delta^k \times [0,1]) \;\to\; X$$

that is defined via σ on $\Delta^k \times \{0\}$ and via the maps $P_{k-1}(\sigma \circ i^k_j)$ on $\partial\Delta^k \times [0,1]$.



Note that by the Cup–Retraction Lemma 140.9 (together with the homeomorphism $\overline{B}^k \cong \Delta^k$) there exists a retraction r from $\Delta^k \times [0,1]$ to the domain of Q. Thus combining this retraction r with Q we obtain an extension of Q to a map $\Delta^k \times [0,1] \to X$. We define $P_k(\sigma)$ to be this extension. It follows basically by construction that all the statements are satisfied. ⊞

Now we turn to the actual proof of the statement that the inclusion map

$$\iota\colon \mathrm{C}^{(n)}_*(X, x_0) \;\to\; \mathrm{C}_*(X, \{x_0\})$$

is a chain homotopy equivalence. To do so we consider the homomorphisms

$$\phi\colon \mathrm{C}_k(X, \{x_0\}) \;\to\; \mathrm{C}^{(n)}_k(X, x_0)$$

that are determined by

$$[\sigma\colon \Delta^k \to X)] \;\mapsto\; \phi([\sigma]) \coloneqq \underbrace{[P_k(\sigma) \circ \eta_1]}_{\Delta^k \to \Delta^k \times [0,1] \to X}.$$

It follows from Claim 1 (b) that ϕ does indeed take values in $\mathrm{C}^{(n)}_*(X, x_0)$ and it follows easily from Claim 1 (d) that ϕ is in fact a chain map. We want to show that ϕ is a chain homotopy inverse to ι. By Claim 1 (c) we have $\phi \circ \iota = \mathrm{id}$. Thus it remains to prove the following claim:

Claim 2. There exists a chain homotopy from $\iota \circ \phi$ to the identity of $\mathrm{C}_*(X, \{x_0\})$.

Proof. Let $k \in \mathbb{N}_0$. We recall that in Notation 144.16 we introduced the singular $(k+1)$-chain

$$\Omega_k \;\coloneqq\; \sum_{j=0}^{k} (-1)^j \cdot \underbrace{[v_0, \ldots, v_j, w_j, \ldots, w_k]}_{\Delta^{k+1} \to \Delta^k \times [0,1]} \;\in\; \mathrm{C}_{k+1}(\Delta^k \times [0,1])$$

where $v_0, \ldots, v_k, w_0, \ldots, w_k$ denote the vertices of $\Delta^k \times [0,1]$, see Notation 144.16 for details. The precise definition of Ω_k is in fact irrelevant to us, all we need to know is that the Prism Boundary Lemma 144.17 says that for any $k \in \mathbb{N}_0$ we have

$$(*) \qquad \partial_{k+1}(\Omega_k) + \sum_{j=0}^{k} (-1)^j \cdot \underbrace{(i^k_j \times \mathrm{id}_{[0,1]})_*}_{\mathrm{C}_*(\Delta^{k-1} \times [0,1]) \to \mathrm{C}_*(\Delta^k \times [0,1])} (\Omega_{k-1}) \;=\; \eta_1 - \eta_0 \;\in\; \mathrm{C}_k(\Delta^k \times [0,1]).$$

For any $k \in \mathbb{N}_0$ we consider the homomorphism

$$D_k\colon \mathrm{C}_k(X, \{x_0\}) \;\to\; \mathrm{C}_{k+1}(X, \{x_0\})$$

that is determined by

$$[\sigma\colon \Delta^k \to X] \;\mapsto\; \Big[\underbrace{P_k(\sigma)_*}_{\mathrm{C}_*(\Delta^k\times[0,1])\to\mathrm{C}_*(X)}\big(\overbrace{\Omega_k}^{\in \mathrm{C}_{k+1}(\Delta^k\times[0,1])}\big)\Big]$$

We claim that these homomorphisms define a chain homotopy from $\iota\circ\phi$ to the identity of $\mathrm{C}_*(X,\{x_0\})$. Thus let $\sigma\colon \Delta^k \to X$ be a singular k-simplex. We need to show that $\partial_{k+1}(D_k(\sigma)) + D_{k-1}(\partial_k(\sigma)) = \phi(\sigma) - \sigma$. Indeed we have the following equalities in $\mathrm{C}_k(X,\{x_0\})$:[1792]

$$\begin{aligned}
\partial_{k+1}(D_k(\sigma)) + D_{k-1}(\partial_k(\sigma)) &\overset{\text{Definition 143.13 of } \partial_k(\sigma)}{=} \partial_{k+1}(D_k(\sigma)) + D_{k-1}\Big(\sum_{j=0}^{k}(-1)^j\cdot\sigma\circ i_j^k\Big) \overset{\text{definition of } D_k \text{ and } D_{k-1}}{=}\\
&= \partial_{k+1}(P_k(\sigma)_*(\Omega_k)) + \sum_{j=0}^{k}(-1)^j\cdot P_{k-1}(\sigma\circ i_j^k)_*(\Omega_{k-1})\\
&\overset{(*)}{=} P_k(\sigma)_*\Big(\partial_{k+1}(\Omega_k) + \sum_{j=0}^{k}(-1)^j\cdot(i_j^k\times \mathrm{id}_{[0,1]})_*(\Omega_{k-1})\Big)
\end{aligned}$$

in the first summand we used that by the C_*-Functor Lemma 143.16 the induced map $P_k(\sigma)_*$ is a chain map and in the second summand we used that by Claim 1 (d) we have $P_{k-1}(\sigma\circ i_j^k) = P_k(\sigma)\circ(i_j^k\times \mathrm{id}_{[0,1]})$ and we used the covariant functoriality of chain groups

$$\overset{(*)}{=} P_k(\sigma)_*(\eta_1-\eta_0) = P_k(\sigma)\circ\eta_1 - P_k(\sigma)\circ\eta_0 \overset{\text{by definition of } \phi \text{ and by Claim 1 (a)}}{=} \phi(\sigma)-\sigma.$$

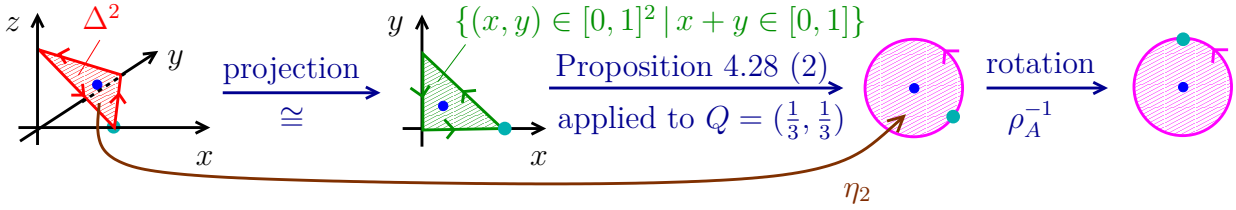
by the above equality (*) ■

167.6. **Proof of the Homotopy-via-Homology Proposition 167.20 .** The proof of the Homotopy-via-Homology Proposition 167.20 requires some preparation. First we introduce the following notation:

Notation 167.21.

(1) We write $* = (0,\dots,0,1) \in S^n$.

(2) By the Orthogonal Action Lemma 2.37 there exists a matrix $A \in \mathrm{SO}(n+1)$ such that $A\cdot * \in \eta_{n+1}((n-1)\text{-skeleton of } \partial\Delta^{n+1})$. We denote by $\rho_A\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ the map that is given by $x \mapsto A\cdot x$.



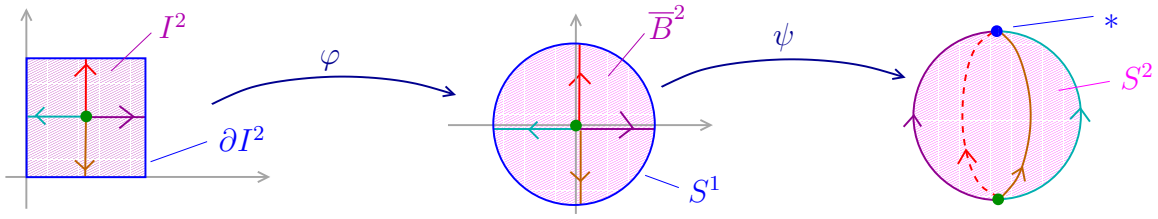
We also recall and introduce more notation:

Notation 167.22.

(1) Let $\varphi\colon (I^n,\partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 2578.

(2) Let $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ be the map from the Ball Quotient-Sphere Lemma 8.10 that induces the standard homeomorphism $\overline{B}^n/S^{n-1} \to S^n$.

[1792] In this calculation we do not distinguish in the notation between a singular chain in $\mathrm{C}_k(X)$ and the element it represents in $\mathrm{C}_k(X,\{x_0\})$.

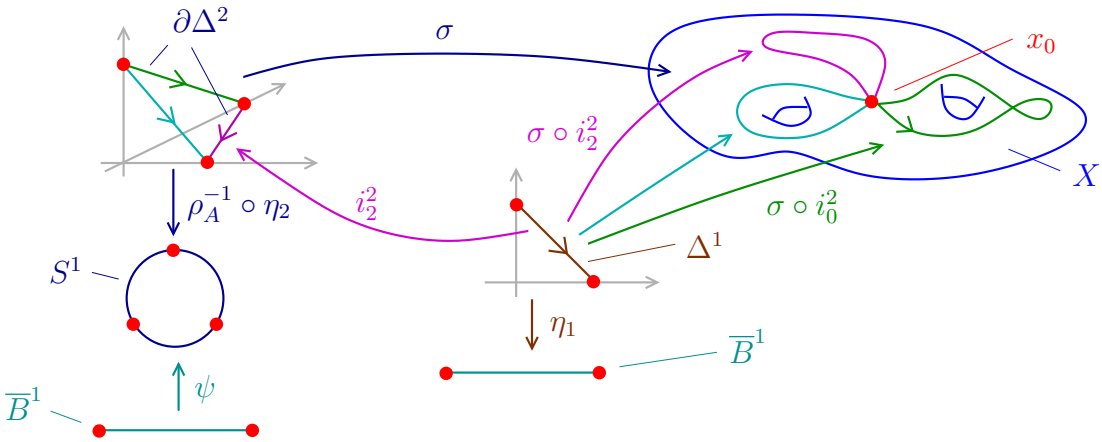


The following lemma is a key technical ingredient in the proof of the Homotopy-via-Homology Proposition 167.20:

Lemma 167.23. (Faces–Addition Lemma) Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}_{\geq 2}$ and let $\sigma\colon \partial\Delta^{n+1} \to X$ be a continuous map that sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0. Then we have the following equality in $\pi_n(X, x_0)$:[1793] [1794]

$$[\underbrace{\sigma \circ (\eta_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)}] \;=\; -\sum_{j=0}^{n+1} (-1)^j \cdot [\underbrace{\sigma \circ i_j^{n+1} \circ \eta_n^{-1}}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)}].$$

We refer to the figure below for an illustration for $n = 1$, even though strictly speaking this case is not allowed.



Proof of the Faces–Addition Lemma 167.23. At this point we are mixing all kinds of conventions and definitions. The proof of this lemma is thus something of a mess, and therefore we outsource the proof to Appendices 167.7 and 167.8. ■

For the reader's convenience we recall the proposition we are supposed to prove:

Proposition 167.20. (Homotopy-via-Homology Proposition) Let $n \in \mathbb{N}$ and let (X, x_0) be a pointed $(n-1)$-connected topological space.

(i) The map

[1793] Note that A is chosen in such a way that $(\sigma \circ (\eta_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi)(S^{n-1}) = \{x_0\}$.

[1794] Note that our hypothesis that σ sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0 implies in particular that for all $j \in \{0, \ldots, n+1\}$ we have $(\sigma \circ i_j^{n+1} \circ \eta_n^{-1})(S^{n-1}) = \{x_0\}$.

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n^{(n-1)}(X,x_0) \\ [f\colon (\overline{B}^n, S^{n-1}) \to (X,x_0)] & \mapsto & [\underbrace{f\circ\eta_n\colon \Delta^n \to X}_{\in \mathrm{C}_n^{(n-1)}(X,x_0)}] \end{array}$$

is well-defined and it is an isomorphism.

(ii) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{tikzcd} \pi_n(X,x_0) \arrow[r, "\Phi_{(X,x_0)}"] \arrow[d, "\widetilde{\Phi}_{(X,x_0)}"'] & \mathrm{H}_n(X) \arrow[d, "\text{я}"', "\cong"] \\ \mathrm{H}_n^{(n-1)}(X,x_0) \arrow[r, "\iota"] & \mathrm{H}_n(X,\{x_0\}), \end{tikzcd}$$

here ι is induced by the inclusion $\mathrm{C}_*^{(n-1)}(X,x_0) \to \mathrm{C}_n(X,\{x_0\})$.

(iii) The right vertical map of the diagram in (ii) is an isomorphism.

Proof. We start out with the following claim:

Claim 1.

(i) The map

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n^{(n-1)}(X,x_0) \\ [f\colon (\overline{B}^n, S^{n-1}) \to (X,x_0)] & \mapsto & [\underbrace{f\circ\eta_n\colon \Delta^n \to X}_{\in \mathrm{C}_n^{(n-1)}(X,x_0)}] \end{array}$$

is well-defined and it is a homomorphism.

(ii) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{tikzcd} \pi_n(X,x_0) \arrow[r, "\Phi_{(X,x_0)}"] \arrow[d, "\widetilde{\Phi}_{(X,x_0)}"'] & \mathrm{H}_n(X) \arrow[d, "\text{я}"', "\cong"] \\ \mathrm{H}_n^{(n-1)}(X,x_0) \arrow[r, "\iota"] & \mathrm{H}_n(X,\{x_0\}) \end{tikzcd}$$

where ι is induced by the inclusion $\mathrm{C}_*^{(n-1)}(X,x_0) \to \mathrm{C}_n(X,\{x_0\})$.

(iii) The right vertical map of the diagram in (ii) is an isomorphism.

Proof. We prove the claim using a somewhat convoluted logic.[1795] We consider the following diagram:

$$\begin{tikzcd} \{\text{set of continuous maps } (\Delta^n, \partial\Delta^n) \to (X,x_0)\} \arrow[d, "f\mapsto[f\circ\eta_n^{-1}]"] \arrow[dd, bend right=60, "f\mapsto[f]"'] & \\ \pi_n(X,x_0) \arrow[r, "\Phi_{(X,x_0)}"] \arrow[d, dashed, "\widetilde{\Phi}_{(X,x_0)}"] \arrow[dr, "{[g]\mapsto g_*([\overline{B}^n])}"] & \mathrm{H}_n(X) \arrow[d, "\text{я}"', "\cong"] \\ \mathrm{H}_n^{(n-1)}(X,x_0) \arrow[r, "\cong", "\iota"'] & \mathrm{H}_n(X,\{x_0\}) \end{tikzcd}$$

[1795]The convoluted logic is forced upon us since proving (i) as a stand-alone statement is actually slightly tricky.

We make the following observations:

(a) It follows from $n \geq 2$, from the long exact sequence of homology groups of the pair $(X, \{x_0\})$ (see the LES-of-Pair Proposition 146.7) and the Singleton-H_*-Lemma 143.26 that the right-hand vertical map я of the diagram is an isomorphism.

(b) Since X is $(n-1)$-connected we obtain from the n-Connected–Chain Group Proposition 167.17 that the bottom horizontal map is an isomorphism.

(c) Let $f\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)$ be a continuous map. The $(n-1)$-skeleton of Δ^n equals $\partial\Delta^n$, thus the map $f\colon \Delta^n \to X$ does indeed define an element in $\mathrm{C}_n^{(n-1)}(X, x_0)$. It follows easily from the definitions that every element in $\mathrm{C}_n^{(n-1)}(X, x_0) \subseteq \mathrm{C}_n(X, \{x_0\})$ is a cycle, thus f defines an element in $\mathrm{H}_n^{(n-1)}(X, x_0)$. This shows that in the above diagram the curved left vertical map is well-defined.

(d) In this proof we think of $\pi_n(X, x_0)$ as the set of homotopy classes of continuous maps $(\overline{B}^n, S^{n-1}) \to (X, x_0)$. It follows immediately from the definitions that the right triangle commutes.

(e) For any continuous map $f\colon (\Delta^n, \partial\Delta^n) \to (X, \{x_0\})$ of pairs of topological spaces we have the following equality in $\mathrm{H}_n(X, \{x_0\})$:

$$(f \circ \eta_n^{-1})_*([\overline{B}^n]) \underset{\uparrow}{=} (-1)^n \cdot (f \circ \eta_n^{-1})_*([\eta_n]) = (-1)^n \cdot [f \circ \eta_n^{-1} \circ \eta_n] = (-1)^n \cdot [f].$$

by the Standard Generator-$\overline{B}^n$-Cycle Lemma 155.28 we have $[\eta_n\colon \Delta^n \to \overline{B}^n] = (-1)^n \cdot [\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$

This shows that the two possibilities of going from the upper left corner to the lower right corner differ by the sign $(-1)^n$.

(f) We want to show that $\widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n^{(n-1)}(X, x_0)$ is well-defined and that it is a homomorphism. By (b) we know that $\iota\colon \mathrm{H}_n^{(n-1)}(X, x_0) \to \mathrm{H}_n(X, \{x_0\})$ is an isomorphism. Thus it suffices to show these statements for $\iota \circ \widetilde{\Phi}_{(X,x_0)}$. We make the following observations:

(α) It follows from (e) and the observation that the top vertical map is a surjection that the rectangle commutes up to the sign $(-1)^n$. In particular the bottom-right map $\iota \circ \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n(X, \{\widetilde{x}_0\})$ is well-defined.

(β) By (d) we know that the right triangle commutes.

(γ) By the Hurewicz Homomorphism Proposition 167.3 we know that $\Phi_{(X,x_0)}$ is a homomorphism. Evidently я is also a homomorphism.

(δ) It follows from (α), (β) and (γ) that $\iota \circ \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n(X, \{\widetilde{x}_0\})$ is a homomorphism. ⊞

By Claim 1 it suffices to show that the homomorphism

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) & \to & \mathrm{H}_n^{(n-1)}(X, x_0) \\ [f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)] & \mapsto & [f \circ \eta_n\colon \Delta^n \to X] \end{array}$$

is in fact an isomorphism. We will do so by defining an explicit inverse.

Claim 2.

(i) The map

$$\begin{array}{rcl} \Xi_{(X,x_0)}\colon \mathrm{C}_n^{(n-1)}(X, x_0) & \to & \pi_n(X, x_0) \\ \sum_{i=1}^{s} a_i \cdot (\sigma_i\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)) & \mapsto & \sum_{i=1}^{s} a_i \cdot [\sigma_i \circ \eta_n^{-1}\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)] \end{array}$$

is a well-defined homomorphism.

(ii) The map $\Xi_{(X,x_0)}$ from (i) descends to a well-defined homomorphism

$$\mathrm{H}_n^{(n-1)}(X,x_0) \;\to\; \pi_n(X,x_0).$$

(iii) The map $\Xi_{(X,x_0)}$ is an inverse to $\widetilde{\Phi}_{(X,x_0)}$.

Proof.

(i) Recall that we know from the π_n-Abelian Proposition 134.11 and our hypothesis $n \geq 2$ that the group $\pi_n(X,x_0)$ on the right is abelian. Therefore it follows from the Free Abelian Group–Existence Lemma 97.8 that the map is well-defined.[1796]

(ii) Let $\sigma\colon \Delta^{n+1} \to X$ be a singular $(n+1)$-simplex that defines an element in the group $\mathrm{C}_{n+1}^{(n-1)}(X,x_0)$. We need to show that $\Xi_{(X,x_0)}(\partial_{n+1}(\sigma)) = 0$. We calculate that we have the following equality in $\pi_n(X,x_0)$:

$$\begin{aligned}\Xi_{(X,x_0)}(\partial_{n+1}(\sigma)) &= \Xi_{(X,x_0)}\Big(\sum_{j=0}^{n+1}(-1)^j\cdot\sigma\circ i_j^n\Big)\\ &\underset{\uparrow}{=} \sum_{j=0}^{n+1}(-1)^j\cdot[\sigma\circ i_j^{n+1}\circ\eta_n^{-1}] \underset{\uparrow}{=} -[\sigma|_{\partial\Delta^{n+1}}\circ\eta_{n+1}^{-1}\circ\rho_A|_{S^n}] \underset{\uparrow}{=} 0.\end{aligned}$$

by definition of $\Xi_{(X,x_0)}$ — follows from the Faces–Addition Lemma 167.23 — by the Trivial-in-π_n-Lemma 134.10 since $\sigma|_{\partial\Delta^{n+1}}\circ\eta_{n+1}^{-1}\circ\rho_A|_{S^n}\colon S^n\to X$ extends to $\sigma\circ\eta_{n+1}^{-1}\circ\rho_A\colon B^{n+1}\to X$

(iii) It follows immediately from the definitions that $\Xi_{(X,x_0)}\circ\widetilde{\Phi}_{(X,x_0)}$ is the identity on $\pi_n(X,x_0)$. Now let us consider the converse. As on page 3209 we see that every element of $\mathrm{C}_n^{(n-1)}(X,x_0)$ is a cycle. It follows from this observation and the definitions that $\widetilde{\Phi}_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is the identity on a generating set for the group $\mathrm{H}_n^{(n-1)}(X,x_0)$. But by Claim 1 and Claim 2 we know that $\widetilde{\Phi}_{(X,x_0)}$ and $\Xi_{(X,x_0)}$ are both homomorphisms. Thus we see that $\widetilde{\Phi}_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is actually the identity on all of $\mathrm{H}_n^{(n-1)}(X,x_0)$.

This concludes the proof of the claim and thus, modulo the proof of the Faces–Addition Lemma 167.23, we are done with the proof of the Hurewicz Theorem 167.9. ■

Remark. The most awkward aspect of the proof of the Hurewicz Theorem 167.9 is that one needs to go back and forth between the different points of view regarding elements in higher homotopy groups, in particular the proof of the Faces–Addition Lemma 167.23 is (as we will see shortly) a technical mess.

In fact there is, in principle, an elegant way around this problem: One could prove the Hurewicz Theorem 167.9 using "cubical singular homology" instead of the usual singular homology. As the name suggests, in this theory the "building blocks" are singular cubes, i.e. maps from $[0,1]^n$ to a topological space X. (This theory is developed in [**Mass1991**, Chapter VII.2].) Since cubes are also used in the definition of the product structure on higher homotopy groups the author would expect that many of the technical issues we had to deal with vanish magically.

The downside is that one still needs to show that "cubical singular homology" is naturally isomorphic to the usual singular homology. This can be shown using the Acyclic Model Theorem 235.7. □

[1796] Put differently, it would be a serious dereliction of duty to use the additive notation for a group structure, if the group was not known to be abelian.

167.7. **Appendix: Proof of the Faces–Addition Lemma 167.23 I.** In this and the following appendix we finally give the proof of the Faces–Addition Lemma 167.23. The proof will require a good understanding of the group structure on the higher homotopy groups. To get this better understanding we need the following new and fairly harmless definition.

Definition 167.24. Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold.

(1) We say that a map $\alpha\colon \overline{B}^n \to M$ is an almost-smooth embedding if α is an embedding and if $\alpha|_{B^n}\colon B^n \to M$ is a smooth embedding.

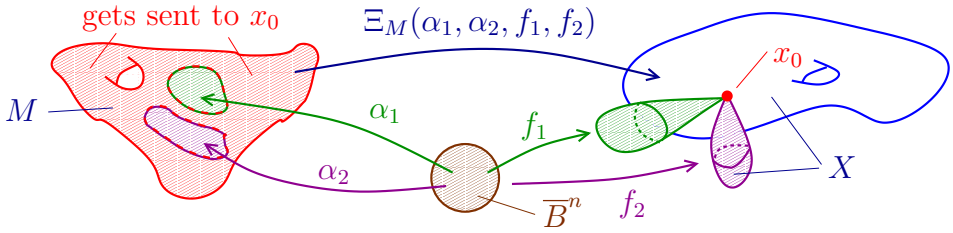
(2) If M is oriented, then we say that an almost-smooth embedding $\alpha\colon \overline{B}^n \to M$ is orientation-preserving if the smooth embedding $\alpha\colon B^n \to M$ is orientation-preserving. The same way we define orientation-reversing.

(3) We say embeddings $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to M$ are (weakly) disjoint if the images (of the interiors) do not intersect.

(4) Let (X, x_0) be a pointed topological space, let $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to M$ be weakly-disjoint almost-smooth embeddings and let $f_1, \dots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ be continuous maps. We set

$$\begin{array}{rccl} \Xi_M(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m)\colon M & \to & X & \\ y & \mapsto & \begin{cases} f_i(z), & \text{if } y = \alpha_i(z) \text{ with } z \in \overline{B}^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

This map is well-defined and continuous[1797].



The following proposition, which is also of independent interest, gives us a versatile tool for dealing with the group structure on the homotopy groups $\pi_n(X, x_0)$:

Proposition 167.25. (Addition-in-π_n-Proposition)[1798] Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given weakly-disjoint almost-smooth embeddings $\alpha_1, \dots, \alpha_m\colon \overline{B}^n \to S^n$ and continuous maps $f_1, \dots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Furthermore let $A \in \mathrm{SO}(n+1)$ such that $A \cdot * \in S^n \setminus (\alpha_1(B^n) \cup \dots \cup \alpha_m(B^n))$. We denote by $\rho_A\colon S^n \to S^n$ the map that is given by $x \mapsto A \cdot x$. Given $i \in \{1, \dots, m\}$ we write $\epsilon_i = 1$ if α_i is orientation-preserving, otherwise we write $\epsilon_i = -1$. Then we have the following

[1797] First note that it follows from the fact that the α_i are weakly disjoint that the given map is well-defined. Next note that it follows easily from our hypothesis and the Smooth Invariance-of-Domain Theorem 24.22 that each $\alpha_i(B^n)$ is an open subset of M. This implies that $M \setminus (\alpha_1(B^n) \cup \dots \cup \alpha_m(B^n))$ is a closed subset of M. Furthermore it follows from the Compact Image Lemma 4.15 and the Closed-Compact Lemma 3.39 that each $\alpha_i(\overline{B}^n)$ is a closed subset of M. It follows from this discussion and the Pasting Proposition 4.7 (2) that the map $\Xi_M(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m)$ is indeed continuous.

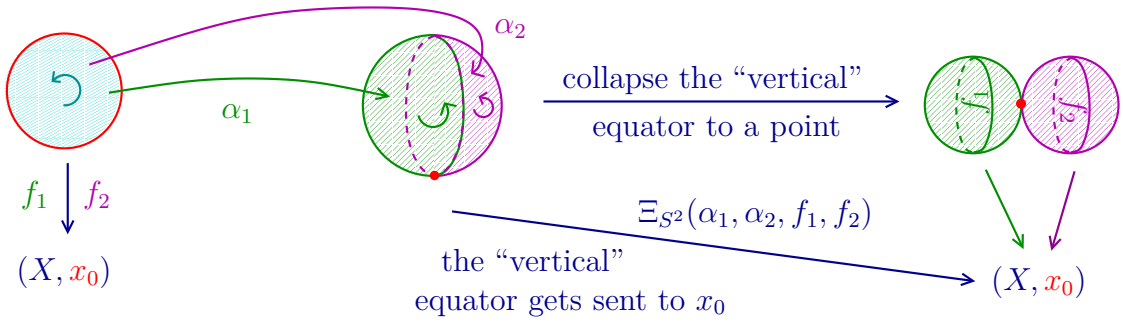
equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Xi_{S^n}(\alpha_1, \ldots, \alpha_m, f_1, \ldots, f_m)\circ\rho_A\circ\psi}_{\text{map } (\overline{B}^n, S^{n-1}) \to (X, x_0)}\big] \;=\; \sum_{i=1}^{m} \epsilon_i \cdot [\, f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)\,].$$

Example. A slight modification of the map $\overline{B}^2 \to S^2$ given in the Ball Quotient-Sphere Lemma 8.10 provides us with an orientation-preserving smooth embedding $\alpha_1 \colon \overline{B}^2 \to S^2$ that sends the disk to the "left hemisphere". Similarly we also obtain an orientation-preserving smooth embedding $\alpha_2 \colon \overline{B}^2 \to S^2$ that sends the disk to the "right hemisphere". It follows from the Addition-in-π_n-Proposition 167.25 that for any two continuous maps $f_1, f_2 \colon (\overline{B}^2, S^1) \to (X, x_0)$ we have the following equality in $\pi_2(X, x_0)$:

$$[\,\Xi_{S^2}(\alpha_1, \alpha_2, f_1, f_2)\,] \;=\; [\,f_1 \colon (\overline{B}^2, S^1) \to (X, x_0)\,] + [\,f_2 \colon (\overline{B}^2, S^1) \to (X, x_0)\,].$$

This situation is illustrated in the figure below.



Note that that is basically the same picture as we had already obtained on page 2579. □

The proof of the Addition-in-π_n-Proposition 167.25 will require the remainder of this appendix. For the proof of the Addition-in-π_n-Proposition 167.25 we will need to formulate and prove two lemmas. The first lemma basically says the following: If we isotope the embeddings α_i, then the resulting Ξ-maps are homotopic.

Lemma 167.26. (Moving Balls Lemma) Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold with $\partial M = \varnothing$ and let Z be a subset of M. Suppose that we are given a pointed topological space (X, x_0) and continuous maps $f_1, \ldots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Finally suppose that for $i = 1, \ldots, m$ we are given homotopies

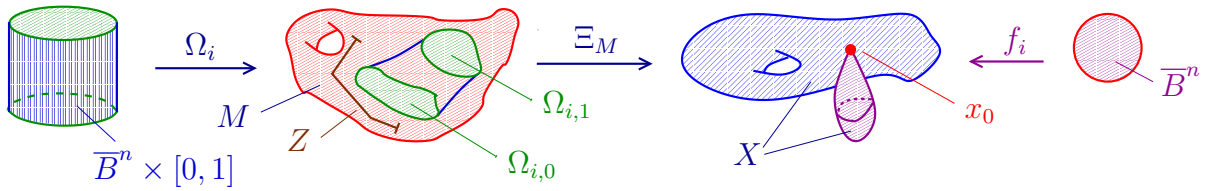
$$\Omega_i \colon \overline{B}^n \times [0,1] \;\to\; M$$

such that for each $t \in [0,1]$ the maps $\Omega_{1,t}, \ldots, \Omega_{m,t} \colon \overline{B}^n \to M$ are weakly-disjoint almost-smooth embeddings and such that for $i = 1, \ldots, m$ the restriction of Ω_i to $B^n \times [0,1]$ is smooth and satisfies $\Phi_i(B^n \times [0,1]) \subseteq M \setminus Z$. Then the map

$$\begin{aligned} H \colon M \times [0,1] &\;\to\; X \\ (P, t) &\;\mapsto\; \Xi_M(\Omega_{1,t}, \ldots, \Omega_{m,t}, f_1, \ldots, f_m)(P) \end{aligned}$$

[1798] The thorough reader will already have digested the Hopf Degree Theorem 46.22 which implies that $\pi_n(S^n) \cong \mathbb{Z}$. If one already knows this fact, then the proof of the Addition-in-π_n-Proposition 167.25 can be simplified quite a bit. But we want to provide a proof that is independent of the Hopf Degree Theorem 46.22. Unfortunately this approach turns the proof into a little bit of a mess.

is a homotopy as maps of pairs of topological spaces $(M, Z) \to (X, x_0)$.



Proof of the Moving Balls Lemma 167.26. Note that it follows easily from our hypotheses and the Compact-Hausdorff Proposition 4.23 (3) that each map

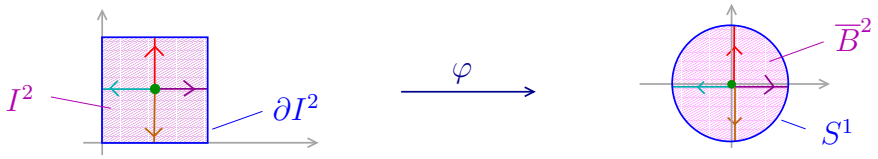
$$\begin{array}{rcl} \overline{B}^n \times [0,1] & \to & M \times [0,1] \\ (x,t) & \mapsto & (\Omega_i(x,t),t) \end{array}$$

is an embedding and that the restriction of each such map to $B^n \times [0,1]$ is a smooth embedding. With this observation it follows again from the Smooth Invariance-of-Domain Theorem 24.22, the Compact Image Lemma 4.15, the Compact-Closed Lemma 3.39 and the Pasting Proposition 4.7 that the given map $H\colon M \times [0,1] \to X$ is continuous. Finally note that one can easily verify that for each $t \in [0,1]$ we have $H_t(Z) = \{x_0\}$. Thus we obtain indeed a homotopy of maps of pairs of topological spaces $(M, Z) \to (X, x_0)$. ∎

We continue with the second lemma which will go into the proof of the Addition-in-π_n-Proposition 167.25.

Lemma 167.27. (Reflection Lemma) Let $n \in \mathbb{N}$, let (X, x_0) be a pointed topological space and let $f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ be a continuous map. If $\rho\colon \overline{B}^n \to \overline{B}^n$ is the reflection in the $(x_1 = 0)$-hyperplane of $\mathbb{R}^n$, then $[f \circ \rho] = -[f] \in \pi_n(X, x_0)$.

Proof of the Reflection Lemma 167.27. Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 2578 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. The map is also illustrated in the figure below:



If one goes back to the actual definition of this homeomorphism, then one sees that the following diagram commutes:

$$\begin{array}{ccc} I^n & \xrightarrow{\varphi} & \overline{B}^n \\ {\scriptstyle (t_1,t_2,\ldots,t_n)\mapsto(1-t_1,t_2,\ldots t_n)}\downarrow & & \downarrow{\scriptstyle \rho} \\ I^n & \xrightarrow{\varphi} & \overline{B}^n. \end{array}$$

The lemma thus follows from the π_n-Group Proposition 134.6 (2). ∎

Sketch of a proof of the Addition-in-π_n-Proposition 167.25. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given weakly-disjoint almost-smooth embeddings $\alpha_1, \ldots, \alpha_m\colon \overline{B}^n \to S^n$ and continuous maps $f_1, \ldots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Furthermore let $A \in \mathrm{SO}(n+1)$ such that $A \cdot * \in S^n \setminus (\alpha_1(B^m) \cup \cdots \cup \alpha_m(B^n))$. We need to prove the following statement:

Statement. We denote by $\rho_A\colon S^n \to S^n$ the map that is given by $x \mapsto A \cdot x$. Given $i \in \{1, \dots, m\}$ we write $\epsilon_i = 1$ if α_i is orientation-preserving otherwise we write $\epsilon_i = -1$. Then we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Xi_{\overline{B}^n}(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m) \circ \rho_A \circ \psi}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)}\big] = \sum_{i=1}^{m} \epsilon_i \cdot \big[\, f_i\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)\,\big].$$

We start out with the following observations:

(1) Let $\rho\colon \overline{B}^n \to \overline{B}^n$ be the reflection in the $(x_1 = 0)$-hyperplane of $\mathbb{R}^n$. We replace every pair (α_i, f_i) for which α_i is orientation-reversing by the pair $(\alpha_i \circ \rho, f_i \circ \rho)$. Note that:
   (a) The corresponding map $\Xi_{\overline{B}^n}(\dots)$ is unchanged.
   (b) The almost-smooth embedding $\alpha_i \circ \rho\colon \overline{B}^n \to S^n$ is now orientation-preserving.
   (c) It follows from the Reflection Lemma 167.27 that $[f_i \circ \rho] = -[f_i] \in \pi_n(X, x_0)$.

   It follows that it suffices to consider the case that the α_i are orientation-preserving.

(2) For $i = 1, \dots, m$ we consider the smooth isotopy

$$\begin{aligned} F_i\colon \overline{B}^n \times [0,1] &\to S^n \\ (P, t) &\mapsto \alpha_i(P \cdot (1 - \tfrac{t}{2})) \end{aligned}$$

   from $\alpha_i = F_{i0}$ to $\widetilde{\alpha}_i := F_{i1}$. We make the following observations:
   (a) Since the images of the α_i are weakly-disjoint we see that the images of the maps $\widetilde{\alpha}_i$ are disjoint.
   (b) Since the α_i are almost-smooth embeddings we see that the maps $\widetilde{\alpha}_i\colon \overline{B}^n \to S^n$ are actually smooth embeddings.

   It follows from this discussion and the Moving Balls Lemma 167.26 that it suffices to consider the case that the α_i are smooth embeddings with disjoint images.

(3) By the Matrix Group-π_0-Proposition 4.54 (3) there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(n+1)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. This path gives rise to an isotopy from id_{S^n} to the map that is given by multiplication by A. Note that

$$\begin{aligned} S^n \times [0,1] &\to X \\ (x, t) &\mapsto \Xi_{S^n}(\rho_{\gamma(t)}^{-1} \circ \alpha_1, \dots, \rho_{\gamma(t)}^{-1} \circ \alpha_m, f_1, \dots, f_m) \circ \rho_{\gamma(t)}^{-1} \circ \rho_A \end{aligned}$$

   is a homotopy of maps of pointed topological spaces $(S^n, *) \to (X, x_0)$. It follows from this discussion that it suffices to consider the case that $A = \mathrm{id}$.

(4) Note that the explicit map $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ from the Ball Quotient-Sphere Lemma 8.10 restricts to a diffeomorphism $B^n \to S^n \setminus \{*\}$.

(5) By definition of Ξ we have the following equality of maps:

$$\underbrace{\Xi_{S^n}(\alpha_1, \dots, \alpha_m, f_1, \dots, f_m)}_{\text{map } S^n \to X} \circ \psi = \underbrace{\Xi_{\overline{B}^n}(\psi \circ \alpha_1, \dots, \psi \circ \alpha_m, f_1, \dots, f_m)}_{\text{map } \overline{B}^n \to X}\colon \overline{B}^n \to X.$$

Keeping the above five observations in mind we obtain from the Moving Balls Lemma 167.26 that it suffices to prove the following statement:

Statement. If $\beta_1, \dots, \beta_m \colon \overline{B}^n \to B^n$ are orientation-preserving smooth embeddings with disjoint embeddings, then the following equality holds in $\pi_n(X, x_0)$:

$$\big[\, \Xi_{\overline{B}^n}(\beta_1, \dots, \beta_m, f_1, \dots, f_m) \colon (\overline{B}^n, S^{n-1}) \to (X, x_0) \,\big] \;=\; \sum_{i=1}^{m} \big[\, f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0) \,\big].$$

Perhaps the key observation in the proof is that the Smooth Ball Embedding Theorem 45.10[1799] together with the Moving Balls Lemma 167.26 implies that we only need to prove the statement for a single choice of orientation-preserving smooth embeddings $\beta_1, \dots, \beta_m$ with disjoint embeddings. Put differently, it remains to prove the following statement:

Statement. There exists a collection $\beta_1, \dots, \beta_m \colon \overline{B}^n \to B^n$ of orientation-preserving smooth embeddings with disjoint embeddings such that

$$\big[\, \Xi_{\overline{B}^n}(\beta_1, \dots, \beta_m, f_1, \dots, f_m) \colon (\overline{B}^n, S^{n-1}) \to (X, x_0) \,\big] \;=\; \sum_{i=1}^{m} \big[\, f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0) \,\big].$$

Proof. By induction it suffices to prove the claim for $m = 2$. Even your famously verbose author does not feel like spelling out all the details. We leave it to the reader to piece together the argument using the figure below and the following observations:

(a) By the Shrinking Lemma 134.8 we can "move cubes" in I^n.
(b) By the Convex–Isotopic-to-Ball Lemma 18.7 the map $\varphi^{-1} \colon \overline{B}^n \to I^n$ is isotopic to the identity.
(c) The restriction of the map $\varphi \colon I^n \to \overline{B}^n$ to each open cone on one of the sides of the cube I^n is smooth.

Using the Moving Balls Lemma 167.26 these moves translate into homotopies of the corresponding maps. ∎

167.8. **Appendix: Proof of the Faces–Addition Lemma 167.23 II.** In this appendix we will use the results from the previous appendix to conclude the proof of the Faces–Addition Lemma 167.23.

In Notation 167.19 we introduced a homeomorphism $\eta_n \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$. Unfortunately the restriction of this homeomorphism to $\mathring{\Delta}^n$ is not smooth. We will rectify this nuisance using the following lemma.[1800]

Lemma 167.28. (Δ^n-Diffeomorphism Lemma) Let $n \in \mathbb{N}$. We set $P := (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ in $\mathbb{R}^n$. We consider the maps

$$\begin{array}{rcl} f \colon \mathring{\Delta}^n & \to & \mathbb{R}^n \\ (t_0, t_1, \dots, t_n) & \mapsto & \dfrac{((t_1, \dots, t_n) - P)}{t_0 \cdot t_1 \cdot \ldots \cdot t_n} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g \colon \mathbb{R}^n & \to & B^n \\ x & \mapsto & \dfrac{x}{\sqrt{1+\|x\|^2}}. \end{array}$$

(1) The map $f \colon \mathring{\Delta}^n \to \mathbb{R}^n$ is a diffeomorphism. It is orientation-preserving[1801] if and only if n is even.
(2) The map $g \colon \mathbb{R}^n \to B^n$ is an orientation-preserving diffeomorphism.

[1799] Here we use our hypothesis that $n \geq 2$ and here we use that we work with smooth embeddings not just embeddings of balls.
[1800] The next lemma is a slight reformulation of the results of Exercise 143.7.

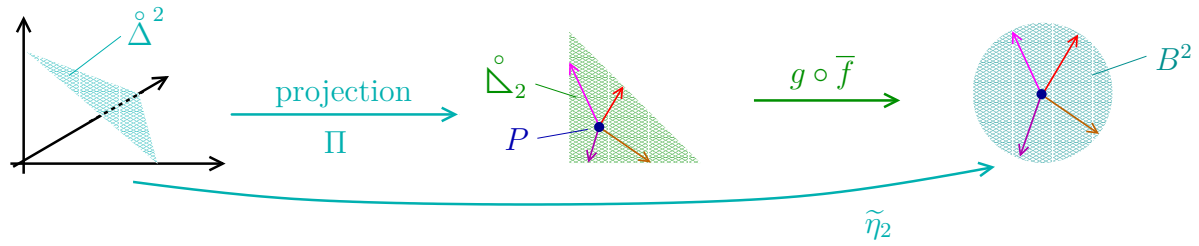
(3) The map $g \circ f \colon \mathring{\Delta}^n \to B^n$ extends to a homeomorphism $\widetilde{\eta}_n \colon \Delta^n \to \overline{B}^n$.
(4) Let $\eta_n \colon \Delta^n \to \overline{B}^n$ be the homeomorphism from Notation 167.19. The homeomorphisms $\eta_n, \widetilde{\eta}_n \colon \Delta^n \to \overline{B}^n$ are isotopic rel $\partial\Delta^n$. In particular they agree on $\partial\Delta^n$.

Sketch of proof. Let $n \in \mathbb{N}$. We introduce some notation:

- As in the Δ^n-Homeomorphism Lemma 143.9 (1) we consider
$$\measuredangle_n \;:=\; \left\{(x_1,\dots,x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \in [0,1]\right\} \;\subseteq\; \mathbb{R}^n.$$
- We denote by $\Pi \colon \Delta^n \to \measuredangle_n$ the projection given by $(t_0,\dots,t_n) \mapsto (t_1,\dots,t_n)$.
- We consider the map
$$\begin{array}{rcl} \overline{f} \colon \mathring{\measuredangle}_n & \to & \mathbb{R}^n \\ (t_1,\dots,t_n) & \mapsto & \dfrac{((t_1,\dots,t_n)-P)}{(1-t_1-\dots-t_n)\cdot t_1 \cdot \ldots \cdot t_n} \end{array}.$$
Evidently $f = \overline{f} \circ \Pi$.

Note that the map $g \circ \overline{f} \colon \mathring{\measuredangle}_n$ has the following property:

(∗) For each $x \in \partial\measuredangle_n$ the map $g \circ \overline{f} \colon \mathring{\measuredangle}_n \to B^n$ sends the half-open Euclidean segment $[P,x)$ bijectively to the half-open Euclidean segment $[0, \frac{x-P}{\|x-P\|})$.



Now we turn to the actual proofs of the four statements:

(1) One can easily verify that $\Pi \colon \mathring{\Delta}^n \to \mathring{\measuredangle}_n$ is a diffeomorphism which is orientation-preserving if and only if n is even. Furthermore, using (∗) one can equally easily show that $g \circ \overline{f} \colon \measuredangle_n \to B^n$ is a bijection. Using this observation and using the Smooth Invariance-of-Domain Theorem 24.22 one can show that the map $g \circ \overline{f} \colon \measuredangle_n \to B^n$ is in fact a diffeomorphism. Using the Orientation Checking-via-Differentials Lemma 31.30 one can easily verify that $g \circ \overline{f} \colon \measuredangle_n \to B^n$ is orientation-preserving.
(2) On page 652 we already pointed out that the map $B^n \to \mathbb{R}^n$, given by $x \mapsto \frac{x}{\sqrt{1-\|x\|^2}}$ is an inverse to $g \colon \mathbb{R}^n \to B^n$. This shows that g is a diffeomorphism.
(3) This statement follows relatively easily from (∗).
(4) Note that $\eta_n \colon \Delta^n \to \overline{B}^n$ can be written as $\eta_n = \overline{\eta}_n \circ \Pi$ where $\overline{\eta}_n \colon \measuredangle_n \to \overline{B}^n$ also has the property (∗). The desired isotopy is thus given by
$$\begin{array}{rcl} \Delta^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \widetilde{\eta}_n(\Pi(x)) \cdot (1-t) + \overline{\eta}_n(\Pi(x)) \cdot t. \end{array}$$
■

Proof of the Faces–Addition Lemma 167.23. Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}_{\geq 2}$ and let $\sigma \colon \partial\Delta^{n+1} \to X$ be a continuous map that sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0. Finally let $A \in \mathrm{SO}(n+1)$ be such that $A \cdot * \in \eta_{n+1}((n-1)\text{-skeleton of } \partial\Delta^{n+1})$. We denote by $\rho_A \colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ the map that is given by $x \mapsto A \cdot x$. Recall that we need to prove the following statement:

[1801] We use Definition 143.8 to view $\mathring{\Delta}^n$ as an oriented smooth manifold.

Statement. The following equality holds in $\pi_n(X, x_0)$:

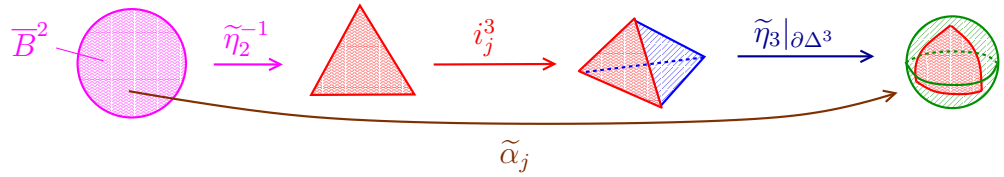
$$[\underbrace{\sigma \circ (\eta_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)}] \;=\; -\sum_{j=0}^{n+1} (-1)^j \cdot [\; \underbrace{\sigma \circ i_j^{n+1} \circ \eta_n^{-1}}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)} \;].$$

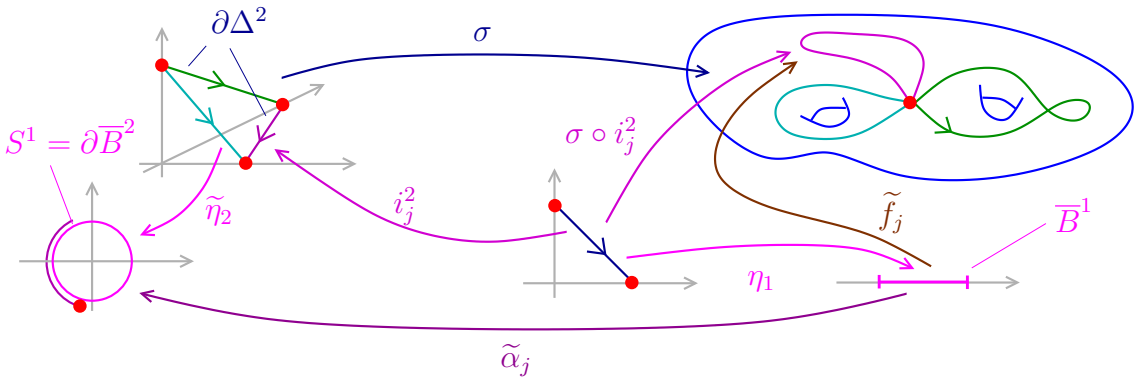
The reader might already have noticed that this statement resembles the statement of the Addition-in-π_n-Proposition 167.25. Indeed, after some massaging we will be able to deduce the desired statement from the Addition-in-π_n-Proposition 167.25.

Let $j \in \{0, \dots, n+1\}$. We consider the maps

$$\begin{aligned} f_j &:= \sigma \circ i_j^{n+1} \circ \eta_n^{-1} \colon \overline{B}^n \;\to\; X, \\ \widetilde{f}_j &:= \sigma \circ i_j^{n+1} \circ \widetilde{\eta}_n^{-1} \colon \overline{B}^n \;\to\; X \end{aligned}$$

and the embedding $\qquad \widetilde{\alpha}_i := \widetilde{\eta}_{n+1} \circ i_j^{n+1} \circ \widetilde{\eta}_n^{-1} \colon \overline{B}^n \;\to\; S^n.$





The above statement follows from the following equalities in $\pi_n(X, x_0)$:

$$\begin{aligned} [\sigma \circ (\eta_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi] &\overset{(6)}{=} [\sigma \circ (\widetilde{\eta}_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi] \\ &\overset{(2)}{=} [\Xi_{S^n}(\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_n, \widetilde{f}_0, \dots, \widetilde{f}_n) \circ \rho_A|_{S^n} \circ \psi] \\ &\overset{(3)}{=} -\sum_{j=0}^{n+1} (-1)^j \cdot [\, \widetilde{f}_j \,] \;\overset{(4)}{=}\; -\sum_{j=0}^{n+1} (-1)^j \cdot [\, f_j \,]. \end{aligned}$$

In the following we explain where the equalities are coming from:

(1) This equality holds since $\eta_{n+1}|_{S^n} = \widetilde{\eta}_{n+1}|_{S^n} \colon S^n \to \partial\Delta^{n+1}$.

(2) By definition we have $\sigma \circ (\widetilde{\eta}_{n+1}|_{\partial\Delta^{n+1}})^{-1} = \Xi_{S^n}(\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_n, \widetilde{f}_0, \dots, \widetilde{f}_n) \colon S^n \to X$.

(3) This equality holds by the Addition-in-π_n-Proposition 167.25 and the following two observations:

  (a) The maps $\widetilde{\alpha}_i \colon \overline{B}^n \to S^n$ are almost-smooth embeddings.

  (b) By the Δ^n-Diffeomorphism Lemma 167.28 we know that the map $\widetilde{\eta}_k \colon \mathring{\Delta}^k \to S^k$ is orientation-preserving if and only if k is even. Furthermore note that it is fairly straightforward to see that the map $i_j^{n+1} \colon \Delta^n \to \partial\Delta^{n+1}$ is orientation-preserving if

and only if j is even. In summary, the almost-smooth embedding $\widetilde{\alpha}_j\colon \widetilde{\eta}_{n+1}\circ i_j^{n+1}\circ\widetilde{\eta}_n^{-1}$ is orientation-preserving if and only if j is odd.[1802]

(4) By the Δ^n-Diffeomorphism Lemma 167.28 (4) the maps $\eta_n, \widetilde{\eta}_n\colon \Delta^n \to \overline{B}^n$ are isotopic rel $\partial\Delta^n$. It follows from Exercise 18.7 that the inverses $\eta_n^{-1}, \widetilde{\eta}_n^{-1}\colon \overline{B}^n \to \Delta^n$ are isotopic rel S^{n-1}. It follows from this discussion that the maps $\widetilde{f}_j$ and f_j are homotopic as maps of pairs $(\overline{B}^n, S^{n-1}) \to (X, x_0)$. ■

167.9. Exercises for Chapter 167 .

Exercise 167.1. Let X be an n-dimensional CW-complex and let Y be a subcomplex of X that is homotopy equivalent to S^n. Show that the inclusion-induced map $\pi_n(Y) \to \pi_n(X)$ is a monomorphism.

Exercise 167.2. We denote by $*$ suitable base points of the spheres. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a continuous map with $f(*) = *$. We suppose that f is a map of degree $k \in \mathbb{Z}$, i.e. f is a map such that the induced map $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by k.

(a) Let $m \in \mathbb{N}_{\geq 2}$ and let $g\colon S^n \to S^m$ be a continuous map. Show that we have the equality $[g \circ f] = k \cdot [g] \in \pi_n(S^m)$.

(b) Show that if $g\colon S^m \to S^n$ is a continuous map, then it is not necessarily the case that the equality $[f \circ g] = k \cdot [g] \in \pi_m(S^n)$ holds.
Hint. We view S^3 as a subset of $\mathbb{C}^2$ and we make the usual identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$. Now consider the following commutative diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{(z,w)\mapsto[(z,w)]} & S^2 \\ {\scriptstyle (z,w)\mapsto(\overline{z},\overline{w})}\downarrow & & \downarrow{\scriptstyle [(z,w)]\mapsto[(\overline{z},\overline{w})]} \\ S^3 & \xrightarrow{(z,w)\mapsto[(z,w)]} & S^2. \end{array}$$

(c) Let $g\colon S^{m-1} \to S^{n-1}$ be a continuous map. We denote by $\Sigma(g)\colon S^m \to S^n$ the suspension of g, see Definition 149.2. Show that $[f \circ \Sigma(g)] = k \cdot [\Sigma(g)] \in \pi_m(S^n)$.

Exercise 167.3. Given $n \in \mathbb{N}$ we denote by $L(n,1) = S^3/(\mathbb{Z}/n\mathbb{Z})$ the lens space from Definition 91.4. For which $n \in \mathbb{N}$ is the Hurewicz homomorphism $\pi_3(L(n,1)) \to \mathrm{H}_3(L(n,1))$ an isomorphism?

Exercise 167.4. Let X be a topological homology n-sphere, i.e. X is an n-dimensional topological manifold such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$. We denote by $\Sigma(X)$ the suspension of X from Definition 20.21. Show that $\pi_i(\Sigma(X)) = 0$ for $i = 1, \ldots, n$.

Exercise 167.5. Let $n \in \mathbb{N}$. Let $\varphi_n\colon S^1 \to S^1$ be the map that is given by $z \mapsto z^n$. We consider the topological space $X_n := \overline{B}^2 \cup_\varphi S^1$. In other words, X_n is the CW-complex obtained by attaching a 2-cell to S^1 via the attaching map φ_n.

(a) Do you recognize X_2?

(b) Compute $\pi_1(X_n)$.

(c) Compute $\pi_2(X_n)$.

[1802] Here we used the deep fact that precisely one out of the two numbers n, $n+1$ is odd.

Exercise 167.6. Let $\varphi\colon S^1 \to S^5$ be a smooth embedding.

(a) Compute $\mathrm{H}_*(S^5 \setminus \varphi(S^1))$.
Hint. You could for example use the Product Tubular Neighborhood Theorem 58.4 (2).

(b) Show that $\pi_3(S^5 \setminus \varphi(S^1)) \neq 0$.

Remark. For (b) you need to use that φ is a smooth embedding, the statement is in general not true, if we only assume that φ is an embedding. More precisely, in [**Haj2019**] it is shown that there exists an embedding $\varphi\colon S^1 \to S^5$ such that $\pi_3(S^5 \setminus \varphi(S^1)) = 0$.

Exercise 167.7.

(a) Show that there exists a closed orientable connected non-empty 3-dimensional smooth manifold M such that $\pi_2(M)$ is not finitely generated.

(b) For which other dimensions can you find smooth manifolds as in (a)?

Exercise 167.8. For $n \geq 2$ determine $\pi_{n-1}(\mathbb{R}\mathrm{P}^n \# \mathbb{R}\mathrm{P}^n)$.

Exercise 167.9. Let X be a connected non-empty countable CW-complex. Show that $\pi_2(X)$ is countable.
Remark. This answers Question 136.24 in the affirmative for $n = 2$.

Exercise 167.10. Determine $\pi_2(\mathbb{R}\mathrm{P}^3 \vee S^2)$.

Exercise 167.11. Let $p, q \in \mathbb{N}$ be coprime. We consider the lens space $L(p,q)$. Determine $\pi_3(L(p,q) \vee S^3)$.

Exercise 167.12. We consider the two topological spaces $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$.

(a) Show that the homology groups of $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$ are isomorphic.

(b) Use suitable homotopy groups to show that $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$ are not homotopy equivalent.

Exercise 167.13. Let X and Y be two simply connected CW-complexes. We form the wedges $X \vee S^1$ and $Y \vee S^1$.

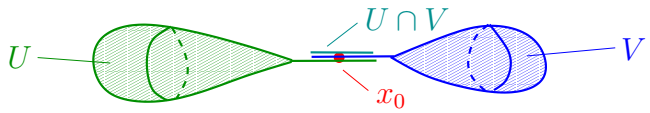
(a) Show that if $\pi_2(X) = 0$, then $\pi_2(X) \cong \pi_2(Y)$ if and only if $\pi_2(X \vee S^1) \cong \pi_2(Y \vee S^1)$.

(b) Show that in (a) we cannot drop the hypothesis that $\pi_2(X) = 0$.

Exercise 167.14. Let X be a topological space and let U and V be open subsets of X. Let $n \in \mathbb{N}_{\geq 2}$. Suppose that U and V are $(n-1)$-connected and that $U \cap V$ is n-connected. We pick a base point $x_0 \in U \cap V$ and we denote by $i\colon U \to X$ and $j\colon V \to X$ the inclusion maps. Show that the map

$$i_* \oplus j_*\colon \pi_n(U, x_0) \oplus \pi_n(V, x_0) \;\to\; \pi_n(X, x_0)$$

is an isomorphism. (An example with $n = 2$ is shown in the figure below.)



Exercise 167.15. Let M and N be two closed oriented n-dimensional smooth manifolds. Suppose that there exists a $k \in \mathbb{N}$ such that M and N are k-connected. Show that the connected sum $M \# N$ is also k-connected.
Hint. Use Exercise 167.14.

Exercise 167.16. We consider the two topological spaces $\mathbb{R}\mathrm{P}^2 \vee S^3$ and $\mathbb{R}\mathrm{P}^3$.

(a) Show that the homology groups of $\mathbb{R}\mathrm{P}^2 \vee S^3$ and $\mathbb{R}\mathrm{P}^3$ are isomorphic.
(b) Use suitable homotopy groups to show that $\mathbb{R}\mathrm{P}^2 \vee S^3$ and $\mathbb{R}\mathrm{P}^3$ are not homotopy equivalent.

Exercise 167.17. Which (if any) of the following statements is true:

(a) We fix a point $Q \in S^1 \times S^2$. Let $f\colon S^1 \times S^2 \to S^1 \times S^2$ be a continuous map such that $f(Q) = Q$. If f_* induces the identity map on H_1 and H_2, then it also induces the identity map on π_1 and π_2.
(b) We fix a point $Q \in S^1 \vee S^2$. Let $f\colon S^1 \vee S^2 \to S^1 \vee S^2$ be a continuous map such that $f(Q) = Q$. If f_* induces the identity map on H_1 and H_2, then it also induces the identity map on π_1 and π_2.

Exercise 167.18. Let X be a simply connected 4-dimensional CW-complex with no 2-cell, $\chi(X) = 5$ and $\pi_3(X) \cong \mathbb{Z}^5 \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$.

(a) Determine the homology groups $\mathrm{H}_n(X; \mathbb{Q})$.
(b) Determine the homology groups $\mathrm{H}_n(X; \mathbb{Z}/2\mathbb{Z})$.
(c) Determine the homology groups $\mathrm{H}_n(X; \mathbb{Z}/6\mathbb{Z})$.

Exercise 167.19. Does there exist a 2-dimensional CW-complex X with $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_2(X) \cong \mathbb{Z}^3$ and $\chi(X) = 1$?

Exercise 167.20.

(a) Show that there exists a finite non-empty path-connected 2-dimensional CW-complex X such that $\mathrm{H}_1(X) = 0$ and such that $\pi_1(X)$ is non-trivial but finite.
(b) Show that there exists a non-empty connected CW-complex such that $\mathrm{H}_1(X) = 0$ but such that $\pi_2(X)$ is not isomorphic to $\mathrm{H}_2(X)$.
Remark. This shows that we cannot loosen the hypothesis of the Hurewicz Theorem 167.9.

For the next three exercises we need to recall Definition 126.11:

Definition 167.29. Let X be a non-empty topological space. We say that X is aspherical if X is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

Exercise 167.21. Let X be a connected non-empty 2-dimensional CW-complex such that $\pi_2(X) = 0$. Show that X is aspherical.

Exercise 167.22. Show that there exists no finite-dimensional aspherical CW-complex with non-trivial finite fundamental group.
Hint. Consider Euler characteristics.

Exercise 167.23. Let X be a CW-complex and let U and V be path-connected subcomplexes of X. Suppose that $U \cap V$ is path-connected and furthermore suppose that the inclusion-induced maps $\pi_1(U \cap V) \to \pi_1(U)$ and $\pi_1(U \cap V) \to \pi_1(V)$ are monomorphisms. Finally suppose that U, V and $U \cap V$ are aspherical. Show that X is also aspherical.
Hint. You might want to use the Amalgamated Product-Monomorphism Proposition 98.26, the Preimage-of-Covering Proposition 129.7, the Mayer–Vietoris Theorem 153.2 (2) and Exercise 167.14.

Exercise 167.24. Recall that according to the definition on Definition 109.5 a link $L \subseteq S^3$ is called *splittable* if there exist two disjoint balls in S^3 each of which contains at least one component of L. Show that for a splittable link $L \subseteq S^3$ we have $\pi_2(S^3 \setminus L) \neq 0$.
Hint. Show that there exists a sphere $S \subseteq S^3 \setminus L$ which is a retract of $S^3 \setminus L$.



Exercise 167.25. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p, q)$ from Definition 111.13. Show that for every $n \in \mathbb{N}_{\geq 2}$ we have $\pi_2(S^3 \setminus T(P, q)) = 0$.
Hint. Use the description of $S^3 \setminus T(p, q)$ that we gave in the proof of the Torus Knot-π_1-Proposition 111.18, use the Covering-π_n-Isomorphism Proposition 134.22 and use the Hurewicz Theorem 167.9

CHAPTER 168

The social choice problem

In this chapter we will discuss a "real-life" problem, namely the so-called "social choice problem". Using the Hurewicz Theorem 167.9, we will see that in many cases of interest there is just no good solution to this "social choice problem". This chapter is supposed to be pure fun. The results of this chapter do not really play any role in any of the other chapters.

This chapter is organized as follows:

- In Section 168.1 we will explain the social choice problem.
- In Section 168.2 we will prove the Social Choice Proposition 168.3, using our calculation of $\pi_n(S^n)$ from the previous chapter.
- In Section 168.3 we will provide exercises for Chapter 168.

In the next Chapter 169 we will return to serious mathematics and we will see that higher homotopy groups should be viewed as modules over group rings.

168.1. **The social choice problem.** In this chapter we will consider the real-life problem that $n \in \mathbb{N}_{\geq 2}$ people have to agree in a democratic and reasonable way on one of the following:

(1) The temperature setting of a joint office.
(2) A vacation spot in Bavaria.
(3) A vacation spot on Earth.

More generally, the n people are supposed to agree on the following:

(4) A point in some topological space X (e.g. $\mathbb{R}$, Bavaria or S^2).

One reasonable approach to solving this problem is the following. Each person gets to pick a point in X. We then need a map f which takes as input the n choices of points in X and which turns these choices into a common choice. In other words, we need a map

$$f\colon X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}} \to X.$$

Our interpretation that the common choice should be chosen "democratically and reasonably" is that the map f should have the following three properties:

(a) A small change in choices of the people involved should correspond to a small change in the common choice. In other words, f should be continuous.
(b) If everybody is of the same opinion, then evidently the common choice should be the point everybody picked.
(c) Everybody should be treated fairly, i.e. the social choice should not depend on the ordering of the n people.

In some cases it is easy to come up with a social choice. For example if X is a convex subset of some $\mathbb{R}^m$, e.g. if $X = \mathbb{R}^m$ itself, then given any $n \in \mathbb{N}$ the map[1803]

$$\begin{array}{rcl} X^n & \to & X \\ (x_1, \ldots, x_n) & \mapsto & \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \end{array}$$

evidently satisfies all of the above conditions. In particular this approach solves the problem for (1). If X is homeomorphic to a convex subset of some $\mathbb{R}^m$, then we can translate our problem into the previous situation and we can find a solution. In particular, Bavaria being to the best of my knowledge homeomorphic to a closed disk, we see that we have a solution to (2).[1804]

In the following we will study the third problem, namely the case $X = S^2$, in greater detail. At this point it is convenient to introduce the following definition from [**Eckm2004**], which is just a formalization of our wish list above.

Definition 168.1. Let X be a topological space and let $n \in \mathbb{N}$. A social choice of type n is a map

$$f \colon X^n = \underbrace{X \times \cdots \times X}_{n \text{ times}} \to X$$

such that the following three axioms are satisfied:

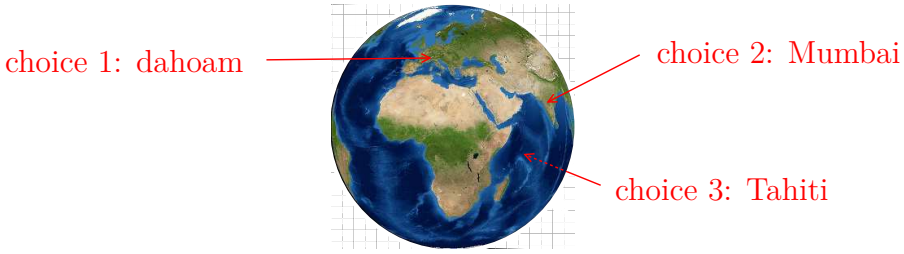
(a) The map f is continuous.
(b) For every $x \in X$ we have $f(x, \ldots, x) = x$.
(c) For every permutation $\sigma \in S_n$ and every $(x_1, \ldots, x_n) \in X^n$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n).$$

We say that a topological space is social, if for every $n \in \mathbb{N}$ it admits a social choice of type n.

Thus the question of whether there exists a "democratic and reasonable" way for n people to settle on a vacation spot on earth can be interpreted as the following question:

Question 168.2. Does there exist a social choice of type n on S^2?



As we will see, this question will be short-lived. The key to answer this question is the following proposition, which is the main technical result of this chapter.

Proposition 168.3. (Social Choice Proposition)

[1803] It follows easily from the convexity of X that the map does indeed take values in X.
[1804] The fact that Bavaria lost the Pfalz in 1946 greatly simplifies finding a solution to (2).

(1) Let X be a path-connected non-empty topological space. If X admits a social choice of type n for some $n \in \mathbb{N}_{\geq 2}$, then given any $k \in \mathbb{N}$ the group[1805] $\pi_k(X)$ is abelian, and it is either finite or it is not finitely generated.
(2) Let X be a path-connected non-empty topological space. If X admits a social choice of type n for every $n \in \mathbb{N}_{\geq 2}$, then given any $k \in \mathbb{N}$ the group[1806] $\pi_k(X)$ is abelian, and it is either zero or it is not finitely generated.
(3) Let X be a path-connected non-empty CW-complex that has the property that for every $k \in \mathbb{N}$ the number of k-cells is finite. If for every $n \in \mathbb{N}_0$ the topological space X admits a social choice of type $n \geq 2$, then for every $k \in \mathbb{N}$ we have $\pi_k(X) = 0$.

Proof. We postpone the proof of the Social Choice Proposition 168.3 to the next section. ■

Remark.

(1) It follows from the Whitehead Theorem 305.1 that in the Social Choice Proposition 168.3 (2) we can upgrade the conclusion to the statement that X is actually contractible.
(2) In the context of sociology, the Social Choice Proposition 168.3 was first formulated in 1980 by the mathematical economist Graciela Chichilnisky [**Chic1980**]. It turns out that the proposition was already proved for some cases by Georg Aumann [**AumG1944**] and in all cases by Beno Eckmann [**Eckm1954**]. The discussion of this section follows mostly [**Eckm2004**]. For more information and some generalizations we refer to [**Eckm2004**] and [**Weinb2004**]. □

We are now already in a position to answer Question 168.2 in the negative.

Corollary 168.4. (Unsocial Sphere Corollary) Let $k \in \mathbb{N}$. The sphere S^k does not admit a social choice for any $n \in \mathbb{N}_{\geq 2}$.

Proof. By the $\pi_n(S^n)$–Proposition 167.10 we know that $\pi_k(S^k) \cong \mathbb{Z}$. Since $\mathbb{Z}$ is finitely generated but not finite we obtain from the Social Choice Proposition 168.3 (1) that S^k does not admit a social choice for any $n \in \mathbb{N}_{\geq 2}$. ■

We conclude this section with the following converse to the Social Choice Proposition 168.3.

Proposition 168.5. (Social Criterion Proposition) Let X be a CW-complex.
(1) If X is homotopy equivalent to a social CW-complex, then X is social.
(2) If X is contractible, then X is social.

Remark. The Social Criterion Proposition 168.5 was first proved in [**ChH**] (see also [**Hor2001**]). □

Proof.

(1) A sketch of a proof of this statement is sketched in [**Weinb2004**, Chapter 1] and [**Eckm2004**, Section 1.2].
(2) This statement follows immediately from (1) together with the fact that any singleton is of course social. ■

[1806] In this section we will only deal with path-connected topological spaces and we will only be interested in the isomorphism type of the higher homotopy groups, thus we will use the Change-of-Base Point Proposition 134.12 as an excuse to ignore base points.

168.2. **Proof of the Social Choice Proposition 168.3 .** The idea behind the proof of the Social Choice Proposition 168.3 is quite simple. We need to see what effect the existence of a social choice on X has on the higher homotopy groups. Perhaps not surprisingly, the properties of a social choice give rise to very similar properties of the homotopy groups. This leads us to the following definition:

Definition 168.6. Let G be a group and let $n \in \mathbb{N}_{\geq 2}$. We say that a map $\varphi\colon G^n \to G$ is a social homomorphism of type n if the following conditions are satisfied:

(a) The map φ is a homomorphism.
(b) For every $g \in G$ we have $\varphi(g, \dots, g) = g$.
(c) For every permutation $\sigma \in S_n$ and every $(g_1, \dots, g_n) \in G^n$ we have

$$\varphi(g_{\sigma(1)}, \dots, g_{\sigma(n)}) \;=\; \varphi(g_1, \dots, g_n).$$

Example. Evidently, if we consider $G = (\mathbb{Q}, +)$ or $G = (\mathbb{R}, +)$, then taking the arithmetic mean $(x_1, \dots, x_n) \mapsto \frac{1}{n} \cdot (x_1 + \dots + x_n)$ defines a social homomorphism. □

The following lemma relates social choices on topological spaces to social homomorphisms.

Lemma 168.7. (Induced Social Homomorphism Lemma) Let X be a path-connected non-empty topological space. If X admits a social choice of some type $n \geq 2$, then for every $k \in \mathbb{N}$ the homotopy group $\pi_k(X)$ admits a social homomorphism of type n.

Proof. To simplify the notation a little bit we only consider the case $n = 2$. Thus let $f\colon X \times X \to X$ be a social choice of type 2. Let $k \in \mathbb{N}$. We fix a base point $x_0 \in X$. We consider the map

$$\begin{array}{rcl} \Theta\colon \pi_k(X, x_0) \times \pi_k(X, x_0) & \to & \pi_k(X \times X, (x_0, x_0)) \\ ([\alpha\colon (I^k, \partial I^k) \to (X, x_0)], [\beta\colon (I^k, \partial I^k) \to (X, x_0)]) & \mapsto & \left[\begin{array}{rcl} (I^k, \partial I^k) & \to & (X \times X, (x_0, x_0)) \\ z & \mapsto & (\alpha(z), \beta(z)) \end{array}\right]. \end{array}$$

Put more succinctly, Θ is given by $\Theta([\alpha], [\beta]) = [(\alpha, \beta)]$. Next we consider the map[1807]

$$\varphi\colon \pi_k(X, x_0) \times \pi_k(X, x_0) \xrightarrow{\Theta} \pi_k(X \times X, (x_0, x_0)) \xrightarrow{f_*} \pi_k(X, x_0).$$

It remains to show that $\varphi = f_* \circ \Theta$ is a social homomorphism of type 2. We do so in the following three steps.

(a) The map Θ is a homomorphism: Indeed, for $\alpha, \alpha', \beta, \beta'\colon (I^k, \partial I^k) \to (X, x_0)$ we have[1808]

$$\begin{array}{ccccccc} & \text{Def. 90.12 of the direct product of two groups} & & \text{Def. 134.4 of the product in } \pi_k(X, x_0) & & \text{definition of } \Theta & \\ \Theta(([\alpha], [\beta]) \cdot ([\alpha'], [\beta'])) & \overset{\downarrow}{=} & \Theta([\alpha] \cdot [\alpha'], [\beta] \cdot [\beta']) & \overset{\downarrow}{=} & \Theta([\alpha * \alpha'], [\beta * \beta']) & \overset{\downarrow}{=} & [(\alpha * \alpha', \beta * \beta')] \\ & = & [(\alpha, \beta) * (\alpha', \beta')] & & = \Theta([\alpha], [\beta]) \cdot \Theta([\alpha'], [\beta']). & & \\ & \uparrow & & & & & \\ & \text{since } (\alpha * \alpha', \beta * \beta') = (\alpha, \beta) * (\alpha', \beta') \text{ as maps } I^k \to X \times X & & & & & \end{array}$$

Since f_* is a homomorphism we see that $\varphi := f_* \circ \Theta$ is a homomorphism.

[1807] Note that here we use that the fact that f is a social choice implies that $f(x_0, x_0) = x_0$ which in turn ensures that we get an induced homomorphism $f_*\colon \pi_k(X \times X, (x_0, x_0)) \to \pi_k(X, x_0)$.

[1808] In fact Θ is an isomorphism. To see this we use the notation from the Product-π_n-Proposition 134.18. It follows easily from the definitions that $\Phi \circ \Theta = \Phi \circ \Psi$, thus $\Theta = \Psi$ is an isomorphism. For the proof of the present lemma we do not need this fact.

(b) For every $g = [\alpha\colon I^k \to X] \in \pi_k(X, x_0)$ we have

$$\Theta(g,g) \;=\; \Theta([\alpha],[\alpha]) \;=\; \left[\begin{array}{rcl} (I^k, \partial I^k) & \to & (X \times X, (x_0, x_0)) \\ z & \mapsto & \underbrace{f(\alpha(z), \alpha(z))}_{=\alpha(z)} \end{array}\right] \;=\; [\alpha] \;=\; g.$$

(c) For every $g = [\alpha\colon I^k \to X] \in \pi_k(X, x_0)$ and $h = [\beta\colon I^k \to X] \in \pi_k(X, x_0)$ we have

$$\Theta(g,h) \;=\; \Theta([\alpha],[\beta]) \;=\; \left[\begin{array}{rcl} (I^k, \partial I^k) & \to & (X \times X, (x_0, x_0)) \\ z & \mapsto & \underbrace{f(\alpha(z), \beta(z))}_{=f(\beta(z),\alpha(z))} \end{array}\right] \;=\; \Theta([\beta],[\alpha]) \;=\; \Theta(h,g).$$

■

In light of the Induced Social Homomorphism Lemma 168.7 we now need to understand, given $n \in \mathbb{N}_{\geq 2}$, which groups admit social homomorphisms of type n.

Definition 168.8. Let G be an abelian group.

(1) Let $n \in \mathbb{N}$. We say that G is strongly divisible by n, if the map $g \mapsto n \cdot g$ is an isomorphism of the group G.
(2) We say that G is strongly divisible, if G is strongly divisible by every $n \in \mathbb{N}$.

Example. Clearly the group $(\mathbb{Q}, +)$ is strongly divisible. □

Remark. Usually an abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. This will also be our definition later in Definition 201.6. Clearly a group which is strongly divisible is also divisible. But the converse does not hold, for example the abelian group $(\mathbb{Q}/\mathbb{Z}, +)$ is divisible but it is not strongly divisible. □

The following lemma gives a satisfying characterization of groups that admit a social homomorphism of type n.

Lemma 168.9. (Social Homomorphism Lemma) Let G be a group and let $n \in \mathbb{N}_{\geq 2}$. The following two statements are equivalent:

(1) The group G admits a social homomorphism of type n.
(2) The group G is abelian and it is strongly divisible by n.

Proof. In the following let G be a group and let $n \in \mathbb{N}_{\geq 2}$.
(2)⇒(1). Suppose that G is abelian and that G is strongly divisible by n. Since G is abelian we use additive notation. Now note that evidently the usual "arithmetic mean formula"

$$(g_1, \ldots, g_n) \;\mapsto\; \frac{1}{n} \cdot (g_1 + \cdots + g_n) \;\coloneqq\; \begin{array}{c} \text{inverse of the isomorphism} \\ G \to G \text{ given by } g \mapsto n \cdot g, \\ \text{applied to } g_1 + \cdots + g_n \in G \end{array}$$

defines a social homomorphism of type n.
(1)⇒(2). We assume that there exists a social homomorphism $\varphi\colon G^n \to G$ of type $n \geq 2$. Recall that this means that the following statements hold:

(a) The map φ is a homomorphism.
(b) For every $g \in G$ we have $\varphi(g, \ldots, g) = g$.
(c) For every $\sigma \in S_n$ and every $(g_1, \ldots, g_n) \in G^n$ we have $\varphi(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \varphi(g_1, \ldots, g_n)$.

We need to show that G is abelian and strongly divisible by n. Since we do not yet know that G is abelian we now use multiplicative notation for G. First note that for any $g \in G$

we have the equalities

$$\varphi(g,e,\dots,e) \underset{\text{by (d)}}{=} \varphi(e,g,e,\dots,e) \underset{\text{by (d)}}{=} \varphi(e,\dots,e,g) \in G.$$

We denote this common value by $\psi(g)$. We make the following observations:

(i) It follows immediately from (a) that $\psi\colon G \to G$ is also a homomorphism.

(ii) For any $g,h \in G$ we have

$$\begin{array}{lll} & \text{by definition of } \psi \text{ and since } n \geq 2 & \text{this follows from (a) since } (g,e,\dots)\cdot(e,h,\dots) = (g,h,\dots) \\ \psi(g)\cdot\psi(h) & \overset{\downarrow}{=} \varphi(g,e,\dots,e)\cdot\varphi(e,h,e,\dots,e) & \overset{\downarrow}{=} \varphi(g,h,e,\dots,e) \\ & \underset{\uparrow}{=} \varphi(e,h,\dots,e)\cdot\varphi(g,e,e,\dots,e) & \underset{\uparrow}{=} \psi(h)\cdot\psi(g). \\ & \text{this follows from (a) since } (e,h,\dots)\cdot(g,e,\dots) = (g,h,\dots) & \text{by definition of } \psi \end{array}$$

(iii) For any $g \in G$ we have

$$\underbrace{\psi(g)\cdot\ldots\cdot\psi(g)}_{=\psi(g)^n} \underset{\text{by definition of } \psi}{=} \varphi(g,e,\dots,e)\cdot\ldots\cdot\varphi(e,\dots,e,g) \underset{\text{by (a)}}{=} \varphi(g,\dots,g) \underset{\text{by (d)}}{=} g.$$

(iv) Let $g,h \in G$. We have

$$\begin{array}{llll} & \text{by (iii)} & \text{by (i)} & \\ g\cdot h & \overset{\downarrow}{=} \psi(g\cdot h)\cdot\ldots\cdot\psi(g\cdot h) & \overset{\downarrow}{=} \psi(g)\cdot\psi(h)\cdot\ldots\cdot\psi(g)\cdot\psi(h) & \\ & \underset{\uparrow}{=} \psi(h)\cdot\psi(g)\cdot\ldots\cdot\psi(h)\cdot\psi(g) & \underset{\uparrow}{=} \psi(h\cdot g)\cdot\ldots\cdot\psi(h\cdot g) & \underset{\uparrow}{=} h\cdot g. \\ & \text{by (ii)} & \text{by (i)} & \text{by (iii)} \end{array}$$

This shows that G is abelian.

Now that we know that G is abelian we switch back to additive notation.

(v) Since G is abelian we know that multiplication by n is a homomorphism of G.

(vi) It follows almost immediately from (i) and (iii) that ψ is an inverse to the map $G \to G$ that is given by $g \mapsto n\cdot g$. Together with (v) this shows that $g \mapsto n\cdot g$ does indeed define an isomorphism of G. ∎

The following lemma basically tells us which finitely generated abelian groups are strongly divisible by some $n \in \mathbb{N}_{\geq 2}$.

Lemma 168.10. (Strongly Divisible Lemma) Let $n \in \mathbb{N}_{\geq 2}$.

(1) The group $\mathbb{Z}$ is not strongly divisible by n.

(2) Let $m \in \mathbb{N}$. The cyclic group $\mathbb{Z}/m\mathbb{Z}$ is strongly divisible by n if and only if m and n are coprime.

(3) If a finitely generated group is strongly divisible by n, then it is isomorphic to a direct sum $\bigoplus_{i=1}^{k} \mathbb{Z}_{m_i}$ where each m_i is coprime to n.

Proof. The first two statements are basically trivial. Finally the last statement follows easily from the previous two statements together with the Abelian Group-Classification Theorem 97.24. ∎

Now we have assembled everything we need for the proof of the Social Choice Proposition 168.3.

Proof of the Social Choice Proposition 168.3.

(1) Let X be a path-connected non-empty topological space that admits a social choice of type n for some $n \in \mathbb{N}_{\geq 2}$ and let $k \in \mathbb{N}$. By the Induced Social Homomorphism Lemma 168.7 we know that the group $\pi_k(X)$ is abelian and that it is strongly divisible by n. It follows almost immediately from the Strongly Divisible Lemma 168.10 that $\pi_k(X)$ is either finite or it is not finitely generated.

(2) Let X be a path-connected non-empty topological space that admits a social choice of type n for every $n \in \mathbb{N}_{\geq 2}$ and let $k \in \mathbb{N}$. By the Induced Social Homomorphism Lemma 168.7 we know that the group $\pi_k(X)$ is abelian and that it is strongly divisible by n. It follows almost immediately from the Strongly Divisible Lemma 168.10 that $\pi_k(X)$ is either zero or it is not finitely generated.

(3) Let X be a path-connected non-empty CW-complex that has the property that for every $k \in \mathbb{N}$ the number of k-cells is finite. We assume that X is social, i.e. we assume that given any $n \geq 2$ there exists a social choice of type n on X. By the Induced Social Homomorphism Lemma 168.7 we know that every homotopy group $\pi_k(X)$ is abelian and that it is strongly divisible by n.

Next note that by the CW-π_1-Finite Presentation–Proposition 136.23 (1) and our hypothesis that X has finitely many 1-cells we know that $\pi_1(X)$ is finitely generated. Therefore it follows from the above discussion together with the Strongly Divisible Lemma 168.10 that $\pi_1(X) = 0$.

Now suppose that we already know that $\pi_i(X) = 0$ for $i = 1, \ldots, k$. By the Hurewicz Theorem 167.9 we know that $\pi_{k+1}(X) \cong \mathrm{H}_{k+1}(X)$. Since X has finitely many $(k+1)$-cells we obtain from the CW-Complex-H_*-Proposition 155.10 (1) that $\pi_{k+1}(X) \cong \mathrm{H}_{k+1}(X)$ is finitely generated. Basically the same argument as above shows that $\pi_{k+1}(X) = 0$. Iterating this procedure shows that all homotopy groups of X vanish. ■

We conclude this section with the following question:

Question 168.11. Does there exist a non-empty CW-complex that is social but that is not contractible?

We will return to this question later on in Section 307.6.

168.3. Exercises for Chapter 168 .

Exercise 168.1. Let $n \in \mathbb{N}$. Show that every discrete topological space admits a social choice of type n.

Exercise 168.2. Let $n \in \mathbb{N}$. Show "by hand" that the infinite-dimensional sphere S^∞ admits a social choice of type n.

Exercise 168.3. For which choices of subsets X of $\mathbb{R}^m$ does the geometric mean

$$(x_1, \ldots, x_n) \mapsto (x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}}$$

define a social choice?

Exercise 168.4. Let $n \in \mathbb{N}_{\geq 2}$. How many examples of social choices of type n on $X = \mathbb{R}$ can you come up with?

CHAPTER 169

Group rings and homotopy groups

In this chapter we will see that one should view the higher homotopy groups $\pi_n(X, x_0)$ with $n \geq 2$ as modules over the "group ring" $\mathbb{Z}[\pi_1(X, x_0)]$. As an example we will use the Hurewicz Theorem 167.9 to calculate and express the homotopy group $\pi_2(S^1 \vee S^2)$ as such a module.

This chapter is organized as follows:

- In Section 169.1 we will classify the self-maps of S^n up to homotopy.
- In Section 169.2 we will see that the fundamental group acts on higher homotopy groups.
- In Section 169.3 we will see that higher homotopy groups can be viewed as modules over the group ring of the fundamental group.
- In Section 169.4 we will determine the $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$-module $\pi_2(S^1 \vee S^2)$.
- In Section 169.5 we will provide exercises for Chapter 169.

In the next Chapter 170 we will prove the Jordan-Brouwer Separation Theorem 170.3.

169.1. The degree of a self-map of a sphere II.

First we recall Definition 151.3:

Definition 169.1. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be a continuous map. We define

$$\deg(f) \;:=\; \deg\left(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\right),$$

i.e. $\deg(f)$ is the unique integer such that f_* is given by multiplication by that integer.

In the Sphere–Degree Lemma 151.4 (3) we showed that any two homotopic maps $S^n \to S^n$ have the same degree. Now we have the tools to prove the converse of this statement. In fact we have the following proposition.

Proposition 169.2. (Degree Isomorphism Proposition) For any $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} \deg\colon \overbrace{[S^n, S^n]}^{\substack{\text{set of homotopy equivalence classes} \\ \text{of continuous maps } S^n \to S^n}} & \to & \mathbb{Z} \\ [f\colon S^n \to S^n] & \to & \deg(f) \end{array}$$

is an isomorphism. (The set $[S^n, S^n]$ is a monoid, in the sense of Definition 2.1, through composition of maps.)

Remark. In the Hopf Degree Theorem 46.22 we proved a much stronger result, using very different techniques. □

Proof. First note that we know by the Sphere–Degree Lemma 151.4 (3) and (4) that the degree map is well-defined and that it is a monoid morphism. It remains to show that the degree map is a bijection. Throughout the proof let $* \in S^n$ be the standard base point.

We consider the following diagram:

$$\begin{array}{ccccc}
\langle (S^n,*),(S^n,*)\rangle & \xrightarrow{\cong} & [S^n,S^n] & \xrightarrow{[f]\mapsto \deg(f)} & \mathbb{Z} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle [f]\mapsto f_*([S^n])} & \nearrow{\scriptstyle \cong}\ {\scriptstyle k\cdot[S^n]\mapsto k} & \\
\pi_n(S^n,*) & \xrightarrow[[f]\mapsto f_*([S^n])]{\cong} & \mathrm{H}_n(S^n). & &
\end{array}$$

We make the following observations and clarifications:

(1) Given two pointed topological spaces (X,x_0) and (Y,y_0) we denote by $\langle (X,x_0),(Y,y_0)\rangle$ the set of homotopy classes of continuous maps $(X,x_0)\to(Y,y_0)$.
(2) Note that by the Sphere-to-Sphere Pointed-Homotopy Corollary 140.16 we know that the natural map $\langle (S^n,*),(S^n,*)\rangle \to [S^n,S^n]$ is a bijection.
(3) Since $n\geq 1$, the sphere S^n is connected. We can thus apply the Hurewicz Theorem 167.9 and we obtain that the bottom horizontal map is an isomorphism.
(4) The diagonal map is an isomorphism by the Sphere-H_*-Proposition 149.4.
(5) Finally note that it is clear that the diagram commutes. It follows from this observation and all of the above that the map $\deg\colon [S^n,S^n]\to\mathbb{Z}$ is indeed a bijection. ■

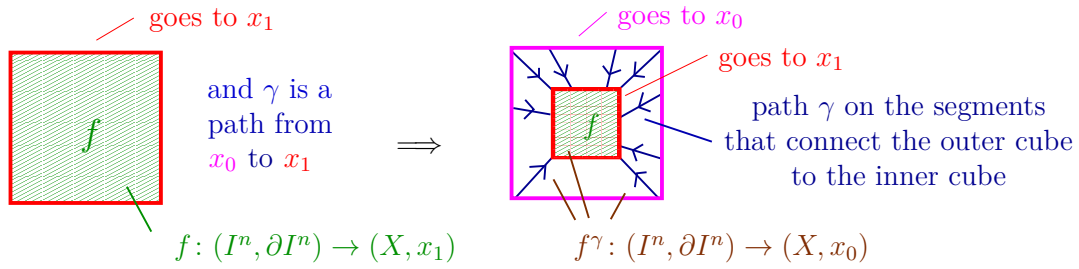
169.2. **Paths acting on homotopy groups.** In the Change-of-Base Point Propositions 88.14 and 134.12 we already saw that for path-connected topological spaces the *isomorphism types* of the homotopy groups do not depend on the choice of the base point.

In this section we want to discuss this statement in greater detail. We recall the following definition which appeared already in the proof of the Change-of-Base Point Proposition 134.12.

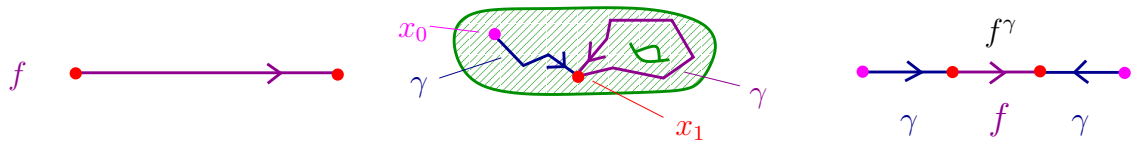
Definition 169.3. Let X be a topological space and let $n\in\mathbb{N}$. Let $\gamma\colon[0,1]\to X$ be a path from a point x_0 to a point x_1. Given a continuous map $f\colon (I^n,\partial I^n)\to(X,x_1)$ we define the map

$$\begin{array}{rcl}
f^\gamma\colon (I^n,\partial I^n) & \to & (X,\{x_0\}) \\
(t_1,\dots,t_n) & \mapsto & \begin{cases} f\big(2t_1-\frac{1}{2},\dots,2t_n-\frac{1}{2}\big), & \text{if } (t_1,\dots,t_n)\in\big[\frac{1}{4},\frac{3}{4}\big]^n, \\ \gamma(2-4t), & \text{if } t:=\max\{|t_1-\frac{1}{2}|,\dots,|t_n-\frac{1}{2}|\}\in[\frac{1}{4},\frac{1}{2}]. \end{cases}
\end{array}$$

Note that it follows easily from the Pasting Proposition 4.7 that this map is continuos.



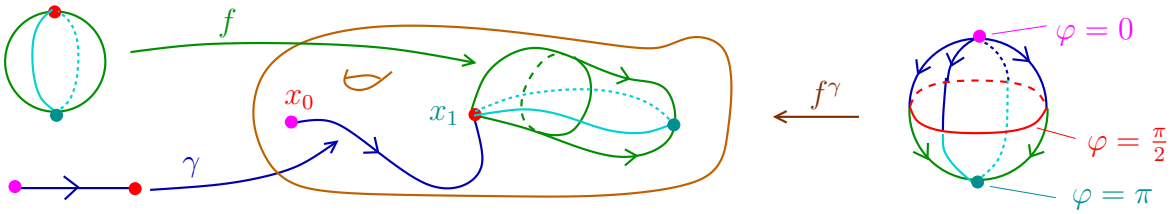
Example. We consider the case $n=1$ in the above Definition 169.3. In this situation the map $f\colon (I,\partial I)\to(X,x_1)$ is a loop.
It follows almost immediately from the definitions that the new loop $f^\gamma\colon (I,\partial I)\to(X,x_0)$ equals, up to an orientation-preserving reparametrization, the loop $\gamma * f * \overline{\gamma}$ that is given by the product of the three paths γ, f and $\overline{\gamma}$. □



Remark. We recall that in Identification 134.9 and in Notation 167.18 we saw that we can interpret elements in a homotopy group $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$ where $* = (0, \ldots, 0, 1)$ is the North Pole. Now let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1 and let $f\colon (S^n, *) \to (X, x_1)$ be a continuous map. We define[1809]

$$\begin{array}{ccc} f^\gamma\colon S^n & \to & X \\ \underbrace{\cos(\varphi)\cdot(0,\ldots,0,1)+\sin(\varphi)\cdot(v,0)}_{\varphi\in[0,\pi],\text{ and }\ v\in\mathbb{R}^n} & \mapsto & \begin{cases} \gamma(\frac{2}{\pi}\cdot t), & \text{if } \varphi\in[0,\frac{\pi}{2}], \\ f(\cos(2\varphi-\pi)\cdot(1,0,\ldots,0)+\sin(2\varphi-\pi)\cdot(v,0)), & \text{if } \varphi\in[\frac{\pi}{2},\pi]. \end{cases} \end{array}$$

Of course one should only work with the formula if one is forced to at gun point. It is much more illuminating to look at the figure below where we try our best to visualize the map $f^\gamma\colon (S^n, *) \to (X, x_0)$.



One can easily verify that, under the various identifications from page 2578, the map f^γ corresponds to the construction in Definition 169.3. □

Before we continue we introduce the following basically self-explanatory definition.

Definition 169.4. Let G be a group and let A be a group (or module or vector space). An action of G on the group A (respectively module or vector space) is an action $G \times A \to A$ in the sense of Definition 2.36 such that for each $g \in G$ the map

$$\begin{array}{ccc} A & \to & A \\ a & \mapsto & g \cdot a \end{array}$$

is a group homomorphism (respectively module or vector space homomorphism).[1810]

The following proposition can be viewed as a more precise version of the Change-of-Base Point Proposition 134.12 and as a generalization of the Change-of-Base Point Proposition 90.4.

[1809] It requires a little bit of thought why this map is continuous: First note that it follows easily from the Pasting Proposition 4.7 (2′) that the given expression defines a continuous map $\alpha\colon [0,1] \times S^{n-1} \to X$ which by the Topological-Quotient Proposition 8.2 descends to a continuous map $\overline{\alpha}\colon ([0,1] \times S^{n-1})/\sim \to X$ where we identify all points in $\{0\} \times S^{n-1}$ and we identify all points in $\{1\} \times S^{n-1}$. We also have an obvious map $\beta\colon ([0,1] \times S^{n-1})/\sim \to S^n$ which is a homeomorphism by the Compact-Hausdorff Proposition 4.23 (3). The map we are interested in is $\overline{\alpha} \circ \beta^{-1}$. So it is continuous by the above discussion.

[1810] In fact, since multiplication by g^{-1} is an inverse we see that these maps are actually isomorphisms.

Proposition 169.5. (Change-of-Base Point Proposition II) Let X be a topological space and let $n \in \mathbb{N}$.

(1) If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is a well-defined isomorphism.

(2) If x_0 and x_1 are two points in X which lie in the same path-component of X, then $\pi_n(X,x_0)$ is isomorphic to $\pi_n(X,x_1)$.

(3) Let $\varphi\colon X \to Y$ be a continuous map between topological spaces.

(a) If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X,x_1) & \xrightarrow{\gamma_*} & \pi_n(X,x_0) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \pi_n(Y,\varphi(x_1)) & \xrightarrow{(\varphi\circ\gamma)_*} & \pi_n(Y,\varphi(x_0)). \end{array}$$

(b) If $\varphi_*\colon \pi_n(X,x_0) \to \pi_n(Y,\varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $\varphi_*\colon \pi_n(X,x_1) \to \pi_n(Y,\varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).[1811]

(4) Let γ and δ be two paths in X from x_0 to x_1. If γ and δ are path-homotopic in X, then the corresponding maps γ_* and δ_* from $\pi_n(X,x_1)$ to $\pi_n(X,x_0)$ agree.

(5) If γ is a path in X from x_0 to x_1 and if δ is a path in X from x_1 to x_2, then[1812]

$$(\underbrace{\gamma * \delta}_{\text{path from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X,x_2) \to \pi_n(X,x_0).$$

(6) The map

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f] \coloneqq [f^\gamma] \end{array}$$

defines an action of the group $\pi_1(X,x_0)$ on the group $\pi_n(X,x_0)$.

(7) Let $f_0, f_1\colon S^n \to X$ be two continuous maps. We write $x_0 = f_0(*)$ and $x_1 = f_1(*)$. The following statement holds:

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff \begin{array}{l} \text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [f_1^\gamma] = [f_0] \in \pi_n(X,x_0). \end{array}$$

Remark. Let X be a topological space, let x_0, x_1 be two points in X and let γ and δ be two paths from x_0 to x_1. If X is simply connected, then the Paths-in-Simply Connected Space Lemma 88.17 tells us that the two paths γ and δ are in fact homotopic. Therefore it follows from the Change-of-Base Point Proposition 169.5 (4) that $\gamma_* = \delta_*\colon \pi_n(X,x_0) \to \pi_n(X,x_1)$. In other words, if X is simply connected, then given any two base points x_0 and x_1 there exists in fact a natural[1813] isomorphism $\pi_n(X,x_0) \xrightarrow{\cong} \pi_n(X,x_1)$. In particular in this case

[1811] For $n = 1$ this is basically the content of the Change-of-Base Point Proposition 90.4.

[1812] Here it is worth remembering that by Definition 88.7 the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.

[1813] As always, when the adjective "natural" gets used it is an amusing exercise to figure out what categories and functors are involved.

it is almost always entirely safe to suppress the base point in the notation. By the Sphere-π_1-Proposition 88.24 this observation applies in particular to $X = S^k$ for $k \geq 2$. □

Proof.

(0) Let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. It is straightforward to see that the map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is well-defined. It is also elementary, albeit potentially slightly painful, to see that γ_* is a group homomorphism. (See the corresponding discussion in the proof of the Change-of-Base Point Proposition 134.12.)

(4),(5) These two statements are basically elementary to verify. Nonetheless it is worth spending a second to convince oneself that in (5) we have the equality $(\gamma * \delta)_* = \gamma_* \circ \delta_*$ instead of $(\gamma * \delta)_* = \delta_* \circ \gamma_*$.

(1) Recall that in the Path-Homotopy Product Proposition 88.10 (3) we saw that for any path δ the composition $\delta * \overline{\delta}$ is path-homotopic to the constant path. It follows easily from this observation together with (4) and (5) that γ_* is a bijection. Together with (0) we see that for any path $\gamma\colon [0,1] \to X$ from x_0 to x_1 the corresponding induced map $\gamma_*\colon \pi_n(X,x_1) \to \pi_n(X,x_0)$ is an isomorphism.

(2) This is an immediate consequence of (1)

(3) Statement (a) follows immediately from the definitions. Statement (b) is a consequence of the commutative diagram of (a) and the fact that the horizontal maps are isomorphisms by (1).

(6) It follows immediately from (4) and (5) that the map defines an action of $\pi_1(X,x_0)$ on the set $\pi_n(X,x_0)$. Furthermore (1) implies that this action is in fact an action on the group $\pi_n(X,x_0)$.

(7) The proof of this statement is a moderately complicated generalization of the proof of the Homotopies-of-Loops Proposition 88.29. We leave it to the reader to fill in the details. ■

169.3. **Homotopy groups as modules.** Before we continue with our study of higher homotopy groups it is convenient to recall Definition 97.30:

Definition 169.6. Let G be a group. We set

$$\mathbb{Z}[G] \;\coloneqq\; \left\{ \sum_{i=1}^{m} r_i \cdot g_i \;\middle|\; r_1,\ldots,r_m \in \mathbb{Z} \text{ and } g_1,\ldots,g_m \in G \right\}.$$

We equip $\mathbb{Z}[G]$ with the multiplication that is given by

$$\left(\sum_{i=1}^{m} r_i \cdot g_i\right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j\right) \;\coloneqq\; \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{r_i \cdot_{\mathbb{Z}} s_j}_{\in\mathbb{Z}} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $\mathbb{Z}[G]$ as the group ring of G.

Example. We consider the infinite cyclic group $\langle t \rangle$. In this case the group ring $\mathbb{Z}[\langle t \rangle]$ is isomorphic to the ring $\mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm 1}] & \to & \mathbb{Z}[\langle t \rangle] \\ \sum_{i=r}^{i=s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{i=s} a_i \cdot t^i \end{array}$$

is clearly a ring isomorphism. □

We recall the following lemma:

Lemma 97.34. (Group Ring–Module Lemma) Let G be a group and let A be an abelian group (with additive notation for the group structure). Suppose we are given a G-action on the group A. Then the map

$$\begin{array}{ccc} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^{m} r_i \cdot g_i, a \right) & \mapsto & \sum_{i=1}^{m} r_i \cdot (\underbrace{g_i \cdot a}_{\in A}) \end{array}$$

defines a natural $\mathbb{Z}[G]$-left module structure on A.

We now return to topology. It is convenient to introduce or recall the following convention:

Convention 169.7. If a homotopy group is abelian, then sometimes we use additive notation for the group operation instead of the multiplicative notation.

Lemma 169.8. (Homotopy Group–Module Lemma) Let (X, x_0) be a pointed topological space. If $n \geq 2$, then the map

$$\mathbb{Z}[\pi_1(X, x_0)] \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$

induced by

$$([\gamma], [f]) \mapsto [f^\gamma]$$

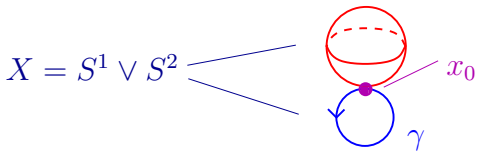
defines a natural $\mathbb{Z}[\pi_1(X, x_0)]$-left module structure on $\pi_n(X, x_0)$.

Proof. The lemma follows almost immediately from the above convention, the Group Ring–Module Lemma 97.34, the π_n-Abelian Proposition 134.11 and the Change-of-Base Point Proposition 169.5 (1) and (6). ■

169.4. **Homotopy groups of $S^1 \vee S^2$.** In this section we want to consider the homotopy groups of $S^1 \vee S^2$ since they are quite instructive. We explicitly construct $X = S^1 \vee S^2$ by identifying the point $\mathrm{i} \in S^1$ with the point $(0, 0, -1) \in S^2$. In the following we evidently use the wedge point as the base point of $X = S^1 \vee S^2$. We refer to it by x_0.

On page 3197 we already saw that $\pi_2(S^1 \vee S^2, x_0)$ is very big, more precisely, we saw that it admits an epimorphism onto $\mathbb{Z}^\infty$. We will now see that if we take the view of the Homotopy Group–Module Lemma 169.8, i.e. if we view $\pi_2(S^1 \vee S^2, x_0)$ as a module over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$, then the apparently messy group $\pi_2(S^1 \vee S^2)$ suddenly gets a simple description.

In the following discussion we denote by $\gamma\colon [0, 1] \to S^1 \subseteq X$ the obvious counterclockwise loop that starts at i. Furthermore we denote by $\mathrm{id}_{S^2}\colon S^2 \to S^2 \subseteq X$ the obvious inclusion map.



Now we can formulate the following proposition which delivers the promised concise description of $\pi_2(S^1 \vee S^2, x_0)$.

Proposition 169.9. **($S^1 \vee S^2$–π_2–Proposition)** The homomorphism[1814]

$$\Psi\colon \mathbb{Z}[t^{\pm 1}] \;\to\; \pi_2(S^1 \vee S^2, x_0)$$

of abelian groups that is determined by

$$\sum_{i=1}^{n} a_i \cdot t^i \;\mapsto\; \sum_{i=1}^{n} a_i \cdot \left[(\mathrm{id}_{S^2})^{\gamma^i}\right]$$

is an isomorphism of $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\langle t\rangle] = \mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$-modules. In particular, as an abelian group, $\pi_2(S^1 \vee S^2, x_0)$ is isomorphic to $\mathbb{Z}^\infty$.

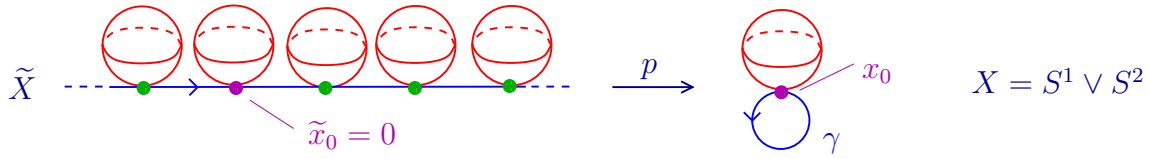
Proof. It follows immediately from the definitions that the map Ψ is indeed a homomorphism of $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\langle t\rangle] = \mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$-modules. Thus it remains to show that Ψ is an isomorphism of abelian groups.

As on page 2466 we see that a universal covering of $X = S^1 \vee S^2$ is given by the map

$$p\colon \overbrace{\big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1))}^{=:\widetilde{X}} \to S^1 \vee S^2$$

$$P \;\mapsto\; \begin{cases} e^{2\pi \mathrm{i}(P+\frac{1}{4})}, & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \text{ with } n \in \mathbb{Z}, Q \in S^2. \end{cases}$$

We pick the base points $x_0 = \mathrm{i} \in S^1 \subseteq X$ and $\widetilde{x}_0 = 0 \in \mathbb{R} \subseteq \widetilde{X}$.



Next we consider the following sequence of maps:

$$(*) \qquad \mathbb{Z}[t^{\pm 1}] \xrightarrow{\Psi} \pi_2(X, x_0) \xrightarrow{\Xi} \pi_2(\widetilde{X}, \widetilde{x}_0) \xrightarrow{\Phi_{(\widetilde{X},\widetilde{x}_0)}} \mathrm{H}_2(\widetilde{X}) \xleftarrow{\Omega} \mathbb{Z}[t^{\pm 1}].$$

Regarding the last three maps we make the following clarifications and observations:

(1) First we recall that given a continuous map $g\colon (S^2, *) \to (X, x_0)$ there exists by the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2 a unique lift $\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)$. Now we consider the map

$$\begin{array}{rcl} \Xi\colon \pi_2(X, x_0) & \to & \pi_2(\widetilde{X}, \widetilde{x}_0) \\ \big[g\colon (S^2, *) \to (X, x_0)\big] & \mapsto & \big[\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)\big]. \end{array}$$

By the Covering-π_n-Isomorphism Proposition 134.22 we know that Ξ is well-defined and that it is an isomorphism.

(2) The map $\Phi_{(\widetilde{X},\widetilde{x}_0)}$ denotes the Hurewicz homomorphism. Since $\widetilde{X}$ is simply connected we obtain from the Hurewicz Theorem 167.9 that it is an isomorphism.

[1814] To simplify the formulation and the proof of the proposition we use the base point $* = (0,0,-1)$ for S^2 and we use this base point to define the second homotopy groups. This convention has the advantage that $\mathrm{id}\colon (S^2, *) \to (S^1 \vee S^2, x_0)$ is a continuous map of based topological spaces and that it defines an element of $\pi_2(S^1 \vee S^2, x_0)$.

(3) Given $i \in \mathbb{Z}$ we denote by $f_i \colon S^2 \to \widetilde{X}$ the "obvious" i-th embedding. In the Infinite String-of-Spheres-H$_*$-Lemma 154.3 we saw that the map

$$\begin{array}{rcl} \Omega \colon \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_2(\widetilde{X}) \\ \sum_{i=r}^{s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{s} a_i \cdot f_{i*}([S^2]). \end{array}$$

is an isomorphism.

We see in particular that in (∗) the three maps to the right are isomorphisms. Therefore it suffices to show that the maps $\Omega^{-1} \circ \Phi_{(\widetilde{X},\widetilde{x}_0)} \circ \Xi \circ \Psi$ is the identity. In other words, it suffices to show the following claim:

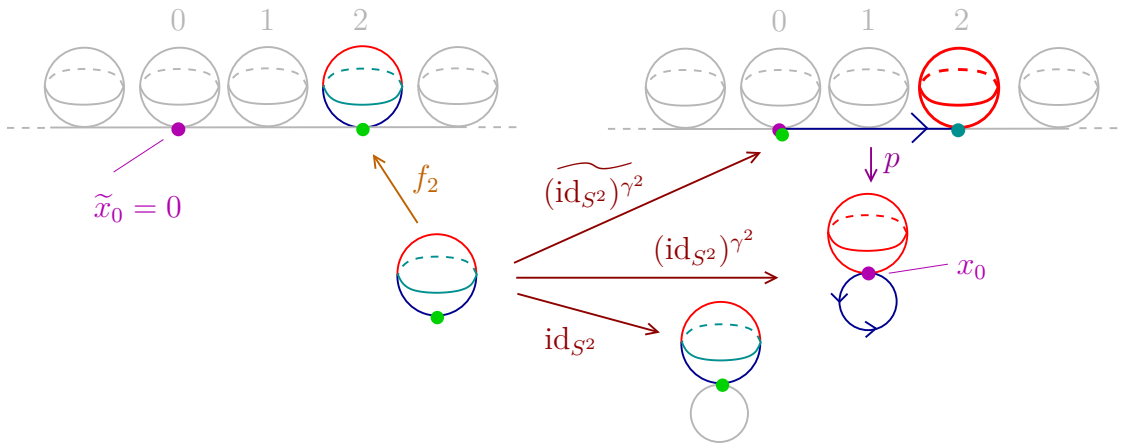
Claim. For every $n \in \mathbb{Z}$ we have

$$(\Phi_{(\widetilde{X},\widetilde{x}_0)} \circ \Xi \circ \Psi)(t^n) \;=\; \Omega(t^n) \;\in\; \mathrm{H}_2(\widetilde{X}).$$

Proof. So let $n \in \mathbb{Z}$. It follows immediately from the definitions of the various maps that the desired statement is equivalent to the statement that

$$\Big(\underbrace{\widetilde{(\mathrm{id}_{S^2})^{\gamma^n}}}_{\text{map } S^2 \to \widetilde{X}}\Big)_*([S^2]) \;=\; \underbrace{f_{n*}}_{\text{map } S^2 \to \widetilde{X}}([S^2]) \;\in\; \mathrm{H}_2(\widetilde{X}).$$

Thus by the Homotopy-H$_*$-Proposition 144.12 it suffices to show that the two maps $\widetilde{(\mathrm{id}_{S^2})^{\gamma^n}}$ and f_n are homotopic maps from S^2 to $\widetilde{X}$. For those readers who are still awake this should be basically obvious.[1815]



As a digestion aid we illustrate both maps in the figure above. ∎

In the CW-Complex-H$_*$-Proposition 155.10 (1) we saw that the homology groups of a finite CW-complex are finitely generated abelian groups. Since the discussion on page 3197 we had known that the corresponding statement is not true for homotopy groups. But we now also know that one should really view the homotopy groups of a pointed topological space (X, x_0) as modules over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Thus the following question arises:

Question 169.10. Let X be a finite CW-complex. Are the homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, finitely generated $\mathbb{Z}[\pi_1(X, x_0)]$-modules?

[1815] In fact this follows from the "⇐"-direction of the Change-of-Base Point Proposition 169.5 (7).

It turns out that the answer is yes if X is simply connected. More precisely, Jean-Pierre Serre [**Ser1953**][1816] proved the following theorem.

Theorem 169.11. (Serre Theorem) If X is a *simply-connected* topological space such that all homology groups are finitely generated (for example X could be a compact topological manifold or a finite CW-complex), then all homotopy groups are also finitely generated abelian groups.[1817]

Proof. A proof is given in [**Ser1953**] or alternatively in [**Nei2010**, Proposition 2.13.7]. ■

Example. It follows in particular from Serre's Theorem 169.11 that the homotopy groups of spheres are finitely generated. Even this statement is highly non-trivial. In particular it is much better than our only result in this direction, namely the Sphere-π_n-Countable Corollary 175.17, which says that homotopy groups of spheres are countable. □

Nonetheless, it turns out that the answer to Question 169.10 is negative. In fact, one does not need to wander off particularly far to find such an example:

Proposition 169.12. (π_n-Not Finitely Generated Module Proposition) The group $\pi_3(S^1 \vee S^2)$ is not finitely generated over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)] = \mathbb{Z}[t^{\pm 1}]$.

Proof. A proof for the proposition is for example sketched in [**Hat2002**, Chapter 4.2, Exercise 38]. ■

Remark. The discussion of $\pi_2(S^1 \vee S^2)$ begs the question what one can say about the homotopy groups of wedges of spheres. The Hilton-Milnor theorem, see [**Hilt1955**, Theorem A],[**Miln1972**] or [**WhdG1978**, Chapter XI], gives a calculation of the homotopy groups of a finite wedge of spheres, more precisely, these homotopy groups are computed in terms of the homotopy groups of the spheres involved. Unfortunately, at least at a first glance, it is not so clear how the module structure over the group ring can be read off from the Hilton-Milnor theorem. □

169.5. Exercises for Chapter 169 .

Exercise 169.1. Is the ring $\mathrm{M}(3 \times 3, \mathbb{R})$ of real (3×3)-matrices isomorphic, as a ring, to the real group ring $\mathbb{R}[G]$ of some group G?

Exercise 169.2. Let G be a finite group. We view $\mathbb{Q}[G]$ as a $\mathbb{Q}[G]$-left module via left multiplication. Show that $\mathbb{Q}[G]$ admits a submodule $V \neq 0$ with $V \neq \mathbb{Q}[G]$ that is isomorphic to the trivial $\mathbb{Q}[G]$-module $\mathbb{Q}$ where every $g \in G$ acts via the identity on $\mathbb{Q}$.

Exercise 169.3. Let G be a finite group. Show that there exists a monomorphism from G to some unitary group $\mathrm{U}(n)$.

Exercise 169.4. Given a group G we define the group ring $\mathbb{R}[G]$ using the obvious modification of the definition of the group ring $\mathbb{Z}[G]$ from Definition 169.6. We consider the ring

[1816] Jean-Pierre Serre (*1926) is a French mathematician. He was, and remains, the youngest person ever to be awarded a Fields Medal.

[1817] It follows immediately from the Hurewicz Theorem 167.9 that $\pi_2(X)$ is finitely generated. The fact that the higher homotopy groups are also finitely generated is much stronger.

$\mathrm{M}(2,\mathbb{R})$. Does there exist a finite set T and an epimorphism from the group ring $\mathbb{R}[\langle T\rangle]$ of the group free group $\langle T\rangle$ onto $\mathrm{M}(2,\mathbb{R})$?

Exercise 169.5. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. We consider the Hurewicz homomorphism $\Phi\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$. Show that for any $g \in \pi_1(X,x_0)$ and any $\sigma \in \pi_n(X,x_0)$ we have

$$\Phi(g\cdot\sigma) \;=\; \Phi(\sigma) \in \mathrm{H}_n(X).$$

Exercise 169.6. Let X be a simply connected topological space and let $f\colon X \to X$ be a self-homeomorphism. Recall that in Definition 21.19 we introduced the mapping torus

$$M \;\coloneqq\; (X\times[0,1])/(x,0)\sim(f(x),1).$$

Express the $\mathbb{Z}[\pi_1(M)]$-module $\pi_2(M)$ in terms of the homology groups of X and the action by f on the homology groups of X.

Exercise 169.7. Show that there exist two path-connected topological spaces X and Y such that all homotopy groups $\pi_n(X,x_0)$ and $\pi_n(Y,y_0)$ are isomorphic as abelian groups but such that $\pi_2(X,x_0)$ and $\pi_2(Y,y_0)$ are not isomorphic as $\mathbb{Z}[\pi_1(X,x_0)]$-modules (under any isomorphism $\pi_1(X,x_0)\cong\pi_1(Y,y_0)$).
Hint. You can take $X = S^1\times S^2$.

Exercise 169.8. Show that there exists a closed smooth manifold M such that $\pi_2(M)$ is not finitely generated as an abelian group.

Exercise 169.9. Given $z \in S^1$ we denote by $\Phi_z\colon S^2 \to S^2$ the map that is given by rotating S^2 along the z-axis by the angle z. The map

$$\begin{array}{rccc} G\colon & S^1\times S^2 & \to & S^1\times S^2 \\ & (z,v) & \mapsto & (z,\Phi_z(v)) \end{array}$$

is called the *Gluck twist*. Let $(x,y)\in S^1\times S^2$. Determine the induced homomorphism $G_*\colon \pi_2(S^1\times S^2,(x,y)) \to \pi_2(S^1\times S^2,(x,y))$ and show that it is an isomorphism.
Remark. The Gluck twist is in so far remarkable as it induces isomorphisms on all invariants we know but it is not homotopic to the identity. This non-trivial fact is proved in [**Glue1962**, p. 320] and [**Wall1970**, p. 232].

CHAPTER 170

The Jordan-Brouwer Separation Theorem

In this chapter we will calculate the homology groups of the complement of any sphere in S^n and in $\mathbb{R}^n$. This result gives us in particular the calculation of the homology groups of the complement of a knot $K \subseteq S^3$. This calculation also allows us to prove the Jordan-Brouwer Separation Theorem 170.3 which says, in a nutshell, that any $(n-1)$-dimensional sphere in $\mathbb{R}^n$ breaks $\mathbb{R}^n$ into precisely two path components.

This chapter is organized as follows:

- In Section 170.1 we will determine the homology groups of complements of spheres and balls. This calculation allows us to prove the Jordan-Brouwer Separation Theorem 170.3.
- In Section 170.2 we will discuss the Schoenflies Theorem 170.6 (which we will not prove).
- In Section 170.3 we will state and prove the Topological Invariance-of-Domain Theorem 170.9.
- In Appendix 170.4 we will discuss a technical point in the definition of subcomplexes of CW-complexes.
- In Section 170.5 we will provide exercises for Chapter 170.

In the next Chapter 171 we will introduce division algebras.

170.1. **The homology groups of complements of spheres and balls.** In this section we want to calculate the homology groups of complements of balls and spheres in the Euclidean space $\mathbb{R}^n$ and the sphere S^n.

First recall that by the Stereographic Projection Lemma 4.24 we can make the identification $S^n =_{\text{i}} \mathbb{R}^n \cup \{\infty\}$ of the sphere with the Euclidean space $\mathbb{R}^n$ together with a "point at infinity". Given a compact subset $X \subseteq \mathbb{R}^n$ the following lemma explains the relationship between the homology groups of $\mathbb{R}^n \setminus X$ and of $S^n \setminus X$:[1818]

Lemma 170.1. (Complement-in-$\mathbb{R}^n$-S^n-Lemma) Let $X \subseteq \mathbb{R}^n$ be a non-empty compact subset.

(a) For every $k \neq n-1$ the inclusion-induced map

$$\widetilde{\mathrm{H}}_k(\mathbb{R}^n \setminus X) \;\to\; \widetilde{\mathrm{H}}_k(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=_{\text{i}} S^n} \setminus X)$$

is an isomorphism.

[1818] The following lemma is basically the content of Exercise 153.21.

(b) There exists a short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus X) \underset{\substack{\uparrow \\ \text{induced by the} \\ \text{inclusion}}}{\to} \widetilde{\mathrm{H}}_{n-1}(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=_{\mathrm{i}} S^n} \setminus X) \to 0$$

which splits.

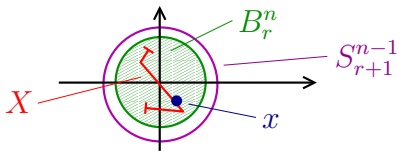
The analogous statements also hold if we replace reduced homology by ordinary homology.

Proof. Since X is compact, we know by the Heine-Borel Theorem 3.44 that there exists an $r > 0$ with $X \subseteq B_r^n$. Note that $A := \mathbb{R}^n \setminus X$ together with $B := (\mathbb{R}^n \cup \{\infty\}) \setminus \overline{B}_r^n$ forms an open cover of $(\mathbb{R}^n \cup \{\infty\}) \setminus X$. We can apply the Mayer–Vietoris Theorem 153.2 (1) to this open cover and we obtain the long exact sequence which is shown on top of the following diagram:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & \widetilde{\mathrm{H}}_k(A \cap B) & \xrightarrow{(i_*, -i_*)} & \widetilde{\mathrm{H}}_k(\mathbb{R}^n \setminus X) \oplus \overbrace{\widetilde{\mathrm{H}}_k(B)}^{=0} & \longrightarrow & \widetilde{\mathrm{H}}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X) & \xrightarrow{\partial_k} & \widetilde{\mathrm{H}}_{k-1}(A \cap B) & \longrightarrow \ldots \\
& \cong \uparrow i_* & & \downarrow i_* & & & & & & \\
& \widetilde{\mathrm{H}}_k(S_{r+1}^{n-1}) & \xrightarrow[\cong]{i_*} & \widetilde{\mathrm{H}}_k(\mathbb{R}^n \setminus \{x\}) & & & & & &
\end{array}$$

We make the following clarifications and observations:

(a) All maps called "i" are the obvious inclusion maps.
(b) It follows easily from the Stereographic Projection Lemma 4.24 that B is homeomorphic to an open n-ball, in particular B is contractible, which implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that the reduced homology groups of B are all zero.
(c) One can easily see that S_{r+1}^{n-1} is a deformation retract of $A \cap B = \mathbb{R}^n \setminus \overline{B}_r^n$. It follows from the Homotopy Equivalence-H_*-Proposition 144.14 that the left vertical map is an isomorphism.
(d) Since X is non-empty, we can pick $x \in X$.
(e) One can easily show that S_{r+1}^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{x\}$. It follows from the Homotopy Equivalence-H_*-Proposition 144.14 that the bottom horizontal map is an isomorphism.
(f) The square in the diagram commutes since all the maps involved are induced by inclusions.
(g) It follows from (b), (e) and (f) that the map $i_* \colon \widetilde{\mathrm{H}}_k(A \cap B) \to \widetilde{\mathrm{H}}_k(\mathbb{R}^n \setminus X)$ is a monomorphism.



It follows from the above and the Exactness Lemma 145.2 that for each $k \in \mathbb{N}_0$ we obtain a short exact sequence of the form

$$0 \to \widetilde{\mathrm{H}}_k(S_{r+1}^{n-1}) \xrightarrow{i_*} \widetilde{\mathrm{H}}_k(\mathbb{R}^n \setminus X) \xrightarrow{i_*} \widetilde{\mathrm{H}}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X) \to 0.$$

Furthermore it follows from the above and the Splitting Lemma 145.5 (2)⇒(1) that the short exact sequence splits. The statement about reduced homology groups now follows from the Sphere-H_*-Proposition 149.4.

The corresponding statements about ordinary homology groups can be proved in a very similar fashion. Alternatively one can deduce them from the reduced statement using the Reduced Homology Lemma 147.3, the $\widetilde{\mathrm{H}}_*$-Functor Lemma 147.2 (2) and the Five Lemma 145.11. We leave it to the reader to fill in the details. ■

We continue with the following proposition, which is the main technical result of this chapter.

Theorem 170.2. (Sphere Complement-H_*-Theorem)

(1) For every $k \in \mathbb{N}_0$ and every embedding[1819] [1820] $h\colon \overline{B}^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(\overline{B}^k)\big) \;=\; 0 \quad \text{for all } i \in \mathbb{N}_0.$$

(2) For every[1821] $k \in \mathbb{Z}_{\geq -1}$ and every embedding $h\colon S^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(S^k)\big) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

Example.

(1) Let $K \subseteq S^3$ be a knot, i.e. let K be a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1. This means in particular that there exists a smooth embedding $h\colon S^1 \to S^3$ with $h(S^1) = K$. The combination of the Sphere Complement-H_*-Theorem 170.2 (2) and the Reduced Homology Lemma 147.3 says that

$$\mathrm{H}_i(S^3 \setminus K) \;=\; \mathrm{H}_i(S^3 \setminus h(S^1)) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that we now have a new, and depending on one's taste, more satisfying proof of the Link Group-Abelianization Proposition 113.9: The Hurewicz Theorem 166.10 together with the above calculation shows that the abelianization of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

(2) In the figure below we see the image of an embedding $h\colon [0,1] \to \mathbb{R}^3$.[1822] We also see a closed curve $C \subseteq \mathbb{R}^3 \setminus h([0,1])$.

In Notation 155.20 we saw that C defines a homology class $[C] \in \mathrm{H}_1(\mathbb{R}^3 \setminus h([0,1]))$. It follows from the Sphere Complement-H_*-Theorem 170.2 (1) and the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1 that $\mathrm{H}_1(\mathbb{R}^3 \setminus h([0,1])) = 0$, in particular C represents the trivial class in $\mathrm{H}_1(\mathbb{R}^3 \setminus h([0,1])) = 0$.[1823] But in [**Stil1993**, p. 152] it is shown that the closed curve C represents a non-trivial element in $\pi_1(\mathbb{R}^3 \setminus h([0,1]))$. □

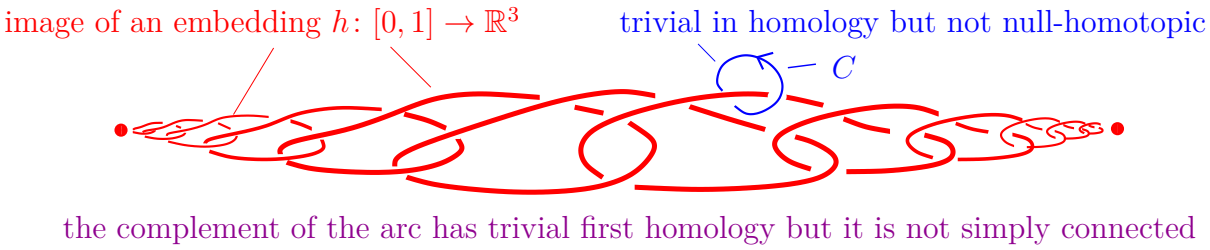
[1819] Since $\overline{B}^k$ is compact and since S^n is Hausdorff we know by the Compact-Hausdorff Proposition 4.23 that any injective continuous map $h\colon \overline{B}^k \to S^n$ is an embedding.

[1820] In Exercise 170.1 we will use the Smooth Ball Embedding Theorem 45.10 to give an alternative proof for smooth embeddings.

[1821] Recall that $S^{-1} = \varnothing$. Conveniently enough the statement makes perfect sense also for $k = -1$.

[1822] Note that h is not smooth at 0 and 1, so h is not a smooth embedding.

[1823] In the Surface-with-One Boundary Component-H_*-Lemma 155.23 we saw that the boundary curve of an orientable surface with one boundary component is null-homologous. In Exercise 170.3 we will see that the closed curve C is in fact the boundary of an orientable surface that is embedded in $\mathbb{R}^3 \setminus h([0,1])$.



Remark.

(1) In the General Alexander Duality Theorem 218.2 we will greatly generalize the Sphere Complement-H_*-Theorem 170.2.
(2) Later, in the Wild Knot-π_1-Proposition 218.20 we will see that there exist embeddings $h\colon S^1 \to \mathbb{R}^3$ such that $\pi_1(S^3 \setminus h(S^1))$ is not finitely generated group, even though by the Complement-H_*-Theorem 170.2 we know that $\mathrm{H}_1(S^3 \setminus h(S^1)) \cong \mathbb{Z}$. □

To make the exposition more readable we prove the two parts of the Sphere Complement-H_*-Theorem 170.2 separately.

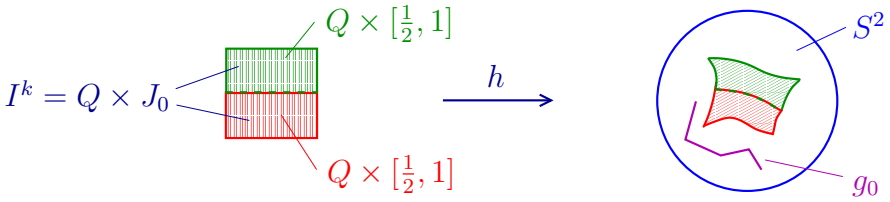
Proof of the Sphere Complement-H_*-Theorem 170.2 (1)**.** We write $I = [-1,1]$. By the Convex-to-Ball Proposition 4.28 (2) there exists a homeomorphism $I^k \xrightarrow{\cong} \overline{B}^k$. In the statement of the Sphere Complement-H_*-Theorem 170.2 (1) we can therefore replace $\overline{B}^k$ by I^k. So it remains to prove the following claim:

Claim. For any $k \in \mathbb{N}_0$ and any embedding $h\colon I^k \to S^n$ we have
$$\widetilde{\mathrm{H}}_i(S^n \setminus h(I^k)) \;=\; 0 \quad \text{for all } i \in \mathbb{N}_0.$$

Proof. We will prove this claim by induction on $k \in \mathbb{N}$. We start out with $k = 0$. So let $h\colon I^0 = \{0\} \to S^n$ be an embedding. It follows from the Stereographic Projection Lemma 4.24 together with the Orthogonal Action Lemma 2.37 (3) that $S^n \setminus h(\{0\})$ is homeomorphic to $\mathbb{R}^n$. Hence we obtain from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that $\widetilde{\mathrm{H}}_*(S^n \setminus h(\{0\})) = 0$.

Now we do the induction step. We assume that we know the desired statement for some $k \in \mathbb{N}$ and for all embeddings $I^{k-1} \to S^n$. Now let $h\colon I^k \to S^n$ be an embedding. We suppose that the desired conclusion does not hold, i.e. we suppose that $\widetilde{\mathrm{H}}_j(S^n \setminus h(I^k)) \neq 0$ for some j. We pick a non-trivial element $g_0 \in \widetilde{\mathrm{H}}_j(S^n \setminus h(I^k))$.

We write $Q := I^{k-1}$, $J_0 := I$ and we write
$$A_1 \;=\; S^n \setminus h(Q \times [0, \tfrac{1}{2}]) \quad \text{and} \qquad A_2 \;=\; S^n \setminus h(Q \times [\tfrac{1}{2}, 1]).$$
Since h is injective, we have
$$A_1 \cap A_2 \;=\; S^n \setminus h(I^k) \qquad \text{and} \quad A_1 \cup A_2 \;=\; S^n \setminus h(Q \times \{\tfrac{1}{2}\}).$$
It follows from the Compact Image Lemma 4.15 (3) and the Compact-Closed Lemma 3.39 that A_1 and A_2 are open subsets of $A_1 \cup A_2$. Therefore we can apply the Mayer–Vietoris



Theorem 153.2 (1) and we obtain the long exact sequence

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_{j+1}(A_1 \cup A_2)}_{\substack{= 0 \text{ by induction, since} \\ A_1 \cup A_2 = S^n \setminus h(Q \times \{\frac{1}{2}\})}} \to \widetilde{\mathrm{H}}_j(A_1 \cap A_2) \xrightarrow{(i_{1*}, -i_{2*})} \widetilde{\mathrm{H}}_j(A_1) \oplus \widetilde{\mathrm{H}}_j(A_2) \to \underbrace{\widetilde{\mathrm{H}}_j(A_1 \cup A_2)}_{\substack{= 0 \text{ by induction, since} \\ A_1 \cup A_2 = S^n \setminus h(Q \times \{\frac{1}{2}\})}} \to \ldots$$

where $i_k \colon A_1 \cap A_2 \to A_k$ is the inclusion. So we see that the middle map is an isomorphism. Thus there exists a $k \in \{1, 2\}$ such that the image of g_0 is non-trivial in $\widetilde{\mathrm{H}}_j(A_k)$. If $k = 1$, then we set $J_1 \coloneqq [0, \frac{1}{2}]$, otherwise we set $J_1 \coloneqq [\frac{1}{2}, 1]$. We denote by

$$\varphi_1 \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0)) \to \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1))$$

the inclusion-induced map. By the above exact sequence φ_1 is an epimorphism. We denote by ρ_1 the inverse of the induced isomorphism

$$\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0)) / \ker(\varphi_1) \to \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1)).$$

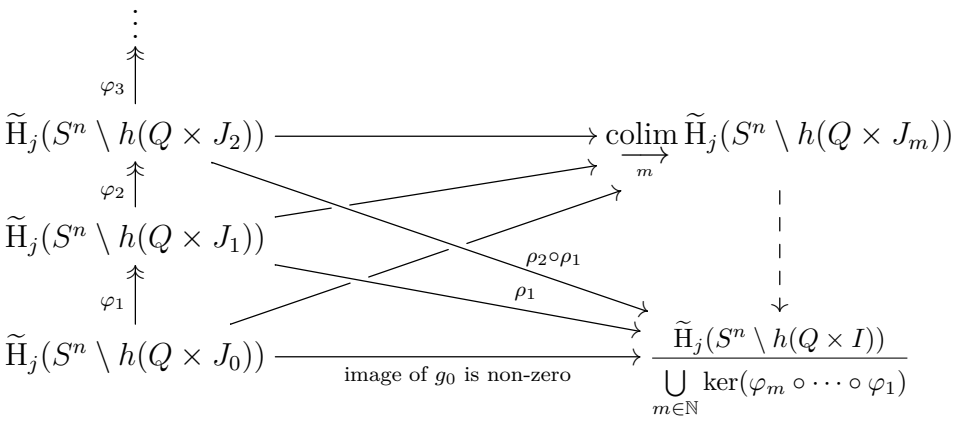
We iterate this process by running the same argument with J_0 replaced by J_1 and so forth. For each $m \in \mathbb{N}$ we can find an interval J_m of length $\frac{1}{2^m}$ that is contained in J_{m-1} such that for the inclusion-induced homomorphism

$$\varphi_m \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_{m-1})) \to \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m))$$

we have that $g_m \coloneqq \varphi_m(g_{m-1})$ is non-zero. We denote by ρ_m the inverse of the induced isomorphism

$$\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_{m-1})) / \ker(\varphi_m) \to \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m)).$$

We consider the following diagram:



We make the following observations:

- Note that we have $\ker(\varphi_1) \subseteq \ker(\varphi_2 \circ \varphi_1) \subseteq \ker(\varphi_3 \circ \varphi_2 \circ \varphi_1) \subseteq \ldots$. It follows that the union of these subgroups (which we consider at the bottom right) is again a subgroup.

- The diagonal maps that we denoted by $\rho_m \circ \cdots \circ \rho_1$ are strictly speaking given by the composition of $\rho_m \circ \cdots \circ \rho_1 \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m)) \to \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times I))/\ker(\varphi_m \circ \cdots \circ \varphi_1)$ with the natural projection.
- We get the vertical map to the right by the definition of the direct colimit.

Since each J_m has half the length of J_{m-1} we conclude, say using Exercise 3.13, that the intersection $\bigcap\limits_{m\in\mathbb{N}} J_m$ contains a unique point $t \in [0,1]$. We now see that

$$\begin{array}{ccccc}
\operatorname*{colim}\limits_{\longrightarrow m} \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m)) & \xrightarrow{\cong} & \widetilde{\mathrm{H}}_j\Big(\bigcup\limits_{m\in\mathbb{N}} (S^n \setminus h(Q \times J_m))\Big) & = & \widetilde{\mathrm{H}}_j\Big(S^n \setminus \bigcap\limits_{m\in\mathbb{N}} h(Q \times J_m)\Big) \\
\uparrow & & \uparrow & & \uparrow \\
& & \text{by the Exhaustions-H}_*\text{-Proposition 154.2,} & & \text{by De Morgan's Law 1.7} \\
& & \text{since the subsets } S^n \setminus h(Q \times J_m) \text{ are open} & & \\
& = & \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times \{t\})) & = & 0. \\
& & \uparrow & & \uparrow \\
& & \text{since } h \text{ is injective, and} & & \text{this follows from induction,} \\
& & \text{by the definition of } t & & \text{since } Q \times \{t\} \cong I^{k-1}
\end{array}$$

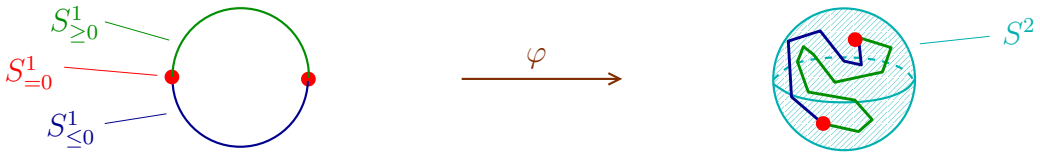
We have now obtained a contradiction, since the bottom horizontal map of the above commutative diagram has a non-trivial image, namely the image of g_0, but it factors through the trivial group. ∎

Proof of the Sphere Complement-H$_*$-Theorem 170.2 (2). Recall that we have to show that for every $k \in \mathbb{Z}_{\geq -1}$ and every embedding $h \colon S^k \to S^n$ we have isomorphisms

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(S^k)\big) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

We prove this statement by induction on k. For $k = -1$ we have $S^{-1} = \varnothing$ and the statement follows from the Sphere-H$_*$-Proposition 149.4.

Now suppose we know the statement for some $k \in \mathbb{Z}_{\geq -1}$. Let $h \colon S^{k+1} \to S^n$ be an embedding. We denote by $S^{k+1}_{\leq 0}$ the lower hemisphere of S^{k+1} and we denote by $S^{k+1}_{\geq 0}$ the upper hemisphere of S^{k+1}. Note that $S^{k+1}_{\leq 0} \cap S^{k+1}_{\geq 0} = S^{k+1}_{=0} \cong S^k$.



We apply the Mayer–Vietoris Theorem 153.2 (1) to the decomposition of $S^n \setminus h(S^{k+1})$ into the open subsets $S^n \setminus h(S^{k+1}_{\leq 0})$ and $S^n \setminus h(S^{k+1}_{\geq 0})$. We obtain a long exact sequence[1824]

$$\cdots \to \widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1})) \to \underbrace{\widetilde{\mathrm{H}}_i((S^n \setminus h(S^{k+1}_{\leq 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\leq 0} \cong B^{k+1}}} \oplus \underbrace{\widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1}_{\geq 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\geq 0} \cong B^{k+1}}} \to \widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1}_{=0})) \xrightarrow{\partial_i} \cdots.$$

Thus we see that

$$\begin{array}{ccccc}
\widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1})) & \xleftarrow{\cong} & \widetilde{\mathrm{H}}_{i+1}(S^n \setminus h(S^{k+1}_{=0})) & \cong & \begin{cases} \mathbb{Z}, & \text{if } i+1 = n-k-1, \\ 0, & \text{otherwise.} \end{cases} \\
\uparrow & & \uparrow & & \\
\text{by the above long} & & \text{by induction,} & & \\
\text{exact sequence} & & \text{since } S^{k+1}_{=0} \cong S^k & &
\end{array}$$

[1824] As in the proof of the Sphere Complement-H$_*$-Theorem 170.2 (1) we use implicitly the injectivity of h.

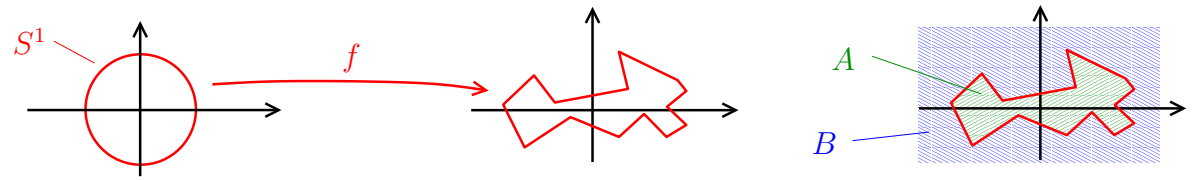
Since $i+1 = n-k-1$ if and only if $i = n-(k+1)-1$ we have proved the induction step. ■

The next result is a fairly immediate consequence of the Sphere Complement-H_*-Theorem 170.2 and the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1.

Theorem 170.3. (Jordan-Brouwer Separation Theorem) Let $n \in \mathbb{N}$. Furthermore let $f\colon S^n \to \mathbb{R}^{n+1}$ be an embedding.

(1) The complement $\mathbb{R}^{n+1} \setminus f(S^n)$ has precisely two path-components, one of which is bounded and one which is unbounded.

(2) If A and B are the two path-components of $\mathbb{R}^{n+1} \setminus f(S^n)$, then $\partial A = \partial B = f(S^n)$.



Proof. Let $n \in \mathbb{N}$ and let $f\colon S^n \to \mathbb{R}^{n+1}$ be an embedding.

(1) We use the embedding $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \cup \{\infty\} =_{\mathrm{i}} S^{n+1}$ to view $\mathbb{R}^{n+1}$ as a subspace of S^{n+1}. Now we see that

$$\widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus f(S^n)) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}_0(S^{n+1} \setminus f(S^n)) \underset{\uparrow}{\cong} \mathbb{Z}.$$

by the Complement-in-$\mathbb{R}^n$-S^n Lemma 170.1

by the Homology-of-Sphere Complement Theorem 170.2 (2)

It follows from the Reduced Homology Lemma 147.3 (0c) that $\mathbb{R}^{n+1} \setminus f(S^n)$ has precisely two path-components. It remains to show that precisely one of them is unbounded.

Since S^n is compact and since f is continuous we know by the Compact Image Lemma 4.15 that $f(S^n)$ is compact. It follows from the Heine-Borel Theorem 3.44 that there exists an $r \in \mathbb{R}$ such that $f(S^n) \subseteq B_r^{n+1}$. It follows easily from $n \geq 1$ that $\mathbb{R}^{n+1} \setminus B_r^{n+1}$ is path-connected. It follows from the Path-Component Lemma 4.44 (4) that $\mathbb{R}^{n+1} \setminus B_r^{n+1}$ is contained in one of the two path-components. This shows that precisely one path-component is unbounded.

(2) We start out with the following claim:

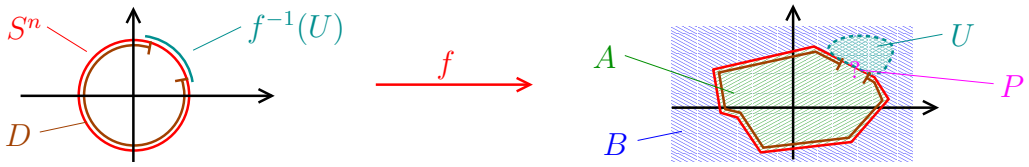
Claim 1. A and B are open subsets of $\mathbb{R}^n$.

Proof. First note that it follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $\mathbb{R}^{n+1} \setminus f(S^n)$ is open. In particular, as we remarked on page 206, $\mathbb{R}^{n+1} \setminus f(S^n)$ is locally path-connected. It follows from the Locally Path-Connected Lemma 4.57 that A and B are both open subsets of $\mathbb{R}^n$. ⊞

Claim 2. We have $\overline{A} = A \cup f(S^n)$ and $\overline{B} = B \cup f(S^n)$.

Proof. In the following we show that $\overline{A} = A \cup f(M)$. The proof that $\overline{B} = B \cup f(M)$ is basically identical. Since B is open we know that $A \cup f(S^n) = \mathbb{R}^n \setminus B$ is closed. By definition of $\overline{A}$ this implies that $\overline{A} \subseteq A \cup f(S^n)$. Now let $P \in A \cup f(S^n)$. We need to show that $P \in \overline{A}$. Suppose that this is not the case. By the Interior-Closure Lemma 3.22 we know that $\overline{A}$ is closed. Thus our assumption that $P \notin \overline{A}$ implies that there exists an open subset $U \subseteq \mathbb{R}^{n+1}$ with $P \in U \subseteq \mathbb{R}^{n+1} \setminus \overline{A}$. By Exercise 170.11 there

exists a subspace $D \subseteq S^n$, which is homeomorphic to $\overline{B}^n$, such that $S^n \setminus D \subseteq f^{-1}(U)$, i.e. such that $f(S^n \setminus D) \subseteq U$.



We now obtain the following equalities:

$$\underbrace{\mathbb{R}^{n+1} \setminus f(D)}_{\substack{\text{path-connected by the} \\ \text{Sphere Complement-H}_*\text{-Theorem 170.2} \\ \text{\& the Complement-in-}\mathbb{R}^n\text{-}S^n\text{-Lemma 170.1} \\ \text{\& the Reduced Homology Lemma 147.3} \\ \text{and the fact that } n > 0}} \overset{\text{since } \mathbb{R}^{n+1} = A \sqcup B \sqcup f(S^n)}{\overset{\downarrow}{=}} A \sqcup B \sqcup (f(S^n) \setminus f(D)) \overset{\text{since } f(S^n) \setminus f(D) = f(S^n \setminus D) \subseteq U}{\overset{\downarrow}{=}} A \sqcup (B \cup (U \cap (\mathbb{R}^{n+1} \setminus f(D)))).$$

$\uparrow$ disjoint union since $A \cap B = \varnothing$ and since $A \cap U = \varnothing$

We have thus written the path-connected space $\mathbb{R}^{n+1} \setminus f(D)$ as the disjoint union of the two open non-empty sets A and $B \cup (U \cap (\mathbb{R}^{n+1} \setminus f(D)))$. But by the Path-Connected-Implies-Connected Corollary 4.33 this is not possible. We have thus obtained a contradiction and we have thus shown that $\overline{A} = A \cup f(S^n)$. ⊞

We now see that

$$\partial A \overset{\text{Definition 3.21 of } \partial A}{\underset{\uparrow}{=}} \overline{A} \setminus \mathring{A} \overset{\text{by Claim 1 and 2}}{\underset{\uparrow}{=}} (A \cup f(S^n)) \setminus A \overset{\text{since } A \cap f(S^n) = \varnothing}{\underset{\uparrow}{=}} f(S^n).$$

The same way we see that $\partial B = f(S^n)$. This gives us the desired result. ■

For historic purposes we state the case $n = 1$ of the Jordan-Brouwer Separation Theorem 170.3 again, with slightly different names. The following definition is a minute variation on the definition of a Jordan curve given in Definition 39.19.

Definition 170.4. A planar Jordan curve is the image of an embedding $S^1 \to \mathbb{R}^2$.

Corollary 170.5. (Jordan Curve Theorem) Let $C \subseteq \mathbb{R}^2$ be a planar Jordan curve.

(1) The complement $\mathbb{R}^2 \setminus C$ has precisely two path-components, one of which is bounded and one which is unbounded.
(2) If A and B are the two path-components of $\mathbb{R}^2 \setminus C$, then $\partial A = \partial B = C$.

Proof. The corollary is just the case $n = 1$ of the Jordan-Brouwer Separation Theorem 170.3. ■

Remark.

(1) The Jordan Curve Theorem 170.5 might appear as "self-evident". To disabuse the reader of this idea recall that in the Injective Curve-but-Positive Area Proposition 10.23 and Exercise 10.15 we saw that there exist planar Jordan curves with non-zero 2-dimensional Lebesgue measure.

(2) The Jordan Curve Theorem 170.5 takes its name from Camille Jordan [**Jor1887**][1825] who formulated it in 1887. It seems like the first convincing proof was given by Oswald Veblen [**Veb1905**] in 1905. The general case of the Jordan-Brouwer Separation Theorem 170.3 was first proved by Luitzen Brouwer [**Brou1912b**] in 1912. □

170.2. **The Schoenflies Theorem.** The Sphere Complement-H_*-Theorem 170.2, together with the Reduced Homology Lemma 147.3, says in particular, that if $h\colon S^{n-1} \to S^n$ is an embedding, then

$$\mathrm{H}_i(S^n \setminus h(S^{n-1})) \cong \begin{cases} \mathbb{Z}^2, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, homologically the topological space $S^n \setminus h(S^{n-1})$ looks like the disjoint union of two open n-dimensional balls. One might now suspect that $S^n \setminus h(S^{n-1})$ is in fact homeomorphic to the union of two open n-dimensional balls. This is in fact true in dimension $n = 2$. More precisely Arthur Schoenflies [**Schoe1906**] proved the following theorem:

Theorem 170.6. (Schoenflies Theorem) Given any embedding $h\colon S^1 \to S^2$ there exists a homeomorphism $\Theta\colon S^2 \to S^2$ such that $(\Theta \circ h)(S^1)$ equals the equatorial sphere $S^1 \times \{0\}$. In particular both components of $S^2 \setminus h(S^1)$ are homeomorphic to open disks.

Remark.

(1) The Schoenflies Theorem is often also called the Jordan-Schoenflies Theorem.
(2) If $h\colon S^1 \to S^2$ is a smooth embedding, then the conclusion of the Schoenflies Theorem 170.6 also follows from the much easier Fill-in-Disk Proposition 107.15. □

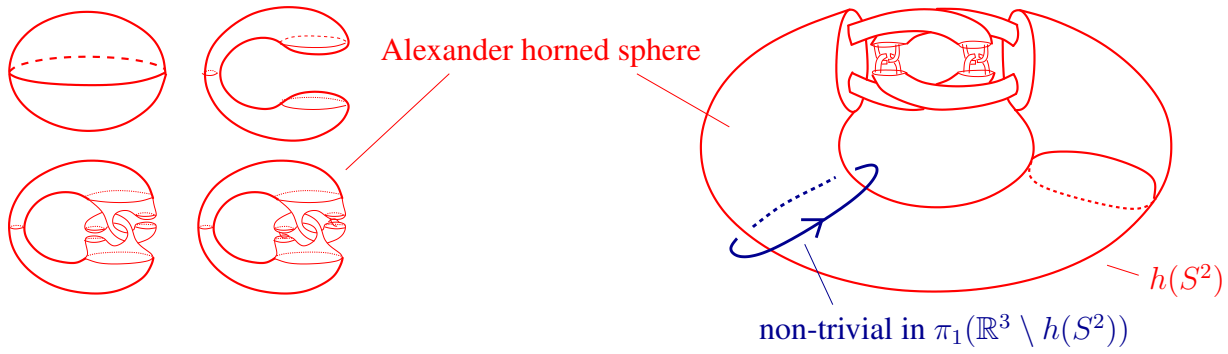
Proof. The first proof of the theorem was given in [**Schoe1906**]. Alternatively, as is explained in [**Pom1992**, p. 25], one can also use the Carathéodory Theorem 39.20 to prove the theorem.

Finally a third approach is the following. Let $h\colon S^1 \to S^2$ be an embedding. In [**Tn1992**, Section 3] it is shown that if $g\colon S^1 \to \mathbb{R}^2$ is an embedding, then g extends to a homeomorphism of $\mathbb{R}^2$. Since $S^2 = \mathbb{R}^2 \cup_i \{\infty\}$ one can easily show that there exists a homeomorphism $\Theta\colon S^2 \to S^2$ such that $(\Theta \circ h)(S^1) = S^1 \times \{0\}$.

Since the two components of $S^2 \setminus i(S^1) = S^2_{>0} \sqcup S^2_{<0}$ are homeomorphic to B^2 we see that both components of $S^2 \setminus h(S^1)$ are homeomorphic to open disks. ■

Example. It is natural to ask whether the conclusion of the Schoenflies Theorem also holds in higher dimensions. But this hope breaks down already one dimension higher. More precisely, James Alexander[1826] found in 1924 the embedding $h\colon S^2 \to \mathbb{R}^3$ that is shown in the figure below. The "outside" component of the complement of $h(S^2)$ is not simply connected. The image $h(S^2)$ is called the *Alexander horned sphere*.
A more detailed description of the Alexander horned sphere and the proof that one of the components of the complement is not simply connected is given in [**Ale1924a**] and alternatively in [**Hat2002**, p. 170] and in [**DV2009**, p. 46]. In Exercise 170.5 we invite the

[1825] Camille Jordan (1838-1922) was a French mathematician who most readers will already have encountered in linear algebra.
[1826] James Alexander (1888-1971) was an American mathematician. He also introduced the Alexander polynomial of a knot.



reader to modify the above example to produce an embedding $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected. □

We have now seen that embeddings $S^2 \to \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ can be pretty "wild". The embedding that defines the Alexander horned sphere is continuous, but it is certainly not smooth. So what happens if we restrict ourselves to smooth embeddings? In fact in the smooth setting we have the following theorem.

Theorem 170.7. (Generalized Smooth Schoenflies Theorem) Let $n \neq 4$. Given any smooth embedding $h\colon S^{n-1} \to S^n$ there exists a diffeomorphism $\Theta\colon S^n \to S^n$ such that $(\Theta \circ h)(S^{n-1})$ equals the equatorial sphere $S^{n-1} \times \{0\}$.

Proof. For $n = 2$ the statement follows fairly easily from the Fill-in-Disk Proposition 107.15 and the Diffeotopies-of-S^1-Proposition 42.8. We will fill in the details in Exercise 170.10. For $n = 3$ this statement was proved by James Alexander [**Ale1924b**]. (See also [**Hat3**, Chapter 1] and [**Schul2014**, Theorem 3.2.5] for an exposition of the proof.) For $n \geq 5$ this statement was independently proved by Barry Mazur [**Maz1959**] in 1959 and by Morton Brown [**BrownM1960**] in 1960. In fact for $n \geq 6$ we will provide a in the High-Dimensional Smooth Schoenflies Theorem 264.10. We also refer to the Low Dimensional Smooth Schoenflies Theorem 264.16 for a few more references. ■

The case $n = 4$ is missing in the Generalized Schoenflies Theorem. In fact it is still unknown whether the statement holds for $n = 4$:

Question 170.8. Let $h\colon S^3 \to S^4$ be a smooth embedding. Does there exist a diffeomorphism $\Theta\colon S^4 \to S^4$ such that $(\Theta \circ h)(S^3)$ equals the equatorial sphere $S^3 \times \{0\}$?

This question is very much open and there is not even a consensus whether one expects the answer to be yes or no. If you can answer this question you would almost certainly obtain a Fields medal.[1827]

170.3. **Invariance of domain.** In this last section before the appendix we will consider two more consequences of the Sphere Complement-H_*-Theorem 170.2. The following theorem was first proved by Luitzen Brouwer [**Brou1911**] in 1911.

[1827] But hurry up, you can only get a Fields medal if you are under 40.

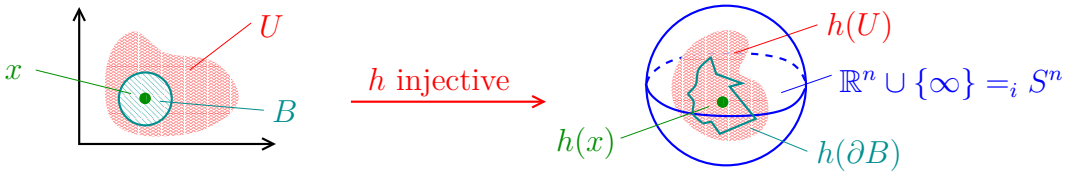
Theorem 170.9. (Topological Invariance-of-Domain Theorem) Let U be an open subset of $\mathbb{R}^n$.

(1) If $h\colon U \to \mathbb{R}^n$ is an injective continuous map, then $h(U)$ is also an open subset of $\mathbb{R}^n$.[1828]
(2) If U is non-empty and if $h\colon U \to \mathbb{R}^m$ is an injective continuous map, then $m \geq n$.

Proof.

(1) First recall that by the homeomorphism from the Stereographic Projection Lemma 4.24 we can view $\mathbb{R}^n$ as an open subset of $\mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$. So we can deal with the following modified situation: We suppose that we are given an injective continuous map $h\colon U \to S^n$ and we need to show that $h(U)$ is an open subset of S^n.

We will use the JH-Openness Criterion 3.3 to show that $h(U)$ is an open subset of S^n. Thus let $x \in U$. We pick a closed Euclidean n-ball B in U that contains x in the interior. It suffices to show that $h(B \setminus \partial B) \subseteq h(U)$ is open in S^n.



Note that it follows from the Compact Image Lemma 4.15 (3) together with the Compact-Closed Lemma 3.39 that $S^n \setminus h(\partial B)$ is an open subset of S^n. It follows from the Locally Path-Connected Lemma 4.57 (1b) and (3) that every path-component of $S^n \setminus h(\partial B)$ is an open subset of S^n. Therefore it suffices to show that $h(B \setminus \partial B)$ is a path-component of $S^n \setminus h(\partial B)$. We will do so in the following claim:

Claim. The topological space $S^n \setminus h(\partial B)$ has precisely two path-components, namely the subsets $h(B \setminus \partial B)$ and $S^n \setminus h(B)$.

Proof. Since $h\colon \partial B \to S^n$ is injective and continuous and since ∂B is compact, we obtain from the Compact-Hausdorff Proposition 4.23 that $h\colon \partial B \to S^n$ is an embedding. Since ∂B is homeomorphic to S^{n-1} we obtain from the Sphere Complement-H_*-Theorem 170.2 (2) and the Reduced Homology Lemma 147.3 that

$$S^n \setminus h(\partial B) \;=\; (h(B) \setminus h(\partial B)) \sqcup (S^n \setminus h(B)) \;\underset{\uparrow}{=}\; h(B \setminus \partial B) \;\sqcup\; (S^n \setminus h(B))$$

by the Image-Preimage Lemma 1.9, since h is injective

has two path-components. By the Path-Component Lemma 4.44 (4b) each path-connected subspace of a topological space is contained in a path-component. It follows from this observation that it suffices to show that the two sets on the right-hand side are path-connected and non-empty:

(i) It follows from the Image-Preimage-Component Lemma 4.50 and the fact that $B \setminus \partial B$ is path-connected, that the first set $h(B \setminus \partial B)$ is path-connected. Since $B \setminus \partial B \neq \varnothing$ we see that $h(B \setminus \partial B) \neq \varnothing$.

[1828] If $h\colon U \to \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ is holomorphic, then the statement also follows from the Non-Constant+Holomorphic–Open Image Proposition 39.13.

(ii) The second set $S^n \setminus h(B)$ is path-connected by the Sphere Complement-H_*-Theorem 170.2 (1) and the Reduced Homology Lemma 147.3. Since $B \subsetneq U$ and since h is injective we see that $h(B) \subsetneq S^n$. In other words, $S^n \setminus h(B) \neq \varnothing$. ⊞

(2) Let $U \subseteq \mathbb{R}^n$ be a non-empty open subset. We need to show that if there exists an injective continuous map $h\colon U \to \mathbb{R}^m$, then $m \geq n$. So let $h\colon U \to \mathbb{R}^m$ be an injective continuous map with $m \leq n$. We need to show that $m = n$. We consider the injective continuous map that is given by composing h with the inclusion $i\colon \mathbb{R}^m \xrightarrow{x \mapsto (x,0)} \mathbb{R}^n$. By (1) we know that $(i \circ h)(U)$ is an open subset of $\mathbb{R}^n$. Since $(i \circ h)(U) \subseteq \{(x,0) \mid x \in \mathbb{R}^m\}$ and since $(i \circ h)(U)$ is non-empty this is only possible if $m = n$. ■

We conclude this section with the following corollary, which in particular gives us a convenient criterion for an injective continuous map to be a homeomorphism.

Corollary 170.10. (Closed Manifold–Injection Corollary) Let M be a closed non-empty n-dimensional topological manifold and let N be a connected n-dimensional topological manifold. Let $h\colon M \to N$ be a continuous map.

(1) If $h\colon M \to N$ is locally injective, then h is also surjective.
(2) If h is injective, then h is already a homeomorphism.

Example. The Closed Manifold–Injection Corollary 170.10 implies in particular that there exists no embedding from the closed smooth manifold S^n into the smooth manifold $\mathbb{R}^n$. □

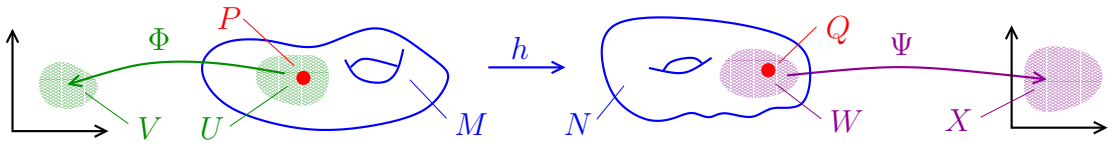
Proof.

(1) Let $h\colon M \to N$ be a locally injective continuous map from a closed non-empty n-dimensional topological manifold M to a connected n-dimensional topological manifold N.

It follows from the fact that M is compact and the fact that N is Hausdorff and from Compact Image Lemma 4.15 together with the Compact-Closed Lemma 3.39 (2) that the image $h(M)$ is closed in N. Since M is non-empty and since N is connected, it suffices to show that $h(M)$ is also open in N.

We will use the JH-Openness Criterion 3.3 to show that $h(M)$ is an open subset of N. Thus let $Q \in h(M)$. We need to show that Q admits an open neighborhood that is contained in $h(M)$. We pick $P \in M$ with $h(P) = Q$. We pick a chart $\Phi\colon U \to V$ around P and we pick a chart $\Psi\colon W \to X$ around Q. Since $\partial M = \varnothing$, we know that V is an open subset of $\mathbb{R}^n$. After possibly replacing U by $U \cap h^{-1}(W)$ we can assume that $h(U) \subseteq W$. Since h is locally injective, we can, after possibly shrinking U even more, assume that $h|_U$ is injective.

Since V is an open subset of $\mathbb{R}^n$ it follows from the Topological Invariance-of-Domain Theorem 170.9 that $(\Psi \circ h \circ \Phi^{-1})(V)$ is an open subset of $\mathbb{R}^n$. Since Ψ is a homeomorphism we see that $h(\Phi^{-1}(V)) = h(U)$ is an open neighborhood of Q that is contained in $h(M)$.



(2) Statement (2) of the corollary is an immediate consequence of Statement (1) and the Compact-Hausdorff Proposition 4.23 (3). ■

170.4. **Appendix: The definition of subcomplexes of CW-complexes.** In Definition 136.4 we introduced the notion of a subcomplex of a CW-complex. In this somewhat technical appendix we will see, using the Topological Invariance-of-Domain Theorem 170.9, that one of the two conditions in the definition of a subcomplex is superfluous.

Before we get to the discussion of the definition of subcomplexes, let us recall Definition 136.2:

Definition 170.11. Let Y be a CW-complex.

(1) An open n-cell of Y is a component of $Y^n \setminus Y^{n-1}$.
(2) The closure of an open n-cell is called an n-cell of Y.

Next let us Definition 136.4 of a subcomplex of a CW-complex.

Definition 170.12. Let X be a CW-complex. We say that a CW-complex A is a subcomplex of X if $A \subseteq X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.
(2) Each n-cell of A is also an n-cell of X.

The following lemma shows that Condition (2) of the definition is implied by the other conditions:

Lemma 170.13. (CW-Subcomplex–Definition Lemma) Let X be a CW-complex and let A be a CW-complex that is a subspace of X. Suppose the following condition is satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.

Then the following condition is also satisfied:

(2) Each n-cell of A is also an n-cell of X.

Proof. We make a few preparations:

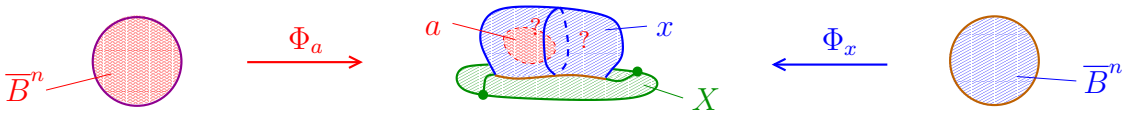
(a) For the purpose of this proof we adopt the following notation: Given a topological space Y and a subset $D \subseteq Y$ we denote by $\mathrm{cl}_Y(D)$ the closure of D in Y.
(b) Let Y be a CW-complex and let d be an open n-cell of Y. Furthermore let $\Phi_d \colon \overline{B}^n \to Y^n$ be a characteristic map (as in Definition 136.7) of the n-cell $\mathrm{cl}_Y(d)$.
  (i) It follows easily from the definitions that $\Phi_d \colon B^n \to d$ is a homeomorphism. In particular $\Phi_d(B^n) = d$.
  (ii) It follows from the Cell-Compact Proposition 136.8 that $\Phi_d(\overline{B}^n) = \mathrm{cl}_Y(d)$.

Now let a be an open n-cell of A, in other words, let a be a component of $A^n \setminus A^{n-1}$. We need to show that $\mathrm{cl}_A(a)$ is an n-cell of X. We start out with the following claim:

Claim 1. The open n-cell a is also an open n-cell of X.

Proof. Since for each $k \in \mathbb{N}_0$ we have $A^k = A \cap X^k$, we see that $a \subseteq X^n \setminus X^{n-1}$. It follows from the Image-Preimage-Component Lemma 4.50 (2) that a is contained in a component x of $X^n \setminus X^{n-1}$. By definition x is an open n-cell of X. We need to show that $a = x$.

We pick a characteristic map $\Phi_a \colon \overline{B}^n \to A$ for a and we pick a characteristic map $\Phi_x \colon \overline{B}^n \to X$ for x. It follows from (i) that the map $(\Phi_x|_{B^n})^{-1} \circ \Phi_a \colon B^n \to B^n$ is defined, injective and continuous. It follows from the Topological Invariance-of-Domain Theorem 170.9 that $((\Phi_x|_{B^n})^{-1} \circ \Phi_a)(B^n) = (\Phi_x|_{B^n})^{-1}(a)$ is an open subset of B^n. By (i) this implies that a is an open subset of $x = \Phi_x(B^n)$.



On the other hand it follows from the Compact Image Lemma 4.15 together with the Compact-Closed Lemma 3.39 (2) and the CW-Complex–Hausdorff Proposition 136.6 that $\Phi_a(\overline{B}^n)$ is a closed subset of X. Since $\Phi_a(S^{n-1}) \subseteq A^{n-1} \subseteq X^{n-1}$ and since $\Phi_a(B^n) \subseteq \Phi_x(B^n)$ we see that $\Phi_a(\overline{B}^n) \cap \Phi_x(B^n) = \Phi_a(B^n) \cap \Phi_x(B^n)$. It follows from this discussion and the Subspace-Characterization Lemma 3.17 (2) that $a = \Phi_a(B^n)$ is a closed subset of $x = \Phi_x(B^n)$.

We have thus shown that $a = \Phi_a(B^n)$ is subset of $x = \Phi_x(B^n)$ that is open and closed. Since $\Phi_x \colon B^n \to \Phi_x(B^n)$ is a homeomorphism and since B^n is connected we see that $\Phi_x(B^n)$ is connected. Since $a \neq \varnothing$ this now implies that $x = \Phi_x(B^n) = a$. ⊞

It remains to prove the following claim:

Claim 2. We have $\mathrm{cl}_A(a) = \mathrm{cl}_X(a)$.

Proof. We prove the two inclusions separately:

"$\subseteq$" By definition of the subspace topology and of the closure we have $\mathrm{cl}_A(a) \subseteq \mathrm{cl}_X(a)$.

"$\supseteq$" As in Claim 1 we see that $\Phi_a(\overline{B}^n)$ is a closed subset of X which contains a. It follows, by Definition 3.21 of the closure$\mathrm{cl}_X(a)$ of $a \subseteq X$, that we have $\mathrm{cl}_X(a) \subseteq \Phi_a(\overline{B}^n)$. But by (ii) we have $\Phi_a(\overline{B}^n) = \mathrm{cl}_A(a)$. ■

170.5. Exercises for Chapter 170 .

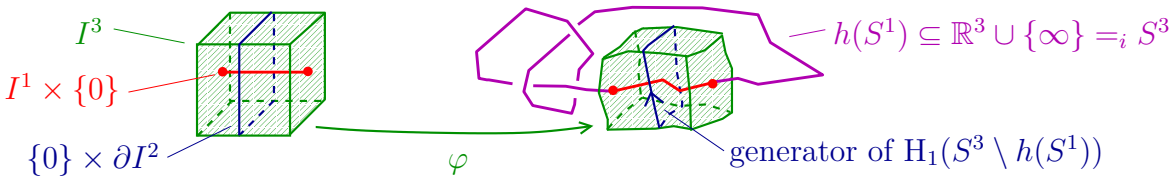
Exercise 170.1. Let $h \colon \overline{B}^k \to S^n$ be a smooth embedding. Use the Smooth Ball Embedding Theorem 45.10 to give a short proof that $\widetilde{\mathrm{H}}_i\big(S^n \setminus h(\overline{B}^k)\big) = 0$ for all $i \in \mathbb{N}_0$.

Exercise 170.2. Let $h \colon S^k \to S^n$ be an embedding. We set $I \coloneqq [-1,1]$. We suppose that there exists an embedding $\varphi \colon I^n \to S^n$ such that $h(S^k) \cap \varphi(I^n) = \varphi(I^k \times \{0\})$. Show that the map

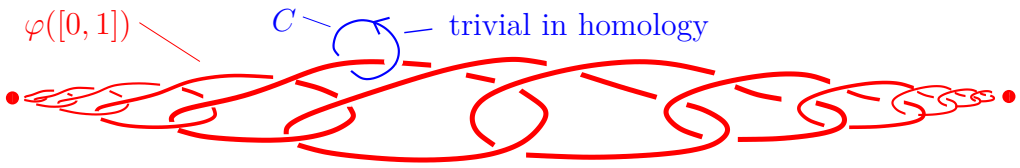
$$\varphi_* \colon \widetilde{\mathrm{H}}_{k-1}(\underbrace{\{0\} \times \partial I^k}_{\cong S^{k-1}}) \to \widetilde{\mathrm{H}}_{k-1}(S^n \setminus h(S^k))$$

is an isomorphism.

Hint. Prove the statement by induction on k.



Exercise 170.3. In the figure below we see the image of an embedding $\varphi \colon [0,1] \to \mathbb{R}^3$ and we see a closed oriented curve $C \subseteq \mathbb{R}^3 \setminus \varphi([0,1])$. Note that it follows from the Sphere Complement-H_*-Theorem 170.2 (1) together with the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1 that $\mathrm{H}_1(\mathbb{R}^3 \setminus \varphi([0,1] = 0$, in particular $[C] = 0$. Show that x is the boundary of an orientable surface Σ that is embedded in $\mathbb{R}^3 \setminus \varphi([0,1])$. (In this context this can be seen as the converse to the Surface-with-One Boundary Component-H_*-Lemma 155.23.)



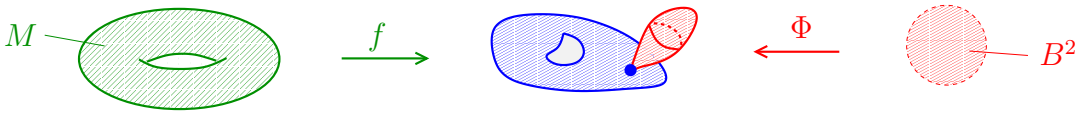
Exercise 170.4. Let $X \subseteq \mathbb{R}^n$ be a possibly non-compact subset. For every $k \in \mathbb{N}$ we consider the inclusion-induced map

$$\widetilde{\mathrm{H}}_k(\mathbb{R}^n \setminus X) \;\to\; \widetilde{\mathrm{H}}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X)$$

(a) Is this map necessarily an epimorphism?
(b) Is the rank of the kernel of this map necessarily bounded?

Exercise 170.5. Describe an embedding $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected.
Remark. You could try to modify the example shown in the figure on page 3249.

Exercise 170.6. Let M be a closed n-dimensional topological manifold, let X be a Hausdorff space, let $f\colon M \to X$ be a continuous map and let $\Phi\colon B^n \to X$ be an embedding. Show that $f(M) \cap \Phi(B^n)$ is either empty or it equals $\Phi(B^n)$.
Remark. A typical example would be that X is a CW-complex and Φ is the characteristic map of an open n-cell.



Exercise 170.7. Let $\varphi\colon S^1 \to \mathbb{R}^2$ be an embedding. Show that there exist distinct three points $x, y, z \in S^1$ such that $\varphi(x), \varphi(y), \varphi(z)$ form an equilateral triangle.
Hint. Use the Jordan-Brouwer Separation Theorem 170.3 for $n = 1$.

Exercise 170.8. Let $I = [a, b]$ be a closed interval and let $\varphi\colon I \to \mathbb{R}^3$ be an embedding. Since I is homeomorphic to $\overline{B}^1$, we obtain from the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1, the Sphere Complement-H_*-Theorem 170.2 and the Reduced Homology Lemma 147.3 that $\mathbb{R}^3 \setminus \varphi(I)$ is path-connected. Does the same conclusion hold if $I = (a, b)$ is an open interval?

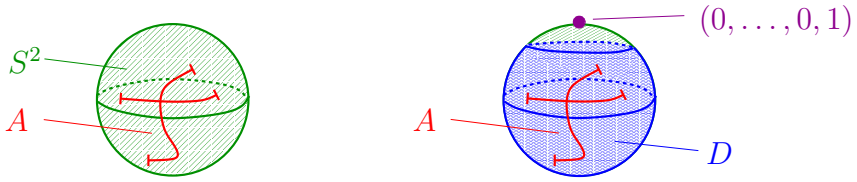
Exercise 170.9.

(a) Let $\varphi, \psi\colon S^1 \to \mathbb{R}^3$ be two embeddings with disjoint images. Determine the homology groups of $\mathbb{R}^3 \setminus (\varphi(S^1) \cup \psi(S^1))$.
Hint. What is the relationship between $\mathbb{R}^3 \setminus (\varphi(S^1) \cup \psi(S^1))$, $\mathbb{R}^3 \setminus \varphi(S^1)$ and $\mathbb{R}^3 \setminus \psi(S^1)$?
(b) Let $\varphi_1, \ldots, \varphi_k\colon S^1 \to \mathbb{R}^3$ be embeddings with disjoint images. Determine the homology groups of $\mathbb{R}^3 \setminus (\varphi_1(S^1) \cup \ldots \varphi_k(S^1))$.
(c) Let $X \subseteq \mathbb{R}^3$ be a finite topological graph. Try to guess the homology groups of $\mathbb{R}^3 \setminus X$ in terms of invariants of X.

Exercise 170.10. Let $h\colon S^1 \to S^2$ be a smooth embedding and let $i\colon S^1 \to S^2$ be the equatorial embedding given by $x \mapsto (x, 0)$. Show that there exists a diffeomorphism $H\colon S^n \to S^n$ with $H \circ i = h$.
Hint. Use the Fill-in-Disk Proposition 107.15 and the Diffeotopies-of-S^1-Proposition 42.8.



Exercise 170.11. Let $n \in \mathbb{N}_0$ and let $A \subseteq S^n$ be a closed subset with $A \neq S^n$. Show that there exists a subspace D of S^n with the following two properties:
(1) D is homeomorphic to $\overline{B}^n$.
(2) $A \subseteq D$.
Hint. First deal with the case that the North Pole $N := (0, \ldots, 0, 1)$ is contained in $S^n \setminus A$.



Exercise 170.12. Provide a proof of the Topological Invariance-of-Domain Theorem 170.9 (1) in the special case $n = 1$ just using the methods from calculus.

CHAPTER 171

Division algebras

We are all familiar with the real numbers and the complex numbers. The complex numbers form a 2-dimensional vector space over $\mathbb{R}$ which admits the structure of a field. In this chapter we will consider somewhat more general concepts, namely algebras over a field and division algebras. In this more fancy language the complex numbers are a 2-dimensional division algebra over $\mathbb{R}$.

We will introduce the quaternions which are a 4-dimensional (non-commutative!) division algebra over $\mathbb{R}$. Furthermore we will show that every finite-dimensional *commutative* division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$. Later in Chapter 214, once we have developed more tools, we will return to division algebras and discuss them in greater detail.

This chapter is organized as follows:

- In Section 171.1 we will introduce division algebras.
- In Section 171.2 we will introduce the quaternions.
- In Section 171.3 we will discuss the quaternions as a complex vector space.
- In Section 171.4 we will classify commutative division algebras.
- In Section 171.5 we will provide exercises for Chapter 171.

In the next Chapter 172 we will determine the covering dimension of open subsets of $\mathbb{R}^n$.

171.1. Division algebras.

We start out with the following purely algebraic definition.

Definition 171.1. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-algebra is an $\mathbb{F}$-vector space A together with an $\mathbb{F}$-bilinear map

$$\begin{array}{ccc} A \times A & \to & A \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

(1) We say that the algebra is commutative if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.
(2) We say that the algebra is associative if for all $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(3) We say that the algebra has a multiplicatively neutral element if there exists an element $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.
(4) If A is an algebra with multiplicatively neutral element 1, then given $a \in A$ we say that $b \in A$ is an inverse of a if $a \cdot b = b \cdot a = 1$.
(5) A division algebra is an algebra such that for all $a \neq 0$ and all $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.

Examples.

(1) The complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i}$ with the usual multiplication are an $\mathbb{R}$-algebra that is commutative and associative, it has a multiplicatively neutral element, and it is a division algebra.
(2) The set of $(n \times n)$-matrices $\mathrm{M}(n \times n, \mathbb{F})$ over a field $\mathbb{F}$, where the multiplication is given by the usual matrix multiplication, is an associative $\mathbb{F}$-algebra with a multiplicatively

neutral element given by the identity matrix id_n. If $n \geq 2$ this algebra is clearly neither commutative nor is it a division algebra.

(3) Finally in Exercise 171.1 we will consider the $\mathbb{R}$-algebra that is given by $\mathbb{R}^3$ together with the cross product and we will determine which of the above properties this algebra satisfies. □

171.2. **The quaternions.** The next definition is one of the most clever definitions in mathematics.

Definition 171.2. We define

$$\text{quaternions } \mathbb{H} \;:=\; \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot \mathrm{i} \oplus \mathbb{R}\cdot \mathrm{j} \oplus \mathbb{R}\cdot \mathrm{k}$$

with the $\mathbb{R}$-bilinear multiplication which is determined by

(1) $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$,

(2) $1\cdot \mathrm{i} = \mathrm{i}\cdot 1 = \mathrm{i}$, $\quad 1\cdot \mathrm{j} = \mathrm{j}\cdot 1 = \mathrm{j}$, $\quad 1\cdot \mathrm{k} = \mathrm{k}\cdot 1 = \mathrm{k}$, and

(3) $\mathrm{i}\cdot \mathrm{j} = k$, $\mathrm{j}\cdot \mathrm{k} = \mathrm{i}$, $\mathrm{k}\cdot \mathrm{i} = \mathrm{j}$ and $\mathrm{j}\cdot \mathrm{i} = -\mathrm{k}$, $\mathrm{k}\cdot \mathrm{j} = -\mathrm{i}$, $\mathrm{i}\cdot \mathrm{k} = -\mathrm{j}$.

For a quaternion

$$z \;:=\; a\cdot 1 + b\cdot \mathrm{i} + c\cdot \mathrm{j} + d\cdot \mathrm{k} \;\in \mathbb{H}$$

we define the conjugate

$$\overline{z} \;:=\; a\cdot 1 - b\cdot \mathrm{i} - c\cdot \mathrm{j} - d\cdot \mathrm{k} \;\in \mathbb{H}$$

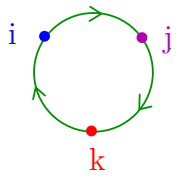
and the norm

$$|z| \;:=\; \sqrt{a^2+b^2+c^2+d^2} \quad \in\; \mathbb{R}_{\geq 0}$$

the same way as we did for complex numbers.

Remark.

(1) The multiplication table for $\mathbb{H}$ is be summarized in the figure below:



$$\begin{aligned} \mathrm{i}\cdot \mathrm{j} &= \mathrm{k} \\ \mathrm{j}\cdot \mathrm{k} &= \mathrm{i} \\ \mathrm{k}\cdot \mathrm{i} &= \mathrm{j} \end{aligned}$$

More precisely, we show the three generators i, j and k. We have $\mathrm{i}\cdot \mathrm{j} = \mathrm{k}$ and the analogous statement holds for any *cyclic* permutation of $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$. On the other hand for an anti-cyclic permutation we pick up a minus sign, e.g. we have $\mathrm{j}\cdot \mathrm{i} = -\mathrm{k}$.

(2) In Chapter 214 we will introduce the Cayley-Dickson construction which gives us a more systematic way to get a new $\mathbb{R}$-algebra out of given one. This construction naturally leads from $\mathbb{R}$ to $\mathbb{C}$ and from $\mathbb{C}$ to $\mathbb{H}$ and it will guide us from the quaternions $\mathbb{H}$ to the even more mysterious octonions $\mathbb{O}$. □

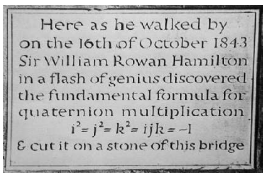
The following lemma summarizes a few key properties of the quaternions. In particular we see that the conjugate and the norm of the quaternions behave in a quite similar fashion as the conjugate and norm of complex numbers.

Lemma 171.3. (Quaternion Lemma)

(1) The quaternions are a non-commutative algebra over $\mathbb{R}$ with multiplicatively neutral element.

(2) The quaternions are associative.
(3) For all $z, w \in \mathbb{H}$ and $\lambda \in \mathbb{R}$ we have
$$\overline{\overline{z}} = z, \quad \overline{\lambda \cdot z} = \lambda \cdot \overline{z}, \quad \overline{z+w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z \cdot w} = \overline{w} \cdot \overline{z}.$$
(4) (a) For any $z \in \mathbb{H}$ we have $\overline{z} + z \in \mathbb{R}$.
(b) We have $\overline{z} \cdot z = z \cdot \overline{z} = |z|^2 \cdot 1$.
(5) If we make the obvious identification $\mathbb{H} = \mathbb{R}^4$, then the norm $|-|$ on $\mathbb{H}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.[1829]
(6) For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} := \frac{\overline{z}}{|z|^2}$.
(7) For all $z, w \in \mathbb{H}$ we have $|z \cdot w| = |z| \cdot |w|$.
(8) The quaternions are a division algebra.





Proof. [1830]

(1) This statement follows easily from the definitions.
(2) There are several approaches to proving associativity:
(a) Since the quaternions are an $\mathbb{R}$-algebra with multiplicatively neutral element 1 we only have to check associativity for products of three of the generators $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ and that can be done easily by hand.
(b) A tedious little calculation (see the $\mathbb{H}=\mathbb{C}^2$-Proposition 171.4) shows that the map
$$\begin{array}{ccc} \mathbb{H} & \to & \mathrm{M}(2 \times 2, \mathbb{C}) \\ \underbrace{z + \mathrm{j} \cdot w}_{\text{with } z, w \in \mathbb{C}} & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array}$$
is a ring homomorphism. Thus we see that $\mathbb{H}$ is isomorphic to a subring of the matrix ring $\mathrm{M}(2 \times 2, \mathbb{C})$, which is of course associative.
(c) Finally one could use the more general result proved in Exercise 171.5.
(3) This statement follows easily from the definitions.
(4) This statement follows easily from a direct calculation.
(5) This statement follows immediately from the definition of $|z|$.
(6) This statement follows immediately from (4b).
(7) Let $z_1 = a_1 \cdot 1 + b_1 \cdot \mathrm{i} + c_1 \cdot \mathrm{j} + d_1 \cdot \mathrm{k}$ and $z_2 = a_2 \cdot 1 + b_2 \cdot \mathrm{i} + c_2 \cdot \mathrm{j} + d_2 \cdot \mathrm{k}$ be two quaternions. We see that
$$\begin{aligned} |z_1 \cdot z_2|^2 &= |(a_1 \cdot 1 + b_1 \cdot \mathrm{i} + c_1 \cdot \mathrm{j} + d_1 \cdot \mathrm{k}) \cdot (a_2 \cdot 1 + b_2 \cdot \mathrm{i} + c_2 \cdot \mathrm{j} + d_2 \cdot \mathrm{k})|^2 \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)^2 + \\ &\quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2 + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)^2 \\ &= (a_1^2 + b_1^2 + c_1^2 + d_1^2) \cdot (a_2^2 + b_2^2 + c_2^2 + d_2^2) = |z_1|^2 \cdot |z_2|^2. \end{aligned}$$

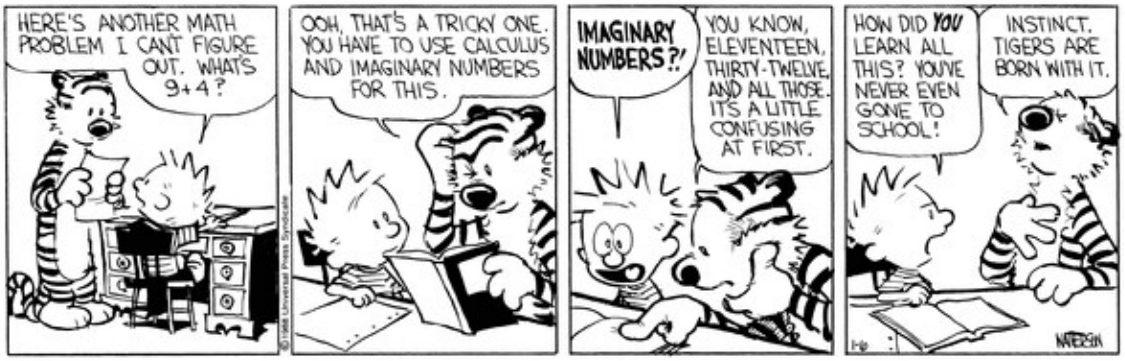
[1829] It follows in particular that the quaternion norm is indeed a norm.
[1830] On page 4104 we will give a less ad hoc proof of some of the statements of this lemma.

(8) It remains to prove that $\mathbb{H}$ is in fact a division algebra. Thus let $a \neq 0$ and b be two elements in $\mathbb{H}$. We set $y := b \cdot a^{-1}$. Then

$$y \cdot a \;=\; (b \cdot a^{-1}) \cdot a \;\underset{\uparrow}{=}\; b \cdot (a^{-1} \cdot a) \;=\; b \cdot 1 \;=\; b.$$

since by (2) we know that $\mathbb{H}$ is associative

The same way we show that there exists an $x \in \mathbb{H}$ with $a \cdot x = b$. ∎



171.3. **Quaternions as a complex vector space.** In this short section we collect a few basic facts about the quaternions viewed as a complex vector space. The reader could also move straight to the next Section 171.4 where we classify commutative division algebras.

Proposition 171.4. ($\mathbb{H}=\mathbb{C}^2$-Proposition)

(1) The map $\mathbb{H} \times \mathbb{C} \to \mathbb{H}$ given by $(z, w) \mapsto z \cdot w$[1831] defines a complex vector space structure on $\mathbb{H}$.
(2) A basis for $\mathbb{H}$ viewed as a complex vector space is given by $\{1, \mathrm{j}\}$.
(3) Let $z + \mathrm{j} \cdot w \in \mathbb{H}$ with $z, w \in \mathbb{C}$. Left-multiplication by $z + \mathrm{j} \cdot w$ defines a complex linear map $\mathbb{H} \to \mathbb{H}$. With respect to the basis $\{1, \mathrm{j}\}$ this linear map is represented by the matrix $\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$.
(4) Left-multiplication by some $h \in \mathbb{H}$ defines a homomorphism of real vector spaces. If $h \neq 0$, then this homomorphism is represented, with respect to the standard basis of $\mathbb{H}$ as a real vector space, by a matrix in $\mathrm{GL}_+(4, \mathbb{R})$.

Proof.

(1) This statement follows immediately from the observation that the obvious map $\mathbb{C} \to \mathbb{H}$ is a ring homomorphism.
(2) It follows from $\mathrm{k} = -\mathrm{j} \cdot \mathrm{i}$ that any quaternion $a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ can be uniquely written as $(a \cdot 1 + b \cdot \mathrm{i}) + \mathrm{j} \cdot (c \cdot 1 - d \cdot \mathrm{i})$ where $a \cdot 1 + b \cdot \mathrm{i}$ and $c \cdot 1 - d \cdot \mathrm{i}$ are complex numbers. This shows that $\{1, j\}$ is a basis for $\mathbb{H}$, viewed as a complex vector space.
(3) First note that it follows from the associativity of quaternions that left-multiplication by a quaternion commutes with right-multiplication by a quaternion, in particular with right-multiplication by a complex number. This shows that left-multiplication by a quaternion is indeed a complex linear map.

[1831] Here we view $\mathbb{C}$ as a subset of $\mathbb{H}$ in the obvious way. Note that the order in the definition is important since $\mathbb{H}$ is not commutative.

Now let us turn to the statement regarding the representative matrix. This statement is elementary, but we think it is nonetheless worth doing. So we consider the basis $\{1, j\}$. Evidently we have $(z + \mathrm{j} \cdot w) \cdot 1 = 1 \cdot z + \mathrm{j} \cdot w$. Thus the first column of the matrix is given by $\begin{pmatrix} z \\ w \end{pmatrix}$. More interestingly, we have

$$(z + j \cdot w) \cdot \mathrm{j} \underset{\uparrow}{=} z \cdot \mathrm{j} + \mathrm{j} \cdot w \cdot \mathrm{j} = \mathrm{j} \cdot \overline{z} + \overline{w} \cdot \mathrm{j} \cdot \mathrm{j} \underset{\uparrow}{=} \mathrm{j} \cdot \overline{z} + 1 \cdot (-\overline{w}).$$

both equalities follow from the fact that $\mathrm{i} \cdot \mathrm{j} = -\mathrm{j} \cdot \mathrm{i}$

Thus the second column of the matrix is given by $\begin{pmatrix} \overline{z} \\ -\overline{w} \end{pmatrix}$.

(4) This statement follows easily from (3). ■

In the SU(2)-SO(3)-Covering Theorem 94.3 we saw that SU(2) is diffeomorphic to S^3. We can now give a refined version of this statement:

Proposition 171.5. (Quaternion Units Proposition)

(1) The set $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ forms a group under multiplication of quaternions.

(2) The map

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z, w) = z + \mathrm{j} \cdot w & \mapsto & \underbrace{\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}} \end{array}$$

matrix that represents left-multiplication by $z + \mathrm{j} \cdot w$ with respect to the complex basis $\{1, \mathrm{j}\}$ of $\mathbb{H}$

is a diffeomorphism and it defines an isomorphism of groups.

Remark. Note that the map $f\colon S^3 \to \mathrm{SU}(2)$ from the Quaternion Unit Proposition 171.5 (2) satisfies in particular

$$f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(\mathrm{i}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f(\mathrm{j}) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \text{and} \quad f(\mathrm{k}) = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}. \quad \square$$

Proof.

(1) This statement follows immediately from the Quaternion Lemma 171.3 (5), (6) and (7).

(2) We saw in the SU(2)-SO(3)-Covering Theorem 94.3 that the map $f\colon S^3 \to \mathrm{SU}(2)$ is a diffeomorphism. It follows from the associativity of quaternion multiplication and the $\mathbb{H}{=}\mathbb{C}^2$-Proposition 171.4 (3), or alternatively from a tedious calculation, that the map

$$\begin{array}{rcl} f\colon \mathbb{H} & \to & \mathrm{M}(2 \times 2, \mathbb{C}) \\ (z, w) = z + \mathrm{j} \cdot w & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array}$$

is a ring homomorphism, in particular it preserves the ring structure. This implies that the given map $f\colon S^3 \to \mathrm{SU}(2)$ is a group homomorphism. ■

171.4. **Commutative division algebras.** In Definition 214.11 we will introduce the octonions, which form a (non-commutative and non-associative) division algebra over $\mathbb{R}^8$. After the discovery of quaternions and the octonions the question arose whether one can classify division algebras, and in particular whether one can find division algebras on $\mathbb{R}^n$ for some $n \neq 1, 2, 4, 8$. The following theorem, which was first proved by Heinz Hopf

[1831] Here we view $\mathbb{C}$ as a subset of $\mathbb{H}$ in the obvious way. Note that the order in the definition is important since $\mathbb{H}$ is not commutative.

[**Hopf1940a, Hopf1940b, Hopf1964**] in 1940, is the best theorem that we can prove in this direction with our present knowledge. Later, once we have developed more tools, we will prove the much more sophisticated Hopf–Division Algebra Dimension Theorem 214.13 which says that division algebras only exist in dimensions 2^n.

Theorem 171.6. (Commutative Division Algebra Theorem) Any finite-dimensional division algebra over $\mathbb{R}$ which is *commutative* and which has a multiplicatively neutral element is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.

Remark.

(1) Note that in the Commutative Division Algebra Theorem 171.6 we do not assume that the division algebra is associative. In Exercise 171.8 we will show, using the Fundamental Theorem of Algebra 32.4, the weaker statement that if V is a finite-dimensional commutative and *associative* division algebra over $\mathbb{R}$, then V is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
(2) The statement of the Commutative Division Algebra Theorem 171.6 does not hold if we drop the "finite-dimension" condition. For example the field $\mathbb{R}(t)$ of rational functions over $\mathbb{R}$ is a commutative division algebra, but it is an infinite-dimensional real vector space. □

In the proof of the Commutative Division Algebra Theorem 171.6 we will need the following basic lemma which might already be familiar from earlier algebra courses.

Lemma 171.7. (No Zero Divisor Lemma) Let A be a *finite-dimensional* algebra over a field $\mathbb{F}$. The following two statements are equivalent:
(1) A is a division algebra,
(2) A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.

Remark. Let A be a division algebra over a field $\mathbb{F}$. If A is associative, then it is clear that A has no zero-divisors. Indeed, if $a \cdot b = 0$ with $a \neq 0$, then we pick x with $x \cdot a = 1$ and we see that $0 = x \cdot 0 = x \cdot (a \cdot b) = (x \cdot a) \cdot b = 1 \cdot b = b$. The point about the No Zero Divisor Lemma 171.7 is that it applies to non-associative algebras, as long as they are finite-dimensional. □

Proof of Lemma 171.7. Let A be a finite-dimensional algebra over a field $\mathbb{F}$. Let $a \in A$ be non-zero. Then the following holds:

$$\begin{aligned} &\text{for any } y \in A \text{ there exists } x \in A \text{ with } a \cdot x = y \\ &\quad \Leftrightarrow \text{the map } A \to A,\ x \mapsto a \cdot x \text{ is surjective} \\ &\quad \Leftrightarrow \text{the map } A \to A,\ x \mapsto a \cdot x \text{ is injective} \\ &\qquad \uparrow \\ &\qquad \text{since } x \mapsto a \cdot x \text{ is a homomorphism of} \\ &\qquad \text{finite-dimensional } \mathbb{F}\text{-vector spaces} \\ &\quad \Leftrightarrow \text{for any } b \neq 0 \text{ we have } ab \neq 0. \end{aligned}$$

Evidently the same statement also holds for left-multiplication. The equivalence of (1) and (2) now follows easily from this observation and the definitions. ■

Now we turn to the proof of the Commutative Division Algebra Theorem 171.6:

Proof of Theorem 171.6. Let $(A, *)$ be a finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has a multiplicatively neutral element. Since A is a finite-dimensional vector space we can, after applying an isomorphism, assume that $A = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

We first consider the case $n = 1$. We set $a := 1 * 1 \in \mathbb{R}$. Since "$*$" is bilinear we have for all $x, y \in \mathbb{R}$ that $x * y = (x \cdot 1) * (y \cdot 1) = x \cdot (1 * 1) \cdot y = xay$. Since the algebra is a division algebra we obtain from the No Zero Divisor Lemma 171.7 that $a \neq 0$. One can easily verify that the map $\varphi(x) = a \cdot x$ defines an isomorphism from the algebra $(\mathbb{R}, *)$ to the algebra $(\mathbb{R}, \cdot)$.[1832]

Now suppose that $n \geq 2$. We first want to show that $n = 2$. We start out with the following claim:

Claim 1. The map
$$\begin{array}{rccc} f\colon & S^{n-1} & \to & S^{n-1} \\ & x & \mapsto & f(x) := \dfrac{x * x}{\|x * x\|} \end{array}$$
is well-defined and continuous.

Proof. This map is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on $(\mathbb{R}^n, *)$ and from the No Zero Divisor Lemma 171.7 that $x * x \neq 0$. The map f is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, hence continuous.[1833] ⊞

Since "$*$" is bilinear we have $f(-x) = f(x)$ for all $x \in \mathbb{R}^n$. It follows from the Topological-Quotient Proposition 8.2 (1b) that the map f descends to a continuous map
$$\begin{array}{rccc} \overline{f}\colon & \mathbb{R}\mathrm{P}^{n-1} & \to & S^{n-1} \\ & [x] & \mapsto & \overline{f}([x]) := f(x). \end{array}$$

Claim 2. The map $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is injective.

Proof. So let $[x], [y] \in \mathbb{R}\mathrm{P}^{n-1} = S^{n-1}/\sim$. We have
$$\underbrace{\overline{f}([x])}_{=f(x)} = \underbrace{\overline{f}([y])}_{=f(y)} \implies f(x) = f(y) \implies x * x = \alpha^2 \cdot (y * y) \text{ for } \alpha := \sqrt{\|x * x\| \cdot \|y * y\|^{-1}}$$
$$\implies 0 = x * x - \alpha^2 \cdot (y * y) \underset{\text{since "}*\text{" is commutative and bilinear}}{=} (x + \alpha y) * (x - \alpha y)$$
$$\underset{\text{by the No Zero Divisor Lemma 171.7}}{\implies} x = \pm\alpha \cdot y \underset{\text{since } x, y \in S^{n-1}}{\implies} x = \pm y \implies [x] = [y].$$
⊞

Since S^{n-1} is closed and connected (here we use that $n \geq 2$) and since $\mathbb{R}\mathrm{P}^{n-1}$ is a closed smooth manifold it follows from the Closed Manifold–Injection Corollary 170.10 that $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is in fact a homeomorphism. In particular the fundamental groups of $\mathbb{R}\mathrm{P}^{n-1}$ and S^{n-1} are isomorphic. But it follows from the Sphere-π_1-Proposition 88.24 and the Real Projective Space–π_1-Proposition 93.3 that this is only possible if $n = 2$.

It remains to show that the algebra $(A, *) = (\mathbb{R}^2, *)$ is isomorphic to the algebra $(\mathbb{C}, \cdot)$. We start out with the following claim:

[1832] Indeed, for $x, y \in \mathbb{R}$ we have $\varphi(x * y) = \varphi(x \cdot a \cdot y) = a \cdot x \cdot a \cdot y = (a \cdot x) \cdot (a \cdot y) = \varphi(x) \cdot \varphi(y)$.

[1833] Indeed, the fact that the map is bilinear implies that there exists a real $(n \times n)$–matrix P such that for every $v, w \in \mathbb{R}^n$ we have $v \cdot w = v^{\mathrm{T}} P w$, hence the map is continuous.

Claim 3. There exists a $k \in A \setminus \mathbb{R} \cdot 1$ with $k * k = c \cdot 1$ for some $c \in \mathbb{R}_{<0}$.

Proof. We first pick a random element $j \in A \setminus \mathbb{R} \cdot 1$. Then 1 and j are a basis for $A = \mathbb{R}^2$ and therefore we can write $j * j = a \cdot 1 + b \cdot \mathrm{j}$ for some $a, b \in \mathbb{R}$. Multiplying out and using this equality we see that

$$\big(\underbrace{j - \tfrac{b}{2} \cdot 1}_{=:k}\big) * \big(\underbrace{j - \tfrac{b}{2} \cdot 1}_{=:k}\big) \;=\; \big(\underbrace{a + \tfrac{1}{4} \cdot b^2}_{=:c}\big) \cdot 1.$$

We set $k \coloneqq j - \frac{b}{2} \cdot 1$ and $c \coloneqq a + \frac{1}{4} b^2$. We still need to show that $c < 0$. Suppose that $c \geq 0$. Then $k * k = c \cdot 1$ implies that $(k + \sqrt{c} \cdot 1) * (k - \sqrt{c} \cdot 1) = k * k - c \cdot 1 = 0$. From the No Zero Divisor Lemma 171.7 we conclude that $k = \pm\sqrt{c} \cdot 1$. But this is not possible since $k = j - \frac{b}{2} \cdot 1$ is linearly independent from 1. ⊞
One can now easily verify that

$$\begin{array}{rcl} A & \to & \mathbb{C} \\ x \cdot 1 + y \cdot \mathrm{k} & \mapsto & x + y \cdot \frac{\mathrm{i}}{\sqrt{-c}} \quad \text{with } x, y \in \mathbb{R} \end{array}$$

is an isomorphism from the algebra $(A, *)$ to the algebra $(\mathbb{C}, \cdot)$. ■

It is natural to ask what other division algebras exist. The following, purely algebraic, theorem classifies *associative* division algebras.

Theorem 171.8. (Frobenius Theorem) Any finite-dimensional division algebra over $\mathbb{R}$ which is associative and which has a multiplicatively neutral element is isomorphic to $\mathbb{R}, \mathbb{C}$ or to $\mathbb{H}$.

Remark. Note that in Exercise 171.7 we will see that there are many more *non-associative* division algebra structures on $\mathbb{R}^4$. □

Proof. The statement was first proved by Frobenius in 1878. A short self-contained proof is for example given in [**Pal1968**]. ■

Finally let us record the following question:

Question 171.9. For which $n \in \mathbb{N}$, besides $n = 1, 2, 4, 8$ does $\mathbb{R}^n$ admit the structure of a division algebra?

We will give a partial answer to this question in the Hopf–Division Algebra Dimension Theorem 214.13 and we will give a complete answer to this question in the Division Algebra–Dimensions Theorem 214.14.

171.5. Exercises for Chapter 171 .

Exercise 171.1. We consider the real vector space $\mathbb{R}^3$ together with the cross product

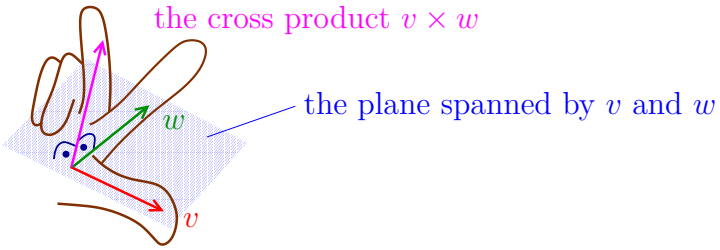
$$\begin{array}{rcl} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} & \mapsto & \begin{pmatrix} v_2 \cdot w_3 - w_2 \cdot v_3 \\ -(v_1 \cdot w_3 - w_1 \cdot v_3) \\ v_1 \cdot w_2 - w_1 \cdot v_2 \end{pmatrix}. \end{array}$$

By the Cross Product Lemma 31.26 a geometric interpretation of the cross product is given as follows: For $v, w \in \mathbb{R}^3$ the cross product $v \times w$ is the unique vector in $\mathbb{R}^3$ that satisfies the following three properties:

- The vector is orthogonal to v and w.
- The length equals the area of the parallelogram spanned by v and w.
- If v and w are non-zero, then t v, w and $v \times w$ form a positive basis for $\mathbb{R}^3$.

Here are your tasks:

(a) Show that the algebra $(\mathbb{R}^3, \times)$ is non-commutative.
(b) Show that the algebra $(\mathbb{R}^3, \times)$ is non-associative.
(c) Show that the algebra $(\mathbb{R}^3, \times)$ does not have a multiplicatively neutral element.
(d) Complete the following sentence: Given $a, b \in \mathbb{R}^3$ there exists a vector $x \in \mathbb{R}^3$ with $a \times x = b$ if and only if ¿. . . ?.



Exercise 171.2. Recall that by definition a division algebra is an algebra such that for all $a \neq 0$ and all $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$. Show that in general we cannot arrange that $a = b$.

Exercise 171.3. Show that the smooth manifolds S^3 admits a Lie group structure, as introduced in Definition 34.2.

Exercise 171.4. Let $q_1, q_2 \in S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$. We consider the map

$$\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 \cdot z \cdot q_2^{-1}. \end{array}$$

Show that the map is linear and that it is represented by a matrix in SO(4).

Exercise 171.5. Let R be a ring. An *involution* on R is a map $R \to R$, written $z \mapsto \overline{z}$ with the following properties:
(1) $\overline{z + w} = \overline{z} + \overline{w}$ for all $z, w \in R$,
(2) $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$ for all $z, w \in R$,
(3) $\overline{\overline{z}} = z$ for all $z \in R$.

Now let R be a ring with involution. We consider $S := R \oplus R$ with the obvious multiplication and the multiplication given by

$$(z_1, z_2) \cdot (w_1, w_2) \;:=\; (z_1 w_1 - \overline{w}_2 z_2, z_2 \overline{w}_1 + w_2 z_1)$$

and the involution given by

$$\overline{(w, z)} \;:=\; (\overline{w}, -z).$$

We say that S is obtained from R via the Cayley-Dickson construction.

(a) Show that S is a ring with involution.

(b) Show that if R is commutative, then S is associative.

Remark. In this chapter we referred to an involution also as a conjugation.

Exercise 171.6. Recall that by the Quaternion Units Proposition 171.5 we know that the set $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ forms a group under multiplication of quaternions. We consider the subgroup G of S^3 that is generated by i, j and k.

(a) How many elements does G have?
(b) What is the abelianization of G?

Exercise 171.7. Let $\mathbb{A} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k}$. We define the multiplication on $\mathbb{A}$ the same way as we did for the quaternions, see Definition 171.2, except that we set $\mathrm{i}^2 = -1 + \epsilon \cdot \mathrm{j}$ for some $\epsilon \in \mathbb{R}$.

(a) Show that $\mathbb{A}$ is a division algebra.
(b) Show that i has a left inverse and a right inverse.
(c) Show that the left inverse and the right inverse of i differ for sufficiently small $\epsilon > 0$.

Exercise 171.8. Show, just using the Fundamental Theorem of Algebra 32.4, that every every finite-dimensional commutative and *associative* division $\mathbb{R}$-algebra over $\mathbb{R}$ is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.

CHAPTER 172

The covering dimension of topological spaces II

Recall that in Chapter 15 we introduced the covering dimension of a topological space. Also recall that in Chapter 15 we managed to give upper bounds on the covering dimension for various topological spaces. For example in the Euclidean Subset–Covering Dimension Proposition 15.14 we showed that for every subspace $X \subseteq \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$. But we failed to give any meaningful lower bounds on the covering dimension. Now, with the homology groups under our belt, we have the tools to rectify this issue. In particular we now have the tools to answer some questions that had occurred naturally, but that back then we could not answer.

This chapter is organized as follows:

- In Section 172.1 we will recall the definition of the covering dimension of a topological space and we will determine the covering dimension of the standard simplex 172.1.
- In Section 172.2 we will determine the covering dimension of topological manifolds.
- In Section 172.3 we will provide exercises for Chapter 172.

In the next Chapter 173 we will introduce the Moore spaces $\mathrm{M}(\pi, n)$.

172.1. **Covering dimension of the standard simplex.** We start out with the following straightforward corollary to the Δ^n-Homeomorphism Lemma 143.9 (2):

First let us recall Definition 15.2:

Definition 172.1. Let X be a topological space.

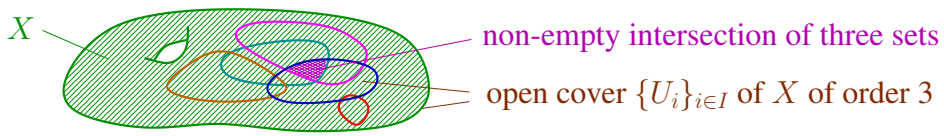
(1) A family $\{U_i\}_{i\in I}$ of subsets of X has order m if the following two statements hold:
   (a) There exists $J \subseteq I$ with $\#J = m$ such that $\bigcap_{j\in J} U_j \neq \varnothing$.
   (b) For every $J \subseteq I$ with $\#J > m$ we have $\bigcap_{j\in J} U_j = \varnothing$.

   If no such m exists, then we say that the family has infinite order.

(2) If X is non-empty, then we refer to

$$\operatorname{cdim}(X) \;:=\; \min\left\{ m \in \mathbb{N}_0 \cup \{\infty\} \;\middle|\; \begin{array}{c}\text{every open cover of } X \text{ admits an}\\ \text{open refinement of order } \leq m+1\end{array} \right\}$$

as the covering dimension of X. If X is empty, then we set $\operatorname{cdim}(X) := -1$.



The following is the main technical result of this section.

Theorem 172.2. (Covering Dimension-of-Δ^n-Theorem) Let $n \in \mathbb{N}_0$. The covering dimension of the standard n-simplex

$$\Delta^n \coloneqq \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \cdots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \ldots, n\}$$

equals n.

The Covering Dimension-of-Δ^n-Theorem 172.2 has many applications. For example, later in the Covering Dimension-of-Manifold Theorem 172.6 we will use it to nail down the covering dimension of topological manifolds. But before we can revel in the applications, we first have to go through the hard slog of proving the Covering Dimension-of-Δ^n-Theorem 172.2. This proof will require the remainder of this section. In the next Section 172.2 we will apply this result to determine the covering dimension of topological manifolds.

The proof of the Covering Dimension-of-Δ^n-Theorem 172.2 requires some preparations.

Definition 172.3. Let $n \in \mathbb{N}_0$. We denote by $\mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1}$ the standard basis vectors of $\mathbb{R}^{n+1}$.

(1) Given a finite subset $K \subseteq \{1, \ldots, n+1\}$ with $r = \#K$ elements we refer to

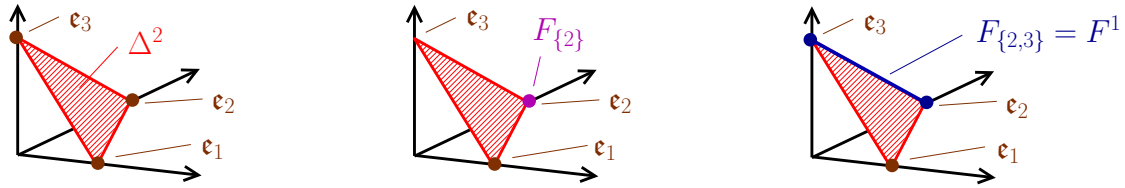
$$F_K \coloneqq \left\{ \sum_{k \in K} \lambda_k \cdot \mathfrak{e}_k \,\middle|\, \sum_{k \in K} \lambda_k = 1 \text{ and each } \lambda_k \in \mathbb{R}_{\geq 0} \right\} \subseteq \Delta^n$$

as an r-dimensional face of the standard n-simplex.

(2) Given $i \in \{1, \ldots, n+1\}$ we denote by F^i the $(n-1)$-dimensional face that is given by the set $\{1, \ldots, \widehat{i}, \ldots, n+1\}$, i.e. we set

$$F^i \coloneqq F_{\{1,\ldots,\widehat{i},\ldots,n+1\}} = \left\{ \sum_{j=1}^{n+1} \lambda_j \cdot \mathfrak{e}_j \in \Delta_n \,|\, \mathfrak{e}_i = 0 \right\}.$$

(3) As in Definition 143.8 we write $\partial\Delta^n = \bigcup_{i=1}^{n+1} F^i$.



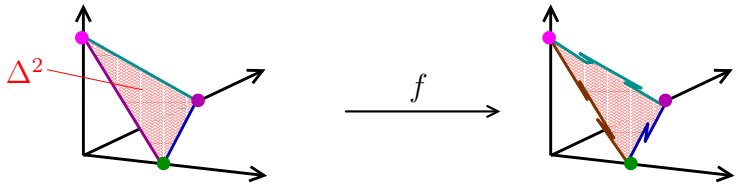
Lemma 172.4. (Face-to-Face Lemma) Let $f\colon \Delta^n \to \Delta^n$ be a continuous map. If f sends each face of Δ^n *into* itself, then[1834] $f_*\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is the identity, in particular it is an isomorphism.

Proof of Lemma 172.4. We prove the lemma by induction on n. The case $n = 0$ is basically trivial, it follows for example from the H_0-Proposition 143.25. So now suppose that the statement holds for some $n-1$. Let $f\colon \Delta^n \to \Delta^n$ be a continuous map that sends each face of Δ^n into itself.

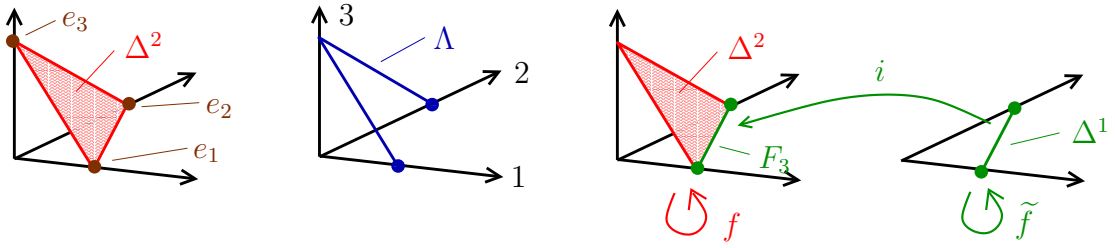
We introduce the following notation:

- As on page 2860 we denote by $\Lambda = F^1 \cup \cdots \cup F^n$ the union of all $(n-1)$-dimensional faces of Δ^n except for the "last one", i.e. except for the $(n-1)$-dimensional face F^{n+1} that is given by $\{1, \ldots, n\}$.

[1834] Note that the hypothesis that f sends each face of Δ^n into itself implies that $f(\partial\Delta^n) \subseteq \partial\Delta^n$. Thus f does indeed induce a self-map of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$.



- We denote by $i\colon \Delta^{n-1} \to F^{n+1} \to \Delta^n$ the inclusion of Δ^{n-1} as the "last n-dimensional face", i.e. $i\colon \Delta^{n-1} \to F^{n+1} \to \Delta^n$ is the map that is given by $i(x) = (x, 0)$.
- The hypothesis that the map $f\colon \Delta^n \to \Delta^n$ sends each face of Δ^n into itself implies that $f(\Lambda) \subseteq \Lambda$, it implies that $f(\partial\Delta^n) \subseteq \partial\Delta^n$ and finally it implies that there exists a continuous self-map $\widetilde{f}\colon \Delta^{n-1} \to \Delta^{n-1}$ such that $f \circ i = i \circ \widetilde{f}$. Note that $\widetilde{f}$ again has the property that it sends each face of Δ^{n-1} into itself.
- We denote by $\partial_n\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ the connecting homomorphism of the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$, see the LES-of-Triple Proposition 146.5 for details.



We obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) & \xleftarrow{i_*} & \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \widetilde{f}_*} \\
\mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) & \xleftarrow{i_*} & \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).
\end{array}$$

We make the following observations:

- The square to the left commutes by the naturality of the connecting homomorphism, see the LES-of-Triple Proposition 146.5. The square to the right commutes by the functoriality of homology groups and the fact that $f \circ i = i \circ \widetilde{f}$.
- As on page 2860 we see that all of the horizontal maps are isomorphisms. For the reader's convenience we recall the argument. First note that the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ looks as follows:

$$\underbrace{\mathrm{H}_n(\Delta^n, \Lambda)}_{\substack{= 0 \text{ by the Deformation} \\ \text{Retract-Relative } \mathrm{H}_*\text{-Corollary 146.9} \\ \text{since } \Lambda \text{ is a deformation} \\ \text{retract of } \Delta^n}} \xrightarrow{p_*} \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{i_*} \underbrace{\mathrm{H}_{n-1}(\Delta^n, \Lambda)}_{\text{again } = 0}.$$

Thus we see that the connecting homomorphism is an isomorphism. Now we turn our attention to the inclusion-induced maps i_*. It follows from the Excision-via-Deformation Theorem 148.5, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = \{(0, \ldots, 0, 1)\}$[1835] that

[1835] One can easily show that U has the property required in the Excision-via-Deformation Theorem 148.5.

the map

$$\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \;=\; \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n\setminus\mathring{\Lambda}}_{=\Delta^{n-1}},\underbrace{\Lambda\setminus\mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{i_*} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$$

is an isomorphism.

- By induction hypothesis the right hand vertical map is the identity.
- It follows from the above that the left hand vertical map is also the identity. ■

Before we head to the actual proof of the Covering Dimension-of-Δ^n-Theorem 172.2 we recall, for the reader's convenience, the following lemma.

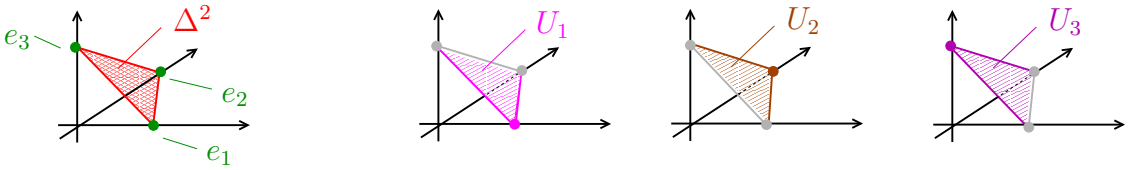
Lemma 15.4. (Covering Dimension–Refinement Lemma) Let X be a topological space with covering dimension $d := \dim(X) \in \mathbb{N}_0$. Given any open cover $\{U_i\}_{i\in I}$ there exists an open cover $\{V_i\}_{i\in I}$ of order $\leq d+1$ such that for each $i \in I$ we have $V_i \subseteq U_i$.

Proof of the Covering Dimension-of-Δ^n-Theorem 172.2. Let $n \in \mathbb{N}_0$. We need to show that $\dim(\Delta^n) = n$. By the Δ^n-Homeomorphism Lemma 143.9 (1) we know that Δ^n is homeomorphic to a compact subset of $\mathbb{R}^n$. Therefore it follows from the Compact Euclidean Subset-Covering Dimension Theorem 15.5 that $\dim(\Delta^n) \leq n$. Thus it remains to show that $\dim(\Delta^n) \geq n$. We assume by contradiction that $\dim(\Delta^n) < n$.

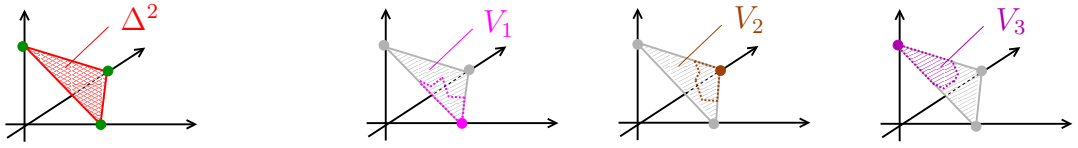
We introduce some notation. Given $i \in \{1,\ldots,n+1\}$ we denote by F^i the $(n-1)$-dimensional face that is given by the set $\{1,\ldots,\widehat{i},\ldots,n+1\}$, i.e. as in Definition 172.3 we set

$$F^i \;:=\; F_{\{1,\ldots,\widehat{i},\ldots,n+1\}} \;=\; \Big\{\sum_{j=1}^{n+1}\lambda_j\cdot e_j \in \Delta_n \,\Big|\, e_i = 0\Big\}.$$

Furthermore for $i = 1,\ldots,n+1$ we set $U_i := \Delta^n \setminus F^i$. Note that e_i is the only vertex of Δ^n that is contained in U_i. It is rather elementary to verify that each U_i is open in Δ^n. Furthermore, note that $U_1 \cup \cdots \cup U_{n+1} = \Delta^n$. In other words, $U_1,\ldots,U_{n+1}$ is an open cover of Δ^n. Since we assume that $\dim(\Delta^n) < n$ we know, by the Covering Dimension–



Refinement Lemma 15.4, that there exists an open refinement $\{V_1,\ldots,V_{n+1}\}$ of the open cover $\{U_1,\ldots,U_{n+1}\}$ of order $\leq n$ such that for each $i \in \{1,\ldots,n+1\}$ we have $V_i \subseteq U_i$.



Claim 1. There exists a partition of unity $\{\Phi_i\colon \Delta^n \to [0,1]\}_{i=1,\ldots,n+1}$ on Δ^n such that for each $i \in \{1,\ldots,n+1\}$ we have $\mathrm{supp}(\Phi_i) \subseteq V_i$.

Proof. In the $\mathbb{R}^n$-Paracompact Lemma 14.4 we saw that it follows from the Paracompact-Criterion Lemma 14.5 that $\mathbb{R}^{n+1}$ is paracompact. We deduce from the Closed Subset-Paracompact Lemma 14.6 that the closed subset Δ^n is also paracompact. The statement now follows from the Paracompact-Partition of Unity Theorem 14.15 (1)⇒(3).[1836] ⊞

Claim 2. The map

$$\begin{aligned} r\colon \Delta^n &\to \mathbb{R}^{n+1} \\ x &\mapsto \sum_{i\in\{1,\dots,n+1\}} \Phi_i(x)\cdot e_i \end{aligned}$$

has the following properties:

(1) it sends each face of Δ^n into itself, in particular it sends Δ^n into itself,
(2) we have $r(\Delta^n)\subseteq\partial\Delta^n$.

Proof.

(1) Let $K\subseteq\{1,\dots,n+1\}$ be a subset. As above we denote by

$$F_K := \left\{ \sum_{k\in K}\lambda_k\cdot e_k \;\middle|\; \sum_{k\in K}\lambda_k = 1 \text{ and each } \lambda_k\in\mathbb{R}_{\geq 0}\right\}$$

the corresponding face. We need to show that $f(F_K)\subseteq F_K$. We start out with the three observations:

(a) We have the following description of the face F_K:

$$F_K = \left\{ \sum_{i=1}^{n+1}\lambda_i\cdot e_i\in\Delta^n \;\middle|\; \lambda_i = 0 \text{ if } i\notin K\right\} = \bigcap_{i\notin K} F^i = \bigcap_{i\notin K}(\Delta^n\backslash U_i) = \Delta^n\backslash\Big(\bigcup_{i\notin K} U_i\Big).$$

(b) It follows from (a) that if for some i we have $x\in F_K\cap U_i$, then $i\in K$.

(c) Let $x\in\Delta^n$. If $\Phi_i(x)\neq 0$, then $x\in\operatorname{supp}(\Phi_i)\subseteq V_i\subseteq U_i$.

Recall that we need to show that $r(F_K)\subseteq F_K$. Let $x\in F_K$. We calculate

$$r(x) \underset{\substack{\uparrow\\ \text{definition of } r}}{=} \sum_{i\in\{1,\dots,n+1\}}\Phi_i(x)\cdot e_i \underset{\substack{\uparrow\\ \text{follows from (c)}}}{=} \sum_{i\,\in\,\{1,\dots,n+1\}\text{ with } x\in U_i}\Phi_i(x)\cdot e_i \underset{\substack{\uparrow\\ \text{by (b) we know that } x\in F_K\cap U_i\\ \text{implies that } i\in K}}{=} \sum_{i\in K}\Phi_i(x)\cdot e_i \in F_K.$$

(2) Since the cover $\{V_i\}_{i\in\{1,\dots,n+1\}}$ has order $\leq n$ we see that for each $x\in\Delta^n$ there exist at most n indices $i\in\{1,\dots,n+1\}$ for which $\Phi_i(x)\neq 0$. But this means that $r(x)$ is the linear combination of at most n vertices, which means that $r(x)\in\bigcup_{i=1}^{n+1}F^i=\partial\Delta^n$. ⊞

We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(\Delta^n,\partial\Delta^n) & \xrightarrow{\quad r_*\quad} & \mathrm{H}_n(\Delta^n,\partial\Delta^n). \\ & {\scriptstyle r_*}\searrow \quad \nearrow & \\ & \mathrm{H}_n(\partial\Delta^n,\partial\Delta^n) & \end{array}$$

We make the following observations:

- By Claim 2 (1) we know that $r(\partial\Delta^n)\subseteq\partial\Delta^n$. Thus we see that the top horizontal map is actually defined. Next with note that it follows from Claim 2 (1) together with the Face-to-Face Lemma 172.4 that the top horizontal map is an isomorphism.

[1836] Alternatively note that by the Δ^n-Homeomorphism Lemma 143.9 (2) we know that Δ^n is homeomorphic to $\overline{B}^n$, which implies easily that Δ^n is an n-dimensional topological manifold. Thus one can deduce Claim 1 from the Manifold–Partitions-of-Unity Theorem 27.2.

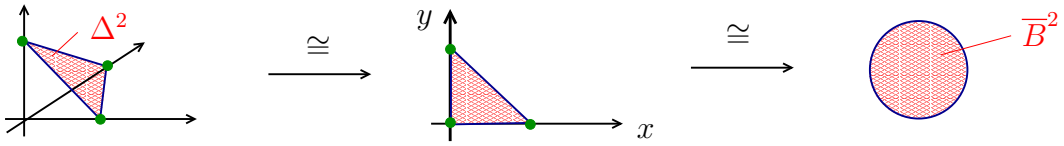
- It follows from Claim 2 (2) that the top horizontal map factors through the bottom group.
- The bottom group is evidently zero.
- We know by the Identity-is-Generator Lemma 150.1 that the top groups are non-trivial.

The above four statements are incompatible. Thus we have found a contradiction to our hypothesis that $\operatorname{cdim}(\Delta^n) < n$. ■

172.2. **Covering dimension of topological manifolds.** We start out with the following straightforward corollary to the Δ^n-Homeomorphism Lemma 143.9 (2):

Corollary 172.5. (Covering Dimension-of-Ball Corollary) For each $n \in \mathbb{N}_0$ we have $\operatorname{cdim}(\overline{B}^n) = n$.

Proof. First recall that by the Δ^n-Homeomorphism Lemma 143.9 (2) we have a homeomorphism $f\colon \overline{B}^n \to \Delta^n$.



Note that

$$\operatorname{cdim}(\overline{B}^n) \underset{\uparrow}{=} \operatorname{cdim}(\Delta^n) \underset{\uparrow}{=} n.$$

by the Covering Dimension Homeomorphism Lemma 15.3 and the above homeomorphism

by the Covering Dimension-of-Δ^n Theorem 172.2

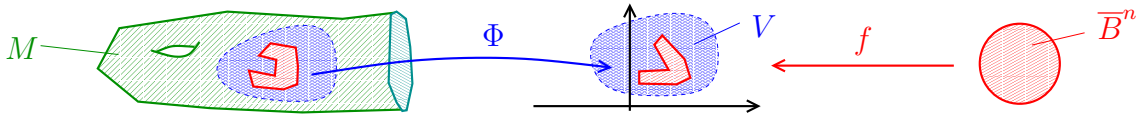
■

The name of the game is now to identify topological spaces that contained closed subsets that are homeomorphic to $\overline{B}^n$ or equivalently to Δ^n.

Theorem 172.6. (Covering Dimension-of-Manifold-Theorem) Let $n \in \mathbb{N}_0$. Given any non-empty n-dimensional topological manifold M we have $\operatorname{cdim}(M) = n$.

Proof. Let M be a non-empty n-dimensional topological manifold. In the Covering Dimension-of-Manifolds Proposition 33.9 we already showed that $\operatorname{cdim}(M) \leq n$. Thus it remains to show that $\operatorname{cdim}(M) \geq n$.

Since M is non-empty there exists a chart $\Phi\colon U \to V$ where V is an open non-empty subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. We pick an embedding $f\colon \overline{B}^n \to V$. Note that $\Phi^{-1} \circ f\colon \overline{B}^n \to M$ is an embedding.



Now we see that

$$\operatorname{cdim}(M) \underset{\uparrow}{\geq} \operatorname{cdim}((\Phi^{-1} \circ f)(\overline{B}^n)) \underset{\downarrow}{=} \operatorname{cdim}(\overline{B}^n) \underset{\uparrow}{\geq} n.$$

this follows from the Closed Subspace Covering Dimension Lemma 15.7, since by the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 we know that $(\Phi^{-1} \circ f)(\overline{B}^n)$ is a closed subset

by the Covering Dimension Homeomorphism Lemma 15.3

by the Covering Dimension-of-Ball Corollary 172.5

■

In the Euclidean Subset–Covering Dimension Proposition 15.14 we showed that for every subset $X \subseteq \mathbb{R}^n$ we have $\operatorname{cdim}(X) \leq n$. We now have the following complementary result, which gives in particular an affirmative answer to Question 15.13.

Corollary 172.7. (Covering Dimension–Non-Empty Interior Corollary) Let $X \subseteq \mathbb{R}^n$ be a subset. If the interior of X is non-empty, then $\operatorname{cdim}(U) = n$.

Proof. Let $X \subseteq \mathbb{R}^n$ be a subset with $\overset{\circ}{X} \neq \varnothing$. Note that by the Interior-Closure Lemma 3.22 we know that $\overset{\circ}{X}$ is an open subset of $\mathbb{R}^n$. It follows that there exists an embedding $f\colon \overline{B}^n \to \overset{\circ}{X}$. Now we see that

$$\begin{array}{ccccccc} & & & & \text{by the Closed Subspace Covering Dimension Lemma 15.7 and since it follows from the Compact-Closed Lemma 3.39 and the Compact Image Lemma 4.15 that } f(\overline{B}^n) \text{ is a closed subset of } M & & \\ & & & & \downarrow & & \\ n & \underset{\uparrow}{\geq} & \operatorname{cdim}(X) & \overset{\downarrow}{\geq} & \operatorname{cdim}(f(\overline{B}^n)) & = & \operatorname{cdim}(\overline{B}^n) & \underset{\uparrow}{\geq} & n. \\ & \text{by the Euclidean Subset–Covering Dimension Proposition 15.14} & & & \text{by the Covering Dimension–Homeomorphism Lemma 15.3} & & & \text{by the Covering Dimension of-Ball Corollary 172.5} & \end{array}$$

■

Interestingly the converse to the Covering Dimension–Non-Empty Interior Corollary 172.7 holds:

Theorem 172.8. (Covering Dimension–Interior Theorem) Given any subset $X \subseteq \mathbb{R}^n$ we have

$$\operatorname{cdim}(X) = n \iff \text{the interior of } X \text{ is non-empty.}$$

Proof. The"$\Leftarrow$"-direction is precisely the content of the Covering Dimension–Non-Empty Interior Corollary 172.7. The proof of the"$\Rightarrow$"-direction is given in [**Enge1978**, Theorem 1.8.10], except that the statement is formulated with the "inductive dimension" instead of the covering dimension. Also note that the definition of "covering dimension" given in [**Enge1978**, Definition 1.6.7] differs slightly from our. But by [**Enge1978**, Theorem 1.7.7] and [**Enge1978**, Exercise 1.7.E] these three different notions of "dimension" agree for subsets of $\mathbb{R}^n$. ■

For completeness' sake we mention the following proposition which we will not make use of:

Proposition 172.9. (Covering Dimension–CW Complex Proposition) Given any CW-complex X we have

$$\dim(X) = \operatorname{cdim}(X).$$

Proof. Let X be a CW-complex that contains an n-cell. Let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of an n-cell. By definition of a characteristic map we know that Φ is continuous. Furthermore, by definition, the restriction of Φ to $\overline{B}^n_{\frac{1}{2}} \to X$ is injective. Furthermore note that by the CW-Complex–Hausdorff Proposition 136.6 we know that X is Hausdorff.

Since $\overline{B}^n_{\frac{1}{2}}$ is compact, since Φ is continuous and since X is Hausdorff we know by the Compact Image Lemma 4.15, the Compact-Closed Lemma 3.39 (2) and the Compact-Hausdorff Proposition 4.23 (2) that $\Phi\colon \overline{B}^n_{\frac{1}{2}} \to X$ is a closed embedding. Thus we see that

$$\operatorname{cdim}(X) \underset{\text{by the Closed Subspace Covering Dimension Lemma 15.7}}{\geq} \operatorname{cdim}(\Phi(\overline{B}^n_{\frac{1}{2}})) \underset{\text{by the Covering Dimension Homeomorphism Lemma 15.3}}{=} \operatorname{cdim}(\overline{B}^n_{\frac{1}{2}}) \overset{\text{by the Covering Dimension-of-Ball Corollary 172.5}}{=} n.$$

This shows that $\operatorname{cdim}(X) \geq \dim(X)$. The reverse inequality is proved in [**FrPi1990b**, Theorem 4.4]. ■

172.3. **Exercises for Chapter 172 .**

Exercise 172.1. We consider the cube $[-1,1]^n$. Given a subset $J = (j_1, \ldots, j_n) \in \{-1,0,1\}^n$ we define the corresponding J-face as the subset

$$F_J := \{(x_1, \ldots, x_n) \in [-1,1]^n \mid \text{for every } i \in \{1, \ldots, n\} \text{ with } j_i \neq 0 \text{ we have } x_i = j_i\}.$$

Let $f\colon [-1,1]^n \to [-1,1]^n$ be a continuous map that sends every J-face into itself. Show that f is surjective.

Exercise 172.2. Let $n \in \mathbb{N}$.

(a) Provide a finite set $\{M_j\}_{j \in J}$ of submanifolds of S^n such that the following statement holds: if $f\colon S^n \to S^n$ is a continuous map such that for every $j \in J$ we have $f(M_j) \subseteq M_j$, then $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is an isomorphism.

(b) What is the minimal cardinality for J that you can achieve?
Remark. The author does not know the answer, but evidently one should strive for minimal cardinality of J. If we allow all submanifolds, in particular open subsets, then the record is $n+2$. If we restrict ourselves to compact submanifolds, then the record is $2n+2$.

Exercise 172.3. Let X be an n-dimensional topological manifold. Show that for every subset $A \subseteq X$ the covering dimension A is $\leq n$.

CHAPTER 173

Moore spaces

This very short chapter is organized as follows:

- In Section 173.1 we will introduce the Moore spaces $\mathrm{M}(\pi, n)$. These can be viewed as some of the simplest building blocks in topology.
- In Section 173.2 we will provide exercises for Chapter 173.

This chapter concludes this part of the lecture notes. In the next Part XIX we will turn our attention to simplicial complexes. We will return to the study of singular homology in Part XX, where we will study homology groups of topological manifolds in greater detail.

173.1. **Definition of Moore spaces.** Now that we have played for so many chapters with homology groups the following question arises: which sequences of abelian groups can appear as the sequence of homology groups $\mathrm{H}_*(X)$ of a non-empty topological space? It turns out that anything goes, there are basically no restrictions, except that we know by the combination of the H_0-Proposition 143.25 and the Path-Component-H_*-Lemma 143.21 that $\mathrm{H}_0(X)$ is necessarily a non-trivial free abelian group.

In other words, we have the following proposition:

Proposition 173.1. (Homology Realization Proposition) If $\{A_n\}_{n\in\mathbb{N}_0}$ is a sequence of abelian groups and if A_0 is a non-trivial free abelian group, then there exists a CW-complex Z such that for every $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(Z) \cong A_n$.

This proposition is hardly the most exciting result of these notes. But the following definition, which we will need in the proof of the Homology Realization Proposition 173.1, will in fact turn out to be convenient on a few other occasions.

Definition 173.2. Let $n \in \mathbb{N}$ and let π be an abelian group. We say that a non-empty topological space X is a Moore[1843] space of type $\mathrm{M}(\pi, n)$ if the following conditions are satisfied:

(1) X admits a CW-structure.
(2) For any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(X) = 0$.
(3) $\mathrm{H}_n(X) \cong \pi$.
(4) If $n \geq 2$, then we also demand that X is simply connected.

Remark. If in the above Definition 173.2 of a Moore space we replace "homology groups" by "homotopy groups", then we obtain the notion of an Eilenberg-Maclane space, which we will introduce in Definition 307.2. □

Example. Let $n \in \mathbb{N}$. It follows from the Sphere-π_1-Proposition 88.24 and the Sphere-H_*-Proposition 149.4 that S^n is a Moore space of type $\mathrm{M}(\mathbb{Z}, n)$. □

[1843] Moore spaces were first studied by John C. Moore in [**MooJ1954**].

Proposition 173.3. (Moore Space Existence Proposition) Given any $n \in \mathbb{N}$ and given any abelian group π there exists CW-complex X with the following properties;

(1) The dimension of the CW-complex X is $\leq n+1$.
(2) X has a single 0-cell.
(3) X has no cells in dimensions $1, \ldots, n-1$.
(4) X is a Moore space of type $\mathrm{M}(\pi, n)$.

Proof of Proposition 173.1 assuming Proposition 173.3. Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence of abelian groups. Furthermore let A_0 be a non-trivial free abelian group, this means that we have $A_0 \cong \mathbb{Z}^{(S)}$ for some non-empty set S.

By the Moore Space Existence Proposition 173.3 we can find for each $n \in \mathbb{N}$ a CW-complex X_n that is a Moore space of type $\mathrm{M}(A_n, n)$. For each $n \in \mathbb{N}$ we pick a point $x_n \in X_n^0$ in the 0-skeleton of X_n. We consider the wedge $Y := \bigvee_{n\in\mathbb{N}} (X_n, x_n)$. By the CW-Wedge Lemma 139.11 we can equip Y with a CW-structure. Next note that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_n(Y) \quad = \quad \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in\mathbb{N}} (X_k, x_k)\Big) \quad \underset{\substack{\uparrow \\ \text{by the Wedge-H}_* \\ \text{Proposition 149.16}}}{\cong} \quad \bigoplus_{k\in\mathbb{N}} \widetilde{\mathrm{H}}_n(X_k) \quad \underset{\substack{\uparrow \\ \text{by Definition 173.2 of a} \\ \text{Moore space}}}{\cong} \quad \begin{cases} A_n, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0. \end{cases}$$

It follows from this calculation and the Reduced Homology Lemma 147.3 that for every $n \in \mathbb{N}$ we have $\mathrm{H}_n(Y) \cong \widetilde{\mathrm{H}}_n(Y) \cong A_n$ and that $\mathrm{H}_0(Y) \cong \mathbb{Z}$. Since S is non-empty we can pick a point $* \in S$. We define $Z := Y \sqcup (S \setminus \{*\})$ where we equip $S \setminus \{*\}$ with the discrete topology. It follows from the CW-Disjoint Union Lemma 139.8 that Z is a CW-complex. Finally note that it follows from the H_0-Proposition 143.25 and the Path-Component-H_*-Lemma 143.21 that Z has the promised homology groups. ∎

Next we want to provide a proof of the Moore Space Existence Proposition 173.3. To simplify the notation in the following discussions we introduce the following convention.

Convention 173.4. Let $n \in \mathbb{N}_0$. Throughout this chapter we equip S^n with the base point given by the North Pole $N = (0, \ldots, 0, 1)$. In particular we use this base point to define the wedges of spheres. Furthermore we equip S^n with the CW-structure from page 2625 which has one 0-cell given by the North Pole, one n-cell and no other cells.

With this convention we can formulate the following elementary auxiliary lemma, which we will need in the proof of the Moore Space Existence Proposition 173.3.

Auxiliary Lemma 173.5.

(1) Let $f\colon S^n \to S^n_1 \vee S^n_2$ be the pinching map that we introduced on page 2579. Furthermore, given $i \in \{1, 2\}$ let $p_i\colon S^n_1 \vee S^n_2 \to S^n_i = S^n$ be the continuous map that is the identity on S^n_i and the constant map $x \mapsto N$ on S^n_{3-i}. With this notation the induced map $(p_i \circ f)_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is the identity.
(2) Given any $k \in \mathbb{Z}$ there exists a continuous map $g_k\colon S^n \to S^n$ with $g_k(N) = N$ such that $g_{k*}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by k.

Proof of Lemma 173.5.

(1) Statement (1) follows easily from the S^n-Local-Global Degree Proposition 152.5.

(2) Statement (2) follows almost immediately from the Degree-of-Power Lemma 151.5 together with the Suspension-H_*-Proposition 149.3. ■

Now we turn to the proof of the Moore Space Existence Proposition 173.3.

Proof of the Moore Space Existence Proposition 173.3. Let $n \in \mathbb{N}$ and let π be an abelian group. Since π is abelian it follows immediately from the Length 1–Free Resolution Lemma 162.9 that there exist sets A and B, a homomorphism $\varphi\colon \mathbb{Z}^{(A)} \to \mathbb{Z}^{(B)}$ and a homomorphism $\rho\colon \mathbb{Z}^{(B)} \to \pi$ such that the following sequence is exact:

$$0 \;\to\; \mathbb{Z}^{(A)} \;\xrightarrow{\varphi}\; \mathbb{Z}^{(B)} \;\xrightarrow{\rho}\; \pi \;\to\; 0.$$

We set $X \coloneqq \bigvee_{a\in A} S^n_a$ and we set $Y \coloneqq \bigvee_{b\in B} S^n_b$. By the CW-Wedge Lemma 139.11 we can equip X with the obvious CW-structure which has precisely one 0-cell and which has precisely one n-cell for each $a \in A$. Similarly we equip Y with the obvious CW-structure. Using the Wedge-H_*-Proposition 149.16 and using the standard generator of $\mathrm{H}_n(S^n)$ we obtain natural isomorphisms $\alpha\colon \mathbb{Z}^{(A)} \to \mathrm{H}_n(X)$ and $\beta\colon \mathbb{Z}^{(B)} \to \mathrm{H}_n(Y)$.

Claim. There exists a cellular map $h\colon X = \bigvee_{a\in A} S^n_a \to Y = \bigvee_{b\in B} S^n_b$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{Z}^{(A)} & \xrightarrow{\quad h_*\quad} & \mathbb{Z}^{(B)} \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\beta} \\ \mathrm{H}_n(X) & \xrightarrow{\quad\varphi\quad} & \mathrm{H}_n(Y). \end{array}$$

Proof. This cellular map h can be constructed explicitly using the following ingredients:

(1) The pinching map $f\colon S^n \to S^n_1 \vee S^n_2$ from Lemma 173.5 (1).
(2) The continuous maps $g_k\colon S^n \to S^n$, $k \in \mathbb{Z}$, from Lemma 173.5 (2).
(3) The natural inclusions $S^n_i \to S^n_1 \vee S^n_2$ from Definition 96.1.
(4) The natural maps $p_i\colon S^n_1 \vee S^n_2 \to S^n_i$ from Lemma 173.5 (1).
(5) The wedge of maps as introduced in the Wedge Proposition 96.2 (1b).

Using the Wedge-H_*-Proposition 149.16, Exercise 149.19 and Lemma 173.5 (1) one can then verify that the map has the desired properties. The notation becomes somewhat cumbersome, thus in a short moment of lassitude we decide to leave it to the reader to fill in the details. A baby example is the content of Exercise 173.1. ⊞

Now let $Z \coloneqq \mathrm{Cone}(h\colon X \to Y)$ be the mapping cone of the cellular map $h\colon X \to Y$ of the claim. It follows from the above claim, the Mapping Cone-H_*-Lemma 153.17 and the Wedge-H_*-Proposition 149.16 that we have the following commutative diagram of long exact sequences

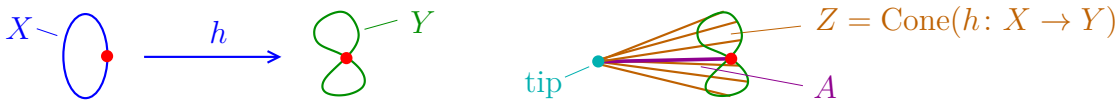
$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathrm{H}_{n+1}(Z) & \longrightarrow & \mathbb{Z}^{(A)} & \xrightarrow{\varphi} & \mathbb{Z}^{(B)} & \longrightarrow & \mathrm{H}_n(Z) & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & {\scriptstyle\cong}\downarrow{\scriptstyle\alpha} & & {\scriptstyle\cong}\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow & & \\ \dots & \longrightarrow & \underbrace{\mathrm{H}_{n+1}(Y)}_{=0} & \longrightarrow & \mathrm{H}_{n+1}(Z) & \longrightarrow & \mathrm{H}_n(X) & \xrightarrow{h_*} & \mathrm{H}_n(Y) & \longrightarrow & \mathrm{H}_n(Z) & \longrightarrow & \underbrace{\mathrm{H}_{n-1}(X)}_{=0} & \longrightarrow & \dots \end{array}$$

It follows that $\mathrm{H}_n(Z) \cong \mathrm{coker}(\varphi) \cong \pi$. Furthermore, since φ is injective we see that $\mathrm{H}_{n+1}(Z) = 0$. Finally note that all other reduced homology groups of Z are also zero.

Since h is cellular we can equip Z with the natural CW-structure coming from the CW-Complex Mapping Cylinder+Cone Proposition 139.25. If one looks at the proof of the

CW-Complex Mapping Cylinder+Cone Proposition 139.25 one sees that this CW-structure of the mapping cone $Z = \operatorname{Cone}(h\colon X \to Y)$ has the following type of cells:

(1) A k-cell for each k-cell of Y.
(2) A $(k+1)$-cell for each k-cell of X.
(3) A 0-cell corresponding to the tip of the mapping cone.



Thus we see that the CW-structure does not quite have the type of cells that we want. Note that by construction there is a subcomplex A of $Z = \operatorname{Cone}(h)$ that consists of the tip of the mapping cone, the 0-cell of Y and the 1-cell corresponding to the 0-cell of X. By the CW-Quotient Lemma 139.10 we can view Z/A as a CW-complex which now has only one 0-cell and otherwise only cells in dimension n and $n+1$. By the discussion on page 2846 we know that A is a good subset of Z. It follows from the Quotient–LES Theorem 149.17 that the reduced homology groups of Z/A are isomorphic to the reduced homology groups of Z. Thus we see that Z/A is the CW-complex of our dreams. ■

Given an abelian group π and given $n \in \mathbb{N}$ there are in general infinitely many pairwise non-homeomorphic Moore spaces of type $\mathrm{M}(\pi, n)$. For example the topological spaces $\overline{B}^k$, $k \in \mathbb{N}_0$ are pairwise non-homeomorphic Moore spaces of type $\mathrm{M}(\{0\}, 1)$. It is slightly less absurd to wonder whether Moore spaces are well-defined up to say homotopy equivalence. We formulate this as a question:[1844]

Question 173.6. Let π be an abelian group and let $n \in \mathbb{N}$. Are any two Moore spaces of type $\mathrm{M}(\pi, n)$ homotopy equivalent?

This concludes our short discussion of Moore spaces for the time being. Moore spaces mostly play a peripheral role in these notes, but they will come in handy in the proof of the Eilenberg-Maclane-H_{n+1}-Lemma 307.35.

173.2. Exercise for Chapter 173.

Exercise 173.1. Let $n \in \mathbb{N}$ and let $X = S^n \vee S^n$ and let $Y = S^n \vee S^n \vee S^n$. As in the proof of the Moore Space Existence Proposition 173.3 we can make the identification $\mathrm{H}_n(X) = \mathbb{Z}^2$ and $\mathrm{H}_n(Y) = \mathbb{Z}^3$. Show that there exists a continuous map $\varphi\colon X \to Y$ such that the map $\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ -5 & 1 \\ 0 & 7 \end{pmatrix}.$$

Exercise 173.2. What is the minimal number of cells in a Moore space of type $\mathrm{M}(\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, 4)$?

Exercise 173.3. We consider the CW-complex $\mathbb{R}\mathrm{P}^3$. What is the minimal number of cells one needs to attach to obtain a Moore space of type $\mathrm{M}(\mathbb{Z}, 3)$?

[1844] We will come back to the question in Exercise 307.8.

Part XIX

Simplicial Complexes

In this part of the lecture notes we introduce the purely combinatorial notion of a simplicial complex. We study many examples and their properties. The connection to topology is given via the topological realization of a simplicial complex. This part of the lecture notes is relatively independent of the other parts of these lecture notes. For the most part it can be read on its own, and for the most part it is also safe to skip it.

This part of the lecture notes is organized as follows:

- In Chapter 174 we will introduce abstract simplicial complexes, we will study examples and basic properties, and we will introduce the topological realization $|K|$ of a simplicial complex K.
- In Chapter 175 we will introduce the barycentric subdivision of an abstract simplicial complex. We will use this notation to state and prove the Simplicial Approximation Theorem 175.15 which says that any continuous map $f\colon |K| \to |L|$ between the topological realizations of finite abstract simplicial complexes K and L is homotopic to a simplicial map (at least after applying barycentric subdivisions to K). One of the many applications of this theorem is the CW–Simplicial Homotopy Type Proposition 175.21 which says that any (finite) CW-complex is homotopy equivalent to the topological realization of a (finite) simplicial complex.
- In Chapter 176 we will introduce the simplicial homology groups of an abstract simplicial complex and we will introduce the (closely related) simplicial homology groups of an ordered simplicial complex. In the Simplicial Künneth Theorem 176.29 we will show that for two finite ordered abstract simplicial complexes K and L we can compute the simplicial homology of $K \times L$ in terms of the simplicial homology groups of K and L.
- In Chapter 177 we will see in the Simplicial–Singular Homology Theorem 177.9 that for any abstract simplicial complex K there exists a natural isomorphism between simplicial homology of K and singular homology of the topological realization $|K|$.
- In Chapter 178 we will state the Nerve Theorem 178.5, and we will sketch a proof (albeit using the Whitehead Theorem 305.1). The Nerve Theorem 178.5 gives a criterion for a topological space to be homotopy equivalent to a simplicial complex. This chapter can be viewed as a slight detour from the main story line.
- In Chapter 179 we will sketch a proof of the Smooth Manifold-Simplicial Structure Theorem 179.5, which says that every smooth manifold admits a smooth simplicial structure.
- In Chapter 180 we will introduce the notion of a PL-structures on manifolds. These are particularly nice simplicial structures on topological manifolds. In the Smooth-implies-PL Theorem 180.16 we will see that the smooth simplicial structures from the Smooth Manifold-Simplicial Structure Theorem 179.5 are in fact PL-structures.
- In next Chapter 181 we will introduce the notion of a combinatorial manifold. Using the Alexander-Newman Theorem 181.6 (which we will not prove) we will see that PL-manifolds are essentially the same as combinatorial manifolds. We will also discuss Pachner moves on combinatorial manifolds.
- In Chapter 182 we will prove the Lefschetz Fixed Point Theorem 182.15 which says that if $\varphi\colon X \to X$ is a continuous self-map of a simplicial complex with non-zero Lefschetz number, then φ has a fixed point. Using the Smooth Manifold-Simplicial Structure Theorem 179.5 we will be able in the Lefschetz Fixed Point Theorem 182.16 to extend this result to self-maps of compact smooth manifolds.

- In Chapter 183 we will introduce the notion of a pseudomanifold and we will see that every smooth manifold is a pseudomanifold. In the Pseudomanifold Homology Theorem 183.12 we will collect several statements about homology groups of pseudomanifolds.
- In Chapter 184 we will introduce the simplicial cohomology groups of an abstract simplicial complex. We will show that these groups are naturally isomorphic to singular cohomology groups (which unfortunately we will only introduce later) and we will introduce the cup product on simplicial cohomology and the cap product that relates simplicial homology and simplicial cohomology.
- In Chapter 185 we will use the Pseudomanifold Homology Theorem 183.12 to introduce, for a compact oriented non-empty n-dimensional smooth manifold M, the fundamental class $[M] \in \mathrm{H}_n(M;\mathbb{Z})$.
- In Chapter 186 we will use the fundamental class from the previous chapter to give an alternative definition of the degree of a continuous map between two closed oriented connected non-empty smooth manifolds. Among many other statements we will show in the Degree One-Epimorphism Proposition 186.10 that if $f\colon M \to N$ is a continuous map between closed oriented connected non-empty n-dimensional smooth manifolds with $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.
- In Chapter 187 we will discuss the question which homology classes can be represented by smooth (sub-)manifolds. We will prove several results using simplicial complexes and using pseudomanifolds.
- In Chapter 188 we will formulate and prove the purely combinatorial Simplicial Poincaré Duality Theorem 188.5 which, given a closed closed n-dimensional simplicial homology manifold relates the simplicial cohomology groups to the simplicial homology group in the "opposite dimension".

CHAPTER 174

Simplicial complexes: Definition, examples and basic properties

In this section we introduce the notion of an abstract simplicial complex. Similar to the notion of an abstract graph these are purely combinatorial objects. The connection to topology comes from the fact that simplicial complexes admit topological realizations which can be used to combinatorially describe many interesting topological spaces. These topological realizations also lead to the notion of a simplicial complex which can be viewed as a special type of CW-complexes with particularly nice properties.

This chapter is organized as follows:

- In Section 174.1 we will introduce abstract simplicial complexes.
- In Section 174.2 we will introduce ordered abstract simplicial complexes.
- In Section 174.3 we will introduce the topological realization of an abstract simplicial complex.
- In Section 174.4 we will "non-abstract" simplicial complexes, these are topological spaces together with a homeomorphism to the topological realization of an abstract simplicial complex.
- In Section 174.5 we will introduce the product of (abstract) simplicial complexes.
- In Section 174.6 we will introduce the simplicial join of (abstract) simplicial complexes.
- In Section 174.7 we will introduce linear simplicial complexes.
- In Section 174.8 we will see that simplicial complexes are in particular CW-complexes.
- In Section 174.9 we will provide exercises for Chapter 174.

In the next Chapter 175 we will state and prove the Simplicial Approximation Theorem 175.15.

174.1. **Abstract simplicial complexes.** Before we do anything else we recall the following notation that we introduced in Notation 1.1.

Notation 174.1. Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

We continue with the following initially rather abstract definition.

Definition 174.2. An abstract simplicial complex is a pair (V, S) where V is a set and S is a subset of the power set $\mathcal{P}(V)$ of V such that the following conditions are satisfied:

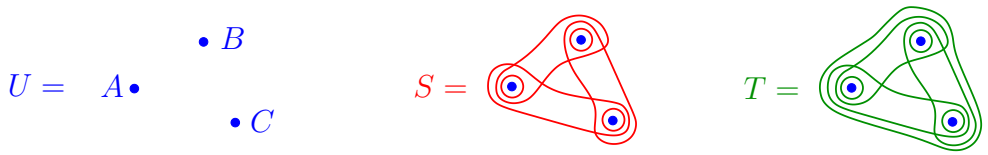
(1) each $s \in S$ is finite and non-empty,
(2) if $s \in S$, then any non-empty subset of s also lies in S,
(3) given any $v \in V$ the set $\{v\}$ lies in S.

We refer to the elements of V as the vertices of the abstract simplicial complex and we refer to the elements of S as the simplices of the abstract simplicial complex.

Examples.

(0) The pair $(\varnothing, \varnothing)$ is an abstract simplicial complex, it is called the *empty abstract simplicial complex*. Every other abstract simplicial complex is called *non-empty*.

(1) We consider the set $U = \{A, B, C\}$. It is elementary to verify that the pair $X := (U, S)$ with $S = \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\}$ and that the pair $Y := (U, T)$ with $T = \mathcal{P}(U) \setminus \{\varnothing\}$ are abstract simplicial complexes. We illustrate these two abstract simplicial complexes in the figure below.



(2) Let $n \in \mathbb{Z}_{\geq -1}$. The pairs
   (a) $D_n := (V, S)$ with $V = \{0, \ldots, n\}$ and $S = \mathcal{P}(V) \setminus \{\varnothing\}$, and
   (b) $S_n := (V, T)$ with $V = \{0, \ldots, n+1\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$,
   are abstract simplicial complexes.

(3) (a) Given $n \in \mathbb{N}_{\geq 3}$ the pair $R_n = (V, S)$ with

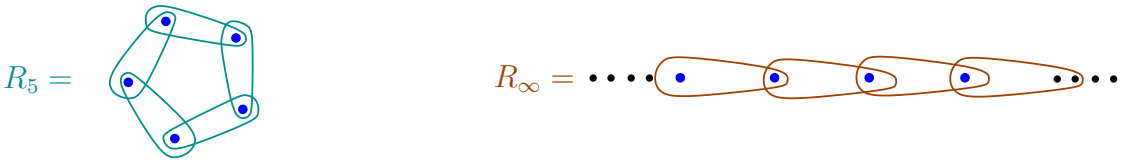
$$V := \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad S := \{\{k\} \,|\, k \in \mathbb{Z}/n\mathbb{Z}\} \cup \{\{k, k+1\} \,|\, k \in \mathbb{Z}/n\mathbb{Z}\}$$

   is an abstract simplicial complex.

   (b) The pair $R_\infty := (V, S)$ with

$$V := \mathbb{Z} \quad \text{and} \quad S := \{\{n\} \,|\, n \in \mathbb{Z}\} \cup \{\{n, n+1\} \,|\, n \in \mathbb{Z}\}$$

   is an abstract simplicial complex.



We illustrate these two abstract simplicial complexes in the figure above.[1845] □

Definition 174.3. Let $K = (V, S)$ be an abstract simplicial complex.

(1) We say that $K = (V, S)$ is finite if V is finite.[1846] Similarly we say that $K = (V, S)$ is countable if V is countable.

(2) If $s \in S$ has $k+1$ elements, then s is called a k-dimensional simplex, or shorter, a k-simplex of (V, S).

(3) If $K = (V, S)$ is the empty abstract simplicial complex, then we define its dimension as -1. Now suppose that $K = (V, S)$ is non-empty. We define

$$\dim(K) := \max\{\#s - 1 \,|\, s \in S\} \in \mathbb{N}_0 \cup \{\infty\}.$$

In other words, the dimension of $K = (V, S)$ is the maximal dimension of a simplex of $K = (V, S)$.

Examples.

[1845] To simplify the figures we will not draw the "trivial" simplices that are given by the vertices.

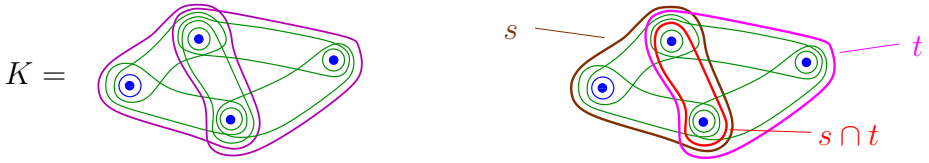
[1846] Note that this implies that S is also finite. In fact since every $v \in V$ defines a simplex we see that an abstract simplicial complex is finite if and only if S is finite.

(1) The above abstract simplicial complex (U, S) is 1-dimensional and the above abstract simplicial complex (U, T) is 2-dimensional.
(2) Let $n \in \mathbb{N}$. The abstract simplicial complex D_n is n-dimensional and the abstract simplicial complex S_{n-1} is $(n-1)$-dimensional
(3) The abstract simplicial complexes R_n and R_∞ are one-dimensional. □

We continue with a few more harmless definitions and statements.

Definition 174.4. Let $K = (V, S)$ be an abstract simplicial complex. Let t be a simplex of K. A face of t is a simplex $s \in S$ with $s \subseteq t$.

Lemma 174.5. (Simplices–Intersection Lemma) Let $K = (V, S)$ be an abstract simplicial complex. Furthermore let s and t be two simplices of K. Then either the intersection $s \cap t$ is empty, or the intersection $s \cap t$ is again a simplex of K and it is a face of s and t.



Proof. Let s and t be two simplices of an abstract simplicial complex $K = (V, S)$. If $s \cap t = \varnothing$ then there is nothing to show. Now suppose that $s \cap t \neq \varnothing$. Evidently $s \cap t$ is a non-empty subset of s and t. By Definition 174.2 of an abstract simplicial complex this means that $s \cap t$ is also a simplex of K. Furthermore, since $s \cap t \subseteq s$ and $s \cap t \subseteq t$ the intersection is a face of s and t. ■

Definition 174.6. Let $L = (W, T)$ be an abstract simplicial complex. An abstract subcomplex of $L = (W, T)$ is a pair $K = (V, S)$ with $V \subseteq W$, with $S \subseteq T$ and such that $K = (V, S)$ is an abstract simplicial complex in its own right. On most occasions we just say "subcomplex" instead of "abstract subcomplex".

Examples.

(1) Basically by definition $X = (U, S)$ is a subcomplex of $Y = (U, T)$.
(2) Let $n \in \mathbb{N}$. Again, basically by definition S_{n-1} is a subcomplex of D_n.
(3) Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. We refer to V together with all simplices of dimension $\leq n$ as the *n-skeleton of the abstract simplicial complex*. It is basically clear from the definitions that the n-skeleton is a subcomplex. For convenience we also write $K^\infty := K$.
(4) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.
   (a) The pair $(s, \mathcal{P}(s) \backslash \{\varnothing\})$ is a subcomplex of K. Often we abbreviate this subcomplex to s.
   (b) The pair $(s, \mathcal{P}(s) \backslash \{\varnothing, s\})$ is a subcomplex of K. Usually we refer to this subcomplex as ∂s. □

The following lemma gives us elementary ways to construct subcomplexes.

Lemma 174.7. (Subcomplexes–Intersection-Union Lemma) Let $\{L_i = (W_i, T_i)\}_{i\in I}$ be a family of abstract simplicial complexes. The following statements hold:

(1) $\bigcap\limits_{i\in I} L_i := \left(\bigcap\limits_{i\in I} W_i, \bigcap\limits_{i\in I} V_i\right)$ and $\bigcup\limits_{i\in I} L_i := \left(\bigcup\limits_{i\in I} W_i, \bigcup\limits_{i\in I} V_i\right)$ are abstract simplicial complexes.

(2) $\bigcap\limits_{i\in I} L_i$ is a subcomplex of each L_i and each L_i in turn is a subcomplex of $\bigcup\limits_{i\in I} L_i$.

(3) If each L_i is a subcomplex of some abstract simplicial complex K, then $\bigcap\limits_{i\in I} L_i$ and $\bigcup\limits_{i\in I} L_i$ are also subcomplexes of K.

Proof. The lemma is close to being a tautology. ■

As the reader will know by now, not only do we want to study objects, we also want to study maps or morphisms between them. The following definition gives us the type of maps we are now interested in.

Definition 174.8. Let $K = (V, S)$ and $L = (W, T)$ be abstract simplicial complexes.

(1) A simplicial map $f\colon K \to L$ is defined as a map $f\colon V \to W$ such that for every $s \in S$ we have $f(s) \in T$. In particular a simplicial map induces a map $S \to T$.

(2) A simplicial isomorphism between $K = (V, S)$ and $L = (W, T)$ is a simplicial map that is a bijection on the set of vertices and on the set of simplices.

Examples.

(1) The identity map of $U = \{A, B, C\}$ defines a simplicial map from $X = (U, S)$ to $Y = (U, T)$.

(2) The map $f\colon \{0, 1, 2\} \to \{A, B, C\}$ given by $f(0) = A$, $f(1) = B$ and $f(2) = C$ defines a simplicial isomorphism between D_2 and (U, T) and it defines a simplicial isomorphism between S_1 and (U, S). Similarly one sees that the abstract simplicial complex R_3 is simplicially isomorphic to S_1 and (U, S).

(3) The inclusion of a subcomplex into an abstract simplicial complex is, basically by definition, a simplicial map. □

It is clear that the composition of simplicial maps is again a simplicial map. Thus it makes sense to introduce the following definition:

Definition 174.9. We call the category $\mathcal{AbsSimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsSimpCplx}) &:= \text{all abstract simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{all simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the category of abstract simplicial complexes.

For future reference let us record the following lemma:

Lemma 174.10. (Simplicial Maps–Cardinality Lemma) Let K and L be two abstract simplicial complexes.

(1) If K and L are finite, then there exist only finitely many simplicial maps from K to L.

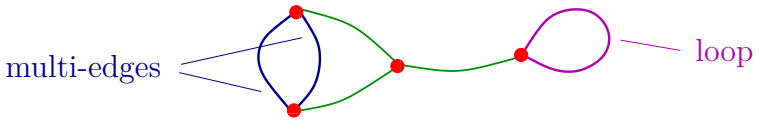
(2) If K is finite and if L is countable, then there exist only countably many simplicial maps from K to L.

Proof.

(1) This statement is obvious.

(2) Let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. We assume that K is finite and that L is countable. It follows from the Countability Lemma 1.37 (6) that there exist only countably many maps $V \to W$. Furthermore, since V is finite we see that given any map $V \to W$ there are only finitely many maps $S \to T$ such that the resulting map is a simplicial map. It follows from the Countability Lemma 1.37 (4) that there are indeed only countably many simplicial maps. ■

Remark. The reader might have a certain sense of déjà vu: the definition of a 1-dimensional abstract simplicial complex sounds very similar to the definition of an abstract graph which we gave in Definition 11.12. In fact some thought shows that the concepts are different since abstract graphs are more flexible. But once we rule out loops and multi-edges, then the concepts are basically the same.



More precisely it follows easily from the definitions that the maps

$$\underbrace{(V, E, \varphi\colon E \to \mathcal{P}(V))}_{\substack{\text{abstract graph}\\ \text{without loops \& multi-edges}}} \longmapsto \underbrace{(V, \varphi(E))}_{\substack{\text{1-dimensional abstract}\\ \text{simplicial complex}}} \quad \text{and} \quad \underbrace{(V, S \subseteq \mathcal{P}(V))}_{\substack{\text{1-dimensional abstract}\\ \text{simplicial complex}}} \longmapsto \underbrace{(V, S, \mathrm{id}\colon S \to \mathcal{P}(V))}_{\substack{\text{abstract graph}\\ \text{without loops \& multi-edges}}}$$

define an isomorphism between the "category of abstract graphs without loops and multi-edges" and the "category of 1-dimensional simplicial complexes". □

174.2. **Ordered abstract simplicial complexes.** We start out by recalling Definition 1.30:

Definition 174.11.

(1) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(2) A partially ordered set is a set together with a partial order.
(3) A partial order that is connex[1847] is called a total order.
(4) A totally ordered set is a set together with a total order.
(5) In a totally ordered set $(P, \leq)$ we write $a < b$ if $a \leq b$ and $a \neq b$.

Examples.

(1) Let X be a set. We consider the power set $\mathcal{P}(X)$. The relation "$\subseteq$" is a partial order on $\mathcal{P}(X)$.
(2) The usual relation "$\leq$" on $\mathbb{Z}$ turns $\mathbb{Z}$ into a totally ordered set.
(3) Any subset of $\{A, B, \ldots, Z\}$ with the usual alphabetical ordering is a totally ordered set. □

We continue with the following definition:

Definition 174.12.

(1) A total order on an abstract simplicial complex (V, S) is a total order on the vertex set V.

[1847] Recall that according to Definition 1.30 a relation "$\leq$" on a set S is called *connex* if for every $a, b \in S$ we have $a \leq b$ or $b \leq a$.

(2) An order on an abstract simplicial complex (V, S) is a partial order on V which has the property that the restriction to each simplex is actually a total order.
(3) A (totally) ordered abstract simplicial complex is an abstract simplicial complex together with a (total) order.

Remark.

(1) Every totally ordered abstract simplicial complex is automatically also an ordered abstract simplicial complex.
(2) By the Well-Ordering Theorem 1.32 we know that every set admits a total ordering. Thus every abstract simplicial complex can be turned into a totally ordered abstract simplicial complex. □

The following definition is the obvious variation on the definition provided in Definition 174.9.

Definition 174.13. We call the category $\mathcal{OrdAbsSimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{OrdAbsSimpCplx}) &:= \text{all ordered abstract simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{all order-preserving simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the category of ordered abstract simplicial complexes.

174.3. **The topological realization of an abstract simplicial complex.** Since we are topologists we want to study abstract simplicial complexes with topological methods. The idea is to associate to an abstract simplicial complex a corresponding topological space. Before we can do so we need to introduce the following notation which is inspired by the definition of the free abelian group $\mathbb{Z}^{(S)}$ that we gave in Definition 97.7.

Notation 174.14.

(1) Given any set V we write

$$\begin{aligned} \mathbb{R}^V &:= \text{all maps from } V \text{ to } \mathbb{R} \\ \mathbb{R}^{(V)} &:= \text{all maps from } V \text{ to } \mathbb{R} \text{ which are non-zero for only finitely many } v \in V. \end{aligned}$$

We have the following conventions and observations:
(a) Given $v \in V$ we sometimes denote by v also the map $V \to \mathbb{R}$ which is given by $v \mapsto 1$ and $w \mapsto 0$ for $w \neq v$.
(b) One can easily verify that with this interpretation the set V forms a basis for the real vector space $\mathbb{R}^{(V)}$.
(c) If $V = \{v_1, \ldots, v_n\}$ is a finite set, then the map $v_i \mapsto e_i$ defines an isomorphism from $\mathbb{R}^V = \mathbb{R}^{(V)}$ to $\mathbb{R}^n$. In particular, if V is a finite set with a total order, then it follows from Exercise 1.5 that we have a natural isomorphism $\mathbb{R}^V \xrightarrow{\cong} \mathbb{R}^{|V|}$.
(2) If V is a subset of some set W, then we view $\mathbb{R}^{(V)}$ as a subspace of $\mathbb{R}^{(W)}$ in the obvious way.

The following definition is the first step in our shift back to topology.

Definition 174.15. Let $K = (V, S)$ be some abstract simplicial complex. For a given n-simplex $s = \{v_0, \ldots, v_n\}$ we define

$$|s| := \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{l} (1)\ \alpha(v) \geq 0 \text{ for all } v \in s \\ (2)\ \alpha(v) = 0 \text{ for all } v \notin s \\ (3)\ \text{we have } \sum_{v \in V} \alpha(v) = 1 \end{array} \right\} = \left\{ \underbrace{\sum_{i=0}^{n} t_i \cdot v_i}_{\in \mathbb{R}^{(V)}} \;\middle|\; \begin{array}{l} (1)\ \text{for each } i \text{ we have } t_i \geq 0 \\ (2)\ t_0 + \cdots + t_n = 1 \end{array} \right\}$$

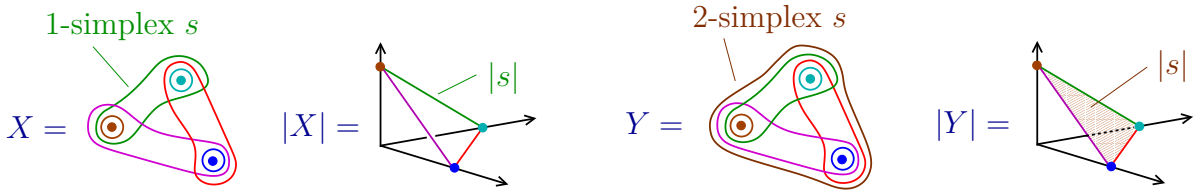
Furthermore we set

$$|K| := \bigcup_{s \in S} |s| \subseteq \mathbb{R}^{(V)}.$$

Sometimes, by a serious abuse of language, we refer to $|s|$ as an n-simplex in $|K|$.

Examples.

(1) We consider the abstract simplicial complexes $X = (U, S)$ and $Y = (U, T)$ as above. The corresponding sets $|X|$ and $|Y|$ are the subsets of $\mathbb{R}^U = \mathbb{R}^{\{A,B,C\}} =_{\mathrm{i}} \mathbb{R}^3$ that are shown in the figure below.



(2) Let $n \in \mathbb{N}$. Given the abstract simplicial complex R_n the corresponding set $|R_n|$ is a subset of $\mathbb{R}^{\mathbb{Z}/n\mathbb{Z}} =_{\mathrm{i}} \mathbb{R}^{n-1}$. This means that for $n \geq 5$ we have serious problems with visualizing the set $|R_n|$. The problem is even bigger for the infinite abstract simplicial complex R_∞. □

Lemma 174.16. (Topological Realization Lemma) Given any abstract simplicial complex $K = (V, S)$ we have

$$|K| = \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{ll} (1) & \text{the set } \{v \in V \,|\, \alpha(v) \neq 0\} \text{ is a simplex of } K \\ (2) & \text{for every } v \in V \text{ we have } \alpha(v) \geq 0 \\ (3) & \text{we have } \sum_{v \in V} \alpha(v) = 1 \end{array} \right\}.$$

Proof. We leave the elementary task of proving the equality to the reader. Note though that one actually needs to use that $K = (V, S)$ is an abstract simplicial complex. ■

Note that as of right now $|K|$ is just a set. But rest assured, we will equip $|K|$ with a suitable topology shortly.[1848] But first let us introduce a convention.

Convention 174.17. Let $K = (V, S)$ be an abstract simplicial complex. As discussed above, we can view V as a subset of $\mathbb{R}^{(V)}$. In fact it follows easily from the definitions that $V \subseteq |K|$. Throughout these notes we use this observation to view V as a subset of $|K|$.

We continue with the following definition:

[1848] A first idea might be to equip $\mathbb{R}^{(V)}$ with a suitable topology and then to endow $|K|$ with the subspace topology. But this is not really a good idea since this topology is artificial and does not reflect the structure of K. For example it would be rather unpleasant to have to show that a map out of K is continuous.

Definition 174.18. Let $K = (V, S)$ be an abstract simplicial complex. Let $k \in \mathbb{N}_0$ and let s be a k-simplex.

(1) If $v_0 \leq \cdots \leq v_k$ is a total order on the set of vertices of s, then we refer to the map

$$\begin{aligned} \Phi_s \colon \Delta^k = \left\{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \,\middle|\, \sum_{i=0}^{k} t_i = 1\right\} &\to |K| \\ (t_0, \ldots, t_k) &\mapsto \sum_{i=0}^{k} t_i \cdot v_i \end{aligned}$$

as a characteristic map of the simplex s.

(2) If K is actually an ordered abstract simplicial complex, then we use the corresponding order on s and we write $\Phi_s^{\leq} \colon \Delta^k \to |K|$. In this case we can refer to $\Phi_s^{\leq}$ unambiguously as the characteristic map of the simplex s.

Remark.

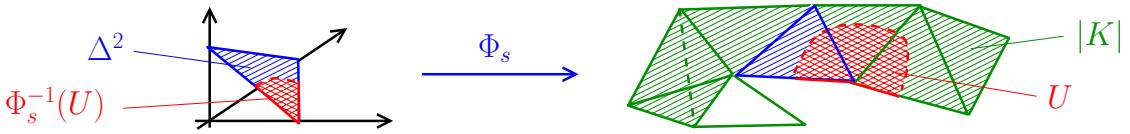
(1) Note that if in the above definition of a characteristic map we use two different orderings of the vertices of s, then we get two maps $\Phi, \Psi \colon \Delta^k \to |K|$ with the same image and such that $\Psi^{-1} \circ \Phi \colon \Delta^k \to \Delta^k$ is a homeomorphism. In other words, any two characteristic maps differ by precomposing by a homeomorphism. In practice this means that any two characteristic maps exhibit the same properties and we do not need to distinguish between different characteristic maps.

(2) Given a k-simplex s we have, by definition, $\Phi_s(\Delta^k) = |s|$. □

Definition 174.19. Let $K = (V, S)$ be an abstract simplicial complex.

(1) We say that a subset $U \subseteq |K|$ is open if for every simplex s of K the preimage $\Phi_s^{-1}(U)$ under a characteristic map[1849] Φ_s is open. An argument as in Exercise 4.45 shows that this defines a topology on $|K|$.

(2) We refer to the set $|K|$, together with this topology, as the topological realization of K.



The following lemma summarizes a few basic properties of the topological realization of an abstract simplicial complex.

Lemma 174.20. (Simplex-Embedding Lemma) Let K be an abstract simplicial complex.

(1) A subset $A \subseteq |K|$ is closed if and only if for every simplex s of K the preimage $\Phi_s^{-1}(A)$ under a characteristic map Φ_s is closed.

(2) The topological realization $|K|$ is Hausdorff.

(3) For every k-simplex $s = \{v_0, \ldots, v_k\}$ of K any characteristic map $\Phi_s \colon \Delta^k \to |K|$ is a closed embedding. In particular the map $\Phi_s \colon \Delta^k \to |s|$ is a homeomorphism.

[1849] It follows from the above remark that the statement that $\Phi_s^{-1}(U)$ is open does not depend on the choice of the characteristic map for the simplex s.

Proof. Let $K = (V, S)$ be an abstract simplicial complex.

(1) This statement follows almost immediately from the definition of the topology on $|K|$ together with the elementary Image-Preimage Lemma 1.9 (8).

(2) Let P and Q be two distinct points on $|K|$. Given $f, g \in \mathbb{R}^{(V)}$ we write

$$d(f,g) \coloneqq \sqrt{\sum_{v \in V} (f(v) - g(v))^2}.$$

One can easily verify that this is a metric on $\mathbb{R}^{(V)}$. By the Metric Space-Hausdorff Proposition 3.28 we know that there exists an $\epsilon > 0$ such that the open ϵ-balls $B_\epsilon(P)$ and $B_\epsilon(Q)$ are disjoint in $\mathbb{R}^{(V)}$. It remains to show that $B_\epsilon(P) \cap |K|$ and $B_\epsilon(Q) \cap |K|$ are open in $|K|$. By definition of the topology on $|K|$ we need to prove the following claim:

Claim. Let $U \coloneqq B_\epsilon(R)$ be an ϵ-ball in $\mathbb{R}^{(V)}$. For every simplex s the preimage $\Phi_s^{-1}(U \cap |K|)$ under a characteristic map Φ_s is open.

Proof. Let $s = \{v_0, \ldots, v_k\}$ be a k-simplex. We make the following observations:

(a) We have $\Phi_s^{-1}(U \cap |K|) = \Phi_s^{-1}(U)$.

(b) It follows from (a) and the observation that U is an open subset of the topological space $(\mathbb{R}^{(V)}, d)$ that it suffices to show that the map $\Phi_s \colon \Delta^k \to (\mathbb{R}^{(V)}, d)$ is continuous.

(c) We view $\Phi_s \colon \Delta^k \to \mathbb{R}^{(V)}$ as the composition of the maps

$$\Delta^k \xrightarrow{\Phi_s} \mathbb{R}^{k+1} =_{\mathrm{i}} \mathbb{R}^{(s)} \hookrightarrow \mathbb{R}^{(V)}.$$

We consider the two maps separately.

(i) Note that $\mathbb{R}^{k+1} = \mathbb{R}^{(s)}$ is a finite-dimensional vector space and that the map $\Phi_s \colon \Delta^k \to \mathbb{R}^{(s)}$ is an affine linear map. It follows almost immediately from the Δ^n-Homeomorphism Lemma 143.9 (2) that this map is continuous.

(ii) It follows easily from the definition of topology on $\mathbb{R}^{k+1}$, via the usual Euclidean metric, and the definition of the metric d on $\mathbb{R}^{(V)}$, together with the simple-minded Exercise 4.17 (a), that the map $\mathbb{R}^{k+1} =_{\mathrm{i}} \mathbb{R}^{(s)} \to \mathbb{R}^{(V)}$ is continuous.

Thus we see that the map $\Phi_s \colon \Delta^k \to (\mathbb{R}^{(S)}, d)$ itself is continuous. ⊞

(3) Let s be a k-simplex of K. It follows almost immediately from the definitions that the map $\Phi_s \colon \Delta^k \to |K|$ defines a bijection $\Phi_s \colon \Delta^k \to |s|$. It follows immediately from the definition of the topology on $|K|$ that Φ_s is continuous. Evidently Δ^k is compact. By (1) we know that $|K|$ is Hausdorff. So once again the Compact-Hausdorff Proposition 4.23 (2) rides to our rescue and tells us that Φ_s is indeed an embedding and that the image is a closed subset. ■

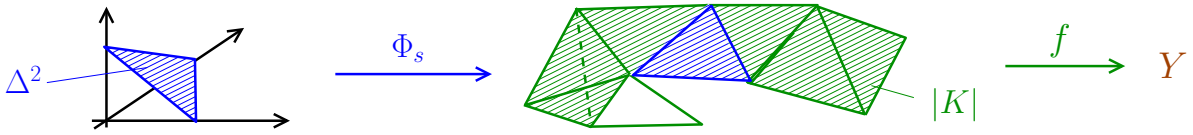
The following is the simplicial analog of the corresponding Characteristic Maps Proposition 136.9 (2) for CW-complexes.

Lemma 174.21. (Restrict-to-Simplex Lemma) Let $K = (V, S)$ be an abstract simplicial complex and let $f \colon |K| \to X$ be a continuous map to some topological space X.

(1) If for each k-simplex $s \in S$ with characteristic map $\Phi_s \colon \Delta^k \to |K|$ the composition $f \circ \Phi_s \colon \Delta^k \to X$ is continuous, then the map f is continuous.

More generally we have the following continuity criterion:

(2) If for each simplex $s \in S$ there exists a k-simplex t with $s \subseteq t$ such that for some characteristic map $\Phi_t \colon \Delta^k \to |K|$ the composition $f \circ \Phi_t \colon \Delta^k \to X$ is continuous, then the map f is continuous.



Proof.

(1) This statement follows immediately from the definition of the topology on $|K|$.

(2) This statement follows immediately from (1), the observation that the face maps $i^k_j \colon \Delta^l \to \Delta^k$ are continuous and the Continuous Composition Lemma 4.2. ■

The following proposition shows that not only do abstract simplicial complexes give rise to topological spaces, but simplicial maps also give rise to continuous maps between the topological realizations.

Proposition 174.22. (Topological Realization–Functor Proposition)

(1) If $f \colon K = (V, S) \to L = (W, T)$ is a simplicial map, then the map

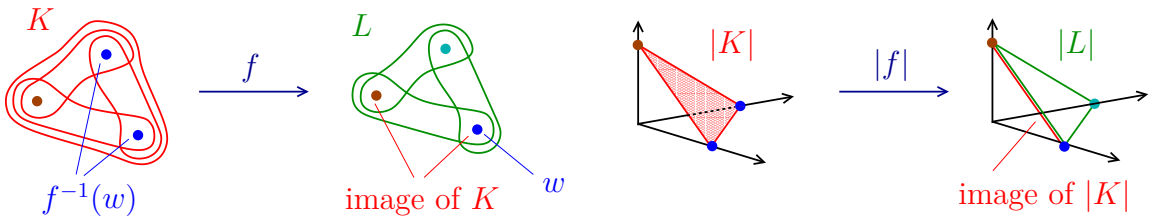
$$\begin{array}{rcl} |f| \colon |K| & \to & |L| \\ (\alpha \colon V \to \mathbb{R}) & \mapsto & \begin{pmatrix} W & \to & \mathbb{R} \\ w & \mapsto & \sum\limits_{v \in f^{-1}(w)} \alpha(v) \end{pmatrix} \end{array}$$

is well-defined and continuous. We refer to the figure below for an illustration.

(2) Let $\mathcal{AbsSimpCplx}$ be the category of abstract simplicial complexes. The maps

$$\begin{array}{rcl} K & \mapsto & |K| \\ (f \colon K \to L) & \mapsto & (|f| \colon |K| \to |L|) \end{array}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{Top}$ of topological spaces.



Remark.

(1) Let $f \colon K = (V, S) \to L = (W, T)$ be a simplicial map. It is elementary to see that, given a simplex $s = \{v_0, \ldots, v_k\}$ of K, the restriction of $|f|$ to $|s| \subseteq |K|$ is the map

$$\begin{array}{rcl} |f| \colon |s| & \to & |f(s)| \\ \sum\limits_{i=0}^{k} t_i \cdot v_i & \mapsto & \sum\limits_{i=0}^{k} t_i \cdot f(v_i). \end{array}$$

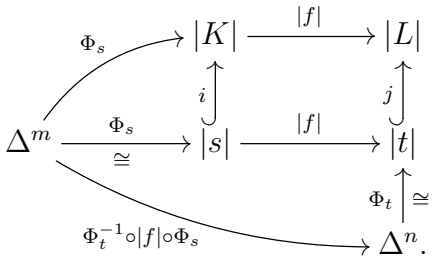
(2) As we will see, the proof of the Topological Realization–Functor Proposition 174.22 (1) is slightly delicate since the topology of the topological realization of a simplicial

complex is designed, see the Restrict-to-Simplex Lemma 174.21, in such a way that it is straightforward to verify that a map *out* of the topological realization is continuous. It is much less clear how one can show that a map *to* a topological realization is continuous. The same health-warning applies also to CW-complexes. □

Proof.

(1) Let $f\colon K = (V, S) \to L = (W, T)$ be a simplicial map. A priori we have written down a map $|f|\colon |K| \to \mathbb{R}^{(W)}$. But it is straightforward to verify, e.g. using the Topological Realization Lemma 174.16, that the map $|f|$ does indeed take values in the subset $|L|$. It remains to show that $|f|\colon |K| \to |L|$ is continuous. By the Restrict-to-Simplex Lemma 174.21 (1) it suffices to show that for each m-simplex $s \in S$ with some characteristic map $\Phi_s\colon \Delta^m \to |K|$ the composition $f \circ \Phi_s\colon \Delta^m \to |L|$ is continuous.

Thus suppose that we are given such s and Φ_s. We write $t := f(s)$. By definition of a simplicial map we know that t is some n-simplex of T. For clarity we consider the following commutative diagram:



We make the following clarifications and observations:

(a) We define $|s| \subseteq |K|$ and $|t| \subseteq |L|$ as in the Simplex-Embedding Lemma 174.20 (3). We equip both with the subspace topology coming from $|K|$ and $|L|$. By the super elementary Subspace Continuity Lemma 4.3 (1) we know that the inclusion maps $i\colon |s| \to |K|$ and $j\colon |t| \to |L|$ are continuous.

(b) It follows immediately from the definition of $|f|$ that the restriction of $|f|$ to $|s|$ takes values in $|t|$.

(c) We picked a characteristic map $\Phi_t\colon \Delta^n \to |t|$ for t.

(d) By the Simplex-Embedding Lemma 174.20 (3) we know that the characteristic map $\Phi_t\colon \Delta^n \to |t|$ is a homeomorphism. In particular the inverse $\Phi_t^{-1}\colon |t| \to \Delta^n$ exists and it is continuous.

(e) It follows almost immediately from the definitions of the maps Φ_s, Φ_t and $|f|$ that the map $\Phi_t^{-1} \circ |f| \circ \Phi_s\colon \Delta^m \to \Delta^n$ is an affine linear map. Hence it follows, say from the Δ^n-Homeomorphism Lemma 143.9 (1), that the map is continuous.

(f) It follows immediately from the above that the map

$$|f| \circ \Phi_s \;=\; j \circ \Phi_t \circ (\Phi_t^{-1} \circ |f| \circ \Phi_s)\colon \Delta^m \;\to\; |L|$$

is indeed continuous.

(2) Let $f\colon K \to L$ and $g\colon L \to M$ be simplicial maps. For any vertex v of M we have $f^{-1}(g^{-1}(v)) = (g \circ f)^{-1}(v)$. It follows immediately from this observation that we have the equality $|g| \circ |f| = |g \circ f|$. It is now straightforward to see that the given continuous maps define a covariant functor. ■

The following lemma can be viewed as an analog of the Subcomplex–Closed Lemma 136.11, which we formulated in the context of CW-complexes.

Lemma 174.23. (Simplicial Subcomplex Lemma) Let $K = (V, S)$ be a subcomplex of some given abstract simplicial complex $L = (W, T)$.

(1) The inclusion $\mathbb{R}^{(V)} \to \mathbb{R}^{(W)}$ restricts to an inclusion $|K| \to |L|$.
(2) The inclusion $|K| \to |L|$ is a closed embedding.
(3) Let $\{K_i\}_{i\in I}$ be a family of subcomplexes of an abstract simplicial complex K. Then

$$\left|\bigcap_{i\in I} K_i\right| = \bigcap_{i\in I} |K_i| \quad \text{and} \quad \left|\bigcup_{i\in I} K_i\right| = \bigcup_{i\in I} |K_i|.$$

are subcomplexes of K.

Remark. The statement and the proof of the Simplicial Subcomplex Lemma 174.23 are closely related to the content of the Subcomplex–Closed Lemma 136.11. □

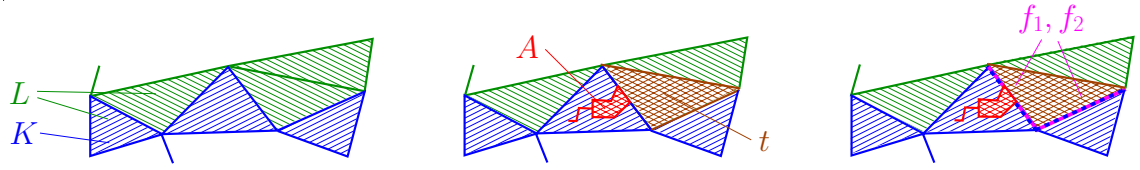
Convention 174.24. Let $K = (V, S)$ be a subcomplex of some given abstract simplicial complex $L = (W, T)$. We will use the Simplicial Subcomplex Lemma 174.23 to view $|K|$ as a subset of $|L|$. By the Simplicial Subcomplex Lemma 174.23 (2) we know that $|K|$ is in fact a closed subset of $|L|$.

Proof. Let $L = (W, T)$ be a simplicial complex and let $K = (V, S)$ be a subcomplex. We denote by $i\colon K \to L$ the inclusion map.

(1) It follows immediately from the definitions that the inclusion $\mathbb{R}^{(V)} \to \mathbb{R}^{(W)}$ restricts to an inclusion $|K| \to |L|$.
(3) This statement also follows immediately from the definitions. For readers who want to fill in the details, perhaps it is clearest to look at the description given in the Topological Realization Lemma 174.16.
(2) Since $i\colon K \to L$ is a simplicial map we obtain from the Topological Realization–Functor Proposition 174.22 (1) that the induced map $|i|\colon |K| \to |L|$ is continuous. Furthermore it follows easily from the definitions that the map $|i|\colon |K| \to |L|$ is in fact just the inclusion. In the remainder of the proof we will view $|K|$ as a subset of $|L|$.

By the Open-Closed–Injective Lemma 4.22 (1) it remains to show that the inclusion $|K| \to |L|$ is a closed map. Thus let $A \subseteq |K|$ be a closed subset. We need to show that A is also a closed subset of $|L|$. By the Simplex-Embedding Lemma 174.20 (1) and (3) it suffices to show that for each $t \in T$ the set $A \cap |t|$ is closed. Note that every simplex, in particular t itself, has only finitely many faces. Let $f_1, \ldots, f_m$ be the faces of t that are contained in K. We consider the subsets $|f_1|, \ldots, |f_m|$ of $|K|$ and the subset $|t|$ of $|L|$ as introduced in Definition 174.15. We make the following observation:

(∗) It follows from the Simplex-Embedding Lemma 174.20 that the subspace topologies on $|f_i|$ coming from $|f_i| \subseteq |K|$ and $|f_i| \subseteq |t| \subseteq |L|$ agree and that $|f_i|$ is a closed subset of $|K|$ and a closed subset of $|t|$.



We have

$$A \cap |t| \underset{\uparrow}{=} A \cap (|K| \cap |t|) \underset{\uparrow}{=} A \cap \bigcup_{i=1}^{m} |f_i| = \bigcup_{i=1}^{m} (A \cap |f_i|)$$

since $A \subseteq |K|$ — this follows from (3) since K is a subcomplex

$$\underset{\uparrow}{=} \bigcup_{i=1}^{m} \text{closed subset of } |f_i| \underset{\uparrow}{=} \bigcup_{i=1}^{m} \text{closed subset of } |t| = \text{closed subset of } |t|.$$

since A is closed in $|K|$ and since $f_1, \ldots, f_m \in S$ are simplices of K — by the Open-Closed Inclusion Lemma 4.13 (2) since by the Simplex-Embedding Lemma 174.20 (3) we know that the $|f_i|$ are closed subsets of $|t|$ ■

Example. Let $n \in \mathbb{N}$. We consider again the simplicial complex D_n and its subcomplex S_{n-1}. It follows almost immediately from the Simplex-Embedding Lemma 174.20 (3) that the map

$$\begin{aligned} \Psi\colon |D_n| &\to \Delta^n \\ \sum_{i=0}^{n} t_i \cdot i &\mapsto (t_0, \ldots, t_n) \end{aligned}$$

is a homeomorphism. By the Topological Realization–Functor Proposition 174.22 (2) the map $|S_{n-1}| \to |D_n|$ is an embedding. Therefore the map Ψ restricts to a homeomorphism $\Psi|_{|S_{n-1}|}\colon |S_{n-1}| \to \Psi(|S_{n-1}|)$ but basically by definition we have $\Psi(|S_{n-1}|) = \partial\Delta^n$. This shows that $|S_{n-1}|$ is homeomorphic to $\partial\Delta^n$. □

Before we consider some examples we want to consider the topology for finite abstract simplicial complexes in greater detail. We recall that given any finite set X the vector space $\mathbb{R}^{(X)}$ is a finite-dimensional vector space. Furthermore recall that by the discussion on page 183 we know that any finite-dimensional real vector space has a natural topology.

Lemma 174.25. (Finite Simplicial Complex Lemma) Let $K = (V, S)$ be an abstract simplicial complex. If K is finite, then the following two conclusions hold:

(1) The topological realization $|K|$ is compact.
(2) The topology of the topological realization $|K|$, introduced in Definition 174.19, agrees with the subspace topology which we obtain from viewing $|K|$ as a subset of the finite-dimensional real vector space $\mathbb{R}^V$.

Proof. Let $K = (V, S)$ be a finite abstract simplicial complex.

(1) By the Simplex-Embedding Lemma 174.20 we know in particular that for each k-simplex s the characteristic map $\Phi_s\colon \Delta^k \to |K|$ is continuous. Since Δ^k is compact we obtain from the Compact Image Lemma 4.15 that $|s| = \Phi_s(\Delta^k)$ is compact. By definition $|K|$ is the union of all the subsets $|s|$. Thus, since K is finite we see that $|K|$ is the union of finitely many compact subsets. By the Compact–Union-Intersection Lemma 3.41 (1) this implies that $|K|$ itself is compact.
(2) We write $V = \{v_1, \ldots, v_n\}$ and we identify the vector spaces $\mathbb{R}^V$ and $\mathbb{R}^n$ in the obvious way. We denote by $\mathcal{T}$ the topology on $|K|$ defined above and we denote by $\mathcal{R}$ the subspace topology which comes from $\mathbb{R}^V$. We need to show that the identity map defines a homeomorphism $(|K|, \mathcal{T}) \to (|K|, \mathcal{R})$. We make the following observations:
   (a) For any k-simplex $s \in S$ and any characteristic map $\Phi_s\colon \Delta^k \to |s|$ the corresponding map $\Delta^k \xrightarrow{\Phi_s} |s| \to |K| \to \mathbb{R}^V = \mathbb{R}^n$ is an affine linear map. It follows from the Δ^n-Homeomorphism Lemma 143.9 (1) that this map is continuous. It follows from

this observation together with the Restrict-to-Simplex Lemma 174.21 (1) that the map $(|K|, \mathcal{T}) \to (|K|, \mathcal{R})$ is continuous.

(b) In (1) we just argued that $(|K|, \mathcal{T})$ is compact.

(c) The topological space $(|K|, \mathcal{R})$ is evidently Hausdorff.

It follows from the Compact-Hausdorff Proposition 4.23 (3) that the identity map is indeed a homeomorphism. ■

As we already noted above, the topological realizations of abstract simplicial complexes have a voracious appetite for dimensions. Fortunately, as we will see shortly, in many cases we can reduce the dimensions drastically.

Lemma 174.26. (Finite Complex–Embedding Lemma) Let $K = (V, S)$ be a finite abstract simplicial complex and let $\varphi\colon \mathbb{R}^V \to \mathbb{R}^n$ be a linear map such that the restriction of φ to $|K|$ is an injection. Then $\varphi\colon |K| \to \mathbb{R}^n$ is an embedding.

Proof. By the Finite Simplicial Complex Lemma 174.25 (1) we know that $|K|$ is compact. Furthermore, note that it follows from the Finite Simplicial Complex Lemma 174.25 (2) that the linear map $\varphi\colon \mathbb{R}^V \to \mathbb{R}^n$ restricts to a continuous map $\varphi\colon |K| \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff it follows from the Compact-Hausdorff Proposition 4.23 (2) that $\varphi\colon |K| \to \mathbb{R}^n$ is indeed an embedding. ■

Example.

(1) Let $n \in \mathbb{N}_{\geq 3}$. As above we consider the abstract simplicial complex R_n. We consider the linear map $\varphi\colon \mathbb{R}^{\mathbb{Z}/n\mathbb{Z}} \to \mathbb{R}^2$ that is given by $[k] \mapsto \exp(2\pi \mathrm{i} k/n)$. One can easily verify that the restriction of φ to $|R_n|$ is an injection. Thus we see that $|R_n|$ is homeomorphic to the regular n-gon that is shown in the figure below.



(2) Using the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 one can formulate a suitable generalization of the Finite Complex–Embedding Lemma 174.26 to infinite abstract simplicial complexes. We will not work this out in detail, but using this approach one can fairly easily show that the topological realization of the abstract simplicial complex R_∞ is homeomorphic to $\mathbb{R}$.



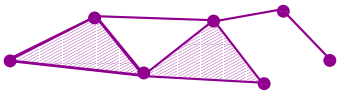
We refer to the figure above for a very convincing illustration of this fact. □

Given an n-simplex s of an abstract simplicial complex $K = (V, S)$ we introduced in Definition 174.18 the corresponding subset $|s| \subseteq |K|$, which, somewhat dangerously, we referred to as an n-simplex of $|K|$. The following definition gives us a convenient variation on that definition.

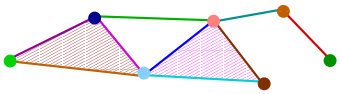
Definition 174.27. Let $K = (V, S)$ be an abstract simplicial complex. If we are given an n-simplex $s = \{v_0, \dots, v_n\} \in S$, then we refer to[1850]

$$\langle s \rangle := \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{l} (1)\ \alpha(v) > 0 \text{ for all } v \in s \\ (2)\ \alpha(v) = 0 \text{ for all } v \notin s \\ (3)\ \text{we have } \sum_{v \in V} \alpha(v) = 1 \end{array} \right\} = \left\{ \underbrace{\sum_{i=0}^{n} t_i \cdot v_i}_{\in \mathbb{R}^{(V)}} \;\middle|\; \begin{array}{l} (1)\ \text{for each } i \text{ we have } t_i > 0 \\ (2)\ t_0 + \dots + t_n = 1 \end{array} \right\}$$

as the corresponding open n-simplex of $|K|$.



simplicial complex



the corresponding open simplices

The following lemma summarizes a few properties of open simplices.

Lemma 174.28. (Open Simplex Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $s \in S$ be a k-simplex.
  (a) The closure of the open simplex $\langle s \rangle$ is given by $|s|$.
  (b) Every characteristic map $\Phi_s \colon \Delta^k \to |s|$ restricts to a homeomorphism $\mathring{\Delta}^k \to \langle s \rangle$.

(2) The topological realization $|K|$ is the disjoint union of the open simplices, i.e.

$$|K| \;=\; \bigsqcup_{s \in S} \langle s \rangle.$$

  Put differently, given any $\alpha \in |K|$ there exists a unique simplex s with $\alpha \in \langle s \rangle$.

(3) For two distinct simplices $s \neq t$ we have $\langle s \rangle \cap \langle t \rangle = \varnothing$.

(4) The union of the open simplices is precisely $|K|$.

(5) If $J = (U, R)$ is a subcomplex of K, then

$$|J| \;=\; \bigsqcup_{r \in R} \langle r \rangle \;=\; \bigcup_{r \in R} |r| \quad \text{and for any } s \in S \setminus R \text{ we have } |J| \cap \langle s \rangle = \varnothing.$$

  In particular, if s is a simplex of K with $\langle s \rangle \cap |J| \neq \varnothing$, then s is a simplex of J which implies that $|s| \subseteq |J|$.

(6) Let s and t be simplices. If $|s| \subseteq |t|$, then $s \subseteq t$. In particular, either $\dim(s) < \dim(t)$ or $s = t$.

(7) Let $s \in S$ be a simplex. If $U \subseteq |K|$ is an open subset with $U \cap |s| \neq \varnothing$, then $U \cap \langle s \rangle \neq \varnothing$.

Proof.

(1) Let $s \in S$ be a k-simplex and let $\Phi_s \colon \Delta^k \to |s|$ be a characteristic map. By the Simplex-Embedding Lemma 174.20 (3) we know that the map $\Phi_s \colon \Delta^k \to |s|$ is a homeomorphism and that $|s|$ is a closed subset of $|K|$. By definition we have $\langle s \rangle \subseteq |s|$.
  (b) It is clear that Φ_s restricts to a bijection $\mathring{\Delta}^k \to \langle s \rangle$. Thus it follows immediately that Φ_s restricts to a homeomorphism $\mathring{\Delta}^k \to \langle s \rangle$.
  (a) Since $|s|$ is a closed subset of $|K|$ we see that the closure of $\langle s \rangle$ in $|K|$ is contained in $|s|$. It is elementary to see that Δ^k is the closure of $\mathring{\Delta}^k$. Since $\Phi_s \colon \Delta^k \to |s|$ is

[1850] Here we use the convention of page 3288 which allows us to view V as a subset of $\mathbb{R}^{(V)}$.

a homeomorphism it follows almost immediately that $|s| = \Phi_s(\Delta^k)$ is indeed the closure of $\langle s\rangle = \Phi_s(\mathring{\Delta}^k)$.

(2) Let $\alpha \in |K|$. We set $s := \{v \in V \,|\, \alpha(v) \neq 0\}$. By the Topological Realization Lemma 174.16 (1) we know that s is a simplex. Evidently $\alpha \in \langle s\rangle$ and evidently s is unique.

(3) This statement follows immediately from (2).

(4) Conveniently enough this statement also follows immediately from (2).

(5) Let J be a subcomplex of K. If $r \in R$, then evidently we have $\langle r\rangle \subseteq |r| \subseteq |J|$. Keeping (2) in mind we see that it remains to show that if we have $s \in S$ with $|J| \cap \langle s\rangle \neq \varnothing$, then $s \in R$. Thus let s be a simplex of K with $|J| \cap \langle s\rangle \neq \varnothing$. It follows from (2), applied to the abstract simplicial complex $J = (U, R)$, that there exists a simplex r of J with $\langle s\rangle \cap \langle r\rangle \neq \varnothing$. But since r is also a simplex of K we obtain from (3), applied to K, that $s = r \in R$.

(6) Let s and t be simplices with $|s| \subseteq |t|$. We apply (5) to the subcomplex that is defined by s. We immediately obtain that $s \subseteq t$. If $s \subsetneq t$, then by definition we have $\dim(s) = \#S - 1 < \#T - 1 = \dim(t)$.

(7) Let $s \in S$ be a k-simplex and let $U \subseteq |K|$ be an open subset with $U \cap |s| \neq \varnothing$. Let $\Phi_s\colon \Delta^k \to |s|$ be a characteristic map. By definition of the topology on $|K|$ and by our hypothesis we know that $\Phi_s^{-1}(U)$ is an open non-empty subset of Δ^k. It is elementary to see that $\Phi_s^{-1}(U) \cap \mathring{\Delta}^k \neq \varnothing$. It now follows from (1b) that $U \cap \langle s\rangle \neq \varnothing$. ■

The following proposition reminds us of the CW-Complex-Finiteness Theorem 136.16 which gives an analogous statement for CW-complexes. The proposition can also be viewed as a converse to the Finite Simplicial Complex Lemma 174.25 (1).

Proposition 174.29. (Simplicial Complex-Finiteness Proposition) Let L be an abstract simplicial complex.

(1) If $X \subseteq |L|$ is a compact subset, then there exists a finite subcomplex K of L such that $X \subseteq |K|$.

(2) If $|L|$ is compact, then L is a finite abstract simplicial complex.

Remark. Let K be an abstract simplicial complex. The Finite Simplicial Complex Lemma 174.25 (1) together with the Simplicial Complex-Finiteness Proposition 174.29 shows that $|K|$ is compact if and only if K is finite. Later on in the Locally Finite–Characterisation Proposition 175.11 we will prove the related result that $|K|$ is regionally compact if and only if K is "locally finite". □

Proof. Let $L = (W, T)$ be an abstract simplicial complex. Clearly (2) is just a special case of (1). Thus it suffices to prove (1). Now let $X \subseteq |L|$ be a compact subset. For each simplex t of L with $\langle t\rangle \cap X \neq \varnothing$ we pick a point $y_t \in \langle t\rangle \cap X$. We denote by Y the set given by all the y_t. We equip Y with the subspace topology coming from $|L|$.

Claim.

(1) Every subset of Y, in particular Y itself, is a closed subset of $|L|$.

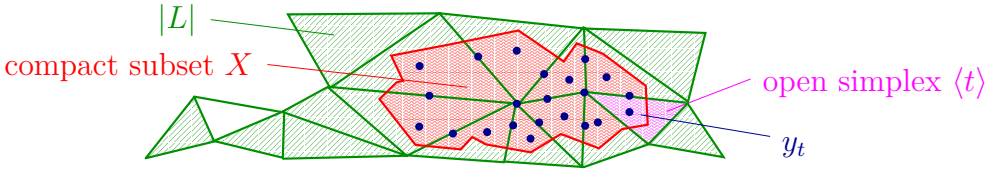
(2) Y is a discrete subset of $|L|$.

Proof.

(1) Let $A \subseteq Y$ be a subset. We need to show that A is a closed subset of $|L|$. By the Simplex-Embedding Lemma 174.20 (1) and (3) it suffices to show that for every simplex t of $|L|$ the intersection $A \cap |t|$ is closed. By the Open Simplex Lemma 174.28 (5) we know that $|t|$ intersects only the interiors of the simplices s with $s \subseteq t$. Since there are only finitely many simplices[1851] contained in t we see that $A \cap |t|$ is finite, in particular it is compact. By the Simplex-Embedding Lemma 174.20 (2) we know that $|t|$ is Hausdorff. Hence we obtain from the Compact-Closed Lemma 3.39 (2) that $A \cap |t|$ is indeed closed.

(2) We need to show that the subspace topology of Y is the discrete topology. It follows from (1) and the Subspace-Characterization Lemma 3.17 that every subset of Y is in fact a closed subset of Y. But this implies that every subset of Y is in fact an open subset of Y. In other words, the subspace topology is indeed the discrete topology. ⊞

It follows from (1) and the definition of the subspace topology that Y is a closed subset of the compact set X. Furthermore by (2) we know that Y is a discrete subset of $|L|$. Since $Y \subseteq X \subseteq |L|$ we see that Y is also a discrete subset of X. Since X is compact, we obtain from the Compact-Discrete Lemma 3.40 that Y is finite.



We define $S := \{t \in T \,|\, X \cap \langle t \rangle \neq \varnothing\}$. Note that $\#S = \#Y$. Thus, by the above discussion we know that S is finite. Now we set K to be the subcomplex that is given the union of all the simplices $t \in S$.[1852] Since t is finite we see that K is a finite subcomplex. Now we see that

$$X \overset{\text{by the Open Simplex Lemma 174.28 (4)}}{=} X \cap \bigcup_{t \in S} \langle t \rangle = \bigcup_{t \in S} X \cap \langle t \rangle \subseteq \bigcup_{t \in S} \langle t \rangle \subseteq \bigcup_{t \in S} |t| \overset{\text{by the Simplicial Subcomplex Lemma 174.23 (3)}}{=} |K|. \qquad \blacksquare$$

We conclude this section with the following lemma which gives a criterion for a subset of a simplicial complex to be a subcomplex.

Lemma 174.30. (Subcomplex Criterion) Let $L = (W, T)$ be an abstract simplicial complex and let $X \subseteq |L|$ be a closed subset. If for every simplex $t \in T$ we have either $\langle t \rangle \cap X = \varnothing$ or $\langle t \rangle \subseteq X$, then there exists a subcomplex K of L with $|K| = X$.

Proof. We consider

$$V := W \cap X \quad \text{and} \quad S := \{t \in T \,|\, \langle t \rangle \subseteq X\}.$$

Claim. We claim that (V, S) is a subcomplex of $L = (W, T)$.

[1851] This finiteness statement makes the proof of this proposition significantly simpler than the technically slightly tricky proof of the CW-Complex-Finiteness Theorem 136.16.

[1852] Strictly speaking we view the simplices as subcomplexes, and then take the union as defined in the Subcomplexes–Intersection-Union Lemma 174.7.

Proof. Let $s \in S$ be a simplex and let r be a face of s. We need to show that $r \in S$, i.e. we need to show that $\langle r \rangle \subseteq X$. Basically by definition we have $\langle r \rangle \subseteq |s|$. Thus it suffices to show that $|s| \subseteq X$.

Now note that by the Open Simplex Lemma 174.28 we know that $|s|$ is the closure of the subset $\langle s \rangle$. Since X is by hypothesis a closed subset of $|L|$ we obtain from elementary point-set topology, see e.g. Exercise 4.30 (a), that $|s| \subseteq X$. ⊞

Now we see that

$$|K| \underset{\text{by the Open Simplex Lemma 174.28 (2)}}{=} \bigcup_{t \in S} \langle t \rangle \overset{\text{definition of } S}{=} \bigcup_{t \in T \text{ with } \langle t \rangle \subseteq X} \langle t \rangle \underset{\text{by the hypothesis that either } \langle t \rangle \cap X = \varnothing \text{ or } \langle t \rangle \subseteq X}{=} \bigcup_{t \in T} (\langle t \rangle \cap X) \overset{\text{by the Intersection Distributive Lemma 1.5}}{=} \left(\bigcup_{t \in T} \langle t \rangle \right) \cap X \underset{\text{by the Open Simplex Lemma 174.28 (5)}}{=} X.$$

■

174.4. **Simplicial complexes.** As on so many other occasions we start out this new section with a definition.

Definition 174.31.

(1) Let X be a topological space. A simplicial structure for X is a pair $(K, \Theta\colon |K| \to X)$ where the following holds:
   (a) K is an abstract simplicial complex,
   (b) Θ is a homeomorphism between the topological realization $|K|$ of K and the topological space X.

(2) A simplicial complex is a pair $(X, (K = (V, S), \Theta\colon |K| \to X))$ given by a topological space X together with a simplicial structure $(K = (V, S), \Theta\colon |K| \to X)$. We refer to X as the underlying topological space of the simplicial complex. To keep the notation at a manageable level we will often not distinguish in the notation between a simplicial complex and its underlying topological space.

(3) We say that a simplicial complex is finite (respectively countable) if the corresponding abstract simplicial complex is *finite* (respectively *countable*).

(4) We say that a simplicial complex is (totally) ordered if the corresponding abstract simplicial complex is (totally) ordered.

(5) Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex. We introduce the following notions which are basically self-explanatory:
   (a) Given $v \in V$ we refer to $\Theta(v)$ as a vertex of the simplicial complex.
   (b) Let $s \in S$ be a k-simplex.
      (a) We refer to $\Theta(|s|) \subseteq X$ as a k-simplex of the simplicial complex. Furthermore we refer to $\Theta(\langle s \rangle) \subseteq X$ as an open k-simplex of the simplicial complex.
      (b) If $\Phi_s\colon \Delta^k \to |s|$ is a characteristic map as in Definition 174.18, then we refer to $\Theta \circ \Phi_s\colon \Delta^k \to X$ as a characteristic map of X.
      (c) If t is a face of s, then we refer to $\Theta(|t|)$ as a face of the simplex $\Theta(|s|)$.
   (c) We define the dimension of the simplicial complex to be the dimension of the corresponding abstract simplicial complex.
   (d) Given $n \in \mathbb{N}_0$ we denote by $|K^n| \subseteq |K|$ the topological realization of the n-skeleton of K. We refer to $X^n := \Theta(|K^n|)$ as the n-skeleton of the simplicial complex X.

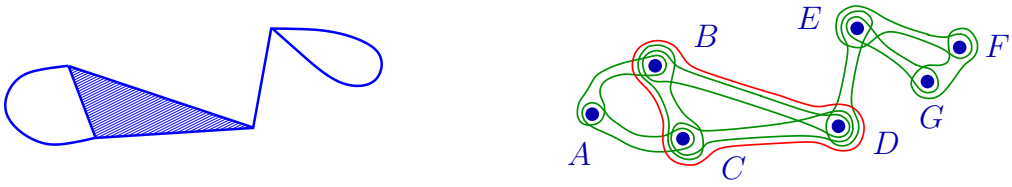
(6) Let $(Y, (L, \Theta\colon |L| \to Y))$ be a simplicial complex. A subcomplex of Y is a simplicial complex $(X, (K, \Omega\colon |K| \to X))$ where X is a subspace of Y, K is a subcomplex of L and where the inclusion $X \to Y$ is precisely the map $\Theta \circ |i| \circ \Omega^{-1}$, where $i\colon K \to L$ is the inclusion map.

Remark.

(1) In many cases the choice of a simplicial structure gets suppressed from the notation.
(2) In the literature a *simplicial structure* is often called a *triangulation*. □

Examples.

(1) Given any abstract simplicial complex K, its topological realization $|K|$ is a simplicial complex in an obvious way.
(2) On page 3294 we wrote down an explicit homeomorphism $|D_n| \xrightarrow{\cong} \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \xrightarrow{\cong} \partial\Delta^n$. These two homeomorphisms endow Δ^n and $\partial\Delta^n$ with a simplicial structure, called the *standard simplicial structure*.
(3) We consider the topological space X to the left of the figure below. To the right we show an abstract simplicial complex K with seven vertices such that X is homeomorphic to $|K|$.[1853]



topological space X abstract simplicial complex K

We have thus found a simplicial structure for X. □

Before we continue with more examples of simplicial complexes we want to formulate the following lemma which summarizes a few basic properties of simplices in a simplicial complex.

Lemma 174.32. (Simplices-in-Complex Lemma) Let Y be a simplicial complex.
(1) Any k-simplex of Y is homeomorphic to Δ^k.
(2) The intersection of two simplices of Y is either empty or it is given by a single face of the two simplices.
(3) Let X be a subcomplex of Y and let s be a simplex of Y. If the corresponding open simplex intersects X, then $s \subseteq X$.
(4) If $X \subseteq Y$ is the union of simplices of X, then Y admits a natural structure of a subcomplex.

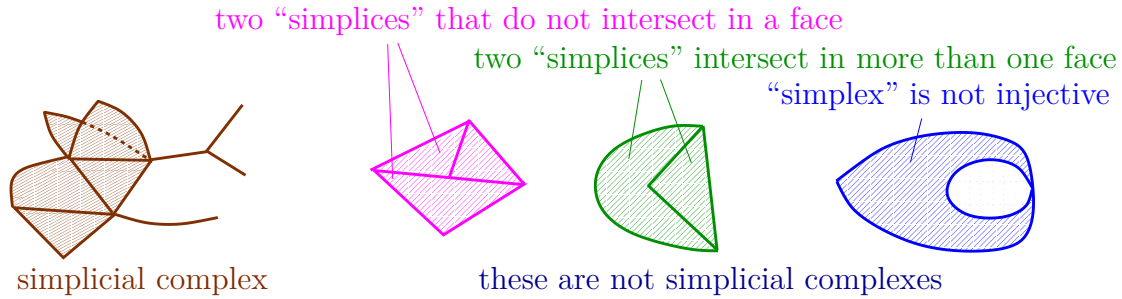
Proof.

(1) This statement is a reformulation of the Simplex-Embedding Lemma 174.20.
(2) This statement is an immediate consequence of the Simplices–Intersection Lemma 174.5.
(3) This follows from the Open Simplex Lemma 174.28 (5).
(4) This statement follows almost immediately from the definitions. ■

[1853] What is the role of the vertices A, F and G? Are they really needed?

Remark.

(1) In the following a simplicial structure for a topological space is often indicated by a suitable decomposition into simplices, i.e. into subsets which (look like they) are homeomorphic to some Δ^k. Some care though needs to be taken to make sure that one actually does define a simplicial structure. In particular one needs to ensure that the statements of the Simplices-in-Complex Lemma 174.32 (1) and (2) are satisfied.

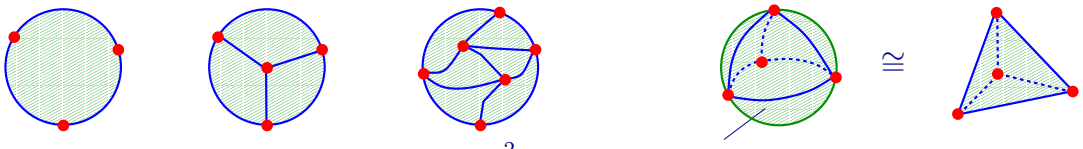


(2) In the following we often draw a simplicial complex and rely on the good will of the reader to read off the corresponding abstract simplicial complex. □

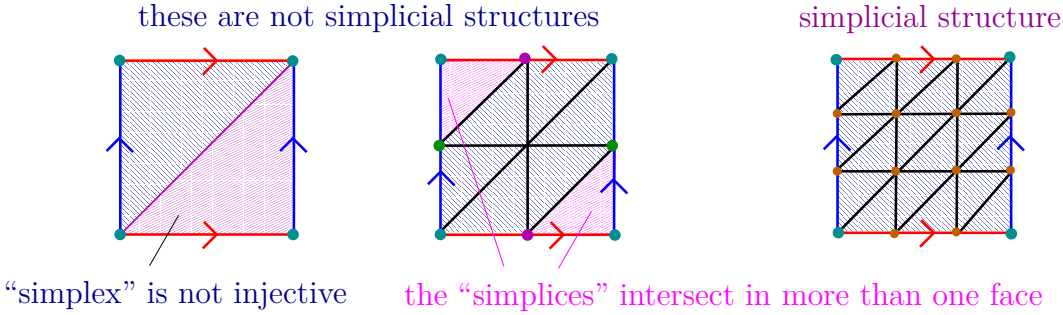
We continue with more examples of simplicial complexes.

Examples.

(1) In the figure below to the left we show several simplicial structures for the disk $\overline{B}^2$. Furthermore, in the figure below to the right we show a simplicial structure for the sphere S^2.



(2) In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$ together with three attempts at endowing it with a simplicial structure. Using the Simplices-in-Complex Lemma 174.32 we see that the two attempts to the left do not define a simplicial structure. Only the fairly complicated attempt to the right actually provides us with a simplicial structure for the torus.



(3) Let $n \in \mathbb{N}$. We will now equip the closed ball $\overline{B}^n$ and the sphere S^n with a standard simplicial structure. To do so we consider the simplicial complex D_n and its

subcomplex S_{n-1} that we introduced on page 3283. On page 3294 we saw that there exists a canonical homeomorphism $f\colon |D_n| \to \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \to \partial\Delta^n$. Furthermore, in the Δ^n-Homeomorphism Lemma 143.9 (2) we gave an explicit homeomorphism $\Phi\colon \overline{B}^n \to \Delta^n$ that restricts to a homeomorphism $S^{n-1} \to \partial\Delta^n$.



We see that $(D_n, \Phi^{-1} \circ f\colon |D_n| \to \overline{B}^n)$ is a simplicial structure for $\overline{B}^n$ and that $(S_{n-1}, \Phi^{-1} \circ f|_{|S_{n-1}|}\colon |S_{n-1}| \to S^{n-1})$ is a simplicial structure for S^{n-1}. We refer to these as the *standard simplicial structures* of $\overline{B}^n$ and S^{n-1}. □

It is by now abundantly clear that most simplicial complexes admit many different simplicial structures. Finally we consider maps between simplicial complexes. What comes next is basically self-explanatory and only written down out of a sense of duty.

Definition 174.33. Let $(X, (K, \Theta\colon |K| \to X))$ and $(Y, (L, \Omega\colon |L| \to Y))$ be two simplicial complexes.

(1) A map $f\colon X \to Y$ is called simplicial if there exists a simplicial map $g\colon K \to L$ such that $f = \Omega \circ |g| \circ \Theta^{-1}$. Note that simplicial maps are automatically continuous.
(2) We say that the two simplicial complexes are simplicially isomorphic if there exist simplicial maps between the two simplicial complexes that are inverses of one another.

Example. If Y is a simplicial complex and X is a subcomplex, then it follows immediately from the definitions that the inclusion $X \to Y$ is a simplicial map. □

It follows immediately from the definitions that the composition of two simplicial maps is again simplicial. This leads us to the following definition:

Definition 174.34. We call the category $\mathcal{SimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SimpCplx}) &:= \text{all simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{all simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the category of simplicial complexes. Similarly we define the category of (totally) ordered simplicial complexes.

We conclude this section with the following basic lemma.

Lemma 174.35. (Induced Simplicial Map Lemma) In the following let $(X, (K = (V, S), \Theta\colon |K| \to X))$ and $(Y, (L = (W, T), \Omega\colon |L| \to Y))$ be two simplicial complexes. Given a simplicial map $\varphi\colon K \to L$ there exists a unique simplicial map $\Phi\colon X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Omega} \\ X & \xrightarrow{\Phi} & Y. \end{array}$$

Proof. The map Φ is given by $\Omega \circ |\varphi| \circ \Theta^{-1}$. Since Ω and Θ are homeomorphisms we see that Φ is also unique. ■

174.5. **The product of (abstract) simplicial complexes.** In this section we want to consider the product of (abstract) simplicial complexes. Let us first consider the simplicial complexes $X = Y = [0,1]$ with the "obvious" simplicial structure. As we see in the figure below, there are two very reasonable looking simplicial structures on the product $[0,1] \times [0,1]$.



It is not clear which of the two simplicial structures shown in the figure above is preferable or "more canonical". In fact to break the tie and to end up with a clear winner we need to work with ordered abstract simplicial complexes, as introduced in Definition 174.2.
First we recall Definition 1.26:

Definition 174.36. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define

$$(a,b) < (a',b') \quad :\Longleftrightarrow \quad \text{either } a < a' \text{ or } a = a' \text{ and } b < b'$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. We refer to "$\leq$" on $A \times B$ as the lexicographic order on $A \times B$.

Remark. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. One can show easily that the lexicographic order is a partial order on the product $A \times B$ and similarly one can show easily that if $(A, \leq)$ and $(B, \leq)$ are totally ordered sets, then the lexicographic order is also a total order. □

Definition 174.37. Let $K = (V, S)$ and $L = (W, T)$ be two (totally) ordered abstract simplicial complexes. We define the product $K \times L$ of the abstract simplicial complexes to be the abstract simplicial complex[1854] that is given by the vertex set $V \times W$ and the simplex set[1855]

$$\left\{ \{(v_0,w_0),\ldots,(v_k,w_k)\} \;\middle|\; \begin{array}{c} \{v_0,\ldots,v_k\} \in S \text{ and } \{w_0,\ldots,w_k\} \in T \text{ and} \\ \text{we have } v_0 \leq \cdots \leq v_k \text{ and } w_0 \leq \cdots \leq w_k \end{array} \right\}.$$

We equip $K \times L$ with the lexicographic order and we view $K \times L$ again as a (totally) ordered abstract simplicial complex.[1856]

Before we discuss examples we first state the following lemma which basically reassures us that we are doing something reasonable.

[1854] One can indeed easily verify that this defines an abstract simplicial complex.
[1855] To rule out any potential confusion, the $v_0, \ldots, v_k$ and also the $w_0, \ldots, w_k$ do not need to be distinct vertices.
[1856] It follows almost immediately from the above remark that with this definition $K \times L$ is in fact a (totally) ordered abstract simplicial complex.

Lemma 174.38. (Simplicial Complex-Product Lemma) Suppose we are given by two ordered abstract simplicial complexes $K = (V, S)$ and $L = (W, T)$.

(1) The two natural projections $p\colon K \times L \to K$ and $q\colon K \times L \to L$ are simplicial maps.
(2) The ordered abstract simplicial complex $K \times L$ together with the natural projections from (1) is the product of K and L in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes, in the sense of Definition 6.1.
(3) The map
$$|p| \times |q|\colon |K \times L| \ \to \ |K| \times |L|$$
is a continuous bijection. If K and L are *finite*, then the map is in fact a homeomorphism.

Thus given two finite simplicial complexes K and L we can view the product also as a simplicial complex.

Remark.

(1) The Product-not-Product Lemma 139.19 teaches us that it is safer to stick to finite abstract simplicial complexes. In fact that is also all we ever need. For products of arbitrary abstract simplicial complexes we refer to [**FrPi1990a**, Proposition 4.3.5] and [**LW1969**, Corollary III.5.2]. Note though that one needs to tread carefully to make sure that the definitions and statements are indeed as desired.
(2) As mentioned earlier, the topology of the topological realization of an abstract simplicial complex is designed to make it easy to show that a map out of the topological realization is continuous. Furthermore the topology of a product of topological spaces makes it easy to find continuous maps *to* the product. This explains why in the Simplicial Complex-Product Lemma 174.38 we have a continuous map from left to right, but why we struggle to find a continuous map from right to left. □

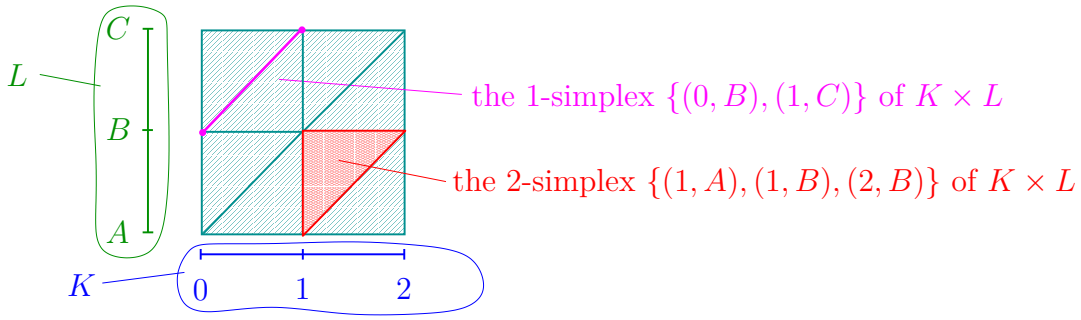
Proof.

(1) It follows immediately from the definitions that the projections are indeed simplicial.
(2) We will not make use of this statement, thus we will outsource it to Exercise 174.11.
(3) First note that it follows immediately from (1) together with the Topological Realization–Functor Proposition 174.22 and the Topological-Product Proposition 7.2 (1a) that the map $|p| \times |q|\colon |K \times L| \to |K| \times |L|$ is continuous. We leave it to the reader to verify that the map is a bijection. Alternatively the details are worked out in [**FriedmG2020**, Lemma B.6.3].

Finally suppose that K and L are finite. It follows from the Compact-Hausdorff Proposition 4.23 (3), together with the Simplex-Embedding Lemma 174.20 (1), the Finite Simplicial Complex Lemma 174.25 (1) and the Product Topology-Properties Proposition 7.9 that the map is indeed a homeomorphism. ■

Examples.

(1) We consider the two totally ordered abstract simplicial complexes K and L that are shown in the figure below. Here we use the obvious total orders given by $0 < 1 < 2$ and $A < B < C$. We also show the corresponding product $K \times L$.
(2) We equip S^1 with the standard simplicial structure defined on page 3302. The product simplicial structure on the torus $S^1 \times S^1$ is simplicially isomorphic to the simplicial structure shown in the figure shown on page 3301 to the right.



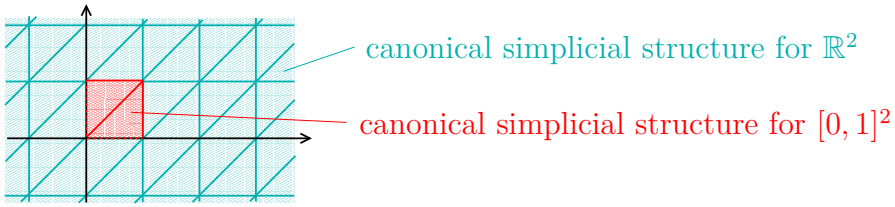
(3) Let I be the abstract simplicial complex given by $(V = \{(0,1)\}, S = \{\{0\},\{1\},\{0,1\}\})$. As in (1) we use the obvious total order given by $0 < 1$. We consider the product $I \times I$. Given any $n \in \mathbb{N}$ we can now view the cube $[0,1]^n$ as a simplicial complex. For $n = 2$ we obtain the simplicial complex drawn in the figure shown on page 3303 on the left. We refer to this simplicial structure as the *standard simplicial structure of* $[0,1]^n$. □

In plain English, the following lemma says that the $\mathbb{Z}^n$-translates of the standard simplicial structure of $[0,1]^n$ give a simplicial structure for $\mathbb{R}^n$.

Lemma 174.39. (Simplicial Structure-on-$\mathbb{R}^n$-Lemma) Let $n \in \mathbb{N}$. We denote by $(K = (V,S), \Theta\colon |K| \to [0,1]^n)$ the standard simplicial structure of $[0,1]^n$. We define a new simplicial complex $\widetilde{K} = (\widetilde{V}, \widetilde{S})$ where

$$\widetilde{V} := \mathbb{Z}^n \quad \text{and} \quad \widetilde{S} := \{\{v_0 + w, \ldots, v_k + w\} \,|\, \{v_0,\ldots,v_k\} \in S \text{ and } w \in \mathbb{Z}^n\}.$$

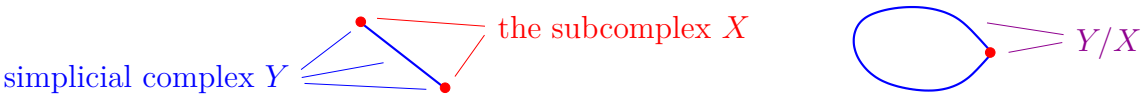
The (hopefully) obvious map $\widetilde{\Theta}\colon |\widetilde{K}| \to \mathbb{R}^n$ is a homeomorphism, in particular it equips $\mathbb{R}^n$ with the structure of a simplicial complex.



Remark. In the Open Euclidean Subset–Simplicial Structure Lemma 179.13 we will show that in fact every open subset of $\mathbb{R}^n$ admits a simplicial structure. □

Proof. It is elementary to see that the map $\widetilde{\Theta}\colon |\widetilde{K}| \to \mathbb{R}^n$, once it is written down properly, is continuous and a bijection. It follows from the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26 that the map is in fact a homeomorphism. ■

Arguably the most important ways to construct new topological spaces out of given ones are by taking products and by taking quotients. Within reasonable limits we saw in the CW-Complex Product Proposition 139.18 and the CW-Quotient Lemma 139.10 that both operations can be done with CW-complexes. Furthermore, we just saw in the Simplicial Complex-Product Lemma 174.38 that we can take products of finite simplicial complexes. Quotients on the other hand are a problem. In general, as is illustrated in the figure below, given a simplicial complex Y and a subcomplex X the simplicial structure of Y does not descend to a simplicial structure on the quotient Y/X.



Let us jot down the following question:

Question 174.40. Let Y be a simplicial complex and let X be a subcomplex. Does the quotient Y/X admit a simplicial structure?

In the Simplicial Complex–Quotient Corollary 174.53 we will give a partial answer to Question 174.40.

174.6. **The simplicial join of (abstract) simplicial complexes.** In this section we will continue with our task of finding useful constructions of (abstract) simplicial complexes that correspond to constructions that we already know from the world of topological spaces. More precisely, we want to consider the join construction from Definition 20.1 that so far we had not seriously used.

Definition 174.41. Let X and Y be two non-empty topological spaces. We define the join $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define $X * Y$ to be the other topological space.

To get into the mood of thinking about joins let us first consider the following two examples.

Examples.

(1) Let X be a non-empty topological space and let $Y = \{1\}$ be the topological space given by a single point 1. We consider the maps

$$\begin{array}{rcl} X * \{1\} = (X \times [0,1] \times \{1\})/\sim & \to & \mathrm{Cone}(X) = (X \times [0,1])/(X \times \{1\}) \\ [(x,t,1)] & \mapsto & [(x,t)] \end{array}$$

One can easily verify that the map is a bijection. Furthermore, using the Topological-Quotient Proposition 8.2 (1b) and the Topological-Quotient Proposition 8.2 (1a) it is straightforward to see that this map and its inverse are continuous. Thus the map is a homeomorphism.

(2) Let X be a non-empty topological space and let $Y = \{-1,1\}$ be the topological space given by the set with two elements ± 1 and the discrete topology. We also consider the suspension

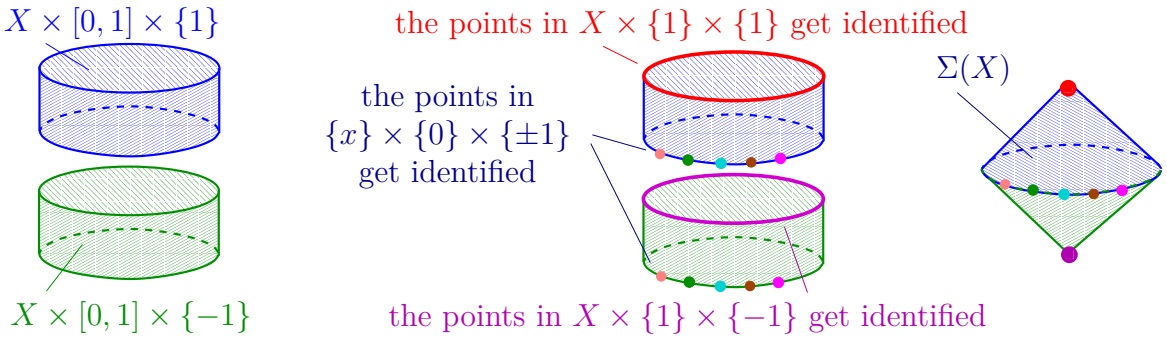
$$\Sigma(X) \ := \ (X \times [-1,1])/\sim$$

which is defined by squashing all points in $X \times \{-1\}$ to a single point and by squashing all points in $X \times \{1\}$ to a single point. We consider the map

$$\begin{array}{rcl} X * \{-1,1\} = (X \times [0,1] \times \{-1,1\})/\sim & \to & \Sigma(X) = (X \times [-1,1])/\sim \\ [(x,t,\epsilon)] & \mapsto & [(x, \epsilon \cdot t)] \end{array}$$

With some minor effort one can show that this map is a bijection. Using the Topological-Quotient Proposition 8.2 (1b) and the Topological-Quotient Proposition 8.2 (1a) it is straightforward to see that this map is continuous. The fact that the inverse is also

continuous needs slightly more effort, in fact this argument also requires the Quotient-to-Quotient Lemma 8.6 and the Regluing Lemma 8.17. We leave it to the reader to fill in the details. We refer to the figure below for $X = S^1$ for an illustration of the homeomorphism.



(3) For completeness we note that for the empty topological space we have, basically by definition, that $\varnothing * \{1\} = \mathrm{Cone}(\varnothing)$ and $\varnothing * \{\pm 1\} = \Sigma(\varnothing)$ □

The following lemma gives us a few examples of joins which will play a role later on.

Lemma 174.42. (Join-Simplex Lemma)

(1) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.[1857]

(2) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \Delta^m * \Delta^n & \to & \Delta^{m+n+1} \\ [(x_0, \ldots, x_m, t, y_0, \ldots, y_n)] & \mapsto & (t \cdot x_0, \ldots, t \cdot x_m, (1-t) \cdot y_0, \ldots, (1-t) \cdot y_n) \end{array}$$

is a homeomorphism.

(3) Given any $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \partial\Delta^n * \{\star\} & \to & \Delta^n \\ [(x,t,\star)] & \mapsto & x \cdot (1-t) + \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \cdot t \end{array}$$

is a homeomorphism.

Proof. The proof of (1), (2) and (3) is basically identical. In each case it is straightforward to verify that the map is well-defined and a bijection. It follows from the Topological-Quotient Proposition 8.2 (1b) that the given map is continuous. Finally it follows from the Join Topology-Properties Proposition 20.2 (1) together with the Compact-Hausdorff Proposition 4.23 (3) that the given map is in fact a homeomorphism. ■

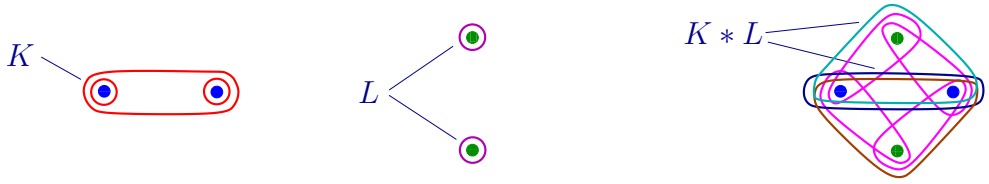
It turns out that in the simplicial setting we have an analogous construction.

[1857] In fact this statement is just the content of the Join-of-Spheres Lemma 20.4.

Definition 174.43. Let $K_i = (V_i, S_i)$, $i = 1, \ldots, m$ be abstract simplicial complexes. We define the simplicial join of $K_1, \ldots, K_m$ to be the abstract simplicial complex[1858] [1859]

$$K_1 * \cdots * K_m := \left(V_1 \sqcup \cdots \sqcup V_m, \left\{ s_1 \sqcup \cdots \sqcup s_m \;\middle|\; \begin{array}{l} \text{for } i = 1, \ldots, m \text{ we have } s_i \in \overbrace{S_i \cup \{\varnothing\}}^{\subseteq \mathcal{P}(V_i)} \\ \text{and there exists at least one } s_i \neq \varnothing \end{array} \right\}\right).$$

This definition is illustrated, with limited success, in the figure below.



Remark. It follows immediately from the definitions that the join construction is associative, i.e. given abstract simplicial complexes K, L and M we have a natural isomorphism $(K * L) * M = K * (L * M)$ of abstract simplicial complexes. □

The examples on page 3306 motivate the following definitions.

Definition 174.44. Let K be an abstract simplicial complex. Furthermore let $\{1\}$ be the abstract simplicial complex consisting of a single vertex 1 and finally let S_0 be the abstract simplicial complex $S_0 = (\{0, 1\}, \{\{0\}, \{1\}\})$ that we introduced on page 3283. We refer to

$$\text{Cone}(K) := K * \{1\} \quad \text{respectively} \quad \Sigma(K) := K * S_0$$

as the simplicial cone on the abstract simplicial complex K respectively the simplicial suspension of the abstract simplicial complex K.

Example. Let $n \in \mathbb{N}_0$. We see that we have an isomorphism

$$\overbrace{\text{Cone}\left(\{0, \ldots, n\}, \mathcal{P}(\{0, \ldots, n\}) \backslash \{\varnothing\}\right)}^{= \text{Cone}(D_n), \text{ see definition of } D_n \text{ on page 3283}} = \left(\{0, \ldots, n, *\}, \mathcal{P}(\{0, \ldots, n, *\} \backslash \{\varnothing\}\right) \xrightarrow{\cong} D_{n+1}.$$

(= follows from Definition 174.44 of the simplicial cone; $\xrightarrow{\cong}$ induced by $* \mapsto n+1$)

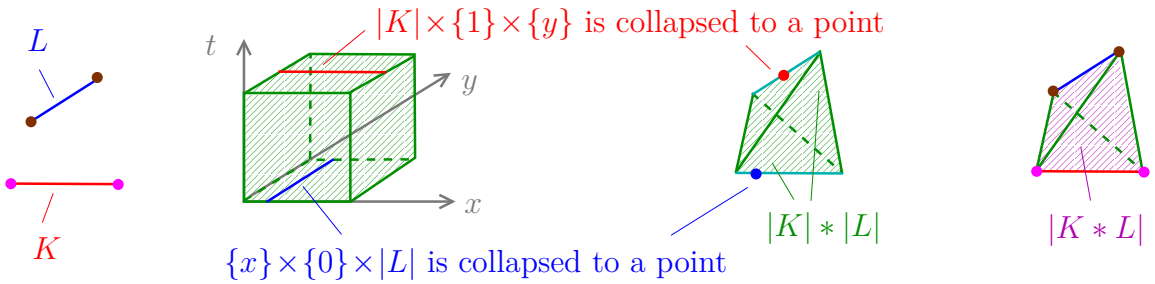
□

Lemma 174.45. (Join Lemma) In the following let K and L be two abstract simplicial complexes. If one of K or L is empty, then by definition we have a natural homeomorphism $|K| * |L| \cong |K * L|$. If K and L are non-empty, then the map

$$\begin{array}{rccl} \Theta\colon & \overbrace{(|K| \times [0,1] \times |L|)/\sim}^{=|K|*|L|} & \to & |K * L| \subseteq \mathbb{R}^{(V \sqcup W)} = \mathbb{R}^{(V)} \times \mathbb{R}^{(W)} \\ & [(x, r, y)] & \mapsto & (\underbrace{r \cdot x}_{\in \mathbb{R}^{(V)}}, \underbrace{(1-r) \cdot y}_{\in \mathbb{R}^{(W)}}) \end{array}$$

is a bijection and its inverse Θ^{-1} is continuous.[1860] Furthermore, if K and L are finite, then Θ is in fact a homeomorphism.

[1858] To avoid confusion it is perhaps worth recalling that the empty set is not a simplex.
[1859] One can easily verify that this is indeed an abstract simplicial complex.



Proof. Let $K=(V,S)$ and $L=(W,T)$ be two abstract simplicial complexes. If one of K or L is empty, then the statement is true basically by definition. Thus let us now assume that K and L are both non-empty. We consider the map

$$\begin{array}{rcl}\Theta\colon \overbrace{(|K|\times[0,1]\times|L|)/\sim}^{=|K|*|L|} & \to & |K*L|\subseteq\mathbb{R}^{(V\sqcup W)}=\mathbb{R}^{(V)}\times\mathbb{R}^{(W)}\\ [(x,r,y)] & \mapsto & (\underbrace{r\cdot x}_{\in\mathbb{R}^{(V)}},\underbrace{(1-r)\cdot y}_{\in\mathbb{R}^{(W)}}).\end{array}$$

We leave it to the reader to verify that the map does indeed take values in $|K*L|$ and that it is a bijection. Next we show that the inverse $\Theta^{-1}\colon |K*L|\to|K|*|L|$ is continuous.

Claim. The map $\Theta^{-1}\colon |K*L|\to|K|*|L|$ is continuous.

Proof. We use the continuity criterion from the Restrict-to-Simplex Lemma 174.21 (2) to show that Θ^{-1} is continuous. Since K and L are non-empty we see that every simplex of $K*L$ is contained in a simplex of the form $s\sqcup t$ where $s\in S$ and $t\in T$ are simplices. Thus let $s\in S$ be a k-simplex and let $t\in T$ be an l-simplex. Note that by the Join-Simplex Lemma 174.42 (2) we know that the map

$$\begin{array}{rcl}\Xi\colon \Delta^k*\Delta^l & \to & \Delta^{k+l+1}\\ [(x_0,\ldots,x_k,r,y_0,\ldots,y_l)] & \mapsto & (r\cdot x_0,\ldots,r\cdot x_k,(1-r)\cdot y_0,\ldots,(1-r)\cdot y_l)\end{array}$$

is a homeomorphism. Next we pick corresponding characteristic maps $\Phi_s\colon\Delta^k\to|K|$ and $\Phi_t\colon\Delta^l\to|L|$. It follows easily from the definitions that the map

$$\begin{array}{rcccl}\Phi_{s\sqcup t}\colon\Delta^{k+l+1} & \xrightarrow{\Xi^{-1}} & \overbrace{(\Delta^k\times[0,1]\times\Delta^l)/\sim}^{\Delta^k*\Delta^l} & \xrightarrow{=:\Omega} & |K*L|\subseteq\mathbb{R}^{(V\sqcup W)}=\mathbb{R}^{(V)}\times\mathbb{R}^{(W)}\\ & & [(x,r,y)] & \mapsto & (\underbrace{r\cdot\Phi_s(x)}_{\in\mathbb{R}^{(V)}},\underbrace{(1-r)\cdot\Phi_t(y)}_{\in\mathbb{R}^{(W)}})\end{array}$$

is a characteristic map for the simplex $s\sqcup t$. We obtain the following diagram:

$$\begin{array}{ccccc}\Delta^{k+l+1} & \xrightarrow{\Phi_{s\sqcup t}} & |K*L| & \xrightarrow{\Theta^{-1}} & |K|*|L|\\ & \searrow^{\Xi^{-1}} & \uparrow{\scriptstyle\Omega} & \nearrow_{\Phi_s*\Phi_t} & \\ & & \Delta^k*\Delta^l & & \end{array}$$

[1860]This might be viewed as an odd formulation, but it is easier to write down the map Θ than to write down its inverse Θ^{-1}.

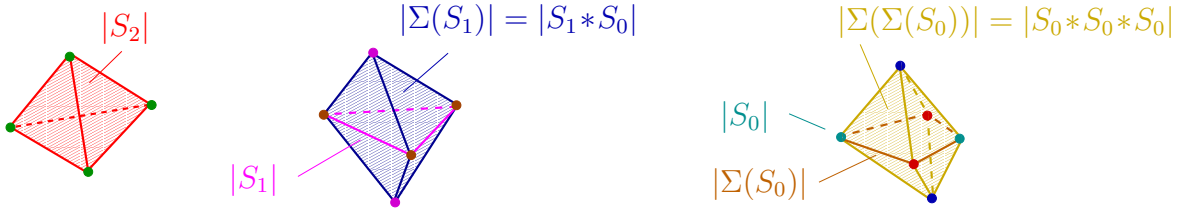
We make the following observations:

(1) It follows immediately from the definitions that the triangle to the left of the diagram commutes. Furthermore it is equally trivial to see that $\Theta \circ (\Phi_s * \Phi_t) = \Omega$, thus the triangle to the right also commutes.
(2) By the Join-Simplex Lemma 174.42 (2) we know that the left diagonal map is continuous.
(3) By the Join–Functor Lemma 20.7 (1) we know that the right diagonal map is continuous.
(4) It follows from the above that $\Theta^{-1} \circ \Phi_{s \sqcup t}$ is continuous.

It now follows from the Restrict-to-Simplex Lemma 174.21 (2) that $\Theta^{-1}\colon |K*L| \to |K|*|L|$ is continuous. ⊞

In the remainder of the proof we now assume that K and L are finite. We need to show that Θ is a homeomorphism. We already know that Θ is a bijection and that the inverse $\Theta^{-1}\colon |K * L| \to |K| * |L|$ is continuous. Furthermore, by the Simplex-Embedding Lemma 174.20 (2) together with the Join Topology-Properties Proposition 20.2 (3) we know that $|K| * |L|$ is Hausdorff. Thus it follows, as so often, from the Compact-Hausdorff Proposition 4.23 (3) that Θ^{-1} is a homeomorphism. Thus Θ is also a homeomorphism. ■

Example. Let $n \in \mathbb{N}_0$. Recall that on page 3302 we gave an explicit simplicial structure $(S_n, \Theta\colon |S_n| \to S^n)$ for S^n. Now let $m, n \in \mathbb{N}$. By the Join-Simplex Lemma 174.42 (1) we know that S^{m+n+1} is homeomorphic to $S^m * S^n$. This observation, together with the Join Lemma 174.45, gives us lots of different simplicial structures on spheres.



We sketch the resulting three simplicial structures for S^2 in the figure below. □

Corollary 174.46. (Simplicial Cone-Suspension Corollary) If $X = (K, \Theta\colon |K| \to X)$ is a finite simplicial complex, then the suspension $\Sigma(X)$ and the cone $\mathrm{Cone}(X)$ admit natural simplicial structures where the corresponding abstract simplicial complexes are given by $\Sigma(K)$ and $\mathrm{Cone}(K)$.

Proof. The corollary follows immediately from the Join Lemma 174.45 together with the two examples on page 3307. ■

Remark. In Section 175.4 we will introduce the simplicial analog of the mapping cylinder and mapping cone. □

174.7. Linear simplicial complexes.

In this short section we discuss an important special case of a simplicial complex.

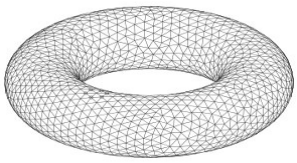
Definition 174.47. A simplicial complex $(X, (K, \Theta\colon |K| \to X))$ is called linear if X is a subspace[1861] of some $\mathbb{R}^n$ and for each each simplex $s \in K$ the map $\Theta\colon |s| \to X$ is an affine linear map.

Example. Let $n \in \mathbb{N}$. The topological space $\mathbb{R}^n$ together with the standard simplicial structure defined in the Simplicial Structure-on-$\mathbb{R}^n$-Lemma 174.39 is clearly a linear simplicial complex. □

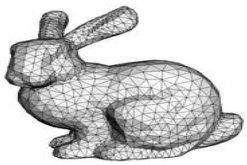
Remark.

(1) Different books have different definitions for "simplicial complexes". Our approach mostly follows the approach taken in [**Spa1995**, Chapter 3]. In many books what we call a "linear simplicial complex" is basically used as the definition of a simplicial complex, see e.g. [**Matv2006**, p. 6] or [**Bre1993**, Chapter 21].

(2) Linear simplicial complexes play an important role in computer visualizations.



linear simplicial structure for the torus



linear simplicial structure for a bunny

More precisely, in computer visualizations surfaces in $\mathbb{R}^3$ are usually approximated by linear simplicial complexes. These have the nice feature that they can be described by a fairly small amount of data. □

Lemma 174.48. (Finite Simplicial Complex–Linear Lemma) Every finite simplicial complex is simplicially isomorphic to a finite linear simplicial complex.

Remark. An (abstract) simplicial complex is called *locally finite* if each simplex is the face of only finitely many simplices. With this definition one can formulate a much stronger version of the Finite Simplicial Complex–Linear Lemma 174.48: By [**FrPi1990a**, Theorem 3.3.15] a simplicial complex is homeomorphic to a linear simplicial complex if and only if it is finite-dimensional, countable and locally finite. We will discuss this statement in Exercise 174.9. □

Proof. By definition a finite simplicial complex is homeomorphic to the topological realization $|K|$ of a finite abstract simplicial complex $K = (V, S)$. In the Finite Simplicial Complex Lemma 174.25 we showed that we can view $|K|$ as a linear simplicial complex in $\mathbb{R}^V = \mathbb{R}^m$. ■

We adopt the following notation:

Notation 174.49. Given[1862] $v_0, \ldots, v_k \in \mathbb{R}^n$ we write

$$|\{v_0, \ldots, v_k\}| := \left\{ \underbrace{\sum_{i=0}^{k} t_i \cdot v_i}_{\in \mathbb{R}^n} \;\middle|\; \begin{array}{l} (1) \text{ for each } i \text{ we have } t_i \geq 0 \\ (2)\ t_0 + \cdots + t_k = 1 \end{array} \right\}.$$

Remark. Let $v_0, \ldots, v_k \in \mathbb{R}^n$. Furthermore let $w_0, \ldots, w_m \in |\{v_0, \ldots, v_k\}|$. An elementary calculation shows that $|\{w_0, \ldots, w_m\}| \subseteq |\{v_0, \ldots, v_k\}|$. This inclusion can also be obtained from Exercise 4.46 (a). □

[1861] In particular we assume that X is endowed with the subspace topology coming from $\mathbb{R}^n$.

[1862] We do not assume that $v_0, \ldots, v_k$ are distinct.

Lemma 174.50. (Linear Simplicial Complex Lemma)

(1) Let K be a linear simplicial complex in $\mathbb{R}^n$.
   (a) Every k-simplex s of K is of the form $s = |\{v_0, \dots, v_k\}|$ for some distinct vertices $v_0, \dots, v_k$.
   (b) Every simplex of K is a convex subset of $\mathbb{R}^n$.
   (c) Let $v_0, \dots, v_k$ be distinct vertices of K. The following three statements are equivalent:
      (i) $|\{v_0, \dots, v_k\}|$ is a simplex of K,
      (ii) $\{v_0, \dots, v_k\}$ is the vertex set of a simplex of K,
      (iii) $|\{v_0, \dots, v_k\}| \subseteq K$.

(2) Let $f\colon K \to L$ be a simplicial map between two linear simplicial complexes. The restriction of f to any k-simplex $|\{v_0, \dots, v_k\}|$ of K is given by

$$\begin{array}{ccc} |\{v_0, \dots, v_k\}| & \to & |\{\varphi(v_0), \dots, \varphi(v_k)\}| \\ \sum_{i=0}^{k} t_i \cdot v_i & \mapsto & \sum_{i=0}^{k} t_i \cdot \varphi(v_i). \end{array}$$

Proof.

(1) Almost all of these statements follow easily from the definitions. The only statement which needs a little bit of thought is the fact that (iii) implies (i). Since we will not make use of this statement we feel comfortable with the thought of leaving the details to the reader.

(2) This statement follows immediately from the definition of a simplicial map, see Definition 174.8 and the definition of the topological realization of a simplicial map between abstract simplicial complexes, see the Topological Realization–Functor Proposition 174.22. ■

174.8. **Simplicial structures and CW-structures I.** It is pretty clear that simplicial complexes and CW-complexes have structural similarities. In particular few readers will be surprised by the following lemma.

Lemma 174.51. (Simplicial-Implies-CW Lemma) Let K be a simplicial complex. The sequence

$$\varnothing \subseteq K^0 \subseteq K^1 \subseteq K^2 \subseteq \dots$$

of the skeleta of K defines a natural CW-structure[1863] on K. This CW-structure has the following properties:

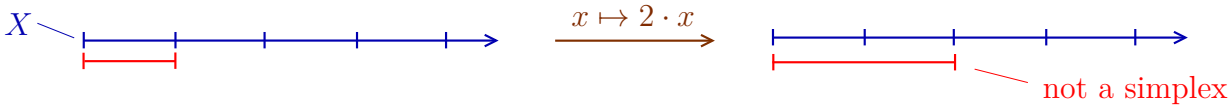
(1) The n-simplices of the simplicial structure are precisely the n-cells[1864] of the CW-structure.
(2) The simplicial subcomplexes are precisely the cellular subcomplexes,
(3) For each $n \in \mathbb{N}_0$ the n-skeleton of the simplicial complex equals the n-skeleton of the CW-complex.
(4) Any simplicial map $f\colon X \to Y$ between simplicial complexes is a cellular map between the corresponding CW-complexes.

[1863] We refer to Definition 136.1 for the definition of a CW-complex.
[1864] Recall that cells and simplices are subsets, so it makes sense to talk of equality.

Remark. The Simplicial-Implies-CW Lemma 174.51 says in particular that there exists a covariant functor

$$F\colon \begin{array}{c}\text{category } \mathcal{OrdAbsSimpCplx} \text{ of}\\ \text{ordered simplicial complexes}\end{array} \to \begin{array}{c}\text{category } \mathcal{CW} \text{ of}\\ \text{CW-complexes}\end{array}$$

which has the property that the functor keeps the underlying topological space fixed. The functor is faithful but not fully faithful. The latter is just a fancy way of saying that there exist cellular maps that are not simplicial. For example, consider the simplicial complex $X = [0, \infty)$ with the vertex set given by $\mathbb{N}_0$. We view X also as a CW-complex with the cellular structure that is induced from the simplicial structure.



The map $f\colon X \to X$ given by $x \mapsto 2x$ is cellular but it is not simplicial. □

Sketch of proof. First note that basically by definition of a simplicial complex it suffices to prove the statement for abstract simplicial complexes and their topological realizations. So let $K = (V, S)$ be an abstract simplicial complex. We write $X := |K|$ and $X^i := |K^i|$. By the definition of a CW-complex in Definition 136.1 we need to prove the following claim:

Claim.

(CW1) We have $X = \operatorname{colim} X^n$, as described in the Topological-Direct Colimit Proposition 104.10. In other words, as a set we have $X = \bigcup_{n\in\mathbb{N}_0} X^n$ and $U \subseteq X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is open in X^n.

(CW2) For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i\colon S^{n-1} \to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism[1865] [1866]

$$X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

Proof.

(CW1) This statement follows basically immediately from the definition of the topology on the topological realization.

(CW2) Let $n \in \mathbb{N}_0$.

(a) Given $n \in \mathbb{N}_0$ let I_n be the set of n-simplices of K. For each $i \in I_n$ let $\Theta_i\colon \Delta^n \to X$ be a characteristic map as defined in Definition 174.18.[1867]

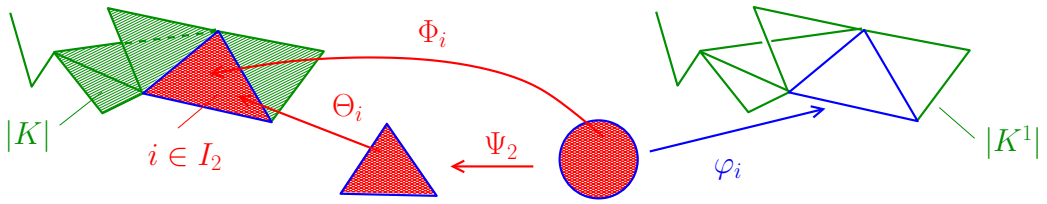
(b) Recall that in the Δ^n-Homeomorphism Lemma 143.9 (2) we had written down a specific homeomorphism $\Psi_n\colon (\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n)$.

(c) Given $i \in I_n$ we write $\Phi_i := \Theta_i \circ \Psi_n\colon \overline{B}^n \to X$ and $\varphi_i := \Phi_i|_{S^{n-1}}\colon S^{n-1} \to X^{n-1}$. Note that φ_i takes values in X^{n-1}.

We now consider the following maps:

$$X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} \partial\Delta^n \,:\, \bigsqcup_{i\in I} \Theta_i} \bigsqcup_{i\in I} \Delta^n \;\to\; X^n$$

[1867] Recall that if K is ordered, then we saw on page 3289 that there is a natural choice of a characteristic map.



Here the map to the left is induced by the identity on X^{n-1} and the maps Ψ_n. Furthermore the map on the right is induced by the identity on X^{n-1} and the map to the right is induced by the characteristic maps Θ_i. It is clear that the maps are bijections.

Finally, using the Topological-Pushout Proposition 8.12 (0) and the Topological-Coproduct Proposition 7.26 (0), together with the definition of the topology on the simplicial complex X^n, one can easily verify that the two maps are homeomorphisms. ⊞

Finally note that it follows almost immediately from the definitions that the CW-structure on X has the promised properties (1)–(4). ■

The Simplicial-Implies-CW Lemma 174.51 allows us to import many previous results about CW-complexes to the context of simplicial complexes. For example, the CW-Complex–Hausdorff Proposition 136.6 gives us a new proof that simplicial complexes are Hausdorff. More interesting of course are new results. The following corollary summarizes some of the nicest and most important ones.

Corollary 174.52. (Simplicial Complex–Topological Properties Corollary)

(1) Simplicial complexes are normal.
(2) Simplicial complexes are locally contractible.[1868]
(3) A simplicial complex is connected if and only if it is path-connected.
(4) Let X be a simplicial complex and let A be a subcomplex. The following statements hold:
  (a) The inclusion $A \to X$ is a cofibration.
  (b) The map

$$\begin{array}{rcl} \varphi\colon \operatorname{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases} \end{array}$$

  is a homotopy equivalence.
  (c) If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.
(5) If X is a finite simplicial complex, then

$$\chi(X) \;=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-simplices of } X.$$

Proof.

[1868] In the rather unlikely event that the reader forgot what these properties are good for, recall that by the Universal Covering Proposition 127.20 this means in particular that any path-connected simplicial complex admits a universal covering. In Exercise 174.10 we will see that any covering of a simplicial complex is again a simplicial complex.

(1),(2) These two statements follow from the Simplicial-Implies-CW Lemma 174.51 together with the CW-Complex–Normal Proposition 136.26 and the CW-Complex–Local Properties Proposition 137.8 (2).

(3) This statement follows from (1) combined with the Locally Path-Connected Lemma 4.57.

(4) (a) This statement follows from the Simplicial-Implies-CW Lemma 174.51 together with the Subcomplex Cofibration Theorem 22.12.

(b) This follows from (a) together with the Cofibration-Quotient Lemma 22.10 (2).

(c) This follows from (a) together with the Cofibration-Quotient Lemma 22.10 (3).

(5) This statement follows from the Simplicial-Implies-CW Lemma 174.51 together with the Homology-χ-Proposition 156.2. ■

Now we can give a partial answer to Question 174.40.

Corollary 174.53. (Simplicial Complex–Quotient Corollary) Let Y be a simplicial complex and let X be a subcomplex. If Y is finite, then there exists a finite simplicial complex, namely $Y \cup_X \operatorname{Cone}(X)$, that is homotopy equivalent to Y/X.



Remark. We leave it as an entertainment for long winter nights to figure out whether the CW×Interval Proposition 137.1 (1) can be used to generalize the Simplicial Complex–Quotient Corollary 174.53 to infinite simplicial complexes. □

Proof. Basically by definition we only need to deal with abstract simplicial complexes and their topological realizations. Thus let $L = (W, T)$ be a finite abstract simplicial complex and let $K = (V, S)$ be a subcomplex. We denote by $i\colon K \to L$ the inclusion map. Recall that i is, basically by definition, simplicial. Since K and L are finite we can consider the abstract simplicial complex $\operatorname{Cone}(i\colon K \to L)$ that we will introduce on Definition 175.18. Now we see that

$$|L|/|K| \xleftarrow[\uparrow]{\simeq} \operatorname{Cone}(|i|\colon |K| \to |L|) \;=\; |L| \cup_{|K|} \operatorname{Cone}(|K|) \xleftarrow[\uparrow]{\cong} \overbrace{|L| \cup_{|K|} |\operatorname{Cone}(K)|}^{=|\operatorname{Cone}(i\colon K \to L)|}$$

by the Simplicial Complex–Topological Properties Corollary 174.52

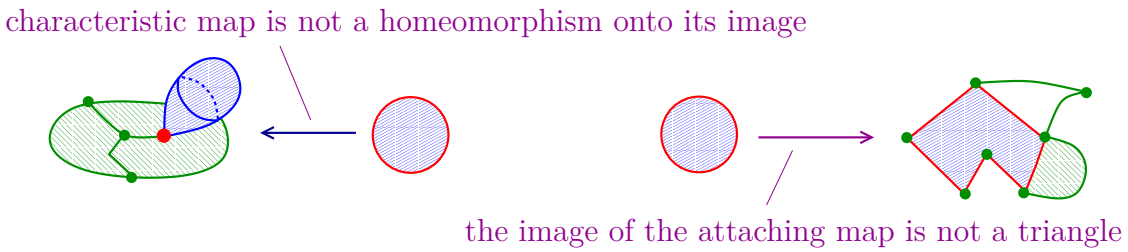
by the Simplicial Cone-Suspension Corollary 174.46 ■

It is pretty clear that there is no naive converse to the Simplicial-Implies-CW Lemma 174.51. For example, the CW-complex constructed in the proof of the Simplicial-Implies-CW Lemma 174.51 has some special properties:

(1) It follows from the Simplex-Embedding Lemma 174.20 (2) that each characteristic map is an embedding.

(2) The CW-structure on the image of a characteristic map of an n-cell is equivalent to the one corresponding to the standard simplicial structure on $\partial\Delta^n$ that we defined on page 3300.

Both types of (counter-) examples are illustrated in the figure below. Thus we see that many, arguably most, CW-structures do not come directly from simplicial structures.



It is perhaps more reasonable to ask, when does a CW-complex admit a simplicial structure? For example it is fairly obvious that the two cellular complexes shown in the figure above admit a simplicial structure. To formulate the relevant proposition we need the following definition:

Definition 174.54. A CW-complex is called regular if every characteristic map is an embedding.

Examples.

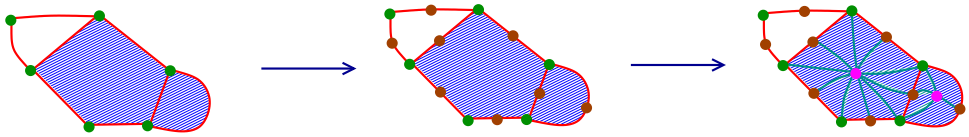
(1) The CW-complex shown in the figure above to the left is non-regular, whereas the CW-complex shown to the right is regular.
(2) Let X be a simplicial complex. We use the Simplicial-Implies-CW Lemma 174.51 to view X as CW-complex. It follows from the Simplex-Embedding Lemma 174.20 (3) that this CW-complex is regular. □

Now we can formulate the promised proposition.

Proposition 174.55. (Regular CW-Complex Proposition) Every (countable) regular CW-complex admits the structure of a (countable) simplicial complex.

Sketch of proof. We will not make use of this theorem, thus we refer to [**Geo2008**, Corollary 5.3.9] or alternatively to [**FrPi1990a**, Theorem 3.4.1] for a proof.

Here we only provide a sketch of a sketch of the proof. Thus let X be a regular CW-complex. We turn inductively the skeleta X^n into simplicial complexes as follows: Suppose we already equipped X^{n-1} with a simplicial structure. For each n-cell of X^n we pick a point in the corresponding open cell and we use it to divide said n-cell into a suitable number of n-simplices.



If the CW-complex X was countable, then it follows from the construction together with the Countability Lemma 1.37 that the simplicial complex is also countable. The reader should have no troubles with turning the above sketch of an argument into a formal proof. ■

The relatively strong hypothesis of the Regular CW-Complex Proposition 174.55 indicates that it is perhaps too much to hope that every CW-complex admits a simplicial structure. Indeed, this fear turns out to be fully justified:

Proposition 174.56. (CW-not-Simplicial Proposition) There exists a finite CW-complex that does not admit a simplicial structure.

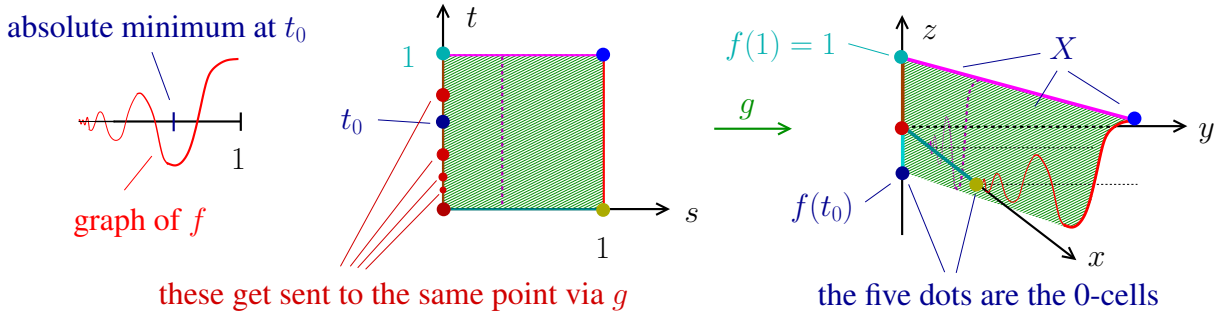
Remark. The CW-not-Simplicial Proposition 174.56 is rather disappointing. Later on in the CW–Simplicial Homotopy Type Proposition 175.21 and the CW-Homotopy Type Theorem 190.9 we will see that every (finite) CW-complex is homotopy equivalent to a (finite) simplicial complex. □

Proof. We consider the map

$$f\colon [0,1] \to \mathbb{R}, \quad t \mapsto \begin{cases} 0, & \text{if } t = 0, \\ t\cdot\cos(\frac{2\pi}{t}) & \text{if } t \in (0,1] \end{cases}$$

and we consider the map $g\colon [0,1]^2 \to \mathbb{R}^3$, $(s,t) \mapsto (s, s\cdot t, f(t))$.

Note that f assumes its maximum at $t = 1$ and that it assumes its minimum at a unique $t_0 \in (0,1)$. We set $X := g([0,1]^2)$. It is not particularly difficult to see that X admits a CW-structure with five 0-cells, five 1-cells and one 2-cell.[1869] We refer to the figure below for an illustration.



In [**FrPi1990a**, p. 128-130] a careful argument is given why X does not admit a simplicial structure. Other examples of finite CW-complexes that do not admit a simplicial structure are given in [**LW1969**, p. 81] (with a very sketchy proof) and [**Met1967**]. ■

174.9. Exercises for Chapter 174 .

Exercise 174.1. For each $g \geq 2$ give an explicit simplicial structure for the surface of genus g.

Exercise 174.2. Give an explicit simplicial structure for $\mathbb{R}P^2$.
Remark. In Exercise 175.5 we will construct an explicit simplicial structure for all $\mathbb{R}P^n$.

Exercise 174.3. In the figure on page 3301 we gave a simplicial structure for the torus $S^1 \times S^1$ with 18 2-simplices. Does there exist a simplicial structure with less than 18 2-simplices?

Exercise 174.4.

(a) Show that the set of isomorphism classes of finite abstract simplicial complexes is countable.

[1869] The five 0-cells and the five 1-cells are clearly indicated in the figure below.

(b) Show that the set of isomorphism classes of countable abstract simplicial complexes is uncountable.

Exercise 174.5. We consider the abstract simplicial complex $K = (V, S)$ which is given by $V = \mathbb{Z} \cup \{*\}$ and $S = \{\{v\} \mid v \in V\} \cup \{\{*, n\} \mid n \in \mathbb{Z}\}$. Show that $|K|$ is not homeomorphic to a linear simplicial complex.
Remark. (V, S) is just the simplicial cone on the abstract simplicial complex with vertex set $\mathbb{Z}$ and which has no simplices of dimension ≥ 1.

Exercise 174.6. Let $m, n \in \mathbb{N}_0$. Show that the simplicial join $D_m * D_n$ of the abstract simplicial complexes D_m and D_n is simplicially isomorphic to D_{m+n+1}.

Exercise 174.7. We consider the simplicial complex $K = (V, S)$ that is given by the vertex set $V = \{\pm 1, \pm 2, \ldots, \pm n\}$ and where S consists of all non-empty subsets of V that do not contain a pair $\{-k, k\}$. What is the homeomorphism type of the topological realization of $K = (V, S)$?

Exercise 174.8. Let K be a finite n-dimensional simplicial complex. Show that K is homeomorphic to a finite linear simplicial complex that is contained in $\mathbb{R}^{2n+1}$.
Remark. In the Non-Planar Graph Proposition 157.19 we already showed that there exist finite 1-dimensional simplicial complexes that cannot be embedded into $\mathbb{R}^2$. In fact the van Kampen-Flores Theorem, which can be proved using the Borsuk-Ulam Theorem 47.12, says that given any $n \in \mathbb{N}$ there exists a finite n-dimensional simplicial complex K that does not admit a map $K \to \mathbb{R}^{2n}$ which is a homeomorphism onto its image. We refer to [**Mato2008**, Theorem 5.1.1] for details.

Exercise 174.9. Let $K = (V, S)$ be an abstract simplicial complex that is finite-dimensional and countable. Furthermore we assume that it is locally finite, which means that each simplex is the face of only finitely many simplices. Show that $|K|$ is homeomorphic to a linear simplicial complex.
Hint. Use the Closed Embedding-to-$\mathbb{R}^n$ Proposition 4.26.

Exercise 174.10. Let K be a simplicial complex and let $p\colon \widetilde{K} \to K$ be a covering in the sense of Definition 91.8. Show that $\widetilde{K}$ admits a structure of a simplicial complex such that p is simplicial.

Exercise 174.11.

(a) Let $K = (V, S)$ and $L = (W, T)$ be two ordered abstract simplicial complexes. Show that the ordered abstract simplicial complex $K \times L$ that we introduced in Definition 174.37, together with the two natural projections $p\colon K \times L \to K$ and $q\colon K \times L \to L$, is the product of K and L in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes.

(b) Is the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes closed under colimits?

Exercise 174.12. Let $K = (V, S)$ be an abstract simplicial complex and let $J = (U, R)$ be a subcomplex such that for every choice of simplices $r \in R$ and $s \in S \setminus R$ we have $r \cap s = \varnothing$. Show that $|J|$ is an open subset of $|K|$.
Remark. Note that by the Simplicial Subcomplex Lemma 174.23 (2) we know that $|J|$ is always a closed subset of $|K|$.

Exercise 174.13. Let $K = (V, S)$ be an abstract simplicial complex. We say that $K = (V, S)$ is *connected* if given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$.[1870] Show that the following statements are equivalent:

(a) The abstract simplicial complex K is connected.

(b) Given any two simplices $t, t' \in S$ there exist simplices $s_0, \ldots, s_k \in S$ such that $t \subseteq s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $t' \in s_k$.

(c) The topological realization $|K|$ is connected.

For completeness we mention that the Simplicial Complex–Topological Properties Corollary 174.52 (3) says that (c) is equivalent to the following statement:

(d) The topological realization $|K|$ is path-connected.

Remark. The equivalence of (a) and (b) is straightforward.

Remark. For the equivalence of (b) and (c) use the Union Connected Lemma 4.39, the Simplicial Subcomplex Lemma 174.23 (2) and Exercise 174.12.

Exercise 174.14. Let $K = (V, S)$ and $L = (W, T)$ be abstract simplicial complexes. We denote by X and Y the corresponding CW-complexes as defined in the Simplicial-Implies-CW Lemma 174.51. Let $\mathcal{CWP}$ be the CW–product topology on $X \times Y$ that we introduced in Definition 139.15. Show that there exists a homeomorphism $|K \times L| \to (X \times Y, \mathcal{CWP})$.

Remark. Note that here we certainly do not assume that K and L are finite.

Exercise 174.15. Let X be a topological space that admits the structure of a 1-dimensional CW-complex. Show that X also admits the structure of a 1-dimensional simplicial complex.

[1870] This definition is modelled on the definition of the connectedness of an abstract graph, see Definition 11.2.

CHAPTER 175

The Simplicial Approximation Theorem

In Chapter 174 we covered in great detail the basics of (abstract) simplicial complexes. In this chapter we will formulate and prove the first major result on simplicial complexes, namely the Simplicial Approximation Theorem.

This chapter is organized as follows:

- In Section 175.1 we will introduce barycentric subdivision of simplicial complexes.
- In Section 175.2 we will introduce stars and links in simplicial complexes.
- In Section 175.3 we will state and prove the Simplicial Approximation Theorem 175.15.
- In Section 175.4 we will introduce mapping cylinders and mapping cones of simplicial maps.
- In Section 175.5 we will see that every CW-complex is homotopy equivalent to a simplicial complex.
- In Section 175.6 we will provide exercises for Chapter 175.

In the next Chapter 176 we will introduce simplicial homology of an abstract simplicial complex.

175.1. **The barycentric subdivision.** In this section we introduce the barycentric subdivision of an abstract simplicial complex. This construction allows us to replace an abstract simplicial complex by a "finer" abstract simplicial complex. This will be a major tool in the proof of several theorems.

Definition 175.1. Let $K = (V, S)$ be an abstract simplicial complex.

(1) For each k-simplex $s = \{v_0, \ldots, v_k\} \in S$ we define the barycenter of s to be the point

$$\underline{s} \coloneqq \sum_{v \in s} \frac{1}{\dim(s)+1} \cdot v = \frac{1}{k+1} \cdot v_0 + \cdots + \frac{1}{k+1} \cdot v_k \in \langle s \rangle \subseteq \mathbb{R}^{(V)}.$$

Note that if $v \in V$ is a vertex, then the barycenter of the corresponding 0-simplex $\{v\}$ is v itself.

(2) Given $s, t \in S$ we write $\underline{s} \leq \underline{t}$ if $s \subseteq t$.

The following lemma summarizes a few properties of the barycenters.

Lemma 175.2. (Barycenter Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) The map $S \mapsto \mathbb{R}^{(V)}$ given by $s \mapsto \underline{s}$ is natural.
(2) For simplices $s, t \in S$ with $s \neq t$ we have $\underline{s} \neq \underline{t}$.
(3) For each 0-simplex s we have $s = \underline{s}$[1871] and for each simplex s of dimension ≥ 1 we have $\underline{s} \notin V$.

[1871] Here we view, as usual, V as a subset of $\mathbb{R}^{(V)}$.

Proof.

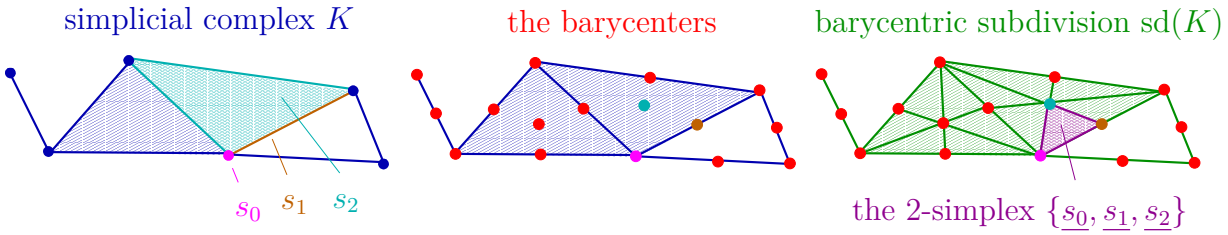
(1) The most difficult part is to figure out what "natural" means in this context. We leave it to the reader to figure this out. Afterwards the statement is basically obvious.
(2) This statement follows from the Open Simplex Lemma 174.28 (2).
(3) This statement is trivial. (But it is nonetheless useful.) ■

The following lemma is much more interesting.

Lemma 175.3. (Barycentric Subdivision Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) The pair
$$\mathrm{sd}(K) \;=\; \big(\{\underline{s}\}_{s\in S}, \{\{\underline{s_0}, \ldots, \underline{s_k}\} \,|\, s_0 \subsetneq \cdots \subsetneq s_k\}\big)$$
is an abstract simplicial complex, called the barycentric subdivision of K.
(2) The relation "$\leq$" on the vertex set $\{\underline{s}\}_{s\in S}$ turns the barycentric subdivision $\mathrm{sd}(K)$ into an *ordered* abstract simplicial complex.

The definition of $\mathrm{sd}(K)$ is illustrated in the figure below.[1872]



Proof. The statements follow immediately from the definitions. ■

Remark.

(1) Perhaps it is helpful to restate the definition of the barycentric subdivision $\mathrm{sd}(K)$ in other words: The vertices of $\mathrm{sd}(K)$ are precisely the barycenters of the original simplices and the simplices of the barycentric subdivision are given by the barycenters of an ascending sequence of simplices in K.
(2) The statement of the Barycentric Subdivision Lemma 175.3 (2) is quite amazing: ex nihilo we created order where there was chaos.
(3) There is an alternative, more formal, but basically equivalent definition of the barycentric subdivision. Namely, given an abstract simplicial complex $K = (V, S)$ one can consider the following abstract simplicial complex:
$$\widetilde{\mathrm{sd}}(K) \;=\; (S, \{\{s_0, \ldots, s_k\} \,|\, s_0 \subsetneq \cdots \subsetneq s_k\}).$$
The map $s \mapsto \underline{s}$ induces a natural isomorphism between the abstract simplicial complex $\widetilde{\mathrm{sd}}(K)$ and the abstract simplicial complex $\mathrm{sd}(K)$. Our first definition is arguably mnemonically superior, the second definition, which is for example used in [**dLon2013**, p. 179], is arguably purer. □

[1872]Here, as on many other occasions, we blur the difference between an abstract simplicial complex, its topological realization, and a linear simplicial complex that is simplicially isomorphic to the topological realization. In particular the illustrations shown in the figure below are dramatically dimensionally reduced.

Notation 175.4. Let $K = (V, S)$ be an abstract simplicial complex. Given any $i \in \mathbb{N}_0$ we refer to

$$\mathrm{sd}^i(K) \coloneqq \underbrace{\mathrm{sd}(\ldots \mathrm{sd}(\mathrm{sd}(K))\ldots)}_{\text{sd applied } i \text{ times}}$$

as the i-th iterated barycentric subdivision of K.

The following lemma summarizes the most important properties of the barycentric subdivision of an abstract simplicial complex.

Lemma 175.5. (SD-Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) If K is finite, then $\mathrm{sd}(K)$ is also finite.

(2) The dimension of $\mathrm{sd}(K)$ equals the dimension of K.

(3) The map

$$\begin{array}{rcl} \mathfrak{V}\colon |\mathrm{sd}(K)| & \to & |K| \\ \alpha & \mapsto & \left(\begin{array}{ccc} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \displaystyle\sum_{s \in S \text{ with } v \in s} \frac{1}{\dim(s)+1} \cdot \alpha(\underline{s}) \end{array} \right) \end{array}$$

($\underline{s}$ is a vertex of $\mathrm{sd}(K)$, hence $\alpha(\underline{s})$ is defined)

has the following properties:

(a) Let σ be a k-simplex of $\mathrm{sd}(K)$. By definition this means that $\sigma = \{\underline{s_0}, \ldots, \underline{s_k}\}$ where $s_0 \subsetneq \cdots \subsetneq s_k$ is an ascending sequence of simplices of K. The restriction of the map $\mathfrak{V}\colon |\mathrm{sd}(K)| \to |K|$ to $|\sigma|$ is given by[1873]

$$\begin{array}{rcl} |\sigma| & \to & |K| \\ \displaystyle\sum_{i=0}^{k} \lambda_i \cdot \underline{s_i} & \mapsto & \displaystyle\sum_{i=0}^{k} \lambda_i \cdot \underline{s_i} \end{array}$$

($\underline{s_i}$: viewed as basis element of $\mathbb{R}^{(\mathrm{sd}(K))}$; $\underline{s_i}$: viewed as vector in $\mathbb{R}^{(V)}$)

In particular we have $\mathfrak{V}(|\sigma|) \subseteq |s_k|$ and $\mathfrak{V}(\langle\sigma\rangle) \subseteq \langle s_k\rangle$.

(b) The map $\mathfrak{V}$ sends each vertex[1874] $\underline{s} \in \mathrm{sd}(K)$ to the corresponding point $\underline{s} \in |K|$.

(c) Given any k-simplex t of K the map $\mathfrak{V}$ restricts to a homeomorphism

$$\bigcup_{\substack{s = \{\underline{a_0}, \ldots, \underline{a_{k-1}}, \underline{t}\} \\ \text{a } k\text{-simplex of } \mathrm{sd}(K) \text{ where} \\ a_0 \subsetneq \cdots \subsetneq a_{k-1} \subsetneq t \\ \text{are simplices of } K}} |s| \xrightarrow{\cong} |t|.$$

(d) The map $\mathfrak{V}$ is a natural homeomorphism.

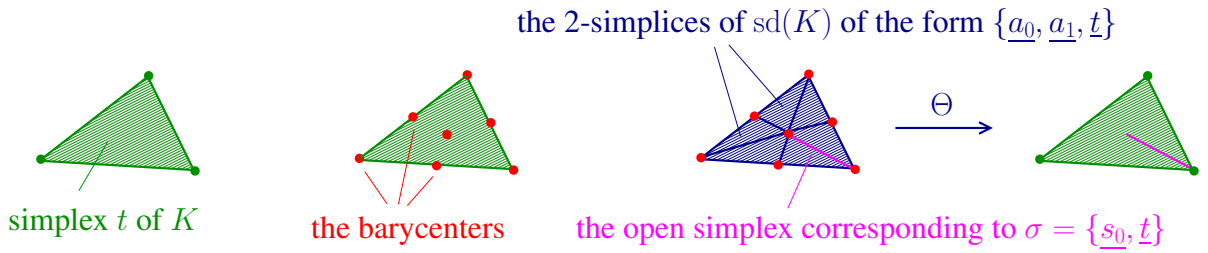
(4) If $f\colon K = (V, S) \to L = (W, T)$ is a simplicial map between two abstract simplicial complexes, then the map

$$\{\underline{s}\} \mapsto \{\underline{f(s)}\}$$

defines a simplicial map $f_*\colon \operatorname{sd}(K) \to \operatorname{sd}(L)$. This map has the property that the following diagram commutes:

$$\begin{array}{ccc} |\operatorname{sd}(K)| & \xrightarrow{\mho_K} & |K| \\ \downarrow{\scriptstyle |f_*|} & & \downarrow{\scriptstyle |f|} \\ |\operatorname{sd}(L)| & \xrightarrow{\mho_L} & |L|. \end{array}$$

(5) The maps from (4) turn $K \mapsto \operatorname{sd}(K)$ into a covariant functor from the category of abstract simplicial complexes to the category of (ordered) abstract simplicial complexes.



Given any abstract simplicial complex K and given any $i \in \mathbb{N}$ we will use the homeomorphism from the SD-Lemma 175.5 (3) to make the identification $|\operatorname{sd}^i(K)| = |K|$.

Proof.

(1) This statement is basically clear.

(2) We will prove this statement in Exercise 175.3.

(3) Now we need to show that $\mho$ has all the promised properties:

(a) Let $\sigma = \{\underline{s_0}, \dots, \underline{s_k}\}$ be a k-simplex of $\operatorname{sd}(K)$. Given a point $P = \sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}$ we calculate that

$$\mho(P) = \mho\Big(\sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}\Big) = \sum_{i=0}^{k} \lambda_i \cdot \mho(\underline{s_i}) = \sum_{i=0}^{k} \lambda_i \cdot \mho\Big(\underbrace{\sum_{v \in s_i} \tfrac{1}{\dim(s_i)+1} \cdot v}_{=\underline{s_i}}\Big) = \sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}.$$

Note that $\underline{s_0}, \dots, \underline{s_k}$ lie in $|s_k|$. Thus it follows from the calculation, together with the remark on page 3311, that $\mho(|\sigma|) \subseteq |s_k|$. Furthermore, if $P \in \langle\sigma\rangle$, then $\lambda_k > 0$, which in turn implies almost immediately that $\mho(P) \in \langle s_k\rangle$.[1875]

(b) This statement follows immediately from (a).

(c) Let t be a k-simplex t of K. We leave it to the reader to verify that $\mho$ restricts to a bijection

$$X := \bigcup_{s=\{\underline{a_0},\dots,\underline{a_{k-1}},\underline{t}\}} |s| \xrightarrow{\cong} |t|.$$

Note that the left-hand side X is actually a finite subcomplex of $|\operatorname{sd}(K)|$. Given a simplex $s = \{\underline{a_0}, \dots, \underline{a_{k-1}}, \underline{t}\}$ we pick a characteristic map $\Phi_s\colon \Delta^k \to |s|$. Furthermore we pick a characteristic map $\Phi_t\colon \Delta^k \to |t|$. It follows almost immediately from the definitions that the map $\Phi_t^{-1} \circ \mho \circ \Phi_s\colon \Delta^k \to \Delta^k$ is an affine linear map, in particular it is continuous. It follows from the Restrict-to-Simplex Lemma 174.21 (1)

[1873] Note that this shows that $\mho$ does take values in $|K|$.

[1874] Here we use the Convention 174.17 that says that we view the vertex of an abstract simplicial complex as a subset of the topological realization.

[1875] For the proof that $\mho(\langle\sigma\rangle) \subseteq \langle s_k\rangle$ it is arguably more efficient to work with the initial description of $\mho$.

and the Simplex-Embedding Lemma 174.20 that the above map $\mho\colon X \to |t|$ is continuous. It now follows, as always, from the Compact-Hausdorff Proposition 4.23 (3) together with the Simplicial Complex-Finiteness Proposition 174.29 and the Simplex-Embedding Lemma 174.20 that the map $\mho\colon X \to |t|$ is a homeomorphism.

(d) Using (c) one can easily verify that $\mho$ is a bijection. Furthermore, it follows from the Restrict-to-Simplex Lemma 174.21 (1) together with (c) that $\mho$ is continuous. Finally, it follows from (c) that for each simplex t of K the map $\mho^{-1}\colon |t| \to |\operatorname{sd}(K)|$ is continuous. By the Restrict-to-Simplex Lemma 174.21 (1) this shows that $\mho^{-1}\colon |K| \to |\operatorname{sd}(K)|$ is continuous. We have thus shown that $\mho$ is indeed a homeomorphism. It follows easily from the definitions and the Barycenter Lemma 175.2 (1) that the homeomorphism is in fact natural.

(4) We leave the elementary task of verifying this statement to the reader.

(5) This statement follows almost immediately from the definitions. The only step which requires a short moment of reflection is to show that if $f\colon K \to L$ is a simplicial map, then the induced map $f_*\colon \operatorname{sd}(K) \to \operatorname{sd}(L)$ is actually order-preserving, i.e. it is a morphism in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes. ■

The following definition is almost self-explanatory.

Definition 175.6. Let $(X, (K, \Theta\colon |K| \to X))$ be a simplicial complex. Furthermore let $i \in \mathbb{N}_0$. We define the i-th barycentric subdivision of (X, Θ) to be the simplicial complex $(X, (\operatorname{sd}^i(K), \Theta\colon |\operatorname{sd}^i(K)| = |K| \to X))$

For completeness we state the following elementary lemma:

Lemma 175.7. (Linear Simplicial Complex–Subdivision Lemma) For every $k \in \mathbb{N}_0$ the barycentric subdivision of a linear simplicial complex in $\mathbb{R}^n$ is again a linear simplicial complex in $\mathbb{R}^n$.

Proof. This statement follows almost immediately from the definition of the barycentric subdivision. ■

175.2. **Stars and links in simplicial complexes.** Given an (abstract) simplicial complex K and a simplex s we introduce and study the corresponding star $\operatorname{St}(K, s)$ and $\operatorname{Lk}(K, s)$. Both concepts are elementary, but both also play a major role in the study of (abstract) simplicial complexes.

Definition 175.8. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $s \in S$ be a simplex. We introduce the following subsets of $|K|$:

$$\text{star } \operatorname{St}(K, s) := \bigcup_{t \in S \text{ with } s \subseteq t} |t| \quad \text{and the} \quad \text{open star } \mathring{\operatorname{St}}(K, s) := \bigcup_{t \in S \text{ with } s \subseteq t} \langle t \rangle.$$

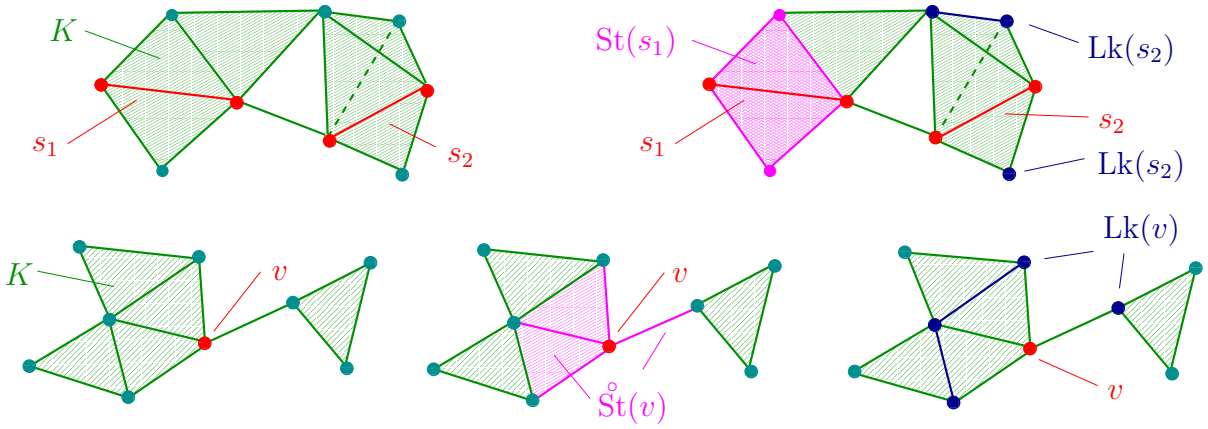
Furthermore we define the link of s to be

$$\operatorname{Lk}(K, s) := \bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} |t|.$$

(2) Given a vertex v we use the obvious shorthand notations

$$\operatorname{St}(K, v) := \operatorname{St}(K, \{v\}), \quad \mathring{\operatorname{St}}(K, v) := \mathring{\operatorname{St}}(K, \{v\}) \text{ and } \operatorname{Lk}(K, v) := \operatorname{Lk}(K, \{v\}).$$

If K is understood, then we drop it from the notation. Furthermore we extend the definition in the obvious way to simplicial complexes.



Example. Let K be an abstract simplicial complex. We consider the simplicial cone $\mathrm{Cone}(K) = K * \{\star\}$ and the simplicial suspension $\Sigma(K) = K * \{\pm 1\}$ as defined on Definition 174.44. Let $\epsilon \in \{-1, 1\}$. It follows easily from the definitions that $\mathrm{St}(\Sigma(K), \epsilon)$ is simplicially isomorphic to $\mathrm{Cone}(K)$ and that $\mathrm{Lk}(\Sigma(K), \epsilon) = K$. □

Lemma 175.9. (Link Star Lemma) Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex.

(1) For every simplex $s \in S$ the following statements hold:
  (a) We have the equality $$\mathrm{Lk}(s) \;=\; \mathrm{St}(s) \setminus \bigcup_{r \in S \text{ with } r \subseteq s} \mathring{\mathrm{St}}(r).$$
  (b) The star $\mathrm{St}(s)$ and the link $\mathrm{Lk}(s)$ are subcomplexes of X. The corresponding abstract simplicial complexes are given by
  $$\Big(\bigcup_{\substack{t \in S \text{ with} \\ s \subseteq t}} t, \{t \in S \mid s \cup t \in S\} \Big) \text{ and } \Big(\bigcup_{\substack{t \in S \text{ with} \\ s \subseteq t}} t \setminus s, \{t \in S \mid s \cup t \in S \text{ and } s \cap t = \varnothing\} \Big).$$
  Furthermore $X \setminus \mathring{\mathrm{St}}(s)$ is a subcomplex, the corresponding abstract simplicial complex is given by $$(V \setminus s, \{t \in S \mid t \cap s = \varnothing\}).$$
  (c) The star $\mathrm{St}(s)$ is naturally simplicially isomorphic to the simplicial join of the link $\mathrm{Lk}(s)$ with the simplex s.[1876] [1877]
  (d) (α) The open star $\mathring{\mathrm{St}}(s)$ is an open neighborhood of every point $P \in \Theta(\langle s \rangle)$, in particular $\mathring{\mathrm{St}}(s)$ is an open subset of X.
  (β) The star $\mathrm{St}(K, s)$ is a closed neighborhood of every point $P \in \Theta(\langle s \rangle)$, in particular $\mathrm{St}(K, s)$ is a closed subset of X.
  (e) (α) The star $\mathrm{St}(s)$ admits a deformation retraction to the simplex $\Theta(|s|)$.
  (β) The open star $\mathring{\mathrm{St}}(s)$ admits a deformation retraction to the open simplex $\Theta(\langle s \rangle)$.
  (f) Both $\mathrm{St}(s)$ and $\mathring{\mathrm{St}}(s)$ are contractible.
  (g) (α) The link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \Theta(|s|)$.

(β) Given any $P \in \Theta(\langle s \rangle)$ the subcomplex $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \{P\}$.

(h) If t is a simplex with $s \subseteq t$, then $\mathrm{St}(t) \subseteq \mathrm{St}(s)$.

(2) (a) Let $s \in S$ be a simplex. For $P \in \Theta(\langle s \rangle)$ and $v \in V$ we have $P \in \mathring{\mathrm{St}}(v)$ if and only if $v \in S$.

(b) The open stars $\{\mathring{\mathrm{St}}(v)\}_{v \in V}$ defined by the vertices are an open cover of X.

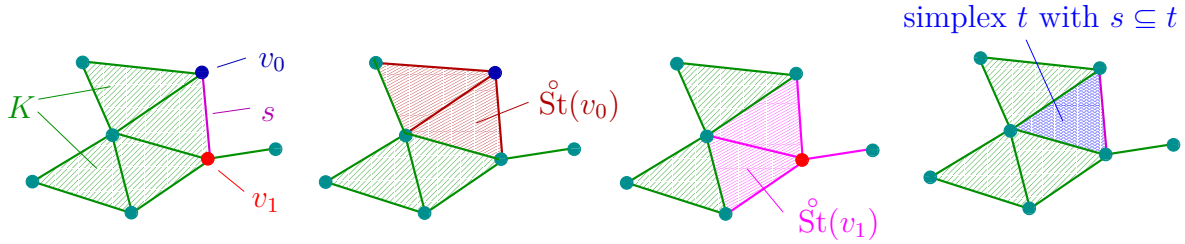
(3) Given simplices $s_0, \ldots, s_n \in S$ the following equality holds:[1878]

$$\mathring{\mathrm{St}}(s_0) \cap \cdots \cap \mathring{\mathrm{St}}(s_n) \;=\; \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \cdots \cup s_n), & \text{if } s_0 \cup \cdots \cup s_n \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

(Note that in Exercise 175.1 we will see that the analogous statement for the stars does *not* hold.)

(4) Given vertices $v_0, \ldots, v_n \in V$ the following statement holds:

$$\mathring{\mathrm{St}}(v_0) \cap \cdots \cap \mathring{\mathrm{St}}(v_n) \neq \varnothing \iff \{v_0, \ldots, v_n\} \text{ is a simplex of } K = (V, S).$$



Proof. Basically by definition we can assume that we are dealing with the topological realization $X = |K|$ of an abstract simplicial complex $K = (V, S)$.

(1) Let $s \in S$ be a k-simplex.

(a) We have the following equalities:

$$\begin{aligned} \mathrm{St}(s) \setminus \bigcup_{r \subseteq s} \mathring{\mathrm{St}}(r) &= \bigcup_{s \subseteq t} |t| \setminus \bigcup_{r \subseteq s} \bigcup_{r \subseteq t} \langle t \rangle \overset{\text{follows from the Open Simplex Lemma 174.28 (5) applied to the subcomplex defined by } t}{=} \bigcup_{s \subseteq t} \bigsqcup_{u \subseteq t} \langle u \rangle \setminus \bigcup_{r \subseteq s} \bigsqcup_{r \subseteq u} \langle u \rangle \\ &= \bigcup_{\substack{u \in S \text{ with } s \cup u \in S \\ \text{but with } s \cap u = \varnothing}} \langle u \rangle = \bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} \bigsqcup_{u \subseteq t} \langle u \rangle = \underbrace{\bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} |t|}_{= \mathrm{Lk}(s)}. \end{aligned}$$

(b) It follows easily from the definitions, (a) and the Simplices-in-Complex Lemma 174.32 (4) that the star $\mathrm{St}(s)$, that the link $\mathrm{Lk}(s)$ and that $|K| \setminus \mathring{\mathrm{St}}(s)$ are subcomplexes of X and that the corresponding abstract simplicial complexes are the ones we wrote down.

[1876] Here we view $\mathrm{St}(s)$, $\mathrm{Lk}(s)$ and s as simplicial complexes.

[1877] In Exercise 175.2 we will prove the related statement that there exists a natural simplicial isomorphism $\mathrm{Lk}(K, s) * \partial s \cong \mathrm{Lk}(\mathrm{sd}(K), \underline{s})$.

[1878] To avoid confusion we point out that here, and in (4), we do not demand that the $s_0, \ldots, s_n$ are distinct.

(c) This statement follows immediately from the description of the abstract simplicial complexes given in (b) and the definition of the simplicial join that we gave in Definition 174.43.

(d) First we show that $\overset{\circ}{\mathrm{St}}(s)$ is an open subset of $|K|$. By the definition of the topology on $|K|$, see Definition 174.19, we need to show that for every k-simplex u, with some characteristic map $\Phi_u\colon \Delta^k \to |u|$, the preimage $\Phi_u^{-1}(\overset{\circ}{\mathrm{St}}(s))$ is an open subset of Δ^k. In fact

$$\Phi_u^{-1}\big(\overset{\circ}{\mathrm{St}}(s)\big) \underset{\text{by Definition 175.8}}{=} \Phi_u^{-1}\Big(\bigcup_{s\subseteq t}\langle t\rangle\Big) \underset{\text{by the Image-Preimage Lemma 1.9 (6)}}{=} \bigcup_{s\subseteq t}\big(\Phi_u^{-1}(\langle u\rangle)\big) \underset{\text{by the Open Simplex Lemma 174.28 (2)}}{=} \begin{cases} \Phi_u^{-1}(\langle t\rangle) = \Delta^k\setminus\partial\Delta^k, & \text{if } s\subseteq u \\ \varnothing, & \text{else.}\end{cases}$$

Since $\Delta^k\setminus\partial\Delta^k$ and $\varnothing$ are open subsets of Δ^k we are done. A very similar argument also shows that $\mathrm{St}(s)$ is a closed subset of $|K|$.
Now let $P\in\Theta(\langle s\rangle)$. By definition $P\in\overset{\circ}{\mathrm{St}}(s)\subseteq\mathrm{St}(s)$. Since $\overset{\circ}{\mathrm{St}}(s)$ is an open subset of $|K|$ we see that $\overset{\circ}{\mathrm{St}}(s)$ and $\mathrm{St}(s)$ are both neighborhoods of P.

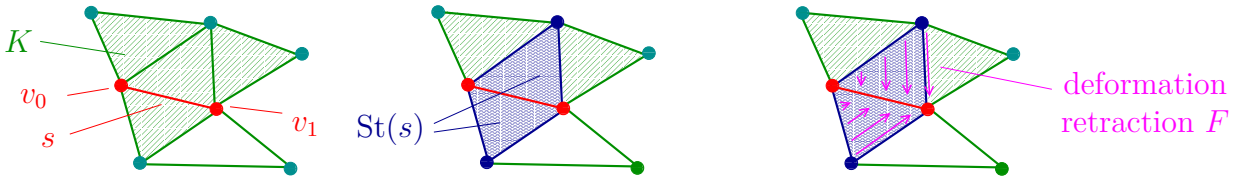
(e) (α) We write $s=\{v_0,\dots,v_k\}$. Note that by (1) and the Simplicial Subcomplex Lemma 174.23 we can view $\mathrm{St}(s)$ as a simplicial complex in its own right. Now let t be a simplex with $s\subseteq t$. We write $t=\{v_0,\dots,v_k,v_{k+1},\dots,v_l\}$. We consider the map

$$\begin{array}{rcl} F_t\colon |t|\times[0,1] & \to & |t| \\ \Big(\sum\limits_{i=0}^{l}\lambda_i\cdot v_i, r\Big) & \mapsto & \sum\limits_{i=0}^{k}\lambda_i\cdot v_i + (1-r)\cdot\sum\limits_{i=k+1}^{l}\lambda_i\cdot v_i + r\cdot\Big(\sum\limits_{i=k+1}^{l}\lambda_i\Big)\cdot v_k. \end{array}$$

It follows immediately from the definitions that these maps F_t combine to give a well-defined map $F\colon \mathrm{St}(s)\times[0,1]\to\mathrm{St}(s)$. Furthermore it is fairly straightforward to deduce from the fact that $\mathrm{St}(s)$ is a simplicial complex, the Simplicial-Implies-CW Lemma 174.51 and the CW×Interval Proposition 137.1 (2) that F is actually continuous. Now that continuity is out of the way it is clear that F is a deformation retraction from $\mathrm{St}(s)$ to $|s|$.

(β) In (α) we just gave an explicit deformation retraction from $\mathrm{St}(s)$ to $|s|$. It follows basically from the definitions that this map restricts to a deformation retraction from $\overset{\circ}{\mathrm{St}}(s)$ to the open simplex $\langle s\rangle$.

(f) This statement follows immediately from (e), together with the Simplex-Embedding Lemma 174.20 (3) and the observation that Δ^k and $\overset{\circ}{\Delta}{}^k$ are contractible.



(g) Let $P\in\Theta(\langle s\rangle)$. The proofs that the link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s)\setminus|s|$ and that the subcomplex $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s)\setminus\{P\}$ are quite similar to the proof of (e). We will fill in the details in Exercise 175.4.

(h) If t is a simplex with $s\subseteq t$, then it follows immediately from the definitions that $\mathrm{St}(t)\subseteq\mathrm{St}(s)$.

(2) (a) Let $s \in S$ be a simplex, let $P \in \langle s \rangle$ and let $v \in V$. We have

$$P \in \mathring{\mathrm{St}}(v) = \bigcup_{t \in S \text{ with } v \in t} \langle t \rangle \iff \langle s \rangle \subseteq \bigcup_{t \in S \text{ with } v \in t} \langle t \rangle \iff v \in s.$$

both statements follow from the Open Simplex Lemma 174.28 (2)

(b) Recall that the Open Simplex Lemma 174.28 (2) says that given any point $P \in K$ there exists an $s \in S$ with $P \in \langle s \rangle$. By (a) we know that for any vertex v of s we have $P \in \langle s \rangle \subseteq \mathring{\mathrm{St}}(v)$. This shows that the open stars corresponding to the vertices cover all of K. By (1d) we know that this cover is in fact an open cover.

(3) Let $s_0, \ldots, s_n \in S$. We have the following equality:

$$\bigcap_{i=0}^{n} \mathring{\mathrm{St}}(s_i) \overset{\text{by Definition 175.8}}{=} \bigcap_{i=0}^{n} \bigcup_{t \in S \text{ with } s_i \subseteq t} \langle t \rangle \overset{\text{by the Open Simplex Lemma 174.28 (5)}}{=} \bigcup_{t \in S \text{ with } s_0, \ldots, s_n \subseteq t} \langle t \rangle \overset{\text{by Definition 175.8 of an open star}}{=} \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \cdots \cup s_n), & \text{if } s_0 \cup \ldots \cup s_n \in S, \\ \varnothing, & \text{otherwise.} \end{cases}$$

(4) Given $v_0, \ldots, v_n \in V$ the following holds:

$$\mathring{\mathrm{St}}(v_0) \cap \cdots \cap \mathring{\mathrm{St}}(v_n) \neq \varnothing \overset{\text{by (3)}}{\iff} \text{there exists a } t \in S \text{ with } v_0, \ldots, v_n \in t \overset{\text{by Definition 174.2 of an abstract simplicial complex}}{\iff} \{v_0, \ldots, v_n\} \text{ is a simplex.}$$

■

Definition 175.10. An abstract simplicial complex $K = (V, S)$ is called locally finite if given any vertex $v \in V$ there exist only finitely many simplices that contain v.

The following proposition can be viewed as an analog of the CW-Complex–Regionally Compact Proposition 136.20.

Proposition 175.11. (Locally Finite–Characterisation Proposition) Let $K = (V, S)$ be an abstract simplicial complex. The following four statements are equivalent:

(1) The abstract simplicial complex K is locally finite.
(2) For every vertex $v \in V$ the star $\mathrm{St}(K, \{v\})$ is a finite simplicial complex.
(3) For every vertex $v \in V$ the link $\mathrm{Lk}(K, \{v\})$ is a finite simplicial complex.
(4) The topological realization $|K|$ is regionally compact.

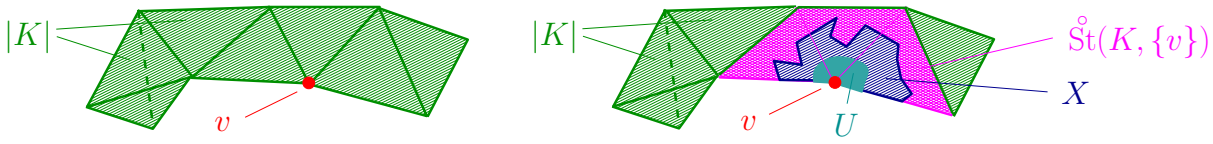
Proof. Let $K = (V, S)$ be an abstract simplicial complex.

We start out with (1)⇔(2)⇔(3). This equivalence of statements follows almost immediately from the definitions and the observation that an abstract simplicial complex is finite if and only if it has finitely many vertices.

We continue with (2)⇒(4). We need to show that $|K|$ is regionally compact. By the Compact Hausdorff-Regionally Compact Lemma 4.58 (2) together with the Simplex-Embedding Lemma 174.20 (2) it suffices to show that every point $P \in |K|$ admits a compact neighborhood. By the Open Simplex Lemma 174.28 there exists a simplex $s \in S$ such that P is contained in the open simplex $\langle s \rangle$. By the Link Star Lemma 175.9 (1d) we know that $\mathrm{St}(K, s)$ is a neighborhood of P. Since $\mathrm{St}(K, s)$ is a finite simplicial complex we obtain from the Finite Simplicial Complex Lemma 174.25 that $\mathrm{St}(K, s)$ is compact.

Finally we prove (4)⇒(2). Now we assume that $|K|$ is regionally compact. Let $v \in V$ be a vertex. By the Link Star Lemma 175.9 (1d) we know that the open star $\mathring{\mathrm{St}}(K, \{v\})$ is an open neighborhood of v. Since $|K|$ is regionally compact there exists a compact neighborhood X of v with $X \subseteq \mathring{\mathrm{St}}(K, \{v\})$. Since X is compact, we know by the Simplicial Complex-Finiteness Proposition 174.29 that there exists a finite subcomplex L of K such that X is contained in $|L|$.

Since X is a neighborhood of v we know that there exists an open neighborhood U of v with $v \in U \subseteq X$.



It follows from the Open Simplex Lemma 174.28 (7) that for any simplex s with $v \in s$ we have $U \cap \langle s \rangle \neq \varnothing$. But by the Open Simplex Lemma 174.28 (5) that implies $|s| \subseteq |L|$. In other words, we see that $\mathrm{St}(K, \{v\}) \subseteq |L|$. Since L is finite we see that $\mathrm{St}(K, \{v\})$ is a finite simplicial complex. ■

We start out by recalling Definition 10.7:

Definition 175.12. The diameter of a non-empty subset A of some $\mathbb{R}^n$ is defined as

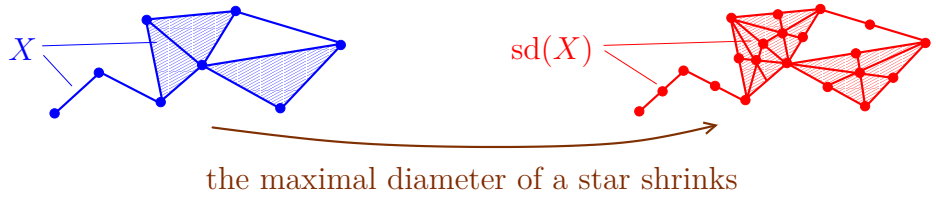
$$\mathrm{diam}(A) \;:=\; \sup\{\|a-b\| \,|\, a,b \in A\} \;\in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

It is pretty clear that the barycentric subdivision of a linear simplicial complex results in more but smaller simplices. The following lemma is one way of formulating precisely that the simplices shrink.

Lemma 175.13. (Diameter Shrinking Lemma) If X is a finite linear non-empty simplicial complex in $\mathbb{R}^n$, then

$$\lim_{k\to\infty} \max\{\mathrm{diam}(\underbrace{\mathrm{St}(\mathrm{sd}^k(X), v)}_{\uparrow}) \,|\, v \text{ is a vertex of } \mathrm{sd}^k(X)\} \;=\; 0.$$

underlying topological space of X and $\mathrm{sd}^k(X)$ is the same



Proof. Let Y be a finite linear non-empty d-dimensional simplicial complex in $\mathbb{R}^n$. We define the *mesh of* Y as

$$\mathrm{mesh}(Y) \;:=\; \max\{\mathrm{diam}(s) \,|\, s \text{ is a simplex of } Y\}.$$

The lemma follows from the SD-Lemma 175.5 (1) and (2), the Linear Simplicial Complex–Subdivision Lemma 175.7 and the following claim:

Claim.

(1) For every vertex v of Y we have $\mathrm{diam}(\mathrm{St}(v)) \leq 2 \cdot \mathrm{mesh}(Y)$.

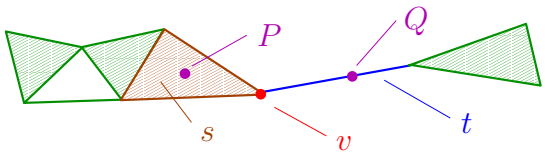
(2) For every $k \in \mathbb{N}_0$ we have

$$\operatorname{mesh}(\operatorname{sd}^k(Y)) \;\leq\; \left(\tfrac{d}{d+1}\right)^k \cdot \operatorname{mesh}(Y).$$

Proof.

(1) Let v be a vertex of Y. We need to show that $\operatorname{diam}(\operatorname{St}(v)) \leq 2 \cdot \operatorname{mesh}(Y)$. Thus let $P, Q \in \operatorname{St}(v)$. By definition of the star $\operatorname{St}(v)$ there exist simplices s and t with $v, P \in s$ and with $v, Q \in t$. We see that

$$\|P-Q\| \underset{\text{triangle inequality}}{\leq} \|P-v\| + \|Q-v\| \underset{\text{since } v,P \in s \text{ and } v,Q \in t}{\leq} \operatorname{diam}(s) + \operatorname{diam}(t) \underset{\text{definition of mesh}}{\leq} 2\operatorname{mesh}(Y).$$



(2) Once the reader has untangled the definitions it will have become clear that this statement is nothing but a reformulation of the Subdivision–Diameter Lemma 148.18. ■

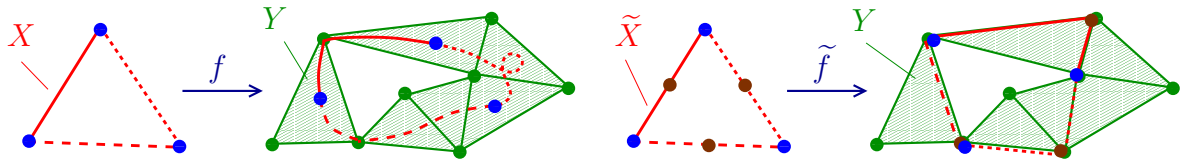
175.3. **The Simplicial Approximation Theorem.** In this section we will state and prove the Simplicial Approximation Theorem which is one of the main tools in the study of simplicial complexes. Before we can state the theorem we need to introduce one last basically self-explanatory notion.

Definition 175.14. Let X be an (abstract) simplicial complex and let A be a subcomplex. We refer to (X, A) as a pair of (abstract) simplicial complexes.

Now we are ready to formulate the main theorem of this section.

Theorem 175.15. (Simplicial Approximation Theorem) Let (X, A) and (Y, B) be pairs of simplicial complexes. Let $f\colon (X, A) \to (Y, B)$ be a continuous map of pairs of topological spaces. If X is finite, then there exists an iterated barycentric subdivision $(\widetilde{X}, \widetilde{A})$ of (X, A) and a simplicial map $\widetilde{f}\colon (\widetilde{X}, \widetilde{A}) \to (Y, B)$ with the following properties:
(1) The map $\widetilde{f}$ is homotopic to f as a map $(X, A) \to (Y, B)$ of pairs of topological spaces.
(2) For every vertex v of $\widetilde{X}$ we have $f(\operatorname{St}(\widetilde{X}, v)) \subseteq \operatorname{St}(Y, \widetilde{f}(v))$.

Example. We consider the simplicial complexes X and Y and the map $f\colon X \to Y$ that are illustrated in the figure below.



This map is actually *not* homotopic to a simplicial map.[1879] But after a single barycentric subdivision we see that f is homotopic to a simplicial map $\widetilde{f}\colon \operatorname{sd}(X) \to Y$. □

Remark.

[1879] Why not?

(1) Statement (1) of the Simplicial Approximation Theorem 175.15 is of course somewhat similar to the statement of the Cellular Approximation Theorem 141.2. But since simplicial complexes and simplicial maps are more rigid than their cellular siblings there are some important differences:
   (a) On the negative side the Simplicial Approximation Theorem 175.15 only applies to finite domains and we need to allow for several barycentric subdivisions.
   (b) On the plus side the output is much stronger since simplicial maps are much more restrictive than cellular maps. We will make use of this shortly.
(2) Statement (2) of the Simplicial Approximation Theorem 175.15 essentially says that the new map $\widetilde{f}$ does not stray too far away from the original map f. □

Before we give a proof of the Simplicial Approximation Theorem 175.15, let us point out an important consequence. On page 3197 we saw that homotopy groups of fairly simple topological spaces, e.g. $S^1 \vee S^2$, can be uncomfortably large, for example we saw that $\pi_2(S^1 \vee S^2)$ is not necessarily finitely generated. In Question 136.24 we asked whether the homotopy groups of a countable CW-complex are actually countable. The following proposition gives us some evidence for a positive answer.

Proposition 175.16. (Simplicial Complex–π_n-Countable Proposition) If X is a simplicial complex with countably many simplices, then for every $n \in \mathbb{N}$ and every $x_0 \in X$ the group $\pi_n(X, x_0)$ is countable.

Proof. Let X be a simplicial complex and let $n \in \mathbb{N}$. By the discussion on page 3302 we can pick a finite simplicial structure for S^n. Let $*$ be a vertex of S^n. By the Change-of-Base Point Proposition 134.12 it suffices to show that for a single vertex x_0 of X the group $\pi_n(X, x_0)$ is countable.

Next note that by the Simplicial Approximation Theorem 175.15, applied to the maps $(S^n, \{*\}) \to (X, x_0)$ we have a surjection

$$\bigcup_{i \in \mathbb{N}_0} \underbrace{\{\text{set of simplicial maps } (\mathrm{sd}^i(S^n), \{*\}) \to (X, x_0)\}}_{\substack{\text{countable set by the Simplicial Maps} \\ \text{Cardinality Lemma 174.10 (2)}}} \twoheadrightarrow \underbrace{[(S^n, *), (X, x_0)]}_{\substack{\text{set of homotopy classes} \\ \text{of continuous maps} \\ (S^n, *) \to (X, x_0)}} = \pi_n(X, x_0).$$

By the Simplicial Maps–Cardinality Lemma 174.10 (2) together with the Countability Lemma 1.37 (4) the set on the left-hand side is countable. It follows from the Countability Lemma 1.37 (4) that the set on the right-hand side, i.e. $\pi_n(X, x_0)$, is also countable. ■

It is not immediately clear how one could possibly generalize the proof of the Simplicial Complex–π_n-Countable Proposition 175.16 from simplicial complexes to CW-complexes. In particular at the moment we still lack the tools to tackle Question 136.24 completely. We will return to Question 136.24 with the CW-Complex–π_n-Countable Proposition 190.8.

Before we turn to the proof of the Simplicial Approximation Theorem 175.15 let us record the following immediate corollary to the Simplicial Complex–π_n-Countable Proposition 175.16.

Corollary 175.17. (Sphere-π_n-Countable Corollary) For every $n, k \in \mathbb{N}$ the group $\pi_n(S^k)$ is countable.

Now we turn to the proof of the Simplicial Approximation Theorem 175.15.

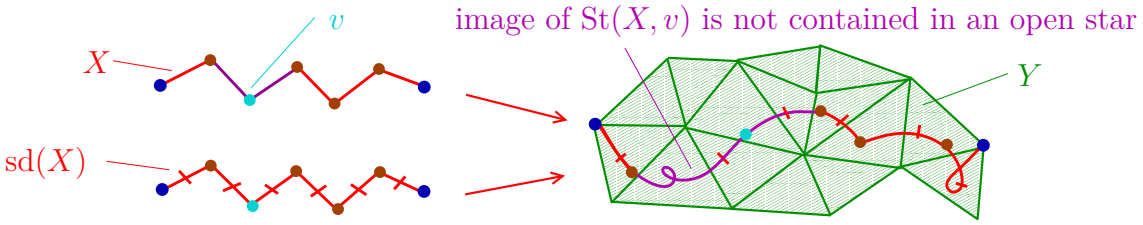
Proof of the Simplicial Approximation Theorem 175.15. In an attempt to keep the notation at a manageable level we will assume for simplicity that $A = B = \varnothing$. We leave it to the reader to modify the subsequent proof to provide a proof for the general case.

Thus let $(X, (K = (V, S), \Theta\colon |K| \to X))$ and $(Y, (L = (W, T), \Omega\colon |L| \to Y))$ be two simplicial complexes. For convenience we identify V with its image $\Theta(V) \subseteq X$ and we identify W with its image $\Omega(W) \subseteq Y$. Furthermore let $f\colon X \to Y$ be a continuous map. We make the following preparations.

(1) Recall that we assume that X is finite. Evidently we can assume that X is non-empty.
(2) It follows from the hypothesis that X is finite and the combination of the Finite Simplicial Complex Lemma 174.25, the Compact Image Lemma 4.15 and the Simplicial Complex-Finiteness Proposition 174.29 that $f(X)$ is contained in a finite subcomplex of Y. Thus we can assume that Y is also a finite simplicial complex.
(3) Since X and Y are finite it follows from the Finite Simplicial Complex–Linear Lemma 174.48 that we can and will assume that X and Y are linear simplicial complexes contained in some $\mathbb{R}^m$ respectively some $\mathbb{R}^n$. Recall that this means, by definition, that X and Y are equipped with the subspace topology coming from $\mathbb{R}^m$ and $\mathbb{R}^n$. In particular we can use the usual Euclidean metric on $\mathbb{R}^m$ to view X as a metric space.

We start out with the following claim:

Claim 1. There exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(X)$ there exists a $w \in W$ with $f(\mathrm{St}(\mathrm{sd}^k(X), v)) \subseteq \mathring{\mathrm{St}}(Y, w)$.



Proof. Recall that by the Link Star Lemma 175.9 (2) we know that the open stars $\{\mathring{\mathrm{St}}(w)\}_{w \in W}$, form an open cover of Y. It follows that the sets $U_w := f^{-1}(\mathring{\mathrm{St}}(w)) \subseteq X$ form an open cover of X. Since X is compact, we obtain from the Lebesgue Lemma 10.8 that there exists an $\epsilon > 0$ such that for every subset A of X with $\mathrm{diam}(A) < \epsilon$ there exists a $w \in W$ with $A \subseteq U_w$. By the Diameter Shrinking Lemma 175.13 there exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(X)$ we have $\mathrm{diam}(\mathrm{St}(\mathrm{sd}^k(X), v)) < \epsilon$. In other words, this k has the desired property. ⊞

By the SD-Lemma 175.5 (1) and the Linear Simplicial Complex–Subdivision Lemma 175.7 we know that $\mathrm{sd}^k(X)$ is again a finite linear simplicial complex in $\mathbb{R}^m$. Thus we can safely replace X by $\mathrm{sd}^k(X)$. We are now in the following setting:

(∗) For every $v \in V$ there exists a $w \in W$ with $f(\mathrm{St}(X, v)) \subseteq \mathring{\mathrm{St}}(Y, w)$. In other words, there exists a map $\varphi\colon V \to W$ with the property that for every vertex $v \in V$ we have $f(\mathrm{St}(X, v)) \subseteq \mathring{\mathrm{St}}(Y, \varphi(v))$.

We continue the proof with the following claim:

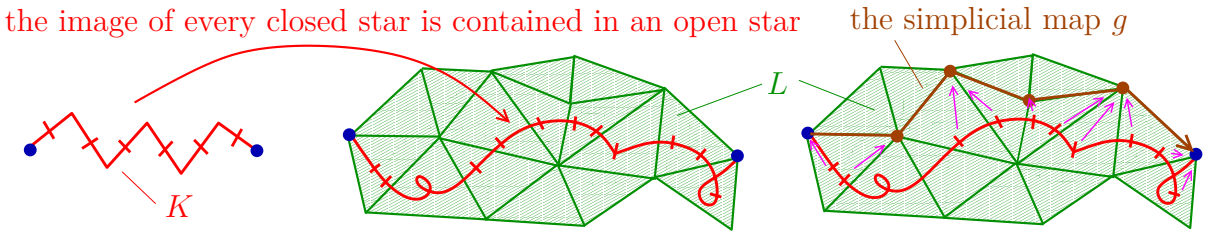
Claim 2. For every simplex $s \in S$ the image $\varphi(s)$ is a simplex in T.

Proof. Let $s = \{v_0, \ldots, v_k\}$ be a simplex of $K = (V, S)$. Note that

$$
\begin{array}{l}
v_0, \ldots, v_k \text{ form} \\
\text{a simplex in } X
\end{array}
\quad
\begin{array}{l}
\text{by the Link Star} \\
\text{Lemma 175.9 (4)} \\
\downarrow \\
\Rightarrow \mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k) \neq \varnothing \\
\Rightarrow f(\mathring{\mathrm{St}}(v_0)) \cap \ldots \cap f(\mathring{\mathrm{St}}(v_k)) \neq \varnothing \\
\uparrow \\
\text{by the Image-Preimage} \\
\text{Lemma 1.9 (4)} \\
\Rightarrow \{\varphi(v_0), \ldots, \varphi(v_k)\} \text{ is a simplex in } T. \\
\uparrow \\
\text{by the Link Star} \\
\text{Lemma 175.9 (4)}
\end{array}
\quad
\begin{array}{l}
\\
\\
\Rightarrow f\big(\mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k)\big) \neq \varnothing \\
\Rightarrow \mathring{\mathrm{St}}(\varphi(v_0)) \cap \ldots \cap \mathring{\mathrm{St}}(\varphi(v_k)) \neq \varnothing \\
\uparrow \\
\text{by choice of } \varphi \text{ we have} \\
f(\mathrm{St}(v_i)) \subseteq \mathring{\mathrm{St}}(\varphi(v_i))
\end{array}
$$

⊞

Claim 2 says that $\varphi\colon V \to W$ defines a simplicial map $K = (V, S) \to L = (W, T)$. It follows from the Induced Simplicial Map Lemma 174.35 that there exists a simplicial map $g\colon X \to Y$ with $g|_V = \varphi$. We claim that g is the desired map.



First note that it follows immediately from the construction of φ and g that for any vertex v of $\mathrm{sd}(X)$ we have $f(\mathrm{St}(X, v)) \subseteq \mathring{\mathrm{St}}(Y, g(v))$. Thus it remains to show that g is in fact homotopic to our original map f. To do so we consider the usual interpolation between the maps f and g:[1880]

$$
\begin{array}{rcl}
F\colon X \times [0,1] & \to & \mathbb{R}^n \\
(x, t) & \mapsto & f(x) \cdot (1 - t) + g(x) \cdot t.
\end{array}
$$

It remains to show the map F actually takes values in the subset $Y \subseteq \mathbb{R}^n$. For each $x \in X$ we know that $f(x) \in Y$ and $g(x) \in Y$. We need to show that for each $t \in [0, 1]$ the point $f(x) \cdot (1 - t) + g(x) \cdot t$ also lies in Y. In our setting this follows from the observation that simplices are convex, see the Linear Simplicial Complex Lemma 174.50 (1b), together with the following claim:

Claim 3. For every $x \in X$ there exists a $t \in T$ such that both $f(x)$ and $g(x)$ are contained in $\Omega(|t|)$.

[1880] Note that we showed in the Convex Homotopy Lemma 18.3 that this map is actually continuous.

Proof. Now let $x \in X$. By the Open Simplex Lemma 174.28 (2) there exists a unique simplex $s = \{v_0, \ldots, v_k\}$ such that $x \in \Theta(\langle s \rangle)$. We see that

$$\begin{array}{l} x \in \Theta(\langle s \rangle) \\ \quad \Rightarrow f(x) \in f\big(\mathring{\mathrm{St}}(v_0)\big) \cap \ldots \cap f\big(\mathring{\mathrm{St}}(v_k)\big) \\ \quad \Rightarrow \text{there exists a } t \in T \text{ which contains } \varphi(s) = \{\varphi(v_0), \ldots, \varphi(v_k)\} \text{ and with } f(x) \in \Omega(\langle t \rangle). \end{array}$$

$\Rightarrow x \in \mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k)$ — by the Link Star Lemma 175.9 (3), since $s = \{v_0, \ldots, v_k\}$

$\Rightarrow f(x) \in f\big(\mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k)\big)$

$\Rightarrow f(x) \in \mathring{\mathrm{St}}(\varphi(v_0)) \cap \ldots \cap \mathring{\mathrm{St}}(\varphi(v_k))$ — by choice of φ we have $f(\mathrm{St}(v_i)) \subseteq \mathring{\mathrm{St}}(\varphi(v_i))$

by the Image-Preimage Lemma 1.9 (4)

by the Link Star Lemma 175.9 (3)

It remains to show that $g(x) \in \Omega(|t|)$. In fact we see that

$$x \in \Theta(\langle s \rangle) \underset{\text{by definition of } \langle s \rangle \text{ and } |s|}{\Longrightarrow} x \in \Theta(|s|) \underset{\text{follows from the discussion on page 3291}}{\Longrightarrow} g(x) \in \Omega(|\varphi(s)|) \underset{\text{since } \varphi(s) \subseteq t}{\Longrightarrow} g(x) \in \Omega(|t|).$$

We have thus found the desired simplex t. ∎

175.4. **Mapping cylinders and mapping cones of simplicial maps.** In this section we will see that there exist simplicial analogs of the mapping cylinder and the mapping cone that we introduced in Definition 21.1 and Definition 21.7. The definitions of these simplicial analogs that we give are taken from [**Kas2014**].

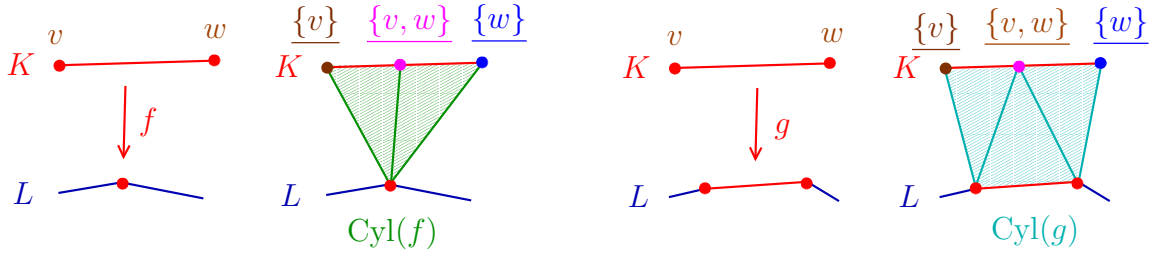
Definition 175.18. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. The simplicial mapping cylinder $\mathrm{Cyl}(f\colon K \to L)$ is the abstract simplicial complex that is given by the following data:

(1) The vertex set is the disjoint union of the vertex set of the barycentric subdivision $\mathrm{sd}(K)$ of K and the vertex set of L.

(2) The set of simplices is given by

$$\left\{ \sigma \sqcup \tau \;\middle|\; \begin{array}{l} \sigma = \varnothing \text{ or } \sigma \text{ is a simplex of } \mathrm{sd}(K) \\ \text{and } \tau = \varnothing \text{ or } \tau \text{ is a simplex of } L \text{ such that for every} \\ \text{vertex } v = \underset{\text{where } s \text{ is a simplex of } K}{\underset{\uparrow}{\underline{s}}} \in \sigma \text{ and every vertex } w \in \tau \text{ we have } w \in f(s) \end{array} \right\} \setminus \{\varnothing\}.$$

Note that $\mathrm{sd}(K)$ and L are naturally subcomplexes of $\mathrm{Cyl}(f\colon K \to L)$.



Let $f\colon K \to L$ be a simplicial map between two finite abstract simplicial complexes. One might expect that the next lemma states, say under the hypothesis that K is finite, that the topological realization $|\operatorname{Cyl}(f\colon K \to L)|$ of the simplicial mapping cylinder is homeomorphic to the mapping cylinder $\operatorname{Cyl}(|f|\colon |K| \to |L|)$ of the map between the topological realizations. We will prove a weaker statement, but which nonetheless says that the simplicial mapping cylinder has all the essential properties we really need.

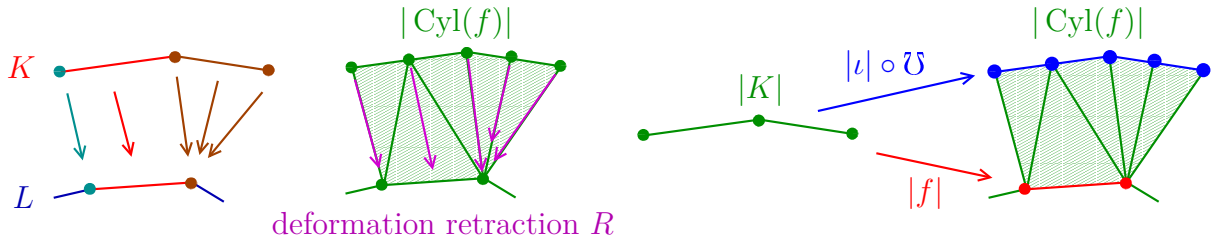
Lemma 175.19. (Simplicial Mapping Cylinder Lemma) Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes.

(1) The simplicial mapping cylinder $\operatorname{Cyl}(f\colon K \to L)$ contains $\operatorname{sd}(K)$ and L as subcomplexes.

We now assume that K is actually a finite abstract simplicial complex. We denote by $\iota\colon \operatorname{sd}(K) \to \operatorname{Cyl}(f\colon K \to L)$ the natural inclusion and we denote by $\mho\colon |K| \to |\operatorname{sd}(K)|$ the natural homeomorphism from the SD-Lemma 175.5.

(2) There exists a deformation retraction $R\colon |\operatorname{Cyl}(f)| \times [0,1] \to |\operatorname{Cyl}(f)|$ from $|\operatorname{Cyl}(f)|$ to the subset $|L|$ such that $R_1 \circ |\iota| \circ \mho = f\colon |K| \to |\operatorname{Cyl}(f)|$.

(3) The map $|f|\colon |K| \to |\operatorname{Cyl}(f)|$ is homotopic to the map $|\iota| \circ \mho\colon |K| \to |\operatorname{Cyl}(f)|$.



Remark. It is not clear to me whether the statement of the Simplicial Mapping Cylinder Lemma 175.19 (2) also holds if K is infinite. For example one might be able to get stronger results using the Infinite Homotopies–CW–Lemma 140.19. But for our purposes it is good enough to know that the statement holds if K is finite. □

Sketch of proof. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes.

(1) By taking $\tau = \varnothing$ respectively $\sigma = \varnothing$ one sees immediately that $\operatorname{sd}(K)$ and L are subcomplexes of $\operatorname{Cyl}(f\colon K \to L)$.

We now assume that K is actually a finite abstract simplicial complex.

(2) Given $r \in \mathbb{N}_0$ we denote by W_r the topological realization of the maximal subcomplex of $\operatorname{Cyl}(f\colon K \to L)$ that contains all vertices of L and that contains all vertices $\operatorname{sd}(K)$ of the form $\underline{s}$ where s is a $\geq r$-simplex of K. We leave it to the reader to verify that for each $r \in \mathbb{N}_0$ there exists a unique homotopy

$$F_r\colon W_r \times [0,1] \;\to\; W_r$$

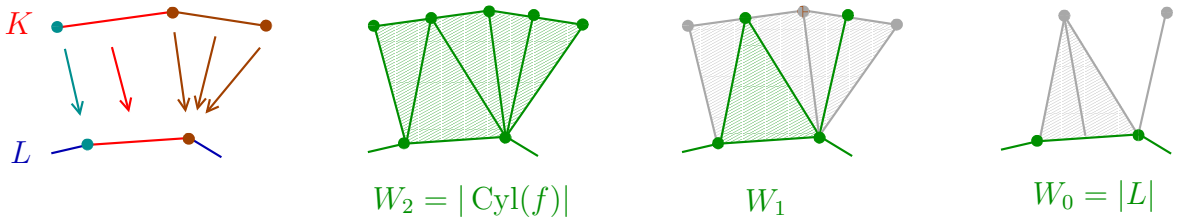
with the following properties:

(a) F_r is a homotopy rel W_{r-1}.

(b) Given any vertex $\underline{s}$ that lies in W_r but that does not lie in W_{r-1} the homotopy is given by $F_r(\underline{s},t) = \underline{s} \cdot (1-t) + \underline{f(s)} \cdot t$.

(c) The restriction of F_r to each simplex takes values in a simplex and it is linear.

Note that F_r is in particular a deformation retraction from W_r to W_{r-1}. The composition of the homotopies $F_{\dim(K)}, \dots, F_1$ and F_0 has the desired properties.

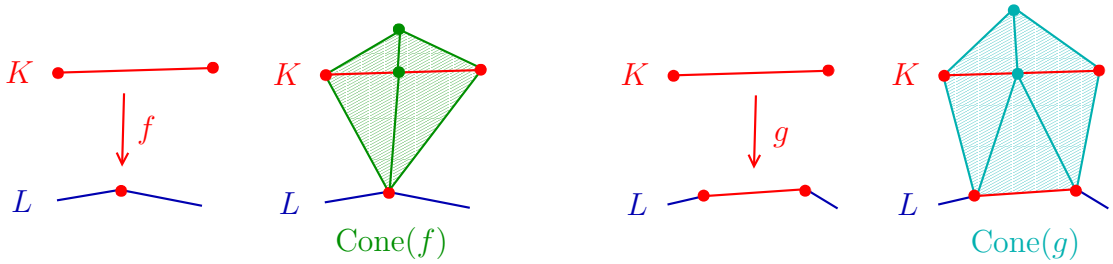


(3) This statement follows immediately from (2). ■

We conclude this section with the following definition:

Definition 175.20. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. The simplicial mapping cone $\operatorname{Cone}(f\colon K \to L)$ is defined as the union of the simplicial mapping cylinder $\operatorname{Cyl}(f\colon K \to L)$ and the simplicial cone $\operatorname{Cone}(\operatorname{sd}(K))$ that we introduced in Definition 174.44. In other words, we set

$$\operatorname{Cone}(f\colon K \to L) \;:=\; \operatorname{Cyl}(f\colon K \to L) \cup_{\operatorname{sd}(K)} \operatorname{Cone}(\operatorname{sd}(K))$$



In Section 175.5 we will see that the concept of a simplicial mapping cone is indeed useful.

175.5. **Simplicial structures and CW-structures II.** Recall that in the Simplicial-Implies-CW Lemma 174.51 we saw that every (ordered) simplicial complex admits a (natural) CW-complex structure where given any $n \in \mathbb{N}$ the n-simplices of the simplicial structure are precisely the n-cells of the CW-structure. The converse of this statement does not hold, in fact in the CW-not-Simplicial Proposition 174.56 we showed that there exists a CW-complex that does not admit a simplicial structure.

Nonetheless, in this section we will prove a partial converse to the Simplicial-Implies-CW Lemma 174.51, namely we will show that every CW-complex is *homotopy equivalent* to a simplicial complex. More precisely, we will prove the following proposition:

Proposition 175.21. (CW–Simplicial Homotopy Type Proposition)

(1) Every finite CW-complex is homotopy equivalent to a finite simplicial complex of the same dimension.
(2) Every countable CW-complex is homotopy equivalent to a countable simplicial complex of the same dimension.
(3) Every CW-complex is homotopy equivalent to a simplicial complex of the same dimension.

Remark. Alternative proofs for the CW–Simplicial Homotopy Type Proposition 175.21 are given in [**Hat2002**, Theorem 2C.5], [**LW1969**, Theorem 6.1], [**WhdJ1949b**, Theorem 13] and [**GrayB1975**, Corollary 16.44]. □

Most of the work in the proof of the CW–Simplicial Homotopy Type Proposition 175.21 is done in the following lemma:

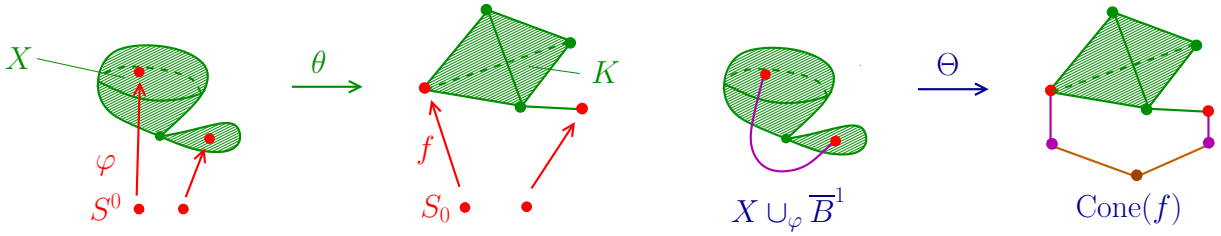
Lemma 175.22. (Simplicial Cell Replacement Lemma) Let X be a CW-complex. We suppose that there exists an abstract simplicial complex K and a continuous map

$$\theta\colon X \to |K|$$

which is a homotopy equivalence. Given any continuous map $\varphi\colon S^{k-1} \to X$ there exists an $i \in \mathbb{N}_0$, a simplicial map $f\colon \mathrm{sd}^i(S_{k-1}) \to K$ and a continuous map

$$\Theta\colon X \cup_\varphi \overline{B}^k \to | \underbrace{\mathrm{Cone}(f\colon \mathrm{sd}^i(S_{k-1}) \to K))}_{=\mathrm{Cyl}(f\colon \mathrm{sd}^i(S_{k-1})\to K)\cup_{\mathrm{sd}^{i+1}(S_{k-1})}\mathrm{Cone}(\mathrm{sd}^{i+1}(S_{k-1}))} |$$

which is a homotopy equivalence and which, restricted to X, is given by θ.



There are various ways how one can prove the Simplicial Cell Replacement Lemma 175.22. One approach would be to use the down-to-earth Homotopy Equivalent Cell Attachment Proposition 135.13. For no particular reason we will prove the lemma using the more general Homotopy Pushout Theorem 22.24, which we now recall for the reader's convenience.

Theorem 22.24. (Homotopy Pushout Theorem) Suppose we are given the following commutative diagram of continuous maps between topological spaces:

$$\begin{array}{ccccc} Y & \xleftarrow{f} & A & \xhookrightarrow{i} & X \\ {\scriptstyle\varphi_Y}\downarrow & & \downarrow{\scriptstyle\varphi_A} & & \downarrow{\scriptstyle\varphi_X} \\ Y' & \xleftarrow{f'} & A' & \xhookrightarrow{i'} & X'. \end{array}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the right are closed cofibrations, then the induced map

$$Y \cup_A X \to Y' \cup_{A'} X'$$

between the pushouts is a homotopy equivalence.

Proof of the Simplicial Cell Replacement Lemma 175.22. We start out with the following preparations:

(1) By the discussion on page 3294 there exists a homeomorphism $g\colon |D_k| \to \overline{B}^k$ which restricts to a homeomorphism $g\colon |S_{k-1}| \to S^{k-1}$.
(2) Given any abstract simplicial complex L and given any $i \in \mathbb{N}_0$ we always denote by $\mho\colon |L| \to |\mathrm{sd}^i(L)|$ the homeomorphism from the SD-Lemma 175.5 (3).

(3) By the Simplicial Approximation Theorem 175.15 there exists an $i \in \mathbb{N}_0$, a simplicial map $f\colon \operatorname{sd}^i(S_{k-1}) \to K$ and a homotopy $F\colon |\operatorname{sd}^i(S_{k-1})| \times [0,1] \to |K|$ such that $F_0 = \theta \circ \varphi \circ g \circ \mathfrak{V}^{-1}$ and such that $F_1 = |f|$.

(4) We denote by $j\colon |K| \to |\operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)|$ the inclusion map. Note that by the Simplicial Mapping Cylinder Lemma 175.19 (2) we know that j is a homotopy equivalence.

(5) It follows immediately from the Simplicial Mapping Cylinder Lemma 175.19 (3) that there exists a homotopy $G\colon |\operatorname{sd}^i(S_{k-1})| \times [0,1] \to |\operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K))|$ with $G_0 = j \circ |f|$ and such that $G_1 = |\iota| \circ \mathfrak{V}$ where $\iota\colon \operatorname{sd}^{i+1}(S_{k-1}) \to \operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)$ is the natural inclusion.

We consider the following commutative diagram:

$$
\begin{array}{ccccc}
X & \xleftarrow{\varphi} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \cong\, g^{-1}} & & \downarrow{\scriptstyle \cong\, g^{-1}} \\
|K| & \xleftarrow{\theta\circ\varphi\circ g} & |S_{k-1}| & \hookrightarrow & |\operatorname{Cone}(S_{k-1})| \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cong\, \mathfrak{v}} & & \downarrow{\scriptstyle \cong\, \mathfrak{v}} \\
|K| & \xleftarrow{\theta\circ\varphi\circ g\circ\mathfrak{V}^{-1}} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle x\mapsto(x,0)} \\
|K| & \xleftarrow{F} & |\operatorname{sd}^i(S_{k-1})| \times [0,1] & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \times [0,1] \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle x\mapsto(x,1)} \\
|K| & \xleftarrow{|f|} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
|\operatorname{Cyl}(f)| & \xleftarrow{j\circ|f|} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle x\mapsto(x,0)} \\
|\operatorname{Cyl}(f)| & \xleftarrow{G} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \times [0,1] \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle x\mapsto(x,1)} \\
|\operatorname{Cyl}(f)| & \xleftarrow{|\iota|\circ\mathfrak{V}} & |\operatorname{sd}^i(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cong\, \mathfrak{v}} & & \downarrow{\scriptstyle \cong\, \mathfrak{v}} \\
|\operatorname{Cyl}(f)| & \xleftarrow{|\iota|} & |\operatorname{sd}^{i+1}(S_{k-1})| & \hookrightarrow & |\operatorname{Cone}(\operatorname{sd}^{i+1}(S_{k-1}))|.
\end{array}
$$

It follows from (4) and elementary observations that all of the vertical maps are homotopy equivalences. Furthermore note that it follows immediately from the Manifold Boundary-Cofibration Proposition 22.11 and the Product-Cofibration Proposition 22.15 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 22.24, applied altogether eight times, that the induced maps between the various pushouts have homotopy inverses. Combining these eight homotopy equivalences, respectively their homotopy inverses, we obtain from Homotopy-Equivalence Basics Lemma 19.14 (2) the promised homotopy equivalence

$$\Theta\colon X \cup_{S^{k-1}} \overline{B}^k \;\to\; \underbrace{|\operatorname{Cyl}(f)| \cup_{|\operatorname{sd}^{i+1}(S_{k-1})|} |\operatorname{Cone}(\operatorname{sd}^{i+1}(S_{k-1}))|}_{=|\operatorname{Cone}(f\colon \operatorname{sd}^i(S_{k-1})\to K)|}.$$

■

Proof of Proposition 175.21 (1). Let X be a *finite* CW-complex. An induction argument over the number of cells, using the Simplicial Cell Replacement Lemma 175.22, shows that X is homotopy equivalent to a finite simplicial complex of the same dimension. ■

Sketch of a proof of Proposition 175.21 (2) and (3). Now let X be any (countable) CW-complex. Without loss of generality we can and will assume that X is connected and non-empty.

Claim 1. There exists an ascending sequence $K_{-1} \subseteq K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ of abstract simplicial complexes and a sequence of homotopy equivalences $\theta_n \colon X^n \to |K_n|$, $n \in \mathbb{Z}_{\geq -1}$ such that for each $n \in \mathbb{N}_0$ the maps $X^n \xrightarrow{\theta_n} |K_n| \to |K_{n+1}|$ and $X^n \to X^{n+1} \xrightarrow{\theta_{n+1}} |K_{n+1}|$ agree.

Proof. We take K_{-1} to be the empty simplicial complex. Assume that we have already constructed $K_{-1}, \ldots, K_n$ and $\theta_{-1}, \ldots, \theta_n$. We denote by $\{\varphi_i \colon S^n \to X^n\}_{i \in I}$ the attaching maps of the $(n+1)$-cells of X. By basically the same argument as in the proof of the Simplicial Cell Replacement Lemma 175.22 we obtain natural numbers $d_i \in \mathbb{N}_0$, we obtain simplicial maps $f_i \colon \mathrm{sd}^{d_i}(S_n) \to K_n$ and we obtain a homotopy equivalence

$$\theta_{n+1} \colon \underbrace{X^n \cup_{\bigsqcup \varphi_i} \bigsqcup_{i \in I} \overline{B}^{n+1}}_{=X^{n+1}} \to \underbrace{\left| \mathrm{Cyl}\left(\bigsqcup_{i \in I} f \colon \bigsqcup_{i \in I} \mathrm{sd}^{d_i}(S_n) \to K_n \right) \cup \bigsqcup_{i \in I} \mathrm{Cone}\left(\mathrm{sd}^{d_i+1}(S_n) \right) \right|}_{=:K_{n+1}}$$

such that the maps $X^n \xrightarrow{\theta_n} |K_n| \to |K_{n+1}|$ and $X^n \to X^{n+1} \xrightarrow{\theta_{n+1}} |K_{n+1}|$ are homotopic. ⊞

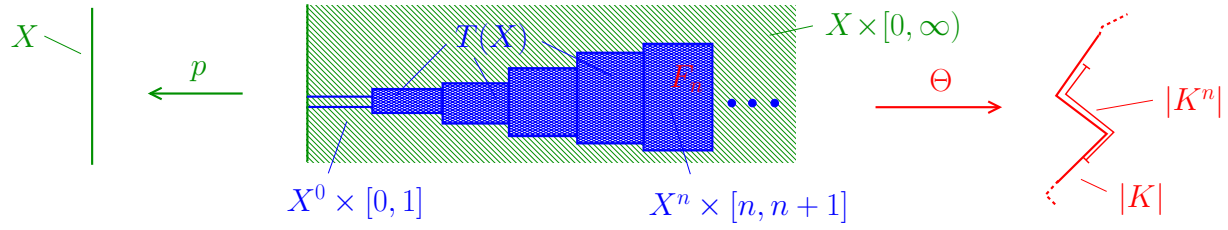
We set $K := \bigcup_{n \in \mathbb{N}_0} K_n$. If X is countable, then K is also countable. Thus it remains to show that X is homotopy equivalent to $|K|$.

(1) As in Definition 140.20 we equip $[0, \infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$. As in Definition 140.20 we define the *telescope of* X as the CW-complex

$$T(X) := \bigcup_{n \in \mathbb{N}_0} (X^n \times [n, n+1]) \quad \subseteq \quad X \times [0, \infty).$$

(2) The maps $X^n \times [n, n+1] \to X^n \xrightarrow{\theta_n} |K|$ define (by the compatibility of the maps θ^n) a well-defined map $\Theta \colon T(X) \to |K|$. It follows almost immediately from the Characteristic Maps Proposition 136.9 (2) that Θ is continuous.

(3) Furthermore the projections $X^n \times [n, n+1] \to X^n$ define a map $p \colon T(X) \to X$ which is also continuous by the Characteristic Maps Proposition 136.9 (2).



By the Homotopy-Equivalence Basics Lemma 19.14 (2) it now remains to prove the following claim:

Claim 2. The maps $\Theta\colon T(X) \to |K|$ and $p\colon T(X) \to X$ are homotopy equivalences.

Proof. In the following we will first deal with the map $\Theta\colon T(X) \to |K|$. For $m \in \mathbb{N}_0$ we set $T_m(X) := \bigcup_{i=0}^{m} (X^i \times [i, i+1])$. We pick $P \in X^0 \times \{0\}$. For each $k \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccc} \operatorname{colim}_m \pi_k(T_m(X), P) & \xrightarrow{\cong} & \pi_k(T(X), P) \\ \downarrow & & \downarrow{\scriptstyle \Theta_*} \\ \operatorname{colim}_m \pi_k(|K^m|, \Theta(P)) & \xrightarrow{\cong} & \pi_k(|K|, \Theta(P)). \end{array}$$

We make the following observations:

(1) It follows from the CW-Direct Colimit-π_n-Proposition 136.25 (which we can apply by the Simplicial-Implies-CW Lemma 174.51 (2)) that the horizontal maps are isomorphisms.

(2) The left vertical map is induced by the maps $\Theta|_{T_m(X)}$. Note that the diagram commutes by the universal properties of direct colimits.

(3) For each m we have an obvious deformation retraction ρ from $\bigcup_{i=0}^{m} (X^i \times [i, i+1])$ to $X^m \times \{m+1\}$. We denote by $\pi\colon X^m \times \{m+1\} \to X^m$ the obvious homeomorphism. Note that $\Theta|_{T_m(X)} = \theta_m \circ \pi \circ \rho$. Since θ_m, π and ρ are homotopy equivalences we see that $\Theta|_{T_m(X)}\colon T_m(X) \to |K^m|$ is a homotopy equivalence. It follows from the Homotopy Equivalence-π_n-Proposition 134.17 that each map $\pi_k(T_m(X), P) \to \pi_k(|K^m|, \Theta(P))$ is an isomorphism. It now follows that the left vertical map is an isomorphism.

It follows from the above discussion that the map to the right is an isomorphism for every $k \in \mathbb{N}_0$. Since we assume that $|K|$ is connected and since $T(X)$ and $|K|$ are CW-complexes it now follows from the Whitehead Theorem 305.1 that the map $\Theta\colon T(X) \to |K|$ is indeed a homotopy equivalence.

The same argument also shows that the map $p\colon T(X) \to X$ is a homotopy equivalence.[1881] ∎

175.6. Exercises for Chapter 175 .

Exercise 175.1. Let $K = (V, S)$ be an abstract simplicial complex. By the Link Star Lemma 175.9 (3) we know that for any choice of simplices $s_0, \ldots, s_n \in S$ the following equality regarding *open* stars holds:

$$\mathring{\mathrm{St}}(s_0) \cap \cdots \cap \mathring{\mathrm{St}}(s_n) = \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \cdots \cup s_n), & \text{if } s_0 \cup \cdots \cup s_n \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Show that in general the analogous statement for stars does *not* hold.

Exercise 175.2. Let K be an abstract simplicial complex and let s be a k-simplex. Show that there exists a natural simplicial isomorphism $\mathrm{Lk}(K, s) * \partial s \cong \mathrm{Lk}(\mathrm{sd}(K), \underline{s})$.

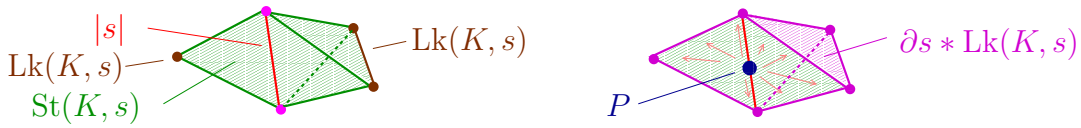
Exercise 175.3. Let $K = (V, S)$ be an abstract simplicial complex. Show that the dimension of the barycentric subdivision $\mathrm{sd}(K)$ equals the dimension of K.

[1881] Alternatively this statement is a consequence of the Telescope Lemma 140.21.

Exercise 175.4. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.

(a) Show that the link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus |s|$.
Hint. Consider the proof of the Link Star Lemma 175.9 (e).

(b) Let $P \in \langle s \rangle$. Show that $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \{P\}$.
Hint. Turn the idea of "pushing away from P" into a rigorous argument.



Exercise 175.5. Let $n \in \mathbb{N}$. As on page 3283 we consider the abstract simplicial complex $S_n := (V, T)$ with $V = \{0, \dots, n+1\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$. Note that given a k-simplex s of S_n the set $V \setminus s$ is an $(n-k)$-simplex. Next we consider the barycentric subdivision $\mathrm{sd}(S_n)$ together with the map

$$\begin{array}{rccc} f\colon & \mathrm{sd}(S_n) & \to & \mathrm{sd}(S_n) \\ & \underline{s} & \mapsto & \underline{V \setminus s}. \end{array}$$

(a) Show that f is a simplicial map.

(b) Show that $S_n/s \sim f(s)$, interpreted in the obvious way, is an abstract simplicial complex with $2^{n+1} - 1$ vertices.

(c) Sketch the abstract simplicial complex $S_n/s \sim f(s)$ for $n = 1$ and $n = 2$.

(d) Show that the topological realization of $S_n/s \sim f(s)$ is homeomorphic to the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$.

Remark. It is not known what is the "smallest" simplicial structure for the real projective spaces $\mathbb{R}\mathrm{P}^n$. It is shown in [**ArM1991**] that any simplicial structure for $\mathbb{R}\mathrm{P}^n$ needs to have at least $\frac{1}{2}(n+1)(n+2)$ vertices. But all known simplicial structures seem to have about 2^n vertices.

Exercise 175.6. Let $n \in \mathbb{N}$ and let $k \in \{1, \dots, n-1\}$. Use the Simplicial Approximation Theorem 175.15 to show that $\pi_k(S^n) = 0$.
Remark. This statement is of course the content of the Sphere–Low-π_n-Proposition 134.19 which we proved using the Cellular Approximation Theorem 141.2.

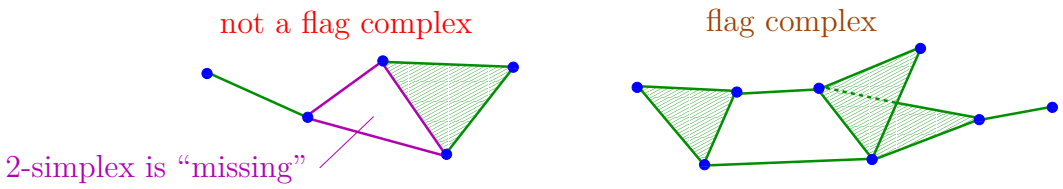
Exercise 175.7. We say that an abstract simplicial complex $K = (V, S)$ is a *flag complex* if the following condition is satisfied:

(∗) If $v_0, \dots, v_n \in V$ have the property that $\{v_i, v_j\} \in S$ for all $i, j \in \{0, \dots, n\}$, then $\{v_0, \dots, v_n\} \in S$.

More loosely speaking, K is a flag complex, if a simplex exists whenever its 1-skeleton exists. We refer to the figure below for an illustration of the definition.

(a) Show that every flag complex is determined by its 1-skeleton. More precisely, let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. Suppose that $\varphi\colon K^1 \xrightarrow{\cong} L^1$ is a simplicial isomorphism between the 1-skeleta. We suppose that K and L are flag complexes. Show that φ can be extended to a simplicial isomorphism $K \xrightarrow{\cong} L$.

(b) Let K be an abstract simplicial complex. Show that the barycentric subdivision $\mathrm{sd}(K)$ is a flag complex.



Exercise 175.8. First we recall Definition 15.1: We say that a family $\{U_i\}_{i\in I}$ of subsets of a set X has *order* $\leq n$ if for any $J \subseteq I$ with $\#J > n$ we have $\bigcap_{j\in J} U_j = \varnothing$.

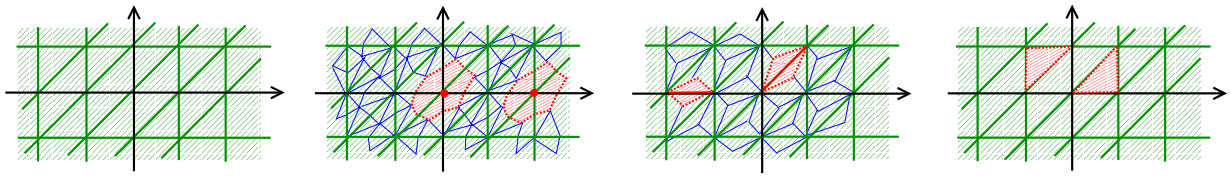
(a) Let $K = (V, S)$ be an n-dimensional abstract simplicial complex. Given $i \in \{0, \dots, n\}$ we denote by K_i the abstract simplicial complex one obtains from barycentric subdivision of all simplices of degree $> i$.[1882] As usual we use (a slight generalization of) the SD-Lemma 175.5 to identify the various topological realizations, i.e. we write $|K| = |K_0| = \cdots = |K_n|$. We denote by $\mathcal{V}_i$ the collection of open stars of the i-simplices of K in $|K_i|$. Show that the sets in $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n$ form an open cover of $|K| = |K_0| = \cdots = |K_n|$ of order $n + 1$.

Next we recall another Definition 15.2. Let X be a non-empty topological space. We say that $\operatorname{cdim}(X) \leq n$ if every open cover of X admits an open refinement of order $n + 1$.

(b) Show that every finite simplicial complex X of dimension n has $\operatorname{cdim}(X) \leq n$.

(c) Show that every compact subspace X of $\mathbb{R}^n$ has $\operatorname{cdim}(X) \leq n$.

Remark. This approach gives a new proof for the Compact Euclidean Subset-Covering Dimension Theorem 15.5.



[1882] We did not define this construction, but the reader should have no troubles making sense of it.

CHAPTER 176

Simplicial homology

In this chapter we will introduce the simplicial homology groups of an (ordered) abstract simplicial complex. As we will see, the definition is refreshingly straightforward. Later in the chapter we discuss the relationship between simplicial and singular homology.

This chapter is organized as follows:

- In Section 176.1 we recall the definition of the sign of a permutation.
- In Section 176.2 we will introduce simplicial homology of an ordered abstract simplicial complex.
- In Section 176.3 we will introduce simplicial homology for "unordered" abstract simplicial complexes.
- In Section 176.4 we will introduce the simplicial cross product and we will state and prove the Simplicial Künneth Theorem 176.29.
- In Section 176.5 we will provide exercises for Chapter 176.

In the next Chapter 177 we will study the relationship between simplicial and singular homology.

176.1. **The sign of a permutation.** Before we get into details it is convenient to recall the following definition from basic group theory which we will need on numerous occasions throughout this chapter:

Definition 176.1. Let $n \in \mathbb{N}$. Given a permutation $\sigma \in S_n$ we refer to[1883]

$$\operatorname{sign}(\sigma) \coloneqq \begin{array}{c}\text{determinant of the } (n \times n)\text{-matrix for which the} \\ (i, \sigma(i))\text{-entries are one and all the other entries are zero}\end{array}$$

as the sign of the permutation.

The following lemma summarizes the two key properties of the sign of a permutation.

Lemma 176.2. (Sign-of-Permutation Lemma) Let $n \in \mathbb{N}$.
(1) The map $\operatorname{sign}\colon S_n \to \{-1, 1\}$ is a homomorphism.
(2) If σ is a transposition, then $\operatorname{sign}(\sigma) = -1$.

Proof. The proof follows immediately from standard properties of the determinant. ■

Definition 176.3. Let X be a finite set and let $\sigma \in \operatorname{Bij}(X)$. We set $n \coloneqq \#X$. We pick a bijection $\varphi\colon \{1, \dots, n\} \to X$. We define $\operatorname{sign}(\sigma) = \operatorname{sign}(\varphi^{-1} \circ \sigma \circ \varphi)$. It follows easily from the Sign-of-Permutation Lemma 176.2 that this definition is independent of the choice of the bijection φ.

[1883] Surely the reader will have seen the definition of the sign of a permutation. We leave it to the reader to show that the previous definition agrees with our definition.

Throughout this chapter we adopt the following convention.

Convention 176.4. Given $n \in \mathbb{N}$ we identify the permutation group $S_n = \mathrm{Bij}\{1, \ldots, n\}$ with the group $\mathrm{Bij}\{0, \ldots, n-1\}$ in the obvious way.

176.2. **Simplicial homology of an ordered abstract simplicial complex.** In this section we will define the simplicial chain complex and the simplicial homology groups of an *ordered* abstract simplicial complex. This is going to be surprisingly easy. In the subsequent section we will then deal with abstract simplicial complexes that are not equipped with an order.

We start out with the following elementary observation which we will use frequently throughout this chapter.

Observation. Let $(T, \leq)$ be a totally ordered set and let $v_0, \ldots, v_n \in T$. By Exercise 1.5 there exists a unique permutation $\sigma \in S_{n+1}$ such that $v_{\sigma(0)} < \cdots < v_{\sigma(n)}$.

After this preliminary remark we can now turn to an interesting definition.

Definition 176.5. Let $K = (V, S)$ be an ordered abstract simplicial complex.[1884] Given $n \in \mathbb{N}_0$ we define

$$\mathrm{C}_n^{\mathrm{simp},\leq}(K) \coloneqq \text{free abelian group generated by the set of } n\text{-simplices of } K.$$

Furthermore we consider the boundary map

$$\begin{aligned} \partial_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) &\to \mathrm{C}_{n-1}^{\mathrm{simp},\leq}(K) \\ \underbrace{\{v_0 < \cdots < v_n\}}_{\substack{\text{any } n\text{-simplex } s \text{ is uniquely} \\ \text{of the form } s = \{v_0, \ldots, v_n\} \\ \text{with } v_0 < \cdots < v_n}} &\mapsto \sum_{i=0}^{n} (-1)^i \cdot \{v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n\}. \end{aligned}$$

Example. Let $K = (V, S)$ be an ordered abstract simplicial complex and let $s = \{v_0, v_1\}$ be a 1-simplex with $v_0 < v_1$. Then

$$\partial_1(\{v_0 < v_1\}) \;=\; \{v_1\} - \{v_0\}. \qquad \square$$

Lemma 176.6. (Simplicial Boundary Lemma) Let $K = (V, S)$ be an ordered abstract simplicial complex. Given any $n \in \mathbb{N}_0$ we have $\partial_{n-1} \circ \partial_n = 0$. In other words, $(C_*^{\mathrm{simp},\leq}(K), \partial_*)$ is a chain complex.

Proof. The proof of the lemma is almost verbatim the same as the proof of the Double Boundary Proposition 143.14. We leave it to the reader to make the necessary minute modifications. ■

The reader is by now well aware of the fact that such a lemma leads straight to the following definition:

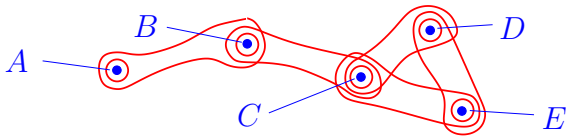
Definition 176.7. Let K be an ordered abstract simplicial complex. We refer to the chain complex $(\mathrm{C}_*^{\mathrm{simp},\leq}(K), \partial_*)$ as the simplicial chain complex. Given $n \in \mathbb{N}_0$ we define the n-th

[1884] Recall that according to Definition 174.12 "ordered" means that the vertex set V is equipped with a partial order such that the restriction to each simplex is a total order.

simplicial homology group of K as the n-th homology group $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$ of the simplicial chain complex, i.e. we define

$$\mathrm{H}_n^{\mathrm{simp},\leq}(K) \;:=\; \frac{\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp},\leq}(K)\big)}{\mathrm{im}\,\big(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(K)\big)}.$$

Example. We consider the 1-dimensional abstract simplicial complex $K = (V, S)$ shown in the figure below. It has five vertices $\{A, B, C, D, E\}$, it has the corresponding five 0-simplices and it has five 1-simplices.



We view $K = (V, S)$ as an ordered abstract simplicial complex in the obvious way, i.e. we have $A < B < C < D < E$. It follows from the example on page 3344 that the corresponding simplicial chain complex is given as follows:

$$0 \;\to\; \begin{matrix}\mathbb{Z}\cdot\{A,B\}\\ \mathbb{Z}\cdot\{B,C\}\\ \mathbb{Z}\cdot\{C,D\}\\ \mathbb{Z}\cdot\{D,E\}\\ \mathbb{Z}\cdot\{C,E\}\end{matrix} \xrightarrow{\begin{pmatrix}-1&0&0&0&0\\1&-1&0&0&0\\0&1&-1&0&-1\\0&0&1&-1&0\\0&0&0&1&1\end{pmatrix}} \begin{matrix}\mathbb{Z}\cdot\{A\}\\ \mathbb{Z}\cdot\{B\}\\ \mathbb{Z}\cdot\{C\}\\ \mathbb{Z}\cdot\{D\}\\ \mathbb{Z}\cdot\{E\}\end{matrix} \;\to\; 0.$$

We leave it as an elementary exercise in linear algebra to show that

$$\mathrm{H}_n^{\mathrm{simp},\leq}(K) \;\cong\; \begin{cases} 0, & \text{if } n > 1,\\ \mathbb{Z}, & \text{if } n = 0, 1.\end{cases}$$

Furthermore one can easily show that

$$\mathrm{H}_1^{\mathrm{simp},\leq}(K) \;=\; \ker\big(\partial_1\colon \mathrm{C}_1^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(K)\big) \;=\; \mathbb{Z}\cdot[\{C,D\} + \{D,E\} - \{C,E\}]. \quad \square$$

As always, not only do we want to associate groups to our given objects, but we really want to construct a covariant functor. This is done in the following lemma:

Lemma 176.8. (Ordered Simplicial Chain Complex–Functor Lemma)

(1) Given an order-preserving simplicial map $f\colon K = (V, S) \to L = (W, T)$ between ordered abstract simplicial complexes we define

$$\begin{aligned} f_*\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) &\to \mathrm{C}_n^{\mathrm{simp},\leq}(L)\\ \{v_0,\ldots,v_n\} &\mapsto \begin{cases}\{f(v_0),\ldots,f(v_n)\}, & \text{if } f(v_0),\ldots,f(v_n) \text{ are pairwise different,}\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

These maps form a chain map, thus they induce maps $f_*\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n^{\mathrm{simp},\leq}(L)$.

(2) The maps

$$\begin{aligned} (K,\leq) &\mapsto \mathrm{C}_*^{\mathrm{simp},\leq}(K)\\ (f\colon (K,\leq) \to (L,\leq)) &\mapsto (f_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*^{\mathrm{simp},\leq}(L))\end{aligned}$$

define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to the category $\mathcal{ChCplx}$ of chain complexes. In particular, for each

$n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl}(K,\leq) & \mapsto & \mathrm{H}_n^{\mathrm{simp},\leq}(K) \\ (f\colon (K,\leq)\to(L,\leq)) & \mapsto & (f_*\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K)\to \mathrm{H}_n^{\mathrm{simp},\leq}(L))\end{array}$$

define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. The statements follow easily from the definitions. We leave it to the reader to fill in the details. ■

Naturally the question arises, to what degree do the simplicial homology groups of an ordered abstract simplicial complex depend on the choice of the order. As the following lemma shows, the answer is, not very much.

Lemma 176.9. (Independence-of-Order Lemma) Let $K = (V, S)$ be an abstract simplicial complex and let "$\leq$" and "$\widetilde{\leq}$" be two orders on K. The maps

$$\begin{array}{rcl}\mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\widetilde{\leq}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & \underbrace{\mathrm{sign}(\sigma)\cdot\{v_{\sigma(0)}\widetilde{\leq}\ldots\widetilde{\leq}v_{\sigma(n)}\}}_{\substack{\sigma\in S_n \text{ is the unique permutation} \\ \text{such that } v_{\sigma(0)}\widetilde{\leq}\ldots\widetilde{\leq}v_{\sigma(n)}}}\end{array}$$

define an isomorphism of chain complexes.

Proof. The proof of the lemma is an appealing little calculation that will be done in Exercise 176.1. ■

176.3. **Simplicial homology.** In the last section we introduced simplicial homology for *ordered* abstract simplicial complexes. In this section we will introduce simplicial homology for "unordered" abstract simplicial complexes. Evidently the definition is going to be very similar, but as we will see, the definition is ever so more subtle.

Definition 176.10. Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$.

(1) An ordered n-simplex of K is an $(n+1)$-tuple $(v_0, \ldots, v_n) \in V^{n+1}$ such that $\{v_0, \ldots, v_n\}$ is an n-simplex of $K = (V, S)$.

(2) Given an ordered n-simplex $(v_0, \ldots, v_n)$ we denote by $\lfloor (v_0, \ldots, v_n) \rfloor := \{v_0, \ldots, v_n\}$ the corresponding n-simplex of K.

(3) Let $n \in \mathbb{N}_0$. We define

$$\widehat{\mathrm{C}}_n^{\mathrm{simp}}(K) := \text{free abelian group generated by the set of ordered } n\text{-simplices of } K.$$

Furthermore we define the n-th simplicial chain group to be

$$\mathrm{C}_n^{\mathrm{simp}}(K) := \widehat{\mathrm{C}}_n^{\mathrm{simp}}(K)/I_n$$

where I_n is the subgroup of $\widehat{\mathrm{C}}_n^{\mathrm{simp}}(K)$ generated by all elements of the form

$$(v_0, \ldots, v_n) - \mathrm{sign}(\sigma)\cdot(v_{\sigma(0)}, \ldots, v_{\sigma(n)}) \quad \text{where } \sigma \in S_{n+1} = \mathrm{Bij}\{0, \ldots, n\}$$

Given an ordered simplex $(v_0, \ldots, v_n)$ we denote its image in $\mathrm{C}_n^{\mathrm{simp}}(K)$ by $[v_0, \ldots, v_n]$.

Remark. Let $K = (V, S)$ be an abstract simplicial complex. Since the set of vertices is in a natural bijection with the set of 0-simplices and since the permutation group S_1 is trivial we see that we have a natural identification $\mathrm{C}_0^{\mathrm{simp}}(K) = \mathbb{Z}^{(V)}$. □

Lemma 176.11. (Simplicial Chain Group–Isomorphism Lemma) Let $K = (V, S)$ be an abstract simplicial complex. If "$\leq$" is an order on $K = (V, S)$, then for each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & [v_0, \ldots, v_n] \end{array}$$

is a natural isomorphism.

Proof. Let $n \in \mathbb{N}_0$. First note that it follows immediately from the definitions that the map is natural.

It remains to show that the given map is an isomorphism. We denote by P_n the set of ordered n-simplices. Note that by definition we have $\widehat{\mathrm{C}}_n^{\mathrm{simp}}(K) = \mathbb{Z}^{(P_n)}$. In the following we denote by Q_n the set of all ordered n-simplices $(v_0, \ldots, v_n)$ such that $v_0 < \cdots < v_n$. By the Free Abelian Group–Existence Lemma 97.8 there exist unique homomorphisms as follows:

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^{(P_n)} & \to & \mathbb{Z}^{(Q_n)} \\ (v_0, \ldots, v_n) & \mapsto & \mathrm{sign}(\sigma) \cdot (v_{\sigma(0)}, \ldots, v_{\sigma(n)}) \end{array}$$

here $\sigma \in S_{n+1}$ is the unique permutation with $v_{\sigma(0)} < \cdots < v_{\sigma(n)}$

and

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}^{(Q_n)} & \to & \mathbb{Z}^{(P_n)} \\ (v_0, \ldots, v_n) & \mapsto & (v_0, \ldots, v_n). \end{array}$$

Furthermore we denote by $\pi\colon \mathbb{Z}^{(P_n)} \to \mathbb{Z}^{(P_n)}/I_n = \mathrm{C}_n^{\mathrm{simp}}(K)$ the obvious projection. It follows easily from the Sign-of-Permutation Lemma 176.2 that $\Phi\colon \mathbb{Z}^{(P_n)} \to \mathbb{Z}^{(Q_n)}$ descends to a homomorphism $\overline{\Phi}\colon \mathbb{Z}^{(P_n)}/I_n \to \mathbb{Z}^{(Q_n)}$. Note that it is an immediate consequence of the definitions that $\overline{\Phi} \circ \pi \circ \Psi$ is the identity on the basis Q_n of $\mathbb{Z}^{(Q_n)}$ and that $\pi \circ \Psi \circ \overline{\Phi}$ is the identity on the generating set $\pi(P_n)$ of $\mathbb{Z}^{(P_n)}/I_n$. This shows that $\pi \circ \Psi$ and $\overline{\Phi}$ are isomorphisms. The promised statements follow readily from this observation. ■

Lemma 176.12. (Simplicial Boundary Lemma) Let K be an abstract simplicial complex.

(1) Given any $n \in \mathbb{N}_0$ the group $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a free abelian group where the rank is given by the cardinality of the set S_n of n-simplices of K. In fact, if we choose for each n-simplex $s \in S_n$ an ordered n-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$, then $\{[\widetilde{s}]\}_{s \in S_n}$ is a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$.

(2) Let $n \in \mathbb{N}$. There exists a unique homomorphism

$$\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$$

with the property that given any ordered n-simplex $(v_0, \ldots, v_n)$ of K we have

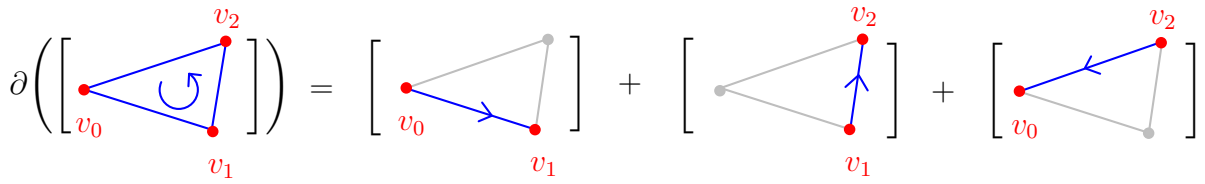
$$\partial_n([v_0, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i \cdot [v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n].$$

(3) For every $n \in \mathbb{N}_0$ we have $\partial_{n-1} \circ \partial_n = 0$.

Remark. Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. The Simplicial Chain Group–Isomorphism Lemma 176.11 and the Simplicial Boundary Lemma 176.12 (1) say in particular that $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a free abelian group. If $K = (V, S)$ is an ordered abstract simplicial complex, then this free abelian group has a natural basis. In the absence of an order there is only a natural basis where each element of the basis is well-defined "up to sign". □

Example. For an ordered simplex (v_0, v_1, v_2) in an abstract simplicial complex K we see that

$$\partial([v_0, v_1, v_2]) \;=\; [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \;=\; [v_0, v_1] + [v_1, v_2] + [v_2, v_0].$$



In other words, if we consider the figure above, then we see that the boundary is exactly what we would naively had thought. □

Proof.

(1) It follows from the Well-Ordering Theorem 1.32 that we can equip $K = (V, S)$ with an order. Thus the desired statement is now an almost immediate consequence of the Simplicial Chain Group–Isomorphism Lemma 176.11

(2) Let $n \in \mathbb{N}$. By the Free Abelian Group–Existence Lemma 97.8 there exists a unique homomorphism

$$\partial_n \colon \widehat{\mathrm{C}}_n^{\mathrm{simp}}(K) \;\to\; \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$$

with the property that given any ordered n-simplex $(v_0, \ldots, v_n)$ of K we have

$$\partial_n((v_0, \ldots, v_n)) \;=\; \sum_{i=0}^{n} (-1)^i \cdot [v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n].$$

It remains to show that ∂_n factors through $\mathrm{C}_n^{\mathrm{simp}}(K) = \widehat{\mathrm{C}}_n^{\mathrm{simp}}(K)/I_n$. In other words, we need to show that ∂_n vanishes on I_n. Since the permutation group S_n is generated by transpositions and since transpositions have the sign -1 it remains to prove the following claim:

Claim. For any $0 \le k < l \le n$ and any ordered n-simplex $(v_0, \ldots, v_n)$ of K we have

$$\partial_n(v_0, \ldots, v_l, \ldots, v_k, \ldots, v_n) \;=\; -\partial_n(v_0, \ldots, v_k, \ldots, v_l, \ldots, v_n).$$

Proof. Let $(v_0, \ldots, v_n)$ be an ordered n-simplex and let $0 \le k < l \le n$. We calculate that

$$\begin{aligned}
&\partial_n(.., v_l, .., v_k, ..) \;= \\
&= \sum_{i \neq k,l} (-1)^i \cdot \underbrace{[.., v_l, .., \widehat{v_i}, .., v_k, ..]}_{= -[.., v_k, .., \widehat{v_i}, .., v_l, ..]} + (-1)^k \cdot \underbrace{[.., \widehat{v_l}, .., v_k, ..]}_{\substack{= (-1)^{l-k-1} \cdot [., v_k, \ldots, \widehat{v_l}.] \\ \text{since } l-k-1 \text{ transpositions} \\ \text{are used}}} + (-1)^l \cdot \underbrace{[.., v_l, .., \widehat{v_k}, ..]}_{\substack{= (-1)^{l-k-1} \cdot [., \widehat{v_k}, \ldots, v_l.] \\ \text{since } l-k-1 \text{ transpositions} \\ \text{are used}}} \\
&= -\sum_{i \neq k,l} (-1)^i \cdot [.., v_k, .., \widehat{v_i}, .., v_l, ..] - (-1)^l \cdot [.., v_k, .., \widehat{v_l}, ..] - (-1)^k \cdot [.., \widehat{v_k}, .., v_l, ..] \\
&= -\partial_n(.., v_k, .., v_l, ..).
\end{aligned}$$

⊞

(3) The proof of this lemma is almost verbatim the same as the proof of the Double Boundary Proposition 143.14. We leave the pleasant task of copy-pasting the proof to the reader. ■

The Simplicial Boundary Lemma 176.12 leads us right away to the following definition:

Definition 176.13. Let K be an abstract simplicial complex. We refer to the chain complex $(\mathrm{C}_*^{\mathrm{simp}}(K), \partial_*)$ as the simplicial chain complex. Given $n \in \mathbb{N}_0$ we define the n-th simplicial homology group of K as the n-th homology group of the simplicial chain complex, i.e. we define

$$\mathrm{H}_n^{\mathrm{simp}}(K) \;:=\; \frac{\ker\left(\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(K)\right)}.$$

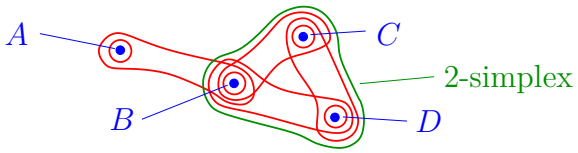
Examples.

(1) We consider the 2-dimensional abstract simplicial complex $K = (V, S)$ shown in the figure below. It has four vertices $\{A, B, C, D\}$, it has the corresponding four 0-simplices, it has four 1-simplices and it has one 2-simplex. The corresponding simplicial chain complex is given as follows:

$$0 \;\to\; \mathbb{Z}\cdot[B,C,D] \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot[A,B] \\ \mathbb{Z}\cdot[B,C] \\ \mathbb{Z}\cdot[B,D] \\ \mathbb{Z}\cdot[C,D] \end{matrix} \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot[A] \\ \mathbb{Z}\cdot[B] \\ \mathbb{Z}\cdot[C] \\ \mathbb{Z}\cdot[D] \end{matrix} \;\to\; 0.$$

Elementary linear algebra shows that

$$\mathrm{H}_n^{\mathrm{simp}}(K) \;\cong\; \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$



(2) If $K = (V, S)$ is an abstract simplicial complex with no simplices of dimension ≥ 1, then it follows immediately from the definitions that for $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{simp}}(K) \;\cong\; \begin{cases} \mathrm{C}_0^{\mathrm{simp}}(K) = \mathbb{Z}^{(V)}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

□

The following is the obvious analog of the Ordered Simplicial Chain Complex–Functor Lemma 176.8.

Lemma 176.14. (Simplicial Chain Complex–Functor Lemma)

(1) Given a simplicial map $f \colon K = (V, S) \to L = (W, T)$ we define

$$\begin{aligned} f_* \colon \mathrm{C}_n^{\mathrm{simp}}(K) &\to \mathrm{C}_n^{\mathrm{simp}}(L) \\ [v_0, \ldots, v_n)] &\mapsto \begin{cases} [f(v_0), \ldots, f(v_n)], & \text{if } f(v_0), \ldots, f(v_n) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

These maps form a chain map, thus they induce maps $f_* \colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)$.

(2) The maps

$$\begin{aligned} K &\mapsto \mathrm{C}_*^{\mathrm{simp}}(K) \\ (f \colon K \to L) &\mapsto (f_* \colon \mathrm{C}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(L)) \end{aligned}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{ChCplx}$ of chain complexes. For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} K & \mapsto & \mathrm{H}_n^{\mathrm{simp}}(K) \\ (f\colon K \to L) & \mapsto & (f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)) \end{array}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. As in the case of the proof of the Ordered Simplicial Chain Complex–Functor Lemma 176.8 we leave it to the reader to provide the proof. ■

For completeness we state the following elementary lemma:

Lemma 176.15. (Two Flavors-of-Simplicial Chain Complexes Lemma) Suppose we are given an ordered abstract simplicial complex $K = (V, S)$. The maps

$$\begin{array}{rcl} \Omega_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & [v_0, \ldots, v_n] \end{array}$$

define a natural isomorphism of chain complexes. In particular they induce natural isomorphisms

$$\Omega_*\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n^{\mathrm{simp}}(K).$$

Remark. In the following we will discuss the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp}}(K)$ in greater detail. Note that it follows from the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 the subsequent statements have obvious analogs for the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$. □

Proof. A very short moment of reflection shows that the given map is a natural chain map. We deduce from the Simplicial Boundary Lemma 176.12 (1) that the chain map is in fact an isomorphism of chain complexes. ■

The next lemma can be viewed as an analog of the CW-Skeleton-H_*-Lemma 155.2 (2) and (3).

Lemma 176.16. (Simplicial Skeleton–Homology Lemma) Let K be an abstract simplicial complex and let $m \in \mathbb{N}_0 \cup \{\infty\}$. As usual we denote by K^m the m-skeleton of K. Furthermore let $n \in \mathbb{N}_0$.

(1) If K has only finitely many n-simplices, then $\mathrm{H}_n^{\mathrm{simp}}(K)$ is finitely generated.
(2) If $n > m$, then $\mathrm{H}_n^{\mathrm{simp}}(K^m) = 0$.
(3) If $n < m$, then the inclusion-induced map $\mathrm{H}_n^{\mathrm{simp}}(K^m) \to \mathrm{H}_n^{\mathrm{simp}}(K)$ is an isomorphism.

Analogous statements also hold for ordered abstract simplicial complexes and the homology groups $\mathrm{H}_n^{\mathrm{simp},\leq}(K^m)$ and $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$.

Proof.

(1) This statement follows easily from the definitions together with the Finite Generation Lemma 97.20.
(2) This statement follows from the observation that $\mathrm{C}_n^{\mathrm{simp}}(K^m) = 0$ for $n > m$.
(3) This statement follows almost immediately from the observation that by definition we have $\mathrm{C}_i^{\mathrm{simp}}(K^m) = \mathrm{C}_i^{\mathrm{simp}}(K)$ for $i = 0, \ldots, m$. ■

It is clear that given any finite abstract simplicial complex one can compute the corresponding homology groups by a brute force calculation. But for large examples this can

be very inefficient. Furthermore, it is rather unsatisfactory to outsource calculations to a computer. It is much more entertaining to try to do calculations cleverly on one's own. In the following we will get to know a few approaches to calculating simplicial homology groups. We will develop more techniques in Exercises 176.3 and 176.7.

As for singular homology, some calculations, statements and proofs become easier if we consider *reduced* simplicial homology.

Definition 176.17. Let K be an abstract simplicial complex.

(1) We refer to the map[1885]
$$\begin{array}{rcl} \epsilon\colon \mathrm{C}_0^{\mathrm{simp}}(K) & \to & \mathbb{Z} \\ \left[\sum_{i=1}^m a_i \cdot [v_i]\right] & \mapsto & \sum_{i=1}^m a_i \end{array}$$
as the augmentation.

(2) We consider the following generalized[1886] chain complex
$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2^{\mathrm{simp}}(K) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{simp}}(K) \xrightarrow{\partial_1} \underset{\text{degree } 0}{\mathrm{C}_0^{\mathrm{simp}}(K)} \xrightarrow{\epsilon} \underset{\text{degree } -1}{\mathbb{Z}} \to 0.$$
We introduce the following two definitions.

(a) We refer to this chain complex as the augmented simplicial chain complex $\widetilde{\mathrm{C}}_*^{\mathrm{simp}}(K)$ of K.

(b) Let $k \in \mathbb{N}_0$. We define the k-th reduced simplicial homology group $\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K)$ of K to be the k-th homology group of the augmented chain complex.

Remark. The above definition also makes sense for the empty abstract simplicial complex $\varnothing$. Furthermore, similar to the Reduced Homology Lemma 147.3 (-1) we see that $\mathrm{H}_{-1}^{\mathrm{simp}}(\varnothing) = \mathbb{Z}$ and $\mathrm{H}_i^{\mathrm{simp}}(\varnothing) = 0$ for $i \neq -1$. □

The following lemma is the analog of the Reduced Homology Lemma 147.3. It basically says that in the following it suffices to calculate reduced simplicial homology groups.

Lemma 176.18. (Reduced Homology Lemma) Let $K = (V, S)$ be an abstract simplicial complex.

(1) Given any vertex $v \in V$ the map
$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0^{\mathrm{simp}}(K) \oplus \mathbb{Z} & \to & \mathrm{H}_0^{\mathrm{simp}}(K) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot v] \end{array}$$
is an isomorphism. In particular, if K is non-empty, then we have an isomorphism $\widetilde{\mathrm{H}}_0^{\mathrm{simp}}(K) \oplus \mathbb{Z} \cong \mathrm{H}_0^{\mathrm{simp}}(K)$.

(2) For every $n \geq 1$ we have the equality $\mathrm{H}_n^{\mathrm{simp}}(K) = \widetilde{\mathrm{H}}_n^{\mathrm{simp}}(K)$.

Proof. The proof is basically identical to the already rather straightforward proof of the Reduced Homology Lemma 147.3. ■

By the Cone Contractability Lemma 20.20 we know that the cone on a topological space is contractible, in particular we know by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that its reduced

[1885] Morally the group $\mathbb{Z}$ is the free abelian group on the "empty" simplex.

[1886] It requires about one second to verify that this is indeed a generalized chain complex.

homology groups vanish. The following lemma can be viewed as a simplicial analog of this observation.

Lemma 176.19. (Cone Homology Lemma) Let K be an abstract simplicial complex. Let $\mathrm{Cone}(K)$ be the simplicial cone from Definition 174.44. Given any $k \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\mathrm{Cone}(K)) \;=\; 0.$$

Proof. Let $K = (V, S)$ be an abstract simplicial complex. We set $L := \mathrm{Cone}(K)$. Note that it follows from the Chain Null-Homotopy Lemma 144.11 that it remains to prove that the identity map $\mathrm{id}\colon \mathrm{C}_*^{\mathrm{simp}}(L) \to \mathrm{C}_*^{\mathrm{simp}}(L)$ is homotopic to the zero map. In other words, it suffices to prove the following claim:

Claim. We can find "diagonal" maps P_n, $n \in \mathbb{Z}_{\geq -1}$, to obtain a diagram

$$\begin{array}{ccccccccccccccc}
\ldots & \longrightarrow & C_3^{\mathrm{simp}}(L) & \xrightarrow{\partial_3} & C_2^{\mathrm{simp}}(L) & \xrightarrow{\partial_2} & C_1^{\mathrm{simp}}(L) & \xrightarrow{\partial_1} & C_0^{\mathrm{simp}}(L) & \xrightarrow{\partial_0 := \epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & \swarrow^{P_3} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_2} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_1} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_0} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow^{P_{-1}} & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & C_3^{\mathrm{simp}}(L) & \xrightarrow[\partial_3]{} & C_2^{\mathrm{simp}}(L) & \xrightarrow[\partial_2]{} & C_1^{\mathrm{simp}}(L) & \xrightarrow[\partial_1]{} & C_0^{\mathrm{simp}}(L) & \xrightarrow[\partial_0 := \epsilon]{} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

such that for every $n \in \mathbb{Z}_{\geq -1}$ we have the equality

$$(*) \qquad \partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \mathrm{id}\,.$$

Proof. In the following we denote the tip of $\mathrm{Cone}(L)$ by "$\star$". To prove the claim we consider the map

$$\begin{array}{rcl} P_{-1}\colon \mathbb{Z} & \to & \mathrm{C}_0^{\mathrm{simp}}(L) \\ n & \mapsto & n \cdot [\star] \end{array}$$

and for $n \geq 0$ we consider the map[1887]

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n^{\mathrm{simp}}(L) & \to & \mathrm{C}_{n+1}^{\mathrm{simp}}(L) \\ [v_0, \ldots, v_n] & \mapsto & \begin{cases} [\star, v_0, \ldots, v_n], & \text{if all } v_i \neq \star, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We claim that $(*)$ holds for all $n \in \mathbb{Z}_{\geq -1}$. The case $n = -1$ can be verified easily. Thus it remains to show that $(*)$ holds if $n \in \mathbb{N}_0$. In fact, given $[v_0, \ldots, v_n] \in \mathrm{C}_n^{\mathrm{simp}}(L)$ with $v_0, \ldots, v_n \in V$ we immediately see that

$$\underbrace{[v_0, \ldots, v_n] + \sum_{i=1}^{n+1} (-1)^i \cdot [\star, v_0, \ldots, \widehat{v_{i-1}}, \ldots, v_n]}_{=(\partial_{n+1} \circ P_n)([v_0, \ldots, v_n])} + \underbrace{\sum_{i=0}^{n} (-1)^i \cdot [\star, v_0, \ldots, \widehat{v_i}, \ldots, v_n]}_{=(P_{n-1} \circ \partial_n)([v_0, \ldots, v_n])} \;=\; [v_0, \ldots, v_n].$$

Furthermore, given $[\star, v_0, \ldots, v_{n-1}] \in \mathrm{C}_n^{\mathrm{simp}}(L)$ with $v_0, \ldots, v_{n-1} \in V$ it is equally straightforward to verify that the equality holds. ∎

Next we want to calculate the simplicial homology groups of the abstract simplicial complexes D_n and S_n that we introduced on page 3283. By the Reduced Simplicial Homology Lemma 176.18 it suffices to compute the reduced simplicial homology groups.

Lemma 176.20. (Simplicial-H_*–Ball-Sphere Lemma) Given any $n \in \mathbb{N}_0$ and given any $k \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(D_n) \;=\; 0 \quad \text{and} \quad \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(S_n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

[1887] One can easily verify that the map P_n is actually well-defined.

Proof.

(1) Given any $n \in \mathbb{N}_0$ and given any $k \in \mathbb{N}_0$ we calculate that

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(D_n) \underset{\text{see page 3308}}{\cong} \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\mathrm{Cone}(D_{n-1})) \underset{\text{by the Cone Homology Lemma 176.19}}{=} 0.$$

(2) Let $n \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{C}_n^{\mathrm{simp}}(S_n) & \xrightarrow{\partial} & \mathrm{C}_{n-1}^{\mathrm{simp}}(S_n) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathrm{C}_0^{\mathrm{simp}}(S_n) & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\
0 & \longrightarrow & \mathrm{C}_{n+1}^{\mathrm{simp}}(D_{n+1}) & \longrightarrow & \mathrm{C}_n^{\mathrm{simp}}(D_{n+1}) & \xrightarrow{\partial} & \mathrm{C}_{n-1}^{\mathrm{simp}}(D_{n+1}) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \mathrm{C}_0^{\mathrm{simp}}(D_{n+1}) & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 \longrightarrow 0.
\end{array}$$

We make the following clarifications and comments:

(a) The top two horizontal sequences are the reduced chain complexes of the abstract simplicial complexes S_n and D_{n+1}.

(b) Recall that by definition the simplices of S_n and D_{n+1} are the same except that D_{n+1} has a single $(n+1)$-simplex whereas S_n has no $(n+1)$-simplices. This observation explains the vertical maps.

(c) The diagram clearly commutes and it is clear that the vertical sequences are exact.

(d) It follows from the above that we can apply the LES-Proposition 145.10 and we obtain a long exact sequence of homology groups. Together with (1) we obtain the reduced simplicial homology groups of S_n. ■

In the Suspension-H_*-Proposition 149.3 we studied the singular homology groups of a suspension. Statement (2) of the following lemma is the simplicial analog of the Suspension-H_*-Proposition 149.3.

Lemma 176.21. (Join-Homology Lemma) Let K be a non-empty abstract simplicial complex.

(1) Let $n \in \mathbb{N}_0$ and let S_n be the abstract simplicial complex that we introduced on page 3283. For every $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * S_n) \cong \widetilde{\mathrm{H}}_{k-(n+1)}^{\mathrm{simp}}(K).$$

(2) We denote by $\Sigma(K)$ the simplicial suspension of K from Definition 174.44. For every $k \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\Sigma(K)) \cong \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K).$$

Example. Let $n \in \mathbb{N}$. Given any $k \in \mathbb{N}_0$ we see that

$$\widetilde{\mathrm{H}}_k\big(\overbrace{\Sigma(\Sigma(\dots\Sigma(S_0)))}^{n\text{-fold simplicial suspension}}\big) \underset{\text{by the Join-Homology Lemma 176.21 applied } n \text{ times}}{\cong} \widetilde{\mathrm{H}}_{k-n}(S_0) \underset{\text{see the discussion on page 3349}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k-n=0, \\ 0, & \text{otherwise.} \end{cases}$$

□

Proof. Since the author has no intention of monopolizing all the fun we will leave the proof of Statement (1) as Exercise 176.2 to the reader. Note that Statement (2) is an immediate consequence of Statement (1) since $\Sigma(K)$ is naturally isomorphic to $K * S_0$. ■

Next we extend the definition of simplicial homology groups in the obvious way to simplicial complexes:

Definition 176.22. Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex. Given $n \in \mathbb{N}_0$ we define

$$\mathrm{H}_n^{\mathrm{simp}}(X) \;:=\; \mathrm{H}_n^{\mathrm{simp}}(K).$$

If K is actually ordered we set $\mathrm{H}_n^{\mathrm{simp},\leq}(K) := \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp},\leq}(K))$. Furthermore, given a simplicial map $f\colon X = (X, \Theta) \to Y = (L, \Omega)$ between two simplicial complexes we define the induced map $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(X) \to \mathrm{H}_n^{\mathrm{simp}}(Y)$ as the induced map $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)$.

Finally we introduce simplicial homology with coefficients in the obvious way. More precisely, we have the following definition which is the obvious variation on the definition given in Section 160.3.

Definition 176.23. Let G be an abelian group and let $n \in \mathbb{N}_0$. Given an abstract simplicial complex K we set

$$\mathrm{H}_n^{\mathrm{simp}}(K; G) \;:=\; \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp}}(K) \otimes G).$$

If K is actually ordered we set $\mathrm{H}_n^{\mathrm{simp},\leq}(K; G) := \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes G)$. Finally given a simplicial complex $(X, (K = (V, S), \Theta\colon |K| \to X))$ we define, following the above convention, $\mathrm{H}_n^{\mathrm{simp}}(X; G) := \mathrm{H}_n^{\mathrm{simp}}(K; G)$.

Surely there is no need to state all the self-evident properties of simplicial homology with coefficients. Let us just mention that it follows from the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that given any abstract simplicial complex K and given any $n \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}_n^{\mathrm{simp}}(K; \mathbb{Z}) & \to & \mathrm{H}_n^{\mathrm{simp}}(K) \\ \left[\sum_{i=0}^{k} s_i \otimes m_i\right] & \mapsto & \left[\sum_{i=0}^{k} s_i \cdot m_i\right] \end{array}$$

is a natural isomorphism. Throughout these notes we will use this isomorphism to make the identification $\mathrm{H}_n^{\mathrm{simp}}(K; \mathbb{Z}) = \mathrm{H}_n^{\mathrm{simp}}(K)$.

176.4. **The simplicial cross product and the Simplicial Künneth Theorem.** Let $(K, \leq)$ and $(L, \leq)$ be two ordered abstract simplicial complexes. Recall that in Definition 174.37 we introduced the corresponding product $K \times L$. In this section we will learn how to determine the simplicial homology groups of $K \times L$ in terms of the simplicial homology groups of K and L.

First we need to recall Definition 164.21:

Definition 176.24. Let $m, n \in \mathbb{N}_0$.

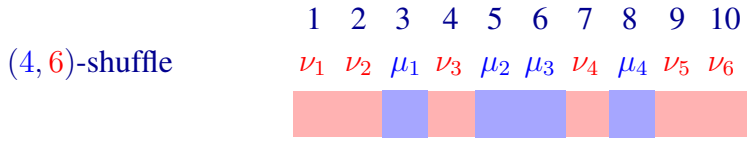
(1) An (m, n)-shuffle is a pair (μ, ν) of disjoint sets of integers

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m + n \quad \text{and} \quad 1 \leq \nu_1 < \nu_2 < \cdots < \nu_n \leq m + n.$$

(2) We denote by $S_{m,n}$ the set of all (m, n)-shuffles.

(3) We denote by $\mathrm{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$.

In Definition 164.23 we used shuffles to introduce a chain map on singular homology. We now perform the analogous construction for simplicial chain complexes:



Definition 176.25.

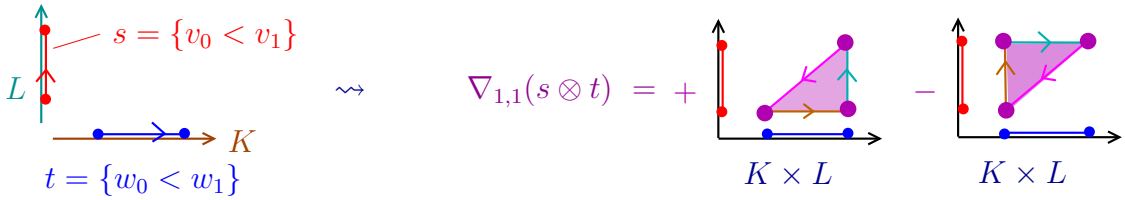
(4) Let μ be a sequence $1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m+n$. We set $\mu_0 := 0$ and we set $\mu_{m+1} := m+n+1$. Given $i \in \{0,\ldots,m+n\}$ we denote by $\mu^\dagger(i)$ the unique $j \in \{0,\ldots,m\}$ with $\mu_j \leq i < \mu_{j+1}$.

(5) Let $(K,\leq)$ and $(L,\leq)$ be two ordered abstract simplicial complexes. We consider the homomorphism

$$\nabla := \nabla_{m,n} \colon \mathrm{C}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L) \to \mathrm{C}_{m+n}^{\mathrm{simp},\leq}(K \times L)$$
$$(\{v_0 < \cdots < v_m\} \otimes \{w_0 < \cdots < w_n\}) \mapsto \sum_{(\mu,\nu)\in S_{m,n}} \mathrm{sign}(\mu,\nu) \cdot \underbrace{\{(v_{\mu^\dagger(i)}, w_{\nu^\dagger(i)})\}_{i=0,\ldots,m+n}}_{(m+n)\text{-simplex in } K\times L}.$$

Example. We consider the case $m = n = 1$. Let $(K,\leq)$ and $(L,\leq)$ be two ordered abstract simplicial complexes and let $s = \{v_0 < v_1\}$ and $t = \{w_0 < w_1\}$ be two ordered 1-simplices of K respectively L. In this case there exist two $(1,1)$-shuffles, namely $\{\mu_1 = 1, \nu_1 = 2\}$ and $\{\mu_1 = 2, \nu_1 = 1\}$. We obtain the following table:

shuffle	sign	corresponding 2-simplex
$\{\mu_1 = 1, \nu_1 = 2\}$	1	$\{(v_0,w_0),(v_1,w_0),(v_1,w_1)\}$
$\{\mu_1 = 2, \nu_1 = 1\}$	-1	$\{(v_0,w_0),(v_0,w_1),(v_1,w_1)\}$.



In summary we see that

$$\nabla_{1,1}(s \otimes t) = \{(v_0,w_0),(v_1,w_0),(v_1,w_1)\} - \{(v_0,w_0),(v_0,w_1),(v_1,w_1)\}. \qquad \square$$

The following proposition is an analog of the Shuffle Proposition 164.24:

Proposition 176.26. (Simplicial Shuffle Proposition) The above homomorphisms

$$\nabla_{m,n} \colon \mathrm{C}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L) \to \mathrm{C}_{m+n}^{\mathrm{simp},\leq}(K \times L)$$

have the following properties:

(1) They form a chain map[1888] $\mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K \times L)$.

(2) They are natural, i.e. if $f \colon K \to K'$ and $g \colon L \to L'$ are two order preserving simplicial maps between ordered simplicial complexes, then for any $m,n \in \mathbb{N}_0$ the following

diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L) & \xrightarrow{\nabla_{m,n}} & \mathrm{C}_{m+n}^{\mathrm{simp},\leq}(K \times L) \\ \downarrow{\scriptstyle f_*} \quad \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ \mathrm{C}_m^{\mathrm{simp},\leq}(K') \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L') & \xrightarrow{\nabla_{m,n}} & \mathrm{C}_{m+n}^{\mathrm{simp},\leq}(K' \times L'). \end{array}$$

Sketch of proof. It follows immediately from the definitions that the maps ∇ are natural. It remains to show that the maps ∇ define a chain map. This statement is an immediate consequence of the following claim:

Claim. Let $(K, \leq)$ and $(L, \leq)$ be two ordered abstract simplicial complexes. For any ordered m-simplex s of K and any ordered n-simplex t of L we have the following equality in $\mathrm{C}_{m+n-1}^{\mathrm{simp},\leq}(K \times L)$:

$$\nabla(\partial_{m+n}^{\otimes}(\underbrace{s \otimes t}_{\in \mathrm{C}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L)})) = \partial_{m+n}(\nabla(\underbrace{s \otimes t}_{\in \mathrm{C}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L)})).$$

Proof. The proof of the claim is basically identical to the proof of the ∇-Chain Map Lemma 164.25. Even though the ∇-Chain Map Lemma 164.25 is a priori a statement about singular simplices, the proof is purely combinatorial and gives us with minor notational adjustments the desired statement. ■

The Simplicial Shuffle Proposition 176.26 allows us to introduce the following simplicial analog of the cross product on singular homology which we first introduced in Definition 164.4.

Definition 176.27. Let $(K, \leq)$ and $(L, \leq)$ be two ordered abstract simplicial complexes. We refer to the map[1889]

$$\times \colon \mathrm{H}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_n^{\mathrm{simp},\leq}(L) \xrightarrow{\mu} \mathrm{H}_{m+n}(\mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L)) \xrightarrow{\nabla_*} \mathrm{H}_{m+n}^{\mathrm{simp},\leq}(K \times L).$$

μ: map from the Cycle Tensor Lemma 163.5

as the (simplicial) cross product.

The following lemma is an analog of the Cross Product Lemma 164.5.

Lemma 176.28. (Simplicial Cross Product Lemma)

(1) Let $(K, \leq)$ and $(L, \leq)$ be two ordered abstract simplicial complexes. The map

$$\times \colon \mathrm{H}_0^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_n^{\mathrm{simp},\leq}(L) \to \mathrm{H}_n^{\mathrm{simp},\leq}(K \times L)$$

is given by

$$[v] \otimes \left[\sum_{i=1}^{k} r_i \cdot \{w_{0,i} < \cdots < w_{n,i}\}\right] \mapsto \left[\sum_{i=1}^{k} r_i \cdot \{(v, w_{0,i}) < \cdots < (v, w_{n,i})\}\right].$$

An analogous statement holds the map

$$\times \colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_n^{\mathrm{simp},\leq}(L) \to \mathrm{H}_0^{\mathrm{simp},\leq}(K \times L)$$

[1888] In Definition 163.2 we introduced the tensor product $(\mathcal{C} \otimes \mathcal{D}, \partial_*^{\otimes})$ of two chain complexes $(\mathcal{C}, \partial_*)$ and $(\mathcal{D}, \partial_*)$.

(2) The cross product is natural, i.e. if $f\colon K \to K'$ and $g\colon L \to L'$ are two order preserving simplicial maps between ordered simplicial complexes, then for any $m, n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_n^{\mathrm{simp},\leq}(L) & \xrightarrow{\times} & \mathrm{H}_{m+n}^{\mathrm{simp},\leq}(K \times L) \\ \downarrow{\scriptstyle f_*} \quad \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ \mathrm{H}_m^{\mathrm{simp},\leq}(K') \otimes \mathrm{H}_n^{\mathrm{simp},\leq}(L') & \xrightarrow{\times} & \mathrm{H}_{m+n}^{\mathrm{simp},\leq}(K' \times L'). \end{array}$$

Proof.

(1) This statement follows easily from the explicit definitions of ∇ and μ.
(2) This statement follows immediately from the naturality of μ and ∇. ■

The following theorem, which can be viewed as the simplicial analog of the Künneth Theorem 164.6, computes the simplicial homology groups of $K \times L$ in terms of the simplicial homology groups of K and L.

Theorem 176.29. (Simplicial Künneth Theorem) For all ordered simplicial complexes $(K, \leq)$ and $(L, \leq)$ there exists a natural homomorphism

$$\widetilde{\nu}\colon \mathrm{H}_n^{\mathrm{simp},\leq}(K \times L) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\leq}(K), \mathrm{H}_q^{\mathrm{simp},\leq}(L))$$

such that the following sequence

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} \mathrm{H}_p^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L) &\xrightarrow{\times} \mathrm{H}_n^{\mathrm{simp},\leq}(K \times L) \\ &\xrightarrow{\widetilde{\nu}} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\leq}(K), \mathrm{H}_q^{\mathrm{simp},\leq}(L)) \to 0 \end{aligned}$$

is exact and such that it splits.[1890]

Sketch of proof. [1891] We start out with the following claim:

Claim. The maps

$$\nabla_{m,n}\colon \mathrm{C}_m^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_n^{\mathrm{simp},\leq}(L) \to \mathrm{C}_{m+n}^{\mathrm{simp},\leq}(K \times L)$$

define a chain map

$$\nabla\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K \times L)$$

which is a chain homotopy equivalence.

Proof. By the Simplicial Shuffle Proposition 176.26 we know that the maps $\nabla_{m,n}$ define a chain map. It remains to show that the chain map is a chain homotopy equivalence. In the following we adopt and adapt the logic of the Eilenberg-Zilber Theorem 164.2.

As in the EZ-Inverse Proposition 164.11 one can show that there exists a chain map

$$\Theta\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K \times L) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L)$$

[1890] As in the case of the Algebraic Universal Coefficient Theorem 162.15, see the No-Natural Splitting Proposition 162.21, one can show that the splitting is not natural.
[1891] In Exercise 177.3 we will give an alternative proof of the theorem.

such that for each vertex v of K and w of L we have $\Theta(\{(v,w)\}) = \{v\} \otimes \{w\}$. The proof of the existence in the simplicial setting is almost line-by-line the same as the proof in the singular setting. Along the line we just need to make the following changes:

- In the original proof we used that given any topological space X, any singular simplex $\sigma\colon \Delta^n \to X$ can be written as $\sigma_*(\mathrm{id}_{\Delta^n})$, where $\mathrm{id}_{\Delta^n} \in \mathrm{C}_n(\Delta^n)$. In our current setting we need to use the simple observation that for any ordered simplicial complex $(M, \leq)$ and any n-simplex s of M the corresponding order preserving simplicial map $s\colon D_n \to M$ has the property that $s_*(\{0,\dots,n\}) = s$, where $\{0,\dots,n\} \in \mathrm{C}_n^{\mathrm{simp},\leq}(D_n)$ is the unique generator.
- We need to replace the Contractible-H_*-Corollary 144.15 by the Simplicial-H_*–Ball-Sphere Lemma 176.20.
- We need to use the (obvious) simplicial analog of the Acyclic Tensor Product Lemma 164.12.[1892]

Finally note that, as in the EZ–Natural Chain Homotopy Proposition 164.13 one can show that the compositions $\Theta \circ \nabla\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L)$ and $\nabla \circ \Theta\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K \times L) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K \times L)$ are chain homotopic to the identity. The proofs in the simplicial setting are almost the same as in the singular setting. The only extra ingredient we need is the calculation of the simplicial homology groups of $D_p \times D_q$ which we outsource to Exercise 176.4. The details of the proof are worked out in [**FePi2011**, Chapter III.6]. ⊞

Finally we consider the following diagram:

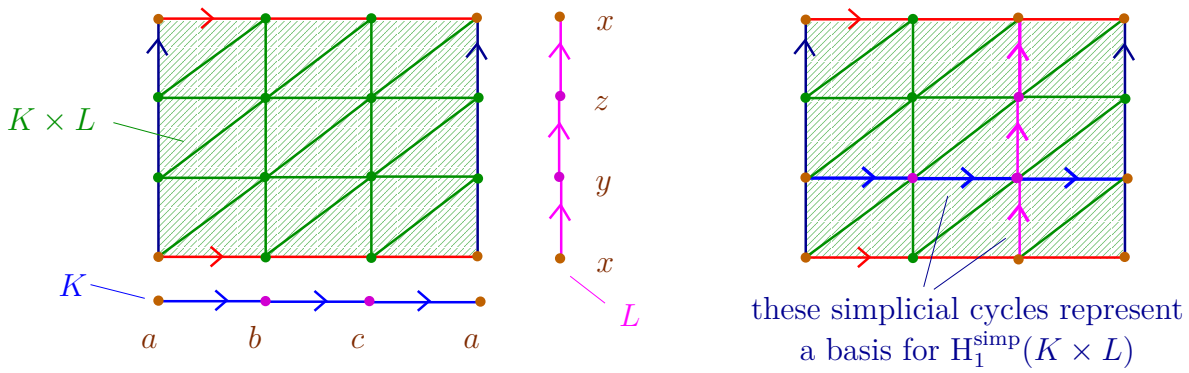
$$\begin{array}{ccccc}
 & & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp},\leq}(K) \otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L)) & & \\
 & \overset{\mu}{\nearrow} & \Big\downarrow \nabla & \overset{\nu}{\searrow} & \\
0 \to \bigoplus_{p+q=n} \mathrm{H}_p^{\mathrm{simp},\leq}(K) \otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L) & & \cong & & \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\leq}(K), \mathrm{H}_q^{\mathrm{simp},\leq}(L)) \to 0. \\
 & \underset{\times}{\searrow} & \Big\downarrow & \overset{=:\widetilde{\nu}}{\nearrow} & \\
 & & \underbrace{\mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp},\leq}(K \times L))}_{=\mathrm{H}_n^{\mathrm{simp},\leq}(K \times L)} & &
\end{array}$$

The theorem is now a consequence of the following clarifications and observations:

(1) The top right diagonal map ν is the natural homomorphism from the Algebraic–Künneth Theorem 163.6.
(2) The Algebraic–Künneth Theorem 163.6 tells us that the sequence on the top is exact and that it splits.
(3) It follows from the above claim together with the Chain Homotopy Equivalence-H_*-Corollary 144.10 that the vertical map is a natural isomorphism.
(4) The triangle to the left commutes by the definition of the simplicial cross product.
(5) We define the bottom right diagonal map $\widetilde{\nu}$ such that the diagram commutes. ■

Example. We consider the ordered abstract simplicial spaces K and L that are shown in the figure below.

[1892] This statement can also be proved using the Acyclic Model Theorem 235.7, which replaces many of the completely formal aspects of the proof of the EZ-Inverse Proposition 164.11 and its various siblings.



Both are simplicially isomorphic to the simplicial circle S_1. Furthermore we consider the "simplicial torus" $K \times L$. It follows from the Simplicial Künneth Theorem 176.29 and the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 that

$$\mathrm{H}_i^{\mathrm{simp}}(K \times L) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i > 2. \end{cases}$$

Furthermore note that it follows easily from the Simplicial Cross Product Lemma 176.28 (1) and the explicit isomorphism provided by the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 that the two singular cycles shown in the figure below form a basis for $\mathrm{H}_1^{\mathrm{simp}}(K \times L)$. In Exercise 176.9 we will determine a generator for $\mathrm{H}_2^{\mathrm{simp}}(K \times L)$. □

176.5. Exercises for Chapter 176 .

Exercise 176.1. Let $K = (V, S)$ be an abstract simplicial complex and let "$\leq$" and "$\widetilde{\leq}$" be two orders on K. Show that the maps

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\widetilde{\leq}}(K) \\ \{v_0 < \cdots < v_n\} & \mapsto & \underbrace{\mathrm{sign}(\sigma) \cdot \{v_{\sigma(0)} \widetilde{\leq} \ldots \widetilde{\leq} v_{\sigma(n)}\}}_{\substack{\sigma \in S_n \text{ is the unique permutation} \\ \text{such that } v_{\sigma(0)} \widetilde{\leq} \ldots \widetilde{\leq} v_{\sigma(n)}}} \end{array}$$

define an isomorphism of chain complexes.

Exercise 176.2. Let K be a non-empty abstract simplicial complex, let $n \in \mathbb{N}_0$ and let S_n be the abstract simplicial complex that we introduced on page 3283. Show that for every $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * S_n) \cong \widetilde{\mathrm{H}}_{k-(n+1)}^{\mathrm{simp}}(K).$$

Exercise 176.3. Let $K = (V, S)$ be an abstract simplicial complex. Recall that in Exercise 174.13 we said that $K = (V, S)$ is *connected* if given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$. Show that if K is a connected non-empty abstract

simplicial complex, then given any abelian group G the map

$$\begin{array}{rcl} \mathrm{H}_0^{\mathrm{simp}}(K;G) & \to & G \\ \Big[\sum_{i=0}^{k} \{v_i\} \otimes g_i\Big] & \mapsto & \sum_{i=0}^{k} g_i \end{array}$$

is a natural isomorphism.

Remark. This is the simplicial analog of the H_0-Proposition 143.25.

Exercise 176.4. Let $p, q \in \mathbb{N}_0$. We consider the product $D_p \times D_q$ of the ordered abstract simplicial complexes defined on page 3283. Show that

$$\mathrm{H}_i^{\mathrm{simp}}(D_p \times D_q) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. This exercise goes into the proof of the Simplicial Künneth Theorem 176.29. In particular you are not allowed to use this theorem.

Exercise 176.5. Let K be an abstract simplicial complex K and let s be a simplex s of K.

(a) Show that $\mathrm{St}(K,s) = \mathrm{Lk}(K,s) * s$.

(b) Show that for every $i \in \mathbb{N}$ we have $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{St}(K,s)) = 0$.
Hint. Make use of Simplicial-H_*–Ball-Sphere Lemma 176.20 and the Join-Homology Lemma 176.21.

Exercise 176.6. Let K be a simplicial complex and let A and B be two subcomplexes with $A \cup B = K$. Show that for each $k \in \mathbb{N}_0$ the natural map $\mathrm{H}_k^{\mathrm{simp}}(A, A \cap B) \to \mathrm{H}_k^{\mathrm{simp}}(X, B)$ is an isomorphism.

Remark. This statement can be viewed as the Simplicial Excision Theorem. The biggest difference to the singular setting is that in the simplicial setting the proof is almost embarrassingly easy.

Exercise 176.7.

(a) Formulate and prove a Mayer–Vietoris style theorem for simplicial homology, along the lines of the Mayer–Vietoris Theorem 153.2.
Remark. We generously provided two different proofs for the Mayer–Vietoris Theorem 153.2. You can easily adapt the first proof in Section 153.1.1 using Exercise 176.6. It is also not hard to modify the second proof which we provided in Section 153.1.2.

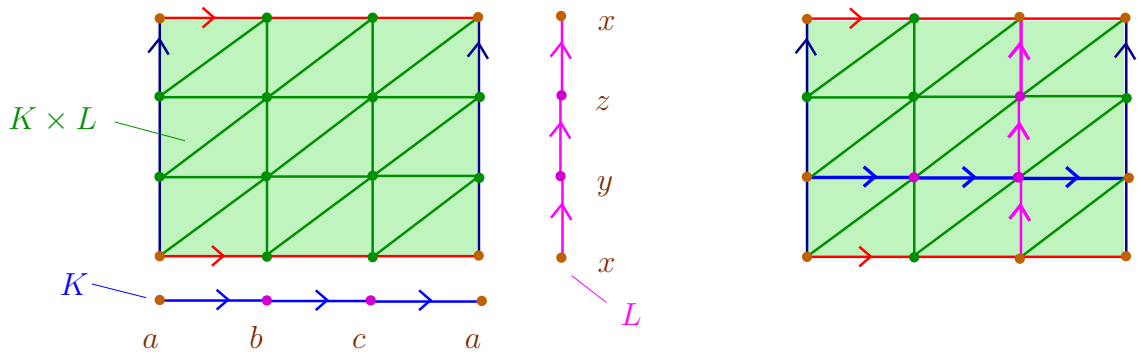
(b) Use the exact sequence from (a) to provide a new solution to Exercise 176.2.

Exercise 176.8. Let K be an abstract simplicial complex and let L be a subcomplex of K. Show that for each $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$\mathrm{H}_k^{\mathrm{simp}}(K,L) \;\cong\; \widetilde{\mathrm{H}}_k^{\mathrm{simp}}\big(\underbrace{K \cup_L \mathrm{Cone}(L)}_{\text{mapping cone of } L \to K}\big).$$

Exercise 176.9. As on page 3359 we consider the ordered abstract simplicial spaces K and L that are shown in the figure below. Provide an explicit 2-cycle that represents a generator of $\mathrm{H}_2^{\mathrm{simp}}(K \times L)$.

Exercise 176.10. Compute the simplicial homology groups of an abstract simplicial complex K such that $|K|$ is homeomorphic to the surface Σ of genus two.



Hint. As we saw in the Surface–Connected Sum Lemma 54.15, we can view Σ as the connected sum of two tori and you could use Exercise 176.7.

Exercise 176.11. Let K and L be two ordered abstract simplicial complexes. Show that for each $k \in \mathbb{N}_0$ there exists a short exact sequence of the following form:

$$0 \;\to\; \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * L) \;\to\; \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K) \oplus \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(L) \;\to\; \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K \times L) \;\to\; 0.$$

Remark. This statement is the simplicial analog of Exercise 164.7. In fact at least if K and L are finite, then we can reduce the simplicial statement to the singular statement. The challenge is to give a purely combinatorial argument.

CHAPTER 177

Simplicial and singular homology

The following question arises naturally.

Question 177.1. Let $(X, (K, \Theta\colon |K| \to X))$ be a simplicial complex and let $n \in \mathbb{N}_0$. Does the isomorphism type of the simplicial homology group $\mathrm{H}_n^{\mathrm{simp}}(X)$ only depend on the homeomorphism type of X?

For example, we know by the discussion on page 3302 and on page 3310 that the topological realizations of S_n and of the n-fold simplicial suspension of S_0 are both homeomorphic to S^n and conveniently enough we know by the Simplicial-H_*–Ball-Sphere Lemma 176.20 and the last example of the previous section that the homology groups are indeed isomorphic.

In this chapter we will discuss two approaches for how to answer this question:

(1) In Section 177.1 we will discuss the "classical" approach through subdivisions. We will discuss some aspects of this approach, but we will not carry it out.
(2) In Section 177.2 we will state and prove the Simplicial-Singular Homology Theorem 177.10 which shows that simplicial homology of an abstract simplicial complex K is naturally isomorphic to the singular homology of the topological realization $|K|$. This result gives of course an affirmative answer to Question 177.1.

This chapter is organized as follows:

- In Section 177.1 we will provide a historical perspective on the topological invariance of simplicial homology.
- In Section 177.2 we will prove that simplicial homology is isomorphic to singular homology. This implies that simplicial homology is a topological invariant.
- In Section 177.3 we will introduce relative simplicial homology.
- In Appendix 177.4 we will introduce the simplicial subdivision map.
- In Section 177.5 we will provide exercises for Chapter 177.

In the next Chapter 178 we will state and prove the Nerve Theorem 178.5.

177.1. The topological invariance of simplicial homology I.

The following "research program" summarizes the traditional approach to tackling Question 177.1.

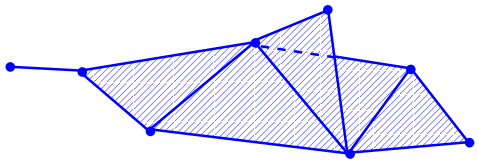
Program 177.2.

(1) Show that any two (let us say finite) abstract simplicial complexes with homeomorphic topological realizations are related by a finite sequence of "moves".
(2) Show that the isomorphism type of simplicial homology groups is unchanged under these "moves".

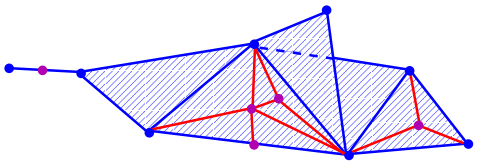
This idea leads straight to the definition of a subdivision of a simplicial complex:

Definition 177.3. We say that a simplicial complex $(X, (K = (V, S), \Omega\colon |K| \to X))$ is a subdivision of some simplicial complex $(X, (L = (W, T), \Theta\colon |L| \to X))$ if the following two conditions are satisfied:

(1) Every simplex of $|K|$ is contained in some simplex of $|L|$. More precisely, given any simplex $s \in S$ of K there exists a simplex $t \in T$ of L such that $\Omega(|s|) \subseteq \Theta(|t|)$.
(2) For every k-simplex s of K and every l-simplex t of L with $\Omega(|s|) \subseteq \Theta(|t|)$ the map $\Phi_t^{-1} \circ \Theta^{-1} \circ \Omega \circ \Phi_s\colon \Delta^k \to |s| \subseteq |t| \to \Delta^l$ is affine linear.



simplicial complex



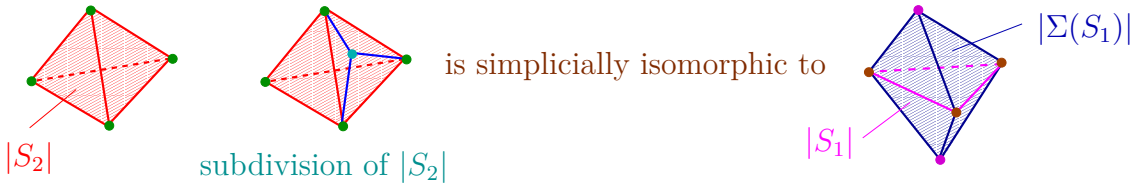
subdivision of the simplicial complex

Remark.

(1) Note that for linear simplicial complexes condition (2) is automatically satisfied.
(2) The definition of a subdivision we give agrees with the definition in most sources, see e.g. [**Hud1969**, p. 5], [**Pon1952**, p. 36], [**RS1972**, p. 15], [**Gla1970**, p. 7], [**Spa1995**, p. 121] and [**Zee1963b**, p. I.4].
(3) In [**Mun1984**, p. 83] the definition of a subcomplex has the extra condition that every simplex of X equals the union of finitely many simplices of Y. It follows from [**Hud1969**, Lemma 1.2] together with the Simplicial Complex-Finiteness Proposition 174.29 (2) that the above definition agrees with the definition provided in [**Mun1984**, p. 83].
(4) The definition of a subdivision is somewhat unsatisfactory. In the spirit that abstract simplicial complexes should be purely combinatorial objects there should be a "topology free" definition of the subdivision of an abstract simplicial complexes, which then mirrors the topological definition given above. We will return to the question of how to work completely combinatorially in Section 181.2. □

Examples.

(1) It follows from the SD-Lemma 175.5 (3a) and (3c) that the barycentric subdivision $\mathrm{sd}(X)$ of a simplicial complex X is indeed a subdivision in the sense of the above definition.
(2) In the figure below we see that a subdivision of the simplicial complex $|S_2|$ is simplicially isomorphic to $\Sigma(S_1)$.



(3) Let $n \in \mathbb{N}_0$. By the Join-Simplex Lemma 174.42 (3) we know that the map

$$\begin{array}{rcl} f\colon \partial\Delta^n * \{\star\} & \to & \Delta^n \\ [(x,t,\star)] & \mapsto & x \cdot (1-t) + \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right) \cdot t \end{array}$$

is a homeomorphism.



Δ^2 subdivision of Δ^2 Δ^3 subdivision of Δ^3

We point out that it follows easily from this observation that the simplicial complex $(S_{n-1} * \{\star\}, |S_{n-1} * \{\star\}| = \partial\Delta^n * \{\star\} \xrightarrow{f} \Delta^n)$ is a subdivision of the simplicial complex $(D_n, \Omega\colon |D_n| \to \Delta^n)$. □

For completeness' sake we mention the following result:

Proposition 177.4. (Linear Simplicial Complexes–Subdivision Proposition)

(1) Let X and Y be two finite linear simplicial complexes. If $|X| = |Y|$, then there exist subdivisions X' of X and Y' of Y which are simplicially isomorphic.
(2) If we are given two subdivisions X' and X'' of some finite linear simplicial complex X, then there exists a subdivision Y of X which is a subdivision of X' and X''.

Proof.

(1) This statement is proved in [**Gla1970**, Corollary I.4] and [**Zee1963b**, Corollary I.1].
(2) This statement follows immediately from (1) together with the Finite Simplicial Complex–Linear Lemma 174.48. ■

The following theorem gives us part (2) of Program 177.2.

Theorem 177.5. (Subdivision–Homology Theorem) Let X be a simplicial complex and let X' be a subdivision of X. If $\varphi\colon X' \to X$ is a simplicial approximation of the identity map $X' \to X$, then the induced map $\varphi_*\colon \mathrm{H}_n^{\mathrm{simp}}(X') \to \mathrm{H}_n^{\mathrm{simp}}(X)$ is an isomorphism for every $n \in \mathbb{N}_0$.

Proof. We refer to [**Mun1966a**, Theorem 17.2] for a detailed proof. ■

So an obvious candidate for the "moves" of Program 177.2 is a combination of subdivisions and simplicial isomorphisms. Now that we have dealt with part (2) of Program 177.2, let us try to tackle part (1). In fact this part is the following conjecture, which came to be known as the "Hauptvermutung". It was originally formulated by Heinrich Tietze [**Tie1908**, p. 13f] in 1908. The conjecture was also raised as a question by Ernst Steinitz [**Steini1908**, p. 32] in 1908. (See also [**?, AlH1974**, p. 152] where it is referred to as the "Hauptvermutung").

Conjecture 177.6. (Hauptvermutung) Let X and Y be two finite simplicial complexes. If the underlying topological spaces are homeomorphic, then there exist subdivisions of X and Y which are simplicially isomorphic.

For better or worse it turned out that the Hauptvermutung was too optimistic. More precisely, the Hauptvermutung was eventually disproved by John Milnor [**Miln1961**, Theorems 1 and 2] in 1961.

Theorem 177.7. (Hauptvermutung Failure Theorem) There exist two finite linear simplicial complex X and Y for which the underlying topological spaces are homeomorphic, but such that there are no subdivisions of X and Y which are simplicially isomorphic.

Remark. In fact a few years later Rob Kirby and Larry Siebenmann [**KSi1977**] (see also [**Ran1996**, Chapter 4]) showed that given any $n \geq 5$ there exist such finite abstract simplicial complexes such that the underlying topological space is a compact n-dimensional topological manifold. □

Proof. The theorem is proved in [**Miln1961**, Theorems 1 and 2]. Alternative proofs are given in [**CoheM1973**, p. 84], [**Stal1965b**, p. 252] and [**Ran1996**, Chapter 2]. ■

Even though Part (1) of Program 177.2 did not work out, it turns out that Question 177.1 nonetheless has a positive answer. More precisely we have the following theorem:

Theorem 177.8. (Simplicial Homology–Homeomorphism Invariant Theorem) Let X and Y be two simplicial complexes. If the underlying topological spaces X and Y are homeomorphic, then for every $n \in \mathbb{N}_0$ the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp}}(X)$ and $\mathrm{H}_n^{\mathrm{simp}}(Y)$ are isomorphic.

Using the Simplicial Approximation Theorem 175.15 and several other tools from the theory of simplicial complexes one can give a direct proof of the Simplicial Homology–Homeomorphism Invariant Theorem 177.8. This approach is for example pursued in [**Mun1966a**, Corollary 18.2], [**Pra2007**, Chapter 2.2] and [**Gib2010**, p. 136].

We will instead take an indirect approach to proving the Simplicial Homology–Homeomorphism Invariant Theorem 177.8, namely we will show that given a simplicial complex the *simplicial* and *singular* homology groups are isomorphic. By the functoriality of *singular* homology the isomorphism type of *singular* homology groups only depends on the homeomorphism type of the topological space. With this observation in mind we see that the Simplicial Homology–Homeomorphism Invariant Theorem 177.8 is a consequence of the following theorem.

Theorem 177.9. (Simplicial–Singular Homology Theorem) Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ there exists a natural[1893] isomorphism $\Phi_X \colon \mathrm{H}_n^{\mathrm{simp}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$.

We postpone the proof of the Simplicial–Singular Homology Theorem 177.9 to the next section. In fact, in the next section we will formulate and prove a more precise version of the Simplicial–Singular Homology Theorem 177.9. But before we delve into the detailed arguments let us first discuss the respective advantages and disadvantages of simplicial homology.

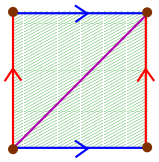
[1893]Here "natural" means that if $f \colon X \to Y$ is a simplicial map, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n^{\mathrm{simp}}(X) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{simp}}(Y) \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_L} \\ \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y). \end{array}$$

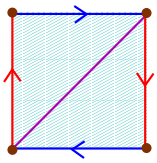
In some textbooks, see e.g. [**Matv2006, Mun1984, SZ1994, HY1988**], homology groups are initially introduced via simplicial homology. This approach has several advantages:

(1) The definition of simplicial homology is much more elementary and intuitive than the definition of singular homology.
(2) One can immediately compute the simplicial homology groups of interesting simplicial complexes, e.g. as we will see in Exercise 176.10, with enough patience and a little bit of cleverness one can compute the simplicial homology groups of abstract simplicial complexes that correspond to the surface of genus g.

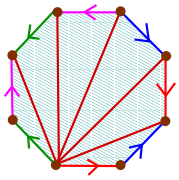
Nonetheless there are serious disadvantages to this approach. A first problem, which is already apparent in the case of the torus, is that even for relatively simple examples the simplicial structures tend to have a large number of simplices, thus the corresponding chain complexes tend to be fairly large and brute force computations by hand become unpleasant. It turns out that there are some ways around this, for example one can extend the theory of simplicial homology from simplicial complexes to "Δ-complexes", which tend to have a significantly smaller number of simplices. Loosely speaking a Δ-complex is a topological space that is decomposed into simplices but now we only demand that the interiors of the simplices are embedded. To give a flavor of the theory we show three Δ-complexes in the figure below. The theory of Δ-complexes is developed in detail in [**Hat2002**, Chapter 2.1].



Δ-complex structure for the torus



Δ-complex structure for $\mathbb{R}\mathrm{P}^2$



Δ-complex structure for the surface of genus 2

Here are more serious reasons why we, and most modern authors, decided against starting out with simplicial homology and why we do not develop the theory of simplicial homology in greater detail:

(1) For slightly more complicated topological spaces it is rather laborious to compute simplicial homology groups. For example we challenge the reader to compute the simplicial homology groups of the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ without reverting to cellular homology.
(2) Most topological spaces one wants to study (e.g. the complement of a knot in S^3), do not come with an obvious simplicial structure, it can be quite cumbersome to show that a simplicial structure exists, let alone to give an explicit simplicial structure.
(3) For a given topological space X which admits a simplicial structure we know by the Simplicial Homology–Homeomorphism Invariant Theorem 177.8 that we have a well-defined *isomorphism type* of simplicial homology groups, but there is no single group that we can associate to X.
(4) Simplicial homology is functorial with respect to simplicial maps. Most maps are not simplicial. We do have the Simplicial Approximation Theorem 175.15 as a tool. But

to use it in a sensible fashion one needs to show that the simplicial approximations of a given continuous map induce the same map on simplicial homology.

It should have become clear by now that simplicial homology comes with a Pandora's box of issues and problems. Overall, the initial extra effort in developing the theory of singular homology groups eventually pays of handily. The theory of singular homology groups is much more natural and much more flexible than the rather pedestrian theory of simplicial homology groups.

We conclude this section with the following piece of wisdom which was handed down to the author from his advisor:

For actual calculations it is best to work with very small objects, for proving general statements it is best to work with very large objects.

177.2. **The topological invariance of simplicial homology II.** In this section we will first show that given an abstract simplicial complex the simplicial homology groups are naturally isomorphic to singular homology groups. Afterwards we will try to find a natural chain homotopy equivalence.

We start out with the first key result of this section.

Theorem 177.10. (Simplicial-Singular Homology Theorem) Let $(K, \Theta\colon |K| \to X)$ be a simplicial complex. For each $n \in \mathbb{N}$ there exists a natural isomorphism

$$\mathrm{H}_n^{\mathrm{simp}}(K) \xrightarrow{\cong} \mathrm{H}_n(X).$$

The proof of the Simplicial-Singular Homology Theorem 177.10 builds on a proposition which relates the simplicial chain complex to the cellular chain complex. Before we state the proposition we recall that given a CW-complex we consider the cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ together with the boundary map ∂_n^{CW} which is given by the composition

$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=:\partial_n^{\mathrm{CW}}}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the natural map from absolute to relative homology.
Now we can formulate the promised proposition.

Proposition 177.11. (Simplicial-Cellular-C_*-Proposition) Let K be an abstract simplicial complex. Given an ordered n-simplex $s = {}_n(v_0, \ldots, v_n)$ of K we denote by

$$\begin{aligned} \Phi_s^{\leq}\colon \Delta^n = \left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \sum_{i=0} t_i = 1\right\} &\to |s| \subseteq |K| = X \\ (t_0, \ldots, t_n) &\mapsto \sum_{i=0}^{n} t_i \cdot v_i \end{aligned}$$

the corresponding characteristic map. The maps[1894]

$$\begin{aligned} \Upsilon_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) &\to \mathrm{C}_n^{\mathrm{CW}}(|K|) \\ [s] &\mapsto (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{aligned}$$

form a natural isomorphism of chain complexes.

[1894]Recall that in the Simplicial-Implies-CW Lemma 174.51 we showed that the simplicial complex $|K|$ admits a natural CW-structure.

Proof of Proposition 177.11. Let $K = (V, S)$ be an abstract simplicial complex. We write $X \coloneqq |K|$. Given $n \in \mathbb{N}_0$ we denote by S_n the set of ordered n-simplices of K. Next we consider the following diagram:

$$\begin{array}{ccc}
\mathrm{C}_n^{\mathrm{simp}}(K) \coloneqq \Big(\bigoplus_{s\in S_n} \mathbb{Z}\cdot s\Big)/I_n & \xrightarrow{[s]\mapsto[\Phi_s^{\leq}]} & \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X) \\
\Big\downarrow{\scriptstyle \partial_n} & \swarrow{\scriptstyle \partial} & \Big\downarrow{\scriptstyle \partial_n^{\mathrm{CW}}} \\
& \mathrm{H}_{n-1}(X^{n-1}) \searrow{\scriptstyle j_{n-1}} & \\
\mathrm{C}_{n-1}^{\mathrm{simp}}(K) \coloneqq \Big(\bigoplus_{s\in S_{n-1}} \mathbb{Z}\cdot s\Big)/I_{n-1} & \xrightarrow{[s]\mapsto[\Phi_s^{\leq}]} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) = \mathrm{C}_{n-1}^{\mathrm{CW}}(X).
\end{array}$$

We make the following clarifications and observations:

(1) If s is an ordered k-simplex, then it follows immediately from the definition of the CW-structure on X that $\Phi_s^{\leq}(\Delta^k) \subseteq X^k$ and that $\Phi_s^{\leq}(\partial\Delta^k) \subseteq X^{k-1}$. In particular this shows that $\Phi_s^{\leq}$ represents an element in $\mathrm{H}_k(X^k, X^{k-1})$.

(2) It follows fairly easily from (1) together with the Free Abelian Group–Existence Lemma 97.8 and Exercise 150.1 (which is an analog of the Sphere–Degree–Matrix Lemma 151.7) that the horizontal maps are well-defined.

(3) Basically the same argument as in the CW-Skeleton-H_*-Lemma 155.2 (1) shows that the horizontal maps are isomorphisms.

(4) As we explained above, the top diagonal map $\partial\colon \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1})$ is the connecting homomorphism of the pair (X^n, X^{n-1}) and the bottom diagonal map $j_{n-1}\colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map from absolute homology to relative homology. The map ∂_n^{CW} is by definition the composition of these two maps. In other words, the triangle commutes by definition.

(5) It follows fairly immediately from the definition of the connecting homomorphism, see the LES-of-Triple Proposition 146.5, and the definition of the maps $\Phi_s^{\leq}$ that the pentagon to the left commutes.

(6) It follows from (4) and (5) that the outer square of the diagram also commutes.

(7) The horizontal maps are basically by definition natural.

The above discussion shows that the horizontal maps define a natural isomorphism of chain complexes. ∎

For the reader's convenience we recall the following proposition:

Proposition 155.8. (Singular-Cellular-H_*-Proposition) Given any CW-complex X and given any $n \in \mathbb{N}_0$ there exists a uniquely determined isomorphism

$$\Phi_X\colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[1895]

$$\begin{array}{ccccc} & & \mathrm{H}_n(X) & & \\ & \nearrow & & \searrow^{\Phi_X} & \\ \mathrm{H}_n(X^n) & & & & \mathrm{H}_n^{\mathrm{CW}}(X). \\ & \searrow_{j_n} & & \nearrow & \\ & & \ker(\partial_n^{\mathrm{CW}}) & & \end{array}$$

Furthermore this map Φ_X is natural.

Now we can put everything together to obtain a proof of the Simplicial-Singular Homology Theorem 177.10.

Proof of Theorem 177.10. It follows immediately from the definitions that we only need to consider the case of the topological realization of an abstract simplicial complex K. We equip $|K|$ with the CW-structure provided by the Simplicial-Implies-CW Lemma 174.51. Now given any $n \in \mathbb{N}_0$ we have the following two natural isomorphisms:

$$\mathrm{H}_n^{\mathrm{simp}}(K) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n^{\mathrm{CW}}(|K|) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(|K|).$$

natural isomorphism Υ_K given by the Simplicial-Cellular-C_*-Proposition 177.11

natural isomorphism $\Phi_{|K|}$ given by the Singular-Cellular-H_*-Proposition 155.8

The natural isomorphism $\Phi_{|K|}^{-1} \circ \Upsilon_K$ is the natural isomorphism of our desires. ∎

One problem with the Simplicial-Singular Homology Theorem 177.10 is that it does not give us an explicit isomorphism between the homology groups. This issue gets mostly resolved by the following more sophisticated theorem.

Theorem 177.12. (Simplicial-Singular Chain Complex Theorem) Let K be an *ordered* abstract simplicial complex. The maps

$$\begin{array}{rcl} \Theta_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq} \colon \Delta^n \to |K|) \end{array}$$

are a natural[1896] chain homotopy equivalence. In particular for each $n \in \mathbb{N}_0$ the induced map

$$\Theta_* \colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \to \mathrm{H}_n(|K|)$$

is a natural isomorphism.

Remark. The Simplicial-Singular Chain Complex Theorem 177.12 can actually be quite useful in practice. Let us elaborate. In practice, given a topological space X one would like to perform the following three types of tasks:

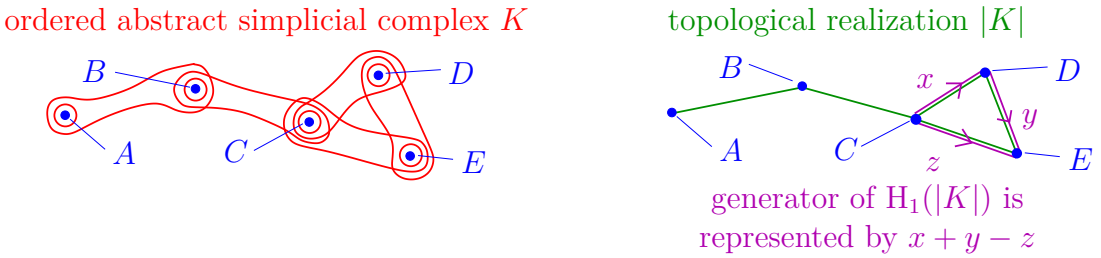
(1) Determine the isomorphism type of the homology groups $\mathrm{H}_k(X)$.
(2) Determine explicit cycles in $\mathrm{C}_k(X)$ that represent a basis or a generating set for the homology groups $\mathrm{H}_k(X)$.
(3) Calculate induced maps $\mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ with respect to, say, given bases or generating sets for the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(Y)$.

[1895] Here we again denote by $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. This implies that the map $j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(\partial_n^{\mathrm{CW}})$.
[1896] Here "natural" means with respect to the category of *ordered* abstract simplicial complexes.

In fact in many situations, e.g. in applications of the Mayer–Vietoris Theorem 153.2, these problems feed on one another. Even though by now we have a long list of tools, the fact that the singular chain groups $\mathrm{C}_*(X)$ are basically always of infinite, in fact mostly of uncountable, rank makes it hard to attack the above problems directly.

Fortunately, in many situations the Simplicial-Singular Chain Complex Theorem 177.12 allows us to translate the above problems into tasks in a simplicial chain complex. For reasonable abstract simplicial complexes, these problems in turn boil down to elementary problems in linear algebra that usually can be solved easily. □

Example. We consider the 1-dimensional ordered abstract simplicial complex K shown in the figure below which we had already considered on page 3345.



By the discussion on page 3345 we know that

$$\mathrm{H}_1^{\mathrm{simp},\leq}(K) \;=\; \ker\big(\partial_1 \colon \mathrm{C}_1^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(K)\big) \;=\; \mathbb{Z}\cdot[\{C,D\}+\{D,E\}-\{C,E\}].$$

We write $x = \Theta_1(\{C,D\})$, $y = \Theta_1(\{D,E\})$ and $z = \Theta_1(\{C,E\})$. It follows from the Simplicial-Singular Chain Complex Theorem 177.12 and the above discussion that $x+y-z$ represents a generator of $\mathrm{H}_1(|K|)$.[1897] □

Proof of the Simplicial-Singular Chain Complex Theorem 177.12. Let K be an ordered abstract simplicial complex. It follows almost immediately from the definitions that the maps

$$\begin{array}{rcl} \Theta_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq} \colon \Delta^n \to |K|) \end{array}$$

define a natural chain map. We leave it to the skeptical reader to verify that this statement is indeed true and basically trivial. It remains to show that the maps Θ_n define a chain homotopy equivalence. By the Quasi Iso–Homotopy Equivalence Proposition 158.8 it now suffices to prove the following claim:

Claim. Let K be an ordered abstract simplicial complex. For every $n \in \mathbb{N}_0$ the induced map

$$\Theta_* \colon \mathrm{H}_n^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{H}_n(|K|)$$

is an isomorphism.

Proof. We start out with two observations. Let K be an ordered abstract simplicial complex, let $m \in \mathbb{N}_0 \cup \{\infty\}$ and let $n \in \mathbb{N}_0$. We write $X = |K|$. It follows from the Simplicial Skeleton–Homology Lemma 176.16 and the CW-Skeleton-H_*-Lemma 155.2 (3) and (2) that the following statements hold:

(i) The two inclusion-induced maps

$$\begin{array}{rcl} \mathrm{H}_n^{\mathrm{simp},\leq}(K^m) & \to & \mathrm{H}_n^{\mathrm{simp},\leq}(K) \\ \mathrm{H}_n(|K^m|) & \to & \mathrm{H}_n(|K|) \end{array}$$

are both isomorphisms whenever $n < m$.

[1897] What other methods do you know for proving this statement?

(ii) If $n > m$, then $\begin{aligned} \mathrm{H}_n^{\mathrm{simp},\leq}(K^m) &= 0 \\ \mathrm{H}_n(|K^m|) &= 0. \end{aligned}$

It follows immediately from (i) that it suffices to prove the claim for finite-dimensional ordered abstract simplicial complexes. As usual, for finite-dimensional ordered abstract simplicial complexes we proceed by induction on the dimension. The case of an empty and of a 0-dimensional ordered abstract simplicial complex is trivial. So suppose we have shown the statement for all ordered abstract simplicial complexes of dimension $\leq m-1$. Now let K be an m-dimensional ordered abstract simplicial complex. We denote by $i\colon K^{m-1} \to K$ and $i\colon X^{m-1} \to X$ the inclusion maps. It follows from (i) and (ii), the naturality of Θ and our induction hypothesis that we only need to worry about the two maps

$$\Theta_*\colon \mathrm{H}_m^{\mathrm{simp},\leq}(K) \to \mathrm{H}_m(X) \quad \text{and} \quad \Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \to \mathrm{H}_{m-1}(X).$$

To take care of this map we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_m^{\mathrm{simp},\leq}(K) & \hookrightarrow & \mathrm{C}_m^{\mathrm{simp},\leq}(K) & \xrightarrow{\partial} & \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1}) & \xrightarrow{i_*} & \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \Theta_*} & & \cong\downarrow{\scriptstyle \Psi_*} & & \downarrow & & \downarrow{\scriptstyle \Theta_*} \\
 & & & & \mathrm{C}_m^{\mathrm{CW}}(X) & & \cong\downarrow{\scriptstyle \Theta_*} & & \\
 & & & & \downarrow{\scriptstyle \mathrm{id}} & & & & \\
0 & \longrightarrow & \mathrm{H}_m(X) & \longrightarrow & \mathrm{H}_m(X, X^{m-1}) & \xrightarrow{\partial} & \mathrm{H}_{m-1}(X^{m-1}) & \xrightarrow{i_*} & \mathrm{H}_{m-1}(X) \longrightarrow 0.
\end{array}$$

We make the following observations and clarifications:

(1) First we show that the horizontal maps on the top actually make sense. Since $K = K^m$ we see that $\mathrm{H}_m^{\mathrm{simp},\leq}(K)$ is a subgroup of $\mathrm{C}_m^{\mathrm{simp},\leq}(K)$. In fact by definition it equals $\ker(\partial\colon \mathrm{C}_m^{\mathrm{simp},\leq}(K) \to \mathrm{C}_{m-1}^{\mathrm{simp},\leq}(K))$. Furthermore note, that basically by definition, given $s \in \mathrm{C}_m^{\mathrm{simp},\leq}(K)$ we have $\partial s \in \mathrm{C}_{m-1}^{\mathrm{simp},\leq}(K^{m-1})$. Since $\partial \circ \partial = 0$ we see that ∂s defines an element in $\mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1})$.
(2) Using (1) and using the explicit definitions of simplicial homology groups it is not hard to see that the top horizontal sequence is exact.
(3) Note that it follows from $X = X^m$ and the CW-Skeleton-H_*-Lemma 155.2 (1) that $\mathrm{H}_i(X, X^{m-1}) = 0$ for $i \neq m$. Thus we see that the bottom horizontal sequence is a segment of the long exact sequence of the pair (X, X^{m-1}).
(4) It follows easily from the definition of the maps Θ and Ψ and the explicit description of the connecting homomorphism $\partial\colon \mathrm{H}_m(X, X^{m-1}) \to \mathrm{H}_{m-1}(X^{m-1})$ that we gave in the LES-of-Triple Proposition 146.5 that the two squares to the left commute. It follows from the naturality of Θ_* that the square to the right commutes.
(5) It follows from induction that the vertical map $\Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1}) \to \mathrm{H}_{m-1}(X^{m-1})$ to the right is an isomorphism.
(6) By the Simplicial-Cellular-C_*-Proposition 177.11 we know that Ψ_* is an isomorphism.
(7) It follows from all of the above together with the Five-Lemma 145.11 that the vertical maps $\Theta_*\colon \mathrm{H}_m^{\mathrm{simp},\leq}(K) \to \mathrm{H}_m(X)$ and $\Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \to \mathrm{H}_{m-1}(X)$ to the left and right are isomorphisms. ■

Given an abstract simplicial complex K together with an order "$\leq$" on K we denote by $\Omega_{\leq,k}\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k^{\mathrm{simp}}(K)$ the natural chain isomorphism given by the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15. Together with the Simplicial-Singular Chain

Complex Theorem 177.12 we obtain an isomorphism

$$\mathrm{H}_k^{\mathrm{simp}}(K) \xleftarrow{\Omega_{\leq,k}^{-1}} \mathrm{H}_k^{\mathrm{simp},\leq}(K) \xrightarrow{\Theta_{\leq,k}} \mathrm{H}_k(|K|).$$

A priori this isomorphism depends on the choice of the order "$\leq$". The next lemma shows that fortunately this is not the case.

Lemma 177.13. (Simplicial-to-Singular Chain Homotopy Lemma) Let K be an abstract simplicial complex and let "$\leq$" and "$\subseteq$" be two orders on K. The two chain maps

$$\begin{aligned} &\mathrm{C}_k^{\mathrm{simp}}(K) \xrightarrow{\Omega_{\leq,k}^{-1}} \mathrm{C}_k^{\mathrm{simp},\leq}(K) \xrightarrow{\Theta_{\leq,k}} \mathrm{C}_k(|K|) \\ \text{and} \quad &\mathrm{C}_k^{\mathrm{simp}}(K) \xrightarrow{\Omega_{\subseteq,k}^{-1}} \mathrm{C}_k^{\mathrm{simp},\subseteq}(K) \xrightarrow{\Theta_{\subseteq,k}} \mathrm{C}_k(|K|) \end{aligned}$$

are chain homotopic. In particular they induce the same maps on homology.

Proof. Let $K = (V, S)$ be an abstract simplicial complex. We start with the following preparations:

(1) For each $i \in \mathbb{N}_0$ we pick once and for all for each i-simplex $s \in S$ an ordered i-simplex $\widetilde{s}$ (in the sense of Definition 176.10) with $\lfloor \widetilde{s} \rfloor = s$.[1898] We denote by $\widetilde{S}_i$ the set given by all these ordered i-simplices.

(2) We introduce the following convention. If for some $i \in \mathbb{N}_0$ we are given for every $\widetilde{s} \in \widetilde{S}_i$ a singular chain $P_i(\widetilde{s}) \in \mathrm{C}_{i+1}(|\widetilde{s}|)$ then we denote by P_i the map

$$\begin{aligned} \mathrm{C}_i^{\mathrm{simp}}(K) &\to \mathrm{C}_{i+1}(|K|) \\ \widetilde{s} &\mapsto P_i(\widetilde{s}) \quad \text{viewed as an element in } \mathrm{C}_{i+1}(|K|) \text{ via the inclusion } |\widetilde{s}| \subseteq |K| \end{aligned}$$

Now let "$\leq$" and "$\subseteq$" be two orders on $K = (V, S)$.

Claim. Let $l \in \mathbb{N}$. Suppose that for every $i \in \{0, \ldots, l-1\}$ and for every $\widetilde{s} \in \widetilde{S}_i$ we are given a singular chain $P_i(\widetilde{s}) \in \mathrm{C}_{i+1}(|\widetilde{s}|)$ such that for every $i \in \{0, \ldots, l-1\}$ we have the equality

$$(*) \qquad \partial_{i+1} \circ P_i + P_{i-1} \circ \partial_i \;=\; \Theta_{\leq,i} \circ \Omega_{\leq,i}^{-1} - \Theta_{\subseteq,i} \circ \Omega_{\subseteq,i}^{-1} \colon \mathrm{C}_i^{\mathrm{simp}}(K) \to \mathrm{C}_i(|K|).$$

Then for every $\widetilde{s} \in \widetilde{S}_l$ we can also find $P_l(s) \in \mathrm{C}_{l+1}(|\widetilde{s}|)$ such that the above equality holds for $i = 0, \ldots, l$.

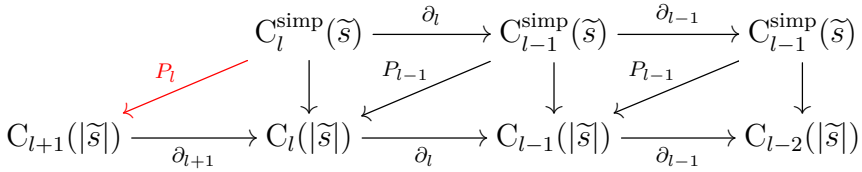
Proof. Let $\widetilde{s} \in \widetilde{S}_l$. Note that we have the following equalities:

$$\begin{aligned} \partial_l((-P_{l-1} \circ \partial_l + \Theta_{\leq,l} \circ \Omega_{\leq,l}^{-1} - \Theta_{\subseteq,l} \circ \Omega_{\subseteq,l}^{-1})(\widetilde{s})) &\overset{\text{since } \Theta_{\leq,i} \circ \Omega_{\leq,i}^{-1} \text{ and } \Theta_{\subseteq,i} \circ \Omega_{\subseteq,i}^{-1} \text{ are chain maps}}{=} \\ &= (-\partial_l \circ P_{l-1} + \Theta_{\leq,l-1} \circ \Omega_{\leq,l-1}^{-1} - \Theta_{\subseteq,l-1} \circ \Omega_{\subseteq,l-1}^{-1})(\partial_l(\widetilde{s})) \\ &= (P_{l-2} \circ \partial_{l-1})(\partial_l(\widetilde{s})) \;=\; P_{l-2}(\partial_{l-1} \circ \partial_l)(\widetilde{s}) \;=\; 0. \end{aligned}$$

$\uparrow$ by equality $(*)$ for $i = l-1$

[1898] Note we just pick a random order for each simplex, the choice is independent of "$\leq$" and "$\subseteq$".

Note that by construction all terms are contained in $\mathrm{C}_l(|\widetilde{s}|) \subseteq \mathrm{C}_l(|K|)$. For orientation it is helpful to consider the following diagram:



Note that we are dealing with $l \in \mathbb{N}$. This implies that $\mathrm{H}_l(|\widetilde{s}|) = 0$. But this means that every cycle is in fact a boundary. In particular the above cycle is the boundary of some $P_{l+1}(\widetilde{s}) \in \mathrm{C}_{l+1}(|\widetilde{s}|)$, i.e. we have

$$(\partial_{l+1}(P_{l+1}(\widetilde{s})) \;=\; (-P_{l-1} \circ \partial_l + \Theta_{\leq,l} \circ \Omega_{\leq,l}^{-1} - \Theta_{\subseteq,l} \circ \Omega_{\subseteq,l}^{-1})(\widetilde{s}).$$

By construction all these $P_{l+1}(\widetilde{s})$ have the desired property. ⊞

Now we can easily find the desired chain homotopy. Indeed, for every 0-simplex s we set $P_0(s) = 0 \in \mathrm{C}_1(|s|)$. Since $\Theta_{\leq,0} \circ \Omega_{\leq,0}^{-1} = \Theta_{\subseteq,0} \circ \Omega_{\subseteq,0}^{-1} \colon \mathrm{C}_0^{\mathrm{simp}}(K) \to \mathrm{C}_0(|K|)$ we see that can start the machine provided by the claim and iteratively we construct maps P_i which are precisely the promised chain homotopy. ■

The following is now an almost immediate corollary to the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15, the Simplicial-Singular Chain Complex Theorem 177.12 and the Simplicial-to-Singular Chain Homotopy Lemma 177.13.

Corollary 177.14. (Simplicial-to-Singular Homology–Order Corollary) Let K be an abstract simplicial complex. We pick an order "$\leq$" on K. The maps

$$\Xi_k := \Theta_{\leq,k} \circ \Omega_{\leq,k}^{-1} \colon \mathrm{H}_k^{\mathrm{simp}}(K) \;\to\; \mathrm{H}_k(|K|),$$

which by the Simplicial-to-Singular Chain Homotopy Lemma 177.13 do not depend on the choice of the order "$\leq$", define a natural isomorphism between functors from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes (the key here is that the word "ordered" is missing) to the category $\mathcal{AbGroup}$ of abelian groups.

It is worth repeating the following:

(1) Given any abstract simplicial complex K and given any $k \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{H}_k^{\mathrm{simp}}(K) \to \mathrm{H}_k(|K|)$ of homology groups.
(2) If K is an ordered abstract simplicial, then there is in fact a natural chain homotopy equivalence $\mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*(|K|)$.

The alert reader will have noticed that given an abstract simplicial complex we gave two natural isomorphisms $\mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(|K|)$. The following proposition shows that the two isomorphisms agree.

Proposition 177.15. (Isomorphism-Independence-of-Order Proposition) Let K be an abstract simplicial complex and let $n_0 \in \mathbb{N}$. We write $X = |K|$. We denote by

$$\Upsilon_* \colon \mathrm{C}_*^{\mathrm{simp}}(K) \;\to\; \mathrm{C}_*^{\mathrm{CW}}(X) \quad \text{and} \quad \Phi_X \colon \mathrm{H}_n(X) \;\to\; \mathrm{H}_n^{\mathrm{CW}}(X)$$

the isomorphisms given by the Simplicial-Cellular-C_*-Proposition 177.11 and the Singular-Cellular-H_*-Proposition 155.8.[1899]Next we pick an order "$\leq$" on K. We consider the chain

homotopy equivalences

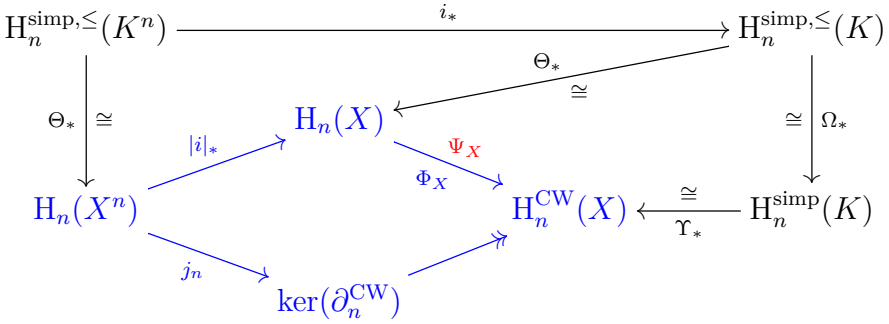
$$\Omega\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*^{\mathrm{simp}}(K) \quad \text{and} \quad \Theta\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*(|K|)$$

from the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 and the Simplicial-Singular Chain Complex Theorem 177.12. With this notation the following equality holds:

$$\Phi_X^{-1} \circ \Upsilon_* = \Theta_* \circ \Omega_*^{-1}\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(X).$$

Notation 177.16. Let K be an abstract simplicial complex and let $n_0 \in \mathbb{N}$. By a slight abuse of notation we denote by $\Theta_K\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(|K|)$ the natural isomorphism given by the Simplicial-Singular Homology Theorem 177.10 and the Isomorphism-Independence-of-Order Proposition 177.15.

Proof. We write $\Psi_X := \Upsilon_* \circ \Omega_* \circ \Theta_*^{-1}$. We need to show that $\Psi_X = \Phi_X$. To do so we consider the following diagram:



We make the following clarifications and observations:

(1) The map $i\colon K^n \to K$ is the inclusion map.
(2) The triangle shaped quadrangle to the upper left commutes by the naturality of Θ_*.
(3) By the definition of Ψ_X the quadrangle to the right commutes.
(4) It follows easily from the definitions of the various maps that the "outer diagram" commutes, i.e. the two maps from $\mathrm{H}_n^{\mathrm{simp},\leq}(K^n)$ to $\mathrm{H}_n^{\mathrm{CW}}(X)$ agree.
(5) Since Υ_*, Ω_* and Θ_* are isomorphisms we see that Ψ_X is an isomorphism and we see, in combination with (2), (3) and (4), that Ψ_X makes the parallelogram commute. But by the characterization of Φ_X in the Singular-Cellular-H_*-Proposition 155.8 this means that $\Psi_X = \Phi_X$. ∎

177.3. **Relative simplicial homology.** In this section, given an (ordered) abstract simplicial complex K together with a subcomplex L we introduce the relative simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(K, L)$ and $\mathrm{H}_k^{\mathrm{simp},\leq}(K, L)$. This section contains no surprises and can be skipped easily. We will make little use of relative simplicial homology groups, except that they will come in handy in the proof of the Simplicial Poincaré Duality Theorem 188.5.

Definition 177.17. Let K be an abstract simplicial complex and let L be a (possibly empty) subcomplex of K.

(1) Given $k \in \mathbb{N}_0$ we define $\mathrm{C}_k^{\mathrm{simp}}(K, L) := \mathrm{C}_k^{\mathrm{simp}}(K)/\mathrm{C}_k^{\mathrm{simp}}(L)$.

[1899] In the Singular-Cellular-H_*-Proposition 155.8 the isomorphism was called Θ_X instead of Φ_X. Unfortunately we are running out of Greek characters.

The usual boundary map $\partial_k\colon \mathrm{C}_k^{\mathrm{simp}}(K) \to \mathrm{C}_{k-1}^{\mathrm{simp}}(K)$ descends to a boundary map $\partial_k\colon \mathrm{C}_k^{\mathrm{simp}}(K,L) \to \mathrm{C}_{k-1}^{\mathrm{simp}}(K,L)$ and we can thus define the corresponding relative simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(K,L)$.

(2) If "$\leq$" is an order on K, then we endow L with the corresponding order and we introduce $\mathrm{C}_k^{\mathrm{simp},\leq}(K,L)$ and $\mathrm{H}_k^{\mathrm{simp},\leq}(K,L)$ in the obvious way.

Lemma 177.18. (Simplicial Homology–LES Lemma) Let K be an abstract simplicial complex and let L be a subcomplex of K. The sequence

$$\begin{array}{ccccccccccc} \ldots & \to & \mathrm{H}_k^{\mathrm{simp}}(L) & \to & \mathrm{H}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp}}(L) & \to & \ldots \\ & & & & & & [\sigma] & \mapsto & [\partial_k(\sigma)] & & \end{array}$$

is exact and the connecting homomorphism is natural. The analogous statement also holds if K is equipped with an order.

Proof. The lemma is an immediate consequence of the LES-Proposition 145.10. ∎

We conclude this short section with the following lemma:

Lemma 177.19. (Simplicial-Singular Homology Lemma) Let K be an abstract simplicial complex and let L be a subcomplex of K.

(1) Let "$\leq$" be an order on K. The natural maps

$$\begin{array}{ccc} \mathrm{C}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{C}_k^{\mathrm{simp}}(K,L) \\ \sigma & \mapsto & \Omega(\sigma) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{C}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{C}_k(|K|,|L|) \\ \sigma & \mapsto & \Theta(\sigma) \end{array}$$

are chain homotopy equivalences. Furthermore the natural maps

$$\begin{array}{ccc} \Omega_*\colon \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{H}_k^{\mathrm{simp}}(K,L) \\ [\sigma] & \mapsto & [\Omega(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \Theta_*\colon \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{H}_k(|K|,|L|) \\ [\sigma] & \mapsto & [\Theta(\sigma)] \end{array}$$

are isomorphisms.

(2) The map $\Xi := \Theta_* \circ \Omega_*^{-1}\colon \mathrm{H}_k^{\mathrm{simp}}(K,L) \to \mathrm{H}_k(|K|,|L|)$ does not depend on the choice of the order "$\leq$" on K, and it defines a natural isomorphism.

(3) The following diagram commutes:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(K) & \longrightarrow & \ldots \\ & & \Xi\downarrow & & \Xi\downarrow & & \Xi\downarrow & & \Xi\downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_k(|K|) & \longrightarrow & \mathrm{H}_k(|K|,|L|) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(|L|) & \longrightarrow & \mathrm{H}_{k-1}(|K|) & \longrightarrow & \ldots \end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(L) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp}}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp}}(K) & \longrightarrow & \ldots \\ & & \Omega_*\uparrow & & \Omega_*\uparrow & & \Omega_*\uparrow & & \Omega_*\uparrow & & \Omega_*\uparrow & & \\ \ldots & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(K) & \longrightarrow & \ldots \\ & & \Theta_*\downarrow & & \Theta_*\downarrow & & \Theta_*\downarrow & & \Theta_*\downarrow & & \Theta_*\downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_k(|L|) & \longrightarrow & \mathrm{H}_k(|K|) & \longrightarrow & \mathrm{H}_k(|K|,|L|) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(|L|) & \longrightarrow & \mathrm{H}_{k-1}(|K|) & \longrightarrow & \ldots \end{array}$$

It follows easily from the definitions and the naturality of the connecting homomorphisms, see the LES-Proposition 145.10, that the diagram commutes. By the Simplicial Homology–LES Lemma 177.18 and by the LES-of-Pair Proposition 146.7 we know that the horizontal sequences are exact. Furthermore by the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 and the Simplicial-Singular Chain Complex Theorem 177.12 we know that the outer two vertical maps are isomorphisms. Now it follows from the Five-Lemma 145.11 that the middle vertical maps are also isomorphisms. It follows from the Quasi Iso–Homotopy Equivalence Proposition 158.8 that the corresponding chain maps are chain homotopy equivalences. This concludes the proof of Statement (1).

The same argument as in the proof of the Simplicial-to-Singular Homology–Order Corollary 177.14 shows that the composition of the middle vertical maps does not depend on the choice of the order "$\leq$".[1900]

Finally Statement (3) follows from the fact that the above diagram commutes. ■

177.4. **Appendix: The simplicial subdivision map.** Let $(K, \leq)$ be an ordered abstract simplicial complex. By the Simplicial-Singular Homology Theorem 177.10 and the SD-Lemma 175.5 we know that the simplicial homology groups of K and of its barycentric subdivision $\mathrm{sd}(K)$ are isomorphic.

In this rather technical appendix we will strengthen this result. More precisely, we will give a natural map $\sigma\colon \mathrm{sd}(K) \to K$ and we will give a natural chain map

$$u_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K))$$

such that the chain maps σ_* and u_* are chain homotopy inverses of one another. These two constructions will prove their worth later on when we will study continuous self-maps $f\colon |K| \to |K|$. More precisely, by the Simplicial Approximation Theorem 175.15 we know that a given continuous map $f\colon |K| \to |K|$ is homotopic to a simplicial map $g\colon |\mathrm{sd}^i(K)| \to |K|$, thus we get an induced chain map $g_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}^i(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$. Unfortunately this is a chain map between *different* chain complexes. But if we combine g_* with an i-fold iteration of u_*, then we obtain an *endomorphism* $\mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}^i(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}^i(K))$ of a chain complex. The reader who is not excited by these results might want to jump to the next Chapter 178 where we discuss the enticingly called "Nerve Theorem".

We start out with the following lemma:

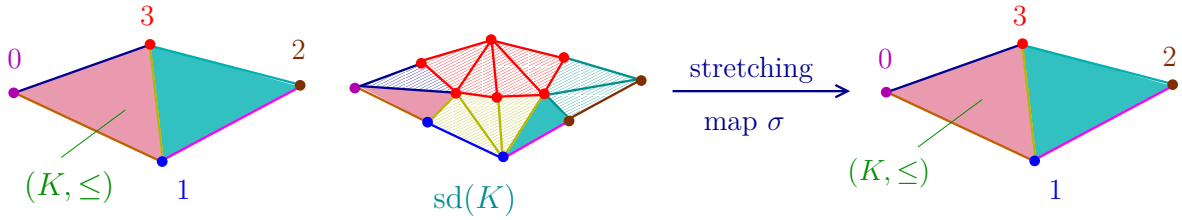
Lemma 177.20. (Stretching Map Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex.

(1) There exists a unique simplicial map $\sigma\colon \mathrm{sd}(K) \to K$, called stretching map, with the following property: Given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K the map σ sends the corresponding vertex $\underline{s}$ of $\mathrm{sd}(K)$ to the vertex v_n of K.
(2) The stretching map $\sigma\colon \mathrm{sd}(K) \to K$ is order-preserving. (Here we equip $\mathrm{sd}(K)$ with the natural order introduced in the Barycentric Subdivision Lemma 175.3 (2).)
(3) The stretching map σ is natural. Here "natural" means that given any order-preserving simplicial map $f\colon (K, \leq) \to (L, \leq)$ between two ordered abstract simplicial complexes

[1900] Note that Exercise 145.19 shows that one cannot reduce this statement "Five Lemma"-style to the fact that we already know that the composition of the outer two vertical maps is independent of the choice of the order "$\leq$".

and given any $n \in \mathbb{N}_0$ the following diagram commutes:[1901]

$$\begin{array}{ccc} \mathrm{sd}(K) & \xrightarrow{f_*} & \mathrm{sd}(L) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ K & \xrightarrow{f} & L. \end{array}$$



Proof. It follows easily from the definitions that the given map on the set of vertices induces a simplicial map, that this map is order-preserving and that this map is natural. We leave it to the reader to fill in the details. ■

In basically all cases the stretching map $\sigma\colon \mathrm{sd}(K) \to K$ is not a simplicial isomorphism, in particular the induced map on the simplicial chain complexes is not an isomorphism. Our next goal is to show the "best possible result", namely that the induced chain map $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$ is a chain homotopy equivalence. We will do so by giving an explicit chain map in the other direction.

Definition 177.21. Let $(K, \leq)$ be an ordered abstract simplicial complex.

(1) Given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K, given[1902] $\sigma \in \mathrm{Bij}\{0, \ldots, n\}$ and given $m \in \{0, \ldots, n\}$ we write

$$s_{\sigma,m} \coloneqq \{v_{\sigma(0)}, \ldots, v_{\sigma(m)}\}.$$

Note that $s_{\sigma,n} = s$. Furthermore, we set

$$s_\sigma \coloneqq \{\overbrace{\underline{s_{\sigma,n}}}^{=\underline{s}}, \underline{s_{\sigma,n-1}}, \ldots, \underline{s_{\sigma,0}}\}.$$

Note that s_σ is an n-simplex of $\mathrm{sd}(K)$.

(2) Given an n-simplex s of K we define

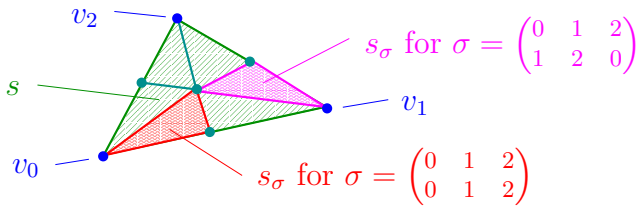
$$u_n(s) \coloneqq \sum_{\sigma \in \mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma) \cdot s_\sigma \in \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)).$$

We refer to the resulting map[1903]

$$\begin{array}{rccc} u_n\colon & \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ & s & \mapsto & u_n(s) \end{array}$$

as the simplicial subdivision map.

[1901] Recall that by the SD-Lemma 175.5 (5) we know that the maps $K \mapsto \mathrm{sd}(K)$ define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to itself.



The following lemma can be viewed as an analog of the Chain Subdivision Lemma 148.11 and the Subdivision–Natural Lemma 148.10.

Lemma 177.22. (Chain Subdivision Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex. The simplicial subdivision maps

$$u_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n \in \mathbb{N}_0$$

are a natural chain map. Here "natural" means that given any order-preserving simplicial map $f \colon (K, \leq) \to (L, \leq)$ between two ordered abstract simplicial complexes and given any $n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{simp},\leq}(L) \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_n} \\ \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(L)). \end{array}$$

Proof. Let $(K, \leq)$ be an ordered abstract simplicial complex. We need to show that the maps

$$u_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n \in \mathbb{N}_0$$

are a natural chain map. In fact it follows immediately from the definitions and the Barycenter Lemma 175.2 (1) that the maps u_n are natural. Thus it remains to prove that the maps u_n are a chain map. We will do so in the following claim:

Claim. For any $n \in \mathbb{N}_0$ and any n-simplex s of K we have the following equality:

$$\partial_n(u_n(s)) \;=\; u_{n-1}(\partial_n(s)).$$

Proof. Throughout the proof of the claim we work with the following notation: Given any ordered n-simplex $t = \{w_0 < \cdots < w_n\}$ and given any $i \in \{0, \ldots, n\}$ we write $t(i) \coloneqq \{w_0 < \cdots < \widehat{w}_i < \cdots < w_n\}$, i.e. $t(i)$ is the $(n-1)$-simplex given by striking out the vertex w_i. Now let $s = \{v_0 < \cdots < v_n\}$ be an n-simplex of K. We perform the following

[1902]Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Bij}\{0, \ldots, n\}$ the group of bijections of the set $\{0, \ldots, n\}$. Note that we have an obvious isomorphism $\mathrm{Bij}\{0, \ldots, n\} \cong S_{n+1}$. Given $\sigma \in \mathrm{Bij}\{0, \ldots, n\}$ we use, as on in Definition 176.1, this isomorphism to define $\mathrm{sign}(\sigma)$.

[1903]Recall that $\mathrm{C}_n^{\mathrm{simp},\leq}(K)$ is by definition the free abelian group generated by the set of simplices. Thus to specify a homomorphism $\varphi \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to A$ to some abelian group we only need to specify $\varphi(s)$ for each n-simplex s of K.

slightly messy calculation:

$$\partial_n(u_n(s)) \overset{\text{by definition of } u_n}{=} \partial_n\Big(\sum_{\sigma\in \mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot s_\sigma\Big) \overset{\text{Definition 176.7 of } \partial_n}{=} \sum_{i=0}^{n}(-1)^i \cdot \sum_{\sigma\in \mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot s_\sigma(i) \overset{\text{split the sum into } i=0 \text{ and } i=1,\dots,n}{=}$$

$$= \sum_{\sigma\in \mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot (s_\sigma)(0) + \sum_{i=1}^{n}(-1)^i \cdot \sum_{\sigma\in \mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot s_\sigma(i)$$

the second summand vanishes since for a fixed $i\in\{1,\dots,n\}$ and for $\mu,\nu\in \mathrm{Bij}\{0,\dots,n\}$ we have $s_\mu(i)=s_\nu(i)$ if $\mu(i-1)=\nu(i)$, if $\mu(i)=\nu(i-1)$ and if all other values of μ and ν are the same, thus given the fixed $i\in\{1,\dots,n\}$ we can pair up the elements in $\mathrm{Bij}\{0,\dots,n\}$ to give the same contribution $s_\sigma(i)$ to the sum, but with opposite sign

$$\overset{}{=} \sum_{\sigma\in \mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot (s_\sigma)(0) = \sum_{\tau\in \mathrm{Bij}\{0,\dots,n-1\}} \sum_{i=0}^{n} (-1)^i \cdot \mathrm{sign}(\tau)\cdot (s(i))_\tau = u_{n-1}(\partial_n(s)).$$

this follows easily from considering the following bijection: $\{0,\dots,n\}\times \mathrm{Bij}\{0,\dots,n-1\} \to \mathrm{Bij}\{0,\dots,n\}$

by Definition 183.11 of ∂_n and u_{n-1}

$$(i,\tau)\mapsto \left(\begin{array}{rcl} \{0,\dots,n\} & \to & \{0,\dots,n\} \\ j & \mapsto & \begin{cases} i, & \text{if } j=0, \\ \tau(j-1), & \text{if } j>0 \text{ and } \tau(j-1)\in\{1,\dots,i-1\}, \\ \tau(j-1)+1, & \text{if } j>0 \text{ and } \tau(j-1)\in\{i,\dots,n-1\}. \end{cases} \end{array}\right)$$

We have thus verified the desired equality. ■

The following lemma finally relates the stretching maps to the simplicial subdivision maps.

Lemma 177.23. (Stretching-Subdivision Lemma) Let $(K,\leq)$ be an ordered abstract simplicial complex. As before we denote by $\sigma\colon \mathrm{sd}(K)\to K$ the stretching map introduced in the Stretching Map Lemma 177.20.

(1) The induced chain map

$$\sigma_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$$

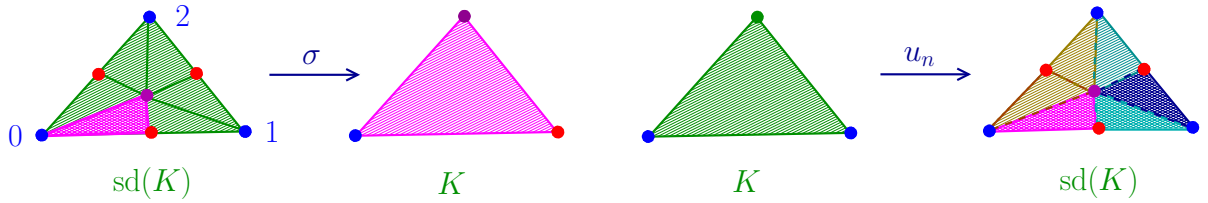
and the simplicial subdivision maps

$$u_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n\in\mathbb{N}_0$$

are chain homotopy inverses of one another.

(2) The maps σ_* and the maps u_* are chain homotopy equivalences.

(3) The maps $\sigma_*\colon \mathrm{H}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K))\to \mathrm{H}_*^{\mathrm{simp},\leq}(K)$ and $u_*\colon \mathrm{H}_*^{\mathrm{simp},\leq}(K)\to \mathrm{H}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K))$ are natural isomorphisms and they are inverses of one another.



Remark. The Stretching-Subdivision Lemma 177.23 (2) can be viewed as an analog of the Subdivision–Chain Homotopy-Lemma 148.12 where we proved a similar statement for the singular subdivision map. □

Proof. Statement (2) of the lemma is an immediate consequence of Statement (1). Furthermore Statement (3) follows from Statements (1) and (2) together with the Chain Homotopy Equivalence-H_*-Corollary 144.10 and the naturality statements of the Stretching Map Lemma 177.20 and the Chain Subdivision Lemma 177.22. Thus it remains to prove Statement (1).

Let $(K, \leq)$ be an ordered abstract simplicial complex. It follows easily from the definitions that for each $n \in \mathbb{N}_0$ the map $\sigma_* \circ u_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(K)$ is the identity. Thus it remains to prove that $u_* \circ \sigma_*$ is chain homotopic to the identity.

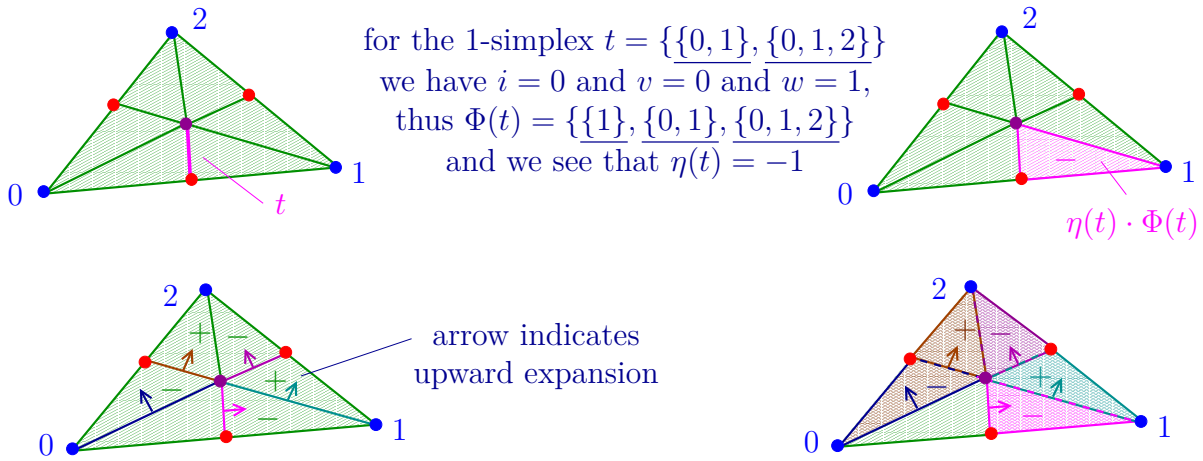
The definition of the chain homotopy requires some preparations and extra definitions. Let $n \in \mathbb{N}_0$ and let $t = \{\underline{s_0}, \ldots, \underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(K)$ with $\#s_n = n+2$. For convenience we set $s_{-1} := \varnothing$. Since $\#s_n = n+2$ and since for each $i \in \{0, \ldots, n\}$ we have $s_{i-1} \subsetneq s_i$ we see that there exists a unique $i \in \{0, \ldots, n\}$ with $\#s_i = \#s_{i-1} + 2$. Thus there exist two vertices $v < w$ of K such that $s_i = s_{i-1} \sqcup \{v, w\}$. We say that t can be *expanded upward* if w is larger than every element in s_{i-1}. In this case we set

$$\Phi(t) := \overbrace{\{\underline{s_0}, \ldots, \underline{s_{i-1}}, \underline{s_{i-1} \cup \{w\}}, \underline{s_i}, \ldots \underline{s_n}\}}^{(n+1)\text{-simplex of } \mathrm{sd}(K)}$$

and

$$\eta(t) := \mathrm{sign}(\sigma) \text{ where } \sigma \in \mathrm{Bij}\{0, \ldots, n\} \text{ with } \Phi(t) = (s_n)_{\sigma}.$$

(Definition 177.21)



For each $n \in \mathbb{N}_0$ we consider the following map

$$\begin{array}{rcl} P_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) & \to & \mathrm{C}_{n+1}^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ t = \{\underline{s_0}, \ldots, \underline{s_n}\} & \mapsto & \begin{cases} \eta(t) \cdot \Phi(t), & \text{if } \#s_n = n+2 \text{ and if } t \text{ can be expanded upward,} \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We leave it to the reader's unrelenting zest for verifying technical details to show that these maps P_n form a chain homotopy between the identity and the chain map $\sigma_* \circ u_*$. ∎

We defined the simplicial subdivision map for *ordered* abstract simplicial complexes. As we will see shortly, the choice of an ordering is irrelevant.

Definition 177.24. Let $(K, \leq)$ be an abstract simplicial complex.

(1) Let $s = (v_0, \ldots, v_n)$ be an ordered n-simplex of K. Given $m \in \{0, \ldots, n\}$ we set

$$F_m(s) \coloneqq \{v_0, \ldots, v_m\}.$$

Furthermore we set

$$F(s) \coloneqq \big(\underline{F_0(s)}, \underline{F_1(s)}, \ldots, \underline{F_n(s)}\big).$$

Note that $F(s)$ is an ordered n-simplex of $\mathrm{sd}(K)$.

(2) Note that any two ordered n-simplices $s = (v_0, \ldots, v_n)$ and $t = (w_0, \ldots, w_n)$ of K with $\{v_0, \ldots, v_n\} = \{w_0, \ldots, w_n\}$ there exist a unique $\sigma \in \mathrm{Bij}\{0, \ldots, n\}$ such that $v_i = w_{\sigma(i)}$ for all $i = 0, \ldots, n$. We set $\mathrm{sign}(s,t) \coloneqq \mathrm{sign}(\sigma)$.

(3) Given an ordered n-simplex $s = (v_0, \ldots, v_n)$ we set

$$u_n(s) \coloneqq \sum_{\substack{t = (w_0, \ldots, w_n) \text{ an ordered} \\ n\text{-simplex of } K \text{ with} \\ \{v_0, \ldots, v_n\} = \{w_0, \ldots, w_n\}}} (-1)^{\mathrm{sign}(s,t)} \cdot F(t)$$

More precisely, we have the following lemma:

Lemma 177.25. (Chain Subdivision Lemma) Let K be an abstract simplicial complex.

(1) For every $n \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \\ [s] & \mapsto & [u_n(s)] \end{array}$$

is well-defined. We refer to this map again as the simplicial subdivision map.

(2) The maps u_n are natural in the sense that given a simplicial map $f\colon K \to L$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{simp}}(L) \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_n} \\ \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(L)). \end{array}$$

(3) If K is equipped with an order "$\leq$", then for every $n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)). \end{array}$$

Here the vertical maps are the natural isomorphisms given by the Simplicial Chain Group–Isomorphism Lemma 176.11.

(4) The maps $u_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K))$ form a chain homotopy equivalence. In fact, if we pick an order on K and we consider the corresponding stretching map $\sigma\colon \mathrm{sd}(K) \to K$, then the maps u_n and the maps $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp}}(K)$ are chain homotopy inverses.

(5) The induced maps $u_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(\mathrm{sd}^{\mathrm{simp}}(K))$ are natural isomorphisms.

Remark. Note that in contrast the stretching maps certainly do depend on the choice of an order. □

Proof. The first three statements follow easily from the definitions. The only statement which perhaps needs a little bit of thought is Statement (1). We leave it to the reader to fill in the details. Let us turn to the proof of Statement (4). As mentioned before, by the Well-Ordering Theorem 1.32 we can pick an order "$\leq$" on K. Let $n \in \mathbb{N}_0$. By (2) we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)). \end{array}$$

By the Stretching-Subdivision Lemma 177.23 we know that the top horizontal maps form a chain homotopy equivalence and that a chain homotopy inverse is given by the stretching maps. It follows that the analogous statements hold for the bottom horizontal maps. Finally Statement (5) follows from (2) and (4) together with the Chain Homotopy Equivalence-H_*-Corollary 144.10. ■

In the remainder of this section we want to discuss the relationship of the stretching map and the simplicial division map with various maps on the topological realizations. First we recall that given an abstract simplicial complex K we gave in the SD-Lemma 175.5 (3) an explicit natural homeomorphism $\mho\colon |\mathrm{sd}(K)| \to |K|$. This allows us to formulate the following lemma.

Lemma 177.26. (Stretching-Homotopy Lemma) Let $(K, \leq)$ be an abstract simplicial complex. We denote by $\sigma\colon \mathrm{sd}(K) \to K$ the stretching map from the Stretching Map Lemma 177.20. The map $|\sigma| \circ \mho^{-1}\colon |K| \to |K|$ is homotopic to the identity.

Proof. Since we will not make use of the Stretching-Homotopy Lemma 177.26, we outsource the proof thereof to Exercise 177.4. ■

We recall some more maps and statements from earlier on.

(1) Let K be an ordered abstract simplicial complex. By the Simplicial-Singular Chain Complex Theorem 177.12 we know that the maps

$$\begin{array}{rcl} \Theta_K\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{array}$$

are a natural chain homotopy equivalence.

(2) Given a topological space X and $n \in \mathbb{N}_0$ we denote by $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ the singular subdivision map as defined on page 2825.

Now we can formulate the following proposition:

Proposition 177.27. (Subdivision–Chain Complexes-Proposition) For every ordered abstract simplicial complex $(K, \leq)$ and every $n \in \mathbb{N}_0$, the following diagram commutes:

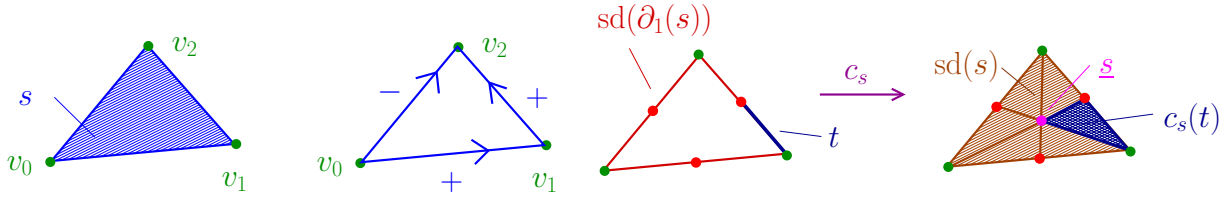
$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{u_n} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ & & \downarrow{\scriptstyle \Theta_{\mathrm{sd}(K)}} \\ \downarrow{\scriptstyle \Theta_K} & & \mathrm{C}_n(|\,\mathrm{sd}(K)|) \\ & & \downarrow{\scriptstyle \upsilon_*} \\ \mathrm{C}_n(|K|) & \xrightarrow{u_n} & \mathrm{C}_n(|K|). \end{array}$$

Furthermore all maps form natural chain homotopy equivalences.

The key step in proving the Subdivision–Chain Complexes-Proposition 177.27 is to give an alternative description of the simplicial subdivision map that resembles the definition of the singular subdivision map which we gave on page 2825. We start out with the following definition:

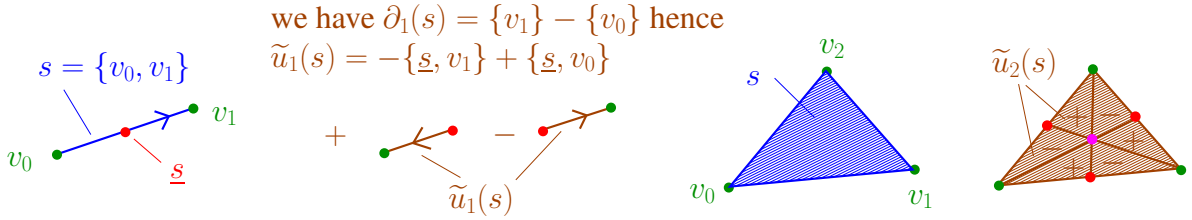
Definition 177.28. Let $(K, \leq)$ be an ordered abstract simplicial complex. Given a k-simplex s of K we define the "coning map"

$$\begin{array}{rccl} c_s \colon & \mathrm{C}_i^{\mathrm{simp},\leq}(\mathrm{sd}(\partial_{k-1}(s))) & \to & \mathrm{C}_{i+1}^{\mathrm{simp},\leq}(\mathrm{sd}(s)) \\ & t & \mapsto & \{\underline{s}\} \cup t. \end{array}$$



Definition 177.29. Let $(K, \leq)$ be an ordered abstract simplicial complex. We define the map [1904] $u_0 \colon \mathrm{C}_0^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(\mathrm{sd}(K))$ to be the obvious inclusion. Furthermore, for $n \geq 1$ we define inductively the following maps:

$$\begin{array}{rccl} \widetilde{u}_n \colon & \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ & s & \mapsto & (-1)^n \cdot c_s\big(\underbrace{u_{n-1}(\partial s)}_{\in \mathrm{C}_{n-1}^{\mathrm{simp},\leq}(\mathrm{sd}(\partial s))}\big). \end{array}$$



The following lemma gives the promised alternative description of the simplicial subdivision maps.

[1904] Recall that in the Barycentric Subdivision Lemma 175.3 (2) we saw that $\mathrm{sd}(K)$ is naturally an ordered abstract simplicial complex.

Lemma 177.30. (Subdivision–Rewriting Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex. Given any n-simplex s of K we have $\widetilde{u}_n(s) = u_n(s)$.

Proof of Lemma 177.30. Let $(K, \leq)$ be an ordered abstract simplicial complex. We prove the lemma by induction on n. For $n = 0$ the statement is basically obvious. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $s = \{v_0 < \cdots < v_{n+1}\}$ be some $(n+1)$-simplex. Given $i \in \{0, \ldots, n+1\}$ we write $s(i) \coloneqq \{v_0, \ldots, \widehat{v}_i, \ldots, v_{n+1}\}$. We calculate that

$$\begin{aligned}
u_{n+1}(s) &\overset{\substack{\text{Definition 177.29 of } \widetilde{u}_{n+1} \\ \downarrow}}{=} (-1)^{n+1} \cdot c_s(\widetilde{u}_n(\partial_n(s))) \overset{\substack{\text{by Definition 176.7 of } \partial_n(s) \\ \downarrow}}{=} (-1)^{n+1} \cdot c_s\Big(\widetilde{u}_n\Big(\sum_{i=0}^{n+1}(-1)^i \cdot s(i)\Big)\Big) \overset{\substack{\text{by induction and definition of } c_s \\ \downarrow}}{=} \\
&= (-1)^{n+1} \cdot \sum_{i=0}^{n+1}(-1)^i \cdot \sum_{\sigma \in \mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma) \cdot \{\underline{s}, s(i)_\sigma\} \\
&\underset{\substack{\uparrow \\ \text{at this point we use the following bijection:}}}{=} \sum_{\tau \in \mathrm{Bij}\{0,\ldots,n+1\}} \mathrm{sign}(\tau) \cdot s_\tau \underset{\substack{\uparrow \\ \text{see Definition 177.21}}}{=} u_{n+1}(s).
\end{aligned}$$

$$\Psi\colon \{0, \ldots, n+1\} \times \mathrm{Bij}\{0, \ldots, n\} \to \mathrm{Bij}\{0, \ldots, n+1\}$$

$$(i, \sigma) \mapsto \left(\begin{array}{rcl} \{0, \ldots, n+1\} & \to & \{0, \ldots, n+1\} \\ j & \mapsto & \begin{cases} \sigma(j), & \text{if } j \leq 0 \text{ and } \sigma(j) \in \{0, \ldots, i-1\}, \\ i, & \text{if } j = n+1, \\ \sigma(j)+1, & \text{if } j \leq n \text{ and } \sigma(j) \in \{i, \ldots, n\} \end{cases} \end{array}\right)$$

and we use that $\mathrm{sign}(\Psi(i, \sigma)) = (-1)^{i+1} \cdot (-1)^{n+2} \cdot \mathrm{sign}(\sigma)$

We have thus proved the induction step. ■

Now we can easily prove the Subdivision–Chain Complexes-Proposition 177.27.

Proof of Proposition 177.27. It follows immediately from the Subdivision–Rewriting Lemma 177.30 together with the definitions of the various maps that the diagram commutes.

By the Stretching-Subdivision Lemma 177.23 and the Subdivision–Chain Homotopy-Lemma 148.12 we know that the horizontal maps form chain homotopy equivalences. Furthermore by the Simplicial-Singular Chain Complex Theorem 177.12 the top vertical maps also form chain homotopy equivalences. Finally the map $\mho$ is a homeomorphism. Thus the bottom vertical maps $\mho_*$ also form a chain homotopy equivalence. Finally note that all maps are defined in such a way that they are obviously natural. ■

We conclude with the following "unordered" version of the Subdivision–Chain Complexes-Proposition 177.27.

Proposition 177.31. (Subdivision–Homologies-Proposition) For every abstract simplicial complex K and every $n \in \mathbb{N}_0$ the following diagram commutes:[1905]

$$\begin{tikzcd}
\mathrm{H}_n^{\mathrm{simp}}(K) \arrow[r, "u_n"] \arrow[dd, "\Theta_K"'] & \mathrm{H}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \arrow[d, "\Theta_{\mathrm{sd}(K)}"] \\
& \mathrm{H}_n(|\,\mathrm{sd}(K)|) \arrow[d, "\mho_*"] \\
\mathrm{H}_n(|K|) \arrow[r, "u_n"] & \mathrm{H}_n(|K|).
\end{tikzcd}$$

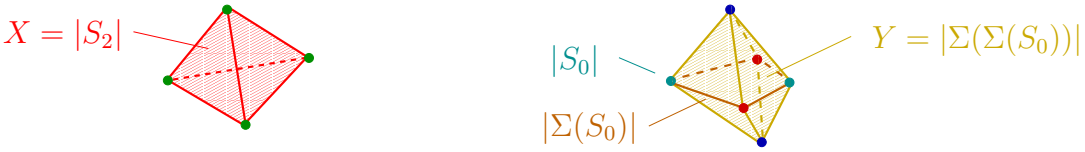
Furthermore all maps are natural.

Proof. The proposition follows easily from the fact that every abstract simplicial complex admits an order together with the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15, the Subdivision–Homologies-Proposition 177.31 and together with the Isomorphism-Independence-of-Order Proposition 177.15. ■

Remark. For the record we point out that some of the definitions and results in this section are at least implicit in [**Deo2018**, Theorem 4.9.1] and [**Mun1966a**, p. 99]. □

177.5. Exercises for Chapter 177 .

Exercise 177.1. Show that the two simplicial complexes $X = |S_2|$ and $Y = |\Sigma(\Sigma(S_0))|$ which are shown in the figure below admit subdivisions X' and Y' which are simplicially isomorphic.



Exercise 177.2. Let K be an abstract simplicial complex. Show that the maps

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{simp}}(K) \otimes \mathbb{Q} & \to & \mathrm{C}_k(|K|) \otimes \mathbb{Q} \\ [v_0, \ldots, v_k] \otimes r & \mapsto & \frac{1}{(k+1)!} \cdot \sum\limits_{\sigma \in S_{k+1}} \cdot \left(\begin{array}{ccc} \Delta^k & \mapsto & |K| \\ (t_0, \ldots, t_k) & \mapsto & \sum\limits_{i=0}^{k} t_i \cdot v_{\sigma(i)} \end{array} \right) \otimes r \end{array}$$

are a natural chain homotopy equivalence.

Exercise 177.3. Use the Simplicial-Singular Chain Complex Theorem 177.12 to deduce the Simplicial Künneth Theorem 176.29 from the Künneth Theorem 164.6.
Remark. Note that in both Künneth Theorems the map to the left is explicitly given.

Exercise 177.4. Let $(K, \leq)$ be an abstract simplicial complex. Let $\sigma\colon \mathrm{sd}(K) \to K$ be the stretching map from the Stretching Map Lemma 177.20. Show that the corresponding map $|\sigma| \circ \mho^{-1}\colon |K| \to |K|$ is homotopic to the identity.
Remark. Use the CW×Interval Proposition 137.1 (2) to overcome any continuity anxieties that might arise.

Exercise 177.5. Let $n \in \mathbb{N}_0$. Let $K = ((V, S), \Theta\colon |K| \to \Delta^n)$ be a subdivision of the standard simplex Δ^n. We denote by $e_0, \ldots, e_n \in V$ the vertices that get sent to the standard vertices of Δ^n. Let Z be a finite set and let $\varphi\colon V \to Z$ be a continuous map. We refer to φ as a *coloring of* K.

(1) We say that φ satisfies *Sperner's condition* if $\varphi\colon V \to Z$ is injective on $\{e_0, \ldots, e_n\} \subseteq V$ and if for each $v \in V$ with $\Theta(v) \in |\Theta(\{e_{r_1}, \ldots, e_{r_s}\})|$ we have $\varphi(v) \in \{\varphi(e_{r_1}), \ldots, \varphi(e_{r_s})\}$. In other words, the vertices $e_0, \ldots, e_n$ have distinct colors, and if a vertex lies on a face spanned by some of the e_i, then the vertex has the color of one of the e_i.

(2) We say that an n-simplex s of K is a *rainbow simplex* if $\varphi\colon s \to Z$ is injective, put differently, if the colors of vertices of s are all different.

[1905] Here the vertical maps Θ are the natural isomorphisms introduced in Notation 177.16.

We want to prove the following statement:

Sperner's Lemma. *If φ satisfies Sperner's condition, then the number of rainbow simplices is odd.*

To prove this statement we introduce some notation:

(3) We set $$\Delta^Z := \left\{ \sum_{z\in Z} t_z \cdot z \,|\, \sum_{z\in Z} t_z = 1 \text{ and each } t_z \geq 0 \right\} \subseteq \mathbb{R}^Z.$$

(4) We denote by $f^\varphi\colon |K| \to \Delta^Z$ the unique map which agrees with φ on V and which is affine linear on each $|s| \subseteq |K|$.

(5) We define $\partial|K| := \Theta^{-1}(\partial\Delta^n)$ and we define $\partial\Delta^Z$ the same way as we defined $\partial\Delta^n$.

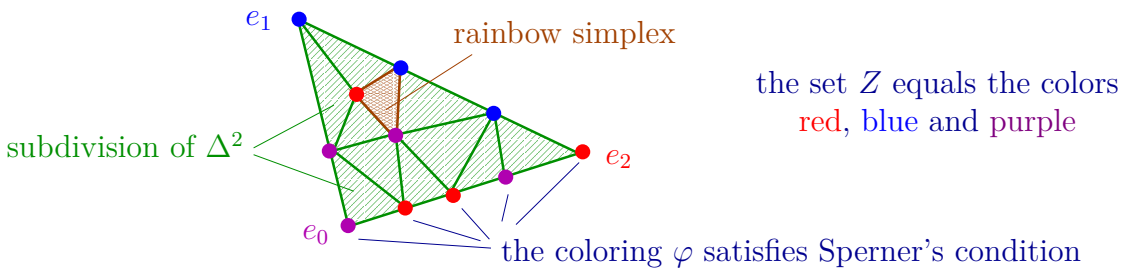
(6) We write $$\deg(\varphi) := \deg\big(\underbrace{\mathrm{H}_n(|K|,\partial|K|;\mathbb{F}_2)}_{\cong\mathbb{F}_2} \to \underbrace{\mathrm{H}_n(\Delta^Z,\partial\Delta^Z;\mathbb{F}_2)}_{\cong\mathbb{F}_2}\big) \in \mathbb{F}_2.$$

Sperner's Lemma follows from solving the following two exercises:

(a) Show that if φ satisfies Sperner's condition, then $\deg(\varphi) \neq 0$.
Hint. Use Exercise 151.18 to prove this statement by induction on n.

(b) Show that $$\deg(\varphi) = \text{number of rainbox simplices} \in \mathbb{F}_2.$$
Hint. The n-simplices of K give rise to a generator of $\mathrm{H}_n(|K|,\partial|K|;\mathbb{F}_2)$.

Remark. Sperner's Lemma was first formulated and proved by Emanuel Sperner [**Spe1928**, p. 267] in 1928. Sperner's Lemma can be proved in a purely combinatorial fashion, see [**AZ2014**, Chapter 27.6], [**Praso2006**, Theorem 2.26] or [**Harz1978**, Satz 12.2]. Sperner's Lemma can be used to give a purely combinatorial proof of the Brouwer Fixed Point Theorem 38.7, we refer to [**Harz1978**, Chapter 12] or [**Praso2006**, p. 81] for details. Sperner's Lemma also has fun applications, for example it can be applied to achieve "rental harmony" [**Su1999**]. Finally we refer to [**DGMM2019**] for a survey on Sperner's Lemma, its cousin Tucker's Lemma, and its many applications and ramifications.



CHAPTER 178

The Nerve Theorem

In this short chapter we will state the "Nerve Theorem". In many situations it gives an effective way for showing that a given topological space is homotopy equivalent to a simplicial complex.

This chapter is organized as follows:

- In Section 178.1 we will state and prove the Nerve Theorem 178.5. The proof unfortunately relies on the later Whitehead Theorem 305.1.
- In Section 178.2 we will use the Nerve Theorem 178.5 to prove Helly's Theorem 178.7.
- In Section 178.3 we will apply the Nerve Theorem 178.5 to smooth manifolds.
- In Section 178.4 we will provide exercises for Chapter 178.

In the next Chapter 179 we will study simplicial structures on manifolds.

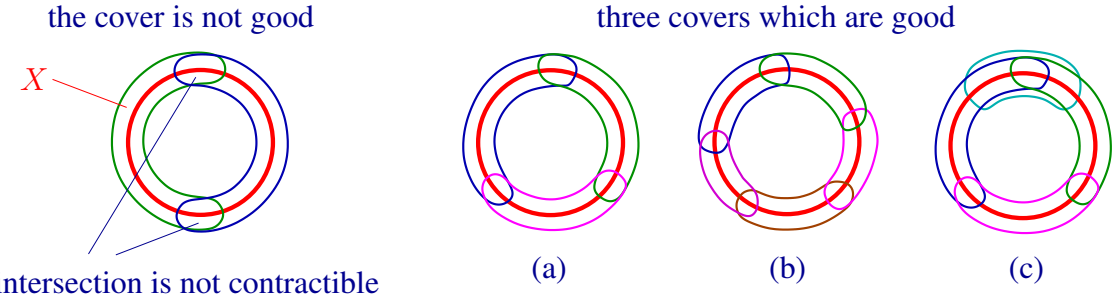
178.1. **The Nerve Theorem.** We start out with the following definition:

Definition 178.1. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of X.

(1) We say that the cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X is good if for every choice of $i_1, \ldots, i_m \in I$ the intersection $U_{i_1} \cap \cdots \cap U_{i_m}$ is either empty or contractible.

(2) We say that the cover $\mathcal{U}$ of X is point-finite if each point $x \in X$ is contained in only finitely many U_i.[1906]

Example. In the figure below we show the topological space $X = S^1$:



We show one cover which is not good and we show three covers which are actually good.□

As we will see, the following lemma is a straightforward consequence of the Link Star Lemma 175.9.

Lemma 178.2. (Open Stars-Good Cover Lemma) Let $K = (V, S)$ be an abstract simplicial complex. The open stars $\{\mathring{\mathrm{St}}(v)\}_{v\in V}$ are an open good point-finite cover.

[1906]Note that this notion is weaker than the notion of a locally finite cover that we introduced in Definition 3.19.



Proof. Let $K = (V, S)$ be an abstract simplicial complex. Let $P \in |K|$. By the Open Simplex Lemma 174.28 (2) we know that there exists a unique $s \in S$ with $P \in \langle s \rangle$. Furthermore, by the Link Star Lemma 175.9 (2) we know that $P \in \mathring{\mathrm{St}}(v)$ if and only if v is a vertex of s. This discussion, together with the Link Star Lemma 175.9 (2), shows that the open stars $\mathring{\mathrm{St}}(v)$ form an open point-finite cover of $|K|$. It remains to show that the cover is good. Thus let $v_0, \ldots, v_n \in V$. We have

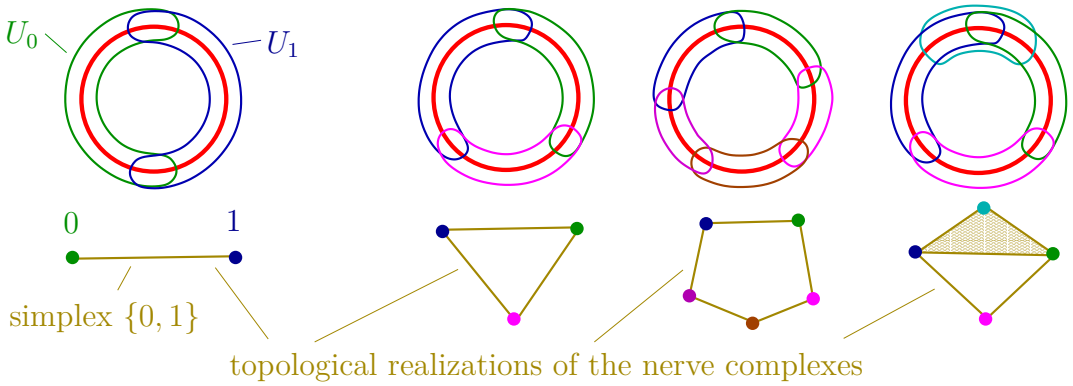
$$\mathring{\mathrm{St}}(v_0) \cap \cdots \cap \mathring{\mathrm{St}}(v_n) \underset{\text{by the Link Star Lemma 175.9 (3)}}{=} \begin{cases} \overbrace{\mathring{\mathrm{St}}(\{v_0, \ldots, v_n\})}^{\text{contractible by the Link Star Lemma 175.9 (1f)}}, & \text{if } \{v_0, \ldots, v_n\} \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

We have thus shown that the open stars $\{\mathring{\mathrm{St}}(v)\}_{v \in V}$ are a good point-finite cover. ■

As we will see shortly, the Open Stars-Good Cover Lemma 178.2 has a partial converse, namely the mysteriously named Nerve Theorem. To state this converse we need the following definition:

Definition 178.3. Let X be a set and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of subsets of X. We refer to the abstract simplicial complex that is given by the vertex set I and the simplex set[1907] $\{\{i_1, \ldots, i_r\} \,|\, i_1, \ldots, i_r \in I \text{ with } U_{i_1} \cap \cdots \cap U_{i_m} \neq \varnothing\}$ as the nerve complex $N(\mathcal{U})$ of the covering.

Example. In the figure below we revisit the four covers of S^1 that we showed in the figure on page 3387. This time we also show the topological realizations of the corresponding nerve complexes.



As is clear from the figure, for the cover to the left, which is not good, the topological realization of the nerve complex does not look at all like S^1. On the other hand, for the

[1907] It follows easily from the definitions that this is indeed an abstract simplicial complex.

three covers to the right, which are good, the topological realization of the nerve complex is homeomorphic or at least homotopy equivalent to S^1. As we will see in a minute, that is not a coincidence. □

To formulate the Nerve Theorem of this section it is convenient to introduce the following definition:

Definition 178.4. We say that a map $f\colon X \to Y$ between path-connected topological spaces is a weak homotopy equivalence if there exists $x_0 \in X$ such that for every $k \in \mathbb{N}_0$ the induced map $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

By the Homotopy Equivalence-π_n-Proposition 134.17 (1a) a homotopy equivalence between path-connected topological spaces is a weak homotopy equivalence. The following theorem says, rather amazingly, that for CW-complexes the converse holds.

Theorem 305.1. (Whitehead Theorem) Let $f\colon X \to Y$ be a continuous map between two connected CW-complexes.[1908] If $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

As is clear from the numbering, we will prove the Whitehead Theorem 305.1 much later in these notes. But, as we will see, this theorem greatly facilitates the discussion of the Nerve Theorem.

Recall that in the Open Stars-Good Cover Lemma 178.2 we showed that every simplicial complex admits a good cover. We can now formulate and prove the Nerve Theorem which can in particular be viewed as a partial converse to the Open Stars-Good Cover Lemma 178.2.

Theorem 178.5. (Nerve Theorem) Let X be a path-connected topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open good point-finite cover of X. The following two statements hold:

(1) There exists a weak homotopy equivalence $|N(\mathcal{U})| \to X$ from the topological realization of the nerve complex $N(\mathcal{U})$ to X. In fact there exists a weak homotopy equivalence $f\colon |N(\mathcal{U})| \to X$ such that for every simplex $s = \{i_0, \dots, i_k\}$ of $N(\mathcal{U})$ the following statements hold:
   - We have $f(|\{i_0, \dots, i_k\}|) \subseteq U_{i_0} \cup \dots \cup U_{i_k}$.
   - We have $f(\underline{s}) \in U_{i_0} \cap \dots \cap U_{i_k}$.

(2) If X is furthermore a CW-complex, then any map $f\colon |N(\mathcal{U})| \to X$ as in (1) is in fact a homotopy equivalence.

Remark.

(1) The Nerve Theorem has its origins in the work of Karol Borsuk [**Bor1948**] from 1948 and Jean Leray [**Ler1950**] from 1950. As stated the theorem was basically proved by André Weil [**Weil1952**, p. 141] in 1952.

(2) Let X be a CW-complex. It follows from the Subcomplex–Neighborhood Proposition 137.9 that the conclusion of the Nerve Theorem 178.5 also holds if $\mathcal{U} = \{U_i\}_{i\in I}$ is a family of *subcomplexes* which form a good point-finite cover of X.

[1908] Note that we do not assume that the map f is cellular.

(3) There are many variations on the statement and on the proof of the Nerve Theorem 178.5.
   (a) In [**Bj1995**, Theorem 10.6] and [**Bj2003003**, Theorem 6] a Nerve Theorem is formulated for the case that X is a regular CW-complex[1909] and the U_i are subcomplexes. In this case the cover does not need to be point-finite.
   (b) Other versions are formulated in [**Nago2007**, Theorem 3.3] and [**BoS1973**, Theorem 8.2.1].
   (c) An elegant proof of the Nerve Theorem 178.5 for the case of a finite cover is given in [**Koz2008**, Theorem 15.21]. □

Proof. Statement (2) of the Nerve Theorem is an immediate consequence of Statement (1) together with the Whitehead Theorem 305.1. Thus it remains to prove Statement (1). In the discussion below we will only sketch an argument. A full proof of Statement (1) is given in [**McCor1967**, Theorem 1].

Thus let X be a topological space and let $\mathcal{U} = \{U_v\}_{v\in V}$ be an open good point-finite cover of X. We adopt the following notation:

(1) We set
$$S \coloneqq \{\{v_0,\dots,v_n\} \,|\, v_0,\dots,v_n \in V \text{ with } U_{v_0}\cap\dots\cap U_{v_n} \neq \varnothing\}.$$
Recall that by definition the nerve complex is given by $N(\mathcal{U}) \coloneqq (V,S)$. For convenience we write $N = N(\mathcal{U})$.

(2) Given $s = \{v_0,\dots,v_n\} \in S$ we set $U_s \coloneqq U_{v_0}\cap\dots\cap U_{v_n}$.

(3) We consider the barycentric subdivision
$$\mathrm{sd}(N) \;=\; \big(\{\underline{s}\}_{s\in S}, \{\{\underline{s_0},\dots,\underline{s_k}\} \,|\, s_0 \subsetneq \dots \subsetneq s_k\}\big).$$
By the SD-Lemma 175.5 we have a natural homeomorphism $|\mathrm{sd}(N)| \to |N|$ which we use to identify these two topological spaces.

Next we define a map $F\colon |\mathrm{sd}(N)| \to X$ which satisfies the following condition:

($\star$) For every simplex $\{\underline{s_0},\underline{s_1},\dots,\underline{s_k}\}$ of $\mathrm{sd}(N)$ we have
$$F\big(\big|\{\underline{s_0},\underline{s_1},\dots,\underline{s_k}\}\big|\big) \;\subseteq\; U_{s_0}.$$

We construct F inductively as follows:

(1) For each $\underline{s}$ we pick a point $F(\underline{s}) \in U_s$.

(2) Suppose that F is defined on the $(n-1)$-skeleton of $|\mathrm{sd}(N)|$. We want to extend F to the n-skeleton of $|\mathrm{sd}(N)|$. Let $t = \{\underline{s_0},\dots,\underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(N)$ and let $\Phi_t^{\leq}\colon \Delta^n \to |\mathrm{sd}(N)|$ be the corresponding characteristic map. Note that
$$(F\circ\Phi_t^{\leq})(\partial\Delta^n) \;=\; \bigcup_{i=0}^{n} \underbrace{F\big(\big|\{\underline{s_0},\dots,\widehat{\underline{s_i}},\dots,\underline{s_n}\}\big|\big)}_{\subseteq U_{s_0} \text{ or } \subseteq U_{s_1} \text{ by } (\star)} \;\subseteq\; U_{s_0}.$$
by definition we have for $s\subseteq t$ that $U_t \subseteq U_s$, in particular we have $U_{s_1}\subseteq U_{s_0}$

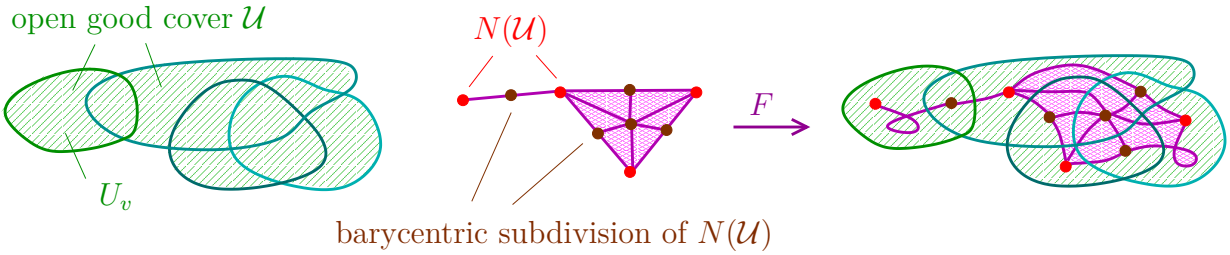
By hypothesis we know that U_{s_0} is contractible. It follows from the Homotopy Equivalence-π_n-Proposition 134.17 and the Trivial-in-π_n-Lemma 134.10 together with the fact, shown in the Δ^n-Homeomorphism Lemma 143.9 (2), that the pair $(\Delta^n,\partial\Delta^n)$

[1909] We refer to Definition 174.54 for the definition of a regular CW-complex, note that simplicial complexes are regular CW-complexes.

is homeomorphic to $(\overline{B}^n, S^{n-1})$ and that the map $F \circ \Phi_t^{\leq} \colon \partial\Delta^n \to U_{s_0}$ extends to a continuous map $G \colon \Delta^n \to U_{s_0}$. Now we define F on $|t| = |\{\underline{s_0}, \ldots, \underline{s_n}\}|$ to be the map $G \circ \Phi_t^{-1}$. Note that this map still satisfies $(\star)$.

Finally note that it follows almost immediately from the SD-Lemma 175.5 and $(\star)$ that $F \colon |N| = |\operatorname{sd}(N)| \to X$ has the desired property that for every simplex $s = \{v_0, \ldots, v_k\}$ of $N = N(\mathcal{U})$ the following statements hold:

- We have $f(|\{i_0, \ldots, i_k\}|) \subseteq U_{i_0} \cup \cdots \cup U_{i_k}$.
- We have $f(\underline{s}) \in U_{i_0} \cap \cdots \cap U_{i_k}$.



It remains to show that F is a weak homotopy equivalence. Unfortunately this is where our discussion of the proof peters out. We suggest to use the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 below to show that F is a weak homotopy equivalence. One should be able to employ this proposition using one of the following two approaches:

(1) One uses the Seifert–van Kampen Theorem 99.1 and the HNN-Seifert–van Kampen Theorem 106.7, and suitable analogs for simplicial complexes, to show that F induces an isomorphism of homotopy groups.

(2) One uses cleverly the naturality of the long exact sequence coming from the Mayer–Vietoris Theorem 153.2 (or more elegantly, the naturality of the Mayer–Vietoris spectral sequence which we alluded to on page 2903) to show that the lift $\widetilde{F}$ of F to the universal coverings of $|\operatorname{sd}(N)|$ and X induces an isomorphism of all homology groups.

We leave it as a rather challenging exercise to the reader to make the above approach work. Even the case of a finite cover is already rather delicate. ■

For completeness we also state the following variation on the Nerve Theorem 178.6.

Theorem 178.6. (Nerve Theorem) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a good open cover of X. If X is paracompact, then there exists a homotopy equivalence $|N(\mathcal{U})| \to X$.

Proof. Since X is paracompact it follows from the Paracompact-Partition of Unity Theorem 14.15 that there exists a partition of unity $\{\varphi_i\}_{i \in I}$ such that for each $i \in I$ we have $\operatorname{supp}(\varphi_i) \subseteq U_i$. We define

$$\begin{array}{rcl} G \colon X & \to & |N(\mathcal{U})| \\ x & \mapsto & \begin{pmatrix} I & \to & \mathbb{R} \\ i & \mapsto & \varphi_i(x) \end{pmatrix}. \end{array}$$

In [**Hat2002**, Proposition 4G.2] it is shown that the map G is a homotopy inverse to the map $F \colon |N(\mathcal{U})| \to X$ that we constructed in the proof of the Nerve Theorem 178.5. ■

178.2. **Helly's Theorem.** One of the many fun applications of the Nerve Theorem 178.5 is the following theorem which was first proved by Eduard Helly [**Hel1923**] in 1923 using

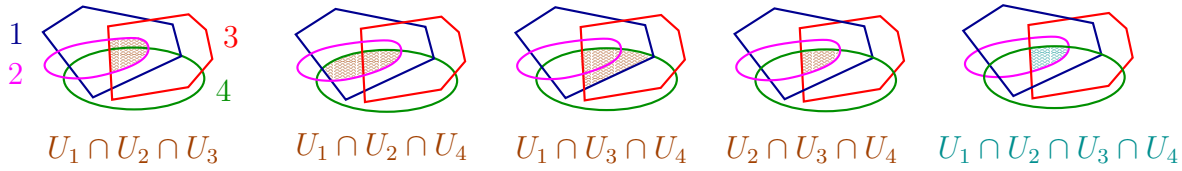
much more elementary techniques. Our proof is arguably technological overkill, but it has the advantage that from our point of view it is very conceptual.

Theorem 178.7. (Helly's Theorem) Let $n \in \mathbb{N}_0$ and let $U_1, \dots, U_m$ be open convex subsets of $\mathbb{R}^n$. If any choice of $n+1$ of those sets has non-empty intersection, then the intersection of all those sets is non-empty, i.e. we also have

$$U_1 \cap \dots \cap U_m \neq \varnothing.$$

Examples.

(1) For $n = 1$ and $m = 3$ Helly's Theorem 178.7 says that if we are given three open intervals such that any two of them have a non-empty intersection, then the intersection of the three open intervals is non-empty.
(2) For $n = 2$ and $m = 4$ Helly's Theorem 178.7 says if we are given four open convex subsets of $\mathbb{R}^2$, such that any three of them have non-empty intersection, then the intersection of all four subsets is also non-empty.



This situation is illustrated in the figure above. □

Remark.

(1) In our formulation of Helly's Theorem 178.7 we require that the sets U_i are open. This is in fact just an artifact of our proof. In fact the statement holds without this assumption. We refer to [**Mato2002**, Theorem 1.3.2] for an elementary proof of this more general statement.
(2) As we mentioned above, our topological proof of Helly's Theorem 178.7 is strictly speaking more difficult than necessary. But the advantage of the topological approach taken in these notes is that it is easier to generalize this approach to non-convex subsets.
(3) We refer to [**Eckh1993**] for a long list of references for proofs of Helly's Theorem and for a discussion of the relevance of Helly's Theorem. □

In our proof of Helly's Theorem we will need the following proposition which is interesting in its own right.

Proposition 178.8. (Euclidean Subset-Homology Proposition) If M is an open subset of $\mathbb{R}^n$, then $\mathrm{H}_k(M) = 0$ for $k \geq n$.

Proof. The proposition follows from a fun argument using the Mayer–Vietoris Theorem 153.2. We invite the reader to fill in the details in Exercise 178.2. Alternatively note that the proposition follows from the more general Manifold-Top-H_*-Theorem 193.1. ■

Proof of Helly's Theorem 178.7. Let $n \in \mathbb{N}_0$. We prove the theorem by induction on m. For $m \leq n+1$ there is nothing to show. Now suppose that the theorem holds for some $m \geq n+1$. Let $\mathcal{U} \coloneqq \{U_1, \dots, U_{m+1}\}$ be a family of open convex subsets of $\mathbb{R}^n$ such that

the intersection of any $n+1$ of those subsets has non-empty intersection. By induction hypothesis we know that the following holds:

(i) The intersection of any m of the sets in $\mathcal{U}$ is non-empty.

We need to show that $U_1 \cap \cdots \cap U_{m+1} \neq \varnothing$. We do a proof by contradiction. So we assume that the following holds:

(ii) $U_1 \cap \cdots \cap U_{m+1} = \varnothing$.

Next we consider the corresponding nerve complex $N(\mathcal{U})$.

(1) It follows immediately from (i) and (ii) that the nerve complex $N(\mathcal{U})$ is simplicially isomorphic to the simplicial complex S_{m-1} that we introduced on page 3283.

(2) As we mentioned before, one of the key properties of convex subsets is that any intersection of convex subsets is again convex, in particular any intersection is contractible. Thus $\mathcal{U}$ is an open good finite cover of $U_1 \cup \cdots \cup U_{m+1}$.

Now we see that

$$\mathbb{Z} \cong \mathrm{H}^{\mathrm{simp}}_{m-1}(S_{m-1}) \overset{\text{by (1)}}{\cong} \mathrm{H}^{\mathrm{simp}}_{m-1}(N(\mathcal{U})) \cong \mathrm{H}_{m-1}(|N(\mathcal{U})|) \overset{\text{by the Nerve Theorem 178.5 which we can apply by (2), together with the Homotopy Equivalence-H}_*\text{-Proposition 144.14 (2)}}{\cong} \mathrm{H}_{m-1}\big(\underbrace{U_1 \cup \cdots \cup U_{m+1}}_{\text{open subset of } \mathbb{R}^n}\big) = 0.$$

by the Simplicial-H_*–Ball Sphere Lemma 176.20

by the Simplicial-Singular Homology Isomorphism Theorem 177.10

by the Euclidean Subset-Homology Proposition 178.8, since $m-1 \geq n$

We have thus obtained a contradiction to (ii). ■

178.3. **The Nerve Theorem 178.5 and smooth manifolds.** The Nerve Theorem 178.5 gives a practical tool for showing that a given topological space is homotopy equivalent to a simplicial complex. It is natural to go through familiar classes of topological spaces and to see whether it applies. Fortunately the next theorem shows that it applies to everybody's favorite topological spaces, namely smooth manifolds.

Theorem 178.9. (Smooth Manifold-Good Cover Theorem)

(1) Every compact smooth manifold admits an open good finite cover.

(2) Every smooth manifold admits an open good point-finite cover.

Remark. To the best of my knowledge it is not known whether every topological manifold admits an open cover that is good, let alone good and point-finite. □

Sketch of proof. Let M be an n-dimensional smooth manifold. There are many approaches to proving the desired statements.[1910]

(1) If M is a smooth manifold with empty boundary, then in the Smoothly Good Cover Theorem 81.2 we proved a stronger statement, namely we showed that one can find a finite (respectively locally finite) smoothly good cover. Here, according to Definition 81.1, "smoothly good" means that in the definition of a good cover we replace "contractible" by "diffeomorphic to an open convex subset of $\mathbb{R}^n$".

(2) In the Smooth Manifold-Simplicial Structure Theorem 179.5 we will see that M admits a smooth simplicial structure, which by the Open Stars-Good Cover Lemma 178.2 gives us the desired result.

[1910] It is not clear to me whether (2) and (3) give smoothly good covers, that is why we did not mention them in the discussion of the Smoothly Good Cover Theorem 81.2.

(3) A detailed proof of the theorem (at least for closed smooth manifolds) is given in [**Weil1952**, p. 120], [**vGl1985**] and [**GuH2019**, Theorem 5.3.2].
(4) Proofs with less detail are given in [**BoT1982**, Theorem 5.1], [**KaL1987**, Chapter VI.3] and [**Lee2009**, Theorem 10.17].

In all approaches, except for the second one, the cases of smooth manifolds with non-empty boundary are not really treated. It is left as a challenge to the reader to extend the results from smooth manifolds with empty boundary to smooth manifolds with boundary. ■

We obtain the following corollary from the combination of the Smooth Manifold-Good Cover Theorem 178.9, the Nerve Theorem 178.5 and the Simplicial-Implies-CW Lemma 174.51.

Corollary 178.10. (Smooth Manifold–Weak Homotopy Type Corollary) Every (compact) smooth manifold is weakly homotopy equivalent to a (finite) simplicial complex, in particular it is weakly homotopy equivalent to a (finite) CW-complex.

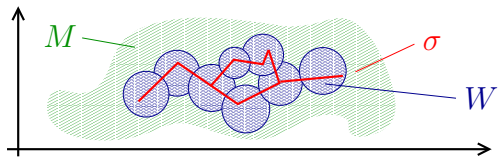
Later, in the Smooth Manifold-CW-Theorem 179.9, we will see that a much stronger version of the Smooth Manifold–Weak Homotopy Type Corollary 178.10 holds.

178.4. Exercises for Chapter 178 .

Exercise 178.1. Let $K = (V, S)$ be an abstract simplicial complex. Let $\mathcal{U} = \{\mathring{\mathrm{St}}(v)\}_{v \in V}$ be the open cover of the topological realization $|K|$ that is given by the open stars. What is the relationship between the original abstract simplicial complex K and the nerve complex $N(\mathcal{U})$?

Exercise 178.2. Let $n \in \mathbb{N}_0$.

(a) Let W be an open subset of $\mathbb{R}^n$ that is the union of finitely many open convex subsets. Show that $\mathrm{H}_k(W) = 0$ for $k \geq n$.
Hint. Use the Mayer–Vietoris Theorem 153.2 (1) and do an induction argument on the number of open convex subsets. As on many other occasions we point out the convenient fact that the intersection of convex subsets of $\mathbb{R}^n$ is again convex.
(b) Let M be an open subset of $\mathbb{R}^n$ and let $\sigma \in \mathrm{C}_n(M)$ be a cycle. Show that there exists an open subset W of $\mathbb{R}^n$ that is the union of finitely many open convex subsets such that $\sigma \in \mathrm{C}_n(W)$ and such that $W \subseteq M$.
(c) Let M be an open subset of $\mathbb{R}^n$. Show that $\mathrm{H}_k(M) = 0$ for $k \geq n$.



Exercise 178.3. Let $P_1, \ldots, P_s$ be points in $\mathbb{R}^2$ with $s \geq 3$. We assume that any three of those points are contained in an open disk of radius r. Show that all the given points are contained in an open disk of radius r.

CHAPTER 179

Simplicial structures on manifolds

In this chapter we will see that smooth manifolds admit a simplicial structure. In fact we will see that they admit particularly nice simplicial structures. There will be many benefits. For example we can almost immediately draw conclusions on the homology groups and homotopy groups of smooth manifolds.

This chapter is organized as follows:

- In Section 179.1 we will see that every compact smooth manifold admits a smooth simplicial structure.
- In Section 179.2 we will use the results on smooth simplicial structures to obtain some results on CW-structures on smooth manifolds.
- In Section 179.3 we will discuss invariants of compact smooth manifolds.
- In Appendix 179.4 we will study invariants of complements of submanifolds.
- In Appendix 179.5 we will discuss simplicial structures on open Euclidean subsets.
- In Section 179.6 we will provide exercises for Chapter 179.

In the next Chapter 180 we will introduce PL-structures on manifolds.

179.1. **Simplicial structures on smooth manifolds.** We start out by recalling Definition 174.31:

Definition 179.1. A simplicial structure for a given topological space X is defined as a pair $(K = (V, S), \Theta\colon |K| \to X)$ where the following holds:

(1) K is an abstract simplicial complex,
(2) Θ is a homeomorphism between the topological realization $|K|$ of K and the topological space X.

Given a simplex $s \in S$ we refer to $\Theta(|s|)$ as a simplex of X and we refer to $\Theta(\langle s \rangle)$ as an open simplex of X.

In this section we want to consider simplicial structures on smooth manifolds. Since the smooth structure on smooth manifolds is important for many arguments and constructions, it is reasonable to consider simplicial structures that reflect the smooth structure. In the following the idea is to consider simplicial structures (K, Θ) such that for each k-simplex the characteristic map $\Theta \circ \Phi_s\colon \Delta^k \to M$ is a smooth embedding. Since Δ^k is not a smooth manifold it takes a little bit of an effort to explain what a "smooth embedding" is supposed to be in this case.

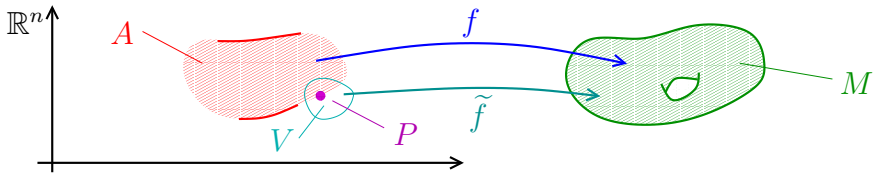
We start out with the following definition: The first part just recalls the definitions introduced on Definition 25.20 and Definition 24.3.

Definition 179.2. Let $n \in \mathbb{N}_0$ and let A be a subset of $\mathbb{R}^n$ for which there exists an open subset U of $\mathbb{R}^n$ with $U \subseteq A \subseteq \overline{U}$. Furthermore let M be a smooth manifold.

(1) We say that a map $f\colon A \to M$ is smooth at a point $P \in A$, if there exists an open neighborhood V of P in $\mathbb{R}^n$ and a smooth map $\widetilde{f}\colon V \to \mathbb{R}^m$ which coincides with f on $A \cap V$. We define $\mathrm{D}f_P := \mathrm{D}\widetilde{f}_P\colon \mathbb{R}^n \to \mathrm{T}_{f(P)}M$.[1911] We refer to the figure below for an illustration.

(2) We say that a map $f\colon A \to M$ is an immersion if f is smooth at every $P \in A$ and if given any $P \in A$ the differential $\mathrm{D}f_P$ has rank n.[1912]

(3) We say that a map $f\colon A \to M$ is a smooth embedding if it is an immersion and an embedding.



Example. Given $k \in \mathbb{N}_0$ we consider the map

$$\begin{aligned} \Xi_k\colon \triangle_k := \left\{(t_1,\dots,t_k) \in [0,1]^k \;\middle|\; \sum_{j=1}^{k} t_j \in [0,1]\right\} &\to \mathbb{R}^{k+1} \\ (t_1,\dots,t_k) &\mapsto \left(t_1,\dots,t_k,1-\sum_{j=1}^{k} t_j\right). \end{aligned}$$

This map has two interesting properties:

(1) By picking the obvious extension of Ξ_k to a map $\mathbb{R}^k \to \mathbb{R}^{k+1}$ we see that Ξ_k is an immersion.

(2) It follows easily from the Compact-Hausdorff Proposition 4.23 (3) that the map Ξ_k defines a homeomorphism $\Xi_k\colon \triangle_k \to \Delta^k$.

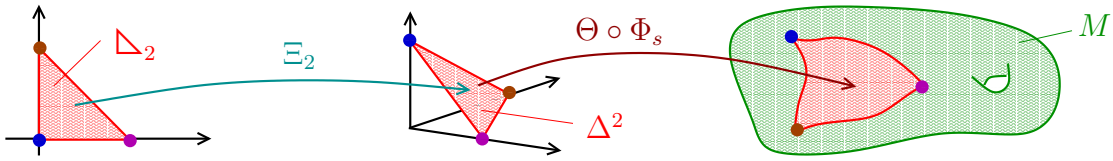
The combination of (1) and (2) shows that Ξ_k is a smooth embedding. □

This leads us to the following definition:

Definition 179.3. Let M be a smooth manifold.

(1) Let $k \in \mathbb{N}_0$. We say that a map $\Phi\colon \Delta^k \to M$ is a smooth embedding if the composition $\Phi \circ \Xi_k\colon \triangle_k \to M$ is a smooth embedding.

(2) We say that a given simplicial structure $(K, \Theta\colon |K| \to M)$ is smooth if for each k-simplex s the corresponding characteristic map $\Theta \circ \Phi_s\colon \Delta^k \to M$ is a smooth embedding.

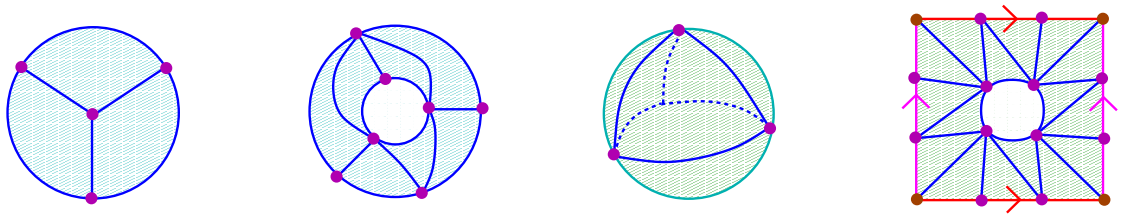


[1911] We leave it to the reader to verify, using our restrictive choice of domain A and using the Maps-agree-on-Closure Lemma 7.21 that the definition of $\mathrm{D}f_P$ does not depend on the choice of the smooth extension $\widetilde{f}$.

[1912] If A is in fact an open subset of $\mathbb{R}^n$, then this definition of an immersion agrees of course with the definition given in Definition 30.9.

Remark. At first glance there are different definitions of "smooth simplicial structure" in the literature. For example our definition of a "smooth simplicial structure" is at first glance different from the one given in [**Mun1966a**, p. 81]. But [**Mun1966a**, Theorem 8.4] shows that our definition is in fact equivalent to the definition used in Munkres' book. Also, it is not difficult to see that our definition is equivalent to the one used in Whitney's book [**Why1957**]. □

Example. In the figure below we show smooth simplicial structures for the closed disk, the annulus, the 2-dimensional sphere and for the torus minus an open disk.



We leave it to the reader to find smooth simplicial structures for all other compact 2-dimensional smooth manifolds. □

The following lemma contains a basic statement about (smooth) simplicial structures of smooth manifolds.

Lemma 179.4. (Manifold Simplex–Dimension Lemma) Let M be a smooth manifold which is equipped with a smooth simplicial structure. If M is n-dimensional, then the dimension of each simplex is at most n.

Remark. In the Manifold-Simplex Proposition 183.2 we will prove several other statements regarding simplicial structures on smooth manifolds. In particular we will give an alternative proof of the Manifold Simplex–Dimension Lemma 179.4. Furthermore we will show that any (smooth) simplicial structure on a manifold M restricts to a (smooth) simplicial structure on the smooth manifold ∂M. □

Proof. Let s be a k-simplex of the given smooth simplicial structure. Combining the characteristic map of the simplex with a chart we obtain a map from an open non-empty subset of $\mathbb{R}^k$ to $\mathbb{R}^n$. If the simplicial structure is smooth this map is in fact an embedding. It follows almost immediately from considering differentials that $k \leq n$. ■

Now we can formulate the following theorem:

Theorem 179.5. (Smooth Manifold-Simplicial Structure Theorem)

(1) Every (compact) smooth manifold admits a countable (compact) smooth simplicial structure.
(2) If M is a smooth manifold with boundary ∂M, then any smooth simplicial structure for ∂M can be *extended* to a smooth simplicial structure for M.
(3) If M is a smooth manifold and if N is a proper submanifold of M, then there exists a smooth simplicial structure for M that *restricts* to a smooth simplicial structure on N.
(4) If $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components

of B, then we can find a smooth simplicial structure on M such that A and B are subcomplexes.

Remark. Note that the conclusion of the Smooth Manifold-Simplicial Structure Theorem 179.5 (3) is somewhat weaker than the conclusion of the Smooth Manifold-Simplicial Structure Theorem 179.5 (2), it does not say that we can extend a given smooth simplicial structure on N to a smooth simplicial structure on M. □

Proof. We start out with giving precise references for the three statements.

(1) This statement is contained in [**Mun1966a**, Theorem 10.6]. For smooth manifolds with empty boundary the statement is also shown in [**Why1957**, Chapter IV.12]. For closed smooth manifolds, with a slightly weaker conclusion, the statement was first proved by Whitehead [**WhdJ1940**, Theorem 7] in 1940, building on earlier work of Cairns [**Cai1934**]. Other proofs for (some types of) smooth manifolds are given in [**Cai1961, Freu1939**] and [**Muk2015**, Theorem 9.5.1]. A sketch of an argument is also given in [**Thu1997**, Theorem 3.10.2].

(2) This statement is also contained in [**Mun1966a**, Theorem 10.6], alternatively see the discussion on [**WhdJ1940**, p. 822f]. Alternatively this result can be deduced from (1), [**Gla1970**, Lemma II.13] (see the discussion on page 3429), the Smooth Simplicial Structure–Uniqueness Theorem 179.7 (applied to smooth singular structures on ∂M) and the Alexander-Newman Theorem 181.6.

(3) If N is a closed submanifold of M, then this statement can be proved by solving [**Mun1966a**, Problem 10.14]. The general case can be deduced from [**Vero1984**, Theorem 7.8], once one has unraveled the statement [**Vero1984**, Theorem 7.8].

(4) First we apply (1) to equip the smooth manifold $A \cap B$ with a simplicial structure. Next we use (2) to extend the smooth simplicial structure to A and B. It is fairly straightforward to verify that putting these two smooth simplicial structures together we obtain a smooth simplicial structure for $M = A \cup B$.

To give the reader a sense of how one can possibly construct smooth simplicial structures on smooth manifolds we provide a sketch for the proof of Statement (1). The sketch is partially inspired by the argument given by Whitney [**Why1957**, Chapter IV.12].

For simplicity let M be a closed n-dimensional smooth manifold. Recall that by the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can view M as a submanifold of $\mathbb{R}^m$ for some $m \geq n$. By the Tubular Neighborhood–Existence Theorem 57.11 we can pick a tubular neighborhood $p\colon W \to M$. (For the purpose of this sketchy proof we might as well assume that the tubular neighborhood is given by $W = \overline{B}^{m-n} \times M$ and that $p\colon \overline{B}^{m-n} \times M \to M$ is the obvious projection.) We equip $\mathbb{R}^m$ with the standard simplicial structure defined in the Simplicial Structure-on-$\mathbb{R}^n$-Lemma 174.39.

We make the following, ever so slightly vague observation.

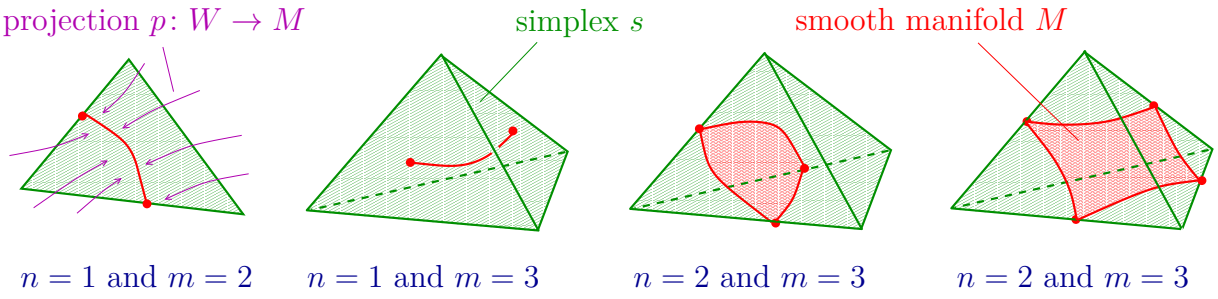
Observation. Using a suitable generalization of the Transversality Theorem 50.9 and using the definition of a smooth map and a smooth submanifold ("locally everything is close to being linear") we can shrink the cubes to a suitably small size and we can find an isotopy

of M such that the simplicial structure (V, S) for $\mathbb{R}^m$ (which we construct as in the proof of the Simplicial Structure-on-$\mathbb{R}^n$-Lemma 174.39) satisfies the following two conditions:[1913]

(a) For every simplex $s \in S$ the intersection of s with M is "close" to the transverse[1914] intersection of an n-dimensional affine space with s.

(b) Every simplex s with $s \cap M \neq \varnothing$ is contained in the tubular neighborhood W and the restriction of the projection $p\colon W \to M$ to s is "close" to being an affine linear map.

We refer to the figure below for an illustration.



We set $k \coloneqq m - n$. In the following we spell out two consequences of (a).

(1) No simplex of dimension less than k intersects M.

(2) If a k-simplex s intersects M, then it does so in a unique point $P = P_s$, which we know by (1) lies necessarily in the interior of s.

Next we make the following constructions.

(3) For each l-simplex t with $l \geq k$ which has the property that the corresponding open simplex $\langle t \rangle$ intersects M we pick a point P_t that lies in the intersection of M with the interior of t.

(4) We consider the abstract simplicial complex

$$L \coloneqq (W, T) \qquad \text{where } W \coloneqq \{s \in S \mid \langle s \rangle \cap M \neq \varnothing\} \text{ and } T \coloneqq \{(w_0, \ldots, w_l) \mid w_0, \ldots, w_l \in S \text{ and } w_0 \subsetneq \ldots \subsetneq w_l\}.$$

(5) Let $\Theta\colon |L| \to \mathbb{R}^m$ be the unique map with $\Theta(w) = P_w$ for $w \in W$ and which is affine linear on each simplex $|t|$.

Some of these definitions are illustrated in the figure below.

We continue with the following observations:

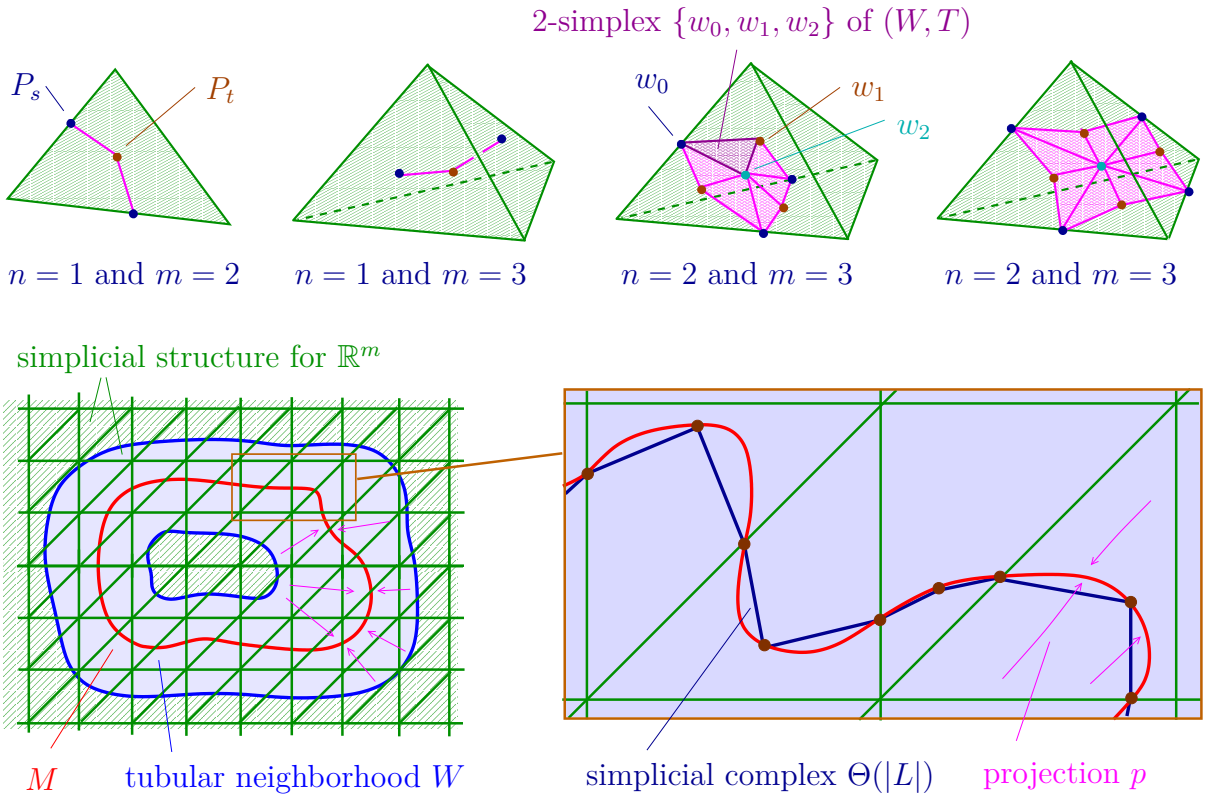
(6) It follows from (a) and (b) that the map $p\colon W \to M$ restricts to a homeomorphism $\Theta(|L|) \to M$. Furthermore, for each simplex $t \in T$ the map $p\colon \Theta(|t|) \to M$ is a submersion, i.e. at each point the differential has full rank.

(7) By (6) we now know that $(L = (W, T), p \circ \Theta\colon |L| \to M)$ is a smooth simplicial structure for M.

We refer to the figure below for a more "global" illustration of the above argument.

[1913] In our notation we blur the difference between the abstract simplices of the simplicial structure and the corresponding simplices in $\mathbb{R}^n$.

[1914] Here by "transverse" we mean the following: If s is an l-dimensional simplex, then it is contained in an l-dimensional affine subspace. The transverse intersection of an n-dimensional affine space with an l-dimensional affine space in $\mathbb{R}^m$ is of dimension $m + l - n$, where negative dimension is understood to be the empty set.



Evidently this outline has several serious issues. For example we never made it clear what we mean by "close". Nonetheless we feel that with enough effort one can turn the above sketch of an argument into a proper proof.

We refer to [**Why1957**, Chapter IV.12] for a very detailed discussion how a variation on the above approach can be made to work. ■

Just for fun we mention that in our sketch of a proof of the Smooth Manifold-Simplicial Structure Theorem 179.5 we basically proved the following proposition which says that any submanifold of $\mathbb{R}^m$ can be "arbitrarily" approximated by a linear simplicial subcomplex of $\mathbb{R}^m$. We will not make use of this proposition, hence we have no qualms about the rather sketchy proof.

Proposition 179.6. (Submanifold-Subcomplex Proposition) Let M be a closed smooth submanifold of $\mathbb{R}^m$. Given any $\epsilon > 0$ there exists a linear simplicial complex $(L = (W, T), \Theta\colon |L| \to X)$ with $X \subseteq \mathbb{R}^m$ and an isotopy $F\colon X \times [0,1] \to \mathbb{R}^m$ with the following properties:

(1) We have $F_0 = \mathrm{id}$.
(2) The map F_1 is a homeomorphism $F_1\colon X \to M$ and it has the property that $F_1 \circ \Theta$ defines a smooth simplicial structure for M.
(3) For every $x \in X$ the map $[0,1] \to \mathbb{R}^m$ given by $t \mapsto F(x,t)$ is affine linear.
(4) For every $x \in X$ we have $\|x - F_1(x)\| < \epsilon$.

Example. The Submanifold-Subcomplex Proposition 179.6 says in particular that we can approximate any knot M in $\mathbb{R}^3$ by a "simplicial knot" X.



We refer to the figure above for an illustration. □

Sketch of proof. We continue with the notation and results in the proof of the Smooth Manifold-Simplicial Structure Theorem 179.5. The $\epsilon > 0$ corresponds to a suitable interpretation of "close" in the formulation of (a) and (b). We set $X := \Theta(|L|)$. The isotopy F is the map which interpolates the identity id_X and the projection $p\colon X \to M$. ■

In the Hauptvermutung Failure Theorem 177.7 we saw that there exist compact spaces which admit two "inequivalent" simplicial structures, in other words, we saw that the Hauptvermutung 177.6 is false in general. Interestingly, the "smooth" analog of the Hauptvermutung 177.6 is actually correct. More precisely, the following theorem holds.

Theorem 179.7. (Smooth Simplicial Structure–Uniqueness Theorem) Let M and N be smooth manifolds and suppose that we are given two smooth simplicial structures $(K, \Theta\colon |K| \to M)$ and $(L, \Omega\colon |L| \to N)$. Furthermore let $\varphi\colon M \to N$ be a diffeomorphism. Then there exist subdivisions K' of K and L' of L and a simplicial isomorphism $\psi\colon K' \to L'$ such that $|\psi|\colon M \to N$ is homeotopic to $\varphi\colon M \to N$.

Proof. This statement is shown in [**WhdJ1940**, Theorem 8] and also alternatively in [**Mun1966a**, Theorem 10.5] (see also [**Mun1975**, Theorem 10.13]). The last half-sentence about "homeotopic to" is not mentioned explicitly in the references, but it is clear from the context and the proofs that this refinement does hold. It is perhaps worth pointing out that we do not need to assume that M is compact. ■

The Smooth Simplicial Structure–Uniqueness Theorem 179.7 will come in handy at a later stage. But for the foreseeable future it is of no real interest to us.

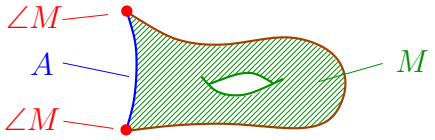
179.2. **CW-structures on smooth manifolds.** In this section we pick up some low-hanging fruit that we obtain almost for free from the Smooth Manifold-Simplicial Structure Theorem 179.5. First we have the following theorem which shows that one can extend simplicial structures from parts of the boundary to the whole manifold:

Theorem 179.8. (Boundary Submanifold–Subcomplex Theorem)

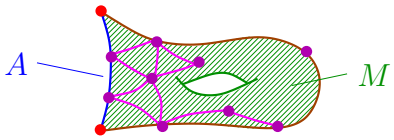
(1) Let M be a compact smooth manifold and let $A \subseteq \partial M$ be a compact codimension-zero submanifold of ∂M. Given any smooth simplicial structure of A there exists a smooth simplicial structure of M such that A is a subcomplex.

(2) Let M be a compact smooth manifold with corner $\angle M$. We assume that the corner $\angle M$ is two-sided in the sense of Definition 60.1. Furthermore let $A \subseteq \partial M$ be a smooth boundary in the sense of Definition 59.4. Given any smooth simplicial structure of A there exists a simplicial structure (not necessarily smooth) of M such that A is a subcomplex.

Since by the Smooth Manifold-Simplicial Structure Theorem 179.5 (1) we know that we can equip A with a smooth simplicial structure we see that we can always find a CW-structure on M such that A is a subcomplex.



A is a smooth boundary of ∂M

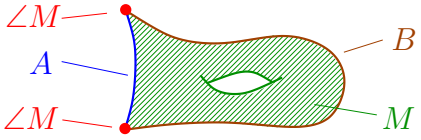


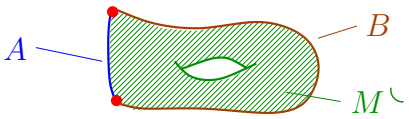
CW-structure on M such that A is a subcomplex

Proof.

(1) We set $B := \partial M \setminus \mathring{A}$. Note that it follows from the Codimension-Zero Submanifold Proposition 25.49 that $A \cap B$ is a union of components of ∂B. It now follows from the Smooth Manifold-Simplicial Structure Theorem 179.5 (2) that we can extend the smooth simplicial structure on $A \cap B$ to a smooth simplicial structure on B. In summary we have a smooth simplicial structure on $\partial M = A \cup B$. It follows again from the Smooth Manifold-Simplicial Structure Theorem 179.5 (2) that we can extend this smooth simplicial structure on ∂M to a smooth simplicial structure on M.

(2) Let $A \subseteq \partial M$ be a smooth boundary in the sense of Definition 59.4. It follows from the Smooth Manifold–Boundary Proposition 59.5 that A is naturally a smooth manifold with $\partial A = \angle M$. Since $\angle M$ is by hypothesis two-sided we can now consider the smooth manifold $M^{\llcorner}$ that we obtain from Rounding Corner Proposition 60.5 by rounding the corner $\angle M$.





It now follows easily from the definitions that we are now in the setting of (1). Thus we obtain a smooth simplicial structure on $M^{\llcorner}$ that extends the smooth simplicial structure of M. But note that on M this simplicial structure is not necessarily smooth. ■

We conclude this section with the following theorem which gets used many times:

Theorem 179.9. (Smooth Manifold-CW-Theorem) Every (compact) n-dimensional smooth manifold M admits a (finite) CW-structure with the following properties:

(1) Every cell has dimension at most n.

(2) The boundary ∂M is a subcomplex.

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW-structure on M which has all the above properties and such that A and B are subcomplexes.

Remark.

(1) The Smooth Manifold-Simplicial Structure Theorem 179.5 and the Smooth Manifold-CW-Theorem 179.9 can be viewed as a significant strengthening of the Smooth Manifold–Weak Homotopy Type Corollary 178.10. Later on, in the Topological Manifold-Simplicial Complex Retract Theorem 189.13 together with the CW-Complex Domination Proposition 305.11, and in the Smooth Manifold-CW-Homotopy Type Proposition 254.4 we will get to know two more methods for equipping a smooth manifold with a CW-structure, at least up to a homotopy equivalence.
(2) Note that the Smooth Manifold-CW-Theorem 179.9 is stated for smooth manifolds and *not* for topological manifolds. The situation for topological manifolds is in fact much more complicated and we will deal with it later on in Section 190.2. □

Proof. The theorem follows almost immediately from the Boundary Submanifold–Subcomplex Theorem 179.8, the Smooth Manifold-Simplicial Structure Theorem 179.5 together with the Manifold Simplex–Dimension Lemma 179.4 and the Simplicial-Implies-CW Lemma 174.51. ■

179.3. **Invariants of smooth manifolds.** Almost for free we now obtain the following proposition which gives us some control over the "size" of invariants of smooth manifolds.

Proposition 179.10. (Smooth Manifolds-Invariants Proposition) Let M be an n-dimensional path-connected non-empty[1915] smooth manifold.

(1) (a) The fundamental group of M is countable.
    (b) If M is compact, then the fundamental group of M is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(M)$ are countably generated abelian groups.[1916]
    (b) If M is compact, then all homology groups $\mathrm{H}_k(M)$ are finitely generated abelian groups.

    Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M;\mathbb{F})$ are of countable dimension and if M is compact, then they are finite-dimensional.
(3) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(M;G) = 0$.
(4) The higher homotopy groups of M are countable.

Remark.

(1) In the Manifold–Countable-π_1-Proposition 95.1 we already showed that fundamental groups of (possibly non-compact) topological manifolds are countable. But it seems difficult to adapt the approach of the Manifold–Countable-π_1-Proposition 95.1 to deal with homology groups.
(2) Note that in Exercise 169.8 we saw that there exists a closed smooth manifold M such that $\pi_2(M)$ is not finitely generated as an abelian group. In other words, the Smooth Manifolds-Invariants Proposition 179.10 (4) is in a sense optimal.
(3) In Exercises 106.7 and 153.12 we used the Smoothly Good Cover Theorem 81.2 to give an alternative proof of Statements (1b) and (2b) of the Smooth Manifolds-Invariants Proposition 179.10.

[1915] We request the adjectives "path-connected" and "empty" to make sure that the isomorphism type of the homotopy groups are well-defined.
[1916] Note that in contrast we will see on page 4455 that the "cohomology groups" of a non-compact smooth manifold can be uncountable.

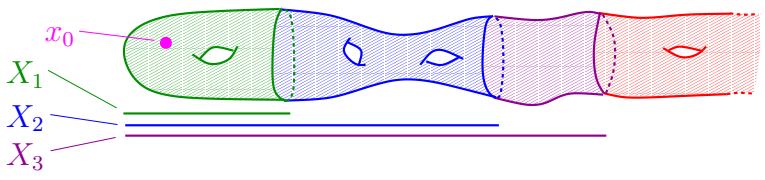
(4) On page 4851 we will give an alternative proof for some of the Statements (1), (2) and (3) of the Smooth Manifolds-Invariants Proposition 179.10 using "handle decompositions". An alternative approach to proving these three statements is also given by using the Smoothly Good Cover Theorem 81.2. We will carry out this idea in Exercise 179.2.
(5) In Chapter 183 we will see that the existence of smooth simplicial structures on smooth manifolds can be used to extract even more information on homology groups of compact smooth manifolds.
(6) In the Topological Manifolds-Invariants Proposition 189.14 we will extend the Smooth Manifolds-Invariants Proposition 179.10 to *topological* manifolds. □

Proof. First we deal with Statements (1b), (2b), (3) and (4). Afterwards we will use a little trick to deduce (1a) from (1b) and (2a) from (2b).

(1) (b) We assume that M is compact. It follows from the Smooth Manifold-CW-Theorem 179.9 together with the CW-π_1-Finite Presentation–Proposition 136.23 (2) that $\pi_1(M)$ is finitely presented.
(2) (b) We assume that M is compact. It follows from the Smooth Manifold-CW-Theorem 179.9 together with the CW-Complex-H_*-Proposition 155.10 (1) and the discussion on page 3081 that the homology groups $\mathrm{H}_k(M)$ are finitely generated (and the ones with field coefficients are finite-dimensional).
(3) It follows from the Smooth Manifold-CW-Theorem 179.9 together with the CW-Complex-H_*-Proposition 155.10 (2) and the discussion on page 3081 that all homology groups of dimension $> k$ are zero.
(4) It follows from the Smooth Manifold-Simplicial Structure Theorem 179.5 together with the Simplicial Complex–π_n-Countable Proposition 175.16 that all homotopy groups of M are countable.

We still need to prove Statements (1a) and (2a). By the Push-Away-from-Boundary Corollary 35.8 we know that M is homotopy equivalent to $M \setminus \partial M$. It follows from this fact, together with the Homotopy Equivalence-π_1-Proposition 90.9 and the Homotopy Equivalence-H_*-Proposition 144.14 (2), that we might as well assume that M has no boundary. Since M has no boundary we know by the Compact Submanifold Exhaustion Proposition 38.2 that there exists a sequence $X_1, X_2, \ldots$ of codimension-zero smooth submanifolds of M with the following three properties:

- For each $i \in \mathbb{N}$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$.
- Each X_i is compact, path-connected and non-empty.
- We have $\bigcup_{i \in \mathbb{N}} \overset{\circ}{X}_i = M$.



We now turn to the missing proofs of (1b) and (2b):

(1) (a) We pick a base point $x_0 \in X_1$. Given $i \leq j$ we denote by $\varphi_{ij} \colon X_i \to X_j$ the inclusion map. We then see that

$$\begin{array}{rcl} \pi_1(M,x_0) & \underset{\text{by the Exhaustion-}\pi_1\text{ Proposition 105.2}}{\cong} & \varinjlim \pi_1(X_i) \\ & \underset{\text{by the Algebraic Direct Colimit Existence Proposition 104.7 (1a)}}{\cong} & \Big(\underset{i\in\mathbb{N}}{*} \pi_1(X_i,x_0) \Big) / \langle\langle \{ \varphi_{ij*}(g) \cdot g^{-1} \mid i \leq j \text{ and } g \in \pi_1(X_i,x_0) \} \rangle\rangle. \end{array}$$

By (1b) we know that each $\pi_1(X_i,x_0)$ is finitely presented, in particular countable. It follows from the above discussion, together with the Countability Lemma 1.37, that $\pi_1(M,x_0)$ is countable.

(2) (a) Similar to (1a) we see that given any $k \in \mathbb{N}_0$ we have the following isomorphisms:

$$\begin{array}{ccccc} \mathrm{H}_k(M) & \underset{\text{by the Exhaustion-H}_*\text{ Proposition 154.2}}{\cong} & \varinjlim \mathrm{H}_k(X_i) & \underset{\text{by the Algebraic Direct Colimit Existence Proposition 104.7 (1b)}}{\cong} & \Big(\bigoplus_{i\in\mathbb{N}} \mathrm{H}_k(X_i) \Big) / \{ \varphi_{ij*}(g) - g \mid \ i \leq j \text{ and } g \in \mathrm{H}_k(X_i) \}. \end{array}$$

By (2b) we know that each $\mathrm{H}_k(X_i)$ is finitely generated. It is now clear that $\mathrm{H}_k(M)$ is countable. A very similar argument also works if we work with coefficients in a field $\mathbb{F}$. ■

On several occasions we will also study relative homology groups of a compact smooth manifold M, for example we will be interested in the relative homology groups $\mathrm{H}_k(M,\partial M)$. The following lemma, together with the Smooth Manifolds-Invariants Proposition 179.10 (4), gives us some control over these invariants.

Lemma 179.11. (Relative-H_*-Finite Generation Lemma) Let X be a topological space, let $A \subseteq X$ be a subspace and let $k \in \mathbb{N}_0$.

(1) If the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_{k-1}(A)$ are finitely generated, then so is the relative homology group $\mathrm{H}_k(X,A)$.

(2) Let $\mathbb{F}$ be a field. If the homology groups $\mathrm{H}_k(X;\mathbb{F})$ and $\mathrm{H}_{k-1}(A;\mathbb{F})$ are finite-dimensional, then so is the relative homology group $\mathrm{H}_k(X,A;\mathbb{F})$.

Proof. Statement (1) follows from the long exact sequence of the pair (X,A) together with the Finite Generation Lemma 97.20. Statement (2) is proved in a very similar fashion. ■

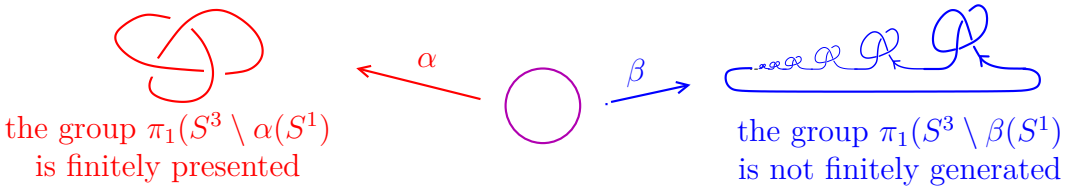
179.4. **Appendix: Invariants of complements of submanifolds.** The goal of this appendix is to prove the following proposition which often gets used unconsciously in geometric topology:

Proposition 179.12. (Complement-of-Submanifold Invariants Proposition) Let M be a compact smooth manifold and let $K \subseteq M$ be a compact submanifold. The following two statements hold:

(3) The fundamental group of $M \setminus K$ (for any base point) is finitely presented.

(4) All homology groups $\mathrm{H}_k(M \setminus K)$ are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M \setminus K; \mathbb{F})$ are finite-dimensional.

Example. Let $\alpha\colon S^1 \to \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be a *smooth* embedding. It follows from the Smooth Embedding Theorem 30.15 that $\alpha(S^1)$ is a compact submanifold of S^3. Thus it follows from the Complement-of-Submanifold Invariants Proposition 179.12 that the fundamental group $\pi_1(S^3 \setminus \alpha(S^1))$ is finitely presented.



It is worth pointing out that this conclusion does *not* hold, if we drop the adjective "smooth". In fact in the Wild Knot-π_1-Proposition 218.20 we will see that there exists an embedding $\beta\colon S^1 \to \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ such that $\pi_1(S^3 \setminus \beta(S^1))$ is not finitely presented. □

We break the proof of the Complement-of-Submanifold Invariants Proposition 179.12 into two parts:

(1) First we consider the case that K is a compact *proper* submanifold.
(2) Afterwards we consider the general case, which, by some clever tricks, we reduce to the first case.

Proof of Proposition 179.12 for proper K. Let M be a compact smooth manifold and let $K \subseteq M$ be a proper smooth submanifold. We proceed with the following steps:

(a) By the Tubular Neighborhood–Existence Theorem 57.11 we know that K admits a tubular neighborhood $(N, \tau\colon N \to K)$. We consider the interior $\mathring{N} := N \setminus \partial_0 N$.



(b) By the Basics-of-Tubular Neighborhood Lemma 57.10 (2) we know that $M \setminus \mathring{N}$ is a compact smooth manifold with corner. It follows from the Corner-Collar Neighborhood Theorem 60.3 and the Rounding Corner Proposition 60.5 that $M \setminus \mathring{N}$ admits the structure of a smooth manifold.[1917]
(c) By the Basics-of-Tubular Neighborhood Lemma 57.10 (3) we know that the inclusion $M \setminus \mathring{N} \to M \setminus K$ is a homotopy equivalence.

[1917]From the Corner–Topological Manifold Lemma 26.5 we obtain the weaker statement that $M \setminus \mathring{N}$ is a compact topological manifold. If we appeal to the Topological Manifolds-Invariants Proposition 189.14 then this piece of information is enough to clinch the deal.

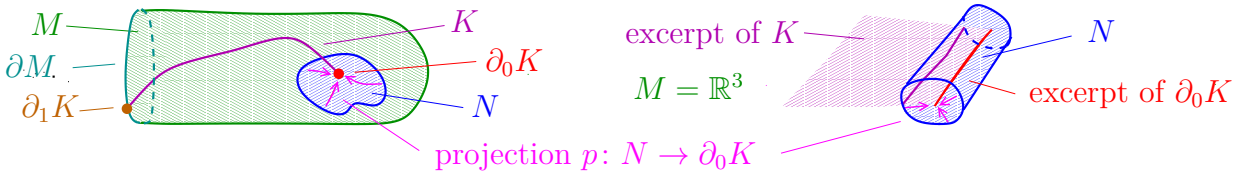
Now let $x_0 \in M \setminus K$. One can easily verify that there exists a $y_0 \in M \setminus \mathring{N}$ such that x_0 and y_0 are connected by a path in $M \setminus K$. We now see that we have the following isomorphisms:

$$\pi_1(M \setminus K, x_0) \underset{\text{by (c) and the Homotopy Equivalence-}\pi_1\text{-Proposition 90.9 (1b)}}{\cong} \pi_1(M \setminus K, y_0) \underset{\text{by (b)}}{\cong} \pi_1(M \setminus \mathring{N}, y_0) = \pi_1(\text{compact smooth manifold}) \underset{\text{by the Smooth Manifolds-Invariants Proposition 179.10 (1)}}{=} \text{finitely presented group.}$$

The argument for homology groups is almost identical, we just need to make use of the Homotopy Equivalence-H_*-Proposition 144.14 (1b) and the Smooth Manifolds-Invariants Proposition 179.10 (2). ■

Proof of Proposition 179.12 for arbitrary compact K. Let M be a smooth manifold and let K be a compact (not necessarily proper) submanifold of M. As in Notation 25.47 we write $\partial_0 K := \partial K \cap (M \setminus \partial M)$ and $\partial_1 K := \partial K \cap \partial M$. By the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33 there exists a tubular neighborhood $(N, p\colon N \to \partial_0 K)$ for the compact proper submanifold $\partial_0 K$ and a diffeomorphism $\Theta\colon N \cap K \to \partial_0 K \times [0,1]$ such that the following diagram commutes:

$$\begin{array}{ccc} N \cap K & \xrightarrow{\Theta} & \partial_0 K \times [0,1] \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle (x,t) \mapsto x} & \\ & \partial_0 K. & \end{array}$$



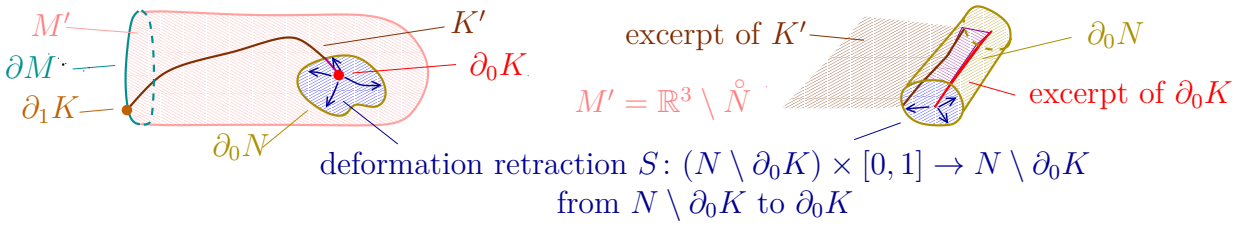
We write $\mathring{N} := N \setminus \partial_0 N$, we set $M' := M \setminus \mathring{N}$ and we set $K' := K \cap M'$. Note that our careful choice of a tubular neighborhood implies that K intersects $\partial_0 N$ transversally. Using the Submanifold Transversal Intersection Theorem 50.4 one can now easily verify that K' is a compact proper submanifold of the compact smooth manifold M'.

Claim. The subspace $M' \setminus K'$ is a deformation retract of $M \setminus K$.

Proof. By the Radial Shrinking Lemma 57.8 (3) there exists a deformation retraction

$$S\colon (N \setminus \partial_0 K) \times [0,1] \to N \setminus \partial_0 K$$

from $N \setminus \partial_0 K$ to $\partial_0 N$ which has the property that for any $(x,t) \in (N \setminus \partial_0 K) \times [0,1]$ we have $p(S(x,t)) = p(x)$.
It follows easily from this property and the above commutative diagram that the deformation retraction S restricts to a deformation retraction from $N \setminus K$ to $\partial_0 N$. Together with the stationary homotopy on $M \setminus \mathring{N}$ this defines a retraction from $M \setminus K$ to $M' \setminus K'$. ⊞



Finally let $x_0 \in M \setminus K$. We set $y_0 \coloneqq S(x_0, 1)$. We now see that

$$\pi_1(M \setminus K, x_0) \;\cong\; \pi_1(M \setminus K, y_0) \;\underset{\uparrow}{\cong}\; \pi_1(M' \setminus K', y_0) \;\underset{\uparrow}{=}\; \text{finitely presented group.}$$

by the claim together with the Homotopy Equivalence-π_1-Proposition 90.9 (1b)

by the case we already considered above

The argument for homology groups is of course almost identical. ■

179.5. **Appendix: Simplicial structures on open Euclidean subsets.** The only goal of this appendix is to prove the following lemma, which on a few rare occasions will be useful:

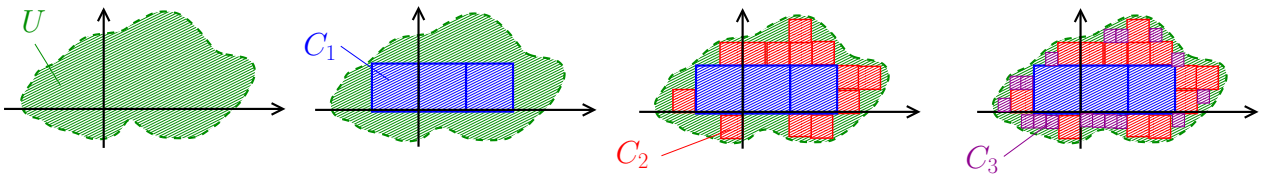
Lemma 179.13. (Open Euclidean Subset–Simplicial Structure Lemma) Every open subset U of $\mathbb{R}^n$ admits the structure of a countable simplicial structure of dimension $\leq n$.

Sketch of proof of Lemma 179.13. For everybody's sanity we only sketch the argument. Thus let $U \subseteq \mathbb{R}^n$ be an open subset. We start out with the following notion: Given $k \in \mathbb{N}_0$ we define a $\frac{1}{2^k}$*-cube* to be a subset C of $\mathbb{R}^n$ of the form

$$C \;=\; \left[\tfrac{a_1}{2^k}, \tfrac{a_1+1}{2^k}\right] \times \cdots \times \left[\tfrac{a_n}{2^k}, \tfrac{a_n+1}{2^k}\right] \quad \text{with } a_1, \ldots, a_n \in \mathbb{Z}.$$

Put differently, a $\frac{1}{2^k}$-cube is a cube in $\mathbb{R}^n$ with side length $\frac{1}{2^k}$ and where the coordinates of all vertices lie in $\frac{1}{2^k} \cdot \mathbb{Z}$.
We now perform the following steps to equip U with a simplicial structure:

(1) We introduce the following subsets of $\mathbb{R}^n$:
- We define C_1 to be the union of all 1-cubes that are contained in U.
- We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\mathring{C}_1$ and that are contained in U.
- We iterate this process, i.e. for any $k \geq 3$ we define C_k to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that are contained in U.



[1917]This lemma also follows from the Smooth Manifold-Simplicial Structure Theorem 179.5 (which we did not prove).
[1917]An even vaguer sketch for the proof is provided by [**ST1980**, Exercise 7, p. 59]. It is unclear to the author whether there is any better reference for a direct proof of this statement.

Claim 1. We have $U = \bigcup_{k\in\mathbb{N}_0} C_k$.

Proof. Let $x \in U$. Since U is open there exists an $\epsilon > 0$ such that $B_\epsilon(x)^n \subseteq U$. We pick $k \in \mathbb{N}$ such that $\frac{1}{2^k} \cdot \sqrt{k} < \epsilon$. It is now elementary to see that $x \in C_k$. ⊞

(2) We equip each $\frac{1}{2^k}$-cube with the standard simplicial structure that we introduced on page 3305.

Note though that these simplicial structures of the cubes certainly do *not* define a simplicial structure on U. We have to modify the simplicial structures to get the desired result.

(3) We start out with the following claim:

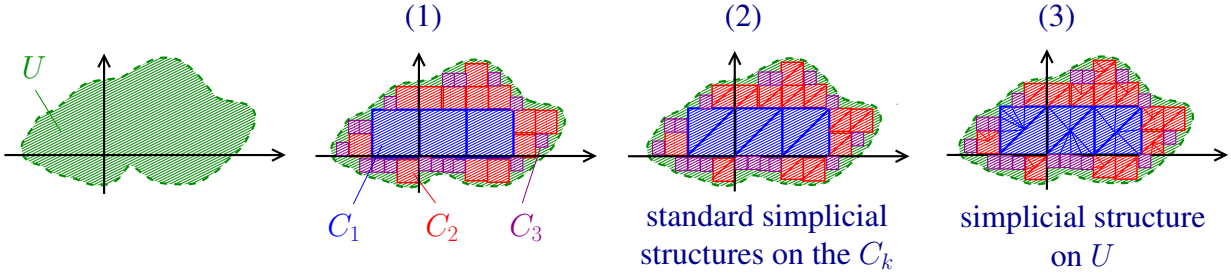
Claim 2. Each cube W in some C_i intersects only finitely many other cubes.

Proof. Since W is compact and disjoint from the closed subset $\mathbb{R}^n \setminus U$ we obtain from the Disjoint Compact-Closed–Distance Lemma 10.5 that $d(W, \mathbb{R}^n \setminus U) > 0$. It follows easily that there exists a $k \in \mathbb{N}$ such that $W \subseteq \overset{\circ}{C}_k$. But now it is clear that W intersects only finitely many other cubes. ⊞

In the following we will, for $k = 0, 1, 2, \ldots$, iteratively subdivide the simplices of the cubes:

(a) We start with the 1-cubes. Let W be a 1-cube. We subdivide the simplicial structure of W in the following way:
- On each side that cobounds another 1-cube we make no changes.
- On each side that does not cobound another 1-cube we subdivide the simplicial structure such that the resulting simplicial structure agrees with the standard structures of the bounding cubes. It follows from Claim 2 that we only need to do finitely many subdivisions. [1918]

(b) Suppose we already fixed the simplicial structures in the cubes of $C_1, \ldots, C_{k-1}$. Now let W be a $\frac{1}{2^k}$-cube. We subdivide the simplicial structure on W in the following way:
- On each side that cobounds a cube in $C_1, \ldots, C_k$ we make no changes.
- On each side that does not cobound a cube in $C_1, \ldots, C_k$ we subdivide the simplicial structure such that the resulting simplicial structure agrees with the standard structures of the bounding cubes in $C_{k+1}, \ldots$. Again it follows from Claim 2 that we only need to do finitely many subdivisions.



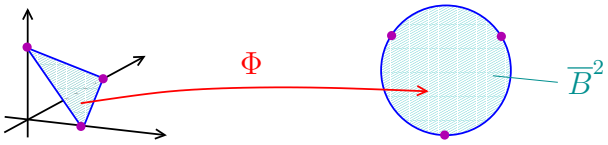
This construction defines an abstract simplicial complex K together with a bijective map $f\colon |K| \to U$. It follows relatively easily from the Pasting Proposition 4.7 that the inverse

[1918] Of course one really should show why that can be done. Since the author doubts that anybody would read that proof, we skip it.

$f^{-1}\colon U \to |K|$ is also continuous. Thus we have shown that $f\colon |K| \to U$ is a homeomorphism. Finally note that it follows easily from the construction that K is in fact a countable simplicial complex and that each simplex has dimension $\leq n$. ■

179.6. Exercises for Chapter 179 .

Exercise 179.1. We equip the smooth manifold $\overline{B}^2$ with the standard simplicial structure $\Phi\colon \Delta^2 \to \overline{B}^2$ introduced on page 3302. Is this a smooth simplicial structure for the smooth manifold $\overline{B}^2$?



Exercise 179.2. Let M be an n-dimensional smooth manifold. In the Smoothly Good Cover Theorem 81.2 we showed that M admits a finite smoothly good open cover. In plain English this means that there exists an open cover $U_1, \ldots, U_k$ of M such that the intersection of any selection of these sets is diffeomorphic to $\mathbb{R}^n$. Use this result to give an alternative proof of Statements (1), (2) and (3) of the Smooth Manifolds-Invariants Proposition 179.10 for M. More precisely, prove the following statements:

(1) The fundamental group of M is finitely presented.
(2) All homology groups of M are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M;\mathbb{F})$ are finite-dimensional.
(3) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(M;G) = 0$.

Remark. You should make use of the calculations of the invariants for $\mathbb{R}^n$, the Mayer–Vietoris Theorem 153.2 (1), the Manifold–Seifert-van Kampen-Theorem 102.1 and the HNN-Seifert-van Kampen Theorem 106.7.

Exercise 179.3. Let M be a compact smooth manifold that is equipped with a smooth simplicial structure S and let $f\colon M \to M$ be a diffeomorphism. Then there exist subdivisions S' and S'' of S and a simplicial isomorphism $\varphi\colon S' \to S''$ such that $|\varphi|\colon M \to M$ is homeotopic to $g\colon M \to M$.

Hint. Apply the Smooth Simplicial Structure–Uniqueness Theorem 179.7 to S and $f^*(S)$.

Remark. This statement is a very useful variation on the Cellular Approximation Theorem 141.2. The Cellular Approximation Theorem 141.2 shows that any continuous maps between CW-complexes is homotopic to a cellular map. But crucially, as we saw on page 2722, given a *homeomorphism*, the Cellular Approximation Theorem 141.2 does not necessarily produce a cellular *homeomorphism*.

CHAPTER 180

PL-structures on manifolds

In this chapter we introduce the non-combinatorial notion of a subdivision of a simplicial complex. This notion allows us to define the notion of a piecewise linear homeomorphism between simplicial complexes. This notion in turn allows us to define the notion of a PL-manifold. These are topological manifolds with a particularly nice simplicial structure. We will see that every smooth manifold admits a PL-structure and we will discuss that the converse does not hold, i.e. there exist PL-manifolds that do not admit a smooth structure.

This chapter is organized as follows:

- In Section 180.1 we will introduce PL-manifolds.
- In Section 180.2 we will discuss regular neighborhoods.
- In Section 180.3 we will discuss the interplay between topological, smooth and PL-manifolds.
- In Section 180.4 we will provide exercises for Chapter 180.

In the next Chapter 181 we will discuss combinatorial manifolds and Pachner moves.

180.1. **PL-manifolds.** Throughout these notes we have so far only encountered two flavors of manifolds, namely topological manifolds and smooth manifolds. In this section we will introduce a new variety of manifolds, namely we will introduce the concept of PL-manifolds. In a nutshell a PL-manifold is a topological manifold together with a well-behaved simplicial structure.

For the reader's convenience we recall Definition 177.3.

Definition 180.1. We say that a simplicial complex Y is a simplicial subdivision[1919] of some simplicial complex X if the following three conditions are satisfied:

(1) The underlying topological spaces are the same.
(2) Every simplex of Y is contained in some simplex of X.
(3) For every l-simplex t of Y and every k-simplex s of X which contains t the map $\Phi_s^{-1} \circ \Phi_t \colon \Delta^l \to \Delta^k$ is affine linear.

Remark. Let X be a simplicial complex and let Y be a simplicial subdivision. In Exercise 180.2 we will see that every k-simplex of X is the union of finitely many simplices of Y of dimension $\leq k$. This shows that our definition of a simplicial subdivision agrees with the definition given in [**Mun1966a**, p. 83]. It also shows that if we use the Simplicial-Implies-CW Lemma 174.51 to view X and Y as CW-complexes, then the identity map $X \to Y$ is cellular. □

We continue with the following definition:

[1919] In the literature a "simplicial subdivision" is usually called just "subdivision". But in Definition 349.16 we will introduce the notion of a cellular subdivision, thus we want to avoid the term "subdivision".

Definition 180.2. Let X and Y be simplicial complexes. We say that a map $f\colon X \to Y$ is piecewise linear, if there exist simplicial subdivisions X' and Y' such that $f\colon X' \to Y'$ is simplicial.[1920]

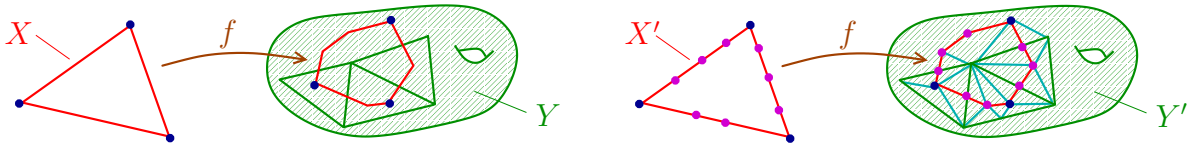
Remark. The definition of a piecewise linear map is the one given in [**Zee1963b**, p. 6] and [**Gla1970**, p. 13]. It follows from [**RS1972**, Theorem 2.14] and [**RS1972**, Example 1.5(4)] that the definition of a piecewise linear map on [**RS1972**, p. 5] is equivalent to the above definition. Furthermore, by [**Zee1963b**, Lemma I.7] the definition of a piecewise linear map on [**Hud1969**, p. 2] is also equivalent to the above definition. □

Warning. Many of our favorite references, e.g. [**Zee1963b**], mostly deal with finite simplicial complexes. Thus on several occasions we will also restrict to the case of finite simplicial complex. We leave it to the undaunted reader to figure out to what degree the results of this section also generalize to infinite simplicial complexes. □

We start out with some examples of piecewise linear maps.

Examples.

(1) In the figure below we show a map $f\colon X \to Y$ between two finite linear simplicial complexes which is not simplicial, but which is piecewise linear.



(2) Let $f\colon X \to Y$ be a continuous map between linear simplicial complexes. If X is a finite linear simplicial complex, then it follows from the Simplicial Approximation Theorem 175.15 that f is homotopic to a piecewise linear map. □

From the definitions it is not entirely clear that the composition of two piecewise linear maps is again piecewise linear. Fortunately, with some effort one can show that this is indeed the case.

Proposition 180.3. (Finite Piecewise Linear-Composition Proposition) The composition of two piecewise linear maps between *finite* simplicial complexes is again piecewise linear.

Proof. The proposition is proved in [**Zee1963b**, Lemma I.2] and [**Gla1970**, Theorem I.6]. ■

Remark. As we hinted at above, both references for the proof of the Finite Piecewise Linear-Composition Proposition 180.3 do indeed assume that the simplicial complexes involved are finite. It is not clear to me whether the statement holds for piecewise linear maps between arbitrary simplicial complexes. □

We move on to the following basically self-explanatory definition.

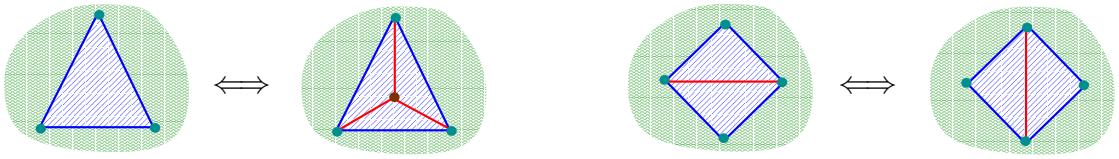
[1920] Colloquially one usually says "PL" instead of "piecewise linear".

Definition 180.4. Let X and Y be two simplicial complexes.

(1) A continuous map $f\colon X \to Y$ is called a PL-homeomorphism if f is piecewise linear, if f is a bijection and if f^{-1} is also piecewise linear.[1921]
(2) We say that X and Y are PL-homeomorphic if there exists a PL-homeomorphism $X \to Y$.

Examples.

(1) If X' is a simplicial subdivision of a simplicial complex X, then it follows immediately from the definition that the map $X' \to X$ is a PL-homeomorphism.
(2) If $f\colon X \to Y$ is a map such that there exist simplicial subdivisions X' of X and Y' of Y such that the map $f\colon X' \to Y'$ is a simplicial isomorphism, then f is evidently a PL-homeomorphism.
(3) (a) In the figure below we show to the left 2-dimensional linear simplicial complexes X_1 and Y_1 which are related by adding a vertex to a 2-dimensional simplex. It follows from (1) that X_1 and Y_1 are PL-homeomorphic.
    (b) In the figure below we show to the right 2-dimensional simplicial complexes X_2 and Y_2. If we subdivide X_2 and Y_2 by adding the intersection point of the two diagonals, then the resulting two linear simplicial complexes are simplicially isomorphic. By (2) this shows that X_2 and Y_2 are PL-homeomorphic.



$|X_1|$ $|Y_1|$ $|X_2|$ $|Y_2|$

(4) Let $n \in \mathbb{N}$. We consider the two linear simplicial complexes $\mathrm{Cone}(\partial\Delta^n)$ and Δ^n. It follows from the discussion on page 3364 that Δ^n admits a simplicial subdivision, given by adding a single vertex, which is simplicially isomorphic to $\mathrm{Cone}(\partial\Delta^n)$. By (1) this shows that $\mathrm{Cone}(\partial\Delta^n)$ and Δ^n are PL-homeomorphic.
(5) By the Linear Simplicial Complexes–Subdivision Proposition 177.4 (1) we know that any two finite linear simplicial complexes X and Y with $|X| = |Y|$ are in fact PL-homeomorphic. □

Later on we will need the following elementary lemma:

Lemma 180.5. (Join-of-PL-Homeomorphisms Lemma) Suppose that we are given two PL-homeomorphisms $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ two between two finite simplicial complexes. Then the map

$$f_1 * f_2\colon X_1 * X_2 \ \to \ Y_1 * Y_2$$

is also a PL-homeomorphism.

Proof. We leave it to the reader to provide the proof. ■

Before we give the definition of a PL-manifold let us introduce the following convention.

[1921] For the record we state that it follows from the Topological Realization–Functor Proposition 174.22 that every piecewise linear map is continuous.

Convention 180.6. Let $n \in \mathbb{N}_0$. On page 3283 we introduced the abstract simplicial complexes

(1) $D_n := (V, S)$ with $V = \{0, \ldots, n\}$ and $S = \mathcal{P}(V) \setminus \{\varnothing\}$, and
(2) $S_{n-1} := (V, T)$ with $V = \{0, \ldots, n\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$.

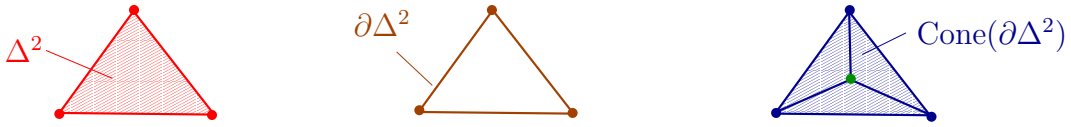
On page 3294 we saw that there exists a canonical homeomorphism $f\colon |D_n| \to \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \to \partial\Delta^n$. Unless we say explicitly something else we will always use these homeomorphisms to view Δ^n and $\partial\Delta^n$ as simplicial complexes.

The above convention leads us to our next definition.

Definition 180.7. Let $n \in \mathbb{N}$.

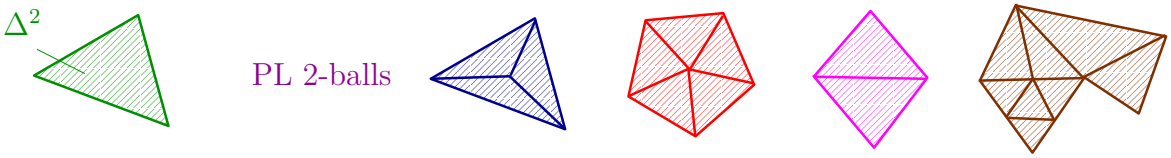
(1) A PL n-ball is defined as a simplicial complex that is PL-homeomorphic to the simplicial complex Δ^n.
(2) A PL n-sphere is defined as a simplicial complex that is PL-homeomorphic to the simplicial complex $\partial\Delta^{n+1}$.

Example. Let $n \in \mathbb{N}_0$.



In Exercise 180.1 we will show that Cone$(\partial\Delta^n)$ is a PL n-ball. □

Remark. In practice it can be a little tricky to show that a given simplicial complex is in fact a PL n-ball or a PL n-sphere. But in most cases one can verify, with enough patience and dexterity, that if a simplicial complex "looks" like an n-ball or an n-sphere, then it is in fact a PL n-ball respectively a PL n-sphere. For example it is entertaining to show that all the simplicial complexes shown in the figure below are in fact PL 2-balls.



Shortly we will get to know a more systematic way for finding PL-homeomorphisms. □

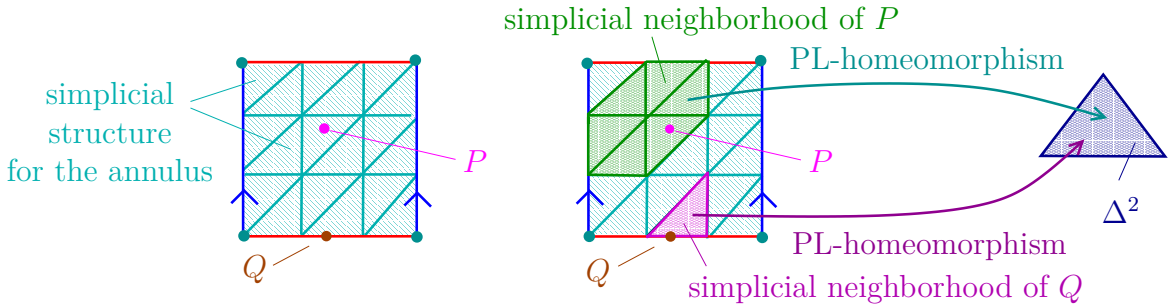
Now we can provide the definition of a PL-manifold. The definition is modelled on the definition of a topological manifold that we gave in Definition 23.2. The key difference is that we replace the words "neighborhood", "ball" and "homeomorphic" by "simplicial neighborhood", "PL-ball" and "PL-homeomorphic".

Definition 180.8.

(1) Let X be a simplicial complex and $P \in X$. A simplicial neighborhood is a subcomplex of X that is a neighborhood of P in the usual sense of Definition 3.2.
(2) An n-dimensional PL-structure on a topological space X is defined as a simplicial structure $(K = (V, S), \Theta\colon |K| \to X)$ such that every point $P \in X$ admits a simplicial neighborhood that is a PL n-ball.
(3) An n-dimensional PL-manifold is a pair $(X, K, \Theta\colon |K| \to X)$ where X is a topological space and Θ is an n-dimensional PL-structure with countably many simplices.

Examples.

(1) In the figure below we see to the left a simplicial structure for the annulus $S^1 \times [0,1]$. The illustrations to the right of the figure should convince the reader that the simplicial structure for the annulus is a PL-structure.



(2) Let $n \in \mathbb{N}_0$. The standard simplicial structure on Δ^n is basically by definition a PL-structure. Recall that in the Δ^n-Homeomorphism Lemma 143.9 (2) we gave an explicit homeomorphism $\Phi\colon \overline{B}^n \to \Delta^n$. Thus $(D_n, \Phi^{-1}\colon \Delta^n = |D_n| \to \overline{B}^n)$ is a PL-structure for $\overline{B}^n$. Similar to the discussion on page 3302 we will refer to it as the *canonical* PL-*structure* of $\overline{B}^n$.

(3) Let $n \in \mathbb{N}_0$. We consider $\partial\Delta^{n+1}$ with the usual simplicial structure from Convention 180.6. Given any vertex v the star $\mathrm{St}(\partial\Delta^{n+1}, v)$ is, basically by definition, simplicially isomorphic to $\mathrm{Cone}(\partial\Delta^{n+1})$. It follows from the example on page 3414 that $\mathrm{St}(\partial\Delta^n, v)$ is a PL n-ball. This shows that $\partial\Delta^n$ is an n-dimensional PL-manifold. Recall that in the Δ^n-Homeomorphism Lemma 143.9 (2) we gave an explicit homeomorphism $\Phi\colon S^n \to \partial\Delta^{n+1}$. This shows that $(S_n, \Phi^{-1}\colon \partial\Delta^{n+1} = |S_n| \to S^n)$ defines a PL-structure on S^n. We refer to it as the *canonical* PL-*structure* on S^n. □

Proposition 180.9. (PL-Manifold–Homeomorphism Proposition) Let X and Y be two simplicial complexes. If X is an n-dimensional PL-manifold and if X and Y are PL-homeomorphic, then Y is also an n-dimensional PL-manifold.

Example. It follows from the PL-Manifold–Homeomorphism Proposition 180.9 that any PL n-ball and any PL n-sphere is indeed an n-dimensional PL-manifold. □

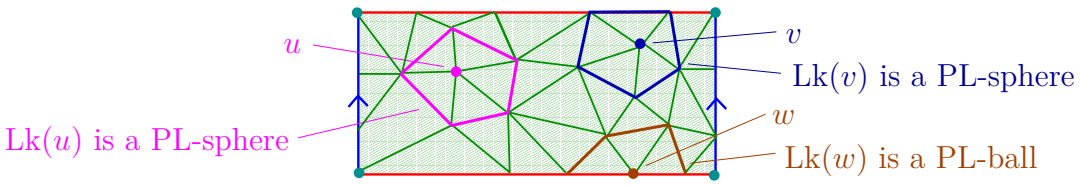
Proof. This proposition is less trivial than it might appear initially. Fortunately a proof is given in [**Gla1970**, Theorem II.2]. ■

The following proposition gives an alternative characterization of PL-manifolds.

Proposition 180.10. (PL-Manifold Characterization Proposition) We consider a simplicial complex $(K = (V, S), \Theta\colon |K| \to X)$. The following statements are equivalent:

(1) (X, Θ) is an n-dimensional PL-manifold.
(2) For every vertex $v \in V$ the link $\mathrm{Lk}(K, v)$ is a PL $(n-1)$-sphere or a PL $(n-1)$-ball.
(3) For every vertex $v \in V$ the star $\mathrm{St}(K, v)$ is a PL n-ball.
(4) For every k-simplex $s \in S$ the link $\mathrm{Lk}(K, s)$ is a PL $(n-k-1)$-sphere or a PL $(n-k-1)$-ball.
(5) For every k-simplex $s \in S$ the star $\mathrm{St}(K, s)$ is a PL $(n-k)$-ball.

Proof.



(1)⇒(2) This implication is shown in [**Hud1969**, p. 26] (alternatively see also [**Zee1963b**, p. III.2]).

(2)⇒(3) Let $v \in V$. First we consider the case that the link $\mathrm{Lk}(v)$ is a PL $(n-1)$-sphere. In this case we have the following PL-homeomorphisms:

$$\mathrm{St}(v) \underset{\uparrow}{\cong} \{v\} * \mathrm{Lk}(v) \underset{\uparrow}{\cong} \{v\} * \text{PL } (n-1)\text{-sphere} \underset{\uparrow}{\cong} \{v\} * \partial\Delta^n \underset{\uparrow}{\cong} \Delta^n.$$

by the Link Star Lemma 175.9 (1c) — since (2) holds — by the Join-of-PL Homeomorphisms Lemma 180.5 — see page 3413

The case that the link $\mathrm{Lk}(v)$ is a PL $(n-1)$-ball is dealt with in the same way, except that this time we use the basically trivial fact, first pointed out on page 3308 that $\{v\} * \Delta^{n-1}$ is in fact simplicially isomorphic to Δ^n. We have thus shown that (2) implies (3).

(3)⇒(1) This implication follows immediately from the Link Star Lemma 175.9 (1).

(2)⇒(4) We prove the statement by induction on k. Note that the case $k = 0$ is precisely Statement (2). Now suppose that the statement holds for some $k-1$. Let $K = (V, S)$ be an abstract simplicial complex and let $t = \{v_0, \ldots, v_k\}$ be a k-simplex. We consider the $(k-1)$-simplex $s = \{v_0, \ldots, v_{k-1}\}$. By induction hypothesis we know that $\mathrm{Lk}(K, s)$ is a PL $(n-k)$-sphere or a PL $(n-k)$-ball. As we pointed out on page 3415, it follows from the PL-Manifold–Homeomorphism Proposition 180.9 that $\mathrm{Lk}(K, s)$ is in fact an $(n-k)$-dimensional PL-manifold. Now we see that

$$\mathrm{Lk}(K,t) \underset{\uparrow}{=} \mathrm{Lk}(\mathrm{Lk}(K,s), v_k) \underset{\uparrow}{=} \mathrm{Lk}\left(\begin{matrix}(n-k)\text{-dimensl.}\\ \text{PL-manifold}\end{matrix}, v_k\right) \underset{\uparrow}{=} \begin{matrix}\text{PL } (n-k-1)\text{-sphere}\\ \text{or a PL } (n-k-1)\text{-ball.}\end{matrix}$$

follows easily from the definitions — by the induction hypothesis — by (1)⇒(2)

(4)⇒(5) The proof of this statement is basically identical to the proof of (2)⇒(3).

(5)⇒(3) This implication holds for the trivial reason that every vertex is in particular a 0-simplex. ■

Remark.

(1) The notion of a PL-manifold gets defined differently by different authors. There is the usual question whether or not one wants a manifold to be second-countable or not which in our case translates into the question whether or not one wants the simplicial complex to be countable. But if we ignore this rather technical issue, then we see the following:

(a) The above definition above is basically the one given in [**Hud1969**, p. 20] and [**Zee1963b**, p. II.10].

(b) It follows from the PL-Manifold Characterization Proposition 180.10 (1)⇔(2) that our definition is equivalent to the definition provided in [**Bry2002**, p. 223] and [**Stal1967**, Definition 4.4.9].
(c) It follows from the PL-Manifold Characterization Proposition 180.10 (1)⇔(5) that our definition is equivalent to the definition given in [**Geo2008**, p. 134].
(d) It is a consequence of [**RS1972**, Theorem 2.2] that our definition is equivalent to the one given in [**RS1972**, p. 7].
(e) In the literature an n-dimensional PL-manifold is sometimes defined as a pair $(X, \{\Phi_i \colon U_i \to V_i\})$ where X is an n-dimensional topological manifold and where $\{\Phi_i \colon U_i \to V_i\}$ is an atlas for X such that given any $i, j \in I$ the corresponding transition map $\Phi_j \circ \Phi_i^{-1} \colon V_i \cap \Phi_i(U_j) \to V_j \cap \Phi_j(U_j)$ is "piecewise linear". Here, in this context we say that a map $f \colon U \to V$ between open subsets of $\mathbb{R}^n$ is *piecewise linear* if any point $P \in U$ there exists a linear simplicial complex $X \subseteq U$ which is a neighborhood of P such that the restriction of f to each simplex of X is linear. This definition is for example used in [**GoS1999**, p. 7], [**Rudy2016**, p. 16] and [**HiM1974**, p. 8]. The discussions on [**Zee1962a**, p. 63], [**Hud1969**, p. 82] and [**Ded1962**, p. 370] show that these two concepts of PL-manifolds basically agree.

(2) A PL-structure for a topological space is sometimes called a *piecewise linear triangulation*, or shorter, a PL-*triangulation*. □

The PL-Manifold Characterization Proposition 180.10, and drawing enough pictures, shows that the following definition captures the right intuition.

Definition 180.11. We say that a PL-manifold is closed if X is compact and for every vertex the link $\mathrm{Lk}(v)$ is a PL-sphere.

We have no intention of going deep into the theory of PL-manifolds. Instead we refer the interested reader to [**Hud1969, Zee1963b, Gla1970, RS1972**] for a much more detailed introduction to PL-manifolds. For example these references contain the definition of the boundary of a PL-manifold and a discussion of PL-submanifolds and the existence of "regular neighborhoods" (see also the next section) of subcomplexes.

As an aside we mention only one result, namely suppose that M is a PL-manifold. Similar to the statements of the Smooth Manifold-Simplicial Structure Theorem 179.5 (2) and (3), one can extend a given PL-structure on the boundary to a PL-structure on M and one can extend a given PL-structure on a proper submanifold to a PL-structure of M. We refer to [**Arm1967**] for the precise statements and proofs.

180.2. **Regular Neighborhoods.** In this short section we introduce the notion of a regular neighborhood of a subcomplex of a simplicial complex. This is an object that gets used commonly in topology and thus we want to give a definition and we want state to some of the main properties of regular neighborhoods. Otherwise we will barely make use of this concept.

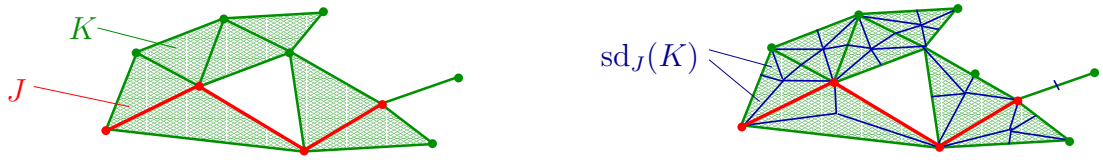
First we give a slight generalization of the concept of the barycentric subdivision of an abstract simplicial complex.

Definition 180.12. Let $K = (V, S)$ be an abstract simplicial complex and let $J = (U, R)$ be a finite subcomplex. We define the barycentric subdivision $\mathrm{sd}_J(K)$ of K away from J

by barycentric subdividing every simplex that is not contained in J. More formally, and much less readably, $\mathrm{sd}_J(K)$ is defined as follows:

$$\begin{aligned}\text{vertices of } \mathrm{sd}_J(K) &:= U \cup \{\underline{s}\}_{s\in S\setminus R}\\ \text{simplices of } \mathrm{sd}_J(K) &:= \left\{ r \cup \{\underline{s_0},\dots,\underline{s_k}\} \;\middle|\; \begin{array}{c}\text{where } r\in R\cup\{\varnothing\} \text{ and } s_0,\dots,s_k\in S \text{ for}\\ \text{some } k\in\mathbb{Z}_{\geq -1} \text{ such that } r\subsetneq s_0\subsetneq\cdots\subsetneq s_k\end{array}\right\}.\end{aligned}$$

It follows immediately from the definitions that $\mathrm{sd}_J(K)$ is an abstract simplicial complex and that for $J=\varnothing$ we obtain the usual barycentric subdivision $\mathrm{sd}(K)$.



The obvious generalization of the SD-Lemma 175.5 holds in the above context. But there is certainly no need for formulating the generalizations. Let us move on to the definition of regular neighborhoods.

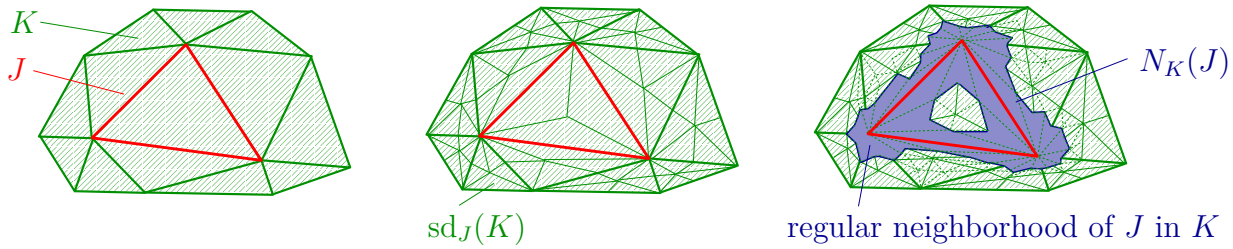
Definition 180.13. Let $(K=(V,S),\Theta\colon |K|\to X)$ be a simplicial complex. Furthermore let $J=(U,R)$ be a subcomplex. We set

$$S_K(J) := \{s\in \mathrm{sd}_J(\mathrm{sd}_J(K)) \,|\, \text{there exists an } r\in R \text{ with } r\subseteq s\}$$

and we set

$$N_K(J) := \bigcup_{s\in S_K(J)} \Theta(|s|) \quad \text{and} \quad \mathring{N}_K(J) := \bigcup_{s\in S_K(J)} \Theta(\langle s\rangle).$$

We refer to $N_K(J)$ as the regular neighborhood of J in K.



Remark. The notion of a regular neighborhood of a subcomplex gets defined differently by different authors. Our definition of *the* regular neighborhood of J in K is an example of *a* regular neighborhood as introduced in [**RS1972**, p. 33]. A more intrinsic definition of a regular neighborhood is given in [**Zee1963b**, Chapter III, p. 22] and [**Gla1970**, p. 77]. It follows from [**Zee1963b**, Chapter III, Theorem 8, p. 22] or alternatively [**Gla1970**, Theorem II.15n] that our definition is consistent with the more intrinsic definition. □

The following proposition states a few key properties of regular neighborhoods of subcomplexes of PL-manifolds.

Proposition 180.14. (Simplicial Regular Neighborhood Proposition) Suppose we are given a compact PL-manifold $(K=(V,S),\Theta\colon |K|\to M)$. Let $J=(U,R)$ be a subcomplex with $\Theta(|J|)\subseteq M\setminus \partial M$.

(1) $N_K(J)$ is a compact PL-manifold and it is a submanifold of the topological manifold M.

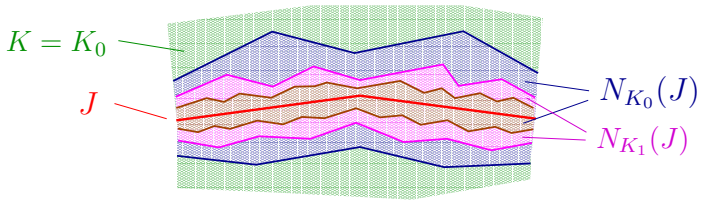
(2) (a) $\mathring{N}_K(J)$ is an open subset of M.
(b) The boundary of $N_K(J)$ as a subset of M equals $N_K(J) \setminus \mathring{N}_K(J)$.
(c) $N_K(J)$ and $\mathring{N}_K(J)$ are both neighborhoods of $\Theta(|J|)$.
(3) $\Theta(|J|)$ is a deformation retract of $N_K(J)$.
(4) $\partial N_K(J)$ is a deformation retract of $N_K(J) \setminus \Theta(|J|)$.

Sketch of proof. Evidently we can just assume that $M = |K|$ and that $\Theta = \mathrm{id}$. As mentioned above, our definition of the regular neighborhood is an example of a regular neighborhood as defined in [**RS1972**, p. 33]. Thus we can use the results from [**RS1972**].

(1) This statement is explicitly and implicitly proved in [**RS1972**, Propostion 3.10].
(2) We leave the fairly routine verification of these statements to the reader.
(3) This statement is proved in [**RS1972**, Corollary 3.30 (ii)], see also [**Gla1970**, p. 77].
(4) We only provide an outline of a proof.[1922] First we define a sequence of abstract simplicial complexes via setting $K_0 := K$ and iteratively via setting $K_{n+1} := \mathrm{sd}_J(K_n)$ for $n \in \mathbb{N}$. Similarly to the SD-Lemma 175.5 (3) we see that these abstract simplicial complexes define again naturally simplicial structures for M. Using a suitable variation on the Diameter Shrinking Lemma 175.13 one can show that $\bigcap_{n\in\mathbb{N}_0} N_{K_n}(J) = |J|$. The key input at this point is [**RS1972**, Corollary 3.18] which implies that for every $n \in \mathbb{N}_0$ there exists a homeomorphism $F_n \colon N_{K_n}(J) \times [0,1] \to N_{K_n}(J) \setminus \mathring{N}_{K_{n+1}}(J)$ such that for every $P \in N_{K_n}(J)$ we have $F_n(P,0) = P$ and $F_n(P,1) \in \partial N_{K_{n+1}}(J)$. Next we want to combine all these maps F_n to obtain a homeomorphism $\partial N_K(J) \times [0,1] \to N_K(J) \setminus |J|$. To do so we set $s_0 = 0$, $s_1 = \frac{1}{2}$, $s_2 = \frac{3}{4}$, $s_3 = \frac{7}{8}$ and so on. Furthermore, given $n \in \mathbb{N}_0$ we define $\Xi_n \colon \partial N_K(J) \to \partial N_{K_n}(J)$ via $\Xi_n(P) = F_{n-1}(\ldots(F_1(F_0(P,1),1),\ldots,1)$. Finally we consider the map

$$\begin{array}{rcl} \partial N_K(J) \times [0,1) & \to & N_K(J) \setminus |J| \\ (P,t) & \mapsto & F_n(\Xi_n(P), 2^{n+1} \cdot (t - s_n)), \quad \text{if } t \in [s_n, s_{n+1}) \text{ for some } n \in \mathbb{N}_0. \end{array}$$

Some thought shows that this map is actually a homeomorphism. The desired statement follows from the observation that $\partial N_K(J) = \partial N_K(J) \times \{0\}$ is a deformation retract of $N_K(J) \times [0,1)$.



The more carefully one looks at this outline, the more little exercises one detects which need to be carried out to turn this argument into a proper proof. ■

This concludes our discussion of regular neighborhoods. This concept is discussed in much greater detail in [**WhdJ1939**], [**RS1972**, Chapter 3], [**Gla1970**, Chapter III.B], [**Hud1969**, Chapter 2] and [**Zee1963b**, Chapter 3]. Some properties are also discussed in [**Hem1976**, p. 7f]. In particular we refer to [**RS1972**, Theorems 3.8 and 3.24] and [**Gla1970**, Theorem II.16n] for some uniqueness theorems which explain to what degree

[1922] A proof of this statement can also be found in [**SZ1994**, Satz 14.8.2].

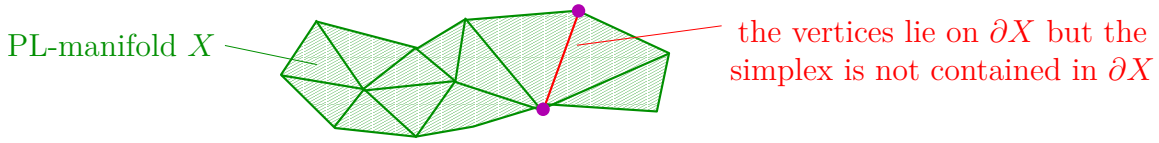
the regular neighborhood depends on the choice of simplicial structures on a given PL-manifold. Unfortunately it is not always immediately apparent that the various definitions in the above literature are mutually compatible.

180.3. **The interplay between topological, smooth and PL-manifolds.** Now that the PL-manifolds joined the playground, let us see how they interact with our old friends, the topological and smooth manifolds. Since morally speaking topological manifolds have the least amount of structure it is not surprising that PL-manifolds are indeed topological manifolds:

Proposition 180.15. (PL–Topological Manifold Proposition) Let X be an *n-dimensional* PL-*manifold*. The following statements hold:

(1) X is an n-dimensional topological manifold.
(2) A vertex v of the PL-structure lies on the boundary of the topological manifold X if and only if the link $\mathrm{Lk}(v)$ is a PL-ball.

Remark. Let X be a PL-manifold and let s be a simplex such that all vertices lie on the boundary ∂X of X, viewed as a topological manifold.



As we see in the figure above, this piece of information does not imply that s is actually contained in ∂X. □

Proof. Let $(K = (V, S), \Theta\colon |K| \to X)$ be an *n-dimensional* PL-*manifold*. Without loss of generality we can assume that $\Theta = \mathrm{id}$, in particular we can assume that $|K| = X$.

(1) In the following we will show that X is an n-dimensional topological manifold.
  (a) By the Simplex-Embedding Lemma 174.20 we know that X is Hausdorff.
  (b) By definition of a PL-manifold we know that S is countable, in particular V is countable. Furthermore note that the PL-Manifold Characterization Proposition 180.10 says that for each $v \in V$ the open star $\mathring{\mathrm{St}}(v)$ is homeomorphic to a subspace of $\mathbb{R}^n$. From the Second Countability Lemma 13.2 (1) and (2) we obtain that each open star is second-countable. Thus it follows from the Link Star Lemma 175.9 (2) that X is the countable union of countably many second-countable open subsets. Hence we obtain from the Second Countability Lemma 13.2 (4) that X itself is second-countable.
  (c) By definition of a PL-manifold we know that for each $P \in X$ there exists in particular a neighborhood U of P and a homeomorphism $\Phi\colon U \to \overline{B}^n$. On page 623 we showed that $\overline{B}^n$ itself is an n-dimensional topological manifold. It it now elementary to see that there exists a chart, in the sense of Definition 23.2, for X at P.

We have thus verified that X is indeed an n-dimensional topological manifold.

(2) Let $v \in V$ be a vertex. We calculate that

$$\mathrm{H}_n(X, X \setminus \{v\}) \underset{\uparrow}{\cong} \mathrm{H}_n(\mathrm{St}(v), \mathrm{St}(v) \setminus \{v\}) \underset{\uparrow}{\cong} \mathrm{H}_n(\mathrm{St}(v), \mathrm{Lk}(v))$$

by the Excision Theorem 148.2 which we can apply by the Link Star Lemma 175.9 (1d)

by the Link Star Lemma 175.9 (1g) and the Homotopy Equivalence-Relative-H_*-Corollary 146.8

$$\underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{n-1}(\mathrm{Lk}(v)) \quad \cong \begin{cases} \mathbb{Z}, & \text{if } \mathrm{Lk}(v) \text{ is a PL } (n-1)\text{-sphere,} \\ 0, & \text{if } \mathrm{Lk}(v) \text{ is a PL } (n-1)\text{-ball.} \end{cases}$$

by the long exact sequence in reduced homology of the pair $(\mathrm{St}(v), \mathrm{Lk}(v))$ together with the Link Star Lemma 175.9 (1f)

It follows from the Manifold-Local-H_*-Lemma 150.9 that $v \in \partial X$ if and only if $\mathrm{Lk}(v)$ is a PL $(n-1)$-ball. ■

Next we turn to the relationship between smooth manifolds and PL-manifolds. The following theorem says that any smooth manifold admits an essentially natural PL-structure.

Theorem 180.16. (Smooth-implies-PL Theorem)

(1) Every smooth simplicial structure on an n-dimensional smooth manifold is an n-dimensional PL-structure.
(2) If M is a closed smooth manifold, then for every smooth simplicial structure the corresponding PL-manifold is also closed.
(3) Every smooth manifold admits the structure of a PL-manifold.
(4) Any two smooth simplicial structures on a smooth manifold are PL-homeomorphic.

Proof.

(1) It follows from [**WhdJ1940**, Theorem 5] (see also the argument of [**Mun1966a**, p. 82]) together with the PL-Manifold Characterization Proposition 180.10 that every smooth simplicial structure on a smooth manifold is a PL-structure in the above sense.
(2) This statement follows immediately from the PL–Topological Manifold Proposition 180.15, together with the fact, proved in the Unique Type-of-Chart Proposition 150.11 (ii), that a smooth manifold is closed as a smooth manifold if and only if it is closed as a topological manifold.
(3) This statement follows from (1) together with the Smooth Manifold-Simplicial Structure Theorem 179.5 where we showed that every smooth manifold admits a countable *smooth* simplicial structure.
(4) This statement follows immediately from the Smooth Simplicial Structure–Uniqueness Theorem 179.7. ■

The Smooth-implies-PL Theorem 180.16 says in particular that given any $n \in \mathbb{N}_0$ the map

$$\mathfrak{w}_n : \left\{ \begin{array}{c} \text{diffeomorphism classes of closed} \\ n\text{-dimensional smooth manifolds} \end{array} \right\} \to \left\{ \begin{array}{c} \text{PL-homeomorphism classes of closed} \\ n\text{-dimensional PL-manifolds.} \end{array} \right\}$$
$$[M] \mapsto [(M, \text{smooth simplicial structure})]$$

is well-defined. It is natural to ask whether $\mathfrak{w}_n$ is a bijection. It turns out that the answer is complicated. More precisely, we have the following theorem.

Theorem 180.17. (PL-Smooth Interplay Theorem)

(1) For $n = 1, \dots, 6$ the map $\mathfrak{w}_n$ is a bijection.
(2) For $n = 7$ (and many other dimensions) the map $\mathfrak{w}_n$ is not a monomorphism.
(3) For $n = 10$ (and many other dimensions) the map $\mathfrak{w}_n$ is not an epimorphism. In fact there exists a closed 10-dimensional PL-manifold that does not admit a single smooth structure.

Proof. The theorem is a combination of several major results in topology. Later in **??** we will discuss this theorem in much greater detail and we will give all the necessary references. In particular we will restate (1) as the PL-Smoothing Theorem **??**. ■

We turn our focus back to the relationship between topological and PL-manifolds. In light of the PL–Topological Manifold Proposition 180.15 it is perhaps tempting to think that a PL-manifold is the same as a topological manifold together with some simplicial structure. But it turns out that the situation is more delicate.

Theorem 180.18. (Simplicial-not-PL-Theorem) There exists a simplicial structure on S^5 that is not a PL-structure.

The key to proving the Simplicial-not-PL-Theorem 180.18 is the following very subtle and very difficult theorem which was proved independently by Jim Cannon [**Cann1979**] and Richard Edwards [**Edw1980, Edw2006**] in the late 1970s. We also refer to [**Dav1986**, Corollary VII.2.C] and [**Lat1979**] for alternative accounts of the proof. It is also worth reading Jim Cannon's reminiscences [**Cann1994**].

Theorem 180.19. (Double Suspension Theorem) Let Y be an n-dimensional PL-manifold. If Y is a homology sphere, i.e. if $\mathrm{H}_k(Y;\mathbb{Z}) \cong \mathrm{H}_k(S^n;\mathbb{Z})$ for all k, then the *double* suspension $\Sigma(\Sigma(Y))$ is homeomorphic to S^{n+2}.

The proof of the Simplicial-not-PL-Theorem 180.18 also requires the following lemma:

Lemma 180.20. (Suspension-of-Non Trivial-π_1-Lemma) Let $n \in \mathbb{N}_{\geq 2}$ and let M be a closed path-connected non-empty n-dimensional topological manifold. If $\pi_1(M)$ is non-trivial, then the suspension $\Sigma(M)$ is not an $(n+1)$-dimensional topological manifold.

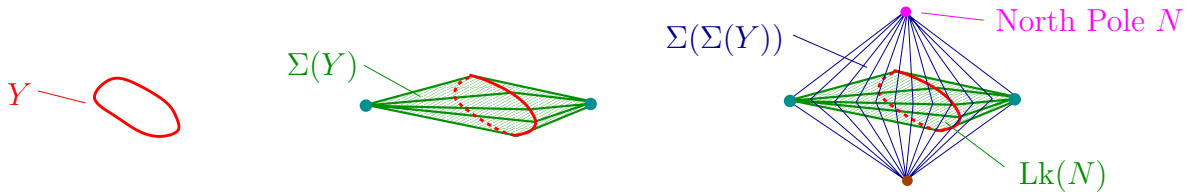
Proof. We outsource the proof to the very amusing Exercise 150.15. ■

With these preparations it is now quite easy to provide a proof of the Simplicial-not-PL-Theorem 180.18.

Proof of the Simplicial-not-PL-Theorem 180.18.

(1) By the Poincaré Homology Sphere Proposition 183.24 we know that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group.
(2) By the Smooth-implies-PL Theorem 180.16 we can equip Y with a PL-structure.
(3) It follows from (2) and the Double Suspension Theorem 180.19 that $\Sigma(\Sigma(Y))$ is homeomorphic to S^5.
(4) By (2) we know that Y is equipped with a simplicial structure. We equip the double suspension $\Sigma(\Sigma(Y))$ with the simplicial structure given by "the double suspension" of the simplicial structure. More precisely, we equip $\Sigma(\Sigma(Y))$ with the simplicial structure given by applying the Simplicial Cone-Suspension Corollary 174.46 twice.
(5) Let N be the "North Pole" of the second suspension. It follows from the discussion on page 3325 that $\mathrm{Lk}(\Sigma(\Sigma(Y)), N) = \Sigma(Y)$.

(6) Since $\pi_1(Y)$ is non-trivial we obtain from the Suspension-of-Non Trivial-π_1-Lemma 180.20 that $\Sigma(Y)$ is not even a 4-dimensional topological manifold. Therefore we obtain from the PL–Topological Manifold Proposition 180.15 that $\mathrm{Lk}(\Sigma(\Sigma(Y)), N) = \Sigma(Y)$ is not a 4-dimensional PL-manifold, let alone a PL 4-sphere.



(7) We obtain from the above, together with the PL-Manifold Characterization Proposition 180.10, that $\Sigma(\Sigma(Y))$ admits a simplicial structure that is not a PL-structure.

(8) In (3) we saw that $\Sigma(\Sigma(Y))$ is homeomorphic to S^5. It follows from (7) that S^5 admits a simplicial structure that is not a PL-structure. ■

Towards the end of these notes, in **??**, we will study the relationship between topological manifolds, smooth manifolds and PL-manifolds in greater detail.

180.4. **Exercises for Chapter 180 .**

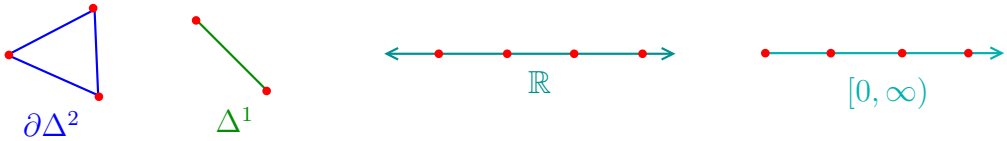
Exercise 180.1. Let $n \in \mathbb{N}_0$. Show that $\mathrm{Cone}(\partial\Delta^n)$ is PL-homeomorphic to Δ^n. We refer to the figure on page 3414 for an illustration.

Exercise 180.2. Let X be a simplicial complex and let Y be a simplicial subdivision. Show that every k-simplex of X is the union of finitely many simplices of Y of dimension $\leq k$.

Exercise 180.3. Let M be a connected non-empty 1-dimensional PL manifold. Show that the PL-homeomorphism type of M is given by the following table:

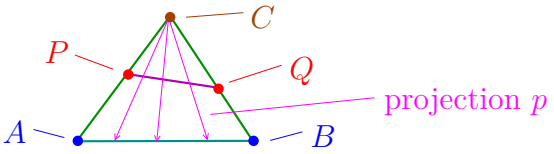
	with empty boundary	with non-empty boundary
compact	the triangle $\partial\Delta^2$	the simplex Δ^1
non-compact	the real line $\mathbb{R}$	the half-open ray $\mathbb{R}_{\geq 0}$.

In the table we give the standard simplicial PL-manifolds ∂^2 and ∂^1 from page 3415. Furthermore, similar to page 3283, we equip $\mathbb{R}$ and $[0,\infty)$ with the obvious simplicial structure where the vertex set is given by $\mathbb{Z}$ respectively $\mathbb{N}_0$.



Exercise 180.4. We consider the triangle Δ_{ABC} shown in the figure below. Let $P \in \overline{AC}$ and $Q \in \overline{BC}$ be two points such that the Euclidean segment $\overline{PQ}$ is not parallel to the Euclidean segment $\overline{AB}$. Let p be the radial projection from C of the Euclidean segment $\overline{PQ}$ to the base $\overline{AB}$ of the triangle. We equip $\overline{PQ}$ and $\overline{AB}$ with the obvious simplicial structures. Show that the map p is not piecewise linear.

Remark. The fact that the radial projection p is not piecewise linear is called the "standard mistake" in [**Zee1963b**, p. I.6] and [**RS1972**, p. 6].



Exercise 180.5. Let X be a topological manifold of dimension $n \geq 3$ with $\partial X = \varnothing$. Furthermore let $(K, \Theta\colon |K| \to X)$ be a simplicial structure for X. Show that for any vertex $v \in K$ we have $\pi_1(\mathrm{Lk}(K, v)) = 0$ and $\mathrm{H}_i(\mathrm{Lk}(K, v)) = 0$ for $i = 1, \ldots, n-2$.
Hint. Note that $\mathrm{Cone}(\mathrm{Lk}(K, v)) \setminus \{v\} \cong \mathrm{Lk}(K, v) \times (0, 1]$ and note that for any open neighborhood U of v there exists an open neighborbood B of v which is homeomorphic to the n-ball B^n, which in turn implies that $U \setminus \{v\}$ is homeomorphic to $S^{n-1} \times (0, 1)$.

CHAPTER 181

Combinatorial manifolds and Pachner moves

The notion of a PL-homeomorphism, and thus also the notion of a PL-manifold, relies on the notion of a simplicial subdivision which, as discussed on page 3363, is not combinatorial. In this chapter we will discuss how one can turn the theory of PL-homeomorphisms and PL-manifolds into a purely combinatorial theory.

The two main statements of this chapter are the Alexander-Newman Theorem 181.6 and Pachner's Theorem 181.16. Unfortunately we will not provide a proof of either statement. This is not a major problem though, since this chapter is meant as a guide to the literature, and since we will not really make use of it.

This chapter is organized as follows:

- In Section 181.1 we will introduce stellar subdivisions and we will discuss the Alexander-Newman Theorem 181.6
- In Section 181.2 we will introduce combinatorial manifolds and we will discuss Pachner's Theorem 181.16.
- In Section 181.3 we will provide exercises for Chapter 181.

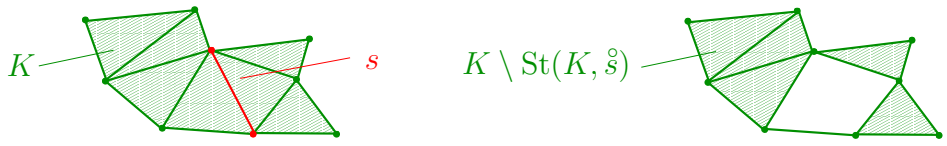
In the next Chapter 182 we will state and prove the Lefschetz Fixed Point Theorem 182.15.

181.1. **Stellar subdivisions and the Alexander-Newman Theorem.** First, given an abstract simplicial complex we introduce, inspired by Definition 175.8 and by the Link Star Lemma 175.9 (1b), the combinatorial notion of the star and the link of a simplex.

Definition 181.1. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex. We introduce the following three subcomplexes[1923] of K:

$$\begin{aligned}
\mathrm{St}(K,s) &:= \Big(\bigcup_{t\in S \text{ with } s\subseteq t} t,\ \{t \in S \,|\, s\cup t \in S\} \Big) \\
\mathrm{Lk}(K,s) &:= \Big(\bigcup_{t\in S \text{ with } s\subseteq t} t \setminus s, \{t \in S \,|\, s\cup t \in S \text{ and } s\cap t = \varnothing\} \Big) \\
K \setminus \mathrm{St}(K,\mathring{s}) &:= \begin{cases} (V\setminus\{v\}, \{t\in S \,|\, s\not\subseteq t\}), & \text{if } s=\{v\} \text{ is a 0-simplex,} \\ (V, \{t\in S \,|\, s\not\subseteq t\}), & \text{otherwise.} \end{cases}
\end{aligned}$$

We refer to the figure below for an illustration of the definition of $K \setminus \mathrm{St}(K,\mathring{s})$.



[1923] One can easily verify that all three are indeed subcomplexes.

Remark. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.

(1) The vigilant reader will have noticed that we defined $\mathrm{St}(K, s)$ and $\mathrm{Lk}(K, s)$ twice, namely on Definition 175.8 as suitable subsets of $|K|$ and now as abstract subcomplexes. The vigilant reader surely will have no troubles in figuring out which of the two definitions we are working with at any given moment.
(2) The notation "$K \setminus \mathrm{St}(K, \mathring{s})$" is meant to be suggestive, but note that on its own $\mathring{s}$ makes no sense. □

Next we recall the following definitions from page 3284 and Definition 175.1.

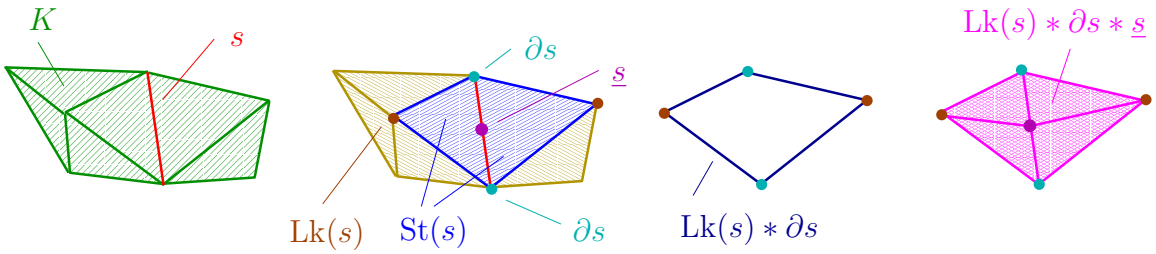
Definition 181.2. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex.

(1) (a) If s is a 0-simplex, then we define $\partial s = (\varnothing, \varnothing)$.
    (b) If s is not a 0-simplex, then we define ∂s to be the abstract simplicial complex $(s, \mathcal{P}(s) \setminus \{\varnothing, s\})$.
(2) We denote by $\underline{s} := \sum_{v \in s} \frac{1}{\dim(s)+1} \cdot v \in |K|$ the barycenter.[1924]

The following lemma summarizes several basic relations between the objects we just introduced.

Lemma 181.3. (Star-Link-Join Lemma) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. The following statements hold:

(1) $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$.
(2) The union of the subcomplexes $K \setminus \mathrm{St}(K, \mathring{s})$ and $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$ equals K and the intersection equals $\mathrm{Lk}(K, s) * \partial s$.
(3) The intersection of the abstract simplicial complexes $K \setminus \mathrm{St}(K, \mathring{s})$ and $\mathrm{Lk}(K, s) * \partial s * \underline{s}$ also equals $\mathrm{Lk}(K, s) * \partial s$.



Proof. We leave it to the reader to verify that all the statements follow indeed immediately from the definitions. For example Statement (1) is already mentioned in Exercise 176.5 (a). ■

Now we define the notion of a "stellar subdivision at a simplex s". The basic idea is to add a vertex in the interior of s and subdivide all the simplices that contain s accordingly. Slightly more precisely, the "stellar subdivision at a simplex s" is defined as replacing the subcomplex $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$ by the abstract simplicial complex $\mathrm{Lk}(K, s) * \partial s * \underline{s}$. Even more precisely, it is defined as follows.

[1924] The meticulous reader will notice that in this section, on several occasions, we will use the Barycenter Lemma 175.2.

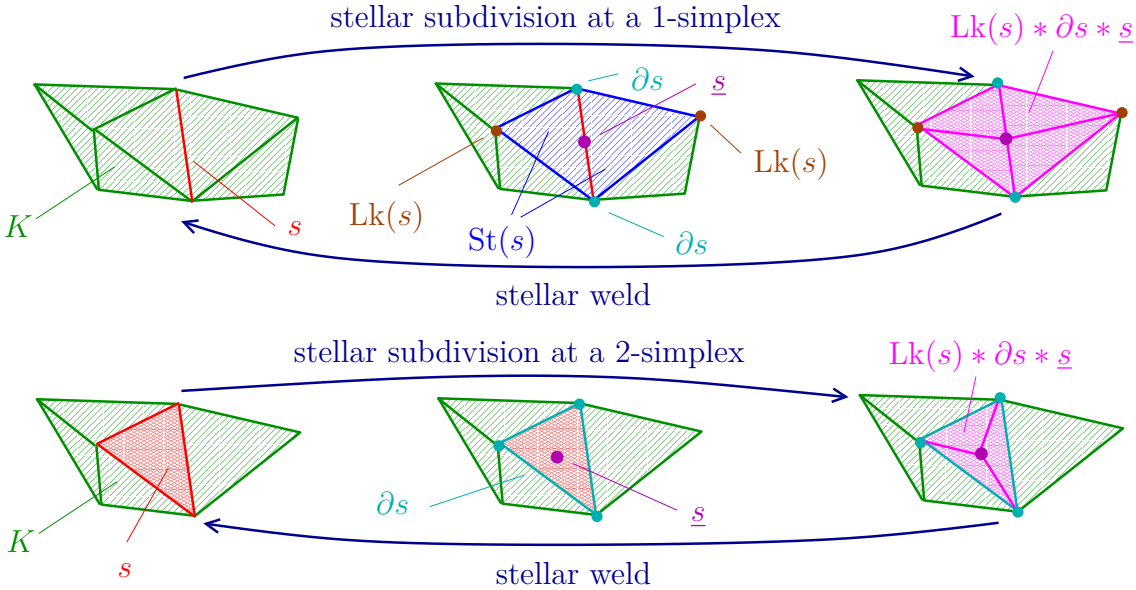
Definition 181.4.

(1) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. We define the stellar subdivision of K at s to be the abstract simplicial complex[19251926]

$$\sigma_s(K) \;:=\; (K \setminus \mathrm{St}(K, \mathring{s})) \cup \mathrm{Lk}(K, s) * \partial s * \{\underline{s}\}.$$

(2) Let K be an abstract simplicial complex. If K' is a stellar subdivision of K, then we say that K is a stellar weld of K'.

(3) We say that two abstract simplicial complexes K and L are stellar equivalent if there exists a finite sequence $K = K_0, K_1, \ldots, K_m = L$ of abstract simplicial complexes such that each K_{i+1} is simplicially isomorphic to either a stellar subdivision or a stellar weld of K_i.



Remark.

(1) We follow the naming conventions used in [**Lic1999**] which differ slightly from the ones used in say [**Hud1969, Zee1963b, RS1972**]. More precisely, what we call a *stellar subdivision* is called an *elementary stellar subdivision* in [**Hud1969, Zee1963b, RS1972**].

(2) It follows easily from the definitions and the Barycenter Lemma 175.2 (3) that the stellar subdivision at a 0-simplex is the identity. □

Lemma 181.5. (Stellar Subdivision–Topological Realization Lemma) Suppose that $K = (V, S)$ is an abstract simplicial complex. Let $s \in S$. If $\mathrm{Lk}(K, s)$ is finite, then there exists a natural PL-homeomorphism $|K| \to |\sigma_s(K)|$.

[1925]It follows from the Subcomplexes–Intersection-Union Lemma 174.7 (1) that $\sigma_s(K)$ is indeed an abstract simplicial complex.

[1926]It follows from Exercise 181.2 that alternatively one could also define

$$\sigma_s(K) \;:=\; (K \setminus \mathrm{St}(K, \mathring{s})) \cup \underbrace{(((K \setminus \mathrm{St}(K, \mathring{s})) \cap \mathrm{St}(K, \mathring{s}))}_{\text{isomorphic to } \mathrm{lk}(K,s)*\partial s} * \{\underline{s}\}.$$

Remark. The proof of the Stellar Subdivision–Topological Realization Lemma 181.5 shows that $|\sigma_s(K)|$ is essentially a simplicial subdivision of $|K|$. This shows that the name "stellar *subdivision*" is justified. □

Proof. Let $K = (V, S)$ be a finite abstract simplicial complex and let $s \in S$. Recall that on page 3413 we saw that there exists a natural PL-homeomorphism $\varphi\colon |s| \to |\partial s * \underline{s}|$ which is the identity on the common subset $|\partial s|$. Note that it follows from the Join Lemma 174.45 and the Join-of-PL-Homeomorphisms Lemma 180.5, together with our hypothesis that $\mathrm{Lk}(K, s)$ is finite, that the maps $\mathrm{id}_{\mathrm{Lk}(K,s)}$ and φ give rise to a natural PL-homeomorphism $f\colon |\mathrm{Lk}(K, s) * s| \to |\mathrm{Lk}(K, s) * \partial s * \underline{s}|$ which has the following property:

(∗) The map f is the identity on the common subset $|\mathrm{Lk}(K, s) * \partial s|$.

Next we consider the two maps

$$\Phi\colon |K| \to |\sigma_s(K)| \qquad P \mapsto \begin{cases} P, & \text{if } P \in |K \setminus \mathrm{St}(K, \mathring{s})|, \\ f(P), & \text{if } P \in |\mathrm{St}(K, s)| \end{cases}$$

$$\Psi\colon |\sigma_s(K)| \to |K| \qquad P \mapsto \begin{cases} P, & \text{if } P \in |K \setminus \mathrm{St}(K, \mathring{s})|, \\ f^{-1}(P), & \text{if } P \in |\mathrm{Lk}(K, s) * \partial s * \underline{s}|. \end{cases}$$

We make the following observations:

(1) It follows from the Star-Link-Join Lemma 181.3 (2) and (3) together with (∗) that both maps are well-defined.
(2) By the Simplicial Subcomplex Lemma 174.23 together with the Pasting Proposition 4.7 we know that both maps are continuous.
(3) Basically by construction the maps are inverses of one another.

In summary we have shown that both maps are homeomorphisms. It remains to show that $\Phi\colon |K| \to |\sigma_s(K)|$ is in fact a PL-homeomorphism. To do so, note that it follows easily from the definition of the map Φ that the simplicial complex $(\sigma_s(K), \Phi^{-1}\colon |\sigma_s(K)| \to |K|)$ is a simplicial subdivision of $|K|$. With respect to this simplicial subdivision the map Φ is evidently a simplicial isomorphism. Thus we see that Φ is in fact a PL-homeomorphism. ■

The following rather amazing theorem is one of the key bridges between topology and combinatorics.

Theorem 181.6. (Alexander-Newman Theorem) Let K and L be two finite abstract simplicial complexes. The following two statements are equivalent:
(1) The abstract simplicial complexes K and L are stellar equivalent.
(2) The simplicial complexes $|K|$ and $|L|$ are PL-homeomorphic.

Proof. The "(1)⇒(2)"-direction is just the combination of the Stellar Subdivision–Topological Realization Lemma 181.5 and the Finite Piecewise Linear-Composition Proposition 180.3. The much harder "(2)⇒(1)"-direction was first proved by James Alexander [**Ale1930**, Theorem 15:1] in 1930, building on work of Max Newman [**New1926**]. More modern expositions of the proof are given in [**Gla1970**, Theorem II.17] and in [**Lic1999**, Theorem 4.5]. ■

According to [**Lic1999**, p. 311] and [**Mel2018**, p. 1] the following question, posed in particular in [**Hud1969**, p. 14] and [**Pac1991**, Problem 3.3], is still open. An affirmative answer for 2-dimensional abstract simplicial complexes was given by Jürgen Ewald [**Ewa1986**] in 1986.

Question 181.7. Let K and L be two finite abstract simplicial complexes such that the simplicial complexes $|K|$ and $|L|$ are PL-homeomorphic. Do there exist iterated stellar subdivisions of K and L which are simplicially isomorphic?

181.2. **Combinatorial manifolds and Pachner's Theorem.** The Alexander-Newman Theorem 181.6, the PL-Manifold Characterization Proposition 180.10 and the PL–Topological Manifold 180.15 (2) motivate the following definitions.

Definition 181.8.

(1) We say that two finite abstract simplicial complexes K and L are combinatorially homeomorphic if K and L are stellar equivalent.
(2) Let $n \in \mathbb{N}_0$. We say that a finite abstract simplicial complex is a combinatorial n-ball if it is combinatorially homeomorphic to D_n and we say that it is a combinatorial n-sphere if it is combinatorially homeomorphic to S_n.[1927]
(3) Let K be a finite abstract simplicial complex. We say that K is an n-dimensional combinatorial manifold if for every vertex v the link $\mathrm{Lk}(K, v)$ is a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-ball.
(4) We say that an n-dimensional combinatorial manifold K is closed if for every vertex v the link $\mathrm{Lk}(K, v)$ is a combinatorial $(n-1)$-sphere.

Examples. Let $n \in \mathbb{N}$. We consider the abstract simplicial complexes D_n and S_n.

(1) For every vertex v of D_n we have, basically by definition, that $\mathrm{Lk}(D_n, v)$ is simplicially isomorphic to D_{n-1}. Thus we see that D_n is an n-dimensional combinatorial manifold.
(2) Similarly, for every vertex v of S_n we have, again basically by definition, that $\mathrm{Lk}(S_n, v)$ is simplicially isomorphic to S_{n-1}. Thus we obtain that S_n is a closed n-dimensional combinatorial manifold.



The combinatorial manifolds D_n and S_n are illustrated in the figure above. □

Remark.

(1) From what I can see, there is no combinatorial analog of the notion that a map between two finite abstract simplicial complexes is piecewise linear.
(2) In the literature, see e.g. [**Hud1969**, p. 26] and [**Gla1970**, p. 18], a *combinatorial manifold* is often defined as a PL-manifold. Even though the concept of a "wrong definition" does not exist, it seems to me that it is more appropriate to define a combinatorial manifold as above, namely to define it immediately as a purely combinatorial object.
(3) The following statement, which is [**Gla1970**, Lemma II.13], gets often used without noticing: Let K^n be an n-dimensional combinatorial manifold and let L^{n-1} be an $(n-1)$-dimensional combinatorial manifold such that ∂K^n is combinatorially homeomorphic to

[1927] We refer to page 3283 for the definition of the abstract simplicial complexes S_n and D_n.

L^{n-1}, then there exists an n-dimensional combinatorial manifold M^n with $\partial M^n = L^{n-1}$, such that K^n is combinatorially homeomorphic to M^n. □

The statement of the following proposition, which can be viewed as an analog of the PL-Manifold–Homeomorphism Proposition 180.9, does not come as a surprise.

Proposition 181.9. (PL Homeomorphism–Combinatorial Manifold Proposition) Let K and L be two finite abstract simplicial complexes. If K is a (closed) n-dimensional combinatorial manifold and if L is combinatorially homeomorphic to K, then L is a (closed) n-dimensional combinatorial manifold.

Example. It follows from the PL Homeomorphism–Combinatorial Manifold Proposition 181.9 and the previous examples that every combinatorial n-ball is indeed an n-dimensional combinatorial manifold and that every combinatorial n-sphere is indeed a closed n-dimensional combinatorial manifold. □

Proof. A short moment's thought shows that this statement does indeed need a proof, it does not follow immediately from the definitions. In fact the proposition is a consequence of [**Gla1970**, Lemma on p. 19]. ■

By definition the link of any 0-simplex in a closed combinatorial manifold is a combinatorial sphere. The following lemma, which is an analog of the PL-Manifold Characterization Proposition 180.10 (2)⇒(4), generalizes this statement to higher dimensional simplices.

Lemma 181.10. (Link-in-Combinatorial Manifold Lemma) Let K be a closed n-dimensional combinatorial manifold. If s is a k-simplex, then the link $\mathrm{Lk}(K, s)$ is a combinatorial $(n-k-1)$-sphere.

Proof. The proof of this lemma is almost the same as the proof of the PL-Manifold Characterization Proposition 180.10 (2)⇒(4). The only slight difference is that now we need to use the fact, mentioned above, that every combinatorial k-sphere is a closed k-dimensional combinatorial manifold. ■

By definition two combinatorially homeomorphic combinatorial manifolds are related by a sequence of stellar subdivisions and stellar welds. In the following we will see that they are also related by the more elegant "bistellar moves".

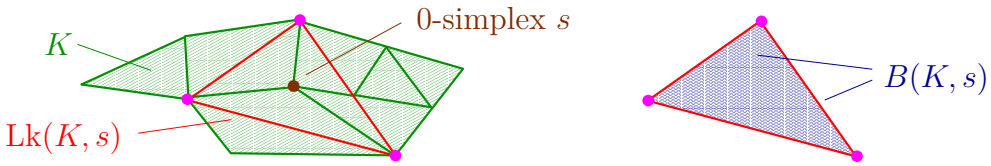
To introduce the notion of a bistellar move we need the following basic definition.

Definition 181.11. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex.

(1) Let $m \in \mathbb{N}_0$. We say that $\mathrm{Lk}(K, s) =: (U, R)$ is a non-trivial m-sphere in K if the following two conditions are satisfied:
   (a) we have $\#U = m + 2$ and $R = \mathcal{P}(U) \setminus \{\varnothing, U\}$,[1928]
   (b) the vertex set U is not a simplex of K.
   In this case we denote by $B(K, s)$ the abstract simplicial complex $(U, \mathcal{P}(U) \setminus \{\varnothing\})$. We illustrate the definition in the figure below.

(2) We say that $\mathrm{Lk}(K, s)$ is a non-trivial (-1)-sphere in K if it is the empty subcomplex. In this case we define $B(K, s)$ to be abstract simplicial complex consisting of the single vertex $\underline{s}$.

As usual, if K is understood then we write $B(s)$ instead of $B(K, s)$.



We move on to the next lemma which can be viewed as the fourth part of the Star-Link-Join Lemma 181.3.

Lemma 181.12. (Second Star-Link-Join Lemma) Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex. We suppose that the link $\mathrm{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. Then the intersection of the abstract simplicial complexes $K \setminus \mathrm{St}(K, \mathring{s})$ and $\partial s * B(K, s)$ equals $\mathrm{Lk}(K, s) * \partial s$.

Proof. Again we leave the task of verifying the lemma to the reader. ■

Suppose we are in the setting of the Second Star-Link-Join Lemma 181.12. Loosely speaking the *bistellar move along* s is defined as the result of replacing the subcomplex

$$\mathrm{St}(K,s) \;=\; \underset{\cong D_k}{s} \;*\; \underset{\cong S_{n-k-1}}{\mathrm{Lk}(K,s)} \quad \text{by the abstract simplicial complex} \quad \underset{\cong S_{k-1}}{\partial s} \;*\; \underset{\cong D_{n-k}}{B(K,s)}.$$

More precisely, we have the following definition:

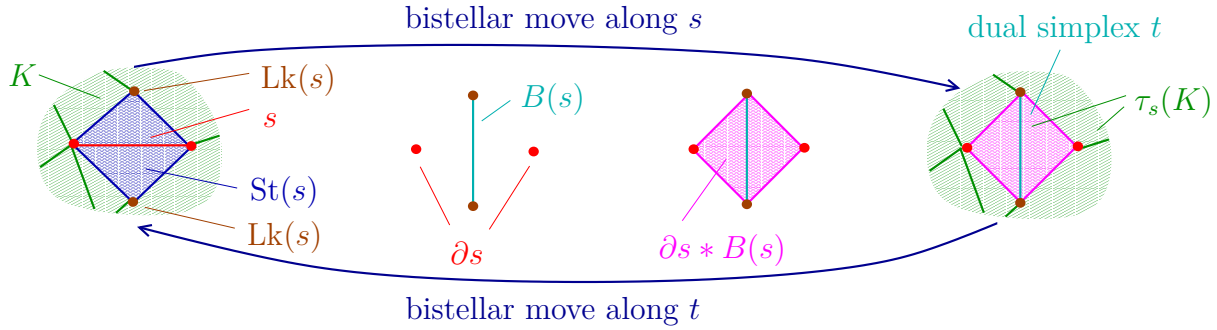
Definition 181.13. Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex such that $\mathrm{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. We define

$$\tau_s(K) \;:=\; (K \setminus \mathrm{St}(K, \mathring{s})) \cup \partial s * B(K, s).$$

We refer to the unique $(n-k)$-simplex in $B(K, s)$ as the dual simplex. We say that $\tau_s(K)$ is obtained from K by a bistellar move along s.

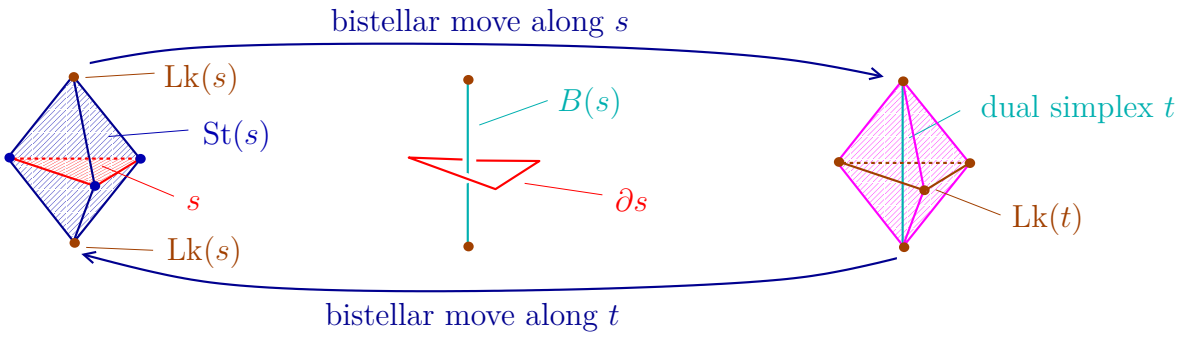
Examples.

(1) In the figure below we show a bistellar move along a 1-simplex s in a 2-dimensional combinatorial manifold.

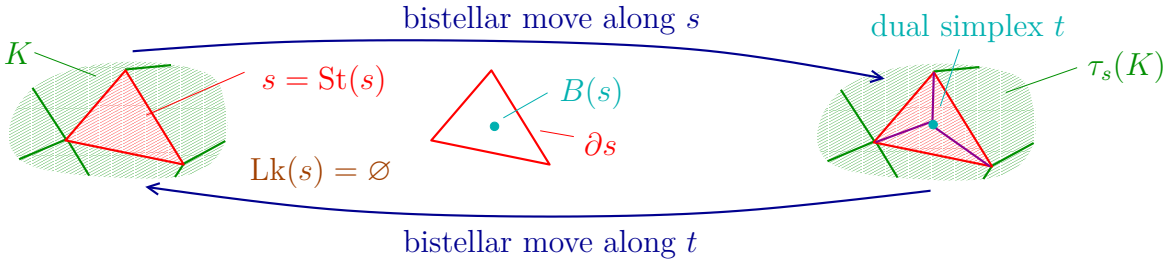


(2) In the figure below we show a bistellar move along a 2-simplex s in a 3-dimensional combinatorial manifold.

[1928]This just means that $\mathrm{Lk}(K, s)$ is simplicially isomorphic to the abstract simplicial complex S_m. It is worth recalling that the Link-in-Combinatorial Manifold Lemma 181.10 says that the link of a k-simplex in a closed n-dimensional combinatorial manifold is combinatorially homeomorphic to S_{n-k-1}, which is a much weaker condition than demanding that the link is simplicially isomorphic to S_{n-k-1}.



(3) In the figure below we show a bistellar move along a 2-simplex s in a 2-dimensional combinatorial manifold.



In this case $\mathrm{Lk}(s) = \varnothing$, which by the above convention, is a non-trivial (-1)-sphere. Note that in each example we see that the bistellar move along the dual simplex actually gets us back to the original abstract simplicial complex. In the next lemma we will see that this is not a coincidence. □

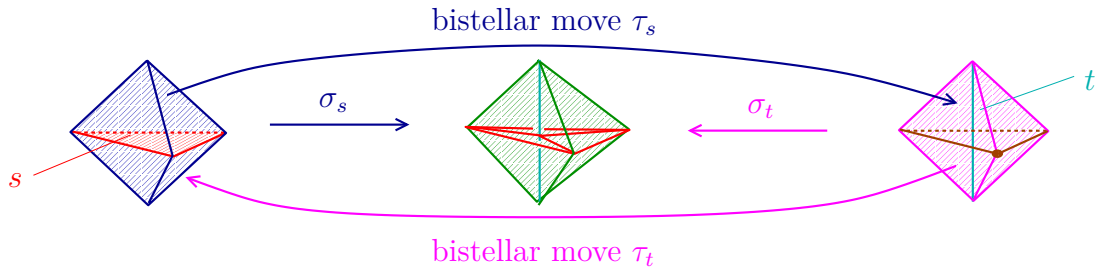
The following lemma summarizes a few key properties of bistellar moves.

Lemma 181.14. (Bistellar Lemma) Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex such that $\mathrm{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. We denote by t the dual simplex. The following statements hold:

(1) The dual simplex t is an $(n-k)$-simplex in $\tau_s(K)$ with the following properties:
  (a) $\partial t = \mathrm{Lk}(K, s)$,
  (b) $\mathrm{Lk}(\tau_s(K), t) = \partial s$,
  (c) (a) if $s = \{v\}$ is a 0-simplex, then $B(\tau_s(K), t) = (\{\underline{t}\}, \{\underline{t}\})$,
    (b) if s is not a 0-simplex, then $B(\tau_s(K), t) = s$,[1929]
  (d) $\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}) = K \setminus \mathrm{St}(K, \mathring{s})$.
  Furthermore ∂s is a non-trivial $(k-1)$-sphere in $\tau_s(K)$.
(2) $\sigma_s(K)$ is naturally simplicially isomorphic to $\sigma_t(\tau_s(K))$.
(3) $\tau_s(K)$ is a closed n-dimensional combinatorial manifold which is naturally combinatorially homeomorphic to K.
(4) $\tau_t(\tau_s(K))$[1930] is naturally simplicially isomorphic to K.

Remark. The Bistellar Lemma 181.14 shows that in particular that if two closed n-dimensional combinatorial manifolds are bistellar equivalent, then they are also stellar equivalent.

[1929] Here we view the simplex s as an abstract simplicial complex in its own right.
[1930] It follows from (1) that we can form $\tau_t(\tau_s(K))$.



Furthermore the Bistellar Lemma 181.14 gives us also the justification for the name "bistellar move". □

Proof.

(1) We leave the elementary verification of these statements to the reader. Note that the slightly awkward fact that we need to deal with 0-simplices separately is due to the fact that already in Definitions 181.11 and 181.1 we were forced to treat 0-simplices slightly differently.

(2) This statement follows easily from (1). Indeed, we see that

$$\sigma_t(\tau_s(K)) \underset{\text{by Definition 181.4}}{=} \overbrace{(\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}))}^{=K\setminus \mathrm{St}(K,s) \text{ by (1)}} \cup \overbrace{\mathrm{Lk}(\tau_s(K), t) * \partial t * \underline{t}}^{=\partial s * \mathrm{Lk}(K,s) * \underline{t} \text{ by (1)}} \underset{\substack{\text{by Definition 181.4 of } \sigma_s(K) \\ \text{the map } \underline{t} \mapsto \underline{s} \text{ defines a} \\ \text{simplicial isomorphism}}}{\overset{\cong}{\to}} \sigma_s(K).$$

(3) It follows from (2) that $\tau_s(K)$ is naturally stellar equivalent to K. In other words, $\tau_s(K)$ is naturally combinatorially homeomorphic to K. It follows from the PL Homeomorphism–Combinatorial Manifold Proposition 181.9 that $\tau_s(K)$ is also a closed n-dimensional combinatorial manifold.

(4) First assume that s is not a 0-simplex. In this case we see that

$$\tau_t(\tau_s(K)) \underset{\text{by Definition 181.13}}{=} \overbrace{(\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}))}^{=K\setminus \mathrm{St}(K,s) \text{ by (1)}} \cup \overbrace{\partial t * B(\tau_s(K), t)}^{=\mathrm{Lk}(K,s) * s \text{ by (1)}} \underset{\substack{\text{by the Star-Link-Join} \\ \text{Lemma 181.3}}}{=} K.$$

Finally we deal with the case that $s = \{v\}$ is a 0-simplex. Note that in this case the vertex set of $\tau_t(\tau_s(K))$ equals $(V \setminus \{v\}) \cup \{\underline{t}\}$. The obvious bijection between the vertex sets of $\tau_t(\tau_s(K))$ and of K, i.e. the map that is given by $\underline{t} \mapsto v$, induces, by the same argument as in (2), a simplicial isomorphism $\tau_t(\tau_s(K)) \to K$. ■

Note that the Bistellar Lemma 181.14 (4) implies in particular that "being related by a sequence of bistellar moves" is an equivalence relation. This observation motivates the following definition:

Definition 181.15. We say that two closed n-dimensional combinatorial manifold are bistellar equivalent if one can be obtained from the other via a finite sequence of bistellar moves and simplicial isomorphisms.

The following theorem can be viewed as a refinement of the Alexander-Newman Theorem 181.6 for closed combinatorial manifolds. The theorem was proved in 1991 by Udo Pachner [**Pac1991**].

Theorem 181.16. (Pachner's Theorem) Let K and L be two closed combinatorial manifolds. The following two statements are equivalent:

(1) The abstract simplicial complexes K and L are bistellar equivalent.
(2) The abstract simplicial complexes K and L are combinatorially homeomorphic (i.e. they are stellar equivalent).

Remark.

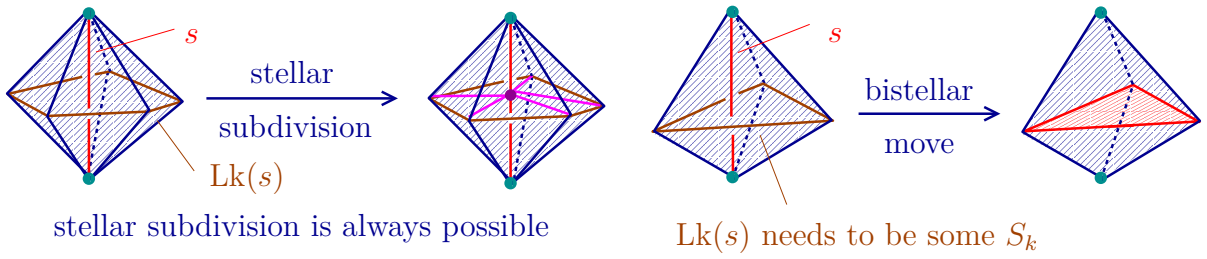
(1) Pachner's Theorem 181.16 is the reason why bistellar moves are often called *Pachner moves*.
(2) An analog of Pachner's Theorem 181.16 for combinatorial manifolds that are not closed is given in [**Pac1991**, Theorem 6.3] and [**Lic1997b**, Theorem 5.10].
(3) The Alexander-Newman Theorem 181.6 and Pachner's Theorem 181.16 can be viewed as an analog of the Reidemeister Moves Theorem 112.10 which says that any two knot diagrams for a given knot can be turned into one another by a finite sequence of Reidemeister moves. □

Proof. The "(1)⇒(2)"-direction is an immediate consequence of the Bistellar Lemma 181.14. The much more interesting "(2)⇒(1)"-direction is proved in [**Pac1991**, Theorem 5.5]. An exposition of the proof is also given in [**Lic1999**, Theorem 5.9]. For combinatorial manifolds of dimension ≤ 3 a more elementary proof is given in [**BW1996**, Theorem 4.11] and implicitly in [**TV1992**]. Furthermore for combinatorial manifolds of dimension ≤ 4 a proof is also given in [**Pac1978**]. ■

Let us quickly compare the statements and roles of the Alexander-Newman Theorem 181.6 and Pachner's Theorem 181.16.

(1) The Alexander-Newman Theorem 181.6 is more general in so far as it applies to any two finite simplicial complexes whereas Pachner's Theorem 181.16 only deals with closed combinatorial manifolds.
(2) Now we restrict ourselves to the class of closed n-dimensional combinatorial manifolds.
   (a) A stellar subdivision consists in replacing $\mathrm{Lk}(K,s)*s$ by $\mathrm{Lk}(K,s)*\partial s*\underline{s}$. As should become clear from considering the figure below, for $n \geq 3$ there are infinitely many simplicial isomorphism types of links of simplices that can appear. Thus we see that even for closed n-dimensional combinatorial manifolds there are infinitely many[1931] "combinatorially distinct" stellar subdivisions.
   (b) On the other hand a Pachner move can only be applied if the link $\mathrm{Lk}(K,s)$ is in fact simplicially isomorphic to some S_k, $k = 0, \ldots, n$. Recall that if a Pachner move is possible, then it replaces $\mathrm{Lk}(K,s) * s$ by $\partial s * B(K,s)$. Thus we see that there are only $n+1$ "combinatorially distinct" moves.

[1931] We leave it to the reader to fill the expression "combinatorially distinct" with meaning.



stellar subdivision is always possible

Lk(s) needs to be some S_k

Now, at the end the end of this chapter we have obtained the following sequence of maps:

$$\begin{array}{c}
\{\text{diffeomorphism classes of closed } n\text{-dimensional smooth manifolds}\} \\
\mathfrak{w}_n \downarrow \text{ by the PL-Smooth Interplay Theorem 180.17 a bijection for } n \leq 6 \\
\{\text{PL-homeomorphism classes of closed } n\text{-dimensional PL-manifolds}\} \\
|K| \updownarrow K \quad \uparrow \text{ a bijection by the Alexander-Newman Theorem 181.6} \\
\left\{ \begin{array}{c} \text{combinatorial homeomorphism classes of} \\ \text{closed } n\text{-dimensional combinatorial manifolds} \end{array} \right\} \\
\uparrow \text{ identity by Pachner's Theorem 181.16} \\
\{\text{bistellar equivalence classes of closed } n\text{-dimensional combinatorial manifolds}\}
\end{array}$$

This shows in particular that for dimensions ≤ 6 the classification of closed smooth manifolds boils down to (very difficult!) combinatorics.

In particular the sequence of maps gives us an approach to defining invariants of closed n-dimensional PL-manifolds (or equivalently smooth manifolds for $n \leq 6$): we "just" need to define an invariant for closed n-dimensional combinatorial manifolds that is invariant under the $n+1$ bistellar moves. This approach to defining invariants has been used by Vladimir Turaev and Oleg Viro [**TV1992**] and John Barrett and Bruce Westbury [**BW1996**] in the 3-dimensional setting and by Christopher Douglas and David Reutter [**DR2018**] in the 4-dimensional setting.

Finally we mention that the significance of Pachner's Theorem 181.16 in mathematics and theoretical physics ("loop quantum gravity theory") is nicely explained in [**Lic1997b, Lic1999**] and [**KleP2018**].

181.3. **Exercises for Chapter 181 .**

Exercise 181.1. Let $K = (V, S)$ be a simplicial complex, let $s \in S$ be a simplex and let $\sigma_s(K)$ be the simplicial complex that is obtained from K by stellar subdivision at s.

(a) Show that there exists a chain map $\mathrm{C}_*(K) \to \mathrm{C}_*(\sigma_s(K))$ that restricts to a chain map $\mathrm{C}_*(\mathrm{Star}(K,s)) \to \mathrm{C}_*(\mathrm{Lk}(K,s) * \partial s)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_*^{\mathrm{simp}}(\mathrm{Star}(K,s)) & \longrightarrow & \mathrm{C}_*^{\mathrm{simp}}(K) & \longrightarrow & \mathrm{C}_*^{\mathrm{simp}}(K, \mathrm{Star}(K,s)) & \longrightarrow & 0 \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\
0 & \longrightarrow & \mathrm{C}_*^{\mathrm{simp}}(\mathrm{Lk}(K,s) * \partial s) & \longrightarrow & \mathrm{C}_*^{\mathrm{simp}}(K) & \longrightarrow & \mathrm{C}_*^{\mathrm{simp}}(K, \mathrm{Lk}(K,s) * \partial s) & \longrightarrow & 0.
\end{array}$$

(b) Show that the vertical map to the right is an isomorphism.

(c) Show that the reduced homology groups of $\mathrm{Star}(K,s)$ and $\mathrm{Lk}(K,s) * \partial s$ are zero.
Hint. Make use of Exercise 176.5 and the Join-Homology Lemma 176.21 (alternatively use Exercise 176.11).
(d) Show that for each $k \in \mathbb{N}_0$ the map φ induces a natural isomorphism

$$\mathrm{H}_k^{\mathrm{simp}}(K) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{simp}}(\sigma_s(K)).$$

Remark. The task is to give a "topology free" proof of (d). In particular you are not supposed to make use of the Simplicial–Singular Homology Theorem 177.9.

Exercise 181.2. Let $K = (V,S)$ be an abstract simplicial complex and let $s \in S$. Show that $(K \setminus \mathrm{St}(K, \mathring{s})) \cap \mathrm{St}(K,s)$ is naturally isomorphic to $\mathrm{lk}(K,s) * \partial s$.

Exercise 181.3. We consider the two abstract simplicial complexes K and L that are sketched in the figure below.
(a) Show that L is not a stellar subdivision of K.
(b) Show "by hand" that K and L are stellar equivalent.



Exercise 181.4. Let $K = (V,S)$ be a combinatorial manifold and let $K' = (V',S')$ be a simplicial subdivision of the simplicial complex $|K| = (V,S)$ in the sense of Definition 177.3. Show that there exists a further simplicial subdivision $K'' = (V'',S'')$ of $K' = (V',S')$ such that $K = (V,S)$ is stellar equivalent to a simplicial complex that is isomorphic to the abstract simplicial complex $K'' = (V'',S'')$.

Exercise 181.5. Use the Alexander-Newman Theorem 181.6 to prove Pachner's Theorem 181.16 in the 2-dimensional case.

Exercise 181.6. Let K be an abstract simplicial complex. Show that if K is a closed n-dimensional combinatorial manifold, then the maximal dimension of a simplex of K is precisely n.
Hint. You could do an induction on n.

Exercise 181.7. The Euler characteristic of a finite simplicial complex $X = (V,S)$ is defined as

$$\chi(X) \;=\; \sum_{k \in \mathbb{N}_0} (-1)^k \cdot \text{number of } k \text{ simplices}.$$

Show that the Euler characteristic is unchanged under stellar subdivision.

CHAPTER 182

The Lefschetz Fixed Point Theorem

In this chapter we will state and prove the Lefschetz Fixed Point Theorem which gives a criterion for a continuous self-map to have a fixed point. This theorem can be viewed as a far reaching generalization of the Brouwer Fixed Point Theorem 38.7.

This chapter is organized as follows:

- In Section 182.1 we will recall properties of the trace of an endomorphism.
- In Section 182.2 we will introduce the Lefschetz number of a self-map of a topological space.
- In Section 182.3 we will formulate and discuss the Lefschetz Fixed Point Theorem 182.15.
- In Section 182.4 we will prove the Lefschetz Fixed Point Theorem 182.15.
- In Section 182.5 we will discuss vector fields on smooth manifolds.
- In Section 182.6 we will provide yet another proof of the Borsuk-Ulam Theorem 47.12.
- In Section 182.7 we will provide exercises for Chapter 182.

In the next Chapter 183 we will discuss homology groups of pseudomanifolds and smooth manifolds.

182.1. **The trace of an endomorphism.** In the formulation of the Lefschetz Fixed Point Theorem we need the notion of the trace of suitable endomorphisms. Even though this notion is surely well-known to the reader we recall the definition and a few properties of the trace of an endomorphism. This short section will facilitate the discussion below. Nonetheless, the author will not hold it against the reader if the reader decides to skip this section.

Definition 182.1. Let R be a ring. Given an $(n \times n)$-matrix $A = (a_{ij})$ over R we define its trace to be

$$\operatorname{tr}(A) \ := \ \sum_{i=1}^{n} a_{ii}.$$

Lemma 182.2. (Trace-Symmetry Lemma) Let R be a ring and let A and B be two $(n \times n)$-matrices over R. If R is commutative, then we have

$$\operatorname{tr}(A \cdot B) \ = \ \operatorname{tr}(B \cdot A).$$

Proof. We perform the following simple calculation:

$$\begin{aligned} \operatorname{tr}(A \cdot B) \ = \ \operatorname{tr}\Big(\text{matrix with } (i,j)\text{-entry } \sum_{k=1}^{n} a_{ik} \cdot b_{kj}\Big) \ &= \ \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \cdot b_{ki} \\ &= \ \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} \cdot a_{ik} \ = \ \operatorname{tr}(B \cdot A). \end{aligned}$$

since the ring R commutative ↑ ; same calculation backwards ↑ ■

Lemma 182.3. (Trace Definition Lemma) Let R be a commutative domain, let V be a free R-module of rank n and let $\varphi\colon V \to V$ be an endomorphism. We pick a basis for V. Then

$$\operatorname{tr}(\varphi) \;:=\; \text{trace of the matrix representing } \varphi \text{ with the respect to the given basis}$$

does not depend on the choice of the basis. We refer to $\operatorname{tr}(\varphi)$ as the trace of the endomorphism.

Notation 182.4. Given a commutative domain R, a free R-module V of finite rank and an endomorphism $\varphi\colon V \to V$, depending on the amount of space we have, we use one of the following three notations:

$$\operatorname{tr}(\varphi) \;=\; \operatorname{tr}(\varphi\colon V \to V) \;=\; \operatorname{tr}(\varphi \circlearrowleft V).$$

Example. Let R be a commutative domain and let V be a free R-module of finite rank. It follows immediately from the definitions that $\operatorname{tr}(\mathrm{id}_V) = \operatorname{rank}(V)$. □

Proof. Let R be a commutative domain. Let $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ be two bases for V. Since R is a commutative domain the rank of V is well-defined, see Definition 2.13. Thus we see that $m = n$. Now let $P = (p_{ij})$ be the transition matrix, i.e. $P = (p_{ij})$ is the matrix that is uniquely determined by the condition that given any $i \in \{1, \ldots, n\}$ we have $v_i = \sum_{j=1}^{n} p_{ji} \cdot w_j$. The "opposite" transition matrix is evidently an inverse to P, in particular we see that P is invertible. We denote by A and B the matrices representing φ with respect to the two bases. We see that

$$\operatorname{tr}(B) \underset{\uparrow}{=} \operatorname{tr}(P \cdot A \cdot P^{-1}) \underset{\uparrow}{=} \operatorname{tr}(P^{-1} \cdot P \cdot A) = \operatorname{tr}(A).$$

an elementary calculation shows that $B \cdot P = P \cdot A$

by the Trace-Symmetry Lemma 182.2 since R is commutative and since all matrices are square matrices of the same size

■

In many of the applications below we will consider the case that R is a field. But on several occasions the case $R = \mathbb{Z}$ is also of interest.

Lemma 182.5. (Trace–Tensor–Direct Sum Lemma) Let $\varphi\colon V \to V$ be an endomorphism of a finitely generated free abelian group and let R be a commutative domain. Let $\gamma\colon \mathbb{Z} \to R$ be the natural ring homomorphism with $\gamma(1_{\mathbb{Z}}) = 1_R$ from page 242. Then

$$\gamma(\operatorname{tr}(\underbrace{\varphi\colon V \to V}_{\text{homomorphism of free } \mathbb{Z}\text{-modules}})) \;=\; \operatorname{tr}(\underbrace{\varphi \otimes \mathrm{id}_R\colon V \otimes R \to V \otimes R}_{\text{homomorphism of free } R\text{-modules by the Tensor-with-Ring Lemma 159.18}}) \;\in\; R.$$

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for the free abelian group V and let A be the $(n \times n)$-matrix that represents φ with respect to this basis. It is straightforward to verify that $v_1 \otimes 1, \ldots, v_n \otimes 1$ is a basis for the R-module $V \otimes R$ and that $\gamma(A)$ is the matrix that represents the endomorphism $\varphi \otimes \mathrm{id}$. Since the trace commutes with ring homomorphisms we obtain the desired equality $\gamma(\operatorname{tr}(A)) = \operatorname{tr}(\gamma(A))$. ■

Sometimes it is also convenient to consider the trace of an endomorphism of a finitely generated abelian group. This leads us to the following definition:

Definition 182.6.

(1) Given a finitely generated abelian group H we refer to
$$\mathrm{F}H \;:=\; H/\operatorname{Tor}(H) \quad \text{where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$
as the maximal torsion-free quotient of H.
(2) Given a homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups we denote the obvious induced map $\mathrm{F}A \to \mathrm{F}B$ by α as well.

Lemma 182.7. (Free Abelian Quotient Lemma) Given a finitely generated abelian group H the maximal torsion-free quotient $\mathrm{F}H = H/\operatorname{Tor}(H)$ of H is a free abelian group.

Proof. A very elementary exercise shows that $\mathrm{F}H = H/\operatorname{Tor}(H)$ is a torsion-free abelian group. Since H is finitely generated, we obtain from the Abelian Group-Classification Theorem 97.24 that $\mathrm{F}H$ is in fact a free abelian group. ■

Remark.

(1) It follows easily from the definition that the map $H \to \mathrm{F}H$ defines a covariant functor from the category of finitely generated abelian groups to the category of free abelian groups.
(2) It might sound silly to mention the Abelian Group-Classification Theorem 97.24 explicitly in the proof of the Free Abelian Quotient Lemma 182.7, but it is worth noting that for commutative rings R which are not PIDs there is usually no meaningful functor from the category of finitely generated R-modules to the category of free R-modules.
(3) The hypothesis that H is finitely generated is of course necessary, for example the maximal torsion-free quotient of the abelian group $(\mathbb{Q},+)$ is again $(\mathbb{Q},+)$, which is certainly not a free abelian group. □

Definition 182.8. Given an endomorphism $\varphi\colon H \to H$ of a finitely generated abelian group we define
$$\operatorname{tr}(\varphi) \;=\; \operatorname{tr}\begin{pmatrix} \mathrm{F}H & \to & \mathrm{F}H \\ [h] & \mapsto & [\varphi(h)] \end{pmatrix}.$$

We conclude this elementary section with the following basic lemma, which is a variation on the Trace–Tensor–Direct Sum Lemma 182.5.

Lemma 182.9. (Trace–Tensor-with-$\mathbb{Q}$-Lemma) Let H be a finitely generated abelian group and let $\varphi\colon H \to H$ be an endomorphism. Then the following equality holds:
$$\operatorname{tr}(\varphi\colon H \to H) \;=\; \operatorname{tr}(\varphi \otimes \mathrm{id}\colon H \otimes \mathbb{Q} \to H \otimes \mathbb{Q}) \;\in\; \mathbb{Q}.$$

Proof. Note that by the Abelian Group-Classification Theorem 97.24 we can write $H = F \oplus T$ where F is a free abelian group and T is a torsion group. We denote by ψ the map
$$F \xrightarrow{f \mapsto f\oplus 0} F \oplus T \xrightarrow{\varphi} F \oplus T \xrightarrow{f\oplus t \mapsto f} F.$$
Then we see that
$$\operatorname{tr}(\varphi\colon H\to H) \underset{\text{follows easily from the definitions}}{=} \operatorname{tr}(\psi\colon F\to F) \underset{\text{by the Trace–Tensor–Direct Sum Lemma 182.5}}{=} \operatorname{tr}(\psi\otimes\mathrm{id}\colon F\otimes\mathbb{Q}\to F\otimes\mathbb{Q}) \underset{\text{by the Tensor–Direct Sum Lemma 159.15}}{=} \operatorname{tr}(\varphi\otimes\mathrm{id}\colon H\otimes\mathbb{Q}\to H\otimes\mathbb{Q}).$$
■

182.2. **The Lefschetz number.**

In this section we return to topology.

Definition 182.10. Let $\mathbb{F}$ be a field.

(1) Let X be a topological space. We say that the homology of X is $\mathbb{F}$-finite if all homology groups $\mathrm{H}_i(X;\mathbb{F})$ with $\mathbb{F}$-coefficients are finite-dimensional vector spaces and if all but finitely many homology groups $\mathrm{H}_i(X;\mathbb{F})$ are zero.

(2) Let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi\colon X \to X$ be a continuous map. We define the $\mathbb{F}$-Lefschetz number of φ to be[1932]

$$\Lambda(\varphi,\mathbb{F}) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(X;\mathbb{F}) \to \mathrm{H}_n(X;\mathbb{F})).$$

Remark. In Exercise 182.1 we will see that the $\mathbb{F}$-Lefschetz number depends only rather mildly on the choice of the field $\mathbb{F}$. □

In general it can be tricky to determine (the trace of) an induced map on homology. The following lemma gives us some basic (partial) calculations of Lefschetz numbers.

Lemma 182.11. (Lefschetz Number–Basics Lemma)

(1) If X is a finite CW-complex, then for any field $\mathbb{F}$ we have $\Lambda(\mathrm{id}_X,\mathbb{F}) = \chi(X)$.

(2) Let X be a path-connected non-empty topological space and let $\varphi\colon X \to X$ be a continuous map. For every field $\mathbb{F}$ we have $\mathrm{tr}(\varphi_*\colon \mathrm{H}_0(X;\mathbb{F}) \to \mathrm{H}_0(X;\mathbb{F})) = 1$.

(3) Let $\mathbb{F}$ be a field and let X be a path-connected non-empty topological space with $\mathbb{F}$-finite homology. If $\varphi\colon X \to X$ is a constant map, then $\Lambda(\varphi,\mathbb{F}) = 1$.

Proof.

(1) We calculate that

$$\Lambda(\mathrm{id}_X,\mathbb{F}) \underset{\text{by Definition 182.10}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\mathrm{id}_* \circlearrowleft \mathrm{H}_n(X;\mathbb{F})) \underset{\text{see the example on page 3438}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(X;\mathbb{F})) \underset{\text{by the Homology-}\chi\text{-Proposition 160.15}}{=} \chi(X).$$

(2) This statement follows immediately from the obvious generalizations of the H_0-Proposition 143.25 to homology with $\mathbb{F}$-coefficients.

(3) This statement follows immediately from (2). ■

Lemma 182.12. (Lefschetz Number-Homotopy Lemma) Let $\mathbb{F}$ be a field and let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi,\psi\colon X \to X$ be two continuous maps. If φ and ψ are homotopic, then the Lefschetz numbers agree, i.e. $\Lambda(\varphi,\mathbb{F}) = \Lambda(\psi,\mathbb{F})$.

Proof. The following lemma is an immediate consequence of the Homotopy-H_*-Proposition 144.12 and the discussion on page 3078. ■

At times it is more efficient to work with the "usual" homology groups, i.e. the homology groups with $\mathbb{Z}$-coefficients. This leads to the following unexciting definition.

Definition 182.13.

[1932]Recall that on page 3078 we pointed out that the induced map $\varphi_*\colon \mathrm{H}_n(X;\mathbb{F}) \to \mathrm{H}_n(X;\mathbb{F})$ is an endomorphism of an $\mathbb{F}$-vector space. Since the vector space $\mathrm{H}_n(X;\mathbb{F})$ is by hypothesis finite-dimensional we see that the trace is in fact defined.

(1) Let X be a topological space. We say that the homology of X is $\mathbb{Z}$-finite if all homology groups $\mathrm{H}_i(X)$ are finitely generated abelian groups and if all but finitely many homology groups $\mathrm{H}_i(X)$ are zero.
(2) Let X be a topological space such that the homology is $\mathbb{Z}$-finite. Let $\varphi\colon X \to X$ be a continuous map. We define the Lefschetz number of φ to be

$$\Lambda(\varphi) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X)).$$

The following lemma relates the above Lefschetz number $\Lambda(\varphi)$ to the Lefschetz number $\Lambda(\varphi, \mathbb{Q})$.

Lemma 182.14. (Lefschetz Number-via-$\mathbb{Z}$-Lemma) Let X be a topological space and let $\varphi\colon X \to X$ be a continuous map.
(1) Let $k \in \mathbb{N}_0$. We have $\mathrm{rank}(\mathrm{H}_k(X)) = \dim_{\mathbb{Q}}(\mathrm{H}_k(X;\mathbb{Q}))$. Furthermore, if $\mathrm{H}_k(X)$ is finitely generated, then we have

$$\mathrm{tr}(\varphi_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(X)) = \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(X;\mathbb{Q}) \to \mathrm{H}_k(X;\mathbb{Q})).$$

(2) If the homology of X is $\mathbb{Z}$-finite, then the homology of X is also $\mathbb{Q}$-finite and we have the equality $\Lambda(\varphi) = \Lambda(\varphi, \mathbb{Q})$.

Proof. Statement (2) is an immediate consequence of Statement (1). Thus let us prove Statement (1). Let $k \in \mathbb{N}_0$. The equality $\mathrm{rank}(\mathrm{H}_k(X)) = \dim_{\mathbb{Q}}(\mathrm{H}_k(X;\mathbb{Q}))$ was shown in the Betti-via-Dimension Lemma 161.12. Now assume that $\mathrm{H}_k(X)$ is finitely generated. We have the following equalities:

$$\mathrm{tr}\left(\mathrm{H}_k(X) \xrightarrow{\varphi_*} \mathrm{H}_k(X)\right) \underset{\uparrow}{=} \mathrm{tr}\left(\mathrm{H}_k(X)\otimes\mathbb{Q} \xrightarrow{\varphi_*\otimes\mathrm{id}} \mathrm{H}_k(X)\otimes\mathbb{Q}\right) \underset{\uparrow}{=} \mathrm{tr}\left(\mathrm{H}_k(X;\mathbb{Q}) \xrightarrow{\varphi_*} \mathrm{H}_k(X;\mathbb{Q})\right).$$

by the Trace–Tensor-with-$\mathbb{Q}$ Lemma 182.9

by the naturality of the short exact sequence of the Universal Coefficient Theorem 162.16 and the observation that by the Tor-Properties Lemma 162.13 (3) the Tor-groups with $\mathbb{Q}$-coefficients vanish ■

182.3. **The formulation of the Lefschetz Fixed Point Theorems.** Now we can finally formulate the long promised Lefschetz Fixed Point Theorem. This theorem, with somewhat different language, was first proved in 1926 by Solomon Lefschetz [**Lef1926**, p. 48].[1933]

Theorem 182.15. (Lefschetz Fixed Point Theorem I) Let X be a finite simplicial complex and let $\varphi\colon X \to X$ be a continuous map. If there exists a field $\mathbb{F}$ such that the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi, \mathbb{F})$ is non-zero, then φ has a fixed point.

We will prove the Lefschetz Fixed Point Theorem 182.15 in the next section. First we will see that one of our earlier great successes, namely the Sphere-Degree Fixed Point Theorem Theorem 46.15, is just a special case of the Lefschetz Fixed Point Theorem 182.15.

Theorem 46.15. (Sphere-Degree Fixed Point Theorem) Let $n \in \mathbb{N}$. Furthermore let $\varphi\colon S^n \to S^n$ be a continuous map. If $\deg(\varphi) \neq (-1)^{n+1}$, then φ has a fixed point.

Proof. We consider the topological space S^n. By the discussion on page 3302 we know that S^n admits the structure of a finite simplicial complex. Let $\varphi\colon S^n \to S^n$ be a continuous

[1933] Solomon Lefschetz (1884-1972) was an American mathematician.

map. We calculate that

$$\Lambda(\varphi,\mathbb{Q}) \overset{\text{by the Lefschetz Number-via-}\mathbb{Z}\text{-Lemma 182.14 we have } \Lambda(\varphi,\mathbb{Q})=\Lambda(\varphi)}{\downarrow} \underbrace{\operatorname{tr}\left(\mathrm{H}_0(S^n)\xrightarrow{\varphi_*}\mathrm{H}_0(S^n)\right)}_{=1 \text{ by the Lefschetz Number–Basics Lemma 182.11 (2)}} + (-1)^n \cdot \underbrace{\operatorname{tr}\left(\mathrm{H}_n(S^n)\xrightarrow{\varphi_*}\mathrm{H}_n(S^n)\right)}_{=\deg(\varphi) \text{ by definition of the degree}} = 1+(-1)^n\cdot\deg(\varphi).$$

The Lefschetz Fixed Point Theorem 182.15 thus says that if $1+(-1)^n\cdot\deg(\varphi)\neq 0$, then φ has a fixed point. But that is exactly what we wanted to show. ■

For many applications the following variation on the Lefschetz Fixed Point Theorem 182.15 is particularly useful.

Theorem 182.16. (Lefschetz Fixed Point Theorem II) Let M be a compact smooth manifold and let $\varphi\colon M\to M$ be a continuous map. If there exists a field $\mathbb{F}$ such that the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi,\mathbb{F})$ is non-zero, then φ has a fixed point.

Remark.

(1) Later on in Theorem 189.21 we will prove the Lefschetz Fixed Point Theorem also for finite CW-complexes and compact topological manifolds. In fact not only is the statement of Theorem 189.21 more general than the statement of Theorem 182.16, the proof of Theorem 189.21 will also be self-contained, since we will not make use of the Smooth Manifold-Simplicial Structure Theorem 179.5 for which we provided only a sketch of a proof.

(2) Reading (1) one might be tempted to think that the conclusion of the Lefschetz Fixed Point Theorem holds for every compact space. But in Exercise 182.5 we will see that this is certainly *not* the case.

(3) We will prove a generalization of the Lefschetz Fixed Point Theorem 182.16 in the Lefschetz-Hopf Theorem 222.9. □

Proof. Let M be a compact smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a finite simplicial structure. Thus we can apply the Lefschetz Fixed Point Theorem 182.15 and we are done. ■

Corollary 182.17. (Acyclic–Fixed Point Corollary) Let M be a compact connected smooth manifold such that $\mathrm{H}_n(M)$ is torsion for all $n\geq 1$. Then every continuous map $\varphi\colon M\to M$ has a fixed point.

Proof. Let M be a compact connected smooth manifold such that $\mathrm{H}_n(M)$ is torsion for all $n\geq 1$. We calculate that

$$\Lambda(\varphi,\mathbb{Q}) \overset{\text{by the Lefschetz Number-via-}\mathbb{Z}\text{-Lemma 182.14}}{\downarrow} \operatorname{tr}\Big(\underbrace{\varphi_*\colon \mathrm{H}_0(M)\to\mathrm{H}_0(M)}_{=1 \text{ by the Lefschetz Number–Basics Lemma 182.11 (2)}}\Big) + \sum_{n\in\mathbb{N}}(-1)^n\cdot\underbrace{\operatorname{tr}(\varphi_*\colon \mathrm{H}_n(M)\to\mathrm{H}_n(M))}_{=0, \text{ since } \mathrm{H}_n(M) \text{ is torsion}} = 1.$$

It follows from the Lefschetz Fixed Point Theorem 182.16 that φ has a fixed point. ■

Examples.

(1) If we apply the Acyclic–Fixed Point Corollary 182.17 to the compact connected smooth manifold $M = \overline{B}^n$, then we recover the Brouwer Fixed Point Theorem 38.7, which we had already proved on page 939 and on page 2843.
(2) In the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we computed the homology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$. These calculations show in particular that the hypotheses of the Acyclic–Fixed Point Corollary 182.17 are satisfied for *even*-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{2n}$. Thus we see that every continuous self-map $f\colon \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point. (This gives an alternative solution of Exercise 151.9.) We will consider the case of *odd*-dimensional real projective spaces in Exercise 182.3. □

In the next section we will provide a proof of the Lefschetz Fixed Point Theorem 182.15 and afterwards we will consider two more elaborate applications of the Lefschetz Fixed Point Theorem 182.15. There are many other pretty consequences of the Lefschetz Fixed Point Theorem 182.15 which we cannot mention in these modest notes. In fact the book by Brown [**BrownR1971**] is completely dedicated to the proof of the Lefschetz Fixed Point Theorem 182.15 and its numerous applications and generalizations.

182.4. **Proof of the Lefschetz Fixed Point Theorem 182.15 .** One of the keys to proving the Lefschetz Fixed Point Theorem 182.15 is the following result in linear algebra which is a not-so-distant relative to the Chain Complex–H_*–χ-Lemma 156.4.

Theorem 182.18. (Hopf Trace Formula)
(1) Let $\mathbb{F}$ be a field. If

$$C_* \;:=\; 0 \;\to\; C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

is a chain complex of finite-dimensional $\mathbb{F}$-vector spaces, then given any chain map $\varphi_*\colon C_* \to C_*$ we have

$$\sum_{i=0}^{k} (-1)^i \cdot \mathrm{tr}(\varphi_i\colon C_i \to C_i) \;=\; \sum_{i=0}^{k} (-1)^i \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_i(C_*) \to \mathrm{H}_i(C_*)).$$

(2) The analog of (1) also holds if we start out with a chain complex of finitely generated free abelian groups.

The key to proving the Hopf Trace Formula 182.18 is the following lemma:

Lemma 182.19. (Trace Additivity Lemma) Let $\mathbb{F}$ be a field. Suppose we are given a commutative diagram of homomorphisms of finite-dimensional $\mathbb{F}$-vector spaces of the following type:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0. \end{array}$$

If the horizontal sequences are exact, then

$$\mathrm{tr}(\beta\colon B \to B) \;=\; \mathrm{tr}(\alpha\colon A \to A) + \mathrm{tr}(\gamma\colon C \to C).$$

Proof. Let $a_1, \dots, a_m$ be a basis for A and let $c_1, \dots, c_n$ be a basis for C. For $i = 1, \dots, n$ we pick a $\overline{c}_i \in B$ with $p(\overline{c}_i) = c_i$. It follows easily from our hypothesis that the horizontal sequences are exact that $i(a_1), \dots, i(a_m), \overline{c}_1, \dots, \overline{c}_n$ form a basis for B. Furthermore it is elementary to show, using the hypothesis that the diagram commutes, that the matrix of

β with respect to this basis is of the following form:

$$\begin{pmatrix} \text{matrix that} & * & * \\ \text{represents } \alpha & * & * \\ 0 \quad 0 & & \text{matrix that} \\ 0 \quad 0 & & \text{represents } \gamma \end{pmatrix}$$

where each "$*$" indicates some element in $\mathbb{F}$.

It is now clear that $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$. ■

Now that the Trace Additivity Lemma 182.19 is in our toolbox the proof of the Hopf Trace Formula 182.18 is almost embarrassingly similar to the proof of the Chain Complex–H_*–χ-Lemma 156.4.

Proof of Hopf Trace Formula 182.18. First we note that Statement (2) follows almost immediately from Statement (1) applied to $\mathbb{F} = \mathbb{Q}$, together with the Trace–Tensor-with-$\mathbb{Q}$-Lemma 182.9 and the Algebraic Universal Coefficient Theorem 162.15. We leave it to the reader to fill in the details.

Thus it remains to prove Statement (1). For the given chain complex (C_*, ∂_*) over some field $\mathbb{F}$ we write for each $n \in \mathbb{N}_0$

$$Z_n := \ker(\partial_n), \quad B_n := \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n / B_n.$$

We write $\varphi_{B_n} := \varphi_n|_{B_n}$ and $\varphi_{Z_n} := \varphi_n|_{Z_n}$. For symmetry reasons we also write $\varphi_{C_n} := \varphi_n$ and $\varphi_{H_n} := \varphi_*\colon H_n \to H_n$. These $\mathbb{F}$-vector spaces and homomorphisms form the following two types of commutative diagrams of short exact sequences

(a)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{Z_n}} & & \downarrow{\scriptstyle \varphi_{C_n}} & & \downarrow{\scriptstyle \varphi_{B_{n-1}}} & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \end{array}$$

and

(b)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{B_n}} & & \downarrow{\scriptstyle \varphi_{Z_n}} & & \downarrow{\scriptstyle \varphi_{H_n}} & & \\ 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \xrightarrow{\partial_n} & H_n & \longrightarrow & 0. \end{array}$$

Now we see that

$$\begin{aligned} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{tr}(\varphi_{C_n}) &\overset{\downarrow}{=} \sum_{n=0}^{k} (-1)^n \cdot (\operatorname{tr}(\varphi_{Z_n}) + \operatorname{tr}(\varphi_{B_{n-1}})) \\ &\overset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot (\operatorname{tr}(\varphi_{B_n}) + \operatorname{tr}(\varphi_{H_n}) + \operatorname{tr}(\varphi_{B_{n-1}})) \\ &= \sum_{n=0}^{k} (-1)^n \cdot \operatorname{tr}(\varphi_{H_n}) + \sum_{n=0}^{k} (-1)^n \cdot \operatorname{tr}(\varphi_{B_n}) + \sum_{n=0}^{k} (-1)^n \cdot \operatorname{tr}(\varphi_{B_{n-1}}) \\ &\overset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{tr}(\varphi_{H_n}) + \sum_{n=0}^{k-1} (-1)^n \cdot \operatorname{tr}(\varphi_{B_n}) - \sum_{m=0}^{k-1} (-1)^m \cdot \operatorname{tr}(\varphi_{B_m}) \\ &= \sum_{n=0}^{k} (-1)^n \cdot \operatorname{tr}(\varphi_{H_n}). \end{aligned}$$

by (a) and the Trace Additivity Lemma 182.19 we have $\operatorname{tr}(\varphi_{C_n}) = \operatorname{tr}(\varphi_{Z_n}) + \operatorname{tr}(\varphi_{B_{n-1}})$

by (b) and the Trace Additivity Lemma 182.19 we have $\operatorname{tr}(\varphi_{Z_n}) = \operatorname{tr}(\varphi_{B_n}) + \operatorname{tr}(\varphi_{H_n})$

we do the substitution $m = n - 1$ and we use that $B_k = 0$ and $B_{-1} = 0$

■

Now we are fully prepared for tackling the proof of the Lefschetz Fixed Point Theorem 182.

Proof of the Lefschetz Fixed Point Theorem 182.15. Let X be a finite simplicial complex and let $\varphi\colon X \to X$ be a continuous map *without* a fixed point. Furthermore let $\mathbb{F}$ be a field. We need to show that

$$\Lambda(\varphi,\mathbb{F}) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(\varphi_*\colon \mathrm{H}_n(X;\mathbb{F}) \to \mathrm{H}_n(X;\mathbb{F})) = 0.$$

The basic idea for the proof is quite simple:

> We need to upgrade the fact that $\varphi(x) \neq x$ for every $x \in X$ to the statement that for a suitable chain complex C_* that calculates $\mathrm{H}_*(X)$ we also have $\varphi(\sigma) \neq \sigma$ for every element σ of a suitable basis. This implies that the trace on the chain group level is zero, hence we get the desired statement from the Hopf Trace Formula 182.18 (1).

For better or worse turning the idea into a proper proof requires a substantial effort. First of all we note that since X is a finite simplicial complex we can use the Finite Simplicial Complex Lemma 174.25 to view X as a subspace of some $\mathbb{R}^n$. In particular we can and will equip X with the usual Euclidean metric $d(x,y) \coloneqq \|x-y\|_{\mathbb{R}^n}$.

Claim 1. There exists an iterated barycentric subdivision Y of X such that for every vertex w of Y we have $\varphi(\mathrm{St}(Y,w)) \cap \mathrm{St}(Y,w) = \varnothing$.

Proof. The claim is proved in several easy steps:

(1) Since X is a finite simplicial complex we know by the Finite Simplicial Complex Lemma 174.25 that X is compact.

(2) From the compactness of X together with our hypothesis that for every $x \in X$ we have $\varphi(x) \neq x$ we obtain through some elementary arguments, as provided in Exercise 10.3, that there exists an $\epsilon > 0$ such that for every $x \in X$ we have $d(\varphi(x),x) \geq \epsilon$.

(3) By the Diameter Shrinking Lemma 175.13 there exists a $j \in \mathbb{N}_0$ such that for every vertex v of the iterated barycentric subdivision $\mathrm{sd}^j(X)$ we have the inequality $\operatorname{diam}(\mathrm{St}(\mathrm{sd}^j(X),v)) < \frac{\epsilon}{2}$.

(4) It follows from (2) and (3) that $\varphi(\mathrm{St}(\mathrm{sd}^j(X),w)) \cap \mathrm{St}(\mathrm{sd}^j(X),w) = \varnothing$. Indeed, let $x,y \in \mathrm{St}(\mathrm{sd}^j(X),w)$. We calculate that

$$d(\varphi(x),y) \underset{\substack{\uparrow\\ \text{by the triangle}\\ \text{inequality}}}{\geq} d(\varphi(x),x) - d(x,y) \underset{\substack{\uparrow\\ \text{by (2) and (3)}}}{\geq} \epsilon - \tfrac{\epsilon}{2} \geq \tfrac{\epsilon}{2} > 0.$$

This implies that $\varphi(\mathrm{St}(\mathrm{sd}^j(X),w)) \cap \mathrm{St}(\mathrm{sd}^j(X),w) = \varnothing$.

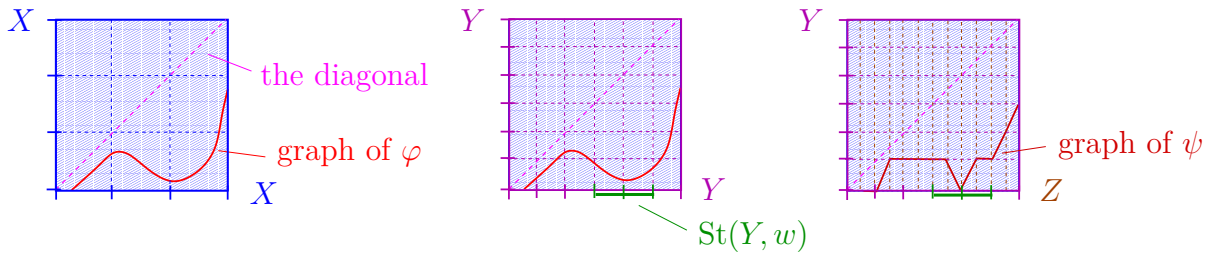
Thus the iterated barycentric subdivision $Y \coloneqq \mathrm{sd}^j(K)$ has the desired property. ⊞

Next we apply the Simplicial Approximation Theorem 175.15 to the map $\varphi\colon Y \to Y$. We obtain an $i \in \mathbb{N}_{\geq 0}$ and a simplicial map $\psi\colon Z \coloneqq \mathrm{sd}^i(Y) \to Y$ with the following properties:

(A) The map ψ is homotopic to φ as a map $Y \to Y$.

(B) For every vertex w of $Z = \mathrm{sd}^i(Y)$ we have $\varphi(\mathrm{St}(\underbrace{\mathrm{sd}^i(Y)}_{=Z},w)) \subseteq \mathrm{St}(Y,\psi(w))$.

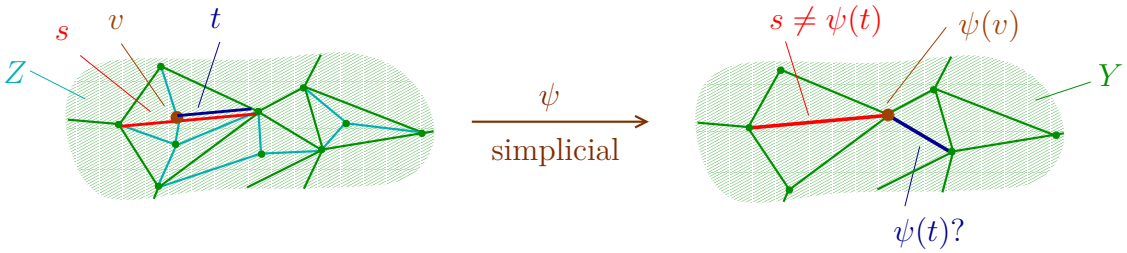
It follows from (A) and the Lefschetz Number-Homotopy Lemma 182.12 that the Lefschetz numbers of φ and ψ agree.



The map $\psi\colon Z \to Y$ is a simplicial map between two different simplicial complexes. The following claim gets as close as possible to saying that ψ does not fix a simplex.

Claim 2. For every simplex s of Y and every simplex t of Z with $t \subseteq s$ we have $\psi(t) \neq s$.

Proof. Let s be a simplex of Y and let t be a simplex of Z with $t \subseteq s$. Let v be a vertex of t.



We make the following deductions:

v is a vertex of the simplex t of Z

$\Longrightarrow t \subseteq \mathrm{St}(Z, v) \Longrightarrow \varphi(t) \subseteq \varphi(\mathrm{St}(Z, v)) \Longrightarrow \varphi(t) \subseteq \mathrm{St}(Y, \psi(v))$

by Definition 175.8 of $\mathrm{St}(Z, v)$; by (B)

$\Longrightarrow \varphi(t) \cap \varphi(\mathrm{St}(Y, \psi(v))) = \varnothing \Longrightarrow t \cap \mathrm{St}(Y, \psi(v)) = \varnothing$

by Claim 1; by the Image-Preimage Lemma 1.9 (4)

$\Longrightarrow t \cap \mathrm{St}(Y, \psi(t)) = \varnothing \Longrightarrow \psi(t) \neq s.$

since $\psi(v)$ is a vertex of $\psi(t)$ we have $\mathrm{St}(Y, \psi(t)) \subseteq \mathrm{St}(Y, \psi(v))$; we have $t \subseteq \mathrm{St}(Y, s)$, indeed this follows from $t \subseteq s$ and from $s \subseteq \mathrm{St}(Y, s)$, which in turn holds by Definition 175.8 of $\mathrm{St}(Y, s)$ ⊞

Next, given $n \in \mathbb{N}_0$ let

$$u_n\colon \mathrm{C}_n^{\mathrm{simp}}(Y) \to \mathrm{C}_n^{\mathrm{simp}}(\underbrace{\mathrm{sd}^i(Y)}_{=Z})$$

be the i-fold iterated simplicial subdivision map that we introduced in the Chain Subdivision Lemma 177.25.

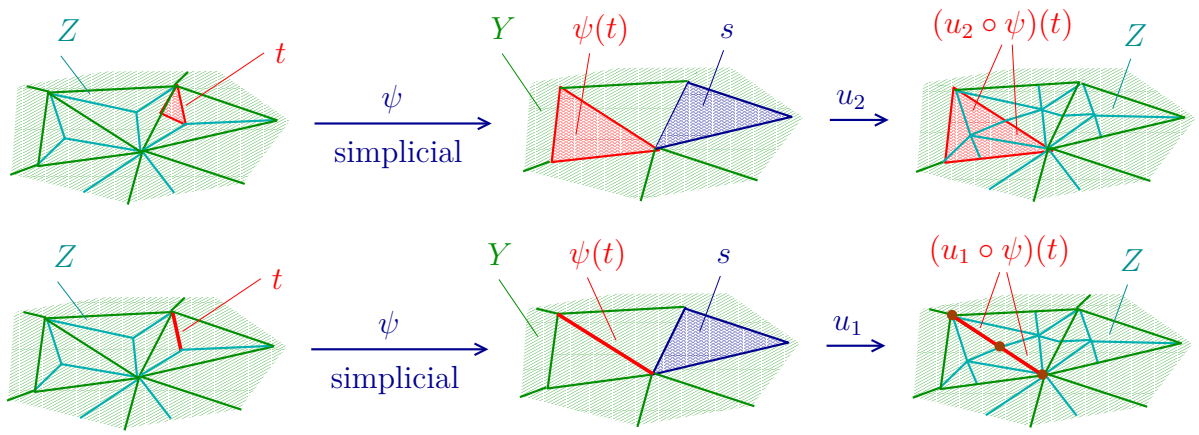
Claim 3. For each $n \in \mathbb{N}_0$ the map $u_n \circ \psi_*\colon \mathrm{C}_n^{\mathrm{simp}}(Z) \to \mathrm{C}_n^{\mathrm{simp}}(Z)$ has trace zero.

Proof. Let T_n be the set of n-simplices of Z. It follows from the Simplicial Chain Group–Isomorphism Lemma 176.11 that T_n, with some choice of ordering for each $t \in T_n$, is a basis of $\mathrm{C}_n^{\mathrm{simp}}(Z)$. Now let $t \in T_n$. If $\psi(t)$ is not an n-simplex of Y, then by definition we

have $\psi_*(t) = 0$. Now suppose that $\psi(t)$ is an n-simplex of Y. Then we see that

$$\begin{aligned}(u_n \circ \psi_*)(t) &\overset{\text{by Definition 177.21 of the simplicial subdivision map}}{=} \text{linear combination of } n\text{-simplices of } Z \text{ contained in } \psi(t) \\ &= \text{linear combination of } n\text{-simplices of } Z \text{ not equal to } t \\ &\overset{\text{here we need to distinguish two cases:}^{1934}}{=} \text{linear combination of elements in } T_n \setminus \{t\}.\end{aligned}$$

(1) If t is contained in some n-simplex s of Y, then the conclusion follows from Claim 2.
(2) If t is not contained in any n-simplex of Y, then it is in particular not contained in the n-simplex $\psi(t)$ of Y



This calculation shows that, with respect to the basis T_n, each diagonal entry of the map $u_n \circ \psi_*$ is zero. In particular the trace of the map $u_n \circ \psi_* \colon \mathrm{C}_n^{\mathrm{simp}}(Z) \to \mathrm{C}_n^{\mathrm{simp}}(Z)$ is zero. ⊞
In the following we will use simplicial homology with $\mathbb{F}$-coefficients. The definition should be clear to the reader and it should be clear that some of the statements which we cite below, which were formulated for homology without coefficients, also have obvious generalizations to homology with $\mathbb{F}$-coefficients.

Claim 4. The following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_n^{\mathrm{simp}}(Z;\mathbb{F}) & \xrightarrow{\psi_*} & \mathrm{H}_n^{\mathrm{simp}}(Y;\mathbb{F}) & \xrightarrow{u_n} & \mathrm{H}_n^{\mathrm{simp}}(Z;\mathbb{F}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_n(Z;\mathbb{F}) & \xrightarrow{\psi_*} & \mathrm{H}_n(Z;\mathbb{F}) & \xrightarrow{u_n} & \mathrm{H}_n(Z;\mathbb{F}). \end{array}$$

Here the vertical maps are the natural isomorphisms from Notation 177.16.

Proof. It turns out that there is not much to say. The rectangle to the left commutes by the naturality of the vertical maps and the rectangle to the right commutes by the Subdivision–Homologies-Proposition 177.31. ⊞

Now we can quickly conclude the proof of the Lefschetz Fixed Point Theorem 182.15. Indeed, we see that

$$\begin{array}{lll}
\Lambda(\varphi,\mathbb{F}) \overset{\text{by the Lefschetz Number Homotopy Lemma 182.12}}{=} \Lambda(\psi,\mathbb{F}) = \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(\psi_*\circlearrowleft \mathrm{H}_i(Z;\mathbb{F})) & \overset{\text{by the Iterated Subdivision Lemma 148.14 the singular singular subdivision maps } u_* \text{ are chain homotopic to the identity, thus the induced map on homology is the identity}}{=} & \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{H}_i(Z;\mathbb{F})) \\
= \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{H}_i^{\mathrm{simp}}(Z;\mathbb{F})) & = & \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(Z;\mathbb{F})) \\
\uparrow \text{by Claim 4} & & \uparrow \text{by the Hopf Trace Formula 182.18 (1)} \\
= \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(Z)) & & = 0. \\
\uparrow \text{by the Trace–Tensor–Direct Sum Lemma 182.5 and since } \mathrm{C}_i^{\mathrm{simp}}(Z;\mathbb{F}) \coloneqq \mathrm{C}_i^{\mathrm{simp}}(Z)\otimes\mathbb{F} & & \uparrow \text{by Claim 3}
\end{array}$$

We have thus proved that if φ has no fixed points, then $\Lambda(\varphi,\mathbb{F}) = 0$. ■

182.5. **Vector fields on smooth manifolds.** In this section we will apply the Lefschetz Fixed Point Theorem 182.15 to the study of the (non-)existence of certain vector fields on smooth manifolds.

For the reader's convenience we recall Definition 29.29:

Definition 182.20. Let M be an n-dimensional smooth manifold.

(1) A vector field on M is a map v which assigns to each $P \in M$ a vector $v(P) \in \mathrm{T}_P M$ such that the following condition is satisfied: For every chart[1935] $\Phi\colon U \to V$ from an open subset U of M to an open subset V of $\mathbb{R}^n$ the map

$$\begin{array}{ccccccc}
V & \to & \mathbb{R}^n & & & & \\
P & \mapsto & \mathrm{D}\Phi_{\Phi^{-1}(P)}\big(\underbrace{v(\Phi^{-1}(P))}_{\in \mathrm{T}_{\Phi^{-1}(P)}M}\big) & \subseteq & \mathrm{T}_P V & \underset{\uparrow}{=} & \mathbb{R}^n
\end{array}$$

identification given by the Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition 29.14

is continuous. We say that the vector field v is smooth if for every chart the above map is smooth.

(2) Given a vector field v on M we say that $x \in M$ is a zero of v if $v(x) = 0$.

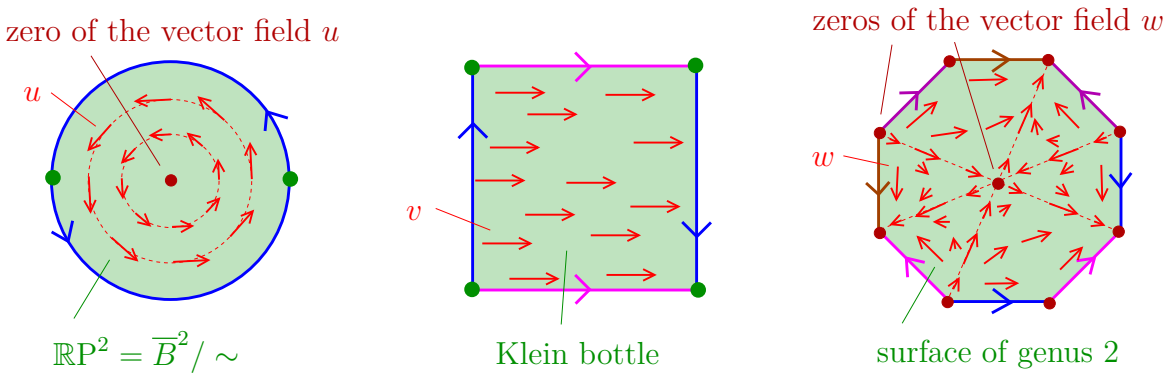
One of the goals of this section is to address the following time-honored question.

Question 29.31 Which closed connected 2-dimensional smooth manifolds admit a nowhere-vanishing vector field?

Example. In the figure below we show vector fields on the projective plane $\mathbb{R}\mathrm{P}^2$, on the Klein bottle and the surface of genus 2.
The vector field on the Klein bottle is nowhere-vanishing, whereas the vector field on $\mathbb{R}\mathrm{P}^2$ and the surface of genus 2 both have zeros. □

The attentive reader will not have failed to notice that it is the right moment to recall one of the highlights from one of the earlier chapters:

[1935] Recall that for a smooth manifold we mean by a "chart" a chart from the given smooth structure.



Theorem 46.18. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere vanishes on at least one point.

Sketch of proof. Let v be a nowhere-vanishing vector field on a sphere S^n. By "traveling along the great circle defined by v" we can find a homotopy $F\colon S^n \times [0,1] \to S^n$ from the identity to the antipodal map $\rho\colon S^n \to S^n$ given by $\rho(x) = -x$. In particular we see that $(-1)^{n+1} = \deg(\rho) = \deg(\mathrm{id}) = 1$. Thus we see that n is odd. ■

Remark. Regarding Question 29.31 the Hairy Ball Theorem 46.18 says in particular that S^2 does not admit a nowhere-vanishing vector field. □

To address Question 29.31 it is also helpful to add a few ways to construct new vector fields to our toolbox.

Lemma 182.21. (Vector Field-Pull Back Lemma)

(1) Let $f\colon M \to N$ be a smooth map between smooth manifolds which is a local diffeomorphism. Let v be a vector field on N. The map

$$P \;\mapsto\; (\mathrm{D}f_P)^{-1}\big(\underbrace{v(f(P))}_{\in \mathrm{T}_{f(P)}(N)}\big)$$

is a vector field on M that we denote by f^*v. A point $P \in M$ is a zero of f^*v if and only if $f(P)$ is a zero of v.

(2) Let M be a smooth manifold and let v be a vector field on M. Suppose that $p\colon \widetilde{M} \to M$ is a covering. Then p^*v is a vector field on $\widetilde{M}$.

(3) Let M be a smooth manifold and let v be a vector field on M. Suppose we are given a group G that acts freely, properly and smoothly on M such that for every $g \in G$ we have $(g\cdot\colon M \to M)^*(v) = v$. Then there exists a unique vector field u on M/G such that $v = p^*u$ where $p\colon M \to M/G$ is the projection. [1936]

Proof.

(1) The first part of the statement follows immediately from Vector Field-Pullback Lemma 29.33. The second part is an immediate consequence of the Tangent Space-Inclusion Proposition 29.20 (3).

(2) This statement follows from (1) and the Manifold Covering Proposition 95.4 (2a).

[1936] Recall that by the Manifold–Quotient Proposition 25.39 we know that M/G is a smooth manifold and that the projection $p\colon M \to M/G$ is a local diffeomorphism.

(3) The proof of this statement is very similar to the proof of (1) and it is also left to the reader. ■

The Vector Field-Pull Back Lemma 182.21 allows us to answer Question 29.31 in some special cases:

Examples.

(1) If v is a vector field on $\mathbb{R}\mathrm{P}^2$ with n zeros, then it follows from the Vector Field-Pull Back Lemma 182.21, applied to the 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ that S^2 admits a vector field with $2n$ zeros. By the Hairy Ball Theorem 46.18 we know that $2n > 0$. In other words, we have just shown that every vector field on $\mathbb{R}\mathrm{P}^2$ has at least one zero.

(2) On the smooth manifold $\mathbb{R}^2$ we now consider the constant "horizontal" vector field $v(x, y) = (1, 0) \in T_{(x,y)}\mathbb{R}^2 = \mathbb{R}^2$. As discussed in the Torus-as-Quotient Lemma 9.2 and on page 835 we can view the torus and the Klein bottle as quotients of $\mathbb{R}^2$ by a group G that acts freely, properly and smoothly. Furthermore, in both cases one can easily verify that the given G-action preserves the vector field v. This shows that the nowhere-vanishing vector field v descends to a nowhere-vanishing vector field on the torus and the Klein bottle. For the Klein bottle we obtain precisely the vector field shown in the figure on page 3449. □

The following theorem is the main result of this section.

Theorem 182.22. (Euler Characteristic–Vector Field Theorem) Let M be a closed smooth manifold. If M admits a nowhere-vanishing vector field, then $\chi(M) = 0$.

Remark.

(1) On page 2987 we saw that for even n the Euler characteristic $\chi(S^n)$ equals 2. Thus we see that the Euler Characteristic–Vector Field Theorem 182.22 is a generalization of the Hairy Ball Theorem 46.18.

(2) In Smooth-χ-Vector Field Theorem 53.29 we proved the same statement for closed orientable smooth manifolds.

(3) For 2-dimensional smooth manifolds a low-tech proof of the Euler Characteristic–Vector Field Theorem 182.22 involving "negative, neutral and positive charges" is given in [**Thu1997**, Proposition 1.3.3].

(4) Let M be a closed connected 2-dimensional smooth manifold that is not diffeomorphic to the torus or the Klein bottle. It follows from the calculations on page 2987 that $\chi(M) \neq 0$. Thus we obtain from the Euler Characteristic–Vector Field Theorem 182.22 that M does not admit a nowhere-vanishing vector field. Together with the above discussion we now have a complete answer to Question 29.31.

(5) In the Nowhere Vanishing Vector Field Existence Theorem 223.7 we will see that the converse to the Euler Characteristic–Vector Field Theorem 182.22 holds. Namely we will see that if M is a closed connected smooth manifold with $\chi(M) = 0$, then M admits a nowhere-vanishing vector field.

(6) In the Euler Characteristic–Vector Field Corollary 223.9 we will prove a suitable extension of the Euler Characteristic–Vector Field Theorem 182.22 to the setting of smooth manifolds with non-empty boundary. □

The key idea behind the proof of the Euler Characteristic–Vector Field Theorem 182.22 is similar to the proof of the Hairy Ball Theorem 46.18, namely we want to use the nowhere-vanishing vector field v on M to construct an interesting continuous self-map of M. This is done in the following proposition.

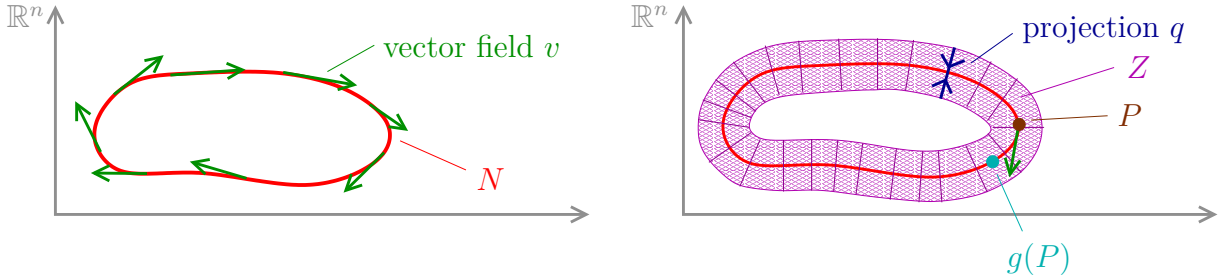
Proposition 182.23. (Non-Zero Vector Field–No Fixed Point Proposition) Let M be a closed smooth manifold. If there exists a nowhere-vanishing vector field on M, then there exists a continuous map $g\colon M \to M$ that is homotopic to the identity and which has no fixed points.

Proof. The proposition follows immediately from the combination of the Replace-by-Smooth Vector Field Proposition 36.7, the Flow Theorem 43.3 and the Flow Fixed Point Theorem 43.9.

Recall that our proof of the Flow Theorem 43.3 was built on the Fundamental Theorem 43.15 for Autonomous ODEs which we did not prove. Therefore, for the reader's peace of mind we will sketch in the following a self-contained proof of the proposition. By the Euclidean Embedding Theorem 33.2 (1) together with the Smooth Embedding Theorem 30.15 we can and will view M as a submanifold of some $\mathbb{R}^n$. It follows from the Neighborhood Projection Theorem 35.2 that there exists an $\epsilon > 0$ such that the following two conditions are satisfied:

(a) The subset $Z := \{P + w \,|\, P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon\}$ of $\mathbb{R}^n$ is a neighborhood of M.

(b) The map
$$\begin{array}{rcll} q\colon Z & \to & M & \\ P + w & \mapsto & P & \text{where } P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon \end{array}$$
is well-defined and continuous.



Next we consider the map
$$\begin{array}{rcl} F\colon M \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^n \\ (P, t) & \mapsto & P + t \cdot v(P). \end{array}$$

We make the following observations:

(1) As in the Convex Homotopy Lemma 18.3 we see that F is, unsurprisingly, continuous.

(2) It follows from (a), together with the fact that M is in particular compact and the Tube Lemma 7.12 that there exists a $\mu > 0$ such that $F(M \times [0, \mu]) \subseteq Z$.

(3) We set $g := q \circ F_\mu\colon M \to M$.

(4) By (1) we know that $q \circ F\colon M \times [0, \mu] \to M$ defines a homotopy between the identity and g.

(5) It remains to show that g has no fixed points. Thus let $P \in M$. By (b) we can write $F_\mu(P) = P + \mu \cdot v(P) =: Q + w$ with $Q \in M$ and $w \in (\widetilde{\mathrm{T}}_Q M)^\perp$ with $\|w\| \leq \epsilon$. Since

$\widetilde{\mathrm{T}}_P M \cap \widetilde{\mathrm{T}}_P M^\perp = \{0\}$ and since $v(P)$ is a non-zero vector in $\widetilde{\mathrm{T}}_P M$ we see that $P \neq Q$. By definition we have $g(P) = Q$. Thus we have shown that $g(P) \neq P$.

This concludes the second proof of the proposition. ■

Now it is straightforward to provide the proof of the Euler Characteristic–Vector Field Theorem 182.22.

Proof of the Euler Characteristic–Vector Field Theorem 182.22. Let M be a closed smooth manifold. Suppose that there exists a nowhere-vanishing vector field v on M. By the Non-Zero Vector Field–No Fixed Point Proposition 182.23 there exists a continuous map $g\colon M \to M$ that is homotopic to the identity and which has no fixed points. Now we calculate that

$$\chi(M) \overset{\text{by Lefschetz Number–Basics Lemma 182.11 (1)}}{=} \Lambda(\mathrm{id}\colon M \to M) \overset{\text{by the Lefschetz Number-Homotopy Lemma 182.12, since } f \text{ and id are homotopic}}{=} \Lambda(f\colon M \to M) \overset{\text{by the Lefschetz Number via-}\mathbb{Z}\text{-Lemma 182.14 (2)}}{=} \Lambda(f\colon M \to M, \mathbb{Q}) \overset{\text{by the claim and the Lefschetz Fixed Point Theorem II 182.16}}{=} 0.$$

■

As a teaser we state the following theorem which we will prove later on. This proposition is basically the converse to the Euler Characteristic–Vector Field Theorem 182.22.

Theorem 223.7. (Nowhere Vanishing Vector Field Existence Theorem) Let M be a closed connected smooth manifold. If $\chi(M) = 0$, then M admits a nowhere-vanishing smooth vector field.

In Chapter 223 we will not only provide a proof of the Nowhere Vanishing Vector Field Existence Theorem 223.7 but we will also formulate and prove a significant generalization of the Euler Characteristic–Vector Field Theorem 182.22, namely the Poincaré-Hopf Theorem 223.8 which, given a vector field, relates the type of zeros which occur to the Euler characteristic of the smooth manifold.

182.6. **The Borsuk-Ulam Theorem II.** In this section we will see how to use the Lefschetz Fixed Point Theorem 182.15 to give an alternative proof of the Borsuk-Ulam Theorem 47.12. First we remind the reader of the statement of the Borsuk-Ulam Theorem 47.12.

Theorem 47.12. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every continuous map $g\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $g(x) = g(-x)$.

Remark. Let us remind the reader that the Borsuk-Ulam Theorem 47.12 has many beautiful applications, in particular the Lusternik-Schnirelmann Theorem 48.2, the Ham-Sandwich Theorem 48.4, Pfister's Level Theorem 48.15 and the Necklace Theorem 48.17. □

The key to our second proof of the Borsuk-Ulam Theorem 47.12 is the following proposition, which is actually just the Odd Map–Odd Degree Proposition 165.6 in slight disguise:

Proposition 182.24. (Odd Map–Even Lefschetz Number Proposition) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a continuous map. If $f(-x) = -f(x)$ for all $x \in S^n$, then the Lefschetz number $\Lambda(f)$ is even.

Examples.

(1) Let $A \in \mathrm{O}(n+1)$. We consider the map $f\colon S^n \to S^n$ that is given by $f(x) \coloneqq A \cdot x$. Evidently we have $f(-x) = -f(x)$ for all $x \in S^n$. Furthermore it follows from the

Sphere–Degree–Matrix Lemma 151.7 together with the Lefschetz Number–Basics Lemma 182.11 (2) that $\Lambda(f) = 1 + (-1)^n \cdot \det(A)$.

(2) In Exercise 182.13 we will see that given any $n \in \mathbb{N}$ and given any *even* $m \in \mathbb{Z}$ there exists a continuous map $f\colon S^n \to S^n$ with $\Lambda(f) = m$ and such that $f(-x) = -f(x)$ for all $x \in S^n$. □

Remark. The proof of the Odd Map–Even Lefschetz Number Proposition 182.24 is an expanded write-up of the arguments in [**Deo2018**, Theorem 4.9.17] and [**Arm1983**, Chapter 9.3]. □

The following lemma gives us a criterion for showing that a trace is even:

Lemma 182.25. (Even Trace Lemma) Let V be a finitely generated free $\mathbb{Z}$-module and let $\rho\colon V \to V$ be an isomorphism with the following two properties:

(i) There exists a subset T of V such that $T \cap \rho(T) = \varnothing$ and such that $T \cup \rho(T)$ is a basis for V.

(ii) We have $\rho \circ \rho = \mathrm{id}_V$.

Let $\varphi\colon V \to V$ be a homomorphism. If φ commutes with ρ, then $\mathrm{tr}(\varphi\colon V \to V)$ is even.

Proof. We equip V with the basis $T \sqcup \rho(T)$. It follows immediately from (ii) that

$$\text{matrix representing } \rho \text{ with respect to the basis } T \sqcup \rho(T) \;=\; \begin{matrix} \\ T \\ \rho(T) \end{matrix} \begin{matrix} \begin{matrix} T & \rho(T) \end{matrix} \\ \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \end{matrix}.$$

Now let $\varphi\colon V \to V$ be a homomorphism that commutes with ρ. We write

$$\text{matrix representing } \varphi \text{ with respect to the basis } T \sqcup \rho(T) \;=:\; \begin{matrix} \\ T \\ \rho(T) \end{matrix} \begin{matrix} \begin{matrix} T & \rho(T) \end{matrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{matrix}.$$

Since φ commutes with g we see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Multiplying out both sides we see in particular that $A = D$. But this implies evidently that the trace of the above matrix representing φ is even. Thus we have shown that $\mathrm{tr}(\varphi)$ is even. ■

Proof of Proposition 182.24. Let $f\colon S^n \to S^n$ be a continuous map which has the property that $f(-x) = -f(x)$ for all $x \in S^n$. We denote by $\rho\colon S^n \to S^n$ the map that is given by $\rho(x) = -x$. Note that our hypothesis on f is equivalent to the statement that f commutes with ρ. We need to show that $\Lambda(f)$ is even. Similar to the proof of the Lefschetz Fixed Point Theorem 182.15 on page 3445 we intend to proceed as follows:

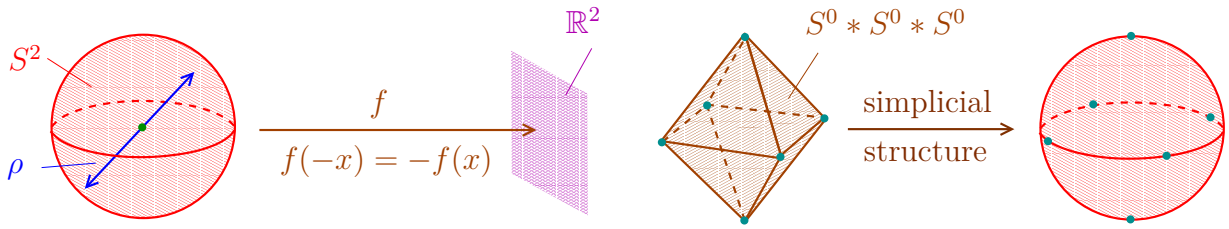
> We want to upgrade the hypothesis that $f \circ \rho = \rho \circ f$ to a statement about an induced map on a suitable simplicial chain complex C_* such that we can conclude from the Even Trace Lemma 182.25 and the Hopf Trace Formula 182.18 (2) that $\Lambda(f)$ is even.

To carry out this program we do the following: We equip S^n with the simplicial structure given by the homeomorphisms

$$S^n \xleftarrow[\text{homeomorphism given by the Join-Simplex Lemma 174.42 (1)}]{\cong} S^0 * \cdots * S^0 \overset{\text{here } S_0 \text{ is the abstract simplicial complex with vertex set } \{0,1\}}{=} |S_0| * \cdots * |S_0| \xrightarrow[\text{homeomorphism given by the Join Lemma 174.45}]{\cong} |S_0 * \cdots * S_0|.$$

We introduce the following notation:

(1) We write $K \coloneqq S_0 * \cdots * S_0$ and we denote by $\Psi\colon |K| \to S^n$ the above homeomorphism.
(2) Given any $k \in \mathbb{N}_0$ we denote by $\mathfrak{V}\colon |\operatorname{sd}^k(K)| \to |K|$ the natural homeomorphism given by the SD-Lemma 175.5 (3).
(3) By a slight abuse of notation we denote by $\rho\colon K \to K$ the map that is induced by the reflection on the simplices of all the copies of S_0.
(4) Note that by the SD-Lemma 175.5 (5) the simplicial map $\rho\colon K \to K$ induces a simplicial map $\operatorname{sd}^k(K) \to \operatorname{sd}^k(K)$ which we also denote by ρ.



Claim 1. There exists a $k \in \mathbb{N}_0$ and a *simplicial* map $\varphi\colon \operatorname{sd}^k(K) \to K$ with the following properties:
(1) The map $\Psi \circ |\varphi| \circ \mathfrak{V}^{-1} \circ \Psi^{-1}\colon S^n \to S^n$ is homotopic to f.
(2) The map φ commutes with the maps ρ, i.e. the following diagram commutes

$$\begin{array}{ccc} \operatorname{sd}^k(K) & \xrightarrow{\varphi} & K \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \operatorname{sd}^k(K) & \xrightarrow{\varphi} & K. \end{array}$$

Proof. As we will see, the proof of Claim 1 follows from a careful reading of the proof of the Simplicial Approximation Theorem 175.15.

(a) First note that Claim 1 of the proof of the Simplicial Approximation Theorem 175.15 shows that there exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\operatorname{sd}^k(K)$ there exists a vertex w of K with $f((\Psi \circ \mathfrak{V})(\operatorname{St}(|\operatorname{sd}^k(K)|, v))) \subseteq \Psi(\mathring{\operatorname{St}}(|K|, w))$. We denote by V the vertex set of $\operatorname{sd}^k(K)$ and we denote by W the vertex set of K. By the above there exists a map $\varphi\colon V \to W$ with the following property:
(∗) For every vertex $v \in V$ we have $f((\Psi \circ \mathfrak{V})(\operatorname{St}(|\operatorname{sd}^k(K)|, v))) \subseteq \Psi(\mathring{\operatorname{St}}(|K|, \varphi(v)))$.
(b) In the proof of the Simplicial Approximation Theorem 175.15 we have shown that every continuous map $\varphi\colon V \to W$ that satisfies (∗) is a simplicial map and that it has the property that $\Psi \circ |\varphi| \circ \mathfrak{V}^{-1} \circ \Psi^{-1}$ is homotopic to f.

(c) It follows immediately from the definition of Ψ and the naturality of $\mathfrak{V}$ that the following two diagrams commute:

$$\begin{array}{ccc} |K| & \xrightarrow{\rho} & |K| \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ S^n & \xrightarrow{\rho} & S^n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} |\operatorname{sd}^k(K)| & \xrightarrow{\rho} & |\operatorname{sd}^k(K)| \\ {\scriptstyle\mathfrak{v}}\downarrow & & \downarrow{\scriptstyle\mathfrak{v}} \\ |K| & \xrightarrow{\rho} & |K|. \end{array}$$

(d) Since all maps called ρ satisfy the condition that $\rho\circ\rho = \mathrm{id}$ we obtain easily from (a) and (c) that we can pick a map $\varphi\colon V \to W$ which satisfies $(*)$ and such that $\varphi\circ\rho = \rho\circ\varphi$.

(e) By (b) we know that φ satisfies (1). Furthermore, by (d) we know that the simplicial map φ also satisfies (2). ⊞

Before we move on to the next claim we introduce and recall some notation:

(1) Now we write $g := \Psi\circ|\varphi|\circ\mathfrak{V}^{-1}\circ\Psi^{-1}$.

(2) We denote by $u_*\colon \mathrm{C}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\operatorname{sd}^k(K))$ the k-fold iterated simplicial subdivision map that we introduced in the Chain Subdivision Lemma 177.25.

Claim 2. For every $j \in \mathbb{N}_0$ there exists an isomorphism $\mu\colon \mathrm{H}_j^{\mathrm{simp}}(\operatorname{sd}^k(K)) \to \mathrm{H}_j(S^n)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_j^{\mathrm{simp}}(\operatorname{sd}^k(K)) & \xrightarrow{u_j\circ\varphi_*} & \mathrm{H}_j^{\mathrm{simp}}(\operatorname{sd}^k(K)) \\ {\scriptstyle\mu}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\mu} \\ \mathrm{H}_j(S^n) & \xrightarrow[(\Psi\circ|\varphi|\circ\mathfrak{V}^{-1}\circ\Psi^{-1})_*]{g_*} & \mathrm{H}_j(S^n). \end{array}$$

Proof. We consider the following much more elaborate diagram:

$$\begin{array}{ccccc} \mathrm{H}_j^{\mathrm{simp}}(\operatorname{sd}^k(K)) & \xrightarrow{\varphi_*} & \mathrm{H}_j^{\mathrm{simp}}(K) & \xrightarrow{u_j} & \mathrm{H}_j^{\mathrm{simp}}(\operatorname{sd}^k(K)) \\ \downarrow{\scriptstyle\Theta_{\operatorname{sd}^k(K)}} & & \downarrow{\scriptstyle\Theta_K} & & \downarrow{\scriptstyle\Theta_{\operatorname{sd}^k(K)}} \\ \mathrm{H}_j(|\operatorname{sd}^k(K)|) & \xrightarrow{|\varphi|_*} & \mathrm{H}_j(|K|) & \xrightarrow{u_j} & \mathrm{H}_j(|\operatorname{sd}^k(K)|) \\ & & & \searrow^{\Psi_*} & \downarrow{\scriptstyle\mathfrak{v}_*} \\ {\scriptstyle(\Psi\circ\mathfrak{v})_*}\downarrow & & & & \mathrm{H}_j(|K|) \\ \mathrm{H}_j(S^n) & & \xrightarrow[(\Psi\circ|\varphi|\circ\mathfrak{V}^{-1}\circ\Psi^{-1})_*]{g_*} & & \downarrow{\scriptstyle\Psi_*} \\ & & & & \mathrm{H}_j(S^n). \end{array}$$

We make the following observations and clarifications:

(1) The maps Θ are the natural isomorphisms introduced in Notation 177.16.

(2) Similar to above, we denote by $u_*\colon \mathrm{C}_*(|K|) \to \mathrm{C}_*(|K|)$ the k-fold iterated singular subdivision map that we introduced on page 2825.

(3) It follows from the Subdivision–Homologies-Proposition 177.31, applied iteratively k times, that the rectangle to the top right commutes.

(4) The quadrilateral at the bottom commutes by definition and the functoriality of homology.

(5) The triangle at the bottom right commutes by the Iterated Subdivision Lemma 148.14.

(6) The square to the top left commutes by the naturality of Θ.

It follows from the above that the isomorphism $\mu := \Psi_* \circ \mho_* \circ \Theta_{\mathrm{sd}^k(K)}$ has the desired property. ⊞

To shorten the notation a little bit we now write $L := \mathrm{sd}^k(K)$. We calculate that

$$\Lambda(f) \overset{\text{by the Lefschetz Number-Homotopy Lemma 182.12 and Claim 1 (1)}}{\downarrow} \Lambda(g) = \sum_{j\in\mathbb{N}_0} (-1)^j \cdot \mathrm{tr}\left(g_* \circlearrowleft \mathrm{H}_i(S^n)\right) \overset{\text{by Claim 2}}{\downarrow} = \sum_{j\in\mathbb{N}_0} (-1)^j \cdot \mathrm{tr}\left((u_j \circ \varphi_*) \circlearrowleft \mathrm{H}_i^{\mathrm{simp}}(L)\right)$$
$$= \sum_{j\in\mathbb{N}_0} (-1)^j \cdot \mathrm{tr}\left((u_j \circ \varphi_*) \circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(L)\right).$$
$$\uparrow \text{ by the Hopf Trace Formula 182.18 (2)}$$

At this point we see that it suffices to prove the following claim:

Claim 3. For each $j \in \mathbb{N}_0$ the trace $\mathrm{tr}\left((u_j \circ \varphi_*) \circlearrowleft \mathrm{C}_j^{\mathrm{simp}}(L)\right)$ is even.

Proof. Evidently the plan is to apply the Even Trace Lemma 182.25. We make a few more preparations:

(1) We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{C}_j^{\mathrm{simp}}(L) & \xrightarrow{g_*} & \mathrm{C}_j^{\mathrm{simp}}(K) & \xrightarrow{u_j} & \mathrm{C}_j^{\mathrm{simp}}(L) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\ \mathrm{C}_j^{\mathrm{simp}}(L) & \xrightarrow{g_*} & \mathrm{C}_j^{\mathrm{simp}}(K) & \xrightarrow{u_j} & \mathrm{C}_j^{\mathrm{simp}}(L). \end{array}$$

It follows from Claim 1 (2) that the square to the left commutes. It follows from the naturality of the simplicial subdivision maps that the square to the right commutes.

(2) It follows immediately from the definition of the map $\rho\colon K \to K$ that $\rho\colon K \to K$ does not fix a single simplex of K and that $\rho \circ \rho = \mathrm{id}_K$. It follows easily from the functorial nature of the map $K \mapsto L = \mathrm{sd}^k(K)$ that $\rho\colon L \to L$ has the same properties, i.e. $\rho\colon L \to L$ does not fix a single simplex of L and it satisfies $\rho \circ \rho = \mathrm{id}_L$.

(3) We denote by S_j the set of j-simplices of L. It follows immediately from (2) that there exists a subset T_j of S_j with the following properties:

$$\text{(i)} \quad T_j \cap \rho(T_j) = \varnothing \qquad \text{and} \qquad \text{(ii)} \quad T_j \cup \rho(T_j) = S_j.$$

We equip each $t \in T_j$ with some order and we use these orders to view T_j as a subset of $\mathrm{C}_j^{\mathrm{simp}}(L)$. It follows from (ii) and the Simplicial Chain Group–Isomorphism Lemma 176.11 that $T_j \cup \rho(T_j)$ is a basis of $\mathrm{C}_j^{\mathrm{simp}}(L)$.

Now we apply the Even Trace Lemma 182.25 to $V = \mathrm{C}_j^{\mathrm{simp}}(L)$, $T = T_j$, $\rho = \rho_*$ and $\varphi = \varphi$. It follows from (1), (2) and (3) that all the hypotheses of the Even Trace Lemma 182.25 are satisfied. Thus we obtain that the trace of $u_j \circ \varphi_*$ is indeed even. ■

The argument for deducing the Borsuk-Ulam Theorem 47.12 from the Odd Map–Even Lefschetz Number Proposition 182.24 is almost identical to the argument on page 3175 for deducing the Borsuk-Ulam Theorem 47.12 from the Odd Map–Odd Degree Proposition 165.6.

Proof of the Borsuk-Ulam Theorem 47.12. Let $g\colon S^n \to \mathbb{R}^n$ be a continuous map. Suppose that there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$.

Put differently, suppose that for any $x \in S^n$ we have $g(x) \neq g(-x)$. We consider the composition of the following two maps

$$\begin{array}{rccccc} \Phi\colon S^{n-1} \times [0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\ & & x & \mapsto & \underbrace{\frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}} \\ & & & & \text{defined since } g(x) \neq g(-x) \end{array}$$

Now we see that

$$\text{even number} \overset{\downarrow}{=} \Lambda(\Phi_0) \underset{\uparrow}{=} \Lambda(\Phi_1) \underset{\uparrow}{=} \Lambda(\text{constant map}) \underset{\uparrow}{=} 1.$$

by definition we have $\Phi(-x,0) = -\Phi(x,0)$, thus it follows from the Odd Map–Even Lefschetz Number Proposition 182.24 that $\Lambda(\Phi_0)$ is even

by the Lefschetz Number Homotopy Lemma 182.12, since Φ_0 and Φ_1 are homotopic

by definition of Φ_1

by the Lefschetz Number Basics Lemma 182.11 (3)

We have thus obtained a contradiction. ■

182.7. Exercises for Chapter 182 .

Exercise 182.1. Let X be a finite simplicial complex, let $\varphi\colon X \to X$ be a continuous map and let $\mathbb{F}$ be a field. We denote by $\gamma\colon \mathbb{Z} \to \mathbb{F}$ the ring homomorphism that is given by $\gamma(n) = n \cdot 1_{\mathbb{F}}$. Show that

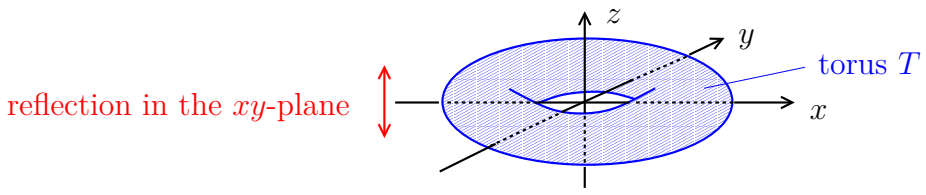
$$\Lambda(X, \mathbb{F}) = \gamma(\Lambda(X)).$$

Hint. Use the Free Module–Tensor-with-Ring Lemma 159.20.

Exercise 182.2. Let $n \in \mathbb{N}$. Show that there exists a compact smooth manifold M and a continuous map $\varphi\colon M \to M$ with Lefschetz number $\Lambda(\varphi) = n$ such that φ has precisely one fixed point.

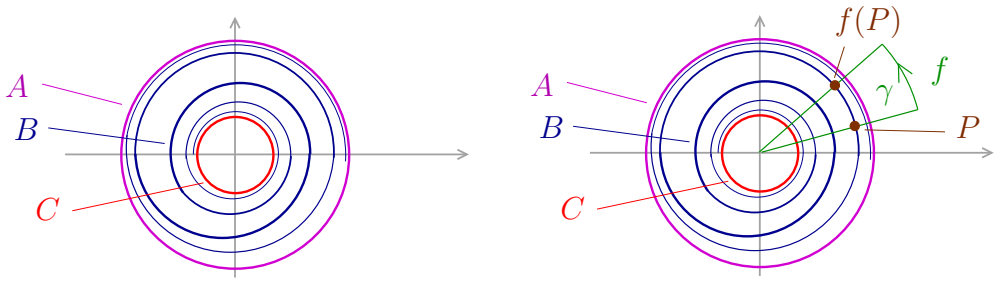
Exercise 182.3. Let $n \in \mathbb{N}$. Does the real projective space $\mathbb{R}\mathrm{P}^{2n+1}$ admit a continuous self-map without fixed points?

Exercise 182.4. We consider the torus $T := S^1 \times S^1$ embedded in $\mathbb{R}^3$ as shown in the figure below. Let $f\colon T \to T$ be the map that is given by reflection in the xy-plane. Is f homotopic to a map without fixed points?



Exercise 182.5. We consider the topological space $X = A \cup B \cup C$ that is shown in the figure below. Here A is a circle of radius 3, C is a circle of radius 1 around the origin and B is the image of an embedding $\mathbb{R} \to \mathbb{R}^2$ which spirals towards A and C.

(a) Give a rigorous description of X.
(b) Show that X is compact.
(c) Compute the homology groups of X.



Fix some "angle" $\gamma \in (0, \pi)$. Let $f\colon X \to X$ be the map that is given by rotation by the angle γ on A and C and which is given by sending a point $s \cdot \exp(\mathrm{i}\varphi)$ on B to the "next" point on B of the form $t \cdot \exp(\mathrm{i}(\varphi + \gamma))$ with $s > t$.

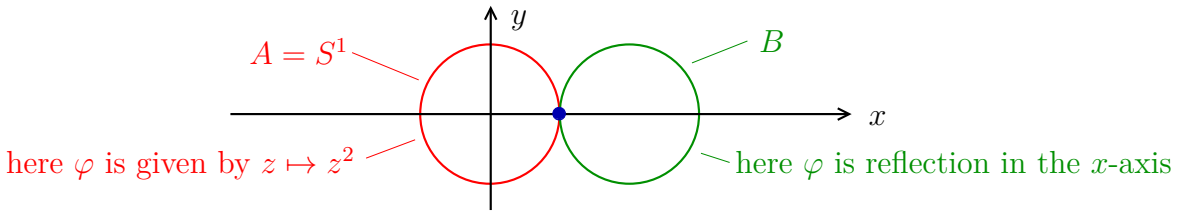
(d) Show that f is continuous.
(e) Show that f has no fixed points.
(f) Show that the Lefschetz number $\Lambda(f) \neq 0$.

This long but fun exercise shows that in the formulation of the Lefschetz Fixed Point Theorems it does not suffice to work with compact spaces that are say $\mathbb{Q}$-homology finite. We need the topological space to be "sufficiently nice".

Exercise 182.6. We consider the topological space $X = A \cup B$ that is shown in the figure below. Let $\varphi\colon X \to X$ be the map that is given by $z \mapsto z^2$ on A and that is given by reflection along the x-axis on B.

(a) Show that the Lefschetz number $\Lambda(\varphi, \mathbb{F}) = 0$ for every field $\mathbb{F}$.
(b) Show that any continuous map ψ that is homotopic to φ actually has a fixed point.

Remark. It is easy to see that X admits a simplicial structure. Thus the exercise shows that the converse to the Lefschetz Fixed Point Theorem 182.15 does not hold.



Exercise 182.7.

(a) Let $A \in \mathrm{GL}(n, \mathbb{C})$ be a matrix. We denote by $\varphi_A\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ the induced map that is given by $\varphi_A([P]) \coloneqq [A \cdot P]$.
  (i) Show that the Lefschetz number $\Lambda(\varphi_A) = n$.
  Hint. Use the Matrix Group-π_0-Proposition 4.54 (4). The super conscientious reader might also want to consult the discussion on page 516 to show that a certain map is indeed continuous.
  (ii) Use (i) to show that A has an eigenvalue.
(b) What does the approach in (a) say about the existence of real eigenvalues of real matrices?

Exercise 182.8. Let $n \in \mathbb{N}_0$ be *even*.

(a) Show that the complex projective space $\mathbb{C}\mathrm{P}^n$ admits a continuous self-map without fixed points.
Remark. In Exercise 213.10 we will prove the much more interesting converse, namely we will show that for every *odd* n every continuous self-map of $\mathbb{C}\mathrm{P}^n$ has a fixed point.

(b) Does there exist a nowhere-vanishing vector field on $\mathbb{C}\mathrm{P}^n$?

Exercise 182.9. For which $n \in \mathbb{N}$ does the real projective space $\mathbb{R}\mathrm{P}^n$ admit a nowhere-vanishing vector field?

Exercise 182.10. Does the Klein bottle K admit two vector fields v and w such that at every point $P \in K$ the two vectors $v(P), w(P) \in \mathrm{T}_P K$ are linearly independent?

Exercise 182.11. In the figure on page 3449 we saw that the surface Σ of genus 2 supports a vector field with two zeros. What does such a vector field look like if we consider Σ as a submanifold of $\mathbb{R}^3$ in the usual way?

Exercise 182.12. Does the surface of genus 2 admit a vector field with a single zero?

Exercise 182.13. Let $m \in \mathbb{Z}$ be even.

(a) Show that there exists a continuous map $f\colon S^1 \to S^1$ with Lefschetz number $\Lambda(f) = m$ and such that for every $x \in S^1$ we have $f(-x) = -f(x)$.

(b) Now prove (a) for higher-dimensional spheres.

Exercise 182.14. Let X be a finite CW-complex and let $\varphi\colon X \to X$ be a continuous self-map. Furthermore let $p\colon \widetilde{X} \to X$ be a finite covering. By the Lift-to-Universal Coverings Proposition 127.26 we know that there exists a continuous map $\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\ f\ } & X. \end{array}$$

Show that

$$\Lambda(\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \Lambda(\varphi\colon X \to X).$$

Hint. Use cellular homology and use the Hopf Trace Formula, i.e. use Theorem 182.18.

Exercise 182.15. Let M be a compact topological manifold. Suppose we are given two compact codimension-zero submanifolds A and B such that $A \cap B = \partial A = \partial B$. Furthermore let $f\colon M \to M$ be a continuous map that restricts to self-maps of A and B. Show that

$$\Lambda(f\colon M \to M) \;=\; \Lambda(f\colon A \to A) + \Lambda(f\colon B \to B) - \Lambda(f\colon A \cap B \to A \cap B).$$

Hint. Use the Mayer–Vietoris Theorem 153.2 (3) and use the Hopf Trace Formula, i.e. use Theorem 182.18.
Remark. In the Topological Manifolds-Invariants Proposition 189.14 we will see that the homology groups of A, B and M are finitely generated and that there exists an n such that they vanish in dimension $> n$.

Exercise 182.16. Let M be a smooth manifold and let v be a vector field on M. Show that the zero set $\{x \in M \mid v(x) = 0\}$ is a closed subset of M.
Hint. Use the Cover-Open Subset Lemma 3.20 (2b).

Exercise 182.17. Recall that according to Definition 34.2 a *Lie group* is a smooth manifold X together with a group structure such that the two maps

$$\begin{array}{ccc} X \times X & \to & X \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth. For example it is not difficult to show that the matrix groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ with the usual multiplication are Lie groups. Show that any Lie group admits a nowhere vanishing smooth vector field.
Remark. It follows from this exercise together with the Euler Characteristic–Vector Field Theorem 182.22 that the Euler characteristic of any closed Lie group of dimension ≥ 1 is zero.
Remark. This exercise is essentially identical to Exercise 53.7.

Exercise 182.18. For which compact connected 2-dimensional smooth manifolds M can you find a nowhere-vanishing vector field v with the additional property that for each $P \in \partial M$ we have $v(P) \in \mathrm{T}_P \partial M$?
Remark. We had posed this question already as Exercise 29.10, but now you actually have the tools to give a complete answer.

CHAPTER 183

Homology groups of pseudomanifolds and smooth manifolds

Let M be an n-dimensional smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a smooth simplicial structure. In the Smooth Manifolds-Invariants Proposition 179.10 we used this fact to show that for every $k > n$ we have $\mathrm{H}_k(M) = 0$. In this chapter we will squeeze more information out of the existence of a simplicial structure. In particular this will allow us to calculate $\mathrm{H}_n(M;\mathbb{Z})$ and it will allow us to make some statements regarding $\mathrm{H}_{n-1}(M;\mathbb{Z})$.

This chapter is organized as follows:

- In Section 183.1 we will introduce the notion of a pseudomanifold. The main example are simplicial complexes whose topological realization is a topological manifold.
- In Section 183.2 we will determine the top-dimensional homology group of a smooth manifold.
- In Section 183.3 we will determine the homology groups of lens spaces.
- In Section 183.4 we will introduce the Poincaré Homology Sphere.
- In Section 183.5 we will discuss deformation retractions of closed smooth manifolds.
- In Section 183.6 we will discuss the topology of smooth manifolds.
- In Section 183.7 we will provide exercises for Chapter 183.

In the next Chapter 184 we will introduce simplicial cohomology.

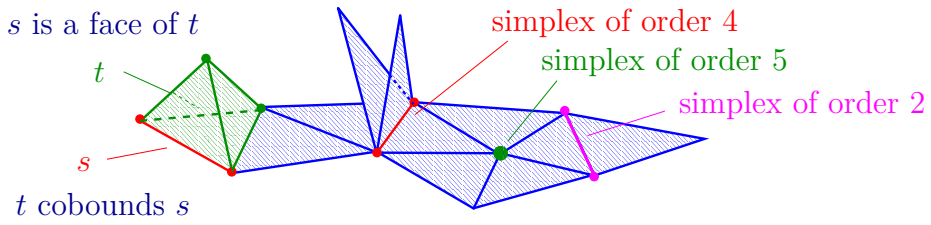
183.1. **The top-dimensional homology group of a pseudomanifold.** In this chapter we will see that any simplicial structure of an n-dimensional topological manifold has several special properties. This will lead us to the notion of an *n-dimensional pseudomanifold* which can be viewed as a simplicial analog of a manifold. We will then prove several statements regarding the simplicial homology groups of an n-dimensional pseudomanifold. In the subsequent section we will then use the calculations from this chapter to prove the promised results on homology groups of smooth manifolds.

First we recall and introduce some definitions.

Definition 183.1. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let t be a simplex of K. As in Definition 174.4 we define a face of t to be a simplex s with $s \subseteq t$. In this setting we also say that t cobounds s.
(2) Let $s \in S$ be a k-simplex of K. We say that s has order r, if there exist precisely r simplices of dimension $k+1$ which contain s as a face.

As before we sometimes use the same terms when we talk about topological realizations and simplicial complexes.

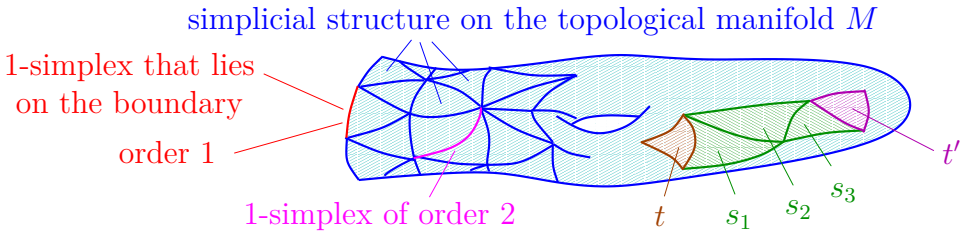


Proposition 183.2. (Manifold-Simplex Proposition) Let M be an n-dimensional topological manifold that is equipped with a simplicial structure.

(1) There is no simplex of dimension $> n$ and every simplex is the face of an n-simplex.
(2) The simplicial structure has finitely many simplices if and only if it has finitely many n-simplices.
(3) The simplicial structure on M restricts to a simplicial structure on the $(n-1)$-dimensional topological manifold ∂M. If M is a smooth manifold and if we are given a smooth simplicial structure, then the restriction of the simplicial structure to the smooth manifold ∂M is also a smooth simplicial structure.
(4) Let s be an $(n-1)$-simplex of M. Then

$$\text{order of } s \;=\; \begin{cases} 1, & \text{if } s \text{ lies on the boundary,} \\ 2, & \text{otherwise.} \end{cases}$$

(5) If M is connected, then given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.



Remark.

(1) In the Manifold Simplex–Dimension Lemma 179.4 we gave a different proof why the maximal dimension of a simplex of a smooth simplicial structure of an n-dimensional smooth manifold equals n.
(2) A different proof of the Manifold-Simplex Proposition 183.2 (5) is given in [**Mun1984**, Corollary 65.2]. □

The key idea behind the proof of the Manifold-Simplex Proposition 183.2 is to squeeze the maximal amount of information out of the local homology groups. To do so we first recall Definition 150.2:

Definition 183.3. Given a topological space X and given a point $P \in X$ the relative homology group $\mathrm{H}_j(X, X \setminus \{P\})$ is called the n-th local homology group of X at the point P.

We recall the calculation of the local homology groups of a topological manifold.

Lemma 150.9. (Manifold-Local-H_*-Lemma) Let M be an n-dimensional topological manifold and let $P \in M$. For every $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(M, M \setminus \{P\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and } j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since in a minute we will calculate a different local homology group it is perhaps helpful to remind us of the proof of the lemma. For simplicity we only consider the case that $P \in M \setminus \partial M$. In this case we can pick a chart $\Phi\colon U \to B^n$ for P with $\Phi(P) = 0$. We then have

$$\mathrm{H}_j(M, M\setminus\{P\}) \xleftarrow[\text{by the Excision Theorem 148.2}]{} \mathrm{H}_j(U, U\setminus\{P\}) \xrightarrow[\cong]{\Phi_*} \mathrm{H}_j(B^n, B^n\setminus\{0\}) \xrightarrow[\text{isomorphism by the long exact sequence in reduced homology of the pair } (B^n, B^n \setminus \{0\}) \text{ and the fact that } \widetilde{\mathrm{H}}_*(B^n) = 0 \text{ by page 2812}]{\partial} \widetilde{\mathrm{H}}_{j-1}(B^n\setminus\{0\}) \xleftarrow[\text{isomorphism since } S^{n-1} \text{ is homotopy equivalent to } B^n \setminus \{0\}]{} \widetilde{\mathrm{H}}_{j-1}(S^{n-1}).$$

The statement now follows from the calculation of the reduced homology groups of spheres given in the Sphere-H_*-Proposition 149.4. ■

The next proposition calculates the local homology groups of a simplicial complex. This calculation is also interesting in its own right.

Proposition 183.4. (Simplicial Complex–Local-H_*-Proposition) Let $K = (V, S)$ be an abstract simplicial complex and let $P \in |K|$. By the Open Simplex Lemma 174.28 there exists a unique simplex $s \in S$ such that P is a point in the open simplex $\langle s \rangle$. We set $m \coloneqq \dim(s)$. If $\mathrm{Lk}(K, s)$ is a finite simplicial complex, then for every $j \in \mathbb{N}_0$ we have an isomorphism[1937] [1938]

$$\mathrm{H}_j(|K|, |K| \setminus \{P\}) \;\cong\; \widetilde{\mathrm{H}}_{j-1-m}(\mathrm{Lk}(K, s)).$$

The proof of the Simplicial Complex–Local-H_*-Proposition 183.4 relies on the following lemma, which is also interesting in its own right.

Lemma 183.5. (Join-with-Sphere-H_*-Lemma) Let X be a topological space that is compact and Hausdorff. Furthermore let $k \in \mathbb{N}_{\geq -1}$. For every $n \in \mathbb{Z}$ there exists a natural isomorphism[1939]

$$\widetilde{\mathrm{H}}_n(X) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_{n+k+1}(S^k \underset{\uparrow}{*} X).$$

join of S^k and X, see Definition 20.1

Proof of the Join-with-Sphere-H_*-Lemma 183.5. First note that the case $X = \varnothing$ follows from the fact that, by definition, $S^k * \varnothing = S^k$, and the calculation of the reduced homology groups of S^k that we gave in the Sphere-H_*-Proposition 149.4.

Now assume that $X \neq \varnothing$. If $k = -1$, then we have, again by definition, the equality $S^{-1} * X = \varnothing * X = X$ which implies the desired statement. Thus it remains to deal with case that $k \geq 0$. To simplify the notation we only consider the case $k = 1$. In this case we have

---

[1937] It can happen that $\mathrm{Lk}(K, s) = \varnothing$. Recall that by definition we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) \cong \mathbb{Z}$.

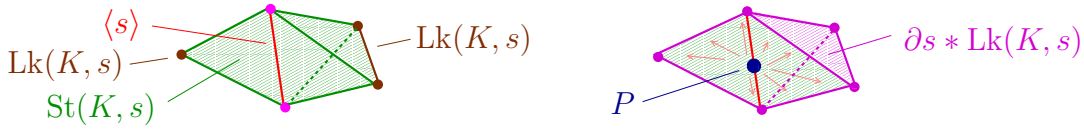
[1938] I do not know whether the statement holds in general without the condition that $\mathrm{Lk}(K, s)$ is finite.

[1939] Here, recall that if X is the empty topological space, then by the Reduced Homology Lemma 147.3 (-1) we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

$$\widetilde{\mathrm{H}}_{n+2}(S^1 * X) \overset{\text{by the Join-of-Spheres Lemma 20.4}}{\cong} \widetilde{\mathrm{H}}_{n+2}((S^0 * S^0) * X) \overset{\text{by the Join Associativity Proposition 20.11, see page 550 for the definition of the homeomorphism}}{\cong} \widetilde{\mathrm{H}}_{n+2}(S^0 * (S^0 * X)) \overset{\text{by the Suspension–Join Lemma 20.24}}{\cong} \widetilde{\mathrm{H}}_{n+2}(\Sigma(\Sigma(X))) \overset{\text{by the Suspension-H}_*\text{-Proposition 149.3 applied twice}}{\cong} \widetilde{\mathrm{H}}_n(X).$$

It follows easily from the definitions of the maps that the isomorphism is in fact natural. ■

Proof of the Simplicial Complex–Local-H$_*$-Proposition 183.4. The following figure might help with understanding the argument that follows.



Using this visual aid we can now perform the following calculation:

$$\mathrm{H}_j(|K|, |K| \setminus \{P\}) \overset{\text{by the long exact sequence in reduced homology of the pair } (\mathrm{St}(s), \mathrm{St}(s) \setminus \{P\}) \text{ and the observation, see the Link Star Lemma 175.9 (1f), that } \mathrm{St}(s) \text{ is contractible, which implies that its reduced homology vanishes}}{\cong} \mathrm{H}_j(\mathrm{St}(s), \mathrm{St}(s) \setminus \{P\}) \overset{\text{by the Link Star Lemma 175.9 (1d) we know that St}(s) \text{ is a neighborhood of } P\text{, thus we can apply the Excision Theorem 148.2 (2)}}{\cong} \widetilde{\mathrm{H}}_{j-1}(\mathrm{St}(s) \setminus \{P\}) \overset{\text{by the Link Star Lemma 175.9 (1g) together with the Homotopy Equivalence-H}_*\text{-Corollary 147.5 (2)}}{\cong} \widetilde{\mathrm{H}}_{j-1}(\partial s * \mathrm{Lk}(s))$$

$$\overset{\text{since } s \text{ is an } m\text{-simplex we know by the Simplex-Embedding Lemma 174.20 and the Standard Simplex Homeomorphism Lemma 143.9 (2) that } \partial s \text{ is homeomorphic to } S^{m-1}}{\cong} \widetilde{\mathrm{H}}_{j-1}(S^{m-1} * \mathrm{Lk}(s)) \overset{\text{by the Join-with-Sphere-H}_*\text{-Lemma 183.5 we have } \widetilde{\mathrm{H}}_{l+m}(S^{m-1} * X) \cong \widetilde{\mathrm{H}}_l(X) \text{ whenever } X \text{ is compact and Hausdorff – we can apply this result since Lk}(s) \text{ is by hypothesis finite}}{\cong} \widetilde{\mathrm{H}}_{j-1-m}(\mathrm{Lk}(s)).$$

■

Now we can finally provide the proof of the Manifold-Simplex Proposition 183.2.

Proof of the Manifold-Simplex Proposition 183.2. Basically by definition of a simplicial structure it suffices to consider the case that the n-dimensional topological manifold M is the topological realization $M = |K|$ of some abstract simplicial complex $K = (V, S)$. Note that by the Manifold–Local Properties Lemma 23.8 we know that the topological manifold $M = |K|$ is regionally compact. Thus it follows from the Locally Finite–Characterisation Proposition 175.11 that the abstract simplicial complex K is locally finite and that the link of any vertex is a finite simplicial complex. This implies in particular that we can apply the Simplicial Complex–Local-H$_*$-Proposition 183.4 without worries.

(1) It suffices to prove the following claim:

Claim. Given any simplex s, the maximal dimension of a simplex that contains s is precisely given by n.

Proof. Let s be a simplex of K. As we mentioned in the beginning of the proof, we know that K is locally finite. This implies that there exists a simplex t of maximal dimension that contains s. We set $m \coloneqq \dim(t)$. Now let $P \in \langle t \rangle$ be a point. We

calculate that

$$\mathbb{Z} \underset{\text{since there are no simplices of dimension } > m \text{ which contain } t \text{ we see that } \mathrm{Lk}(K,t)=\varnothing}{\overset{\uparrow}{\cong}} \widetilde{\mathrm{H}}_{-1}(\mathrm{Lk}(K,t)) \underset{\text{by the Simplicial Complex–Local-H}_*\text{-Proposition 183.4 applied to the } m\text{-simplex } t}{\overset{\downarrow}{\cong}} \mathrm{H}_m(|K|,|K|\backslash\{P\}) \underset{\text{by the Manifold-Local-H}_*\text{-Lemma 150.9 applied to } |K|=M}{\overset{\downarrow}{\cong}} \begin{cases} \mathbb{Z}, & \text{if } P \in M\backslash\partial M \text{ and } m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the left-hand side with the right-hand side shows that $m = n$. ⊞

(2) This statement is an almost immediate consequence of (1). Indeed, we only have to show that if K has finitely many n-simplices, then it has only finitely many simplices. But every n-simplex has only finitely many faces and by the proof of (1) we see that every simplex of K is the face of an n-simplex. Thus we are done.

(3) First note that if we are given a smooth simplicial structure on a smooth manifold that restricts to a simplicial structure on ∂M, then it follows easily from the definitions that this simplicial structure on ∂M is actually a smooth simplicial structure. Thus, to prove (3), it remains to show that the simplicial structure on $|K| = M$ restricts to a simplicial structure on the topological manifold ∂M.

If M is a smooth manifold, then we know by the Smooth Manifold–Boundary Proposition 25.27 (2a) that ∂M is a closed subset of M. In the general case that M is a topological manifold the same statement follows from the Topological Manifold–Boundary Proposition 23.15 (2a).

Now that we know that ∂M is a closed subset of M we obtain the desired statement from the Subcomplex Criterion 174.30 together with the following claim:

Claim. Let s be an m-simplex of K. Either $\langle s\rangle \cap \partial M = \varnothing$ or $\langle s\rangle \subseteq \partial M$.

Proof. Let s be an m-simplex. For any point $P \in \langle s\rangle$ we perform almost the same calculation as in (1):

$$\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) \underset{\text{by the Simplicial Complex Local-H}_*\text{-Proposition 183.4 applied to the } m\text{-simplex } s}{\overset{\downarrow}{\cong}} \mathrm{H}_n(|K|,|K|\backslash\{P\}) \underset{\text{by the Manifold-Local-H}_*\text{-Lemma 150.9 applied to } |K|=M}{\overset{\downarrow}{\cong}} \begin{cases} \mathbb{Z}, & \text{if } P \in M\backslash\partial M, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from this calculation that $\langle s\rangle \cap \partial M = \varnothing$ if $\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}$ and that $\langle s\rangle \subseteq \partial M$ if $\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) = 0$. ⊞

(4) Let s be an $(n-1)$-simplex of M. We denote by r the order of s. We pick a point $P \in \langle s\rangle$. Similar to (1) we calculate that

$$\mathbb{Z}^{r-1} \underset{\text{since } s \text{ is an } (n-1)\text{-simplex and since there are no simplices of dimension } > n \text{ we see that } \mathrm{Lk}(K,s) \text{ consists of } r \text{ points, the isomorphism now follows from the Reduced Homology Lemma 147.3 (0c)}}{\overset{\uparrow}{\cong}} \widetilde{\mathrm{H}}_0(\mathrm{Lk}(K,s)) \underset{\text{the Simplicial Complex Local-H}_*\text{-Proposition 183.4 applied to the } (n-1)\text{-simplex } s}{\overset{\downarrow}{\cong}} \mathrm{H}_n(|K|,|K|\setminus\{P\}) \underset{\text{the Manifold-Local-H}_*\text{-Lemma 150.9 applied to } |K|=M}{\overset{\downarrow}{\cong}} \begin{cases} \mathbb{Z}, & \text{if } P \in M\setminus\partial M, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the left-hand side with the right-hand side gives us the desired statement.

(5) Now we assume that M is in fact connected. We say that two n-simplices t and t' of K are *equivalent* if there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face. Recall that our task is to show that any two n-simplices are equivalent in the above sense.

The key to doing so is the following claim:

Claim. Let t and t' be two n-simplices. If $t \cap t' \neq \varnothing$, then t and t' are actually equivalent in the above sense.

Proof. Given $m \in \{0, \ldots, n-1\}$ we say that two n-simplices x and x' are *m-equivalent* if there exists a sequence $x = s_0, s_1, \ldots, s_k = x'$ of n-simplices such that any two consecutive simplices have a common face of dimension $\geq m$. It suffices to show that if we are given $m \in \{0, \ldots, n-2\}$ such that any two n-simplices in K are m-equivalent, then any two n-simplices in K are also $(m+1)$-equivalent. In fact evidently it suffices to show that if we are given two n-simplices x and x' which have an m-simplex s in common, then x and x' are $(m+1)$-equivalent. To do so we first note that since $\dim(x) > \dim(s)$ we can pick a vertex w of $x \cap \mathrm{Lk}(K, s)$ and similarly we can pick a vertex w' of $x' \cap \mathrm{Lk}(K, s)$.

Now we want to show that $\mathrm{Lk}(K, s)$ is connected. To do so we pick a point $P \in \langle s \rangle$ and we perform basically the same calculation as in the previous arguments:

$$\widetilde{\mathrm{H}}_0(\mathrm{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{m+1}(|K|, |K| \setminus \{P\}) = \widetilde{\mathrm{H}}_{m+1}(M, M \setminus \{P\}) \underset{\uparrow}{=} 0.$$

the Simplicial Complex–Local-H_*-Proposition 183.4 applied to the m-simplex s and $j = m+1$

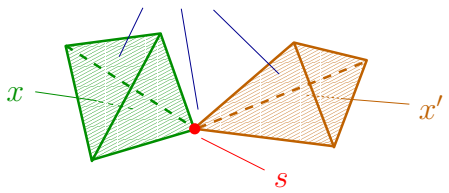
by the Manifold Local-H_*-Lemma 150.9 since $m \leq n-2$

It follows from this calculation together with the Reduced Homology Lemma 147.3 and Exercise 174.13 that the abstract simplicial complex $\mathrm{Lk}(K, s)$ is in fact connected in the sense of Exercise 174.13. This means in our setting that there exist 1-simplices $e_1, \ldots, e_k \in S$ of $\mathrm{Lk}(K, s)$ such that the following holds:

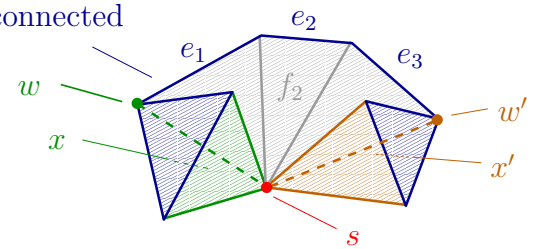
(∗) We have $w \in e_1$, for every $i \in \{1, \ldots, k-1\}$ we have $e_i \cap e_{i+1} \neq \varnothing$ and we have $w' \in e_k$.

Let $i \in \{1, \ldots, k\}$. Since e_i is a simplex in $\mathrm{Lk}(K, s)$ there exists a simplex f_i of K with $e_i \cup s \subseteq f_i$.

x and x' are 0-equivalent



$\mathrm{Lk}(K, s)$ is connected



By (1) we can in fact find an n-simplex $\widetilde{f}_i$ of K with $e_i \cup s \subseteq f_i \subseteq \widetilde{f}_i$. We set $\widetilde{f}_0 := x$ and $\widetilde{f}_{k+1} := x'$. Note that it follows from (∗) that any two consecutive $\widetilde{f}_i$ and $\widetilde{f}_{i+1}$ have s and a vertex of $\mathrm{Lk}(K, s)$ in common, i.e. they have an $(m+1)$-simplex in common. This shows that x and x' are in fact $(m+1)$-connected. ⊞

We pick an equivalence class of n-simplices of K and we denote by J the subcomplex of K given by the union of the n-simplices in said equivalence class. Since J is a subcomplex we know by the Simplicial Subcomplex Lemma 174.23 that $|J|$ is a closed subset of $|K| = M$. Furthermore it follows from the claim together with Exercise 174.12 that $|J|$ is also an open subset of $|K| = M$. By hypothesis $|K| = M$ is connected. Thus we see that $K = J$, which implies that any two n-simplices in K are in fact equivalent. ■

Our main goal is to say something meaningful about homology groups of smooth manifolds. The idea is to use the Manifold-Simplex Proposition 183.2 and the fact that we can calculate singular homology groups via simplicial homology groups. Since we will only use the information given by the Manifold-Simplex Proposition 183.2 it is convenient to introduce the following definition:

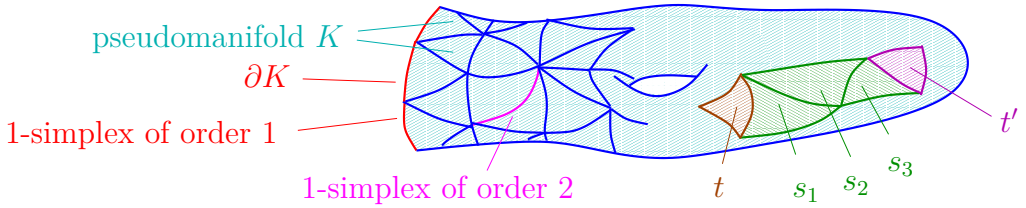
Definition 183.6. Let $n \in \mathbb{N}_0$. An n-dimensional pseudomanifold is a non-empty abstract simplicial complex K such that the following conditions are satisfied:

(1) Every simplex of K is a face of an n-simplex of K.[1940]
(2) Every $(n-1)$-simplex has order one or two.
(3) Given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.[1941]

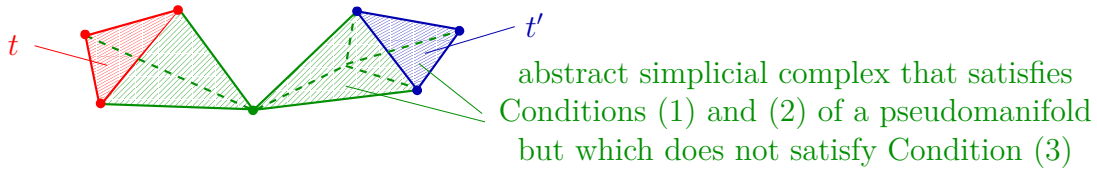
Given a pseudomanifold K we define ∂K to be the subcomplex that is given by the union of all $(n-1)$-simplices of order 1. We say that a pseudomanifold K is closed if $\partial K = \varnothing$ and if K is finite.

Examples.

(1) In the figure below we illustrate the definition of a 2-dimensional pseudomanifold.



(2) In the figure below we show an abstract simplicial complex that satisfies (1) and (2) but that does not satisfy (3).



This example shows that Condition (3) is stronger than requiring ordinary connectedness. □

Remark.

[1940] Note that this condition implies that K is necessarily n-dimensional.
[1941] Note that this condition implies that K is connected.

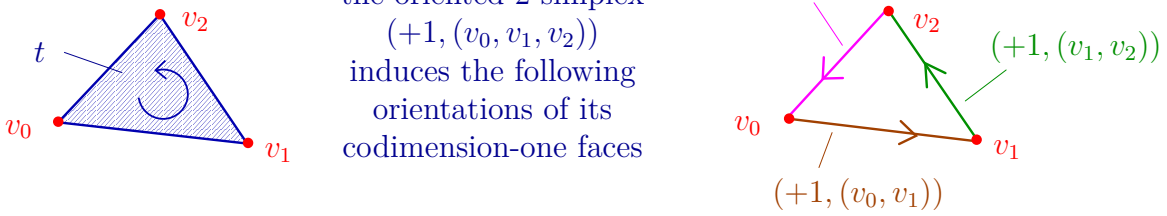
(1) The definition of a pseudomanifold we give seems to be the one which is most frequently used, it goes back to work of Luitzen Brouwer [**Brou1976**, p. 477] in 1912.
(2) The Manifold-Simplex Proposition 183.2 says that if we are given a simplicial structure $(K, \Theta\colon |K| \to M)$ of a connected non-empty n-dimensional topological manifold M, then K is an n-dimensional pseudomanifold.
(3) In Exercise 183.4 we will show that if K is an n-dimensional pseudomanifold where $n \in \{1, 2\}$, then $|K|$ admits the structure of an n-dimensional smooth manifold.
(4) Let K be a finite n-dimensional pseudomanifold. As usual we denote by $\Sigma(K)$ the suspension from Definition 174.44. As we will see in Exercise 183.1, it follows very easily from the definitions that $\Sigma(K)$ is an $(n+1)$-dimensional pseudomanifold with $\partial\Sigma(K) = \Sigma(\partial K)$.

    Now assume that the fundamental group $\pi_1(|K|)$ of the topological realization $|K|$ is non-trivial. It follows from the Simplicial Cone-Suspension Corollary 174.46 together with the Suspension-of-Non Trivial-π_1-Lemma 180.20 that the topological realization $|\Sigma(K)|$ of the suspension $\Sigma(K)$ is not a topological manifold. This shows that for $n \geq 3$ not all n-dimensional pseudomanifolds arise from topological manifolds.
(5) In the literature there exist several, not always entirely equivalent, notions of a pseudomanifold. In general, as we just saw in (3), the topological realizations of pseudomanifolds are not topological manifolds. But many tools and results on topological manifolds can be generalized to the context of (topological realizations of) pseudomanifolds, see e.g. [**KiW2006**, Chapter 4]. Also note that in the literature an n-dimensional pseudomanifold is sometimes called an *n-circuit*. □

Next we recall and expand on Definition 176.10.

Definition 183.7. Let $K = (V, S)$ be an abstract simplicial complex.
(1) Let $n \in \mathbb{N}_0$. An ordered n-simplex of K is an $(n+1)$-tuple $(v_0, \dots, v_n) \in V^{n+1}$ such that $\{v_0, \dots, v_n\} \in S$. We write $\lfloor (v_0, \dots, v_n) \rfloor \coloneqq \{v_0, \dots, v_n\}$.
(2) An oriented n-simplex is a pair (ϵ, s) where $\epsilon \in \{\pm 1\}$ and s is an ordered n-simplex of K. As in (1) we write $\lfloor (\epsilon, s) \rfloor \coloneqq \lfloor s \rfloor$ and we refer to $\lfloor (\epsilon, s) \rfloor$ as the underlying simplex. Sometimes we also say that (ϵ, s) is an orientation of the n-simplex s.
(3) We say that two oriented n-simplices $(\mu, s = (v_0, \dots, v_n))$ and $(\nu, t = (w_0, \dots, w_n))$ are equivalent if
$$\mu \;=\; \nu \cdot \underset{\uparrow}{\operatorname{sign}}(\sigma) \qquad \text{where } \sigma \in \operatorname{Bij}\{0, \dots, n\} \text{ with } v_i = w_{\sigma(i)} \text{ for } i = 0, \dots, n.$$
sign of σ from Definition 176.3
    It follows immediately from the Sign-of-Permutation Lemma 176.2 that this is indeed an equivalence relation on the set of oriented n-simplices.
(4) An orientation of an n-simplex s is an equivalence class of oriented n-simplices with underlying simplex s.

Definition 183.8. Let $K = (V, S)$ be an abstract simplicial complex. Furthermore let $(\epsilon, t = (v_0, \dots, v_n))$ be an oriented n-simplex. Let s be a codimension-one face of t. If v_i is the one vertex of t that is missing in s, then we define the induced orientation of s to be
$$((-1)^i \cdot \epsilon, (v_0, \dots, \widehat{v_i}, \dots, v_n)).$$



Example. In the figure below we show to the left the oriented 2-simplex $(+1,(v_0,v_1,v_2))$. To the right we see the orientations induced on all of the three codimension-one faces. □

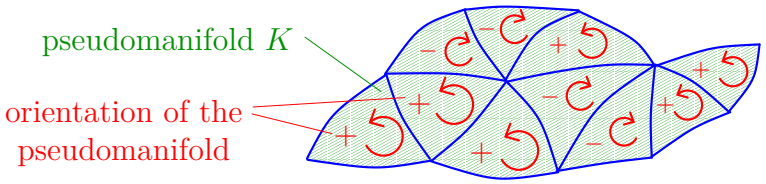
Lemma 183.9. (Induced Orientation–Well-Defined Lemma) Let $K=(V,S)$ be an abstract simplicial complex, let t be a simplex of K and let s be a codimension-one face of t. Any two equivalent orientations of t induce equivalent orientations of s.

Proof. This statement follows easily from the properties of the sign of a bijection, see the Sign-of-Permutation Lemma 176.2. We leave it to the reader to fill in the details. ■

Definition 183.10. Let K be an n-dimensional pseudomanifold.

(1) An orientation is a choice of an orientation for each n-simplex of K such that for every $(n-1)$-simplex s of order 2 the two cobounding n-simplices induce opposite orientations on s.

(2) We say that K is orientable if K admits an orientation.



In the following we will study the simplicial homology groups of pseudomanifolds. Before we do so, let us recall the definition of simplicial homology that we gave in Definition 176.5 and in the Simplicial Boundary Lemma 176.12.

Definition 183.11. Let $K=(V,S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. We set

$$\mathrm{C}_n^{\mathrm{simp}}(K) := \left\{ \begin{array}{c} \text{free abelian group generated} \\ \text{by all ordered } n\text{-simplices of } K \end{array} \right\} \Big/ (v_0,\ldots,v_n) \sim \mathrm{sign}(\sigma)\cdot(v_{\sigma(0)},\ldots,v_{\sigma(n)}).$$

In the discussion below, we will often not distinguish in the notation between an ordered simplex and the element it represents in $\mathrm{C}_k^{\mathrm{simp}}(K)$. The boundary map is given by

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \\ (v_0,\ldots,v_n) & \mapsto & \sum_{i=0}^{n} (-1)^i \cdot (v_0,\ldots,v_{i-1},\widehat{v_i},v_{i+1},\ldots,v_n). \end{array}$$

Now we can provide the promised theorem.

Theorem 183.12. (Pseudomanifold Homology Theorem) Let K be an n-dimensional pseudomanifold. The following statements hold:

(1) We have an isomorphism as follows:

$$\mathrm{H}_n^{\mathrm{simp}}(K) \cong \begin{cases} \mathbb{Z}, & \text{if } K \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if K is closed and orientable, and if $\{(\epsilon_w, Y_w)\}_{w\in W}$ is an orientation of K (here W denotes the set of n-simplices of K), then a generator of $\mathrm{H}_n^{\mathrm{simp}}(K) \cong \mathbb{Z}$ is represented by the simplicial cycle $\sum_{w\in W} \epsilon_w \cdot Y_w \in \mathrm{C}_n^{\mathrm{simp}}(K)$.

(2) If K is finite, then we also have an isomorphism as follows:

$$\mathrm{H}_{n-1}^{\mathrm{simp}}(K) \cong \begin{cases} \text{free abelian group}, & \text{if } K \text{ is orientable or if } \partial K \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } K \text{ is non-orientable and } \partial K = \varnothing. \end{cases}$$

(3) For $\mathbb{F}_2$-coefficients we have

$$\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } K \text{ is closed, regardless whether or not } K \text{ is orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if K is in fact closed and if $Y_1, \ldots, Y_m$ is a set of ordered n-simplices of K such that for each n-simplex s there exists a unique $i \in \{1, \ldots, m\}$ with $s = \lfloor Y_i \rfloor$, then the unique non-zero element of $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \cong \mathbb{F}_2$ is represented by the simplicial cycle $Y_1 \otimes 1 + \cdots + Y_m \otimes 1 \in \mathrm{C}_n^{\mathrm{simp}}(K;\mathbb{F}_2) = \mathrm{C}_n^{\mathrm{simp}}(K) \otimes \mathbb{F}_2$.

The Pseudomanifold Homology Theorem 183.12 leads us to the following definition:

Definition 183.13. Let K be a closed n-dimensional pseudomanifold.

(1) If K is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w\in W}$, then we refer to the simplicial homology class

$$[K] := \left[\sum_{w\in W} \epsilon_w \cdot Y_w\right] \in \mathrm{H}_n^{\mathrm{simp}}(K)$$

as the fundamental class of K.

(2) We refer to the unique non-zero element in $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \cong \mathbb{F}_2$ as the $\mathbb{F}_2$-fundamental class $[K]_{\mathbb{F}_2}$.

For clarity we break the proof of the Pseudomanifold Homology Theorem 183.12 into three parts. For the proof of the first part it is convenient to introduce the following definitions which we will use on a few other occasions.

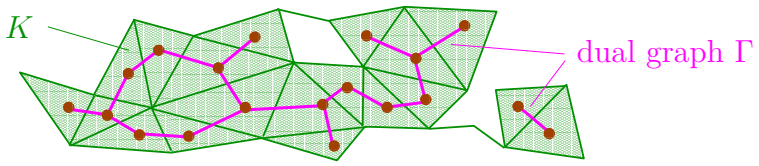
Definition 183.14. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex such that every $(n-1)$-simplex has order ≤ 2.

(1) We denote by W the set of n-simplices of K.

(2) We denote by E_2 the set of $(n-1)$-simplices of order 2.

(3) Let $\Gamma = (W, E_2, \varphi)$ be the abstract graph with vertex set W, with edge set E_2 and where $\varphi\colon E_2 \to \mathcal{P}(W)$ is the map that associates to each edge $e \in E_2$ the set $\{w \in W \mid e \text{ is a face of } w\}$. Note that for $e \in E_2$ the set $\varphi(e)$ has precisely two elements. This means that Γ is indeed an abstract graph in the sense of Definition 11.12. We call Γ the dual graph of the abstract simplicial complex K. We refer to the figure below for an illustration.

Proof of the Pseudomanifold Homology Theorem 183.12 (1). Let K be an n-dimensional pseudomanifold. Our goal is to calculate $\mathrm{H}_n^{\mathrm{simp}}(K)$. We start out with the



following easy but useful observation which we will use throughout this proof without referring to it again explicitly:

$$\mathrm{H}_n^{\mathrm{simp}}(K) \underset{\uparrow}{=} \ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big).$$

K has no simplices of dimension $n+1$, hence $\mathrm{C}_{n+1}^{\mathrm{simp}}(K) = 0$

To determine the right-hand side of the above equality we need to introduce some notation.

(1) We denote by E the set of $(n-1)$-simplices of K. Furthermore, as above, we denote by E_1 the set of $(n-1)$-simplices of order 1 and similarly we denote by E_2 the set of $(n-1)$-simplices of order 2. Note that it follows from the hypothesis that K is an n-dimensional pseudomanifold that $E = E_1 \sqcup E_2$.

(2) As above we denote by W the set of n-simplices of K.

(3) Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. Note that part (3) of the definition of a pseudomanifold implies that the abstract graph Γ is connected in the sense of Definition 11.15.

(4) For each $(n-1)$-simplex $e \in E$ we pick, once and for all, an ordered $(n-1)$-simplex X_e with $\lfloor X_e \rfloor = e$. We use the Simplicial Chain Group–Isomorphism Lemma 176.11 to view $\{X_e\}_{e\in E}$ as a basis for $\mathrm{C}_{n-1}^{\mathrm{simp}}(K)$.

(5) We consider the two obvious projections

$$p_1\colon \overbrace{\bigoplus_{e\in E} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \bigoplus_{e\in E_1} \mathbb{Z}\cdot X_e \quad \text{and} \quad p_2\colon \overbrace{\bigoplus_{e\in E} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \bigoplus_{e\in E_2} \mathbb{Z}\cdot X_e.$$

Since $E = E_1 \sqcup E_2$ we see for $s \in \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$ that $s = 0$ if and only if $p_1(s) = 0$ and $p_2(s) = 0$.

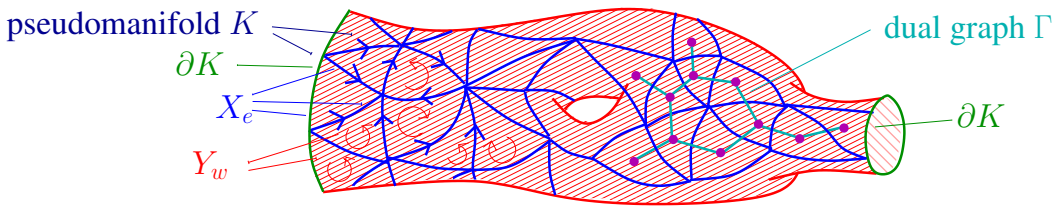
Now suppose that for each $w \in W$ we are given an oriented n-simplex Y_w with $\lfloor Y_w \rfloor = w$. We introduce the following conventions.

(6) As above we use the Simplicial Chain Group–Isomorphism Lemma 176.11 to view $\{Y_w\}_{w\in W}$ as a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$.

(7) Given $e \in E$ and $w \in W$ we set[1942]

$$\mu_{ew} = \begin{cases} 0, & \text{if } e \text{ is not a face of } w, \\ +1, & \text{if } e \text{ is a face of } w \text{ and } [(+1, Y_w)] \text{ induces } [(+1, X_e)], \\ -1, & \text{if } e \text{ is a face of } w \text{ and } [(+1, Y_w)] \text{ induces } [(-1, X_e)]. \end{cases}$$

[1942]Note that if e is a face of w, then precisely one of the two cases does indeed occur.



Claim 1. Given $\sum_{w\in W} a_w \cdot Y_w \in \mathrm{C}_n^{\mathrm{simp}}(K)$ we have

(a) $p_1\Big(\partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big)\Big) = 0 \quad\Leftrightarrow\quad$ for all $w\in W$ that cobound an $e\in E_1$ we have $a_w = 0$

(b) $p_2\Big(\partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big)\Big) = 0 \quad\Leftrightarrow\quad$ for all $w \neq w' \in W$ that cobound some $e \in E_2$ we have $\mu_{ew}\cdot a_w = -\mu_{ew'}\cdot a_{w'}$.

Proof. We start out our proof of Claim 1 with the following calculation:

$$\partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big) = \sum_{w\in W} a_w\cdot\partial_n(Y_w) = \sum_{w\in W} a_w\cdot \sum_{\substack{e\in E \text{ such that}\\ e \text{ is a face of } w}} \mu_{ew}\cdot X_e = \sum_{e\in E} X_e\cdot \sum_{\substack{w\in W \text{ such that}\\ e \text{ is a face of } w}} \mu_{ew}\cdot a_w.$$

this follows immediately from Definition 183.11 of ∂_n the definition of induced orientations, see Definition 183.8, and the above definition of μ_{ew}

Now we consider the two projections.

(a) Note that an element $s \in \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$ lies in the kernel of p_1 if and only if for all $e \in E_1$ the X_e-coefficient of s vanishes. Thus let $e \in E_1$. Let $w \in W$ be the unique n-simplex that cobounds e. Using the above calculation we see that

$$X_e\text{–coefficient of } \partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big) \;=\; \mu_{ew}\cdot a_w.$$

The promised result follows from this discussion.

(b) Next let $e \in E_2$. Let $w \neq w'$ be the two n-simplices that cobound e. Similar to the above we see that

$$X_e\text{–coefficient of } \partial_n\Big(\sum_{w\in W} a_w\cdot Y_w\Big) \;=\; \mu_{ew}\cdot a_w + \mu_{ew'}\cdot a_{w'}.$$

As in (a) we see that this calculation gives us the desired result.

Now note that Claim 1 follows almost immediately from the above discussion. ⊞

Claim 2. Every element $\sum_{w\in W} a_w\cdot Y_w$ that lies in the kernel of the map

$$\mathrm{C}_n^{\mathrm{simp}}(K) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K) = \bigoplus_{e\in E}\mathbb{Z}\cdot X_e \xrightarrow{p_2} \bigoplus_{e\in E_2}\mathbb{Z}\cdot X_e$$

has the property that for all $w, w' \in W$ we have $|a_w| = |a_{w'}|$.

Proof. We consider the map

$$\begin{aligned} f\colon W &\to \mathbb{N}_0\\ w &\mapsto |a_w|. \end{aligned}$$

It follows from Claim 1 that for every $e \in E_2$ and $w, w' \in \varphi(e)$ we have $|a_w| = |a_{w'}|$. Since the abstract graph (W, E_2, φ) is connected we obtain from the Combinatorial-Connected-to-Discrete Lemma 11.16 that $f\colon W \to \mathbb{N}_0$ is constant. ⊞

Claim 3. Given any $w_0 \in W$ the map

$$\begin{array}{rccc} \Theta_{w_0}\colon \ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right) & \to & \mathbb{Z} \\ \sum\limits_{w\in W} a_w \cdot Y_w & \mapsto & a_{w_0} \end{array}$$

is a monomorphism.

Proof. It is clear that the given map is a homomorphism. It follows immediately from Claim 2 that the map Θ_{w_0} is a monomorphism. ⊞

Now we can prove Statement (1) of the proposition.

(i) We need to show that if K is infinite, then $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. So suppose that K is infinite. It follows from the fact that K is finite-dimensional and the fact that every simplex is contained in an n-simplex that W is also infinite. Now let $\sum_{w\in W} a_w \cdot Y_w$ be an element in $\ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)$. Since W is infinite and since by definition every element in $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a finite linear combination of ordered simplices we see that there exists a $w_0 \in W$ with $a_{w_0} = 0$. But then it follows from Claim 2 that $a_w = 0$ for all $w \in W$.

(ii) We need to show that if $\partial K \neq \varnothing$, then $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. Thus suppose that $\partial K \neq \varnothing$. By definition of ∂K this implies that $E_1 \neq \varnothing$. We pick some $e_0 \in E_1$. Let w_0 be the unique n-simplex that cobounds e_0. Now let $\sum_{w\in W} a_w \cdot Y_w$ be an element in $\ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)$. By Claim 1 we have $a_{w_0} = 0$. But then it follows from Claim 2 that $a_w = 0$ for all $w \in W$.

(iii) Now assume that $\mathrm{H}_n^{\mathrm{simp}}(K) \neq 0$. We need to show that K is actually orientable. To do so it suffices to prove the following claim:

Claim. If $\sum_{w\in W} a_w \cdot Y_w$ is a non-zero element in $\ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)$, then the orientations[1943] $\{(\mathrm{sign}(a_w), Y_w)\}_{w\in W}$ define an orientation of K.

Proof. Let $\sum_{w\in W} a_w \cdot Y_w$ be a non-zero element in $\ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)$. Let e be an $(n-1)$-simplex of order 2, i.e. let $e \in E_2$. Let $w \neq w'$ be the two cobounding n-simplices of e. By Claim 1 we know that $\mu_{ew} \cdot a_w = -\mu_{ew'} \cdot a_{w'}$. It follows from Claim 2 that we also have the equality $\mu_{ew} \cdot \mathrm{sign}(a_w) = -\mu_{ew'} \cdot \mathrm{sign}(a_{w'})$. But basically by definition that means that $(\mathrm{sign}(a_w), Y_w)$ and $(\mathrm{sign}(a_{w'}), Y_{w'})$ induce opposite orientations on the $(n-1)$-simplex e. ⊞

(iv) Finally we suppose that K is closed and orientable. Let $\{(\mu_w, Y_w)\}_{w\in W}$ be an orientation of K. We need to show that $\mathrm{H}_n^{\mathrm{simp}}(K) \cong \mathbb{Z}$ and that $\sum_{w\in W} \mu_w \cdot Y_w$ is a generator of $\mathrm{H}_n^{\mathrm{simp}}(K)$.

Since K is by definition non-empty we can pick some $w_0 \in W$. We consider the maps

$$\begin{array}{rccccc} \mathbb{Z} & \xrightarrow{\Psi} & \ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)) & \xrightarrow{\Theta_{w_0}} & \mathbb{Z} \\ r & \mapsto & r \cdot \sum\limits_{w\in W} \mu_w \cdot Y_w & & \end{array}$$

[1943]Here recall that given a non-zero integer $a \in \mathbb{Z} \setminus \{0\}$ we denote by $\mathrm{sign}(a) := \frac{a}{|a|} \in \{\pm 1\}$ its sign. Also note that we know by hypothesis that at least one $a_w \neq 0$, but then it follows from Claim 2 that all $a_w \neq 0$, in particular it makes sense to consider the signs of the a_w.

It follows almost immediately from Claim 1 that the map Ψ to the left is well-defined, in the sense that it takes values in $\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K))$. Furthermore it is clear that the composition $\Theta_{w_0} \circ \Psi$ is given by multiplication by μ_{w_0}, in particular the composition $\Theta_{w_0} \circ \Psi$ is an isomorphism. By Claim 3 we know that Θ_{w_0} is a monomorphism. It follows from this discussion that Ψ is an isomorphism. But that is exactly what we needed to show. ■

The proof of Statement (2) of the Pseudomanifold Homology Theorem 183.12 will require the following elementary lemma, which incidentally also solves Exercise 145.17.

Lemma 183.15. (Direct Sum-Induced Isomorphism Lemma) Let $A, \widetilde{A}, B, \widetilde{B}$ and C be abelian groups. Now suppose that we are given a sequence

$$A \oplus \widetilde{A} \xrightarrow{\varphi} B \oplus \widetilde{B} \xrightarrow{\psi} C$$

of homomorphisms such that the following conditions are satisfied:

(1) We have $\psi \circ \varphi = 0$.

(2) The map $\varphi \circ i_A\colon A \to B \oplus \widetilde{B}$ is of the form $i_B \circ \alpha$ for some homomorphism $\alpha\colon A \to B$.

(3) The map $p_B \circ \varphi \circ i_{\widetilde{A}}\colon \widetilde{A} \to B$ is an isomorphism.

(Here and in the following i indicates a natural inclusion and p indicates a natural projection.) Then the map[1944]

$$\begin{array}{ccc} \dfrac{\ker(\psi \circ i_B\colon B \to C)}{\mathrm{im}(\alpha\colon A \to B)} & \to & \dfrac{\ker(\psi\colon B \oplus \widetilde{B} \to C)}{\mathrm{im}(\alpha\colon A \to B \oplus \widetilde{B})} \\ [\widetilde{b}] & \mapsto & [i_{\widetilde{B}}(\widetilde{b})] \end{array}$$

is well-defined and it is an isomorphism.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i_A} & A \oplus \widetilde{A} & \xrightarrow{p_{\widetilde{A}}} & \widetilde{A} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \cong\ p_B \circ \varphi \circ i_{\widetilde{A}}} & & \\ 0 & \longrightarrow & B & \xrightarrow{i_B} & B \oplus \widetilde{B} & \xrightarrow{p_B} & \widetilde{B} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi \circ i_B} & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\ 0 & \longrightarrow & C & \xrightarrow{\mathrm{id}} & C & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

It follows easily from hypothesis (2) that the diagram commutes. It is clear that the horizontal sequences are exact. It follows from hypothesis (1) that the vertical sequences are chain complexes. It follows from these observations and the LES-Proposition 145.10 that we obtain the following exact sequence:

$$0 \to \underbrace{\frac{\ker(\psi \circ i_B\colon B \to C)}{\mathrm{im}(\alpha\colon A \to B)}}_{=0 \text{ by hypothesis (3)}} \to \frac{\ker(\psi\colon B \oplus \widetilde{B} \to C)}{\mathrm{im}(\alpha\colon A \to B \oplus \widetilde{B})} \to \underbrace{\frac{B}{(\mathrm{im}(p_B \circ \varphi \circ i_{\widetilde{A}})(A)}}_{=0 \text{ by hypothesis (3)}}.$$

Since the groups to the left and right are zero, we obtain the promised statement. ■

[1944]The group on the right can be viewed as the middle homology of the above sequence, which by (1) we can view as a chain complex.

We will also need another lemma, which is again a consequence of the LES-Proposition 145.10:

Lemma 183.16. (Kernel-mod-Image Lemma) Let A and H be abelian groups and let F be a free abelian group. We suppose that we are given a sequence

$$A \xrightarrow{\alpha} H \xrightarrow{\varphi} F$$

of homomorphisms such that $\varphi \circ \alpha = 0$. Then we have an isomorphism

$$\frac{H}{\mathrm{im}(\alpha)} \cong \frac{\ker(\varphi)}{\mathrm{im}(\alpha)} \oplus \widetilde{F}$$

where $\widetilde{F}$ is a free abelian group.

Proof. We denote by $i\colon \ker(\varphi) \to H$ the inclusion. We set $\widetilde{F} := \mathrm{im}(\varphi)$. Note that it follows from the Free Abelian Subgroup Theorem 97.14 that $\widetilde{F}$, as a subgroup of the free abelian group F, is itself a free abelian group. Next we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ 0 & \longrightarrow & \ker(\varphi) & \xrightarrow{i} & H & \xrightarrow{\varphi} & \widetilde{F} & \longrightarrow & 0. \end{array}$$

Note that it follows from our hypotheses that this diagram commutes. Clearly this is a short exact sequence of (rather dull) vertical chain complexes. It follows from the LES-Proposition 145.10 that the following sequence is exact:

$$0 \to \frac{\ker(\varphi)}{\mathrm{im}(\alpha)} \to \frac{H}{\mathrm{im}(\alpha)} \to \widetilde{F} \to 0.$$

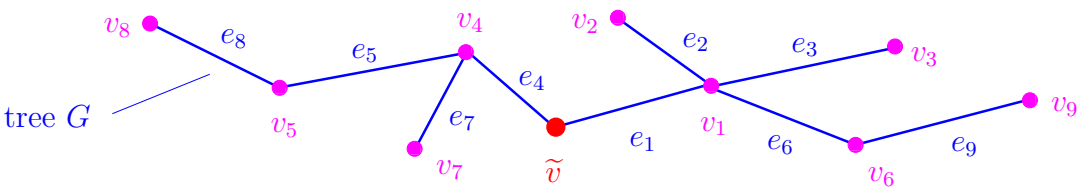
Since $\widetilde{F}$ is a free abelian group it follows from the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 that the middle group is isomorphic to the direct sum of the two outer groups. ■

We move towards the proof of Statement (2) of the Pseudomanifold Homology Theorem 183.12. That proof relies on the following lemma from graph theory.

Lemma 183.17. (Tree–Edge Numbering Lemma) Let $G = (V, E, i, t)$ be an abstract graph and let $\widetilde{v} \in V$ be a vertex. If G is a tree, then there exist distinct edges $e_1, \ldots, e_k$ and distinct vertices $v_1, \ldots, v_k$ with the following properties:

(1) We have $V = \{\widetilde{v}, v_1, \ldots, v_k\}$ and $E = \{e_1, \ldots, e_k\}$.
(2) For each $i \in \{1, \ldots, k\}$ the vertex v_i is an endpoint of the edge e_i.
(3) If some vertex v_i is an endpoint of some e_j, then $i \leq j$.

The same statement also holds for abstract graphs.



Proof. For abstract graphs we provided the proof of the lemma in Exercise 11.11. For abstract graphs the proof is verbatim the same. ■

Proof of the Pseudomanifold Homology Theorem 183.12 (2). Let K be a finite n-dimensional pseudomanifold. Recall that we need to prove the following statements:

(a_1) If K is orientable and closed, then $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
(a_2) If $\partial K \neq \varnothing$, then $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
(b) If K is non-orientable and if $\partial K = \varnothing$, then $\mathrm{H}_{n-1}^{\mathrm{simp}}(K) \cong F \oplus \mathbb{Z}/2\mathbb{Z}$ where F is a free abelian group.

In the following we will use freely the notation that we introduced in the proof of Statement (1). In particular, as on page 3471, we consider again the dual graph $\Gamma = (W, E_2, \varphi)$, where W denotes the set of n-simplices of K and where E_2 is the set of $(n-1)$-simplices of order 2.

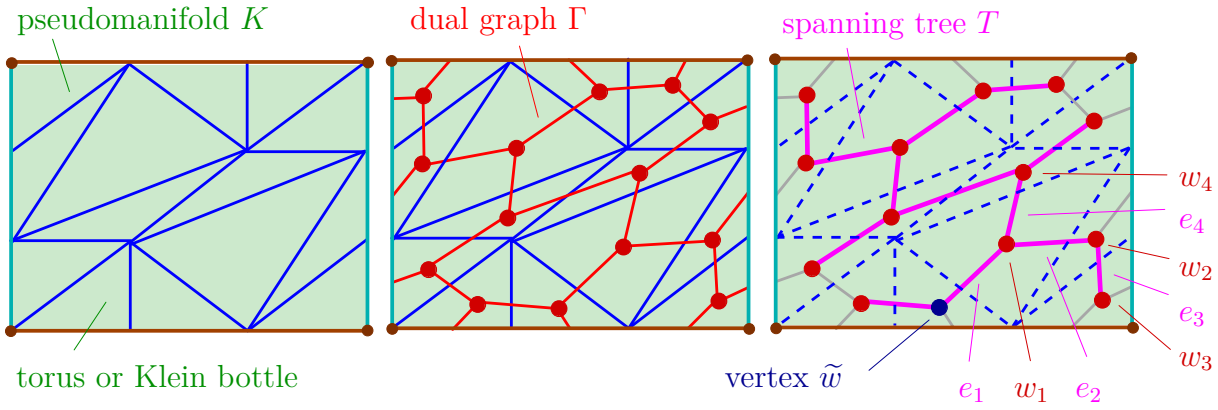
As we pointed out in Definition 183.6, a pseudomanifold is necessarily non-empty and connected. Since we assume that K is finite, we know furthermore that Γ is finite. Hence it follows from the Spanning Tree–Existence Proposition 11.26 that there exists a spanning tree T of Γ. Recall that this means that T is a tree and that T contains every vertex. We pick a "base n-simplex" $\widetilde{w} \in W$. By the Tree–Edge Numbering Lemma 183.17 we know that we can write $W = \{\widetilde{w}, w_1, \dots, w_k\}$ and $\{e_1, \dots, e_k\} \subseteq E_2$ such that the following statements hold:

- For each $i \in \{1, \dots, k\}$ the vertex w_i is an endpoint of the edge e_i.
- If some vertex w_i is an endpoint of some e_j, then $i \leq j$.

We denote by

$$q_1 \colon \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \to \bigoplus_{i=1}^{k} \mathbb{Z} \cdot e_i \quad \text{and} \quad q_2 \colon \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \to \bigoplus_{e \in E \setminus \{e_1, \dots, e_k\}} \mathbb{Z} \cdot e_i$$

the natural projections.



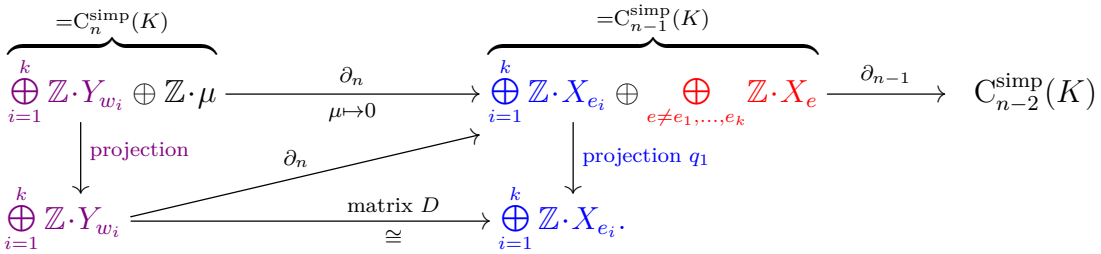
The key to the proof of the Pseudomanifold Homology Theorem 183.12 (2) is the following observation, which is an immediate consequence of Claim 1 together with the clever choice of $w_1, \dots, w_k$ and $e_1, \dots, e_k$.

Observation. The map $\bigoplus_{i=1}^{k} \mathbb{Z} \cdot w_i \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{q_1} \bigoplus_{i=1}^{k} \mathbb{Z} \cdot e_i$ is represented by a matrix of the form

$$D \coloneqq \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ * & \pm 1 & 0 & \vdots \\ * & * & \pm 1 & 0 \\ * & \dots & * & \pm 1 \end{pmatrix}$$

After these preparations we turn to the actual proof of the above statements.

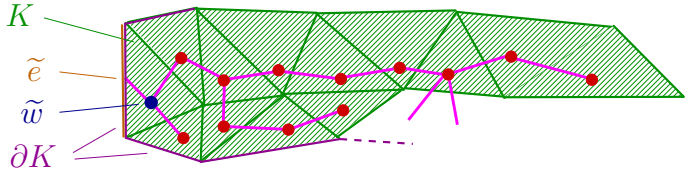
(a$_1$) First we assume that K is orientable and closed. Let $\{(\epsilon_w, Y_w)\}_{w \in W}$ be an orientation of K. We set $\mu \coloneqq \sum_{w \in W} \epsilon_w \cdot Y_w$. Since $\epsilon_{\widetilde{w}} \in \{\pm 1\}$ we see that $Y_{w_1}, \dots, Y_{w_k}, \mu$ form a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$. This time we consider the following maps:



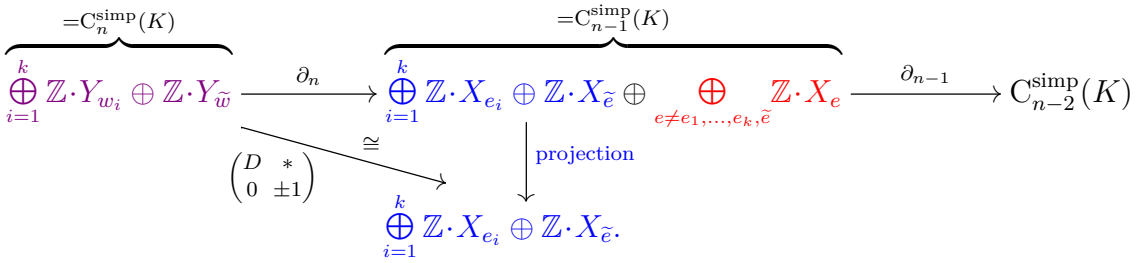
As we already discussed in the proof of Statement (1), since K is closed we obtain from Claim 1 that $\partial_n(\mu) = 0$. Thus the image of ∂_n equals the image of the diagonal map. Furthermore, the bottom horizontal map is represented by the matrix D from the observation, in particular we see that the bottom horizontal map is an isomorphism. From the Direct Sum-Induced Isomorphism Lemma 183.15 (with $A = 0$) we now obtain the following isomorphism:

$$\underbrace{\ker\left(\bigoplus_{e \neq e_1,\dots,e_k} \mathbb{Z}\cdot X_e \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}_{\substack{\text{subgroup of a free abelian group, thus by} \\ \text{the Free Abelian Subgroup Theorem 97.14 it} \\ \text{is a free abelian group}}} \xrightarrow{\cong} \underbrace{\frac{\ker\left(\mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\mathrm{C}_n^{\mathrm{simp}}(K) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

(a$_2$) Now we assume that $\partial K \neq \varnothing$, i.e. we assume that $E_1 \neq \varnothing$. We pick an $\widetilde{e} \in E_1$. Let $\widetilde{w} \in W$ be the unique n-simplex that cobounds $\widetilde{e}$. Now we use $\widetilde{w}$ as the "base n-simplex" and we apply again the Tree–Edge Numbering Lemma 183.17 to the maximal tree Γ to pick $w_1, \dots, w_k$ and $e_1, \dots, e_k$ accordingly.



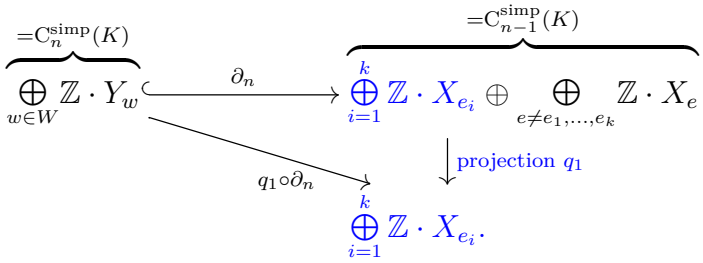
We consider the following maps:



It follows easily from the Observation and Claim 1 that the diagonal map is an isomorphism. It now follows from the Direct Sum-Induced Isomorphism Lemma 183.15 (again with $A = 0$) that

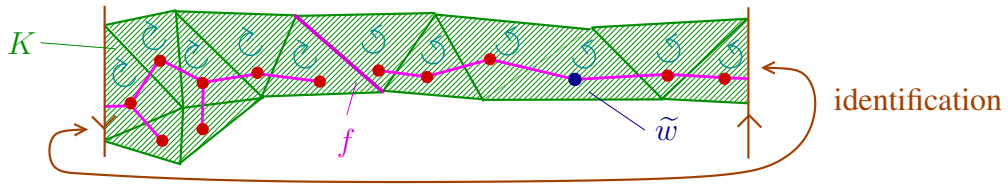
$$\underbrace{\ker\left(\bigoplus_{e\neq e_1,\dots,e_k,\widetilde{e}} \mathbb{Z}\cdot X_e \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}_{\substack{\text{subgroup of a free abelian group thus by the} \\ \text{Free Abelian Subgroup Theorem 97.14 it} \\ \text{is a free abelian group}}} \xrightarrow{\cong} \underbrace{\frac{\ker\left(\mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\mathrm{C}_n^{\mathrm{simp}}(K) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

(b) Finally we assume that K is closed and non-orientable. First we consider the following maps:



By our hypothesis that K is closed and non-orientable, together with Statement (1), we know that $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. This means that the horizontal map is a monomorphism. On the other hand, for rank reasons the diagonal map $q_1 \circ \partial_n$ has a kernel. In other words, there exists some non-zero $\mu = \sum_{w\in W} a_w \cdot Y_w \in \mathrm{C}_n^{\mathrm{simp}}(K)$ with $q_1(\partial_n(\mu)) = 0$. By the Claim 2 on page 3472 we know that the value $|a_w|$ is constant. Thus after dividing by this constant (which is non-zero since μ is non-zero) we can arrange that each $|a_w| = 1$.

Since K is by hypothesis closed, i.e. since each edge has order 2, we see, say by the calculation in the proof of Claim 1 on page 3471 or by direct inspection, that for each $e \in E$ the X_e-coefficient of $\partial_n(\mu)$ lies in $\{0, \pm 2\}$. Furthermore since ∂_n is a monomorphism we know that $\partial_n(\mu) \neq 0$. Thus there exists some $f \in E$ such that the f-coefficient of $\partial_n(\mu)$ equals ± 2. From $q_1(\partial_n(\mu)) = 0$ we obtain that $f \neq e_1, \dots, e_k$.



Next we consider the following maps:

$$\begin{array}{ccccc} \overbrace{\mathbb{Z}\cdot\mu \oplus \bigoplus_{i=1}^{k} \mathbb{Z}\cdot Y_{w_i}}^{=\mathrm{C}_n^{\mathrm{simp}}(K)} & \xhookrightarrow{\partial_n} & \overbrace{\bigoplus_{i=1}^{k} \mathbb{Z}\cdot X_{e_i} \oplus \mathbb{Z}\cdot X_f \oplus \bigoplus_{e\neq e_1,\dots,e_k,f} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} & \xrightarrow{\partial_{n-1}} & \mathrm{C}_{n-2}^{\mathrm{simp}}(K) \\ \downarrow \text{projection} & & \downarrow \text{projection} & & \\ \bigoplus_{i=1}^{k} \mathbb{Z}\cdot Y_{w_i} & \xrightarrow[\cong]{D} & \bigoplus_{i=1}^{k} \mathbb{Z}\cdot X_{e_i}. & & \end{array}$$

Now note that it follows from the above together with the Direct Sum-Induced Isomorphism Lemma 183.15 that we have an isomorphism:

$$\frac{\ker\left(\mathbb{Z}\cdot X_f \oplus \bigoplus_{e\neq e_1,\dots,e_k,f} \mathbb{Z}\cdot X_e \to \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathbb{Z}\cdot\partial_n(\mu)} \to \underbrace{\frac{\ker\left(\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\to \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathrm{im}\left(\mathrm{C}_{n}^{\mathrm{simp}}(K)\to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

It now follows from the Kernel-mod-Image Lemma 183.16 that there exists a free abelian group F such that we have an isomorphism

$$\underbrace{\frac{\mathbb{Z}\cdot X_f \oplus \bigoplus_{e\neq e_1,\dots,e_k,f} \mathbb{Z}\cdot X_e}{\mathbb{Z}\cdot\partial_n(\mu)}}_{\substack{\text{it follows from the above discussion}\\ \text{that this is isomorphic to the direct sum}\\ \text{of } \mathbb{Z}/2\mathbb{Z} \text{ and a free abelian group}}} \cong \frac{\ker\left(\mathbb{Z}\cdot X_f \oplus \bigoplus_{e\neq e_1,\dots,e_k,f} \mathbb{Z}\cdot X_e \to \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\right)}{\mathbb{Z}\cdot\partial_n(\mu)} \oplus F.$$

It follows from this discussion and the Cancellation Proposition 97.26 that $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is isomorphic to the direct sum of $\mathbb{Z}/2\mathbb{Z}$ with a free abelian group. We have thus finished the proof of the Pseudomanifold Homology Theorem 183.12 (2). ■

Finally we get to the proof of the last statement of the Pseudomanifold Homology Theorem 183.12 (3).

Proof of the Pseudomanifold Homology Theorem 183.12 (3). Let K be an n-dimensional pseudomanifold. If K is infinite, then it follows from the same argument as in (1) that $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) = 0$.

Now assume that K is finite. In this case it is also a fun exercise to modify the proof of (1) to obtain the promised result. But in these notes we will not pursue this approach. Instead we will do a little trick. First we consider the case that K is closed. In this case we perform the following calculation:

$$\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \underset{\substack{\uparrow\\ \text{by the Algebraic Universal Coefficient Theorem 162.15 applied to the chain complex } \mathrm{C}_*^{\mathrm{simp}}(K)}}{\cong} \mathrm{H}_n^{\mathrm{simp}}(K)\otimes\mathbb{F}_2 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}^{\mathrm{simp}}(K),\mathbb{F}_2) \underset{\substack{\uparrow\\ \text{by (1) and (2)}}}{\cong} \begin{cases} \overbrace{\mathbb{Z}\otimes\mathbb{F}_2 \oplus \mathrm{Tor}(\mathbb{Z}^l,\mathbb{F}_2)}^{\text{if } K \text{ is orientable}} \\ \underbrace{0\otimes\mathbb{F}_2 \oplus \mathrm{Tor}(\mathbb{Z}^l\oplus\mathbb{Z}/2\mathbb{Z},\mathbb{F}_2)}_{\text{if } K \text{ is non-orientable}} \end{cases} \underset{\substack{\uparrow\\ \text{by the Tensor-with-}\mathbb{Z}\text{–Lemma 159.9, the Tor-Properties Lemma 162.13 and the } \mathbb{Z}/k\mathbb{Z}\text{-Tor-Lemma 162.12}}}{\cong} \mathbb{F}_2.$$

Now suppose that $Y_1, \dots, Y_m$ is a set of ordered n-simplices of K such that for each n-simplex s of K there exists a unique $i \in \{1, \dots, m\}$ with $s = \lfloor Y_i \rfloor$. One easily verifies that the simplicial chain $Y_1 \otimes 1 + \cdots + Y_m \otimes 1 \in \mathrm{C}_n^{\mathrm{simp}}(K; \mathbb{F}_2) = \mathrm{C}_n^{\mathrm{simp}}(K) \otimes \mathbb{F}_2$ is a simplicial cycle. Since it is non-zero we see that it equals necessarily the unique non-zero element of $\mathrm{H}_n^{\mathrm{simp}}(K; \mathbb{F}_2) = \ker(\mathrm{H}_n^{\mathrm{simp}}(K; \mathbb{F}_2) \to \mathrm{H}_{n-1}^{\mathrm{simp}}(K; \mathbb{F}_2)) \cong \mathbb{F}_2$.

Finally if the pseudomanifold K is not closed, then the above approach shows almost immediately that $\mathrm{H}_n^{\mathrm{simp}}(K; \mathbb{F}_2) = 0$. With a slight sigh of relief we note that we have now finally managed to prove all three statements of the Pseudomanifold Homology Theorem 183.12. ■

183.2. **The top-dimensional homology group of a smooth manifold.** As the reader surely expected, the main application of the Pseudomanifold Homology Theorem 183.12 is the following theorem which gives us lots of interesting extra information on the homology groups of smooth manifolds.

Theorem 183.18. (Manifold-Top-H$_*$-Theorem) Let M be a connected non-empty n-dimensional smooth manifold. The following statements hold:

(1) We have an isomorphism as follows:

$$\mathrm{H}_n(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If M is compact, then we also have an isomorphism as follows:

$$\mathrm{H}_{n-1}(M; \mathbb{Z}) \cong \begin{cases} \text{free abelian group,} & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

In both cases the free abelian group is finitely generated.

(3) With $\mathbb{F}_2$-coefficients we have

$$\mathrm{H}_n(M; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } M \text{ is closed, regardless whether or not } M \text{ is orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Examples.

(1) Let $k \in \mathbb{N}$. We consider $k \cdot \mathbb{R}\mathrm{P}^2$, i.e. we consider the connected sum of k copies of $\mathbb{R}\mathrm{P}^2$. By Exercise 54.8 we know that $k \cdot \mathbb{R}\mathrm{P}^2$ is non-orientable. In the Surface-H$_*$-Proposition 155.17 we showed that

$$\mathrm{H}_1(\underbrace{k \cdot \mathbb{R}\mathrm{P}^2}_{=N_k}; \mathbb{Z}) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{H}_2(k \cdot \mathbb{R}\mathrm{P}^2; \mathbb{Z}) = 0.$$

This is reassuringly consistent with the Manifold-Top-H$_*$-Theorem 183.18.

(2) Recall that the Möbius band M is a compact non-orientable 2-dimensional smooth manifold with non-empty boundary. Since M is homotopy equivalent to S^1 we see that $\mathrm{H}_1(M; \mathbb{Z}) \cong \mathrm{H}_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$. This example illustrates that in the Manifold-Top-H$_*$-Theorem 183.18 the "$\mathbb{Z}/2\mathbb{Z}$"-term for a non-orientable smooth manifold only appears if M is closed.

(3) In Chapter 191 we will make sense of the notion of an orientable topological manifold. We will prove the generalization of the Manifold-Top-H$_*$-Theorem 183.18 to the

setting of topological manifolds in the Orientation-Top-H_*-Theorem 193.3, the Top-$\mathbb{F}_2$-Homology Theorem 193.2 and the Manifold-Top-H_*-Theorem 193.1 and the Manifold-H_{n-1}-Theorem 193.4. These new results completely subsume the Manifold-Top-H_*-Theorem 183.18. Nonetheless it is worth covering the above proof since it is fairly geometric, whereas the latter proofs for topological manifolds are rather unintuitive. □

Proof of the Manifold-Top-H_*-Theorem 183.18. Let M be a connected non-empty n-dimensional smooth manifold. Note that by the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that the smooth manifold M admits a smooth simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. By the Manifold-Simplex Proposition 183.2 we know that K is an n-dimensional pseudomanifold. Evidently now we would like to apply the Pseudomanifold Homology Theorem 183.12 and call it a day.

But to do so we need to show that the hypotheses on the smooth manifolds do indeed match the corresponding hypotheses for pseudomanifolds. Let us start with the good news:

(1) By the Finite Simplicial Complex Lemma 174.25 (1) and the Simplicial Complex-Finiteness Proposition 174.29 we know that M is compact if and only if K is finite.
(2) From the Manifold-Simplex Proposition 183.2 (1), (3) and (4) we obtain that $\partial M = \varnothing$ if and only if $\partial K = \varnothing$.

Here is the bad news: We still have to relate the (non-) orientability of the smooth manifold M to the (non-) orientability of the abstract simplicial complex K. This will be done in the subsequent Proposition 183.19. ■

The following proposition gives us the last piece of the proof of the Manifold-Top-H_*-Theorem 183.18.

Proposition 183.19. (Pseudomanifold-Orientation Proposition) Let $n \in \mathbb{N}$ and let M be a connected non-empty n-dimensional smooth manifold.[1945] Furthermore let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. The map

$$\left\{\begin{array}{c}\text{orientations of the smooth}\\ \text{manifold } M \text{ in the sense of the}\\ \text{Definition 31.19}\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{orientations of the}\\ \text{pseudomanifold } K \text{ in the sense}\\ \text{of Definition 183.10}\end{array}\right\}.$$

$$\begin{array}{rl} & \text{for each } n\text{-simplex } s \text{ we pick a corresponding}\\ & \text{ordered simplex } Y_s \text{ and we consider}\\ \text{orientation of } M \mapsto & \text{the oriented } n\text{-simplex } [(\epsilon_s, Y_s)] \text{ where}\\ & \epsilon_s = +1 \text{ if } \Theta \circ \Phi^{\leq}_{Y_s} : \mathring{\Delta}^n \to M \text{ is orientation-pres.}\\ & \epsilon_s = -1 \text{ if } \Theta \circ \Phi^{\leq}_{Y_s} : \mathring{\Delta}^n \to M \text{ is orientation-rev.}\end{array}$$

is a natural bijection.[1946] In particular the smooth manifold M is orientable if and only if the pseudomanifold K is orientable.

Proof of Proposition 183.19. In the following we will provide a proof of Proposition 183.19. To preserve the author's sanity we will skip a few technical calculations. We leave it to the reader to fill in the details.

[1945] We do not impose any other conditions, in particular neither do we assume that M is compact nor do we assume that $\partial M = \varnothing$.
[1946] We leave it to the reader to figure out what "natural" means in this context.

Throughout the proof let M be a non-empty n-dimensional smooth manifold and let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. First note that the case $n = 0$ is trivial, since every 0-dimensional smooth manifold and every 0-dimensional pseudomanifold is orientable. Thus in the following we assume that $n > 0$. Before we do anything else we recall and introduce a few definitions and we make a few basic comments regarding these definitions.

(a) Let $k \in \mathbb{N}_0$. In Definition 143.8 we introduced

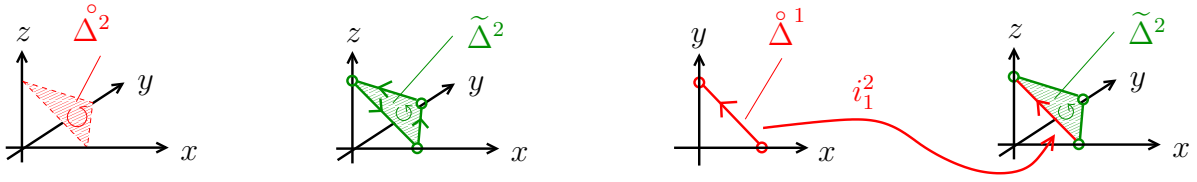
$$\mathring{\Delta}^k \;:=\; \{(t_0, \dots, t_k) \in \Delta^k \,|\, \text{all coordinates are non-zero}\}.$$

Now we also define

$$\widetilde{\Delta}^k \;:=\; \{(t_0, \dots, t_k) \in \Delta^k \,|\, \text{at least two coordinates are non-zero}\}.$$

(b) Similar to the discussion on Definition 143.8 we view $\mathring{\Delta}^k$ and $\widetilde{\Delta}^k$ in the obvious way[1947] as smooth manifolds. Note that $\mathring{\Delta}^k$ has no boundary and that the boundary of $\widetilde{\Delta}^k$ equals $\widetilde{\Delta}^k \cap \partial\Delta^k$. We equip the smooth manifolds $\mathring{\Delta}^k$ and $\widetilde{\Delta}^k$ with the orientation where for each $P \in \mathring{\Delta}^k$ a basis $v_1, \dots, v_k$ of $\mathrm{T}_P\mathring{\Delta}^k$ is positive if the vectors $\{(1, \dots, 1), v_1, \dots, v_k\}$ are a positive basis for $\mathbb{R}^{k+1}$. As always we endow the boundary $\partial\widetilde{\Delta}^k$ of the oriented smooth manifold $\widetilde{\Delta}^k$ with the orientation given by the Boundary Orientation Lemma 31.38.

(c) Note that for any $j \in \{0, \dots, k\}$ the usual j-th face map $i^k_j\colon \Delta^{k-1} \to \Delta^k$ restricts to an embedding $i^k_j\colon \mathring{\Delta}^{k-1} \to \partial\widetilde{\Delta}^k \subseteq \widetilde{\Delta}^k$. Using the Manifold–Orientation Basics Lemma 31.31 (2) it is straightforward to verify that the map $i^k_j\colon \mathring{\Delta}^{k-1} \to \partial\widetilde{\Delta}^k$ is orientation-preserving if and only if j is even.



(d) Let $k \in \mathbb{N}_0$ and let $Y = (v_0, \dots, v_k)$ be an ordered k-simplex of $K = (V, S)$. In Definition 174.18 we introduced the corresponding characteristic map

$$\begin{aligned} \Phi^{\leq}_Y\colon \Delta^k \;=\; \left\{(t_0, \dots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \,\middle|\, \sum_{i=0}^{k} t_i = 1\right\} &\to |K| \\ (t_0, \dots, t_k) &\mapsto \sum_{i=0}^{k} t_i \cdot v_i. \end{aligned}$$

(e) Note that it follows almost immediately from the definition of a "smooth" simplicial structure, see Definition 179.3, that for every ordered k-simplex $Y = (v_0, \dots, v_k)$ the map $\Theta \circ \Phi^{\leq}_Y\colon \widetilde{\Delta}^n \to M$ is an immersion.

Now we turn to the actual proof of the proposition.

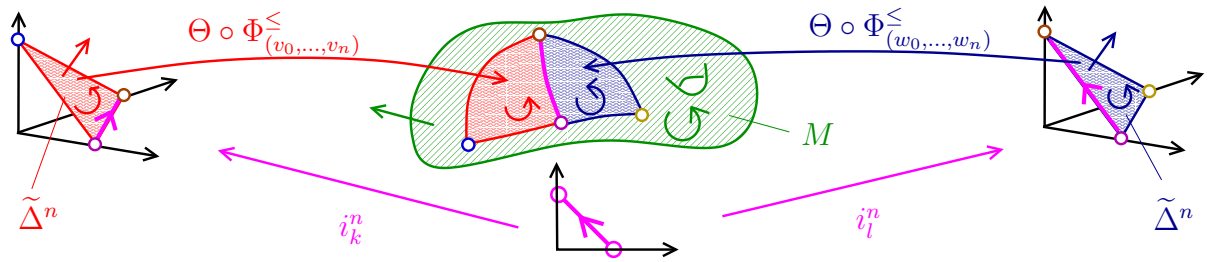
First we assume that we are given an orientation of the smooth manifold M. We need to equip the pseudomanifold $K = (V, S)$ with a natural orientation. We say that an ordered n-simplex $Y = (v_0, \dots, v_n)$ is *positive* if the map $\Theta \circ \Phi^{\leq}_Y\colon \mathring{\Delta}^n \to M$ is orientation-preserving

[1947] Slightly more precisely, a smooth atlas for $\mathring{\Delta}^k$ is given by projection to the first k coordinates, and for $\widetilde{\Delta}^k$ we obtain a smooth atlas by the same projection and an affine linear map that sends points on a component of $\widetilde{\Delta}^k \cap \partial\Delta^k$ into the $(k-1)$-hyperplane.

in the sense of Definition 31.29. Since $n \geq 1$ we see easily that, by possibly swapping the order of two vertices, we can find for each n-simplex $s \in S$ a positive ordered n-simplex Y_s with $\lfloor Y_s \rfloor = s$. It remains to prove the following claim:

Claim 1. The oriented n-simplices $[(+1, Y_s)]$ form an orientation of the pseudomanifold K.

Proof. Let e be an $(n-1)$-simplex of K of order 2 and let s, t be the two n-simplices that cobound e. By the definition of an orientation of a pseudomanifold (see Definition 183.10) we need to show that the orientations $[(+1, Y_s)]$ and $[(+1, Y_t)]$ induce opposite orientations on e. We write $Y_s = (v_0, \ldots, v_n)$ and we write $Y_t = (w_0, \ldots, w_n)$. Let v_k be the vertex missing in e and let w_l be the vertex missing in e.



We consider the maps

$$\mathring{\Delta}^{n-1} \xrightarrow{i^n_k} \widetilde{\Delta}^n \xrightarrow{\Theta \circ \Phi^{\leq}_{(v_0,\ldots,v_n)}} M \xleftarrow{\Theta \circ \Phi^{\leq}_{(w_0,\ldots,w_n)}} \widetilde{\Delta}^n \xleftarrow{i^n_l} \mathring{\Delta}^{n-1}.$$

The two maps $\mathring{\Delta}^{n-1} \to M$ are immersions and the images coincide, thus we obtain a self-diffeomorphism of $\mathring{\Delta}^{n-1}$. A little bit of thought, using (c) and using the Manifold–Orientation Basics Lemma 31.31 (2), shows that this map is orientation-preserving if and only if $k + l + 1$ is even.[1948] It follows from this discussion and the above Observation (c) that the orientations $[(+1, Y_s)]$ and $[(+1, Y_t)]$ induce opposite orientations on e. ⊞

Next we assume that we are given an orientation of the pseudomanifold K. We need to associate to it an orientation of the smooth manifold M. Thus suppose that we are given an orientation $\{(\mu_s, Y_s)\}_{s \in S}$ of the pseudomanifold K. Since $n > 0$ we can and will assume that for each n-simplex s we have $\mu_s = 1$.

Claim 2. The smooth manifold $M \setminus \Theta(K^{n-2})$ has a natural orientation.[1949]

Proof. Let $P \in M \setminus \Theta(K^{n-2})$. Since every $(n-1)$-simplex is the face of an n-simplex we see that there exists an n-simplex s with $P \in \Theta(|s|)$. Since $P \notin \Theta(K^{n-2})$ we see that there exists a $Q \in \widetilde{\Delta}^n$ such that $P = \Phi^{\leq}_{Y_s}(Q)$. Since $\Phi^{\leq}_{Y_s} \colon \widetilde{\Delta}^n \to M$ is an immersion we know that the induced map $\mathrm{T}_Q \widetilde{\Delta}^n \to \mathrm{T}_P M$ is an isomorphism. Thus it makes sense to equip $\mathrm{T}_P M$ with the image of the positive orientation of Δ^n under the above isomorphism. We need to show that the definition of the orientation does not depend on the choice of s. If $P \in M \setminus \Theta(K^{n-1})$, then we know from the Open Simplex Lemma 174.28 that s is unique, hence we are done. Otherwise there exists an $(n-1)$-simplex e such that $P \in \Theta(\langle e \rangle)$. If the order of e equals one, then s is again unique and again we are done. Finally if the

[1948] Also note that the fact that $\mathring{\Delta}^n$ is dense in $\widetilde{\Delta}^n$ implies that the map $\Theta \circ \Phi^{\leq}_{Y} \colon \widetilde{\Delta}^n \to M$ is also orientation-preserving.

[1949] Note that by the Simplicial Subcomplex Lemma 174.23 we know that $M \setminus \Theta(K^{n-2})$ is an open subset of M, in particular it is a smooth submanifold of M.

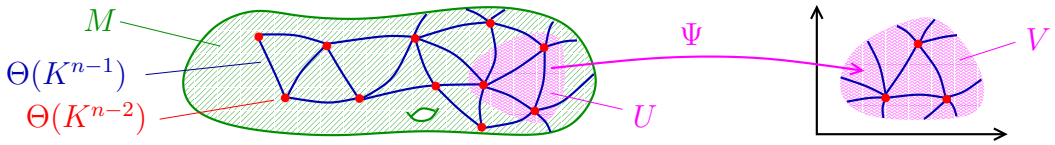
order of e equals two, then a reasonably elementary calculation (using the hypothesis that $\{(\mu_s, Y_s)\}_{s\in S}$ is an orientation of K), shows that the orientation on $\mathrm{T}_P M$ does not depend on the choice of the cobounding n-simplex. Finally we note that it is fairly straightforward to show that these orientations of the tangent spaces define an orientation, in the sense of Definition 31.19, of the smooth submanifold $M \setminus \Theta(K^{n-2})$. ⊞

We move on to the next claim.

Claim 3. For every open connected subset U of M the following holds:
(a) The set $U \cap (M \setminus \Theta(K^{n-2}))$ is connected.
(b) The closure of $U \cap (M \setminus \Theta(K^{n-2}))$ equals U.

Proof. For $i = 0, \ldots, n-2$ we set $N_i = U \cap (M \setminus \Theta(K^{i-1}))$ and we set $Y_i = \Theta(K^i) \cap N_i$. Note that for each $i \in \{1, \ldots, n-2\}$ we have $N_i = N_{i-1} \setminus Y_{i-1}$. It follows easily from the definition of a smooth simplicial structure together with the Simplicial Subcomplex Lemma 174.23 that each N_i is an n-dimensional smooth manifold and that each Y_i is a proper i-dimensional smooth submanifold of N_i. Since $i \leq n-2$ we obtain iteratively from the Complement-of-Submanifold Connected Corollary 50.11 and the hypothesis that $N_0 = U$ is path-connected that each N_i is path-connected. Finally note that we know by Exercise 25.14 and the fact that $\dim(Y_i) = i < n = \dim(N_i)$ that the closure of $N_i \setminus Y_i$ is N_i. Note that the statements of Claim 3 follow almost immediately from the above considerations. ⊞

Finally let $P \in M$. It follows quite easily from Claim (3a) that we can find a chart $\Psi\colon U \to V$ around P such that V is a connected open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and such that Ψ is orientation-preserving for some $Q \in M \setminus \Theta(K^{n-2})$. Furthermore it follows from Claim (3a) and the Manifold–Orientation Basics Lemma 31.31 (2) that the restriction of Ψ to $U \setminus \Theta(K^{n-2})$ is orientation-preserving at every point.



Now we equip all $Q \in U$ with the orientation induced by Ψ and the standard orientation of $\mathbb{R}^n$ respectively $\mathbb{H}^n$. It follows easily from Claim (3b) that these orientations do not depend on the choice of Ψ and that these orientations of the tangent spaces define an orientation of the smooth manifold M.

In the above discussion we constructed natural maps

$$\{\text{orientations of the smooth manifold } M\} \leftrightarrow \{\text{orientations of the pseudomanifold } K\}$$

in both directions. It follows reasonably easily from the definitions that the two maps are inverses of one another, thus both maps are natural bijections. ■

For manifolds with non-empty boundary the following theorem contains lots of extra information.

Theorem 183.20. (Manifold-Top-H_*-Theorem) Let M be a connected non-empty n-dimensional smooth manifold. The following statements hold:

(1) We have an isomorphism as follows:

$$\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and compact,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If M is compact, then we also have an isomorphism as follows:

$$\mathrm{H}_{n-1}(M, \partial M; \mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable.} \end{cases}$$

In both cases the free abelian group is finitely generated.

(3) If M is compact, then we have $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.

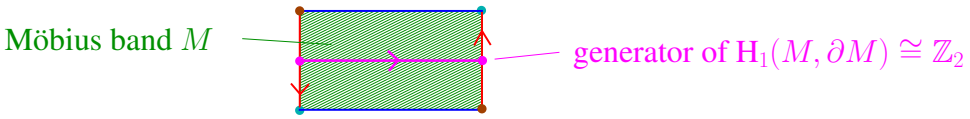
Remark.

(1) Note that if M is a connected non-empty smooth manifold with $\partial M = \varnothing$, then the statements of the Manifold-Top-H_*-Theorem 183.20 are contained in the Manifold-Top-H_*-Theorem 183.18.

(2) We will prove the generalization of the Manifold-Top-H_*-Theorem 183.18 to the setting of topological manifolds in the Manifold-Top-H_*-Theorem 193.1, the Top-$\mathbb{F}_2$-Homology Theorem 193.2 and the Manifold-H_{n-1}-Theorem 193.4. □

Example. Let M be the Möbius band. Recall that M is a compact non-orientable 2-dimensional smooth manifold. In Exercise 151.4 we already saw that

$$\mathrm{H}_k(M, \partial M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$



This calculation is of course consistent with the Manifold-Top-H_*-Theorem 183.20. □

Proof. Let M be a connected non-empty n-dimensional smooth manifold. We dealt with the case $\partial M = \varnothing$ in the Manifold-Top-H_*-Theorem 183.18 so we might as well assume that $\partial M \neq \varnothing$.

In the following we first sketch what we think should be the "correct" proof of the Manifold-Top-H_*-Theorem 183.20 (1). Afterwards we will give the full details for an alternative proof.

(a) As before one uses the Smooth Manifold-Simplicial Structure Theorem 179.5 to equip M with some smooth simplicial structure $(K, \Theta\colon |K| \to M)$. It follows from the Manifold-Simplex Proposition 183.2 that K is an n-dimensional pseudomanifold such that ∂M corresponds precisely to ∂K.

(b) By the Simplicial-Singular Homology Lemma 177.19 we have a natural isomorphism $\mathrm{H}_n^{\mathrm{simp}}(K, \partial K) \xrightarrow{\cong} \mathrm{H}_n(|K|, |\partial K|; \mathbb{Z})$.

(c) It is fairly straightforward to modify the argument of the Pseudomanifold Homology Theorem 183.12 to show that one has an isomorphism

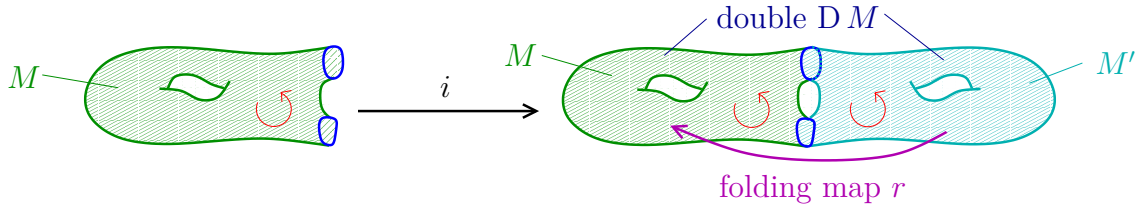
$$\mathrm{H}_n^{\mathrm{simp}}(K, \partial K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } K \text{ is orientable and finite,} \\ 0, & \text{otherwise.} \end{cases}$$

(d) By the Pseudomanifold-Orientation Proposition 183.19 we know that M is orientable if and only if K is orientable. The desired Statement (1) now follows from the combination of all of the above.

The above approach is surely convincing and it is relatively straightforward to work out the details. Furthermore the same approach can also be employed to settle Statements (2) and (3).

Now we turn to our second approach to proving the proposition. This second approach has the advantage that this time we can provide all the details. Given M we consider its double $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ where M' is a second copy of M. We refer to Definition 54.8 for details and we refer to the figure below for an illustration. Recall that by the Doubling-Manifold Lemma 54.10 we know the following:

(a) The double $\mathrm{D}M$ is also an n-dimensional smooth manifold such that $M \subseteq \mathrm{D}M$ is a smooth submanifold.
(b) The boundary of the double $\mathrm{D}M$ is empty.
(c) M is orientable if and only if $\mathrm{D}M$ is orientable.
(d) M is connected if and only if $\mathrm{D}M$ is connected.
(e) M is compact if and only if $\mathrm{D}M$ is compact.



In the following we denote by $i\colon M \to \mathrm{D}M$ the natural inclusion and we denote by $r\colon \mathrm{D}M \to M$ the folding map that we introduced in Definition 54.9. Given $k \in \mathbb{N}_0$ we consider the following maps:

$$\begin{array}{ccccccccccc} & & & & & & \mathrm{H}_k(M, \partial M) & & & & \\ & & & & & & \downarrow \cong & & & & \\ \ldots & \longrightarrow & \mathrm{H}_k(M) & \underset{r_*}{\overset{i_*}{\hookrightarrow}} & \mathrm{H}_k(\mathrm{D}M) & \longrightarrow & \mathrm{H}_k(\mathrm{D}M, M) & \xrightarrow{\partial} & \mathrm{H}_{k-1}(M) & \underset{r_*}{\overset{i_*}{\hookrightarrow}} & \mathrm{H}_{k-1}(\mathrm{D}M) \longrightarrow \ldots \end{array}$$

We make the following observations:

(f) The horizontal sequence is the long exact sequence in homology corresponding to the pair $(\mathrm{D}M, M)$.
(g) Since $r \circ i = \mathrm{id}_M$ we have $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_k(M)}$, which implies that $i_*\colon \mathrm{H}_k(M) \to \mathrm{H}_k(\mathrm{D}M)$ is a monomorphism.
(h) Recall that we know by the Doubling-Manifold Lemma 54.10 that M is a closed subset of $\mathrm{D}M$ and that M is a codimension-zero submanifold of $\mathrm{D}M$. Therefore it follows from the Manifold–Excision Theorem 148.3 that the vertical map is an isomorphism.

Next we recall some of our earlier calculations:

(i) Since $\partial M \neq \varnothing$, we obtain from the Manifold-Top-H_*-Theorem 183.18 that $\mathrm{H}_n(M) = 0$ and that $\mathrm{H}_{n-1}(M)$ is a free abelian group.

(j) By the Manifold-Top-H_*-Theorem 183.18 we have

$$\mathrm{H}_n(\mathrm{D}M) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } \mathrm{D}M \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(k) By the Manifold-Top-H_*-Theorem 183.18 and since $\partial(\mathrm{D}M) = \varnothing$ we have

$$\mathrm{H}_{n-1}(\mathrm{D}M) \;\cong\; \begin{cases} \text{free abelian group}, & \text{if } \mathrm{D}M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } \mathrm{D}M \text{ is non-orientable.} \end{cases}$$

Now we turn to the actual proofs of the three statements of the proposition.

(1) By the above we have an exact sequence

$$0 \longrightarrow \underbrace{\mathrm{H}_n(M)}_{=0} \underset{r_*}{\overset{i_*}{\hookrightarrow}} \mathrm{H}_n(\mathrm{D}M) \longrightarrow \underbrace{\mathrm{H}_n(\mathrm{D}M, M)}_{\cong \mathrm{H}_n(M,\partial M)} \xrightarrow{\partial} 0.$$

The desired statement now follows from (a)–(h) together with (i) and -(j).

(2) By the above we have an exact sequence:

$$0 \longrightarrow \underbrace{\mathrm{H}_{n-1}(M)}_{\text{free abelian}} \underset{r_*}{\overset{i_*}{\hookrightarrow}} \mathrm{H}_{n-1}(\mathrm{D}M) \longrightarrow \underbrace{\mathrm{H}_{n-1}(\mathrm{D}M, M)}_{\cong \mathrm{H}_{n-1}(M,\partial M)} \xrightarrow{\partial} 0.$$

Since $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(M)}$ we obtain from the Splitting Lemma 145.5 (3)⇒(2) and the above discussion that $\mathrm{H}_{n-1}(\mathrm{D}M) \cong \mathrm{H}_{n-1}(M, \partial M) \oplus$ free abelian group. The desired statement follows from (a)–(h) together with -(k) and the Cancellation Proposition 97.26.

(3) We can deduce this statement from the Manifold-Top-H_*-Theorem 183.18 (3) using a slight variation on the above arguments. Alternatively we can deduce (3) from (1) and (2) using the Universal Coefficient Theorem 162.16, the same way that we deduced the Manifold-Top-H_*-Theorem 183.18 (3) from the Manifold-Top-H_*-Theorem 183.18 (1) and (2). ■

183.3. **Homology groups of lens spaces.** In this section we will apply the results from the previous sections to complete the long overdue calculation of the homology groups of lens spaces.

Lemma 183.21. (Lens Space-H_*-Lemma) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. For the corresponding lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ (from Definition 91.4) we have

$$\mathrm{H}_n(L(p,q)) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n = 1. \end{cases}$$

Remark. An alternative approach to computing the homology groups of the lens spaces $L(p,q)$ will be given in Exercise **??** and later in the Generalized Lens Space-H_*-Proposition 306.7. □

The calculation of this lemma is rather disappointing, it shows that homology groups cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. In particular we have made no progress at all on Question 91.6.

Proof. We start out with the following observations:

(1) We already know from the H_0-Proposition 143.25 that $H_0(L(p,q)) \cong \mathbb{Z}$.
(2) By the Lens Space-H_1-Corollary 166.11 we know that $H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.
(3) In Definition 91.4 we pointed out that every lens space $L(p,q)$ is a closed orientable connected 3-dimensional smooth manifold.
(4) By the Smooth Manifolds-Invariants Proposition 179.10 we know that $H_n(L(p,q)) = 0$ for $n \geq 4$.
(5) By the Manifold-Top-H_*-Theorem 183.18 (1) we know that $H_3(L(p,q)) \cong \mathbb{Z}$.

Thus at the moment we are still missing $H_2(L(p,q))$. To obtain this homology group we make two more observations:

(6) By the Smooth Manifold-CW-Theorem 179.9 we know that $L(p,q)$ admits the structure of a finite CW-complex.
(7) It follows from the Action–Covering Proposition 91.11 and the definition of a lens space that $L(p,q)$ is finitely covered by S^3. It follows from $\chi(S^3) = 0$ together with the Covering-χ-Proposition 156.13 that $\chi(L(p,q)) = 0$.

Now note that

$$\operatorname{rank} H_2(L(p,q)) \overset{\text{by the Homology-}\chi\text{-Proposition 156.2}}{=} \underbrace{\chi(L(p,q))}_{=0} - \underbrace{\operatorname{rank} H_0(L(p,q))}_{=1} + \underbrace{\operatorname{rank} H_1(L(p,q))}_{=0} + \underbrace{\operatorname{rank} H_3(L(p,q))}_{=1} = 0.$$

We know by the Manifold-Top-H_*-Theorem 183.18 (2) that $H_2(L(p,q))$ is a finitely generated free abelian group. Therefore we see that $H_2(L(p,q)) = 0$. ∎

183.4. **The Poincaré Homology Sphere.** We start out with the following almost self-explanatory definition of a homology sphere.

Definition 183.22.

(1) We say that an n-dimensional topological manifold X is a topological homology n-sphere if for every $k \in \mathbb{N}_0$ we have $H_k(X;\mathbb{Z}) \cong H_k(S^n;\mathbb{Z})$. Sometimes we also just say homology n-sphere.
(2) Sometimes, for disambiguation we also refer to such a topological manifold as an integral homology n-sphere.
(3) If X is actually a smooth manifold, then we refer to X as a smooth homology n-sphere.

Lemma 183.23. (Homology Sphere Lemma) Let M be a smooth manifold. If M is a homology n-sphere, then M is closed, orientable and connected.

Proof. The lemma is an immediate consequence of the H_0-Proposition 143.25 and the Manifold-Top-H_*-Theorem 183.18. ∎

It follows from the Topological 1-Manifold Classification Theorem 28.1 and the Surface Classification Theorem 101.16 and the calculation of homology groups in the Surface-H_*-Proposition 155.17 that for $n = 1, 2$ every homology n-sphere is homeomorphic to the standard sphere S^n. In 1900 Henri Poincaré [**Poi1900**],[**PGL1953**, p. 370] claimed that

the same statement also holds for $n = 3$, i.e. he claimed that every homology 3-sphere is in fact homeomorphic to S^3.[1950] Shortly afterwards, in 1904, Henri Poincaré [**Poi1904, PGL1953**] himself found a counterexample to this claim. More precisely, he proved the following proposition.

Proposition 183.24. (Poincaré Homology Sphere Proposition) There exists a 3-dimensional smooth manifold M that is a homology 3-sphere, but such that $\pi_1(M)$ is a non-trivial group.

Remark. In Exercise 183.17 and in the Homology Sphere-Existence Proposition 256.9 we will prove the higher-dimensional analog of the Poincaré Homology Sphere Proposition 183.24. □

Now we turn to the proof of the Poincaré Homology Sphere Proposition 183.24. By now there are many proofs. In the following we will give Poincaré's example of such a smooth manifold. Later on in Exercise 185.5 we will give a different proof of the Poincaré Homology Sphere Proposition 183.24.

To construct Poincaré's example we need to recall a few definitions and results from earlier on.

Definition 183.25.

(1) Let $z := 1+\frac{\sqrt{5}}{2}$ be the golden ratio. As on page 3013 let D be the regular dodecahedron that is the convex hull of the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

Furthermore let $\operatorname{Sym}(D)$ be the subgroup of $\mathrm{SO}(3)$ that preserves the regular dodecahedron, i.e. that preserves the set of these 20 vertices.

(2) Let $T := \{A \in \mathrm{M}(2\times 2, \mathbb{C}) \,|\, A = \overline{A}^{\mathrm{T}} \text{ and } \operatorname{tr}(A) = 0\}$. This is a 3-dimensional real vector space which we equip with the basis that is given by the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Furthermore we equip T with the form $(A, B) \mapsto g(A, B) := \frac{1}{2}\operatorname{tr}\left(A\overline{B}^{\mathrm{T}}\right)$. Next recall that on page 1824 we showed that the map $T \to \mathbb{R}^3$ given by $E_i \to e_i$ defines an isometry $(T, g) \to (\mathbb{R}^3, \langle\ ,\ \rangle)$ which we now use to make the identification $\mathrm{SIsom}(T, g) = \mathrm{SO}(3)$. Furthermore, in the SU(2)-Action Lemma 94.2 we showed that the map

$$\begin{array}{rccl} q\colon & \mathrm{SU}(2) & \to & \mathrm{SIsom}(T, g) = \mathrm{SO}(3) \\ & A & \mapsto & (B \mapsto ABA^{-1}) \end{array}$$

is a well-defined group homomorphism, that it is an epimorphism and that the kernel is given by $\pm\,\mathrm{id}$.

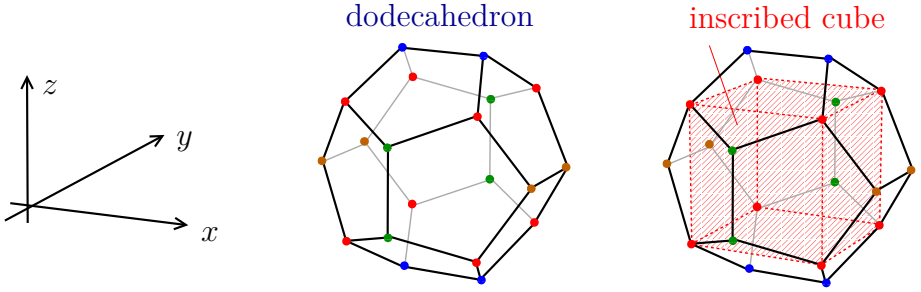
(3) We make the identification $S^3 =_{\mathrm{i}} \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$, i.e. we view S^3 as the set of quaternions of length 1. In the Quaternion Unit Proposition 171.5 (2) we showed that

[1950] In fact in [**PGL1953**, p. 498] he wrote "Pour ne pas trop allonger ce travail, je me bornerai à énoncer le théorème suivant dont la démonstration demanderait quelques développements: Tout polyèdre qui a tous ses nombres de Betti égaux à 1 si et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire homéomorphe à l'hypersphère."

the following map is a diffeomorphism and a group isomorphism:

$$\begin{aligned} f\colon S^3 &\to \mathrm{SU}(2) \\ (z,w) = z + \mathrm{j}\cdot w &\mapsto \underbrace{\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}}_{\substack{\text{matrix that represents} \\ \text{left-multiplication by } z+\mathrm{j}\cdot w \text{ with} \\ \text{respect to the complex basis } \{1,\mathrm{j}\} \text{ of } \mathbb{H}}} \end{aligned}$$

(4) We refer to $\Gamma \coloneqq (q\circ f)^{-1}(\mathrm{Sym}(D)) \subseteq S^3$ as the binary icosahedral group.



Remark. It might be a little confusing that we refer to Γ as the binary *icosahedral* group. As we discussed on page 3013, the symmetry group of a regular icosahedron I equals the symmetry group of a regular dodecahedron D. Therefore in the above definition of the binary icosahedral group we could replace $\mathrm{Sym}(D)$ by $\mathrm{Sym}(I)$, which is more logical and more common. But since it is a little easier to describe $\mathrm{Sym}(D)$ geometrically, we prefer to work with the regular dodecahedron. □

Proposition 183.26. (Poincaré Homology Sphere Proposition)

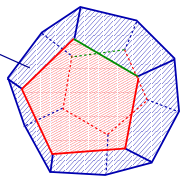
(1) The binary icosahedral group Γ is a group with 120 elements.
(2) The binary icosahedral group Γ has trivial abelianization.
(3) The quotient $M \coloneqq S^3/\Gamma$ is a smooth homology 3-sphere with non-trivial fundamental group.

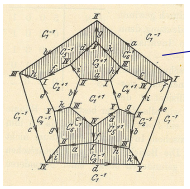
Definition 183.27. The smooth manifold S^3/Γ is called the Poincaré Homology Sphere.

Remark. The Poincaré Homology Sphere is ubiquitous in topology and can be described in many different ways. The different avatars and also the history of the Poincaré Homology Sphere are discussed in detail in [**SeW1933**], [**Rolf1990**, Chapter 10.E], [**Vol2013**] and [**KSc1979**]. In fact [**KSc1979**] provides altogether *eight* different ways to construct the Poincaré Homology Sphere. In the following we outline a few of these alternative descriptions of the Poincaré Homology Sphere.

(1) In 1904, when Henri Poincaré discovered the eponymous sphere, he gave a different description in terms of gluing two "handlebodies". It is shown in [**SeW1933**] that this topological space is homeomorphic to the topological space we describe next in (2).
(2) Let D be the "solid" regular dodecahedron. Now identify each pentagon on the boundary with the opposite pentagon via a $\pi/5$ twist in the "clockwise" direction. As is explained in [**KSc1979**], the resulting topological space is in fact homeomorphic to the Poincaré Homology Sphere as we defined it above.

dodecahedron





description of the Poincaré homology sphere in [**ST1931**]

(3) In the Poincaré-$\mathbb{Z}$HS^3–E_8-Manifold Proposition 259.17 and on page **??** we will give two descriptions of the Poincaré Homology Sphere as the result of a "surgery" on a knot respectively a link. □

Proof.

(1) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. Recall that D is the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

In the figures on page 3013 and on page 3490 we show that one can inscribe five cubes $C_1, \dots, C_5$ into the regular dodecahedron. As we already mentioned on page 3013, it is shown in [**Arm1988**, p. 40] that the map

$$\begin{array}{rcl} \mathrm{Sym}(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \dots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

is an isomorphism. We consider the maps : $\mathrm{SU}(2) \to \mathrm{SIsom}(T, g) = \mathrm{SO}(3)$ and $f\colon S^3 \to \mathrm{SU}(2)$ that we defined above and we recall that the binary icosahedral group is defined as $\Gamma := (q \circ f)^{-1}(\mathrm{Sym}(D)) \subseteq S^3$. As we mentioned above, in the SU(2)-Action Lemma 94.2 we showed that $\ker(q \circ f) = \pm 1 \in S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$. Furthermore it follows from the SU(2)-SO(3)-Covering Theorem 94.3 (1) that $q \circ f$ is an epimorphism. Since $\mathrm{Sym}(D) \cong A_5$ has $\frac{1}{2}5! = 60$ elements and since $q \circ f$ has a kernel with two elements we see that Γ is a group with 120 elements.

(2) We need to show that the abelianization of Γ is trivial.[1951] By (1) we know that the group $\Gamma \subseteq S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ fits into a short exact sequence

$$1 \to \{\pm 1\} \xrightarrow{\rho} \Gamma \xrightarrow{q \circ f} \mathrm{Sym}(D) \to 1.$$

It follows from the Abelianization Functor Proposition 98.16, together with the Abelianization-Epimorphism Lemma 98.17, that we get an induced exact sequence

$$\{\pm 1\} \xrightarrow{\rho} \Gamma_{\mathrm{ab}} \xrightarrow{q \circ f} (\mathrm{Sym}(D))_{\mathrm{ab}} \to 1.$$

Almost every algebra book worth its money shows that $A_5 \cong \mathrm{Sym}(D)$ is a simple group, see e.g. [**Bog2008**, Theorem 11.3]. This fact implies that the abelianization of $\mathrm{Sym}(D)$ is trivial. Thus it suffices to prove the following claim:

Claim. The quaternion $-1 \in S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ represents the trivial element in Γ_{ab}.

[1951] An argument similar to our proof of (2) is also provided in [**Bre1993**, p. 354] and in [**Toen2017**, p. 133].

Proof. First recall that a straightforward explicit calculation, which you and I carried out in Exercise 94.3 and in Definition 171.2, shows that

$$(q\circ f)(\mathrm{i}) = q\begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}}_{=:A} \text{ and } (q\circ f)(\mathrm{j}) = q\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \underbrace{\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{=:B}.$$

It follows immediately from the explicit description of the vertices of the regular dodecahedron D in (1) that the two matrices A and B to the right actually lie in $\mathrm{Sym}(D)$. Next note that it follows from the above calculation, that $\mathrm{i} = (q\circ f)^{-1}(A)$ and $\mathrm{j} = (q\circ f)^{-1}(B)$. In particular we see that $\mathrm{i},\mathrm{j}\in(q\circ f)^{-1}(\mathrm{Sym}(D)) = \Gamma$. By definition of the quaternion multiplication in Definition 171.2 we have

$$[\mathrm{i},\mathrm{j}] \;=\; \underbrace{\mathrm{i}\cdot\mathrm{j}}_{=-\mathrm{j}\cdot\mathrm{i}}\cdot\mathrm{i}^{-1}\cdot\mathrm{j}^{-1} \;=\; -\mathrm{j}\cdot\mathrm{i}\cdot\mathrm{i}^{-1}\cdot\mathrm{j}^{-1} \;=\; -1.$$

We have thus shown that -1 is a commutator of elements in Γ. By definition of Γ_{ab} this implies that -1 represents indeed the trivial element in Γ_{ab}. ⊞

(3) We point out that it follows from the Quaternion Unit Proposition 171.5 (2) that $\Gamma \coloneqq (q\circ f)^{-1}(\mathrm{Sym}(D))$ is a finite subgroup of the *group* $S^3 = \{z\in\mathbb{H}=\mathbb{R}^4 \,|\, |z|=1\}$. Since subgroups act freely on groups, we see that Γ acts freely on S^3. Since Γ is a finite subgroup, it now follows from the Manifold–Quotient Proposition 25.39 that $M \coloneqq S^3/\Gamma$ is a closed 3-dimensional smooth manifold. It remains to show that $M = S^3/\Gamma$ is a homology sphere. First we perform the following calculation:

$$\mathrm{H}_1(S^3/\Gamma) \overset{\text{by the Hurewicz Theorem 166.10}}{\cong} \pi_1(S^3/\Gamma)_{\mathrm{ab}} \overset{\text{by the Sphere-}\pi_1\text{-Proposition 88.24 and the Action–}\pi_1\text{-Theorem 93.1}}{\cong} \Gamma_{\mathrm{ab}} \overset{\text{by (2)}}{=} 0.$$

Next we point out that by the Smooth Manifolds-Invariants Proposition 179.10 we have $\mathrm{H}_i(M) = 0$ for $i\geq 4$. Furthermore note that by construction Γ is contained in $\mathrm{SU}(2)$. Thus we see that the action of Γ on S^3 is orientation-preserving.[1952] Therefore it follows from the Orientation-Action Proposition 31.44 that $M = S^3/\Gamma$ is actually orientable. This implies in turn by the Manifold-Top-H_*-Theorem 183.18 that $\mathrm{H}_3(M)\cong\mathbb{Z}$. Thus it remains to show that $\mathrm{H}_2(M) = 0$. By the Manifold-Top-H_*-Theorem 183.18 (2) we know that $\mathrm{H}_2(M)$ is a free abelian group. So it suffices to show that $b_2(M) = 0$. In fact we have

$$b_2(M) \overset{\text{since } b_0(M)=b_3(M)=1 \text{ and } b_1(M)=0}{=} \chi(M) \overset{\text{by the Covering-}\chi\text{-Proposition 156.13 and the Smooth Manifold-CW-Theorem 179.9}}{=} \frac{1}{120}\cdot\chi(S^3) \overset{\text{see page 2987}}{=} 0.$$

■

Remark. Let M be a homology 3-sphere with *finite* fundamental group. As is pointed out in [**Kerv1969**, p. 67], it follows from the work of Michio Suzuki [**Suz1955**] or alternatively

[1952] Since the action of Γ is free, it follows from Exercise 9.1 that the action is orientation-preserving.

John Milnor [**Miln1957b**] that $\pi_1(M)$ is isomorphic to the binary icosahedral group. In fact, it follows from the much more fancy Elliptisation Theorem **??** that such M is actually diffeomorphic to the Poincaré Homology Sphere. □

We conclude this section with the statement of a conjecture that was formulated by Henri Poincaré [**PGL1953**, p. 498] [**Poi1904**].

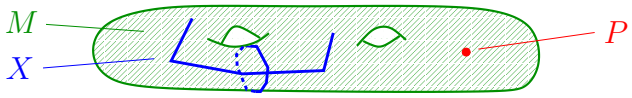
Conjecture 183.28. (Poincaré Conjecture) If M is a homology 3-sphere that is simply connected, then M is homeomorphic to S^3.

We will discuss the Poincaré Conjecture and various variations thereof in **??**.

183.5. **Deformation retractions of closed smooth manifolds.** In this section we will use the Manifold-Top-H_*-Theorem 183.18 to prove some general results. We start out with the following proposition:

Proposition 183.29. (Manifold-Non Deformation Retraction Proposition) Let M be a closed connected smooth manifold. There is no proper subspace $X \subsetneq M$ which is a deformation retract of M.

Proof. Let M be a closed connected n-dimensional smooth manifold and let $X \subsetneq M$ be a proper subspace. This means that there exists a $P \in M \setminus X$. Let $i\colon X \to M \setminus \{P\}$ and $j\colon M \setminus \{P\} \to M$ be the inclusions.



If X was a deformation retract, then it would follow from the Deformation Retract–Homotopy Equivalence Lemma 19.10 that $j \circ i\colon X \to M$ would be a homotopy equivalence. This in turn would, by the Homotopy Equivalence-H_*-Proposition 144.14 (1a), imply that the induced map $(j \circ i)_*\colon \mathrm{H}_n(X;\mathbb{F}_2) \to \mathrm{H}_n(M;\mathbb{F}_2)$ would be an isomorphism. In other words, to prove the proposition it suffices to prove the following claim:

Claim. The map $(j \circ i)_*\colon \mathrm{H}_n(X;\mathbb{F}_2) \to \mathrm{H}_n(M;\mathbb{F}_2)$ is *not* an isomorphism.

Proof. By functoriality we have the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow{\quad (j\circ i)_* \quad} & \mathrm{H}_n(M;\mathbb{F}_2) \\ & {\scriptstyle i_*}\searrow \quad \nearrow{\scriptstyle j_*} & \\ & \mathrm{H}_n(M\setminus\{P\};\mathbb{F}_2). & \end{array}$$

We make the following observations:

(1) Since M is a closed connected and non-empty n-dimensional smooth manifold, we know by the Manifold-Top-H_*-Theorem 183.18 (3) that $\mathrm{H}_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$.
(2) Note that it follows from Exercise 23.4 that $M \setminus \{P\}$ is a connected *non-compact* non-empty n-dimensional smooth manifold. Thus it follows from the Manifold-Top-H_*-Theorem 183.18 (3) that $\mathrm{H}_n(M \setminus \{P\};\mathbb{F}_2) = 0$.
(3) The diagram commutes by the functoriality of homology.

It follows from the above that the top map cannot be an isomorphism. ■

Recall that in the Matrix Groups Deformation Retract Proposition 19.4 we showed that $\mathrm{SO}(n)$ is a deformation retract of $\mathrm{GL}_+(n,\mathbb{R})$ and in Question 19.6 we asked whether

$\mathrm{GL}_+(n,\mathbb{R})$ admits a "smaller deformation" retract, i.e. we asked whether $\mathrm{SO}(n)$ admits a proper subspace that is a deformation retract of $\mathrm{SO}(n)$. The following corollary to the Manifold-Non Deformation Retraction Proposition 183.29 now gives a negative answer to Question 19.6:

Corollary 183.30. ($\mathrm{SO}(n)$-No Retraction Corollary) The matrix group $\mathrm{SO}(n)$ does not admit a proper subspace that is a deformation retract.

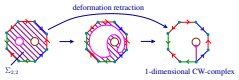
Proof. By the Matrix Group-Smooth Manifold Proposition 32.10 and the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n)$ is a closed connected smooth manifold. Thus the desired statement follows from the Manifold-Non Deformation Retraction Proposition 183.29. ■

183.6. **The topology of smooth manifolds.** In this section we provide two results on the topology of smooth manifolds. Let us formulate our first result.

Proposition 183.31. (No Top-Cell Proposition) Let M be a connected non-empty n-dimensional smooth manifold. If M is non-compact or if M has non-empty boundary, then M is homotopy equivalent to an $(n-1)$-dimensional simplicial complex, in particular M is homotopy equivalent to an $(n-1)$-dimensional CW-complex (which can be assumed to be compact if M is compact).

Examples.

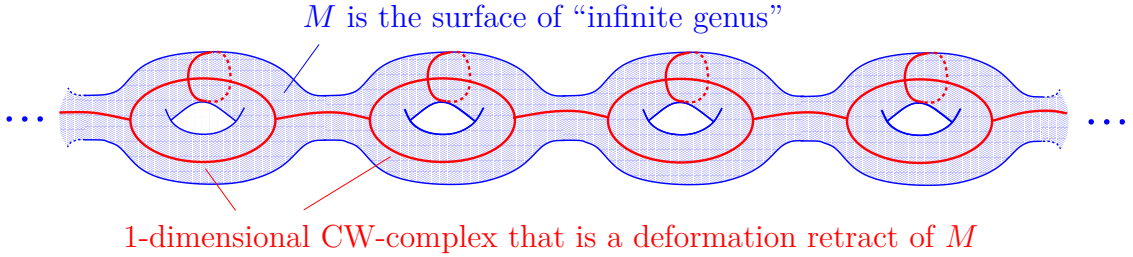
(1) Given $g \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we denote as usual by $\Sigma_{g,k}$ the surface of genus g with k open disks removed. In the figure below we sketch a deformation retraction from $\Sigma_{2,2}$ to a CW-complex of Euler characteristic -4.



More generally, almost the same argument shows that $\Sigma_{g,k}$ with $k \geq 1$ admits a deformation retraction to a CW-complex with k 0-cells and $2g + 2(k-1)$ 1-cells, i.e. a 1-dimensional CW-complex of Euler characteristic $k - (2g + 2(k-1)) = -2g - k + 2$. Using the 1-Complex-$\pi_1$-Theorem 137.6 we also see that $\Sigma_{g,k}$ is homotopy equivalent to the wedge of $2g + k - 1$ circles.

(2) The non-compact smooth manifold that is given by removing a point from the torus is easily seen to be homotopy equivalent to the wedge of two circles.

(3) In the figure below we show the surface M of infinite genus, which is non-compact, and a 1-dimensional CW-complex that is deformation retract of M.



It is admittedly not overly obvious why the 1-dimensional CW-complex is indeed a deformation retract of M. □

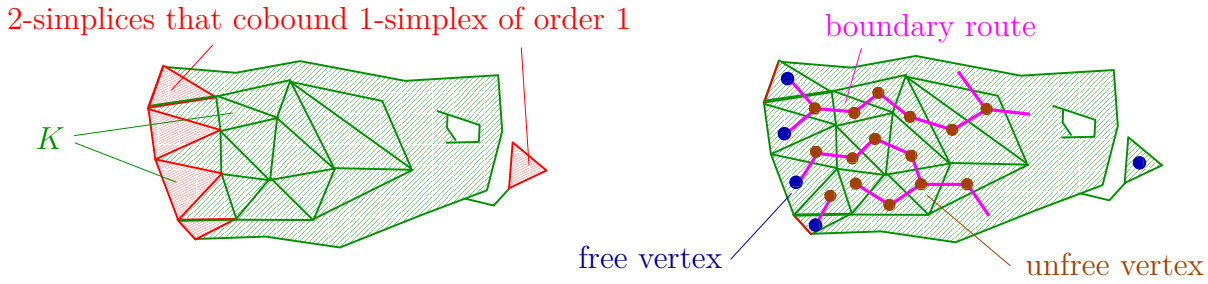
Remark. Let M be a compact connected n-dimensional smooth manifold with non-empty boundary. By the No Top-Cell Proposition 183.31 we now know that M is homotopy equivalent to a finite $(n-1)$-dimensional CW-complex which implies by the CW-Complex-H_*-Proposition 155.10 (2) and (3) that $\mathrm{H}_n(M) = 0$ and that $\mathrm{H}_{n-1}(M)$ is a finitely generated free abelian group. This is of course consistent with the Manifold-Top-H_*-Theorem 183.18. □

The proof of the No Top-Cell Proposition 183.31 requires some preparations.

Definition 183.32. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex such that every $(n-1)$-simplex has order ≤ 2. We denote by W the set of n-simplices and we denote by E the set of $(n-1)$-simplices.

(1) We say that an n-simplex is free if it admits a codimension one face of order 1. Otherwise we say that the n-simplex is unfree.

(2) A boundary route is a subgraph $\widetilde{\Gamma} = (W, \widetilde{F}, \varphi)$ of the dual graph[1953] $\Gamma = (W, E_2, \varphi)$ with the following property: every n-simplex of K, i.e. every vertex of the dual graph, is connected in $\widetilde{\Gamma}$ to a free n-simplex.



Lemma 183.33. (Boundary Route Lemma) Let $K = (V, S)$ be an n-dimensional abstract simplicial complex that admits a boundary route. There exists a subcomplex $K' = (V', S')$ of $K = (V, S)$ with the following three properties:

(1) $K' = (V', S')$ admits a boundary route.

(2) $|K'|$ is a deformation retract of $|K|$.

(3) The number of n-simplices of K' is less than the number of n-simplices of K.

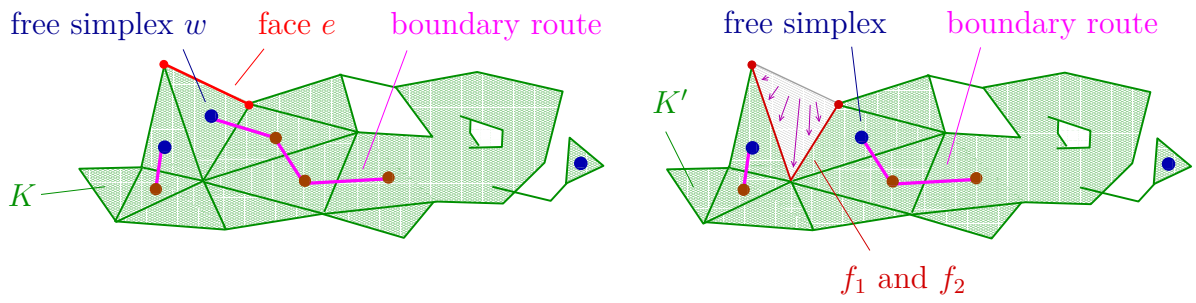
Proof of Lemma 183.33. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex that admits a boundary route $\widetilde{\Gamma} = (W, \widetilde{F}, \varphi)$. Since K is n-dimensional we know that it contains at least one n-simplex and since K admits a boundary route we see that there exists an n-simplex w that is free. By definition this means that w admits a codimension-one face e of order 1. We denote by $f_1, \ldots, f_n$ the remaining codimension-one faces of w. We set

$$K' \coloneqq (V, S \setminus \{w, e\}).$$

Note that K' is indeed a subcomplex of K. A slight generalization of Exercise 19.24 (b) shows that there exists a deformation retraction from $|w|$ to $\bigcup_{i=1}^{n} |f_i|$.

It follows from the Simplicial Subcomplex Lemma 174.23 together with the Homotopy Combination Lemma 18.8 (1) that this deformation retraction together with the stationary homotopy on $|K'|$ defines a deformation retraction from $|K|$ to $|K'|$.

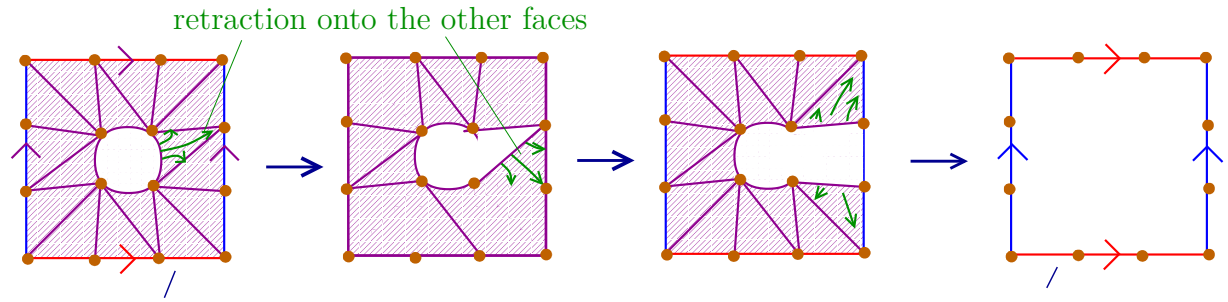
[1953] We refer to Definition 183.14 for the definition of the dual graph.



Since every $(n-1)$-simplex of K has order ≤ 2 we see see that every n-simplex of K' that contains one of the f_i as a face, is in fact a free n-simplex of K'. Using this observation it is now rather straightforward to see that a boundary route for the abstract simplicial complex K' is given by $(W \setminus \{w\}, \{\widetilde{f} \in \widetilde{F} \,|\, \widetilde{f} \not\subseteq w\}, \varphi)$. ■

Proof of the No Top-Cell Proposition 183.31 in the compact case. Let M be a compact connected non-empty n-dimensional smooth manifold with non-empty boundary. Recall that by the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$ such that ∂M is a subcomplex. We need to show that $|K|$ is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.

Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. By the Manifold-Simplex Proposition 183.2 (5) we know that Γ is connected. Thus it follows from the Spanning Tree–Existence Proposition 11.26 that Γ admits a spanning tree T. By definition T is connected and T contains all vertices. Since $\partial M \neq \varnothing$ we know by the Manifold-Simplex Proposition 183.2 that $\partial K \neq \varnothing$.



Thus we see that T is in fact a boundary route for K. This means that we can start up the machine given by the Boundary Route Lemma 183.33. Iteratively applying the Boundary Route Lemma 183.33 we get the desired deformation retraction. ■

Proof of the No Top-Cell Proposition 183.31 in the non-compact case. Let M be a non-compact connected n-dimensional smooth manifold. It follows from [**NR2004**, Theorem 2.2], or alternatively [**WhdJ1961b**, Lemma 2.1], together with the CW–Simplicial Homotopy Type Proposition 175.21 that M is homotopy equivalent to an $(n-1)$-dimensional simplicial complex.

We will not attempt to give a full proof of the proposition of our own, but we will sketch an outline of a proof. Evidently we again use the fact that by the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that we can equip M with a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. It is fairly straightforward to show that there exists a sequence $Y_1, Y_2, \dots$ of non-empty subcomplexes of K with the following properties:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $Y_i \subseteq Y_{i+1}$.
(2) Each Y_i is a finite n-dimensional pseudomanifold.
(3) We have $\bigcup_{i\in\mathbb{N}} Y_i = K$.

Now, with a considerable effort one can modify the above arguments used in the compact case to show that there exists a sequence of $(n-1)$-dimensional subcomplexes $X_i \subseteq Y_i$ with the following properties:

(4) For each $i \in \mathbb{N}$ we have $X_i \subseteq X_{i+1}$.
(5) Each $|X_i|$ is a deformation retract of $|Y_i|$.

Finally we set $X \coloneqq \bigcup_{i\in\mathbb{N}} X_i$. Note that X is an $(n-1)$-dimensional subcomplex of K. It remains to show that X is a deformation retract of K.

We pick a base point $* \in$. Using the above one can show, reasonably easily, that the inclusion-induced map $\pi_n(X, *) \to \pi_n(K, *)$ is an isomorphism for every $n \in \mathbb{N}$. It follows from the Whitehead Deformation Retract Proposition 305.2[1954], which is a variation on the Whitehead Theorem 305.1, that X is indeed a deformation retract of K. We leave it as a charming challenge to the reader to turn this idea into a rigorous proof. ■

We give a new proof of the following proposition:

Proposition 130.6. (Non-Compact Surface-π_1-Proposition) Let M be a connected 2-dimensional smooth manifold. If M is non-compact, then $\pi_1(M)$ is a free group.

Proof. Let M be a connected 2-dimensional smooth manifold that is non-compact. By the No Top-Cell Proposition 183.31 we know that M is homotopy equivalent to a 1-dimensional CW-complex. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 (1) and the 1-Complex-π_1-Theorem 137.6 (1) that $\pi_1(M)$ is a free group. ■

We conclude this chapter with the following proposition:

Proposition 183.34. (Single Top-Cell Proposition) Let M be a compact connected non-empty n-dimensional smooth manifold. There exists a CW-structure for M with a single n-cell

Remark. For fans of small CW-complexes we also recall the CW-Complex–Spanning Tree Proposition 140.5 which says that any path-connected non-empty CW-complex X is homotopy equivalent to a CW-complex Y with a single 0-cell which has the property that for any $n \geq 2$ the number of n-cells of X equals the number of n-cells of Y. □

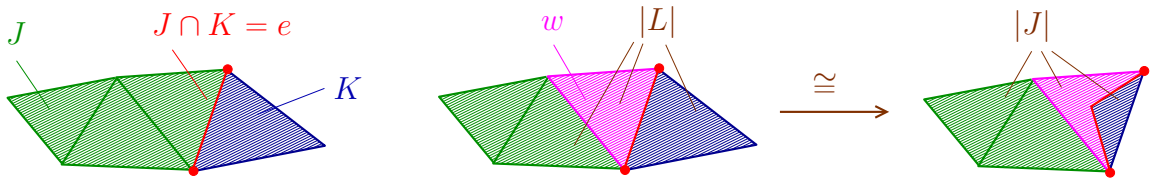
In the proof of the Single Top-Cell Proposition 183.34 we will make use of the following lemma:

Lemma 183.35. (Glue-on-Simplex Lemma) Let L be an n-dimensional pseudomanifold that contains two subcomplexes J and K. If K consists of a single n-simplex, if

[1954] My apologies for referring to a future result.

$J \cap K$ consists of a single $(n-1)$-simplex e and if e has order 1 in J, then there exists a homeomorphism $(|L|, |\partial L|) \to (|J|, |\partial J|)$ of pairs of topological spaces.

Sketch of a proof of the Glue-on-Simplex Lemma 183.35. Let w be the one n-simplex of J that cobounds e. It is elementary, albeit ever-so-slightly painful, to write down an explicit homeomorphism $|w| \cup |K| \to |w|$ that is the identity on $|\partial w| \setminus (|\partial w| \cap |K|)$.



Together with the identity on $|J| \setminus |w|$ this homeomorphism gives us the desired homeomorphism from $|L|$ to $|J|$. It follows almost immediately from the definitions that this homeomorphism restricts to a homeomorphism $|\partial L| \to |\partial J|$. We leave it to the reader to fill in the details. ■

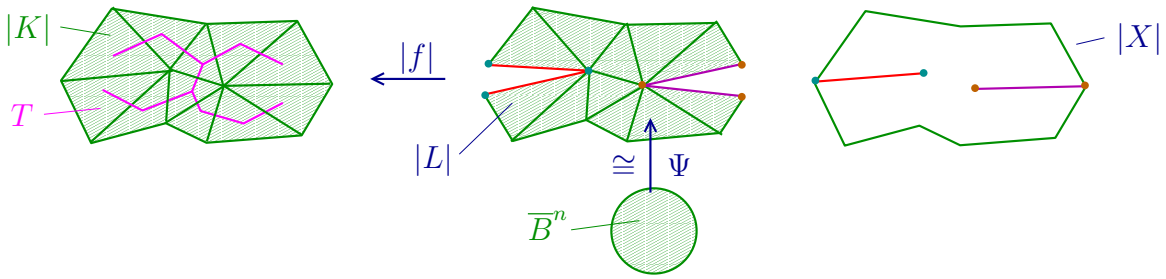
Sketch of a proof of the Single Top-Cell Proposition 183.34. We sketch a proof of the proposition. The statement is also proved with fewer details but more authority in [**McMu2002**, Theorem 5.1].

Let M be a closed connected non-empty n-dimensional smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. We need to show that $|K|$ admits a CW-structure with a single n-cell.

Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. By the Manifold-Simplex Proposition 183.2 (5) we know that Γ is connected. Thus it follows from the Spanning Tree–Existence Proposition 11.26 that Γ admits a spanning tree T. Note that by definition this means that T contains every vertex of Γ.

Let X be the subcomplex of K that is given by removing all n-simplices and by removing all $(n-1)$-simplices that do not correspond to an edge of the spanning tree T. Note that X is an $(n-1)$-dimensional simplicial complex. Furthermore let L be the abstract simplicial complex that is given by the "disjoint union of the n-simplices of K and gluing them along the $(n-1)$-simplices corresponding to the edges of T". It is not difficult, but also not particularly instructive, to write down a completely rigorous definition of L. Note that L comes with a natural simplicial map $f\colon L \to K$.

Next note that using an induction argument based on the Glue-on-Simplex Lemma 183.35, and which also makes use of the Δ^n-Homeomorphism Lemma 143.9 (2), one can easily show that there exists a homeomorphism $\Psi\colon (\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n) \to (|L|, |\partial L|)$.



Finally, using the Compact-Hausdorff Proposition 4.23 (3) one can verify that the map

$$\begin{array}{rcl} (\overline{B}^n \sqcup |X|)/\overset{\text{where } x \sim (\Psi \circ |f|)(x) \text{ for every } x \in S^{n-1}}{\overset{\downarrow}{\sim}} & \to & |K| \\ [P] & \mapsto & \begin{cases} (\Psi \circ |f|)(P), & \text{if } P \in \overline{B}^n, \\ P, & \text{if } P \in |X| \subseteq |K| \end{cases} \end{array}$$

is a homeomorphism. The left-hand side admits an obvious CW-structure with a single n-cell. Thus we have shown that $|K|$ admits a CW-structure with a single n-cell. ■

183.7. Exercises for Chapter 183 .

Exercise 183.1. Let K be a finite n-dimensional pseudomanifold.

(a) Show that the suspension $\Sigma(K)$, as introduced in Definition 174.44, is an $(n+1)$-dimensional pseudomanifold.
(b) Show that if K is closed, then $\Sigma(K)$ is closed.
(c) Show that if K is orientable in the sense of Definition 183.10, then $\Sigma(K)$ is orientable.

Exercise 183.2. Let K be an n-dimensional pseudomanifold such that ∂K is an $(n-1)$-dimensional pseudomanifold. Let $i\colon \partial K \to K$ be the inclusion map. We consider the cone $\mathrm{Cone}(i\colon \partial K \to K) := K \cup_{\partial K} \mathrm{Cone}(\partial K)$. Show that $\mathrm{Cone}(i\colon \partial K \to K)$ is a closed n-dimensional pseudomanifold.

Exercise 183.3. Let K be an m-dimensional pseudomanifold and let L be an n-dimensional pseudomanifold. We equip K and L with a total order and we define the product $K \times L$ as in Definition 174.37.

(a) Show that $K \times L$ is an $(m+n)$-dimensional pseudomanifold.
(b) Show that $K \times L$ is closed if and only if both K and L are closed.
(c) Show that $K \times L$ is orientable if and only if both K and L are orientable.

Exercise 183.4.

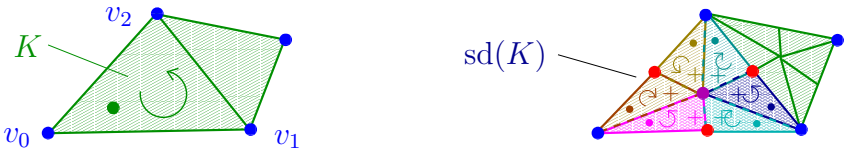
(a) Let K be a 1-dimensional pseudomanifold. Show that $|K|$ admits the structure of a 1-dimensional smooth manifold.
(b) Let K be a 2-dimensional pseudomanifold. Show that $|K|$ admits the structure of a 2-dimensional smooth manifold.

Remark. Part (a) is fairly straightforward. The proof of part (b) is more delicate. One way to prove part (b) is to imitate the proof, given in the Surface-is-Topological Manifold Proposition 23.7 and the Standard Smooth Structures Proposition 25.6, that the surface Σ_g of genus g is a closed 2-dimensional smooth manifold.

Exercise 183.5. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(a) Show that the barycentric subdivision $\mathrm{sd}(K)$ is an n-dimensional pseudomanifold.
(b) Suppose that K is equipped with an orientation. Let $t = \{\underline{s_0}, \ldots, \underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(K)$. For $i = 0, \ldots, n$ we denote by v_i the vertex of s_i that is not contained in s_{i-1}. Let $\epsilon \in \{-1, 1\}$ be the sign such that $(\epsilon, (v_0, \ldots, v_n))$ matches the given orientation of K. We equip the n-simplex t of $\mathrm{sd}(K)$ with the orientation given by $(\epsilon, (\underline{s_0}, \ldots, \underline{s_n}))$.

Show that the above orientations of the n-simplices of $\mathrm{sd}(K)$ define an orientation for $\mathrm{sd}(K)$.

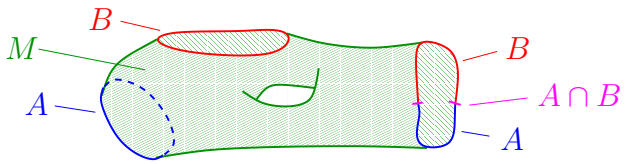


Exercise 183.6.

(a) Let M be a closed smooth manifold and let $X \subsetneq M$ be a proper subset of M. Show that X cannot be a deformation retract of M.
(b) Show that in (a) we cannot replace "closed" by "compact".
(c) Show that in (a) we cannot replace "closed" by "$\partial M = \varnothing$".

Exercise 183.7. Let M be a compact connected non-empty smooth manifold. Furthermore let $(K = (V, S), \Theta\colon |K| \to M)$ be a simplicial structure for M. Let Γ be the dual graph from Definition 183.14. Is there a relationship between the homology groups $\mathrm{H}_1(\Gamma; \mathbb{Z})$ and the groups $\mathrm{H}_1(M; \mathbb{Z})$ or $\mathrm{H}_1(M, \partial M; \mathbb{Z})$? For example is there a relationship between the ranks or the minimal number of generators?

Exercise 183.8. Let M be a compact orientable n-dimensional smooth manifold. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Show that $\mathrm{H}_{n-1}(M, A)$ is a free abelian group.



Exercise 183.9. Let M be a compact connected n-dimensional smooth manifold and let $p \neq 2$ be a prime. Determine $\mathrm{H}_n(M, \partial M; \mathbb{F}_p)$ and $\mathrm{H}_n(M; \mathbb{F}_p)$.

Exercise 183.10. Let $k, l \in \mathbb{N}$ and let $N_{k,l}$ be $N_k = k \cdot \mathbb{R}\mathrm{P}^2$ minus l open disks. Determine $\mathrm{H}_1(N_{k,l}, \partial N_{k,l}; \mathbb{Z})$.

Exercise 183.11. Let K be an infinite n-dimensional pseudomanifold such that $\partial K = \varnothing$. Show that $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
Remark. This exercise will require a serious effort.

Exercise 183.12. Let $p, q \in \mathbb{N}$ be coprime. Show that every orientation-reversing diffeomorphism of the lens space $L(p, q)$ has at least one fixed point.

Exercise 183.13. Let $p, q \in \mathbb{N}$ be coprime.

(a) Determine $\mathrm{H}_k(L(p, q); \mathbb{Q})$.
(b) Let m be a prime. Determine $\mathrm{H}_k(L(p, q); \mathbb{F}_m)$.

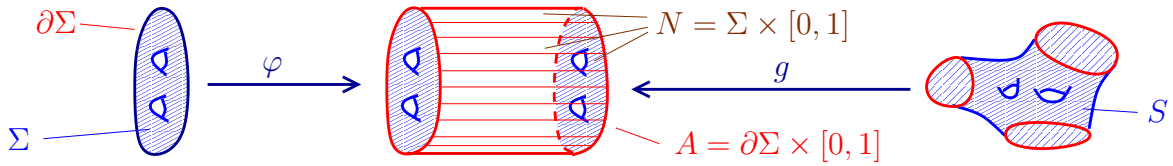
Exercise 183.14. Let $p, q \in \mathbb{N}$ be coprime and let $\widetilde{p}, \widetilde{q} \in \mathbb{N}$ be coprime. Determine $\mathrm{H}_k(L(p, q) \times L(\widetilde{p}, \widetilde{q}))$.

Exercise 183.15. Let Γ be a finite group that acts freely and smoothly, in an orientation-preserving way on S^3. Determine the homology groups of S^3/Γ in terms of suitable invariants of the group Γ.

Exercise 183.16. Let X be a connected 2-dimensional smooth manifold with infinite fundamental group. (We do not assume that X is compact.) Show that X is aspherical, i.e. show that $\pi_i(X) = 0$ for $i \geq 2$.
Hint. First use the Hurewicz Theorem 167.9 and the results from this section to show that $\pi_2(X) = 0$.
Remark. This exercise gives in particular a new proof of the fact that given a surface Σ_g of genus $g \geq 1$ we have $\pi_i(\Sigma_g) = 0$ for $i \geq 2$. We had initially proved this statement on page 2587 using the rather difficult results from Chapter 132.

Exercise 183.17. Let M be a 3-dimensional smooth homology 3-sphere with non-trivial fundamental group.[1955] Furthermore let $n \geq 4$. We pick a smooth embedding $\varphi\colon \overline{B}^3 \to M$ and we consider $W := (M \setminus \varphi(B^3)) \times \overline{B}^{n-3}$. By the Smooth Manifold Rounded Product Proposition 60.7 we can equip W with the structure of an orientable n-dimensional smooth manifold. Show that the double $\mathrm{D}W$ is an n-dimensional smooth homology n-sphere with non-trivial fundamental group.

Exercise 183.18. Let Σ be a compact oriented connected 2-dimensional smooth manifold. We consider the product $N := \Sigma \times [0,1]$ and we consider the subset $A := \partial\Sigma \times [0,1] \subseteq N$. Let $\varphi\colon \Sigma \to \Sigma \times \{0\} \subseteq N$ be the obvious inclusion map. Now let $g\colon S \to N$ be a continuous map from a compact oriented connected 2-dimensional smooth manifold with $g(\partial S) \subseteq A$ such that $\varphi_*([\Sigma]) = g_*([S])$. Show that $\chi(S) \leq \chi(\Sigma)$.
Remark. In a slightly simplified version the exercise says that the surface $\Sigma \times \{0\}$ is the "simplest" surface in N representing the homology class $[\Sigma \times \{0\}]$.



[1955] By the Poincaré Homology Sphere Proposition 183.24 we know that such M exists.

CHAPTER 184

Simplicial cohomology

In this chapter we will introduce the simplicial cohomology groups of an (ordered) abstract simplicial complex. This chapter can be seen as a continuation of our discussions of abstract simplicial complexes in Chapters 174, 175 and 176. The objects in this chapter are for the most part purely combinatorial and for the most part this chapter can be read independently of the chapters on singular cohomology. We will only make use of the purely algebraic statements in Sections 197.1, 197.2 and 197.3.

As for simplicial homology, the study of simplicial cohomology is great fun in its own right. But as we will see, it is also a very convenient tool for calculating singular cohomology groups and for determining the often intractable cup and cap products.
In Chapter 188 we will prove the Simplicial Poincaré Duality Theorem 188.5 for pseudo-manifolds, which gives us in particular a Poincaré Duality theorem for smooth manifolds.

This chapter is organized as follows:

- In Section 184.1 we will recall the definition of simplicial chain complexes.
- In Section 184.2 we will introduce the simplicial cohomology groups.
- In Section 184.3 we will discuss properties of simplicial cohomology groups.
- In Section 184.4 we will introduce the simplicial cup product.
- In Section 184.5 we will introduce the simplicial cap product.
- In Appendix 184.6 we will discuss quasi-isomorphisms and splittings.
- In Section 184.7 we will provide exercises for Chapter 184.

In the next Chapter 185 we will introduce the fundamental class of smooth manifolds.

184.1. **Simplicial chain complexes.** In the following section we will introduce simplicial cohomology groups. Before we do so it is convenient to recall several definitions and results on simplicial chain complexes.

We start out with Definition 176.5.

Definition 184.1. Let $K = (V, S)$ be an ordered abstract simplicial complex.[1956] Given $k \in \mathbb{N}_0$ we define

$$\mathrm{C}_k^{\mathrm{simp},\leq}(K) \coloneqq \text{free abelian group generated by the set of } k\text{-simplices of } K.$$

Furthermore we consider the boundary map

$$\begin{aligned} \partial_k \colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) &\to \mathrm{C}_{k-1}^{\mathrm{simp},\leq}(K) \\ \underbrace{\{v_0 < \cdots < v_k\}}_{\substack{\text{any } k\text{-simplex } s \text{ is uniquely} \\ \text{of the form } s = \{v_0, \ldots, v_k\} \\ \text{with } v_0 < \cdots < v_k}} &\mapsto \sum_{i=0}^{k} (-1)^i \cdot \{v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_k\}. \end{aligned}$$

By the Simplicial Boundary Lemma 176.6 we know that $(\mathrm{C}_*^{\mathrm{simp},\leq}(K), \partial_*)$ is a chain complex, thus we can define the corresponding simplicial homology groups $\mathrm{H}_*^{\mathrm{simp},\leq}(K)$.

The above definition is straightforward, but it does have the major disadvantage that it requires the choice of an order on K. In most situations there is no natural choice of an order. In the absence of an order we are led to the following more elaborate definition which we initially gave in Definition 176.10 and in the Simplicial Boundary Lemma 176.12.

Definition 184.2. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $k \in \mathbb{N}_0$. An ordered k-simplex of K is a $(k+1)$-tuple $(v_0, \ldots, v_k) \in V^{k+1}$ such that $\{v_0, \ldots, v_k\}$ is a k-simplex of $K = (V, S)$. We write $\lfloor (v_0, \ldots, v_k) \rfloor := \{v_0, \ldots, v_k\}$.

(2) Let $k \in \mathbb{N}_0$. We define the k-th simplicial chain group

$$\mathrm{C}_k^{\mathrm{simp}}(K) := \begin{array}{c}\text{free abelian group on the set} \\ \text{of ordered } k\text{-simplices of } K\end{array} \Big/ \begin{array}{c}(v_0, \ldots, v_k) - \mathrm{sign}(\sigma) \cdot (v_{\sigma(0)}, \ldots, v_{\sigma(k)}) \\ \text{where } \sigma \in S_{k+1} = \mathrm{Bij}\{0, \ldots, k\}\end{array}$$

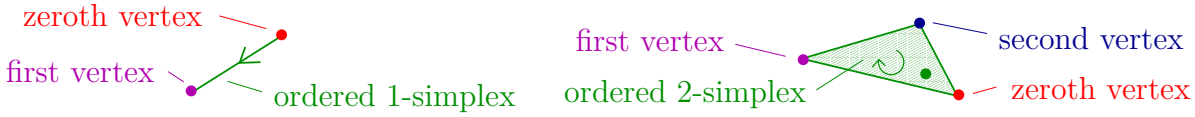
(3) Given an ordered simplex $(v_0, \ldots, v_k)$ we denote its image in $\mathrm{C}_k^{\mathrm{simp}}(K)$ by $[v_0, \ldots, v_k]$.

(4) We consider the map

$$\begin{aligned} \partial_k \colon \mathrm{C}_k^{\mathrm{simp}}(K) &\to \mathrm{C}_{k-1}^{\mathrm{simp}}(K) \\ [v_0, \ldots, v_k] &\mapsto \sum_{i=0}^{k} (-1)^i \cdot [v_0, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_k]. \end{aligned}$$

(5) By the Simplicial Boundary Lemma 176.12 we know that $(\mathrm{C}_*^{\mathrm{simp}}(K), \partial_*)$ is a chain complex, thus we can define the corresponding simplicial homology groups $\mathrm{H}_*^{\mathrm{simp}}(K)$.

Notation 184.3. In pictures we denote the ordering of the simplices of a 1-simplex in the obvious way by an arrow pointing from the first to the second vertex. Furthermore, given an ordered 2-simplex we indicate the first vertex by a dot and the ordering of the remaining vertices by an oriented 3/4-circle. We refer to the figure below for an illustration.



We recall the following lemma:

Lemma 176.12. (Simplicial Boundary Lemma) Let K be an abstract simplicial complex. If we choose for each k-simplex $s \in S$ an ordered k-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$, then the $[\widetilde{s}]$ form a basis for $\mathrm{C}_k^{\mathrm{simp}}(K)$.

We conclude this short section with the following proposition that combines several earlier statements on the relationship between the above two simplicial chain complexes and also the singular chain complex of the topological realization.

Proposition 184.4. (Simplicial–Singular Chain Complex Proposition)

[1956] Recall that, according to Definition 174.12, an *order* on an abstract simplicial complex (V, S) is a partial order on V that has the property that the restriction to each simplex is actually a total order.

(1) Let $(K = (V, S), \leq)$ be an ordered abstract simplicial complex.
   (a) We consider the following maps:

$$\begin{array}{ccccc} \mathrm{C}_k^{\mathrm{simp}}(K) & \xleftarrow{\Omega_{\leq,k}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|) \\ [v_0, \dots, v_k] & \longleftarrow\!\shortmid & s = \{v_0 < \cdots < v_k\} & \longmapsto & (\Phi_s^{\leq} \colon \Delta^k \to |K|). \end{array}$$

map that is given by sending $\mathfrak{e}_i$ to v_i

   The maps $\Omega_{\leq,k}$ to the left define a natural isomorphism of chain complexes whereas the maps $\Theta_{\leq,k}$ to the right define a natural chain homotopy equivalence.
   (b) There exist maps

$$\pi_k \colon \mathrm{C}_k(|K|) \to \mathrm{C}_k^{\mathrm{simp},\leq}(K)$$

   which have the following two properties:
      (a) The maps π_k form a chain map which form a chain homotopy equivalence.
      (b) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \Theta_k = \mathrm{id}_{\mathrm{C}_k^{\mathrm{simp},\leq}(K)}$.

(2) Let K be an abstract simplicial complex and let "$\leq$" and "$\subseteq$" be two orders on K. The two chain maps

$$\begin{array}{ccccc} & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega_{\leq,k}^{-1}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|) \\ \text{and} & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega_{\subseteq,k}^{-1}} & \mathrm{C}_k^{\mathrm{simp},\subseteq}(K) & \xrightarrow{\Theta_{\subseteq,k}} & \mathrm{C}_k(|K|) \end{array}$$

   are chain homotopic.

(3) Let G be an abelian group. The chain homotopy equivalence class of (2) induces a natural isomorphism

$$\Xi_{k*} \colon \mathrm{H}_k^{\mathrm{simp}}(K; G) \to \mathrm{H}_k(|K|; G).$$

Proof.

(1) (a) This statement is the combination of the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 and the Simplicial-Singular Chain Complex Theorem 177.12.
   (b) It follows almost immediately from the definitions that for each $k \in \mathbb{N}_0$ the map $\mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ is a monomorphism and that the image is a summand. The desired statement is now an immediate consequence of (a) and the Chain Homotopy Equivalence Criterion 184.31, which we prove in the appendix to this chapter.

(2) This statement is the content of the Simplicial-to-Singular Chain Homotopy Lemma 177.13.

(3) It follows immediately from (2) and a slight generalization of the Chain Homotopy–H_*-Lemma 144.5 that any choice of order on K induces the same map on homology with G-coefficients, i.e. the map Ξ_{k*} is well-defined. It follows easily from (1) and the Chain Homotopy Equivalence-H_*-Corollary 144.10 that the map Ξ_{k*} is a natural isomorphism. ∎

184.2. **The definition of the simplicial cohomology groups.** In this section we will introduce the simplicial cohomology groups of an (ordered) abstract simplicial complex. The discussion in the previous section also shows that it will be necessary to consider two slightly different types of simplicial cohomology groups.
We start out with a decidedly dull definition.

Definition 184.5. Let K be an (ordered) abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ we write

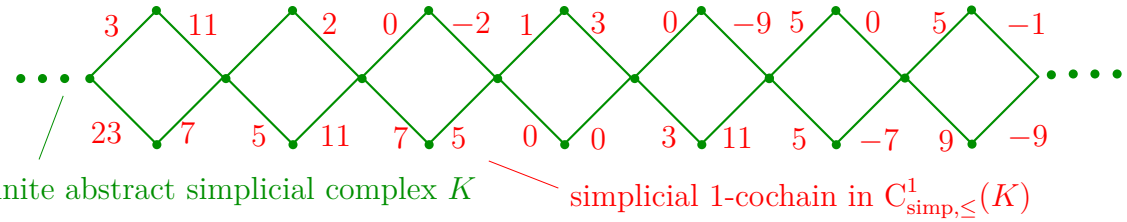
$$\mathrm{C}^k_{\mathrm{simp}}(K;G) \coloneqq \mathrm{Hom}(\mathrm{C}^{\mathrm{simp}}_k(K), G) \quad \text{and} \quad \mathrm{C}^k_{\mathrm{simp},\leq}(K;G) \coloneqq \mathrm{Hom}(\mathrm{C}^{\mathrm{simp},\leq}_k(K), G).$$

Elements of $\mathrm{C}^k_{\mathrm{simp}}(K;G)$ and $\mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ are called simplicial k-cochains.

Remark. Let G be an abelian group, let K be an ordered abstract simplicial complex and let $k \in \mathbb{N}_0$.

(1) It follows from the Free Abelian Group–Existence Lemma 97.8 that a simplicial k-cochain in $\mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ is the same data as the assignment of an element of G to *every* k-simplex. Note that in contrast to the setting of singular cochains one can actually hope to "see" simplicial cochains. For example see the figure below for a simple-minded illustration.

(2) In contrast, it follows easily from the Tensor–Direct Sum Lemma 159.15 and the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that a chain in $\mathrm{C}^{\mathrm{simp},\leq}_k(K) \otimes G$ is the same data as assigning an element of G to *finitely many* k-simplices.



These observations are illustrated in the figure above. □

Now we introduce the following notation:

Notation 184.6.

(1) Let K be an order abstract simplicial complex and let s be an ordered k-simplex. We denote by $s^*\colon \mathrm{C}^{\mathrm{simp},\leq}_k(K) \to \mathbb{Z}$ the map that is given by $s^*(s) = 1$ and $s^*(t) = 0$ for any k-simplex $t \neq s$.

(2) Let K be an abstract simplicial complex and let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex. It follows immediately from the definitions that

$$\begin{array}{rcl} s^*\colon \mathrm{C}^{\mathrm{simp}}_k(K) & \to & \mathbb{Z} \\ [w_0, \ldots, w_k] & \mapsto & \begin{cases} 0, & \text{if } \{v_0, \ldots, v_k\} \neq \{w_0, \ldots, w_k\}, \\ \mathrm{sign}(\sigma), & \text{if there exists a } \sigma \in \mathrm{Bij}\{0, \ldots, k\} \\ & \qquad \text{with } w_i = v_{\sigma(i)} \text{ for } = 0, \ldots, k \end{cases} \end{array}$$

is well-defined, in other words, s^* is a well-defined cochain in $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z})$.

In the following we will mostly study the simplicial cohomology of finite (ordered) abstract simplicial complexes. In this context the following lemma is very helpful.

Lemma 184.7. (Cochains–Basis Lemma) Let $k \in \mathbb{N}_0$.

(1) Let $K = (V, S)$ be an ordered abstract simplicial complex. If K is *finite*, then the set $\{s^* \mid s \text{ a } k\text{-simplex}\}$ is a basis for $\mathrm{C}^k_{\mathrm{simp},\leq}(K;\mathbb{Z})$.

(2) Let $K = (V, S)$ be an abstract simplicial complex. We choose for each k-simplex s an ordered k-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$. If K is finite, then the $\widetilde{s}^*$ form a basis for $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z})$.

Proof. This follows immediately from the Free Abelian Group–Existence Lemma 97.8 together with the definition of $\mathrm{C}_k^{\mathrm{simp},\leq}(K)$ and together with the Simplicial Boundary Lemma 176.12. ■

Now that we have gained a little bit of familiarity with cochains let us move on to the definition of the simplicial cochain complex and the simplicial cohomology groups.

Definition 184.8. Let K be an abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ we refer to

$$\delta_k := \partial_{k+1}^* \colon \underbrace{\mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K), G)}_{=\mathrm{C}^k_{\mathrm{simp}}(K;G)} \to \underbrace{\mathrm{Hom}(\mathrm{C}_{k+1}^{\mathrm{simp}}(K), G)}_{=\mathrm{C}^{k+1}_{\mathrm{simp}}(K;G)}$$

as the coboundary map. Any simplicial k-cochain in $\ker(\delta_k \colon \mathrm{C}^k_{\mathrm{simp}}(K;G) \to \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G))$ is called a simplicial k-cocycle. We refer to the cochain complex

$$\ldots \leftarrow \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G) \xleftarrow{\delta_k} \mathrm{C}^k_{\mathrm{simp}}(K;G) \xleftarrow{\delta_{k-1}} \mathrm{C}^{k-1}_{\mathrm{simp}}(K;G) \leftarrow \ldots \xleftarrow{\delta_0} \mathrm{C}^0_{\mathrm{simp}}(K;G) \leftarrow 0$$

as the simplicial cochain complex of K with G-coefficients. We define the k-th simplicial cohomology group of K with G-coefficients as follows:

$$\mathrm{H}^k_{\mathrm{simp}}(K;G) := \mathrm{H}^k(\mathrm{C}^{\mathrm{simp}}_*(K);G) = \frac{\ker(\delta_k \colon \mathrm{C}^k_{\mathrm{simp}}(K;G) \to \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G))}{\mathrm{im}(\delta_{k-1} \colon \mathrm{C}^{k-1}_{\mathrm{simp}}(K;G) \to \mathrm{C}^k_{\mathrm{simp}}(K;G))}.$$

If K is actually an ordered abstract simplicial complex, then we define the cochain complex $\mathrm{C}^*_{\mathrm{simp},\leq}(K;G)$ and the simplicial cohomology groups $\mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$ in the obvious way.

Given a finite abstract simplicial complex it is just an elementary, albeit at times surprisingly intricate, problem in linear algebra to determine the simplicial cohomology groups and to give explicit examples of simplicial cocycles.

Example. We consider the "simplicial circle" C with vertex set $V = \{a, b, c\}$ shown in the figure below. We consider the ordered 1-simplices $s := (a, b)$, $t := (b, c)$ and $u := (c, a)$. We see that the simplicial chain complex $\mathrm{C}^{\mathrm{simp}}_*(C;\mathbb{Z})$ is of the following form:[1957]

$$\begin{array}{ccccc} 0 \to \mathrm{C}^{\mathrm{simp}}_1(C) \xrightarrow{\partial_1} \mathrm{C}^{\mathrm{simp}}_0(C) \to 0 & & \text{thus the dual} & & 0 \leftarrow \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z}) \xleftarrow{\delta_0} \mathrm{C}^0_{\mathrm{simp}}(C;\mathbb{Z}) \leftarrow 0 \\ s \mapsto b - a & & \text{chain complex} & & u^* - s^* \mapsfrom a^* \\ t \mapsto c - b & & \mathrm{C}^*_{\mathrm{simp}}(C;\mathbb{Z}) & & s^* - t^* \mapsfrom b^* \\ u \mapsto a - c. & & \text{is given by} & & t^* - u^* \mapsfrom c^*. \end{array}$$

[1957] Usually it takes a while to get used to coboundaries. Thus let us show in detail that we have the equality $\delta_0(a^*) = u^* - s^* \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}^{\mathrm{simp}}_1(C), \mathbb{Z})$. We need to show that $\delta_0(a^*) = u^* - s^*$ are the same homomorphism. By the Simplicial Boundary Lemma 176.12 we know that $\{s, t, u\}$ is a basis for $\mathrm{C}^{\mathrm{simp}}_1(C)$. So we need to show that the two homomorphisms take the same values on s, t and u. We calculate that

$$\delta_0(a^*)(s) \underset{\text{Definition 183.11 of } \delta_0}{=} a^*(\partial_1 s) \underset{\text{Definition 183.11 of } \partial_1}{=} a^*(b - a) = a^*(b) - a^*(a) = -1 = u^*(s) - s^*(s) = (u^* - s^*)(s).$$

The same way we proceed with t and u.

An elementary calculation shows that

$$\mathrm{H}^0_{\mathrm{simp}}(C;\mathbb{Z}) \;=\; \ker(\delta_0) \;=\; \mathbb{Z}\cdot(a^*+b^*+c^*).$$

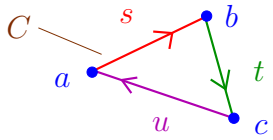
We turn to the slightly more tricky calculation of $\mathrm{H}^1_{\mathrm{simp}}(C;\mathbb{Z})$. First note that it is elementary to show that

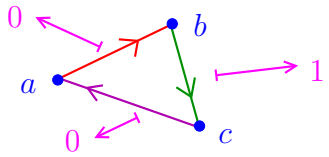
$$\mathrm{im}\left(\delta_0\colon \mathrm{C}^0_{\mathrm{simp}}(C;\mathbb{Z}) \to \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})\right) \;=\; \{k\cdot s^* + l\cdot t^* + m\cdot u^* \,|\, k+l+m=0\}.$$

Now let us consider the cochain $\theta = t^*\colon \mathrm{C}^{\mathrm{simp}}_1(C) \to \mathbb{Z}$. (In other words, θ is the cochain that is given by $\theta(s)=0$, $\theta(t)=1$ and $\theta(u)=0$.) An elementary argument, see e.g. Exercise 143.3, shows that the map

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\cong} & \mathrm{H}^1_{\mathrm{simp}}(C;\mathbb{Z}) & = & \mathrm{coker}(\delta_0) \\ n & \mapsto & n\cdot[\theta] & & \end{array}$$

is an isomorphism.[1958]





cocycle $\theta\colon \mathrm{C}^{\mathrm{simp}}_1(K) \to \mathbb{Z}$ represents a generator of $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z})$

Note that on several occasions throughout this chapter we will make use of the above simplicial cocycle $\theta = t^* \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$. □

We state the next example as a lemma.

Lemma 184.9. (Simplicial-H^0-Lemma) Let G be an abelian group and let $K=(V,S)$ be an abstract simplicial complex.

(1) Given any $g\in G$ the constant cochain $\varphi_g\colon \mathrm{C}^{\mathrm{simp}}_0(K)\to G$ given by $\varphi_g(v)=g$ for every $v\in V$ is a cocycle.
(2) If K is non-empty and connected in the sense of Exercise 174.13, then we have $\mathrm{H}^0_{\mathrm{simp}}(K;G) = \{\varphi_g \,|\, g\in G\}$. In particular $\mathrm{H}^0_{\mathrm{simp}}(K;G)\cong G$.

Proof.

(1) Let $g\in G$. We need to show that $\delta_0(\varphi_g) = 0 \in \mathrm{C}^1_{\mathrm{simp}}(K) = \mathrm{Hom}(\mathrm{C}^{\mathrm{simp}}_1(K), G)$. In other words, by definition of $\mathrm{C}^{\mathrm{simp}}_1(K)$ we need to show that for every ordered 1-simplex (v_0,v_1) of K we have $(\delta_0\varphi_g)([v_0,v_1]) = 0$. Indeed, we calculate that

$$\delta_0(\varphi_g)([v_0,v_1]) \underset{\text{Definition 184.8 of } \delta_0}{=} \varphi_g(\partial_1([v_0,v_1])) \underset{\text{Definition 176.7 of } \partial_1}{=} \varphi_g(v_1-v_0) \;=\; \varphi_g(v_1)-\varphi_g(v_0) \;=\; g-g \;=\; 0.$$

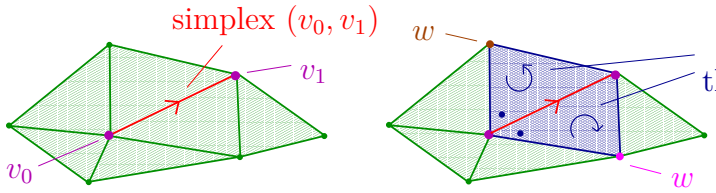
(2) We will prove Statement (2) in Exercise 184.2. ■

We conclude this section with the following explicit calculation of the coboundary map. This result will come in handy in the following chapter.

[1958] Of course there are many other choices of a cochain θ that would work. We picked this particular one since later on it works well in the pictures.

Lemma 184.10. (Dual Simplex–Coboundary Lemma) Let $K = (V, S)$ be an abstract simplicial complex and let $(v_0, \ldots, v_k)$ be an ordered k-simplex of K. We have

$$\delta_k([v_0, \ldots, v_k]^*) \;=\; (-1)^{k+1} \cdot \sum_{\substack{w \in V \text{ such that} \\ \{v_0, \ldots, v_k, w\} \text{ is} \\ \text{a } (k+1)\text{-simplex}}} [v_0, \ldots, v_k, w]^* \;\in\; \mathrm{C}^{k+1}_{\mathrm{simp}}(K; \mathbb{Z}).$$



Proof. We need to show that both sides define the same map $\mathrm{C}^{\mathrm{simp}}_{k+1}(K) \to \mathbb{Z}$. Basically by definition of $\mathrm{C}^{\mathrm{simp}}_{k+1}(K)$ this means that we need to show both sides provide the same value for any given $[y_0, \ldots, y_{k+1}]$. So suppose that we are given an ordered $(k+1)$-simplex $(y_0, \ldots, y_{k+1})$ of K. Note that after possibly reordering the vertices y_i we can arrange that there exists an $m \in \{0, \ldots, k\}$ such that $y_i = v_i$ for $i = 0, \ldots, m-1$ and such that $\{y_m, \ldots, y_{k+1}\} \cap \{v_m, \ldots, v_k\} = \varnothing$. We calculate that

$$\begin{aligned}
(\delta_k([v_0, \ldots, v_k]^*)([y_0, \ldots, y_{k+1}]) &\overset{\text{Definition 184.8 of } \delta_k}{=} [v_0, \ldots, v_k]^*(\partial_{k+1}([y_0, \ldots, y_{k+1}])) \overset{\text{Definition 176.7 of } \partial_{k+1}}{=} \\
&= \sum_{i=0}^{k+1} (-1)^i \cdot [v_0, \ldots, v_k]^*([y_0, \ldots, \widehat{y_i}, \ldots, y_{k+1}]) \\
&\overset{\text{follows from the choice of } m}{=} \sum_{i=0}^{k+1} \begin{cases} 0, & \text{if } i < k+1 \text{ or if } m \neq k \\ (-1)^{k+1}, & \text{if } i = k+1 \text{ and if } m = k. \end{cases}
\end{aligned}$$

It follows easily from the definitions that the cochain on the right-hand side of the lemma leads to the same result. ∎

184.3. **Properties of simplicial cohomology groups.** In the following lemma we will see that simplicial cohomology groups define contravariant functors. Before we can formulate that lemma let us recall a definition from the Ordered Simplicial Chain Complex–Functor Lemma 176.8 and the Simplicial Chain Complex–Functor Lemma 176.14. Namely, given a simplicial map $f\colon K = (V, S) \to L = (W, T)$ between abstract simplicial complexes we consider the following chain map:

$$\begin{aligned}
f_*\colon \mathrm{C}^{\mathrm{simp}}_k(K) &\to \mathrm{C}^{\mathrm{simp}}_k(L) \\
[v_0, \ldots, v_k] &\mapsto \begin{cases} [f(v_0), \ldots, f(v_k)], & \text{if } f(v_0), \ldots, f(v_k) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

When we are dealing with an order-preserving simplicial map between ordered abstract simplicial complexes, then we define the induced map $\mathrm{C}^{\mathrm{simp},\leq}_k(K) \to \mathrm{C}^{\mathrm{simp},\leq}_k(L)$ in the analogous way.

Lemma 184.11. (Simplicial Cohomology–Functor Lemma) Let $k \in \mathbb{N}_0$ and let G be an abelian group.

(1) The maps

$$\begin{array}{rcl} (K,\leq) & \mapsto & \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) \\ (f\colon (K,\leq)\to(L,\leq)) & \mapsto & \left(\begin{array}{rcl} f^*\colon \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) & \to & \mathrm{H}^k_{\mathrm{simp},\leq}(L;G) \\ [\varphi\colon \mathrm{C}^{\mathrm{simp}}_k(K)\to G] & \mapsto & [\varphi\circ f_*] \end{array}\right) \end{array}$$

define a contravariant functor from the category *OrdAbsSimpCplx* of ordered abstract simplicial complexes to the category *AbGroup* of abelian groups.

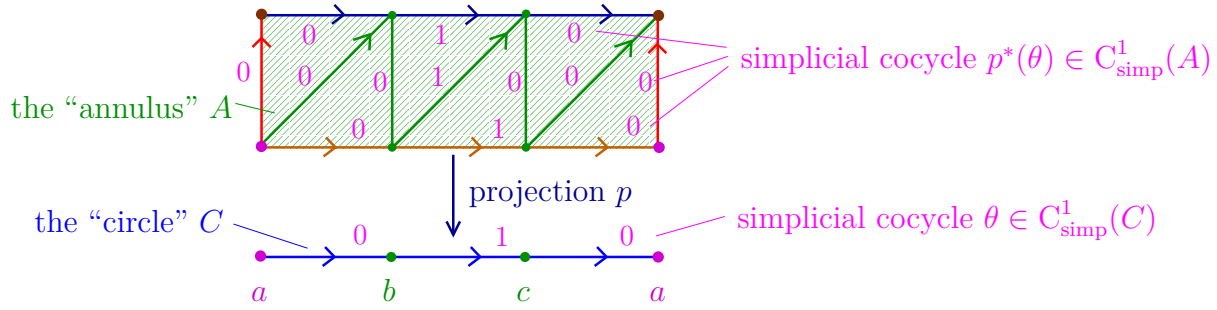
(2) The maps

$$\begin{array}{rcl} K & \mapsto & \mathrm{H}^k_{\mathrm{simp}}(K;G) \\ (f\colon K\to L) & \mapsto & \left(\begin{array}{rcl} f^*\colon \mathrm{H}^k_{\mathrm{simp}}(K;G) & \to & \mathrm{H}^k_{\mathrm{simp}}(L;G) \\ [\varphi\colon \mathrm{C}^{\mathrm{simp}}_k(K)\to G] & \mapsto & [\varphi\circ f_*] \end{array}\right) \end{array}$$

define a contravariant functor from the category *AbsSimpCplx* of abstract simplicial complexes to the category *AbGroup* of abelian groups.

Proof. The lemma is an immediate consequence of the Ordered Simplicial Chain Complex–Functor Lemma 176.8, the Simplicial Chain Complex–Functor Lemma 176.14 and the Dual Cochain Map Lemma 197.20. ■

Example. In the figure below we show the "simplicial annulus" A together with the natural projection p to the simplicial circle C.



Furthermore we also show the simplicial cocycle $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$ that we introduced on page 3507 and we show the corresponding cocycle $p^*(\theta) \in \mathrm{C}^1_{\mathrm{simp}}(A)$. □

The following proposition relates the two flavors of simplicial cohomology groups to one another and it relates them to the singular cohomology groups of the topological realization.

Proposition 184.12. (Simplicial-Singular Cohomology Proposition) Let G be an abelian group.

(1) Let $(K,\leq)$ be an ordered abstract simplicial complex.

(a) The natural maps

$$\begin{array}{rcl} \Theta^*_{\leq,k}\colon \mathrm{C}^k(|K|;G) & \to & \mathrm{C}^k_{\mathrm{simp},\leq}(K;G) \\ (\varphi\colon \mathrm{C}_k(|K|)\to G) & \mapsto & \left(\mathrm{C}^{\mathrm{simp}}_{k,\leq}(K) \xrightarrow{\Theta_{\leq,k}} \mathrm{C}_k(|K|) \xrightarrow{\varphi} G\right) \end{array}$$

are a cochain homotopy equivalence. In particular the maps induce natural isomorphisms

$$\Theta^*_{\leq,k}\colon \mathrm{H}^k(|K|;G) \to \mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$$

(b) Every simplicial cocycle $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ "extends" to a singular cocycle of $\mathrm{C}^k(|K|;G)$. More precisely, there exists a cocycle $\psi \in \mathrm{C}^k(|K|;G)$ such that $\Theta^*_{\leq,k}(\psi) = \varphi$, i.e. such that $\psi \circ \Theta_{\leq,k} = \varphi \colon \mathrm{C}^{\mathrm{simp}}_k(K) \to G$.

(2) Let K be an abstract simplicial complex. We pick an order "$\leq$" for K. The composition of the two maps

$$\mathrm{H}^k(|K|;G) \xrightarrow{\Theta^*_{\leq}} \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) \xrightarrow{\Omega^{-1}_{\leq,*}} \mathrm{H}^k_{\mathrm{simp}}(K;G)$$

does not depend on the choice of "$\leq$" and it defines a natural isomorphism

$$\Xi^*_k \colon \mathrm{H}^k(|K|;G) \to \mathrm{H}^k_{\mathrm{simp}}(K;G).$$

Proof.

(1) (a) This statement follows immediately from the Simplicial–Singular Chain Complex Proposition 184.4 (1) and the Dual Cochain Map Lemma 197.20.

(b) Let $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ be a simplicial cocycle. By the Simplicial–Singular Chain Complex Proposition 184.4 (2) there exist homomorphisms $\pi_k \colon \mathrm{C}_k(|K|) \to \mathrm{C}^{\mathrm{simp},\leq}_k(K)$ which have the following two properties:

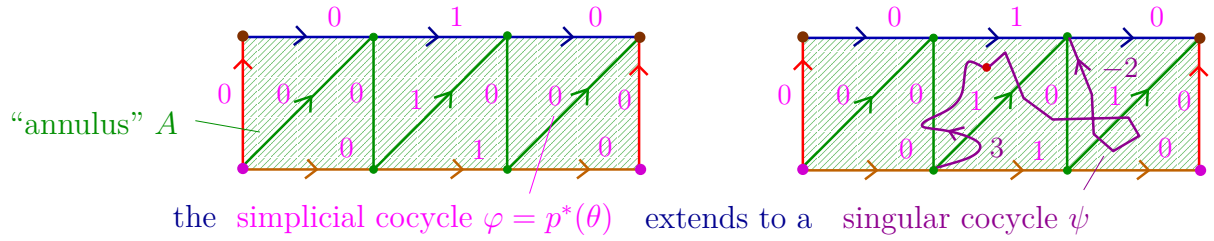
(i) The maps π_k form a chain map.

(ii) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \Theta_k = \mathrm{id}_{\mathrm{C}^{\mathrm{simp},\leq}_k(K)}$.

We set $\psi \coloneqq \pi^*_k(\varphi)$. By (i) this is a cocycle and by (ii) we have $\psi \circ \Theta_{\leq,k} = \varphi$.

(2) This statement follows immediately from the Simplicial–Singular Chain Complex Proposition 184.4 (1) and the Dual Cochain Map Lemma 197.20. ■

Example. Let us rephrase the statement of the Simplicial-Singular Cohomology Proposition 184.12 (1b). It says that given any simplicial cocycle $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ there exists a singular cocycle $\psi \in \mathrm{C}^k(|K|;G)$ with $[\Theta^*_{\leq,k}(\psi)] = [\varphi] \in \mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$ and which, restricted to the singular simplices provided by the simplices of K, coincides with φ.



We try to illustrate this statement in the figure above. □

We continue with the following lemma which is a special case of the future Kronecker Pairing Lemma 199.7 and the Kronecker Pairing–Natural Lemma 199.9.

Lemma 184.13. (Kronecker Pairing Lemma) Let G be an abelian group and let $k \in \mathbb{N}_0$.

(1) Given any abstract simplicial complex K the map

$$\begin{aligned} \langle\ ,\ \rangle \colon \mathrm{H}^k_{\mathrm{simp}}(K;G) \times \mathrm{H}^{\mathrm{simp}}_k(K) &\to G \\ ([\varphi],[\sigma]) &\mapsto \langle [\varphi],[c]\rangle_K \coloneqq \varphi(\sigma), \end{aligned}$$

called the *Kronecker pairing*, is well-defined and bilinear.

(2) Given any simplicial map $f\colon K \to L$ between two abstract simplicial complexes, given any $\sigma \in \mathrm{H}_k^{\mathrm{simp}}(K)$ and given any $\varphi \in \mathrm{H}^k_{\mathrm{simp}}(L;G)$ we have

$$\langle f^*(\varphi), \sigma\rangle_K \;=\; \langle \varphi, f_*(\sigma)\rangle_L.$$

With the above lemma we can now formulate the following theorem which is an immediate consequence of the Algebraic Universal Coefficient-H*-Theorem 201.22.

Theorem 184.14. (Universal Coefficient–Simplicial-H*-Theorem) Let K be an abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ there exists a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(\mathrm{H}^{\mathrm{simp}}_{k-1}(K), G) & \longrightarrow & \mathrm{H}^k_{\mathrm{simp}}(K;G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_k(K), G) & \longrightarrow & 0 \\ & & & & \varphi & \mapsto & \begin{pmatrix} \mathrm{H}^{\mathrm{simp}}_k(K) \to G \\ \sigma \mapsto \langle \varphi, \sigma\rangle_K \end{pmatrix} & & \end{array}$$

which is natural in the abstract simplicial complex K and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^k_{\mathrm{simp}}(K;G) \;\cong\; \mathrm{Ext}(\mathrm{H}^{\mathrm{simp}}_{k-1}(K), G) \oplus \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_k(K), G).$$

The analogous statement for ordered abstract simplicial complexes and the corresponding simplicial homology and cohomology groups also holds.

Remark.

(1) The Universal Coefficient–Simplicial-H*-Theorem 184.14 is a double-edged sword. It allows us to compute the isomorphism types of simplicial cohomology groups in terms of the much more friendly simplicial homology groups. But it also says, somewhat disappointingly, that the isomorphism types of simplicial cohomology groups do not contain any information that is not already contained in the isomorphism types of the simplicial homology groups.

(2) The Universal Coefficient–Simplicial-H*-Theorem 184.14 is also somewhat less powerful than it might appear. The theorem can be very convenient for determining the isomorphism types of simplicial cohomology groups. But if the Ext-terms are non-zero, then the theorem does not provide an approach to finding explicit cocycles representing cohomology classes. □

184.4. **The simplicial cup product.** The reader who is already familiar with singular cohomology will remember that one the remarkable features of singular cohomology is that it comes with a cup product and that it interacts with singular homology via the cap product. Before we attempt to introduce simplicial analogs let us remind us of the definitions from Definition 143.8, Definition 203.11 and Definition 206.2.

Definition 184.15.

(1) For $i = 0, \ldots, n$ we write $\mathfrak{v}_i := (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

(2) For $a_0, \ldots, a_k \in \Delta^n$ we consider the map

$$\begin{array}{rcl} [a_0, \ldots, a_k]\colon \Delta^k & \to & \Delta^n \\ (\lambda_0, \ldots, \lambda_k) & \mapsto & \sum\limits_{j=0}^{k} \lambda_j \cdot a_j. \end{array}$$

(3) Let X be a topological space and let R be a commutative ring.

(a) Let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We define the cup product of φ and ψ to be the singular cochain

$$\begin{array}{rcl} \varphi \cup \psi \colon \mathrm{C}_{p+q}(X) & \to & R \\ (\sigma \colon \Delta^{p+q} \to X) & \mapsto & \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])}_{\in R}. \end{array}$$

(b) Given a singular cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma \colon \Delta^n \to X$ with $k \leq n$ we define the cap product of φ and σ to be the singular chain

$$\varphi \cap \sigma \ := \ \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k])}_{\in R} \cdot \underbrace{\sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_n]}_{\text{map } \Delta^{n-k} \to X}.$$

For $k > n$ we define $\varphi \cap \sigma = 0$.

In this and in the coming section we will introduce simplicial analogs of the above definitions. Evidently the idea is to imitate the above definitions and to define the cup and cap products on the level of simplicial (co-) chains. A short moment's thought shows that on the (co-) chain level one needs to work with *ordered* abstract simplicial complexes.

First we have the following definition which is clearly inspired by our second definition of the cup product in singular cohomology that we gave in Definition 203.11.

Definition 184.16. Let $(K, \leq)$ be an ordered abstract simplicial complex and let R be a commutative ring. Given $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq}(K;R)$ we consider the cup product

$$\begin{array}{rcl} \varphi \cup \psi \colon \mathrm{C}^{\mathrm{simp},\leq}_{p+q}(K) & \to & R \\ \{\mathfrak{v}_0 < \cdots < \mathfrak{v}_p < \cdots < \mathfrak{v}_{p+q}\} & \mapsto & \underbrace{\varphi(\{\mathfrak{v}_0 < \ldots < \mathfrak{v}_p\})}_{\in R} \cdot \underbrace{\psi(\{\mathfrak{v}_p < \ldots < \mathfrak{v}_{p+q}\})}_{\in R}. \end{array}$$

The following lemma is the analog of the Cup Product–Coboundary Lemma 203.12.

Lemma 184.17. (Cup Product–Coboundary Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex and let R be a commutative ring. Also let $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq}(K;R)$. Then

$$\delta(\varphi \cup \psi) \ = \ \delta\varphi \cup \psi \ + (-1)^p \cdot \varphi \cup \delta\psi \ \in \ \mathrm{C}^{p+q+1}_{\mathrm{simp},\leq}(K;R).$$

In the following we will state several lemmas that are the simplicial counterparts of earlier results on the singular cup product. In most cases the proofs are, with an appropriate change of notation, basically identical. For the sake of illustration we provide the proof of the Cup Product–Coboundary Lemma 184.17 and we invite the reader to compare the proof to the proof of the original Cup Product–Coboundary Lemma 203.12.

Proof. Let $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq}(K;R)$ be simplicial cochains. We have to prove the following equality:

$$\delta(\varphi \cup \psi) \ = \ \delta\varphi \cup \psi \ + (-1)^p \cdot \varphi \cup \delta\psi \ \in \ \mathrm{C}^{p+q+1}_{\mathrm{simp},\leq}(K;R) = \mathrm{Hom}(\mathrm{C}^{\mathrm{simp},\leq}_{p+q+1}(K), R).$$

So let $s = \{\mathfrak{v}_0 < \cdots < \mathfrak{v}_{p+q+1}\}$ be a simplicial $(p+q+1)$-simplex. We have the following three equalities in R:[1959]

$$\begin{aligned}
\text{(a)}\ (\delta(\varphi\cup\psi))(s) &= (\varphi\cup\psi)\Big(\sum_{j=0}^{p+q+1}(-1)^j\cdot\{\mathfrak{v}_0<\ldots<\widehat{\mathfrak{v}_j}<\ldots<\mathfrak{v}_{p+q+1}\}\Big)\\
&= \sum_{j=0}^{p}(-1)^j\cdot(\varphi\cup\psi)(\{\mathfrak{v}_0<\ldots<\widehat{\mathfrak{v}_j}<\ldots<\mathfrak{v}_{p+q+1}\})\\
&\quad+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot(\varphi\cup\psi)(\{\mathfrak{v}_0<\ldots<\widehat{\mathfrak{v}_{k+p}}<\ldots<\mathfrak{v}_{p+q+1}\})\\
&= \sum_{j=0}^{p}(-1)^j\cdot\varphi(\{\mathfrak{v}_0<\ldots<\widehat{\mathfrak{v}_j}<\ldots<\mathfrak{v}_{p+1}\})\cdot\psi(\{\mathfrak{v}_p<\ldots<\mathfrak{v}_{p+q+1}\})\\
&\quad+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot\varphi(\{\mathfrak{v}_0<\ldots<\mathfrak{v}_p\})\cdot\psi(\{\mathfrak{v}_p<\ldots<\widehat{\mathfrak{v}_k}<\ldots<\mathfrak{v}_{p+q+1}\}),\\
\text{(b)}\ (\delta\varphi\cup\psi)(s) &= (\delta\varphi)(\{\mathfrak{v}_0<\ldots<\mathfrak{v}_{p+1}\})\cdot\psi(\{\mathfrak{v}_p<\ldots<\mathfrak{v}_{p+q+1}\})\\
&= \varphi\Big(\sum_{j=0}^{p+1}(-1)^j\cdot\{\mathfrak{v}_0<\ldots<\widehat{\mathfrak{v}_j}<\ldots<\mathfrak{v}_{p+1}\}\Big)\cdot\psi(\{\mathfrak{v}_p<\ldots<\mathfrak{v}_{p+q+1}\})\\
&= \sum_{j=0}^{p+1}(-1)^j\cdot\varphi(\{\mathfrak{v}_0<\ldots<\widehat{\mathfrak{v}_j}<\ldots<\mathfrak{v}_{p+1}\})\cdot\psi(\{\mathfrak{v}_p<\ldots<\mathfrak{v}_{p+q+1}\})
\end{aligned}$$

and

$$\begin{aligned}
\text{(c)}\ (-1)^p\cdot(\varphi\cup\delta\psi)(s) &= (-1)^p\cdot\varphi(\{\mathfrak{v}_0<\ldots<\mathfrak{v}_p\})\cdot(\delta\psi)(\{\mathfrak{v}_p<\ldots<\mathfrak{v}_{p+q+1}\})\\
&= (-1)^p\cdot\varphi(\{\mathfrak{v}_0<\ldots<\mathfrak{v}_p\})\cdot\psi\Big(\sum_{k=0}^{q+1}(-1)^k\cdot\{\mathfrak{v}_p<\ldots<\widehat{\mathfrak{v}_k}<\ldots<\mathfrak{v}_{p+q+1}\}\Big)\\
&= (-1)^p\cdot\sum_{k=0}^{q+1}(-1)^k\cdot\varphi(\{\mathfrak{v}_0<\ldots<\mathfrak{v}_p\})\cdot\psi(\{\mathfrak{v}_p<\ldots<\widehat{\mathfrak{v}_k}<\ldots<\mathfrak{v}_{p+q+1}\}).
\end{aligned}$$

Now we observe that if we take the sum of (b) and (c), then the $(j=p+1)$-summand in (b) cancels with the $(k=0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ∎

In light of the Cup Product–Coboundary Lemma 184.17 it is now straightforward to see that the cup product descends to a map on simplicial cohomology.

Lemma 184.18. (Simplicial Cup Product–Well Defined Lemma) Let R be a commutative ring. For every ordered abstract simplicial complex $(K,\leq)$ the map

$$\begin{aligned}
\cup\colon \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)\times\mathrm{H}^q_{\mathrm{simp},\leq}(K;R) &\to \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R)\\
([\varphi],[\psi]) &\mapsto [\varphi]\cup[\psi] := [\varphi\cup\psi]
\end{aligned}$$

is well-defined. We refer to this map again as the cup product.

Proof. This lemma follows almost immediately from the Cup Product–Coboundary Lemma 184.17. For more details we refer to the proof of the Cup Product–Definition Lemma 203.13. In that proof we just need to replace the Cup Product–Coboundary Lemma 203.12 by the Cup Product–Coboundary Lemma 184.17. ∎

The simplicial cup product is an interesting invariant in its own right and as we will see shortly, for finite abstract simplicial complexes it can be computed effectively. As for

[1959] Here in the argument we use on several occasions that for a simplicial n-cochain ϕ and an $(n+1)$-simplex $t=\{w_0<\ldots<w_{n+1}\}$ we have by definition

$$(\delta_{n+1}\phi)(t) \;=\; \phi(\partial_n t) \;=\; \phi\Big(\sum_{l=0}^{n+1}(-1)^l\cdot\{w_0<\ldots<\widehat{w_l}<\ldots<w_{n+1}\}\Big).$$

simplicial homology the question arises whether the isomorphism type of the simplicial cup product is an invariant of the topological realization. As we discussed in Sections 177.1 and 177.2, arguably the cleanest way of showing this is by proving that the simplicial cup product corresponds to the singular cup product. This is precisely what we will do in the first part of the following proposition, which can be viewed as an addendum to the Simplicial-Singular Cohomology Proposition 184.12.

Proposition 184.19. (Simplicial Cup Product Proposition) Let K be an abstract simplicial complex and let R be a commutative ring.

(1) Let "$\leq$" be an order on K. Let $\Theta_{\leq,k}\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ be the natural chain map that we introduced in the Simplicial-Singular Chain Complex Theorem 177.12 and the Simplicial–Singular Chain Complex Proposition 184.4. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(|K|;R) \\ {\scriptstyle \Theta^*_\leq}\downarrow\cong \quad \cong\downarrow{\scriptstyle \Theta^*_\leq} & & \cong\downarrow{\scriptstyle \Theta^*_\leq} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R). \end{array}$$

(2) There exists a unique map

$$\mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) \xrightarrow{\cup} \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R)$$

such that for every choice of order "$\leq$" for K the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R) \\ {\scriptstyle \Omega^*_\leq}\downarrow\cong \quad \cong\downarrow{\scriptstyle \Omega^*_\leq} & & \cong\downarrow{\scriptstyle \Omega^*_\leq} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R). \end{array}$$

In other words, we can define the cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ in a way that is independent of the choice of an order "$\leq$".

(3) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(|K|;R) \\ {\scriptstyle \Xi^*_p}\downarrow\cong \quad \cong\downarrow{\scriptstyle \Xi^*_q} & & \cong\downarrow{\scriptstyle \Xi^*_{p+q}} \\ \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R). \end{array}$$

Remark. Let K be an abstract simplicial complex. In the presence of an order on K we have, by definition, a natural simplicial cup product on the cochain groups $\mathrm{C}^*_{\mathrm{simp},\leq}(K;R)$. In the absence of an order we do not have a natural simplicial cup product on the cochain groups $\mathrm{C}^*_{\mathrm{simp}}(K;R)$. But the Simplicial Cup Product Proposition 184.19 says that we do have a natural simplicial cup product on the cohomology groups $\mathrm{H}^*_{\mathrm{simp}}(K;R)$. □

Proof.

(1) It follows immediately from the definitions of the simplicial cup product and the definition of the singular cup product that we gave in Definition 203.11 and that we recalled

in Definition 184.15 that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}^p(|K|;R) \times \mathrm{C}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{C}^{p+q}(|K|;R) \\ {\scriptstyle \Theta^*_\leq}\downarrow \quad \downarrow{\scriptstyle \Theta^*_\leq} & & \downarrow{\scriptstyle \Theta^*_\leq} \\ \mathrm{C}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{C}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\cup} & \mathrm{C}^{p+q}_{\mathrm{simp},\leq}(K;R). \end{array}$$

The desired statement follows immediately from this observation.

(2) Surely there is a heroic proof for this statement that does not involve singular cohomology. But to lessen the pain let us give a proof that makes the most use of our previous results. Thus we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R) \\ {\scriptstyle (\Omega^{-1}_\leq)^*}\uparrow\cong \quad \cong\uparrow{\scriptstyle (\Omega^{-1}_\leq)^*} & & \cong\uparrow{\scriptstyle (\Omega^{-1}_\leq)^*} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\leq}(K;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R) \\ {\scriptstyle \Theta^*_\leq}\uparrow\cong \quad \cong\uparrow{\scriptstyle \Theta^*_\leq} & & \cong\uparrow{\scriptstyle \Theta^*_\leq} \\ \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(|K|;R). \end{array}$$

By the Simplicial–Singular Chain Complex Proposition 184.4 together with the Cochain Homotopy-H^*-Lemma 197.14 (1) we see that the composition of the two vertical maps does not depend on the choice of "$\leq$". By (1) we know that the lower square commutes. Now we define the cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ via the singular cup product and we see that it makes the upper square commute for any choice of "$\leq$".

(3) This statement follows immediately from the definition of the natural isomorphisms Ξ^*, see the Simplicial-Singular Cohomology Proposition 184.12 (3). ■

Remark. The Simplicial Cup Product Proposition 184.19 shows in particular that if we are given a simplicial complex $(X, \Theta\colon |K| \to X)$, then we can compute the initially intractable singular cup product on $\mathrm{H}^*(X;R)$ via the simplicial cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;\mathbb{R})$. If K is finite and explicitly given, then at least in principle one can compute the simplicial cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ by brute force. This approach though is not a cure-all. For example given any $n \in \mathbb{N}$ we will want to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. By the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the real projective spaces $\mathbb{R}\mathrm{P}^n$ admit simplicial structures. But we run into the following issues:

(1) It seems to be difficult to find explicit simplicial structures for the complex projective spaces $\mathbb{C}\mathrm{P}^n$. So we cannot even get started.

(2) Explicit simplicial structures for smooth manifolds tend to have many simplices. For example in Exercise 175.5 we gave an explicit simplicial structure for the real projective space $\mathbb{R}\mathrm{P}^n$ with $2^{n+1}-1$ vertices and it seems difficult to find simplicial structures with a significantly smaller number of vertices.

Later on, in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 we will compute these cup products using the Simplicial Poincaré Duality Theorem 188.5. □

To do calculations of cup products we need to deal with specific cohomology classes and in fact also with explicit cocycles. In the next lemma we will see that closed oriented pseudomanifolds are a rich source of interesting explicit cocycles. Before we can state this lemma it is convenient to recall Definitions 183.10 and 183.13.

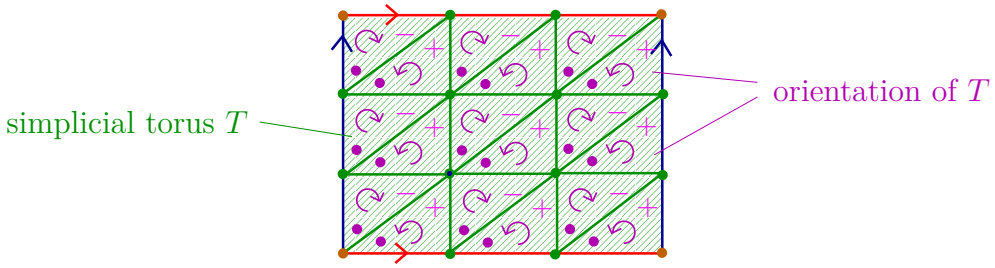
Definition 184.20. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(1) We denote by S_n the set of n-simplices of K.
(2) An orientation of an n-simplex s of K is an equivalence class of a pair $(\epsilon, \widetilde{s})$ where $\epsilon \in \{-1, 1\}$ and $\widetilde{s}$ is an ordered n-simplex with $\lfloor \widetilde{s} \rfloor = s$.
(3) An orientation of K is a choice of an orientation for each n-simplex of K such that for every $(n-1)$-simplex s of order 2 the two cobounding n-simplices induce opposite orientations on s.
(4) If K is a closed pseudomanifold that is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in S_n}$, then we refer to the simplicial homology class

$$[K] \;:=\; \Big[\sum_{w \in S_n} \epsilon_w \cdot Y_w \Big] \;\in\; \mathrm{H}_n^{\mathrm{simp}}(K)$$

as the fundamental class of K.

We illustrate the definitions for the simplicial torus in the figure below.



Now we can formulate the following lemma:

Lemma 184.21. (Dual Fundamental Class Lemma) Let $K = (V, S)$ be a closed connected n-dimensional pseudomanifold that is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in S_n}$ in the sense of Definition 183.10.

(1) (a) For any $s, t \in S_n$ we have $[\epsilon_s \cdot Y_s^*] = [\epsilon_t \cdot Y_t^*] \in \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z})$. We refer to this cohomology class as the dual fundamental class $[K]^* \in \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z})$.
    (b) We have $\mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z}) = \mathbb{Z} \cdot [K]^*$.
(2) We have $\langle [K]^*, [K] \rangle_K = 1$.

Proof.

(1) It follows from the Pseudomanifold Homology Theorem 183.12, and the discussion on page 3477 that the simplicial chain complex of K is of the form

$$0 \;\to\; \underbrace{\mathbb{Z} \cdot \Big[\sum_{w \in S_n} \epsilon_w \cdot Y_w \Big] \oplus \widetilde{C}_n}_{=\mathrm{C}_n^{\mathrm{simp}}(K)} \;\xrightarrow[\partial_n]{\begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}}\; \underbrace{\widetilde{C}_{n-1} \oplus \widehat{\mathrm{C}}_{n-1}}_{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \;\to\; \dots$$

such that $D \colon \widetilde{C}_n \to \widetilde{C}_{n-1}$ is an isomorphism. It follows easily that the map

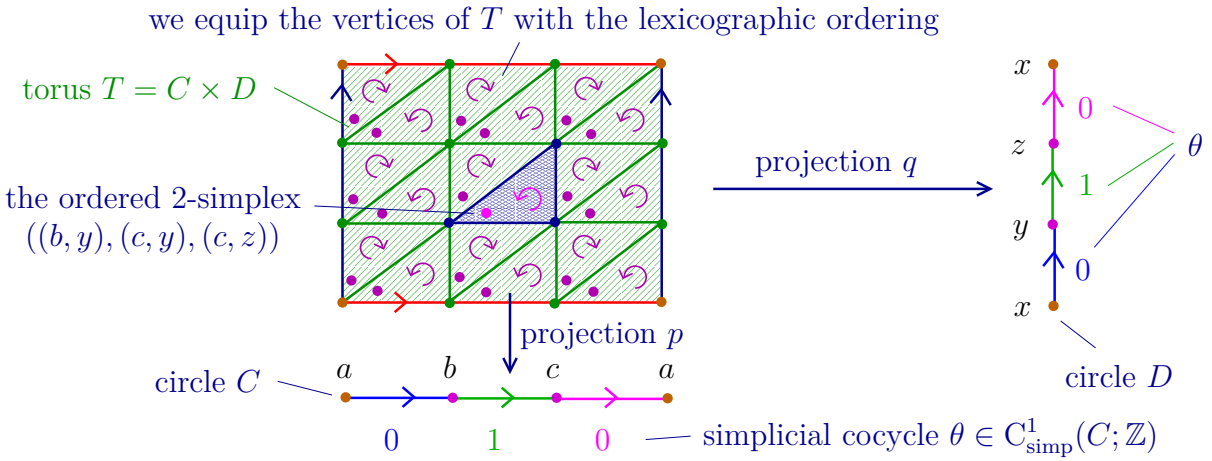
$$\begin{array}{rcccc} \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z}) = \mathrm{coker}(\delta_{n-1}) & \to & \mathrm{Hom}\Big(\mathbb{Z} \cdot \Big[\sum_{w \in S_n} \epsilon_w \cdot Y_w \Big], \mathbb{Z}\Big) & \to & \mathbb{Z} \\ \varphi & \mapsto & \text{restriction of } \varphi & & \\ & & \varphi & \mapsto & \varphi\Big(\sum_{w \in S_n} \epsilon_w \cdot Y_w \Big) \end{array}$$

is an isomorphism. The desired statements follow almost immediately from this observation.[1960]

(2) This statement follows immediately from the definitions. ∎

Now let us try to determine the simplicial cup product of the simplicial torus.

Example. We consider the simplicial circle C with vertex set $V = \{a, b, c\}$ and the simplicial circle $D = \{x, y, z\}$ with the obvious total orders on the vertex sets. Furthermore we consider the simplicial torus $T = C \times D$, where we define the product as on Definition 174.37. We equip T with the orientation shown in the figure on page 3516. We denote by $p\colon C \times D \to C$ and $q\colon C \times D \to D$ the natural projections. We equip the vertex set of $T = C \times D$ with the lexicographic ordering.



Let us first consider the simplicial cohomology of T. Let $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C)$ be the simplicial cocycle that we introduced on page 3507. We have the following equalities:

$$\mathrm{H}^i_{\mathrm{simp}}(T; \mathbb{Z}) = \begin{cases} \mathbb{Z} \cdot [\text{constant cochain } v \mapsto 1], & \text{if } i = 0, \\ \mathbb{Z} \cdot [p^*(\theta)] \oplus \mathbb{Z} \cdot [q^*(\theta)], & \text{if } i = 1, \\ \mathbb{Z} \cdot [T]^*, & \text{if } i = 2, \\ 0, & \text{if } i > 2. \end{cases}$$

This can be obtained either through a slightly heroic linear algebra exercise, or, more systematically, through the combination of the Dual Fundamental Class Lemma 184.21, the Simplicial Künneth Theorem 176.29 and the Universal Coefficient–Simplicial-H^*-Theorem 184.14.

Now we can calculate our first interesting simplicial cup product:

$$[p^*(\theta)] \cup [q^*(\theta)] = [p^*(\theta) \cup q^*(\theta)] \underset{\text{by inspection}}{=} [\overbrace{((b, y), (c, y), (c, z))}^{\text{ordered 2-simplex}}{}^*] \underset{\substack{\text{by the Dual Fundamental} \\ \text{Class Lemma 184.21}}}{=} [T]^* \in \mathrm{H}^2_{\mathrm{simp}}(T; \mathbb{Z}).$$

[1960] The above argument is related to the content of Exercise 197.6.

Similar calculations give us the following table for the cup product:[1961]

$$\begin{pmatrix} [p^*(\theta)] \cup [p^*(\theta)] & [p^*(\theta)] \cup [q^*(\theta)] \\ [q^*(\theta)] \cup [p^*(\theta)] & [q^*(\theta)] \cup [q^*(\theta)] \end{pmatrix} = \begin{pmatrix} 0 & [T]^* \\ -[T]^* & 0 \end{pmatrix}.$$

This is the simplicial analog of the calculation performed in the Torus–Cup Product Lemma 203.28. □

In the following we state several properties of the simplicial cup product that are the analogs of the corresponding properties of the singular cup product.

Proposition 184.22. (Cup Product-Ring Proposition) Let $K = (V, S)$ be an abstract simplicial complex and let R be a commutative ring. The following statements hold:

(1) The cup product on $\mathrm{H}^*_{\mathrm{simp}}(K; R)$ is R-bilinear and associative.
(2) The abelian group $\mathrm{H}^*_{\mathrm{simp}}(K; R) = \bigoplus_{n \in \mathbb{N}_0} \mathrm{H}^n_{\mathrm{simp}}(K; R)$ together with the map
$$\cup \colon \mathrm{H}^*_{\mathrm{simp}}(K; R) \times \mathrm{H}^*_{\mathrm{simp}}(K; R) \to \mathrm{H}^*_{\mathrm{simp}}(K; R)$$
is a ring where the multiplicatively neutral element is given by
$$1_K \coloneqq \big[\text{map } V \to R \text{ given by } v \mapsto 1\big] \in \mathrm{H}^0_{\mathrm{simp}}(K; R).$$

Proof. We pick an order "$\leq$" for K. As in the proof of the Cup Product-Ring Proposition 203.25 we see that the cup product on the cochain level is associative. It now follows easily that all the given statements hold for $(\mathrm{H}^*_{\mathrm{simp},\leq}(K; R), \cup)$. By the Simplicial Cup Product Proposition 184.19 (2) the statements now also hold for $(\mathrm{H}^*_{\mathrm{simp}}(K; R), \cup)$. ■

Proposition 184.23. (Simplicial Cup Product-Commutativity Proposition) Let K be an abstract simplicial complex and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K; R)$ and any $\psi \in \mathrm{H}^q_{\mathrm{simp}}(K; R)$ we have
$$\varphi \cup \psi = (-1)^{pq} \cdot \psi \cup \varphi \in \mathrm{H}^{p+q}_{\mathrm{simp}}(K; R).$$

Proof. We pick an order "$\leq$" for K. By the Simplicial Cup Product Proposition 184.19 (2) it suffices to prove the following claim:

Claim. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K; R)$ and $\psi \in \mathrm{H}^q_{\mathrm{simp},\leq}(K; R)$ we have
$$\varphi \cup \psi = (-1)^{pq} \cdot \psi \cup \varphi \in \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K; R).$$

Proof. There are two possible approaches to proving the claim:

(1) The claim follows immediately from the corresponding singular statement by applying the Simplicial-Singular Cohomology Proposition 184.12, the Simplicial Cup Product Proposition 184.19 and the Cup Product-Commutativity Proposition 203.20
(2) One can also prove the claim without making recourse to singular cohomology. In fact the claim can be proved directly, by modifying the argument in the proof of the Cup Product-Commutativity Proposition 203.20. The only real difference is that one needs to replace the product topological space $\Delta^n \times [0, 1]$ by the product of the ordered abstract simplicial complexes $D_n = (\{0, \ldots, n\}, \mathcal{P}(\{0, \ldots, n\} \setminus \{\varnothing\}))$ and

[1961] In fact the remaining three entries of the matrix can also be obtained easily from the Simplicial Cup Product-Commutativity Proposition 184.23 and the Simplicial Cohomology Ring–Functor Lemma 184.24.

$D_1 = (\{0,1\}, \{\{0\},\{1\}\})$, as defined in the Simplicial Complex-Product Lemma 174.38. We leave it to the reader to make the necessary modifications to the argument. ■

Lemma 184.24. (Simplicial Cohomology Ring–Functor Lemma) Let R be a commutative ring.

(1) Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(L;R)$ and $\psi \in \mathrm{H}^q_{\mathrm{simp}}(L;R)$ we have

$$f^*(\varphi) \cup f^*(\psi) \quad = \quad f^*(\varphi \cup \psi).$$

In particular the map

$$f^*\colon \mathrm{H}^*_{\mathrm{simp}}(L;R) \quad \to \quad \mathrm{H}^*_{\mathrm{simp}}(K;R)$$

is a homomorphism of graded rings which satisfies $f^*(1_L) = 1_K$.

(2) The maps

$$\begin{array}{rcl} K & \mapsto & (\mathrm{H}^*_{\mathrm{simp}}(K;R), \cup) \\ (f\colon K \to L) & \mapsto & (f^*\colon \mathrm{H}^*_{\mathrm{simp}}(L;R) \to \mathrm{H}^*_{\mathrm{simp}}(K;R)) \end{array}$$

define a contravariant functor from the category *AbsSimpCplx* of abstract simplicial complexes to the category *GrRing* of graded rings.

Proof.

(1) Let $f\colon K = (V,S) \to L = (W,T)$ be a simplicial map between two abstract simplicial complexes. By Exercise 1.6 we can pick total orders "$\leq_K$" on K and "$\leq_L$" on L such that the map $f\colon V \to W$ is order-preserving. It follows basically immediately from the definitions that for $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq_L}(L;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\leq_L}(L;R)$ we have the equality $f^*(\varphi) \cup f^*(\psi) = f^*(\varphi \cup \psi) \in \mathrm{C}^{p+q}_{\mathrm{simp},\leq_K}(K)$. The statements now follow from the Simplicial Cup Product Proposition 184.19 (2).

(2) This statement follows from (1) together with the Simplicial Cohomology–Functor Lemma 184.11 (2). ■

184.5. **The simplicial cap product.** In this section we will introduce the simplicial cap product. In contrast to the simplicial cup product it is perhaps less immediately clear what the simplicial cap product is good for. But it will have its moment of glory in Chapter 185 when we state and prove the Simplicial Poincaré Duality Theorem 188.5.

Definition 184.25. Let $(K,\leq)$ be an ordered abstract simplicial complex and let R be a commutative ring. Given an n-simplex $s = \{\mathfrak{v}_0 < \cdots < \mathfrak{v}_n\}$, given $k \leq n$ and given a simplicial cochain $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;R)$ we define

$$\varphi \cap s \;:=\; \underbrace{\varphi(\{\mathfrak{v}_0 < \cdots < \mathfrak{v}_k\})}_{\in R} \cdot \underbrace{\{\mathfrak{v}_k < \cdots < \mathfrak{v}_n\}}_{\in \mathrm{C}^{\mathrm{simp},\leq}_{n-k}(K)} \;\in\; \mathrm{C}^{\mathrm{simp},\leq}_{n-k}(K;R).$$

We extend this definition to simplicial R-chains and we obtain the cap product

$$\begin{array}{rcl} \cap\colon \mathrm{C}^k_{\mathrm{simp},\leq}(X;R) \times \mathrm{C}^{\mathrm{simp},\leq}_n(X;R) & \to & \mathrm{C}^{\mathrm{simp},\leq}_{n-k}(X;R) \\ \varphi \cap r \cdot \sigma & \mapsto & r \cdot \underbrace{(\varphi \cap \sigma)}_{\in R}. \end{array}$$

For $k > n$ we define the cap product to be the zero map.

The next lemma says that the cap product defines a map on simplicial (co-) homology.

Lemma 184.26. (Simplicial Cap Product–Definition Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex. Given any commutative ring R and given any $k, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \to & \mathrm{H}_{n-k}^{\mathrm{simp},\leq}(K;R) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi \cap \sigma] \end{array}$$

is well-defined.

Proof. The same argument as in the proof of the Cap Product–Boundary Lemma 206.3 shows that given an n-simplex $s = \{\mathfrak{v}_0 < \cdots < \mathfrak{v}_n\}$, given $k \leq n$ and given any simplicial cochain $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;R)$ we have the following equality:

$$\partial_{n-k}(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta_k(\varphi) \cap \sigma \;+\; \varphi \cap \partial_n(\sigma)) \;\in\; \mathrm{C}_{n-k-1}(X;R).$$

As in the proof of the Cap Product–Definition Lemma 206.4 we see that the lemma follows easily from this equality. ∎

The next proposition is an analog of the Simplicial Cup Product Proposition 184.19.

Proposition 184.27. (Simplicial Cap Product Proposition) Let K be an abstract simplicial complex and let R be a commutative ring.

(1) Let "$\leq$" be an order on K. Let $\Theta_{\leq,k} \colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ be the natural chain map that we introduced in the Simplicial–Singular Chain Complex Proposition 184.4. The following diagram commutes:[1962]

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}_n(|K|;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}(|K|;R) \\ {\scriptstyle\Theta^*_{\leq}}\downarrow\cong \qquad \cong\uparrow{\scriptstyle\Theta_{\leq,*}} & & \cong\uparrow{\scriptstyle\Theta_{\leq,*}} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp},\leq}(K;R). \end{array}$$

(2) There exists unique map

$$\mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}_n^{\mathrm{simp}}(K;R) \xrightarrow{\quad\cap\quad} \mathrm{H}_{n-p}^{\mathrm{simp}}(K;R)$$

such that for every choice of order "$\leq$" for K the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}_n^{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp}}(K;R) \\ {\scriptstyle\Omega^*_{\leq}}\downarrow\cong \qquad \cong\uparrow{\scriptstyle\Omega_{\leq,*}} & & \cong\uparrow{\scriptstyle\Omega_{\leq,*}} \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R) \times \mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp},\leq}(K;R). \end{array}$$

In other words, we can define the cap product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ and $\mathrm{H}_*^{\mathrm{simp}}(K;R)$ in a way that is independent of the choice of an order "$\leq$".

(3) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}_n(|K|;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}(|K|;R) \\ {\scriptstyle\Xi^*}\downarrow\cong \qquad \cong\uparrow{\scriptstyle\Xi_*} & & \cong\uparrow{\scriptstyle\Xi_*} \\ \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}_n^{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}^{\mathrm{simp}}(K;R). \end{array}$$

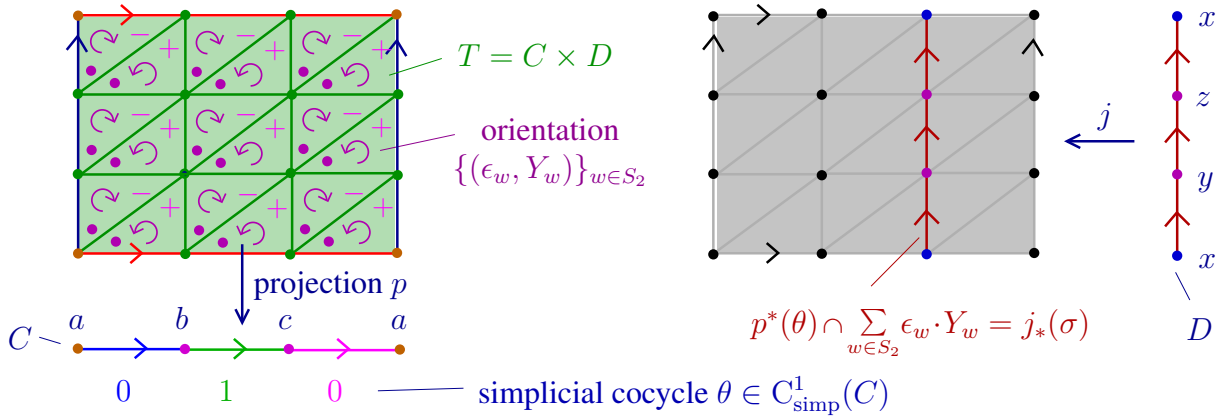
Sketch of proof.

[1962] Here the expression "the diagram commutes" is understood in the sense of the Convention 204.17.

(1) It follows almost immediately from the definition that the corresponding diagram on the level of simplicial (co-) chains commutes. This implies that the given diagram commutes.
(2) As in the proof of the Simplicial Cup Product Proposition 184.19 (2) this statement follows from (1) together with the Simplicial–Singular Chain Complex Proposition 184.4, the Homotopy Equivalence-H_*-Proposition 144.14 and the Cochain Homotopy-H^*-Lemma 197.14.
(3) This statement follows immediately from (1), (2), the definition of the natural isomorphisms Ξ_* and Ξ^* (see the Simplicial–Singular Chain Complex Proposition 184.4 (3)) and the Simplicial-Singular Cohomology Proposition 184.12 (3). ■

Example. As on page 3517 we consider again the simplicial torus $T = C \times D$. We equip $T = (V, S)$ with the orientation $\{(\epsilon_w, Y_w)\}_{w \in S_2}$ shown in the figures on page 3516 and 3521 and we denote by $p\colon C \times D \to C$ the natural projection. Furthermore, we again make use of the simplicial cocycle $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C; \mathbb{Z})$ that we first encountered on page 3507. We introduce the following extra notation:

(1) We denote by $j\colon D \to C \times D$ the simplicial map that is given by $v \mapsto (c, v)$.
(2) We denote by $\sigma := \{x, y\} + \{y, z\} + \{z, x\} \in \mathrm{C}_1^{\mathrm{simp},\leq}(D)$ the "obvious" simplicial cycle. Note that it represents the simplicial fundamental class $[D] \in \mathrm{H}_1^{\mathrm{simp},\leq}(D)$.



It follows immediately from the definitions that

$$p^*(\theta) \cap \sum_{w \in S_2} \epsilon_w \cdot Y_w \;=\; j_*(\sigma) \;\in\; \mathrm{C}_1^{\mathrm{simp},\leq}(C \times D),$$

thus in terms of (co-) homology we have

$$p^*([\theta]) \cap [T] \;=\; j_*([D]) \;\in\; \mathrm{H}_1^{\mathrm{simp},\leq}(C \times D). \qquad \square$$

In the following we state three lemmas that summarize the key properties of the simplicial cap product. All of these statements are analogs of singular statements. First, the following lemma is an analog of the Cap Product-Kronecker Lemma 206.12.

Lemma 184.28. (Simplicial Cap Product-Kronecker Lemma) Let K be a connected non-empty abstract simplicial complex. Given any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K; R)$ and $\sigma \in$

$\mathrm{H}_p^{\mathrm{simp}}(K;R)$ the following equality holds:

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma \rangle_K \;\in\; \mathrm{H}_0^{\mathrm{simp}}(K;R) \;\underset{\text{as in Exercise 176.3 there is a natural identification}}{=}\; R.$$

Proof. First we pick an order "$\leq$" for the abstract simplicial complex K. Next we observe that the statement for $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$ and $\sigma \in \mathrm{H}_p^{\mathrm{simp},\leq}(K;R)$ follows immediately from the definitions. Now the statement follows from the Simplicial Cap Product Proposition 184.27 (2). ■

The following lemma relates the simplicial cup product to the simplicial cap product. It can be viewed as an analog of the Cup-Cap Lemma 206.15.

Lemma 184.29. (Simplicial Cup-Cap Lemma) Let K be an abstract simplicial complex and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$, any $\psi \in \mathrm{H}^q_{\mathrm{simp}}(K;R)$ and any $\sigma \in \mathrm{H}_n^{\mathrm{simp}}(K;R)$ we have

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}^{\mathrm{simp}}_{n-p-q}(K;R).$$

Sketch of proof. We pick an order "$\leq$" for K. With basically the same, quite elementary, argument as in the proof of the Cup-Cap Lemma 206.15 we see that for any $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$, any $\psi \in \mathrm{H}^q_{\mathrm{simp},\leq}(K;R)$ and any $\sigma \in \mathrm{H}_n^{\mathrm{simp},\leq}(K;R)$ we have

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}^{\mathrm{simp},\leq}_{n-p-q}(K;R).$$

The desired statement now follows from the Simplicial Cap Product Proposition 184.27 (2). ■

Finally we have the following analog of the Cap Product-Naturality Lemma 206.13.

Lemma 184.30. (Simplicial Cap Product-Naturality Lemma) Let R be a commutative ring and let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. Given any $p, n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^p_{\mathrm{simp}}(K;R) & \times & \mathrm{H}_n^{\mathrm{simp}}(K;R) & \xrightarrow{\cap} & \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R) \\
{\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{H}^p_{\mathrm{simp}}(L;R) & \times & \mathrm{H}_n^{\mathrm{simp}}(L;R) & \xrightarrow{\cap} & \mathrm{H}^{\mathrm{simp}}_{n-p}(L;R),
\end{array}$$

i.e. for any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(L;R)$ and any $\sigma \in \mathrm{H}_n^{\mathrm{simp}}(K;R)$ we have

$$\varphi \cap f_*(\sigma) \;=\; f_*(\underbrace{f^*(\varphi) \cap \sigma}_{\in \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R)}) \;\in\; \mathrm{H}^{\mathrm{simp}}_{n-p}(L;R).$$

Proof. Let $f\colon K = (V,S) \to L = (W,T)$ be a simplicial map between two abstract simplicial complexes. By Exercise 1.6 we can pick total orders "$\leq_K$" on K and "$\leq_L$" on L such that the map $f\colon V \to W$ is order-preserving. It follows basically immediately from the definitions that for any $\varphi \in \mathrm{C}^p_{\mathrm{simp},\leq_L}(L;R)$ and $\sigma \in \mathrm{C}_n^{\mathrm{simp},\leq_K}(K;R)$ we have the equality $f_*(\sigma) = f_*(f^*(\varphi) \cap \sigma)$. The promised statement now follows from the Simplicial Cap Product Proposition 184.27 (2). ■

We conclude this chapter with a short calculation which in particular shows that at times simplicial cup products can be computed using simplicial cap products.

Example. Let us look one last time at the simplicial torus $T = C \times D$ with the natural projections $p\colon C \times D \to C$ and $q\colon C \times D \to D$. We will now see how we can obtain the simplicial cup product $p^*(\theta) \cup q^*(\theta)$, that we initially computed on page 3518, from the calculation of the simplicial cap product on page 3521. Indeed, under the natural identification $\mathrm{H}_0^{\mathrm{simp}}(T;R) = R = \mathrm{H}_0^{\mathrm{simp}}(D;R)$ from Exercise 176.3 we have the following equalities:

$$\begin{array}{lllll}
\langle p^*([\theta]) \cup q^*([\theta]), [T]\rangle_T & \overset{\text{by the Simplicial Cap Product Kronecker Lemma 184.28}}{=} & (p^*([\theta]) \cup q^*([\theta])) \cap [T] & \overset{\text{by the Simplicial Cup-Cap Lemma 184.29}}{=} & q^*([\theta]) \cap (p^*([\theta]) \cap [T]) \\
= \langle q^*([\theta]), j_*([D])\rangle_T & = & \langle [\theta], (q_* \circ j_*)([D])\rangle_D & = & \langle [\theta], [D]\rangle_D \\
\overset{\text{by the Simplicial Cap Product Kronecker Lemma 184.28}}{=} & & \overset{\text{by the Kronecker Pairing Lemma 184.13 (2)}}{=} & & \overset{\text{since } q\circ j = \mathrm{id}_D}{=} \\
& & & & \overset{\text{by definition of } \theta \text{ and } [D]}{=} 1.
\end{array}$$

$$\overset{\text{see the calculation on page 3521}}{=} q^*([\theta]) \cap j_*([D])$$

In Chapter 185 we will state the Simplicial Poincaré Duality Theorem 188.5. Given a closed oriented PL-manifold K it allows for the calculation of simplicial cap products of the form $\varphi \cap [K]$. With tricks as above one can use this calculation of a simplicial cap product to determine several interesting simplicial cup products. In particular in Chapter 213 we will use this approach to calculate the (simplicial) cup products of the real and complex projective spaces. □

184.6. **Appendix: Quasi-Isomorphisms and Splittings.** In the Quasi Iso–Homotopy Equivalence Proposition 158.8 we showed that if $f\colon C_* \to D_*$ is a chain map between chain complexes of *free abelian groups* and if f is a quasi-isomorphism, i.e. if f induces an isomorphism of all homology groups, then f is in fact a chain homotopy equivalence. The following proposition is a convenient variation on this result.

Proposition 184.31. (Chain Homotopy Equivalence Criterion) Let C_* and D_* be two chain complexes of free abelian groups and let $\iota\colon C_* \to D_*$ be a chain map with the following two properties:

(a) The chain map $\iota\colon C_* \to D_*$ is a quasi-isomorphism, i.e. it induces an isomorphism of homology groups.
(b) For each $k \in \mathbb{N}_0$ the map $\iota\colon C_k \to D_k$ is a monomorphism and the image is a subsummand of D_k.

Then there exists a chain map $\pi\colon D_* \to C_*$ with the following two properties:

(1) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \iota_k = \mathrm{id}_{C_k}$.
(2) The maps ι and π are chain homotopy equivalences.

Proof. Let C_* and D_* be two chain complexes of free abelian groups and let $\iota\colon C_* \to D_*$ be a chain map with the following two properties:

(a) The chain map $\iota\colon C_* \to D_*$ induces an isomorphism of homology groups.
(b) For each $k \in \mathbb{N}_0$ the map $\iota\colon C_k \to D_k$ is a monomorphism and the image is a subsummand of D_k.

We need to show that there exists a chain map $\pi\colon D_* \to C_*$ with the following two properties:

(1) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \iota_k = \mathrm{id}_{C_k}$.
(2) The maps ι and π are chain homotopy equivalences.

The key to proving the proposition is the following claim:

Claim. We set $C_{-1} = D_{-1} = 0$ and we define π_{-1} to be the zero homomorphism. Let $k \in \mathbb{Z}_{\geq -1}$. Suppose that for $j = -1, \dots, k$ we are given homomorphisms $\pi_j \colon D_j \to C_j$ such that for each $j \in \{0, \dots, k\}$ we have the following equalities:
(1) $\pi_j \circ \iota_j = \mathrm{id}_{C_j}$.
(2) $\partial_j \circ \pi_j = \pi_{j-1} \circ \partial_j$.
Then there exists a homomorphism $\pi_{k+1} \colon D_{k+1} \to C_{k+1}$ such that the maps $\pi_0, \dots, \pi_{k+1}$ have the above properties.

Proof. First note that by hypothesis we can pick for each $m \in \mathbb{N}$ a submodule $C_m^\dagger$ of D_m such that $D_m = \iota(C_m) \oplus C_m^\dagger$. For orientation it is perhaps convenient to consider the following maps:

$$\begin{array}{ccc} D_{k+1} = \iota(C_{k+1}) \oplus C_{k+1}^\dagger & \xrightarrow{\partial_D} & \iota(C_k) \oplus C_k^\dagger = D_k \\ {\scriptstyle \iota_{k+1}}\uparrow \quad \downarrow {\scriptstyle ?\pi_{k+1}?} & & {\scriptstyle \iota_k}\uparrow \quad \downarrow {\scriptstyle \pi_k} \\ C_{k+1} & \xrightarrow[\partial_C]{} & C_k. \end{array}$$

Since $C_{k+1}^\dagger$ is a subgroup of the free abelian group D_{k+1}, we know by the Free Abelian Subgroup Theorem 97.14 that $C_{k+1}^\dagger$ is also a free abelian group. We pick a basis T for the free abelian group $C_{k+1}^\dagger$. Let $t \in T$.

Since ι induces an isomorphism of homology groups, we know that there exists a $c_{k+1} \in C_{k+1}$ such that $t \in C_{k+1}^\dagger$ and $\iota(c_{k+1}) \in D_{k+1}$ are homologous in D_*. We calculate that

$$\pi_k(\partial_D(t_k)) \underset{\substack{\uparrow \\ \text{since } t_{k+1} \text{ and } \iota(c_{k+1}) \text{ are homologous} \\ \text{we know that } \partial_D(t_{k+1}) = \partial_D(\iota(c_{k+1}))}}{=} \pi_k(\partial_D(\iota_{k+1}(c_{k+1}))) \underset{\substack{\uparrow \\ \text{since } \iota \text{ is a} \\ \text{chain map}}}{=} \pi_k(\iota_k(\partial_C(c_{k+1}))) \underset{\substack{\uparrow \\ \text{since } \pi_k \circ \iota_k = \mathrm{id}_{C_k}}}{=} \partial_C(c_{k+1}).$$

We set $\varphi(t) \coloneqq c_{k+1}$. Since T_{k+1} is a basis for the free abelian group $C_{k+1}^\dagger$ we obtain a homomorphism $\varphi \colon C_{k+1}^\dagger \to C_{k+1}$. It is now straightforward to verify that the map

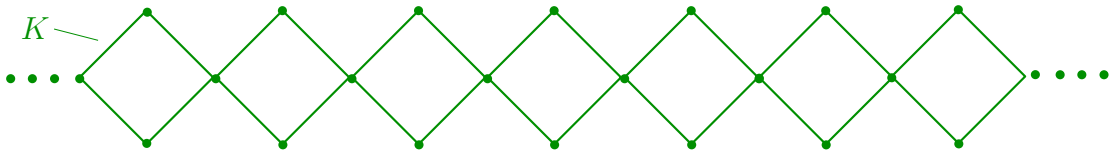
$$\begin{array}{rcl} \pi_{k+1} \colon D_{k+1} = \iota(C_{k+1}) \oplus C_{k+1}^\dagger & \to & C_{k+1} \\ \iota(c_{k+1}) + d_{k+1} & \mapsto & c_{k+1} + \varphi(d_{k+1}) \end{array}$$

has all the desired properties. ⊞

Since C_* and D_* are chain complexes of free abelian groups and since ι induces an isomorphism of homology groups, we know by the Quasi Iso–Homotopy Equivalence Proposition 158.8 that ι is a chain homotopy equivalence. By Property (2) of the claim the maps $\{\pi_n \colon D_n \to C_n\}_{n \in \mathbb{N}_0}$ define a chain map. By Property (1) of the claim we have for each $k \in \mathbb{N}_0$ that $\pi_* \circ \iota_* = \mathrm{id}_{\mathrm{H}_k(C_*)}$. Thus we see that the chain map π_* also induces isomorphisms of the homology groups. So once again we appeal to the Quasi Iso–Homotopy Equivalence Proposition 158.8 to conclude that the maps π_* form a chain homotopy equivalence. ■

184.7. Exercises for Chapter 184 .

Exercise 184.1. We consider the abstract simplicial complex K shown in the figure below. Determine the isomorphism type of $\mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{Z})$.



Exercise 184.2. Let $K = (V, S)$ be a non-empty abstract simplicial complex. We assume that K is connected. Recall that by Exercise 174.13 this means that given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$. Now let G be an abelian group. Given any $g \in G$ we consider the simplicial cochain $\varphi_g \colon \mathrm{C}_0^{\mathrm{simp}}(K) \to G$ that is given by $\varphi_g(v) = g$ for every vertex $v \in V$. Show that the map

$$\begin{array}{ccc} G & \to & \mathrm{H}^0_{\mathrm{simp}}(K; G) \\ g & \mapsto & \varphi_g \end{array}$$

is a natural isomorphism.

Remark. This statement can be viewed is the cohomological analog of Exercise 176.3.

Exercise 184.3. Let K be an abstract simplicial complex and let L be a subcomplex of K. Given an abelian group G and given $k \in \mathbb{N}_0$ we define

$$\mathrm{H}^k_{\mathrm{simp}}(K, L; G) \;:=\; \mathrm{H}^k(\mathrm{C}^{\mathrm{simp}}_*(K, L), G).$$

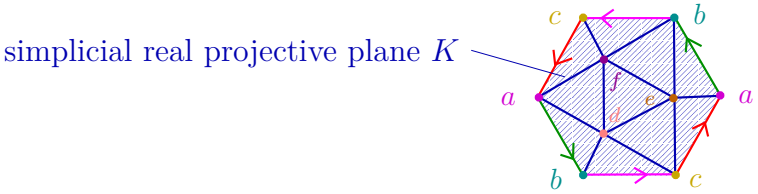
Show that there exists a natural isomorphism

$$\mathrm{H}^k_{\mathrm{simp}}(K, L; G) \;\cong\; \widetilde{\mathrm{H}}^k_{\mathrm{simp}}(\underbrace{K \cup_L \mathrm{Cone}(L)}_{\text{mapping cone of } L \to K}; G).$$

Exercise 184.4. In the figure below we show an abstract simplicial complex K whose topological realization $|K|$ is homeomorphic to the real projective plane $\mathbb{R}\mathrm{P}^2$.

(a) Show that for $i = 0, 1, 2$ we have $\mathrm{H}^i_{\mathrm{simp}}(K; \mathbb{F}_2) \cong \mathbb{F}_2$ and identify in each dimension a cocycle that represents the unique non-zero element.

(b) Compute the simplicial cup product $\mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{F}_2) \times \mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{F}_2) \xrightarrow{\cup} \mathrm{H}^2_{\mathrm{simp}}(K; \mathbb{F}_2)$. In other words, determine whether it is trivial or non-trivial.



Exercise 184.5. Let $g \in \mathbb{N}$. First pick a suitable abstract simplicial complex K that has the property that its topological realization is homeomorphic to the surface of genus g. Afterwards compute the cup product

$$\mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{Z}) \times \mathrm{H}^1_{\mathrm{simp}}(K; \mathbb{Z}) \;\to\; \mathrm{H}^2_{\mathrm{simp}}(K; \mathbb{Z}).$$

Exercise 184.6. We consider the topological space

$$X = \overline{B}^2 / \sim \qquad \text{where } z \sim z \cdot \exp(2\pi \mathrm{i}/3) \text{ for } z \in S^1.$$

(a) Determine the isomorphism types of $\mathrm{H}_i(X; \mathbb{F}_3)$ and $\mathrm{H}^i(X; \mathbb{F}_3)$.

(b) Give an explicit simplicial structure K for X.
(c) Compute the simplicial cup product $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{F}_3) \times \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{F}_3) \xrightarrow{\cup} \mathrm{H}^2_{\mathrm{simp}}(K;\mathbb{F}_3)$.

CHAPTER 185

The fundamental class of smooth manifolds

Let M be a compact *orientable* connected non-empty n-dimensional smooth manifold. By the Manifold-Top-H_*-Theorem 183.20 we now know that $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$. In this chapter we will see that a *choice of an orientation* naturally determines a generator $[M]$ of $H_n(M, \partial M; \mathbb{Z})$, called the fundamental class of M. As the name already suggests, this homology class is of fundamental importance in the study of smooth manifolds.

This chapter is organized as follows:

- In Section 185.1 we will introduce the fundamental class of a smooth manifold.
- In Section 185.2 we will provide examples of fundamental classes.
- In Section 185.3 we will discuss properties of the fundamental class.
- In Section 185.4 we will determine the homology groups of the connected sum of two smooth manifolds.
- In Section 185.5 we will determine the homology groups of knot complements.
- In Section 185.6 we will introduce the $\mathbb{F}_2$-fundamental class.
- In Section 185.7 we will provide exercises for Chapter 185.

In the next Chapter 186 we will (re-) introduce the degree of a continuous map between oriented smooth manifolds.

185.1. **The definition of the fundamental class.** For our definition of fundamental classes we need to recall the following notation:

Notation 185.1. Let $n \in \mathbb{N}_0$. As in Definition 143.8 we consider

$$\begin{aligned} \partial\Delta^n &:= \{(t_0, \dots, t_n) \in \Delta^n \mid \text{at least one coordinate is zero}\} \\ \mathring{\Delta}^n &:= \{(t_0, \dots, t_n) \in \Delta^n \mid \text{all coordinates are non-zero}\}. \end{aligned}$$

We view $\mathring{\Delta}^n$ as a smooth manifold and we equip it with the orientation from Definition 143.8. Loosely speaking it is the orientation given by the "outward pointing vector $(1, \dots, 1)$".

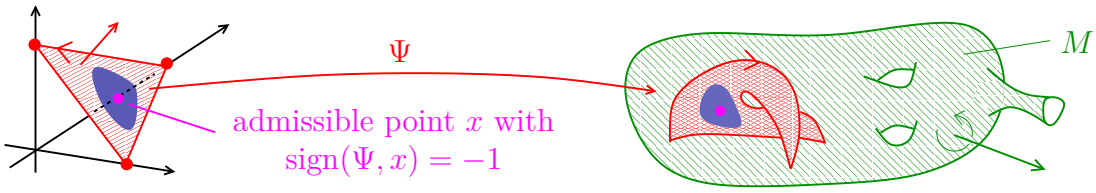
We move on to the following somewhat technical definition:

Definition 185.2. Let M be an oriented n-dimensional smooth manifold. Furthermore let $\Psi\colon \Delta^n \to M$ be a continuous map.

(1) We say that a point $x \in \mathring{\Delta}^n$ is admissible for Ψ if $\Psi^{-1}(\Psi(x)) = \{x\}$ and if there exists an open connected neighborhood U of $x \in \mathring{\Delta}^n$ such that the map $\Psi|_U\colon U \to M$ is a smooth embedding.

(2) Given an admissible point x for Ψ we define

$$\operatorname{sign}(\Psi, x) = \begin{cases} +1, & \text{if } \Psi \text{ is orientation-preserving at } x, \\ -1, & \text{if } \Psi \text{ is orientation-reversing at } x. \end{cases}$$



Example. In many cases we will consider embeddings $\Psi\colon \Delta^n \to M$ such that the restriction of Ψ to $\mathring{\Delta}^n$ is a smooth embedding. In this case every point $x \in \mathring{\Delta}^n$ is admissible for Ψ, and we have

$$\operatorname{sign}(\Psi, x) = \begin{cases} +1, & \text{if } \Psi\colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \Psi\colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

Also note that it is clear that any non-empty smooth manifold admits such a smooth embedding. □

The following theorem is the main technical result of this chapter.

Theorem 185.3. (Manifold-Top-H_*-Theorem) Let $n \in \mathbb{N}$ and let M be a compact oriented connected n-dimensional smooth manifold.

(1) For every $x \in M \setminus \partial M$ the inclusion-induced map

$$\omega_x\colon \quad \underbrace{\mathrm{H}_n(M, \partial M)}_{\substack{\cong \mathbb{Z} \text{ by the Top Homology} \\ \text{Theorem 183.20}}} \quad \to \quad \underbrace{\mathrm{H}_n(M, M \setminus \{x\})}_{\substack{\cong \mathbb{Z} \text{ by the Manifold-Local} \\ \text{Homology Lemma 150.9}}}$$

is an isomorphism.

(2) There exists a unique $[M] \in \mathrm{H}_n(M, \partial M)$ with the following property: For every continuous map $\Psi\colon \Delta^n \to M$ and every admissible $x \in \mathring{\Delta}^n$ for Ψ we have the following equality in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:[1963] [1964]

$$\underset{\substack{\uparrow \\ \text{map } \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M, M\setminus\{\Psi(x)\}) \text{ as in (1)}}}{\omega_{\Psi(x)}([M])} = \operatorname{sign}(\Psi, x) \cdot [\, \underbrace{\Psi\colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})} \,].$$

(3) The uniquely determined element $[M] \in \mathrm{H}_n(M, \partial M)$ from (2) is a generator, in particular we have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$.

Proof of Theorem 185.3 (1) for closed manifolds. Let $n \in \mathbb{N}$ and let M be a closed oriented connected n-dimensional smooth manifold. Recall that it follows from the Smooth Manifold-Simplicial Structure Theorem 179.5 that we can pick a finite smooth simplicial structure $(K = (V, S), \lambda\colon |K| \to M)$ for the smooth manifold M. Since V is finite, we can easily pick a total order "$\leq$" on the vertex set V.

In the following we will use the results from Section 183.1 to construct an explicit singular cycle in $\mathrm{C}_n(M)$ which represents a generator of $\mathrm{H}_n(M)$. We will use this cycle to prove (1). Afterwards we will show that the homology class $[M]$ that it represents actually has all the properties stated in (2).

[1963] Note that by definition of a smooth embedding, see Definition 30.9, we know that $\Psi(x) \subseteq M \setminus \partial M$. This implies that we have an induced map $\mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(M, M \setminus \{\Psi(x)\}$.

[1964] Since x is an admissible point for Ψ, we know that $\Psi^{-1}(\Psi(x)) = \{x\}$. Note that this implies in particular that $\Psi(\partial\Delta^n) \subseteq M \setminus \{\Psi(x)\}$, which in turn implies that Ψ is a cycle in $\mathrm{C}_*(M, M \setminus \{\Psi(x)\})$.

In the following we will make heavy use of the objects we introduced in Section 183.1. Even if the reader might no longer be completely familiar with all the technical definitions, the overall flow of the argument should still be easy to follow. Without further ado, let us construct the promised singular cycle.

(a) Given an m-simplex s of K we consider the embedding

$$\begin{array}{rccl} \Phi_s^{\leq}\colon & \Delta^m & \to & |K| \\ & (t_0,\dots,t_m) & \mapsto & \sum_{i=0}^{m} t_i \cdot i\text{-th vertex of } s. \end{array}$$

here we use the total order on V to order the vertices of s

(b) We denote by $s_1,\dots,s_k$ the n-simplices of K.

(c) Let $i \in \{1,\dots,k\}$. It follows from the definition of a smooth simplicial structure, see Definition 179.3, that the map $\lambda \circ \Phi_{s_i}^{\leq}\colon \mathring{\Delta}^n \to M$ is a smooth embedding. Now we set

$$\epsilon_i \coloneqq \begin{cases} +1, & \text{if } \lambda \circ \Phi_{s_i}^{\leq}\colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi_{s_i}^{\leq}\colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

(d) By the Pseudomanifold-Orientation Proposition 183.19 we know that the oriented n-simplices $[(\epsilon_1, s_1)],\dots,[(\epsilon_k, s_k)]$ form an orientation, in the sense of Definition 183.10, for the pseudo-manifold K.

(e) Next we consider the maps

isomorphism by the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 ↓ — chain homotopy equivalence by the Simplicial-Singular Chain Complex Theorem 177.12 ↓

$$\begin{array}{ccccccc} \mathrm{C}_m^{\mathrm{simp}}(K) & \leftarrow & \mathrm{C}_m^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_m(|K|) & \xrightarrow{\lambda_*} & \mathrm{C}_m(M) \\ \underbrace{[\mathfrak{v}_0,\dots,\mathfrak{v}_m]}_{=[s]} & \mapsto & \underbrace{\{\mathfrak{v}_0 < \dots < \mathfrak{v}_m\}}_{=s} & \mapsto & \Phi_s^{\leq} & & \end{array}$$

(f) It follows from (d) together with the Pseudomanifold Homology Theorem 183.12 (1) that $\epsilon_1 \cdot [s_1] + \dots + \epsilon_k \cdot [s_k]$ is a cycle in the simplicial chain complex $\mathrm{C}_*^{\mathrm{simp}}(K)$ and we know that it represents a generator of $\mathrm{H}_n^{\mathrm{simp}}(K)$.

(g) It now follows from (e) and (f) that $c \coloneqq \epsilon_1 \cdot \lambda \circ \Phi_{s_1}^{\leq} + \dots + \epsilon_k \cdot \lambda \circ \Phi_{s_k}^{\leq} \in \mathrm{C}_n(M)$ is a singular cycle in $\mathrm{C}_*(M)$ and that it represents a generator of $\mathrm{H}_n(M)$.

Now we turn to the actual proof of (1). As a reminder, recall that we need to prove the following claim:

Claim. For every $x \in M$ the map

$$\omega_x\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, M \setminus \{x\})$$

is an isomorphism.

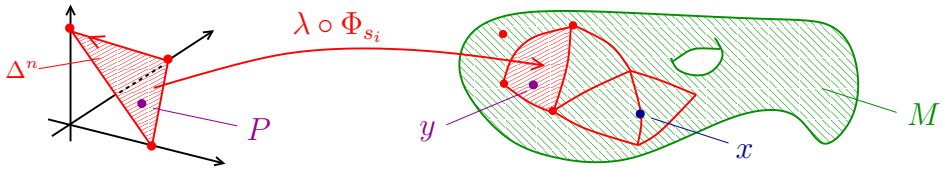
Proof. First we take care of the special case of a point $y \in M$ which can be written as $y = (\lambda \circ \Phi_{s_i}^{\leq})(P)$ for some $i \in \{1,\dots,k\}$ and some $P \in \mathring{\Delta}^n$. We consider the following two maps:

$$\begin{array}{ccccccc} & & & & \text{isomorphism by the Manifold-Local-H}_*\text{e Lemma 150.9} & & \text{follows easily from the Identity-is-Generator Lemma 150.1} \\ & & & & \downarrow & & \downarrow \\ \mathrm{H}_n(M) & \xrightarrow{\omega_y} & \mathrm{H}_n(M, M\setminus\{y\}) & \xleftarrow[\cong]{(\lambda\circ\Phi^{\leq}_{s_i})_*} & \mathrm{H}_n(\Delta^n, \Delta^n\setminus\{P\}) & = & \mathbb{Z}\cdot[\mathrm{id}_{\Delta^n}]. \\ [c] & \mapsto & [c] = [\lambda\circ\Phi^{\leq}_{s_i}] & \longleftarrow & [\mathrm{id}_{\Delta^n}] & & \end{array}$$

since all the other $\lambda\circ\Phi^{\leq}_{s_j}$ are zero in $\mathrm{C}_n(M, M\setminus\{y\})$

The above data shows that the image of $[c]$ in $\mathrm{H}_n(M, M\setminus\{y\})$ equals the image of a generator of $\mathrm{H}_n(\Delta^n, \Delta^n\setminus\{P\})$. Since the map to the right is an isomorphism we see that the map $\omega_y\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, M\setminus\{y\})$ is an epimorphism. Since M is closed orientable connected and non-empty we know by the Manifold-Top-H$_*$-Theorem 183.20 that the group to the left is also isomorphic to $\mathbb{Z}$. Thus we see that both groups are isomorphic to $\mathbb{Z}$. It follows, e.g. from the Epi-Automorphism Lemma 97.27, that the map $\omega_y\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, M\setminus\{y\})$ is actually an isomorphism.

Now let $x \in M$ be any point. Since M is connected we obtain from the Homogeneous-via-Diffeotopies Proposition 44.6 a diffeomorphism $f\colon M \to M$ with $f(x) = y$.



Now we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M) & \xrightarrow{\omega_x} & \mathrm{H}_n(M, M\setminus\{x\}) \\ {\scriptstyle f_*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(M) & \xrightarrow[\cong]{\omega_y} & \mathrm{H}_n(M, M\setminus\{y\}). \end{array}$$

By the above we know that the bottom map is an isomorphism. Since f is a diffeomorphism the vertical maps are isomorphisms. Since the diagram commutes we see that the top horizontal map is also an isomorphism. ∎

The proof of the Manifold-Top-H$_*$-Theorem 185.3 (2) is a technically more intricate variation on the proof of the Manifold-Top-H$_*$-Theorem 185.3 (1).

Proof of Theorem 185.3 (2) and (3) for closed manifolds. We continue with the notation of the above proof of the Manifold-Top-H$_*$-Theorem 185.3 (1). Note that the uniqueness part of Statement (2) is basically an immediate consequence of Statement (1). Thus to prove Statement (2) it remains to prove existence. In the following we will now show that

$$[M] := [c] = [\epsilon_1\cdot\lambda\circ\Phi^{\leq}_{s_1} + \cdots + \epsilon_k\cdot\lambda\circ\Phi^{\leq}_{s_k}] \in \mathrm{H}_n(M)$$

has the desired properties. Before we do so, note though that in (1) we already saw that $[c]$ is a generator of $\mathrm{H}_n(M)$. Thus once we have shown that $[c]$ has the desired properties we have also shown Statement (3).

This discussion shows that it now suffices to prove the following claim:

Claim. For every continuous map $\Psi\colon \Delta^n \to M$ and every admissible $x \in \mathring{\Delta}^n$ we have

$$\omega_{\Psi(x)}([M]) \;=\; \operatorname{sign}(\Psi, x) \cdot [\, \underbrace{\Psi\colon \Delta^n \to M}_{\in C_n(M, M\setminus\{\Psi(x)\})} \,].$$

Proof. Similar to the proof of the claim in Statement (1) we first deal with a special case. We pick an n-simplex s_i and we consider the corresponding map $\Upsilon \coloneqq \lambda \circ \Phi^{\leq}_{s_i}\colon \Delta^n \to M$. Note that since we are given a smooth simplicial structure we know that the restriction of $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}\colon \Delta^n \to M$ to $\mathring{\Delta}^n$ is a smooth embedding. In particular any $x \in \mathring{\Delta}^n$ is admissible. Next note that we have the following equalities in $\mathrm{H}_n(M, M \setminus \{\Upsilon(x)\})$:

$$\omega_{\Upsilon(x)}([M]) \;\underset{\text{by definition of } [M]}{=}\; [\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k}] \;\underset{\substack{\text{since all other } \Phi^{\leq}_{s_j} \text{ are} \\ \text{zero in } C_n(M, M\setminus\{\Upsilon(x)\})}}{=}\; [\epsilon_i \cdot \underbrace{\lambda \circ \Phi^{\leq}_{s_i}}_{=\Upsilon}] \;\underset{\text{since } \epsilon_i = \operatorname{sign}(\Upsilon, x)}{=}\; \operatorname{sign}(\Upsilon, x) \cdot [\Upsilon].$$

This concludes the proof of the claim for the special case $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}$ and any $x \in \mathring{\Delta}^n$.

Now we turn to the general case. Let $\Psi\colon \Delta^n \to M$ be a continuous map and let $x \in \mathring{\Delta}^n$ be an admissible point. By definition we have $\Psi^{-1}(\Psi(x)) = \{x\}$ and by definition there exists an open connected neighborhood U of $x \in \mathring{\Delta}^n$ such that the map $\Psi|_U\colon U \to M$ is a smooth embedding. We pick an orientation-preserving embedding $\theta\colon \overline{B}^n \to U$ with $\theta(0) = x$.

Next we consider the embedding $\widetilde{\Upsilon} \coloneqq \Upsilon \circ \theta\colon \overline{B}^n \to M$, with $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}$ as above, and we consider the embedding $\widetilde{\Psi} = \Psi \circ \theta\colon \overline{B}^n \to M$. First we make the following assumption:

(∗) The maps $\widetilde{\Upsilon}$ and $\widetilde{\Psi}$ are both orientation-preserving embeddings or they are both orientation-reversing embeddings.

In this case it follows from the Smooth Ball Embedding Theorem 45.10 that there exists a diffeomorphism f of M with the following two properties:

(i) the map f is homotopic to the identity,
(ii) $f \circ \widetilde{\Upsilon} = \widetilde{\Psi}$.

We consider the following commutative diagram:[1965]

$$\begin{array}{ccccc}
 & \overset{\Upsilon_*}{\nearrow} & \mathrm{H}_n(M, M\setminus\{\Upsilon(x)\}) & \xleftarrow[\cong]{\omega_{\Upsilon(x)}} & \mathrm{H}_n(M) \\
\mathrm{H}_n(\Delta^n, \Delta^n\setminus\{x\}) & \xleftarrow[\theta_*]{\cong} & \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) & & \\
 & \underset{\Psi_*}{\searrow} & \mathrm{H}_n(M, M\setminus\{\Psi(x)\}) & \xleftarrow[\omega_{\Psi(x)}]{\cong} & \mathrm{H}_n(M).
\end{array}$$

with $\widetilde{\Upsilon}_*$ and $\widetilde{\Psi}_*$ from the middle to the top and bottom rows, $\cong\ f_*$ between the two middle-row entries, and $f_* = \mathrm{id}$ between the two right-column entries.

We make the following observations and clarifications:

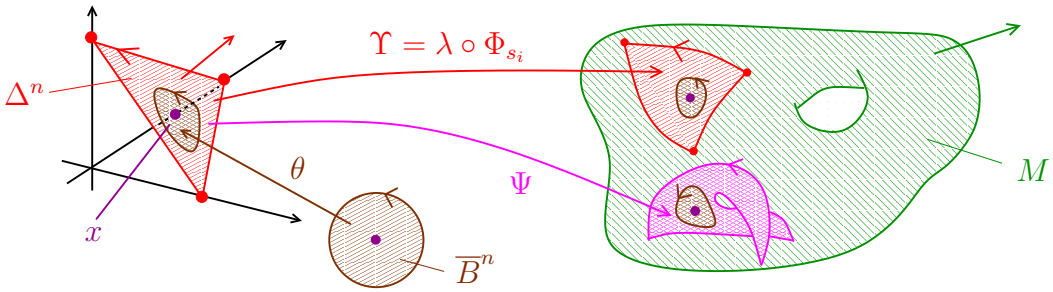
(a) It follows from (i) and the Homotopy-H_*-Proposition 144.12 that the induced map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is the identity.

[1965] Since Ψ is admissible, we know that $\Psi^{-1}(\Psi(x)) = \{x\}$. This implies that Ψ does indeed induce a map $\Psi_*\colon \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\}) \to \mathrm{H}_n(M, M \setminus \{\Psi(x)\})$.

(b) The map $\theta_*\colon \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \to \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\})$ is an isomorphism by the Excision Theorem 148.2.
(c) By (b) we can now set $\sigma \coloneqq \theta_*^{-1}([\mathrm{id}_{\Delta^n}]) \in \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})$.

We make the following observations:

$$\begin{array}{rrcll}
 & \omega_{\Upsilon(x)}([M]) & = & \mathrm{sign}(\Upsilon, x) \cdot [\Upsilon\colon \Delta^n \to M] & \text{by the above special case} \\
\Longrightarrow & \omega_{\Upsilon(x)}([M]) & = & \mathrm{sign}(\Upsilon, x) \cdot \widetilde{\Upsilon}_*(\sigma) & \text{by (c), since } \widetilde{\Upsilon}_* = \Upsilon_* \circ \theta_* \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Upsilon, x) \cdot \widetilde{\Psi}_*(\sigma) & \text{by (a) and functoriality of } f_* \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi, x) \cdot \widetilde{\Psi}_*(\sigma) & \text{by } (*) \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi, x) \cdot [\Psi\colon \Delta^n \to M] & \text{by (c), since } \widetilde{\Psi}_* = \Psi_* \circ \theta_*.
\end{array}$$



Finally we suppose that $(*)$ does not hold. Let

$$\begin{array}{rcl} \rho\colon \Delta^n & \to & \Delta^n \\ (t_0, t_1, t_2, \ldots, t_n) & \mapsto & (t_1, t_0, t_2, \ldots, t_n) \end{array}$$

be the map that swaps the first two coordinates. Since ρ is orientation-reversing we now see that $\widetilde{\Upsilon}$ and $\widetilde{\Psi} \circ \rho$ satisfy $(*)$. Now we see that

$$\begin{array}{rrcll}
 & \omega_{(\Psi\circ\rho)(\rho(x))}([M]) & = & \mathrm{sign}(\Psi\circ\rho, \rho(x)) \cdot \overbrace{[\Psi\circ\rho\colon \Delta^n \to M]}^{=\Psi_*(\rho_*[\mathrm{id}_{\Delta^n}])} & \text{since } \widetilde{\Upsilon} \text{ and } \widetilde{\Psi}\circ\rho \text{ satisfy } (*) \\
 & & & & \text{we can apply the above} \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi \circ \rho, \rho(x)) \cdot \Psi_*(-[\mathrm{id}_{\Delta^n}]) & \text{by Exercise 150.1} \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & (-\mathrm{sign}(\Psi, x)) \cdot \Psi_*(-[\mathrm{id}_{\Delta^n}]) & \text{since } \rho \text{ is orientation-reversing} \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi, x) \cdot \underbrace{[\Psi\colon \Delta^n \to M]}_{=\Psi_*([\mathrm{id}_{\Delta^n}])}. & \blacksquare
\end{array}$$

We have now proved the Manifold-Top-H_*-Theorem 185.3 for all closed smooth manifolds. It remains to prove the Manifold-Top-H_*-Theorem 185.3 for smooth manifolds with non-empty boundary.

Proof of Theorem 185.3 for manifolds with non-empty boundary. Let M be a compact oriented connected n-dimensional smooth manifold with non-empty boundary. Let M' be another copy of M. We consider the double $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ of M from Definition 54.8. Since $\partial M \neq \varnothing$, we obtain from the Doubling-Manifold Lemma 54.10 that $\mathrm{D}M$

is a closed oriented n-dimensional smooth manifold such that the map $M \to \mathrm{D}M$ is an orientation-preserving embedding.

(1) Let $x \in M \setminus \partial M$. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_n(M, \partial M) & \xrightarrow{\cong} & \mathrm{H}_n(\mathrm{D}M, M') & \xleftarrow{\cong} & \mathrm{H}_n(\mathrm{D}M) \\ \downarrow & & & \searrow & \downarrow \cong \\ \mathrm{H}_n(M, M \setminus \{x\}) & & \xrightarrow{\cong} & & \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{x\}). \end{array}$$

First note that it follows from the functoriality of relative homology groups that both parts of the diagram commute. Next note that it follows from the Doubling-Manifold Lemma 54.10 and the Manifold–Excision Theorem 148.3 that the top left horizontal map is an isomorphism. Basically the same argument shows that the bottom horizontal map is an isomorphism. Furthermore note that on page 3487, in the proof of the Manifold-Top-H_*-Theorem 183.20, we saw that the top right horizontal map is an isomorphism. By the above discussion of closed smooth manifolds we know that the right vertical map is an isomorphism. It now follows, as desired, that the left vertical map is also an isomorphism.

(2) By the discussion of the closed case we can consider the class $[\mathrm{D}M] \in \mathrm{H}_n(\mathrm{D}M)$. We define $[M] \in \mathrm{H}_n(M, \partial M)$ to be the preimage of $[\mathrm{D}M]$ under the composition of the above two horizontal isomorphisms. A slight variation on the argument provided in (1), using the properties of $[\mathrm{D}M] \in \mathrm{H}_n(\mathrm{D}M)$ proved above, now shows that $[M]$ has the desired property. We leave the details to the reader.

(3) It follows from the discussion in (1), and the properties of $[\mathrm{D}M] \in \mathrm{H}_n(\mathrm{D}M)$ proved above, that the class $[M] \in \mathrm{H}_n(M, \partial M)$, defined in (2), is a generator of $\mathrm{H}_n(M, \partial M)$. Again we leave the details to the reader. ■

The Manifold-Top-H_*-Theorem 185.3 leads us to the following definition:

Definition 185.4. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold.

(1) We denote by $N_1, \ldots, N_k$ the components of M. Furthermore for $i = 1, \ldots, k$ we now denote by $[N_i] \in \mathrm{H}_n(N_i, \partial N_i)$ the homology class from the Manifold-Top-H_*-Theorem 185.3. We refer to

$$[M] \coloneqq \sum_{i=1}^{k} [N_i] \in \mathrm{H}_n(M, \partial M) \underset{\uparrow}{=_{\mathrm{i}}} \bigoplus_{i=1}^{k} \mathrm{H}_n(N_i, \partial N_i).$$

isomorphism given by the Path-Component–H_*–Lemma 143.21

as the fundamental class of M.

(2) We refer to any cycle in $\mathrm{C}_n(M, \partial M)$ that represents the fundamental class $[M]$ as a fundamental cycle.

Remark. In Chapter 191 we will introduce the notion of an orientation on a topological manifold and in Definition 194.3 we will introduce fundamental classes for oriented topological manifolds. After a few reinterpretations one sees that the Smooth-Topological Orientation Proposition 191.25 together with the Orientation-Top-H_*-Theorem 193.3 provide in particular an alternative proof of the Manifold-Top-H_*-Theorem 185.3. □

Conveniently enough, the proof of the Manifold-Top-H_*-Theorem 185.3 actually gives an algorithm for determining explicit fundamental cycles.

Algorithm 185.5. (Fundamental Class Algorithm) Let $n \in \mathbb{N}$ and let M be a compact oriented non-empty n-dimensional smooth manifold.

(1) By the Smooth Manifold-Simplicial Structure Theorem 179.5 we can pick a smooth simplicial structure $(K = (V, S), |K| \xrightarrow{\lambda} M)$. Since V is finite we can pick a total order "$\leq$" on V.

(2) We denote by $s_1, \ldots, s_k$ the n-simplices of K.

(3) For $i \in \{1, \ldots, k\}$ we consider the characteristic map

$$\begin{array}{rccl} \Phi^{\leq}_{s_i}\colon & \Delta^n & \to & |K| \\ & (t_0, \ldots, t_n) & \mapsto & \sum\limits_{j=0}^{n} t_j \cdot \underset{\uparrow}{j}\text{-th vertex of } s_i \end{array}$$

we use the total order on V to order the vertices of s_i

and we set

$$\epsilon_i \coloneqq \begin{cases} +1, & \text{if } \lambda \circ \Phi^{\leq}_{s_i}\colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi^{\leq}_{s_i}\colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

(4) The cycle $\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k} \in \mathrm{C}_n(M)$ is a fundamental cycle of M.

Proof. It is basically clear that it suffices to prove the validity of the algorithm for the *connected* case. Next note that if M is closed, then it follows immediately from the proof of the Manifold-Top-H_*-Theorem 185.3 (1) that $\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k} \in \mathrm{C}_n(M)$ is a cycle. Furthermore, in the proof of the Manifold-Top-H_*-Theorem 185.3 (2) we saw that this cycle represents the fundamental class $[M] \in \mathrm{H}_n(M)$ of M. If M has non-empty boundary, then we equip the double $\mathrm{D}M$ with the simplicial structure given by "doubling" the simplicial structure of M. It is now straightforward to verify that the above algorithm also works in that case. We leave it to the reader to fill in the details. ■

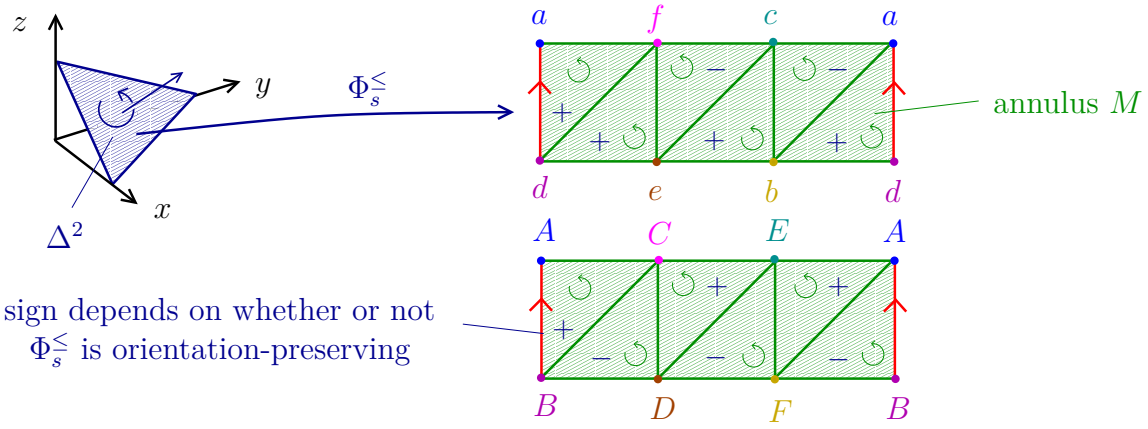
Example. In the figure below we consider the annulus $M = ([0,3] \times [0,1])/(0,y) \sim (3,y)$ equipped with an orientation and equipped with a smooth simplicial structure. In the figure below we use two different ways to name the vertices, namely once the vertex set is $V = \{a, \ldots, f\}$ and once it is $V = \{A, \ldots, F\}$. In both cases we equip these sets with the obvious total order.

It follows from the Fundamental Class Algorithm 185.5 that the singular chains

$$\begin{aligned} \mu = {} & +\Phi^{\leq}_{\{a,d,f\}} + \Phi^{\leq}_{\{d,e,f\}} - \Phi^{\leq}_{\{c,e,f\}} + \Phi^{\leq}_{\{b,c,e\}} - \Phi^{\leq}_{\{a,b,c\}} + \Phi^{\leq}_{\{a,b,d\}} \in \mathrm{C}_n(M, \partial M) \\ \nu = {} & +\Phi^{\leq}_{\{A,B,C\}} - \Phi^{\leq}_{\{B,C,D\}} + \Phi^{\leq}_{\{C,D,E\}} - \Phi^{\leq}_{\{D,E,F\}} + \Phi^{\leq}_{\{A,E,F\}} - \Phi^{\leq}_{\{A,B,F\}} \in \mathrm{C}_n(M, \partial M) \end{aligned}$$

are both fundamental cycles of M. □

Even though the Fundamental Class Algorithm 185.5 is very explicit, it also has some drawbacks. For example simplicial structures tend to have many simplices, which in the above algorithm leads to cycles with many summands. In a minute we will state a result that addresses this issue. But first let us prove a lemma which in particular says that, in order to show that we are given the fundamental class of a compact oriented *connected*



smooth manifold, we only have to verify the defining property for a single choice of a map $\Psi\colon \Delta^n \to M$ and an admissible point $x \in \overset{\circ}{\Delta}{}^n$.

Lemma 185.6. (Fundamental Class-Characterization Lemma) Let M be a compact oriented non-empty n-dimensional smooth manifold and let $\varphi \in \mathrm{H}_n(M, \partial M)$. We denote by $N_1, \ldots, N_k$ the components of M. The following statements are equivalent:

(1) φ is the fundamental class of M.

(2) For each $i \in \{1, \ldots, k\}$ there exists a continuous map $\Psi\colon \Delta^n \to N_i$ and an admissible point $x \in \overset{\circ}{\Delta}{}^n$ such that the following equality holds in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:

$$\underset{\text{map } \mathrm{H}_n(M,\partial M)\to\mathrm{H}_n(M,M\setminus\{\Psi(x)\})}{\omega_{\Psi(x)}(\varphi)} \;=\; \mathrm{sign}(\Psi, x) \cdot \underbrace{[\Psi\colon \Delta^n \to N_i \to M]}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})}.$$

(3) For each $i \in \{1, \ldots, k\}$ there exists a point $y \in N_i \setminus \partial_i N$ such that

$$\omega_y(\varphi) \;=\; \omega_y([M]).$$

Proof. A short moment of reflection shows that it suffices to prove the lemma in the case that M is connected. Next we note that the (1)⇒(2)-direction and (1)⇒(3)-direction are of course trivial. We turn to the proof of the (2)⇒(1)-direction. We have the following two equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\omega_{\Psi(x)}(\varphi) \underset{\text{by hypothesis}}{=} \mathrm{sign}(\Psi, x) \cdot [\Psi\colon \Delta^n \to M] \underset{\text{by Definition 185.4 of } [M]}{=} \omega_{\Psi(x)}([M]).$$

Since we assume that M is connected we know by the Manifold-Top-H_*-Theorem 185.3 that $\omega_{\Psi(x)}$ is an isomorphism. Thus it follows from the above that $\varphi = [M]$.

Finally we consider the (3)⇒(1)-direction. This statement follows again from the fact that ω_y is an isomorphism. ∎

The following proposition now gives us a practical criterion for showing that a given cycle represents the fundamental class.

Proposition 185.7. (Fundamental Class-Criterion)[1966] Let M be a compact oriented connected n-dimensional smooth manifold. Let $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ be a

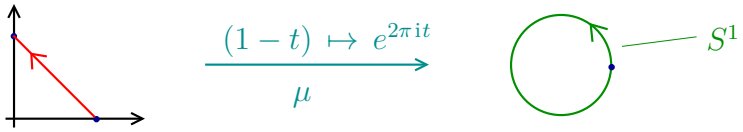
cycle. Suppose that there exists a $j \in \{1, \dots, m\}$ such that the following three conditions are satisfied:

(1) The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is an orientation-preserving smooth embedding.[1967]
(2) There exists an $x \in \sigma_j(\mathring{\Delta}^n)$ which is not contained in any $\sigma_i(\Delta^n)$, $i \neq j$.
(3) $r_j = 1$.

Then the cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m$ represents the fundamental class of the oriented smooth manifold, i.e. we have[1968]

$$[r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m] \;=\; [M] \;\in\; \mathrm{H}_n(M, \partial M).$$

Example. We consider the circle S^1 with the standard orientation.



It follows easily from the Fundamental Class-Criterion 185.7 and our orientation convention, see page 817, that the singular 1-simplex $\mu \colon \Delta^1 \to S^1$ that is given by $\mu(1-t, t) = \exp(2\pi i t)$ represents the fundamental class of the oriented smooth manifold S^1. □

Proof of the Fundamental Class-Criterion 185.7. Let M be a compact oriented connected n-dimensional smooth manifold, let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in \mathrm{C}_n(M, \partial M)$ be a cycle and let $j \in \{1, \dots, m\}$ be such that conditions (1)–(3) above are satisfied. We pick $x \in \sigma_j(\mathring{\Delta}^n)$ as in (2). Now we see that we have the following equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\omega_x([r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m]) \;=\; [\overbrace{r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m}^{\in \mathrm{C}_n(M, M\setminus\{x\})}] \;\underset{\substack{\uparrow \\ \text{since (2) implies that for } i \neq j \text{ we} \\ \text{have } \sigma_i(\Delta^n) \subseteq M \setminus \{x\}}}{=}\; [r_j \cdot \sigma_j] \;=\; [\sigma_j \colon \Delta^n \to M].$$

Next recall that by hypothesis $\sigma_j \colon \Delta^n \to M$ restricts to an orientation-preserving embedding $\sigma_j \colon \mathring{\Delta}^n \to M$. Since $x \in \sigma_j(\mathring{\Delta}^n)$, we see that x is an admissible point for σ_j. Thus all the hypotheses of the Fundamental Class-Characterization Lemma 185.6 hold, and therefore we conclude from the Fundamental Class-Characterization Lemma 185.6 that the given cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ is indeed a fundamental cycle of M. ■

185.2. **Examples of fundamental classes.** In this section we will give explicit cycles representing the fundamental classes of $\overline{B}^n$, of S^n and of surfaces of genus ≥ 1. At this stage it is convenient to recall the following notation:

Notation 185.8.

[1966] Note that we do *not* place any conditions on the restriction of σ_j to $\partial\Delta^n$.
[1967] For $M = S^n$ this proposition is closely related to the Standard Generator-at-Point Lemma 155.31.
[1968] This statement and its proof are closely related to the Detect-Cycle Proposition 150.5.

(1) We consider the projection

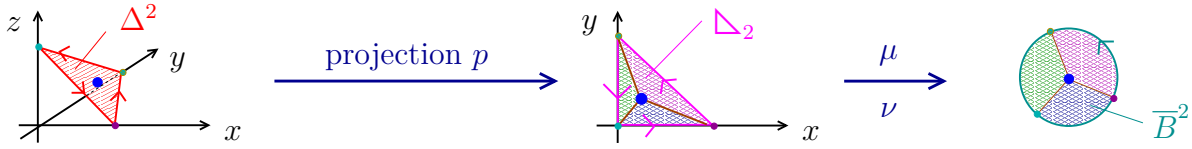
$$\begin{array}{rcl} p\colon \Delta^n & \to & \triangle_n \coloneqq \left\{(t_1,\dots,t_n)\in[0,1]^n \;\middle|\; \sum_{j=1}^n t_j \in [0,1]\right\} \\ (t_0,\dots,t_n) & \mapsto & (t_0,\dots,t_{n-1}). \end{array}$$

One can easily verify that the restriction $p\colon \overset{\circ}{\triangle}_n \to \overset{\circ}{\Delta}^n$ is orientation-preserving if and only if n is even.

(2)
- Let $\mu\colon \triangle_n \to \overline{B}^n$ be the explicit homeomorphism with $\mu(\frac{1}{n+1},\dots,\frac{1}{n+1}) = 0$ that is given by the Convex-to-Ball Proposition 4.28 (2). Note that the restriction of μ to a map $\triangle_n \to B^n$ is not smooth. But it is easily seen to be an orientation-preserving smooth embedding on an open dense subset of $\overset{\circ}{\triangle}_n$.
- We denote by $\Phi\colon \Delta^n \to \triangle_n$ the natural projection. Given $P \in \overset{\circ}{\triangle}_n$ we consider the maps

$$\begin{array}{rcl} f_P\colon \overset{\circ}{\triangle}_n & \to & \mathbb{R}^n \\ (x_1,\dots,x_n) & \mapsto & \frac{((x_1,\dots,x_n)-P)}{x_1\cdot\ldots\cdot x_n\cdot(1-x_1-\dots-x_n)} \end{array} \quad \& \quad \begin{array}{rcl} g\colon \mathbb{R}^n & \to & B^n \\ (x_1,\dots,x_n) & \mapsto & \frac{x_1^2+\dots+x_n^2}{1+x_1^2+\dots+x_n^2}\cdot(x_1,\dots,x_n) \end{array}$$

In Exercise 143.7 we saw that the map $g\circ f_P\colon \overset{\circ}{\triangle}_n \to B^n$ is actually an orientation-preserving diffeomorphism and that it extends to a homeomorphism by $\triangle_n \to \overline{B}^n$. For $P = (\frac{1}{n+1},\dots,\frac{1}{n+1})$ denote the resulting map $\triangle_n \to \overline{B}^n$ by ν.



Lemma 185.9. (Fundamental Class-of-Ball Lemma) Let $n \in \mathbb{N}$. The cycles $(-1)^n \cdot (\mu\circ p\colon \Delta^n \to \overline{B}^n) \in \mathrm{C}_n(\overline{B}^n, S^{n-1})$ and $(-1)^n \cdot (\nu\circ p\colon \Delta^n \to \overline{B}^n) \in \mathrm{C}_n(\overline{B}^n, S^{n-1})$ are both representatives of the fundamental class $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.

Proof. The lemma is an immediate consequence of the discussion preceding the lemma and the Fundamental Class-Characterization Lemma 185.6. ∎

Let $n \in \mathbb{N}$. The eagle-eyed reader will have noticed that by now we have introduced the symbol $[S^n] \in \mathrm{H}_n(S^n)$ twice:

(1) In Definition 150.8 we introduced the "standard generator" $[S^n] = [\alpha - \beta] \in \mathrm{H}_n(S^n)$. Here α is given by

$$\begin{array}{rcccl} \Delta^n & \xrightarrow{\mu\circ p} & \overline{B}^n & \to & S^n \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n\times\mathbb{R} \\ & & x & \mapsto & \left(x, \sqrt{1-\|x\|^2}\right) \end{array}$$

and $\beta\colon \Delta^n \to S^n$ is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$.

(2) We endow S^n with the orientation introduced on page 817, namely given $P \in S^n$ we say that a basis $v_1,\dots,v_n$ of $\mathrm{T}_P M = \widetilde{\mathrm{T}}_P M$ is positive if $P, v_1,\dots,v_n$ is a positive basis for $\mathbb{R}^{n+1}$. With this orientation of S^n we now also have the corresponding fundamental class $[S^n] \in \mathrm{H}_n(S^n)$.

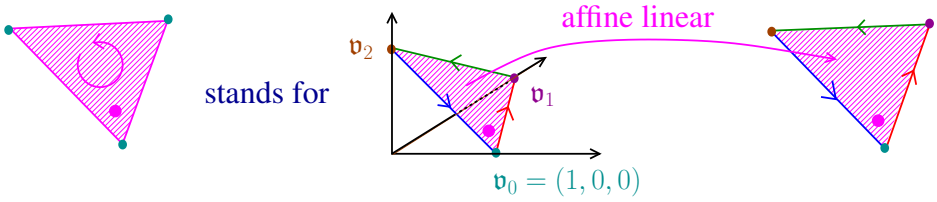
Fortunately the next lemma says that our various sign conventions work out in such a way that the two definitions of $[S^n]$ agree.

Lemma 185.10. (Generator-of-$H_n(S^n)$-Lemma) Let $n \in \mathbb{N}$. The standard generator $[S^n] \in \mathrm{H}_n(S^n)$ that we introduced on Definition 150.8 agrees with the fundamental class of the oriented smooth manifold S^n.

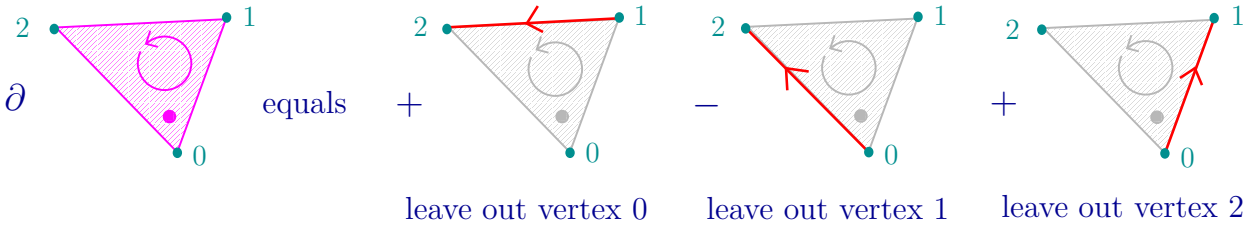
Proof. It follows from the above discussion, and some elementary arguments that the map $\alpha\colon \Delta^n \to S^n$ is an embedding and that it is an orientation-preserving embedding on an open subset of $\mathring{\Delta}^n$. The desired equality now follows almost immediately from the Fundamental Class-Criterion 185.7 applied to the cycle $\alpha - \beta$. ■

We move on to giving explicit representatives of fundamental classes for surfaces. This will require the introduction of more notation.

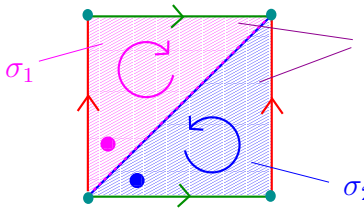
Notation 185.11. We will give many examples of fundamental classes of 2-dimensional smooth manifolds. Since we want to be very careful what singular simplex we mean by a given picture we adopt the following convention: A triangle T with a dot in a corner and a directional arrow in the center means the unique affine linear map $\Delta^2 \to T$ which sends $(1,0,0)$ to the vertex decorated with the dot and which respects the orientation of the vertices. We illustrate this notation in the figure below.



Convention 185.12. We also take some artistic liberties and allow triangles where the boundary is not given by segments and where the boundary consists of curves. In this case we expect that the boundary curves are parametrized with constant speed. Since the precise definition of the boundary of a singular 2-simplex is often a source of confusion we recall the definition in the figure below.



Example. In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$. We equip T with the orientation coming from the standard orientation on $(0,1) \times (0,1) \subseteq T$, which we can also view as an open subset of $\mathbb{R}^2$. In the figure we also show two singular 2-simplices σ_1 and σ_2. It is straightforward to verify that $-\sigma_1 + \sigma_2$ is a cycle.
The summand σ_2 satisfies the conditions imposed in the Fundamental Class-Criterion 185.7, thus we see that $-\sigma_1 + \sigma_2$ represents a fundamental class of T, in particular it represents a generator of $\mathrm{H}_2(T)$. This is of course the same cycle that we had already initially encountered on page 2755. We already saw on page 2914, using a similar argument, that it represents a generator of $\mathrm{H}_2(T)$. □



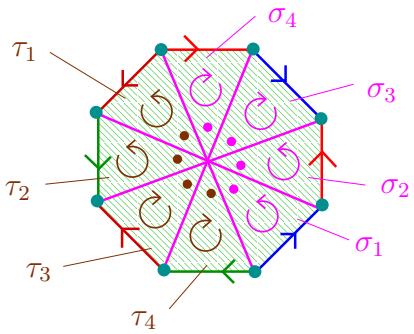
the torus T

the 2-cycle $-\sigma_1 + \sigma_2 \in \mathrm{C}_2(T)$ represents the fundamental class of the torus

Example. In the figure below we show the surface of genus 2 with eight singular 2-simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\tau_1, \tau_2, \tau_3, \tau_4$. The interior of the octagon is an open subset of $\mathbb{R}^2$ and we equip the surface of genus 2 with the orientation coming from the standard orientation of $\mathbb{R}^2$. It is a fun exercise to verify that the singular 2-chain

$$\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 \; + \; \tau_1 + \tau_2 - \tau_3 - \tau_4$$

is indeed a cycle.



$\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4$
$+ \quad \tau_1 + \tau_2 - \tau_3 - \tau_4$
is a cycle and represents the fundamental class

It follows easily from the Fundamental Class-Criterion 185.7 that the singular 2-chain is a fundamental cycle, i.e. it represents a fundamental class of the surface of genus 2. □

185.3. **Properties of the fundamental class.** In this section we will studies properties of the fundamental class. Before we can do so we will have fill in a few holes in our definitions, namely we have not yet dealt with non-empty 0-dimensional manifolds and with the empty manifold.

First let us recall the following remark from page 810.

Definition 185.13. An orientation for a 0-dimensional smooth manifold M is defined as a map $M \to \{-1, 1\}$.

Now we define the corresponding fundamental class.

Definition 185.14. Let $(M = \{P_1, \dots, P_k\}, \epsilon\colon M \to \{\pm 1\})$ be a compact oriented non-empty 0-dimensional smooth manifold. The corresponding fundamental class $[M]$ is defined as the homology class

$$[M] \;:=\; \sum_{i=1}^{k} \epsilon(P_i) \cdot [P_i] \;\in\; \mathrm{H}_0(M).$$

Finally let us consider the empty manifold. As we mentioned on page 624, basically by definition the empty set is a topological and smooth manifold of any dimension. Note that for the empty manifold $\varnothing$ we have the *equality* $\mathrm{H}_0(\varnothing) = \mathrm{H}_1(\varnothing) = \mathrm{H}_2(\varnothing) = \cdots = \{0\}$. This leads us to the following definition:

Definition 185.15. We define the fundamental class $[\varnothing]$ of the empty manifold to be $0 \in \{0\}$. Note that by the above the fundamental class $[\varnothing]$ is an element in every homology group of $\varnothing$.

Remark. The oriented 0-dimensional smooth manifolds and the empty manifold are a nuisance for a conscientious writer. In the following we will not always treat these cases explicitly and rely on the good will of the reader to deal with these on their own. □

Now let us continue with serious mathematics. The following lemma tells us the behavior of the fundamental class under diffeomorphisms.

Lemma 185.16. (Fundamental Class-Diffeomorphism Lemma) Let $n \in \mathbb{N}_0$, let M and N be compact oriented connected n-dimensional smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We have[1969]

$$f_*([M]) \;=\; \begin{cases} [N], & \text{if } f \text{ is orientation-preserving,} \\ -[N], & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof. If $n = 0$, then the statement follows immediately from the definitions. As we will see, for $n \in \mathbb{N}$ the statement follows basically immediately from the defining property of the fundamental class. Indeed, we set $\epsilon \coloneqq +1$ if f is orientation-preserving, otherwise we set $\epsilon \coloneqq -1$. Let $\Psi\colon \Delta^n \to M$ be an embedding such that the map $\Psi\colon \mathring{\Delta}^n \to M$ is orientation-preserving. We pick some $x \in \mathring{\Delta}^n$. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow{\omega_{\Psi(x)}} & \mathrm{H}_n(M, M \setminus \{\Psi(x)\}) \\ \downarrow f_* & & \downarrow f_* \\ \mathrm{H}_n(N, \partial N) & \xrightarrow{\omega_{(f\circ\Psi)(x)}} & \mathrm{H}_n(N, N \setminus \{(f\circ\Psi)(x)\}). \end{array}$$

Now we see that

$$\begin{array}{rrcll} & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Psi, x) \cdot [\Psi], & \text{by Definition 185.4 of } [M] \\ \Longrightarrow & \omega_{(f\circ\Psi)(x)}(f_*([M])) & = & \operatorname{sign}(\Psi, x) \cdot [f \circ \Psi], & \text{since the diagram commutes} \\ \Longrightarrow & \omega_{(f\circ\Psi)(x)}(f_*([M])) & = & \epsilon \cdot \operatorname{sign}(f \circ \Psi, x) \cdot [f \circ \Psi], & \text{by definition of } \epsilon \\ \Longrightarrow & f_*([M]) & = & \epsilon \cdot [N], & \text{by Lemma 185.6.} \end{array}$$

■

Given an oriented smooth manifold M we denote, as usual, by $-M$ the same smooth manifold but with the opposite orientation. Now we can formulate the following lemma:

Lemma 185.17. (Fundamental Class-Opposite Orientation Lemma) For every compact oriented n-dimensional smooth manifold M we have

$$[-M] \;=\; -[M] \;\in\; \mathrm{H}_n(M, \partial M).$$

Proof. This statement follows basically immediately from the Fundamental Class-Diffeomorphism Lemma 185.16 since the identity map $M \to -M$ is orientation-reversing. ■

In Convention 31.39 we saw that the boundary of an oriented smooth manifold inherits a natural orientation. For compact smooth manifolds the following proposition gives the relationship between the fundamental class of the smooth manifold and of its boundary.

[1969] Note that this statement now also makes sense for the empty manifold.

Proposition 185.18. (Fundamental Class-of-∂-Proposition) Let M be a compact oriented n-dimensional smooth manifold with boundary. The connecting homomorphism

$$\partial_n \colon \mathrm{H}_n(M, \partial M) \quad \to \quad \mathrm{H}_{n-1}(\partial M)$$

of the long exact sequence in homology of the pair $(M, \partial M)$ has the property that

$$[M] \quad \mapsto \quad [\partial M].$$



Remark. In the Fundamental Class-of-∂-Proposition 194.13 II we will prove the analog of the Fundamental Class-of-∂-Proposition 185.18 for topological manifolds. □

Proof. Let M be a compact oriented n-dimensional smooth manifold with boundary components $N_1, \ldots, N_k$. By the Smooth Manifold-Simplicial Structure Theorem 179.5 the smooth manifold M admits a smooth simplicial structure $(K = (V, S), \lambda \colon |K| \to M)$. By the Manifold-Simplex Proposition 183.2 (3) we know that there exists a subcomplex $J = (U, R)$ of (V, S) such that $(J = (U, R), \lambda \colon |J| \to \partial M)$ is a smooth simplicial structure for ∂M. We pick some total order "$\leq$" on V. Next we introduce the following notation for K:

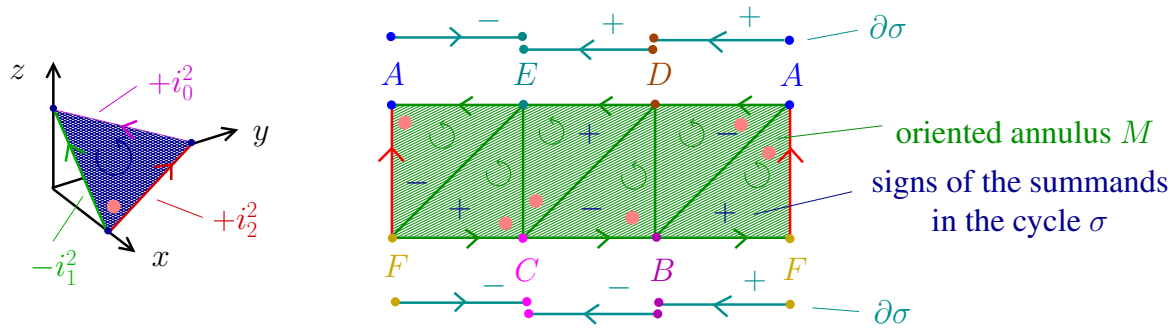
(1) We denote by S_n the set of n-simplices of K.
(2) For each $s \in S_n$ we denote by $\Phi_s^{\leq} \colon \Delta^n \to |K|$ the corresponding characteristic map.
(3) For each $s \in S_n$ we set

$$\epsilon_s \ := \ \begin{cases} +1, & \text{if } \lambda \circ \Phi_s^{\leq} \colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi_s^{\leq} \colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

Next we turn to the subcomplex J which, by restricting the total order "$\leq$" to $U \subseteq V$ we can and will also view as an ordered abstract simplicial complex. This leads us to the following notation:

(4) We denote by R_{n-1} the set of $(n-1)$-simplices of R.
(5) For each $r \in R_{n-1}$ we denote by $\Psi_r^{\leq} \colon \Delta^{n-1} \to |J|$ the corresponding characteristic map.
(6) For each $r \in R_{n-1}$ we set

$$\mu_r \ := \ \begin{cases} +1, & \text{if } \lambda \circ \Psi_r^{\leq} \colon \mathring{\Delta}^{n-1} \to \partial M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Psi_r^{\leq} \colon \mathring{\Delta}^{n-1} \to \partial M \text{ is orientation-reversing.} \end{cases}$$



Now we see that

$$\partial[M] \overset{\text{by the Fundamental Class Algorithm 185.5}}{\underset{\downarrow}{=}} \partial\Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \Phi_s^{\leq}\Big] \overset{\text{LES-of-Triple Proposition 146.5}}{\underset{\downarrow}{=}} \Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \partial_n(\Phi_s^{\leq})\Big] \overset{\text{Definition 176.7 of } \partial_n \colon \mathrm{C}_n(M) \to \mathrm{C}_{n-1}(M)}{\underset{\downarrow}{=}} \Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \sum_{j=0}^{n} (-1)^j \cdot \Phi_s^{\leq} \circ i_j^n\Big]$$

since we are given a cycle in $\mathrm{C}_n(M, \partial M)$ the contributions of singular simplices for which the image does not lie in ∂M cancel

$$\underset{\downarrow}{=} \Big[\sum_{\substack{s\in S_n \text{ and } j\in\{0,\dots,n\} \\ \text{such that} \\ (\Phi_s^{\leq} \circ i_j^n)(\Delta^{n-1}) \subseteq \partial M}} (-1)^j \cdot \epsilon_s \cdot \lambda \circ \Phi_s^{\leq} \circ i_j^n\Big] \underset{\uparrow}{=} \Big[\sum_{r\in R} \mu_r \cdot \lambda \circ \Psi_r^{\leq}\Big] \underset{\uparrow}{=} [\partial M].$$

by the Manifold-Simplex Proposition 183.2 (4) we have a bijection of singular simplices that appear in the sums, furthermore the corresponding signs match by the discussion on page 3482

follows from the Fundamental Class Algorithm 185.5

■

The following straightforward corollary says that we can "extend" fundamental cycles from the boundary to the total manifold.

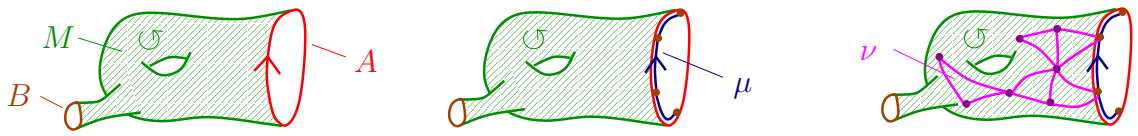
Corollary 185.19. (Extend Fundamental Cycle Lemma) Let M be a compact oriented n-dimensional smooth manifold. Let A be a union of components of ∂M. We write $B \coloneqq \partial M \setminus A$. We consider the map

$$\mathrm{C}_n(M, \partial M) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(\partial M) \xrightarrow{\mathrm{id}} \mathrm{C}_{n-1}(A \sqcup B) \xrightarrow{\cong} \mathrm{C}_{n-1}(A) \oplus \mathrm{C}_{n-1}(B) \twoheadrightarrow \mathrm{C}_{n-1}(A).$$

$$=: \Theta_A$$

Given any fundamental cycle $\mu \in \mathrm{C}_{n-1}(A)$ there exists a chain $\nu \in \mathrm{C}_n(M)$ such that ν represents a fundamental cycle in $\mathrm{C}_n(M, \partial M)$ and such that $\Theta_A(\nu) = \mu$.

Proof. First note that it is basically clear that it suffices to prove the statement for the case $A = \partial M$. Next, pick a singular chain $\widetilde{\nu} \in \mathrm{C}_n(M)$ that represents a fundamental cycle in $\mathrm{C}_n(M, \partial M)$. By the Fundamental Class-of-∂-Proposition 185.18 and the LES-of-Triple Proposition 146.5 we know that the cycles $\partial_n(\widetilde{\nu})$ and μ are homologous in $\mathrm{C}_{n-1}(\partial M)$. This means that there exists a $\sigma \in \mathrm{C}_n(\partial M)$ with $\partial\sigma = \partial_n(\widetilde{\nu}) - \mu$. Now we set $\nu \coloneqq \widetilde{\nu} - \sigma \in \mathrm{C}_n(M)$.



Since ν and $\widetilde{\nu}$ represent the same element in $\mathrm{C}_n(M, \partial M)$, we see that ν is also a fundamental cycle. Finally we note that by construction we have $\partial_n(\nu) = \partial_n(\widetilde{\nu}) - \partial_n(\sigma) = \mu$. ■

Next we record the following rather useful corollary to the Fundamental Class-of-∂-Proposition 185.18.

Corollary 185.20. (Kernel-of-Boundary Inclusion Corollary) Let M be a compact oriented connected n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map and we denote by $N_1, \ldots, N_k$ the components of ∂M.

(1) We have
$$\ker\big(i_*\colon \underbrace{\mathrm{H}_{n-1}(\partial M)}_{=\mathbb{Z}\cdot[N_1]\oplus\cdots\oplus\mathbb{Z}\cdot[N_k]} \to \mathrm{H}_{n-1}(M)\big) = \mathbb{Z}\cdot\underbrace{[\partial M]}_{\substack{\uparrow \\ \text{we have } [\partial M] = [N_1]+\cdots+[N_k]}} \subseteq \mathrm{H}_{n-1}(\partial M).$$

(2) Let $j \in \{1, \ldots, k\}$. We consider the following maps:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_{n-1}(\partial M)/\mathbb{Z}\cdot[\partial M] & \xrightarrow{i_*} & \mathrm{H}_{n-1}(M) & \longrightarrow & \mathrm{coker}(\mathrm{H}_{n-1}(\partial M) \xrightarrow{i_*} \mathrm{H}_{n-1}(M)) & \longrightarrow & 0. \\ & & \cong\uparrow & & & & & & \\ & & \mathbb{Z}\cdot[N_1]\oplus\ldots\oplus\overbrace{\mathbb{Z}\cdot[N_j]}\oplus\ldots\oplus\mathbb{Z}\cdot[N_k] & & & & & & \end{array}$$
The following two statements hold:
(a) The horizontal sequence is exact and it splits.
(b) The vertical map is an isomorphism.

(3) If M has precisely one boundary component, then $i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)$ is the zero map.

(4) The boundary ∂M is not a retract of M.

Remark.

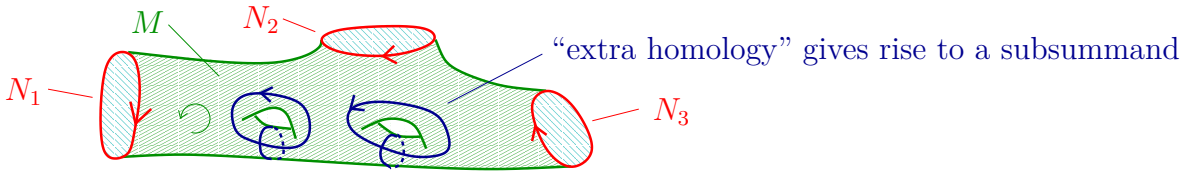
(1) The special case that M is a 2-dimensional smooth manifold is foreshadowed by the Surface-with-Boundary-H_*-Lemma 155.21.

(2) Recall that in the Sphere-Non Retract Proposition 149.9 we showed that the sphere $S^{n-1} = \partial\overline{B}^{n-1}$ is not a retract of the closed ball $\overline{B}^n$. The Kernel-of-Boundary Inclusion Corollary 185.20 (4) can thus be viewed as a generalization of the Sphere-Non Retract Proposition 149.9 from $\overline{B}^n$ to the case of compact oriented connected smooth manifolds with non-empty boundary.

(3) In the Topological Kernel-of-Boundary Inclusion Corollary 194.14 we will extend the statement of the Kernel-of-Boundary Inclusion Corollary 185.20 to topological manifolds. Also note that in the Boundary Non-Retraction Proposition 38.6 we gave a very different proof of the Kernel-of-Boundary Inclusion Corollary 185.20.

(4) Let N be a smooth manifold and let F be a closed oriented smooth submanifold. We denote by $i\colon F \to N$ the inclusion. If F is the boundary of a compact oriented smooth submanifold $W \subseteq N$, then it follows easily from the Kernel-of-Boundary Inclusion Corollary 185.20 (1) that $i_*([F]) = 0 \in \mathrm{H}_k(N)$. In the Null Homologous Boundary Proposition 185.22 we will prove a converse to this statement. □

Example. In the figure below we show a compact oriented n-dimensional smooth manifold with three boundary components N_1, N_2 and N_3.



The Kernel-of-Boundary Inclusion Corollary 185.20 says that *any two* boundary components generate a *subsummand* of $\mathrm{H}_{n-1}(M)$. Furthermore it says that in $\mathrm{H}_{n-1}(M)$ we have the equality $[N_1]+[N_2]+[N_3]=0$. □

Proof. Let M be a compact oriented n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) Using the Fundamental Class-of-∂-Proposition 185.18 and the long exact sequence in homology of the pair $(M,\partial M)$ we obtain that the following sequence is exact:

$$\underbrace{\mathrm{H}_n(M,\partial M)}_{=\mathbb{Z}\cdot[M] \text{ by the Top Homology Theorem 185.3}} \xrightarrow{[M]\mapsto[\partial M]} \mathrm{H}_{n-1}(\partial M) \xrightarrow{i_*} \mathrm{H}_{n-1}(M).$$

Statement (1) is now an immediate consequence.

(2) (a) It follows from (1) that the horizontal sequence is exact. Recall that by the Manifold-Top-H_*-Theorem 183.18 (2) the group $\mathrm{H}_{n-1}(M)$ is free abelian. Next note that it follows immediately from the Free Abelian Subgroup Theorem 97.14 that the group

$$\mathrm{im}(i_*\colon \mathrm{H}_{n-1}(\partial M)\to \mathrm{H}_{n-1}(M)) \underset{\text{isomorphism from the long exact sequence of the pair}}{\cong} \ker(\mathrm{H}_{n-1}(M)\to \mathrm{H}_{n-1}(M,\partial M))$$

is also free abelian. It now follows from the Free Abelian Group–Splitting Lemma 97.11 together with the Splitting Lemma 145.5 (1)⇒(3) that the short exact sequence splits.

(b) It follows immediately from $[\partial M]=[N_1]+\cdots+[N_k]$ that the vertical map is an isomorphism.

(3) Now we assume that ∂M has precisely one component. By the Manifold-Top-H_*-Theorem 185.3 we now have $\mathrm{H}_{n-1}(\partial M)=\mathbb{Z}\cdot[\partial M]$. Thus this statement is now an immediate consequence of (1).

(4) Suppose that there exists a retraction $r\colon M\to\partial M$, i.e. a continuous map which satisfies $r\circ i=\mathrm{id}_{\partial M}$. We consider the following commutative diagram:

$$\begin{array}{ccc} & \mathrm{H}_{n-1}(M) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \mathrm{H}_{n-1}(\partial M) & \xrightarrow{(r\circ i)_*=\mathrm{id}} & \mathrm{H}_{n-1}(\partial M). \end{array}$$

We just saw in (1) that i_* is not injective. This leads immediately to a contradiction. ■

In the next lemma we study the fundamental classes of codimension zero submanifolds.

Lemma 185.21. (Codimension Zero-Fundamental Class Lemma) Let M be a compact oriented n-dimensional smooth manifold. Furthermore let $W \subseteq M$ be a compact n-dimensional submanifold. We write $\overset{\circ}{W} = W \setminus \partial W$ and we equip W with the orientation given by the Manifold–Orientation Basics Lemma 31.31 (6). The following two statements hold:

(1) The inclusion-induced map $\mathrm{H}_n(W, \partial W) \to \mathrm{H}_n(M, M \setminus \overset{\circ}{W})$ is an isomorphism.

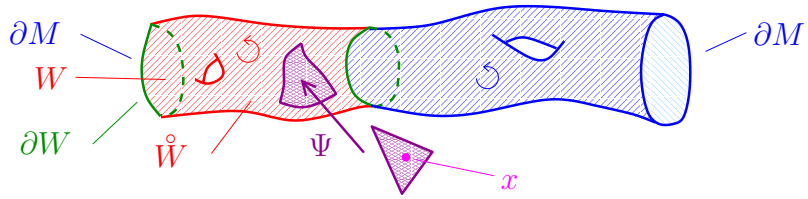
(2) The images of the fundamental classes $[M] \in \mathrm{H}_n(M, \partial M)$ and $[W] \in \mathrm{H}_n(W, \partial W)$ under the inclusion-induced maps

$$\mathrm{H}_n(M, \partial M) \longrightarrow \mathrm{H}_n(M, M \setminus \overset{\circ}{W}) \xleftarrow{\cong} \mathrm{H}_n(W, \partial W)$$

it follows from the Manifold–Excision Theorem 148.3 that this is an isomorphism

isomorphism by (1)

agree.



Proof.

(1) Statement (1) is an immediate consequence of the Manifold–Excision Theorem 148.3.

(2) We denote by $\Phi\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(W, \partial W)$ the map given in Statement (2). We have to show that $\Phi([M])$ is the fundamental class for W. So let $\Psi\colon \Delta^n \to W$ be a continuous map and let $x \in \overset{\circ}{\Delta}{}^n$ be an admissible point. We consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{\Phi}{\frown} & & \\
\mathrm{H}_n(M, \partial M) & \longrightarrow & \mathrm{H}_n(M, M \setminus \overset{\circ}{W}) & \xleftarrow{\cong} & \mathrm{H}_n(W, \partial W) \\
\downarrow{\scriptstyle \omega_{\Psi(x)}} & \swarrow & & & \downarrow{\scriptstyle \omega_{\Psi(x)}} \\
\mathrm{H}_n(M, M \setminus \{\Psi(x)\}) & & \xleftarrow[\text{isomorphism by the Excision Theorem 148.2}]{\cong} & & \mathrm{H}_n(W, W \setminus \{\Psi(x)\}) \\
 & {\scriptstyle \Psi_*}\nwarrow & & \nearrow{\scriptstyle \Psi_*} & \\
 & & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\}). & &
\end{array}$$

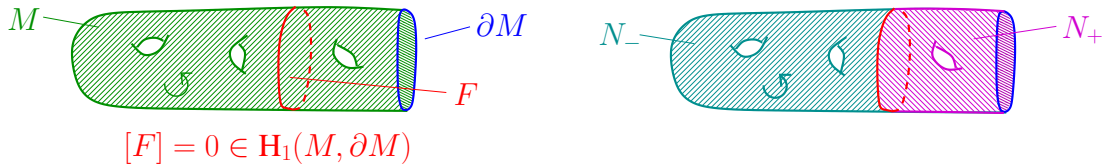
The desired statement now follows easily from the defining properties of fundamental classes and the above discussions. We leave the few remaining details to the reader. ■

One of the most popular approaches to computing the homology groups of some topological space X is to write X as the union $X = A \cup B$ of "simpler topological spaces A and B" in a suitable way and to use the Mayer–Vietoris Theorem 153.2. The catch of the Mayer–Vietoris Theorem 153.2 is that not only does one need to understand the homology groups of A, B and $A \cap B$, one also needs to understand the inclusion-induced maps on homology groups, which can be quite difficult. Fortunately the Kernel-of-Boundary Inclusion Corollary 185.20 (2) now gives an interesting calculation of an inclusion-induced map of homology groups. In the next two sections we will use the Kernel-of-Boundary Inclusion

Corollary 185.20 (2) to study the homology groups of the connected sum of two smooth manifolds and to study the homology groups of knot complements.

The following proposition can be viewed as a partial converse to the Kernel-of-Boundary Inclusion Corollary 185.20 (1):

Proposition 185.22. (Null Homologous Boundary Proposition) Let M be a compact oriented n-dimensional smooth manifold and let $F \subseteq M$ be a closed oriented *connected* smooth submanifold of codimension one. If $[F] = 0 \in \mathrm{H}_{n-1}(M, \partial M)$, then there exist precisely two codimension-zero submanifolds $N_\pm \subseteq M$ such that $\partial N_\pm = F \cup (N_\pm \cap \partial M)$ and these satisfy $M = N_- \cup N_+$ and $N_- \cap N_+ = F$.



Proof. [1970] Let M be a compact oriented n-dimensional smooth manifold and let $F \subseteq M$ be a closed oriented *connected* smooth submanifold of codimension one. To simplify the notation a little bit we now assume that M is closed. We leave it to the reader to deal with the case $\partial M \neq \varnothing$.

Since M and F are oriented we obtain from the Product Tubular Neighborhood–Existence Theorem 58.4 (3) a tubular map $\tau\colon [-1,1] \times F \to M$. In the following we consider the smooth manifold $W := M \setminus \tau((-1,1) \times F)$. Note that $\partial W = \tau(\{-1,1\} \times F)$.

Let W_- be the component of W that contains the *connected* subspace $\tau(\{-1\} \times F)$ and similarly we define W_+. Thus, if $W_- \neq W_+$, then we can take $N_- = W_- \cup \tau([-1,0] \times F)$ and $N_+ = W_+ \cup \tau([0,1] \times F)$. We leave it to the reader to verify that these are the only two options. It remains to prove the following claim:

Claim. If $W_- = W_+$, then $[F] \neq 0 \in \mathrm{H}_{n-1}(M)$.

Proof. We consider the following little diagram:

$$\begin{array}{ccccccccc} \mathrm{H}_n(M) & \xrightarrow{\text{я}} & \mathrm{H}_n(M, \tau([-1,1] \times F) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\tau([-1,1] \times F)) & \longrightarrow & \mathrm{H}_{n-1}(M) & \longrightarrow & \dots \\ & & \uparrow{\scriptstyle i_*} \cong & & & & & & \\ & & \mathrm{H}_n(W, \partial W) & & & & & & \end{array}$$

We make the following clarifications and observations:

(a) Note that $\partial W = \tau(\{\pm 1\} \times F)$. It follows from this observation together with the Manifold–Excision Theorem 148.3 that the vertical map is an isomorphism.
(b) Since W is connected we see that the bottom group is isomorphic to $\mathbb{Z}$.
(c) It follows from (b) and the Codimension Zero-Fundamental Class Lemma 185.21 that the image of i_* is contained in the image of я.
(d) It follows from (c) that the connected homomorphism ∂_n is the zero map. But this gives us the desired result. ∎

[1970] In Exercise 212.3 we will prove a generalization of the Null Homologous Boundary Proposition 185.22 and in Exercise 291.7 we will give an alternative proof of the proposition.

185.4. **The homology groups of the connected sum of two smooth manifolds.** Let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. We recall the Definition 54.12 of the connected sum $M\#N$: First we pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N \setminus \partial N$. Then we defined the connected sum of M and N as follows:

$$M\#N \;:=\; (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n)) \,/\, \sim$$

where $\varphi(P) \sim \psi(P)$ for all $P \in S^{n-1}$. We refer to the Smooth Connected Sum-Proposition 54.13 for some key properties of $M\#N$. Recall that in the Connected Sum-π_1-Proposition 102.4 we showed that $\pi_1(M\#N)$ is isomorphic to the free product of $\pi_1(M)$ and $\pi_1(N)$. The following proposition says that a very similar result holds for the homology groups of $M\#N$.

Proposition 185.23. (Connected Sum-H_*-Proposition) Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. For any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(M\#N) \;\cong\; \begin{cases} \mathrm{H}_k(M) \oplus \mathrm{H}_k(N), & \text{if } k \neq 0, n, \\ \mathbb{Z}, & \text{if } k = 0, n. \end{cases}$$

Moreover, for $k \neq 0, n$ the inclusion maps induce isomorphisms

$$\mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \;\xleftarrow{\cong}\; \mathrm{H}_k(M \setminus B^n) \oplus \mathrm{H}_k(N \setminus B^n) \;\xrightarrow{\cong}\; \mathrm{H}_k(M\#N)$$

where $B^n \subseteq M$ and $B^n \subseteq N$ denote the open balls used in the definition of the connected sum of M and N.

Remark. In Exercise 185.10 we will consider the homology groups of $M\#N$ if we drop the condition that the smooth manifolds are closed. □

The key to the proof of the Connected Sum-H_*-Proposition 185.23 is the following lemma:

Lemma 185.24. (Glue-along-Spheres-H_*-Lemma) Let X and Y be two compact orientable connected n-dimensional smooth manifolds such that $\partial X \cong S^{n-1}$ and $\partial Y \cong S^{n-1}$. We pick a diffeomorphism $\varphi\colon \partial X \to \partial Y$.

(1) There exists an isomorphism

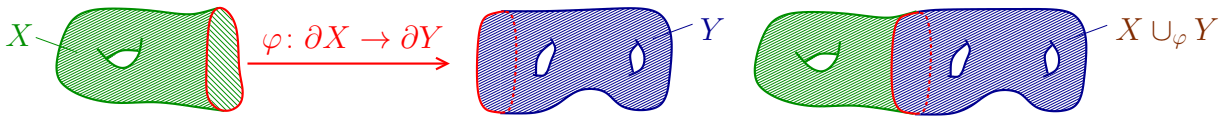
$$\mathrm{H}_n(X \cup_\varphi Y) \;\cong \mathbb{Z}.$$

Furthermore for every $k < n$ the inclusion maps $X \to X \cup_\varphi Y$ and $Y \to X \cup_\varphi Y$ induce an isomorphism

$$\widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_k(X \cup_\varphi Y).$$

(2) In particular for the case that $Y = \overline{B}^n$ we obtain that for any $k < n$ the inclusion map $X \to X \cup_\varphi \overline{B}^n$ induces an isomorphism

$$\widetilde{\mathrm{H}}_k(X) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_k(X \cup_\varphi \overline{B}^n).$$



Proof of Lemma 185.24. Statement (2) is an immediate consequence of Statement (1) and the fact that for every $k < n$ we have $\widetilde{\mathrm{H}}_k(\overline{B}^n) = 0$. Thus it suffices to prove Statement (1). To simplify the notation we write $S^{n-1} = \partial X = \partial Y$. We also write $Z = X \cup_\varphi Y$.

It follows from the Mayer-Vietoris Theorem 153.2 (3)[1971] that there exists a long exact sequence

$$\to \widetilde{\mathrm{H}}_k(S^{n-1}) \xrightarrow{(i_*,-i_*)} \widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \xrightarrow{i_*\oplus i_*} \widetilde{\mathrm{H}}_k(Z) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \to \dots$$

where i always stands for the corresponding inclusion map. By the Sphere-H_*-Proposition 149.4 we have $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{\mathrm{H}}_k(S^n) = 0$ for $k \neq n$. We see immediately that for any $k < n-1$ the inclusion maps induce an isomorphism $\widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \to \widetilde{\mathrm{H}}_k(Z)$.

Now we turn to the $(n-1)$-st and the n-th homology of Z. First note that by the Manifold-Top-H_*-Theorem 183.20 we have $\widetilde{\mathrm{H}}_n(X) = \widetilde{\mathrm{H}}_n(Y) = 0$. Therefore the above long exact sequence gives rise to an exact sequence

since S^{n-1} is the unique boundary component of X, we deduce from the Kernel-of-Boundary Inclusion Corollary 185.20 that the map $i_*\colon \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(X)$ is the zero map, the same way we also see that the map $i_*\colon \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(Y)$ is the zero map

$$\underbrace{\widetilde{\mathrm{H}}_n(X)\oplus\widetilde{\mathrm{H}}_n(Y)}_{=0 \text{ by the Smooth Manifold Manifold-Top-H}_*\text{-Theorem 183.20}} \to \widetilde{\mathrm{H}}_n(Z) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(S^{n-1})}_{=\mathbb{Z}\cdot[S^{n-1}]} \xrightarrow{(i_*,-i_*)} \widetilde{\mathrm{H}}_{n-1}(X)\oplus\widetilde{\mathrm{H}}_{n-1}(Y) \to \widetilde{\mathrm{H}}_{n-1}(Z) \to 0.$$

∂_n: $\uparrow$ isomorphism by the above

All the desired statements follow from the above data. ∎

Now we can easily provide the proof of the Connected Sum-H_*-Proposition 185.23.

Proof of the Connected Sum-H_*-Proposition 185.23. Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. As required in the definition of the connected sum of M and N we pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N$. Recall that

$$M\#N \;=\; (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n)) \,/\, \sim$$

where $\varphi(P) \sim \psi(P)$ for all $P \in S^{n-1}$. We write $X = M \setminus \varphi(B^n)$ and $Y = N \setminus \psi(B^n)$. Note that it follows immediately from the Glue-along-Spheres-H_*-Lemma 185.24 (1) that $\mathrm{H}_n(M\#N) = \mathrm{H}_n(X \cup_{S^{n-1}} Y) \cong \mathbb{Z}$. Now let $k < n$. Note that in this case we have

$$\mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \xleftarrow{\cong} \mathrm{H}_k(\underbrace{M \setminus \varphi(B^n)}_{=X}) \oplus \mathrm{H}_k(\underbrace{N \setminus \psi(B^n)}_{=Y}) \xrightarrow{\cong} \mathrm{H}_k(M\#N)$$

$\uparrow$ by the Glue-along-Spheres-H_*-Lemma 185.24 (1) applied to $M = X \cup_{S^{n-1}} \overline{B}^n$ and $N = Y \cup_{S^{n-1}} \overline{B}^n$

$\uparrow$ by the Glue-along-Spheres-H_*-Lemma 185.24 (1) applied to $M\#N = X \cup_{S^{n-1}} Y$ ∎

Examples.

(1) Let $T = S^1 \times S^1$ be the torus. In the Surface–Connected Sum Lemma 54.15 we showed that the connected sum $T\#T$ is diffeomorphic to the surface Σ_2 of genus 2. More generally, the same argument shows that the connected sum of g copies of the torus T is diffeomorphic to the surface Σ_g of genus g. Together with the calculation of the

[1971] Note that here secretly we also used the Gluing-Smooth Manifolds-Proposition 54.5

homology groups of the torus in the Torus-H_*-Proposition 153.9 (1) we now see that

$$\mathrm{H}_k(\Sigma_g) \cong \mathrm{H}_k(\underbrace{T\#\ldots\#T}_{g \text{ copies}}) \cong \begin{cases} \mathbb{Z}, & \text{if } k=0,2,\\ \mathbb{Z}^{2g}, & \text{if } k=1,\\ 0, & \text{if } k>2. \end{cases}$$

This agrees of course with the result obtained in the Surface-H_*-Proposition 155.17.

(2) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. By the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 we know that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k=0,2,4,\\ 0, & \text{otherwise.} \end{cases}$$

As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from the Complex Manifold-Orientation Proposition 40.18. We can now form the connected sum $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$. It follows from the Connected Sum-H_*-Proposition 185.23 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k=0,4,\\ \mathbb{Z}\oplus\mathbb{Z}, & \text{if } k=2,\\ 0, & \text{otherwise.} \end{cases}$$

For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. Since the homology groups do not depend on the orientation we have the same homology groups for $\overline{\mathbb{C}\mathrm{P}}^2$ and the calculation shows that the homology groups of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ agree. □

Recall that on page 3141 we showed, using the Künneth Theorem 164.6 that

$$\mathrm{H}_k(S^2\times S^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k=0,4,\\ \mathbb{Z}\oplus\mathbb{Z}, & \text{if } k=2,\\ 0, & \text{otherwise.} \end{cases}$$

This fact and our above calculations shows: The homology groups of the connected sums $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ and of $S^2\times S^2$ are all isomorphic. Furthermore, note that by the Sphere-π_1-Proposition 88.24, the Product-π_1-Proposition 90.13, the Projective Space–π_1-Proposition 135.8 and the Connected Sum-π_1 Proposition 102.4 we know that all of these topological spaces are simply connected. So this raises the following intriguing question.

Question 185.25. Are the closed 4-dimensional smooth manifolds $S^2\times S^2$, $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ homeomorphic?

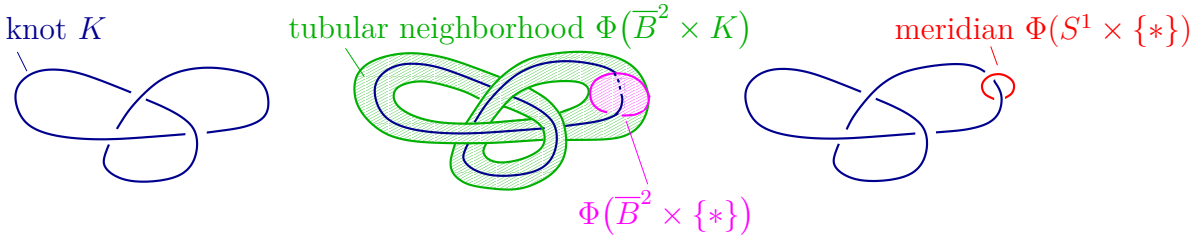
We will answer this question in the $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$-Homotopy Type Proposition 226.18 and the $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$-$S^2\times S^2$-Proposition 226.26.

185.5. **Homology groups of knot complements.** We return to the ever-popular study of knots in S^3. We recall some definitions from Definition 108.3 and Definition 109.1 and we introduce a new concept:

Definition 185.26.

(1) An (oriented) knot is an (oriented) 1-dimensional submanifold of S^3 that is diffeomorphic to S^1.

(2) Let K be an (oriented) knot in S^3. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) (together with the Tubular Map Lemma 58.3) there exists an (orientation-preserving) smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0,P) = P$ for all $P \in K$.
   (a) Let $* \in S^1$ we refer to the (oriented) submanifold $\Phi(S^1 \times \{*\})$ as a meridian of the (oriented) knot K.[1972]
   (b) As in Definition 108.30 we refer to
$$X_K \;:=\; S^3 \setminus \Phi(B^2 \times K)$$
   as the exterior of K.



Lemma 185.27. (Knot Exterior Lemma) For any knot $K \subseteq S^3$ the following statements hold:
(1) The exterior X_K of the knot K is a compact orientable connected 3-dimensional smooth manifold and the boundary ∂X_K is diffeomorphic to the 2-dimensional torus.
(2) The diffeomorphism type of X_K as an oriented smooth manifold does not depend on the choice of Φ.
(3) The exterior X_K is a deformation retract of the knot complement $S^3 \setminus K$.[1973]

Proof. Let $K \subseteq S^3$ be a knot.
(1) This statement is basically a consequence of the Codimension-Zero Smooth Embedding Proposition 30.21.
(2) This statement is an immediate consequence of the Tubular Map–Uniqueness Theorem 58.5.
(3) Let $\Phi\colon \overline{B}^2 \times K \to S^3$ be a smooth embedding with $\Phi(0,P) = P$ for all $P \in K$. A deformation retraction from $S^3 \setminus K$ to X_K is given by the map
$$\begin{array}{rcl} (S^3 \setminus K) \times [0,1] & \to & X_K = S^3 \setminus \Phi(B^2 \times K) \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \in S^3 \setminus \Phi(\overline{B}^2 \times K), \\ \Phi(\frac{w}{1-t+t\cdot|z|}, z), & \text{if } P = \Phi(w,z) \in \Phi((\overline{B}^2 \setminus \{0\}) \times K). \end{cases} \end{array}$$
   It follows easily from the Homotopy Combination Lemma 18.8 (1) and the Codimension-Zero Smooth Embedding Proposition 30.21 that this map is indeed continuous. ■

[1972] By the Meridian Proposition 109.2 (2) this definition of a meridian is consistent with Definition 109.1.
[1973] This statement can be viewed as a special case of the Basics-of-Tubular Neighborhood Lemma 57.10 or of the Link Exterior Lemma 108.31.

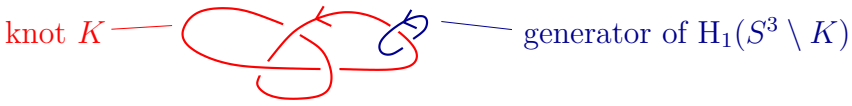
Notation 185.28. Let M be a smooth manifold and let C be an oriented smoothly embedded closed curve in M, in the sense of Definition 107.2. We pick an orientation-preserving diffeomorphism $S^1 \to C$. Sometimes we denote by $[C]$ the element in $\mathrm{H}_1(M)$ that is given by the image of the standard generator $[S^1]$ of $\mathrm{H}_1(S^1)$ under the map $\mathrm{H}_1(S^1) \xrightarrow{\Phi_*} \mathrm{H}_1(C) \to \mathrm{H}_1(M)$.[1974]

The following proposition gives us in particular the long-awaited calculation of the homology groups of knot complements, together with explicit generators for the homology groups.

Proposition 185.29. (Knot Complement-H_*-Proposition) Let $K \subseteq S^3$ be an oriented knot. For every $n \in \mathbb{N}_0$ we have[1975]

$$\mathrm{H}_n(X_K) \underset{\text{induced by the inclusion } X_K \to S^3 \setminus K}{\xrightarrow{\cong}} \mathrm{H}_n(S^3 \setminus K) = \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \cdot [\mu_K], & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

where μ_K denotes a meridian of K.



Remark.

(1) We already computed the isomorphism types of the homology groups of $S^3 \setminus K$ in the Sphere Complement-H_*-Theorem 170.2. The proof we now give is quite different and it has the nice feature that it gives an explicit generator of $\mathrm{H}_1(S^3 \setminus K)$.

(2) The result of the Knot Complement-H_*-Proposition 185.29 is easy to state but also slightly disappointing since it shows that the first homology groups cannot be used to distinguish different knots. In contrast to this fact we saw in the Torus Knot-π_1-Proposition 111.18 and the Trefoil–Non-Trivial Proposition 111.19 that the fundamental group can be used to distinguish the trivial knot from the trefoil.

(3) In the Link Complement-H_*–Basis-Proposition 283.4 we will prove a generalization of Knot Complement-H_*-Proposition 185.29 for links and in the Manifold–Knot Complement-H_*-Proposition 217.18 we will prove a generalization to knots in arbitrary closed orientable 3-dimensional smooth manifolds. □

Proof. Let $K \subseteq S^3$ be an oriented knot. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) there exists a smooth embedding $\tau\colon \overline{B}^2 \times K \to S^3$ with $\tau(0, P) = P$ for all $P \in K$. We set $X_K := S^3 \setminus \tau(B^2 \times K)$.

First note that it follows from the Knot Exterior Lemma 185.27 together with the Homotopy Equivalence-H_*-Proposition 144.14 (1b) that the inclusion $X_K \to S^3 \setminus K$ induces an

[1974] It follows from the Diffeotopies-of-S^1-Proposition 42.8 and the Homotopy-H_*-Proposition 144.12 that, up to a sign, the homology class does not depend on the choice of the orientation-preserving diffeomorphism Φ.

[1975] We calculated some of the homology groups much earlier in Exercise 153.19. Furthermore we will give an alternative approach to calculating these homology groups in Exercise 185.6.

isomorphism on all homology groups. Thus it remains to determine the homology groups of X_K.

We pick $P, Q \in S^1$. We write $\mu_K = \tau(S^1 \times \{P\})$ and $\gamma = \tau(\{Q\} \times K)$. We consider the decomposition

$$S^3 = \underbrace{(S^3 \setminus \tau(B^2 \times K))}_{=X_K} \cup_{S^1 \times K} (\tau(\overline{B}^2 \times K)).$$

It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that we can apply the Mayer-Vietoris Theorem 153.2 (3) with reduced homology groups to the above decomposition. We obtain the following long exact sequence:

$$\begin{aligned}
&\to \mathrm{H}_3(\partial X_K) \xrightarrow{(i_*,-i_*)} \mathrm{H}_3(X_K) \oplus \mathrm{H}_3(\tau(\overline{B}^2 \times K)) \xrightarrow{i_* \oplus i_*} \mathrm{H}_3(S^3) \xrightarrow{\partial_3} \\
&\to \mathrm{H}_2(\partial X_K) \xrightarrow{(i_*,-i_*)} \mathrm{H}_2(X_K) \oplus \mathrm{H}_2(\tau(\overline{B}^2 \times K)) \xrightarrow{i_* \oplus i_*} \mathrm{H}_2(S^3) \xrightarrow{\partial_2} \\
&\to \mathrm{H}_1(\partial X_K) \xrightarrow{(i_*,-i_*)} \mathrm{H}_1(X_K) \oplus \mathrm{H}_1(\tau(\overline{B}^2 \times K)) \xrightarrow{i_* \oplus i_*} \mathrm{H}_1(S^3) \xrightarrow{\partial_1} \\
&\to \widetilde{\mathrm{H}}_0(\partial X_K) \xrightarrow{(i_*,-i_*)} \widetilde{\mathrm{H}}_0(X_K) \oplus \widetilde{\mathrm{H}}_0(\tau(\overline{B}^2 \times K)) \xrightarrow{i_* \oplus i_*} \widetilde{\mathrm{H}}_0(S^3) \xrightarrow{\partial_0} 0.
\end{aligned}$$

Now we throw in everything we know about the above setting:

$$\begin{aligned}
&\to \underbrace{\mathrm{H}_3(\partial X_K)}_{=0} \xrightarrow{(i_*,-i_*)} \mathrm{H}_3(X_K) \oplus \underbrace{\mathrm{H}_3(\tau(\overline{B}^2 \times K))}_{=0} \xrightarrow{i_* \oplus i_*} \underbrace{\mathrm{H}_3(S^3)}_{=\mathbb{Z}\cdot[S^3]} \xrightarrow{\partial_3} \\
&\to \underbrace{\mathrm{H}_2(\partial X_K)}_{=\mathbb{Z}\cdot[\partial X_K]} \xrightarrow{(i_*,-i_*)} \mathrm{H}_2(X_K) \oplus \underbrace{\mathrm{H}_2(\tau(\overline{B}^2 \times K))}_{=0} \xrightarrow{i_* \oplus i_*} \underbrace{\mathrm{H}_2(S^3)}_{=0} \xrightarrow{\partial_2} \\
&\qquad \uparrow \text{ by the Kernel-of-Boundary Inclusion Corollary 185.20 (2) this is the zero map} \\
&\to \underbrace{\mathrm{H}_1(\partial X_K)}_{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\gamma]} \xrightarrow{(i_*,-i_*)} \mathrm{H}_1(X_K) \oplus \underbrace{\mathrm{H}_1(\tau(\overline{B}^2 \times K))}_{=\mathbb{Z}\cdot[\gamma]} \xrightarrow{i_* \oplus i_*} \underbrace{\mathrm{H}_1(S^3)}_{=0} \xrightarrow{\partial_1} \\
&\qquad \uparrow \text{ we have } [\mu_K] \mapsto ([\mu_K], 0) \text{ and and we have } [\gamma] \mapsto (?, -[\gamma]) \\
&\to \underbrace{\widetilde{\mathrm{H}}_0(\partial X_K)}_{=0} \xrightarrow{(i_*,-i_*)} \widetilde{\mathrm{H}}_0(X_K) \oplus \underbrace{\widetilde{\mathrm{H}}_0(\tau(\overline{B}^2 \times K))}_{=0} \xrightarrow{i_* \oplus i_*} \underbrace{\widetilde{\mathrm{H}}_0(S^3)}_{=0} \xrightarrow{\partial_0} 0.
\end{aligned}$$

We make the following observations:

(3) First note that we obtain from the above discussion that the connecting homomorphism $\partial_3 \colon \mathbb{Z} \cdot [S^3] = \mathrm{H}_3(S^3) \to \mathrm{H}_2(\partial X_K) = \mathbb{Z} \cdot [\partial X_K]$ is an epimorphism. Since every epimorphism $\mathbb{Z} \to \mathbb{Z}$ is necessarily an isomorphism we see that the connecting homomorphism $\partial_3 \colon \mathrm{H}_3(S^3) \to \mathrm{H}_2(\partial X_K)$ is in fact an isomorphism. It follows that $\mathrm{H}_3(X_K) = 0$.

(2) The discussion in (3) and the exact sequence also gives us that $\mathrm{H}_2(X_K) = 0$.

(1) The lower part of the exact sequence gives us an isomorphism of the form

$$\mathbb{Z} \cdot [\mu_K] \oplus \mathbb{Z} \cdot [\gamma] \xrightarrow[\cong]{\substack{[\mu_K] \mapsto ([\mu_K], 0) \\ [\gamma] \mapsto (?, -[\gamma])}} \mathrm{H}_1(X_K) \oplus \mathbb{Z} \cdot [\gamma]$$

It follows from the Cancellation Proposition 97.26 that $\mathrm{H}_1(X_K) \cong \mathbb{Z}$. Since a matrix of the form $\begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix}$ is invertible if and only if $a = \pm 1$ we see that $[\mu_K]$ is a generator of $\mathrm{H}_1(X_K) \cong \mathbb{Z}$.

(0) Finally note that the very bottom of the above exact sequence implies immediately that $\widetilde{\mathrm{H}}_0(X_K) = 0$. ■

185.6. **The $\mathbb{F}_2$-fundamental class.** In this final section we introduce the $\mathbb{F}_2$-fundamental class for any compact (not necessarily orientable) smooth manifold. Truth be told, this notion will only play a fairly minor role in the subsequent chapters.

Now let M be a compact connected n-dimensional smooth manifold. Recall that by the Manifold-Top-H_*-Theorem 183.20 (3) we know that, regardless of whether or not M is orientable, we have $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$. This leads us to the following definition:

Definition 185.30. Let M be a compact n-dimensional smooth manifold. We denote by $N_1, \dots, N_k$ the components of M. We refer to

$$[M]_{\mathbb{F}_2} \coloneqq \sum_{i=1}^{k} \text{the unique non-zero element of } \mathrm{H}_n(N_i, \partial N_i; \mathbb{F}_2) \in \underbrace{\mathrm{H}_n(M, \partial M; \mathbb{F}_2)}_{= \bigoplus_{i=1}^{k} \mathrm{H}_n(N_i, \partial N_i; \mathbb{F}_2)}$$

as the $\mathbb{F}_2$-fundamental class of M.

Remark. In the $\mathbb{F}_2$-Orientability Proposition 191.22 together with the Fundamental Class-Proposition 194.2 we will generalize this notion to compact *topological* manifolds. □

The following proposition is the "$\mathbb{F}_2$-analog" of some of the results from Section 185.1.

Proposition 185.31. ($\mathbb{F}_2$-Fundamental Class Proposition) Let M be a compact n-dimensional smooth manifold.

(1) Let $(K = (V, S), \lambda\colon |K| \to M)$ be a smooth simplicial structure for M. We pick a total order "$\leq$" on V. We denote by $s_1, \dots, s_k$ the n-simplices of K. Furthermore for $i \in \{1, \dots, k\}$ we consider the characteristic map $\Phi^{\leq}_{s_i}\colon \Delta^n \to |K|$. With this notation the singular chain

$$\lambda \circ \Phi^{\leq}_{s_1} \otimes 1 + \cdots + \lambda \circ \Phi^{\leq}_{s_k} \otimes 1 \in \mathrm{C}_n(M) \otimes \mathbb{F}_2 = \mathrm{C}_n(M; \mathbb{F}_2)$$

is a cycle which represents the $\mathbb{F}_2$-fundamental class of M.

(2) Let $x \in M \setminus \partial M$.
   (a) The image of the $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ under the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is the unique non-trivial element.
   (b) If M is connected, then the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is an isomorphism.

Sketch of proof. It is straightforward to see that it suffices to consider the case that M is connected. Furthermore, using a doubling argument one can fairly easily reduce the proof of the statement to the case of a closed n-dimensional smooth manifold M. Next note that a slight variation on the argument of the Pseudomanifold Homology Theorem 183.12 shows that

$$\sigma \coloneqq \lambda \circ \Phi^{\leq}_{s_1} \otimes 1 + \cdots + \lambda \circ \Phi^{\leq}_{s_k} \otimes 1 \in \mathrm{C}_n(M; \mathbb{F}_2)$$

is indeed a cycle in the chain complex $\mathrm{C}_*(M; \mathbb{F}_2) = \mathrm{C}_*(M) \otimes \mathbb{F}_2$. Next note that the argument of the proof of the Manifold-Top-H_*-Theorem 185.3 (1) shows that for every

$x \in M$ the natural map $\mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\};\mathbb{F}_2)$ sends $[\sigma]$ to the unique non-zero element. Furthermore note that by the Manifold-Top-H_*-Theorem 183.20 (3) we know that $\mathrm{H}_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$. Finally note that this implies that the natural map $\mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\};\mathbb{F}_2)$ is indeed an isomorphism and that $[\sigma]$ is indeed a generator of $\mathrm{H}_n(M;\mathbb{F}_2)$. ■

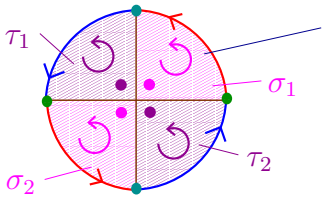
The following proposition is the $\mathbb{F}_2$-analog of the Fundamental Class-Criterion 185.7.

Proposition 185.32. ($\mathbb{F}_2$-Fundamental Class-Criterion) In the following let M be a compact connected n-dimensional smooth manifold. Furthermore let $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M;\mathbb{F}_2) = \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2$ be a cycle. Suppose that there exists a $j \in \{1, \ldots, m\}$ such that the following two conditions are satisfied:
(1) The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is a smooth embedding.
(2) There exists an $x \in \sigma_j(\mathring{\Delta}^n)$ which is not contained in any $\sigma_i(\Delta^n)$, $i \neq j$.
Then the cycle $(\sigma_1 + \cdots + \sigma_m) \otimes 1$ represents the $\mathbb{F}_2$-fundamental class of M.

Proof. The proof of this proposition is almost the same as the proof of the Fundamental Class-Criterion 185.7. We leave it to the reader to make the necessary modifications. ■

Examples.

(1) Now we will give a representative for the $\mathbb{F}_2$-fundamental class for the real projective plane $\mathbb{R}\mathrm{P}^2$. More precisely, we consider the four singular simplices $\sigma_1, \sigma_2, \tau_1$ and τ_2 of $\mathbb{R}\mathrm{P}^2$ shown in the figure below, where we use a hopefully obvious variation on the notation introduced in the figure on page 3538. It is straightforward to see that the singular 2-chain $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a cycle. Using the $\mathbb{F}_2$-Fundamental Class-Criterion 185.32 we see that it represents the $\mathbb{F}_2$-fundamental class of the real projective plane.



projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2 / \sim$

the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class of the real projective plane

(2) In Exercise 185.15 we will write down a representative for the $\mathbb{F}_2$-fundamental class of the Klein bottle. □

The following proposition is the analog of the Fundamental Class-of-∂-Proposition 185.18.

Proposition 185.33. ($\mathbb{F}_2$-Fundamental Class-of-∂-Proposition) Let M be a compact n-dimensional smooth manifold. The connecting homomorphism

$$\partial \colon \mathrm{H}_n(M, \partial M;\mathbb{F}_2) \quad \to \quad \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2)$$

of the long exact sequence in homology with $\mathbb{F}_2$-coefficients of the pair $(M, \partial M)$ has the property that

$$[M]_{\mathbb{F}_2} \quad \mapsto \quad [\partial M]_{\mathbb{F}_2}.$$

Proof. The proof is the obvious modification of the proof of the Fundamental Class-of-∂-Proposition 185.18. ■

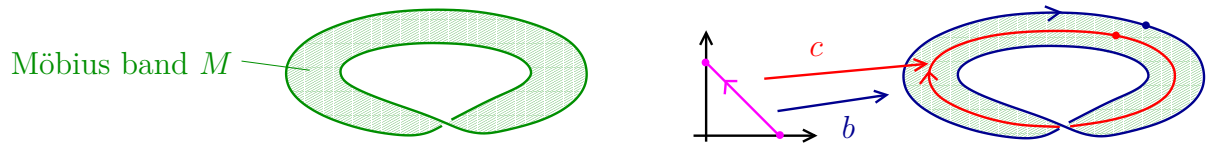
Finally we have the following analog of the Kernel-of-Boundary Inclusion Corollary 185.20.

Corollary 185.34. ($\mathbb{F}_2$-Kernel-of-Boundary Inclusion Corollary) Let M be a compact n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) We have
$$\ker(i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)) = \mathbb{Z}\cdot[\partial M]_{\mathbb{F}_2} \in \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2).$$
(2) If M has precisely one boundary component, then $i_*\colon \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2) \to \mathrm{H}_{n-1}(M;\mathbb{F}_2)$ is the zero map.
(3) The boundary ∂M is not a retract of M.

Proof. Once again the proof is basically the same as the proof of the analogous statement for ordinary homology groups. ■

Remark. It is perhaps instructive to consider the good old Möbius band M. We denote by $i\colon \partial M \to M$ the inclusion map. Furthermore let $[b] \in \mathrm{H}_1(M)$ and $[c] \in \mathrm{H}_1(\partial M)$ be the generators shown in the figure below.

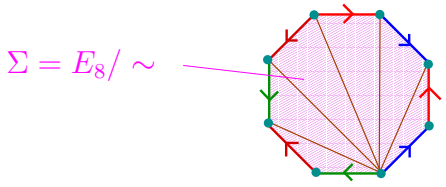


We end up with the following situation:
$$\begin{array}{ccc} \mathrm{H}_1(\partial M) & \xrightarrow{[b]\mapsto 2\cdot[c]} & \mathrm{H}_1(M) \\ \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma\otimes 1]} & & \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma\otimes 1]} \\ \mathrm{H}_1(\partial M;\mathbb{F}_2) & \xrightarrow{[b\otimes 1]\mapsto[2\cdot c\otimes 1]=[c\otimes 2]=0} & \mathrm{H}_1(M;\mathbb{F}_2). \end{array}$$

Note that the vertical maps are isomorphisms by the Universal Coefficient Theorem 162.16. We see that the top horizontal map is an injective map, in fact it is basically given by multiplication by 2. On the other hand we see that the bottom horizontal map is the zero map. □

185.7. Exercises for Chapter 185 .

Exercise 185.1. We consider the surface $\Sigma = E_8/\sim$ of genus two as shown in the figure below. Show that there exists a cycle of the form $\epsilon_1\cdot\sigma_1 + \cdots + \epsilon_6\cdot\sigma_6 \in \mathrm{C}_2(E_8/\sim)$ such that the images of the six singular simplices $\sigma_1,\ldots,\sigma_6$ are precisely the six triangles shown in the figure.



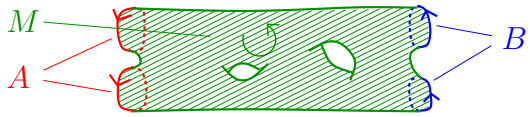
Exercise 185.2. Let M be a compact oriented connected n-dimensional smooth manifold with *at least two* boundary components. Let C be one of the boundary components. By the Kernel-of-Boundary Inclusion Corollary 185.20 we know that the inclusion-induced

map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left inverse, i.e. there exists a homomorphism $\varphi\colon \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(C)$ such that $\varphi \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(C)}$. Also note that in the Surface Boundary–Retraction Proposition 107.11 we showed that in the case $n = 2$ a stronger statement holds, namely C is a retract of M. Is it true in general that C is necessarily a retract of M?

Exercise 185.3. Let M be a compact oriented connected n-dimensional smooth manifold. Furthermore let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Show that

$$\ker(i_*\colon \mathrm{H}_{n-1}(A) \to \mathrm{H}_{n-1}(M, B)) = \mathbb{Z} \cdot [A] \quad \in \mathrm{H}_{n-1}(A).$$

Remark. The case $A = \partial M$ is the statement of the Kernel-of-Boundary Inclusion Corollary 185.20 (1).



Exercise 185.4. Let M be a closed oriented connected n-dimensional smooth manifold and let $C \subseteq M$ be a closed oriented connected $(n-1)$-dimensional submanifold of M. We assume that C is non-separating. Recall that according to Definition 107.4 this means that $M \setminus C$ is connected. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can pick a tubular neighborhood $[-1, 1] \times C$. We write $W := M \setminus ((-1, 1) \times C)$.

(a) Show that there exists an exact sequence of the form

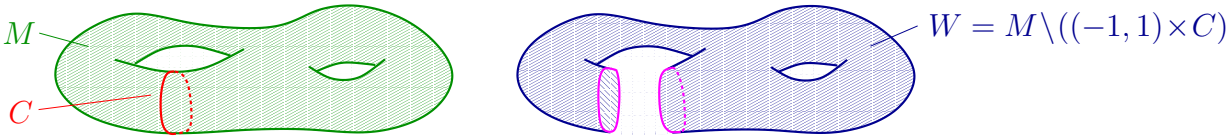
$$0 \to \mathrm{H}_{n-1}(\underbrace{M \setminus ((-1,1) \times C)}_{=W}) \to \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-2}(C) \to \dots$$

Hint. Consider a Mayer–Vietoris sequence.

(b) Use (a) and Exercise 185.2 to show that the map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left inverse.

Remark. This implies in particular that $i_*([C])$ is a non-trivial element of $\mathrm{H}_{n-1}(M)$.

(c) Does the conclusion of (b) also hold if we replace the hypothesis that M is closed by the hypothesis that M is compact?



Exercise 185.5. Let $K = \gamma(S^1) \subseteq S^3$ be a knot and let $\Phi\colon \overline{B}^2 \times K \to S^3$ be a tubular map for K (which exists by the Product Tubular Neighborhood–Existence Theorem 58.4 (2)). As in Definition 108.30 we refer to $X_K = S^3 \setminus \Phi(B^2 \times K)$ as the exterior of the knot K. Now let $P \in K$. By the Meridian Proposition 109.2 (2) we know that $\mu_K = \Phi(S^1 \times \{P\})$ is a meridian of K. In the Knot Complement-H_*-Proposition 185.29 we saw that $[\mu_K]$, viewed as an element in $\mathrm{H}_1(X_K)$, is a generator of $\mathrm{H}_1(X_K)$.

(a) Show that there exists an $n \in \mathbb{Z}$ such that the closed curve

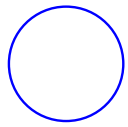
$$\begin{array}{ccc} S^1 & \to & \partial X_K = \Phi(S^1 \times K) \\ z & \mapsto & \Phi(z^n, \gamma(z)) \end{array}$$

represents a generator of $\ker(\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K))$. We refer to the image of this curve as a *longitude* λ_K of K.

(b) Sketch longitudes for the knots that are shown in the figure below.

(c) Now suppose that we are given two knots K and J. We define X_K, X_J, μ_K, μ_J and λ_K and λ_J as above. We pick a diffeomorphism $\varphi\colon \partial X_K \to \partial X_J$ with $\varphi_*(\mu_K) = \lambda_J$ and $\varphi_*(\lambda_K) = \mu_K$. We denote by $Y = X_K \cup_\varphi X_J$ the result of gluing X_K to X_J along the boundary. Show that Y is a homology sphere, i.e. show that $\mathrm{H}_1(Y) = \mathrm{H}_2(Y) = 0$ and $\mathrm{H}_3(Y) \cong \mathbb{Z}$.

Remark. If K and J are two non-trivial knots, then it follows from the Trivial Knot–Characterisation Proposition **??** (which in turn is a consequence of the famous Loop Theorem **??**) together with the Amalgamated Product-Monomorphism Proposition 98.26, that the fundamental group of Y is always non-trivial. This shows that Y is a homology sphere with non-trivial fundamental group. In other words we have now found an alternative proof of the Poincaré Homology Sphere Proposition 183.24.


trivial knot


(left-handed) trefoil

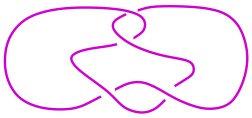


figure-8 knot


stevedore knot

Exercise 185.6. In this exercise we will consider an alternative approach to computing the homology groups of a knot complement. Let $K \subseteq S^3$ be a knot. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) we can pick a tubular neighborhood $\overline{B}^2 \times K$ and we consider the exterior $X_K = S^3 \setminus (B^2 \times K)$. Now do the following:

(a) Show that the inclusion-induced maps

$$\mathrm{H}_*(S^3, K) \xrightarrow{\cong} \mathrm{H}_*(S^3, \overline{B}^2 \times K) \xleftarrow{\cong} \mathrm{H}_*(X_K, \partial X_K)$$

are both isomorphisms.

(b) Compute the homology groups of the pair $(X_K, \partial X_K)$.

(c) Determine, using (b), the homology groups of X_K.

Exercise 185.7. We define an *m-component link* to be a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) we can pick a tubular neighborhood $\overline{B}^2 \times L$ and we define the exterior of L as $X_L \coloneqq S^3 \setminus (B^2 \times L)$. We denote the components of L by $L_1, \dots, L_m$. For each $i \in \{1, \dots, m\}$ we pick a point $P_i \in L_i$ and we define $\mu_i \coloneqq S^1 \times \{P_i\}$.

(a) Show that $\mathrm{H}_1(X_L) = \mathbb{Z} \cdot [\mu_1] \oplus \cdots \oplus \mathbb{Z} \cdot [\mu_m]$.

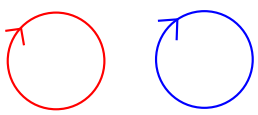
(b) Determine $\mathrm{H}_2(X_L)$.

Remark. There are basically three ways to go about doing this exercise, you can modify the proof of the Knot Complement-H_*-Proposition 185.29, you could generalize the approach taken in Exercise 185.6, or you could write $X_{L_1 \sqcup L_2} = X_{L_1} \cap X_{L_2}$ and use a Mayer-Vietoris sequence.

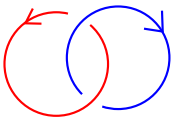
Exercise 185.8.

(a) In the figure below we see two 2-component links, namely the unlink and the Hopf link. Are these two links isotopic?

(b) Let $L \subseteq S^3$ be an m-component link. Show that the weight of the group $\pi_1(S^3 \setminus L)$, as defined in Exercise 111.31, equals m.



2-component trivial link



Hopf link

Exercise 185.9. Let $p, q \in \mathbb{N}$ be coprime. We consider the corresponding lens space $L(p,q)$. Let $K \subseteq L(p,q)$ be a knot, i.e. K is a closed connected non-empty one-dimensional smooth submanifold of $L(p,q)$. Determine the homology groups $\mathrm{H}_*(L(p,q) \setminus K)$.
Remark. Recall that in the Lens Space-H_*-Lemma 183.21 we showed that we have an isomorphism $\mathrm{H}_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$. The isomorphism type of $\mathrm{H}_*(L(p,q) \setminus K)$ will depend on what homology class K represents in $\mathrm{H}_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.

Exercise 185.10. Let M and N be two oriented, connected n-dimensional smooth manifolds. We suppose that M is closed and that N has *non-empty* boundary. Express the homology groups of $M \# N$ in terms of the homology groups of M and N.

Exercise 185.11. Let $f\colon M \to N$ be a continuous map between two compact orientable connected n-dimensional smooth manifolds with non-empty boundary. We assume that $f\colon \partial M \to \partial N$ is a homeomorphism. Show that f is a surjection.

Exercise 185.12. Let M be a compact oriented n-dimensional smooth manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. By the Manifold Covering Proposition 95.4 we can equip $\widetilde{M}$ with the structure of a compact oriented smooth manifold such that $p\colon \widetilde{M} \to M$ is an orientation-preserving map. Let $p^*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(\widetilde{M})$ be the transfer map that we introduced in Definition 165.1. Show that $p^*([M]) = \left[\widetilde{M}\right] \in \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M})$.

Exercise 185.13. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $f\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the corresponding mapping torus $M := \mathrm{Tor}(\Sigma, f)$. Note that we know by the Manifold-Mapping Torus Lemma 59.11 that M is a closed connected 3-dimensional smooth manifold.
(a) Suppose that if f is orientation-preserving, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M))$.
(b) Suppose that if f is orientation-reversing, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M)) - 1$.
Remark. You might want to use the Mapping Torus-H_*-Proposition 153.6.

Exercise 185.14. Give examples of closed oriented connected non-empty smooth manifolds M and N of the same dimension such that $\pi_2(M \# N)$ is not isomorphic to $\pi_2(M) \oplus \pi_2(N)$.

Exercise 185.15. Give an example of a representative for the $\mathbb{F}_2$-fundamental class for the Klein bottle.

Exercise 185.16. Can you find a representative for the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$ with two or three summands?

Exercise 185.17. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold. The image of the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ under the natural map

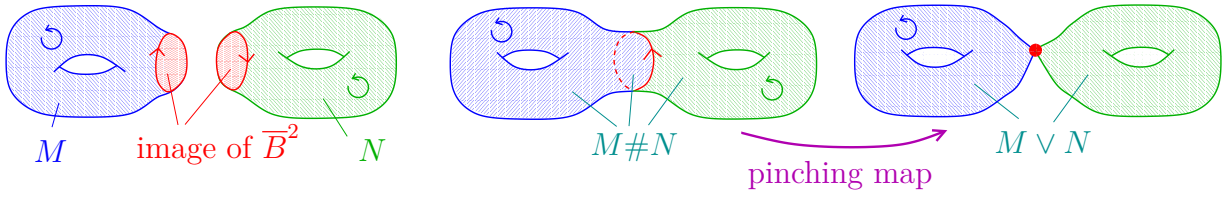
$\mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M,\partial M;\mathbb{R})$ is called the *real fundamental class*, denoted by $[M]_{\mathbb{R}}$. Let $(K=(V,S),\Theta\colon |K| \to M)$ be a smooth simplicial structure. Given an ordered n-simplex $s=(v_0,\dots,v_n)$ of K we denote by $\Phi_s\colon \Delta^n \to |K|$ the corresponding characteristic map that is given by $(t_0,\dots,t_n) \mapsto t_0\cdot v_0 + \dots + t_n \cdot v_n$.

(a) Show that
$$\sigma \coloneqq \sum_{\substack{s=(v_0,\dots,v_n)\ \text{ordered } n\text{-simplex}\\ \text{such that } \Theta\circ\Phi_s\colon \mathring{\Delta}^n \to M\\ \text{is orientation-preserving}}} (\Theta\circ\Phi_s\colon \Delta^n \to M) \ \in\ \mathrm{C}_n(M)$$
represents a cycle in $\mathrm{C}_n(M,\partial M)$.

(b) Show that $[\sigma]\otimes\frac{2}{n!} \in \mathrm{C}_n(M,\partial M;\mathbb{R}) = \mathrm{C}_n(M,\partial M)\otimes\mathbb{R}$ represents $[M]_{\mathbb{R}}$.

Exercise 185.18. Let $n\in\mathbb{N}$ and let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n_2 \to M\setminus\partial M$ and furthermore we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n_2 \to N\setminus\partial N$. We use φ and ψ to define the connected sum
$$M\#N \coloneqq \big(M\setminus\varphi(B^n)\big) \sqcup \big(N\setminus\psi(B^n)\big) \,/\, \big\{\varphi(P)=\psi(P)\,\big|\,P\in S^{n-1}\big\}.$$
We refer to the map
$$p\colon M\#N \to M\vee N,\quad P \mapsto \begin{cases} \varphi(x\cdot 2(t-1)), & \text{if } P=\varphi(x\cdot t) \text{ with } x\in S^{n-1} \text{ and } t\in[1,2]\\ \psi(x\cdot 2(t-1)), & \text{if } P=\psi(x\cdot t) \text{ with } x\in S^{n-1} \text{ and } t\in[1,2]\\ P, & \text{otherwise.}\end{cases}$$
as the *pinching map*. Show that for $i=0,\dots,n-1$ the induced map
$$p_*\colon \mathrm{H}_i(M\#N) \to \mathrm{H}_i(M\vee N)$$
is an isomorphism and that $p_*([M\#N]) = [M]+[N] \in \mathrm{H}_n(M\vee N) = \mathrm{H}_n(M)\oplus\mathrm{H}_n(N)$.

Remark. We leave it to the reader to show that for $M=N=S^n$ this pinching map is homotopic to the pinching map that we introduced on page 2579.



CHAPTER 186

The degree of a map between oriented smooth manifolds II

In Chapters 46 and 47 we used Sard's Theorem 38.1 and several other tricks from differential topology to introduce the degree of a continuous map between closed oriented connected non-empty smooth manifolds of the same dimension. In this chapter we will use a completely different approach to define the degree. The new approach is perhaps less intuitive, but since it builds on the machinery we have developed over the last chapters we can now quickly develop the theory of degrees.

This chapter is organized as follows:

- In Section 186.1 we will give a (new) definition of the degree of a map between compact smooth manifolds.
- In Section 186.2 we will discuss properties of the degree.
- In Section 186.3 we will provide exercises for Chapter 186.

In the next Chapter 187 we will discuss the question whether homology classes can be represented by smooth (sub-)manifolds.

186.1. The definition of the degree of a map between compact smooth manifolds.

Let $n \in \mathbb{N}$ and let M be a compact oriented connected non-empty n-dimensional smooth manifold. Recall that in the Manifold-Top-H_*-Theorem 185.3 and the Fundamental Class-Characterization Lemma 185.6 we proved that there exists a unique class $[M] \in \mathrm{H}_n(M, \partial M)$ such that the following holds:

(1) We have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$.

(2) For every smooth embedding[1976] $\Psi \colon \Delta^n \to M$ and for every point $x \in \mathring{\Delta}^n$ with $\Psi^{-1}(\Psi(x)) = \{x\}$ we have the following equality in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:

$$\underbrace{\omega_{\Psi(x)}([M])}_{\substack{\uparrow \\ \text{this is the natural map } \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M, M\setminus\{\Psi(x)\}), \text{ which} \\ \text{is in fact an isomorphism by the Manifold-Top-H}_*\text{-Theorem 185.3 (1)}}} = \operatorname{sign}(\Psi, x) \cdot [\, \underbrace{\Psi \colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})} \,].$$

In this short chapter we will explore the following definition:[1977]

Definition 186.1. Let $f \colon M \to N$ be a continuous map between two compact oriented connected non-empty n-dimensional smooth manifolds. We assume that $f(\partial M) \subseteq \partial N$.

[1976] Recall that in Definition 179.3 we defined what it means for a map $\Delta^n \to M$ to be a smooth embedding.

[1977] The definition of degree given below is very similar in style to the definition given in Definition 86.2 in terms of de Rham cohomology.

We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$f_*([M]) \;=\; \deg(f) \cdot [N] \;\in\; \mathrm{H}_n(N, \partial N) = \mathbb{Z} \cdot [N].$$

since $f(\partial M) \subseteq \partial N$ we have an induced map $f_* \colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(N, \partial N)$

We refer to $\deg(f)$ as the degree of f.

Remark.

(1) For a continuous map $f \colon S^n \to S^n$ the above degree is, by definition, exactly the same as the degree that we introduced in Definition 151.3. Furthermore, if f is smooth and M and N are closed, then it follows from the Local-Global Degree Proposition 186.8 below that the above definition of the degree agrees with the definitions given on Definition 46.3, Definition 47.6 and Definition 86.2.

(2) In Definition 195.1 we will generalize the notion of degree from smooth manifolds to topological manifolds. □

The following lemma generalizes several statements of the Sphere–Degree Lemma 151.4.

Lemma 186.2. (Manifold Degree Lemma) Let $n \in \mathbb{N}$ and let L, M and N be compact oriented connected non-empty n-dimensional smooth manifolds.

(1) $\deg(\mathrm{id}_M) = 1$.

(2) For every continuous map $f \colon M \to N$ such that $f(\partial M) \subseteq \partial N$ we have the following equality:[1978]

$$f_*([\partial M]) \;=\; \deg(f \colon M \to N) \cdot [\partial N] \;\in\; \mathrm{H}_{n-1}(\partial N).$$

In particular, if ∂M and ∂N are connected and non-empty, then we obtain the equality $\deg(f \colon \partial M \to \partial N) = \deg(f \colon M \to N)$.

(3) Let $f \colon M \to N$ be a continuous map with $f(\partial M) \subseteq \partial N$. If f is not surjective, then $\deg(f) = 0$.

(4) Let $f, g \colon M \to N$ be two continuous maps with $f(\partial M) \subseteq \partial N$ and $g(\partial M) \subseteq \partial N$. If f is homotopic to g, as a map of pairs $(M, \partial M) \to (N, \partial N)$, then $\deg(f) = \deg(g)$.

(5) For any two continuous maps $f \colon L \to M$ and $g \colon M \to N$ with $f(\partial L) \subseteq \partial M$ and $g(\partial M) \subseteq \partial N$ we have $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Proof.

(1) This statement is trivial.

(2) We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow[[M]\mapsto[\partial M], \text{ by the Fundamental Class-of-}\partial\text{-Proposition 185.18}]{\partial_n} & \mathrm{H}_{n-1}(\partial M) \\ \downarrow f_* & & f_* \downarrow \\ \mathrm{H}_n(N, \partial N) & \xrightarrow[[N]\mapsto[\partial N], \text{ by the Fundamental Class-of-}\partial\text{-Proposition 185.18}]{\partial_n} & \mathrm{H}_{n-1}(\partial N) \end{array}$$

where the horizontal maps are given by the connecting homomorphisms of the pairs $(M, \partial M)$ and $(N, \partial N)$. By the LES-of-Triple Proposition 146.5 we know that the connecting homomorphisms are natural, in other words, the diagram commutes. The statement now follows from the Fundamental Class-of-∂-Proposition 185.18.

[1978] This statement can be viewed as a generalization of the Boundary Degree Proposition 47.9.

(3) So let $f\colon M \to N$ be a non-surjective map with $f(\partial M) \subseteq \partial N$. We pick a point P in N that is not in the image of f. Note that $N \setminus \{P\}$ is an open subset of N. Thus, by the discussion on page 672, we can view it as an n-dimensional smooth manifold with $\partial(N \setminus \{P\}) = \partial N \cap (N \setminus \{P\}) = \partial N \setminus \{P\}$. Also note that it follows easily from $n \geq 1$ that $N \setminus \{P\}$ is non-compact.

Now we consider the following very simple commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_n(N, \partial N) \\ & {\scriptstyle f_*} \searrow \quad \nearrow & \\ & \underbrace{\mathrm{H}_n(N \setminus \{P\}, \partial(N \setminus \{P\}))}_{\substack{= 0 \text{ by the Manifold-Top-H}*\text{-Theorem 183.18,} \\ \text{since } N \setminus \{P\} \text{ is non-compact and connected}}} & \end{array}$$

It follows immediately that $f_*\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(N, \partial N)$ is the zero map.

(4) This statement follows immediately from the Homotopy-H$_*$-Proposition 144.12.

(5) Exactly as in the proof of the Sphere–Degree Lemma 151.4 (4) we have

$$(g \circ f)_*([L]) \;=\; g_*(f_*([L])) \;=\; g_*(\deg(f) \cdot [M]) \;=\; \deg(g) \cdot \deg(f) \cdot [N].$$

This shows that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. ■

We move on to the following definition:

Definition 186.3. Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension. Let $P \in M \setminus \partial M$. We say that f is a local diffeomorphism at P if there exists an open neighborhood U of P such that $f|_U$ is an embedding.[1979]

The following proposition is much more interesting than the previous lemma. It gives in particular a practical criterion for a continuous map to have degree ± 1.

Proposition 186.4. (Local-Global Degree Proposition 186.4 Let $n \in \mathbb{N}$. Furthermore let $f\colon M \to N$ be a continuous map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$. We suppose that there exists an $x \in M \setminus \partial M$ such that $f^{-1}(f(x)) = \{x\}$ and we suppose that f is a local diffeomorphism at x. Then

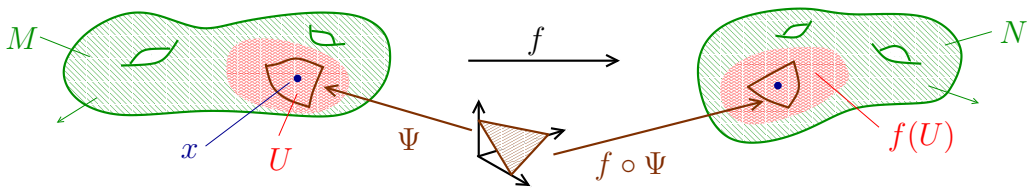
$$\deg(f) \;=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Proof. Let U be an open connected neighborhood of x such that $f\colon U \to N$ is an embedding. We set

$$\epsilon \;:=\; \begin{cases} +1, & \text{if } f\colon U \to N \text{ is orientation-preserving} \\ -1, & \text{if } f\colon U \to N \text{ is orientation-reversing.} \end{cases}$$

Note that we need to show that $\deg(f) = \epsilon$. Since $f^{-1}(\partial N) = \partial M$ we can pick a smooth embedding $\Psi\colon \Delta^n \to U$ such that $x \in \Psi(\mathring{\Delta}^n)$. Note that $\Psi^{-1}(x)$ is an admissible point for Ψ and note that it follows from our hypothesis $f^{-1}(f(x)) = \{x\}$ that $\Psi^{-1}(x)$ is also an admissible point for $f \circ \Psi$.

[1979] Note that it follows from the Differential-of-Diffeomorphism Theorem 24.22 that this definition of a "local diffeomorphism at a point" is consistent with the definition given in Definition 25.18.



We consider the following commutative diagram:[1980]

$$\begin{array}{ccc} \mathrm{H}_n(M,\partial M) & \xrightarrow[\cong]{\omega_x} & \mathrm{H}_n(M, M\setminus\{x\}) \\ \downarrow f_* & & f_* \downarrow \\ \mathrm{H}_n(N,\partial N) & \xrightarrow[\cong]{\omega_{f(x)}} & \mathrm{H}_n(N, N\setminus\{f(x)\}). \end{array}$$

After these preparations we can now turn to the actual proof:

since $x \in \Psi(\mathring{\Delta}^n)$ and since Ψ is an embedding we see that $\Psi\colon \Delta^n \to M$ defines an element of $\mathrm{H}_n(M, M\setminus\{x\})$

$$\begin{array}{rrcll} & \omega_x([M]) & = & \mathrm{sign}(\Psi,x)\cdot[\Psi\colon\Delta^n\to M] & \text{by Definition 185.4 of } [M] \\ \Rightarrow & f_*(\omega_x([M])) & = & \mathrm{sign}(\Psi,x)\cdot f_*([\Psi]) & \text{applying } f_* \\ \Rightarrow & \omega_{f(x)}(f_*([M])) & = & \mathrm{sign}(\Psi,x)\cdot[f\circ\Psi] & \text{since the diagram commutes} \\ \Rightarrow & \omega_{f(x)}(f_*([M])) & = & \epsilon\cdot\mathrm{sign}(f\circ\Psi,x)\cdot[f\circ\Psi] & \text{since } \mathrm{sign}(f\circ\Psi,x)=\epsilon\cdot\mathrm{sign}(\Psi,x) \\ \Rightarrow & \omega_{f(x)}(\deg(f)\cdot[N]) & = & \epsilon\cdot\omega_{f(x)}([N]) & \text{by definition of } \deg(f) \text{ and } [N] \\ \Rightarrow & \deg(f)\cdot[N] & = & \epsilon\cdot[N] & \text{by the Manifold-Top-H}_*\text{-Theorem 185.3 (1)} \\ \Rightarrow & \deg(f) & = & \epsilon & \text{since } \mathrm{H}_n(N)=\mathbb{Z}\cdot[N]. \quad \blacksquare \end{array}$$

186.2. **Properties of the degree.** We can now restate and reprove the Degree Realization Proposition 46.19 from earlier on.

Proposition 46.19 (Degree Realization Proposition) Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Given any $k \in \mathbb{Z}$ there exists a smooth map $f\colon M \to S^n$ with $\deg(f) = k$.

Proof. [1981] Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $k \in \mathbb{Z}$. It follows from the Degree-of-Power Lemma 151.5 together with the Suspension-Degree Lemma 151.6 that there exists a continuous map $g\colon S^n \to S^n$ of degree k. It follows from this fact and the Manifold Degree Lemma 186.2 (5) that it suffices to prove the following claim:

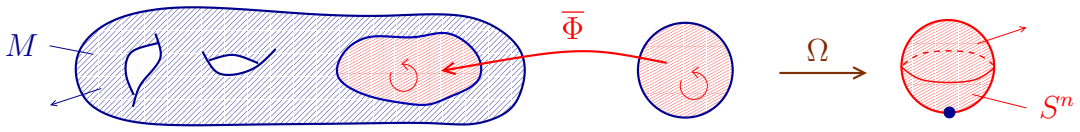
Claim. There exists a continuous map $f\colon M \to S^n$ of degree $+1$.

Proof. First we pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n \to M$. It follows fairly easily from the Quotient-to-Quotient Lemma 8.6, and the Compact-Hausdorff Proposition 4.23 (3) together with the Compact Image Lemma 4.15 and the Quotient Topology-Properties Proposition 8.5 (3), that the induced map $\overline{\Phi}\colon \overline{B}^n/S^{n-1} \to M/(M\setminus\Phi(B^n))$ is a homeomorphism. We denote by $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ the explicit homeomorphism from

[1980] Note that our hypothesis that $f^{-1}(f(x)) = \{x\}$ implies that $f(M\setminus\{x\}) \subseteq N\setminus\{f(x)\}$. Also note that $x \in M\setminus\partial M$ and $f(x) \in N\setminus\partial N$. In particular this implies that the various inclusion-induced maps actually make sense.

[1981] The proof is almost identical to the proof we had given on page 1084.

the Ball Quotient-Sphere Lemma 8.10. By the Ball Quotient-Sphere Lemma 8.10 (3) this homeomorphism has the property that the restriction of Ω to B^n is in fact an orientation-preserving smooth embedding.



We consider the map $f\colon M \to S^n$ that is given by

$$M \longrightarrow M/(M \setminus \Phi(B^n)) \xrightarrow{\overline{\Phi}^{-1}} \overline{B}/S^{n-1} \xrightarrow{\Omega} S^n.$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{=:f}$$

It follows almost immediately from the Local-Global Degree Proposition 186.4 that f is a map of degree $+1$. ■

To put the previous proposition into context we recall the following theorem. It says that for continuous maps to S^n the converse to the Manifold Degree Lemma 186.2 (4) holds.

Theorem 46.22. (Hopf Degree Theorem) Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two continuous maps. If[1982] $\deg(f) = \deg(g)$, then the maps f and g are actually homotopic.

For the record we also state the following open question, see [**Kir1997**, Problem 5.26], which was initially posed by Heinz Hopf.

Question 186.5. (Hopf) Let M be a closed orientable smooth manifold M. Is every continuous map $f\colon M \to M$ of degree ± 1 a homotopy equivalence?

In the following discussion, given $g \in \mathbb{N}_0$, we denote as usual by Σ_g the surface of genus g. We move on to our next list of interesting examples of continuous maps of non-zero degree.

Lemma 186.6. (Larger-to-Smaller Genus) Let $k \in \mathbb{N}_0$. For every $g \geq k$ there exists a degree-one map from Σ_g to Σ_k.

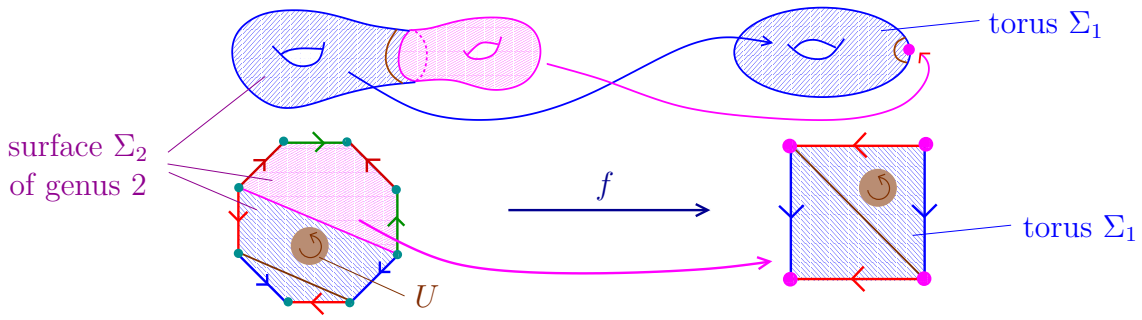
Proof. We prove the lemma for $g = 2$ and $k = 1$. It should be clear how to modify the discussion for other values of $g \geq k$. Let us consider the map from the surface Σ_2 of genus two to the torus Σ_1 that is shown in the figure below.
In fact in the figure we show the map twice, on top we show a slightly informal description of the map, at the bottom we give a more formal description. Note that the hypotheses of the Local-Global Degree Proposition 186.4 are satisfied. Thus we see that $\deg(f) = 1$. ■

The following question, which extends Question 46.21, arises naturally.

Question 186.7. Suppose that $g < k$. Does there exist a degree-one map from Σ_g to Σ_k?

We will be able to give an answer to this question later on in the Smaller-to-Larger Genus Proposition 186.11.

[1982] When we formulated the Hopf Degree Theorem 46.22 we used the definition of the degree that we gave in Definition 46.3. On page 3561 we pointed out that it follows from the Local-Global Degree Proposition 186.8 that the earlier definition of the degree agrees with the definition of the degree in terms of fundamental classes that we are using right now.



But first let us move on to the next proposition, which is a generalization of the Local-Global Degree Proposition 186.4.

Proposition 186.8. (Local-Global Degree Proposition) Let $f\colon M \to N$ be a continuous map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$.

(1) Let $y \in N \setminus \partial N$ such that the following condition is satisfied:
   ($*$) The preimage $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$ and f is a local diffeomorphism at each x_i.
   Then
$$\deg(f) \;=\; \sum_{i=1}^{m} \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x_i \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$
(2) If f is smooth, then every regular value in $N \setminus \partial N$ satisfies ($*$).
(3) If f is smooth, then the set of regular values in $N \setminus \partial N$ has full measure in N.[1983]

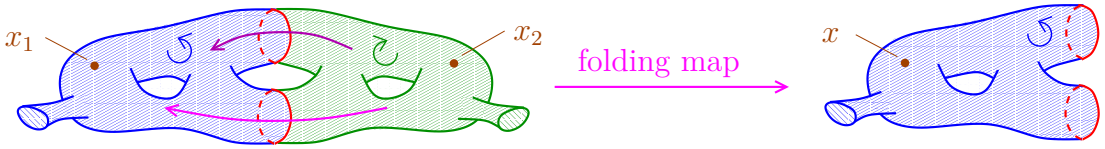
Remark.

(1) It follows from the Local-Global Degree Proposition 186.8 that the definitions of the degree from Definition 46.5 agrees with the notion defined above in Definition 186.1. Also recall that in the Degrees Agree Proposition 86.7 we showed that the notion of a degree from Definition 46.5 agrees with the degree defined via de Rham cohomology in Definition 86.2.
(2) In the Local-Global Degree Proposition 195.6 we state a result that generalizes the Local-Global Degree Proposition 186.8 (1) in the sense that we no longer demand that f is a local diffeomorphism at each x_i. □

Example. Let M be a compact oriented n-dimensional smooth manifold and let W be the union of some components of ∂M. As in Definition 54.8 we consider the double
$$\mathrm{D}_W M \;=\; (M \times \{1\}) \cup_{W\times\{1\}=W\times\{2\}} (M \times \{2\})$$
of M along W. By the Doubling-Manifold Lemma 54.10 we know that $\mathrm{D}_W M$ is naturally a compact oriented n-dimensional smooth manifold. We consider the folding map
$$\begin{array}{rcl} f\colon \mathrm{D}_W M & \to & M \\ [(P,i)] & \mapsto & P. \end{array}$$

[1983] We refer to Definition 37.1 for the definition of a subset of full measure in N and, more usefully, we refer to the Measure-Zero Properties Proposition 37.3 and the Full Measure-Density Proposition 37.4 for the main properties of subsets of full measure.

If we take any $x \in M \setminus \partial M$, then we see that $f^{-1}(P)$ consists of precisely two points, namely $x_1 = [(x,1)]$ and $x_2 = [(x,2)]$.



The folding map is an orientation-preserving local diffeomorphism at x_1 and it is an orientation-reversing local diffeomorphism at x_2. Thus we obtain from the Local-Global Degree Proposition 186.8 that the degree of the folding map is zero. □

Since the three parts of the Local-Global Degree Proposition 186.8 are of very different nature we break the proof of the Local-Global Degree Proposition 186.8 into several parts.

Proof of the Local-Global Degree Proposition 186.8 (1). Let $f\colon M \to N$ be a continuous map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$. Let $y \in N \setminus \partial N$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$ and such that f is a local diffeomorphism at each x_i.

For $i = 1, \ldots, m$ we pick an open connected neighborhood U_i of x_i such that $f\colon U_i \to N$ is an embedding. Note that we can arrange that for any $i \neq j$ we have $x_j \notin U_i$. For $i = 1, \ldots, m$ we set

$$\epsilon_i := \begin{cases} +1, & \text{if } f\colon U_i \to N \text{ is orientation-preserving,} \\ -1, & \text{if } f\colon U_i \to N \text{ is orientation-reversing.} \end{cases}$$

Note that we need to show that $\deg(f) = \epsilon_1 + \cdots + \epsilon_m$.

Let $i \in \{1, \ldots, m\}$. Since $y \in N \setminus \partial N$ and since by hypothesis we have $f^{-1}(\partial N) = \partial M$ we know that $x_i \in M \setminus \partial M$. Thus we can pick a smooth embedding $\Psi_i\colon \Delta^n \to U_i$ such that $x_i \in \Psi_i(\mathring{\Delta}^n)$. We consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}_n(M, \partial M) & & \xrightarrow{\quad =:\mu \quad} & & \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_m\}) \\ \Big\downarrow{\scriptstyle f_*} & \searrow^{\oplus \omega_{x_i}} & & \swarrow_{\oplus \nu_i}^{\cong} & \Big\downarrow{\scriptstyle f_*} \\ & & \bigoplus_{i=1}^{m} \mathrm{H}_n(M, M \setminus \{x_i\}) & & \\ \mathrm{H}_n(N, \partial N) & & \xrightarrow[\cong]{\omega_{f(x)}} & & \mathrm{H}_n(N, N \setminus \{y\}). \end{array}$$

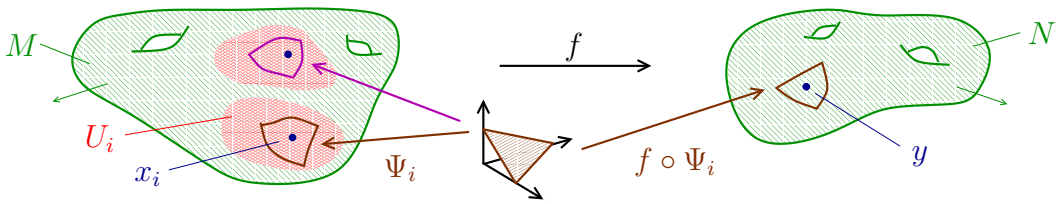
We make the following clarifications and observations:

(1) The map $\mu\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_m\})$ and furthermore all the maps $\nu_i\colon \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_m\}) \to \mathrm{H}_n(M, M \setminus \{x_i\})$ are induced by the natural maps on the pairs of topological spaces. Note that here we use that $x_1, \ldots, x_m \in M \setminus \partial M$.

(2) It follows from $f(\partial M) \subseteq \partial N$ that the left vertical map is defined.

(3) It follows easily from the Excision Theorem 148.2, see also Exercise 150.11, that the diagonal map $\bigoplus_{i=1}^{m} \nu_i\colon \mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_m\}) \to \bigoplus_{i=1}^{m} \mathrm{H}_n(M, M \setminus \{x_i\})$ is an isomorphism.

(4) Let $i \in \{1, \ldots, m\}$. Note that we arranged that $\Psi_i(\partial \Delta^n) \subseteq M \setminus \{x_1, \ldots, x_m\}$. In particular $\Psi_i\colon \Delta^n \to M$ defines an element in $\mathrm{H}_n(M, M \setminus \{x_1, \ldots, x_m\})$.



Now we see that

$$\begin{aligned}
& \Big(\bigoplus_{i=1}^m \omega_{x_i}\Big)([M]) = \bigoplus_{i=1}^m \operatorname{sign}(\Psi_i, x_i)\cdot[\Psi_i] && \text{by Definition 185.4 of } [M]\\
\Rightarrow\ & \Big(\bigoplus_{i=1}^m \omega_{x_i}\Big)([M]) = \Big(\bigoplus_{i=1}^m \nu_i\Big)\Big(\sum_{j=1}^m \operatorname{sign}(\Psi_j, x_j)\cdot[\Psi_j]\Big) && \text{since } \nu_i([\Psi_j]) = \delta_{ij}\cdot[\Psi_i]\\
\Rightarrow\ & \mu([M]) = \sum_{j=1}^m \operatorname{sign}(\Psi_j, x_j)\cdot[\Psi_j] && \text{since } \oplus\nu_i \text{ is an isomorphism and since the triangle commutes}\\
\Rightarrow\ & f_*(\mu([M])) = \sum_{j=1}^m \operatorname{sign}(\Psi_j, x_j)\cdot f_*([\Psi_j]) && \text{applying } f_*\\
\Rightarrow\ & \omega_{f(x)}(f_*([M])) = \sum_{j=1}^m \operatorname{sign}(\Psi_j, x_j)\cdot[f\circ\Psi_j] && \text{since the diagram commutes}\\
\Rightarrow\ & \omega_{f(x)}(f_*([M])) = \sum_{j=1}^m \epsilon_j\cdot\operatorname{sign}(f\circ\Psi_j, x_j)\cdot[f\circ\Psi_j] && \text{since } \operatorname{sign}(f\circ\Psi_j, x_j) = \epsilon_j\cdot\operatorname{sign}(\Psi_j, x_j)\\
\Rightarrow\ & \omega_{f(x)}(\deg(f)\cdot[N]) = \sum_{j=1}^m \epsilon_j\cdot\omega_{f(x)}([N]) && \text{by definition of } \deg(f) \text{ and } [N]\\
\Rightarrow\ & \deg(f)\cdot[N] = \sum_{j=1}^m \epsilon_j\cdot[N] && \text{by the Manifold-Top-H}_*\text{-Theorem 185.3 (1)}\\
\Rightarrow\ & \deg(f) = \sum_{j=1}^m \epsilon_j && \text{since } \mathrm{H}_n(N,\partial N) = \mathbb{Z}\cdot[N]. \qquad \blacksquare
\end{aligned}$$

Proof of the Local-Global Degree Proposition 186.8 (2) and (3). Let $f\colon M \to N$ be a smooth map between two compact oriented connected non-empty n-dimensional smooth manifolds M and N such that $f^{-1}(\partial N) = \partial M$.

(2) Let $y \in N \setminus \partial N$ be a regular value. It follows from the Stack-of-Records Lemma 32.3 (2) that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Let $i \in \{1, \ldots, m\}$. Note that it follows from $f^{-1}(\partial N) = \partial M$ that $x_i \in M \setminus \partial M$. Since x_i is a regular point we obtain from the Stack-of-Records Lemma 32.3 (1) that f is a local diffeomorphism at x_i.

(3) By Sard's Theorem 38.1 together with the Measure-Zero Properties Proposition 37.3 we know that the set of regular values of f that are contained in $N \setminus \partial N$ has full measure in N. ■

The following proposition gives us many examples of degrees of continuous maps between smooth manifolds.

Proposition 186.9. (Covering Map-Degree Proposition) Let $p\colon \widetilde{M} \to M$ be a finite covering of compact oriented connected non-empty smooth manifolds. If p is orientation-preserving, then

$$\deg(p) \ = \ [\widetilde{M} : M].$$

Proof. It turns out that there are two quick proofs for the proposition. In fact the desired statement follows immediately from the Local-Global Degree Proposition 186.8. Alternatively, if we denote by $p^*\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M})$ the transfer map that we introduced

in Definition 165.1, then we also see that

$$p_*([\widetilde{M}]) \underset{\text{by Exercise 185.12}}{=} p_*(p^*([M])) \underset{\text{by the Transfer–Projection Proposition 165.4}}{=} [\widetilde{M} : M] \cdot [M].$$

■

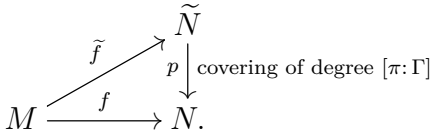
The following proposition is the key to answering our ancient Question 46.21 and our much more recent Question 186.7.

Proposition 186.10. (Degree One-Epimorphism Proposition) Let $f\colon M \to N$ be a continuous map between compact oriented connected non-empty n-dimensional smooth manifolds such that $f(\partial M) \subseteq \partial N$. If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.

The statement of the Degree One-Epimorphism Proposition 186.10 is somewhat surprising, in so far as it takes as input fairly simple information, namely the induced map on n-th homology, which is an abelian group, and gives as output information about fundamental groups, which are usually much harder to deal with. Also it is a priori not clear why the homology in the maximal dimension should have any impact on fundamental groups.

Proof. Let $f\colon M \to N$ be a continuous map between compact oriented connected non-empty n-dimensional smooth manifolds such that $f(\partial M) \subseteq \partial N$. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N, Q)$ and $\Gamma = f_*(\pi_1(M, P))$.

Note that by the Covering Existence Theorem 127.10[1984] there exists a path-connected covering $p\colon (\widetilde{N}, \widetilde{Q}) \to (N, Q)$ such that $p_*(\pi_1(\widetilde{N}, \widetilde{Q})) = \Gamma$ and such that $[\widetilde{N} : N] = [\pi : \Gamma]$. By the Manifold Covering Proposition 95.4 we can equip the covering space $\widetilde{N}$ with the structure of a compact oriented smooth manifold such that the projection $p\colon \widetilde{N} \to N$ is an orientation-preserving map and such that we have $\partial \widetilde{N} = p^{-1}(\partial N)$. Since by design we know that $p_*(\pi_1(\widetilde{N}, \widetilde{Q})) = \Gamma = f_*(\pi_1(M, P))$ we obtain from the Map Lifting Criterion 127.5 that the map $f\colon (M, P) \to (N, Q)$ lifts to a continuous map $\widetilde{f}\colon (M, P) \to (\widetilde{N}, \widetilde{Q})$. We can summarize the situation in the following simple commutative diagram



First suppose that Γ is a subgroup of π of infinite index. It follows immediately from $[\widetilde{N} : N] = [\pi : \Gamma]$ and the Covering–Basics Lemma 91.12 (4) that $\widetilde{N}$ is non-compact. Therefore we obtain from the Manifold-Top-H_*-Theorem 183.18 that $\mathrm{H}_n(\widetilde{N}) = 0$. But this shows that $f_* = p_* \circ \widetilde{f}_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is necessarily a finite-index subgroup of π. It follows from $[\widetilde{N} : N] = [\pi : \Gamma]$ and the Covering–Basics Lemma 91.12 (4) that $\widetilde{N}$ is compact. Therefore, by the above we now know that $\widetilde{N}$ is a compact oriented connected n-dimensional smooth

[1984]Here we use that according to the discussion on page 2442 any smooth manifold is locally simply connected.

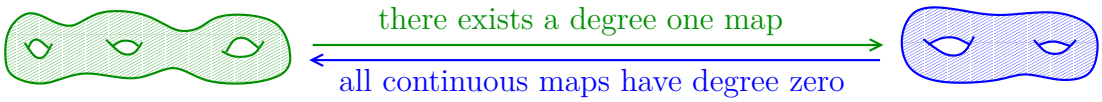
manifold. We deduce that

$$\pm 1 \;=\; \deg(f) \underset{\text{by the Manifold Degree Lemma 186.2 (5)}}{=} \deg(\widetilde{f}) \cdot \deg(p) \underset{\text{by the Covering Map-Degree Proposition 186.9}}{=} \deg(\widetilde{f}) \cdot [\widetilde{N} : N] \;=\; \deg(\widetilde{f}) \cdot [\Gamma : \pi].$$

This shows that $[\Gamma : \pi] = 1$, i.e. $\Gamma = \pi$. ■

Now we can give a negative answer to Questions 46.21 and 186.7.

Proposition 186.11. (Smaller-to-Larger Genus Proposition) Let $g, k \in \mathbb{N}_0$. If we have a strict inequality $g < k$, then every continuous map $\Sigma_g \to \Sigma_k$ has degree zero.



Proof. Let $g < k$ and let $f : \Sigma_g \to \Sigma_k$ be a continuous map. We start out with the following claim:

Claim. $\deg(f) \neq 1$.

Proof. Suppose that $\deg(f) = 1$. By the Degree One-Epimorphism Proposition 186.10 we obtain an epimorphism $\pi_1(\Sigma_g) \to \pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By the Surface-π_1-Proposition 101.1 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows from the Rank Comparison Lemma 97.16 that $g \geq k$. ⊞

The proof that $\deg(f)$ cannot be non-zero is a modification of the above argument which we outsource to Exercise 186.4. ■

We conclude this chapter with several questions. Recall that it follows from the Degree Realization Proposition 46.19 that there exists a degree-one map $S^2 \times S^2 \to S^4$. It is natural to ask whether the converse holds. This leads us to the following question which we will answer in the Sphere-to-Product-of-Spheres Lemma 207.5.

Question 186.12. Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$?

Later, in the No Degree-1-Map Corollary 210.17, we will also address the following question:

Question 186.13. Does there exist a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$?

It is also interesting to study degrees of continuous self-maps of smooth manifolds. For example note that it follows from the Degree-of-Power Lemma 151.5 and the Suspension-Degree Lemma 151.6 that given any $k \in \mathbb{Z}$ and given any $n \in \mathbb{N}$ there exists a continuous map $f : S^n \to S^n$ of degree k. Also note that it follows easily from the Covering Map-Degree Proposition 186.9 that the torus $T = S^1 \times S^1$ admits a continuous self-map of any degree. This naturally leads us to the question whether similar phenomena hold for surfaces of higher genus.

Question 186.14.

(1) Let $g \geq 2$. Does there exist a continuous self-map $f : \Sigma_g \to \Sigma_g$ of degree -1?
(2) Let $g \geq 2$ and let $k \geq 2$. Does there exist a continuous self-map $f : \Sigma_g \to \Sigma_g$ of degree k?

(3) Does there exist a continuous self-map $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ of degree -1?

We will address the first question in Exercise 186.3 and we will address the last two questions later on in the Surface Flexibility Proposition 203.35 and in the $\mathbb{C}\mathrm{P}^n$-Amphichiral Corollary 213.13.

Finally we conclude this chapter with two conjectures that were formulated by Heinz Hopf in 1931. We refer to [**Hausm1987, NeumB1953**] for some more information and partial results.

Conjecture 186.15. (Hopf Conjecture) Let M be a closed oriented connected smooth manifold and let $f\colon M \to M$ be a continuous map.

(1) If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(M)$ is an isomorphism.
(2) If $\deg(f) = \pm 1$, then f is a homotopy equivalence.

Remark. Let M be a closed oriented connected smooth manifold and let $f\colon M \to M$ be a continuous map with $\deg(f) = \pm 1$.

(1) By the Degree One-Epimorphism Proposition 186.10 we know that f_* is an epimorphism. Thus the Hopf Conjecture 186.15 (2) leads to the group-theoretic question, under what circumstances is a surjective automorphism of a group also injective. In particular this leads to the definition of a Hopfian group that we gave in Definition 130.13.
(2) In Proposition **??** we will see that in general f is not homotopic to a homeomorphism. Thus in general "homotopy equivalence" is the best we can hope for. Note though that if M is in fact aspherical, see Definition 167.29 for the definition, then the Borel Conjecture **??** predicts that f is actually homotopic to a homeomorphism. □

186.3. Exercises for Chapter 186.

Exercise 186.1. Let M be a compact oriented n-dimensional smooth manifold with non-empty boundary. Show that there exists a continuous map $f\colon M \to \overline{B}^n$ with $f(\partial M) \subseteq S^{n-1}$ of degree $+1$.

Remark. For $n \geq 2$ one can combine this result with Exercise 151.5 to get maps of arbitrary degree.

Exercise 186.2. Give an example of a continuous map $f\colon M \to N$ between closed oriented connected non-empty n-dimensional smooth manifolds such that $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism, but such that the degree of f is zero.

Exercise 186.3. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g.

(a) Let $g \geq 2$. Does there exist a continuous self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?
(b) Let $g \geq 1$ and let $k \geq 2$. Does there exist a continuous self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?

Exercise 186.4. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. Show that if there exists a continuous map $f\colon \Sigma_g \to \Sigma_k$ of non-zero degree, then $g \geq k$.

Exercise 186.5. Let $f\colon M \to N$ be a continuous map between two closed oriented non-empty n-dimensional smooth manifolds. Suppose that there exists a compact smooth manifold W with $\partial W = M$ and a smooth map $g\colon W \to N$ with $g|_M = f$. Show that

$\deg(f) = 0$.

Remark. This exercise is almost identical to Exercise 46.12, but now you should be able to give a much shorter proof.

Exercise 186.6. Let $f\colon M \to N$ be a continuous map between two closed connected non-empty n-dimensional smooth manifolds. Furthermore let $p\colon \widetilde{N} \to N$ be a finite covering. As in the Topological-Pullback Proposition 7.22 we consider the pullback

$$\widetilde{M} \;:=\; f^*\widetilde{N} \;=\; \{(x,y) \in M \times \widetilde{N} \,|\, f(x) = p(y)\}$$

together with the following commutative diagram:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{(x,y)\mapsto y} & \widetilde{N} \\ {\scriptstyle =:q}\downarrow{\scriptstyle (x,y)\mapsto x} & \overset{=:F}{} & \downarrow{\scriptstyle p} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

Note that by the Pullback-Covering Lemma 92.12 we know that q is again a covering with $\deg(q) = \deg(p)$.

(a) We suppose that M and N are oriented. We equip the smooth manifolds $\widetilde{M}$ and $\widetilde{N}$ with the corresponding orientations given by the Manifold Covering Proposition 95.4. Show that $\deg(F\colon \widetilde{M} \to \widetilde{N}) = \deg(f\colon M \to N)$.

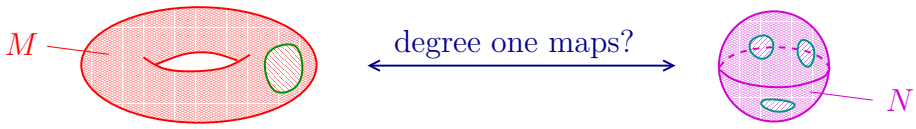
(b) Suppose that the induced map $f_*\colon \mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(N;\mathbb{F}_2)$ is an isomorphism. Show that $F_*\colon \mathrm{H}_n(\widetilde{M};\mathbb{F}_2) \to \mathrm{H}_n(\widetilde{N};\mathbb{F}_2)$ is also an isomorphism.

Exercise 186.7. Let M be the torus minus one open disk and let N be S^2 minus three open disks.

(a) Show that there is no degree one map $f\colon M \to N$.

(b) Show that there is no degree one map $g\colon N \to M$.

(c) Bonus question: Why is the map $g\colon N \to M$ given by the gluing of two boundary components not a counter example to (b)?



Exercise 186.8. Given $m \in \mathbb{N}$ we denote by N_m the non-orientable surface of genus m from Definition 9.19. Let $g \geq k$. Show that there exists a continuous map $f\colon N_g \to N_k$ such that the induced map $\mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$ is an isomorphism.

Exercise 186.9. Let X be a topological space. Given $\varphi \in \mathrm{H}_k(X;\mathbb{R})$ we define

$$\|\varphi\|_{\mathbb{R}} \;:=\; \inf\Big\{ \sum_{i=1}^{n} |r_i| \,\Big|\, \sigma = \underbrace{\sum_{i=1}^{n} \sigma_i \otimes r_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i \neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X;\mathbb{R}) \Big\}.$$

In the following let M be a closed oriented connected non-empty n-dimensional smooth manifold M. As usual we denote by $[M] \in \mathrm{H}_n(M)$ the fundamental class of M. Let $i\colon \mathrm{H}_1(M) \xrightarrow{\cong} \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{R})$ be the homomorphism from the Homology-$\mathbb{Z}$-Coefficients

Lemma 160.10 and the Homology-Coefficients-Functor Lemma 160.11. We define the *simplicial volume of* M as

$$\|M\|_{\mathbb{R}} \;:=\; \|i_*([M])\| \;\in \mathbb{R}_{\geq 0}.$$

(a) Show that $\|S^1\|_{\mathbb{R}} = 0$.

(b) Let $n \in \mathbb{N}_{\geq 2}$. What is $\|S^n\|_{\mathbb{R}}$?

(c) Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional smooth manifolds. Show that

$$\|N\|_{\mathbb{R}} \;\leq\; \frac{1}{\deg(f)} \cdot \|M\|_{\mathbb{R}},$$

where $\deg(f)$ denotes the degree of f from Definition 186.1.

(d) Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. We suppose that M and N are in fact homotopy equivalent. Show that $\|M\|_{\mathbb{R}} = \|N\|_{\mathbb{R}}$.

(e) Let N be a closed oriented non-empty n-dimensional smooth manifold. Furthermore let $p\colon M \to N$ be a d-fold covering. Show that $\|M\|_{\mathbb{R}} = d \cdot \|N\|_{\mathbb{R}}$.

Hint. Use the transfer map that we introduced in Definition 165.1.

(f) Let $g \in \mathbb{N}_0$. As usual we denote by Σ_g the surface of genus $g \geq 1$. Show that $\|\Sigma_g\|_{\mathbb{R}} \leq 2g - 2$.

Hint. Make use of (e).

Remark. In [**Frig2017**, Corollary 7.5] or alternatively [**Thu1978**, Theorem 6.2] it is shown that we also have the reverse inequality, i.e. we have $\|\Sigma_g\|_{\mathbb{R}} = 2g - 2$.

(g) Let $m, n \in \mathbb{N}$. Show that there exists a $C_{m,n}$ such that for any closed oriented connected smooth manifolds M and N, of dimensions m and n, we have

$$\|M \times N\|_{\mathbb{R}} \;\leq\; C_{m,n} \cdot \|M\|_{\mathbb{R}} \cdot \|N\|_{\mathbb{R}}.$$

Hint. Use the shuffle map ∇ from Definition 164.23.

Remark. The invariant $\|\varphi\|_{\mathbb{R}}$ is the "real analog" of the invariant $\|\varphi\|_{\mathbb{Z}}$ introduced in Exercise 151.11.

CHAPTER 187

Homology classes and smooth (sub-)manifolds

In Chapter 185 we saw that every compact oriented n-dimensional smooth manifold N comes naturally with a fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$. In Chapters 185 and 186 we already saw several interesting applications of the fundamental class and we now feel like we have a good understanding of fundamental classes.

In this chapter we want to study to which degree homology classes can be understood in terms of fundamental classes. More precisely, we want to study the following two questions:

(1) Given a topological space X and a homology class $\varphi \in \mathrm{H}_n(X)$, can we always find a closed oriented n-dimensional smooth manifold N and a continuous map $f\colon N \to X$ such that $f_*([N]) = \varphi$?

(2) If X is in fact a smooth manifold, can we find an $f\colon N \to X$ as in (1) that is actually a smooth embedding?

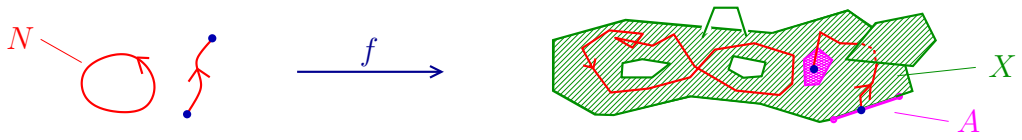
This chapter is organized as follows:

- In Section 187.1 we will define what it means to realize homology classes by smooth manifolds. In the H_2-via-Surface Proposition 187.4 we will see that every second homology class is realized by a surface. Furthermore we will state the Realize-by-Connected Manifold Proposition 187.5 which says that if one can realize a homology class by a smooth manifold, then one can also realize it by a connected smooth manifold.
- In Section 187.2 we will prove the H_2-via-Surface Proposition 187.4.
- In Section 187.3 we will prove the Realize-by-Connected Manifold Proposition 187.5
- In Section 187.4 we will define what it means to represent a homology class by submanifolds. We will state the Homotopic-to-Embedding Proposition 187.18 which explains under what restrictions on dimensions one can upgrade a map of smooth manifolds to a smooth embedding. We will also formulate the Represent-by-Connected Submanifold Proposition 187.21.
- In Section 187.5 we will prove several technical statements which we will need for the proofs of the Homotopic-to-Embedding Proposition 187.18 and the Represent-by-Connected Submanifold Proposition 187.21.
- In Section 187.6 we will prove the Homotopic-to-Embedding Proposition 187.18
- In Section 187.7 we will introduce the internal connected sum and we will use it to prove the Represent-by-Connected Submanifold Proposition 187.21.
- In Section 187.8 we will provide exercises for Chapter 187.

In the next Chapter 188 we will state and prove the Simplicial Poincaré Duality Theorem 188.5.

187.1. **Realizing homology classes by smooth manifolds.** Let us start out with the following friendly definition.

Definition 187.1. Let X be a topological space, let $A \subseteq X$ be a subset and let $\varphi \in \mathrm{H}_n(X, A)$ be a homology class. We say that φ can be realized by a smooth manifold (or Steenrod representable) if there exists a compact oriented n-dimensional smooth manifold N and a continuous map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$.



Example. Let (X, x_0) be a path-connected pointed topological space and let $n \in \mathbb{N}$. Recall that the Hurewicz homomorphism is defined as the map

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_n(X, x_0) & \to & \mathrm{H}_n(X) \\ [f\colon S^n \to X] & \mapsto & f_*([S^n]). \end{array}$$

Thus we see that every homology class that lies in the image of the Hurewicz homomorphism is realized by a smooth manifold. But as we remarked on page 3197, in general the Hurewicz homomorphism is certainly not an epimorphism. □

In this and the subsequent section we want to study the following question:

Question 187.2. Let $n \in \mathbb{N}_0$. Is it possible to realize every relative homology class in every relative homology group $\mathrm{H}_n(X, A)$ by a smooth manifold?

The case $n = 0$ is evidently trivial. So let us continue with the case $n = 1$.

Proposition 187.3. (H_1-via-Curve Proposition) Let X be a topological space. Every class in $\mathrm{H}_1(X)$ is realized by a smooth manifold.

Proof. The proposition is basically an immediate consequence of the Hurewicz Epimorphism Proposition 166.6. Nonetheless we prefer to do prove the proposition here "by hand", since this proof will also give us some ideas for how to approach the case of higher-dimensional homology classes.

Thus suppose that we are given a homology class $\varphi \in \mathrm{H}_1(X)$. We pick a singular chain $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_k(X)$ that represents φ. Note that we do not necessarily demand that the σ_j are different. In fact note that by "taking multiple copies of the singular simplices σ_j" we can and will arrange that $a_j \in \{-1, 1\}$, $j = 1, \ldots, k$. Next we will see that we can arrange that all $a_j = 1$. To do so, given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}\colon \Delta^1 \to X$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) \coloneqq \sigma(1-t, t)$. In the Singular 1-Simplex Lemma 143.28 we saw that $-\sigma$ and $\overline{\sigma}$ are homologous. Thus, after replacing each summand $-\sigma_j$ by $\overline{\sigma}_j$ we can assume that all $a_j = 1$.

Next note that we have

$$0 \underset{\uparrow}{=} \partial_1(c) = \partial_1\Big(\sum_{j=1}^{k} \sigma_j\Big) \underset{\uparrow}{=} \sum_{j=1}^{k} \sigma_j(0,1) - \sigma_j(1,0) = \sum_{j=1}^{k} \sigma_j(0,1) + \sum_{j=1}^{k} -\sigma_j(1,0).$$

since c is a cycle (see also page 2751) — by Definition 176.7 of ∂_1,

On the right-hand side each "positive summand" needs to cancel with a "negative summand". As in the proof of the Hurewicz Epimorphism Proposition 166.6 this means, see also the Matching Lemma 166.7, that there exists a permutation $\theta \in S_k$ such that for each

$j \in \{1, \dots, k\}$ the "endpoint" $\sigma_j(0,1)$ of σ_j is just the "starting point" $\sigma_{\theta(j)}(1,0)$ of $\sigma_{\theta(j)}$. We introduce the following notation:

(1) We consider the topological space

$$N \coloneqq (\{1, \dots, k\} \times \Delta^1)/\sim \quad \text{where for } j \in \{1, \dots, k\} \text{ we have } \sigma_j(0,1) \sim \sigma_{\theta(j)}(1,0).$$

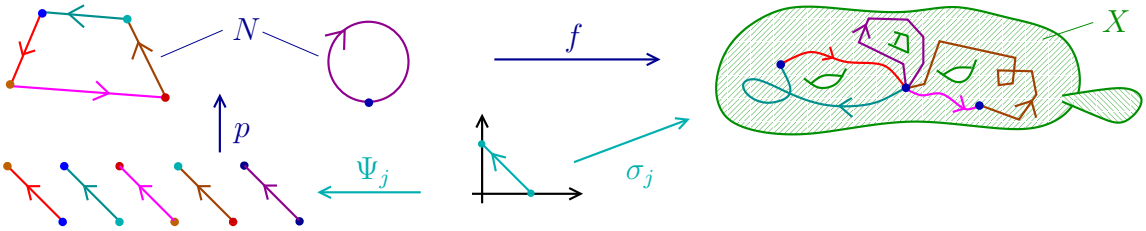
(2) For $j = 1, \dots, k$ we denote by $\Psi_j \colon \Delta^1 \xrightarrow{x \mapsto (j,x)} \{1, \dots, k\} \times \Delta^1$ the j-th inclusion map.

(3) We denote by $p \colon \{1, \dots, k\} \times \Delta^1 \to N$ the projection.

(4) It follows immediately from the definitions and the Topological-Quotient Proposition 8.2 (1b) that the map

$$\begin{array}{rccc} f \colon N = (\{1, \dots, k\} \times \Delta^1)/\sim & \to & X \\ [(j,z)] & \mapsto & \sigma_j(z) \end{array}$$

is well-defined and continuous.



It is now straightforward to verify that N can be equipped with the structure of a closed 1-dimensional smooth manifold and that we can equip N with an orientation such that for each $j \in \{1, \dots, k\}$ the map $p \circ \Psi_j \colon \mathring{\Delta}^1 \to N$ is an orientation-preserving embedding. Finally we see that

$$f_*([N]) \underset{\uparrow}{=} f_*([p \circ \Psi_1 + \dots + p \circ \Psi_k]) \underset{\uparrow}{=} [\sigma_1 + \dots + \sigma_k] = c.$$

this follows from the Fundamental Class Criterion 185.7 applied to any of the maps $p \circ \Psi_j$

by definition of f and $p \circ \Psi_j$

■

Evidently the idea now is to work our way up to higher dimensions. In fact we have the following proposition:

Proposition 187.4. (H_2-via-Surface Proposition) Let X be a topological space. Every class in $H_2(X)$ is realized by a smooth manifold.

Proof. The basic idea of the proof of the H_2-via-Surface Proposition 187.4 is similar to the proof of the H_1-via-Curve Proposition 187.3. Nonetheless, ironing out the details is much more messy, hence we postpone the proof of the case $n = 2$ to Section 187.2. Alternatively see also [**Cal2009**, p. 3] for a quick sketch of the proof. ■

Remark. In the $Y(\Theta)$-Not Necessarily Manifold Lemma 187.11 we will see that any naive attempt of generalizing the ideas behind the proofs of the H_1-via-Curve Proposition 187.3 and the H_2-via-Surface Proposition 187.4 to higher dimensions runs into very serious problems. □

We move on to the following two propositions that summarizes a few generalities on realizing homology classes by smooth manifolds.

Proposition 187.5. (Realize-by-Connected Manifold Proposition) In the following let $k \in \mathbb{N}$, let X be a topological space and let $A \subseteq X$ be a subset. We suppose that X is

path-connected. If $A = \varnothing$ or if $k \geq 2$, then every homology class in $\mathrm{H}_k(X, A)$ that can be realized by a smooth manifold can also be realized by a *connected* smooth manifold.

Proposition 187.6. (Represent Relative Homology Class Proposition) Let $k \in \mathbb{N}$. Suppose that for every topological space X every homology class in $\mathrm{H}_k(X)$ can be realized by a smooth manifold. Then also for every pair (X, A) of topological spaces, where $A \subseteq X$ is a closed subset, every relative homology class in $\mathrm{H}_k(X, A)$ can be realized by a smooth manifold.

Proof. In order not to disrupt the flow of the conversation too much we postpone the somewhat technical proofs of the Realize-by-Connected Manifold Proposition 187.5 and the Represent Relative Homology Class Proposition 187.6 to Section 187.3. ■

Note that the H_1-via-Curve Proposition 187.3 and the H_2-via-Surface Proposition 187.4 together with the Realize-by-Connected Manifold Proposition 187.5 and the Represent Relative Homology Class Proposition 187.6 give us a complete, and affirmative, answer to Question 187.2 for the dimensions $k = 1$ and $k = 2$.

We move on to the following much more fancy theorem which was first proved by René Thom [**Tho1954b, Tho2007**] in 1954.[1985] The proof of the theorem uses methods that go well beyond our present abilities. We will not make use of the theorem.

Theorem 187.7. (Thom's Theorem I)

(1) If X is a topological space, then every homology class of dimension ≤ 6 can be realized by a smooth manifold.
(2) Given any $n \geq 7$ there exists a topological space X and a homology class in $\mathrm{H}_n(X)$ that is not realized by a smooth manifold.
(3) Given any $n \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}$ such that for every topological space X and any homology class $\sigma \in \mathrm{H}_n(X)$ the class $k \cdot \sigma$ can be realized by a smooth manifold.

Remark. The statement of Thom's Theorem 187.7 solved what was known as the *Steenrod problem*, see [**Eil1949**, p. 257]. □

Proof. As mentioned before, the theorem is proved in [**Tho1954b**, Theorem III.3, III.4 and III.9]. A more modern account of the proof is given in Theorems IV.7.35, IV.7.36, IV.7.37 and IV.7.38 together with Remark IV.7.40 of [**Rudy1998**]. In fact the topological space X in (2) is given by the $(n-7)$-fold suspension of the Eilenberg-Maclane space $\mathrm{K}(\mathbb{Z}/3\mathbb{Z}, 2)$, which we will introduce later in Chapter 307. ■

For completeness we also mention $\mathbb{F}_2$-coefficients.

Theorem 187.8. (Thom's Theorem II) Let X be a path-connected topological space and let $A \subseteq X$ be a subset. Given any $\varphi \in \mathrm{H}_n(X, A; \mathbb{F}_2)$ there exists a compact connected n-dimensional smooth manifold N and a continuous map $f\colon N \to X$ with $f(\partial N) \subseteq A$ such that $f_*([N]_{\mathbb{F}_2}) = \varphi$.

Proof. First let us consider the case $A = \varnothing$ and for the time being let us allow possibly disconnected smooth manifolds. In this setting, for $n = 1$ and $n = 2$ the statement follows from straightforward modifications of the proofs of the H_1-via-Curve Proposition 187.3 and

[1985] René Thom (1923-2002) was a French mathematician who received a Fields Medal in 1958. A short outline of the key ideas and contributions of René Thom can be found in [**Sul2004**]

the H_2-via-Surface Proposition 187.4. In fact, much more interestingly, the statement now holds in all dimensions by the above work of René Thom [**Tho1954b**, Theorem III.2], see also [**Rudy1998**, Theorem IV.7.33]. The statements about relative homology and connectedness follow from the obvious variations on the Realize-by-Connected Manifold Proposition 187.5 and the Represent Relative Homology Class Proposition 187.6. ■

We conclude this section with the following remark.

Remark.

(1) In Chapter 191 we will introduce the notion of an orientation of a topological manifold. Furthermore in Chapter 193 we will introduce the notion of the fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$ of a compact oriented n-dimensional topological manifold N. It now makes sense to ask whether a homology class can be realized by a topological manifold. By [**Rudy1998**, Theorem IV.7.35 and Remark IV.3.740] there exist homology classes in any dimension ≥ 7 that cannot by realized by topological manifolds. But by [**Rudy1998**, Theorem IV.7.38] there do exist homology classes that can be realized by topological manifolds but which cannot be realized by smooth manifolds.

(2) Lowell Jones [**JonL1973**, Proposition 7.3] (see also Norman Levitt [**Levit1972**, Theorem 7.1]) showed that every homology class can be represented by a "Poincaré Duality space". □

187.2. **Proof of the H_2-via-Surface Proposition 187.4.** In this section we prove the following proposition from the last section.

Proposition 187.4. (H_2-via-Surface Proposition) Let X be a topological space. Every class in $\mathrm{H}_2(X)$ is realized by a smooth manifold.

The proof of the H_2-via-Surface Proposition 187.4 is modelled on the proof of the H_1-via-Curve Proposition 187.3. But, as we will see, the proof is technically more difficult.

We start out our proof with the more general setting of some n-dimensional class. Later on we will specialize to the case that $n = 2$. It is even worthwhile keeping the case $n = 1$ in mind, since even for $n = 1$ the argument below differs slightly from the argument provided in the proof of the H_1-via-Curve Proposition 187.3.

Thus let X be a topological space, let $n \in \mathbb{N}$ and let $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k$ be an n-dimensional singular chain in X. We do not necessarily demand that the σ_j are different. In fact note that by "taking multiple copies of the singular simplices σ_j" we can and will arrange that $a_j \in \{-1, 1\}$, $j = 1, \ldots, k$.[1986] We set $T := \{1, \ldots, k\} \times \{0, \ldots, n\}$.

Now assume that c is a cycle. We see that we have the following equality in the free abelian group $\mathrm{C}_{n-1}(X)$:

$$0 \underset{\uparrow}{=} \partial_1(c) = \partial_1\Big(\sum_{j=1}^{k} a_j \cdot \sigma_j\Big) = \sum_{j=1}^{k}\sum_{r=0}^{n} (-1)^r \cdot a_j \cdot \sigma_j \circ i_r^n \underset{\uparrow}{=} \sum_{(j,r)\in T} (-1)^r \cdot a_j \cdot \sigma_j \circ i_r^n.$$

since c is a cycle

here $i_r^n \colon \Delta^{n-1} \to \Delta^n$ is the r-th face map

[1986] Note that if n is even, then it follows from Exercise 143.13 that we cannot arrange that all a_j are of the same sign. This is one, but not the main, reason why the proof of the H_2-via-Surface Proposition 187.4 is more difficult than the proof of the H_1-via-Curve Proposition 187.3.

It follows from the Matching Lemma 166.7 that there exists a decomposition $T = T_1 \sqcup T_2$ into two disjoint subsets together with a bijection

$$\begin{array}{rcl} \Theta = (\theta_1, \theta_2) \colon T_1 & \to & T_2 \\ (j,r) & \mapsto & \Theta(j,r) = (\theta_1(j,r), \theta_2(j,r)) \end{array}$$

such that the following condition is satisfied: For every $(j,r) \in T_1$ we have

$$\begin{array}{rcl} \sigma_j \circ i_r^n & = & \sigma_{\theta_1(j,r)} \circ i_{\theta_2(j,r)}^n \\ \text{and} \quad (-1)^r \cdot a_j & = & -(-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)}. \end{array}$$

Next we introduce some notation:

(1) We consider the topological space

$$Y(\Theta) := (\{1,\dots,k\} \times \Delta^n)/\sim \quad \begin{array}{l} \text{where for each } (j,r) \in T_1 \text{ and each } P \in \Delta^{n-1} \\ \text{we have } (j, i_r^n(P)) \sim (\theta_1(j,r), i_{\theta_2(j,r)}^n(P)). \end{array}$$

Informally speaking, $Y(\Theta)$ is given by taking one n-simplex for each singular simplex σ_j and gluing these simplices along $(n-1)$-dimensional faces.[1987]

(2) For $j = 1,\dots,k$ we denote by $\Psi_j \colon \Delta^n \xrightarrow{x \mapsto (j,x)} \{1,\dots,k\} \times \Delta^n$ the j-th inclusion map.

(3) We denote by $p \colon \{1,\dots,k\} \times \Delta^n \to Y(\Theta)$ the projection.

(4) Given $j \in \{1,\dots,k\}$ we write $\Delta_j := (p \circ \Psi_j)(\Delta^n) \subseteq Y(\Theta)$. Similarly we define $\mathring{\Delta}_j$.

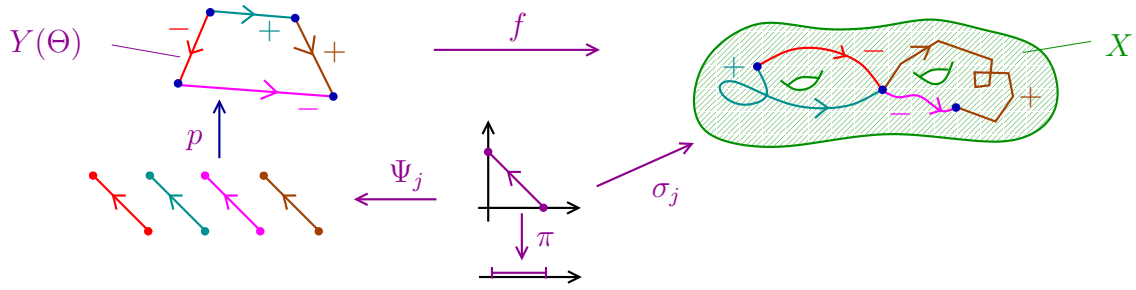
(5) It follows immediately from the definitions and the Topological-Quotient Proposition 8.2 (1b) that the map

$$\begin{array}{rcl} f \colon Y(\Theta) = (\{1,\dots,k\} \times \Delta^n)/\sim & \to & X \\ [(j,P)] & \mapsto & \sigma_j(P) \end{array}$$

is well-defined and continuous.

(6) We set

$$\mu := \sum_{j=1}^{k} a_j \cdot p \circ \Psi_j \in \mathrm{C}_n(Y(\Theta)).$$



We can now formulate the following lemma:

Lemma 187.9. (μ-Cycle Lemma)
(1) $\mu \in \mathrm{C}_n(Y(\Theta))$ is a cycle.
(2) $f_*([\mu]) = [c] \in \mathrm{H}_n(X)$.

[1987] It might seem that $Y(\Theta)$ has an "obvious simplicial structure", but that is not necessarily the case, for example after gluing there might be two n-dimensional simplices that intersect in more than just one $(n-1)$-dimensional face.

Proof. First we show that μ is a cycle. This follows from the following straightforward calculation:

$$\begin{aligned}\partial\mu &= \partial\Big(\sum_{j=1}^{k} a_j \cdot p\circ\Psi_j\Big) = \sum_{j=1}^{k}\sum_{r=0}^{n}(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n = \sum_{(j,r)\in T}(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n\\
&= \sum_{(j,r)\in T_1}(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n + \sum_{(j,r)\in T_1}(-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)}\cdot p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}\\
&\quad\uparrow\\
&\quad\text{since } T = T_1\sqcup T_2 \text{ and}\\
&\quad\text{since } \Theta=(\theta_1,\theta_2)\colon T_1\to T_2 \text{ is a bijection}\\
&= \sum_{(j,r)\in T_1}\underbrace{(-1)^r\cdot a_j\cdot p\circ\Psi_j\circ i_r^n + (-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)}\cdot p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}}_{=0 \text{ since } (-1)^r\cdot a_j=-(-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)} \text{ and } p\circ\Psi_j\circ i_r^n = p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}\colon \Delta^{n-1}\to Y(\Theta)} = 0.\end{aligned}$$

It follows immediately from the definitions that $f_*(\mu) = c \in \mathrm{C}_n(X)$. In particular we have $f_*([\mu]) = [c] \in \mathrm{H}_n(X)$. ∎

We move on to the next lemma, which together with the μ-Cycle Lemma 187.9 provides the long-desired proof of the H_2-via-Surface Proposition 187.4. Note that, as a bonus, we also obtain a new proof of the H_1-via-Curve Proposition 187.3.

Lemma 187.10. ($Y(\Theta)$-Manifold Lemma) We continue with the above notation. If $n = 1$ or if $n = 2$, then the topological space $Y(\Theta)$ admits the structure of a closed oriented 2-dimensional smooth manifold such that $[\mu] \in \mathrm{H}_2(Y(\Theta))$ is the corresponding fundamental class.

Remark. In the $Y(\Theta)$-Not Necessarily Manifold Lemma 187.11 below we will see that for $n = 3$ the topological space $Y(\Theta)$ is in general not a topological manifold. □

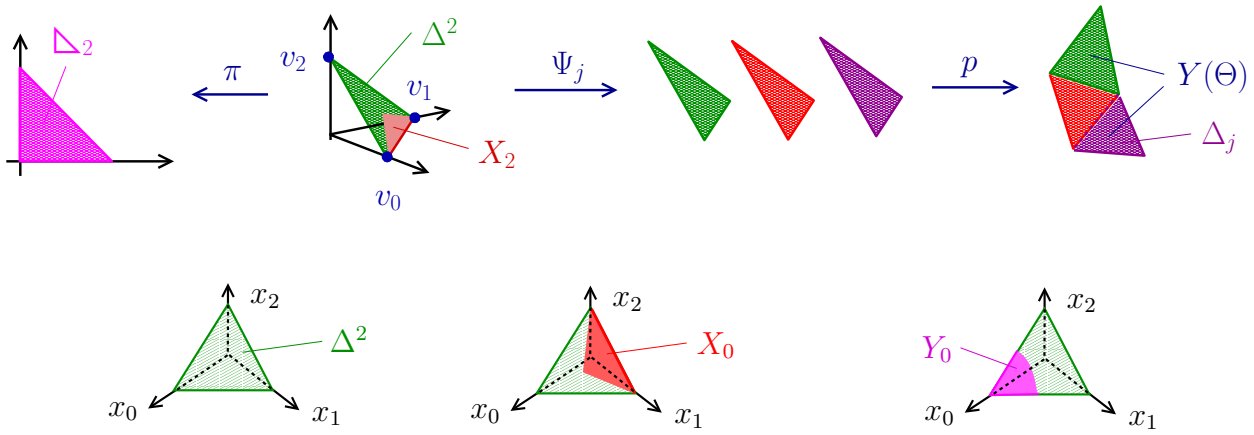
Sketch of a proof of the $Y(\Theta)$-Manifold Lemma 187.10. The proof of the $Y(\Theta)$-Manifold Lemma 187.10 for the case $n = 1$ is very elementary and thus left to the reader.

In the following we sketch a proof of the $Y(\Theta)$-Manifold Lemma 187.10 for the case $n = 2$. As we will see, the proof is quite similar to the proof, given in the Surface-is-Topological Manifold Proposition 23.7 and the Standard Smooth Structures Proposition 25.6, that the surface Σ_g of genus g is indeed a closed orientable 2-dimensional smooth manifold.

First note that it follows easily from the Quotient Topology-Properties Proposition 8.5 (1) that $Y(\Theta)$ is compact. We leave it to the reader to verify, say using the Quotient Topology-Properties Proposition 8.5 (3), that $Y(\Theta)$ is Hausdorff.

Next we construct a finite atlas for $Y(\Theta)$. To do so we introduce the following notation:

- Let $\pi\colon \Delta^2 \to \bigtriangleup_2$ be the projection. Recall that in the Δ^n-Homeomorphism Lemma 143.9 (1) we saw that π is a homeomorphism.
- We denote by $v_0 = (1, 0, 0)$, $v_1 = (0, 1, 0)$ and $v_2 = (0, 0, 1)$ the three vertices of Δ^2.
- Let $j \in \{0, 1, 2\}$. In the following will define two open subsets X_j and Y_j of Δ^2. In the subsequent definition we denote by w, w' the two vertices that are not equal to v_j. Now we introduce the following notation:
  - ▷ We denote by X_j the union of the open interval from w to w' and the open triangle in Δ^2 spanned by w, w' and the barycenter $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of Δ^2.
  - ▷ We denote by Y_j the intersection of $\Delta^2 \subseteq \mathbb{R}^3$ with the open ball in $\mathbb{R}^3$ of radius $\frac{1}{2}$ around the j-th vertex v_j.



Next we sketch the construction of three types of charts for the topological space $Y(\Theta)$. We leave it to the reader to fill in the details in the discussions below.

(1) For each $j \in \{1, \ldots, k\}$ we let $\alpha_j := \pi \circ \Psi_j^{-1} \circ p^{-1} \colon U_j := p(\mathring{\Delta}_j) \to V_j := \mathring{\Delta}_2 \subseteq \mathbb{R}^2$ be the "obvious" chart.

(2) Let $(j, r) \in T_1$. We write $\Theta(j, r) =: (j', r')$. Similar to the proof of the Surface-is-Topological Manifold Proposition 23.7 one can easily write down an explicit chart

$$\beta_{j,r} \colon U_{j,r} := \underbrace{p(\Psi_i(X_r)) \cup p(\Psi_{j'}(X_{r'}))}_{\subseteq Y(\Theta)} \quad \to \quad V_{j,r} \subseteq \mathbb{R}^2$$

such that the maps $\beta_{j,r} \circ p \circ \Psi_j^{-1} \circ \pi^{-1} \colon \Delta_2 \to \mathbb{R}^2$ and $\beta_{j,r} \circ p \circ \Psi_{j'}^{-1} \circ \pi^{-1} \colon \Delta_2 \to \mathbb{R}^2$ are both affine linear and such that they are both orientation-preserving if and only if we have $(-1)^r \cdot a_j = 1$ and such that they are both orientation-reversing if and only if we have $(-1)^r \cdot a_j = -1$.

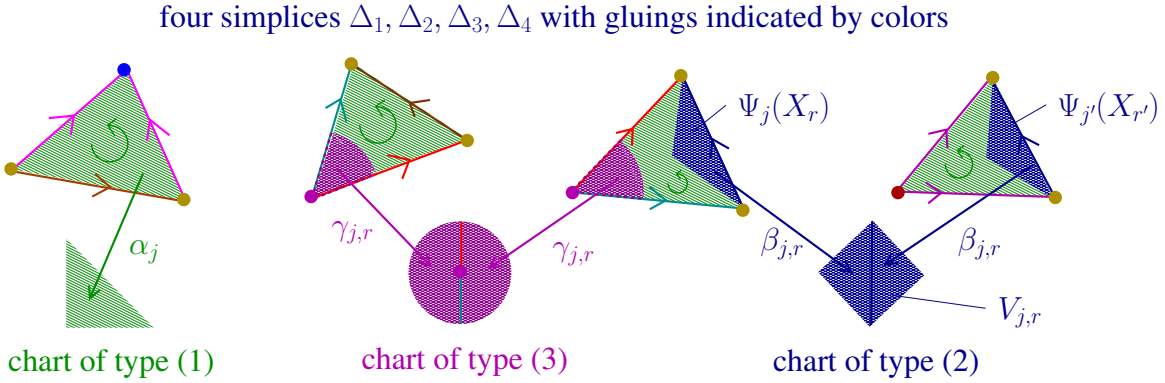
(3) Let $j \in \{1, \ldots, k\}$ and let $r \in \{0, 1, 2\}$. An open neighborhood of $p(\Psi_j(v_r))$ is given by

$$\widetilde{U}_{j,r} \quad := \bigcup_{\substack{(j', r') \in T \text{ such that} \\ p(\Psi_{j'}(v_{r'})) = p(\Psi_j(v_r))}} p(\Psi_{j'}(Y_{r'})).$$

Again similar to the proof of the Surface-is-Topological Manifold Proposition 23.7 one can now write down an explicit chart $\gamma_{j,r} \colon \widetilde{U}_{j,r} \to B^2_{\frac{1}{2}}$ such that on each $p(\Psi_r(Y, s))$ the map is a concatenation of π, $(p \circ \Psi_r)^{-1}$, translation, rotation and a continuous map of the form $\exp(it) \mapsto \exp(iwt)$ for some $w \in \mathbb{R}_{\geq 0}$. In fact we can take $w = \frac{2\pi}{n}$ where $n \in \mathbb{N}$ is the number of terms (j', r') that appear in the above definition of $\widetilde{U}_{j,r}$.

We leave it to the reader to verify that all of these maps are homeomorphisms and that they form an atlas for $Y(\Theta)$. Note that by the Countable Atlas-Second Countable Lemma 23.4 (2) we now also know that $Y(\Theta)$ is second-countable. We have thus verified that $Y(\Theta)$ is a closed 2-dimensional topological manifold.

Next note that, as in the proof of the Standard Smooth Structures Proposition 25.6, one can now verify easily that the transition maps between the given charts are indeed smooth and that they are orientation-preserving. Not surprisingly, once again we leave the details to the reader. Thus in summary we obtain from the above discussion, together with the Countable Atlas-Second Countable Lemma 23.4 and the Orientations-via-Transition Maps



Lemma 31.35, that $Y(\Theta)$ is a smooth manifold that is endowed with an orientation such that all charts are orientation-preserving.

Recall that in the μ-Cycle Lemma 187.9 we showed that

$$\mu \;=\; a_1 \cdot p \circ \Psi_1 + \cdots + a_k \cdot p \circ \Psi_k \;\in\; \mathrm{C}_2(Y(\Theta))$$

is a cycle. It follows almost immediately from the Fundamental Class-Criterion 185.7 that μ represents the fundamental class of the closed oriented smooth manifold $Y(\Theta)$. ■

We conclude this section with the following lemma which shows that the above approach will not work in higher dimensions.

Lemma 187.11. ($Y(\Theta)$-Not Necessarily Manifold Lemma) Given any $n \in \mathbb{N}_{\geq 3}$ there exists a topological space X together with a cycle $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_n(X)$ such that for every choice of Θ the topological space $Y(\Theta)$ is not a closed topological manifold, let alone a closed smooth manifold.

Sketch of proof. Let $n \in \mathbb{N}_{\geq 3}$ and let M be the $(n-1)$-dimensional torus. Furthermore let $(K, \Xi\colon |K| \to M)$ be a smooth simplicial structure for M, which we know exists by the Smooth Manifold-Simplicial Structure Theorem 179.5, or by some more down-to-earth argument.

By the Manifold-Simplex Proposition 183.2 and the Pseudomanifold-Orientation Proposition 183.19 we know that K is a closed orientable $(n-1)$-dimensional pseudomanifold. Next we consider the suspension $\Sigma(K)$ from Definition 174.44. Note that by the Simplicial Cone-Suspension Corollary 174.46 we know that $X \coloneqq |\Sigma(K)|$ is homeomorphic to $\Sigma(M)$.

Next recall that by the fairly straightforward Exercise 183.1 we know that $\Sigma(K)$ is a closed n-dimensional pseudomanifold that admits an orientation $\{(\epsilon_1, Y_1), \ldots, (\epsilon_k, Y_k)\}$. Now let $\varphi = \epsilon_1 \cdot Y_1 + \cdots + \epsilon_k \cdot Y_k \;\in \mathrm{C}_n^{\mathrm{simp}}(\Sigma(K))$ be the cycle from the Pseudomanifold Homology Theorem 183.12. Next note that after picking some total order on the vertex set of $\Sigma(K)$ we obtain from the Two Flavors-of-Simplicial Chain Complexes Lemma 176.15 and the Simplicial-Singular Chain Complex Theorem 177.12 a corresponding singular cycle $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_n(X) = \mathrm{C}_n(|\Sigma(K)|)$. We set $T \coloneqq \{1, \ldots, k\} \times \{0, \ldots, n\}$. We leave it to the dedicated reader to verify that there is essentially only one choice of a decomposition $T = T_1 \sqcup T_2$ and a bijection $\Theta = (\theta_1, \theta_2)\colon T_1 \to T_2$ such that the condition

from Section 187.2 holds, i.e. such that for every $(j,r) \in T_1$ we have

$$\begin{aligned} \sigma_j \circ i_r^n &= \sigma_{\theta_1(j,r)} \circ i^n_{\theta_2(j,r)} \\ \text{and} \quad (-1)^r \cdot a_j &= -(-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)}. \end{aligned}$$

Furthermore we leave it to the reader to verify that $Y(\Theta)$ is actually homeomorphic to $|\Sigma(K)| \cong \Sigma(M)$.

Finally we point out that it follows from Exercise 150.14, together with the fact that $\mathrm{H}_1(M) = \mathrm{H}_1((S^1)^n) \neq 0$ and together with our hypothesis $n \geq 3$ that $X = |\Sigma(K)| \cong \Sigma(M)$ is not a closed topological manifold. ■

187.3. **Proof of Propositions 187.5 and 187.6.** This section is completely dedicated to the proofs of the Realize-by-Connected Manifold Proposition 187.5 and of the Represent Relative Homology Class Proposition 187.6.

187.3.1. **Proof of the Realize-by-Connected Manifold Proposition 187.5.**

Proposition 187.5. (Realize-by-Connected Manifold Proposition Let $k \in \mathbb{N}$, let X be a topological space and let $A \subseteq X$ be a subset. We suppose that X is path-connected. If $A = \varnothing$ or if $k \geq 2$, then every homology class in $\mathrm{H}_k(X,A)$ that can be realized by a smooth manifold can also be realized by a *connected* smooth manifold.

Proof of Proposition 187.5. To simplify the discussion we only consider the case that $A = \varnothing$. The case $A \neq \varnothing$ is proved in a completely analogous fashion, the only difference is that the notation becomes even more messy. We will point out though explicitly where we need that $k \geq 2$ for $A \neq \varnothing$.

Let X be a path-connected topological space and let $\theta \in \mathrm{H}_k(X)$ be a homology class. Suppose that there exists a closed oriented k-dimensional smooth manifold W and a continuous map $f\colon W \to X$ with $f_*([W]) = \theta$. If W is connected, then there is nothing to show. In the following we will deal with the case that W has precisely two components M and N. The general case is dealt with in the same way.

We make the following preparations.

(1) As usual we denote by $\overline{B}_3^k \subseteq \mathbb{R}^k$ the closed ball of radius 3 around the origin. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}_3^k \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}_3^k \to N$. We write $P \coloneqq \varphi(0)$ and $Q \coloneqq \psi(0)$.

(2) We use the restrictions of φ and ψ to $\overline{B}^k$ to form, following Definition 54.12, the connected sum $M\#N$, i.e. we consider

$$M\#N \;=\; ((M \setminus \varphi(B^k)) \sqcup (N \setminus \psi(B^k)))/\sim \qquad \text{where } \varphi(z) \sim \psi(z) \text{ for } z \in S^{k-1}.$$

(3) We recall a few statements from the Smooth Connected Sum-Proposition 54.13:
   (a) The connected sum $M\#N$ is a closed k-dimensional smooth manifold which can be oriented in such a way that the two natural inclusions $M \setminus \varphi(B^k) \to M\#N$ and $N \setminus \psi(B^k) \to M\#N$ are both orientation-preserving smooth embeddings.
   (b) If $k \geq 2$ or if at least one of M or N is closed, then $M\#N$ is connected.
   Note though that for $A \neq \varnothing$ we have to deal with the connected sum of smooth compact manifolds M and N that are not necessarily closed. So here the argument breaks down for $A \neq \varnothing$ and $k = 1$, since the connected sum of two copies of $\overline{B}^1$ is actually not connected.

(4) We consider the topological space $Z := (M \sqcup [-1,1] \sqcup N)/\sim$ where we make the identifications $P \sim -1$ and $1 \sim Q$.

(5) We denote by $i\colon M \to Z$ and $j\colon N \to Z$ the obvious maps. It is easy to see that they are in fact embeddings. Next we denote by $p\colon Z \to M$ the map that is given by the identity on M and which sends all other points to P. Similarly we define $q\colon Z \to N$.

(6) We consider the following maps:

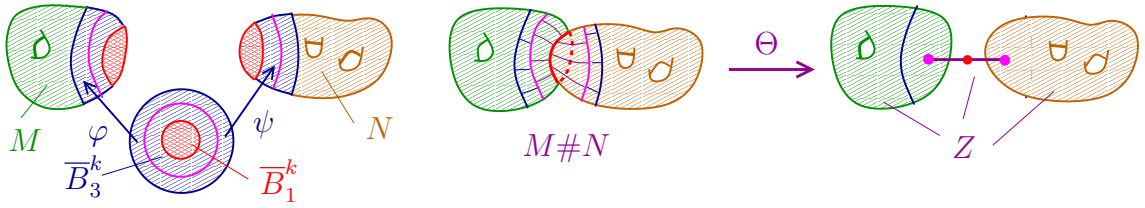
$$\begin{array}{ccc} & & \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \\ & \overset{\text{isomorphism since the inclusions are homotopy equivalences}}{\swarrow} & \\ \mathrm{H}_k((M \sqcup [-1,1))/\sim) \oplus \mathrm{H}_k((N \sqcup (-1,1])/\sim) & \xrightarrow[\text{Mayer–Vietoris Theorem 153.2}]{\text{isomorphism by the}} & \mathrm{H}_k(Z). \end{array} \qquad i_* \oplus j_* \searrow$$

It follows that the right diagonal map $i_* \oplus j_*\colon \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \to \mathrm{H}_k(Z)$ is an isomorphism. It is now straightforward to see that the inverse of $i_* \oplus j_*$ is given by $(p_*, q_*)\colon \mathrm{H}_k(Z) \to \mathrm{H}_k(M) \oplus \mathrm{H}_k(N)$.

(7) We consider the map

$$\begin{array}{rcl} \Theta\colon M\#N & \to & Z \\ x & \mapsto & \begin{cases} [x], & \text{if } x \in M \setminus \varphi(B_3^k), \\ [\varphi(3(r-2)\cdot y)], & \text{if } x = \varphi(r \cdot y) \text{ with } r \in [2,3] \text{ and } y \in S^{k-1}, \\ [-r+1], & \text{if } x = \varphi(r \cdot y) \text{ with } r \in [1,2] \text{ and } y \in S^{k-1}, \\ [r-1], & \text{if } x = \psi(r \cdot y) \text{ with } r \in [1,2] \text{ and } y \in S^{k-1}, \\ [\psi(3(r-2)\cdot y)], & \text{if } x = \psi(r \cdot y) \text{ with } r \in [2,3] \text{ and } y \in S^{k-1}, \\ [x], & \text{if } x \in N \setminus \psi(B_3^k). \end{cases} \end{array}$$

It follows easily from the Pasting Proposition 4.7 (2) that $\Theta\colon M\#N \to Z$ is actually continuous.



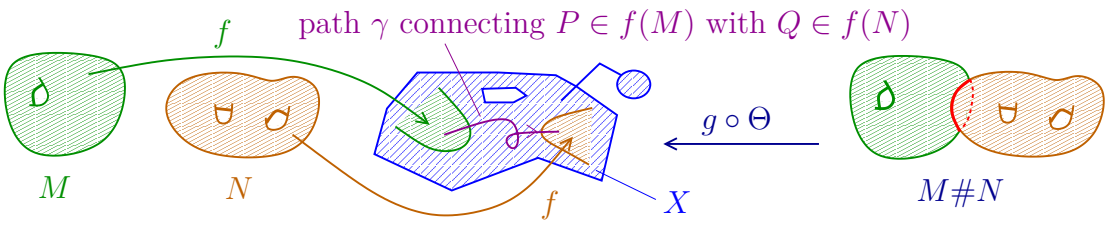
(8) It follows almost immediately from the Local-Global Degree Proposition 186.4 together with (3a) that the maps $p \circ \Theta\colon M\#N \to M$ and $q \circ \Theta\colon M\#N \to N$ both have degree one. In other words, we have the equality

$$(p_*, q_*)(\Theta_*([M\#N])) = [M] + [N] \in \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) = \mathrm{H}_k(M \sqcup N) = \mathrm{H}_k(W).$$

(9) By hypothesis the topological space X is path-connected. This implies that there exists a path $\gamma\colon [-1,1] \to X$ with $\gamma(-1) = P$ and $\gamma(1) = Q$. Using γ we can extend the map $f\colon W = M \sqcup N \to X$ to a continuous map $g\colon Z \to X$ in the obvious way.

By (3) the connected sum $M\#N$ is a closed oriented *connected* smooth manifold. Thus it remains to prove the following claim:

Claim. We have $(g \circ \Theta)_*([M\#N]) = \theta \in \mathrm{H}_k(X)$.



Proof. It follows from the above that we have the following diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_k(M)\oplus\mathrm{H}_k(N) & \xrightarrow{\mathrm{id}} & \mathrm{H}_k(M\sqcup N)=\mathrm{H}_k(W) \\ & & {\scriptstyle (p_*,q_*)}\uparrow\cong \quad \cong\downarrow{\scriptstyle i_*\oplus j_*} & \searrow^{f_*} & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_k(M\#N) & \xrightarrow{\Theta_*} & \mathrm{H}_k(Z) & \xrightarrow{g_*} & \mathrm{H}_k(X). \end{array}$$

It follows from the definition of g together with (6) that the diagram commutes. Now we see that we have the following equalities in $\mathrm{H}_k(X)$:

$$(g\circ\Theta)_*([M\#N]) \underset{\text{by (8) and (6)}}{\overset{\uparrow}{=}} g_*(i_*([M])+j_*([N])) \underset{\text{since the diagram commutes}}{\overset{\uparrow}{=}} f_*([M])+f_*([N]) = f_*(\underbrace{[M\sqcup N]}_{=[W]}) = \theta.$$

■

187.3.2. **Proof of the Represent Relative Homology Class Proposition 187.6.** We recall the proposition that we want to prove:

Proposition 187.6. (Represent Relative Homology Class Proposition) Let $k\in\mathbb{N}$. Suppose that for every topological space X every homology class in $\mathrm{H}_k(X)$ can be realized by a smooth manifold. Then also for every pair (X,A) of topological spaces (where $A\subseteq X$ is a closed subset), every relative homology class in $\mathrm{H}_k(X,A)$ can be realized by a smooth manifold.

Proof. Let $k\in\mathbb{N}$. We make the following hypothesis:

(∗) For every topological space Y every homology class in $\mathrm{H}_k(Y)$ can be realized by a smooth manifold.

Now let (X,A) be a pair of topological spaces and let $\varphi\in\mathrm{H}_k(X,A)$ be a relative homology class. We denote by $i\colon A\to X$ the inclusion map. We consider the mapping cone

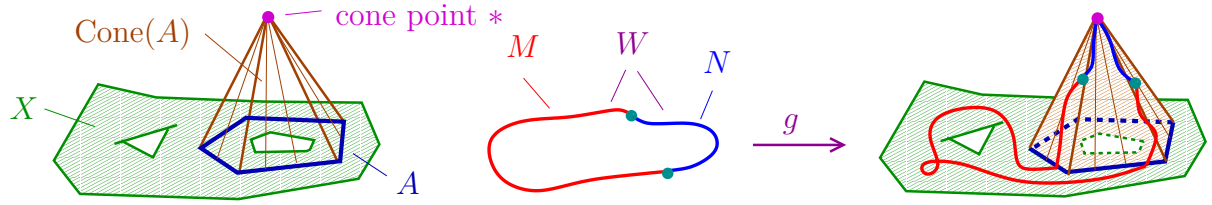
$$\mathrm{Cone}(i\colon A\to X) = (\underbrace{\mathrm{Cone}(A)}_{=(A\times[0,1])/A\times\{0\}}\sqcup X)/\sim \text{ where } [(a,1)]\sim i(a)=a \text{ for } a\in A$$

which we introduced in Definition 21.7. We denote by $*:=[A\times\{0\}]\in\mathrm{Cone}(i\colon A\to X)$ the tip of the mapping cone. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_k(M,\partial M) & \xrightarrow{\cong} & \mathrm{H}_k(W,N) & \xleftarrow{\text{я}} & \mathrm{H}_k(W) \\ \downarrow{\scriptstyle (r\circ g)_*} & \searrow^{g_*} & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ & & \mathrm{H}_k(\mathrm{Cone}(i\colon A\to X),\mathrm{Cone}(A)) & \xleftarrow{\text{я}}\cong & \\ & \swarrow_{g_*} & \uparrow\cong & & \\ \mathrm{H}_k(X,A) & \xrightarrow{\cong} & \mathrm{H}_k(\mathrm{Cone}(i\colon A\to X)\setminus\{*\},\mathrm{Cone}(A)\setminus\{*\}) & & \mathrm{H}_k(\mathrm{Cone}(i\colon A\to X)). \\ & \overset{r_*}{\underset{\cong}{\leftarrow}} & & & \end{array}$$

We make the following explanations, clarifications and observations:

(1) By the Mapping Cone–Embedding Lemma 21.11 (2) and (3) we can and will view $\mathrm{Cone}(A)$ and X as closed subspaces of $\mathrm{Cone}(i\colon A \to X)$. (Note that here we use our hypothesis that A is a closed subset of X.)
(2) All maps that are not specified explicitly are inclusion-induced maps.
(3) The maps я are the natural maps from absolute to relative homology.
(4) Similar to the Mapping Cylinder–Homotopy Lemma 21.5 (1) one sees that there exists a natural deformation retraction from $\mathrm{Cone}(i\colon A \to X) \setminus \{*\}$ to X. We denote by $r\colon \mathrm{Cone}(i\colon A \to X) \setminus \{*\} \to X$ the corresponding retraction. Note that it restricts to a retraction from $\mathrm{Cone}(A) \setminus \{*\}$ to A.
(5) By the proof of the Mapping Cone-of-Inclusion-$\widetilde{\mathrm{H}}_*$-Lemma 153.15 we know that the four maps at the bottom, shown in blue, are isomorphisms.
(6) By (5) together with our hypothesis (∗) we know that there exists a closed oriented k-dimensional smooth manifold W and a continuous map $g\colon W \to \mathrm{Cone}(i\colon A \to X)$ such that $я(g_*([W]))$ equals the image of our given class φ in the central red group $\mathrm{H}_k(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A))$.
(7) Since $\mathrm{Cone}(i\colon A \to X) \setminus \{*\})$ and $\mathrm{Cone}(A) \setminus A$ are open subsets that cover $\mathrm{Cone}(i)$ we obtain from Exercise 38.3 that there exist two compact codimension-zero submanifolds M and N of W such that $M \cup N = W$, $M \cap N = \partial M = \partial N$, such that $M \subseteq g^{-1}(\mathrm{Cone}(i\colon A \to X) \setminus \{*\})$ and such that $N \subseteq g^{-1}(\mathrm{Cone}(A) \setminus A)$.
(8) By (7) we have the three inclusions $g(N) \subseteq \mathrm{Cone}(A)$, $g(\partial M) \subseteq \mathrm{Cone}(A) \setminus \{*\}$ and $g(M) \subseteq \mathrm{Cone}(i\colon A \to X) \setminus \{*\})$. This implies that the three maps in relative homology denoted by g_*, namely the two diagonal maps to the left and the vertical map in the middle, actually do exist.
(9) By the Codimension Zero-Fundamental Class Lemma 185.21 (2) we know that the images of $[M]$ and $[W]$ in $\mathrm{H}_k(W, N)$ agree.



It is clear that the diagram commutes. It is now straightforward to verify that the compact oriented n-dimensional smooth manifold M together with the map $r \circ g\colon (M, \partial M) \to (X, A)$ has all the desired properties. ■

187.4. **Representing homology classes by submanifolds I.** In the previous sections we studied the following definition:

Definition 187.12. Let X be a topological space, let $A \subseteq X$ be a subset and let $\varphi \in \mathrm{H}_k(X, A)$ be a homology class. We say that φ can be realized by a smooth manifold if there exists a compact oriented k-dimensional smooth manifold N and a continuous map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$.

If the topological space X we are interested in is in fact a smooth manifold, then it is natural to ask whether one can actually find maps f that are smooth embeddings. This leads us to the following definition:

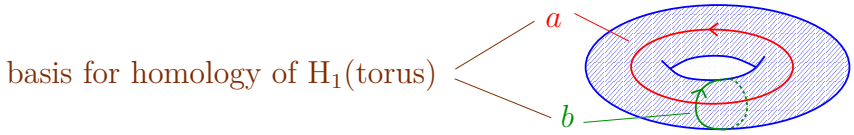
Definition 187.13. Let M be a smooth manifold and let A be a union of components of ∂M.

(1) Let N be a compact oriented proper k-dimensional submanifold of M with $\partial N \subseteq A$. By a slight abuse of notation, we denote by $[N]$ the image of the fundamental class $[N]$ under the inclusion-induced map $\mathrm{H}_k(N, \partial N) \to \mathrm{H}_k(M, A)$.

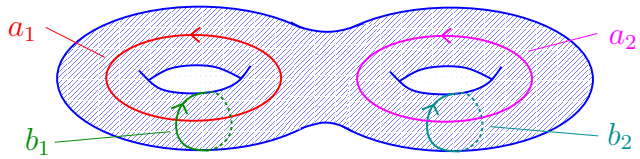
(2) We say that a homology class $\varphi \in \mathrm{H}_k(M, A)$ is represented by a submanifold if there exists a compact oriented proper k-dimensional submanifold N with $\partial N \subseteq A$ such that $[N] = \varphi$.

Examples.

(1) We first consider the 2-dimensional torus $T = S^1 \times S^1$. Under the usual isomorphism $\mathrm{H}_1(S^1 \times S^1) \cong \mathbb{Z}^2$, the basis element $(1, 0)$ is represented by the oriented submanifold $S^1 \times \{1\}$ and the basis element $(0, 1)$ is represented by the oriented submanifold $\{1\} \times S^1$.



(2) We consider the surface Σ_2 of genus 2. As we showed in the Surface–Connected Sum Lemma 54.15 we can view Σ_2 as the connected sum $T \# T$ of two tori. If we use the calculation on page 3549 to determine an isomorphism $\mathrm{H}_1(\Sigma_2) \cong \mathbb{Z}^4$, then it follows from (1) that the basis elements are represented by the submanifolds a_1, b_1, a_2, b_2 that are illustrated in the figure below.[1988] [1989]



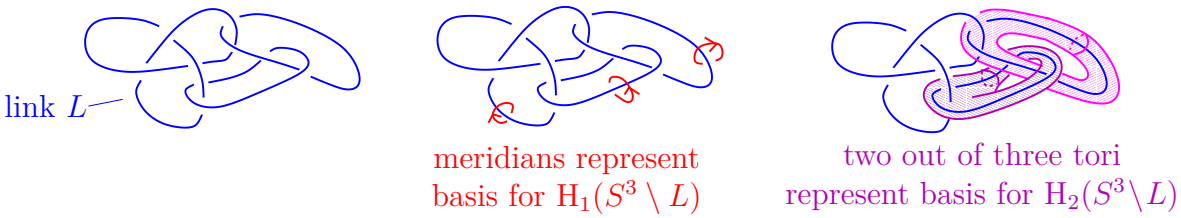
basis for homology of H_1(surface of genus 2)

(3) Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link in S^3. Those readers who solved Exercise 185.7 will see that the m meridians represent a basis for $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^m$ and that any choice of $m - 1$ tori $S^1 \times L_i$ represent a basis for $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^{m-1}$.

(4) Any zero class $0 \in \mathrm{H}_k(M)$ is represented by the empty submanifold.

[1988] Another point of view is that the submanifolds a_1, b_1, a_2, b_2 are given by deforming the closed curves x_1, y_1, x_2, y_2 that are shown in the figure on page 1979.

[1989] What do these submanifolds look like in the formal picture of the surface of genus 2 as the quotient $E_8/\sim$ of the Euclidean octagon?



(5) We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. By the Projective Space–Smooth Embedding Lemma 30.11 we know that for $k \leq n$ the map

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$

is a smooth embedding. We use this map to view $\mathbb{C}\mathrm{P}^k$ as a submanifold of $\mathbb{C}\mathrm{P}^n$. It follows from the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 and the natural isomorphism of the Singular-Cellular-H_*-Proposition 155.8 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^i], & \text{if } k = 2i \text{ for some } i \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, the generators of the non-trivial homology groups of the complex projective space $\mathbb{C}\mathrm{P}^n$ are represented by the complex projective spaces $\mathbb{C}\mathrm{P}^i$ with $i = 0, \ldots, n$.

(6) Let M be a compact oriented connected k-dimensional smooth manifold. By the Manifold-Top-H_*-Theorem 183.20 we have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$. It follows from the Fundamental Class-Opposite Orientation Lemma 185.17 that the classes $\pm[M]$ are represented by M, viewed as its own submanifold, equipped with the two possible orientations. Since M is connected, we know by Exercise 25.17 that M admits no other proper non-empty n-dimensional submanifold. Thus we see that the classes $k \cdot [M]$ with $k \notin \{\pm 1\}$ are not represented by submanifolds.

Later in the Cobordant–Same Homology Class Proposition 212.12 we will give a criterion for when two compact oriented proper submanifolds represent the same homology class. □

For completeness' sake we state the following lemma.

Lemma 187.14. (Boundary Represents Boundary Lemma) Let M be a smooth manifold, let A be a union of components of ∂M and let $\varphi \in \mathrm{H}_k(M, A)$. If N is a proper submanifold that represents φ, then ∂N represents the class $\partial_k(\varphi) \in \mathrm{H}_{k-1}(A)$, where $\partial_k \colon \mathrm{H}_k(M, A) \to \mathrm{H}_{k-1}(A)$ is the connecting homomorphism of the long exact sequence of the pair (M, A).

Proof. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_k(M, A) & \xrightarrow[\varphi \mapsto \partial_k(\varphi)]{\partial_k} & \mathrm{H}_{k-1}(A) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathrm{H}_k(N, \partial N) & \xrightarrow[[N] \mapsto [\partial N]]{\partial_k} & \mathrm{H}_{k-1}(\partial N). \end{array}$$

Here the horizontal maps are given by the connecting homomorphisms and where the vertical maps are induced by the inclusion maps. By the LES-of-Triple Proposition 146.5 we

know that the connecting homomorphisms are natural, in other words, the diagram commutes. The statement now follows from the Fundamental Class-of-∂-Proposition 185.18. ■

We are naturally lead to the following question which can be viewed as a refinement of Question 187.2 in the context of smooth manifolds.

Question 187.15. Let M be a smooth manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_k(M, A)$ be represented by submanifolds?

The following proposition, which can be viewed as analog of the Represent Relative Homology Class Proposition 187.6, says that the relative problem is equivalent to the absolute problem.

Proposition 187.16. (Represent Relative Homology Class-by-Submanifold Proposition) Let $k \in \mathbb{N}$. Suppose that for every (orientable) n-dimensional smooth manifold every homology class in $\mathrm{H}_k(M)$ can be represented by a submanifold. Then also for every (orientable) n-dimensional smooth manifold M and every union A of components of ∂M every relative homology class in $\mathrm{H}_k(M, A)$ can be represented by a submanifold.

In our proof of the Represent Relative Homology Class-by-Submanifold Proposition 187.16 we will make use of the following lemma, which is also of independent interest:

Lemma 187.17. (Doubling-Homology Lemma) Let M be a topological manifold and let W be a union of components of ∂M. For $i \in \{1, 2\}$ let $j_i \colon M \to M \times \{i\} \to \mathrm{D}_W M$ be the obvious map.

(1) For $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{D} \colon \mathrm{C}_k(M, W) & \to & \mathrm{C}_k(\mathrm{D}_W M) \\ \big[\underbrace{\sigma}_{\in \mathrm{C}_k(M)} \big] & \mapsto & j_{1*}(\sigma) - j_{2*}(\sigma) \end{array}$$

is well-defined.

(2) The maps from (1) define a natural chain map, in particular for every $k \in \mathbb{N}_0$ these maps induce a map

$$\mathrm{D} \colon \mathrm{H}_k(M, W) \to \mathrm{H}_k(\mathrm{D}_W M)$$

which is called the doubling homomorphism.

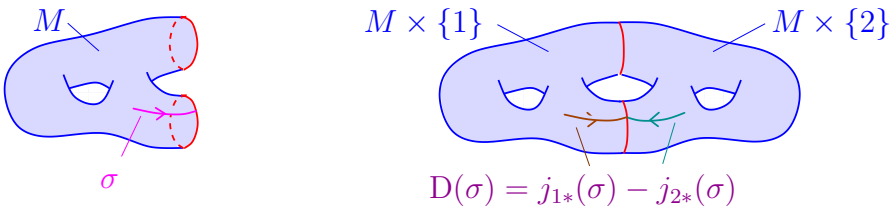
(3) We make the identification $M = j_1(M \times \{1\})$ and we write $M' = j_2(M \times \{2\})$. The doubling homomorphism has the property that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(M, W) & \xrightarrow{\quad \mathrm{D} \quad} & \mathrm{H}_k(\mathrm{D}_W M) \\ & \searrow \quad \swarrow & \\ & \mathrm{H}_k(\mathrm{D}_W M, M'), & \end{array}$$

where the diagonal maps are the obvious maps. (As an aside we mention that it follows from the Manifold–Excision Theorem 148.3 that the left diagonal map is an isomorphism.)

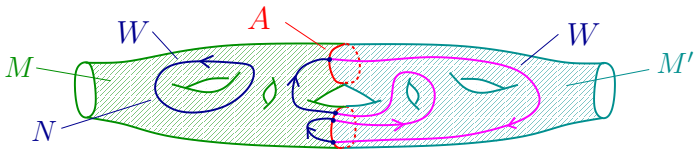
Proof of Lemma 187.17. The lemma follows almost immediately from the definitions, hence we leave it to the conscientious reader to fill in the details. ■

Remark. In the Homology-of-Double Lemma 256.6 we will calculate the homology groups of $\mathrm{D}M$ in terms of the homology groups of M. □



Proposition 187.16. Let M be an (orientable) n-dimensional smooth manifold, let A be the union of some components of ∂M and let $\sigma \in \mathrm{H}_k(M, A)$. Let M' be a second copy of M. As in Definition 54.8 we consider the double of M along A, i.e. we consider $\mathrm{D}_A M = M \cup_{A=A'} M'$. Note that by the Doubling-Manifold Lemma 54.10 we know that $\mathrm{D}_A M$ is an (orientable) n-dimensional smooth manifold which contains A as a submanifold.

Next let $\mathrm{D}\colon \mathrm{H}_k(M, A) \to \mathrm{H}_k(\mathrm{D}_A M)$ be the doubling homomorphism given by the Doubling-Homology Lemma 187.17. By our hypothesis there exists a closed oriented submanifold W of $\mathrm{D}_A M$ such that $[W] = \mathrm{D}(\sigma) \in \mathrm{H}_k(\mathrm{D}_A M)$. It follows from the Transversality Theorem 50.9 and the Homotopy-H_*-Proposition 144.12 that we can arrange that W is transverse to A. Now we set $N := W \cap M$. It follows from Exercise 50.9, which is a variation on the Submanifold Transversal Intersection Theorem 50.4, that N is a proper submanifold of W.



It remains to prove the following claim:

Claim. We have $[N] = \sigma \in \mathrm{H}_k(M, A)$.

Proof. We denote by $i\colon W \to \mathrm{D}_A M$ the inclusion map. Now we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_k(N, \partial N) & \xrightarrow[\cong]{\alpha} & \mathrm{H}_n(W, W \setminus \mathring{N}) & \xleftarrow{\beta} & \mathrm{H}_k(W) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}_k(M, A) & \xrightarrow[\cong]{\gamma} & \mathrm{H}_k(\mathrm{D}_A M, M') & \xleftarrow{\delta} & \mathrm{H}_k(\mathrm{D}_A M).
\end{array}$$

$$\mathrm{H}_k(\mathrm{D}_A M) \xrightarrow{\mathrm{D}} \mathrm{H}_k(M, A)$$

Here α, β, γ and δ are the obvious maps. Now we make the following observations:

$$\begin{array}{rrcll}
 & i_*([W]) & = & \mathrm{D}(\sigma) & \text{by choice of } W \\
\Longrightarrow & i_*(\beta([W])) & = & \delta(\mathrm{D}(\sigma)) & \text{since the diagram commutes} \\
\Longrightarrow & i_*(\alpha([N])) & = & \gamma(\sigma) & \text{by the Codimension Zero-Fundamental Class Lemma 185.21} \\
 & & & & \text{and the Doubling-Homology Lemma 187.17 (3)} \\
\Longrightarrow & \gamma(i_*([N])) & = & \gamma(\sigma) & \text{since the diagram commutes} \\
\Longrightarrow & i_*([N]) & = & \sigma & \text{since } \gamma \text{ is an isomorphism by the} \\
 & & & & \text{Manifold–Excision Theorem 148.3.}
\end{array}$$

∎

One obvious approach to Question 187.15 is as follows: in Section 187.1 we already saw that in many cases, given a class $\varphi \in \mathrm{H}_k(M, A)$, there exists a compact oriented k-dimensional smooth manifold N together with a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$. The

question now is the following: can we replace f by some proper embedding? The following proposition, which we will prove in Section 187.6, addresses this issue:

Proposition 187.18. (Homotopic-to-Embedding Proposition) Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold, let A be a union of components of ∂M, let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a continuous map.

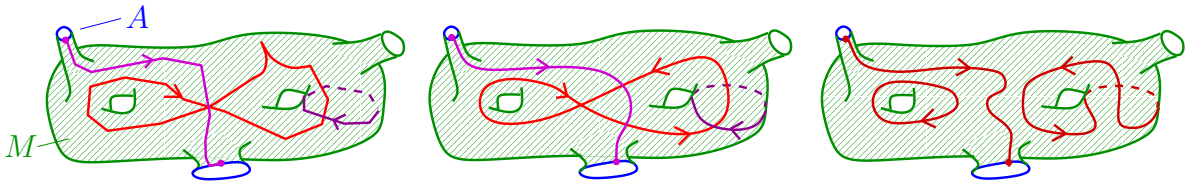
(1) If $n > 2k$, then $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. In particular by the Smooth Embedding Theorem 30.15 (2) $\psi(N)$ is a proper submanifold of M and by the Homotopy-H_*-Proposition 146.11 we have $\varphi_*([N]) = [\psi(N)] \in \mathrm{H}_k(M, A)$.

(2) If $n = 2k$, then there exists a compact oriented proper k-dimensional submanifold W of N with $\partial W \subseteq A$ such that $\varphi_*([N]) = [W] \in \mathrm{H}_k(M, A)$.[1990]

It is worth stating the following immediate corollary to the H_1-via-Curve Proposition 187.3 and the Homotopic-to-Embedding Proposition 187.18.

Corollary 187.19. (Represent-by-1-Manifold Corollary) Let M be an m-dimensional smooth manifold with $m \geq 2$ and let A be the union of some components of ∂M. Given any homology class in $\sigma \in \mathrm{H}_1(M, A)$ there exists a compact oriented proper 1-dimensional submanifold P with $\partial P \subseteq A$ such that $[P] = \sigma$.

Example. Let M be a 2-dimensional smooth manifold and let $\sigma \in \mathrm{H}_1(M)$ be a homology class.



by Proposition 187.3 we can represent the homology class by some map $\varphi\colon N \to M$

isotope to get smooth proper embedding without triple points

resolve intersections

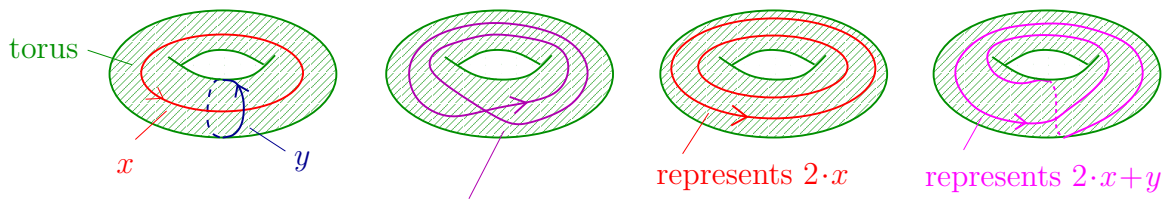
Just for fun we illustrate in the figure above the three main steps in the proof of the Homotopic-to-Embedding Proposition 187.18 in this special case. □

Next we want to consider the question whether we can represent a homology class by a *connected* submanifold. For tori we have the following ambiguous answer.

Lemma 187.20. (Represent-on-Torus-by-Connected Curve Lemma) Let $T = S^1 \times S^1$ be the torus. We denote by $x = [S^1 \times \{1\}]$ and $y = [\{1\} \times S^1]$ the obvious basis for $\mathrm{H}_1(T)$. A non-zero class $r \cdot x + s \cdot y$ can be represented by a connected submanifold if and only if $\gcd(r, s) = 1$.

Proof. We, i.e. you, will provide the proof in Exercise 187.4. For fun we will deal with the analogous problem for surfaces of higher genus in Exercise 187.5. ■

[1990] Note that we do not claim that W is diffeomorphic to N.



The following proposition, which can be viewed as an analog of the Realize-by-Connected Manifold Proposition 187.5, shows that in most settings we will not have troubles finding connected submanifolds. As for the previous proposition we postpone the proof to Section 187.7.

Proposition 187.21. (Represent-by-Connected Submanifold Proposition) Let $n \in \mathbb{N}$, let M be a connected n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional submanifold with $\partial N \subseteq A$. We suppose that $n - k \geq 2$. If $n \geq 2$ or if $A = \varnothing$, then there exists a compact oriented *connected* submanifold W of M with $\partial W \subseteq A$ such that $[W] = [N] \in \mathrm{H}_k(M, A)$.

Remark. As we just saw in the Represent-on-Torus-by-Connected Curve Lemma 187.20, in general the conclusion of the Represent-by-Connected Submanifold Proposition 187.21 fails to hold if $k = n - 1$. Note though that in this setting some control over connectedness can be still obtained from the Connected Codimension-One Representative Proposition 234.8. □

The combination of the H_2-via-Surface Proposition 187.4, the Homotopic-to-Embedding Proposition 187.18 and the Represent-by-Connected Submanifold Proposition 187.21 gives us the following pretty corollary.

Corollary 187.22. (Represent-by-Connected 2-Manifold Corollary) Let M be an n-dimensional smooth manifold with $n \geq 4$ and let $A \subseteq \partial M$ be a union of boundary components. Every class in $\mathrm{H}_2(M, A)$ is represented by a connected submanifold.

Remark. Let M be a compact 4-dimensional smooth manifold and let $\sigma \in \mathrm{H}_2(M)$. By the Represent-by-Connected 2-Manifold Corollary 187.22 it makes sense to the define the *genus of* σ:

$$\mathrm{genus}(\sigma) \;:=\; \begin{array}{c}\text{minimal genus of a closed oriented connected}\\ \text{smooth submanifold } \Sigma \text{ of } M \text{ with } [\Sigma] = \sigma.\end{array}$$

As an example let us consider the 4-dimensional smooth manifold $\mathbb{C}\mathrm{P}^2$. By the discussion on page 3587 we know that $\mathrm{H}_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^1]$. Let $n \in \mathbb{Z}$. We consider

$$W_n \;:=\; \{[x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}.$$

In Exercise 40.8 we showed that W_n is a closed 1-dimensional complex submanifold of the 2-dimensional complex manifold $\mathbb{C}\mathrm{P}^2$. In particular this shows that W_n is a closed 2-dimensional submanifold of $\mathbb{C}\mathrm{P}^2$. In Exercise 220.19 we will show that $[W_n] = n \cdot [\mathbb{C}\mathrm{P}^1]$. As is pointed out in [**GoS1999**, p. 38], it is a consequence of the "adjunction formula", that W_n is diffeomorphic to the surface of genus $\frac{1}{2}(n-1)(n-2)$. In particular we have the inequality $\mathrm{genus}(n \cdot [\mathbb{C}\mathrm{P}^1]) \leq \frac{1}{2}(n-1)(n-2)$. The Thom Conjecture predicted that

equality holds. This conjecture was proved in 1994 by Peter Kronheimer and Tom Mrowka [**KroM1994**] using "Seiberg-Witten Theory". In general though it is still a very difficult problem to determine the genus of the homology class of a given smooth manifold. □

The Represent-by-Connected 2-Manifold Corollary 187.22 suffers from the rather annoying defect that it does not apply to the case of 3-dimensional smooth manifolds. Perhaps rather surprisingly this gets taken care of by coming "from the other direction", namely this case gets taken care of by the following proposition.

Proposition 234.6. (Codimension-One Submanifold Existence Proposition) Let M be a compact orientable n-dimensional smooth manifold and let $A \subseteq \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A)$ can be represented by a submanifold.

Sketch of proof. As the number already suggests, we will not prove this statement in this chapter. In the following we will just provide the flavor of the argument which is meant to serve as an amuse-bouche. For simplicity we assume that M is closed. Now suppose that we are given a homology class $\sigma \in \mathrm{H}_{n-1}(M) = \mathrm{H}_{n-1}(M; \mathbb{Z})$. By the Poincaré Duality Theorem 210.1 we have an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-1}(M; \mathbb{Z}) \to \mathrm{H}^1(M; \mathbb{Z})$. Thus we obtain the Poincaré dual $\mathrm{PD}_M(\sigma) \in \mathrm{H}^1(M; \mathbb{Z})$. We will see that cohomology classes in $\mathrm{H}^1(M; \mathbb{Z})$ give rise to smooth maps $f \colon M \to K(\mathbb{Z}, 1) = S^1$. (Here $K(G, n)$ stands for an Eilenberg-Maclane space of type $K(G, n)$, the precise definition is irrelevant right now, except that in the discussion below it indicates some similarities between initially quite different looking arguments.) The desired submanifold is then given by the preimage of a regular value of f. ■

The analog for codimension-two homology classes also holds:

Proposition ??. (Codimension-Two Submanifold Existence Existence Proposition) Let M be a compact orientable n-dimensional smooth manifold and let $A \subseteq \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-2}(M, A)$ can be represented by a submanifold.

Sketch of proof. The logic of the proof is somewhat similar to the idea behind the proof of the Codimension-One Submanifold Existence Proposition 234.6. This time the Poincaré Duality Theorem 210.1 gives us an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-2}(M; \mathbb{Z}) \to \mathrm{H}^2(M; \mathbb{Z})$. Thus the given homology class in $\mathrm{H}_{n-2}(M; \mathbb{Z})$ corresponds to a cohomology class in $\mathrm{H}^2(M; \mathbb{Z})$. We will see that such cohomology classes give rise to maps $f \colon M \to K(\mathbb{Z}, 2) = \mathbb{C}\mathrm{P}^\infty$. Since M is compact, we obtain a map $f \colon M \to \mathbb{C}\mathrm{P}^m$ for sufficiently large m. In this occasion the desired submanifold of M is given by the preimage of $\mathbb{C}\mathrm{P}^{m-1} \subseteq \mathbb{C}\mathrm{P}^m$. ■

Optimistically one might hope that the pattern of the previous two propositions extends to codimension-three homology classes. But Statement (2) of the following theorem shows that this is not the case.

Theorem 187.23. (Thom's Theorem III)

(1) Let M be a compact orientable n-dimensional smooth manifold and let A be a union of boundary components. Given $k \leq 6$ with $k < n$ every class in $\mathrm{H}_k(M, A)$ can be represented by a submanifold.

(2) There exists a 10-dimensional closed smooth manifold[1991] and a class in $\mathrm{H}_7(M)$ that cannot be represented by a submanifold.

Proof.

(1) First let us deal with the case that $A = \varnothing$. In this case, for $k \leq 5$ the statement can be found in [**Tho1954a**, Théorème 6] and also in [**Tho1954b**, Théorème II.27]. The statement for $k = 6$ can be found in the footnote on page 173 of [**Tho2007**]. The relative case now follows from the above together with the Represent Relative Homology Class-by-Submanifold Proposition 187.16.

(2) This statement was proved by Christian Bohr, Bernhard Hanke and Dieter Kotschick, see [**BHK2002**, Theorem 1]. Their work builds on earlier work of René Thom [**Tho1954b**, p. 62f]. ■

For completeness' sake we now also discuss homology with $\mathbb{F}_2$-coefficients. Given a compact (possibly non-orientable) proper k-dimensional submanifold N of a manifold M we denote by $[N]_{\mathbb{F}_2}$ the image of the $\mathbb{F}_2$-fundamental class N under the inclusion-induced map $\mathrm{H}_k(N, \partial N; \mathbb{F}_2) \to \mathrm{H}_k(M, \partial M; \mathbb{F}_2)$. We say N *represents the homology class* $[N]_{\mathbb{F}_2}$.

Surely it does not come as a surprise that we have the following analog of the Represent-by-1-Manifold Corollary 187.19 and the Represent-by-Connected 2-Manifold Corollary 187.22 for $\mathbb{F}_2$-coefficients.

Proposition 187.24. (Represent-$\mathbb{F}_2$-Homology-by-Manifold Proposition) Let M be an n-dimensional smooth manifold and let $A \subseteq \partial M$ be a union of boundary components.

(1) If $n \geq 2$, then every class in $\mathrm{H}_1(M, A; \mathbb{F}_2)$ can be represented by a submanifold.

(2) If $n \geq 4$, then every class in $\mathrm{H}_2(M, A; \mathbb{F}_2)$ can be represented by a connected submanifold.

Proof. The proof is basically the same as the proof of Represent-by-1-Manifold Corollary 187.19 and the Represent-by-Connected 2-Manifold Corollary 187.22 for $\mathbb{F}_2$-coefficients. There is surely no need to go over the argument again. ■

The following proposition is an analog of the Codimension-One Submanifold Existence Proposition 234.6.

Proposition ??. ($\mathbb{F}_2$-Codimension-One Submanifold Existence Proposition) Let M be a compact n-dimensional smooth manifold and let A be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A; \mathbb{F}_2)$ can be represented by a submanifold.

Remark. This result will have its moment of glory when we will employ it in our proof, provided in **??**, that $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. □

Sketch of proof. The proof of this proposition is quite similar to the proof of the Codimension-One Submanifold Existence Proposition 234.6. The Homology Symmetry Theorem 210.6 with $\mathbb{F}_2$-coefficients gives us an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-1}(M; \mathbb{F}_2) \to \mathrm{H}^1(M; \mathbb{F}_2)$. Thus given the homology class in $\mathrm{H}_{n-1}(M; \mathbb{F}_2)$ corresponds to a cohomology class in $\mathrm{H}^1(M; \mathbb{F}_2)$. We will see that such cohomology classes give rise to maps $f \colon M \to K(\mathbb{F}_2, 1) = \mathbb{R}\mathrm{P}^\infty$. Since M is compact, we obtain a map $M \to \mathbb{R}\mathrm{P}^m$ for sufficiently large m. Now the desired submanifold is given by the preimage of $\mathbb{R}\mathrm{P}^{m-1} \subseteq \mathbb{R}\mathrm{P}^m$. ■

[1991]It follows from Statement (1) together with the Codimension-One Submanifold Existence Proposition 234.6 and the Codimension-Two Submanifold Existence Existence Proposition **??** that for a compact orientable n-dimensional smooth manifold of dimension $n \leq 9$ any homology class of dimension $< n$ can be represented by a submanifold. Thus the example in Statement (2) is "dimensionally minimal".

Remark. Note that this author is not aware of an analog of the Codimension-Two Submanifold Existence Proposition **??**. □

Finally let us see once again what René Thom has to say:

Theorem 187.25. (Thom's Theorem IV) Let M be a compact n-dimensional smooth manifold and let A be a union of boundary components.
(1) If $n \leq 5$, then any class in $\mathrm{H}_{n-2}(X, A; \mathbb{F}_2)$ can be represented by a submanifold.
(2) If $n \leq 7$, then any class in $\mathrm{H}_{n-3}(X, A; \mathbb{F}_2)$ can be represented by a submanifold.
(3) For $k \leq \lfloor \frac{n}{2} \rfloor$, any class in $\mathrm{H}_k(X, A; \mathbb{F}_2)$ can be represented by a submanifold.

Proof. In the case $A = \varnothing$ the three statements are precisely the content of [**Tho2007**, Théorème II.26]. The relative case follows from the $\mathbb{F}_2$-analog of the Represent Relative Homology Class-by-Submanifold Proposition 187.16. Finally note that Statement (3) also follows from Thom's Theorem 187.7 together with the $\mathbb{F}_2$-analog of the Homotopic-to-Embedding Proposition 187.18 (2). ■

We conclude this section with the following remark:

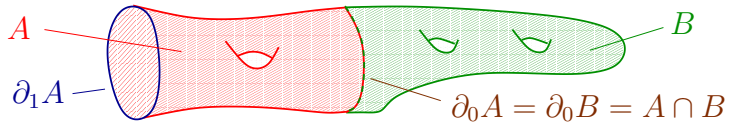
Remark. As we already mentioned on page 3577, later in Chapter 191 we will introduce the notion of an orientation of a topological manifold and in Chapter 193 we will introduce the notion of the fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$ of a compact oriented n-dimensional topological manifold N. Thus it makes sense to ask which homology classes of a given topological manifold can be represented by submanifolds. This question is discussed in [**FNOP2025**]. □

In the remainder of this chapter we will provide the proofs of the Homotopic-to-Embedding Proposition 187.18 and of the Represent-by-Connected Submanifold Proposition 187.21.

187.5. **Rearranging maps from smooth manifolds.** Let (X, A) be a pair of topological spaces. In this section we will state and prove a proposition which allows us to replace certain compact oriented smooth manifolds M and continuous maps $f\colon (M, \partial M) \to (X, A)$ by a different compact oriented smooth manifold N together with a map $g\colon (N, \partial N) \to (X, A)$ such that $f_*([M]) = g_*([N])$. This procedure will come in handy in the proofs of the Homotopic-to-Embedding Proposition 187.18 and the Represent-by-Connected Submanifold Proposition 187.21.

The following definition is a special case of a more general concept introduced in Definition 25.50.

Definition 187.26. Let M be a compact n-dimensional smooth manifold. We say that M is decomposed into submanifolds A and B if the following conditions hold:
(1) $M = A \cup B$.
(2) A and B are both compact n-dimensional submanifolds of M.
(3) We have $A \cap B = \partial_0 A = \partial_0 B$.



Now we can state the one result of this section.

Proposition 187.27. (Recombine Fundamental Class Proposition) Let M and $\widetilde{M}$ be two compact oriented n-dimensional smooth manifolds. Suppose that M is decomposed into two codimension-zero submanifolds A and B and that $\widetilde{M}$ is decomposed into two codimension-zero submanifolds $\widetilde{A}$ and $\widetilde{B}$. We equip A, B with the orientations coming from M and we equip $\widetilde{A}, \widetilde{B}$ with the orientations coming from $\widetilde{M}$. Furthermore we equip $A \cap B$ and $\widetilde{A} \cap \widetilde{B}$ with the orientation coming from A respectively $\widetilde{A}$. Suppose that we are given an orientation-preserving diffeomorphism $\Theta \colon A \cap B \to \widetilde{A} \cap \widetilde{B}$. Next we set

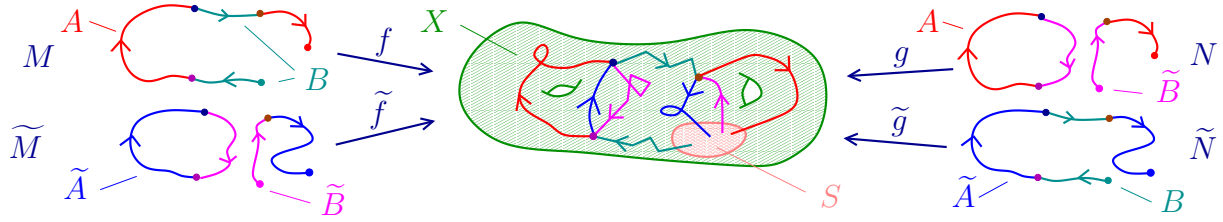
$$N \coloneqq (A \sqcup \widetilde{B})/x \sim \Theta(x) \text{ for } x \in A \cap B \quad \text{and} \quad \widetilde{N} \coloneqq (\widetilde{A} \sqcup B)/\Theta(y) \sim y \text{ for } y \in A \cap B.$$

Note that it follows from the Gluing-Smooth Manifolds-Proposition 54.5 that N and $\widetilde{N}$ are both compact oriented n-dimensional smooth manifolds such that the obvious maps $A \to N$ and $\widetilde{A} \to \widetilde{N}$ are orientation-preserving embeddings. Next let (X, S) be a pair of topological space and furthermore let $f \colon (M, \partial M) \to (X, S)$ and $\widetilde{f} \colon (\widetilde{M}, \partial\widetilde{M}) \to (X, S)$ be two continuous maps such that we have $\widetilde{f} \circ \Theta = f \colon A \cap B \to X$. We define[1992]

$$\begin{array}{rcl} g \colon N = (A \sqcup \widetilde{B})/\sim & \to & X \\ [P] & \mapsto & \begin{cases} f(P), & \text{if } P \in A, \\ \widetilde{f}(P), & \text{if } P \in \widetilde{B} \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{g} \colon \widetilde{N} = (\widetilde{A} \sqcup B)/\sim & \to & X \\ [P] & \mapsto & \begin{cases} \widetilde{f}(P), & \text{if } P \in \widetilde{A}, \\ f(P), & \text{if } P \in B. \end{cases} \end{array}$$

With this notation we have

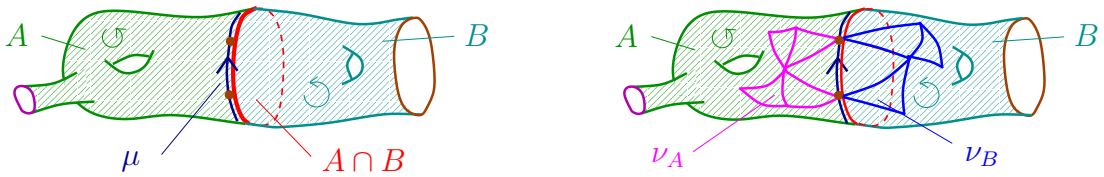
$$f_*([M]) + \widetilde{f}_*([\widetilde{M}]) \;=\; g_*([N]) + \widetilde{g}_*([\widetilde{N}]) \;\in\; \mathrm{H}_n(X, S).$$



Proof. We pick a fundamental cycle $\mu \in \mathrm{C}_{n-1}(A \cap B)$ for the closed oriented $(n-1)$-dimensional smooth manifold $A \cap B$. We consider the map

$$\mathrm{C}_n(A, \partial A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(\overbrace{\partial_0 A \sqcup \partial_1 A}^{=\partial A}) \xrightarrow{\cong} \mathrm{C}_{n-1}(\overbrace{A \cap B}^{=\partial_0 A}) \oplus \mathrm{C}_{n-1}(\partial_1 A) \twoheadrightarrow \mathrm{C}_{n-1}(A \cap B).$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{=:\omega_A}$$

By the Extend Fundamental Cycle Corollary 185.19 we know that there exists a singular chain $\nu_A \in \mathrm{C}_n(A)$ such that ν_A represents a fundamental cycle in $\mathrm{C}_n(A, \partial A)$ and such that $\omega_A(\nu_A) = \mu$. With the same notation we also obtain some $\nu_B \in \mathrm{C}_n(B)$ with $\omega_B(\nu_B) = \mu$.



[1992] It follows from $\widetilde{f} \circ \Theta = f$ that both maps g and $\widetilde{g}$ are continuous.

Next we consider $\widetilde{\mu} := \Theta_*(\mu) \in \mathrm{C}_{n-1}(\widetilde{A} \cap \widetilde{B})$. As above we pick corresponding singular chains $\widetilde{\nu}_{\widetilde{A}} \in \mathrm{C}_n(\widetilde{A})$ and $\widetilde{\nu}_{\widetilde{B}} \in \mathrm{C}_n(\widetilde{B})$. Now we see that we have the following equalities in $\mathrm{H}_n(X, S)$:

$$
\begin{aligned}
f_*([M]) + \widetilde{f}_*([\widetilde{M}]) &\overset{\text{it follows from } \omega_A(\nu_A) = \mu = \omega_B(\nu_B) \text{ that } \nu_A - \nu_B \text{ is a cycle in } \mathrm{C}_n(M, \partial M); \text{ using the Fundamental Class-Characterization Lemma 185.6 (3)}\Rightarrow\text{(1) one now can easily verify that } \nu_A - \nu_B \text{ is a fundamental cycle of } M, \text{ the same argument shows that } \widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_{\widetilde{B}} \text{ is a fundamental cycle of } \widetilde{M}}{=} f_*([\nu_A - \nu_B]) + \widetilde{f}_*([\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_{\widetilde{B}}]) \\
&= [f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}}) + \widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)] \overset{\text{it follows fairly easily from } \widetilde{f} \circ \Theta = f \text{ and } \widetilde{\mu} = \Theta_*(\mu) \text{ that } f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}}) \text{ and } \widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B) \text{ are both cycles in } \mathrm{C}_n(X, S)}{=} [f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})] + [\widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)] \\
&\overset{\text{by definition of } g \text{ and } \widetilde{g}}{=} [g_*(\nu_A) - g_*(\widetilde{\nu}_{\widetilde{B}})] + [\widetilde{g}_*(\widetilde{\nu}_{\widetilde{A}}) - \widetilde{g}_*(\nu_B)] \;=\; g_*([\nu_A - \widetilde{\nu}_B]) + \widetilde{g}_*([\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_B]). \\
&\overset{\text{by an argument as in the beginning of this calculation we see that } [\nu_A - \widetilde{\nu}_B] = [N] \text{ and } [\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_B] = [\widetilde{N}]}{=} g_*([N]) - \widetilde{g}_*([\widetilde{N}]).
\end{aligned}
$$

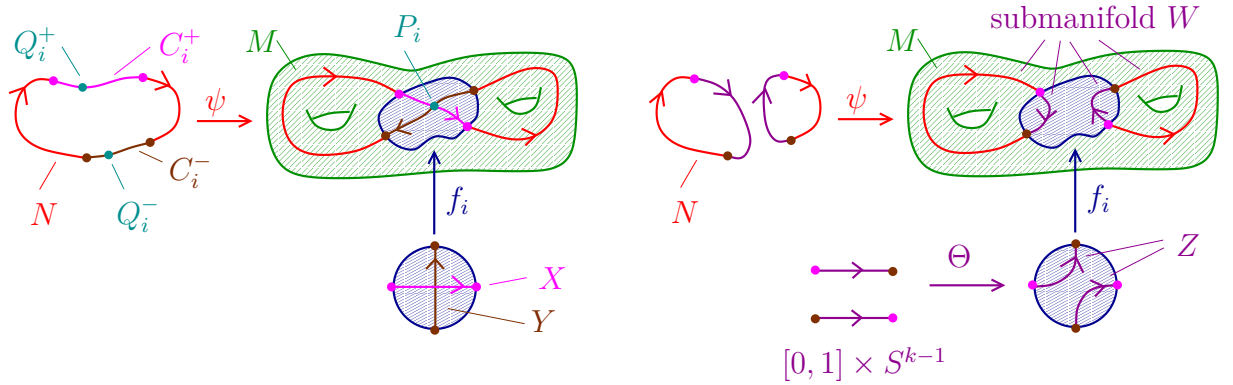
■

187.6. **Proof of the Homotopic-to-Embedding Proposition 187.18.** For the reader's convenience we recall the statement of the Homotopic-to-Embedding Proposition 187.18.

Proposition 187.18. (Homotopic-to-Embedding Proposition) Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and finally let $\varphi\colon (N, \partial N) \to (M, A)$ be a continuous map.

(1) If $n > 2k$, then $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. In particular $\psi(N)$ is a proper submanifold of M and we have $\varphi_*([N]) = [\psi(N)] \in \mathrm{H}_k(M, A)$.

(2) If $n = 2k$, then there exists a compact oriented proper k-dimensional submanifold W of N with $\partial W \subseteq A$ such that $\varphi_*([N]) = [W] \in \mathrm{H}_k(M, A)$.



Proof. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a continuous map.

First let us consider the case $n > 2k$. In this case it follows from the Whitney Approximation Theorem 36.1 (3) together with the Deform-to-Embedding Theorem **??** (2)

that $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper smooth embedding $\psi\colon (N, \partial N) \to (M, A)$. It follows from the Smooth Embedding Theorem 30.15 (2) that $\psi(N)$ is a proper submanifold of M and that $\psi\colon N \to \psi(N)$ is a diffeomorphism. Thus we see that

$$\varphi_*([N]) \underset{\substack{\uparrow \\ \text{by the Homotopy-H}_* \\ \text{Proposition 146.11}}}{=} \psi_*([N]) \underset{\substack{\uparrow \\ \text{since } \psi \text{ is a} \\ \text{diffeomorphism}}}{=} [\psi(N)] \in \mathrm{H}_k(M, A).$$

Now let us deal with the much more subtle case $n = 2k$. From the Whitney Approximation Theorem 36.1 (3) together with the Deform-to-Embedding Theorem **??** (1) we now obtain that the map φ is homotopic, as a map of pairs of topological spaces, to a proper immersion $\psi\colon N \to M$ which has furthermore the property that there exist points $P_1, \dots, P_m$ in $M \setminus \partial M$ with the following properties:

(1) For each $i \in \{1, \dots, m\}$ the preimage $\psi^{-1}(P_i)$ consists of precisely two points $Q_i^\pm$ which furthermore satisfy that $\psi_*(\mathrm{T}_{Q_i^+} N) + \psi_*(\mathrm{T}_{Q_i^-} N) = \mathrm{T}_{P_i}(M)$.
(2) The restriction of ψ to $N \setminus \{Q_1^\pm, \dots, Q_m^\pm\}$ is injective.

In the following we write

$$X \coloneqq \left\{(x, 0) \in \overline{B}^{2k} \,\middle|\, x \in \overline{B}^k\right\} \quad \text{and} \quad Y \coloneqq \left\{(0, y) \in \overline{B}^{2k} \,\middle|\, y \in \overline{B}^k\right\}.$$

We view X and Y as submanifolds of $\overline{B}^{2k}$. We equip X and Y with the obvious orientations. It follows easily from the Smooth Embedding Theorem 30.15 (2) together with the Submanifold Transversal Intersection Theorem 50.4 (1) that we can find the following:

(1) Disjoint k-dimensional submanifolds $\mathrm{C}_1^\pm, \dots, \mathrm{C}_m^\pm$ of N such that for each $i \in \{1, \dots, m\}$ we have $Q_i^\pm \in \mathring{C}_i^\pm = \mathrm{C}_i^\pm$. Note that the restriction of ψ to any $\mathring{C}_i^\pm$ is in fact a smooth embedding.
(2) Disjoint embeddings $f_1, \dots, f_m\colon \overline{B}^{2k} \to M \setminus \partial M$ such that for each $i \in \{1, \dots, m\}$ we have

$$\psi(\mathrm{C}_i^+) = f_i(X), \quad \psi(\mathrm{C}_i^-) = f_i(Y), \quad \text{and } f_i(\overline{B}^{2k}) \cap \psi(N) = f_i(X) \cup f_i(Y).$$

Furthermore, after possibly precomposing the maps f_i with reflections in hyperplanes we can assume that the maps $X \to f_i(X) \overset{\psi}{\leftarrow} \mathrm{C}_i^+$ and $Y \to f_i(Y) \overset{\psi}{\leftarrow} \mathrm{C}_i^-$ are orientation-preserving.

Claim 1. There exists a proper smooth embedding $\Theta\colon [0, 1] \times S^{k-1} \to \overline{B}^{2k}$ with the following two properties:

(1) For $t \in [0, \frac{1}{4}]$ and $z \in S^{k-1}$ we have $\Theta(t, z) = ((1-t)z, 0) \in \mathbb{R}^k \times \mathbb{R}^k$. Furthermore the map $\Theta_0\colon \{0\} \times S^{k-1} \to S^{k-1} \times \{0\}$ is orientation-reversing.
(2) For $t \in [\frac{3}{4}, 1]$ and $z \in S^{k-1}$ we have $\Theta(t, z) = (0, t \cdot z) \in \mathbb{R}^k \times \mathbb{R}^k$. Furthermore the map $\Theta_0\colon \{1\} \times S^{k-1} \to \{0\} \times S^{k-1}$ is orientation-preserving.

Proof. To prove the claim first note that, using the Smooth Transition Function Lemma 24.9 one can easily find a smooth function $\mu\colon [0, 1] \to [0, 1]$ such that $\mu(t) = t$ for $t \in [0, \frac{1}{4}]$, and such that $\mu(t) = 1$ for $t \in [\frac{3}{4}, 1]$ and such that $\mu'(t) > 0$ for any $t \in (0, \frac{3}{4})$.

We leave it to the reader to verify that the following map has all the desired properties:

$$\begin{aligned} \Theta\colon [0, 1] \times S^{k-1} &\to \overline{B}^{2k} \subseteq \mathbb{R}^k \times \mathbb{R}^k \\ (t, z) &\mapsto (\underbrace{(1 - \mu(t)) \cdot z}_{\in \mathbb{R}^k}, \underbrace{(1 - \mu(1-t)) \cdot z}_{\in \mathbb{R}^k}). \end{aligned}$$



⊞

Next we set $Z := \Theta([0,1] \times S^{k-1})$ and we set

$$W := \psi\Big(\underbrace{N \setminus \bigcup_{i=1}^{m} (\mathring{C}_i^+ \cup \mathring{C}_i^-)}_{=:\widehat{N}}\Big) \cup \bigcup_{i=1}^{m} f_i(Z).$$

We leave it to the reader to verify that W is in fact a compact proper k-dimensional submanifold of M with $\partial W \subseteq A$ which can be oriented in such a way that the embedding $\psi\colon \widehat{N} \to W$ is orientation-preserving. It remains to prove the following claim:

Claim 2. We have $\varphi_*([N]) = [W] \in \mathrm{H}_k(X, A)$.

Proof. As usual we denote by $-W$ the same smooth manifold but with the opposite orientation. Note that by the Fundamental Class-Opposite Orientation Lemma 185.17 we have $[-W] = -[W]$. Thus it suffices to show that $\varphi_*([N]) + [-W] = 0$. Now we perform the following calculation:

$$\varphi_*([N])+[-W] \overset{\text{by the Homotopy-H}_*\text{-Proposition 146.11. since } \psi \text{ is homotopic to } \varphi}{\downarrow} \psi_*([N])+[-W] = \psi_*\Big(\underbrace{\Big[\widehat{N} \cup \bigcup_{i=1}^{m} (C_i^+ \cup C_i^-)\Big]}_{N}\Big) + \underbrace{\Big[-\psi(\widehat{N}) \cup -\bigcup_{i=1}^{m} f_i(Z)\Big]}_{=W}$$

$$\overset{\text{by the Recombine Fundamental Class Proposition 187.27}}{\downarrow} = \underbrace{(\psi \cup \psi)_*([\overbrace{\widehat{N} \cup_{\partial_0 \widehat{N} = \partial_0(-\widehat{N})} -\widehat{N}}^{\text{the double } \mathrm{D}_{\partial_0 \widehat{N}} \widehat{N}}])}_{\substack{\text{the map } \psi \cup \psi \text{ factors through the} \\ \text{folding map } \mathrm{D}_{\partial_0 \widehat{N}} \widehat{N} \to \widehat{N} \\ \text{which has degree zero by} \\ \text{the discussion on page 3566}}} + \sum_{i=1}^{m} \underbrace{(\psi \cup f_i)_*((Q_i^+ \cup Q_i^-) \cup Z)}_{\substack{= 0, \text{since } \psi \cup f_i \text{ takes values in } f_i(\overline{B}^{2k}) \\ \text{and since } \mathrm{H}_k(f_i(\overline{B}^{2k})) = 0}} = 0.$$

■

187.7. **The internal connected sum and the proof of Proposition 187.21.** We start with the slightly technical construction of an internal connected sum of two oriented submanifolds of a given smooth manifold. This construction is interesting in its own right, and it will be the key ingredient for the proof of the Represent-by-Connected Submanifold Proposition 187.21.

Construction 187.28. (Internal Connected Sum Construction) Let M be a connected n-dimensional smooth manifold and let N_0 and N_1 be two compact oriented proper k-dimensional submanifolds of M. We suppose that $n - k \geq 2$. We pick two points $P_0 \in N_0 \setminus \partial N_0$ and $P_1 \in N_1 \setminus \partial N_1$. We proceed with the following steps:

(1) Suppose we are given a smooth embedding $\gamma\colon [0,1] \to M \setminus \partial M$ with the following properties:
   (a) $\gamma(0) = P_0$ and $\gamma(1) = P_1$.

(b) If we set $C \coloneqq \gamma([0,1])$, then $C \cap N_0 = \{P_0\}$ and $C \cap N_1 = \{P_1\}$.
(c) C hits N_0 and N_1 transversally, in the sense that $\mathrm{T}_{P_0}C \cap \mathrm{T}_{P_0}N_0 = \{0\}$ and that $\mathrm{T}_{P_1}C \cap \mathrm{T}_{P_1}N_1 = \{0\}$.

Note that it follows from the Complement-of-Submanifold Connected Corollary 50.11 together with the Curve–Smooth Embedding Theorem **??** that it follows from our hypothesis $n-k \geq 2$ that such γ always exists.

(2) Next we write $D \coloneqq \{(x,0) \in \mathbb{R}^k \times \mathbb{R}^{n-1-k} \mid x \in \overline{B}^k\} \subseteq \overline{B}^{n-1}$. As we will essentially see in the Internal Band Proposition 260.11 one can show, using the Tubular Neighborhood–Boundary Extension Theorem 57.29 and the Product Tubular Neighborhood–Existence Theorem 58.4 (2), that there exists an embedding $\Phi\colon \overline{B}^{n-1} \times C \to M$ such that $\Phi(\overline{B}^{n-1} \times C) \cap N_0 = \Phi(D \times \{P_0\})$ and such that $\Phi(\overline{B}^{n-1} \times C) \cap N_1 = \Phi(D \times \{P_1\})$.

(3) Using Exercise 24.4 one can easily find a smooth embedding $\Theta\colon S^{k-1} \times [0,1] \to \overline{B}^k \times C$ with the following two properties:
(a) For $z \in S^{k-1}$ and $t \in [0, \frac{1}{4}]$ we have $\Theta(z,t) = ((1-t)\cdot z, P_0)$.
(b) For $z \in S^{k-1}$ and $t \in [\frac{3}{4}, 1]$ we have $\Theta(z,t) = (t \cdot z, P_1)$.

We set $Z \coloneqq \Theta(S^{k-1} \times [0,1])$.

There are now two cases to consider.

Case (1) Among the two maps $\Phi\colon D \times \{P_0\} \to N_0$ and $\Phi\colon D \times \{P_1\} \to N_1$ one is orientation-preserving and one is orientation-reversing. In this case we set

$$N_0 \#_\gamma N_1 \;\coloneqq\; (N_0 \setminus \Phi(D \times \{P_0\})) \cup (N_1 \setminus \Phi(D \times \{P_1\}))) \cup \Phi(Z).$$

Case (2) If both maps $\Phi\colon D \times \{P_0\} \to N_0$ and $\Phi\colon D \times \{P_1\} \to N_1$ are orientation-preserving or if both are orientation-reversing then we add a "half-twist" to get us back into the game. More precisely, we consider the map [1993]

$$\begin{aligned} \Omega\colon D \times C &\to \overline{B}^{n-1} \times C \\ (\underbrace{(x_1, x_2, \dots, x_k)}_{\in \mathbb{R}^k}, \underbrace{(0,0)}_{\in \mathbb{R} \times \mathbb{R}^{n-k-2}}), \gamma(t)) &\mapsto (\underbrace{(\cos(\pi t)\cdot x_1, x_2, \dots, x_k)}_{\in \mathbb{R}^k}, \underbrace{(\sin(\pi t)\cdot x_1, 0)}_{\in \mathbb{R} \times \mathbb{R}^{n-k-2}}), \gamma(t)). \end{aligned}$$

Note that Ω is the identity on $D \times \{P_0\}$ and that Ω restricts to an orientation-reversing self-diffeomorphism of $D \times \{P_1\}$. Now we set

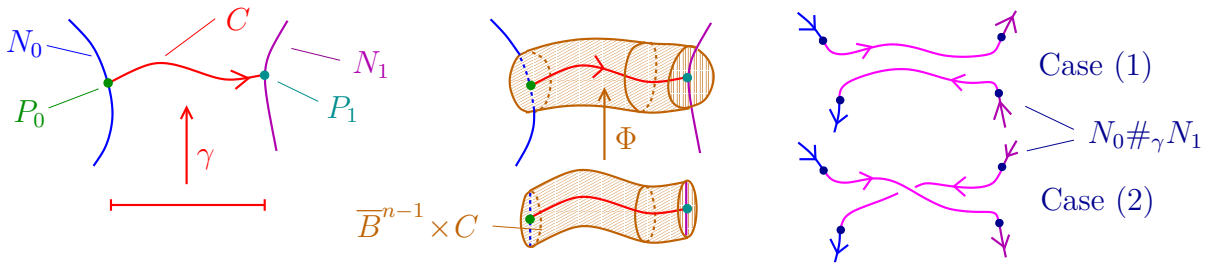
$$N_0 \#_\gamma N_1 \;\coloneqq\; (N_0 \setminus \Phi(D \times \{P_0\})) \cup (N_1 \setminus \Phi(D \times \{P_1\})) \cup (\Phi \circ \Omega)(Z).$$

It is straightforward to verify that in both cases $N_0 \#_\gamma N_1$ is a k-dimensional submanifold of M. Furthermore one sees easily that in both cases we can equip the smooth manifold $N_0 \#_\gamma N_1$ with an orientation in such a way that the inclusions $N_0 \setminus \Phi(D \times \{P_0\}) \to N_0 \#_\gamma N_1$ and $N_1 \setminus \Phi(D \times \{P_1\}) \to N_0 \#_\gamma N_1$ are both orientation-preserving. We refer to $N_0 \#_\gamma N_1$ as an internal connected sum of N_0 and N_1 along γ or as a tubing of N_0 and N_1 along γ.[1994]

The following lemma justifies the name "internal connected sum".

[1993] Note that here we use again, in a subtle way, our hypothesis $n-k \geq 2$.

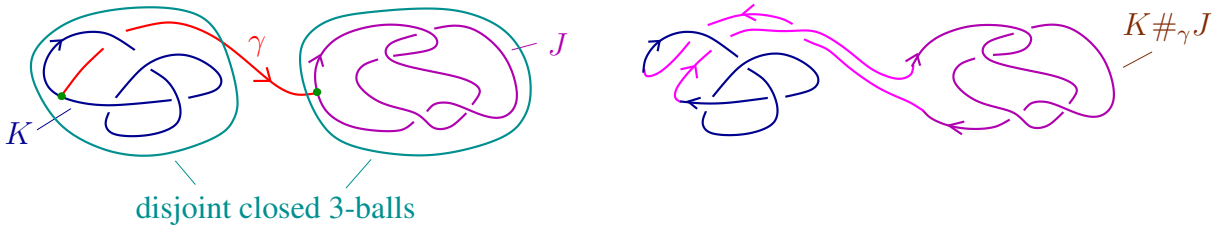
[1994] The construction also depends on the choice of Φ and Θ. We do not claim that the isotopy type of $N_0 \#_\gamma N_1$ is well-defined.



Lemma 187.29. (Internal Connected Sum Lemma) We continue with the above notation. The internal connected sum $N_0 \#_\gamma N_1$, viewed as a smooth manifold in its own right, is diffeomorphic to the connected sum $N_0 \# N_1$.

Proof. The lemma can be proved fairly easily using the Smooth Collar Neighborhood Theorem 35.6 applied to $N_0 \setminus \Phi(D \times \{P_0\})$ and $N_1 \setminus \Phi(D \times \{P_1\})$. We leave it to the reader to fill in the details. ■

Example. In the figure below we show two disjoint oriented knots K and J. By definition both are submanifolds of S^3. We also show a path γ that connects K to J and we show a corresponding internal connected sum $K \#_\gamma J$. In Definition 109.12 we introduced the connected sum $K \# J$. The resulting image, see e.g. the figure on page 2121 looks quite similar to what we see in the figure below.



With some effort one can show that if we are given two oriented knots, which are contained in the interiors of two disjoint smoothly embedded closed 3-dimensional balls, then the corresponding connected sum $K \# J$ is smoothly isotopic to any internal connected sum $K \#_\gamma J$. In fact sometimes this approach is used as the definition of the connected sum of two oriented knots. □

We recall the statement of the proposition that we are actually supposed to prove.

Proposition 187.21. (Represent-by-Connected Submanifold Proposition) Let $n \in \mathbb{N}$, let M be a connected n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional submanifold with $\partial N \subseteq A$. We suppose that $n - k \geq 2$. If $n \geq 2$ or if $A = \varnothing$, then there exists a compact oriented *connected* submanifold W of M with $\partial W \subseteq A$ such that $[W] = [N] \in \mathrm{H}_k(M, A)$.

Sketch of proof. To simplify the notation we assume that N has precisely two components N_0 and N_1. As explained in the Internal Connected Sum Construction 187.28, since $n - k \geq 2$ we can perform an internal connected sum $N_0 \#_\gamma N_1$. As in the proof of the Homotopic-to-Embedding Proposition 187.18 one can deduce, using the Recombine Fundamental Class Proposition 187.27, that $[N_0 \#_\gamma N_1] = [N_0] + [N_1] \in \mathrm{H}_k(M, A)$.

Next, as we pointed out in the Internal Connected Sum Lemma 187.29, the internal connected sum $N_0 \#_\gamma N_1$ is diffeomorphic to the connected sum $N_0 \# N_1$. If $n \geq 2$, then it follows immediately from the Smooth Connected Sum-Proposition 54.13 that the connected sum $N_0 \# N_1$ is connected. Finally, if $A = \varnothing$, then we know that both N_0 and N_1 are closed. Thus it follows again from the Smooth Connected Sum-Proposition 54.13 that $N_0 \# N_1$ is connected. ■

187.8. Exercises for Chapter 187 .

Exercise 187.1. Let M be a closed 3-dimensional smooth manifold and let $\sigma \in \mathrm{H}_2(M)$ be a homology class. By the H_2-via-Surface Proposition 187.4 there exists a closed oriented 2-dimensional smooth manifold F and a continuous map $\varphi\colon F \to M$ such that $\varphi_*([F]) = \varphi$. Can you modify the proof of the Homotopic-to-Embedding Proposition 187.18 to show that there exists a closed oriented 2-dimensional smooth manifold G and an *embedding* $\psi\colon G \to M$ such that $\psi_*([G]) = \sigma$? Identify the issues you run into and try to resolve them.

Exercise 187.2. Let $n \in \mathbb{N}$. As in Definition 9.11 we consider the complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$. By the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 we know that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k \in \{0, 2, \ldots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the map $f\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ given by $f([z_0 : \cdots : z_n]) = [\overline{z}_0 : \cdots : \overline{z}_n]$. For each $k \in \{0, 2, \ldots, 2n\}$ determine the degree of the induced map $f_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n)$.

Exercise 187.3. Let $g \in \mathbb{N}_0$. As usual we denote by Σ_g the surface of genus g. Let M be a smooth manifold and let $P \in M$. Now let $k \in \mathbb{N}_0$. Show that if there exists a continuous map $f\colon \Sigma_k \to M \times \Sigma_g$ with $f_*([\Sigma_k]) = [\{P\} \times \Sigma_g] \in \mathrm{H}_2(M \times \Sigma_g)$, then $k \geq g$.

Exercise 187.4. Let $T = S^1 \times S^1 =_{\mathrm{i}} \mathbb{R}^2/\mathbb{Z}^2$ be the torus. We denote by $x = [S^1 \times \{1\}]$ and $y = [\{1\} \times S^1]$ the obvious basis for $\mathrm{H}_1(T)$. Let $(r, s) \neq (0, 0) \in \mathbb{Z}^2$. Show that the following two statements are equivalent:

(a) The homology class $r \cdot x + s \cdot y$ can be represented by a connected submanifold of T.
(b) $\gcd(r, s) = 1$.

Hint. For the "(a)⇒(b)"-direction you might want to use the classification of 2-dimensional smooth manifolds.

Exercise 187.5. Let $g \in \mathbb{N}$. We consider the surface Σ of genus g. Let $\varphi \neq 0 \in \mathrm{H}_1(\Sigma)$. Show that the following two statements are equivalent:

(a) φ can be represented by a connected submanifold of Σ.
(b) φ is a primitive class in $\mathrm{H}_1(\Sigma)$.

Remark. The case $g = 1$ is evidently just the content of Exercise 187.4.

Exercise 187.6. Let M and N be topological manifolds and let $\varphi \in \mathrm{H}_k(M)$ and $\psi \in \mathrm{H}_l(N)$. Prove the following: If φ is realized by a closed oriented k-dimensional submanifold A and if ψ is realized by a closed oriented l-dimensional submanifold B, then the cross product $\varphi \times \psi \in \mathrm{H}_{k+l}(M \times N)$ is realized by the closed $(k + l)$-dimensional submanifold $A \times B$.

CHAPTER 188

The Simplicial Poincaré Duality Theorem

In this chapter we will prove one of the most beautiful and powerful theorems in the theory of abstract simplicial complexes, namely the Simplicial Poincaré Duality Theorem 188.5. This theorem has many applications, arguably the most interesting ones concern the singular (co-) homology groups of smooth manifolds.

This chapter is organized as follows:

- In Section 188.1 we will introduce simplicial homology manifolds.
- In Section 188.2 we will state the Simplicial Poincaré Duality Theorem 188.5 and we will see that it implies the Poincaré Duality Theorem 188.6 for closed smooth manifolds.
- In Section 188.3 we will recall the barycentric subdivision and we will prove several new statements about the barycentric subdivision.
- In Section 188.4 we will introduce the dual chain complex.
- In Section 188.5 we will show that the dual chain complex is chain homotopy equivalent to the simplicial chain complex.
- In Section 188.6 we will conclude the proof of the Simplicial Poincaré Duality Theorem 188.5.
- In Section 188.7 we will introduce the dual CW-structure of a smooth manifold that is equipped with a smooth simplicial structure.
- In Section 188.8 we will provide exercises for Chapter 188.

188.1. **Simplicial Homology Manifolds.** To state the Simplicial Poincaré Duality Theorem we will need the following notion:

Definition 188.1.

(1) We say that an abstract simplicial complex L is a simplicial homology m-sphere if L is m-dimensional, if L is finite and if for every $i \in \mathbb{N}_0$ we have

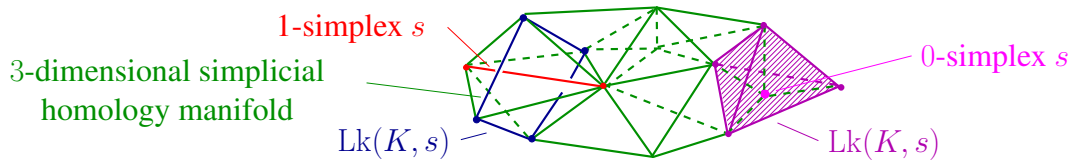
$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(L) \cong \begin{cases} \mathbb{Z}, & \text{if } i = m, \\ 0, & \text{else.} \end{cases}$$

Furthermore we say that L is a simplicial homology ball if $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(L) = 0$ for all $i \in \mathbb{N}_0$.

(2) We say that an abstract simplicial complex K is an n-dimensional simplicial homology manifold if K is n-dimensional, if K is finite and if for every $k \in \mathbb{N}_0$ and every k-simplex s of K the link $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-1-k)$-sphere or if it is a simplicial homology ball.

(3) We say that a simplicial homology manifold K is closed, if K is finite and if for every k-simplex s of K the link $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-1-k)$-sphere.

Remark. Note that by the discussion on page 3351 the empty abstract simplicial complex is a simplicial homology (-1)-sphere. Furthermore it is a straightforward consequence of



the Reduced Simplicial Homology Lemma 176.18 (1) that the empty abstract simplicial complex is in fact the only simplicial homology (-1)-sphere. It follows in particular that every simplex in a closed n-dimensional simplicial homology manifold has dimension at most n. □

The following lemma gives a reformulation of the definition of a simplicial homology manifold.

Lemma 188.2. (Simplicial Homology Manifold–Characterization Lemma) Let K be a finite abstract simplicial complex. As always we denote by $\mathrm{sd}(K)$ its barycentric subdivision. The following two statements are equivalent:

(1) K is a closed n-dimensional simplicial homology manifold.
(2) For every vertex v[1995] of the barycentric subdivision $\mathrm{sd}(K)$ the link $\mathrm{Lk}(\mathrm{sd}(K),v)$ is a simplicial homology $(n-1)$-sphere.

Proof. The lemma follows immediately from the second part of the following claim:

Claim. Let K be an abstract simplicial complex and let s be a k-simplex of K.

(a) There is a simplicial isomorphism $\mathrm{Lk}(K,s)*\partial s \cong \mathrm{Lk}(\mathrm{sd}(K),\underline{s})$.
(b) For each $i\in\mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \cong \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\mathrm{Lk}(\mathrm{sd}(K),\underline{s}))$.

Proof.

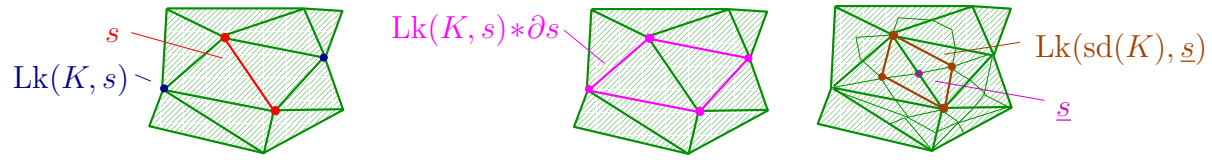
(a) This statement is actually the content of Exercise 175.2. For the reader's convenience we sketch the proof. So let us consider the map

$$\begin{array}{rcl} \mathrm{Lk}(K,s)*\partial s & \to & \mathrm{Lk}(\mathrm{sd}(K),\underline{s}) \\ v & \mapsto & \begin{cases} \underline{s\setminus\{v\}}, & \text{if } v\in\partial s, \\ \underline{s\sqcup\{v\}}, & \text{if } v\in\mathrm{Lk}(K,s). \end{cases} \end{array}$$

It is elementary to verify that this map on the level of vertices induces an isomorphism of abstract simplicial complexes.

(b) For any $i\in\mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \underset{\text{by the Join-Homology Lemma 176.21 (1)}}{\overset{\uparrow}{\cong}} \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\mathrm{Lk}(K,s)*\partial s) \underset{\text{by (a)}}{\overset{\uparrow}{\cong}} \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\mathrm{Lk}(\mathrm{sd}(K),\underline{s})).$$



[1995] It is worth remembering that by definition the vertices of $\mathrm{sd}(K)$ are precisely the barycenters of the simplices of K.

This is precisely the statement we were hoping for. ■

Slightly annoyingly the notion of a "simplicial homology manifold" is now the third definition of a "manifold-like" structure on an abstract simplicial complex. Namely in Definition 183.6 we introduced the notion of a pseudomanifold and in Definition 180.8 we introduced the notion of a PL-manifold. For the reader's and also the author's convenience we now recall these two earlier notions. Thus let K be a finite non-empty abstract simplicial complex.

(1) K is a closed n-dimensional pseudomanifold if the following conditions are satisfied:
   (a) Every simplex of K is a face of an n-simplex of K.
   (b) Every $(n-1)$-simplex has order two, i.e. it is the face of precisely two n-dimensional faces.
   (c) Given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.
(2) By the PL-Manifold Characterization Proposition 180.10 we know that K is a closed n-dimensional PL-manifold if and only if for every k-simplex $s \in S$ the link $\mathrm{Lk}(K, s)$ is a PL $(n-k-1)$-sphere, i.e. $|\mathrm{Lk}(K, s)|$ is PL-homeomorphic to the simplicial complex $\partial\Delta^{n-k}$.[1996]

The following proposition summarizes everything anybody ever wanted to know about the relationship between this new concept and our two earlier concepts.

Proposition 188.3. (PL-Simplicial-Pseudo-Manifold Proposition)
(1) A closed connected non-empty n-dimensional simplicial homology manifold K is a closed n-dimensional pseudomanifold with $\partial K = \varnothing$.
(2) There exists a closed pseudomanifold that is not a simplicial homology manifold.
(3) Every closed non-empty n-dimensional PL-manifold is a closed n-dimensional simplicial homology manifold.
(4) There exists a closed simplicial homology manifold that is not a PL-manifold.

Remark. Slightly more informally the PL-Simplicial-Pseudo-Manifold Proposition 188.3 can be summarized as follows:

PL-manifold $\rightleftarrows$ simplicial homology manifold $\rightleftarrows$ pseudomanifold. □

Proof.

(1) Let K be a closed connected non-empty n-dimensional simplicial homology manifold. We want to show that K is a closed n-dimensional pseudomanifold. The well-versed reader will easily spot that arguments below are at times almost identical to the arguments provided in the proof of the Manifold-Simplex Proposition 183.2.

[1996]By the Alexander-Newman Theorem 181.6 the notion of a PL-manifold is basically equivalent to the notion of a combinatorial manifold that we introduced in Definition 181.8. Since the notion of a PL-manifold is more commonly used we stick in the discussion to PL-manifold even though the notion of a combinatorial manifold would be slightly more appropriate.

Let us prove that the three conditions of a pseudomanifold, which we recalled above, are satisfied.

(a) Let s be a k-simplex of K. By hypothesis $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{n-k-1}(\mathrm{Lk}(K,s)) \neq 0$. This implies immediately that s is the face of an n-simplex.

(b) Let s be an $(n-1)$-simplex of K. By hypothesis we have $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}$. Note that by the Reduced Simplicial Homology Lemma 176.18 this implies that $\mathrm{H}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}^2$. Recall that on page 3603 we pointed out that the maximal dimension of a simplex of K is n. Since s is $(n-1)$-dimensional we see that $\mathrm{Lk}(K,s)$ is a 0-dimensional simplicial complex. Now we see that it consists of precisely two vertices, which implies that there are precisely two n-simplices which contain s as a face.

(c) We say that two n-simplices t and t' of K are *equivalent* if there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face. Recall that our task is to show that any two n-simplices are equivalent in the above sense.

The key to doing so is the following claim:

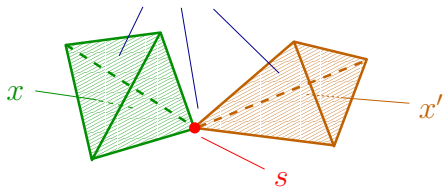
Claim. Let t and t' be two n-simplices. If $t \cap t' \neq \varnothing$, then t and t' are actually equivalent in the above sense.

Proof. Given $m \in \{0, \ldots, n-1\}$ we say that two n-simplices x and x' are *m-equivalent* if there exists a sequence $x = s_0, s_1, \ldots, s_k = x'$ of n-simplices such that any two consecutive simplices have a common face of dimension $\geq m$. It suffices to show that if we are given $m \in \{0, \ldots, n-2\}$ such that any two n-simplices in K are m-equivalent, then any two n-simplices in K are also $(m+1)$-equivalent. In fact evidently it suffices to show that if we are given two n-simplices x and x' which have an m-simplex s in common, then x and x' are $(m+1)$-equivalent. To do so we first note that since $\dim(x) > \dim(s)$ we can pick a vertex w of $x \cap \mathrm{Lk}(K,s)$ and similarly we can pick a vertex w' of $x' \cap \mathrm{Lk}(K,s)$.

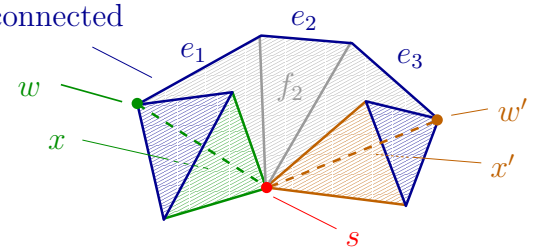
By hypothesis we know that $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) = 0$. It follows from this fact together with the Reduced Homology Lemma 147.3 and Exercise 174.13 that the abstract simplicial complex $\mathrm{Lk}(K,s)$ is in fact connected. This means in our setting that there exist 1-simplices $e_1, \ldots, e_k \in S$ of $\mathrm{Lk}(K,s)$ such that the following holds:

(∗) We have $w \in e_1$, for every $i \in \{1, \ldots, k-1\}$ we have $e_i \cap e_{i+1} \neq \varnothing$ and we have $w' \in e_k$.

x and x' are 0-equivalent



$\mathrm{Lk}(K,s)$ is connected



Let $i \in \{1, \ldots, k\}$. Since e_i is a simplex in $\mathrm{Lk}(K,s)$ there exists a simplex f_i of K with $e_i \cup s \subseteq f_i$. By (a) we can in fact find an n-simplex $\widetilde{f}_i$ of K with $e_i \cup s \subseteq f_i \subseteq \widetilde{f}_i$. We set $\widetilde{f}_0 := x$ and $\widetilde{f}_{k+1} := x'$. Note that it follows from (∗) that any two consecutive $\widetilde{f}_i$ and $\widetilde{f}_{i+1}$ have s and a vertex of $\mathrm{Lk}(K,s)$ in common, i.e.

they have an $(m+1)$-simplex in common. This shows that x and x' are in fact $(m+1)$-connected. ⊞

Now let t and t' be any two n-simplices of K. By hypothesis K is connected. Together with (a) this implies easily that there exists a sequence of n-simplices $t = u_0, u_1, \ldots, u_k = t'$ such that any two consecutive n-simplices share at least one vertex. By the claim any two consecutive n-simplices are equivalent. Hence t and t' are equivalent.

(2) Let K be a closed n-dimensional pseudomanifold. It follows from the discussion on page 3468 that the suspension $\Sigma(K) = K * \{\pm 1\}$ is a closed $(n+1)$-dimensional pseudomanifold. By the discussion on page 3325 we know that $\operatorname{Lk}(\Sigma(K), \{-1\}) = K$. Thus if $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(K) \neq 0$ for two different dimensions, e.g. if $K = S_1 \times S_1$ is the simplicial torus, then $\Sigma(K)$ is not a simplicial homology manifold.

(3) Let K be a closed n-dimensional PL-manifold and let s be a k-simplex. We see that

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\operatorname{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_i(|\operatorname{Lk}(K,s)|) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_i(S^{n-k-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{else.} \end{cases}$$

by the Simplicial-Singular Homology Isomorphism Theorem 177.10

since $\operatorname{Lk}(K,s)$ is a PL $(n-k-1)$-sphere we know that $|\operatorname{Lk}(K,s)|$ is homeomorphic to $\partial\Delta^{n-k}$, in particular $|\operatorname{Lk}(K,s)|$ is homeomorphic to S^{n-k-1}

(4) By the Poincaré Homology Sphere Proposition 183.24 we know that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group. By the Smooth Manifold-Simplicial Structure Theorem 179.5 and the Smooth-implies-PL Theorem 180.16 we can equip Y with a PL-structure $(K, \Theta\colon |K| \to Y)$. We consider the suspension $\Sigma(K) = K * \{\pm 1\}$. We want to show that $\Sigma(K)$ is a closed 4-dimensional simplicial homology manifold but that it is not a PL-manifold. It turns out that the simplices of $\Sigma(K) = K * \{\pm 1\}$ come in the following three flavors:

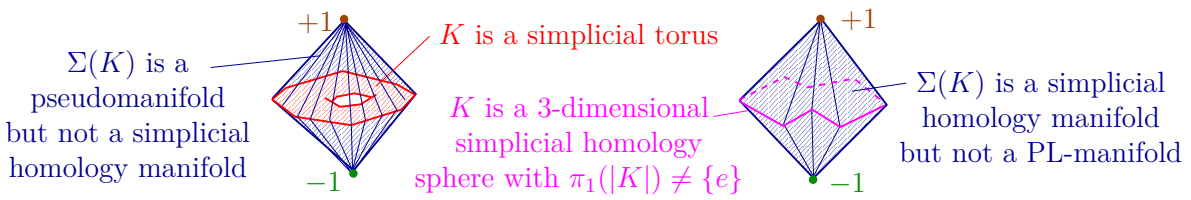
(a) Any k-simplex s of K is also a k-simplex of $\Sigma(K)$. We have, almost by definition, that $\operatorname{Lk}(\Sigma(K), s) = \Sigma(\operatorname{Lk}(K,s))$. It follows from this observation, together with (1), (3) and the Join-Homology Lemma 176.21 (2) that $\operatorname{Lk}(\Sigma(K), s)$ is a simplicial homology $(3-k)$-sphere.

(b) Given a k-simplex s of K we obtain the $(k+1)$-simplices $s \cup \{-1\}$ and $s \cup \{1\}$ of $\Sigma(K)$. Note that it follows almost immediately from the various definitions that $\operatorname{Lk}(\Sigma(K), s \cup \{-1\}) = \operatorname{Lk}(K,s) = \operatorname{Lk}(\Sigma(K), s \cup \{1\})$. Thus we obtain from (1) that $\operatorname{Lk}(\Sigma(K), s \cup \{-1\})$ and $\operatorname{Lk}(\Sigma(K), s \cup \{1\})$ are both simplicial homology $(3-(k+1))$-spheres.

(c) Finally we need to consider the 0-simplices ± 1 of $\Sigma(K) = K * \{\pm 1\}$. As we discussed on page 3325, we have $\operatorname{Lk}(\Sigma(K), \{\pm 1\}) = K$. Since K is a homology 3-sphere we see that $\operatorname{Lk}(\Sigma(K), \{\pm 1\}) = K$ are simplicial homology 3-spheres. But since the fundamental group $\pi_1(|K|) = \pi_1(Y)$ is by construction a non-trivial group we see that $|\operatorname{Lk}(\Sigma(K), \{\pm 1\})| = |K|$ are not homeomorphic to the 3-sphere. This implies that $\operatorname{Lk}(\Sigma(K), \{\pm 1\})$ is neither a PL 3-spheres nor is it a PL-ball.

Thus we see that $\Sigma(K)$ is a 4-dimensional simplicial homology manifold, but $\Sigma(K)$ is not a PL-manifold. ■

We conclude this section with the following lemma:



Lemma 188.4. (Simplicial Complex–Topological Manifold Lemma) Let K be a finite abstract simplicial complex. If $|K|$ is an n-dimensional topological manifold with $\partial(|K|) = \varnothing$, then K is a closed n-dimensional simplicial homology manifold.

Proof. Let s be a k-simplex of K. We need to show that $\mathrm{Lk}(K,s)$ is a simplicial homology $(n-k-1)$-sphere. We pick a point P in the corresponding open simplex $\langle s \rangle$. We perform the following calculation:

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{i+k+1}(|K|, |K| \setminus \{P\}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i+k+1 = n, \\ 0, & \text{else.} \end{cases}$$

by the Simplicial Complex Local-H_*-Proposition 183.4

by the Manifold-Local-H_*-Lemma 150.9, since $|K|$ is an n-dimensional topological manifold with $\partial(|K|) = \varnothing$

Since K is finite it follows from the above discussion that K is a closed n-dimensional simplicial homology manifold.[1997] ■

188.2. **The statement of the Simplicial Poincaré Duality Theorem.** Let K be a closed connected n-dimensional simplicial homology manifold. By the PL-Simplicial-Pseudo-Manifold Proposition 188.3 (1) we know that K is in particular a closed n-dimensional pseudomanifold. In particular it makes sense to talk of orientations of K. We recall the notion of a fundamental class.

(1) If K is oriented, then by Definition 183.13 we have a fundamental class $[K] \in \mathrm{H}_n^{\mathrm{simp}}(K)$. In the following, given a commutative ring R we denote by $[K]$ also the image of $[K]$ under the natural map $\mathrm{H}_n^{\mathrm{simp}}(K) \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1_R]} \mathrm{H}_n^{\mathrm{simp}}(K;R)$.

(2) Regardless of whether or not K is orientable we introduced in Definition 183.13 the $\mathbb{F}_2$-fundamental class $[K]_{\mathbb{F}_2} \in \mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)$.

The following theorem is the main theorem on simplicial (co-) homology groups of simplicial homology manifolds.

Theorem 188.5. (Simplicial Poincaré Duality Theorem) Let K be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$. If K is orientable, then for every choice of orientation and every commutative ring R the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ \varphi & \mapsto & \varphi \cap [K] \end{array}$$

[1997] As a parenthetical remark we point out that alternatively we could have proved the lemma using the Simplicial Homology Manifold–Characterization Lemma 188.2 instead of using the Simplicial Complex–Local-H_*-Proposition 183.4.

is an isomorphism. Regardless of whether or not K is orientable the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{F}_2) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [K]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Remark. One might think or hope that the Simplicial Poincaré Duality Theorem 188.5 also holds for closed (orientable) pseudomanifolds, but as we will see in Exercise 188.1 this is not the case. Nonetheless, it turns out that on pseudomanifolds one can introduce more refined (co-) homology groups, namely intersection homology, that do satisfy an analog of the Poincaré Duality Theorem. We refer to [**Ban2007**] for details. □

The proof of the Simplicial Poincaré Duality Theorem 188.5 is evidently non-trivial and somewhat lengthy. Thus we will first draw several interesting conclusions before we head to the proof.

Example. Let K be a closed orientable n-dimensional simplicial homology manifold. Let us see what the Simplicial Poincaré Duality Theorem 188.5 tells us for $k = 1$. We have the following isomorphisms:

$$\begin{array}{ccccc} \mathrm{H}^{\mathrm{simp}}_{n-1}(K) & \cong & \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z}) & \cong & \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_1(K),\mathbb{Z}) \oplus \mathrm{Ext}(\overbrace{\mathrm{H}^{\mathrm{simp}}_0(K)}^{\text{free abelian group by Exercise 176.3}},\mathbb{Z}) \\ & & \uparrow & & \uparrow \\ & & \text{by the Poincaré Duality Theorem 188.5} & & \text{by the Universal Coefficient Simplicial-H}^*\text{-Theorem 184.14} \\ & \cong & \mathrm{Hom}(\mathrm{H}^{\mathrm{simp}}_1(K),\mathbb{Z}) & \cong & \mathrm{Hom}(\mathbb{Z}^d \oplus T,\mathbb{Z}) \cong \mathbb{Z}^k. \\ & & \uparrow & & \uparrow \\ & & \text{by the Ext-Properties Lemma 201.19 (3)} & & \text{by the Simplicial Skeleton–Homology Lemma 176.16 (1) we know that the group } \mathrm{H}^{\mathrm{simp}}_1(K) \text{ is finitely generated, hence it is isomorphic to } \mathbb{Z}^d \oplus T \text{ for some } d \in \mathbb{N} \text{ and some finite group } T \end{array}$$

We have thus shown that $\mathrm{H}^{\mathrm{simp}}_{n-1}(K)$ is a free abelian group. In the more general setting of closed orientable pseudomanifolds we had obtained the same result in the Pseudomanifold Homology Theorem 183.12, but using a rather different argument. □

Our richest source of simplicial homology manifolds are smooth manifolds. We obtain the following corollary to the Simplicial Poincaré Duality Theorem 188.5.

Theorem 188.6. (Poincaré Duality Theorem) Let M be a closed n-dimensional smooth manifold and let $k \in \mathbb{N}_0$. If M is orientable, then for every choice of orientation and every commutative ring R the map

$$\begin{array}{rcl} \mathrm{H}^k(M;R) & \to & \mathrm{H}_{n-k}(M;R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. Regardless of whether or not M is orientable the map

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{F}_2) & \to & \mathrm{H}_{n-k}(M;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [M]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Proof. One can show easily (see e.g. the elementary future Cup-Cap Disjoint Union Lemma 204.16) that we can assume that M is actually connected. Recall that by the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a smooth simplicial

structure $(K = (V, S), f\colon |K| \to M)$. Without loss of generality we can assume that $|K| = M$. We denote by $\Xi_*\colon \mathrm{H}_k^{\mathrm{simp}}(K;R) \to \mathrm{H}_k(M;R)$ and $\Xi^*\colon \mathrm{H}^k(M;R) \to \mathrm{H}_k^{\mathrm{simp}}(K;R)$ the natural isomorphisms provided by the Simplicial–Singular Chain Complex Proposition 184.4 (1) and the Simplicial-Singular Cohomology Proposition 184.12 (3). By the Simplicial Complex–Topological Manifold Lemma 188.4 we know that K is a closed n-dimensional simplicial homology manifold.

Let us first deal with the case that M is equipped with a natural orientation. By the Pseudomanifold-Orientation Proposition 183.19 we know that the simplicial homology manifold K inherits an orientation. Next we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \xrightarrow[\cong]{\cap[K]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ {\scriptstyle \Xi^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Xi_*} \\ \mathrm{H}^k(M;R) & \xrightarrow{\cap[M]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(M;R). \end{array}$$

We make the following observations:

(1) By the Fundamental Class Algorithm 185.5 we know that $\Xi_*([K]) = [M]$.
(2) It follows from (1) and the Simplicial Cap Product Proposition 184.27 (3) that the diagram commutes.
(3) The top horizontal map is an isomorphism by the Simplicial Poincaré Duality Theorem 188.5.

Since the vertical maps are isomorphisms we obtain from the above that the bottom horizontal map is, as promised, an isomorphism.

The proof for the statement regarding $\mathbb{F}_2$-coefficients is almost identical. We leave it to the reader to make the necessary straightforward modifications to the above argument. ■

There are many applications of the Poincaré Duality Theorem 188.6. In the following we will state two results. Since we have a predilection for smooth manifolds, we state them in the context of smooth manifolds, but the statements also hold in the admittedly more general setting of simplicial homology manifolds.

Theorem 210.6. (Homology Symmetry Theorem) Let M be a closed n-dimensional smooth manifold and let $k \in \mathbb{N}_0$.

(1) We have $\dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) = \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M;\mathbb{F}_2))$.
(2) If M is orientable, then for any field $\mathbb{F}$ we have $\dim_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M;\mathbb{F}))$.

Proposition 210.9. (Odd-Dimensional Manifold-χ-Proposition) The Euler characteristic of any closed odd-dimensional smooth manifold is zero.

As the reader will have noticed, the numbers of these results correspond to Chapter 210. In fact, in Chapter 210 we will prove a more general Poincaré Duality Theorem 210.1 that applies also to topological manifolds, possibly with boundary. In Section 210.3 we discuss many consequences of the Poincaré Duality Theorem 210.1. For closed smooth manifolds all statements in Section 210.3 can also be deduced from the above Poincaré Duality Theorem 188.6. In particular this way we obtain the above results. To avoid repetition, and for the sake of brevity, we do not spell out the proofs twice.

There are many other applications of the Poincaré Duality Theorem 188.6. Arguably one of the most interesting applications is that the Poincaré Duality Theorem 188.6 allows us

to compute the cup product of the real and complex projective spaces. This will be done in Chapter 213.

Remark. The Poincaré Duality Theorem 188.6 has two significant limitations and one aesthetic issue.

(1) The theorem only deals with closed smooth manifolds. In fact the approach taken in the proof of Poincaré Duality Theorem 188.5 can be generalized, with some effort, to deal with compact (oriented) smooth manifolds with boundary. We refer to [**SZ1994**, Satz 14.8.8] for details.
(2) The Poincaré Duality Theorem 188.6 applies only to smooth manifolds. There is though some room for improvement. More precisely, the Simplicial Poincaré Duality Theorem 188.5 together with the Simplicial Complex–Topological Manifold Lemma 188.4 shows that we have Poincaré Duality for any closed oriented topological manifold that admits a simplicial structure. But by the Manolescu Theorem 190.21 we know that for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure. Thus the Poincaré Duality Theorem 188.5 does not imply that Poincaré Duality holds for all closed topological manifolds.
(3) Even for smooth manifolds the proof of the Poincaré Duality Theorem 188.6 is aesthetically problematic since it makes use of the somewhat artificial choice of a simplicial structure.

All three issues will be addressed by the Poincaré Duality Theorem 210.1 that we will state and prove in Chapter 210. Nonetheless, the (proof) of the Poincaré Duality Theorems 188.5 and 188.6 also have two advantages over the Poincaré Duality Theorem 210.1:

(1) The proofs in this chapter are much more visual, whereas the proof of the Poincaré Duality Theorem 210.1 is rather formal and it is difficult to get an intuition for "why" and "how" it works.
(2) More importantly, much later in Chapter 348 we will introduce a new invariant, namely Reidemeister torsion. The intermediate results in our proof of Poincaré Duality Theorem 188.5 will help us in showing that Reidemeister torsion of closed smooth manifolds is symmetric. □

188.3. **The barycentric subdivision.** The proof of the Simplicial Poincaré Duality Theorem 188.5 relies on a clever use of the barycentric subdivision of the simplicial homology manifold. In this section we will recall the main definitions and statements on barycentric subdivisions which we will need later on in this chapter.

We recall the following definitions from Definition 175.1 and Definition 177.24.

Definition 188.7. Let $K = (V, S)$ be an abstract simplicial complex.

(1) For each k-simplex $s = \{v_0, \dots, v_k\} \in S$ we define the barycenter of s to be the point
$$\underline{s} \coloneqq \sum_{v \in s} \frac{1}{\dim(s)+1} \cdot v = \frac{1}{k+1} \cdot v_0 + \cdots + \frac{1}{k+1} \cdot v_k \in \mathbb{R}^{(V)}.$$
(2) The barycentric subdivision of K is defined to be the following abstract simplicial complex:
$$\operatorname{sd}(K) = \big(\{\underline{s}\}_{s \in S}, \{\{\underline{s_0}, \dots, \underline{s_k}\} \mid s_0 \subsetneq \cdots \subsetneq s_k\}\big).$$

(3) Given an ordered k-simplex $(v_0, \dots, v_k)$ of K we consider the following ordered k-simplex of $\mathrm{sd}(K)$:

$$F(v_0, \dots, v_k) \;:=\; \big(\underline{\{v_0\}}, \underline{\{v_0, v_1\}}, \dots, \underline{\{v_0, \dots, v_k\}}\big).$$

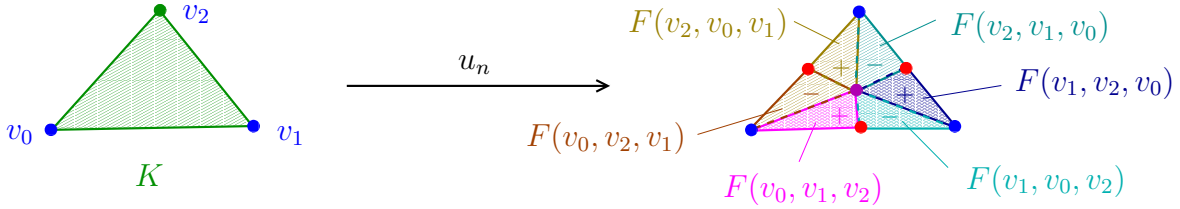
(4) Given an ordered k-simplex $s = (v_0, \dots, v_k)$ of K we set

$$u_k(s) \;:=\; \sum_{\sigma \in \mathrm{Bij}\{0,\dots,k\}} (-1)^{\mathrm{sign}(\sigma)} \cdot [F(v_{\sigma(0)}, \dots, v_{\sigma(k)})] \;\in\; \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K)).$$

We refer to the resulting map

$$\begin{array}{rcl} u_k \colon \mathrm{C}_k^{\mathrm{simp}}(K) & \to & \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K)) \\ [s] & \mapsto & u_k(s) \end{array}$$

as the simplicial subdivision map. By the Chain Subdivision Lemma 177.25 (1) we know that these subdivision maps are actually well-defined and that they form a chain map.

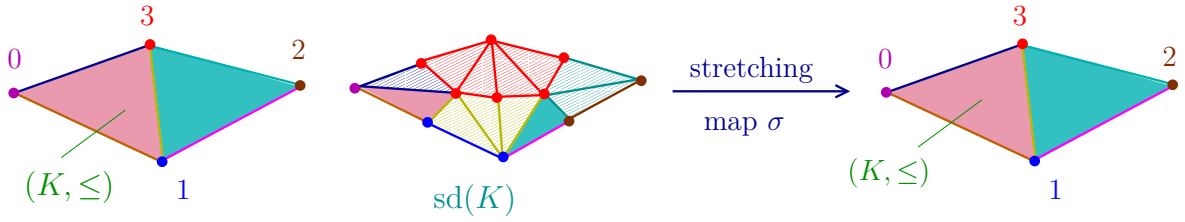


Remark. Let $K = (V, S)$ be an abstract simplicial complex. Note that it follows easily from the definitions that given any n-simplex t of $\mathrm{sd}(K)$ there exists a unique ordered n-simplex $(v_0, \dots, v_n)$ of K with $\lfloor F(v_0, \dots, v_n) \rfloor = t$. □

The following lemma shows that subdivision maps induce isomorphisms on simplicial (co-) homology groups.

Lemma 177.20. (Stretching Map Lemma) Let $(K, \leq)$ be an ordered abstract simplicial complex.

(1) There exists a unique simplicial map $\sigma \colon \mathrm{sd}(K) \to K$, called the *stretching map*, with the following property: Given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K the map σ sends the corresponding vertex $\underline{s}$ of $\mathrm{sd}(K)$ to the vertex v_n of K.



The following lemma is essentially a slight generalization of the Chain Subdivision Lemma 177.25 (5):

Lemma 177.25. (Chain Subdivision Lemma) Let K be an abstract simplicial complex. We pick an order on the simplicial complex K and we consider the corresponding stretching map $\sigma\colon \mathrm{sd}(K) \to K$.

(5) For every choice of some $k \in \mathbb{N}_0$ and of some abelian group G the two induced maps $u_*\colon \mathrm{H}_k^{\mathrm{simp}}(K;G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K);G)$ and $u^*\colon \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);G) \to \mathrm{H}^k_{\mathrm{simp}}(K;G)$ are isomorphisms and the inverses are given by σ_* and σ^*.

Proof. First note that by the Chain Subdivision Lemma 177.25 (4) we know that the maps $u_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K))$ form a chain homotopy equivalence and that are chain homotopy inverse is given by the maps $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp}}(K)$. The statements about simplicial (co-) homology with G-coefficients now follow from the following purely algebraic lemmas: The Functor-applied-to-Chain Complex Lemma 160.3, the Dual Cochain Map Lemma 197.20 (3) and the Cochain Homotopy-H*-Lemma 197.14. ■

Next we turn to barycentric subdivisions of pseudomanifolds. At this point it is convenient to introduce the following definition:

Definition 188.8. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold. Given an ordered n-simplex $(v_0, \ldots, v_n)$ of K we write

$$\mathrm{sign}((v_0,\ldots,v_n)) \coloneqq \begin{cases} +1, & \text{if the oriented } n\text{-simplex } [(+1,(v_0,\ldots,v_n))] \text{ corresponds} \\ & \qquad \text{to the orientation,} \\ -1, & \text{otherwise.} \end{cases}$$

The following lemma contains everything anybody ever wanted to know about barycentric subdivisions of closed pseudomanifolds.

Lemma 188.9. (Pseudomanifold–Subdivision Lemma) Let $K = (V, S)$ be a closed n-dimensional pseudomanifold.

(1) The barycentric subdivision $\mathrm{sd}(K)$ is a closed n-dimensional pseudomanifold.

(2) Suppose that K is equipped with an orientation. Let t be an n-simplex of $\mathrm{sd}(K)$. As we pointed out on page 3611, there exists a unique ordered n-simplex $(v_0, \ldots, v_n)$ of K with $\lfloor F(v_0, \ldots, v_n) \rfloor = t$. We equip the n-simplex t of $\mathrm{sd}(K)$ with the orientation given by $(\mathrm{sign}((v_0, \ldots, v_n)), F(v_0, \ldots, v_n))$.
   (a) The above orientations of the n-simplices define an orientation for $\mathrm{sd}(K)$.
   (b) We have $u_*([K]) = [\mathrm{sd}(K)] \in \mathrm{H}_n^{\mathrm{simp}}(\mathrm{sd}(K))$.

(3) If K is oriented, then given any $k \in \mathbb{N}_0$ and given any commutative ring R the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \xrightarrow{\cap[K]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ {\scriptstyle u^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle u_*} \\ \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);R) & \xrightarrow{\cap[\mathrm{sd}(K)]} & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);R). \end{array}$$

The analogs of (2) and (3) hold for arbitrary closed n-dimensional pseudomanifolds if we work with $\mathbb{F}_2$-coefficients and the simplicial $\mathbb{F}_2$-fundamental classes $[K]_{\mathbb{F}_2}$ and $[\mathrm{sd}(K)]_{\mathbb{F}_2}$.

Sketch of proof. Statements (1) and (2a) follow almost immediately from the definitions, the proof is outsourced to Exercise 183.5. Furthermore Statement (2b) can be deduced easily from the definition of the fundamental class in Definition 183.13 and the definition

of the subdivision map $u_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K))$. We leave it to the reader to fill in the details.

Next we turn to the proof of (3). We pick an order "$\leq$" on K and we denote by $\sigma\colon \mathrm{sd}(K) \to K$ the corresponding stretching map. Note that by the above Chain Subdivision Lemma 177.25 (5) we know that $u_* = \sigma_*^{-1}$ and $u^* = (\sigma^*)^{-1}$. It follows that it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \xrightarrow{\ \cap[K]\ } & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ {\scriptstyle\sigma^*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\sigma_*} \\ \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);R) & \xrightarrow{\ \cap[\mathrm{sd}(K)]\ } & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);R). \end{array}$$

Now let $\varphi \in \mathrm{H}^k_{\mathrm{simp}}(K;R)$. We see that we have the following equalities:

$$\begin{array}{ccccc} \sigma_*(\sigma^*(\varphi) \cap [\mathrm{sd}(K)]) & = & \varphi \cap \sigma_*([\mathrm{sd}(K)]) & = & \varphi \cap [K]. \\ & \uparrow & & \uparrow & \end{array}$$

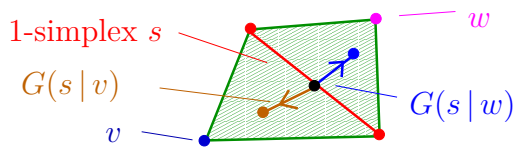
by the Simplicial Cap Product-Naturality Lemma 184.30, here we use that $\sigma\colon \mathrm{sd}(K) \to K$ is a simplicial map — by (2b)

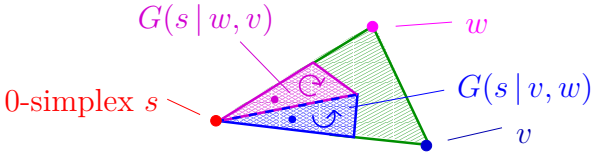
Finally we leave it to the reader to verify that the $\mathbb{F}_2$-analogs also hold. ■

188.4. **The dual chain complex I.** In this section, given a pseudomanifold K, we will introduce the dual chain complex $\mathrm{D}_*^{\mathrm{simp}}(K)$. This is the main new concept needed for the proof of the Simplicial Poincaré Duality Theorem 188.5.

Definition 188.10. Let $K = (V, S)$, let $s \in S$ be a k-simplex and let $v_1, \ldots, v_l \in V$ be such that $s \cup \{v_1, \ldots, v_l\}$ is a $(k+l)$-simplex of K. We set

$$G(s \,|\, v_1, \ldots, v_l) \;=\; \overbrace{\big(\underline{s}, \underline{s \cup \{v_1\}}, \ldots, \underline{s \cup \{v_1, \ldots, v_l\}}\big)}^{\text{ordered } l\text{-simplex of } \mathrm{sd}(K)}.$$





Remark. Note that for an ordered k-simplex $s = (v_0, \ldots, v_k)$ of K we have by definition $G(\{v_0\} \,|\, v_1, \ldots, v_k) = F(v_0, \ldots, v_k)$. □

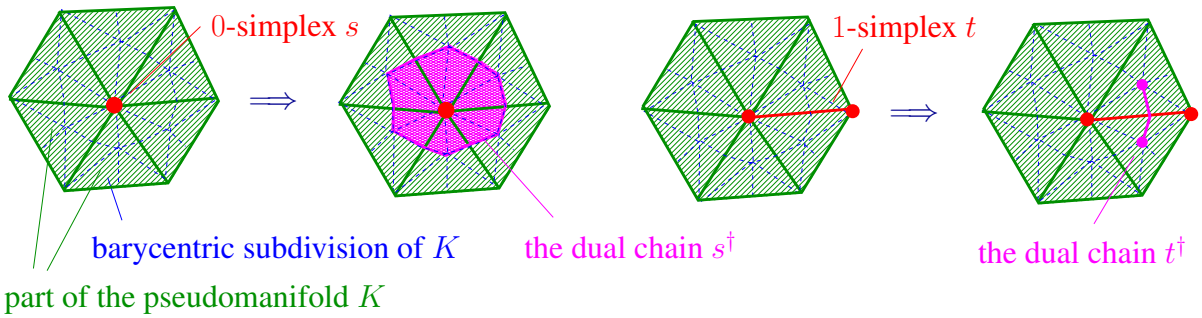
Definition 188.11. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold.

(1) Let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex of K. We define the dual chain of s to be the following simplicial chain in $\mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K))$:

$$s^\dagger \;:=\; \sum_{\substack{(v_{k+1}, \ldots, v_n) \in V^{n-k} \text{ such that} \\ \{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n}} \mathrm{sign}((v_0, \ldots, v_k, v_{k+1}, \ldots, v_n)) \cdot [\; \underbrace{G(s \,|\, v_{k+1}, \ldots, v_n)}_{\text{ordered } (n\text{-}k)\text{-simplex of } \mathrm{sd}(K)} \;].$$

(2) We define

$$\mathrm{D}^{\mathrm{simp}}_{n-k}(K) \;=\; \begin{array}{c}\text{subgroup of } \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)) \text{ generated by the } s^\dagger, \\ \text{where } s \text{ runs over all } k\text{-simplices of } K.\end{array}$$



Lemma 188.12. (Dual Chain Lemma) Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold and let $k \in \mathbb{N}_0$.

(1) Given two ordered k-simplices s and $\widetilde{s}$ with $[s] = [\widetilde{s}] \in \mathrm{C}_k^{\mathrm{simp}}(K)$ we have $s^\dagger = \widetilde{s}^\dagger$.

(2) Let "$\leq$" be an order on $K = (V, S)$. The dual chains

$$\{s^\dagger \,|\, s = (v_0, \ldots, v_k) \text{ an ordered } k\text{-simplex of } K \text{ with } v_0 < \cdots < v_k\}$$

form a basis for $\mathrm{D}_{n-k}^{\mathrm{simp}}(K)$.

Proof. First note that Statement (1) follows easily from the definitions. Thus let us turn to the proof of Statement (2). We pick an order "$\leq$" on $K = (V, S)$. We set

$$Y \;:=\; \{(v_0, \ldots, v_k) \in V^{k+1} \,|\, \{v_0, \ldots, v_k\} \in S_k \text{ and } v_0 < \cdots < v_k\}.$$

It remains to prove the following claim:

Claim. The map

$$\begin{array}{rccc} \alpha\colon & \bigoplus\limits_{y \in Y} \mathbb{Z} \cdot y & \to & \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \\ & y & \mapsto & y^\dagger \end{array}$$

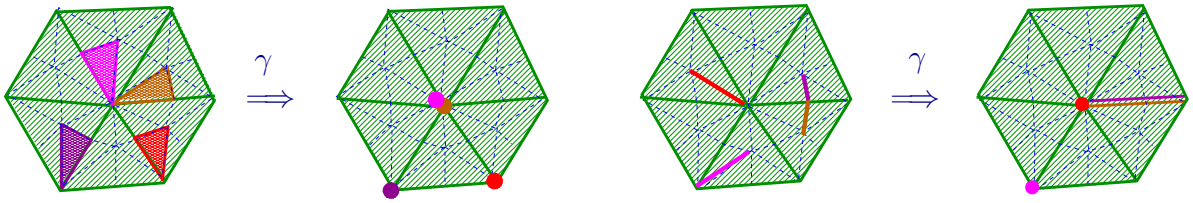
is an isomorphism.

Proof. By definition the map α is an epimorphism. Thus it remains to show that the map α is a monomorphism. To do so we consider a bunch of extra homomorphisms:

- We denote by $\beta\colon \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \to \mathrm{C}_{n-k}^{\mathrm{simp}}(\mathrm{sd}(K))$ the inclusion.
- Recall that by definition $\mathrm{C}_{n-k}^{\mathrm{simp}}(\mathrm{sd}(K))$ is the free abelian group generated by the $(n-k)$-simplices of $\mathrm{sd}(K)$. Also recall that each $(n-k)$-simplex of $\mathrm{sd}(K)$ can be written uniquely in the form $[\overline{s_0}, \ldots, \overline{s_{n-k}}]$ where $s_0 \subsetneq \cdots \subsetneq s_{n-k}$ is an increasing sequence of simplices of K. Thus there exists a unique homomorphism

$$\begin{array}{rccc} \gamma\colon & \mathrm{C}_{n-k}^{\mathrm{simp}}(\mathrm{sd}(K)) & \to & \mathbb{Z}^{\bigsqcup\limits_{i \in \mathbb{N}_0} S_i(K)} \\ & \underbrace{[\overline{s_0}, \ldots, \overline{s_{n-k}}]}_{\text{with } s_0 \subsetneq \cdots \subsetneq s_{n-k}} & \mapsto & s_0. \end{array}$$

(Here $S_i(K)$ denotes the set of i-simplices of K.)



- Next we denote by $\delta\colon \mathbb{Z}^{\bigsqcup_{i\in\mathbb{N}_0} S_i(K)} \to \mathbb{Z}^{S_k(K)}$ the obvious map.
- Finally we denote by $\epsilon\colon \mathbb{Z}^Y \to \mathbb{Z}^{S_k(K)}$ the isomorphism that is induced by the obvious bijection $Y \to S_k(K)$.

We consider the following sequence of homomorphisms:

$$\bigoplus_{y\in Y}\mathbb{Z}\cdot y \xrightarrow{\alpha} \mathrm{D}^{\mathrm{simp}}_{n-k}(K) \xrightarrow{\beta} \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)) \xrightarrow{\gamma} \mathbb{Z}^{\bigsqcup_{i\in\mathbb{N}_0} S_i(K)} \xrightarrow{\delta} \mathbb{Z}^{S_k(K)} \xleftarrow{\epsilon} \bigoplus_{y\in Y}\mathbb{Z}\cdot y.$$

It follows easily from the definitions that for each $y \in Y$ the map $\epsilon^{-1}\circ\delta\circ\gamma\circ\beta\circ\alpha$ is just given by multiplication by the number of n-simplices that contain y. By definition of an n-dimensional pseudomanifold this number is non-zero. Thus this composition is a monomorphism. It follows that α is also a monomorphism. ■

To fully appreciate our next lemma it is worth recalling the following lemma:

Lemma 184.10. (Dual Simplex–Coboundary Lemma) Let $K = (V, S)$ be an abstract simplicial complex. Given any ordered k-simplex $s = (v_0, \ldots, v_k)$ we have the following equality:

$$\delta_k([v_0,\ldots,v_k]^*) \;=\; (-1)^{k+1}\cdot \sum_{\substack{w\in V \text{ such that}\\ \{v_0,\ldots,v_k,w\} \text{ is}\\ \text{a } (k+1)\text{-simplex}}} [v_0,\ldots,v_k,w]^* \;\in\; \mathrm{C}^{k+1}_{\mathrm{simp}}(K;\mathbb{Z}).$$

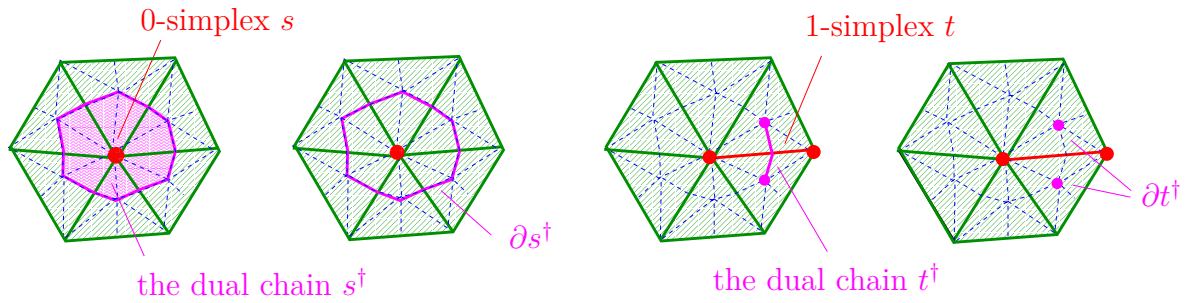
Now we can state the promised lemma. Its statement perfectly matches the statement of the Dual Simplex–Coboundary Lemma 184.10.

Lemma 188.13. (Dual Chain–Boundary Lemma) Let $K = (V, S)$ be an abstract simplicial complex that is an orientable n-dimensional pseudomanifold.

(1) Given any ordered k-simplex $s = (v_0, \ldots, v_k)$ we have the following equality:

$$\partial_{n-k}(\underbrace{[v_0,\ldots,v_k]^\dagger}_{\in \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K))}) \;=\; \sum_{\substack{w\in V \text{ such that}\\ \{v_0,\ldots,v_k,w\}\in S_{k+1}}} [v_0,\ldots,v_k,w]^\dagger \;\in\; \mathrm{C}^{\mathrm{simp}}_{n-k-1}(\mathrm{sd}(K)).$$

(2) The groups $\mathrm{D}^{\mathrm{simp}}_*(K)$ form a subcomplex of $\mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K))$.



Proof. Statement (2) is an immediate consequence of Statement (1). Thus we just need to prove Statement (1). We start out with the following elementary observation.

Observation. Let $s = (v_0, \ldots, v_k)$ be some ordered k-simplex of K and furthermore let $(v_{k+1}, \ldots, v_n) \in V^{n-k}$ be such that $s \cup \{v_{k+1}, \ldots, v_n\} = \{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n$.

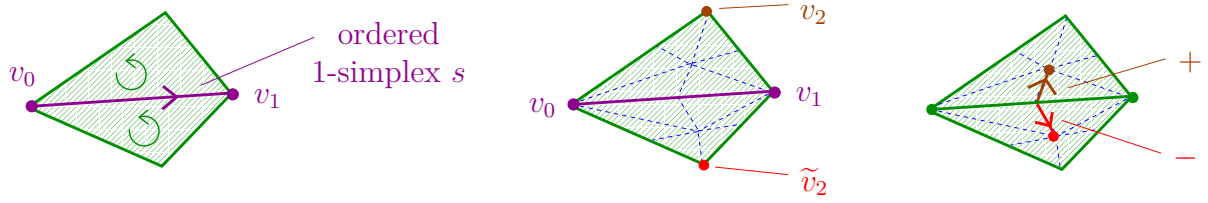
Given any $i \in \{1, \ldots, n-k-1\}$ we have by definition

$$\begin{matrix} G(s|v_{k+1}, .., v_n) \text{ with} \\ i\text{-th vertex removed} \end{matrix} = (\underline{s}, .., \underline{s \cup \{v_{k+1}, .., v_{k+i-1}\}}, \underline{s \cup \{v_{k+1}, .., v_{k+i+1}\}}, .., \underline{s \cup \{v_{k+1}, .., v_n\}}).$$

Furthermore it follows immediately from the definitions that we have

$$\begin{aligned} G(s \,|\, v_{k+1}, .., v_n) \text{ with 0-th vertex removed} &= G(s \cup \{v_{k+1}\} \,|\, v_{k+2}, \ldots, v_n) \\ G(s \,|\, v_{k+1}, .., v_n) \text{ with } (n-k)\text{-th vertex removed} &= G(s \,|\, v_{k+1}, v_{k+2}, \ldots, v_{n-1}). \end{aligned}$$

Now we turn to the actual proof of Statement (1). Thus let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex of K.



We see that

by Definition 188.11 of $s^\dagger$ and
by Definition 176.7 of the simplicial boundary map

$$\partial s^\dagger \overset{\downarrow}{=} \sum_{\substack{(v_{k+1}, \ldots, v_n) \in V^{n-k} \text{ such that} \\ \{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n}} \sum_{i=0}^{n-k} (-1)^i \cdot \mathrm{sign}(v_0, \ldots, v_n) \cdot \begin{bmatrix} G(s \,|\, v_{k+1}, \ldots, v_n) \text{ with} \\ i\text{-th vertex removed} \end{bmatrix}$$

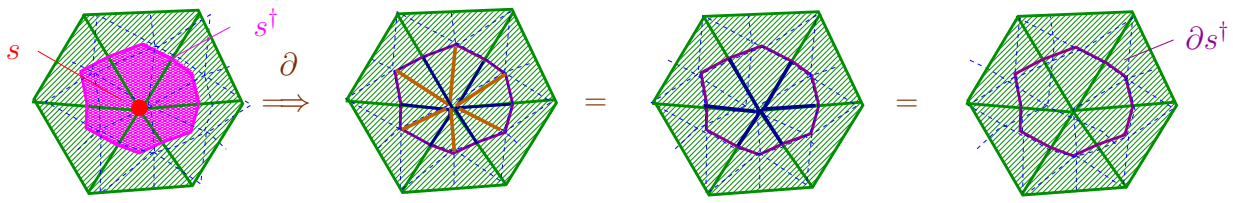
$$\overset{\text{(i)}}{=} \sum_{\substack{(v_{k+1}, \ldots, v_n) \in V^{n-k} \text{ such that} \\ \{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n}} \sum_{i=0 \text{ or } i=n-k} (-1)^i \cdot \mathrm{sign}(v_0, \ldots, v_n) \cdot \begin{bmatrix} G(s \,|\, v_{k+1}, \ldots, v_n) \text{ with} \\ i\text{-th vertex removed} \end{bmatrix}$$

given a fixed $(v_{k+1}, \ldots, v_n) \in V^{n-k}$ with $\{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n$ and given $i \in \{1, \ldots, n-k-1\}$
it follows almost immediately from the above observation that the contributions
by $((v_{k+1}, \ldots, v_{k+i}, v_{k+i+1}, \ldots, v_n), i)$ and $((v_{k+1}, \ldots, v_{k+i+1}, v_{k+i}, \ldots, v_n), i+1)$ cancel,
in other words, for $i = 1, \ldots, n-k-1$ all terms cancel

$$\overset{\text{(ii)}}{=} \sum_{\substack{(v_{k+1}, \ldots, v_n) \in V^{n-k} \text{ such that} \\ \{v_0, \ldots, v_k, v_{k+1}, \ldots, v_n\} \in S_n}} \mathrm{sign}(v_0, \ldots, v_n) \cdot \begin{bmatrix} G(s \,|\, v_{k+1}, \ldots, v_n) \text{ with} \\ 0\text{-th vertex removed} \end{bmatrix} = (*)$$

by Definition 183.6 of a closed pseudomanifold every $(n-1)$-simplex has order 2, since the pseudomanifold is oriented we obtain from the above observation that all contributions for $i = n-k$ cancel.

We take a short break from the calculation and we show a figure that illustrates the two types of cancellations seen above:



After this illuminating break we continue with the argument:

$$(*) \quad = \sum_{\substack{v_{k+1}\in V \text{ such that} \\ \{v_0,\ldots,v_k,v_{k+1}\}\in S_{k+1}}} \; \sum_{\substack{(v_{k+2},\ldots,v_n)\in V^{n-k-1} \text{ such that} \\ \{v_0,\ldots,v_k,v_{k+1},v_{k+2},\ldots,v_n\}\in S_n}} \operatorname{sign}(v_0,..,v_n)\cdot[G(s\cup\{v_{k+1}\}\,|\,v_{k+2},..,v_n)]$$

this follows from the fact that any subset of a simplex is again a simplex, we also use the above observation to rewrite the summand

$$= \sum_{\substack{w\in V \text{ such that} \\ \{v_0,\ldots,v_k,w\}\in S_{k+1}}} [v_0,\ldots,v_k,w]^\dagger. \qquad \blacksquare$$

Using the Dual Chain–Boundary Lemma 188.13 we can now easily prove the following proposition which is one of the key steps in the proof of the Simplicial Poincaré Duality Theorem 188.5.

Proposition 188.14. (Duality-via-Dual Chains Proposition) Let $K=(V,S)$ be a closed oriented n-dimensional pseudomanifold.

(1) For each $k\in\mathbb{N}_0$ the map

$$\begin{aligned} \Lambda\colon \operatorname{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K),\mathbb{Z}) &\to \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \\ \sum_{i=1}^m a_i\cdot s_i^* &\mapsto \sum_{i=1}^m a_i\cdot s_i^\dagger \end{aligned}$$

here s_i is a k-simplex of K and we define $s_i^*\colon \mathrm{C}^k_{\mathrm{simp}}(K)\to\mathbb{Z}$ as in Notation 184.6

is a natural isomorphism of abelian groups.

(2) For each $k\in\mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K),\mathbb{Z}) & \xrightarrow[\Lambda]{s^*\mapsto s^\dagger} & \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \\ \downarrow{\scriptstyle (-1)^{k+1}\cdot\delta_k=(-1)^{k+1}\cdot\partial^*_{k+1}} & & \downarrow{\scriptstyle \partial_{n-k}} \\ \operatorname{Hom}(\mathrm{C}_{\mathrm{simp}}^{k+1}(K),\mathbb{Z}) & \xrightarrow[\Lambda]{s^*\mapsto s^\dagger} & \mathrm{D}_{n-k-1}^{\mathrm{simp}}(K). \end{array}$$

(3) The maps

$$\begin{aligned} \Lambda_*\colon \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) &\to \mathrm{H}_{n-k}(\mathrm{D}_*^{\mathrm{simp}}(K)) \\ \left[\sum_{i=1}^m a_i\cdot s_i^*\right] &\mapsto \left[\sum_{i=1}^m a_i\cdot s_i^\dagger\right] \end{aligned}$$

are well-defined and they are natural isomorphisms.

Proof.

(1) First note that K is by definition a finite abstract simplicial complex. Thus it follows immediately from the Cochains–Basis Lemma 184.7 and the Dual Chain Lemma 188.12 (2) that the maps Λ are isomorphisms. Basically by definition the maps Λ are natural.

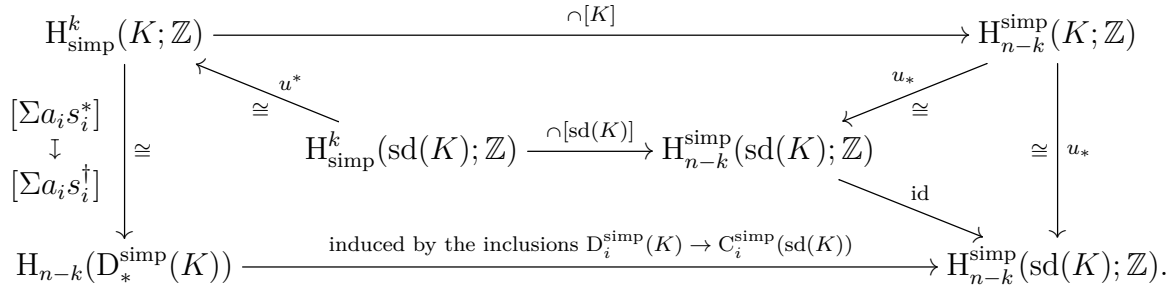
(2) Let $k\in\mathbb{N}_0$ and let $s=(v_0,\ldots,v_k)$ be an ordered k-simplex of K. We calculate that

$$\partial_{n-k}(\Lambda(s)) = \partial_{n-k}([v_0,..,v_k]^\dagger) \overset{\text{by the Dual Chain–Boundary Lemma 188.13}}{=} \sum_{\substack{w\in V \text{ such that} \\ \{v_0,\ldots,v_k,w\}\in S_{k+1}}} [v_0,\ldots,v_k,w]^\dagger \overset{\text{by the Dual Simplex–Coboundary Lemma 184.10}}{=} (-1)^{k+1}\cdot\Lambda(\delta_k([v_0,..,v_k]^*).$$

(3) In (2) we showed that the maps Λ are cochain maps "up to sign". One can easily verify that this sign issue is completely irrelevant and that we still get a well-defined

map on (co-) homology groups. Furthermore, by (1) we know that the maps on each cochain level are isomorphisms, thus we get in fact an isomorphism of (co-) homology groups. ■

It is a good moment to keep stock of where we are in the proof of the Simplicial Poincaré Duality Theorem 188.5. Let K be a closed oriented n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$. Let us consider the following diagram:



Our main goal is to show that the top horizontal map is an isomorphism. By now we already know the following:

(1) By the Duality-via-Dual Chains Proposition 188.14 we now know that the left vertical map is an isomorphism.
(2) By the Chain Subdivision Lemma 177.25 we know that the subdivision maps u_* and u^* are isomorphisms.
(3) By the Pseudomanifold–Subdivision Lemma 188.9 (3) we know that the upper quadrilateral commutes.

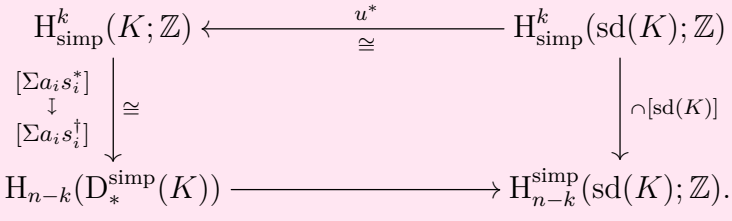
Thus to prove the core of the Simplicial Poincaré Duality Theorem 188.5 it remains to carry out the following steps:

(4) We need to show that the lower left part of the diagram commutes.
(5) We need to show that the bottom horizontal map is an isomorphism.

As we will see in a second, Step (4) follows from a fairly straightforward calculation. Step (5) is much more difficult and we postpone Step (5) to the next section. Note though that Step (5) is particularly interesting since so far we have only used that K is a closed n-dimensional pseudomanifold. We did not yet use the hypothesis that we are dealing with a simplicial homology manifold.

As promised, here is the lemma that takes care of Step (4).

Lemma 188.15. (Subdivision-Duality-Cap-Lemma) Given any closed oriented n-dimensional pseudomanifold K the following diagram commutes:



Proof. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold. Given two k-simplices s and t of K we write $\underline{s} \leq \underline{t}$ if $s \subseteq t$. As we pointed out in the Barycentric

Subdivision Lemma 175.3, this order on the vertex set of $\mathrm{sd}(K)$ turns $\mathrm{sd}(K)$ naturally into an ordered abstract simplicial complex. In the following we use the natural isomorphisms $\mathrm{C}_k^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \to \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K))$ and $\mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \to \mathrm{C}^k_{\mathrm{simp},\leq}(\mathrm{sd}(K);\mathbb{Z})$ to identify these two (co-) chain complexes and throughout the proof we will work with $\mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K))$ and $\mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K))$.

We set

$$\mu \coloneqq \sum_{\substack{(u_0,\dots,u_n) \text{ ordered} \\ n\text{-simplex of } K}} \mathrm{sign}(u_0,\dots,u_n) \cdot [\underline{\{u_0\}}, \underline{\{u_0,u_1\}}, \dots, \underline{\{u_0,\dots,u_n\}}] \;\in\; \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)).$$

It follows immediately from the Pseudomanifold–Subdivision Lemma 188.9 and the Pseudomanifold Homology Theorem 183.12 that μ is a simplicial cycle that represents the simplicial fundamental class $[\mathrm{sd}(K)] \in \mathrm{H}_n^{\mathrm{simp}}(\mathrm{sd}(K))$.

Next we pick an order "$\leq$" on K and we denote by $\sigma\colon \mathrm{sd}(K) \to K$ the corresponding stretching map, as defined in the Stretching Map Lemma 177.20. Recall that by the Chain Subdivision Lemma 177.25 we know that $\sigma^*\colon \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) \to \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z})$ and $u^*\colon \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \to \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z})$ are inverses of one another. Thus it remains to prove the following claim:[1998]

Claim. The following diagram commutes:[1999]

$$\begin{array}{ccc} \mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z}) & \xrightarrow{\quad\sigma^*\quad} & \mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \\ \Big\downarrow \begin{array}{c}\Sigma a_i s_i^* \\ \downarrow \\ \Sigma a_i s_i^\dagger\end{array} \cong & & \Big\downarrow \cap\mu \\ \mathrm{D}^{\mathrm{simp}}_{n-k}(K) & \hookrightarrow & \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)). \end{array}$$

Proof. The verification of the claim consists of a careful, but at the end of the day elementary, calculation. Note that by the Cochains–Basis Lemma 184.7 (2) it suffices to show that for every ordered k-simplex $s = (v_0,\dots,v_k)$ of K with $v_0 < \dots < v_k$ we have the following equality:

$$\sigma^*(s^*) \cap \mu \;=\; s^\dagger \;\in\; \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)).$$

To verify this equality it is helpful to first do one specific calculation separately. Namely let $(z_0,\dots,z_k)$ be some other ordered k-simplex of S. We perform the following little calculation:

$$\begin{array}{rcl} & & \overset{\text{definition of } \sigma^*}{\downarrow} \\ \sigma^*(s^*)([\underline{\{z_0\}},\dots,\underline{\{z_0,\dots,z_k\}}]) & \overset{\downarrow}{=} & s^*(\sigma_*([\underline{\{z_0\}},\dots,\underline{\{z_0,\dots,z_k\}}])) \\ \uparrow = \begin{cases} (v_0,\dots,v_k)^*(z_0,\dots,z_k), & \text{if } z_0<\dots<z_k, \\ 0, & \text{otherwise} \end{cases} & = & \begin{cases} 1, & \text{if } (v_0,\dots,v_k)=(z_0,\dots,z_k), \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

by the definition of the stretching map we see that $\{\sigma(\underline{\{z_0\}}),\dots,\sigma(\underline{\{z_0,\dots,z_k\}})\}$ is a k-simplex only if $z_0 < \dots < z_k$, furthermore if $z_0 < \dots < z_k$, then $(\sigma(\underline{\{z_0\}}),\dots,\sigma(\underline{\{z_0,\dots,z_k\}})) = (z_0,\dots,z_k)$

[1998] Here, as on many other occasions we will use the remark on page 3354 to make the identification $\mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K)) = \mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}(K);\mathbb{Z})$.

[1999] It follows from the Duality-via-Dual Chains Proposition 188.14 and the formula in the proof of the Simplicial Cap Product–Definition Lemma 184.26 that the two vertical maps are (co-) chain maps "up to the sign $(-1)^k$".

Now we can verify that $\sigma^*(s^*) \cap \mu = s^\dagger$. Indeed:

$$\sigma^*(s^*) \cap \mu \underset{\substack{\uparrow \\ \text{definition of } \mu}}{=} \sigma^*(s^*) \cap \Big(\sum_{\substack{(z_0,..,z_n) \text{ ordered} \\ n\text{-simplex of } K}} \operatorname{sign}(z_0,..,z_n) \cdot [\underline{\{z_0\}}, \underline{\{z_0,z_1\}}, .., \underline{\{z_0,..,z_n\}}] \Big)$$

by Definition 184.25 of the cap product, here we use that $\underline{\{z_0\}} < \underline{\{z_0,z_1\}} < \ldots < \underline{\{z_0,..,z_n\}}$

$$\downarrow$$

$$= \sum_{\substack{(z_0,..,z_n) \text{ ordered} \\ n\text{-simplex of } K}} \operatorname{sign}(z_0,..,z_n) \cdot \sigma^*(s^*)([\underline{\{z_0\}}, .., \underline{\{z_0,..,z_k\}}]) \cdot \underbrace{[\underline{\{z_0,..,z_{k+1}\}}, .., \underline{\{z_0,..,z_n\}}]}_{=G(\{z_0,..,z_k\} \,|\, z_{k+1},..,z_n)}$$

$$= \sum_{\substack{(z_0,..,z_n) \text{ ordered} \\ n\text{-simplex of } K \text{ with} \\ (z_0,..,z_k)=(v_0,..,v_k)}} \operatorname{sign}(z_0,..,z_n) \cdot G(\underbrace{\{v_0,..,v_k\}}_{=s} \,|\, z_{k+1},..,z_n) \underset{\substack{\uparrow \\ \text{by Definition 188.11 of } s^\dagger}}{=} s^\dagger.$$

$\uparrow$ by the above calculation

■

188.5. **The dual chain complex II.** As mentioned in the last section, the following proposition is the last major puzzle-piece in the proof of the Simplicial Poincaré Duality Theorem 188.5.

Proposition 188.16. (Dual Complex–Subdivision–Homotopy Proposition) Let K be a closed n-dimensional simplicial homology manifold.

(1) The inclusion map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ is a chain homotopy equivalence.
(2) For every abelian group G the induced maps $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K) \otimes G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K); G)$ are isomorphisms.

Remark.

(1) Note that in the Dual Complex–Subdivision–Homotopy Proposition 188.16 we finally work with simplicial homology manifolds and not just with pseudomanifolds.
(2) Note that in the Dual Complex–Subdivision–Homotopy Proposition 188.16 we do not demand that K is orientable.
(3) In Section 188.7 we will see that the dual chain complex $\mathrm{D}_*^{\mathrm{simp}}(K)$ can be viewed as the cellular chain complex of a "dual CW-structure" on $|K|$. The proof of the Dual Complex–Subdivision–Homotopy Proposition 188.16 is in fact similar in spirit to the proof, given in the Singular-Cellular-H_*-Proposition 155.8 and the Singular-Cellular Chain Complex Proposition 158.16 (2), that the cellular chain complex of a CW-complex X is chain homotopy equivalent to the singular chain complex of X. □

Definition 188.17. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex.

(1) Given $k \in \mathbb{N}_0$ we define

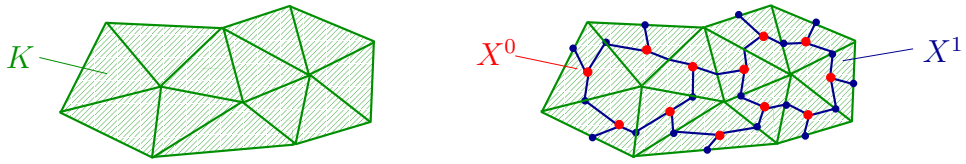
$$X^k \coloneqq \begin{array}{c} \text{all simplices of } \mathrm{sd}(K) \text{ of the form } G(s \,|\, v_1, \ldots, v_l) \\ \text{where } s \text{ is a simplex of } S \text{ of dimension at least } n-k \end{array}$$

It follows immediately from the definitions that X^k can be viewed as a k-dimensional subcomplex of $\mathrm{sd}(K)$ and that $X^n = \mathrm{sd}(K)$.

(2) Let $k \in \mathbb{N}_0$. Similar to the discussion on Definition 155.3 we consider the map

$$\begin{array}{rcl} d_k \colon \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^{k-1}, X^{k-2}) \\ [\sigma] & \mapsto & [\partial_k(\sigma)]. \end{array}$$

where $\sigma \in \mathrm{C}_k^{\mathrm{simp}}(X^k)$



The following lemma can be viewed as a partial analog of the CW-Skeleton-H_*-Lemma 155.2.

Lemma 188.18. (Dual–CW-Skeleton-H_*-Lemma) Let $K = (V, S)$ be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$.

(1) For $i \neq k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X^k, X^{k-1}) = 0$.

(2) The map

$$\begin{array}{rcl} \Theta \colon \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \\ \sigma & \mapsto & [\sigma] \end{array}$$

is well-defined, i.e. every chain $\sigma \in \mathrm{D}_k^{\mathrm{simp}}(K)$ represents a cycle in $\mathrm{C}_k^{\mathrm{simp}}(X^k, X^{k-1})$. Furthermore the map Θ is an isomorphism.

(3) The maps Θ introduced in (2) define an isomorphism of chain complexes

$$(\mathrm{D}_k^{\mathrm{simp}}(K), \partial_k) \to (\mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}), d_k).$$

Proof. Let $K = (V, S)$ be a closed n-dimensional simplicial homology manifold. Given $l \in \mathbb{N}_0$ we denote by S_l the set of l-simplices of K. Given an $(n-k)$-simplex $s \in S_{n-k}$ we define the following:[2000]

$$\begin{array}{rcl} Z(s) & = & \{ \text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t \,|\, v_1, \ldots, v_l) \text{ with } s \subseteq t \} \subseteq X^k \\ \partial Z(s) & = & \{ \text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t \,|\, v_1, \ldots, v_l) \text{ with } s \subsetneq t \} \subseteq X^{k-1}. \end{array}$$



Claim 1. Let s be an $(n-k)$-simplex of K. Given any $l \in \mathbb{N}$ we have

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(Z(s)) = 0 \quad \text{and} \quad \widetilde{\mathrm{H}}_l^{\mathrm{simp}}(\partial Z(s)) \cong \begin{cases} \mathbb{Z}, & \text{if } l = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular $\partial Z(s)$ is a simplicial homology $(k-1)$-sphere.

[2000] Note that it follows immediately from the definitions that $Z(s)$ are precisely the simplices that appear as summands in $s^\dagger \in \mathrm{C}_k^{\mathrm{simp}}(K)$ and that $\partial Z(s)$ are precisely the simplices that appear as summands in $\partial s^\dagger \in \mathrm{C}_{k-1}^{\mathrm{simp}}(K)$.

Proof. Let s be an $(n-k)$-simplex of K. We start out with the following little observation:

(∗) Note that $Z(s)$ and $\mathrm{sd}(\partial s)$ are subcomplexes of $\mathrm{St}(\mathrm{sd}(K), \underline{s})$ and that $\partial Z(s)$ and $\mathrm{sd}(\partial s)$ are subcomplexes of $\mathrm{Lk}(\mathrm{sd}(K), \underline{s})$. It is now elementary to verify that the natural maps on the vertex sets induce simplicial isomorphisms

$$Z(s) * \mathrm{sd}(\partial s) \xrightarrow{\cong} \mathrm{St}(\mathrm{sd}(K), \underline{s}) \quad \text{and} \quad \partial Z(s) * \mathrm{sd}(\partial s) \xrightarrow{\cong} \mathrm{Lk}(\mathrm{sd}(K), \underline{s}).$$

We leave it to the reader to go over the details. It might be helpful to have a look at the figure on page 3621 to do a reality check.

Now let us turn to the actual calculation of the simplicial homology groups of $Z(s)$ and $\partial Z(s)$. Thus let $l \in \mathbb{N}_0$. We perform the following two calculations:

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(Z(s)) \underset{\uparrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(Z(s) * \mathrm{sd}(\partial s)) \underset{\uparrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(\overbrace{\mathrm{St}(\mathrm{sd}(K), \underline{s})}^{=\mathrm{Cone}(\mathrm{Lk}(\mathrm{sd}(K), \underline{s}))}) \underset{\uparrow}{=} 0.$$

since s is an $(n-k)$-simplex we obtain from the Chain Subdivision Lemma 177.25 that $\widetilde{\mathrm{H}}_{n-k-1}^{\mathrm{simp}}(\mathrm{sd}(\partial s)) \cong \mathbb{Z}$ and that $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{sd}(\partial s)) = 0$ for $i \neq n-k-1$, the statement now follows from Exercise 176.11

by (∗)

by the Link Star Lemma 175.9 together with the Cone Homology Lemma 176.19

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(\partial Z(s)) \underset{\downarrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(\partial Z(s) * \mathrm{sd}(\partial s)) \underset{\downarrow}{=} \widetilde{\mathrm{H}}_{l+n-k}^{\mathrm{simp}}(\mathrm{Lk}(\mathrm{sd}(K), \underline{s})) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } l+n-k=n-1, \\ 0, & \text{otherwise.} \end{cases}$$

by (∗)

by our hypothesis that K is an n-dimensional simplicial homology manifold together with the Simplicial Homology Manifold–Characterization Lemma 188.2 we know that $\mathrm{Lk}(\mathrm{sd}(K), \underline{s})$ is an $(n-1)$-dimensional simplicial homology sphere ⊞

Claim 2. Let $k \in \mathbb{N}_0$. The map

$$\bigoplus_{s \in S_{n-k}} j_s \colon \bigoplus_{s \in S_{n-k}} \mathrm{C}_*^{\mathrm{simp}}(Z(s), \partial Z(s)) \to \mathrm{C}_*^{\mathrm{simp}}(X^k, X^{k-1}),$$

induced by the inclusions, is an isomorphism of chain complexes.

Proof. Since each of the inclusion-induced maps $\mathrm{C}_*^{\mathrm{simp}}(Z(s), \partial Z(s)) \to \mathrm{C}_*^{\mathrm{simp}}(X^k, X^{k-1})$ is a chain map we see that the given map is indeed a chain map. Thus it suffices to show that the map is an isomorphism in each degree. Now let $l \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus\limits_{s \in S_{n-k}} \mathrm{C}_l^{\mathrm{simp}}(\partial Z(s)) & \xrightarrow{\alpha} & \bigoplus\limits_{s \in S_{n-k}} \mathrm{C}_l^{\mathrm{simp}}(Z(s)) & \longrightarrow & \bigoplus\limits_{s \in S_{n-k}} \mathrm{C}_l^{\mathrm{simp}}(Z(s), \partial Z(s)) & \longrightarrow & 0 \\
 & & \downarrow =:\varphi & & \downarrow =:\psi & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_l^{\mathrm{simp}}(X^{k-1}) & \longrightarrow & \mathrm{C}_l^{\mathrm{simp}}(X^k) & \longrightarrow & \mathrm{C}_l^{\mathrm{simp}}(X^k, X^{k-1}) & \longrightarrow & 0.
\end{array}$$

$$\mathrm{C}_l^{\mathrm{simp}}(X^k) \xrightarrow{p} \bigoplus_{s \in S_{n-k}} \bigoplus_{(v_0, \ldots, v_l)} \mathbb{Z} \cdot G(s | v_0, \ldots, v_l)$$

We make the following clarifications and observations:

(a) The horizontal sequences are evidently exact.
(b) The vertical maps between the first and second row are induced by the inclusions $\partial Z(s) \to X^{k-1}$ and $Z(s) \to X^k$.
(c) It is basically clear that the middle vertical map ψ is an epimorphism.

(d) Let $G(t \mid v_0, \dots, v_l)$ be an l-simplex in X^{k-1}. By definition of X^{k-1} this means that t is a simplex of K of dimension $\geq n-(k-1)$. For any $(n-k)$-simplex s with $s \subseteq t$ we have by definition that $G(t \mid v_0, \dots, v_l)$ is a simplex of $\partial Z(s)$. This shows that the left vertical map φ is also an epimorphism.
(e) The vertical map p is the natural projection given by the fact that the bottom group is generated by a subset of the set of l-simplices of X^k.
(f) It follows easily from the definitions that $\ker(p \circ \psi) \subseteq \operatorname{im}(\alpha)$.
(g) It follows from (f) that $\ker(\psi) \subseteq \operatorname{im}(\alpha)$.

It now follows from (a), (c), (d) and (g) together with an elementary diagram chase, see Exercise 145.9, that the right vertical map is indeed an isomorphism. ⊞

Claim 3. For every ordered $(n-k)$-simplex s of K we have $\mathbb{Z} \cdot [s^\dagger] = \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s))$.

Proof. Let $s = (v_0, \dots, v_{n-k})$ be an ordered $(n-k)$-simplex of K. By the PL-Simplicial-Pseudo-Manifold Proposition 188.3 we know that K is a closed n-dimensional pseudomanifold. By definition this implies that s is the face of an n-simplex. We pick vertices $w_1, \dots, w_k$ of K such that $(v_0, \dots, v_{n-k}, w_1, \dots, w_k)$ is an n-simplex of K. Next we consider the following diagram:

$$\begin{array}{ccccccc}
\overbrace{\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(Z(s))}^{=0 \text{ by Claim 1}} & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s)) & \xrightarrow[\cong]{\partial_k} & \overbrace{\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s))}^{\cong \mathbb{Z} \text{ by Claim 1}} & \longrightarrow & \overbrace{\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(Z(s))}^{=0 \text{ by Claim 1}} \\
 & & \Big\uparrow{\scriptstyle n \mapsto n \cdot [s^\dagger]} \quad \Big\uparrow{\scriptstyle =:\varphi} & & \Big\downarrow{\scriptstyle \iota} & & \\
 & & \mathbb{Z} & \xleftarrow[\text{projection } p]{} & \widetilde{\mathrm{C}}_{k-1}^{\mathrm{simp}}(\partial Z(s)). & &
\end{array}$$

We make the following clarifications and observations:

(a) It follows from Claim 1 that the connecting homomorphism ∂_k on the top is an isomorphism.
(b) Note that $\partial Z(s)$ is a $(k-1)$-dimensional simplicial complex. Therefore we have the *inclusion* $\iota \colon \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s)) \to \widetilde{\mathrm{C}}_{k-1}^{\mathrm{simp}}(\partial Z(s))$.
(c) It follows immediately from the definitions that $\partial s^\dagger$ lies in $\mathrm{C}_{k-1}^{\mathrm{simp}}(\partial Z(s))$. In particular $s^\dagger$ defines an element $\mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s))$. In other words, the vertical map φ is defined.
(d) In the following we denote by $p \colon \mathrm{C}_{k-1}^{\mathrm{simp}}(\partial Z(s)) \to \mathbb{Z}$ the projection from the free abelian group $\mathrm{C}_{k-1}^{\mathrm{simp}}(\partial Z(s))$ onto the $G(s \cup \{w_{k+1}\} \mid w_{k+2}, \dots, w_n)$-coefficient.
(e) It follows easily from the Dual Chain–Boundary Lemma 188.13 (1) together with the various definitions that the map $p \circ \iota \circ \partial_k \circ \varphi \colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by ± 1.
(f) By Claim 1 we have $\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s)) \cong \mathbb{Z}$.
(g) It follows easily from (e) and (f) that $\partial_k \circ \varphi$ is an isomorphism. Together with (a) we see that φ is an isomorphism. ⊞

As the reader will have noticed, the above claims basically take care of the three statements of the proposition:

(1) It follows from Claim 1 and 2 that for $i \neq k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X^k, X^{k-1}) = 0$.

(2),(3) We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{D}_k^{\mathrm{simp}}(K) & \xrightarrow[\cong]{s^\dagger \mapsto [s^\dagger]} & \bigoplus_{s\in S_{n-k}} \mathrm{H}_k^{\mathrm{simp}}(Z(s),\partial Z(s)) & \xrightarrow[\cong]{} & \mathrm{H}_k^{\mathrm{simp}}(X^k,X^{k-1}) \\ \downarrow{\scriptstyle \partial_k} & & & & \downarrow{\scriptstyle d_k} \\ \mathrm{D}_{k-1}^{\mathrm{simp}}(K) & \xrightarrow[\cong]{s^\dagger \mapsto [s^\dagger]} & \bigoplus_{s\in S_{n-k+1}} \mathrm{H}_{k-1}^{\mathrm{simp}}(Z(s),\partial Z(s)) & \xrightarrow[\cong]{} & \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1},X^{k-2}). \end{array}$$

We make the following clarifications and observations:

(a) By the Dual Chain Lemma 188.12 we know that the $\{s^\dagger\}_{s\in S_{n-l}}$ form a basis for each $\mathrm{D}_l^{\mathrm{simp}}(K)$.

(b) The composition of the horizontal maps equals the map Θ that, for $j = k-1, k$ is induced by the maps $\mathrm{D}_j^{\mathrm{simp}}(K) \to \mathrm{C}_j^{\mathrm{simp}}(X^k) \to \mathrm{C}_j^{\mathrm{simp}}(X^k, X^{k-1})$.

(c) It follows immediately from the definitions that the diagram commutes.

(d) By Claim 3 we know that the left horizontal maps are isomorphisms and by Claim 2 we know that the right horizontal maps are isomorphisms.

This discussion shows that the maps

$$\begin{array}{rccc} \Theta\colon & \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k,X^{k-1}) \\ & \sigma & \mapsto & [\sigma] \end{array}$$

define an isomorphism $(\mathrm{D}_*^{\mathrm{simp}}(K),\partial_*) \to (\mathrm{H}_k^{\mathrm{simp}}(X^k,X^{k-1}),d_k)_{k\in\mathbb{N}_0}$ of chain complexes. ■

The following lemma is also a partial analog of the CW-Skeleton-H_*-Lemma 155.2.

Lemma 188.19. (Simplicial CW-Skeleton-H_*-Lemma) Let $K = (V,S)$ be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$.

(1) For every $i > k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X^k) = 0$.

(2) For every $i < k$ the inclusion-induced map $\mathrm{H}_i^{\mathrm{simp}}(X^k) \to \mathrm{H}_i^{\mathrm{simp}}(\mathrm{sd}(K))$ is an isomorphism.

Proof.

(1) Note that X^k is an abstract simplicial complex of dimension k. It follows immediately from the definition of simplicial homology that $\mathrm{H}_i^{\mathrm{simp}}(K) = 0$ for $i > k$.

(2) We fix $i \in \mathbb{N}_0$. Let $m > i$. We consider the following excerpt from the long exact sequence in simplicial homology of the pair (X^{m+1}, X^m):

$$\ldots \to \underbrace{\mathrm{H}_{i+1}^{\mathrm{simp}}(X^{m+1},X^m)}_{\substack{=0 \text{ by the Dual–CW} \\ \text{Skeleton-H}_*\text{-Lemma 188.18 (1)} \\ \text{since } i<m}} \to \mathrm{H}_i^{\mathrm{simp}}(X^m) \to \mathrm{H}_i^{\mathrm{simp}}(X^{m+1}) \to \underbrace{\mathrm{H}_i^{\mathrm{simp}}(X^{m+1},X^m)}_{\substack{=0 \text{ by the Dual–CW} \\ \text{Skeleton-H}_*\text{-Lemma 188.18 (1)} \\ \text{since } i<m}} \to \ldots$$

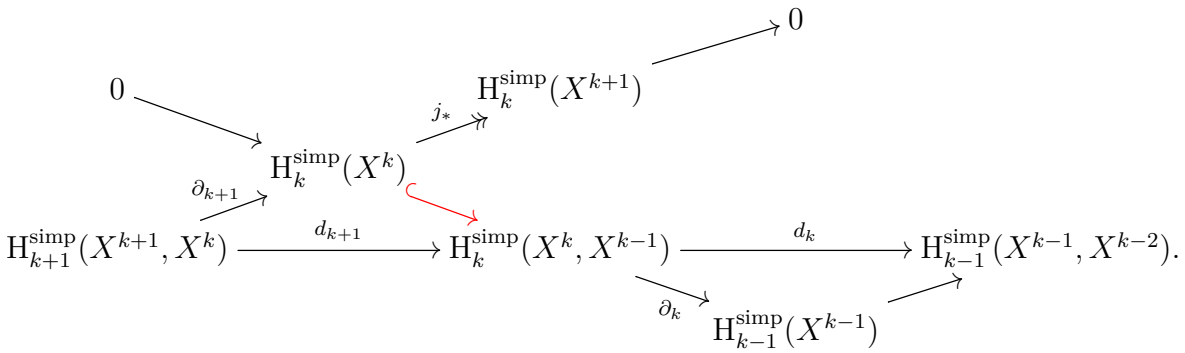
Now let $k > i$. The above discussion immediately implies that the following inclusion-induced maps are isomorphisms: $\mathrm{H}_i^{\mathrm{simp}}(X^k) \xrightarrow{\cong} \mathrm{H}_i^{\mathrm{simp}}(X^{k+1}) \xrightarrow{\cong} \ldots \xrightarrow{\cong} \mathrm{H}_i^{\mathrm{simp}}(X^n)$. The desired statement follows from the fact that $X^n = \mathrm{sd}(K)$. ■

Now we are finally in a position to provide the long overdue proof of the Dual Complex–Subdivision–Homotopy Proposition 188.16.

Proof of the Dual Complex–Subdivision–Homotopy Proposition 188.16. Let K be a closed n-dimensional simplicial homology manifold and let G be an abelian group. Recall that we need to prove the following two statements:

(1) The inclusion map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ is a chain homotopy equivalence.
(2) The induced maps $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K) \otimes G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K); G)$ are isomorphisms.

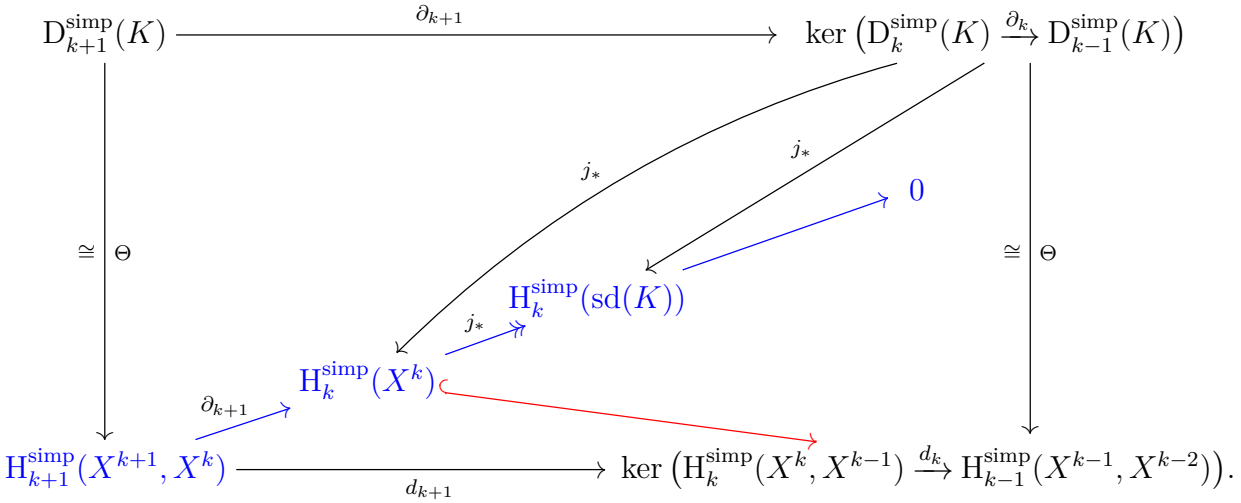
Similarly to the proof of the Singular-Cellular-H_*-Proposition 155.8 we consider the following diagram:



We make the following observations:

(1) It follows from the Simplicial Homology–LES Lemma 177.18, i.e. from the long exact sequence in simplicial homology, together with the Dual–CW-Skeleton-H_*-Lemma 188.18 (1) and the Simplicial CW-Skeleton-H_*-Lemma 188.19 (1) that all diagonal sequences are exact.
(2) It follows from the definition of d_k that the map $\mathrm{H}_k^{\mathrm{simp}}(X^k) \to \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1})$ takes values in $\ker(d_k \colon \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \to \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}))$.

We move on to the main diagram of this proof:



We make the following clarification and observations:

(3) All the maps called j_* are induced by inclusions on the level of chain groups. In fact note that every map in the diagram is either induced by taking boundaries or it is

induced by an inclusion. It follows almost immediately from this observation that the diagram commutes.

(4) The vertical maps are isomorphisms by the Dual–CW-Skeleton-H_*-Lemma 188.18.

(5) By (1) the map $\mathrm{H}_k^{\mathrm{simp}}(X^k) \to \ker(d_k \colon \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \to \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}))$ is a monomorphism.

(6) It follows from (1), together with the Simplicial CW-Skeleton-H_*-Lemma 188.19, that the diagonal blue sequence

$$\mathrm{H}_{k+1}^{\mathrm{simp}}(X^{k+1}, X^k) \xrightarrow{\partial_{k+1}} \mathrm{H}_k^{\mathrm{simp}}(X^k) \xrightarrow{j_*} \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K)) \to 0$$

is exact.

An amusing little diagram chase now shows that the sequence

$$\mathrm{D}_{k+1}^{\mathrm{simp}}(K) \xrightarrow{\partial_{k+1}} \ker\left(\partial_k \colon \mathrm{D}_k^{\mathrm{simp}}(K) \to \mathrm{D}_{k-1}^{\mathrm{simp}}(K)\right) \xrightarrow{j_*} \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K)) \to 0$$

is also exact. In other words we have proved the following statement:

(0) The inclusion map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ induces for each $k \in \mathbb{N}_0$ an isomorphism $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K)) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K))$.

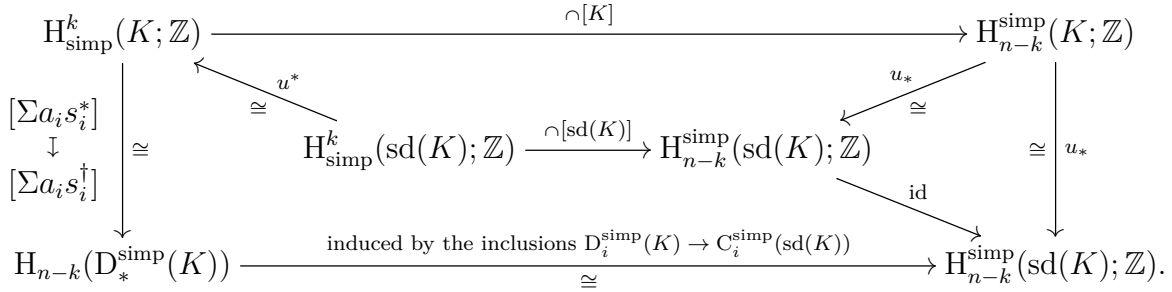
Now let us turn to the two statements we actually need to prove.

(1) It follows from (0) and the Quasi Iso–Homotopy Equivalence Proposition 158.8 that the chain map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ is in fact a chain homotopy equivalence.

(2) Finally note that it follows either from (1), or from (0) together with the Algebraic Universal Coefficient Theorem 162.15 and the Five Lemma 145.11, that the homomorphisms $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K); G) \to \mathrm{H}_*^{\mathrm{simp}}(\mathrm{sd}(K); G)$ are also isomorphisms. Alternatively we could have carried through the above arguments with G-coefficients without any hiccup. ∎

188.6. **The proof of the Simplicial Poincaré Duality Theorem 188.5 .** In this section we combine the results from the previous sections to finally provide the proof of the Simplicial Poincaré Duality Theorem 188.5. For the most part we are now just saying that the program, set out on page 3618, has been completed successfully.

Let K be a closed n-dimensional pseudomanifold. First we assume that K is orientable and that we are given an orientation for K. In the following we will only work with the commutative ring $R = \mathbb{Z}$. The proof for the general case is basically the same, except that one has to carry slightly more notation around. As on page 3618 we consider the following diagram:



In the last sections we have shown the following:

(1) By the Duality-via-Dual Chains Proposition 188.14 we now know that the left vertical map is an isomorphism.
(2) By the Chain Subdivision Lemma 177.25 we know that the subdivision maps u_* and u^* are isomorphisms.
(3) By the Pseudomanifold–Subdivision Lemma 188.9 (3) we know that the upper quadrilateral commutes.
(4) By the Subdivision-Duality-Cap-Lemma 188.15 we know that the lower left part of the diagram commutes.
(5) Finally we now know by the Dual Complex–Subdivision–Homotopy Proposition 188.16 that the bottom horizontal map is an isomorphism.

The combination of the above five statements implies that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [K] \end{array}$$

is, as promised and desired, an isomorphism.

Finally suppose that we are dealing with any closed n-dimensional pseudomanifold, in other words we drop the hypothesis that K is orientable. We need to show that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{F}_2) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [K]_{\mathbb{F}_2} \end{array}$$

is an isomorphism. The argument is almost identical to the previous argument. In fact the only place where we ever really used the orientation was in the definition of $s^\dagger$ in Definition 188.11 and the resulting homomorphism $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z}) \to \mathrm{D}^{\mathrm{simp}}_k(K)$ that is given by $s^* \mapsto s^\dagger$. A little bit of thought shows that the same recipe defines a homomorphism $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{F}_2) \to \mathrm{D}^{\mathrm{simp}}_k(K) \otimes \mathbb{F}_2$ and the argument of the Duality-via-Dual Chains Proposition 188.14 shows that this map is actually an isomorphism of cochain complexes. In the remainder of the argument we did not really use the orientation, so these parts stay virtually untouched. For example the Dual Complex–Subdivision–Homotopy Proposition 188.16 says that the maps $\mathrm{H}_{n-k}(\mathrm{D}^{\mathrm{simp}}_*(K) \otimes \mathbb{F}_2) \to \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{F}_2)$ are isomorphisms. We leave it to the reader to turn this sketch of a proof into a proper argument.

188.7. **The dual CW-structure of a smooth manifold.** Let M be a closed smooth manifold that is equipped with a smooth simplicial structure $(K = (V,S), \Theta\colon |K| \to M)$. Some of the figures above, e.g. the figures on pages 3614 and 3621, suggest that the dual chains correspond to an alternative decomposition of M into cells. As we will see shortly, that is indeed the case.

First we recall some of the notation that we introduced earlier.

Notation 188.20. Let $K = (V,S)$ be an n-dimensional pseudomanifold.

(1) Given $m \in \mathbb{N}_0$ we denote by S_m the set of m-simplices of K.
(2) Let $s \in S_m$ be an m-simplex and let $v_1, \ldots, v_l \in V$ such that $s \cup \{v_1, \ldots, v_l\} \in S_{m+l}$. As in Definition 188.10 we consider the following ordered l-simplex of $\mathrm{sd}(K)$:

$$G(s \,|\, v_1, \ldots, v_l) \;\coloneqq\; \big(\underline{s}, \underline{s \cup \{v_1\}}, \ldots, \underline{s \cup \{v_1, \ldots, v_l\}}\big).$$

(3) Let $k \in \mathbb{N}_0$ and let $s \in S_{n-k}$ be an $(n-k)$-simplex. As in Section 188.5 we consider

$$Z(s) \coloneqq \{ \text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t \,|\, v_1, \ldots, v_l) \text{ with } s \subseteq t \}.$$

Furthermore, as in Definition 188.17 we consider the following:

$$X^k \coloneqq \text{ the subcomplex of } \mathrm{sd}(K) \text{ given by all } Z(s) \text{ with } s \in S_{n-k}.$$

Now we can formulate the main result of this section.

Proposition 188.21. (Dual CW-Structure Proposition) Let M be a closed n-dimensional smooth manifold. Furthermore let $(K = (V, S), \Theta \colon |K| \to M)$ be a smooth simplicial structure.

(1) For each $(n-k)$-simplex s we have a homeomorphism $|Z(s)| \cong \overline{B}^k$.
(2) There exists a CW-structure on M such that for each $k \in \mathbb{N}_0$ the following holds:
  (a) The k-cells are given by the $|Z(s)|$ with $s \in S_{n-k}$.
  (b) The k-skeleton is given by $|X^k|$.
(3) If we equip M with the CW-structure from (2), then there exists a natural isomorphism $\mathrm{D}_k^{\mathrm{simp}}(K) \to \mathrm{C}_k^{\mathrm{CW}}(M)$ of chain complexes.

Definition 188.22. We refer to the CW-structure described in the Dual CW-Structure Proposition 188.21 as the dual CW-structure of the smooth manifold.



Sketch of proof. Let M be a closed n-dimensional smooth manifold. Furthermore let $(K = (V, S), \Theta \colon |K| \to M)$ be a smooth simplicial structure. Note that we know by the Smooth-implies-PL Theorem 180.16 that $(K = (V, S), \Theta \colon |K| \to M)$ is a closed n-dimensional PL-manifold. This implies that we might as well prove the statement under the hypothesis that we are dealing with a closed n-dimensional PL-manifold. Furthermore note that it follows immediately from the definition of a PL-manifold, see Definition 180.8, together with the SD-Lemma 175.5 that the barycentric subdivision $\mathrm{sd}(K)$ is also a closed n-dimensional PL-manifold.

Now let us turn to the proof of the three statement.

(1) One could prove this statement in one go, but it seems didactically opportune to first consider the two simplest cases.
  (a) Let $s = \{v_0\}$ be a 0-simplex of K. We have the following homeomorphisms:

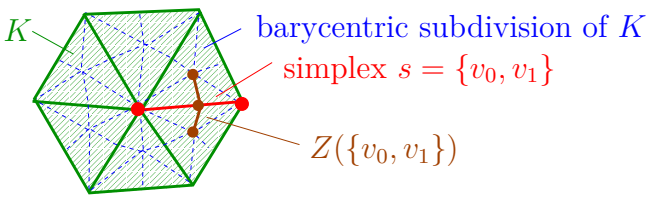
$$|Z(\{v_0\})| \underset{\text{follows immediately from the definitions}}{=} |\mathrm{St}(\mathrm{sd}(K), \underline{\{v_0\}})| \underset{\substack{\text{by the PL-Manifold Characterization Proposition 180.10 since } \mathrm{sd}(K) \text{ is an } n\text{-dimensional PL-manifold}}}{=} |\,\text{PL } n\text{-ball}\,| \underset{\text{since PL-homeomorphic implies homeomorphic}}{\cong} \overline{B}^n.$$

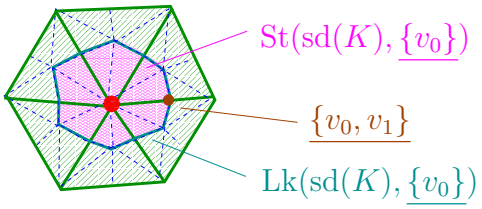
(b) Next let $s = \{v_0, v_1\}$ be a 1-simplex of K. Note that

$$\begin{array}{ccccc} & & \text{follows easily from the definitions} & & \text{by the PL-Manifold Characterization Proposition 180.10 since } \mathrm{sd}(K) \text{ is a closed } n\text{-dimensional PL-manifold} \\ |Z(\{v_0,v_1\})| & \overset{\downarrow}{=} & |\operatorname{St}(\operatorname{Lk}(\mathrm{sd}(K), \underline{\{v_0\}}), \underline{\{v_0,v_1\}})| & \overset{\downarrow}{=} & |\operatorname{St}(\text{PL } (n-1)\text{-sphere}, \underline{\{v_0,v_1\}})| \\ & = & |\operatorname{St}(\text{closed } (n-1)\text{-dimensional PL-manifold}, \underline{\{v_0,v_1\}})| & & \\ & \overset{\uparrow}{\cong} & |\text{PL } (n-1)\text{-ball}| & \overset{\uparrow}{\cong} & \overline{B}^{n-1}. \end{array}$$

as we pointed out on page 3415, it follows from the PL-Manifold–Homeomorphism Proposition 180.9 that a PL $(n-1)$-sphere is a closed $(n-1)$-dimensional PL-manifold

follows from the PL-Manifold Characterization Proposition 180.10 — since PL-homeomorphic implies homeomorphic





(c) Finally let $s = \{v_0, \ldots, v_k\}$ be any k-simplex of K. Similar to (b) we see that $Z(s)$ is the star of an iterated sequence of k links. Applying the PL-Manifold Characterization Proposition 180.10 multiple times we see that $Z(s)$ is homeomorphic to $\overline{B}^{n-k}$. We leave it to the reader to write down this notationally messy argument.

(2) This statement follows fairly immediately from (1). We leave it to the reader to fill in the details.

(3) Given each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{ccccc} \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) & \overset{\downarrow}{\underset{\Xi}{\longrightarrow}} & \mathrm{H}_k(|X^k|, |X^{k-1}|) = \mathrm{C}_k^{\mathrm{CW}}(M). \\ \sigma & \mapsto & [\sigma] & & \end{array}$$

natural isomorphism given by the Simplicial-Singular Homology Lemma 177.19

Note that by the Dual–CW-Skeleton-H_*-Lemma 188.18 we know that the left maps are well-defined and that they form an isomorphism of chain complexes. We obtain immediately from the Simplicial-Singular Homology Lemma 177.19 (3) that the right maps also form an isomorphism of chain complexes. The composition of the two chain isomorphisms thus gives us the promised natural isomorphism of chain complexes. ■

188.8. Exercises for Chapter 188 .

Exercise 188.1. Let T be the simplicial 2-dimensional torus. By Exercise 183.1 we know that the suspension $\Sigma(T)$ is a closed orientable 3-dimensional pseudomanifold. Show that the conclusion of the Simplicial Poincaré Duality Theorem 188.5 does not hold for $\Sigma(T)$.

Exercise 188.2. Let K be a simplicial complex. Show that K is an n-dimensional simplicial homology manifold if and only if the topological realization $|K|$ is an n-dimensional homology manifold. Recall by the definition on page 2873 a topological space X is called

an n-dimensional homology manifold if for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 188.3.

(a) Let M be a closed connected 3-dimensional smooth manifold. Show that if M is non-orientable, then $\mathrm{H}_1(M; \mathbb{Z})$ is infinite.
(b) Is the hypothesis in (a) that M is closed necessary?

Exercise 188.4. Let M and N be two closed orientable connected non-empty 3-dimensional smooth manifolds such that $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Show that for all $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(M; \mathbb{Z}) \cong \mathrm{H}_i(N; \mathbb{Z})$.

Exercise 188.5. Let W be a closed orientable connected 4-dimensional smooth manifold with Euler characteristic $\chi(W) = -3$.

(a) Show that W admits a finite connected covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.
(b) Is the hypothesis in (a) that W is orientable necessary?

Exercise 188.6. Let M be a closed orientable connected odd-dimensional smooth manifold and let $f\colon M \to M$ be a continuous map such that the induced map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is the identity. Show that the Lefschetz number

$$\Lambda(\varphi) \;=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)),$$

equals zero.

Exercise 188.7. Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Suppose that there exists a continuous map $f\colon S^n \to M$ of degree ± 1, i.e. such that $f_*([S^n]) = [M] \in \mathrm{H}_n(M)$. Show that $\mathrm{H}_i(M) = 0$ for $i = 1, \ldots, n-1$.
Remark. In the Degree One-Epimorphism Proposition 186.10 we showed under the same hypotheses that we also have $\pi_1(M) = 0$.

Exercise 188.8.

(1) Give a suitable definition of an n-dimensional simplicial homology manifold *with boundary*.
(2) Let K be a connected orientable n-dimensional simplicial homology manifold with non-empty boundary ∂K. We consider $M \coloneqq K \cup_{\partial K} \mathrm{Cone}(\partial K)$, i.e. we consider the simplicial mapping cone of the inclusion $L \to K$.
   (a) Show that $M = K \cup_{\partial K} \mathrm{Cone}(\partial K)$ is a *closed* orientable n-dimensional pseudomanifold.
   (b) Show that
   $$\mathrm{H}_i(\mathrm{D}_k(M)) \;\cong\; \begin{cases} 0, & \text{if } k = n, \\ \mathrm{H}_i^{\mathrm{simp}}(\mathrm{sd}(K)), & \text{otherwise.} \end{cases}$$
   Remark. This is a non-trivial variation on the Dual Complex–Subdivision–Homotopy Proposition 188.16.
   (c) Use the above together with the Duality-via-Dual Chains Proposition 188.14 and Exercise 184.3 to show that for every $i \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{H}^i_{\mathrm{simp}}(K, \partial K; \mathbb{Z}) \to \mathrm{H}^{\mathrm{simp}}_{n-i}(K; \mathbb{Z})$.

Part XX

Homology groups of topological manifolds

At least this particular author is very interested in topological and smooth manifolds. The theory of smooth manifolds is often times easier than the theory of topological manifolds. For example, using smooth structures, we already introduced and proved the following notions and results:

- In Chapter 29 we introduced the tangent spaces of smooth manifolds.
- In Chapter 31 we used tangent spaces to introduce the notion of an orientation of a smooth manifold.
- In the Smooth Manifold-Simplicial Structure Theorem 179.5 we showed that every smooth manifold admits a (finite) smooth simplicial structure. This result allowed us to prove the Smooth Manifolds-Invariants Proposition 179.10 which says in particular, that for any compact connected smooth manifold M the fundamental group $\pi_1(M)$ is finitely presented and that the homology groups $\mathrm{H}_k(M)$ are finitely generated.
- Let M be an n-dimensional smooth manifold. In the Manifold-Top-H_*-Theorems 183.18 and 183.20 we used the Smooth Manifold-Simplicial Structure Theorem 179.5 to calculate $\mathrm{H}_n(M)$ and $\mathrm{H}_n(M, \partial M)$ in terms of the topology of M.
- In Chapter 185 we used the Manifold-Top-H_*-Theorems 183.18 and 183.20 to introduce the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ of a compact oriented n-dimensional smooth manifold M.
- In Chapter 186 we used fundamental classes to introduce and study the degree of a continuous map between compact oriented smooth manifolds.

In this part of the lecture notes we now turn to the study of the homology groups of *topological* manifolds. We will manage to extend most of the earlier notions and results from smooth manifolds to topological manifolds.

This part of the lecture notes is organized as follows:

- In Chapter 189 we will formulate and state Borsuk's Theorem 189.7. Among many other statements this theorem is essential in the proof of the Topological Manifold-Simplicial Complex Retract Theorem 189.13 which says that every compact topological manifold is the retract of a finite simplicial complex. This result allows us to prove a generalization of the Smooth Manifolds-Invariants Proposition 179.10. More precisely, in the Topological Manifolds-Invariants Proposition 189.14 we will show that for any compact connected topological manifold X the fundamental group $\pi_1(X)$ is finitely presented and that the homology groups $\mathrm{H}_k(X)$ are finitely generated.
- In Chapter 190 we will give a survey on further results on topological manifolds. In particular we will discuss the question whether topological manifolds are actually homotopy equivalent to CW-complexes, we will introduce the Euler characteristic of compact topological manifolds and we will introduce the notion of the tangent microbundle.
- In Chapter 191 we will introduce the notion of an orientation of a topological manifold. We will see that for smooth manifolds this new notion of an orientation is consistent with the earlier notion.
- In Chapter 192 we will study induced orientations. For example we will see, that for an oriented topological manifold, the boundary, any covering and any codimension-zero submanifold inherit a natural orientation. Furthermore we will use the cross product on homology to introduce the product orientation.

- In Chapter 193 we will study the homology groups of topological manifolds. More precisely, given an n-dimensional topological manifold X we will determine in the Manifold-Top-H_*-Theorem 193.1 the homology groups $\mathrm{H}_n(X)$ and $\mathrm{H}_n(X, \partial X)$ in terms of the topology of X.
- In Chapter 194 we will use the Manifold-Top-H_*-Theorem 193.1 to introduce the fundamental class $[X] \in \mathrm{H}_n(X, \partial X)$ of a compact oriented n-dimensional topological manifold.
- In Chapter 195 we will use fundamental classes to introduce and study the degree of a continuous map between compact oriented topological manifolds. We will see that this notion is consistent with the earlier notions of degrees.

CHAPTER 189

Neighborhood retracts and Borsuk's Theorem

In this chapter we will state and prove Borsuk's Theorem 189.7 which will give us many goodies:

(1) Using Borsuk's Theorem we can get some control over the invariants of compact topological manifolds. For example we will be able to show that the homology groups of a compact topological manifold are finitely generated and that the fundamental group is finitely presented.

(2) Borsuk's Theorem will allow us to prove the Lefschetz Fixed Point Theorem 182.15 also for CW-complexes and topological manifolds.[2001]

(3) We will use Borsuk's Theorem in the proof of the Generalized Sphere–Alexander Duality Theorem 218.2 which allows us in particular to calculate the homology groups of topological spaces of the form $S^n \setminus X$ where X is a compact topological manifold.

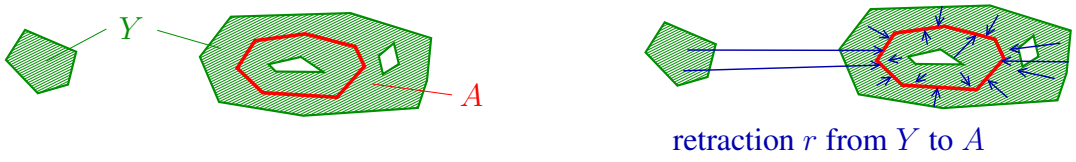
This chapter is organized as follows:

- In Section 189.1 we will introduce and discuss neighborhood retracts.
- In Section 189.2 we will state Borsuk's Theorem 189.7.
- In Section 189.3 we will prove Borsuk's Theorem 189.7.
- In Section 189.4 we will provide applications to topological manifolds.
- In Section 189.5 we will discuss invariants of topological manifolds.
- In Appendix 189.6 we will discuss Euclidean Neighborhood Retracts (ENRs).
- In Appendix 189.7 we will use Borsuk's Theorem 189.7 to extend the Lefschetz Fixed Point Theorem to topological manifolds.
- In Section 189.8 we will provide exercises for Chapter 189.

In the next Chapter 190 we will discuss several advanced topics on topological manifolds.

189.1. **Neighborhood retracts.** We start out by recalling the following excerpts from Definition 19.1 and Definition 90.5:

Definition 189.1. Let Y be a topological space. We say that a subspace $A \subseteq Y$ is a retract of Y if there exists a retraction from Y to A, i.e. if there exists a continuous map $r\colon Y \to A$ with $r(a) = a$ for all $a \in A$.



The following lemma gives the most basic necessary condition for a subspace to be a retract:

[2001] So far, in Chapter 182 we only proved the Lefschetz Fixed Point Theorem for simplicial complexes and smooth manifolds.

Lemma 189.2. (Retract Closed Lemma) Let Y be a topological space and let $A \subseteq Y$ be a retract. If Y is Hausdorff, then $A \subseteq Y$ is closed.

Proof. We already proved this lemma in Exercises 4.31 and 7.24. But lest the reader gets the impression that the author intends to offload all the work to the reader, let us provide a proof.

So let Y be a Hausdorff space, let $A \subseteq Y$ be a retract and let $r\colon Y \to A$ be a retraction. We consider the map

$$\begin{array}{rccc} f\colon & Y & \to & Y \times Y \\ & x & \mapsto & (x, r(x)). \end{array}$$

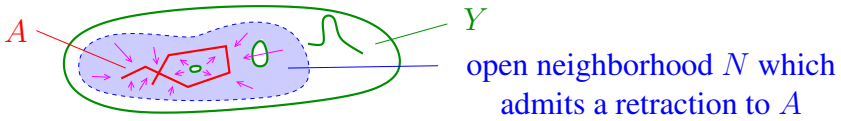
We make the following observations:

(1) Since r is continuous it follows easily from the Topological-Product Proposition 7.2 (1∗) that f is continuous.
(2) We consider the diagonal $\Delta = \{(y, y) \mid y \in Y\} \subseteq Y \times Y$. Since Y is Hausdorff we obtain from the Hausdorff-Diagonal Lemma 7.20 that Δ is a closed subset of $Y \times Y$.
(3) It follows easily from the definition of a retraction that $f^{-1}(\Delta) = A$.
(4) Since f is continuous and since Δ is closed we obtain from the Continuity-via-Closed Subsets Lemma 4.5 that $A = f^{-1}(\Delta)$ is indeed a closed subset of Y. ■

The functoriality of the fundamental group, of higher homotopy groups and of homology groups gave us many tools for showing that a given subspace is *not* a retract. For example in the Circle–Non Retract Proposition 90.6 we used the functoriality of the fundamental group to show that S^1 is not a retract of $\overline{B}^2$.

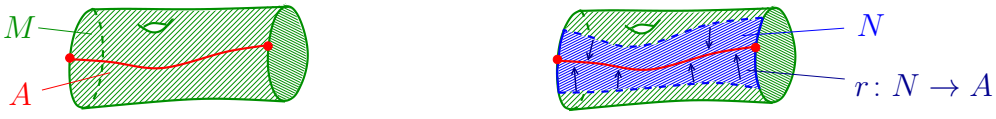
In the following we will not be interested in "global retracts" but in more "local retracts". This leads us to the following definition:

Definition 189.3. Let Y be a topological space and let A be a subset. We denote by $\overline{A}$ the closure of A in Y. We say that A is a neighborhood retract in Y if there exists an open neighborhood N of $\overline{A}$ in Y such that A is a retract of N.[2002]



Examples.

(1) Let M be a smooth manifold and let $A \subseteq M$ be a compact proper smooth submanifold of M. It follows from the Tubular Neighborhood–Existence Theorem 57.11, the Radial Shrinking Lemma 57.8 and the Basics-of-Tubular Neighborhood Lemma 57.10that A is a neighborhood retract in M.



[2002]Note that if there exists a neighborhood N of $\overline{A}$ that admits a retraction to A, then the restriction to the interior of N is also a retraction. This shows that our definition of a neighborhood retract is equivalent to demanding that there exists any neighborhood (not necessarily open) of $\overline{A}$ that admits a retract to A. This formulation is often the definition of a neighborhood retract that is used in the literature. For our discussion it is convenient to already assume that the neighborhood is open.

(2) In the elementary Exercise 4.32 we showed that $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, viewed as a subset of $Y = \mathbb{R}$, is not a neighborhood retract. □

Here is a moderately exciting question:

Question 189.4. Which subsets of topological manifolds are neighborhood retracts?

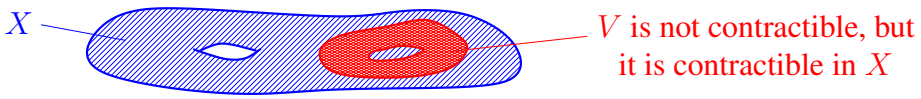
Recall that every topological manifold is by definition Hausdorff and recall that a subspace of a Hausdorff space is again Hausdorff. As we will see in Exercise 189.1, it follows easily from these observations and the Retract Closed Lemma 189.2 that every neighborhood retract A of a topological manifold X is necessarily a closed subset of X.

The above example of $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ shows that being closed is not a sufficient condition for being a neighborhood retract. We will discuss further conditions in the next section.

189.2. **Borsuk's Theorem.** Our attempt to answer Question 189.4 leads us, after some serious pondering (which we skip for time reasons), to the notion of a "weakly locally contractible" topological space. To introduce this new notion we first need to recall two definitions from Definition 4.55 and Definition 19.18 and we need to introduce two related definitions.

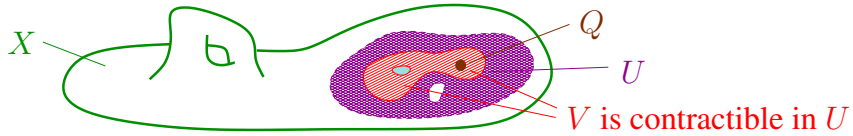
Definition 189.5. Let X be a topological space.

(1) We say that X is contractible if X is homotopy equivalent to a point.
(2) It follows from the Definition 4.55 of a local property of a topological space, that X is locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible.
(3) We say that a subspace V of X is contractible in X if the inclusion map $V \to X$ is homotopic to a constant map.[2003]



Definition 189.5.

(4) We say that X is weakly locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible in U.[2004]



[2003] Note that the Contractability Criterion Lemma 19.20 says that a topological space X is contractible if and only if the identity map is homotopic to a constant map. Thus we see that the subspace X is contractible in X if and only if X itself is contractible.

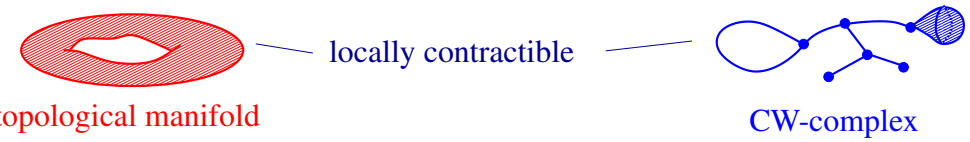
[2004] What we call a "weakly locally contractible" topological space is referred to in many books, see e.g. [**Bor1967, Saka2013**], as a "locally contractible" topological space. Since in Definition 4.55 we defined once and for all what it means that a topological space is "locally P" we stick with our more cumbersome name.

Examples.

(1) (a) It follows almost immediately from the definitions that a topological space that is locally contractible is also weakly locally contractible.

(b) By the Manifold–Local Properties Lemma 23.8 (1) every topological manifold is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. Since non-empty convex subsets of $\mathbb{R}^n$ are contractible we see that every topological manifold is locally contractible and thus weakly locally contractible.

(c) In the CW-Complex–Local Properties Proposition 137.8 (2) we showed that CW-complexes are locally contractible and thus weakly locally contractible. In particular topological graphs are (weakly) locally contractible.[2005]

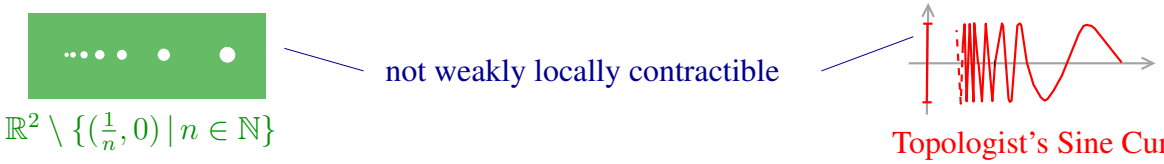


(2) (a) It is easy to show that $\mathbb{R}^2 \setminus \{(\frac{1}{n}, 0) \,|\, n \in \mathbb{N}\}$ is not weakly locally contractible.

(b) In Exercise 189.2 we will see that the Topologist's Sine Curve

$$\{(0, y) \,|\, y \in [-1, 1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \subseteq \mathbb{R}^2,$$

which we had initially introduced on page 195, is not weakly locally contractible.



(c) It is an equally amusing (but not entirely trivial) exercise to come up with a topological space that is weakly locally contractible but not locally contractible. □

The following lemma gives a partial explanation why we introduced the notion of a weakly locally contractible topological space:

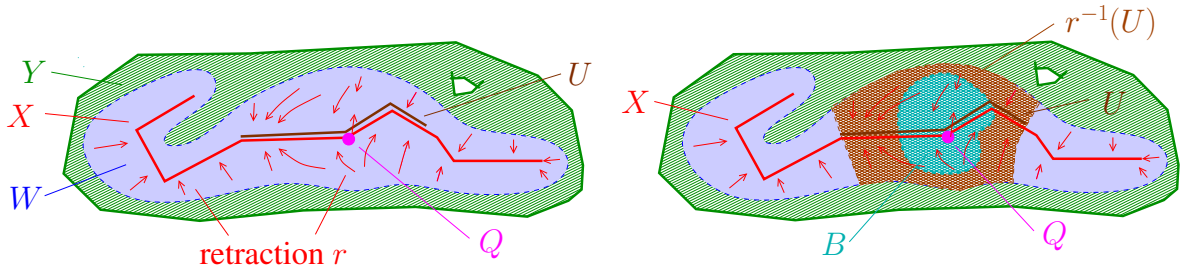
Lemma 189.6. (Manifold Neighborhood Retract–Contractibility Lemma) Let Y be a topological manifold and let X be a subspace of Y. If X is a neighborhood retract in Y, then the topological space X is weakly locally contractible.

Proof. Let Y be a topological manifold and let X be a subspace of Y. Furthermore let W be an open neighborhood of X that admits a retraction $r\colon W \to X$. Let $Q \in X$ and let U be a neighborhood of Q in X.

Claim. There exists an open neighborhood B of $Q \in Y$ with $B \subseteq r^{-1}(U)$ and a homotopy $F\colon B \times [0, 1] \to B$ from the identity to a constant map.

Proof. Recall that by the Manifold–Local Properties Lemma 23.8 we know that Y is locally contractible. It follows from this information, applied to the open neighborhood $r^{-1}(U)$ of Q, that there exists an open contractible neighborhood B of $Q \in Y$ with $B \subseteq r^{-1}(U)$. Since B is contractible we thus obtain from the Contractability Criterion 19.20 the desired homotopy. ⊞

[2005]Note that in Exercise 127.1 we gave a more elementary proof that topological graphs are locally contractible.



Next we consider the map $\begin{array}{rcl}(B\cap U)\times[0,1] & \to & U\\ (x,t) & \mapsto & r(\underbrace{F(x,t)}_{\in B\subseteq r^{-1}(U)}).\end{array}$

Evidently this map is a homotopy from the identity to a constant map. In particular $V := B \cap U$ is contractible in U. ■

As we will see in a minute, rather stunningly for subsets of topological manifolds the converse to the Retract Closed Lemma 189.2 and the Manifold Neighborhood Retract–Contractibility Lemma 189.6 holds.

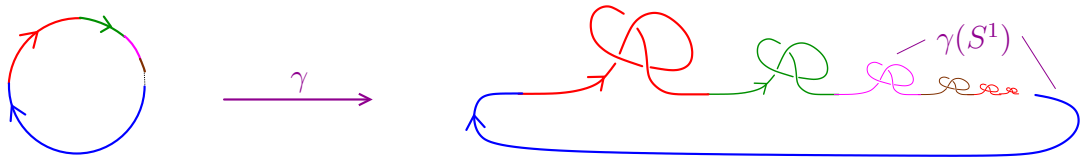
We start out with the important special case of subsets of $\mathbb{R}^n$. The following theorem was first proved by Karol Borsuk [**Bor1932**] in 1932:

Theorem 189.7. (Borsuk's Theorem) Let $n \in \mathbb{N}_0$ and let X be a closed subset of $\mathbb{R}^n$. If the topological space X is weakly locally contractible, then X is a neighborhood retract in $\mathbb{R}^n$.[2006] [2007]

We now proceed as follows:

(1) First we discuss an example.
(2) In the Manifold–Neighborhood Retract Theorem 189.8 we use Borsuk's Theorem 189.7 to also deal with subspaces of arbitrary topological manifolds.
(3) In Section 189.3 we provide the tricky proof of Borsuk's Theorem 189.7.
(4) In Section 189.4 and 189.5 we will discuss applications of Borsuk's Theorem 189.7.
(5) In Appendices 189.6 and 189.7 we will introduce "Euclidean Neighborhood Retracts (ENR)", we will see that topological manifolds are ENRs and, using Borsuk's Theorem 189.7, we will prove the Lefschetz Fixed Point Theorem 189.21 for compact ENRs.

Example. We consider the map $\gamma\colon S^1 \to \mathbb{R}^3$ that is shown in the figure below:



As we discussed on page 786, using the Compact-Hausdorff Proposition 4.23 (3) it is not difficult to show that γ is an embedding. Evidently S^1 is locally contractible and by the

[2006] Is is perhaps worth stressing that the theorem does *not* claim that X is a *deformation retract* of a neighborhood of X.
[2007] Borsuk's Theorem 189.7 can be viewed as a topological analog of the Neighborhood Projection Theorem 35.2.

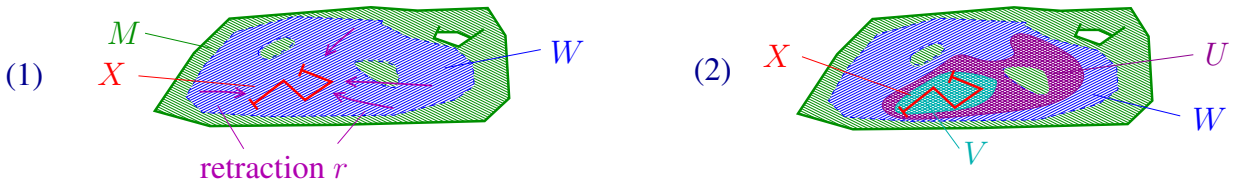
Compact Image Lemma 4.15 together with the Compact-Closed Lemma 3.39 we know that $\gamma(S^1)$ is a closed subset of $\mathbb{R}^3$. Thus it follows from Borsuk's Theorem 189.7 that $\gamma(S^1)$ is a neighborhood retract in $\mathbb{R}^3$. It is a mindboggling exercise to try to visualize an (open) neighborhood of $\gamma(S^1) \subseteq \mathbb{R}^3$ together with a retraction to $\gamma(S^1)$.[2008] □

We conclude this section with a theorem which generalizes the statement of Borsuk's Theorem 189.7 from $\mathbb{R}^n$ to topological manifolds and that furthermore gives another rather technical conclusion. This second conclusion will play a significant role later on in our proof of the General Alexander Duality Theorem 218.2.

Theorem 189.8. (Manifold–Neighborhood Retract Theorem) Let M be a topological manifold and let $X \subseteq M$ be a closed subset that is weakly locally contractible.

(1) X is a neighborhood retract, i.e. there exists an open neighborhood $W \subseteq M$ of X for which there exists a retraction $r\colon W \to X$.[2009]

(2) We can find W and r such that furthermore the following condition is satisfied: Given any open neighborhood U of X there exists an open neighborhood V such that $X \subseteq V \subseteq U \cap W$ with the property that the inclusion $i\colon V \to U$ and the retraction $r\colon V \to X \subseteq U$ are homotopic maps from V to U.



The proof of the Manifold–Neighborhood Retract Theorem 189.8 relies on the following theorem:

Theorem 33.2. (Euclidean Embedding Theorem) Given any topological manifold M (not necessarily compact) there exists an $n \in \mathbb{N}$ and a proper embedding $f\colon M \to \mathbb{H}^n$.

Proof of the Manifold–Neighborhood Retract Theorem 189.8. Let M be a topological manifold and let $X \subseteq M$ be a closed subset that is weakly locally contractible. By the Euclidean Embedding Theorem 33.2 there exists an $n \in \mathbb{N}$ and a proper embedding $f\colon M \to \mathbb{R}^n$ of topological manifolds.[2010] By the definition of a proper map between topological manifolds, see Definition 30.7, we know that $f(M)$ is a closed subset of $\mathbb{R}^n$. Since $X \subseteq M$ is by hypothesis closed it follows from the elementary Open-Closed Inclusion Lemma 4.13 that $f(X)$ is a closed subset of $\mathbb{R}^n$.

The above discussion shows that we only need to consider the case that M is a closed subset of $\mathbb{R}^n$ and that $X \subseteq M$ is a closed subset of $\mathbb{R}^n$.

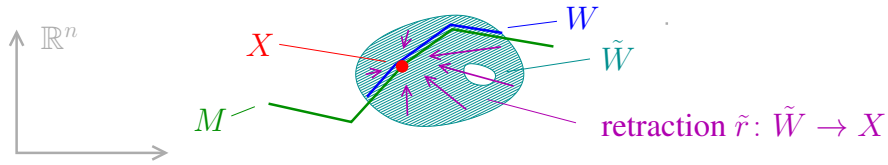
(1) Since X is by hypothesis weakly locally contractible we know by Borsuk's Theorem 189.7 that there exists an open neighborhood $\widetilde{W} \subseteq \mathbb{R}^n$ of $X \subseteq \mathbb{R}^n$ for which

[2008] Note that in Exercise 30.21 we showed that $\gamma(S^1)$ is not a smooth submanifold of $\mathbb{R}^3$ and in the Wild Knot Existence Theorem 218.13 we will show that K is not even a topological submanifold of S^3.

[2009] This statement together with the Manifold Neighborhood Retract–Contractibility Lemma 189.6 and the discussion on page 3636 shows that a subspace X of a topological manifold is a neighborhood retract if and only if X is a closed subset and if X is weakly locally contractible.

[2010] Note that one could also use the Topological Space–Euclidean Embedding Theorem 15.16 together with the Covering Dimension-of-Manifolds Proposition 33.9 to obtain this conclusion.

there exists a retraction $\widetilde{r}\colon \widetilde{W} \to X$. We set $W := \widetilde{W} \cap M$ and $r := \widetilde{r}|_W \colon W \to X$. By the definition of the subspace topology we know that W is indeed an open neighborhood of $X \subseteq M$. Furthermore clearly $r\colon W \to X$ is a retraction. Thus we have shown that X is a neighborhood retract in M.

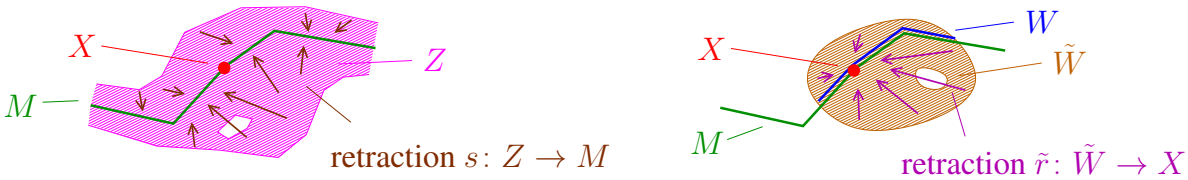


(2) To get the desired more refined statement we now apply Borsuk's Theorem 189.7 not once, but twice:

(a) Since M is a topological manifold we know by the Manifold–Local Properties Lemma 23.8 that M is weakly locally contractible. Since M is a closed subset of $\mathbb{R}^n$ we know by Borsuk's Theorem 189.7 that there exists an open neighborhood $Z \subseteq \mathbb{R}^n$ of $M \subseteq \mathbb{R}^n$ for which there exists a retraction $s\colon Z \to M$.

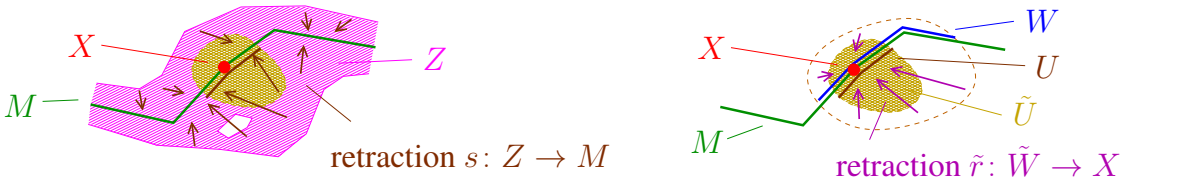
(b) Since X is weakly locally contractible we know again by Borsuk's Theorem 189.7 that there exists an open neighborhood $\widetilde{W} \subseteq \mathbb{R}^n$ of $X \subseteq \mathbb{R}^n$ for which there exists a retraction $\widetilde{r}\colon \widetilde{W} \to X$. After possibly replacing $\widetilde{W}$ by $\widetilde{W} \cap Z$ we can assume that $\widetilde{W} \subseteq Z$.

We set $W := \widetilde{W} \cap M$ and $r := \widetilde{r}|_W \colon W \to X$. By the definition of the subspace topology we know that $\widetilde{W}$ is indeed an open neighborhood of $X \subseteq M$.



Now let us show that the above W and r actually satisfy Statement (2). Thus let U be an open neighborhood of $X \subseteq M$. We need to show that there exists an open neighborhood V with $X \subseteq V \subseteq U \cap W$ such that the inclusion $i\colon V \to U$ and the retraction $r\colon V \to X \subseteq U$ are homotopic maps from V to U.

Claim 1. There exists an open subset $\widetilde{U}$ of $\mathbb{R}^n$ with $X \subseteq \widetilde{U} \cap \widetilde{W}$ and with $s(\widetilde{U}) \subseteq U$.



Proof. By definition of the subspace topology there exists an open subset U' of $\mathbb{R}^n$ with $U' = \widetilde{U} \cap M$. We set $\widetilde{U} := U' \cap \widetilde{W} \cap s^{-1}(U)$. Since clearly $U \subseteq \widetilde{W}$ and $U \subseteq s^{-1}(U)$ we see that $\widetilde{U}$ has all the desired properties. ⊞

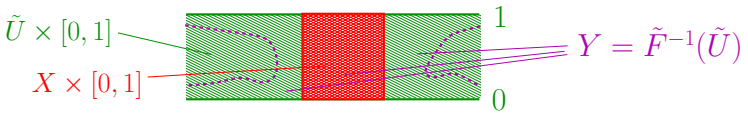
We consider the homotopy

$$\begin{array}{rcl} \widetilde{F}\colon \widetilde{U} \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & (1-t)\cdot x + t \cdot \widetilde{r}(x) \end{array}$$

from the identity $\mathrm{id}_{\widetilde{U}}$ to the retraction $\widetilde{r}\colon \widetilde{U} \to X$. Note that this is a homotopy rel X.

Claim 2. There exists an open neighborhood $\widetilde{V}$ of $X \subseteq \mathbb{R}^n$, that is contained in $\widetilde{U}$, such that $\widetilde{F}(\widetilde{V} \times [0,1]) \subseteq \widetilde{U}$.

Proof. We consider the open subset $Y := \widetilde{F}^{-1}(\widetilde{U}) \subseteq \widetilde{U} \times [0,1]$. Since $X \subseteq \widetilde{U}$ and since $\widetilde{F}$ is a homotopy rel X we see that Y contains $X \times [0,1]$. (For the record we note that Y contains $\widetilde{U} \times \{0,1\}$.)



Since $[0,1]$ is compact and since Y is open we obtain from the Tube Lemma 7.12 that for every $x \in X$ there exists an open neighborhood R_x of $x \in \widetilde{U}$ such that $R_x \times [0,1] \subseteq Y$. We set $\widetilde{V} := \bigcup_{x \in X} R_x$. This is an open neighborhood of $X \subseteq \mathbb{R}^n$ that is contained in $\widetilde{U}$ such that $\widetilde{F}(\widetilde{V} \times [0,1]) \subseteq \widetilde{U}$. ⊞

We set $V := \widetilde{V} \cap M \cap W$. By definition of the subspace topology this is an open neighborhood of $X \subseteq M$. Evidently $X \subseteq V \subseteq U \cap W$. Finally we consider the map

$$\begin{array}{rcl} F\colon V \times [0,1] & \to & M \\ (x,t) & \mapsto & \underbrace{s(\underbrace{\widetilde{F}(x,t)}_{\in \widetilde{U} \subseteq Z})}_{\in U}. \end{array}$$

Note that this map takes values in $s(\widetilde{U}) = U$. Furthermore note that this is a homotopy from $F_0 = i\colon V \to U$ to $F_1 = r\colon V \to U$. We have thus shown that W and r satisfy the desired condition. ■

189.3. **Proof of Borsuk's Theorem 189.7 .** Before we provide the proof of Borsuk's Theorem 189.7 it is convenient to introduce one definition and to prove one lemma.

Definition 189.9. Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. A $\frac{1}{2^k}$-cube is a subset C of $\mathbb{R}^n$ of the form

$$C = \left[\tfrac{a_1}{2^k}, \tfrac{a_1+1}{2^k}\right] \times \cdots \times \left[\tfrac{a_n}{2^k}, \tfrac{a_n+1}{2^k}\right] \quad \text{with } a_1, \ldots, a_n \in \mathbb{Z}.$$

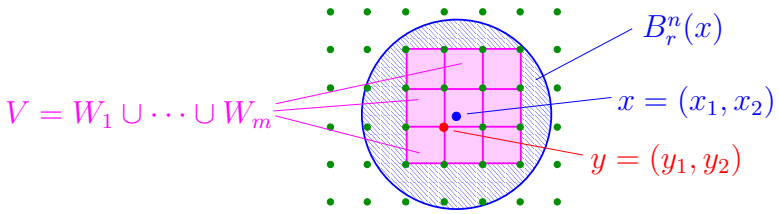
Put differently, a closed $\frac{1}{2^k}$-cube is a cube in $\mathbb{R}^n$ with side length $\frac{1}{2^k}$ and where the coordinates of all vertices lie in $\frac{1}{2^k} \cdot \mathbb{Z}$.

Lemma 189.10. (Point-in-Cubes-in-Ball Lemma) Let $x \in \mathbb{R}^n$ and let $r > 0$. If $k \in \mathbb{N}_0$ is chosen such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$, then there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_m$ such that

$$x \in \text{interior of } (W_1 \cup \cdots \cup W_m) \subseteq W_1 \cup \cdots \cup W_m \subseteq B_r^n(x).$$

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $r > 0$. Furthermore let $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$. We write $\epsilon := \frac{1}{2^k}$. For $i = 1, \ldots, n$ we denote by y_i the result of rounding x_i down to the closest number in $\mathbb{Z} \cdot \epsilon$. Clearly we have

$$\begin{array}{rcl} (x_1, \ldots, x_n) \in [y_1, y_1+\epsilon] \times \cdots \times [y_n, y_n+\epsilon] & \subseteq & (y_1-\epsilon, y_1+2\epsilon) \times \ldots \times (y_n-\epsilon, y_n+2\epsilon) \\ & = & \text{interior of } \underbrace{[y_1-\epsilon, y_1+2\epsilon] \times \ldots \times [y_n-\epsilon, y_n+2\epsilon]}_{=:V}. \end{array}$$



Since V is the union of finitely many ϵ-cubes it suffices to show that V is contained in $B_r^n(x)$. Thus let $y \in V$. We have

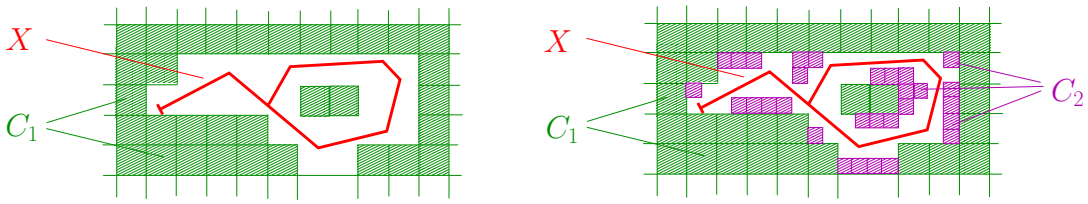
$$d(y,x) \underset{\text{since } x \in V}{\leq} \text{diameter of } V \underset{\text{see page 356}}{=} \sqrt{n} \cdot 3\epsilon = \sqrt{n} \cdot \frac{3}{2^k} \underset{\text{choice of } k}{<} r$$

which means by definition that $y \in B_r^n(x)$. Thus we have shown that $V \subseteq B_r^n(x)$. ■

Now we turn to the actual proof of Borsuk's Theorem 189.7:

Proof of Borsuk's Theorem 189.7. Let X be a closed, weakly locally contractible subset of $\mathbb{R}^n$. It follows from the discussion in Footnote 2002 that it suffices to show that there exists a neighborhood (not necessarily open) W of X in $\mathbb{R}^n$ that admits a retraction $W \to X$. Evidently we only have to consider the case that X is non-empty. We introduce the following notation:

(1) We define C_1 to be the union of all 1-cubes that do not intersect X.
(2) We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\overset{\circ}{C}_1$ and that do not intersect X.
(3) We iterate this process, i.e. for any $k \geq 3$ we define C_k to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that do not intersect X.



Claim 1.
(1) We have $\bigcup_{k \in \mathbb{N}} C_k = \mathbb{R}^n \setminus X$.
(2) Given any $x_0 \in \mathbb{R}^n \setminus X$ there exists an open neighborhood U that intersects only finitely many cubes of $C_1, C_2, \ldots$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus X$. Since X is closed we know that $\mathbb{R}^n \setminus X$ is open. This means that there exists an $r > 0$ such that $B_r^n(x_0) \subseteq \mathbb{R}^n \setminus X$. We pick $k \in \mathbb{N}_0$ such that $\frac{3 \cdot \sqrt{n}}{2^k} < r$. By the Point-in-Cubes-in-Ball Lemma 189.10 we can find finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_d$ such that

$$x_0 \in \underbrace{\text{interior of } W_1 \cup \cdots \cup W_d}_{:=U} \subseteq B_r^n(x).$$

This implies in particular that x_0 is contained in C_k or it is contained already in one of $C_1, \ldots, C_{k-1}$.

Subclaim. If there exists an $m \geq k$ and a $\frac{1}{2^m}$-cube in $\mathbb{R}^n \setminus X$ which intersects U, there exists already a $\frac{1}{2^k}$-cube in $\mathbb{R}^n \setminus X$ which intersects U.

Proof of subclaim. First note that an elementary argument shows that

$$U \;=\; (W_1 \cup \cdots \cup W_d) \setminus \text{the union of all other } \tfrac{1}{2^k}\text{-cubes.}$$

Now let Y be a $\frac{1}{2^m}$-cube in $\mathbb{R}^n \setminus X$ which intersects U, where $m \geq k$. Note that there exists a unique $\frac{1}{2^k}$-cube Z such that $Y \subseteq \mathring{Z}$. Note that this implies that $Y \subseteq Z$.



It follows from this discussion and the above description of U that Z is one of the W_i. The subclaim now follows from the observation that each W_i intersects U. □

Finally note that it follows from this subclaim and the iterative definition of the C_i that U intersects only finitely many cubes of $C_1, C_2, \ldots$. ⊞

Claim 2. The topological space $\mathbb{R}^n \setminus X = C \coloneqq \bigcup_{k\in\mathbb{N}} C_k$ admits a CW-structure that restricts to the standard CW-structure on each cube.

Proof. We view each C_i as a CW-complex in the obvious way: The j-skeleton of C_i is given by all points of C_i such that at most j coordinates lie outside $\mathbb{Z} \cdot \frac{1}{2^k}$. One can easily verify that with this CW-structure the n-cells correspond precisely to the $\frac{1}{2^i}$-cubes and the lower dimensional cells correspond precisely to the faces of the cubes. Furthermore one can easily show, say using the Regluing Lemma 8.17 and the CW-Gluing Proposition 139.9 that for each k the union $C_1 \cup \cdots \cup C_k$ is also a CW-complex in an obvious way and finally note that we can also view the union $C \coloneqq \bigcup_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$ as a CW-complex with the obvious filtration.[2011] ⊞

We continue with the following definition:

- Given $x, y \in \mathbb{R}^n$ we write $d(x, y) \coloneqq \|x - y\|$.
- Given $x \in X$ and given a subset $\varnothing \neq Y \subseteq \mathbb{R}^n$ we write $d(x, Y) \coloneqq \inf\{d(x, y) \mid y \in Y\}$.
- Given a cell σ of C and a continuous map $f\colon \sigma \to X$ we set[2012] [2013]

$$\rho(f) \;\coloneqq\; \max\{d(z, f(z)) \mid z \in \sigma\}.$$

We think of $\rho(f)$ as the maximal displacement of f.

We start with the iterative definition of a subset of $\mathbb{R}^n$ together with a map to X.

(a) We denote by A_0 the 0-skeleton of C. We define a map $r_0\colon A_0 \to X$. Let $a \in A_0$. Since X is a closed subset of $\mathbb{R}^n$ it follows from the Distance-Minimizer Lemma 10.6 that we can find a point $r_0(a) \in X$ with $d(a, r_0(a)) = d(a, X)$.

[2011] We leave it to the reader to verify, using Statement (2) of the claim (which says that the collection of cubes involved is locally finite) and using the Cover-Open Subset Lemma 3.20 (2), that condition (CW1) of the definition of a CW-complex, see Definition 136.1, is satisfied.

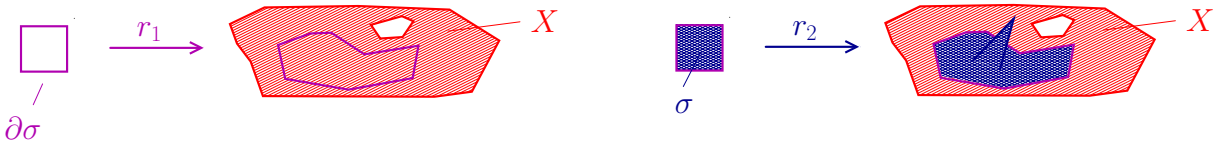
[2012] Given $a, b \in \mathbb{R}^n$ we denote by $d(a, b) = \|a - b\|$ the usual Euclidean distance.

[2013] Note that σ is compact, hence the maximum exists by the Compact Image Lemma 4.15 (2).

(b) Now suppose inductively that we have defined a map $r_i\colon A_i \to X$ on a union A_i of some of the i-cells of C. We define the next set A_{i+1} and the corresponding map $r_{i+1}\colon A_{i+1} \to X$ as follows:

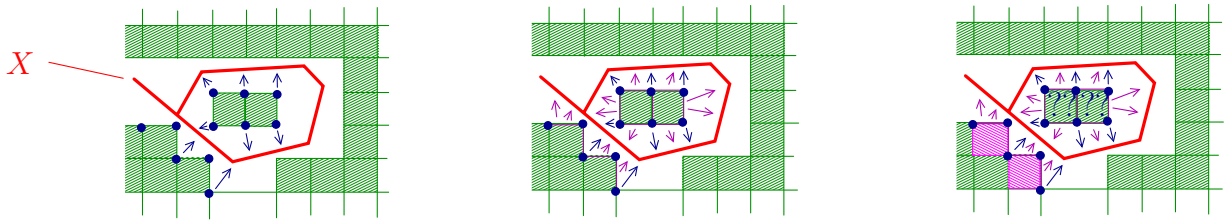
(b_1) Let σ be an $(i+1)$-cell of C for which r_i is defined on $\partial\sigma$. Suppose that r_i can be extended to a continuous map $g\colon \sigma \to X$ that agrees with r_i on $\partial\sigma$. Then we pick a continuous map $r_{i+1}\colon \sigma \to X$ with $r_{i+1}|_{\partial\sigma} = r_i$ such that r_{i+1} has "small displacement", more precisely, such that

$$\rho(r_{i+1}) \;\leq\; 2 \cdot \underbrace{\inf\{\rho(g) \,|\, g\colon \sigma \to X \text{ is a map with } g|_{\partial\sigma} = r_i|_{\partial\sigma}\}}_{\geq d(\sigma, X) \text{ which in turn is } > 0 \text{ by the Disjoint Compact-Closed–Distance Lemma 10.5, here we use again that } X \subseteq \mathbb{R}^n \text{ is closed}}.$$



(b_2) We denote by A_{i+1} the union of all $(i+1)$-cells as in (b_1) and we define r_{i+1} on A_{i+1} as in (b_1).

(c) We set $A \coloneqq \bigcup_{i\in\mathbb{N}} A_i \cup X$ and we define $r\colon A \to X$ to be the unique map that equals r_i on each A_i and that is the identity on X.



It remains to show that the set A is a neighborhood of X and that the map $r\colon A \to X$ is continuous.

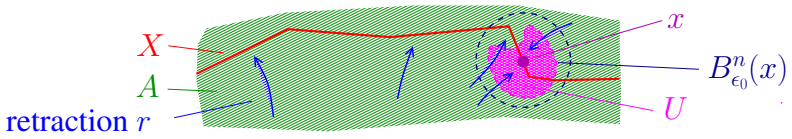
Let us first discuss the idea behind the proof. We start out with two observations:

- We can extend the map r_i over an $(i+1)$-cube σ if the image $r_i|_{\partial\sigma}$ is contractible.
- The extension we picked (assuming it exists) is "close" to the original map if $r_i|_{\partial\sigma}$ is contractible in a small neighborhood of $r_i(\partial\sigma)$.

To get a suitable extension and to get an extension that is close to the original map we will use that X is weakly locally contractible.

We now have to make this idea work. We start out with the following claim, which implies already that A is indeed a neighborhood of X and which will be essential in showing that the map $r\colon A \to X$ is indeed continuous:

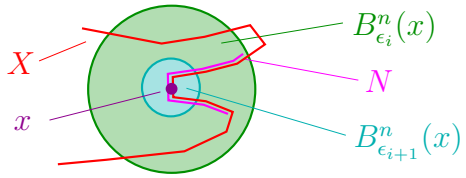
Claim 3. Given any $x \in X$ and any $\epsilon_0 > 0$ there exists an open neighborhood U of x that is contained in A and such that $\|\underbrace{r(x)}_{=x} - r(y)\| < \epsilon_0$ for all $y \in U$, i.e. such that $r(U) \subseteq B^n_{\epsilon_0}(x)$.



Proof. So let $x \in X$ and let $\epsilon_0 > 0$. We adopt the following notation: Given $\mu > 0$ we write $K(\mu) \coloneqq X \cap B^n_\mu(x)$.

Subclaim 1. We can find $\epsilon_0, \dots, \epsilon_{2n} \in \mathbb{R}_{>0}$[2014] with the following two properties:
(1) For each $i \in \{0, \dots, 2n-1\}$ we have $\epsilon_{i+1} < \frac{1}{3} \cdot \epsilon_i$.
(2) For every $i = 1, \dots, 2n$ the subset $K(\epsilon_i)$ is contractible in $K(\epsilon_{i-1})$.

Proof of subclaim. First we set $\epsilon_0 \coloneqq 1$. We now suppose that we have already found $\epsilon_0, \dots, \epsilon_i$ for some $i \in \{0, \dots, 2n-1\}$ with the desired properties. Since X is weakly locally contractible we can find an open neighborhood N of x in X that is contained in $K(\epsilon_i)$ and that is contractible in $K(\epsilon_i)$. Since N is open there exists an $\epsilon_{i+1} < \frac{1}{3}\epsilon_i$ such that $K(\epsilon_{i+1}) \subseteq N$.

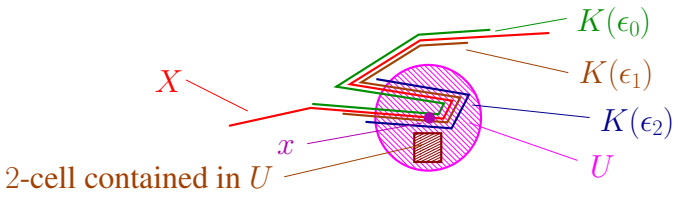


Since N is contractible in $K(\epsilon_i)$ the subset $K(\epsilon_{i+1})$ is also contractible in $K(\epsilon_i)$. □

We set $U \coloneqq B^n_{\frac{1}{4}\epsilon_{2n}}(x)$. It suffices to prove the following two subclaims:

Subclaim 2. For each $i = 0, \dots, n$ the map r is defined for each i-cell σ of the CW-complex C that lies in U and for each such σ we have $r(\sigma) \subseteq K(\epsilon_{2n-2i})$.

Subclaim 3. The union of X with all cells of C that lie in U is a neighborhood of x.



Proof of subclaim 2. We prove Subclaim 2 iteratively for $i = 0, 1, \dots, n$. We start out with $i = 0$. Thus we have to show that $r(U \cap A_0) \subseteq K(\epsilon_{2n})$. Therefore let $y \in U \cap A_0$. Then we have

$$d(x, r(y)) \underset{\uparrow}{\leq} d(x, y) + d(y, r_0(y)) \underset{\uparrow}{\leq} d(x, y) + d(y, x) = 2d(x, y) \underset{\uparrow}{<} 2 \cdot \frac{1}{4} \cdot \epsilon_{2n} = \frac{1}{2} \cdot \epsilon_{2n}.$$

by the triangle inequality and since $r(y) = r_0(y)$ — by (a) since $x \in X$ — since $y \in U = B^n_{\frac{1}{4}\epsilon_{2n}}(x)$

[2014] We only stop at the index $2n$ since that is what we need later on. One could also find an infinite sequence of ϵ_i with the stated properties.

But since $r(y) \in X$ this means precisely that $r(y) \in K(\epsilon_{2n}) = X \cap B^n_{\epsilon_{2n}}(x)$. This shows that Subclaim 1 holds for $i = 0$.

Now suppose that Subclaim 1 holds for some $i \in \{0, \dots, n-1\}$. Let σ be an $(i+1)$-cell of the CW-complex C that is contained in U. Since $r(\partial\sigma) \subseteq K(\epsilon_{2n-2i})$ and since $K(\epsilon_{2n-2i})$ is contractible in $K(2n-2i-1)$ it follows from the Convex-to-Ball Proposition 4.28 and the Homotopy Equivalence-π_n-Proposition 134.17 (2) that we can extend $r|_{\partial\sigma} = r_i|_{\partial\sigma}$ to a continuous map $s\colon \sigma \to K(\epsilon_{2n-2i-1})$. By definition of A_{i+1} this implies that $\sigma \subseteq A_{i+1}$. In particular r is defined on σ. It remains to show that $r(\sigma) \subseteq K(\epsilon_{2n-2i-2})$. Thus let $y \in \sigma$.

We have the following inequalities:

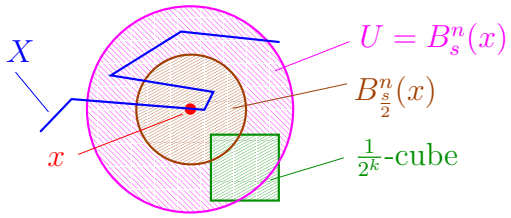
$$d(x, r(y)) \underset{\substack{\uparrow\\ \text{by the triangle inequality and}\\ \text{since } r(y) = r_{i+1}(y)}}{\leq} d(x,y) + d(y, r_{i+1}(y)) \underset{\substack{\downarrow\\ \text{since } y \in \sigma \subseteq U = B^n_{\frac{1}{4}\epsilon_{2n}}(x) \text{ and}\\ \text{by definition of } \rho(r_{i+1})}}{\leq} \tfrac{1}{4}\epsilon_{2n} + \rho(r_{i+1}) \underset{\substack{\uparrow\\ \text{by } (b_1)}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \inf\left\{ \rho(g) \,\middle|\, \begin{array}{c} g\colon \sigma \to X \text{ with} \\ g|_{\partial\sigma} = r_i|_{\partial\sigma} \end{array} \right\}.$$

$$\underset{\substack{\uparrow\\ \text{since } s \text{ is}\\ \text{such a map}}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \underbrace{\max\{d(z, s(z)) \mid z \in \sigma\}}_{=\rho(s)} \underset{\substack{\uparrow\\ \text{by the triangle}\\ \text{inequality}}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \max\{d(z,x) + d(x, s(z)) \mid z \in \sigma\}$$

$$\underset{\substack{\uparrow\\ \text{since } \sigma \subseteq U = B^n_{\frac{1}{4}\epsilon_{2n}}(x) \text{ and}\\ \text{since for each } z \in \sigma \text{ we have}\\ s(z) \in K(\epsilon_{2n-2i-1}) \subseteq B^n_{\epsilon_{2n-2i-1}}(x)}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2\left(\tfrac{1}{4}\epsilon_{2n} + \epsilon_{2n-2i-1}\right) = \tfrac{3}{4}\epsilon_{2n} + 2\epsilon_{2n-2i-1} \underset{\substack{\uparrow\\ \text{since for each } j \text{ we}\\ \text{have } \epsilon_{j+1} \leq \frac{1}{3}\epsilon_j}}{\leq} \tfrac{1}{12}\epsilon_{2n-2i-2} + \tfrac{2}{3}\epsilon_{2n-2i-2} \leq \epsilon_{2n-2i-2}.$$

This concludes the proof that $r(\sigma) \subseteq K(\epsilon_{2n-2i-2})$. □

Proof of subclaim 3. We set $s = \frac{1}{4}\epsilon_{2n}$. As a reminder, we have to show that the union of X with all cells of C that lie in $B^n_s(x)$ is a neighborhood of x. Let W be a $\frac{1}{2^k}$-cube with $W \cap B^n_{\frac{s}{2}}(x) \neq \varnothing$ and $W \cap (\mathbb{R}^n \setminus B^n_s(x)) \neq \varnothing$.



It follows that the diameter of W is at least $\frac{s}{2}$ which in turn implies that $\frac{\sqrt{n}}{2^k} \geq \frac{s}{2}$. We set $k_0 = \lfloor \ln_2(\frac{2\sqrt{n}}{s}) \rfloor$. We see that

$$
\begin{array}{ll}
& \text{union of } X \text{ with all cells of } C \text{ that lie in } B^n_s(x) \\
\supseteq & \text{union of } X \text{ with all cubes of } C \text{ that lie in } B^n_s(x) \\
\supseteq & B^n_s(x) \setminus \text{union of all cubes of } C \text{ that are not completely contained in } B^n_s(x) \\
\uparrow & \\
\multicolumn{2}{l}{\text{since } \mathbb{R}^n \setminus X = C} \\
\supseteq & B^n_s(x) \setminus \text{union of all cubes of } C \text{ that are not completely contained in } B^n_s(x) \\
& \quad \text{which do not contain the point } x \text{ but which intersect } B^n_{\frac{s}{2}}(x) \\
\supseteq & B^n_s(x) \setminus \underbrace{\text{union of all } \tfrac{1}{2^k}\text{-cubes with } k \leq k_0 \text{ of } C \text{ that are not contained in } B^n_s(x)}_{\text{finite union of cubes, hence a closed subset}} \\
\uparrow & \quad \underbrace{\text{which do not contain the original point } x \text{ but which do intersect } B^n_{\frac{s}{2}}(x).}_{\text{finite union of cubes, hence a closed subset}} \\
\multicolumn{2}{l}{\text{definition of } k_0}
\end{array}
$$

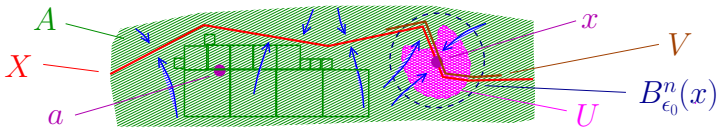
The last set is clearly an (open) neighborhood of x. But then the first set is also a neighborhood of x. This concludes the proof of Subclaim 3 and thus also the proof of Claim 3. ⊞

Recall that we need to show that A is indeed a neighborhood of X and that we need to show that the map $r\colon A \to X$ is indeed continuous. Fortunately it follows immediately from Claim 3 that A is a neighborhood of X. So it remains to prove the following claim:

Claim 4. The map $r\colon A \to X$ is continuous.

Proof. Let V be an open subset of X. We need to show that $r^{-1}(V) \subseteq A$ is open. It follows from the JH-Openness Criterion 3.3 that it suffices to show that for each $a \in A$ there exists an open neighborhood U_a of $a \in A$ with $r(U_a) \subseteq W$.

First we consider the case that $a \notin X$. In this case a is in the interior of the union C of finitely many cubes. By construction the restriction of r to each cube is continuous. It follows from the Pasting Proposition 4.7 that $r|_C\colon C \to X$ is continuous. In particular $r^{-1}(W) \subseteq C$ is open. It follows that $U_a := r^{-1}(W) \cap \mathring{C}$ has the desired properties.



Finally we need to deal with the case $a \in X$. Since V is an open neighborhood of a we see that there exists an $\epsilon_0 > 0$ such that $B_{\epsilon_0}(a) \cap V \subseteq V$. We now obtain the desired open neighborhood U_a of a by appealing to Claim 3. ■

189.4. **Applications to topological manifolds.** In the following we want to study applications of Borsuk's Theorem 189.7. The key to many applications is the following proposition:

Proposition 189.11. (Simplicial Complex Retract Proposition) Let $K \subseteq \mathbb{R}^n$ be a non-empty subspace.

(1) If K is a neighborhood retract, then K is also the retract of a countable n-dimensional (possibly infinite) simplicial complex.
(2) If $K \subseteq \mathbb{R}^n$ is compact and if it is a neighborhood retract, then K is the retract of a finite n-dimensional simplicial complex.

Proof. We prove the proposition in the reverse order:

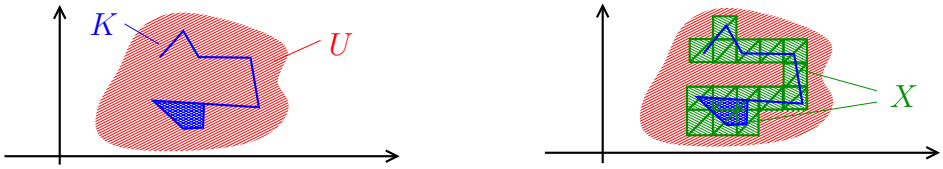
(2) Let K be a compact non-empty subspace of $\mathbb{R}^n$ which is a retract of some open neighborhood $U \subseteq \mathbb{R}^n$. Clearly it suffices to prove the following claim:

Claim. There exists a finite n-dimensional simplicial complex X with $K \subseteq X \subseteq U$.

Proof. The idea is to construct a suitable simplicial complex $X \subseteq \mathbb{R}^n$ as the union of finitely many sufficiently small cubes. To do so we set

$$r \;:=\; d(K, \mathbb{R}^n \setminus U) \;=\; \inf\{d(x,y) \,|\, x \in K, y \in \mathbb{R}^n \setminus U\}.$$

Since K is compact and since $\mathbb{R}^n \setminus U$ is closed it follows from the Disjoint Compact-Closed–Distance Lemma 10.5 that $r > 0$. We pick $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{\sqrt{n}} \cdot r$. We denote by X the union of all $\frac{1}{2^k}$-cubes that intersect K. (See Definition 189.9 for the definition of a $\frac{1}{2^k}$-cube.) It follows easily from the hypothesis that K is compact that X is the union of finitely many $\frac{1}{2^k}$-cubes. The same argument as in the proof of the Point-in-Cubes-in-Ball Lemma 189.10 shows, by our choice of k, that each $\frac{1}{2^k}$-cube is contained in U.



We equip each cube with the standard simplicial structure that we introduced on page 3305. It follows easily from the Compact-Hausdorff Proposition 4.23 (3) that these simplicial structures on the cubes define an n-dimensional simplicial structure on X. ⊞

(1) Let $K \subseteq \mathbb{R}^n$ be a non-empty neighborhood retract. By definition K is a retract of an open non-empty neighborhood U. Thus it remains to prove the following claim:

Claim. Every open non-empty subset U of $\mathbb{R}^n$ admits the structure of a countable n-dimensional simplicial complex.

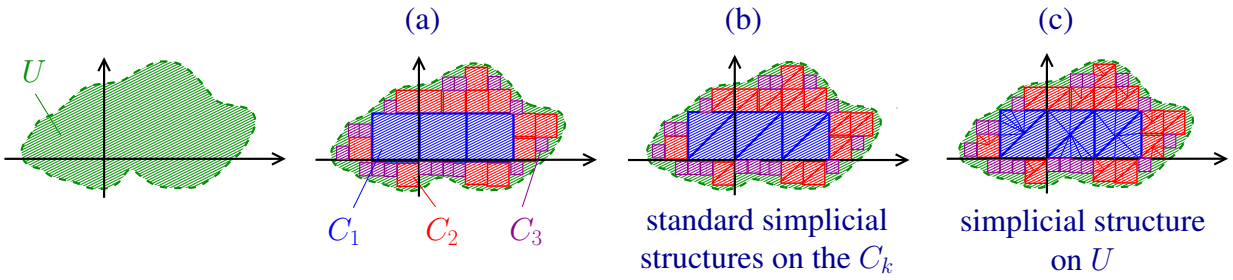
Proof. This claim is precisely the content of the Open Euclidean Subset–Simplicial Structure Lemma 179.13. For the reader's convenience we give an outline of the proof:

(a) We introduce the following subsets of $\mathbb{R}^n$:
- We define C_1 to be the union of all 1-cubes that are contained in U.
- We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\mathring{C}_1$ and that are contained in U.
- We iterate this process, i.e. for any $k \geq 3$ we define C_k to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that are contained in U.

One can easily show (using the same logic as in the proof of Claim 1 on page 3642) that $U = \bigcup_{k \in \mathbb{N}_0} C_k$.

(b) We equip each cube with the standard simplicial structure.

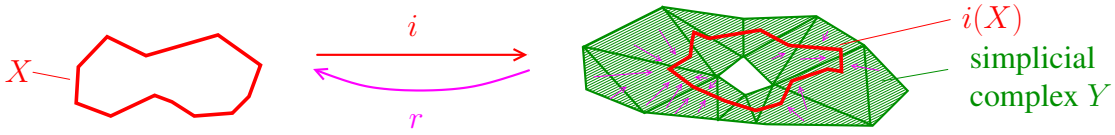
(c) We iteratively subdivide the simplicial structures on $C_1, C_2, \ldots$ so that the simplicial structures match.



This construction defines an abstract simplicial complex K together with a bijective map $f\colon |K| \to U$. It follows relatively easily from the Pasting Proposition 4.7 that the inverse $f^{-1}\colon U \to |K|$ is also continuous. Thus we have shown that $f\colon |K| \to U$ is a homeomorphism. Finally note that it follows easily from the construction that K is in fact a countable simplicial complex and that each simplex has dimension $\leq n$. ■

Before we state the next theorem we need to introduce the following elementary definition:

Definition 189.12. Let X be a topological space and let $\mathcal{C}$ be a class of topological spaces (e.g. $\mathcal{C}$ could be the class of simplicial complexes or CW-complexes). We say that X is a retract of an element in $\mathcal{C}$ if there exists a topological space $Y \in \mathcal{C}$, a continuous map $i\colon X \to Y$ and a continuous map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$.[2015]



We can now formulate the following theorem:

Theorem 189.13. (Topological Manifold-Simplicial Complex Retract Theorem)

(1) Every topological manifold is a retract of a countable finite-dimensional simplicial complex.
(2) Every compact topological manifold is a retract of a finite simplicial complex.

Proof. Let M be a topological manifold. By the Euclidean Embedding Theorem 33.2 we know that there exists an embedding $i\colon M \to \mathbb{R}^n$ such that $i(M)$ is a closed subset of $\mathbb{R}^n$. As we discussed on page 3637, every topological manifold is weakly locally contractible. Thus we can apply Borsuk's Theorem 189.7 to deduce that $i(M) \subseteq \mathbb{R}^n$ is a neighborhood retract. The desired statement now follows from the Simplicial Complex Retract Proposition 189.11. ■

189.5. **Invariants of topological manifolds.** There are several ways how one can get a grip on the topology and the invariants of a (compact) *smooth* manifold M:

(i) In the Smoothly Good Cover Theorem 81.2 we showed that every closed smooth manifold admits a finite smoothly good cover in the sense of Definition 81.1. In the Smooth Manifold–Weak Homotopy Type Corollary 178.10 we applied the Nerve Theorem 178.5

[2015] Note that Exercise 4.51 implies that such a continuous map $i\colon X \to Y$ is necessarily an embedding.

to these smoothly good covers and we obtained that M is weakly homotopy equivalent to a finite simplicial complex.

(ii) In the Smooth Manifold-Simplicial Structure Theorem 179.5 we used the fact that M is diffeomorphic to a smooth submanifold of some $\mathbb{R}^n$ to show that M actually admits a (finite) smooth simplicial structure.

(iii) Later on, in the Manifold-CW-Homotopy Type 254.4, we will use "Morse theory" to show that any compact smooth manifold M is homotopy equivalent to a (finite) CW-complex X with $\dim(X) \leq \dim(M)$.

These results, together with basic facts about invariants of finite CW-complexes, see e.g. the CW-π_1-Finite Presentation–Proposition 136.23 (2) and the CW-Complex-H_*-Proposition 155.10 (1), imply (most of the statements of) the following proposition:

Proposition 179.10. (Smooth Manifolds-Invariants Proposition) Let M be a path-connected non-empty[2016] n-dimensional smooth manifold.

(1) (a) The fundamental group of M is countable.
(b) If M is compact, then the fundamental group of M is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(M)$ are countably generated abelian groups.
(b) If M is compact, then all homology groups $\mathrm{H}_k(M)$ are finitely generated abelian groups.
Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M;\mathbb{F})$ are of countable dimension and if M is compact, then they are finite-dimensional.
(3) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(M;G) = 0$.
(4) The higher homotopy groups of M are countable.

To the best of the author's understanding none of the above approaches (i), (ii) and (iii) can be carried over to topological manifolds.

Fortunately we now have the Topological Manifold-Simplicial Complex Retract Theorem 189.13 at our disposal, which is strong enough to extend most of the results of the Smooth Manifolds-Invariants Proposition 179.10 from smooth manifolds to topological manifolds. This leads us to the following proposition:

Proposition 189.14. (Topological Manifolds-Invariants Proposition) Let M be a connected non-empty topological manifold. The following statement holds:

(1) (a) The fundamental group of M is countable.[2017]
(b) If M is compact, then the fundamental group of M is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(M)$ are countably generated abelian groups.
(b) If M is compact, then all homology groups $\mathrm{H}_k(M)$ are finitely generated abelian groups.
Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M;\mathbb{F})$ are of countable dimension and if M is compact, then they are finite-dimensional.
(3) There exists an $l \in \mathbb{N}_0$ such that for every $k > l$ and for every abelian group G we have $\mathrm{H}_k(M;G) = 0$.[2018]
(4) The higher homotopy groups of M are countable.

[2016] We request the adjectives "path-connected" and "empty" to make sure that the isomorphism type of the homotopy groups are well-defined.

Remark. The Topological Manifolds-Invariants Proposition 189.14 together with the Relative-H_*-Finite Generation Lemma 179.11 often allows us to control the "size" of relative homology groups of a compact topological manifold. For example it follows from these two results and the Topological Manifold–Boundary Proposition 23.15 (1) and (2) that for a compact topological manifold M all relative homology groups $\mathrm{H}_k(M, \partial M)$ are finitely generated. □

The following lemma is the one extra input we need to prove the Topological Manifolds-Invariants Proposition 189.14.

Lemma 189.15. (Simplicial Complex-Retract-Invariants Lemma) Let X be a path-connected non-empty topological space that is a retract of a countable n-dimensional simplicial complex Y. The following statement holds:

(1) (a) The fundamental group of X is countable.
    (b) If X is compact, then the fundamental group of X is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(X)$ are countably generated abelian groups.
    (b) If X is compact, then all homology groups $\mathrm{H}_k(X)$ are finitely generated abelian groups.

    Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(X; \mathbb{F})$ are of countable dimension and if X is compact, then they are finite-dimensional.
(3) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(X; G) = 0$.
(4) The higher homotopy groups of X are countable.

Proof of Proposition 189.14 assuming Lemma 189.15. Let M be a path-connected non-empty n-dimensional topological manifold. By the Topological Manifold-Simplicial Complex Retract Theorem 189.13 we know that M is the retract of a countable finite-dimensional simplicial complex Y. The theorem also tells us that if M is compact, then we can arrange that Y is finite. Thus the desired statements follow immediately from the Simplicial Complex-Retract-Invariants Lemma 189.15. ■

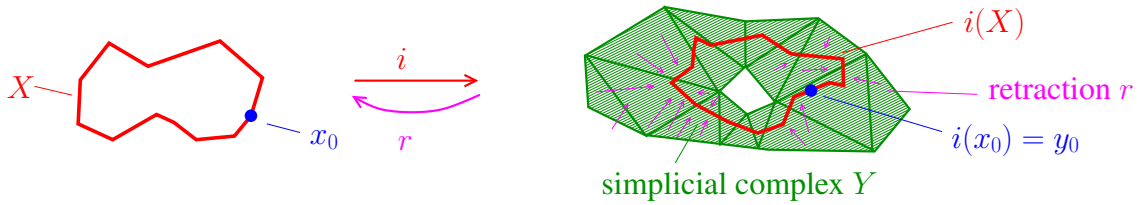
Proof of the Simplicial Complex-Retract-Invariants Lemma 189.15. Let X be a path-connected non-empty topological space that is a retract of a countable n-dimensional simplicial complex Y. Recall that this means that there exists a continuous map $i\colon X \to Y$ and a continuous map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. By considering the path-component of Y that contains $i(X)$ we can assume that Y is path-connected. Furthermore let $\mathbb{F}$ be a field and let G be an abelian group. We pick a base point $x_0 \in X$. We write $y_0 \coloneqq i(x_0)$.

If X is compact, then it follows from the Compact Image Lemma 4.15, the Simplicial-Implies-CW Lemma 174.51 and the CW-Complex-Finiteness Theorem 136.16 that X is contained in a finite subcomplex. So if X is compact, we can arrange that Y is a finite simplicial complex.

We start out with the following claim:

[2017] In the Manifold–Countable-π_1-Proposition 95.1 we already showed that fundamental groups of (possibly non-compact) topological manifolds are countable. But it seems difficult to adapt the approach of the Manifold–Countable-π_1-Proposition 95.1 to deal with homology groups and higher homotopy groups.

[2018] Let M be an n-dimensional topological manifold. In the Manifold-Top-H_*-Theorem 193.1 we will see that we can take $l = \dim(M)$ and we will in fact be able compute the homology group $\mathrm{H}_n(M)$ in terms of the topology of M.



Claim.

(1) (a) $\pi_1(Y)$ is countable.
(b) If Y is a finite simplicial complex, then $\pi_1(Y)$ is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(Y)$ are countable and for any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(Y;\mathbb{F})$ are of countable dimension.
(b) If Y is finite, then all homology groups $\mathrm{H}_k(Y)$ are finitely generated and for any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(Y;\mathbb{F})$ are finite-dimensional.
(3) For every $k > n$ we have $\mathrm{H}_k(Y;G) = 0$.
(4) The higher homotopy groups of Y are countable.

Proof.

(1) This statement follows immediately from the Simplicial-Implies-CW Lemma 174.51, the CW-Skeleton-π_1-Proposition 136.22 and the 1-Complex-π_1-Theorem 137.6.
(2),(3) These two statements follow from the Simplicial-Implies-CW Lemma 174.51 together with the CW-Complex-H_*-Proposition 155.10 (1) and (2) and the discussion on page 3081.
(4)) This statement follows immediately from the Simplicial Complex–π_n-Countable Proposition 175.16.[2019] It is perhaps worth recalling that this statement is a consequence of the Simplicial Approximation Theorem 175.15 ⊞

Now we turn to the invariants of X. Note that we deal with them in a slightly unexpected order:

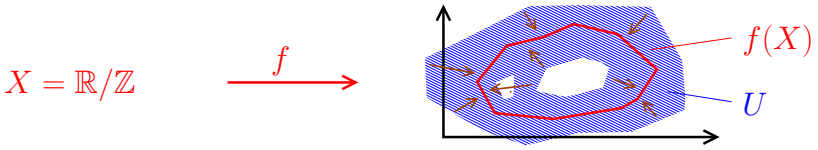
(2) Let $k \in \mathbb{N}_0$. Note that it follows from the functoriality of homology groups and the fact that $r \circ i = \mathrm{id}_X$ that the maps $r_*\colon \mathrm{H}_k(Y) \to \mathrm{H}_k(X)$ and $r_*\colon \mathrm{H}_k(Y;G) \to \mathrm{H}_k(X;G)$ are epimorphisms. In particular if one of the groups to the left is countable (finitely generated, finite-dimensional, of countable dimension), then so is the corresponding group to the right. This observation together with Claim (2) implies the desired Statement (2).
(3) In the proof of (2) we just saw that the induced maps $r_*\colon \mathrm{H}_k(Y) \to \mathrm{H}_k(X)$ and $r_*\colon \mathrm{H}_k(Y;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F})$ are epimorphisms. In particular if the groups to the left vanish, then so do the groups to the right. This observation together with Claim (3) implies the desired Statement (3).
(4) As in (2) we see that $r_*\colon \pi_n(Y, y_0) \to \pi_n(X, x_0)$ is an epimorphism. It follows from Claim (4) and the Countability Lemma 1.37 that $\pi_n(X, x_0)$ is also countable.
(1) (a) The logic of (2), (3) and (4) together with Claim (1) shows immediately that $\pi_1(X, x_0)$ is countable.

[2019] Note that this is the one place where we use that Y is a simplicial complex and not just a CW-complex.

(b) Now assume that X is compact. As we discussed above, this means that we can assume that Y is a finite simplicial complex. The logic of (2), (3) and (4) together with Claim (1) shows immediately that $\pi_1(X, x_0)$ is *finitely generated*. The statement that $\pi_1(X, x_0)$ is *finitely presented* requires a little more thought. More precisely, by the functoriality of fundamental groups we know that $\pi_1(X, x_0)$ is a *retract* (in the sense of Definition 100.22) of $\pi_1(Y, y_0)$. By Claim (1) we know that the group $\pi_1(Y, y_0)$ is finitely presented. Thus we obtain from the purely group-theoretic Finite Presentation-Retract Lemma 100.23 that $\pi_1(X, x_0)$ is also finitely presented. ■

189.6. **Appendix: Euclidean Neighborhood Retracts.** In this appendix we will formalize and generalize some of the results from the previous sections. The key to doing so is the following definition:

Definition 189.16. A topological space X is called an euclidean neighborhood retract (or short ENR) if there exists an embedding $f\colon X \to \mathbb{R}^n$ and an open neighborhood U of $f(X)$ such that $f(X)$ is a retract of U.[2020]



Example. Let X be a weakly contractible topological space. If X admits an embedding $f\colon X \to \mathbb{R}^n$ such that $f(X)$ is a closed subset of $\mathbb{R}^n$, then it follows from Borsuk's Theorem 189.7 that $f(X)$ is an ENR. □

The following proposition shows that many of our favorite topological spaces are ENRs.

Proposition 189.17. (Manifolds-CW–ENR Proposition)
(1) Every topological manifold is an ENR.
(2) Every countable CW-complex that is regionally compact[2021] and finite-dimensional is an ENR.[2022]

The proof of the Manifolds-CW–ENR Proposition 189.17 (2) relies very much on the following lemma which is perhaps interesting in its own right.

Lemma 189.18. (CW-Complex–Euclidean Embedding Lemma) Let X be a CW-complex. If X is countable, regionally compact and finite-dimensional, then X admits an embedding into some $\mathbb{R}^n$.

[2020]Let $i\colon X \to \mathbb{R}^n$ be an embedding of a topological space. The property of X being a retract of an open neighborhood, rather surprisingly, does not depend on the choice of the embedding. More precisely, [**tD2008**, Proposition 18.4.1] says the following: If X is an ENR, then given any embedding $f\colon X \to Y$ into a metric space Y there exists an open neighborhood U of $f(X)$ such that X is a retract of U. We will not make use of this pretty statement.

[2021]By the CW-Complex–Regionally Compact Proposition 136.20 we know that a CW-complex is regionally compact if and only if it is locally finite.

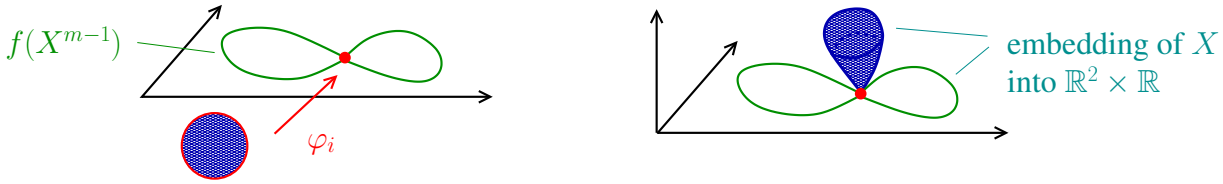
[2022]By the CW-Complex Compactness Corollary 136.17 we know that a finite CW-complex is compact, in particular (by the Compact Hausdorff-Regionally Compact Lemma 4.58) regionally compact. Thus the Manifolds-CW–ENR Proposition 189.17 (2) implies that every finite CW-complex is an ENR.

Proof of the CW-Complex–Euclidean Embedding Lemma 189.18. The theorem in its full generality is proved in [**FrPi1990b**, Theorem A]. In the discussion below we will only deal with the much simpler case that X is actually a finite CW-complex.[2023] We prove this case by induction on the dimension of the CW-complex X. The case $\dim(X) = -1$ is of course trivial. So suppose we already know the statement for all finite CW-complexes of dimension $\leq m-1$. Now let X be a finite m-dimensional CW-complex. As usual we denote by X^{m-1} the m-skeleton of X. Let $\varphi_1, \ldots, \varphi_k \colon S^{m-1} \to X^{m-1}$ be a full set of attaching maps[2024] of the m-cells of X. By induction there exists an $n \in \mathbb{N}_0$ and an embedding $f \colon X^{m-1} \to \mathbb{R}^n$. It remains to prove the following claim:

Claim. The following map is well-defined and it is an embedding: map

$$\overbrace{\left(X^{m-1} \sqcup \bigsqcup_{i=1}^{k} \overline{B}_i^m\right)/\sim}^{\cong X} \to \mathbb{R}^n \times \overbrace{\mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1}}^{k \text{ copies}}$$

$$[x] \mapsto \begin{cases} (f(x), 0, \ldots, 0), & \text{if } x \in X^{m-1}, \\ (r \cdot f(\varphi_i(v)), 0, \ldots, \underbrace{rv\cdot(1-r), 1-r}_{\text{in the } i\text{-th } \mathbb{R}^{m+1}\text{-factor}}, \ldots, 0), & \text{if } x = r\cdot v \in \overline{B}_i^m \text{ where } r \in [0,1] \text{ and } v \in S^{m-1}. \end{cases}$$



Proof. First note that it follows from the Cylinder Quotient-Ball Lemma 8.9 (2) that the somewhat risqué notation for points in $\overline{B}_i^m$ actually leads to a meaningful and continuous map on each $\overline{B}_i^m$. Next note that it follows from the Characteristic Maps Proposition 136.9 (2) that the overall map is continuous. Our hypothesis that X is a finite CW-complex implies by the CW-Complex Compactness Corollary 136.17 that X is compact. Finally it is elementary to verify that the map is actually injective. Thus, in summary, we can appeal to everybody's best friend, i.e. the Compact-Hausdorff Proposition 4.23 (2), to conclude the proof that this map is in fact an embedding. ■

Proof of the Manifolds-CW–ENR Proposition 189.17.

(1) Let M be a topological manifold. By the Euclidean Embedding Theorem 33.2 we know that there exists an embedding $f \colon M \to \mathbb{R}^n$ such that $f(M)$ is a closed subset of $\mathbb{R}^n$. As we discussed on page 3637, every topological manifold is weakly locally contractible. Thus we can apply Borsuk's Theorem 189.7 to deduce that M is indeed an ENR.[2025]

(2) Let X be a CW-complex that is countable, regionally compact and finite-dimensional. By the CW-Complex–Euclidean Embedding Lemma 189.18 we know that there exists an embedding $f \colon X \to \mathbb{R}^n$. Furthermore note, as we discussed on page 3637, that it

[2023] This special case is precisely the content of Exercise 136.15.

[2024] Recall that in Definition 136.7 we defined what we mean by a full set of attaching maps.

[2025] Here is a separate, more down to earth argument, for the case of a closed smooth manifold M: By the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can view M as a closed submanifold of some $\mathbb{R}^n$. Now we apply the Neighborhood Projection Theorem 35.2 to conclude that $M \subseteq \mathbb{R}^n$ admits an open neighborhood that retracts to M.

follows almost immediately from the CW-Complex–Local Properties Proposition 137.8 (2) that the CW-complex X is weakly locally contractible. The following claim, together with the above discussion and Borsuk's Theorem 189.7 implies that X is indeed an ENR.

Claim. There exists an embedding $\widetilde{f}\colon X \to \mathbb{R}^{n+1}$ such that $\widetilde{f}(X)$ is closed.

Proof. [2026] Since X is homeomorphic to a subspace of $\mathbb{R}^n$, we know by the Second Countability Lemma 13.2 (2) and the Subspace-Hausdorff Lemma 3.31 that X is also second countable and Hausdorff. Since X is by hypothesis regionally compact we obtain from the Exhaustion Function Lemma 14.12 that there exists a continuous proper map $\varphi\colon X \to \mathbb{R}$. Next we consider the map

$$\begin{array}{rcl} \widetilde{f}\colon X & \to & \mathbb{R}^n \times \mathbb{R} \\ x & \mapsto & (f(x), \varphi(x)). \end{array}$$

Using the fact that φ is proper one can easily verify (e.g. using the elementary Exercise 7.13) that $\widetilde{f}$ is proper, i.e. the preimage of every compact subspace is again compact. It now follows from the Proper-Closed Proposition 4.64 that $\widetilde{f}$ is an embedding such that $\widetilde{f}(X)$ is a closed subset of $\mathbb{R}^{n+1}$. ■

We conclude this section with the following proposition:

Proposition 189.19. (ENR-Invariants Proposition) Let X be a connected non-empty ENR. The following statement holds:

(1) (a) The fundamental group of X is countable.
    (b) If X is compact, then the fundamental group of X is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(X)$ are countably generated abelian groups.
    (b) If X is compact, then all homology groups $\mathrm{H}_k(X)$ are finitely generated abelian groups.

    Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(X;\mathbb{F})$ are of countable dimension and if X is compact, then they are finite-dimensional.
(3) There exists an $l \in \mathbb{N}_0$ such that for every $k > l$ and for every abelian group G we have $\mathrm{H}_k(X;G) = 0$.
(4) The higher homotopy groups of X are countable.

Proof. Let X be a (compact) connected non-empty ENR. By definition of an ENR there exists an embedding $f\colon X \to \mathbb{R}^n$ and an open neighborhood U of $f(X)$ such that $f(X)$ is a retract of U. It follows from the Simplicial Complex Retract Proposition 189.11 that X is a retract of a countable finite-dimensional simplicial complex. The desired statements now follow from the Simplicial Complex-Retract-Invariants Lemma 189.15. ■

189.7. Appendix: The Lefschetz Fixed Point Theorem for topological manifolds.

Recall that in Theorem 182.15 and 182.16 we formulated and proved the Lefschetz Fixed Point Theorem in the realm of finite simplicial complexes and compact smooth manifolds. Our aim now is to extend the Lefschetz Fixed Point Theorem to compact ENRs.

First let us recall the relevant concepts from Definition 182.10:

[2026] We will sketch a more self-contained proof of the claim in Exercise 189.4.

Definition 189.20. Let $\mathbb{F}$ be a field.

(1) Let X be a topological space. We say that the homology of X is $\mathbb{F}$-finite if all homology groups $\mathrm{H}_i(X;\mathbb{F})$ with $\mathbb{F}$-coefficients are finite-dimensional vector spaces and if all but finitely many homology groups $\mathrm{H}_i(X;\mathbb{F})$ are zero.

(2) Let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi\colon X \to X$ be a continuous map. We define the $\mathbb{F}$-Lefschetz number of φ to be

$$\Lambda(\varphi,\mathbb{F}) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(X;\mathbb{F}) \to \mathrm{H}_n(X;\mathbb{F})).$$

the trace of the endomorphism φ_*

We now want to extend the Lefschetz Fixed Point Theorem to the setting of compact ENRs. Note that it follows from the ENR-Invariants Proposition 189.19 that every compact ENR is $\mathbb{F}$-finite for every field $\mathbb{F}$. This shows that the Lefschetz number of every continuous self-map of a compact ENR is actually defined. Thus is makes sense to formulate the following theorem:

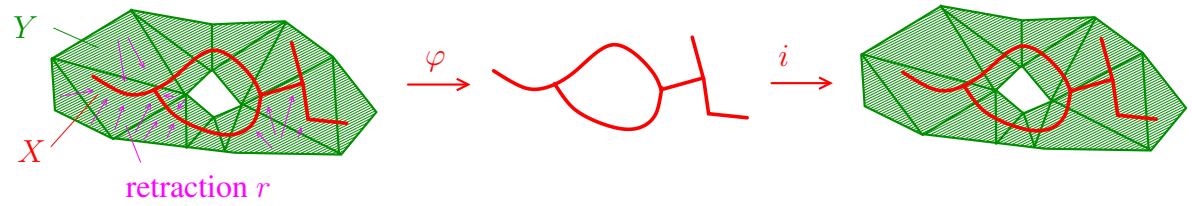
Theorem 189.21. (Lefschetz Fixed Point Theorem III) Let X be a compact ENR (e.g. X could be a compact topological manifold or a finite CW-complex) and let $\varphi\colon X \to X$ be a continuous map. If the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi,\mathbb{F})$ is non-zero for some field $\mathbb{F}$, then φ has a fixed point.

Remark.

(1) The above Lefschetz Fixed Point Theorem 189.21 together with the Manifolds-CW–ENR Proposition 189.17 also gives a different, now totally self-contained[2027], proof of the Lefschetz Fixed Point Theorem 182.16 for compact smooth manifolds.

(2) For compact orientable topological manifolds we will provide a generalization of the above Lefschetz Fixed Point Theorem 189.21 in the Lefschetz-Hopf Theorem 222.14.□

Proof. Let X be a compact ENR. By the Simplicial Complex Retract Proposition 189.11 (2) we know that X is a retract of a finite simplicial complex Y. Recall that this means that there exists a continuous map $i\colon X \to Y$ and a continuous map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. By the very harmless Exercise 4.51 we know that i is an embedding. Thus in the following we will view X as a subspace of Y. Now let $\varphi\colon X \to X$ be a continuous map. We consider the map $i \circ \varphi \circ r\colon Y \to Y$.



Claim. We have $\Lambda(i \circ \varphi \circ r,\mathbb{F}) = \Lambda(\varphi,\mathbb{F})$.

[2027] Our original proof rests on the Smooth Manifold-Simplicial Structure Theorem 179.5, for which we only sketched a proof.

Proof. Let $k \in \mathbb{N}_0$. We consider the following sequence of homomorphisms:

$$0 \longrightarrow \ker(r_*) \hookrightarrow \mathrm{H}_k(Y;\mathbb{F}) \underset{i_*}{\overset{r_*}{\rightleftarrows}} \mathrm{H}_k(X;\mathbb{F}) \longrightarrow 0.$$

Note that we have the equality $r_* \circ i_* = (r \circ i)_* = \mathrm{id}_* = \mathrm{id}_{\mathrm{H}_k(X;\mathbb{F})}$. This observation implies that the horizontal sequence is exact and that it splits. As in the Splitting Lemma 145.5 (1)⇒(2) we see that this implies that we can write $\mathrm{H}_k(Y;\mathbb{F}) = i_*(\mathrm{H}_k(X;\mathbb{F})) \oplus \ker(r_*)$. With slightly suggestive notation we see that

$$\mathrm{tr}(\mathrm{H}_k(Y;\mathbb{F}) \xrightarrow{i_* \circ \varphi_* \circ r_*} \mathrm{H}_k(Y;\mathbb{F})) = \mathrm{tr}\Bigg(\underbrace{\begin{pmatrix}\mathrm{id}\\0\end{pmatrix}(\varphi_*)\begin{pmatrix}\mathrm{id} & 0\end{pmatrix}}_{=\begin{pmatrix}\varphi_* & 0\\0 & 0\end{pmatrix}}\Bigg) = \mathrm{tr}(\mathrm{H}_k(X;\mathbb{F}) \xrightarrow{\varphi_*} \mathrm{H}_k(X;\mathbb{F})).$$

The desired formula for the Lefschetz numbers follows immediately from this claim. ⊞

From our hypothesis, the claim and the Lefschetz Fixed Point Theorem 182.15 for simplicial complexes we obtain that $i \circ \varphi \circ r \colon Y \to Y$ admits a fixed point $y \in Y$. This fixed point y lies necessarily in the image of the last map, i.e. it lies in $i(X) = X$. It is clear that the point y is also a fixed point for φ. ■

Remark. In the possibly unlikely event that the reader wants to know more about ENRs, we refer to [**Dol1980**, Chapter IV.8] for more information. □

189.8. Exercises for Chapter 189 .

Exercise 189.1. Let M be a topological manifold. Show that if $A \subseteq M$ is a neighborhood retract, then A is a closed subset of M.

Exercise 189.2. Show that the Topologist's Sine Curve

$$\{(0,y) \mid y \in [-1,1]\} \cup \{(x, \sin(\tfrac{1}{x})) \mid x \in (0, \tfrac{1}{\pi}]\} \subseteq \mathbb{R}^2$$

is not weakly locally contractible.

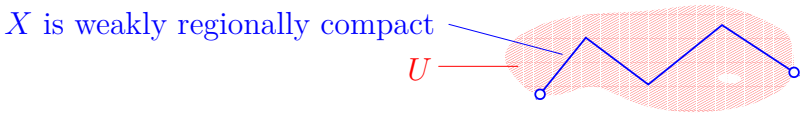
Exercise 189.3. Is every contractible topological space also weakly locally contractible?

Exercise 189.4. We say that a topological space X is *weakly regionally compact* if every point $x \in X$ admits a compact neighborhood. Now let $X \subseteq \mathbb{R}^n$ be a subspace that is weakly regionally compact. In the following we will show that there exists an embedding $g \colon X \to \mathbb{R}^{n+1}$ such that $g(X)$ is a *closed* subset of $\mathbb{R}^{n+1}$.

(a) Show that there exists an open subset $U \subseteq \mathbb{R}^n$ with $U \cap \overline{X} = X$.

(b) We set $C := \overline{X} \setminus X = \overline{X} \setminus U$. Note that it follows from the Interior-Closure Lemma 3.22 that C is a closed subset of $\mathbb{R}^n$ and note that it follows from the Disjoint Compact-Closed–Distance Lemma 10.5 that $d(x,C) > 0$ for every $x \in X$. Now we consider the map $f \colon X \to \mathbb{R}$ given by $x \mapsto d(x,C)$. Show that the map $g \colon X \to \mathbb{R}^{n+1}$ given by $x \mapsto (x, \frac{1}{f(x)})$ is an embedding such that $g(X)$ is a closed subset of $\mathbb{R}^{n+1}$.

Exercise 189.5. Let M be a topological manifold and let $X \subseteq M$ be a compact subset that is contractible. Is X necessarily a neighborhood retract?



Exercise 189.6. Show that for every $n \in \mathbb{N}_{\geq 2}$ there exists a compact path-connected non-empty smooth manifold M such that $\pi_n(M)$ is not finitely generated as an abelian group.

Exercise 189.7. Let $K \subseteq \mathbb{R}^n$ be a compact neighborhood retract. Show that K is the retract of a compact n-dimensional smooth submanifold of $\mathbb{R}^n$.
Hint. Make use of the Compact Submanifold Exhaustion Proposition 38.2

CHAPTER 190

Topological manifolds: Advanced topics

In this chapter we will cover several more advanced topics on topological manifolds. In particular we will do the following:

(1) We will discuss the question whether topological manifolds are actually homotopy equivalent to (countable, finite) CW-complexes and simplicial complexes.
(2) We will introduce the Euler characteristic of compact topological manifolds and we will discuss the key properties of this invariant.
(3) For topological manifolds we will introduce the notion of a microbundle, in particular the tangent microbundle and a normal microbundle.

This chapter contains few proofs and should be viewed as a survey. We will not really make use of the results of this chapter.

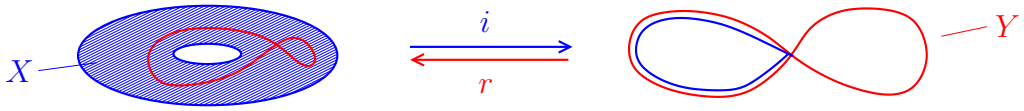
This chapter is organized as follows:

- In Section 190.1 we will discuss Absolute Neighborhood Retracts (ANRs) and we will discuss the homotopy type of topological manifolds.
- In Section 190.2 we will discuss the question whether topological manifolds admit CW-structures.
- In Section 190.3 we will study the Euler characteristic of compact topological manifolds.
- In Appendix 190.4 we will introduce microbundles.
- In Section 190.5 we will provide exercises for Chapter 190.

In the next Chapter 191 we will introduce orientations of topological manifolds.

190.1. **ANRs and the homotopy type of topological manifolds.** In this section we will see in particular that all topological manifolds are homotopy equivalent to locally finite simplicial complexes, in particular, they are homotopy equivalent to CW-complexes.
We start out with the following definition:

Definition 190.1. We say that a topological space X is dominated by a topological space Y if there exist continuous maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X.



Example.

(1) Let X and Y be topological spaces.
   (a) If X is a retract of Y, then evidently X is dominated by Y.
   (b) If X is homotopy equivalent to Y, then clearly X is dominated by Y.

(2) It follows from the Simplicial Complex Retract Proposition 189.11 (2) together with the Manifolds-CW–ENR Proposition 189.17 that every compact topological manifold and every finite CW-complex is dominated by a finite simplicial complex. □

The following proposition is the most powerful result on dominations that we actually prove in these notes.

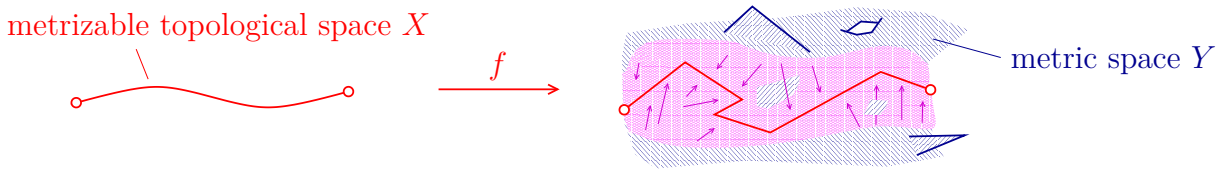
Proposition 305.11 (CW-Complex Domination Proposition). Let X be a topological space. If X is dominated by a CW-complex, then X is homotopy equivalent to a CW-complex.

Remark. Suppose a topological space X is a retract of some CW-complex Y. By the CW-Complex Domination Proposition 305.11 we now know that X is homotopy equivalent to a CW-complex. One might hope that X should actually admit the structure of a CW-complex. But in Exercise 190.4 we will see that in general this is *not* the case. □

Proof. We will prove the CW-Complex Domination Proposition 305.11 in Section 305.3, using the Whitehead Theorem 305.1. Alternatively see the proof of the Countable CW-Homotopy Type Theorem 190.6, the CW-Homotopy Type Theorem 190.9 and the Finite CW-Homotopy Type Theorem 190.10 for other proofs. ■

It turns out that there exist several powerful generalizations of the CW-Complex Domination Proposition 305.11. To formulate these generalizations it is helpful to introduce the following definition:

Definition 190.2. A topological space X is called an absolute neighborhood retract (or short ANR) if X is metrizable[2028] and if the following condition is satisfied: Whenever $f\colon X \to Y$ is an embedding into a metric space Y such that $f(X)$ is a closed subset of Y, then $f(X)$ is a neighborhood retract in Y.



Remark.

(1) Let X be an ANR. If X admits a closed embedding into some $\mathbb{R}^n$, then it follows immediately from the definitions that X is an ENR, as introduced in Definition 189.16. Nonetheless, not every ANR is a Euclidean neighborhood retract ENR. For example we will see shortly that the disjoint union $X = \bigsqcup_{k\in\mathbb{N}} \overline{B}^k$ is an ANR, but it follows easily from the Topological Invariance-of-Domain Theorem 170.9 that X does not admit an embedding into any $\mathbb{R}^n$.

(2) Like in many other settings in the literature there exist various equivalent approaches to defining an ANR, see e.g. [**MaS1982**, Chaper I.3.1(ii)] and [**FrPi1990a**, Proposition A.6.2]. The definition above is the one given in [**FrPi1990a**, p. 281].

(3) Some references use a definition of an ANR which differs slightly from ours. More precisely, in [**Han1951**, p. 390] and [**Miln1959a**, p. 272] an ANR is supposed to

[2028] Recall that in Definition 3.12 we called a topological space X metrizable if there exists a metric d on X such that the given topology agrees with the topology coming from d.

satisfy the extra condition that it is separable, i.e. that it contains a countable subset that is dense. Note that in Exercise 13.8 we saw that a metrizable topological space is separable if and only if it is second-countable. Thus, since in these notes we are more familiar with the concept of second-countable topological spaces we will use that notion in the remainder of our discussion of ANRs.

(4) The theory of ANRs is developed in detail in [**Hu1965, Bor1967, Chig1996**]. □

We will what ANRs are good for. But first let us quote two theorems which give us many examples of ANRs.

Theorem 190.3. (Dugundji Extension Theorem) Let V be a real vector space (not necessarily finite-dimensional) and $\| - \| \colon V \to \mathbb{R}_{\geq 0}$ a norm on V. We equip V with the corresponding metric and we use this metric to view V as a topological space. With respect to this topology every convex subset of V is an ANR.[2029]

Example. The only consequence of the Dugundji Extension Theorem 190.3 that we will use is that every convex subset of some $\mathbb{R}^n$ is an ANR. □

Proof. This theorem was proved by James Dugundji [**Dug1951**, Corollary 4.2] in 1952. A proof is also given in [**MaS1982**, Theorem I.3.1], [**MayeK1989**, Theorem I.5.7] and [**Saka2013**, Theorem 6.1.1]. ■

Theorem 190.4. (Hanner's Theorem) Let X be a topological space which is second-countable and metrizable. If X is the union of a family $\{U_i\}_{i \in I}$ of open subsets such that each U_i is an ANR, then X itself is an ANR.

Proof. This theorem was proved by Olof Hanner [**Han1951**, Theorem 3.3] in 1950. An exposition of the proof is also given in [**Fera2004**, Theorem 2.77] and in [**Hu1965**, Theorem III.8.1]. ■

Corollary 190.5. (Topological Manifold–ANR Corollary) Every topological manifold is an ANR.

Proof. Let X be an n-dimensional topological manifold. By definition X is second-countable. By the Manifold Metrization Proposition 23.10 we know that X is metrizable. In the Manifold–Local Properties Lemma 23.8 (1) we showed that X is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. By the Dugundji Extension Theorem 190.3 every non-empty convex subset of $\mathbb{R}^n$ is an ANR. In other words, we have shown that X admits an open cover by ANRs. Thus we obtain from Hanner's Theorem 190.4 that X is indeed an ANR. ■

The following amazing theorem now tells us what ANRs are good for.

Theorem 190.6. (Countable CW-Homotopy Type Theorem) Let X be a topological space. The following statements are equivalent:

(1) X is dominated by a countable CW-complex.
(2) X is homotopy equivalent to a countable CW-complex.
(3) X is homotopy equivalent to a countable simplicial complex.

[2029] Note that there are no other hidden conditions on the subset except for convexity, in particular we do not demand that the subset is open, closed or compact.

(3′) X is homotopy equivalent to a countable simplicial complex that is locally finite.
(4) X is homotopy equivalent to an ANR which is second-countable.

Furthermore the following refinement of the equivalence of (2) and (3) holds: If X is homotopy equivalent to an n-dimensional countable CW-complex, then it is also homotopy equivalent to an n-dimensional countable simplicial complex, and vice versa.

Proof. The fact that all the statements are equivalent goes well beyond the scope of these notes. Thus we just give the necessary references: The implications (3′)⇒(3)⇒(2)⇒(1) are of course trivial. Next note that the equivalence of (2) and (3) is a consequence of the Simplicial-Implies-CW Lemma 174.51 and the CW–Simplicial Homotopy Type Proposition 175.21. As is explained in [**Miln1959a**, Theorem 1], the implication (1)⇒(3′) follows from work of James Whitehead [**WhdJ1950a**, Theorem 24] and the implication (3′)⇒(4)⇒(1) follows from the work of Olof Hanner [**Han1951**, Corollary 3.5 and Theorem 6.1]. Alternatively we refer to [**FrPi1990a**, Theorem 5.2.1] or [**LW1969**, p. 137] for a proof that the five statements are equivalent. ■

The above results now give us several corollaries on topological manifolds.

Corollary 190.7. (Manifold–Simplicial Complex Homotopy Type Corollary) Every topological manifold is homotopy equivalent to a countable locally finite simplicial complex.

Remark. Later, in the Manifold–Simplicial Complex Homotopy Type Theorem 336.3 we will be able to refine some of the conclusions of the Manifold–Simplicial Complex Homotopy Type Corollary 190.7. □

Proof. This statement follows from the Topological Manifold–ANR Corollary 190.5 together with the Countable CW-Homotopy Type Theorem 190.6 (4)⇒ (3′) and the fact that topological manifolds are by definition second-countable. ■

By the $S^1 \vee S^2$–π_2–Proposition 169.9 we know that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\mathbb{Z}^\infty$, in particular it is not finitely generated. This result, foreshadowed by our discussion on page 2588, shows that homotopy groups of finite CW-complexes are not necessarily finitely generated. In Question 136.24 we asked, somewhat apprehensively, whether the homotopy groups of a countable CW-complex are necessarily countable. With a slight sigh of relief we can now give an affirmative answer to this question.

Proposition 190.8. (CW-Complex–π_n-Countable Proposition) If X is a CW-complex with countably many cells, then for any $n \in \mathbb{N}$ and any $x_0 \in X$ the group $\pi_n(X, x_0)$ is countable.

Proof. This follows from the combination of the Countable CW-Homotopy Type Theorem 190.6 (1)⇒(4) together with the Homotopy Equivalence-π_n-Proposition 134.17 (1) and the Simplicial Complex–π_n-Countable Proposition 175.16. ■

The Countable CW-Homotopy Type Theorem 190.6 deals with (second-) countable objects. The following theorem is the analog of the Countable Theorem 190.6 without any cardinality restrictions and, on the flip side, without any cardinality information.

Theorem 190.9. (CW-Homotopy Type Theorem) If X is a topological space, then the following statements are equivalent:

(1) X is dominated by a CW-complex.
(2) X is homotopy equivalent to a CW-complex.
(3) X is homotopy equivalent to a simplicial complex.
(4) X is homotopy equivalent to an ANR.

Proof. The implication (1)⇒(2) is precisely the statement of the CW-Complex Domination Proposition 305.11. The fact that (2) and (3) are equivalent is a consequence of the Simplicial-Implies-CW Lemma 174.51 and the CW–Simplicial Homotopy Type Proposition 175.21. The statement that in fact all four statements are equivalent is proved in [**FrPi1990a**, Theorem 5.2.1]. ■

The CW-Homotopy Type Theorem 190.9, which gives no control of the "size" of the objects is not all that helpful. It is much more interesting to ask whether the analog of the Countable Theorem 190.6 holds if we replace "countable" by "finite". In fact the following theorem holds.

Theorem 190.10. (Finite CW-Homotopy Type Theorem) Let X be a topological space. The following statements are equivalent:

(2) X is homotopy equivalent to a finite CW-complex.
(3) X is homotopy equivalent to a finite simplicial complex.
(4) X is homotopy equivalent to a compact ANR.

Furthermore the following refinement of the equivalence of (2) and (3) holds: If X is homotopy equivalent to a finite n-dimensional CW-complex, then it is also homotopy equivalent to a finite n-dimensional simplicial complex, and vice versa.

Proof. The fact that (2) and (3) are equivalent is a consequence of the Simplicial-Implies-CW Lemma 174.51 and the CW–Simplicial Homotopy Type Proposition 175.21. Next note that one can show, with some effort, that it is a consequence of the Dugundji Extension Theorem 190.3 that every finite simplicial complex is an ANR, we refer to [**LW1969**, p. 210] and [**HNV2004**, p. 472] for details. This observation gives us the implication (3)⇒(4). Finally the implication (4)⇒(2) was shown by James West [**Wes1977**] in 1977. ■

The attentive reader will have noticed that the dog that did not bark is the (1)⇒(2) implication. It turns out that there is a good reason why it did not bark. The following theorem, proved by C.T.C. Wall shows that in general the statement one would hope for is actually incorrect.

Theorem 190.11. (Dominated-but-not-Homotopy Type Theorem) There exists a topological space that is dominated by a finite CW-complex but that is not homotopy equivalent to a finite CW-complex.

Proof. The statement is proved in [**Wall1965a**, p. 66]. Alternatively see [**Var1989**, p. 163] and [**FerR2001**, Theorem 3.1]. More precisely, in [**Wall1965a, Var1989**] it is shown that to a topological space X that is dominated by a finite CW-complex one can associate an invariant in the "K-group $\widetilde{\mathrm{K}}_0(\mathbb{Z}[\pi_1(X)])$" which vanishes if and only if X is homotopy equivalent to a finite CW-complex. It is shown in the above references that there exist examples where the invariant is indeed non-zero. Interestingly it is conjectured, see e.g.

[**KrL2005**, p. 179], that the group $\widetilde{\mathrm{K}}_0(\mathbb{Z}[\pi_1(X)])$ is zero, whenever $\pi_1(X)$ is a torsion-free group. ■

From this discussion we obtain the following corollary:

Corollary 190.12. (Compact Manifold–Homotopy Type Corollary) Every compact topological manifold is homotopy equivalent to a finite simplicial complex.

Proof. This statement follows immediately from the Topological Manifold–ANR Corollary 190.5 together with the Finite CW-Homotopy Type Theorem 190.10 (4)⇒(3). A very different proof for the corollary was given by Rob Kirby and Larry Siebenmann in [**KSi1969**, Chapter III.2] and [**KSi1977**, p. 744]. ■

Let M be a (compact) n-dimensional topological manifold. By the above we now know that M is homotopy equivalent to a (finite) simplicial complex X. But the above results do not say anything about the dimension of the simplicial complex X. But it turns out, that at least for compact topological manifolds this can mostly be fixed. More precisely, later in the lecture notes we will prove (with some major input from the literature) the following theorem:

Theorem 336.3. (Manifold–Simplicial Complex Homotopy Type Theorem)
(1) Every closed n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional simplicial complex.
(2) Every compact connected n-dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.

We conclude this section with the following proposition which can be viewed as an analog of the Single Top-Cell Proposition 183.34.

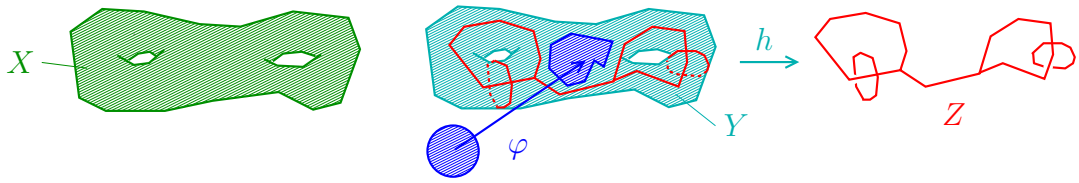
Proposition 190.13. (Single Zero-and-Top Cell Proposition) Every closed connected non-empty n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional CW-complex which has a single 0-cell and which has a single n-cell.

Remark. For smooth manifolds we will prove closely related statements in the Minimal Handle Proposition 248.5 and in the Smooth Manifold-CW-Homotopy Type Proposition 254.4. □

Proof. Let X be a closed connected non-empty n-dimensional topological manifold. In the following we assume that $n \geq 2$. (The case $n = 1$ can be dealt with by slightly adjusting the argument or by appealing to the Topological 1-Manifold Classification Theorem 28.1.) We pick an embedding $\varphi\colon \overline{B}^n \to X$ such that $\varphi(\overline{B}^n)$ is a submanifold. It follows from the Codimension-Zero Submanifold Proposition 150.16 that $Y \coloneqq X \setminus \varphi(B^n)$ is a compact non-empty n-dimensional topological manifold with $\partial Y = \varphi(S^{n-1})$. Since $n \geq 2$ we know that $\varphi(S^{n-1})$ is connected. Since X is also connected it follows from Exercise 23.9 that Y is also connected.

Recall that by the Manifold–Simplicial Complex Homotopy Type Theorem 336.3 we know that there exists a finite $(n-1)$-dimensional CW-complex Z and a homotopy equivalence $h\colon Y \to Z$.

Claim. The pushouts $\overline{B}^n \cup_{S^{n-1}} Y$ and $\overline{B}^n \cup_{S^{n-1}} Z$ are homotopy equivalent.



Proof. We give two slightly different proofs:

- The statement follows from the Homotopy Equivalent Cell Attachment Proposition 135.13.
- We consider the following diagram:

$$\begin{array}{ccccc} \overline{B}^n & \xleftarrow{\quad i \quad} & S^{n-1} & \xrightarrow{\quad \varphi \quad} & Y \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h} \\ \overline{B}^n & \xleftarrow{\quad i \quad} & S^{n-1} & \xrightarrow{\quad h\circ\varphi \quad} & Z. \end{array}$$

  Note that the diagram commutes, note that the vertical maps are homotopy equivalences and note that it follows from the Manifold Boundary-Cofibration Proposition 22.11 that the inclusion map $i\colon S^{n-1} \to \overline{B}^n$ is a closed cofibration. Thus the claim follows from the Homotopy Pushout Theorem 22.24. ⊞

By the Regluing Lemma 8.17 the former pushout is homeomorphic to X. Furthermore by the CW-Gluing Proposition 139.9 the latter pushout admits a CW-structure with a single n-cell and no cells of dimension $> n$. Finally note that it follows from the CW-Complex–Spanning Tree Proposition 140.5 that we can in fact find an n-dimensional CW-complex that is homotopy equivalent to X that contains not only a single n-cell but that also contains only a single 0-cell. ■

190.2. **Topological manifolds and CW-structures.** Recall that by the Smooth Manifold-CW-Theorem 179.9 we know that every smooth manifold admits a simplicial structure. By the Manifold–Simplicial Complex Homotopy Type Corollary 190.7 and the Compact Manifold–Homotopy Type Corollary 190.12 we now know that every (compact) topological manifold is homotopy equivalent to a (finite) CW-complex. The following question naturally arises.

Question 190.14.
(1) Does every topological manifold admit a simplicial structure?
(2) Does every topological manifold admit a CW-structure?

In the following we will survey what is known regarding Question 190.14. Let us start out with some positive statements.

Theorem 190.15. (Radó's Theorem) Every 2-dimensional topological manifold admits a PL-structure, in particular it admits a simplicial and a CW-structure.

Proof. In 1926 it was shown by Tibor Radó [**Rad1926**] that every 2-dimensional topological manifold admits a simplicial structure. It is elementary to see that the simplicial structure of any 1-dimensional topological manifold is in fact a PL-structure. It follows from the PL-Manifold Characterization Proposition 180.10 that the simplicial structure of any 2-dimensional topological manifold is in fact a PL-structure.

Note that we already mentioned the above result of Tibor Radó in the proof of the Surface-Smooth Structure Existence Theorem 101.17. We refer to that proof for alternative references. ■

Theorem 190.16. (Moise's Theorem) Every 3-dimensional topological manifold admits a PL-structure, in particular it admits a simplicial and a CW-structure.

Proof. This theorem was first proved in 1952 by Edwin Moise [**Moi1952**]. Alternative proofs are given in [**Moi1977**, p. 252 and 253], [**Moi1952**, Theorem 6], [**Bin1983**, Theorem XVIII.3.B], [**GNR2021**], [**Hami1976**, Theorem 2], [**Shal1984**]. ■

Now we turn to the higher dimensional setting. In this setting the stories between CW-structures and simplicial structures diverge. In the following we first consider CW-structures on high-dimensional topological manifolds. It turns out that the 4-dimensional case is a no-show, in the sense that the following question is still open:

Question 190.17. Does every closed 4-dimensional topological manifold admit a CW-structure?

Once we move beyond dimension 4, the situation becomes somewhat clearer again.

Theorem 190.18. (Kirby-Quinn-Siebenmann Theorem) Every *closed* topological manifold of dimension ≥ 5 has a CW-structure.

Proof. For dimensions ≥ 6 this theorem was proved in the 1970s by Rob Kirby and Larry Siebenmann [**KSi1977**, Essay III.2]. In 1982 this result was extended to the 5-dimensional case by Frank Quinn [**Quin1982**, Theorem 2.3.1]. ■

Note that the Kirby-Quinn-Siebenmann Theorem 190.18 only deals with *closed* topological manifolds. The following question is still open.

Question 190.19. Let $n \in \mathbb{N}_{\geq 5}$. Does every *compact* n-dimensional topological manifold admit a CW-structure?

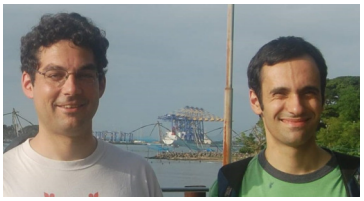
Now what about simplicial structures?

Theorem 190.20. (Casson Theorem) There exists a closed 4-dimensional topological manifold that does not admit a simplicial structure.

Proof. This theorem was proved by Andrew Casson, see [**AkM1990**, page xvi], in the 1980s, building on the work of Mike Freedman [**Free1982**]. An account of the proof is also given in [**Sav2012**, Theorem 18.3]. ■

The question whether every topological manifolds of dimension ≥ 5 admit a simplicial structure had been open for a very long time and it was finally resolved in 2013 by Ciprian Manolescu [**Mano2016a**].

Theorem 190.21. (Manolescu Theorem) For every $n \in \mathbb{N}_{\geq 5}$ there exists a closed n-dimensional topological manifold that does not admit a simplicial structure.



Ciprian Manolescu (on the right)
with somebody else

Proof. The theorem is proved in [**Mano2016a**]. A more relaxed exposition of the key ideas of the proof is also given in [**Mano2016b**]. Note that, as is explained in [**Mano2016a, Mano2016b**], in dimension 5 the examples are necessarily non-orientable, whereas for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure.[2030] ■

In the unlikely event that the reader got slightly confused by the various theorems and counterexamples we now summarize some of the key statements about the topology of compact topological manifolds in the following table:

	$n=1,2,3$	$n=4$	$n\geq 5$
each closed top. n-mfd. is homotopy equiv. to a finite CW-cplx.	✓	✓	✓
each closed top. n-mfd. admits a CW-structure	✓	?	✓
each compact top. n-mfd. is homotopy equ. to a finite CW-cplx.	✓	✓	✓
each compact top. n-manifold admits a CW-structure	✓	?	?
each compact top. n-manifold admits a simplicial structure	✓	✗	✗

Finally we note that we will discuss the existence and uniqueness of PL-structures and smooth structures on topological manifolds in greater detail in **??**.

190.3. **The Euler characteristic of compact topological manifolds.** The Euler characteristic of a finite CW-complex is one of the most essential invariants and often it gives a convenient approach to calculating invariants of CW-complexes. As we just saw, it is not clear whether (compact) topological manifolds admit CW-structures. But fortunately we assembled just enough results on compact topological manifolds to be able to develop the theory of Euler characteristics of compact topological manifolds.

Next we recall Definition 156.6:

Definition 190.22. Let X be a topological space and let $A \subseteq X$ be a subspace.

(1) We define the n-th Betti[2037] number of (X, A) to be

$$b_n(X,A) \coloneqq \operatorname{rank}(\mathrm{H}_n(X,A)) \quad \in \mathbb{N}_0 \cup \{\infty\}$$

(2) If $\sum_{n\in\mathbb{N}_0} b_n(X,A)$ is finite, then we define the Euler characteristic of (X, A) to be[2038]

$$\chi(X,A) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X,A).$$

(3) Let X be a compact topological manifold and let A be a compact submanifold or let A be a union of components of ∂X. It follows from the Topological Manifold–Boundary Proposition 23.15 (1) and (2) and the Topological Manifolds-Invariants Proposition 189.14 (2) and (3), together with the Relative-H_*-Finite Generation Lemma 179.11, that the Euler characteristic $\chi(X, A)$ is defined.

Evidently we write $b_n(X) \coloneqq b_n(X, \varnothing)$ and $\chi(X) \coloneqq \chi(X, \varnothing)$.

In Section 156.3 we already discussed some properties of this Euler characteristic. For instance, in the Homotopy Type-χ-Lemma 156.7 we pointed out that the Euler characteristic

[2030] We will explain later on in Chapter 191 what it means for a topological manifold to be orientable.

[2038] It follows from the Homology-χ-Proposition 156.2 that for a finite CW-complex the above definition agrees with the original definition of the Euler characteristic of a finite CW-complex that we gave in Definition 156.1.

is a homotopy type invariant. Now we will add some more properties. For example we have the following generalization of the Product-χ-Proposition 156.12:

Proposition 190.23. (Product-χ-Proposition) Let X and Y be two topological spaces. If $\chi(X)$ and $\chi(Y)$ are defined and if all homology groups of X and Y are finitely generated, then

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Proof. [2039] Since all homology groups of X and Y are finitely generated we see that for any $n \in \mathbb{N}_0$ we have the following equalities:

$$\begin{aligned} b_n(X \times Y) &= \operatorname{rank}(\mathrm{H}_n(X \times Y)) \overset{\text{by the Künneth Theorem 164.6}}{=} \\ &= \operatorname{rank}\Big(\bigoplus_{i=0}^{n} \mathrm{H}_i(X) \otimes \mathrm{H}_{n-i}(Y) \oplus \bigoplus_{i=0}^{n-1} \operatorname{Tor}(\mathrm{H}_i(X), \mathrm{H}_{n-1-i}(Y)) \Big) \\ &= \sum_{i=0}^{n} \operatorname{rank}(\mathrm{H}_i(X) \otimes \mathrm{H}_{n-i}(Y)) + \sum_{i=0}^{n-1} \operatorname{rank}\big(\underbrace{\operatorname{Tor}(\mathrm{H}_i(X), \mathrm{H}_{n-1-i}(Y))}_{\substack{\text{finite by the Tor-Properties Lemma 162.13,} \\ \text{the } \mathbb{Z}/k\mathbb{Z}\text{-Tor-Lemma 162.12 and the} \\ \text{classification of finitely} \\ \text{generated abelian groups}}} \big) \\ &\quad \uparrow \text{by the Rank Additivity Proposition 97.18} \\ &= \sum_{i=0}^{n} b_i(X) \cdot b_{n-i}(X). \\ &\quad \uparrow \text{follows from the properties of the tensor product that we discussed in Chapter 159 and the Abelian Group-Classification Theorem 97.24} \end{aligned}$$

Almost exactly as in the proof of the Product-χ-Proposition 156.12 we now see that

$$\begin{aligned} \chi(X) \cdot \chi(Y) &= \Big(\sum_{k \in \mathbb{N}_0} (-1)^k \cdot b_k(X) \Big) \cdot \Big(\sum_{l \in \mathbb{N}_0} (-1)^l \cdot b_l(Y) \Big) \\ &= \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} (-1)^{k+l} \cdot b_k(X) \cdot b_l(Y) \;=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \sum_{k+l=n} \cdot\, b_k(X) \cdot b_l(Y) \\ &\quad \uparrow \text{by multiplying out} \\ &= \sum_{n \in \mathbb{N}_0} (-1)^n \cdot b_k(X \times Y) \qquad\qquad = \chi(X \times Y). \\ &\quad \uparrow \text{by the above discussion} \end{aligned}$$

■

In the following proposition we summarize some more properties of the Euler characteristic, with a particular focus on the case of compact topological manifolds. Some of these results are related to earlier results, like the Union-χ-Proposition 156.10 and the Covering-χ-Proposition 156.13:

Proposition 190.24. (Manifold-χ-Proposition)

(1) Let X be and Y be two compact topological manifolds, then

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

(2) If $X = Y \cup Z$ is a decomposition of a compact topological manifold into two compact codimension-zero submanifolds Y and Z such that $Y \cap Z$ is a union of components of ∂X and of ∂Y, then

[2039] This proof is essentially the solution to Exercise 164.5.

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

(3) Let X be a compact topological manifold and let $A \subseteq X$ be a subspace that is a compact topological manifold in its own right.[2040] Then

$$\chi(X, A) \;=\; \chi(X) \;-\; \chi(A).$$

(4) For any field $\mathbb{F}$ we have

$$\chi(X, A) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X, A; \mathbb{F})).$$

(5) Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Then

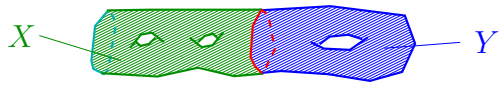
$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Remark. Most of the statements of the Manifold-χ-Proposition 190.24 also hold, with essentially the same proofs, for topological spaces with suitable restrictions on their homology groups. Note though that our proof of the multiplicativity of the Euler characteristics requires that compact topological manifolds are homotopy equivalent to finite CW-complexes. To the best of my knowledge there is no naive way to prove this multiplicativity. □

Sketch of proof.

(1) Let X and Y be two compact topological manifolds. By the Topological Manifolds-Invariants Proposition 189.14 we know that all homology groups are finitely generated abelian groups. Thus the statement follows from the above Product-χ-Proposition 190.23.

(2) As we already mentioned in the proof of the Union-χ-Proposition 156.10, this statement follows easily from the Mayer–Vietoris Theorem 153.2 (3) applied to the decomposition $X = Y \cup Z$ and the Sequence-χ-Lemma 156.5.



$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$$

(3) This statement is a special case of the Relative-χ-Lemma 156.8. As we pointed out back then, this statement follows in fact almost immediately from the LES-Proposition 145.10 and the Exact Sequence-χ-Lemma 156.5.

(4) Let $\mathbb{F}$ be a field. We start out with a special case:

Claim. For every compact topological manifold Y we have

$$\chi(Y) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Y; \mathbb{F})).$$

[2040] For example A could be compact submanifold or A could be a union of boundary components.

Proof. From the Compact Manifold–Homotopy Type Corollary 190.12 we know that Y is homotopy equivalent to a finite CW-complex Z. We observe that

$$\chi(Y) \overset{\text{by the Homotopy Equivalence-H}_*\text{-Proposition 144.14 (2)}}{=} \chi(Z) \overset{\text{by the Homology-}\chi\text{ Proposition 160.15}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Z;\mathbb{F})) \overset{\text{by the generalization of the Homotopy Equivalence-H}_*\text{-Proposition 144.14 (2a) to } \mathbb{F}\text{-coefficients, see page 3078}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Y;\mathbb{F})).$$

Let X be a compact topological manifold and let $A \subseteq X$ be a subspace that is a compact topological manifold in its own right. Using this observation we can now prove that[2041]

$$\chi(X,A) \overset{\text{by (4)}}{=} \chi(X) - \chi(A) \overset{\text{by the above claim}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X;\mathbb{F})) - \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(A;\mathbb{F}))$$
$$\overset{\text{by the long exact sequence of the pair } (X,A) \text{ with } \mathbb{F}\text{-coefficients and the Chain Complex–H}_*\text{–}\chi\text{-Lemma 156.4}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X,A;\mathbb{F})).$$

(5) Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Without loss of generality we can assume that X and $\widetilde{X}$ are connected. By the Compact Manifold–Homotopy Type Corollary 190.12 there exists a finite CW-complex Y and a homotopy equivalence $f\colon X \to Y$. It follows from the CW-Complex–Local Properties Proposition 137.8 and the Covering Existence Theorem 127.10 there exists a connected covering $q\colon \widetilde{Y} \to Y$ such that $q_*(\pi_1(\widetilde{Y})) = f_*(p_*(\pi_1(\widetilde{X})))$. Using the Map Lifting Criterion 127.5 one can now easily show that $f\colon X \to Y$ lifts to a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ and that this map is again a homotopy equivalence[2042].[2043] Now we obtain that

$$\chi(\widetilde{X}) \overset{\text{by the Homotopy Type-}\chi\text{-Lemma 156.7, since } \widetilde{X} \text{ and } \widetilde{Y} \text{ are homotopy equivalent}}{=} \chi(\widetilde{Y}) \overset{\text{by the Covering-}\chi\text{-Proposition 156.13 and the Covering-Degree Proposition 92.10}}{=} [\pi_1(Y) : q_*(\pi_1(\widetilde{Y}))] \cdot \chi(Y) \overset{\substack{\text{we have } [\pi_1(Y) : q_*(\pi_1(\widetilde{Y}))] = [\pi_1(Y) : q_*(\pi_1(\widetilde{Y}))] \text{ since } f_* \text{ is an isomorphism} \\ \text{we have } \chi(Y) = \chi(X), \text{ by the Homotopy Type-}\chi\text{-Lemma 156.7,} \\ \text{and since the topological spaces } X \text{ and } Y \text{ are homotopy equivalent}}{=} [\pi_1(X) : p_*(\pi_1(\widetilde{X}))] \cdot \chi(X)$$
$$\overset{\text{by the Covering-Degree Proposition 92.10}}{=} [\widetilde{X} : X] \cdot \chi(X).$$

■

[2041] Alternatively one can prove this statement using the Universal Coefficient Theorem 162.16, using that the homology groups are all finitely generated and using the calculations of tensor products and Tor-groups. Note that for that approach one needs to distinguish between the case that $\mathbb{F}$ has characteristic zero and the case that $\mathbb{F}$ has finite characteristic. But surely the reader is full aware of this, since this is precisely the approach taken in Exercise 162.16.

[2042] Here the point is that not only the homotopy equivalences but also the homotopies lift to the covering spaces. We leave the details to the reader.

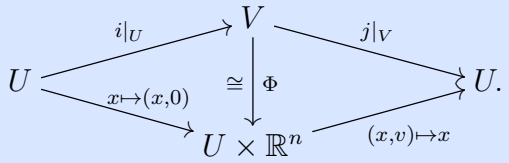
[2043] Note that we worked out the details in Exercise 127.23.

190.4. **Appendix: Microbundles.** In this appendix we will give a short introduction to the concept of a normal microbundle of a submanifold of a topological manifold. A normal microbundle can be viewed as an analog of the normal vector bundle of a smooth submanifold of a smooth manifold. We will not make use of microbundles. So this section we can safely be skipped.

We start out with the definition of a microbundle, which was introduced by John Milnor [**Miln1964a**, Chapter 2].

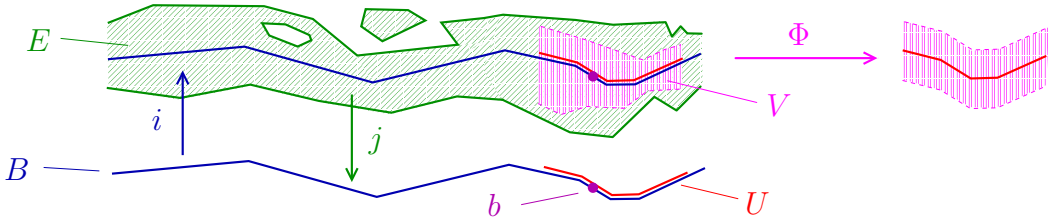
Definition 190.25. An n-dimensional microbundle consists of two topological spaces B and E and two continuous maps $i\colon B \to E$ and $j\colon E \to B$ such that the following two conditions are satisfied:

(1) We have $j \circ i = \mathrm{id}_B$.
(2) For every $b \in B$ there exists an open neighborhood U of $b \in B$ and an open neighborhood V of $i(b) \in E$ with $i(U) \subseteq V$ and $j(V) \subseteq U$ and there exists a homeomorphism $\Phi\colon V \to U \times \mathbb{R}^n$ such that the following diagram commutes:



Such a map Φ is called a local trivialization.

We refer to B as the base space of the microbundle, we refer to E as the total spaces of the microbundle and refer to n as the fiber dimension of the microbundle.[2044]



Examples.

(1) Let B be a topological space. Given any $n \in \mathbb{N}$ we consider $B \times \mathbb{R}^n$ with the projection $j\colon B \times \mathbb{R}^n \to B$ and the embedding $i\colon B \to B \times \mathbb{R}^n$ that is given by $b \mapsto (b, 0)$. Clearly this defines an n-dimensional microbundle.
(2) We will now generalize the example from (a). Let $p\colon X \to B$ be a continuous map and let V be a topological space.
   (a) An V-trivialization of the map $p\colon X \to B$ for an open subset $U \subseteq B$ is a homeomorphism $\Phi\colon p^{-1}(U) \to U \times V$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & U \times V \\ & \searrow^{p} \quad \swarrow_{(x,v)\mapsto x} & \\ & U. & \end{array}$$

[2044] In Exercise 190.7 we will show, using the local homology groups $\mathrm{H}_k(j^{-1}(b), j^{-1}(b), i(b))$, that n is well-defined.

(b) A V-fiber atlas for the map $p\colon X \to B$ consists of a family of V-trivializations $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times V\}_{i\in I}$ such that the U_i form an open cover of X.
(c) We say that $p\colon X \to B$ is a V-bundle if there exists a V-fiber atlas.
(d) Let $v_0 \in V$.[2045] We say that the map $p\colon X \to B$ is a (V, v_0)-bundle if there exists a V-fiber atlas $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times V\}_{i\in I}$ such that for every $i, j \in I$ the map $\Phi_j \circ \Phi_i^{-1}\colon (U_i \cap U_j) \times V \to (U_i \cap U_j) \times V$ preserves v_0, i.e. for each $x \in U_i \cap U_j$ the point (x, v_0) gets sent to (x, v_0).

Now suppose that $p\colon X \to B$ is an $(\mathbb{R}^n, 0)$-bundle. It is straightforward to see that there exists a unique map $i\colon B \to X$ such that for every $b \in B$ we have $p(i(b)) = b$ and b gets sent to 0 via the trivializations of the atlas.

After these lengthy preparations it is elementary to see that the pair $i\colon B \to X$ and $p\colon X \to B$ is an n-dimensional microbundle.[2046] But the notion of a microbundle is much less restrictive. For example we only have control over a neighborhood of $i(B) \subseteq E$ and the local trivializations do not have to be compatible in any way.

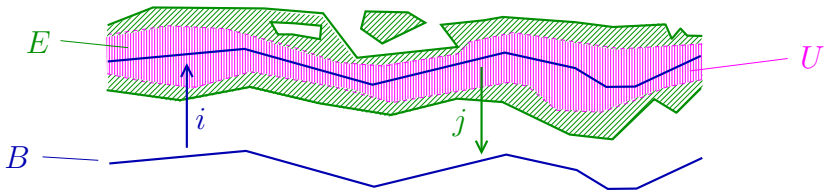
(3) Let M be an n-dimensional topological manifold with $\partial M = \varnothing$. One can easily show, see Exercise 190.8 or alternatively [**Miln1964a**, Lemma 2.1], that the maps

$$\begin{array}{rcl} i\colon M & \to & M \times M \\ x & \mapsto & (x, x) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon M \times M & \to & M \\ (x, y) & \mapsto & x \end{array}$$

form an n-dimensional microbundle. This microbundle is called the *tangent microbundle of* M.[2047] □

As we just discussed, $(\mathbb{R}^n, 0)$-bundles are microbundles. Somewhat surprisingly, in the settings we are interested in, every microbundle is essentially an $\mathbb{R}^n$-bundle:

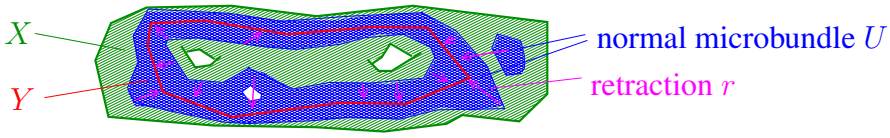
Theorem 190.26. (Kister-Mazur Theorem) Let $i\colon B \to E$ and $j\colon E \to B$ be an n-dimensional microbundle. If B is a locally finite, finite-dimensional simplicial complex or if B is a topological manifold, then there exists an open neighborhood U of $i(B) \subseteq E$ such that the restriction of j to U is an $(\mathbb{R}^n, 0)$-bundle.



Proof. This theorem was proved by James Kister [**Kis1964**, Corollary 1 and Theorem 2] (see also [**Kis1963**]) and independently by Barry Mazur. The proof is also written up as [**Kupe2017b**, Theorem 3.7]. ■

[2045] We will study this concept in much greater detail in Definition 298.2.
[2046] This example is of relevance since one can easily show that any $\mathbb{R}^n$-vector bundle naturally defines an $(\mathbb{R}^n, 0)$-bundle.
[2047] This definition and this name is inspired by the Total Tangent Space-Diagonal Embedding Proposition 53.23 which says that given a closed smooth submanifold of $\mathbb{R}^n$ there exists a smooth embedding of the tangent bundle $\widetilde{\mathrm{T}}M$ into the product $M \times M$ with $(x, 0) \mapsto (x, x)$ for each $x \in M$.

Definition 190.27. Let X be a topological manifold and let Y be a submanifold of X. A normal microbundle of Y in X consists of a neighborhood U of $Y \subseteq X$ and a retraction $j\colon U \to Y$ such that the inclusion $i\colon Y \to U$ and the retraction $j\colon U \to Y$ form a microbundle.



Example. Let M be a closed topological manifold. We consider the topological manifold $M \times M$ and the submanifold $\Delta_M := \{(x,x) \in M \times M \,|\, x \in M\}$ that is given by the diagonal. It follows easily from the above discussion that the map $j\colon M \times M \to \Delta_M$ that is given by $j(x,y) = (x,x)$ is a normal microbundle of $\Delta_M \subseteq M$. □

Remark.

(1) Note that if Y admits a normal microbundle, then it follows from the elementary discussion in Footnote 3635 that Y is in particular a neighborhood retract in X in the sense of Definition 189.3.

(2) A normal microbundle can be viewed as an analog of the normal vector bundle of a smooth submanifold of a smooth manifold. □

Colin Rourke and Brian Sanderson [**RS1968**, Theorem 3.12] gave an example of a topological submanifold of $S^9 \times S^{19}$ that is homeomorphic to S^{19} and that does not admit a normal microbundle. But fortunately the following theorem says that normal microbundles exist if the codimension of the submanifold is either small or sufficiently large.

Theorem 190.28. (Microbundle Existence-Uniqueness Theorem) Let X be a compact $(n+q)$-dimensional topological manifold and let Y be a compact n-dimensional submanifold of X with $\partial Y = \varnothing$.

(1) Let $q = 1$. We now assume that X and Y are oriented.
   (a) There exists an orientation-preserving embedding $\beta\colon [-1,1] \times Y \to X$ with the following two properties:
      (i) For each $y \in Y$ we have $\beta(0,y) = y$.
      (ii) The image $\beta([-1,1] \times Y)$ is a submanifold of X.
   (b) Given any two maps $\beta_0, \beta_1\colon [-1,1] \times Y \to X$ as in (a) there exists a homeotopy $F\colon X \times [0,1] \to X$ rel Y with $F_0 = \mathrm{id}$ and such that $F_1 \circ \beta_0 = \beta_1\colon Y \times [-1,1] \to X$.

(2) Let $q = 2$. We assume that Y is a closed submanifold and we assume that X and Y are oriented.
   (a) There exists an orientation-preserving embedding $\tau\colon \overline{B}^2 \times Y \to X$ with the following two properties:
      (i) For each $y \in Y$ we have $\tau(0,y) = y$.
      (ii) The image $\tau(\overline{B}^2 \times Y)$ is a submanifold of X.
   (b) Given any two maps $\tau_0, \tau_1\colon \overline{B}^2 \times Y \to X$ as in (a) there exists a continuous map $g\colon Y \to \mathrm{SO}(2)$ and a homeotopy $F\colon X \times [0,1] \to X$ rel Y with $F_0 = \mathrm{id}$ and such that for each $(x,y) \in \overline{B}^2 \times Y$ we have $F_1(\tau_0(x,y)) = \tau_1(g(y) \cdot x, y)$.

(3) If there exists a $j \in \{0,1,2\}$ with $n \leq q+1+j$ and $q \geq 5+j$, then Y admits a normal microbundle restricting to a normal microbundle of $\partial N \subseteq \partial M$.

Proof.

(1) The existence was proved by Morton Brown [**BrownM1962b**]. (In fact the existence can be deduced from the Topological Collar Neighborhood Theorem 35.14.) The uniqueness statement follows from [**Arm1970**, Theorem 2] (see also [**KSi1977**, Essay I, Theorem A.2, p. 40]).

(2) For $n \neq 2$ this statement was proved by Rob Kirby and Larry Siebenmann [**KSi1975**, Theorem A and p. 311], using the Kister-Mazur Theorem 190.26. The case $n = 2$ was proved by Mike Freedman and Frank Quinn [**FQ1990**, Section 9.3].

(3) This statement was proved by Ron Stern [**Ster1975**, Theorem 4.5], also using the Kister-Mazur Theorem 190.26. (If $n \leq q+j$, then this normal microbundle is unique in a suitable sense, we refer to [**Ster1975**, Theorem 4.5] for the precise formulation.) ■

We will not develop the theory of microbundles any further. We refer to [**Scor2005, FNOP2025**] and [**KSi1977**] for more information and the role they play in the study of topological manifolds.

190.5. Exercises for Chapter 190 .

Exercise 190.1. Give an example of a topological space that is weakly locally contractible but not locally contractible.

Exercise 190.2.

(1) Show that there are only countably many homotopy types of finite CW-complexes.
Hint. Use the Finite CW-Homotopy Type Theorem 190.10.

(2) Show that there are uncountably many homotopy types of countable CW-complexes.

(3) Let $k, n \in \mathbb{N}$.
   (a) Are there only finitely many homotopy types of k-dimensional CW-complexes with at most n cells?
   (b) Are there only finitely many homotopy types of k-dimensional simplicial complexes with at most n simplices?

Exercise 190.3. Show that there exist embeddings $f\colon S^1 \to S^3$ which do not admit a tubular neighborhood, i.e. for which there is no continuous map $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ with $f(P) = \Phi(P,0)$ for all $P \in S^1$.

Exercise 190.4. Let C be the Cantor set from Definition 3.45. Note that basically by construction the complement of C in $[0,1]$ is the disjoint union of countably many open intervals $\{(a_n - \epsilon_n, a_n + \epsilon_n)\}_{n\in\mathbb{N}}$. We set

$$X \;:=\; \underbrace{\{(c,0) \in \mathbb{R}^2 \,|\, c \in C\}}_{=C\times\{0\}\subseteq\mathbb{R}^2} \cup \bigcup_{n\in\mathbb{N}} \overline{B}^2_{\epsilon_n}((a_n,0)) \;\subseteq\; \mathbb{R}^2.$$

In the following we will show that X is a retract of a finite CW-complex but that X itself is not a CW-complex.

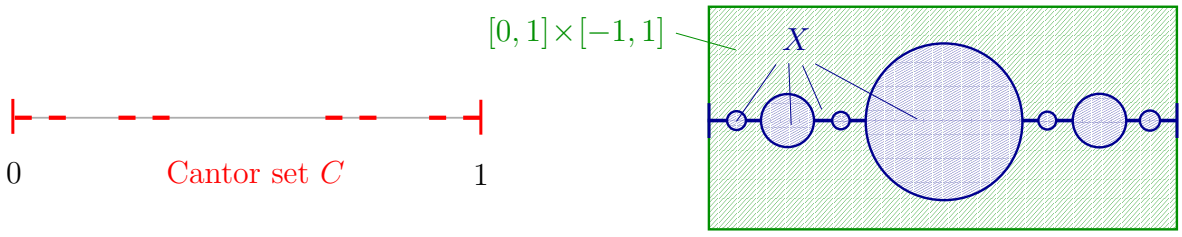
(a) Show that X is a deformation retract of the CW-complex $[0,1] \times [-1,1] \subseteq \mathbb{R}^2$.

(b) Show that X is compact.

Given a path-connected topological space Y we define

$$S(Y) \coloneqq \{y \in Y \,|\, Y \setminus \{y\} \text{ is not path-connected}\}.$$

(c) Show that $S(X) = \{(c, 0) \,|\, c \in C \setminus \{0, 1\}\}$.
(d) Let Y be a finite CW-complex. Show that $S(Y)$ has only finitely many path-components.
(e) Show that X does not admit the structure of a CW-complex.



Exercise 190.5. Let X be a compact space that is a retract of a finite simplicial complex X. Show that X is an ENR.
Remark. This is the, rather easy, converse to the Simplicial Complex Retract Proposition 189.11 (2).

Exercise 190.6. Let X be a compact space. Is $\mathrm{H}_1(X)$ necessarily finitely generated?

Exercise 190.7. Suppose we are given a microbundle consisting of two continuous maps $i\colon B \to E$ and $j\colon E \to B$. Show that the dimension of the microbundle is well-defined.
Hint. Use local homology groups.

Exercise 190.8. Let M be an n-dimensional topological manifold with $\partial M = \varnothing$. Show that the maps

$$\begin{array}{rcl} i\colon M & \to & M \times M \\ x & \mapsto & (x, x) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon M \times M & \to & M \\ (x, y) & \mapsto & x \end{array}$$

form an n-dimensional microbundle.

Exercise 190.9. Let X be a compact $(n+1)$-dimensional topological manifold and let Y be a compact n-dimensional submanifold of X.

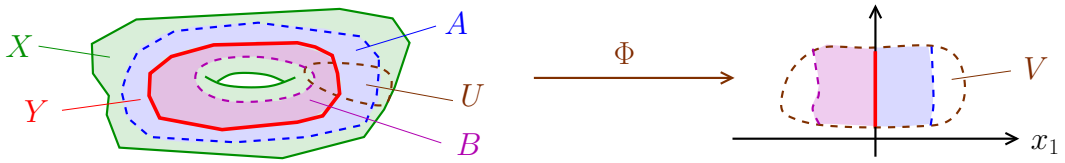
(a) We use Definition 59.14:
  (i) A *submanifold–two-coloring for* Y is a pair (A, B) of disjoint open subsets A and B of X such that the following conditions are satisfied:
    - $A \cup Y \cup B$ is an open subset of X,
    - For every submanifold chart $\Phi\colon U \to V$ of type (α), in the sense of Definition 23.18, one of the following holds:
      ▷ $\Phi(A \cap U) \subseteq \{(x_1, \dots, x_k) \,|\, x_1 < 0\}$ and $\Phi(B \cap U) \subseteq \{(x_1, \dots, x_k) \,|\, x_1 > 0\}$, or
      ▷ $\Phi(B \cap U) \subseteq \{(x_1, \dots, x_k) \,|\, x_1 > 0\}$ and $\Phi(A \cap U) \subseteq \{(x_1, \dots, x_k) \,|\, x_1 < 0\}$.
  (ii) We say that the submanifold Y is *two-sided* if Y admits a submanifold–two-coloring.

We now assume that X and Y are oriented.

(a) Show that Y is two-sided.

(b) Use the Topological Collar Neighborhood Theorem 35.14 to show that there exists an orientation-preserving embedding $\beta\colon [-1,1]\times Y \to X$ with the following two properties:

- For each $y \in Y$ we have $\beta(0,y) = y$.
- The image $\beta([-1,1]\times Y)$ is a submanifold of X.



CHAPTER 191

Orientations of topological manifolds

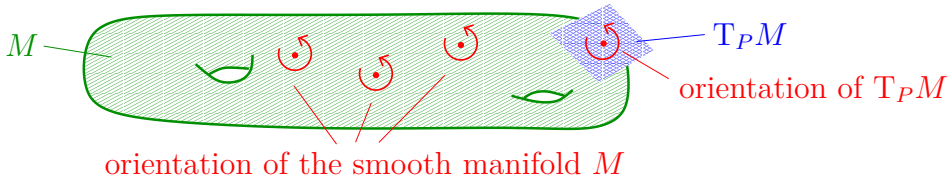
In Chapter 31 we introduced the notion of an orientation of a smooth manifold M. In this chapter we will see that one can extend the notion of an orientation in a sensible way to topological manifolds.

This chapter is organized as follows:

- In Section 191.1 we will recall the notion of an orientation of a smooth manifold.
- In Section 191.2 we will study local homology of topological manifolds and we will formulate the Local-H_*-Basics Lemma 191.7.
- In Section 191.3 we will prove the Local-H_*-Basics Lemma 191.7.
- In Section 191.4 we will study R-sections on topological manifolds and we will state the Sections-on-Connected Proposition 191.13.
- In Section 191.5 we will prove the Sections-on-Connected Proposition 191.13.
- In Section 191.6 we will introduce orientations of topological manifolds.
- In Section 191.7 we will study orientation-preserving local homeomorphisms.
- In Section 191.8 we will compare the notions of smooth and topological orientations.
- In Appendix 191.9 we will introduce the orientation covering of topological manifolds.
- In Section 191.10 we will provide exercises for Chapter 191.

In the next Chapter 192 we will discuss several instances of induced orientations.

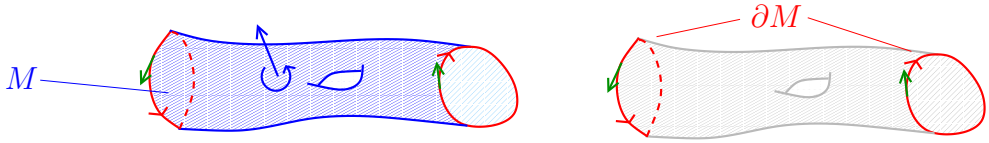
191.1. **Orientations of smooth manifolds.** In Chapter 31 we introduced the notion of an orientation of a smooth manifold M. The basic idea is that an orientation of a smooth manifold M is an orientation of each tangent space $\mathrm{T}_P M$ such that the orientations are "continuous in P".



We showed that orientations have various "inheritance properties":

(1) In the Manifold–Orientation Basics Lemma 31.31 (6) we showed that codimension-zero submanifolds and coverings of oriented smooth manifolds inherit a natural orientation.

(2) In the Boundary Orientation Lemma 31.38 we showed that the boundary of an oriented smooth manifold inherits a natural orientation.



(3) In the Product Orientation Proposition 31.48 we showed that the product of two oriented smooth manifolds admits a natural orientation.

But what is an orientation actually good for? In the earlier chapters we have already given several answers:

(a) Let M be a closed oriented non-empty n-dimensional smooth manifold. In the Form Integration Proposition 74.15 we showed that we can integrate any smooth n-form ω over M. Furthermore, in the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 we showed, using Stokes' Theorem 75.11, that the map

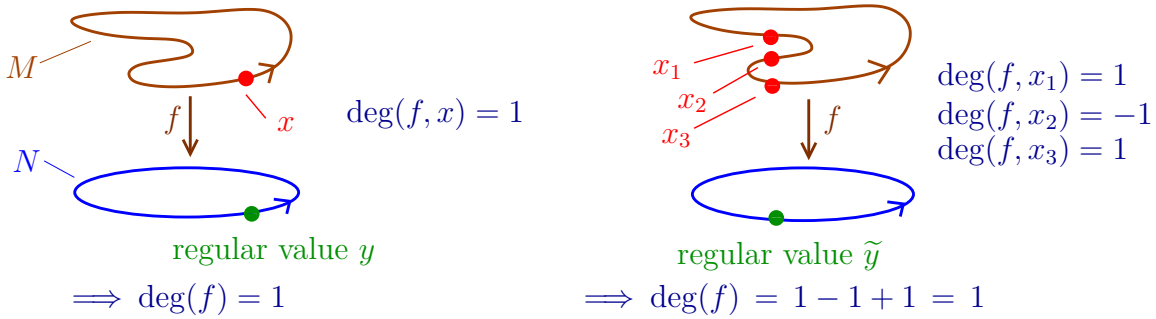
$$\begin{array}{ccc} \overbrace{\mathrm{H}^n_{\mathrm{dR}}(M)}^{\text{de Rham cohomology}} & \to & \mathbb{R} \\ [\omega] & \mapsto & \int\limits_M \omega \\ \uparrow & & \\ \text{smooth } n\text{-form} & & \end{array}$$

is well-defined and that it is an epimorphism. Finally, in the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 84.7 we showed that if M is furthermore connected, then the above map $\mathrm{H}^n_{\mathrm{dR}}(M) \to \mathbb{R}$ is even an isomorphism.

(b) In Definition 46.5 we showed that to every smooth map $f\colon M \to N$ between two closed oriented connected non-empty n-dimensional smooth manifolds we can associate a degree $\deg(f) \in \mathbb{Z}$. For the reader's convenience we quickly recall the definition: We pick a regular value $y \in N$ (which exists by Sard's Theorem 38.1) and we define

$$\deg(f) \coloneqq \sum_{x \in f^{-1}(y)} \begin{cases} +1, & \text{if } \mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_yN \text{ is orientation-preserving,} \\ -1, & \text{if } \mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_yN \text{ is orientation-reversing.} \end{cases}$$

In the Degree–Well-Defined Theorem 46.2 we showed that the degree is well-defined and in Chapter 46 we saw that this concept has many applications.

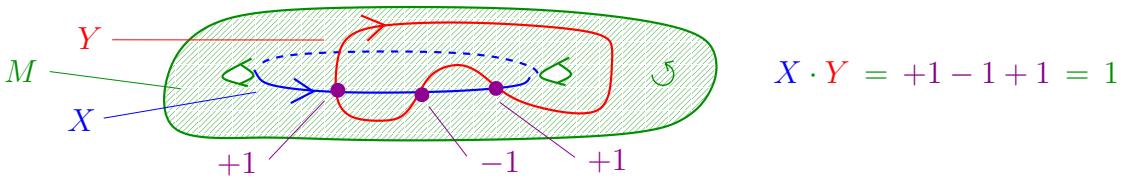


(c) Let M be an oriented smooth manifold and let $X, Y \subseteq M$ be two closed oriented submanifolds of M that intersect transversally and with $\dim(X) + \dim(Y) = \dim(M)$. Given $P \in X \cap Y$ we define $\mathrm{sign}(P)$ as follows: If a positive basis of T_PX followed by a positive basis of T_PY is a positive basis of T_PM, then we set $\mathrm{sign}(P) = +1$. Otherwise we set $\mathrm{sign}(P) = -1$.

In Definition 51.3 we defined the algebraic intersection number of X and Y as follows:

$$X \cdot Y \coloneqq \sum_{P \in X \cap Y} \mathrm{sign}(P).$$

In Chapter 51 we saw that this concept has many fun applications.



Our goal in this chapter is to introduce the notion of an orientation of a *topological* manifold M which has the following features:

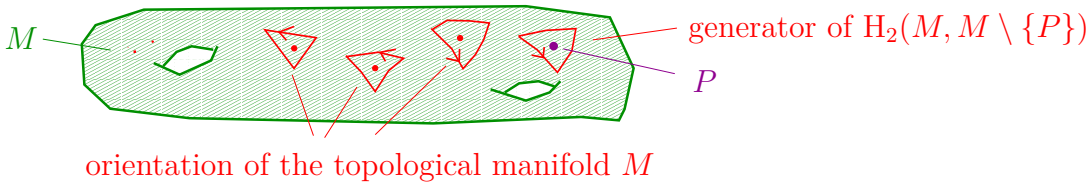
(0) It extends the previous notion in the sense that an oriented smooth manifold is naturally also an oriented topological manifold.
(1) Codimension-zero submanifolds and coverings of oriented topological manifolds have natural orientations.
(2) Boundaries of oriented topological manifolds have natural orientations.
(3) Products of oriented topological manifolds have natural orientations.

In the following chapters we will see that we also have the following analogs and generalizations of the above statements (a), (b) and (c):

(a) In the Fundamental Class-Proposition 194.2 we will see that every compact oriented connected non-empty n-dimensional topological manifold admits a natural isomorphism $\mathrm{H}_n(M, \partial M) \to \mathbb{Z}$.
(b) In Definition 151.3 we introduced the degree of any continuous map $S^n \to S^n$. In Definition 195.1 we will use (a) to extend this concept. More precisely, we will assign to every continuous map $f\colon M \to N$ between two closed oriented connected non-empty n-dimensional topological manifolds a degree $\deg(f) \in \mathbb{Z}$.
(c) In Definition 224.12 we will introduce the intersection form of any compact oriented topological manifold. It will follow from the Cup Product–Intersection Number Theorem 220.10 that the intersection form is a generalization of the algebraic intersection number.

So how can we achieve this? As we just mentioned, an orientation of a smooth manifold is a "continuous" choice of orientations for each tangent space. Since topological manifolds do not come with tangent spaces this definition does not make sense for topological manifolds.

What we can do is the following: If M is an n-dimensional topological manifold, then given any $P \in M$ we can consider the local homology group $\mathrm{H}_n(M, M \setminus \{P\})$. If $P \notin \partial M$, then one can show easily that $\mathrm{H}_n(M, M \setminus \{P\}) \cong \mathbb{Z}$. Instead of picking one of the two orientations of the tangent space at P we now pick one of the two generators of the local homology group $\mathrm{H}_n(M, M \setminus \{P\})$. We then have to make sense of the notion that this choice is "continuous in P".



In the following sections we will develop this idea.

191.2. **Local homology of topological manifolds.** We start out by recalling Definition 150.2:

Definition 191.1. Given a pointed topological space (X, x_0) and given a commutative ring R the homology group $\mathrm{H}_n(X, X \setminus \{x_0\}; R)$ is called the n-th local homology group with R-coefficients of X at the point x_0.

In this section we will study the local homology of topological manifolds. We start out with the following basic lemma which is almost identical to the Manifold-Local-H_*-Lemma 150.9.

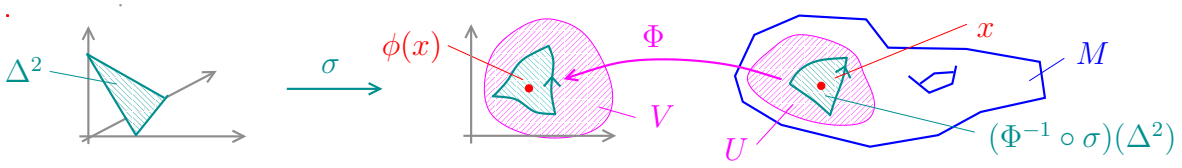
Lemma 191.2. (Manifold-Local R-Homology Lemma) Let M be an n-dimensional topological manifold and let R be a commutative ring. For every $k \in \mathbb{N}$ and every $x \in M$ the following statements hold:

(1) We have
$$\mathrm{H}_k(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } x \in M \setminus \partial M \text{ and if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The following map is an isomorphism of R-modules:
$$\begin{array}{ccc} \mathrm{H}_k(M, M \setminus \{x\}) \otimes R & \xrightarrow{\mu} & \mathrm{H}_k(M, M \setminus \{x\}; R) \\ \sum_{i=1}^{m} [\sigma_i] \otimes r_i & \mapsto & \sum_{i=1}^{m} [\sigma_i \otimes r_i]. \end{array}$$

(3) We have
$$\mathrm{H}_k(M, M \setminus \{x\}; R) \cong \begin{cases} R, & \text{if } x \in M \setminus \partial M \text{ and if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $x \in M \setminus \partial M$, if $\Phi\colon U \to V$ is a chart around x and if $\sigma\colon \Delta^n \to V$ represents a generator of $\mathrm{H}_n(V, V \setminus \{\Phi(x)\})$, then $(\Phi^{-1} \circ \sigma) \otimes 1_R$ represents a generator of $\mathrm{H}_n(M, M \setminus \{x\}; R)$.



Proof.

(1) We proved this statement in the Manifold-Local-H_*-Lemma 150.9. For the reader's convenience we recall the argument. First we consider the case that $x \in M \setminus \partial M$. Since $x \notin \partial M$ we can pick a chart $\Phi\colon U \to V$ around x such that V is an open subset of $\mathbb{R}^n$. We write $y = \Phi(x)$. We pick $r > 0$ with $\overline{B}^n_r(y) \subseteq V$. For any $k \in \mathbb{N}_0$ we then have
$$\mathrm{H}_k(M, M \setminus \{x\}) \xleftarrow[\text{by the Excision Theorem 148.2}]{\cong} \mathrm{H}_k(U, U \setminus \{x\}) \xrightarrow[\text{isomorphism since } \Phi \text{ is a homeomorphism}]{\Phi_*} \mathrm{H}_k(V, V \setminus \{y\}) \xleftarrow[\text{by the Excision Theorem 148.2}]{\cong} \mathrm{H}_k(\overline{B}^n_r(y), \overline{B}^n_r(y) \setminus \{y\})$$
$$\xrightarrow[\text{isomorphism since } \widetilde{\mathrm{H}}_*(\overline{B}^n_r(y)) = 0]{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\overline{B}^n_r(y) \setminus \{y\}) \xrightarrow[\text{since } S^{n-1}_r(y) \text{ is a deformation retract of } \overline{B}^n_r(y) \setminus \{y\}]{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}_r(y)) \xrightarrow[\text{by the Sphere-H}_* \text{ Proposition 160.13}]{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$
The case $x \in \partial M$ is dealt with by a similar argument which we leave to the reader.

(2) This statement follows easily from (1), the Universal Coefficient Theorem 162.16 and the Tor-Properties Lemma 162.13 (4).
(3) The calculation of $\mathrm{H}_k(M, M \setminus \{x\}; R)$ is an almost immediate consequence of (1), (2) and the Tensor-with-$\mathbb{Z}$-Lemma 159.9. The statement about the generator follows easily from the argument in (1). ■

On a few occasions the following lemma will be useful.

Lemma 191.3. (Local-H_*-Reflection Lemma) Let $n \in \mathbb{N}_0$ and let $P \in \mathbb{R}^n$. Furthermore let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane through P. For every commutative ring R the induced map

$$\rho_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is given by multiplication by minus -1.

Proof. Let $\rho_V\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane V through a point P. We denote by $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by addition by P and furthermore we write $W = f^{-1}(V)$. Note that W goes through the origin. We denote by $\rho_W\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the hyperplane W. We consider the following diagram:

$$\begin{array}{ccccccc}
\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \otimes R & \xrightarrow[\cong]{\mu} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \xleftarrow[\cong]{\partial_n} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \\
\downarrow{\scriptstyle \rho_{W*}\otimes \mathrm{id}} & & \downarrow{\scriptstyle \rho_{W*}} & & \downarrow{\scriptstyle \rho_{V*}} & & \downarrow{\scriptstyle \rho_{V*}} \\
\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \otimes R & \xrightarrow[\cong]{\mu} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \xleftarrow[\cong]{\partial_n} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R).
\end{array}$$

where μ is the natural map from Footnote 1707 and where the right-hand horizontal maps are the connecting homomorphisms from the Reduced LES-of-Pair Proposition 147.8.

It follows from the Sphere-Degree-Matrix Lemma 151.7 and the Matrix-Tensor-with-Ring Lemma 159.20 that the vertical map on the left is given by multiplication by -1. It follows from the naturality of μ and of the connecting homomorphism ∂_n, and the functoriality of homology, that the diagram commutes. Furthermore it follows from the Universal Coefficient Theorem 162.16 and other standard arguments that all horizontal maps are isomorphisms. Furthermore the horizontal maps on the top and bottom are identical. It thus follows that the vertical map on the right is also given by multiplication by -1. ■

Next we want to find explicit generators of local homology groups $\mathrm{H}_n(M, M \setminus \{x\})$ for n-dimensional topological manifolds M and ideally we would like to find "standard generators". To do so it is worth recalling Definition 143.8.

Definition 191.4. Let $n \in \mathbb{N}$. We write

$$\partial\Delta^n := \bigcup_{i=0}^{n} \{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0\}.$$

and we refer to $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$ as the interior of Δ^n.

The following lemma can be viewed as a mild extension of the Identity-is-Generator Lemma 150.1.

Lemma 191.5. (Identity-is-Generator Lemma) For any $k \in \mathbb{N}_0$ and any commutative ring R we have

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ R \cdot [\mathrm{id}_{\Delta^n} \otimes 1_R], & \text{if } k = n. \end{cases}$$

Proof. This lemma follows easily from the following two observations:

(1) By the "original" Identity-is-Generator Lemma 150.1 we know that

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ \mathbb{Z} \cdot [\mathrm{id}_{\Delta^n}], & \text{if } k = n. \end{cases}$$

(2) By the Universal Coefficient Theorem 162.16 we have the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}_k(\Delta^n, \partial\Delta^n) \otimes R & \xrightarrow{\mu} & \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\nu} & \underbrace{\mathrm{Tor}(\mathrm{H}_{k-1}(\Delta^n, \partial\Delta^n), R)} & \to & 0. \\ & & \sum_{i=1}^m [\sigma_i] \otimes r_i & \mapsto & \sum_{i=1}^m [\sigma_i \otimes r_i]. & & = 0, \text{ by (1) and the Tor-Properties Lemma 162.13 (4)} & & \end{array}$$

■

Next we recall more concepts from Definition 143.8.

Definition 191.6. Let $n \in \mathbb{N}$.

(1) We equip $\mathbb{R}^n$ with the orientation given by the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$. With this orientation we can view any open subset of $\mathbb{R}^n$ as an oriented smooth manifold.

(2) As in Definition 143.8 we say that a basis $\{v_1, \ldots, v_n\}$ for $\mathrm{T}_P\mathring{\Delta}^n$ is positive, if the ordered set of vectors $\{(1, \ldots, 1), v_1, \ldots, v_n\}$ is a positive basis for $\mathbb{R}^{n+1}$. This equips $\mathring{\Delta}^n$ with the structure of an oriented smooth manifold.

(3) Given a subset V of $\mathbb{R}^n$ we say that a map $f\colon V \to \mathbb{R}^k$ is affine linear if there exists a $P \in \mathbb{R}^k$ and a $(k \times n)$–matrix A such that $f(v) = P + A \cdot v$ for all $v \in V$.

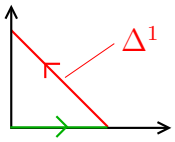


orientations of the interiors of the standard simplices

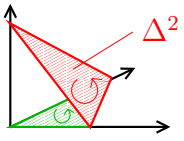
Example. The projection

$$\begin{array}{ccc} p\colon \Delta^n & \to & \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) & \mapsto & (x_1, \ldots, x_n) \end{array}$$

is an injective affine linear map.



projection is orientation-reversing



projection is orientation-preserving

One can easily verify that the restriction of this map to the interior $\mathring{\Delta}^n$ is orientation-preserving if and only if n is even. □

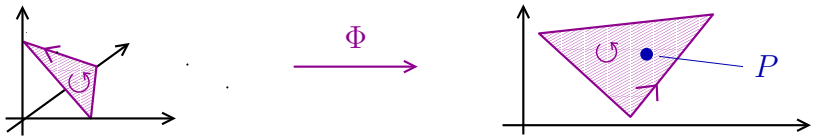
Lemma 191.7. (Local-H_*-Basics Lemma) Let R be a commutative ring, let $U \subseteq \mathbb{R}^n$ be an open subset and let $P \in \mathbb{R}^n$. Let $\Phi\colon \Delta^n \to U$ be an injective affine linear map such that P lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is *orientation-preserving*.

(1) For any $k \in \mathbb{N}_0$ the induced map

$$\Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;\to\; \mathrm{H}_k(U, U \setminus \{P\}; R)$$

is an isomorphism

(2) The map in (1) does not depend on the choice of the orientation-preserving map Φ.



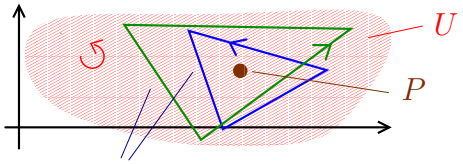
Proof. We delay the proof of the lemma to Section 191.3. ■

Definition 191.8. Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$ and let $P \in U$. Let $\Phi\colon \Delta^n \to U$ be an injective affine linear map such that P lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is *orientation-preserving*. By the Local-H_*-Basics Lemma 191.7 the induced map

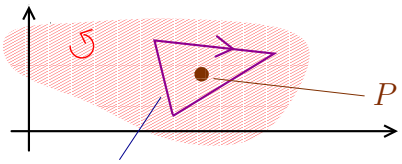
$$\Phi_*\colon \underbrace{\mathrm{H}_n(\Delta^n, \partial\Delta^n; R)}_{=R\cdot[\mathrm{id}_{\Delta^n}\otimes 1_R]} \xrightarrow{\cong} \mathrm{H}_n(U, U \setminus \{P\}; R)$$

is an isomorphism and this isomorphism does not depend on the choice of Φ. By the Identity-is-Generator Lemma 191.5 we know that $\mathrm{H}_n(\Delta^n, \partial\Delta^n; R) = R \cdot [\mathrm{id}_{\Delta^n} \otimes 1_R]$. We refer to the image of the generator $[\mathrm{id}_{\Delta^n} \otimes 1_R] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n; R)$ as the standard generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.

Example. We consider the following figure:



standard generator of $\mathrm{H}_2(U, U\setminus\{P\}; \mathbb{Z})$



minus the standard generator

To the left we see representatives of the standard generators.[2048] Here, for simplicity, given a singular 2-simplex $\Delta^2 \to \mathbb{R}^2$ we only sketch the image of $\partial\Delta^2$ with the given orientation. Also note that it follows from the Local-H_*-Reflection Lemma 191.3 that the singular simplex illustrated in the right of the figure below represents *minus* the standard generator. □

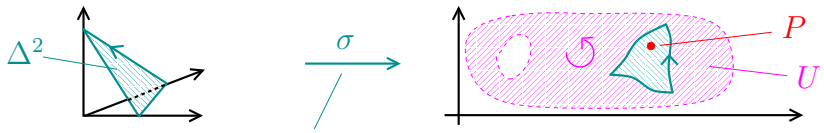
The following lemma gives a convenient criterion for a singular simplex to represent a standard generator:

[2048] At this occasion it is good to remember that if $\sigma\colon \Delta^k \to X$ is a continuous map with $\sigma(\partial\Delta^{k-1}) \subseteq A$, then σ is a cycle in $\mathrm{C}_k(X, A)$, i.e. it represents a class in $\mathrm{H}_k(X, A)$.

Lemma 191.9. (Local-H_*–Generator Lemma) Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$, let $P \in U$ and let $\sigma\colon \Delta^n \to U$ be a singular n-simplex with the following properties:

(1) The point P lies in the image of $\mathring{\Delta}^n$.
(2) The restriction of σ to a map $\mathring{\Delta}^n \to U$ is a diffeomorphism onto its image.[2049]
(3) The restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving (as a smooth map).

Then $\sigma \otimes 1_R$ represents the standard generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.



$\sigma \otimes 1_R$ represents the standard generator of $\mathrm{H}_2(U, U \setminus \{P\}; R)$

Proof. To simplify the notation a little bit we work with homology without coefficients. Also note that it follows immediately from the definition of the standard generator that it suffices to deal with the case $U = \mathbb{R}^n$.

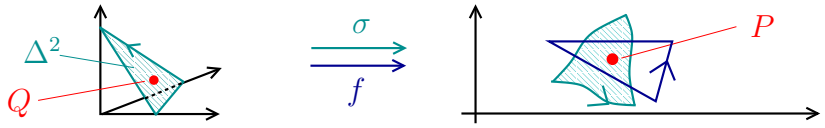
Thus let $P \in \mathbb{R}^n$ and let $\sigma\colon \Delta^n \to U$ be a singular n-simplex that satisfies (1), (2) and (3). We denote by $Q \in \mathring{\Delta}^n$ the point with $\sigma(Q) = P$.

As always we denote by $\mathrm{D}\sigma_Q\colon \mathrm{T}_Q\mathring{\Delta}^n \to \mathrm{T}_P U = \mathbb{R}^n$ the differential of σ at the point Q. We denote by

$$\begin{array}{rccl} f\colon \Delta^n & \to & \mathbb{R}^n \\ x & \mapsto & (\mathrm{D}\sigma_Q)\underbrace{(x-Q)}_{\in \mathrm{T}_Q\Delta^n} + P \end{array}$$

the linear approximation of σ at the point Q. We make the following two observations:

(a) Since $\sigma\colon \mathring{\Delta}^n \to U$ is a diffeomorphism onto its image we obtain that $\mathrm{D}\sigma_Q$ is an isomorphism of vector spaces, in particular f is an injective affine linear map.
(b) Basically the same argument as in the proof of the Linearization–Homotopy Lemma 152.7 shows that there exists an open neighborhood V of Q in $\mathring{\Delta}^n$ such that the restrictions of σ and f to maps $\sigma, f\colon (V, V \setminus \{Q\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$ are homotopic.



Now consider the following diagram:

$$\begin{array}{ccccccccc} \mathrm{H}_n(\Delta^n,\partial\Delta^n) & \to & \mathrm{H}_n(\Delta^n,\Delta^n\setminus\{Q\}) & \longleftarrow & \mathrm{H}_n(V,V\setminus\{Q\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(V,V\setminus\{Q\}) & \to & \mathrm{H}_n(\Delta^n,\Delta^n\setminus\{Q\}) \\ & {\scriptstyle \sigma_*}\searrow & & {\scriptstyle \sigma_*}\searrow & \downarrow{\scriptstyle \sigma_*} & \swarrow{\scriptstyle f_*} & & \swarrow{\scriptstyle f_*} & \\ & & & & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{P\}). & & & & \end{array}$$

It is clear that the first two triangles and the last triangle commute. Furthermore the third triangle commutes by the above observation (b) together with the Homotopy-H_*-Proposition 146.11. It follows easily from the Excision Theorem 148.2 that the last diagonal

[2049] Note that we do not make any assumptions on the behavior of σ restricted to $\partial\Delta^n$. If one looks very carefully, then one realizes that one has to convince oneself that $\sigma(\partial\Delta^n)$ is a subset of $U \setminus \{P\}$, i.e. one has to show that it follows from (1) and (2), that $P \notin \sigma(\partial\Delta^n)$. How can one prove this statement?

map is an isomorphism. It now follows easily that $\sigma \in \mathrm{C}_n(\mathbb{R}^n)$ represents a generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$.

Finally recall that we assume that the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving. This implies, by the definition of being orientation-preserving, that the map f is also orientation-preserving. It follows from the above commutative diagram that the singular n-simplex σ does indeed represent the standard generator of $\mathrm{H}_n(U, U \setminus \{P\})$. ■

191.3. Proof of the Local-H_*-Basics Lemma 191.7.

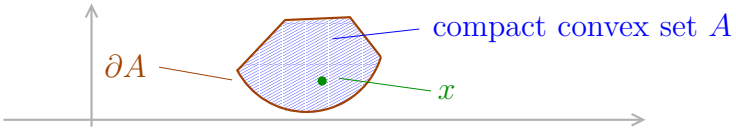
In this section we will provide the proof of the Local-H_*-Basics Lemma 191.7. The general discussion of this chapter continues in Section 191.4.

We start out with the following lemma:

Lemma 191.10. (Convex-Rel-Boundary Homology Lemma) Let $n \in \mathbb{N}$, let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in \mathring{A}$. For each abelian group G the inclusion-induced maps

$$\mathrm{H}_i(A, \partial A; G) \;\to\; \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G)$$

are isomorphisms.



Proof. Let $n \in \mathbb{N}$, let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in \mathring{A}$. To simplify the notation we drop the coefficients G from the notation and we assume that $x = 0$.

Claim. The subspace ∂A is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Proof. First note that since $0 \in \mathring{A}$ we can find $\epsilon > 0$ with $\overline{B}^n_\epsilon(0) \subseteq A$. Next note that it follows from the Convex-to-Ball Proposition 4.28 that the following map is continuous:

$$\begin{array}{rcl} \rho\colon S^{n-1}_\epsilon(0) & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sup\{\|r \cdot x\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot x \in A\}. \end{array}$$

Using this fact and using the convexity of A one can easily verify that the map

$$\begin{array}{rcl} (\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ (x, t) & \mapsto & x \cdot ((1-t) + \frac{t}{\|x\|} \cdot \rho(\epsilon \cdot \frac{x}{\|x\|})) \end{array}$$

is a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to ∂A. ⊞

Now let $i \in \mathbb{N}_0$. From the Reduced LES-of-Pair Proposition 147.8 we obtain the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \widetilde{\mathrm{H}}_i(\partial A) & \longrightarrow & \widetilde{\mathrm{H}}_i(A) & \xrightarrow{\text{я}} & \mathrm{H}_i(A, \partial A) & \xrightarrow{\partial_i} & \widetilde{\mathrm{H}}_{i-1}(\partial A) & \longrightarrow & \widetilde{\mathrm{H}}_{i-1}(A) \longrightarrow \ldots \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ \ldots & \longrightarrow & \widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus \{x\}) & \longrightarrow & \widetilde{\mathrm{H}}_i(\mathbb{R}^n) & \xrightarrow{\text{я}} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) & \xrightarrow{\partial_i} & \widetilde{\mathrm{H}}_{i-1}(\mathbb{R}^n \setminus \{x\}) & \longrightarrow & \widetilde{\mathrm{H}}_{i-1}(\mathbb{R}^n) \longrightarrow \ldots \end{array}$$

We make the following observations:

(1) It follows from the claim and the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 147.5 (1b) that the first and fourth vertical maps are isomorphisms.

(2) Since A and $\mathbb{R}^n$ are contractible we see that the second and fifth vertical maps are isomorphisms.
(3) It follows from (1) and (2) together with the Five Lemma 145.11 that the middle vertical map is also an isomorphism. ■

Proof of the Local-H_*-Basics Lemma 191.7. Let R be a commutative ring. Furthermore let $V \subseteq \mathbb{R}^n$ be an open subset and let $P \in \mathbb{R}^n$. Note that it follows from the Excision Theorem 148.1 and the discussion on page 3079 that the inclusion-induced map

$$\mathrm{H}_k(V, V \setminus \{P\}; R) \;\to\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is an isomorphism. It follows almost immediately from this observation that we can restrict ourselves to the case that $V = \mathbb{R}^n$.

Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that P lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving.

(1) We need to show that the induced map $\Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$ is an isomorphism. We have the following commutative diagram:

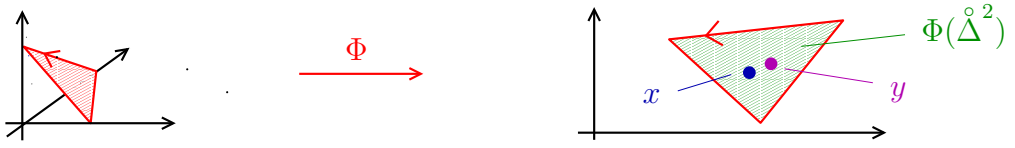
$$\begin{array}{ccc} \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\quad\Phi_*\quad} & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R). \\ & {\scriptstyle\Phi_*}\searrow \quad \nearrow & \\ & \mathrm{H}_k(\Phi(\Delta^n), \Phi(\partial\Delta^n); R) & \end{array}$$

where the right diagonal map is induced by the inclusions. Note that the left diagonal map is an isomorphism since Φ is an embedding[2050]. Since Φ is an injective affine linear map we see that $\Phi(\Delta^n)$ is a convex subset of $\mathbb{R}^n$ with interior $\Phi(\mathring{\Delta}^n)$ and boundary $\Phi(\partial\Delta^n)$. By our hypothesis the point P lies in the interior of $\Phi(\Delta^n)$. It follows from the Convex-Rel-Boundary Homology Lemma 191.10 that the right diagonal map is an isomorphism. It follows from this discussion that the top horizontal map is an isomorphism.

(2) Now we need to show that the homomorphism of (1) does not depend on the choice of Φ. To simplify the notation a little bit we now assume that $P = 0$. Thus let $\Psi\colon \Delta^n \to \mathbb{R}^n$ be another such injective affine linear orientation-preserving map. It follows from elementary linear algebra that there exists a $P \in \mathbb{R}^n$ and a matrix A with *positive* determinant such that

$$\Phi \;=\; (v \mapsto P + A \cdot v) \circ \Psi\colon \Delta^n \to \mathbb{R}^n.$$

We write $\Omega = (v \mapsto A \cdot v) \circ \Psi$.



It suffices to show that $\Psi_* = \Omega_* = \Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$.

(a) Recall that it follows from the Matrix Group-π_0-Proposition 4.54 (1) that the group $\{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) > 0\}$ is path-connected. This implies in particular that there exists a path $\gamma\colon [0, 1] \to \mathrm{GL}(n, \mathbb{R})$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. We consider the map

[2050]Here we implicitly use the ever popular Compact-Hausdorff Proposition 4.23 (2).

$$\begin{array}{rcl} F\colon \Delta^n \times [0,1] & \to & \mathbb{R}^n \\ (Q,t) & \mapsto & \gamma(t)\cdot\Psi(Q). \end{array}$$

Since by hypothesis we have $0 \notin \Phi(\partial\Delta^n)$ and since $\gamma(t)\cdot 0 = 0$ we see that $\gamma(t)(\partial\Delta^n) \subseteq \mathbb{R}^n \setminus \{0\}$ for all t. In particular we deduce that F defines a homotopy between Ψ and Ω as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But by the Homotopy-$\mathrm{H}_*$-Proposition 146.11 this means that $\Psi_* = \Omega_*$ as maps on relative homology.

(b) Now we compare Ω and $\Phi = (w \mapsto P + w) \circ \Omega$. We consider the map

$$\begin{array}{rcl} G\colon \Delta^n \times [0,1] & \to & \mathbb{R}^n \\ (w,t) & \mapsto & (P\cdot t + \Omega(w)). \end{array}$$

Note that $0 \in \Omega(\mathring{\Delta}^n) = A(\Psi(\mathring{\Delta}^n))$ and that $0 \in \Phi(\mathring{\Delta}^n)$ which implies by $\Phi = P + \Omega$ that $-P \in \Omega(\mathring{\Delta}^n)$. Since $\Omega(\mathring{\Delta}^n)$ is convex we see that G defines a homotopy between Ω and Φ as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. By the Homotopy-$\mathrm{H}_*$-Proposition 146.11 this means that $\Omega_* = \Phi_*$ as maps on relative homology. ■

191.4. ***R*-Sections on topological manifolds.** Recall the idea from page 3679: We want to define an orientation of an n-dimensional topological manifold M to consist of a generator of each local homology group $\mathrm{H}_n(M, M \setminus \{x\})$ for $x \in M \setminus \partial M$, such that the generators change "continuously" in x.

In this section we will define more generally what it means for a choice of elements in the local homology groups $\mathrm{H}_n(M, M \setminus \{x\})$ to be "continuous in x". This discussion will have many benefits, in particular in the next section we will be able to finally define the notion of an orientation of a topological manifold.

We start out with the following, initially rather obscure definition.

Definition 191.11. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-section for M at $x \in M \setminus \partial M$ is an element $\mu_x \in \underbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}_{\cong R}$.

(2) Let $W \subseteq M$ and let $x \in W$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the natural map

$$\rho_x\colon \mathrm{H}_k(M, M \setminus W; R) \to \mathrm{H}_k(M, M \setminus \{x\}; R).$$

Given $\varphi \in \mathrm{H}_k(M, M \setminus W; R)$ we often write $\varphi_x := \rho_x(\varphi)$, i.e. the above map is also written as

$$\varphi \mapsto \varphi_x.$$

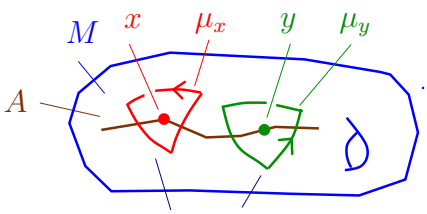
(3) Let A be a subset of $M \setminus \partial M$. (In the following we will mostly be interested in the case $A = M \setminus \partial M$). An R-section for M along A is a choice for each $x \in A$ of an R-section $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ at x such that the following "continuity condition" is satisfied:

(∗) For each $x \in A$ there exists an open neighborhood U of x in A and a homology class $\nu \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ the image of ν under the natural map

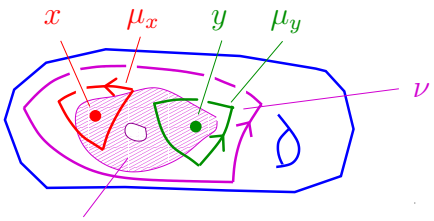
$$\rho_y\colon \mathrm{H}_n(M, M \setminus U; R) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$$

equals μ_y.

(4) We denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A.



here the $\mathbb{Z}$-sections at the points correspond to a $\mathbb{Z}$-section on M



open set U

The above Definition 191.11 allows us to formulate the following lemma:

Lemma 191.12. (Homology-defines-Section Lemma) Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) Given any subset $A \subseteq M \setminus \partial M$ the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y \in A$ the map $\Gamma_R^A(M) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ that is given by $\{\alpha_x\}_{x\in A} \mapsto \alpha_y$ is an R-module homomorphism.

(2) For every open subset $A \subseteq M \setminus \partial M$ (often we will take $A = M \setminus \partial M$) the map

$$\begin{array}{rcl} \mathrm{H}_n(M, M \setminus A; R) & \to & \Gamma_R^A(M) \\ \alpha & \mapsto & \{\alpha_x\}_{x\in A} \end{array}$$

is a well-defined homomorphism of R-modules.

Remark. In the Homology-Section Theorem 193.6 we will see that in many settings the converse to the Homology-defines-Section Lemma 191.12 (2) holds. □

Proof.

(1) This statement follows easily from the definitions and a few basic considerations. We outsource the proof to Exercise 191.3.

(2) Let $A \subseteq M \setminus \partial M$ be an open subset and let $\alpha \in \mathrm{H}_n(M, M \setminus A; R)$. Let $x \in A$. We need to show that there exists an open neighborhood U of x in A and a homology class $\nu \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ we have $\rho_x(\nu) = \alpha_x$. But this is trivial, we just take $U = A$ and $\nu = \alpha$. ■

The following proposition contains much more interesting results on $\Gamma_R^A(M)$.

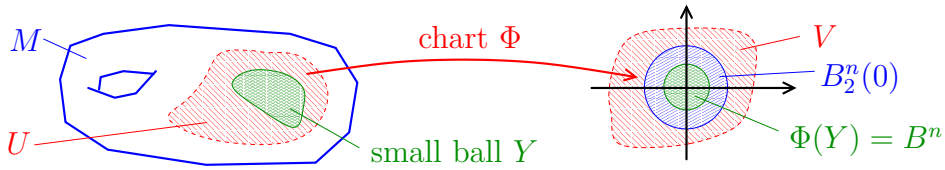
Proposition 191.13. (Sections-on-Connected Proposition) Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let A be a *connected* subset of $M \setminus \partial M$[2051] and let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. If there exists an $x \in A$ with $\alpha_x = \beta_x$, then the two R-sections agree.

Proof. We outsource the proof of the Sections-on-Connected Proposition 191.13 to the next section. The reader who wants to skip the proof can also go straight to Section 191.6. ■

191.5. **Proof of the Sections-on-Connected Proposition 191.13 .** Before we provide the proof of the Sections-on-Connected Proposition 191.13 we need to prove one little lemma which in turn requires the following definition.

[2051] In some applications we will use that by Exercise 23.23 we know that if M is a connected topological manifold, then $M \setminus \partial M$ is also connected.

Definition 191.14. Let M be an n-dimensional topological manifold. We say that $Y \subseteq M$ is a small ball in M if there exists a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subseteq V$ and such that $\Phi(Y) = B^n$.



Lemma 191.15. (Small Ball-Lemma) Let M be an n-dimensional topological manifold.

(0) Given any point $x \in M \setminus \partial M$ and any neighborhood U of x there exists a small ball Y with $x \in Y \subseteq U$.

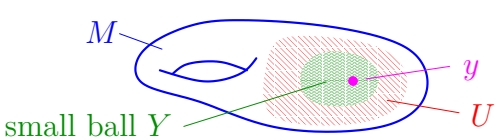
Let R be a commutative ring.

(1) For any point y in a small ball Y and any $k \in \mathbb{N}_0$ the map

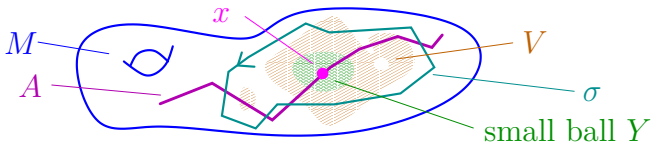
$$\rho_y\colon \mathrm{H}_k(M, M \setminus Y; R) \;\to\; \mathrm{H}_k(M, M \setminus \{y\}; R)$$

is an isomorphism.

(2) Let A be a subset of $M \setminus \partial M$, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. There exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.



Statement (1)



Statement (2)

Proof. Let M be an n-dimensional topological manifold.

(0) The proof of Statement (1) is elementary and is thus left to the reader.

(1) Let y be a point that is contained in a small ball Y. To simplify the notation we drop the commutative ring R from the notation. We consider the long exact sequence of the triple $(M, M \setminus \{y\}, M \setminus Y)$ given by the LES-of-Triple Proposition 146.5:

$$\to \mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \to \mathrm{H}_k(M, M\setminus Y) \to \mathrm{H}_k(M, M\setminus\{y\}) \to \mathrm{H}_{k-1}(M\setminus\{y\}, M\setminus Y) \to$$

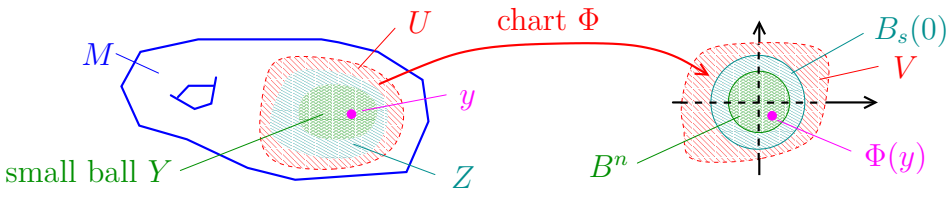
We see immediately that it suffices to show that all the relative homology groups $\mathrm{H}_k(M \setminus \{y\}, M \setminus Y)$ vanish. Since Y is a small ball we can pick a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subseteq V$ and such that $\Phi(Y) = B^n$. We pick $s \in (1, 2)$ and we write $Z = \Phi^{-1}(B_s^n(0))$. Then

$$\mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \underset{\uparrow}{\overset{\cong}{\leftarrow}} \mathrm{H}_k(Z\setminus\{y\}, Z\setminus Y) \underset{\cong}{\overset{\Phi_*}{\to}} \mathrm{H}_k(B_s^n(0)\setminus\{\Phi(y)\}, B_s^n(0)\setminus B^n) \underset{\uparrow}{=} 0.$$

by the Excision Theorem 148.2

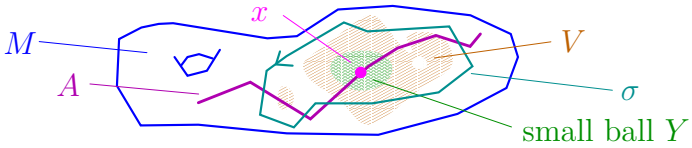
by the Deformation Retract-Relative H_*-Corollary 146.9, since $B_s^n(0) \setminus B^n$ is a deformation retract of $B_s^n(0) \setminus \{\Phi(y)\}$

(2) Let A be a subset of M, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. By definition of an R-section along A and by definition of the subspace topology there exists an open neighborhood V of $x \in M \setminus \partial M$ and a singular chain $\sigma \in \mathrm{C}_n(M; R)$



that represents a cycle in $\mathrm{C}_n(M, M \setminus (V \cap A); R)$ such that for each $y \in V \cap A$ we have $\alpha_y = \rho_y([\sigma]) \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.

Since σ represents a cycle in the relative chain complex $\mathrm{C}_*(M, M \setminus (V \cap A); R)$ we know that $\partial\sigma \in \mathrm{C}_{n-1}(M \setminus (V \cap A); R)$. It follows pretty easily from the fact that Δ^{n-1} is compact together with the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that there exists an open neighborhood $U \subseteq M$ of x with $U \subseteq V$ such that $\partial\sigma \in \mathrm{C}_{n-1}(M \setminus U; R)$. In other words, σ represents a cycle in $\mathrm{C}_n(M, M \setminus U; R)$.

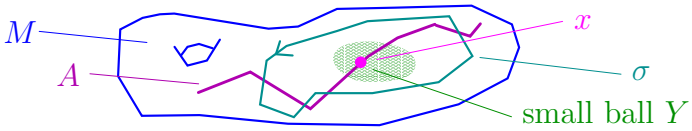


By (0) there exists a small ball Y with $x \in Y \subseteq U$. It follows immediately from the functoriality of homology groups that the homology class $[\sigma] \in \mathrm{H}_n(M, M \setminus Y; R)$ has all the desired properties. ∎

Proof of the Sections-on-Connected Proposition 191.13. Let M be an n-dimensional topological manifold, let A be a connected subset of $M \setminus \partial M$ and let R be a commutative ring. Let $\{\alpha_x\}_{x \in A}$ and $\{\beta_x\}_{x \in A}$ be two R-sections for M along A. We assume that there exists a $z \in A$ with $\alpha_z = \beta_z$. By the Homology-defines-Section Lemma 191.12 (1) we can consider the R-section along A that is given by $\gamma_x = \alpha_x - \beta_x$ for $x \in A$. Next we consider the subsets $V_{=0} := \{y \in A \,|\, \gamma_y = 0\}$ and $V_{\neq 0} := \{y \in A \,|\, \gamma_y \neq 0\}$. We need to show that $V_{=0} = A$. Since $V_{=0}$ is by hypothesis non-empty and since A is connected it suffices to show that $V_{=0}$ and $V_{\neq 0}$ are both open. In other words, it suffices to prove the following claim:

Claim. Let $x \in A$. There exists an open neighborhood U of x in A such that either γ_y is zero for every $z \in U$ or γ_y is non-zero for every $z \in U$.

Proof. Let $x \in A$. It follows immediately from the Small Ball-Lemma 191.15 (0) and (2) that there exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \gamma_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.



Note that by the Small Ball-Lemma 191.15 (1) we know that for every $y \in Y$ the map $\rho_y \colon \mathrm{H}_n(M, M \setminus Y; R) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ is an isomorphism. But this implies that $\gamma_x = 0$ if and only if $\gamma_y = 0$ for all $y \in Y$. Therefore the intersection $U := Y \cap A$ has all the desired properties. ∎

191.6. **Orientations of topological manifolds.** We start out with the following addendum to Definition 191.11 of a section along a subspace A.

Definition 191.16. Let M be an n-dimensional topological manifold and let R be a commutative ring. For $A = M \setminus \partial M$ we say "section" instead of "section along A" and we write $\Gamma_R(M)$ instead of $\Gamma_R^{M\setminus\partial M}(M)$.

After the technical preparations from the last section we can now finally introduce the notion of an orientation of a topological manifold.

Definition 191.17. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-orientation for M at a point $x \in M \setminus \partial M$ is a generator of the R-module $\underbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}_{\cong R}$.
(2) An R-section for M that is an R-orientation at each point $x \in M \setminus \partial M$ is called an R-orientation of M.
(3) The topological manifold M is called R-orientable if it admits an R-orientation.
(4) An R-oriented topological manifold is a topological manifold that is equipped with an R-orientation.

If we work with homology groups $\mathrm{H}_n(M, M \setminus \{x\})$ "without coefficients", then we say orientation, section and orientable. Note that it follows from the discussion on page 3077 that a $\mathbb{Z}$-orientation is naturally the same as an orientation.



two different orientations for the torus

Example. Let M be a 0-dimensional topological manifold. For each $x \in M$ we have a natural isomorphism

$$\mathrm{H}_0(M, M \setminus \{x\}) \xleftarrow{\cong} \mathrm{H}_0(\{x\}) \xleftarrow{n \mapsto n \cdot [x]} \mathbb{Z}.$$

this is an isomorphism since M has the discrete topology

by the H_0-Proposition 143.25 this is an isomorphism

Since M has the discrete topology we see that any assignment of generators satisfies the continuity condition of a section. Thus we see that an orientation of a 0-dimensional topological manifold is basically the same as a map $M \to \{\pm 1\}$. In other words, we see that we have the same concept of an orientation as we remarked on page 810 for 0-dimensional smooth manifolds. We refer to the orientation which assigns $1 = 1 \cdot [x]$ to each point $x \in M$ as the standard orientation of the 0-dimensional topological manifold M. □

Before we move on to other examples let us get the following basic lemma out of the way:

Lemma 191.18. (Number-of-Orientations Lemma) Let M be a topological manifold.

(1) For any point $x \in M \setminus \partial M$ there exist precisely two orientations for M at x.
(2) If M is orientable, connected and non-empty, then M admits precisely two orientations.

Proof.

(1) This statement is an immediate consequence of the fact, proved in the Manifold-Local R-Homology Lemma 191.2 (1), that for any $x \in M \setminus \partial M$ we have $\mathrm{H}_n(M, M \setminus \{x\}) \cong \mathbb{Z}$.

(2) This statement follows easily from the Sections-on-Connected Proposition 191.13 and the various definitions. ■

We formulate our first interesting example of an orientation of a topological manifold as a lemma:

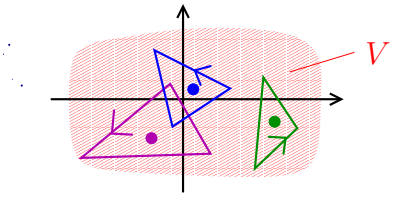
Lemma 191.19. (Standard Euclidean Orientation Lemma) Let V be an open subset of $\mathbb{R}^n$.

(1) For any $x \in V$ we set

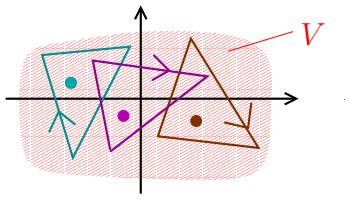
$$\mu_x \;:=\; \text{the standard generator of } \mathrm{H}_n(V, V \setminus \{x\}) \text{ as defined on page 3683.}$$

These homological classes μ_x define an orientation for V. We refer to this orientation as the standard orientation of V.

(2) If V is connected and non-empty, then V admits precisely two orientations, namely the one from (1) and the one where we take minus the standard generator.[2052]



standard generators for $\mathrm{H}_2(V, V \setminus \{x\})$, these define the standard orientation of V



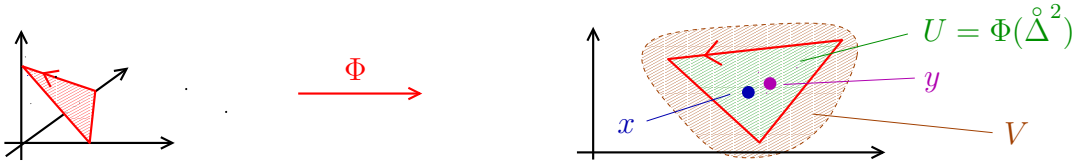
the negatives of the standard generators, these define the opposite of the standard orientation

Proof.

(1) Let V be an open subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We pick an orientation-preserving injective affine linear map $\Phi\colon \Delta^n \to V$ such that $x \in \Phi(\mathring{\Delta}^n)$. We set $U := \Phi(\mathring{\Delta}^n)$ and we set $\nu := [\Phi] \in \mathrm{H}_n(V, V \setminus U)$. It remains to prove the following claim:

Claim. For every $y \in U$ the class $\nu_y \in \mathrm{H}_n(V, V \setminus \{y\})$ is the standard generator.

Proof. It follows immediately from the Local-H_*-Basics Lemma 191.7 (2), the definition of the standard generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ and the fact that y is contained in $\Phi(\mathring{\Delta}^n)$ that $\rho_y([\Phi]) = [\Phi] \in \mathrm{H}_n(V, V \setminus \{y\})$ represents the standard generator of $\mathrm{H}_n(V, V \setminus \{y\})$. ⊞



(2) This statement follows immediately from (1) and the Number-of-Orientations Lemma 191.18 (2). ■

[2052] By the Local-H_*-Reflection Lemma 191.3 the other one is given by a reflection of the standard generators in hyperplanes.

The following lemma gives another essential example of an orientable topological manifold.

Lemma 191.20. (Sphere Orientation Lemma) Let $n \in \mathbb{N}$. In the following we denote by $[S^n] \in \mathrm{H}_n(S^n)$ the standard generator that we introduced in Definition 150.8. For each $x \in S^n$ we consider the natural map

$$\rho_x \colon \mathrm{H}_n(S^n) \;\to\; \mathrm{H}_n(S^n, S^n \setminus \{x\}).$$

The images of $[S^n]$ in the local homology groups $\mathrm{H}_n(S^n, S^n \setminus \{x\})$ define an orientation of S^n. We refer to it as the standard orientation of S^n.

Proof. First note that it follows easily from the long exact sequence of homology groups of the pair $(S^n, S^n \setminus \{x\})$ and the fact that $S^n \setminus \{x\} \cong \mathbb{R}^n$ is contractible, that each map ρ_x is an isomorphism. Since $[S^n]$ is a generator of $\mathrm{H}_n(S^n)$ we see that each $\rho_x([S^n])$ is a generator of $\mathrm{H}_n(S^n, S^n \setminus \{x\})$.

Now we show that the homology classes $\rho_x([S^n])$ define a section. Thus let $x \in S^n$. In Definition 191.11 we just need to set $U = S^n$ and $\nu = [S^n] \in \mathrm{H}_n(S^n, \underbrace{S^n \setminus S^n}_{=\varnothing})$. ■

Lemma 191.21. (Orientable-for-all-Rings Lemma) Let M be a topological manifold and let R be a commutative ring. Every orientation on M defines naturally an R-orientation on M.

Proof. Let M be an orientable n-dimensional topological manifold and let R be a commutative ring. Furthermore let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an orientation for M. By the Manifold-Local R-Homology Lemma 191.2 (2) we know that for each $x \in M \setminus \partial M$ the natural map

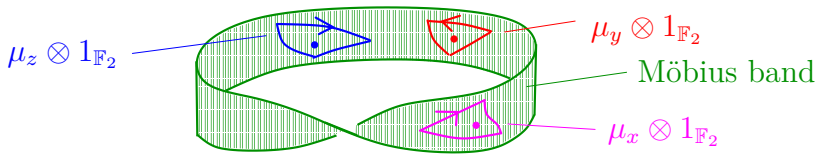
$$\begin{aligned} \mu \colon \mathrm{H}_n(M, M \setminus \{x\}) \otimes R \;&\to\; \mathrm{H}_n(M, M \setminus \{x\}; R) \\ \sum_{i=1}^{k} [\sigma_i] \otimes r_i \;&\mapsto\; \Big[\sum_{i=1}^{k} \sigma_i \otimes r_i \Big] \end{aligned}$$

is an isomorphism. For each $x \in M \setminus \partial M$ we denote by $\widetilde{\alpha}_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ the image of $\alpha_x \otimes 1_R$ under the above isomorphism. It is straightforward to verify that these generators $\{\widetilde{\alpha}_x\}_{x \in M \setminus \partial M}$ define an R-orientation for the topological manifold M. ■

The following proposition shows that the notion of R-orientability is much more flexible than the previous notion of orientability of smooth manifolds.

Proposition 191.22. ($\mathbb{F}_2$-Orientability Proposition) Every topological manifold is $\mathbb{F}_2$-orientable and it admits a unique $\mathbb{F}_2$-orientation.

Example. The $\mathbb{F}_2$-Orientability Proposition 191.22 says in particular that the real projective plane, the Klein bottle and the Möbius band are $\mathbb{F}_2$-orientable.



This observation will turn out to be more useful than one might think right now. □

Proof. Let M be an n-dimensional topological manifold. Let $x \in M \setminus \partial M$. By the Manifold-Local R-Homology Lemma 191.2 we know that $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. We define

φ_x to be the unique non-trivial element in $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. Using the Small Ball-Lemma 191.15 (0) and (1) it is straightforward to show that the continuity condition is satisfied, in particular we have defined an $\mathbb{F}_2$-orientation on M. ■

191.7. **Orientation-preserving local homeomorphisms.** The following definition is the topological analog of the concept introduced in Definition 31.29.

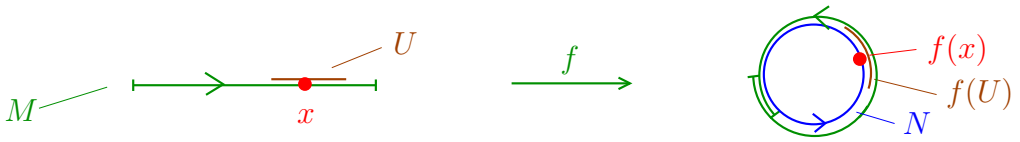
Definition 191.23. Let R be a commutative ring. Let M and N be two n-dimensional topological manifolds that are equipped with R-orientations $\{\alpha_x\}_{x \in M \setminus \partial M}$ and $\{\beta_x\}_{x \in N \setminus \partial N}$. Let $f\colon M \to N$ be a continuous map that is a local homeomorphism.

- We say that the map f is orientation-preserving at a point $x \in M \setminus \partial M$ if for every open neighborhood U of x, such that $f\colon U \to f(U)$ is a homeomorphism and such that $f(U)$ is open, the image of α_x under the map[2055320542055]

$$\mathrm{H}_n(M, M\setminus\{x\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(U, U\setminus\{x\}) \xrightarrow[\cong]{f_*} \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n(N, N\setminus\{f(x)\}).$$

  it follows from the Excision Theorem 148.2 that these two maps are isomorphisms

  equals $\beta_{f(x)}$. Otherwise we say that f is orientation-reversing at x.
- We say that $f\colon M \to N$ is orientation-preserving (respectively orientation-reversing) if it is orientation-preserving (respectively orientation-reversing) at all $x \in M \setminus \partial M$.



Example. Let $n \in \mathbb{N}_0$, $v \in \mathbb{R}^n$, let $A \in \mathrm{GL}(n, \mathbb{R})$ and let $U \subseteq \mathbb{R}^n$ be an open subset. We consider the embedding

$$\begin{array}{ccc} U & \to & \mathbb{R}^n \\ x & \mapsto & A \cdot x + v. \end{array}$$

If $\det(A) > 0$, then it follows almost immediately from the definition of the standard orientation (see page 3683) that the above map is orientation-preserving. If $\det(A) < 0$, then it follows easily from the Local-H_*-Reflection Lemma 191.3 that the above map is orientation-reversing. □

The following lemma is the analog of some statements of the Basics-of-Orientations Lemma 31.31.

Lemma 191.24. (Orientation Preserving Lemma) Let R be a commutative ring.

(1) Let $f\colon M \to N$ be a local homeomorphism between R-oriented n-dimensional manifolds.
   (a) To check whether f is orientation-preserving at a point $x \in M \setminus \partial M$ it suffices to check the above condition for a single open neighborhood of x.

[2053] Since $f\colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$.
[2054] It follows from the fact that $f\colon U \to f(U)$ is a homeomorphism, the fact that $x \in U \setminus \partial U$ and the Unique Type-of-Chart Proposition 150.11 that $f(x) \notin \partial N$.
[2055] We suppress the R-coefficients from the notation for space reasons.

(b) If f is orientation-preserving at some point $x \in M \setminus \partial M$ and if M is connected, then $f\colon M \to N$ is orientation-preserving.

(2) The composition of k orientation-preserving and l orientation-reversing local homeomorphisms is orientation-preserving if and only if l is even.

Sketch of proof.

(1) (a) This statement follows easily from the functoriality of homology.
(b) This statement follows from the Sections-on-Connected Proposition 191.13.

(2) This statement is basically clear. ■

191.8. **Smooth and topological orientations.** The following proposition shows in particular that for smooth manifolds the notion of orientability from Definition 31.19 agrees with the above notion of orientability.

Proposition 191.25. (Smooth-Topological Orientation Proposition)

(1) Give any smooth manifold M there exists a unique natural bijection

$$\left\{\begin{array}{c}\text{orientations of the}\\ \text{smooth manifold } M\\ \text{in the sense from Definition 31.19}\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{orientations of the}\\ \text{topological manifold } M \text{ in}\\ \text{the sense from Definition 191.17}\end{array}\right\}$$

such that the following two statements hold:

(a) For any open subset of some $\mathbb{R}^n$ the above bijection sends the smooth standard orientation from Definition 31.19 to the above topological standard orientation.

(b) If $f\colon M \to N$ is a local diffeomorphism between oriented connected smooth manifolds, then the following two statements are equivalent:
- f preserves the orientations on M and N in the sense from Definition 31.19,
- f preserves the orientations on M and N in the sense from Definition 191.17.

(2) Given an n-dimensional smooth manifold M we have

$$M \text{ is orientable in the sense from Definition 31.19} \iff M \text{ is orientable.}$$

In the proof of the Smooth-Topological Orientation Proposition 191.25 we will use the following lemma:

Lemma 191.26. (Orientations–Preserved Lemma) Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. For any $x \in V$ we have

$$\left\{\begin{array}{c}\text{at the point } x \text{ the map } \Psi \text{ preserves}\\ \text{the standard orientation of the}\\ \text{topological manifolds } V \text{ and } V'\end{array}\right\} \iff \left\{\begin{array}{c}\text{at the point } x \text{ the map } \Psi \text{ preserves}\\ \text{the standard orientation of the}\\ \text{smooth manifolds } V \text{ and } V'\end{array}\right\}$$

Proof. Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ and let $x \in V$. We write $x' = \Psi(x)$. Given $w \in \mathbb{R}^n$ we denote by t_w the translation $\mathbb{R}^n \to \mathbb{R}^n$ that is given by $v \mapsto v + w$. We set $\epsilon := 1$ if Ψ preserves the orientation of the smooth manifold V at the point x. Otherwise we set $\epsilon := -1$. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(V, V\setminus\{x\}) & \xleftarrow{(t_x)_*} & \mathrm{H}_n(t_{-x}(V), t_{-x}(V)\setminus\{0\}) & \xrightarrow{\cong} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
\downarrow{\scriptstyle \Psi_*} & & \downarrow{\scriptstyle (t_x\circ\Psi\circ t_{-x'})_*} & & \downarrow{\scriptstyle \cdot\epsilon} \\
\mathrm{H}_n(V', V'\setminus\{x'\}) & \xleftarrow[\cong]{(t_{x'})_*} & \mathrm{H}_n(t_{-x}(V), t_{-x}(V)\setminus\{0\}) & \xrightarrow{\cong} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{array}$$

We make the following observations:

(1) It follows easily from the Orientation Checking-via-Differentials Lemma 31.30 that each translation t_v is an orientation-preserving diffeomorphism of smooth manifolds. Furthermore it follows from the Orientation Preserving Lemma 191.24 that each translation t_v is an orientation-preserving homeomorphism of topological manifolds.
(2) It follows from the Excision Theorem 148.2 that the horizontal homomorphisms to the right are isomorphisms.
(3) It follows from the Local Diffeomorphism-Degree Proposition 152.6 together with (1) that the square at the right commutes.
(4) It is clear that the square at the left commutes.
(5) The statement of the lemma follows from (2)–(4) together with the Orientation Preserving Lemma 191.24 (2) and the Basics-of-Orientations Lemma 31.31 (5). ■

Sketch of the proof of Proposition 191.25. First, as we saw on page 3691, the case of 0-dimensional smooth manifolds is basically a tautology, at least as long as all statements are interpreted in a reasonable way. Thus in the following we assume that we are dealing with smooth manifolds of dimension ≥ 1.

Next note that Statement (2) is just a reader-friendly condensation of Statement (1). Thus we need to prove Statement (1). We define the promised natural bijections. Thus let M be an n-dimensional smooth manifold where $n \geq 1$. We first suppose that M is oriented as a smooth manifold, i.e. in the sense from Definition 31.19. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x that is orientation-preserving as a diffeomorphism between smooth manifolds. We define

$$\mu_x \coloneqq \Phi_*^{-1}(\text{standard generator of } \mathrm{H}_n(V, V \setminus \{\Phi(x)\}) \text{ as defined on page 3683}).$$

It follows from the Orientations–Preserved Lemma 191.26 that this definition does not depend on the choice of the orientation-preserving chart Φ. It now follows easily from the Standard Euclidean Orientation Lemma 191.19 that the classes $\{\mu_x\}_{x \in M \setminus \partial M}$ form an orientation for the topological manifold M.

Now we define the inverse map. So suppose that we are given an orientation $\{\mu_x\}_{x \in M \setminus \partial M}$ for the topological manifold M. Note that by the Manifold–Orientation Basics Lemma 31.31 we only need to find an orientation for $M \setminus \partial M$. Thus let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x such that V is connected. It follows from the Orientation Preserving Lemma 191.24 that, after possibly reflecting V in a hyperplane, we can assume that Φ preserves the orientation at x, i.e. it sends the given orientation μ_x to the standard orientation of the open subset V of $\mathbb{R}^n$ that we introduced in Lemma 191.19. We define

$$\text{orientation of } \mathrm{T}_x M \coloneqq \Phi^*(\text{standard orientation of } \mathrm{T}_{\Phi(x)} V = \mathbb{R}^n).$$

It follows from the Orientations–Preserved Lemma 191.26 that this definition does not depend on the choice of Φ. It is now straightforward to verify that these orientations define an orientation of the smooth manifold M.

It is basically clear, that by construction, the bijections have the desired properties (a) and (b). It also straightforward to see that the map between the two types of orientations is determined uniquely by properties (a) and (b). This concludes the proof of Statement (1). ■

We formulate one essential example as a lemma:

Lemma 191.27. (Sphere Orientations Agree Lemma) Let $n \in \mathbb{N}$. We consider the smooth manifold S^n. Under the natural bijection from the Smooth-Topological Orientation Proposition 191.25 the smooth standard orientation of $S^n$from page 817 gets matched to the topological standard orientation that we introduced in Lemma 191.20.

Proof. First we have to recall the two definitions of the orientations:

(1) Given $P \in S^n$ we say that a basis $(v_1, \ldots, v_n)$ of the tangent space $\widetilde{\mathrm{T}}_P S^n \subseteq \mathbb{R}^{n+1}$ is positive if $\det(P\, v_1 \ldots v_n) > 0$. These orientations on the tangent spaces give us the orientation from page 817.

(2) (a) We denote by $\pi\colon \Delta^n \to \triangle_n$ the projection onto the first n coordinates. Note that $\pi\colon \Delta^n \to \triangle_n$ is given by an affine linear map which is orientation-preserving if and only if n is even.

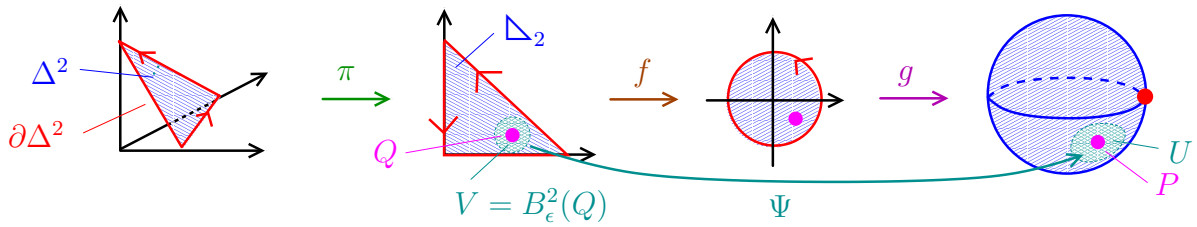
(b) We denote by $f\colon \triangle_n \to \overline{B}^n$ the explicit homeomorphism that we obtain from the Convex-to-Ball Proposition 4.28 applied to the point $(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$. Note that it follows easily from the explicit definition of the homeomorphism $f\colon \Delta^n \to \overline{B}^n$ in the Δ^n-Homeomorphism Lemma 143.9 (2) that there exists an open ball $B^n_\epsilon(Q) \subseteq \triangle_n$ such that the restriction of f to $B^n_\epsilon(Q) \to \overline{B}^n$ is an orientation-preserving smooth embedding.

(c) We consider the map

$$\begin{array}{rccccc} g\colon \overline{B}^n & \to & S^n & \subseteq & \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ r \cdot v & \mapsto & \begin{pmatrix} -\cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}. \end{array}$$

An elementary argument shows that $g\colon \overline{B}^n \to S^n$, viewed as a smooth embedding between smooth manifolds, is orientation-reversing for all n.

We set $\alpha_n := g \circ f \circ \pi\colon \Delta^n \to S^n$ and we denote by $\beta_n\colon \Delta^n \to S^n$ the constant map that sends all points to the "East Pole" $(1, 0, \ldots, 0)$. As in Definition 150.8 we set $[S^n] := (-1)^{n-1} \cdot [\alpha_n - \beta_n] \in \widetilde{\mathrm{H}}_n(S^n)$. For each $P \in S^n$ we now equip $\mathrm{H}_n(S^n, S^n \setminus \{P\})$ with the generator which is given by the image of $[S^n]$ under the natural map from absolute to relative homology.



We make the following further preparations:

(1) Note that the restriction of $g \circ f$ to V is a smooth embedding.

(2) We set $U := (g \circ f)(V)$ and $P := (g \circ f)(Q)$.

(3) We set $\Psi := (g \circ f)|_V\colon V \to U$. Note that $\Phi := \Psi^{-1}\colon U \to V$ is a chart around $P \in S^n$.

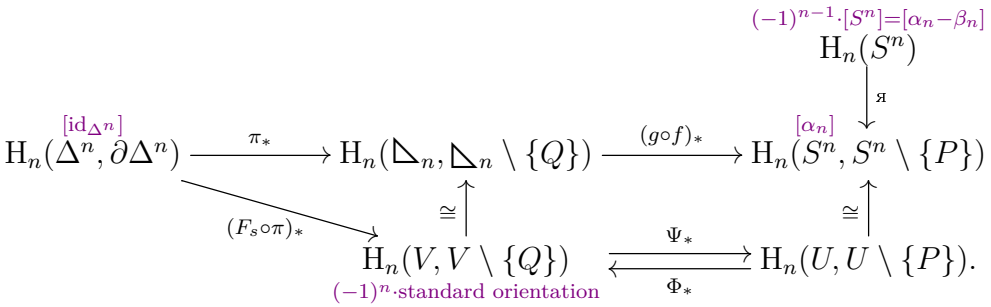
It follows from the Manifold–Orientation Basics Lemma 31.31 (1) and the Sections-on-Connected Proposition 191.13 that it suffices to show that the orientations match at a single point on S^n. Evidently we will consider P. By our definition of the bijection between the two types of orientations it suffices to prove the following claim:

Claim. For both notions of orientations the chart $\Phi\colon U \to V$ is orientation-reversing at P.

Proof. For the smooth setting this follows from (2b) and (2c). So it remains to consider the topological setting. We consider the map

$$\begin{array}{rcl} F\colon \triangle_n \times [0,1] & \to & \triangle_n \\ (x,t) & \mapsto & x\cdot(1-t)+Q\cdot t. \end{array}$$

Since $B^n_\epsilon(Q)$ is an open neighborhood of Q we see that there exists an $s \in (0,1)$ such that $F_s(\triangle_n) \subseteq V = B^n_\epsilon(Q)$. Next we consider the following diagram:



We make the following observations:

(1) The vertical maps are induced by inclusions and they are isomorphisms by the Excision Theorem 148.2.

(2) Clearly F gives rise to a homotopy between π and $F_s \circ \pi$. It follows from the Homotopy-H_*-Proposition 146.11 that the triangle commutes.

(3) Since $g \circ f \circ \pi = \alpha_n$ and since $\beta_n = 0 \in \mathrm{C}_n(S^n, S^n \setminus \{P\})$ we see that the image of id_{Δ^n} under the two horizontal maps equals $я([\alpha_n - \beta_n])$. This equals $я([S^n])$ if and only if n is odd.

(4) By (2a) we know that the affine linear map $F_s \circ \pi\colon \Delta^n \to V$ is orientation-preserving if and only if n is even.

It follows from the above discussion that the chart $\Phi\colon U \to V$ is orientation-reversing at P for any choice of n. ∎

The following lemma is the analog of the Smooth Isotopy-Orientation Proposition 42.4.

Lemma 191.28. (Isotopy-Orientation Lemma) Let M and N be two n-dimensional topological manifolds and furthermore let $f, g\colon M \to N$ be two homeomorphisms. If f and g are isotopic, then f is orientation-preserving if and only if g is orientation-preserving.

Proof. We will not make use of the lemma, thus we outsource the proof to Exercise 191.13. ∎

191.9. **Appendix: The orientation covering of topological manifolds.** In the appendix we discuss the following proposition which is a variation on the Orientation Covering–Existence Proposition 95.8 and the Orientation Covering Proposition 95.10, which were formulated in the context of smooth manifolds.

Proposition 191.29. (Orientation Covering–Properties-Proposition)

(1) To each topological manifold M we can naturally[2056] associate an oriented topological manifold $\widetilde{M}$ together with a 2-fold covering $p\colon \widetilde{M} \to M$ and to each local homeomorphism $f\colon M \to N$ we can associate an orientation-preserving local homeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & N. \end{array}$$

(2) Let M be a connected topological manifold and let $p\colon \widetilde{M} \to M$ be the 2-fold covering constructed in (1).

(a) If M is orientable, then there exists a unique orientation-preserving homeomorphism to $\Theta\colon \widetilde{M} \to M \sqcup -M$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Theta} & M \sqcup -M \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & M & \end{array}$$

where q is the obvious map that is given by the identity on each component of $M \sqcup -M$.

(b) If M is non-orientable, then $\widetilde{M}$ is connected.

(3) Let M be a topological manifold. The map

$$\begin{array}{rccl} \Xi_M\colon & \widetilde{M} & \to & \widetilde{M} \\ & x & \mapsto & \text{the unique other points of } p^{-1}(p(x)) \end{array}$$

is the unique orientation-reversing self-homeomorphism $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{\Xi_M} & \widetilde{M} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & M. & \end{array}$$

Furthermore, given a local homeomorphism $f\colon M \to N$ we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(4) Let M be a non-orientable connected topological manifold. Suppose that we are given two connected 2-fold coverings $p\colon \widetilde{M} \to M$ and $q\colon \widehat{M} \to M$ such that $\widetilde{M}$ and $\widehat{M}$ are orientable. There exists a homeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\Phi} & \widehat{M} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & M. & \end{array}$$

If $\widetilde{M}$ and $\widehat{M}$ are oriented, then there exists a unique orientation-preserving map Φ as above.

The Orientation Covering–Properties-Proposition 191.29 leads us to the following analog of Definition 95.9:

[2056]Since the definition of $\widetilde{M}$ is a little more messy than in the case of smooth manifolds we prefer to hide the definition of $\widetilde{M}$ in the proof.

Definition 191.30. Let M be a non-orientable connected topological manifold. We refer to the connected 2-fold covering $p\colon \widetilde{M} \to M$ which we construct in the proof of the Orientation Covering–Properties-Proposition 191.29 as the orientation covering of M.

Example. It follows easily from the uniqueness statement of the Orientation Covering–Properties-Proposition 191.29 (4) that the orientation covering of the Möbius band is homeomorphic to the annulus.

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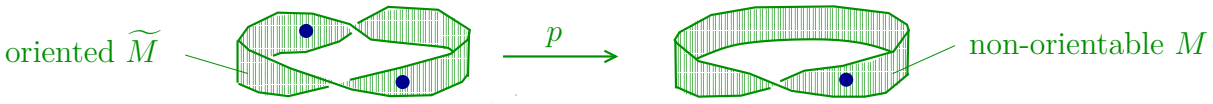


Similarly one sees that the orientation covering of the Klein bottle is homeomorphic to the torus and that for every even $n \in \mathbb{N}$ the orientation covering of the real projective space $\mathbb{R}\mathrm{P}^n$ is homeomorphic to S^n. □

Given a non-orientable connected smooth manifold M with corresponding orientation covering $p\colon \widetilde{M} \to M$ we discussed in the Orientation Character Proposition 95.14 at length the subgroup of $\pi_1(M)$ defined by $\pi_1(\widetilde{M})$. These results generalize without problems to the present setting of topological manifolds. In particular we now have the following generalization of Definition 95.15.

Definition 191.31. Let M be a connected topological manifold and let $x_0 \in M$.

(1) If M is non-orientable, then we consider the orientation covering $p\colon \widetilde{M} \to M$ and we pick $\widetilde{x}_0 \in \widetilde{M}$ with $p(\widetilde{x}_0) = x_0$. We then refer to the epimorphism
$$w_1^{x_0}(M) := w_1^{x_0}\colon \pi_1(M, x_0) \ \to \ \pi_1(M, x_0)/p_*(\pi_1(\widetilde{M}, \widetilde{x}_0)) =_{\mathrm{i}} \mathbb{Z}/2\mathbb{Z},$$
which is given by the obvious analog of the Orientation Character Proposition 95.14, as the orientation character of M. Note that it follows from the analog of the Orientation Character Proposition 95.14 (1b) that this definition does not depend on the choice of the point $\widetilde{x}_0$.

(2) If M is orientable, then we define the orientation character of M to be the trivial homomorphism $w_1\colon \pi_1(M, x_0) \to \mathbb{Z}/2\mathbb{Z}$.

Sketch of proof. The proof is very similar to the proof of the Orientation Covering–Existence Proposition 95.8 and the Orientation Covering Proposition 95.10.

(1) Let M be a topological manifold.
   (a) For $x \in M \setminus \partial M$ we define an orientation at x to be a $\mathbb{Z}$-orientation for M at the point x.
   (b) Let $x \in \partial M$. An orientation-neighborhood for x is a triple (U, μ, x) where U is an open neighborhood of $x \in M$ and μ is a $\mathbb{Z}$-orientation on U. We say that two orientation-neighborhoods (U, μ, x) and (V, ν, x) are equivalent if the orientations μ and ν agree on $U \cap V$. For the purpose of this proof we define an orientation for x to be an equivalence class of orientation-neighborhoods for x.

   We consider the set
$$\widetilde{M} \ := \ \{\mu_x \mid x \in M \text{ and } \mu_x \text{ is an orientation for } M \text{ at the point } x\}$$

and the projection $p\colon \widetilde{M} \to M$ that is given by $\mu_x \mapsto x$ for $x \in M \setminus \partial M$ and that is given by $[(U, \mu, x)] \mapsto x$ for $x \in \partial M$. It follows from the Number-of-Orientations Lemma 191.18 (1) that for every $x \in M \setminus \partial M$ the preimage $p^{-1}(x)$ has precisely two elements. We leave it to the reader to verify that for every $x \in \partial M$ the preimage $p^{-1}(x)$ also has precisely two elements.

We equip $\widetilde{M}$ with a topology similar to the proof of the Orientation Covering–Existence Proposition 95.8. It is then straightforward to see that $p\colon \widetilde{M} \to M$ is a 2-fold covering. Similar to the proof of the Orientation Covering–Existence Proposition 95.8 one can show that $\widetilde{M}$ admits a natural orientation.

Now let $f\colon M \to N$ be a local homeomorphism. We leave it to the reader to define the map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ that has the promised properties.[2057]

(2) The proof of this statement is almost identical to the proof of the Orientation Covering–Properties-Proposition 95.10 (2).

(3) The proof of this statement is identical to the very terse proof of the Orientation Covering–Properties-Proposition 95.10 (3).

(4) Not surprisingly the proof of this statement is also identical to the proof of the Orientation Covering–Properties-Proposition 95.10 (4).

We leave it to the reader to flesh out all the details. ■

We continue with the following corollary which is closely related to the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16.

Corollary 191.32. (Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary) Let M be a connected non-empty topological manifold. If M is non-orientable, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$.

Proof. The proof of the corollary is basically identical to the proof of the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16, we just need to replace the Orientation Covering–Properties-Proposition 95.10 (2b) by the Orientation Covering–Properties-Proposition 191.29 (2b). ■

Remark. Many other results on orientations of smooth manifolds can now be extended to topological manifolds. For example the proof of the Orientation-Flipping Proposition 45.6 that we gave in Section 95.6 also applies verbatim to the setting of topological manifolds. We leave it to the reader to work out the precise statements and proofs. □

Finally we conclude this section with the following lemma which is the topological analog of the w_1-Functorial Lemma 95.17:

Lemma 191.33. (w_1-Functorial Lemma) Let $f\colon M \to N$ be a local homeomorphism between connected topological manifolds and let $x_0 \in M$. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(M, x_0) & \xrightarrow{\quad f_* \quad} & \pi_1(N, f(x_0)) \\ & {\scriptstyle w_1^{x_0}(M)} \searrow \quad \swarrow {\scriptstyle w_1^{f(x_0)}(N)} & \\ & \mathbb{Z}/2\mathbb{Z}. & \end{array}$$

[2057] It is obvious how to define the map if $\partial M = \partial N = \varnothing$. Dealing with the boundaries is slightly awkward.

Sketch of proof. This statement follows easily from the naturality statement of the Orientation Covering–Properties-Proposition 191.29 (1). We leave it to the reader to fill in the details. ■

191.10. Exercises for Chapter 191 .

Exercise 191.1. Let M be an oriented smooth manifold and let $X, Y \subseteq M$ be two closed oriented submanifolds that intersect transversally and with $\dim(X) + \dim(Y) = \dim(M)$. We consider the algebraic intersection number as defined on Definition 51.3 and Item (c).

(a) What is the relationship between $X \cdot Y$ and $Y \cdot X$?
(b) Let $p\colon \widetilde{M} \to M$ be a finite covering. What is the relationship between $p^{-1}(X) \cdot p^{-1}(Y)$ and $X \cdot Y$?

Exercise 191.2.

(a) What are the local homology groups of $S^1 \vee S^2$?
(b) Let X be the Topologist's Sine Curve that we introduced on page 195. What are the local homology groups of X?
(c) Let X be a topological space such that for every $x \in X$ we have

$$\mathrm{H}_n(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{for } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Does X have the discrete topology?

Exercise 191.3. Let M be an n-dimensional topological manifold, let R be a commutative ring and let $A \subseteq M \setminus \partial M$ be a subset.

(a) Show that for R-sections $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ along A the collection $\{\alpha_x + \beta_x\}_{x\in A}$ is also an R-section along A.
(b) Show that the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y \in A$ the map $\Gamma_R^A(M) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ given by $\{\alpha_x\}_{x\in A} \mapsto \alpha_y$ is an R-module homomorphism.

Exercise 191.4. Let M be an-dimensional topological manifold.

(a) Let R be a commutative ring and let $A \subseteq M \setminus \partial M$ be a *connected* subset. We assume that M is R-orientable. Show that for every $x \in M \setminus \partial M$ the map

$$\begin{array}{rcl} \Phi\colon \Gamma_R^A(M) & \to & \mathrm{H}_n(M, M \setminus \{x\}; R) \\ \{\alpha_y\}_{y\in A} & \mapsto & \alpha_x \end{array}$$

is an isomorphism of R-modules.
(b) Show that if M is non-orientable and connected, then $\Gamma_{\mathbb{Z}}(M) = 0$.

Exercise 191.5. An n-dimensional *non-Hausdorff-manifold* is a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$. Note that the notion of an orientation makes sense for non-Hausdorff-manifolds. Show that there exists a non-orientable 1-dimensional non-Hausdorff-manifold.
Remark. This exercise is almost identical to Exercise 31.23.

Exercise 191.6.

(a) Let M be a topological manifold and let $A, B \subseteq M$ be two open subsets. We suppose that A and B are orientable and that $A \cap B$ is connected. Show that M is also orientable.
(b) Use (a) to give a new proof that each S^n is orientable.

Exercise 191.7. Let $f\colon S^n \to S^n$ be a homeomorphism. Show that f is orientation-preserving if and only if $\deg(f) = 1$.

Exercise 191.8. Let R be a commutative ring and let M be a compact R-orientable topological manifold. How many different R-orientations does M admit?

Exercise 191.9. Let $c\colon [0,1] \to [0,1]$ be the Cantor function that we introduced on page 216. We consider the map

$$\begin{array}{rccc} f\colon & (0,1) & \to & (0,2) \\ & x & \mapsto & x + c(x). \end{array}$$

(a) Show that f is a homeomorphism.
(b) We equip $(0,1)$ and $(0,2)$ with the standard orientations as topological manifolds. Show that f is orientation-preserving.

Remark. This exercise is identical to Exercise 37.2.

Exercise 191.10. We view $\mathbb{R}$ as a topological manifold and we equip it with the standard $\mathbb{Z}$-orientation that we had introduced in Lemma 191.19. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous map that is a local homeomorphism. Show that the following two statements are equivalent:

(1) f is orientation-preserving.
(2) f is strictly monotonously increasing.

Exercise 191.11. Show that every orientation-reversing self-homeomorphism of $\mathbb{R}$ has a fixed point.

Exercise 191.12. Show, without making of the Smooth-Topological Orientation Proposition 191.25, that the 2-dimensional torus $S^1 \times S^1$ is an orientable topological manifold.

Exercise 191.13. Let M be an oriented n-dimensional topological manifold. Furthermore let $f, g\colon M \to M$ be two homeomorphisms. Show that if f is orientation-preserving and if g is homeotopic to f, then g is also orientation-preserving.
Remark. In the Smooth Isotopy-Orientation Proposition 42.4 we proved the analog in the setting of smooth manifolds.

Exercise 191.14. Let M be the Möbius band. Show that there is no embedding $M \to \mathbb{R}^2$.
Remark. Note that we do not assume that the embedding is a smooth embedding.

Exercise 191.15. Let $f\colon \overline{B}^n \to \mathbb{R}^n$ be an embedding with $0 \in f(B^n)$. Show that the map $f_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\})$ is an isomorphism.

Exercise 191.16. Let M be an n-dimensional topological manifold with $\partial M = \varnothing$ and let R be a commutative ring. We consider the set

$$\widetilde{M}^R := \bigsqcup_{P \in M} \mathrm{H}_n(M, M \setminus \{P\}; R).$$

We want to equip $\widetilde{M}^R$ with a topology. To do so we introduce the following notation:

- For an open subset $V \subseteq \mathbb{R}^n$ and $y \in V$ we denote by $\varphi_y \in \mathrm{H}_n(V, V \setminus \{y\})$ the standard generator.
- Given $U \subseteq M$ we denote by $i\colon U \to M$ the inclusion.
- Given a chart $\Phi\colon U \to V$ and given $r \in R$ we set

$$X(\Phi, r) \coloneqq \bigcup_{P \in U} \{i_*((\Phi_*)^{-1}(\underbrace{\mu(\varphi_{\Phi(Q)} \otimes r)}_{\in \mathrm{H}_n(V, V\setminus\{y\}, R)}))\}.$$

We now equip $\widetilde{M}^R$ with the topology that is generated by the sets $X(\Phi, r)$.

(a) Let M be the "open" Möbius band, i.e. M is the Möbius band without the boundary. What is the homeomorphism type of $\widetilde{M}^{\mathbb{Z}}$?
(b) Is $\widetilde{M}^R$ a topological manifold?
(c) Show that we have a covering $\widetilde{M}^R \to M$.
(d) How can you modify the above construction to always get a 2-fold covering that is orientable?

Exercise 191.17. In the Orientation Covering–Properties-Proposition 191.29 we saw in particular that every closed non-orientable topological manifold admits a 2-fold covering that is orientable. Is every closed orientable topological manifold the 2-fold covering of a closed non-orientable topological manifold?

Exercise 191.18. Let $k \in \mathbb{N}_{\geq 3}$. Show that if a topological manifold is $\mathbb{Z}/k\mathbb{Z}$-orientable, then it is $\mathbb{Z}$-orientable.

Exercise 191.19.

(a) Let M be a simply connected n-dimensional topological manifold. Show that M is orientable.
Hint. Pick $x_0 \in M$ and pick a generator for $\mathrm{H}_n(M, M \setminus \{x\})$. Use paths to equip each point x with a generator for $\mathrm{H}_n(M, M \setminus \{x_0\})$ and use the hypothesis that M is simply connected to show that this generator does not depend on the choice of the path.
(b) Let H be a group which acts continuously on an oriented topological manifold M. Show that there exists a subgroup $G \subseteq H$ of index ≤ 2 such that G acts in an orientation-preserving way.
(c) Let N be a non-orientable connected non-empty n-dimensional topological manifold. Use (b) to show that N admits a 2-fold covering that is orientable.

Exercise 191.20. Let M be an n-dimensional topological manifold with $\partial M = \varnothing$ and let

$$\widetilde{M} \coloneqq \{\mu_x \mid x \in M \text{ and } \mu_x \text{ is an orientation for } M \text{ at the point } x\}$$

be the orientation-covering from the proof of the Orientation Covering–Properties-Proposition 191.29 on page 3700. We consider the topological space

$$\widehat{M} \coloneqq (\widetilde{M} \times \mathbb{R})/(\mu_x, t) \sim /(-\mu_x, -t).$$

(a) Show that $\widehat{M}$ is an $(n+1)$-dimensional topological manifold.
(b) Show that $\widehat{M}$ is orientable.
(c) Show that there exists an embedding $M \to \widehat{M}$.

This exercise so far shows that any n-dimensional topological manifold with empty boundary can be embedded into an orientable $(n+1)$-dimensional topological manifold.

(d) Show that any closed n-dimensional topological manifold can be embedded into a closed orientable $(n+1)$-dimensional topological manifold.

Remark. This exercise is the topological analog of Exercise 95.15.

CHAPTER 192

Induced orientations

In the last chapter we introduced the notion of an orientation of a topological manifold. In the Smooth-Topological Orientation Proposition 191.25 we showed that an orientation on a smooth manifold naturally defines an orientation of the underlying topological manifold.

For smooth manifolds we showed in the Manifold–Orientation Basics Lemma 31.31, the Boundary Orientation Lemma 31.38 and the Product Orientation Proposition 31.48 that if we start with oriented smooth manifolds, then various constructions of smooth manifolds can naturally be endowed with an orientation.

Similar to the setting of smooth manifolds we will now see that various constructions of topological manifolds out of given given oriented topological manifolds can also be endowed with natural orientations. For the most part the statements sound reasonable and familiar from the setting of smooth manifolds. Nonetheless, the constructions of the induced representations for topological manifolds are at times quite different from the constructions of induced orientations for smooth manifolds.

This chapter is organized as follows:

- In Section 192.1 we will study various induced orientations.
- In Section 192.2 we will consider the operation of gluing topological manifolds.
- In Section 192.3 we will introduce product orientations of topological manifolds.
- In Section 192.4 we will provide exercises for Chapter 192.

In the next Chapter 193 we will determine the top homology groups of topological manifolds.

192.1. **Induced orientations.** The following basic lemma is the topological analog of the Manifold–Orientation Basics Lemma 31.31 (6) and the Manifold Covering Proposition 95.4 (4).

Lemma 192.1. (Topological Manifold-Induced Orientation Lemma) Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation.

(1) Let N be a topological manifold and let $f\colon N \to M$ be a local homeomorphism, then there exists a unique R-orientation on N such that f is orientation-preserving.
(2) If N is a submanifold of codimension zero (for example N could be an open subset of M), then N admits a unique R-orientation such that the inclusion map $N \to M$ is orientation-preserving.[2058]
(3) Let $p\colon \widetilde{M} \to M$ be a covering of topological manifolds. There exists a unique R-orientation on $\widetilde{M}$ that turns p into an orientation-preserving map.

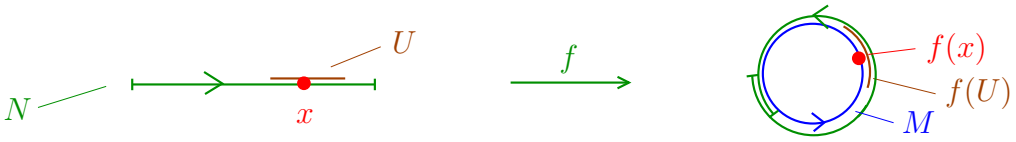
[2058]Later, in the Submanifold Orientation Lemma 212.5 we will generalize this statement to codimension-zero submanifolds with corner.

Proof. To simplify the notation we suppress the ring R from the notation. Thus let M be an n-dimensional topological manifold that is equipped with an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$.

(1) Let N be a topological manifold and let $f\colon N\to M$ be a local homeomorphism. Let $x\in N\setminus\partial N$. We pick an open neighborhood U of x such that $f\colon U\to f(U)$ is a homeomorphism and such that the image $f(U)$ is an open subset of M. We define $\beta_x\in \mathrm{H}_n(N,N\setminus\{x\};R)$ to be the image of α_x under the map[2059]

$$\mathrm{H}_n(M,M\setminus\{f(x)\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(f(U),f(U)\setminus\{f(x)\}) \xrightarrow[\cong]{(f_*)^{-1}} \mathrm{H}_n(U,U\setminus\{x\}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n(N,N\setminus\{x\}).$$

it follows from the Excision Theorem 148.2 that these two maps are isomorphisms



We leave it to the reader to verify that these classes β_x define an orientation on N. By definition the map f is orientation-preserving with respect to the orientations on M and N. It is elementary to verify that the orientation on N is uniquely determined.

(2) This statement follows from (1) since the inclusion map of a codimension-zero submanifold is, basically by definition, a local homeomorphism.

(3) This statement also follows from (1) since a covering map is a local homeomorphism. ■

We continue with the following analog of the Orientation-Action Proposition 31.44.

Proposition 192.2. (Topological Orientation-Action Proposition) Let M be a topological manifold. Furthermore let G be a group that acts freely, properly and continuously on M. By the Manifold–Quotient Proposition 25.39 (1) the quotient M/G is a topological manifold and the projection $p\colon M\to M/G$ is a local homeomorphism.

(1) If M is oriented and if the G-action is orientation-preserving, then the topological manifold M/G admits a unique orientation such that the projection $p\colon M\to M/G$ is orientation-preserving.

(2) Conversely, if the topological manifold M/G admits an orientation, then there exists a unique orientation on M such that the projection $p\colon M\to M/G$ is orientation-preserving. Furthermore this orientation has the property that the G-action on M is orientation-preserving.

Example. We consider the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ that is given by addition. It follows from the discussion on page 3694 that this action is orientation-preserving. It follows from the Orientation-Action Proposition 192.2 that $\mathbb{R}^n/\mathbb{Z}^n$ is an orientable topological manifold. □

Proof. The proof follows the same overall logic of the proof of the Orientation-Action Proposition 31.44. We leave it to the reader to fill in the details. ■

In the Boundary Orientation Lemma 31.38 we showed that the boundary of an oriented smooth manifold can be equipped with a natural orientation. The following proposition is the topological analog:

[2059]Since $f\colon U\to f(U)$ is a homeomorphism we have $f(U\setminus\{x\}) = f(U)\setminus\{f(x)\}$. Furthermore it follows from the fact that $x\in U\setminus\partial U$ and the Unique Type-of-Chart Proposition 150.11 that $f(x)\notin\partial N$.

Proposition 192.3. (Boundary Orientation Proposition)

(1) Let R be a commutative ring. Given any topological manifold M there exists a natural map

$$\{R\text{-orientations of } M\} \;\to\; \{R\text{-orientations of } \partial M\}.$$

Here natural means that if $f\colon M \to N$ is a local homeomorphism, then the following diagram commutes:

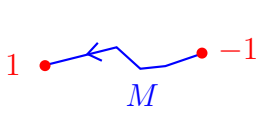
$$\begin{array}{ccc} \{R\text{-orientations of } M\} & \longrightarrow & \{R\text{-orientations of } \partial M\} \\ \downarrow f_* & & \downarrow f_* \\ \{R\text{-orientations of } N\} & \longrightarrow & \{R\text{-orientations of } \partial N\}. \end{array}$$

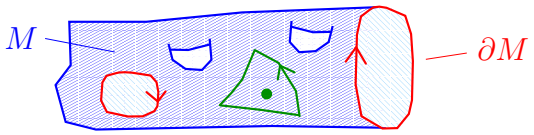
This orientation has the property that $\partial(-M) = -\partial M$. The same results also hold for homology without coefficients.

(2) For homology without coefficients this boundary orientation can be chosen to be compatible with the boundary orientation in the smooth setting. More precisely: Let M be an n-dimensional smooth manifold that is equipped with an orientation in the sense of Section 31.2. In Convention 31.39 and page 825 we equipped the $(n-1)$-dimensional smooth manifold ∂M with an orientation in the sense of Section 31.2. With these orientations we have the following commutative "diagram"

$$\begin{array}{ccc} \text{oriented smooth manifold } M & \overset{\text{take boundary}}{\rightsquigarrow} & \text{oriented smooth manifold } \partial M \\ \Big\downarrow \text{Smooth-Topological Orientation Proposition 191.25} & & \Big\downarrow \text{Smooth-Topological Orientation Proposition 191.25} \\ \text{oriented topological manifold } M & \overset{\text{take boundary}}{\rightsquigarrow} & \text{oriented topological manifold } \partial M. \end{array}$$

Convention 192.4. If M is an R-oriented topological manifold, then invariably we will equip ∂M with the R-orientation provided by the construction in the proof of the Boundary Orientation Proposition 192.3.



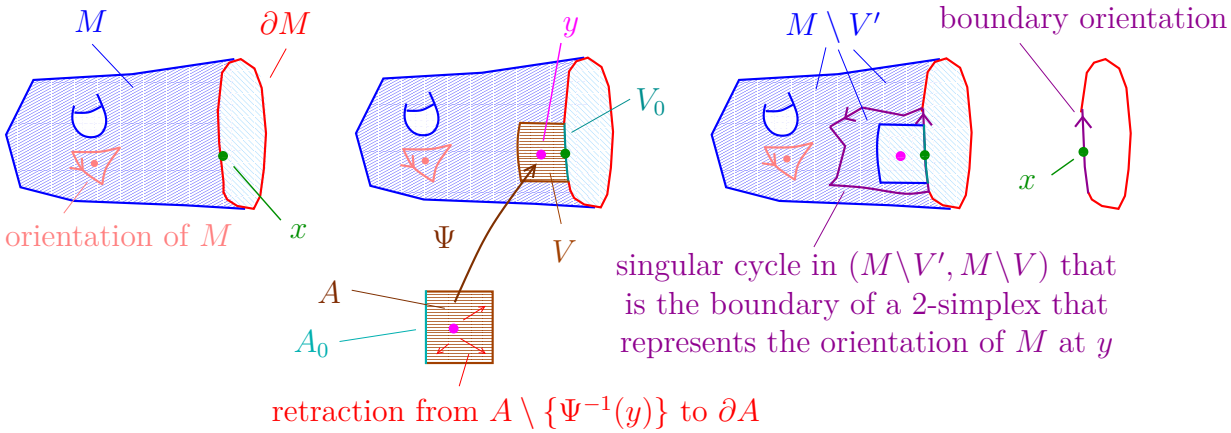


Example. Let $n \in \mathbb{N}$. On page 826 we saw that the smooth standard orientation of S^{n-1} is the boundary orientation of the smooth standard orientation on $\overline{B}^n$. It follows from the Boundary Orientation Proposition 192.3 and the Sphere Orientations Agree Lemma 191.27 that the same statement holds for the topological setting. □

Proof of Statement (1). To simplify the notation we will only deal with case that we have no coefficients. That is anyway the only case we care about. We suppose that we are given an orientation $\{\mu_y\}_{y\in M\setminus\partial M}$ for M. In the following we will define a natural orientation for the topological manifold ∂M.

We set $n := \dim(M)$ and we write $A = [0,1]\times[-1,1]^{n-1}$, we write $A_0 := \{0\}\times[-1,1]^{n-1}$ and we write $A' = (0,1)\times(-1,1)^{n-1}$. Now let $x \in \partial M$. Since M is a topological manifold and since $x \in \partial M$ we can pick a chart $\Phi\colon U \to [0,2)\times(-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We

write $\Psi := \Phi^{-1}$. Furthermore we write $V = \Psi(A)$, $V_0 = \Psi(A_0)$ and we write $V' = \Psi(A')$. Finally we pick a point $y \in V'$.



We consider the following sequence of maps:

$$\mathrm{H}_n(M, M\setminus\{y\}) \xleftarrow{(1)} \mathrm{H}_n\big(M, M\setminus V'\big) \xrightarrow[(2)]{\partial_n} \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\})$$
$$\uparrow{\scriptstyle (3)} \qquad\qquad \nwarrow^{(5)}$$
$$\mathrm{H}_{n-1}(V_0, V_0\setminus\{x\}) \xrightarrow[(4)]{} \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}).$$

Here all the maps are induced by the obvious inclusions of pairs of topological spaces, except for the second map which is the connecting homomorphism of the long exact sequence of the triple $(M, M\setminus V', (M\setminus V')\setminus\{x\})$ that is provided by the LES-of-Triple Proposition 146.5.

Claim 1. All the maps in the above diagram are isomorphisms.

Proof. In the following we treat the five maps separately.

(1) It is elementary to see that $A \setminus \Psi^{-1}(y)$ is a deformation retract of ∂A. It follows that $M \setminus V'$ is a deformation retract of $M \setminus \{y\}$.[2060] We obtain from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that (1) an isomorphism.

(2) We consider the long exact sequence of the triple $(M, M \setminus V', (M\setminus V')\setminus\{x\})$ that is provided by the LES-of-Triple Proposition 146.5:

$$\mathrm{H}_n(M, (M\setminus V')\setminus\{x\}) \to \mathrm{H}_n(M, M\setminus V') \xrightarrow{\partial_n} \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) \to$$
$$\to \mathrm{H}_{n-1}(M, (M\setminus V')\setminus\{x\}).$$

Similarly to (1) one sees that $(M \setminus V') \setminus \{x\}$ is a deformation retract of M. Therefore it follows from the Deformation Retract-Relative H_*-Corollary 146.9 that the groups

[2060] Looking at the figure on page 3709 this statement sounds reasonable. But the statement nonetheless requires a proof. So let r be a deformation retraction from $A \setminus \Psi^{-1}(y)$ to ∂A. We consider the map

$$\begin{array}{rcl} s\colon (M\setminus\{y\})\times[0,1] & \to & M\setminus\{y\} \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin V\setminus\{y\}, \\ \Psi(r(\Psi^{-1}(P),t)), & \text{if } P \in V\setminus\{y\}. \end{cases} \end{array}$$

Note that $V \setminus \{y\}$ and $M \setminus \Psi(A')$ are closed subsets of $M \setminus \{y\}$. (Hmm, why is that?). It follows from the Homotopy Combination Lemma 18.8 (1) that the map s is continuous.

at the two extremes are zero. In other words, the connecting homomorphism is an isomorphism.
(3) It follows quite easily from the Excision Theorem 148.2 that (3) is an isomorphism.
(4) It follows equally easily from the Excision Theorem 148.2 that (4) is an isomorphism.
(5) The triangle is given by inclusion-induced maps, hence it commutes. It now follows from (3) and (4) that (5) is also an isomorphism. ⊞

Given $x \in \partial M$ we now define $\nu_x \in \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ to be the image of the generator $\mu_y \in \mathrm{H}_n(M, M \setminus \{y\})$ under the above sequence of isomorphisms. We continue with our next claim.

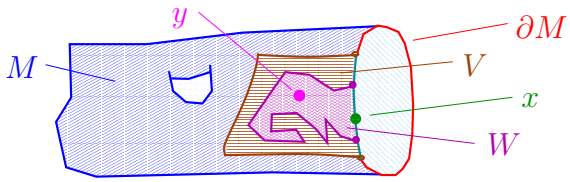
Claim 2.
(1) The definition of ν_x is independent of the choice of y.
(2) The definition of ν_x is independent of the choice of Φ.

Proof.

(1) We need to show that for every $z \in \Psi(V')$ we have

$$\begin{array}{c}\text{preimage of } \mu_y \text{ under the map} \\ \mathrm{H}_n(M, M\setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{y\})\end{array} = \begin{array}{c}\text{preimage of } \mu_z \text{ under the map} \\ \mathrm{H}_n(M, M\setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{z\}).\end{array}$$

But this statement can be shown in a similar fashion as the Sections-on-Connected Proposition 191.13. We leave it to the reader to fill in the details.
(2) Suppose we are given a different chart than Φ. We define W, W_0, W' in the obvious way. By possibly taking a third chart whose image lies in the intersection of V and W we can assume, without loss of generality, that $W \subseteq V$.



Furthermore by (1) we can work with a point $y \in W$ for both charts. We consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}_n(M, M\setminus\{y\}) & \xleftarrow[(1)]{\cong} & \mathrm{H}_n\big(M, M\setminus V'\big) \\
{\scriptstyle\cong}\uparrow{\scriptstyle(1)} & & {\scriptstyle(2)}\downarrow{\scriptstyle\partial_n} \\
\mathrm{H}_n\big(M, M\setminus W'\big) & \dashleftarrow & \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) \\
{\scriptstyle(2)}\downarrow{\scriptstyle\partial_n} & & {\scriptstyle(3)}\uparrow{\scriptstyle\cong} \\
\mathrm{H}_{n-1}(M\setminus W', (M\setminus W')\setminus\{x\}) & \dashleftarrow & \mathrm{H}_{n-1}(V_0, V_0\setminus\{x\}) \\
{\scriptstyle(3)}\uparrow{\scriptstyle\cong} & & {\scriptstyle(4)}\downarrow \\
\mathrm{H}_{n-1}(W_0, W_0\setminus\{x\}) & \xrightarrow[(4)]{} & \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}).
\end{array}$$

Here the two "outer paths" from the top left to the lower right correspond to the two "competing" definitions of ν_x. The dashed arrows are given by inclusions. We make the following observations:
(a) The triangle on top commutes since all maps are induced by inclusions.

(b) The upper parallelogram commutes since the connecting homomorphism is natural, see the LES-of-Triple Proposition 146.5 (3) for details.
(c) The lower parallelogram commutes since all the maps are induced by inclusions. (Note that the reversal of the direction of the arrow is not a typo.)
(d) The triangle on the bottom commutes since all maps are induced by inclusions.

We saw above that all maps (1), (2), (3) and (4) are isomorphisms. By going from the top to the bottom we see that all the dashed arrows are isomorphisms.

Since all pieces of the diagram commute and since all maps are isomorphisms we see that the total diagram commutes. This shows that the two "competing" definitions of ν_x agree. ⊞

Using Claim 1 it is now straightforward to show that the $\{\nu_x\}_{x\in\partial M}$ form an orientation for ∂M, that this orientation is natural and that $\partial(-M) = -\partial M$. ■

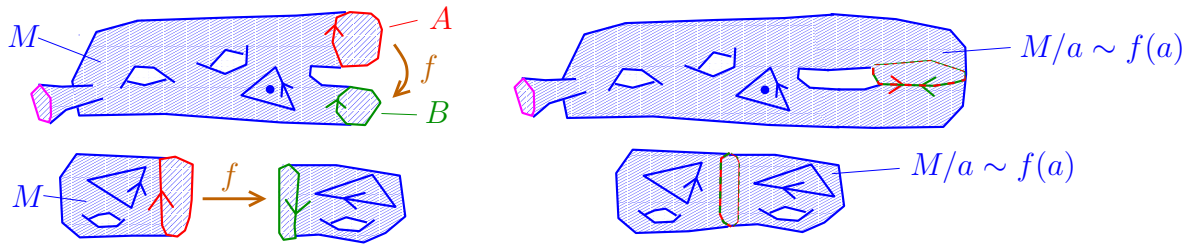
Proof of Statement (2). It follows from (1) and the Smooth-Topological Orientation Proposition 191.25 that it suffices to deal with the case of the smooth manifold $M = \mathbb{H}^n$ with the standard orientation. In this case one can prove by hand that the boundary orientations from Convention 31.39 and from the proof of the Boundary Orientation Proposition 192.3 are compatible. We leave it to the reader to fill in the details. ■

192.2. **Gluing topological manifolds.** We recall the following proposition:

Proposition 54.3. (Gluing-Topological Manifolds-Proposition) Let M be an n-dimensional topological manifold (possibly disconnected), let A and B be disjoint unions of boundary components of M and let $f\colon A \to B$ be a homeomorphism. Then the following statements hold:

(1) The topological space
$$M/a \sim f(a) \quad \text{for } a \in A$$
is an n-dimensional topological manifold.
(2) The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.
(3) (a) The image of A in $M/a \sim f(a)$ is a codimension-one submanifold of $M/a \sim f(a)$.
(b) If M consists of two components X and Y with $A \subseteq X$ and $B \subseteq Y$, then the following statements hold:
(i) The obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are embeddings.
(ii) The images $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$ and they are codimension-zero submanifolds of $M/a \sim f(a)$.
(4) Let R be a commutative ring. Assume that M is R-oriented. We equip A and B with the boundary orientations, as defined in the Boundary Orientation Proposition 192.3.
(a) If f is orientation-reversing, then $M/a \sim f(a)$ admits an R-orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.
(b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is not R-orientable.

Proof. Recall that we already proved Statements (1), (2) and (3) on page 1226. We still need to prove Statement (4). We will not make use of Statement (4b), thus we only provide a proof of Statement (4a), and even that proof will only be sketch.



We now assume that M is equipped with an R-orientation. We set $\widetilde{M} = M/a \sim f(a)$ and we denote by $p\colon M \to \widetilde{M}$ the projection. We need to show that $\widetilde{M}$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.

Now let $x \in \widetilde{M} \setminus \partial\widetilde{M}$. We need to distinguish two cases:

(α) We assume that $x = p(y)$ for some $y \in M \setminus \partial M$. In this case we have natural isomorphisms $\mathrm{H}_n(\widetilde{M}, \widetilde{M}\setminus\{y\}; R) \xleftarrow{\cong} \mathrm{H}_n(M\setminus\partial M, (M\setminus\partial M)\setminus\{y\}; R) \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{y\}; R)$ and we equip the local homology group of $\widetilde{M}$ with the generator given by the R-orientation of M.

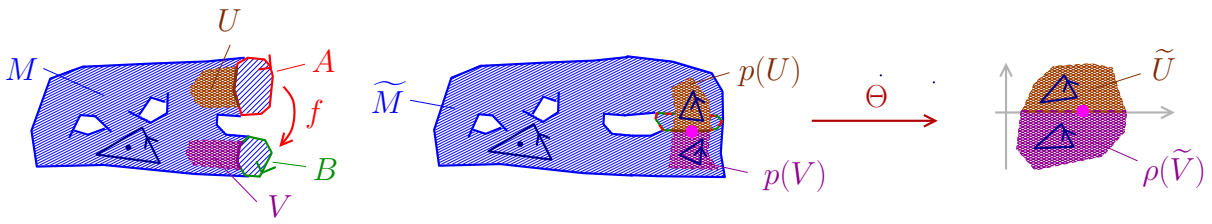
(β) If (α) does not hold, then there exist $a \in A$ and $b \in B$ with $x = p(a) = p(b)$. We pick an orientation-preserving chart $\Phi\colon U \to \widetilde{U}$ of type (ii) for $a \in M$ and we pick an orientation-reversing chart $\Psi\colon V \to \widetilde{V}$ of type (ii) for $b \in M$. After possibly shrinking U and V we can assume that $f(U \cap A) = V \cap B$ and we can assume that $U \cap V = \varnothing$. We introduce the following notation:

- We denote by $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the $(x_n{=}0)$-hyperplane.
- We write $\widetilde{U}' := \{x \in \mathbb{R}^{n-1} \mid (x, 0) \in \widetilde{U}\}$ and we denote by $\iota\colon \widetilde{U}' \to \widetilde{U}$ the inclusion.
- We denote by $\pi\colon \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^{n-1}$ the obvious diffeomorphism.
- We write $\widetilde{f} := \pi \circ \Psi|_{V\cap B} \circ f \circ (\Phi_{U\cap A})^{-1} \circ \iota\colon \widetilde{U}' \to \mathbb{R}^{n-1}$.

As we discussed in the proof of the Gluing-Topological Manifolds-Proposition 54.3 the map

$$\begin{array}{rcl} \Theta\colon p(U) \cup p(V) & \to & \widetilde{U} \cup \rho(\widetilde{V}) \\ x & \mapsto & \begin{cases} ((\widetilde{f}, \mathrm{id}_{\mathbb{R}})(\Phi(u)), & \text{if } x = p(u) \text{ for some } u \in U, \\ \rho(\Psi(v)), & \text{if } x = p(v) \text{ for some } v \in V \end{cases} \end{array}$$

is a chart of type (i) for x.



It follows from our choice of Φ and Ψ, the example on page 3694 and the Orientation Preserving Lemma 191.24 that this map Θ is orientation-preserving on the subset $p(U \setminus A) \cup p(V \setminus B)$. But then we can use this chart to also equip the points on $p(U \cap A) = p(V \cap B)$ with a coherent R-orientation.

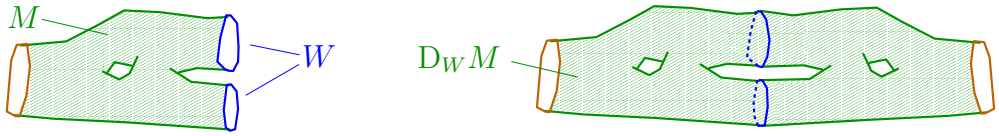
We leave it to the reader to verify that these choices do indeed define an R-orientation on $\widetilde{M}$ with the desired properties. ∎

We now apply the Gluing-Topological Manifolds-Proposition 54.3 to the double of a topological manifold. Let us recall Definition 54.8:

Definition 192.5. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to

$$\mathrm{D}_W M \;:=\; (M \times \{1\}) \cup_{W\times\{1\}=W\times\{2\}} (M \times \{2\})$$

as the double of M along W. We refer to $\mathrm{D}M := \mathrm{D}_{\partial M} M$ as the double of M.



The following lemma can be viewed as an analog of the Doubling-Manifold Lemma 54.10 (4):

Lemma 192.6. (Double-Orientability Lemma) Let M be a connected topological manifold and let W be the union of some components of ∂M. Furthermore let R be a commutative ring. Then

$$M \text{ is } R\text{-orientable} \iff \text{the double } \mathrm{D}_W M \text{ is } R\text{-orientable.}$$

Furthermore, if M is equipped with an R-orientation, then $\mathrm{D}_W M$ admits a unique R-orientation such that the "first" inclusion map $M \to \mathrm{D}_W M$ is orientation-preserving and such that the "second" inclusion map $M \to \mathrm{D}_W M$ is orientation-reversing.

Proof. The"$\Rightarrow$"–direction follows from the Gluing-Topological Manifolds-Proposition 54.3 applied to the following data:

- We consider the topological manifold $(M \times \{1\}) \sqcup (M \times \{2\})$. We equip it with an R-orientation on $M \times \{1\}$ and with the opposite R-orientation on $M \times \{2\}$.
- We set $A = W \times \{1\}$ and $B = W \times \{2\}$.
- We let $f\colon A \to B$ be the obvious homeomorphism. Note that f is orientation-reversing.

The"$\Leftarrow$"–direction is an immediate consequence of the Topological Manifold-Induced Orientation Lemma 192.1. ∎

192.3. **Product orientations of topological manifolds.** In this last section we want to consider products of topological manifolds.

First let us recall a few facts. If M is an m-dimensional topological manifold and N is an n-dimensional topological manifold, then it follows from the Topological Manifold Product Proposition 150.12 that the following two statements hold:

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold.
(2) We have $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.



The following proposition can be viewed as an analog of the Product Orientation Proposition 31.48.

Proposition 192.7. (Product Orientation Proposition)

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.

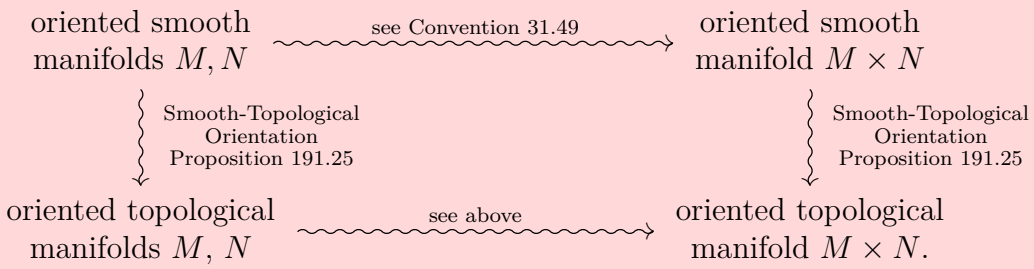
(a) For any $x \in M$ and any $y \in N$ the cross product

$$\mathrm{H}_m(M, M \setminus \{x\}) \otimes \mathrm{H}_n(N, N \setminus \{y\}) \xrightarrow{\times} \mathrm{H}_{m+n}(M \times N, (M \times N) \setminus \{(x,y)\})$$

is an isomorphism.

(b) If $\{\mu_x\}_{x \in M \setminus \partial M}$ is an orientation of M and if $\{\nu_y\}_{y \in N \setminus \partial N}$ is an orientation on N, then the cross products $\mu_x \times \nu_y$ define a natural[2061] orientation on $M \times N$.

(2) The product orientation of (1) is compatible with the product orientation of smooth manifolds. More precisely, for any two smooth manifolds M and N (where at most one has non-empty boundary) we have the following commutative diagram:



Example. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open non-empty subsets. We equip U and V with the smooth and topological standard orientations. In the smooth setting the product orientation on $U \times V$ is clearly the orientation coming from viewing $U \times V$ as a subspace of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. It follows from the Product Orientation Proposition 192.7 that the same conclusion also holds in the context of orientations of topological manifolds. □

Definition 192.8. Let M and N be oriented topological manifolds. We refer to the orientation on $M \times N$ defined in the Product Orientation Proposition 192.7 (1) as the product orientation.

The remainder of this section is occupied by the proof of the Product Orientation Proposition 192.7. The reader who is comfortable without seeing the full details of the proof can also jump straight to Chapter 193, where we consider homology groups of topological manifolds.

The proof of the Product Orientation Proposition 192.7 (2) requires that we work with an explicit Eilenberg-Zilber map. Fortunately in the Shuffle Proposition 164.24 we already gave an explicit example of an Eilenberg-Zilber map. It is worthwhile to recall the statement of the Shuffle Proposition 164.24. To do so it is necessary to recall Definition 164.21.

Definition 192.9. Let $m, n \in \mathbb{N}_0$.

(1) An (m,n)-shuffle is a pair (μ, ν) of disjoint sets of integers

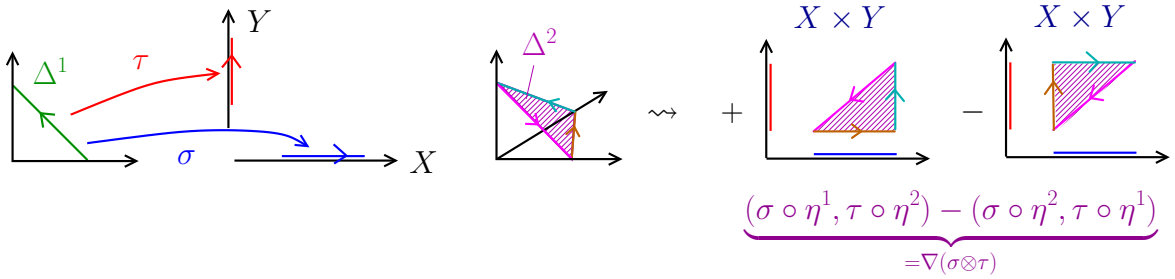
$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m+n \quad \text{and} \quad 1 \leq \nu_1 < \nu_2 < \cdots < \nu_n \leq m+n.$$

[2061] Here "natural" refers to local homeomorphisms between topological manifolds.

(2) We denote by $S_{m,n}$ the set of all (m,n)-shuffles.
(3) We denote by $\operatorname{sign}(\mu,\nu)$ the sign of the permutation $(\mu_1,\dots,\mu_m,\nu_1,\dots,\nu_n)$.
(4) Given a sequence $1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m+n$ we denote by $\eta^\mu\colon \Delta^{m+n} \to \Delta^m$ the unique affine linear map which satisfies $\eta^\mu(\mathfrak{v}_i) = \mathfrak{v}_j$ for $\mu_j \leq i < \mu_{j+1}$.[2062]
(5) Given two topological spaces X and Y we consider the homomorphism[2063]

$$\begin{aligned} \nabla := \nabla_{m,n}\colon \mathrm{C}_m(X) \otimes \mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X\times Y) \\ (\sigma\otimes\tau) &\mapsto \sum_{(\mu,\nu)\in S_{m,n}} \operatorname{sign}(\mu,\nu)\cdot(\overbrace{\overbrace{(\sigma\circ\eta^\mu)}^{\Delta^{m+n}\to X}, \overbrace{(\tau\circ\eta^\nu)}^{\Delta^{m+n}\to Y}}^{\text{map } \Delta^{m+n}\to X\times Y}). \end{aligned}$$

Example. We consider the case $X = Y = \mathbb{R}$ and $m = n = 1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1 = 1, \nu_1 = 2\}$ and $\{\mu_1 = 2, \nu_1 = 1\}$. Note that $\eta^1\colon \Delta^2 \to \Delta^1$ equals the map[2064] $[\mathfrak{v}_0,\mathfrak{v}_1,\mathfrak{v}_1]$ and that $\eta^2\colon \Delta^2 \to \Delta^1$ equals the map $[\mathfrak{v}_0,\mathfrak{v}_0,\mathfrak{v}_1]$.



In the figure above we illustrate the singular 2-simplex $\nabla_{1,1}(\sigma\otimes\tau)$ for two affine linear singular 1-simplices $\sigma\colon \Delta^1 \to X = \mathbb{R}$ and $\tau\colon \Delta^1 \to Y = \mathbb{R}$. □

With this notation we can now recall the following result:

Proposition 164.24 (Shuffle Proposition) The maps $\nabla_{m,n}$ that we had just defined have all the properties that we had mentioned in the EZ-Construction Proposition 164.10, in other words, the maps $\nabla_{m,n}\colon \mathrm{C}_m(X)\otimes\mathrm{C}_n(Y) \to \mathrm{C}_{m+n}(X\times Y)$ have the following properties:

(1) On the 0-level they equal the canonical maps that we introduced in Definition 164.1. In other words, for all $x \in X$ and $y \in Y$ we have $\nabla_{0,0}(x\otimes y) = (x,y) \in \mathrm{C}_0(X\times Y)$.
(2) They form a chain map $\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \to \mathrm{C}_*(X\times Y)$.
(3) They are natural.

Now we can provide the proof of the Product Orientation Proposition 192.7:

Proof of the Product Orientation Proposition 192.7.

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.
   (a) First note that we have $(M\times N)\setminus\{(x,y)\} = ((M\setminus\{x\})\times N)\cup(M\times(N\setminus\{y\}))$. This shows that the cross product does indeed take values in the relative homology group

[2062] Here we implicitly use that $\mu_0 = 0$ and $\mu_{m+1} = m+n+1$.
[2063] Historically this definition is inspired by the definition of the product of two ordered simplicial complexes, see Definition 174.37.
[2064] As in Definition 143.8, given points $a_0,\dots,a_k \in \mathbb{R}^n$ we denote by $[a_0,\dots,a_k]\colon \Delta^k \to \mathbb{R}^n$ the unique affine linear map that sends the i-th standard basis vector e_i to a_i.

$\mathrm{H}_{m+n}(M\times N,(M\times N)\setminus\{(x,y)\})$. It follows from the Künneth Theorem 164.15 (2), the Manifold-Local-H_*-Lemma 150.9 and the Tor-Properties Lemma 162.13 (4) that the given cross product is indeed an isomorphism.

(b) Let $(x,y)\in(M\times N)\setminus\partial(M\times N)$. Recall that this means that $x\in M\setminus\partial M$ and $y\in N\setminus\partial N$. We pick an open neighborhood U of $x\in M\setminus\partial M$ and a class $\mu_U\in\mathrm{H}_m(M,M\setminus U)$ and an open neighborhood V of $y\in N\setminus\partial N$ and a class $\nu_V\in\mathrm{H}_n(N,N\setminus V)$ as given by Definition 191.11. By the naturality of the cross product, see the Cross Product Lemma 164.5 (2), we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_m(M,M\setminus U)\otimes\mathrm{H}_n(N,N\setminus V) & \xrightarrow{\quad\times\quad} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus(U\times V)) \\
{\scriptstyle \mu_U\mapsto\mu_x}\big\downarrow{\scriptstyle \nu_V\mapsto\nu_y} & & \big\downarrow \\
\mathrm{H}_m(M,M\setminus\{x\})\otimes\mathrm{H}_n(N,N\setminus\{y\}) & \xrightarrow[(\mu_x,\nu_y)\mapsto\mu_x\times\nu_y]{\quad\times\quad} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus\{(x,y)\}).
\end{array}$$

It is now straightforward to verify that $\phi_{U\times V}\coloneqq\mu_U\times\nu_V$ has the properties desired by Definition 191.11. Finally note that it follows again from the naturality of the cross product that the product orientation is natural.

(2) It follows easily from the naturality of the product orientations of smooth and topological statements that it suffices to prove the statement for the smooth manifolds $M=\mathbb{R}^m$ and for $N=\mathbb{R}^n$ and only for a suitable point $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}$. Given any $k\in\mathbb{N}_0$ we consider the singular chain $\sigma_k=(-1)^k\cdot\pi_k$ where[2065] $\pi_k=[\mathfrak{v}_0,\ldots,\mathfrak{v}_{k-1},0]$. (In plain English the singular simplex $\pi_k=[\mathfrak{v}_0,\ldots,\mathfrak{v}_{k-1},0]\colon\Delta^k\to\mathbb{R}^k$ is just given by the obvious projection onto the first k coordinates.) We make the following observation:

(∗) Let $z\in\mathbb{R}^k$ be any point that lies in the interior of the image of σ_k. It follows from the discussion on page 3682 that the singular chain $\sigma_k=(-1)^k\cdot\pi_k$ represents the standard generator of $\mathrm{H}_k(\mathbb{R}^k,\mathbb{R}^k\setminus\{z\})$.

Next we consider the (m,n)-shuffle that is given by the sets $\mu_0=\{1,\ldots,m\}$ and by $\nu_0=\{m+1,\ldots,m+n\}$. It follows easily from the definitions that

$$\eta^{\mu_0}=[\mathfrak{v}_0,\ldots,\mathfrak{v}_m,\ldots,\mathfrak{v}_m]\colon\Delta^{m+n}\to\Delta^m\quad\&\quad\eta^{\nu_0}=[\mathfrak{v}_0,\ldots,\mathfrak{v}_0,\ldots,\mathfrak{v}_n]\colon\Delta^{m+n}\to\Delta^n.$$

Also note that it is a straightforward consequence of the definitions that we have the following equality of singular $(m+n)$-chains in $\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}$:

$$(\pi_m\circ\eta^{\mu_0},\pi_n\circ\eta^{\nu_0})\;=\;[\mathfrak{v}_0,\ldots,\mathfrak{v}_{m-1},\mathfrak{v}_m,\ldots,\mathfrak{v}_{m+n-1},0]\;=\;\pi_{m+n}.$$

We pick a point $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$ that lies in the interior of the image of π_{m+n}. Note that x lies in the interior of the image of π_m and that y lies in the interior of the image of π_n.

It follows from the observation (∗), the definition of the product orientation in Definition 192.8 and the Sections-on-Connected Proposition 191.13 that it suffices to prove the following claim:

Claim. The following equality holds in $\mathrm{H}_{m+n}(\mathbb{R}^{m+n},\mathbb{R}^{m+n}\setminus\{(x,y)\})$:

$$[\sigma_m]\times[\sigma_n]\;=\;[\sigma_{m+n}].$$

[2065] As in Definition 143.8, given points $a_0,\ldots,a_k\in\mathbb{R}^n$ we denote by $[a_0,\ldots,a_k]\colon\Delta^k\to\mathbb{R}^n$ the unique affine linear map that sends the i-th standard basis vector e_i to a_i.

Proof. We have

$$[\sigma_m]\times[\sigma_n] \underset{\text{by the Shuffle Proposition 164.24 and the EZ–Natural Chain Homotopy Proposition 164.13 (d)}}{=} [\nabla_{m,n}(\sigma_m\otimes\sigma_n)] \underset{\text{by Definition 192.9 of } \nabla_{m,n}}{=} \Big[\sum_{(\mu,\nu)}(-1)^m\cdot(-1)^n\cdot\operatorname{sign}(\mu,\nu)\cdot((\pi_m\circ\eta^\mu),(\pi_n\circ\eta^\nu))\Big]$$

$$\underset{\text{for all other shuffles } (\mu,\nu) \text{ the point } (x,y) \text{ does not lie in the image of } (\pi_m\circ\eta^\mu,\pi_n\circ\eta^\nu)}{=} \big[(-1)^{m+n}\cdot\underbrace{\operatorname{sign}(\mu_0,\nu_0)}_{=1}\cdot\underbrace{(\pi_m\circ\eta^{\mu_0}\times\pi_n\circ\eta^{\nu_0})}_{=\pi_{m+n}}\big] = [\sigma_{m+n}].$$

■

192.4. Exercises for Chapter 192 .

Exercise 192.1. Let $a<b$ be two real numbers. We equip $[a,b]$ with the standard orientation. Use the explicit description of the boundary orientation in the Boundary Orientation Proposition 192.3 (1) to show that $\partial([a,b]) = \{b\}\cup(-\{a\})$.

Exercise 192.2. Let M be a compact oriented 1-dimensional topological manifold. We equip ∂M with the orientation coming from the Boundary Orientation Proposition 192.3. Recall that on page 3691 we saw that an orientation on the 0-dimensional topological manifold ∂M is naturally the same as a map $\epsilon\colon \partial M\to\{\pm1\}$. Show that

$$\sum_{P\in\partial M}\epsilon(P) = 0.$$

Remark. This exercise can be viewed as the analog of the Boundary-of-Compact Smooth 1-Manifold Proposition 31.42 for topological manifolds.

Exercise 192.3. Let M be the Möbius band. We consider the 3-dimensional topological manifold $M\times[0,1]$.

(a) What is the homeomorphism type of $\partial(M\times[0,1])$?
(b) Is the topological manifold $M\times[0,1]$ orientable?

Exercise 192.4. Let $n\in\mathbb{N}$.

(a) Show that the action of $G=\{\pm1\}$ on the topological manifold S^n is orientation-preserving if and only if n is odd.
(b) Show that the real projective space $\mathbb{R}\mathrm{P}^n$, viewed as a topological manifold, is orientable if and only if n is odd.

Exercise 192.5. Let M and N be two topological manifolds of dimension ≥ 1. Show that $M\times N$ is orientable if and only if M and N are both orientable.
Remark. This exercise is the topological analog of Exercise 31.20. The present exercise can be viewed as a converse to the Product Orientation Proposition 192.7.

Exercise 192.6. Let $n\in\mathbb{N}$ and let $A\in\mathrm{GL}(n,\mathbb{Z})$. By the Maps-on-Torus Lemma 25.46 we know that the map

$$\begin{aligned} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n &\to \mathbb{R}^n/\mathbb{Z}^n \\ [v] &\mapsto [A\cdot v] \end{aligned}$$

is a homeomorphism. Show that this homeomorphism is orientation-preserving if and only if $A\in\mathrm{SL}(n,\mathbb{Z})$, i.e. if and only if $\det(A)=1$.
Remark. This exercise is the topological analog of Exercise 31.7.

Exercise 192.7. Can a closed non-orientable connected topological manifold be the boundary of a compact topological manifold?

Exercise 192.8. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. We equip the product topological manifolds $M \times N$ and $N \times M$ with the product orientations as defined in the Product Orientation Proposition 192.7.

(a) Show that the homeomorphism
$$\begin{array}{ccc} M \times N & \to & N \times M \\ (x, y) & \mapsto & (y, x) \end{array}$$
is orientation-preserving if and only if $m \cdot n$ is even.

(b) We suppose that $\partial N = \varnothing$. We equip ∂M and $N \times M$ with the boundary orientation and we equip $N \times \partial M$ with the product orientation. Show that we have the following equality of *oriented* topological manifolds:
$$\partial(N \times M) \;=\; \begin{cases} N \times \partial M, & \text{if } n \text{ is even,} \\ -(N \times \partial M), & \text{if } n \text{ is odd.} \end{cases}$$

Hint. These are local statements.

Exercise 192.9. Does there exist an oriented connected 1-dimensional topological manifold that does not admit an orientation-reversing homeomorphism?

Exercise 192.10. Let $f\colon M \to M'$ and $g\colon N \to N'$ be two homeomorphisms of oriented connected topological manifolds. Show that $f \times g\colon M \times N \to M' \times N'$ is orientation-preserving if and only if both f and g are orientation-preserving or if both are orientation-reversing.

Exercise 192.11. Let $n \in \mathbb{N}$. We introduce the following notation:

- Given $a_0, \ldots, a_n \in [0, 1]$ we consider, following the Definition 148.17, the singular n-simplex
$$[a_0, \ldots, a_n]\colon \Delta^n \;\to\; [0, 1]^n$$
which is the unique affine linear map that sends the i-th vertex $\mathfrak{v}_i$ of Δ^n to a_i.
- Let $\sim$ be the equivalence relation on $[0, 1]^n$ which is generated by
$$(x_1, \ldots, 0, \ldots, x_n) \;\sim\; (x_1, \ldots, 1, \ldots, x_n).$$
- We denote by $p\colon [0, 1]^n \to [0, 1]^n/\!\sim$ the natural projection.

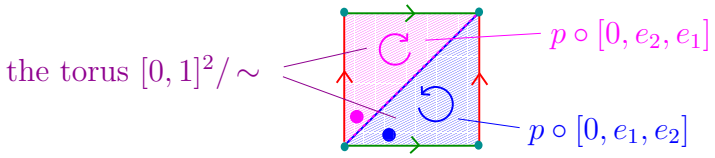
With this notation we can pose the following problems:

(a) Show that $[0, 1]^n/\!\sim$ is homeomorphic to $(S^1)^n$.

(b) Show that
$$\omega \;:=\; \sum_{\sigma \in S_n} (-1)^{\operatorname{sign}(\sigma)} \cdot p \circ [0, e_{\sigma(1)}, \ldots, e_{\sigma(n)}] \qquad \in\; \mathrm{C}_n([0, 1]^n/\!\sim)$$
is a cycle.

Hint. Either do it by hand or use the Shuffle Proposition 164.24.

(c) Show that $\mathrm{H}_n([0, 1]^n/\!\sim) = \mathbb{Z} \cdot [\omega]$.


the torus $[0,1]^2/\sim$
$p \circ [0, e_2, e_1]$
$p \circ [0, e_1, e_2]$

CHAPTER 193

The top homology groups of topological manifolds

In several previous chapters, e.g. in Chapter 179 and 183, we assembled lots of information on homology groups of smooth manifolds. Let us recall some of the results. Thus let M be a connected n-dimensional smooth manifold.

(1) In the Smooth Manifolds-Invariants Proposition 179.10 (4) we saw, using the Smooth Manifold-CW-Theorem 179.9, that $\mathrm{H}_i(M;\mathbb{Z}) = 0$ for $i > n$.
(2) Furthermore, in the Manifold-Top-H_*-Theorems 183.18 and 183.20 we showed that

$$\mathrm{H}_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is closed and orientable,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if M is compact and orientable, then we used (2) to define the corresponding fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$ which turned out to be a very useful tool.

The key tool in proving the above statements was the fact, established in the Smooth Manifold-Simplicial Structure Theorem 179.5, that smooth manifolds admit simplicial structures. As we discussed in Section 190.2, in general topological manifolds do not admit simplicial structures, so if we want to replicate the above results for topological manifolds we will have to use a very different approach.

In this chapter we will indeed manage to reprove all of the above statements for topological manifolds. The advantage of the new approach is that it is more general, it covers topological manifolds, and that it does not require any extra topological input, i.e. it does not require the existence of a simplicial or cellular structure. The flip side of this chapter's approach is that the proofs are arguably less intuitive.

This chapter is organized as follows:

- In Section 193.1 we will discuss the relationship between homology and orientability. In particular we will formulate the Manifold-Top-H_*-Theorem 193.1 and the Manifold-H_{n-1}-Theorem 193.4.
- In Section 193.2 we will use the above results to calculate the homology groups of lens spaces.
- In Section 193.3 we will formulate the all important Homology-Section Theorem 193.6.
- The intricate proof of the Homology-Section Theorem 193.6 will take place in Subsections 193.3.1, 193.3.2 and 193.3.3.
- In Section 193.4 we will use the Homology-Section Theorem 193.6 to provide the proof of the Manifold-Top-H_*-Theorem 193.1.

- In Section 193.5 we will deduce the Manifold-H_{n-1}-Theorem 193.4 from the Manifold-Top-H_*-Theorem 193.1.
- In Section 193.7 we will provide exercises for Chapter 193.

Readers who are not all that interested in lens spaces and the proofs of the Manifold-Top-H_*-Theorem 193.1 can jump straight for Chapter 194 where we apply the above theorems.

193.1. **Homology and orientability.** The following is the main theorem of this chapter:

Theorem 193.1. (Manifold-Top-H_*-Theorem) Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. The following three statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M;R) = 0$.
(b) For every $i > n$ we have $\mathrm{H}_i(M, \partial M; R) = 0$.

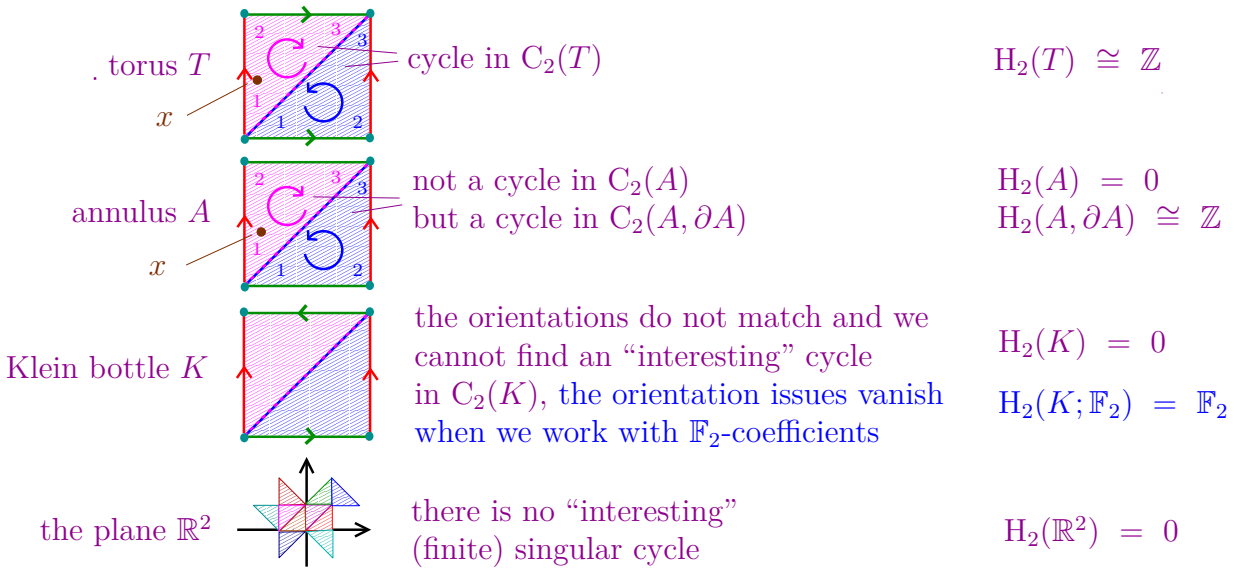
(2) (a)
$$\mathrm{H}_n(M;R) \cong \begin{cases} R, & \text{if } M \text{ is closed and } R\text{-orientable,} \\ \{r \in R \,|\, 2r = 0\}, & \text{if } M \text{ is closed and not } R\text{-orientable,} \\ 0, & \text{if } M \text{ is not closed.} \end{cases}$$

(b)
$$\mathrm{H}_n(M, \partial M; R) \cong \begin{cases} R, & \text{if } M \text{ is compact and } R\text{-orientable,} \\ \{r \in R \,|\, 2r = 0\}, & \text{if } M \text{ is compact and not } R\text{-orientable,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases}$$

(3) If M is compact and R-orientable, then for every $x \in M \setminus \partial M$ the natural map
$$\mathrm{H}_n(M, \partial M; R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$$
is an isomorphism.

Proof. We will prove the theorem in Section 193.4 after long and arduous preparations. ■

Example. In the following figure we show some examples which hopefully illustrate the statement of the theorem.



Hopefully the figure works as a mnemonic device to remember the many statements. □

For the most part we will be interested in the rings $R = \mathbb{Z}$ and $R = \mathbb{F}_2$. In these two cases we obtain two specialized theorems, which are worth recording:

Theorem 193.2. (Top-$\mathbb{F}_2$-Homology Theorem) Let M be a compact connected non-empty n-dimensional topological manifold M. The following two statements hold:

(1) $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.
(2) For every $x \in M \setminus \partial M$ the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is an isomorphism.

Proof. Recall that in the $\mathbb{F}_2$-Orientability Proposition 191.22 we showed that every topological manifold admits an $\mathbb{F}_2$-orientation. This theorem is thus an immediate consequence of the Manifold-Top-H_*-Theorem 193.1 (2b) and (3) applied to $R = \mathbb{F}_2$. ■

Not every topological manifold is orientable. The following theorem gives us several criteria for orientability.

Theorem 193.3. (Orientation-Top-H_*-Theorem) Let M be a compact connected non-empty n-dimensional topological manifold. The following statements are equivalent:

(1) The topological manifold M is orientable.
(2) $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \neq 0$.
(3) $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.
(4) For every $x \in M \setminus \partial M$ the natural map $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z})$ is an isomorphism.

Proof of the Orientation-Top-H_*-Theorem 193.3. Let M be a compact connected non-empty n-dimensional topological manifold. We have (1)⇒(4) by the Manifold-Top-H_*-Theorem 193.1 (3) applied with $R = \mathbb{Z}$. Since M is non-empty we obtain from the Interior-Non Compact Lemma 23.16 that there exists a point $x \in M \setminus \partial M$. Therefore we have (4)⇒(3) by the Manifold-Local R-Homology Lemma 191.2. Evidently we have (3)⇒(2). Furthermore we have (2)⇒(1) by the Manifold-Top-H_*-Theorem 193.1 (2b) and the little observation that $\{r \in \mathbb{Z} \mid 2r = 0\} = 0$. ■

Remark.

(1) As mentioned before, the above theorems generalize most of the results of the Smooth Manifolds-Invariants Proposition 179.10 (4), the Manifold-Top-H_*-Theorem 183.18 and the Manifold-Top-H_*-Theorem 183.20 from smooth manifolds to topological manifolds.
(2) One might be tempted to consider the Manifold-Top-H_*-Theorem 193.1 (1) as "intuitively clear". After all, how could an "n-dimensional object" possibly have homology groups in higher dimensions? To shatter this belief, fix $n \in \mathbb{N}_{\geq 2}$. We consider the following subset of $\mathbb{R}^n$:[2066]

$$\begin{aligned} X &:= \bigcup_{n \in \mathbb{N}} n\text{-dimensional sphere of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n}, 0, \ldots, 0) \\ &= \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R}^n \mid \|x - (\tfrac{1}{n}, 0, \ldots, 0)\| = \tfrac{1}{n^2}\}. \end{aligned}$$

In [**BaM1962**] it is shown, rather shockingly, that for every q with $q \equiv 1 \bmod (r-1)$ the group $\mathrm{H}_q(X; \mathbb{Q})$ is not only non-zero, it is even uncountable. □

[2066] As the reader will surely have noticed, this example can be viewed as a high-dimensional analog of the Hawaiian Earings that we introduced in Exercise 105.1.

Let M be a compact n-dimensional topological manifold. With the Manifold-Top-H_*-Theorem 193.1 we now completely understand the n-dimensional homology of M for all rings. We will use these results to obtain partial information on the $(n-1)$-st homology groups with $\mathbb{Z}$-coefficients:

Theorem 193.4. (Manifold-H_{n-1}-Theorem) Let M be a compact connected n-dimensional topological manifold.

(1) $$H_{n-1}(M;\mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

(2) $$H_{n-1}(M,\partial M;\mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proof. Much later in Section 193.5 we will provide a proof of the Manifold-H_{n-1}-Theorem 193.4. As we will see, for the most part it is a fairly formal consequence of the Manifold-Top-H_*-Theorem 193.1 applied to $R = \mathbb{Z}$ and $R = \mathbb{F}_p$ for all primes p together with the Universal Coefficient Theorem 162.16. ■

Remark. The Manifold-H_{n-1}-Theorem 193.4 generalizes the results of the Manifold-Top-H_*-Theorem 183.18 (2) and the Manifold-Top-H_*-Theorem 183.20 (2) from smooth manifolds to topological manifolds. □

Example. As a reality check it is good to recall the calculation of the homology groups of closed 2-dimensional topological manifolds that we gave in the Surface-H_*-Proposition 155.17 and of the real projective spaces $\mathbb{R}P^n$ in the $\mathbb{R}P^n$-H_*-Proposition 155.18:

topological manifold M	orientable	$H_{\dim(M)}(M)$	$H_{\dim(M)-1}(M)$
surface Σ_g of genus g	yes	$\mathbb{Z}$	$\mathbb{Z}^{2g}$
non-orientable surface N_g of genus g	no	0	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{g-1}$
real projective space $\mathbb{R}P^n$ with n odd	yes	$\mathbb{Z}$	0
real projective space $\mathbb{R}P^n$ with n even	no	0	$\mathbb{Z}/2\mathbb{Z}$.

Fortunately these calculations are coherent with the Manifold-Top-H_*-Theorem 193.1 and the Manifold-H_{n-1}-Theorem 193.4. □

193.2. **Homology groups of lens spaces.** In this section we will apply the results from the previous section to calculate the homology groups of lens spaces. The results and the type of proof are basically just a repetition of what we did in Section 183.3.

Lemma 183.21 (Lens Space-H_*-Lemma) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. For the corresponding lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ we have

$$H_n(L(p,q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n = 1. \end{cases}$$

Remark. An alternative approach to computing the homology groups of the lens spaces $L(p,q)$ will be given in Exercise **??** and later in the Generalized Lens Space-H_*-Proposition 306.7. □

The calculation of this lemma is rather disappointing: the q-parameter is once again hiding. Thus we have made no progress at all on Question 91.6.

Proof. We start out with the following observations:

(1) We already know from the H_0-Proposition 143.25 that $H_0(L(p,q)) \cong \mathbb{Z}$.
(2) By the Lens Space-H_1-Corollary 166.11 we know that $H_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.
(3) In Definition 91.4 we pointed out that every lens space $L(p,q)$ is a closed orientable connected 3-dimensional smooth manifold.
(4) By the Manifold-Top-H_*-Theorem 193.1 (1) we know that $H_n(L(p,q)) = 0$ for $n \geq 4$.
(5) By the Orientation-Top-H_*-Theorem 193.3 we know that $H_3(L(p,q)) \cong \mathbb{Z}$.

Thus at the moment we are still missing $H_2(L(p,q))$. To obtain this homology group we make two more observations:

(6) By the Smooth Manifold-CW-Theorem 179.9 we know that $L(p,q)$ admits the structure of a finite CW-complex.
(7) It follows from the Action–Covering Proposition 91.11 and the definition of a lens space that $L(p,q)$ is finitely covered by S^3. It follows from $\chi(S^3) = 0$ together with the Covering-χ-Proposition 156.13 that $\chi(L(p,q)) = 0$.

Now note that

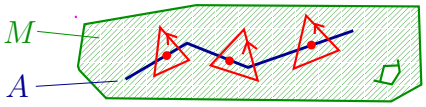
$$\operatorname{rank} H_2(L(p,q)) \overset{\text{by the Homology-}\chi\text{ Proposition 156.2}}{=} \underbrace{\chi(L(p,q))}_{=0} - \underbrace{\operatorname{rank} H_0(L(p,q))}_{=1} + \underbrace{\operatorname{rank} H_1(L(p,q))}_{=0} + \underbrace{\operatorname{rank} H_3(L(p,q))}_{=1} = 0.$$

We know by the Manifold-H_{n-1}-Theorem 193.4 that $H_2(L(p,q))$ is a finitely generated free abelian group. Therefore we see that $H_2(L(p,q)) = 0$. ∎

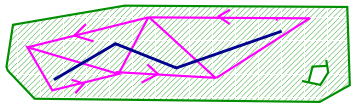
193.3. **Lifts of sections on topological manifolds.** In this section we will state and prove a fundamental theorem of topological manifolds which lies at the heart of the proof of the Manifold-Top-H_*-Theorem 193.1. Before we can state the theorem we introduce the following helpful notation.

Notation 193.5. Let M be a topological manifold with $\partial M = \varnothing$, let R be a commutative ring and let $A \subseteq M$ be a subset.

(1) We write $H_n(M|A;R) := H_n(M, M \setminus A; R)$. Given a point $x \in M$ we use the obvious abbreviation $H_n(M|x;R) := H_n(M|\{x\};R)$.
(2) Given a homology class $\alpha \in H_n(M|A;R)$ and a point $x \in A$ we denote by α_x the image of α under the obvious map $H_n(M|A;R) \to H_n(M|x;R)$.
(3) Let $A \subseteq M$ be a subset and let $\{\alpha_x\}_{x\in A}$ be an R-section of M along A. We call a class $\beta \in H_n(M|A;R)$ a lift to A if for all $x \in A$ we have $\beta_x = \alpha_x$.
(4) As in Definition 191.11 we denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A. Recall that by the Homology-defines-Section Lemma 191.12 we know that $\Gamma_R^A(M)$ is naturally an R-module.



section $\{\alpha_x\}_{x\in A}$ along A



lift β of the section to $H_2(X, A)$

The following theorem is the main technical result of this chapter.

Theorem 193.6. (Homology-Section Theorem) Let M be an n-dimensional topological manifold with $\partial M = \varnothing$ and let $A \subseteq M$ be a compact subset. For every commutative ring R the following statements hold:

(i) The map

$$\begin{array}{rcl} \mathrm{H}_n(M|A;R) & \to & \Gamma_R^A(M) \\ \varphi & \mapsto & \{\varphi_x\}_{x\in A} \end{array}$$

is an isomorphism of R-modules. In other words, if $\{\alpha_x\}_{x\in A}$ is an R-section for M along A, then there exists a unique lift $\alpha_A^M = \alpha_A$ to A, i.e. there exists a unique homology class $\alpha_A^M = \alpha_A \in \mathrm{H}_n(M|A;R)$ such that for each $x \in A$ we have $\alpha_{A,x} = \alpha_x$.

(ii) For $i > n$ we have $\mathrm{H}_i(M|A;R) = 0$.

Remark.

(1) To avoid any nervousness later on we remark that the statements (i) and (ii) hold trivially for $A = \varnothing$ since $\mathrm{H}_n(M|\varnothing;R) = \mathrm{H}_n(M,M;R) = 0$.

(2) In the Boundary Homology–Section Theorem 193.20 we extend the Homology-Section Theorem 193.6 to topological manifolds with non-empty boundary. □

How can we prove the Homology-Section Theorem 193.6? We need to show two statements for all topological manifolds M with $\partial M = \varnothing$ and all compact subsets $A \subseteq M$. We proceed as follows:

(∗) First we show in the Homology Section–Union Lemma 193.7 the following: If we are given a topological manifold M and two compact subsets A and B such that the statements (i) and (ii) hold for M and compact subsets A, B and $A \cap B$, then the statement holds for $A \cup B$.

We then work our way up from simple cases to the general cases:

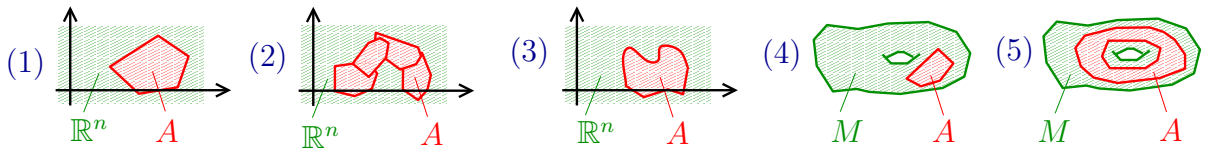
(1) We show the statements for $M = \mathbb{R}^n$ and A a convex compact subset.

(2) We use (1) and (∗) to show the statements for $M = \mathbb{R}^n$ and A a finite union of convex compact subsets.

(3) We use (2) to show the statements for $M = \mathbb{R}^n$ and any compact subset A.

(4) We use (3) to show the statements for any M and any compact subset $A \subseteq M$ contained in a chart.

(5) We use (4) and (∗) to show the general case.



193.3.1. **Proof of the Homology-Section Theorem 193.6 I.** We start out with the following lemma which corresponds to the statement (∗) above.

Lemma 193.7. (Homology Section–Union Lemma) Let M be an n-dimensional topological manifold with $\partial M = \varnothing$, let R be a commutative ring and let A and B be two compact subsets of M. If the statements (i) and (ii) from the Homology-Section Theorem 193.6 hold for A, B and $A \cap B$, then they also hold for $A \cup B$.

The proof of the Homology Section–Union Lemma 193.7 requires the following variation on the Mayer–Vietoris Theorem 153.2.

Theorem 193.8. (Inverted Mayer–Vietoris Theorem)[2067] Let X be a topological space and let U and V be open subsets. Then for all $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n \colon \mathrm{H}_n(X, U \cup V) \to \mathrm{H}_{n-1}(X, U \cap V)$ such that

$$.. \to \mathrm{H}_n(X, U \cap V) \xrightarrow{(i,-i)} \mathrm{H}_n(X,U) \oplus \mathrm{H}_n(X,V) \xrightarrow{i \oplus i} \mathrm{H}_n(X, U \cup V) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X, U \cap V) \to ..$$

is an exact sequence. The same statement also holds for coefficients in any commutative ring R.

Proof of the Inverted Mayer–Vietoris Theorem 193.8. Let X be a topological space and let U and V be open subsets. We recall Definition 148.8:

$$\mathrm{C}_n^{\{U,V\}}(X) \;:=\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(U) \text{ and } \tau \in \mathrm{C}_n(V)\}.$$

As in the second proof of the Mayer–Vietoris Theorem 153.2 on page 2905 we have a short exact sequence

$$0 \;\to\; \mathrm{C}_n(X, U \cap V) \;\xrightarrow{(i,-i)}\; \mathrm{C}_n(X,U) \oplus \mathrm{C}_n(X,V) \;\xrightarrow{i \oplus i}\; \mathrm{C}_n(X)/\mathrm{C}_n^{\{U,V\}}(U \cup V) \;\to\; 0.$$

Thus we get a corresponding long exact sequence of homology groups. It suffices to prove the following claim:

Claim. There exists an isomorphism $\mathrm{H}_n(\mathrm{C}_*(X)/\mathrm{C}_*^{\{U,V\}}(U \cup V)) \to \mathrm{H}_n(X, U \cup V)$.

Proof. We consider the following commutative pair of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_*^{\{U,V\}}(U \cup V) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X)/\mathrm{C}_*^{\{U,V\}}(U \cup V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_*(U \cup V) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X, U \cup V) & \longrightarrow & 0.
\end{array}$$

We obtain the corresponding long exact sequence of homology groups:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n^{\{U,V\}}(U \cup V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n\big(\mathrm{C}_*(X)/\mathrm{C}_*^{\{U,V\}}(U \cup V)\big) & \longrightarrow & \mathrm{H}_{n-1}^{\{U,V\}}(U \cup V) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \mathrm{H}_n(U \cup V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, U \cup V) & \longrightarrow & \mathrm{H}_{n-1}(U \cup V) & \longrightarrow & \ldots
\end{array}$$

It follows from the Comfortable Cover Proposition 148.9 that the first and fourth vertical map are isomorphisms. The second map and the fifth map (which for space reasons is not shown) are the identity. Hence it follows from the Five Lemma 145.11 that the third vertical map is also an isomorphism. ∎

Proof of the Homology Section–Union Lemma 193.7. Let M be an n-dimensional topological manifold with $\partial M = \varnothing$ and let R be a commutative ring. We start out with a little observation:

Observation. Let $X \subseteq Y \subseteq M$ be two subsets. Let $\{\alpha_x\}_{x \in Y}$ be an R-section on M along Y. If $\mu_Y \in \mathrm{H}_n(M|Y; R)$ is a lift of this R-section to Y, then it follows immediately from the functoriality of relative homology groups that the image of μ_Y under the natural map $\mathrm{H}_n(M|Y; R) \to \mathrm{H}_n(M|X; R)$ is a lift of the given R-section to the subset X.

Now we start with the actual proof of the lemma. Thus let A and B be two compact subsets of M such that A, B and $A \cap B$ satisfy statements (i) and (ii). To simplify the notation we drop R from the notation. We consider the inverted Mayer–Vietoris Sequence

[2067] This theorem is also the content of Exercise 153.2.

from the Inverted Mayer–Vietoris Theorem 193.8

$$\cdots \to \overbrace{\mathrm{H}_{n+1}(M|A\cap B)}^{=0,\text{ by hypothesis on } A\cap B} \to \mathrm{H}_n(M|A\cup B) \xrightarrow{\Phi} \mathrm{H}_n(M|A)\oplus \mathrm{H}_n(M|B) \xrightarrow{\Psi} \mathrm{H}_n(M|A\cap B)\to \dots$$
$$\alpha \mapsto (\alpha,-\alpha) \qquad (\alpha,\beta)\mapsto \alpha+\beta$$

Now we can prove that $A\cup B$ also satisfies statements (i) and (ii).

(ii) For $i>n$ the group $\mathrm{H}_i(M|A\cup B)$ is stuck, by our hypothesis on A, B and $A\cap B$, between groups that are zero, thus we see that $\mathrm{H}_i(M|A\cup B)=0$ for $i>n$.

(i) Now suppose that we are given a section $\{\alpha_x\}_{x\in A\cup B}$ on M along $A\cup B$. We denote by α_A, α_B and $\alpha_{A\cap B}$ the unique lifts to A, B and $A\cap B$ given by our hypothesis. By the above observation the images of α_A and α_B in $\mathrm{H}_n(M|A\cap B)$ are lifts to $A\cap B$. By the uniqueness of the lift $\alpha_{A\cap B}$ these images thus equal $\alpha_{A\cap B}$. By the exactness of the above sequence we see that $(\alpha_A,-\alpha_B)=\Phi(\alpha_{A\cup B})$ for some $\alpha_{A\cup B}\in \mathrm{H}_n(M|A\cup B)$. We claim that $\alpha_{A\cup B}$ is indeed a lift to $A\cup B$. So let $x\in A\cup B$. First we consider the case that $x\in A$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M|A\cup B) & \xrightarrow{[\alpha_{A\cup B}\mapsto \alpha_A]} & \mathrm{H}_n(M|A) \\ & \searrow & \downarrow \\ & & \mathrm{H}_n(M|x). \end{array}$$

We see that the image of $\alpha_{A\cup B}$ in $\mathrm{H}_n(M|x)$ is the image of α_A in $\mathrm{H}_n(M|x)$, hence it equals α_x. The same argument applies to the case that $x\in B$.

It remains to prove the uniqueness of the lift $\alpha_{A\cup B}$. If $\beta\in \mathrm{H}_n(M|A\cup B)$ is another lift, then it follows immediately from the above observation and the uniqueness of α_A and α_B that $\Phi(\beta)=(\alpha_A,-\alpha_B)$. By the exactness of the sequence we see that we already have the equality $\alpha_{A\cup B}=\beta$ in $\mathrm{H}_n(M|A\cup B)$. ■

Later on we will need the following variation on the Homology Section–Union Lemma 193.7.

Lemma 193.9. (Homology Section–Finite Union Lemma) Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let $\mathcal{A}$ be a family of compact subsets of M with the following two properties:

(1) Statements (i) and (ii) from the Homology-Section Theorem 193.6 hold for each $A\in\mathcal{A}$.
(2) The family $\mathcal{A}$ is closed under intersections, i.e. whenever $A, B\in\mathcal{A}$, then $A\cap B\in\mathcal{A}$.

Then the statements hold for all finite unions of elements in $\mathcal{A}$.

Proof. Given $k\in\mathbb{N}$ we write

$$\mathcal{U}_k \coloneqq \{\text{all subsets of } M \text{ that are the union of at most } k \text{ subsets in } \mathcal{A}\}.$$

We claim that for every $k\in\mathbb{N}$ statements (i) and (ii) hold for all elements in $\mathcal{U}_k$. We prove this claim by induction on k.

(1) If $k=1$, then the claim holds by hypothesis (1).

(2) We now suppose that the claim holds for some $k \in \mathbb{N}$. Let $A_1, \ldots, A_k, A_{k+1} \in \mathcal{A}$. We write $B = A_2 \cup \cdots \cup A_{k+1}$. Note that

$$A_1 \cap B \;=\; A_1 \cap (A_2 \cup \cdots \cup A_{k+1}) \;=\; \bigcup_{i=2}^{k+1} \underbrace{(A_1 \cap A_i)}_{\in \mathcal{A} \text{ by (2)}} \;\in \mathcal{U}_k.$$

Thus by induction hypothesis we see that the statements hold for A_1, B and also for $A_1 \cap B$. So the statements also hold for $A_1 \cup B = A_1 \cup \cdots \cup A_{k+1}$ by the Homology Section–Union Lemma 193.7. ■

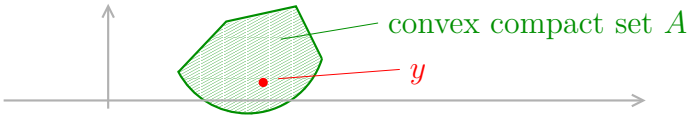
193.3.2. **Proof of the Homology-Section Theorem 193.6 II: $\mathbb{R}^n$.** We now turn to the five steps in the proof of the Homology-Section Theorem 193.6 that we had formulated on page 3725. In this section we provide the proofs of the first three steps, namely the ones, which cover the case $M = \mathbb{R}^n$.

Lemma 193.10. (Homology Section–Convex Compact Lemma) Statements (i) and (ii) from the Homology-Section Theorem 193.6 hold for any commutative ring R, $M = \mathbb{R}^n$ and any convex compact subset A of $\mathbb{R}^n$.

Proof. We start out with the following general claim:

Claim. Let A be a convex compact subset of $\mathbb{R}^n$. For every $y \in A$ and every $i \in \mathbb{N}_0$ the following inclusion-induced map is an isomorphism:

$$\mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A; R) \;\to\; \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}; R).$$



Proof. To simplify the notation we drop the coefficients R from the notation and we assume that $y = 0$. Now let $i \in \mathbb{N}_0$. In the following we will prove that the following inclusion-induced maps are isomorphisms:

$$\begin{array}{lrcl} \text{(a)} & \mathrm{H}_i(\mathbb{R}^n \setminus A) & \to & \mathrm{H}_i(\mathbb{R}^n \setminus \{y\}) \\ \text{(b)} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}). \end{array}$$

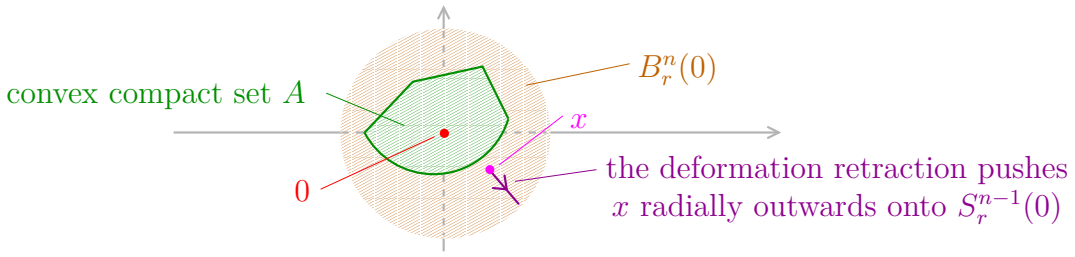
So let us turn to the proofs: [2068]

(a) Since A is compact we know by the Heine-Borel Theorem 3.44 that A is bounded. Thus we can pick an $r > 0$ with $A \subseteq B_r^n(0)$. We consider the following map:

$$\begin{array}{rcl} F\colon (\mathbb{R}^n \setminus A) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R}^n \setminus B_r^n(0), \\ x \cdot \underbrace{((1-t) + t \cdot \frac{r}{\|x\|})}_{\geq 1 \text{ since } r > \|x\|}, & \text{if } x \in \overline{B}_r^n(0) \setminus A. \end{cases} \end{array}$$

Note that if $x \notin A$, then it follows from the convexity of A and the fact that $0 \in A$, that for any $s \geq 1$ the point $x \cdot s$ cannot lie in A either. This shows that the above map actually takes values in $\mathbb{R}^n \setminus A$.

[2068] Statement (a) is the content of Exercise 144.3.



Now it is pretty obvious that the map F defines a deformation retraction from $\mathbb{R}^n \setminus A$ to $\mathbb{R}^n \setminus B_r^n(0)$. [2069] Clearly $\mathbb{R}^n \setminus B_r^n(0)$ is also a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Next we consider the following commutative diagram of inclusion-induced maps:

$$\begin{array}{ccc} \mathrm{H}_i(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}) \\ & \nwarrow \quad \nearrow & \\ & \mathrm{H}_i(\mathbb{R}^n \setminus B_r^n(0)). & \end{array}$$

It follows from the Homotopy Equivalence-H_*-Proposition 144.14 (1a) and the above discussion that the two diagonal maps are isomorphisms. It follows that the horizontal map is also an isomorphism.

(b) From the LES-of-Pair Proposition 146.7 we obtain the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n) & \xrightarrow{\text{я}} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \xrightarrow{\partial_i} & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_{i-1}(\mathbb{R}^n) \longrightarrow \ldots \\ & & \downarrow \cong & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \cong & & \downarrow \mathrm{id} \\ \ldots & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus \{y\}) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n) & \xrightarrow{\text{я}} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) & \xrightarrow{\partial_i} & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus \{y\}) & \longrightarrow & \mathrm{H}_{i-1}(\mathbb{R}^n) \longrightarrow \ldots \end{array}$$

By (a) the first and the fourth vertical maps are isomorphisms. The second and the fifth vertical map are the identity. It follows from the Five Lemma 145.11 that the middle vertical map is also an isomorphism. ⊞

Now we turn to the actual proof of the lemma. Let A be a convex compact subset of $\mathbb{R}^n$ and let R be a commutative ring. As pointed out above, if $A = \varnothing$, then there is nothing to prove. Thus we can assume that there exists a $y \in A$. Now let us prove that A satisfies the statements (i) and (ii) of the Homology-Section Theorem 193.6:

(i) Let $\{\alpha_x\}_{x \in \mathbb{R}^n}$ be a section for $\mathbb{R}^n$. We denote by $\alpha_A \in \mathrm{H}_n(\mathbb{R}^n | A)$ the preimage of α_y under the isomorphism

$$\mathrm{H}_n(\mathbb{R}^n | A; R) \;\to\; \mathrm{H}_n(\mathbb{R}^n | \{y\}; R)$$

given by the claim. Since A is in particular connected it follows from the Sections-on-Connected Proposition 191.13 (applied to the sections $\{\alpha_{A,x}\}_{x \in A}$ and $\{\alpha_x\}_{x \in A}$) that $\alpha_{A,x} = \alpha_x$ for all $x \in A$, i.e. we have shown that α_A is a lift to A. The uniqueness of the lift α_A follows again from the above isomorphism.

(ii) For every $i > n$ we have

$$\mathrm{H}_i(\mathbb{R}^n | A; R) \;\underset{\text{by the claim}}{\cong}\; \mathrm{H}_i(\mathbb{R}^n | \{y\}; R) \;=\; \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) \;\underset{\text{by the Manifold Local-H}_*\text{-Lemma 150.9}}{=}\; 0.$$

[2069] As always, checking that a map is a deformation retraction also requires showing that the map is continuous. In this case the map is continuous by the Pasting Proposition 4.7 and the fact that $B_r^n(0) \setminus A$ and $\mathbb{R}^n \setminus B_r^n(0)$ are closed subsets of $\mathbb{R}^n \setminus A$.

■

We move on to another technical lemma:

Lemma 193.11. (Homology Section–Finite Union-of-Convex Compact Lemma) Statements (i) and (ii) from the Homology-Section Theorem 193.6 hold for any commutative ring R, $M = \mathbb{R}^n$ and any compact subset A of $\mathbb{R}^n$ that is the union of finitely many convex compact sets.

Proof. We consider

$$\mathcal{A} := \{\text{all convex compact subsets of } \mathbb{R}^n\}.$$

This family of subsets of $\mathbb{R}^n$ is closed under intersections.[2070] By Lemma 193.10 the two statements hold for each $A \in \mathcal{A}$. The lemma is now an immediate consequence of the Homology Section–Finite Union Lemma 193.9. ■

The following lemma is arguably the hardest step towards proving the fundamental theorem of topological manifolds.

Lemma 193.12. (Homology Section–Compact-in-$\mathbb{R}^n$-Lemma) Statements (i) and (ii) from the Homology-Section Theorem 193.6 hold for any commutative ring R, $M = \mathbb{R}^n$ and any compact subspace A of $\mathbb{R}^n$.

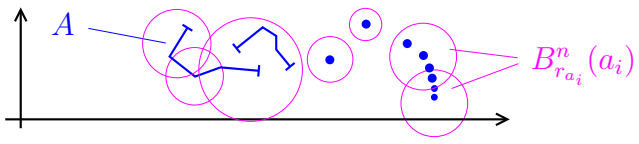
Proof. To simplify the notation we again mostly drop the R from the notation. Let $A \subseteq \mathbb{R}^n$ be a compact subspace. If $A = \varnothing$, then there is nothing really to show. So we assume that $A \neq \varnothing$.

(i) Recall that we need to show that the map

$$\mathrm{H}_n(\mathbb{R}^n | A; R) \ \to \ \Gamma_R^A(\mathbb{R}^n)$$

is an isomorphism.

We first show that the map is an epimorphism. Let $\{\alpha_a \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})\}_{a\in A}$ in $\Gamma_R^A(\mathbb{R}^n)$ be an R-section along A. It follows easily from the definition of an R-section and the naturality of homology that for each each $a \in A$ we can find an $r_a > 0$ and a $\varphi_a \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_{2r_a}(a))$ such that for each $b \in A \cap \overline{B}^n_{2r_a}(a)$ the image of φ_a in $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{b\})$ equals α_b. Next, since A is compact we see that there exist $a_1, \ldots, a_m \in A$ with $A \subseteq B^n_{r_{a_1}}(a_1) \cup \cdots \cup B^n_{r_{a_m}}(a_m)$.



Claim 1. We consider the subset

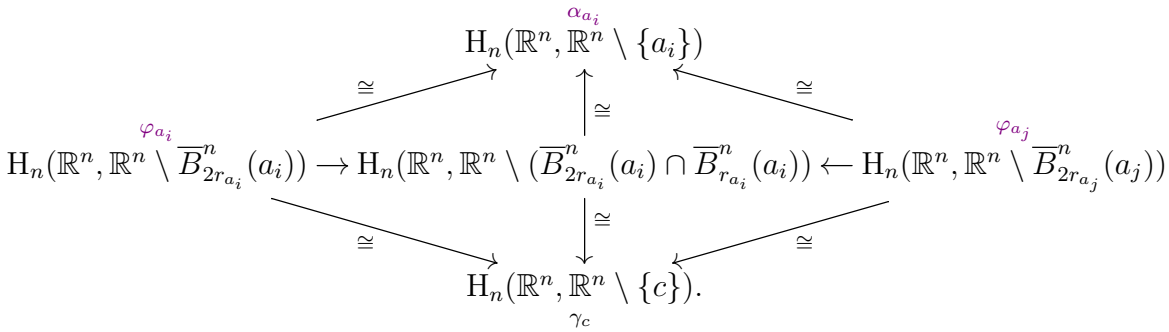
$$C := \overline{B}^n_{r_{a_1}}(a_1) \cup \cdots \cup \overline{B}^n_{r_{a_m}}(a_m) \subseteq \mathbb{R}^n.$$

There exists an R-section $\{\gamma_c \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{c\})\}_{c\in C} \in \Gamma_R^C(\mathbb{R}^n)$ along C which restricts to the given R-section $\{\alpha_a \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})\}_{a\in A}$ on A.

Proof. Let $c \in C$. We pick $i \in \{1, \ldots, m\}$ with $c \in B^n_{r_{a_i}}(a_i)$. Let $\gamma_c \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{c\})$ be the image of φ_{a_i} under the natural map $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_{2r_{a_i}}(a_i)) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{c\})$.

[2070] Here we use one of the most convenient properties of convex sets: the intersection of two convex sets is again convex.

We need to show that this definition is independent of the choice of i. Thus suppose that we also have $c \in B^n_{r_{a_j}}(a_j)$. Note that it follows from the triangle equality that $a_i \in \overline{B}^n_{2r_{a_j}}(a_j)$. This observation allows us to consider the following diagram of inclusion induced homomorphisms:



Note that the diagram commutes by functoriality of homology. Since closed balls, and finite intersections of closed balls, are compact and convex we obtain from the claim on page 3728 that the maps to the top and bottom are isomorphisms. By definition the images of φ_i and φ_j at the top agree. Thus the images agree at the bottom. This shows that $\gamma_c \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{c\})$ is well-defined.

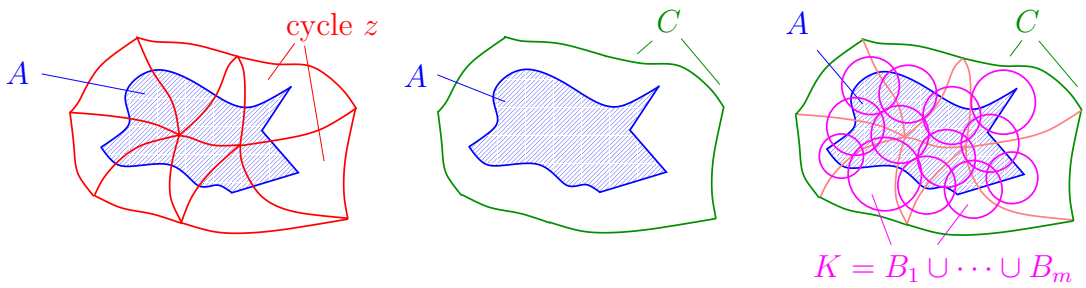
It remains to verify that these homology classes form an R-section along C. Thus let $c \in C$. We pick $i \in \{1, \dots, m\}$ with $c \in B^n_{r_{a_i}}(a_i)$. One can easily verify that the image of φ_i in $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_{r_{a_i}}(a_i))$ has the desired properties. ⊞

Since $C := \overline{B}^n_{r_{a_1}}(a_1) \cup \dots \cup \overline{B}^n_{r_{a_m}}(a_m)$ is the union of finitely many compact convex subset of $\mathbb{R}^n$ we obtain from the Homology Section–Finite Union-of-Convex Compact Lemma 193.11 a lift $\varphi \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus C)$ of $\{\gamma_c \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{c\})\}_{c \in C}$. It follows from the naturality of homology that the image of $\varphi \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus C)$ under the natural map $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus C) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ is a lift of $\{\alpha_a\}_{a \in A}$. We have now finally shown that the map $\mathrm{H}_n(\mathbb{R}^n | A; R) \to \Gamma^A_R(\mathbb{R}^n)$ is an epimorphism.

Now we show that the above map $\mathrm{H}_n(\mathbb{R}^n | A; R) \to \Gamma^A_R(\mathbb{R}^n)$ is a monomorphism. Thus we need to show that if $\beta \in \mathrm{H}_n(\mathbb{R}^n | A)$ is a class such that $\beta_x = 0$ for all $x \in A$, then β is already zero. So let $\beta \in \mathrm{H}_n(\mathbb{R}^n | A)$. We pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We denote by $C \subseteq \mathbb{R}^n \setminus A$ the union of the images of the singular simplices appearing in $\partial z \in \mathrm{C}_{n-1}(\mathbb{R}^n \setminus A)$. Note that C is the union of finitely many compact sets, hence C itself is compact.

Claim 2. There exist finitely many closed balls $B_1, \dots, B_m$ with the following three properties:
(1) Each B_i is contained in $\mathbb{R}^n \setminus C$.
(2) Each B_i contains a point $x_i \in A$.
(3) We have $A \subseteq B_1 \cup \dots \cup B_m$.

Proof. Since C is compact and since $A \cap C = \varnothing$ we can find for each $x \in A$ an $r_x > 0$ with $B_{2r_x}(x) \subseteq \mathbb{R}^n \setminus C$. Since A is compact there exist finitely many points $x_1, \dots, x_m$ in A such that A is contained in $B_{r_{x_1}}(x_1) \cup \dots \cup B_{r_{x_m}}(x_m)$.
It is now clear that $B_i := \overline{B}_{r_{x_i}}(x_i)$, $i = 1, \dots, m$ have the desired property. ⊞



We write $K := B_1 \cup \cdots \cup B_m$. By (1) the singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ also represents an element $\gamma \in \mathrm{H}_n(\mathbb{R}^n|K)$. Note that for each $x \in A$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(\mathbb{R}^n|K) & \xrightarrow{\gamma=[z]\mapsto\beta=[z]} & \mathrm{H}_n(\mathbb{R}^n|A) \\ & {\scriptstyle \gamma\mapsto\gamma_x} \searrow \quad \swarrow {\scriptstyle \beta\mapsto\beta_x} & \\ & \mathrm{H}_n(\mathbb{R}^n|x). & \end{array}$$

It follows in particular from the commutative diagram that it suffices to show that we have $\gamma = 0 \in \mathrm{H}_n(\mathbb{R}^n|K)$. We do so as follows:

(1) By our hypothesis on β and by the above commutative diagram we know that for each $i \in \{1, \ldots, m\}$ we have $\gamma_{x_i} = \beta_{x_i} = 0$.

(2) Let $i \in \{1, \ldots, m\}$. Since $\gamma_{x_i} = 0$ and since B_i is connected it follows from the Sections-on-Connected Proposition 191.13 (applied to the 0-section and the section γ_x) that $\gamma_x = 0$ for all $x \in B_i$.

(3) By (2) we know that $\gamma_x = 0$ for all $x \in K = B_1 \cup \cdots \cup B_m$.

(4) But now it follows from the Homology Section–Finite Union-of-Convex Compact Lemma 193.11 that $\gamma = 0$.

(ii) We need to show that $\mathrm{H}_i(\mathbb{R}^n|A) = 0$ for $i > n$. So let $\beta \in \mathrm{H}_i(\mathbb{R}^n|A)$. As above we pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We define C, K and γ as above. By the Homology Section–Finite Union-of-Convex Compact Lemma 193.11 we have $\gamma = 0 \in \mathrm{H}_i(\mathbb{R}^n|K)$. But as before this implies that also $\beta = 0 \in \mathrm{H}_i(\mathbb{R}^n|A)$. ■

We concludes this section with an elementary lemma:

Lemma 193.13. (Homology Section–Compact-in-Chart Lemma) Statements (i) and (ii) from the Homology-Section Theorem 193.6 hold for any commutative ring R, any n-dimensional topological manifold with empty boundary and any compact subset A of M such that A is contained in the domain of a chart $\Phi\colon U \to \mathbb{R}^n$.

Proof. Let M be an n-dimensional topological manifold and let $A \subseteq M$ be a compact subset that is contained in the domain of a chart $\Phi\colon U \to \mathbb{R}^n$. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_n(M, M \setminus A; R) & \xleftarrow{\iota_*} & \mathrm{H}_n(U, U \setminus A; R) & \xrightarrow{\Phi_*} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Phi(A); R) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_R^A(M) & \xleftarrow{\iota_*} & \Gamma_R^A(U) & \xrightarrow{\Phi_*} & \Gamma_R^{\Phi(A)}(\mathbb{R}^n). \end{array}$$

The lemma follows from the following clarifications and observations:

(1) The homeomorphism $\Phi\colon U \to \mathbb{R}^n$ induces in an obvious way the two isomorphisms to the right.
(2) We denote by $\iota\colon U \to M$ the inclusion.
(3) It follows from the Excision Theorem 148.2 that the top left horizontal map is an isomorphism.
(4) Note that for each $a \in A$ we obtain from the Excision Theorem 148.2 that the map $\iota_*\colon \mathrm{H}_n(U, U \setminus \{a\}) \to \mathrm{H}_n(M, M \setminus \{a\})$ is an isomorphism. Using this observation one can easily define the bottom left horizontal map and one sees that the bottom left horizontal map is an isomorphism.
(5) The vertical maps are the natural maps from the Homology-Section Theorem 193.6. It follows almost immediately from the naturality of homology that these maps produce commuting diagrams.
(6) By the Homology Section–Compact-in-$\mathbb{R}^n$-Lemma 193.12 we know that the right vertical map is an isomorphism. ■

193.3.3. **Proof of the Homology-Section Theorem 193.6 III.** Now we can finally complete the proof of the Homology-Section Theorem 193.6.

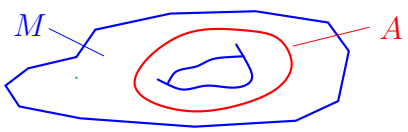
Proof of the Homology-Section Theorem 193.6. Let M be an n-dimensional topological manifold with empty boundary. We consider

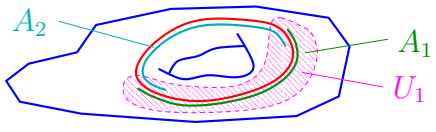
$\mathcal{A} :=$ all compact subsets A of M for which there exists a chart $\Phi\colon U \to \mathbb{R}^n$ with $A \subseteq U$.

By the Homology Section–Compact-in-Chart Lemma 193.13 statements (i) and (ii) hold for all subsets in $\mathcal{A}$. It is clear that $\mathcal{A}$ is closed under intersections. Thus it follows from the Homology Section–Finite Union Lemma 193.9 that the statements (i) and (ii) hold for all finite unions of sets in $\mathcal{A}$. Therefore it suffices to prove the following claim:

Claim. Every compact subset A of M is the union of finitely many sets in $\mathcal{A}$.

Proof. Let A be a compact subset of M. It is clear that given any point $x \in A$ there exists a chart $\Phi_x\colon U_x \to V_x$ around x such that the image is some open ball in $\mathbb{R}^n$. But since each open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ we can find around each $x \in A$ a chart around x such that the image V_x equals $\mathbb{R}^n$. Now we write $A_x := A \cap \Phi_x^{-1}(\overline{B}_1^n(0))$ and $U'_x := \Phi_x^{-1}(B_1^n(0))$. Since A is compact we can find finitely many $x_1, \dots, x_m$ in A such that A is contained in $U'_{x_1} \cup \dots \cup U'_{x_m}$.





By construction the corresponding charts $\Phi_i := \Phi_{x_i}\colon U_{x_i} \to \mathbb{R}^n$ and the sets $A_i := A_{x_i}$ have the desired properties. ⊞

It follows from the Homology Section–Compact-in-Chart Lemma 193.13 that statements (i) and (ii) hold for all the A_i and for all arbitrary intersections of the A_i. But then the Homology Section–Union Lemma 193.7 implies that they also hold for $A = A_1 \cup \dots \cup A_m$. ■

193.4. **Proof of the Manifold-Top-H$_*$-Theorem 193.1.** In this section we will finally provide the long-delayed proof of Manifold-Top-H$_*$-Theorem 193.1. We break the proof into three parts:

(1) The case that M is closed.
(2) The case that M is non-compact.
(3) The case that M is compact but not closed.

These three cases are dealt with in the following three subsections.

193.4.1. **Proof of the Manifold-Top-H$_*$-Theorem 193.1: Closed case.** The following is the statement of the Manifold-Top-H$_*$-Theorem 193.1 for closed topological manifolds:

Theorem 193.1 (Manifold-Top-H$_*$-Theorem) Let M be a closed connected non-empty n-dimensional topological manifold and let R be a commutative ring. The following statements hold:

(1) For every $i > n$ we have $\mathrm{H}_i(M;R) = 0$.

(2)
$$\mathrm{H}_n(M;R) \cong \begin{cases} R, & \text{if } M \text{ is } R\text{-orientable,} \\ \{r \in R \,|\, 2r = 0\}, & \text{if } M \text{ is not } R\text{-orientable.} \end{cases}$$

(3) (a) If M is R-orientable, then for every $x \in M$ the map
$$\mathrm{H}_n(M;R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$$
is an isomorphism.

(b) If M is not R-orientable, then for every $x \in M$ the natural map
$$\mathrm{H}_n(M;R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$$
is a monomorphism and the image equals $\{r \in \mathrm{H}_n(M, M \setminus \{x\}; R) \,|\, 2 \cdot r = 0\}$.

Proof. Let M be a closed connected non-empty n-dimensional topological manifold and let R be a commutative ring.

(1) Let $i > n$. We have
$$\mathrm{H}_i(M;R) = \mathrm{H}_i(M, \overbrace{M \setminus M}^{=\varnothing}; R) = \mathrm{H}_i(M|M;R) \underset{\uparrow}{\cong} 0.$$
by the Homology-Section Theorem 193.6 (ii) since M is compact and since M has no boundary

(3) Let $x \in M$.

(a) We assume that M is R-orientable. We start out with the following claim:

Claim. The map
$$\begin{array}{rcl} \Phi\colon \Gamma_R(M) & \to & \overbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}^{\cong R} \\ \{\alpha_y\}_{y \in M} & \mapsto & \alpha_x \end{array}$$
is an isomorphism of R-modules.

Proof. Since M is R-orientable we can pick an R-orientation $\{\mu_y\}_{y \in M}$ for M. By definition μ_x is a generator of the R-module $\mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$. This means that every element in $\mathrm{H}_n(M, M \setminus \{x\}; R)$ is of the form $r \cdot \mu_x$ for some unique $r \in R$.

We consider the map[2071]

$$\begin{array}{rcl} \Psi\colon \mathrm{H}_n(M, M\setminus\{x\}; R) & \to & \Gamma_R(M) \\ r\cdot\mu_x & \mapsto & \{r\cdot\mu_y\}_{y\in M}. \end{array}$$

Evidently we have $\Phi\circ\Psi = \mathrm{id}$. It follows from our hypothesis that M is connected and from the Sections-on-Connected Proposition 191.13 that $\Psi\circ\Phi$ is also the identity. ⊞

We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M;R) & \xrightarrow{\quad\varphi\mapsto\varphi_x\quad} & \mathrm{H}_n(M, M\setminus\{x\}; R) \\ & {\scriptstyle \varphi\mapsto\{\varphi_x\}_{x\in M}} \searrow \quad \nearrow {\scriptstyle \{\varphi_x\}_{x\in M}\mapsto\varphi_x} & \\ & \Gamma_R(M) & \end{array}$$

The diagram commutes by definition. Since M is compact, we know by the Homology-Section Theorem 193.6 (i) that the left diagonal map is an isomorphism. Furthermore, since M is connected and since M is R-orientable we obtain from the claim that the right diagonal map is an isomorphism. Thus we see that the horizontal map is an isomorphism, as promised.

(b) We now assume that M is not R-orientable. We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} \Phi_x\colon \Gamma_R(M) & \to & \mathrm{H}_n(M, M\setminus\{x\}; R) \\ \{\alpha_y\}_{y\in M} & \mapsto & \alpha_x \end{array}$$

is a monomorphism of R-modules with image $\{\gamma\in \mathrm{H}_n(M, M\setminus\{x\}; R)\,|\,2\cdot\gamma = 0\}$.

Proof. It follows from our hypothesis that M is connected and from the Sections-on-Connected Proposition 191.13 that Φ_x is a monomorphism. It remains to prove that the following equality holds:

$$\mathrm{im}(\Phi_x) \;=\; \{\gamma\in \mathrm{H}_n(M, M\setminus\{x\}; R)\,|\,2\cdot\gamma = 0\}.$$

Before we prove equality we make two preparations:

- For each $x\in M\setminus\partial M$ we choose a generator $\varphi_x\in \mathrm{H}_n(M, M\setminus\{x\})$. Note that each φ_x is well-defined up to sign. Also note that we do not make any claims about continuity.
- Given any $x\in M\setminus\partial M$ we consider the isomorphisms

$$\begin{array}{rcl} R & \to & \mathrm{H}_n(M, M\setminus\{x\})\otimes R \\ r & \mapsto & \varphi_x\otimes r \end{array} \quad \& \quad \begin{array}{rcl} \mathrm{H}_n(M, M\setminus\{x\})\otimes R & \xrightarrow{\mu} & \mathrm{H}_n(M, M\setminus\{x\}; R) \\ [\varphi]\otimes r & \mapsto & [\varphi\otimes r] \end{array}$$

of R-modules which are given by the Manifold-Local R-Homology Lemma 191.2.

We want to prove the description of $\mathrm{im}(\Phi_x)$. We will show the two inclusions "$\subseteq$" and "$\supseteq$" separately. First we show "$\supseteq$". Let $\gamma\in \mathrm{H}_n(M, M\setminus\{x\}; R)$ with $2\cdot\gamma = 0$. We pick $r\in R$ with $\mu(\varphi_x\otimes r) = \gamma$. Since $2\cdot\gamma = 0$ we see that $2\cdot r = 0$. For each $x\in M\setminus\partial M$ we set $\alpha_x := \mu(\varphi_x\otimes r)$. Since $2\cdot r = 0$ we have $r = -r$. We see that $\mu((-\varphi_x)\otimes r) = \mu(\varphi_x\otimes(-r)) = \mu(\varphi_x\otimes r)$. Thus we see that α_x does not depend on the choice of φ_x. Using charts one can now easily verify that $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an R-section. We leave it to the reader to fill in the details.

[2071] It follows easily from the Homology-defines-Section Lemma 191.12 (1) that this map is well-defined.

Now we show "$\subseteq$". We show the following equivalent statement: If $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an R-section such that $2\cdot\alpha_y\neq 0$, then M is orientable (which by the Orientable-for-all-Rings Lemma 191.21 implies that M is also R-orientable). So suppose that $2\cdot\alpha_y\neq 0$. We pick $r\in R$ such that $[\varphi_y\otimes r]=\alpha_y$. Note that $2\cdot r\neq 0$. Using the fact that $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an R-section one can show fairly easily that for each $x\in M\setminus\partial M$ there exists an $\epsilon_x\in\{\pm 1\}$ with $[\epsilon_x\cdot\varphi_x\otimes r]=\alpha_x$. Since $2\cdot r\neq 0$ we see that ϵ_x is uniquely determined and one can now show without any real problems that $\{\epsilon_x\cdot\varphi_x\}_{x\in M\setminus\partial M}$ is a section. Since each $\epsilon_x\cdot\varphi_x$ is a generator we see that this is in fact an orientation.[2072] ⊞

As in (a) we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M;R) & \xrightarrow{\quad\varphi\mapsto\varphi_x\quad} & \mathrm{H}_n(M,M\setminus\{x\};R) \\ & {\scriptstyle\varphi\mapsto\{\varphi_x\}_{x\in M}}\searrow \quad \nearrow{\scriptstyle\{\varphi_x\}_{x\in M}\mapsto\varphi_x} & \\ & \Gamma_R(M) & \end{array}$$

As in (a) we note that the diagram commutes and that the left diagonal map is an isomorphism. Furthermore, since M is connected and since M is R-orientable we obtain from the claim that the right diagonal map is a monomorphism with image $\{\gamma\in\mathrm{H}_n(M,M\setminus\{x\},R)\,|\,2\cdot\gamma=0\}$. Thus we see that the horizontal map has the promised properties.

(2) This statement follows immediately from (3) and the Manifold-Local R-Homology Lemma 191.2 which tells us that we have an isomorphism $\mathrm{H}_n(M,M\setminus\{x\};R)\cong R$ of R-modules. ■

193.4.2. **Proof of the Manifold-Top-H$_*$-Theorem 193.1: Non-compact case.** The key extra ingredient in the proof of the Manifold-Top-H$_*$-Theorem 193.1 is the following proposition.

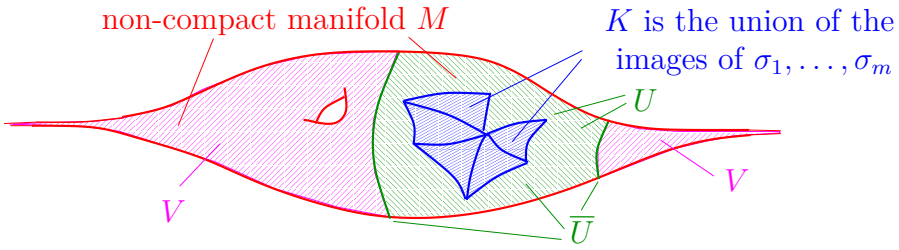
Proposition 193.14. (Non Compact–Top Homology Proposition) Let M be a connected n-dimensional topological manifold and let R be a commutative ring. If M is non-compact and if $\partial M=\varnothing$, then $\mathrm{H}_k(M;R)=0$ for every $k\geq n$.

Proof. Let M be a connected non-compact n-dimensional topological manifold such that $\partial M=\varnothing$, let R be a commutative ring and let $k\geq n$. Let $\alpha\in\mathrm{H}_k(M;R)$. We need to show that $\alpha=0$. We represent α by a cycle $z=\sum_{i=1}^{m}\sigma_i\otimes r_i\in\mathrm{C}_k(M;R)=\mathrm{C}_k(M)\otimes R$. The union of the images of $\sigma_1,\ldots,\sigma_m$ is a compact subset K of M. Since K is compact it follows from Exercise 193.10 that there exists an open subset $U\subseteq M$ that contains K and such that the closure $\overline{U}$ is compact. We consider the open subset $V=M\setminus\overline{U}$ and we consider the long exact sequence of the triple $(M,U\cup V,V)$, see the LES-of-Triple Proposition 146.5. It fits into the following commutative diagram:

$$\begin{array}{ccccccc} \ldots\longrightarrow\mathrm{H}_{k+1}(M,U\cup V;R) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(U\cup V,V;R) & \longrightarrow & \mathrm{H}_k(M,V;R) & \longrightarrow\ldots \\ & & \cong\uparrow & & \uparrow & \\ & & \mathrm{H}_k(U;R) & \xrightarrow{[z]\mapsto[z]=\alpha} & \mathrm{H}_k(M;R). & \end{array}$$

[2072] This claim gives in particular a solution of Exercise 191.18.

Here the left vertical map is an isomorphism since $U \cup V$ is the disjoint union of the open subsets U and V.[2073]



Now we consider the modules to the top-left and top-right of the diagram. We have

$$\begin{array}{rcl} \mathrm{H}_{k+1}(M, U \cup V; R) & = & \mathrm{H}_{k+1}(M|\overbrace{M \setminus (U \cup V)}^{=\overline{U} \setminus U}; R) \\ & & \uparrow \\ & & \text{by the Homology-Section Theorem 193.6 (ii) since } M \text{ has} \\ & & \text{no boundary and since } \overline{U} \text{ and } \overline{U} \setminus U \text{ are compact} \\ & & \downarrow \\ \mathrm{H}_k(M, V; R) & = & \mathrm{H}_k(M|\underbrace{M \setminus V}_{=\overline{U}}; R) \end{array} \qquad \begin{array}{l} = 0 \quad \text{if } k > n-1 \\ \\ \\ \\ = 0 \quad \text{if } k > n. \end{array}$$

After these preparations we come to the actual proof that $\alpha = [z] = 0 \in \mathrm{H}_k(M; R)$. We first consider the case $k > n$. We deduce from the above exact sequence and the above calculations that $\mathrm{H}_k(U; R) = 0$. But z represents an element in $\mathrm{H}_k(U; R)$. It follows that $[z] = 0$ in $\mathrm{H}_k(M; R)$.

Now we consider the case $k = n$. It follows from the above discussion that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(U \cup V, V; R) & \hookrightarrow & \mathrm{H}_n(M, V; R) \\ \cong\uparrow & & \uparrow \\ \mathrm{H}_n(U; R) & \xrightarrow{[z] \mapsto [z] = \alpha} & \mathrm{H}_n(M; R). \end{array}$$

Thus it suffices to show that the image of α in $\mathrm{H}_n(M, V; R)$ is zero. We showed in the Homology-defines-Section Lemma 191.12 (2) that $\{\alpha_x\}_{x \in M}$ is an R-section of M.[2074] Since M is non-compact we can find a point x outside of the compact set K. Note that the class $\alpha_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ is again represented by z which lies in $K \subseteq M \setminus \{x\}$. It is thus clear that $\alpha_x = 0 \in \mathrm{H}_n(M, M \setminus \{x\}; R)$. But since M is connected it follows from the Sections-on-Connected Proposition 191.13 that $\{\alpha_x\}_{x \in M}$ is the zero section. Both the image of α in $\mathrm{H}_n(M, V; R)$ and the zero-class are lifts of the zero section to $\mathrm{H}_n(M, V; R)$. By the Homology-Section Theorem 193.6 (i) both lifts agree, i.e. the image of α in $\mathrm{H}_n(M, V; R)$ is zero. ∎

In the proof of the non-compact case of the Manifold-Top-H_*-Theorem 193.1 we also need the following two little lemmas:

[2073] One could at this point refer to the Excision Theorem 148.1 but this isomorphism follows already from the observation that $\mathrm{C}_*(U \cup V) = \mathrm{C}_*(U) \oplus \mathrm{C}_*(V)$ as chain complexes, since U and V are both open subsets of $U \cup V$.

[2074] Here we used a second time that M has no boundary.

Lemma 193.15. (Manifold-minus-Boundary Lemma) Let M be an n-dimensional topological manifold.

(1) The inclusion $M \setminus \partial M \to M$ is a homotopy equivalence.
(2) The topological manifold $M \setminus \partial M$ has no boundary.
(3) If $\partial M \neq \varnothing$ or if M itself is non-compact, then $M \setminus \partial M$ is non-compact.

Proof. This lemma follows by the argument of the Push-Away-from-Boundary Corollary 35.8. We just need to replace the Smooth Collar Neighborhood Theorem 35.6 by the Topological Collar Neighborhood Theorem 35.14. ■

Lemma 193.16. (Double Homology Lemma) Let M be an n-dimensional topological manifold. If $\partial M \neq \varnothing$, then $\mathrm{H}_n(\mathrm{D}M; R)$ and $\mathrm{H}_n(M, \partial M; R)$ are isomorphic R-modules.

Proof. As before we think of the double as $\mathrm{D}M = (M \sqcup M')/\sim$ where M' is another copy of M. We consider the following diagram:

$$\begin{array}{ccccccccc} \underbrace{\mathrm{H}_n(M'; R)}_{=0} & \longrightarrow & \mathrm{H}_n(\mathrm{D}M; R) & \longrightarrow & \mathrm{H}_n(\mathrm{D}M, M'; R) & \longrightarrow & \mathrm{H}_{n-1}(M'; R) & \hookrightarrow & \mathrm{H}_{n-1}(\mathrm{D}M; R) \\ & & & & \uparrow \cong & & & & \\ & & & & \mathrm{H}_n(M, \partial M; R), & & & & \end{array}$$

We make the following clarifications and observations:

- On the top we have written down the long exact sequence of the pair $(\mathrm{D}M, M')$.
- Since $\partial M' \neq \varnothing$ we obtain from the Manifold-minus-Boundary Lemma 193.15 and the Non Compact–Top Homology Proposition 193.14 that $\mathrm{H}_n(M'; R) = 0$.
- The vertical map is an isomorphism by the Manifold–Excision Theorem 148.3.
- Let $i\colon M' \to \mathrm{D}M$ the natural embedding. By the Doubling-Manifold Lemma 54.10 (3) we know that there exists a retraction $r\colon \mathrm{D}M \to M'$. Since $r \circ i = \mathrm{id}_{M'}$ we obtain from a functoriality argument that the homomorphism $i_*\colon \mathrm{H}_{n-1}(M'; R) \to \mathrm{H}_{n-1}(\mathrm{D}M; R)$ is a monomorphism.

It follows immediately from the above information that $\mathrm{H}_n(\mathrm{D}M; R)$ and $\mathrm{H}_n(M, \partial M; R)$ are isomorphic R-modules. ■

The following is the statement of the Manifold-Top-H_*-Theorem 193.1 for non-compact topological manifolds:

Theorem 193.1 (Manifold-Top-H_*-Theorem) Let M be a non-compact connected n-dimensional topological manifold and let R be a commutative ring. The following statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M; R) = 0$.
    (b) For every $i > n$ we have $\mathrm{H}_i(M, \partial M; R) = 0$.
(2) (a) $\mathrm{H}_n(M; R) = 0$.
    (b) $\mathrm{H}_n(M, \partial M; R) = 0$.

Proof. Let M be a non-compact connected n-dimensional topological manifold and let R be a commutative ring.

(1) (a) Let $i > n$. If $\partial M = \varnothing$, then this statement follows immediately from the Non Compact–Top Homology Proposition 193.14. If $\partial M \neq \varnothing$, then the statement follows via the extra detour given by the Manifold-minus-Boundary Lemma 193.15.

(b) By (a) we know that for $i > n$ we have $\mathrm{H}_i(M; R) = 0$. Note that ∂M is an $(n-1)$-dimensional topological manifold with empty boundary. Thus it follows from (a) and the discussion on page 3734 of the closed case that $\mathrm{H}_i(\partial M; R) = 0$ for all $i > n-1$. It now follows from the long exact sequence of the pair $(M, \partial M)$ that for every $i > n$ we have $\mathrm{H}_i(M, \partial M; R) = 0$.

(2) (a) It follows from the Manifold-minus-Boundary Lemma 193.15 and the Non Compact–Top Homology Proposition 193.14 that $\mathrm{H}_n(M; R) = 0$.

(b) Note that $\mathrm{D}M$ is the union of two copies of M. It follows that if M is not compact, then $\mathrm{D}M$ is not compact. It now follows from (a) and the Double Homology Lemma 193.16 that $\mathrm{H}_n(M, \partial M; R) = 0$. ■

193.4.3. **Proof of the Manifold-Top-H_*-Theorem 193.1: Compact but not closed.** The following theorem gives us the remaining statements of the Manifold-Top-H_*-Theorem 193.1, namely it gives us the statements for compact topological manifolds that are not closed:

Theorem 193.1 (Manifold-Top-H_*-Theorem) Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. We assume that M is compact but not closed. The following statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M; R) = 0$.

(b) For every $i > n$ we have $\mathrm{H}_i(M, \partial M; R) = 0$.

(2) (a) $\mathrm{H}_n(M; R) = 0$.

(b) $$\mathrm{H}_n(M, \partial M; R) \cong \begin{cases} R, & \text{if } M \text{ is compact and } R\text{-orientable,} \\ \{r \in R \mid 2 \cdot r = 0\}, & \text{if } M \text{ is compact and not } R\text{-orientable.} \end{cases}$$

(3) If M is R-orientable, then for every $x \in M \setminus \partial M$ the natural map
$$\mathrm{H}_n(M, \partial M; R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$$
is an isomorphism.

Proof. Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. We assume that M is compact but not closed.

(1) Let $i > n$.

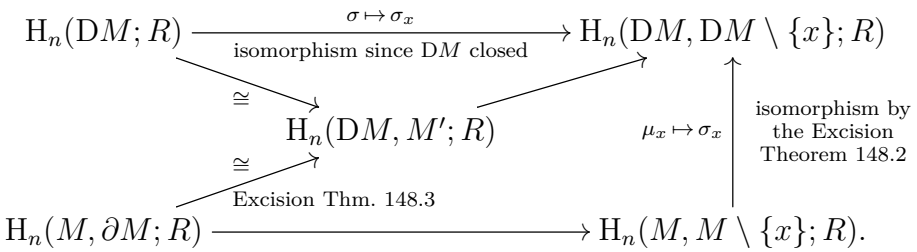
(a) It follows from the Manifold-minus-Boundary Lemma 193.15 and the non-compact case that $\mathrm{H}_i(M; R) = 0$ for $i > n$.

(b) Note that (b) is an immediate consequence of (a), applied to M and ∂M, and the long exact sequence in homology of the pair $(M, \partial M)$.

(2) (a) This statement follows from the Manifold-minus-Boundary Lemma 193.15 and the Non Compact–Top Homology Proposition 193.14.

(b) We consider the double $\mathrm{D}M$. By the Doubling-Manifold Lemma 54.10 and the Double-Orientability Lemma 192.6 we know that M is compact and R-orientable if and only if $\mathrm{D}M$ is closed and R-orientable. Thus the promised result follows from the closed case.

(3) We suppose that M is R-orientable. Let $x \in M \setminus \partial M$. We consider the following diagram:



Here all maps are the obvious inclusion-induced maps. This implies that the diagram commutes. Since $\mathrm{D}M$ is closed and since $\mathrm{D}M$ is R-orientable by the Double-Orientability Lemma 192.6 we know from the first part of the proof, i.e. the from the discussion on page 3734, that the top horizontal map is an isomorphism. Next note that in the proof of the Double Homology Lemma 193.16 we showed that the top-left diagonal map is an isomorphism. Finally two of the maps are isomorphisms by the Manifold–Excision Theorem 148.3. It now follows immediately that the bottom horizontal map is also an isomorphism. But that is exactly what we were supposed to show. ■

193.5. **Proof of the Manifold-H_{n-1}-Theorem 193.4.** As we discussed on page 3077, homology with $\mathbb{Z}$-coefficients is naturally isomorphic to homology without coefficients. Thus the theorem that we want to prove can be formulated as follows:

Theorem 193.4 (Manifold-H_{n-1}-Theorem) Let M be a compact connected n-dimensional topological manifold.

(1) $$\mathrm{H}_{n-1}(M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

(2) $$\mathrm{H}_{n-1}(M, \partial M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable.} \end{cases}$$

In the proof of the Manifold-H_{n-1}-Theorem 193.4 we will use the following lemma:

Lemma 193.17. (Free Abelian–Characterization Lemma) Let A be a finitely generated abelian group. Then the following holds:

$$A \text{ is a free abelian group} \iff \mathrm{Tor}(A, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for all primes } p.$$

Proof. The lemma is an immediate consequence of the Abelian Group-Classification Theorem 97.24, the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12 and the Tor-Properties Lemma 162.13. ■

Now we can provide the proof of the Manifold-H_{n-1}-Theorem 193.4. First we consider Statement (1). For that statement we consider the two cases that the topological manifold is closed and that it has non-empty boundary separately.

Proof of the Manifold-H_{n-1}-Theorem 193.4 (1) I. Let M be a closed connected n-dimensional topological manifold. In the proof we constantly use that by the Topological Manifolds-Invariants Proposition 189.14 we know that $\mathrm{H}_{n-1}(M)$ is finitely generated.

We first consider the case that M is orientable. We want to show that $\mathrm{H}_{n-1}(M)$ is a free abelian group. It follows from the Free Abelian–Characterization Lemma 193.17 that it suffices to prove the following claim:

Claim 1. For every prime p we have $\mathrm{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_p) = 0$.

Proof. So let p be a prime. We have

$$\mathbb{F}_p \overset{\text{by the Manifold-Top-H}_*\text{-Theorem 193.1 (2a), since by the Orientable-for-all-Rings Lemma 191.21 we know that } M \text{ is also } \mathbb{F}_p\text{-orientable}}{\cong} \mathrm{H}_n(M;\mathbb{F}_p) \overset{\text{by the UCT-Splitting Theorem 162.18}}{\cong} \mathrm{H}_n(M)\otimes\mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M),\mathbb{F}_p) \overset{\text{by the Orientation-Top-H}_*\text{-Theorem 193.3 (3) we have } \mathrm{H}_n(M)\cong\mathbb{Z}}{\cong} \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M),\mathbb{F}_p).$$

It follows from the Cancellation Proposition 97.26 that $\mathrm{Tor}(\mathrm{H}_{n-1}(M),\mathbb{F}_p)=0$. ⊞

Now we turn to the case that M is non-orientable. By the Orientation Covering–Properties-Proposition 191.29 there exists a connected 2-fold covering $q\colon \widetilde{M}\to M$ such that $\widetilde{M}$ is orientable. We consider the maps

$$\mathrm{H}_{n-1}(M) \xrightarrow{q^*} \mathrm{H}_{n-1}(\widetilde{M}) \xrightarrow{q_*} \mathrm{H}_{n-1}(M), \qquad \mathrm{H}_{n-1}(M) \xrightarrow{\cdot 2} \mathrm{H}_{n-1}(M).$$

Here q^* denotes the transfer map that we introduced in Definition 165.1 and q_* denotes of course the map induced by the projection q. By the Transfer–Projection Proposition 165.4 the composition of q^* and q_* is given by multiplication by 2. Put differently, the above diagram commutes.

Since $\widetilde{M}$ is orientable we obtain from the case that we had just proved that $\mathrm{H}_{n-1}(\widetilde{M})$ is torsion-free. Now we see that

$$\text{torsion subgroup of } \mathrm{H}_{n-1}(M) \overset{\text{since } \mathrm{H}_{n-1}(\widetilde{M}) \text{ is torsion-free}}{\subseteq} \ker(q^*) \subseteq \ker(q_*\circ q^*) \overset{\text{by the above commutative diagram}}{\subseteq} \{a\in \mathrm{H}_{n-1}(M)\,|\,2a=0\}.$$

From these inclusions and the classification of finitely generated abelian groups we obtain easily that $\mathrm{H}_{n-1}(M)\cong(\mathbb{Z}/2\mathbb{Z})^r\oplus\mathbb{Z}^k$ for some $r\in\mathbb{N}_0$ and some $k\in\mathbb{N}_0$.

It remains to show that $r=1$. In light of the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12 and the Tor-Properties Lemma 162.13 it suffices to prove the following claim:

Claim 2. We have $\mathrm{Tor}(\mathrm{H}_{n-1}(M),\mathbb{F}_2)\cong\mathbb{F}_2$.

Proof. The argument is very similar to the proof of Claim 1. More precisely, we have

$$\mathbb{F}_2 \overset{\text{by the } \mathbb{F}_2\text{-Orientability Proposition 191.22 and the Top-}\mathbb{F}_2\text{-Homology Theorem 193.2}}{\cong} \mathrm{H}_n(M;\mathbb{F}_2) \overset{\text{by the UCT-Splitting Theorem 162.18}}{\cong} \mathrm{H}_n(M)\otimes\mathbb{F}_2 \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M),\mathbb{F}_2) \overset{\text{by the Orientation-Top-H}_* \text{ Theorem 193.3 (3) we have } \mathrm{H}_n(M)=0}{\cong} 0\oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M),\mathbb{F}_2).$$

■

Now we deal with topological manifolds with non-empty boundary.

Proof of the Manifold-H$_{n-1}$-Theorem 193.4 (1) II. Let M be a connected, compact n-dimensional topological manifold with non-empty boundary. We need to show that $\mathrm{H}_{n-1}(M)$ is a free abelian group. There are two approaches to the proof of this statement:

(1) We can imitate the proof of the closed case, or

(2) we reduce the proof of the proposition to the case of closed topological manifolds that we had just dealt with.

We will pursue the second approach. In the proof we again constantly use that by the Topological Manifolds-Invariants Proposition 189.14 we know that $\mathrm{H}_{n-1}(M)$ is finitely generated. First recall that by the Double-Orientability Lemma 192.6 the double $\mathrm{D}M$ is orientable if and only if M is orientable.

We first consider the case that M is orientable. By the Manifold-H_{n-1}-Theorem 193.4 we know that $\mathrm{H}_{n-1}(\mathrm{D}M)$ is a finitely generated free abelian group. By the Doubling-Manifold Lemma 54.10 (3) we know that M is a retract of $\mathrm{D}M$. It follows that the group $\mathrm{H}_{n-1}(M)$ is isomorphic to a subgroup of $\mathrm{H}_{n-1}(\mathrm{D}M)$. It follows from the Free Abelian Subgroup Theorem 97.14 that $\mathrm{H}_{n-1}(M)$ is also a finitely generated free abelian group.

Now we consider the case that M is non-orientable. We denote by M' a copy of M and as usual we denote by $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ the double. From the Mayer–Vietoris Theorem 153.2 we obtain the following long exact sequence:

$$\dots \to \mathrm{H}_n(\mathrm{D}M) \to \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M) \oplus \mathrm{H}_{n-1}(M') \to \mathrm{H}_{n-1}(\mathrm{D}M) \to \dots$$

We make the following observations:

(1) By the Topological Manifold–Boundary Proposition 23.15 (1) and (2) we know that ∂M is a closed $(n-1)$-dimensional topological manifold. Thus we obtain from the Orientation-Top-H_*-Theorem 193.3 that $\mathrm{H}_{n-1}(\partial M)$ is isomorphic to a free abelian group $\mathbb{Z}^r$ for some $r \in \mathbb{N}_0$. (We do not have any control over the orientability of ∂M and the number of components of ∂M but this does not matter.)
(2) Since $\mathrm{D}M$ is non-orientable we know from the Orientation-Top-H_*-Theorem 193.3 that $\mathrm{H}_n(\mathrm{D}M) = 0$. Since $\mathrm{D}M$ is closed we know from the closed case of the Manifold-H_{n-1}-Theorem 193.4 (1) that $\mathrm{H}_{n-1}(\mathrm{D}M) \cong \mathbb{Z}^s \oplus \mathbb{Z}/2\mathbb{Z}$ for some $s \in \mathbb{N}_0$.
(3) Since $M = M'$ we have $\mathrm{H}_{n-1}(M) = \mathrm{H}_{n-1}(M')$. By the classification of finitely generated abelian groups we can pick an isomorphism $\mathrm{H}_{n-1}(M) \cong F \oplus T$ where F is a free abelian group and T is a torsion group.

We can thus rewrite the above sequence as follows:

$$0 \;\to\; \mathbb{Z}^r \;\xrightarrow{\varphi}\; F \oplus F \oplus T \oplus T \;\to\; \mathbb{Z}^s \oplus \mathbb{Z}/2\mathbb{Z} \;\to\; \dots$$

The map $\varphi\colon \mathbb{Z}^r \to F \oplus F \oplus T \oplus T$ is a monomorphism, but since $\mathbb{Z}^r$ is torsion-free and since T is finite we see that $\mathrm{im}(\varphi) \cap (T \oplus T)$ is trivial. But this implies that $T \oplus T$ is isomorphic to a subgroup of $\mathbb{Z}^s \oplus \mathbb{Z}/2\mathbb{Z}$. But this is only possible if $T = 0$. This concludes the proof that $\mathrm{H}_{n-1}(M)$ is a free abelian group. ■

Proof of the Manifold-H_{n-1}-Theorem 193.4 (2). There are two approaches to proving Statement (2):

(1) We can use basically the argument as in the proof of the Manifold-Top-H_*-Theorem 183.20 (2) to reduce, via a doubling construction, the study of relative homology groups to the study of absolute homology groups. The only difference is that we need to replace the Manifold-Top-H_*-Theorem 183.18 by Statement (1) of the present proposition.
(2) Alternatively we can adopt the argument of Statement (1). First we assume that M is orientable. By the Free Abelian–Characterization Lemma 193.17 it suffices to show

that $\mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M),\mathbb{F}_p)=0$ for all primes p. So let p be a prime. We have

$$\begin{array}{ccccc}
\mathbb{F}_p & \cong & \mathrm{H}_n(M,\partial M;\mathbb{F}_p) & \overset{\text{by the UCT-Splitting Theorem 162.18}}{\cong} & \mathrm{H}_n(M,\partial M)\otimes\mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M),\mathbb{F}_p) \\
 & & \uparrow & & \cong\ \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M),\mathbb{F}_p). \\
 & & \text{by the Manifold-Top-H}_*\text{-Theorem 193.1 (2a), since by the Orientable-for-all-Rings Lemma 191.21 } M \text{ is also } \mathbb{F}_p\text{-orientable} & & \uparrow \\
 & & & & \text{by the Orientation-Top Homology Theorem 193.3 (3) we have } \mathrm{H}_n(M,\partial M)\cong\mathbb{Z}
\end{array}$$

It follows from the Cancellation Proposition 97.26 that $\mathrm{Tor}(\mathrm{H}_{n-1}(M,\partial M),\mathbb{F}_p)=0$. The case that M is non-orientable is very similar, we leave it to the reader to make the necessary modifications. ■

193.6. **Appendix: Homology-Section Theorem 193.6 with boundary.** In this very technical section we generalize the Homology-Section Theorem 193.6 (i) to the case of topological manifolds with boundary. We will only make use of this result once, namely in the proof of the ∂-Cycles–Cup Product-Intersection Number Theorem 221.26. Unless the reader is blessed with a surplus of energy it is best to skip this appendix.

Definition 193.18. Let M be a topological manifold. In the following we say that a compact subset $A\subseteq M$ is a product close to the boundary if there exists a collar neighborhood $[0,1]\times\partial M$ for the boundary ∂M and a compact subset $K\subseteq\partial M$ such that we have the equality $A\cap([0,1]\times\partial M)=[0,1]\times K$.



Remark. If M is a topological manifold with *empty boundary*, then every compact subset is in fact a product close to the boundary. □

Throughout this section we will need the following notation:

Notation 193.19. Given a subset A of a topological manifold M we write

$$A_\partial \ :=\ A\cap(M\setminus\partial M).$$

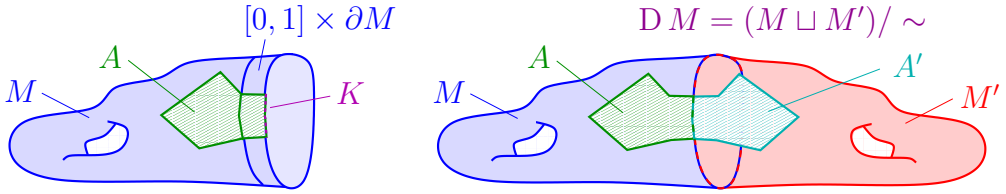
In other words, we have $A_\partial = A\setminus\partial M$. Note that $(M\setminus A)\cup\partial M = M\setminus A_\partial$.

Now we can formulate the main result of this section.

Theorem 193.20. (Boundary Homology–Section Theorem) Let M be an n-dimensional topological manifold and let $A\subseteq M$ be a compact subset that is a product close to the boundary. Let R be a commutative ring. If $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an R-section for M, then there exists a unique lift $\alpha_A^M=\alpha_A$ to A_∂, i.e. there exists a unique class $\alpha_A^M=\alpha_A\in\mathrm{H}_n(M,M\setminus A_\partial;R)$ such that for each $x\in A_\partial$ we have $\alpha_{A,x}=\alpha_x$.

Proof. Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let $A \subseteq M$ be a compact subset that is a product close to the boundary. We introduce the following notation and conventions:

(1) As usual we denote by $\mathrm{D}M$ the double of M. As before we think of the double as $\mathrm{D}M = (M \sqcup M')/\sim$ where M' is another copy of M. We recall that by the Doubling-Manifold Lemma 54.10 we know that $\mathrm{D}M$ is an n-dimensional topological manifold with empty boundary such that $M \subseteq \mathrm{D}M$ is a submanifold.
(2) By definition there exists a collar neighborhood $[0,1] \times \partial M$ for the boundary ∂M and a compact subset $K \subseteq \partial M$ such that $A \cap ([0,1] \times \partial M) = [0,1] \times K$.
(3) We denote by $DA = A \cup A' \subseteq \mathrm{D}M$ the double of A. As DA is the union of two compact sets we see that DA itself is compact.
(4) Given $t \in (0,1]$ we write $A_t \coloneqq A \setminus ([0,t) \times K) = A \cap (M \setminus ([0,t) \times \partial M))$. It follows easily from the Compact-Closed Lemma 3.39 that A_t is again compact.
(5) It follows from the Manifold–Excision Theorem 148.3 that for any $P \in M \setminus \partial M$ the inclusion-induced map
$$\mathrm{H}_n(M, M \setminus \{P\}; R) \;\to\; \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{P\}; R)$$
is an isomorphism. We will use this isomorphism to identify these two homology groups.



We will prove the desired statement for M by reducing it to a corresponding statement for the topological manifold $\mathrm{D}M$ that has no boundary.

Claim 1. The inclusion-induced map
$$\beta\colon \mathrm{H}_n(M, M \setminus A_\partial; R) \;\to\; \mathrm{H}_n(\mathrm{D}M, (M \setminus A) \cup M'; R)$$
is an isomorphism.

Proof. Recall that A is a product close to the boundary. Using this fact and using a gentle generalization of the Manifold–Excision Theorem 148.3 one can easily show that the inclusion-induced map is an isomorphism. We leave it to the reader to fill in the details. ⊞

We denote by Φ the composition of the following two maps
$$\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus DA; R) \;\to\; \mathrm{H}_n(\mathrm{D}M, (M \setminus A) \cup M'; R) \xrightarrow{\beta^{-1}} \mathrm{H}_n(M, M \setminus A_\partial; R).$$
After all these preparations we now turn to the actual proof of the theorem. Thus let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-section for M. As in the proof of the Double-Orientability Lemma 192.6 we see that there exists an R-section $\{\alpha_x\}_{x \in \mathrm{D}M}$ for $\mathrm{D}M$ that restricts to the given R-section on M. By the Homology-Section Theorem 193.6 (i) there exists a lift $\mu_{DA} \in \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus DA; R)$ of the R-section on $\mathrm{D}M$. We claim that $\Phi(\mu_{DA})$ is a lift for

M. So let $P \in A_\partial$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{\Phi}{\frown} & & \\
\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus DA; R) & \longrightarrow & \mathrm{H}_n(\mathrm{D}M, (M \setminus A) \cup M'; R) & \underset{\cong}{\overset{\beta}{\longleftarrow}} & \mathrm{H}_n(M, M \setminus A_\partial; R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{P\}; R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{P\}; R) & \xleftarrow{\cong} & \mathrm{H}_n(M, M \setminus \{P\}; R).
\end{array}$$

Since all maps are induced by inclusions of pairs of topological spaces we see that the diagram commutes. By definition of μ_{DA} the left-hand vertical map sends μ_{DA} to α_P. But this implies that the right-hand vertical map sends $\Phi(\mu_{DA})$ to $\alpha_P \in \mathrm{H}_n(M, M \setminus \{P\}; R)$.

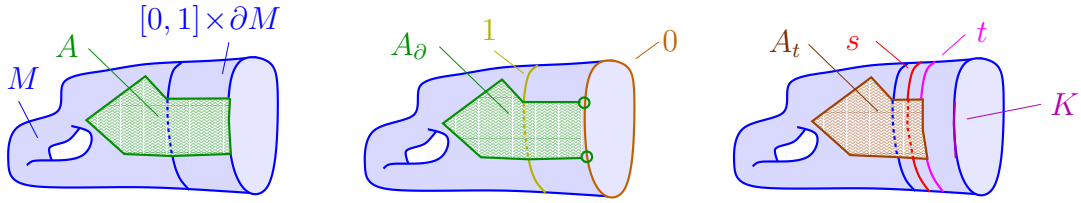
Now we turn to the proof of the uniqueness of a lift. The following claim is the key to proving uniqueness.

Claim 2. Let $t \in (0,1)$. Note that $A_t \subseteq A_\partial$. The two inclusion-induced maps

$$\begin{aligned} \gamma\colon \mathrm{H}_n(M, M \setminus A_\partial; R) &\to \mathrm{H}_n(M, M \setminus A_t; R) \\ \delta\colon \mathrm{H}_n(M, M \setminus A_t; R) &\to \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus A_t; R) \end{aligned}$$

are isomorphisms.

Proof. It follows immediately from the Manifold–Excision Theorem 148.3 that the map δ is an isomorphism. Regarding the map γ, note that it follows from the long exact sequence of the triple $(M, M \setminus A_t, M \setminus A_\partial)$, see the LES-of-Triple Proposition 146.5, that it suffices to show that $\mathrm{H}_i(M \setminus A_t, M \setminus A_\partial; R) = 0$ for all i. We pick $s \in (t,1)$.



We see that indeed we have

$$\mathrm{H}_i(M \setminus A_t, M \setminus A_\partial; R) \underset{\cong}{\overset{\downarrow}{\longleftarrow}} \mathrm{H}_i(([0,s) \times \partial M) \setminus (K \times [t,s)), ([0,s) \times \partial M) \setminus ((0,s) \times K); R) \underset{\uparrow}{=} 0.$$

by the Excision Theorem 148.1, here we excise everything in the complement of $([0,s) \times \partial M) \cap A$

this follows from the Deformation Retract-Relative H_*-Corollary 146.9 since $([0,s) \times \partial M) \setminus ((0,s) \times K)$ is a deformation retract of $([0,s) \times \partial M) \setminus ([t,s) \times K)$ ⊞

Now let $\mu, \nu \in \mathrm{H}_n(M, M \setminus A_\partial; R)$ be two lifts of the R-section $\{\alpha_x\}_{x \in M \setminus \partial M}$. With the same arguments as before one can convince oneself easily of the fact that $\delta^{-1}(\gamma(\mu))$ and $\delta^{-1}(\gamma(\nu))$ are lifts of the R-section $\{\alpha_x\}_{x \in \mathrm{D}M}$. By the Homology-Section Theorem 193.6 these two lifts are identical. But then it follows from Claim 2 that also $\mu = \nu$. ■

193.7. Exercises for Chapter 193 .

Exercise 193.1. Show that for any closed orientable n-dimensional topological manifold and any abelian group G we have $\mathrm{H}_n(M; G) \cong G$.

Exercise 193.2. Let $g \in \mathbb{N}_0$, let $k \in \mathbb{N}$ and let $m, n \in \mathbb{N}_0$. As in Notation 101.12 we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks and we denote by $N_{k,n}$ the non-orientable surface of genus k minus n open disks.

(a) Determine the relative homology groups $\mathrm{H}_*(\Sigma_{g,m}, \partial\Sigma_{g,m})$.
(b) Determine the relative homology groups $\mathrm{H}_*(N_{k,n}, \partial N_{k,n})$.
(c) Use (a) and (b) to show that the topological spaces $\Sigma_{g,m}$ and $N_{k,n}$ are pairwise non-homeomorphic.

Hint. Make use of the Orientation-Top-H_*-Theorem 193.3 and the Manifold-H_{n-1}-Theorem 193.4. You could also make use of the Euler characteristic to determine ranks of homology groups.
Remark. This gives a new proof for the Surfaces-Non Homeomorphic Theorem 101.18.

Exercise 193.3.

(a) Show that $\mathrm{SO}(n)$ is an orientable topological manifold.
(b) By the Matrix Groups Deformation Retract Proposition 19.4 we know that $\mathrm{SO}(n)$ is a deformation retract of $\mathrm{GL}(n, \mathbb{R})$. Does there exist a proper submanifold M of $\mathrm{SO}(n)$ such that M is a deformation retract of $\mathrm{GL}(n, \mathbb{R})$?

Exercise 193.4. Which homology groups of $\mathrm{O}(n)$ can you calculate?

Exercise 193.5. Let $p, q \in \mathbb{N}$ be coprime. Show that every orientation-reversing diffeomorphism of the lens space $L(p, q)$ has at least one fixed point.

Exercise 193.6. Let $p, q \in \mathbb{N}$ be coprime.

(a) Determine $\mathrm{H}_k(L(p, q); \mathbb{Q})$.
(b) Let m be a prime. Determine $\mathrm{H}_k(L(p, q); \mathbb{F}_m)$.

Exercise 193.7. Let $p, q \in \mathbb{N}$ be coprime and let $\widetilde{p}, \widetilde{q} \in \mathbb{N}$ be coprime. Determine the homology groups $\mathrm{H}_k(L(p, q) \times L(\widetilde{p}, \widetilde{q}))$.

Exercise 193.8. Let Γ be a finite group that acts freely and smoothly, in an orientation-preserving way on S^3. Determine the homology groups of S^3/Γ in terms of suitable invariants of the group Γ.

Exercise 193.9. Let X be a connected 2-dimensional topological manifold with infinite fundamental group (We do not assume that X is compact.) Show that X is aspherical, i.e. show that $\pi_i(X) = 0$ for $i \geq 2$.
Hint. First use the Hurewicz Theorem 167.9 and the results from this section to show that $\pi_2(X) = 0$.
Remark. This exercise is a copy of Exercise 183.16.

Exercise 193.10. Let M be a connected n-dimensional topological manifold with $\partial M = \varnothing$ and let K be a compact subset. Show that there exists an open subset $U \subseteq M$ that contains K and such that the closure $\overline{U}$ is compact.
Remark. This charming exercise played a role in the Non Compact–Top Homology Proposition 193.14.

Exercise 193.11. Let M be a closed orientable connected non-empty topological manifold of dimension $n \geq 3$. Now we know that $\mathrm{H}_n(M; \mathbb{Z}) \cong \mathbb{Z}$ and that $\mathrm{H}_{n-1}(M; \mathbb{Z})$ is torsion-free.

Furthermore we know that $\mathrm{H}_0(M;\mathbb{Z}) \cong \mathbb{Z}$. Can we also say something about the other homology groups? More precisely, are there any restrictions on $\mathrm{H}_{n-2}(M;\mathbb{Z})$ or can any finitely generated abelian group be isomorphic to $\mathrm{H}_{n-2}(M;\mathbb{Z})$ for some such topological manifold?

Exercise 193.12. Let $d \in \mathbb{N}$. By the Manifold-H_{n-1}-Theorem 193.4 we know that for every closed orientable d-dimensional topological manifold M the group $\mathrm{H}_{n-1}(M)$ is a finitely generated free abelian group. Which ranks can appear? (The answer might depend on d.)

Exercise 193.13. Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We also assume that $B \neq \varnothing$, in other words $A \neq \partial M$.

(a) Show that for each commutative ring R we have $\mathrm{H}_n(M, A; R) = 0$.
Hint. You could consider the long exact sequence of the triple $(M, \partial M, A)$.
(b) Show that $\mathrm{H}_{n-1}(M, A)$ is a free abelian group.

Exercise 193.14. Let $X = S^1 \cup \{*\}$ be the topological space that is given by the circle "with two 1's", where the topology is defined in the same way as we defined the topology on the "line with two 0's" on page 152. It is straightforward to see that X is locally homeomorphic to open subsets of $\mathbb{R}$, in fact one can easily find a finite atlas for X. Furthermore we know by the Second Countability Lemma 13.2 (4) that X is second-countable. But X is not a topological manifold since it is not Hausdorff.

(a) Show that $\mathrm{H}_1(X;\mathbb{F}_2) \cong \mathbb{F}_2^2$.
Remark. This part is identical to Exercise 153.6.
(b) Part (a) shows that in the proof of the Top-$\mathbb{F}_2$-Homology Theorem 193.2 at some point we must have used that topological manifolds are Hausdorff. Where did we make use of this fact?

Exercise 193.15. Let M be a closed *orientable* connected non-empty n-dimensional topological manifold and let N be a closed *non-orientable* connected non-empty n-dimensional topological manifold. Show that for every continuous map $f\colon M \to N$ the induced map $f_*\colon \mathrm{H}_n(M;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_n(N;\mathbb{Z}/2\mathbb{Z})$ is the trivial map.

Exercise 193.16.

(a) Let M be a closed topological manifold and let $X \subsetneq M$ be a proper subset of M. Show that X cannot be a deformation retract of M.
(b) Show that in (a) we cannot replace "closed" by "compact".
(c) Show that in (a) we cannot replace "closed" by "$\partial M = \varnothing$".

Remark. This exercise is almost identical to Exercise 183.6.

Exercise 193.17.

(a) Let F be a non-compact connected 2-dimensional smooth manifold. Show that for each boundary component C of F the induced map $\mathrm{H}_1(C) \to \mathrm{H}_1(F)$ is a monomorphism.
(b) Let F be a non-compact connected 2-dimensional smooth manifold with $\mathrm{H}_1(F) = 0$. Show that F is diffeomorphic to $\mathbb{R}^2$.

Hint. In both statements make use of the Compact Submanifold Exhaustion Proposition 38.2. For the second problem show that we can find an exhaustion by compact submanifolds, each of which is diffeomorphic to a closed disk.

Exercise 193.18. Let M be an orientable n-dimensional topological manifold and let $K \subseteq M \setminus \partial M$ be a compact connected non-empty subspace. Show that for every $x \in K$ the natural map $\mathrm{H}_n(M, M \setminus K) \to \mathrm{H}_n(M, M \setminus \{x\})$ is an isomorphism.
Hint. Imitate the proof on page 3734.

CHAPTER 194

The fundamental class of topological manifolds

In this chapter we introduce the fundamental class of a compact oriented topological manifold. This concept is extremely useful for calculations of homology groups and it will play an essential role in the formulation of the Poincaré Duality Theorem 210.1.

This chapter is organized as follows:

- In Section 194.1 we will introduce the fundamental class of topological manifolds.
- In Section 194.2 we will compare the notions of fundamental classes of smooth and topological manifolds.
- In Section 194.3 we will find explicit fundamental classes
- In Section 194.4 we will discuss how induced orientations interact with fundamental classes.
- In Section 194.5 we will point to several earlier sections, which we initially proved using the notion of a fundamental class of a smooth manifold, but which we can now (re-) prove using the notion of the fundamental class of a topological manifold.
- In Section 194.6 we will provide exercises for Chapter 194.

In the next Chapter 195 we will introduce the degree of a map between closed oriented topological manifolds.

194.1. **The definition of the fundamental class.** First we recall Definition 191.11.

Notation 194.1. Let M be an n-dimensional topological manifold and let R be a commutative ring. Given any $k \in \mathbb{N}_0$ and given any $x \in M \setminus \partial M$ we denote by ρ_x the natural homomorphism

$$\rho_x \colon \mathrm{H}_k(M, M \setminus \partial M; R) \quad \to \quad \mathrm{H}_k(M, M \setminus \{x\}; R).$$

Given $\varphi \in \mathrm{H}_k(M, M \setminus \partial M; R)$ we often write $\varphi_x := \rho_x(\varphi)$, i.e. the above map is also written as

$$\varphi \quad \mapsto \quad \varphi_x.$$

The following proposition is the starting point for this chapter:

Proposition 194.2. (Fundamental Class-Proposition) Let M be a compact n-dimensional topological manifold, let R be a commutative ring and let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-orientation for M.

(1) (a) There exists a unique $[M]_R \in \mathrm{H}_n(M, \partial M; R)$ with $\rho_x([M]_R) = \alpha_x$ for all points $x \in M \setminus \partial M$.

(b) If M has components $M_1, \dots, M_k$, then

$$[M] \;=\; \sum_{i=1}^{k} [M_i] \;\in\; \mathrm{H}_n(M, \partial M) \;\underset{\uparrow}{=_{\mathrm{i}}}\; \bigoplus_{i=1}^{k} \mathrm{H}_n(M_i, \partial M_i).$$

isomorphism given by the Path-Component–H_*–Lemma 143.21

(2) We now suppose that M is connected, R-orientable and non-empty.
(a) We have $\mathrm{H}_n(M, \partial M; R) = R \cdot [M]_R$.[2075]
(b) Let $\Theta \in \mathrm{H}_n(M, \partial M; R)$. If there exists a single $x \in M \setminus \partial M$ with $\rho_x(\Theta) = \alpha_x$, then $\Theta = [M]_R$.

The analogous statements hold for homology without coefficients.

Proof. Let M be a compact n-dimensional topological manifold, let R be a commutative ring and let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-orientation for M.

(1) We first consider the case that M is connected and non-empty. In this setting it follows from the Manifold-Top-H_*-Theorem 193.1 (3) that for every $x \in M \setminus \partial M$ the map

$$\rho_x \colon \mathrm{H}_n(M, \partial M; R) \;\to\; \mathrm{H}_n(M, M \setminus \{x\}; R)$$

is an isomorphism of R-modules.

We pick a[2076] $y \in M \setminus \partial M$ and we set $[M]_R \coloneqq \rho_y^{-1}(\alpha_y) \in \mathrm{H}_n(M, \partial M; R)$. Since ρ_y is an isomorphism the class $[M]_R$ is uniquely determined. Thus it remains to show that $\rho_x([M]_R) = \alpha_x$ for all $x \in M \setminus \partial M$. By the Homology-defines-Section Lemma 191.12 (2) the classes $\{\rho_x([M]_R)\}_{x \in M \setminus \partial M}$ form an R-section. By construction we have $\rho_y([M]_R) = \alpha_y$. Since M is connected it follows from the Sections-on-Connected Proposition 191.13 that $\rho_x([M]_R) = \alpha_x$ for all $x \in M \setminus \partial M$.

Now we consider the general case. Let $\{M_i\}_{i=1,\dots,k}$ be the components of M. By the Path-Component-H_*-Lemma 143.21 we have a natural isomorphism

$$\bigoplus_{i=1}^{k} \mathrm{H}_n(M_i, \partial M_i; R) \;\to\; \mathrm{H}_n(M, \partial M; R).$$

One can easily verify that the image of $([M_1]_R, \dots, [M_k]_R)$ under this map has all the desired and promised properties and that it is unique.

(2) We now assume that M is connected. Let $x \in M \setminus \partial M$. Since M is connected and orientable we know by the Manifold-Top-H_*-Theorem 193.1 (3) that the map

$$\rho_x \colon \mathrm{H}_n(M, \partial M; R) \;\to\; \underbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}_{=R \cdot \alpha_x}$$

is an isomorphism of R-modules. The two promised statements follow immediately from this observation and the definition of $[M]_R$. ∎

The Fundamental Class-Proposition 194.2 motivates the following definition:

[2075] It follows from the Homology-defines-Section Lemma 191.12 (2) together with the Manifold-Top-H_*-Theorem 193.1 that given any generator Θ of $\mathrm{H}_n(M, \partial M; R)$ the homology classes $\{\Theta_x\}_{x \in M \setminus \partial M}$ form an R-orientation of M. Thus we see that in the connected cases there is a bijection between generators of $\mathrm{H}_n(M, \partial M; R)$ and R-orientations on M.

[2076] It follows from the Interior-Non Compact Lemma 23.16 that there exists a $y \in M \setminus \partial M$.

Definition 194.3. Let M be a compact n-dimensional topological manifold.

(1) Let R be a commutative ring and let $\{\alpha_x\}_{x\in M\setminus\partial M}$ be an R-orientation for M. Recall that we know by the Fundamental Class-Proposition 194.2 that there exists a unique $[M]_R \in \mathrm{H}_n(M,\partial M;R)$ with $\rho_x([M]_R)=\alpha_x$ for all $x\in M\setminus\partial M$. We refer to $[M]_R$ as the fundamental class of the R-oriented topological manifold M.

(2) If we consider homology without coefficients, then we write $[M]\in \mathrm{H}_n(M,\partial M)$ for this uniquely determined element.

Example.

(1) Let $n\in\mathbb{N}$. We equip the topological manifold S^n with the standard orientation that we introduced in the Sphere Orientation Lemma 191.20. It follows immediately from the definition of this orientation that the corresponding fundamental class $[S^n]\in \mathrm{H}_n(S^n)$ equals the standard generator that we introduced in Definition 150.8.

(2) Let M be a compact *oriented* n-dimensional topological manifold and let R be a commutative ring. If follows from the Orientable-for-all-Rings Lemma 191.21 that the orientation of M also defines an R-orientation of M. Let $\mu\colon \mathrm{H}_k(X,A)\otimes R\to \mathrm{H}_k(X,A;R)$ be the natural map from the Homology-Multiplication Lemma 161.7. It follows easily from the naturality of μ that $\mu([M]\otimes 1_R)=[M]_R$.[2077]

(3) Let M be a compact n-dimensional topological manifold with components $M_1,\dots,M_k$. We assume that M is equipped with an R-orientation. It follows easily from a relative version of the Path-Component–H_*–Lemma 143.21 that

$$[M]_R \;=\; [M_1]_R+\cdots+[M_k]_R \;=\; \mathrm{H}_n(M,\partial M;R) \;=_{\mathrm{i}}\; \bigoplus_{i=1}^{k}\mathrm{H}_n(M_i,\partial M_i;R).$$

identification is given by a relative version of the Path-Component–H_*–Lemma 143.21

(4) We consider the empty topological manifold $\varnothing$. As we mentioned on page 624, basically by definition the empty set is a topological manifold of any dimension. Now let R be a commutative ring. Evidently $\varnothing$ has a unique R-orientation. Note that we have the *equality* of R-modules $\mathrm{H}_0(\varnothing;R)=\mathrm{H}_1(\varnothing;R)=\mathrm{H}_2(\varnothing;R)=\cdots=\{0\}$. As in Definition 185.15 we see that the fundamental class $[\varnothing]$ of the empty topological manifold $\varnothing$ is the unique element in the R-module $\mathrm{H}_0(\varnothing;R)=\mathrm{H}_1(\varnothing;R)=\cdots=\{0\}$. □

The case $R=\mathbb{F}_2$ is particularly interesting:

Example. Let M be a compact n-dimensional topological manifold. In the $\mathbb{F}_2$-Orientability Proposition 191.22 we showed that M admits a unique $\mathbb{F}_2$-orientation. This implies that M has a natural $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2}\in \mathrm{H}_n(M,\partial M;\mathbb{F}_2)$. If M is connected and non-empty, then it follows immediately from the Top-$\mathbb{F}_2$-Homology Theorem 193.2 (2) that

$$[M]_{\mathbb{F}_2} \;=\; \text{the unique non-zero element of } \mathrm{H}_n(M,\partial M;\mathbb{F}_2)\cong\mathbb{F}_2. \qquad \square$$

[2077]This can be seen as follows: Let $\{\alpha_x\}_{x\in M\setminus\partial M}$ be the given orientation of the topological manifold M. By definition $\{\mu(\alpha_x\otimes 1_R)\}_{x\in M\setminus\partial M}$ is the corresponding R-orientation. It follows immediately from the naturality of μ that for each $x\in M\setminus\partial M$ we have $\rho_x(\mu([M]\otimes 1_R))=\mu(\rho_x([M])\otimes 1_R)=\mu(\alpha_x\otimes 1_R)$. By definition this means that $\mu([M]\otimes 1_R)=[M]_R$.

Given an oriented topological manifold M we denote, as usual, by $-M$ the same topological manifold but with the opposite orientation. Now we can formulate the following lemma, which is the topological analog of the Fundamental Class-Opposite Orientation Lemma 185.17:

Lemma 194.4. (Fundamental Class-Opposite Orientation Lemma) For every compact oriented n-dimensional topological manifold M we have

$$[-M] \;=\; -[M] \;\in\; \mathrm{H}_n(M, \partial M).$$

Proof. This statement follows immediately from the definitions. ■

194.2. **Fundamental class of smooth and topological manifolds.** Let M be a compact non-empty n-dimensional *smooth* manifold. In Definition 185.30 we introduced the corresponding $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$. Furthermore, if M is in fact oriented, then in Definition 185.4 we introduced the corresponding fundamental class $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$. We will now see that those previous definitions of fundamental classes are consistent with the definitions from this chapter.

First let us consider the case of $\mathbb{F}_2$-fundamental classes. This case is easy: It follows basically immediately from the $\mathbb{F}_2$-Fundamental Class Proposition 185.31 (2a) that the two definitions actually agree.

Now let us assume that the smooth manifold M is oriented as a smooth manifold, in the sense of Section 31.2. In the Manifold-Top-H_*-Theorem 185.3 we defined the fundamental class $[M]$ in $\mathrm{H}_n(M, \partial M; \mathbb{Z})$. On the other hand, in the Smooth-Topological Orientation Proposition 191.25 we saw that the orientation of M also determines a $\mathbb{Z}$-orientation and thus we obtain a fundamental class $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ by the Fundamental Class-Proposition 194.2. Not surprisingly we will now see that these two definitions agree.

Lemma 194.5. (Smooth-Topological Fundamental Class Lemma) The two definitions of the fundamental class of a compact oriented non-empty smooth manifold agree.

Remark. Let M be a compact oriented non-empty smooth manifold. In the Fundamental Class Algorithm 185.5 and the Fundamental Class-Criterion 185.7 we developed some techniques for finding explicit cycles representing the fundamental class of the oriented smooth manifold M. By the Smooth-Topological Fundamental Class Lemma 194.5 these approaches also work for finding explicit cycles representing the fundamental class of a smooth manifold that is equipped with a $\mathbb{Z}$-orientation. □

Proof. Let M be a compact n-dimensional smooth manifold that is oriented in the sense of Section 31.2. The case $n = 0$ is basically trivial. Hence we now only consider the case $n \geq 1$. As we will see, the lemma follows almost immediately from the definitions.

Let us first recall the construction, in the proof of the Smooth-Topological Orientation Proposition 191.25, of the corresponding $\mathbb{Z}$-orientation of M: Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x that is orientation-preserving in the sense from Definition 31.19. We set

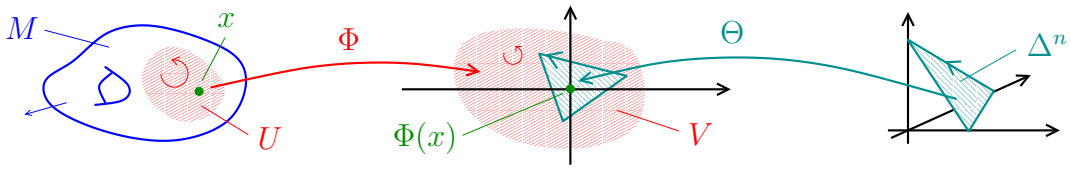
$$\mu_x \;:=\; \Phi_*^{-1}\big(\text{standard generator of } \mathrm{H}_n(V, V \setminus \{\Phi(x)\}; \mathbb{Z}), \text{ as defined on page 3683}\big).$$

In the proof of the Smooth-Topological Orientation Proposition 191.25 we saw that this definition of μ_x does not depend on the choice of the orientation-preserving chart. Furthermore we saw that these classes $\{\mu_x\}_{x \in M \setminus \partial M}$ define a $\mathbb{Z}$-orientation for M.

We denote by $\alpha \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ the fundamental class of the Manifold-Top-H_*-Theorem 185.3. By definition of the fundamental class from the Fundamental Class-Proposition 194.2 it now remains to prove the following claim:

Claim. For every $x \in M \setminus \partial M$ we have $\alpha_x = \mu_x \in \mathrm{H}_n(M, M \setminus \{x\})$.

Proof. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x with $\Phi(x) = 0$ that is orientation-preserving. Furthermore let $\Theta\colon \Delta^n \to V$ be an affine linear map such that 0 lies in the image of $\Theta(\mathring{\Delta}^n)$ and which has the property that $\Theta\colon \mathring{\Delta}^n \to V$ is orientation-preserving.



We see that

since $[\Theta\colon \Delta^n \to V]$ is by definition a standard generator of $\mathrm{H}_n(V, V \setminus \{0\}; \mathbb{Z})$

$$\alpha_x \underset{\text{follows from the Manifold-Top-H}_*\text{-Theorem 185.3 and the fact that } \Phi^{-1}\circ\Theta \text{ is orientation-preserving}}{=} [\Phi^{-1} \circ \Theta\colon \Delta^n \to M] \overset{\downarrow}{=} \Phi_*^{-1}([\text{standard generator of } \mathrm{H}_n(V, V \setminus \{0\}; \mathbb{Z})]) = \mu_x.$$

■

194.3. **Finding explicit fundamental classes.** In this section we will formulate and prove two propositions which can be viewed as generalizations of two earlier results in the smooth setting, namely of the Fundamental Class-Criterion 185.7 and the $\mathbb{F}_2$-Fundamental Class-Criterion 185.32.

Before we can state these two propositions we need to talk about Δ^n viewed as an oriented topological manifold:

Convention 194.6. We equip the topological manifold Δ^n with the orientation that is given by the images of[2078] $[\mathrm{id}_{\Delta^n}] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ in $\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\})$, $x \in \Delta^n \setminus \partial\Delta^n$.

Lemma 194.7. (Simplex-Orientation Lemma) Let $n \in \mathbb{N}_0$. We consider the oriented topological manifold Δ^n. We equip the codimension-zero submanifold $\mathring{\Delta}^n$ with the corresponding orientation coming from the Topological Manifold-Induced Orientation Lemma 192.1. Under the bijection from the Smooth-Topological Orientation Proposition 191.25 this orientation matches the orientation of the oriented smooth manifold $\mathring{\Delta}^n$ that we introduced in Definition 143.8.



Sketch of proof. The lemma follows easily from the following observations:

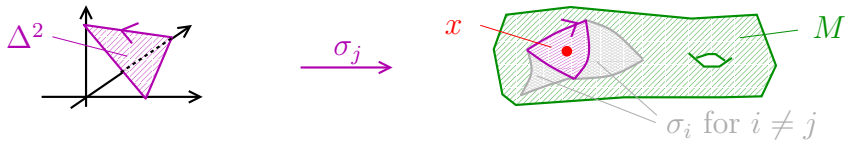
(1) Let $\pi\colon \Delta^n \to \mathbb{R}^n$ be the projection onto the first n coordinates. This map π is given by an affine linear map that is orientation-preserving if and only if n is even.

[2078] Recall that by the Identity-is-Generator Lemma 150.1 we know that $\mathbb{Z} \cdot [\mathrm{id}_{\Delta^n}] = \mathrm{H}_n(\Delta^n, \partial\Delta^n)$.

(2) It follows fairly easily from (1) and the definition of the standard orientation of the topological manifold $\mathbb{R}^n$ that the map $\pi\colon \mathring{\Delta}^n \to \mathbb{R}^n$ is orientation-preserving, as a local homeomorphism between oriented topological manifolds, if and only if n is even.
(3) It follows from the definition of the orientation of the smooth manifold $\mathring{\Delta}^n$ in Definition 143.8 that the map $\pi\colon \mathring{\Delta}^n \to \mathbb{R}^n$ is orientation-preserving, as a local diffeomorphism between oriented smooth manifolds, if and only if n is even.
(4) By the Smooth-Topological Orientation Proposition 191.25 (1a) the bijection matches the standard topological and smooth orientations of $\mathbb{R}^n$. ■

The following proposition is the topological analog of the Fundamental Class-Criterion 185.7.

Proposition 194.8. (Fundamental Class-Criterion)[2079] [2080] Let M be a compact oriented connected n-dimensional topological manifold. Let $r_1\cdot\sigma_1+\cdots+r_m\cdot\sigma_m \in \mathrm{C}_n(M,\partial M)$ be a cycle. Suppose that there exists a $j \in \{1,\dots,m\}$ such that the following three conditions are satisfied:
(1) The map $\sigma_j\colon \mathring{\Delta}^n \to M$ is an orientation-preserving embedding.
(2) There exists an $x \in \sigma_j(\mathring{\Delta}^n)$ which is not contained in any $\sigma_i(\Delta^n)$, $i \neq j$.
(3) $r_j = 1$.
Then the cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m$ represents the fundamental class $[M] \in \mathrm{H}_n(M,\partial M)$ of the oriented topological manifold M.



Proof. Let $\{\alpha_y\}_{y\in M\setminus\partial M}$ be the given orientation of M, let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in \mathrm{C}_n(M,\partial M)$ be a cycle and let $j \in \{1,\dots,m\}$ such that conditions (1)–(3) above are satisfied. Since M is connected it follows from the Fundamental Class-Proposition 194.2 (2b) that it suffices to prove the following claim:

Claim. There exists an $x \in M \setminus \partial M$ such that $\rho_x([r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m]) = \alpha_x$.

Proof. We make a few preparations:

- We pick $x \in \sigma_j(\mathring{\Delta}^n)$ as in hypothesis (2).
- Let $y \in \mathring{\Delta}^n$ be the unique point with $\sigma_j(y) = x$.
- We denote by $i\colon \mathring{\Delta}^n \to \Delta^n$ the inclusion.
- It follows from the Excision Theorem 148.2 that $i_*\colon \mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n\setminus\{y\}) \to \mathrm{H}_n(\Delta^n, \Delta^n\setminus\{y\})$ is an isomorphism.
- We denote by $\gamma_y \in \mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n \setminus \{y\})$ the generator corresponding to the orientation of $\mathring{\Delta}^n$. By definition of the orientations of the topological manifolds Δ^n and $\mathring{\Delta}^n$ we have $i_*(\gamma_y) = \rho_y([\mathrm{id}_{\Delta^n}])) \in \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{y\})$.
- Since $\sigma_j\colon \mathring{\Delta}^n \to M$ is orientation-preserving we have $\sigma_{j*}(\gamma_y) = \alpha_x$.

[2079] For $M = S^n$ this proposition is closely related to the Standard Generator-at-Point Lemma 155.31.
[2080] This statement and its proof are closely related to the Detect-Cycle Proposition 150.5.

We summarize some of the maps in the following basic commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\rho_y} & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{y\}) & \xleftarrow[\cong]{i_*} & \mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n \setminus \{y\}) \\ & \searrow^{\sigma_{j*}} & \downarrow{\scriptstyle \sigma_{j*}} & \swarrow_{\sigma_{j*}} & \\ & & \mathrm{H}_n(M, M \setminus \{x\}). & & \end{array}$$

($[\mathrm{id}_{\Delta^n}]$ over $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$; $\rho_y([\mathrm{id}_{\Delta^n}])$ over $\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{y\})$; γ_y over $\mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n \setminus \{y\})$; α_x under $\mathrm{H}_n(M, M \setminus \{x\})$.)

We obtain the following equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\rho_x([r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m]) = [\overbrace{r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m}^{\in \mathrm{C}_n(M, M\setminus\{x\})}] = [r_j \cdot \sigma_j] = [\sigma_j \colon \Delta^n \to M].$$

(since $r_j = 1$)

$$= \sigma_{j*}([\mathrm{id}_{\Delta^n}]) = \sigma_{j*}(\gamma_y) = \alpha_x.$$

(by definition of σ_{j*}; since (2) implies that for $i \neq j$ we have $\sigma_i(\Delta^n) \subseteq M \setminus \{x\}$; by the above diagram; since $\sigma_j \colon \mathring{\Delta}^n \to M$ is orientation-preserving) ■

Not particularly surprisingly we continue with the topological analog of the $\mathbb{F}_2$-Fundamental Class-Criterion 185.32:

Proposition 194.9. ($\mathbb{F}_2$-Fundamental Class-Criterion) Let M be a compact connected n-dimensional topological manifold. Let $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2$ be a cycle. Suppose that there exists a $j \in \{1, \ldots, m\}$ such that the following two conditions are satisfied:

(1) The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is an embedding.

(2) There exists an $x \in \sigma_j(\mathring{\Delta}^n)$ which is not contained in any $\sigma_i(\Delta^n)$, $i \neq j$.

Then the cycle $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2 = \mathrm{C}_n(M, \partial M; \mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class in $\mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ of M.

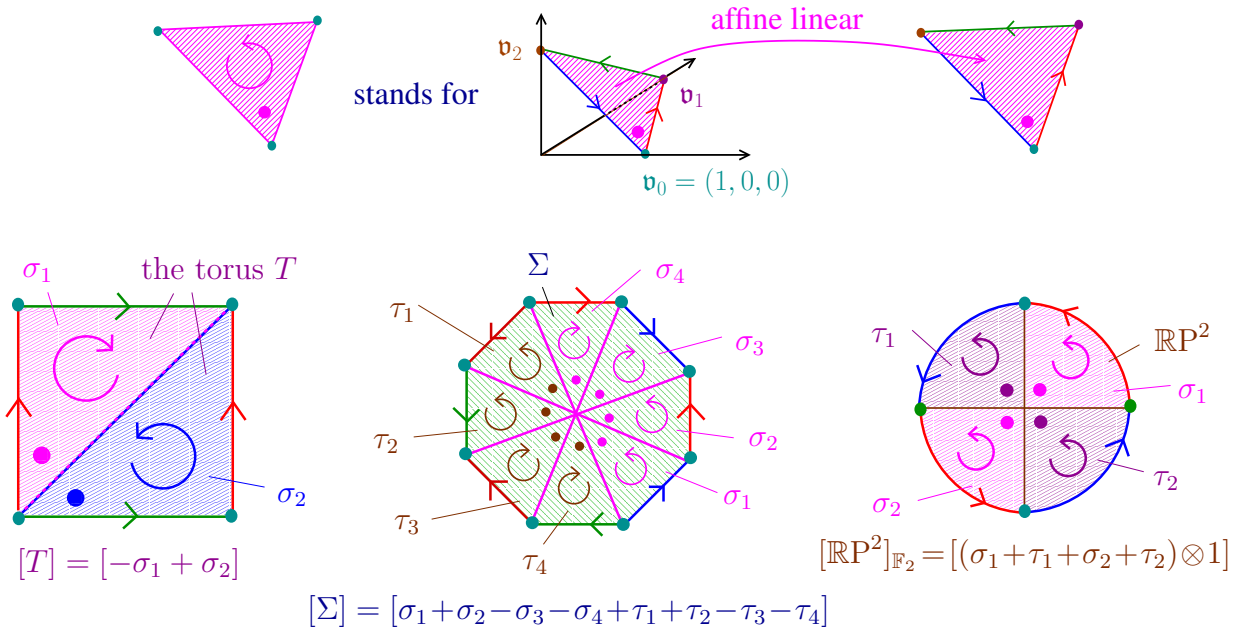
Proof. We leave it to the reader to simplify the proof of the Fundamental Class-Criterion 194.8 to obtain the desired result. ■

Example. We consider the topological manifold S^1 with the standard orientation. Note that it follows quite easily from the Fundamental Class-Criterion 194.8 that the singular simplex $\mu \colon \Delta^1 \to S^1$ that is given by $(1-t, t) \mapsto \exp(2\pi i t)$ represents the fundamental class of S^1. □

Example. [2081] We will give some examples of fundamental classes of 2-dimensional topological manifolds. We use the notation from the figure below to indicate a singular simplex.

In the figure below we now show various cycles.
It follows from the Fundamental Class-Criterion 194.8 and the $\mathbb{F}_2$-Fundamental Class-Criterion 194.9 that these cycles represent the fundamental classes of the torus and the surface of genus 2 and the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z$. □

[2081] We discussed these examples already in the figures on pages 3539 and 3539. But there is always the outside chance that a reader decided to give Chapter 185 a pass, so it is worth recalling the examples.



194.4. **The fundamental class and induced orientations.** In Section 192.1 we saw the following:

(1) An orientation on a topological manifold induces orientations on codimension-zero submanifolds and on coverings.
(2) An orientation on a topological manifold induces an orientation on the boundary.
(3) The product of two oriented topological manifolds has a natural orientation.

If the topological manifolds are compact, then the orientations give rise to fundamental classes. In the following three subsections we will see how the fundamental classes of the original compact oriented topological manifolds are related to the fundamental classes of the resulting compact oriented topological manifolds.

194.4.1. **The fundamental class and induced orientations: Local homeomorphisms.** In the Topological Manifold-Induced Orientation Lemma 192.1 we saw that a codimension-zero submanifold W of an R-oriented topological manifold M inherits an R-orientation. For compact topological manifolds the following lemma, which is the analog of the Codimension Zero-Fundamental Class Lemma 185.21, shows the relationship between the corresponding fundamental classes:

Lemma 194.10. (Codimension Zero-Fundamental Class Lemma) Let M be a compact n-dimensional topological manifold and let R be a commutative ring. Furthermore let $W \subseteq M$ be a compact codimension-zero submanifold. We write $\mathring{W} = W \setminus \partial W$. The following two statements hold:

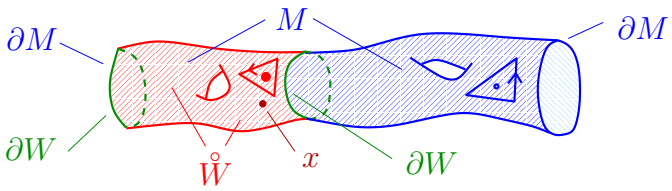
(1) The inclusion-induced map $i_*\colon \mathrm{H}_n(W, \partial W; R) \to \mathrm{H}_n(M, M \setminus \mathring{W}; R)$ is an isomorphism.
(2) We assume that M is R-oriented. We equip W with the R-orientation defined in the Topological Manifold-Induced Orientation Lemma 192.1 (2). The image of the

fundamental classes $[M]_R \in \mathrm{H}_n(M, \partial M; R)$ and $[W]_R \in \mathrm{H}_n(W, \partial W; R)$ under the maps

$$\mathrm{H}_n(M, \partial M; R) \overset{[M]_R}{\xrightarrow{\quad j_* \quad}} \mathrm{H}_n(M, M \setminus \mathring{W}; R) \xleftarrow{\quad i_* \quad} \mathrm{H}_n(W, \partial W; R) \overset{[W]_R}{}$$

agree.

The analogous statement holds for homology without coefficients.



Proof.

(1) This statement is an immediate consequence of the Manifold–Excision Theorem 148.3.

(2) We have to show that $(i_*)^{-1}(j_*([M]_R))$ is the fundamental class for W. Thus we pick a point $x \in W \setminus \partial W$. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(M, \partial M; R) & \xrightarrow{\quad j_* \quad} & \mathrm{H}_n(M, M \setminus \mathring{W}; R) & \xleftarrow[\cong]{\quad i_* \quad} & \mathrm{H}_n(W, \partial W; R) \\
\downarrow{\scriptstyle \rho_x^M} & \swarrow & & & \downarrow{\scriptstyle \rho_x^W} \\
\mathrm{H}_n(M, M \setminus \{x\}; R) & & \xleftarrow[\substack{\text{isomorphism by the} \\ \text{Excision Theorem 148.2}}]{\quad \cong \quad} & & \mathrm{H}_n(W, W \setminus \{x\}; R).
\end{array}$$

Since all maps are induced by inclusions we see that the diagram commutes. Since $[M]_R$ is the fundamental class we see that $\rho_x^M([M]_R)$ equals the generator given by the R-orientation of M. It follows from the above diagram that ρ_x^W sends $(i_*)^{-1}(j_*([M]_R))$ to the generator corresponding to the induced R-orientation of W. ■

In the Topological Manifold-Induced Orientation Lemma 192.1 (3) we showed that a covering space $\widetilde{M}$ of an R-oriented topological manifold M inherits an R-orientation from M. The following proposition can now be viewed as a generalization of Exercise 185.12 and the Covering Map-Degree Proposition 186.9.

Proposition 194.11. (Fundamental Class-Covering Proposition) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. We equip $\widetilde{M}$ with the induced R-orientation coming from the Topological Manifold-Induced Orientation Lemma 192.1 (3). We denote by $[M]_R$ the fundamental class of M and we denote by $p^*\colon \mathrm{H}_n(M, \partial M; R) \to \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M}; R)$ the transfer map that we introduced in Definition 165.1. Then

$$p^*([M]_R) \;=\; [\widetilde{M}]_R \;\in\; \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M}; R).$$

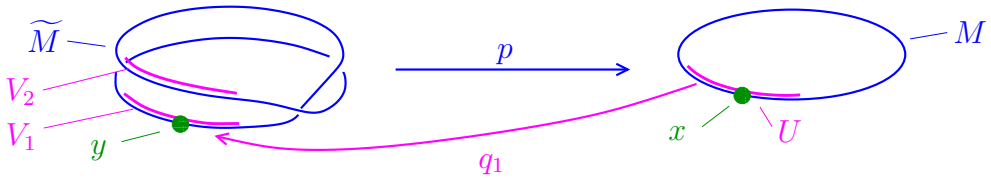
The analogous statement holds for homology without coefficients.

Proof. To simplify the notation we work with homology without coefficients. Let M be a compact n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. Let $\{\nu_x\}_{x \in M \setminus \partial M}$ be an orientation on M. We equip $\widetilde{M}$ with the orientation $\{\widetilde{\nu}_y\}_{y \in \widetilde{M} \setminus \partial\widetilde{M}}$ that we defined in the Topological Manifold-Induced Orientation Lemma 192.1 (3).

We want to show that $p^*([M])$ equals the fundamental class $[\widetilde{M}]$. By definition of the fundamental class we need to prove the following claim:

Claim. For all $y \in \widetilde{M}$ we have $\rho_y(p^*([M])) = \widetilde{\nu}_y \in \mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus \{y\})$.

Proof. Let $y \in \widetilde{M}$. We set $x := p(y)$. We pick an open neighborhood U around x such that U is uniformly covered. Recall that this means that we can write $p^{-1}(U) = V_1 \sqcup \cdots \sqcup V_d$ where $d = [\widetilde{M} : M]$, where each $V_i \subseteq \widetilde{M}$ is open and such that the restriction of p to each V_i is a homeomorphism. We denote by $q_i \colon U \to V_i$ the inverses. We can assume that $y \in V_1$.



We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(M, \partial M) \arrow[r, "\rho_x"] \arrow[d, "p^*"] & \mathrm{H}_n(M, M \setminus \{x\}) \arrow[d, "p^*"] & \mathrm{H}_n(U, U \setminus \{x\}) \arrow[l, "\cong"'] \arrow[d, "p^*", "q_{1*}+\cdots+q_{d*}"'] \arrow[ddd, bend left=60, "q_{1*}"] \\
\mathrm{H}_n(\widetilde{M}, \partial\widetilde{M}) \arrow[r] \arrow[dr, "\rho_y"'] & \mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus p^{-1}(\{x\})) \arrow[d, "\pi_*"] & \mathrm{H}_n(p^{-1}(U), p^{-1}(U) \setminus p^{-1}(\{x\})) \arrow[l, "\cong"'] \arrow[d, "\pi_*"] \\
 & \mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus \{y\}) & \mathrm{H}_n(p^{-1}(U), p^{-1}(U) \setminus \{y\}) \arrow[l, "\cong"'] \\
 & & \mathrm{H}_n(V_1, V_1 \setminus \{y\}) \arrow[u, "i_*", "\cong"']
\end{tikzcd}$$

We make the following clarifications and observations;

(1) The undecorated maps are induced by the obvious maps of pairs of topological spaces.
(2) The diagram commutes by the functoriality of homology and since it follows from the Transfer Map Lemma 165.2 that the transfer map is natural.
(3) It follows easily from the definition of the transfer map that the right hand map $p^* \colon \mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(p^{-1}(U), p^{-1}(U) \setminus p^{-1}(\{x\}))$ is given by $q_{1*} + \cdots + q_{d*}$.
(4) We denote by $i \colon V_1 \to p^{-1}(U)$ the inclusion. Note that the corresponding bottom vertical map is an isomorphism by the Excision Theorem 148.2.
(5) We denote by $\pi \colon (p^{-1}(U), p^{-1}(U) \setminus p^{-1}(\{x\})) \to (p^{-1}(U), p^{-1}(U) \setminus \{y\})$ the obvious map. It follows easily from (3) that $\pi_* \circ p^* = i_* \circ q_{1*}$.
(6) If we apply the "outside route" from $\mathrm{H}_n(M, M \setminus \{x\})$ to $\mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus \{y\})$ via q_{1*} we send, by definition, ν_x to $\widetilde{\nu}_y$.

We leave it to the reader to verify that the above information implies the desired statement. ■

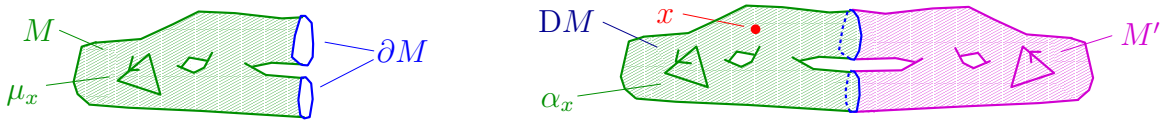
In the Double-Orientability Lemma 192.6 we saw that the double of an oriented topological manifold comes with a natural orientation. The following lemma following relates the various fundamental classes in this construction.

Lemma 194.12. (Fundamental Class-Doubling Lemma) Let R be a commutative ring and let M be a compact R-oriented n-dimensional topological manifold. If we equip $\mathrm{D}M$ with the R-orientation coming from the Double-Orientability Lemma 192.6, then

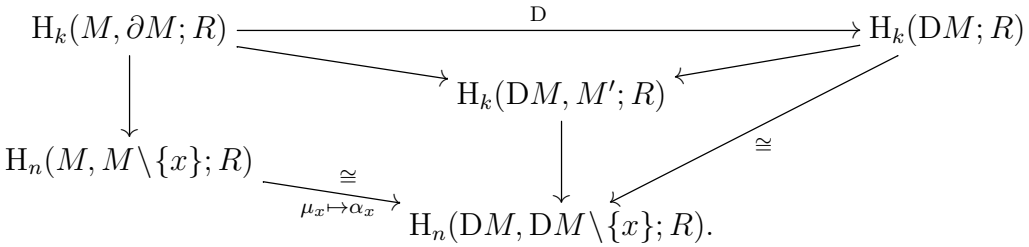
$$[\mathrm{D}M]_R \;=\; \mathrm{D}([M]_R) \;\in\; \mathrm{H}_n(\mathrm{D}M;R),$$

where $\mathrm{D}\colon \mathrm{H}_n(M,\partial M;R) \to \mathrm{H}_n(\mathrm{D}M;R)$ denotes the doubling homomorphism that we introduced in the Doubling-Homology Lemma 187.17.

Proof. Let $\{\mu_x\}_{x\in M\setminus\partial M}$ be an orientation for M. We denote by $\{\alpha_x\}_{x\in \mathrm{D}M}$ the corresponding orientation for $\mathrm{D}M$ that we had constructed in the Double-Orientability Lemma 192.6. To simplify the notation we write $\mathrm{D}M = (M \sqcup M')/\sim$ where M' is a copy of M and we view M as a submanifold of the double.



We pick $x \in M \setminus \partial M$. We consider the following diagram:



Here all maps are the obvious maps for pairs of topological spaces. The lemma follows from the following observations:

(1) The top triangle commutes by the Doubling-Homology Lemma 187.17 (3). The parallelogram on the left and the triangle on the right commute by the functoriality of homology.
(2) The homomorphism to the bottom left is an isomorphism by the Manifold–Excision Theorem 148.3.
(3) The right diagonal map is an isomorphism by the Orientation-Top-H_*-Theorem 193.3.
(4) By definition $[M]_R$ gets sent to $\mu_x \in \mathrm{H}_n(M, M\setminus\{x\};R)$ and by definition μ_x gets sent to α_x. It follows from the commutativity of the diagram that $\mathrm{D}([M]_R)$ gets sent to α_x. But since the right diagonal map is an isomorphism it follows, from the Fundamental Class-Proposition 194.2 (2b) that $[\mathrm{D}M]_R = \mathrm{D}([M]_R)$. ∎

194.4.2. **The fundamental class and induced orientations: Boundaries.** In the Boundary Orientation Proposition 192.3 we saw that the boundary of an oriented topological manifold inherits a natural orientation. The following proposition can now be seen as a generalization of the Fundamental Class-of-∂-Proposition 185.18 from smooth manifolds to topological manifolds.

Proposition 194.13. (Fundamental Class-of-∂-Proposition) Let R be a commutative ring and let M be a compact R-oriented n-dimensional topological manifold. If

we equip ∂M with the R-orientation coming from the Boundary Orientation Proposition 192.3, then we have

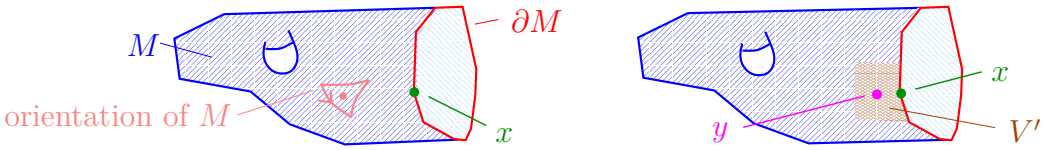
$$\underset{\substack{\uparrow\\ \text{connecting homomorphism}\\ \text{of the long exact exact}\\ \text{sequence of the pair } (M,\partial M)}}{\partial_n}([M]_R) \;=\; [\partial M]_R \quad \in\; \mathrm{H}_{n-1}(\partial M; R).$$

The analogous statement also holds if we consider homology without coefficients.

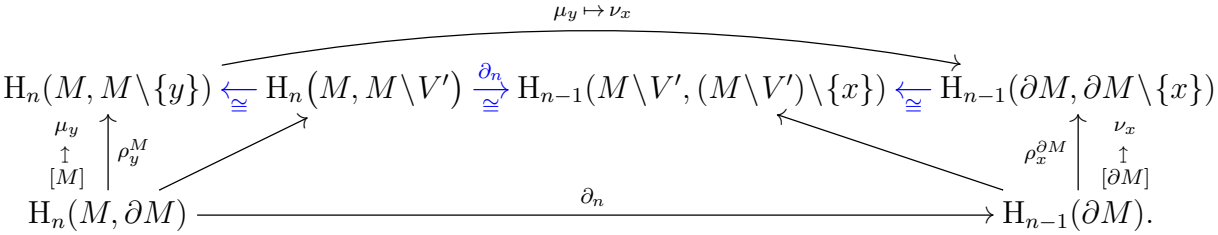
Remark. In Definition 155.11 we defined the standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ to be the unique element with $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$. It follows from the Fundamental Class-of-∂-Proposition 194.13 and the discussion on page 3708 that $[\overline{B}^n]$ is the fundamental class of the n-dimensional topological manifold $\overline{B}^n$ that we equip with the standard orientation coming from $\mathbb{R}^n$. □

Proof. To simplify the notation we consider in the proof only homology without coefficients. As a health warning we point out that in this proof we make heavy use of the notation introduced in the Boundary Orientation Proposition 192.3.

Let $\{\mu_y\}_{y\in M\setminus\partial M}$ be the orientation of M. First let $x \in \partial M$. Next we pick a chart $\Phi\colon U \to [0,2)\times(-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We write $V' = \Phi^{-1}((0,1)\times(-1,1)^{n-1})$. Finally we pick a point $y \in V'$.



We consider the following diagram:

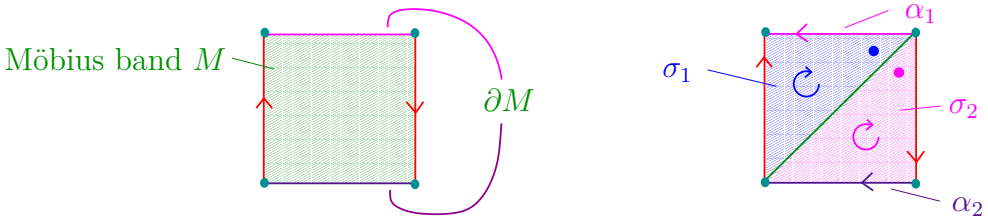


We make the following observations and clarifications:

(1) All the unmarked maps are induced by the obvious inclusions of pairs of topological space.
(2) Recall that in the proof of the Boundary Orientation Proposition 192.3 we showed that the three horizontal blue maps on the top, connecting the group $\mathrm{H}_n(M, M\setminus\{y\})$ to the group $\mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\})$, are isomorphisms.
(3) By definition of the orientation $\{\nu_x\}_{x\in\partial M}$ of ∂M the composition of the top horizontal maps sends μ_y to ν_y.
(4) The maps ∂_n are the connecting homomorphism of the long exact sequence of the triple $(M, M\setminus V', (M\setminus V')\setminus\{x\})$ respectively of the pair $(M, \partial M)$.
(5) The triangles are given by inclusion-induced maps, hence they both commute.
(6) Finally note that it follows from the naturality of the connecting homomorphisms, see the LES-of-Triple Proposition 146.5 (3), that the quadrilateral commutes.

It follows from the above and the defining property of $[M]$ that for any $x \in \partial M$ the image of $\partial_n([M]) \in \mathrm{H}_{n-1}(M)$ under the map $\mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ equals ν_x. But that means that $\partial_n([M]) = [\partial M]$. ■

Example. In the figure below we consider the Möbius band M with two singular 1-simplices α_1, α_2 and two singular 2-simplices σ_1, σ_2.



In this case we have $$\partial_2(\underbrace{\sigma_1 + \sigma_2}_{=:\mu}) = \underbrace{\alpha_1 + \alpha_2}_{\in \mathrm{C}_1(\partial M)}.$$

In particular, $\mu \otimes 1$ is a cycle in $\mathrm{C}_2(M, \partial M; \mathbb{F}_2)$. It follows easily from the $\mathbb{F}_2$-Fundamental Class-Criterion 194.9 that $\mu \otimes 1$ represents the $\mathbb{F}_2$-fundamental class in $\mathrm{H}_2(M, \partial M; \mathbb{F}_2)$. We now see that

$$\partial_2([M]_{\mathbb{F}_2}) = \partial_2([\mu \otimes 1]) \underset{\substack{\uparrow \\ \text{by the LES-of-Triple} \\ \text{Proposition 146.5}}}{=} [\partial\mu \otimes 1] \underset{\substack{\uparrow \\ \text{by the above}}}{=} [(\alpha_1 + \alpha_2) \otimes 1].$$

It follows again from $\mathbb{F}_2$-Fundamental Class-Criterion 194.9 that the last class is the $\mathbb{F}_2$-fundamental class for the circle ∂M. □

The following corollary generalizes many statements of the Kernel-of-Boundary Inclusion Corollary 185.20 to the case of topological manifolds.

Corollary 194.14. (Kernel-of-Boundary Inclusion Corollary) Let R be a commutative ring and let M be a compact R-oriented connected n-dimensional topological manifold.

(1) We denote by $i\colon \partial M \to M$ the inclusion map. Then
$$\ker(i_*\colon \mathrm{H}_{n-1}(\partial M; R) \to \mathrm{H}_{n-1}(M; R)) = R \cdot [\partial M]_R \in \mathrm{H}_{n-1}(\partial M; R).$$

(2) Let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Then[2082] [2083]
$$\ker(i_*\colon \mathrm{H}_{n-1}(A; R) \to \mathrm{H}_{n-1}(M, B; R)) = R \cdot [A]_R \in \mathrm{H}_{n-1}(A; R).$$



Remark.

(1) Note that this corollary is quite similar to the $\mathrm{H}^*_{\mathrm{dR}}$-Kernel-of-Boundary Inclusion Lemma 85.11.

(2) Let N be a topological manifold and let F be a closed oriented submanifold. We denote by $i\colon F \to N$ the inclusion. If F is the boundary of a compact oriented topological

[2082] The case $A = \partial M$ is of course just the statement of (1).

[2083] This gives also a solution to Exercise 185.3.

submanifold $W \subseteq N$, then it follows easily from the Kernel-of-Boundary Inclusion Corollary 194.14 (1) that $i_*([F]) = 0 \in \mathrm{H}_k(N)$. Finally note that in the Null Homologous Boundary Proposition 185.22 we proved a partial converse to this statement in the smooth context. □

Proof.

(1) It follows from the Fundamental Class-of-∂-Proposition 194.13 and the long exact sequence in homology of the pair $(M, \partial M)$ that the following sequence is exact:

$$\underbrace{\mathrm{H}_n(M, \partial M; R)}_{\substack{= R \cdot [M] \text{ by the} \\ \text{Fundamental Class} \\ \text{Proposition 194.2}}} \xrightarrow{[M]_R \mapsto [\partial M]_R} \mathrm{H}_{n-1}(\partial M; R) \xrightarrow{i_*} \mathrm{H}_{n-1}(M; R).$$

We are done.

(2) We consider the following diagram

$$\begin{array}{ccccc} & & \mathrm{H}_{n-1}(A; R) & \longrightarrow & \mathrm{H}_{n-1}(M, B; R) \\ & & \downarrow & & \uparrow \\ \mathbb{Z} \cdot [\partial M]_R & \longrightarrow & \mathrm{H}_{n-1}(A; R) \oplus \mathrm{H}_{n-1}(B; R) & \longrightarrow & \mathrm{H}_{n-1}(M; R) \\ & & \uparrow & & \uparrow \\ & & \mathrm{H}_{n-1}(B; R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-1}(B; R) \end{array}$$

We make the following observations and clarifications:

(a) It follows from the Path-Component-H_*-Lemma 143.21 that we have a natural isomorphism $\mathrm{H}_{n-1}(\partial M; R) \to \mathrm{H}_{n-1}(A; R) \oplus \mathrm{H}_{n-1}(B; R)$.

(b) It follows from (1) and (a) that the middle horizontal sequence is exact.

(c) It follows from (1) that the right bottom vertical map is a monomorphism.

(d) The right vertical sequence is, as a small snippet of the long exact sequence of a pair, exact.

(e) By definition we have $[A]_R + [B]_R = [\partial M]_R$.

The promised statement now follows from a little diagram chase. ■

The following proposition is the culmination of a long line of results, see the Boundary Non-Retraction Proposition 38.6, the Circle–Non Retract Proposition 90.6 and the Sphere-Non Retract Proposition 149.9.

Proposition 194.15. (Boundary Non-Retraction Proposition) Let M be a *compact* non-empty topological manifold. There is no retraction from M to ∂M.

Example. It follows that the Möbius band does not retract to its boundary. What other proofs do you know for this statement? □

Proof. The proof of the proposition follows the same logic as the proof of the Circle–Non Retract Proposition 90.6: We denote by $i\colon \partial M \to M$ the inclusion. We now suppose that there exists a retraction $r\colon M \to \partial M$, i.e. a continuous map such that $r \circ i = \mathrm{id}_{\partial M}$. We assume that M is non-empty, which implies in our case that $\partial M \neq \varnothing$. For $n := \dim(M)$

we consider the following commutative diagram:

$$\begin{array}{ccc} & \mathrm{H}_{n-1}(M;\mathbb{F}_2) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2) & \xrightarrow{(r\circ i)_*=\mathrm{id}} & \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2). \end{array}$$

Next recall that we know by the $\mathbb{F}_2$-Orientability Proposition 191.22 that M is $\mathbb{F}_2$-orientable. It follows from this observation, the hypothesis that M is compact, the above observation that $\partial M \neq \varnothing$ and the Kernel-of-Boundary Inclusion Corollary 194.14 that i_* is not injective. But this leads immediately to a contradiction. ■

The results of this and the previous subsection now allow us to perform a calculation of a connecting homomorphism in the Mayer–Vietoris sequence:

Corollary 194.16. (Manifold–MV-Connecting Homomorphism Corollary) Let X be a closed oriented connected non-n-dimensional topological manifold and let $A, B \subseteq X$ be two compact codimension-zero submanifolds of M such that the following conditions are satisfied:

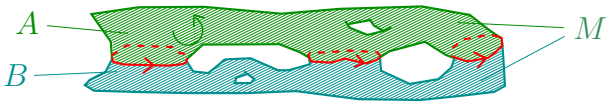
- $X = A \cup B$.
- $A \cap B = \partial A = \partial B$.

We equip A with the codimension-zero orientation from the Topological Manifold-Induced Orientation Lemma 192.1 (2) and we equip $\partial A = A \cap B$ with the corresponding boundary orientation given by the Boundary Orientation Proposition 192.3. The n-th connecting homomorphism

$$\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$$

from the Mayer–Vietoris Theorem 153.2 is given by

$$\partial_n([X]) = [A \cap B].$$



Proof. We denote by $\iota \colon \mathrm{H}_n(A, A \cap B) \to \mathrm{H}_n(X, B)$ the inclusion induced map. By the Manifold–Excision Theorem 148.3 we know that this map is an isomorphism. By the definition in the Mayer–Vietoris Theorem 153.2 the connecting homomorphism is given as the composition of the following homomorphisms:

$$\mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X, B) \xrightarrow{\iota^{-1}} \mathrm{H}_n(A, A \cap B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A \cap B)$, provided by the LES-of-Pair Proposition 146.7

It now follows easily from the Codimension Zero-Fundamental Class Lemma 194.10 and the Fundamental Class-of-∂-Proposition 194.13 that the connecting homomorphism sends $[X]$ to $[A \cap B]$. ■

194.4.3. **The fundamental class and induced orientations: Products.** We conclude this section with the last construction of topological manifolds, namely products:

Proposition 194.17. (Fundamental Class-of-Product Proposition) Let M be a compact oriented m-dimensional topological manifold and let N be a compact oriented n-dimensional topological manifold. We consider the cross product

$$\mathrm{H}_m(M,\partial M)\otimes \mathrm{H}_n(N,\partial N) \xrightarrow{\times} \mathrm{H}_{m+n}(M\times N,\partial(M\times N))$$

which we introduced in Definition 164.14.

(1) We have
$$[M]\times[N] \;=\; [M\times N] \;\in\; \mathrm{H}_{m+n}(M\times N,\partial(M\times N))$$
where $[M\times N]$ denotes the fundamental class of the compact topological manifold $M\times N$ equipped with the orientation defined in the Product Orientation Proposition 192.7 (1).

(2) If M and N are connected, then the above cross product is an isomorphism.

Remark. In the Shuffle Proposition 164.24 we gave an explicit description of the cross product on the chain level. As an example we consider $M=N=[0,1]/0\sim 1$ with the standard orientation and the cycles $\mu\colon \Delta^1\to M$ and $\nu\colon \Delta^1\to M$ that are both given by $(1-t,t)\mapsto t$. It follows from the discussion on page 3158 that $\mu\times\nu\in \mathrm{C}_2(([0,1]\times[0,1])/\sim)$ is precisely the singular cycle that we considered on page 3756. □

Proof. Let M be a compact oriented m-dimensional topological manifold and let N be a compact oriented n-dimensional topological manifold. First note that by the Product Topology-Properties Proposition 7.9 the product $M\times N$ is also compact. Furthermore note that by the Topological Manifold Product Proposition 150.12 we know that

$$\partial(M\times N) \;=\; (\partial M\times N)\cup(M\times\partial N).$$

(1) Let $x\in M\setminus\partial M$ and let $y\in N\setminus\partial N$. It follows from the above description of $\partial(M\times N)$ that $(x,y)\in(M\times N)\setminus\partial(M\times N)$. Note that, by the (relative version of the) Cross Product Lemma 164.5 (2) we have the following *commutative* diagram

$$\begin{array}{ccc}
\mathrm{H}_m(M,\partial M)\otimes \mathrm{H}_n(N,\partial N) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,\partial(M\times N)) \\
\downarrow{\scriptstyle \rho_x\otimes\rho_y} & & \downarrow{\scriptstyle \rho_{(x,y)}} \\
\mathrm{H}_m(M,M\setminus\{x\})\otimes \mathrm{H}_n(N,N\setminus\{y\}) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,(M\times N)\setminus\{(x,y)\}).
\end{array}$$

It follows immediately from the fact that the diagram commutes and the definition of the product orientation that $\rho_{(x,y)}([M]\times[N])$ is the generator given by the product orientation.

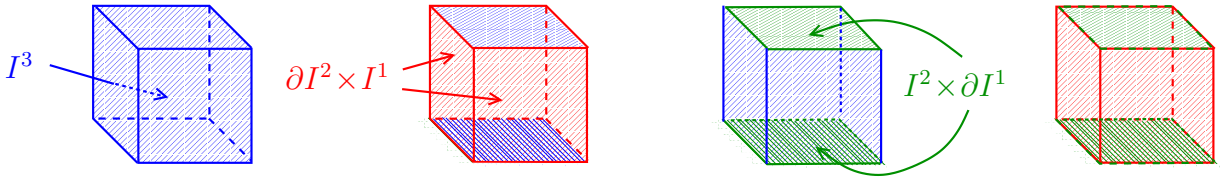
(2) We now assume that M and N are connected. If one of them is empty, then there is nothing to show. Otherwise we pick $x\in M\setminus\partial M$ and $y\in N\setminus\partial N$. We consider again the commutative diagram from (1). The vertical maps are isomorphisms by the Orientation-Top-H_*-Theorem 193.3 (3). In the Product Orientation Proposition 192.7 (1) we showed that the bottom horizontal map is an isomorphism. It follows, as claimed, that the top horizontal map is also an isomorphism. ■

Example. We write $I=[-1,1]$. We view I as a 1-dimensional topological manifold and we equip I with the orientation coming from $\mathbb{R}$. Given $k\in\mathbb{N}_0$ we write

$$\partial I^k \;:=\; \{(x_1,\ldots,x_k)\,|\,\text{there exists an } i \text{ with } x_i=-1 \text{ or } x_i=1\}.$$

Given any $m, n \in \mathbb{N}$ one sees easily that

$$\partial(I^m \times I^n) \;=\; (\partial I^m \times I^n) \cup (I^m \times \partial I^n) \;=\; \partial(I^{m+n}).$$



It follows from the Topological Manifold Product Proposition 23.6 and the above discussion that we can view each $I^k = I \times \cdots \times I$ as a k-dimensional topological manifold with boundary given by the above subset ∂I^k. We equip I^k with the product orientation. It follows immediately from the Product Orientation Proposition 192.7 (2) that the orientation on the open subset $I^k \setminus \partial I^k \subseteq \mathbb{R}^k$ is the standard orientation.[2084] As usual we denote by $[I^k] \in \mathrm{H}_k(I^k, \partial I^k; \mathbb{Z})$ the fundamental class.

Now let $m, n \in \mathbb{N}_0$. We get a cross product

$$\overbrace{\mathrm{H}_m(I^m, \partial I^m)}^{=\mathbb{Z}\cdot[I^m]} \otimes \overbrace{\mathrm{H}_n(I^n, \partial I^n)}^{=\mathbb{Z}\cdot[I^n]} \;\to\; \overbrace{\mathrm{H}_{m+n}(I^{m+n}, \partial I^{m+n})}^{=\mathbb{Z}\cdot[I^{m+n}]}$$

which by the Fundamental Class-of-Product Proposition 194.17 has the property that

$$[I^m] \times [I^n] \;=\; [I^{m+n}]. \qquad \square$$

194.5. **Applications.** The results from this chapter makes it possible to read several sections which are placed in the part on simplicial complexes. For example, with the results from this chapter one can now read the following sections:

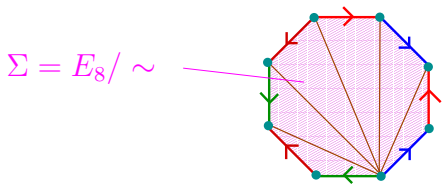
(1) Section 183.5 where we show in particular that $\mathrm{SO}(n)$ does not admit a deformation retraction.
(2) Section 185.4, in which we calculate the homology groups of the connected sum of two closed oriented smooth manifolds.
(3) Section 185.5, in which we calculate the homology groups of a knot complement.
(4) Chapter 187 where we discuss the question, which homology classes can be represented by (sub-) manifolds.

194.6. **Exercises for Chapter 194 .**

Exercise 194.1. We consider the surface $\Sigma = E_8/\sim$ of genus two as shown in the figure below. Show that there exists a cycle of the form $\epsilon_1 \cdot \sigma_1 + \cdots + \epsilon_6 \cdot \sigma_6 \in \mathrm{C}_2(E_8/\sim)$ such that the images of the six singular simplices $\sigma_1, \ldots, \sigma_6$ are precisely the six triangles shown in the figure.

Remark. This exercise is identical to Exercise 185.1.

[2084] Note that the homeomorphism $f\colon (I^k, \partial I^k) \to (\overline{B}^k, S^{k-1})$ from the Convex-to-Ball Proposition 4.28 (2) has the property that the restriction to a dense open subset of $I^k \setminus \partial I^k \to B^k$ is an orientation-preserving diffeomorphism. In particular this homeomorphism is thus an orientation-preserving homeomorphism from the oriented topological manifold I^k to the oriented topological manifold $\overline{B}^k$.



Exercise 194.2. Give an example of a representative for the $\mathbb{F}_2$-fundamental class for the Klein bottle.
Remark. This exercise is identical to Exercise 185.15.

Exercise 194.3. Can you find a representative for the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$ with one, two or three summands?

Exercise 194.4. Let M be a compact oriented connected n-dimensional topological manifold with *at least two* boundary components. Let C be one of the boundary components. By the Kernel-of-Boundary Inclusion Corollary 194.14 we know that the inclusion-induced map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left inverse, i.e. there exists a homomorphism $\varphi\colon \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(C)$ such that $\varphi \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(C)}$. Also note that in the Surface Boundary–Retraction Proposition 107.11 we showed that in the case $n = 2$ a stronger statement holds, namely C is a retract of M. Is it true in general that C is necessarily a retract of M?
Remark. This exercise is basically the same as Exercise 185.2.

Exercise 194.5. Let M be a closed oriented connected n-dimensional smooth manifold and let $C \subseteq M$ be a closed oriented connected $(n-1)$-dimensional submanifold of M. We assume that C is non-separating. Recall that according to Definition 107.4 this means that $M \setminus C$ is connected. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can pick a tubular neighborhood $[-1,1] \times C$. We write $W \coloneqq M \setminus ((-1,1) \times C)$.

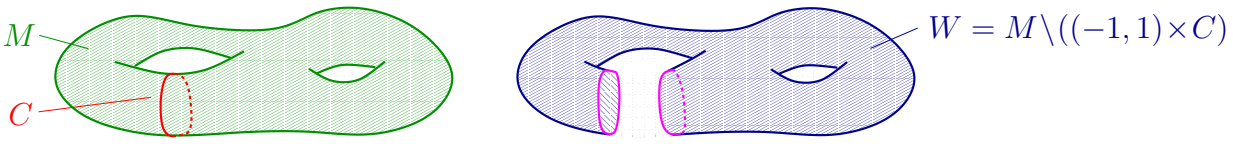
(a) Show that there exists an exact sequence of the form

$$0 \to \mathrm{H}_{n-1}(\underbrace{M \setminus ((-1,1) \times C)}_{=W}) \to \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-2}(C) \to \dots$$

Hint. Consider a Mayer–Vietoris sequence.

(b) Use (a) and Exercise 194.4 to show that the map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left inverse.
Remark. This implies in particular that $i_*([C])$ is a non-trivial element of $\mathrm{H}_{n-1}(M)$.

(c) Does the conclusion of (b) also hold if we replace the hypothesis that M is closed by the hypothesis that M is compact?



Exercise 194.6. We define an *m-component link* to be a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1. By the Product Tubular Neighborhood–Existence Theorem 58.4 we can pick a tubular neighborhood $\overline{B}^2 \times L$. We

define the exterior of L as $X_L \coloneqq S^3 \setminus (B^2 \times L)$. We denote the components of L by $L_1, \dots, L_m$. For each $i \in \{1, \dots, m\}$ we pick a point $P_i \in L_i$ and we define $\mu_i \coloneqq S^1 \times \{P_i\}$. Use the decomposition $S^3 = X_L \cup (\overline{B}^2 \times L)$ to solve the following problems:

(a) Show that $\mathrm{H}_1(X_L) = \mathbb{Z} \cdot [\mu_1] \oplus \cdots \oplus \mathbb{Z} \cdot [\mu_m]$.
(b) Determine $\mathrm{H}_2(X_L)$.

Remark. There are basically three ways to go about doing this exercise, you can modify the proof of the Knot Complement-H_*-Proposition 185.29, you could generalize the approach taken in Exercise 185.6, or you could write $X_{L_1 \sqcup L_2} = X_{L_1} \cap X_{L_2}$ and use a Mayer-Vietoris sequence.
Remark. This exercise is identical to Exercise 185.7.

Exercise 194.7. In this exercise we will consider an alternative approach to computing the homology groups of a link complement. Let $L \subseteq S^3$ be a link. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) we can pick a tubular neighborhood $\overline{B}^2 \times L$ and we consider the exterior $X_L \coloneqq S^3 \setminus (B^2 \times L)$. Now do the following:

(a) Show that the inclusion-induced maps

$$\mathrm{H}_*(S^3, L) \xrightarrow{\cong} \mathrm{H}_*(S^3, \overline{B}^2 \times L) \xleftarrow{\cong} \mathrm{H}_*(X_L, \partial X_L)$$

are both isomorphisms.
(b) Compute the homology groups of the pair $(X_L, \partial X_L)$.
(c) Determine, using (b), the homology groups of X_L.

Remark. This exercise is a mild generalization of Exercise 185.6.

Exercise 194.8. Let $K = \gamma(S^1) \subseteq S^3$ be a knot and let $\Phi \colon \overline{B}^2 \times K \to S^3$ be a tubular map for K (which exists by the Product Tubular Neighborhood–Existence Theorem 58.4 (2)). As in Definition 108.30 we refer to $X_K = S^3 \setminus \Phi(B^2 \times K)$ as the exterior of the knot K. Let $K \in P$. By the Meridian Proposition 109.2 (2) we know that $\mu_K = \Phi(S^1 \times \{P\})$ is a meridian of K is a meridian of K. In the Knot Complement-H_*-Proposition 185.29 (or alternatively in Exercise 185.7) we showed that $[\mu_K]$, viewed as an element in $\mathrm{H}_1(X_K)$, is a generator of $\mathrm{H}_1(X_K)$.

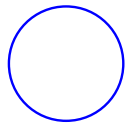
(a) Show that there exists an $n \in \mathbb{Z}$ such that the closed curve

$$\begin{aligned} S^1 &\to \partial X_K = \Phi(S^1 \times K) \\ z &\mapsto \Phi(z^n, \gamma(z)) \end{aligned}$$

represents a generator of $\ker(\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K))$. We refer to the image of this curve as a *longitude* λ_K of K.
(b) Sketch longitudes for the three knots that are shown in the figure below.
(c) Now suppose that we are given two knots K and J. We define X_K, X_J, μ_K, μ_J and λ_K and λ_J as above. We pick a diffeomorphism $\varphi \colon \partial X_K \to \partial X_J$ with $\varphi_*(\mu_K) = \lambda_J$ and $\varphi_*(\lambda_K) = \mu_K$. We denote by $Y = X_K \cup_\varphi X_J$ the result of gluing X_K to X_J along the boundary. Show that Y is a homology sphere, i.e. show that $\mathrm{H}_1(Y) = \mathrm{H}_2(Y) = 0$ and $\mathrm{H}_3(Y) \cong \mathbb{Z}$.

Remark. If K and J are two non-trivial knots, then it follows from the Trivial Knot–Characterisation Proposition **??** (which in turn is a consequence of the famous Loop Theorem **??**) together with the Amalgamated Product-Monomorphism Proposition 98.26, that

the fundamental group of Y is always non-trivial. This shows that Y is a homology sphere with non-trivial fundamental group. In other words we have now found an alternative proof of the Poincaré Homology Sphere Proposition 183.24.
Remark. This exercise is identical to Exercise 185.5.


trivial knot


(left-handed) trefoil

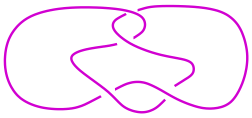


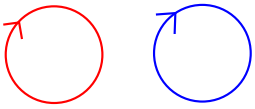
figure-8 knot

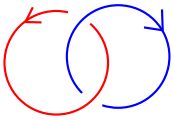

stevedore knot

Exercise 194.9.

(a) In the figure below we see two 2-component links, namely the unlink and the Hopf link. Are these two links isotopic?
(b) Let $L \subseteq S^3$ be an m-component link. Show that the weight of the group $\pi_1(S^3 \setminus L)$, as defined in Exercise 111.31, equals m.

Remark. This exercise is identical to Exercise 185.8.


2-component trivial link


Hopf link

Exercise 194.10. Let M and N be two oriented, connected n-dimensional smooth manifolds. We suppose that M is closed and that N has *non-empty* boundary. Express the homology groups of $M\#N$ in terms of the homology groups of M and N.
Remark. This exercise is identical to Exercise 185.10.

Exercise 194.11. Let M be a smooth manifold. Furthermore let $C \subseteq M$ be an oriented smooth submanifold that is diffeomorphic to S^1. Note that C represents a homology class $[C] \in \mathrm{H}_1(M)$. Show that if C is the boundary of a compact orientable 2-dimensional smooth submanifold $\Sigma \subseteq M$, the $[C] = 0 \in \mathrm{H}_1(M)$.

Exercise 194.12. Let $f\colon M \to N$ be a continuous map between two compact orientable connected n-dimensional smooth manifolds with non-empty boundary. We assume that $f\colon \partial M \to \partial N$ is a homeomorphism. Show that f is a surjection.

Exercise 194.13. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $f\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the corresponding mapping torus $M := \mathrm{Tor}(\Sigma, f)$. Note that we know by the Manifold-Mapping Torus Lemma 59.11 that M is a closed connected 3-dimensional smooth manifold.

(a) Show that if f is orientation-preserving, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M))$.
(b) Show that if f is orientation-reversing, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M)) - 1$.

Remark. You might want to use the Mapping Torus-H_*-Proposition 153.6.

Exercise 194.14. Give examples of two closed oriented connected non-empty smooth manifolds M and N of the same dimension such that $\pi_2(M\#N)$ is not isomorphic to the direct sum $\pi_2(M) \oplus \pi_2(N)$.

Exercise 194.15. Let M be a compact oriented connected non-empty n-dimensional topological manifold. Show that there exists a cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ which satisfies the three properties stated in the Fundamental Class-Criterion 194.8.

Exercise 194.16. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold. The image of the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ under the natural map $\mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(M, \partial M; \mathbb{R})$ is called the *real fundamental class*, denoted by $[M]_{\mathbb{R}}$. Let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. Given an ordered n-simplex $s = (v_0, \ldots, v_n)$ of K we denote by $\Phi_s\colon \Delta^n \to |K|$ the corresponding characteristic map that is given by $(t_0, \ldots, t_n) \mapsto t_0 \cdot v_0 + \cdots + t_n \cdot v_n$.

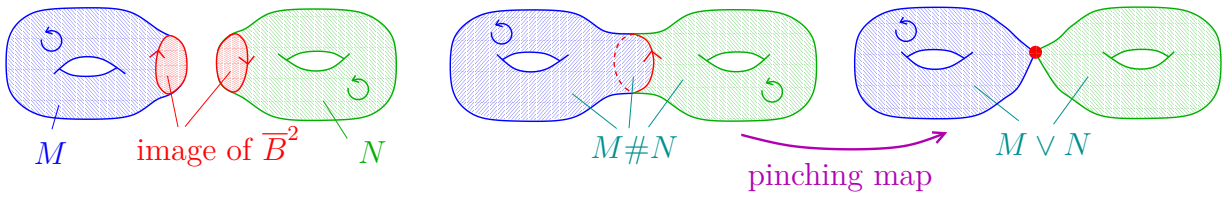
(a) Show that
$$\sigma \coloneqq \sum_{\substack{s = (v_0, \ldots, v_n) \text{ ordered } n\text{-simplex} \\ \text{such that } \Theta \circ \Phi_s\colon \mathring{\Delta}^n \to M \\ \text{is orientation-preserving}}} (\Theta \circ \Phi_s\colon \Delta^n \to M) \in \mathrm{C}_n(M)$$
represents a cycle in $\mathrm{C}_n(M, \partial M)$.

(b) Show that $[\sigma] \otimes \frac{2}{n!} \in \mathrm{C}_n(M, \partial M; \mathbb{R}) = \mathrm{C}_n(M, \partial M) \otimes \mathbb{R}$ represents $[M]_{\mathbb{R}}$.

Exercise 194.17. Let $n \in \mathbb{N}$ and let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n_2 \to M \setminus \partial M$ and furthermore we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n_2 \to N \setminus \partial N$. We use φ and ψ to define the connected sum
$$M \# N \coloneqq \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$
We refer to the map
$$p\colon M\#N \to M \vee N, \quad P \mapsto \begin{cases} \varphi(x \cdot 2(t-1)), & \text{if } P = \varphi(x \cdot t) \text{ with } x \in S^{n-1} \text{ and } t \in [1,2] \\ \psi(x \cdot 2(t-1)), & \text{if } P = \psi(x \cdot t) \text{ with } x \in S^{n-1} \text{ and } t \in [1,2] \\ P, & \text{otherwise.} \end{cases}$$
as the *pinching map*. Show that for $i = 0, \ldots, n-1$ the induced map
$$p_*\colon \mathrm{H}_i(M\#N) \to \mathrm{H}_i(M \vee N)$$
is an isomorphism and that $p_*([M\#N]) = [M] + [N] \in \mathrm{H}_n(M \vee N) = \mathrm{H}_n(M) \oplus \mathrm{H}_n(N)$.

Remark. We leave it to the reader to show that for $M = N = S^n$ this pinching map is homotopic to the pinching map that we introduced on page 2579.

Remark. This exercise is identical to Exercise 185.18.



Exercise 194.18. Let X be a topological space, let $n \in \mathbb{N}_0$ and let G be an abelian group.

(1) A *locally finite singular n-chain* is a (possibly infinite) sum $\sum_{i \in I} a_i \cdot (\sigma_i\colon \Delta^n \to X)$ with $a_i \in \mathbb{Z}$, such that for every $x \in X$ there exists an open neighborhood U of x such that $\{i \in I \,|\, \sigma_i(\Delta^n) \cap U \neq \varnothing\}$ is finite. We define $\mathrm{C}_n^{\mathrm{lf}}(X)$ as the abelian group given by

all locally finite singular n-chains. Note that the usual boundary map defines in an obvious way a boundary map $\mathrm{C}_n^{\mathrm{lf}}(X) \otimes G \to \mathrm{C}_{n-1}^{\mathrm{lf}}(X) \otimes G$. We define the *$n$-th locally finite homology group* $\mathrm{H}_n^{\mathrm{lf}}(X;G)$ to be the n-th homology of the corresponding chain complex.

(2) We consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subseteq$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Given any two compact subsets $K \subseteq L$ we get an induced map $\mathrm{H}_n(X, X \setminus L; G) \to \mathrm{H}_n(X, X \setminus K; G)$. These homology groups form an *inverse* system over the directed set $(\mathcal{K}(X), \subseteq)$ and we can form the inverse limit[2085]

$$\overleftarrow{\mathrm{H}}_n(X;G) \coloneqq \varprojlim_{K \in \mathcal{K}(X)} \mathrm{H}_n(X, X \setminus K; G).$$

In the following we will see that these two types of homology groups are often naturally isomorphic.

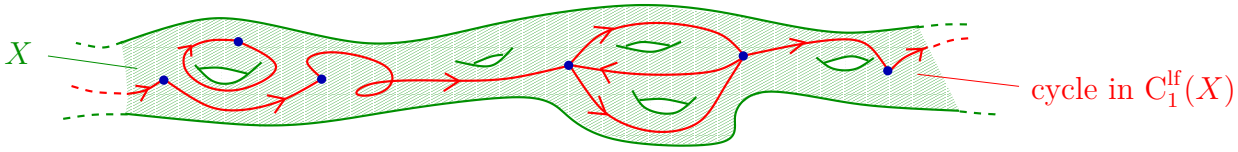
(a) Let *TopPropMap* be the category of topological spaces with proper maps, i.e. the objects are all topological spaces and the morphisms are continuous maps that are proper in the sense of Definition 4.63, i.e. maps such that preimage of compact sets are compact. Show that both homology groups define a covariant functor from *TopPropMap* to the category *AbGroup* of abelian groups.

(b) Give an explicit natural map $\mathrm{H}_n^{\mathrm{lf}}(X;G) \to \overleftarrow{\mathrm{H}}_n(X;G)$ which is the identity for all compact spaces.

(c) We assume that there exists a nested sequence $K_1 \subseteq K_2 \subseteq \dots$ of compact subsets of X such that $X = \bigcup_{i \in \mathbb{N}} \overset{\circ}{K}_n$.[2086] Show that the natural map $\mathrm{H}_n^{\mathrm{lf}}(X;G) \to \overleftarrow{\mathrm{H}}_n(X;G)$ from (b) is an isomorphism.

(d) Let $k \in \mathbb{N}$. Compute $\mathrm{H}_n^{\mathrm{lf}}(\mathbb{R}^k;\mathbb{Z})$.

Remark. Locally finite homology is discussed in detail in[**HR1996**, p. 30-34] and [**Spa1993**]. Note that statement (c) is [**Spa1993**, Theorem 10.1], also see [**HR1996**, p 34] for a related result. For completeness sake we note that it follows with some effort from [**Bre1997**, Corollary V.12.21 and pages 288-292] that for suitable topological spaces these two homology groups are naturally isomorphic to Borel-Moore homology.



Exercise 194.19. Let M be a connected orientable n-dimensional topological manifold with $\partial M = \varnothing$, let $K \subseteq M$ be a compact subset and let $x_0 \in K$ be a point.

(a) Show that there exists a *connected* compact subset $L \subseteq M$ with $K \subseteq L$.

[2085] To the best of my knowledge there is no established name for this natural object.

[2086] Note that it follows from the Topological Manifold Exhaustion Lemma 23.12 that such a sequence exists if X is a topological manifold.

(b) Show that the map

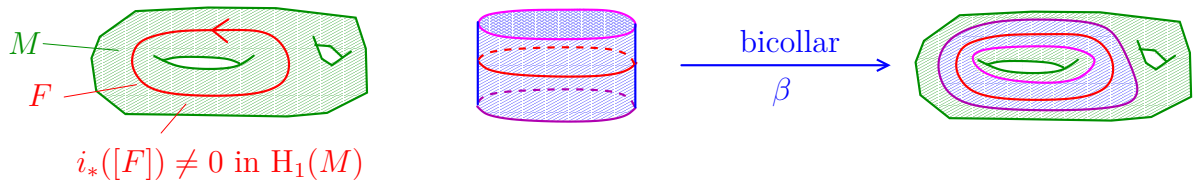
$$\begin{array}{rcl} \mathrm{Im}(\mathrm{H}_n(M, M\setminus L;\mathbb{Z}) \to \mathrm{H}_n(M, M\setminus K;\mathbb{Z})) & \to & \mathrm{H}_n(M, M\setminus\{x_0\};\mathbb{Z}) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

is an isomorphism.

Exercise 194.20. Let M be a connected orientable non-empty n-dimensional topological manifold such that $\partial M = \varnothing$. As in the above Exercise 194.18 we now consider the group $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) \coloneqq \varprojlim_{K\in\mathcal{K}(M)} \mathrm{H}_n(M, M\setminus K;\mathbb{Z})$. Show that in our situation this group is naturally isomorphic to $\mathbb{Z}$. We refer to the corresponding natural generator as the fundamental class $[M] \in \overleftarrow{\mathrm{H}}_n(M;\mathbb{Z})$. Note that by Exercise 194.18 we can view this fundamental class as an element of locally finite homology.
Hint. Use Exercise 194.19.

Exercise 194.21. Let X be a *countable* CW-complex. Is locally finite homology of X necessarily *countable*?

Exercise 194.22. Let M be a compact orientable connected n-dimensional topological manifold and let $F \subseteq M$ be a closed oriented connected codimension-one submanifold of M which is non-seperating, i.e. we assume that $M \setminus F$ is connected. We suppose that F admits a *bicollar*, i.e. we suppose that there exists an embedding $\beta\colon [-1,1]\times F \to M\setminus\partial M$ such that $\beta(0,x) = x$ for all $x \in F$.[2087] We denote by $i\colon F \to M$ the inclusion. Show that $\mathbb{Z}\cdot i_*([F])$ is a summand of $\mathrm{H}_{n-1}(M)$.



Exercise 194.23. Let M be a 3-dimensional smooth homology 3-sphere with non-trivial fundamental group.[2088] Furthermore let $n \geq 4$. We pick a smooth embedding $\varphi\colon \overline{B}^3 \to M$ and we consider $W \coloneqq (M \setminus \varphi(B^3)) \times \overline{B}^{n-3}$. By the Smooth Manifold Rounded Product Proposition 60.7 we can equip W with the structure of an orientable n-dimensional smooth manifold. Show that the double $\mathrm{D}W$ is an n-dimensional smooth homology n-sphere with non-trivial fundamental group.
Remark. This exercise is identical to Exercise 183.17.

Exercise 194.24. We consider the 3-dimensional torus $T \coloneqq S^1 \times S^1 \times S^1 = [0,1]^3/\sim$. Find an explicit 3-cycle that generates $\mathrm{H}_3(T)$.
Hint. Use the Fundamental Class-of-Product Proposition 194.17 and the shuffle product from Definition 164.23 and apply it to the usual singular simplex $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$.

[2087] If M is smooth and F is a smooth submanifold, then such a bicollar exists by the Bicollar Neighborhood Theorem 55.2.
[2088] By the Poincaré Homology Sphere Proposition 183.24 we know that such M exists.

CHAPTER 195

The degree of a map between oriented topological manifolds

Let $f\colon M \to N$ be a continuous map between two closed oriented *smooth* manifolds of the same dimension. In Chapter 46 we introduced the degree $\deg(f) \in \mathbb{Z}$ of f by looking at preimages of regular values of f. We then used this down-to-earth approach to prove various properties of degrees.

In Chapter 186 we reinterpreted the degree in terms of homology and fundamental classes of compact oriented *smooth* manifolds.

In this chapter we will generalize the approach of Chapter 186 to the case of continuous maps between compact oriented *topological* manifolds. For the reader who is familiar with Chapter 186 the present chapter will hold no surprises. In fact many statements and proofs of the current chapter are self-plagiarized from Chapter 186 with minimal editing.

This chapter is organized as follows:

- In Section 195.1 we will introduce the degree of a map between oriented topological manifolds.
- In Section 195.2 we will introduce local degrees and we will discuss the relationship between local and global degrees.
- In Section 195.3 we will discuss several properties of the degree.
- In Section 195.4 we will provide exercises for Chapter 195.

195.1. **The degree of a map between oriented topological manifolds.** Recall that in Definition 186.1 we used the fundamental class to define the degree of a continuous map between smooth manifolds. Now that we also managed to introduce the fundamental class for topological manifolds we can generalize the earlier definition to the setting of topological manifolds. More precisely, we have the following definition:

Definition 195.1. Let $f\colon M \to N$ be a continuous map between two compact oriented connected non-empty n-dimensional topological manifolds. We assume that $f(\partial M) \subseteq \partial N$. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$\underset{\uparrow}{f_*}([M]) \;=\; \deg(f)\cdot[N] \;\in\; \mathrm{H}_n(N,\partial N) \;\underset{\uparrow}{=}\; \mathbb{Z}\cdot[N].$$

since $f(\partial M)\subseteq \partial N$ we have an induced map $f_*\colon \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(N,\partial N)$

by the Fundamental Class-Proposition 194.2

We refer to $\deg(f)$ as the degree of f.

Remark.

(1) For a continuous map $f\colon S^n \to S^n$ the above degree is, by definition, exactly the same as the degree that we introduced in Definition 151.3.

(2) Given a continuous map between *smooth* manifolds it follows from the Smooth-Topological Fundamental Class Lemma 194.5 that the above definition of the degree agrees with the definition given in Definition 186.1.
(3) It follows from (2) and the discussion on page 3561 (or alternatively the discussion on page 3776) that the above definition of degree is also consistent with the definition given on Definition 46.3 and Definition 47.6.
(4) In the Degrees-Agree Proposition 231.19 we will see that the above definition is consistent with the definition of degree using de Rham cohomology which we gave in Definition 86.2. □

Example. Let $f\colon M \to N$ be a *homeomorphism* be tween two compact oriented connected non-empty n-dimensional topological manifolds. It follows easily from the defining properties of the fundamental class, see the Fundamental Class-Proposition 194.2, that

$$\deg(f) \;=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving,} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$ □

The following lemma generalizes several statements of the Sphere–Degree Lemma 151.4 and it can be viewed as the topological analog of the Manifold Degree Lemma 186.2.

Lemma 195.2. (Manifold Degree Lemma) Let $n \in \mathbb{N}$ and let L, M and N be compact oriented connected non-empty n-dimensional topological manifolds.

(1) $\deg(\mathrm{id}_M) = 1$.
(2) For every continuous map $f\colon M \to N$ such that $f(\partial M) \subseteq \partial N$ we have the following equality:[2089] $f_*([\partial M]) \;=\; \deg(f\colon M \to N) \cdot [\partial N] \;\in\; \mathrm{H}_{n-1}(\partial N)$.
In particular, if ∂M and ∂N are connected and non-empty, then we obtain the equality $\deg(f\colon \partial M \to \partial N) = \deg(f\colon M \to N)$.
(3) Let $f\colon M \to N$ be a continuous map with $f(\partial M) \subseteq \partial N$. If f is not surjective, then $\deg(f) = 0$.
(4) Let $f, g\colon M \to N$ be two continuous maps with $f(\partial M) \subseteq \partial N$ and $g(\partial M) \subseteq \partial N$. If f is homotopic to g, as a map of pairs $(M, \partial M) \to (N, \partial N)$, then $\deg(f) = \deg(g)$.
(5) For any two continuous maps $f\colon L \to M$ and $g\colon M \to N$ with $f(\partial L) \subseteq \partial M$ and $g(\partial M) \subseteq \partial N$ we have $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Proof.

(1) This statement is trivial.
(2) The proof of this statement is basically the same as the proof of the Manifold Degree Lemma 186.2 (2), we just need to replace the Fundamental Class-of-∂-Proposition 185.18 by the Fundamental Class-of-∂-Proposition 194.13.
(3) The proof of this statement is basically the same as the proof of the Manifold Degree Lemma 186.2 (2), we just need to replace the Manifold-Top-H_*-Theorem 183.18 by the Manifold-Top-H_*-Theorem 193.1.
(4) This statement follows immediately from the Homotopy-H_*-Proposition 144.12.
(5) Exactly as in the proof of the Sphere–Degree Lemma 151.4 (4) we have

$$(g \circ f)_*([L]) \;=\; g_*(f_*([L])) \;=\; g_*(\deg(f) \cdot [M]) \;=\; \deg(g) \cdot \deg(f) \cdot [N].$$

It follows that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. ■

[2089] This statement can be viewed as a generalization of the Boundary Degree Proposition 47.9.

195.2. **Local and global degrees.** In the S^n-Local-Global Degree Proposition 152.5, in the Degree-via-Local Orientations Proposition 152.11 and in the Local-Global Degree Proposition 186.8 we already saw that in some settings we can calculate the "global" degree of a continuous map by "local" degrees. We now want to generalize these results to our setting.

Before we can formulate this result we need to define the notion of a local degree. At this point it is convenient to recall Definition 152.2:

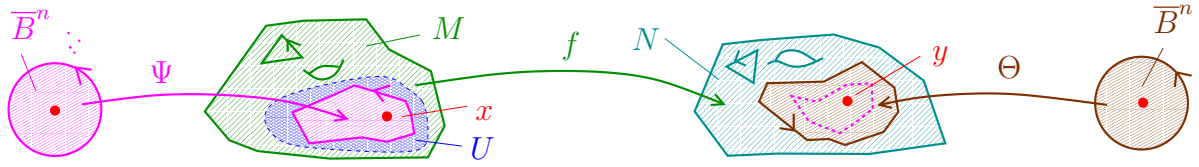
Definition 195.3. We say that a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

With this definition we can now introduce our new notion of a local degree:

Definition 195.4. Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional topological manifolds and let $x \in M \setminus \partial M$ such that $f(x) \in N \setminus \partial N$. We assume that x admits a nice neighborhood U. We set $y := f(x)$. We pick an orientation-preserving embedding $\Theta\colon \overline{B}^n \to N$ with $\Theta(0) = y$ and we pick an orientation-preserving embedding $\Psi\colon \overline{B}^n \to M$ with $\Psi(0) = x$ and such that $\Psi(\overline{B}^n) \subseteq U \cap f^{-1}(\Theta(\overline{B}^n))$. We define the local degree $\deg(f,x)$ as follows:

$$\deg(f,x) := \deg\big((\Theta^{-1} \circ f \circ \Psi)_*\colon \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \to \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})\big).$$

In the Local Degree Lemma 195.5 below we will see that $\deg(f,x)$ is well-defined.



Lemma 195.5. (Local Degree Lemma) Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional topological manifolds such that $f(\partial M) \subseteq \partial N$.

(1) The local degree, as defined above, is well-defined, i.e. independent of the choice of Ψ and Θ.

(2) Let $x \in M \setminus \partial M$. If f is a local homeomorphism around x, then x is nice and

$$\deg(f,x) = \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

(3) If $f\colon M \to N$ is actually a smooth map between smooth manifolds and if $x \in M \setminus \partial M$ is a point at which f is a local diffeomorphism, then the above definition of the local degree agrees with the definition given in Definition 46.1.

(4) For $M = N = S^n$ the above local degree agrees with the definition given in Definition 152.3.

Proof.

(1) Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional topological manifolds and let $x \in M \setminus \partial M$ such that $f(x) \in N \setminus \partial N$. We assume that x admits a nice neighborhood U. We set $y := f(x)$. Let $\Theta\colon \overline{B}^n \to N$ be an orientation-preserving embedding with $\Theta(0) = y$ and let $\Psi\colon \overline{B}^n \to M$ an orientation-preserving embedding

with $\Psi(0) = x$ and such that $\Psi(\overline{B}^n) \subseteq U \cap f^{-1}(\Theta(\overline{B}^n))$. We now suppose that we have alternative orientation-preserving embeddings $\widetilde{\Psi}$ and $\widetilde{\Theta}$. By possibly comparing with embeddings with smaller images we might as well assume that $\Psi(\overline{B}^n) \subseteq \widetilde{\Psi}(\overline{B}^n)$ and that $\Theta(\overline{B}^n) \subseteq \widetilde{\Theta}(\overline{B}^n)$.

We now have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{H}_n(\Psi(\overline{B}^n), \Psi(\overline{B}^n)\setminus\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(\Theta(\overline{B}^n), \Theta(\overline{B}^n)\setminus\{x\}) \\
 & \overset{\Psi_*}{\nearrow}_{\cong} & \downarrow\cong & & \downarrow\cong \quad \overset{\Theta_*}{\nwarrow}_{\cong} \\
\mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) & & & & \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) \\
 & \underset{\Psi_*}{\searrow} & & & \quad \underset{\Theta_*}{\swarrow} \\
\downarrow{\scriptstyle (\widetilde{\Psi}^{-1}\circ\Psi)_*} & & \mathrm{H}_n(U, U\setminus\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(N, N\setminus\{y\}) \quad \downarrow{\scriptstyle (\widetilde{\Theta}^{-1}\circ\Theta)_*} \\
 & \overset{\widetilde{\Psi}_*}{\nearrow} & \uparrow\cong & & \uparrow\cong \quad \overset{\widetilde{\Theta}_*}{\nwarrow} \\
\mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) & & & & \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}). \\
 & \underset{\cong}{\searrow}^{\widetilde{\Psi}_*} & & & \quad \underset{\cong}{\swarrow}^{\widetilde{\Theta}_*} \\
 & & \mathrm{H}_n(\widetilde{\Psi}(\overline{B}^n), \widetilde{\Psi}(\overline{B}^n)\setminus\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(\widetilde{\Theta}(\overline{B}^n), \widetilde{\Theta}(\overline{B}^n)\setminus\{x\})
\end{array}$$

Since $\Psi, \widetilde{\Psi}, \Theta$ and $\widetilde{\Theta}$ are orientation-preserving we see that the vertical maps to the left and right are the identity. It now follows that the definitions of the local degree with the two different choices of orientation-preserving embeddings agree.

(2) This statement follows easily from the definitions.

(3) This statement follows easily from Statement (2), the Smooth-Topological Orientation Proposition 191.25 and the Orientations–Preserved Lemma 191.26.

(4) Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. We make a few preparations:

- We pick a nice neighborhood U of x and we set $y = f(x)$.
- We pick an orientation-preserving embedding $\Theta\colon \overline{B}^n \to S^n$ with $\Theta(0) = y$ and we pick an orientation-preserving embedding $\Psi\colon \overline{B}^n \to S^n$ with $\Psi(0) = x$ and such that $\Psi(\overline{B}^n) \subseteq U \cap f^{-1}(\Theta(\overline{B}^n))$.
- We denote by $\xi \in \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})$ the generator corresponding to the standard orientation of $\overline{B}^n$.

We consider the following diagram:

$$\begin{array}{ccccccc}
\overset{[S^n]}{\mathrm{H}_n(S^n)} & \underset{\cong}{\overset{\rho_x}{\to}} & \mathrm{H}_n(S^n, S^n\setminus\{x\}) & \xleftarrow{\cong} & \mathrm{H}_n(U, U\setminus\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\setminus\{y\}) \quad \underset{\cong}{\overset{\rho_y}{\leftarrow}} \quad \overset{[S^n]}{\mathrm{H}_n(S^n)}. \\
 & & & & \uparrow{\scriptstyle \Psi_*} & & \uparrow{\scriptstyle \Theta_*} \\
 & & & & \mathrm{H}_n(\overline{B}^n, \underset{\xi}{\overline{B}^n} \setminus \{0\}) & \xrightarrow{(\Theta^{-1}\circ f\circ\Psi)_*} & \mathrm{H}_n(\overline{B}^n, \underset{\xi}{\overline{B}^n} \setminus \{0\})
\end{array}$$

The statement follows easily from the following observations:

- The composition of the top horizontal maps is, by definition, given by the multiplication of the local degree from Definition 152.3.
- The bottom horizontal map is given by multiplication by the "new" local degree.
- It follows from the definition of the standard orientation of S^n (see the Sphere Orientation Lemma 191.20) and the fact that Ψ and Θ are orientation-preserving

that the images of ξ and $[S^n]$ in $\mathrm{H}_n(S^n, S^n \setminus \{x\})$ and $\mathrm{H}_n(S^n, S^n \setminus \{y\})$ agree. From this observation it follows that the two definitions of local degree agree. ■

It follows from the Local Degree Lemma 195.5 (4) that the following proposition is a generalization of the S^n-Local-Global Degree Proposition 152.5 and the Local-Global Degree Proposition 186.8.

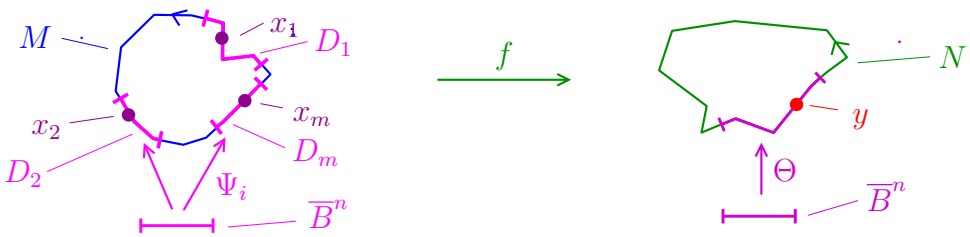
Proposition 195.6. (Local-Global Degree Proposition) Let $f\colon M \to N$ be a continuous map between two closed oriented connected n-dimensional topological manifolds M and N. Let $y \in N$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Then all the points $x_1, \ldots, x_m$ are nice and we obtain the equality

$$\deg(f\colon M \to N) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

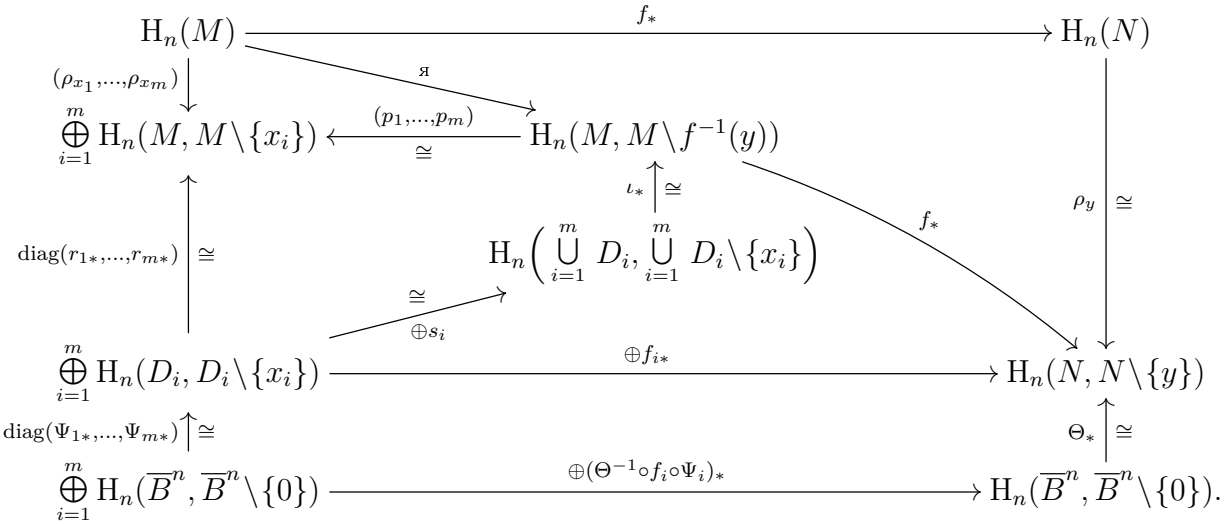
Remark. The Local-Global Degree Proposition 195.6 together with the Local Degree Lemma 195.5 (3) gives us another proof that the original definition of the degree of a smooth map from Definition 46.3 is consistent with the current definition. Also recall that by the Degrees Agree Proposition 86.7 we know that the original (and the new) definition of degree agree with the definition of the degree in Definition 86.2 via de Rham cohomology. □

Proof. Let $n \in \mathbb{N}$, let $f\colon M \to N$ be a continuous map and let $y \in M$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$.

- As we saw in Exercise 3.12, it follows easily from the fact that M is Hausdorff that we can choose pairwise disjoint open neighborhoods $U_1, \ldots, U_m$ for $x_1, \ldots, x_m$. Note that this observation implies already that all the x_i are nice.
- We pick an orientation-preserving embedding $\Theta\colon \overline{B}^n \to N$ with $\Theta(0) = y$.
- For $i = 1, \ldots, m$ we pick orientation-preserving embeddings $\Psi_i\colon \overline{B}^n \to M$ such that $\Psi_i(0) = x_i$ and such that $\Psi_i(\overline{B}^n) \subseteq U_i \cap f^{-1}(\Theta(\overline{B}^n))$.
- For $i = 1, \ldots, m$ we write $D_i := \Psi_i(\overline{B}^n)$.



We consider the following diagram:[2090]



We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i\colon (D_i, D_i\setminus\{x_i\}) \to (N, N\setminus\{y\})$ are given by the restriction of f to D_i.
(2) It follows pretty easily from the functoriality of relative homology groups and elementary considerations that the diagram commutes.
(3) It follows easily from the fact that the D_i are contained in disjoint open subsets of M, the Disjoint Union Embedding Lemma 7.29, the Path-Component–H_*–Lemma 143.21 and the Component Lemma 4.48 (4) and (5) that the diagonal map $\bigoplus_{i=1}^m s_i$ to the center left is an isomorphism.
(4) It follows from the Excision Theorem 148.2 that the vertical map ι_* and the maps r_{i*} are isomorphisms.
(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $(p_1, \dots, p_m)$ is also an isomorphism.

By the Fundamental Class-Proposition 194.2 (2b) it suffices to prove the following claim:

Claim. We have $$\rho_y(f_*([M])) \;=\; \sum_{i=1}^m \deg(f, x_i)\cdot \rho_y([N]) \;\in\; \mathrm{H}_n(N, N\setminus\{y\}).$$

[2090] We use some of the formal notation that we introduced in Section 152.2 for maps out of and into direct sums.

Proof. We verify the claim by performing the following calculation:

$$\begin{aligned}
\rho_y(f_*([M])) &\overset{\text{since the diagram commutes and since the map } \operatorname{diag}(r_{1*},\dots,r_{m*}) \text{ is an isomorphism}}{=} \Big(\bigoplus_{i=1}^m f_{i*}\Big) \circ \operatorname{diag}(r_{1*},\dots,r_{m*})^{-1} \circ (\rho_{x_1},\dots,\rho_{x_m})([M]) \overset{\text{since each } r_{i*} \text{ is an isomorphism}}{=} \\
&= \Big(\bigoplus_{i=1}^m f_{i*}\Big) \circ (r_{1*}^{-1} \circ \rho_{x_1},\dots,r_{m*}^{-1} \circ \rho_{x_m})([M]) \\
&= \sum_{i=1}^m (f_i \circ r_i^{-1} \circ \rho_{x_i})([M]) \overset{\text{by Definition 195.4 of the local degree and since } \Psi_i \text{ and } \Theta \text{ are orientation-preserving}}{=} \sum_{i=1}^m \deg(f,x_i) \cdot \rho_y([N]).
\end{aligned}$$

since we always have the equality $\Big(\Big(\bigoplus_{i=1}^m \varphi_i\Big) \circ (\psi_1,\dots,\psi_m)\Big)(x) = \sum_{i=1}^m (\varphi_i \circ \psi_i)(x)$

■

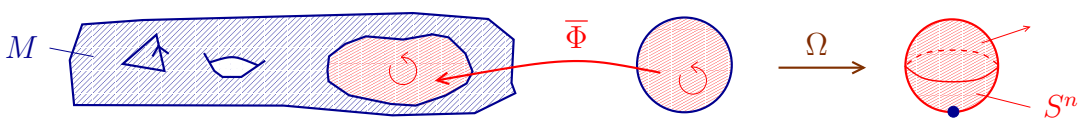
195.3. **Properties of the degree.** We can now restate and reprove the Degree Realization Proposition 46.19 from earlier on.

Proposition 46.19 (Degree Realization Proposition) Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional topological manifold. Given any $k \in \mathbb{Z}$ there exists a continuous map $f\colon M \to S^n$ with $\deg(f) = k$.

Proof. [2091] Let M be a closed oriented connected non-empty n-dimensional topological manifold and let $k \in \mathbb{Z}$. It follows from the Degree-of-Power Lemma 151.5 together with the Suspension-Degree Lemma 151.6 that there exists a continuous map $g\colon S^n \to S^n$ of degree k. It follows from this fact and the Manifold Degree Lemma 195.2 (5) that it suffices to prove the following claim:

Claim. There exists a continuous map $f\colon M \to S^n$ of degree $+1$.

Proof. We pick an orientation-preserving embedding $\Phi\colon \overline{B}^n \to M$. It follows fairly easily from the Quotient-to-Quotient Lemma 8.6, the Compact-Hausdorff Proposition 4.23 (3) together with the Compact Image Lemma 4.15 and the Quotient Topology-Properties Proposition 8.5 (3), that the induced map $\overline{\Phi}\colon \overline{B}^n/S^{n-1} \to M/(M\setminus\Phi(B^n))$ is a homeomorphism. We denote by $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 8.10.



We consider the map $f\colon M \to S^n$ that is given by

$$M \longrightarrow M/(M \setminus \Phi(B^n)) \xrightarrow{\overline{\Phi}^{-1}} \overline{B}/S^{n-1} \xrightarrow{\Omega} S^n.$$
$$=:f$$

It follows almost immediately from the Local-Global Degree Proposition 195.6 together with the Local Degree Lemma 195.5 (2) that f is a map of degree $+1$. ■

[2091] Rather embarrassingly the proof is almost identical to the proofs we had given on Section 46.5 and page 3563.

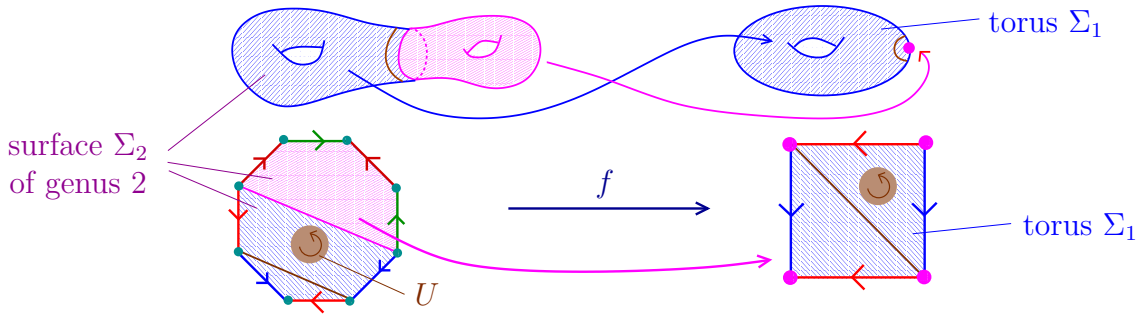
To put the previous proposition into context we recall the following theorem. It says that for continuous maps to S^n the converse to the Manifold Degree Lemma 195.2 (3) holds.

Theorem 46.22. (Hopf Degree Theorem) Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two continuous maps. If[2092] $\deg(f) = \deg(g)$, then the maps f and g are actually homotopic.

In the following discussion, given $g \in \mathbb{N}_0$, we denote as usual by Σ_g the surface of genus g. For the reader's convenience we recall the following lemma and its proof.

Lemma 186.6. (Larger-to-Smaller Genus Lemma) Let $k \in \mathbb{N}_0$. For every $g \geq k$ there exists a degree-one map from Σ_g to Σ_k.

Proof. We prove the lemma for $g = 2$ and $k = 1$. It should be clear how to modify the discussion for other values of $g \geq k$. Let us consider the map from the surface Σ_2 of genus two to the torus Σ_1 that is shown in the figure below.



In fact in the figure we show the map twice, on top we show a slightly informal description of the map, at the bottom we give a more formal description. Note that the hypotheses of the Local-Global Degree Proposition 195.6 are satisfied. Thus we see that $\deg(f) = 1$. ■

The following proposition is the topological analog of the Covering Map-Degree Proposition 186.9.

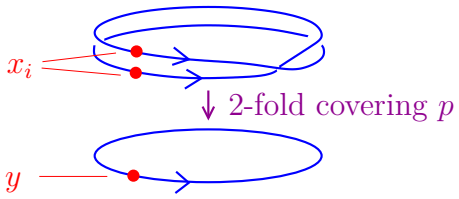
Proposition 195.7. (Covering Map-Degree Proposition) Let $p\colon \widetilde{M} \to M$ be a finite covering between compact oriented connected non-empty topological manifolds. If p is orientation-preserving, then

$$\deg(p) \;=\; [\widetilde{M} : M].$$

Proof. It turns out that there are two quick proofs for the proposition:

(1) We pick $y \in M$. Note that $p^{-1}(y)$ consists of precisely $[\widetilde{M} : M]$ points and at each point in $p^{-1}(y)$ the projection is an orientation-preserving local homeomorphism. Thus each local degree is $+1$ and the desired statement follows from the Local-Global Degree Proposition 195.6.

[2092]When we formulated the Hopf Degree Theorem 46.22 we used the definition of the degree that we gave in Definition 46.3. On page 3776 we pointed out that it follows from the Local-Global Degree Proposition 195.6 that the earlier definition of the degree agrees with the definition of the degree in terms of fundamental classes that we are using right now.



(2) We denote by $p^*\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(\widetilde{M},\partial\widetilde{M})$ the map that we introduced in Definition 165.1, then we see that

$$p_*([\widetilde{M}]) \underset{\text{by the Fundamental Class Covering Proposition 194.11}}{=} p_*(p^*([M])) \underset{\text{by the Transfer–Projection Proposition 165.4}}{=} [\widetilde{M} : M] \cdot [M].$$

■

The following proposition is the topological analog of the Degree One-Epimorphism Proposition 186.10:

Proposition 195.8. (Degree One-Epimorphism Proposition) Let $f\colon M \to N$ be a continuous map between compact oriented connected non-empty n-dimensional topological manifolds such that $f(\partial M) \subseteq \partial N$. If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.

Proof. Let $f\colon M \to N$ be a continuous map between compact oriented connected non-empty n-dimensional topological manifolds such that $f(\partial M) \subseteq \partial N$. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N,Q)$ and $\Gamma = f_*(\pi_1(M,P))$.

Note that by the Covering Existence Theorem 127.10[2093] there exists a path-connected covering $p\colon (\widetilde{N},\widetilde{Q}) \to (N,Q)$ such that $p_*(\pi_1(\widetilde{N},\widetilde{Q})) = \Gamma$ and such that the degree of the covering is given by $[\widetilde{N} : N] = [\pi : \Gamma]$. By the Manifold Covering Proposition 95.4 we know that $\widetilde{N}$ is a topological manifold and by the Topological Orientation-Action Proposition 192.2 (3) we know that we can equip $\widetilde{N}$ with an orientation such that the projection $p\colon \widetilde{N} \to N$ is orientation-preserving. Since p is a local homeomorphism we have $\partial\widetilde{N} = p^{-1}(\partial N)$. Since by design we know that $p_*(\pi_1(\widetilde{N},\widetilde{Q})) = \Gamma = f_*(\pi_1(M,P))$ we obtain from the Map Lifting Criterion 127.5 that the map $f\colon (M,P) \to (N,Q)$ lifts to a continuous map $\widetilde{f}\colon (M,P) \to (\widetilde{N},\widetilde{Q})$. We can summarize the situation in the following simple commutative diagram

$$\begin{array}{ccc} & & \widetilde{N} \\ & \overset{\widetilde{f}}{\nearrow} & \Big\downarrow{\scriptstyle p}\ \text{covering of degree } [\pi\colon\Gamma] \\ M & \xrightarrow[f]{} & N. \end{array}$$

First suppose that Γ is a subgroup of π of infinite index. It follows immediately from $[\widetilde{N} : N] = [\pi : \Gamma]$ and the Covering–Basics Lemma 91.12 (4) that $\widetilde{N}$ is non-compact. Therefore we obtain from the Manifold-Top-H_*-Theorem 193.1 that $\mathrm{H}_n(\widetilde{N}) = 0$. But this shows that $f_* = p_* \circ \widetilde{f}_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is necessarily a finite-index subgroup of the group π. It follows from $[\widetilde{N} : N] = [\pi : \Gamma]$ and the Covering–Basics Lemma 91.12 (4) that $\widetilde{N}$ is compact.

[2093]Here we use that according to the discussion on page 2442 any topological manifold is locally simply connected.

Therefore, by the above we now know that $\widetilde{N}$ is a compact oriented connected n-dimensional topological manifold. We deduce that

$$\pm 1 \;=\; \deg(f) \underset{\substack{\uparrow \\ \text{by the Manifold Degree Lemma 195.2}}}{=} \deg(\widetilde{f}) \cdot \deg(p) \underset{\substack{\uparrow \\ \text{by the Covering Map-Degree Proposition 195.7 (5)}}}{=} \deg(\widetilde{f}) \cdot [\widetilde{N} : N] \;=\; \deg(\widetilde{f}) \cdot [\Gamma : \pi].$$

This shows that $[\Gamma : \pi] = 1$, i.e. $\Gamma = \pi$. ■

We recall the following proposition:

Proposition 186.11. (Smaller-to-Larger Genus Proposition) Let $g, k \in \mathbb{N}_0$. If $g < k$, then there is no degree-one map from Σ_g to Σ_k.

Proof. Suppose that there exists a degree-one map from Σ_g to Σ_k. By the Degree One-Epimorphism Proposition 195.8 we obtain an epimorphism $\pi_1(\Sigma_g) \to \pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By the Surface-π_1-Proposition 101.1 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows from the Rank Comparison Lemma 97.16 that $g \geq k$. ■

We conclude this chapter with several questions. Recall that it follows from the Degree Realization Proposition 46.19 that there exists a degree-one map $S^2 \times S^2 \to S^4$. It is natural to ask whether the converse holds. We remind the reader of the following question which we will answer in the Sphere-to-Product-of-Spheres Lemma 207.5.

Question 186.12. Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$?

Later, in the No Degree-1-Map Corollary 210.17, we will also address the following question:

Question 186.13 Does there exist a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$?

195.4. Exercises for Chapter 195 .

Exercise 195.1. Let $n \in \mathbb{Z}$. We consider the map

$$\begin{array}{rcl} f\colon \overline{B}^2 & \to & \overline{B}^2 \\ z & \mapsto & z^n. \end{array}$$

For each $x \in B^2$ determine the local degree $\deg(f, x)$.

Exercise 195.2. Let M be a compact oriented n-dimensional topological manifold with non-empty boundary. Show that there exists a continuous map $f\colon M \to \overline{B}^n$ such htat $f(\partial M) \subseteq S^{n-1}$ of degree $+1$.
Remark. For $n \geq 2$ one can combine this result with Exercise 195.1 to get maps of arbitrary degree.

Exercise 195.3. Let $f\colon A \to M$ and $g\colon B \to N$ be continuous maps between closed oriented connected non-empty topological manifolds with $\dim(A) = \dim(M)$ and with $\dim(B) = \dim(N)$. We consider the degree of continuous maps between closed oriented connected non-empty topological manifolds of the same dimension as introduced in Definition 195.1. Show that

$$\deg(f \times g\colon A \times B) \;=\; \deg(f\colon A \to M) \cdot \deg(g\colon B \to N).$$

Remark. For smooth manifolds we gave a very very different proof in Exercise 46.18.

Exercise 195.4. Does the converse to the Degree One-Epimorphism Proposition 195.8 hold? More precisely, let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds such that $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism. Does it follow that $\deg(f) = \pm 1$?

Exercise 195.5. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g.

(a) Let $g \geq 2$. Does there exist a continuous self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?
(b) Let $g \geq 1$ and let $k \geq 2$. Does there exist a continuous self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?

Remark. This exercise is identical to Exercise 186.3.

Exercise 195.6. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. Show that if there exists a continuous map $f\colon \Sigma_g \to \Sigma_k$ of *non-zero degree*, then $g \geq k$.
Remark. This exercise (which is identical to Exercise 195.6) concludes the proof of the Smaller-to-Larger Genus Proposition 186.11.

Exercise 195.7. Let $f\colon M \to N$ be a continuous map between two closed oriented non-empty n-dimensional topological manifolds. Suppose that there exists a compact topological manifold W with $\partial W = M$ and a topological space $g\colon W \to N$ with $g|_M = f$. Show that $\deg(f) = 0$.
Remark. This exercise is almost identical to Exercise 46.12, but now you should be able to give a much shorter proof.

Exercise 195.8. Let $f\colon M \to N$ be a continuous map between two closed connected non-empty n-dimensional topological manifolds. Furthermore let $p\colon \widetilde{N} \to N$ be a finite covering. As in the Topological-Pullback Proposition 7.22 we consider the pullback

$$\widetilde{M} \;:=\; f^*\widetilde{N} \;=\; \{(x,y) \in M \times \widetilde{N} \mid f(x) = p(y)\}$$

together with the following commutative diagram:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[=:F]{(x,y)\mapsto y} & \widetilde{N} \\ {\scriptstyle =:q}\big\downarrow{\scriptstyle (x,y)\mapsto x} & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

Note that by the Pullback-Covering Lemma 92.12 we know that q is again a covering with $\deg(q) = \deg(p)$.

(a) We suppose that M and N are oriented. We equip the topological manifolds $\widetilde{M}$ and $\widetilde{N}$ with the corresponding orientations given by the Manifold Covering Proposition 95.4. Show that $\deg(F\colon \widetilde{M} \to \widetilde{N}) = \deg(f\colon M \to N)$.
(b) Suppose that the induced map $f_*\colon \mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(N;\mathbb{F}_2)$ is an isomorphism. Show that $F_*\colon \mathrm{H}_n(\widetilde{M};\mathbb{F}_2) \to \mathrm{H}_n(\widetilde{N};\mathbb{F}_2)$ is also an isomorphism.

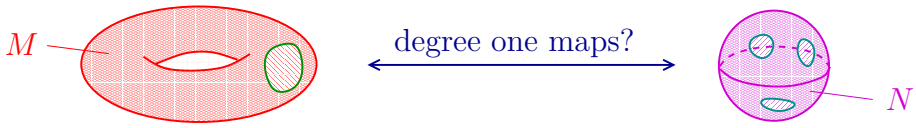
Remark. This exercise is almost identical to Exercise 186.6.

Exercise 195.9. Let M be the torus minus one open disk and let N be S^2 minus three open disks.

(a) Show that there is no degree one map $f\colon M \to N$.

(b) Show that there is no degree one map $g\colon N \to M$.
(c) Bonus question: Why is the map $g\colon N \to M$ given by the gluing of two boundary components not a counter example to (b)?

Remark. This exercise is identical to Exercise 186.7.



Exercise 195.10. Given $m \in \mathbb{N}$ we denote by N_m the non-orientable surface of genus m from Definition 9.19. Let $g \geq k$. Show that there exists a continuous map $f\colon N_g \to N_k$ such that the induced map $\mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$ is an isomorphism.
Remark. This exercise is identical to Exercise 186.8.

Exercise 195.11. Let $f\colon M \to N$ be a continuous map between two closed *non-orientable* connected non-empty n-dimensional topological manifolds such that the induced map $f_*\colon \mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(N;\mathbb{F}_2)$ is an isomorphism. Does it follow that the induced map $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism?

Exercise 195.12. Let X be a topological space. Given $\varphi \in \mathrm{H}_k(X;\mathbb{R})$ we define

$$\|\varphi\|_{\mathbb{R}} := \inf\left\{ \sum_{i=1}^{n} |r_i| \,\middle|\, \sigma = \underbrace{\sum_{i=1}^{n} \sigma_i \otimes r_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i \neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X;\mathbb{R}) \right\}.$$

Let M be a closed oriented connected non-empty n-dimensional topological manifold M. Let $[M] \in \mathrm{H}_n(M)$ be the fundamental class and let $i\colon \mathrm{H}_1(M) \xrightarrow{\cong} \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{R})$ be the homomorphism from the Homology-$\mathbb{Z}$-Coefficients Lemma 160.10 and the Homology-Coefficients-Functor Lemma 160.11. We define the *simplicial volume of* M as

$$\|M\|_{\mathbb{R}} := \|i_*([M])\| \in \mathbb{R}_{\geq 0}.$$

(a) Show that $\|S^1\|_{\mathbb{R}} = 0$.
(b) Let $n \in \mathbb{N}_{\geq 2}$. What is $\|S^n\|_{\mathbb{R}}$?
(c) Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. Show that

$$\|N\|_{\mathbb{R}} \leq \frac{1}{\deg(f)} \cdot \|M\|_{\mathbb{R}}.$$

(d) Let M and N be two closed oriented connected non-empty n-dimensional topological manifolds. We suppose that M and N are in fact homotopy equivalent. Show that $\|M\|_{\mathbb{R}} = \|N\|_{\mathbb{R}}$.
(e) Let N be a closed oriented non-empty n-dimensional topological manifold. Furthermore let $p\colon M \to N$ be a d-fold covering. Show that $\|M\|_{\mathbb{R}} = d \cdot \|N\|_{\mathbb{R}}$.
Hint. Use the transfer map that we introduced in Definition 165.1.
(f) Let $g \in \mathbb{N}_0$. As usual we denote by Σ_g the surface of genus $g \geq 1$. Show that $\|\Sigma_g\|_{\mathbb{R}} \leq 2g - 2$.
Hint. Make use of (e).
Remark. In [**Frig2017**, Corollary 7.5] or alternatively [**Thu1978**, Theorem 6.2] it is shown that we also have the reverse inequality, i.e. we have $\|\Sigma_g\|_{\mathbb{R}} = 2g - 2$.

(g) Let $m, n \in \mathbb{N}$. Show that there exists a $C_{m,n}$ such that for any closed oriented connected topological manifolds M and N, of dimensions m and n, we have

$$\|M \times N\|_{\mathbb{R}} \leq C_{m,n} \cdot \|M\|_{\mathbb{R}} \cdot \|N\|_{\mathbb{R}}.$$

Hint. Use the shuffle map ∇ from Definition 164.23.

Remark. The invariant $\|\varphi\|_{\mathbb{R}}$ is the "real analog" of the invariant $\|\varphi\|_{\mathbb{Z}}$ introduced in Exercise 151.11.

Remark. This exercise is almost identical to Exercise 186.9.

Exercise 195.13. What is the degree of the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_n] & \mapsto & [z_0^2 : \cdots : z_n^2] \end{array}?$$

Exercise 195.14. Let $n \in \mathbb{N}$ be odd. Recall that by the Real Projective Space Orientability Lemma 31.46 we know that $\mathbb{R}\mathrm{P}^n$ is a closed orientable smooth manifold. Determine the degree of the orientation-preserving map $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ that is given by $x \mapsto [x]$.

Exercise 195.15. Let $n \in \mathbb{N}_{\geq 2}$. In the Matrix Group-Smooth Manifold Proposition 32.10 we showed that $\mathrm{SU}(n)$ is a closed orientable smooth manifold of dimension $\frac{1}{2}n \cdot (n-1)$. Show that for any $d \in \mathbb{Z}$ we have

$$\deg \begin{pmatrix} \mathrm{SU}(n) & \to & \mathrm{SU}(n) \\ A & \mapsto & A^d \end{pmatrix} = d^{n-1}.$$

Remark. This exercise is a special case of a much more general calculation of the degree of a power map on a Lie group in terms of the dimension of a maximal torus. We refer to [**HoM2020**, Lemma 6.92] for details.

Part XXI

Singular Cohomology

In this part of the lecture notes we introduce the singular cohomology groups $\mathrm{H}^i(X;G)$ of a topological space X with coefficients in an abelian group G. As groups they are only mildly interesting, since they are fully determined by the ordinary homology groups of X. What makes cohomology interesting is that if we work with cohomology groups with coefficients in a commutative ring R, then we obtain the cup product

$$\cup\colon \mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) \ \to \ \mathrm{H}^{k+l}(X;R),$$

which gives us the tools to handle many problems, which cannot be dealt with by homology.

This part of the lecture notes is organized as follows:

- In the chatty Chapter 196 we will outline several problems which homology cannot deal with, but which will eventually be dealt with using cohomology.
- In the purely algebraic Chapter 197 we will introduce the algebraic notions of cochain complexes and dual complexes.
- In Chapter 198 we will use the tools from Chapter 197 to introduce the cohomology groups $\mathrm{H}^i(X;G)$ of a topological space X with coefficients in an abelian group G.
- In Chapter 199 we will introduce the Kronecker pairing which relates homology to cohomology. We will also introduce cellular cohomology which allows us to calculate cohomology groups of CW-complexes.
- In Chapter 200 we will state and prove the Excision-in-H^*-Theorem 200.1 and the Mayer–Vietoris-H^*-Theorem 200.11 which are the cohomological analogs of the earlier similar sounding statements in homology.
- In Chapter 201 we will introduce the Ext-groups $\mathrm{Ext}(G,H)$ of two abelian groups and we will use them to state and prove the Universal Coefficient-H^*-Theorem 201.23 which allows us to calculate the cohomology $\mathrm{H}^k(X;G)$ in terms of the homology groups of X.
- In Chapter 202 we prove two other Universal Coefficient Theorem for Cohomology Groups, which are not entirely main stream, but which will play a role on a few occasions.
- In Chapter 203, given a commutative ring R, we will introduce the all important cup product
$$\cup\colon \mathrm{H}^k(M;R) \times \mathrm{H}^l(X;R) \ \to \ \mathrm{H}^{k+l}(X;R)$$
on the cohomology groups of a topological space X. In fact we will give two definitions: The first definition using the Eilenberg-Zilber Theorem 164.2 is more systematic, the second one is more ad hoc, but more elementary and works well for most purposes. Of course we will show that the two definitions agree.
- In Chapter 204 we will introduce the relative cup product for suitable relative cohomology groups.
- In Chapter 205 we will apply the relative cup product to give lower bounds on the Lusternik-Schnirelmann category.
- In Chapter 206 we will introduce the cap product
$$\cap\colon \mathrm{H}^k(M;R) \times \mathrm{H}_l(X;R) \ \to \ \mathrm{H}_{l-k}(X;R)$$
which relates homology and cohomology groups.
- In Chapter 207 we will prove the Cross-Cup-Cap Theorem 207.1 which, as the name suggests, relates the cross product, the cup product and the cap product.

- In Chapter 208 we will prove the Künneth-H*-Theorem 208.1 which calculates, under mild hypotheses, the cohomology groups of a product $X \times Y$ in terms of the cohomology groups of X and Y.

CHAPTER 196

Chirality

In this section we will take a short break from overly abstract concepts and we study chirality of objects in $\mathbb{R}^3$, knots and topological manifolds. We will run into several questions which will motivate us to develop new tools in the study of topological spaces.

This chapter is organized as follows:

- In Section 196.1 we will discuss chiral objects.
- In Section 196.2 we will discuss chirality in the context of knots.
- In Section 196.3 we will discuss chiral topological manifolds.
- In Section 196.4 we will provide exercises for Chapter 196.

In the next Chapter 197 we will start the actual theory by introducing cochain complexes and dual complexes.

196.1. **Chiral objects.** In physics or chemistry an object X in $\mathbb{R}^3$ is called *amphichiral* (or often short *achiral*) if it agrees with its mirror image, otherwise it is called *chiral*. To make this definition more precise we have to explain what we mean by "mirror image" and what we mean by "agree".

First of all, given a hyperplane $H \subseteq \mathbb{R}^3$ through the origin we denote by $\rho_H \colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the hyperplane H.[2094] Sometimes in this chapter we will extend ρ_H to a continuous map from $S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ to itself by setting $\rho_H(\infty) = \infty$.

The following lemma says that any two reflections in a hyperplane through the origin differ by multiplication by a matrix in $\mathrm{SO}(3)$.

Lemma 196.1. (Hyperplane-Reflection Lemma) If H and G are two hyperplanes in $\mathbb{R}^3$ through the origin, then there exists a matrix $A \in \mathrm{SO}(3)$ such that for all $v \in \mathbb{R}^3$ we have

$$\rho_G(v) \;=\; A \cdot \rho_H(v).$$

Sketch of proof. With respect to the standard basis of $\mathbb{R}^3$ the maps ρ_G and ρ_H are represented by orthogonal matrices P and Q with determinant -1. The matrix $A = PQ^{-1}$ then lies in $\mathrm{SO}(3)$ and for any $v \in \mathbb{R}^3$ we have $\rho_G(v) = Pv = A \cdot Qv = A \cdot \rho_H(v)$. ■

The above discussion leads us to the following definitions.

Definition 196.2. Let X be a subset of $\mathbb{R}^3$.

(1) We say another subset Y of $\mathbb{R}^3$ agrees with X if there exists a matrix $A \in \mathrm{SO}(3)$ and a point $P \in \mathbb{R}^3$ such that $X = A \cdot Y + P = \{A \cdot y + P \,|\, y \in Y\}$.

(2) For every hyperplane $H \subseteq \mathbb{R}^3$ we refer to $\rho_H(X)$ as the mirror image of X.[2095]

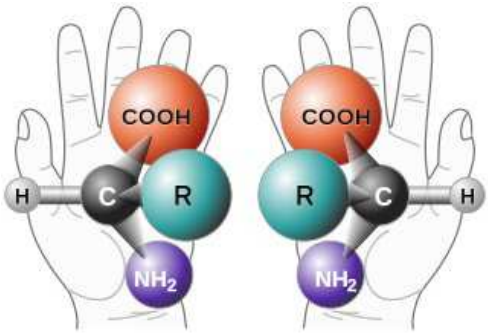
[2094] How is the reflection in a hyperplane defined?

(3) We say X is amphichiral if its mirror image agrees with X, otherwise we say X is chiral.

Examples.

(1) Human hands are chiral, indeed, the left-hand does not agree with the right-hand. In fact the term chirality is derived from the Greek word for hand, $\chi\epsilon\iota\rho$.[2096]

(2) Many molecules are chiral, see e.g. the figure below for an example of a chiral molecule.[2097]



source: `https://en.wikipedia.org/wiki/Chirality`

(3) Any sphere or ball is of course amphichiral. □

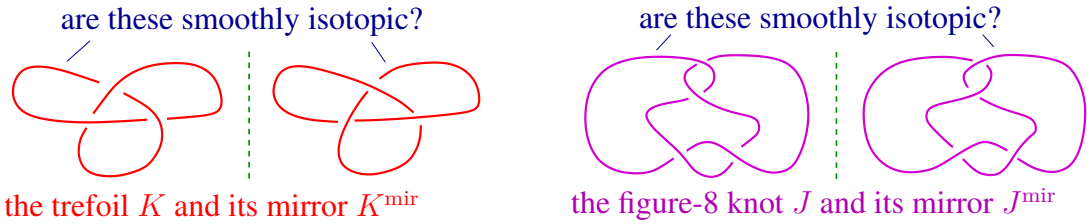
196.2. **Chiral knots.** Now we turn our focus to knots. We recall some of the relevant definitions from Definition 108.3 and Definition 108.23.

Definition 196.3.

(1) A knot is a 1-dimensional submanifold of $S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.

(2) Given a knot $K \subseteq S^3$ the reflection of K in any hyperplane of $\mathbb{R}^4$ is called mirror of K and denoted by K^{mir}. Note that by the S^3-Link Mirror Lemma 108.24 (1) we know that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.

Example. In the figure below we show the trefoil and its mirror and we also show the figure-8 knot and its mirror.



the trefoil K and its mirror K^{mir} the figure-8 knot J and its mirror J^{mir}

In both cases the question arises whether the original knot and its mirror are smoothly isotopic. □

[2095] To be precise, the definition of the mirror image depends on the choice of the hyperplane. By the Hyperplane-Reflection Lemma 196.1 reflections in two different hyperplanes differ by multiplication by a matrix in SO(3), i.e. the mirror images with respect to different hyperplanes agree in the above sense.

[2096] The same word is also the etymological root for the German word "Chirurg".

[2097] In fact, how can you show that the molecule is chiral?

Next we recall Definition 108.26:

Definition 196.4. We say that a knot K in $\mathbb{R}^3$ is amphichiral if it is smoothly isotopic to its mirror image, otherwise we call the knot K chiral.

Example. In the figure on page 2107 we showed that the figure-8 knot J is smoothly isotopic to its mirror image J^*, i.e. the figure-8 knot is amphichiral. □

We recall the following question on which so far we have made zero progress.

Question 108.27. Is the trefoil chiral?

The question that naturally arises is, what does this discussion have to do with homology groups? How can we address this question using the tools of algebraic topology? In the following section we will turn Question 108.27 into a question about homology groups.

196.3. **Chiral topological manifolds.** We now introduce a notion of chirality for oriented topological manifolds;

Definition 196.5. Let M be an oriented topological manifold. We say that M is amphichiral if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is chiral.

In the next proposition, which is a variation on the Fundamental Class-Diffeomorphism Lemma 185.16, we will give several criteria for a self-homeomorphism to be orientation-preserving.

Proposition 196.6. (Orientation Preserving–Characterisation Proposition) Let M be a compact oriented connected n-dimensional topological manifold. Furthermore let $f\colon M \to M$ be a homeomorphism. Then the following three statements are equivalent:

(1) The map f is orientation-preserving in the sense of Definition 191.23.

(2) There exists a point $x \in M \setminus \partial M$ and an open neighborhood U of x, e.g. $U = M$, such that[2098]
$$f_*(\underbrace{\text{orientation at } x}_{\in \mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z})}) = \underbrace{\text{orientation at } f(x)}_{\in \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\};\mathbb{Z})}.$$

(3) We have $f_*([M]) = [M]$.

Suppose that M is in fact a smooth manifold and that f is a diffeomorphism. We equip M with the orientation in the sense of Section 31.2 coming from the Smooth-Topological Orientation Proposition 191.25. Then the above are equivalent to the following statement:

(4) The map f is orientation-preserving in the sense of Section 31.2.

Analogously the above statements also hold if in (1) and (4) we replace "orientation-preserving" by "orientation-reversing" and if in (2) and (3) we add a minus sign.

For the most part the proof of the Orientation Preserving–Characterisation Proposition 196.6 boils down to the following lemma which is also useful in its own right.

Lemma 196.7. (Local Degree Lemma) Let $f\colon M \to N$ be a continuous map between compact oriented connected n-dimensional topological manifolds such that $f(\partial M) \subseteq \partial N$.

[2098] Recall that by the construction of the Topological Manifold-Induced Orientation Lemma 192.1 the orientation on M induces an orientation on any open subset.

Let $x \in M \setminus \partial M$ and let U be an open neighborhood of x with the property that f restricts to a homeomorphism $f\colon U \to f(U)$ and such that $f^{-1}(f(x)) = \{x\}$. Then

$$\begin{aligned} \deg(f\colon M \to N) &= \deg(f_*\colon \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \to \mathrm{H}_n(N, N \setminus \{f(x)\}; \mathbb{Z}) \\ &= \deg(f_*\colon \mathrm{H}_n(U, U \setminus \{x\}; \mathbb{Z}) \to \mathrm{H}_n(f(U), f(U) \setminus \{f(x)\}; \mathbb{Z}). \end{aligned}$$

Proof. Let M and N be compact connected n-dimensional topological manifolds that are equipped with orientations $\{\mu_x\}_{x \in M \setminus \partial M}$ and $\{\nu_x\}_{x \in N \setminus \partial N}$. Furthermore let f, $x \in M \setminus \partial M$ and U as above. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, \partial M; \mathbb{Z}) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_n(N, \partial N; \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_n(N, N \setminus \{f(x)\}; \mathbb{Z}) \\ \uparrow & & \uparrow \\ \mathrm{H}_n(U, U \setminus \{x\}; \mathbb{Z}) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_n(f(U), f(U) \setminus \{f(x)\}; \mathbb{Z}). \end{array}$$

We make the following observations:

(1) The top vertical maps are isomorphisms by the Orientation-Top-H_*-Theorem 193.3.
(2) The bottom vertical maps are isomorphisms by the Excision Theorem 148.2.
(3) It follows from the definition of the fundamental class that the vertical maps send the fundamental classes $[M]$ and $[N]$ to the orientations μ_x respectively $\nu_{f(x)}$.

The desired equalities follow easily from these observations.[2099] ■

Now we can also provide a proof for the Orientation Preserving–Characterisation Proposition 196.6.

Sketch of a proof. The equivalence of (1), (2) and (3) follows quite easily from the Local Degree Lemma 196.7. Now suppose M that is in fact a smooth manifold and that f is a diffeomorphism. The equivalence of (2) and (4) is a straightforward consequence of the Orientations–Preserved Lemma 191.26 and the Excision Theorem 148.2. Once again we leave it to the reader to fill in the details. ■

Examples. Let $n \in \mathbb{N}_0$.

(1) It follows from the Sphere–Degree–Matrix Lemma 151.7 that the map

$$\begin{aligned} \rho\colon S^n &\to S^n \\ (x_0, \dots, x_{n-1}, x_n) &\mapsto (x_0, \dots, x_{n-1}, -x_n), \end{aligned}$$

i.e. the map that is given by reflection in the $x_n = 0$ hyperplane, is orientation-reversing. This shows that the sphere S^n is amphichiral.

(2) Let M be a closed, oriented topological manifold. It follows from (1) and the naturality of the maps in the Künneth Theorem 164.6 that the map

$$\begin{aligned} S^n \times M &\to S^n \times M \\ (x, P) &\mapsto (\rho(x), P), \end{aligned}$$

is orientation-reversing. This shows that the product of the sphere S^n with any closed, oriented topological manifold is amphichiral. □

[2099] Where on earth did we use the hypothesis that $f^{-1}(f(x)) = \{x\}$?

We continue with the following proposition:

Proposition 196.8. (Sphere-Homeomorphism-Orientation Proposition) Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. Then the following equality holds in $\pi_n(S^n, *)$:

$$[f] = \begin{cases} [\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-preserving,} \\ -[\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Remark. Let $n \in \mathbb{N}$. The Sphere-Homeomorphism-Orientation Proposition 196.8 says that every orientation-preserving self-homeomorphism f of S^n with $f(*) = *$ is *homotopic*, rel the base point $*$, to the identity. It is not clear to me whether f is *isotopic*, rel the base point $*$, to the identity. Put differently, can one find a homotopy F from f to id such that each $F_t\colon S^n \to S^n$ is also a homeomorphism? □

Proof. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. By the naturality of the Hurewicz homomorphism, see the Hurewicz Homomorphism Proposition 167.3 (3), we have the following commutative diagram:

$$\begin{array}{ccc} \pi_n(S^n, *) & \xrightarrow{f_*} & \pi_n(S^n, *) \\ {\scriptstyle \Phi_{(S^n,*)}}\downarrow & & \downarrow{\scriptstyle \Phi_{(S^n,*)}} \\ \mathrm{H}_n(S^n) & \xrightarrow{f_*} & \mathrm{H}_n(S^n). \end{array}$$

By the Sphere–Low-π_n-Proposition 134.19 we know that S^n is $(n-1)$-connected. Therefore the vertical maps are isomorphisms by the Hurewicz Theorems 166.10 and 167.9. The corollary now follows from the Orientation Preserving–Characterisation Proposition 196.6 and the simple observation that $[f] = [f \circ \mathrm{id}_{S^n}] = f_*([\mathrm{id}_{S^n}]) \in \pi_n(S^n, *)$. ■

Now we want to relate the notion of amphichiral knots to the notion of amphichiral smooth manifolds. Before we can do so we recall the following definition:

Definition 196.9. Let $K \subseteq \mathbb{R}^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2 \times K$ as provided by the Product Tubular Neighborhood–Existence Theorem 58.4 (2). We refer to $X_K := S^3 \setminus (B^2 \times K)$ as the exterior of K.[2100]



The following proposition together with the Orientation Preserving–Characterisation Proposition 196.6 now turns Question 108.27 into a question about topological spaces.

Proposition 196.10. (Amphichiral Knot Exterior Proposition) Let $K \subseteq \mathbb{R}^3$ be a knot. If K is amphichiral, then the knot exterior X_K is also amphichiral.

[2100] As we pointed out in the Knot Exterior Lemma 185.27, the exterior X_K is a compact orientable smooth manifold and the diffeomorphism type of X_K as an oriented smooth manifold is well-defined, i.e. it is independent of the choice of the tubular neighborhood.

Remark. We had just seen in the Amphichiral Knot Exterior Proposition 196.10 that the exterior of an amphichiral knot is amphichiral. The converse to that statement is also true, i.e. if the exterior of a knot is amphichiral, then the knot is also amphichiral. But this direction is much harder to prove. To the best of my knowledge it follows from the Gordon-Luecke Theorem 126.8. □

Proof. Let $K \subseteq \mathbb{R}^3$ be an amphichiral knot. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) we can pick a tubular neighborhood $\nu K = \overline{B}^2 \times K$. As in Definition 108.30 we consider the knot exterior $X_K := S^3 \setminus (B^2 \times K)$. By the Orientation Preserving–Characterisation Proposition 196.6 it suffices to show that there exists a homeomorphism $f\colon X_K \to X_K$ with $f_*([X_K]) = -[X_K]$.

We denote by ρ the reflection in a hyperplane of $\mathbb{R}^3$. As usual we extend ρ to a self-homeomorphism of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ by setting $\rho(\infty) = \infty$. We write $K^{\mathrm{mir}} = \rho(K)$ and $\nu K^{\mathrm{mir}} = \rho(\nu K)$. By hypothesis there exists a smooth isotopy from K^{mir} to K. By the Isotopy Extension Theorem 44.2 this smooth isotopy extends to a diffeotopy of S^3. This means that we have a smooth map

$$F\colon S^3 \times [0,1] \;\to\; S^3$$

such that $F_0 = \mathrm{id}$, such that each F_t is a diffeomorphism and such that we have the equality $F_1(K^{\mathrm{mir}}) = K$. We write $\Phi := F_1\colon S^3 \to S^3$. Then $\Phi(\nu K^{\mathrm{mir}})$ is also a tubular neighborhood for K and by the Tubular Map–Uniqueness Theorem 58.5 there exists an orientation-preserving diffeomorphism $\Psi\colon S^3 \to S^3$ with $\Psi(\Phi(\nu K^{\mathrm{mir}})) = \nu K$. We write $f := \Psi \circ \Phi \circ \rho$.

Now we consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_3(S^3) & \xrightarrow{\rho_*} & \mathrm{H}_3(S^3) & \xrightarrow{\Phi_*} & \mathrm{H}_3(S^3) & \xrightarrow{\Psi_*} & \mathrm{H}_3(S^3) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathrm{H}_3(S^3, \nu K) & \xrightarrow{\rho_*} & \mathrm{H}_3(S^3, \nu K^{\mathrm{mir}}) & \xrightarrow{\Phi_*} & \mathrm{H}_3(S^3, \Phi(\nu K^{\mathrm{mir}})) & \xrightarrow{\Psi_*} & \mathrm{H}_3(S^3, \nu K) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
\mathrm{H}_3(X_K, \partial X_K) & \xrightarrow{\rho_*} & \mathrm{H}_3(X_{K^{\mathrm{mir}}}, \partial X_{K^{\mathrm{mir}}}) & \xrightarrow{\Phi_*} & \mathrm{H}_3(\Phi(X_{K^{\mathrm{mir}}}), \Phi(\partial X_{K^{\mathrm{mir}}})) & \xrightarrow{\Psi_*} & \mathrm{H}_3(X_K, \partial X_K).
\end{array}$$

$$\mathrm{H}_3(X_K, \partial X_K) \xrightarrow{f_* = (\Psi \circ \Phi \circ \rho)_*} \mathrm{H}_3(X_K, \partial X_K)$$

Here the upper left vertical map is an isomorphism by the long exact sequence of the pair $(S^3, \nu K)$ and the fact that $\mathrm{H}_i(\nu K) = 0$ for $i = 2, 3$ since $\nu K = K \times B^2$ is homotopy equivalent to S^1. Exactly the same argument shows that in fact all upper vertical maps are isomorphisms. The lower vertical maps are induced by the inclusion of pairs of topological spaces. All of them are isomorphisms by the Manifold-Excision Theorem 148.3.

By the Orientation Preserving–Characterisation Proposition 196.6 we are done once we have shown that the composition of three bottom horizontal maps is given by multiplication by -1. By the commutative diagram it suffices to prove the analogous statement for the composition of the top three horizontal maps. It follows from the Sphere–Degree–Matrix Lemma 151.7 that the top-left horizontal map is given by multiplication by -1. It is a consequence of the Homotopy-H_*-Proposition 144.12 that the top-middle horizontal map is

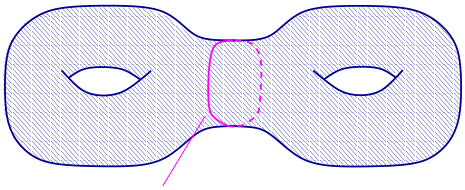
the identity, since Φ is homotopic to the identity. Finally using the Orientation Preserving–Characterisation Proposition 196.6 we see that the top-right horizontal map is the identity since Ψ is orientation-preserving. ■

Now we continue with our discussion of examples of amphichiral smooth manifolds.

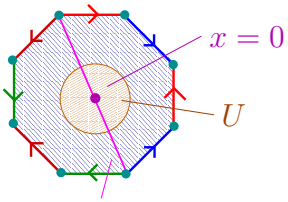
Examples.

(1) We had just seen on page 3791 that topological manifolds of the type $S^n \times M$ are amphichiral. It follows that the torus $S^1 \times S^1$ is amphichiral.

(2) The surface of genus 2 is amphichiral. Pictorially an orientation-reversing self-homeomorphism is given by "reflection in the yz-plane", see the figure below. In the "octagon-picture" an orientation-reversing self-homeomorphism is given by the reflection f in the line illustrated in the figure below. It follows easily from the Orientation Preserving–Characterisation Proposition 196.6 applied to the point $x = 0$ and U an open disk around x together with the Orientation Preserving Lemma 191.24 (3) that the map f is indeed orientation-reversing.

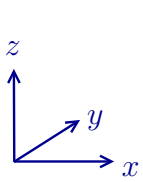


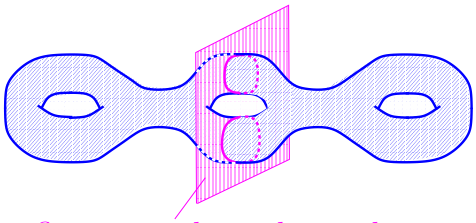
reflection in the yz-hyperplane
is orientation reversing



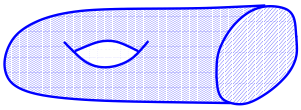
reflection in this line
is orientation reversing

(3) All other compact oriented 2-dimensional smooth manifolds are also amphichiral. We show some of the orientation-reversing maps in the figure below. In Exercise 196.2 we will discuss how these maps can be described rigorously.





reflection in the yz-hyperplane
is orientation reversing



reflection in the xy-hyperplane
is orientation reversing

(4) Example (2) can also be generalized as follows: If M is a compact oriented topological manifold, then its double $\mathrm{D}M = (M \sqcup M')/\sim$ is amphichiral. Indeed, the map $f\colon \mathrm{D}M \to \mathrm{D}M$ that swaps the two copies of M is orientation-reversing. This follows immediately from the definition of the orientation of $\mathrm{D}M$ that we gave in the Double-Orientability Lemma 192.6. □

The long list of examples of amphichiral smooth manifolds raises the following question:

Question 196.11. Are there compact oriented topological manifolds that are chiral?

We end this chapter with a short discussion of lens spaces. We recall the following definition:

Definition 196.12. Let $p \in \mathbb{N}$ and let $q \in \mathbb{N}$ be coprime to p. In Definition 91.4 we defined the lens space

$$L(p,q) \coloneqq S^3/(\mathbb{Z}/p\mathbb{Z}) = \big\{(z_1,z_2) \in \mathbb{C}^2 \,\big|\, |z_1|^2+|z_2|^2 = 1\big\}/(z_1,z_2) \sim (e^{2\pi \mathrm{i}/p}\cdot z_1, e^{2\pi \mathrm{i} q/p}\cdot z_2).$$

Furthermore, in Definition 91.4 we had equipped $L(p,q)$ with the orientation which turns the natural projection $S^3 \to L(p,q)$ into an orientation-preserving map.

Now let $q,r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. In the Lens Space-Diffeomorphism Lemma 91.5 we showed that if $q \equiv \pm r^{\pm 1} \bmod p$, then $L(p,q)$ and $L(p,r)$ are diffeomorphic.[2101] The following lemma is now a slight refinement of the Lens Space-Diffeomorphism Lemma 91.5.

Lemma 196.13. (Lens Space–Diffeomorphisms Lemma) Let $p \in \mathbb{N}$ and let $q,r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$.

(1) In the case that $q \equiv r^{\pm 1} \bmod p$ there exists an orientation-preserving diffeomorphism $L(p,q) \to L(p,r)$.
(2) On the other hand, if $q \equiv -r^{\pm 1} \bmod p$, then there exists an orientation-reversing diffeomorphism $L(p,q) \to L(p,r)$.

Proof. Let $p \in \mathbb{N}$ and let $q,r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \bmod p$. As in the proof of the Lens Space-Diffeomorphism Lemma 91.5 we define a diffeomorphism

$$\Phi\colon S^3 \to S^3$$

as follows:

$$\begin{array}{lllll}
\text{Case 1: } q \equiv r \bmod p. & (z_1,z_2) & \mapsto & (z_1,z_2) \\
\text{Case 2: } q \equiv -r \bmod p. & (z_1,z_2) & \mapsto & (z_1,\overline{z_2}) \\
\text{Case 3: } q \equiv r^{-1} \bmod p. & (z_1,z_2) & \mapsto & (z_2,z_1) \\
\text{Case 4: } q \equiv -r^{-1} \bmod p. & (z_1,z_2) & \mapsto & (z_2,\overline{z_1})
\end{array}$$

In all four cases it is straightforward to verify that the map descends to a diffeomorphism $L(p,q) \to L(p,r)$. It follows fairly easily from the Sphere-Degree-Matrix Lemma 151.7 and the Orientation Preserving–Characterisation Proposition 196.6 that the diffeomorphism Φ is orientation-preserving in cases 1 and 3 and that it is orientation-reversing in cases 2 and 4.[2102] The lemma now follows from the fact that the projections $S^3 \to L(p,q)$ and $S^3 \to L(p,r)$ are orientation-preserving. ∎

Example. Note that $-2^{-1} \equiv -3 \equiv 2 \bmod 5$. Thus we obtain from the Lens Space–Diffeomorphisms Lemma 196.13 that there exists an orientation-reversing diffeomorphism $L(5,2) \to L(5,2)$, i.e. the lens space $L(5,2)$ is amphichiral. □

The argument of the previous example does not work for the lens space $L(3,1)$. So the following question arises:

Question 196.14. Is the lens space $L(3,1)$ chiral?

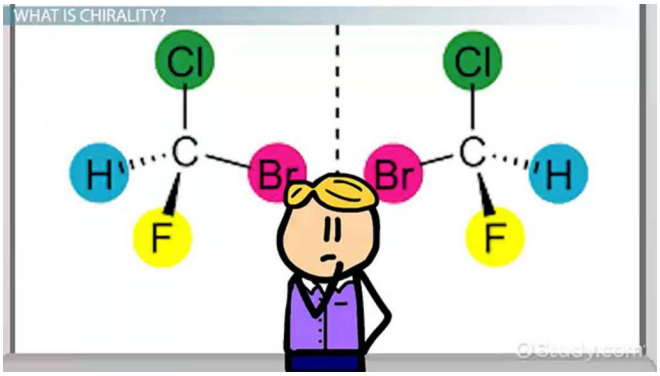
[2101] We still don't know whether the converse holds.

[2102] For example in case 3 the map is given by $(x_1,x_2,x_3,x_4) \mapsto (x_3,x_4,x_1,x_2)$ which is the composition of two reflections in hyperplanes, namely in the $x_1 = x_3$ and the $x_2 = x_4$ hyperplane.

With our present knowledge we are not able to answer Questions 108.27, 196.11 and 196.14. The key to answering these three questions will be the cohomology groups that we introduce in Chapter 198.

196.4. Exercises for Chapter 196 .

Exercise 196.1. Why is the molecule shown in the figure below chiral? How can you show rigorously that the two molecules left and right cannot be turned into one another using a translation and a matrix in SO(3)?



Exercise 196.2.

(a) Give a rigorous proof for the statement that the surface of genus 3 is amphichiral. More precisely, give a precise description of the surface of genus 3 (e.g. as a dodecagon with appropriate identifications) and indicate carefully an orientation-reversing self-homeomorphism.

(b) Do the same exercise for the genus-2 surface with two open disks removed.

Exercise 196.3. Which connected 1-dimensional topological manifolds are chiral?
Remark. In Exercise 51.16 we studied the 2-dimensional case.

CHAPTER 197

Cochain complexes and dual complexes

In the following chapter we will introduce the cohomology groups $\mathrm{H}^k(X;G)$ of a topological space X together with a choice of an abelian group G. The idea is quite similar to the idea for homology with coefficients from Chapter 160:

(1) We consider the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{C}h\mathcal{C}pl\mathcal{X}$ of chain complexes.
(2) We apply a suitable "intermediate'' algebraic functor.
(3) Finally we take homology.

For homology with coefficients the "intermediate" functor we chose was the *covariant* functor $A \mapsto A \otimes G$ for a fixed abelian group G. This idea lead to the homology groups $\mathrm{H}_k(X;G) = \mathrm{H}_k(\mathrm{C}_*(X) \otimes G)$ with coefficients. This time we will use the *contravariant* functor $A \mapsto \mathrm{Hom}(A,G)$ for a fixed abelian group G. This leads us to the cohomology groups $\mathrm{H}^k(X;G)$.

In this chapter we lay the algebraic foundations for carrying out this idea: We will study the Hom-functor $A \mapsto \mathrm{Hom}(A,G)$, we will study cochain complexes (which are basically chain complexes where the boundary maps go upward instead of downward) and dual chain complexes. The reader who is familiar with these notions can also move straight to Chapter 198.

This chapter is organized as follows:

- In Section 197.1 we will introduce the Hom-functor.
- In Section 197.2 we will introduce cochain complexes.
- In Section 197.3 we will introduce dual cochain complexes.
- In Section 197.4 we will provide exercises for Chapter 197.

In the next Chapter 198 we will finally introduce the cohomology groups of a topological space.

197.1. **The Hom-functor.** We start out with the following obvious notation.

Notation 197.1. Given a group A and an abelian group G we write

$$\mathrm{Hom}(A,G) \;=\; \{\text{all group homomorphisms } A \to G\}.$$

We make two observations:

(1) The set of homomorphisms $\mathrm{Hom}(A,G)$ has the structure of an abelian group with the sum of two homomorphisms $\varphi, \psi \colon A \to G$ given by the group homomorphism that is defined by $(\varphi+\psi)(a) \coloneqq \varphi(a) + \psi(a)$.

(2) If R is in fact a commutative ring, then for any group A the set of homomorphisms $\mathrm{Hom}(A,R)$ forms an R-module in an obvious way. More precisely, given a homomorphism $\varphi\colon A\to R$ and $r\in R$ we define $r\cdot\varphi$ to be the homomorphism that is given by $a\mapsto r\cdot\varphi(a)$.

We move straight to induced maps:

Definition 197.2. Let G be an abelian group and let $f\colon A\to B$ be a group homomorphism. Then f induces a homomorphism

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}(B,G) & \to & \mathrm{Hom}(A,G) \\ (\varphi\colon B\to G) & \mapsto & \left(\begin{array}{rcl} \varphi\circ f\colon A & \to & G \\ a & \mapsto & \varphi(f(a)) \end{array}\right). \end{array}$$

Example. Let G be an abelian group. If A is a subgroup of a group B and if $i\colon A\to B$ is the inclusion map, then for each homomorphism $\varphi\colon B\to G$ the homomorphism $i^*\varphi\colon A\to G$ is just the restriction of φ to $A\subseteq B$. □

In Definition 160.1 we introduced the notion of an additive covariant functor. The following definition gives us the obvious analog for contravariant functors:

Definition 197.3. We say that a contravariant functor $F\colon \mathcal{AbGroup}\to\mathcal{AbGroup}$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is additive if for any two abelian groups A and B and any two homomorphisms $\varphi,\psi\colon A\to B$ we have

$$F(\varphi+\psi)=F(\varphi)+F(\psi)\colon F(B)\ \to\ F(A).$$

With this definition we can now formulate the following analog of the Tensor-Functor Lemma 160.4:

Lemma 197.4. (Hom–Contravariant Functor Lemma)

(1) Given an abelian group G the maps

$$\begin{array}{rcl} A & \mapsto & \mathrm{Hom}(A,G) \\ (f\colon A\to B) & \mapsto & (f^*\colon \mathrm{Hom}(B,G)\to\mathrm{Hom}(A,G)) \end{array}$$

define an additive *contravariant* functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{AbGroup}$ of abelian groups. In particular we have

$$\begin{array}{rcll} (\mathrm{id}_A)_* & = & \mathrm{id}_{\mathrm{Hom}(A,G)}, & \text{for all groups } A \\ (g\circ f)^* & = & f^*\circ g^* & \text{for all homomorphisms } f\colon A\to B \text{ and } g\colon B\to C. \end{array}$$

(2) If R is a commutative ring, then the above maps $A\mapsto\mathrm{Hom}(A,R)$ and $f\mapsto f^*$ define a contravariant functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{Mod}_R$ of R-modules.

Proof. Both statements follow easily from the definitions. ■

Lemma 197.5. (Abelian Group-Homomorphism Lemma) Let G be an abelian group.

(1) If F is a free abelian group with basis S, then the homomorphism[2103]

$$\begin{array}{rcl} \mathrm{Hom}(F,G) & \to & \{\text{all maps from the set } S \text{ to } G\} \\ (\varphi\colon F \to G) & \mapsto & \begin{pmatrix} S & \to & G \\ s & \mapsto & \varphi(s) \end{pmatrix} \end{array}$$

is an isomorphism of abelian groups.

(2) For any $n \in \mathbb{N}_0$ and any abelian group G the natural map

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G) & \cong & \ker(G \xrightarrow{\cdot n} G) \\ \varphi & \mapsto & \varphi([1]) \end{array}$$

is an isomorphism of abelian groups.

Proof.

(1) This statement follows immediately from the Definition 97.5 of a free abelian group.

(2) We leave the elementary proof of this statement to the reader. ■

Lemma 197.6. (Hom–Direct Sum-Product Lemma) Let G be an abelian group.

(1) For any family $\{A_i\}_{i\in I}$ of groups the natural map[2104]

$$\begin{array}{rcl} \mathrm{Hom}\left(\bigoplus\limits_{i\in I} A_i, G\right) & \to & \prod\limits_{i\in I} \mathrm{Hom}(A_i, G) \\ f & \mapsto & \prod\limits_{i\in I} f|_{A_i} \end{array}$$

is an isomorphism of abelian groups.

(2) For any group A and any family $\{G_i\}_{i\in I}$ of abelian groups the natural map

$$\begin{array}{rcl} \mathrm{Hom}\left(A, \prod\limits_{i\in I} G_i\right) & \to & \prod\limits_{i\in I} \mathrm{Hom}(A, G_i) \\ f & \mapsto & \prod\limits_{i\in I} (\mathrm{pr}_i \circ f) \\ & & \uparrow \\ & & \text{natural projection to } G_i \end{array}$$

is an isomorphism of abelian groups.

Proof.

(1) Note that for each $j \in I$ we have, by definition of a direct sum, a natural inclusion $\iota_j\colon A_j \to \bigoplus\limits_{i\in I} A_i$. We get an induced map $\iota_j^*\colon \mathrm{Hom}\left(\bigoplus\limits_{i\in I} A_i, G\right) \to \mathrm{Hom}(A_j, G)$. By definition of the product these homomorphisms give rise to a homomorphism

$$\mathrm{Hom}\left(\bigoplus_{i\in I} A_i, G\right) \to \prod_{j\in I} \mathrm{Hom}(A_j, G).$$

Using the Coproduct-of-Abelian Groups Lemma 97.2 one can easily verify that this map is an isomorphism.

(2) This statement follows easily from the definitions. ■

[2103]Note that the group structure on G turns both sides of the statement into a group, so the given map clearly respects this group structure, i.e. the given map is a homomorphism.

[2104]This statement can be viewed as the analog of the Tensor-Direct Sum Lemma 159.15.

Example.

(1) Let $m, n \in \mathbb{N}$. It follows from the Abelian Group-Homomorphism Lemma 197.5 (2) and some elementary arguments that $\operatorname{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$.

(2) Let S be a set. In the Free Abelian Group–Existence Lemma 97.8 we showed that $\mathbb{Z}^{(S)}$ is a free abelian group on the set S. It follows from the Abelian Group-Homomorphism Lemma 197.5 (1) that the homomorphism

$$\begin{array}{rcl} \operatorname{Hom}(\mathbb{Z}^{(S)}, G) & \to & \{\text{all maps from the set } S \text{ to } G\} \\ (\varphi\colon \mathbb{Z}^{(S)} \to G) & \mapsto & \begin{pmatrix} S & \to & G \\ s & \mapsto & \varphi(s) \end{pmatrix} \end{array}$$

is an isomorphism.

(3) Let A be a finitely generated abelian group. Recall that it follows from the Abelian Group-Classification Theorem 97.24 that A is the direct sum of finitely many cyclic groups. Using the Abelian Group-Homomorphism Lemma 197.5 and the Hom–Direct Sum-Product Lemma 197.6 we now fully understand $\operatorname{Hom}(A, G)$.

(4) Let F be a free abelian group.

(a) If F is finitely generated, then it follows from the Abelian Group-Homomorphism Lemma 197.5 and the Hom–Direct Sum-Product Lemma 197.6 that $\operatorname{Hom}(F, \mathbb{Z})$ is again a free abelian group.

(b) If $F = \mathbb{Z}^{(\mathbb{N})}$, then it follows from the Abelian Group-Homomorphism Lemma 197.5 and the Hom–Direct Sum-Product Lemma 197.6 that $\operatorname{Hom}(\mathbb{Z}^{(\mathbb{N})}, \mathbb{Z}) \cong \prod_{i \in \mathbb{N}} \mathbb{Z}$. Note that this group is *not* a free abelian group. This surprising statement is known as Baer's Theorem and was first proved by Reinhold Baer [**Baer1937**, Theorem 12.4]. We refer to [**Schröe2008**] and [**Rot2009**, Theorem 4.17] for two more recent accounts. □

The following lemma is an analog of the Free Module–Tensor-with-Ring Lemma 159.20.

Lemma 197.7. (Free Module–Hom-to-Ring Lemma) Let R be a commutative ring.

(1) Let $k \in \mathbb{N}_0$. The map

$$\begin{array}{rcl} \Theta_k\colon \operatorname{Hom}(\mathbb{Z}^k, R) & \to & R^k \\ (f\colon \mathbb{Z}^k \to R) & \mapsto & (f(\mathfrak{e}_1), \ldots, f(\mathfrak{e}_k)) \end{array}$$

is an isomorphism of R-modules.

(2) Let A be an $(m \times n)$-matrix over $\mathbb{Z}$. We denote by $\varphi\colon \mathbb{Z} \to R$ the ring homomorphism that is given by $\varphi(k) = k \cdot 1_R$. With this notation the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}(\mathbb{Z}^m, R) & \xrightarrow[\cong]{\Theta_m} & R^m \\ \downarrow{\scriptstyle (v \mapsto A \cdot v)^*} & & \downarrow{\scriptstyle w \mapsto \varphi(A^{\mathrm{T}}) \cdot w} \\ \operatorname{Hom}(\mathbb{Z}^n, R) & \xrightarrow[\cong]{\Theta_n} & R^n. \end{array}$$

Proof.

(1) As in the case of the Abelian Group-Homomorphism Lemma 197.5 this statement follows almost immediately from the fact that $\mathbb{Z}^k$ is a free abelian group on the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_k$.

(2) We introduce some notation:
- We denote by $\widetilde{\mathfrak{e}}_1, \dots, \widetilde{\mathfrak{e}}_k$ the standard R-basis of R^k.
- We denote by $\pi_i \colon \mathbb{Z}^m \to \mathbb{Z}$ the projection onto the i-th coordinate.
- Let $\Psi \colon \mathbb{Z}^m \to \mathbb{Z}^n$ be the map that is given by multiplication by $A = (a_{ij})$.

For $i \in \{1, \dots, m\}$ we calculate that we have the following equality in R^n:

$$\begin{array}{lllll} (\Theta_n \circ \Psi^* \circ \Theta_m^{-1})(\widetilde{\mathfrak{e}}_i) & \overset{\text{since } \Theta_m(\pi_i) = \widetilde{\mathfrak{e}}_i}{=} & \Theta_n(\Psi^*(\pi_i)) & \overset{\text{definition of } \Theta_n}{=} & \sum_{j=1}^{n} (\Psi^*\pi_i)(\mathfrak{e}_j) \cdot \widetilde{\mathfrak{e}}_j \\ & \overset{\text{definition of } \Psi}{=} & \sum_{j=1}^{n} \pi_i\Big(\sum_{k=1}^{n} a_{kj} \cdot \mathfrak{e}_k \Big) \cdot \widetilde{\mathfrak{e}}_j & \overset{\substack{\text{by definition of } \pi_i \text{ and} \\ \text{since for } a \in R,\, v \in S^n \text{ we} \\ \text{have } a \cdot v = \varphi(a) \cdot v}}{=} & \sum_{j=1}^{n} \varphi(a_{ij}) \cdot \widetilde{\mathfrak{e}}_j \end{array} \quad \begin{array}{l} \overset{\text{definition of } \Psi^*}{=} \sum_{j=1}^{n} \pi_i(\Psi(\mathfrak{e}_j)) \cdot \widetilde{\mathfrak{e}}_j \\ \overset{\text{definition of } A^{\mathrm{T}}}{=} \varphi(A^{\mathrm{T}}) \cdot \widetilde{\mathfrak{e}}_i. \end{array}$$

■

The following lemma is similar to the Tensor-Split-SES Lemma 160.9.

Lemma 197.8. (Hom-Split-SES Lemma) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups and let G be an abelian group. If the short exact sequence splits in the sense of Definition 145.4[2105], then the dual sequence

$$0 \to \operatorname{Hom}(C, G) \xrightarrow{p^*} \operatorname{Hom}(B, G) \xrightarrow{i^*} \operatorname{Hom}(A, G) \to 0$$

is also exact.

Proof. We start out with the following claim, which already appeared in the proof of the Tensor-Split-SES Lemma 160.9 and which is the content of Exercise 145.10:

Claim. Let $0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between abelian groups. The following two statements are equivalent:

(1) The sequence is exact and the sequence splits.

(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma \colon G \to F$ and $\tau \colon H \to G$ such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_G$.

Proof. The (1)⇒(2) Statement follows almost immediately from the Splitting Lemma 145.5. Conversely the (2)⇒(1)-Statement can easily be verified by hand. ⊞

Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups that splits and let G be an abelian group. We pick homomorphisms $\sigma \colon B \to A$ and $\tau \colon C \to B$ as in the claim. By the Hom–Contravariant Functor Lemma 197.4 we know that $D \mapsto \operatorname{Hom}(D, G)$ is an *additive contravariant* functor. It follows that $\alpha^* \circ \sigma^* = \mathrm{id}_{\operatorname{Hom}(A,G)}$, $\tau^* \circ \beta^* = \mathrm{id}_{\operatorname{Hom}(C,G)}$ and $\sigma^* \circ \alpha^* + \beta^* \circ \tau^* = \mathrm{id}_{\operatorname{Hom}(B,G)}$. Note that this shows that the two homomorphisms $\tau^* \colon \operatorname{Hom}(B, G) \to \operatorname{Hom}(C, G)$ and $\sigma^* \colon \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G)$ have the desired properties for the sequence

$$0 \to \operatorname{Hom}(G, C) \xrightarrow{p^*} \operatorname{Hom}(B, G) \xrightarrow{i^*} \operatorname{Hom}(A, G) \to 0.$$

It follows from the claim that this sequence is exact and that it splits. ■

[2105] Recall that by the Free Abelian Group–Splitting Lemma 97.11 this happens if C is free abelian.

197.2. **Cochain complexes.** Now we introduce the notion of a cochain complex and of its cohomology groups. A cochain complex is exactly the same as a chain complex except that the boundary map now increases the degree by one instead of lowering it by one.

Definition 197.9.

(1) A cochain complex is a sequence[2106]

$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \dots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$

of homomorphisms between abelian groups such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as coboundary maps.

(2) We define the n-th cohomology group of a cochain complex (D^*, δ_*) to be[2107]

$$\mathrm{H}^n(D^*) \coloneqq \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

Example. The readers who studied Part IX on de Rham cohomology are already familiar with a cochain complex: Recall that given a smooth manifold M and given $n \in \mathbb{N}_0$ we defined on Notation 74.5

$$\Omega^n(M) \coloneqq \{\text{all smooth } n\text{-forms on } M\}.$$

Furthermore in Definition 75.5 we introduced the differentials $d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)$. By the Differential-Natural Proposition 75.8 we now obtain a cochain complex. The corresponding cohomology groups are called the de Rham cohomology groups of M. We studied this cochain complex in great detail in Chapter 78. □

Now we will see that most of the definitions and algebraic results for chain complexes can be reinterpreted with minuscule changes for cochain complexes. We start out with the following definition which is modelled on the definition of a chain map in Definition 143.2.

Definition 197.10. A cochain map $f \colon C^* \to D^*$ between two cochain complexes consists of a family $\{f_n \colon \mathrm{C}^n \to D^n\}_{n \in \mathbb{N}_0}$ of homomorphisms that commute with the coboundary maps, i.e. we demand that the following diagram commutes:

$$\begin{array}{ccccccccc}
\xleftarrow{\delta_{n+1}} & C^{n+1} & \xleftarrow{\delta_n} & C^n & \xleftarrow{\delta_{n-1}} & C^{n-1} & \xleftarrow{\delta_{n-2}} & & \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & \\
\xleftarrow[\delta_{n+1}]{} & D^{n+1} & \xleftarrow[\delta_n]{} & D^n & \xleftarrow[\delta_{n-1}]{} & D^{n-1} & \xleftarrow[\delta_{n-2}]{} & &
\end{array}$$

The above definition naturally leads us to the following definition:

Definition 197.11. We refer to category $\mathcal{CoChCplx}$ that is given by

$$\mathrm{Ob}(\mathcal{CoChCplx}) \coloneqq \text{all cochain chain complexes,}$$

and with morphisms

$$\mathrm{Mor}(C^*, D^*) \coloneqq \text{all cochain maps from } C^* \text{ to } D^*.$$

[2106] The reader who conscientiously already went through the part on de Rham cohomology will find that this definition, and many later results, are slight variations on concepts that we already introduced in Chapter 77.

[2107] Evidently it follows from $\delta_n \circ \delta_{n-1} = 0$ that $\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n) \subseteq \ker(\delta_n \colon D^n \to D^{n+1})$.

with the obvious composition of morphism as the category of cochain complexes.

The following lemma is a generalization of the earlier Cochain Complex-H*-Lemma 77.4 and it is the obvious analog of the Chain Complex-H$_*$-Functor Lemma 143.4.

Lemma 197.12. (Cochain Complex-H*-Functor Lemma)

(1) If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^*, then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}^n(C^*) & \to & \mathrm{H}^n(D^*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined homomorphism.

(2) Given $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C^* & \mapsto & \mathrm{H}^n(C^*) \\ (f\colon C^* \to D^*) & \mapsto & (f_*\colon \mathrm{H}^n(C^*) \to \mathrm{H}^n(D^*)) \end{array}$$

define a covariant functor from the category $\mathcal{CoChCplx}$ of cochain complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. The proof is basically identical to the proof of the Cochain Complex-H*-Lemma 77.4 and of the Chain Complex-H$_*$-Functor Lemma 143.4. ■

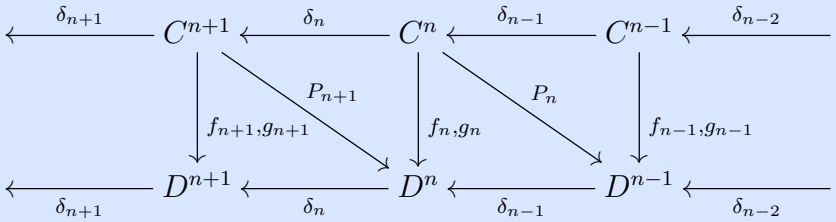
We continue with the following definition which is modelled on the Definitions 144.4 and (2).

Definition 197.13.

(1) Let $f, g\colon C^* \to D^*$ be two cochain maps between two cochain complexes C^* and D^*. A cochain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n\colon \mathrm{C}^n \to D^{n-1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n \;=\; f_n - g_n.$$

The maps are illustrated in the following diagram:



If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are cochain homotopic.[2108]

(2) A cochain map $f\colon C^* \to D^*$ between two cochain complexes C^* and D^* is called a cochain homotopy equivalence if there exists a cochain map $g\colon D^* \to C^*$ such that $f \circ g$ is cochain homotopic to id_{C^*} and $g \circ f$ is cochain homotopic to id_{D^*}.

The following lemma is the cohomological analog of the Chain Homotopy–H$_*$-Lemma 144.5 and the Chain Homotopy Equivalence-H$_*$-Corollary 144.10.

Lemma 197.14. (Cochain Homotopy-H*-Lemma)

[2108]The argument of the Chain Homotopy-Lemma 144.6 shows of course that being cochain homotopic is an equivalence relation on the set of all cochains $C^* \to D^*$.

(1) If two cochain maps are cochain homotopic, then they induce the same map on cohomology groups.
(2) Every cochain homotopy equivalence induces an isomorphism of cohomology groups.

Proof. Statement (1) is proved the same way as the Chain Homotopy–H_*-Lemma 144.5. The proof of Statement (2) is basically the same as the proof of the Chain Homotopy Equivalence-H_*-Corollary 144.10. ■

One of the most important algebraic constructions in homology theory was that we could associate to a short exact sequence of chain complexes a long exact sequence of homology groups. We have an almost identical construction for short exact sequences of cochain complexes and their cohomology groups. We will discuss this construction in detail:

Definition 197.15. Let

$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$

be a short exact sequence of cochain complexes. This hypothesis means that for each $n \in \mathbb{N}_0$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{n+1} & \xrightarrow{f_{n+1}} & B^{n+1} & \xrightarrow{g_{n+1}} & A^{n+1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \\ 0 & \longrightarrow & C^{n} & \xrightarrow{f_{n}} & B^{n} & \xrightarrow{g_{n}} & A^{n} & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact. Now we will define a connecting homomorphism

$$\delta_n \colon \mathrm{H}^n(A) \to \mathrm{H}^{n+1}(C).$$

The definition is almost verbatim the same as Definition 145.8 of the connecting homomorphism in homology. More precisely, we perform the following steps:

(1) Let $z \in \mathrm{H}^n(A)$.
(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) \coloneqq [\gamma] \in \mathrm{H}^{n+1}(C)$.

The following proposition is the obvious analog of the Connecting Homomorphism Lemma 145.9:

Lemma 197.16. (Cohomology–Connecting Homomorphism Lemma)

(1) The connecting homomorphisms are well-defined and they are indeed homomorphisms.[2109]
(2) The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of cochain complexes give rise to commuting diagrams of cohomology groups involving the connecting homomorphisms.

Proof. The proof is almost identical to the proof of the Connecting Homomorphism Lemma 145.9. We leave it to the reader to fill in the details. ■

The following proposition is the obvious analog of the LES-Proposition 145.10:

[2109] Furthermore, if the cochain complexes are cochain complexes over a commutative ring R, then the connecting homomorphism is in fact an R-module homomorphism.

Proposition 197.17. (Cohomology–LES Proposition) If

$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$

is a short exact sequence of cochain complexes, then

$$\ldots \to \mathrm{H}^n(C) \xrightarrow{f^*} \mathrm{H}^n(B) \xrightarrow{g^*} \mathrm{H}^n(A) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(C) \xrightarrow{f^*} \mathrm{H}^{n+1}(B) \xrightarrow{g^*} \ldots$$

forms an exact sequence.

Proof. The proof is almost identical to the proofs of the LES-Proposition 145.10, we just have to flip the indices. We leave it to the reader to fill in the details. ■

Finally the following lemma says that cohomology commutes with products. This lemma can be viewed as the obvious generalization of the earlier Product–H^*-Lemma 77.7 and a cohomological analog of the Direct Sum-H_*-Lemma 143.7:

Lemma 197.18. (Product–H^*-Lemma) If $\{C_a\}_{a\in A}$ is a family of cochain complexes, then the natural projections $\prod_{a\in A} C_a \to C_b$, $b \in A$ (which come from the definition of a product) induce for each $n \in \mathbb{N}_0$ an isomorphism[2110]

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \xrightarrow{\cong} \prod_{a\in A} \mathrm{H}^n(C_a).$$

Proof. Let $n \in \mathbb{N}_0$. Note that for each $b \in A$ the projection $\prod_{a\in A} C_a \to C_b$ induces a homomorphism

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \to \mathrm{H}^n(C_b).$$

It follows from the Definition 6.1 of a product that we get a well-defined homomorphism

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \to \prod_{b\in A} \mathrm{H}^n(C_b).$$

We leave it to the reader to verify that this map is in fact an isomorphism. ■

197.3. **Dual cochain complexes.** Now we will present the arguably most important source of cochain complexes:

Definition 197.19. Let

$$(C_*, \partial_*)\colon \quad \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex and let G be an abelian group. For each $n \in \mathbb{N}_0$ we write

$$\delta_n \coloneqq \partial_{n+1}^*\colon \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G).$$

We obtain the following cochain complex[2111]

$$\leftarrow \mathrm{Hom}(C_{n+1}, G) \xleftarrow{\delta_n} \mathrm{Hom}(C_n, G) \xleftarrow{\delta_{n-1}} \ldots \xleftarrow{\delta_1} \mathrm{Hom}(C_1, G) \xleftarrow{\delta_0} \mathrm{Hom}(C_0, G) \leftarrow 0.$$

We refer to it as cochain complex dual to (C_*, ∂_*) and we denote it by $\mathrm{Hom}(C_*, G)$. We define the n-th cohomology group of the chain complex (C_*, ∂_*) with G-coefficients as the cohomology of the cochain complex $(\mathrm{Hom}(C_*, G), \delta_*)$, i.e. we define

$$\mathrm{H}^n(C; G) \coloneqq \frac{\ker(\delta_n\colon \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{Hom}(C_{n-1}, G) \to \mathrm{Hom}(C_n, G))}.$$

[2110] The same statement also holds with direct sums instead of products. The reason we formulated it for direct products is that that is what we will study later on.

[2111] By the Hom–Contravariant Functor Lemma 197.4 we have $\delta_n \circ \delta_{n-1} = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0$, i.e. this is indeed a cochain complex.

Remark.

(1) We had just defined $\delta_n := \partial_{n+1}^*$, i.e. the coboundary map in the cochain complex is the dual of the boundary map in the chain complex. This is the same definition that is used in most textbooks on algebraic topology, see e.g. [**Mun1984**, p. 251] or [**Hat2002**, p. 189]. In the book by Bredon [**Bre1993**, p. 321] the coboundary is defined as $\delta_n := (-1)^{n+1} \cdot \partial_{n+1}^*$.

The different conventions do not affect the definition of the cohomology groups, but they will affect the sign conventions in some formulas later on. In particular if one cites results from other books one needs to be careful about the sign conventions used.

(2) If R is a commutative ring and if (C_*, ∂_*) is a chain complex, then it follows immediately from the Hom–Contravariant Functor Lemma 197.4 that the corresponding cohomology groups $\mathrm{H}^n(C; R)$ come naturally with the structure of R-modules. □

The following lemma is related in spirit to the Functor-applied-to-Chain Complex Lemma 160.3.

Lemma 197.20. (Dual Cochain Map Lemma) Let G be an abelian group.

(1) Let $f = \{f_n \colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map between chain complexes C_* and D_*, then the maps

$$\begin{array}{rcl} f^* \colon \mathrm{Hom}(D_n, G) & \to & \mathrm{Hom}(C_n, G) \\ (\varphi \colon D_n \to G) & \mapsto & \underbrace{\left(C_n \xrightarrow{f_n} D_n \xrightarrow{\varphi} G\right)}_{=\varphi\circ f_n = f_n^*(\varphi)} \end{array}$$

form a cochain map.

(2) The maps

$$\begin{array}{rcl} C_* & \mapsto & \mathrm{Hom}(C_*, G) \\ f & \mapsto & f^* \end{array}$$

define a *contravariant* functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{CoChCplx}$ of cochain complex.

(3) (a) Let $f, g \colon C_* \to D_*$ be two chain maps. If they are chain homotopic, then the induced cochain maps $f^*, g^* \colon \mathrm{Hom}(D_*, G) \to \mathrm{Hom}(C_*, G)$ are cochain homotopic.

(b) Let $f \colon C_* \to D_*$ be a chain map. If f is a chain homotopy equivalence, then $f^* \colon \mathrm{Hom}(D_*, G) \to \mathrm{Hom}(C_*, G)$ is a cochain homotopy equivalence.

Proof.

(1) Let $f = \{f_n \colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map. This means that for each $n \in \mathbb{N}_0$ we have

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n.$$

It follows from the Hom–Contravariant Functor Lemma 197.4 that

$$\underbrace{\partial_n^*}_{=\delta_{n-1}} \circ f_{n-1}^* \;=\; f_n^* \circ \underbrace{\partial_n^*}_{=\delta_{n-1}}.$$

We have thus shown that f^* is a cochain map.

(2) This statement follows easily from the definitions.

(3) (a) Let $\{P_n \colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g. Recall that this means that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n.$$

We set $Q_n := P^*_{n-1}\colon \operatorname{Hom}(D_n, G) \to \operatorname{Hom}(C_{n-1}, G)$. By the Hom–Contravariant Functor Lemma 197.4 we know that $A \mapsto \operatorname{Hom}(A, G)$ is an *additive contravariant* functor. This implies that

$$\underbrace{P^*_n}_{=Q_{n+1}} \circ \underbrace{\partial^*_{n+1}}_{=\delta_n} + \underbrace{\partial^*_n}_{=\delta_{n-1}} \circ \underbrace{P^*_{n-1}}_{=Q_n} = g^*_n - f^*_n.$$

This shows that the maps $Q_n, n \in \mathbb{N}_0$ form a cochain homotopy.

(b) This statement follows easily from (a). ■

The following lemma plays the role of the Homology-Coefficients-Functor Lemma 160.11. In a nutshell it says that "cohomology is covariant in the coefficients".

Lemma 197.21. (Cohomology-Coefficients-Functor Lemma) Let $f\colon A \to B$ be a homomorphism of abelian groups.

(1) Given a chain complex C_* and $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} A & \mapsto & \mathrm{H}^n(C_*; A) \\ (f\colon A \to B) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^n(C_*; A) & \to & \mathrm{H}^n(C_*; B) \\ [\varphi\colon C_n \to A] & \mapsto & [f \circ \varphi\colon C_n \to B] \end{array} \right) \end{array}$$

form a covariant functor from the category *AbGroup* of abelian groups to the category *AbGroup* of abelian groups.

(2) The induced maps f_* define a natural transformation from the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; A)$ to the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; B)$.

Proof. Let C_* be a chain complex and let $f\colon A \to B$ be a homomorphism of abelian groups. One can easily verify that the maps

$$\begin{array}{rcl} f_*\colon \operatorname{Hom}(C_n, A) & \to & \operatorname{Hom}(C_n, B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map. The two statements of the lemma now follow basically immediately from the definitions. ■

197.4. Exercises for Chapter 197 .

Exercise 197.1.

(a) What is $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z})$?

(b) What is $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q})$?

Exercise 197.2. Is $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ countable or uncountable?

Exercise 197.3. Provide an example of a short exact sequence

$$0 \to C^* \xrightarrow{\varphi} B^* \xrightarrow{\psi} A^* \to 0$$

of cochain complexes such that there exists an $n \in \mathbb{N}_0$ such that the connecting homomorphism $\delta_n\colon \mathrm{H}^n(A) \to \mathrm{H}^{n+1}(C)$ is non-zero.

Exercise 197.4. Provide an example of a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of abelian groups and of an abelian group G such that the dual sequence

$$0 \to \operatorname{Hom}(C,G) \xrightarrow{p^*} \operatorname{Hom}(B,G) \xrightarrow{i^*} \operatorname{Hom}(A,G) \to 0$$

is not exact.

Exercise 197.5. We consider the chain complex

$$C_* \quad = \quad 0 \to \underbrace{\mathbb{Z}^4}_{=C_2} \xrightarrow{\begin{pmatrix} 5 & 0 & 2 & -2 \\ 5 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}} \underbrace{\mathbb{Z}^3}_{=C_1} \xrightarrow{\begin{pmatrix} 3 & -3 & 6 \\ -3 & 3 & -6 \end{pmatrix}} \underbrace{\mathbb{Z}^2}_{=C_0} \to 0.$$

(a) Compute the cohomology groups $\mathrm{H}^n(C_*;\mathbb{Q})$.
(b) Compute the cohomology groups $\mathrm{H}^n(C_*;\mathbb{Z})$.
(c) Compute the cohomology groups $\mathrm{H}^n(C_*;\mathbb{Z}/3\mathbb{Z})$.
(d) Compute the cohomology groups $\mathrm{H}^n(C_*;\mathbb{Q}/\mathbb{Z})$.

Exercise 197.6. Let $f\colon B \to C$ be a homomorphism between abelian groups such that $\operatorname{im}(f)$ is a summand of C. Let $\{k_1,\dots,k_m\}$ be a generating set for $\ker(f)$. Finally suppose that we are given homomorphisms $\varphi_1,\dots,\varphi_m\colon B \to \mathbb{Z}$ such that for all $i,j \in \{1,\dots,m\}$ we have $\varphi_i(k_j) = \delta_{ij}$. Show that the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^m & \to & \operatorname{coker}(f\colon \operatorname{Hom}(C,\mathbb{Z}) \to \operatorname{Hom}(B,\mathbb{Z})) \\ e_i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

Exercise 197.7. Let $\varphi\colon C_1 \to C_0$ be a homomorphism between two finitely generated free abelian groups.

(a) Suppose that A is an $(n \times m)$-matrix that represents φ. Show that the transpose A^{T} represents the map $\varphi^*\colon \operatorname{Hom}(C_0,\mathbb{Z}) \to \operatorname{Hom}(C_1,\mathbb{Z})$.
(b) We consider the chain complex $C_* \coloneqq 0 \to C_1 \xrightarrow{\varphi} C_0 \to 0$. Show that

$$\operatorname{Tor}(\mathrm{H}^1(C_*;\mathbb{Z})) \;\cong\; \operatorname{Tor}(\mathrm{H}_1(C)) \quad \text{and} \quad \mathrm{FH}^1(C_*;\mathbb{Z}) \;\cong\; \mathrm{FH}_1(C).$$

Remark. Statement (b) can be seen as a special case of the Algebraic Universal Coefficient-H*-Theorem 201.22 that we will prove later on.

Exercise 197.8. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of abelian groups and let F_* be a chain complex of free abelian groups.

(a) Show that for each $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\delta_n\colon \mathrm{H}^n(F;C) \;\to\; \mathrm{H}^{n+1}(F;A)$$

such that the following sequence is exact:

$$\dots \;\to\; \mathrm{H}^n(F;A) \xrightarrow{\alpha_*} \mathrm{H}^n(F;B) \xrightarrow{\beta_*} \mathrm{H}^n(F;C) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(F;A) \;\to\; \dots$$

Remark. Such a sequence is sometimes called *Bockstein sequence*.
(b) We consider the short exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{[x]\mapsto[2x]} \mathbb{Z}/4\mathbb{Z} \xrightarrow{[y]\mapsto[y]} \mathbb{Z}/2\mathbb{Z} \to 0$. Determine whether or not the map $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta_1} \mathrm{H}^2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}/2\mathbb{Z})$ in the corresponding Bockstein sequence of $\mathbb{R}\mathrm{P}^3$ is trivial.

CHAPTER 198

The cohomology groups of a topological space

In this chapter we finally introduce the cohomology groups $\mathrm{H}^k(X;G)$ of a topological space X together with a choice of an abelian group G. As we mentioned in the beginning of Chapter 197, the idea is similar to the idea for homology with coefficients:

(1) We consider the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{C}h\mathcal{C}pl\mathcal{X}$ of chain complexes.
(2) We fix an abelian group G. But instead of applying the covariant functor $A \mapsto A \otimes G$ we now apply the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$.
(3) Finally we take (co-) homology of the resulting (co-) chain complex.

The effort to introduce homology with coefficients only produced moderately interesting results. Initially it might appear that this new idea is not particularly promising either. But we will eventually see that cohomology is much more exciting, since it has a product structure which turns cohomology into a ring.

The study of cohomology will keep us busy for a very long time. For many mathematicians it takes a long time till they feel fully comfortable with cohomology groups since elements of cohomology groups are much less visual than elements of homology groups. Perhaps, before we embark on our new adventure it is perhaps good to quote the great John von Neumann:



In mathematics you don't understand things.
You just get used to them.

This chapter is organized as follows:

- In Section 198.1 we will introduce the singular cohomology groups $\mathrm{H}^k(X;G)$ of a topological space X with coefficients in an abelian group G.
- In Section 198.2 we will give examples of explicit cochains and cocycles.
- In Section 198.3 we will provide a few basic calculations of cohomology.
- In Section 198.4 we will study some basic properties of cohomology groups.
- In Section 198.5 we will introduce reduced cohomology groups.
- In Section 198.6 we will provide exercises for Chapter 198.

In the next Chapter 199 we will introduce cellular cohomology and the Kronecker pairing.

198.1. **The definition of singular cohomology groups.** Now we apply the somewhat dry algebra from the last chapter to our favorite chain complex, namely the singular chain complex of a pair of topological spaces:

Definition 198.1. Let (X, A) be a pair of topological spaces and let G be an abelian group.

(1) For each $n \in \mathbb{N}_0$ we write

$$\mathrm{C}^n(X, A; G) \coloneqq \mathrm{Hom}(\mathrm{C}_n(X, A), G) \text{ and } \delta_n \coloneqq \partial^*_{n+1} \colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G).$$

We use the following notation:

(a) We refer to the elements of $\mathrm{C}^n(X, A; G)$ as singular n-cochains, or shorter, as n-cochain.

(b) We refer to the elements in the kernel of $\delta_n \colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G)$ as singular n-cocycles.

(c) We refer to the elements in the image of $\delta_{n-1} \colon \mathrm{C}^{n-1}(X, A; G) \to \mathrm{C}^n(X, A; G)$ as singular n-coboundaries.

When $n \in \mathbb{N}_0$ is understood then sometimes we just write δ instead of δ_n.

(2) By the discussion in Definition 197.19 we obtain a cochain complex

$$\ldots \leftarrow \mathrm{C}^{n+1}(X, A; G) \xleftarrow{\delta_n} \mathrm{C}^n(X, A; G) \xleftarrow{\delta_{n-1}} \mathrm{C}^{n-1}(X, A; G) \leftarrow \ldots \xleftarrow{\delta_0} C^0(X, A; G) \leftarrow 0.$$

We refer to it as the singular cochain complex of (X, A) with G-coefficients. We define the n-th singular cohomology group of (X, A) with G-coefficients[2112] as follows:

$$\mathrm{H}^n(X, A; G) \coloneqq \mathrm{H}^n(\mathrm{C}_*(X, A); G) = \frac{\ker(\delta_n \colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G))}{\mathrm{im}(\delta_{n-1} \colon \mathrm{C}^{n-1}(X, A; G) \to \mathrm{C}^n(X, A; G))}.$$

(3) If $A = \varnothing$ then we just write[2113] $\mathrm{C}^n(X; G) = \mathrm{C}^n(X, \varnothing; G)$ and similar for cohomology groups.

Before we try to get a feeling for these cohomology groups, let us first get an important formality out to the way: The following definition shows that continuous maps between pairs of topological spaces induce cochain maps and homomorphisms on cohomology groups:

Definition 198.2. Let G be an abelian group and let $f \colon (X, A) \to (Y, B)$ be a continuous map between pairs of topological spaces. By the C_*–Functor Lemma 143.16 we know that f induces a chain map $f_* \colon \mathrm{C}_*(X, A) \to \mathrm{C}_*(Y, B)$. It follows from the Dual Cochain Map Lemma 197.20 (1) that we get an induced cochain map

$$f^* \colon \mathrm{C}^*(Y, B; G) \to \mathrm{C}^*(X, A; G)$$

and it follows from the Cochain Complex-H^*-Functor Lemma 197.12 (1) that for each $n \in \mathbb{N}_0$ we get an induced homomorphism

$$f^* \colon \mathrm{H}^n(Y, B; G) \to \mathrm{H}^n(X, A; G).$$

[2112] In most cases we drop the adjective "singular" and we just talk of the n-th cohomology group of (X, A) with coefficients in G. We only refer to it as singular cohomology when we want to distinguish it from de Rham cohomology or cellular cohomology.

[2113] As we had already pointed out on page 3806, if R is a commutative ring, e.g. if $R = \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $R = \mathbb{C}$, then the cohomology groups $\mathrm{H}^n(X, A; R)$ can in fact be viewed as R-modules and the coboundary maps are R-module homomorphisms.

Example. Let X be a topological space, let A be a subspace, let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) The inclusion $i\colon A \to X$ induces a homomorphism $i^*\colon \mathrm{H}^n(X;G) \to \mathrm{H}^n(A;G)$ which is sometimes called the *restriction map*.
(2) The natural map $(X, \varnothing) \to (X, A)$ induces a homomorphism $\mathrm{H}^n(X, A; G) \to \mathrm{H}^n(X; G)$ which is denoted by я. □

We have the following analog of the C_*–Functor Lemma 143.16 and the H_n–Functor Lemma 143.20:

Lemma 198.3. (H^n–Functor Lemma) Let G be an abelian group.

(1) The maps
$$\begin{aligned} (X, A) &\mapsto \mathrm{C}^*(X, A; G) \\ (f\colon (X, A) \to (Y, B)) &\mapsto (f^*\colon \mathrm{C}^*(Y, B; G) \to \mathrm{C}^*(X, A; G)) \end{aligned}$$
define a contravariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{CoChCplx}$ of cochain complexes.
(2) For any $n \in \mathbb{N}_0$ the maps
$$\begin{aligned} (X, A) &\mapsto \mathrm{H}^n(X, A; G) \\ (f\colon (X, A) \to (Y, B)) &\mapsto (f^*\colon \mathrm{H}^n(Y, B; G) \to \mathrm{H}^n(X, A; G)) \end{aligned}$$
define a contravariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups. (If G is a commutative ring, then this is in fact a contravariant functor to the category $\mathcal{Mod}_G$ of G-modules.)

Remark. It follows immediately from the Cohomology-Coefficients-Functor Lemma 197.21 that the singular cochain complex and the singular cohomology groups of a pair of topological spaces are covariantly functorial in the abelian coefficient group G. □

Proof.

(1) This statement follows immediately from the C_*–Functor Lemma 143.16 (2) and the Dual Cochain Map Lemma 197.20.
(2) This statement follows immediately from Statement (1) and the Cochain Complex-H^*-Functor Lemma 197.12 (2). ■

198.2. **Explicit cochains and cocycles.** In this section we will consider a few explicit examples of cochains and cocycles. These examples gives us hopefully some intuition what cohomology does and later they will be useful to exhibit various statements on cohomology groups. We will continue with developing the theory of cohomology groups in the next Section 198.3.

Let X be a topological space. Recall that a homology class in dimension k is represented by a singular chain of the form $\sum_{i=1}^{m} a_i \cdot (\sigma_i\colon \Delta^k \to X)$. Thus homology classes can be described using "finite" data and in explicit cases one can often visualize a homology class.

A cohomology class in dimension k is a homomorphism $\mathrm{C}_k(X) \to G$. In almost all cases it is very hard to explicitly write down a cohomology class. Nonetheless, in this section we will present a few explicit cocycles in dimension 1 which are interesting in their own right, and which will be our go-to examples throughout the following chapters.

First we discuss examples in "absolute" cohomology groups. We start this discussion with the following remark.

Identification 198.4. Let G be an abelian group.

(1) Let X be a topological space and let $n \in \mathbb{N}_0$. We denote by $\mathrm{S}_n(X)$ the set of singular n-simplices of X. We have the following identities and natural identification

$$\mathrm{C}^n(X;G) \;:=\; \mathrm{Hom}(\mathrm{C}_n(X),G) \;\underset{\uparrow}{=}\; \mathrm{Hom}\left(\mathbb{Z}^{(\mathrm{S}_n(X))},G\right) \;\underset{\uparrow}{=_{\mathrm{i}}}\; \{\text{maps from } \mathrm{S}_n(X) \text{ to } G\}.$$

by Definition 143.11 we have $\mathrm{C}_n(X) = \mathbb{Z}^{(\mathrm{S}_n(X))}$ — identification given by the Abelian Group-Homomorphism Lemma 197.5 (1)

Put differently, a singular n-cochain is precisely the same as assigning to each singular n-simplex an element in G. We will use this bijection throughout the course without explicitly referring to it.

(2) Let X be a topological space. We recall that, following the discussion on page 2750, we can identify the singular 0-simplices in X with the points in X. According to the discussion in (1) we can now make the identification

$$\mathrm{C}^0(X;G) \;=_{\mathrm{i}}\; \{\text{all maps from } X \text{ to } G\}.$$

We now consider explicit cochains and cocycles on the topological spaces $\mathbb{R}$ and S^1:

Examples.

(1) We start out with the topological space $X = \mathbb{R}$. We consider the two maps

$$\begin{array}{rcl} \alpha_{\mathbb{R}}\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_{\mathbb{Z}}\colon \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \le x\}. \end{array}$$

By the above identification, these two maps define singular 0-cochains on $\mathbb{R}$. We denote these 0-cochains again by $\alpha_{\mathbb{R}}$ and $\alpha_{\mathbb{Z}}$. Note that for any singular 1-simplex $\mu\colon \Delta^1 \to \mathbb{R}$ we have

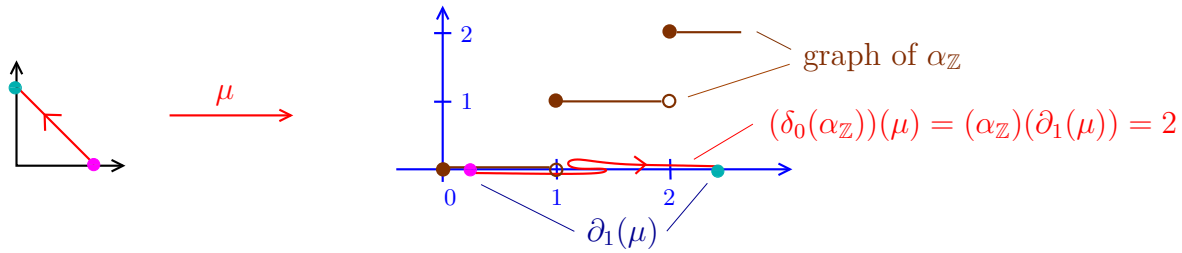
$$(\delta_0(\alpha_{\mathbb{R}}))(\mu) \;\underset{\uparrow}{=}\; \alpha_{\mathbb{R}}(\partial_1(\mu)) \;\underset{\uparrow}{=}\; \alpha_{\mathbb{R}}(\mu(0,1)-\mu(1,0)) \;\underset{\uparrow}{=}\; \mu(0,1)-\mu(1,0).$$

Definition 198.1 of $\delta_0 = (\partial_1)^*$ — see page 2751 for the description of ∂_1 — definition of $\alpha_{\mathbb{R}}$

Similarly we have[2114]

$$(\delta_0(\alpha_{\mathbb{Z}}))(\mu) \;=\; \alpha_{\mathbb{Z}}(\partial_1(\mu)) \;=\; \alpha_{\mathbb{Z}}(\mu(0,1)-\mu(1,0)) \;=\; \lfloor \mu(0,1) \rfloor - \lfloor \mu(1,0) \rfloor.$$

In a somewhat informal way we can say that the cochain $\delta_0(\alpha_{\mathbb{Z}})\colon \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex μ measures how many integers the simplex μ "crosses from left to right". We refer to the figure below for an illustration of $\delta_0(\alpha_{\mathbb{Z}})$.



(2) We consider the topological space $X = S^1$. Let $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ be the universal covering of S^1. Let $\sigma\colon \Delta^1 \to S^1$ be a singular 1-simplex. Since Δ^1 is simply

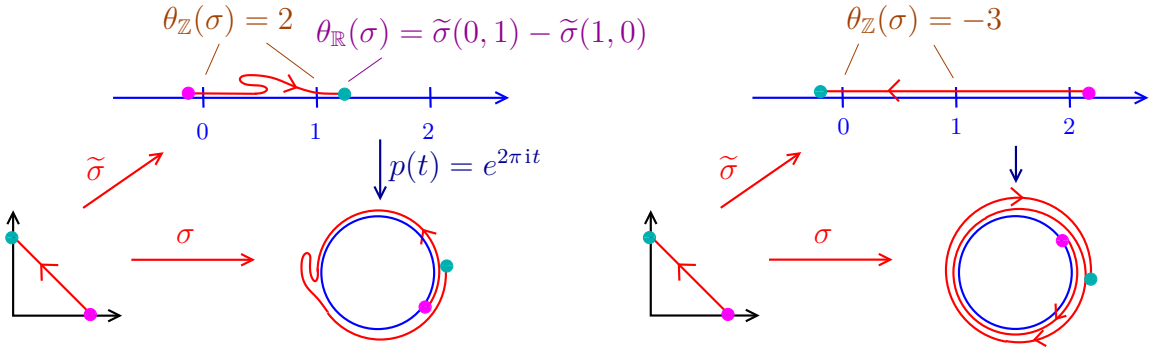
[2114] In other words:

$$(\delta_0(\alpha_{\mathbb{Z}})(\mu)) \;=\; \begin{cases} \#\{n \in \mathbb{Z} \mid \mu(1,0) < n \le \mu(0,1)\}, & \text{if } \mu(0,1) \ge \mu(1,0) \\ -\#\{n \in \mathbb{Z} \mid \mu(0,1) < n \le \mu(1,0)\}, & \text{if } \mu(1,0) \ge \mu(0,1). \end{cases}$$

connected it follows from the Covering Existence Theorem 127.10 that σ lifts to a continuous map $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$. We set

$$\theta_{\mathbb{R}}(\sigma) \;:=\; (\delta_1(\alpha_{\mathbb{R}}))(\widetilde{\sigma}) \;=\; \widetilde{\sigma}(0,1) - \widetilde{\sigma}(1,0) \quad \in \mathbb{R}$$
$$\uparrow \in \mathrm{C}_1(\mathbb{R}) \downarrow$$
$$\theta_{\mathbb{Z}}(\sigma) \;:=\; (\delta_1(\alpha_{\mathbb{Z}}))(\widetilde{\sigma}) \;=\; \lfloor \widetilde{\sigma}(0,1) \rfloor - \lfloor \widetilde{\sigma}(1,0) \rfloor \quad \in \mathbb{Z}.$$

Evidently the lift $\widetilde{\sigma}$ is not unique, but it follows almost immediately from the Lift Uniqueness Lemma 88.19 (or the more general Lift Uniqueness Proposition 92.2) that any two lifts differ by addition by an integer. Using this observation one can easily show that $\theta_{\mathbb{R}}(\sigma)$ and $\theta_{\mathbb{Z}}(\sigma)$ are both well-defined, i.e. independent of the choice of the lift $\widetilde{\sigma}$. The definitions of $\theta_{\mathbb{R}} \in \mathrm{C}_1(S^1, \mathbb{R})$ and $\theta_{\mathbb{Z}} \in \mathrm{C}_1(S^1, \mathbb{Z})$ are illustrated in the figure below.



It follows from the discussion on page 3812 that these two maps $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ actually define singular 1-cochains in $\mathrm{C}^1(S^1; \mathbb{R})$ and $\mathrm{C}^1(S^1; \mathbb{Z})$. As an example we consider the singular 1-simplex

$$\mu\colon \Delta^1 \;\to\; S^1 \quad \text{that is defined by} \quad \mu(1-t,t) \;=\; \exp(2\pi i t).$$

A lift of μ to the above universal covering $\mathbb{R} \to S^1$ is given by the singular 1-simplex

$$\widetilde{\mu}\colon \Delta^1 \;\to\; \mathbb{R} \quad \text{that is defined by} \quad \widetilde{\mu}(1-t,t) \;=\; t.$$

It follows immediately from the definitions that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$. □

The following lemma shows that $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1; \mathbb{R})$ and $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1; \mathbb{Z})$ are cocycles; in fact even better, it shows that these cocycles represent non-zero elements in cohomology:

Lemma 198.5. (θ-Cocycle Lemma) We continue with the notation from the previous example.

(1) The singular 1-cochains $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1; \mathbb{R})$ and $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1; \mathbb{Z})$ are cocycles, in particular $\theta_{\mathbb{R}}$ defines an element in $\mathrm{H}^1(S^1; \mathbb{R})$ and $\theta_{\mathbb{Z}}$ defines an element in $\mathrm{H}^1(S^1; \mathbb{Z})$.

(2) The singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain, i.e. the cohomology classes $[\theta_{\mathbb{R}}] \in \mathrm{H}^1(S^1; \mathbb{R})$ and $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1; \mathbb{Z})$ are non-trivial.

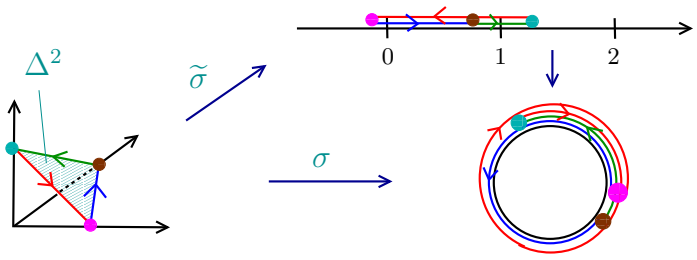
Remark. Note that in the θ-Cocycle Lemma 198.5 we have just given a mysteriously simple proof that $\mathrm{H}^1(S^1; \mathbb{R}) \neq 0$ and also that $\mathrm{H}^1(S^1; \mathbb{Z}) \neq 0$. □

Proof.

(1) We will first prove that the singular 1-cochain $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1; \mathbb{R})$ is a cocycle:

Claim. We have $\delta_1(\theta_{\mathbb{R}}) = 0 \in \mathrm{C}^2(S^1, \mathbb{R}) = \mathrm{Hom}(\mathrm{C}_2(S^1), \mathbb{R})$.

Proof. So let $\sigma\colon \Delta^2 \to S^1$ be a singular 2-simplex. Note that we have to show that $(\delta_1(\theta_{\mathbb{R}}))(\sigma) = \theta_{\mathbb{R}}(\partial_2(\sigma)) = 0$. As before we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi\,\mathrm{i}t)$ the universal covering of S^1. Since Δ^2 is simply connected we can apply the Map Lifting Criterion 127.5 to obtain a lift of $\sigma\colon \Delta^2 \to S^1$ to a continuous map $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$.



Then

$$(\delta_1(\theta_{\mathbb{R}}))(\sigma) \underset{\text{Definition 198.1 of } \delta_1}{=} \theta_{\mathbb{R}}(\partial_2(\sigma)) \underset{\text{Definition 143.8 of } \partial_2}{=} \theta_{\mathbb{R}}\Big(\sum_{j=0}^{2}(-1)^j \cdot \sigma \circ i_j^2\Big) \underset{\substack{\text{by definition of } \theta_{\mathbb{R}} \text{ on page 3813 and} \\ \text{since } \widetilde{\sigma} \circ i_j^2 \text{ is a lift of } \sigma \circ i_j^2}}{=} (\delta_1(\alpha_{\mathbb{R}}))\Big(\overbrace{\sum_{j=0}^{2}(-1)^j \cdot \widetilde{\sigma} \circ i_j^2}^{=\partial_2(\widetilde{\sigma})}\Big)$$

$$\underset{\substack{\text{by Definition 198.1} \\ \text{of } \delta_1(\alpha_{\mathbb{R}}) \text{ and } \partial_2(\widetilde{\sigma})}}{=} \alpha_{\mathbb{R}}(\partial_1(\partial_2\widetilde{\sigma})) \underset{\text{since } \partial_2 \circ \partial_1 = 0}{=} 0.$$

⊞

Exactly the same argument can be used to show that $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1; \mathbb{Z})$ is a cocycle.

(2) We still need to show that the singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain. So let β be a singular 0-cochain. As usual we view β as a function $S^1 \to \mathbb{R}$. We consider again the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ that is defined by $\mu(1-t, t) = \exp(2\pi\,\mathrm{i}t)$. Then

$$(\delta_0(\beta))(\mu) \underset{\text{Definition 198.1 of } \delta_0}{=} \beta(\partial_1(\mu)) = \beta(\underbrace{\mu(0,1)}_{=1\in S^1} - \underbrace{\mu(1,0)}_{=1\in S^1}) = 0 \neq 1 \underset{\substack{\text{by the example} \\ \text{preceding the lemma}}}{=} \theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu).$$

■

We continue with another example of cochains.

Example. Let $I \subseteq \mathbb{R}$ be an interval, e.g. $I = [0,1]$ or $I = \mathbb{R}$. We fix a point $y \in I$. We consider the cochain

$$\varphi_y\colon \mathrm{C}_0(I) \to \mathbb{Z}$$

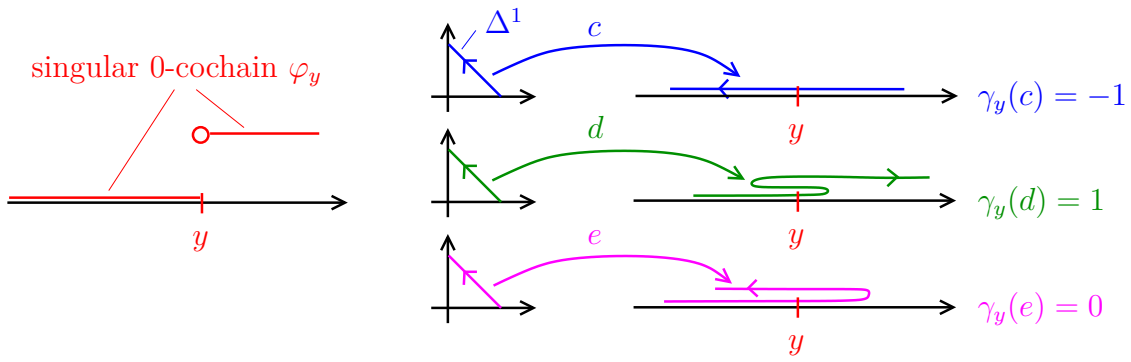
that is given by

$$\begin{aligned} I &\to \mathbb{Z} \\ x &\mapsto \begin{cases} 0, & \text{if } x \leq y, \\ 1, & \text{else.} \end{cases} \end{aligned}$$

For any singular 1-simplex $d\colon \Delta^1 \to X = I$ we have

$$(\delta_0(\varphi_y))(d) = \varphi_y(\partial_1(d)) \underset{\substack{\text{since } \partial_1(d) = d(0,1) - d(1,0) \text{ and} \\ \text{since } \varphi_y\colon \mathrm{C}_0(I) \to \mathbb{Z} \text{ is linear}}}{=} \varphi_y(d(0,1)) - \varphi_y(d(1,0)) = \begin{cases} 0, & \text{if } d(0,1), d(1,0) \leq y, \\ 0, & \text{if } d(0,1), d(1,0) \geq y, \\ 1, & \text{if } d(1,0) \leq y < d(0,1) \\ -1, & \text{if } d(0,1) \leq y < d(1,0). \end{cases}$$

In a slightly informal way we can say that the homomorphism $\delta_0(\varphi_y)\colon \mathrm{C}_1(I) \to \mathbb{Z}$ applied to a singular 1-simplex d equals -1 if the simplex "crosses y from right to left" and it equals $+1$ if the simplex "crosses y from left to right".



We will make use of this singular 1-cochain on several occasions, hence we give it its own notation, namely we write

$$\gamma_y \;:=\; \delta_0(\varphi_y) \;\in\; \mathrm{C}^1(I;\mathbb{Z}). \qquad \square$$

We want to conclude this section with an explicit cocycle in a relative cohomology group. Before we do so, it is again helpful to work with a few identifications:

Identification 198.6.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(C/D, G) & \xrightarrow{\cong} & \text{all homomorphisms } C \to G \text{ that vanish on } D \\ (\varphi\colon C/D \to G) & \mapsto & (C \xrightarrow{c\mapsto[c]} C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) If S is a set and T is a subset, then for any abelian group G the identification

$$\mathrm{Hom}(\mathbb{Z}^{(S)}, G) \;=_{\mathrm{i}}\; \{\text{all maps from } S \text{ to } G\}$$

from the Abelian Group-Homomorphism Lemma 197.5 (1) restricts to an identification

$$\left\{ f \in \mathrm{Hom}(\mathbb{Z}^{(S)}, G) \,\middle|\, f \text{ vanishes on } \mathbb{Z}^{(T)} \right\} \;=_{\mathrm{i}}\; \{\text{all maps } S \to G \text{ that are trivial on } T\}.$$

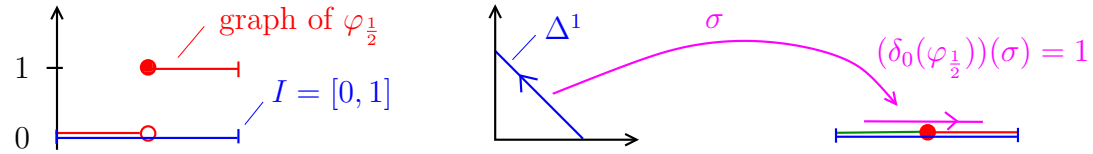
(3) It follows immediately from the Abelian Group-Homomorphism Lemma 197.5 (1) that given a pair of topological spaces (X, A) and an abelian group G we have for every $n \in \mathbb{N}_0$ an identification

$$\underbrace{\mathrm{Hom}(\mathrm{C}_n(X,A), G)}_{=\mathrm{C}^n(X,A;G) =_{\mathrm{i}} \mathrm{Hom}(\mathrm{C}_n(X)/\mathrm{C}_n(A), G)} \;=\; \text{all maps } \mathrm{S}_n(X) \to G \text{ that are trivial on } \mathrm{S}_n(A).$$

Example. We write $I = [0,1]$. We want to construct a potentially interesting element in $\mathrm{H}^1(I, \partial I; \mathbb{Z})$. As in the previous example we consider the map

$$\begin{array}{rcl} \varphi_{\frac{1}{2}}\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

which we use to define the singular 1-cochain $\gamma_{\frac{1}{2}} \coloneqq \delta_0(\varphi_{\frac{1}{2}}) \in \mathrm{C}^1(I;\mathbb{Z})$.



Note that for any singular simplex $\sigma\colon \Delta^1 \to I$ we have

$$\gamma_{\frac{1}{2}}(\sigma) \;=\; (\delta_0(\varphi_{\frac{1}{2}}))(\sigma) \;=\; \varphi_{\frac{1}{2}}(\partial_1(\sigma)) \;=\; \varphi_{\frac{1}{2}}(\sigma(0,1)) - \varphi_{\frac{1}{2}}(\sigma(0,1)).$$

We make two observations:

(1) Let $\sigma\colon \Delta^1 \to \partial I = \{0,1\}$ be a singular simplex. Since Δ^1 is connected and since $\{0,1\}$ is discrete it follows from the Connected-to-Discrete Lemma 4.37 that the map $\sigma\colon \Delta^1 \to \partial I = \{0,1\}$ is constant. It follows from the above calculation that $\gamma_{\frac{1}{2}}(\sigma) = 0$. This shows that $\gamma_{\frac{1}{2}}$ defines a cochain $\overline{\gamma}_{\frac{1}{2}}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$.
(2) In Exercise 198.8 we will see that the cochain $\overline{\gamma}_{\frac{1}{2}}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$ is a cocycle. Finally on page 3821 we will see that $\mathrm{H}^1(I,\partial I;\mathbb{Z}) \cong \mathbb{Z}$ and that the cocycle $\overline{\gamma}_{\frac{1}{2}}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$ represents a generator of $\mathrm{H}_1(I,\partial I;\mathbb{Z})$. □

198.3. **First calculations of cohomology.** After this long discussion of explicit cochains and cohomology classes we want to turn to more general results and calculations. Recall that the only long hanging fruit for calculating homology groups were the following:

(1) The Empty Space-H_*-Lemma 143.22.
(2) The Path-Component–H_*–Lemma 143.21.
(3) The calculation of the 0-th homology group of a path-connected space in the H_0-Proposition 143.25.
(4) The homology groups of a singleton in the Singleton-H_*-Lemma 143.26.

Not surprisingly we can also deal with the same situations with cohomology groups. First of all the following lemma is the analog of the Empty Space-H_*-Lemma 143.22.

Lemma 198.7. (Empty Space-H^*-Lemma) For the empty topological space $\varnothing$ we have $\mathrm{H}^k(\varnothing;G) = 0$ for all abelian groups G and all $k \in \mathbb{N}_0$.

Proof. This lemma follows immediately from the observation that in this case we have $\mathrm{C}_k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$. ■

We continue with the somewhat more interesting analog of the Path-Component-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.7 and the Path-Component–H_*–Lemma 143.21:

Lemma 198.8. (Path-Component-H^*-Lemma) Let X be a topological space and let G be an abelian group. We denote by $\{X_j\}_{j\in J}$ the path-components of X. The inclusion maps $\iota_j\colon X_j \to X$, $j \in J$, induce an isomorphism

$$\prod_{j\in J} \iota_j^*\colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \prod_{j\in J} \mathrm{H}^n(X_j;G).$$

Proof. Let X be a topological space with path-components $\{X_j\}_{j\in J}$ and let G be an abelian group. We have the following isomorphisms of cochain complexes:

$$\mathrm{Hom}(\mathrm{C}_*(X),G) \xrightarrow[\text{by the Path-Component–}\mathrm{C}_*\text{–Lemma 143.17 this map is an isomorphism}]{\left(\bigoplus_{j\in J}\iota_{j*}\right)^*} \mathrm{Hom}\left(\bigoplus_{j\in J}\mathrm{C}_*(X_j),G\right) \xrightarrow[\text{by the Direct Sum-Product Homomorphism Lemma 197.6 this is an isomorphism}]{f\mapsto \prod_{j\in J} f\circ i_j} \prod_{j\in J}\mathrm{Hom}(\mathrm{C}_*(X_j),G).$$

It follows immediately from the definitions that the composition of these two maps is precisely the map $\prod_{j\in J}\iota_j^*\colon \mathrm{Hom}(\mathrm{C}_*(X),G)\to\prod_{j\in J}\mathrm{Hom}(\mathrm{C}_*(X_j),G)$. The lemma now follows from these observations together with the Product–H^*-Lemma 197.18, which says that cohomology commutes with products. ∎

We continue with the following analog of the $\mathrm{H}^0_{\mathrm{dR}}$-Proposition 78.8 and the H_0-Proposition 143.25.

Proposition 198.9. (H^0-Proposition) Let G be an abelian group.

(1) Given any topological space the map

$$\begin{aligned}\zeta_X\colon G &\to \mathrm{H}^0(X;G)\\ g &\mapsto [\text{0-cochain that assigns to each } x\in X \text{ the value } g]\end{aligned}$$

is a well-defined homomorphism and it is natural. Here by "natural" we mean that for any continuous map $f\colon X\to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id}} & G \\ \downarrow{\scriptstyle \zeta_Y} & & \downarrow{\scriptstyle \zeta_X} \\ \mathrm{H}^0(Y;G) & \xrightarrow{f^*} & \mathrm{H}^0(X;G). \end{array}$$

(2) If X is a path-connected non-empty topological space, then the map $\zeta_X\colon G\to\mathrm{H}^0(X;G)$ from (1) is an isomorphism. For any point $P\in X$ an inverse is given by the map

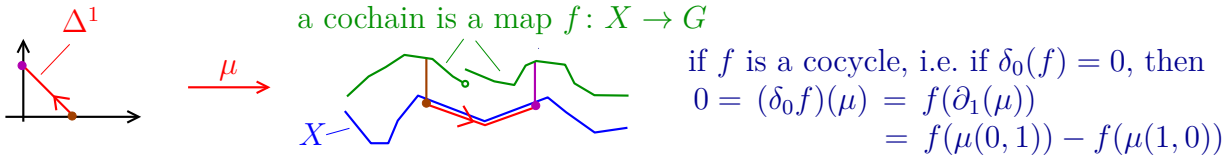
$$\begin{aligned}\mathrm{H}^0(X;G) &\to G\\ [f] &\mapsto f(P).\end{aligned}$$

Proof. Statement (1) follows easily from the definitions. Thus it remains to prove Statement (2). We consider the "right-hand end" of the chain complex of $\mathrm{C}_*(X)$ and of the cochain complex $\mathrm{Hom}(\mathrm{C}_*(X),G)$:

$$\mathrm{C}_1(X)\xrightarrow{\partial_1}\mathrm{C}_0(X)\to 0 \quad\text{and}\quad \mathrm{Hom}(\mathrm{C}_1(X),G)\xleftarrow{\delta_0=\partial_1^*}\mathrm{Hom}(\mathrm{C}_0(X),G)\leftarrow 0.$$

Now we calculate that

$$\begin{array}{l} \mathrm{H}^0(X;G) \\ \quad \downarrow \\ \quad = \ \ker \left\{ \begin{array}{rcl} \delta_1 \colon \mathrm{Hom}(\mathrm{C}_0(X),G) & \to & \mathrm{Hom}(\mathrm{C}_1(X),G) \\ (f \colon \mathrm{C}_0(X) \to G) & \mapsto & (f \circ \partial_1 \colon \mathrm{C}_1(X) \to G) \end{array} \right\} \\ \qquad \text{see Definition 198.1} \\ \quad \downarrow \\ \quad = \ \{\text{maps } f \colon X \to G \mid \underbrace{f(\partial_1(\mu))}_{=f(\mu(0,1))-f(\mu(1,0))} = 0 \text{ for all singular 1-simplices } \mu \colon \Delta^1 \to X\} \\ \qquad \text{since the singular 1-simplices form a basis of } \mathrm{C}_1(X) \\ \quad \uparrow \\ \quad = \ \text{all constant maps } X \to G \ =_{\mathrm{i}} \ G. \\ \qquad \text{by the Abelian Group-Homomorphism Lemma 197.5 (1) we have the natural identification } \mathrm{Hom}(\mathrm{C}_0(X),G) = \text{maps from } X \text{ to } G \\ \qquad \text{since } X \text{ is path-connected and since } \Delta^1 \cong [0,1] \qquad \text{since } X \text{ is non-empty} \end{array}$$



The remaining statements of the proposition can be verified easily. ■

The following lemma is an analog of the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10 and the Singleton-H_*-Lemma 143.26.

Lemma 198.10. (Singleton-H^*-Lemma) Let $X = \{*\}$ be a topological space that consists of a single point $*$. For any abelian group G we have

$$\mathrm{H}^n(X;G) \ \cong \ \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

Proof. We outsource the proof to Exercise 198.2. ■

198.4. **Basic properties of cohomology groups.** The following proposition is the analog of the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2, the Homotopy-H_*-Proposition 144.12, and the Homotopy-H_*-Proposition 146.11:

Proposition 198.11. (Homotopy-H^*-Proposition) Let G be an abelian group and let $n \in \mathbb{N}_0$. Let $f, g \colon (X,A) \to (Y,B)$ be continuous maps between pairs of topological spaces. If f and g are homotopic, then they induce the same maps

$$f^* \ = \ g^* \colon \mathrm{H}^n(Y,B;G) \ \to \ \mathrm{H}^n(X,A;G).$$

Proof. The statement is an immediate consequence of the Homotopy-H_*-Proposition 146.11 together with the Dual Cochain Map Lemma 197.20 (3a) and the Cochain Homotopy-H^*-Lemma 197.14 (1). ■

The following proposition is the analog of the Homotopy Equivalence-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 79.8, the Homotopy Equivalence-H_*-Proposition 144.14 and the Homotopy Equivalence-Relative-H_*-Corollary 146.8:

Proposition 198.12. (Homotopy Equivalence-H*-Proposition) Let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces, then the induced map
$$f^*\colon \mathrm{H}^n(Y;G) \xrightarrow{\cong} \mathrm{H}^n(X;G)$$
is an isomorphism.

(2) The cohomology groups of homotopy equivalent topological spaces are isomorphic.

Proof. This statement follows immediately from the Homotopy-H*-Proposition 198.11 and the H^n–Functor Lemma 198.3. ■

Example. Let X be a contractible topological space (e.g. $X = \mathbb{R}^k, \overline{B}^k, B^k, \Delta^k$), then for any abelian group G and any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(X;G) \underset{\text{by the Homotopy Equivalence H}^*\text{-Proposition 198.12 (2)}}{\cong} \mathrm{H}^n(\{*\};G) \underset{\text{by the Singleton-H}^*\text{ Lemma 198.10}}{\cong} \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

□

The following proposition is the analog of the LES-of-Triple Proposition 146.5 in homology theory.

Proposition 198.13. (LES-of-Triple Proposition) Given a triple (X, B, A) of topological spaces and an abelian group G we obtain a long exact sequence
$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \mathrm{H}^n(X,A;G) \xrightarrow{i^*} \mathrm{H}^n(B,A;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \ldots$$
hereby the maps are given as follows:

(1) p^* is the map induced by the map $p\colon (X,A) \to (X,B)$.

(2) i^* is the map induced by the map $i\colon (B,A) \to (X,A)$.

(3) The connecting homomorphism $\delta_n\colon \mathrm{H}^n(B,A;G) \to \mathrm{H}^{n+1}(X,B;G)$ is the natural homomorphism that is given by the map
$$\begin{array}{ccc} \mathrm{H}^n(B,A;G) & \to & \mathrm{H}^{n+1}(X,B;G) \\ [f\colon \mathrm{C}_n(B,A) \to G] & \mapsto & \left[\begin{array}{ccc} \mathrm{C}_{n+1}(X,B) & \to & G \\ [\sigma] & \mapsto & \widetilde{f}(\partial_{n+1}(\sigma)) \end{array}\right]. \\ & & \uparrow \quad\quad\quad\quad \uparrow \\ & & \text{with } \sigma \in \mathrm{C}_{n+1}(X) \quad \text{where } \widetilde{f}\colon \mathrm{C}_{n+1}(X,A) \to G \text{ is an extension of } f \end{array}$$

Remark.

(1) In a slightly more algorithmic language the connecting homomorphism can be described as follows:
   (a) take a cocycle $f \in \mathrm{C}^n(B,A;G) = \mathrm{Hom}(\mathrm{C}_n(B,A),G)$,
   (b) denote by $\pi\colon \mathrm{C}_n(B) \to \mathrm{C}_n(B,A)$ the projection,
   (c) pick an extension $\mathrm{C}_n(X) \to G$ of $f \circ \pi\colon \mathrm{C}_n(B) \to G$,[2115]

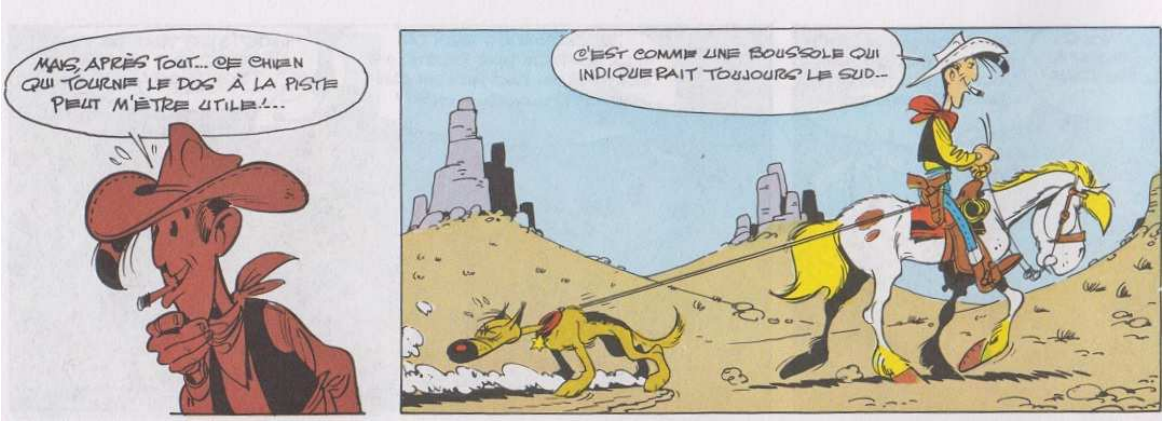
[2115]For example we could consider the extension $\mathrm{C}_n(X) \to G$ that is given by sending all singular simplices that do not lie in B to zero. This extension we sometimes call the *trivial extension* of $f \circ \pi$.

(d) then consider the coboundary $\delta_n(\widetilde{f}) \in \mathrm{C}^{n+1}(X;G)$, since f is a cocycle it follows that $\delta_n(\widetilde{f})$ actually defines an element in $\mathrm{C}^{n+1}(X,B;G)$ that we also denote by $\delta_n(\widetilde{f})$, and which turns out to be a cocycle,
(e) then $\delta_n([f]) = [\delta_n(\widetilde{f})] \in \mathrm{H}^{n+1}(X,B;G)$.

(2) The most frequent application of the LES-of-Triple Proposition 198.13 is that given a pair (X,B) of topological spaces, i.e. given the triple $(X,B,\varnothing)$ and given an abelian group G we obtain a natural homomorphism $\delta_n\colon \mathrm{H}^n(B;G) \to \mathrm{H}^{n+1}(X,B;G)$ which in turn fits into an exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \mathrm{H}^n(X;G) \xrightarrow{i^*} \mathrm{H}^n(B;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \ldots$$

(3) One of the difficulties with the long exact sequence of cohomology groups is that all maps seem to point the wrong way.



You should use the Lucky Luke approach to cohomology: If Rantanplan (i.e. your intuition) points one way, you should take the opposite direction. □

Proof. Let (X,B,A) be a triple of topological spaces. As in the proof of the LES-of-Triple Proposition 146.5 we note that

$$0 \to \mathrm{C}_*(B,A) \xrightarrow{i_*} \mathrm{C}_*(X,A) \xrightarrow{p_*} \mathrm{C}_*(X,B) \to 0$$

is a short exact sequence of chain complexes. Now let G be an abelian group. Recall that we pointed out on page 2798 that the chain groups $\mathrm{C}_*(X,B)$ are free abelian groups. Thus we deduce from the Hom-Split-SES Lemma 197.8 that the following induced sequence of cochain complexes is also exact:

$$0 \to \mathrm{Hom}(\mathrm{C}_*(X,B),G) \xrightarrow{p^*} \mathrm{Hom}(\mathrm{C}_*(X,A),G) \xrightarrow{i^*} \mathrm{Hom}(\mathrm{C}_*(B,A),G) \to 0.$$

It now follows from the Cohomology–Connecting Homomorphism Lemma 197.16 and the Cohomology–LES Proposition 197.17 that we get a connecting homomorphism that gives rise to a long exact sequence. To determine the connecting homomorphism we have to look a bit more under the hood. Thus let $[f] \in \mathrm{H}^n(B,A;G)$ where $f\colon \mathrm{C}_n(B,A) \to G$ is a

homomorphism. Similar to Definition 145.8 we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}(\mathrm{C}_{n+1}(X,B),G) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_{n+1}(X,A),G) & \xrightarrow{i^*} & \mathrm{Hom}(\mathrm{C}_{n+1}(B,A),G) & \to & 0 \\
 & & \uparrow{\scriptstyle \delta_n=\partial^*_{n+1}} & & \uparrow{\scriptstyle \delta_n=\partial^*_{n+1}} & & \uparrow{\scriptstyle \delta_n=\partial^*_{n+1}} & & \\
0 & \to & \mathrm{Hom}(\mathrm{C}_n(X,B),G) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_n(X,A),G) & \xrightarrow{i^*} & \mathrm{Hom}(\mathrm{C}_n(B,A),G) & \to & 0.
\end{array}$$

An extension $\widetilde{f}\colon \mathrm{C}_N(X,A,G) \to G$ of f gives us a preimage of f under the epimorphism i^*. It now follows basically immediately from Definition 145.8 that the connecting homomorphism is as described in the proposition. ∎

Examples.

(1) Let X be a path-connected topological space and let A be any *non-empty* subset of X. We denote by $i\colon A \to X$ the inclusion map. By the LES-of-Triple Proposition 198.13 we have a long exact sequence

$$0 \to \mathrm{H}^0(X,A;G) \to \mathrm{H}^0(X;G) \xrightarrow[\uparrow]{i^*} \mathrm{H}^0(A;G) \to \dots$$

it follows from the H^0-Proposition 198.9 and our hypothesis on X and A that this map is a monomorphism

Thus it follows that $\mathrm{H}^0(X,A;G) = 0$. In Exercise 198.3 we will provide an alternative proof for this fact using only the definitions of cohomology groups.

(2) We consider the beginning of the long exact sequence in cohomology with $\mathbb{Z}$-coefficients of the pair $([0,1],\{0,1\})$:

$$\dots \to \mathrm{H}^0([0,1];\mathbb{Z}) \xrightarrow{i^*} \mathrm{H}^0(\{0,1\};\mathbb{Z}) \xrightarrow{\delta_0} \mathrm{H}^1([0,1],\{0,1\};\mathbb{Z}) \to \underbrace{\mathrm{H}^1([0,1];\mathbb{Z})}_{=0}.$$

$\overline{\gamma}_{\frac{1}{2}}$

In the above sequence we already indicated the cocycle $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0,1],\{0,1\})$ that we introduced on page 3816. It follows easily from the Path-Component-H^*-Lemma 198.8 and the H^0-Proposition 198.9 that the cokernel of the homomorphism $i^*\colon \mathrm{H}^0([0,1];\mathbb{Z}) \to \mathrm{H}^0(\{0,1\};\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and that it is represented by the singular 0-cochain that is given by

$$\begin{array}{rcl} \psi\colon \{0,1\} & \to & \mathbb{Z} \\ x & \mapsto & x \end{array}$$

The following claim now shows that $\mathrm{H}^1([0,1],\{0,1\};\mathbb{Z}) = \mathbb{Z}\cdot[\overline{\gamma}_{\frac{1}{2}}]$:

Claim. The class $\delta_0([\psi]) \in \mathrm{H}^1([0,1],\{0,1\};\mathbb{Z})$ is represented by $[\overline{\gamma}_{\frac{1}{2}}]$.

Proof. We need to determine $\delta_0([\psi]) \in \mathrm{H}^1([0,1],\{0,1\};\mathbb{Z})$. It is perhaps helpful to consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}(\mathrm{C}_1([0,1],\{0,1\}),\mathbb{Z}) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_1([0,1]),\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{Hom}(\mathrm{C}_1(\{0,1\}),\mathbb{Z}) & \to & 0 \\
 & & \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \delta_0} & & \\
0 & \to & \mathrm{Hom}(\mathrm{C}_0([0,1],\{0,1\},\mathbb{Z}) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_0([0,1]),\mathbb{Z}) & \xrightarrow{i^*} & \underbrace{\mathrm{Hom}(\mathrm{C}_0(\{0,1\},\mathbb{Z})}_{\text{contains } \psi} & \to & 0.
\end{array}$$

We start with the map $\psi\colon \{0,1\} \to \mathbb{Z}$. Next we need to pick an extension to a map $[0,1] \to \mathbb{Z}$. We might as well take the map $\varphi_{\frac{1}{2}}\colon [0,1] \to \mathbb{Z}$ from page 3815. Note that the coboundary $\gamma_{\frac{1}{2}} := \delta_0(\varphi_{\frac{1}{2}})$ defines the cocycle $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0,1],\{0,1\};\mathbb{Z})$. The claim now follows from the definition of the connecting homomorphism. ⊞

We conclude this section with the cohomological analog of the popular Homotopy Equivalence-Relative-H_*-Corollary 146.8:

Corollary 198.14. (Homotopy Equivalence-Relative-H^*-Corollary) In the following let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces. If $f\colon X \to Y$ is a homotopy equivalence and if $f\colon A \to B$ is a homotopy equivalence, then for every $n \in \mathbb{N}_0$ and every abelian group G the induced map $f^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G)$ is an isomorphism.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & \mathrm{H}^n(Y;G) & \longrightarrow & \mathrm{H}^n(B;G) & \longrightarrow & \mathrm{H}^n(Y,B;G) & \xrightarrow{\delta_n} & \mathrm{H}^{n+1}(Y;G) & \longrightarrow & \mathrm{H}^{n+1}(B;G) & \longrightarrow \\
 & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & \\
\longrightarrow & \mathrm{H}^n(X;G) & \longrightarrow & \mathrm{H}^n(A;G) & \longrightarrow & \mathrm{H}^n(X,A;G) & \xrightarrow{\delta_n} & \mathrm{H}^{n+1}(X;G) & \longrightarrow & \mathrm{H}^{n+1}(A;G) & \longrightarrow
\end{array}$$

By the LES-of-Triple Proposition 198.13 we know that the diagram commutes and that the horizontal sequences are exact. Furthermore note that it follows from our hypothesis on f and the Homotopy Equivalence-H^*-Proposition 198.12 (1) that the two vertical maps to the left and the two vertical maps to the right are isomorphisms. It follows from the Five Lemma 145.11 that the middle vertical map is, as promised, an isomorphism. ■

198.5. **Reduced cohomology groups.** We saw on several occasions that the reduced homology groups $\widetilde{\mathrm{H}}_k(X)$ of a topological space can be convenient for expressing calculations and results. In this last section we quickly introduce the reduced cohomology groups. None of the concepts or results of this section are in anyway surprising. So the reader might as well move on to the next chapter.

Definition 198.15. Let X be a topological space. By the Augmentation Lemma 143.24 (1) we know that the following is a chain complex

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\
 & & & & & & \sum_{i=1}^k n_i \cdot \sigma_i & \mapsto & \sum_{i=1}^k n_i. & &
\end{array}$$

We introduce the following two definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.
(2) Let G be an abelian group. Given $n \in \mathbb{Z}_{\geq -1}$ we call $\widetilde{\mathrm{H}}^n(X;G) := \mathrm{H}^n(\mathrm{Hom}(\widetilde{\mathrm{C}}_*(X),G))$ the n-th reduced cohomology group of X with G-coefficients.

The following lemma is the analog of the $\widetilde{\mathrm{H}}_n$–Functor Lemma 147.2 and the H^n–Functor Lemma 198.3:

Lemma 198.16. ($\widetilde{\mathrm{H}}^n$–Functor Lemma) Let G be an abelian group.

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and

that is the identity on the final $\mathbb{Z}$-term. This map is a chain map and thus induces homomorphisms $f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)$ on the reduced cohomology groups.

(2) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}^n(X;G) \\ (f\colon X \to Y) & \mapsto & (f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)) \end{array}$$

define a contravariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups.

(3) The maps $\mathrm{id}\colon \widetilde{\mathrm{C}}_n(X) \to \mathrm{C}_n(X)$ for $n \geq 0$ and the zero map $\widetilde{\mathrm{C}}_{-1}(X) \to \mathrm{C}_{-1}(X)$ define a natural chain map and for any $n \geq -1$ they induce a natural transformation

$$\text{ж}\colon \mathrm{H}^n(X;G) \to \widetilde{\mathrm{H}}^n(X;G)$$

from the functor $X \mapsto \mathrm{H}^n(X;G)$ to the functor $X \mapsto \widetilde{\mathrm{H}}^n(X;G)$.

Proof. The lemma follows easily from the $\widetilde{\mathrm{H}}_n$–Functor Lemma 147.2 together with the Dual Cochain Map Lemma 197.20. ■

The following lemma is the analog of the Reduced Homology Lemma 147.3:

Lemma 198.17. (Reduced Cohomology Lemma) Let G be an abelian group. For every topological space X the following statements hold:

(-1) We have $\widetilde{\mathrm{H}}^{-1}(X;G) = \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathrm{Hom}(\mathbb{Z},G) =_{\mathrm{i}} G, & \text{if } X = \varnothing. \end{cases}$

(0) (a) If X is non-empty, then the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & G & \xrightarrow{\zeta_X} & \mathrm{H}^0(X;G) & \xrightarrow{\text{ж}} & \widetilde{\mathrm{H}}^0(X;G) & \to & 0. \\ & & g & \mapsto & [0\text{-simplex} \mapsto g] & & & & \end{array}$$

(b) For any $P \in X$ the map

$$\begin{array}{rcl} \mathrm{H}^0(X;G) & \to & \widetilde{\mathrm{H}}^0(X;G) \oplus G \\ [\sigma] & \mapsto & \text{ж}([\sigma]) + \sigma(P) \end{array}$$

is an isomorphism.

(1) For every $n \geq 1$ the natural map $\text{ж}\colon \mathrm{H}^n(X;G) \to \widetilde{\mathrm{H}}^n(X;G)$ is the identity.

Proof. The proof is elementary and left to the reader. ■

We also have the following analog of the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 147.4 and the Homotopy-H^*-Proposition 198.11.

Proposition 198.18. (Homotopy-$\widetilde{\mathrm{H}}^*$-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then for each $n \in \mathbb{N}_0$ and each abelian group G we have $f^* = g^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)$.

Proof. This proposition follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 147.4 (1) together with the Dual Cochain Map Lemma 197.20 (3a) and the Cochain Homotopy-H^*-Lemma 197.14 (1). ■

Next the following proposition plays the role of the Reduced LES-of-Pair Proposition 147.8 and the LES-of-Triple Proposition 198.13.

Proposition 198.19. (Reduced LES-of-Pair Proposition) Let X be a topological space and let $B \subseteq X$ be a non-empty subset. Given any abelian group G there exist natural homomorphisms $\delta_n \colon \widetilde{\mathrm{H}}^n(B;G) \to \mathrm{H}^{n+1}(X,B;G)$ which give rise to a long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \widetilde{\mathrm{H}}^n(X;G) \xrightarrow{i^*} \widetilde{\mathrm{H}}^n(B;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \ldots$$

Proof. In the proof of the Reduced LES-of-Pair Proposition 147.8 we showed that we have a short exact sequence $0 \to \widetilde{\mathrm{C}}_*(B) \to \widetilde{\mathrm{C}}_*(X) \to \mathrm{C}_*(X,B) \to 0$ of chain complexes. By the discussion on page 2798 we know that each $\mathrm{C}_n(X,B)$ is a free abelian group. The proposition now follows from the Hom-Split-SES Lemma 197.8, the Cohomology–Connecting Homomorphism Lemma 197.16 and the Cohomology–LES Proposition 197.17. ■

We conclude this section with the following analog of the Reduced-to-Relative-H_*-Lemma 147.9:

Lemma 198.20. (Reduced-to-Relative-H^*-Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. For each $x_0 \in X$ and each abelian group G there exists an explicit natural isomorphism

$$\text{ж}_{X,\{x_0\}} \colon \mathrm{H}^n(X,\{x_0\};G) \to \widetilde{\mathrm{H}}^n(X;G).$$

Proof. Similar to the proof of the Reduced-to-Relative-H_*-Lemma 147.9 one can easily obtain a proof using the Reduced LES-of-Pair Proposition 198.19. We leave it to the reader to fill in the details. ■

198.6. Exercises for Chapter 198 .

Exercise 198.1. On page 3813 we introduced a singular 1-cochain $\theta \in \mathrm{C}^1(S^1;\mathbb{R})$. For fans of complex analysis we can give an alternative description of this singular 1-cochain: We denote by $\Phi \colon [0,1] \to \Delta^1$ the homeomorphism that is given by $\Phi(t) = (1-t,t)$. Show that for any singular 1-simplex $\sigma \colon \Delta^1 \to S^1$ we have

$$\theta_{\mathbb{R}}(\sigma) = \frac{1}{2\pi\,\mathrm{i}} \int\limits_{\sigma\circ\Phi} \frac{1}{z}\,dz,$$

where the right-hand side denotes the path integral of the holomorphic function $\frac{1}{z}$ over the continuous path $\sigma \circ \Phi \colon [0,1] \to S^1$.

Exercise 198.2. Let $X = \{x\}$ be a topological space that consists of a single point x. Show that for any abelian group G we have

$$\mathrm{H}^n(X;G) \cong \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

Exercise 198.3. Let X be a path-connected topological space and let A be a non-empty subset of X. Show that

$$\mathrm{H}^0(X,A;G) = 0,$$

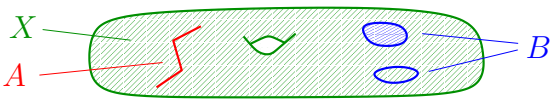
using only the definition of cohomology groups.

Exercise 198.4. Let X be a compact topological space. Are the cohomology groups $\mathrm{H}^k(X;\mathbb{Z})$ necessarily countable?

Exercise 198.5. Let X be a path-connected topological space and let A and B be two disjoint non-empty subsets of X. Provide an explicit example of a map $f\colon X \to \mathbb{Z}$ such that the map

$$\begin{array}{rcl} \delta_1(f)\colon \mathrm{C}_1(X, A\cup B) & \to & \mathbb{Z} \\ [\sigma\colon \Delta^1 \to X] & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)) \end{array}$$

is cocycle that represents a non-zero element in $\mathrm{H}^1(X, A\cup B; \mathbb{Z})$.



Exercise 198.6. Let $f\colon X \to Y$ be a continuous map between two topological spaces.

(a) Show that if X and Y are path-connected non-empty topological spaces, then f induces an isomorphism $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$.

(b) Show that if Y is path-connected and X is non-empty, then $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$ is a monomorphism.

Exercise 198.7. Let $X = \mathbb{R}^2$. We consider the singular 2-cochain $\varphi\colon \mathrm{C}_2(\mathbb{R}^2;\mathbb{F}_2)$ that is given by

$$\varphi(\sigma) \coloneqq \begin{cases} 1, & \text{if } (0,0) \text{ lies in the interior of the convex hull} \\ & \text{of the three points } \sigma(1,0,0), \sigma(0,1,0), \sigma(0,0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Is φ a cocycle?

Exercise 198.8. We write $I = [0,1]$. We consider the map

$$\begin{array}{rcl} \varphi_{\frac{1}{2}}\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

which we use to define the singular 1-cochain $\gamma_{\frac{1}{2}} \coloneqq \delta_0(\varphi_{\frac{1}{2}}) \in \mathrm{C}^1(I;\mathbb{Z})$. On page 3816 we showed that $\gamma_{\frac{1}{2}}$ descends to a cochain $\overline{\gamma}_{\frac{1}{2}}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$. Show that $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1(I, \partial I; \mathbb{Z})$ is a cocycle

Exercise 198.9. Let G be an abelian group and let X be a topological space. For $k \in \mathbb{N}_0$ we consider $\mathrm{Hom}(G, \mathrm{C}_k(X))$.

(a) Show that the abelian groups $\mathrm{Hom}(G, \mathrm{C}_k(X))$ give rise to a (co-) chain complex.

(b) Are the (co-) homology groups obtained in (a) new invariants of topological spaces?

Exercise 198.10. Let (X, x_0) be a pointed topological and let G be an abelian group. We denote by $\theta\colon \Delta^1 \to [0,1]$ the homeomorphism that is given by $(1-t, t) \mapsto t$.

(a) Show that the map

$$\begin{array}{rcl} \Phi\colon \mathrm{H}^1(X;G) & \to & \mathrm{Hom}(\pi_1(X,x_0), G) \\ [f\colon \mathrm{C}_1(X) \to G] & \mapsto & \left(\begin{array}{rcl} \pi_1(X,x_0) & \to & G \\ [\gamma\colon [0,1] \to X] & \mapsto & f(\gamma \circ \theta) \end{array} \right) \end{array}$$

is well-defined.

(b) Show that if X is path-connected, then the map $\Phi\colon \mathrm{H}^1(X;G) \to \mathrm{Hom}(\pi_1(X,x_0), G)$ is an isomorphism.

Exercise 198.11. Let X be a topological space and let $A \subseteq B$ be subsets of X. Show that if A is a deformation retract of B, then for any abelian group G and any $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}^n(X, B; G) \ \to \ \mathrm{H}^n(X, A; G)$$

is an isomorphism.

Exercise 198.12. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a topological space X. Let

$$\begin{array}{rcl} \tau\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(\widetilde{X}) \\ (\sigma\colon \Delta^n \to X) & \mapsto & \text{sum of the two lifts of } \sigma \text{ to } \Delta^n \to \widetilde{X} \end{array}$$

be the transfer map from Definition 165.1. Note that we showed in the Transfer Map Lemma 165.2 that τ is a chain map. Show that we have a long exact sequence of the form

$$\ldots \to \mathrm{H}^n(X; \mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}^n(\widetilde{X}; \mathbb{F}_2) \xrightarrow{\tau^*} \mathrm{H}^n(X; \mathbb{F}_2) \to \mathrm{H}^{n+1}(X; \mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}^{n+1}(X; \mathbb{F}_2) \to \ldots$$

Remark. This is an analog of the Double Covering-LES-Proposition 165.5.

CHAPTER 199

Cellular cohomology and the Kronecker pairing

After all this theory we now want to calculate cohomology groups. In Chapter 155 we saw that in many cases the most efficient way to calculate homology groups is to go through cellular homology. In this chapter we will repeat the trick and we will see that we can calculate singular cohomology via cellular cohomology.

This allows us to calculate the *isomorphism types* of the cohomology groups of many topological spaces, e.g. of surfaces and of the real and complex projective spaces. We also introduce the Kronecker pairing between homology and cohomology. This simple minded object will be surprisingly helpful at showing that explicit cycles and cocycles represent non-trivial elements and bases of (co-) homology.

This chapter is organized as follows:

- In Section 199.1 we will introduce cellular cohomology and we will see that it is naturally isomorphic to singular cohomology.
- In Section 199.2 we will perform some calculations of (cellular) cohomology.
- In Section 199.3 we will introduce the Kronecker pairing between singular homology and singular cohomology.
- In Section 199.4 we will provide exercises for Chapter 199.

In the next Chapter 200 we will prove cohomological analogs of the Excision Theorem and we will provide Mayer–Vietoris for cohomology.

199.1. **Cellular cohomology.** First we recall some definitions and some notation from the context of cellular chain complexes.

Definition 199.1. Let X be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X.

(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$. We refer to elements of $\mathrm{C}_n^{\mathrm{CW}}(X)$ as a cellular chain.

(3) (a) A charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c \in \mathrm{Cell}(X)}$ of characteristic maps of the cells of X.

(b) For an n-cell α we write $\Xi_\alpha := \Phi_{\alpha *}([\overline{B}^n]) \in \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X)$.

By the Cellular Chain Complex–Basis Lemma 155.13 we know that for every $n \in \mathbb{N}_0$ the set $\{\Xi_c\}_{c \in \mathrm{Cell}_n(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.

(4) We denote by $\partial_n^{CW} \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from Definition 155.3.

(5) We refer to

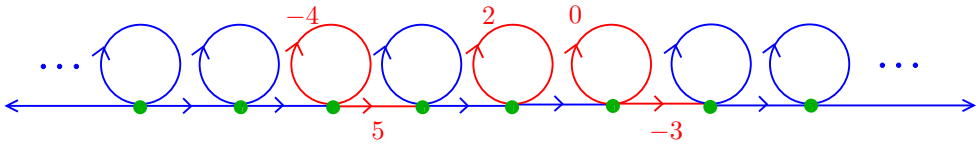
$$\ldots \xrightarrow{\partial_{n+2}^{CW}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{\partial_{n+1}^{CW}} \mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{\partial_n^{CW}} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{\partial_{n-1}^{CW}} \ldots$$

as the cellular chain complex.

(6) The homology groups of the cellular chain complex are the cellular homology groups $\mathrm{H}_n^{\mathrm{CW}}(X)$.

Example. Given a charted CW-complex X a cellular 1-chain in $\mathrm{C}_1^{\mathrm{CW}}(X)$ can be viewed as a finite linear combination of 1-cells.

a cellular 1-chain in $\mathrm{C}_1(X)$ is a finite linear combination of 1-cells



the charted CW-complex X

We illustrate this concept in the above infinite CW-complex X. □

Next we turn to a new concept, namely cellular cohomology:

Definition 199.2. Let X be a CW-complex and let G be an abelian group.

(1) Let $n \in \mathbb{N}_0$.
  (a) We refer to the elements of $\mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G)$ as cellular n-cochains.
  (b) If X is charted, then we have

$$\mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \underset{\uparrow}{=_{\mathrm{i}}} \text{ maps from the set of } n\text{-cells to } G \;=\; G^{\#\, n\text{-cells}}.$$

by the CW-Skeleton-H_*-Lemma 155.2 (1) and
the Abelian Group-Homomorphism Lemma 197.5 (1)

(2) Given any $n \in \mathbb{N}_0$ we write $\delta_n := (\partial_{n+1}^{CW})^*$. We refer to

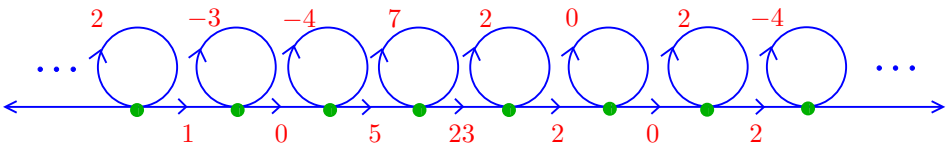
$$\ldots \leftarrow \mathrm{Hom}(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G) \xleftarrow{\delta_n} \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \xleftarrow{\delta_{n-1}} \mathrm{Hom}(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G) \leftarrow \ldots$$

as the cellular cochain complex. We define the n-th cellular cohomology group of X with G-coefficients as

$$\begin{aligned} \mathrm{H}_{\mathrm{CW}}^n(X; G) &:= \mathrm{H}^n(\mathrm{Hom}(\mathrm{C}_*^{\mathrm{CW}}(X), G)) \\ &= \frac{\ker(\delta_n \colon \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G))}{\mathrm{im}(\delta_{n-1} \colon \mathrm{Hom}(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G))}. \end{aligned}$$

Example. As we discussed above, given a charted CW-complex X an integral cellular 1-cochain in $\mathrm{Hom}(\mathrm{C}_1^{\mathrm{CW}}(X), \mathbb{Z})$ assigns to *every* 1-cell an integer.

a cellular 1-cochain in $\mathrm{C}^1(X; \mathbb{Z})$ assigns to *every* 1-cell an integer



the charted CW-complex X

So naively speaking a cellular 1-cochain can be viewed as an infinite linear combination of 1-cells. □

Definition 199.3. Let $f\colon X \to Y$ be a cellular[2116] map between two CW-complexes. It follows from the Cellular Homology-Functor Lemma 155.7 that we get an induced chain map $f_*\colon \mathrm{C}_*^{\mathrm{CW}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(Y)$. Thus we obtain from the Dual Cochain Map Lemma 197.20 (1) and the Cochain Homotopy-H^*-Lemma 197.14 for any abelian group G and any $n \in \mathbb{N}_0$ an induced homomorphism

$$f^*\colon \mathrm{H}^n_{\mathrm{CW}}(Y;G) \ \to\ \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

For completeness' sake we jot down the following fairly self-evident lemma.

Lemma 199.4. (Cellular Cohomology-Functor Lemma) Let G be an abelian group and let $n \in \mathbb{N}_0$. The maps

$$\begin{aligned} X &\mapsto \mathrm{H}^n_{\mathrm{CW}}(X;G) \\ (f\colon X \to Y) &\mapsto (f^*\colon \mathrm{H}^n_{\mathrm{CW}}(Y;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G)) \end{aligned}$$

define a contravariant functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups. If G is a commutative ring, then this defines a contravariant functor to the category of $\mathcal{Mod}_G$ of G-modules.

Proof. This lemma follows immediately from the Cellular Homology-Functor Lemma 155.7, the Dual Cochain Map Lemma 197.20 (2) and the Cochain Complex-H^*-Functor Lemma 197.12. ■

Perhaps not surprisingly we have the following theorem which plays a similar role as the Singular-Cellular-H_*-Proposition 155.8.

Proposition 199.5. (Singular-Cellular-H^*-Isomorphism Proposition) For any CW-complex X, any abelian group G and any $n \in \mathbb{N}_0$ there exists a natural isomorphism[2117]

$$\mathrm{H}^n(X;G) \ \xrightarrow{\cong}\ \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

Remark. An alternative proof of this theorem is given in [**Hat2002**, p. 203]. □

First proof of Proposition 199.5. Let X be a CW-complex. Recall that in Definition 158.13 we introduced the intermediate cellular chain complex that is given by

$$\mathrm{C}_n^{\mathrm{CW,int}}(X) \ :=\ \ker\left(\mathrm{C}_n(X^n) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})\right) \quad \text{for } n \in \mathbb{N}_0.$$

We consider the two *natural* chain maps

$$\mathrm{C}_*(X) \ \xleftarrow{i}\ \mathrm{C}_*^{\mathrm{CW,int}}(X) \ \xrightarrow{p}\ \mathrm{C}_*^{\mathrm{CW}}(X),$$

which are given by the obvious inclusion and the obvious projection. In the Singular-Cellular Chain Complex Proposition 158.16 we had seen that these two chain maps are chain homotopy equivalences. Thus it follows from the Dual Cochain Map Lemma 197.20 (3b) together with the Cochain Homotopy-H^*-Lemma 197.14 that for any abelian group G we get isomorphisms

$$\mathrm{H}^n(X;G) \ =\ \mathrm{H}^n(\mathrm{C}_*(X);G) \ \underset{\cong}{\xrightarrow{i^*}}\ \mathrm{H}^n(\mathrm{C}_*^{\mathrm{CW,int}}(X);G) \ \underset{\cong}{\xleftarrow{p^*}}\ \mathrm{H}^n(\mathrm{C}_*^{\mathrm{CW}}(X);G) \ =\ \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

[2116] According to Definition 139.2 a map $f\colon X \to Y$ between two CW-complexes is called *cellular* if f is continuous and if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subseteq Y^n$.

[2117] Here "natural" means that there exists a natural transformation between the contravariant functors $X \mapsto \mathrm{H}^n(X;G)$ and $X \mapsto \mathrm{H}^n_{\mathrm{CW}}(X;G)$ from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups.

As pointed out above, the two chain maps i and p are natural. Therefore we see that the above isomorphism $\mathrm{H}^n(X;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G)$ is in fact natural. ■

Alternative proof of Proposition 199.5 without "natural". We start out with the following general claim:

Claim. Suppose that (C_*, ∂_*) and (C'_*, ∂'_*) are two chain complexes of free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $\mathrm{H}_n(C_*)$ and $\mathrm{H}_n(C'_*)$ are isomorphic. Then for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $\mathrm{H}^n(C;G)$ and $\mathrm{H}^n(C';G)$ are also isomorphic.

Proof. By our hypothesis there exist isomorphisms $\gamma_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$. By the Lift-to-Chain Map-Proposition 158.21 these isomorphisms can be realized by a chain map $f \colon C_* \to C'_*$. But by the Quasi Iso–Homotopy Equivalence Proposition 158.8 this chain map is a chain homotopy equivalence. The statement of the claim follows immediately from the Dual Cochain Map Lemma 197.20. ⊞

Let X be a CW-complex. We denote by $C_* = \mathrm{C}_*(X)$ the singular chain complex and we denote by $D_* = \mathrm{C}^{\mathrm{CW}}_*(X)$ the cellular chain complex. In the Singular-Cellular-H_*-Proposition 155.8 we showed that all the homology groups of the chain complexes C_* and D_* are isomorphic. It follows from the claim that for any abelian group G and any $n \in \mathbb{N}_0$ the cohomology groups $\mathrm{H}^n(X;G) = \mathrm{H}^n(C_*;G)$ and $\mathrm{H}^n_{\mathrm{CW}}(X;G) = \mathrm{H}^n(D_*;G)$ are isomorphic. This approach to proving the Singular-Cellular-H^*-Isomorphism Proposition 199.5 circumvents the use of the Singular-Cellular Chain Complex Proposition 158.16. But it is not clear how this approach can be used to show that there exists a *natural* isomorphism. Of course for many applications this is not an issue. ■

199.2. **Calculations of (cellular) cohomology.** In this section we will use our expertise on cellular chain complexes to calculate many cellular cohomology groups. By the Singular-Cellular-H^*-Isomorphism Proposition 199.5 these calculations also give us the isomorphism types of singular cohomology.

Examples. Let G be an abelian group.

(1) Let $n \in \mathbb{N}$. On page 2625 we showed that we can view S^n as a CW-complex with one 0-cell and one other cell of dimension n. Furthermore the cellular boundary maps are the zero maps.[2118] Thus we see that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \to 0.$$

The dual cochain complex is then given by

$$0 \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z},G)}_{\cong G} \xleftarrow{0} \cdots \xleftarrow{0} \underbrace{\mathrm{Hom}(\mathbb{Z},G)}_{\cong G} \leftarrow 0.$$

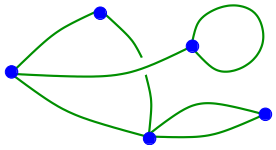
We obtain that $\mathrm{H}^k(S^n;G) \cong \begin{cases} G, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$ Similarly we see that $\mathrm{H}^k(S^0;G) \cong \begin{cases} G^2, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$

(2) Let Γ be a connected finite topological graph. It follows from the Graph-Homotopy Type Corollary 96.12 (1) that Γ is homotopy equivalent to a CW-complex X with one

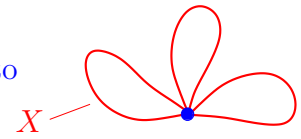
[2118] For $n \geq 2$ this follows by definition, for $n = 1$ this is a consequence of the Cellular-First Boundary Map Lemma 155.14.

0-cell and $1-\chi(\Gamma)$ 1-cells. We see that for any $n \in \mathbb{N}_0$ we have

the topological graph Γ



is homotopy equivalent to



by the Singular-Cellular-H*-Isomorphism Proposition 199.5

$$\mathrm{H}^n(\Gamma;G) \underset{\text{by the Homotopy Equivalence-H}^* \text{ Proposition 198.12 (2)}}{\cong} \mathrm{H}^n(X;G) \cong \mathrm{H}^n_{\mathrm{CW}}(X;G) \underset{\text{it follows from the Cellular-First Boundary Map Lemma 155.14 that the cellular coboundary map is zero}}{\cong} \mathrm{Hom}(\mathrm{C}^{\mathrm{CW}}_n(X);G) \cong \begin{cases} G, & \text{if } n=0, \\ G^{1-\chi(\Gamma)}, & \text{if } n=1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

(3) Given $g \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks. On page 3494 we had argued that for $m \geq 1$ the topological space $\Sigma_{g,m}$ is homotopy equivalent to the wedge of $2g+m-1$ circles. The previous example, together with the Homotopy Equivalence-H*-Proposition 198.12 (2), thus gives us the calculation of the cohomology groups of $\Sigma_{g,m}$ for $m \geq 1$.

(4) Let Σ_g be the surface of genus g. In the figure on page 2626 we showed that we can view Σ_g as a CW-complex with one 0-cell, $2g$ 1-cells and one 2-cell. On page 2964 we saw that the cellular boundary maps are all zero. Hence for any abelian group G and any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(\Sigma_g;G) \underset{\text{by the Singular-Cellular-H}^* \text{ Proposition 199.5}}{\cong} \mathrm{H}^n_{\mathrm{CW}}(\Sigma_g;G) = \mathrm{Hom}\left(\mathrm{C}^{\mathrm{CW}}_n(\Sigma_g);G\right) \cong \begin{cases} G, & \text{if } n=0 \text{ or } n=2, \\ G^{2g}, & \text{if } n=1, \text{ and} \\ 0, & \text{if } n \neq 0,1,2. \end{cases}$$

□

Next we consider CW-complexes with interesting cellular boundary maps:

Examples.

(1) We consider the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$. In the proof of the $\mathbb{R}\mathrm{P}^n$-H$_*$-Proposition 155.18 we saw that there exists a CW-structure on $\mathbb{R}\mathrm{P}^n$ such that the corresponding cellular chain complex is of the following form:

$$0 \to \underset{\text{degree } n}{\mathbb{Z}} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \to \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

which allowed us to compute the homology groups of $\mathbb{R}\mathrm{P}^n$. Now let G be an abelian group. We dualize this chain complex, i.e. we apply the functor $\mathrm{Hom}(-,G)$. It follows from the Free Module–Hom-to-Ring Lemma 197.7 that the resulting cochain complex is isomorphic to the following cochain complex:

$$0 \leftarrow G \xleftarrow{\cdot(1+(-1)^n)} G \leftarrow \ldots \xleftarrow{0} G \xleftarrow{\cdot 2} G \xleftarrow{0} G \leftarrow 0.$$

For $G = \mathbb{Z}$ we thus obtain from the Singular-Cellular-H^*-Isomorphism Proposition 199.5 that for any $k \in \{0, \dots, n\}$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{if } k \text{ is odd and } k \neq n, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases} \quad \text{but } \mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } k \text{ is odd and } k \neq n, \\ 0, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases}$$

where on the right-hand side we recall the calculation of the homology groups using the above cellular chain complex, see also the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18. Note that the calculations for homology and cohomology are mysteriously similar and different at the same time.

(2) To add to the confusion we also compute the cohomology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. It follows from the Free Module–Hom-to-Ring Lemma 197.7 that the corresponding dual cellular cochain complex is then given by

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \leftarrow \dots \leftarrow \mathbb{F}_2 \xleftarrow{\cdot 2 = 0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \leftarrow 0$$

and we see that

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

One can easily modify the above argument and apply it to $\mathbb{R}\mathrm{P}^\infty$, which we can view as a CW-complex with precisely one cell in each dimension, see 2039. We obtain that for each $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2$$

Also note the following: Given any $k < l \in \mathbb{N} \cup \{\infty\}$ we have the standard embedding $\mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$. Since the isomorphism of the Singular-Cellular-H^*-Isomorphism Proposition 199.5 is natural and since the cellular chain complexes of $\mathbb{R}\mathrm{P}^k$ and $\mathbb{R}\mathrm{P}^l$ agree up to the dimension k and since, with $\mathbb{F}_2$-coefficients the boundary map ∂_{k+1} is zero, we see that the inclusion-induced maps $\mathrm{H}^n(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2) \to \mathrm{H}^n(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ are actually isomorphisms up to dimension k.

(3) We recall that on page 2628 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with exactly one cell in dimensions $0, 2, \dots, 2n$ and with no odd-dimensional cells. It follows easily from the Singular-Cellular-H^*-Isomorphism Proposition 199.5 that

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \dots, 2n, \\ 0, & \text{otherwise} \end{cases} \quad \begin{matrix} \text{Similarly} \\ \text{we obtain} \end{matrix} \quad \mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{f } k \text{ even}, \\ 0, & \text{if } k \text{ odd}. \end{cases}$$

Again we note the following: Given $k < l \in \mathbb{N} \cup \{\infty\}$ we have an embedding $\mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^l$. Similar to (2) we see that the inclusion-induced maps $\mathrm{H}^n(\mathbb{C}\mathrm{P}^l;\mathbb{Z}) \to \mathrm{H}^n(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ are actually isomorphisms up to dimension $2k$. □

We conclude this section with the following fairly immediate corollary to the Singular-Cellular-H^*-Isomorphism Proposition 199.5. This corollary can be viewed as an analog of the CW-Complex-H_*-Proposition 155.10 (1) and the Smooth Manifolds-Invariants Proposition 179.10.

Corollary 199.6. (Finitely Generated-H^*-Corollary) Let X be a topological space. Suppose we are in one of the following two situations:

(1) X is homotopy equivalent to a retract of a finite CW-complex,

(2) X is a compact topological manifold.
Then for every $n \in \mathbb{N}_0$ the cohomology group $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated group.

Proof. Statement (2) is an immediate consequence of Statement (1) and the Topological Manifold-Simplicial Complex Retract Theorem 189.13. In Exercise 199.8 we will use the Singular-Cellular-H^*-Isomorphism Proposition 199.5 to prove Statement (1). ■

199.3. **The Kronecker pairing.** In the previous section we used cellular cohomology to compute the isomorphism types of the cohomology groups of many different topological spaces. But this calculation of the isomorphism types does not give us any explicit generators of the cohomology groups. For example on page 3813 we introduced a singular 1-cochain $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ and we showed in the θ-Cocycle Lemma 198.5 that $\theta_{\mathbb{Z}}$ defines a non-zero cohomology class $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$. By the discussion on page 3830 we now also know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$, but we do not know whether $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is a generator.

On the other hand in earlier chapters we had obtained a pretty good understanding of generators of homology groups. The idea now is to translate information on homology groups to information on cohomology groups. The key tool for relating homology and cohomology is given by the Kronecker pairing that we now introduce.

Lemma 199.7. (Kronecker Pairing Lemma) Let C_* be a chain complex and let G be an abelian group.

(1) The map

$$\begin{array}{rccl} \langle\ ,\ \rangle\colon \mathrm{H}^n(C;G) \times \mathrm{H}_n(C) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle [\varphi],[c]\rangle \coloneqq \underset{\uparrow}{\varphi(c)} \end{array}$$

here $\varphi \in \mathrm{Hom}(C_n,G)$ and $c \in C_n$, so $\varphi(c) \in G$ is defined

is well-defined and bilinear.[2119]

(2) If (X,A) is a pair of topological spaces, then the map

$$\begin{array}{rccl} \langle\ ,\ \rangle_X \coloneqq \langle\ ,\ \rangle_{(X,A)}\colon \mathrm{H}^n(X,A;G) \times \mathrm{H}_n(X,A) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle [\varphi],[c]\rangle_X \coloneqq \varphi(c) \end{array}$$

is well-defined and bilinear.

Definition 199.8. We refer to both maps in the Kronecker Pairing Lemma 199.7 as the Kronecker pairing.[2120]

[2119] In general, given a commutative ring R a map $\Theta\colon U \times V \to W$ between R-modules is called *bilinear*, if for all $u,u' \in U$, $v,v' \in V$ and $r \in R$ we have

$$\begin{array}{lllllll} \Theta(u+u',v) & = & \Theta(u,v)+\Theta(u',v) & \text{and} & \Theta(r\cdot u,v) & = & r\cdot\Theta(u,v), \quad \text{and} \\ \Theta(u,v+v') & = & \Theta(u,v)+\Theta(u,v') & \text{and} & \Theta(u,r\cdot v) & = & r\cdot\Theta(u,v). \end{array}$$

In our special case we consider the case of $\mathbb{Z}$-modules.

[2120] Leopold Kronecker (1823–1891) was a German mathematician who worked on number theory. I don't know how his name got attached to this pairing.

Proof. Let C_* be a chain complex and let G be an abelian group.

(1) Let $\varphi \in \operatorname{Hom}(C_n, G)$ be some cocycle and let $c \in C_n$ be some cycle. Furthermore let $\alpha \in \operatorname{Hom}(C_{n-1}, G)$ and let $d \in C_{n+1}$. Note that the following equalities hold in G:

$$\begin{array}{rcccl} (\varphi + \delta_{n-1}(\alpha))(c) & \underset{\uparrow}{=} & \varphi(c) + \alpha(\partial_n(c)) & \underset{\uparrow}{=} & \varphi(c), \\ & \text{Definition 198.1 of } \delta_{n-1} & & \text{since } c \text{ is a cycle} & \\ \varphi(c + \partial_n(d)) & \underset{\uparrow}{=} & \varphi(c) + (\delta_{n-1}(\varphi))(d) & \underset{\uparrow}{=} & \varphi(c). \\ & \text{Definition 198.1 of } \delta_{n-1} & & \text{since } \varphi \text{ is a cocycle} & \end{array}$$

This calculation shows that the definition of the Kronecker pairing does not depend on the choices of the representatives of the cohomology and the homology classes. Finally note that it is clear that the Kronecker pairing is bilinear.

(2) This is just a special case of (a). ∎

The following lemma shows that the Kronecker pairing is natural with a suitable interpretation of the word "natural":

Lemma 199.9. (Kronecker Pairing–Natural Lemma) Let C_* be a chain complex and let G be an abelian group.

(1) (a) Let $f\colon C_* \to D_*$ be a chain map between two chain complexes and let G be an abelian group. For any $c \in \mathrm{H}_n(C)$ and any $\varphi \in \mathrm{H}^n(D; G)$ we have

$$\langle f^*(\varphi), c\rangle \;=\; \langle \varphi, f_*(c)\rangle.$$

(b) Let $f\colon (X, A) \to (Y, B)$ be a continuous map between two pairs of topological spaces. For any $c \in \mathrm{H}_n(X, A)$ and any $\varphi \in \mathrm{H}^n(Y, B; G)$ we have

$$\langle f^*(\varphi), c\rangle_X \;=\; \langle \varphi, f_*(c)\rangle_Y.$$

(2) If $\varphi\colon G_1 \to G_2$ is a homomorphism between abelian groups then the following diagram commutes:[2121]

$$\begin{array}{ccc} \mathrm{H}^n(C; G_1) \times \mathrm{H}_n(C) & \xrightarrow{\langle\,,\,\rangle} & G_1 \\ \downarrow{\scriptstyle \varphi_* \times \mathrm{id}} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{H}^n(C; G_2) \times \mathrm{H}_n(C) & \xrightarrow{\langle\,,\,\rangle} & G_2. \end{array}$$

Proof. Let C_* be a chain complex and let G be an abelian group.

(1) (a) The statement also follows immediately from the definitions: Indeed, if $[c] \in \mathrm{H}_n(C)$ and $[\varphi] \in \mathrm{H}^n(D; G)$, then

$$\langle f^*([\varphi]), [c]\rangle \;=\; \langle [f^*\varphi], [c]\rangle \;=\; (f^*\varphi)(c) \underset{\uparrow}{=} \varphi(f_*(c)) \;=\; \langle [\varphi], [f(c)]\rangle \;=\; \langle [\varphi], f([c])\rangle.$$

definition of $f^*\varphi$

(b) This statement is just a special case of (a).

(2) This statement follows basically immediately from the definition of the homomorphism $\varphi_*\colon \mathrm{H}^n(C; G_1) \to \mathrm{H}^n(C; G_2)$ in the Cohomology-Coefficients-Functor Lemma 197.21. ∎

Now we state the lemma which will allow us in many situations to show that a given collection of cocycles forms a basis for a given cohomology group.

[2121] The vertical map $\varphi_*\colon \mathrm{H}^n(C; G_1) \to \mathrm{H}^n(C; G_2)$ is provided by the Cohomology-Coefficients-Functor Lemma 197.21.

Lemma 199.10. (Kronecker-Basis Lemma)

(1) Let A and B be free abelian groups of rank k, let $a_1, \ldots, a_k \in A$ and let $b_1, \ldots, b_k \in B$. Furthermore let $\langle\ ,\ \rangle\colon A \times B \to \mathbb{Z}$ be a bilinear pairing. If

$$\det\Big(\underbrace{\langle a_i, b_j\rangle_{i,j=1,\ldots,k}}_{=(k\times k)\text{–matrix}}\Big) \;=\; \pm 1,$$

then both $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are bases.

(2) Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}_n(X;\mathbb{Z})$ are free abelian groups of rank k and suppose that we are given $\varphi_1, \ldots, \varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1, \ldots, c_k \in \mathrm{H}_n(X)$. If

$$\det\Big(\underbrace{\langle \varphi_i, c_j\rangle_{i,j=1,\ldots,k}}_{=(k\times k)\text{–matrix}}\Big) \;=\; \pm 1,$$

then $\varphi_1, \ldots, \varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1, \ldots, c_k \in \mathrm{H}_n(X)$ are both bases.

Remark. The Kronecker-Basis Lemma 199.10 (2) is really quite generous, it gives us as output that both sets are bases. In particular to apply the lemma we do not even have to verify that the c_i form a basis. □

Example. Let $X = S^1$. We denote by $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 3813. We consider the singular cycle $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$. We now see that

$$\langle [\theta_{\mathbb{Z}}], [\mu]\rangle \;\underset{\substack{\uparrow\\ \text{Definition 199.8}\\ \text{of } \langle\ ,\ \rangle_{S^1}}}{=}\; \theta_{\mathbb{Z}}(\mu) \;\underset{\substack{\uparrow\\ \text{definition of } \theta_{\mathbb{Z}}\\ \text{from page 3813}}}{=}\; \alpha_{\mathbb{Z}}\Big(\text{boundary of the lift } \begin{array}{ccc}\widetilde{\mu}\colon \Delta^1 & \to & \mathbb{R}\\ (1-t,t) & \mapsto & t\end{array} \text{ of } \mu\Big) \;\underset{\substack{\uparrow\\ \text{see the discussion}\\ \text{on page 3813}}}{=}\; 1.$$

From the Sphere-H_*-Proposition 149.4 we know that $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ and by the discussion on page 3830 we know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Therefore it follows from the Kronecker-Basis Lemma 199.10 (2) that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is in fact a generator.[2122] In particular we have an equality $\mathrm{H}^1(S^1;\mathbb{Z}) = \mathbb{Z}\cdot[\theta_{\mathbb{Z}}]$. □

Proof.

(1) We denote by $i\colon \mathbb{Z}^k \to A$ and by $j\colon \mathbb{Z}^k \to B$ the homomorphisms that are given by $e_i \mapsto a_i$ and by $e_i \mapsto b_i$. We denote by $\Phi\colon A \to \mathrm{Hom}(B,\mathbb{Z})$ the homomorphism that is given by $a \mapsto (b \mapsto \langle a, b\rangle)$.[2123] The desired statement follows from the following three observations:

(a) Let $\gamma\colon G \to H$ be an epimorphism between two finitely generated free abelian groups of the same rank. The Epi-Automorphism Lemma 97.27 says that if γ is an epimorphism, then γ it is already an isomorphism.

(b) We consider the following sequence of homomorphisms

$$\mathbb{Z}^k \;\xrightarrow{\ i\ }\; A \;\xrightarrow{\ \Phi\ }\; \mathrm{Hom}(B,\mathbb{Z}) \;\xrightarrow{\ j^*\ }\; \mathrm{Hom}(\mathbb{Z}^k,\mathbb{Z}).$$

[2122] It also follows from the Kronecker-Basis Lemma 199.10 (2) that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ represents a generator for $\mathrm{H}_1(S^1) \cong \mathbb{Z}$. We had obtained that result on page 2866.

[2123] The fact that $\langle\ ,\ \rangle$ is bilinear implies that $b \mapsto \langle a, b\rangle$ is a homomorphism and that $\Phi\colon A \to \mathrm{Hom}(B,\mathbb{Z})$ is a homomorphism.

With respect to the basis $\mathfrak{e}_1, \dots, \mathfrak{e}_k$ of $\mathbb{Z}^k$ and the dual basis $\mathfrak{e}_1^*, \dots, \mathfrak{e}_k^*$ of $\operatorname{Hom}(\mathbb{Z}^k, \mathbb{Z})$ this linear map is represented by the matrix $\langle a_i, b_j \rangle_{i,j=1,\dots,k}$. Thus by our hypotheses this map is an isomorphism. Furthermore by our hypotheses each of the four individual groups is a free abelian group of rank k.

(c) Since the map in (b) is an isomorphism it is in particular an epimorphism. Starting from the right and using (a) three times in a row we see that all maps are in fact isomorphisms. In particular we see that $i\colon \mathbb{Z}^k \to A$ is an isomorphism. But this implies that $a_1, \dots, a_k$ is a basis for A. The same argument turned around shows that $b_1, \dots, b_k$ is a basis for B.

(2) This statement follows immediately from (1) and the fact, proved in the Kronecker Pairing Lemma 199.7, that the Kronecker pairing is bilinear. ■

Now we will use the Kronecker-Basis Lemma 199.10 (2) to determine explicit bases for the first cohomology groups of surfaces. We start out with the torus.

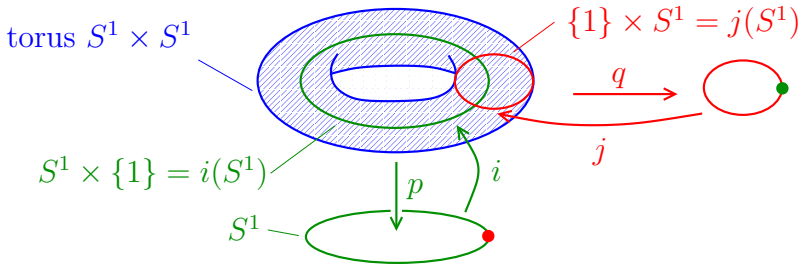
Lemma 199.11. (Torus-H^1-Lemma) We denote by $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 3813. Furthermore we consider the two projections

$$\begin{array}{rcl} p\colon S^1 \times S^1 & \to & S^1 \\ (v,w) & \mapsto & v \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon S^1 \times S^1 & \to & S^1 \\ (v,w) & \mapsto & w. \end{array}$$

Then $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2$.

Proof. First we introduce some notation:

- Let $\mu\colon \Delta^1 \to S^1$ be the singular 1-cycle that is given by $\mu(1-t,t) = \exp(2\pi \mathrm{i} t)$.
- Let $c\colon \Delta^1 \to S^1$ be the constant singular 1-cycle that is given by $(1-t,t) \mapsto 1$.
- We write $\theta = \theta_{\mathbb{Z}}$.
- We denote by $\begin{array}{rcl} i\colon S^1 & \to & S^1 \times S^1 \\ z & \mapsto & (z,1) \end{array}$ and $\begin{array}{rcl} j\colon S^1 & \to & S^1 \times S^1 \\ z & \mapsto & (1,z) \end{array}$ the two obvious inclusion maps.



We calculate that

$$\begin{pmatrix} \langle [p^*\theta], [i_*(\mu)] \rangle & \langle [p^*\theta], [j_*(\mu)] \rangle \\ \langle [q^*\theta], [i_*(\mu)] \rangle & \langle [q^*\theta], [j_*(\mu)] \rangle \end{pmatrix} \overset{\downarrow}{=} \begin{pmatrix} \theta(p_*(i_*(\mu))) & \theta(p_*(j_*(\mu))) \\ \theta(q_*(i_*(\mu))) & \theta(q_*(j_*(\mu))) \end{pmatrix} = \begin{pmatrix} \theta(\mu) & \theta(c) \\ \theta(c) & \theta(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

it follows from the definition of the Kronecker pairing that $\langle [p^*\theta], [i_*(\mu)] \rangle = (p^*\theta)(i_*(\mu)) = \theta(p_*(i_*(\mu)))$, the other terms are dealt with in a similar fashion

$\uparrow$

since $p \circ i = \mathrm{id}_{S^1}$ and $q \circ j = \mathrm{id}_{S^1}$ and since $p \circ j = c\colon \Delta^1 \to S^1$ and $q \circ i = c\colon \Delta^1 \to S^1$

We know from the Torus-H_*-Proposition 153.9 (1) that $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$. Recall that we saw on page 3831 that $H^1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2$. The above calculation, together with the Kronecker-Basis Lemma 199.10 (2) shows that $p^*([\theta_\mathbb{Z}])$ and $q^*([\theta_\mathbb{Z}])$ form a basis for $H^1(S^1 \times S^1; \mathbb{Z})$. ■

Together with the H^0-Proposition 198.9 we have now found explicit generators for the cohomology groups of the torus in dimensions 0 and 1. The following question arises:

Question 199.12. Can we find an explicit generator of $H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$?

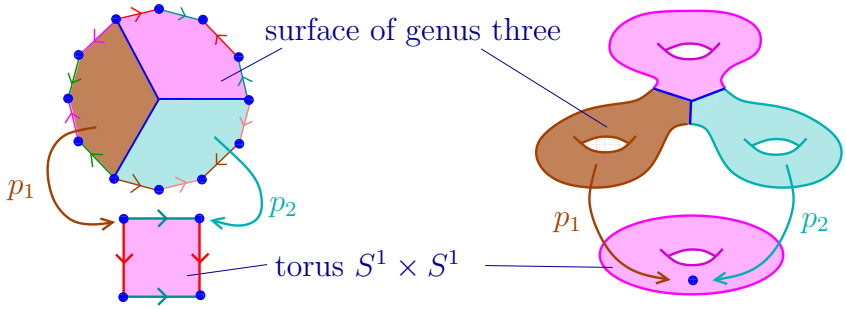
We will answer the question in the Torus–Cup Product Lemma 203.28.

Now we turn to surfaces of higher genus. As usual, given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. The following lemma is a generalization of the Torus-H^1-Lemma 199.11.

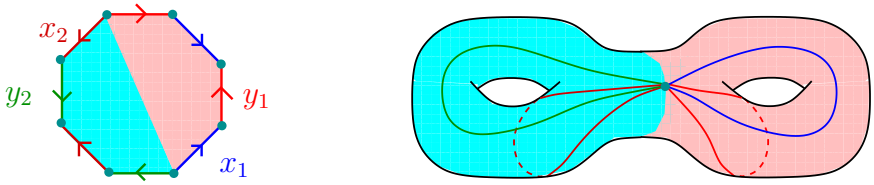
Lemma 199.13. (Surface-H^1-Lemma) Let $g \in \mathbb{N}_{\geq 2}$. We introduce the following notation:

- In the figure below we show projections $p_i \colon \Sigma_g \to S^1 \times S^1$, $i = 1, \ldots, g$.[2124]
- Let $\alpha, \beta \in H^1(S^1 \times S^1; \mathbb{Z})$ be the basis that is given by the Torus-H^1-Lemma 199.11.

With this notation a basis for $H^1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is given by $p_i^*(\alpha), p_i^*(\beta)$, $i = 1, \ldots, g$.



Proof. As we will see, the proof of the lemma is very similar to the proof of the Torus-H^1-Lemma 199.11. For $i = 1, \ldots, g$ we write $\varphi_i = p_i^*(\alpha)$ and $\psi_i := p_i^*(\beta)$. We consider the singular 1-cycles $x_1, \ldots, x_g$ and $y_1, \ldots, y_g$ that are indicated in the figure below.



As in the proof of the Torus-H^1-Lemma 199.11 it follows easily from the definitions that (for $g = 2$) we have

$$\begin{pmatrix} \langle \varphi_1, x_1 \rangle & \langle \varphi_1, y_1 \rangle & \langle \varphi_1, x_2 \rangle & \langle \varphi_1, y_2 \rangle \\ \langle \psi_1, x_1 \rangle & \langle \psi_1, y_1 \rangle & \langle \psi_1, x_2 \rangle & \langle \psi_1, y_2 \rangle \\ \langle \varphi_2, x_1 \rangle & \langle \varphi_2, y_1 \rangle & \langle \varphi_2, x_2 \rangle & \langle \varphi_2, y_2 \rangle \\ \langle \psi_2, x_1 \rangle & \langle \psi_2, y_1 \rangle & \langle \psi_2, x_2 \rangle & \langle \psi_2, y_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The lemma now follows immediately from the Kronecker-Basis Lemma 199.10 (2). ■

[2124] In fact for $g = 2$ we already saw this map in the figure on page 1771.

199.4. Exercises for Chapter 199 .

Exercise 199.1. Let X be a CW-complex and let $n \in \mathbb{N}_0$. As we discussed on page 3828 we can think of elements in $\mathrm{C}_n(X;\mathbb{Z}) = \mathrm{C}_n(X) \otimes \mathbb{Z}$ as finite linear combinations of cells and we can think of elements in $\mathrm{C}^n(X;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_n(X),\mathbb{Z})$ as arbitrary (possibly infinite) linear combinations of cells.

(a) Why does it not make sense to apply the usual cellular boundary map to arbitrary linear combination of cells?
(b) Does the coboundary map restrict to a coboundary map on finite linear combination of cells?

Exercise 199.2. Let X be a finite CW-complex.

(a) Can we determine the Euler characteristic of X from the cohomology groups $\mathrm{H}^k(X;\mathbb{Z})$?
(b) Let R be a commutative ring. Can we determine the Euler characteristic of X from the cohomology modules $\mathrm{H}^k(X;R)$?

Exercise 199.3. Let K be the Klein bottle.

(a) Determine the cohomology groups $\mathrm{H}^*(K;\mathbb{Z})$.
(b) Determine the cohomology groups $\mathrm{H}^*(K;\mathbb{F}_2)$.
(c) Determine the cohomology groups $\mathrm{H}^*(K;\mathbb{F}_3)$.

Exercise 199.4. Let M be the Möbius band.

(a) Compute the cohomology groups $\mathrm{H}^*(M,\partial M;\mathbb{Z})$.
(b) Compute the cohomology groups $\mathrm{H}^*(M,\partial M;\mathbb{F}_2)$.

Exercise 199.5. Let $n \in \mathbb{N}$. Determine the cohomology groups $\mathrm{H}^i(\mathbb{R}\mathrm{P}^n;\mathbb{F}_3)$.

Exercise 199.6. Let G be an abelian group.

(a) Let (X,A) be a pair of topological spaces. Show that if A is a retract of X, then for each $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}^n(X;G) \;\cong\; \mathrm{H}^n(A;G) \oplus \mathrm{H}^n(X,A;G).$$

(b) Show that the isomorphism from (a) cannot be chosen to be natural. More precisely, provide a pair (X,A) of topological space, a continuous map $f\colon X \to Y$ with $f(A) \subseteq B$ and an abelian group G such that there is no commutative diagram of the form

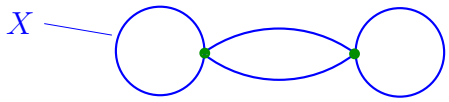
$$\begin{array}{ccc} \mathrm{H}^n(X;G) & \xrightarrow[\cong]{\Phi} & \mathrm{H}^n(A;G) \oplus \mathrm{H}^n(X,A;G) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} \\ \mathrm{H}^n(X;G) & \xrightarrow[\cong]{\Phi} & \mathrm{H}^n(A;G) \oplus \mathrm{H}^n(X,A;G). \end{array}$$

Exercise 199.7. Does there exist a CW-complex with $\mathrm{H}^2(X;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and which admits S^2 as a retract?

Exercise 199.8. Let X be a topological space which is homotopy equivalent to a retract of a finite CW-complex. Show that for every $n \in \mathbb{N}_0$ the cohomology group $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated group.

Exercise 199.9. On page 3816 we introduced an explicit cocycle $\overline{\gamma}_{\frac{1}{2}} : \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$. Show that $\overline{\gamma}_{\frac{1}{2}}$ represents a generator of $\mathrm{H}^1(I,\partial I;\mathbb{Z})$.

Exercise 199.10. We consider the (fairly explicit) topological graph X shown in the figure below. Give explicit 1-cocycles that represent a basis for $\mathrm{H}^1(X;\mathbb{Z})$.



Exercise 199.11. Let $n \geq 1$ and let $f\colon S^n \to S^n$ be a continuous map. In Definition 151.3 we defined the degree $\deg(f)$ of f to be the unique integer such that $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by $\deg(f)$. Show that for any abelian group G the induced map $f^*\colon \mathrm{H}^n(S^n;G) \to \mathrm{H}^n(S^n;G)$ is also given by multiplication by $\deg(f)$.

Exercise 199.12. Let (X,A) be a pair of topological spaces, let G be an abelian group, let $\sigma \in \mathrm{H}_n(X,A)$ and let $\varphi \in \mathrm{H}^{n-1}(A;G)$. Show that we have the following equality in G:

$$\underset{\substack{\uparrow\\ \text{connecting homomorphism}\\ \delta_{n-1}\colon \mathrm{H}^{n-1}(A;G)\to\mathrm{H}^n(X,A;G)}}{\langle \delta_{n-1}(\varphi), \sigma\rangle_{(X,A)}} \quad = \quad \underset{\substack{\uparrow\\ \text{connecting homomorphism}\\ \partial_n\colon \mathrm{H}_n(X,A)\to\mathrm{H}_{n-1}(A)}}{\langle \varphi, \partial_n(\sigma)\rangle_A}.$$

Exercise 199.13. Let $k \in \mathbb{Z}$ and let G be an abelian group. Show that given any topological space X we have a natural isomorphism

$$\Sigma_X\colon \widetilde{\mathrm{H}}^k(\Sigma(X);G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{k+1}(X;G)$$

which has the property that for any $\varphi \in \mathrm{H}^k(X;G)$ and any $\sigma \in \mathrm{H}_k(X)$ we have

$$\langle \Sigma_X(\varphi), \Sigma_X(\sigma)\rangle_{\Sigma(X)} \;=\; \langle \varphi, \sigma\rangle_X \;\in\; G.$$

Hint. Consider the proof of the Suspension-H_*-Proposition 149.3 and make use of Exercise 199.12.

Exercise 199.14. Let $p\colon S^2 \to \mathbb{R}\mathrm{P}^2$ be the obvious projection. What is the induced map $p^*\colon \mathrm{H}^2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \to \mathrm{H}^2(S^2;\mathbb{F}_2)$?

Exercise 199.15. How many cohomology groups of the topological space $\mathrm{GL}(3,\mathbb{R})$ can you determine?

Exercise 199.16. Let M be a connected topological manifold (not necessarily compact). Are the cohomology groups $\mathrm{H}^k(M;\mathbb{Z})$ necessarily countable?

Exercise 199.17. Let R be a commutative ring. Define for every chain complex C_* a Kronecker type pairing

$$\langle -,- \rangle\colon \mathrm{H}_n(C_*;R) \times \mathrm{H}^n(C_*;R) \;\to\; R.$$

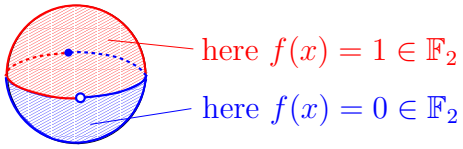
Exercise 199.18. Let $f\colon S^2 \to \mathbb{F}_2$ be a map such that $f(-x) = 1 - f(x)$ for all $x \in S^2$. (An example is illustrated in the figure below.) We denote by $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ the projection. We consider the following cochain

$$\begin{array}{rcl} \varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) & \to & \mathbb{F}_2 \\ (\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2) & \mapsto & \text{pick a lift } \widetilde{\sigma}\colon \Delta^1 \to S^2 \text{ of } \sigma \text{ and calculate} \\ & & f(\widetilde{\sigma}(0,1)) - f(\widetilde{\sigma}(0,1)). \end{array}$$

(a) Show that φ is well-defined, i.e. show that $\varphi(\sigma)$ does not depend on the choice of the lift $\widetilde{\sigma}$.

(b) Show that φ is a cocycle in $\mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$.

(c) Show that φ represents a non-zero cohomology class.
Hint. Use Exercise 199.17.



CHAPTER 200

Excision and Mayer–Vietoris for cohomology

In this chapter we want to extend the following deep theorems from the homological setup to the cohomological setting:

(1) The Excision Theorem 148.1.
(2) The Mayer-Vietoris Theorem 153.2.

These theorems will allow us in particular to calculate the cohomology groups of suspensions and wedges.

This chapter is organized as follows:

- In Section 200.1 we will state and prove the Excision-in-H^*-Theorem 200.1.
- In Section 200.2 we will calculate the cohomology groups of suspensions.
- In Section 200.3 we will recall good subsets and we will study relative cohomology groups.
- In Section 200.4 we will calculate the cohomology groups of wedges.
- In Section 200.5 we will state and prove the Mayer–Vietoris-H^*-Theorem 200.11.
- In Section 200.6 we will provide exercises for Chapter 200.

In the next Chapter 201 we will formulate and prove a Universal Coefficient Theorem for cohomology groups.

200.1. **Excision theorems for cohomology groups.** We start out with the following analog of the Excision Theorems 148.1 and 148.2, the Manifold–Excision Theorem 148.3 and the CW-Excision Theorem 148.4.

Theorem 200.1. (Excision-in-H^*-Theorem) Let X be a topological space, let G be an abelian group and let $n \in \mathbb{N}_0$.

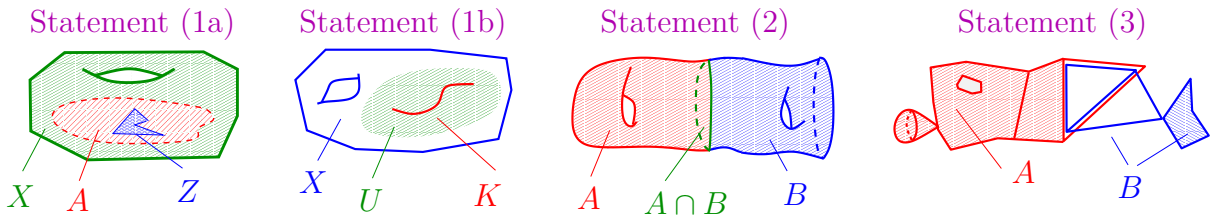
(1) (a) Let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. The inclusion $i\colon (X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism
$$i^*\colon \mathrm{H}^n(X, A; G) \xrightarrow{\cong} \mathrm{H}^n(X \setminus Z, A \setminus Z; G).$$
(b) Let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $i\colon (U, U \setminus K) \to (X, X \setminus K)$ induces an isomorphism
$$i^*\colon \mathrm{H}^n(X, X \setminus K; G) \xrightarrow{\cong} \mathrm{H}^n(U, U \setminus K; G).$$
(2) We suppose that X is a topological manifold. Let $A, B \subseteq X$ be two codimension-zero submanifolds such that the following statements hold:
(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.

The following inclusion-induced map is an isomorphism:

$$\mathrm{H}^n(X, A; G) \xrightarrow{\cong} \mathrm{H}^n(B, A \cap B; G).$$

(3) If X is a CW-complex and if A and B are two subcomplexes of X with $X = A \cup B$, then the following inclusion-induced map is an isomorphism:

$$\mathrm{H}^n(X, A; G) \xrightarrow{\cong} \mathrm{H}^n(B, A \cap B; G).$$



The proof of the Excision-in-H*-Theorem 200.1 relies on the following wonderful proposition:

Proposition 200.2. (Also-Iso-on-Cohomology Proposition) Let $f\colon (X, A) \to (Y, B)$ be a continuous map between pairs of topological spaces. If for all $n \in \mathbb{N}_0$ the induced map

$$f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$$

is an isomorphism, then for all abelian groups G and all $n \in \mathbb{N}_0$ the induced map

$$f^*\colon \mathrm{H}^n(Y, B; G) \to \mathrm{H}^n(X, A; G)$$

is also an isomorphism for all $n \in \mathbb{N}_0$.

Remark. In the Vanishing-of-(Co-) Homology Proposition 201.29 (2) we will see that the converse to the Also-Iso-on-Cohomology Proposition 200.2 holds. □

Proof of the Also-Iso-on-Cohomology Proposition 200.2. Let $f\colon (X, A) \to (Y, B)$ be a continuous map between pairs of topological spaces and let G be an abelian group. We suppose that $f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$ is an isomorphism for all $n \in \mathbb{N}_0$. By the discussion on page 2798 we know that all chain groups $\mathrm{C}_n(X, A)$ and $\mathrm{C}_n(Y, B)$ are free abelian groups. It thus follows from the Quasi Iso–Homotopy Equivalence Proposition 158.8 that the induced map $f_*\colon \mathrm{C}_*(X, A) \to \mathrm{C}_*(Y, B)$ is a chain homotopy equivalence. Now we get the desired conclusion from the Dual Cochain Map Lemma 197.20 (3b) together with the Cochain Homotopy-H*-Lemma 197.14 (2). ■

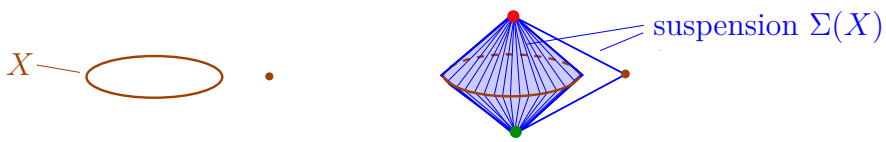
Proof of the Excision-in-H*-Theorem 200.1. It follows from Excision Theorems 148.1 and 148.2, the Manifold–Excision Theorem 148.3 and the CW-Excision Theorem 148.4 that in each of the four situations under consideration the inclusion-induced map on homology is an isomorphism for all $n \in \mathbb{N}_0$. It follows from the Also-Iso-on-Cohomology Proposition 200.2 that in all cases for all abelian groups G all inclusion-induced maps on cohomology groups with G-coefficients are isomorphisms. ■

200.2. Cohomology groups of suspensions.

We start out by recalling Definition 20.21:

Definition 200.3. Let X be a topological space. We define the suspension of X to be the topological space

$$\Sigma(X) := ((X \times [-1, 1])/(X \times \{-1\}))/(X \times \{1\}).$$



Recall that in the Suspension-H_*-Proposition 149.3 we showed that for any $k \in \mathbb{Z}$ and any topological space X we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)).$$

Not surprisingly we have a totally analogous statement in cohomology:

Lemma 200.4. (Suspension-H^*-Lemma) Let $k \in \mathbb{Z}$ and let G be an abelian group. Given any topological space X we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}^k(X;G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{k+1}(\Sigma(X);G)$$

which has the property that for any $\varphi \in \mathrm{H}^k(X;G)$ and any $\sigma \in \mathrm{H}_k(X)$ we have

$$\langle \Sigma_X(\varphi), \Sigma_X(\sigma)\rangle_{\Sigma(X)} \;=\; \langle \varphi, \sigma\rangle_X \;\in\; G.$$

Proof. The construction of the isomorphism Σ_X is basically the same as the construction of the corresponding isomorphism in homology. In the proof of the Suspension-H_*-Proposition 149.3 we just need to replace the Excision Theorems 148.1 by the Excision-in-H^*-Theorem 200.1. Using Exercise 199.12 one can easily verify that this isomorphism satisfies the desired equality for the Kronecker pairings. The details will surely be filled in in Exercise 199.13. ■

200.3. **Good subsets and relative cohomology.** Psychologically it is often helpful to replace relative (co-) homology groups by the (co-) homology groups of a quotient space. In the Relative-to-Quotient-H_*-Proposition 149.13 we had seen that in the homological setting we can perform this switch for reasonable subsets. In this section we will consider the cohomological setting. We start out with the following cohomological analog of Definition 149.10.

Definition 200.5. Let X be a topological space and let $A \subseteq X$ be a subset. We denote by $p\colon (X,A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ and any abelian group G we define

$$\begin{tikzcd} \widetilde{\mathrm{H}}^n(X/A;G) \arrow[rr, bend right, "=:\,\text{ю}"'] & \mathrm{H}^n(X/A, A/A;G) \arrow[l, "\substack{\text{isomorphism ж from the}\\ \text{Reduced-to-Relative}\\ \text{Lemma 198.20}}"', "\cong"] \arrow[r, "p^*"] & \mathrm{H}^n(X,A;G) \end{tikzcd}$$

The following lemma, which is the cohomological analog of the Relative-to-Quotient-H_*-Lemma 149.11, summarizes two properties of the maps ю.

Lemma 200.6. (Quotient-to-Relative-H*-Lemma) Let G be an abelian group.

(1) Let X be a topological space and let $A \subseteq X$ be a subset. We denote by $p\colon X \to X/A$ the natural projection. The following diagram commutes:[2125]

$$\begin{array}{ccc} & \mathrm{H}^n(X,A;G) & \\ {\scriptstyle \text{ю}}\nearrow & & \searrow{\scriptstyle \text{ж}} \\ \widetilde{\mathrm{H}}^n(X/A;G) & \xrightarrow{\quad p_* \quad} & \widetilde{\mathrm{H}}^n(X;G). \end{array}$$

(2) We denote by *PairTop* the category of pairs of topological spaces that we introduced in Definition 19.23. Furthermore as usual we denote by *AbGroup* the category *AbGroup* of abelian groups. The above maps $\text{ю}\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ define a natural transformation from the functor

$$\begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X,A) & \mapsto & \widetilde{\mathrm{H}}^n(X/A;G) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathit{PairTop} & \to & \mathit{AbGroup} \\ (X,A) & \mapsto & \mathrm{H}^n(X,A;G). \end{array}$$

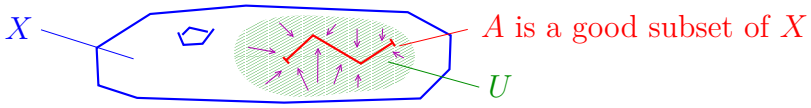
Proof. Both statements follow easily from the definition. Alternatively one can easily modify the proof of the Relative-to-Quotient-H$_*$-Lemma 149.11 to obtain a proof for the current lemma. ■

Next we recall Definition 149.12:

Definition 200.7. Let X be a topological space.

(1) We say that a subset $A \subseteq X$ is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.

(2) We say that a point $a \in A$ is good, if $\{a\} \subseteq X$ is good.



Example. Note that it follows from the Subcomplex–Closed Lemma 136.11 and the Subcomplex–Neighborhood Proposition 137.9 (1) that every subcomplex A of a CW-complex X is good. □

The following proposition is the cohomological analog of the Relative-to-Quotient-H$_*$-Proposition 149.13.

Proposition 200.8. (Quotient-to-Relative-H*-Proposition) Let X be a topological space, let $A \subseteq X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset, then for any abelian group G the natural homomorphism $\text{ю}\colon \widetilde{\mathrm{H}}^n(X/A;G) \to \mathrm{H}^n(X,A;G)$ is an isomorphism.

Proof. At this stage we have all the tools to rewrite the proof of the Relative-to-Quotient-H$_*$-Proposition 149.13. We just need to replace the Reduced-to-Relative-H$_*$-Lemma 147.9 by the Reduced-to-Relative-H*-Lemma 198.20 and (more importantly) we need to replace the Excision Theorem 148.1 by the Excision-in-H*-Theorem 200.1 (1a). We leave it to the reader to verify that nothing goes wrong. ■

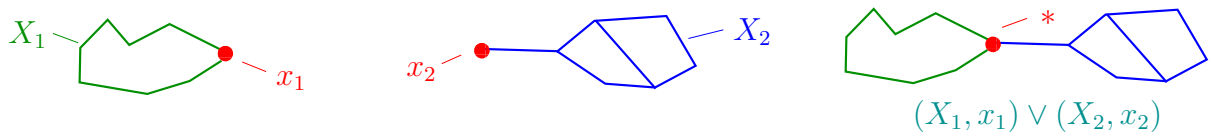
[2125] The right diagonal map is the composition of the natural map $\text{я}\colon \mathrm{H}^n(X,A;G) \to \mathrm{H}^n(X;G)$ with the natural map $\mathrm{H}^n(X;G) \to \widetilde{\mathrm{H}}_n(X;G)$ from the Reduced Cohomology Lemma 198.17.

200.4. **The cohomology groups of wedges.** We start out by recalling Definition 96.1:

Definition 200.9. Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces. We refer to

$$\bigvee_{k\in K}(X_k, x_k) \coloneqq \Big(\bigsqcup_{k\in K} X_k\Big)\Big/\bigsqcup_{k\in k}\{x_k\}$$

as the wedge of the pointed topological spaces (X_k, x_k).



The following proposition is the analog of the Wedge-H$_*$-Proposition 149.16.

Proposition 200.10. (Wedge-H*-Proposition) Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces such that each $x_k \in X_k$ is good. We denote the wedge point of $\bigvee_{k\in K}(X_k, x_k)$ by $*$. Let $n \in \mathbb{N}_0$.

(1) Given $j \in K$ we denote by $i_j\colon X_j \to \bigvee_{k\in K}(X_k, x_k)$ the natural inclusion that we introduced in Definition 96.1. The natural map

$$\prod_{k\in K} i_k^*\colon \mathrm{H}^n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}; G\Big) \to \prod_{k\in K}\mathrm{H}^n(X_k, \{x_k\}; G)$$

is an isomorphism.

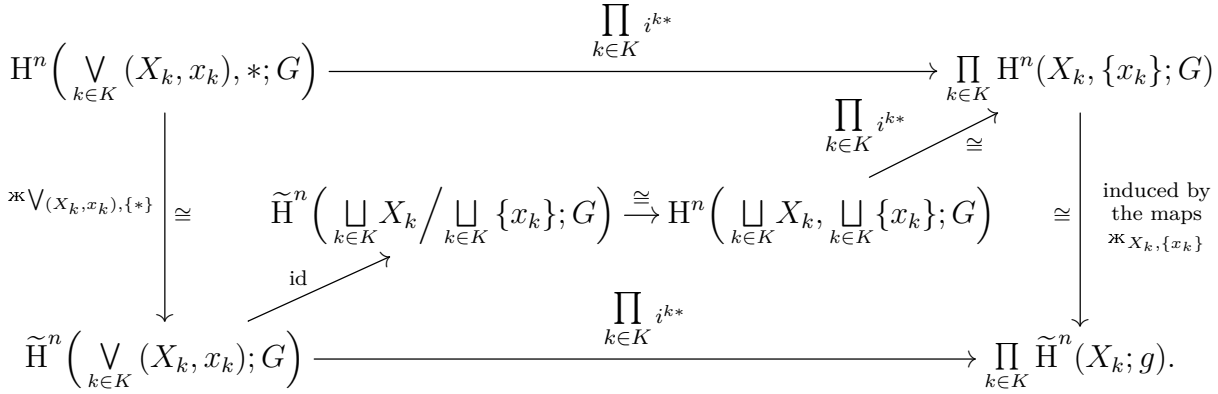
(2) For $j \in K$ let $p_j\colon \bigvee_{k\in K} X_k \to X_j$ be the natural map that we introduced in Definition 96.1. If K is finite, then the map

$$\bigoplus_{k\in K} p_k^*\colon \bigoplus_{k\in K}\mathrm{H}^n(X_k, \{x_k\}; G) = \underset{\text{since } K \text{ is finite}}{\prod_{k\in K}}\mathrm{H}^n(X_k, \{x_k\}; G) \to \mathrm{H}^n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}; G\Big)$$

is defined and it is the inverse to the isomorphism from (1).

Both statements also hold for reduced cohomology instead of relative cohomology.

Proof of the Wedge-H*-Proposition 200.10 (1). [2126] We consider the following diagram:



The proposition follows from the following clarifications and observations:

(a) It follows from the Reduced-to-Relative-H*-Lemma 198.20 that the vertical maps to the left and right are isomorphisms.

(b) It follows from the obvious generalization of the Path-Component-H*-Lemma 198.8 that the top right diagonal map is an isomorphism.

(c) By hypothesis each $x_k \in X_k$ is good. It follows easily from the definitions that $\bigsqcup_{k\in K}\{x_k\}$ is a good subset of $\bigsqcup_{k\in K} X_k$. Thus we can apply the Quotient-to-Relative-H*-Proposition 200.8 to see that the middle horizontal map is an isomorphism.

(d) The bottom left diagonal map is by definition of the wedge indeed the identity.

(e) One can easily verify that the upper triangle commutes. This implies the desired statement for relative homology groups.

(f) One can also easily verify that the lower triangle commutes. Thus we also get the desired statement on reduced homology groups. ∎

Proof of the Wedge-H*-Proposition 200.10 (2). By (1) it suffices to prove the following claim:

Claim. The composition

$$\prod_{k\in K}\mathrm{H}^n(X_k;\{x_k\};G) \xrightarrow{\bigoplus_{k\in K} p_k^*} \mathrm{H}^n\Big(\bigvee_{k\in K}(X_k,x_k),\{*\};G\Big) \xrightarrow{\prod_{k\in K} i_k^*} \prod_{k\in K}\mathrm{H}^n(X_k,\{x_k\};G)$$

is the identity.

Proof. This claim follows immediately from the observation that $p_k \circ i_k = \mathrm{id}_{X_k}$ and that for $k \neq l$ the map $p_k \circ i_l$ is the constant map $x \mapsto x_k$ which induces the zero map on relative cohomology. ∎

200.5. **The Mayer–Vietoris Theorem for cohomology groups.** We conclude this chapter with the following analog of the Mayer–Vietoris-H$^*_{\mathrm{dR}}$-Theorem 80.2 and the Mayer–Vietoris Theorem 153.2:

[2126] The proof we provide is obtained from the proof of the Wedge-H$_*$-Proposition 149.16 by making all the obvious modifications. But we figured, that every now and then it does not hurt to write down the modified proof.

Theorem 200.11. (Mayer–Vietoris-H*-Theorem) Let X be a topological space and let $A, B \subseteq X$ be subspaces such that one of the following three conditions holds:

(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold[2127] and $A, B \subseteq X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.

It follows from the Excision-in-H*-Theorem 200.1 that in each of the three cases the inclusion-induced map $i^{(X,B)}_{(A,A\cap B)} \colon \mathrm{H}^{n+1}(X, B; G) \to \mathrm{H}^{n+1}(A, A \cap B; G)$ is an isomorphism. We refer to the homomorphism

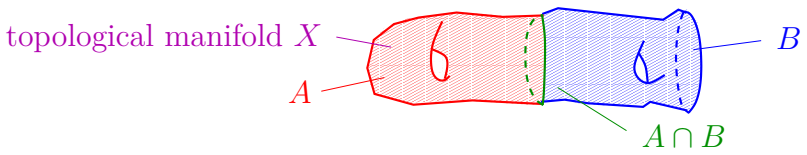
$$\mathrm{H}^n(A \cap B; G) \xrightarrow[\uparrow]{\delta_n} \mathrm{H}^{n+1}(A, A \cap B) \xrightarrow{\left(i^{(X,B)}_{(A,A\cap B;G)}\right)^{-1}} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{\text{я}} \mathrm{H}^{n+1}(X; G)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A \cap B)$, provided by the LES-of-Triple Proposition 198.13

as the connecting homomorphism $\delta_n \colon \mathrm{H}^n(A \cap B; G) \to \mathrm{H}^{n+1}(X; G)$ in cohomology corresponding to (X, A, B). The connecting homomorphism is natural in a suitable sense and with this notation the following sequence is exact:

$$\to \mathrm{H}^n(X; G) \xrightarrow{(i^X_A, -i^X_B)} \mathrm{H}^n(A; G) \oplus \mathrm{H}^n(B; G) \xrightarrow{i^A_{A\cap B} \oplus -i^B_{A\cap B}} \mathrm{H}^n(A \cap B; G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X; G) \to$$

Totally analogous statements also hold if we replace the cohomology groups of $A \cap B, A, B$ and X by the corresponding reduced cohomology groups.



Proof. We gave two slightly different proofs for the Mayer–Vietoris Theorem 153.2. Both proofs can be rewritten easily for the current context:

(1) In the first proof of the Mayer–Vietoris Theorem 153.2, which we gave in Subsection 153.1.1, we just need to replace the LES-of-Pair Proposition 146.7 by the LES-of-Triple Proposition 198.13 and (more importantly), we need to replace the Excision Theorem 148.1, the Manifold–Excision Theorem 148.3 and the CW-Excision Theorem 148.4 by the Excision-in-H*-Theorem 200.1.
(2) Using the Hom-Split-SES Lemma 197.8, the Quasi Iso–Homotopy Equivalence Proposition 158.8 and the Dual Cochain Map Lemma 197.20 one can easily modify the second

[2128]For smooth manifolds there is also a slight and useful variation: If X is a smooth manifold and if A, B are codimension-zero submanifolds with corner (in the sense of Definition 26.14) and if $A \cap B$ is a smooth boundary of A and B (in the sense of Definition 59.4), then the desired statements also hold. The only difference is that in the proof we need to replace the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7.

proof of the Mayer–Vietoris Theorem 153.2, which we gave in Section 153.1.2, to obtain the desired result.

We leave it to the reader to fill in the details. ■

200.6. **Exercises for Chapter 200 .**

Exercise 200.1. Let $\{A_k\}_{k\in K}$ be a family of CW-complexes. Suppose that for each $k \in K$ suppose that we are given a point in the 0-skeleton of A_k. Provide a proof for the Wedge-H*-Proposition 200.10 using cellular cohomology. More precisely, provide a proof along the following lines:

(a) Equip $X \coloneqq \bigvee_{k\in K} A_k$ with the CW-structure provided by the CW-Wedge Lemma 139.11.
(b) Determine the relationship between the reduced cellular chain complexes of X and of the A_i.
(c) Make use of the Singular-Cellular Chain Complex Proposition 158.16.
(d) Make use of the Hom–Direct Sum-Product Lemma 197.6 and the Product–H*-Lemma 197.18.

Exercise 200.2. Use the Mayer–Vietoris-H*-Theorem 200.11 to give an alternative proof of the Suspension-H*-Lemma 200.4.

Exercise 200.3. Let X be a topological space and let $f\colon X \to X$ be a continuous map. As in Definition 21.19 we consider the mapping torus

$$\mathrm{Tor}(X, f) \;\coloneqq\; (X \times [0, 1])/(x, 0) \sim (f(x), 1).$$

We denote by $i\colon X \to \mathrm{Tor}(X, f)$ the natural map that is given by $x \mapsto [(x, 0)]$. In the Mapping Torus-H$_*$-Proposition 153.6 we showed that this data gives rise to an exact sequence which relates the homology groups of the mapping torus $\mathrm{Tor}(X, f)$ to the homology groups of X via the induced maps i_* and f_*. Formulate a corresponding exact sequence in cohomology.

CHAPTER 201

The Universal Coefficient Theorem for Cohomology Groups

One of the questions that arises from the last chapter is, how can we explain the erratic relationship between homology groups and cohomology groups that we saw on page 3832. Furthermore it would be interesting to know whether given a chain complex one can determine the cohomology groups with G-coefficients from the usual homology groups.

These questions are purely in the realm of algebra. More precisely, let (C_*, ∂_*) be a chain complex of free abelian groups and let G an abelian group. We want to find a connection between the cohomology groups $\mathrm{H}^*(C; G) = \mathrm{H}^*(\mathrm{Hom}(C_*, G))$ and the homology groups $\mathrm{H}_*(C)$. This problem, and later on its solution, is very similar to the problem we faced in Chapter 160, where we showed that we can determine the homology groups $\mathrm{H}_*(C_* \otimes G)$ of a chain complex with G-coefficients from the usual homology groups $\mathrm{H}_*(C_*)$ of the chain complex.

This chapter is organized as follows:

- In Section 201.1 we will discuss contravariant functors and exactness.
- In Section 201.2 we will introduce the Ext groups $\mathrm{Ext}(H, G)$ of an abelian group H with coefficients in an abelian group G.
- In Section 201.3 we will state and prove the Universal Coefficient-H^*-Theorem 201.23.
- In Section 201.4 we will discuss applications of the Universal Coefficient-H^*-Theorem 201.23.
- In Section 201.5 we will consider cohomology groups of topological manifolds.
- In Appendix 201.6 we will show that homology vanishes if and only if cohomology vanishes.
- In Section 201.7 we will provide exercises for Chapter 201.

In the next Chapter 202 we will state and prove two other, less frequently used, Universal Coefficient Theorem for cohomology groups.

201.1. **Contravariant functors and exactness.** In the following we denote, as usual, by $\mathcal{AbGroup}$ the category $\mathcal{AbGroup}$ of abelian groups. Recall that in Definition 161.2 we defined what it means for an additive *covariant* functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ to be left-exact, right-exact and exact. Basically the same way we can introduce these notions for additive *contravariant* functors $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$:

Definition 201.1. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive contravariant functor.

(1) We say that F is left-exact[2129] if for every exact sequence[2130]

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact.

(2) We say that F is right-exact, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

of abelian groups the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.

(3) We say that F is exact if it is left-exact and right-exact.

The following lemma is the analog of the Exact Functor–Homology Lemma 161.4:

Lemma 201.2. (Exact Functor–Cohomology Lemma) Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive contravariant functor and let (C_*, ∂_*) be a chain complex. If F is exact, then for any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{H}^k(\underbrace{F(C_*), F(\partial_*)}_{\text{cochain complex}}) \to F(\mathrm{H}_k(\underbrace{C_*, \partial_*}_{\text{chain complex}})).$$

Proof. We leave it to the reader to modify the proof of the Exact Functor–Homology Lemma 161.4 to obtain our new lemma. ■

Recall that in the Tensor–Right-Exact Lemma 161.5 we saw that the covariant functor $A \mapsto A \otimes G$ is right-exact but that in general it is not left-exact. With the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ the situation is exactly the opposite. More precisely, first we give an example that shows that the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ is in general not right-exact and then we prove a lemma that shows that this functor is left-exact.

[2129] Admittedly one could also refer to this definition as "right"-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.

[2130] As in the Additive Functor–Basics Lemma 160.2 we see that for an additive contravariant functor we have $F(0) = 0$.

Example. We consider the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z}$$

where $p\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ denotes the natural projection. We apply the contravariant functor $A \mapsto \mathrm{Hom}(A, \mathbb{Z})$. By the Free Module–Hom-to-Ring Lemma 197.7 the resulting cochain complex is of the form

$$\underbrace{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})}_{=0} \xrightarrow{p^*} \underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}} \xrightarrow[\cdot 2]{(\cdot 2)^*} \underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}} \to 0.$$

We see that the rightmost map is not surjective. Thus this cochain complex is not exact. This shows that the functor $A \mapsto \mathrm{Hom}(A, \mathbb{Z})$ is not right-exact. □

The following lemma is the promised analog of the Tensor–Right-Exact Lemma 161.5.

Lemma 201.3. (Hom-Left-Exact Lemma) For any abelian group G the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ is left-exact.

Proof. [2131]

Let G be an abelian group and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

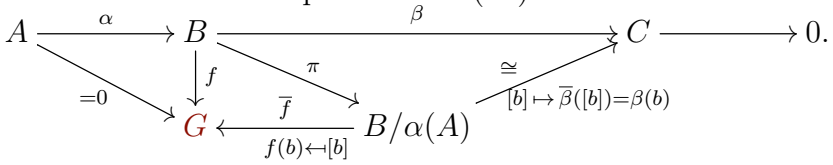
be an exact sequence of abelian groups. We need to show that the sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \xrightarrow{\beta^*} \mathrm{Hom}(B, G) \xrightarrow{\alpha^*} \mathrm{Hom}(A, G)$$

is also exact. It is clear that β^* is injective. Next note that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$, i.e. we have $\mathrm{im}(\beta^*) \subseteq \ker(\alpha^*)$. Thus it remains to prove the following claim:

Claim. We have $\ker(\alpha^*) \subseteq \mathrm{im}(\beta^*)$.

Proof. Let $f\colon B \to G$ be a homomorphism in $\ker(\alpha^*)$. We consider the following diagram:



We make the following observations:

(1) Since $f \in \ker(\alpha^*)$ we know that $\alpha^*(f) = 0$ which in turn implies that f vanishes on $\alpha(A)$. We denote by $\overline{f}\colon B/\alpha(A) \to G$ the resulting homomorphism given by $[b] \mapsto f(b)$ which we obtain from the Quotient Group Lemma 2.4.

(2) We denote by $\pi\colon B \to B/\alpha(A)$ the projection.

(3) It follows from the exactness of the original sequence and the Exactness Lemma 145.2 (3) that the homomorphism β induces an isomorphism $\overline{\beta}\colon B/\alpha(A) \to C$.

[2131] For the Tensor–Right-Exact Lemma 161.5 we gave two proofs, one was down-to-earth, the other one used that the covariant tensor functor $A \mapsto A \otimes G$ is a left adjoint, and thus preserves limits. Here one could also argue in a similar fashion: First one needs to define what one means for a contravariant functor to be an adjoint, and then one needs to show that for a fixed abelian group G the contravariant Hom-functor $A \mapsto \mathrm{Hom}(A, G)$ is an adjoint functor. We will not pursue this approach. The categorically inclined reader might want to look this argument.

(4) By construction the above diagram commutes.
(5) Now we see that $f = \overline{f} \circ \overline{\beta}^{-1} \circ \beta$, i.e. we have $f = \beta^*(\overline{f} \circ \overline{\beta}^{-1})$. ■

In the Tensor-with-Torsion Free Proposition 161.6 we showed that for any torsion-free abelian group G the covariant functor $A \mapsto A \otimes G$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is exact. The question is, what condition on G do we have to impose so that $A \mapsto \mathrm{Hom}(A, G)$ is exact?

The following definition gives us the answer:

Definition 201.4. An abelian group G is called injective[2132] if for all abelian groups A and B, every injective homomorphism $\alpha\colon A \to B$ and every homomorphism $\mu\colon A \to G$ there exists a homomorphism $\nu\colon B \to G$ such that the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \overset{\alpha}{\hookrightarrow} & B \\ & & {\scriptstyle\mu}\downarrow & \swarrow_{\exists\nu} & \\ & & G. & & \end{array}$$

The following lemma shows that we found the correct adjective:

Lemma 201.5. (Hom-to-Injective Lemma) If G is an injective abelian group, then the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is exact.

Proof. As in the Exact Functor Lemma 161.3 it suffices to show that for every short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of abelian groups the resulting sequence

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{\beta^*} \mathrm{Hom}(B, G) \xrightarrow{\alpha^*} \mathrm{Hom}(A, G) \to 0$$

is also exact. Note that by the Hom-Left-Exact Lemma 201.3 it remains to prove that $\alpha^*\colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is an epimorphism. But this follows immediately from the fact that α is a monomorphism and the hypothesis that G is injective. ■

Of course the Hom-to-Injective Lemma 201.5 is not all that interesting without examples of injective abelian groups. The key to finding interesting examples of injective abelian groups is the following definition:

Definition 201.6. An abelian group G is called divisible if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$.

Examples.

(1) The abelian groups $(\mathbb{Q}, +), (\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are clearly divisible.

[2132] The notion of begin "injective" also makes sense for modules over a commutative ring R. In this setting the notion "injective" comes with the twin notion of being "projective": An R-module P is called *projective* if for all R-modules A and B, every epimorphism $\alpha\colon A \to B$ and every homomorphism $\mu\colon P \to B$ there exists a homomorphism $\nu\colon P \to A$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & & & P \\ & & & \swarrow_{\nu} & \downarrow{\scriptstyle\mu} \\ A & \xleftarrow{\alpha} & B & \longrightarrow & 0. \end{array}$$

We will return to this notion in Definition 309.25.

(2) Every quotient of a divisible group is easily seen to be divisible again. For example the quotient group $\mathbb{Q}/\mathbb{Z}$ is also divisible.[2133] This example also shows, using the notation of the definition, that h is not necessarily unique. For example, for $g = [0]$ and $n = 3$ we could take $h = [0]$ or $h = [\frac{1}{3}]$.
(3) The group $\mathbb{Z}$ is not divisible, and the only finite group that is divisible is the trivial group. □

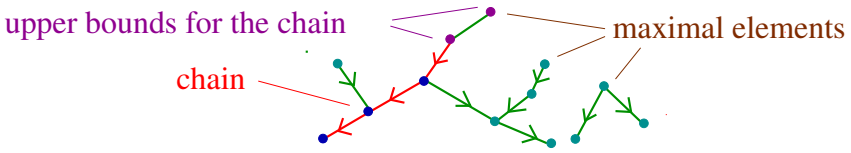
The previous examples together with the following proposition give us many examples of injective abelian groups.

Proposition 201.7. (Divisible Group-implies-Injective Proposition) Every abelian group that is divisible is also injective.

Remark. In Exercise 201.1 we will prove the converse to the Divisible Group-implies-Injective Proposition 201.7. More precisely, we will show that every injective abelian group is also divisible. Also note that in the Divisible Group-implies-Injective Proposition 315.31 we will prove a generalization of the above Divisible-implies-Injective Proposition 201.7 to PIDs. □

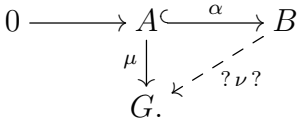
In the proof of the Divisible Group-implies-Injective Proposition 201.7 we will need Zorn's Lemma which we now recall for the reader's convenience:

Lemma 1.29. (Zorn's Lemma)[2134] Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. If P is non-empty, then the set P contains at least one maximal element.



Now we can give the proof of the Divisible Group-implies-Injective Proposition 201.7.

Proof of Proposition 201.7. Let G be an abelian group that is divisible. Suppose we are given a diagram



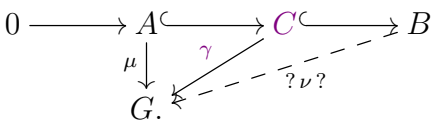
Since α is a monomorphism we can assume that A is in fact a subgroup of B and that α is the inclusion map.

Let P be the set of all pairs (C, γ) where C is a subgroup of B which contains A and where $\gamma\colon C \to G$ is a homomorphism such that $\gamma|_A = \mu$. Note that such pairs fit into the following diagram:

[2133] Odd as it may sound, this example will actually be important later in **??**.

[2134] We recall the definitions in the statement of Zorn's Lemma:

(1) A *partially ordered set* $(P, \leq)$ is a set together with a relation $\leq$ that is reflexive, transitive and antisymmetric, i.e. that has the property that $a \leq b$ and $b \leq a$ implies that $a = b$.
(2) A *chain* in P is a subset Q such that for every two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.
(3) An *upper bound* of a subset Q is an element $p \in P$ such that $q \leq p$ for all $q \in Q$.
(4) A *maximal element* of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.



We define $(C_i, \gamma_i) \leq (C_j, \gamma_j)$ if $C_i \subseteq C_j$ and if $\gamma_j|_{C_i} = \gamma_i$. One can easily verify that this defines a partial order on P.

Claim 1. Every chain $(C_j, \gamma_j)_{j\in J}$ in P admits an upper bound.

Proof. Let $(C_j, \gamma_j)_{j\in J}$ be a chain in P. If we are dealing with the empty chain, then an upper bound is given by the pair $(A, \mu) \in P$. Now suppose that the chain $(C_j, \gamma_j)_{j\in J}$ in P is non-empty. We get an upper bound by setting

$$C := \bigcup_{j\in J} C_j \qquad \text{and} \qquad \begin{array}{rccl} \gamma\colon & C & \to & G \\ & c & \mapsto & \gamma_j(c) \\ & & & \uparrow \\ & & & \text{where we pick } j \in J \\ & & & \text{with } c \in C_j \end{array}$$

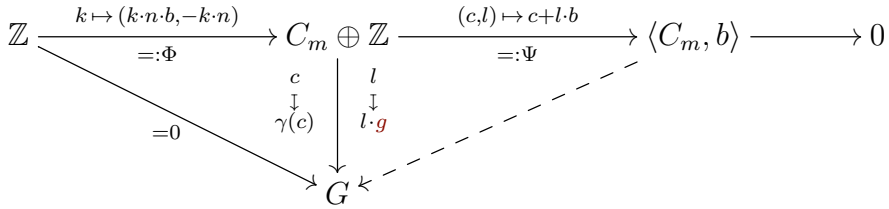
Since $(C_j, \gamma_j)_{j\in J}$ is a non-empty chain one can easily verify that C is in fact a subgroup of B which contains A. Furthermore it follows easily from the definitions that γ is a homomorphism with $\gamma|_A = \mu$. ⊞

It follows from Claim 1 that we can apply Zorn's Lemma 1.29 to this partially ordered set $(P, \leq)$ and we obtain a maximal pair (C_m, γ_m). It remains to prove the following claim:

Claim 2. We have $C_m = B$.

Proof. Suppose that this is not the case. Then there exists a $b \in B \setminus C_m$. We denote by $\langle C_m, b\rangle$ the subgroup of B generated by C_m and b, i.e. $\langle C_m, b\rangle$ is the smallest subgroup of B that contains C_m and b. To obtain a contradiction it suffices to show that there exists a homomorphism $\delta\colon \langle C_m, b\rangle \to G$ that extends γ_m. We distinguish two cases:

(1) If there exists no $n \in \mathbb{N}$ such that $n \cdot b \in C_m$, then one can show easily that we have $C_m \oplus \langle b\rangle = \langle C_m, b\rangle$. We extend γ_m to $C_m \oplus \langle b\rangle = \langle C_m, b\rangle$ by setting $\delta(b) = 0$.
(2) Otherwise there exists a minimal $n \in \mathbb{N}$ such that $n \cdot b \in C_m$. Since G is divisible there exists a $g \in G$ such that $n \cdot g = \gamma(n \cdot b)$. Next we consider the following little diagram:



We make the following observations:

(a) The horizontal sequence is exact: Indeed, it is clear that $\Psi \circ \Phi = 0$. Next note that the image of Ψ lies a priori in B, but it is straightforward to see that the image of Ψ is precisely $\langle C_m, b\rangle$. Finally note that it follows easily from the choice of n that we have $\ker(\Psi) \subseteq \mathrm{im}(\Phi)$ (note that here we use the minimality of n.)
(b) It follows from the Coproduct-of-Abelian Groups Lemma 97.2 that the vertical homomorphism is well-defined.
(c) One can easily verify that $\mathrm{im}(\Phi)$ is contained in the kernel of the vertical map.

(d) It follows from (a) and (c) that we get the desired extension $\langle C_m, b\rangle \to G$ of $\gamma\colon C_m \to G$.

This concludes the proof of the claim and thus of the lemma. ■

It follows from the Exact Functor–Cohomology Lemma 201.2 and the Hom-to-Injective Lemma 201.5 that for any chain complex C_* and any injective abelian group G we have an isomorphism $\mathrm{H}^k(\mathrm{Hom}(C_*), G) \to \mathrm{Hom}(\mathrm{H}_k(C_*), G)$. Our next goal is to give an explicit isomorphism. This leads us to the following lemma, which is an analog of the Homology-Multiplication Lemma 161.7.

Lemma 201.8. (Evaluation Map Lemma) Let (C_*, ∂_*) be a chain complex and let G be an abelian group. The map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C_*; G) & \to & \mathrm{Hom}(\mathrm{H}_n(C_*), G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*) & \to & G \\ [c] & \mapsto & \underset{\text{Kronecker Pairing from Definition 199.8}}{\langle [\varphi], [c]\rangle} = \varphi(c) \end{array} \right) \end{array}$$

is a well-defined natural[2135] homomorphism.

Definition 201.9. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. We refer to the map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C_*; G) & \to & \mathrm{Hom}(\mathrm{H}_n(C_*), G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*) & \to & G \\ [c] & \mapsto & \langle [\varphi], [c]\rangle = \varphi(c) \end{array} \right) \end{array}$$

from the Evaluation Map Lemma 201.8 as the evaluation homomorphism.

Proof. It follows from the Kronecker Pairing Lemma 199.7 that this map is well-defined, that it is a homomorphism and that it is natural. ■

The following proposition is the cohomological analog of the Torsion–Free Coefficients Proposition 161.8.

Proposition 201.10. (Injective Coefficients Proposition) Let (C_*, ∂_*) be a chain complex, let G be an abelian group and let $n \in \mathbb{N}_0$. If G is an injective abelian group,

[2135]Here "natural" means two statements. If $f\colon C_* \to D_*$ is a chain map, then for any $n \in \mathbb{N}_0$ and any abelian group G the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(D_*; G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(D_*), G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f_*)^*} \\ \mathrm{H}^n(C_*; G) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(C_*), G). \end{array}$$

Furthermore, if $\varphi\colon G_1 \to G_2$ is a homomorphism of abelian groups, then for any $n \in \mathbb{N}_0$ and any chain complex C_* the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(C_*; G_1) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(C_*), G_1) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^n(C_*; G_2) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_n(C_*), G_2). \end{array}$$

Here the map on the left is the one we had introduced in the Cohomology-Coefficients-Functor Lemma 197.21.

then the map

$$\mathrm{ev}\colon \mathrm{H}^n(C_*;G) \;\to\; \mathrm{Hom}(\mathrm{H}_n(C_*),G)$$

from Definition 201.9 is a natural isomorphism. In particular for any pair (X,A) of topological spaces the map

$$\mathrm{ev}\colon \mathrm{H}^n(X,A;G) \;\to\; \mathrm{Hom}(\mathrm{H}_n(X,A);G)$$

is an isomorphism.

Proof. Similar to the proof of the Torsion–Free Coefficients Proposition 161.8 this proposition follows quite easily from the Exact Functor–Cohomology Lemma 201.2 and the Hom-to-Injective Lemma 201.5. We leave it to the reader to fill in the details. ■

201.2. **The Ext group.** In Section 162.1 we introduced the torsion groups $\mathrm{Tor}(H,G)$ which "measure" to what degree the functor $A \mapsto A \otimes G$ fails to be a left-exact functor. The same way we now introduce the Ext-groups $\mathrm{Ext}(H,G)$ which "measure" to what degree the functor $A \mapsto \mathrm{Hom}(A,G)$ fails to be right-exact:

Definition 201.11. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the natural free resolution from Definition Definition 162.2. We define the n-th G-Ext-group of H as

$$\mathrm{Ext}^n(H,G) \;:=\; \mathrm{H}^n(\mathrm{Hom}(F_*,G)).$$

On page 3107 we saw that the groups $\mathrm{Tor}_n(H,G)$ are covariantly functorial in both G and H. Now we will see that the groups $\mathrm{Ext}^n(H,G)$ are contravariantly functorial in H and covariantly functorial in G:

Definition 201.12.

(1) We fix an abelian group G. Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. It follows from the Homological Algebra–Fundamental Theorem 158.4 together with the definition of the G-Ext groups that α induces a homomorphism

$$\alpha^*\colon \mathrm{Ext}^n(H',G) \;\to\; \mathrm{Ext}^n(H,G).$$

It is now straightforward to show that the maps

$$\begin{aligned} H &\mapsto \mathrm{Ext}^n(H,G) \\ (\alpha\colon H \to H') &\mapsto (\alpha^*\colon \mathrm{Ext}^n(H',G) \to \mathrm{Ext}^n(H,G)) \end{aligned}$$

define a *contravariant* functor from the category *AbGroup* of abelian groups to itself.

(2) We fix an abelian group H. If $\beta\colon G \to G'$ is a homomorphism between abelian groups then one sees easily, using the Cohomology-Coefficients-Functor Lemma 197.21, that β induces a homomorphism

$$\beta_*\colon \mathrm{Ext}^n(H,G) \;\to\; \mathrm{Ext}^n(H,G')$$

Furthermore one can easily prove that the maps

$$\begin{aligned} G &\mapsto \mathrm{Ext}^n(H,G) \\ (\beta\colon G \to G') &\mapsto (\beta_*\colon \mathrm{Ext}^n(H,G) \to \mathrm{Ext}^n(H,G')) \end{aligned}$$

define a *covariant* functor from the category *AbGroup* of abelian groups to itself.

The following proposition is the analog of the Tor-Free Resolution Proposition 162.6.

Proposition 201.13. (Ext-Free Resolution Proposition) Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. For every $n \in \mathbb{N}_0$ there exists an isomorphism

$$\operatorname{Ext}^n(H,G) \xrightarrow{\cong} \mathrm{H}^n(F_*,G)$$

which is natural in G.

Proof. The proof is almost the same as the proof of the Tor-Free Resolution Proposition 162.6. For the most part we just need to replace the Homology-with-Coefficients Lemma 160.6 (2) by the Dual Cochain Map Lemma 197.20 (3b) and at some point we need to make use of the Cochain Homotopy-H*-Lemma 197.14. We leave it to the reader to fill in the details. ■

In the Tor_0–Lemma 162.7 we saw that the 0-th Torsion functor is an old acquaintance. More precisely we saw that for all abelian groups G and H there exists a natural isomorphism $\mathrm{Tor}_0(H,G) \xrightarrow{\cong} H \otimes G$. The following lemma gives us the analogous result for the 0-th Ext functor:

Lemma 201.14. (Ext^0–Lemma) For any choice of abelian groups G and H there exists a natural[2136] isomorphism

$$\operatorname{Ext}^0(H,G) \xrightarrow{\cong} \operatorname{Hom}(H,G).$$

Identification 201.15. In the remainder of this course we will use the natural isomorphisms from the Ext^0–Lemma 201.14 to make the identification $\operatorname{Ext}^0(H,G) =_{\mathrm{i}} \operatorname{Hom}(H,G)$.

Proof. Let G and H be abelian groups. We consider the natural free resolution of H:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

We know from the Hom-Left-Exact Lemma 201.3 that the sequence

$$(*) \qquad 0 \to \operatorname{Hom}(H,G) \xrightarrow{f_0^*} \operatorname{Hom}(F_0,G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1,G)$$

is exact. Now we obtain the following isomorphisms

$$\begin{aligned} \operatorname{Ext}^0(H,G) &\overset{\text{by Definition 201.12}}{=} \mathrm{H}^0\big(\cdots \leftarrow \operatorname{Hom}(F_1,G) \xleftarrow{f_1^*} \operatorname{Hom}(F_0,G) \leftarrow 0\big) \\ &= \ker\big(\operatorname{Hom}(F_1,G) \xleftarrow{f_1^*} \operatorname{Hom}(F_0,G)\big) \xrightarrow[\text{isomorphism since } (*) \text{ is exact}]{f_0^*} \operatorname{Hom}(H,G). \end{aligned}$$

It is straightforward to verify that these isomorphisms are in fact natural. ■

In the Higher Tor Lemma 162.10 we saw that the higher torsion groups of abelian groups vanish. The same statement, with basically the same proof, also holds for Ext groups:

[2136] Here by "natural" we mean that these isomorphisms define a natural isomorphism from the contravariant functor $H \mapsto \operatorname{Ext}^0(H,G)$ to the contravariant functor $H \mapsto \operatorname{Hom}(H,G)$ and that they also define a natural transformation from the covariant functor $G \mapsto \operatorname{Ext}^0(H,G)$ to the covariant functor $G \mapsto \operatorname{Hom}(H,G)$.

Lemma 201.16. (Higher Ext Lemma) For every choice of abelian groups G and H we have $\mathrm{Ext}^n(H,G) = 0$ for $n \geq 2$.

Proof. By the Length 1–Free Resolution Lemma 162.9 we know that the abelian group H admits a free resolution of length 1:

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

It now follows that for $n \geq 2$ we have

$$\mathrm{Ext}^n(H,G) \underset{\text{by the Ext-Free Resolution Proposition 201.13}}{\cong} \mathrm{H}^n(\mathrm{Hom}(F_*,G)) \underset{\text{since } F_n = 0 \text{ for } n \geq 2}{=} 0.$$

■

The Ext^0–Lemma 201.14 and the Higher Ext Lemma 201.16 justify the following definition:

Definition 201.17. Given abelian groups G and H we refer to

$$\mathrm{Ext}(H,G) := \mathrm{Ext}^1(H,G)$$

as *the* G-Ext-group of H.[2137]

In light of the Ext^0–Lemma 201.14 and the Higher Ext Lemma 201.16 the remaining task is to compute the Ext-group $\mathrm{Ext}(H,G) = \mathrm{Ext}^1(H,G)$. We start out with the following lemma which is the analog of the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12.

Lemma 201.18. ($\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma) Let $k \in \mathbb{N}$.

(1) For any abelian group G we have a natural isomorphism

$$\mathrm{Ext}(\mathbb{Z}/k\mathbb{Z}, G) \xrightarrow{\cong} G/k \cdot G.$$

(2) Given any $m \in \mathbb{N}$ we have $\mathrm{Ext}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(k,m)\mathbb{Z}$.

Proof.

(1) We consider the free resolution $0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0$ of $\mathbb{Z}/k\mathbb{Z}$. Given any abelian group G this gives rise to the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathbb{Z},G) & \xrightarrow{(n\mapsto n\cdot k)^*} & \mathrm{Hom}(\mathbb{Z},G) & \longrightarrow & \overbrace{\mathrm{coker}\left(\mathrm{Hom}(\mathbb{Z},G) \xrightarrow{(n\mapsto n\cdot k)^*} \mathrm{Hom}(\mathbb{Z},G)\right)}^{\cong \mathrm{Ext}(\mathbb{Z}/k\mathbb{Z},G)} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi \mapsto \varphi(1)}\,{\scriptstyle\cong} & & \downarrow{\scriptstyle \varphi \mapsto \varphi(1)}\,{\scriptstyle\cong} & & \vdots & & \\
G & \xrightarrow{g \mapsto g\cdot k} & G & \longrightarrow & \mathrm{coker}(G \xrightarrow{g\mapsto g\cdot k} G) & \longrightarrow & 0.
\end{array}$$

[2137] Let us explain where the name "Ext" comes from. Let G and H be abelian groups. An *extension of* H *by* G is a short exact sequence of the form

$$0 \longrightarrow G \xrightarrow{\alpha} K \xrightarrow{\beta} H \longrightarrow 0$$

where K is a (possibly non-abelian) group. We say that two such extensions given by K and K' are *equivalent* if there exists an isomorphism $\varphi\colon K \to K'$ that makes the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha'} & K' & \xrightarrow{\beta'} & H & \longrightarrow & 0.
\end{array}$$

In [**HS1997**, Theorem II.3.4] it is shown that there is a natural bijection from the set of equivalence classes of extensions of H by G with the set $\mathrm{Ext}(H,G)$. This fact is the origin of the name "Ext-group".

We make the following observations:
(a) Since $0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/k\mathbb{Z}$ we obtain from the Ext-Free Resolution Proposition 201.13 the isomorphism in the top right corner.
(b) The top horizontal sequence is exact by definition.
(c) The two vertical maps to the left are the natural isomorphisms given by the Abelian Group-Homomorphism Lemma 197.5. It follows immediately from the definitions that the square to the left commutes.

It now follows from a mild diagram chase that from the above we get an induced isomorphism $\operatorname{coker}\left(G \xrightarrow{g \mapsto g \cdot k} G\right) \xrightarrow{\cong} \operatorname{Ext}(\mathbb{Z}/k\mathbb{Z}, G)$. One can easily verify that this isomorphism is natural.

(2) By (1) we know that $\operatorname{Ext}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \operatorname{coker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{h \mapsto h \cdot k} \mathbb{Z}/m\mathbb{Z})$. Thus it suffices to prove the following claim:

Claim. The map

$$\begin{array}{rcl} \operatorname{coker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{h \mapsto h \cdot k} \mathbb{Z}/m\mathbb{Z}) & \to & \mathbb{Z}/\gcd(m,k)\mathbb{Z} \\ [a] & \mapsto & [a] \end{array}$$

is well-defined and it is an isomorphism.

Proof. Clearly any element of the form $[k \cdot b]$ gets sent to zero in $\mathbb{Z}/\gcd(m,k)\mathbb{Z}$, which implies that the map is well-defined. It is also clear that the map is a surjective. Finally, if we are given $a \in \mathbb{Z}$ with $[a] = 0 \in \mathbb{Z}/\gcd(m,k)\mathbb{Z}$, then $a = x \cdot m + y \cdot k$ for some $x, y \in \mathbb{Z}$. But then we see that $[a] = [x \cdot m + y \cdot k] = [y \cdot k] \in \mathbb{Z}/m\mathbb{Z}$ lies in the image of the map $\mathbb{Z}/m\mathbb{Z} \xrightarrow{h \mapsto h \cdot k} \mathbb{Z}/m\mathbb{Z}$. This shows that the map of the claim is also injective. ■

The following lemma plays the role of the Tor-Properties Lemma 162.13:

Lemma 201.19. (Ext-Properties Lemma) Let G, H and $\{G_j\}_{j \in J}$ and $\{H_i\}_{i \in I}$ be abelian groups. The following statements hold:
(1) There exists a natural isomorphism $\operatorname{Ext}\left(\bigoplus_{i \in I} H_i, G\right) \cong \prod_{i \in I} \operatorname{Ext}(H_i, G)$.
(2) There exists a natural isomorphism $\operatorname{Ext}\left(H, \prod_{j \in J} G_j\right) \cong \prod_{j \in J} \operatorname{Ext}(H, G_j)$.
(3) If G is injective (e.g. if G is divisible), then $\operatorname{Ext}(H, G) = 0$.
(4) If H is a free abelian group, then $\operatorname{Ext}(H, G) = 0$.

Example. Let H be a finitely generated abelian group. By the Abelian Group-Classification Theorem 97.24 we know that H is isomorphic to a direct sum of cyclic groups. It follows easily from this observation, the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 and the Ext-Properties Lemma 201.19 (1) and (4) that we have an isomorphism

$$\operatorname{Ext}(H, \mathbb{Z}) \cong \text{ torsion subgroup of } H := \{h \in H \mid \text{there exists an } n \in \mathbb{N} \text{ with } n \cdot h = 0\}.$$

In particular $\operatorname{Ext}(H, \mathbb{Z})$ is a finite group. □

Proof.

(1) For each $i \in I$ we let F_*^i be the natural free resolution of H_i. One can easily verify that $\bigoplus_{i\in I} F_*^i$ is a free resolution of $\bigoplus_{i\in I} H_i$. We have the following natural isomorphisms:

$$\begin{array}{ccccc} \operatorname{Ext}\left(\bigoplus_{i\in I} H_i, G\right) & \overset{\text{by the Ext-Free Resolution Proposition 201.13}}{\underset{\uparrow}{\cong}} & \mathrm{H}^1\left(\operatorname{Hom}\left(\bigoplus_{i\in I} F_*^i, G\right)\right) & \overset{\text{by the Direct Sum-Product Homomorphism Lemma 197.6 (1)}}{\underset{\uparrow}{\cong}} & \mathrm{H}^1\left(\prod_{i\in I} \operatorname{Hom}(F_*^i, G)\right) \\ & \overset{\cong}{\underset{\text{by the Product–H}^* \text{ Lemma 197.18}}{\longleftarrow}} & \prod_{i\in I} \mathrm{H}^1(\operatorname{Hom}(F_*^i, G)) & \overset{\text{by Definition 201.12}}{\underset{\uparrow}{=}} & \prod_{i\in I} \operatorname{Ext}(H_i, G). \end{array}$$

(2) Let F_* be the natural resolution for H. We have the following natural isomorphisms:

$$\begin{array}{ccccc} \operatorname{Ext}\left(H, \prod_{j\in J} G_j\right) & \overset{\text{by Definition 201.12}}{\underset{\uparrow}{=}} & \mathrm{H}^1\left(\operatorname{Hom}\left(F_*, \prod_{j\in J} G_j\right)\right) & \overset{\text{by the Direct Sum-Product Homomorphism Lemma 197.6 (2)}}{\underset{\uparrow}{\cong}} & \mathrm{H}^1\left(\prod_{j\in J} \operatorname{Hom}(F_*, G_j)\right) \\ & \overset{\cong}{\underset{\text{by the Product–H}^* \text{ Lemma 197.18}}{\longleftarrow}} & \prod_{j\in J} \mathrm{H}^1(\operatorname{Hom}(F_*, G_j)) & \overset{\text{by Definition 201.12}}{\underset{\uparrow}{=}} & \prod_{j\in J} \operatorname{Ext}(H, G_j). \end{array}$$

(3) We need to show that if G is an injective abelian group, then for any abelian group N we have $\operatorname{Ext}(N, G) = 0$. To do so we consider the natural free resolution of N:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0.$$

We denote by F'_* the above sequence *with* N, i.e. $F'_i = F_i$ for $i \in \mathbb{N}_0$ and $F_{-1} = N$. Note that for each $k \in \mathbb{N}$ we have the following isomorphisms:

$$\begin{array}{ccccccc} \operatorname{Ext}(N, G) & \overset{\text{by Definition 201.11}}{\underset{\uparrow}{=}} & \mathrm{H}^1(\operatorname{Hom}(F_*, G)) & \overset{\text{since } 1>0}{\underset{\uparrow}{=}} & \mathrm{H}^1(F'_*, G) & & \\ & & & \overset{\text{by the Exact Functor–Cohomology Lemma 201.2 and the Hom-to-Injective Lemma 201.5}}{\underset{\uparrow}{\cong}} & \operatorname{Hom}(\mathrm{H}^1(F_*), G) & \overset{\text{since } F'_* \text{ is exact}}{\underset{\uparrow}{=}} & \operatorname{Hom}(0, G) = 0. \end{array}$$

(4) Let H be a *free* abelian group. Note that in this case $0 \to F_0 := H \xrightarrow{\mathrm{id}} H \to 0$ is already a free resolution. It follows immediately from the Ext-Free Resolution Proposition 201.13 that $\operatorname{Ext}(H, G) = 0$. ■

We conclude this section with two more exotic propositions on Ext-groups. We will make use of them in rather peripheral situations.

Proposition 201.20. (Ext$(\mathbb{Q},\mathbb{Z})$-Proposition) The group $\operatorname{Ext}(\mathbb{Q},\mathbb{Z})$ is isomorphic to the additive group $(\mathbb{R},+)$.

Remark. Note that this proposition is not totally pointless, for example it follows from the existence of Moore spaces, see the Moore Space Existence Proposition 173.3, that there exist topological spaces X with $\mathrm{H}_n(X) \cong (\mathbb{Q},+)$. Thus even unexpected abelian groups can appear in topology. We will make of use these observations in Exercise 201.11. □

Proof. This proposition is proved in [**Wie1969**]. Alternatively the proposition is almost proved in [**Bre1997**, Lemma 14.8]. We also refer to [**Boar2010**] for the calculation of $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$. ■

201.3. **The Universal Coefficient Theorem for Cohomology Groups.** In this section we want to explore the connection between the cohomology groups of a chain complex (C_*,∂_*) and the usual homology groups of the chain complex. The key to relating homology groups to cohomology groups is given by the evaluation homomorphism which from Definition 201.9:

Definition 201.21. Let (C_*,∂_*) be a chain complex and let G be an abelian group. We refer to the map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C_*;G) & \to & \mathrm{Hom}(\mathrm{H}_n(C_*),G) \\ [\varphi\colon C_n\to G] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*) & \to & G \\ [c] & \mapsto & \langle[\varphi],[c]\rangle=\varphi(c)\end{array}\right)\end{array}$$

as the evaluation homomorphism.

The following theorem relates homology groups and cohomology groups via the evaluation homomorphism and the Ext-groups.

Theorem 201.22. (Algebraic Universal Coefficient-H*-Theorem) Let (C_*,∂_*) be a chain complex of free abelian groups and let G be an abelian group. For each $n\in\mathbb{N}_0$ there exists a homomorphism

$$\epsilon\colon\ \mathrm{Ext}(\mathrm{H}_{n-1}(C_*),G)\ \to\ \mathrm{H}^n(C_*;G)$$

which is natural in the chain complex C_* and natural in the abelian group G and which has the property that the following sequence is exact:

$$0\ \to\ \mathrm{Ext}(\mathrm{H}_{n-1}(C_*),G)\ \xrightarrow{\epsilon}\ \mathrm{H}^n(C_*;G)\ \xrightarrow{\mathrm{ev}}\ \mathrm{Hom}(\mathrm{H}_n(C_*),G)\ \to\ 0.$$

This short exact sequence splits and there exists therefore an isomorphism[2138]

$$\mathrm{H}^n(C_*;G)\ \cong\ \mathrm{Ext}(\mathrm{H}_{n-1}(C_*),G)\oplus\mathrm{Hom}(\mathrm{H}_n(C_*),G).$$

Remark. The Algebraic Universal Coefficient-H*-Theorem 201.22 is the analog of the Algebraic Universal Coefficient Theorem 162.15 together with the UCT-Splitting Theorem 162.18. □

Before we provide the proof of the above Algebraic Universal Coefficient-H*-Theorem 201.22 let us formulate the most important application, namely let us apply it to the singular chain complex of a pair of topological spaces:

Theorem 201.23. (Universal Coefficient-H*-Theorem) Let (X,A) be a pair of topological spaces and let G be an abelian group. For each $n\in\mathbb{N}_0$ there exists a homomorphism

$$\epsilon\colon\ \mathrm{Ext}(\mathrm{H}_{n-1}(X,A),G)\ \to\ \mathrm{H}^n(X,A;G)$$

[2138] Once again the two maps in the short exact sequence are natural, whereas the isomorphism is not natural. More precisely, the discussion on page 3123 can easily be modified to show that there exists no *natural* isomorphism $\mathrm{H}^n(C_*;G)\ \cong\ \mathrm{Ext}(\mathrm{H}_{n-1}(C_*),G)\oplus\mathrm{Hom}(\mathrm{H}_n(C_*),G)$.
We will discuss this issue in Exercise 201.6.

which is natural in (X, A), which is natural in the abelian group G and which has the property that the following sequence is exact:

$$0 \to \operatorname{Ext}(\mathrm{H}_{n-1}(X, A), G) \xrightarrow{\epsilon} \mathrm{H}^n(X, A; G) \xrightarrow{\mathrm{ev}} \operatorname{Hom}(\mathrm{H}_n(X, A), G) \to 0.$$

This short exact sequence splits (in the sense of Definition 145.4) and there exists therefore an isomorphism

$$\mathrm{H}^n(X, A; G) \cong \operatorname{Ext}(\mathrm{H}_{n-1}(X, A), G) \oplus \operatorname{Hom}(\mathrm{H}_n(X, A), G).$$

Proof of Theorem 201.23 assuming Theorem 201.22. As the reader will have noticed there is really not much to say, except that we recall that we pointed out on page 2798 that the chain groups $\mathrm{C}_n(X, A)$, $n \in \mathbb{N}_0$ are free abelian groups. ■

Remark.

(1) The Universal Coefficient-H*-Theorem 201.23 implies in particular that topological spaces with isomorphic homology groups also have isomorphic cohomology groups. In other words, if our only goal is to distinguish homeomorphism types of topological spaces, then cohomology groups will not be of much help.
(2) The fact that cohomology groups are determined by homology groups is also an immediate consequence of the combination of the Lift-to-Chain Map-Proposition 158.21 and the Quasi Iso–Homotopy Equivalence Proposition 158.8 together with the Cochain Homotopy-H*-Lemma 197.14 (2).
(3) In the Universal Coefficient-H*–Field Theorem 202.2 we will prove a variation and in the In the Algebraic Universal Coefficient-H*-Theorem 315.44 we will prove a generalization of the Universal Coefficient-H*-Theorem 201.23. Finally in the Cohomological UCSS Theorem 316.10 we will state (but not prove) a generalization to modules over arbitrary rings. □

In the next Section 201.4 we will study many examples and applications. The remainder of this section is occupied with the proof of the Algebraic Universal Coefficient-H*-Theorem 201.22. We break the proofs into two parts:

(1) First we show that the short exact sequences exist.
(2) Afterwards we show that the short exact sequences split.

The proof of the two parts are, perhaps not surprisingly, rather similar to the proofs of the Algebraic Universal Coefficient Theorem 162.15 and the Algebraic UCT-Splitting Proposition 162.17.

Proof of the existence of short exact sequence. Let (C_*, ∂_*) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$. Note that it follows from the Free Abelian Subgroup Theorem 97.14 that B_n and Z_n are free abelian groups.

As in the proof of the Universal Coefficient Theorem 162.16 we consider the following short exact sequence of vertical chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Now let G be an abelian group. We apply the functor $A \mapsto \mathrm{Hom}(A, G)$ to the above short exact sequence of chain complexes. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" cochain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B_{n-1}, G) & \xrightarrow{\partial_n^*} & \mathrm{Hom}(C_n, G) & \longrightarrow & \mathrm{Hom}(Z_n, G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Hom}(B_{n-2}, G) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}(C_{n-1}, G) & \longrightarrow & \mathrm{Hom}(Z_{n-1}, G) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Dual Cochain Map Lemma 197.20 that this is a sequence of cochain complexes.

(2) Recall that the groups B_n are free abelian groups. Note that it thus follows from the Free Abelian Group–Splitting Lemma 97.11 that each of the horizontal sequences $0 \to Z_k \to C_k \to B_{k-1} \to 0$ in the original diagram splits. It follows from the Hom-Split-SES Lemma 197.8 that the above horizontal sequences of cochain complexes are still exact. ⊞

We apply the Cohomology–Connecting Homomorphism Lemma 197.16 to the short exact sequence of cochain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{\mathrm{Hom}(Z_n, G)}_{=\mathrm{H}^n(Z_*;G)} \to \underbrace{\mathrm{Hom}(B_n, G)}_{=\mathrm{H}^n(B_*;G)}.$$

In fact, it follows quite easily from Definition 197.15 that the connecting homomorphism $\mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G)$ is the map that is induced by the inclusion $i_n \colon B_n \to Z_n$. By the Cohomology–LES Proposition 197.17 this connecting homomorphism gives us the following long exact sequence:

$$\to \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{i_n^*} \mathrm{Hom}(B_{n-1}, G) \to \mathrm{H}^n(C; G) \to \mathrm{Hom}(Z_n, G) \xrightarrow{d_{n-1}} \mathrm{Hom}(B_n, G) \to \dots$$

By the Break-into-SES Lemma 145.6 we obtain the following short exact sequence

$$\begin{aligned} 0 \to \mathrm{coker}(\mathrm{Hom}(Z_{n-1}, G) \to \mathrm{Hom}(B_{n-1}, G)) &\to \mathrm{H}^n(C; G) \\ &\to \ker(\mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G)) \to 0. \end{aligned}$$

We have to show that the expressions left and right coincide with the desired expressions. Both equalities follow immediately from the Ext-Free Resolution Proposition 201.13 (together with the Ext^0–Lemma 201.14 for the right-hand term) and the observation that for each $l \in \mathbb{N}_0$ a free resolution of $\mathrm{H}_l(C)$ is given by $0 \to B_l \xrightarrow{i_l} Z_l \to \mathrm{H}_l(C_*)$. We have thus shown the existence of the desired short exact sequence

$$0 \;\to\; \mathrm{Ext}(\mathrm{H}_{n-1}(C_*), G) \;\xrightarrow{\epsilon}\; \mathrm{H}^n(C_*; G) \;\xrightarrow{\mathrm{ev}}\; \mathrm{Hom}(\mathrm{H}_n(C_*), G) \;\to\; 0.$$

The statement that the homomorphism ϵ to the left is natural follows easily from the definitions and the Homological Algebra–Fundamental Theorem 158.4 (2). We leave the verification of the details to the reader. ■

Proof that the short exact sequence splits. Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. By Definition 145.4 we need to construct a homomorphism $\eta\colon \mathrm{Hom}(\mathrm{H}_n(C_*), G) \to \mathrm{H}^n(C_*; G)$ such that $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}(\mathrm{H}_n(C_*), G)$. We summarize this problem as follows:

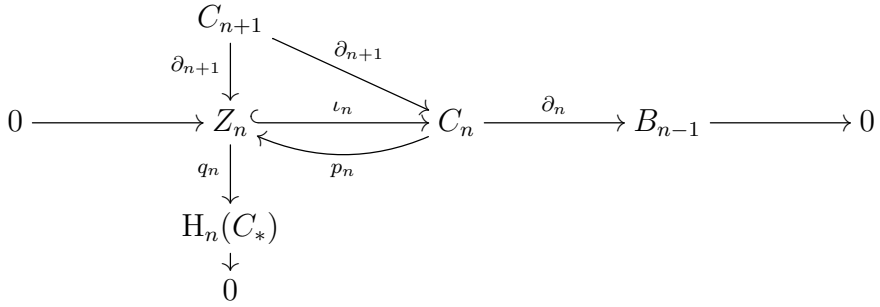
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(\mathrm{H}_{n-1}(C_*), G) \arrow[r] & \mathrm{H}^n(\mathrm{Hom}(C_*, G)) \arrow[r, "\mathrm{ev}"'] & \mathrm{Hom}(\mathrm{H}_n(C_*), G) \arrow[l, dashed, bend right, "?\,\eta\,?"'] \arrow[r] & 0.
\end{tikzcd}$$

For each $n \in \mathbb{N}_0$ we consider, as on page 3114, $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \xhookrightarrow{\iota_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

The group $B_{n-1} \subseteq C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by the Free Abelian Subgroup Theorem 97.14. Therefore we can deduce from the Free Abelian Group–Splitting Lemma 97.11 that the above short exact sequence splits. In particular, according to the Splitting Lemma 145.5 (1)⇒(2), for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. Again we summarize the setting as follows:

$$\begin{tikzcd}
 & C_{n+1} \arrow[d, "\partial_{n+1}"'] \arrow[dr, "\partial_{n+1}"] & & & \\
0 \arrow[r] & Z_n \arrow[r, hook, "\iota_n"] \arrow[d, "q_n"'] & C_n \arrow[l, bend left, "p_n"] \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & 0 \\
 & \mathrm{H}_n(C_*) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$



where q_n is just the natural projection.

Claim 1. The following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "q_{n+1}\circ p_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "q_n\circ p_n"] & C_{n-1} \arrow[r] \arrow[d, "q_{n-1}\circ p_{n-1}"] & \dots \\
\dots \arrow[r] & \mathrm{H}_{n+1}(C_*) \arrow[r, "0"] & \mathrm{H}_n(C_*) \arrow[r, "0"] & \mathrm{H}_{n-1}(C_*) \arrow[r] & \dots
\end{tikzcd}$$

Proof. The map down-right is evidently zero. On the other hand we also have

$$(q_n \circ p_n) \circ \partial_{n+1} \underset{\text{since } \iota_n \circ \partial_{n+1} = \partial_{n+1}}{=} q_n \circ p_n \circ \iota_n \circ \partial_{n+1} \underset{\text{since } p_n \circ \iota_n = \mathrm{id}_{Z_n}}{=} q_n \circ \partial_{n+1} \underset{\text{since } \mathrm{H}_n(C_*) = Z_n/\operatorname{im}(\partial_{n+1})}{=} 0.$$

⊞

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps. Now we apply the $\mathrm{Hom}(-, G)$ functor to the top and the bottom horizontal sequences. From the Dual Cochain Map Lemma 197.20 we obtain the following diagram of cochain complexes:

$$\begin{array}{ccccccccc}
\ldots & \longleftarrow & \mathrm{Hom}(C_{n+1}, G) & \xleftarrow{\partial^*_{n+1}} & \mathrm{Hom}(C_n, G) & \xleftarrow{\partial^*_n} & \mathrm{Hom}(C_{n-1}, G) & \longleftarrow & \ldots \\
 & & \uparrow{\scriptstyle (q_{n+1}\circ p_{n+1})^*} & & \uparrow{\scriptstyle (q_n \circ p_n)^*} & & \uparrow{\scriptstyle (q_{n-1}\circ p_{n-1})^*} & & \\
\ldots & \longleftarrow & \mathrm{Hom}(\mathrm{H}_{n+1}(C_*), G) & \xleftarrow{0} & \mathrm{Hom}(\mathrm{H}_n(C_*), G) & \xleftarrow{0} & \mathrm{Hom}(\mathrm{H}_{n-1}(C_*), G) & \longleftarrow & \ldots
\end{array}$$

The induced maps on the cohomology of these two cochain complexes give us a homomorphism

$$\eta\colon \mathrm{H}^n(\text{lower sequence}) = \mathrm{Hom}(\mathrm{H}_n(C_*), G) \;\to\; \mathrm{H}_n(\mathrm{Hom}(C_*, G)).$$

It remains to prove the following claim:

Claim 2. The composition $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}(\mathrm{H}_n(C_*), G)$.

Proof. Let $\varphi\colon \mathrm{H}_n(C_*) \to G$ be a homomorphism. We need to show that $(\mathrm{ev} \circ \eta)(\varphi) = \varphi$ in $\mathrm{Hom}(\mathrm{H}_n(C_*), G)$. So let $[\sigma] \in \mathrm{H}_n(C_*)$ with $\sigma \in Z_n$. Then

$$\big((\mathrm{ev} \circ \eta)(\varphi)\big)([\sigma]) \underset{\text{Definition 201.9 of ev}}{=} \langle \overbrace{\eta(\varphi)}^{\in \mathrm{H}^n(C;G)}, [\sigma] \rangle \underset{\text{definition of } \eta \text{ and } \langle\,,\,\rangle}{=} (\overbrace{\varphi \circ q_n \circ p_n}^{\in \mathrm{Hom}(C_n, G)})(\sigma) \underset{\text{since } p_n = \mathrm{id} \text{ on } Z_n}{=} (\varphi \circ q_n)(\sigma) \underset{\text{since } q_n(\sigma) = [\sigma]}{=} \varphi([\sigma]).$$

■

201.4. **Applications of the Universal Coefficient-H*-Theorem 201.23.** We now apply the Universal Coefficient-H*-Theorem 201.23 to some of our favorite topological spaces:

Example. We consider the real projective space $P := \mathbb{R}\mathrm{P}^2$. As a reminder, in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we showed that $\mathrm{H}_2(P) = 0$, $\mathrm{H}_1(P) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{H}_0(P) \cong \mathbb{Z}$. It follows from the Universal Coefficient-H*-Theorem 201.23 and from the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 and the Ext-Properties Lemma 201.19 (4) that

$$\begin{array}{lllllll}
\mathrm{H}^2(P;\mathbb{Z}) & \cong & \mathrm{Hom}(\mathrm{H}_2(P), \mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_1(P), \mathbb{Z}) & \cong & \mathrm{Hom}(0, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) & = & 0 \oplus \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \\
\mathrm{H}^1(P;\mathbb{Z}) & \cong & \mathrm{Hom}(\mathrm{H}_1(P), \mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_0(P), \mathbb{Z}) & \cong & \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) & = & 0 \oplus 0 \;\; = 0 \\
\mathrm{H}^0(P;\mathbb{Z}) & \cong & \mathrm{Hom}(\mathrm{H}_0(P), \mathbb{Z}) & \cong & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) & & \cong \mathbb{Z}
\end{array}$$

and

$$\begin{array}{lllll}
\mathrm{H}^2(P;\mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{H}_2(P), \mathbb{F}_2) \oplus \mathrm{Ext}(\mathrm{H}_1(P), \mathbb{F}_2) & \cong \mathrm{Hom}(0, \mathbb{F}_2) \oplus \mathrm{Ext}(\mathbb{F}_2, \mathbb{F}_2) & = 0 \oplus \mathbb{F}_2 = \mathbb{F}_2 \\
\mathrm{H}^1(P;\mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{H}_1(P), \mathbb{F}_2) \oplus \mathrm{Ext}(\mathrm{H}_0(P), \mathbb{F}_2) & \cong \mathrm{Hom}(\mathbb{F}_2, \mathbb{F}_2) \oplus \mathrm{Ext}(\mathbb{Z}, \mathbb{F}_2) & = \mathbb{F}_2 \oplus 0 = \mathbb{F}_2 \\
\mathrm{H}^0(P;\mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{H}_0(P), \mathbb{F}_2) & \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{F}_2) & = \mathbb{F}_2.
\end{array}$$

Fortunately this coincides with the calculations from page 3832. □

We move on to the ever mysterious lens spaces:

Example. Let $p, q \in \mathbb{N}$ be coprime. We consider the corresponding lens space $L(p,q)$. In the Lens Space-H_*-Lemma 183.21 we saw that

$$\mathrm{H}_n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n = 1. \end{cases}$$

We can now use the Universal Coefficient-H^*-Theorem 201.23, the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 and the Ext-Properties Lemma 201.19 (4) to compute the cohomology groups of $L(p,q)$. We leave it to the reader to verify that[2139]

$$\mathrm{H}^n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3 \\ 0, & \text{if } n = 1 \text{ or } n > 3, \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n = 2. \end{cases}$$

This shows that cohomology groups also cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. In particular we have not made any progress on Question 91.6 since the introduction of the fundamental group. All the extra invariants, namely higher homotopy groups, homology groups and cohomology groups (with any coefficients) have proved useless. □

The following proposition gives us a reinterpretation of the first cohomology group:

Proposition 201.24. (H^1-Hom-Proposition) Let X be a path-connected topological space, let $x_0 \in X$ and let G be an abelian group. We have natural isomorphisms

$$\mathrm{H}^1(X;G) \xrightarrow[\uparrow]{\cong} \mathrm{Hom}(\mathrm{H}_1(X);G) \xrightarrow[\uparrow]{\cong} \mathrm{Hom}(\pi_1(X,x_0),G).$$

given by the evaluation homomorphism ev

induced by the Hurewicz homomorphism $\pi_1(X,x_0) \to \mathrm{H}_1(X)$ from Definition 166.5

Here, "natural" means that for any continuous map $(X,x_0) \to (Y,y_0)$ of pointed topological spaces the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^1(Y;G) & \xrightarrow{\cong} & \mathrm{Hom}(\mathrm{H}_1(Y);G) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(Y,y_0),G) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^1(X;G) & \xrightarrow{\cong} & \mathrm{Hom}(\mathrm{H}_1(X);G) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(X,x_0),G). \end{array}$$

Proof. As we pointed out in the Evaluation Map Lemma 201.8 and the Hurewicz Homomorphism Proposition 166.3 (3), both maps are natural. So it remains to show that both maps are isomorphisms.

[2139] Note that the $\mathbb{Z}/p\mathbb{Z}$ shifted from $n = 1$ in homology to $n = 2$ in cohomology.

We insert two extra maps into the above sequence of maps to see that the maps are indeed isomorphisms:

$$\begin{array}{ccccc} \mathrm{H}^1(X;G) & \underset{\uparrow}{\cong} & \overbrace{\mathrm{Ext}(\mathrm{H}_0(X),G)}^{=0} \oplus \mathrm{Hom}(\mathrm{H}_1(X),G) & \underset{\uparrow}{=} & \mathrm{Hom}(\mathrm{H}_1(X),G). \\ & \text{by the Universal Coefficient Theorem 201.23} & & \text{by the } \mathrm{H}_0\text{-Proposition 143.25 we know that } \mathrm{H}_0(X)\cong\mathbb{Z}\text{, it follows from the Ext-Properties Lemma 201.19 that the Ext-Term is zero} & \\ & \underset{\uparrow}{\overset{\cong}{\to}} & \mathrm{Hom}(\pi_1(X,x_0)_{\mathrm{ab}},G) & \underset{\uparrow}{\overset{\cong}{\to}} & \mathrm{Hom}(\pi_1(X,x_0),G). \\ & \text{by the Hurewicz Theorem 166.10} & & \text{by the Commutator Subgroup Lemma 98.14 (3), since } G \text{ is abelian} & \end{array}$$

■

201.5. **Cohomology groups of topological manifolds.** In this section we consider cohomology groups of compact topological manifolds. Before we delve into the topic it is convenient to recall Definition 194.3:

Definition 201.25. Let M be a compact n-dimensional topological manifold. Furthermore let $\{\alpha_x\}_{x\in M\setminus\partial M}$ be an orientation for M. We know by the Fundamental Class-Proposition 194.2 that there exists a unique $[M]\in \mathrm{H}_n(M,\partial M)$ with $\rho_x([M])=\alpha_x$ for all $x\in M\setminus\partial M$. We refer to $[M]$ as the fundamental class of the oriented topological manifold M. If M is non-empty and connected, then $\mathrm{H}_n(M,\partial M)=\mathbb{Z}\cdot[M]$.

The following theorem is the cohomological analog of the Manifold-Top-H_*-Theorem 193.1:

Theorem 201.26. (Top Cohomology Theorem) Let M be a compact[2140] connected non-empty n-dimensional topological manifold and let G be an abelian group. The following statements hold:

(1) (a) For every $i>n$ we have $\mathrm{H}^i(M;G)=0$.
(b) For every $i>n$ we have $\mathrm{H}^i(M,\partial M;G)=0$.

(2) (a)
$$\mathrm{H}^n(M;G)\ \cong\ \begin{cases} G, & \text{if } M \text{ is closed and orientable,}\\ G/2\cdot G, & \text{if } M \text{ is closed and not orientable,}\\ 0, & \text{if } M \text{ is not closed.}\end{cases}$$

(b)
$$\mathrm{H}^n(M,\partial M;G)\ \cong\ \begin{cases} G, & \text{if } M \text{ is orientable,}\\ G/2\cdot G, & \text{if } M \text{ is not orientable.}\end{cases}$$

Furthermore, if M is orientable, then the evaluation homomorphism

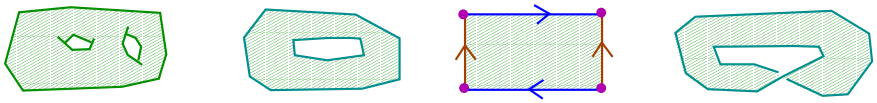
$$\mathrm{ev}\colon \mathrm{H}^n(M,\partial M;G)\ \to\ \mathrm{Hom}(\underbrace{\mathrm{H}_n(M,\partial M)}_{=\mathbb{Z}\cdot[M]\text{ if oriented}},G)$$

is an isomorphism.

Proof. Let M be a compact connected non-empty n-dimensional topological manifold. By the Manifold-Top-H_*-Theorem 193.1 we know that the following statements hold:

(1) (a) For every $i>n$ we have $\mathrm{H}_i(M)=0$.
(b) For every $i>n$ we have $\mathrm{H}_i(M,\partial M)=0$.

[2140]Note that we restrict ourselves to the compact case. One would expect that for a non-compact connected n-dimensional topological manifold M we have $\mathrm{H}^n(M;G)=0$ for any abelian group G. With the methods at our disposal it is not possible to prove such a statement.



$\mathrm{H}^2(M;G)$	G	0	$G/2\cdot G$	0
$\mathrm{H}^2(M,\partial M;G)$	G	G	$G/2\cdot G$	$G/2\cdot G$

(2) (a)
$$\mathrm{H}_n(M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is closed and orientable,} \\ 0, & \text{if } M \text{ is closed and not orientable,} \\ 0, & \text{if } M \text{ is not closed.} \end{cases}$$

(b)
$$\mathrm{H}_n(M,\partial M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable,} \\ 0, & \text{if } M \text{ is not orientable} \end{cases}$$

By the Manifold-H_{n-1}-Theorem 193.4 we also know the following:

(3) (a)
$$\mathrm{H}_{n-1}(M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

(b)
$$\mathrm{H}_{n-1}(M,\partial M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } M \text{ is non-orientable.} \end{cases}$$

All the statements of the theorem now follow easily from the above facts together with the Universal Coefficient-H^*-Theorem 201.23, the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 and the Ext-Properties Lemma 201.19. For example, if M is closed and not orientable, then we have

$$\mathrm{H}^n(M;G) \overset{\text{by the Universal Coefficient Theorem 201.23}}{\cong} \mathrm{Hom}(\mathrm{H}_n(M),G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(M),G) \overset{\text{by (2a) we have } \mathrm{H}_n(M)=0 \text{ and by (3a) we have } \mathrm{H}_{n-1}(M)\cong\mathbb{Z}^k\oplus\mathbb{Z}/2\mathbb{Z} \text{ for some } k\in\mathbb{N}_0}{\cong} \mathrm{Ext}(\mathbb{Z}^k\oplus\mathbb{Z}/2\mathbb{Z},G) \overset{\text{by the Ext-Properties Lemma 201.19 and the } \mathbb{Z}/k\mathbb{Z}\text{-Ext-Lemma 201.18}}{\cong} G/2\cdot G.$$

All the other statements of the theorem can be deduced in a similar fashion. ■

The Manifold-Top-H^*-Theorem 201.26 (2b) now allows us to introduce the dual fundamental class of a compact oriented topological manifold:

Definition 201.27.

(1) Let M be a compact oriented connected non-empty n-dimensional topological manifold M. The Manifold-Top-H^*-Theorem 201.26 (2b) says that the evaluation homomorphism
$$\begin{aligned} \mathrm{ev}\colon \mathrm{H}^n(M,\partial M;\mathbb{Z}) &\to \mathrm{Hom}(\overbrace{\mathrm{H}_n(M,\partial M)}^{=\mathbb{Z}\cdot[M]},\mathbb{Z}) \\ \varphi &\mapsto (\sigma \mapsto \langle \varphi,\sigma\rangle) \end{aligned}$$
is an isomorphism. This implies that there exists a generator $[M]^*$ of $\mathrm{H}^n(M,\partial M;\mathbb{Z})$ with $\langle [M]^*,[M]\rangle = 1$. We refer to $[M]^*$ as the dual fundamental class of M.

(2) Given a compact oriented non-empty n-dimensional topological manifold M with finitely many components $M_1,\dots,M_k$ we define its dual fundamental class to be
$$[M]^* := [M_1]^* + \dots + [M_k]^* \in \underbrace{\mathrm{H}^n(M_1,\partial M_1;\mathbb{Z}) \oplus \dots \oplus \mathrm{H}^n(M_k,\partial M_k;\mathbb{Z})}_{=\mathrm{H}^n(M,\partial M;\mathbb{Z}),\text{ by the Path-Component-}\mathrm{H}^*\text{-Lemma 198.8}}.$$

Example.

(1) We consider the topological manifold S^1 with the standard orientation. Recall that on page 3536 we saw that the singular simplex

$$\begin{array}{rcl} \mu\colon \Delta^1 & \to & S^1 \\ (1-t,t) & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

represents the fundamental class of S^1. Also recall that on page 3813 we introduced a cocycle $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$ and we explicitly calculated that $\langle [\theta_{\mathbb{Z}}], [S^1]\rangle = \theta_{\mathbb{Z}}(\mu) = 1$. This shows immediately that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the dual fundamental class of S^1.

(2) Note that many results from Section 194.4 have analogs in the setting of the dual fundamental class: For example let M be a compact oriented n-dimensional topological manifold. It follows quite easily from the Fundamental Class-of-∂-Proposition 194.13 together with Exercise 199.12 that we have the equality $\delta_{n-1}([\partial M]^*) = [M]^*$, where as usual $\delta_{n-1}\colon \mathrm{H}^{n-1}(\partial M;\mathbb{Z}) \to \mathrm{H}^n(M,\partial M;\mathbb{Z})$ is the connecting homomorphism. □

Later on we will need the following lemma:

Lemma 201.28. (Dual Fundamental Class–Degree Lemma) Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. Then

$$f^*([N]^*) \;=\; \underbrace{\deg(f)}_{\text{see Definition 195.1}} \cdot [M]^*.$$

Proof. Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. It follows from the discussion in Definition 201.27 that any class in $\varphi = \mathrm{H}^n(M;\mathbb{Z})$ is of the form $\varphi = k\cdot[M]^*$ for some $k\in\mathbb{Z}$ and it follows from $\langle [M]^*, [M]\rangle = 1$ and the bilinearity of the Kronecker pairing that $k = \langle \varphi, [M]\rangle_M$. In our case we have

$$\begin{array}{rcl@{\qquad}rcl} \langle f^*([N]^*), [M]\rangle_M & \overset{\text{by the Kronecker Pairing Natural Lemma 199.9}}{\downarrow} & & & \overset{\text{by Definition 195.1 of } \deg(f)}{\downarrow} & \\ & = & \langle [N]^*, f_*([M])\rangle_N & & = & \langle [N]^*, \deg(f)\cdot[N]\rangle_N \\ & = & \deg(f)\cdot \underbrace{\langle [N]^*, [N]\rangle_N}_{=1} & & = & \deg(f). \\ & \underset{\text{since } \langle\,,\,\rangle_N \text{ is bilinear}}{\uparrow} & & & & \end{array}$$

It follows from the above discussion that $f^*([N]^*) = \deg(f)\cdot[M]^*$. ■

201.6. **Appendix: Vanishing of homology and cohomology.** In this appendix we will prove the following proposition that gives us in particular a converse to the Also-Iso-on-Cohomology Proposition 200.2. We will rarely make use of it.

Proposition 201.29. (Vanishing-of-(Co-) Homology Proposition)

(1) Let C_* be a chain complex. The following two statements are equivalent:
   (a) $\mathrm{H}_k(C_*) = 0$ for all $k\in\mathbb{N}_0$.
   (b) $\mathrm{H}^k(C_*;\mathbb{Z}) = 0$ for all $k\in\mathbb{N}_0$.

(2) Let $f\colon D_* \to E_*$ be a chain map. The following two statements are equivalent:
   (a) All the induced maps $f_*\colon \mathrm{H}_k(D_*) \to \mathrm{H}_k(E_*)$ are isomorphisms.
   (b) All the induced maps $f^*\colon \mathrm{H}^k(E_*;\mathbb{Z}) \to \mathrm{H}^k(D_*;\mathbb{Z})$ are isomorphisms.

Proof of Proposition 201.29 if all homology groups are finitely generated.

(1) We start out with the following claim:

Claim. Let A be a finitely generated abelian group. If A is non-trivial, then $\mathrm{Hom}(A,\mathbb{Z})$ or $\mathrm{Ext}(A,\mathbb{Z})$ is non-trivial.

Proof. Let A be a non-trivial abelian group. It follows easily from the Abelian Group-Classification Theorem 97.24 of Finitely Generated Abelian Groups together with the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 that $\mathrm{Hom}(A,\mathbb{Z})$ or $\mathrm{Ext}(A,\mathbb{Z})$ is non-trivial. ⊞

The equivalence of (a) and (b) follows immediately from the claim together with the Algebraic Universal Coefficient-H*-Theorem 201.22.

(2) Let $f\colon D_* \to E_*$ be a chain map. We denote by $(\mathrm{M}(f),\partial_*)$ the corresponding algebraic mapping cone from Definition 158.10. By the Mapping Cone-LES Lemma 158.11 there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_k(D_*) \xrightarrow{f_*} \mathrm{H}_k(E_*) \to \mathrm{H}_k(\mathrm{M}(f)) \to \mathrm{H}_{k-1}(D_*) \xrightarrow{f_*} \ldots$$

Basically the same argument also shows that there exists a long exact sequence of the form

$$\ldots \leftarrow \mathrm{H}^k(D_*;\mathbb{Z}) \xleftarrow{f^*} \mathrm{H}_k(E_*;\mathbb{Z}) \leftarrow \mathrm{H}_n(\mathrm{M}(f);\mathbb{Z}) \leftarrow \mathrm{H}_{n-1}(D_*;\mathbb{Z}) \xleftarrow{f^*} \ldots$$

Thus we see that the equivalence of (2a) and (2b) is an immediate consequence of the equivalence of (1a) and (1b) applied to $C_* = \mathrm{M}(f)$. ■

To provide a proof of Proposition 201.29 in the case that the homology groups are not necessarily finitely generated we need the following extra input:

Proposition 201.30. (Uncountable-Hom-Ext-Proposition) Let H be an abelian group. If H is not a finitely generated group, then $\mathrm{Hom}(H,\mathbb{Z})$ is uncountable or $\mathrm{Ext}(H,\mathbb{Z})$ is uncountable.

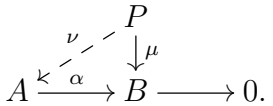
Proof of the Uncountable-Hom-Ext-Proposition 201.30. We refer to [**Hat2002**, Proposition 3.12] for a proof. ■

Proof of Proposition 201.29 in the general case. The proof is exactly the same as before except that in the above proof we get the conclusion of the claim for all abelian groups that are not finitely generated from the Uncountable-Hom-Ext-Proposition 201.30. ■

201.7. Exercises for Chapter 201 .

Exercise 201.1. Show that every injective abelian group is also divisible.

Exercise 201.2. Let R be a commutative ring R. An R-module P is called *projective* if for all R-modules A and B, every epimorphism $\alpha\colon A \to B$ and every homomorphism $\mu\colon P \to B$ there exists a homomorphism $\nu\colon P \to A$ such that the following diagram commutes:



(a) Show that a finitely generated projective $\mathbb{Z}$-module is free.

(b) Give an example of a commutative ring and a finitely generated R-module that is projective but not free.

Exercise 201.3. Let

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$$

be a short exact sequence of chain complexes. We consider the following diagram

$$\begin{array}{ccccccccc}
\to \mathrm{H}^k(\mathrm{Hom}(C_*, G)) & \xrightarrow{g^*} & \mathrm{H}^k(\mathrm{Hom}(B_*, G)) & \xrightarrow{f^*} & \mathrm{H}^k(\mathrm{Hom}(A_*, G)) & \xrightarrow{(\partial_k)^*} & \mathrm{H}^{k+1}(\mathrm{Hom}(C_*, G)) \to \\
\downarrow \mathrm{ev} & & \downarrow \mathrm{ev} & & \downarrow \mathrm{ev} & & \downarrow \mathrm{ev} \\
\to \mathrm{Hom}(\mathrm{H}_k(C_*), G) & \xrightarrow{g^*} & \mathrm{Hom}(\mathrm{H}_k(B_*), G) & \xrightarrow{f^*} & \mathrm{Hom}(\mathrm{H}_k(A_*), G) & \xrightarrow{(\partial_k)^*} & \mathrm{Hom}(\mathrm{H}^{k+1}(C_*), G) \to
\end{array}$$

where the vertical maps are given by the evaluation map from Definition 201.9. Show that the diagram commutes.
Remark. This exercise is almost identical to Exercise 77.6.

Exercise 201.4. Let

$$\ldots \to A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \xrightarrow{\alpha_{k-1}} \ldots$$

be an exact sequence of abelian groups and let B be an abelian group. Show that the sequence

$$\ldots \leftarrow \mathrm{Hom}(A_{k+1}, B) \xleftarrow{\alpha_{k+1}^*} \mathrm{Hom}(A_k, B) \xleftarrow{\alpha_k^*} \mathrm{Hom}(A_{k-1}, B) \xleftarrow{\alpha_{k-1}^*} \ldots$$

is also exact.

Exercise 201.5. Let G be an abelian group. Furthermore let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of abelian groups. Show that there exists a natural homomorphism $\delta\colon \mathrm{Hom}(G, C) \to \mathrm{Ext}(G, A)$ such that the following sequence is exact:

$$\begin{aligned}
0 \to \mathrm{Hom}(G, A) &\xrightarrow{\alpha_*} \mathrm{Hom}(G, B) \xrightarrow{\beta_*} \mathrm{Hom}(G, C) \\
&\xrightarrow{\delta} \mathrm{Ext}(G, A) \xrightarrow{\alpha_*} \mathrm{Ext}(G, B) \xrightarrow{\beta_*} \mathrm{Ext}(G, C) \to 0.
\end{aligned}$$

Exercise 201.6. Let C_* be a chain complex of free abelian groups and let $n \in \mathbb{N}$. Show that there does not exist a *natural* isomorphism

$$\mathrm{H}^n(C_*; G) \cong \mathrm{Ext}(\mathrm{H}_{n-1}(C_*), G) \oplus \mathrm{Hom}(\mathrm{H}_n(C_*), G).$$

Exercise 201.7. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_0 and A_1 are free abelian groups. Show that there exists a CW-complex X such that for every $n \in \mathbb{N}$ we have $\mathrm{H}^n(X; \mathbb{Z}) \cong A_n$.
Hint. Use the Homology Realization Proposition 173.1.
Remark. The statement is not true if we drop the hypothesis that the groups are finitely generated, in fact as is shown in [**Luc1978**] (see also [**KaW1961**]) there is no topological space X with $\mathrm{H}^n(X; \mathbb{Z}) \cong \mathbb{Q}$ for some $n \in \mathbb{N}_0$.

Exercise 201.8. Let X and Y are topological spaces. Suppose that $\mathrm{H}_n(X; \mathbb{Z}) \cong \mathrm{H}_n(Y; \mathbb{Z})$ for some $n \in \mathbb{N}_0$. Does it follow that $\mathrm{H}^n(X; \mathbb{Z})$ and $\mathrm{H}^n(Y; \mathbb{Z})$ are also isomorphic?

Exercise 201.9. Redo Exercise 199.3. More precisely, use the Universal Coefficient-H^*-Theorem 201.23 to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 201.10. Let M be a compact oriented connected n-dimensional topological manifold with connected non-empty boundary. Show that $\delta_{n-1}([\partial M]^*) = [M]^* \in \mathrm{H}^n(M, \partial M; \mathbb{Z})$

where $\delta_{n-1}\colon \mathrm{H}^{n-1}(\partial M;\mathbb{Z}) \to \mathrm{H}^n(M,\partial M;\mathbb{Z})$ denotes the connecting homomorphism in the long exact sequence of cohomology groups of the pair $(M,\partial M)$.
Hint. Use the Fundamental Class-of-∂-Proposition 194.13.

Exercise 201.11.

(a) Let G and H be $\mathbb{Q}$-vector spaces. Let $\varphi\colon G \to H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of $\mathbb{Q}$-vector spaces.
(b) Show that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic.
(c) Show that the abelian groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.
(d) Use (a) and (b), together with the existence of Moore spaces, see the Moore Space Existence Proposition 173.3, and together with the Universal Coefficient-H*-Theorem 201.23 and the Ext$(\mathbb{Q},\mathbb{Z})$-Proposition 201.20 to conclude that there exist topological spaces X and Y such that $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathrm{H}^n(Y;\mathbb{Z})$ for all $n \in \mathbb{N}_0$ but such that $\mathrm{H}_1(X;\mathbb{Z})$ is not isomorphic to $\mathrm{H}_1(Y;\mathbb{Z})$.

Remark. It follows from the Universal Coefficient Theorem for Cohomology Groups 201.23 that the isomorphism types of cohomology groups with $\mathbb{Z}$-coefficients are completely determined by the isomorphism types of the homology groups with $\mathbb{Z}$-coefficients. This exercise shows that in this statement the roles of homology and cohomology are not symmetric.

Exercise 201.12. Let X be a topological space which is $\mathbb{Z}$-finite in the sense of the definition on Definition 182.13, i.e. all homology groups $\mathrm{H}_i(X)$ are finitely generated abelian groups and all but finitely many homology groups $\mathrm{H}_i(X)$ are zero. Let $\varphi\colon X \to X$ be a continuous map. Show that

$$\underbrace{\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(X;\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z}))}_{\text{Lefschetz number } \Lambda(\varphi) \text{ from Definition 182.13}} = \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{tr}(\varphi^*\colon \mathrm{H}^k(X;\mathbb{Z}) \to \mathrm{H}^k(X;\mathbb{Z})).$$

Exercise 201.13. Let (X,x_0) be a pointed topological space that is k-connected, i.e. we assume that $\pi_i(X,x_0) = 0$ for $i = 0,\ldots,k$. Furthermore let G be an abelian group. Show that there exists a natural isomorphism

$$\mathrm{H}^{k+1}(X;G) \xrightarrow{\cong} \mathrm{Hom}(\pi_{k+1}(X,x_0),G).$$

Remark. In other words, your task is to prove a cohomological analog of the Hurewicz Theorem 166.10.

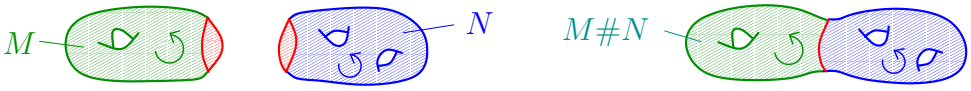
Exercise 201.14. Suppose that (C_*,∂_*) and (C'_*,∂'_*) are two chain complexes. We do *not* assume that the chain groups are necessarily free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $\mathrm{H}_n(C_*)$ and $\mathrm{H}_n(C'_*)$ are isomorphic. Does it follow that for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $\mathrm{H}^n(C_*;G)$ and $\mathrm{H}^n(C'_*;G)$ are also isomorphic?

Exercise 201.15. Let X be a topological space, let G be an abelian group and let $n \in \mathbb{N}_0$. Furthermore let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. Use the Excision Theorem 148.1 and the Universal Coefficient Theorem for Cohomology Groups 201.23 to show that the inclusion $i\colon (X\setminus Z, A\setminus Z) \to (X,A)$ induces an isomorphism

$$i^*\colon \mathrm{H}^n(X,A;G) \xrightarrow{\cong} \mathrm{H}^n(X\setminus Z, A\setminus Z;G).$$

Remark. This gives an alternative proof of the Excision-in-H*-Theorem 200.1 (1).

Exercise 201.16. Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. We consider the connected sum $M\#N$ from Definition 54.12. Determine for every commutative ring R the cohomology groups $\mathrm{H}^*(M\#N;R)$ in terms of the cohomology groups $\mathrm{H}^*(M;R)$ and $\mathrm{H}^*(N;R)$.
Remark. This exercise is the analog of the Connected Sum-H_*-Proposition 185.23.



Exercise 201.17. Show that given any finitely generated abelian group G there exists a closed orientable connected 3-dimensional smooth manifold M with $\mathrm{H}^2(M;\mathbb{Z}) \cong G$.
Hint. You could consider connected sums.
Remark. This exercise shows that, in contrast to the Manifold-H_{n-1}-Theorem 193.4, there are no real restriction on cohomology groups in codimension one.

Exercise 201.18. How many cohomology groups of $\mathrm{GL}(4,\mathbb{R})$ can you determine?

Exercise 201.19. Does there exist a finite 5-dimensional CW-complex X with no 3-cells such that $\mathrm{H}^2(X) \cong \mathbb{Z}/3\mathbb{Z}$, $\mathrm{H}^4(X;\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$?

Exercise 201.20. Does there exist a CW-complex X with the following properties:

(a) X has one 5-cell.
(b) $\mathrm{H}_4(X;\mathbb{F}_2) = 0$.
(c) $\mathrm{H}^3(X;\mathbb{Z}) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$.
(d) $\mathrm{H}_2(X;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
(e) There exists a covering $p\colon S^5 \to X$.

Exercise 201.21. Does there exist a CW-complex X with the following properties:

(a) X has precisely two 3-cells and precisely one 2-cell.
(b) $\mathrm{H}^2(X;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
(c) $\mathrm{H}^2(X;\mathbb{F}_2) \cong \mathbb{Z}/2\mathbb{Z}$.
(d) There exists a covering $p\colon X \to \mathbb{R}\mathrm{P}^3$.

Exercise 201.22. Let $L \subseteq S^3$ be an m-component link. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) (together with the Tubular Map Lemma 58.3) there exists a smooth embedding $\Phi\colon \overline{B}^2 \times L \to S^3$ such that $\Phi(0,P) = P$ for all $P \in L$. We refer to $X_K = S^3 \setminus \Phi(B^2 \times L)$ as the *exterior of* L. Let G be an abelian group.

(a) Determine the relative cohomology groups $\mathrm{H}^*(X_L, \partial X_L; G)$.
(b) Determine the cohomology groups $\mathrm{H}^*(X_L;G)$.

Exercise 201.23. Let $f, g\colon X \to Y$ be continuous maps between topological spaces X and Y. We assume that for all $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$. Does it follow that for all abelian groups G and all $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}^n(Y;G) \to \mathrm{H}^n(X;G)$?

Exercise 201.24. Let X be a topological space and let $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X)$ and $\mathrm{H}_{k-1}(X)$ are finitely generated.

(a) Let $\mathbb{F}$ be a field of characteristic zero. Use the Abelian Group-Classification Theorem 97.24, Universal Coefficient-H*-Theorem 201.23 and the Universal Coefficient Theorem 162.16 to calculate $\dim(\mathrm{H}_k(X;\mathbb{F}))$ and $\dim(\mathrm{H}^k(X;\mathbb{F}))$ in terms of suitable invariants of the abelian groups $\mathrm{H}_k(X)$ and $\mathrm{H}_{k-1}(X)$.

(b) Repeat (a) for a field of characteristic a prime p.

Remark. If you did your job well, you will have found out that for any field $\mathbb{F}$ we have $\dim(\mathrm{H}_k(X;\mathbb{F})) = \dim(\mathrm{H}^k(X;\mathbb{F}))$.

Exercise 201.25. Let $T = S^1 \times S^1$ be the torus and let K be the Klein bottle.

(a) Show that $\mathrm{H}^2(T;\mathbb{F}_2) \cong \mathbb{F}_2$ and that $\mathrm{H}^2(K;\mathbb{F}_2) \cong \mathbb{F}_2$.

(b) Show that for every continuous map $f\colon T \to K$ the corresponding induced homomorphism $f^*\colon \mathrm{H}^2(K;\mathbb{F}_2) \to \mathrm{H}^2(T;\mathbb{F}_2)$ is the zero map.

(c) Show that for every continuous map $f\colon K \to T$ the corresponding induced homomorphism $f^*\colon \mathrm{H}^2(T;\mathbb{F}_2) \to \mathrm{H}^2(K;\mathbb{F}_2)$ is the zero map.

Remark. This exercise is almost identical to Exercise 162.22.

CHAPTER 202

Two other Universal Coefficient Theorem for Cohomology Groups

In this chapter we prove two variations on the Algebraic Universal Coefficient-H^*-Theorem 201.22:

(1) Let C_* be a chain complex and let $\mathbb{F}$ be a field. In the Universal Coefficient-H^*–Field Theorem 202.2 we relate the homology groups $\mathrm{H}_*(C_* \otimes \mathbb{F})$ of C_* with $\mathbb{F}$-coefficients to the cohomology groups $\mathrm{H}^*(C_*; \mathbb{F})$ of C_* with $\mathbb{F}$-coefficients.

(2) Let C_* be a chain complex of free abelian groups and let R be a commutative ring. In the Algebraic H^*–Change-of-Coefficients Theorem 202.7 we show (under suitable hypotheses) how one can determine the cohomology groups $\mathrm{H}^k(C_*; R)$ from the cohomology groups $\mathrm{H}^k(C_*; \mathbb{Z})$.

Both results are not especially essential in what comes next. The reader might want to go straight to Chapter 203, where we introduce the cup product on cohomology.

This chapter is organized as follows:

- In Section 202.1 we will study cohomology with field coefficients and we will state the Universal Coefficient-H^*–Field Theorem 202.2.
- In Section 202.2 we will prove the Universal Coefficient-H^*–Field Theorem 202.2.
- In Section 202.3 we will state and prove the Algebraic H^*–Change-of-Coefficients Theorem 202.7.
- In Section 202.4 we will provide exercises for Chapter 202.

In the next Chapter 203 we will introduce the cup product on cohomology.

202.1. **Cohomology with field coefficients.** We start out by introducing a slight variation on the Kronecker pairing defined in the Kronecker Pairing Lemma 199.7:

Lemma 202.1. (Ring–Kronecker Pairing Lemma) Let C_* be a chain complex and let R be a commutative ring.

(1) The map

$$\begin{aligned} \langle\ ,\ \rangle\colon \overbrace{\mathrm{H}^n(C;R)}^{=\mathrm{H}^n(\mathrm{Hom}(C_*,R))} \times \overbrace{\mathrm{H}_n(C;R)}^{=\mathrm{H}_n(C_*\otimes R)} &\to R \\ \Big([\varphi], \Big[\sum_{i=1}^{k} c_i \otimes a_i\Big]\Big) &\mapsto \sum_{i=1}^{k} \varphi(c_i)\cdot a_i. \end{aligned}$$

is well-defined, bilinear and natural. We refer to this pairing also as the Kronecker pairing.

(2) For any $\varphi \in \mathrm{H}^n(C;R)$ and $\sigma \in \mathrm{H}_n(C)$ we have

$$\underbrace{\langle \varphi, [\sigma \otimes 1] \rangle}_{\text{as defined in (1)}} = \underbrace{\langle \varphi, \sigma \rangle}_{\text{see Definition 199.8}}.$$

Proof. [2141] In the following we show that the Kronecker pairing is well-defined. Once this is out of the way it is basically clear that the Kronecker pairing is bilinear and natural and that statement (2) holds.

Let $\varphi \in \mathrm{Hom}(C_n, R)$ be a cocycle. We consider the map

$$\begin{array}{rcl} C_n \times R & \to & R \\ (c_i, r_i) & \to & \varphi(c_i) \cdot r_i. \end{array}$$

One can easily verify that this map is bilinear. It follows from the Tensor Product–Universal Property Lemma 159.4 (2) that we get an induced map $C_n \otimes R \to R$. This shows that the map

$$\begin{array}{rccl} \langle\ ,\ \rangle\colon & \mathrm{Hom}(C_n, R) \times (C_n \otimes R) & \to & R \\ & \Big(\varphi, \sum\limits_{i=1}^{k} c_i \otimes r_i\Big) & \to & \sum\limits_{i=1}^{k} \varphi(c_i) \cdot r_i \end{array}$$

is well-defined. One can easily verify that this map is bilinear. It remains to show that this map defines a map on (co-) homology.

First let $\varphi \in \mathrm{Hom}(C_n, R)$, let $\sigma \in C_n \otimes R$ and let $\sum\limits_{i=1}^{k} \widetilde{c}_i \otimes \widetilde{r}_i \in C_{n+1} \otimes R$. We see that

$$\Big\langle \varphi, \sigma + \partial_{n+1}\Big(\sum_{i=1}^{k} \widetilde{c}_i \otimes \widetilde{r}_i\Big)\Big\rangle \underset{\substack{\uparrow \\ \text{by definition of} \\ \partial_* \text{ on } C_* \otimes R}}{=} \Big\langle \varphi, \sigma + \sum_{i=1}^{k} \partial_{n+1}(\widetilde{c}_i) \otimes \widetilde{r}_i\Big\rangle \underset{\substack{\uparrow \\ \text{by Definition 199.8} \\ \text{of } \langle\ ,\ \rangle}}{=} \langle \varphi, \sigma\rangle + \sum_{i=1}^{k} \underbrace{\langle \varphi, \partial_{n+1}(\widetilde{c}_i)\rangle}_{\substack{= 0, \text{ since } \varphi \text{ cocycle}}} \cdot \widetilde{r}_i.$$

Next let $\varphi \in \mathrm{Hom}(C_n, R)$ be a cochain, let $\mu \in \mathrm{Hom}(C_{n-1}, R)$ and let $\sigma = \sum\limits_{i=1}^{k} c_i \otimes r_i \in C_n \otimes R$ be a cycle. Then we have the following equalities:

$$\begin{array}{lll} & \underset{\text{since } \langle\ ,\ \rangle \text{ is bilinear}}{\downarrow} & \underset{\text{Definition 199.8 of } \langle\ ,\ \rangle}{\downarrow} \\ \langle \varphi + \delta_{n-1}(\mu), \sigma\rangle & = \langle \varphi, \sigma\rangle + \Big\langle \delta_{n-1}(\mu), \sum\limits_{i=1}^{k} c_i \otimes r_i\Big\rangle & = \langle \varphi, \sigma\rangle + \sum\limits_{i=1}^{k} (\delta_{n-1}(\mu))(c_i) \cdot r_i \\ & = \underset{\uparrow}{\langle \varphi, \sigma\rangle} + \sum\limits_{i=1}^{k} \mu(\partial_n(c_i)) \cdot r_i & = \underset{\uparrow}{\langle \varphi, \sigma\rangle} + \Big\langle \mu, \sum\limits_{i=1}^{k} \partial_n(c_i) \otimes r_i\Big\rangle. \\ & \underset{\text{Definition 198.1 of } \delta_{n-1}}{\uparrow} & \underset{\text{Definition 199.8 of } \langle\ ,\ \rangle}{\uparrow} \\ & = \langle \varphi, \sigma\rangle + \Big\langle \mu, \partial_n\Big(\sum\limits_{i=1}^{k} c_i \otimes r_i\Big)\Big\rangle & = \langle \varphi, \sigma\rangle + \langle \mu, \partial_n(\sigma)\rangle \underset{\substack{\uparrow \\ \text{since } \sigma \text{ is a cycle}}}{=} \langle \varphi, \sigma\rangle + 0. \\ & \underset{\text{definition of } \partial_* \text{ on } C_* \otimes R}{\uparrow} & \end{array}$$

■

In the Universal Coefficient-H*-Theorem 201.23 we determined, given an abelian group G, the relationship between the ordinary homology groups $\mathrm{H}_*(X, A)$ and the cohomology groups $\mathrm{H}^*(X, A; G)$. In the following theorem we prove a similar statement: Given a field $\mathbb{F}$ we determine the relationship between the homology groups $\mathrm{H}_*(X, A; \mathbb{F})$ and the cohomology groups $\mathrm{H}^*(X, A; \mathbb{F})$:

[2141] As we will see, the proof of the lemma is elementary. Nonetheless, the calculations involved are much longer than the calculations in the proof of the original Kronecker Pairing Lemma 199.7.

Theorem 202.2. (Universal Coefficient-H*–Field Theorem) Let $\mathbb{F}$ be a field.

(1) For any chain complex C_* and any $n \in \mathbb{N}_0$ the evaluation homomorphism

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(\mathrm{Hom}(C_*,\mathbb{F})) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(C_*\otimes\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*\otimes\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle\varphi,\sigma\rangle \end{array}\right) \end{array}$$

Kronecker pairing from the Ring–Kronecker Pairing Lemma 202.1

is a natural isomorphism of $\mathbb{F}$-vector spaces.

(2) Let (X,A) be a pair of topological spaces. For every $n \in \mathbb{N}_0$ the evaluation map

$$\mathrm{ev}\colon \mathrm{H}^n(X,A;\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(X,A;\mathbb{F}),\mathbb{F})$$

from (1) is a natural isomorphism of $\mathbb{F}$-vector spaces.

Remark. In the Algebraic Universal Coefficient-H*-Theorem 315.44 we will prove a generalization of the Universal Coefficient-H*–Field Theorem 202.2. □

The proof of the Universal Coefficient-H*–Field Theorem 202.2 consists of two steps:

(1) In the Evaluation-Isomorphism Theorem 77.18 we formulate and prove a "Universal Coefficient Type"-Theorem for chain complexes of $\mathbb{F}$-vector spaces.
(2) We need to massage the setting from the Universal Coefficient-H*–Field Theorem 202.2 to be able to apply (1).

We carry out this program in the next section.

202.2. **Proof of the Universal Coefficient-H*–Field Theorem 202.2.** In this section we give the proof of the Universal Coefficient-H*–Field Theorem 202.2. Before we can provide the proof we need to prepare the ground with some almost self-explanatory definitions:

Definition 202.3. Let $\mathbb{F}$ be a field.

(1) We define a chain complex over $\mathbb{F}$ to be a chain complex (V_*,v_*) where all chain groups are $\mathbb{F}$-vector spaces and all boundary maps are $\mathbb{F}$-homomorphisms.
(2) Given a chain complex (V_*,v_*) over the field $\mathbb{F}$ and given $n \in \mathbb{N}_0$ its n-th homology

$$\mathrm{H}_n(V_*) := \frac{\ker(v_n\colon V_n \to V_{n-1})}{\mathrm{im}(v_{n+1}\colon V_{n+1} \to V_n)}.$$

is an $\mathbb{F}$-vector space in a natural way.
(3) We define cochain complexes and cochain maps over $\mathbb{F}$ in the obvious way.
(4) Given a chain complex (V_*,v_*) over $\mathbb{F}$ we obtain a cochain complex by considering $(\mathrm{Hom}_{\mathbb{F}}(V_*,\mathbb{F}),v^*)$ and we define its n-th cohomology group to be the $\mathbb{F}$-vector space

$$\mathrm{H}^n(V;\mathbb{F}) := \frac{\ker(v_{n+1}^*\colon \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(V_{n+1},\mathbb{F}))}{\mathrm{im}(v_n^*\colon \mathrm{Hom}_{\mathbb{F}}(V_{n-1},\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F}))}.$$

Fortunately we already proved the following theorem, which can be viewed as an analog of the Algebraic Universal Coefficient-H*-Theorem 201.22 for chain complexes over fields:

Theorem 77.18 (Evaluation-Isomorphism Theorem) Let $\mathbb{F}$ be a field and let (V_*, v_*) be a chain complex over $\mathbb{F}$. The map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}}\colon \mathrm{H}^k(V_*;\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*),\mathbb{F}). \\ (\varphi\colon V_k \to \mathbb{F}) & \mapsto & \begin{pmatrix} \mathrm{H}_k(V_*) & \to & \mathbb{F} \\ [c] & \mapsto & \varphi(c) \end{pmatrix} \end{array}$$

is well-defined and it is a natural isomorphism of $\mathbb{F}$-vector spaces.

For the reader's convenience we recall the proof of the Evaluation-Isomorphism Theorem 77.18. The proof rests on the following elementary lemma, which happens to be identical to the Exactness Preservation Lemma 77.19:

Auxiliary Lemma 202.4. Let $\mathbb{F}$ be a field and let $0 \to B \xrightarrow{i} Z \xrightarrow{p} H \to 0$ be a short exact sequence of $\mathbb{F}$-vector spaces. Then the sequence

$$0 \to \mathrm{Hom}_{\mathbb{F}}(H,\mathbb{F}) \xrightarrow{p^*} \mathrm{Hom}_{\mathbb{F}}(Z,\mathbb{F}) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{F}}(B,\mathbb{F}) \to 0$$

is also exact.

Proof of Lemma 202.4. We already prove the lemma on page 1586. For the reader's convenience we summarize the argument: As in the Free Abelian Group–Splitting Lemma 97.11 we see that the original short exact sequence splits. The desired statement then follows from the analog of the Hom-Split-SES Lemma 197.8 for chain complexes over a field. ■

Proof of the Evaluation-Isomorphism Theorem 77.18. [2142] Let $\mathbb{F}$ be a field and let (V_*, ∂_*) be a chain complex of $\mathbb{F}$-vector spaces. First note that it follows, as in the Evaluation Map Lemma 201.8, from an elementary argument that the evaluation homomorphism $\mathrm{ev}_{\mathbb{F}}\colon \mathrm{H}^k(V_*;\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*),\mathbb{F})$ is actually well-defined and that it is natural. So it remains to show that it is an isomorphism.

For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. We consider the following short exact sequence of vertical chain complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & V_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & V_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

We apply the contravariant functor $A \mapsto \mathrm{Hom}_{\mathbb{F}}(A,\mathbb{F})$ to the above short exact sequence of chain complexes. This leads us to the following claim:

[2142] The following proof is very similar to the proof of the Algebraic Universal Coefficient-H*-Theorem 201.22, which in turn is very similar to the proof of the Algebraic Universal Coefficient Theorem 162.15.

Claim. The following maps define a short exact sequence of "vertical" cochain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}) & \xrightarrow{\partial_n^*} & \mathrm{Hom}_{\mathbb{F}}(V_n,\mathbb{F}) & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(B_{n-2},\mathbb{F}) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}_{\mathbb{F}}(V_{n-1},\mathbb{F}) & \longrightarrow & \mathrm{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Dual Cochain Map Lemma 197.20 that this is a sequence of cochain complexes.

(2) Since we work over a field we obtain from Lemma 202.4 that the above horizontal sequences of cochain complexes are still exact. ⊞

We apply the Cohomology–Connecting Homomorphism Lemma 197.16 to the short exact sequence of cochain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F})}_{=\mathrm{H}^n(Z_*;\mathbb{F})} \to \underbrace{\mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F})}_{=\mathrm{H}^n(B_*;\mathbb{F})}.$$

In fact, it follows easily from Definition 197.15 that the connecting homomorphism $\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F})$ is the map that is induced by the inclusion $i_n\colon B_n \to Z_n$. By the Cohomology–LES Proposition 197.17 this connecting homomorphism gives us the following long exact sequence:

$$\begin{aligned}
&\to \mathrm{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) \xrightarrow{i_n^*} \mathrm{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}) \to \mathrm{H}^n(\mathrm{Hom}(V_*,\mathbb{F})) \\
&\qquad\qquad \to \mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \xrightarrow{i_{n-1}^*} \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F}) \to \dots
\end{aligned}$$

By the Break-into-SES Lemma 145.6 we obtain the following short exact sequence

$$\begin{aligned}
0 \to \overbrace{\mathrm{coker}(\mathrm{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}))}^{\substack{=0 \text{ by Lemma 202.4 applied to the short} \\ \text{exact sequence } 0 \to B_l \to Z_l \to \mathrm{H}_l(V_*) \to 0}} &\to \mathrm{H}^n(\mathrm{Hom}_{\mathbb{F}}(V_*,\mathbb{F})) \\
&\to \underbrace{\ker(\mathrm{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(B_n,\mathbb{F}))}_{\substack{\cong \mathrm{Hom}(\mathrm{H}_n(V_*),\mathbb{F}) \text{ by Lemma 202.4} \\ \text{applied to the short exact sequence} \\ 0 \to B_l \to Z_l \to \mathrm{H}_l(V_*) \to 0}} \to 0.
\end{aligned}$$

■

Now we can finally provide the proof of the Universal Coefficient-H*–Field Theorem 202.2.

Proof of the Universal Coefficient-H*–Field Theorem 202.2. Statement (2) of the theorem is of course just a special case of Statement (1). So let us provide a proof of Statement (1). Let C_* be a chain complex and let $\mathbb{F}$ be a field. We need to show that for any $n \in \mathbb{N}_0$ the evaluation homomorphism

$$\begin{array}{rcl}
\mathrm{ev}\colon \mathrm{H}^n(\mathrm{Hom}(C_*,\mathbb{F})) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(C_*\otimes\mathbb{F}),\mathbb{F}) \\
\varphi & \mapsto & \begin{pmatrix} \mathrm{H}_n(C_*\otimes\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle\varphi,\sigma\rangle \end{pmatrix}
\end{array}$$

from the Ring–Kronecker Pairing Lemma 202.1 is a natural isomorphism of $\mathbb{F}$-vector spaces. Naturality follows from the naturality of the Kronecker pairing which we discussed in the Ring–Kronecker Pairing Lemma 202.1. So it remains to show that the map is an isomorphism.

One might think that the desired statement follows immediately from the Evaluation-Isomorphism Theorem 77.18, but that is not quite the case since we first have to translate our setting into a problem about a chain complex over the field $\mathbb{F}$. To do so we need the following elementary claim.

Claim.

(1) Let C be an abelian group. The map[2143] [2144]

$$\begin{array}{rcl} \Phi_C\colon \operatorname{Hom}(C,\mathbb{F}) & \to & \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) \\ (\varphi\colon C\to\mathbb{F}) & \mapsto & \left(\begin{array}{ccc} C\otimes\mathbb{F} & \to & \mathbb{F} \\ \sum\limits_{i=1}^{k} c_i\otimes a_i & \mapsto & \sum\limits_{i=1}^{k}\varphi(c_i)\cdot a_i \end{array}\right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces.

(2) The maps

$$\Phi_{C_k}\colon \operatorname{Hom}(C_k,\mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(C_k\otimes\mathbb{F},\mathbb{F})$$

from (1) define an isomorphism of cochain complexes of $\mathbb{F}$-vector spaces.

Proof.

(1) First note that one can verify using the Tensor Product–Universal Property Lemma 159.4 (2) that the map Φ_C is well-defined. Next note that one can easily check that the map is natural. Finally note that it is straightforward to verify that an inverse map to Φ_C is given by the following homomorphism of $\mathbb{F}$-vector spaces:

$$\begin{array}{rcl} \Psi_C\colon \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) & \to & \operatorname{Hom}(C,\mathbb{F}) \\ (\varphi\colon C\otimes\mathbb{F}\to\mathbb{F}) & \mapsto & \big(C \xrightarrow{c\mapsto c\otimes 1} C\otimes\mathbb{F} \xrightarrow{\varphi} \mathbb{F}\big). \end{array}$$

(2) It follows from the naturality of Φ_C that the maps in (1) actually form a cochain map. By (1) the maps Φ_{C_k}, $k\in\mathbb{N}_0$, are isomorphisms of the cochain complexes of $\mathbb{F}$-vector spaces. ⊞

Given $n\in\mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} & \mathrm{H}^n(\operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F})) & \\ {\scriptstyle\Phi}\nearrow & & \searrow{\scriptstyle \mathrm{ev}_{\mathbb{F}}} \\ \mathrm{H}^n(\operatorname{Hom}(C_*,\mathbb{F})) & \xrightarrow[\mathrm{ev}]{} & \operatorname{Hom}_{\mathbb{F}}(\mathrm{H}_n(C\otimes\mathbb{F}),\mathbb{F}). \end{array}$$

We make the following observations:

- It follows easily from the definitions that for any representative in $\mathrm{H}^n(\operatorname{Hom}(C_*,\mathbb{F}))$ the image under the composition of the two diagonal maps agrees with the image under the map ev. This shows that the diagram commutes.[2145]
- It follows from the claim that the left diagonal map is an isomorphism.

[2143] It follows from the Tensor-with-Ring Lemma 159.18 that we can view $C\otimes\mathbb{F}$ as an $\mathbb{F}$-vector space.

[2144] Note that to the left we consider homomorphisms of abelian groups whereas to the right we consider homomorphisms of $\mathbb{F}$-vector spaces.

[2145] In fact this also gives a new proof that ev is well-defined.

- It follows from the Evaluation-Isomorphism Theorem 77.18 that the right diagonal map is an isomorphism.

It follows from the above observations that the bottom map is an isomorphism. But that is exactly what we had wanted to show. ■

202.3. **Change of coefficients in cohomology.** Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. By the Algebraic Universal Coefficient Theorem 162.15 we know that the homology groups $\mathrm{H}_n(C_* \otimes G)$ of C_* with G-coefficients are determined by the ordinary homology groups of C_*. Similarly, by the Universal Algebraic Universal Coefficient-H^*-Theorem 201.22 we know that the cohomology groups $\mathrm{H}^*(C_*; G)$ are also determined by ordinary the homology groups $\mathrm{H}_*(C_*)$. The following question arises.

Question 202.5. Let G be an abelian group. Are the cohomology groups $\mathrm{H}^*(C; G)$ determined by the cohomology groups $\mathrm{H}^*(C; \mathbb{Z})$?

It turns out that the question is slightly subtle. To formulate the answer we need the following lemma:

Lemma 202.6. (H^*–Change-of-Coefficients Lemma) Let G be an abelian group.

(1) For any abelian group H the map

$$\begin{array}{rcl} \mu\colon \mathrm{Hom}(H,\mathbb{Z}) \otimes G & \to & \mathrm{Hom}(H,G) \\ \sum_{i=1}^{m} \varphi_i \otimes g_i & \mapsto & \begin{pmatrix} H & \to & G \\ h & \mapsto & \sum_{i=1}^{m} \varphi_i(h) \cdot g_i \end{pmatrix} \end{array}$$

is a well-defined *natural* homomorphism.

(2) Given a chain complex (C_*, ∂_*) the map

$$\begin{array}{rcl} \mathrm{H}^n(C_*;\mathbb{Z}) \otimes G & \stackrel{\mu}{\to} & \mathrm{H}^n(C_*;G) \\ \sum_{i=1}^{m} [\varphi_i] \otimes g_i & \mapsto & \sum_{i=1}^{m} [\mu(\varphi_i \otimes g_i)] \end{array}$$

is well-defined.

Proof.

(1) It follows easily from the Tensor Product–Universal Property Lemma 159.4 (2) that the map is well-defined. Naturality follows immediately from the definitions.

(2) Since the maps in (1) are natural we see that they respect boundary maps. Statement (2) is a straightforward consequence of this observation. ■

The following theorem gives an affirmative answer to Question 202.5, albeit only under an extra hypothesis.

Theorem 202.7. (Algebraic H^*–Change-of-Coefficients Theorem) Let (C_*, ∂_*) be a chain complex of free abelian groups. We suppose that one of the following two conditions holds:

(a) All cohomology groups $\mathrm{H}^n(C_*; \mathbb{Z})$ are finitely generated.

(b) All homology groups $\mathrm{H}_n(C_*)$ are finitely generated.

For every abelian group G and for every $n \in \mathbb{N}_0$ there exists a homomorphism

$$\nu\colon \mathrm{H}^n(C_*;G) \ \to\ \mathrm{Tor}(\mathrm{H}^{n+1}(C_*;\mathbb{Z}),G)$$

such that the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^n(C_*;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(C_*;G) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}^{n+1}(C_*;\mathbb{Z}),G) & \to & 0. \\ & & [\varphi]\otimes g & \mapsto & [\mu(\varphi\otimes g)] & & & & \end{array}$$

Furthermore the short exact sequence splits.[2146]

The key to proving the Algebraic H*–Change-of-Coefficients Theorem 202.7 is the following elementary but subtle lemma.

Auxiliary Lemma 202.8. (Dual-of-Free Abelian Group Lemma) Let G be an abelian group. If H is a free abelian group of *finite rank*, then the map

$$\begin{array}{rcl} \mu\colon \mathrm{Hom}(H,\mathbb{Z})\otimes G & \to & \mathrm{Hom}(H,G) \\ \displaystyle\sum_{i=1}^m \varphi_i\otimes g_i & \mapsto & \begin{pmatrix} H & \to & G \\ h & \mapsto & \displaystyle\sum_{i=1}^m \varphi_i(h)\cdot g_i \end{pmatrix} \end{array}$$

is an isomorphism.

Proof of Lemma 202.8. Let $v_1,\dots,v_n$ be a basis for the finitely generated free abelian group H. We denote by $\varphi_1,\dots,\varphi_n$ the dual basis of $\mathrm{Hom}(H,\mathbb{Z})$. In other words, these are the homomorphisms that are uniquely determined by $\varphi_i(v_j)=\delta_{ij}$. We leave it to the reader to verify that the map

$$\begin{array}{rcl} \nu\colon \mathrm{Hom}(H,G) & \to & \mathrm{Hom}(H,\mathbb{Z})\otimes G \\ \psi & \mapsto & \displaystyle\sum_{i=1}^n \varphi_i\otimes\psi(v_i) \end{array}$$

is an inverse to the homomorphism μ. ■

The following lemma shows that the two hypotheses of the Algebraic H*–Change-of-Coefficients Theorem 202.7 are actually equivalent:

Lemma 202.9. (Finitely Generated (Co-) Homology Lemma) Let (C_*,∂_*) be a chain complex of free abelian groups. The following two statements are equivalent:

(a) All cohomology groups $\mathrm{H}^n(C_*;\mathbb{Z})$ are finitely generated.
(b) All homology groups $\mathrm{H}_n(C_*)$ are finitely generated.

Proof of Lemma 202.9. Let (C_*,∂_*) be a chain complex of free abelian groups. By the Algebraic Universal Coefficient-H*-Theorem 201.22 we know that for each $n\in\mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}^n(C_*;\mathbb{Z}) \ \cong\ \mathrm{Ext}(\mathrm{H}_{n-1}(C_*),\mathbb{Z})\oplus\mathrm{Hom}(\mathrm{H}_n(C_*),\mathbb{Z}).$$

If the homology groups $\mathrm{H}_*(C_*)$ are all finitely generated, then it follows easily from the Abelian Group-Classification Theorem 97.24, the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 and the Ext-Properties Lemma 201.19 (1) and (4) that the cohomology groups are also finitely generated.

[2146] As always, the splitting is not natural. In fact in this theorem we do *not* even claim that the map ν is natural. The reason is that the construction of the short exact sequence involves a choice of the chain complex D_* and we do not intend to discuss the question how that choice affects the map on the right-hand side.

Conversely, if one of the homology groups $\mathrm{H}_k(C_*)$ is not finitely generated, then it follows from the Uncountable-Hom-Ext-Proposition 201.30 that at least one of the groups $\mathrm{Ext}(\mathrm{H}_k(C_*),\mathbb{Z})$ or $\mathrm{Hom}(\mathrm{H}_k(C_*),\mathbb{Z})$ is uncountable. Thus one of the cohomology groups is also uncountable, in particular not finitely generated. ■

Proof of Theorem 202.7. Let (C_*,∂_*) be a chain complex of free abelian groups. By the Finitely Generated (Co-) Homology Lemma 202.9 it suffices to deal with the case that all homology groups $\mathrm{H}_n(C_*)$ are finitely generated. Thus we obtain from the Finite-Rank Chain Complex Lemma 158.22 that there exists a chain homotopy equivalence $h\colon C_* \to D_*$ from (C_*,∂_*) to a chain complex (D_*,∂_*) such that each chain group D_n is a finitely generated free abelian group.

For any abelian group G we consider the following diagram:

$$\begin{array}{ccccccccc}
 & & \overbrace{\phantom{\mathrm{H}_n(\mathrm{Hom}(D_*,\mathbb{Z}))}}^{=\mathrm{H}^n(D;\mathbb{Z})\text{ by definition}} & & & & \overbrace{\phantom{\mathrm{H}_{n+1}(\mathrm{Hom}(D_*,\mathbb{Z}))}}^{=\mathrm{H}^{n+1}(D_*;\mathbb{Z})\text{ by definition}} & & \\
0 & \to & \mathrm{H}_n(\mathrm{Hom}(D_*,\mathbb{Z}))\otimes G & \xrightarrow{[\varphi]\otimes g\mapsto[\varphi\otimes g]} & \mathrm{H}_n(\mathrm{Hom}(D_*,\mathbb{Z})\otimes G) & \to & \mathrm{Tor}(\mathrm{H}_{n+1}(\mathrm{Hom}(D_*,\mathbb{Z})),G) & \to & 0. \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cong}\ {\scriptstyle \mu} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
 & & \mathrm{H}^n(D_*;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}_n(\mathrm{Hom}(D_*,G)) & & \mathrm{Tor}(\mathrm{H}^{n+1}(D_*;\mathbb{Z}),G) & & \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
 & & \mathrm{H}^n(D_*;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(D_*;G) & & \mathrm{Tor}(\mathrm{H}^{n+1}(D_*;\mathbb{Z}),G) & & \\
 & & \downarrow{\scriptstyle \cong}\ {\scriptstyle h^*} & & \downarrow{\scriptstyle \cong}\ {\scriptstyle h^*} & & \downarrow{\scriptstyle \cong}\ {\scriptstyle h^*} & & \\
 & & \mathrm{H}^n(C_*;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(C_*;G) & & \mathrm{Tor}(\mathrm{H}^{n+1}(C_*;\mathbb{Z}),G). & &
\end{array}$$

We make the following clarifications:

(1) The horizontal sequence on the top is the short exact sequence obtained from the Algebraic Universal Coefficient Theorem 162.15 which we applied to the chain complex $E_* = \mathrm{Hom}(D_{-*},\mathbb{Z})$. We need to invert the sign of the degrees twice, thus we end up with the degree $n+1 = -((-n)-1)$ on the right-hand side. We know by the Algebraic UCT-Splitting Proposition 162.17 that the short exact sequence splits.

(2) It follows from the Dual-of-Free Abelian Group Lemma 202.8 that the middle vertical map between the first two rows is an isomorphism. Here we use that each chain group of D_* is a free abelian group of finite rank.

(3) One can easily verify that the diagram commutes.

(4) It follows from the Dual Cochain Map Lemma 197.20 (3b) together with the Cochain Homotopy-H^*-Lemma 197.14 that the bottom vertical maps are isomorphisms.

It is now clear that we obtain the desired short exact sequence. ■

The following theorem gives a practical "topological" version of the Algebraic H^*–Change-of-Coefficients Theorem 202.7.

Theorem 202.10. (H^*–Change-of-Coefficients Theorem) Let X be a topological space, let $A \subseteq X$ be a subspace and let G be an abelian group. If all homology groups $\mathrm{H}_n(X,A)$ of X are finitely generated, then for each $n \in \mathbb{N}_0$ there exists a short exact

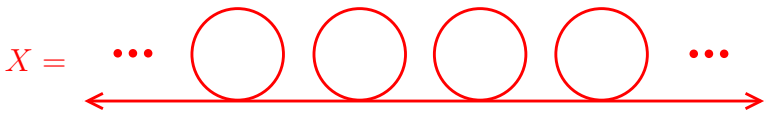
sequence of the form

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^n(X,A;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(X,A;G) & \to & \mathrm{Tor}(\mathrm{H}^{n+1}(X,A;\mathbb{Z}),G) & \to & 0. \\ & & [\varphi]\otimes g & \mapsto & [\mu(\varphi\otimes g)] & & & & \end{array}$$

Furthermore the short exact sequence splits.

Proof. This theorem follows immediately from applying the Algebraic H*–Change-of-Coefficients Theorem 202.7 to the singular chain complex of X. ■

Remark. In general the hypothesis on X that we put into the H*–Change-of-Coefficients Theorem 202.10 cannot be dropped. For example consider the infinite string of circles X as shown in the figure below and we take $G = \mathbb{Q}$.



the groups $\mathrm{H}^1(X;\mathbb{Z})\otimes\mathbb{Q}$ and $\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic

By the Tor-Properties Lemma 162.13 (3) we know that all Tor-groups $\mathrm{Tor}(H,\mathbb{Q})$ vanish. In Exercise 202.3 we will see that the groups $\mathrm{H}^1(X;\mathbb{Z})\otimes\mathbb{Q}$ and $\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic. This shows that in this setting the sequence stated in the H*–Change-of-Coefficients Theorem 202.10 is not exact. □

The following lemma gives a convenient interpretation of the map μ appearing in the H*–Change-of-Coefficients Theorem 202.10.

Lemma 202.11. (H*–Coefficient Change Lemma) Let X be a topological space, let $A \subseteq X$ be a subspace, let G be an abelian group and let $f\colon \mathbb{Z} \to G$ be a homomorphism. As usual we denote by $f_*\colon \mathrm{H}^n(X,A;\mathbb{Z}) \to \mathrm{H}^n(X,A;G)$ the induced homomorphism from the Cohomology-Coefficients-Functor Lemma 197.21. With this notation the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(X,A;\mathbb{Z}) & & \\ {\scriptstyle a\mapsto a\otimes f(1)}\downarrow & \searrow^{f_*} & \\ \mathrm{H}^n(X,A;\mathbb{Z})\otimes G & \xrightarrow[\mu]{} & \mathrm{H}^n(X,A;G). \end{array}$$

Proof. The statement follows immediately from explicitly writing down the homomorphisms. ■

Example. Let $n \in \mathbb{N}\cup\{\infty\}$. We denote by $f\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ the obvious group homomorphism. It follows from the H*–Change-of-Coefficients Theorem 202.10, from the H*–Coefficient Change Lemma 202.11, from the calculations of the cohomology groups $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ on page 3832, and from the properties of Tor-groups, see the Tor-Properties Lemma 162.13, that for any $k \in \mathbb{N}_0$ the map $f_*\colon \mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}/2\mathbb{Z})$ is an epimorphism. □

We conclude this chapter with the following lemma which unwittingly gets used frequently.

[2146] In many examples we consider a pair $(X,A) = (X,\varnothing)$ where X is a compact topological manifold or X is a CW-complex that has only finitely many cells in each dimension. It follows from the Topological Manifolds-Invariants Proposition 189.14 (2) and the CW-Complex-H_*-Proposition 155.10 (1) that such topological spaces qualify.

Lemma 202.12. (Integral-Rational Cohomology Lemma) We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ the obvious inclusion. Let $n \in \mathbb{N}_0$.

(1) Let X be a topological space such that all homology groups are finitely generated. We suppose that $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated free abelian group. If $v_1, \ldots, v_m$ is a basis of $\mathrm{H}^n(X;\mathbb{Z})$, then $\varphi_*(v_1), \ldots, \varphi_*(v_m)$ is a basis of $\mathrm{H}^n(X;\mathbb{Q})$.

(2) Let $f\colon X \to Y$ be a continuous map between topological spaces. We suppose that all homology groups of X and Y are finitely generated and we suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}^n(Y;\mathbb{Z})$ are free abelian groups. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^n(Y;\mathbb{Z}) & \xrightarrow{f^*} & \mathrm{H}^n(X;\mathbb{Z}) \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \mathrm{H}^n(Y;\mathbb{Q}) & \xrightarrow{f^*} & \mathrm{H}^n(X;\mathbb{Q}). \end{array}$$

Suppose that $v_1, \ldots, v_r$ is a basis for $\mathrm{H}^n(X;\mathbb{Z})$ and that $w_1, \ldots, w_s$ is a basis for $\mathrm{H}^n(Y;\mathbb{Z})$. The matrix that represents $f^*\colon \mathrm{H}^n(Y;\mathbb{Z}) \to \mathrm{H}^n(X;\mathbb{Z})$ with respect to the given bases also represents $f^*\colon \mathrm{H}^n(Y;\mathbb{Q}) \to \mathrm{H}^n(X;\mathbb{Q})$ with respect to the bases $\varphi_*(v_1), \ldots, \varphi_*(v_r)$ and $\varphi_*(w_1), \ldots, \varphi_*(w_s)$.

Proof.

(1) This statement follows easily from the H^*–Change-of-Coefficients Theorem 202.10 together with the Tor-Properties Lemma 162.13 (3) and the H^*–Coefficient Change Lemma 202.11.

(2) The fact that the diagram commutes is an immediate consequence of the Cohomology-Coefficients-Functor Lemma 197.21. The remaining statement is just a psychologically helpful reinterpretation of the fact that the diagram commutes. ■

202.4. Exercises for Chapter 202 .

Exercise 202.1. Let (X, A) be a pair of topological space and let $k \in \mathbb{N}_0$.

(a) We assume that $\mathrm{H}_k(X, A)$ and $\mathrm{H}_{k-1}(X, A)$ are finitely generated. Show that for any field $\mathbb{F}$ the homology and cohomology groups $\mathrm{H}_k(X, A; \mathbb{F})$ and $\mathrm{H}^k(X, A; \mathbb{F})$ are finite-dimensional $\mathbb{F}$-vector spaces with

$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, A; \mathbb{F})) \;=\; \dim_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F})).$$

Remark. This statement is essentially the same as in Exercise 201.24. But with the tools of this chapter we can prove this equality of dimensions with less pain.

(b) Is it always true that $\mathrm{H}_k(X, A; \mathbb{F})$ and $\mathrm{H}^k(X, A; \mathbb{F})$ are isomorphic?

Exercise 202.2. Let H be a free abelian group and let G be an abelian group. As in the Dual-of-Free Abelian Group Lemma 202.8 we consider the map

$$\begin{array}{ccc} \mathrm{Hom}(H, \mathbb{Z}) \otimes G & \to & \mathrm{Hom}(H, G) \\ \sum_{i=1}^{m} \varphi_i \otimes g_i & \mapsto & \left(\begin{array}{ccc} H & \to & G \\ h & \mapsto & \sum_{i=1}^{m} \varphi_i(h) \cdot g_i \end{array} \right). \end{array}$$

Show that this map is in general not an isomorphism.

Exercise 202.3. We consider the topological space X shown in the figure on page 3884. Show that the map

$$\mu\colon \mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q} \ \to \ \mathrm{H}^1(X;\mathbb{Q})$$

from the H^*–Change-of-Coefficients Lemma 202.6 is not an isomorphism.

Exercise 202.4. Show that there exists a topological space X such that the two groups $\mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic.
Hint. You could make use of the Homology Realization Proposition 173.1 to find unusual topological spaces.

Exercise 202.5. Let X be a topological space, let $\mathbb{F}$ be a field and let $i \in \mathbb{N}_0$. Show that $\mathrm{H}^i(X;\mathbb{F})$ is either finite-dimensional or the dimension is uncountable.

CHAPTER 203

The cup product

The previous chapters on cohomology produced mostly disappointing results. For example we showed in the Universal Coefficient-H*-Theorem 201.23 that the cohomology groups are completely determined by the usual humble singular homology groups. In this chapter we will finally get to see something new: We will define the cup product on cohomology, which has no real analog for homology.

This chapter is organized as follows:

- In Section 203.1 we will first introduce the *homological* cross product (which is not so interesting). Afterwards we will introduce the *cohomological* cross product, which is much more interesting, since it leads to a first definition of the cup product.
- In Section 203.2 we will give an alternative definition of the cup product.
- In Section 203.3 we will show that the two definitions of the cup product agree.
- In Section 203.4 we will prove the graded commutativity of the cup product.
- In Section 203.5 we will discuss the ring structure on cohomology.
- In Section 203.6 we will perform some calculations of the cup product.
- In Section 203.7 we will consider applications of the cup product.
- In Appendix 203.8 we will express the cohomological cross product in terms of the cup product.
- In Section 203.9 we will provide exercises for Chapter 203.

In the next Chapter 204 we will introduce the cup product on certain relative cohomology groups.

203.1. A first definition of the cup product.

We start out by recalling some definitions from Chapters 163 and 164. First of all let us recall Definition 163.2:

Definition 203.1. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{D} = (D_*, \partial_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we set

$$(\mathcal{C} \otimes \mathcal{D})_n \coloneqq \bigoplus_{p+q=n} C_p \otimes D_q.$$

The homomorphisms

$$\begin{aligned} \partial_n^{\otimes} \colon (\mathcal{C} \otimes \mathcal{D})_n &\to (\mathcal{C} \otimes \mathcal{D})_{n-1} \\ c_p \otimes d_q &\mapsto \underbrace{\partial_p(c_p) \otimes d_q + (-1)^p \cdot c_p \otimes \partial_q(d_q)}_{=(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial_q)(c_p \otimes d_q)} \end{aligned}$$

which are given by

turn the groups $(\mathcal{C} \otimes \mathcal{D})_n$ into a chain complex, called the tensor product $\mathcal{C} \otimes \mathcal{D}$.

Visualization. During the following discussions it might be convenient to keep the following visualization of $\mathcal{C} \otimes \mathcal{D}$ from page 3129 in mind:

$$\begin{array}{ccccccc}
C_0 \otimes D_3 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_3 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_3 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_3 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial_3} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_3} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial_3} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_3} \\
C_0 \otimes D_2 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_2 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_2 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_2 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_2} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_2} \\
C_0 \otimes D_1 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_1 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_1 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_1 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_1} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_1} \\
C_0 \otimes D_0 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_0 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_0 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_0.
\end{array}$$

Note that here the "antidiagonal" groups of the same color "add up" to the chain groups of $\mathcal{C} \otimes \mathcal{D}$. □

Next we recall the following purely algebraic lemma:

Lemma 163.5. (Cycle Tensor Lemma) Let $\mathcal{C}$ and $\mathcal{D}$ be chain complexes. The map

$$\begin{array}{rcl}
\mu \colon \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{D}) & \to & \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{D}) \\
\sum_{i=1}^{r} [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^{r} [c_i \otimes d_i]
\end{array}$$

is a well-defined natural homomorphism.

We turn to recalling topological concepts:

Notation 203.2. Given topological spaces X and Y we denote by

$$\begin{array}{rcl}
\Upsilon := \Upsilon_{X,Y} \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \to & \mathrm{C}_*(X \times Y) \\
\Theta := \Theta_{X,Y} \colon \mathrm{C}_*(X \times Y) & \to & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)
\end{array}$$

the natural chain homotopy equivalences from the Eilenberg-Zilber Theorem 164.2.[2147]

As in Definition 164.4 the above concepts allow us to define the cross product on homology:

Definition 203.3. Let X and Y be topological spaces. We refer to the homomorphism[2148]

$$\times \colon \overbrace{\mathrm{H}_p(\mathrm{C}_*(X)) \otimes \mathrm{H}_q(\mathrm{C}_*(X))}^{=\mathrm{H}_p(X) \otimes \mathrm{H}_q(Y)} \xrightarrow{\mu} \mathrm{H}_{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \xrightarrow{\Upsilon_*} \overbrace{\mathrm{H}_{p+q}(\mathrm{C}_*(X \times Y))}^{=\mathrm{H}_{p+q}(X \times Y)}$$

as the homological cross product (or just *cross product*).

The cross product applied to the same topological space $X = Y$ gives us in particular the cross product

$$\times \colon \mathrm{H}_p(X) \otimes \mathrm{H}_q(X) \to \mathrm{H}_{p+q}(X \times X).$$

This is very charming, but this construction does not give us a product on the homology groups of X itself, since we end up in the homology of a different topological space.

Now we try our luck with cohomology groups. Our first goal is to formulate a cohomological version of the Cycle Tensor Lemma 163.5. This requires a little bit of preparation:

[2147] In the Eilenberg-Zilber Theorem 164.2 we saw that these maps are unique up to a natural chain homotopy equivalence. So for the most part the precise choice of these maps is irrelevant.

[2148] As we just mentioned in Footnote 2147, it follows from the Eilenberg-Zilber Theorem 164.2 (2) together with the Chain Homotopy–H_*-Lemma 144.5 that this map is independent of the choice of Υ.

Definition 203.4. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes.

(1) Given $p, q \in \mathbb{N}_0$ we denote by

$$\Pi_{p,q} \colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} = \bigoplus_{i+j=p+q} \mathcal{C}_i \otimes \mathcal{D}_j \ \to \ \mathcal{C}_p \otimes \mathcal{D}_q$$

the natural projection.

(2) For any commutative ring R we consider the multiplication map[2149]

$$\begin{array}{rcl} \mu_R \colon R \otimes R & \to & R \\ \sum_{i=1}^{m} r_i \otimes s_i & \mapsto & \sum_{i=1}^{m} r_i \cdot s_i. \end{array}$$

The following lemma plays the role of the Cycle Tensor Lemma 163.5:

Lemma 203.5. (Cocycle Tensor Lemma) Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. For any commutative ring R the maps

$$\begin{array}{rcl} \operatorname{Hom}(\mathcal{C}_p, R) \times \operatorname{Hom}(\mathcal{D}_q, R) & \to & \operatorname{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\ (\varphi, \psi) & \mapsto & ((\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\Pi_{p,q}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi \otimes \psi} R \otimes R \xrightarrow{\mu_R} R) \end{array}$$

descend to a well-defined homomorphism

$$\begin{array}{rcl} \Xi \colon \mathrm{H}^p(\mathcal{C}_*; R) \otimes \mathrm{H}^q(\mathcal{D}_*; R) & \to & \mathrm{H}^{p+q}(\mathcal{C}_* \otimes \mathcal{D}_*; R) \\ [\varphi] \otimes [\psi] & \mapsto & [\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}]. \end{array}$$

Proof. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. We start out with the following claim:

Claim 1. If $\varphi \colon C_p \to R$ and $\psi \colon D_q \to R$ are cocycles, then

$$\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q} \colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \ \to \ R$$

is a cocycle.

Proof. We need to show that $\delta_{p+q}(\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}) = 0 \in \operatorname{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q+1}, R)$. In other words, we need to show that $(\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}) \circ \partial^{\otimes}_{p+q+1} \colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q+1} \to R$ is the zero map. This is indeed the case:

$$\begin{aligned} (\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}) \circ \partial^{\otimes}_{p+q+1} &\overset{\text{Definition 203.1 of } \partial^{\otimes}}{=} (\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}) \Big(\bigoplus_{r+s=p+q+1} (\partial_r \otimes \mathrm{id}_{D_s} + (-1)^r \cdot \mathrm{id}_{C_r} \otimes \partial_s) \Big) \\ &= (\mu_R \circ (\varphi \otimes \psi)) (\partial_{p+1} \otimes \mathrm{id}_{D_q} + (-1)^p \cdot \mathrm{id}_{C_p} \otimes \partial_{q+1}) \\ &= (\mu_R \circ (\underbrace{(\varphi \circ \partial_{p+1})}_{\substack{=0, \text{ since } \varphi \\ \text{is a cocycle}}} \otimes \psi)) + (-1)^p \cdot (\mu_R \circ (\varphi \otimes \underbrace{(\psi \circ \partial_{q+1})}_{\substack{=0, \text{ since } \psi \\ \text{is a cocycle}}})) \ = \ 0. \end{aligned}$$

⊞

Claim 2. The map

$$\begin{array}{rcl} \mathrm{H}^p(\mathcal{C}_*; R) \otimes \mathrm{H}^q(\mathcal{D}_*; R) & \to & \mathrm{H}^{p+q}(\mathcal{C}_* \otimes \mathcal{D}_*; R) \\ [\varphi] \otimes [\psi] & \mapsto & [\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}] \end{array}$$

is well-defined, i.e. it does not depend on the choice of representatives of the cohomology classes.

[2149] Using the Tensor Product–Universal Property Lemma 159.4 (2) one can easily verify that μ_R is well-defined.

Proof. Let $\varphi\colon C_p \to R$ and $\psi\colon D_q \to R$ be cocycles and let $\alpha\colon C_{p-1} \to R$ be a cochain. Note that we have the following equalities in $\mathrm{H}^{p+q}(\mathcal{C}_* \otimes \mathcal{D}_*; R)$:

$$\begin{aligned}
[(\mu_R \circ ((\varphi + \delta_{p-1}(\alpha)) \otimes \psi) \circ \Pi_{p,q})] - [(\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}] &= [\mu_R \circ (\delta_{p-1}(\alpha) \otimes \psi) \circ \Pi_{p,q}] \\
&= [\mu_R \circ (\underbrace{(\alpha \circ \partial_p)}_{=\delta_{p-1}(\alpha)} \otimes \psi) \circ \Pi_{p,q}] + (-1)^p \cdot [\mu_R \circ (\alpha \otimes \underbrace{(\psi \circ \partial_{q+1})}_{\substack{=0,\text{ since } \psi \\ \text{is a cocycle}}}) \circ \Pi_{p-1,q+1}] \\
&= [(\mu_R \circ (\alpha \otimes \psi) \circ \Pi_{p-1,q}) \circ (\partial_p \otimes \mathrm{id}_{D_q} + (-1)^p \cdot \mathrm{id}_{C_{p-1}} \otimes \partial_{q+1})] \\
&= [(\mu_R \circ (\alpha \otimes \psi) \circ \Pi_{p-1,q}) \circ \partial^{\otimes}_{p+q}] = 0.
\end{aligned}$$

$=$ Definition 163.2 of $\partial^{\otimes}_{p+q}$; $=$ since cocycles represent the zero class in cohomology

The same argument also shows that the definition does not depend on the choice of the representative of the element in $\mathrm{H}^q(\mathcal{D}_*; R)$. ⊞

It follows easily from the definitions that the map in Claim 2 is bilinear. Finally note that it follows from the Tensor Product–Universal Property Lemma 159.4 (2) that we get the promised homomorphism Ξ. ■

We return to topology. The following definition is the cohomological analog of the above cross product:

Definition 203.6. Let X and Y be topological spaces and let R be a commutative ring. We refer to the homomorphism[2150] [2151]

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}^p(X;R) \otimes \mathrm{H}^q(\mathrm{C}_*(Y); R)}^{=\mathrm{H}^p(X;R) \otimes \mathrm{H}^q(Y;R)} & \xrightarrow{\Xi} & \mathrm{H}^{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y); R) & \xrightarrow{\Theta^*_{X,Y}} & \overbrace{\mathrm{H}^{p+q}(\mathrm{C}_*(X \times Y); R)}^{=\mathrm{H}^{p+q}(X \times Y; R)}. \\
[\varphi] \otimes [\psi] & \mapsto & [\mu_R \circ (\varphi \otimes \psi) \circ \Pi_{p,q}]. & &
\end{array}$$

as the cohomological cross product (or just cross product for short).

The cross product on cohomology gives us in particular a homomorphism

$$\times\colon \mathrm{H}^p(X; R) \otimes \mathrm{H}^q(X; R) \to \mathrm{H}^{p+q}(X \times X; R).$$

Again, this is nice, but exactly as in the case of the cross product on homology, we end up in the cohomology of $X \times X$, but we really would like to end up in the cohomology of X. Now we have to find a way how to go back from the topological space $X \times X$ to the original space X. In general there does not exist an "interesting"[2152] map $X \times X \to X$. But, as we saw before, we still have one ace up in our sleeves: There does exist an interesting map in the opposite direction, namely the diagonal map:

[2150] The first map is provided by the Cocycle Tensor Lemma 203.5.

[2151] It follows from the Eilenberg-Zilber Theorem 164.2 (2), the Dual Cochain Map Lemma 197.20 (3) and the Cochain Homotopy-H*-Lemma 197.14 that the definition below does not depend on the choice of Θ.

[2152] The projection from $\pi\colon X \times X \to X$ onto say the first factor does not count as "interesting", since it does not "mix" the two factors. In fact in Exercise 164.16 we showed that the map

$$\mathrm{H}_p(X) \otimes \mathrm{H}_q(X) \xrightarrow{\times} \mathrm{H}_{p+q}(X \times X) \xrightarrow{\pi_*} \mathrm{H}_*(X)$$

is the zero map unless $q = 0$.

Definition 203.7. Given a topological space X we refer to[2153]

$$\begin{array}{rcl} d_X\colon X & \to & X\times X \\ x & \mapsto & (x,x) \end{array}$$

as the diagonal map d_X.

Note that the diagonal map induces a map

$$d_{X*}\colon \mathrm{H}_*(X;R) \ \to \ \mathrm{H}_*(X\times X;R)$$

on homology, and, much more interesting in our context, it also induces a map

$$d_X^*\colon \mathrm{H}^*(X\times X;R) \ \to \ \mathrm{H}^*(X;R).$$

on cohomology groups that goes in the "opposite direction". This observation shows that we can "marry" the cohomological cross product with d_X^*, and as an offspring we obtain the following wonderful definition:

Definition 203.8. Let X be a topological space and let R be a commutative ring. We define the cup product[2154] on the cohomology groups of X as the composition of the following maps:

$$\mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) \ \to \ \mathrm{H}^p(X;R)\otimes\mathrm{H}^q(X;R) \ \xrightarrow{\Theta^*_{X,X}\circ\Xi} \ \mathrm{H}^{p+q}(X\times X;R) \ \xrightarrow{d_X^*} \ \mathrm{H}^{p+q}(X;R).$$

For the resulting pairing we use the notation

$$\begin{array}{rcl} \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) & \to & \mathrm{H}^{p+q}(X;R) \\ (a,b) & \mapsto & a\mathbin{\widetilde{\cup}} b \end{array}$$

and we refer to[2155] $a\mathbin{\widetilde{\cup}} b$ as the cup product of a and b.

The following lemma shows that the cup product is natural:

Lemma 203.9. (Cup Product-Naturality Lemma) Let R be a commutative ring and let $p,q\in\mathbb{N}_0$. For any continuous map $f\colon X\to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(Y;R)\times\mathrm{H}^q(Y;R) & \xrightarrow{\widetilde{\cup}} & \mathrm{H}^{p+q}(Y;R) \\ \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) & \xrightarrow{\widetilde{\cup}} & \mathrm{H}^{p+q}(X;R). \end{array}$$

In other words, for any $c\in\mathrm{H}^p(Y;R)$ and any $d\in\mathrm{H}^q(Y;R)$ we have

$$f^*(c)\mathbin{\widetilde{\cup}} f^*(d) \ = \ f^*(c\mathbin{\widetilde{\cup}} d).$$

[2153]It follows from the Topological-Product Proposition 7.2 (1a) and (1b) that the diagonal map d_X is continuous.

[2154]The name "cup product" comes from the notation "$a\mathbin{\widetilde{\cup}} b$" for the composition since $\cup$ looks like a cup.

[2155]The weird looking "$\sim$" will be removed shortly, once we show that this definition is equivalent to an alternative definition of the cup product.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^p(Y;R) \times \mathrm{H}^q(Y;R) & \longrightarrow & \mathrm{H}^p(Y;R) \otimes \mathrm{H}^q(Y;R) & \xrightarrow{\Theta^*_{Y,Y} \circ \Xi} & \mathrm{H}^{p+q}(Y \times Y;R) & \xrightarrow{d_Y^*} & \mathrm{H}^{p+q}(Y;R) \\
\downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^* \otimes f^*} & & \downarrow{\scriptstyle (f,f)^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \longrightarrow & \mathrm{H}^p(X;R) \otimes \mathrm{H}^q(X;R) & \xrightarrow{\Theta^*_{X,X} \circ \Xi} & \mathrm{H}^{p+q}(X \times X;R) & \xrightarrow{d_X^*} & \mathrm{H}^{p+q}(X;R).
\end{array}$$

It follows easily from the Tensor–Map Composition Lemma 159.7 and the naturality of $\Theta_{X,X}$, Ξ and d_X that the diagram commutes. ∎

The above definition of the cup product is very natural (no pun intended). But the problem with the above definition is that it is not explicit at all. In the next section we will give an alternative and utterly explicit description of the cup product. This definition is much more suitable for calculations of the cup product.

Later, when we study properties of the cup product and when we develop the theory we will make use of the fact that we have two alternative approaches, and we will choose whatever definition is more practical for the given task.

203.2. **An alternative definition of the cup product.** In this section we give the promised alternative definition of the cup product which is more ad hoc, but which works surprisingly well in practice. We will see in the Cup Product–Definitions Proposition 203.15 that both approaches lead to the same concept.

First we recall some notation from Definitions 148.17 and 143.8:

Notation 203.10.

(1) For $i = 0, \dots, n$ we write $\mathfrak{v}_i := (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.
(2) For $a_0, \dots, a_k \in \Delta^n$ we denote by $[a_0, \dots, a_k]\colon \Delta^k \to \Delta^n$ the unique affine linear map that sends $\mathfrak{v}_i$ to a_i.

With this notation we can now give our second definition of the cup product:

Definition 203.11. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We define the cup product of φ and ψ to be the $(p+q)$-cochain

$$\begin{aligned}
\varphi \cup \psi\colon \mathrm{C}_{p+q}(X) &\to R \\
(\sigma\colon \Delta^{p+q} \to X) &\mapsto \underbrace{\varphi(\underbrace{\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p]}_{\text{map } \Delta^p \to X})}_{\in R} \cdot \underbrace{\psi(\underbrace{\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]}_{\text{map } \Delta^q \to X})}_{\in R}.
\end{aligned}$$

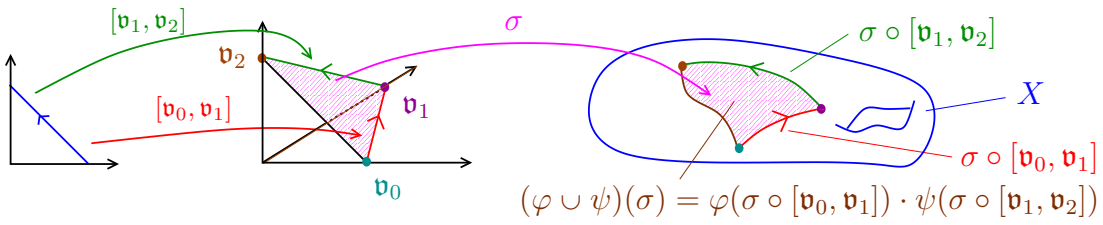
that is given by[2156]

The cup product thus defines a map

$$\cup\colon \mathrm{C}^p(X;R) \times \mathrm{C}^q(X;R) \to \mathrm{C}^{p+q}(X;R).$$

Remark. The sign conventions for the cup product for cohomology classes $\varphi \in \mathrm{H}^p(X;R)$ and $\psi \in \mathrm{H}^q(X;R)$ differ in the literature. More precisely, given a topological space X and

[2156] In the literature, given a topological space X and a singular n-simplex $\sigma\colon \Delta^n \to X$ one often uses the shorthand notation ${}_p\lfloor\sigma = \sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p]$ and $\sigma\rfloor_p = \sigma \circ [\mathfrak{v}_{n-p}, \dots, \mathfrak{v}_n]$.



cochains $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ the cup product $\varphi \cup \psi$, applied to a singular $(p+q)$-simplex $\sigma\colon \Delta^{p+q} \to X$ is defined as follows:

		$(\varphi \cup \sigma)(\sigma)$
(a)	Hatcher [**Hat2002**, p. 206]	$\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]) \cdot \psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])$
(b)	Bredon [**Bre1993**, p. 328]	$(-1)^{pq} \cdot \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]) \cdot \psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])$
(c)	Dold [**Dol1956**, p. 222]	$(-1)^{pq} \cdot \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]) \cdot \psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])$
(d)	Spanier [**Spa1995**, p. 251]	$\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]) \cdot \psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])$
(e)	Munkres [**Mun1984**, p. 288]	$\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]) \cdot \psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])$
(f)	Greenberg-Harper [**GrH1981**, p. 195]	$\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]) \cdot \psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])$.

So we see that our sign convention agrees with the conventions of Hatcher, Spanier, Munkres and Greenberg-Harper, but they differ by the sign $(-1)^{pq}$ from the conventions used in Bredon and Dold. □

The definition of the cup product on cochains might look strange and arbitrary. But the following lemma shows that it behaves well with respect to coboundary maps.

Lemma 203.12. (Cup Product–Coboundary Lemma) Let X be a topological space and let R be a commutative ring. Given any $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ we have

$$\delta_{p+q}(\varphi \cup \psi) \;=\; \delta_p(\varphi) \cup \psi \;+ (-1)^p \cdot \varphi \cup \delta_q(\psi) \quad \in \; \mathrm{C}^{p+q+1}(X;R).$$

Remark. Note that a similar looking statement holds for differential forms. More precisely, given a smooth manifold M and two differential forms $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ it follows from the Product Rule of the Differentials-on-Manifolds Proposition 75.7 (3) that

$$d(\omega \wedge \eta) \;=\; d\omega \wedge \eta \;+\; (-1)^p \cdot \omega \wedge d\eta \;\in\; \Omega^{p+q+1}(M).$$

The (co-) boundary formula of the Cup Product–Coboundary Lemma 203.12 is of course also very reminiscent of the definition of the boundary map in the tensor product of two chain complexes which we gave in Definition 163.2. □

Proof. The lemma follows from an elementary, albeit slightly confusing, calculation. Indeed, let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We have to prove the following equality:

$$\delta_{p+q}(\varphi \cup \psi) \;=\; \delta_p(\varphi) \cup \psi \;+ (-1)^p \cdot \varphi \cup \delta_q(\psi) \;\in\; \mathrm{C}^{p+q+1}(X;R) = \mathrm{Hom}(\mathrm{C}_{p+q+1}(X), R).$$

So let $\sigma\colon \Delta^{p+q+1} \to X$ be a singular $(p+q+1)$-simplex. We have the following three equalities in R:[2157]

$$\begin{aligned}
\text{(a)}\quad (\delta_{p+q}(\varphi\cup\psi))(\sigma) &= (\varphi\cup\psi)(\partial_{p+q+1}(\sigma)) \;=\; (\varphi\cup\psi)\Big(\sum_{j=0}^{p+q+1}(-1)^j\cdot\sigma\circ i_j^{p+q+1}\Big)\\
&= (\varphi\cup\psi)\Big(\sum_{j=0}^{p}(-1)^j\cdot\sigma\circ i_j^{p+q+1}\Big) + (\varphi\cup\psi)\Big(\sum_{j=p+1}^{p+q+1}(-1)^j\cdot\sigma\circ i_j^{p+q+1}\Big)\\
&= \sum_{j=0}^{p}(-1)^j\cdot(\varphi\cup\psi)(\sigma\circ i_j^{p+q+1}) + (-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot(\varphi\cup\psi)(\sigma\circ i_{k+p}^{p+q+1})\\
&= \sum_{j=0}^{p}(-1)^j\cdot\varphi(\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}]\circ i_j^{p+1}}_{\text{map }\Delta^p\to X})\cdot\psi(\underbrace{\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}]}_{\text{map }\Delta^q\to X})\\
&\quad + (-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot\varphi(\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]}_{\text{map }\Delta^p\to X})\cdot\psi(\underbrace{\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}]\circ i_k^{q+1}}_{\text{map }\Delta^q\to X}).\\
\text{(b)}\quad (\delta_p(\varphi)\cup\psi)(\sigma) &= (\delta_p(\varphi))(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}])\cdot\psi(\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}])\\
&= \varphi\Big(\sum_{j=0}^{p+1}(-1)^j\cdot\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}]\circ i_j^{p+1}}_{\text{map }\Delta^p\to X}\Big)\cdot\psi(\underbrace{\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}]}_{\text{map }\Delta^q\to X})\\
&= \sum_{j=0}^{p+1}(-1)^j\cdot\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}]\circ i_j^{p+1})\cdot\psi(\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}])
\end{aligned}$$

and

$$\begin{aligned}
\text{(c)}\quad (-1)^p\cdot(\varphi\cup\delta_q(\psi))(\sigma) &= (-1)^p\cdot\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot(\delta_q(\psi))(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}])\\
&= (-1)^p\cdot\varphi(\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]}_{\text{map }\Delta^p\to X})\cdot\psi\Big(\sum_{k=0}^{q+1}(-1)^k\cdot\underbrace{\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}]\circ i_k^{q+1}}_{\text{map }\Delta^q\to X}\Big)\\
&= (-1)^p\cdot\sum_{k=0}^{q+1}(-1)^k\cdot\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}]\circ i_k^{q+1}).
\end{aligned}$$

Now we observe that if we take the sum of (b) and (c), then the $(j=p+1)$-summand in (b) cancels with the $(k=0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ∎

The Cup Product–Coboundary Lemma 203.12 leads us to the following lemma:

Lemma 203.13. (Cup Product–Definition Lemma) Let X be a topological space. For any commutative ring R the map

$$\begin{aligned}
\cup\colon \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) &\to \mathrm{H}^{p+q}(X;R)\\
([\varphi],[\psi]) &\mapsto [\varphi]\cup[\psi] := [\varphi\cup\psi]
\end{aligned}$$

is well-defined.

Definition 203.14. We refer to the pairing of the Cup Product–Definition Lemma 203.13 also as the cup product.

[2157] On several occasions we use that for a singular n-cochain ϕ and a singular $(n+1)$-simplex μ_R we have by definition

$$(\delta_n(\phi))(\mu_R) \;=\; \phi(\partial_{n+1}(\mu_R)) \;=\; \phi\Big(\sum_{l=0}^{n+1}(-1)^l\cdot\mu_R\circ i_l^{n+1}\Big).$$

Proof. [2158] Let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cocycles.

(1) It follows immediately from the Cup Product–Coboundary Lemma 203.12 that $\varphi \cup \psi$ is again a cocycle.
(2) If $\varphi' = \varphi + \delta_{p-1}(\tau)$ is another representative of $[\varphi]$, then

$$\varphi' \cup \psi - \varphi \cup \psi \underset{\substack{\uparrow \\ \text{since "}\cup\text{" is} \\ \text{clearly bilinear}}}{=} (\varphi' - \varphi) \cup \psi = \delta_{p-1}(\tau) \cup \psi \underset{\substack{\uparrow \\ \text{by the Cup Product–Coboundary} \\ \text{Lemma 203.12 and since } \delta_q(\psi) = 0}}{=} \delta_{p+q-1}(\tau \cup \psi).$$

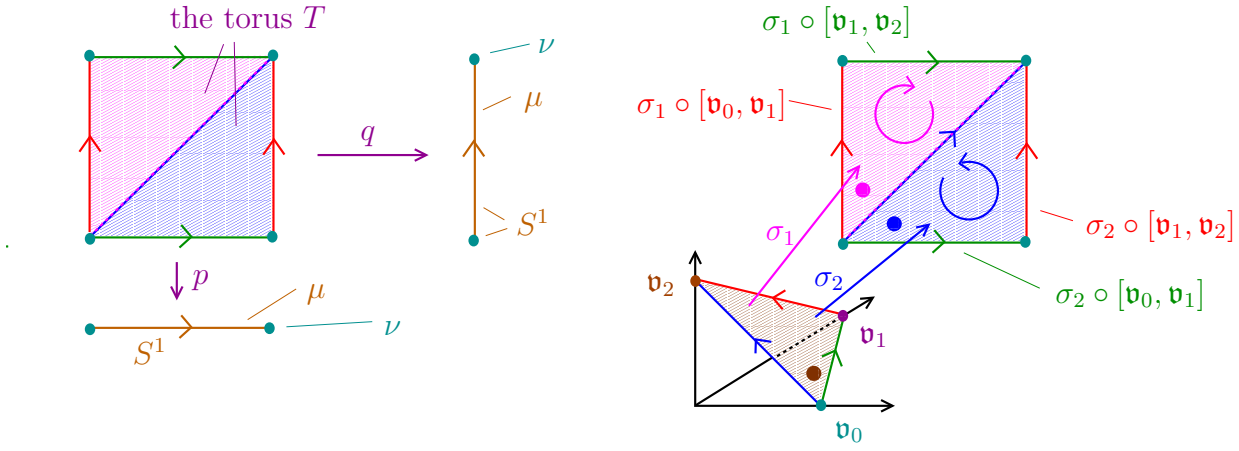
This shows that $\varphi \cup \psi$ and $\varphi' \cup \psi$ represent the same cohomology class, i.e. the cup product does not depend on the choice of the representative of the cohomology class of the first entry.
(3) The proof that the cup product does not depend on the choice of the representative of the second cohomology class is basically the same as the proof of (2). ∎

Example. In the following we consider the torus $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$. We recall the following notation:

(1) We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
(2) On page 3813 we introduced an explicit cochain $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$. In the θ-Cocycle Lemma 198.5 we showed that $\theta_{\mathbb{Z}}$ is actually a cocycle.
(3) We consider the cohomology classes $p^*([\theta_{\mathbb{Z}}]), q^*([\theta_{\mathbb{Z}}]) \in \mathrm{H}^2(T;\mathbb{Z})$.

We want to show that the cup product $p^*([\theta_{\mathbb{Z}}]) \cup q^*([\theta_{\mathbb{Z}}]) \neq 0 \in \mathrm{H}^2(T;\mathbb{Z})$. To do so we need to introduce one more piece of notation:

(4) Let $\sigma_1, \sigma_2 \colon \Delta^2 \to T$ be the singular 2-simplices that are shown in the figure below.



[2158] The proof of the Cup Product–Definition Lemma 203.13 is basically identical to the proof of the Wedge-on-de Rham Cohomology Proposition 78.12.

First we want to determine the values of $p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}) \in \mathrm{C}^2(T;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_2(T),\mathbb{Z})$ on the singular 2-simplices σ_1 and σ_2. We calculate that

$$\begin{aligned}(p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_1) &= p^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_0,\mathfrak{v}_1]) \cdot q^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_1,\mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_1 \circ [\mathfrak{v}_0,\mathfrak{v}_1]}_{\text{constant simplex } \nu}) \cdot \theta_{\mathbb{Z}}(\underbrace{q \circ \sigma_1 \circ [\mathfrak{v}_1,\mathfrak{v}_2]}_{\text{constant simplex } \nu}) \underset{\text{clearly } \theta_{\mathbb{Z}}(\nu)=0}{\uparrow} = 0 \cdot 0 = 0.\end{aligned}$$

Much more interestingly we see that

$$\begin{aligned}(p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_2) &= p^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_0,\mathfrak{v}_1]) \cdot q^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_1,\mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_2 \circ [\mathfrak{v}_0,\mathfrak{v}_1]}_{\substack{\text{the simplex } \mu\colon \Delta^1 \to S^1 \text{ that} \\ \text{is given by } (1\text{–}t,t) \mapsto \exp(2\pi i t)}}) \cdot \theta_{\mathbb{Z}}(\underbrace{q \circ \sigma_2 \circ [\mathfrak{v}_1,\mathfrak{v}_2]}_{=\mu}) \underset{\substack{\text{on page 3813 we saw} \\ \text{that } \theta_{\mathbb{Z}}(\mu)=1}}{\uparrow} = 1 \cdot 1 = 1.\end{aligned}$$

Now we consider the Kronecker pairing $\langle\ ,\ \rangle\colon \mathrm{H}^2(T;\mathbb{Z}) \times \mathrm{H}_2(T) \to \mathbb{Z}$ and we see that

$$\begin{aligned}\langle p^*([\theta_{\mathbb{Z}}]) \cup q^*([\theta_{\mathbb{Z}}]), [T]\rangle &\underset{\substack{\text{on page 3756 we showed that } \sigma_2-\sigma_1 \text{ is} \\ \text{a cycle that represents } [T] \in \mathrm{H}_2(T)}}{\downarrow} = \langle [p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}})], [\sigma_1 - \sigma_2]\rangle \\ &\underset{\substack{\text{Definition 199.8 of the} \\ \text{Kronecker pairing}}}{=} (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_2) - (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_1) \underset{\substack{\text{see the above} \\ \text{calculation}}}{=} 1 - 0.\end{aligned}$$

This calculation shows that the cup product $p^*([\theta_{\mathbb{Z}}]) \cup q^*([\theta_{\mathbb{Z}}]) \in \mathrm{H}^2(T;\mathbb{Z})$ is non-zero, in fact we have shown that it is a generator of $\mathrm{H}^2(T;\mathbb{Z})$. In other words, we have just shown that the explicit 2-cocycle $p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}) \in \mathrm{C}^2(T;\mathbb{Z})$ represents a generator of $\mathrm{H}^2(T;\mathbb{Z})$. We have thus answered Question 199.12. □

203.3. **The two definitions of the cup product.** The following proposition says that the original definition of the cup product and the new definition actually agree.

Proposition 203.15. (Cup Product–Definitions Proposition) Let X be a topological space and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p(X;R)$ and any $\psi \in \mathrm{H}^q(X;R)$ we have

$$\underbrace{\varphi \mathbin{\widetilde{\cup}} \psi}_{\substack{\text{defined in} \\ \text{Definition 203.8}}} = \underbrace{\varphi \cup \psi}_{\substack{\text{defined in} \\ \text{Definition 203.11}}} \in \mathrm{H}^{p+q}(X;R).$$

We now know that our two definitions of the cup product agree. Beginning with the next section we will go back and forth between the two definitions, at each given moment we will pick the definition which is easier to work with.

The proof of the Cup Product–Definitions Proposition 203.15 will require the remainder of this section. In the next Section 203.4 we will then study the question whether or not the cup product is commutative.

The proof of the Cup Product–Definitions Proposition 203.15 rests on the notion of a "diagonal approximation":

Definition 203.16. A diagonal approximation consists of a natural chain map

$$\Phi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds:

$$\Phi(x) \;=\; x \otimes x \quad \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X).$$

Example. It follows easily from the definitions that for any topological space X the maps

$$\Theta_{X,X} \circ d_{X*} \colon \mathrm{C}_*(X) \xrightarrow[\uparrow]{d_{X*}} \mathrm{C}_*(X \times X) \xrightarrow[\uparrow]{\Theta_{X,X}} \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

d_X is the diagonal map; Eilenberg-Zilber map

form a diagonal approximation in the above sense. □

The plan for the remainder of this section is as follows:

(1) In the Alexander-Whitney Lemma 203.17 we will provide an explicit diagonal approximation.
(2) In the Diagonal Approximation–Uniqueness Proposition 203.19 we show that diagonal approximations are essentially unique.
(3) We use (1), (2) and the above example to prove the Cup Product–Definitions Proposition 203.15.

The following lemma gives a completely explicit diagonal approximation which will be very useful for doing actual calculations:

Lemma 203.17. (Alexander-Whitney Lemma) Given any topological space X we consider the map

$$\Delta \colon \mathrm{C}_*(X) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

which is given by

$$(\sigma \colon \Delta^n \to X) \;\mapsto\; \sum_{p=0}^{n} \underbrace{\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]}_{\text{map } \Delta^p \to X} \otimes \underbrace{\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_n]}_{\text{map } \Delta^{n-p} \to X}.$$

These maps form a diagonal approximation.

Definition 203.18. The diagonal approximation

$$\begin{aligned} \Delta \colon \mathrm{C}_*(X) &\to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X) \\ (\sigma \colon \Delta^n \to X) &\mapsto \sum_{p=0}^{n} \sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p] \otimes \sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_n] \end{aligned}$$

from the Alexander-Whitney Lemma 203.17 is called the[2159] [2160] Alexander-Whitney diagonal approximation.

Proof. It follows almost immediately from the definition that the maps Δ are natural. Furthermore it is obvious that on the 0-chains the maps are of the desired form. It remains to prove the following claim:

[2160] James Alexander (1888-1971) was an American mathematician who we had already met on page 3248.
[2160] Hassler Whitney (1907–1989) was an American mathematician. We will encounter him again later on.

Claim. For any topological space X the map

$$\begin{array}{rcl}\Delta\colon \mathrm{C}_*(X) & \to & \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)\\ (\sigma\colon \Delta^n\to X) & \mapsto & \sum_{p=0}^{n}\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]\otimes\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_n]\end{array}$$

which is given by

is a chain map.

Proof. Let $\sigma\colon\Delta^n\to X$ be a singular n-simplex. Given $i_0,i_1,\dots,i_r$ we use the shorthand $[i_0,\dots,i_r]\coloneqq\sigma\circ[\mathfrak{v}_{i_0},\mathfrak{v}_{i_1},\dots,\mathfrak{v}_{i_r}]\colon\Delta^r\to X$. Furthermore we use the convention that the empty singular simplex $[\,]$ is 0. We calculate that

$$\begin{aligned}\Delta(\partial_n([0,\dots,n])) &\overset{\text{Definition 143.8 of }\partial_n}{=} \Delta\Big(\sum_{i=0}^{n}(-1)^i\cdot[0,\dots,\widehat{i},\dots,n]\Big)\\ &= \sum_{0\le p<i\le n}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n]+\sum_{0\le i<q\le n}(-1)^i\cdot[0,\dots,\widehat{i},\dots,q]\otimes[q,\dots,n]\\ &\uparrow \text{ in the second sum we secretly used the substitution } q=p+1\end{aligned}$$

and

$$\begin{aligned}\partial_n^{\otimes}(\Delta([0,\dots,n])) &= \partial_n^{\otimes}\Big(\sum_{p=0}^{n}[0,\dots,p]\otimes[p,\dots,n]\Big)\\ &= \sum_{p=0}^{n}\partial_p([0,\dots,p])\otimes[p,\dots,n]+\sum_{p=0}^{n}(-1)^p\cdot[0,\dots,p]\otimes\partial_{n-p}([p,\dots,n])\\ &= \sum_{0\le i\le p\le n}(-1)^i\cdot[0,\dots,\widehat{i},\dots,p]\otimes[p,\dots,n]+\sum_{0\le p\le i\le n}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n].\end{aligned}$$

We see that the difference between the two terms equals

$$\begin{aligned}&\sum_{i=p=0}^{n}(-1)^i\cdot[0,\dots,\widehat{i},\dots,p]\otimes[p,\dots,n]+\sum_{i=p=0}^{n}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n]\\ &= \sum_{i=1}^{n}(-1)^i\cdot[0,\dots,i-1]\otimes[i,\dots,n]+\sum_{i=0}^{n-1}(-1)^i\cdot[0,\dots,i]\otimes[i+1,\dots,n]\\ &\uparrow \text{ here we use that } [\,]=0\\ &= \sum_{i=1}^{n}(-1)^i\cdot[0,\dots,i-1]\otimes[i,\dots,n]+\sum_{j=1}^{n}(-1)^{j-1}\cdot[0,\dots,j-1]\otimes[j,\dots,n] \;=\; 0.\\ &\uparrow \text{ substitution } j=i+1\end{aligned}$$

We have thus shown that Δ is indeed a chain map. ∎

The following proposition shows that diagonal approximations are essentially unique:

Proposition 203.19. (Diagonal Approximation–Uniqueness Proposition) Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy between Φ and Ψ.

Proof. [2161] Let $\Phi,\Psi\colon\mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ be two natural chain maps such that for each $x\in X$ we have

$$\Phi(x)\;=\;\Psi(x)\;=\;x\otimes x\quad\in(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X))_0=\mathrm{C}_0(X)\otimes\mathrm{C}_0(X).$$

[2161] The proof is a slight variation on the proof of the EZ–Natural Chain Homotopy Proposition 164.13. The reader who is tired of getting served the same proof time and again can also deduce it from the Acyclic Model Theorem 235.7 (2), which formalizes all these identical looking proofs.

For all $k \in \mathbb{Z}_{\geq -1}$ we have to find a homomorphism

$$P_k \colon \mathrm{C}_k(X) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+1}$$

that satisfies the following three properties:

(1) The map P_{-1} is the zero map.
(2) We have $\partial^{\otimes}_{k+1} \circ P_k + P_{k-1} \circ \partial_k = \Phi_k - \Psi_k \colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_k$.
(3) The map P_k is natural in X.

So suppose we have already defined the maps $P_{-1}, \ldots, P_{k-1}$ with $k \geq 0$. Next we have to define the map P_k. We first consider a special case, namely we first want to define $P_k(\mathrm{id}_{\Delta^k}) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)_{k+1}$.

Claim 1. There exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with

$$\partial^{\otimes}_{k+1}(c) \;=\; (\Phi_k - \Psi_k - P_{k-1} \circ \partial_k)(\mathrm{id}_{\Delta^k}).$$

Proof. In the proof of the claim we use the abbreviation $\mathrm{id}_k \coloneqq \mathrm{id}_{\Delta^k}$. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\begin{aligned}
\partial^{\otimes}_k((\Phi_k - \Psi_k - P_{k-1} \circ \partial_k)(\mathrm{id}_k)) \;&=\; \partial^{\otimes}_k(\Phi_k(\mathrm{id}_k)) - \partial^{\otimes}_k(\Psi_k(\mathrm{id}_k)) - (\partial^{\otimes}_k \circ P_{k-1})(\partial_k(\mathrm{id}_k)) \\
&= \Phi_{k-1}(\partial_k(\mathrm{id}_k)) - \Psi_{k-1}(\partial_k(\mathrm{id}_k)) - (\Phi_{k-1} - \Psi_{k-1} - (P_{k-2} \circ \partial_{k-1}))(\partial_k(\mathrm{id}_k)) \\
&\quad \uparrow \\
&\quad \text{since } \Phi \text{ and } \Psi \text{ are chain maps and since inductively} \\
&\quad \text{we know by (2) that } \partial^{\otimes}_k \circ P_{k-1} + P_{k-2} \circ \partial_{k-1} = \Phi_{k-1} - \Psi_{k-1} \\
&= P_{k-2}(\underbrace{(\partial_{k-1} \circ \partial_k)}_{=0}(\mathrm{id}_k)) \;=\; 0.
\end{aligned}$$

In the following we first consider the case that $k \geq 1$. It follows in this case from the above calculation and the Acyclic Tensor Product Lemma 164.12 that there exists a $(k+1)$-chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}_{k+1}(c) = (\Phi_k - \Psi_k - P_{k-1} \circ \partial_k)(\mathrm{id}_k)$. Now we consider the case $k = 0$. Using the fact that Φ and Ψ agree on 0-simplices one can easily verify that for the augmentation homomorphism $\epsilon \colon \mathrm{C}_0(X) \otimes \mathrm{C}_0(X) \to \mathbb{Z}$ from the Acyclic Tensor Product Lemma 164.12 we have $\epsilon((\Phi_0 - \Psi_0)(\mathrm{id}_0)) = 0$. It follows again from the Acyclic Tensor Product Lemma 164.12 that there exists a 1-chain $c \in (\mathrm{C}_*(\Delta^0) \otimes \mathrm{C}_*(\Delta^0))_1$ with $\partial^{\otimes}_{k+1}(c) = (\Phi_0 - \Psi_0)(\mathrm{id}_0)$. ⊞

We set $P_k(\mathrm{id}_k) \coloneqq c$. Next let X be a topological space and let $\sigma \colon \Delta^k \to X$. We set

$$P_k(\sigma) \;\coloneqq\; (\sigma_* \otimes \sigma_*)(\underbrace{P_k(\mathrm{id}_{\Delta^k})}_{=c}) \;\in\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+1}.$$

Claim 2. The map $P_k \colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_k$ has all the desired properties.

Proof. First note that basically by definition this map P_k is natural in X. Also note that, as in the proof of the EZ-Construction Proposition 164.10, one easily verifies that condition (2) is satisfied. We leave the details to the reader. ■

Now we are ready to provide the proof of the Cup Product–Definitions Proposition 203.15.

Proof of Proposition 203.15. Let X be a topological space and let R be a commutative ring. We need to show that for any $\varphi \in \mathrm{H}^p(X; R)$ and any $\psi \in \mathrm{H}^q(X; R)$ we have

$$\underbrace{\varphi \mathbin{\widetilde{\cup}} \psi}_{\text{Definition 203.8}} \;=\; \underbrace{\varphi \cup \psi}_{\text{Definition 203.11}} \;\in\; \mathrm{H}^{p+q}(X; R).$$

In the following prove we use the following standard notation:

(1) We denote by $\Pi_{p,q}\colon (\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_{p+q} \to \mathrm{C}_p(X)\otimes \mathrm{C}_q(X)$ the projection.
(2) We denote by $\mu_R\colon R\otimes R\to R$ the multiplication map.
(3) We denote by $\Delta\colon \mathrm{C}_{p+q}(X)\to (\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_{p+q}$ the above Alexander-Whitney diagonalization.
(4) We denote by $d_X\colon X\to X\times X$ the diagonal map.

Now we can formulate the following claim:

Claim. For any two cochains $\varphi\in \mathrm{C}^p(X;R)$ and $\psi\in \mathrm{C}^q(X;R)$ we have

$$\varphi\cup\psi \;=\; \mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta \;\in\; \mathrm{C}^{p+q}(X;R)=\mathrm{Hom}(\mathrm{C}_{p+q}(X),R).$$

Proof. We have to show that the two sides are the same maps $\mathrm{C}_{p+q}(X)\to R$. Therefore let $\sigma\colon \Delta^{p+q}\to X$ be a singular $(p+q)$-simplex. We calculate that

$$\begin{aligned}(\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta)(\sigma) &\overset{\text{Definition 203.18 of the Alexander-Whitney diagonalization}}{=} (\mu_R\circ(\varphi\otimes\psi))\big(\Pi_{p,q}\big(\sum_{i+j=p+q}\overbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_i]\otimes\sigma\circ[\mathfrak{v}_i,\dots,\mathfrak{v}_{p+q}]}^{\in \mathrm{C}_i(X)\otimes\mathrm{C}_j(X)}\big)\big)\\ &= (\mu_R\circ(\varphi\otimes\psi))(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]\otimes\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}])\\ &= \varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \;=\; (\varphi\cup\psi)(\sigma).\end{aligned}$$

(definition of $\mu_R\circ(\varphi\otimes\psi)$; Definition 203.11 of $\cup$) ⊞

Now suppose that we are given cocycles $\varphi\in \mathrm{C}^p(X;R)$ and $\psi\in \mathrm{C}^q(X;R)$. We obtain the following equality in $\mathrm{H}^{p+q}(X;R)$:

$$\begin{aligned}[\varphi\mathbin{\widetilde{\cup}}\psi] &\overset{\text{by Definition 203.8}}{=} [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ(\Theta\circ d_{X*})] &&= (\Theta\circ d_{X*})^*\big(\overbrace{[\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}]}^{\in \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X);R)}\big)\\ &= \Delta^*([\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}]) &&= [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta] \;=\; [\varphi\cup\psi].\end{aligned}$$

(by the Diagonal Approximation–Uniqueness Proposition 203.19 the maps $\Theta\circ d_{X*}, \Delta\colon \mathrm{C}_*(X)\to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ are chain homotopic, so by the Dual Cochain Map Lemma 197.20 (3a) together with the Cochain Homotopy-H*-Lemma 197.14 they induce the same map on cohomology; by the claim)

We have thus obtained the desired equality of cohomology classes. ■

203.4. **Graded commutativity of the cup product.** The following proposition shows that the cup product is (up to a sign) commutative:

Proposition 203.20. (Cup Product-Commutativity Proposition) Let X be a topological space and let R be a commutative ring. For any $a\in \mathrm{H}^k(X;R)$ and $b\in \mathrm{H}^l(X;R)$ we have

$$b\cup a \;=\; (-1)^{kl}\cdot a\cup b.$$

Remark.

(1) The type of (non-) commutativity we encounter in the Cup Product-Commutativity Proposition 203.20 is the same as the one we encountered when we studied the wedge product of alternating forms and the wedge product on de Rham cohomology, see the Wedge Product Properties Lemma 72.10 and the Wedge-on-de Rham Cohomology Proposition 78.12.

(2) If $R = \mathbb{F}_2$, then we are in the pretty situation that $-1 = 1$, i.e. the cup product is commutative. □

The proof of the Cup Product-Commutativity Proposition 203.20 will require the remainder of this section. In the next Section 203.5 we will see that the cup product defines a ring structure on cohomology.

In fact, in the following we give not one, but two proofs of the Cup Product-Commutativity Proposition 203.20, namely one proof for each of the two possible definitions of the cup product.

Proof of Proposition 203.20 using the first definition. We start out with the following claim:

Claim 1. Let A_* and B_* be chain complexes. The unique map

$$\tau\colon A_* \otimes B_* \quad \to \quad B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \quad \mapsto \quad (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Proof. Given $a \in A_k$ and $b \in B_l$ we see that

$$\begin{aligned} \tau(\partial^{\otimes}_{k+l}(a \otimes b)) &= \tau(\partial_k(a) \otimes b + (-1)^k \cdot a \otimes \partial_l(b)) \\ &= \tau(\partial_k(a) \otimes b) + (-1)^k \cdot \tau(a \otimes \partial_l(b)) \\ &= (-1)^{(k-1)l} \cdot b \otimes \partial_k(a) + (-1)^{k+k(l-1)} \cdot \partial_l(b) \otimes a \\ &= (-1)^{kl} \cdot \partial_l(b) \otimes a + (-1)^{l+kl} \cdot b \otimes \partial_k(a) \\ &= \partial^{\otimes}((-1)^{kl} \cdot b \otimes a) \quad = \quad \partial^{\otimes}(\tau(a \otimes b)). \end{aligned}$$

⊞

Now we turn to topology. Let X be a topological space and let R be a commutative ring. It now remains to prove the following claim:

Claim 2. The following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \xrightarrow{([a],[b]) \mapsto [\mu_R \circ (a \otimes b) \circ \Pi_{p,q}]} & \mathrm{H}^{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X);R) & \xrightarrow{d_X^* \circ \Theta^*} & \mathrm{H}^{p+q}(X;R) \\ \downarrow{\scriptstyle (a,b) \mapsto (-1)^{kl} \cdot (b,a)} & & \downarrow{\scriptstyle \tau} \text{ induced by the chain map } [a \otimes b] \mapsto [(-1)^{kl} \cdot b \otimes a] & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{H}^q(X;R) \times \mathrm{H}^p(X;R) & \xrightarrow{([c],[d]) \mapsto [\mu_R \circ (c \otimes d) \circ \Pi_{q,p}]} & \mathrm{H}^{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X);R) & \xrightarrow{d_X^* \circ \Theta^*} & \mathrm{H}^{p+q}(X;R). \end{array}$$

$$(a,b) \mapsto a \cup b \qquad\qquad (c,d) \mapsto c \cup d$$

Proof. We make the following clarifications and observations:

(1) The maps $\mu_R\colon R \otimes R \to R$, $\Pi_{p,q}$, $d_X\colon X \to X \times X$ and $\Theta\colon \mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ are the maps appearing in the first definition of the cup product. In fact the upper and lower horizontal sequence give us the first definition of the cup product.

(2) It follows from Claim 1 together with the Dual Cochain Map Lemma 197.20 (1) that the second vertical map is defined.

(3) It is clear that the square to the left commutes.

(4) Note that the maps $\mathrm{id}\circ\Theta\circ d_X, \tau\circ\Theta\circ d_X\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ are both diagonal approximations. It follows from the Diagonal Approximation–Uniqueness Proposition 203.19 that these two chain maps are actually chain homotopic. It follows from this observation, the Dual Cochain Map Lemma 197.20 (3a) and the Cochain Homotopy-H*-Lemma 197.14 (2) that both maps induce the same map on cohomology. This implies that the square to the right commutes. ■

Proof of Proposition 203.20 using the second definition. [2162] Given a singular n-simplex $\sigma\colon\Delta^n\to X$ we denote by $\overline{\sigma}$ the singular n-simplex obtained from σ by precomposing σ with the affine linear map $[\mathfrak{v}_n,\dots,\mathfrak{v}_0]\colon\Delta^n\to\Delta^n$ that reverses the order of the vertices. Furthermore we write $\epsilon_n=(-1)^{n(n+1)/2}$.

Claim. The maps
$$\begin{array}{rccc}\theta\colon & \mathrm{C}_n(X) & \to & \mathrm{C}_n(X)\\ & \sigma & \mapsto & \epsilon_n\cdot\overline{\sigma}\end{array}$$
form a chain map that is chain homotopic to the identity.

We first show that the claim actually implies the statement of the proposition. Thus let $[\varphi]\in\mathrm{H}^k(X;R)$ and $[\psi]\in\mathrm{H}^l(X;R)$. Since by the claim θ is chain homotopic to the identity it follows from the Dual Cochain Map Lemma 197.20 and the Cochain Homotopy-H*-Lemma 197.14 (1) that the induced map $\theta^*\colon\mathrm{C}^*(X;R)\to\mathrm{C}^*(X;R)$ induces the identity on cohomology groups. So to deduce the statement of the proposition it suffices to show that
$$\theta^*(\psi\cup\varphi)\;=\;(-1)^{kl}\cdot\theta^*(\varphi)\cup\theta^*(\psi)\;\in\;\mathrm{C}^{k+l}(X;R).$$
Thus let $\sigma\colon\Delta^{k+l}\to X$ be a singular simplex. Then
$$\begin{aligned}(\theta^*(\psi\cup\varphi))(\sigma)\;&=\;(\psi\cup\varphi)(\theta_*(\sigma))\;=\;(\psi\cup\varphi)(\epsilon_{k+l}\cdot\sigma\circ[\mathfrak{v}_{k+l},\dots,\mathfrak{v}_0])\\&=\;\epsilon_{k+l}\cdot\psi(\sigma\circ[\mathfrak{v}_{k+l},\dots,\mathfrak{v}_k])\cdot\varphi(\sigma\circ[\mathfrak{v}_k,\dots,\mathfrak{v}_0])\\&=\;(-1)^{kl}\cdot\varphi(\epsilon_k\cdot\sigma\circ[\mathfrak{v}_k,\dots,\mathfrak{v}_0])\cdot\psi(\epsilon_l\cdot\sigma\circ[\mathfrak{v}_{k+l},\dots,\mathfrak{v}_k])\\&\quad\uparrow\\&\quad\text{we use the elementary observation that }\epsilon_{k+l}=(-1)^{kl}\cdot\epsilon_k\cdot\epsilon_l\text{ and}\\&\quad\text{we use that }R\text{ is a }\textit{commutative}\text{ ring}\\&=\;(-1)^{kl}\cdot(\theta^*(\varphi)\cup\theta^*(\psi))(\sigma).\end{aligned}$$
We have thus shown the statement of the proposition modulo the claim.

Proof. It is straightforward to verify that θ is a chain map, indeed, for any singular n-simplex σ we have
$$\begin{aligned}\partial_n(\theta(\sigma))\;&=\;\partial_n(\epsilon_n\cdot\sigma\circ[\mathfrak{v}_n,\dots,\mathfrak{v}_0])\;=\;\epsilon_n\cdot\sum_{i=0}^{n}(-1)^i\cdot\sigma\circ[\mathfrak{v}_n,\dots,\widehat{\mathfrak{v}_{n-i}},\dots,\mathfrak{v}_0],\\\theta(\partial_n(\sigma))\;&=\;\theta\Big(\sum_{i=0}^{n}(-1)^i\cdot\sigma\circ[\mathfrak{v}_0,\dots,\widehat{\mathfrak{v}_i},\dots,\mathfrak{v}_n]\Big)\;\underset{\uparrow}{=}\;\epsilon_{n-1}\cdot\sum_{j=0}^{n}(-1)^{n-j}\cdot\sigma\circ[\mathfrak{v}_n,\dots,\widehat{\mathfrak{v}_{n-j}},\dots,\mathfrak{v}_0].\end{aligned}$$
definition of θ and by the substitution $j=n-i$

From $\epsilon_n=(-1)^n\cdot\epsilon_{n-1}$ we now obtain that $\partial_n\circ\theta=\theta\circ\partial_n$, i.e. we have shown that θ is a chain map.

We still have to show that θ is chain homotopic to the identity. We proceed with ideas similar to the ideas used in the proof of the Homotopy-H$_*$-Proposition 144.12. As in that

[2162] We give a proof of the proposition following [**Hat2002**, p. 216-217]. In Hatcher's book more motivations for the steps in the proof are given.

proof, given $i \in \{0,\dots,n\}$ we write

$$x_i \coloneqq ((0,\dots,1,0,\dots,0),0) \in \mathbb{R}^{n+1}\times\{0\},$$

and

$$y_i \coloneqq ((0,\dots,1,0,\dots,0),1) \in \mathbb{R}^{n+1}\times\{1\}.$$

We denote by $\pi\colon \Delta^n\times[0,1]\to\Delta^n$ the projection. We consider the map

$$\begin{aligned} P_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(X)\\ (\sigma\colon\Delta^n\to X) &\mapsto \sum_{i=0}^{n}(-1)^i\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ\underbrace{[x_0,\dots,x_i,y_n,\dots,y_i]}_{\text{map }\Delta^{n+1}\to\Delta^n\times[0,1]}. \end{aligned}$$

Let $n\in\mathbb{N}_0$. We want to show that $\partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n = \theta - \mathrm{id}$. We calculate

$$\begin{aligned} (P_{n-1}\circ\partial_n)(\sigma) &= \sum_{i<j}(-1)^i\cdot(-1)^j\cdot\epsilon_{n-i-1}\cdot(\sigma\circ\pi)\circ[x_0,\dots,x_i,y_n,\dots,\widehat{y_j},\dots,y_i]\\ &\quad+\sum_{i>j}(-1)^{i-1}\cdot(-1)^j\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[x_0,\dots,\widehat{x_j},\dots,x_i,y_n,\dots,y_i], \end{aligned}$$

On the other hand we have

$$\begin{aligned} (\partial_{n+1}\circ P_n)(\sigma) &= \sum_{j\le i}(-1)^i\cdot(-1)^j\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[x_0,\dots,\widehat{x_j},\dots,x_i,y_n,\dots,y_i]\\ &\quad+\sum_{j\ge i}(-1)^i\cdot(-1)^{i+1+n-j}\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[x_0,\dots,x_i,y_n,\dots,\widehat{y_j},\dots,y_i]. \end{aligned}$$

Now using that $\epsilon_{n-i} = (-1)^{n-i}\cdot\epsilon_{n-i-1}$ we see that the sum over the $j\neq i$ terms agrees precisely with $-P_{n-1}\circ\partial_n$ calculated above (with the roles of i and j reversed). It remains to show that the sum over the $i=j$ terms equals $(\theta-\mathrm{id})(\sigma)$. This is indeed the case since

$$\begin{aligned} &\text{sum of the } i=j \text{ summands in the above formula for } (\partial_{n+1}\circ P_n)(\sigma)\\ &= \epsilon_n\cdot(\sigma\circ\pi)\circ[y_n,\dots,y_0] + \sum_{i=1}^{n}\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[x_0,\dots,x_{i-1},y_n,\dots,y_i]\\ &\quad+\sum_{i=0}^{n-1}(-1)^{n+i+1}\cdot\epsilon_{n-i}\cdot(\sigma\circ\pi)\circ[x_0,\dots,x_i,y_n,\dots,y_{i+1}] - (\sigma\circ\pi)\circ[x_0,\dots,x_n]\\ &\underset{\uparrow}{=} \epsilon_n\cdot(\sigma\circ\pi)\circ[y_n,\dots,y_0] - (\sigma\circ\pi)\circ[x_0,\dots,x_n] = (\theta-\mathrm{id})(\sigma). \end{aligned}$$

in the second sum we substitute $j=i+1$, we obtain a new sign $(-1)^{n+j}\cdot\epsilon_{n-j+1} = -\epsilon_{n-j}$, thus we see that the two sums over i cancel

We have now completed the proof that $\partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n = \theta - \mathrm{id}$, in particular we have shown the claim and thus also the proposition. ∎

We conclude this section with a short addendum to the Cup Product-Commutativity Proposition 203.20: Let $k\in\mathbb{N}$ be odd. Note that given any topological space X, given any commutative ring R and given any $\varphi\in\mathrm{H}^k(X;\mathbb{Z})$ we have

$$\varphi\cup\varphi \underset{\uparrow}{=} (-1)^{k\cdot k}\cdot(\varphi\cup\varphi) \underset{\uparrow}{=} -(\varphi\cup\varphi).$$

by the Cup Product Commutativity Proposition 203.20 — since k is odd

This shows that we have $2\cdot(\varphi\cup\varphi)=0$. In other words, we see that $\varphi\cup\varphi$ is 2-torsion. The next lemma makes the amusing observation that for $k=1$ and $R=\mathbb{Z}$ this "square" is in fact not only 2-torsion, but even always zero:

Lemma 203.21. (Cup Product–Square Zero Lemma) Let X be a topological space. For any $\varphi \in \mathrm{H}^1(X;\mathbb{Z})$ we have $\varphi \cup \varphi = 0$.

Remark.

(1) Note that there is no higher-dimensional analog of the Cup Product–Square Zero Lemma 203.21. For example in Exercise 213.22 we will show that given any odd k with $k \geq 3$ there exists a cohomology class $\gamma \in \mathrm{H}^k(\mathbb{RP}^\infty \times \mathbb{RP}^\infty;\mathbb{Z})$ such that $\gamma^2 \neq 0$. (See also [**Hat2002**, p. 403] for a discussion of such examples.)

(2) Note that we will see shortly in the Projective Plane–Cup Product Lemma 203.32 that the statement of the Cup Product–Square Zero Lemma 203.21 does not hold for $\mathbb{F}_2$-coefficients. □

Proof. Let X be a topological space and let $\varphi \in \mathrm{C}^1(X;\mathbb{Z})$ be a cocycle. We need to show that $\varphi \cup \varphi \in \mathrm{C}^2(X;\mathbb{Z})$ is a coboundary. We consider the 1-dimensional cochain

$$\begin{array}{rcl} \xi\colon \mathrm{C}_1(X) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to X) & \mapsto & \frac{1}{2}\cdot(\underbrace{\varphi(\sigma)-\varphi(\sigma)^2}_{\in 2\mathbb{Z}}). \end{array}$$

that is determined by

It remains to prove the following claim:

Claim. We have $\varphi \cup \varphi = \delta_1(\xi)\colon \mathrm{C}_2(X) \to \mathbb{Z}$.

Proof. So suppose that we are given a singular simplex $\tau\colon \Delta^2 \to X$. Given $j \in \{0,1,2\}$ we write $i_j := i_j^2\colon \Delta^1 \to \Delta^2$. Note that

$$(*) \qquad \varphi(\tau\circ i_0)-\varphi(\tau\circ i_1)+\varphi(\tau\circ i_2) = \varphi(\partial_2(\tau)) \underset{\text{Definition 198.1 of } \delta_1(\varphi)}{=} (\delta_1(\varphi))(\tau) \underset{\text{since } \varphi \text{ is a cocycle}}{=} 0.$$

Now we compute that

$$\begin{array}{rl} & \overset{\text{Definition 198.1 of } \delta_1}{=} \xi(\partial_2(\tau)) \overset{\text{Definition 143.8 of } \partial_2}{=} \xi((\tau\circ i_0)-(\tau\circ i_1)+(\tau\circ i_0)) \overset{\text{definition of } \xi}{=} \\ (\delta_1(\xi))(\tau) & = \frac{1}{2}(\varphi(\tau\circ i_0)-\varphi(\tau\circ i_1)+\varphi(\tau\circ i_2)+\varphi(\tau\circ i_0)^2-\varphi(\tau\circ i_1)^2+\varphi(\tau\circ i_2)^2) \\ & \underset{\text{by } (*) \text{ we have } \varphi(\tau\circ i_1)=\varphi(\tau\circ i_0)+\varphi(\tau\circ i_2)}{=} \frac{1}{2}(\varphi(\tau\circ i_0)^2-(\varphi(\tau\circ i_0)+\varphi(\tau\circ i_2))^2+\varphi(\tau\circ i_2)^2) \\ & \underset{\text{elementary algebra}}{=} \varphi(\tau\circ i_2)\cdot\varphi(\tau\circ i_0) \underset{\substack{\text{by Definition 203.11 of} \\ \text{the cup product on cochains}}}{=} (\varphi\cup\varphi)(\tau). \end{array}$$

■

203.5. **The ring structure on cohomology.** In this section we will see that the cup product defines a ring structure on cohomology. To formulate this result it is helpful to introduce the following algebraic concepts:

Definition 203.22.

(1) Recall that in Definition 2.6 we defined a ring to be an abelian group together with a multiplicative structure which is associative, distributive and that admits a multiplicatively neutral element. We do not demand that a ring is commutative.
(2) A graded ring is a ring R together with a direct sum decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ into subgroups such that for every $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map
$$R_m \times R_n \to R_{m+n}.$$
(3) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be two graded rings. A morphism between the two graded rings is a ring homomorphism $\varphi\colon R \to S$ such that for every $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subseteq S_n$.
(4) We define the category $\mathcal{GrRing}$ of graded rings to be the category where the objects are graded rings and the morphisms are morphisms of graded rings.

We mostly care about a special type of graded ring:

Definition 203.23. A superalgebra is a graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for every $a \in R_k$ and $b \in R_l$ we have[2163]
$$b \cdot a = (-1)^{k\cdot l} \cdot a \cdot b.$$

We now return to topology:

Notation 203.24. Given a topological space X and an abelian group G we write
$$\mathrm{H}^*(X; G) := \bigoplus_{k\in\mathbb{N}_0} \mathrm{H}^k(X; G).$$

Now we can formulate the promised result:

Proposition 203.25. (Cup Product-Ring Proposition) Let X be a topological space and let R be a commutative ring.
(1) The cup product on $\mathrm{H}^*(X; R)$ is R-bilinear.
(2) The abelian group $\mathrm{H}^*(X; R) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}^n(X; R)$ together with the map
$$\cup\colon \mathrm{H}^*(X; R) \times \mathrm{H}^*(X; R) \to \mathrm{H}^*(X; R)$$
is a superalgebra where the multiplicatively neutral element is given by
$$1_X := \text{constant map } X \to R \text{ given by } x \mapsto 1 \in \mathrm{H}^0(X; R).$$
(3) The maps
$$\begin{aligned} X &\mapsto (\mathrm{H}^*(X; R), \cup) \\ (f\colon X \to Y) &\mapsto (f^*\colon \mathrm{H}^*(Y; R) \to \mathrm{H}^*(X; R)) \end{aligned}$$
define a contravariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{GrRing}$ of graded rings.

Definition 203.26. Given a topological space X and given a commutative ring R we refer to $(\mathrm{H}^*(X; R), \cup)$ as the R-cohomology ring of X.[2164]

[2163] The word "algebra" in our context is justified by the fact that any ring is a $\mathbb{Z}$-algebra. We could also have referred to such an object as a superring, but this terminology seems to be uncommon.

[2164] Note that the cup product also turns the direct product $\prod_{k\in\mathbb{N}_0} \mathrm{H}^k(X; R)$ into a ring. But in most cases it is more convenient to work with the much smaller ring given by $\bigoplus_{k\in\mathbb{N}_0} \mathrm{H}^k(X; R)$.

Remark. Let R be a commutative ring and let $f\colon X \to Y$ be a continuous map between topological spaces. If f is in fact homotopy equivalence, then it follows from the Homotopy Equivalence-H^*-Proposition 198.12 (1) that the induced maps $f^*\colon \mathrm{H}^n(Y;R) \to \mathrm{H}^n(X;R)$ on cohomology *groups* are isomorphisms. It follows from the Cup Product-Ring Proposition 203.25 (2) that these isomorphisms preserve the ring structure, i.e. the homomorphism $f^*\colon \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$ is in fact an isomorphism of graded rings. □

Proof. Let X be a topological space and let R be a commutative ring.

(1) It is clear from both definitions of the cup product that it is R-bilinear.
(2) By (1) and by the Cup Product-Commutativity Proposition 203.20 it remains to show the following claim:

Claim.
(a) The cup product is associative.
(b) The 0-cochain 1_X is the multiplicatively neutral element.

Proof. Both statements follow easily from the second definition of the cup product:
(a) Let $\alpha \in \mathrm{C}^p(X;R)$, $\beta \in \mathrm{C}^q(X;R)$ and $\gamma \in \mathrm{C}^r(X;R)$ be cochains. Given any singular simplex $\sigma\colon \Delta^{p+q+r} \to X$ we have

$$\begin{aligned}((\alpha \cup \beta) \cup \gamma)(\sigma) &= (\alpha \cup \beta)(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_{p+q}]) \cdot \gamma(\sigma \circ [\mathfrak{v}_{p+q}, \dots, \mathfrak{v}_{p+q+r}]) \\ &= \alpha(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p]) \cdot \beta(\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]) \cdot \gamma(\sigma \circ [\mathfrak{v}_{p+q}, \dots, \mathfrak{v}_{p+q+r}]) \\ &= (\alpha \cup (\beta \cup \gamma))(\sigma).\end{aligned}$$

↑ same argument backwards

Thus we see that the cup product is already associative for cochains, so it is in particular associative for cohomology classes.[2165]
(b) Let $\eta\colon X \to R$ be the map that assigns to each point in X the value $1 \in R$. As usual we view η as a singular 0-cochain. It follows from the H^0-Proposition 198.9 that η is a 0-cocycle and that we obtain an element $1_X \coloneqq \eta \in \mathrm{H}^0(X;R)$. It is clear from the second definition of the cup product that 1_X is the neutral element with respect to the cup product. ⊞

(3) By the H^n–Functor Lemma 198.3 it remains to prove the following claim:

Claim. Given any continuous map $f\colon X \to Y$ between topological spaces the induced map $f^*\colon \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$ is a ring homomorphism.

Proof. The claim follows from the following observations:
(a) In the Cup Product-Naturality Lemma 203.9 we showed that for any $c \in \mathrm{H}^p(Y;R)$ and any $d \in \mathrm{H}^q(Y;R)$ we have $f^*(c) \cup f^*(d) = f^*(c \cup d)$.

Purely for fun we now also verify this equality using the second definition of the cup product. Thus let $c = [\varphi] \in \mathrm{H}^p(Y;R)$ and let $d = [\psi] \in \mathrm{H}^q(Y;R)$. Furthermore

[2165] In Exercise 203.1 we will see that one can also prove associativity just using the first definition of the cup product.

let $\sigma\colon \Delta^{p+q} \to X$ be a singular $(p+q)$-chain. Then we see that

$$\begin{aligned}(f^*\varphi \cup f^*\psi)(\sigma) &= (f^*\varphi)(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p]) \cdot (f^*\psi)(\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]) \\ &= \varphi(f \circ (\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p])) \cdot \psi(f \circ (\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}])) \\ &= \varphi((f \circ \sigma) \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p)) \cdot \psi((f \circ \sigma) \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]) \\ &= (\varphi \cup \psi)(f \circ \sigma) \;=\; (f^*(\varphi \cup \psi))(\sigma).\end{aligned}$$

(b) It follows immediately from the definitions of 1_X, 1_Y and f^* that we have the equality $f^*(1_Y) = 1_X$. ∎

203.6. **Calculations of the cup product.** In this section we calculate the cup product for the following topological spaces:

(1)	The torus	(with $\mathbb{Z}$-coefficients).
(2)	Any finite wedge of spheres	(with any coefficients).
(3)	The surface of genus 2	(with $\mathbb{Z}$-coefficients).
(4)	The real projective plane $\mathbb{R}\mathrm{P}^2$	(with $\mathbb{F}_2$-coefficients).

In the following calculations we will make use of the following elementary observations regarding the cup product:

Observation 203.27. Let R be a commutative ring.

(1) Let X be a path-connected non-empty topological space. It follows from the H^0-Proposition 198.9 that $\mathrm{H}^0(X;R) = R \cdot 1_X$. For any $r \cdot 1_X \in \mathrm{H}^0(X;R)$ and $\varphi \in \mathrm{H}^k(X;R)$ we have

$$(r \cdot 1_X) \cup \varphi \underset{\substack{\uparrow \\ \text{since } \cup \text{ is} \\ R\text{-bilinear}}}{=} r \cdot (1_X \cup \varphi) \underset{\substack{\uparrow \\ \text{Cup Product-Ring} \\ \text{Proposition 203.25}}}{=} r \cdot \varphi \underset{\substack{\uparrow \\ \text{same argument} \\ \text{backwards}}}{=} \varphi \cup (r \cdot 1_X).$$

(2) By the Cup Product-Ring Proposition 203.25 the cup product is R-bilinear, so it suffices to determine the cup product on a generating set.

(3) Let M be an n-dimensional topological manifold.

(a) For any p, q with $p+q > n$ the cup product on $\mathrm{H}^p(M;R) \times \mathrm{H}^q(M;R)$ is zero, since it takes values in the group $\mathrm{H}^{p+q}(M;R)$, which is zero by the Manifold-Top-H^*-Theorem 201.26 (1).

(b) We assume that M is closed connected and non-empty.

- If M is oriented, then we know by the Top Cohomology Theorem 201.26 (2b) that the map
$$\begin{aligned}\mathrm{H}^n(M;\mathbb{Z}) &\to \mathbb{Z} \\ [\varphi] &\mapsto \langle \varphi, [M] \rangle\end{aligned}$$
is an isomorphism.
- We know by the $\mathbb{F}_2$-Orientability Proposition 191.22, the Manifold-Top-H_*-Theorem 193.1 and the Universal Coefficient-H^*–Field Theorem 202.2 that the map
$$\begin{aligned}\mathrm{H}^n(M;\mathbb{F}_2) &\to \mathbb{F}_2 \\ [\varphi] &\mapsto \langle \varphi, [M]_{\mathbb{F}_2} \rangle\end{aligned}$$
is an isomorphism.

203.6.1. **The cup product of the torus.** In this section we want to determine the cup product on the torus $T = S^1 \times S^1$. In fact the following lemma, together with Observation 203.27, gives us the complete calculation of cup products on the torus:

Lemma 203.28. (Torus–Cup Product Lemma) Let $T = S^1 \times S^1$ be the 2-dimensional torus.

(1) We recall some notations from before:
- We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
- On page 3813 we introduced a cochain $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$. In the $\theta_{\mathbb{Z}}$-Cocycle Lemma 198.5 we showed that $\theta_{\mathbb{Z}}$ is a cocycle.

The cohomology classes $\alpha := p^*([\theta_{\mathbb{Z}}])$ and $\beta := q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1 \times S^1;\mathbb{Z})$.

(2) We consider the pairing

$$\begin{array}{rcl} \mathrm{H}^1(T;\mathbb{Z}) \times \mathrm{H}^1(T;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi,\psi) & \mapsto & \langle \varphi \cup \psi, [T]\rangle. \end{array}$$

With respect to the basis α, β of $\mathrm{H}^1(T;\mathbb{Z})$ this pairing is given by the matrix

$$\begin{pmatrix} \langle \alpha \cup \alpha, [T]\rangle & \langle \alpha \cup \beta, [T]\rangle \\ \langle \beta \cup \alpha, [T]\rangle & \langle \beta \cup \beta, [T]\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof.

(1) We proved this statement in the Torus-H^1-Lemma 199.11.

(2) We determine these four entries separately:

(a) We first consider the entry $\langle \alpha \cup \beta, [T]\rangle$. Fortunately we already did all the work on page 3896: More precisely, we have

$$\begin{aligned} \langle \alpha \cup \beta, [T]\rangle &= (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(-\sigma_1 + \sigma_2) \\ &= (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(-\sigma_1) + (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_2) \underset{\uparrow}{=} 0 + 1 = 1. \end{aligned}$$

see the calculations on page 3896

(b) We could do the same argument as in (a) with α and β reversed. But it is much easier to apply the Cup Product-Commutativity Proposition 203.20 which gives us that $\beta \cup \alpha = (-1)^{1\cdot 1} \cdot \alpha \cup \beta = -\alpha \cup \beta$.

(c) There are many different ways for showing that $\alpha \cup \alpha = 0$:
- We could do the calculation as in (a), but that is the least efficient approach.
- We could use the symmetry from the Cup Product-Commutativity Proposition 203.20 to argue that $\alpha \cup \alpha = -\alpha \cup \alpha$, and hence $\alpha \cup \alpha = 0$ since $\mathrm{H}^2(T;\mathbb{Z})$ is torsion-free.
- We could use the Cup Product–Square Zero Lemma 203.21 and remove all the fun.
- We can also argue that

$$\alpha \cup \alpha = [p^*(\theta_{\mathbb{Z}})] \cup [p^*(\theta_{\mathbb{Z}})] \underset{\uparrow}{=} p^*([\theta_{\mathbb{Z}}] \cup [\theta_{\mathbb{Z}}]) \underset{\uparrow}{=} 0.$$

by the Cup Product Naturality Lemma 203.9 — since we have $[\theta_{\mathbb{Z}}] \cup [\theta_{\mathbb{Z}}] \in \mathrm{H}^2(S^1;\mathbb{Z}) = 0$

(d) The same way as in (c) we see that $\beta \cup \beta = 0$. ∎

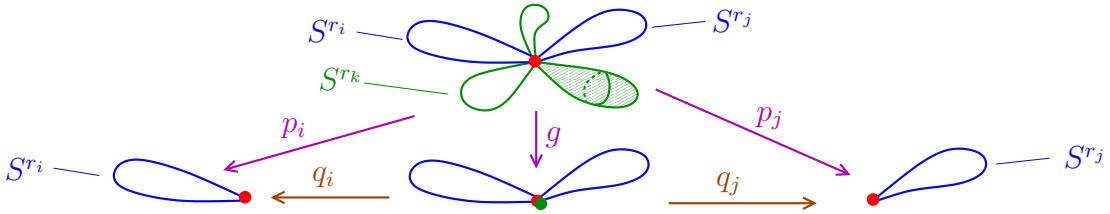
203.6.2. **The cup product of a wedge of spheres.** Next we calculate the cup product of the wedge of finitely many spheres:

Lemma 203.29. (Wedge-of-Spheres–Cup Product Lemma) For any wedge of finitely many spheres all cup products (with respect to any commutative ring) of cohomology classes in degrees ≥ 1 are zero.

Proof. Let $S^{r_1} \vee \cdots \vee S^{r_k}$ be a wedge of finitely many spheres and let R be a commutative ring. For each $i \in \{1, \ldots, k\}$ we denote by $p_i \colon S^{r_1} \vee \cdots \vee S^{r_k} \to S^{r_i}$ the natural projection that we introduced in Definition 96.1. By the Wedge-H*-Proposition 200.10 we know that for every $n \in \mathbb{N}$ the map

$$p_1^* \oplus \cdots \oplus p_k^* \colon \mathrm{H}^n(S^{r_1}; R) \oplus \cdots \oplus \mathrm{H}^n(S^{r_k}; R) \xrightarrow{\cong} \mathrm{H}^n(S^{r_1} \vee \cdots \vee S^{r_k}; R)$$

is an isomorphism. It follows from the calculations of the cohomology groups of spheres on page 3830 and from Observation 203.27 that it remains to show that for any $i, j \in \{1, \ldots, k\}$ with $r_i > 0$ and $r_j > 0$ we have $p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) = 0$. We prove the statement for $i \neq j$. The proof of the statement for $i = j$ is basically the same. So suppose that $i \neq j$. We denote by $g \colon S^{r_1} \vee \cdots \vee S^{r_k} \to S^{r_i} \vee S^{r_j}$ the natural projection and we denote by $q_i \colon S^{r_i} \vee S^{r_j} \to S^{r_i}$ and $q_j \colon S^{r_j} \vee S^{r_j} \to S^{r_j}$ the natural projections.



Then

$$p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) \overset{\text{since } q_i \circ g = p_i \text{ and } q_j \circ g = p_j}{=} g^*(q_i^*([S^{r_i}]^*)) \cup g^*(q_j^*([S^{r_j}]^*)) \overset{\text{by the Cup Product Naturality Lemma 203.9}}{=} g^*(\underbrace{q_i^*([S^{r_i}]^*) \cup q_j^*([S^{r_j}]^*)}_{\substack{\in \mathrm{H}^{r_i + r_j}(S^{r_i} \vee S^{r_j}; R) \\ \text{which is } = 0 \text{ since } r_i, r_j > 0}}) = 0.$$

We have thus proved that all cup products in degrees ≥ 1 are zero. ∎

Remark. We have now performed enough calculations to see that the cohomology rings contain more information than the homology or the cohomology groups. More precisely, it follows from the Torus-H$_*$-Proposition 153.9 (1), the Sphere-H$_*$-Proposition 149.4 and the Wedge-H$_*$-Proposition 149.16 that for the 2-dimensional torus $S^1 \times S^1$ and for $S^1 \vee S^1 \vee S^2$ we have

$$\mathrm{H}_i(S^1 \times S^1) \cong \mathrm{H}_i(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i \geq 3. \end{cases}$$

We deduce from the Universal Coefficient-H*-Theorem 201.23 that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ also have isomorphic cohomology groups. In the Torus–Cup Product Lemma 203.28 we saw that the cup product

$$\mathrm{H}^1(S^1 \times S^1; \mathbb{Z}) \times \mathrm{H}^1(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}^2(S^1 \times S^1; \mathbb{Z})$$

is non-zero, whereas we showed in the Wedge-of-Spheres–Cup Product Lemma 203.29 that the cup product

$$\mathrm{H}^1(S^1 \vee S^1 \vee S^2; \mathbb{Z}) \times \mathrm{H}^1(S^1 \vee S^1 \vee S^2; \mathbb{Z}) \to \mathrm{H}^2(S^1 \vee S^1 \vee S^2; \mathbb{Z})$$

is zero.[2166] □

203.6.3. **The cup product of the surface of genus 2.** In this section we want to determine the cup product on the surface of genus 2. In fact the following lemma, together with Observation 203.27, gives us the complete calculation of cup products on the surface of genus 2:

Lemma 203.30. (Surface–Cup Product Lemma) Let Σ be the surface of genus 2.

(1) We recall the following notation:
- In Lemma 199.13 we introduced for $i = 1, 2$ a continuous map $p_i \colon \Sigma \to T = S^1 \times S^1$. We refer to the figure below for a reminder of the definition of the maps.
- As on page 3908 let $\alpha, \beta \in \mathrm{H}^1(T; \mathbb{Z})$ be the basis from the Torus-H^1-Lemma 199.11.

A basis for $\mathrm{H}^1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^4$ is given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1, 2$.

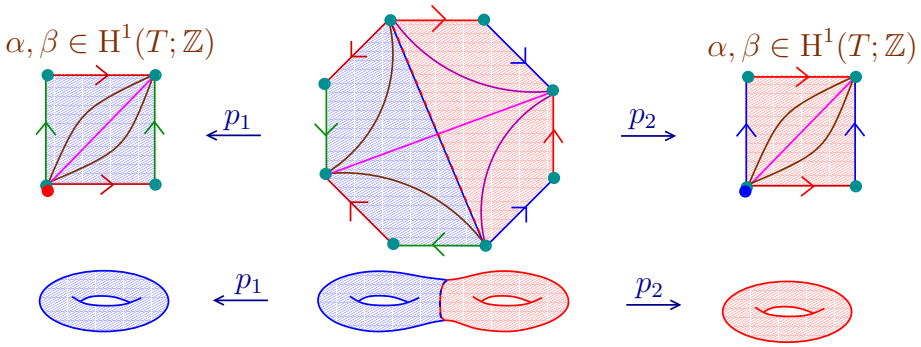
(2) We consider the pairing

$$\begin{array}{rcl} \mathrm{H}^1(\Sigma; \mathbb{Z}) \times \mathrm{H}^1(\Sigma; \mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [\Sigma] \rangle. \end{array}$$

With respect to the above basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $\mathrm{H}^1(\Sigma; \mathbb{Z})$ this pairing is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

An analogous statement holds for any surface of genus $g \geq 1$, i.e. the cup product on the first cohomology of the surface of genus $g \geq 1$ can be represented by a block diagonal matrix consisting of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.



Proof.

(1) In the Surface-H^1-Lemma 199.13 we saw that the map

$$p_1^* \oplus p_2^* \colon \mathrm{H}^1(T; \mathbb{Z}) \oplus \mathrm{H}^1(T; \mathbb{Z}) \to \mathrm{H}^1(\Sigma; \mathbb{Z})$$

[2166] These calculations, together with the Homotopy Equivalence-H^*-Proposition 198.12 (2) and the Cup Product-Ring Proposition 203.25 (2), give us a new proof that the torus $S^1 \times S^1$ and the wedge $S^1 \vee S^1 \vee S^2$ are not homotopy equivalent. This fact can of course already be seen by considering fundamental groups. Nonetheless it is encouraging to see that we found a new way to prove this statement.

is an isomorphism. It follows from this fact together with the Torus-H^1-Lemma 199.11 that a basis for $H^1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ is indeed given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1, 2$.

(2) We need to calculate the 16 entries of the matrix. In fact there are several shortcuts we can make use of:

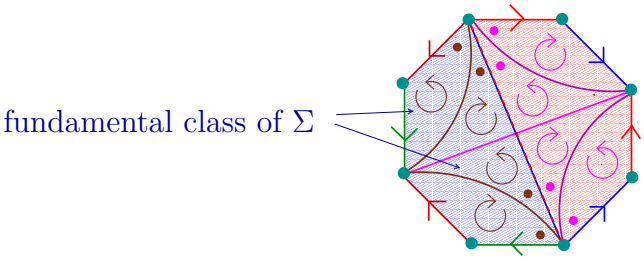
(a) By the Cup Product-Commutativity Proposition 203.20 we know that the matrix is skew symmetric. In particular all diagonal entries are zero.

(b) For $i = 1, 2$ we see that

$$\begin{array}{lllll} \langle \alpha_i \cup \beta_i, [\Sigma]\rangle_\Sigma & = & \langle p_i^*(\alpha) \cup p_i^*(\beta), [\Sigma]\rangle_\Sigma & \overset{\text{by the Cup Product-Naturality Lemma 203.9}}{=} & \langle p_i^*(\alpha \cup \beta), [\Sigma]\rangle_\Sigma \\ & \overset{\text{by the Kronecker Pairing Natural Lemma 199.9 (1b)}}{=} & \langle \alpha \cup \beta, p_{i*}([\Sigma])\rangle_T & \overset{\text{Definition 195.1 of } \deg(p_i)}{=} & \langle \alpha \cup \beta, \deg(p_i) \cdot [T]\rangle_T \overset{\text{it follows easily from the Local-Global Degree Proposition 195.6 that } \deg(p_i) = 1}{=} \langle \alpha \cup \beta, [T]\rangle_T. \end{array}$$

This shows that the top-left 2×2-submatrix and the bottom-right 2×2-submatrix are exactly given by the matrix from the Torus–Cup Product Lemma 203.28.

(c) It remain to show that the top-right 2×2-submatrix is zero. This we have to do "by hand". In the figure below we sketch the representative of the fundamental class $[\Sigma] \in H_2(\Sigma)$ which we introduced in the figure on page 3539.



A rather dull calculation now lets us calculate all the 4 entries of the matrix in the upper right corner. In fact it is fairly easy to see that all entries are zero. In the next chapter, on page 3933, we will give a more conceptual proof why these entries are zero. ■

203.6.4. **The cup product of the real projective plane.** In this subsection we want to understand the cup product of the real projective plane $\mathbb{R}P^2$ in cohomology with $\mathbb{F}_2$-coefficients. Recall that on page 3832 we showed that $H^i(\mathbb{R}P^2;\mathbb{F}_2) \cong \mathbb{F}_2$ for $i = 0, 1, 2$ and $= 0$ otherwise.

Similar to the case of the torus and of the surfaces of genus ≥ 2 we need an explicit 1-cocycle that represents the unique non-zero class in $H^1(\mathbb{R}P^2;\mathbb{F}_2)$. We will find such a representative in an analogous fashion to the construction of the popular singular 1-cocycle $\theta_\mathbb{Z} \in C^1(S^1;\mathbb{Z})$ on page 3813:

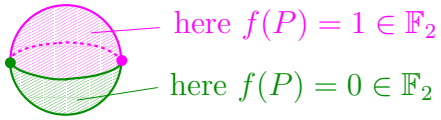
Lemma 203.31. (Generator-of-$H^1(\mathbb{R}P^2;\mathbb{F}_2)$-Lemma) Let $f\colon S^2 \to \mathbb{F}_2$ be any map such that for all $P \in S^2$ we have $f(-P) = 1 - f(P)$. For example we could consider the map

$$\begin{array}{rcl} f\colon S^2 & \to & \mathbb{F}_2 \\ (x,y,z) & \mapsto & \begin{cases} 1, & \text{if } (z > 0) \text{ or } (z = 0 \text{ and } y > 0) \text{ or } (z = y = 0 \text{ and } x = 1), \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We denote by $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ the projection. For a singular 1-simplex $\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to S^2$ of σ and we set

$$\varphi(\sigma) \coloneqq \underbrace{f(\widetilde{\sigma}(0,1)) - f(\widetilde{\sigma}(1,0))}_{=f(\partial_1(\widetilde{\sigma}))} \in \mathbb{F}_2.$$

The map $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ is a cocycle which represents the unique non-trivial element of $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$.



Proof. [2167] We start out with the following claim:

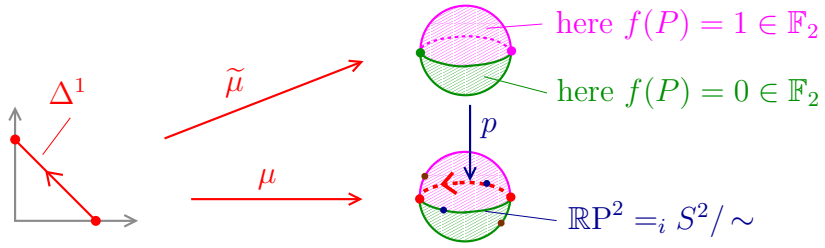
Claim. The map φ is well-defined.

Proof. Let $\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2$ be a singular 1-simplex. Note that it follows from the Lift Uniqueness Proposition 92.2 that there are precisely two lifts, and these two lifts differ by reflection in the origin. It follows from $f(-P) = 1 - f(P) \in \mathbb{F}_2$ that the difference of the evaluations at the endpoints is unaffected by the reflection in the origin. (Here we use that we work with $\mathbb{F}_2$-coefficients instead of $\mathbb{Z}$-coefficients.) ⊞

Next note that the argument of the θ-Cocycle Lemma 198.5 can easily be adapted to show that the singular 1-cochain $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ is a cocycle. It remains to show that $\varphi \neq 0 \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. To do so we consider the Kronecker pairing

$$\langle\ ,\ \rangle\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \times \mathrm{H}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$$

from the Ring–Kronecker Pairing Lemma 202.1. We need a mate for φ: We consider the singular 1-simplex $\mu\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} S^2/\sim$ that is given by $\mu(1-t,t) = [(\cos(\pi t), \sin(\pi t), 0)]$.



Note that μ is a cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$. An explicit lift $\widetilde{\mu}\colon \Delta^1 \to S^2$ of μ is of course given by $(1-t,t) \mapsto (\cos(\pi t), \sin(\pi t), 0)$. We now consider the above Kronecker pairing and we calculate that

$$\langle [\varphi], [\mu] \rangle \underset{\text{Definition 199.8 of } \langle\ ,\ \rangle}{=} \varphi(\mu) \underset{\text{definition of } \varphi}{=} f(\widetilde{\mu}(0,1)) - f(\widetilde{\mu}(1,0)) \underset{\text{since } \widetilde{\mu}(0,1) \text{ and } \widetilde{\mu}(1,0) \text{ are antipodal}}{=} 1 \in \mathbb{F}_2.$$

This calculation implies that $\varphi \neq 0 \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. ■

Observation 203.27 together with the following lemma gives us a complete understanding of the cup product on $\mathbb{R}\mathrm{P}^2$ with $\mathbb{F}_2$-coefficients:

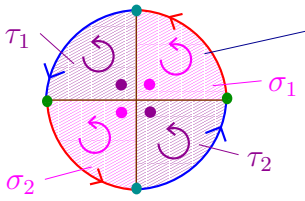
[2167] Note that this proof gives the solution to Exercise 199.18.

Lemma 203.32. (Projective Plane–Cup Product Lemma) Let $[\varphi] \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the unique non-trivial element. We have

$$\underbrace{\langle [\varphi] \cup [\varphi], [\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2} \rangle}_{\substack{\text{pairing from the} \\ \text{Ring–Kronecker Pairing} \\ \text{Lemma 202.1}}} \;=\; 1 \;\in\; \mathbb{F}_2.$$

Proof. We introduce the following notation:

- Let $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the cocycle from the Generator-of-$\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$-Lemma 203.31.
- We consider the singular simplices $\sigma_1, \sigma_2, \tau_1, \tau_2$ that are shown in the figure below. It follows easily from the $\mathbb{F}_2$-Fundamental Class-Criterion 194.9 (see also the discussion on page 3554) that $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a representative of the $\mathbb{F}_2$-fundamental class $[\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2}$ of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$.



projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$

the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class of the real projective plane

By the definition of the Kronecker pairing in the Ring–Kronecker Pairing 202.1 it remains to prove the following claim:

Claim. We have

$$(\varphi \cup \varphi)(\sigma_1) + (\varphi \cup \varphi)(\sigma_2) + (\varphi \cup \varphi)(\tau_1) + (\varphi \cup \varphi)(\tau_2) \;=\; 1 \;\in\; \mathbb{F}_2.$$

Proof. In Exercise 203.7 we will determine all four summands and we will see that the sum is indeed 1. ∎

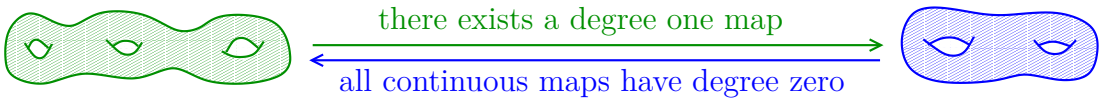
The above calculation of the cup product of the real projective plane $\mathbb{R}\mathrm{P}^2$ is very explicit and hands on, but not very conceptual. Later on, in the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19, we will give a very different calculation of the cup product on $\mathbb{F}_2$-cohomology of all real projective spaces $\mathbb{R}\mathrm{P}^n$.

203.7. **Applications of the cup product.** Now that we had some fun with calculating the cup product, the question arises, what is it actually good for? Of course the cohomology together with the cup product is a stronger invariant, than the cohomology on its own. For example we saw that the torus $S^1 \times S^1$ and the wedge $S^1 \vee S^1 \vee S^2$ have isomorphic cohomology groups, but the cohomology rings are not isomorphic. Once we have developed more tools for calculating cup products, this theme will become more important, for example we will be able to distinguish $S^2 \times S^2$ from $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$.

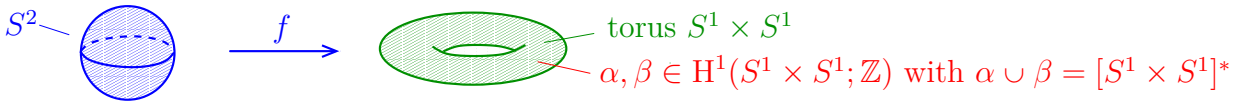
There are also many other questions which can be attacked using the cup product. For example we can give a new proof of the following proposition:

Proposition 186.11. (Smaller-to-Larger Genus Proposition) Let $g, k \in \mathbb{N}_0$. If $g < k$, then every continuous map $f\colon \Sigma_g \to \Sigma_k$ has degree zero.

Proof. In the following we will just deal with the case $g = 0$ and $k = 1$. In other words, we will show that every continuous map $f\colon S^2 \to S^1 \times S^1$ has degree zero. As we



already gave an alternative proof of the proposition, it is fair if we outsource the other cases to Exercise 203.10. So let $f\colon S^2 \to S^1 \times S^1$ be a continuous map. By the Torus–Cup Product Lemma 203.28 we know that there exist $\alpha, \beta \in \mathrm{H}^1(S^1 \times S^1; \mathbb{Z})$ such that $[S^1 \times S^1]^* = \alpha \cup \beta \in \mathrm{H}^2(S^1 \times S^1; \mathbb{Z})$.



We now see that

$$\begin{array}{ccccc} & \text{by Definition 201.27 of } [S^2]^* & & \text{by the Dual Fundamental Class–Degree Lemma 201.28} & & \text{choice of } \alpha \text{ and } \beta \\ \deg(f) & \overset{\downarrow}{=} & \langle \deg(f) \cdot [S^2]^*, [S^2] \rangle & \overset{\downarrow}{=} & \langle f^*([S^1 \times S^1]^*), [S^2] \rangle & \overset{\downarrow}{=} & \langle f^*(\alpha \cup \beta), [S^2] \rangle \\ & = & \langle f^*(\alpha) \cup f^*(\beta), [S^2] \rangle & = & \langle 0 \cup 0, [S^2] \rangle & = & 0. \\ & \text{by the Cup Product Naturality Lemma 203.9} & \overset{\uparrow}{} & \text{since } \mathrm{H}^1(S^2; \mathbb{Z}) = 0 & \overset{\uparrow}{} & & \end{array}$$

■

As we just saw in the previous proof, one interesting aspect of the cup product is that it relates cohomology groups of different dimensions. This feature plays a role in many applications. A typical example is given in the following lemma:

Lemma 203.33. (Orientation-Torus-H^1-Lemma) Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. If f is orientation-preserving, then

$$\deg(f_*\colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)) \;=\; \det(f^*\colon \mathrm{H}^1(T; \mathbb{Z}) \to \mathrm{H}^1(T; \mathbb{Z})) \;=\; +1.$$

Conversely, if f is orientation-reversing, then both determinants equal -1.

Proof. In Exercise 203.4 we will use the Torus–Cup Product Lemma 203.28 to prove the lemma. ■

We are now also in a position to settle the following question that we had asked earlier on.

Question 186.14.

(2) Let $g \geq 2$ and let $k \geq 2$. Does there exist a continuous self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?[2168]

The cup product is the tool to answer this question. But before we address the question it is useful to introduce the following definition:

Definition 203.34. A closed oriented connected non-empty topological manifold M is called flexible if given any $k \in \mathbb{Z}$ there exists a continuous map $f\colon M \to M$ of degree k.

Example.

[2168] Recall that if $f\colon M \to N$ is a continuous map between two closed, oriented connected, non-empty n-dimensional topological manifolds, then the degree $\deg(f) \in \mathbb{Z}$ is defined as the unique integer $\deg(f)$ that satisfies $f_*([M]) = \deg(f) \cdot [N]$.

(1) In the Degree-of-Power Lemma 151.5 we saw that S^1 is flexible. The combination of this fact with the Degree-Suspension Lemma 151.6 shows that in fact any sphere S^n, $n \geq 1$ is flexible.

(2) Let M and N be closed oriented connected non-empty topological manifolds. If M is flexible, then so is the product $M \times N$. This can be seen as follows. Let $k \in \mathbb{Z}$. Since M is flexible there exists a continuous map $f\colon M \to M$ of degree k. We claim that $\operatorname{diag}(f, \mathrm{id}_N)\colon M \times N \to M \times N$ has degree k. We set $m := \dim(M)$ and $n := \dim(N)$. By the naturality of the cross product we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_m(M) \otimes \mathrm{H}_n(N) & \xrightarrow[\cong]{\times} & \mathrm{H}_{m+n}(M \times N) \\ \downarrow{\scriptstyle f_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle \operatorname{diag}(f, \mathrm{id}_N)_*} \\ \mathrm{H}_m(M) \otimes \mathrm{H}_n(N) & \xrightarrow[\cong]{\times} & \mathrm{H}_{m+n}(M \times N). \end{array}$$

It follows from the Künneth-Theorem 164.6, together with the Manifold-H_{n-1}-Theorem 193.4 and the Tor-Properties Lemma 162.13 (4) that the horizontal maps are isomorphisms. Since the left vertical map is given by multiplication by k, so is the right vertical map.[2169] □

Now we can determine which closed oriented connected 2-dimensional smooth manifolds are flexible. More precisely, we have the following proposition which gives in particular a negative answer to Question 186.14 (2).

Proposition 203.35. (Surface Flexibility Proposition) Let $g \in \mathbb{N}_0$. We denote by Σ_g the surface of genus g.

(1) The sphere $\Sigma_0 = S^2$ and the torus $\Sigma_1 = S^1 \times S^1$ are flexible.

(2) For $g \geq 2$ the surface Σ_g is not flexible. In fact if $k \neq -1, 0, 1$, then there is no continuous map $\Sigma_g \to \Sigma_g$ of degree k.

Proof.

(1) It follows from the example preceding the proposition that the sphere S^2 and the torus $S^1 \times S^1$ are flexible.

(2) Now let $g \geq 2$ and let $f\colon \Sigma_g \to \Sigma_g$ be a continuous map of degree $k \neq 0$. We have to show that $k \in \{\pm 1\}$. We write $\pi := \pi_1(\Sigma_g)$. We start out with the following claim:

Claim 1. The map $f_*\colon \pi \to \pi$ is an epimorphism.

Proof. [2170] We start out with the following observations:

- Since $k \neq 0$ it follows from the argument in the proof of the Degree One-Epimorphism Proposition 195.8 that $f_*(\pi)$ is a finite-index subgroup of π. We denote by $n \in \mathbb{N}$ the index of the subgroup $f_*(\pi) \subseteq \pi$.
- It follows from the Covering Existence Theorem 127.10 and the Covering-Degree Proposition 92.10 that there exists a connected covering $p\colon \widetilde{\Sigma} \to \Sigma$ of degree n with $p_*(\pi_1(\widetilde{\Sigma})) = f_*(\pi)$.
- By the Covering-π_1-Monomorphism Proposition 92.7 we know that the induced map $p_*\colon \pi_1(\widetilde{\Sigma}) \to \pi_1(\Sigma)$ is a monomorphism.

[2169] This argument is essentially the proof of Exercise 195.3.

[2170] This proof shares some features with the proof of the Surface-π_1–Not Free Proposition 130.21 on page 2999.

• The Genus-of-Covering Lemma 156.18 implies that $\widetilde{\Sigma}$ is a surface of genus $n(g-1)+1$. Now we consider the following little diagram:

$$\begin{array}{ccccccccc}
\pi_1(\Sigma_g) & \xrightarrow{\mathrm{id}} & \pi & \xrightarrow{f_*} & f_*(\pi) = p_*(\pi_1(\widetilde{\Sigma})) & \xleftarrow[p_*]{\cong} & \pi_1(\widetilde{\Sigma}) & \xrightarrow{\cong} & \pi_1(\Sigma_{n(g-1)+1}) \\
\downarrow & & & & & & & & \downarrow \\
\underbrace{\pi_1(\Sigma_g)_{\mathrm{ab}}}_{\cong \mathbb{Z}^{2g}} & & & & & & & & \underbrace{\pi_1(\Sigma_{n(g-1)+1})_{\mathrm{ab}}}_{\cong \mathbb{Z}^{2n\cdot(g-1)+2}}.
\end{array}$$

Since the top horizontal map is an epimorphism it follows from the Abelianization-Epimorphism Lemma 98.17 that the bottom horizontal map is also an epimorphism. It follows that there exists an epimorphism from $\mathbb{Z}^{2g}$ onto $\mathbb{Z}^{2n\cdot(g-1)+2}$. By the Rank Comparison Lemma 97.16 this implies that $2g \geq 2n \cdot (g-1) + 2$. Since $g > 1$, this is only possible if $n = 1$. ⊞

Claim 2. The map $f^*\colon \mathrm{H}^1(\Sigma_g) \to \mathrm{H}^1(\Sigma_g)$ is an isomorphism.

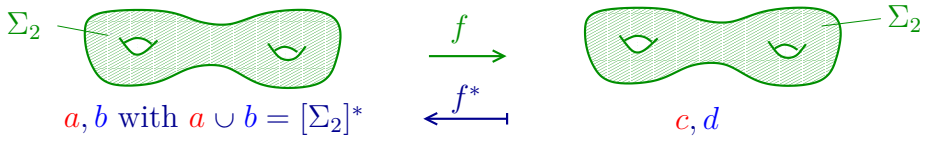
Proof. By Claim 1 we know that the map $f_*\colon \pi \to \pi$ is an epimorphism. It follows from the Hurewicz Epimorphism Proposition 166.6 and the Hurewicz Homomorphism Proposition 166.3 (3) that the induced map $f_*\colon \mathrm{H}_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g)$ is also an epimorphism. By the Epi-Automorphism Lemma 97.27 we know that every epimorphism from a finitely generated abelian group to itself is in fact an isomorphism, so we see that $f_*\colon \mathrm{H}_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g)$ is in fact an isomorphism. By the H^1-Hom-Proposition 201.24 the evaluation homomorphism $\mathrm{ev}\colon \mathrm{H}^1(\Sigma_g) \to \mathrm{Hom}(\mathrm{H}_1(\Sigma_g), \mathbb{Z})$ is a natural isomorphism, i.e. we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1(\Sigma_g) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(\Sigma_g), \mathbb{Z}) \\
\downarrow{\scriptstyle f^*} & & \cong \downarrow{\scriptstyle (f_*)^*} \\
\mathrm{H}^1(\Sigma_g) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(\Sigma_g), \mathbb{Z}).
\end{array}$$

The right-hand map is an isomorphism, it follows from the commutative diagram that the left-hand map $f^*\colon \mathrm{H}^1(\Sigma_g; \mathbb{Z}) \to \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ is also an isomorphism. ⊞

We make two observations:

- By the Surface–Cup Product Lemma 203.30 we know that there exist $a, b \in \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ with $a \cup b = [\Sigma_g]^*$.
- By Claim 2 we know that $f^*\colon \mathrm{H}^1(\Sigma_g; \mathbb{Z}) \to \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ is an isomorphism. This implies in particular that we can find cohomology classes $c, d \in \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ with $f^*(c) = a$ and $f^*(d) = b$.



Since $c \cup d \in \mathrm{H}^2(\Sigma_g;\mathbb{Z})$ and since $[\Sigma_g]^* \in \mathrm{H}^2(\Sigma_g;\mathbb{Z})$ is a generator there exists an $m \in \mathbb{Z}$ with $c \cup d = m \cdot [\Sigma_g]^*$. Now we see that

$$\begin{array}{ccccc} & \overset{\text{Definition 201.27 of } [\Sigma_g]^*}{\downarrow} & & \overset{\text{by the Dual Fundamental Class–Degree Lemma 201.28}}{\downarrow} & & \overset{\text{definition of } m}{\downarrow} & \\ m \cdot \deg(f) & = & \langle m \cdot \deg(f) \cdot [\Sigma_g]^*, [\Sigma_g] \rangle & = & \langle m \cdot f^*([\Sigma_g]^*), [\Sigma_g] \rangle & = & \langle f^*(c \cup d), [\Sigma_g] \rangle \\ & = & \langle f^*(c) \cup f^*(d), [\Sigma_g] \rangle & = & \langle a \cup b, [\Sigma_g]^* & = & \langle [\Sigma_g]^*, [\Sigma_g] \rangle \quad = \quad 1. \\ & \underset{\text{by the Cup Product-Naturality Lemma 203.9}}{\uparrow} & & \underset{\text{definition of } c \text{ and } d}{\uparrow} & & \underset{\text{definition of } a, b}{\uparrow} & \underset{\text{Definition 201.27 of } [\Sigma_g]^*}{\uparrow} \end{array}$$

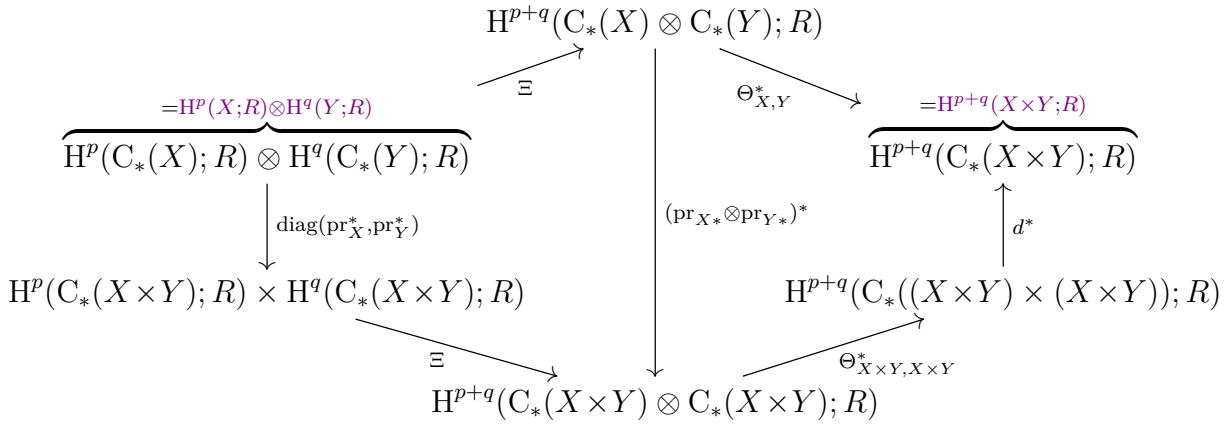
Evidently this is only possible if $\deg(f) \in \{\pm 1\}$. ∎

203.8. **Appendix: The cohomological cross product.** In Definition 203.6 we introduced the cohomological cross product. In this appendix we will express this cohomological cross product in terms of cup products:

Proposition 203.36. (Cohomological Cross Product Proposition) Let X and Y be topological spaces and let R be a commutative ring. We denote by $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ the two natural projections. For any $\varphi \in \mathrm{H}^k(X;R)$ and any $\psi \in \mathrm{H}^l(Y;R)$ we have the equality:

$$\varphi \times \psi \;=\; \mathrm{pr}_X^*(\varphi) \cup \mathrm{pr}_Y^*(\psi) \;\in\; \mathrm{H}^{k+l}(X \times Y;R).$$

Proof. Let X and Y be topological spaces and let R be a commutative ring. We consider the following diagram:



We make the following clarifications:

(1) The maps Ξ are the natural maps from the Cocycle Tensor Lemma 203.5.
(2) The maps Θ are the natural maps from Notation 203.2, which in turn come straight from the Eilenberg-Zilber Theorem 164.2.

First note that the top route from $\mathrm{H}^p(X;R) \otimes \mathrm{H}^q(Y;R)$ to $\mathrm{H}^{p+q}(X \times Y;R)$ describes the cohomological cross product from Definition 203.6. Next note that the bottom route describes the map $(\varphi, \psi) \mapsto \mathrm{pr}_X^*(\varphi) \cup \mathrm{pr}_Y^*(\psi)$, where we use Definition 203.8 of the cup product. Thus it remains to prove the following claim:

Claim. The above diagram commutes.

Proof. First note that it follows from the naturality of Ξ that the left quadrilateral commutes. Now let us turn to the right quadrilateral. To do so we introduce the map

$$\begin{array}{rcl} \alpha\colon (X\times Y)\times(X\times Y) & \to & X\times Y \\ ((x_1,y_1),(x_2,y_2)) & \mapsto & (x_1,y_2). \end{array}$$

Next we consider the following diagram:

$$\begin{array}{ccc} \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) & \xleftarrow{\Theta_{X,Y}} & \mathrm{C}_*(X\times Y) \\ \uparrow{\scriptstyle \mathrm{pr}_{X*}\otimes\mathrm{pr}_{Y*}} & & \uparrow{\scriptstyle \alpha_*} \\ \mathrm{C}_*(X\times Y)\otimes\mathrm{C}_*(X\times Y) & \xleftarrow{\Theta_{X\times Y,X\times Y}} & \mathrm{C}_*((X\times Y)\times(X\times Y)) \end{array}$$

We make the following clarifications and observations:

(1) The map
(2) All maps are chain maps.
(3) All maps are natural.
(4) On the 0-level the two routes from $\mathrm{C}_*((X\times Y)\times(X\times Y))$ $\mathrm{C}_*(X\times Y)$ to are the same.

It now follows from the same argument as in the EZ–Natural Chain Homotopy Proposition 164.13 that the diagram commutes up to a chain homotopy.[2171] But by the Cochain Homotopy-H^*-Lemma 197.14 this implies that the following cohomological version of the diagram also commutes:

$$\begin{array}{ccc} \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y);) & \xrightarrow{\Theta^*_{X,Y}} & \mathrm{H}^{p+q}(\mathrm{C}_*(X\times Y);R) \\ {\scriptstyle (\mathrm{pr}_{X*}\otimes\mathrm{pr}_{Y*})^*}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ \mathrm{H}^{p+q}(\mathrm{C}_*(X\times Y)\otimes\mathrm{C}_*(X\times Y);R) & \xrightarrow[\Theta^*_{X\times Y,X\times Y}]{} & \mathrm{H}^{p+q}(\mathrm{C}_*((X\times Y)\times(X\times Y));R) \end{array}$$

Finally note that clearly $\alpha\circ d=\mathrm{id}_{X\times Y}$. This implies that $d^*\circ\alpha^*=\mathrm{id}$. This observation, together with the commutativity of the above rectangle shows that the above right quadrilateral commutes. ∎

203.9. Exercises for Chapter 203 .

Exercise 203.1. Let X be a topological space and let R be a commutative ring. Show, just using Definition 203.8, that the cup product on $\mathrm{H}^*(X;R)$ is associative.
Hint. You should formulate and prove an analog of the EZ–Natural Chain Homotopy Proposition 164.13. You could use the Acyclic Model Theorem 235.7 to simplify your life.
Remark. The point of the exercise is to show that one can prove the Cup Product-Ring Proposition 203.25 with the original definition of the cup product.

Exercise 203.2. Let X be a topological space and let R be a commutative ring. In the Cup Product-Commutativity Proposition 203.20 and the Cup Product-Ring Proposition 203.25 we showed that the cup product $\cup\colon \mathrm{H}^k(X;R)\times\mathrm{H}^l(X;R)\to\mathrm{H}^{k+l}(X;R)$ on *cohomology* is (anti-) commutative and associative.
(a) Is the cup product $\cup\colon \mathrm{C}^k(X;R)\times\mathrm{C}^l(X;R)\to\mathrm{C}^{k+l}(X;R)$ (anti-) commutative?

[2171] Alternatively one can deduce this statement from the more abstract Acyclic Model Theorem 235.7.

(b) Is the cup product $\cup\colon \mathrm{C}^k(X;R) \times \mathrm{C}^l(X;R) \to \mathrm{C}^{k+l}(X;R)$ associative?

Exercise 203.3. Let X be a CW-complex and let $n, d \in \mathbb{N}$. Given $\varphi \in \mathrm{H}^d(X;\mathbb{Z})$ we write $\varphi^n := \underbrace{\varphi \cup \cdots \cup \varphi}_{n \text{ factors}}$.

(a) We suppose that there exists a $\varphi \in \mathrm{H}^d(X;\mathbb{Z})$ with $\varphi^n \neq 0$. Show that X has at least $n+1$ cells.
(b) Again we suppose that there exists a $\varphi \in \mathrm{H}^d(X;\mathbb{Z})$ with $\varphi^n \neq 0$. This time we also assume that $\varphi \in \mathrm{H}^d(X;\mathbb{Z})$ is torsion, i.e. we assume that there exists an $m \in \mathbb{N}$ such that $m \cdot \varphi = 0 \in \mathrm{H}^d(X;\mathbb{Z})$. What is the best lower bound on the number of cells of X that you can provide?

Exercise 203.4. Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. Show that if f is orientation-preserving, then the following two statements hold:

(a) $f^*\colon \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^1(T;\mathbb{Z})$ has determinant one.
(b) $f_*\colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)$ has determinant one.

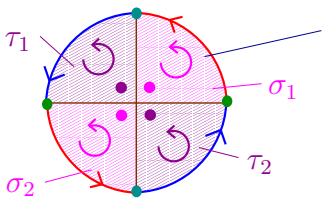
Exercise 203.5.

(a) Let S be a ring and let $a, b \in S$. If $a \cdot b = 1$, does it follow that $b \cdot a = 1$?
(b) Let X be a topological space and let R be a commutative ring. As in the Inverse-in-Cohomology Ring 275.5 we consider the cohomology ring $(\mathrm{H}^*(X;R), \cup)$ from the Cup Product-Ring Proposition 203.25. Let $\alpha, \beta \in \mathrm{H}^*(X;R)$ such that $\alpha \cup \beta = 1$. Show that $\beta \cup \alpha = 1$.
Hint. Make use of the Cup Product-Commutativity Proposition 203.20.

Exercise 203.6. For the purpose of this exercise let $\mathbb{Z}\{x_1, \ldots, x_k\}$ be the "monoid ring" on the free monoid $\{x_1, \ldots, x_k\}$. Show that the ring $(\mathrm{H}^*(S^1 \times S^1;\mathbb{Z}), \cup)$ is isomorphic to the quotient of the monoid ring $\mathbb{Z}\{x_1, \ldots, x_k\}$. What is the minimal k you need?

Exercise 203.7. We consider $\mathbb{R}\mathrm{P}^2 = S^2/\sim$ which we identify with $\overline{B}^2/\sim$ via the homeomorphism that is induced by the projection $S^2 \to \overline{B}^2$. Let $\sigma_1, \sigma_2, \tau_1, \tau_2$ be the singular simplices that are sketched in the figure below. Furthermore let $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the cocycle that we defined in the Generator-of-$\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$-Lemma 203.31. Determine *each of the four summands* in the sum

$$(\varphi \cup \varphi)(\sigma_1) + (\varphi \cup \varphi)(\sigma_2) + (\varphi \cup \varphi)(\tau_1) + (\varphi \cup \varphi)(\tau_2).$$



projective plane $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim$

the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$
represents the $\mathbb{F}_2$-fundamental class
of the real projective plane

Exercise 203.8. Let R be a commutative ring. A R-bilinear pairing $\varphi\colon V \times \widetilde{V} \to R$ on R-modules V and $\widetilde{V}$ is called *non-degenerate* if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(\widetilde{V}, R) \\ v & \mapsto & \left(\begin{array}{ccc} \widetilde{V} & \to & R \\ \widetilde{v} & \mapsto & \varphi(v, \widetilde{v}) \end{array} \right) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{V} & \to & \mathrm{Hom}_R(V, R) \\ \widetilde{v} & \mapsto & \left(\begin{array}{ccc} V & \to & R \\ v & \mapsto & \varphi(v, \widetilde{v}) \end{array} \right) \end{array}$$

are monomorphisms. Now let V and W be finitely generated free R-modules and let $\varphi\colon V \times V \to R$ and $\psi\colon W \times W \to R$ be R-bilinear pairings. Let $\theta\colon V \to W$ be an isometry, i.e. θ is a homomorphism such that for all $v, v' \in V$ we have $\psi(\theta(v), \theta(v')) = \varphi(v, v')$. Show that if φ is non-degenerate, then $\theta\colon V \to W$ is a monomorphism.

Exercise 203.9. Let X be a topological space and let R be a commutative ring. For the purpose of this exercise we consider the "extended" cohomology ring

$$\mathrm{H}^{**}(X; R) \;:=\; \prod_{k \in \mathbb{N}_0} \mathrm{H}^k(X; R).$$

The proof of the Cup Product-Ring Proposition 203.25 shows that the cup product also turns $\mathrm{H}^{**}(X; R)$ into a ring. Let $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathrm{H}^{**}(X; R)$. Show that the following two statements are equivalent:

(1) There exists an inverse $\beta = (\beta_0, \beta_1, \dots) \in \mathrm{H}^{**}(X; R)$ in the ring $(\mathrm{H}^*(X; R), \cup)$.
(2) The 0-th term α_0 is a unit in $\mathrm{H}^0(X; R) =_{\mathrm{i}} R$.

Exercise 203.10. Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g. Let $g < k$. Use cup products and Exercise 203.8 to show that every continuous map $f\colon \Sigma_g \to \Sigma_k$ has degree zero.

Exercise 203.11. Let K be the Klein bottle.

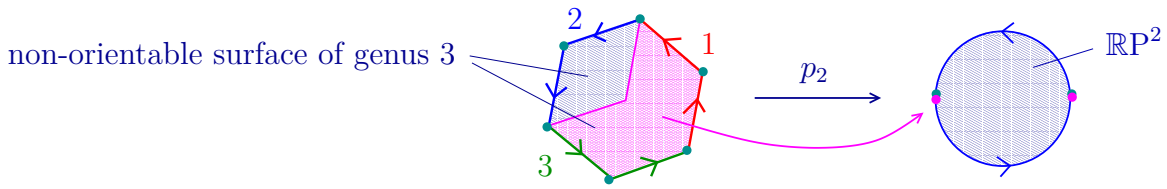
(a) Is the cup product $\mathrm{H}^1(K; \mathbb{Z}) \times \mathrm{H}^1(K; \mathbb{Z}) \to \mathrm{H}^2(K; \mathbb{Z})$ trivial?
(b) Show that the cup product $\mathrm{H}^1(K; \mathbb{F}_2) \times \mathrm{H}^1(K; \mathbb{F}_2) \to \mathrm{H}^2(K; \mathbb{F}_2)$ is non-trivial.

Exercise 203.12. Let $k \in \mathbb{N}$. We denote by N_k the non-orientable surface of genus k from Definition 9.19. For $i = 1, \dots, k$ we denote by $p_i\colon N_k \to \mathbb{R}\mathrm{P}^2$ the map that is illustrated in the figure below.

(a) Show that the map $p_1^* \oplus \dots \oplus p_k^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) \oplus \dots \oplus \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) \to \mathrm{H}^1(N_k; \mathbb{F}_2)$ is an isomorphism.
Hint. You could use a variation on the Kronecker-Basis Lemma 199.10.
(b) Let $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ be the unique non-trivial element. For $i = 1, \dots, k$ we write $y_i = p_i^*(x)$. Furthermore we denote by $[N_k]^*_{\mathbb{F}_2} \in \mathrm{H}^2(N_k; \mathbb{F}_2)$ the unique non-trivial element. Show that for any $i, j \in \{1, \dots, k\}$ we have $y_i \cup y_j = \delta_{ij} \cdot [N_k]^*_{\mathbb{F}_2}$.
(c) Let $g < k$. Use cup products and Exercise 203.8 to show that there is no continuous map $f\colon N_g \to N_k$ that induces an isomorphism $f_*\colon \mathrm{H}_2(N_g; \mathbb{F}_2) \to \mathrm{H}_2(N_k; \mathbb{F}_2)$.

Exercise 203.13. Given a matrix $A \in \mathrm{M}(n \times n, \mathbb{F}_2)$ we define the bilinear form

$$\begin{array}{rccc} \lambda(A)\colon \mathbb{F}_2^n \times \mathbb{F}_2^n & \to & \mathbb{F}_2 \\ (v, w) & \mapsto & v^{\mathrm{T}} \cdot A \cdot w. \end{array}$$



We consider the two matrices $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ over $\mathbb{F}_2$. Next we consider the torus $T = S^1 \times S^1$ and the Klein bottle $K = N_2$. By a slight variation on the Torus–Cup Product Lemma 203.28 and by Exercise 203.12 we know that the cup products $\mathrm{H}^1(T;\mathbb{F}_2) \times \mathrm{H}^1(T;\mathbb{F}_2) \to \mathrm{H}^2(T;\mathbb{F}_2)$ and $\mathrm{H}^1(K;\mathbb{F}_2) \times \mathrm{H}^1(K;\mathbb{F}_2) \to \mathrm{H}^2(K;\mathbb{F}_2)$ are isometric to $\lambda(P)$ respectively $\lambda(Q)$. Show that the forms $\lambda(P)$ and $\lambda(Q)$ are not isometric.
Remark. This gives a new, rather fancy proof, that T and K are not homotopy equivalent.

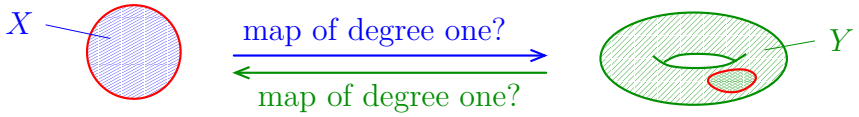
Exercise 203.14. Let $m \in \mathbb{N}$. We define an *m-component link* to be a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1. It follows from Exercise 185.7 or Exercise 194.6, the Basics-of-Tubular Neighborhood Lemma 57.10 (3a) and the Universal Coefficient-H*-Theorem 201.23 that for any m-component link $L \subseteq S^3$ we have

$$\mathrm{H}^k(S^3 \setminus L;\mathbb{Z}) \;\cong\; \begin{cases} 0, & \text{if } k \geq 3, \\ \mathbb{Z}^{m-1}, & \text{if } k = 2, \\ \mathbb{Z}^m, & \text{if } k = 1, \\ \mathbb{Z}, & \text{if } k = 0. \end{cases}$$

Thus cohomology with $\mathbb{Z}$-coefficients cannot be used to distinguish two links with the same number of components. Are there links with the same number of components but such that the complements have non-isomorphic cohomology rings?

Exercise 203.15. Let $T = S^1 \times S^1$ be the torus and let $f\colon T \to T$ be a continuous map of degree $k \neq 0$. Show that there exists a covering $p\colon \widetilde{T} \to T$ of degree k such that f lifts to a continuous map $\widetilde{f}\colon T \to \widetilde{T}$ of degree one.

Exercise 203.16. Let $\Phi\colon \overline{B}^2 \to S^1 \times S^1$ be a smooth embedding. We set $X := \overline{B}^2$ and we set $Y := (S^1 \times S^1) \setminus \Phi(B^2)$.
(a) Does there exist a continuous map $X \to Y$ of degree one?
(b) Does there exist a continuous map $Y \to X$ of degree one?



Exercise 203.17. Let $g \in \mathbb{N}$. We consider the surface Σ_g of genus g. In the following let $\varphi\colon \mathrm{H}_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g)$ be an isomorphism. Does there necessarily exist a continuous map $f\colon \Sigma_g \to \Sigma_g$ such that $f_* = \varphi$?

Exercise 203.18. Does there exist a topological space X such that the cup product

$$\mathrm{H}^1(X;\mathbb{Z}) \times \mathrm{H}^1(X;\mathbb{Z}) \;\to\; \mathrm{H}^2(X;\mathbb{Z})$$

is zero but such that the cup product

$$\mathrm{H}^1(X;\mathbb{Z}/3\mathbb{Z}) \times \mathrm{H}^1(X;\mathbb{Z}/3\mathbb{Z}) \ \to \ \mathrm{H}^2(X;\mathbb{Z}/3\mathbb{Z})$$

is non-zero?

CHAPTER 204

The relative cup product

Our goal in this chapter is to introduce a relative version of the cup product. More precisely, given an excisive triad (X, A, B) and a commutative ring R we will introduce a cup product

$$\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \xrightarrow{\cup} \mathrm{H}^{p+q}(X, A \cup B; R)$$

that is natural in an appropriate sense and that equals the previous definition of the cup product for the special case that $A = B = \varnothing$. This relative cup has many applications:

(1) It will help us in determining the absolute cup products of suspensions of topological spaces and of wedges of topological spaces.
(2) It can be used to define invariants of topological manifolds with boundary.
(3) In the next chapter we will see that it gives us lower bounds on the mysterious sounding Lusternik-Schnirelmann category.

This chapter is organized as follows:

- In Section 204.1 we will (re-) introduce excisive triads.
- In Section 204.2 we will provide the definition of the relative cup product.
- In Section 204.3 we will study properties of the relative cup product.
- In Section 204.4 we will perform some calculations of the "absolute" cup product using the relative cup product.
- In Section 204.5 we will study the relationship between the cup product and connecting homomorphisms.
- In Section 204.6 we will provide exercises for Chapter 204.

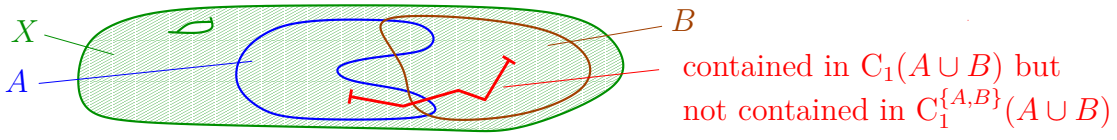
In the next Chapter 205 we will introduce the weirdly named Lusternik-Schnirelmann category of a topological space.

204.1. **Excisive triads.** In the definition of the relative cup product, and later, in the definition of the relative cap product, we will have to work with the notion of an excisive triad which we introduced in Definition 158.18. We start out by recalling Definition 148.8.

Definition 204.1. Let X be a topological space and let $A, B \subseteq X$ be subsets of X. We consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\{A,B\}}(X) := \mathrm{C}_n(A) + \mathrm{C}_n(B) = \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \begin{array}{l}\text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } A \text{ or } B\end{array} \right\}.$$

In Definition 148.8 we saw that these groups define a subcomplex of $\mathrm{C}_*(X)$.



Next we recall the technical, but very useful, definition of an (excisive) triad of topological spaces from Definition 158.18:

Definition 204.2.
(1) A triad of topological spaces is a triple (X, A, B) where X is a topological space and where both A and B are subspaces of X.
(2) We say that a triad (X, A, B) is excisive if the inclusion map
$$\mathrm{C}_*^{\{A,B\}}(A \cup B) \ \to \ \mathrm{C}_*(A \cup B)$$
is a chain homotopy equivalence.

Next we recall the following proposition which gives us many examples of excisive triads:

Proposition 158.20. (Excisive Triad-Proposition) Let (X, A, B) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:
(0) (a) $A \subseteq B$ or $B \subseteq A$.
(b) $A = \varnothing$ or $B = \varnothing$ or $A = B$.
(c) A_1 and A_2 are both unions of path components of $A_1 \cup A_2$.
(1) (a) $A \cup B = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A and B in $A \cup B$.
(b) A and B are both open subsets of $A \cup B$.
(2) $A \cup B$ is a topological manifold and A and B are codimension-zero submanifolds which are both closed subsets and such that each component of $A \cap B$ is a boundary component of A and a boundary component of B.
(3) X is a CW-complex and A and B are both subcomplexes.
(4) Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$ be good points in the sense of Definition 99.8. Then $((A, a_0) \vee (B, b_0), A, B)$ is an excisive triad.

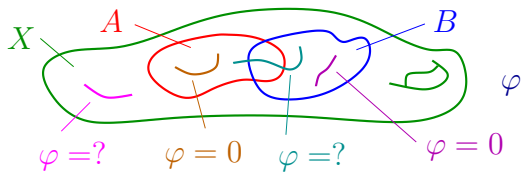


(1) (2) (3) (4)

Definition 204.3. Let (X, A, B) be a triad of topological spaces. For each $k \in \mathbb{N}_0$ we write
$$\mathrm{C}_k(X, \{A, B\}) \ := \ \frac{\mathrm{C}_k(X)}{\mathrm{C}_k^{\{A,B\}}(A \cup B)} \ = \ \frac{\mathrm{C}_k(X)}{\mathrm{C}_k(A) + \mathrm{C}_k(B)}.$$
Since $\mathrm{C}_*^{\{A,B\}}(A \cup B)$ is a subcomplex of $\mathrm{C}_*(X)$ we can view $\mathrm{C}_*(X, \{A, B\})$ also as a chain complex. For any abelian group G and any $k \in \mathbb{N}_0$ we define
$$\begin{aligned}
\text{the chain complex} \quad & \mathrm{C}_k(X, \{A, B\}; G) \ := \ \mathrm{C}_k(X, \{A, B\}) \otimes G \\
\text{with homology groups} \quad & \mathrm{H}_k(X, \{A, B\}; G) \ := \ \mathrm{H}_k(\mathrm{C}_*(X, \{A, B\}; G)) \\
\text{the cochain complex} \quad & \mathrm{C}^k(X, \{A, B\}; G) \ := \ \mathrm{Hom}(\mathrm{C}_k(X, \{A, B\}), G) \\
\text{with cohomology groups} \quad & \mathrm{H}^k(X, \{A, B\}; G) \ := \ \mathrm{H}^k(\mathrm{C}^*(X, \{A, B\}; G)).
\end{aligned}$$

Remark. Let (X, A, B) be a triad of topological spaces.
Note that any element in $\mathrm{C}^k(X, \{A, B\}; G)$ corresponds to a cochain $\varphi\colon \mathrm{C}_k(X) \to G$ such that $\varphi(\sigma) = 0$ for all singular simplices $\sigma\colon \Delta^k \to X$ that take values in A or B. □

The definition of the relative cup product (and later the relative cap product) relies very much on the following lemma:



$$\varphi \in \mathrm{C}^1(X, \{A, B\}; G) = \mathrm{Hom}\left(\frac{\mathrm{C}_1(X)}{\mathrm{C}_1(A) + \mathrm{C}_1(B)}, G\right)$$

Lemma 204.4. (Excisive Triad–Invariants Lemma) Let (X, A, B) be an excisive triad of topological spaces.

(1) The natural chain map $\quad \iota\colon \mathrm{C}_*(X, \{A, B\}) \to \mathrm{C}_*(X, A \cup B)$

is a chain homotopy equivalence.

(2) For any abelian group G and any $k \in \mathbb{N}_0$ the natural maps

(a) $\quad \iota_*\colon \mathrm{H}_k(X, \{A, B\}; G) \to \mathrm{H}_k(X, A \cup B; G)$

(b) $\quad \iota^*\colon \mathrm{H}^k(X, A \cup B; G) \to \mathrm{H}^k(X, \{A, B\}; G)$

are isomorphisms.

Proof.

(1) Note that we have the following commutative diagram of short exact sequences of chain complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_*^{\{A,B\}}(A \cup B) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \overbrace{\dfrac{\mathrm{C}_*(X)}{\mathrm{C}_*^{\{A,B\}}(A \cup B)}}^{=\mathrm{C}_*(X,\{A,B\})} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \iota & & \\
0 & \longrightarrow & \mathrm{C}_*(A \cup B) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X, A \cup B) & \longrightarrow & 0.
\end{array}$$

Claim. All groups involved are free abelian.

Proof. By definition each $\mathrm{C}_k(X)$ and $\mathrm{C}_k(A \cup B)$ is free abelian. Next note that $\mathrm{C}_k^{\{A,B\}}(A \cup B)$ is the free abelian group on the set of singular k-simplices that lie in A or B. On page 2798 we already remarked that relative chain groups $\mathrm{C}_k(X, A \cup B)$ are free abelian. Finally note that the same same argument shows that the top right group is also free abelian, in fact a basis is given by all singular simplices of X that are not contained in A or B. ⊞

By our hypothesis the triad (X, A, B) is excisive. By definition this means that the left vertical map is a chain homotopy equivalence. The middle vertical map is the identity. It follows from this discussion together with the Two-Out-Of-Three-Homotopy Equivalence Corollary 158.9 that the right vertical map is also a chain homotopy equivalence.

(2) (a) This statement follows from (1) together with the Homology-with-Coefficients Lemma 160.6 (2b).

(b) This statement follows from (1) together with the Dual Cochain Map Lemma 197.20 (3b) and the Cochain Homotopy-H*-Lemma 197.14. ■

204.2. **Definition of the relative cup product.** In this section we will apply the rather dry discussion from the last section to introduce the relative cup product.

The relative cup product is defined on relative cohomology. In this section and the following sections we will thus work a lot with relative cochains. We will adopt the following sanity preserving notation:

Notation 204.5. Let (X, A) be a pair of topological spaces and let G be an abelian group. Given $\varphi \in \mathrm{C}^k(X, A; G) = \mathrm{Hom}(\mathrm{C}_k(X, A), G)$ we denote by φ also the corresponding homomorphism $\mathrm{C}_k(X) \to \mathrm{C}_k(X, A) \to G$.

The following lemma gives us a first version of a relative cup product:

Lemma 204.6. (Triad–Cup Product Lemma) Let (X, A, B) be a triad of topological spaces and let R be a commutative ring.

(1) The cup product

$$\begin{aligned} \cup\colon \mathrm{C}^p(X;R)\times\mathrm{C}^q(X;R) &\to \mathrm{C}^{p+q}(X;R) \\ (\varphi,\psi) &\mapsto \begin{pmatrix} \mathrm{C}_{p+q}(X) & \to & R \\ \sigma & \mapsto & \varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \end{pmatrix} \end{aligned}$$

from Definition 203.11 descends to a well-defined R-bilinear map

$$\begin{aligned} \cup\colon \mathrm{C}^p(X,A;R)\times\mathrm{C}^q(X,B;R) &\to \mathrm{C}^{p+q}(X,\{A,B\};R) \\ (\varphi,\psi) &\mapsto \begin{pmatrix} \frac{\mathrm{C}_{p+q}(X)}{\mathrm{C}_{p+q}(A)+\mathrm{C}_{p+q}(B)} & \to & R \\ [\sigma] & \mapsto & (\varphi\cup\psi)(\sigma) \end{pmatrix}. \end{aligned}$$

(2) The map from (1) defines an R-bilinear map

$$\cup\colon \mathrm{H}^p(X,A;R)\times\mathrm{H}^q(X,B;R) \to \mathrm{H}^{p+q}(X,\{A,B\};R).$$

Proof.

(1) This statement follows easily from the following claim:

Claim. Let $\varphi \in \mathrm{C}^p(X, A; R)$ and $\psi \in \mathrm{C}^q(X, B; R)$. The homomorphism

$$\begin{aligned} \mathrm{C}_{p+q}(X) &\to R \\ (\sigma\colon \Delta^{p+q}\to X) &\mapsto \varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \end{aligned}$$

vanishes on the subgroup $\mathrm{C}_{p+q}(A) + \mathrm{C}_{p+q}(B)$.

Proof. We have to show that the above cochain vanishes on all singular $(p+q)$-simplices that lie entirely in A or that lie entirely in B. But this is clear:

(a) Given $\sigma\colon \Delta^{p+q} \to A$ the first factor $\varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p])$ is necessarily zero since we have $\varphi \in \mathrm{C}^p(X, A; R)$.

(b) Given $\sigma\colon \Delta^{p+q} \to B$ the second factor $\psi(\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}])$ is necessarily zero since we have $\psi \in \mathrm{C}^q(X, B; R)$. ⊞

(2) Using the Cup Product–Coboundary Lemma 203.12 one can easily show that we get an induced cup product on cohomology groups. ■

The relative cup product from the Triad–Cup Product Lemma 204.6 is not all that interesting, since it takes values in the weird cohomology group $\mathrm{H}^{p+q}(X, \{A, B\}; R)$. Fortunately, if we are dealing with an excisive triad, then the Excisive Triad–Invariants Lemma 204.4 (2b) lets us move on to a much more reasonable relative cohomology group. More precisely, the Triad–Cup Product Lemma 204.6 together with the Excisive Triad–Invariants Lemma 204.4 (2b) allows us to introduce the following definition:

Definition 204.7. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We refer to the composition

$$\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \xrightarrow[\uparrow]{(c,d)\mapsto c\cup d} \mathrm{H}^{p+q}(X, \{A, B\}; R) \xleftarrow[\uparrow]{\cong} \mathrm{H}^{p+q}(X, A \cup B; R)$$

well-defined by the Triad–Cup Product Lemma 204.6

isomorphism by the Excisive Triad Invariants Lemma 204.4 (2b), here we use that (X, A, B) is excisive

again as a cup product (or sometimes as a relative cup product) and we denote it as usual by the "$\cup$"-symbol.

Remark. Let X be a topological space, let $A, B \subseteq X$ be subsets and let R be a commutative ring. Suppose that $\mathrm{C}_*^{\{A,B\}}(A \cup B) = \mathrm{C}_*(A \cup B)$. (This is for example the case if we are in the setting of Excisive Triad-Proposition 158.20 (0), e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$.) Then in the above definition the horizontal map to the right is the identity and we see that the cup product is just given by the "naive" definition

$$\begin{aligned} \mathrm{C}^p(X, A; R) \times \mathrm{C}^q(X, B; R) &\to \mathrm{C}^{p+q}(X, A \cup B; R) \\ (\varphi, \psi) &\mapsto \Big(\mathrm{C}_{p+q}(X, A \cup B) \to R, \\ &\qquad \sigma \mapsto \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}])}_{\in R} \Big). \end{aligned}$$

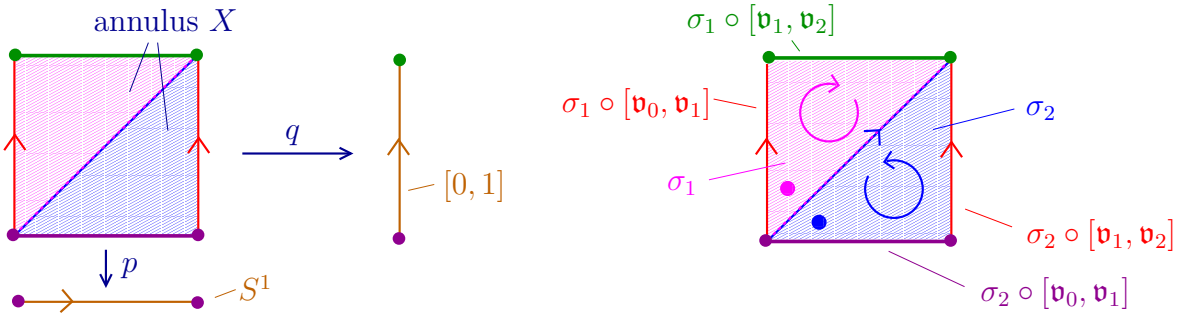
□

Example. We consider the annulus $X = S^1 \times [0, 1]$, the subspace $A = \partial X = S^1 \times \{0, 1\}$ and the subspace $B = \varnothing$. We want to understand the relative cup product

$$\begin{array}{ccccc} \mathrm{H}^1(X; \mathbb{Z}) \times \mathrm{H}^1(X, \partial X; \mathbb{Z}) & \to & \mathrm{H}^2(X, \partial X; \mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \varphi \cup \psi & & \\ & & \theta & \mapsto & \langle \theta, [X] \rangle. \end{array}$$

To perform this calculation we need explicit cocycles and an explicit representative of the fundamental class $[X] \in \mathrm{H}_2(X, \partial X)$. We introduce and recall the following notation:

(1) We denote by $p\colon X \to S^1$ and $q\colon X \to [0, 1]$ the projections.
(2) Let $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1, \mathbb{Z})$ be the 1-cocycle from page 3813.
(3) Let $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0, 1], \{0, 1\}, \mathbb{Z})$ be the relative 1-cocycle from page 3816 and 3821.
(4) Let $\sigma_1, \sigma_2\colon \Delta^2 \to X$ be the singular simplices shown in the figure below.



An explicit calculation, as on page 3896, shows that

$$\begin{aligned}(p^*(\theta_{\mathbb{Z}}) \cup q^*(\overline{\gamma}_{\frac{1}{2}}))(\sigma_1) &= p^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) \cdot q^*(\overline{\gamma}_{\frac{1}{2}})(\sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1]}_{\text{constant simplex}}) \cdot \overline{\gamma}_{\frac{1}{2}}(\underbrace{q \circ \sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2]}_{\text{constant simplex}}) = 0 \cdot 0 = 0.\end{aligned}$$

$$\begin{aligned}(p^*(\theta_{\mathbb{Z}}) \cup q^*(\overline{\gamma}_{\frac{1}{2}}))(\sigma_2) &= p^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) \cdot q^*(\overline{\gamma}_{\frac{1}{2}})(\sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1]}_{\substack{\text{the simplex } \Delta^1 \to S^1 \\ \text{that is given} \\ \text{by } (1\text{–}t, t) \mapsto \exp(2\pi i t)}}) \cdot \overline{\gamma}_{\frac{1}{2}}(\underbrace{q \circ \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2]}_{\substack{\text{the simplex } \Delta^1 \to [0,1] \\ \text{that is given} \\ \text{by } (1\text{–}t, t) \mapsto t}}) \overset{\substack{\uparrow \\ \text{by the explicit} \\ \text{definitions} \\ \text{of } \theta_{\mathbb{Z}} \text{ and } \overline{\gamma}_{\frac{1}{2}}}}{=} 1 \cdot 1 = 1.\end{aligned}$$

We now see that

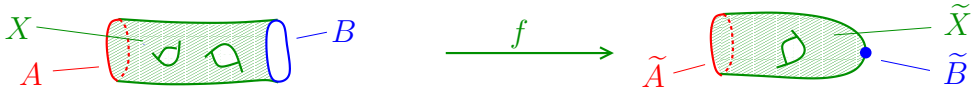
$$\langle p^*([\theta_{\mathbb{Z}}]) \cup q^*([\overline{\gamma}_{\frac{1}{2}}]), [X]\rangle \overset{\substack{\uparrow \\ \text{it follows easily from the} \\ \text{Fundamental Class-Criterion 194.8} \\ \text{that } [-\sigma_1 + \sigma_2] = [X] \in \mathrm{H}_2(X, \partial X)}}{=} \langle [p^*(\theta_{\mathbb{Z}}) \cup q^*(\overline{\gamma}_{\frac{1}{2}})], [-\sigma_1 + \sigma_2]\rangle \overset{\substack{\uparrow \\ \text{by the above calculations} \\ \text{and the above remark on} \\ \text{relative cup products}}}{=} -0 + 1 = 1.$$

□

204.3. **Properties of the relative cup product.** In the following we summarize a few properties of the relative cup product. We start out with the following lemma, which is a generalization of the Cup Product-Naturality Lemma 203.9.

Lemma 204.8. (Cup Product-Naturality Lemma) Let (X, A, B) and $(\widetilde{X}, \widetilde{A}, \widetilde{B})$ be two excisive triads and let R be a commutative ring. Let $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ be a continuous map with $f(A) \subseteq \widetilde{A}$ and $f(B) \subseteq \widetilde{B}$. For all $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(\widetilde{X}, \widetilde{A}; R) \times \mathrm{H}^q(\widetilde{X}, \widetilde{B}; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\widetilde{X}, \widetilde{A} \cup \widetilde{B}; R) \\ \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A \cup B; R). \end{array}$$



Proof of Lemma 204.8. We have to prove the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^p(\widetilde{X}, \widetilde{A}; R) \times \mathrm{H}^q(\widetilde{X}, \widetilde{B}; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\widetilde{X}, \{\widetilde{A}, \widetilde{B}\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(\widetilde{X}, \widetilde{A} \cup \widetilde{B}; R) \\ \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, \{A, B\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(X, A \cup B; R). \end{array}$$

Proof. The second argument in the proof of the Cup Product-Ring Proposition 203.25 (3) shows that the square to the left commutes. It follows immediately from the functoriality of cohomology that the square to the right commutes. ■

The following lemma says that the cup product is natural in the coefficient ring:

Lemma 204.9. (Cup Product-Coefficients Lemma) Let (X, A, B) be an excisive triad. If $\varphi\colon R \to S$ is a homomorphism between commutative rings, then for all $p, q \in \mathbb{N}_0$

the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(X,A;R)\times\mathrm{H}^q(X,B;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X,A\cup B;R) \\ \downarrow{\scriptstyle\varphi_*}\quad\downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\varphi_*} \\ \mathrm{H}^p(X,A;S)\times\mathrm{H}^q(X,B;S) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X,A\cup B;S). \end{array}$$

Here the vertical maps are the homomorphism defined in the Cohomology-Coefficients-Functor Lemma 197.21.

Proof. This proposition follows, similar to the Cup Product-Naturality Lemma 204.8, basically immediately from the definitions. ■

We continue with the obvious generalization of the Cup Product-Commutativity Proposition 203.20:

Proposition 204.10. (Cup Product-Commutativity Proposition) Let (X,A,B) be an excisive triad of topological spaces and let R be a commutative ring. For any cohomological classes $\varphi\in\mathrm{H}^p(X,A;R)$ and $\psi\in\mathrm{H}^q(X,B;R)$ we have

$$\varphi\cup\psi \;=\; (-1)^{pq}\cdot\psi\cup\varphi \;\in\; \mathrm{H}^{p+q}(X,A\cup B;R).$$

Proof. Let (X,A,B) be an excisive triad of topological spaces and let R be a commutative ring. Given a singular n-simplex $\sigma\colon\Delta^n\to X$ we denote, as on page 3902, by $\overline{\sigma}\colon\Delta^n\to X$ the singular n-simplex that is obtained from σ by precomposing σ with the affine linear map $[\mathfrak{v}_n,\ldots,\mathfrak{v}_0]\colon\Delta^n\to\Delta^n$ that reverses the order of the vertices. Furthermore we write $\epsilon_n=(-1)^{n(n+1)/2}$.

Claim. The maps

$$\begin{array}{rccc} \theta\colon & \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ & [\sigma] & \mapsto & [\epsilon_n\cdot\overline{\sigma}] \end{array}$$

form a chain map and there exists a chain homotopy $P_n\colon\mathrm{C}_n(X)\to\mathrm{C}_{n+1}(X)$ from θ to the identity with the property, that for any subspace $S\subseteq X$ we have $P_n(\mathrm{C}_n(S))\subseteq\mathrm{C}_{n+1}(S)$.

Proof. On page 3902 we showed that θ is a chain map and we gave an explicit description of a chain homotopy $P_n\colon\mathrm{C}_n(X)\to\mathrm{C}_{n+1}(X)$ from θ to the identity. A quick look at the definition shows that it has the desired extra property, namely for any subspace $S\subseteq X$ we have $P_n(\mathrm{C}_n(S))\subseteq\mathrm{C}_{n+1}(S)$. ⊞

We now consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^p(X,A;R)\times\mathrm{H}^q(X,B;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X,\{A,B\};R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(X,A\cup B;R) \\ \downarrow{\scriptstyle\theta^*}\quad\downarrow{\scriptstyle\theta^*} & & \downarrow{\scriptstyle\theta^*} & & \downarrow{\scriptstyle\theta^*} \\ \mathrm{H}^p(X,A;R)\times\mathrm{H}^q(X,B;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X,\{A,B\};R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(X,A\cup B;R). \end{array}$$

Note that it follows from the claim that θ descends to the various vertical chain maps and that these are chain homotopic to the identity. It follows from this observation and the Dual Cochain Map Lemma 197.20 that the vertical maps are in fact the identity. So to deduce the statement of the proposition it suffices to show that for all $\varphi\in\mathrm{C}^p(X,A;R)$ and $\psi\in\mathrm{C}^q(X,B;R)$ we have

$$\theta^*(\psi\cup\varphi) \;=\; (-1)^{pq}\cdot\theta^*(\varphi)\cup\theta^*(\psi) \;\in\; \mathrm{C}^{p+q}(X,\{A,B\};R).$$

But this follows, exactly as on page 3902, immediately from the definitions. For peace of mind we recall the argument: Let $\sigma\colon \Delta^{p+q} \to X$ be a singular simplex. Then

$$\begin{aligned}(\theta^*(\psi \cup \varphi))(\sigma) &= (\psi \cup \varphi)(\theta_*(\sigma)) = (\psi \cup \varphi)(\epsilon_{p+q} \cdot \sigma \circ [\mathfrak{v}_{p+q}, \ldots, \mathfrak{v}_0]) \\ &= \epsilon_{p+q} \cdot \psi(\sigma \circ [\mathfrak{v}_{p+q}, \ldots, \mathfrak{v}_p]) \cdot \varphi(\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_0]) \\ &= (-1)^{pq} \cdot \varphi(\epsilon_p \cdot \sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_0]) \cdot \psi(\epsilon_q \cdot \sigma \circ [\mathfrak{v}_{p+q}, \ldots, \mathfrak{v}_p]) \\ &\quad \uparrow \\ &\quad \text{since } \epsilon_{p+q} = (-1)^{pq} \cdot \epsilon_p \cdot \epsilon_q \text{ and} \\ &\quad \text{since } R \text{ is a } \textit{commutative} \text{ ring} \\ &= (-1)^{pq} \cdot (\theta^*(\varphi) \cup \theta^*(\psi))(\sigma). \end{aligned}$$

■

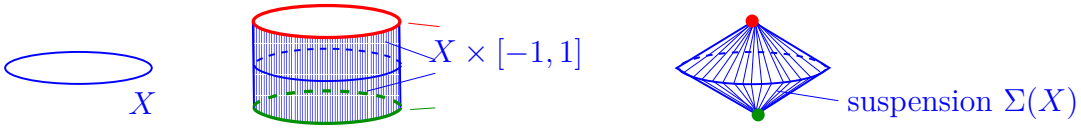
204.4. **Calculations using the relative cup product.** So what is the relative cup product good for? In this section we will give a surprising answer, namely we will see that we can use it to calculate "absolute" cup products. The key to doing so is the Cup Product-Naturality Lemma 204.8. To be more precise, in this section we will use relative cup products to calculate the cup products of the following types of topological spaces:

(1) Suspensions.
(2) Wedges of topological spaces.
(3) The surface of genus two (for which we give a new calculation).
(4) Disjoint unions.

204.4.1. **The cup product of suspensions.** In this subsection we study the cup product of the suspension $\Sigma(X)$ of a topological space X. For the reader's convenience we recall the definition of a suspension from Definition 20.21:

Definition 204.11. Given a topological space X we define its suspension $\Sigma(X)$ to be

$$\Sigma(X) \coloneqq ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\}).$$



In the Suspension-H*-Lemma 200.4 we calculated the cohomology groups of suspensions, more precisely, we showed that given any topological space X and any abelian group G we have a natural isomorphism

$$\Sigma_X\colon \widetilde{\mathrm{H}}^k(X;G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{k+1}(\Sigma(X);G).$$

The following lemma shows that the cup product of the suspension of a topological space is quite dull.

Lemma 204.12. (Suspension-Cup Product Lemma) Let X be a topological space and let R be a commutative ring. For any $k, l \geq 1$ the cup product

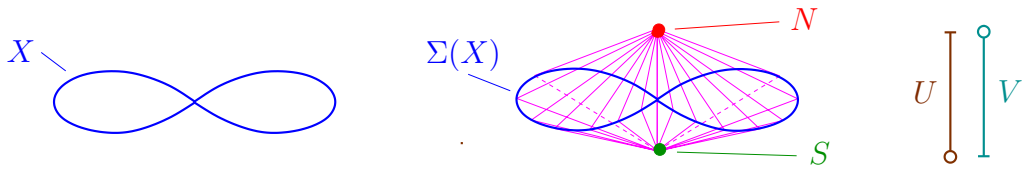
$$\mathrm{H}^k(\Sigma(X);R) \times \mathrm{H}^l(\Sigma(X);R) \to \mathrm{H}^{k+l}(\Sigma(X);R)$$

is the zero map.

Proof. We introduce the following notation:

(1) We denote by $N \coloneqq [X \times \{1\}]$ the "North Pole" and we denote by $S \coloneqq [X \times \{-1\}]$ the "South Pole" of $\Sigma(X)$.
(2) We write $U = \Sigma(X) \setminus \{S\}$ and $V = \Sigma(X) \setminus \{N\}$. By the same argument as in the Cone Contractability Lemma 20.20 we see that U admits deformation retraction to the

North Pole N. This shows that U is contractible. The same argument shows that V is contractible.



Now let $p, q \geq 1$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^p(\Sigma(X), U; R) \times \mathrm{H}^q(\Sigma(X), V; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\Sigma(X), \overbrace{U \cup V}^{=\Sigma(X)}; R) \\ \downarrow \cong & & \downarrow \\ \mathrm{H}^p(\Sigma(X); R) \times \mathrm{H}^q(\Sigma(X); R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\Sigma(X); R). \end{array}$$

We make the following observations and clarifications:

(1) It follows from the Excisive Triad-Proposition 158.20 (1) that the triad $(\Sigma(X), U, V)$ is excisive. It follows that the relative cup product on top is defined and it follows from the Cup Product-Naturality Lemma 204.8 that the diagram commutes.

(2) It follows immediately from the long exact sequence of cohomology groups corresponding to the pair $(\Sigma(X), U)$, the fact that U is homotopy equivalent to a point (together with the Singleton-H*-Lemma 198.10 and the Homotopy Equivalence-H*-Proposition 198.12 (1)) and the hypothesis that $p \geq 1$ that the inclusion-induced map $\mathrm{H}^p(\Sigma(X), U; R) \to \mathrm{H}^p(\Sigma(X); R)$ is an isomorphism. Evidently the same also holds if we replace U by V.

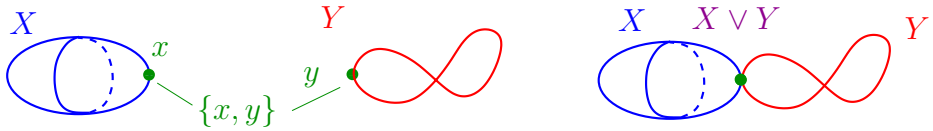
(3) Since $U \cup V = \Sigma(X)$ we see that the cohomology group on the top-right is zero.

It follows from the above discussion that the cup product on the bottom is zero. ■

204.4.2. **The cup product of wedges.** In this subsection we want to calculate the cup product of wedges. For the reader's convenience we recall the relevant Definition 96.1:

Definition 204.13. Let X and Y be two topological spaces and let $x \in X$ and $y \in Y$. We define the wedge

$$(X, x) \vee (Y, y) \coloneqq (X \sqcup Y)/\underbrace{\{x, y\}}_{\subseteq X \sqcup Y}.$$



The following lemma determines the cup product of $X \vee Y$ in terms of the cup products of X and Y. It can be viewed as a generalization of the Wedge-of-Spheres–Cup Product Lemma 203.29.

Lemma 204.14. (Wedge-Cup Product Lemma) Let X and Y be two topological spaces, let $x \in X$ and $y \in Y$ be good[2172] points and let R be a commutative ring. We

denote by $p\colon X \vee Y \to X$ and $q\colon X \vee Y \to Y$ the natural projections which we introduced in Definition 96.1.

(1) By the Wedge-H*-Proposition 200.10 we know that for every $n \in \mathbb{N}$ the map
$$p^* \oplus q^*\colon \mathrm{H}^n(X;R) \oplus \mathrm{H}^n(Y;R) \xrightarrow{\cong} \mathrm{H}^n(X \vee Y;R)$$
is an isomorphism.

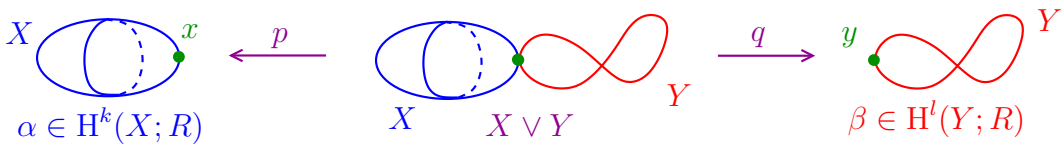
(2) For any $\alpha_1, \alpha_2 \in \mathrm{H}^*(X;R)$ and $\beta_1, \beta_2 \in \mathrm{H}^*(Y;R)$ in degrees ≥ 1 the following statements hold in $\mathrm{H}^*(X \vee Y;R)$:

$\cup$	$p^*(\alpha_2)$	$q^*(\beta_2)$
$p^*(\alpha_1)$	$p^*(\alpha_1 \cup \alpha_2)$	0
$q^*(\beta_1)$	0	$q^*(\beta_1 \cup \beta_2)$

Proof. Note that we proved Statement (1) already in the Wedge-H*-Proposition 200.10. So it remains to prove Statement (2).

The statement regarding the "diagonal" terms of the table follows immediately from the Cup Product-Naturality Lemma 203.9. It remains to prove that the off-diagonal terms are zero. We show that the top-right entry is zero. The proof that the bottom-left entry is zero is basically identical.

Thus let $\alpha \in \mathrm{H}^k(X;R)$ and $\beta \in \mathrm{H}^l(Y;R)$ with $k, l \geq 1$. We want to show that $p^*(\alpha) \cup q^*(\beta) = 0$.



We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k(X \vee Y, Y;R) \times \mathrm{H}^l(X \vee Y, X;R) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(X \vee Y, \overbrace{X \cup Y}^{=X \vee Y};R) \\
\downarrow{\scriptstyle g^*} & & \downarrow \\
\mathrm{H}^k(X \vee Y;R) \times \mathrm{H}^l(X \vee Y;R) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(X \vee Y;R). \\
{\scriptstyle p^*(\alpha)} \quad {\scriptstyle q^*(\beta)} & &
\end{array}$$

We make the following observations and clarifications:

(1) As we discussed on page 1864, we can view X and Y as subspaces of $X \vee Y$.

(2) We denote by $f\colon (X \vee Y, \varnothing) \to (X \vee Y, X)$ and $g\colon (X \vee Y, \varnothing) \to (X \vee Y, Y)$ the two inclusion maps of pairs of topological spaces.

(3) It follows from the hypothesis that $x \in X$ and $y \in Y$ are good and the Excisive Triad-Proposition 158.20 (4) that the triad $(X \vee Y, X, Y)$ is excisive. It follows that the relative cup product on top is defined and it follows from the Cup Product-Naturality Lemma 204.8 that the diagram commutes.

Since the top right group is clearly zero it remains to prove the following claim:

Claim.

(a) There exists a $\widetilde{\alpha} \in \mathrm{H}^k(X \vee Y, Y;R)$ with $g^*(\widetilde{\alpha}) = p^*(\alpha)$.

[2172] We refer to Definition 99.8 for the definition of a good point.

(b) There exists a $\widetilde{\beta} \in \mathrm{H}^l(X \vee Y, X; R)$ with $f^*(\widetilde{\beta}) = q^*(\beta)$.

Proof. We provide a proof of (a). The proof of (b) is of course basically identical. To prove (a) we consider the following diagrams[2173]

$$\begin{array}{ccc} \mathrm{H}^k(X, \{x_0\}; R) & \xrightarrow{p^*} & \overset{?\,\widetilde{\alpha}\,?}{\mathrm{H}^k(X \vee Y, Y; R)} \\ \cong\big\downarrow \text{я} & & \big\downarrow g^* \\ \underset{\alpha}{\mathrm{H}^k(X; R)} & \xrightarrow{p^*} & \underset{p^*(\alpha)}{\mathrm{H}^k(X \vee Y; R)}. \end{array}$$

The claim is an immediate consequence of the following two observations:

- It follows from the functoriality of cohomology that the diagram commutes.
- It follows easily from the long exact sequence in cohomology of a pair of topological spaces and the fact that $k \geq 1$ that the left vertical map is an isomorphism. ■

204.4.3. **The cup product of surfaces.** The following lemma, together with the Torus–Cup Product Lemma 203.28, gives us a new proof of the Surface–Cup Product Lemma 203.30.

Lemma 204.15. (Surface–Torus Cup Product Lemma) Let Σ be the surface of genus 2. Recall that in Lemma 199.13 we introduced two projections $p_i \colon \Sigma \to T_i = S^1 \times S^1$ for $i = 1, 2$.

(1) The map $$p_1^* \oplus p_2^* \colon \mathrm{H}^1(T_1; \mathbb{Z}) \oplus \mathrm{H}^1(T_2; \mathbb{Z}) \;\to\; \mathrm{H}^1(\Sigma; \mathbb{Z})$$ is an isomorphism.

(2) The maps $p_i^* \colon \mathrm{H}^2(T_i; \mathbb{Z}) \to \mathrm{H}^2(\Sigma; \mathbb{Z})$ are isomorphisms.

(3) For any $\alpha_1, \beta_1 \in \mathrm{H}^1(T_1; R)$ and $\alpha_2, \beta_2 \in \mathrm{H}^1(T_2; R)$ the equality hold in $\mathrm{H}^2(\Sigma; R)$:

$\cup$	$p_1^*(\beta_1)$	$p_2^*(\beta_2)$
$p_1^*(\alpha_1)$	$p_1^*(\alpha_1 \cup \beta_1)$	0
$p_2^*(\alpha_2)$	0	$p_2^*(\alpha_2 \cup \beta_2)$

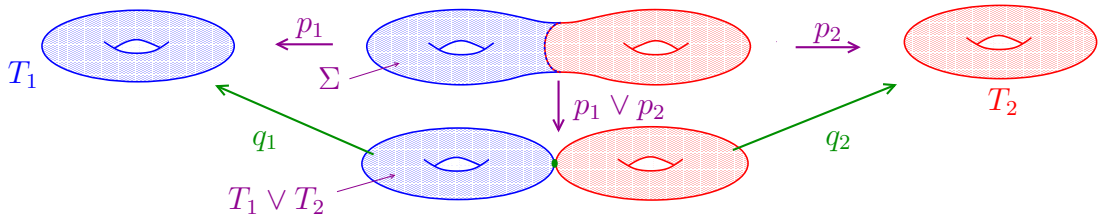
Proof.

(1) We proved this statement in Surface-H^1-Lemma 199.13.

(2) It follows easily from the Local-Global Degree Proposition 195.6 that the projections $p_i \colon \Sigma \to T_i$ are degree-one maps. It is then a consequence of the Dual Fundamental Class–Degree Lemma 201.28 that the maps $p_i^* \colon \mathrm{H}^2(T_i; \mathbb{Z}) \to \mathrm{H}^2(\Sigma; \mathbb{Z})$ are isomorphisms.

(3) We make the following observations:
   (a) The diagonal entries of the matrix are just a special case of the Cup Product-Naturality Lemma 203.9.
   (b) We denote by $q_i \colon T_1 \vee T_2 \to T_i$, $i = 1, 2$ the obvious projections. Note that the two projections p_1, p_2 give rise to a continuous map $p_1 \vee p_2 \colon \Sigma \to T_1 \vee T_2$ which has the property that $q_i \circ (p_1 \vee p_2) = p_i$, $i = 1, 2$.

[2173] Recall that in $X \vee Y$ the points x and y get identified, so we can view x as a point in Y and we can view y as a point in X.



For any $\varphi_1 \in \mathrm{H}^1(T_1;\mathbb{Z})$ and $\varphi_2 \in \mathrm{H}^1(T_2;\mathbb{Z})$ we calculate that

$$p_1^*(\varphi_1) \cup p_2^*(\varphi_2) \underset{\uparrow}{=} (p_1 \vee p_2)^*(q_1^*(\varphi_1) \cup q_2^*(\varphi_2)) \underset{\uparrow}{=} (p_1 \vee p_2)^*(0) = 0.$$

follows from the Cup Product Naturality Lemma 203.9, since $q_i \circ (p_1 \vee p_2) = p_i$

by the Wedge-Cup Product Lemma 204.14

Of course the same argument works with the roles of the first and second torus swapped. ■

204.4.4. **The cup product of disjoint unions.** The next lemma says that the relative cup product behaves in the obvious way with respect to the path-components.

Lemma 204.16. (Cup Product–Disjoint Union Lemma) Let X be a topological space and let R be a commutative ring. We denote by $\{X_j\}_{j\in J}$ the path-components of X. By the Path-Component-H*-Lemma 198.8 the inclusion maps $\iota_j\colon X_j \to X$, $j \in J$, induce for any $n \in \mathbb{N}_0$ an isomorphism

$$\Psi := \prod_{j\in J} \iota_j^*\colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \prod_{j\in J} \mathrm{H}^n(X_j;G).$$

For any $i \neq j$ and any $\alpha \in \mathrm{H}^m(X_i;R)$ and $\beta \in \mathrm{H}^n(X_j;R)$ we have

$$\Psi^{-1}(\alpha) \cup \Psi^{-1}(\beta) = 0.$$

The same type of statement also holds for cup products on relative cohomology groups.

Proof. The statements follow easily from the explicit second definition of the cup product. For fun we also give a slightly more formal proof, which is in the spirit of the Wedge-Cup Product Lemma 204.14. This proof does not go back to the actual definitions and it uses just the general properties that we have already found.

Let $\alpha \in \mathrm{H}^m(X_i;R)$ and $\beta \in \mathrm{H}^n(X_j;R)$. We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^m\big(X, \bigsqcup_{k\neq i} X_k; R\big) \times \mathrm{H}^n\big(X, \bigsqcup_{k\neq j} X_k; R\big) & \xrightarrow{\cup} & \mathrm{H}^{m+n}\big(X, \overbrace{\bigsqcup_k X_k}^{=X}; R\big) \\
\downarrow \quad \downarrow & & \downarrow \\
\mathrm{H}^m(X;R) \times \mathrm{H}^n(X;R) & \xrightarrow{\cup} & \mathrm{H}^{m+n}(X;R).
\end{array}$$

We make the following observations:

(1) Note that for any $l \in I$ and any $d \in \mathbb{N}_0$ we have the following commutative diagram:

$$\begin{array}{ccc}
 & \mathrm{H}^d\big(X, \bigsqcup_{k\neq l} X_k; R\big) & \\
\swarrow & & \searrow^{\cong} \\
\mathrm{H}^d(X;R) & \xrightarrow{\iota_l^*} & \mathrm{H}^d(X_l;R).
\end{array}$$

One can easily verify that the right diagonal map is an isomorphism.

(2) It follows easily from (1) that $\Psi^{-1}(\alpha)$ and $\Psi^{-1}(\beta)$ lie in the image of the two vertical maps in the above diagram.

(3) It follows from the Cup Product-Naturality Lemma 204.8, the Components-are-Closed Lemma 4.49 and the Excisive Triad-Proposition 158.20 (0c) that the above rectangular shaped diagram commutes.

(4) On the top right we have the zero group.

It follows from the above discussion that $\Psi^{-1}(\alpha) \cup \Psi^{-1}(\beta) = 0$. ■

204.5. **The cup product and connecting homomorphisms.** In this last section we will study the connection between cup products and connecting homomorphisms. It is perhaps best to look up the lemmas, when they actually get used, and to move straight to Chapter 205.

Before we can state the relevant lemmas let us introduce the following convention:

Convention 204.17. Let A, B, C, D, X and Y be sets, let $f\colon C \to A$, $g\colon B \to D$ and $h\colon X \to Y$ be maps and let $\varphi\colon A \times B \to X$ and $\psi\colon C \times D \to Y$ be maps. We say that the diagram

$$\begin{array}{ccc} A \times \overset{b}{B} & \xrightarrow{\varphi} & X \\ {\scriptstyle f}\uparrow \quad \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \underset{c}{C} \times D & \xrightarrow{\psi} & Y \end{array}$$

commutes, if for all $c \in C$ and $b \in B$ we have

$$\psi(c, g(b)) \;=\; h(\varphi(f(c), b)).$$

A similar convention holds with the directions of f and g reversed. Furthermore, if we are dealing with homomorphisms of abelian groups, then there is an obvious notion of "commutes up to a given sign".

Using this convention we can now formulate the first lemma of this section:

Lemma 204.18. (Cup Product-Connecting Homomorphism Lemma) Let (X, A) be a pair of topological spaces and let R be a commutative ring. We denote by $i\colon A \to X$ the inclusion map and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A). For any $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(A; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A; R) \\ {\scriptstyle i^*}\uparrow \quad \downarrow{\scriptstyle \delta_q} & & \downarrow{\scriptstyle \delta_{p+q}} \\ \mathrm{H}^p(X; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, A; R) \end{array}$$

up to the sign $(-1)^p$. In other words, for any $\varphi \in \mathrm{H}^p(X; R)$ and any $\psi \in \mathrm{H}^q(A; R)$ we have

$$\varphi \cup \delta_q(\psi) \;=\; (-1)^p \cdot \delta_{p+q}(i^*(\varphi) \cup \psi) \;\in\; \mathrm{H}^{p+q+1}(X, A; R).$$

Proof. We pick cocycles $\varphi \in \mathrm{C}^p(X; R)$ and $\psi \in \mathrm{C}^q(A; R)$. In the following we denote by $\widetilde{\psi}\colon \mathrm{C}^q(X) \to R$ the extension of ψ that is defined to be zero on all singular q-simplices that

do not lie in the subset A. We have the following equalities:

$$\begin{aligned}
\delta_{p+q}([i^*\varphi]\cup[\psi]) &= \delta_{p+q}([i^*\varphi\cup\psi]) \overset{\text{by the discussion of the connecting homomorphism in the remark on page 3819 and since } \varphi\cup\widetilde{\psi} \text{ is an extension}^{2174} \text{ of } i^*\varphi\cup\psi\colon \mathrm{C}_{p+q}(A)\to R \text{ to a homomorphism } \mathrm{C}_{p+q}(X)\to R}{=} [\delta_{p+q}(\varphi\cup\widetilde{\psi})] \\
&\overset{\text{by the Cup Product–Coboundary Lemma 203.12}}{=} [\delta_p(\varphi)\cup\widetilde{\psi}+(-1)^p\cdot(\varphi\cup\delta_q(\widetilde{\psi}))] \\
&\overset{\text{since } \varphi \text{ is a cocycle}}{=} (-1)^p\cdot[\varphi\cup\delta_q(\widetilde{\psi})] = (-1)^p\cdot([\varphi]\cup[\delta_q(\widetilde{\psi})]) \overset{\text{again by the remark on page 3819}}{=} (-1)^p\cdot([\varphi]\cup\delta_q([\psi])).
\end{aligned}$$

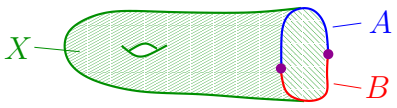
This equality that the given diagram commutes up to the sign $(-1)^p$. ∎

The Cup Product-Connecting Homomorphism Lemma 204.18 is in fact just the reader-friendly special case of the following lemma with $B=\varnothing$:

Lemma 204.19. (Cup Product-Connecting Homomorphism Lemma) Let (X,A,B) be an excisive triad of topological spaces and let R be a commutative ring. We denote by $i\colon (A,A\cap B)\to(X,B)$ and $j\colon (A,A\cap B)\to(A\cup B,B)$ the inclusion maps and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X,A) and the triple $(X,A\cup B,B)$. Note that it follows from the Excisive Characterization Lemma 158.19 (1)⇒(3) together with the Also-Iso-on-Cohomology Proposition 200.2 that $j^*\colon \mathrm{H}^{p+q}(A\cup B,B;R)\to\mathrm{H}^{p+q}(A,A\cap B;R)$ is an isomorphism. The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^p(A,A\cap B;R)\times\mathrm{H}^q(A;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A,A\cap B;R) \\
\uparrow{\scriptstyle i^*} \quad \downarrow{\scriptstyle \delta_q} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} \\
 & & \mathrm{H}^{p+q}(A\cup B,B;R) \\
 & & \downarrow{\scriptstyle \delta_{p+q}} \\
\mathrm{H}^p(X,B;R)\times\mathrm{H}^{q+1}(X,A;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X,A\cup B;R)
\end{array}$$

up to the sign $(-1)^p$.



excisive triad to which we will apply the Cup Product-Connecting Homomorphism Lemma 204.19

Sketch of proof. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^p(A,A\cap B;R)\times\mathrm{H}^q(A;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A,A\cap B;R) & \xleftarrow{\mathrm{id}} & \mathrm{H}^{p+q}(A,A\cap B;R) \\
\uparrow{\scriptstyle i^*} \quad \downarrow{\scriptstyle \delta_q} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} \\
 & & \mathrm{H}^{p+q}(\{A,B\},B;R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(A\cup B,B;R) \\
 & & \downarrow{\scriptstyle \delta_{p+q}} & & \downarrow{\scriptstyle \delta_{p+q}} \\
\mathrm{H}^p(X,B;R)\times\mathrm{H}^{q+1}(X,A;R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X,\{A,B\};R) & \xleftarrow{\cong} & \mathrm{H}^{p+q+1}(X,A\cup B;R).
\end{array}$$

[2174] Is this the trivial extension of $i^*\varphi\cup\psi$?

We make the following observations and clarifications:

(1) (a) We denote by $\mathrm{H}^k(\{A,B\},B;R)$ the cohomology of the dual cochain complex corresponding to the chain complex $\mathrm{C}_*^{\{A,B\}}(A\cup B)/\mathrm{C}_*(B)$.
(b) Clearly the obvious map $j_*\colon \mathrm{C}_k(A,A\cap B)\to \mathrm{C}_*^{\{A,B\}}(A\cup B)/\mathrm{C}_*(B)$ is an isomorphism of chain complexes. It follows that on cohomology we have $(j^*)^{-1}=((j_*)^{-1})^*$.
(2) The cup products on the left are given by the "naive" formula on the cochain level.
(3) It follows easily from (1), (2) and the argument of the proof of the Cup Product-Connecting Homomorphism Lemma 204.18 that the diagram to the left commutes up to the sign $(-1)^p$.
(4) The right part of the diagram commutes by functoriality of cohomology and the fact, shown in the Cohomology–Connecting Homomorphism Lemma 197.16 (2) that connecting homomorphisms are natural.

It follows from the above observations that the diagram of the lemma commutes. ■

204.6. Exercises for Chapter 204 .

Exercise 204.1. Let $X=S^1\times[0,1]$ be the annulus. What is the cup product

$$\underbrace{\mathrm{H}^1(X,\partial X;\mathbb{Z})}_{\cong\mathbb{Z}}\times\underbrace{\mathrm{H}^1(X,\partial X;\mathbb{Z})}_{\cong\mathbb{Z}}\ \to\ \underbrace{\mathrm{H}^2(X,\partial X;\mathbb{Z})}_{\cong\mathbb{Z}}?$$

Exercise 204.2. Let X be a topological space, let $A\subseteq X$ and let R be a commutative ring. Which axioms of a ring does the relative cup product on $\bigoplus_{k\in\mathbb{N}_0}\mathrm{H}^k(X,A;R)$ satisfy?

Exercise 204.3. Let (X,A,B) be an excisive triad and let R be a commutative ring. We saw that we have a natural cup product

$$\mathrm{H}^k(X,A;R)\times\mathrm{H}^l(X,B;R)\ \to\ \mathrm{H}^{k+l}(X,A\cup B;R).$$

Do we also have a natural cup product

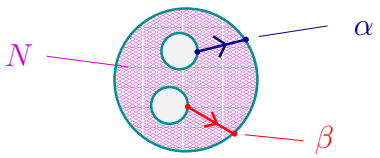
$$\mathrm{H}^k(X,A;R)\times\mathrm{H}^l(X,B;R)\ \to\ \mathrm{H}^{k+l}(X;R)?$$

Exercise 204.4. Let M be the torus minus one open disk. Show that the cup product $\mathrm{H}^1(M,\partial M;\mathbb{Z})\times\mathrm{H}^1(M,\partial M;\mathbb{Z})\to\mathrm{H}^2(M,\partial M;\mathbb{Z})$ is non-trivial.
Hint. Reduce this statement to the calculation of the cup product of the torus.

Exercise 204.5. Let N be the disk $\overline{B}^2$ minus two open disks.

(a) Show that the two homology classes $\alpha,\beta\in\mathrm{H}_1(N,\partial N)$ shown in the figure below are a basis for $\mathrm{H}_1(N,\partial N)\cong\mathbb{Z}^2$.
(b) We denote by $\varphi,\psi\in\mathrm{H}^1(N,\partial N;\mathbb{Z})=_{\mathrm{i}}\mathrm{Hom}(\mathrm{H}_1(N,\partial N),\mathbb{Z})$ the basis dual to α,β. Show that $\alpha\cup\beta=0\in\mathrm{H}^2(N,\partial N;\mathbb{Z})$.
Hint. Show that there exist open subsets A and B with the following properties:
(i) $\partial N\subseteq A$ and $\partial N\subseteq B$.
(ii) $A\cup B=N$.
(iii) φ lies in the image of $\mathrm{H}^1(N,A;\mathbb{Z})\to\mathrm{H}^1(N,\partial N;\mathbb{Z})$ and ψ lies in the image of $\mathrm{H}^1(N,B;\mathbb{Z})\to\mathrm{H}^1(N,\partial N;\mathbb{Z})$.

(c) Show that the cup product $\mathrm{H}^1(N, \partial N; \mathbb{Z}) \times \mathrm{H}^1(N, \partial N; \mathbb{Z}) \to \mathrm{H}^2(N, \partial N; \mathbb{Z})$ is the zero map.



Exercise 204.6. Let M be the torus minus one open disk and let N be the disk $\overline{B}^2$ minus two open disks. Use the calculations from Exercise 204.4 and 204.5 to show that there is no degree one map $g\colon N \to M$.
Remark. This gives an alternative solution to Exercise 195.9 (b).

Exercise 204.7. Let M be the Möbius band.

(a) Is the cup product

$$\mathrm{H}^1(M; \mathbb{F}_2) \times \mathrm{H}^1(M, \partial M; \mathbb{F}_2) \ \to \ \mathrm{H}^2(M, \partial M; \mathbb{F}_2)$$

trivial or non-trivial?

(b) Is the cup product

$$\mathrm{H}^1(M, \partial M; \mathbb{F}_2) \times \mathrm{H}^1(M, \partial M; \mathbb{F}_2) \ \to \ \mathrm{H}^2(M, \partial M; \mathbb{F}_2)$$

trivial or non-trivial?

Exercise 204.8. We consider $X = [0,1] \times [0,1]$ and the subsets $A = \{0,1\} \times [0,1]$ and $B = [0,1] \times \{0,1\}$. Determine the relative cup product

$$\underbrace{\mathrm{H}^1(X, A; \mathbb{Z})}_{\cong \mathbb{Z}} \times \underbrace{\mathrm{H}^1(X, B; \mathbb{Z})}_{\cong \mathbb{Z}} \ \to \ \underbrace{\mathrm{H}^2(X, \partial X; \mathbb{Z})}_{\cong \mathbb{Z}}.$$



Exercise 204.9. Let (X, A) be a pair of topological spaces and let R be a commutative ring. We denote by $i\colon A \to X$ the inclusion map and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A). Show that for any $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(A; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(A; R) \\ \uparrow{\scriptstyle i^*} \quad \downarrow{\scriptstyle \delta_q} & & \downarrow{\scriptstyle \delta_{p+q}} \\ \mathrm{H}^p(X; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q+1}(X, A; R) \end{array}$$

up to a sign and determine this sign in terms of p and q.

Exercise 204.10. We consider the Klein bottle K. Provide a topological space X such that for all $n \in \mathbb{N}_0$ we have an isomorphism $\mathrm{H}^n(K; \mathbb{Z}) \cong \mathrm{H}^n(X; \mathbb{Z})$ but such that the cohomology rings of X and F with suitable ring coefficients are non-isomorphic as graded rings.

CHAPTER 205

The Lusternik-Schnirelmann category

In this chapter we introduce and study the Lusternik-Schnirelmann category of a topological space. This is a fun topic, but it is also a slight detour from developing the general theory of cup products and cohomology. In the next Chapter 206 we will continue the development of the general theory with the introduction of the cap product.

This chapter is organized as follows:

- In Section 205.1 we will introduce the parametrization number of a smooth manifold.
- In Section 205.2 we will introduce the Lusternik-Schnirelmann category of a topological space.
- In Section 205.3 we will provide some upper bounds on the Lusternik-Schnirelmann category.
- In Section 205.4 we will use the relative cup produce to provide lower bounds on the Lusternik-Schnirelmann category.
- In Section 205.5 we will provide exercises for Chapter 205.

In the next Chapter 206 we will introduce the cap product on (co-) homology.

205.1. **The parametrization number.** We start out with the following definition:

Definition 205.1. Let M be an m-dimensional smooth manifold.

(1) A parametrization for M is a smooth map $\Psi\colon V \to M$ with the following properties:
   (a) The domain V is either an open m-dimensional rectangle or all of $\mathbb{R}^m$.[2175]
   (b) The map Ψ is a diffeomorphism onto its image.

(2) We say that a family $\{\Psi_i\colon V_i \to M\}_{i\in I}$ of parametrizations covers M if $\bigcup_{i\in I} \Psi_i(V_i) = M$.

If M is a smooth manifold with empty boundary, then M can clearly be covered by a family of parametrizations. This leads us to the following definition:

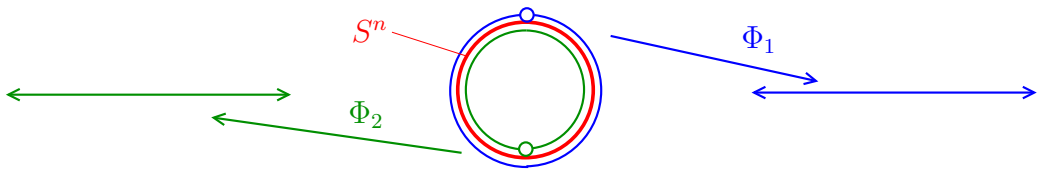
Definition 205.2. Given a smooth manifold M with empty boundary we define its parametrization number as

$$\text{param}(M) \;:=\; \text{minimal cardinality of a family of parametrizations that cover } M.$$

Examples.

(1) Let $N = (0, \ldots, 0, 1)$ be the North Pole and let $S = (0, \ldots, 0, -1)$ be the South Pole of the n-dimensional sphere S^n. We can cover S^n by the two open subsets $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$. Stereographic projection gives rise to two diffeomorphisms $\Phi_1\colon S^n \setminus \{N\} \to \mathbb{R}^n$ and $\Phi_2\colon S^n \setminus \{S\} \to \mathbb{R}^n$. The inverses of these two maps define parametrizations that cover S^n.

[2175] The condition on V sounds slightly arbitrary, and it is, but it is reasonable that for practical purposes one prefers domains that are "easy to describe".

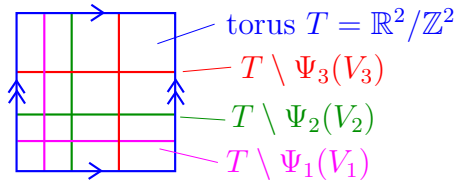


Since S^n is not diffeomorphic to any of the admissible domains of a parametrization we see that the above collection of parametrizations is optimal in the sense that we cannot parametrize S^n with fewer parametrizations. Thus we have shown that $\operatorname{param}(S^n) = 2$.

(2) We consider the n-torus $(S^1)^n =_{\mathrm{i}} \mathbb{R}^n/\mathbb{Z}^n$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection. We pick real numbers $0 < s_1 < s_2 < \cdots < s_{n+1} < 1$. For $i = 1, \ldots, n+1$ we consider the map

$$\begin{array}{rcl} \Psi_i\colon \overbrace{(s_i, s_i+1)\times\ldots\times(s_i, s_i+1)}^{=:V_i} & \to & \mathbb{R}^n/\mathbb{Z}^n \\ (t_1, \ldots, t_n) & \mapsto & p(t_1, \ldots, t_n). \end{array}$$

It is straightforward to see that the maps $\Psi_1, \ldots, \Psi_{n+1}$ are parametrizations that cover $(S^1)^n =_{\mathrm{i}} \mathbb{R}^n/\mathbb{Z}^n$. Note that this discussion shows that $\operatorname{param}((S^1)^n) \leq n+1$.



(3) For $i = 1, \ldots, n+1$ we consider the map

$$\begin{array}{rcl} \Psi_i\colon \mathbb{R}^n & \to & \mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim \\ (x_1, \ldots, x_n) & \mapsto & \underbrace{[x_1 : \ldots : x_{i-1} : 1 : x_i : \ldots : x_n]}_{\substack{\text{point in } \mathbb{RP}^n \text{ represented} \\ \text{by } (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_n)}}. \end{array}$$

Note that it follows from the Projective Space–Manifold Proposition 25.13 that the maps $\Psi_1, \ldots, \Psi_{n+1}$ are parametrizations that cover the real projective space $\mathbb{RP}^n$. This discussion shows that $\operatorname{param}(\mathbb{RP}^n) \leq n+1$.

(4) Almost the same construction as in (3) shows that we can cover the complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ by $n+1$ parametrizations such that the domain of each parametrization is diffeomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This discussion shows that $\operatorname{param}(\mathbb{CP}^n) \leq n+1$. □

The above discussion leads us to the following question:

Question 205.3.

(1) Can we cover $(S^1)^n$ by fewer than $n+1$ parametrizations?
(2) Can we cover $\mathbb{RP}^n$ by fewer than $n+1$ parametrizations?
(3) Can we cover $\mathbb{CP}^n$ by fewer than $n+1$ parametrizations?

In the next section we will introduce a variation on the parametrization number which is defined for all topological spaces. We will then see how relative cup products can be used to give lower bounds on the parametrization number and its more general sibling.

On page 3946 we will then use this approach to answer Question 205.3 for the 2-dimensional torus $S^1 \times S^1$ and the real projective plane $\mathbb{R}\mathrm{P}^2$. The other cases will have to wait till the LS Category–Torus Corollary 208.14 (or for impatient readers till Exercise 205.8) and the LS Category–Projective Space Proposition 213.21.

205.2. **The definition of the Lusternik-Schnirelmann category.** In the previous section we introduced the parametrization number of a smooth manifold. We now introduce a variation on this concept, which makes sense for arbitrary topological spaces:

Definition 205.4.

(1) Recall that according to Definition 18.2 a map $f\colon U \to X$ is called null-homotopic if it is homotopic to a constant map.

(2) Let X be a topological space.

(a) A Lusternik-Schnirelmann cover of X is an open cover $\{U_i\}_{i\in I}$ of X such that each inclusion map $U_i \to X$ is null-homotopic.

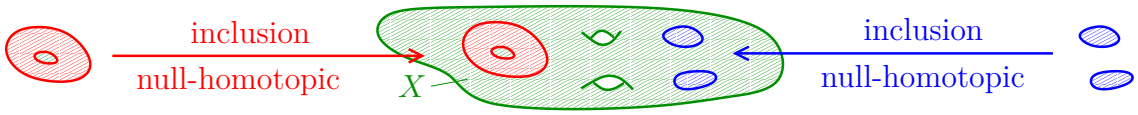
(b) If X admits a finite Lusternik-Schnirelmann cover, then we define the Lusternik-Schnirelmann category[2182] $\mathrm{cat}(X) \in \mathbb{Z}_{\geq -1}$ as[2183]

$$\mathrm{cat}(X) \;:=\; \min\left\{ n \in \mathbb{Z}_{\geq -1} \;\middle|\; \begin{array}{c}\text{there exists a Lusternik-Schnirelmann}\\ \text{cover with } n+1 \text{ sets}\end{array} \right\}.$$

Otherwise we set $\mathrm{cat}(X) := \infty$.

Remark.

(1) The sets U_i in the definition of the Lusternik-Schnirelmann category do not have to be contractible. For example if X itself is contractible, then for any subset $U \subseteq X$ the inclusion map $U \to X$ is null-homotopic.



(2) The Lusternik-Schnirelmann category plays an important role in topological robotics, see e.g. [**Far2008**] for more details. □

Examples.

(1) We have $\mathrm{cat}(X) = -1$ if and only if $X = \varnothing$.

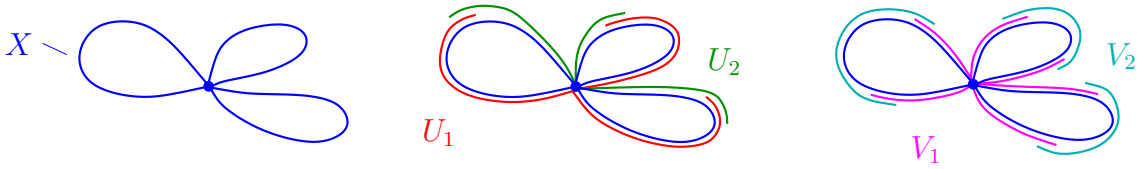
(2) For a non-empty topological space X we have $\mathrm{cat}(X) = 0$ if and only if $\mathrm{id}\colon X \to X$ is null-homotopic. By the Contractability Criterion Lemma 19.20 this holds precisely if X is contractible.

(3) (a) In the figure below we consider the wedge X of $n \geq 1$ circles together with two open covers $X = U_1 \cup U_2$ and also $X = V_1 \cup V_2$. Note that all the inclusion maps $U_i \to X$ and also $V_i \to X$ are null-homotopic. So this gives us two separate proofs

[2182] The word "category" with its modern meaning was first used by Eilenberg-Maclane [**EMc1945**] in 1945. On the other hand the term "Lusternik-Schnirelmann category" was already established by 1941, see [**Fox1941**]. The concept goes back to the work of Lazar Lusternik and Lev Schnirelmann [**LuS1930, LuS1934**].

[2183] Note the term "+1" in the definition. Sometimes, especially in the older literature the Lusternik-Schnirelmann category is defined without the adjustment given by the "+1".

that $\operatorname{cat}(X) \leq 1$. Together with Statement (2) this shows that we have $\operatorname{cat}(X) = 1$.



(b) Using the obvious generalization of the sets V_1 and V_2 shown in the figure above one can easily show that the Lusternik-Schnirelmann category of the wedge of a non-empty family of spheres of dimension ≥ 1 has Lusternik-Schnirelmann category equal to 1. □

The following lemma relates the notions of parametrization number and Lusternik-Schnirelmann category:

Lemma 205.5. (Parametrization–LS Category Lemma) For any smooth manifold M we have $\operatorname{cat}(M) \leq \operatorname{param}(M) - 1$.

Proof. Let $\Psi_i \colon V_i \to M$, $i = 1, \ldots, n+1$ be a family of parametrizations that cover M. Note that it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that each $U_i := \Psi_i(V_i)$ is an open subset of M. In particular $U_1, \ldots, U_{n+1}$ is an open cover of M.

Since the domain of a parametrization is contractible and since a parametrization is a diffeomorphism onto its image we see that each U_i is contractible. It follows from the Contractability Criterion Lemma 19.20 that each inclusion $U_i \to M$ is null-homotopic.

We have thus shown that $\operatorname{cat}(M) \leq (n+1) - 1 = n$. ■

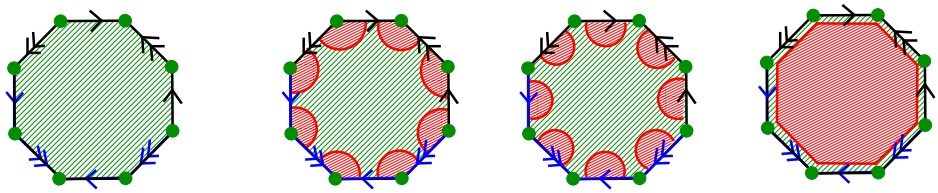
The Parametrization–LS Category Lemma 205.5 allows us to bring our expertise on parametrizations to bear:

Examples.

(4) The discussion of parametrizations of S^n on page 3940 together with the above Example (2) and the Parametrization–LS Category Lemma 205.5 shows that $\operatorname{cat}(S^n) = 1$.

(5) As in (4) it follows from the above discussion of parametrizations that $\operatorname{cat}((S^1)^n) \leq n$, $\operatorname{cat}(\mathbb{R}\mathrm{P}^n) \leq n$ and also $\operatorname{cat}(\mathbb{C}\mathrm{P}^n) \leq n$.

(6) Let $g \geq 2$. The figure below convincingly shows that $\operatorname{cat}(\Sigma) \leq 2$.



Similarly one can show that for any non-orientable surface N_k of some genus $k \in \mathbb{N}$ we have $\operatorname{cat}(N_k) \leq 2$. □

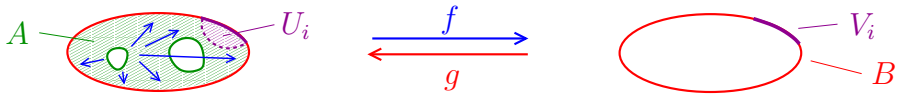
We move on to more general statements:

Proposition 205.6. (LS Category–Homotopy Proposition) Let X and Y be topological spaces. If X and Y are homotopy equivalent, then $\mathrm{cat}(X) = \mathrm{cat}(Y)$.[2184]

Proof. It suffices to prove the following claim:

Claim. Let $f\colon A \to B$ be a continuous map between topological spaces. If there exists a continuous map $g\colon B \to A$ such that $f \circ g\colon B \to B$ is homotopic to the identity, then $\mathrm{cat}(A) \geq \mathrm{cat}(B)$.

Proof. Let $\mathrm{cat}(A) = n$. Let $U_1, \ldots, U_n$ be an open cover of A such that each inclusion $U_i \to X$ is null-homotopic. For $i = 1, \ldots, n$ we set $V_i \coloneqq g^{-1}(U_i)$. Clearly this is an open cover of B.



It remains to show that each inclusion $V_i \to B$ is null-homotopic. So let $i \in \{1, \ldots, n\}$. By hypothesis we can pick a homotopy $G\colon B \times [0,1] \to B$ with $G_0 = \mathrm{id}_B$ and $G_1 = f \circ g$ and we can pick a homotopy $H\colon U_i \times [0,1] \to A$ such that H_0 is the inclusion and such that H_1 is a constant map. We now consider the map

$$\begin{array}{rcl} V_i \times [0,1] & \to & B \\ (y,t) & \mapsto & \begin{cases} G(y,2t), & \text{if } t \in [0,\frac{1}{2}], \\ f(H(g(y),2t-1)), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

First note that one can easily verify that this map is well-defined. Furthermore note that it follows from the Homotopy Combination Lemma 18.8 (1) that this map is continuous. It is now clear that this map defines a homotopy from the inclusion $V_i \to B$ to a constant map. ∎

205.3. **Upper bounds on the Lusternik-Schnirelmann category.** The following proposition gives interesting upper bounds for many examples. In particular it gives a unified approach to the above Examples -(4)–(6).

Proposition 205.7. (LS Category–Upper Bound Proposition)
(1) If X is a connected n-dimensional simplicial complex X, then $\mathrm{cat}(X) \leq n$.
(2) If X is a connected n-dimensional CW-complex, then $\mathrm{cat}(X) \leq n$.
(3) If X is a connected n-dimensional smooth manifold, then $\mathrm{cat}(X) \leq n$.
(4) If X is a connected n-dimensional topological manifold, then $\mathrm{cat}(X) \leq n$.

In the proof of the LS Category–Upper Bound Proposition 205.7 we make use of the following lemma:

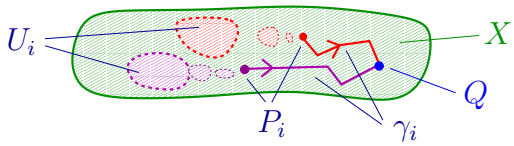
Lemma 205.8. (Contractible Components–Null-Homotopy Lemma) Let X be a topological space and let $U \subseteq X$ be a subset with components $\{U_i\}_{i \in I}$. We suppose that the following statements hold:
(1) Each U_i is an open subset of U.

[2184]The definition of the Lusternik-Schnirelmann category is perhaps not the most natural definition, but it is chosen in such a way that it leads to clean statements, like the statement of this lemma.

(2) Each inclusion $U_i \to X$ is null-homotopic.
If X is path-connected, then the inclusion $U \to X$ is null-homotopic.

Proof of Lemma 205.8. By hypothesis we know that for each $i \in I$ there exists a point $P_i \in X$ and a homotopy $G_i \colon U_i \times [0,1] \to U_i$ such that $G_{i,0} \colon U_i \to X$ is the inclusion and such that $G_{i,1}$ is the constant map $x \mapsto P_i$.

Now pick $Q \in X$. Since X is path-connected we know that for each $i \in I$ there exists a continuous map $\gamma_i \colon [0,1] \to X$ with $\gamma_i(0) = P_i$ and $\gamma_i(1) = Q$.



It remains to prove the following claim:

Claim. The following map is well-defined and continuous:

$$\begin{array}{rcl} F \colon U \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} G_i(x,2t), & \text{if } x \in U_i \text{ and } t \in [0,\frac{1}{2}], \\ \gamma_i(2t-1), & \text{if } x \in U_i \text{ and } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

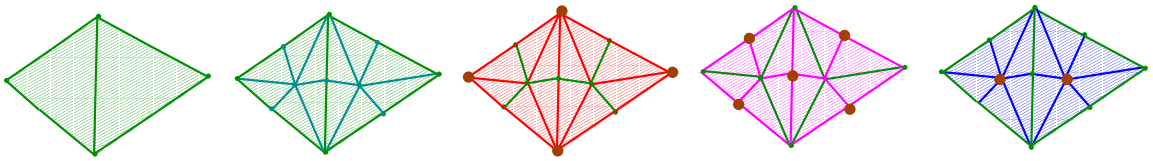
Proof. First note that the map is clearly well-defined. Next note that it follows from the fact that each U_i is open in U and the Definition 7.3 of the product topology that each $U_i \times [0,1]$ is open in $U \times [0,1]$. By the Disjoint Union Embedding Lemma 7.29 this in turn implies that $U \times [0,1]$ is the coproduct of the $U_i \times [0,1]$. By the Topological-Coproduct Proposition 7.26 (1b) it remains to show that the restriction of F to each $U_i \times [0,1]$ is continuous. But this is an easy consequence of the Homotopy Combination Lemma 18.8. ∎

Proof of Proposition 205.7.

(1) We only have to deal with the case that $X = |K|$ is the topological realization of a connected n-dimensional abstract simplicial complex. We consider the barycentric subdivision $\mathrm{sd}(K)$ as defined in the Barycentric Subdivision Lemma 175.3. By the SD-Lemma 175.5 (3d) we know that the topological realization $|\mathrm{sd}(K)|$ is homeomorphic to $|K|$. Thus it suffices to show that $\mathrm{cat}(|\mathrm{sd}(K)|) \leq n$. For $i = 0, \ldots, n$ we set

$$S_i := \text{set of } i\text{-simplices of } K \qquad \text{and} \qquad U_i = \bigcup_{s \in S_i} \underbrace{\mathring{\mathrm{St}}(\mathrm{sd}(K), \underline{s})}_{\substack{\text{open star as defined in} \\ \text{the Link Star Lemma 175.9}}}.$$

In other words, U_i is the union of the open stars in the barycentric subdivision corresponding to the vertices of the barycentric subdivision that are given by the barycenters of the i-simplices of K.



We fix $i \in \{0, \dots, n\}$. We make the following observations:

(a) By the Link Star Lemma 175.9 (1d) we know that each $\mathring{\mathrm{St}}(\mathrm{sd}(K), \underline{s})$ is an open subset of X.

(b) By the Link Star Lemma 175.9 (1f) we know that each $\mathring{\mathrm{St}}(\mathrm{sd}(K), \underline{s})$ is contractible. Noe that it follows from the elementary Contractability Criterion Lemma 19.20 that each inclusion $\mathring{\mathrm{St}}(\mathrm{sd}(K), \underline{s}) \to X$ is null-homotopic.

(c) It follows easily from the Link Star Lemma 175.9 (3) that for any two $s \neq t \in S_i$ we have $\mathring{\mathrm{St}}(\mathrm{sd}(K), s) \cap \mathring{\mathrm{St}}(\mathrm{sd}(K), t) = \varnothing$.

It follows almost immediately from (a), (b) and (c), together with the Contractible Components–Null-Homotopy Lemma 205.8 that the inclusion $U_i \to |\mathrm{sd}(K)|$ is null-homotopic.

Furthermore it follows from the Link Star Lemma 175.9 (2b) that the U_i, $i = 0, \dots, n$ form an open cover of $|\mathrm{sd}(K)|$. Thus we have shown that $\mathrm{cat}(|\mathrm{sd}(K)|) \leq n$.

(2) By the CW–Simplicial Homotopy Type Proposition 175.21 we know that every connected n-dimensional CW-complex is homotopy equivalent to a connected n-dimensional simplicial complex. Thus the statement follows from (1) together with the LS Category–Homotopy Proposition 205.6.[2185]

(3) Let X be a connected n-dimensional smooth manifold. There are several different approaches to showing that $\mathrm{cat}(X) \leq n$:

(a) By the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that X has the structure of an n-dimensional simplicial complex. Thus the statement follows from (1).

(b) In the proof of the Small Atlas Proposition 33.10 we implicitly showed that X admits an atlas $\{\Phi_i \colon U_i \to V_i\}_{i=1,\dots,n+1}$ with $n+1$ charts with the property that for each component C of any U_i there exists a $y_C \in C$ such that the map $\Phi_i|_C \colon C \to V_i$ is homotopic to the constant map $x \mapsto y_C$. Note that it follows from the Locally Path-Connected Lemma 4.57 that each component C of U_i is open. Since X is connected it follows easily from the Contractible Components–Null-Homotopy Lemma 205.8 that each Φ_i is homotopic to a constant map.

(c) In Exercise 250.2 we will use handle decompositions to show that if X is closed, then $\mathrm{cat}(X) \leq n$.

(4) Let X be a connected n-dimensional topological manifold. There are again several different approaches to showing that $\mathrm{cat}(X) \leq n$:

(a) At least if X is compact, then by the Manifold–Simplicial Complex Homotopy Type Theorem 336.3 we know that X is homotopy equivalent to a connected n-dimensional simplicial complex. So the statement follows again from (1) together with the LS Category–Homotopy Proposition 205.6.

(b) The approach of (3b) also works, without any changes, in the topological context. ■

205.4. **Lower bounds on the Lusternik-Schnirelmann category.** As we saw in the above examples and the LS Category–Upper Bound Proposition 205.7, one can obtain an upper bound on the Lusternik-Schnirelmann category of a given topological space by

[2185] The intrepid reader could also try to use the constructions from the proof of the CW-Complex–Local Properties Proposition 137.8 to prove this statement.

writing down explicit open subsets. To obtain lower bounds on the Lusternik-Schnirelmann category we introduce the following definition:

Definition 205.9. Given a non-empty topological space X we define its cup length $\mathrm{cl}(X)$ in $\mathbb{N}_0 \cup \{\infty\}$ as follows:[2186]

$$\mathrm{cl}(X) \coloneqq \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c}\text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } \varphi_1, \dots, \varphi_n \text{ in degrees} \geq 1 \text{ with } \varphi_1 \cup \dots \cup \varphi_n \neq 0\end{array} \right\}.$$

Examples.

(1) It follows easily from the Wedge-of-Spheres–Cup Product Lemma 203.29 that for any wedge $X = S^{r_1} \vee \dots \vee S^{r_k}$ of finitely many spheres, where at least one sphere has dimension ≥ 1, the cup length equals 1.

(2) If X is an n-dimensional CW-complex, then it follows easily from the Singular-Cellular-H^*-Isomorphism Proposition 199.5 that for any commutative ring R and any $i > n$ we have $\mathrm{H}^i(X; R) = 0$. This implies that the cup product of $\geq n+1$ cohomology classes of degree ≥ 1 is necessarily zero. In other words, this implies that $\mathrm{cl}(X) \leq n$.

(3) By the same logic as in (1) it follows from the Manifold-Top-H^*-Theorem 201.26 that for any n-dimensional topological manifold X we have $\mathrm{cl}(X) \leq n$.

(4) In the Torus–Cup Product Lemma 203.28 and the Surface–Cup Product Lemma 203.30 we saw that for any surface Σ of genus $g \geq 1$ there exist $\varphi, \psi \in \mathrm{H}^1(\Sigma; \mathbb{Z})$ with $\varphi \cup \psi \neq 0$, which together with (2) shows that $\mathrm{cl}(\Sigma) = 2$.

(5) Recall that in the Projective Plane–Cup Product Lemma 203.32 we showed that there exists a $\varphi \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ with $\varphi \cup \varphi \neq 0$, which shows by (2) that $\mathrm{cl}(\mathbb{R}\mathrm{P}^2) = 2$. Similarly it follows from Exercise 203.12 that for any $k \in \mathbb{N}$ we have for the non-orientable surface of genus k that $\mathrm{cl}(N_k) = 2$. □

The following proposition shows that the cup length is a lower bound on the Lusternik-Schnirelmann category.

Proposition 205.10. (Cup Length–LS Category Proposition) For every topological space X we have

$$\mathrm{cl}(X) \;\leq\; \mathrm{cat}(X).$$

Example. It follows immediately from the Cup Length–LS Category Proposition 205.10, the above calculations of cup lengths and the discussion on page 3942 that

$$\mathrm{cat}(S^1 \times S^1) \;=\; \mathrm{cat}(\text{surface of genus } g \geq 2) \;=\; \mathrm{cat}(\mathbb{R}\mathrm{P}^2) \;=\; \mathrm{cat}(N_k) \;=\; 2.$$

Furthermore, it follows from the Parametrization–LS Category Lemma 205.5 and the examples on page 3940 that $\mathrm{param}(S^1 \times S^1) = 3$ and $\mathrm{param}(\mathbb{R}\mathrm{P}^2) = 3$. □

Proof. Let X be a topological space, let R be a commutative ring and let $\varphi_j \in \mathrm{H}^{s_j}(X; R)$, $j = 1, \dots, n$ be cohomology classes in degrees $s_j \geq 1$ such that $\varphi_1 \cup \varphi_2 \cup \dots \cup \varphi_n \neq 0$. We have to show that $\mathrm{cat}(X) \geq n$.

Thus let $k \leq n$ and let $U_1, \dots, U_k$ be open subsets in X such that each inclusion map $\iota_i \colon U_i \to X$ is null-homotopic. We have to show that $U_1 \cup \dots \cup U_k$ is a *proper* subset of X.

[2186] As usual we follow the convention that in a ring (in our case the cohomology ring), the product of the empty set of elements is the multiplicative neutral element. If X is non-empty we have $1_X \neq 0$. This shows that for a non-empty topological space X the definition actually makes sense.

Claim. For any $j \in \{1, \dots, k\}$ the natural map $\text{я}_j \colon \mathrm{H}^{s_j}(X, U_j; R) \to \mathrm{H}^{s_j}(X; R)$ is an epimorphism.

Proof. We consider the long exact sequence in cohomology:

$$\mathrm{H}^{s_j}(X, U_j; R) \xrightarrow{\text{я}_j} \mathrm{H}^{s_j}(X; R) \xrightarrow[\uparrow]{\iota_j^*} \mathrm{H}^{s_j}(U_j; R).$$

since ι_j is homotopic to a constant map and since $s_j \geq 1$ it follows from the argument on page 2771 that this map is the zero map

Since the right map is zero, the map to the left has to be an epimorphism. ⊞

It follows from the claim that for each $j \in \{1 \dots, k\}$ we can pick $\psi_j \in \mathrm{H}^{s_j}(X, U_j; R)$ with $\text{я}_j(\psi_j) = \varphi_j$. It follows from the Excisive Triad-Proposition 158.20 (1) and the Cup Product-Naturality Lemma 204.8 (1) (applied iteratively) that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^{s_1}(X, U_1; R) \times \cdots \times \mathrm{H}^{s_k}(X, U_k; R) & \xrightarrow{\cup} & \mathrm{H}^{s_1+\cdots+s_k}(X, U_1 \cup \cdots \cup U_k; R) \\ \downarrow (\psi_1, \dots, \psi_k) \mapsto (\varphi_1, \dots, \varphi_k) & & \downarrow \\ \mathrm{H}^{s_1}(X; R) \times \cdots \times \mathrm{H}^{s_k}(X; R) & \xrightarrow{\cup} & \mathrm{H}^{s_1+\cdots+s_k}(X; R). \end{array}$$

By our hypotheses the cup product $\varphi_1 \cup \cdots \cup \varphi_k$ is non-zero. This implies, by the commutativity of the diagram, that the cup product $\psi_1 \cup \cdots \cup \psi_k$ is also non-zero. But this implies that the group $\mathrm{H}^{s_1+\cdots+s_k}(X, U_1 \cup \cdots \cup U_k; R)$ is non-zero, which in turn implies that $U_1 \cup \cdots \cup U_k$ is a proper subset of X. ■

The Cup Length–LS Category Proposition 205.10 together with the Parametrization–LS Category Lemma 205.5 gives us a tool to attack our original Question 205.3 on the parametrization number of $(S^1)^n$, $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. To make this approach work we need to answer the following questions:

Question 205.11.
(1) What is the $\mathbb{Z}$-cohomology ring of $(S^1)^n$?
(2) What is the $\mathbb{F}_2$-cohomology ring of $\mathbb{R}\mathrm{P}^n$?
(3) What is the $\mathbb{Z}$-cohomology ring of $\mathbb{C}\mathrm{P}^n$?

We will answer these questions in the Torus–Cup Product-Proposition 208.13, the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 and the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7.

There are many other question on the Lusternik-Schnirelmann category that are still open. As an example we conclude this introduction to the Lusternik-Schnirelmann category of a topological space with the following conjecture which is formulated as a question in [**Rudy2017**, Question 1.1].

Conjecture 205.12. (Rudyak Conjecture) Let M and N be two closed oriented connected n-dimensional topological manifolds. If there exists a continuous map $f \colon M \to N$ of degree one, then $\mathrm{cat}(M) \geq \mathrm{cat}(N)$.

Remark. In Exercise 213.19 we will prove some evidence supporting the Rudyak Conjecture 205.12. □

205.5. Exercises for Chapter 205 .

Exercise 205.1. Let X be a topological space and let $U \subseteq X$ be a subset with components $\{U_i\}_{i\in I}$. We suppose that for each $i \in I$ the inclusion $U_i \to X$ is null-homotopic and we assume that X is path-connected. Does it follow that the inclusion $U \to X$ is null-homotopic?

Exercise 205.2. Can you determine the parametrization number of the surface of genus 2?

Exercise 205.3. Show that if M is a closed connected topological manifold of dimension ≥ 2 with non-trivial $\pi_1(M)$, then M cannot be covered by two contractible open subsets.

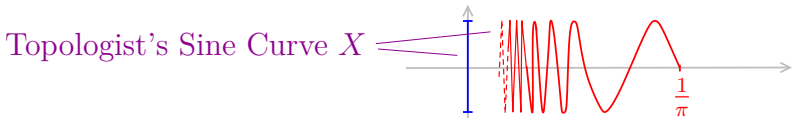
Exercise 205.4.

(a) What is the minimal number of charts in an atlas for the 2-dimensional torus $S^1 \times S^1$?
(b) Let $g \geq 2$. What is the minimal number of charts in an atlas for the surface of genus g?

Exercise 205.5. What is the Lusternik-Schnirelmann category of the Topologist's Sine Curve X. Recall that

$$X \;:=\; \{(0,y)\,|\,y\in[-1,1]\} \cup \{(x,\sin(\tfrac{1}{x}))\,|\,x\in(0,\tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2,$$

where we equip X with the subspace topology.



Exercise 205.6.

(a) Let X be a topological space and let $A \subseteq X$ be a subspace. Show that if A is a retract of X, then $\mathrm{cl}(A) \leq \mathrm{cl}(X)$.
(b) Let X and Y be non-empty topological spaces. Show that we have the following lower bound on the cup length of $X \times Y$:

$$\mathrm{cl}(X\times Y) \;\geq\; \max\{\mathrm{cl}(X),\mathrm{cl}(Y)\}.$$

Exercise 205.7. Let $p\colon \widetilde{X} \to X$ be a covering of topological spaces.

(a) What is the best general inequalities between $\mathrm{cat}(\widetilde{X})$ and $\mathrm{cat}(X)$ that you can prove?
(b) Are there examples with $\mathrm{cat}(\widetilde{X}) > \mathrm{cat}(X)$?
(c) Are there examples with $\mathrm{cat}(\widetilde{X}) < \mathrm{cat}(X)$?

Exercise 205.8. Let $n \in \mathbb{N}$. We introduce the following notation:

- Given $a_0,\ldots,a_n \in [0,1]$ we consider, following the Definition 148.17, the singular n-simplex
$$[a_0,\ldots,a_n]\colon \Delta^n \;\to\; [0,1]^n$$
which is the unique affine linear map that sends the i-th vertex $\mathfrak{v}_i$ of Δ^n to a_i.
- We consider the map
$$\begin{array}{rcl} f\colon [0,1]^n & \to & (S^1)^n \\ (x_1,\ldots,x_n) & \mapsto & (\exp(2\pi\,\mathrm{i}x_1),\ldots,\exp(2\pi\,\mathrm{i}x_n)). \end{array}$$
- For $i=1,\ldots,n$ we denote by $p_i\colon (S^1)^n \to S^1$ the i-th projection.

With this notation we can pose the following problems:

(a) Show that
$$\omega \;:=\; \sum_{\sigma\in S_n} (-1)^{\mathrm{sign}(\sigma)} \cdot f\circ[0, e_{\sigma(1)}, \dots, e_{\sigma(n)}] \qquad \in\; \mathrm{C}_n((S^1)^n)$$
is a cycle.

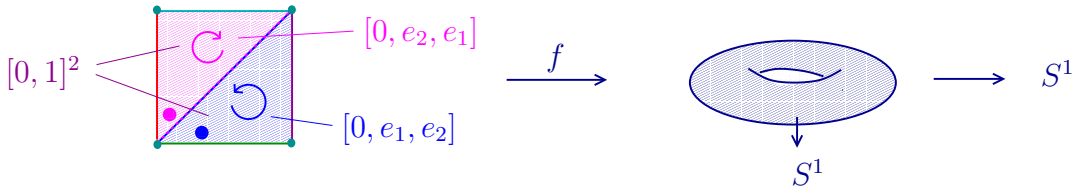
Hint. Either do it by hand or use the Shuffle Proposition 164.24.

(b) We consider the Kronecker pairing $\langle\ ,\ \rangle\colon \mathrm{H}^n((S^1)^n;\mathbb{Z}) \times \mathrm{H}_n((S^1)^n) \to \mathbb{Z}$. Determine
$$\langle p_1^*([\theta_{\mathbb{Z}}]) \cap \dots \cap p_n^*([\theta_{\mathbb{Z}}]), [\omega]\rangle.$$

(c) Show that ω represents a generator of $\mathrm{H}_n((S^1)^n)$.

(d) Determine the cup length and the Lusternik-Schnirelmann category of $(S^1)^n$.

Remark. The first parts of the exercise have a significant overlap with Exercise 192.11.



CHAPTER 206

The cap product

In this chapter we introduce the cap product that relates homology and cohomology groups. It can be viewed as a generalization of the Kronecker pairing $\mathrm{H}^k(X;R) \times \mathrm{H}_k(X;R) \to R$. Initially it might not be clear what the cap product is good for, but later on we will see that it plays an essential role in many calculations and in the formulation of the Poincaré Duality Theorem.

This chapter is organized as follows:

- In Section 206.1 we will define the cap product.
- In Section 206.2 we will perform some calculations of cap products.
- In Section 206.3 we will introduce the relative cap product.
- In Section 206.4 we will study properties of the cap product.
- In Section 206.5 we will see that we can view homology as a module over the cohomology ring.
- In Section 206.6 we will study the relationship between the cap product and connecting homomorphisms.
- In Section 206.7 we will provide exercises for Chapter 206.

In the next Chapter 207 we will study the interaction between cross-, cup- and cap-products.

206.1. **The definition of the cap product.** First we recall some notation from Definitions 148.17 and 143.8:

Notation 206.1.

(1) For $i = 0, \dots, n$ we write $\mathfrak{v}_i \coloneqq (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

(2) For $a_0, \dots, a_k \in \Delta^n$ we denote by $[a_0, \dots, a_k] \colon \Delta^k \to \Delta^n$ the unique affine linear map that sends $\mathfrak{v}_i$ to a_i.

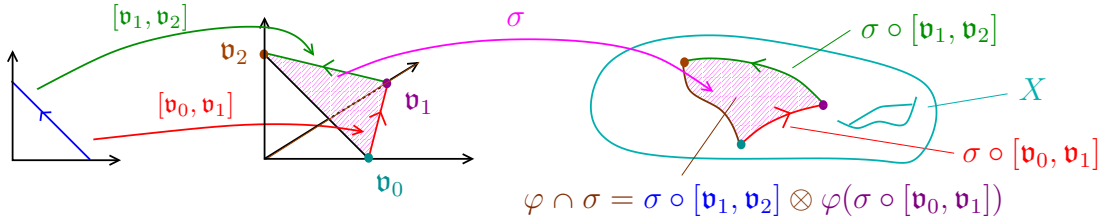
The following definition is inspired by the second definition of the cup product that we gave in Definition 203.11.

Definition 206.2. Let X be a topological space and let R be a commutative ring. Given a cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma \colon \Delta^l \to X$ with $k \leq l$ we define the cap product as[2187]

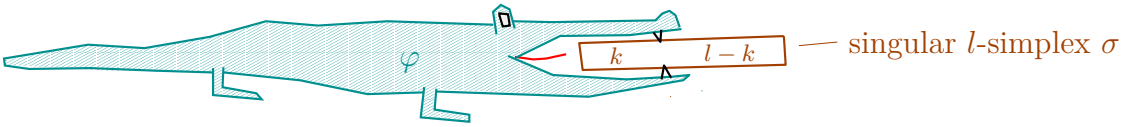
$$\varphi \cap \sigma \coloneqq \underbrace{\sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l]}_{\text{map } \Delta^{l-k} \to X} \otimes \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k])}_{\in R} \in \mathrm{C}_{l-k}(X;R) = \mathrm{C}_{l-k}(X) \otimes R.$$

For $k > l$ we define the cap product to be the zero map. Since $\mathrm{C}^l(X)$ is the free abelian group on the set of singular l-simplices we get an induced map

$$\begin{aligned} \cap\colon \mathrm{C}^k(X;R) \times \mathrm{C}_l(X) &\to \mathrm{C}_{l-k}(X;R) \\ \Big(\varphi, \sum_{i=1}^d a_i \cdot \sigma_i\Big) &\mapsto \sum_{i=1}^d a_i \cdot \varphi \cap \sigma_i. \end{aligned}$$



Remark. The definition of the cap product can also be visualized by the cap product crocodile.



This crocodile has the following eating habit: It bites the first k coordinates off a singular simplex, spits out $\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k])$ and combines it with the "remaining" simplex $\sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_l]$. □

As for the cup product, the first thing we want to verify is that this map on the cochain and chain level defines a map on the level of cohomology groups and homology groups. The key to doing so is the following elementary lemma, which can be viewed as an analog of the Cup Product–Coboundary Lemma 203.12.

Lemma 206.3. (Cap Product–Boundary Lemma) Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^k(X;R)$ and let $\sigma\colon \Delta^l \to X$ be a singular simplex. Then

$$\partial_{l-k}(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta_k(\varphi) \cap \sigma \,+\, \varphi \cap \partial_l(\sigma)) \;\in\; \mathrm{C}_{l-k-1}(X;R).$$

In particular this implies that $\varphi \cap \partial_l(\sigma)$ and $\delta_k(\varphi) \cap \sigma$ are homologous.

[2187]The notation is unfortunately slightly awkward: In the notation of the cap product $\varphi \cap \sigma$ in the literature the cochain is always to the left and the chain is always to the right. But in the tensor product $\mathrm{C}_{l-k}(X) \otimes R$ the chain is to the left and the result of the cochain is to the right.

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. Let $\varphi \in \mathrm{C}^k(X;R)$ and let $\sigma\colon \Delta^l \to X$ be a singular simplex. Note that

$$\begin{aligned}
\text{(a)} \qquad \varphi \cap \partial_l(\sigma) &= \sum_{i=0}^{l} (-1)^i \cdot \varphi \cap \sigma \circ [\mathfrak{v}_0, \ldots, \widehat{\mathfrak{v}}_i, \ldots, \mathfrak{v}_l] \\
&= \sum_{i=0}^{k} (-1)^i \cdot \sigma \circ [\mathfrak{v}_{k+1}, \ldots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \widehat{\mathfrak{v}}_i, \ldots, \mathfrak{v}_{k+1}]) \\
&\qquad + \sum_{i=k+1}^{l} (-1)^i \cdot \sigma \circ [\mathfrak{v}_k, \ldots, \widehat{\mathfrak{v}}_i, \ldots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]) \\
\text{(b)} \qquad \delta_k(\varphi) \cap \sigma &= \sum_{i=0}^{k+1} (-1)^i \cdot \sigma \circ [\mathfrak{v}_{k+1}, \ldots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \widehat{\mathfrak{v}}_i, \ldots, \mathfrak{v}_{k+1}]) \\
\text{(c)} \qquad (-1)^k \cdot \partial_{l-k}(\varphi \cap \sigma) &= \sum_{i=k}^{l} (-1)^i \cdot \sigma \circ [\mathfrak{v}_k, \ldots, \widehat{\mathfrak{v}}_i, \ldots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]).
\end{aligned}$$

Note that the summand of (b) corresponding to $i = k+1$ equals *minus* the summand of (c) corresponding to $i = k$. Adding (b) and (c) and comparing it to (a) we see that

$$\varphi \cap \partial_l(\sigma) = \delta_k(\varphi) \cap \sigma + (-1)^k \cdot \partial_{l-k}(\varphi \cap \sigma)$$

which gives the desired equality. ∎

The following lemma, which is the analog of the Cup Product–Definition Lemma 203.13, says that "$\cap$" descends to a map on homology and cohomology.

Lemma 206.4. (Cap Product–Definition Lemma) Let X be a topological space and let R be a commutative ring. For any $k, l \in \mathbb{N}_0$ the map

$$\begin{aligned}
\mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) &\to \mathrm{H}_{l-k}(X;R) \\
\left([\varphi], \left[\sum_{i=1}^{m} \sigma_i \otimes r_i\right]\right) &\mapsto \left[\sum_{i=1}^{m} \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i\right]
\end{aligned}$$

is well-defined and R-bilinear.

Definition 206.5. We refer to the pairing

$$\begin{aligned}
\cap\colon \mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) &\to \mathrm{H}_{l-k}(X;R) \\
\left([\varphi], \left[\sum_{i=1}^{m} \sigma_i \otimes r_i\right]\right) &\mapsto \left[\sum_{i=1}^{m} \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i\right]
\end{aligned}$$

from the Cap Product–Definition Lemma 206.4 as the cap product.[2188]

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. As we will see, this case follows from the Cap Product–Boundary Lemma 206.3 together with routine arguments. But for peace of mind, let us go through the steps. First note that using the Tensor

[2188] The name "cap product" comes from the notation "$a \cap b$" for the composition since $\cap$ looks like, just a little bit, like the "a" in cap. The other reason is that the cup product was first introduced, and mathematicians needed a different name for a somewhat related object. Also mathematicians are well-known for their great sense of humor.

Product–Universal Property Lemma 159.4 (2) one can easily verify that the map

$$\begin{array}{rcl} \cap\colon \mathrm{C}^k(X;R) \times \mathrm{C}_l(X;R) & \to & \mathrm{C}_{l-k}(X;R) \\ \left(\varphi, \sum_{i=1}^m \sigma_i \otimes r_i\right) & \mapsto & \sum_{i=1}^m \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i \end{array}$$

is well-defined. It remains to show that the above map descends to a well-defined map on (co-) homology. We start out with the following familiar looking claim.

Claim. For any $\varphi \in \mathrm{C}^k(X;R)$ and any $b \in \mathrm{C}_k(X;R)$ we have

$$\partial_{l-k}(\varphi \cap b) \;=\; (-1)^k \cdot (-\delta_k(\varphi) \cap b \,+\, \varphi \cap \partial_l(b)) \;\in\; \mathrm{C}_{l-k-1}(X;R).$$

Proof. Since "$\cap$" is evidently bilinear it suffices to show the desired equality for any $\varphi \in \mathrm{C}^k(X;R)$ and any element of $\mathrm{C}_k(X;R)$ of the form $b = \sigma \otimes r$ where $\sigma \in \mathrm{C}_k(X)$ and $r \in R$. For such element we have the following equality in $\mathrm{C}_{l-k-1}(X;R)$:

$$\begin{array}{lcll} \varphi \cap (\sigma \otimes r) & \overset{\text{by Definition 206.5 of } \cap}{\overset{\downarrow}{=}} & (\varphi \cap \sigma) \cdot r & \overset{\text{by the Cap Product–Boundary Lemma 206.3}}{\overset{\downarrow}{=}} \\ & & & ((-1)^k \cdot (-\delta_k(\varphi) \cap \sigma \,+\, \varphi \cap \partial_l(\sigma))) \cdot r \\ & & & = \quad (-1)^k \cdot (-\delta_k(\varphi) \cap (\sigma \otimes r) \,+\, \varphi \cap \underbrace{\partial_l(\sigma \otimes r)}_{=\partial_l(\sigma)\otimes r}). \\ & & & \overset{\text{by Definition 206.5 of } \cap}{\uparrow} \end{array}$$

⊞

Now let $\varphi \in \mathrm{C}^k(X;R)$ be a cocycle and let $b \in \mathrm{C}_l(X;R)$ be a cycle.

(1) It follows from the claim that $\varphi \cap b$ is a cycle, i.e. it represents a homology class.
(2) If $b' = b + \partial_{l+1}(\tau)$ is another representative of $[b]$, then

$$\varphi \cap b - \varphi \cap b' \;=\; \varphi \cap (b - b') \;=\; \varphi \cap \partial_{l+1}(\tau) \;\underset{\text{by the claim}}{\overset{\uparrow}{=}}\; (-1)^k \cdot \partial_{l+1-k}(\varphi \cap \tau).$$

This shows that $\varphi \cap b$ and $\varphi \cap b'$ represent the same homology class, i.e. the cap product does not depend on the choice of the representative of the homology class.
(3) The proof that the cap product does not depend on the choice of representative of the cohomology class is basically the same as the proof of (2).

This shows that the map is well-defined. It is clear that the map is indeed R-bilinear. ■

Remark. We already saw on page 3893 that the definition of the cup product in the literature is not uniform. Rather depressingly, the situation is even worse for the cap product, there is an endless variety of definitions of the cap product in the literature. More precisely, given a cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma\colon \Delta^l \to X$ the following

definitions of $\varphi \cap \sigma$ can be found in the literature:[2189]

		$\varphi \cap \sigma$
(a)	Hatcher [**Hat2002**, p. 239]	$\sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k])$
(b)	Bredon [**Bre1993**, p. 335]	$(-1)^{k(l-k)} \cdot \sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k])$
(c)	Dold [**Dol1956**, p. 335]	$(-1)^{k(l-k)} \cdot \sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_{l-k}] \otimes \varphi(\sigma \circ [\mathfrak{v}_{l-k}, \dots, \mathfrak{v}_l])$
(d)	Spanier [**Spa1995**, p. 254]	$\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_{l-k}] \otimes \varphi(\sigma \circ [\mathfrak{v}_{l-k}, \dots, \mathfrak{v}_l])$
(e)	Munkres [**Mun1984**, p. 389]	$\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_{l-k}] \otimes \varphi(\sigma \circ [\mathfrak{v}_{l-k}, \dots, \mathfrak{v}_l])$

To the best of my understanding, see [**Spa1995**, p. 250], the singular chains

$$\sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]) \quad \text{and} \quad (-1)^{k(l-k)} \cdot \sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_{l-k}] \otimes \varphi(\sigma \circ [\mathfrak{v}_{l-k}, \dots, \mathfrak{v}_l])$$

are homologous. This shows that on the level of homology and cohomology groups the definition (a), which is also our definition, agrees with (c), but that the sign differs for (b), (d) and (e). For the most part this is not an issue, but one needs to be careful when one cites formulas from the literature. For example, as we had already mentioned on page 3806, Bredon [**Bre1993**] defined the coboundary map as

$$\delta_k^{\mathrm{Br}} \coloneqq (-1)^{k+1} \cdot \partial_{k+1}^* = (-1)^{k+1} \cdot \delta_k$$

and as we saw above, the cap product is also defined differently. Thus our Cap Product–Boundary Lemma 206.3, which says that for $\varphi \in \mathrm{C}^k(X; R)$ and $\sigma \colon \Delta^l \to X$ we have

$$\partial_{l-k}(\varphi \cap \sigma) = (-1)^k \cdot (-\delta_k(\varphi) \cap \sigma) + \varphi \cap \partial_l(\sigma))$$

becomes the following (admittedly neater) equality in [**Bre1993**, Proposition VI.5.1]:[2190]

$$\partial_{l-k}(\varphi \cap \sigma) = \delta_k^{\mathrm{Br}}(\varphi) \cap \sigma + (-1)^k \cdot \varphi \cap \partial_l(\sigma). \qquad \square$$

206.2. **Calculations of cap products.** In this section we consider the cap product on the following topological spaces:

(1) The torus with $\mathbb{Z}$-coefficients.
(2) The real projective plane with $\mathbb{F}_2$-coefficients.

206.2.1. **Cap product on the torus.** In this subsection we will calculate the cap product on the torus:

Lemma 206.6. (Torus–Cap Product Lemma) Let $T = S^1 \times S^1$ be the torus. We recall the following notations and results:

- Let p and q be the projections $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
- We denote by $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1; \mathbb{Z})$ the singular 1-cocycle that we introduced on page 3813.
- We denote by $[\{1\} \times S^1]$ the image of the fundamental class of $\{1\} \times S^1$ under the monomorphism $\mathrm{H}_1(\{1\} \times S^1; \mathbb{Z}) \to \mathrm{H}_1(S^1 \times S^1; \mathbb{Z})$.

We have

[2189] The definition of the cap product given by Greenberg-Harper in [**GrH1981**, p. 205] agrees with our definition, *except*, that they use a different notation, namely they cap a homology class with a cohomology class instead of capping a cohomology class with a homology class. In other words, in our setting they write $\sigma \cap \varphi$ instead of $\varphi \cap \sigma$.

[2190] The lesson you should draw is that if you ever have to write algebraic topology lectures notes, stick to a single book and do not try to mix books.

$$\begin{array}{rcl} [p^*(\theta_{\mathbb{Z}})] \cap [T] & = & [\{1\} \times S^1] \\ [q^*(\theta_{\mathbb{Z}})] \cap [T] & = & -[S^1 \times \{1\}] \end{array}$$

In particular the map

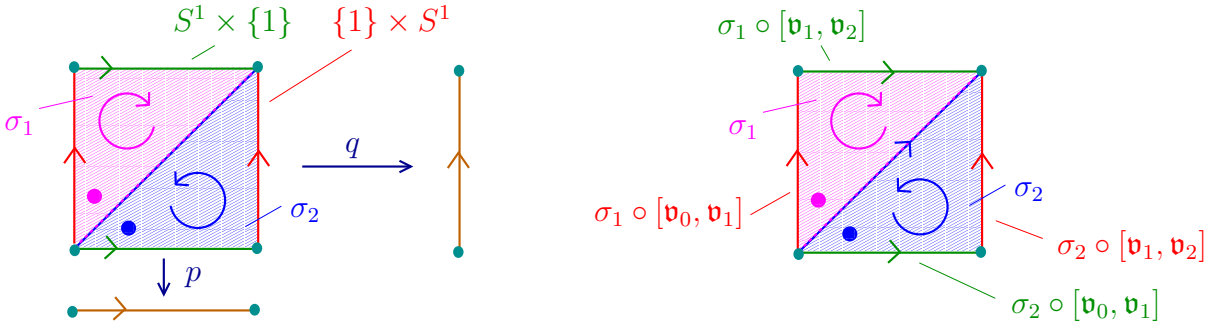
$$\begin{array}{rcl} \mathrm{H}^1(T;\mathbb{Z}) & \to & \mathrm{H}_1(T;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [T] \end{array}$$

is an isomorphism.

Proof. We consider $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\!\sim$. We introduce some more notation:

(1) We denote by σ_1, σ_2 the singular 2-simplices from the figure below. As we pointed out on page 3756, the singular 2-chain $-\sigma_1 + \sigma_2$ is a representative of the fundamental class $[T]$.

(2) We denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t,t) = \exp(2\pi\,\mathrm{i}t)$. By the discussion on page 3536 we know that μ represents the fundamental class $[S^1]$ of S^1 equipped with the standard orientation.



We calculate that

$$p^*(\theta_{\mathbb{Z}}) \cap \sigma_1 \;=\; \sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \otimes \overbrace{p^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}^{=\theta_{\mathbb{Z}}(p \circ \sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) = 0} \;=\; 0$$

and

$$p^*(\theta_{\mathbb{Z}}) \cap \sigma_2 \;=\; \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \otimes \underbrace{p^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}_{=\theta_{\mathbb{Z}}(p \circ \sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) = 1} \;=\; \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \;=\; 1 \times \mu.$$

Putting these calculations together we obtain that

$$[p^*(\theta_{\mathbb{Z}})] \cap [T] \;=\; [p^*(\theta_{\mathbb{Z}}) \cap (-\sigma_1 + \sigma_2)] \;=\; [\{1\} \times S^1].$$

Basically the same calculation shows that

$$[q^*(\theta_{\mathbb{Z}})] \cap [T] \;=\; [q^*(\theta_{\mathbb{Z}}) \cap (-\sigma_1 + \sigma_2)] \;=\; -[S^1 \times \{1\}].$$

Since $[p^*(\theta_{\mathbb{Z}})]$ and $[q^*(\theta_{\mathbb{Z}})]$ form a basis of $\mathrm{H}^1(T;\mathbb{Z})$ and since $[\{1\} \times S^1]$ and $[S^1 \times \{1\}]$ form a basis of $\mathrm{H}_1(T;\mathbb{Z})$ we have thus shown that the map $\varphi \mapsto \varphi \cap [T]$ is indeed an isomorphism. ■

Remark. In Exercise 206.1 we will deal with the surface Σ_2 of genus 2 and there we will see that, similar to the Torus–Cap Product Lemma 206.6, the map

$$\begin{array}{rcl} \mathrm{H}^1(\Sigma_2;\mathbb{Z}) & \to & \mathrm{H}_1(\Sigma_2;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [\Sigma_2] \end{array}$$

is an isomorphism. □

206.2.2. **The cap product on the real projective plane.** In this subsection we consider the real projective plane $\mathbb{R}\mathrm{P}^2$:

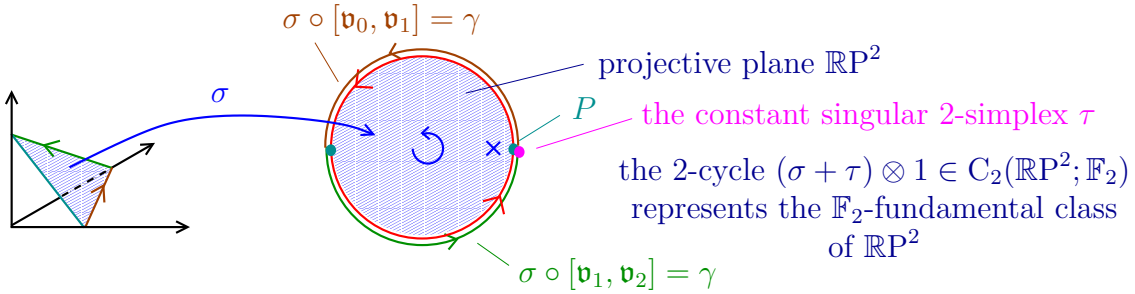
Lemma 206.7. (Projective Plane–Cap Product Lemma) The map

$$\begin{array}{ccc} \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \to & \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Proof. We consider $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$. We introduce the following notation:

- We denote by $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ the cocycle that we introduced in the Generator-of-$\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$-Lemma 203.31.
- Let $\gamma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ be the singular 1-simplex that is given by $\gamma(t,1-t) = [e^{\pi i t}]$. Note that γ is a cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$ and note that $\varphi(\gamma) = 1$.
- In the figure below we see the singular 2-simplex σ that sends the face $[\mathfrak{v}_0,\mathfrak{v}_2]$ to a point P and the singular 2-simplex τ that sends all points on Δ^2 to P. It is straightforward to see that $(\sigma+\tau)\otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a cycle. Using the $\mathbb{F}_2$-Fundamental Class-Criterion 194.9 we see that it represents the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$.



It is straightforward to see that

$$[\varphi] \cap [\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2} \;=\; [\varphi] \cap [(\sigma+\tau)\otimes 1] \;=\; [\,(\underbrace{\varphi\cap\sigma}_{=\gamma})\otimes 1 + (\underbrace{\varphi\cap\tau}_{=0})\otimes 1] \;=\; [\gamma\otimes 1].$$

It follows from $\varphi(\gamma) = 1$ that $[\gamma\otimes 1]$ is the (unique) non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. ■

Remark. In the three examples above we saw that for the closed 2-dimensional smooth manifolds $F = S^1\times S^1$, $F = \Sigma_2$, $F = \mathbb{R}\mathrm{P}^2$ and for $R = \mathbb{Z}$ respectively $R = \mathbb{F}_2$ the map

$$\begin{array}{ccc} \mathrm{H}^1(F;R) & \to & \mathrm{H}_{2-1}(F;R) \\ \varphi & \mapsto & \varphi \cap [F]_R \end{array}$$

is an isomorphism. In Chapter 210 we will see that this is not a coincidence. More precisely we will see that this observation can be generalized to all closed topological manifolds. □

206.3. **The relative cap product.** After considering the various examples of the cap product we continue with developing the theory.

Similar to the case of the cup product, we intend to extend the notion of cap product to relative homology and cohomology groups. As in the case of the cup product, this requires some delicate footwork to get to the right notion of a relative cap product. We start out with the following rather technical definition.

Definition 206.8. Let (X, A, B) be a triad of topological spaces and let R be a commutative ring.

(1) We write $$\mathrm{C}_n^{\{A,B\}}(A \cup B) \coloneqq \mathrm{C}_n(A) + \mathrm{C}_n(B) \subseteq \mathrm{C}_n(X).$$

(2) We write $$\mathrm{C}_n(X, \{A, B\}; R) \coloneqq \mathrm{C}_n(X, \{A, B\}) \otimes R = \left(\frac{\mathrm{C}_n(X)}{\mathrm{C}_n^{\{A,B\}}(A \cup B)}\right) \otimes R.$$

It follows from the Functor-applied-to-Chain Complex Lemma 160.3 and the Tensor–Functor Lemma 160.4 that the usual boundary map on the chain complex $\mathrm{C}_*(X; R)$ defines a boundary map on $\mathrm{C}_*(X, \{A, B\}; R)$. We denote the corresponding homology groups by $\mathrm{H}_*(X, \{A, B\}; R)$.

This technical definition allows us to formulate the following lemma, which is a generalization of the Cap Product–Definition Lemma 206.4 and which can be viewed as an analog of the Triad–Cup Product Lemma 204.6:

Lemma 206.9. (Triad–Cap Product Lemma) Let (X, A, B) be a triad of topological spaces, let $k \leq l$ and let R be a commutative ring.

(1) The map
$$\begin{aligned} \cap \colon \mathrm{C}^k(X, A; R) \times \mathrm{C}_l(X, \{A, B\}) &\to \mathrm{C}_{l-k}(X, B; R) \\ (\varphi, [\sigma \colon \Delta^l \to X]) &\mapsto \underbrace{\sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_l]}_{\text{map } \Delta^{l-k} \to X} \otimes \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k])}_{\in R} \end{aligned}$$
is well-defined.

(2) The map
$$\begin{aligned} \mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, \{A, B\}; R) &\to \mathrm{H}_{l-k}(X, B; R) \\ \left([\varphi], \left[\sum_{i=1}^m \sigma_i \otimes r_i\right]\right) &\mapsto \left[\sum_{i=1}^m \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i\right] \end{aligned}$$
is well-defined and R-bilinear.

Proof.

(1) By definition of $\mathrm{C}_l(X, \{A, B\}) \coloneqq \frac{\mathrm{C}_l(X)}{\mathrm{C}_l^{\{A,B\}}(A \cup B)} = \frac{\mathrm{C}_l(X)}{\mathrm{C}_l(A) + \mathrm{C}_l(B)}$ it remains to prove the following claim:

Claim. If we are given a singular l-simplex $\sigma \colon \Delta^l \to X$ such that
(a) the image of σ lies in A, or
(b) the image of σ lies in B,
then $\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k])$ is zero in $\mathrm{C}_{l-k}(X, B; R) = \mathrm{C}_{l-k}(X, B) \otimes R$.

Proof. But this is indeed the case:
(a) If the image σ lies in A, then it follows from the hypothesis that $\varphi \in \mathrm{C}^k(X, A; R)$, that $\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]) = 0$.
(b) Now we assume that the image of σ lies in B. This implies that the image of $\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]$Flies in B. Thus the factor $\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]$ is zero in $\mathrm{C}_{l-k}(X, B)$. ⊞

(2) The argument that the map on (co-) chain complexes descends to a map on the level of homology and cohomology groups is the same as in the proof of the Cap Product–Definition Lemma 206.4. Finally note that it is obvious that the map $\cap$ is R-bilinear. ■

The Excisive Triad–Invariants Lemma 204.4 (2a) leads to the following definition:

Definition 206.10. Let (X, A, B) be an excisive triad of topological spaces, let $k \leq l$ and let R again be a commutative ring. We refer to the map

$$\mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, A \cup B; R) \xleftarrow[\uparrow]{\cong} \mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, \{A, B\}; R) \xrightarrow[\uparrow]{\cap} \mathrm{H}_{l-k}(X, B; R)$$

isomorphism given by the Excisive Triad–Invariants Lemma 204.4 (2a), here we use that (X, A, B) is excisive

map from the Triad–Cap Product Lemma 206.9

as the cap product (sometimes we refer to it as the relative cap product) that we also denote by "$\cap$". For $k > l$ we define the cap product to be the zero map.

Remark. [2191] Let X be a topological space, let $A, B \subseteq X$ be subsets and let R be a commutative ring. Suppose that $\mathrm{C}_*^{\{A,B\}}(A \cup B) = \mathrm{C}_*(A \cup B)$. (This is for example the case if we are in the setting of the Excisive Triad-Proposition 158.20 (0), e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$.) Then in the above definition the horizontal map to the left is the identity and we see that the relative cap product is just given by the "naive" definition

$$\begin{array}{rcl} \mathrm{C}^k(X, A; R) \times \mathrm{C}_l(X, A \cup B; R) & \to & \mathrm{C}^{l-k}(X, B; R) \\ (\varphi, \sigma\colon \Delta^l \to X) & \mapsto & \sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]). \end{array}$$

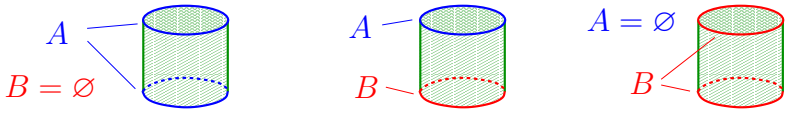
□

Again we formulate our main example as a lemma:

Lemma 206.11. (Annulus–Cap Product Lemma) Let $X = S^1 \times [0, 1]$ be the annulus. Let $A, B \subseteq \partial X$ be unions of boundary components such that $A \cup B = \partial X$ and $A \cap B = \varnothing$. Then the map

$$\begin{array}{rcl} \mathrm{H}^1(X, A; \mathbb{Z}) & \to & \mathrm{H}_1(X, B; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is an isomorphism.

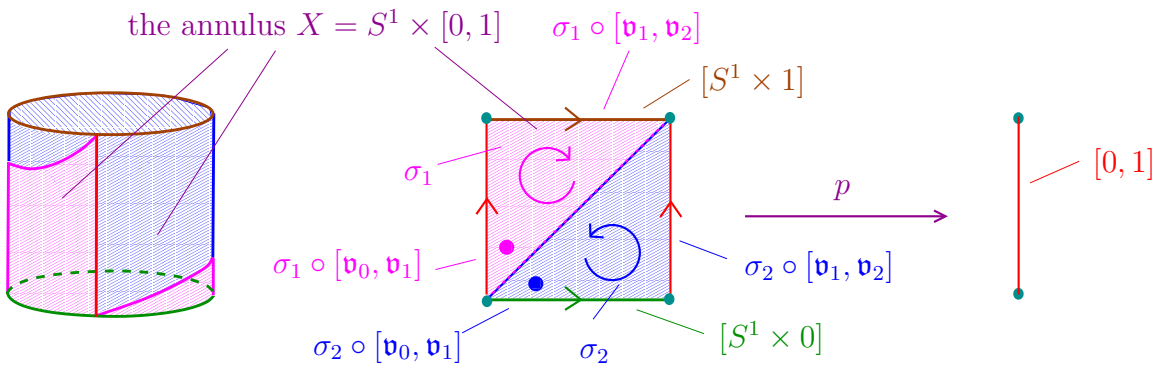


Proof. Let $X = S^1 \times [0, 1]$ be the annulus. There are several options for A and B:

(1) First we assume that $A = \partial X$ and $B = \varnothing$. We will use the following notation:
- In the figure below we show two singular 2-simplices σ_1, σ_2. It follows easily from the Fundamental Class-Criterion 194.8 that $(-\sigma_1 + \sigma_2) \otimes 1$ represents the fundamental class $[X] \in \mathrm{H}_2(X, \partial X; \mathbb{Z})$ of the annulus X with the usual orientation.
- We denote by $p\colon X \to [0, 1]$ the projection.
- Let $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0, 1], \{0, 1\}, \mathbb{Z})$ be the relative 1-cocycle from page 3816 and 3821.

We obtain that

[2191] This remark is the analog of the remark on page 3927.



$$\begin{array}{rcl} [p^*(\overline{\gamma}_{\frac{1}{2}})] \cap \sigma_1 & = & \overbrace{\sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2]}^{=[S^1\times\{1\}]} \otimes \overbrace{\overline{\gamma}_{\frac{1}{2}}(p \circ \sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}^{=1} = [S^1 \times \{1\}] \\ [p^*(\overline{\gamma}_{\frac{1}{2}})] \cap \sigma_2 & = & \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \otimes \underbrace{\overline{\gamma}_{\frac{1}{2}}(p \circ \sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}_{=0} = 0. \end{array}$$

Since $[X] = (-\sigma_1 + \sigma_2) \otimes 1$ we obtain from these calculations that

$$[p^*(\overline{\gamma}_{\frac{1}{2}})] \cap [X] = [p^*(\overline{\gamma}_{\frac{1}{2}}) \cap (-\sigma_1 + \sigma_2)] = -[S^1 \times \{1\}].$$

Since the inclusion $S^1 \times \{1\} \to X$ is a homotopy equivalence we see that $\mathrm{H}_1(X;\mathbb{Z}) \cong \mathbb{Z}$ and that $[S^1 \times 1]$ is a generator. A straightforward argument, e.g. using the long exact sequence of the pair $(X, \partial X)$, shows that $\mathrm{H}^1(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$. The above calculation thus shows that the map

$$\begin{array}{rcl} \mathrm{H}^1(X, \partial X; \mathbb{Z}) & \to & \mathrm{H}_1(X; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is an epimorphism. Since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism and as a bonus we obtain that $[p^*(\overline{\gamma}_{\frac{1}{2}})] \in \mathrm{H}^1(X, \partial X; \mathbb{Z})$ is a generator.

(2) The case $A = \varnothing$ and $B = \partial X$ follows from a similar argument as in (1). We will provide the details in Exercise 206.2 (a).

(3) The case that A and B both consist of a single boundary component will be dealt with in Exercise 206.2 (b). ■

206.4. **Properties of the cap product.** The following lemma shows that for $k = l$ the cap product is essentially the same as the Kronecker pairing.

Lemma 206.12. (Cap Product-Kronecker Lemma) Let X be a topological space, let $A \subseteq X$ and let R be a commutative ring. The composition

$$\mathrm{H}^k(X, A; R) \times \mathrm{H}_k(X, A; R) \xrightarrow{\cap} \mathrm{H}_0(X; R) \xrightarrow[\text{from page 3079}]{\text{augmentation } \epsilon_X} R$$

is precisely the Kronecker pairing $\langle\ ,\ \rangle_{(X,A)}$ from Definition 199.8 and Item (1). In other words, for all $\varphi \in \mathrm{H}^k(X, A; R)$ and $\sigma \in \mathrm{H}_k(X, A; R)$ we have

$$\epsilon_X(\varphi \cap \sigma) = \langle \varphi, \sigma \rangle_{(X,A)}.$$

Proof. The lemma follows almost immediately from the definitions. For peace of mind, let u provide the details. Let $\varphi \in \mathrm{C}^k(X, A; R)$ be a cocycle and let $b = \sum_{i=1}^{m} \sigma_i \otimes r_i \in \mathrm{C}_k(X, A; R)$

be a cycle. We see that

$$\begin{aligned}\epsilon_X([\varphi]\cap[b]) &\overset{\text{follows from the discussion on page 3958}}{=} \epsilon_X\Big(\sum_{i=1}^{k}(\varphi\cap\sigma_i)\cdot r_i\Big) &&= \epsilon_X\Big(\sum_{i=1}^{k}\sigma([\mathfrak{v}_0])\otimes\varphi(\sigma_i\circ\underbrace{[\mathfrak{v}_0,\dots,\mathfrak{v}_k]}_{=\mathrm{id}_{\Delta^k}})\cdot r_i\Big)\\ &= \sum_{i=1}^{k}\varphi(\sigma_i)\cdot r_i &&= \langle[\varphi],[b]\rangle_{(X,A)}.\end{aligned}$$

■

Example. Let M be a compact oriented path-connected non-empty n-dimensional topological manifold. We calculate that

$$\epsilon_M([M]^*\cap[M]) \underset{\substack{\uparrow\\ \text{by the Cap Product-Kronecker Lemma 206.12}}}{=} \langle[M]^*,[M]\rangle \underset{\substack{\uparrow\\ \text{by Definition 201.27 of } [M]^*}}{=} 1.$$

Since M is path-connected we know that $\epsilon_M\colon \mathrm{H}_0(M;\mathbb{Z})\to\mathbb{Z}$ is an isomorphism. Since for any $P\in M$ we have $\epsilon_M(\{P\}\otimes 1)=1$ we now see that $[M]^*\cap[M]=[\{P\}\otimes 1]$. Running the argument backwards we see that $[M]^*$ is the only element in $\mathrm{H}^n(M,\partial M;\mathbb{Z})\cong\mathbb{Z}$ that has that property. □

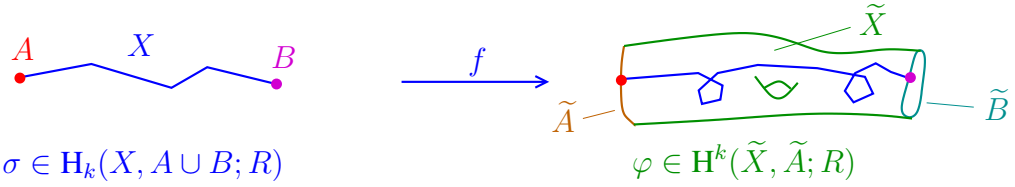
The following lemma can be viewed as an analog of the Cup Product-Naturality Lemmas 203.9 and 204.8:

Lemma 206.13. (Cap Product-Naturality Lemma) Let R be a commutative ring. If $f\colon (X,A,B)\to(\widetilde{X},\widetilde{A},\widetilde{B})$ is a continuous map between two excisive triads, then for any $k,l\in\mathbb{N}_0$ the following diagram commutes:[2192]

$$\begin{array}{ccccc}\mathrm{H}^k(X,A;R) & \times & \mathrm{H}_l(X,A\cup B;R) & \xrightarrow{\cap} & \mathrm{H}_{l-k}(X,B;R)\\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*}\\ \mathrm{H}^k(\widetilde{X},\widetilde{A};R) & \times & \mathrm{H}_l(\widetilde{X},\widetilde{A}\cup\widetilde{B};R) & \xrightarrow{\cap} & \mathrm{H}_{l-k}(\widetilde{X},\widetilde{B};R),\end{array}$$

i.e. for any $\varphi\in\mathrm{H}^k(\widetilde{X},\widetilde{A};R)$ and any $\sigma\in\mathrm{H}_l(X,A\cup B;R)$ we have

$$\varphi\cap f_*(\sigma) = f_*(\underbrace{f^*(\varphi)\cap\sigma}_{\in\mathrm{H}_{l-k}(X,B;R)})\in\mathrm{H}_{l-k}(\widetilde{X},\widetilde{B};R).$$

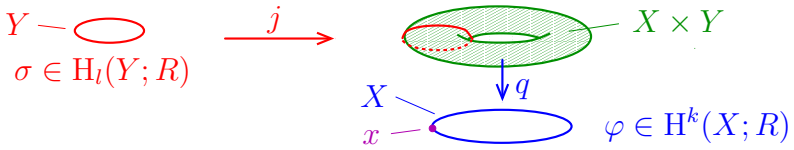


Proof. Similar to the proof of the Cup Product-Naturality Lemma 204.8 the statement follows easily from the definitions. We leave it to the reader to fill in the details. ■

In Section 204.4 we saw that if we apply the Cup Product-Naturality Lemmas 203.9 and 204.8 judiciously, then it can be quite useful for calculating cup products. In a similar vein the Cap Product-Naturality Lemma 206.13 can also be helpful in the calculation of cap products:

[2192]Here "the diagram commutes" is understood in the sense of Convention 204.17.

Example. Let X, Y be topological spaces, let $x \in X$ and let R be a commutative ring. We denote by $p\colon X \times Y \to X$ the natural projection and we denote by $j\colon Y \to X \times Y$ the embedding that is given by $y \mapsto (x, y)$. Let $\varphi \in \mathrm{H}^k(X; R)$ and let $\sigma \in \mathrm{H}_l(Y; R)$.



By the Cap Product-Naturality Lemma 206.13 we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(Y;R) & \times & \overset{\sigma}{\mathrm{H}_l(Y;R)} & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{l-k}(Y;R) \\ {\scriptstyle j^*}\uparrow & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \mathrm{H}^k(X\times Y;R) & \times & \mathrm{H}_l(X\times Y;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{l-k}(X\times Y;R). \\ {\scriptstyle p^*(\varphi)} & & & & \end{array}$$

It follows that for $k \geq 1$ we have

$$p^*(\varphi) \cap j_*(\sigma) \underset{\substack{\uparrow \\ \text{by the above} \\ \text{commutative diagram}}}{=} j_*(j^*(p^*(\varphi)) \cap \sigma) = j_*(\underbrace{(\underbrace{p \circ j}_{\text{constant map}})^*}_{\text{zero map since } k \geq 1}(\varphi) \cap \sigma) = 0.$$

□

The Cap Product-Naturality Lemma 206.13 also allows us to prove the following lemma, which is an analog of the Cup Product–Disjoint Union Lemma 204.16.

Lemma 206.14. (Cap Product–Disjoint Union Lemma) Let X be a topological space and let R be a commutative ring. We denote by $\{X_j\}_{j\in J}$ the path-components of X. By Path-Component-H*-Lemma 198.8 the inclusion maps $\iota_j\colon X_j \to X$, $j \in J$, induce for any $n \in \mathbb{N}_0$ an isomorphism

$$\Psi := \prod_{j\in J} \iota_j^*\colon \mathrm{H}^n(X;R) \xrightarrow{\cong} \prod_{j\in J} \mathrm{H}^n(X_j;R).$$

For any $i \neq j$, any $\alpha \in \mathrm{H}^m(X_i; R)$ and any $\beta \in \mathrm{H}_n(X_j; R)$ we have

$$\Psi^{-1}(\alpha) \cap \iota_{j*}(\beta) = 0.$$

The same type of statements hold for cap products on relative (co-) homology groups.

Proof. The statements follow easily from the explicit definition of the cap product. For fun we also give a slightly more formal proof, which only needs the general statements we already proved. Let $\alpha \in \mathrm{H}^m(X_i; R)$ and $\beta \in \mathrm{H}_n(X_j; R)$. We consider the following diagram:

$$\begin{array}{ccccc} \overset{0}{\mathrm{H}^m(X_j;R)} & \times & \overset{\beta}{\mathrm{H}_n(X_j;R)} & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-m}(X_j;R) \\ {\scriptstyle \iota_{j*}}\uparrow & & \downarrow{\scriptstyle \iota_j^*} & & \downarrow{\scriptstyle \iota_{j*}} \\ \mathrm{H}^m(X;R) & \times & \mathrm{H}_n(X;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-m}(X;R). \\ {\scriptstyle \Psi^{-1}(\alpha)} & & {\scriptstyle \iota_{j*}(\alpha)} & & \end{array}$$

Note that $\iota_j^*(\Psi^{-1}(\alpha)) = 0$ since $i \neq j$. Thus the cap product on top is zero. But since the diagram commutes we see that the cap product on the bottom is also zero. ■

206.5. **Homology as a module over the cohomology ring.** The following lemma relates the cup product to the cap product.

Lemma 206.15. (Cup-Cap Lemma) Let X be a topological space and let R be a commutative ring. Let $A \subseteq X$ be a subset and let $C, D \in \{\varnothing, A\}$. For any $\varphi \in \mathrm{H}^k(X, C; R)$, any $\psi \in \mathrm{H}^l(X, D; R)$ and any $\sigma \in \mathrm{H}_n(X, C \cup D; R)$ we have[2193]

$$\varphi \cap (\underbrace{\psi \cap \sigma}_{\in \mathrm{H}_{n-l}(X,C;R)}) \;=\; (\underbrace{\psi \cup \varphi}_{\in \mathrm{H}^{k+l}(X,C\cup D;R)}) \cap \sigma \quad \in \mathrm{H}_{n-k-l}(X; R).$$

Proof. Let X be a topological space and let R be a commutative ring. We start out with a claim on the (co-) chain level:

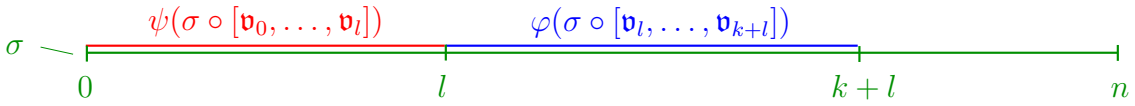
Claim. Let $\varphi \in \mathrm{C}^k(X; R)$ and let $\psi \in \mathrm{C}^l(X; R)$. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{C}_{n-k-l}(X; R).$$

Proof. If $k + l > n$, then both sides are by definition 0. So suppose that $k + l \leq n$. We then see that

$$\begin{aligned}
\varphi \cap (\psi \cap \sigma) &= \varphi \cap (\sigma \circ [\mathfrak{v}_l, \ldots, \mathfrak{v}_n] \otimes \psi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_l])) \\
&= (\sigma \circ [\mathfrak{v}_{l+k}, \ldots, \mathfrak{v}_n] \otimes \varphi(\sigma \circ [\mathfrak{v}_l, \ldots, \mathfrak{v}_{l+k}])) \cdot \psi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_l]) \\
&= F\sigma \circ [\mathfrak{v}_{l+k}, \ldots, \mathfrak{v}_n] \otimes \underbrace{\psi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_l]) \cdot \varphi(\sigma \circ [\mathfrak{v}_l, \ldots, \mathfrak{v}_{l+k}])}_{=(\psi\cup\varphi)(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_{l+k}])} \\
&\quad \uparrow \text{ since } R \text{ is commutative} \\
&= (\psi \cup \varphi) \cap \sigma.
\end{aligned}$$

The above argument gets summarized in the following figure:



We hope this helps. ⊞

Now we turn to the proof of the desired statement. Let $A \subseteq X$ be a subset and let $C, D \in \{\varnothing, A\}$. Furthermore let $[\varphi] \in \mathrm{H}^k(X, C; R)$, $[\psi] \in \mathrm{H}^l(X, D; R)$ and finally let $\left[\sum_{i=1}^{d} a_i \cdot \sigma_i\right] \in \mathrm{H}_n(X, C \cup D; R)$. Since $C, D \in \{\varnothing, A\}$ we obtain from the discussions on pages 3958 and 3927 that in all four calculations of cup and cap products we can deal with the "naive" definition on the (co-) chain level. The desired then follows immediately from the claim. ■

Lemma 206.16. (Cap-with-1-Lemma) Let X be a topological space, let $A \subseteq X$ be a subspace and let R be a commutative ring. We denote by $1_X \in \mathrm{H}^0(X; R)$ the multiplicatively neutral element from the Cup Product-Ring Proposition 203.25. For any $\sigma \in \mathrm{H}_k(X; R)$ we have

$$1_X \cap \sigma \;=\; \sigma.$$

[2193] Note that on the right-hand side it says $(\psi \cup \varphi) \cap \sigma$ instead of $(\varphi \cup \psi) \cap \sigma$. In fact if one picks a different sign convention for the cap product this confusing issue goes away. For example with the sign conventions used by Bredon, see [**Bre1993**, Chapter 5], the equation becomes

$$\varphi \cap (\psi \cap \sigma) \;=\; (\varphi \cup \psi) \cap \sigma$$

which looks much more appealing. It follows from the Cup Product-Commutativity Proposition 204.10 that the difference between the two expressions $(\psi \cup \varphi) \cap \sigma$ and $(\varphi \cup \psi) \cap \sigma$ is just a factor of $(-1)^{kl}$.

Proof. The lemma follows immediately from the definitions. ■

One can summarize the above two lemmas in slightly more fancy language. To do so we need to introduce the notion of an S-right module:

Definition 206.17. Let S be a ring. An S-right module is an abelian group A together with a bilinear map

$$\begin{array}{rcl} S \times A & \to & A \\ (s, a) & \mapsto & a \cdot s \end{array}$$

such that the following conditions are satisfied:

$$\begin{array}{lrcll} \text{(a)} & (a \cdot s) \cdot \widetilde{s} & = & a \cdot (s \cdot \widetilde{s}) & \text{for all } s, \widetilde{s} \in S \text{ and } a \in A, \\ \text{(b)} & a \cdot 1 & = & a & \text{for all } a \in A. \end{array}$$

Lemma 206.18. (Homology-as-Module Lemma) Let X be a topological space, let $A \subseteq X$ be a subspace and let R be a commutative ring. The map

$$\begin{array}{rcl} \mathrm{H}^*(X; R) \times \mathrm{H}_*(X, A; R) & \to & \mathrm{H}_*(X, A; R) \\ (\varphi, \sigma) & \mapsto & \varphi \cap \sigma \end{array}$$

turns $\mathrm{H}_*(X, A; R) := \bigoplus_{n \in \mathbb{N}_0} \mathrm{H}_*(X, A; R)$ into a right module over the ring $(\mathrm{H}^*(X; R), \cup)$.

Proof. This statement follows immediately from the bilinearity of the cap product, the Cup-Cap Lemma 206.15 and the Cap-with-1-Lemma 206.16. ■

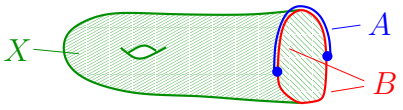
206.6. **Cap product and connecting homomorphisms.** In this last section we will relate in two lemmas the relative cap product to (co-) boundary maps on (co-) homology. These results are rather technical, and perhaps it is best to look them up, once they get actually used.

We start out with the first of the two lemmas:

Lemma 206.19. (Cap Product-Triple-Connecting Homomorphism Lemma) Let R be a commutative ring and let (X, B, A) be a triple of topological spaces. For any $n, k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(B, A; R) \times \mathrm{H}_{n-1}(B, A; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B; R) \\ \downarrow{\scriptstyle \delta_k} & \uparrow{\scriptstyle \partial_n} & \downarrow{\scriptstyle i_*} \\ \mathrm{H}^{k+1}(X, B; R) \times \mathrm{H}_n(X, B; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(X; R). \end{array}$$

Here $i\colon B \to X$ denotes the inclusion map and δ_k and ∂_n denote the connecting homomorphisms in the long exact sequences in (co-) homology corresponding to the triple (X, B, A).



triple to which we will apply the Cap Product-Triple-Connecting Homomorphism Lemma 206.19

Proof. Let $\psi \in \mathrm{H}^k(B, A; R)$ and $\tau \in \mathrm{H}_n(B, A; R)$. We make the following preparations:

- We pick $\sigma \in \mathrm{C}_n(B; R)$ such that $[\sigma] \in \mathrm{C}_n(B, A; R)$ is a cycle that represents τ.
- We pick $\varphi \in \mathrm{C}^k(B; R) = \mathrm{Hom}(\mathrm{C}_k(B), R)$ that vanishes on $\mathrm{C}_k(A)$ and such that the corresponding element in $\mathrm{C}^k(B, A; R)$ represents ψ. We denote by $\widetilde{\varphi}\colon \mathrm{C}_k(X) \to R$ the

extension of φ that is defined to be zero on all singular k-simplices whose image does not lie in B.

We obtain that

follows from the discussion of relative cap products on page 3958

$$\delta\psi \cap \tau \underset{\uparrow}{=} [\delta\widetilde{\varphi}] \cap [\sigma] \overset{\downarrow}{=} [\delta\widetilde{\varphi} \cap \sigma] \underset{\uparrow}{=} [\widetilde{\varphi} \cap \partial\sigma] = [\varphi \cap \partial\sigma] \overset{\downarrow}{=} [\varphi] \cap [\partial\sigma] \underset{\uparrow}{=} \psi \cap \partial\tau.$$

by the LES-of-Triple Proposition 198.13 and the discussion on page 3819

by the Cap Product–Boundary Lemma 206.3 the classes $\delta\widetilde{\varphi} \cap \sigma$ and $\widetilde{\varphi} \cap \partial\sigma$ are homologous

by the Topological LES Proposition 146.5

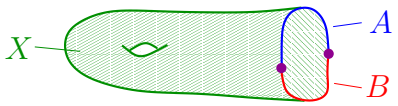
This equality implies that the diagram commutes.[2194] ■

The following lemma is similar in spirit to the Cup Product-Connecting Homomorphism Lemma 204.18:

Lemma 206.20. (Cap Product-Triad-Connecting Homomorphism Lemma) Let R be a commutative ring, let (X, A, B) be an excisive triad and let $n \in \mathbb{N}_0$. We denote by $i\colon (A \cup B, A) \to (X, A)$ and $j\colon (B, A \cap B) \to (A \cup B, A)$ the inclusion maps. It follows from the Excisive Characterization Lemma 158.19 (1)⇒(3), the Quasi Iso–Homotopy Equivalence Proposition 158.8 and the Homology-with-Coefficients Lemma 160.6 (2b) that the map $j_*\colon \mathrm{H}_{n-1}(B, A \cap B; R) \to \mathrm{H}_{n-1}(A \cup B, A; R)$ is an isomorphism. For any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc}
\mathrm{H}^k(X, A; R) \times \mathrm{H}_n(X, A \cup B; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(X, B; R) \\
{\scriptstyle i^*}\downarrow \quad {\scriptstyle \partial_n}\downarrow & & \\
\mathrm{H}^k(A \cup B, A; R) \times \mathrm{H}_{n-1}(A \cup B, A; R) & & \Big\downarrow {\scriptstyle \partial_{n-k}} \\
{\scriptstyle j^*}\downarrow \cong \quad {\scriptstyle (j_*)^{-1}}\downarrow \cong & & \\
\mathrm{H}^k(B, A \cap B; R) \times \mathrm{H}_{n-1}(B, A \cap B; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B; R).
\end{array}$$

Here the ∂_i denote the connecting homomorphisms in the long exact sequences in homology corresponding to the triple $(X, A \cup B, A)$ respectively the pair (X, B).



excisive triad to which we will apply the Cap Product-Triad-Connecting Homomorphism Lemma 206.20

Proof. Let $\psi \in \mathrm{H}^k(X, A; R)$ and $\tau \in \mathrm{H}_n(X, A \cup B; R)$. We make the following observations:

(a) It follows immediately from our hypothesis that the triad (X, A, B) is excisive together with the Excisive Triad–Invariants Lemma 204.4 (2a) that we can find $\sigma \in \mathrm{C}_n(X; R)$ such that $\partial_n(\sigma) = \alpha + \beta$ with $\alpha \in \mathrm{C}_{n-1}(A; R)$ and $\beta \in \mathrm{C}_{n-1}(B; R)$ and such that the image of σ in $\mathrm{C}_n(X, A \cup B; R)$ is a cycle that represents τ.

(b) We pick a cocycle $\varphi \in \mathrm{C}^k(X, A; R) = \mathrm{Hom}(\mathrm{C}_k(X, A), R)$ that represents ψ. We denote by φ also the induced homomorphism $\mathrm{C}_k(X) \to \mathrm{C}_k(X, A) \to R$. Note that φ vanishes on $\mathrm{C}_k(A)$.

[2194] With the various quotient groups, lifts, extensions, inclusions and so on it is admittedly quite tricky to figure out in between where any of the equalities hold.

We have the following equalities:

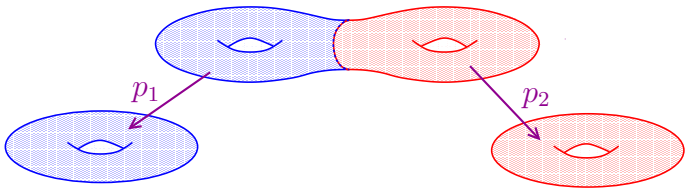
$$\begin{array}{lllll}
\partial_{n-k}(\psi\cap\tau) & \overset{\substack{\text{choice of } \sigma \text{ and definition}\\ \text{of relative cap product}}}{\overset{\downarrow}{=}} \partial_{n-k}([\varphi\cap\sigma]) & \overset{\substack{\text{by the Topological}\\ \text{LES-Proposition 146.5}}}{\overset{\downarrow}{=}} [\partial_{n-k}(\varphi\cap\sigma)] & \overset{\substack{\text{by the Cap Product–Boundary}\\ \text{Lemma 206.3}}}{\overset{\downarrow}{=}} (-1)^k\cdot[-\delta_k(\varphi)\cap\sigma+\varphi\cap\partial_n(\sigma)] \\
 & \underset{\substack{\uparrow\\ \text{since } \delta_k(\varphi)=0 \text{ and } \partial_n(\sigma)=\alpha+\beta}}{=} (-1)^k\cdot[\varphi\cap\alpha+\varphi\cap\beta]) & & \underset{\substack{\uparrow\\ \text{since } \varphi \text{ vanishes on } \mathrm{C}_k(A) \text{ and}\\ \text{since } \alpha\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_k]\in\mathrm{C}_k(A)}}{=} (-1)^k\cdot[\varphi\cap\beta] \\
 & \underset{\substack{\uparrow\\ \text{by the LES-of-Triple Proposition 146.5}\\ \text{we have } j_*([\beta])=\partial_n([\sigma])}}{=} (-1)^k\cdot([j^*i^*\varphi]\cap j_*^{-1}(\partial_n([\sigma]))). & & &
\end{array}$$

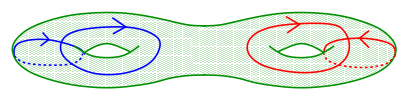
This equality implies that the diagram commutes up to the sign $(-1)^k$. ■

206.7. Exercises for Chapter 206 .

Exercise 206.1. We denote by Σ_2 the surface of genus two. Furthermore we denote by $p_1, p_2\colon \Sigma_2 \to T = S^1\times S^1$ the usual two projections and we denote by $p, q\colon S^1\times S^1\to S^1$ the projection onto the first respectively the second factor. As usual we write $\alpha = p^*(\theta_{\mathbb{Z}})$ and $\beta = q^*(\theta_{\mathbb{Z}})$. We saw that $[p_1^*(\alpha)], [p_1^*(\beta)], [p_2^*(\alpha)], [p_2^*(\beta)]$ form a basis for $\mathrm{H}^1(\Sigma_2;\mathbb{Z})$. Furthermore we denote by $x_1, y_1, x_2, y_2 \in \mathrm{H}_1(\Sigma_2;\mathbb{Z})$ the standard basis. For each of the four basis elements of $\mathrm{H}^1(\Sigma_2;\mathbb{Z})$ determine the cap product with $[\Sigma_2]$ in terms of the basis for $\mathrm{H}_1(\Sigma_2;\mathbb{Z})$.

Remark. You could steal some ideas from the proof of the Surface–Torus Cup Product Lemma 204.15 on page 3933.





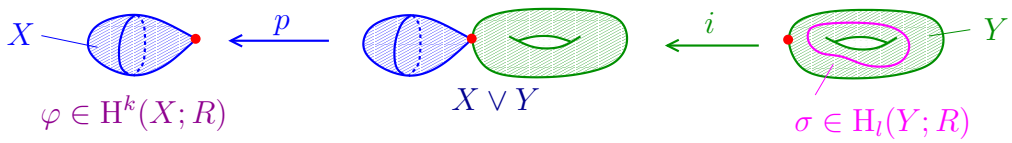
basis x_1, x_2, y_1, y_2 of $\mathrm{H}_1(\Sigma_2;\mathbb{Z})$

Exercise 206.2. We consider the annulus $X = S^1\times[0,1]$.

(a) Show that the map
$$\begin{array}{rcl}\mathrm{H}^1(X;\mathbb{Z}) & \to & \mathrm{H}_1(X,\partial X;\mathbb{Z})\\ \varphi & \mapsto & \varphi\cap[X]\end{array}$$
is an isomorphism.

(b) Let A be one of the components of ∂X and let B be the other component. Show that the map
$$\begin{array}{rcl}\mathrm{H}^1(X,A;\mathbb{Z}) & \to & \mathrm{H}_1(X,B;\mathbb{Z})\\ \varphi & \mapsto & \varphi\cap[X]\end{array}$$
is an isomorphism.

Exercise 206.3. Let X and Y be topological spaces and let $x\in X$ and $y\in Y$ be good points. We consider the wedge $X\vee Y := (X,x)\vee(Y,y)$. We denote by $p\colon X\vee Y\to Y$ the natural projection and we denote by $i\colon X\to X\vee Y$ the natural inclusion. Show that for any commutative ring, any $\varphi\in\mathrm{H}^k(X;R)$ and any $\sigma\in\mathrm{H}_l(Y;R)$ with $k,l\geq 1$ we have $p^*(\varphi)\cap i_*(\sigma) = 0\in\mathrm{H}_{l-k}(X\vee Y;R)$.



Exercise 206.4. Let M be the Möbius band. Show that the map

$$\begin{array}{ccc} \mathrm{H}^1(M;\mathbb{F}_2) & \to & \mathrm{H}_1(M,\partial M;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [M]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.
Remark. You could use the Cap Product-Naturality Lemma 206.13 to reduce the calculation to the Projective Plane–Cap Product Lemma 206.7.

Exercise 206.5. Let X be a topological space and let R be a commutative ring. Let $A \subseteq X$ be a subset and let $C, D \in \{\varnothing, A\}$. Finally let $\varphi \in \mathrm{H}^k(X, C; R)$, $\psi \in \mathrm{H}^l(X, D; R)$ and $\sigma \in \mathrm{H}_{k+l}(X, C \cup D; R)$. Show that we have the following equality of Kronecker pairings:

$$\langle \varphi, \psi \cap \sigma \rangle_{(X,C)} \;=\; \langle \psi \cup \varphi, \sigma \rangle_{(X,C\cup D)} \;\in\; R.$$

Exercise 206.6. We consider $X = [0,1] \times [0,1]$ and the subsets $A = \{0,1\} \times [0,1]$ and $B = [0,1] \times \{0,1\}$. Determine the relative cup product

$$\underbrace{\mathrm{H}^1(X, A; \mathbb{Z})}_{\cong \mathbb{Z}} \times \underbrace{\mathrm{H}_2(X, \partial X; \mathbb{Z})}_{\cong \mathbb{Z}} \to \underbrace{\mathrm{H}_1(X, B; \mathbb{Z})}_{\cong \mathbb{Z}}.$$



Exercise 206.7. We consider the 3-torus $T^3 := S^1 \times S^1 \times S^1$. In Exercise 205.8 we used the Shuffle Proposition 164.24 to write down an explicit 3-cycle $\omega \in \mathrm{C}_3(T^3)$ that represents the fundamental class $[T^3] \in \mathrm{H}_3(T; \mathbb{Z})$.

(a) Let $p\colon T^3 = S^1 \times S^1 \times S^1 \to S^1$ be the projection onto the first coordinate. Determine $[p^*(\theta_{\mathbb{Z}})] \cap [\omega] \in \mathrm{H}_2(T^3; \mathbb{Z})$.

(b) Show that the map

$$\begin{array}{ccc} \mathrm{H}^1(T^3;\mathbb{Z}) & \to & \mathrm{H}_2(T^3;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [T^3] \end{array}$$

is an isomorphism.

Exercise 206.8. Let X be a topological space and let R be a commutative ring. By the Homology-as-Module Lemma 206.18 we know that $\mathrm{H}_*(X; R) := \bigoplus_{n \in \mathbb{N}_0} \mathrm{H}_*(X; R)$ is a right module over the ring $(\mathrm{H}^*(X; R), \cup)$.

(a) Let $X = S^1 \times S^1$. What is the minimal cardinality of a generating in $\mathrm{H}_*(X; \mathbb{Z})$ of this module?

(b) Repeat exercise (a) with $X = \mathbb{R}\mathrm{P}^2$ and $R = \mathbb{F}_2$.

(c) Provide an example of a topological space where the minimal cardinality of such a generating set is at least 5.

Exercise 206.9. Let X be a topological space such that all homology groups are finitely generated. Let $n \in \mathbb{N}$ and let $i \in \mathbb{N}_0$. We assume that there exists a $\sigma \in \mathrm{H}_n(X; \mathbb{Z})$ such

that the map

$$\begin{array}{ccc} \mathrm{H}^i(X;\mathbb{Z}) & \to & \mathrm{H}_{n-i}(X;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap \sigma \end{array}$$

is an isomorphism. Show that for each $\varphi \in \mathrm{H}^i(X;\mathbb{Z})$ which is not torsion there exists a $\psi \in \mathrm{H}^{n-i}(X;\mathbb{Z})$ such that $\varphi \cup \psi \neq 0$.
Remark. The hypotheses sound artificial, but we will see that they are satisfied if X is a closed orientable connected non-empty n-dimensional topological manifold.

Exercise 206.10. We start out with two preparations:

- Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write
$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix}$$
and we consider the orientation-reversing diffeomorphism
$$\begin{array}{rccc} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$
We consider two copies of the solid torus $U = V = S^1 \times \overline{B}^2$. Under the usual identification $S^1 \times S^1 =_{\mathrm{i}} \mathbb{R}^2/\mathbb{Z}^2$ we can use φ to glue ∂U to ∂V. We denote the result by $L(p,q)$. (Later, in the Lens Space–Heegaard Splitting Proposition **??** we will see that $L(p,q)$ admit an orientation-preserving diffeomorphism to the lens space $L(p,q)$. Thus our notation is justified.)
- Let $p \in \mathbb{N}$ be coprime. We consider the short exact sequence
$$0\ \mathbb{Z}/p\mathbb{Z} \ \to \ \xrightarrow[=:\varphi]{n \mapsto n \cdot p} \ \mathbb{Z}/p^2\mathbb{Z} \ \xrightarrow{\pi} \ \mathbb{Z}/p\mathbb{Z} \ \to \ 0.$$
In Exercise 197.8 we constructed, given any topological space X, the corresponding Bockstein sequence
$$\ldots \ \to \ \mathrm{H}^i(X;\mathbb{Z}/p\mathbb{Z}) \ \xrightarrow{\varphi_*} \ \mathrm{H}^i(X;\mathbb{Z}/p^2\mathbb{Z}) \ \xrightarrow{\pi_*} \ \mathrm{H}^i(X;\mathbb{Z}/p\mathbb{Z}) \ \xrightarrow{\beta} \ \mathrm{H}^{i+1}(X;\mathbb{Z}/p\mathbb{Z}) \ \to \ \ldots$$
- The image of $[L(p,q)] \in \mathrm{H}_3(L(p,q);\mathbb{Z})$ in $\mathrm{H}_3(L(p,q);\mathbb{Z}/p\mathbb{Z})$ defines a homology class $\sigma \in \mathrm{H}_3(L(p,q);\mathbb{Z}/p\mathbb{Z})$.

We can now formulate the tasks:

(a) Show that for any two generators $\varphi, \psi \in \mathrm{H}^1(L(p,q);\mathbb{Z}/p\mathbb{Z})$ there exists an $m \in \mathbb{Z}$ with
$$\langle \underbrace{\varphi \cup \beta(\varphi)}_{\in \mathrm{H}^3(L(p,q);\mathbb{Z}/p\mathbb{Z})}, \sigma \rangle \ = \ m^2 \cdot \langle \underbrace{\psi \cup \beta(\psi)}_{\in \mathrm{H}^3(L(p,q);\mathbb{Z}/p\mathbb{Z})}, \sigma \rangle \ \in \ \mathbb{Z}/p\mathbb{Z}.$$

(b) Given a generator $\varphi \in \mathrm{H}^1(L(p,q);\mathbb{Z}/p\mathbb{Z})$ determine $\langle \varphi \cup \beta(\varphi), \sigma \rangle$ (up to multiplication by a square).
Hint. Consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^3(L(p,q);\mathbb{Z}/p\mathbb{Z}) & \times & \mathrm{H}_3(L(p,q);\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}_0(L(p,q);\mathbb{Z}/p\mathbb{Z}) \\
\uparrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^3(L(p,q),V;\mathbb{Z}/p\mathbb{Z}) & \times & \mathrm{H}_3(L(p,q),V;\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}_0(L(p,q);\mathbb{Z}/p\mathbb{Z}) \\
\downarrow{\scriptstyle j^* \cong} & & \uparrow{\scriptstyle \cong j_*} & & \uparrow{\scriptstyle \cong j_*} \\
\mathrm{H}^3(U,\partial U;\mathbb{Z}/p\mathbb{Z}) & \times & \mathrm{H}_3(U,\partial U;\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}_0(U;\mathbb{Z}/p\mathbb{Z}) \\
\uparrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \partial_3} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^2(\partial U;\mathbb{Z}/p\mathbb{Z}) & \times & \mathrm{H}_2(\partial U;\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}_0(U;\mathbb{Z}/p\mathbb{Z}).
\end{array}$$

This diagram commutes (in the usual slightly weird sense) by the Cap Product-Naturality Lemma 206.13 and the Cap Product-Triple-Connecting Homomorphism Lemma 206.19. Also make use of the calculation of the cup product on the torus ∂U.

Remark. A similar argument is given in [**Bre1993**, p. 363] and [**Mun1984**, Theorem 69.3].

CHAPTER 207

The interaction of cross-, cup- and cap-products

In this chapter we will study the interaction of cross-, cup- and cap-products which we introduced in Definitions 164.14, 204.7 and 206.10. It is perhaps worth recalling, that these products are (for suitable subspaces) of the form

$$\begin{aligned} \times\colon \mathrm{H}_k(X,A)\times\mathrm{H}_l(Y,B) &\to \mathrm{H}_{k+l}(X\times Y,(A\times Y)\cup(X\times B)) \\ \cup\colon \mathrm{H}^k(X,A;\mathbb{Z})\times\mathrm{H}^l(X,B;\mathbb{Z}) &\to \mathrm{H}^{k+l}(X,A\cup B;\mathbb{Z}) \\ \cap\colon \mathrm{H}^k(X,A;\mathbb{Z})\times\mathrm{H}_l(X,A\cup B;\mathbb{Z}) &\to \mathrm{H}_{l-k}(X,B;\mathbb{Z}). \end{aligned}$$

The main result of this chapter will allow us to determine the cup product of the product $S^m\times S^n$ of two spheres. The results of this chapter will also be essential in the next chapter when we calculate the cup product of the n-dimensional torus $(S^1)^n$.

This chapter is organized as follows:

- In Section 207.1 we will formulate the Cross-Cup-Cap Theorem 207.1 and we will discuss some applications
- In Section 207.2 we will prove the Cross-Cup-Cap Theorem 207.1 .
- In Section 207.3 we will provide exercises for Chapter 207.

In the next Chapter 208 we will state and prove a Künneth Theorem for cohomology which allows us to calculate the cohomology groups of some products in terms of the individual cohomology groups.

207.1. **The formulation of the main theorem.** The following theorem relates the cross product to the cap product and the cup product:

Theorem 207.1. (Cross-Cup-Cap Theorem) Let (X,A) and (Y,B) be two pairs of topological spaces such that the triad $(X\times Y, A\times Y, X\times B)$ is excisive. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the natural projections. Let[2195] $A_1,A_2\in\{\varnothing,A\}$ and let $B_1,B_2\in\{\varnothing,B\}$. For any $\alpha\in\mathrm{H}^k(X,A_1;\mathbb{Z})$, $\mu\in\mathrm{H}_m(X,A_1\cup A_2;\mathbb{Z})$, $\beta\in\mathrm{H}^l(Y,B_1;\mathbb{Z})$ and $\nu\in\mathrm{H}_n(Y,B_1\cup B_2;\mathbb{Z})$ we have the equality[2196]

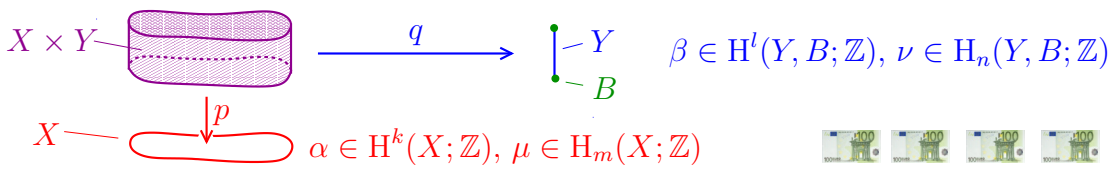
$$(p^*(\alpha)\cup q^*(\beta))\cap(\mu\times\nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)}\cdot(\alpha\cap\mu)\times(\beta\cap\nu)$$

in $\mathrm{H}_{m+n-k-l}(X\times Y,(X\times B_2)\cup(A_2\times Y);\mathbb{Z})$.

Remark. Even though the equality in the Cross-Cup-Cap Theorem 207.1 is easy to write down, it is much harder to figure out in which (co-) homology groups the various objects

[2195] By $A_1,A_2\in\{\varnothing,A\}$ we mean that $A_1=\varnothing$ or $A_1=A$ and that $A_2=\varnothing$ or $A_2=A$.

[2196] Here, and throughout this section, we denote by p also the map of pairs $(X\times Y,A\times Y)\to(X,A)$ given by projection onto the first factor and we denote by q the map of pairs $(X\times Y,X\times B)\to(Y,B)$ given by projection onto the second factor.



actually live. In fact the equality is understood as follows:

$$\underbrace{\Big(\underbrace{p^*(\alpha)}_{\in \mathrm{H}^k(X\times Y, A_1\times Y)} \cup \underbrace{q^*(\beta)}_{\in \mathrm{H}^l(X\times Y, X\times B_1)}\Big)}_{\in \mathrm{H}^{k+l}(X\times Y, (A_1\times Y)\cup(X\times B_2))} \cap \overbrace{(\mu\times\nu)}^{\text{lies in } \mathrm{H}_{m+n}(X\times Y, ((A_1\cup A_2)\times Y)\cup(X\times(B_1\cup B_2)))} = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot \underbrace{(\alpha\cap\mu)}_{\in \mathrm{H}_{m-k}(X, A_2)} \times \underbrace{(\beta\cap\nu)}_{\in \mathrm{H}_{n-l}(Y, B_2)}.$$

$$\in \mathrm{H}_{m+n-k-l}(X\times Y, (X\times B_2)\cup(A_2\times Y))$$

In this formula we have to ensure that all products are in fact defined. This follows easily from the Excisive Triad-Proposition 158.20, our hypothesis that $A_1, A_2 \in \{\varnothing, A\}$ and $B_1, B_2 \in \{\varnothing, B\}$ and our hypothesis that the triad $(X \times Y, A \times Y, X \times B)$ is excisive. □

Remark.

(1) The condition in the Cross-Cup-Cap Theorem 207.1 that the triad $(X\times Y, A\times Y, X\times B)$ needs to be excisive is arguably a little awkward. By the Excisive Triad-Proposition 158.20 this condition is satisfied if we are in one of the following situations:
   (a) $A = \varnothing$ or $B = \varnothing$,
   (b) (X, A) and (Y, B) are pairs of CW-complexes,
   (c) $(X, A) = (X, \partial X)$ and $(Y, B) = (Y, \partial Y)$ where X and Y are topological manifolds,[2197]
   (d) A and B are open subsets of X respectively Y.

(2) In fact a more general statement of the Cross-Cup-Cap Theorem 207.1 holds if we deal with two excisive triads (X, A_1, A_2) and (Y, B_1, B_2) and (co-) homology classes $\alpha \in \mathrm{H}^k(X, A_1; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X, A_2; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y, B_1; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y, B_2; \mathbb{Z})$. But some care needs to be taken to see which other triads need to be excisive to ensure that all the cap and cup products are defined. We will not attempt to formulate this more general statement and we will not make use of it. (See also [**Dol1956**, p. 240 and p. 244] for a few more details.)

(3) The cross product is defined the same way in all of the usual textbooks in algebraic topology. But we saw on pages 3893 and 3954 that in the literature different sign conventions are used for the cup product and the cap product. These also lead to different formulations of the Cross-Cup-Cap Theorem 207.1. More precisely, given topological spaces X and Y and $\alpha \in \mathrm{H}^k(X; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y; \mathbb{Z})$ and

[2197] Let X be an m-dimensional topological manifold and let Y be an n-dimensional smooth manifold. It follows from the Topological Manifold Product Proposition 150.12 that $W := \partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$ is an $(m + n - 1)$-dimensional topological manifold. It is straightforward to verify that $\partial X \times Y$ and $X \times \partial Y$ are submanifolds of W such that

$$(\partial X \times Y) \cap (X \times \partial Y) \;=\; \partial X \times \partial Y = \partial(\partial X \times Y) \;=\; \partial(X \times \partial Y).$$

Therefore it follows from the Excisive Triad-Proposition 158.20 (2) that the triad $(X \times Y, \partial X \times Y, X \times \partial Y)$ is excisive.

$\nu \in \mathrm{H}_n(Y;\mathbb{Z})$ the following equations can be found in the literature:

[**Bre1993**, p. 337] $(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) = (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$
[**Dol1956**, p. 240] $(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) = (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$
[**Spa1995**, p. 255] $(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) = (-1)^{\deg(\alpha)\cdot(\deg(\nu)-\deg(\beta))} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$.

With the aforementioned sign conventions the formulas of Dold and Spanier are consistent with our formula, whereas the formula of Bredon differs from the formula of Dold and Spanier by the term $(-1)^{\deg(\alpha)\cdot\deg(\nu)+\deg(\beta)\cdot\deg(\mu)}$.[21982199]

(4) The conclusion of the Cross-Cup-Cap Theorem 207.1 also holds, with verbatim the same proof, if throughout we replace $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients. □

The proof of the Cross-Cup-Cap Theorem 207.1 is somewhat lengthy and technical. Therefore we first discuss some consequences of the Cross-Cup-Cap Theorem 207.1 which will then hopefully convince us that it is worth the effort to prove the Cross-Cup-Cap Theorem 207.1.

The following is one of the key calculations of the cup product and the cap product. As we will see later, many other calculations can be reduced to this calculation.

Proposition 207.2. (Fundamental Class-Product Proposition) Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the natural projections. The following two statements hold:

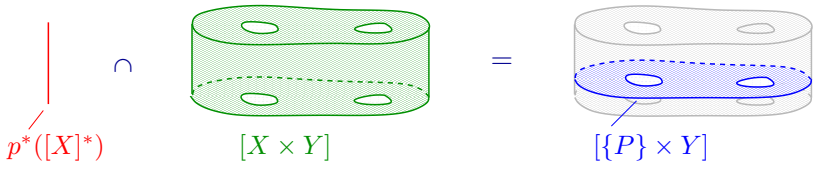
(1)
$$\underbrace{p^*([X]^*)}_{\in \mathrm{H}^m(X\times Y, \partial X \times Y)} \cup \underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y, X\times \partial Y)} = \underbrace{[X \times Y]^*}_{\in \mathrm{H}^{m+n}(X\times Y, \partial(X\times Y))}.$$

(2) If $P \in X$ and $Q \in Y$ are points, then[2200]

$$\overbrace{p^*([X]^*)}^{\in \mathrm{H}^m(X\times Y, \partial X\times Y)} \cap \overbrace{[X \times Y]}^{\in \mathrm{H}_{m+n}(X\times Y, \partial(X\times Y))} = \overbrace{[\{P\} \times Y]}^{\in \mathrm{H}_n(X\times Y, X\times \partial Y)}$$

and similarly

$$\underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y, X\times \partial Y)} \cap \underbrace{[X \times Y]}_{\in \mathrm{H}_{m+n}(X\times Y, \partial(X\times Y))} = (-1)^{mn} \cdot \underbrace{[X \times \{Q\}]}_{\in \mathrm{H}_m(X\times Y, \partial X\times Y)}.$$



[2198] Mathematicians with a long experience of teaching cohomology groups sidestep all issues with signs by working with $\mathbb{F}_2$-coefficients, see [**Hau2014**].

[2199] We leave it to the never-flagging enthusiasm of the reader to figure out whether the formulas agree with the one given by Greenberg-Harper [**GrH1981**, p. 262].

[2200] Here we use the following suggestive notation: We can view $\{P\} \times Y$ as a submanifold of $X \times Y$. It is naturally homeomorphic to Y, thus can view $\{P\} \times Y$ as a compact oriented topological manifold. We denote by $[\{P\} \times Y]$ its fundamental class, and by a slight abuse of notation we also denote by $[\{P\} \times Y]$ the image of the fundamental class under the inclusion $\{P\} \times Y \to X \times Y$.

Remark. As we pointed out above, the conclusion of the Cross-Cup-Cap Theorem 207.1 also holds if we work with $\mathbb{F}_2$-coefficients. It follows easily that there is also an $\mathbb{F}_2$-coefficient analog of the Fundamental Class-Product Proposition 207.2. □

Example. If we apply the Fundamental Class-Product Proposition 207.2 (2) to $X = S^1$ and $Y = S^1$ we see that

$$p^*([S^1]^*) \cap [S^1 \times S^1] = [\{1\} \times S^1] \quad \text{and} \quad q^*([S^1]^*) \cap [S^1 \times S^1] = -[S^1 \times \{1\}].$$

This matches our down-to-earth calculation from the Torus–Cap Product Lemma 206.6. □

Proof. Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Furthermore let P be a point in X and let Q be a point in Y. We start out with the following two observations:

(a) For any compact oriented connected k-dimensional topological manifold W and any $w_0 \in W$ we obtain from the discussion on page 3960 that $[W]^* \cap [W] = [w_0]$ and that $[W]^*$ is the unique element in $\mathrm{H}^k(W, \partial W; \mathbb{Z}) \cong \mathbb{Z}$ with $[W]^* \cap [W] = [w_0]$ in $\mathrm{H}_0(W; \mathbb{Z}) = \mathbb{Z} \cdot [w_0]$.

(b) By the Fundamental Class-of-Product Proposition 194.17 we have $[X \times Y] = [X] \times [Y]$.

Now we turn to the actual proof of the two statements.

(1) We perform the following calculation in $\mathrm{H}_0(X \times Y; \mathbb{Z})$:

$$\begin{array}{rcl} & & \text{observation (b)} \\ & & \downarrow \\ (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X \times Y]) & = & (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\ = & (-1)^{mn+mn} \cdot ([X]^* \cap [X]) \times ([Y]^* \cap [Y]) & = & [P] \times [Q] \quad = \quad [(P,Q)]. \\ \uparrow & & \uparrow & & \uparrow \end{array}$$

by the Cross-Cup-Cap Theorem 207.1 applied to $(X, A_1, A_2) = (X, \partial X, \partial X)$ and $(Y, B_1, B_2) = (Y, \partial Y, \partial Y)$ — observation (a) — by the Cross Product Lemma 164.5 (1)

This implies, by observation (a), that $p^*([X]^*) \cup q^*([Y]^*) = [X \times Y]^*$.

(2) Let $P \in X$. We have the following equality in $\mathrm{H}_n(X \times Y, X \times \partial Y; \mathbb{Z})$:

$$p^*([X]^*) \cap ([X \times Y]) \quad = \quad (p^*([X]^*) \cup q^*(1_Y)) \cap ([X] \times [Y])$$

by the Cup Product-Ring Proposition 203.25 (3) we have $q^*(1_Y) = 1_{X \times Y}$ and by observation (b) we have $[X \times Y] = [X] \times [Y]$

$$\begin{array}{rcl} & = & ([X]^* \cap [X]) \times (1_Y \cap [Y]) \quad = \quad [P] \times [Y] \quad = \quad [\{P\} \times Y]. \\ & \uparrow & \uparrow \qquad\qquad\qquad\qquad\quad \uparrow \end{array}$$

by the Cross-Cup-Cap Theorem 207.1 applied to $(Y, B_1, B_2) = (Y, \varnothing, \partial Y)$, note that here we use that 1_Y lies in $\mathrm{H}^0(Y; \mathbb{Z})$ — this equality follows from the Cap-with-1-Lemma 206.16 and observation (a) — by the obvious relative version of the Cross Product Lemma 164.5 (1)

This proves the first equality of (2). Now let $Q \in Y$. Then, very similar to the above, we have

$$\begin{array}{rcl} q^*([Y]^*) \cap ([X \times Y]) & = & (p^*(1_X) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\ & = & (-1)^{mn} \cdot (1_X \cap [X]) \times ([Y]^* \cap [Y]) \\ & = & (-1)^{mn} \cdot [X] \times [Q] \quad = \quad (-1)^{mn} \cdot [X \times \{Q\}]. \end{array}$$

■

Using the Fundamental Class-Product Proposition 207.2 (1) we can now easily prove the following lemma, which can be viewed as a generalization of the Torus–Cup Product Lemma 203.28.

Lemma 207.3. (Product-of-Spheres–Cup Product Lemma) Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Let $1 \in \mathrm{H}^0(S^m \times S^n; \mathbb{Z})$ be the neutral element from the Cup Product-Ring Proposition 203.25.

(1) The cohomology of $S^m \times S^n$ is given as follows:

$$\mathrm{H}^*(S^m \times S^n; \mathbb{Z}) \;=\; \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*([S^m]^*) \oplus \mathbb{Z} \cdot q^*([S^n]^*) \oplus \mathbb{Z} \cdot [S^m \times S^n]^*.$$

(2) The multiplication table for the cup product is given by

$\cup$	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
1	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
$p^*([S^m]^*)$	$p^*([S^m]^*)$	0	$[S^m \times S^n]^*$	0
$q^*([S^n]^*)$	$q^*([S^n]^*)$	$(-1)^{mn} \cdot [S^m \times S^n]^*$	0	0
$[S^m \times S^n]^*$	$[S^m \times S^n]^*$	0	0	0.

Here the table shows the cup product of the term in the first column with the term in the first row.

Remark. If we use the Fundamental Class-Product Proposition 207.2 (1) with $\mathbb{F}_2$-coefficients then we can also determine the cup product on $\mathrm{H}^*(S^m \times S^n; \mathbb{F}_2)$ and we obtain basically the same result. □

Proof.

(1) Using the $(S^k \times S^l)$-H_*-Lemma 164.7 Universal Coefficient-H^*-Theorem 201.23 one can easily determine the isomorphism type of $\mathrm{H}^*(S^m \times S^n; \mathbb{Z})$. Furthermore, using the Kronecker-Basis Lemma 199.10 (2) and using the explicit basis of the $(S^k \times S^l)$-H_*-Lemma 164.7 one can easily verify that the given cohomology classes do indeed form a basis for $\mathrm{H}^*(S^m \times S^n; \mathbb{Z})$.

(2) By Observation 203.27 it remains to show that the red-colored and the blue-colored entries are correct.

(a) First we consider the red-colored entries. We have

$$p^*([S^m]^*) \cup p^*([S^m]^*) \underset{\uparrow}{=} p^*([S^m]^* \cup [S^m]^*) \underset{\uparrow}{=} p^*(0) = 0.$$

by the Cup Product-Naturality Lemma 203.9 — since we have the equality $[S^m]^* \cup [S^m]^* \in \mathrm{H}^{2m}(S^m; \mathbb{Z}) = 0$

The same argument also shows that $q^*([S^n]^*) \cup q^*([S^n]^*) = 0$.

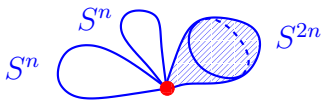
(b) We turn to the blue-colored entries. We calculate that

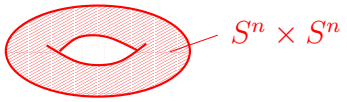
$$(-1)^{mn} \cdot q^*([S^n]^*) \cup p^*([S^m]^*) \underset{\uparrow}{=} p^*([S^m]^*) \cup q^*([S^n]^*) \underset{\uparrow}{=} [S^m \times S^n]^*.$$

by the Cup Product-Commutativity Proposition 203.20 — by the Fundamental Class Product Proposition 207.2 (1)

■

In the following we consider the wedge $S^n \vee S^n \vee S^{2n}$ and we consider the product $S^n \times S^n$. We point out that it follows from the discussion on page 3141, the Sphere-H_*-Proposition 149.4, the Wedge-H_*-Proposition 149.16 and the Universal Coefficient-H^*-Theorem 201.23 that $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ have isomorphic homology and cohomology groups. Furthermore, if $n \geq 2$, then one obtains easily from the Sphere-π_1-Proposition 88.24, the Product-π_1-Proposition 90.13 and the Wedge-π_1-Proposition 99.10 that both $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ are simply connected.





It is quite easy to show that $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ are not homeomorphic.[2201] The following lemma gives us the much more interesting result, that these two topological spaces are not even homotopy equivalent:

Lemma 207.4. (Wedge–Product-of-Spheres Lemma) The wedge $S^n \vee S^n \vee S^{2n}$ and the product $S^n \times S^n$ are not homotopy equivalent.

Proof. We had just seen in the Product-of-Spheres–Cup Product Lemma 207.3 that there exist cohomology classes α and β in $\mathrm{H}^n(S^n \times S^n; \mathbb{Z})$ with $\alpha \cup \beta \neq 0$. On the other hand in the Wedge-of-Spheres–Cup Product Lemma 203.29 we saw that the cup product

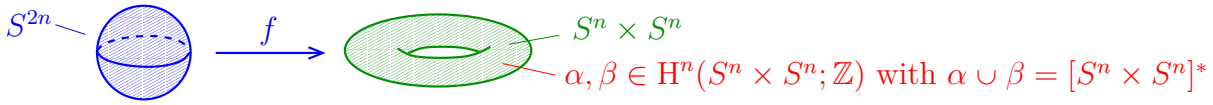
$$\cup \colon \mathrm{H}^n(S^n \vee S^n \vee S^{2n}; \mathbb{Z}) \times \mathrm{H}^n(S^n \vee S^n \vee S^{2n}; \mathbb{Z}) \ \to \ \mathrm{H}^{2n}(S^n \vee S^n \vee S^{2n}; \mathbb{Z})$$

is the zero-map. It follows from the Homotopy Equivalence-H^*-Proposition 198.12 (2) together with the Cup Product-Ring Proposition 203.25 (2) that the topological spaces $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ are not homotopy equivalent. ∎

Let $n \in \mathbb{N}$. In the Degree Realization Proposition 46.19 we saw that there exists a degree-one map from $S^n \times S^n$ onto the $2n$-dimensional sphere S^{2n}. In Question 186.12 we had asked whether there exists a degree-one map from the $2n$-dimensional sphere S^{2n} to $S^n \times S^n$. Now we can answer this question in the negative. In fact we have the following more general lemma, which is a generalization of some parts of the Smaller-to-Larger Genus Proposition 186.11.

Lemma 207.5. (Sphere-to-Product-of-Spheres Lemma) Let $n \in \mathbb{N}$. There is no continuous map $f \colon S^{2n} \to S^n \times S^n$ of non-zero degree.

Proof. [2202] Let $f \colon S^{2n} \to S^n \times S^n$ be a continuous map. By the Product-of-Spheres–Cup Product Lemma 207.3 we know that there exist $\alpha, \beta \in \mathrm{H}^n(S^n \times S^n; \mathbb{Z})$ such that $[S^n \times S^n]^* = \alpha \cup \beta \in \mathrm{H}^{2n}(S^n \times S^n; \mathbb{Z})$.



We now see that

$$\begin{array}{ccccc} & \overset{\text{by definition of } [S^{2n}]^*}{\downarrow} & & \overset{\text{by the Dual Fundamental Class–Degree Lemma 201.28}}{\downarrow} & & \overset{\text{choice of } \alpha \text{ and } \beta}{\downarrow} \\ \deg(f) & = & \langle \deg(f) \cdot [S^{2n}]^*, [S^{2n}] \rangle & = & \langle f^*([S^n \times S^n]^*), [S^{2n}] \rangle & = & \langle f^*(\alpha \cup \beta), [S^{2n}] \rangle \\ & = & \langle f^*(\alpha) \cup f^*(\beta), [S^{2n}] \rangle & = & \langle 0 \cup 0, [S^{2n}] \rangle & = & 0. \\ & \underset{\text{by the Cup Product Naturality Lemma 203.9}}{\uparrow} & & \underset{\text{since } \mathrm{H}^n(S^{2n}; \mathbb{Z}) = 0}{\uparrow} & & & \end{array}$$

[2201] Indeed, suppose that there exists a homeomorphism $\phi \colon S^n \vee S^n \vee S^{2n} \to S^n \times S^n$. We denote by P the wedge point of $S^n \vee S^n \vee S^{2n}$ and we write $Q = f(P)$. Then $(S^n \vee S^n \vee S^{2n}) \setminus \{P\}$ and $(S^n \times S^n) \setminus \{Q\}$ would also be homeomorphic. But the former topological space has three path-components whereas the latter has, since it is a 4-dimensional topological manifold, just one path-component.

[2202] The proof is basically identical to the proof of the corresponding statement in the Smaller-to-Larger Genus Proposition 186.11 that we gave on page 3913. We just need to replace the Torus–Cup Product Lemma 203.28 by the Product-of-Spheres–Cup Product Lemma 207.3.

■

207.2. **Proof of the Cross-Cup-Cap Theorem 207.1 .** Now that we are fully convinced of the value of the Cross-Cup-Cap Theorem 207.1 it is time to actually provide a proof thereof. This, very lengthy, proof will require the remainder of this chapter. The impatient reader can also move to Chapter 208 for the Künneth Theorem in cohomology.

207.2.1. **Proof of the Cross-Cup-Cap Theorem 207.1: Algebraic preparations.** Initially we introduced only the tensor product $C_* \otimes D_*$ of two chain complexes C_* and D_*. The definition of the boundary map in the tensor product $C_* \otimes D_*$ is *not* symmetric in the two terms. But it is straightforward to see that the tensor product of chain complexes is associative. This leads us to the following definition:

Definition 207.6. Given chain complexes $C_*^1, \ldots, C_*^k$ the boundary map on the tensor product $C_*^1 \otimes \cdots \otimes C_*^k$ is always determined by

$$\partial(c_1 \otimes c_2 \otimes \cdots \otimes c_k) \;=\; \sum_{i=1}^{k} (-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})} \cdot c_1 \otimes \cdots \otimes \partial c_i \otimes \cdots \otimes c_k,$$

regardless of the way how one breaks the tensor product into the tensor product of pairs of two chain complexes. This observation lets us define the chain complex $C_*^1 \otimes \cdots \otimes C_*^k$ (regardless of parentheses).

The following lemma is basically the same as the Tensoring Chain Maps Lemma 163.4, but we state it here for the convenience of the reader.

Lemma 207.7. (Tensoring Chain Maps Lemma) Let $f_i \colon C_*^i \to D_*^i$, $i = 1, \ldots, k$ be chain maps between chain complexes. The map

$$f_1 \otimes \cdots \otimes f_k \colon C_*^1 \otimes \cdots \otimes C_*^k \;\to\; D_*^1 \otimes \cdots \otimes D_*^k$$

that is determined by

$$c_1 \otimes \cdots \otimes c_k \;\mapsto\; f(c_1) \otimes \cdots \otimes f(c_k)$$

is also a chain map.

Remark. Even the definition of $f_1 \otimes \cdots \otimes f_k$ is, oddly enough, not the same everywhere. For example [**Bre1993**, p. 315] adds a sign to the map. □

Proof. Since later on, in the proof of the Cross-Cup-Cap Theorem 207.1, there are lots of steps which can make us nervous, let us convince ourselves that the statement is indeed correct. We have

$$\begin{aligned}
\partial((f_1 \otimes \ldots \otimes f_k)(c_1 \otimes \ldots \otimes c_k)) &= \partial(f_1(c_1) \otimes \ldots \otimes f_k(c_k)) \\
&= \sum_{i=1}^{k} (-1)^{\deg(f_1(c_1))+\cdots+\deg(f_{i-1}(c_{i-1}))} \cdot f_1(c_1) \otimes \ldots \otimes \overbrace{f_i(\partial c_i)}^{=\partial f_i(c_i)} \otimes \ldots \otimes f_k(c_k) \\
&= (f_1 \otimes \ldots \otimes f_k)\Big(\sum_{i=1}^{k} (-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})} \cdot c_1 \otimes \ldots \partial c_i \otimes \ldots \otimes c_k \Big) \\
&\quad \uparrow \\
&\quad \text{since } \deg(f_i(c_i)) = \deg(c_i) \\
&= (f_1 \otimes \ldots \otimes f_k)(\partial(c_1 \otimes \ldots \otimes c_k)).
\end{aligned}$$

■

Note though that the naive swapping of two factors in the tensor product of two chain complexes is not necessarily a chain map. But we have the following lemma:

Lemma 207.8. (Swap Factors–Chain Map Lemma) Let A_* and B_* be chain complexes. The unique map

$$\tau\colon A_* \otimes B_* \quad \to \quad B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \quad \mapsto \quad (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Proof. [2203] The difficult bit about the lemma is coming up with the statement. The verification is now almost trivial. Indeed, given $a \in A_k$ and $b \in B_l$ we see that

$$\begin{aligned}
\tau(\partial^{\otimes}_{k+l}(a \otimes b)) &= \tau(\partial_k(a) \otimes b + (-1)^k \cdot a \otimes \partial_l(b)) \\
&= \tau(\partial_k(a) \otimes b) + (-1)^k \cdot \tau(a \otimes \partial_l(b)) \\
&= (-1)^{(k-1)l} \cdot b \otimes \partial_k(a) + (-1)^{k+k(l-1)} \cdot \partial_l(b) \otimes a \\
&= (-1)^{kl} \cdot \partial_l(b) \otimes a + (-1)^{l+kl} \cdot b \otimes \partial_k(a) \\
&= \partial^{\otimes}((-1)^{kl} \cdot b \otimes a) \quad = \quad \partial^{\otimes}(\tau(a \otimes b)).
\end{aligned}$$

■

Now we have to introduce a plethora of new definitions and notations:

Notation 207.9.

(1) We denote by $\mathbb{Z}$ also the chain complex given by the group $\mathbb{Z}$ in degree zero and where all other chain groups are zero.
(2) Given a chain complex C_* and $k \in \mathbb{N}_0$ we denote by $C_*[k]$ the chain complex which is the result of "shifting C_* by k to the left", i.e. it is the chain complex for which the m-th chain groups $(C_*[k])_m$ is given by C_{m-k}.[2204] [2205]
(3) If $\alpha\colon C_* \to D_*$ is a chain map, then for any $k \in \mathbb{N}_0$ we denote by $\alpha[k]\colon C_*[k] \to D_*[k]$ the obvious shift of the chain map. Evidently this is again a chain map.
(4) Let C_* be a chain complex and let $k \in \mathbb{N}_0$.
   (a) Given a cocycle $\varphi\colon C_k \to \mathbb{Z}$ we denote by $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ the map that is given by φ on C_k and zero on all other chain groups.
   (b) Conversely, given a chain map $\psi\colon C_* \to \mathbb{Z}[k]$ we denote by $\psi_k\colon C_k \to \mathbb{Z}$ the map that is the k-th map of the chain map.

Lemma 207.10. (Projection–Cocycle Lemma) Let C_* be a chain complex. Given any $k \in \mathbb{N}_0$ the following statements hold:

(1) Let $\varphi\colon C_k \to \mathbb{Z}$ be a cocycle. The map $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ is a chain map.
(2) Let $\psi\colon C_* \to \mathbb{Z}[k]$ be a chain map. The map $\psi_k\colon C_k \to \mathbb{Z}$ is a cocycle.
(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps. If α and β are chain homotopic, then α_k and β_k are cohomologous.

Proof.

[2203] We proved this statement already on page 3901. We repeat the proof for the reader's convenience.
[2204] Once again, conventions in the literature differ. It is fairly common to define the boundary maps on $C_*[k]$ to be the boundary maps of C_* multiplied by $(-1)^k$, see e.g. [**Weib1994**, p. 10].
[2205] For example, $\mathbb{Z}[3]$ is the following chain complex:

$$\ldots \quad \to \quad 0 \quad \to \quad \underset{\substack{\uparrow \\ \text{degree } 3}}{\mathbb{Z}} \quad \to \quad 0 \quad \to \quad 0 \quad \to \quad 0.$$

(1) Let $\varphi\colon C_k \to \mathbb{Z}$ be a cocycle. We consider the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots . \end{array}$$

The hypothesis that φ is a cocycle means that $\delta_k(\varphi) = \varphi \circ \partial_{k+1} = 0$, which implies immediately that $\Pi_k(\varphi)$ is a chain map.

(2) The proof of Statement (2) is almost the same as the proof of Statement (1).

(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps and let $\{P_m\colon C_m \to \mathbb{Z}[k]_{m+1}\}_{m\in\mathbb{N}_0}$ be a chain homotopy between α and β. We summarize all relevant maps in the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \longrightarrow & \dots \\ & & \downarrow & \swarrow{\scriptstyle P_k} & \downarrow{\scriptstyle \alpha_k-\beta_k} & \swarrow{\scriptstyle P_{k-1}} & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots . \end{array}$$

By definition of a chain homotopy we obtain that $\alpha_k - \beta_k = P_{k-1} \circ \partial_k$. But this means that $\alpha_k - \beta_k = \delta_{k-1}(P_{k-1})$, i.e. α_k and β_k are cohomologous. ■

Lemma 207.11. (Chain Complex–Shift Lemma) Let C_* be a chain complex and let $k \in \mathbb{Z}$. The maps

$$\begin{array}{rcl} \eta_m\colon (\mathbb{Z}[k] \otimes C_*)_m = \mathbb{Z} \otimes C_{m-k} & \to & (C_*[k])_m = C_{m-k} \\ a \otimes c_{m-k} & \mapsto & (-1)^{k(m-k)} \cdot a \cdot c_{m-k} \end{array}$$

form an isomorphism

$$\eta\colon \mathbb{Z}[k] \otimes C_* \xrightarrow{\cong} C_*[k]$$

of chain complexes.

Example. Let $k, l \in \mathbb{N}_0$. The chain map

$$\eta\colon \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l]$$

is determined by the fact that for $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and $b \in (\mathbb{Z}[l])_l = \mathbb{Z}$ we have

$$\eta(a \otimes b) = (-1)^{kl} \cdot a \cdot b.$$ □

Proof. It follows from the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that each η_m is an isomorphism. But we still need to verify that the maps η_m form a chain map $\eta\colon \mathbb{Z}[k] \otimes C_* \to C_*[k]$. Thus let $k \in \mathbb{N}_0$, let $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and let $c_{m-k} \in C_{m-k}$. Then

$$\begin{array}{rcl} & \overset{\text{since } a \text{ is of degree } k \text{ and since } \partial_k(a)=0}{\downarrow} & \overset{\text{since } \partial_{m-k}(c_{m-k}) \text{ is of degree } m-k-1}{\downarrow} \\ \eta_{m-1}(\partial^{\otimes}_m(a \otimes c_{m-k})) & = \eta_{m-1}((-1)^k \cdot a \otimes \partial_{m-k}(c_{m-k})) & = (-1)^k \cdot (-1)^{k(m-1-k)} \cdot a \cdot \partial_{m-k}(c_{m-k}) \\ & = \partial_{m-k}((-1)^{k(m-k)} \cdot a \cdot c_{m-k}) & = \partial_m(\eta_m(a \otimes c_{m-k})). \end{array}$$

It follows immediately that the maps η_m form a chain map. ■

207.2.2. **Proof of the Cross-Cup-Cap Theorem 207.1: Topological preparations.** After the purely algebraic notions from the last subsection we now return to topological concepts:

Notation 207.12.

(1) Given topological spaces X and Y we denote by

$$\begin{aligned} \Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) &\to \mathrm{C}_*(X\times Y)\\ \Theta\colon \mathrm{C}_*(X\times Y) &\to \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\end{aligned}$$

the chain maps from the Eilenberg-Zilber Theorem 164.2.

(2) Given a topological space W we denote by $\Delta_W\colon \mathrm{C}_*(W)\to\mathrm{C}_*(W)\otimes\mathrm{C}_*(W)$ the Alexander-Whitney diagonal approximation from Definition 203.18. Recall that Δ_W is also a chain map.

In the following two lemmas we will reinterpret the cup product and the cap product as chain maps.

Lemma 207.13. (Cup Product–Reinterpretation Lemma) Let W be a topological space and let $\alpha\in\mathrm{C}^k(W;\mathbb{Z})$ and $\beta\in\mathrm{C}^l(W;\mathbb{Z})$ be cocycles. Then we have the following equality of chain maps:

$$(-1)^{kl}\cdot\Pi_{k+l}\big(\underbrace{\alpha\cup\beta}_{\text{cocycle in }\mathrm{C}^{k+l}(W;\mathbb{Z})}\big) \;=\; \eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Delta_W\colon \mathrm{C}_*(W)\to\mathbb{Z}[k+l].$$

Proof. We introduce the following notation:

(1) We denote by $\mu_{\mathbb{Z}}\colon\mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Z}$ the obvious isomorphism of abelian groups given by $\mu_{\mathbb{Z}}(a\otimes b)=a\cdot b$.

(2) We denote by $\Pi_{k,l}\colon(\mathrm{C}_*(W)\otimes\mathrm{C}_*(W))_{k+l}\to\mathrm{C}_k(W)\otimes\mathrm{C}_l(W)$ the natural projection.

In the proof of the Cup Product–Definitions Proposition 203.15 on page 3900 we showed that

$$\alpha\cup\beta \;=\; \mu_{\mathbb{Z}}\circ(\alpha\otimes\beta)\circ\Pi_{k,l}\circ\Delta_W\colon \mathrm{C}_{k+l}(W) \;\to\; \mathbb{Z}.$$

The lemma follows immediately from this equality and the following claim:

Claim.

$$(-1)^{kl}\cdot\mu_{\mathbb{Z}}\circ(\alpha\otimes\beta)\circ\Pi_{k,l} \;=\; \underbrace{(\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l}}_{\substack{\text{the restriction to the } (k+l)\text{-th chain group of the}\\ \text{chain map } \mathrm{C}_*(W)\otimes\mathrm{C}_*(W)\to\mathbb{Z}[k]\otimes\mathbb{Z}[l]\to\mathbb{Z}[k+l]\\ a\otimes b\mapsto(-1)^{kl}\cdot ab}} \;:(\mathrm{C}_*(W)\otimes\mathrm{C}_*(W))_{k+l} \;\to\; \mathbb{Z}.$$

Proof. Let $\sigma_i\in\mathrm{C}_i(W)$ and $\sigma_j\in\mathrm{C}_j(W)$ with $i+j=k+l$. If $(i,j)\neq(k,l)$, then both maps are zero. So now suppose that $(i,j)=(k,l)$. Then it is clear that both maps send $\sigma_k\otimes\sigma_l$ to $(-1)^{kl}\cdot\alpha(\sigma_k)\cdot\beta(\sigma_l)$. (Note that the sign $(-1)^{kl}$ for the map on the right-hand side stems from the definition of the chain map η.) ∎

Given a topological space W and given $\varphi\in\mathrm{C}^k(W;\mathbb{Z})$ we obtain for each $m\in\mathbb{N}_0$ a map[2206]

$$\begin{aligned} \varphi\cap-\colon \mathrm{C}_m(W) &\to \mathrm{C}_{m-k}(W)=(\mathrm{C}_*(W)[k])_m\\ \sigma &\mapsto \varphi\cap\sigma.\end{aligned}$$

Unfortunately in general these maps do *not* form a chain map from the chain complex $\mathrm{C}_*(W)$ to the chain complex $\mathrm{C}_*(W)[k]$. This motivates the introduction of the following slight variation on the cap product:

[2206]Here we use the natural isomorphism $\mathrm{C}_n(W)\otimes\mathbb{Z}=_{\mathrm{i}}\mathrm{C}_n(W)$. This way we can arrange that the cap product takes values in $\mathrm{C}_n(W)$ instead of $\mathrm{C}_n(W)\otimes\mathbb{Z}$.

Definition 207.14. Let W be a topological space. Given $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ and given $\sigma \in \mathrm{C}_n(W)$ we define[2207]

$$\varphi \mathbin{\widetilde{\cap}} \sigma \coloneqq (-1)^{m(n-m)} \cdot \varphi \cap \sigma.$$

In particular for $m = n$ we have $\varphi \mathbin{\widetilde{\cap}} \sigma = \varphi \cap \sigma$.

Lemma 207.15. (Cap-as-Chain Map Lemma) Let W be a topological space.

(1) If $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ is a cocycle, then the map

$$\begin{aligned} \varphi \mathbin{\widetilde{\cap}} -\colon \mathrm{C}_*(W) &\to \mathrm{C}_*(W)[m] \\ \sigma &\mapsto \varphi \mathbin{\widetilde{\cap}} \sigma \end{aligned}$$

is a chain map.

(2) If $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ are cohomologous cocycles, then the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$ are chain homotopic.

Proof. Let W be a topological space. We start out with the following claim:

Claim. For any cochain $\tau \in \mathrm{C}^l(W;\mathbb{Z})$ and any chain $\mu \in \mathrm{C}_n(W;\mathbb{Z})$ we have

$$\partial_{n-l}(\tau \mathbin{\widetilde{\cap}} \mu) = (-1)^{l+n} \cdot \delta_l(\tau) \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial_n(\mu).$$

Proof. The claim is an immediate consequence of the Cap Product–Boundary Lemma 206.3. Since we are worried about signs we write down the detailed calculation. We have

$$\begin{aligned} \partial_{n-l}(\tau \mathbin{\widetilde{\cap}} \mu) &\overset{\text{Definition 207.14 of } \widetilde{\cap}}{=} (-1)^{l(n-l)} \cdot \partial_{n-l}(\tau \cap \mu) \overset{\text{by the Cap Product–Boundary Lemma 206.3}}{=} (-1)^{l(n-l)} \cdot (-1)^l \cdot (-\delta_l(\tau) \cap \mu + \tau \cap \partial_n(\mu)) \\ &= (-1)^{l(n-l)+l} \cdot (-(-1)^{(l+1)\cdot(n-l-1)} \cdot \delta_l(\tau) \mathbin{\widetilde{\cap}} \mu + (-1)^{l\cdot(n-l-1)} \cdot \tau \mathbin{\widetilde{\cap}} \partial_n(\mu)) \\ &= (-1)^{l+n} \cdot \delta_l(\tau) \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial_n(\mu). \end{aligned}$$

⊞

Now we turn to the actual proof of the two statements.

(1) Let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. Furthermore let $\sigma \in \mathrm{C}_n(W)$. Note that by the definition of the cap product and by definition of the shifting of chain complexes we have $\varphi \cap \sigma \in \mathrm{C}_{n-m}(W) = (\mathrm{C}_*(W)[m])_n$. This shows that $\varphi \mathbin{\widetilde{\cap}} -$ preserves degrees. It remains to show that the map $\varphi \mathbin{\widetilde{\cap}} -$ commutes with the boundary maps. But this is an immediate consequence of the above claim and the fact that φ is a cocycle.

(2) So let $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ be cocycles. We assume they are cohomologous which means that there exists a $\tau \in \mathrm{C}^{m-1}(W;\mathbb{Z})$ such that $\varphi = \psi + \delta_{m-1}(\tau)$. For any $k \in \mathbb{N}_0$ we consider the map

$$\begin{aligned} P_k\colon \mathrm{C}_k(W) &\to (\mathrm{C}_*(W)[m])_{k+1} = \mathrm{C}_{k-m+1}(W) \\ \sigma &\mapsto (-1)^{m+k-1} \cdot \tau \mathbin{\widetilde{\cap}} \sigma. \end{aligned}$$

For $\sigma \in \mathrm{C}_k(W)$ we see that

$$\begin{aligned} \varphi \mathbin{\widetilde{\cap}} \sigma - \psi \mathbin{\widetilde{\cap}} \sigma = \delta_{m-1}(\tau) \mathbin{\widetilde{\cap}} \sigma &\overset{\text{by the claim with } l=m-1 \text{ and } n=k \text{ we have } \partial_{k-m+1}(\tau \mathbin{\widetilde{\cap}} \sigma) = (-1)^{k+m-1} \cdot \delta_{m-1}(\tau) \mathbin{\widetilde{\cap}} \sigma + \tau \mathbin{\widetilde{\cap}} \partial_k(\sigma)}{=} (-1)^{m+k-1} \cdot \partial_{k-m+1}(\tau \mathbin{\widetilde{\cap}} \sigma) + (-1)^{m+k} \cdot \tau \mathbin{\widetilde{\cap}} \partial_k(\sigma) \\ &= (\partial_{k+1} \circ P_k)(\sigma) + (P_{k-1} \circ \partial_k)(\sigma). \end{aligned}$$

This shows that the maps $\{P_k\}_{k\in\mathbb{N}_0}$ form a chain homotopy between the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$. ■

[2207] If we compare this definition with the discussion on page 3953 we see that the definition of $\widetilde{\cap}$ is precisely the definition of the cap product in Bredon [**Bre1993**] and Dold [**Dol1956**].

In the proof of the Cross-Cup-Cap Theorem 207.1 we will also need the following reformulation of the cap product.

Proposition 207.16. (Cap Product–Reinterpretation Lemma) Let W be a topological space and let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. We have the following equality of chain maps $\varphi \mathbin{\widetilde{\cap}} - \;=\; \eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W \colon \mathrm{C}_*(W) \;\to\; \mathrm{C}_*(W)[m]$.

Proof. Let W be a topological space, let $\varphi \in \mathrm{C}^m(W)$ be a cocycle and let $\sigma\colon \Delta^n \to W$ be a singular n-simplex. We have the following equalities in $\mathrm{C}_*(W)[m]_n = \mathrm{C}_{n-m}(W)$:

$$
\begin{aligned}
\big(\eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W\big)(\sigma)
&\overset{\text{Definition 203.18 of the Alexander-Whitney diagonal approximation}}{=} \eta\Big((\Pi_m(\varphi) \otimes \mathrm{id})\Big(\sum_{j=0}^{n} \sigma \circ [\mathfrak{v}_0,\dots,\mathfrak{v}_j] \otimes \sigma \circ [\mathfrak{v}_j,\dots,\mathfrak{v}_n]\Big)\Big) \\
&= \eta\Big(\underbrace{\varphi(\sigma \circ [\mathfrak{v}_0,\dots,\mathfrak{v}_m])}_{\in \mathbb{Z}[m]_m} \otimes \underbrace{\sigma \circ [\mathfrak{v}_m,\dots,\mathfrak{v}_n]}_{\in \mathrm{C}_{n-m}}\Big) \\
&\overset{\text{by definition of the chain map } \Pi_m(\varphi) \otimes \mathrm{id}\colon \mathrm{C}_*(W) \otimes \mathrm{C}_*(W) \to \mathbb{Z}[m] \otimes \mathrm{C}_*(W)}{=} (-1)^{m(n-m)} \cdot \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0,\dots,\mathfrak{v}_m]) \cdot \sigma \circ [\mathfrak{v}_m,\dots,\mathfrak{v}_n]}_{=\varphi \cap \sigma} \;=\; \varphi \mathbin{\widetilde{\cap}} \sigma.
\end{aligned}
$$

(definition of η in the Chain Complex–Shift Lemma 207.11; Definition 207.14 of $\widetilde{\cap}$)

We have thus verified that the desired equality holds. ∎

The following statement is just the definition of the cross product.

Tautology 207.17. Let X and Y be two topological spaces. For any $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$ we have $[\mu] \times [\nu] \;=\; [\Upsilon(\mu \otimes \nu)]$.

We need one more slightly delicate chain map before we can finally turn to the proof of the Cross-Cup-Cap Theorem 207.1. More precisely, we will need the following lemma, which can be viewed as a generalization of the Chain Complex–Shift Lemma 207.11:

Lemma 207.18. (Tensor-and-Shift Lemma) Let C_* and D_* be chain complexes and let $k, l \in \mathbb{N}_0$. The map

$$\Psi_{k,l}\colon C_*[k] \otimes D_*[l] \;\to\; (C_* \otimes D_*)[k+l]$$

which is defined by the property that for $c_m \in C_m$ and $d_n \in D_n$ we have

$$c_m \otimes d_n \;\mapsto\; (-1)^{k(l+n)} \cdot c_m \otimes d_n.$$

is an isomorphism of chain complexes.

Proof. First note that it is clear that $\Psi_{k,l}$ preserves degrees and that on each degree it is an isomorphism. It remains to show that the map $\Psi_{k,l}$ is a chain map. Let $c_m \in C_m$ and

$d_n \in D_n$. We calculate that

$$\begin{aligned}
\Psi_{k,l}(\partial^{\otimes}_{m+n}(c_m \otimes d_n)) &\overset{\text{since } c_m \text{ has degree } m+k \text{ in } C_*[k]}{=} \Psi_{k,l}(\partial_m(c_m) \otimes d_n + (-1)^{m+k} \cdot c_m \otimes \partial_n(d_n)) \\
&= (-1)^{k(l+n)} \cdot \partial_m(c_m) \otimes d_n + (-1)^{m+k} \cdot (-1)^{k(l+n-1)} \cdot c_m \otimes \partial_n(d_n) \\
&= (-1)^{k(l+n)} \cdot (\partial_m(c_m) \otimes d_n + (-1)^m \cdot c_m \otimes \partial_n(d_n)) \\
&= (-1)^{k(l+n)} \cdot \partial^{\otimes}_{m+n}(c_m \otimes d_n) \\
&\overset{\text{since } c_m \text{ has degree } m \text{ in } C_*}{=} \partial^{\otimes}_{m+n}((-1)^{k(l+n)} \cdot c_m \otimes d_n)) \;=\; \partial^{\otimes}_{m+n}(\Psi_{k,l}(c_m \otimes d_n)).
\end{aligned}$$

This calculation implies that $\Psi_{k,l}$ is a chain map. ■

207.2.3. **Proof of the Cross-Cup-Cap Theorem 207.1: The main step.** The following theorem is the main technical result of this section. As we will see in the next subsection, the Cross-Cup-Cap Theorem 207.1 is a straightforward consequence of the following Cross-Cup-Cap-on-Chain Level Theorem 207.19.

Theorem 207.19. (Cross-Cup-Cap-on-Chain Level Theorem) Let X and Y be topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projections. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. Then the chain maps

$$(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon \quad \text{and} \quad \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)$$

from $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ to $\mathrm{C}_*(X \times Y)[k+l]$ are chain homotopic.

Proof. Let X and Y be topological spaces and let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We consider the following diagram of chain maps

$$\begin{array}{ccccc}
\mathrm{C}_*X \otimes \mathrm{C}_*Y & \xrightarrow{\Upsilon} & \mathrm{C}_*(X\times Y) & \xrightarrow{\Delta_{X\times Y}} & \mathrm{C}_*(X\times Y)\otimes \mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \Delta_X\otimes\Delta_Y} & & & & \downarrow{\scriptstyle \Theta\otimes\mathrm{id}} \\
\mathrm{C}_*X\otimes \mathrm{C}_*X\otimes \mathrm{C}_*Y\otimes \mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathrm{C}_*X\otimes \mathrm{C}_*Y\otimes \mathrm{C}_*X\otimes \mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon} & \mathrm{C}_*X\otimes \mathrm{C}_*Y\otimes \mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \Pi_k(\alpha)\otimes\mathrm{id}\otimes\Pi_l(\beta)\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}} \\
\mathbb{Z}[k]\otimes \mathrm{C}_*X\otimes \mathbb{Z}[l]\otimes \mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes \mathrm{C}_*X\otimes \mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon} & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes \mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}} \\
 & & \mathbb{Z}[k+l]\otimes(\mathrm{C}_*X\otimes \mathrm{C}_*Y) & \xrightarrow{\mathrm{id}\otimes\Upsilon} & \mathbb{Z}[k+l]\otimes \mathrm{C}_*(X\times Y) \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
\mathrm{C}_*X[k]\otimes \mathrm{C}_*Y[l] & \xrightarrow{\Psi_{k,l}} & (\mathrm{C}_*X\otimes \mathrm{C}_*Y)[k+l] & \xrightarrow{\Upsilon[k+l]} & \mathrm{C}_*(X\times Y)[k+l].
\end{array}$$

Claim 1. The above diagram commutes up to chain homotopies.

Proof. We consider the various smaller squares and rectangles.

(1) The two paths along the top rectangle from the top-left corner to the bottom-right corner are both natural chain maps. Furthermore, both have the property that for $x \in X$ and $y \in Y$ the element $x \otimes y \in \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ gets sent to $x \otimes y \otimes (x,y)$ in $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \otimes \mathrm{C}_*(X \times Y)$. In particular both paths agree on the 0-level.

Almost the same argument as in the proof of the EZ–Natural Chain Homotopy Proposition 164.13 (a) (which is essentially the same argument as in the proof of the Diagonal Approximation–Uniqueness Proposition 203.19) shows that the rectangle commutes up to a chain homotopy.[2208]

As we will see the other squares commute "on the nose" for formal reasons:

(2) It is clear that the two squares between the second and the third row commute.
(3) It is straightforward to verify that the two squares at the bottom right commute.
(4) It remains to consider the big square to the bottom left of the diagram. Starting with $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$, $b \in \mathrm{C}_m(X)$, $c \in (\mathbb{Z}[l])_l = \mathbb{Z}$ and $d \in \mathrm{C}_n(Y)$ and keeping in mind the sign conventions of the Swap Factors–Chain Map Lemma 207.8, the Chain Complex–Shift Lemma 207.11 and the Tensor-and-Shift Lemma 207.18 we get the following results for $a \otimes b \otimes c \otimes d$:
   (a) going down-right gives $(-1)^{km} \cdot (-1)^{ln} \cdot (-1)^{k(l+n)} \cdot ac \cdot (b \otimes d)$,
   (b) going right-down-down gives $(-1)^{ml} \cdot (-1)^{kl} \cdot (-1)^{(k+l)(m+n)} \cdot ac \cdot (b \otimes d)$.
   One easily verifies that the signs agree. So we see that the bottom left square also commutes. ⊞

In light of the previous claim it suffices to prove the following claim:

Claim 2.
(1) The "left-bottom route" equals the chain map $\Upsilon[k+l] \circ \Psi_{k,l} \circ ((\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -))$.
(2) The "top-right route" is chain homotopic to $(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon$.

Proof. We start out with the proof of Statement (1). First we note that

$$(\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -) \overset{\text{by the Cap Product–Reinterpretation Proposition 207.16}}{=} (\eta \circ (\Pi_k(\alpha) \otimes \mathrm{id}) \circ \Delta_X) \otimes (\eta \circ (\Pi_l(\beta) \otimes \mathrm{id}) \circ \Delta_Y) = (\eta \otimes \eta) \circ ((\Pi_k(\alpha) \otimes \mathrm{id} \otimes \Pi_l(\beta) \otimes \mathrm{id})) \circ (\Delta_X \otimes \Delta_Y).$$

(the last equality follows from the observation that for chain maps a, b, c and d we have $(a \circ b) \otimes (c \circ d) = (a \otimes b) \circ (c \otimes d)$)

If we compose this equality with $\Upsilon[k+l] \circ \Psi_{k,l}$ then we obtain the "bottom left route" in the above diagram. This concludes the proof of Statement (1).

We continue with the proof of Statement (2). In that proof we will make use of the following subclaim.

Subclaim. The two cocycles[2209]

$$(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \quad \text{and} \quad (-1)^{kl} \cdot p^*(\alpha) \cup q^*(\beta)$$

in $\mathrm{C}^{k+l}(X \times Y; \mathbb{Z})$ are cohomologous.

[2208] There is an alternative approach: It also follows from the future Acyclic Model Theorem 235.7 (2) that the rectangle commutes up to a chain homotopy. More precisely, we apply the Acyclic Model Theorem 235.7 as follows:
(a) We take $\mathcal{C}$ to be the category of tuples (X, Y) of topological spaces.
(b) We consider the functor $F\colon (X, Y) \mapsto (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y), \epsilon_X \otimes \epsilon_Y)$ and we consider also the functor $G\colon (X, Y) \mapsto (\mathrm{C}_*(X \times Y), \epsilon_{X \times Y})$.
(c) We take $\mathcal{M} = \{\Delta^k \times \Delta^l\}_{k,l \in \mathbb{N}_0}$.
(d) We consider the two natural transformations that are given by the two extremal paths in the diagram (shifted downward in degrees).

Proof of subclaim. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{C}_{k+l}(X\times Y) & \hookrightarrow & \mathrm{C}_*(X\times Y) & \xrightarrow{\Theta} & \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \\
 & & \downarrow{\scriptstyle \Delta_{X\times Y}} & \nearrow{\scriptstyle p_*\otimes q_*} & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)} \\
 & & \mathrm{C}_*(X\times Y)\otimes\mathrm{C}_*(X\times Y) & \xrightarrow[\Pi_k(p^*(\alpha))\otimes\Pi_l(q^*(\beta))]{} & \mathbb{Z}[k]\otimes\mathbb{Z}[l] \xrightarrow[\eta]{} \mathbb{Z}[k+l].
\end{array}$$

Note that by the above discussion all maps except for the top left map are chain maps. It follows immediately from the definitions that the bottom triangle of chain maps commutes "on the nose". The top triangle of chain maps commutes, up to a chain homotopy, by the EZ–Natural Chain Homotopy Proposition 164.13 (a).[2210] Note that it follows from the Projection–Cocycle Lemma 207.10 (3) that the restrictions of these two chain maps to maps $\mathrm{C}_{k+l}(X\times Y)\to\mathbb{Z}=\mathbb{Z}[k+l]$ are cohomologous cochains. But the restriction of the "top route" equals evidently the left-hand side of the subclaim. On the other hand, by the Cup Product–Reinterpretation Lemma 207.13 the restriction of the "bottom route" equals the right-hand side of the subclaim. □

Now we see that

$$\begin{aligned}
(-1)^{kl}\cdot((p^*\alpha\cup q^*\beta)\mathbin{\widetilde{\cap}}-)\circ\Upsilon
&\overset{\text{by the above subclaim and the Cap-as-Chain Map Lemma 207.15 (2)}}{\cong}
\big((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l}\mathbin{\widetilde{\cap}}-\big)\circ\Upsilon \\
&= \eta\circ\big(\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l})\otimes\mathrm{id}\big)\circ\Delta_{X\times Y}\circ\Upsilon \\
&\quad\uparrow \text{ by the Cap Product–Reinterpretation Proposition 207.16 applied to the } (k+l)\text{-cocycle } (\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l} \\
&= \eta\circ\big(\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l})\circ\Theta\otimes\mathrm{id}\big)\circ\Delta_{X\times Y}\circ\Upsilon \\
&\quad\uparrow \text{ since for every chain map } \Omega\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathbb{Z}[k+l] \text{ we have } \Pi_{k+l}((\Omega\circ\Theta)_{k+l})=\Pi_{k+l}(\Omega_{k+l})\circ\Theta \text{ as maps } \mathrm{C}_*(X\times Y)\to\mathbb{Z}[k+l], \\
&\quad\quad \text{applies this observation to } \Omega=\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)) \\
&= \eta\circ\big((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)\otimes\mathrm{id}\big)\circ\Delta_{X\times Y}\circ\Upsilon \\
&\quad\uparrow \text{ here we use the equality of chain maps} \\
&\quad\quad \Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l})=\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathbb{Z}[k+l] \\
&\quad\quad \text{which is obvious once one unravels the definitions} \\
&= \text{the "top right route" in the above diagram.} \\
&\quad\uparrow \text{ follows from } (f\otimes\mathrm{id})\circ(g\otimes\mathrm{id})=(g\circ f)\otimes\mathrm{id}
\end{aligned}$$

This concludes the proof of the claim and thus also of the theorem. ■

207.2.4. **Proof of the Cross-Cup-Cap Theorem 207.1: Conclusion.** Now we can finally provide the proof of the Cross-Cup-Cap Theorem 207.1:

Proof of the Cross-Cup-Cap Theorem 207.1. To simplify the notation we only consider the special case that $A=B=\varnothing$. For the general case one needs to take more care, we refer to [**Dol1956**, p. 240 and p. 244] for details. So let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projections. Let $[\alpha]\in\mathrm{H}^k(X;\mathbb{Z})$,

[2209] Recall that $\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta$ is a chain map $\mathrm{C}_*(X\times Y)\to\mathbb{Z}[k+l]$. Furthermore recall that the map $(\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l}\colon \mathrm{C}_{k+l}(X\times Y)\to(\mathbb{Z}[k+l])_{k+l}=\mathbb{Z}$ is the $(k+l)$-th map of the chain map.

[2210] More precisely, given $x\in X$ and $y\in Y$ both routes along the triangle send (x,y) to $x\otimes y$. Furthermore both routes are natural chain maps. Thus we can apply the EZ–Natural Chain Homotopy Proposition 164.13 (a).

$[\beta] \in \mathrm{H}^l(Y;\mathbb{Z})$ and let $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$. We have to show that

$$(\underbrace{[p^*(\alpha)]}_{\in \mathrm{H}^k(X\times Y)} \cup \underbrace{[q^*(\beta)]}_{\in \mathrm{H}^l(X\times Y)}) \cap \underbrace{([\mu]\times[\nu])}_{\in \mathrm{H}_{m+n}(X\times Y)} = (-1)^{kl+lm} \cdot \underbrace{([\alpha]\cap[\mu])}_{\in \mathrm{H}_{m-k}(X)} \times \underbrace{([\beta]\cap[\nu])}_{\in \mathrm{H}_{n-l}(Y)}$$

We have the following equality in $\mathrm{H}_{m+n-k-l}(X\times Y)$:

$$\begin{aligned}
(p^*([\alpha]) \cup q^*([\beta])) \cap ([\mu]\times[\nu]) &\overset{\text{Tautology 207.17 and Definition 207.14 of } \widetilde{\cap}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ([p^*(\alpha)\cup q^*(\beta)]) \mathbin{\widetilde{\cap}} [\Upsilon(\mu\otimes\nu)] \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (((p^*(\alpha)\cup q^*(\beta)) \mathbin{\widetilde{\cap}} -) \circ \Upsilon)_*([\mu\otimes\nu]) \\
&\overset{\text{by the Cross-Cup-Cap-on-Chain Level Theorem 207.19 and the fact that chain homotopic maps induce the same map on homology groups}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ((-1)^{kl} \cdot \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -))_*([\mu\otimes\nu]) \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot [\Upsilon[k+l] \circ \Psi_{k,l} \circ (\underbrace{\alpha \mathbin{\widetilde{\cap}} \mu}_{\substack{=(-1)^{k(m-k)}\cdot\alpha\cap\mu \\ \in \mathrm{C}_{m-k}(X)[k]}} \otimes \underbrace{\beta \mathbin{\widetilde{\cap}} \nu}_{\substack{=(-1)^{l(n-l)}\cdot\beta\cap\nu \\ \in \mathrm{C}_{n-l}(Y)[l]}})] \\
&\overset{\text{definition of } \Psi_{k,l}\text{, see Lemma 207.18, and of the shifting from } \Upsilon \text{ to } \Upsilon[k+l]}{=} (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot (-1)^{k(l+n-l)} \cdot (-1)^{k(m-k)} \cdot (-1)^{l(n-l)} \cdot [\Upsilon(\alpha\cap\mu\otimes\beta\cap\nu)] \\
&\overset{\text{cleaning up signs and Tautology 207.17}}{=} (-1)^{kl+lm} \cdot [\alpha\cap\mu \times \beta\cap\nu] = (-1)^{kl+lm} \cdot ([\alpha]\cap[\mu] \times [\beta]\cap[\nu]).
\end{aligned}$$

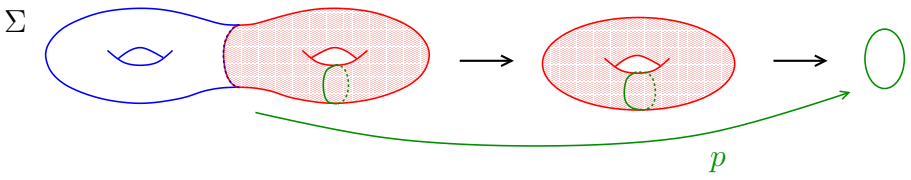
■

207.3. Exercises for Chapter 207 .

Exercise 207.1. Let Σ be the surface of genus 2, let $q\colon \Sigma\times S^1 \to \Sigma$ be the projection and let $p\colon \Sigma \to S^1$ be the map shown in the figure below. Determine the cap product

$$q(p^*([S^1]^*)) \cap [\Sigma\times S^1] \in \mathrm{H}_2(\Sigma\times S^1;\mathbb{Z})$$

up to sign.



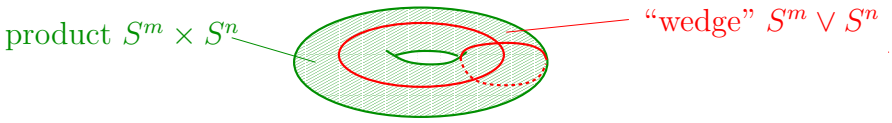
Exercise 207.2. Given $n\in\mathbb{N}$ we denote by $p\colon S^n\times S^n \to S^n$ and $q\colon S^n\times S^n\to S^n$ the two obvious projections. Let $\varphi \in \mathrm{H}^n(S^n;\mathbb{Z})$ be a generator. We know, for example by the Product-of-Spheres–Cup Product Lemma 207.3, that $p^*(\varphi)$ and $q^*(\varphi)$ form a basis for $\mathrm{H}^n(S^n\times S^n;\mathbb{Z})$. Given a continuous map $f\colon S^n\times S^n \to S^n\times S^n$ we denote by $A(f)$ the matrix representing the automorphism f^* of $\mathrm{H}^n(S^n\times S^n;\mathbb{Z})$ with respect to the basis $p^*(\varphi)$ and $q^*(\varphi)$.

(a) Let $B\in \mathrm{SL}(2,\mathbb{Z})$. Show that there exists a homeomorphism $f\colon S^1\times S^1 \to S^1\times S^1$ with $A(f)=B$.
Hint. Use the Action-on-Torus Homology Lemma 166.12.

(b) Now let $f\colon S^n \times S^n \to S^n \times S^n$ be a homeomorphism. Show that if n is even, then $A(f)$ is given by one of the following eight matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Exercise 207.3. Let $m, n \in \mathbb{N}$. We pick $P \in S^m$ and $Q \in S^n$. We consider the product $S^m \times S^n$ and the wedge $S^m \vee S^n = (S^m \times \{Q\}) \cup (\{P\} \times S^n)$. Is $S^m \vee S^n$ a retract of $S^m \times S^n$?



Exercise 207.4. We consider the two topological spaces $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$.
(a) Show that the homology groups of $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$ are isomorphic.
(b) Use cup products to show that $S^1 \vee S^2 \vee S^3$ and $S^1 \times S^2$ are not homotopy equivalent.

Exercise 207.5. We consider the two topological spaces $S^2 \vee S^4 \vee S^6$. Give an example of a simply connected topological space with the same homology and cohomology groups, that is not homotopy equivalent to $S^2 \vee S^4 \vee S^6$.

Exercise 207.6. Let $X = S^1 \times [0,1]$ be the annulus. Let $A, B \subseteq \partial X$ be unions of boundary components such that $A \cup B = \partial X$ and $A \cap B = \varnothing$. Then the map

$$\begin{array}{ccc} \mathrm{H}^1(X, A; \mathbb{Z}) & \to & \mathrm{H}_1(X, B; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is an isomorphism.
Remark. This gives a new proof of the Annulus–Cap Product Lemma 206.11.

Exercise 207.7. Show that there exists a topological space with infinite cup length.

CHAPTER 208

The Künneth Theorem for cohomology

Given two topological spaces X and Y the Künneth Theorem 164.15 determines the homology groups of $X \times Y$ in terms of the homology groups of X and Y. In this chapter we prove a cohomological analog of this result.

Together with results from the last chapter we will be able to fully calculate the cup product of the n-dimensional torus $(S^1)^n$. In particular this will allow us to determine the Lusternik-Schnirelmann category of the n-dimensional torus.

This chapter is organized as follows:

- In Section 208.1 we will state the Künneth-H^*-Theorem 208.1 and we will discuss some applications.
- In Section 208.2 we will prove the Künneth-H^*-Theorem 208.1.
- In Section 208.3 we will discuss the cohomology ring of the product of two topological spaces.
- In Section 208.4 we will discuss cohomology with $\mathbb{F}_2$-coefficients.
- In Section 208.5 we will provide exercises for Chapter 208.

208.1. **The Künneth Theorem for cohomology groups.** The following theorem can be viewed as the singular analog of the Künneth-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 81.21 and as the cohomological analog of the Künneth Theorem 164.15.

Theorem 208.1. (Künneth-H^*-Theorem) Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. We suppose that X satisfies the following technical hypothesis:

(⁕) The singular chain complex of X is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

Then given any $n \in \mathbb{N}_0$ there exists a[2211] homomorphism

$$\nu\colon \mathrm{H}^n(X \times Y; \mathbb{Z}) \to \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z}))$$

such that the following sequence is exact:[2212]

$$\begin{array}{ccccccc} 0 \to \displaystyle\bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) & \to & \mathrm{H}^n(X \times Y;\mathbb{Z}) & \xrightarrow{\nu} & \displaystyle\bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0. \\ \varphi \otimes \psi & \mapsto & p^*(\varphi) \cup q^*(\psi) & & \end{array}$$

If Y also satisfies (⁕), then the short exact sequence splits (albeit not naturally).[2213]

The obvious question is, when is (⁕) satisfied? We will now give two answers, one is topological, one is purely algebraic:

Lemma 208.2. (CW–Chain Complex Condition Lemma) Let X be a topological space. If X is homotopy equivalent to a CW-complex $\widetilde{X}$ such that in each dimension $\widetilde{X}$ has only finitely many cells, then X satisfies Condition (⁕) from the Künneth-H*-Theorem 208.1.

Proof. Let X be a topological space that is homotopy equivalent to a CW-complex $\widetilde{X}$ such that in each dimension $\widetilde{X}$ has only finitely many cells. The lemma follows immediately from the following observations:

(1) By the Chain Homotopy-Lemma 144.6 being "chain homotopy equivalent" is indeed an equivalence relation.
(2) It follows easily from the Homotopy-H$_*$-Proposition 144.12 (1) that the singular chain complex $\mathrm{C}_*(X)$ is chain homotopy equivalent to the singular chain complex $\mathrm{C}_*(\widetilde{X})$.
(3) By the Singular-Cellular Chain Complex Proposition 158.16 we know that the singular chain complex $\mathrm{C}_*(\widetilde{X})$ is chain homotopy equivalent to the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$.
(4) It follows from our hypothesis on $\widetilde{X}$ together with the CW-Skeleton-H$_*$-Lemma 155.2 that each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(X)$ is a finitely generated free abelian group. ∎

Lemma 208.3. (Algebraic–Chain Complex Condition Lemma) Let X be a topological space. If all the homology groups $\mathrm{H}_k(X)$ are finitely generated abelian groups, then X satisfies condition (⁕) from the Künneth-H*-Theorem 208.1.

Remark. It follows from the CW-Complex-H$_*$-Proposition 155.10 that the CW–Chain Complex Condition Lemma 208.2 can in fact be viewed as a consequence of the Algebraic–Chain Complex Condition Lemma 208.3.

Proof. This lemma is an immediate consequence of the fact that all chain groups $\mathrm{C}_k(X)$ are free abelian and the Finite-Rank Chain Complex Lemma 158.22. ∎

208.2. **Proof of the Künneth-H*-Theorem 208.1.** This section is completely dedicated to the proof of the Künneth-H*-Theorem 208.1. In the next Section 208.3 we will study applications.

208.2.1. **The Algebraic–Künneth-H*-Theorem.** The proof of the Künneth-H*-Theorem 208.1 rests on a purely algebraic theorem that, given two chain complexes $\mathcal{C}$ and $\mathcal{C}'$, relates the cohomology groups of $\mathcal{C}$ and $\mathcal{C}'$ to the cohomology groups of the tensor product[2214] $\mathcal{C} \otimes \mathcal{C}'$. Before we can state the theorem we need to recall Definition 203.4:

[2211] The dog that did not bark is that here we do not claim that the map is natural. It is conceivable that with more effort one can arrange this. But our proof does not give this straightaway and we will not need such a statement anyway.

[2212] The map

$$\begin{aligned} \mathrm{H}^k(X;\mathbb{Z}) \times \mathrm{H}^l(Y;\mathbb{Z}) &\to \mathrm{H}^{k+l}(X \times Y;\mathbb{Z}) \\ (\varphi, \psi) &\mapsto p^*(\varphi) \cup q^*(\psi) \end{aligned}$$

is often referred to as the *cohomology cross product* or *cohomological cross product* and it is written as $\varphi \times \psi := p^*(\varphi) \cup q^*(\psi)$. We will not make use of this notation.

[2213] Of course the roles of X and Y can be flipped in the statements of the hypothesis.

[2214] We refer to Definition 163.2 for the definition of the tensor product of two chain complexes.

Definition 208.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes and let $p, q \in \mathbb{N}_0$.

(1) We denote by
$$\Pi_{p,q} \colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} = \bigoplus_{i+j=p+q} C_i \otimes C'_j \to C_p \otimes C'_q$$
the natural projection.

(2) We denote by $\mu_{\mathbb{Z}} \colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism given by $\mu_{\mathbb{Z}}(a \otimes b) = ab$.

(3) It follows from the Cocycle Tensor Lemma 203.5 that the following map is well-defined:[2215]
$$\begin{aligned} \Xi_{p,q} \colon \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) &\to \mathrm{H}^{p+q}(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}) \\ ([\varphi_p] \otimes [\varphi'_q]) &\mapsto [\mu_{\mathbb{Z}} \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q}]. \end{aligned}$$
Furthermore it follows immediately from the definition that $\Xi_{p,q}$ is natural with respect to $\mathcal{C}$ and $\mathcal{C}'$.

Now we can state the promised purely algebraic theorem, which can be viewed as the cohomological analog of the Algebraic–Künneth Theorem 163.6.

Theorem 208.5. (Algebraic–Künneth-H*-Theorem) Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. We suppose that $\mathcal{C}$ satisfies the following hypothesis:

(⋇) The chain complex $\mathcal{C}$ is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

Then for every $n \in \mathbb{N}_0$ there exists a[2216] homomorphism
$$\nu \colon \mathrm{H}^n(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}) \to \bigoplus_{p+q=n+1} \mathrm{Tor}(\mathrm{H}^p(\mathcal{C}; \mathbb{Z}), \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}))$$
such that the following sequence is exact
$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) \xrightarrow{\Xi_{p,q}} \mathrm{H}^n(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}) \xrightarrow{\nu} \bigoplus_{p+q=n+1} \mathrm{Tor}(\mathrm{H}^p(\mathcal{C}; \mathbb{Z}), \mathrm{H}^q(\mathcal{C}'; \mathbb{Z})) \to 0.$$
If the second chain complex $\mathcal{C}'$ also satisfies (⋇), then the short exact sequence splits (albeit not naturally).

The idea behind the proof of the Algebraic–Künneth-H*-Theorem 208.5 is quite simple: Since a cochain complex is essentially the same as a chain complex, just with opposite directions for boundary maps, it seems realistic that we can reduce the above theorem, with some massaging, to the original Algebraic–Künneth Theorem 163.6.

To make this idea work we need to recall Definition 143.1:

Definition 208.6. A generalized chain complex (C_*, ∂_*) is a sequence
$$\ldots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \ldots$$
of homomorphisms between abelian groups such that for every $i \in \mathbb{Z}$ we have the equality $\partial_{i-1} \circ \partial_i = 0$.

Example. Let (C_*, ∂_*) be a chain complex. Given $n \in \mathbb{Z}_{\leq 0}$ we write $E_n := \mathrm{Hom}(C_{-n}, \mathbb{Z})$ and $e_n := \partial^*_{-n+1} \colon E_n \to E_{n-1}$. Then (E_*, e_*) is clearly a generalized chain complex. □

[2215] Here we use the Definition 159.6: Given two homomorphism $f \colon A \to A'$ and $g \colon B \to B'$ be tween abelian groups we consider the induced homomorphism $f \otimes g \colon A \otimes B \to A' \otimes B'$. Thus in this setting we have the homomorphism $\varphi_p \otimes \varphi'_q \colon C_p \otimes C'_q \to \mathbb{Z} \otimes \mathbb{Z}$.

[2216] Again we do not claim that this homomorphism is natural.

Theorem 163.6. (Algebraic–Künneth Theorem) Let $\mathcal{E}$ and $\mathcal{E}'$ be two generalized chain complexes. If all chain groups of the chain complex $\mathcal{E}$ are free abelian groups, then there exists a natural homomorphism

$$\nu\colon \mathrm{H}_n(\mathcal{E}\otimes\mathcal{E}') \;\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{E}),\mathrm{H}_q(\mathcal{E}'))$$

such that the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & \displaystyle\bigoplus_{p+q=m} \mathrm{H}_p(\mathcal{E})\otimes\mathrm{H}_q(\mathcal{E}') & \xrightarrow{\mu} & \mathrm{H}_m(\mathcal{E}\otimes\mathcal{E}') & \xrightarrow{\nu} & \displaystyle\bigoplus_{p+q=m-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{E}),\mathrm{H}_q(\mathcal{E}')) & \to & 0. \\ & & [\sigma_p]\otimes[\tau_q] & \mapsto & [\sigma_p\otimes\tau_q] & & & & \end{array}$$

If the chain groups of $\mathcal{E}'$ are also free abelian, then this short exact sequence splits.

Proof. We proved the Algebraic–Künneth Theorem 163.6 for "ordinary" chain complexes. The proof for generalized chain complexes is basically the same. ∎

Now let us try to prove the Algebraic–Künneth-H*-Theorem 208.5. Thus let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes and let $n\in\mathbb{N}_0$. The idea is to apply the Algebraic–Künneth Theorem 163.6 to the generalized chain complexes $\mathcal{E}_* := \mathrm{Hom}(\mathcal{C}_{-*},\mathbb{Z})$ and $\mathcal{E}' := \mathrm{Hom}(\mathcal{C}'_{-*},\mathbb{Z})$ and to $m=-n$. If we compare the short exact sequence from the Algebraic–Künneth Theorem 163.6 with the desired short exact sequence from the Algebraic–Künneth-H*-Theorem 208.5 we see that the groups to the left and right look about the same. But the middle group is different. In fact the following question arises: Under what circumstances do we have an isomorphism

$$\mathrm{Hom}(\mathcal{C}_*,\mathbb{Z})\otimes\mathrm{Hom}(\mathcal{C}'_*,\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{C}_*\otimes\mathcal{C}'_*,\mathbb{Z})\,?$$

The following lemma is the key to answering this question:

Lemma 208.7. (Tensor-of-Hom Lemma)[2217] Let A and B be abelian groups. We consider the homomorphism[2218]

$$\begin{array}{rcl} \Phi\colon \mathrm{Hom}(A,\mathbb{Z})\otimes\mathrm{Hom}(B,\mathbb{Z}) & \to & \mathrm{Hom}(A\otimes B,\mathbb{Z}) \\ \alpha\otimes\beta & \mapsto & \mu_{\mathbb{Z}}\circ\left(\begin{array}{rcl} \alpha\otimes\beta\colon A\otimes B & \to & \mathbb{Z}\otimes\mathbb{Z} \\ \displaystyle\sum_{i=1}^{m} a_i\otimes b_i & \mapsto & \displaystyle\sum_{i=1}^{m}\alpha(a_i)\otimes\beta(b_i) \end{array}\right). \end{array}$$

If A is a finitely generated free abelian group, then above map is an isomorphism.

Proof. If A is a finitely generated free abelian group, then we can find a basis $a_1,\ldots,a_k$ for A. We denote by $a_1^*,\ldots,a_k^*$ the dual basis of $\mathrm{Hom}(A,\mathbb{Z})$. We consider the homomorphism

$$\begin{array}{rcl} \Psi\colon \mathrm{Hom}(A\otimes B,\mathbb{Z}) & \to & \mathrm{Hom}(A,\mathbb{Z})\otimes\mathrm{Hom}(B,\mathbb{Z}) \\ \theta & \mapsto & \displaystyle\sum_{i=1}^{k} a_i^*\otimes(b\mapsto\theta(a_i\otimes b)). \end{array}$$

It remains to prove the following claim:

Claim. The map Φ and Ψ are inverses of one another.

[2217] Note that this lemma is the solution to Exercise 159.16 (b).

[2218] It follows easily from the Tensor Product–Universal Property Lemma 159.4 (2) that this map Φ is a well-defined homomorphism.

Proof. First we show that $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Hom}(A,\mathbb{Z})\otimes\mathrm{Hom}(B,\mathbb{Z})}$. Note that by the Tensor Product–Basics Lemma 159.3 (2) it suffices to prove that $\Psi \circ \Phi$ is the identity on any element of the form $a_j^* \otimes \beta \in \mathrm{Hom}(A,\mathbb{Z}) \otimes \mathrm{Hom}(B,\mathbb{Z})$. But for those elements we have indeed

$$(\Psi \circ \Phi)(a_j^* \otimes \beta) \underset{\text{definition of } \Phi}{\overset{\uparrow}{=}} \Psi(\mu_\mathbb{Z} \circ (a_j^* \otimes \beta)) \underset{\text{definition of } \Psi}{\overset{\uparrow}{=}} \sum_{i=1}^{k} a_i^* \otimes (b \mapsto \underbrace{(\mu_\mathbb{Z} \circ (a_j^* \otimes \beta))(a_i \otimes b)}_{= a_j^*(a_i)\cdot\beta(b) = \delta_{ij}\cdot\beta(b)}) = a_j^* \otimes \beta.$$

It remains to show that $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Hom}(A\otimes B,\mathbb{Z})}$. So let $\theta \in \mathrm{Hom}(A \otimes B, \mathbb{Z})$. We need to show that $(\Phi \circ \Psi)(\theta) = \theta\colon A \otimes B \to \mathbb{Z}$. Again by the Tensor Product–Basics Lemma 159.3 (2) it suffices to show that these two homomorphisms agree on any element of the form $a_j \otimes \widetilde{b} \in A \otimes B$. For such elements we calculate

$$\begin{aligned}((\Phi \circ \Psi)(\theta))(a_j \otimes \widetilde{b}) &\overset{\text{definition of } \Psi}{\overset{\downarrow}{=}} \Phi\Big(\sum_{i=1}^{k} a_i^* \otimes (b \mapsto \theta(a_i \otimes b))\Big)(a_j \otimes \widetilde{b}) \overset{\text{definition of } \Phi}{\overset{\downarrow}{=}} \\ &= \sum_{i=1}^{k} \underbrace{(\mu_\mathbb{Z} \circ (a_i^* \otimes (b \mapsto \theta(a_i \otimes b))))(a_j \otimes \widetilde{b})}_{= \mu_\mathbb{Z}(a_i^*(a_j)\otimes\theta(a_i\otimes\widetilde{b})) = \delta_{ij}\cdot\theta(a_i\otimes\widetilde{b})} = \theta(a_j \otimes \widetilde{b}).\end{aligned}$$

■

Lemma 208.8. (Cochain–Tensor Lemma) Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes. If all chain groups of $\mathcal{C}$ are finitely generated free abelian groups, then the map

$$\begin{aligned}\Omega\colon \mathrm{Hom}(\mathcal{C}_*,\mathbb{Z}) \otimes \mathrm{Hom}(\mathcal{C}'_*,\mathbb{Z}) &\to \mathrm{Hom}(\mathcal{C}_* \otimes \mathcal{C}'_*,\mathbb{Z}) \\ (\varphi_p\colon C_p \to \mathbb{Z}) \otimes (\varphi'_q\colon C'_q \to \mathbb{Z}) &\mapsto (\mu_\mathbb{Z} \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} \to \mathbb{Z})\end{aligned}$$

is an isomorphism of cochain complexes.

Proof. We start out with the following claim:

Claim 1. The map Ω is a cochain map.

Proof. This statement follows easily from the definitions. In fact, we basically proved this statement in the proof of the Cocycle Tensor Lemma 203.5. We leave it to the reader to fill in the details. ⊞

It follows from Claim 1 that it remains to show that for every degree $n \in \mathbb{N}_0$ the given map Ω is an isomorphism. In other words, we need to prove the following claim:

Claim 2. For every $n \in \mathbb{N}_0$ the following map is an isomorphism:

$$\Omega\colon \bigoplus_{p+q=n} \mathrm{Hom}(C_p,\mathbb{Z}) \otimes \mathrm{Hom}(C'_q,\mathbb{Z}) \to \mathrm{Hom}\Big(\bigoplus_{p+q=n} C_p \otimes C'_q, \mathbb{Z}\Big).$$

Proof. We consider the following two maps:

$$\begin{aligned}\bigoplus_{p+q=n} \mathrm{Hom}(C_p,\mathbb{Z}) \otimes \mathrm{Hom}(C'_q,\mathbb{Z}) &\overset{\Omega}{\to} \mathrm{Hom}\Big(\bigoplus_{p+q=n} C_p \otimes C'_q, \mathbb{Z}\Big) \to \bigoplus_{p+q=n} \mathrm{Hom}(C_p \otimes C'_q,\mathbb{Z}) \\ \varphi_p \otimes \varphi'_q &\mapsto \mu_\mathbb{Z} \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q} \\ \theta &\mapsto \sum_{p+q=n} \theta \circ \iota_{p,q}.\end{aligned}$$

We make the following clarifications and observations:

(1) The map to the left is precisely the map Ω for which we want to show that it is an isomorphism.

(2) For $p, q \in \mathbb{N}_0$ with $p+q=n$ let $\iota_{p,q}\colon C_p \otimes C'_q \to \bigoplus_{r+s=n} C_r \otimes C'_s$ the natural inclusion.

(3) It follows pretty easily from the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 that the map to the right is an isomorphism.

(4) The composition of the two maps is, for each $p, q \in \mathbb{N}_0$, given by the map

$$\begin{array}{rcl} \operatorname{Hom}(C_p, \mathbb{Z}) \otimes \operatorname{Hom}(C'_q, \mathbb{Z}) & \to & \operatorname{Hom}(C_p \otimes C'_q, \mathbb{Z}) \\ \varphi \otimes \varphi' & \mapsto & \mu_{\mathbb{Z}} \circ (\varphi \otimes \varphi'). \end{array}$$

from the Tensor-of-Hom Lemma 208.7. By hypothesis all groups C_p are finitely generated free abelian groups. Thus it follows from the Cochain–Tensor Lemma 208.8 that the above map is an isomorphism.

In summary we have shown that the right map and the composition of the two maps is an isomorphism. It follows that the left map Ω is an isomorphism. ■

Now we can finally prove the Algebraic–Künneth-H*-Theorem 208.5.

Proof of the Algebraic–Künneth-H*-Theorem 208.5. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be chain complexes. We assume that $\mathcal{C}$ satisfies (⁕), i.e. we assume that $\mathcal{C}$ is chain homotopy equivalent to a chain complex such that all chain groups are finitely generated abelian groups.

The proof proceeds in three steps:

(1) It follows from the Tensoring Chain Maps Lemma 163.4 (3), the Cochain Homotopy-H*-Lemma 197.14 and the Dual Cochain Map Lemma 197.20 (3b) that we can replace $\mathcal{C}$ by the chain complex where all chain groups are finitely generated free abelian groups. (Note though that at this point we lose naturality.)

(2) Given $n \in \mathbb{Z}_{\leq 0}$ we write

$$\begin{array}{rclcrcl} E_n & \coloneqq & \operatorname{Hom}(C_{-n}, \mathbb{Z}), & & e_n & \coloneqq & \partial^*_{-n+1}\colon E_n \to E_{n-1}, \\ E'_n & \coloneqq & \operatorname{Hom}(C'_{-n}, \mathbb{Z}), & & e'_n & \coloneqq & \partial^*_{-n+1}\colon E'_n \to E'_{n-1}. \end{array}$$

It follows immediately from the definitions that $\mathcal{E} \coloneqq (E_*, e_*)$ and $\mathcal{E}' \coloneqq (E'_*, e'_*)$ form chain complexes in negative degrees. Since the groups C_* are finitely generated free abelian groups we see that the groups E_* are free abelian.[2219] Therefore we can apply the Algebraic–Künneth Theorem 163.6 and we obtain for each $m \in \mathbb{Z}_{\leq 0}$ a natural homomorphism

$$\nu\colon \mathrm{H}_m(\mathcal{E} \otimes \mathcal{E}') \to \bigoplus_{p+q=m-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{E}), \mathrm{H}_q(\mathcal{E}'))$$

such that the following sequence is exact:

$$0 \to \bigoplus_{p+q=m} \mathrm{H}_p(\mathcal{E}) \otimes \mathrm{H}_q(\mathcal{E}') \to \mathrm{H}_m(\mathcal{E} \otimes \mathcal{E}') \xrightarrow{\nu} \bigoplus_{p+q=m-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{E}), \mathrm{H}_q(\mathcal{E}')) \to 0.$$

$$[\varphi] \otimes [\psi] \mapsto [\varphi \otimes \psi]$$

[2219] Here we use the following observation: If D is a finitely generated free abelian group, then it follows from the Abelian Group-Homomorphism Lemma 197.5 and the Direct Sum–Product-Homomorphism Lemma 197.6 that $\operatorname{Hom}(D, \mathbb{Z})$ is again a finitely generated free abelian group. As we discussed on page 3800, the analogous statement for arbitrary free abelian groups does *not* hold.

Put differently, for each $n \in \mathbb{N}_0$ we can apply the above discussion to $m = -n$ and we obtain the following exact sequence:

$$0 \to \bigoplus_{-p-q=-n} \underbrace{\mathrm{H}_{-p}(\mathcal{E})}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})} \otimes \underbrace{\mathrm{H}_{-q}(E')}_{=\mathrm{H}^q(\mathcal{C}';\mathbb{Z})} \to \mathrm{H}_{-n}(E \otimes E') \xrightarrow{\nu} \bigoplus_{-p-q=-n-1} \mathrm{Tor}\left(\underbrace{\mathrm{H}_{-p}(E)}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})}, \underbrace{\mathrm{H}_{-q}(E')}_{=\mathrm{H}^p(\mathcal{C}';\mathbb{Z})}\right) \to 0.$$

(3) For each $n \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}_{-n}(E_* \otimes E'_*) \underset{\text{by definition}}{=} \mathrm{H}^n(E_{-*} \otimes E'_{-*}) \underset{\text{by definition}}{=} \mathrm{H}^n(\overbrace{\mathrm{Hom}(C_*, \mathbb{Z})}^{=\mathcal{E}_{-*}} \otimes \overbrace{\mathrm{Hom}(C'_*, \mathbb{Z})}^{=\mathcal{E}'_{-*}}) \underset{\substack{\text{by the Cochain–Tensor Lemma 208.8,} \\ \text{since all chain groups of } \mathcal{C}_* \text{ are} \\ \text{finitely generated free abelian groups}}}{\xrightarrow{\Omega}} \mathrm{H}^n(C_* \otimes C'_*; \mathbb{Z}).$$

We use this isomorphism to replace the middle term of the second short exact sequence in (2) and we obtain the desired short exact sequence.

Finally suppose that $\mathcal{C}'$ also satisfies (⁕). In other words, we assume that $\mathcal{C}'$ is chain homotopy equivalent to a chain complex such that all chain groups are finitely generated abelian groups. By the same discussion as above this allows us to replace $\mathcal{C}'$ by a chain complex such that all chain groups are finitely generated free abelian groups. This observation implies that $\mathcal{E}'_*$ is a chain complex consisting of free abelian groups. This then implies, by the last statement of the Algebraic–Künneth Theorem 163.6, that the short exact sequence splits. ■

208.2.2. **Proof of the Künneth-H*-Theorem 208.1.** Now we are in a position to prove the Künneth-H*-Theorem 208.1:

Proof of the Künneth-H*-Theorem 208.1. Let X and Y be two topological spaces. We assume that X satisfies the following hypothesis:

(⁕) The singular chain complex of X is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. Throughout the proof we use all the definitions that we introduced in Section 207.2. We consider the following diagram:

$$\begin{array}{ccccccc} & & & & \mathrm{H}^n(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y); \mathbb{Z}) & & \\ & & & \nearrow^{[\alpha]\otimes[\beta]\mapsto[\mu_{\mathbb{Z}}\circ(\alpha\otimes\beta)\circ\Pi_{k,l}]} & \cong \downarrow \Theta^* & \searrow^{\nu} & \\ 0 \to & \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) & \xrightarrow{[\alpha]\otimes[\beta]\mapsto[p^*\alpha\cup q^*\beta]} & \underbrace{\mathrm{H}^n(X \times Y; \mathbb{Z})}_{=\mathrm{H}^n(\mathrm{C}_*(X\times Y);\mathbb{Z})} & \to & \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0. \end{array}$$

We make the following clarifications:

(1) The goal is to show that the "lower" sequence is exact.
(2) The "upper" sequence is the short exact sequence which we obtain from the Algebraic–Künneth-H*-Theorem 208.5, where we use that X satisfies (⁕).
(3) The map $\Theta\colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ is one of the chain homotopy equivalences provided by the Eilenberg-Zilber Theorem 164.2. It follows from the Dual Cochain

Map Lemma 197.20 together with the Cochain Homotopy-H*-Lemma 197.14 that the vertical map is an isomorphism.

(4) We define the bottom right map to be $\nu \circ (\Theta^*)^{-1}$.

We see that it remains to prove that the triangle to the left commutes. In other words, we need to prove the following claim:

Claim. For every $k, l \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) & \to & \mathrm{H}^{k+l}(X \times Y;\mathbb{Z}) \\ [\alpha] \otimes [\beta] & \mapsto & [p^*(\alpha) \cup q^*(\beta)] \end{array}$$

agrees with the map that is given by

$$[\alpha] \otimes [\beta] \quad \mapsto \quad [\mu_{\mathbb{Z}} \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Theta].$$

Proof. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We have to show that the two cocycles

$$\underbrace{p^*(\alpha) \cup q^*(\beta)}_{=:\aleph} \quad \text{and} \quad \underbrace{\mu_{\mathbb{Z}} \circ (\alpha \otimes \beta) \circ \Pi_{k,l}}_{=:\beth} \circ \Theta$$

are cohomologous.

As usual, given a topological space W we denote by $\epsilon_W \colon \mathrm{C}_0(W) \to \mathbb{Z}$ the augmentation map. Clearly ϵ_W is a cocycle, so by the Projection–Cocycle Lemma 207.10 (1) we can extend ϵ_W to a chain map $\mathrm{C}_*(W) \to \mathbb{Z}[0]$ which, by a slight abuse of notation, we also denote by ϵ_W. It follows from the Eilenberg-Zilber Theorem 164.2, the Cross-Cup-Cap-on-Chain Level Theorem 207.19 and the Projection–Cocycle Lemma 207.10 (2) that the maps

$$\underbrace{(-1)^{kl} \cdot \epsilon_{X\times Y}[k+l] \circ ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -)}_{=:\aleph'} \colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$$

and

$$\underbrace{\epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)}_{=:\beth'} \circ \Theta \colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$$

are cocycles that are cohomologous. It remains to prove the following two subclaims.

Subclaim. We have $\aleph' = (-1)^{kl} \cdot \aleph \colon \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}$.

Proof of subclaim. Let $\sigma \in \mathrm{C}_{k+l}(X \times Y)$. Then

$$\aleph'(\sigma) \underset{\uparrow}{=} (-1)^{kl} \cdot \epsilon_{X\times Y}((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} \sigma) \underset{\uparrow}{=} (-1)^{kl} \cdot (p^*\alpha \cup q^*\beta)(\sigma) = (-1)^{kl} \cdot \aleph(\sigma).$$

since we have $(\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$ we have $\epsilon_{X\times Y}[k+l] = \epsilon_{X\times Y}$

by definition of $\widetilde{\cap}$, see Definition 207.14, together with the Cap Product-Kronecker Lemma 206.12 □

Subclaim. We have $\beth' = (-1)^{kl} \cdot \beth \colon (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+l} \to \mathbb{Z}$.

Proof of subclaim. We have to show that both sides take the same value on any tensor product of the form $\sigma_i \otimes \sigma_j$ with $i + j = k + l$ and $\sigma_i \in \mathrm{C}_i(X)$ and let $\sigma_j \in \mathrm{C}_j(Y)$. It follows easily from the definition that both sides are zero for $\sigma_i \otimes \sigma_j$ with $(i, j) \neq (k, l)$. So

let $\sigma_k \in \mathrm{C}_k(X)$ and let $\sigma_l \in \mathrm{C}_l(Y)$. Then

$$\begin{aligned}
\beth'(\sigma_k \otimes \sigma_l) &= (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \circ \Psi_{k,l} \circ (\alpha \frown \sigma_k \otimes \beta \frown \sigma_l) \\
&\quad \uparrow \text{ since } \epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] = (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \\
&= (-1)^{kl} \cdot \mu_{\mathbb{Z}} \circ (\epsilon_X \otimes \epsilon_Y) \circ (\alpha \cap \sigma_k \otimes \beta \cap \sigma_l) \qquad = \quad (-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l) \\
&\quad \uparrow \text{ we use } \epsilon_{X\times Y} \circ \Upsilon = \mu_{\mathbb{Z}} \circ \epsilon_X \otimes \epsilon_Y \colon (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_0 \to \mathbb{Z}, \text{ definition of } \Psi_{k,l} \text{ in the Tensor-and-Shift Lemma 207.18 and the definition of } \frown \text{ in Definition 207.14} \qquad \uparrow \text{ by the Cap Product-Kronecker Lemma 206.12} \\
&= (-1)^{kl} \cdot (\mu_{\mathbb{Z}} \circ (\alpha \otimes \beta) \circ \Pi_{k,l})(\sigma_k \otimes \sigma_l) \qquad = \quad (-1)^{kl} \cdot \beth(\sigma_k \otimes \sigma_l).
\end{aligned}$$

This concludes the proof of the second subclaim, thus also of the claim and therefore also of the theorem. ∎

208.3. **The cohomology ring of product topological spaces.** Let X and Y be two topological spaces. In the Künneth-H*-Theorem 208.1 we just showed that (under some mild hypothesis) we have a short exact sequence

$$\begin{array}{ccccccccc}
0 \to \bigoplus\limits_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) & \to & \mathrm{H}^n(X\times Y;\mathbb{Z}) & \xrightarrow{\nu} & \bigoplus\limits_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0. \\
\varphi \otimes \psi & \mapsto & p^*(\varphi) \cup q^*(\psi) & &
\end{array}$$

In many cases this result allows us to calculate the cohomology *groups* of a product. But our real goal is to calculate the cohomology *rings* of a product. To get to this point, we need to make some algebraic preparations. First we recall Definition 203.22 and Definition 203.23:

Definition 208.9.

(1) A graded ring is a ring R together with a decomposition $R = \bigoplus\limits_{n\in\mathbb{N}_0} R_n$ such that for every $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map $R_m \times R_n \to R_{m+n}$.
(2) A homomorphism $\varphi\colon A \to B$ between graded rings is a ring homomorphism that preserves the gradings, i.e. that satisfies $\varphi(A_n) \subseteq B_n$ for all $n \in \mathbb{N}_0$.
(3) We say that an element r of R that lies in some R_n is homogeneous and we refer to $\deg(r) := n$ as the degree of r.
(4) A graded ring $R = \bigoplus\limits_{n\in\mathbb{N}_0} R_n$ such that for every two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} \cdot b \cdot a$ holds is called a superalgebra.

Examples.

(1) In the Cup Product-Ring Proposition 203.25 we saw that for any topological space X the cohomology ring $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is a superalgebra. Furthermore the Cup Product-Ring Proposition 203.25 (3) implies that for any continuous map $f\colon X \to Y$ the induced map $f^*\colon \mathrm{H}^*(Y;\mathbb{Z}) \to \mathrm{H}^*(X;\mathbb{Z})$ is a homomorphism of superalgebras.
(2) The polynomial ring $\mathbb{F}_2[a] = \bigoplus\limits_{i\in\mathbb{N}_0} \mathbb{F}_2 \cdot a^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(a^i) := i$.
(3) The polynomial ring $\mathbb{Z}[x] = \bigoplus\limits_{i\in\mathbb{N}_0} \mathbb{Z} \cdot x^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(x^i) := 2 \cdot i$.
(4) More generally, the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ in k variables $x_1, \ldots, x_k$ can be viewed as a superalgebra by setting $\deg(x_1^{r_1} \cdot \ldots \cdot x_k^{r_k}) = 2 \cdot (r_1 + \cdots + r_k)$.

(5) The ring $\mathbb{Z}[x]/(x^2) = \mathbb{Z} \oplus \mathbb{Z} \cdot x$ is a superalgebra by viewing elements in $\mathbb{Z}$ as elements of degree 0 and by viewing elements in $\mathbb{Z} \cdot x$ as elements of degree 1. Note that the map

$$\begin{array}{rcl} \mathbb{Z}[x]/(x^2) & \to & \mathrm{H}^*(S^1; \mathbb{Z}) \\ a \cdot 1 + b \cdot x & \mapsto & a \cdot 1_{S^1} + b \cdot [S^1]^* \end{array}$$

is an isomorphism of superalgebras.

(6) We denote by $\langle x_1, \ldots, x_k \rangle$ the free group on the generating set $\{x_1, \ldots, x_k\}$ as introduced in Definition 98.8. We consider the corresponding group ring $\mathbb{Z}\langle x_1, \ldots, x_k \rangle$ and we consider the ring

$$\Lambda[x_1, \ldots, x_k] \ := \ \mathbb{Z}\langle x_1, \ldots, x_k \rangle / \text{ideal generated by } x_1^2, \ldots, x_k^2, \{x_i x_j + x_j x_i\}_{i,j=1,\ldots,k}.$$

In this ring we use the symbol "$\wedge$" for multiplication instead of "$\cdot$". For example for any $i, j \in \{1, \ldots, k\}$ we have $x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$. This ring can be viewed in a unique way as a superalgebra such that $\deg(x_j) = 1$ for $j = 1, \ldots, k$.[22202221] In Exercise 208.9 we will verify that $\Lambda[x_1, \ldots, x_k]$ is a free abelian group of rank 2^k where a basis is given by the set $\{x_{n_1} \wedge \cdots \wedge x_{n_l} \,|\, 1 \leq n_1 < \cdots < n_l \leq k\}$.[2222] □

Definition 208.10. The tensor product of two superalgebras $A = \bigoplus_{n \in \mathbb{N}_0} A_n$ and $B = \bigoplus_{n \in \mathbb{N}_0} B_n$ is defined as the graded abelian group

$$A \otimes B \ := \ \bigoplus_{n \in \mathbb{N}_0} \underbrace{\bigoplus_{p+q=n} A_p \otimes B_q}_{=:(A \otimes B)_n}$$

where for $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$ the multiplication is given by[2223]

$$(a_k \otimes b_l) \cdot (a_m \otimes b_n) \ := \ (-1)^{lm} \cdot a_k \cdot a_m \otimes b_l \cdot b_n.$$

We continue with the following lemma which is somewhat related to Exercise 159.8 and the Tensor Product-of-Rings Lemma 311.7:

Lemma 208.11. (Super Algebra–Tensor Product Lemma)

(1) The tensor product of two superalgebras is again a superalgebra.

(2) The tensor product of superalgebras is associative, i.e. for any three superalgebras A, B and C the map[2224]

[2220] The ring $\Lambda[x_1, \ldots, x_k]$ is often referred to as the *exterior algebra on generators* $x_1, \ldots, x_k$, see e.g. [**Hat2002**, p. 217] and [**Mac1975**, p. 183].

[2221] If we replace the $\mathbb{Z}$-coefficients by $\mathbb{R}$-coefficients we obtain a ring that we denote by $\Lambda_{\mathbb{R}}[x_1, \ldots, x_k]$. This ring is isomorphic to the ring $\wedge^*(\mathbb{R}^k)^* = \bigoplus_{j=0}^{k} \wedge^j (\mathbb{R}^k)^*$ of alternating forms on $\mathbb{R}^k$, which we introduced in Notation 72.6, and where the ring structure on $\wedge^*(\mathbb{R}^k)^*$ is given by the wedge product of alternating forms. More precisely a ring isomorphism is given by

$$\begin{array}{rcl} \Lambda_{\mathbb{R}}[x_1, \ldots, x_k] & \to & \wedge^*(\mathbb{R}^k)^* \\ x_{i_1} \wedge \cdots \wedge x_{i_r} & \to & dy_{i_1} \wedge \cdots \wedge dy_{i_r} \end{array}$$

where $dy_i \colon \mathbb{R}^k \to \mathbb{R}$ is the projection onto the i-th coordinate.

[2222] A very similar statement is also proved in the Alternating Forms-Basis Proposition 72.7.

[2223] The mnemonic for the multiplication rule is that swapping b_l and a_m introduces the (by now familiar) sign $(-1)^{lm}$.

$$\begin{array}{rcl}(A\otimes B)\otimes C & \to & A\otimes(B\otimes C)\\ (a\otimes b)\otimes c & \mapsto & a\otimes(b\otimes c)\end{array}$$

that is induced by

is an isomorphism of superalgebras.

Proof. Let A, B and C be superalgebras and let $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$.

(1) We have the following equalities:

$$\begin{array}{rcl}(a_k\otimes b_l)\cdot(a_m\otimes b_n) & = & (-1)^{lm}\cdot a_k\cdot a_m\otimes b_l\cdot b_n \;=\; (-1)^{lm+km+ln}\cdot a_m\cdot a_k\otimes b_n\cdot b_l\\ & = & (-1)^{lm+km+ln+kn}\cdot(a_m\otimes b_n)\cdot(a_k\otimes b_l)\\ & = & (-1)^{(m+n)(k+l)}\cdot(a_m\otimes b_n)\cdot(a_k\otimes b_l)\\ & = & (-1)^{\deg(a_m\otimes b_n)\cdot\deg(a_k\otimes b_l)}\cdot(a_m\otimes b_n)\cdot(a_k\otimes b_l).\end{array}$$

It follows immediately that $A\otimes B$ is again a superalgebra.

(2) It suffices to show that the indicated map is a homomorphism of superalgebras. Thus let $c_r \in C_r$ and $c_s \in C_s$. Then

$$\begin{array}{rcl}((a_k\otimes b_l)\otimes c_r)\cdot((a_m\otimes b_n)\otimes c_s) & = & (-1)^{r(m+n)}\cdot((a_k\otimes b_l)\cdot(a_m\otimes b_n))\otimes(c_r\cdot c_s)\\ & = & (-1)^{r(m+n)+lm}\cdot((a_k\cdot a_m)\otimes(b_l\cdot b_n)\otimes(c_r\cdot c_s))\\ & = & (-1)^{m(l+r)}\cdot(a_k\cdot a_m)\otimes((b_l\otimes c_r)\cdot(b_n\otimes c_s))\\ & = & (a_k\otimes(b_l\otimes c_r))\cdot(a_m\otimes(b_n\otimes c_s)).\end{array}$$

■

Examples.

(1) As above we view the polynomial rings $\mathbb{F}_2[a]$ and $\mathbb{F}_2[b]$ as superalgebras by setting $\deg(a^i) = \deg(b^i) = i$. We consider the map

$$\begin{array}{rcl}\mathbb{F}_2[a]\otimes\mathbb{F}_2[b] & \to & \mathbb{F}_2[a,b]\\ a^r\otimes b^s & \mapsto & a^r\cdot b^s.\end{array}$$

Using the Tensor–Direct Sum Lemma 159.15 it is straightforward to show that this map is an isomorphism of abelian groups. One can easily verify that the map is a homomorphism of superalgebras. Summarizing we have convinced ourselves that this map is in fact an isomorphism of superalgebras.

(2) As before we view the polynomial rings $\mathbb{Z}[x_j]$, $j = 1,\dots,k$ as superalgebras with $\deg(x_j^i) = 2i$. We consider the map

$$\begin{array}{rcl}\mathbb{Z}[x_1]\otimes\cdots\otimes\mathbb{Z}[x_k] & \to & \mathbb{Z}[x_1,\dots,x_k]\\ x_1^{r_1}\otimes\cdots\otimes x_k^{r_k} & \mapsto & x_1^{r_1}\cdot\ldots\cdot x_k^{r_k}.\end{array}$$

The same argument as in (1) shows that this map is an isomorphism of superalgebras.

(3) As above we view the rings $\mathbb{Z}[x_j]/(x_j^2)$ as superalgebras with $\deg(x_j) = 1$. We leave it as an amusing exercise to verify that the map

$$\begin{array}{rcl}\mathbb{Z}[x_1]/(x_1^2)\otimes\cdots\otimes\mathbb{Z}[x_k]/(x_k^2) & \to & \Lambda[x_1,\dots,x_k]\\ x_1^{r_1}\otimes\cdots\otimes x_k^{r_k} & \mapsto & x_1^{r_1}\wedge\cdots\wedge x_k^{r_k}\end{array}$$

is an isomorphism of superalgebras. □

The following proposition explains our sudden interest in tensor products of superalgebras.

[2224] It follows from this statement that henceforth we can ignore parentheses when we consider the tensor product of several superalgebras.

Proposition 208.12. (Cohomology Ring-of-Product Proposition) Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections.

(1) The map
$$\begin{aligned} \Phi\colon \mathrm{H}^*(X;\mathbb{Z}) \otimes \mathrm{H}^*(Y;\mathbb{Z}) &\to \mathrm{H}^*(X \times Y;\mathbb{Z}) \\ \varphi \otimes \psi &\mapsto p^*(\varphi) \cup q^*(\psi) \end{aligned}$$
from the Künneth-H*-Theorem 208.1 is a homomorphism of superalgebras.

(2) If X satisfies the condition (⁕) from the Künneth-H*-Theorem 208.1, and if all homology groups of X are torsion-free, then the homomorphism from (1) is in fact an isomorphism.

Proof. Let X and Y be two topological spaces.

(1) Let $\varphi_k \in \mathrm{H}^k(X;\mathbb{Z})$, $\psi_l \in \mathrm{H}^l(Y;\mathbb{Z})$, $\varphi_m \in \mathrm{H}^m(X;\mathbb{Z})$ and $\psi_n \in \mathrm{H}^n(Y;\mathbb{Z})$. Then
$$\begin{array}{lll} \overbrace{\Phi((-1)^{lm}\cdot(\varphi_k \cup \varphi_m) \otimes (\psi_l \cup \psi_n))}^{=\Phi((\varphi_k \otimes \psi_l)\cdot(\varphi_m \otimes \psi_n))} & & (-1)^{lm}\cdot p^*(\varphi_k \cup \varphi_m) \cup q^*(\psi_l \cup \psi_n) \\ = (-1)^{lm}\cdot(p^*(\varphi_k) \cup p^*(\varphi_m) \cup q^*(\psi_l) \cup q^*(\psi_n)) & = & \underbrace{p^*(\varphi_k) \cup q^*(\psi_l) \cup p^*(\varphi_m) \cup q^*(\psi_n)}_{=\Phi(\varphi_k \otimes \psi_l) \cup \Phi(\varphi_m \otimes \psi_n))}. \\ \uparrow & & \uparrow \\ \text{by the Cup Product-Naturality} & & \text{by the Cup Product} \\ \text{Proposition 203.20} & & \text{Commutativity} \\ & & \text{Lemma 203.9} \end{array}$$
It follows immediately that Φ is a homomorphism of superalgebras.

(2) By our hypothesis on X we can apply the Künneth-H*-Theorem 208.1 and we obtain the following short exact sequence:
$$0 \to \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) \to \mathrm{H}^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+l=n+1} \underbrace{\mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z}))}_{=0, \text{ see below}} \to 0.$$
$$\varphi \otimes \psi \mapsto p^*(\varphi) \cup q^*(\psi)$$
Since the homology groups of X are torsion-free we obtain from the Universal Coefficient-H*-Theorem 201.23 together with the Ext-Properties Lemma 201.19 (4) that the cohomology groups $\mathrm{H}^k(X;\mathbb{Z})$ are also torsion-free. But by the Tor-Properties Lemma 162.13 (3) this implies that all the Tor-terms on the right-hand side of the short exact sequence vanish. ∎

The following proposition in particular provides an answer to Question 205.11 (1).

Proposition 208.13. (Torus–Cup Product Proposition) Let $n \in \mathbb{N}_0$. We denote by $(S^1)^n$ the n-dimensional torus. For $i = 1, \ldots, n$ we denote by $p_i\colon (S^1)^n \to S^1$ the projection onto the i-th factor. There exists a unique isomorphism
$$\psi\colon \Lambda[x_1, \ldots, x_n] \to \mathrm{H}^*((S^1)^n;\mathbb{Z})$$
of superalgebras which has the property that for each $i \in \{1, \ldots, n\}$ we have
$$\psi(x_i) = p_i^*([S^1]^*).$$

Proof. [2225]In Exercise 244.1 we will give a much more geometric proof of this proposition. We consider the following isomorphism of superalgebras

$$\Lambda[x_1,\ldots,x_n] \underset{\uparrow}{\overset{\cong}{\leftarrow}} \mathbb{Z}[x_1]/(x_1^2)\otimes\ldots\otimes\mathbb{Z}[x_n]/(x_n^2) \underset{\uparrow}{\overset{\cong}{\rightarrow}} \mathrm{H}^*(S^1;\mathbb{Z})\otimes\ldots\otimes\mathrm{H}^*(S^1;\mathbb{Z}) \underset{\uparrow}{\overset{\cong}{\rightarrow}} \mathrm{H}^*((S^1)^n;\mathbb{Z}).$$

isomorphism page 3996 — isomorphism from page 3995 — by the Cohomology Ring-of-Product Proposition 208.12, which we can apply by the CW–Chain Complex Condition Lemma 208.2

It is clear from the definitions of the various maps that the map from left to right sends each x_i to $p_i^*([S^1]^*)$. The composition of the isomorphisms thus gives us the desired isomorphism. The uniqueness follows immediately from the fact that the x_i generate $\Lambda[x_1,\ldots,x_n]$ as a ring. ■

The following corollary gives in particular a negative answer to Question 205.3.

Corollary 208.14. (LS Category–Torus Corollary) Let $n\in\mathbb{N}$. For the n-dimensional torus $(S^1)^n$ the Lusternik-Schnirelmann category $\mathrm{cat}((S^1)^n)$ equals n.[2226]

Proof. Given any $n\in\mathbb{N}_0$ we have

see page 3942 — by the Cup Length–LS Category Proposition 205.10

$$n \underset{\downarrow}{\geq} \text{Lusternik-Schnirelmann category } \mathrm{cat}((S^1)^n) \underset{\downarrow}{\geq} \text{cup length } \mathrm{cl}((S^1)^n) \underset{\uparrow}{\geq} n.$$

by the Torus–Cup Product Proposition 208.13 there exists an isomorphism $\psi\colon \Lambda[x_1,\ldots,x_n]\to\mathrm{H}^*((S^1)^n;\mathbb{Z})$ of graded rings, furthermore by the discussion on page 3995 we know that $x_1\wedge\cdots\wedge x_n\neq 0$, since ψ is an isomorphism we deduce that $\psi(x_1)\cup\cdots\cup\psi(x_n)\neq 0$, i.e. $\mathrm{cl}((S^1)^n)\geq n$ ■

We conclude this section with a short discussion of realization results. In Exercise 201.7 we saw that any sequence $\{A_n\}_{n\geq 1}$ of finitely generated abelian groups, with A_1 torsion-free, can appear as the sequence of cohomology groups $\mathrm{H}^*(X;\mathbb{Z})$ of some CW-complex. It is natural to ask whether any superalgebra structure can be realized as the cohomology ring of a topological space. More precisely, we have the following question:

Question 208.15. Let $R=\bigoplus_{n\in\mathbb{N}_0} R_n$ be superalgebra such that $R_0\cong\mathbb{Z}$, R_1 is torsion-free and such that all R_n are finitely generated. Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}),\cup)$ is isomorphic to R?

We will return to this question later with the Cohomology Ring–Non-Realization Proposition 237.7.

208.4. **Cohomology with $\mathbb{F}_2$-coefficients.** So far in this chapter we only considered cohomology with $\mathbb{Z}$-coefficients. In this final section we study cohomology with $\mathbb{F}_2$-coefficients. This case will be of (rather mild) interest when later on we study the cohomology ring of $\mathbb{R}\mathrm{P}^n\times\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients.

The following proposition is the $\mathbb{F}_2$-analog of the Cohomology Ring-of-Product Proposition 208.12.

Proposition 208.16. ($\mathbb{F}_2$-Cohomology Ring-of-Product Proposition) Let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the natural

[2226]We obtained the same result in Exercise 205.8.

projections. If X satisfies Condition (⋇) from the Künneth-H*-Theorem 208.1, then the map

$$\begin{array}{rcl} \Phi\colon \mathrm{H}^*(X;\mathbb{F}_2)\otimes \mathrm{H}^*(Y;\mathbb{F}_2) & \to & \mathrm{H}^*(X\times Y;\mathbb{F}_2) \\ \varphi\otimes\psi & \mapsto & p^*(\varphi)\cup q^*(\psi) \end{array}$$

is an isomorphism of superalgebras.

In the proof of the $\mathbb{F}_2$-Cohomology Ring-of-Product Proposition 208.16 we will need the following theorem which is the $\mathbb{F}_2$-analog of the Algebraic–Künneth-H*-Theorem 208.5.

Theorem 208.17. (Algebraic–Cohomological $\mathbb{F}_2$-Künneth Theorem) Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. We suppose that $\mathcal{C}$ satisfies the following hypothesis:

(⋇) The chain complex $\mathcal{C}$ is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

Then for every $n\in\mathbb{N}_0$ there exists a natural isomorphism

$$\bigoplus_{k+l=n} \mathrm{H}^k(\mathcal{C};\mathbb{F}_2)\otimes \mathrm{H}^l(\mathcal{C}';\mathbb{F}_2) \xrightarrow{\Xi} \mathrm{H}^n(\mathcal{C}\otimes\mathcal{C}';\mathbb{F}_2).$$

Proof of the Algebraic–Cohomological $\mathbb{F}_2$-Theorem 208.17. The proof of the Algebraic–Cohomological $\mathbb{F}_2$-Theorem 208.17 is evidently very similar to the proof of the Algebraic–Künneth-H*-Theorem 208.5. We refer to [**Mun1984**, Theorem 60.6] for details. ■

Proof of the $\mathbb{F}_2$-Cohomology Ring-of-Product Proposition 208.16. The proof is almost the same as the proof of the Cohomology Ring-of-Product Proposition 208.12. More precisely, we first note that if in the proof of the Künneth-H*-Theorem 208.1 we replace the Algebraic–Künneth-H*-Theorem 208.5 by the Algebraic–Cohomological $\mathbb{F}_2$-Theorem 208.17 we obtain that the map

$$\begin{array}{rcl} \displaystyle\bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{F}_2)\otimes \mathrm{H}^l(Y;\mathbb{F}_2) & \to & \mathrm{H}^n(X\times Y;\mathbb{F}_2) \\ \varphi\otimes\psi & \mapsto & p^*(\varphi)\cup q^*(\psi) \end{array}$$

is an isomorphism. Finally note that the same argument as in the proof of the Cohomology Ring-of-Product Proposition 208.12 shows that this map is in fact a homomorphism of superalgebras. ■

208.5. Exercises for Chapter 208 .

Exercise 208.1. Let A and B be abelian groups. We consider the homomorphism

$$\begin{array}{rcl} \Phi\colon \mathrm{Hom}(A,\mathbb{Z})\otimes \mathrm{Hom}(B,\mathbb{Z}) & \to & \mathrm{Hom}(A\otimes B,\mathbb{Z}) \\ \alpha\otimes\beta & \mapsto & \mu_{\mathbb{Z}}\circ(\alpha\otimes\beta) \end{array}$$

from the Tensor-of-Hom Lemma 208.7. Show that in general this map is not an isomorphism.

Remark. This exercise is the same as Exercise 159.16 (a).

Exercise 208.2. Let $g\in\mathbb{N}$. We denote by Σ_g the surface of genus g. Show that the Lusternik-Schnirelmann category of $S^1\times\Sigma_g$ is greater or equal than three.

Exercise 208.3. Let $n\in\mathbb{N}_0$. We write $T := (S^1)^n$. Let $\varphi\colon T\to T$ be a continuous map. Show that

$$\deg(\varphi) \;=\; \det(\mathrm{H}_1(T)\to \mathrm{H}_1(T)).$$

Exercise 208.4. Let $n \in \mathbb{N}_0$. We write $T \coloneqq (S^1)^n$. Let $\varphi\colon T \to T$ be a continuous map. We pick a basis for $\mathrm{H}_1(T) \cong \mathbb{Z}^n$ and we denote by A the $(n \times n)$-matrix that represents the map $\varphi_*\colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)$ with respect to the given basis. Show that

$$\underbrace{\sum_{k=0}^{n} (-1)^k \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(T) \to \mathrm{H}_k(T))}_{\text{Lefschetz number } \Lambda(\varphi)} \;=\; \det(\mathrm{id} - A).$$

Hint. Use the cohomological reformulation of the Lefschetz number given in Exercise 201.12 and use the Torus–Cup Product Proposition 208.13.
Remark. It follows from this calculation together with the Lefschetz Fixed Point Theorem 182.15 that unless 1 is an eigenvalue of A the map φ has a fixed point.

Exercise 208.5. Let $n \in \mathbb{N}$ and let W be a compact orientable $(n+1)$-dimensional topological manifold such that ∂W is homeomorphic to the n-torus $(S^1)^n$. Show that the inclusion-induced map $\mathrm{H}_1(\partial W; \mathbb{Z}) \to \mathrm{H}_1(W; \mathbb{Z})$ is not an injection.
Hint. Consider the cup product on $(S^1)^n$ and consider the interplay of the fundamental classes of ∂W and W.

Exercise 208.6. We say that a topological space X is an *H-space* if there exists a continuous map

$$\mu\colon X \times X \;\to\; X$$

and an element $e \in X$ such that $\mu(e,e) = e$ and such that the maps

$$\begin{array}{rcl} (X,e) & \to & (X,e) \\ x & \mapsto & \mu(x,e) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} (X,e) & \to & (X,e) \\ x & \mapsto & \mu(e,x) \end{array}$$

of pairs of topological spaces are homotopic to the identity.

(a) Show that S^1 and S^3 are H-spaces.
(b) Show that if $n \in \mathbb{N}$ is even, then S^n is not an H-space.
Hint. Denote by $p, q\colon S^n \times S^n \to S^n$ the two obvious projections and denote by $i, j\colon S^n \to S^n \times S^n$ the embeddings $x \mapsto (x,e)$ and $x \mapsto (e,x)$. Consider $q^*([S^n])$, $\mu^*([S^n]^*)$, $p^*([S^n]^*)$ and $p^*([S^n]^* \cup [S^n]^*)$.

Exercise 208.7. Given a topological space X we define its $\mathbb{Z}$-*cup length* as follows:

$$\mathrm{cl}_{\mathbb{Z}}(X) \coloneqq \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exist cohomology classes } \varphi_1, \dots, \varphi_n \text{ with } \mathbb{Z}\text{-coefficients} \\ \text{in degrees} \geq 1 \text{ such that } \varphi_1 \cup \dots \cup \varphi_n \neq 0 \end{array} \right\}.$$

Let $X_1, \dots, X_m$ be compact non-empty topological manifolds. Show that

$$\mathrm{cl}_{\mathbb{Z}}(X_1 \times \dots \times X_m) \;\geq\; \mathrm{cl}_{\mathbb{Z}}(X_1) + \dots + \mathrm{cl}_{\mathbb{Z}}(X_m).$$

Exercise 208.8. We consider the topological space $S^2 \vee S^4 \vee S^6 \vee S^8$. Give an example of a simply connected topological space with the same homology and cohomology groups, that is not homotopy equivalent to $S^2 \vee S^4 \vee S^6 \vee S^8$.

Exercise 208.9. We consider the ring

$$\Lambda[x_1, \dots, x_k] \;\coloneqq\; \mathbb{Z}[x_1, \dots, x_k]/\text{ideal generated by } x_1^2, \dots, x_k^2, \{x_i x_j + x_j x_i\}_{i,j=1,\dots,k}.$$

In this ring we use the symbol "$\wedge$" for multiplication instead of "$\cdot$". Show that the abelian group $\Lambda[x_1, \ldots, x_k]$ is a free abelian group of rank 2^k where a basis is given by the set $\{x_{n_1} \wedge \cdots \wedge x_{n_l} \,|\, 1 \leq n_1 < \cdots < n_l \leq k\}$.

Part XXII

Poincaré Duality

In this part of the lecture notes we will state and prove the Poincaré Duality Theorem 210.1 which states (in simplified form) that given any closed oriented n-dimensional topological manifold M and given any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) & \to & \mathrm{H}_{n-k}(M) \\ \varphi & \mapsto & \varphi \cap [M], \end{array}$$

which is given by capping with the fundamental class $[M] \in \mathrm{H}_n(M)$ of M, is an isomorphism. This part of the lecture notes contains the following chapters:

- In Chapter 209 we will introduce cohomology with compact support of any topological space. This invariant is interesting in its own right, but for us the most interesting aspect is that it plays an essential role in the proof of the Poincaré Duality Theorem 210.1.
- In **??** we will introduce a variation on ordinary homology, namely locally finite homology.
- In Chapter 210 we will state the Poincaré Duality Theorem 210.1 in full generality and we discuss many of the pretty immediate consequences.
- In Chapter 211 we provide the proof of the Poincaré Duality Theorem 210.1. The proof is long, but the main idea, is at the end not that hard.
- In Chapter 212 we discuss the interaction between Poincaré Duality and codimension-zero submanifolds. This chapter is rather technical and it is best to skip it for the time being.

CHAPTER 209

Cohomology with compact support and bounded cohomology

The Poincaré Duality Theorem 210.1 will say, in a slightly simplified version, that if M is a closed oriented n-dimensional topological manifold, then for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} -\cap[M]\colon \mathrm{H}^k(M;\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;\mathbb{Z}) \\ \varphi & \mapsto & \varphi\cap[M] \end{array}$$

that is given by capping with the fundamental class $[M] \in \mathrm{H}_n(M)$, is an isomorphism. So the good news is that we now have all the tools to formulate the most important theorem in (co-) homology theory.

The idea for the proof is somewhat similar to the proof of the Homology-Section Theorem 193.6, namely we will do a five step process where we work up our way from a trivial case, through cleverly arranged Mayer–Vietoris type arguments, to the general statement.

For better or worse this five step process involves non-compact topological manifolds and we will need to work with an analog of the above map $-\cap[M]$ for non-compact topological manifolds. To get a sensible statement we will have to replace ordinary cohomology by "cohomology with compact support". In this chapter we will introduce and study "cohomology with compact support". As we will see, this is an interesting invariant in its own right.

In an appendix we will introduce "bounded cohomology". This is another variation on ordinary singular cohomology. We will not make use of this variation. But the reader might draw pleasure from seeing that slight modifications of the definition of usual cohomology lead to new, interesting and dramatically different invariants.

This chapter is organized as follows:

- In Section 209.1 we will give the definition of cohomology with compact support and we will discuss basic properties.
- In Section 209.2 we will relate cohomology with compact support to relative cohomology.
- In Section 209.3 we will recall direct colimits of direct systems.
- In Section 209.4 we will express cohomology with compact support as a direct colimit.
- In Section 209.5 we will discuss the covariant functoriality of cohomology with compact support under inclusions of open subsets.
- In Section 209.6 we will study the cup product on cohomology with compact support.
- In Appendix 209.7 we will give a quick introduction to bounded cohomology.
- In Section 209.8 we will provide exercises for Chapter 209.

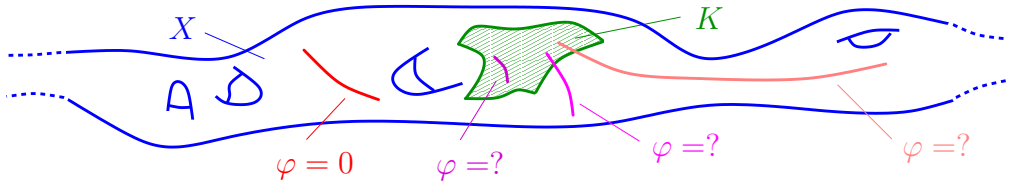
In the next **??** we will introduce a variation on ordinary homology, namely locally finite homology.

209.1. **The definition of cohomology with compact support and basic properties.** In this section we introduce "cohomology with compact support". We start out with the following definition:

Definition 209.1. Let X be a topological space and let G be an abelian group.

(1) Given a subset $A \subseteq X$ we say that a cochain $\varphi \in \mathrm{C}^n(X;G) = \mathrm{Hom}(\mathrm{C}_n(X), G)$ vanishes on A if $\varphi(c) = 0$ for all singular n-simplices c which take values in A, i.e. if φ vanishes on the subgroup $\mathrm{C}_n(A) \subseteq \mathrm{C}_n(X)$.

(2) Given a subset $B \subseteq X$ we say that a cochain vanishes outside B if the cochain vanishes on $X \setminus B$.



φ is a cohomology class that vanishes outside of the compact subset K

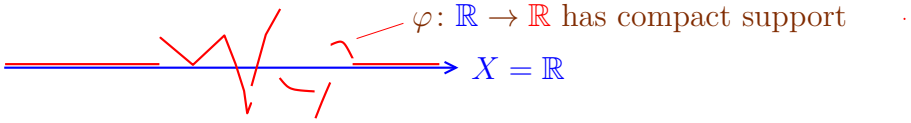
Definition 209.1.

(3) We say that a cochain $\varphi \in \mathrm{C}^n(X;G) = \mathrm{Hom}(\mathrm{C}_n(X), G)$ has compact support if there exists a compact set $K \subseteq X$ such that φ vanishes outside of K.

(4) We denote by $\mathrm{C}^n_{\mathrm{c}}(X;G)$ the set of all n-cochains with compact support.

Example. Let X be a topological space and let G be an abelian group. On page 3812 we made the identification

$$\mathrm{C}^0(X;G) \ =_{\mathrm{i}} \ \{\text{maps from } X \text{ to } G\}.$$



With this identification we can now write

$$\mathrm{C}^0_{\mathrm{c}}(X;G) \ = \ \{\text{maps } f\colon X \to G \,|\, \text{there exists a compact subset } K \subseteq X \text{ with } f|_{X\setminus K} \equiv 0\}. \ \square$$

The first question that arises is whether cochains with compact support form a cochain complex. The following lemma shows in particular that this is indeed the case.

Lemma 209.2. (Cochain-with-Compact Support Lemma) Let X be a topological space and let G be an abelian group. The set of cochains that have compact support form a subcomplex $(\mathrm{C}^*_{\mathrm{c}}(X;G);\delta_*)$ of the cochain complex $(\mathrm{C}^*(X;G);\delta_*)$.

Proof. The lemma is an immediate consequence of the following claim and the observation, shown in the Compact–Union-Intersection Lemma 3.41 (1), that the union of two compact subspaces is again compact.

Claim. Let $A, B \subseteq X$ be two subsets.

(a) If $\varphi \in \mathrm{C}^n(X;G)$ vanishes outside A and if $\psi \in \mathrm{C}^n(X;G)$ vanishes outside B, then for any $r, s \in \mathbb{Z}$ the linear combination $r \cdot \varphi + s \cdot \psi \in \mathrm{C}^n(X;G)$ vanishes outside the union $A \cup B$.

(b) If $\varphi \in \mathrm{C}^n(X;G)$ vanishes outside A, then $\delta_n(\varphi) \in \mathrm{C}^{n+1}(X;G)$ also vanishes outside A.

Proof. Statement (1) is obvious. Now we turn to the proof of Statement (2). Suppose we are given a cochain $\varphi \in \mathrm{C}^n(X;G)$ that vanishes outside A. We need to show that $\delta_n(\varphi) \in \mathrm{C}^{n+1}(X;G)$ also vanishes outside A. So let c be a singular $(n+1)$-simplex in $X \setminus A$. Then

$$(\delta_n(\varphi))(c) \underset{\uparrow}{=} \varphi(\partial_{n+1}(c)) \underset{\uparrow}{=} \sum_{j=0}^{n+1} (-1)^n \cdot \varphi(c \circ i_j^{n+1}) \underset{\uparrow}{=} 0.$$

Definition 198.1 of δ_n — Definition 143.8 of ∂_{n+1} — since the image of each $c \circ i_j^{n+1}$ also lies in $X \setminus A$ and since φ vanishes outside A ■

The Cochain-with-Compact Support Lemma 209.2 allows us to introduce the following definition:

Definition 209.3. Let X be a topological space and let G be an abelian group. We define the n-th cohomology group with compact support $\mathrm{H}_{\mathrm{c}}^n(X;G)$ to be the cohomology of the cochain complex given by cochains with compact support, i.e. we define

$$\mathrm{H}_{\mathrm{c}}^n(X;G) \coloneqq \mathrm{H}^n(\mathrm{C}_{\mathrm{c}}^*(X;G)) \coloneqq \frac{\ker(\delta_n \colon \mathrm{C}_{\mathrm{c}}^n(X;G) \to \mathrm{C}_{\mathrm{c}}^{n+1}(X;G))}{\mathrm{im}(\delta_{n-1} \colon \mathrm{C}_{\mathrm{c}}^{n-1}(X;G) \to \mathrm{C}_{\mathrm{c}}^n(X;G))}.$$

Remark. If X is a compact space, then every cochain has compact support, in particular the cohomology groups with compact support are just the usual cohomology groups. □

Example. Let X be a non-empty path-connected topological space. Then

$$\mathrm{H}_{\mathrm{c}}^0(X;G) \underset{\uparrow}{=} \left\{ X \xrightarrow{f} G \;\middle|\; \begin{array}{l} f \text{ is constant and there exists a} \\ \text{compact subset } K \text{ with } f|_{X \setminus K} \equiv 0 \end{array} \right\} \cong \begin{cases} 0, & \text{if } X \text{ is non-compact,} \\ G, & \text{if } X \text{ is compact.} \end{cases}$$

see the proof of the H^0-Proposition 198.9 and the description of $\mathrm{C}_{\mathrm{c}}^0(X;G)$ on page 4006.

This simple example already indicates that cohomology with compact support can, for non-compact topological spaces, be quite different from usual cohomology. □

Example. We consider the topological space $X = \mathbb{R}$. As on page 3815 we consider the singular 0-cochain

$$\varphi_0 \colon \mathrm{C}_0(X) \to \mathbb{Z}$$

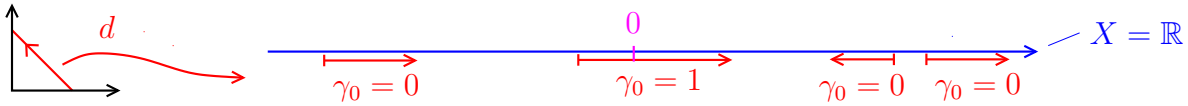
that is uniquely determined by the property that for any singular 0-simplex $x \in X$ we have

$$x \mapsto \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

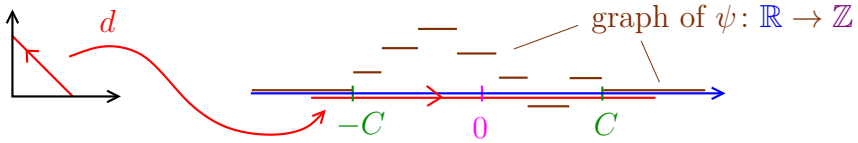
It is clear that this 0-cochain does *not have* compact support. As on page 3815 we also consider the coboundary $\gamma_0 \coloneqq \delta_0(\varphi_0) \in \mathrm{C}^1(\mathbb{R};\mathbb{Z})$. Since γ_0 is a coboundary, it is in particular a cocycle. By definition $\gamma_0 = \delta_0(\varphi_0)$ is uniquely determined by the property that for any singular 1-simplex $d \colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta_0(\varphi_0))(d) = \varphi_0(\partial_1(d)) = \varphi_0(d(0,1)) - \varphi_0(d(1,0)).$$

One can easily verify that $\gamma_0 = \delta_0(\varphi_0)$ vanishes outside of the compact subset $K \coloneqq \{0\}$.[2227] Thus we see that $\gamma_0 = \delta_0(\varphi_0)$ defines an element in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R})$.



Now we claim that $[\gamma_0] \in \mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ is non-zero. This means that we have to show that there is no $\psi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ with $\delta_0(\psi) = \gamma_0$. So let $\psi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. Following the discussion on page 4006 we view ψ as a function $\mathbb{R} \to \mathbb{Z}$ which is zero outside of a compact subset of $\mathbb{R}$. This implies that there exists a $C \geq 0$ such that $\psi(x) = 0$ for all $x \notin [-C, C]$. Now we consider the singular 1-chain $d\colon \Delta^1 \to \mathbb{R}$ that is given by $d(1-t,t) = -2C + 4C \cdot t$ with $t \in [0,1]$.



Then we have

$$\gamma_0(d) \;=\; (\delta_0(\varphi_0))(d) \;=\; \varphi_0(d(0,1)) - \varphi_0(d(1,0)) \;=\; \varphi_0(-2C) - \varphi_0(2C) \;=\; 0 - 1 \;=\; -1$$

but on the other hand we have

$$(\delta_0(\psi))(d) \;=\; \psi(d(0,1)) - \psi(d(1,0)) \;=\; \psi(-2C) - \psi(2C) \underset{\uparrow}{=} 0 - 0 \;=\; 0.$$

since ψ is zero outside $[-C, C]$

This concludes the proof that $\gamma_0 = \delta_0(\varphi_0)$ is non-trivial in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. □

We conclude this section with the discussion of induced maps on cohomology groups with compact support. Let $f\colon X \to Y$ be a continuous map between topological spaces and let G be an abelian group. We saw that f induces a homomorphism $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$. We make the following simple observation:

$$\begin{aligned}
&\varphi \in \mathrm{C}^k(Y;G) \text{ vanishes outside of a subset } B \subseteq Y \\
\Longrightarrow\;& \varphi \in \mathrm{C}^k(Y;G) \text{ vanishes on } Y \setminus B \\
\Longrightarrow\;& f^*(\varphi) \in \mathrm{C}^k(X;G) \text{ vanishes on } f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \\
\Longrightarrow\;& f^*(\varphi) \in \mathrm{C}^k(X;G) \text{ vanishes outside of } f^{-1}(B) \subseteq X.
\end{aligned}$$

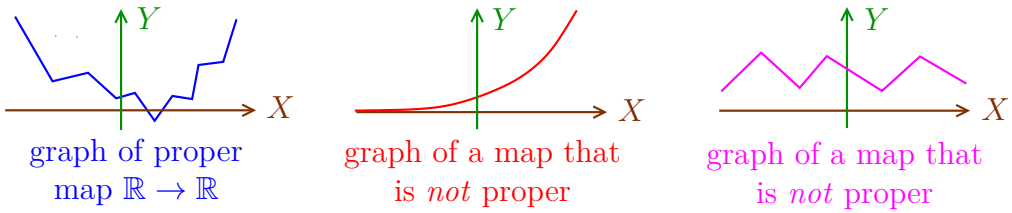
Since the preimage of a compact subspace is in general not compact we see that we have a problem with a "naive" approach to functoriality of cohomology with compact support.

This discussions leads us to the following definition, which we encountered already in Definition 4.63.

Definition 209.4. A continuous map $f\colon X \to Y$ between topological spaces is called proper if the preimage of every compact subset of Y is a compact subset of X.

We can now formulate the following slightly subtle analog of the H^*-Functor Lemma 198.3:

[2227] Indeed, if $d\colon \Delta^1 \to \mathbb{R}\setminus\{0\}$ is a singular 1-simplex, then it follows from the fact that Δ^1 is homeomorphic to the interval $[0,1]$ and the intermediate value theorem that either $d(0,1)$ and $d(1,0)$ both lie to the left of 0 or both lie to the right of 0. In both cases we obtain from the above formula that $\gamma_0(d) = (\delta_0(\varphi_0))(d) = 0$.



Lemma 209.5. (H_c^n-Functor Lemma) Let G be an abelian group.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a *proper* map, then the cochain map $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$ restricts to a cochain map

$$f^*\colon \mathrm{C}_c^*(Y;G) \ \to \ \mathrm{C}_c^*(X;G).$$

In particular f^* induces a well-defined homomorphism

$$\mathrm{H}_c^*(Y;G) \ \to \ \mathrm{H}_c^*(X;G).$$

(2) For any $n \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}_c^n(X;G)$ and $f \mapsto f^*$ define a contravariant functor

$$\text{category } \mathcal{TopPropMap} \text{ where objects are topological spaces and morphisms are proper maps}^{2228} \ \to \ \text{category } \mathcal{AbGroup} \text{ of abelian groups.}$$

In particular homeomorphic topological spaces have isomorphic cohomology groups with compact support.

Proof. Let G be an abelian group.

(1) Let $f\colon X \to Y$ be a proper map. Let $\varphi\colon \mathrm{C}_n(Y) \to G$ be a cocycle with compact support. We pick a compact subset K such that φ vanishes outside of K. It follows from the discussion above that $f^*(\varphi)$ vanishes outside of $f^{-1}(K)$. Since f is proper we see that $f^{-1}(K)$ is compact. This shows that $f^*\colon \mathrm{C}_c^*(Y;G) \to \mathrm{C}^*(X;G)$ takes values in $\mathrm{C}_c^*(X;G)$.

(2) Statement (2) follows easily from (1). ∎

209.2. **Cohomology with compact support and relative cohomology.** In this section we want to relate cohomology with compact support to relative cohomology groups. To do so it is worth introducing the following notation:

Notation 209.6. Let X be a topological space and let G be an abelian group. Given a subspace $A \subseteq X$ we denote by

$$я\colon \mathrm{C}^n(X,A;G) \ \to \ \mathrm{C}^n(X;G)$$

the natural chain map that is induced by the natural projection $\mathrm{C}_n(X) \to \mathrm{C}_n(X,A)$.

Lemma 209.7. (Compact Support–Relative Cochain Lemma) Let X be a topological space and let G be an abelian group.

(1) (a) A cochain $\varphi \in \mathrm{C}^n(X;G)$ vanishes on a subset $A \subseteq X$ if and only if φ lies in the image of the natural map $я\colon \mathrm{C}^n(X,A;G) \to \mathrm{C}^n(X;G)$.

[2228] Clearly the composition of proper maps is again a proper map, thus we see that we do indeed have a category in our hands.

(b) A cochain $\varphi \in \mathrm{C}^n(X;G)$ has compact support if and only if there exists a compact subset $K \subseteq X$ such that φ is contained in the image of the natural homomorphism $\text{я}\colon \mathrm{C}^n(X, X \setminus K; G) \to \mathrm{C}^n(X;G)$.

(2) (a) For any compact subset K of X the natural map $\text{я}\colon \mathrm{C}^n(X, X \setminus K; G) \to \mathrm{C}^n(X;G)$ restricts to a natural cochain map $\text{я}\colon \mathrm{C}^n(X, X \setminus K; G) \to \mathrm{C}^n_{\mathrm{c}}(X;G)$.

(b) Given any $\varphi \in \mathrm{H}^n_{\mathrm{c}}(X;G)$ there exists a compact subset K of X such that φ lies in the image of the natural map $\text{я}\colon \mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X;G)$.

Proof. Let X be a topological space and let G be an abelian group.

(1) (a) We consider the following little diagram:

$$\begin{array}{ccccc} \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X, A) \\ & \searrow^{=0} & \downarrow & \swarrow & \\ & & G. & & \end{array}$$

The desired conclusion is a consequence of the following elementary observation: It follows immediately from the Quotient Group Lemma 2.4 and the definition $\mathrm{C}_n(X, A) \coloneqq \mathrm{C}_n(X)/\mathrm{C}_n(A)$ that a homomorphism $\varphi\colon \mathrm{C}_n(X) \to G$ vanishes on $\mathrm{C}_n(A)$ if and only if φ descends to a homomorphism $\mathrm{C}_n(X, A) \to G$.

(b) This statement follows immediately from (a) and the definition of "compact support".

(2) (a) This statement is an immediate consequence of the definitions.

(b) Let $[\varphi] \in \mathrm{H}^n_{\mathrm{c}}(X;G)$. It follows from (1b) that there exists a compact subset K and a $\psi \in \mathrm{C}^n(X, X \setminus K; G)$ such that $\text{я}(\psi) = \varphi$. We consider the following little diagram:

$$\begin{array}{ccc} \mathrm{C}^{n+1}(X, X \setminus K; G) & \xhookrightarrow{\text{я}} & \mathrm{C}^{n+1}_{\mathrm{c}}(X;G) \\ \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} \\ \mathrm{C}^n(X, X \setminus K; G) & \xhookrightarrow{\text{я}} & \mathrm{C}^n_{\mathrm{c}}(X;G). \\ \psi & & \varphi \end{array}$$

Since the horizontal maps are monomorphisms we see that ψ is actually a cocycle in $\mathrm{C}^*(X, X \setminus K; G)$. This discussion implies that $[\varphi]$ lies in the image of the map $\mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X;G)$. ∎

The Compact Support–Relative Cochain Lemma 209.7 suggests that cohomology $\mathrm{H}^n_{\mathrm{c}}(X;G)$ with compact support is related to the relative cohomology groups $\mathrm{H}^n(X, X \setminus K; G)$ where $K \subseteq X$ is compact. In fact, it seems reasonable that for "larger and larger" compact $K \subseteq X$ one might be able to "approximate" cohomology with compact support. The notion of a direct colimit, which we discuss in the next section, will help us in turning this vague idea into proper mathematics.

209.3. **Interlude: Direct colimits of direct systems.** In the next section we will give a reinterpretation of cohomology with compact support as a direct colimit of the relative cohomology groups $\mathrm{H}^n(X, X \setminus K; G)$ where K runs over all compact subsets of X.

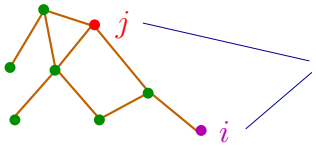
In this section we recall the basic definition of direct systems and direct colimits from Chapter 104 and we generalize a few concepts. First we recall Definition 104.1:

Definition 209.8. A directed set is a set I together with a relation "$\leq$" on I which satisfies the following three conditions:

(1) The relation is reflexive, i.e. for every $i \in I$ we have $i \leq i$.
(2) The relation is transitive, i.e. for every $i, j, k \in I$ with $i \leq j$ and $j \leq k$ we have $i \leq k$.
(3) For every $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.

Examples.

(a) Evidently the set $\mathbb{N}$ together with the usual "$\leq$" relation forms a directed set $(\mathbb{N}, \leq)$.
(b) In the figure we show a directed set $(I, \leq)$:



we have $i \leq j$ if we can go from i to j via an upward pointing path

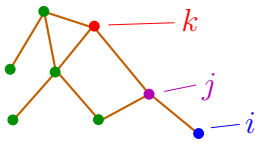
(c) Let X be a topological space. We denote by $\mathcal{K}(X)$ the set of all compact subsets of X. The set $\mathcal{K}(X)$, together with the relation "$\subseteq$" defined by inclusion, is a directed set since the Compact–Union-Intersection Lemma 3.41 tells us that the union of two compact subsets is again compact.
(d) Given a metric space (X, d) the set of all closed balls in X, together with the relation "$\subseteq$" defined by inclusion, is a directed set since the union of two closed balls is clearly contained in a closed ball. □

Next we recall Definition 104.2:

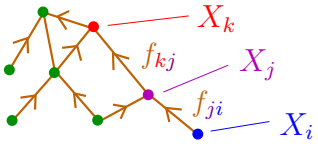
Definition 209.9. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system over the directed set $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,
(2) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

directed set:



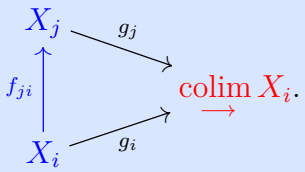
direct system:



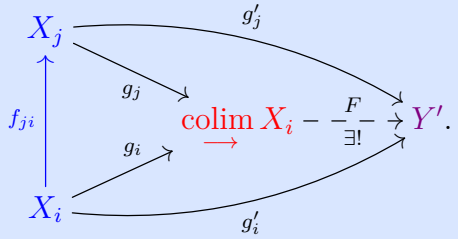
Next we recall the definition of a direct colimit from Definition 104.4.

Definition 209.10. Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ in the category $\mathcal{C}$. A direct colimit of the direct system is a colimit, in the sense of Definition 6.21, of the direct system viewed as a diagram. In other words, a direct colimit is an object $\operatorname{colim} X_i$ in $\mathcal{C}$ together with a family of morphisms $g_i \colon X_i \to \operatorname{colim} X_i$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ji} = g_i \colon X_i \to \operatorname{colim} X_i$, i.e. for all $i \leq j$ the following diagram commutes:



(2) If we are given another object Y' and a family of morphisms $\{g'_i \colon X_i \to Y'\}_{i \in I}$ that satisfy (1), then there exists a unique morphism $F \colon \operatorname{colim} X_i \to Y'$ such that for all $i \in I$ we have $g'_i = F \circ g_i \colon X_i \to Y'$. In particular for all $i \leq j$ the following diagram commutes:



By the Colimit Uniqueness Lemma 6.22 we know that if a direct colimit exists, then it is unique in the usual sense.

We recall the following existence result:

Proposition 104.7. (Algebraic–Direct Colimit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ be a direct system in the category *AbGroup* of abelian groups. A direct colimit is given by

$$\varinjlim X_i \;=\; \Big(\bigoplus_{i \in I} X_i\Big) / \langle f_{ji}(x) - x \mid \; i \leq j \text{ and } x \in X_i \rangle.$$

We also recall the following lemma, which follows easily from the explicit description of the colimit in the Algebraic–Direct Colimit Existence Proposition 104.7.

Lemma 104.8. (Elements-in-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ be a direct system over I in the category *AbGroup* of abelian groups. The following two statements hold:

(1) Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.
(2) Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{lj}(a_j) = f_{lk}(a_k) \in X_l$.

The following definition is a generalization of Definition 104.9.

Definition 209.11. Let $(I, \leq)$ and $(J, \leq)$ be directed sets.

(1) A morphism between the directed sets $(I, \leq)$ and $(J, \leq)$ is a map $\varphi \colon I \to J$ such that for every $i_1 \leq i_2$ in I we have $\varphi(i_1) \leq \varphi(i_2)$.
(2) Let $(\{K_i\}_{i \in I}, \{\chi_{i_2,i_1}\}_{i_1 \leq i_2})$ and $(\{L_j\}_{j \in J}, \{\lambda_{j_2,j_1}\}_{j_1 \leq j_2})$ be direct systems in the category *AbGroup* of abelian groups. A morphism between the direct systems, modelled on a morphism $\varphi \colon I \to J$, is a family $\{f_i \colon K_i \to L_{\varphi(i)}\}_{i \in I}$ of homomorphisms that has the

property that for any $i_1 \leq i_2$ the following diagram commutes:

$$\begin{array}{ccc} K_{i_2} & \xrightarrow{f_{i_2}} & L_{\varphi(i_2)} \\ {\scriptstyle \chi_{i_2,i_1}}\uparrow & & \uparrow{\scriptstyle \lambda_{\varphi(i_2),\varphi(i_1)}} \\ K_{i_1} & \xrightarrow{f_{i_1}} & L_{\varphi(i_1)}. \end{array}$$

(3) Suppose that we are in the setting of (2). The homomorphisms $K_i \xrightarrow{f_i} L_{\varphi(i)} \to \varinjlim L_j$ induce, by the universal property of a direct colimit, a homomorphism

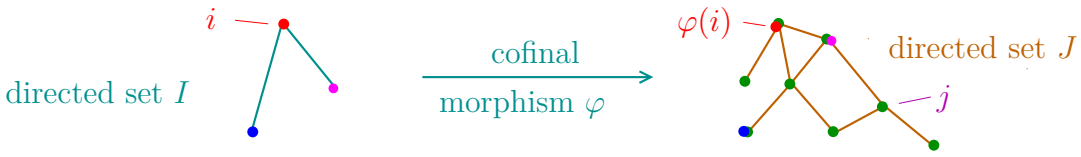
$$\varinjlim_{i \in I} K_i \to \varinjlim_{j \in J} L_j$$

which is uniquely determined by the property that for any $i \in I$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_{\varphi(i)} \\ \downarrow & & \downarrow \\ \varinjlim_{i \in I} K_i & \longrightarrow & \varinjlim_{j \in J} L_j. \end{array}$$

Next we want to give a criterion for an induced map of direct colimits to be an isomorphism. To formulate the lemma we need the following definition:

Definition 209.12. Let $(I, \leq)$ and $(J, \leq)$ be directed sets. A morphism $\varphi\colon I \to J$ is called cofinal if given any $j \in J$ there exists an $i \in I$ with $\varphi(i) \geq j$.



Lemma 209.13. (Cofinal-Isomorphism Lemma) Let $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i \in I}$ be a morphism between two direct systems $(\{K_i\}_{i \in I}, \{\chi_{i_2,i_1}\}_{i_1 \leq i_2})$ and $(\{L_j\}_{j \in J}, \{\lambda_{j_2 j_1}\}_{j_1 \leq j_2})$ in the category $\mathcal{AbGroup}$ of abelian groups. We suppose that the following two conditions are satisfied:

(1) The morphism $\varphi\colon I \to J$ is cofinal.
(2) For each $i \in I$ the homomorphism $f_i\colon K_i \to L_{\varphi(i)}$ is an isomorphism.

Then the induced homomorphism

$$\Phi\colon \varinjlim_{i \in I} K_i \to \varinjlim_{j \in J} L_j$$

is an isomorphism.

Proof. Our goal is to construct an inverse $\Psi\colon \varinjlim L_j \to \varinjlim K_i$ to the given map. First let $j \in J$. Since $\varphi\colon I \to J$ is cofinal we can pick $i \in I$ with $\varphi(i) \geq j$. We consider the

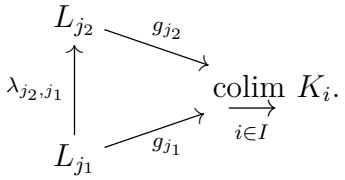
homomorphism g_j that is given by

$$L_j \xrightarrow{\lambda_{\varphi(i),j}} L_{\varphi(i)} \xrightarrow[\cong]{(f_i)^{-1}} K_i \to \varinjlim_{i\in I} K_i.$$

It is easy to verify that this homomorphism does *not* depend on the choice of i.



Using this observation it is elementary to check that for any $j_1 \le j_2$ the following diagram commutes:



By the universal property of the direct colimit this implies that we get an induced homomorphism

$$\varinjlim_{j\in J} L_j \to \varinjlim_{i\in I} K_i.$$

Using the fact that direct colimits are unique up to a unique isomorphism one can now easily show that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity. We leave it to the reader to fill in the details. ■

Example. Let $(J, \le)$ be a directed set and let $(\{X_j\}_{j\in J}, \{f_{kj} \colon X_j \to X_k\}_{j\le k})$ be a direct system over J in the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups. Let I be a subset of J such that $(I, \le)$ is again a directed set. The inclusion $I \to J$ is obviously a morphism of directed sets. By the above definition there exists a corresponding natural homomorphism

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i.$$

If given any $i \in I$ there exists a $j \in J$ with $i \le j$, then the inclusion $I \to J$ is cofinal and it follows from the Cofinal-Isomorphism Lemma 209.13 that the above homomorphism is an isomorphism. □

209.4. **Cohomology with compact support and direct colimits.** After this refresher on direct colimits we are now able to give the promised reinterpretation of cohomology with compact support.

Definition 209.14. Let X be a topological space and let G be an abelian group.

(1) We denote by $\mathcal{K}(X)$ the set of all compact subsets of X. As we discussed above, the set $\mathcal{K}(X)$ together with the relation "$\subseteq$" defined by inclusion is a directed set.

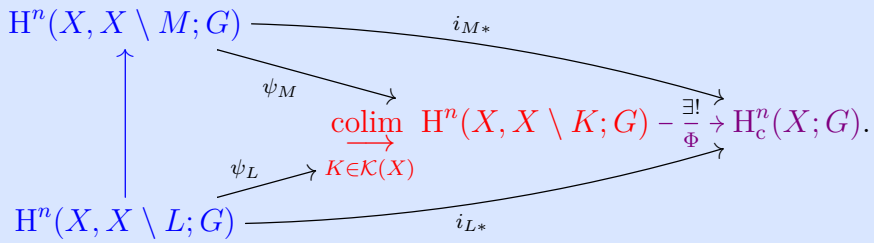
(2) Given any two compact subsets $K \subseteq L$ of X and given any $i \in \mathbb{N}_0$ we get an induced homomorphism[2229]

$$\mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n(X, X \setminus L; G).$$

Since cohomology groups are contravariantly functorial we see that these cohomology groups form a *direct* system over the directed set $(\mathcal{K}(X), \subseteq)$. It follows from the Algebraic–Direct Colimit Existence Proposition 104.7 that we can form the direct colimit

$$\varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K; G).$$

(3) Given any compact subset $K \subseteq X$ we have by the Compact Support–Relative Cochain Lemma 209.7 (2a) a natural homomorphism $i_{K*} \colon \mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$. Given compact subsets $L \subseteq M$ these homomorphisms give rise to the following diagram:



It follows immediately from the functoriality of cochain complexes that the outer triangle commutes for all compact subsets $L \subseteq M \subseteq X$. Thus we get the induced dotted arrow by the universal property of the direct colimit.

The following proposition says that the above dotted arrow is in fact an isomorphism. This gives in particular an often convenient reformulation of the concept of cohomology with compact support.

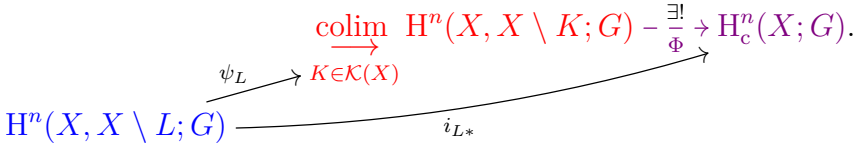
Proposition 209.15. ($\mathrm{H}^n_{\mathrm{c}}$-via-Direct Colimits Proposition) If X is a topological space and G is an abelian group, then the above map

$$\Phi \colon \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$$

defines a natural isomorphism.[2230]

Proof. We need to show that the map Φ in the proposition is a bijection.

We first show that Φ is surjective. So let $\varphi \in \mathrm{H}^n_{\mathrm{c}}(X; G)$. By the Compact Support–Relative Cochain Lemma 209.7 (2b) there exists a compact subset $L \subseteq X$ such that φ lies in the image of $\mathrm{H}^n(X, X \setminus L; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$. It follows immediately from the commutative diagram

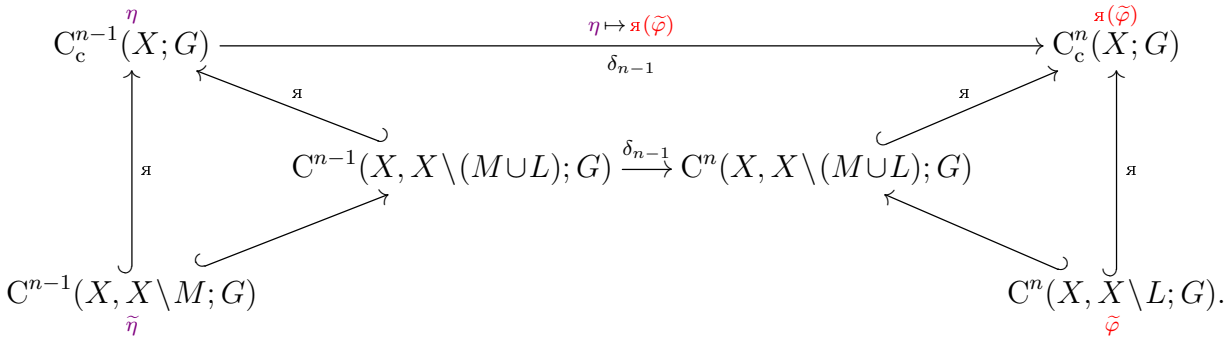


[2229] The most confusing part about cohomology is to figure out which way the maps go. In this case $K \subseteq L$ implies that $X \setminus L \subseteq X \setminus K$ which means that we have a map of pairs $(X, X \setminus L) \to (X, X \setminus K)$ given by inclusion, which by the discussion on page 3810 implies that we do indeed get an induced homomorphism $\mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n(X, X \setminus L; G)$.

[2230] In Exercise 209.3 we will see that the left-hand side defines a contravariant functor from the category *TopPropMap* where objects are topological spaces and morphisms are proper maps to the category *AbGroup* of abelian groups. Thus it makes sense to say that the maps are "natural".

that φ also lies in the image of $\Phi\colon \operatorname{colim} \mathrm{H}^n(X, X\setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$.

Next we will show that the map is in fact Φ is injective. Suppose we are given an element c in the kernel of $\Phi\colon \operatorname{colim} \mathrm{H}^n(X, X\setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$. By the Elements-in-Direct Colimit Lemma 104.8 (1) we know that the element c lies in the image of some element in $\mathrm{H}^n(X, X\setminus L; G)$. We pick a representative $\widetilde{\varphi} \in \mathrm{C}^n(X, X\setminus L; G)$. Since $\text{я}(\widetilde{\varphi})$ represents the zero class in $\mathrm{H}^n_{\mathrm{c}}(X; G)$ there exists a compactly supported $(n-1)$-cocycle η with $\delta_{n-1}(\eta) = \text{я}(\widetilde{\varphi})$. We pick a compact subset M of X such that η vanishes on $X\setminus M$. Next note that it follows from the Compact Support–Relative Cochain Lemma 209.7 (1a) that there exists an $\widetilde{\eta} \in \mathrm{C}^{n-1}(X, X\setminus M; G)$ with $\text{я}(\widetilde{\eta}) = \eta$. We consider the following diagram:



Note that the diagram commutes by the functoriality of cochain complexes.

Next we denote by $\varphi \in \mathrm{C}^n(X, X\setminus(M\cup L); G)$ the image of $\widetilde{\varphi}$. It follows immediately from the diagram that in $\mathrm{C}^n(X, X\setminus(M\cup L); G)$ we have $\delta_{n-1}(\eta) = \varphi$, which shows that $[\varphi]$ represents the zero class in $\mathrm{H}^n(X, X\setminus(M\cup L); G)$. It follows from the Elements-in-Direct Colimit Lemma 104.8 (2) or alternatively from the commutative diagram from Definition 209.14 that $\psi_L([\varphi]) = c$ is zero. ∎

Example. Now we turn to a key example, namely for each $k \in \mathbb{N}_0$ we consider the cohomology with compact support $\mathrm{H}^k_{\mathrm{c}}(\mathbb{R}^n; G)$ of $\mathbb{R}^n$. We see that

$$
\begin{array}{ccccc}
 & & \text{this follows from the Cofinal-Isomorphism Lemma 209.13 since every compact subspace } K \subseteq \mathbb{R}^n \text{ is contained in a disk } \overline{B}^n_s(0) \text{ with } s \in \mathbb{N}_0 & & \\
 & & \downarrow & & \vdots \\
 & & & & \uparrow \\
\text{by the } \mathrm{H}^n_{\mathrm{c}}\text{-via-Direct Limits Proposition 209.15} & & & & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n\setminus \overline{B}^n_2(0); G) \\
\downarrow & & & & \uparrow \\
\mathrm{H}^k_{\mathrm{c}}(\mathbb{R}^n; G) & \xleftarrow{\cong} & \operatorname*{colim}_{K\in\mathcal{K}(\mathbb{R}^n)} \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n\setminus K; G) & \xleftarrow{\cong} & \operatorname*{colim}_{s\in\mathbb{N}_0} \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n\setminus \overline{B}^n_1(0); G) \\
 & & & & \uparrow \\
 & & & & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n\setminus \overline{B}^n_0(0); G) \\
 & & \downarrow\cong & & \\
 & & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n\setminus \overline{B}^n_0(0); G) & \cong & \begin{cases} G, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\
 & & \uparrow & & \uparrow \\
\text{by the Homotopy Equivalence-Relative-H}_*\text{-Corollary 146.8 all vertical maps in the direct system are isomorphisms, hence we can apply the Direct Colimit–Stabilization Lemma 104.5} & & & & \text{follows from the Manifold-Local } R\text{-Homology Lemma 191.2}
\end{array}
$$

We draw three interesting conclusions from this example:

(1) This example shows that cohomology with compact support can be used to show that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.

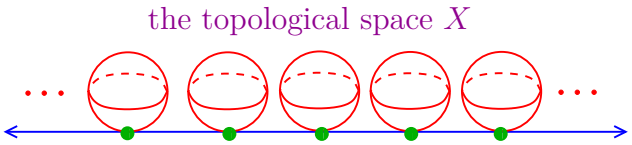
(2) The example also shows that cohomology with compact support, in contrast to basically all other invariants we have studied so far, is not the same for homotopy equivalent topological spaces.
(3) Finally we consider the case $n = 1$ in more detail. We consider again the singular 1-cochain $\gamma_0 := \delta_0(\varphi_0)$ from page 3815. The argument from page 3815 shows that γ_0 defines an element in each $\mathrm{H}^1(\mathbb{R}, \mathbb{R}\backslash[-s,s]; \mathbb{Z})$. Furthermore the discussion on page 4008 shows that γ_0 also defines an element in $\mathrm{H}^1_c(\mathbb{R}; \mathbb{Z})$. Furthermore the argument from page 3821 shows that for each $s \in N$ the class $[\gamma_0] \in \mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-s,s]; \mathbb{Z})$ is a generator. The above argument now implies that $[\gamma_0] \in \mathrm{H}^1_c(\mathbb{R}; \mathbb{Z})$ is also a generator. □

In the previous examples cohomology with compact support turns out to be "bigger" than ordinary cohomology. But there are also examples "in the other direction":

Example. We consider again the topological space

$$X := (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subseteq \mathbb{R}$ with $((0,0,-1), i) \in S^2 \times \mathbb{Z}$.



In Exercise 209.9 we will see that

$$\mathrm{H}^2_c(X;\mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})},$$

in particular $\mathrm{H}^2_c(X)$ is countable, whereas on page 4455 we saw that

$$\mathrm{H}^2(X;\mathbb{Z}) \cong \mathbb{Z}^{\mathbb{N}}$$

is uncountable. □

209.5. **Cohomology with compact support and inclusions.** In this section we will make the rather surprising observation that under fairly mild hypotheses an inclusion $U \to V$ of topological spaces induces for every $n \in \mathbb{N}_0$ a homomorphism $\mathrm{H}^n_c(U;G) \to \mathrm{H}^n_c(V;G)$ which "goes the same way".

Definition 209.16. Let U be an open subset of a Hausdorff space V and let G be an abelian group. We denote by $i_{VU*} = i_* \colon \mathrm{H}^n_c(U;G) \to \mathrm{H}^n_c(V;G)$ the composition of the following homomorphisms (for space reasons we ignore the coefficients)

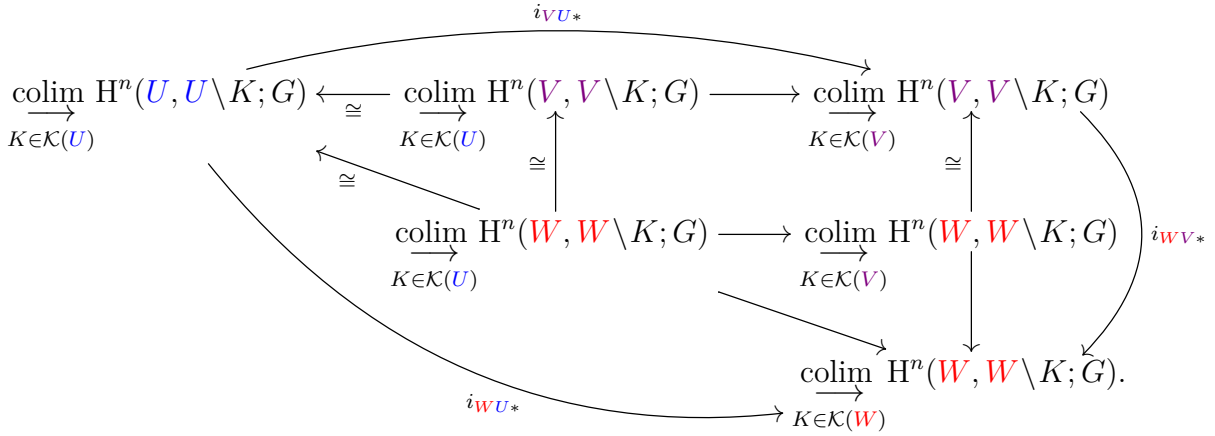
$$\mathrm{H}^n_c(U) \overset{\text{by the } \mathrm{H}^n_c\text{-via-Direct Colimits Proposition 209.15}}{=} \operatorname*{colim}_{K \in \mathcal{K}(U)} \mathrm{H}^n(U, U\setminus K) \overset{\text{it follows from the Compact-Closed Lemma 3.39 (2) together with the Excision-in-H}^*\text{-Theorem 200.1 (1b) that the map } \mathrm{H}^n(V,V\setminus K) \to \mathrm{H}^n(U,U\setminus K) \text{ is an isomorphism for every compact subset } K \text{ of } U}{\leftarrow} \operatorname*{colim}_{K \in \mathcal{K}(U)} \mathrm{H}^n(V, V\setminus K) \overset{\text{given by the functoriality of direct colimits, see Definition 104.9, and the fact that by the Compact Image Lemma 4.15 we have } \mathcal{K}(U) \subseteq \mathcal{K}(V)}{\to} \operatorname*{colim}_{K \in \mathcal{K}(V)} \mathrm{H}^n(V, V\setminus K) \overset{\text{by the } \mathrm{H}^n_c\text{-via-Direct Colimits Proposition 209.15}}{=} \mathrm{H}^n_c(V).$$

Note that, confusingly, the induced map on cohomology with compact support "goes the same direction" as the inclusion map $i\colon U \to V$.

Lemma 209.17. (Covariant Cohomology Lemma) Let G be an abelian group. Given a topological space X that is Hausdorff the above construction defines for each $n \in \mathbb{N}_0$ a *covariant* functor

$$\begin{array}{c}\text{category where the objects are open subsets}\\ \text{of } X \text{ and where the morphisms are inclusions}\end{array} \to \begin{array}{c}\text{category of}\\ \text{abelian groups.}\end{array}$$

Proof. Let X be a topological space that is Hausdorff and let $U \subseteq V \subseteq W$ be open subsets of X. Let $n \in \mathbb{N}_0$. We have to show that $i_{WU*} = i_{WV*} \circ i_{VU*}\colon \mathrm{H}^n_{\mathrm{c}}(U;G) \to \mathrm{H}^n_{\mathrm{c}}(W;G)$. We consider the following diagram:



It follows pretty easily from the universal property of direct colimits that the two smaller triangles and the rectangle commute. Hence the diagram commutes and we get the desired statement. ∎

We conclude this section with the following proposition.

Proposition 209.18. (Exhaustion-$\mathrm{H}^*_{\mathrm{c}}$-Proposition) Let X be a Hausdorff space, let G be an abelian group and let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X such that the following holds:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subseteq X_{i+1}$.
(2) Each X_i is open in X.
(3) $X = \bigcup_{i\in\mathbb{N}} X_i$.

Then for any $n \in \mathbb{N}_0$ the above inclusion-induced maps $\mathrm{H}^n_{\mathrm{c}}(X_i;G) \to \mathrm{H}^n_{\mathrm{c}}(X;G)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \underbrace{\mathrm{H}^n_{\mathrm{c}}(X_i;G)}_{\substack{\uparrow\\ \text{by the Covariant Cohomology Lemma 209.17 this is a}\\ \text{direct system with the above inclusion-induced maps}}} \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{c}}(X;G).$$

Proof. Using the $\mathrm{H}^n_{\mathrm{c}}$-via-Direct Colimits Proposition 209.15 we see that we have to show that the map

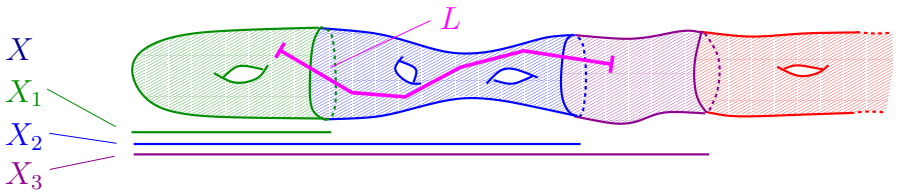
$$\Phi\colon \operatorname*{colim}_{i\in\mathbb{N}} \operatorname*{colim}_{K\in\mathcal{K}(X_i)} \mathrm{H}^n(X_i, X_i\setminus K;G) \xrightarrow{\cong} \operatorname*{colim}_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X\setminus K;G)$$

is an isomorphism. We need to show that Φ is injective and surjective. The key to the proof of these two statements are the following observations:

(a) It follows from (1) and (2), together with the Compact Subspace Covering Lemma 3.38, that every compact subset L of X is already contained in some X_i.
(b) It follows from the Excision-in-H*-Theorem 200.1 (1b) together with the Compact-Closed Lemma 3.39 (2) that for each compact subset $L \subseteq X_i$ we get an inclusion-induced isomorphism $\mathrm{H}^n(X, X \setminus L; G) \to \mathrm{H}^n(X_i, X_i \setminus L; G)$.

Now we prove that Φ is surjective, the proof that Φ is injective is fairly similar and we leave that part of the proof to the reader.

So let c be an element of the limit on the right-hand side. By the Elements-in-Direct Colimit Lemma 104.8 (1) there exists a compact subset L of X and some $\varphi \in \mathrm{H}^n(X, X \setminus L; G)$ such that c is the image of φ under the map $\mathrm{H}^n(X, X \setminus L; G) \to \varinjlim \mathrm{H}^n(X, X \setminus K; G)$.



By (a) there exists an $i \in \mathbb{N}$ with $L \subseteq X_i$. We consider the following commutative diagram:

$$\begin{array}{ccc} \varinjlim_{i\in\mathbb{N}} \varinjlim_{K\in\mathcal{K}(X_i)} \mathrm{H}^n(X_i, X_i \setminus K; G) & \xrightarrow{\Phi} & \varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K; G) \\ \uparrow{\scriptstyle =:\beta} & & \uparrow{\scriptstyle c} \\ \mathrm{H}^n(X_i, X_i \setminus L; G) & \xleftarrow[\cong]{=:\alpha} & \mathrm{H}^n(X, X \setminus L; G) \\ & & \uparrow{\scriptstyle \varphi} \end{array}$$

The bottom horizontal map is an isomorphism by (b). Since the diagram is commutative we now see that $\Phi(\beta(\alpha(\varphi))) = c$. ∎

209.6. **The cup product on cohomology with compact support.** For the insatiable reader we also provide the definition of the cup product on cohomology with compact support.

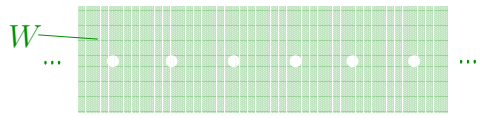
Definition 209.19. Let X be a Hausdorff space, let $\varphi \in \mathrm{H}^k_{\mathrm{c}}(X; \mathbb{Z})$ and let $\psi \in \mathrm{H}^l_{\mathrm{c}}(X; \mathbb{Z})$. Recall that by the Compact Support–Relative Cochain Lemma 209.7 (2b) there exists a compact subset $K \subseteq X$ and a cohomology class $\widetilde{\varphi} \in \mathrm{H}^k(X, X \setminus K; \mathbb{Z})$ such that the image of $\widetilde{\varphi}$ in $\mathrm{H}^k_{\mathrm{c}}(X; \mathbb{Z})$ equals φ. Analogously we find a compact set L and $\widetilde{\psi} \in \mathrm{H}^l(X, X \setminus L; \mathbb{Z})$. We define[2231]

$$\varphi \cup \psi \;\coloneqq\; \text{the image of } \varphi \cup \psi \in \mathrm{H}^{k+l}(X, X \setminus (K \cup L); \mathbb{Z}) \text{ in } \mathrm{H}^{k+l}_{\mathrm{c}}(X; \mathbb{Z}).$$

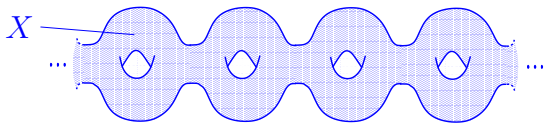
It follows easily from the Cup Product-Naturality Lemma 204.8 that this definition does not depend on any of the choices. We refer to $\varphi \cup \psi$ as the cup product in cohomology with compact support.

[2231]Since X is Hausdorff we know by the Compact-Closed Lemma 3.39 (2) that $K, L \subseteq X$ are closed, which implies that $X \setminus K$ and $X \setminus L$ are open. It thus follows from the Excisive Triad-Proposition 158.20 (1) that

The cup product on cohomology with compact support can be very useful for tackling non-compact topological spaces. For example in Exercise 209.8 we will make of use of the cup product on cohomology with compact support to show that the topological manifold $W = \mathbb{R}^2 \setminus \{(n,0) \,|\, n \in \mathbb{Z}\}$ is not homeomorphic to the surface X of infinite genus that is shown in the figure below.



cup product on $\mathrm{H}^*_{\mathrm{c}}(W;\mathbb{Z})$ is trivial



cup product on $\mathrm{H}^*_{\mathrm{c}}(X;\mathbb{Z})$ is non-trivial

On a more theoretical side we will make of the above Definition 209.19 in the Cycles–Cup Product-Intersection Number Theorem 221.20 when we will learn how to compute cup products on topological manifolds by "intersecting cycles".

209.7. **Appendix: Bounded cohomology.** We had just seen that with a modest modification of the original definition of cohomology we obtain a new invariant which for non-compact spaces behaves quite differently from the usual cohomology groups.

In this appendix we will introduce another variation on cohomology, namely "bounded cohomology". We will not make use of it later on, but it does play an important role in topology. A much more detail discussion of bounded cohomology is for example provided in [**Löh2010**].

Definition 209.20. Let X be a topological space.

(1) Let $C \in \mathbb{R}$. We say that a cochain $\rho \in \mathrm{C}_n(X;\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X),\mathbb{R})$ is C-bounded if for all singular n-simplices $\sigma\colon \Delta^n \to X$ we have $|\rho(\sigma)| \leq C$.

(2) We say that a cochain $\rho \in \mathrm{C}_n(X;\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X),\mathbb{R})$ is bounded if it is C-bounded for some $C \in \mathbb{R}$.

(3) For $i \in \mathbb{N}_0$ we set
$$\mathrm{C}^n_{\mathrm{b}}(X;\mathbb{R}) \;:=\; \{\text{all bounded cochains in } \mathrm{C}^n(X;\mathbb{R})\}.$$

Examples.

(1) Let X be a topological space. The 0-cochain that assigns to each point $x \in X$ the same value $c \in \mathbb{R}$ is bounded.

(2) The singular 1-cochain $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1;\mathbb{R})$ that we defined on page 3813 is *not* bounded. Indeed, given $n \in \mathbb{N}$ we denote by $\mu_n\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}nt)$. It follows easily from the definition of $\theta_{\mathbb{R}}$ (see page 3813) that $\theta_{\mathbb{R}}(\mu) = n$. □

Lemma 209.21. (Bounded H^*–Cochain Complex Lemma) Let X be a topological space. The bounded cochains $\mathrm{C}^*_{\mathrm{b}}(X)$ form a subcomplex of the cochain complex $(\mathrm{C}^*(X), \delta_*)$.

---

the triad $(X, X \setminus K, X \setminus L)$ is excisive, which implies that the relative cup product
$$\cup\colon \mathrm{H}^k(X, X \setminus K;\mathbb{Z}) \times \mathrm{H}^l(X, X \setminus L;\mathbb{Z}) \;\to\; \mathrm{H}^{k+l}(X, X \setminus (K \cup L);\mathbb{Z})$$
is actually defined.

Proof. Suppose that $\rho \in \mathrm{C}^n(X;\mathbb{R})$ is a cochain that is C-bounded. For any singular $(n+1)$-simplex $\varphi\colon \Delta^{n+1} \to X$ we obtain that

$$|(\delta_n(\rho))(\varphi)| \;=\; |\rho(\partial_{n+1}(\varphi))| \;\underset{\text{Definition 143.13 of } \partial_{n+1} \text{ and linearity of } \rho}{=} \left|\sum_{j=0}^{n+1}(-1)^j\cdot\rho(\varphi\circ i_j^{n+1})\right| \;\underset{\text{by the triangle triangle}}{\leq} \sum_{j=0}^{n+1}|\rho(\varphi\circ i_j^{n+1})| \;\underset{\text{since } \rho \text{ is } C\text{-bounded } C\text{-bounded}}{\leq} (n+2)\cdot C.$$

This shows that $\delta_n(\rho) \in \mathrm{C}^{n+1}(X;\mathbb{R})$ is $C\cdot(n+2)$-bounded, in particular it is bounded. This implies that the bounded cochains $\mathrm{C}_\mathrm{b}^*(X)$ form a subcomplex of the cochain complex $(\mathrm{C}^*(X),\delta_*)$. ■

This lemma leads us straight to the following definition:

Definition 209.22. Let X be a topological space and let $n \in \mathbb{N}_0$. The *n-th bounded cohomology group of X* is defined as

$$\mathrm{H}_\mathrm{b}^n(X;\mathbb{R}) \;:=\; \mathrm{H}^n(\mathrm{C}_\mathrm{b}^*(X;\mathbb{R})) \;:=\; \frac{\ker(\delta_n\colon \mathrm{C}_\mathrm{b}^n(X;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^{n+1}(X;\mathbb{R}))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}_\mathrm{b}^{n-1}(X;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^n(X;\mathbb{R}))}.$$

Examples.

(1) The proof of the H^0-Proposition 198.9 can be used, without any changes, to show that 0-th bounded cohomology of a path-connected topological space agrees with 0-th cohomology with real coefficients.[2232]

(2) Every cochain on a topological space with finitely many points is bounded. Thus it follows from the Singleton-H^*-Lemma 198.10 that for a topological space $\{x_0\}$ consisting of a single point x_0 we have

$$\mathrm{H}_\mathrm{b}^n(\{x_0\};\mathbb{R}) \;=\; \mathrm{H}^n(\{x_0\};\mathbb{R}) \;\cong\; \begin{cases} 0, & \text{if } n \geq 1, \\ \mathbb{R}, & \text{if } n = 0. \end{cases}$$ □

The following lemma is the analog of the H^*-Functor Lemma 198.3:

Lemma 209.23. (H_b^*-Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. The induced cochain map $f^*\colon \mathrm{C}^*(Y;\mathbb{R}) \to \mathrm{C}^*(X;\mathbb{R})$ on singular cochains restricts to a cochain map $f^*\colon \mathrm{C}_\mathrm{b}^*(Y;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^*(X;\mathbb{R})$. In particular the map f^* induces a homomorphism

$$f^*\colon \mathrm{H}_\mathrm{b}^n(Y;\mathbb{R}) \;\to\; \mathrm{H}_\mathrm{b}^n(X;\mathbb{R}).$$

(2) For each $n \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}_\mathrm{b}^n(X;\mathbb{R})$ and $f \mapsto f^*$ define a contravariant functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{V}ec_\mathbb{R}$ of real vector spaces.

Proof.

(1) It is obvious that the pullback of a bounded cohomology class is again bounded. Statement (1) is an immediate consequence of this observation.

(2) This statement is obvious. ■

For this variation on the usual cohomology groups we actually do have an analog of the Homotopy-H^*-Proposition 198.11 and the Homotopy Equivalence-H^*-Proposition 198.12:

[2232] Is this statement also true for topological spaces that are not path-connected?

Proposition 209.24. (Homotopy-H_b^*-Proposition)
(1) Homotopic maps induce the same maps on bounded cohomology.
(2) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map

$$f^*\colon \mathrm{H}_\mathrm{b}^n(Y;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}_\mathrm{b}^n(X;\mathbb{R})$$

is in fact an isomorphism.

Proof.

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces X and Y. Furthermore let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. As on page 2776 we define

$$P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)$$

to be the homomorphism that is given by

$$(\sigma\colon \Delta^n \to X) \mapsto \sum_{j=0}^{n} (-1)^j \cdot F \circ (\sigma \times \mathrm{id}_{[0,1]}) \circ \underbrace{[\mathfrak{v}_0, \dots, \mathfrak{v}_j, \mathfrak{w}_j, \dots, \mathfrak{w}_n]}_{\Delta^{n+1} \to \Delta^n \times [0,1]}.$$

In the proof of the Homotopy-H_*-Proposition 144.12 we saw that the maps P_n define a chain homotopy between f_* and g_*. Note that it follows easily from this explicit description of P_n that for any cochain $\varphi \in \mathrm{C}^{n+1}(Y;\mathbb{R})$ that is C-bounded the chain $P_n^*(\varphi) \in \mathrm{C}^n(X;\mathbb{R})$ is $(n+1)\cdot C$-bounded. This shows that the map $P^*\colon \mathrm{C}^{n+1}(Y;\mathbb{R}) \to \mathrm{C}^n(X;\mathbb{R})$ restricts to a map $\mathrm{C}_\mathrm{b}^{n+1}(Y;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^n(X;\mathbb{R})$. As in the proof of the Dual Cochain Map Lemma 197.20 (3) one sees rather easily that the maps P^* form a cochain homotopy between $f^*, g^*\colon \mathrm{C}_\mathrm{b}^*(Y;\mathbb{R}) \to \mathrm{C}_\mathrm{b}^*(X;\mathbb{R})$. Statement (3) now follows from the Cochain Homotopy-H^*-Lemma 197.14 (1).

(2) This statement follows immediately from (2) and (3). ∎

So far it seems like bounded cohomology might be quite similar to usual cohomology with real coefficients. It turns out that this guess is dramatically misguided.

For example the following theorem was proved by Mikhail Gromov.[2233]

Theorem 209.25. (Bounded Cohomology–Isomorphism Theorem) Let X and Y be two connected non-empty CW-complexes and let $f\colon X \to Y$ be a continuous map. If the induced map $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an epimorphism of fundamental groups such that the kernel $\ker(f_*\colon \pi_1(X) \to \pi_1(Y))$ is an "amenable group", then for any $n \in \mathbb{N}_0$ the induced map

$$f^*\colon \mathrm{H}_\mathrm{b}^n(Y;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}_\mathrm{b}^n(X;\mathbb{R})$$

is an isomorphism.

Proof. A proof of this theorem is given in [**Grom1983**, p. 40] or alternatively in [**Iv1987**, Theorem 4.3]. ∎

Remark. In the formulation of the Bounded Cohomology–Isomorphism Theorem 209.25 we used the notion of an "amenable group". We will not give the definition of an "amenable

[2233] Mikhail Gromov (*1943) is a Russian-French mathematician who is easily one of the most influential geometers alive. He won the Abel prize in 2009.

group", instead we refer to [**Pie1984**] or [**CSC2010**, Section 4] or [**Löh2015**, Chapter 9] for the definition. What matters to us are the following two facts:

(1) Groups that admit a finite index subgroup that is solvable are amenable, see [**Pie1984**, Propositions 13.3 and 13.4]. In particular finite groups and solvable groups are amenable.
(2) Groups that contain a free group on two generators are not amenable, see [**Pie1984**, Proposition 14.1]. □

Now we can formulate the following rather surprising corollary.

Corollary 209.26. (Bounded Cohomology–Vanishing Corollary) Let X be a connected non-empty CW-complex such that $\pi_1(X)$ is amenable, e.g. such that $\pi_1(X)$ is solvable. Then $\mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R}) = 0$ for any $n \geq 1$.

Example. It follows from the Bounded Cohomology–Vanishing Corollary 209.26 that the higher bounded cohomology groups of the n-tori $(S^1)^n$, the spheres S^n and lens spaces all vanish. In particular this new invariant is also not useful in our attempt to answer Question 91.6. □

Proof. We consider the map $f\colon X \to \{x_0\}$ that is given by sending every point in X to the point x_0. Then for any $n \geq 1$ we have

$$\begin{array}{ccccc} \mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R}) & \xrightarrow{f^*} & \mathrm{H}^n_{\mathrm{b}}(\{x_0\};\mathbb{R}) & = & 0. \\ \uparrow & & \uparrow & & \\ \text{isomorphism by the Bounded} & & \text{see the above} & & \\ \text{Cohomology–Isomorphism Theorem 209.25} & & \text{example} & & \\ \text{since } \ker(f_*) = \pi_1(X) \text{ is amenable} & & & & \end{array}$$

■

We continue with a completely general calculation:

Lemma 209.27. (First Bounded Cohomology Lemma) For any CW-complex X we have $\mathrm{H}^1_{\mathrm{b}}(X;\mathbb{R}) = 0$.

Proof. The proof of this lemma is pretty elementary, we refer to [**Frig2017**, Chapter 2] for details. ■

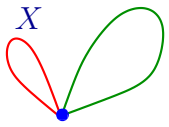
On the other hand, if the fundamental group is not amenable, then we also see some totally unexpected behavior as is shown in the following proposition:

Proposition 209.28. (Bounded Cohomology-of-Wedge-of-Circles Proposition) Let X be the wedge of two circles. Then $\mathrm{H}^2_{\mathrm{b}}(X;\mathbb{R})$ and $\mathrm{H}^3_{\mathrm{b}}(X;\mathbb{R})$ are vector spaces of uncountable dimension.

Proof. The statement regarding $\mathrm{H}^2_{\mathrm{b}}(X;\mathbb{R})$ is proved in [**EF1997**]. An alternative fairly straightforward proof, using quasi-morphisms as introduced on page 1918, is given in [**Roll2009**, Corollary 2.3]. Furthermore the statement regarding $\mathrm{H}^3_{\mathrm{b}}(X;\mathbb{R})$ is proved in [**So1997**, Theorem 3]. ■

Somewhat shockingly the following question is still open:

Question 209.29. Let X be the wedge of two circles. What are the bounded cohomology groups $\mathrm{H}^k_{\mathrm{b}}(X;\mathbb{R}) \neq 0$ for $k \geq 4$? Is any of the groups non-zero? Are all of them non-zero? Are all of them of uncountable dimension?



$$\dim(\mathrm{H}^1_{\mathrm{b}}(X;\mathbb{R})) = 0$$
$$\dim(\mathrm{H}^2_{\mathrm{b}}(X;\mathbb{R})) = \infty$$
$$\dim(\mathrm{H}^3_{\mathrm{b}}(X;\mathbb{R})) = \infty$$
$$\dim(\mathrm{H}^4_{\mathrm{b}}(X;\mathbb{R})) = ???$$

209.8. Exercises for Chapter 209.

Exercise 209.1. Let (X, d) be a metric space. Does it make sense to talk of "cohomology of X with bounded support"?

Exercise 209.2. Provide an example of a continuous map $f\colon X \to Y$ between topological spaces and of a cochain $\varphi \in \mathrm{H}^k(Y;G)$ of compact support, such that $f^*(\varphi) \in \mathrm{H}^k(X;G)$ does not have compact support.

Exercise 209.3. Let G be an abelian group and let $i \in \mathbb{N}_0$. Explain why the map

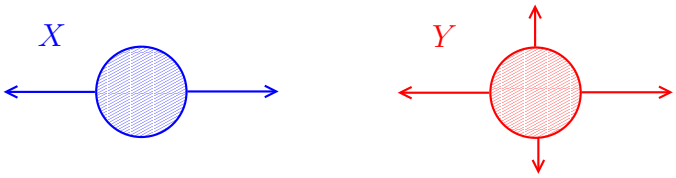
$$X \;\mapsto\; \operatorname*{colim}_{K\in\mathcal{K}(X)} \mathrm{H}^i(X, X\setminus K;G)$$

can be used to define a contravariant functor from the category *TopPropMap* where objects are topological spaces and morphisms are proper maps to the category *AbGroup* of abelian groups.

Exercise 209.4. What are the cohomology groups $\mathrm{H}^k_{\mathrm{c}}(-;\mathbb{Z})$ with compact support of the topological spaces

$$X \;:=\; \overline{B}^2 \cup x\text{-axis} \quad \text{and} \quad Y \;:=\; \overline{B}^2 \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in the figure below.



Exercise 209.5.

(a) Give an example of a path-connected topological space X such that $\mathrm{H}^1(X;\mathbb{Z}) = 0$ but such that $\mathrm{H}^1_{\mathrm{c}}(X;\mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})}$.

(b) Give an example of a path-connected topological space X such that $\mathrm{H}^2(X;\mathbb{Z}) = 0$ but such that $\mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z}) \cong \mathbb{Z}^3$.

Exercise 209.6. Let M be a *compact* topological manifold. We write $\mathring{M} = M \setminus \partial M$. For each $k \in \mathbb{N}_0$ and each abelian group G the cohomology group $\mathrm{H}^n_{\mathrm{c}}(\mathring{M};G)$ is homeomorphic to some other cohomology group that might look more familiar. What is it?

Exercise 209.7. Let M be a non-compact orientable connected n-dimensional smooth manifold with $\partial M = \varnothing$.

(a) Show that for every compact subset $K \subseteq M$ there exists a compact connected codimension-zero submanifold N of M with $K \subseteq N \setminus \partial N \subseteq M$.
Remark. This exercise is essentially the same as Exercise 38.5.

(b) Determine $\mathrm{H}_c^n(M;\mathbb{Z})$.

Exercise 209.8. As in Exercise 209.8 we consider $W = \mathbb{R}^2 \setminus \{(n,0) \mid n \in \mathbb{Z}\}$ and we consider the surface X of infinite genus that is shown in the figure below.

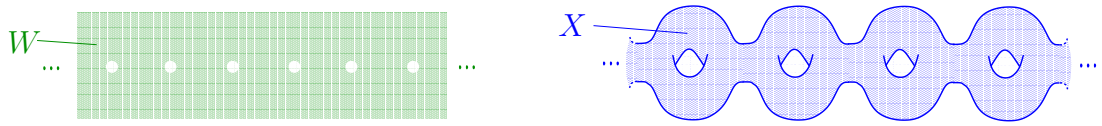
(a) Show that the cup product $\mathrm{H}_c^1(W;\mathbb{Z}) \times \mathrm{H}_c^1(W;\mathbb{Z}) \to \mathrm{H}_c^2(W;\mathbb{Z})$, that we introduced in Definition 209.19, is the zero map.

(b) Show that the cup product $\mathrm{H}_c^1(X;\mathbb{Z}) \times \mathrm{H}_c^1(X;\mathbb{Z}) \to \mathrm{H}_c^2(X;\mathbb{Z})$ is non-zero.
Hint. You could use the pairing $\mathrm{H}_c^2(X;\mathbb{Z}) \times \mathrm{H}_2(X;\mathbb{Z}) \to \mathbb{Z}$ from Exercise **??** to detect that an element in $\mathrm{H}_c^2(X;\mathbb{Z})$ is non-zero.

(c) Show that X and W are not homeomorphic.

(d) Are X and W homotopy equivalent?

Remark. In Exercise 101.22 we showed the weaker statement that W and X, viewed as 2-dimensional smooth manifolds, are not diffeomorphic.



Exercise 209.9. Let us once again consider the topological space

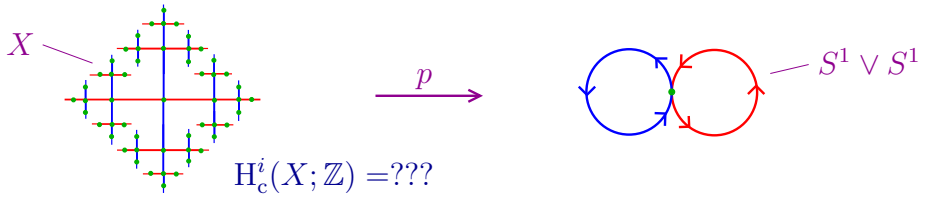
$$X := (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subseteq \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. On page 4455 we saw that $\mathrm{H}^2(X;\mathbb{Z}) \cong \mathbb{Z}^{\mathbb{N}}$. Show that $\mathrm{H}_c^2(X;\mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})}$.

Exercise 209.10. The examples on page 4016 show that cohomology with compact support is *not* a homotopy invariant. Formulate and prove a statement along the following lines: If $f,g\colon X \to Y$ are two proper maps which are "properly homotopic", then for each $k \in \mathbb{N}_0$ we have $f^* = g^*\colon \mathrm{H}_c^k(Y;\mathbb{Z}) \to \mathrm{H}_c^k(X;\mathbb{Z})$.

Exercise 209.11. Show that for every Hausdorff space X, every $n \in \mathbb{N}_0$ and every abelian group G we have a natural isomorphism $\mathrm{H}_c^n(X;G) \to \mathrm{H}_c^{n+1}(X \times \mathbb{R};G)$.

Exercise 209.12. Let X be the universal covering of $S^1 \vee S^1$. Determine the cohomology groups $\mathrm{H}_c^i(X;\mathbb{Z})$ with compact support.



Exercise 209.13. Let $X = \mathbb{R}^2 \setminus \{(\frac{1}{n},0) \mid n \in \mathbb{N}\}$. Determine the cohomology groups $\mathrm{H}_c^i(X;\mathbb{Z})$ with compact support.

Exercise 209.14. Let X be a topological space and let $A,B \subseteq X$ be open subsets with $X = A \cup B$. Does the obvious analog of the Mayer–Vietoris-H*-Theorem 200.11 hold for cohomology with compact support?

Exercise 209.15. Show that for every smooth manifold M and any $n \in \mathbb{N}_0$ the group $\mathrm{H}_c^n(M;\mathbb{Z})$ is countable.

Exercise 209.16. Show that for every path-connected topological space X the bounded cohomology $\mathrm{H}_{\mathrm{b}}^1(X;\mathbb{R}) = 0$.

Exercise 209.17. Let X be a Hausdorff space with path-components $\{X_j\}_{j\in J}$ and let G be an abelian group. We suppose that each X_j is an open subset of X. Show that for each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{j\in J} i_{j*} \colon \bigoplus_{j\in J} \mathrm{H}_{\mathrm{c}}^n(X_j;G) \;\to\; \mathrm{H}_{\mathrm{c}}^n(X;G)$$

is an isomorphism.

Exercise 209.18. Let X be a topological space and let $U \subseteq V \subseteq X$ be open subsets. In Definition 209.16 we constructed for each abelian group G and any $n \in \mathbb{N}_0$ a homomorphism $\mathrm{H}_{\mathrm{c}}^n(U;G) \to \mathrm{H}_{\mathrm{c}}^n(V;G)$. Can you define this homomorphism directly, without going through direct colimits?

Exercise 209.19. Show that the inclusion-induced map $\mathrm{H}_{\mathrm{c}}^n(B^n;\mathbb{Z}) \to \mathrm{H}_{\mathrm{c}}^n(\mathbb{R}^n;\mathbb{Z})$ is an isomorphism.

Exercise 209.20. Let X be a topological space and let $A, B \subseteq X$ be open subsets with $X = A \cup B$. Does the obvious analog of the Mayer–Vietoris-H*-Theorem 200.11 hold for bounded cohomology?

CHAPTER 210

The Topological Poincaré Duality Theorem

In Chapter 188 we stated and proved the Simplicial Poincaré Duality Theorem 188.5 and we saw that it gives rise to the Poincaré Duality Theorem 188.6 for closed smooth manifolds. As discussed on page 3610, that approach has its limitations. We will rectify this by stating and proving in the following two chapters the Poincaré Duality Theorem 210.1, which in particular has the advantage that it also applies to topological manifolds.

This chapter is organized as follows:

- In Section 210.1 we will formulate the Poincaré Duality Theorem 210.1
- In Section 210.2 we will study the Poincaré Duality isomorphisms.
- In Section 210.3 we will discuss several applications of the Poincaré Duality Theorem 210.1.
- In Section 210.4 we will study consequence of the Poincaré Duality Theorem 210.1 applied with coefficients in a field.
- In Section 210.5 we will discuss Euler characteristics of topological manifolds.
- In Section 210.6 we will introduce the umkehr map.
- In Section 210.7 we will discuss various characterizations of homotopy spheres.
- In Section 210.8 we will provide exercises for Chapter 210.

In the next Chapter 211 we will prove the Poincaré Duality Theorem 210.1.

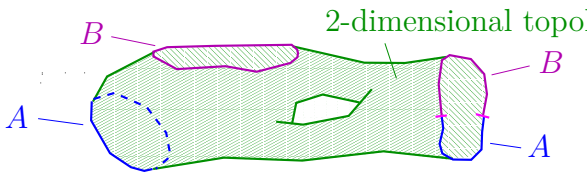
210.1. **The Poincaré Duality Theorem.** The following theorem is by far the most important theorem regarding homology and cohomology groups of topological manifolds:

Theorem 210.1. (Poincaré Duality Theorem)[2234] Let R be a commutative ring, let M be a compact R-oriented[2235] n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the map[2236] [2237] [2238]

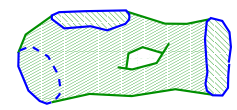
$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

$[M] \in \mathrm{H}_n(M, \partial M; R)$ is the R-fundamental class

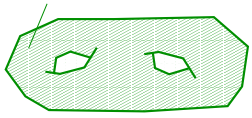
is an isomorphism.



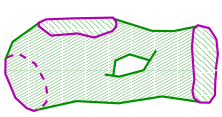
for all commutative rings R we have
$\mathrm{H}^k(M, A; R) \xrightarrow{\cong} \mathrm{H}_{2-k}(M, B; R)$



$\mathrm{H}^k(M, \partial M; R) \xrightarrow{\cong} \mathrm{H}_{n-k}(M; R)$



$\mathrm{H}^k(M; R) \xrightarrow{\cong} \mathrm{H}_{n-k}(M; R)$



$\mathrm{H}^k(M; R) \xrightarrow{\cong} \mathrm{H}_{n-k}(M, \partial M; R)$

We proceed as follows:

(1) In the remainder of this chapter we discuss several special cases and applications of the Poincaré Duality Theorem 210.1. For the most part we will only make use of the fact that the theorem gives us an isomorphism between cohomology and homology groups.
(2) In the next Chapter 211 we will provide the proof of the Poincaré Duality Theorem 210.1.
(3) In Chapter 212 we collect several technical statements about Poincaré Duality and codimension-zero submanifolds. It is best to skip this chapter in a first reading.
(4) In Chapter 213 we will use the fact that the isomorphism is given by capping with the fundamental class to calculate the cohomology rings of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients and of $\mathbb{C}\mathrm{P}^n$ with $\mathbb{Z}$-coefficients.

210.2. **The Poincaré Duality isomorphisms.** The Poincaré Duality Theorem 210.1 leads us to the following definition:

Definition 210.2. We continue with the notation from the Poincaré Duality Theorem 210.1. The isomorphism

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is called Poincaré Duality isomorphism. The corresponding inverse is denoted by

$$\mathrm{PD}_M \colon \mathrm{H}_{n-k}(M, B; R) \xrightarrow{\cong} \mathrm{H}^k(M, A; R)$$

and it will also be called Poincaré Duality isomorphism.

[2234]For topological manifolds with non-empty boundary the Poincaré Duality Theorem is sometimes also called the "Poincaré-Lefschetz Duality Theorem".
[2235]The $\mathbb{F}_2$-Orientability Proposition 191.22 says that any topological manifold M is $\mathbb{F}_2$-orientable.
[2236]If M is in fact orientable, then it follows from the Orientable-for-all-Rings Lemma 191.21 that M is R-orientable for any commutative ring R.
[2237]The Excisive Triad-Proposition 158.20 (2) implies that (M, A, B) is an excisive triad. By the definition of the relative cap product in Definition 206.10, we thus obtain indeed a relative cap product

$$\mathrm{H}^k(M, A; R) \times \underbrace{\mathrm{H}_n(M, \overbrace{A \cup B}^{=\partial M}; R)}_{\text{contains } [M]} \xrightarrow{\cap} \mathrm{H}_{n-k}(X, B; R).$$

[2237]In the Torus–Cap Product Lemma 206.6, the Projective Plane–Cap Product Lemma 206.7 and the Annulus–Cap Product Lemma 206.11 we verified special case of this isomorphism "by hand".

Example. Let M be a compact oriented connected n-dimensional topological manifold and let $P \in M$. As we pointed out on page 3960, we have $[M]^* \cap [M] = [P] \in \mathrm{H}_0(M;\mathbb{Z})$. This implies that $\mathrm{PD}_M([P]) = [M]^* \in \mathrm{H}^n(M,\partial M;\mathbb{Z})$. □

We formulate our second example as a proposition:

Proposition 210.3. (Poincaré Duality-Products Proposition) Let M and N be two compact oriented connected topological manifolds. Furthermore let $x_0 \in M$ and $y_0 \in N$.

(a) We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the obvious projections.
(b) We denote by $i\colon M \xrightarrow{x\mapsto(x,y_0)} M\times N$ and $j\colon N \xrightarrow{y\mapsto(x_0,y)} M\times N$ the obvious embeddings.
(c) We equip $M \times N$ with the product orientation introduced in the Product Orientation Proposition 192.7 (1).
(d) We work with the decomposition $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.
(e) We write $m = \dim(M)$ and $n = \dim(N)$.

The following equalities hold:

$$\begin{aligned} \mathrm{PD}_{M\times N}(i_*([M])) &= (-1)^{mn}\cdot q^*([N]^*) &&\in \mathrm{H}^m(M\times N, M\times\partial N;\mathbb{Z}) \\ \mathrm{PD}_{M\times N}(j_*([N])) &= p^*([M]^*) &&\in \mathrm{H}^n(M\times N, \partial M\times N;\mathbb{Z}) \\ \mathrm{PD}_{M\times N}(i_*([M])) \cup \mathrm{PD}_{M\times N}(j_*([N])) &= [M\times N]^* &&\in \mathrm{H}^{m+n}(M\times N, \partial(M\times N);\mathbb{Z}). \end{aligned}$$

Proof. By the Fundamental Class-Product Proposition 207.2 (2) we have

$$p^*([M]^*) \cap [M\times N] = j_*([N]) \quad \text{and} \quad q^*([N]^*) \cap [M\times N] = (-1)^{mn}\cdot i_*([M]).$$

The first two promised equalities follow from these two calculations and the definition of the Poincaré Duality isomorphisms. The third equality is verified in the following little calculation:

$$\begin{aligned} \mathrm{PD}_{M\times N}(i_*([M])) \cup \mathrm{PD}_{M\times N}(j_*([N])) &\overset{\text{by the first two equalities}}{=} (-1)^{mn}\cdot q^*([N]^*) \cup p^*([M]^*) \\ &= (-1)^{mn}\cdot(-1)^{mn}\cdot p^*([M]^*)\cup q^*([N]^*) \overset{\text{by the Fundamental Class-Product Proposition 207.2 (2)}}{=} [M\times N]^*. \end{aligned}$$

by the Cup Product Commutativity Proposition 204.10

■

210.3. **Poincaré Duality Theorem 210.1applied with $\mathbb{Z}$-coefficients.** In this and the following sections we give several more or less immediate corollaries to the Poincaré Duality Theorem 210.1. If we restrict ourselves to closed smooth manifolds, then the statements below can also be deduced from the Poincaré Duality Theorem 188.6 that we proved earlier.

We start out with the following proposition, which combines the Poincaré Duality Theorem 210.1 with the Universal Coefficient-H^*-Theorem 201.23.

Proposition 210.4. (Poincaré Duality–Non Natural Isomorphism Proposition) Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For every $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_{n-k}(M,A) \cong \underset{\uparrow}{\mathrm{FH}_k(M,B)} \oplus \underset{\uparrow}{\mathrm{Tor}}(\mathrm{H}_{k-1}(M,B)).$$

given an abelian group G we denote by $\mathrm{Tor}(G)$ its torsion-subgroup and we denote by $\mathrm{F}G \coloneqq G/\mathrm{Tor}(G)$ the maximal torsion-free quotient

Proof. We have the following isomorphisms:

$$\mathrm{H}_{n-k}(M,A)\overset{\text{by the Homology-}\mathbb{Z}\text{ Coefficients Lemma 160.10}}{\cong}\mathrm{H}_{n-k}(M,A;\mathbb{Z})\overset{\text{by the Poincaré Duality Theorem 210.1}}{\cong}\mathrm{H}^k(M,B;\mathbb{Z})\overset{\text{by the non-natural isomorphism from the Universal Coefficient-H}^*\text{-Theorem 201.23}}{\cong}\mathrm{Hom}(\mathrm{H}_k(M,B),\mathbb{Z})\oplus\mathrm{Ext}(\mathrm{H}_{k-1}(M,B),\mathbb{Z})$$
$$\overset{\text{follows from the } \mathbb{Z}/k\mathbb{Z}\text{-Ext-Lemma 201.18, the Ext-Properties Lemma 201.19 (4) and the fact that by the Relative-H}_*\text{-Finite Generation Lemma 179.11 together with the Topological Manifolds-Invariants Proposition 189.14 (2) we know that the homology groups } \mathrm{H}_i(M,B) \text{ are finitely generated abelian groups}}{\cong}\mathrm{FH}_k(M,B)\oplus\mathrm{Tor}(\mathrm{H}_{k-1}(M,B)).$$

■

In the following corollary we apply the above proposition to the cases $k=0$ and $k=1$:

Corollary 210.5. (Homology-rel-Partial Boundary Corollary) Let M be a compact orientable connected non-empty n-dimensional topological manifold and let A be a compact codimension-zero submanifold of ∂M.

(1) We have[2238]
$$\mathrm{H}_n(M,A)\cong\begin{cases}\mathbb{Z}, & \text{if } A=\partial M,\\ 0, & \text{otherwise.}\end{cases}$$

(2) The homology group $\mathrm{H}_{n-1}(M,A)$ is a free abelian group.[2239]

Proof. We set $B:=(\partial M)\backslash A$. Note that it follows from the Codimension-Zero Submanifold Proposition 150.16 (3) that A and B satisfy the hypothesis of the Poincaré Duality–Non Natural Isomorphism Proposition 210.4.

(1) Note that
$$\mathrm{H}_n(M,A)\overset{\text{by the Poincaré Duality Non Natural Isomorphism Proposition 210.4}}{\cong}\mathrm{FH}_0(M,B)\overset{\text{follows from the H}_0\text{-Proposition 143.25 and the discussion on page 2798, here we use that } M \text{ is path-connected and non-empty}}{\cong}\begin{cases}\mathbb{Z}, & \text{if } B=\varnothing, \text{ i.e. if } A=\partial M,\\ 0, & \text{otherwise.}\end{cases}$$

(2) Note that
$$\mathrm{H}_{n-1}(M,A)\overset{\text{by the Poincaré Duality–Non Natural Isomorphism Proposition 210.4}}{\cong}\underbrace{\mathrm{FH}_1(M,B)}_{\text{free abelian group}}\oplus\underbrace{\mathrm{Ext}(\overbrace{\mathrm{H}_0(M,B)}^{\text{free abelian group}},\mathbb{Z})}_{=0\text{ by the Ext-Properties Lemma 201.19 (4)}}$$

Note that here we used that it follows from the Topological Manifolds-Invariants Proposition 189.14 together with the Relative-H$_*$-Finite Generation Lemma 179.11 that $\mathrm{H}_1(M,B)$ is finitely generated. ■

210.4. **Poincaré Duality Theorem 210.1 applied with field coefficients.** In this section we consider (co-) homology groups with field coefficients. The following proposition combines the Poincaré Duality Theorem 210.1 with the Universal Coefficient-H*–Field Theorem 202.2:

[2238] This statement is consistent with the statements of the Manifold-Top-H$_*$-Theorem 193.1.

[2239] This statement gives a new proof of some statements of the Manifold-H$_{n-1}$-Theorem 193.4 (2) and it gives a (partial) solution to Exercise 193.13.

Theorem 210.6. (Homology Symmetry Theorem) Let M be a compact n-dimensional topological manifold and let $k \in \mathbb{N}_0$. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$.

(1) We have[2240]
$$\dim_{\mathbb{F}_2}(\mathrm{H}_k(M, A; \mathbb{F}_2)) \;=\; \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)).$$

(2) If M is orientable, then for any field $\mathbb{F}$ we have
$$\dim_{\mathbb{F}}(\mathrm{H}_k(M, A; \mathbb{F})) \;=\; \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M, B; \mathbb{F})).$$

By the Poincaré Duality Theorem 210.1 the analogous statements also hold for cohomology instead of homology.

Remark. In the De Rham Cohomology Symmetry Corollary 84.6 we proved the analogous statement in terms of dimensions of de Rham cohomology groups. □

Proof.

(1) We note that
$$\dim_{\mathbb{F}_2}(\mathrm{H}_k(M, A; \mathbb{F}_2)) \underset{\uparrow}{=} \dim_{\mathbb{F}_2}(\mathrm{H}^{n-k}(M, B; \mathbb{F}_2)) \underset{\uparrow}{=} \dim_{\mathbb{F}_2}(\mathrm{Hom}(\mathrm{H}_{n-k}(M, B; \mathbb{F}_2), \mathbb{F}_2))$$
by the Poincaré Duality Theorem 210.1, which we can apply since by the $\mathbb{F}_2$-Orientability Proposition 191.22 we know that M is $\mathbb{F}_2$-orientable

by the Universal Coefficient-H*–Field Theorem 202.2
$$= \underset{\uparrow}{} \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)).$$
since the Topological Manifolds-Invariants Proposition 189.14 (2) and the Topological Manifold–Boundary Proposition 23.15 (1) and (2) together with the Relative-H$_*$-Finite Generation Lemma 179.11 imply that $\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)$ is finite-dimensional

(2) Let $\mathbb{F}$ be any field. We now assume that M is orientable. It follows from the Orientable-for-all-Rings Lemma 191.21 that M is also $\mathbb{F}$-orientable. The proof of (2) is now basically identical to the proof of (1). ■

Example. We consider the real projective space $\mathbb{R}\mathrm{P}^n$. On page 3117 we showed that
$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$
So the homology groups with $\mathbb{F}_2$-coefficients are in particular symmetric, which is exactly what the above Homology Symmetry Theorem 210.6 with $\mathbb{F}_2$-coefficients predicts. □

Next we recall Definition 156.6:

Definition 210.7. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th Betti number of X as[2241]
$$b_n(X) \;:=\; \mathrm{rank}(\mathrm{H}_n(X)) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(\mathrm{H}_n(X; \mathbb{Q})) \;\in\; \mathbb{N}_0 \cup \{\infty\}.$$
by the Betti-via-Dimension Lemma 161.12

With this definition we can formulate the following straightforward corollary to the Homology Symmetry Theorem 210.6:

[2240] The statement implies of course that $\mathrm{H}_k(M, A; \mathbb{F}_2)$ and $\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)$ are isomorphic. We prefer to state the result in terms of dimensions to stress that the isomorphism is not natural.

[2241] We refer to Definition 97.15 for the definition of the rank of an abelian group.

Corollary 210.8. (Betti Number Symmetry Corollary) Let M be a closed orientable n-dimensional topological manifold. Given any $k \in \mathbb{N}_0$ we have $b_{n-k}(M) = b_k(M)$.

Proof. Let M be a closed orientable n-dimensional topological manifold. We have

$$b_{n-k}(M) \quad = \quad \dim_{\mathbb{Q}}(\mathrm{H}_{n-k}(M;\mathbb{Q})) \quad \underset{\uparrow}{=} \quad \dim_{\mathbb{Q}}(\mathrm{H}_k(M;\mathbb{Q})) \quad = \quad b_k(M).$$

by the Homology Symmetry Theorem 210.6

■

Remark. Henri Poincaré [**Poi1895, Poi2010, PGL1953**, Chapter 9], formulated the conclusion of the Betti Number Symmetry Corollary 210.8 in a distinctly more elegant way: in his seminal paper "analysis situs", published in 1895, he wrote

> "[. . .] pour une variété fermé les nombres de Betti
> également distants des extrêmes sont égaux".

In fact when Henri Poincaré first developed what became known as Poincaré Duality he thought that for any closed orientable n-dimensional smooth manifold the groups $\mathrm{H}_k(M)$ and $\mathrm{H}_{n-k}(M)$ are isomorphic. A Danish mathematician, Poul Heegaard [**Hee1898**], pointed out to him that this is not true in general, for example for the lens space $L(p,q)$ we have by the Lens Space-H_*-Lemma 183.21 that $\mathrm{H}_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$ but $\mathrm{H}_2(L(p,q)) = 0$. Shortly afterwards Henri Poincaré [**Poi1900**] came up with the correct formulation of Poincaré Duality, namely the one given in the above Poincaré Duality–Non Natural Isomorphism Proposition 210.4. □

We conclude this section on Poincaré Duality applied to field coefficients with a curious observation:

Observation. Let M be a compact n-dimensional topological manifold and let $\mathbb{F}$ be a field. In the Topological Manifolds-Invariants Proposition 189.14 (2) we had mentioned that all homology groups $\mathrm{H}_k(M;\mathbb{F})$ are finite-dimensional. But this proof relied on the slightly mysterious Topological Manifold-Simplicial Complex Retract Theorem 189.13. Now we can give an alternative argument if M is closed and orientable. Indeed, we have

$$\mathrm{H}_k(M;\mathbb{F}) \underset{\uparrow}{\cong} \mathrm{H}^{n-k}(M;\mathbb{F}) \quad \overset{\downarrow}{\cong} \mathrm{H}_{n-k}(M;\mathbb{F})^* \underset{\uparrow}{\cong} \mathrm{H}^k(M;\mathbb{F})^* \overset{\downarrow}{\cong} (\mathrm{H}_k(M;\mathbb{F})^*)^*.$$

by the Universal Coefficient-H^*–Field Theorem 202.2

this follows from the Orientable-for-all-Rings Lemma 191.21 together with the Poincaré Duality Theorem 210.1

(Here, as always given an $\mathbb{F}$-vector space V we denote by $V^* = \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ its dual vector space.) But by [**Jacb1953**, p. 244-248] (see also [**DF2004**, Chapter 11.4]) we know that an $\mathbb{F}$-vector space V is isomorphic to its double dual $(V^*)^*$ if and only if the vector space is finite-dimensional. □

210.5. **Euler characteristics of topological manifolds.** Using the Homology Symmetry Theorem 210.6 we obtain the following pretty proposition:

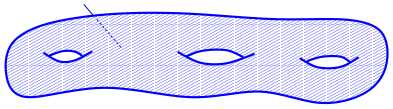
Proposition 210.9. (Odd-Dimensional Manifold-χ-Proposition) For every compact odd-dimensional topological manifold M we have

$$\chi(M) \ = \ \tfrac{1}{2} \cdot \chi(\partial M).$$

In particular if M is a closed odd-dimensional topological manifold, then $\chi(M) = 0$.

$$\begin{array}{rcl} \mathrm{H}_2(M) & = & 0 \\ \mathrm{H}_1(M) & \cong & \mathbb{Z}^3 \\ \mathrm{H}_0(M) & \cong & \mathbb{Z} \\ \Longrightarrow \chi(M) & = & -2 \end{array}$$

M is a 3-dimensional "handlebody"



∂M is a surface of genus 3

$$\begin{array}{rcl} \mathrm{H}_2(\partial M) & \cong & \mathbb{Z} \\ \mathrm{H}_1(\partial M) & \cong & \mathbb{Z}^6 \\ \mathrm{H}_0(\partial M) & \cong & \mathbb{Z} \\ \Longrightarrow \chi(\partial M) & = & -4 \end{array}$$

Remark. In the Odd-Dimensional Manifold-χ-Proposition 53.6 we showed the weaker statement that the smooth Euler characteristic of a closed *orientable* odd-dimensional smooth manifold is zero. In fact, in the Odd-Dimensional Manifold-χ-Proposition 250.9 we will use handle decompositions to give an alternative proof of this weaker statement on the Euler characteristic of closed odd-dimensional smooth manifolds. □

Proof. Let M be a compact topological manifold of dimension $2n+1$. Then

$$\begin{array}{rcl} \chi(M) & \underset{\uparrow}{=} & \sum_{i=0}^{2n+1} (-1)^i \cdot \dim_{\mathbb{F}_2}(\mathrm{H}_i(M;\mathbb{F}_2)) \\ & & \text{by the Manifold-}\chi\text{ Proposition 190.24 (4)} \\ & \underset{\uparrow}{=} & -\sum_{j=0}^{2n+1} (-1)^j \cdot \dim_{\mathbb{F}_2}(\mathrm{H}_j(M,\partial M;\mathbb{F}_2)) \\ & & \text{follows from the substitution } j = 2n+1-i \text{ together with the fact that } (-1)^{2n+1} = -1 \\ & \underset{\uparrow}{=} & -\chi(M,\partial M) \\ & & \text{by the Manifold-}\chi\text{ Proposition 190.24 (3)} \\ & \underset{\uparrow}{=} & -\chi(M) + \chi(\partial M). \\ & & \text{follows from the Relative-}\chi\text{-Lemma 156.8} \end{array}$$

$$\begin{array}{rcl} & \underset{\uparrow}{=} & \sum_{i=0}^{2n+1} (-1)^i \cdot \dim_{\mathbb{F}_2}(\mathrm{H}_{2n+1-i}(M,\partial M;\mathbb{F}_2)) \\ & & \text{by the Homology Symmetry Theorem 210.6} \end{array}$$

Summarizing we have shown that $\chi(M) = -\chi(M) + \chi(\partial M)$. The desired equality follows from solving for $\chi(M)$. ■

Using the Odd-Dimensional Manifold-χ-Proposition 210.9 we easily obtain the following surprising (?) corollary.

Corollary 210.10. (Boundary-χ-Corollary) If W is a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold, then $\chi(W)$ is even.

Example. Recall that on page 2988 we showed that $\chi(\mathbb{R}\mathrm{P}^{2n}) = \frac{1}{2}(1 + (-1)^{2n}) = 1$ and that $\chi(\mathbb{C}\mathrm{P}^{2n}) = 2n+1$. It thus follows from the Boundary-χ-Corollary 210.10 that the closed topological manifolds $\mathbb{R}\mathrm{P}^{2n}$ and $\mathbb{C}\mathrm{P}^{2n}$ are not homeomorphic to the boundary of a compact topological manifold. □

Proof. Let W be a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold M. Evidently M is odd-dimensional. Therefore we obtain immediately from the Odd-Dimensional Manifold-χ-Proposition 210.9 that $\chi(W) = \chi(\partial M) = 2 \cdot \chi(M) \in 2 \cdot \mathbb{Z}$. ■

The Odd-Dimensional Manifold-χ-Proposition 210.9 raises the following question: What can we say about Euler characteristics of closed even-dimensional topological manifolds? By the classification of closed oriented connected 2-dimensional topological manifolds, see the Surface Classification Theorem 101.16, we know that the Euler characteristic of such a topological manifold is even and ≤ 2. In higher dimensions there is evidently much more freedom for constructing topological manifolds. This raises the following question:

Question 210.11. Let $n \geq 4$ be even. Is every integer the Euler characteristic of a closed orientable connected n-dimensional topological manifold?

We will answer this question in the Euler Characteristic–Realization Proposition 226.3.

We conclude this section with a surprising statement about closed non-orientable 3-dimensional manifolds. To put this result in context, recall that in the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16 we showed that if M is a non-orientable topological manifold, then $\pi_1(M)$ admits an epimorphism onto $\mathbb{Z}/2\mathbb{Z}$.[2242] It follows from the Hurewicz Theorem 166.10 that there exists also an epimorphism $\mathrm{H}_1(M) \to \mathbb{Z}/2\mathbb{Z}$. For closed 3-dimensional topological manifolds the following lemma is thus a considerable strengthening of the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16.

Lemma 210.12. (Non-Orientable 3-Manifold-H_1-Lemma) Let M be a closed connected 3-dimensional topological manifold. If M is non-orientable, then $\mathrm{H}_1(M)$ is infinite.

Proof. Let M be a closed non-orientable connected 3-dimensional topological manifold. We have

$$0 \overset{\text{by the Odd-Dimensional Manifold } \chi\text{-Proposition 210.9}}{=} \chi(M) \overset{\text{see Definition 156.6}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(M) \overset{\text{by the Orientation-Top-H}_*\text{-Theorem 193.3 and the Manifold-Top-H}_*\text{-Theorem 193.1}}{=} b_0(M) - b_1(M) + b_2(M) \overset{\text{by the H}_0\text{-Proposition 143.25}}{=} 1 - b_1(M) + b_2(M).$$

This equality can only hold if $b_1(M) > 0$, i.e. if $\mathrm{H}_1(M)$ is infinite. ∎

210.6. **The umkehr map.** The Poincaré Duality Theorem 210.1 allows us to introduce the weirdly named "umkehr map":

Definition 210.13. Let $f\colon M \to N$ be a continuous map between closed oriented n-dimensional topological manifolds. We consider the following diagram of maps:

$$\begin{array}{ccc} \mathrm{H}_k(M;\mathbb{Z}) & \underset{\cap[M]}{\overset{\mathrm{PD}_M}{\rightleftarrows}} & \mathrm{H}^{n-k}(M;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{H}_k(N;\mathbb{Z}) & \underset{\cap[N]}{\overset{\mathrm{PD}_N}{\rightleftarrows}} & \mathrm{H}^{n-k}(N;\mathbb{Z}). \end{array}$$

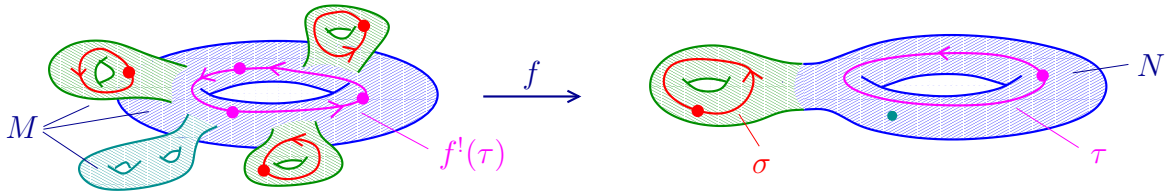
Here the horizontal maps are the Poincaré Duality isomorphisms provided by the Poincaré Duality Theorem 210.1. We refer to

$$\begin{array}{rccl} f^!\colon \mathrm{H}_k(N;\mathbb{Z}) & \to & \mathrm{H}_k(M;\mathbb{Z}) \\ \sigma & \mapsto & f^*(\mathrm{PD}_N(\sigma)) \cap [M] \end{array}$$

as the umkehr map of f.[2243] [2244]

[2242] If in the proof of the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16 we replace the Orientation Covering–Properties-Proposition 95.10 (2b) by the Orientation Covering–Properties-Proposition 191.29 (2b) we see that the statement also holds for non-orientable topological manifolds.

Example. In the figure below we show a continuous map $f\colon M \to N$ which is given by first collapsing the turquoise bit to a point, followed by a 3-fold covering map.



$f^!(\sigma)$ is the sum of the red terms

With some effort one can show, say using the Umkehr–Transfer Map Lemma 210.14, that the umkehr map is of the form as described in the example. □

The following lemma shows that in the right circumstances the umkehr map is just the transfer map that we introduced in Definition 165.1.

Lemma 210.14. (Umkehr–Transfer Map Lemma) Let $p\colon \widetilde{N} \to N$ be a finite covering of a closed oriented connected n-dimensional topological manifold. If we equip $\widetilde{N}$ with the orientation given by the Topological Manifold-Induced Orientation Lemma 192.1 (3), then for every $k \in \mathbb{N}_0$ the umkehr map $p^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.

Proof. We start our proof with the following claim:

Claim. Let $r \leq s$. Given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in \mathrm{C}^r(N;\mathbb{Z})$ we have

$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma) \;\in\; \mathrm{C}_{s-r}(\widetilde{N};\mathbb{Z}).$$

Proof. We write $m = [\widetilde{N} : N]$. We denote by $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_m\colon \Delta^s \to \widetilde{N}$ the lifts of $\sigma\colon \Delta^s \to N$. We have the following equality in $\mathrm{C}_{s-r}(\widetilde{N};\mathbb{Z}) = \mathrm{C}_{s-r}(\widetilde{N}) \otimes \mathbb{Z}$:

$$\begin{aligned}
p^*(\phi) \cap p^*(\sigma) &\overset{\text{definition of } p^*(\sigma)}{=} p^*(\phi) \cap \sum_{i=1}^m \widetilde{\sigma}_i \overset{\text{Definition 206.2 of the cap product}}{=} \sum_{i=1}^m \widetilde{\sigma}_i \circ [\mathfrak{v}_r, \ldots, \mathfrak{v}_s] \otimes p^*(\phi)(\widetilde{\sigma}_i \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_r]) \\
&= \sum_{i=1}^m \widetilde{\sigma}_i \circ [\mathfrak{v}_r, \ldots, \mathfrak{v}_s] \otimes \phi((p_* \circ \widetilde{\sigma}_i) \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_r]) \\
&= p^*(\sigma \circ [\mathfrak{v}_r, \ldots, \mathfrak{v}_s] \otimes \phi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_r])) \overset{\text{Definition 206.2 of the cap product}}{=} p^*(\phi \cap \sigma)
\end{aligned}$$

since $p_* \circ \widetilde{\sigma}_i = \sigma$ and since $\widetilde{\sigma}_i \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_r]$, $i = 1, \ldots, m$ are the lifts of $\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_r]$

⊞

Now we turn to the actual proof of the lemma. We pick an n-cycle τ that is a representative for the fundamental class $[N] \in \mathrm{H}_n(N;\mathbb{Z})$ of N. We now see that for any $\sigma \in \mathrm{H}_k(N;\mathbb{Z})$ we have

$$p^!(\sigma) \overset{\text{Definition 210.13 of the umkehr map}}{=} p^*(\mathrm{PD}_N(\sigma)) \cap [\widetilde{N}] \underset{\text{by the Fundamental Class-Covering Proposition 194.11 we know that } p^*(\tau) \text{ represents the fundamental class of } \widetilde{N}}{=} p^*(\mathrm{PD}_N(\sigma) \cap p^*([\tau])) \overset{\text{follows from the claim and from } [N] = [\tau]}{=} p^*(\mathrm{PD}_N(\sigma) \cap [N]) \underset{\text{by Definition 210.2}}{=} p^*(\sigma).$$

[2243] This map is indeed called "umkehr map" in English. The name stems from the fact that it is a map on homology that "goes the other way".

[2244] Wikipedia writes: "In category theory [...] certain unusual functors are denoted $f^!$ and $f_!$, with the exclamation mark used to indicate that they are exceptional in some way. They are thus accordingly sometimes called shriek maps, with "shriek" being slang for an exclamation mark [...]."

■

To formulate our main result on umkehr maps let us recall Definition 195.1:

Definition 210.15. Given a continuous map $f\colon M \to N$ between closed oriented connected non-empty n-dimensional topological manifolds we define the degree of f to be the unique integer $\deg(f) \in \mathbb{Z}$ with $f_*([M]) = \deg(f) \cdot [N]$.

Proposition 210.16. (Umkehr Map Proposition) Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. Let $k \in \mathbb{N}_0$. We consider the following two maps:

$$\mathrm{H}_k(N;\mathbb{Z}) \xrightarrow{\text{umkehr map } f^!} \mathrm{H}_k(M;\mathbb{Z}) \xrightarrow{\text{induced map } f_*} \mathrm{H}_k(N;\mathbb{Z}).$$

The following statements hold:

(1) The map $f_* \circ f^!$ is multiplication by $\deg(f)$.
(2) If $\deg(f) = \pm 1$, then there exists an isomorphism $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(N;\mathbb{Z}) \oplus Q$ for some finitely generated abelian group Q.

Proof. Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds.

(1) Let $\sigma \in \mathrm{H}_k(N;\mathbb{Z})$. We perform the following calculation in $\mathrm{H}_k(N;\mathbb{Z})$:

$$\begin{array}{lccc} & \text{Definition 210.13 of } f^! & & \text{by the Cap Product Naturality Lemma 206.13} \\ (f_* \circ f^!)(\sigma) & \overset{\downarrow}{=} & f_*(f^*(\mathrm{PD}_N(\sigma)) \cap [M]) & \overset{\downarrow}{=} \quad \mathrm{PD}_N(\sigma) \cap f_*([M]) \\ & = & \deg(f) \cdot \mathrm{PD}_N(\sigma) \cap [N] & = \quad \deg(f) \cdot \sigma. \\ & \overset{\uparrow}{} & & \overset{\uparrow}{} \\ & \text{by Definition 195.1 of } \deg(f) & & \text{Definition 210.2 of } \mathrm{PD}_N \end{array}$$

(2) We now assume that $\deg(f) = \pm 1$. We consider the sequence

$$0 \to \mathrm{H}_k(N;\mathbb{Z}) \xrightarrow{f^!} \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{coker}(f^!) \to 0.$$

By (1) and our hypothesis we know that $f_* \circ f^!$ is an isomorphism. This shows that the umkehr map $f^!$ admits a left-inverse. In particular the left map is a monomorphism. It is now clear that the sequence is exact. Finally note that it follows from the Splitting Lemma 145.5 (2)⇒(3) that $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(N;\mathbb{Z}) \oplus \mathrm{coker}(f^!)$. ■

On several occasions we were wondering whether there exists a continuous map of degree one between two given topological manifolds. We had obtained the following results:

(1) In the Degree Realization Proposition 46.19 (in the formulation from page 3778) we showed that for any closed oriented non-empty n-dimensional topological manifold M there exists a degree-one map $M \to S^n$.
(2) In the Degree One-Epimorphism Proposition 195.8 we saw that for a continuous map $f\colon M \to N$ of degree ± 1 the induced map $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.

For example, it follows from remark (2) above that there is no degree-one map from $\mathbb{R}\mathrm{P}^3$ to $S^1 \times S^2$: Indeed this follows, since evidently there is no epimorphism from $\pi_1(\mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}/2\mathbb{Z}$ onto $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$. In Question 186.13 we had asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$. The above remark (2) has no objections to the existence of such a map. But now we can give a negative answer:

Corollary 210.17. (No Degree-1-Map Corollary) There is no degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$.

Proof. The corollary follows immediately from the Umkehr Map Proposition 210.16 and the observation that the group $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is evidently not a subsummand of the group $\mathrm{H}_1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. ■

210.7. **Homotopy spheres.** We conclude this chapter with a short discussion of "homotopy spheres". In the literature these get defined in many different ways. But the following proposition shows that many formulations are equivalent:

Proposition 210.18. (Homotopy Sphere-Characterization Proposition) Let $n \in \mathbb{N}$. For every closed connected non-empty n-dimensional topological manifold X the following statements are equivalent:

(1) X is $\lfloor \frac{n}{2} \rfloor$-connected.
(2) X is $(n-1)$-connected.
(3) There exists a continuous map $f\colon S^n \to X$ of degree one, in the sense of Definition 195.1.

If $n \geq 2$, then the above are also equivalent to the following statements:

(4) $\pi_1(X)$ is trivial and $\mathrm{H}_i(X) = 0$ for $i = 2, \ldots, n-1$.
(5) $\pi_1(X)$ is trivial and $\mathrm{H}_i(X) = 0$ for $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Definition 210.19. Let $n \in \mathbb{N}$. A closed connected n-dimensional topological manifold M that satisfies any of the five equivalent statements in the Homotopy Sphere-Characterization Proposition 210.18 is called a homotopy n-sphere. If M is actually a smooth manifold, then we refer to M as a smooth homotopy n-sphere.

Remark. In the Homotopy Sphere-Characterization Proposition **??** we will give even more characterizations of homotopy spheres. □

Proof. The statements for $n = 1$ are either trivial or they follow from the Topological 1-Manifold Classification Theorem 28.1. We thus now assume that $n \geq 2$. Let X be a closed connected non-empty n-dimensional topological manifold. We start out with a basic observation:

(∗) If $\pi_1(X) = 0$, then it follows from the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 191.32 that X is actually orientable.

The equivalence of the five statements now follows from the following implications:
(1)⇒(5). This follows from $n \geq 2$ and the Hurewicz Corollary 167.13.
(5)⇒(4). This follows easily from the Poincaré Duality Theorem 210.1 (here we implicitly use (∗)) and the Universal Coefficient-H*-Theorem 201.23.
(4)⇒(2). This follows again from the Hurewicz Corollary 167.13.
(2)⇒(1). This statement is trivial.
(2)⇒(3). Since X is $(n-1)$-connected and since $n \geq 2$ we obtain from (∗) that X is orientable. Next note that it follows from the Hurewicz Theorem 167.9 that the Hurewicz map

$$\begin{array}{rcl} \Phi\colon \pi_n(X) & \to & \mathrm{H}_n(X) \\ [f\colon S^n \to X] & \mapsto & f_*([S^n]) \end{array}$$

is an isomorphism. In particular there exists an $[f\colon S^n \to X] \in \pi_n(X)$ with $f_*([S^n]) = [X]$. But this means that we have found a continuous map $f\colon S^n \to X$ of degree one.
(3)⇒(4). We suppose that X is orientable and we suppose that there exists a continuous map $f\colon S^n \to X$ of degree one. Recall that by the Degree One-Epimorphism Proposition 195.8 this implies that $f_*\colon \pi_1(S^n) \to \pi_1(X)$ is an epimorphism. Since $n \geq 2$ it follows from the Sphere–Low-π_n-Proposition 134.19 that X is simply connected. Furthermore it follows from the Umkehr Map Proposition 210.16 (which we can apply since it follows from (∗) that X is orientable) that for $i = 2, \ldots, n-1$ the homology group $\mathrm{H}_i(X)$ is isomorphic to a subgroup of $\mathrm{H}_i(S^n)$. By the Sphere-H_*-Proposition 149.4 we know that the latter is zero. Thus we obtain (4). ■

210.8. Exercises for Chapter 210.

Exercise 210.1. Let $g \in \mathbb{N}_0$, let $k \in \mathbb{N}$ and let $m, n \in \mathbb{N}_0$. As in Notation 101.12 we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks and we denote by $N_{k,n}$ the non-orientable surface of genus k minus n open disks. Use the Poincaré Duality Theorem 210.1 and the Universal Coefficient-H^*-Theorem 201.23 to perform the following calculations:
(a) Determine the relative homology groups $\mathrm{H}_*(\Sigma_{g,m}, \partial\Sigma_{g,m})$.
(b) Determine the relative homology groups $\mathrm{H}_*(N_{k,n}, \partial N_{k,n})$.

Exercise 210.2. Let X and Y be topological spaces. We say that X is a *homology suspension of* Y if for every $i \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_i(X) \cong \widetilde{\mathrm{H}}_{i-1}(Y)$. Let X be a closed oriented connected non-empty n-dimensional topological manifold. Show that if X is the homology suspension of a closed oriented topological manifold Y, then X is a homology n-sphere, i.e. show that $\mathrm{H}_i(X) \cong \mathrm{H}_i(S^n)$ for all $i \in \mathbb{N}_0$.
Remark. In view of the Suspension-H_*-Proposition 149.3 this exercise can be viewed as a variation on Exercise 150.14.

Exercise 210.3. Let M and N be two closed connected non-empty 3-dimensional topological manifolds such that $\mathrm{H}_1(M;\mathbb{Z}) \cong \mathrm{H}_1(N;\mathbb{Z})$.
(a) We assume that M and N are orientable. Show that for all $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(M;\mathbb{Z}) \cong \mathrm{H}_i(N;\mathbb{Z})$.
(b) Does the conclusion of (a) also hold if we drop the orientability hypothesis?

Exercise 210.4. Let M be an n-dimensional closed orientable connected topological manifold such that $\pi_1(M)$ is isomorphic to the permutation group S_5.
(a) What can you say about $\mathrm{H}^{n-1}(M;\mathbb{Z})$?
(b) Does the answer depend on whether or not M has non-empty boundary?

Exercise 210.5. Let M be a compact non-orientable topological manifold. Show that the group $\mathrm{H}_{n-1}(M, \partial M; \mathbb{F}_2)$ is non-zero.

Exercise 210.6. Is the Klein bottle the boundary of a compact 3-dimensional topological manifold?

Exercise 210.7. Let M be a compact orientable 3-dimensional topological manifold such that $\mathrm{H}_1(M;\mathbb{Z})$ is torsion-free. We denote by $i\colon \partial M \to M$ the inclusion. Show that there exists a subgroup G of $\mathrm{H}_1(M;\mathbb{Z})$ such that $\mathrm{H}_1(M;\mathbb{Z}) = i_*(\mathrm{H}_1(\partial M;\mathbb{Z})) \oplus G$.

Exercise 210.8. Let M be a compact orientable 3-dimensional topological manifold such that the boundary ∂M is a surface of genus g. Show that $\operatorname{rank}(\mathrm{H}_1(M;\mathbb{Z})) \geq g$.
Hint. Look at the long exact sequence in homology of the pair $(M, \partial M)$.

Exercise 210.9.

(a) Let W be a closed orientable connected 4-dimensional topological manifold such that $\chi(W) = -3$. Show that W admits a finite connected covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.
(b) Is the hypothesis in (a) that W is orientable necessary?

Remark. This exercise is almost identical to Exercise 188.5.

Exercise 210.10. Let M be a closed connected non-empty 4-dimensional topological manifold that is simply connected. Does there exist a $k \in \mathbb{Z}$ such that $\chi(M) = k$ implies that $\pi_3(M) = 0$?

Exercise 210.11. Let M be a closed 4-dimensional topological manifold.

(a) Show that if M is simply connected, then there exist at most finitely many isomorphism classes of finite groups which can act discretely on M.
(b) Is the hypothesis in (a) that M is simply connected necessary?

Exercise 210.12. Let M be a compact orientable 4-dimensional topological manifold with $\partial M = A \sqcup B$ where A and B are two disjoint closed connected 3-dimensional topological manifolds with $\mathrm{H}_1(A;\mathbb{Z})$ finite. We suppose that the inclusion-induced map $\mathrm{H}_1(A;\mathbb{Z}) \to \mathrm{H}_1(M)$ is an epimorphism. Show that the inclusion-induced map $\mathrm{H}_1(B;\mathbb{Z}) \to \mathrm{H}_1(M)$ is a monomorphism.

Exercise 210.13. Let M be a closed simply connected 4-dimensional topological manifold. Show that $\mathrm{H}_1(M;\mathbb{Z}) = 0$, $\mathrm{H}_3(M;\mathbb{Z}) = 0$ and show that $\mathrm{H}_2(M;\mathbb{Z})$ is torsion-free.

Exercise 210.14. Let M be a closed connected non-empty 4-dimensional topological manifold. Show that if $\mathrm{H}_1(M;\mathbb{Z})$ is torsion-free, then so is $\mathrm{H}_2(M;\mathbb{Z})$.

Exercise 210.15. Let M be a closed orientable connected 4-dimensional topological manifold such that $\chi(M) = 10$ and $\mathrm{H}_1(M) \cong \mathbb{Z}/3\mathbb{Z}$. Compute all the homology and cohomology groups of M.

Exercise 210.16. Let M be a closed orientable 6-dimensional topological manifold with $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, $\mathrm{H}_2(M) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathrm{H}_3(M) \cong \mathbb{Z}^4$. Is it possible that M is the product of two non-simply connected topological manifolds?

Exercise 210.17. Let M be a closed orientable 6-dimensional topological manifold with $\pi_1(M) \cong \mathbb{Z}^2$, $\mathrm{H}_2(M) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathrm{H}_3(M) \cong \mathbb{Z}^2$. Is it possible that M is the product of two non-simply connected topological manifolds?

Exercise 210.18. Let M be a compact orientable connected 3-dimensional topological manifold such that $\mathrm{H}_1(M) \cong \mathbb{Z}$ and such that ∂M is homeomorphic to the 2-dimensional torus $S^1 \times S^1$. Show that the inclusion-induced homomorphism $\mathrm{H}_1(\partial M) \to \mathrm{H}_1(M)$ is an epimorphism.

Exercise 210.19. For which $n \in \mathbb{N}$ do there exist at least four different degrees i with $\mathrm{H}_i(\mathrm{GL}(n,\mathbb{R});\mathbb{F}_2) \neq 0$?

Exercise 210.20. Let M be a closed orientable connected odd-dimensional topological manifold and let $f\colon M \to M$ be a continuous map such that the induced homomorphism $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is an isomorphism. Show that the Lefschetz number

$$\Lambda(\varphi) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M))$$

equals zero.
Remark. This exercise is almost identical to Exercise 188.6.

Exercise 210.21. Let M be a closed oriented connected n-dimensional topological manifold. Furthermore let $i \in \{0,\dots,n\}$ and let $v_1,\dots,v_m$ be a basis of $\mathrm{FH}^i(M;\mathbb{Z})$. Show that there exists a basis $\widetilde{v}_1,\dots,\widetilde{v}_m$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for every $k,l \in \{1,\dots,m\}$ we have $\langle v_k \cup \widetilde{v}_l, [M]\rangle = \delta_{kl}$.

Exercise 210.22. We introduce the following concepts:

- A *topological homology n-ball* is a compact n-dimensional topological manifold X such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(\overline{B}^n)$.
- A *topological homology m-sphere* is a closed m-dimensional topological manifold X such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^m)$.

Show that if X is a topological homology n-ball, then ∂X is a topological homology $(n-1)$-sphere.

Exercise 210.23. Let M be a compact oriented connected n-dimensional topological manifold. We say that M is an *n-dimensional homotopy ball* if $\pi_i(M) = 0$ for $i \geq 1$. Show that if M is an n-dimensional homotopy ball with $\pi_1(\partial M) = \{e\}$, then ∂M is a homotopy $(n-1)$-sphere, as defined in Exercise 210.23.
Remark. We will see in the Mazur Manifold Proposition 260.18 that in general we cannot drop the hypothesis that $\pi_1(\partial M)$ is trivial.

Exercise 210.24. Let $n \in \mathbb{N}_{\geq 2}$ and let M be a compact oriented connected n-dimensional topological manifold. Show that if there exists a degree one map $f\colon (\overline{B}^n, S^{n-1}) \to (M, \partial M)$, then M is an n-dimensional homotopy ball, as defined in Exercise 210.23.

Exercise 210.25. Let M and N be two compact oriented n-dimensional topological manifolds and let $f\colon \partial M \to \partial N$ be a homeomorphism. Show that if M and N are both contractible, then

$$(M \sqcup N)/x \sim f(x) \quad \text{ with } x \in \partial M$$

is a homotopy sphere.

Exercise 210.26. Let M be a compact oriented connected n-dimensional topological manifold. Let Y be a boundary component of M which is homeomorphic to S^{n-1}. Show that either $[Y] \neq 0 \in \pi_{n-1}(M)$ or M is a homotopy ball, as defined in Exercise 210.23.
Hint. Show that if $[Y] = 0$, then there exists a degree one map $(\overline{B}^n, S^{n-1}) \to (M, Y)$. Now use Exercise 210.24.

Exercise 210.27. Let $f\colon M \to N$ be a homeomorphism between two compact oriented connected n-dimensional topological manifolds and let A, B be two compact $(n-1)$-dimensional submanifolds of ∂M with $A \cup B = \partial M$, $A \cap B = \partial A = \partial B$. Show that for any

$\sigma \in \mathrm{H}_k(M, A)$ we have

$$f^*(\mathrm{PD}_N(f_*(\sigma)) \;=\; \begin{cases} \mathrm{PD}_M(\sigma), & \text{if } \sigma \text{ is orientation-preserving,} \\ -\mathrm{PD}_M(\sigma), & \text{if } \sigma \text{ is orientation-reversing.} \end{cases}$$

Remark. This exercise consists mostly of collecting fairly standard facts.

Exercise 210.28. Let X and Y be compact oriented connected non-empty n-dimensional topological manifolds and let $f\colon X \to Y$ be a continuous map of degree one. Does it follow that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(X;\mathbb{Z}) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_k(Y;\mathbb{Z}) \\ {\scriptstyle \mathrm{PD}_X}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{PD}_Y} \\ \mathrm{H}^{n-k}(X;\mathbb{Z}) & \xleftarrow{\quad f^* \quad} & \mathrm{H}^{n-k}(Y;\mathbb{Z})\,? \end{array}$$

Exercise 210.29. Let M be a compact orientable n-dimensional topological manifold. Determine the homology groups of the double $\mathrm{D}M$ in terms of the homology groups of M. *Remark.* It is tempting to use Mayer–Vietoris sequences, but this will most likely not lead very far. Better look at the long exact sequence of the pair $(\mathrm{D}M, M)$, use excision and use the Poincaré Duality Theorem 210.1.

Exercise 210.30. Given a ring $S \subseteq \mathbb{R}$ we consider the *Heisenberg group*

$$H_S \;\coloneqq\; \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{R}) \;\middle|\; x, y, z \in S \right\}.$$

(a) Show that the *Heisenberg manifold* $M \coloneqq H_{\mathbb{R}}/H_{\mathbb{Z}}$ is a closed orientable 3-dimensional smooth manifold.
(b) Determine all homology groups of the Heisenberg manifold.

Exercise 210.31. Determine all sequences of finitely generated abelian groups A_0, A_1, A_2, A_3 such that there exists a closed orientable 3-dimensional topological manifold M such that for all $i \in \{0,1,2,3\}$ the homology group $\mathrm{H}_i(M)$ is isomorphic to A_i.

Exercise 210.32. Let A, B, C be three closed oriented connected non-empty topological manifolds with $\dim(B) = \dim(C)$. Let $c \in C$ and let $i\colon A \to A \times C$ be the embedding that is given by $a \mapsto (a, c)$. Finally let $f\colon B \to C$ be a continuous map. Determine $(f \times \mathrm{id})^!(i_*([A])) \in \mathrm{H}_*(B \times C;\mathbb{Z})$.

Exercise 210.33. Let M be a closed orientable connected non-empty n-dimensional topological manifold. Show that if there exists a continuous map $f\colon S^n \to M$ of non-zero degree, then $\mathrm{H}_i(M;\mathbb{Q}) = 0$ for $i = 1, \ldots, n-1$.

Exercise 210.34. Let (X, A) and (Y, B) be two pairs of topological spaces and let $n \in \mathbb{N}_0$ such that the following statements hold:

(1) All relative homology groups of (X, A) and (Y, B) are finitely generated.
(2) For every $j \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_j(X, A) \cong \mathrm{H}^{n-j}(Y, B;\mathbb{Z})$.

Show that for every abelian group G and every $j \in \mathbb{N}_0$ there exists also an isomorphism $\mathrm{H}^j(X, A; G) \cong \mathrm{H}_{n-j}(Y, B; G)$.

Remark. This exercise implies that we have a version of Poincaré Duality with G-coefficients. In Exercise 211.3 we will give a more satisfactory formulation of such a statement.

Exercise 210.35. Let $n \in \mathbb{N}_{\geq 2}$ and let X be a compact n-dimensional topological manifold such that for each $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^{n-1})$. Show that ∂X has two boundary components A and B.

CHAPTER 211

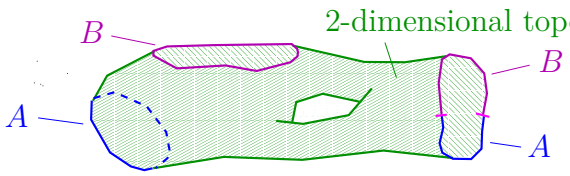
Proof of the Poincaré Duality Theorem 210.1

In the last chapter we formulated the following theorem:

Theorem 210.1. (Poincaré Duality Theorem) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism.



for all commutative rings R we have
$\mathrm{H}^k(M, A; R) \xrightarrow{\cong} \mathrm{H}_{2-k}(M, B; R)$

In the last chapter we had a lot of fun applying the Poincaré Duality Theorem 210.1 to various settings. We obtained many interesting, sometimes perhaps also surprising, results. But now party time is over, we have to provide a proof of the theorem. In fact the whole chapter is completely dedicated to the proof of the theorem.

This chapter is organized as follows:

- In Section 211.1 we will use cohomology with compact support to formulate the Poincaré Duality–Non-Compact Theorem 211.4 which is a version of the Poincaré Duality Theorem 210.1 for topological manifolds with empty boundary but which are possibly non-compact.
- In Section 211.2 we provide the rather lengthy proof of the Poincaré Duality–Non-Compact Theorem 211.4. This gives us in particular a proof of the Poincaré Duality Theorem 210.1 for compact topological manifolds with empty boundary. This is certainly the key section of the whole chapter.
- In Section 211.3 we will use the Poincaré Duality–Non-Compact Theorem 211.4, and some formal tricks, to provide a proof of the Poincaré Duality Theorem 210.1 for the general case.
- In Appendix 211.4 we will relate Poincaré Duality on a manifold to Poincaré Duality on its boundary.
- In Section 211.5 we will provide exercises for Chapter 211.

In the next Chapter 212 we will discuss the interaction between Poincaré Duality and codimension-zero submanifolds.

211.1. **Poincaré Duality for non-compact manifolds with empty boundary.** In this section we will formulate a version of Poincaré Duality for every oriented topological manifold (not necessarily compact) with empty boundary.

Why do we do this? Here are a few answers:

(1) The key idea for the proof of Poincaré Duality Theorem 210.1 for *closed* topological manifolds is somewhat similar to the strategy of the proof of the Homology-Section Theorem 193.6: Using Mayer–Vietoris arguments and using limit arguments we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$.

(2) To make sense of the approach in (1) we need to allow in between topological manifolds that are non-compact. Thus we need to make sense of Poincaré Duality for non-compact manifolds.

(3) Later, when we study Poincaré Duality for topological manifolds we will see that relative cohomology $\mathrm{H}^k(M, \partial M; R)$ is actually isomorphic to $\mathrm{H}^k_c(M \setminus \partial M; R)$. Thus Poincaré Duality for non-compact manifolds will also allow us to deal with some cases of Poincaré Duality for compact manifolds with non-empty boundary.

The formulation of Poincaré Duality for (possibly non-compact) topological manifolds with empty boundary requires some preparations. First we recall some definitions and results from earlier on:

Notation 211.1. Let R be a commutative ring. Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. By the H^n_c-via-Direct Colimits Proposition 209.15 we have for every $k \in \mathbb{N}_0$ a natural isomorphism

$$\operatorname*{colim}_{K \in \mathcal{K}(X)} \mathrm{H}^k(X, X \setminus K; R) \xrightarrow{\cong} \mathrm{H}^k_c(X; R)$$

which we will use to identify these two groups. In fact in this chapter we will only work with the group on the left and we will not make use at all of the original definition of cohomology with compact support.

Notation 211.2. Let R be a commutative ring and let M be an n-dimensional topological manifold with empty boundary. If the topological manifold M is equipped with an R-orientation $\{\alpha_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)\}_{x \in M}$, then we saw in the Homology-Section Theorem 193.6 (i) that given any compact subset K of M there exists a unique homology class $\mu_K^M \in \mathrm{H}_n(M, M \setminus K; R)$ such that for every point $x \in K$ the image of the homology class μ_K^M under the natural homomorphism $\mathrm{H}_n(M, M \setminus K; R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$ equals the given orientation α_x at x. If M is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ_K^M.



With this notation we can now formulate the following lemma:

Lemma 211.3. (Cap–Non Compact Manifold Lemma) Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary.

(1) Let $K \subseteq L \subseteq M$ be compact subsets. We denote by $f_{KL} \colon (M, M \setminus L) \to (M, M \setminus K)$ the inclusion map of pairs.
   (a) The map $f_{KL*} \colon \mathrm{H}_n(M, M \setminus L; R) \to \mathrm{H}_n(M, M \setminus K; R)$ sends μ_L to μ_K.
   (b) For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(M, M \setminus L; R) & \xrightarrow{\cap \mu_L} & \mathrm{H}_{n-k}(M; R). \\ \uparrow{\scriptstyle f_{KL}^*} & & \\ \mathrm{H}^k(M, M \setminus K; R) & \xrightarrow{\cap \mu_K} & \end{array}$$

(2) For every $k \in \mathbb{N}_0$ there exists a unique homomorphism

$$- \cap M \colon \mathrm{H}^k_{\mathrm{c}}(M; R) \to \mathrm{H}_{n-k}(M; R)$$

such that for each compact subset $K \subseteq M$ the following diagram commutes:[2245]

$$\begin{array}{ccc} \mathrm{H}^k(M, M \setminus K; R) & \xrightarrow{\cap \mu_K} & \mathrm{H}_{n-k}(M; R). \\ & \searrow \quad \nearrow_{\cap M} & \\ & \mathrm{H}^k_{\mathrm{c}}(M; R) & \end{array}$$

Proof. Let M be an n-dimensional topological manifold with empty boundary that is equipped with an R-orientation $\{\alpha_x\}_{x \in M}$.

(1) (a) We need to show that $f_{KL*}(\mu_L)$ has the defining property of μ_K. But this is indeed the case since for every $x \in K$ we have, by functoriality of homology, the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, M \setminus L; R) & \xrightarrow{f_{KL*}} & \mathrm{H}_n(M, M \setminus K; R) \\ & {\scriptstyle \mu_L \mapsto \alpha_x} \searrow \quad \swarrow & \\ & \mathrm{H}_n(M, M \setminus \{x\}; R). & \end{array}$$

   (b) Let $\varphi \in \mathrm{H}^k(M, M \setminus K; R)$. We have the following equality in $\mathrm{H}_{n-k}(M; R)$:

$$\varphi \cap \mu_K \underset{\text{part (1a)}}{=} \varphi \cap f_{KL*}(\mu_L) \underset{\text{by the Cap Product-Naturality Lemma 206.13}}{=} f_{KL}^*(\varphi) \cap \mu_L.$$

   But this means precisely that the diagram under consideration commutes.

(2) We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^n(M, M \setminus L; R) & & \xrightarrow{\cap \mu_L} & & \\ \uparrow{\scriptstyle f_{KL}^*} & \searrow & & & \\ & \mathrm{H}^n_{\mathrm{c}}(M; R) =_{\mathrm{i}} \underset{K \in \mathcal{K}(M)}{\operatorname{colim}} \mathrm{H}^n(M, M \setminus K; R) & \xrightarrow[\cap M]{\exists!} & \mathrm{H}_{n-k}(M; R). \\ \mathrm{H}^n(M, M \setminus K; R) & \nearrow & \xrightarrow{\cap \mu_K} & & \end{array}$$

[2245] Note that for a class $\varphi \in \mathrm{H}^k(M, M \setminus K; R)$ and $\mu_K \in \mathrm{H}_n(M, M \setminus K; R)$ the cap product $\varphi \cap \mu_K$ does indeed lie in $\mathrm{H}_{n-k}(M; R)$.

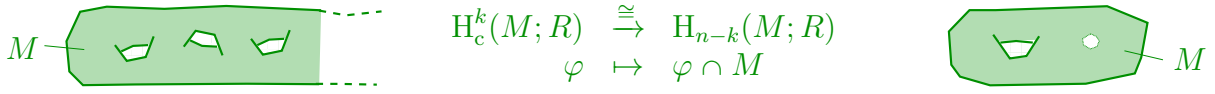
The outer diagram commutes by (1b). It now follows from the universal property of the direct colimit that we get a uniquely defined homomorphism $- \cap M$ which makes the diagram commute. ■

We can now formulate the promised result on Poincaré Duality for possibly non-compact topological manifolds:

Theorem 211.4. (Poincaré Duality–Non-Compact Theorem) Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} - \cap M \colon & \mathrm{H}_{\mathrm{c}}^k(M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;R) \\ & \varphi & \mapsto & \varphi \cap M \end{array}$$

from the Cap–Non Compact Manifold Lemma 211.3 (2) is an isomorphism.



Proof. We provide the proof of the theorem in Section 211.2. ■

Example. Let $n \in \mathbb{N}_0$.

(1) Let M be an oriented connected non-empty n-dimensional topological manifold with empty boundary. Then

$$\mathrm{H}_n(M;\mathbb{Z}) \underset{\uparrow}{\overset{\cong}{\longleftarrow}} \mathrm{H}_{\mathrm{c}}^0(M;\mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases}$$

by the Poincaré Duality Non-Compact Theorem 211.4 — see page 4007

This calculation is consistent with the results obtained in the Orientation-Top-H_*-Theorem 193.3 and the Manifold-Top-H_*-Theorem 193.1.

(2) We note that

$$\mathrm{H}_{\mathrm{c}}^n(\mathbb{R}^n;\mathbb{Z}) \underset{\uparrow}{\xrightarrow{\cong}} \mathrm{H}_0(\mathbb{R}^n;\mathbb{Z}) \underset{\uparrow}{\cong} \mathbb{Z}.$$

by the Poincaré Duality Non-Compact Theorem 211.4 — by the H_0-Proposition 143.25

This is consistent with the calculation on page 4016. □

Remark. Given a possibly non-compact oriented topological manifold with empty boundary the Poincaré Duality–Non-Compact Theorem 211.4 allows us to relate singular homology to cohomology with compact support. There is also a different version of Poincaré Duality that relates singular cohomology to a different flavor of homology, namely Borel-Moore homology. For topological manifolds Borel-Moore homology seems to be isomorphic to locally finite homology which we introduced and studied in Exercises 194.18 and **??**. We refer to [**Greenb1967**], [**Bre1997**, Chapter V.9] and [**PeS2008**, p. 332] for details. □

On a few occasions we will need a naturality statement regarding the map in the above Poincaré Duality–Non-Compact Theorem 211.4. To formulate the lemma let us recall Definition 209.16:

Notation 211.5. Let X be a Hausdorff space, let U be an open subset of X, let $k \in \mathbb{N}_0$ and let G be an abelian group. As in Definition 209.16 let $\iota_* = \iota_{XU*} \colon \mathrm{H}^k_{\mathrm{c}}(U;G) \to \mathrm{H}^k_{\mathrm{c}}(X;G)$ be the homomorphism that is defined by

$$\mathrm{H}^k_{\mathrm{c}}(U) =_{\mathrm{i}} \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(U, U \setminus K) \xleftarrow{\cong} \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(X, X \setminus K) \to \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^k(X, X \setminus K) =_{\mathrm{i}} \mathrm{H}^k_{\mathrm{c}}(X).$$

(We dropped "G" for space reasons.)

Now we are ready to formulate the promised naturality statement:

Lemma 211.6. (Poincaré Duality–Open Subset Lemma) Let R be a commutative ring, let M be an R-oriented n-dimensional topological manifold with empty boundary and let U be an open subset of M. We denote by $\iota \colon U \to M$ the inclusion. For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{c}}(U;R) & \xrightarrow{\iota_*} & \mathrm{H}^k_{\mathrm{c}}(M;R) \\ \downarrow{\scriptstyle \cap U} & & \downarrow{\scriptstyle \cap M} \\ \mathrm{H}_{n-k}(U;R) & \xrightarrow{\iota_*} & \mathrm{H}_{n-k}(M;R). \end{array}$$

Proof. Let $U \subseteq M$ be an open subset.

Claim. For every compact subset $K \subseteq U$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(U, U \setminus K; R) & \xrightarrow[\cong]{(\iota^*)^{-1}} & \mathrm{H}^k(M, M \setminus K; R) \\ \downarrow{\scriptstyle \cap \mu^U_K} & & \downarrow{\scriptstyle \cap \mu^M_K} \\ \mathrm{H}_k(U;R) & \xrightarrow{\iota_*} & \mathrm{H}_{n-k}(M;R). \end{array}$$

Proof. Given any $\phi \in \mathrm{H}^k(M, M \setminus K; R)$ we calculate that

$$\begin{array}{ccccc} \iota_*(\iota^*(\phi) \cap \mu^U_K) & = & \phi \cap \iota_*(\mu^U_K) & = & \phi \cap \mu^M_K. \\ & \uparrow & & \uparrow & \\ & \text{by the Cap Product Naturality Lemma 206.13} & & \text{by the functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 193.6 we have } \iota_*(\mu^U_K) = \mu^M_K & \end{array}$$

⊞

Next we consider the following diagram (we drop the R-coefficients for space reasons):

$$\begin{array}{ccccccccc} \mathrm{H}^k_{\mathrm{c}}(U;\mathbb{Z}) & \xleftarrow{\cong} & \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(U, U \setminus K) & \xrightarrow[\cong]{(\iota^*)^{-1}} & \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(M, M \setminus K) & \longrightarrow & \varinjlim_{K \in \mathcal{K}(M)} \mathrm{H}^k(M, M \setminus K) & \xrightarrow{\cong} & \mathrm{H}^k_{\mathrm{c}}(M) \\ \downarrow{\scriptstyle \cap U} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cap M} \\ \mathrm{H}_{n-k}(U) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(U) & \xrightarrow{\iota_*} & \mathrm{H}_{n-k}(M) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(M) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(M). \end{array}$$

(Here the unnamed vertical maps are induced by capping with μ^U_K and μ^M_K.) The squares to the left and right commute by the definition of the maps "$\cap U$" and "$\cap M$". It follows from the claim that the second square commutes. Finally note that it follows easily from the universal property of direct colimits that the third square commutes. ■

211.2. **Proof of the Poincaré Duality–Non-Compact Theorem 211.4 .** The proof of the Poincaré Duality–Non-Compact Theorem 211.4 takes place in the following three subsections:

(1) In Subsection 211.2.1 we prove the desired statement for $M = B^n$.

(2) In Subsection 211.2.2 we will state and prove the Poincaré Duality–Mayer Vietoris Proposition 211.8 which allows us to deal with unions of open subsets which satisfy Poincaré Duality. This subsection contains technically the hardest argument for the proof of the Poincaré Duality–Non-Compact Theorem 211.4.
(3) In Subsection 211.2.3 we will use (1) and (2) to conclude the proof of the Poincaré Duality–Non-Compact Theorem 211.4. In contrast to the proof of (2), this argument is again quite pleasant to read.

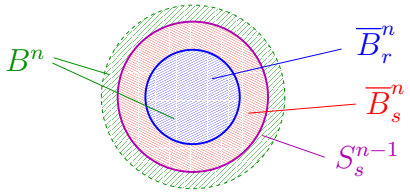
211.2.1. **Poincaré Duality for Balls.** As a warm up for the proof of the Poincaré Duality–Non-Compact Theorem 211.4 we will deal with the case $M = B^n$ by hand. More precisely, we will prove the following lemma:

Lemma 211.7. (PD-for-Ball Lemma) Let R be a commutative ring. For each $k \in \mathbb{N}_0$ the map[2246]

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{c}}(B^n;R) & \to & \mathrm{H}_{n-k}(B^n;R) \\ \varphi & \mapsto & \varphi \cap B^n \end{array}$$

from the Cap–Non Compact Manifold Lemma 211.3 (2) is an isomorphism.

Proof. Let $k \in \mathbb{N}_0$. Given any $s \in (0,1)$ we write $\overline{B}^n_s := \{x \in \mathbb{R}^n \mid \|x\| \leq s\}$ and we also write $S^{n-1}_s := \partial \overline{B}^n_s$. For the remainder of the proof we fix $0 < r < s < 1$. We denote by $i\colon (\overline{B}^n_s, S^{n-1}_s) \to (B^n, B^n \setminus \overline{B}^n_r)$ the inclusion map of pairs.



We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(\overline{B}^n_s, S^{n-1}_s;R) & \xleftarrow{i^*} & \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r;R) & \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(B^n;R) \\ \downarrow{\scriptstyle \cap [\overline{B}^n_s]} & & \downarrow{\scriptstyle \cap \mu_{\overline{B}^n_r}} & & \downarrow{\scriptstyle \cap B^n} \\ \mathrm{H}_{n-k}(\overline{B}^n_s;R) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(B^n;R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(B^n;R). \end{array}$$

We need to show that the vertical map to the right is an isomorphism. The lemma is an immediate consequence of the following claim:

Claim.
(1) The diagram commutes.
(2) The left vertical map is an isomorphism.
(3) All horizontal maps are isomorphisms.

Proof.

[2246] We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation which is provided by the Standard Euclidean Orientation Lemma 191.19.

(1) The square on the right commutes by definition of the map $- \cap B^n$. So it remains to show that the square on the left commutes. Thus let $\varphi \in \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r; R)$. We have

$$i_*(i^*\varphi \cap [\overline{B}^n_s]) \underset{\uparrow}{=} \varphi \cap i_*([\overline{B}^n_s]) \underset{\uparrow}{=} \varphi \cap \mu_{\overline{B}^n_r} \in \mathrm{H}_{n-k}(B^n; R).$$

by the Cap Product-Naturality Lemma 206.13

we have $i_*([\overline{B}^n_s]) = \mu_{\overline{B}^n}$ since for each $x \in \overline{B}^n_r$ the image of $i_*([\overline{B}^n_s])$ in $\mathrm{H}_n(B^n, B^n \setminus \{x\})$ equals the generator given by the orientation

(2) We consider the left vertical map of the diagram. Clearly both groups are zero for $k \neq n$. So it remains to consider the case $k = n$. It follows from the Cap Product-Kronecker Lemma 206.12 together with the discussion on page 3079 that it remains to show that the map

$$\begin{array}{rcl} \mathrm{H}^n(\overline{B}^n_s, S^{n-1}_s; R) & \to & R \\ \varphi & \mapsto & \langle \varphi, [\overline{B}^n_s] \rangle = \epsilon_{\overline{B}^n_s}(\varphi \cap [\overline{B}^n_s]) \end{array}$$

augmentation homomorphism from Definition 143.23

is an isomorphism. But this statement is an immediate consequence of the Manifold-Top-H*-Theorem 201.26 together with the Ring–Kronecker Pairing Lemma 202.1 (2) and the discussion on page 3751.

(3) It follows easily from the Homotopy Equivalence-Relative-H$_*$-Corollary 146.8 (1) and the Homotopy Equivalence-H*-Proposition 198.12 (2) (say using the intermediate pair (B^n, S^{n-1}_s)) that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r; R) & \longrightarrow & \varinjlim_{K \in \mathcal{K}(B^n)} \mathrm{H}^k(B^n, B^n \setminus K; R) =_{\mathrm{i}} \mathrm{H}^k_{\mathrm{c}}(B^n; R). \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t \in (0,1)} \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_t; R) & \end{array}$$

We make the following three observations:

(a) It follows easily from the definitions that the diagram commutes.

(b) Every compact subset of B^n is contained in a closed ball $\overline{B}^n_t$ with $t \in (0, 1)$. Therefore it follows from the Cofinal-Isomorphism Lemma 209.13 that the right diagonal map is an isomorphism.

(c) It follows easily from the earlier Homotopy Equivalence-H*-Proposition 198.12 (1) that for each $t > r$ the inclusion-induced map $\mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r; R) \to \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_t; R)$ is an isomorphism. It follows from this observation together with the Direct Colimit–Stabilization Lemma 104.5 that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map in the above triangular diagram is also an isomorphism. ■

211.2.2. **Poincaré Duality for unions.** The key to the proof of the Poincaré Duality–Non-Compact Theorem 211.4 is the following proposition:

Proposition 211.8. (Poincaré Duality–Mayer Vietoris Proposition) Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with

empty boundary that is the union of two open subsets U and V. Then there exists a diagram of homomorphisms (where we drop the "R" for space reasons)

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \mathrm{H}_{\mathrm{c}}^k(U\cap V) & \xrightarrow{(\iota^*,-\iota^*)} & \mathrm{H}_{\mathrm{c}}^k(U)\oplus\mathrm{H}_{\mathrm{c}}^k(V) & \xrightarrow{\iota^*\oplus\iota^*} & \mathrm{H}_{\mathrm{c}}^k(M) & \longrightarrow & \mathrm{H}_{\mathrm{c}}^{k+1}(U\cap V) & \longrightarrow \dots \\
& \downarrow{\scriptstyle \cap(U\cap V)} & & \downarrow{\scriptstyle \cap U\oplus\cap V} & & \downarrow{\scriptstyle \cap M} & & \downarrow{\scriptstyle \cap(U\cap V)} & \\
\dots \to & \mathrm{H}_{n-k}(U\cap V) & \xrightarrow{(\iota_*,-\iota_*)} & \mathrm{H}_{n-k}(U)\oplus\mathrm{H}_{n-k}(V) & \xrightarrow{\iota_*\oplus\iota_*} & \mathrm{H}_{n-k}(M) & \to & \mathrm{H}_{n-k-1}(U\cap V) & \to \dots
\end{array}$$

with the following properties:

(1) The first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$.
(2) The lower exact sequence is the long exact Mayer–Vietoris sequence for $M = U \cup V$.
(3) The upper horizontal sequence is also exact.

The remainder of this subsection is concerned with the proof of the Poincaré Duality–Mayer Vietoris Proposition 211.8. This proof is rather long and technical and, frankly, not particularly enlightening. For most readers the best thing to do is to jump straight to Subsection 211.2.3 where we conclude the proof of the Poincaré Duality–Non-Compact Theorem 211.4.

Since you decided to stick around for the proof of the Poincaré Duality–Mayer Vietoris Proposition 211.8, let us get started. Among many other things we will have to show that the top horizontal sequence is exact. The key tool is an "inverted" Mayer–Vietoris Theorem, which is very similar to the Inverted Mayer–Vietoris Theorem 193.8.

Before we can state this result we need to recall Definition 204.3:

Notation 211.9. Let (X, A, B) be a triad of topological spaces. For each $k \in \mathbb{N}_0$ we write

$$\mathrm{C}_k(X,\{A,B\}) \;:=\; \frac{\mathrm{C}_*(X)}{\mathrm{C}_k^{\{A,B\}}(A\cup B)} \;=\; \frac{\mathrm{C}_*(X)}{\mathrm{C}_k(A)+\mathrm{C}_k(B)}.$$

Given any abelian group G we obtain the corresponding cochain complex $\mathrm{C}^*(X,\{A,B\};G)$ and the corresponding cohomology groups $\mathrm{H}^k(X,\{A,B\};G)$.

We also make use of the following convention:

Convention 211.10. Given a topological space X, given subspaces $A \subseteq B \subseteq X$ and given $\varphi \in \mathrm{C}^k(X,B;G)$ we denote by φ also the image of φ under the natural homomorphism $\mathrm{C}^k(X,B;G) \to \mathrm{C}^k(X,A;G)$.

Theorem 211.11. (Inverted Mayer–Vietoris Theorem) Let X be a topological space, let A and B be open subsets and let G be an abelian group.

(1) For every $n \in \mathbb{N}_0$ and every cochain $\varphi \in \mathrm{C}^n(X, A\cap B; G)$ there exist $\varphi_A \in \mathrm{C}^n(X,A;G)$ and $\varphi_B \in \mathrm{C}^n(X,B;G)$ such that[2247] $\varphi_A + \varphi_B = \varphi$.
(2) For all $n \in \mathbb{N}_0$ there exists a unique homomorphism

$$\delta_n\colon \mathrm{H}^n(X,A\cap B;G) \;\to\; \mathrm{H}^{n+1}(X,\{A,B\};G)$$

which has the following property: For every $\varphi_A \in \mathrm{C}^n(X,A;G)$ and $\varphi_B \in \mathrm{C}^n(X,B;G)$ such that $\varphi_A+\varphi_B$ is a cocycle in $\mathrm{C}^n(X,A\cap B;G)$ we have

$$\delta_n([\varphi_A+\varphi_B]) \;=\; [\delta_n(\varphi_A)] \;=\; -[\delta_n(\varphi_B)] \;\in\; \mathrm{H}^{n+1}(X,\{A,B\};G).$$

(3) The natural map $\xi\colon \mathrm{H}^k(X,A\cup B;G) \to \mathrm{H}^k(X,\{A,B\};G)$ is an isomorphism.

(4) The sequence

$$\cdots \to \mathrm{H}^n(X, A\cup B; G) \xrightarrow{(i^*,-i^*)} \mathrm{H}^n(X, A; G)\oplus \mathrm{H}^n(X, B; G) \xrightarrow{i^*\oplus i^*} \mathrm{H}^n(X, A\cap B; G) \xrightarrow{\xi^{-1}\circ\delta_n} \mathrm{H}^{n+1}(X, A\cup B; G) \to \dots$$

is exact.

Sketch of proof. First note that it is pretty clear that the following sequence of chain maps between chain complexes is exact:

$$0 \to \mathrm{C}_*(X, A\cap B) \xrightarrow{(i_*,i_*)} \mathrm{C}_*(X, A)\oplus \mathrm{C}_*(X, B) \xrightarrow{i_*\oplus(-i_*)} \mathrm{C}_*(X, \{A, B\}) \to 0.$$

Note that the group to the right is free abelian. It follows from the Free Abelian Group–Splitting Lemma 97.11 together with Hom-Split-SES Lemma 197.8 that the following induced sequence of cochain maps between cochain complexes is also exact:

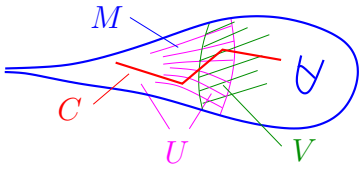
$$0 \to \mathrm{C}^*(X, \{A, B\}; G) \xrightarrow{(i^*,-i^*)} \mathrm{C}^*(X, A; G)\oplus \mathrm{C}^*(X, B; G) \xrightarrow{i^*\oplus i^*} \mathrm{C}^*(X, A\cap B; G) \to 0.$$

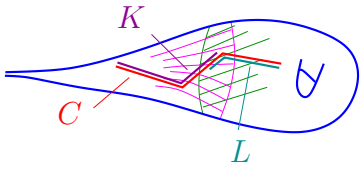
We now turn to the proof of the four statements:

(1) This statement follows immediately from the above short exact sequence.
(2) This statement follows from the above short exact sequence and the explicit description of the connecting homomorphism in Definition 145.8.
(3) This statement follows from the Excisive Triad–Invariants Lemma 204.4 (2b).
(4) As in the LES-Proposition 145.10 and in the Cohomology–LES Proposition 197.17 we get a long exact sequence in cohomology involving the group $\mathrm{H}^*(X; \{A, B\}; G)$. Using (3) we obtain the desired long exact sequence. ■

In the proof of the Poincaré Duality–Non-Compact Theorem 211.4 we will also need the following two lemmas.

Lemma 211.12. (Compact Subset Decomposition Lemma) Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Then given any compact subset $C \subseteq M$ there exist compact subsets $K \subseteq U$ and $L \subseteq V$ such that $C = K \cup L$.





Proof of Lemma 211.12. Let M be an n-dimensional topological manifold and let U, V be open subsets with $M = U \cup V$. Furthermore let C be a compact subset of M. Using charts one can easily show that given any point $x \in C$ there exists a compact set A_x with the following properties:

(1) The point x lies in the interior of A_x.
(2) If $x \in U$ then the set A_x is contained in U.
(3) If $x \in V$ then the set A_x is contained in V.

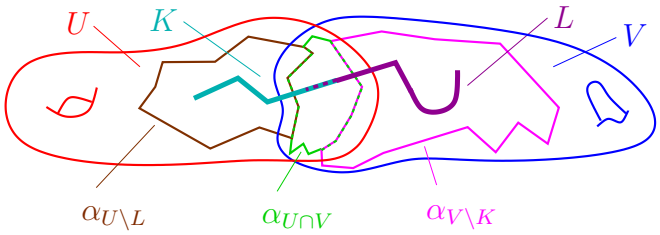
It follows from (1), the fact that the sets A_x cover all of C together with the hypothesis that C is compact that there exist finitely many $y_1, \dots, y_r \in C$ with $C \subseteq \mathring{A}_{y_1} \cup \cdots \cup \mathring{A}_{y_r}$.

[2247] Here we make use of the above convention.

One can now easily verify that $K := \bigcup_{y_i \in U} (A_{y_i} \cap C)$ and $L := \bigcup_{y_i \in V} (A_{y_i} \cap C)$ have the desired properties. ■

Lemma 211.13. (Fundamental Cycle Decomposition Lemma) Let R be a commutative ring, let M be an R-oriented n-dimensional topological manifold, let U, V be open subsets of M such that $M = U \cup V$ and let $K \subseteq U$ and $L \subseteq V$ be compact subsets. Then there exist singular n-chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:

(a)	$\mu^M_{K\cup L}$	$=$	$[\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$	in $\mathrm{H}_n(M, M \setminus (K \cup L); R)$
(b)	$\mu^{U\cap V}_{K\cap L}$	$=$	$[\alpha_{U\cap V}]$	in $\mathrm{H}_n(U \cap V, (U \cap V) \setminus (K \cap L); R)$
(c)	μ^U_K	$=$	$[\alpha_{U\setminus L} + \alpha_{U\cap V}]$	in $\mathrm{H}_n(U, U \setminus K; R)$.



Proof of Lemma 211.13. We pick a representative $\alpha \in \mathrm{C}_n(M; R)$ for $\mu^M_{K\cup L}$. Note that the open subsets $U \setminus L, U \cap V$ and $V \setminus K$ cover M. It follows easily from the Iterated Subdivision Lemma 148.14 and the Simplex–Reduction Lemma 148.20 that there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that α is homologous in $\mathrm{C}_k(M; R)$ to $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$. The chains are in particular homologous in the relative chain complex $\mathrm{C}_k(M, M \setminus (K \cup L); R)$ which implies that (a) holds. Next note that we have

$$\mu^M_{K\cap L} \underset{\uparrow}{=} \text{image of } \mu^M_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] \underset{\uparrow}{=} [\alpha_{U\cap V}] \in \mathrm{H}_n(M, M \setminus (K \cap L); R).$$

by the Cap–Non Compact Manifold Lemma 211.3 (1)

since $\alpha_{U\setminus L}, \alpha_{V\setminus K}$ lie in $\mathrm{C}_n(M \setminus (K \cap L); R)$

But $\alpha_{U\cap V}$ lies in $U \cap V$, therefore it follows from functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 193.6 that $\mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U \cap V, (U \cap V) \setminus (K \cap L); R)$. We have thus verified (b).

Finally the proof of equality (c) is basically the same as the proof of equality (b). ■

Now we are finally in a position to prove the Poincaré Duality–Mayer Vietoris Proposition 211.8:

Proof of the Poincaré Duality–Mayer Vietoris Proposition 211.8. Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary. For space reasons we will drop the "R" throughout the proof of this proposition.

Given any two subsets $X \subseteq Y$ of M and given $i \in \mathbb{N}_0$ we write, as in Notation 193.5, $\mathrm{H}^i(Y|X) := \mathrm{H}^i(Y, Y \setminus X)$. We note that if U is open and $K \subseteq U$ is compact, then we obtain a homomorphism

$$- \cap \mu^U_K \colon \mathrm{H}^k(U|K) \to \mathrm{H}_{n-k}(U).$$

Now let U and V be open subsets with $M = U \cup V$. Furthermore let $K \subseteq U$ and $L \subseteq V$ be compact subsets. We consider the following diagram:

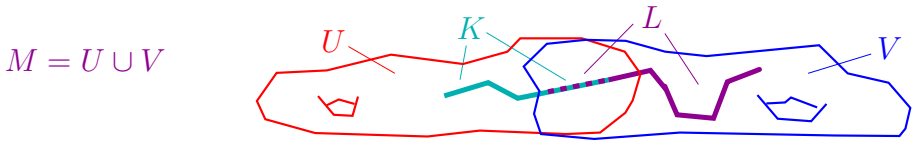
$$\begin{array}{ccccccccc}
\ldots \to \mathrm{H}^k(M|K\cap L) & \to & \mathrm{H}^k(M|K)\oplus\mathrm{H}^k(M|L) & \longrightarrow & \mathrm{H}^k(M|K\cup L) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M|K\cap L) & \to \ldots \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & \\
\mathrm{H}^k(U\cap V|K\cap L) & & \mathrm{H}^k(U|K)\oplus\mathrm{H}^k(V|L) & & \downarrow\cap\mu^M_{K\cup L} & & \mathrm{H}^{k+1}(U\cap V|K\cap L) & \\
\downarrow\cap\mu^{U\cap V}_{K\cap L} & & \downarrow\cap\mu^U_K\oplus\cap\mu^V_L & & & & \downarrow\cap\mu^{U\cap V}_{K\cap L} & \\
\ldots \to \mathrm{H}_{n-k}(U\cap V) & \longrightarrow & \mathrm{H}_{n-k}(U)\oplus\mathrm{H}_{n-k}(V) & \longrightarrow & \mathrm{H}_{n-k}(M) & \xrightarrow{\partial_{n-k}} & \mathrm{H}_{n-k-1}(U\cap V) & \to \ldots
\end{array}$$

which we now explain in more detail:

- We write $A = M \setminus K$ and $B = M \setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology groups from the Inverted Mayer–Vietoris Theorem 211.11.
- The lower horizontal sequence is the Mayer-Vietoris sequence coming from the Mayer–Vietoris Theorem 153.2 (1), corresponding to the decomposition $M = U \cup V$.
- The three vertical maps on the top are induced by inclusions of pairs of topological spaces, they are isomorphisms by the Excision-in-H*-Theorem 200.1 (1a).

The following is the key claim of the proof of the proposition:

Claim. The first two rectangles commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.



Before we prove the claim we show that it implies our proposition. For each group appearing in the above diagram we take the direct colimit over the directed set[2248]

$$\mathcal{K}(U,V) \;:=\; \{(K,L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

and we consider the induced maps of direct colimits that were introduced in Definition 104.9. We make the following remarks regarding this new diagram:

(1) Using the Cofinal-Isomorphism Lemma 209.13 one can show that the groups that we obtain taking the direct colimit over $\mathcal{K}(U,V)$ are naturally isomorphic to the corresponding cohomology groups with compact support. We carry out the argument for two groups:

$$\varinjlim_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k(U\cap V|K\cap L) \;=\; \varinjlim_{C\in\mathcal{K}(U\cap V)} \mathrm{H}^k(U\cap V|C) \;=\; \mathrm{H}^k_{\mathrm{c}}(U\cap V).$$

we apply the Cofinal-Isomorphism Lemma 209.13 to the cofinal morphism of directed sets that is given by $(K,L) \mapsto K \cap L$

and

[2248]We view $\mathcal{K}(U,V)$ as a directed set given by the relation $(K,L) \leq (K',L')$ if $K \subseteq K'$ and $L \subseteq L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

$$\operatorname*{colim}_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k(M|K\cup L) \;=\; \operatorname*{colim}_{C\in\mathcal{K}(M)} \mathrm{H}^k(M|C) \;=\; \mathrm{H}^k_{\mathrm{c}}(M;\mathbb{Z}).$$

we apply the Cofinal-Isomorphism Lemma 209.13 to the morphism of directed sets that is given by $(K, L) \mapsto K \cup L$, this morphism is cofinal by the Compact Subset Decomposition Lemma 211.12

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct colimits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in the Cap–Non Compact Manifold Lemma 211.3 (2) one can easily verify that the direct colimit of the diagram does indeed give rise to the diagram specified in the statement of the lemma.

(4) The Direct Colimit-Exactness Proposition 154.6 says that the direct colimit of exact sequences is again exact.

All these remarks put together show that the direct colimit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the proposition.

Proof. Now we turn to the actual proof of the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of the Cap Product-Naturality Lemma 206.13. The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in (co-) homology.

Subclaim 1. The first rectangle of the diagram on page 4053 commutes.

So we need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^k(M, M\setminus(K\cap L)) & \xrightarrow{(i^*,-i^*)} & \mathrm{H}^k(M, M\setminus K)\oplus \mathrm{H}^k(M, M\setminus L;\mathbb{Z}) \\
\downarrow\cong & \searrow^{(i^*,-i^*)} & \downarrow\cong \\
\mathrm{H}^k(U\cap V, (U\cap V)\setminus(K\cap L)) & & \mathrm{H}^k(U, U\setminus K)\oplus \mathrm{H}^k(V, V\setminus L) \\
\downarrow\cap\mu^{U\cap V}_{K\cap L} & & \downarrow\cap\mu^U_K\oplus\cap\mu^V_L \\
\mathrm{H}_{n-k}(U\cap V) & \xrightarrow{(i_*,-i_*)} & \mathrm{H}_{n-k}(U)\oplus \mathrm{H}_{n-k}(V).
\end{array}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand summands. So let $\phi \in \mathrm{H}^k(M, M\setminus(K\cap L))$. We consider the following diagram of continuous maps between pairs of topological spaces

$$\begin{array}{ccc}
(U\cap V, (U\cap V)\setminus(K\cap L)) & \xrightarrow{\alpha} & (U, U\setminus(K\cap L)) \\
\downarrow{\scriptstyle f\circ\alpha} & \swarrow_{f} & \uparrow{\scriptstyle \beta} \\
(M, M\setminus(K\cap L)) & \xleftarrow[f\circ\beta]{} & (U, U\setminus K).
\end{array}$$

With this notation we need to show that the following equality holds in $\mathrm{H}_{n-k}(U)$:

$$\alpha_*((f\circ\alpha)^*(\phi)\cap\mu^{U\cap V}_{K\cap L}) \;=\; (f\circ\beta)^*(\phi)\cap\mu^U_K.$$

But this is indeed the case, since

$$\alpha_*(\underbrace{(f\circ\alpha)^*}_{=\alpha^*\circ f^*}(\phi)\cap\mu_{K\cap L}^{U\cap V}) \overset{\text{by the Cap Product-Naturality Lemma 206.13 applied to the map } \alpha\colon (U\cap V,(U\cap V)\setminus(K\cap L),\varnothing)\to(U,U\setminus(K\cap L),\varnothing) \text{ of triples of topological spaces}}{=} f^*(\phi)\cap\alpha_*(\mu_{K\cap L}^{U\cap V}) \overset{\text{by the functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 193.6 we have } \alpha_*(\mu_{K\cap L}^{U\cap V})=\mu_{K\cap L}^{U}}{=} f^*(\phi)\cap\mu_{K\cap L}^{U}$$

$$\overset{\text{same argument backwards with the roles of } \alpha \text{ and } \beta \text{ exchanged}}{=} \beta_*((f\circ\beta)^*(\phi)\cap\mu_K^U) \overset{\text{since the map } \beta \text{ is the identity on the first entry of the pair}}{=} (f\circ\beta)^*(\phi)\cap\mu_K^U$$

□

Subclaim 2. The second rectangle of the diagram on page 4053 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. Thus we leave it to the reader to fill in the details. □

Subclaim 3. The third rectangle of the diagram on page 4053 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{array}{ccccc}
\mathrm{H}^k(M,M\setminus(K\cup L)) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M,M\setminus(K\cap L)) & \xrightarrow{\cong} & \mathrm{H}^{k+1}(U\cap V,(U\cap V)\setminus(K\cap L)) \\
\downarrow{\scriptstyle \cap\mu_{K\cup L}} & & & & \downarrow{\scriptstyle \cap\mu_{K\cap L}} \\
\mathrm{H}_{n-k}(M) & & \xrightarrow{\qquad\qquad\partial_{n-k}\qquad\qquad} & & \mathrm{H}_{n-k-1}(U\cap V).
\end{array}$$

So let $[\varphi]\in\mathrm{H}^k(M,M\setminus(K\cup L))=\mathrm{H}^k(M,A\cap B)$. We start out with the following preparations:

(1) By the Inverted Mayer–Vietoris Theorem 211.11 (1) we can find $\varphi_A\in\mathrm{C}^k(M,A)$ and $\varphi_B\in\mathrm{C}^k(M,B)$ such that $\varphi=\varphi_A+\varphi_B$.

(2) By the Fundamental Cycle Decomposition Lemma 211.13 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

(2a) $\mu_{K\cup L} = [\alpha_{U\setminus L}+\alpha_{U\cap V}+\alpha_{V\setminus K}]$ in $\mathrm{H}_n(M,M\setminus(K\cup L))$

(2b) $\mu_{K\cap L}^{U\cap V} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V,(U\cap V)\setminus(K\cap L))$

(2c) $\mu_K^U = [\alpha_{U\setminus L}+\alpha_{U\cap V}]$ in $\mathrm{H}_n(U,U\setminus K)$.

After these preparations we obtain the following equalities in $\mathrm{H}_{n-k-1}(U \cap V)$:[2249]

$$\begin{array}{l}
\text{maps in the rectangle applied clockwise to some } [\varphi] \in \mathrm{H}^k(M, M \setminus (K \cup L)) \\
\quad = \delta_k([\varphi]) \cap \mu_{K \cap L}^{U \cap V} \overset{\text{by the Inverted Mayer–Vietoris Theorem 211.11 and (2b)}}{=} [\delta_k(\varphi_A) \cap \alpha_{U \cap V}] \\
= [\varphi_A \cap \partial_n(\alpha_{U \cap V})] \overset{\text{note that (2c) implies that } \partial_n(\alpha_{U\cap V} + \alpha_{U \setminus L}) \text{ is a singular chain in } U \setminus K, \text{ but } \varphi_A \text{ vanishes on singular chains in } A = M \setminus K}{=} [\varphi_A \cap \partial_n(\alpha_{U \cap V}) - \underbrace{\varphi_A \cap \partial_n(\alpha_{U \cap V} + \alpha_{U \setminus L})}_{=0}] = [-\varphi_A \cap \partial_n(\alpha_{U \setminus L})] \\
\quad \overset{\text{by the Cap Product Boundary Lemma 206.3}}{=} [-\varphi_A \cap \partial_n(\alpha_{U \setminus L}) - \underbrace{\varphi_B \cap \partial_n(\alpha_{U \setminus L})}_{=0}] = [-(\varphi_A + \varphi_B) \cap \partial_n(\alpha_{U \setminus L})] \\
\quad \overset{\text{since } \varphi = \varphi_A + \varphi_B}{=} [-\varphi \cap \partial_n(\alpha_{U \setminus L})] \\
\quad \overset{\text{since } \varphi_B \text{ vanishes on singular chains in } B = M \setminus L}{=} [(-1)^{k+1} \cdot \partial_{n-k}(\varphi \cap \alpha_{U \setminus L})] \\
\quad \overset{\text{by the Cap Product–Boundary Lemma 206.3 and since } \delta_k(\varphi) = 0}{=} (-1)^{k+1} \cdot \partial_{n-k}[\underbrace{(\varphi \cap \alpha_{U \setminus L})}_{\in \mathrm{C}_{n-k}(U)} + \underbrace{(\varphi \cap \alpha_{U \cap V} + \varphi \cap \alpha_{V \setminus K})}_{\in \mathrm{C}_{n-k}(V)}] \\
\quad \overset{\text{by the definition of the connecting homomorphism } \partial_{n-k} \text{ in the Mayer–Vietoris sequence, see the Mayer–Vietoris Theorem 153.2 and the LES-of-Pair Proposition 146.7}}{=} \\
\quad \overset{\text{by equality (2a)}}{=} (-1)^{k+1} \cdot \partial_{n-k}([\varphi] \cap \mu_{K \cup L}^{M}) = (-1)^{k+1} \cdot \text{ maps in the rectangle applied counterclockwise to } [\varphi].
\end{array}$$

This concludes the proof of Subclaim 3, thus of the claim and thus also of the proposition. ∎

211.2.3. **Proof of the Poincaré Duality–Non-Compact Theorem 211.4.** Now we turn to the actual proof of the Poincaré Duality–Non-Compact Theorem 211.4.

Proof of the Poincaré Duality–Non-Compact Theorem 211.4. Let R be a commutative ring. We fix a dimension n. Throughout this proof by a "manifold" we mean an R-oriented n-dimensional topological manifold with empty boundary. We say *Poincaré Duality holds for a manifold* M if for each $k \in \mathbb{N}_0$ the map

$$\cap M \colon \mathrm{H}_{\mathrm{c}}^k(M; R) \to \mathrm{H}_{n-k}(M; R)$$

from the Cap–Non Compact Manifold Lemma 211.3 (2) is an isomorphism. Our goal is to show that Poincaré Duality holds for all manifolds.

We start out with the following claim:

Claim 1. Let W be a manifold.

(A) If Poincaré Duality holds for W equipped with an orientation, then it also holds for any other orientation of W.

(B) If W is homeomorphic to a manifold for which Poincaré Duality holds, then Poincaré Duality also holds for W.

(C) Suppose that W is the union of two open subsets U and V. If Poincaré Duality holds for U, V and $U \cap V$, then it also holds for W.

[2249] The maps δ and ∂ written in blue are the connecting homomorphisms in the Mayer–Vietoris sequences.

(D) If W is the union of a nested sequence $U_1 \subseteq U_2 \subseteq \dots$ of open subsets such that Poincaré Duality holds for each U_i, then it also holds for W.

Proof. So let W be a manifold.

(A) It follows easily from the Cup Product–Disjoint Union Lemma 204.16 and the Cap Product–Disjoint Union Lemma 206.14 that it suffices to consider the case that W is path-connected. By the Number-of-Orientations Lemma 191.18 the path-connected oriented topological manifold W admits only one other orientation, namely the reverse orientation. Evidently Poincaré Duality also holds for that orientation.

(B) If we have an orientation-preserving homeomorphism, then this statement is obvious. For a non-orientation-preserving homeomorphism this statement follows from (A).

(C) We consider the following commutative diagram from the Poincaré Duality–Mayer Vietoris Proposition 211.8:

$$\begin{array}{ccccccccc}
.. & \longrightarrow & \mathrm{H}_{\mathrm{c}}^k(U\cap V) & \xrightarrow{(\iota_*,-\iota_*)} & \mathrm{H}_{\mathrm{c}}^k(U)\oplus\mathrm{H}_{\mathrm{c}}^k(V) & \xrightarrow{\iota_*\oplus\iota_*} & \mathrm{H}_{\mathrm{c}}^k(W) & \longrightarrow & \mathrm{H}_{\mathrm{c}}^{k+1}(U\cap V) \longrightarrow .. \\
 & & \downarrow{\scriptstyle \cap(U\cap V)} & & \downarrow{\scriptstyle \cap U\oplus\cap V} & & \downarrow{\scriptstyle \cap W} & & \downarrow{\scriptstyle \cap(U\cap V)} \\
.. & \to & \mathrm{H}_{n-k}(U\cap V) & \xrightarrow{(\iota_*,-\iota_*)} & \mathrm{H}_{n-k}(U)\oplus\mathrm{H}_{n-k}(V) & \xrightarrow{\iota_*\oplus\iota_*} & \mathrm{H}_{n-k}(W) & \to & \mathrm{H}_{n-k-1}(U\cap V) \to ..
\end{array}$$

(Here we again drop the "R" for space reasons.) If Poincaré Duality holds for U, V and $U\cap V$, then in the above commutative diagram the corresponding vertical maps are isomorphisms. Since the horizontal sequences are exact it follows from the Five Lemma 145.11 that the vertical maps corresponding to W are also isomorphisms, i.e. Poincaré Duality holds for $W = U\cup V$.

(D) Let $k\in\mathbb{N}_0$. We suppose that for every $i\in\mathbb{N}$ the map $\cap U_i\colon \mathrm{H}_{\mathrm{c}}^k(U_i;R)\to\mathrm{H}_{n-k}(U_i;R)$ is an isomorphism. We first point out that it follows from the Poincaré Duality–Open Subset Lemma 211.6 that for any $i\le j$ the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_{\mathrm{c}}^k(U_i;R) & \xrightarrow{\cap U_i} & \mathrm{H}_{n-k}(U_i;R) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
\mathrm{H}_{\mathrm{c}}^k(U_j;R) & \xrightarrow{\cap U_j} & \mathrm{H}_{n-k}(U_j;R).
\end{array}$$

We obtain isomorphisms

$$\mathrm{H}_{\mathrm{c}}^k(W;R) \xrightarrow{\cong} \varinjlim \mathrm{H}_{\mathrm{c}}^k(U_i;R) \xrightarrow{\varinjlim(\cap U_i)} \varinjlim \mathrm{H}_{n-k}(U_i;R) \xleftarrow{\cong} \mathrm{H}_{n-k}(W;R).$$

isomorphism by the Exhaustion-$\mathrm{H}_{\mathrm{c}}^*$ Proposition 209.18

isomorphism by our hypothesis and since the above diagram commutes

isomorphism by the Exhaustions-H_* Proposition 154.2

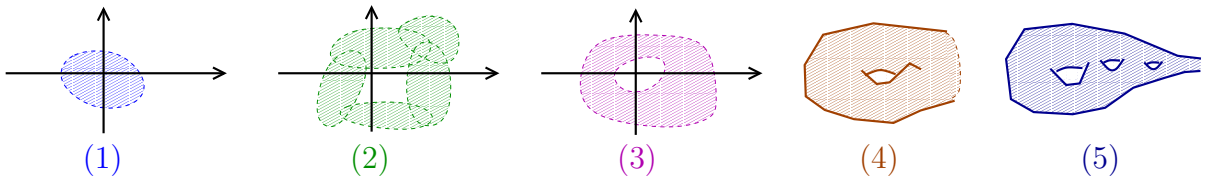
It follows easily from the definitions that for each compact subset $K\subseteq W$ the corresponding map $\mathrm{H}^k(W, W\setminus K;R)\to\mathrm{H}_{\mathrm{c}}^k(W;R)\to\mathrm{H}_{n-k}(W;R)$ is given by capping with μ_K^W. It follows that the above map $\mathrm{H}_{\mathrm{c}}^k(W;R)\to\mathrm{H}_{n-k}(W;R)$ is indeed given by $\cap W$.⊞

The following claim completes the proof of the theorem.

Claim 2. [2250]

(1) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.

(2) For every $m \in \mathbb{N}$ Poincaré Duality holds for the union of m open bounded convex subsets of $\mathbb{R}^n$.
(3) Poincaré Duality holds for every open subset of $\mathbb{R}^n$.
(4) For every $m \in \mathbb{N}$ Poincaré Duality holds for all manifolds that admit an atlas with m charts.
(5) Poincaré Duality holds for all manifolds.



Proof.

(1) Let M be an open bounded convex subset of $\mathbb{R}^n$. If $M = \varnothing$, then Poincaré Duality holds for trivial reasons. If $M \neq \varnothing$, then it follows from the Convex-to-Ball Proposition 4.28 (1) there exists a homeomorphism $M \to B^n$. It follows from the PD-for-Ball Lemma 211.7 and (B) that Poincaré Duality also holds for M.
(2) We prove the claim by induction on m. By (1) we know that the statement holds for $m = 1$. Now suppose that it holds for all unions of $m-1$ open bounded convex subsets of $\mathbb{R}^n$. Let M be the union of m open bounded convex subsets $U_1, \ldots, U_m$. We set $V = U_1 \cup \cdots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for $V \cap U_m = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$, which is the union of $(m-1)$ open bounded convex subsets.[2251] Hence by (C) the statement also holds for

$$M \;=\; V \cup U_m \;=\; (U_1 \cup \cdots \cup U_{m-1}) \cup U_m.$$

(3) Let M be an arbitrary open subset of $\mathbb{R}^n$. By the Rational Open Balls Lemma 231.7 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup_{j \leq i} B_j$. By (2) Poincaré Duality holds for each $U_i \in \mathbb{N}$. But then it follows from (D) that Poincaré Duality also holds for M.
(4) We prove this statement by induction on m. The case $m = 1$ is taken care of by (3) and (B). Now suppose Poincaré Duality holds for all manifolds that admit an atlas with $m-1$ charts. Let M be a manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\ldots,m}$ with m charts. We write $U = U_1 \cup \cdots \cup U_{m-1}$ and $V = U_m$. Both are open subsets of a manifold, hence they are also manifolds. Furthermore U and V are manifolds that admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \cdots \cup (U_{m-1} \cap U_m)$ is a manifold that admits an atlas with $m-1$ charts. By induction Poincaré Duality holds for U, V and $U \cap V$. Thus by (C) Poincaré Duality also holds for M itself.
(5) Finally let M be any manifold. By the Countable Cover Lemma 13.4 we know that M admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j \in \mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup_{j \leq i} U_j$. By (4) Poincaré Duality holds for each X_i. It follows from (D) that Poincaré Duality also holds for M. ∎

[2250] The five steps in the claim correspond precisely to the five steps in the proof of the Homology-Section Theorem 193.6.
[2251] Here we once again use the convenient fact that the intersection of two convex sets is convex.

211.3. **The proof of Poincaré Duality for topological manifolds with boundary.** Now we will finally prove the general statement of the Poincaré Duality Theorem 210.1. For the reader's convenience we recall the statement.

Theorem 210.1. (Poincaré Duality Theorem) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism.

In the following we first consider an important special case of the Poincaré Duality Theorem 210.1, namely the case $A = \partial M$ and $B = \varnothing$. Immediately afterwards we will use this special case to deal with the general case.

Proof of Theorem 210.1 for $A = \partial M$ and $B = \varnothing$. In this proof we consider the case that $A = \partial M$ and $B = \varnothing$. We need to show that for each $k \in \mathbb{N}_0$ the map

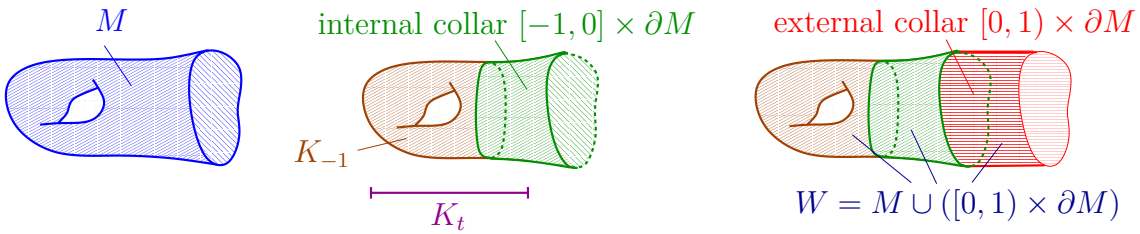
$$\begin{array}{ccc} \mathrm{H}^k(M, \partial M; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. We introduce the following objects:

(i) Recall that from the Topological Collar Neighborhood Theorem 35.14 we obtain an embedding $\chi\colon [-1,0] \times \partial M \to M$ such that $\chi(0,P) = P$ for all $P \in \partial M$. We identify $[-1,0] \times \partial M$ with its image under χ. We refer to $[-1,0] \times \partial M \subseteq M$ as the "internal collar" of ∂M.

(ii) We set $$W \coloneqq M \cup ([0,1) \times \partial M)/\sim \qquad \text{with } x \sim (x,0) \text{ for every } x \in \partial M.$$ Sometimes $[0,1) \times \partial M$ is called an *external collar of* M. Furthermore, given any $t \in [-1,1)$ we write $$K_t \coloneqq (M \setminus ([-1,0] \times \partial M)) \cup ([-1,t] \times \partial M)/\sim .$$

(iii) We denote by $i\colon M \to W$ the obvious map. By the Topological Pushout-Maps Lemma 8.16 (3) and (4) this map is a closed embedding. We will use this to view M as a closed subset of W.



By elementary arguments and using the Unique Type-of-Chart Proposition 150.11 (i) and the Gluing-Topological Manifolds-Proposition 54.3 one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary.

(B) W admits an orientation such that the inclusion $M \to W$ is orientation-preserving.

We denote by $i\colon (M, \partial M) \to (W, W \setminus K_{-1})$ the obvious inclusion map. We obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(M, \partial M; R) & \xleftarrow{i^*} & \mathrm{H}^k(W, W \setminus K_{-1}; R) & \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(W; R) \\ \downarrow{\scriptstyle \cap [M]} & & \downarrow{\scriptstyle \cap \mu_{K_t}} & & \downarrow{\scriptstyle \cap W} \\ \mathrm{H}_{n-k}(M; R) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(W; R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(W; R). \end{array}$$

We want to show that the vertical map to the left is an isomorphism. It remains to prove the following claim which resembles closely the claim in the proof of the PD-for-Ball Lemma 211.7 on page 4048.

Claim. The following three statements hold:
(1) The diagram commutes.
(2) The right vertical map is an isomorphism.
(3) All horizontal maps are isomorphisms.

Proof.

(1) The right-hand square commutes by Definition 206.5 of $\cap W$. Using (B) we can apply the same argument as in the proof of the claim in the PD-for-Ball Lemma 211.7 to show that the left-hand square commutes.
(2) The Poincaré Duality–Non-Compact Theorem 211.4 implies that the right vertical map is an isomorphism.
(3) First note that it follows easily from the Homotopy Pushout Lemma 18.13 that $M = i(M)$ is a deformation retract of W. Next note that it follows easily from this discussion together with the Homotopy Equivalence-Relative-H_*-Corollary 146.8 and the Homotopy Equivalence-H^*-Proposition 198.12 (1) that the left horizontal maps are isomorphisms. Note that the bottom right horizontal map is the identity. Thus it remains to show that the top right horizontal map is an isomorphism. Similar to the proof of the PD-for-Ball Lemma 211.7 we consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^k(W, W \setminus K_{-1}; R) & \longrightarrow & \varinjlim_{K \in \mathcal{K}(W)} \mathrm{H}^k(W, W \setminus K; R) =_{\mathrm{i}} \mathrm{H}^k_{\mathrm{c}}(W; R). \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t \in [-1,1)} \mathrm{H}^k(W, W \setminus K_t; R) & \end{array}$$

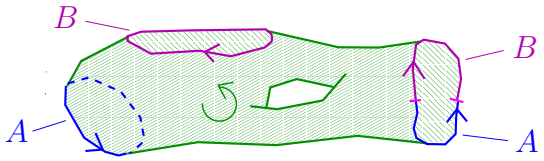
We make the following two observations:
(a) One can easily verify that every compact subset of W is contained in some K_t. Therefore it follows from the Cofinal-Isomorphism Lemma 209.13 that the right diagonal map is an isomorphism.
(b) Note that it follows quite easily from the Homotopy Equivalence-H^*-Proposition 198.12 (1) that for each $t \in [-1, 1)$ the inclusion-induced homomorphism $\mathrm{H}^k(W, W \setminus K_{-1}; R) \to \mathrm{H}^k(W, W \setminus K_t; R)$ is an isomorphism. It follows from this observation and the Direct Colimit–Stabilization Lemma 104.5 that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim and it concludes the proof of the Poincaré Duality Theorem 210.1 in the special case that $A = \partial M$ and $B = \varnothing$. ■

Now we finally consider the general case of the Poincaré Duality Theorem 210.1:

Proof of the Poincaré Duality Theorem 210.1 for the general case. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We recall that by the Excisive Triad-Proposition 158.20 (2) we know that the triad (M, A, B) is excisive.



we need to show that the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism

We equip ∂M and then also B with the orientations coming from the Boundary Orientation Proposition 192.3 and the Topological Manifold-Induced Orientation Lemma 192.1. We consider the following diagram:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{H}^k(M, \partial M; R) & \longrightarrow & \mathrm{H}^k(M, A; R) & \longrightarrow & \mathrm{H}^k(\partial M, A; R) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M, \partial M; R) & \longrightarrow \ldots \\
& \cong \downarrow -\cap[M] & & \downarrow -\cap[M] & & \cong \downarrow j^* & & \cong \downarrow -\cap[M] & \\
& & & & & \mathrm{H}^k(B, \partial B; R) & & & \\
& & & & & \cong \downarrow -\cap[B] & & & \\
\ldots \longrightarrow & \mathrm{H}_{n-k}(M; R) & \longrightarrow & \mathrm{H}_{n-k}(M, B; R) & \xrightarrow{\partial_{n-k}} & \mathrm{H}_{n-k-1}(B; R) & \longrightarrow & \mathrm{H}_{n-k-1}(M; R) & \longrightarrow \ldots
\end{array}$$

We make the following clarifications and observations:

(1) The top horizontal sequence is the long exact sequence in cohomology of the triple $(M, \partial M, A)$ provided by the LES-of-Triple Proposition 198.13.
(2) The bottom horizontal sequence is the long exact sequence in homology of the pair (M, B) provided by the LES-of-Pair Proposition 146.7.
(3) We denote by $j\colon (B, \partial B) \to (\partial M, A)$ the inclusion. We obtain the vertical map $j^*\colon \mathrm{H}^k(\partial M, A; R) \to \mathrm{H}^k(B, \partial B; R)$. Note that it follows from the Excisive Triad-Proposition 158.20 (2) together with the Excisive Characterization Lemma 158.19 (1)⇒(3) and the Also-Iso-on-Cohomology Proposition 200.2 that this map j^* is an isomorphism.
(4) The other maps decorated with "≅" are isomorphisms by the special case of the Poincaré Duality Theorem 210.1 that we have already dealt with on page 4059.

By the Five Lemma 145.11 it remains to prove the following claim:

Claim. The above diagram commutes up to signs, more precisely:
(1) The first rectangle commutes.
(2) The second rectangle commutes up to the sign $(-1)^{k+1}$.
(3) The third rectangle commutes.

Proof. We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k(M,\partial M;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M;R) \\
{\scriptstyle \mathrm{id}}\uparrow \quad\quad \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\mathrm{H}^k(M,\partial M;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M,B;R) \\
\downarrow \quad\quad \uparrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \mathrm{id}}\uparrow \\
\mathrm{H}^k(M,A;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M,B;R).
\end{array}$$

Note that it follows from the Cap Product-Naturality Lemma 206.13, applied to the map $(M,\partial M,\varnothing)\to(M,\partial M,B)$ and to the map $(M,A,B)\to(M,\partial M,B)$ of excisive triads, that the diagram commutes. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $\partial_n\colon \mathrm{H}_n(M,\partial M;R)\to\mathrm{H}_{n-1}(\partial M,A;R)$ the connecting homomorphism in the long exact sequence of the triple $(M,\partial M,A)$. It follows from the Fundamental Class-of-∂-Proposition 194.13, the Codimension Zero-Fundamental Class Lemma 194.10 and the naturality of the connecting homomorphism that the following equality holds

$$(*)\qquad j_*([B]) \;=\; \partial_n([M]) \;\in\; \mathrm{H}_{n-1}(\partial M,A;R).$$

Now we continue with the proof of the claim.

(2) We apply the Cap Product-Triad-Connecting Homomorphism Lemma 206.20 to the excisive triad (M,A,B). (Recall that $A\cup B=\partial M$ and $A\cap B=\partial B$.) We obtain that for any $k\in\mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{array}{ccc}
\mathrm{H}^k(M,A;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M,B;R) \\
{\scriptstyle i^*}\downarrow \quad\quad \downarrow{\scriptstyle \partial_n} & & \\
\mathrm{H}^k(\partial M,A;R)\times\mathrm{H}_{n-1}(\partial M,A;R) & & \Big\downarrow{\scriptstyle \partial_{n-k}} \\
{\scriptstyle j^*}\downarrow\cong \quad\quad {\scriptstyle (j_*)^{-1}}\downarrow\cong & & \\
\mathrm{H}^k(B,\partial B;R)\times\mathrm{H}_{n-1}(B,\partial B;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B;R).
\end{array}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^{k+1}$.

(3) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k(B,\partial B;R)\times\mathrm{H}_{n-1}(B,\partial B;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B;R) \\
{\scriptstyle j^*}\uparrow \quad\quad \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\
\mathrm{H}^k(\partial M,A;R)\times\mathrm{H}_{n-1}(\partial M,A;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(\partial M;R) \\
{\scriptstyle \delta_k}\downarrow \quad\quad \uparrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^{k+1}(M,\partial M;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(M;R).
\end{array}$$

The upper part of this diagram commutes by the Cap Product-Naturality Lemma 206.13 applied to the map $(B,\partial B,\varnothing)\to(\partial M,A,\varnothing)$ between excisive triads. The lower part

commutes by the Cap Product-Triple-Connecting Homomorphism Lemma 206.19 applied to the triple $(M, \partial M, A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim commutes.

This concludes the proof of the claim and thus also of the Poincaré Duality Theorem 210.1. ■

211.4. **Appendix: Poincaré Duality on a manifold and its boundary.** We conclude this chapter with two propositions that relate Poincaré Duality on a manifold to Poincaré Duality on its boundary. It is best to ignore this appendix for the time being and to return to it, when these two propositions get actually used.

Proposition 211.14. (Poincaré Duality–Boundary Proposition) Let R be a commutative ring and let M be a compact R-oriented n-dimensional topological manifold. We equip the boundary ∂M with the R-orientation given by the Boundary Orientation Proposition 192.3. As usual we denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ and $[\partial M] \in \mathrm{H}_{n-1}(\partial M; R)$ the corresponding R-fundamental classes. We consider the following diagram:

$$\begin{array}{ccccccccc}
\ldots \to \mathrm{H}^k(M, \partial M; R) & \xrightarrow{\text{я}} & \mathrm{H}^k(M; R) & \longrightarrow & \mathrm{H}^k(\partial M; R) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M, \partial M; R) & \to \ldots \\
\cong \downarrow -\cap[M] & & \cong \downarrow -\cap[M] & & \cong \downarrow -\cap[\partial M] & & \cong \downarrow -\cap[M] & \\
\ldots \longrightarrow \mathrm{H}_{n-k}(M; R) & \xrightarrow{\text{я}} & \mathrm{H}_{n-k}(M, \partial M; R) & \xrightarrow{\partial_{n-k}} & \mathrm{H}_{n-k-1}(\partial M; R) & \longrightarrow & \mathrm{H}_{n-k-1}(M; R) & \longrightarrow \ldots
\end{array}$$

where the horizontal sequences are the long exact sequences in (co-) homology of the pair $(M, \partial M)$. (Note that the vertical maps are isomorphisms by the Poincaré Duality Theorem 210.1.) The first and the third square commute, furthermore the second square commutes up to the sign $(-1)^k$.

Proof. We had implicitly proved the proposition in the above proof of the Poincaré Duality Theorem 210.1, see the discussion on page 4061, applied to the special case that $A = \varnothing$ and $B = \partial M$. ■

Finally we have the following proposition which relates the Poincaré Duality isomorphism to connecting homomorphisms:

Proposition 211.15. (Poincaré Duality–Connecting Homomorphism Proposition) Let R be a commutative ring and let M be an R-oriented compact n-dimensional topological manifold. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We consider the following two diagrams:

$$\begin{array}{ccc}
\mathrm{H}_k(M, A; R) & \xrightarrow{\partial_k} & \mathrm{H}_k(A; R) \\
{\scriptstyle \mathrm{PD}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_A} \\
\mathrm{H}^{n-k}(M, B; R) & \xrightarrow{w^*} & \mathrm{H}^{n+1-k}(A, \partial A; R)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathrm{H}_k(A, \partial A; R) & \xrightarrow{w_*} & \mathrm{H}_k(M, B; R) \\
{\scriptstyle \mathrm{PD}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}_M} \\
\mathrm{H}^{n-k}(A; R) & \xrightarrow{\delta_{n-k}} & \mathrm{H}^{n+1-k}(M, A; R)
\end{array}$$

where $w\colon (A, \partial A) \to (M, B)$ is the natural inclusion. The first diagram commutes up to multiplication by $(-1)^k$ whereas the second commutes as it is.

Proof. We already showed on page 4062 that the first diagram commutes up to multiplication by $(-1)^k$. The fact that the second diagram commutes is a fairly straightforward consequence of the Cap Product-Triple-Connecting Homomorphism Lemma 206.19 and the Fundamental Class-of-∂-Proposition 194.13. We leave it to the reader to fill in the details. ∎

211.5. Exercises for Chapter 211.

Exercise 211.1. Let M be an oriented connected non-empty n-dimensional topological manifold. As in Exercise 194.18 we consider the group

$$\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) \coloneqq \varprojlim_{K\in\mathcal{K}(M)} \mathrm{H}_n(M, M\setminus K;\mathbb{Z}).$$

In Exercise 194.20 we showed that $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and we showed that there exists a natural generator $[M]$. Let $k\in\mathbb{N}_0$.

(a) Let $\cap\colon \mathrm{H}_{\mathrm{c}}^{n-k}(M;\mathbb{Z}) \times \overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ be the natural cap product that we introduced in Exercise **??**. Show that the map

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{c}}^{n-k}(M;\mathbb{Z}) & \to & \mathrm{H}_k(M;\mathbb{Z}) \\ \varphi & \mapsto & \varphi\cap[M] \end{array}$$

is an isomorphism.

Remark. Use the Cap–Non Compact Manifold Lemma 211.3.

(b) Now let $f\colon M\to N$ be a proper map between two connected orientable non-empty n-dimensional topological manifolds. Since f is proper we obtain by Exercise 194.18 an induced homomorphism $f_*\colon \overleftarrow{\mathrm{H}}_k(M;\mathbb{Z}) \to \overleftarrow{\mathrm{H}}_k(N;\mathbb{Z})$. We define $\deg(f)\in\mathbb{Z}$ via the equality $f_*([M]) = \deg(f)\cdot[N]$. Furthermore, using (a), we can define the umkehr map $f^!\colon \overleftarrow{\mathrm{H}}_k(N;\mathbb{Z}) \to \overleftarrow{\mathrm{H}}_k(M;\mathbb{Z})$. Show that the map $f^!\circ f_*\colon \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ is given by multiplication by $\deg(f)$.

Remark. The goal of this exercise is to generalize the ideas and concepts behind the umkehr Map Proposition 210.16 to the setting of non-compact topological manifolds.

Exercise 211.2. Let M be a non-orientable connected n-dimensional smooth manifold with $\partial M=\varnothing$. Show that $\mathrm{H}_{\mathrm{c}}^n(M;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 211.3. Let G be an abelian group.

(a) Define for every excisive triple (X,A,B) and every $n,k\in\mathbb{N}_0$ a suitable cap product

$$\cap\colon \mathrm{H}^k(X,A;G) \times \mathrm{H}_n(X, A\cup B) \to \mathrm{H}_{n-k}(X,B;G).$$

(b) Let M be a compact oriented n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A\cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A\cap B = \partial A = \partial B$. Show that for each $k\in\mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M,A;G) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M,B;G) \\ \varphi & \mapsto & \varphi\cap[M] \end{array}$$

$[M]\in\mathrm{H}_n(M,\partial M)$ is the fundamental class

is an isomorphism.

Remark. The task is to identify in the proof of the Poincaré Duality Theorem 210.1 the points where you need to make changes.

Exercise 211.4. Let M be a connected orientable non-empty n-dimensional topological manifold such that $\partial M = \varnothing$. We denote by $[M] \in \mathrm{H}_n^{\mathrm{lf}}(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{H}_n(M;\mathbb{Z})$ the fundamental class from Exercise 194.20.

(a) Show that for each $k \in \{0,\ldots,n\}$ the map

$$\begin{array}{rcl} \mathrm{H}_{\mathrm{c}}^k(M;\mathbb{Z}) & \to & \mathrm{H}_{n-k}(M;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

from Exercise **??** is an isomorphism.

(b) Show that for each $k \in \{0,\ldots,n\}$ the map

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) & \to & \mathrm{H}_{n-k}^{\mathrm{lf}}(M;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

from Exercise **??** is an isomorphism.

Remark. This exercise might need some work.

Remark. Implicitly we are using Exercise 194.18 to go between different notions of locally finite homology.

CHAPTER 212

Poincaré Duality and codimension-zero submanifolds

In this short chapter we will study the interplay between Poincaré Duality on a given compact topological manifold M and on a given codimension-zero submanifold W of M. Later on, in some of the more fancy applications of Poincaré duality, we will also deal with submanifolds with corner, so we will use this opportunity to finally introduce the notion of a submanifold with corner. The results and concepts of this chapter we will be used on several occasions. Nonetheless it is most likely best to skip this chapter and to move on to the next, significantly more entertaining, Chapter 213 where we determine the cup products on projective spaces.

This chapter is organized as follows:

- In Section 212.1 we will introduce submanifolds with corner.
- In Section 212.2 we will discuss the relationship between Poincaré Duality and codimension-zero submanifolds.
- In Section 212.3 we will show that if we are given two compact oriented proper submanifolds that are cobordant, then they represent the same homology class.
- In Section 212.4 we will provide exercises for Chapter 212.

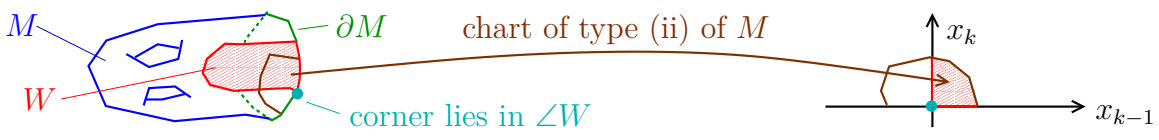
In the next Chapter 213 we will calculate the cup product of projective spaces.

212.1. **Submanifolds with corner.** Recall that in Definition 23.18 we defined what it means for a subset W of a topological manifold M to be a submanifold. Furthermore in Definition 26.14 we introduced the notion of a submanifold with corner. Since the reader is most likely less familiar with this notion let us recall the definition:

Definition 212.1. Let M be an n-dimensional topological manifold. We say that a subset $W \subseteq M$ is a k-dimensional submanifold with corner if for every $P \in W$ there exists a chart of the type (α), (β) and (γ) from Definition 23.18 or if
(δ) there exists a chart $\Phi\colon U \to V$ of type (ii)[2252] for M such that

$$\Phi(U \cap W) \subseteq \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_1,\dots,x_k \in \mathbb{R} \text{ with } x_{k-1} \geq 0 \text{ and } x_k \geq 0\}$$

and with $\Phi(P) \in \{(0,\dots,0,x_1,\dots,x_{k-2},0,0) \,|\, x_1,\dots,x_{k-2} \in \mathbb{R}\}$.
We refer to the set of points in W that admit a chart of type (δ) as the corner set of W and we denote it by $\angle W$.

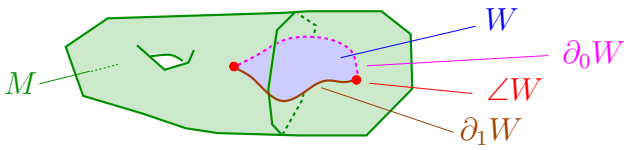


[2252] See Definition 23.2 for the definition of a chart of type (ii).

The following definition generalizes some concepts from Notation 25.47, Definition 26.15 and Notation 150.14.

Definition 212.2. Let M be a topological manifold and let W be a submanifold with corner of M. We write

$$\begin{aligned} \partial_0 W &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\gamma) \text{ or } (\delta)\ , \\ \partial_1 W &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\beta) \text{ or } (\delta), \\ \angle W &:= \text{the set of all } P \in X \text{ that admit a submanifold chart of type } (\delta). \end{aligned}$$



The following lemma can be viewed as a generalization of the Codimension-Zero Submanifold Proposition 150.16 and the topological analog of the Smooth Submanifold-with-Corner Lemma 26.16.

Lemma 212.3. (Topological Submanifold-with-Corner Lemma) Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold with corner. The following statements hold:

(0) (a) $\partial_0 W = \overline{\partial W \setminus \partial M}$,
(b) $\partial_1 W = \partial W \cap \partial M$,
(c) $\angle W = \partial_0 W \cap \partial_1 W$.

(1) The subset W itself is an n-dimensional topological manifold.

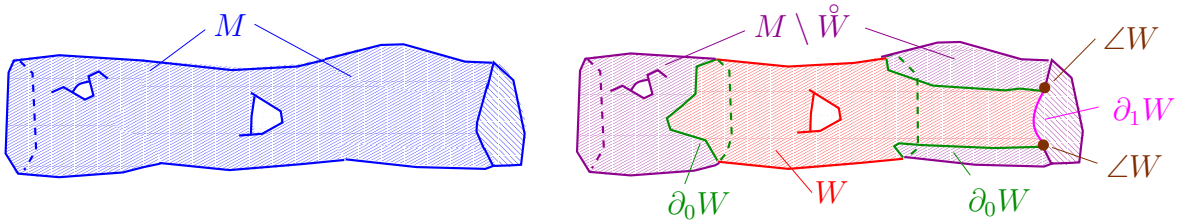
(2) The boundary of the topological manifold W equals $\partial_0 W \cup \partial_1 W$. Furthermore $\partial_0 W$ and $\partial_1 W$ are $(n-1)$-dimensional submanifolds of the boundary ∂W such that

$$\partial_0 W \cap \partial_1 W \;=\; \partial(\partial_0 W) \;=\; \partial(\partial_1 W) \;=\; \angle W.$$

(3) The set $\mathring{W} := W \setminus \partial_0 W$ is the interior of W in M, in particular $\mathring{W}$ is an open subset of M.

(4) The subset $M \setminus \mathring{W}$ is a codimension-zero submanifold of M with corner given by $\angle(M \setminus \mathring{W}) = \angle W$. Furthermore $\partial_0(M \setminus \mathring{W}) = \partial_0 W$ and $\partial_1(M \setminus \mathring{W}) = \overline{\partial M \setminus \partial_1 W}$.

(5) $\partial_0 W$ is a proper submanifold of M, in the sense of Definition 23.18.



Sketch of proof. Statement (1) follows from the Corner Smooth Lemma 26.2 (1). The other statements follow either immediately from the definitions or by the arguments of the Codimension-Zero Submanifold Proposition 150.16. Once again we leave the slightly annoying task of filling in the details to the reader. ■

In the following we will generalize a few results on codimension-zero submanifolds to codimension-zero submanifolds with corner. It is unlikely that the reader's pulse will quicken while reading the remainder of this section.

First we have the following variation on the Manifold–Excision Theorem 148.3.

Theorem 212.4. (Manifold–Excision Theorem II) Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. We write $\mathring{W} := W \setminus \partial_0 W$. For every $k \in \mathbb{N}_0$ and every abelian group G the inclusion-induced maps

$$\begin{array}{lrcl} \text{(a)} & \mathrm{H}_k(W, \partial_0 W; G) & \xrightarrow{\cong} & \mathrm{H}_k(M, M \setminus \mathring{W}; G) \\ \text{(b)} & \mathrm{H}_k(W, \partial W; G) & \xrightarrow{\cong} & \mathrm{H}_k(M, \underbrace{(M \setminus \mathring{W}) \cup \partial M}_{=(M\setminus\mathring{W})\cup\partial_1 W}; G) \\ \text{(c)} & \mathrm{H}_k(W, \partial_1 W; G) & \xrightarrow{\cong} & \mathrm{H}_k(W \cup \partial M, \partial M; G) \end{array}$$

are isomorphisms. Similarly, all maps in cohomology "in the other direction" are also isomorphisms.

Proof. Recall that the proof of the original Manifold–Excision Theorem 148.3 made use of the Topological Collar Neighborhood Theorem 35.14. In Exercise 212.1 we will prove a suitable generalization of the Topological Collar Neighborhood Theorem 35.14. Using this extension we will provide a proof of the new Excision Theorem for homology groups with $\mathbb{Z}$-coefficients in Exercise 212.2. The statement for homology with G-coefficients and the statement for cohomology with G-coefficients are an immediate consequence of the above, together with the Universal Coefficient Theorem 162.16, the Universal Coefficient-H^*-Theorem 201.23, the fact that the homomorphisms in the Universal Coefficient Theorems are natural, and the Five Lemma 145.11. ■

We turn to the topic of orientations and fundamental classes. First we have the following lemma which generalizes the Topological Manifold-Induced Orientation Lemma 192.1 (2).

Lemma 212.5. (Submanifold Orientation Lemma) Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation. If W is a codimension-zero submanifold of M with corner, then W admits a unique R-orientation such that the inclusion map $W \to M$ is orientation-preserving.

Proof. This statement follows from a very mild modification of the argument of the Topological Manifold-Induced Orientation Lemma 192.1 (2). ■

Convention 212.6. Let M be a topological manifold and let W be a codimension-zero submanifold with corner. If M is oriented, then we equip W with the orientation coming from the Submanifold Orientation Lemma 212.5.

The following lemma is a generalization of the Codimension Zero-Fundamental Class Lemma 194.10.

Lemma 212.7. (Codimension Zero-Fundamental Class Lemma II) Let M be a compact oriented n-dimensional topological manifold. Furthermore let $W \subseteq M$ be a

compact non-empty codimension-zero submanifold *with corner*. We write $\mathring{W} = W \setminus \partial W$. The following two statements hold:

(1) The inclusion-induced map $\mathrm{H}_n(W, \partial W; \mathbb{Z}) \to \mathrm{H}_n(M, (M \setminus \mathring{W}) \cup \partial_1 W; \mathbb{Z})$ is an isomorphism.

(2) The image of the fundamental classes $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ and $[W] \in \mathrm{H}_n(W, \partial W; \mathbb{Z})$ under the maps

$$\mathrm{H}_n(M, \partial M; \mathbb{Z}) \longrightarrow \mathrm{H}_n(M, (M \setminus \mathring{W}) \cup \partial_1 W; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{isomorphism by the} \\ \text{Manifold–Excision} \\ \text{Theorem 212.4 (2)}}}{\longleftarrow} \mathrm{H}_n(W, \partial W; \mathbb{Z})$$

agree.

Proof. The proof of the Codimension Zero-Fundamental Class Lemma 212.7 is basically identical to the proof of the Codimension Zero-Fundamental Class Lemma 194.10. ∎

212.2. **Poincaré Duality and codimension-zero submanifolds.** The following lemma says that given a homology class in a codimension-zero submanifold W of some topological manifold M we can apply Poincaré Duality in M or W and essentially we end up with the same result.

Lemma 212.8. (Small H_*-to-Large H^*-PD Lemma) Let M be a compact oriented n-dimensional topological manifold and let W be a compact codimension-zero with corner. We equip W with the induced orientation from the Submanifold Orientation Lemma 212.5. We write $\mathring{W} = W \setminus \partial_0 W$.[2253] Furthermore let A and B be disjoint unions of components of ∂M with $\partial M = A \cup B$ and $\partial_1 W \subseteq A$. We fix the following notation:

(1) We denote by w the obvious map $(W, \partial_1 W) \to (M, A)$ and also the obvious map $(W, \partial_0 W) \to (M, M \setminus \mathring{W})$.

(2) We denote by $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious map.

Then for every $k \in \mathbb{N}_0$ the following diagram commutes:[2254] [2255]

$$\begin{array}{ccccc} \mathrm{H}_k(W, \partial_1 W; \mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W, \partial_0 W; \mathbb{Z}) & \xrightarrow[\cong]{(w^*)^{-1}} & \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M, A; \mathbb{Z}) & & \xrightarrow{\qquad\qquad\mathrm{PD}_M\qquad\qquad} & & \mathrm{H}^{n-k}(M, B; \mathbb{Z}). \end{array}$$

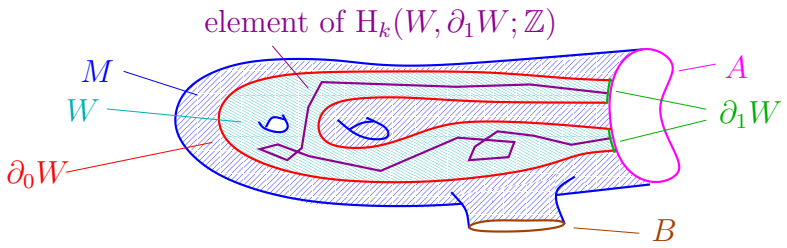
The analogous statement of course also holds with $\mathbb{F}_2$-coefficients.

In the proofs of the Small H_*-to-Large H^*-PD Lemma 212.8 and of the Large H_*-to-Small H^*-PD Lemma 212.9 will use the full force of the Cap Product-Naturality Lemma 206.13 applied to excisive triads. Since this can be seriously confusing we recall the general statement of the lemma, before we provide the proof of the Small H_*-to-Large H^*-PD Lemma 212.8.

[2253] Recall that by the Topological Submanifold-with-Corner Lemma 212.3 (3) the set $\mathring{W} = W \setminus \partial_0 W$ is indeed the interior of W in M.

[2254] Here we use that by the Manifold–Excision Theorem 212.4 we know that $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on cohomology groups.

[2255] Note that by the Topological Submanifold-with-Corner Lemma 212.3 (2) we can apply the Poincaré Duality Theorem 210.1 to obtain that $\cap[W]\colon \mathrm{H}^{n-k}(W, \partial_1 W; \mathbb{Z}) \to \mathrm{H}_k(W, \partial_0 W; \mathbb{Z})$ is an isomorphism. As usual we denote its inverse by PD_W.



Lemma 206.13. (Cap Product-Naturality Lemma) Let $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ be a continuous map between two excisive triads. Then for any choice of $\varphi \in \mathrm{H}^{n-k}(\widetilde{X}, \widetilde{A}; \mathbb{Z})$ and $\sigma \in \mathrm{H}_m(X, A \cup B; \mathbb{Z})$ we have

$$\begin{array}{ccccc} f_*\colon \mathrm{H}_m(X, A\cup B) \to \mathrm{H}_m(\widetilde{X}, \widetilde{A}\cup \widetilde{B}) & & f^*\colon \mathrm{H}^{n-k}(\widetilde{X}, \widetilde{A}) \to \mathrm{H}^{n-k}(X, A) & & \\ \downarrow & & \downarrow & & \\ \varphi \cap f_*(\sigma) & = & f_*(f^*(\varphi) \cap \sigma) & \in & \mathrm{H}_k(\widetilde{X}, \widetilde{B}; \mathbb{Z}). \\ & & \uparrow & & \\ & & f_*\colon \mathrm{H}_k(X, B) \to \mathrm{H}_k(\widetilde{X}, \widetilde{B}) & & \end{array}$$

Proof of the Small H_*-to-Large H^*-PD Lemma 212.8. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(W, \partial_1 W; \mathbb{Z}) & \xleftarrow[\cong]{\cap [W]} & \mathrm{H}^{n-k}(W, \partial_0 W; \mathbb{Z}) & \xleftarrow[w^*]{\cong} & \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M, A) & & \xleftarrow[\cong]{\cap [M]} & & \mathrm{H}^{n-k}(M, B; \mathbb{Z}). \end{array}$$

In the following we use the following notation:

(1) We write $\widehat{W} \coloneqq W \setminus \partial W$.
(2) We denote the map $(W, \partial W) \to (M, M \setminus \widehat{W})$ by w as well
(3) We denote the map $(M, \partial M) \to (M, M \setminus \widehat{W})$ by p as well.

Now let $\beta \in \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z})$. We have the following equality in $\mathrm{H}_k(M, A)$:

$$\begin{array}{ccccccc} w_*(w^*(\beta) \cap [W]) & = & \beta \cap w_*([W]) & = & \beta \cap p_*([M]) & = & p_*(\overbrace{p^*(\beta) \cap [M]}^{\in \mathrm{H}_k(M,A)}) \\ \uparrow & & \downarrow & & \uparrow & & \uparrow \\ \text{by the Cap Product-Naturality} & & \text{by the Codimension Zero-Fundamental} & & \text{by the Cap Product-Naturality} & & \text{since } p\colon (M, A) \to (M, A) \\ \text{Lemma 206.13 applied to the map} & & \text{Class Lemma 212.7 we} & & \text{Lemma 206.13 applied to the map} & & \text{is id we have } p_* = \mathrm{id}_* = \mathrm{id} \\ w\colon (W, \partial_0 W, \partial W) \to (M, M\setminus \mathring{W}, M \setminus \widehat{W}) & & \text{have } p_*([M]) = w_*([W]) \in \mathrm{H}_m(M, M \setminus \widehat{W}; \mathbb{Z}) & & p\colon (M, B, A) \to (M, M\setminus \mathring{W}, A) & & \end{array} \quad = \; p^*(\beta) \cap [M].$$

We have thus shown that the above diagram does indeed commute. ∎

We conclude this discussion of Poincaré Duality and submanifolds (with corner) with the following lemma which also relates Poincaré Duality in a topological manifold M to Poincaré Duality in a compact submanifold W with corner.

Lemma 212.9. (Large H_*-to-Small H^*-PD Lemma) Let M be a compact oriented n-dimensional topological manifold and let W be a compact codimension-zero submanifold

with corner. Furthermore let A and B be disjoint unions of components of ∂M with $\partial M = A \cup B$. We fix the following notation:

(1) We set $C \coloneqq \partial_1 W \cap A$ and $D \coloneqq (\partial_1 W \cap B) \cup \partial_0 W$.
(2) We set $\mathring{W} \coloneqq W \setminus \partial_0 W$.
(3) The two natural maps $(W, C) \to (M, A)$ and $(W, D) \to (M, (M \setminus \mathring{W}) \cup B)$ are both denoted by w.
(4) We denote by $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious map.

Then for every $k \in \mathbb{N}_0$ the following diagram commutes: [22562257]

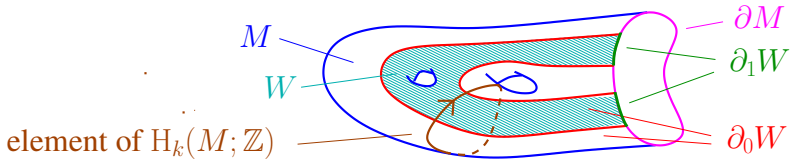
$$\begin{array}{ccccc} \mathrm{H}_k(M, B; \mathbb{Z}) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{n-k}(M, A; \mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, (M \setminus \mathring{W}) \cup B; \mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W, D; \mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W, C; \mathbb{Z}). \end{array}$$

The analogous statement of course also holds with $\mathbb{F}_2$-coefficients.

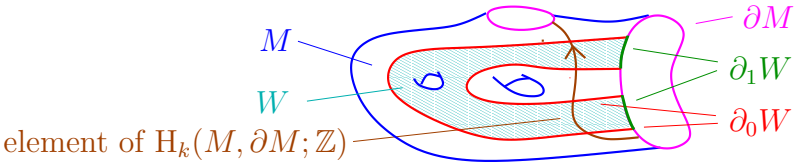
Example. We consider two special cases of from the above lemma:

(a) If we set $A \coloneqq \partial M$ and $B \coloneqq \varnothing$, then $C = \partial_1 W$ and $D = \partial_0 W$ and we obtain that the following diagram commutes:

$$\text{(a)} \qquad \begin{array}{ccccc} \mathrm{H}_k(M; \mathbb{Z}) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{n-k}(M, \partial M; \mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M \setminus \mathring{W}; \mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W, \partial_0 W; \mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W, \partial_1 W; \mathbb{Z}). \end{array}$$



(b) We now set $A \coloneqq \varnothing$ and $B \coloneqq \partial M$. Note that this implies that $C = \varnothing$ and $D = \partial W$.



[2256] We use that $w\colon (W, D) \to (M, (M \setminus \mathring{W}) \cup B)$ induces an isomorphism on cohomology. This follows from a slight generalization of the Manifold–Excision Theorem 212.4.

[2257] If we are given by a homology class in $\mathrm{H}_k(M, B; \mathbb{Z})$ that is represented by a submanifold, then the map $\mathrm{H}_k(M, B; \mathbb{Z}) \xrightarrow{p_*} \mathrm{H}_k(M, (M \setminus \mathring{W}) \cup B; \mathbb{Z}) \xrightarrow{(w_*)^{-1}} \mathrm{H}_k(W, D; \mathbb{Z})$ can be often be calculated via the Codimension Zero-Fundamental Class Lemma 194.10.

We see that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M,\partial M;\mathbb{Z}) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{n-k}(M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M,(M\setminus \mathring{\widehat{W}})\cup\partial M;\mathbb{Z}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W,\partial W;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W;\mathbb{Z}). \end{array} \tag{b}$$

□

Proof. We set $\widehat{W} := W \setminus (\partial_0 W \cup C)$. Note that $M \setminus \widehat{W} = (M \setminus \mathring{\widehat{W}}) \cup B$. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M,B;\mathbb{Z}) & & \xleftarrow[\cong]{\quad\cap[M]\quad} & & \mathrm{H}^{n-k}(M,A;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M,M\setminus\widehat{W};\mathbb{Z}) & \xleftarrow[\cong]{w_*} & \mathrm{H}_k(W,D;\mathbb{Z}) & \xleftarrow{\cap[W]} & \mathrm{H}^{n-k}(W,C;\mathbb{Z}). \end{array}$$

So let $\beta \in \mathrm{H}^{n-k}(M,A;\mathbb{Z})$. We have the following equalities in $\mathrm{H}_k(M, M\setminus\widehat{W};\mathbb{Z})$:

$$\begin{array}{ccccccc} p_*(\beta\cap[M]) & \underset{\uparrow}{=} & \beta\cap p_*([M]) & \overset{\downarrow}{=} & \beta\cap w_*([W]) & \underset{\uparrow}{=} & w_*(w^*(\beta)\cap[W]). \\ \text{by the Cap Product-Naturality Lemma 206.13 applied to the map } p\colon (M,A,B)\to(M,A,M\setminus\widehat{W}) \text{ of excisive triads} & & & \text{by the Codimension Zero-Fundamental Class Lemma 212.7 we have } p_*([M]) = w_*([W]) \in \mathrm{H}_n(M,(M\setminus\mathring{\widehat{W}})\cup\partial M) & & \text{by the Cap Product-Naturality Lemma 206.13 applied to the map } w\colon (W,C,D)\to(M,A,M\setminus\widehat{W}) \text{ of excisive triads} & & \end{array}$$

■

212.3. **Cobordisms and fundamental classes.** In Definition 187.13 we saw that compact oriented proper submanifolds give rise to homology classes. Now that we are familiar with topological manifolds with corner it is good moment to return to this topic. We start out with the following intuitive notation:

Notation 212.10. Let M be a topological manifold and let S be a compact oriented proper k-dimensional submanifold of M. Let $[S] \in \mathrm{H}_k(S,\partial S)$ be the fundamental class from Definition 194.3. We denote by $i\colon S \to M$ the inclusion. By a slight abuse of notation we also write

$$[S] \;:=\; \underset{\uparrow}{i_*}([S]) \;\in\; \mathrm{H}_k(M,\partial M).$$

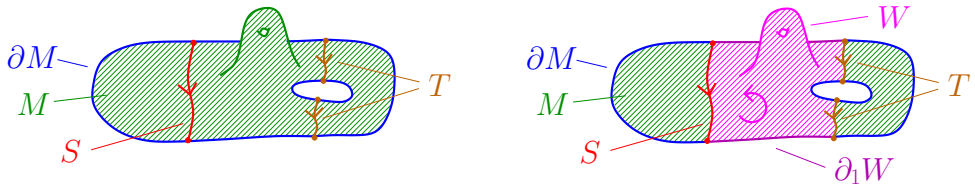
here we consider the map $i_*\colon \mathrm{H}_k(S,\partial S)\to\mathrm{H}_k(M,\partial M)$

It is natural to ask, when do two compact oriented proper k-dimensional submanifolds of M represent the same homology class? To give a partial answer to this question we need to introduce the following concept:

Definition 212.11. Let M be a topological manifold and let S and T be two compact oriented proper k-dimensional submanifolds of M. We say that S and T are cobordant in M if there exists a cobordism from S to T in M, i.e. a compact oriented $(k+1)$-dimensional

submanifold W with corner of M such that

$$\underset{\substack{\uparrow \\ \text{see Definition 212.2}}}{\partial_0 W} \;=\; S \sqcup -T.$$



Remark. Later in **??** and **??** we will introduce what it means for two closed oriented n-dimensional smooth manifolds to be "abstractly" cobordant. □

The following proposition often gets used subconsciously:

Proposition 212.12. (Cobordant–Same Homology Class Proposition) Let M be a topological manifold and let S and T be two compact oriented proper k-dimensional submanifolds of M. If S and T are cobordant in M, then

$$[S] \;=\; [T] \;\in\; \mathrm{H}_k(M, \partial M).$$

Remark. In the Surface Cobordism Proposition 291.23 we will prove a partial converse to the Cobordant–Same Homology Class Proposition 212.12. □

Proof. Let W be a cobordism in M from S to T. The following claim is the key to proving the proposition:

Claim 1. The image of $[\partial_0 W] \in \mathrm{H}_k(\partial_0, \angle W)$ in $\mathrm{H}_k(W, \partial_1 W)$ is zero.[2258]

Proof. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_{k+1}(W, \partial W) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(\partial W) & \longrightarrow & \mathrm{H}_k(W) \\
\overset{[W]}{\uparrow} & & \overset{[\partial W]}{\uparrow} & & \overset{0}{\uparrow} \\
\downarrow \mathrm{id} & & \downarrow \text{я} & & \downarrow \text{я} \\
\mathrm{H}_{k+1}(W, \partial W) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(\partial W, \partial_1 W) & \longrightarrow & \mathrm{H}_k(W, \partial_1 W) \\
 & & \uparrow \cong & \nearrow & \\
 & & \mathrm{H}_k(\partial_0 W, \angle W). & & \\
 & & \underset{[\partial_0 W]}{\uparrow} & &
\end{array}$$

We make the following clarifications and observations:

(a) All unnamed maps are induced by inclusions.
(b) The horizontal sequences are the long exact sequences of the pair $(W, \partial W)$ and of the triple $(W, \partial W, \partial_1 W)$.
(c) As we pointed out in the LES-of-Triple Proposition 146.5, it follows from the naturality of the connecting homomorphism and the functoriality of homology that the diagram commutes.

[2258] Note that it follows from the Topological Submanifold-with-Corner Lemma 212.3 and the Boundary Orientation Proposition 192.3 that $\partial_0 W$ is indeed a compact oriented k-dimensional manifold with boundary $\angle W$. Thus it makes sense to talk about the fundamental class $[\partial_0 W] \in \mathrm{H}_k(\partial_0, \angle W)$.

(d) It follows immediately from the Fundamental Class-of-∂-Proposition 194.13 that we have the equality $\partial_{k+1}([W]) = [\partial W] \in \mathrm{H}_k(\partial W)$.
(e) It follows from the Codimension Zero-Fundamental Class Lemma 194.10 that the image of $[\partial_0 W]$ in $\mathrm{H}_k(\partial W, \partial_1 W)$ equals $\text{я}([\partial W])$.

It follows from the exactness of the top sequence that the image of $[W]$ in $\mathrm{H}_k(W)$ is zero. But by the above this gives us the desired statement. ⊞

Now we can easily prove the actual statement of the proposition:

Claim 2. $[S] = [T] \in \mathrm{H}_k(M, \partial M)$.

Proof. We denote by $\iota\colon W \to M$ the inclusion. We have the following diagram:

$$\begin{array}{ccccc} & & \overset{[\partial_0 W]}{\mathrm{H}_k(\partial_0 W, \angle W)} & \longrightarrow & \overset{0}{\mathrm{H}_k(W, \partial_1 W)} \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota_*} \\ \underset{[S]-[T]}{\mathrm{H}_k(S, \partial S) \oplus \mathrm{H}_k(T, \partial T)} & \xrightarrow{\cong} & \underset{[S\cup -T]}{\mathrm{H}_k(S \cup T, \partial(S \cup T))} & \longrightarrow & \mathrm{H}_k(M, \partial M). \end{array}$$

We make the following clarifications and observations:

(a) The map to the bottom left is the isomorphism from the Path-Component–H_*–Lemma 143.21.
(b) It follows from the Fundamental Class-Proposition 194.2 (1b) that the map to the bottom left sends $[S] - [T]$ to $[S \cup -T]$.
(c) It follows from the functoriality of homology that the diagram commutes.
(d) By Claim 1 we know that $[\partial_0 W]$ gets sent to zero.

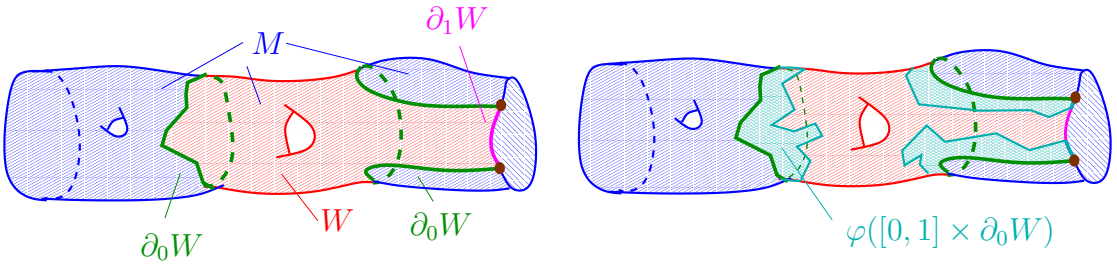
It follows from the above that $[S] - [T] = 0 \in \mathrm{H}_k(M, \partial M)$. ■

212.4. Exercises for Chapter 212 .

Exercise 212.1. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. Show that there exists an embedding $\varphi\colon [0,1] \times \partial_0 W \to W$ with the following two properties:

(1) For every $x \in \partial_0 W$ we have $\varphi(0, x) = x$.
(2) For every $t \in [0,1]$ and every $x \in \angle W = \partial(\partial_0 W)$ we have $\varphi(t, x) \in \partial_1 W$.

Hint. Adapt the proof of the Topological Collar Neighborhood Theorem 35.14.



Exercise 212.2. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. We write $\mathring{W} := W \setminus \partial_0 W$. Show that for

every $k \in \mathbb{N}_0$ the inclusion-induced maps

$$\text{(a)} \qquad \mathrm{H}_k(W, \partial_0 W) \xrightarrow{\cong} \mathrm{H}_k(M, M \setminus \mathring{W})$$

$$\text{(b)} \qquad \mathrm{H}_k(W, \partial W) \xrightarrow{\cong} \mathrm{H}_k(M, (M \setminus \mathring{W}) \cup \partial M)$$

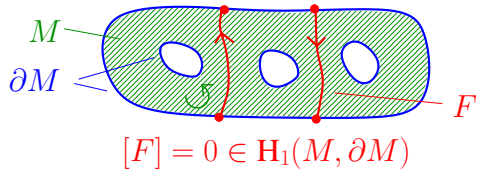
$$\text{(c)} \qquad \mathrm{H}_k(W, \partial_1 W) \xrightarrow{\cong} \mathrm{H}_k(W \cup \partial M, \partial M)$$

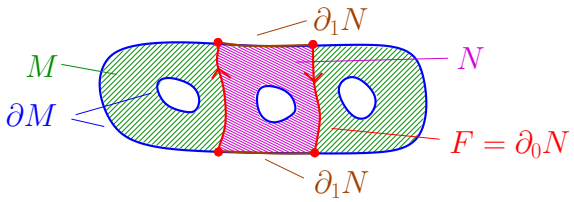
are isomorphisms.

Remark. This exercise is a variation on the Manifold–Excision Theorem 148.3. You might want to use Exercise 212.1.

Exercise 212.3. Let M be a compact oriented n-dimensional smooth manifold and let $F \subseteq M$ be a proper oriented *connected* smooth submanifold of codimension one. We suppose that $[F] = 0 \in \mathrm{H}_{n-1}(M, \partial M)$. Show that there exists a codimension-zero submanifold $N \subseteq M$ with corner such that $\partial_0 N = F$.[2259]

Remark. This exercise is a generalization of the Null Homologous Boundary Proposition 185.22. It thus also makes sense to look at the proof of the earlier proposition. The results of this chapter can be used to generalize the earlier proof.





[2259]Recall that in Definition 26.15 we defined, for a codimension-zero submanifold $N \subseteq M$ with corner, $\partial_0 N$ and $\partial_1 N$.

Part XXIII

Applications of Poincaré Duality

In this part of the lecture notes we explore the full power of the Poincaré Duality Theorem 210.1. This part consists of the following chapters:

- In Chapter 213 we use the Poincaré Duality Theorem 210.1 to calculate the cup products on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$.
- In Chapter 214 we will introduce division algebras. Obvious examples of division algebras are given by $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$. We also introduce two more interesting division algebras, namely the quaternions $\mathbb{H} \cong \mathbb{R}^4$ and the octonions $\mathbb{O} \cong \mathbb{R}^8$. The main result of the chapter is the Hopf–Division Algebra Dimension Theorem 214.13 which says that if $\mathbb{R}^n$ admits the structure of a division algebra, then n is a power of 2.
- In Chapter 215 we will introduce for each $n \in \mathbb{N}$ the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ and we introduce the octonionic projective spaces $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$. We also compute the cohomology rings of these projective spaces.
- In Chapter 216 we use the cup product to introduce the Hopf invariant which is a homomorphism $\pi_{4n-1}(S^n) \to \mathbb{Z}$. Using the calculations of the cup products of $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ we see that this homomorphism is an epimorphism for $n = 2, 4, 8$. Furthermore in the Bidegree–Realization Proposition 216.22 we will show that the homomorphism is non-trivial for all even n.
- In Chapter 217 we will state and prove the Basic Sphere–Alexander Duality Theorem 217.8 which says, in a simplified version, that for any compact submanifold $K \subseteq S^n$ and any $j \in \mathbb{N}_0$ we have an isomorphism
$$\widetilde{\mathrm{H}}_j(S^n \setminus K) \;\cong\; \mathrm{H}^{n-j-1}(K;\mathbb{Z}).$$
- In Chapter 218 we will state and prove the General Alexander Duality Theorem 218.2 which says, again in a simplified version, that for any compact subspace $K \subseteq S^n$ which is "weakly locally contractible" and any $j \in \mathbb{N}_0$ we have a natural isomorphism
$$\widetilde{\mathrm{H}}_j(S^n \setminus K) \;\xrightarrow{\cong}\; \mathrm{H}^{n-j-1}(K;\mathbb{Z}).$$
- In Chapter 219 we will state and prove two fun "real life"-applications of cohomology. First of all, in the Table Theorem 219.3 we will show that, under suitable hypothesis, we can always arrange a table to be horizontal. Secondly, we will use the General Alexander Duality Theorem 218.2 to prove the Inscribed Rectangle Proposition 219.25 which says that every Jordan curve admits an inscribed rectangle.
- In Chapter 220, given a closed smooth manifold M, we will (re-) introduce the algebraic intersection number of two transverse submanifolds of M of opposite dimension. Furthermore, in the Cup Product–Intersection Number Theorem 220.10 we will relate the algebraic intersection number to the cup product. We will see that if two smooth manifolds have non-zero algebraic intersection number, then
- In Chapter 221 we generalize some results and concepts from Chapter 220. More precisely, we will introduce the notion of transverse cycles, we will introduce the algebraic intersection number of transverse cycles and in the Maps-Cup Product–Intersection Number Theorem 221.16 we will relate this algebraic intersection number to the cup product.
- In Chapter 222 we will use the Cup Product–Intersection Number Theorem 220.10 to prove the Lefschetz-Hopf Theorem 222.9 and we will use the more powerful Maps-Cup

Product Intersection Number 221.16 to prove the more powerful Lefschetz-Hopf Theorem 222.14. Both Lefschetz-Hopf Theorems relate, given a suitable self-map $f\colon X \to X$ of a topological space, the Lefschetz number of f to the sum of the indices of the fixed points of X.

- In Chapter 223 we will use the Lefschetz-Hopf Theorem 222.14 to prove the Poincaré-Hopf Theorem 223.8 which relates, given a smooth manifold with a vector field v, the Euler characteristic of M to the sum of the indices of the zeros of the vector field.

CHAPTER 213

The cup product of projective spaces

In this chapter we will use the Poincaré Duality Theorem 210.1 to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. These calculations will allow us to prove several interesting consequences which at first glance have very little to do with cup products.

This chapter is organized as follows:

- In Section 213.1 we will discuss non-singular pairings and applications to the cup product.
- In Section 213.2 we will determine the cup product of $\mathbb{C}\mathrm{P}^n$ and we will study applications of this calculation.
- In Section 213.3 we will determine the cup product of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients and we will study applications of this calculation.
- In Appendix 213.4 we will give yet another proof of the Borsuk-Ulam Theorem 47.12, this time via cup products.
- In Section 213.5 we will provide exercises for Chapter 213.

In the next Chapter 214 we will discuss quaternions, octonions and other division algebras.

213.1. **Non-singular pairings and the cup product.** Before we start drawing more conclusions from the Poincaré Duality Theorem 210.1 we need to introduce some purely algebraic definitions:

Definition 213.1. Let R be a commutative ring and let V, W and S be R-modules.

(1) A pairing on $V \times W$ is a map
$$\varphi\colon V \times W \to S$$
such that for all $v, v' \in V$ and $w, w' \in W$ we have $\varphi(v+v', w) = \varphi(v,w) + \varphi(v',w)$ and $\varphi(v, w+w') = \varphi(v,w) + \varphi(v,w')$.

(2) A pairing is called R-bilinear if it is R-linear in V and if it is R-linear in W.[2260]

(3) We say that an R-bilinear pairing $\langle\ ,\ \rangle\colon V \times W \to S$ is non-singular if the induced maps
$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \begin{pmatrix} W & \to & S \\ w & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \begin{pmatrix} V & \to & S \\ v & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array}$$
are isomorphisms. Otherwise we say that the R-bilinear pairing is singular.

We start out with the following elementary lemma:

Lemma 213.2. (Non-Singularity Lemma) Let R be a commutative ring. Let V and W be R-modules and let
$$\langle\ ,\ \rangle\colon V \times W \to R$$

[2260] Note that it follows from the elementary discussion on page 122 that a pairing is always $\mathbb{Z}$-bilinear.

be an R-bilinear pairing.

(1) We suppose that V and W are finitely generated free R-modules. Let $v_1, \dots, v_m$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W. Then the following statement holds:[2261]

$$\text{The pairing } \langle\,,\,\rangle \text{ is non-singular} \iff \begin{array}{ll} \text{(i)} & m = n \text{ and} \\ \text{(ii)} & \det(\langle v_i, w_j\rangle)_{i,j=1,\dots,n}) \text{ is a unit of } R. \end{array}$$

(2) If V and W are free R-modules of finite rank, then the pairing is non-singular if and only if *one* of the two maps $V \to \operatorname{Hom}(W, R)$ or $W \to \operatorname{Hom}(V, R)$ is an isomorphism.

Proof.

(1) Let $v_1, \dots, v_m$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W. We denote by $v_1^*, \dots, v_m^* \in \operatorname{Hom}(V, R)$ and $w_1^*, \dots, w_n^* \in \operatorname{Hom}(W, R)$ the dual bases. We consider the maps

$$\begin{array}{rcl} V & \to & \operatorname{Hom}_R(W, R) \\ v & \mapsto & \left(\begin{array}{rcl} W & \to & R \\ w & \mapsto & \langle v, w\rangle \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{rcl} W & \to & \operatorname{Hom}_R(V, R) \\ w & \mapsto & \left(\begin{array}{rcl} V & \to & R \\ v & \mapsto & \langle v, w\rangle \end{array} \right). \end{array}$$

Since $\langle\,,\,\rangle$ is R-bilinear, both maps are homomorphisms of R-modules. Furthermore, it is straightforward to show that with respect to these bases the homomorphisms of R-modules are represented by the (transpose of the) matrix $(\langle v_i, w_j\rangle_{i=1,\dots,m, j=1,\dots,n})$. The promised statement is a straightforward consequence of these observations and the fact that for a commutative ring a matrix is invertible only if its determinant is a unit.[2262]

(2) This statement follows immediately from Statement (1) and the fact that a square matrix and its transpose have the same determinant. ∎

Now we return to topology:

Proposition 213.3. (Manifold-Non Singularity Proposition) Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) If M is oriented and if $\mathrm{H}^k(M;\mathbb{Z})$ and $\mathrm{H}^{n-k}(M;\mathbb{Z})$ are torsion-free, then the pairing

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) \times \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$

is non-singular.

(2) Let $\mathbb{F}$ be a field. If M is $\mathbb{F}$-oriented, then the pairing

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{F}) \times \mathrm{H}^{n-k}(M;\mathbb{F}) & \to & \mathbb{F} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]_{\mathbb{F}}\rangle \end{array}$$

is non-singular.

[2261] Here is a useful special case. Suppose that V and W are of rank one. If v and w are generators of V respectively W, then the following statement holds:

$$\text{the pairing } \langle\,,\,\rangle \text{ is non-singular} \iff \langle v, w\rangle \text{ is a unit of the ring } R.$$

[2262] If R is a domain, then this fact can be seen by going from R to its quotient field. For an arbitrary non-zero commutative ring R this fact can be proved by picking a maximal ideal $\mathfrak{m}$ and considering the field $R/\mathfrak{m}$. We refer to [**Hun1980**, Corollary IV.2.12] or alternatively to [**Eisen1995**, Corollary 4.4] for details.

Proof. Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) We suppose that M is oriented and that $\mathrm{H}^k(M;\mathbb{Z})$ and $\mathrm{H}^{n-k}(M;\mathbb{Z})$ are torsion-free. By the Non-Singularity Lemma 213.2 (2) and Definition 213.1 it suffices to show that the map

$$\begin{aligned} \Omega\colon \mathrm{H}^{n-k}(M;\mathbb{Z}) &\to \mathrm{Hom}(\mathrm{H}^k(M;\mathbb{Z}),\mathbb{Z}) \\ \psi &\mapsto (\varphi \mapsto \langle \varphi \cup \psi, [M]\rangle) \end{aligned}$$

is an isomorphism. We start out with the following slightly confusing claim:

Claim. The following diagram commutes:[2263]

$$\begin{array}{ccc} \mathrm{H}^{n-k}(M;\mathbb{Z}) & \xrightarrow{\quad\Omega\quad} & \mathrm{Hom}(\mathrm{H}^k(M;\mathbb{Z}),\mathbb{Z}). \\ & {\scriptstyle \mathrm{ev}}\searrow \quad \nearrow{\scriptstyle (\cap[M])^*} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{n-k}(M),\mathbb{Z}) & \end{array}$$

Proof. Note that for any $\psi \in \mathrm{H}^{n-k}(M;\mathbb{Z})$ and any $\varphi \in \mathrm{H}^k(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:[2264]

$$\Omega(\psi)(\varphi) \underset{\text{by definition of } \Omega \text{ and by the Cap Product-Kronecker Lemma 206.12}}{=} \epsilon_M((\varphi\cup\psi)\cap[M]) \underset{\text{by the Cup-Cap Lemma 206.15}}{=} \epsilon_M(\psi\cap(\varphi\cap[M])) \underset{\text{by the Cap Product Kronecker Lemma 206.12}}{=} \langle\psi,\varphi\cap[M]\rangle = \mathrm{ev}(\psi)(\varphi\cap[M]).$$

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms:

- By the Universal Coefficient-H*-Theorem 201.23 we have a short exact sequence

$$0 \to \underbrace{\mathrm{Ext}(\mathrm{H}_{n-k-1}(M),\mathbb{Z})}_{\substack{\text{torsion by the Ext-Properties Lemma 201.19} \\ \text{and the } \mathbb{Z}/k\mathbb{Z}\text{-Ext-Lemma 201.18}}} \to \underbrace{\mathrm{H}^{n-k}(M;\mathbb{Z})}_{\substack{\text{torsion-free by} \\ \text{hypothesis}}} \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{n-k}(M),\mathbb{Z}) \to 0.$$

  It follows from the above data that the left homomorphism is trivial, thus the right homomorphism is, as desired, an isomorphism.
- The diagonal map to the right is an isomorphism by the Poincaré Duality Theorem 210.1.

(2) Now let $\mathbb{F}$ be a field. We suppose that M is $\mathbb{F}$-oriented. It follows from the Ring–Kronecker Pairing Lemma 202.1 and the Universal Coefficient-H*–Field Theorem 202.2 that the map

$$\begin{aligned} \mathrm{ev}\colon \mathrm{H}^k(M;\mathbb{F}) &\to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F}),\mathbb{F}) \\ \varphi &\mapsto \begin{pmatrix} \mathrm{H}_k(M;\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle\varphi,\sigma\rangle \end{pmatrix} \end{aligned}$$

is well-defined and that it is a natural isomorphism of $\mathbb{F}$-vector spaces. The proof of (2) is now basically the same as the proof of (1). ■

[2263] The right diagonal map is the dual map corresponding to the map $\mathrm{H}^k(M;\mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{H}_{n-k}(M)$.

[2264] Here $\epsilon_M\colon \mathrm{H}_0(M) \to \mathbb{Z}$ is the augmentation homomorphism from Definition 143.23.

Example. Let $n \in \mathbb{N}$. We want to determine the pairing

$$\begin{array}{rcl} \mathrm{H}^n(S^n \times S^n; \mathbb{Z}) \times \mathrm{H}^n(S^n \times S^n; \mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [S^n \times S^n] \rangle. \end{array}$$

We start out with the following preparations:

(1) We denote by $p\colon S^n \times S^n \to S^n$ and by $q\colon S^n \times S^n \to S^n$ the two natural projections.

(2) By the Product-of-Spheres–Cup Product Lemma 207.3 we know that two pullbacks $p^*([S^n]^*)$, $q^*([S^n]^*)$ form a basis of $\mathrm{H}^n(S^n \times S^n; \mathbb{Z})$.

(3) We have

$$p^*([S^n]^*) \cup p^*([S^n]^*) \underset{\substack{\uparrow \\ \text{by the Cup Product} \\ \text{Naturality Lemma 203.9}}}{=} p^*([S^n]^* \cup [S^n]^*) \underset{\substack{\uparrow \\ \text{since we have} \\ [S^n]^* \cup [S^n]^* \in \mathrm{H}^{2n}(S^n;\mathbb{Z}) = 0}}{=} 0.$$

The same argument shows of course that $q^*([S^n]^*) \cup q^*([S^n]^*) = 0$.

(4) It follows from the Cup Product-Commutativity Proposition 203.20 that the above pairing is symmetric if n is even and it is antisymmetric if n is odd.

It follows from the above discussion together with the Non-Singularity Lemma 213.2 that for some $\epsilon \in \mathbb{Z}$ we obtain the following table:

$\langle \varphi \cup \psi, [S^n \times S^n] \rangle$	$p^*([S^n]^*)$	$q^*([S^n]^*)$
$p^*([S^n]^*)$	0	ϵ
$q^*([S^n]^*)$	$(-1)^n \cdot \epsilon$	0.

Finally note that by the Manifold-Non Singularity Proposition 213.3 we know that the pairing is non-singular. By the Non-Singularity Lemma 213.2 this implies that the above matrix is invertible, i.e. we see that $\epsilon \in \{-1, 1\}$.

This result recovers (up to the sign indeterminacy) the calculation of the non-trivial cup product on $S^n \times S^n$ that we had initially performed in the Product-of-Spheres–Cup Product Lemma 207.3 using the somewhat mysterious Cross-Cup-Cap Theorem 207.1. □

213.2. **The cup product of $\mathbb{C}\mathrm{P}^n$ and applications.** In this section we apply our acquired wisdom to the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

Definition 213.4.

(1) Given $m \leq n$ we consider the embedding[2265]

$$\begin{array}{rcl} i\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_m] & \mapsto & [z_0 : \cdots : z_m : 0 : \cdots : 0]. \end{array}$$

(2) We set $\mathbb{C}\mathrm{P}^\infty := \operatorname{colim}_{\longrightarrow} \mathbb{C}\mathrm{P}^n$.

We prepare the ground for calculating the cup product of projective spaces with the following elementary lemma:

Lemma 213.5. ($\mathbb{C}\mathrm{P}^n$-H^*-Lemma)

(1) For every $n \in \mathbb{N} \cup \{\infty\}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \in \{0, 2, 4, \ldots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

[2265] On page 338 we showed that the given map $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ is indeed an embedding.

(2) Let $m \leq n \leq \infty$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. For every $k \in \mathbb{N}_0$ with $k \leq 2m$ the induced homomorphism $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ is an isomorphism.

$$\begin{array}{ccc} 0 & & 0 \\ 0 & & \mathrm{H}^6(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \cong \mathbb{Z} \\ \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z} & \xleftarrow[\cong]{i^*} & \mathrm{H}^4(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \cong \mathbb{Z} \\ \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z} & \xleftarrow[\cong]{i^*} & \mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \cong \mathbb{Z} \end{array}$$

Proof. We first consider the analogous statements for homology groups:

(1) Let $n \in \mathbb{N}_0 \cup \{\infty\}$. In the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 we showed that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Recall that on page 2628 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with one cell in all even dimensions up to $2n$ and with no cells in any other dimension. Furthermore we saw that the CW-structure can be chosen in such a way that for $k \in \{0, \ldots, n\}$ the $(2k+1)$-skeleton on $\mathbb{C}\mathrm{P}^n$ equals $i(\mathbb{C}\mathrm{P}^k)$. It follows from the CW-Skeleton-H_*-Lemma 155.2 (3) that for any $k \in \mathbb{N}_0$ with $k \leq 2m$ the induced map $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n)$ is an isomorphism.

The statements regarding cohomology groups follow from the above discussion of homology groups together with the observation that from the Universal Coefficient-H^*-Theorem 201.23 and the Ext-Properties Lemma 201.19 we obtain for any $m \leq n \leq \infty$ the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) & \xrightarrow{\quad i^* \quad} & \mathrm{H}^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \\ {\scriptstyle \mathrm{ev}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{ev}}\downarrow{\scriptstyle \cong} \\ \mathrm{Hom}(\mathrm{H}_k(\mathbb{C}\mathrm{P}^n),\mathbb{Z}) & \xrightarrow[\substack{\text{isomorphism by the above,} \\ \text{for all } k \in \mathbb{N}_0 \text{ with } k \leq 2m}]{\quad i^* \quad} & \mathrm{Hom}(\mathrm{H}_k(\mathbb{C}\mathrm{P}^m),\mathbb{Z}). \end{array}$$

■

We also introduce the following notation:

Notation 213.6.

(1) Let $n \in \mathbb{N}_0$. In the Projective Space–Manifold Proposition 25.13 and the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we showed that $\mathbb{C}\mathrm{P}^n$ is a closed $2n$-dimensional smooth manifold that comes with a natural orientation. Thus we will always view $\mathbb{C}\mathrm{P}^n$ as an *oriented* smooth manifold. In particular $\mathbb{C}\mathrm{P}^n$ comes with a canonical fundamental class $[\mathbb{C}\mathrm{P}^n] \in \mathrm{H}_{2n}(\mathbb{C}\mathrm{P}^n)$ and it comes with a canonical dual fundamental class $[\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

(2) For any $n \in \mathbb{N} \cup \{\infty\}$ we denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the above embedding. By the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 we can define $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. We refer to $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ as the standard generator. Note that it follows from the functoriality of cohomology that for any $1 \in k \leq \infty$ the restriction of $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^l;\mathbb{Z})$ to $\mathrm{H}^2(\mathbb{C}\mathrm{P}^l;\mathbb{Z})$ equals x.

Now we have the tools to determine the cup product of all complex projective spaces. More precisely, we can now prove the following proposition:[2266]

Proposition 213.7. (Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition)

(1) Let $n \in \mathbb{N}_0 \cup \{\infty\}$. For every $k \in \mathbb{N}_0$ with $k \leq n$ the k-th power[2267] x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$.

(2) (a) Let $n \in \mathbb{N}_0$. If we view $\mathbb{Z}[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map

$$\mathbb{Z}[x]/(x^{n+1}) \ \to \ (\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}),\cup)$$

is an isomorphism of superalgebras.

(b) If we view $\mathbb{Z}[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map

$$\mathbb{Z}[x] \ \to \ (\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}),\cup)$$

is an isomorphism of superalgebras.[2268]

(c) Let $m \in \mathbb{N}$. Given $i \in \{1,\ldots,m\}$ we denote by $p_i\colon (\mathbb{C}\mathrm{P}^\infty)^m \to \mathbb{C}\mathrm{P}^\infty$ the i-th projection. If we view $\mathbb{Z}[x_1,\ldots,x_m]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x_i) = 2$, then the map

$$\mathbb{Z}[x_1,\ldots,x_m] \ \to \ (\mathrm{H}^*(\underbrace{\mathbb{C}\mathrm{P}^\infty\times\ldots\times\mathbb{C}\mathrm{P}^\infty}_{m \text{ copies}};\mathbb{Z}),\cup)$$

which sends x_i to $p_i^*(x)$ is an isomorphism of superalgebras. We have obvious variations if we take finite products of arbitrary complex projective spaces.

Proof.

(1) (a) We first deal with the case $n \in \mathbb{N}_0$. We prove this statement by induction on n. Evidently the statement holds for $n = 0$. Now suppose that it holds for some $n-1$ with $n \geq 2$. We need to show that for every $k \in \{0,\ldots,n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. In the following two claims we deal with the cases $k \in \{0,\ldots,n-1\}$ and $k = n$ separately:

Claim 1. For $k \in \{0,\ldots,n-1\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator.

Proof. Let $i\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ be the usual embedding and let $\widetilde{x} \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$ and $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ be the standard generators. It follows easily from the functoriality

[2266] The proposition is the "singular cohomology" analog of the Wedge Product-$\mathbb{C}\mathrm{P}^n$-Proposition 85.2. In fact the proofs are formally almost identical, both proofs reduce the calculation to a Poincaré Duality theorem.

[2267] Let X be a topological space, let R be a commutative ring and let $\varphi \in \mathrm{H}^d(X;R)$. As in any ring we define $\varphi^0 = 1_X$ and given any $k \in \mathbb{N}$ we define $\varphi^k := \underbrace{\varphi \cup \cdots \cup \varphi}_{k \text{ times}} \in \mathrm{H}^{kd}(X;R)$.

[2268] In Exercise 244.2 we will give a much more geometric proof of this proposition.

of cohomology that we have $i^*(x) = \widetilde{x}$. We consider the following diagram:

$$\begin{array}{ccc} \overset{\widetilde{x}^k}{\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})} & \xleftarrow[\cong]{i^*} & \overset{x^k}{\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^{n};\mathbb{Z})} \\ {\scriptstyle \varphi \mapsto \varphi^k}\uparrow & & \uparrow{\scriptstyle \varphi \mapsto \varphi^k} \\ \underset{\widetilde{x}}{\mathrm{H}^{2}(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})} & \xleftarrow[\cong]{i^*} & \underset{x}{\mathrm{H}^{2}(\mathbb{C}\mathrm{P}^{n};\mathbb{Z})}. \end{array}$$

It follows from the Cup Product-Naturality Lemma 203.9 that the diagram commutes and it follows from the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 (2) that the horizontal maps are isomorphisms. Since $k \leq n-1$ we obtain that $\widetilde{x}^k$ is by induction hypothesis a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$ we deduce that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^{n};\mathbb{Z})$. ⊞

It remains to prove the following claim:

Claim 2. The n-th power $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator.

Proof. We consider the pairing

$$\begin{array}{rcl} \overbrace{\mathrm{H}^{2}(\mathbb{C}\mathrm{P}^{n};\mathbb{Z})}^{=\mathbb{Z}\cdot x} \times \overbrace{\mathrm{H}^{2n-2}(\mathbb{C}\mathrm{P}^{n};\mathbb{Z})}^{=\mathbb{Z}\cdot x^{n-1}} & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [\mathbb{C}\mathrm{P}^n] \rangle. \end{array}$$

As we discussed above, $\mathbb{C}\mathrm{P}^n$ is a closed orientable $2n$-dimensional topological manifold. It thus follows from the Manifold-Non Singularity Proposition 213.3 (1) that the above pairing is non-singular. It follows from this observation, the fact that $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $x^{n-1} \in \mathrm{H}^{2n-2}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are basis elements and the Non-Singularity Lemma 213.2 (1) that $\langle x \cup x^{n-1}, [\mathbb{C}\mathrm{P}^n] \rangle = \pm 1$. By the Top Cohomology Theorem 201.26 this implies that $x \cup x^{n-1} = \pm [\mathbb{C}\mathrm{P}^n]^*$. ⊞

(b) Now we consider the case $n = \infty$. Let $k \in \mathbb{N}_0$. By (1) we know that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$. As in the proof of Claim 1 above we obtain from the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 (2) together with the Cup Product-Naturality Lemma 203.9 that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. ■

For completeness we also deal with the cup product of products of complex projective spaces:

Proposition 213.8. (Cup Product-Multiple-$\mathbb{C}\mathrm{P}^n$-Proposition) Let $m \in \mathbb{N}$. Given any $i \in \{1, \ldots, m\}$ we denote by $p_i \colon (\mathbb{C}\mathrm{P}^\infty)^m \to \mathbb{C}\mathrm{P}^\infty$ the i-th projection. If we view $\mathbb{Z}[x_1, \ldots, x_m]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x_i) = 2$, then the map

$$\mathbb{Z}[x_1, \ldots, x_m] \to (\mathrm{H}^*(\underbrace{\mathbb{C}\mathrm{P}^\infty \times \ldots \times \mathbb{C}\mathrm{P}^\infty}_{m \text{ copies}};\mathbb{Z}), \cup)$$

which sends x_i to $p_i^*(x)$ is an isomorphism of superalgebras. We have obvious variations if we take finite products of arbitrary complex projective spaces.

Proof. First note that it follows immediately from the definitions and the Cup Product-Commutativity Proposition 203.20 that the maps in (2) are homomorphisms of superalgebras. This observation, together with the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5, shows that (2a) and (2b) are a consequence of (1).

Finally note that Statement (2c) follows from (2b) and from iteratively applying the Cohomology Ring-of-Product Proposition 208.12 (which we can apply by the CW–Chain Complex Condition Lemma 208.2 since $\mathbb{C}\mathrm{P}^\infty$ has a CW-structure with at most one cell in each dimension) to the product $\mathbb{C}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty \times (\cdots \times \mathbb{C}\mathrm{P}^\infty)\dots)$ and from the discussion of tensor products of polynomial rings on page 3996. ■

The proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 leaves it open whether nor not $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equals the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$. We formulate this as a question which we will answer later in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20, when we will have obtained more tools for calculating cup products.

Question 213.9. Let $n \in \mathbb{N}$. Does the n-th power $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equal the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$?

Using the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 we can now distinguish pairs of topological spaces which had so far eluded all our attempts to tell them apart. For the purpose of this discussion we introduce the following definition:

Definition 213.10. We say that two path-connected non-empty topological spaces X and X' are doppelgänger of one another if the fundamental groups $\pi_1(X)$ and $\pi_1(X')$ are isomorphic and if for all $k \in \mathbb{N}_0$ the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(X')$ are isomorphic.[2269]

Pairs of doppelgänger are notoriously hard to distinguish. The following corollary reminds us of two doppelgänger and it shows, using the cup product, that they are not homotopy equivalent. The corollary gives in particular an answer to Question 164.8.

Corollary 213.11. (Doppelgänger Corollary)
(1) $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are doppelgänger but they are not homotopy equivalent.
(2) $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are doppelgänger but they are not homotopy equivalent.

Proof.

(1) (a) It follows from the Sphere-π_1-Proposition 88.24, the Wedge-π_1-Proposition 99.10 and the Projective Space–π_1-Proposition 135.8 (2) that $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are both simply connected. Furthermore it follows from the Sphere-H_*-Proposition 149.4, the Wedge-H_*-Proposition 149.16 and the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(S^2 \vee S^4;\mathbb{Z}) \;\cong\; \mathrm{H}_k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(b) In the Wedge-of-Spheres–Cup Product Lemma 203.29 we saw that the cup product

$$\cup\colon \mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \times \mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \;\to\; \mathrm{H}^4(S^2 \vee S^4;\mathbb{Z})$$

is the zero-map. On the other hand we saw in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 (1) that the corresponding cup product for $\mathbb{C}\mathrm{P}^2$ is non-trivial. It follows from these observations and the Homotopy Equivalence-H^*-Proposition 198.12 (2) together with the Cup Product-Ring Proposition 203.25 (2) that $S^2 \vee S^4$ and X are not homotopy equivalent.

[2269] It follows from the Universal Coefficient-H^*-Theorem 201.23 that all cohomology groups are also isomorphic.

(2) (a) On page 3143 we already pointed out that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ cannot be distinguished using the fundamental group and using homology groups. In other words, we showed that they are doppelgänger. As on page 4083 it follows from the Cup Product-Naturality Lemma 203.9 that the cup product

$$\underbrace{\mathrm{H}^2(S^2 \times S^4; \mathbb{Z})}_{=p^*([S^2]^*)} \times \underbrace{\mathrm{H}^2(S^2 \times S^4; \mathbb{Z})}_{=p^*([S^2]^*)} \to \mathrm{H}^4(S^2 \times S^4; \mathbb{Z})$$

is zero. On the other hand we saw in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 that the cup product

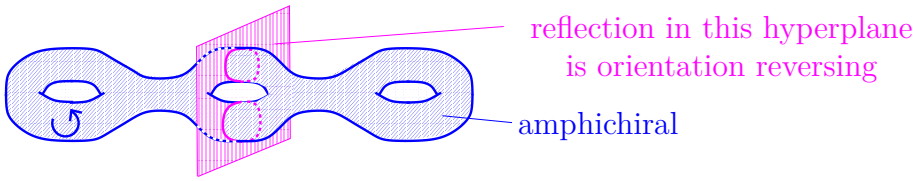
$$\mathrm{H}^2(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \times \mathrm{H}^2(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \to \mathrm{H}^4(\mathbb{C}\mathrm{P}^3; \mathbb{Z})$$

is non-zero. As in (1b) it follows that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent. ■

Next we recall Definition 196.5:

Definition 213.12. Let M be a compact oriented topological manifold. We say that M is amphichiral if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is chiral.

On page 3794 we saw that many smooth manifolds are amphichiral, for example we showed that all compact oriented 2-dimensional topological manifolds are amphichiral. In particular $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ is amphichiral.



In Question 196.11 we asked whether there exist compact oriented topological manifolds that are chiral. Now we will see that the complex projective space $\mathbb{C}\mathrm{P}^2$ is chiral. This gives an affirmative answer to Question 196.11. Note that the following corollary also gives a negative answer to Question 186.14 (3).

Corollary 213.13. ($\mathbb{C}\mathrm{P}^n$-Amphichiral Corollary) Given any $n \in \mathbb{N}$ we have

$$\text{the smooth manifold } \mathbb{C}\mathrm{P}^n \text{ is amphichiral} \iff n \text{ is odd.}$$

Proof. First we prove the "⇒"-direction. Suppose that there exists an orientation-reversing self-homeomorphism $f\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$. We make a few observations and preparations:

(1) It follows from the discussion on page 3773 and the Dual Fundamental Class–Degree Lemma 201.28 that $f^*([\mathbb{C}\mathrm{P}^n]^*) = -[\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$.
(2) As usual let $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ be the standard generator. Since f is a homeomorphism we see that $f^*(x) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is a generator. This implies that there exists an $\eta \in \{-1, 1\}$ with $f^*(x) = \eta \cdot x$.
(3) By the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 we know that $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is a generator. This implies that $x^n = \epsilon \cdot [\mathbb{C}\mathrm{P}^n]^*$ for some $\epsilon \in \{-1, 1\}$.

[2269] and it gives a complete solution to Exercise 31.26

Now we see that

$$-[\mathbb{C}\mathrm{P}^n]^* = f^*([\mathbb{C}\mathrm{P}^n]^*) = \epsilon\cdot f^*(x^n) \underset{\uparrow}{=} \epsilon\cdot(f^*(x))^n = \epsilon\cdot(\eta\cdot x)^n \underset{\uparrow}{=} \epsilon\cdot\eta^n\cdot x^n = \eta^n\cdot[\mathbb{C}\mathrm{P}^n]^*.$$

by the Cup Product Naturality Lemma 203.9 — since $\cup$ is bilinear

Thus we have shown that for some $\eta \in \{-1, 1\}$ we have $\eta^n = -1$. But this is only possible if n is odd.

Now we prove the "$\Rightarrow$"-direction. It remains to prove the following claim:

Claim. For any odd $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_n] & \mapsto & [\overline{z}_0 : \cdots : \overline{z}_n] \end{array}$$

is an orientation-reversing diffeomorphism.

Proof. First note that this map is indeed well-defined and that it is its own inverse. One can easily verify that f is smooth, thus f is in fact a diffeomorphism. Next we consider the chart

$$\begin{array}{rcl} \Phi\colon \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_0 \neq 0\} & \to & \mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n} \\ [z_0 : z_1 : \cdots : z_n] & \mapsto & \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) \end{array}$$

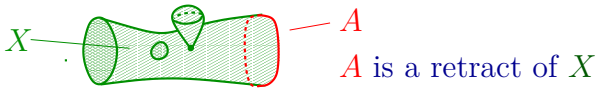
from the Projective Space–Manifold Proposition 25.13. Note that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{(w_1,\ldots,w_n)\mapsto(\overline{w}_1,\ldots,\overline{w}_n)} & \mathbb{C}^n \\ {\scriptstyle\Phi^{-1}}\downarrow & & \downarrow{\scriptstyle\Phi^{-1}} \\ \mathbb{C}\mathrm{P}^n & \xrightarrow{\quad f \quad} & \mathbb{C}\mathrm{P}^n. \end{array}$$

Since n is odd one sees that the top horizontal map is orientation-reversing. But this implies that the bottom horizontal map is also orientation-reversing. ∎

Towards the end of this section we recall the definition of a retract.

Definition 213.14. Let A be a subspace of a topological space X. We denote by $i\colon A \to X$ the inclusion map. Recall that we say that A is a retract of X if there exists a continuous map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. (We recall that it follows immediately from the functoriality of homology groups and the Splitting Lemma 145.5 (3)⇒(2) that for each k the group $\mathrm{H}_k(A)$ is a subsummand of $\mathrm{H}_k(X)$.)



Let $m < n$. In Question 167.7 we had asked whether $i(\mathbb{C}\mathrm{P}^m) \subseteq \mathbb{C}\mathrm{P}^n$ is a retract of $\mathbb{C}\mathrm{P}^n$. Note that the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 shows that a study of homology groups does not preclude that for $m < n$ the complex projective space $\mathbb{C}\mathrm{P}^m$ is a retract of $\mathbb{C}\mathrm{P}^n$. Nonetheless, the final result of the section gives a negative answer to Question 167.7, i.e. we show that $i(\mathbb{C}\mathrm{P}^m)$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proposition 213.15. ($\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition) If $1 \leq m < n \leq \infty$, then $i(\mathbb{C}\mathrm{P}^m) \subseteq \mathbb{C}\mathrm{P}^n$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proof. Let $1 \leq m < n \leq \infty$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Suppose that there exists a retraction $r\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$. We denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ the standard

generator. We have

$$\begin{array}{ccccccccc} x^n & = & (r^*(i^*(x)))^n & \overset{\text{by the Cup Product-Naturality Lemma 203.9 applied to } r}{=} & r^*(i^*(x)^n) & = & r^*(0) & = & 0 \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}). \\ & \uparrow & & & & \uparrow & & & \end{array}$$

by the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 the induced map $i^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ is an isomorphism, since $i^* \circ r^* = (r \circ i)^* = \mathrm{id}$ we obtain that $r^* \circ i^* = \mathrm{id}$

note that $n > m$ implies that $i^*(x)^n$ lies in the trivial group $\mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = 0$

$$\begin{array}{lll} i^*(x)^3 = 0 & 0 & \xrightarrow{r^*} \mathrm{H}^6(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \quad x^3 = (r^*(i^*(x)))^3 = r^*(i^*(x)^3) \\ \uparrow & \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) & \underset{r^*}{\overset{i^* \cong}{\rightleftarrows}} \mathrm{H}^4(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \quad \uparrow \\ i^*(x) & \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) & \mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \quad x = r^*(i^*(x)) \end{array}$$

But this calculation contradicts the fact shown in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 that $x^n \neq 0 \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. ■

213.3. **The cup product of $\mathbb{R}\mathrm{P}^n$ and applications.** In this section we intend to determine the cup product of the real projective space $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. Evidently the approach will be very similar to the one taken in the previous section.

Definition 213.16.

(1) Given $m \leq n$ we consider the embedding[2270]

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 : \cdots : x_m] & \mapsto & [x_0 : \cdots : x_m : 0 : \cdots : 0]. \end{array}$$

(2) We set $\mathbb{R}\mathrm{P}^\infty := \operatorname{colim}_{\longrightarrow} \mathbb{R}\mathrm{P}^n$.

The following lemma plays the role of the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5:

Lemma 213.17. ($\mathbb{R}\mathrm{P}^n$-H^*-Lemma)

(1) For every $n \in \mathbb{N} \cup \{\infty\}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $m \leq n \leq \infty$. For any $k \leq m$ the induced map $i^*\colon \mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \mathrm{H}^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$ is an isomorphism.

Proof. The proof is very similar to the proof of $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5. We first show, by basically the same approach, that the analogous statements hold for homology with $\mathbb{F}_2$-coefficients. The statements regarding cohomology then follow from the Universal Coefficient-H^*–Field Theorem 202.2. We leave it to the reader to fill in the details. ■

Notation 213.18. Given $n \in \mathbb{N} \cup \{\infty\}$ let $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ be the unique non-trivial element.

The following proposition is the analog of the Cup Product-$\mathbb{C}\mathrm{P}^n$ Proposition 213.7:

Proposition 213.19. (Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition)

[2270] On page 338 we showed that the given map $i\colon \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ is indeed an embedding.

(1) Let $n \in \mathbb{N}_0 \cup \{\infty\}$. For every $k \in \mathbb{N}_0$ with $k \leq n$ the k-th power x^k is the unique non-trivial element of $\mathrm{H}^k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \mathbb{F}_2$.

(2) (a) Let $n \in \mathbb{N}_0$. If we view $\mathbb{F}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map

$$\mathbb{F}_2[x]/(x^{n+1}) \ \to \ (\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2), \cup)$$

is an isomorphism of superalgebras.

(b) If we view $\mathbb{F}_2[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map

$$\mathbb{F}_2[x] \ \to \ (\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2), \cup)$$

is an isomorphism of superalgebras.

Proof.

(1) Recall that in the Projective Space–Manifold Proposition 25.13 we saw that the real projective space $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional smooth manifold. Next recall that the $\mathbb{F}_2$-Orientability Proposition 191.22 implies that any topological manifold, in particular $\mathbb{R}\mathrm{P}^n$, is $\mathbb{F}_2$-orientable. With this observation the proof of the proposition is basically identical to the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 (1). We just need to replace the Manifold-Non Singularity Proposition 213.3 (1) by the corresponding "part (2)-statement". We leave it to the reader to fill in the details.

(2) Note that it follows immediately from the definitions and the Cup Product-Commutativity Proposition 203.20 that the maps in (2) are homomorphisms of superalgebras. This observation, together with the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 213.17, shows that (2a) and (2b) are a consequence of (1). ■

We also have the following analog of the Cup Product-Multiple-$\mathbb{C}\mathrm{P}^n$-Proposition 213.8:

Proposition 213.20. (Cup Product-Multiple-$\mathbb{R}\mathrm{P}^n$-Proposition) Let $m \in \mathbb{N}$. Given $i \in \{1, \ldots, m\}$ we denote by $p_i \colon (\mathbb{R}\mathrm{P}^\infty)^m \to \mathbb{R}\mathrm{P}^\infty$ the i-th projection. If we view $\mathbb{F}_2[x_1, \ldots, x_m]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x_i) = 1$, then the map

$$\mathbb{F}_2[x_1, \ldots, x_k] \ \to \ (\mathrm{H}^*(\underbrace{\mathbb{R}\mathrm{P}^\infty \times \ldots \times \mathbb{R}\mathrm{P}^\infty}_{m \text{ copies}}; \mathbb{F}_2), \cup)$$

which sends x_i to $p_i^*(x)$ is an isomorphism of superalgebras. We have obvious variations if we take finite products of arbitrary real projective spaces.

Proof. This proposition follows almost immediately from the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 (2b) and from iteratively applying the $\mathbb{F}_2$-Cohomology Ring-of-Product Proposition 208.16 (which we can apply by the CW–Chain Complex Condition Lemma 208.2 since $\mathbb{R}\mathrm{P}^\infty$ has a CW-structure with at most one cell in each dimension) to the product $\mathbb{R}\mathrm{P}^\infty \times (\mathbb{R}\mathrm{P}^\infty \times (\cdots \times \mathbb{R}\mathrm{P}^\infty) \ldots)$ and from the discussion of tensor products of polynomial rings on page 3996. ■

For completeness' sake we point out that we can now give a new proof of the following proposition, which is the "real" analog of the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 213.15:

Proposition 167.6 ($\mathbb{R}\mathrm{P}^n$-Non-Retraction Proposition) If $1 \leq m < n \leq \infty$, then the subspace $i(\mathbb{R}\mathrm{P}^m) \subseteq \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

Proof. The proof is totally analogous to the proof of the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 213.15. In the proof we just need to replace $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients and we need to replace the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 by the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 213.17. We leave it to the reader to fill in the details. ■

Finally recall that in Definition 205.2 we introduced the parametrization number $\mathrm{param}(M)$ of a smooth manifold and that in Definition 205.4 we introduced the Lusternik-Schnirelmann category $\mathrm{cat}(X)$ of a topological space X. One of the many consequences of our calculations from this chapter is that we can now calculate the parametrization number and the Lusternik-Schnirelmann category for $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. More precisely, we have the following proposition which gives a complete answer to Question 205.3.

Proposition 213.21. (LS Category–Projective Space Proposition) Let $n \in \mathbb{N}_0$.
(1) We have $\mathrm{param}(\mathbb{R}\mathrm{P}^n) = \mathrm{param}(\mathbb{C}\mathrm{P}^n) = n+1$.
(2) We have $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) = \mathrm{cat}(\mathbb{C}\mathrm{P}^n) = n$.

Proof. Given any $n \in \mathbb{N}_0$ we have

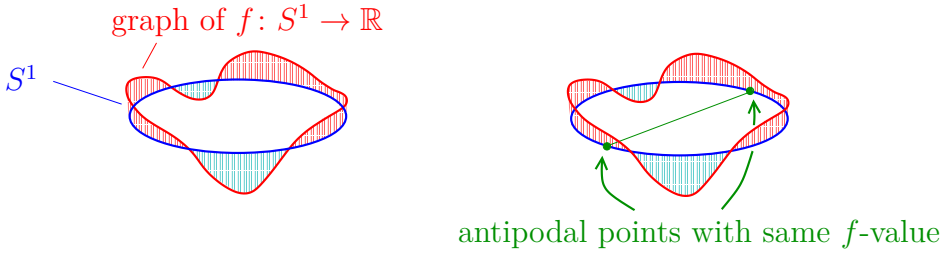
$$n \underset{\uparrow}{\geq} \mathrm{param}(\mathbb{R}\mathrm{P}^n) - 1 \underset{\uparrow}{\geq} \mathrm{cat}(\mathbb{R}\mathrm{P}^n) \underset{\uparrow}{\geq} \overbrace{\text{cup length cl}(\mathbb{R}\mathrm{P}^n)}^{\text{see Definition 205.9}} \underset{\uparrow}{\geq} n.$$

see page 3940 — by the Parametrization LS Category Lemma 205.5 — by the Cup Length–LS Category Proposition 205.10 — by the Cup Product-$\mathbb{R}\mathrm{P}^n$ Proposition 213.19

The same argument also works for $\mathbb{C}\mathrm{P}^n$, we just need to replace the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19, by the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7. ■

213.4. **Appendix: The Borsuk-Ulam Theorem via cup products.** In this appendix we will give an alternative proof for the Borsuk-Ulam Theorem. First we recall the statement of the Borsuk-Ulam Theorem.

Theorem 47.12. (Borsuk-Ulam Theorem) For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.

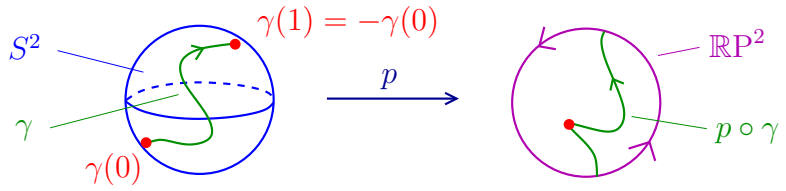


Remark. As we have pointed out on numerous occasions, the Borsuk-Ulam Theorem 47.12 has many beautiful applications, in particular the Lusternik-Schnirelmann Theorem 48.2, the Ham-Sandwich Theorem 48.4, Pfister's Level Theorem 48.15 and the Necklace Theorem 48.17. □

In the new proof of the Borsuk-Ulam Theorem below we will need two elementary lemmas.

Lemma 213.22. (Loop-in-$\mathbb{R}\mathrm{P}^n$-Lemma) Let $k \in \mathbb{N}$. We denote by $p\colon S^k \to \mathbb{R}\mathrm{P}^k$ the natural projection. Let $\gamma\colon [0,1] \to S^k$ be a path with $\gamma(1) = -\gamma(0)$.
(1) If $k = 1$, then $p \circ \gamma$ represent an odd multiple of a generator of $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$.

(2) If $k \geq 2$, then $p \circ \gamma$ represents the unique non-trivial element in[2271] $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}/2\mathbb{Z}$.



Proof of Lemma 213.22. Let $\gamma\colon [0,1] \to S^k$ be a path with $\gamma(1) = -\gamma(0)$. Note that $p \circ \gamma$ is a loop in $(\mathbb{R}\mathrm{P}^k, \gamma(0))$ and note that γ is a lift of this loop to the covering $p\colon S^k \to \mathbb{R}\mathrm{P}^k$. Since the lift γ of $p \circ \gamma$ is not a loop we obtain from the Loop–Lifting Proposition 92.8 that $[p \circ \gamma] \notin p_*(\pi_1(S^k), \gamma(0))$. This observation already deals with the case $k \geq 2$. If $k = 1$, then the desired statement follows from the following observation: Using the explicit homeomorphism $S^1 \to \mathbb{R}\mathrm{P}^1$ on page 338 and using the Circle-π_1-Theorem 88.22 one can easily show that the image $p_*(\pi_1(S^1), \gamma(0))$ is given by the even elements of $\pi_1(\mathbb{R}\mathrm{P}^1, [\gamma(0)]) \cong \mathbb{Z}$. ■

Lemma 213.23. (Isomorphism-on-H^1-Lemma) Let $k, l \in \mathbb{N}$, let $h\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$ be a continuous map and let $x_0 \in \mathbb{R}\mathrm{P}^k$. We make the following assumption:

(1) If $l = 1$, then we assume that the image of $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k, x_0) \to \pi_1(\mathbb{R}\mathrm{P}^l, h(x_0)) \cong \mathbb{Z}$ contains an odd element.

(2) If $l \geq 2$, then we assume that $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k, x_0) \to \pi_1(\mathbb{R}\mathrm{P}^l, h(x_0)) \cong \mathbb{Z}/2\mathbb{Z}$ has non-trivial image.

Then $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$ is an isomorphism.

Proof of Lemma 213.23. By the H^1-Hom-Proposition 201.24 we have a commutative diagram with vertical isomorphisms of the following form:

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^l, h(x_0)), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{h^*} & \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^k, x_0), \mathbb{Z}/2\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^1(\mathbb{R}\mathrm{P}^l; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{h^*} & \mathrm{H}^1(\mathbb{R}\mathrm{P}^k; \mathbb{Z}/2\mathbb{Z}). \end{array}$$

It follows easily from our hypothesis that the top horizontal map has non-trivial image. But since the vertical maps are isomorphisms and since all four groups in the diagram are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ we see that the top and bottom horizontal maps are isomorphisms. ■

Proof of the Borsuk-Ulam Theorem 47.12. In the following argument we deal with the case that $n \geq 3$. The case $n = 2$ can be proved the same way, we just need to replace the Loop-in-$\mathbb{R}\mathrm{P}^n$-Lemma 213.22 (2) and the Isomorphism-on-H^1-Lemma 213.23 (2) by the corresponding "(1)-Statements". Finally we leave it to the reader to modify the argument below to deal with the case $n = 1$.

[2271] Recall that for $k \geq 2$ we know from the Real Projective Space–π_1-Proposition 93.3 that we have an isomorphism $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}/2\mathbb{Z}$.

Let us suppose that there exists a continuous map $f\colon S^n \to \mathbb{R}^n$ such that for every $x \in S^n$ we have that $f(x) \neq f(-x)$. This hypothesis allows us to consider the map

$$\begin{array}{rcl} g\colon S^n & \to & S^{n-1} \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

It is clear that $g(-x) = -g(x)$ for all $x \in S^n$. Therefore this map g descends to a map

$$\begin{array}{rcl} h\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [x] & \mapsto & [g(x)]. \end{array}$$

Note that it follows from the Quotient-to-Quotient Lemma 8.6 that the map h is also continuous. In summary we get the following commutative diagram:

$$\begin{array}{ccc} S^n & \xrightarrow{\quad g \quad} & S^{n-1} \\ \downarrow{\scriptstyle p} & & {\scriptstyle q}\downarrow \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{[x]\mapsto h([x]):=[g(x)]} & \mathbb{R}\mathrm{P}^{n-1} \end{array}$$

where p and q denote the natural projections.

Claim. The map $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is an isomorphism.

Proof. Since $n \geq 3$ it follows from the Isomorphism-on-H^1-Lemma 213.23 (1) that we only have to show that $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1})$ has non-trivial image. Let $\gamma\colon [0,1] \to S^n$ be a path with $\gamma(1) = -\gamma(0)$. Note that $p \circ \gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^n$ is a loop. We calculate that

$$h_*([p\circ\gamma]) \;=\; [h\circ p\circ\gamma] \;\overset{\text{follows from the above commutative diagram}}{=}\; [q\circ(g\circ\gamma)] \;\overset{\text{by the Loop-in-}\mathbb{R}\mathrm{P}^n\text{-Lemma 213.22 (2) since } (g\circ\gamma)(1) = g(\gamma(1)) = g(-\gamma(0)) = -g(\gamma(0)) = -(g\circ\gamma)(0)}{\neq}\; e \;\in \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

We denote by $y \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ the unique non-trivial elements. We have the following equality in $\mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$:[2272]

$$0 \;\overset{\text{by the Cup Product-}\mathbb{R}\mathrm{P}^n\text{ Proposition 213.19}}{\neq}\; x^n \;\overset{\text{by the claim we have } h^*(y)=x}{=}\; (h^*(y))^n \;\overset{\text{by the Cup Product-Naturality Lemma 203.9}}{=}\; h^*(y^n) \;\overset{\text{by the Cup Product-}\mathbb{R}\mathrm{P}^n\text{ Proposition 213.19 applied to } \mathbb{R}\mathrm{P}^{n-1}}{=}\; h^*(0) \;=\; 0.$$

We have thus obtained the desired contradiction. ∎

213.5. Exercises for Chapter 213 .

Exercise 213.1. Let $m \neq n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Use Poincaré Duality to show that

$$p^*([S^m]^*) \cup q^*([S^n]^*) \;=\; \pm[S^m \times S^n]^*.$$

[2272] The argument is very similar to the argument in the proof of the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 213.15.

Put differently, you should show the equality without using the Cross-Cup-Cap Theorem 207.1.

Exercise 213.2. Let $m \in \mathbb{N}_0$ and let $k \in \{0, \ldots, m\}$. As in the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 we write $\mathrm{H}^*(\mathbb{R}\mathrm{P}^m; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{m+1})$. Show that

$$\mathrm{PD}_{\mathbb{R}\mathrm{P}^m}([\mathbb{R}\mathrm{P}^k]) \;=\; x^{m-k} \;\in\; \mathrm{H}^{m-k}(\mathbb{R}\mathrm{P}^m; \mathbb{F}_2).$$

Exercise 213.3. Does there exist a degree one map from $\mathbb{C}\mathrm{P}^2$ to $S^2 \times S^2$?

Exercise 213.4. Show that every continuous map $f\colon S^2 \times S^2 \to \mathbb{C}\mathrm{P}^2$ has even degree.

Exercise 213.5. We will consider the question whether the topological space $S^2 \vee S^4$ is homotopy equivalent to a topological manifold with various adjectives involved:

(a) Is $S^2 \vee S^4$ homotopy equivalent to a closed orientable topological manifold?
(b) Is $S^2 \vee S^4$ homotopy equivalent to a closed non-orientable topological manifold?
(c) Is $S^2 \vee S^4$ homotopy equivalent to a compact orientable topological manifold?
(d) Is $S^2 \vee S^4$ homotopy equivalent to a non-compact orientable topological manifold?
(e) Is $S^2 \vee S^4$ homotopy equivalent to a non-compact non-orientable topological manifold?

Exercise 213.6. We consider the two topological spaces $\mathbb{R}\mathrm{P}^2 \vee S^3$ and $\mathbb{R}\mathrm{P}^3$.

(a) Show that the homology groups of $\mathbb{R}\mathrm{P}^2 \vee S^3$ and $\mathbb{R}\mathrm{P}^3$ are isomorphic.
(b) Use cup products to show that $\mathbb{R}\mathrm{P}^2 \vee S^3$ and $\mathbb{R}\mathrm{P}^3$ are not homotopy equivalent.

Exercise 213.7. Is $S^3 \vee \mathbb{R}\mathrm{P}^2$ homotopy equivalent to a closed smooth manifold?

Exercise 213.8. Show that there exists a connected CW-complex X with base point x_0 and a homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z}/2\mathbb{Z} = \pi_1(\mathbb{R}\mathrm{P}^2, *)$ which cannot be realized by a continuous map $g\colon X \to \mathbb{R}\mathrm{P}^2$, i.e. such that there is no continuous map $f\colon X \to \mathbb{R}\mathrm{P}^2$ with $f(x_0) = *$ such that $f_* = \varphi$.
Remark. This gives a negative answer to Question 233.5 (3).

Exercise 213.9.

(a) Let $n \in \mathbb{N}$ and let M be a closed orientable $2n$-dimensional smooth manifold. Suppose that M admits a CW-structure with precisely three cells. Show that M has precisely one cell in dimensions 0, n and $2n$.
(b) Show that there is no closed orientable 6-dimensional smooth manifold that has only three cells.
(c) In (a) and (b), do we really need to assume that M is orientable?

Exercise 213.10. Let $n \in \mathbb{N}_0$ and let $\varphi\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be a continuous map.

(a) Show that there exists a $d \in \mathbb{Z}$ such that

$$\underbrace{\sum_{k=0}^{2n} (-1)^k \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n))}_{\text{Lefschetz number } \Lambda(\varphi)} \;=\; 1 + \sum_{l=1}^{n} d^l.$$

Hint. Use the cohomological reformulation of the Lefschetz number given in Exercise 201.12.
(b) Suppose that n is odd. Use the Lefschetz Fixed Point Theorem 182.15 to show that φ admits a fixed point.

Remark. In Exercise 182.8 we showed that for any even $n \in \mathbb{N}_0$ there exists a continuous self-map of $\mathbb{C}\mathrm{P}^n$ with no fixed points.

Exercise 213.11.
(a) For which $n \in \mathbb{N}_0$ is $\mathbb{C}\mathrm{P}^n$ homeomorphic to the suspension of a topological space?
(b) For which $n \in \mathbb{N}_0$ is $\mathbb{R}\mathrm{P}^n$ homeomorphic to the suspension of a topological space?

Exercise 213.12. Let X be a closed oriented connected non-empty n-dimensional topological manifold. Use cup products to show that if X is the homeomorphic to the suspension of a topological space, then X is a homology sphere, i.e. show that $\mathrm{H}_i(X) \cong \mathrm{H}_i(S^n)$ for all $i \in \mathbb{N}_0$.
Remark. This exercise is related to Exercise 210.2.

Exercise 213.13. In the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 we computed the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$. Modify the proof to determine the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z})$.
Remark. This cup product is also determined, using a somewhat different approach, in [**Hat2002**, p. 214].

Exercise 213.14. Use the calculation of the cup product of $S^1 \times S^2$ and of $\mathbb{R}\mathrm{P}^3$ on page 3973 and in the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 and the argument in the Sphere-to-Product-of-Spheres Lemma 207.5 to show that there is no continuous map $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$ of degree one.
Remark. This gives us an alternative proof of the No Degree-1-Map Corollary 210.17.

Exercise 213.15. Let $n \in \mathbb{N}$. Determine the image of the map

$$\begin{array}{rcl} \text{set of continuous self-maps of } \mathbb{C}\mathrm{P}^n & \to & \mathbb{Z} \\ f & \mapsto & \deg(f). \end{array}$$

Exercise 213.16. Is every 1-dimensional topological manifold amphichiral?

Exercise 213.17. For which $d \in \mathbb{Z}$ can you find a continuous map $S^2 \times S^2 \times S^2 \to \mathbb{C}\mathrm{P}^3$ of degree d?

Exercise 213.18. Recall that given a topological space X we introduced in Definition 205.9 its *cup length* $\mathrm{cl}(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows

$$\mathrm{cl}(X) \;:=\; \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } \varphi_1, \ldots, \varphi_n \text{ in degrees} \geq 1 \text{ with } \varphi_1 \cup \cdots \cup \varphi_n \neq 0 \end{array} \right\}.$$

(a) Let X be a closed connected *non-orientable* topological manifold. Show that $\mathrm{cl}(X) \geq 2$.
(b) Show that for every $n \geq 2$ we have $\mathrm{cl}(\mathbb{R}\mathrm{P}^n) \geq 2$.

Exercise 213.19. Recall that given a topological space X we introduced in Definition 205.9 its *cup length* $\mathrm{cl}(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows

$$\mathrm{cl}(X) \;:=\; \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } \varphi_1, \ldots, \varphi_n \text{ in degrees} \geq 1 \text{ with } \varphi_1 \cup \cdots \cup \varphi_n \neq 0 \end{array} \right\}.$$

Let M and N be closed orientable connected non-empty n-dimensional topological manifold. Show that if there exists a continuous map $f\colon M \to N$ of degree one, then

$$\mathrm{cl}(M) \;\geq\; \mathrm{cl}(N).$$

Remark. At least in spirit this exercise is related to the Rudyak Conjecture 205.12.

Exercise 213.20. We know that for odd n the real projective space $\mathbb{R}\mathrm{P}^n$ is orientable. For which odd $n \in \mathbb{N}$ can you find an orientation-reversing diffeomorphism $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$?

Exercise 213.21. Let X be a CW-complex which admits a continuous map $f\colon X \to \mathbb{R}\mathrm{P}^n$ which induces an isomorphism $f_*\colon \mathrm{H}_n(X;\mathbb{F}_2) \to \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. Show that X has at least $n+1$ cells.

Exercise 213.22. Show that given any odd $k \geq 3$ there exists a $\varphi \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ with $\varphi \cup \varphi \neq 0$.
Hint. By the Cup Product-Multiple-$\mathbb{R}\mathrm{P}^n$-Proposition 213.20 we understand the cohomology ring of $\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty$ with $\mathbb{F}_2$-coefficients. Now make use of the Cup Product-Coefficients Lemma 204.9, the H^*–Coefficient Change Lemma 202.11 and the H^*–Change-of-Coefficients Lemma 202.6 to get enough control over cup products with $\mathbb{Z}$-coefficients.
Remark. This exercise shows that the Cup Product–Square Zero Lemma 203.21 cannot be generalized to higher degrees.

Exercise 213.23. Let $p, q \in \mathbb{N}$ be coprime with $p \geq 2$. We consider the lens space $L(p,q)$. Does there exist a commutative ring R such that the cup product

$$\mathrm{H}^1(L(p,q);R) \times \mathrm{H}^2(L(p,q);R) \;\to\; \mathrm{H}^3(L(p,q);R)$$

is non-trivial?

Exercise 213.24. For which $n \in \mathbb{N}_0$ does there exist a closed oriented connected topological manifold X of dimension $4n+2$ such that $\mathrm{H}_{n+1}(X) \cong \mathbb{Z}$?

Exercise 213.25. Let X be a topological space. Show that if there exists an $n \in \mathbb{N}$ such that the wedge $X \vee S^n$ is homotopy equivalent to a closed orientable topological manifold, then $\mathrm{H}_i(X) = 0$ for all $i \geq 1$.

CHAPTER 214

Quaternions, octonions and Division algebras

In this chapter we do the following:

(1) We introduce the Cayley-Dickson construction which turns an $\mathbb{R}$-algebra with conjugation into a new $\mathbb{R}$-algebra, with double the dimension, with conjugation.
(2) The Cayley-Dickson construction turns $\mathbb{R}$ into the complex numbers $\mathbb{C}$, it turns $\mathbb{C}$ into the quaternions $\mathbb{H}$, it turns $\mathbb{H}$ into the octonions $\mathbb{O}$ and it turns $\mathbb{O}$ into the utterly useless sedenions $\mathbb{S}$.
(3) We will discuss the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ in detail and we will see that they are division algebras.
(4) We use cup products to prove the Hopf–Division Algebra Dimension Theorem 214.13 which says that if $\mathbb{R}^n$ admits the structure of division algebras, then n is a power of 2.

In the next Chapter 215 we will use the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ to introduce for each $n \in \mathbb{N}_0$ the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ and to introduce the octonionic projective spaces $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$.

This chapter is organized as follows:

- In Section 214.1 we will study real algebras.
- In Section 214.2 we will explain the Cayley-Dickson construction.
- In Section 214.3 we will discuss the quaternions.
- In Section 214.4 we will introduce the octonions.
- In Section 214.5 we will discuss dimensions of division algebras.
- In Section 214.6 we will provide exercises for Chapter 214.

In the next Chapter 215 we will introduce quaternionic and octonionic projective spaces.

214.1. **Real algebras.** We start out by recalling Definition 171.1:

Definition 214.1. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-algebra is an $\mathbb{F}$-vector space A together with an $\mathbb{F}$-bilinear map

$$\begin{array}{ccc} A \times A & \to & A \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

(1) We say that the algebra is commutative if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.
(2) (a) We say that the algebra is associative if for all $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
    (b) We say that the algebra is alternative if given any $a, b \in A$ we have the equalities $(a \cdot b) \cdot b = a \cdot (b \cdot b)$ and $(a \cdot a) \cdot b = a \cdot (a \cdot b)$.
(3) We say that the algebra has a multiplicatively neutral element if there exists an element $1_A \in A$ with $1_A \neq 0$ such that $1_A \cdot a = a \cdot 1_A = a$ for all $a \in A$.
(4) If A is an algebra with a multiplicatively neutral element 1_A, then given $a \in A$ we say that $b \in A$ is an inverse of a if $a \cdot b = b \cdot a = 1_A$.

(5) A division algebra is an algebra such that for all $a \neq 0$ and all $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.

Example. The real numbers $\mathbb{R}$ are evidently an $\mathbb{R}$-algebra that is commutative and associative, it has a multiplicatively neutral element, and it is a division algebra. □

Most of the above definitions are expected and standard, except perhaps for "alternative'. The following theorem gives an important property of alternative algebras:

Theorem 214.2. (Alternative Algebra Theorem) Let $\mathbb{F}$ be a field and let A be an $\mathbb{F}$-algebra. If A is alternative, then given any two $x, y \in A$ the algebra generated by 1, x and y is associative.[2273]

Proof. This theorem is due to Emil Artin. Textbook proofs are given in [**SchafR1966**, Theorem 3.1] and [**ZSSS1982**, p. 36]. ■

We move on to the concept of a conjugation on an algebra:

Definition 214.3. Let $\mathbb{F}$ be a field and let A be an $\mathbb{F}$-algebra. A conjugation on A is an $\mathbb{F}$-linear map

$$\begin{array}{ccc} A & \to & A \\ a & \mapsto & \overline{a} \end{array}$$

with the following two properties:

(1) For every $a \in A$ we have $\overline{\overline{a}} = a$.
(2) For every $a, b \in A$ we have $\overline{a \cdot b} = \overline{b} \cdot \overline{a}$.

As always there are various adjectives that one can apply to a new mathematical object:

Definition 214.4. Let A be an $\mathbb{R}$-algebra with a multiplicatively neutral element 1_A and with conjugation.

(1) We say that the conjugation is imaginary if $\overline{1}_A = 1_A$ and if for every $a \in A$ we have $a + \overline{a} \in \mathbb{R} \cdot 1_A$.
(2) (a) We say that the conjugation is positive if for every non-zero $a \in A$ we have $a \cdot \overline{a} \in \mathbb{R}_{>0} \cdot 1_A$.
    (b) If the conjugation is positive then we denote by $|a|$ the unique non-negative real number with
$$a \cdot \overline{a} = |a|^2 \cdot 1_A.$$
(3) We say that the conjugation is symmetric if for every $a \in A$ we have $a \cdot \overline{a} = \overline{a} \cdot a$.

The following lemma shows why positive symmetric conjugations are interesting and what the property of being "alternative" is good for:

Lemma 214.5. (Involution-Norm-Algebra Lemma) Let A be an $\mathbb{R}$-algebra with a multiplicatively neutral element 1_A. We assume that A is equipped with a conjugation that is positive and symmetric.

(1) Given any $a \in A \setminus \{0\}$ there exists a multiplicative inverse, namely $\frac{1}{|a|^2} \cdot \overline{a}$.
(2) If A is alternative (in particular if A is associative) and if the conjugation is imaginary, then for any $a, b \in A$ we have $|a \cdot b| = |a| \cdot |b|$.

[2273] Our main application of the Alternative Algebra Theorem 214.2 will be the octonions. For octonions an alternative proof is given in [**CSm2003**, p. 76].

Proof.

(1) Let $a \in A \setminus \{0\}$. We calculate that

$$a \cdot \left(\tfrac{1}{|a|^2} \cdot \overline{a}\right) \;=\; \tfrac{1}{|a|^2} \cdot a \cdot \overline{a} \;=\; \tfrac{1}{|a|^2} \cdot |a|^2 \cdot 1_A \;=\; 1_A.$$

Since the conjugation is symmetric we have $a \cdot \overline{a} = \overline{a} \cdot a$. It follows that we also have $(\frac{1}{|a|^2} \cdot \overline{a}) \cdot a = 1_A$.

(2) We now assume that A is alternative and that the conjugation is imaginary. Let $a, b \in A$.

Claim. The elements $a, \overline{a}, b, \overline{b}$ are contained in an associative subalgebra.

Proof. We define $\mathrm{Re}(a) := \frac{1}{2}(a + \overline{a})$ and $\mathrm{Im}(a) := \frac{1}{2}(a - \overline{a})$. Similarly we define $\mathrm{Re}(b)$ and $\mathrm{Im}(b)$. Since the conjugation is imaginary we see that $\mathrm{Re}(a), \mathrm{Re}(b) \in \mathbb{R} \cdot 1_A$. Thus we see that $a, \overline{a}, b, \overline{b}$ are contained in the subalgebra that is generated by $1, \mathrm{Im}(a)$ and $\mathrm{Im}(b)$. It follows from our hypothesis that A is alternative and the Alternative Algebra Theorem 214.2 that this subalgebra is associative. ⊞

For any $a, b \in A$ we see that we have the following equalities in A:

$$\begin{array}{lcllcll}
|a \cdot b|^2 \cdot 1_A & = & (a \cdot b) \cdot \overline{(a \cdot b)} & = & (a \cdot b) \cdot (\overline{b} \cdot \overline{a}) & \overset{\text{by the above claim}}{=} & a \cdot (b \cdot \overline{b}) \cdot \overline{a} \\
& \overset{\text{since the conjugation is positive}}{=} & a \cdot (|b|^2 \cdot 1_A) \cdot \overline{a} & \overset{\text{since multiplication on } A \text{ is } \mathbb{R}\text{-bilinear}}{=} & |b|^2 \cdot (a \cdot \overline{a}) & = & |a|^2 \cdot |b|^2 \cdot 1_A.
\end{array}$$

It follows that $|a \cdot b|^2 = |a|^2 \cdot |b|^2$ which implies that $|a \cdot b| = |a| \cdot |b|$. ■

214.2. **The Cayley-Dickson construction.** In this section we discuss the Cayley-Dickson construction which, as we already mentioned, iteratively produces a long string of (increasingly less interesting) $\mathbb{R}$-algebras.

Lemma 214.6. (Cayley-Dickson Construction Lemma) Let V be an $\mathbb{R}$-algebra with conjugation. We consider the $\mathbb{R}$-vector space $V \oplus V$ with the multiplication that is given by

$$\begin{array}{rcl}
(V \oplus V) \times (V \oplus V) & \to & V \oplus V \\
((v_1, v_2), (w_1, w_2)) & \mapsto & (v_1 \cdot w_1 - \overline{w_2} \cdot v_2, v_2 \cdot \overline{w_1} + w_2 \cdot v_1)
\end{array}$$

and the conjugation

$$\begin{array}{rcl}
V \oplus V & \to & V \oplus V \\
(v, w) & \mapsto & \overline{(v, w)} := (\overline{v}, -w).
\end{array}$$

The following statements hold:

(1) The multiplication is $\mathbb{R}$-bilinear. In particular $(V \oplus V, +, \cdot)$ is an algebra over $\mathbb{R}$.

(2) (a) If V has a multiplicatively neutral element 1_V then $(1_V, 0) \in V \oplus V$ is a multiplicatively neutral element.

(b) If V is commutative and if the conjugation on V is trivial, then the multiplication on $V \oplus V$ is commutative.

(c) If V is commutative and associative, then $V \oplus V$ is associative.

(3) (a) The above conjugation on $V \oplus V$ is indeed a conjugation.

(b) For any $(v, w) \in V \oplus V$ we have $(v, w) + \overline{(v, w)} = (v + \overline{v}, 0)$.

(c) For any $(v, w) \in V \oplus V$ we have $(v, w) \cdot \overline{(v, w)} = (v \cdot \overline{v} + \overline{w} \cdot w, 0)$.

(4) (a) If the conjugation on V is imaginary, then so is the conjugation on $V \oplus V$.
(b) If the conjugation on V is positive, then so is the conjugation on $V \oplus V$.
(c) If the conjugation on V is symmetric, then so is the conjugation on $V \oplus V$.
(5) The map
$$\begin{array}{ccc} V & \to & V \oplus V \\ v & \mapsto & (v, 0) \end{array}$$
is a monomorphism of $\mathbb{R}$-algebras that preserves the multiplicatively neutral element and that preserves the conjugation.

Definition 214.7. Let V be an $\mathbb{R}$-algebra with conjugation. We say that the above $\mathbb{R}$-algebra $V \oplus V$ is obtained from V via the Cayley-Dickson construction.

Proof.

(1) This statement follows easily from the definitions.
(2) (a) This statement also follows immediately from the definitions.
(b) Fortunately this statement also follows immediately from the definitions.
(c) We assume that V is commutative and associative. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ be three elements of $V \oplus V$. Using the hypothesis that V is associative we see that
$$\begin{array}{rcl} (u \cdot v) \cdot w & = & (u_1 \cdot v_1 - \overline{v_2} \cdot u_2, u_2 \cdot \overline{v_1} + v_2 \cdot u_1) \cdot (w_1, w_2) \\ & = & (u_1 \cdot v_1 \cdot w_1 - \overline{v_2} \cdot u_2 \cdot w_1 - \overline{w_2} \cdot u_2 \cdot \overline{v_1} - \overline{w_2} \cdot v_2 \cdot u_1, \\ & & u_2 \cdot \overline{v_1} \cdot \overline{w_1} + v_2 \cdot u_1 \cdot \overline{w_1} + w_2 \cdot u_1 \cdot v_1 - w_2 \cdot \overline{v_2} \cdot u_2) \\ & = & (u_1 \cdot v_1 \cdot w_1 - u_1 \cdot \overline{w_2} \cdot v_2 - w_1 \cdot \overline{v_2} \cdot u_2 - \overline{v_1} \cdot \overline{w_2} \cdot u_2, \\ & \uparrow & u_2 \cdot \overline{w_1} \cdot \overline{v_1} - u_2 \cdot \overline{v_2} \cdot w_2 + v_2 \cdot \overline{w_1} \cdot u_1 + w_2 \cdot v_1 \cdot u_1) \end{array}$$
since V is commutative
$$= (u_1, u_2) \cdot (v_1 \cdot w_1 - \overline{w_2} \cdot v_2, v_2 \cdot \overline{w_1} + w_2 \cdot v_1) = u \cdot (v \cdot w).$$

(3) (a) It is clear that the map $(v, w) \mapsto \overline{(v, w)} := (\overline{v}, -w)$ is $\mathbb{R}$-linear and it is clear that for any $(v, w) \in V \oplus V$ we have $\overline{(v, w)} = (\overline{v}, -w) = (v, w)$. Finally let $v = (v_1, v_2), w = (w_1, w_2) \in V \oplus V$. We see that
$$\begin{array}{rcl} \overline{v \cdot w} & = & \overline{(v_1 \cdot w_1 - \overline{w}_2 \cdot v_2, v_2 \cdot \overline{w}_1 + w_2 \cdot v_1)} \\ & = & (\overline{w_1} \cdot \overline{v_1} - \overline{v}_2 \cdot w_2, -v_2 \cdot \overline{w}_1 - w_2 \cdot v_1) \\ & = & (\overline{w}_1, -w_2) \cdot (\overline{v}_1, -v_2) \end{array} \quad \begin{array}{rcl} & = & (\overline{v_1 \cdot w_1 - \overline{w}_2 \cdot v_2}, -v_2 \cdot \overline{w}_1 - w_2 \cdot v_1) \\ & = & (\overline{w_1} \cdot \overline{v_1} - \overline{v}_2 \cdot w_2, -w_2 \cdot v_1 - v_2 \cdot \overline{w}_1) \\ & = & \overline{w} \cdot \overline{v} \end{array}$$
(b) This statement follows immediately from the definitions.
(c) This statement also follows immediately from the definitions.
(4) These three statements can be verified easily using (3b) and (3c).
(5) This statement also follows immediately from the definitions. ∎

Example.

(1) We apply the Cayley-Dickson construction to $\mathbb{R}$, viewed as an $\mathbb{R}$-algebra in the obvious way, with the trivial conjugation and we obtain an algebra $\mathbb{R} \oplus \mathbb{R}$ with conjugation. Clearly the map
$$\begin{array}{ccc} \mathbb{R} \oplus \mathbb{R} & \to & \mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i} \\ (v, w) & \mapsto & v + w \cdot \mathrm{i} \end{array}$$
is an isomorphism of $\mathbb{R}$-algebras with involution. Note that it follows from the Cayley-Dickson Construction Lemma 214.6 that $\mathbb{C}$ is commutative and associative.

(2) We apply the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{C}$ and we obtain the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} = \mathbb{R}^4$. We will study these in greater detail in Section 214.3. We will see that the quaternions are a division algebra that is non-commutative but associative.
(3) We apply the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{H}$ and we obtain the octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^8$. We will study these in greater detail in Section 214.4. We will see that the octonions are a division algebra that is non-commutative and non-associative, but it is at least alternative.
(4) If we apply the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{O}$, then we obtain the sedenion $\mathbb{S} = \mathbb{O} \oplus \mathbb{O} = \mathbb{R}^{16}$. In Exercise 214.5 we will see that the sedenions are *not* a division algebra.
(5) We can iterate this procedure and we obtain algebras which are creatively called pathions (32-dimensional), chingons (64-dimensional), routons (128-dimensional), voudons (256-dimensional). We challenge the reader to come up with the explanations for these names. Alternatively the reader can look at [**dMar2002**, p. 7]. We refer to [**Baez2001**, Section 2.2] for a more detailed discussion of the Cayley-Dickson construction and we refer to [**More1997**] for more information on the failure of the sedenions to be a division algebra. □

In the following table we summarize the key properties of $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$ and the sedenions $\mathbb{S}$:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{S}$
conjugation is trivial	✓	✗	✗	✗	✗
algebra is commutative	✓	✓	✗	✗	✗
algebra is associative	✓	✓	✓	✗	✗
algebra is alternative	✓	✓	✓	✓	✗
non-zero elements have inverses	✓	✓	✓	✓	✓
division algebra	✓	✓	✓	✓	✗

In the following sections, and Exercises 214.5 and 214.4, we explain where the ✓ and ✗ come from.

214.3. **The quaternions.** As we already mentioned in the previous section, we can now introduce the quaternions using the Cayley-Dickson construction:

Definition 214.8. We define the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ to be the $\mathbb{R}$-algebra with conjugation that we obtain from applying the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{C}$ (with the usual conjugation). Recall that this means that the multiplication on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ is given by

$$(z_1, z_2) \underset{\mathbb{H}}{\cdot} (w_1, w_2) := (z_1 \underset{\mathbb{C}}{\cdot} w_1 - \overline{w}_2 \underset{\mathbb{C}}{\cdot} z_2,\ z_2 \underset{\mathbb{C}}{\cdot} \overline{w}_1 + w_2 \underset{\mathbb{C}}{\cdot} z_1)$$

and the conjugation by

$$\overline{(w, z)} := (\overline{w}, -z).$$

Before we discuss the properties of quaternions let us remind the reader of an alternative definition of quaternions which we gave in Definition 171.2:

Definition 214.9. We define

$$\mathbb{H} := \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k}$$

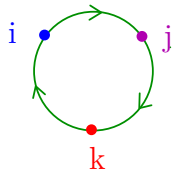
with the $\mathbb{R}$-bilinear multiplication that is determined by
(1) $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$,
(2) $1 \cdot \mathrm{i} = \mathrm{i} \cdot 1 = \mathrm{i}$, $1 \cdot \mathrm{j} = \mathrm{j} \cdot 1 = \mathrm{j}$, $1 \cdot \mathrm{k} = \mathrm{k} \cdot 1 = \mathrm{k}$, and
(3) $\mathrm{i} \cdot \mathrm{j} = k$, $\mathrm{j} \cdot \mathrm{k} = \mathrm{i}$, $\mathrm{k} \cdot \mathrm{i} = \mathrm{j}$ and $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$, $\mathrm{k} \cdot \mathrm{j} = -\mathrm{i}$, $\mathrm{i} \cdot \mathrm{k} = -\mathrm{j}$.
For a quaternion

$$z \;:=\; a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k} \;\in \mathbb{H}$$

we define the conjugate

$$\overline{z} \;:=\; a \cdot 1 - b \cdot \mathrm{i} - c \cdot \mathrm{j} - d \cdot \mathrm{k} \;\in \mathbb{H}.$$

Remark. The multiplication table for $\mathbb{H}$ is be summarized in the following figure:



$$\begin{aligned} \mathrm{i} \cdot \mathrm{j} &= \mathrm{k} \\ \mathrm{j} \cdot \mathrm{k} &= \mathrm{i} \\ \mathrm{k} \cdot \mathrm{i} &= \mathrm{j} \end{aligned}$$

More precisely, we show the three generators i, j and k. We have $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ and the analogous statement holds for any *cyclic* permutation of $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$. On the other hand for an anti-cyclic permutation we pick up a minus sign, e.g. we have $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$. □

In the following we will use the following lemma to go back and forth between the two definitions of quaternions:

Lemma 214.10. (Equivalence-of-Quaternions Lemma) The map

$$\begin{array}{rcl} \mathbb{C} \oplus \mathbb{C} & \to & \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k} \\ ((a + b \cdot \mathrm{i}), (c + d \cdot \mathrm{i})) & \mapsto & a + b \cdot \mathrm{i} + \underbrace{c \cdot \mathrm{j} + d \cdot \mathrm{k}}_{=(c + d \cdot \mathrm{i}) \cdot \mathrm{j}} \end{array}$$

is an isomorphism of $\mathbb{R}$-algebras that preserves the conjugation.

Proof. The lemma follows immediately from the definitions. We leave the details to the reader. ■

Now we can summarize the key properties of quaternions:

Lemma 171.3 (Quaternion Lemma)
(1) The quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ are a non-commutative $\mathbb{R}$-algebra with multiplicatively neutral element $1_{\mathbb{H}}$.
(2) The quaternions are associative.
(3) The map $(z, w) \mapsto \overline{(z, w)} := (\overline{z}, -w)$ is a conjugation.
(4) The conjugation is positive, symmetric and imaginary.
It follows from (4) and Definition 214.4 that for each $z \in \mathbb{H}$ there exists a unique $|z| \in \mathbb{R}_{\geq 0}$ which satisfies $z \cdot \overline{z} = |z|^2 \cdot 1_{\mathbb{H}}$.
(5) If we make the obvious identification $\mathbb{H} = \mathbb{R}^4$, then the norm $|-|$ on $\mathbb{H}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.[2274]
(6) For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{1}{|z|^2} \cdot \overline{z}$.
(7) For all $z, w \in \mathbb{H}$ we have $|z \cdot w| = |z| \cdot |w|$.
(8) The quaternions are a division algebra.

Proof. [2275]

(1) It follows from the Cayley-Dickson Construction Lemma 214.6 (1) and (2a) that $\mathbb{H}$ is an $\mathbb{R}$-algebra with multiplicatively neutral element. One can easily verify by example that $\mathbb{H}$ is not commutative, for example we already saw that $\mathrm{i} \cdot \mathrm{j} = \mathrm{k} \neq -\mathrm{k} = \mathrm{j} \cdot \mathrm{i}$.
(2) This statement follows from the Cayley-Dickson Construction Lemma 214.6 (2c) and the fact that $\mathbb{C}$ is commutative.
(3) This statement follows from the Cayley-Dickson Construction Lemma 214.6 (3).
(4) This statement follows from the Cayley-Dickson Construction Lemma 214.6 (4).
(5) This statement follows easily from the Cayley-Dickson Construction Lemma 214.6 (3c).
(6) This statement follows from (4) and the Involution-Norm-Algebra Lemma 214.5 (1).
(7) This statement follows from (2) and the Involution-Norm-Algebra Lemma 214.5 (2).
(8) Let $a \neq 0$ and b be two elements in $\mathbb{H}$. We set $y := b \cdot a^{-1}$. Then

$$y \cdot a \;=\; (b \cdot a^{-1}) \cdot a \;\underset{\substack{\uparrow \\ \text{since by (2) we know} \\ \text{that } \mathbb{H} \text{ is associative}}}{=}\; b \cdot (a^{-1} \cdot a) \;=\; b \cdot 1 \;=\; b.$$

The same way we show that there exists an $x \in \mathbb{H}$ with $a \cdot x = b$. ∎

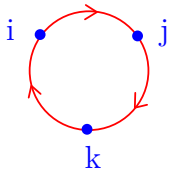
214.4. **The octonions.** Since applying the Cayley-Dickson construction to $\mathbb{C}$ was so much fun, it is natural to iterate this procedure and to apply the Cayley-Dickson construction to the quaternions $\mathbb{H}$. This leads us to the following definition:

Definition 214.11. We define the Cayley octonions $\mathbb{O}$ (or usually just octonions) to be the $\mathbb{R}$-algebra with conjugation that we obtain from applying the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{H}$ (with the above conjugation). Recall that this means that the multiplication on $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ is given by

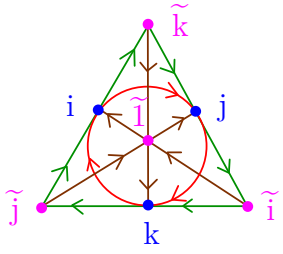
$$(z_1, z_2) \underset{\mathbb{O}}{\cdot} (w_1, w_2) \;:=\; (z_1 \underset{\mathbb{H}}{\cdot} w_1 - \overline{w}_2 \underset{\mathbb{H}}{\cdot} z_2,\; z_2 \underset{\mathbb{H}}{\cdot} \overline{w}_1 + w_2 \underset{\mathbb{H}}{\cdot} z_1)$$

and the conjugation by $\overline{(w, z)} \;:=\; (\overline{w}, -z)$.

Remark. We equip the 8-dimensional real vector space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^4 \oplus \mathbb{R}^4$ with the basis that is given by $1 = (1, 0)$, $\mathrm{i} = (\mathrm{i}, 0)$, $\mathrm{j} = (\mathrm{j}, 0)$, $\mathrm{k} = (\mathrm{k}, 0)$ and the almost identical twins $\widetilde{1} = (0, 1)$, $\widetilde{\mathrm{i}} = (0, \mathrm{i})$, $\widetilde{\mathrm{j}} = (0, \mathrm{j})$, $\widetilde{\mathrm{k}} = (0, \mathrm{k})$ and $\widetilde{1} = (0, 1)$. With respect to this basis octonionic multiplication is summarized the right-part of the figure below.



quaternionic multiplication



octonionic multiplication

[2274] This shows in particular that the quaternion norm is indeed a norm.
[2275] On page 3258 we gave a more direct proof of this lemma.

To the right of the figure below we show the *Fano plane*, which is a diagram with 7 points and 7 directed "lines". Similar to the discussion on page 4103 it encodes the multiplication where the product of two generators equals the third generator on the line with the positive sign if they are cyclic permutation of the given ordering and with a negative sign otherwise. For example we have

$$(\widetilde{\mathrm{j}} \cdot \mathrm{i}) \cdot \mathrm{j} = \widetilde{\mathrm{k}} \cdot \mathrm{j} = \widetilde{\mathrm{i}} \qquad \text{and} \qquad \widetilde{\mathrm{j}} \cdot (\mathrm{i} \cdot \mathrm{j}) = \widetilde{\mathrm{j}} \cdot \mathrm{k} = -\widetilde{\mathrm{i}}.$$

Note that this calculation also shows that the octonions are not associative. We refer to [**Baez2001**, Chapter 2.1] for a more detailed discussion of the role of the Fano plane. □

The following lemma is the analog of the Quaternion Lemma 171.3.

Lemma 214.12. (Octonion Lemma)

(1) The octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ are a non-commutative $\mathbb{R}$-algebra with multiplicatively neutral element $1_{\mathbb{O}}$.
(2) The octonions are not associative[2276] but they are alternative, recall that this means that given any $x, y \in \mathbb{O}$ we have $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ and $(x \cdot x) \cdot y = x \cdot (x \cdot y)$.
(3) The map $(z, w) \mapsto \overline{(z, w)} := (\overline{z}, -w)$ is a conjugation.
(4) The conjugation is positive, symmetric and imaginary.

It follows from (4) and Definition 214.4 that for each $z \in \mathbb{O}$ there exists a unique $|z| \in \mathbb{R}_{\geq 0}$ which satisfies $z \cdot \overline{z} = |z|^2 \cdot 1_{\mathbb{O}}$.

(5) If we make the obvious identification $\mathbb{O} = \mathbb{R}^8$, then the norm $|-|$ on $\mathbb{O}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^8$.
(6) For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{1}{|z|^2} \cdot \overline{z}$.
(7) For all $z, w \in \mathbb{O}$ we have $|z \cdot w| = |z| \cdot |w|$.
(8) The octonions are a division algebra.

Proof. As we will see, the proof has many similarities with the proof of the Quaternion Lemma 171.3. But as we will see, there are also some important differences. In particular we will prove the statements in a different order:

(1) It follows from the Cayley-Dickson Construction Lemma 214.6 (1) and (2a) that $\mathbb{H}$ is an $\mathbb{R}$-algebra with multiplicatively neutral element. Since $\mathbb{H}$ is not commutative one sees easily that $\mathbb{O}$ is not commutative either.
(3) This statement follows from the Cayley-Dickson Construction Lemma 214.6 (3).
(4) This statement follows from the Cayley-Dickson Construction Lemma 214.6 (4).
(2) In the previous remark we saw that the octonions are not associative. But we will now see that the octonions are alternative: Let $x, y \in \mathbb{O}$. We start out with the following claim:

Claim. We have the equalities $(x \cdot y) \cdot \overline{y} = x \cdot (y \cdot \overline{y})$ and $(x \cdot \overline{x}) \cdot y = x \cdot (\overline{x} \cdot y)$.

[2276] By the remark on page 115 we know that in an associative ring we can ignore parentheses when it comes to multiplication. Now that we are dealing with the octonions, which are not associative, we have to carefully keep track of parentheses.

Proof. Let us first prove the first equality. We write $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_1, x_2, y_1, y_2 \in \mathbb{H}$. Heroically we calculate that

$$\begin{aligned}(x\cdot y)\cdot\overline{y} &= ((x_1,x_2)(y_1,y_2))\cdot(\overline{y}_1,-y_2) \;=\; (x_1y_1-\overline{y}_2x_2, y_2x_1+x_2\overline{y}_1)\cdot(\overline{y}_1,-y_2)\\ &= ((x_1y_1-\overline{y}_2x_2)\overline{y}_1+\overline{y}_2(y_2x_1+x_2\overline{y}_1), (-y_2)(x_1y_1-\overline{y}_2x_2)+(y_2x_1+x_2\overline{y}_1)y_1)\\ &= ((|y_1|^2+|y_2|^2)x_1, (|y_1|^2+|y_2|^2)x_2) \;=\; (|y_1|^2+|y_2|^2)\cdot(x_1,x_2)\end{aligned}$$

since $\mathbb{H}$ is associative we have $(\overline{y}_2x_2)\overline{y}_1 = \overline{y}_2(x_2\overline{y}_1)$ and $y_2(x_1y_1) = (y_2x_1)y_1$
we also use that $\overline{y}_1y_1 = |y_1|^2\cdot 1 \in \mathbb{R}$, hence $\overline{y}_1y_1$ commutes with any $z \in \mathbb{H}$

$$= |y|^2\cdot x \;=\; x\cdot|y|^2 \;=\; x\cdot(y\cdot\overline{y}).$$

The proof of the second equality is basically the same. ⊞

By (4) we know that the involution is imaginary. This implies that $x+\overline{x} \in \mathbb{R}\cdot 1$ and $y+\overline{y} \in \mathbb{R}\cdot 1$. In other words, there exist $a, b \in \mathbb{R}$ with $\overline{x} = -x+2a\cdot 1$ and $\overline{y} = -y+2b\cdot 1$. We plug these equalities into the equalities of the claim and, since we are dealing with an algebra over the real numbers, we obtain the desired equalities $(x\cdot y)\cdot y = x\cdot(y\cdot y)$ and $(x\cdot x)\cdot y = x\cdot(x\cdot y)$.

(5) This statement follows easily from the Cayley-Dickson Construction Lemma 214.6 (3c).

(6) This statement follows from (4) and the Involution-Norm-Algebra Lemma 214.5 (1).

(7) This statement follows from (2) and the Involution-Norm-Algebra Lemma 214.5 (2).[2277]

(8) Finally we want to prove that $\mathbb{O}$ is in fact a division algebra. Since the octonions are not associative we cannot copy-paste the proof of the Quaternion Lemma 171.3 (8), since the latter proof used the associativity of the quaternions. Fortunately the spartan associativity result given in (2) is enough to give us the desired result. Let $a, b \in \mathbb{O}$ with $a \neq 0$. We need to show that there exists an $x \in \mathbb{O}$ with $a\cdot x = b$ and a $y \in \mathbb{O}$

[2277] For the sake of being as self-contained as possible we provide an alternative proof that does not make use of the Alternative Algebra Theorem 214.2. Let $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{O}$.

Claim. Given $a \in \mathbb{H}$ we set $S(a) := z_2\overline{w}_1\overline{z}_1\overline{a} + az_1w_1\overline{z}_2 - z_1w_1\overline{z}_2a - \overline{a}z_2\overline{w}_1\overline{z}_1$. We have $S(w_2) = 0$.

Proof. Evidently, if w_2 is a real number, then we have $S(w_2) = 0$. Now suppose that w_2 is a purely imaginary quaternion, i.e. $w_2 = a\cdot\mathrm{i} + b\cdot\mathrm{j} + c\cdot\mathrm{j}$ for some $a, b, c \in \mathbb{R}$. In this case we calculate that

$$S(w_2) \overset{\text{since } w_2 \text{ is purely imaginary we have } \overline{w}_2 = -w_2}{=} w_2\underbrace{(z_2\overline{w}_1\overline{z}_1 + z_1w_1\overline{z}_2)}_{=:c} - \underbrace{(z_2\overline{w}_1\overline{z}_1 + z_1w_1\overline{z}_2)}_{=:c}w_2 \;=\; w_2\cdot c - c\cdot w_2 \overset{\text{we have } \overline{c} = c, \text{ thus } c \text{ is a real number}}{=} 0.$$

Note that for any $a, b \in \mathbb{H}$ we have $S(a+b) = S(a) + S(b)$. Since any quaternion is the sum of a real number and a purely imaginary quaternion we see that $S(w_2) = 0$ for any quaternion w_2. ⊞

We obtain that

$$\begin{aligned}|z\cdot w|^2 &= |(z_1,z_2)(w_1,w_2)|^2 \;=\; |(z_1w_1-\overline{w}_2z_2, z_2\overline{w}_1+w_2z_1)|^2\\ &= (z_1w_1-\overline{w}_2z_2)\overline{(z_1w_1-\overline{w}_2z_2)} + (z_2\overline{w}_1+w_2z_1)\overline{(z_2\overline{w}_1+w_2z_1)}\\ &= (z_1w_1-\overline{w}_2z_2)(\overline{w}_1\overline{z}_1-\overline{z}_2w_2) + (z_2\overline{w}_1+w_2z_1)(w_1\overline{z}_2+\overline{z}_1\overline{w}_2)\\ &= (z_1w_1)(\overline{w}_1\overline{z}_1) - (z_1w_1)(\overline{z}_2w_2) - (\overline{w}_2z_2)(\overline{w}_1\overline{z}_1) + (\overline{w}_2z_2)(\overline{z}_2w_2)\\ &\quad + (z_2\overline{w}_1)(w_1\overline{z}_2) + (z_2\overline{w}_1)(\overline{z}_1\overline{w}_2) + (w_2z_1)(w_1\overline{z}_2) + (w_2z_1)(\overline{z}_1\overline{w}_2)\\ &\overset{(*)}{=} (z_1w_1)(\overline{w}_1\overline{z}_1) + (\overline{w}_2z_2)(\overline{z}_2w_2) + (z_2\overline{w}_1)(w_1\overline{z}_2) + (w_2z_1)(\overline{z}_1\overline{w}_2)\\ &= (z_1\overline{z}_1 + z_2\overline{z}_2)(w_1\overline{w}_1 + w_2\overline{w}_2) \;=\; |(z_1,z_2)|^2|(w_1,w_2)|^2 \;=\; |z|^2\cdot|w|^2.\end{aligned}$$

where $(*)$ follows from the claim and the fact that the quaternions are associative.

with $y \cdot a = b$. We set $x := a^{-1} \cdot b$. We calculate that

$$a \cdot x \;=\; a \cdot (a^{-1} \cdot b) \underset{\text{by (6)}}{\overset{\uparrow}{=}} a \cdot \left(\tfrac{1}{|a|^2} \cdot \overline{a} \cdot b\right) \;=\; \tfrac{1}{|a|^2} \cdot a \cdot (\overline{a} \cdot b) \underset{\text{by the claim in the proof of (2)}}{\overset{\uparrow}{=}} \tfrac{1}{|a|^2} \cdot (a \cdot \overline{a}) \cdot b \;=\; b.$$

Basically the same calculation shows that we have $y \cdot a = b$. ■

Remark. The octonions were discovered by Arthur Cayley in 1845, shortly after Hamilton found the quaternions. Since they are not associative they are hard to deal with.[2278] The octonions are discussed in much greater detail in [**Baez2001**], [**CSm2003**, Chapter 6] and [**KaS1989**, Chapter 6]. □

214.5. **Dimensions of division algebras.** We are now aware of the division algebras $\mathbb{R}, \mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, i.e. we are aware of division algebras that are defined over $\mathbb{R}^n$ with $n = 1, 2, 4$ and 8. These dimensions are powers of 2 and the following theorem, which was first proved by Heinz Hopf [**Hopf1940a**] in 1940, shows that this is not a coincidence:

Theorem 214.13. (Hopf–Division Algebra Dimension Theorem) If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.

Remark. On Section 237.3 and 5305 we will give an alternative proof of the Hopf–Division Algebra Dimension Theorem 214.13. The former will use Steenrod operations and the latter proof will use the Stiefel-Whitney classes of real vector bundles. □

In the proof of the Hopf–Division Algebra Dimension Theorem 214.13 we will need the following earlier lemma:

Lemma 171.7. (No Zero Divisor Lemma) Let A be a *finite-dimensional* algebra over a field $\mathbb{F}$. The following two statements are equivalent:

(1) A is a division algebra,
(2) A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.

Proof of Theorem 214.13. Let "$*$" be a real division algebra structure on $\mathbb{R}^n$. We want to show that n is a power of 2. Since 1 and 2 are powers of 2 we can assume that $n \geq 3$. As in the proof of the Commutative Division Algebra Theorem 171.6 we start out with the following claim:

Claim 1. The map

$$\begin{array}{rccl} \widetilde{f}\colon & S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ & (x, y) & \mapsto & \widetilde{f}(x, y) := \dfrac{x * y}{\|x * y\|} \end{array}$$

is well-defined and continuous.

Proof. This map is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on $(\mathbb{R}^n, *)$ and from the No Zero Divisor Lemma 171.7 that $x * y \neq 0$. The map $\widetilde{f}$ is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, hence continuous.[2279] ⊞

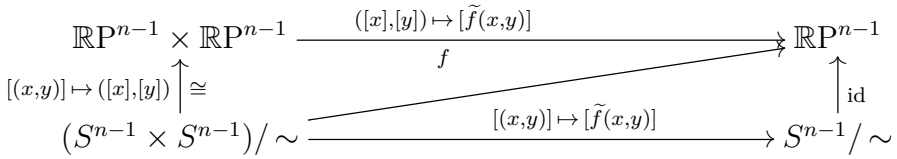
Claim 2.

[2278] To quote from [**Frie2012**, p. 5] the immortal words: "here, associativity is obviously a concern".
[2279] Indeed, the fact that the map is bilinear implies that there exists a real $(n \times n)$–matrix P such that for every $v, w \in \mathbb{R}^n$ we have $v * w = v^{\mathrm{T}} P w$, hence the map is continuous.

(1) For every $x, y \in S^{n-1}$ we have $\widetilde{f}(-x, y) = -\widetilde{f}(x, y)$ and $\widetilde{f}(x, -y) = -\widetilde{f}(x, y)$.
(2) The map
$$\begin{array}{rccc} f\colon & \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ & ([x],[y]) & \mapsto & [\widetilde{f}(x,y)]. \end{array}$$
is well-defined and continuous.

Proof.

(1) This statement follows immediately from the fact that "$*$" defines an algebra.
(2) It follows from (1) that the map f is well-defined. It remains to show that f is continuous. This requires a little bit of thought. We consider the following diagram:



We make the following clarifications and observations:
(a) Let $\sim$ be the equivalence relation on $S^{n-1} \times S^{n-1}$ that is given by $(x, y) \sim (\pm x, \pm y)$.
(b) One can easily verify that the rectangle commutes. The diagonal map is defined in such a way that the two triangles commute.
(c) It follows immediately from (b) and the Quotient-to-Quotient Lemma 8.6 that in the above diagram the diagonal map $(S^{n-1} \times S^{n-1})/\sim \to S^n/\sim = \mathbb{R}\mathrm{P}^{n-1}$ is continuous.
(d) It follows from the Topological-Quotient Proposition 8.2 (1a) and (1b) and the Topological-Product Proposition 7.2 (1a) and (1b) (applied in the correct order) that the left vertical map is continuous. The left vertical map is clearly a bijection.
(e) It follows from (d) and the Projective Spaces-Compact+Hausdorff Proposition 9.15 together with the Compact-Hausdorff Proposition 4.23 (3) that the left vertical map is a homeomorphism.

It follows from this discussion that the top horizontal map f is indeed continuous. ⊞

Next we pick $\widetilde{s}_0 \in S^{n-1}$. We write $s_0 \coloneqq [\widetilde{s}_0] \in \mathbb{R}\mathrm{P}^{n-1}$. We consider the embeddings
$$\begin{array}{rccc} i\colon & \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \\ & x & \mapsto & (x, s_0) \end{array} \quad \text{and} \quad \begin{array}{rccc} j\colon & \mathbb{R}\mathrm{P}^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \\ & x & \mapsto & (s_0, x). \end{array}$$

Claim 3. The map $(f \circ i)_*\colon \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0))$ is an isomorphism. The same conclusion holds with i replaced by j.

Proof. We provide the proof for Statement (1). The proof with i replaced by j is basically the same. Since $\pi_1(\mathbb{R}\mathrm{P}^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ it suffices to show that the map
$$(f \circ i)_*\colon \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0)$$
has non-trivial image. We introduce the following notation:

(1) We denote by $\pi\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ the natural projection.
(2) We pick a path $\gamma\colon [0, 1] \to S^{n-1}$ with $\gamma(0) = \widetilde{s}_0$ and with $\gamma(1) = -\gamma(0) = -\widetilde{s}_0$. Note that $\pi \circ \gamma\colon [0, 1] \to \mathbb{R}\mathrm{P}^{n-1}$ is a loop in s_0.
(3) We denote by $\widetilde{i}$ the embedding $S^{n-1} \to S^{n-1} \times S^{n-1}$ that is given by $\widetilde{i}(x) = (x, \widetilde{s}_0)$.

Note that we get the following commutative diagram:

$$\begin{array}{ccccccc} [0,1] & \xrightarrow{\gamma} & S^{n-1} & \xrightarrow{\widetilde{i}} & S^{n-1}\times S^{n-1} & \xrightarrow{\widetilde{f}} & S^{n-1} \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{i} & \mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{f} & \mathbb{R}\mathrm{P}^{n-1}. \end{array}$$

Finally we note that

$$(f\circ i)_*([\pi\circ\gamma]) \;=\; [f\circ i\circ\pi\circ\gamma] \overset{\text{by the above commutative diagram}}{\downarrow} = \; [\pi\circ(\widetilde{f}\circ\widetilde{i}\circ\gamma)] \;\overset{\uparrow}{\neq}\; e \;\in \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0).$$

this follows from the Loop-in-$\mathbb{R}\mathrm{P}^n$-Lemma 213.22 since the following equality holds:
$(\widetilde{f}\circ\widetilde{i}\circ\gamma)(1) = \widetilde{f}(\gamma(1),\widetilde{s}_0) = \widetilde{f}(-\gamma(0),\widetilde{s}_0) = -\widetilde{f}(\gamma(0),\widetilde{s}_0) = -(\widetilde{f}\circ\widetilde{i}\circ\gamma)(0)$ ⊞

We denote by $\varphi\in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ the unique non-trivial element. Furthermore, as so often we denote by p and q the two natural projections $\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1}\to\mathbb{R}\mathrm{P}^{n-1}$. Finally we write $\alpha \coloneqq p^*(\varphi)$ and $\beta\coloneqq q^*(\varphi)\in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$.

Claim 4. The following equality holds

$$f^*(\varphi) \;=\; \alpha+\beta \;\in\; \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2).$$

Proof. We start out the proof of the claim with the following three observations:

(1) By the H^1-Hom-Proposition 201.24 we have for each $t_0\in\mathbb{R}\mathrm{P}^{n-1}$ a natural isomorphism
$$\mathrm{ev}\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2) \;\to\; \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^{n-1},t_0),\mathbb{F}_2).$$
(2) Let $\gamma\in\pi_1(\mathbb{R}\mathrm{P}^{n-1},s_0)$ be the unique non-trivial element. It follows from the Product-π_1-Proposition 90.13 that $\pi_1(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1},(s_0,s_0))$ is generated by $i_*(\gamma)$ and $j_*(\gamma)$.
(3) Since $\mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and $\pi_1(\mathbb{R}\mathrm{P}^{n-1})$ both contain a unique non-trivial element, namely φ and γ, we see that $\mathrm{ev}(\varphi)(\gamma)=1\in\mathbb{F}_2$.

It follows from (1) and (2) that it suffices to show that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(\gamma)$ and $j_*(\gamma)$. In fact we have

$$\mathrm{ev}(f^*(\varphi))(i_*(\gamma)) \;\overset{\text{by the naturality of ev}}{\uparrow}= \mathrm{ev}(\varphi)(f_*(i_*(\gamma))) \;=\; \mathrm{ev}(\varphi)((f\circ i)_*(\gamma))) \;\overset{\text{by Claim 3}}{\uparrow}= \mathrm{ev}(\varphi)(\gamma) \;\overset{\text{by (3)}}{\uparrow}= 1.$$

We also see that

$$\begin{aligned} \mathrm{ev}(\alpha+\beta)(i_*(\gamma)) &\overset{\text{by definition of } \alpha \text{ and } \beta}{\downarrow}= \mathrm{ev}(p^*(\varphi)+q^*(\varphi))(i_*(\gamma)) \overset{\text{since ev is a homomorphism}}{\downarrow}= \mathrm{ev}(p^*(\varphi))(i_*(\gamma))+\mathrm{ev}(q^*(\varphi))(i_*(\gamma)) \\ &= \mathrm{ev}(\varphi)((p\circ i)_*(\gamma))+\mathrm{ev}(\varphi)((q\circ i)_*(\gamma)) \;\overset{\text{since } p\circ i=\mathrm{id} \text{ and since } q\circ i \text{ is a constant map}}{\uparrow}= \mathrm{ev}(\varphi)(\gamma) \;\overset{\text{by (3)}}{\uparrow}= 1. \end{aligned}$$

We have thus shown that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(\gamma)$. The same argument also works with $i_*(\gamma)$ replaced by $j_*(y)$. ⊞

Next we make the following observations:

(1) We note that

$$0 = f^*(0) \underset{\uparrow}{=} f^*(\varphi^n) \overset{\downarrow}{=} (f^*(\varphi))^n \underset{\uparrow}{=} (\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot \alpha^k \cdot \beta^{n-k} \underset{\uparrow}{=} \sum_{k=1}^{n-1} \binom{n}{k} \cdot \alpha^k \cdot \beta^{n-k}.$$

by the Cup Product-Naturality Lemma 203.9

follows from $\varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19

by Claim 4

since $\alpha^n = \beta^n = 0$

(2) By the Cup Product-Multiple-$\mathbb{R}\mathrm{P}^n$-Proposition 213.20 the obvious map

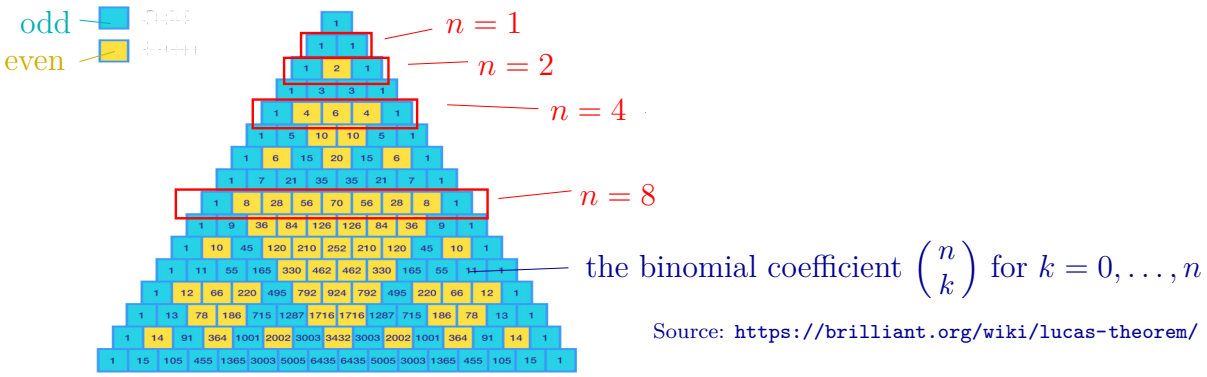
$$\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n) \;\to\; \mathrm{H}^*(\mathbb{R}\mathrm{P}^{n-1}\times\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$$

is an isomorphism.

(3) In the polynomial ring $\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n)$ the $n-1$ monomials $\alpha\beta^{n-1}, \alpha^2\beta^{n-2}, \ldots, \alpha^{n-1}\beta$ are linearly independent over the field $\mathbb{F}_2$.

It follows from these three observations that for $k = 1, \ldots, n-1$ we have $\binom{n}{k} \equiv 0 \bmod 2$. Thus it remains to prove the following purely algebraic claim:

Claim 5. Let $n \in \mathbb{N}$. If $\binom{n}{k} \equiv 0 \bmod 2$ for $k = 1, \ldots, n-1$, then n is a power of 2.[2280]



Proof. We can and will write n as a sum $n = n_1 + \cdots + n_r$ where each n_i is a power of 2 and with $n_1 < \cdots < n_r$. In the polynomial ring $\mathbb{F}_2[x]$ we obtain that

$$1 + x^n \overset{\downarrow}{=} \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \overset{\downarrow}{=} (1+x)^n \overset{\downarrow}{=} (1+x)^{n_1} \cdot \ldots \cdot (1+x)^{n_r}$$
$$= (1+x^{n_1}) \cdot \ldots \cdot (1+x^{n_r}) \underset{\uparrow}{=} 1 + x^{n_1} + \text{higher powers}.$$

follows from our hypothesis

by the generalized binomial theorem

since $n = n_1 + \cdots + n_r$

since each n_i is a power of 2 and since in $\mathbb{F}_2[x]$ we have $(a+b)^2 = a^2 + b^2$

since $n_1 < \cdots < n_r$

But this is only possible if $n = n_1$, i.e. if n itself is a power of 2. ∎

In fact, with more fancy topological methods one can actually strengthen the previous theorem to obtain the following theorem which gives a complete answer to Question 171.9. Unfortunately the proof is well beyond the scope of these modest lecture notes.

Theorem 214.14. (Division Algebra–Dimensions Theorem) Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.

[2280] In the Binomial Coefficient–Mod 2-Lemma 237.3 we will prove a generalization of the claim.

Proof. The theorem was first proved by Michel Kervaire [**Kerv1958**] and John Milnor [**Miln1958a**, Corollary 1], building on the Bott Periodicity Theorem 299.22 proved by Raoul Bott [**Bot1957, Bot1959**]. Full details for the proof can also be found in [**Hat2**, Theorem 2.16].

An alternative proof of the Division Algebra–Dimensions Theorem 214.14 is given by combining the Hopf Invariant Proposition 237.10 (3) (which we prove) with the Adams Theorem 237.12 (which is proved using Steenrod operations, but which we do not prove).

Finally note that as of right now there is no purely algebraic or analytic proof for the theorem. ■

Remark. One might ask whether any finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. As we will see in Exercises 214.8 and 214.9, it is not hard to find division algebras that are not isomorphic to any of our standard division algebras. We refer to [**Bruc1944, OsbJ1962**] for more examples. But uniqueness statements do hold under stronger hypotheses. For example in the Commutative Division Algebra Theorem 171.6 we already proved that every commutative division algebra is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Similar uniqueness statements hold under the stronger hypothesis that the division algebra is "normed" or alternative. We refer to [**BD1973**, p. 73], [**EHHKMNPR1991**, p. 262] and [**Baez2001**] for results in this direction. □

214.6. Exercises for Chapter 214 .

Exercise 214.1. We consider the $\mathbb{R}$-algebra $\mathbb{C}$ equipped with the *identity conjugation*, i.e. with the conjugation that is given by $\overline{z} = z$ for all $z \in \mathbb{C}$. We can still apply the Cayley-Dickson construction. Is the resulting $\mathbb{R}$-algebra isomorphic to a well-known $\mathbb{R}$-algebra?

Exercise 214.2.

(a) If an element x in $\mathbb{O}$ has a left-inverse y, is y also a right-inverse?

(b) Let $x \in \mathbb{O}$. Are left-inverses of x, if they exist, unique?

Remark. Since octonions are *not* associative the usual elementary argument from Exercise 2.3 does not apply.

Exercise 214.3. Show that the octonions $\mathbb{O}$ are not associative.

Exercise 214.4. Show that the sedenions $\mathbb{S}$ are not alternative.

Exercise 214.5. Show that the sedenions are *not* a division algebra.

Exercise 214.6. Let $q_1, q_2 \in S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$. We consider the map

$$\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 \cdot z \cdot q_2^{-1}. \end{array}$$

Show that the map is linear and that it is represented by a matrix in SO(4).

Exercise 214.7. Let $n \in \mathbb{Z}$. Show that the degree of the map

$$\begin{array}{rcl} S^3 = \{v \in \mathbb{H} \,|\, \|v\| = 1\} & \to & S^3 \\ z & \mapsto & z^n \end{array}$$

equals n.

Hint. You could use the Degree-via-Local Orientations Proposition 152.11.

Exercise 214.8. Let $\mathbb{A} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k}$. We define the multiplication on $\mathbb{A}$ the same way as we did for the quaternions, see Definition 171.2, except that we set $\mathrm{i}^2 = -1 + \epsilon \cdot \mathrm{j}$.

(a) Show that $\mathbb{A}$ is a division algebra.
(b) Show that i has a left inverse and a right inverse.
(c) Show that the left inverse and the right inverse of i differ for sufficiently small $\epsilon > 0$.

Remark. This exercise is identical to Exercise 171.7.

Exercise 214.9. Let $(V, +, *)$ be a division algebra over $\mathbb{R}$.

(a) Let $\varphi \colon V \to V$ be an isomorphism of the real vector space V. Show that the map

$$\begin{array}{ccc} V \times V & \to & V \\ (v, w) & \mapsto & \varphi(v * w) \end{array}$$

also defines a division algebra on V (which might or might not be isomorphic to the original division algebra).
(b) Show that there exists a division algebra $(V, +, \widetilde{*})$ and a $v \in V$ with $v \mathbin{\widetilde{*}} v = v$.
(c) Show that there exists a division algebra $(V, +, \widehat{*})$ that admits a multiplicatively neutral element.
(d) Show that there exists a division algebra $(V, +, \overline{*})$ that does not admit a multiplicatively neutral element.

Exercise 214.10. Show that there exists a bijection between the Fano plane and the non-zero points of $\mathbb{Z}_2^3$ such that the lines shown in Figure correspond precisely to the subgroups of $\mathbb{Z}_2^3$ that are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

CHAPTER 215

Quaternionic and octonionic projective spaces

In Chapter 214 we just learned about the quaternions $\mathbb{H}$ and the rather mysterious octonions $\mathbb{O}$, which can both be viewed as generalizations of the complex numbers.

In this section we introduce the $4n$-dimensional quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ and we introduce the significantly more subtle 16-dimensional octonionic projective space $\mathbb{O}\mathrm{P}^2$. These are closed orientable smooth manifolds which now get added to our ever expanding list of amusing topological spaces. Note though that in the octonionic setting we fail at producing higher dimensional projective spaces.

In this chapter we will also calculate the cohomology rings of the quaternionic and octonionic projective spaces and we will see that these are again polynomial rings. These examples raise several interesting questions which we discuss at the end of the chapter and which we will partially answer later in the lecture notes. The calculation of the cup product on $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ will also be useful in the next chapter where we will use these calculations to show that the homotopy group $\pi_7(S^4)$ and $\pi_{15}(S^8)$ are infinite.

This chapter is organized as follows:

- In Section 215.1 we will introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$.
- In Section 215.2 we will introduce the octonionic projective spaces $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$.
- In Section 215.3 we will formulate some questions inspired by quaternionic and octonionic projective spaces.
- In Section 215.4 we will provide exercises for Chapter 215.

In the next Chapter 216 we will introduce the Hopf maps on higher homotopy groups.

215.1. **Quaternionic projective spaces.** In Definition 135.6, given $n \in \mathbb{N}_0$, we introduced the real projective space $\mathbb{R}\mathrm{P}^n$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In the following we will introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$. Since the quaternions $\mathbb{H}$ are non-commutative we better do the constructions carefully.

Lemma 215.1. (Quaternion Equivalence Relation Lemma) If we are given two points $v, w \in \mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$, then we write $v \sim w$ if there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. This defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$.

Proof.

(1) It follows from the fact that $\mathbb{H}$ contains a multiplicatively neutral element that the relation $\sim$ is reflexive.

(2) We want to show that $\sim$ is symmetric. Thus let $v, w \in \mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$ with $v \sim w$. By definition this means that there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. Since $\mathbb{H}$ is a division algebra we can consider the multiplicative inverse h^{-1} and since $\mathbb{H}$ is associative we see that $h^{-1} \cdot w = h^{-1} \cdot (h \cdot v) = (h^{-1} \cdot h) \cdot v = 1_{\mathbb{H}} \cdot v = v$. Thus we have shown that $w \sim v$.

(3) Finally suppose that $u \sim v$ and $v \sim w$. Thus there exist $g, h \in \mathbb{H} \setminus \{0\}$ with $g \cdot u = v$ and $h \cdot v = w$. Since $\mathbb{H}$ is associative we see that $(h \cdot g) \cdot u = h \cdot (g \cdot u) = h \cdot v = w$. ■

Now we can define the quaternionic projective spaces:

Definition 215.2. Let $n \in \mathbb{N}_0$.

(1) We denote by "$\sim$" the equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$ from the Quaternion Equivalence Relation Lemma 215.1.

(2) For $(h_0, \ldots, h_n) \in \mathbb{H}^{n+1} \setminus \{0\}$ let $[h_0 : \ldots : h_n]$ be the corresponding equivalence class.

(3) Given $n \in \mathbb{N}$ we define the quaternionic projective space

$$\mathbb{H}\mathrm{P}^n \;:=\; (\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\})/\sim .$$

As on page 338 we see that the maps $[z] \mapsto [(z, 0)]$ define embeddings $\mathbb{H}\mathrm{P}^n \to \mathbb{H}\mathrm{P}^{n+1}$. Sometimes, given $m \leq n$, we use these embeddings to view $\mathbb{H}\mathrm{P}^m$ as a subspace of $\mathbb{H}\mathrm{P}^n$.

(4) We can use the embeddings from (3) to form the corresponding direct colimit

$$\mathbb{H}\mathrm{P}^\infty \;:=\; \operatorname*{colim}_{\longrightarrow} \mathbb{H}\mathrm{P}^n.$$

We refer to $\mathbb{H}\mathrm{P}^\infty$ as the infinite-dimensional quaternionic projective space.[2281]

The following lemma summarizes some key properties of quaternionic projective spaces:

Lemma 215.3. (Quaternion-Projective Space Lemma)

(1) The map

$$\begin{array}{rcl} \mathbb{H}\mathrm{P}^1 & \to & \mathbb{H} \cup \{\infty\} = \mathbb{R}^4 \cup \{\infty\} \overset{\downarrow}{=}_{\mathrm{i}} S^4 \\ [z_0 : z_1] & \mapsto & \begin{cases} z_1^{-1} \cdot z_0, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$

by the Stereographic Projection Lemma 4.24

is a homeomorphism.

(2) For each $n \in \mathbb{N}_0$ the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ is a closed orientable $4n$-dimensional smooth manifold.

(3) (a) Let $n \in \mathbb{N}_0$. The filtration $\varnothing \subseteq \mathbb{H}\mathrm{P}^0 \subseteq \mathbb{H}\mathrm{P}^1 \subseteq \ldots \subseteq \mathbb{H}\mathrm{P}^n$ is a CW-structure which has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$ and which has no other cells.

(b) We can equip $\mathbb{H}\mathrm{P}^\infty$ with a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots$ and which has no other cells.

Identification 215.4. We use the homeomorphism from the Quaternion-Projective Space Lemma 215.3 (1) to to make the identification $S^4 =_{\mathrm{i}} \mathbb{H}\mathrm{P}^1$.

Sketch of proof. We start out with the following claim:

Claim. Let $n \in \mathbb{N}_0$. It follows from the Quaternion Lemma 171.3 (5)–(7) that quaternion multiplication turns the sphere $S^3 = \{v \in \mathbb{H} \,|\, |z| = 1\}$ into a group and that this group acts continuously on $S^{4n+3} \subseteq \mathbb{H}^{4n+4}$ by left multiplication.

(a) We have a homeomorphism $S^{4n+3}/S^3 \to \mathbb{H}\mathrm{P}^n$.

(b) $\mathbb{H}\mathrm{P}^n$ is compact and Hausdorff.

[2281] It follows quite easily from the Direct Colimit–Action Lemma 104.20 that we have a homeomorphism $\operatorname*{colim}_{\longrightarrow} \mathbb{H}\mathrm{P}^n \cong (\mathbb{H}^\infty \setminus \{0\})/\sim$.

Proof. As in the Projective Space-Sphere Quotient Lemma 9.13 we see that the maps

$$\begin{array}{ccc} S^{4n+3}/S^3 & \to & \mathbb{H}\mathrm{P}^n \\ [z] & \mapsto & [z] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{H}\mathrm{P}^n & \to & S^{4n+3}/S^3 \\ [z_1 : \ldots : z_{n+1}] & \mapsto & \left[\frac{(z_1, \ldots, z_{n+1})}{\sqrt{|z_1|^2 + \ldots + |z_{n+1}|^2}}\right] \end{array}$$

are continuous and that they are inverses of one another. This shows that $\mathbb{H}\mathrm{P}^n$ is homeomorphic to S^{4n+3}/S^3. As in the Projective Spaces-Compact+Hausdorff Proposition 9.15 one can now show that $\mathbb{H}\mathrm{P}^n \cong S^{4n+3}/S^3$ is compact and Hausdorff. ⊞

Now we turn to the actual statements of the lemma.

(1) Now that we know that $\mathbb{H}\mathrm{P}^1$ is compact we can mostly copy-paste the proof of the Projective Space-Avatar Lemma 9.16 (1) and we can show, using the Compact-Hausdorff Proposition 4.23 (3), that the given map $\mathbb{H}\mathrm{P}^1 \to \mathbb{H} \cup \{\infty\}$ is a homeomorphism.

(2) By the claim we already know that $\mathbb{H}\mathrm{P}^n \cong S^{4n+3}/S^3$ is compact and Hausdorff. Next note that one can easily verify that for $i = 0, \ldots, n$ the map

$$\begin{array}{rcl} \Phi_i \colon \{[z_0 : \cdots : z_n] \in \mathbb{H}\mathrm{P}^n \mid z_i \neq 0\} & \to & \mathbb{H}^n =_{\mathrm{i}} \mathbb{R}^{4n} \\ [z_0 : \cdots : z_n] & \mapsto & \left(z_i^{-1} \cdot z_0, \ldots, z_i^{-1} \cdot z_{i-1}, z_i^{-1} \cdot z_{i+1}, \ldots, z_i^{-1} \cdot z_n\right) \end{array}$$

is a homeomorphism and that the transition maps are orientation-preserving diffeomorphisms. It follows from this discussion, the Countable Atlas-Second Countable Lemma 23.4 and the Orientation-via-Atlas Lemma 31.34 that $\mathbb{H}\mathrm{P}^n$ is a topological manifold, that the above maps define a smooth atlas and that the above atlas endows $\mathbb{H}\mathrm{P}^n$ with an orientation.[2282]

(3) First let $n \in \mathbb{N}_0$. As in the Projective Space–Cell Attachment Lemma 135.7 we consider the continuous map

$$\begin{array}{rcl} \varphi \colon S^{4n-1} & \to & \mathbb{H}\mathrm{P}^{n-1} \\ (z_1, \ldots, z_n) & \mapsto & [z_1 : \ldots : z_n] \end{array}$$

and we consider the map

$$\begin{array}{rcl} \Theta \colon \mathbb{H}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^{4n} & \to & \mathbb{H}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [z_1 : \cdots : z_n : 0], & \text{if } P = [z_1 : \cdots : z_n] \in \mathbb{H}\mathrm{P}^{n-1}, \\ [P : \sqrt{1 - \|P\|^2}], & \text{if } P \in \overline{B}^{4n}. \end{cases} \end{array}$$

Using the fact that $\mathbb{H}\mathrm{P}^n$ is Hausdorff and using that $\mathbb{H}\mathrm{P}^{n-1}$ is compact we can show again that Θ is a homeomorphism.

The statements about the CW-structure of $\mathbb{H}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^\infty$ now follow, as on page 2628, from the above observations and the definitions. ■

We conclude this section with the calculation of the cup product on quaternionic projective spaces. The statement is of course, in light of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7, entirely predictable.

Proposition 215.5. (Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition)

(1) Given any $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathbb{Z}[z]/(z^{n+1}) \to (\mathrm{H}^*(\mathbb{H}\mathrm{P}^n; \mathbb{Z}), \cup)$$

[2282] Note though that $\mathbb{H}\mathrm{P}^n$ does in general not have a complex structure, for example $\mathbb{H}\mathrm{P}^1 = S^4$ does not have a complex structure, see [**Stee1951**, Corollary 41.20] and [**Hsi1995**, p. 233].

of superalgebras, where z is an element of degree 4.

(2) There exists an isomorphism
$$\mathbb{Z}[z] \;\to\; (\mathrm{H}^*(\mathbb{H}\mathrm{P}^\infty;\mathbb{Z}),\cup)$$
of superalgebras, where z is an element of degree 4.

Proof.

(1) Let $n \in \mathbb{N}_0$. By the above Quaternion-Projective Space Lemma 215.3 (2) and (3a) we know that $\mathbb{H}\mathrm{P}^n$ is a $4n$-dimensional closed orientable smooth manifold which admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$ and such that for $k = 0, \ldots, n$ the $(4k+1)$-skeleton of $\mathbb{H}\mathrm{P}^n$ is given by $\mathbb{H}\mathrm{P}^k$. Using these observations the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 can be used, with very minor modifications, to prove the desired result.

(2) It should by now be clear to the reader how to prove (2). ■

215.2. **Octonionic projective spaces.** Since the octonions $\mathbb{O}$ are not associative, the definition of projective spaces on Definition 135.6 and Definition 215.2 cannot be applied to the octonions. But there is some hope in low dimensions. More precisely, as we will see shortly, one can use the fact that octonions are alternative to construct smooth manifolds that have all the properties that one would naively expect $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ to have.

Definition 215.6.

(1) We equip[2283]
$$T_2 \;:=\; \mathbb{O}^2 \setminus \{(0,0)\}$$
with the equivalence relation[2284 2285]
$$(x_1,y_1) \sim (x_2,y_2) \quad\Longleftrightarrow\quad \begin{array}{l}\text{there exists a } \lambda \in \mathbb{R}_{>0} \text{ such that}\\ y_1\overline{y_1} = \lambda \cdot y_2\overline{y_2} \text{ and } x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}.\end{array}$$
We set
$$\mathbb{O}\mathrm{P}^1 \;:=\; T_2/\!\sim$$
and we refer to $\mathbb{O}\mathrm{P}^1$ as the octonionic projective line.

(2) We equip
$$T_3 \;:=\; \{(x,y,z) \in \mathbb{O}^3 \setminus \{(0,0,0)\} \mid 1, x, y, z \text{ generate an associative algebra}\}$$
with the equivalence relation[2286]
$$(x_1,y_1,z_1) \sim (x_2,y_2,z_2) \iff \begin{array}{l}\text{there exists a } \lambda \in \mathbb{R}_{>0} \text{ such that}\\ x_1\overline{x_1} = \lambda\cdot x_2\overline{x_2}, \quad x_1\overline{y_1} = \lambda\cdot x_2\overline{y_2}, \quad x_1\overline{z_1} = \lambda\cdot x_2\overline{z_2},\\ y_1\overline{y_1} = \lambda\cdot y_2\overline{y_2}, \quad y_1\overline{z_1} = \lambda\cdot y_2\overline{z_2}, \quad z_1\overline{z_1} = \lambda\cdot z_2\overline{z_2}.\end{array}$$
We set[2287]
$$\mathbb{O}\mathrm{P}^2 \;:=\; T_3/\!\sim$$
and we refer to $\mathbb{O}\mathrm{P}^2$ as the octonionic projective plane. In the literature the octonionic projective plane is often also called the Cayley projective plane.

(3) We denote the equivalence class of a point $(x,y) \in T_2$ by $[x:y]$ and we denote the equivalence class of a point $(x,y,z) \in T_3$ by $[x:y:z]$.

The following proposition hopefully convinces the reader that the octonion projective line $\mathbb{O}\mathrm{P}^1$ and the octonion projective plane $\mathbb{O}\mathrm{P}^2$ deserve their names.

Proposition 215.7. (Octonion Plane Proposition)

(1) The map

by the Stereographic Projection Lemma 4.24

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\cup\{\infty\} = \mathbb{R}^8\cup\{\infty\} \overset{\downarrow}{=}_{\mathrm{i}} S^8 \\ [x:y] & \mapsto & \begin{cases} x\cdot y^{-1}, & \text{if } y\neq 0, \\ \infty, & \text{if } [x:y]=[1:0] \end{cases} \end{array}$$

is well-defined and it is a homeomorphism.

(2) The octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold.

(3) (a) The map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2 \\ [x:y] & \mapsto & [x:y:0] \end{array}$$

is well-defined and it is an embedding. We use it to view $\mathbb{O}\mathrm{P}^1$ as a subspace of $\mathbb{O}\mathrm{P}^2$.

(b) The octonionic projective plane $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Furthermore this CW-structure can be chosen such that the 8-skeleton is precisely $\mathbb{O}\mathrm{P}^1$.

Identification 215.8. We use the homeomorphism from the Octonion Plane Proposition 215.7 (1) to to make the identification $S^8 =_{\mathrm{i}} \mathbb{O}\mathrm{P}^1$.

Proof. We start out with the following claim:

Claim 1. The octonionic projective line $\mathbb{O}\mathrm{P}^1$ and the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ are both compact and Hausdorff.

Proof. It is straightforward to define equivalence relations on S^{15} and on S^{23} such that the octonionic projective line $\mathbb{O}\mathrm{P}^1$ is homeomorphic to $S^{15}/\sim$ and such that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is homeomorphic to $S^{23}/\sim$. It follows from the Compact Image Lemma 4.15 that $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ are both compact. Furthermore, using the Quotient Topology-Properties Proposition 8.5 (3) one can show fairly easily that $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ are both Hausdorff. We leave the details to Exercise 215.5.[2288] ⊞

[2283] What happens if we use these definitions of $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$, but if we replace the octonions by our more familiar friends $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$?

[2284] Note that this is obviously an equivalence relation.

[2285] The definition of the equivalence relation is a priori asymmetric in the first and the second variable. But this asymmetry is an illusion, in fact if $y_1\overline{y_1} = \lambda\cdot y_2\overline{y_2}$ and $x_1\overline{y_1} = \lambda\cdot x_2\overline{y_2}$, then using the Alternative Algebra Theorem 214.2 and the Octonion Lemma 214.12 (2) and (4) one can argue that the following calculation makes sense:

$$x_1\overline{x}_1 = x_1\underbrace{((\overline{y}_1\overline{y}_1^{-1})(y_1^{-1}y_1))}_{=1}\overline{x}_1 = x_1\overline{y}_1(y_1\overline{y}_1)^{-1}\overline{x_1\overline{y}_1} = \lambda x_2\overline{y}_2(\lambda y_2\overline{y}_2)^{-1}\overline{\lambda x_2\overline{y}_2} = \lambda x_2\overline{y}_2\overline{y}_2^{-1}y_2^{-1}y_2\overline{x}_2 = \lambda x_2\overline{x}_2.$$

We will provide the details in Exercise 215.4.

[2286] As for $\sim$ on T_2, this is basically by definition an equivalence relation.

[2287] We equip $\mathbb{O} = \mathbb{R}^8$ with the usual topology, equip $\mathbb{O}^3$ with the product topology, we equip T_3 with the subspace topology and we equip $\mathbb{O}\mathrm{P}^2$ with the quotient topology.

[2288] An arguably more elegant proof that $\mathbb{O}\mathrm{P}^2$ is Hausdorff is given in [**Lackm2012**].

Now we turn to the proofs of the actual statements we are interested in:

(1) First we need to show that the given map $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O} \cup \{\infty\}$ is well-defined. Thus suppose that we are given two points (x_1, y_1) and (x_2, y_2) in $\mathbb{O}^2 \setminus \{(0,0)\}$ that are equivalent. One can verify easily that $y_1 \neq 0$ if and only if $y_2 \neq 0$. By definition of Θ we only need to worry about the case that $y_1, y_2 \neq 0$. In this case we do indeed have

$$x_1 \cdot y_1^{-1} \underset{\uparrow}{=} x_1 \cdot \frac{1}{|y_1|^2} \cdot \overline{y}_1 \overset{\downarrow}{=} \frac{1}{y_1 \cdot \overline{y}_1} \cdot (x_1 \cdot \overline{y}_1) \underset{\uparrow}{=} \frac{1}{y_2 \cdot \overline{y}_2} \cdot (x_2 \cdot \overline{y}_2) \overset{\downarrow}{=} \ldots = x_2 \cdot y_2^{-1}.$$

by definition of $|y_1|$; same steps backwards; by the Octonion Lemma 214.12 (6); by definition of "$\sim$" we have $y_1 \cdot \overline{y_1} = \lambda \cdot y_2 \cdot \overline{y_2}$ and $x_1 \cdot \overline{y_1} = \lambda \cdot x_2 \cdot \overline{y_2}$

We have thus shown that Θ is well-defined. It is clear that the map Θ is a surjection. In Exercise 215.3 we will verify that Θ is also an injection. Next note that the argument in the proof of the Projective Space-Avatar Lemma 9.16 (2) shows that Θ is continuous. We showed in the above claim that $\mathbb{O}\mathrm{P}^1$ is compact. Evidently $\mathbb{O} \cup \{\infty\} =_{\mathrm{i}} S^8$ is Hausdorff. In summary we obtain from the Compact-Hausdorff Proposition 4.23 (3) that $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O} \cup \{\infty\}$ is a homeomorphism.

(2) We start out with the following claim:

Claim 2. The map

$$\begin{array}{rcl} \Phi_z\colon \{[x:y:z] \in \mathbb{O}\mathrm{P}^2 \mid z \neq 0\} & \to & \mathbb{O}^2 \\ [x:y:z] & \mapsto & (xz^{-1}, yz^{-1}) \end{array}$$

is a homeomorphism between an open subset of $\mathbb{O}\mathrm{P}^2$ and $\mathbb{O}^2 = \mathbb{R}^{16}$.

Proof. We prove the claim in three mostly elementary steps.

(a) Basically the same argument as in (1) shows that the map Φ_z is well-defined.

(b) It follows easily from the Definition 8.3 of the quotient topology that the domain of the map Φ_z is indeed an open subset of $\mathbb{O}\mathrm{P}^2$.

(c) We consider the map

$$\begin{array}{rcl} \Psi_z\colon \mathbb{O}^2 & \to & \{[x:y:z] \in \mathbb{O}\mathrm{P}^2 \mid z \neq 0\} \\ (x,y) & \mapsto & [x:y:1]. \end{array}$$

We first have to argue that this map makes sense, namely that for any $x, y \in \mathbb{O}$ we have $(x, y, 1) \in T_3$. To do so, recall that by the Octonion Lemma 214.12 (2) we know that $\mathbb{O}$ is alternative. By the Alternative Algebra Theorem 214.2 this implies the algebra generated by $1, x$ and y is associative. In other words, we have indeed $(x, y, 1) \in T_3$.

Furthermore note that it follows from the Topological-Quotient Proposition 8.2 (1b) that Ψ_z is continuous. It is clear that $\Phi_z \circ \Psi_z = \mathrm{id}$. Finally note that given $[x:y:z] \in \mathbb{O}\mathrm{P}^2$ with $z \neq 0$ we have

$$\Psi_z(\Phi_z([x:y:z])) = [x \cdot z^{-1} : y \cdot z^{-1} : 1] \underset{\uparrow}{=} [x:y:z]$$

follows with $\lambda = |z|^2$ and from the observation that the hypothesis $(x, y, z) \in T_3$ together with the Octonion Lemma 214.12 (4) (which implies that we always have $w + \overline{w} \in \mathbb{R} \cdot 1_{\mathbb{O}}$) implies that the elements $x, y, z, \overline{x}, \overline{y}, \overline{z}$ generate an associative algebra

We have thus shown that Ψ_z is an inverse to Φ_z. This implies that Φ_z is indeed a homeomorphism. ⊞

Similarly we define charts Φ_x and Φ_y. It is straightforward to verify that the transition maps between the three charts are smooth and orientation-preserving. In particular these three charts define a smooth atlas for $\mathbb{O}\mathrm{P}^2$. We obtain from the Countable Atlas-Second Countable Lemma 23.4 that $\mathbb{O}\mathrm{P}^2$ is second-countable. In Claim 1 we showed that $\mathbb{O}\mathrm{P}^2$ is compact and Hausdorff. The combination of all of the above shows that $\mathbb{O}\mathrm{P}^2$ is indeed a closed 16-dimensional smooth manifold. Finally, since the transition maps between the three charts are orientation-preserving we obtain from the Orientations-via-Transition Maps Lemma 31.35 that $\mathbb{O}\mathrm{P}^2$ is actually orientable.

(3) (a) As in the proof of (2) we use the non-trivial fact that for any $x, y \in \mathbb{O}$ we have $(x, y, 0) \in T_3$. This implies that the map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2 \\ [x : y] & \mapsto & [x : y : 0] \end{array}$$

is well-defined. It follows easily from the Quotient-to-Quotient Lemma 8.6 that the map is continuous. It follows basically immediately from the discussion in Footnote 2285 that the given map $\mathbb{O}\mathrm{P}^1 \to \mathbb{O}\mathrm{P}^2$ is an injection. By Claim 1 we know that $\mathbb{O}\mathrm{P}^1$ is compact and that $\mathbb{O}\mathrm{P}^2$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 4.23 (2) that the given map $\mathbb{O}\mathrm{P}^1 \to \mathbb{O}\mathrm{P}^2$ is an embedding.

(b) We view $\mathbb{O}\mathrm{P}^1 = S^8$ as a CW-complex with one 0-cell and one 8-cell in the obvious way. Next we consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{15} = \{(x_1, x_2) \in \mathbb{O}^2 \,|\, |x_1|^2 + |x_2|^2 = 1\} & \to & \mathbb{O}\mathrm{P}^1 \\ (x_1, x_2) & \mapsto & [x_1 : x_2]. \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{O}\mathrm{P}^1 \cup_\varphi \overline{B}^{16} & \to & \mathbb{O}\mathrm{P}^2 \\ [P] & \mapsto & \begin{cases} [x_1 : x_2 : 0], & \text{if } P = [x_1 : x_2] \in \mathbb{O}\mathrm{P}^1, \\ [P : 1 - \|P\|], & \text{if } P \in \overline{B}^{16}. \end{cases} \end{array}$$

As before we note that it follows from the Octonion Lemma 214.12 (2) together with the Alternative Algebra Theorem 214.2 that the map actually does take values in $\mathbb{O}\mathrm{P}^2$. We leave the slightly tedious task of verifying that the map is well-defined and a bijection to the reader. It follows easily from Claim 1 that the left-hand side is compact and it follows from Claim 1 that the right-hand side is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map is a homeomorphism. But this implies that $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely three cells, one in dimensions 0, 8 and 16. ∎

We continue with the following predictable analog of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 and the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5:

Proposition 215.9. (Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition) There exists an isomorphism

$$\mathbb{Z}[z]/(z^3) \to (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2; \mathbb{Z}), \cup)$$

of superalgebras, where z is an element of degree 8.

Proof. In the Octonion Plane Proposition 215.7 (2) we showed that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold which admits a

CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Once again the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 clinches the deal. ■

215.3. **Questions inspired by quaternionic and octonionic projective spaces.** The discussion of quaternionic and octonionic projective spaces raises at least as many questions as it answers. In this section we will collect a few questions. Some of them will be addressed later in the lecture notes.

First note that it follows from the Octonion Plane Proposition 215.7 (3) together with the CW-Skeleton-π_n-Proposition 141.3 that the octonion plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold that is 7-connected and that has odd Euler characteristic. The following question is, to the best of my knowledge, open:

Question 215.10. Does there exist a closed 8-connected orientable smooth manifold with odd Euler characteristic?

In [**Hoe2020**, Theorem 1.2 together with Proposition 4.2] it is shown that the dimension of a closed 8-connected orientable smooth manifold with odd Euler characteristic, if it exists, has to be divisible by 16.

It is natural to ask whether there exists a smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ for $n \geq 3$. In the proof of the Octonion Plane Proposition 215.7 (2) we used that $n = 2$ has the property, provided by the Octonion Lemma 214.12 (2) and the Alternative Algebra Theorem 214.2, that for any $x_1, \dots, x_n \in \mathbb{O}$ the elements $1, x_1, \dots, x_n$ generate an associative algebra. It is straightforward to see that this statement is not true for $n \geq 3$. Nonetheless, one can wonder whether there exists an alternative way to define a meaningful higher dimensional octonionic projective space. This slightly vague question can be formulated in many different ways. Here is one possible interpretation of this question.

Question 215.11. Let $n \geq 3$. Does there exist a closed orientable $8n$-dimensional smooth manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \dots, 8n$ and which has no other cells?

We will answer this question in Exercise 237.2, using the No-$\mathbb{O}\mathrm{P}^3$-Proposition 237.15, which we will also mention in a second.

Now let us turn to cohomology rings. In the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7, the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5 and the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 215.9 we saw that certain polynomials rings $\mathbb{Z}[x]/(x^{n+1})$, where x has degree $2, 4, 8$, are realized as cohomology rings of topological spaces. This raises the following question, which can be viewed as a more specific version of Question 208.15:

Question 215.12. For which even numbers m does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^3)$, where z is now an element of degree m?

We will give a partial answer to this question in the Cohomology Ring–Non-Realization Proposition 237.7. The following question is in a similar vein, namely it asks whether there exists a "cohomological $\mathbb{O}\mathrm{P}^3$".

Question 215.13. Let $n \in \mathbb{N}_{\geq 3}$. Does there exist a topological space X such that the cohomology ring $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^{n+1})$, where z is now an element of degree 8?

In Exercise 215.7 we will see that a negative answer to Question 215.13 would also tragically give a negative answer to Question 215.11. We will settle Question 215.13 much later in the No-$\mathbb{O}\mathrm{P}^3$-Proposition 237.15 using Steenrod operations.

215.4. **Exercises for Chapter 215 .**

Exercise 215.1. For which $n \in \mathbb{N}_0$ does there exist an orientation-reversing diffeomorphism $\mathbb{H}\mathrm{P}^n \to \mathbb{H}\mathrm{P}^n$?

Exercise 215.2. Let $1 \leq m < n$. Show that $\mathbb{H}\mathrm{P}^m \subseteq \mathbb{H}\mathrm{P}^n$ is not a retract of $\mathbb{H}\mathrm{P}^n$.

Exercise 215.3. Show that the map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O} \cup \{\infty\} \\ [x:y] & \mapsto & \begin{cases} x \cdot y^{-1}, & \text{if } y \neq 0, \\ \infty, & \text{if } [x:y] = [1:0] \end{cases} \end{array}$$

is an injection.

Exercise 215.4. Let $x_1, y_1, x_2, x_2 \in \mathbb{O}$ and $\lambda \in \mathbb{R}_{>0}$ such that $y_1\overline{y_1} = \lambda \cdot y_2\overline{y_2}$ and such that $x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}$. Show that $x_1\overline{x_1} = \lambda \cdot x_2\overline{x_2}$.
Remark. The idea for the proof is to do the following calculation:

$$\begin{array}{rclcrcl} x_1\overline{x}_1 & = & x_1((\overline{y}_1\overline{y}_1^{-1})(y_1^{-1}y_1))\overline{x}_1 & = & x_1\overline{y}_1(y_1\overline{y}_1)^{-1}\overline{x_1\overline{y}_1} & = & \lambda x_2\overline{y}_2(\lambda y_2\overline{y}_2)^{-1}\overline{\lambda x_2\overline{y}_2} \\ & = & \lambda x_2\overline{y}_2\overline{y}_2^{-1}y_2^{-1}y_2\overline{x}_2 & & = & & \lambda x_2\overline{x}_2. \end{array}$$

The problem with this calculation is that we cannot just drop all the parentheses, since the octonions are not associative. Your task is to use the Alternative Algebra Theorem 214.2 and the Octonion Lemma 214.12 (2) and (4) to make the argument work. If anything else, this exercise will give the reader a good respect for associativity.

Exercise 215.5. Show that $\mathbb{O}\mathrm{P}^2$ is Hausdorff.
Hint. First show that $\mathbb{O}\mathrm{P}^2$ is in fact homeomorphic to a suitable quotient of the sphere $S^{23} = \{(x,y,z) \in \mathbb{O}^3 \,|\, |x|^2 + |y|^2 + |z|^2 = 1\}$ and then apply the Quotient Topology-Properties Proposition 8.5 (3).

Exercise 215.6. For which $n \in \mathbb{N}$ does there exist a continuous map $f\colon \mathbb{H}\mathrm{P}^n \to \mathbb{H}\mathrm{P}^n$ of degree 7?

Exercise 215.7. Let $d \in \mathbb{N}$ be even and let $k \in \mathbb{N}$. Let M be a closed $d \cdot k$-dimensional topological manifold that admits a CW-structure with precisely one cell in dimensions $0, d, 2d, \ldots, kd$ and which has no other cells. Show that the cohomology ring $(\mathrm{H}^*(M;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[x]/(x^{k+1})$.
Remark. This exercise shows that a negative answer to Question 215.13 entails a negative answer to Question 215.11.

CHAPTER 216

The Hopf invariant

In previous chapters we have seen that the calculation of the cup product, in particular on projective spaces, allowed us to answer many questions on topological spaces that we could not deal with before. In this chapter we want to use the cup product to address the following question that we formulated a long time ago:

Question 134.25. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?

In this chapter we will use the calculation of the cup product of $\mathbb{C}\mathrm{P}^2$ to give an affirmative answer to this question. Furthermore we will use the quaternions and octonions to define higher dimensional analogs of the Hopf map which, as we will see, also represent non-trivial elements in homotopy groups of spheres.

Furthermore we will use cup products to introduce for each $n \in \mathbb{N}_{\geq 2}$ the Hopf invariant $\pi_{2n-1}(S^n, *) \to \mathbb{Z}$ which turns out to be a homomorphism. We will see that for each even n this invariant has non-zero image. In other words, we will show that for each even n the group $\pi_{2n-1}(S^n, *) \to \mathbb{Z}$ admits an epimorphism onto $\mathbb{Z}$.

This chapter is organized as follows:

- In Section 216.1 we will see that Hopf map is not null-homotopic.
- In Section 216.2 we will define the Hopf invariant.
- In Section 216.3 we will discuss the relationship between the Hopf invariant and multiplication maps.
- In Section 216.4 we will provide examples of maps of non-trivial bidegrees.
- In Section 216.5 we will provide exercises for Chapter 216.

In the next Chapter 217 we will formulate and prove a basic version of Alexander duality for submanifolds of spheres.

216.1. The Hopf map is not null-homotopic.

First we recall the definition of the Hopf map from Definition 134.24:

Definition 216.1. We refer to the map[2289]

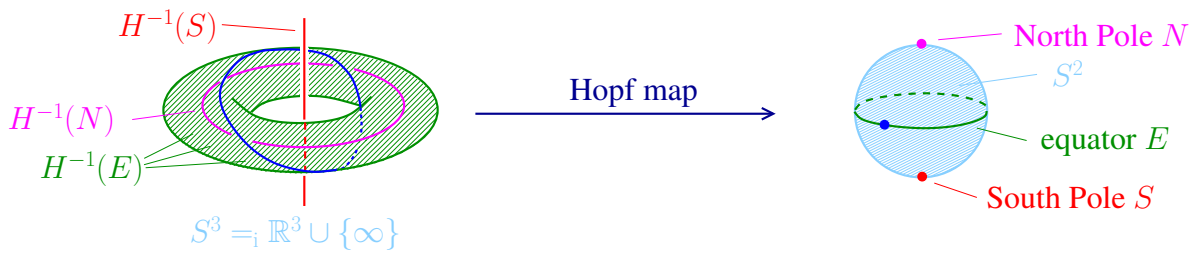
see page 340

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 \overset{\downarrow}{=}_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the Hopf map.

Since the Hopf map is so much fun we will introduce two higher-dimensional analogs of the Hopf map. More precisely, we will repeat the trick with $\mathbb{C}$ replaced by the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. First we point out that we can use the homeomorphisms from the

[2289] See also the figure on page 2590 for an illustration of the map.



Quaternion-Projective Space Lemma 215.3 (1) and the Octonion Plane Proposition 215.7 (1) to make the identifications $\mathbb{H}\mathrm{P}^1 =_{\mathrm{i}} S^4$ and $\mathbb{O}\mathrm{P}^1 =_{\mathrm{i}} S^8$. Now we can introduce the quaternionic and octonionic Hopf maps.

Definition 216.2. We refer to

$$\begin{array}{rcl} H_{\mathbb{H}} \colon S^7 = \{(z_1, z_2) \in \mathbb{H}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^4 =_{\mathrm{i}} \mathbb{H}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the quaternionic Hopf map and we refer to

$$\begin{array}{rcl} H_{\mathbb{O}} \colon S^{15} = \{(z_1, z_2) \in \mathbb{O}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^8 =_{\mathrm{i}} \mathbb{O}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the octonionic Hopf map.

The idea for answering Question 134.25 is to translate this question about maps to a question about topological spaces for which we can then bring our new invariants to bear. To carry out this approach we recall the following definitions from earlier on:

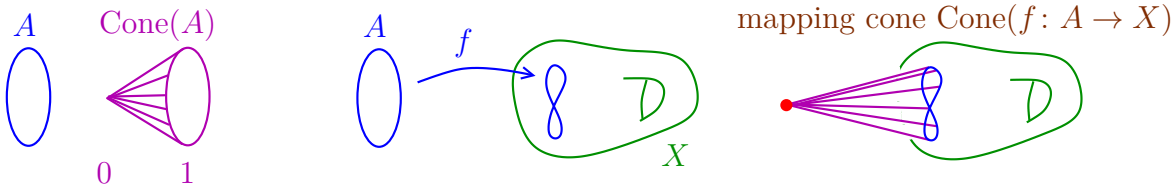
Definition 216.3.

(1) In Definition 20.12 we defined the cone of a topological space A to be the topological space $\mathrm{Cone}(A) := (A \times [0,1])/(A \times \{0\})$.

(2) In Definition 21.7 we defined the mapping cone of a continuous map $f \colon A \to X$ between topological spaces as

$$\mathrm{Cone}(f \colon A \to X) \;:=\; (\mathrm{Cone}(A) \cup X)/\sim$$

where $(a, 1) \sim f(a)$ for all $a \in A$.



Example. It follows easily from the Compact-Hausdorff Proposition 4.23 (3) and the Topological-Quotient Proposition 8.2 (1b) that the map

$$\begin{array}{rcl} \mathrm{Cone}(S^{n-1}) & \to & \overline{B}^n \\ [(P, t)] & \mapsto & t \cdot P \end{array}$$

is a homeomorphism. If X is a topological space and if $f \colon S^{n-1} \to X$ is a continuous map, then we can use the above homeomorphism to get a homeomorphism

$$\mathrm{Cone}(f \colon S^{n-1} \to X) \;=\; \overline{B}^n \cup_f X.$$

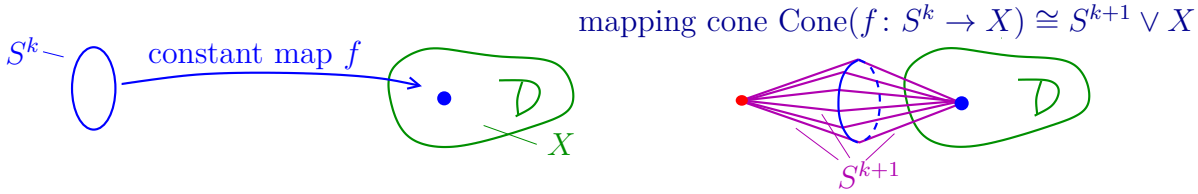
It follows immediately from this observation that any finite CW-complex can be viewed as an iterated mapping cone (where we start out with the empty topological space). □

The next lemma gives us another elementary but oddly useful example.

Lemma 216.4. (Mapping Cone-of-Constant Map Lemma) If X is a topological space and if $f\colon S^k \to X$ is a constant map, then there exists a homeomorphism

$$\mathrm{Cone}(f\colon S^k \to X) \xrightarrow{\cong} S^{k+1} \vee X,$$

where we perform the wedge for suitable points in S^{k+1} and X.



Proof. The Mapping Cone-of-Constant Map Lemma 216.4 is essentially a special case of the earlier Mapping Cone-of-Constant Map Lemma 21.10 which says that if $f\colon A \to X$ is a constant map between topological spaces, then there exists a homeomorphism

$$\mathrm{Cone}(f\colon A \to X) \xrightarrow{\cong} \Sigma(A) \vee X.$$

The current lemma follows from this general fact and the observation from page 556 that the suspension of S^k is homeomorphic to S^{k+1}. ■

The following lemma gives particularly interesting examples of mapping cones.

Lemma 216.5. (Mapping Cone-of-Hopf Map Lemma) There exist homeomorphisms

(1) $\mathrm{Cone}(H\colon S^3 \to S^2) \cong \mathbb{C}\mathrm{P}^2$
(2) $\mathrm{Cone}(H_{\mathbb{H}}\colon S^7 \to S^4) \cong \mathbb{H}\mathrm{P}^2$
(3) $\mathrm{Cone}(H_{\mathbb{O}}\colon S^{15} \to S^8) \cong \mathbb{O}\mathrm{P}^2$.

Proof.

(1) As we discussed in the Projective Space-Cell Attachment Lemma 135.7 on page 2628 we can view $\mathbb{C}\mathrm{P}^2$ as a CW-complex that is obtained from the CW-complex $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ by attaching a 4-cell such that the attaching map $S^3 \to \mathbb{C}\mathrm{P}^1$ of the 4-cell is precisely the Hopf map. This observation, together with the discussion on page 4123, already implies the statement.For completeness' sake we give the argument more explicitly and in more detail. We consider the maps[2290]

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^3) & \to & \mathbb{C}\mathrm{P}^2 \\ [(z_0,z_1),r] & \mapsto & [r\cdot z_0 : r\cdot z_1 : 1-r] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^2 \\ [z_0:z_1] & \mapsto & [z_0:z_1:0]. \end{array}$$

It follows immediately from the definitions together with the Topological-Quotient Proposition 8.2 (1b) that these two maps define a continuous map

$$\underbrace{\mathrm{Cone}(H\colon S^3 \to S^2)}_{=(\mathrm{Cone}(S^3)\cup\mathbb{C}\mathrm{P}^1)/\sim} \to \mathbb{C}\mathrm{P}^2.$$

[2290]It follows from the Topological-Quotient Proposition 8.2 (1b) and the Topological-Quotient Proposition 8.2 (1a) that these two maps are continuous.

One can easily verify that this map is a bijection. Furthermore, using the Compact-Hausdorff Proposition 4.23 (3) and the Projective Spaces-Compact+Hausdorff Proposition 9.15 one sees that the map is in fact a homeomorphism.

(2) In the proof of Quaternion-Projective Space Lemma 215.3 we showed that $\mathbb{H}\mathrm{P}^2$ is homeomorphic to the result of attaching an 8-cell to $\mathbb{H}\mathrm{P}^1 = S^4$ via the quaternion Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$. With this observation the proof of (2) is basically identical to the proof of (1).

(3) Implicitly we mostly proved this statement in the proof of the Octonion Plane Proposition 215.7. Alternatively we also refer to [**Lackm2012**, p. 4] for details. ■

Now we can finally answer Question 134.25 in the affirmative. More precisely we have the following theorem:

Theorem 216.6. (Hopf Map–Homotopy Theorem)[2291]

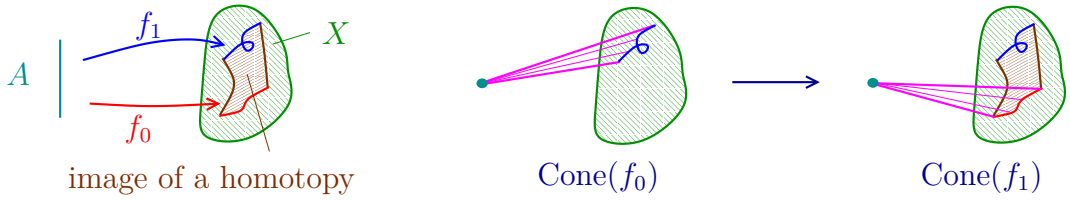
(1) The Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in $\pi_3(S^2)$, i.e. we have $[H] \neq 0 \in \pi_3(S^2)$.

(2) We have $[H_{\mathbb{H}}] \neq 0 \in \pi_7(S^4)$.

(3) We have $[H_{\mathbb{O}}] \neq 0 \in \pi_{15}(S^8)$.

The proof of the Hopf Map–Homotopy Theorem 216.6 builds on the following earlier proposition.

Proposition 21.15. (Homotopic Map–Mapping Cone Proposition) Suppose we are given continuous maps $f_0, f_1\colon A \to X$ between topological spaces. If f_0, f_1 are homotopic, then $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ are homotopy equivalent.



Proof of the Hopf Map–Homotopy Theorem 216.6. First we prove (1). Recall that for two topological spaces X and Y we write $X \simeq Y$ if they are homotopy equivalent. We now see that

$$\mathrm{Cone}(\text{Hopf map } H) \underset{\text{by the Mapping Cone-of-Hopf Map Lemma 216.5 (1)}}{\cong} \mathbb{C}\mathrm{P}^2 \underset{\text{by the Doppelgänger Corollary 213.11}}{\not\simeq} S^2 \vee S^4 \underset{\text{by the Mapping Cone-of-Constant Map Lemma 216.4}}{\cong} \mathrm{Cone}(\text{constant map}).$$

But by the Homotopic Map–Mapping Cone Proposition 21.15 this means that the Hopf map is not homotopic to the constant map. In particular the Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the homotopy group $\pi_3(S^2)$.

The proof of the other two statements is quite similar. Again we use the Mapping Cone-of-Hopf Map Lemma 216.5 and we use the calculations of the cup products of $\mathbb{H}\mathrm{P}^2$

[2291] The statement regarding the "classical" Hopf map was first proved by Heinz Hopf [**Hopf1931**] in 1931.

and $\mathbb{O}\mathrm{P}^2$, see the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5 and the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 215.9, to conclude that the mapping cones of the quaternionic and octonionic Hopf maps are not homotopy equivalent to a wedge of spheres. ∎

216.2. **The Hopf invariant.** It is natural to ask how far we can push the methods of the Hopf Map–Homotopy Theorem 216.6 to gain more insight into homotopy groups of spheres. In this section, given $n \geq 2$ we will introduce the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \;\to\; \mathbb{Z}$$

which satisfies $\mathrm{Hopf}(H) = \pm 1$, $\mathrm{Hopf}(H_{\mathbb{H}}) = \pm 1$ and $\mathrm{Hopf}(H_{\mathbb{O}}) = \pm 1$. Furthermore we will show that the Hopf invariant is in fact a homomorphism.

The idea behind the definition of the Hopf invariant is straightforward. Namely, given $n \in \mathbb{N}_{\geq 2}$ and given a continuous map $f\colon S^{2n-1} \to S^n$ we consider again the corresponding mapping cone $\mathrm{Cone}(f\colon S^{2n-1} \to S^n)$. We will see in the Mapping Cone-of-Spheres-H*-Lemma 216.9 that it follows from the fact that $n \geq 2$ that we have $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$. We will equip $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z})$ with a "preferred generator". We pick a generator $\varphi \in \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})$ and we define $\mathrm{Hopf}(f)$ to be the unique integer that satisfies

$$\varphi \cup \varphi \;=\; \mathrm{Hopf}(f) \cdot \text{preferred generator} \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

To carry out this program we need to carefully study the cohomology groups of a mapping cone. Recall that in the Mapping Cone-H$_*$-Lemma 153.17 we saw that the homology groups of a mapping cone fit naturally into a long exact sequence of homology groups. The following lemma shows that the analogous statement also holds for the cohomology groups of a mapping cone.

Lemma 216.7. (Mapping Cone-H*-Lemma) Let G be an abelian group.

(1) Let $f\colon A \to X$ be a continuous map between topological spaces.
   (a) We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\sim$ the obvious inclusion map.
   (b) We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also the figure below.) We consider the following diagram:

$$\begin{array}{ccccccc} \ldots \longrightarrow \mathrm{H}^{k-1}(U \cap V; G) & \xrightarrow{\delta_k} & \mathrm{H}^k(\mathrm{Cone}(f); G) & \longrightarrow & \mathrm{H}^k(U;G) \oplus \mathrm{H}^k(V;G) & \longrightarrow & \ldots \\ \downarrow \cong & \nearrow_{=:\delta_{k-1}(f)} & & & & & \\ \mathrm{H}^{k-1}(A; G). & & & & & & \end{array}$$

   Here the top sequence is the long exact sequence arising from the Mayer–Vietoris-H*-Theorem 200.11 (1)[2292] and the vertical map comes from the natural isomorphisms $\mathrm{H}^{k-1}(U \cap V; G) \xleftarrow{\cong} \mathrm{H}^{k-1}(A \times \frac{1}{2}; G) \xrightarrow{\cong} \mathrm{H}^{k-1}(A; G)$. From the above diagram we obtain the map

$$\delta_{k-1}(f)\colon \mathrm{H}^{k-1}(A; G) \;\to\; \mathrm{H}^k(\mathrm{Cone}(f); G).$$

With this notation the following sequence is exact

$$\ldots \longrightarrow \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f); G) \xrightarrow{j(f)^*} \mathrm{H}^k(X; G) \xrightarrow{f^*} \mathrm{H}^k(A; G) \xrightarrow{\delta_k(f)} \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f); G) \longrightarrow \ldots$$

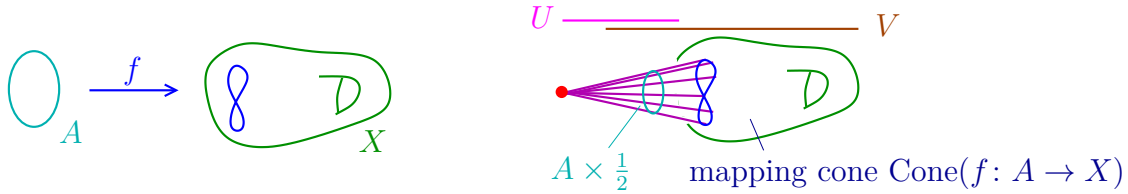
(2) The connecting homomorphism in (1) is natural. In particular given any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{X} \end{array}$$

of continuous maps between topological spaces the following diagram commutes:

$$\begin{array}{ccccccccc} \dots \longrightarrow \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f);G) & \xrightarrow{j(f)^*} & \mathrm{H}^k(X;G) & \xrightarrow{f^*} & \mathrm{H}^k(A;G) & \xrightarrow{\delta_k(f)} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f);G) \longrightarrow \dots \\ \uparrow & & \uparrow{\scriptstyle\psi^*} & & \uparrow{\scriptstyle\varphi^*} & & \uparrow \\ \dots \longrightarrow \widetilde{\mathrm{H}}^k(\mathrm{Cone}(\widetilde{f});G) & \xrightarrow{j(\widetilde{f})^*} & \mathrm{H}^k(\widetilde{X};G) & \xrightarrow{\widetilde{f}^*} & \mathrm{H}^k(\widetilde{A};G) & \xrightarrow{\delta_k(\widetilde{f})} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(\widetilde{f});G) \longrightarrow \dots \end{array}$$

Here the vertical maps between the cohomology groups of the mapping cones are induced by the continuous maps from the Mapping Cone–Functor Lemma 21.13.



Proof. The proof is basically identical to the proof of the Mapping Cone-H$_*$-Lemma 153.17. The meticulous reader who filled in the details of the proof of the Mapping Cone-H$_*$-Lemma 153.17 will have no troubles working out the details of the present proof. ■

Notation 216.8. Throughout this section we use the notation that we had already introduced in the formulation of the Mapping Cone-H*-Lemma 216.7. More precisely, given a continuous map $f\colon A \to X$ between two topological spaces we denote by

$$j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \cup X)/\sim$$

the natural embedding and furthermore we denote by

$$\delta_{k-1}(f)\colon \mathrm{H}^{k-1}(A;\mathbb{Z}) \to \mathrm{H}^k(\mathrm{Cone}(f);\mathbb{Z})$$

the coboundary map in the long exact sequence in cohomology of the Mapping Cone-H*-Lemma 216.7.

In the following we consider a special case of the Mapping Cone-H*-Lemma 216.7 which will be particularly interesting for us:

Lemma 216.9. (Mapping Cone-of-Spheres-H*-Lemma) Let $f\colon S^m \to S^n$ be a continuous map. If $m \geq n+1$ and $n \geq 1$, then the maps

$$\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \xrightarrow{j(f)^*} \mathrm{H}^n(S^n;\mathbb{Z})$$

and

$$\mathrm{H}^m(S^m;\mathbb{Z}) \xrightarrow{\delta_m(f)} \mathrm{H}^{m+1}(\mathrm{Cone}(f);\mathbb{Z})$$

[2292] Note that U and V are open subsets of $\mathrm{Cone}(f)$.

are isomorphisms and all other reduced cohomology groups of $\mathrm{Cone}(f)$ are zero. In particular we have isomorphisms

$$\mathrm{H}^k(\mathrm{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, n, m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The lemma is an almost immediate consequence of the Mapping Cone-H^*-Lemma 216.7 and the calculation of the cohomology groups of spheres, see page 3830. For $n = 1$ one also needs to invoke Exercise 198.6. ■

Now we are finally in a position to turn the idea from page 4126 into a proper definition.

Definition 216.10. Let $n \in \mathbb{N}_{\geq 2}$ and let $f\colon S^{2n-1} \to S^n$ be a continuous map. The Hopf invariant $\mathrm{Hopf}(f)$ is defined to be the unique integer that satisfies the following equality in $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$:[2293] [2294]

$$\underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of} \\ \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \\ \text{by the above lemma}}} \cup \underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of} \\ \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \\ \text{by the above lemma}}} = \mathrm{Hopf}(f) \cdot \underbrace{\delta_{2n-1}(f)([S^{2n-1}]^*)}_{\substack{\text{generator of} \\ \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \\ \text{by the above lemma}}}$$

Remark. Note that it follows immediately from the (anti-) commutativity of the cup product, see the Cup Product-Commutativity Proposition 203.20, that the Hopf invariant is zero for all odd $n \in \mathbb{N}$. □

We formulate our first example as a lemma:

Lemma 216.11. (Hopf Map-Hopf Invariant Lemma) The Hopf maps $H\colon S^3 \to S^2$, $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ all have Hopf invariant ± 1.[2295]

Proof. In the Mapping Cone-of-Hopf Map Lemma 216.5 we showed that the mapping cone $\mathrm{Cone}(H)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$. By the Mapping Cone-of-Spheres-H^*-Lemma 216.9 we know that $(j(H)^*)^{-1}([S^2]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and $\delta_3(f)([S^3]^*) \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ are generators. It follows from the calculation of the cup product on $\mathbb{C}\mathrm{P}^2$, see the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7, that $\mathrm{Hopf}(H) = -1$ or $\mathrm{Hopf}(H) = 1$. By the same argument it follows from the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5 and the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 215.9 that the Hopf invariants of $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ are ± 1. ■

Lemma 216.12. (Hopf Invariant–Homotopy Lemma) Let $n \geq 2$. Suppose we are given two continuous maps $f_0, f_1\colon S^{2n-1} \to S^n$. If the two maps f_0 and f_1 are homotopic, then $\mathrm{Hopf}(f_0) = \mathrm{Hopf}(f_1)$.

Proof. Let $H\colon S^{2n-1} \times [0,1] \to S^n$ be a homotopy between f_0 and f_1. We denote by $i_0, i_1\colon S^{2n-1} \to S^{2n-1} \times [0,1]$ the two obvious closed embeddings. We obtain the following

[2293] Here we implicitly use that in the Mapping Cone-of-Spheres-H^*-Lemma 216.9 we showed that $j(f)^*$ and $\delta_{2n-1}(f)$ are isomorphisms.

[2294] Instead of the generator $(j(f)^*)^{-1}([S^n]^*) \in$ we could also have used the other generator of $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})$, since the sign ambiguity goes away with the square.

[2295] With more care one can show that the Hopf invariant in each case is $+1$. But this extra information is irrelevant for our purpose.

two commutative diagrams

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f_0} & S^n \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ S^{2n-1}\times[0,1] & \xrightarrow{H} & S^n \end{array} \quad \text{and} \quad \begin{array}{ccc} S^{2n-1} & \xrightarrow{f_1} & S^n \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ S^{2n-1}\times[0,1] & \xrightarrow{H} & S^n. \end{array}$$

Next that it follows from the Mapping Cone–Functor Lemma 21.13 and the Mapping Cone–Closed Embedding Lemma 21.16 that the diagrams give rise to closed embeddings $\Xi_0\colon \mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$ and $\Xi_1\colon \mathrm{Cone}(f_1) \to \mathrm{Cone}(H)$. From the Mapping Cone-H*-Lemma 216.7 we obtain the following commutative diagram of long exact sequences (where we drop the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccccccc}
\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f_0)) & \xrightarrow{j(f_0)^*} & \mathrm{H}^k(S^n) & \xrightarrow{f_0^*} & \mathrm{H}^k(S^{2n-1}) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f_0)) \to \ldots \\
{\scriptstyle \Xi_0^*}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \cong}\uparrow{\scriptstyle i_0^*} & & {\scriptstyle \Xi_0^*}\uparrow \\
\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(H)) & \xrightarrow{j(H)^*} & \mathrm{H}^k(S^n) & \xrightarrow{H^*} & \mathrm{H}^k(S^{2n-1}\times[0,1]) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(H)) \to \ldots \\
{\scriptstyle \Xi_1^*}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle i_1^*} & & {\scriptstyle \Xi_1^*}\downarrow \\
\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f_1)) & \xrightarrow{j(f_1)^*} & \mathrm{H}^k(S^n) & \xrightarrow{f_1^*} & \mathrm{H}^k(S^{2n-1}) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f_1)) \to \ldots
\end{array}$$

Note that the vertical maps between the cohomology groups of the mapping cones are isomorphisms. This can be shown using the Five Lemma 145.11, or alternatively, in a more fancy way, using the Mapping Cone-of-Homotopy Lemma 21.17 together with the Homotopy Equivalence-H*-Proposition 198.12.

Next we consider the generator $\Theta := (i_0^*)^{-1}([S^{2n-1}]^*) = (i_1^*)^{-1}([S^{2n-1}]^*)$ of the group $\mathrm{H}^{2n-1}(S^{2n-1}\times[0,1];\mathbb{Z})$. We define a number $\mathrm{Hopf}(H)$ via the equality

$$(j(H)^*)^{-1}([S^n]^*) \cup (j(H)^*)^{-1}([S^n]^*) \;=\; \mathrm{Hopf}(H)\cdot\delta_{2n-1}(H)(\Theta)$$

in $\mathrm{H}^{2n}(\mathrm{Cone}(H);\mathbb{Z}) \cong \mathbb{Z}$. Finally we apply the above commutative diagram and we apply the Cup Product-Naturality Lemma 203.9 to the maps $\Xi_0\colon \mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$ and $\Xi_1\colon \mathrm{Cone}(f_1) \to \mathrm{Cone}(H)$ and we see that $\mathrm{Hopf}(f_0) = \mathrm{Hopf}(H) = \mathrm{Hopf}(f_1)$. ■

Definition 216.13. It follows from the Hopf Invariant–Homotopy Lemma 216.12 that for each $n \geq 2$ the map

$$\begin{array}{rccc} \mathrm{Hopf}\colon & \pi_{2n-1}(S^n,*) & \to & \mathbb{Z} \\ & [f] & \mapsto & \mathrm{Hopf}(f) \end{array}$$

is well-defined. This map is called the Hopf invariant.

Our next proposition says that the Hopf invariant is a homomorphism. This is arguably somewhat surprising since at first glance the definition of the Hopf variant, given by squaring an element, does not look particularly like a linear map.

Proposition 216.14. (Hopf Invariant–Homomorphism Proposition) For any $n \geq 2$ the Hopf invariant $\mathrm{Hopf}\colon \pi_{2n-1}(S^n,*) \to \mathbb{Z}$ is a homomorphism.

Before we talk about the proof of the Hopf Invariant–Homomorphism Proposition 216.14 let us formulate the following immediate corollary:

Corollary 216.15. (Hopf Invariant One-Corollary) The Hopf invariants

$$\begin{array}{lrcl} (1) & \mathrm{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ (2) & \mathrm{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ (3) & \mathrm{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms. In particular all three homotopy groups to the left are infinite.

Proof. The corollary follows immediately from the Hopf Invariant–Homomorphism Proposition 216.14 and the calculation of the Hopf invariant for the Hopf maps H, $H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ in the Hopf Map-Hopf Invariant Lemma 216.11. ■

The technically slightly dicey proof of the Hopf Invariant–Homomorphism Proposition 216.14 will require the remainder of this section. The reader is cordially invited to skip the argument and to jump to the much more entertaining content of the next Section 216.3.

Our proof[2296] of the Hopf Invariant–Homomorphism Proposition 216.14 will require a few preparations. First of all, on several occasions we will need the following construction.

Construction 216.16. Suppose we are given the following commutative diagram of continuous maps between topological spaces:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y}. \end{array}$$

Then, as discussed in the Mapping Cone–Functor Lemma 21.13, we obtain an induced continuous map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y) & \to & \mathrm{Cone}(f\colon \widetilde{X} \to \widetilde{Y}) \\ [P] & \mapsto & \begin{cases} [(\varphi(x),t)], & \text{if } P = (x,t) \text{ with } x \in X,\ t \in [0,1], \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array}$$

We consider the following diagram:

$$\begin{array}{ccccc} Y & \xhookrightarrow{j} & \mathrm{Cone}(f\colon X \to Y) & \xrightarrow{p} & \Sigma(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma(\varphi)} \\ \widetilde{Y} & \xhookrightarrow{j} & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) & \xrightarrow{p} & \Sigma(\widetilde{X}). \end{array}$$

Here the horizontal maps to the left are the natural embeddings i that we introduced in the Mapping Cone–Embedding Lemma 21.11 (2) and the horizontal maps p on the right are given by the natural projection and the natural homeomorphism from the Mapping Cone–Quotient Lemma 21.14. Since these maps are natural we see that the diagram commutes.

We will also need the following slight reinterpretation of the Hopf invariant.

Lemma 216.17. (Hopf Invariant–Reinterpretation Lemma) Let $f\colon S^{2n-1} \to S^n$ be a continuous map. We denote by $j\colon S^n \to \mathrm{Cone}(f)$ the natural embedding and we denote by $p\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f)/S^n = \Sigma(S^{2n-1})$ the natural map that is given by the projection

[2296] Our proof is based on the proof in [**MTa1968**, p. 35]. An alternative proof is sketched in [**Stro2011**, p. 557].

and the homeomorphism from the Mapping Cone–Quotient Lemma 21.14. Then

$$(j^*)^{-1}([S^n]^*) \cup (j^*)^{-1}([S^n]^*) \;=\; \mathrm{Hopf}(f) \cdot p^*(\underset{\substack{\uparrow \\ \text{isomorphism provided by} \\ \text{the Suspension-H}^*\text{-Lemma 200.4}}}{\Sigma}([S^{2n-1}]^*)) \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

Proof of Lemma 216.17. We start out with the following claim:

Claim. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 556. We have

$$\Sigma_{S^n}([S^n]^*) \;=\; (-1)^{n+1} \cdot [S^{n+1}]^* \;\in\; \mathrm{H}^{n+1}(S^{n+1};\mathbb{Z}).$$

Proof. Recall that on page 2978 we saw that $\Sigma_{S^n}([S^n]) = (-1)^{n+1} \cdot [S^{n+1}] \in \mathrm{H}_{n+1}(S^{n+1})$. The claim now follows from Statement (2) of the Suspension-H*-Lemma 200.4. ⊞

The statement follows basically immediately from the claim[2297] and the cohomological analog of the Cone–Connecting Homomorphism Lemma 153.18. ■

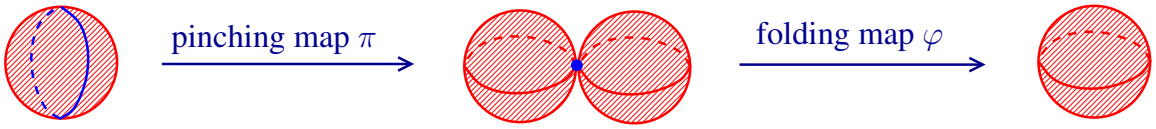
Now we are finally in a position to prove the Hopf Invariant–Homomorphism Proposition 216.14:

Proof of Proposition 216.14. We start out with a few preparations.

(1) Given a pointed topological space (X, x_0) we define $X \vee X$ by identifying two copies of X along the base point. We refer to the map $\varphi\colon X \vee X \to X$ which is given by the identity on both copies as the *folding map*.

(2) Let $k \in \mathbb{N}_{\geq 2}$. On page 2579 we introduced a pinching map $\pi\colon S^k \to S^k \vee S^k$. The precise definition is rather irrelevant to us right now. What matters is that we saw on page 2579 that given any pointed topological space (X, x_0) and given any two continuous maps $f_1, f_2\colon (S^k, *) \to (X, x_0)$ we have

$$[f_1 * f_2] \;=\; [\, S^k \xrightarrow{\pi} S^k \vee S^k \xrightarrow{f_1 \vee f_2} X \vee X \xrightarrow{\varphi} X \,] \;\in\; \pi_k(X, x_0).$$

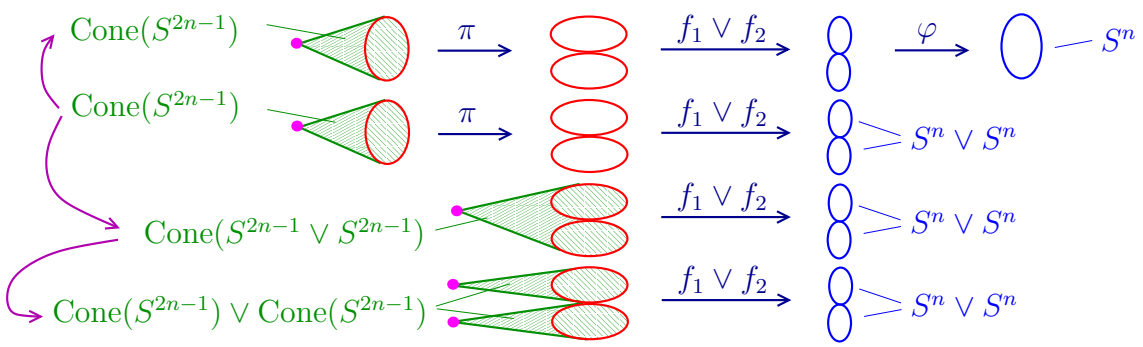


Now we turn to our actual situation. Let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given two continuous maps $f_1, f_2\colon (S^{2n-1}, *) \to (S^n, *)$. For $i = 1, 2$ we consider the following diagram:

$$\begin{array}{ccccc}
S^n & \xrightarrow{j} & \mathrm{Cone}(\varphi \circ (f_1 \vee f_2) \circ \pi\colon S^{2n-1} \to S^n) & \xrightarrow{p} & \Sigma S^{2n-1} \\
{\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\mathrm{id}} \\
S^n \vee S^n & \xrightarrow{j} & \mathrm{Cone}((f_1 \vee f_2) \circ \pi\colon S^{2n-1} \to S^n \vee S^n) & \xrightarrow{p} & \Sigma S^{2n-1} \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\Sigma(\pi)} \\
S^n \vee S^n & \xrightarrow{j} & \mathrm{Cone}(f_1 \vee f_2\colon S^{2n-1} \vee S^{2n-1} \to S^n \vee S^n) & \xrightarrow{p} & \Sigma(S^{2n-1} \vee S^{2n-1}) \\
{\scriptstyle\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\Xi} & & \uparrow{\scriptstyle\Omega} \\
S^n \vee S^n & \xrightarrow{j \vee j} & \mathrm{Cone}(f_1\colon S^{2n-1} \to S^n) \vee \mathrm{Cone}(f_2\colon S^{2n-1} \to S^n) & \xrightarrow{p \vee p} & \Sigma S^{2n-1} \vee \Sigma S^{2n-1} \\
{\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_i} \\
S^n & \xrightarrow{j} & \mathrm{Cone}(f_i\colon S^{2n-1} \to S^n) & \xrightarrow{p} & \Sigma S^{2n-1}.
\end{array}$$

The top three vertical maps are illustrated in the figure below. We make the following

[2297] Note that here we use the convenient fact that $(-1)^{(2n-1)+1} = +1$. In reality the sign is largely irrelevant, as long as we can be sure that the sign depends only on n.



clarifications:

(1) The horizontal maps to the left are the embeddings j introduced in the Mapping Cone–Embedding Lemma 21.11 (2).

(2) In the Mapping Cone–Quotient Lemma 21.14 we showed that for any mapping cone $\mathrm{Cone}(g\colon A \to X)$ we have a natural homeomorphism $\mathrm{Cone}(g\colon A \to X)/X \to \Sigma(A)$. In the above diagram the horizontal maps p on the right are given by the composition of a projection and the homeomorphism we just mentioned.

(3) The vertical maps between the first and the second and between the second and the third row are obtained from Construction 216.16 applied to the following two commutative diagrams

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{(f_1\vee f_2)\circ\pi} & S^n \vee S^n \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ S^{2n-1} & \xrightarrow{\varphi\circ(f_1\vee f_2)\circ\pi} & S^n \end{array} \quad \text{and} \quad \begin{array}{ccc} S^{2n-1} & \xrightarrow{(f_1\vee f_2)\circ\pi} & S^n \vee S^n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1\vee f_2} & S^n \vee S^n. \end{array}$$

In particular we see that the four upper squares commute.

(4) For $i = 1, 2$ we consider the commutative diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\qquad f_i \qquad} & S^n \\ \downarrow & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1\vee f_2} & S^n \vee S^n \end{array}$$

which is given by the embedding into the first respectively second factor. By Construction 216.16 these diagrams give rise to two sequences of maps. We "wedge" the two sequences and we obtain the vertical maps from the fourth to the third row.

(5) At the bottom, in each column the map q_i is the projections onto the i-th factor.

Next we apply the cohomology functor $X \mapsto \mathrm{H}^k(X;\mathbb{Z})$ with $k = n$ and $k = 2n$ to the above diagram (except that at the bottom we consider both maps q_1 and q_2). We obtain

the following commutative diagram (for space reasons we dropped the $\mathbb{Z}$-coefficients):[2298]

$$\begin{array}{ccccc}
\mathrm{H}^k(S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}(\varphi\circ(f_1\vee f_2)\circ\pi : S^{2n-1}\to S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma S^{2n-1}) \\
{\scriptstyle\varphi^*}\uparrow & & \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle\mathrm{id}} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}((f_1\vee f_2)\circ\pi : S^{2n-1}\to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma S^{2n-1}) \\
{\scriptstyle\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\psi^*} & & \uparrow{\scriptstyle\Sigma(\pi)^*} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}(f_1\vee f_2 : S^{2n-1}\vee S^{2n-1}\to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma(S^{2n-1}\vee S^{2n-1})) \\
{\scriptstyle\mathrm{id}}\downarrow & & \cong\downarrow{\scriptstyle\Xi^*} & & \cong\downarrow{\scriptstyle\Omega^*} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{(j\vee j)^*} & \mathrm{H}^k(\mathrm{Cone}(f_1)\vee\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{(p\vee p)^*} & \mathrm{H}^k(\Sigma S^{2n-1}\vee\Sigma S^{2n-1}) \\
{\scriptstyle q_1^*\oplus q_2^*}\uparrow\cong & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} \\
\mathrm{H}^k(S^n)\oplus\mathrm{H}^k(S^n) & \xleftarrow[\cong]{j_1^*\oplus j_2^*} & \mathrm{H}^k(\mathrm{Cone}(f_1))\oplus\mathrm{H}^k(\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{p_1^*\oplus p_2^*} & \mathrm{H}^k(\Sigma S^{2n-1})\oplus\mathrm{H}^k(\Sigma S^{2n-1}).
\end{array}$$

We make the following observations:

(1) It follows almost immediately from the Mapping Cone-H*-Lemma 216.7 that the horizontal maps to the left are isomorphisms in degree n. Since these isomorphisms are only in degree n they are colored in red.
(2) It follows easily from the Mapping Cone-H*-Lemma 216.7 and the cohomological analog of the Cone–Connecting Homomorphism Lemma 153.18 that the horizontal maps to the right are isomorphisms for $k = 2n$. Since these isomorphisms are only in degree $2n$ they are colored in green.
(3) Similar to the calculation in Exercise 149.18 we see that the map Ω^* is an isomorphism.
(4) It follows from (1), (2) and (3) that the map Ξ^* is an isomorphism in degrees n and $2n$, thus in all dimensions we care about.
(5) The vertical maps at the very bottom are induced by the natural projections. The maps are isomorphisms by the Wedge-H*-Proposition 200.10.
(6) Note that the folding map $\varphi\colon S^n\vee S^n\to S^n$ has a right-inverse given by any one of the two obvious embeddings $S^n\to S^n\vee S^n$. Note that it follows from this observation and the functoriality of cohomology groups that the induced homomorphisms $\varphi^*\colon \mathrm{H}^k(S^n;\mathbb{Z})\to\mathrm{H}^k(S^n\vee S^n;\mathbb{Z})$ are monomorphisms.

Next we denote by $\alpha := [S^n]^*\in\mathrm{H}^n(S^n;\mathbb{Z})$ the dual fundamental class. Furthermore we write $\beta := [\Sigma([S^{2n-1}]^*)]\in\mathrm{H}^{2n}(\Sigma S^{2n-1};\mathbb{Z})$.

Claim 1. We have $\varphi^*(\alpha) = q_1^*(\alpha)+q_2^*(\alpha)$ and $(\Sigma(\pi)^*\circ(\Omega^*)^{-1})(q_1^*(\beta)+q_2^*(\beta)) = \beta$.

Proof. We leave it to the reader to verify this claim.

The key calculation takes place in the fourth row. More precisely, we have the following claim:

Claim 2. The following equality holds in $\mathrm{H}^{2n}(\mathrm{Cone}(f_1)\vee\mathrm{Cone}(f_2);\mathbb{Z})$:

$$(((j\vee j)^*)^{-1}(q_1^*(\alpha)+q_2^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1)+\mathrm{Hopf}(f_2))\cdot(p\vee p)^*(q_1^*(\beta)+q_2^*(\beta)).$$

Proof. We note that

[2298] Any reader who knows of a program which can automatically turn the latex-code for a diagram of topological spaces into the latex-code for the corresponding diagram of cohomology groups should contact the exasperated author urgently.

$$\begin{array}{l}
\qquad\qquad\qquad

Examples.

(1) The map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication has bidegree $(1,1)$. This can be seen as follows: Taking $x_0 = y_0 = 1$, we see that the two maps $S^1 \to S^1$ given by $z \mapsto f(z,1)$ and $z \mapsto f(1,z)$ are in fact the identity.

(2) The same argument as in (1) shows that the quaternion multiplication $S^3 \times S^3 \to S^3$ is a map of bidegree $(1,1)$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ is also a map of bidegree $(1,1)$.[2300]

(3) Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a continuous map of bidegree (p,q). Furthermore let $\varphi, \psi\colon S^n \to S^n$ be two continuous maps. It follows almost immediately from the definitions that the bidegree of the continuous map $(x,y) \mapsto f(\varphi(x), \psi(y))$ is given by $(p \cdot \deg(\varphi), q \cdot \deg(\psi))$. □

The previous example shows that the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ give rise to maps of bidegree $(1,1)$. For general division algebras we have the following closely related non-triviality result:

Lemma 216.19. (Division Algebra–Bidegree-Lemma) Let $n \in \mathbb{N}_{\geq 2}$. If the real vector space $\mathbb{R}^n$ admits the structure of a division algebra[2301], then there exists a continuous map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (p,q) where both p and q are odd.

Proof. Let $*$ be the multiplication map of a division algebra on $\mathbb{R}^n$. As in the proof of the Commutative Division Algebra Theorem 171.6 and the proof of the Hopf–Division Algebra Dimension Theorem 214.13 we consider the map

$$\begin{array}{rcl} f\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & f(x,y) := \dfrac{x * y}{\|x * y\|}. \end{array}$$

In the proof of the Commutative Division Algebra Theorem 171.6 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined and that it is continuous. Now we pick a point $\star \in S^{n-1}$ and we consider the maps

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & S^{n-1} \\ x & \mapsto & f(x,\star) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon S^{n-1} & \to & S^{n-1} \\ y & \mapsto & f(\star,y). \end{array}$$

It follows easily from the fact that $*$ is the multiplication of a real algebra that for all $z \in S^{n-1}$ we have $\varphi(-z) = -\varphi(z)$ and $\psi(-z) = -\psi(z)$. By the Odd Map–Odd Degree Proposition 165.6 this implies that $\deg(\varphi)$ and $\deg(\psi)$ are odd. Conveniently enough that is exactly what we had set out to show. ■

Before we continue we recall a few conventions that we had introduced along the way. Most of them are also illustrated in the figure below.

(1) Given a topological space we consider the cone $\mathrm{Cone}(X)$, the suspension $\Sigma(X)$ and we consider the two copies $\mathrm{Cone}_{\pm}(X)$ of the cone in the suspension.[2302]

[2299] In Exercise 151.17 (a) we saw that the definition does not depend on the choice of x_0 and y_0.

[2300] Here we used implicitly the Quaternion Lemma 171.3 (7) and the Octonion Lemma 214.12 (7) to ensure that these maps do indeed take values in S^3 and S^7.

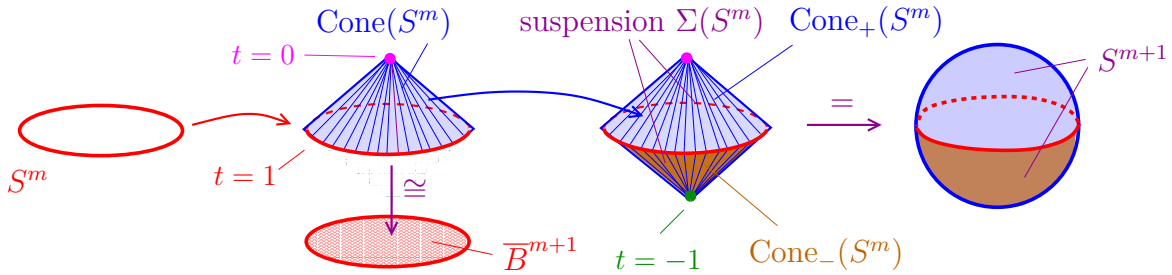
[2301] Recall that by Definition 171.1 a division algebra is an algebra that has left and right inverses. We do not assume that division algebras are associative and we do not assume that they preserve the standard inner product.

[2302] Here we implicitly use the Cone-Suspension Embedding Lemma 20.27.

(2) Let $m \in \mathbb{N}_0$. Note that by the discussion on page 552 we can make the identification $\mathrm{Cone}(S^m) =_{\mathrm{i}} \overline{B}^{m+1}$ and by the discussion on page 556 we can make the identification $\Sigma(S^m) =_{\mathrm{i}} S^{m+1}$.

(3) Let $m, n \in \mathbb{N}_0$. By the discussion on page 323 we can make the identification

$$(S^m \times \overline{B}^{n+1}) \cup_{\mathrm{id}_{S^m \times S^n}} (\overline{B}^{m+1} \times S^n) \xrightarrow{=} S^{m+n+1}.$$



Now we can move on to the promised construction of potentially interesting maps between spheres.

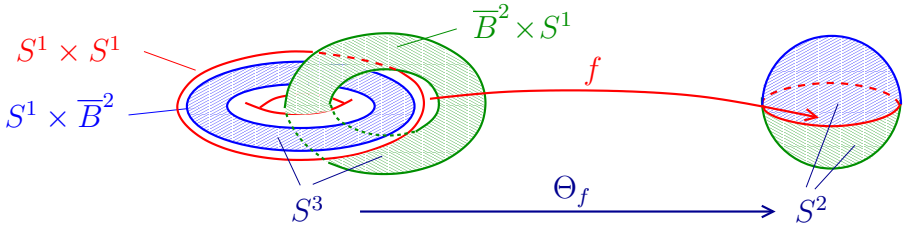
Definition 216.20. Let $f\colon S^p \times S^q \to S^r$ be a continuous map. We consider the maps

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}_+(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q), 1-t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(S^p) \times S^q & \to & \mathrm{Cone}_-(S^r) \\ ([(P,t)], Q) & \mapsto & [(f(P,Q), t-1)]. \end{array}$$

We will see in Exercise 216.4 that these two maps are continuous. Evidently the two maps agree on the overlap $S^p \times S^q$. Thus by the Topological Pushout Proposition 8.12 (1b) these two maps define a continuous map

$$\Theta_f\colon \underbrace{(S^p \times \mathrm{Cone}(S^q)) \cup_{S^p \times S^q} (\mathrm{Cone}(S^p) \times S^q) = S^{p+q+1}}_{=(S^p \times \overline{B}^{q+1}) \cup_{S^p \times S^q} (\overline{B}^{p+1} \times S^q)} \to \underbrace{\Sigma(S^r)}_{=S^{r+1}}.$$

We refer to the figure below for a moderately successful attempt at illustrating the above map Θ_f.



Example.

(1) Again we consider the map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication. We obtain by the above construction a continuous map $\Theta_f\colon S^3 \to S^2$. The heroically inclined reader will surely rise to the challenge of showing that this map is homotopic to the Hopf map.[2303]

[2303] To be on the safe side, let us more cautiously say that the map Θ_f is either homotopic to the Hopf map or that it is homotopic to the composition of the Hopf map with a reflection in a hyperplane.

(2) Similarly to (1) one can show that the quaternion multiplication $S^3 \times S^3 \to S^3$ gives rise to the quaternionic Hopf map $S^7 \to S^4$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ gives rise to the quaternionic Hopf map $S^{15} \to S^8$. □

Given a continuous map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ the next proposition computes the Hopf invariant of the corresponding map $\Theta_f\colon S^{2n-1} \to S^n$ in terms of the bidegree of f.

Proposition 216.21. (Bidegree–Hopf Invariant Proposition) Let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous map of bidegree (r, s). The induced map $\Theta_f\colon S^{2n-1} \to S^n$ has Hopf invariant $\pm r \cdot s$.[2304]

The proof of the Bidegree–Hopf Invariant Proposition 216.21 will require the remainder of this section. In the last Section 216.4 we will go back to examples and applications to homotopy groups.

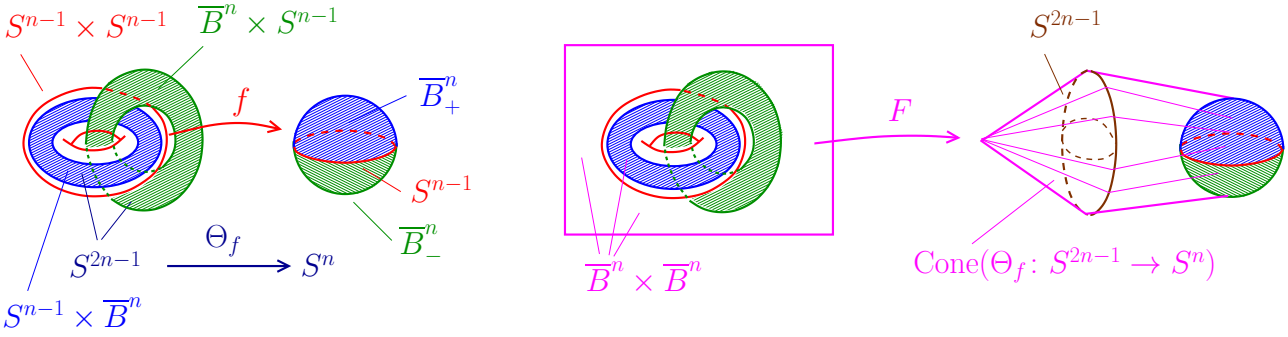
Proof. We make the following preparations:

(1) To simplify the notation we write $\mathrm{C}(\Theta_f)$ instead of $\mathrm{Cone}(\Theta_f)$.
(2) Recall that by the Mapping Cylinder–Embedding Lemma 21.4 we know that the inclusion $S^n \to \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n)$ is actually an embedding. Thus we can identify S^n with its image in $\mathrm{C}(\Theta_f)$.
(3) We denote by $\overline{B}^n_+$ the upper hemisphere of S^n and we denote by $\overline{B}^n_-$ the lower hemisphere of S^n.
(4) By the discussion on page 552 we can make the identification

$$\overline{B}^n \times \overline{B}^n \;=\; \overline{B}^{2n} \;=\; \mathrm{Cone}\big(\underbrace{(S^{n-1} \times \overline{B}^n) \cup_{S^{n-1}\times S^{n-1}} (\overline{B}^n \times S^{n-1})}_{=S^{2n-1}}\big).$$

Let $$F\colon \overline{B}^n \times \overline{B}^n \;\to\; \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n) \;=\; \underbrace{\mathrm{Cone}(S^{2n-1})}_{=\overline{B}^n\times\overline{B}^n=\overline{B}^{2n}} \cup_{\Theta_f} S^n$$

the obvious map which is given by the identification $\overline{B}^n \times \overline{B}^n = \mathrm{Cone}(S^{2n-1})$ followed by the projection.



Next we pick generators for various cohomology groups:

(1) We denote by $x \in \mathrm{H}^n(\mathrm{C}(\Theta_f); \mathbb{Z})$ and $z \in \mathrm{H}^{2n}(\mathrm{C}(\Theta_f); \mathbb{Z})$ the generators that are given in the definition of the Hopf invariant, see Definition 216.10.

[2304] We leave it to the sign aficionados to determine the correct sign.

(2) The long exact sequence in cohomology of the pair $(\mathrm{C}(\Theta_f), \overline{B}^n_\pm)$ shows that the maps $\mathfrak{я}\colon \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm; \mathbb{Z}) \to \mathrm{H}^n(\mathrm{C}(\Theta_f); \mathbb{Z})$ are isomorphisms. Let $x_\pm \in \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm; \mathbb{Z})$ be the generators that correspond to x under the given isomorphisms.

(3) We denote by $[S^{n-1}]^* \in \mathrm{H}^{n-1}(S^{n-1}; \mathbb{Z})$ and $[\overline{B}^n]^* \in \mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{Z})$ the standard dual generators. For $i = 1, 2$ we denote by $p_i\colon \overline{B}^n \times \overline{B}^n \to \overline{B}^n$ the projection onto the first respectively second coordinate. We write

$$\begin{aligned} y_1 &:= p_1^*([\overline{B}^n]^*) \in \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n; \mathbb{Z}) \quad \text{and} \\ y_2 &:= p_2^*([\overline{B}^n]^*) \in \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}; \mathbb{Z}). \end{aligned}$$

The following claim contains the key calculation of the proof.

Claim. The map F induces the following maps: on relative cohomology:

$$\begin{array}{lccc} \text{(i)} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+; \mathbb{Z}) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n; \mathbb{Z}) \\ & x_+ & \mapsto & \pm r \cdot y_1 \\ \text{(ii)} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_-; \mathbb{Z}) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}; \mathbb{Z}) \\ & x_- & \mapsto & \pm s \cdot y_2. \end{array}$$

Proof. Before we turn to the actual calculation we make the following two preparations:

(a) The restrictions of F to various subspaces gives us the following commutative diagram:

$$\begin{array}{ccc} \overline{B}^n \times \overline{B}^n & \xrightarrow{F} & \mathrm{C}(\Theta_f) \\ \uparrow & & \uparrow \\ \overline{B}^n \times S^{n-1} & \xrightarrow{\Theta_f} & \overline{B}^n_- \\ \uparrow & & \uparrow \\ S^{n-1} \times S^{n-1} & \xrightarrow{f} & S^{n-1}. \end{array}$$

(b) We pick a point $* \in S^{n-1}$.

To provide the actual proof of the claim we consider the following diagram (where once again we drop the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccc} \mathrm{H}^n(\mathrm{C}(\Theta_f)) & \xleftarrow{\cong} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ \downarrow \cong & \swarrow & \downarrow & & \downarrow \cong \\ \mathrm{H}^n(S^n) & \xleftarrow{\cong} \mathrm{H}^n(S^n, \overline{B}^n_+) & \xrightarrow[\text{exc.}]{\cong} \mathrm{H}^n(\overline{B}^n_-, S^{n-1}) & \xrightarrow{\Theta_f^*} & \mathrm{H}^n(\overline{B}^n \times *, S^{n-1} \times *) \\ & & \cong \uparrow \delta & & \delta \uparrow \cong \\ & & \mathrm{H}^{n-1}(S^{n-1}) & \xrightarrow[[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*]{f^*} & \mathrm{H}^{n-1}(S^{n-1} \times *). \end{array}$$

We make the following clarifications and observations:

(1) The maps that are not specified are induced by the obvious inclusions of (pairs) of topological spaces.

(2) The inclusion-induced maps that are decorated by "$\cong$" are isomorphisms for the following reasons:
   (a) by the long exact sequence in cohomology for the corresponding pair,
   (b) the maps are induced by a homotopy equivalence,
   (c) by the long exact sequence provided by the Mapping Cone-H*-Lemma 216.7,
   (d) by the Excision-in-H*-Theorem 200.1 (2).

(3) The quadrilateral to the left commutes since all maps are induced by inclusions.
(4) It follows from (2) and (3) that the diagonal map is an isomorphism.
(5) The triangle towards the center commutes since all maps are induced by inclusions.
(6) It follows from (2), (4) and (5) that the middle vertical map is an isomorphism.
(7) The squares to the right commute by the commutative diagram shown in (a) and by the naturality of the connecting homomorphisms.
(8) The connecting homomorphisms δ are easily seen to be isomorphisms since the other groups in the corresponding long exact sequence are zero.
(9) Basically by the definition of the bidegree of the map f, the bottom horizontal map is given by $[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*$.

The above arguments show that the vertical maps of the blue part of the diagram are isomorphisms. Since the bottom map is given by multiplication by r we see that the top map, with whatever choice of generators, is also given by multiplication by $\pm r$. This concludes the proof of part (i) of the claim. The proof of part (ii) is evidently basically the same. ⊞

Finally we consider the following diagram (again we drop the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccc}
\begin{array}{c}\mathrm{H}^n(\mathrm{C}(\Theta_f)) \\ \times \\ \mathrm{H}^n(\mathrm{C}(\Theta_f))\end{array} & \xleftarrow[\cong]{\substack{x \leftarrow\!\shortmid x_- \\ x \leftarrow\!\shortmid x_+}} & \begin{array}{c}\mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_-) \\ \times \\ \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+)\end{array} & \xrightarrow[F^*]{\substack{x_- \mapsto \pm r \cdot y_1 \\ x_+ \mapsto \pm s \cdot y_2}} & \begin{array}{c}\mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ \times \\ \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1})\end{array} \\
\cup \downarrow {\scriptstyle (x,x) \mapsto x^2 = \mathrm{Hopf}(\Theta_f) \cdot z} & & \downarrow \cup & & \cup \downarrow {\scriptstyle (y_1, y_2) \mapsto y_1 \cup y_2 = [\overline{B}^n \times \overline{B}^n]^*} \\
\mathrm{H}^{2n}(\mathrm{C}(\Theta_f)) & \xleftarrow[\cong]{} & \mathrm{H}^{2n}(\mathrm{C}(\Theta_f), S^n) & \xrightarrow[\cong]{F^*} & \mathrm{H}^{2n}(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1} \cup S^{n-1} \times \overline{B}^n).
\end{array}$$

$$\pm z \leftarrow\!\shortmid [\overline{B}^n \times \overline{B}^n]^*$$

We make the following comments:

(1) It follows easily from the Excisive Triad-Proposition 158.20 (2) that $(\mathrm{C}(\Theta_f), \overline{B}^n_-, \overline{B}^n_+)$ and $(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n, \overline{B}^n \times S^{n-1})$ are excisive triads. Thus we see that the relative cup product in the center and to the right are actually defined.
(2) It follows from the naturality of the cup product, see the Cup Product-Naturality Lemma 204.8, that the two squares commute.
(3) By the Fundamental Class-Product Proposition 207.2 (1) we have the following equality: $y_1 \cup y_2 = p_1^*([\overline{B}^n]) \cup p_2^*([\overline{B}^n]) = [\overline{B}^n \times \overline{B}^n]^*$.
(4) By the above claim we know that the top right horizontal map is in fact given by $x_- \mapsto \pm r \cdot y_1$ and $x_+ \mapsto \pm s \mapsto s \cdot y_2$.
(5) Since the bottom horizontal maps are isomorphisms we see that going right to left the generator $[\overline{B}^n \times \overline{B}^n]^*$ gets sent to $\pm z$.

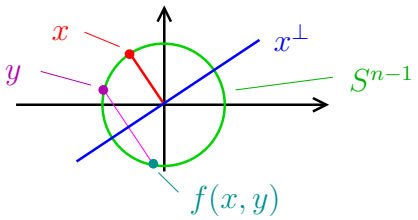
It follows immediately from the above comments that $\mathrm{Hopf}(\Theta_f) = \pm r \cdot s$. ■

216.4. **Examples of non-trivial bidegrees.** In light of the Hopf Invariant–Homomorphism Proposition 216.14 and the Bidegree–Hopf Invariant Proposition 216.21 we are now very eager to find maps $S^{n-1} \times S^{n-1} \to S^{n-1}$ with bidegree (r, s) where $r \cdot s \neq 0$. Perhaps somewhat surprisingly it is not that hard to find such maps:

Proposition 216.22. (Bidegree–Realization Proposition) Given any *even* $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} f\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & y - 2 \cdot \langle x,y \rangle \cdot x \ = \text{ reflection of } y \text{ in the hyperplane } x^\perp \end{array}$$

has bidegree $(2,-1)$.



Proof. We need to show that the given map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ has bidegree $(2,-1)$. The proof of this statement naturally breaks up into two parts.

(1) We take $y_0 = (1,0,\ldots,0)$. We consider the map $\varphi\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto f(x,y_0)$. Note that $\varphi^{-1}((1,0,\ldots,0)) = (\pm 1,0,\ldots,0)$. Using this observation together with the Degree-via-Local Orientations Proposition 152.11 one can easily show that the degree of the map $x \mapsto (x,y_0)$ equals $1+(-1)^{n-2}$. Since n is by hypothesis even we see that the degree equals two.

(2) We take $x_0 = (1,0,\ldots,0)$. The corresponding map $S^{n-1} \to S^{n-1}$, $y \mapsto f(x_0,y)$ is the reflection in a hyperplane, hence it follows from the Sphere-Degree-Matrix Lemma 151.7 that the map has degree -1. ■

We can now easily prove the following amazing theorem:

Theorem 216.23. ($\pi_{4n-1}(S^{2n})$–Infinite Theorem) Given any $n \in \mathbb{N}$ there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$.

Proof. It follows from the Hopf Invariant–Homomorphism Proposition 216.14 together with the Bidegree–Realization Proposition 216.22 that the Hopf map defines a homomorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$ with non-zero image. Since every non-zero subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ this implies that there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$. ■

As always in mathematics, every result opens up lots of new questions. Here are a few:

Question 216.24. By the Hopf Invariant One-Corollary 216.15 the Hopf maps

$$\begin{array}{rcl} \text{(1)} \quad \text{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ \text{(2)} \quad \text{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ \text{(3)} \quad \text{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms. Are they also isomorphisms?

We will provide the answer to Question 216.24 (1) in the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12 we will see the answer to Question 216.24 (2) in the $\pi_{n+3}(S^n)$-Theorem 302.29 (1) and we will see the answer to (3) on page 5838.

Question 216.24 points to a problem: We have now developed some tools for showing that certain homotopy groups of spheres are non-trivial, but we have no tools yet for computing any interesting homotopy groups. We will partially rectify this problem in Chapter 302.

Here is another question:

Question 216.25. Given any $n \in \mathbb{N}$ we know by the Hurewicz Sphere Epimorphism Lemma 167.5 and the $\pi_{4n-1}(S^{2n})$–Infinite Theorem 216.23 that $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$ admit an epimorphism onto $\mathbb{Z}$. Are there are any other homotopy groups of spheres that admit an epimorphism onto $\mathbb{Z}$?

We delay the answer to the $\pi_*(S^n)$-Finiteness Theorem 303.1.

In light of the Division Algebra–Bidegree-Lemma 216.19 and the Bidegree–Realization Proposition 216.22 the following question becomes interesting:

Question 216.26. For which $n \in \mathbb{N}_{\geq 2}$ does there exist a continuous map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f) = 1$?[2305]

We will see the complete answer to Question 216.26 in the Hopf Invariant–Power-of-2 Theorem 237.11 and the Adams Theorem 237.12.

Initially one might not appreciate Question 216.26. But rather mysteriously it is not that hard to come up with maps of Hopf degree 2, but it is strikingly difficult to come up with maps of Hopf degree 1. To underline our point we conclude this chapter with another construction of a continuous map of Hopf degree 2.

Example. Let $n \in \mathbb{N}$. We start out with the following two reminders:

(1) We can view S^n as a CW-complex with one 0-cell and one n-cell. In particular, if we denote by S^n_a and S^n_b two copies of S^n then we view S^n_a as a CW-complex with one 0-cell P and one n-cell that we denote by S^n_a. Similarly we view S^n_b as a CW-complex with one 0-cell Q and one n-cell S^n_b.

(2) Given two finite CW-complexes X and Y we introduced in the CW-Complex Product Proposition 139.18 together with the CW-Category–Product Proposition 139.16 the product CW-structure on $X \times Y$. Recall that the n-cells of $X \times Y$ correspond precisely to products $e_x \times f_y$ where e_x is a p-cell of X and f_y is an $(n-p)$-cell of Y.

It follows from (1) and (2) that we can view $S^n_a \times S^n_b$ as a CW-complex with one 0-cell (P, Q), two n-cells $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$ and one $2n$-cell $S^n_a \times S^n_b$. We consider the maps

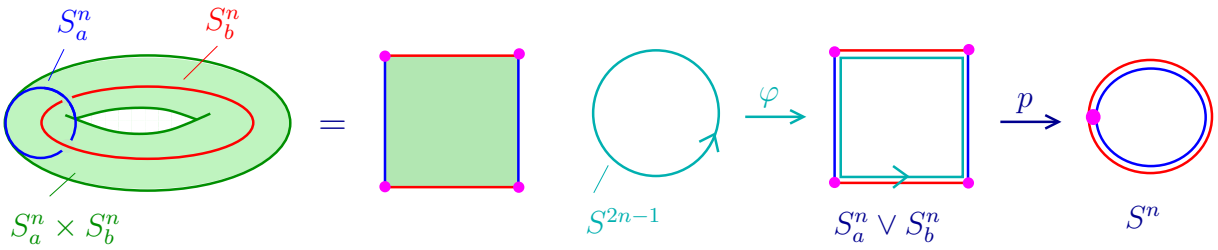
$$S^{2n-1} \xrightarrow{\varphi} \overbrace{(S^n_a \times \{Q\}) \cup (\{P\} \times S^n_b)}^{=S^n_a \vee S^n_b} \xrightarrow{p} S^n.$$

φ: attaching map of the $2n$-cell to the $(2n-1)$-skeleton of $S^n_a \times S^n_b$ which is precisely $S^n_a \times \{Q\} \cup \{P\} \times S^n_b$

p: obvious projection on $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$

The composition of the maps φ and p defines a continuous map $f\colon S^{2n-1} \to S^n$.
It is clear from the figure above that this map is not very interesting for $n = 1$. But using fairly elementary arguments one can show that for all even n we have $\mathrm{Hopf}(f) = \pm 2$. We refer to [**Hat2002**, p. 428] or [**Vic1994**, p. 138] for details. Alternatively we leave it as a brain teaser for the reader to perform this calculation. □

[2305] Note that the Hopf Invariant–Homomorphism Proposition 216.14 and the Bidegree–Realization Proposition 216.22 tell us that this question is equivalent to the question, for which $n \in \mathbb{N}$ does there exist a continuous map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f)$ is odd?



216.5. Exercises for Chapter 216 .

Exercise 216.1. Let $n \in \mathbb{N}_{\geq 2}$ and let $\varphi\colon S^2 \to S^2$ be a continuous map of degree n. We consider $X \coloneqq \mathrm{Cone}(S^2 \xrightarrow{\varphi} S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2)$ and $Y \coloneqq \mathrm{Cone}(S^2 \xrightarrow{\varphi} S^2) \vee S^4$.

(a) Show that the cohomology groups of X and Y with $\mathbb{Z}$-coefficients are isomorphic.
(b) Show that the cohomology rings of X and Y with $\mathbb{Z}$-coefficients are not isomorphic.

Exercise 216.2. What is the homeomorphism type of the mapping cone $\mathrm{Cone}(f)$ of the map $f\colon S^1 \to S^1$ given by $z \mapsto z^2$?
Hint. It is a topological space that we are quite familiar with.

Exercise 216.3. By the Quaternion Units Proposition 171.5 we know that quaternion multiplication turns $S^3 = \{z \in \mathbb{H} \,|\, |z| = 1\}$ into a group. What is the homeomorphism type of the mapping cone $\mathrm{Cone}(f)$ of the map $f\colon S^3 \to S^3$ given by $z \mapsto z^2$?

Exercise 216.4. Let $f\colon S^p \times S^q \to S^r$ be a continuous map. Show that the map

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q),t)] \end{array}$$

is continuous.
Hint. Use the Whitehead Quotient Theorem 17.6 and the Quotient Map-Composition Lemma 17.7 (2).

Exercise 216.5. Let $f\colon S^k \times S^k \to S^k$ be the projection onto the first factor. Show that the corresponding map $\Theta_f\colon S^{2k+1} \to S^{k+1}$, that is introduced in Definition 216.20, is homotopic to a constant map.
Hint. The exercise is very easy.

Exercise 216.6. Let $n \in \mathbb{N}$ be even and let $a, b \in \mathbb{Z}$. Show that if at least one of a or b is even, then there exists a continuous map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (a, b).

CHAPTER 217

The Basic Sphere–Alexander Duality Theorem

In this and the following chapter we will study two versions of the Sphere–Alexander Duality Theorem which both, given a "reasonable" subspace X, of S^n relate the homology groups of $S^n \setminus X$ to the cohomology groups of X. More precisely, we will prove the following results:

(1) In this chapter we will prove the "basic" Sphere–Alexander Duality Theorem 217.8 which says the following: If K is a subspace of S^n and if K admits a "regular neighborhood" (e.g. if K is a closed smooth submanifold of S^n), then for every $i \in \mathbb{N}_0$ there exists an isomorphism
$$\widetilde{\mathrm{H}}_i(S^n \setminus K \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z}).$$
As we will see, once one has digested the definition of a regular neighborhood the above isomorphism follows from a short and pleasant argument involving the Poincaré Duality Theorem 210.1.

(2) In the following chapter we will prove the General Alexander Duality Theorem 218.2 which says that if $K \subseteq S^n$ is a closed subset that is "weakly locally contractible", then there exists a *natural* isomorphism
$$\widetilde{\mathrm{H}}_i(S^n \setminus K) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z}).$$
This statement is stronger than the statement in (1), but at the price, that the proof is much more involved. In particular it will build on Borsuk's Theorem 189.7.

This chapter is organized as follows:

- In Section 217.1 we will define regular neighborhoods and we will discuss some basic properties.
- In Section 217.2 we will introduce spatial graphs.
- In Section 217.3 we will prove the Sphere–Alexander Duality Theorem 217.8.
- In Section 217.4 we will study immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$.
- In Section 217.5 we will state and prove the Alexander Duality–Boundary Theorem 217.16.
- In Appendix 217.6 we will study embeddings of non-orientable topological manifolds into the Euclidean space $\mathbb{R}^n$.
- In Appendix 217.7 we will study the homology groups of complements of knots in arbitrary 3-manifolds.
- In Section 217.8 we will provide exercises for Chapter 217.

In the next Chapter 218 we will formulate and prove the General Alexander Duality Theorem 218.2.

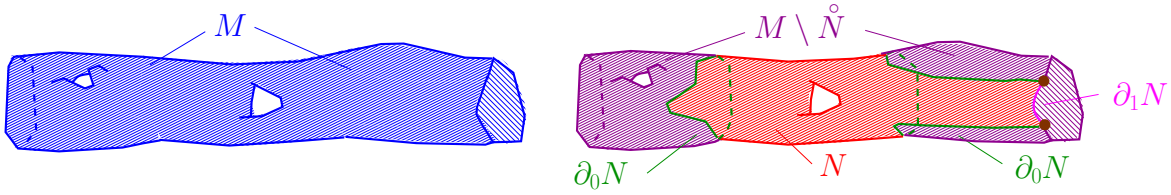
217.1. **Definition and properties of regular neighborhoods.** In this section we will introduce the notion of a regular neighborhood of a compact subspace of a topological

manifold. This notion is a variation on the eponymous concept that we introduced in Definition 180.13 and it is a variation on the notion of a tubular neighborhood that we encountered in Definition 58.2.

Before we can introduce the notion of a regular neighborhood we need to recall the following definitions from the Smooth Submanifold-with-Corner Lemma 26.16 (0), Definition 212.2 and the Topological Submanifold-with-Corner Lemma 212.3 (0):

Notation 217.1. Let M be an m-dimensional topological manifold and let $N \subseteq M$ be a codimension-zero submanifold of N with corner. We write

$$\partial_0 N \coloneqq N \cap (\overline{M \setminus N}) \quad \text{and} \quad \partial_1 N \coloneqq N \cap \partial M.$$

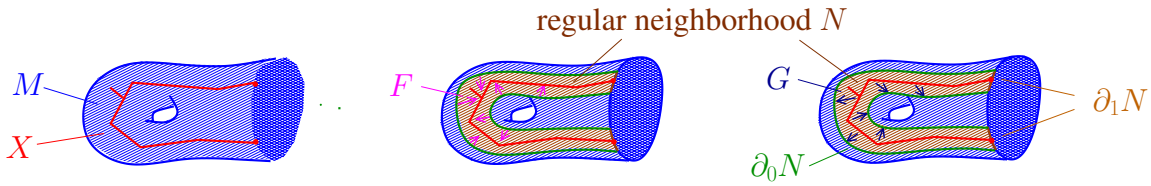


Now we can provide the definition of a regular neighborhood:

Definition 217.2. Let M be an m-dimensional topological manifold and let X be a compact subspace. A regular neighborhood of X is a compact codimension-zero submanifold N with corner of M with the following properties:[2306] [2307]

(1) X is contained in $N \setminus \partial_0 N$.
(2) There exists a deformation retraction $F\colon N \times [0,1] \to N$ from N to X such that $F(\partial_1 N \times [0,1]) \subseteq \partial_1 N$.
(3) There exists a deformation retraction $G\colon (N \setminus X) \times [0,1] \to N \setminus X$ from $N \setminus X$ to $\partial_0 N$ such that $G((\partial_1 N \setminus X) \times [0,1]) \subseteq \partial_1 N \setminus X$.

We refer to $N \setminus \partial_0 N$ as an open regular neighborhood of X.

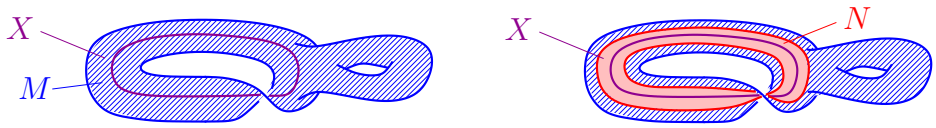


The Definition 217.2 of a regular neighborhood surely reminds the reader of the notion of a tubular neighborhood, which we introduced in Definition 57.9. In fact the following proposition shows in particular that tubular neighborhoods are regular neighborhoods:

Proposition 217.3. (Tubular–Regular Neighborhood Proposition) Suppose that M is a smooth manifold. Let X be a compact proper smooth submanifold of M.

(1) X admits a tubular neighborhood.
(2) Every tubular neighborhood N of X is a regular neighborhood of X.

[2306] We will see in the Basics-of-Regular Neighborhood Lemma 217.4 that a regular neighborhood is indeed a neighborhood of X.
[2307] Very loosely speaking a regular neighborhood can be thought of as a codimension-zero thickening of X.



Remark. We will now discuss whether there exist versions of the Tubular–Regular Neighborhood Proposition 217.3 for other types of manifolds:

(1) Let M be a PL-manifold and let $X \subseteq M \setminus \partial M$ be a finite subcomplex. In Section 180.2 we defined the regular neighborhood $N_X(M)$ of X in M. By the Simplicial Regular Neighborhood Proposition 180.14 we know that $N_X(M)$ does indeed have the properties demanded in the above Definition 217.2 of a regular neighborhood. In other words, by the Simplicial Regular Neighborhood Proposition 180.14 we know that X has a regular neighborhood in the above sense.

(2) The Microbundle Existence-Uniqueness Theorem 190.28 shows that some submanifolds of topological manifolds admit regular neighborhoods. It is not clear to me whether every compact proper submanifold of a topological manifold admits a regular neighborhood. □

Proof.

(1) This statement is just the content of the Tubular Neighborhood–Existence Theorem 57.11.

(2) This statement is an immediate consequence of the definition of a tubular neighborhood, the Basics-of-Tubular Neighborhood Lemma 57.10 and the 0-Section Lemma 57.5. ■

Let M be a topological manifold. Note that in the Definition 217.2 of a regular neighborhood we did not demand that the subspace $X \subseteq M$ is a submanifold, in particular X could a priori be any compact subspace. It will surely not come as a surprise that not even every compact subspace admits a regular neighborhood. For example we will show in Exercise 217.1 that the compact subspace $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ does not admit a regular neighborhood. But there are many interesting examples of subspaces, which are not submanifolds, but which nonetheless admit regular neighborhoods. For example we will show in the Spatial Graph–Regular Neighborhood Proposition 217.7 that every "spatial graph" in $\mathbb{R}^3$ admits a regular neighborhood.

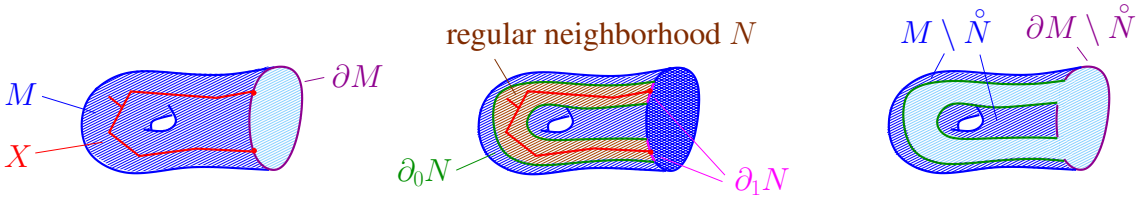
The following lemma can be viewed as a generalization of the Basics-of-Tubular Neighborhood Lemma 57.10.

Lemma 217.4. (Basics-of-Regular Neighborhood Lemma) Let M be a topological manifold, let X be a compact subspace and let N be a regular neighborhood of X. As always we set $\partial_0 N := N \cap (\overline{M \setminus N})$. The following statements hold:

(1) (a) $\mathring{N} := N \setminus \partial_0 N$ is the interior of $N \subseteq M$.
(b) N is a neighborhood of $X \subseteq M$.

(2) (a) $M \setminus \mathring{N}$ is a codimension-zero submanifold with corner of M where the boundary is given by $\partial(M \setminus \mathring{N}) = \partial_0 N \cup (\partial M \setminus \mathring{N})$.
(b) $\partial_0 N$ and $\partial M \setminus \mathring{N}$ are codimension-zero submanifolds of the topological manifold $\partial(M \setminus \mathring{N})$ and they are closed subsets of $\partial(M \setminus \mathring{N})$.

(3) (a) $M \setminus \mathring{N}$ is a deformation retract of $M \setminus X$ and $\partial M \setminus \mathring{N}$ is a deformation retract of $\partial M \setminus X$.
(b) The inclusions $M \setminus \mathring{N} \to M \setminus X$ and $\partial M \setminus \mathring{N} \to \partial M \setminus X$ are homotopy equivalences.



Proof.

(1) By definition N is a codimension-zero submanifold of M with corner. Statement (a) thus follows from the Topological Submanifold-with-Corner Lemma 212.3. Statement (b) follows immediately from Statement (a).
(2) This statement follows from the Topological Submanifold-with-Corner Lemma 212.3 (2b) and (4).
(3) (a) First we recall that by Definition 217.2 of a regular neighborhood there exists a deformation retraction $G\colon (N \setminus X) \times [0,1] \to N \setminus X$ from $N \setminus X$ to $\partial_0 N$ such that $G((\partial_1 N \setminus X) \times [0,1]) \subseteq \partial_1 N \setminus X$. We extend this deformation retraction to all of $M \setminus X$ in the obvious way. More precisely, we consider the map

$$\begin{aligned} G\colon (M \setminus X) \times [0,1] &\to M \setminus X \\ (P,t) &\mapsto \begin{cases} G(P,t), & \text{if } P \in N \setminus X, \\ P, & \text{if } P \notin \mathring{N}. \end{cases} \end{aligned}$$

It follows from (1a) and the Compact-Closed Lemma 3.39 that $N \setminus X$ and $M \setminus \mathring{N}$ are closed subsets of $M \setminus X$. It thus follows from the Homotopy Combination Lemma 18.8 (1) that this map G is indeed continuous. Once continuity is verified it is straightforward to show that G is indeed a deformation retraction from $M \setminus X$ to $M \setminus \mathring{N}$ and that it restricts to a deformation retraction from $\partial M \setminus X$ to $\partial M \setminus \mathring{N}$.
(b) It follows from (a) and the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the inclusions $M \setminus X \to M \setminus \mathring{N}$ and $\partial M \setminus \mathring{N} \to \partial M \setminus X$ are indeed homotopy equivalences. ■

The following corollary, which in the literature is often used implicitly or explicitly, gives one application of the theory of regular neighborhoods.

Lemma 217.5. (Submanifold Complement–Invariants Lemma) Let M be a compact topological manifold and let $X \subseteq M$ be a compact subspace. If X admits a regular neighborhood, then the following two statements hold:
(1) The fundamental group $\pi_1(M \setminus X)$ is finitely presented.
(2) For each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(M \setminus X)$ is finitely generated.[2308]

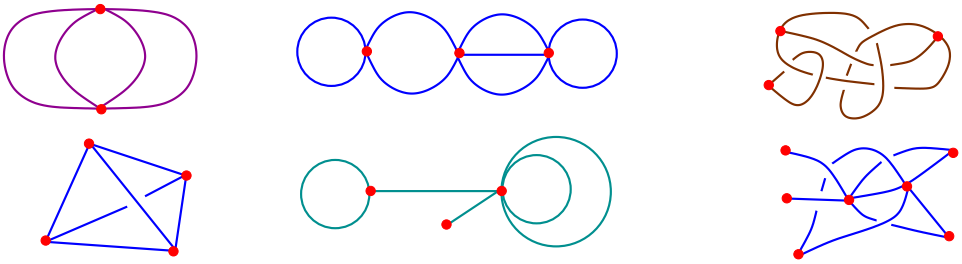
[2308] In Exercise 217.2 we will see that if we drop the hypothesis that X admits a regular neighborhood, then things can go completely haywire.

Proof. Let N be a regular neighborhood of X. In the following we give two different arguments, one which deals with the general case, and one which deals with the most important special case:

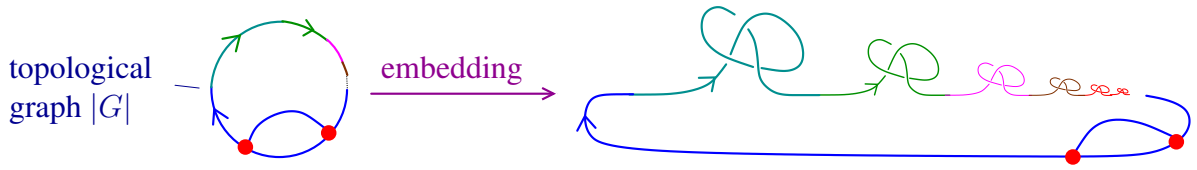
(i) The desired statements follow from the following observations:
  (a) By the Basics-of-Regular Neighborhood Lemma 217.4 the topological space $M \setminus X$ is homotopy equivalent to $M \setminus \overset{\circ}{N}$.
  (b) Since M is compact it follows from the Topological Submanifold-with-Corner Lemma 212.3 (3) that $M \setminus \overset{\circ}{N}$ is also compact. Furthermore, by the Topological Submanifold-with-Corner Lemma 212.3 (4) we know that $M \setminus \overset{\circ}{N}$ is an n-dimensional topological manifold. Summarizing we have seen that $M \setminus \overset{\circ}{N}$ is a compact n-dimensional topological manifold.
  (c) By (b) and the Topological Manifolds-Invariants Proposition 189.14 we know that $\pi_1(M \setminus \overset{\circ}{N})$ is finitely presented and we know that the homology groups $\mathrm{H}_k(M \setminus \overset{\circ}{N})$ are finitely generated.
  (d) It follows from (a), the Homotopy Equivalence-π_1-Proposition 90.9 (1) and the Homotopy Equivalence-H_*-Proposition 144.14 that the invariants of $M \setminus \overset{\circ}{N}$ are isomorphic to the invariants of $M \setminus X$.

(ii) If M is a smooth manifold, X is a compact proper submanifold and N is a tubular neighborhood, then we can also argue slightly differently. By the Basics-of-Tubular Neighborhood Lemma 57.10 the topological space $M \setminus X$ is homotopy equivalent to $M \setminus \overset{\circ}{N}$. By the Smooth Submanifold-with-Corner Lemma 26.16 together with the Rounding Corner Proposition 60.5 we know that $M \setminus \overset{\circ}{N}$ admits the structure of a compact smooth manifold. The promised statements now follow from the Smooth Manifolds-Invariants Proposition 179.10. ∎

217.2. **Spatial graphs.** On several occasions we already considered knots and links in the 3-dimensional sphere $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$, i.e. we considered closed 1-dimensional smooth submanifolds of S^3. Playing with pen and paper one quickly ends up with generalizations of this concept, for example with the following "spatial graphs" in $\mathbb{R}^3 \subseteq S^3$:



We now want to give a formal definition of a "spatial graph" which captures the above examples. A first attempt might be to attempt to define "a spatial graph" as the image of an embedding of a (say finite) topological graph into $\mathbb{R}^3$. But on page 3638 and in Exercise 190.3 we learned of wild knots and in the figure below we show a "wild" spatial graph, which is perhaps not what we had initially in mind:
The last example shows that we need some control over such embeddings to end up with reasonable objects.

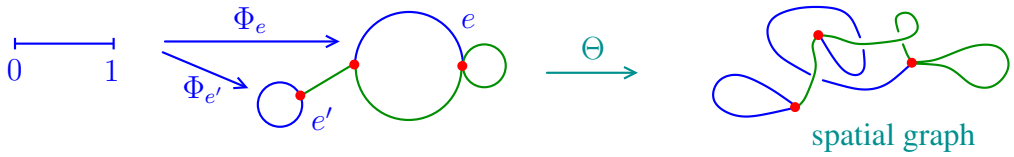


Some leisurely thinking leads us to the following definition, which is modelled on the notion of a topological graph that we introduced in Definition 11.9 and which is modelled on the notion of a smooth embedding from Definition 30.9:

Definition 217.6. A spatial graph is a pair $(G = (V, E, i, t), \Theta\colon |G| \to \mathbb{R}^3)$ where the following holds:

(1) The quadruple $G = (V, E, i\colon E \to V, t\colon E \to V)$ is a finite abstract graph.
(2) The map[2309] $\Theta\colon |G| \to \mathbb{R}^3$ is an embedding of the topological realization $|G|$ into $\mathbb{R}^3$.
(3) For every edge $e \in E$ with characteristic map $\Phi_e\colon [0,1] \to |G|$ the following statements hold:
   (a) The map $f_e \coloneqq \Theta \circ \Phi_e\colon [0,1] \to \mathbb{R}^3$ is smooth in the sense of Definition 24.3.[2310 2311]
   (b) The first derivative[2312] $f_e'(t)$ is non-zero for all $t \in [0,1]$,

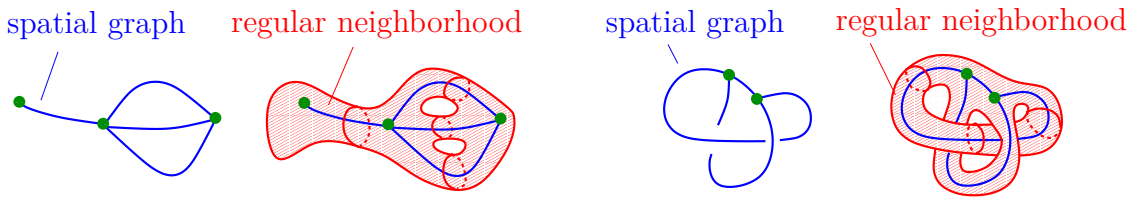
Sometimes we just refer to $\Theta(|G|)$ as a "spatial graph" and we ignore the extra data involved in the proper definition of a spatial graph.



Example. It should hopefully be clear that the examples sketched in the figure on page 4147 are spatial graphs. □

Now we can formulate the following proposition:

Proposition 217.7. (Spatial Graph–Regular Neighborhood Proposition) Every spatial graph G in $\mathbb{R}^3$ admits a regular neighborhood.

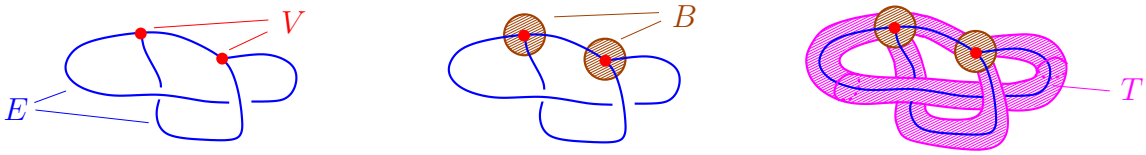


[2309] We refer to Definition 11.5 for the definition of the topological realization $|G|$ of an abstract graph G.
[2310] Note that it follows easily from Definition 24.3 that a map $f\colon [a,b] \to \mathbb{R}^3$ on a closed interval is smooth if and only if there exists an $\epsilon > 0$ and a smooth map $\widetilde{f}\colon (a-\epsilon, b+\epsilon) \to \mathbb{R}^3$ (in the sense from Definition 24.1) on the open interval such that $\widetilde{f}|_{[a,b]} = f$.
[2311] In the Smooth Functions-on-Compact Interval Proposition 24.10 we gave an alternative characterization of smooth maps on closed intervals.
[2311] A priori the derivative f' is defined on only on $(0,1)$. But by the Regular Closed–∂_*-Lemma 24.5 we know that it extends continuously to a continuous map $f'\colon [0,1] \to \mathbb{R}^3$.

Sketch of proof. Let $G = (V, E) \subseteq \mathbb{R}^3$ be a spatial graph. The idea of the proof is quite straightforward: First one shows that one can pick $\epsilon > 0$ with the following properties:

(1) For each $v \in V$ and each $t \in (0, \epsilon]$ the sphere $S^2_t(v)$ intersects each edge transversally.[2312]
(2) The closed balls $\{\overline{B}^3_\epsilon(v)\}_{v\in V}$ are disjoint.

We set $B \coloneqq \bigcup_{v\in V} \overline{B}^3_\epsilon(v)$. It follows from (1) and the Submanifold Transversal Intersection Theorem 50.4 that $E \cap (\mathbb{R}^3 \setminus \mathring{B})$ is a proper 1-dimensional smooth submanifold of $\mathbb{R}^3 \setminus \mathring{B}$. Thus it follows from the Tubular Neighborhood–Existence Theorem 57.11 that it admits a tubular neighborhood $T \subseteq \mathbb{R}^3 \setminus \mathring{B}$.



Then $N \coloneqq B \cup T$ has (after rounding corners) the desired properties. The details are worked out in [**FrH2021a, FrH2021b**].[2313] ■

217.3. **The Sphere–Alexander Duality Theorem.** In this and the following chapter we will prove several theorems which are called "Sphere–Alexander Duality Theorem". We start out with the most basic Sphere–Alexander Duality Theorem. This version is surprisingly easy to prove and it is already quite helpful:

Theorem 217.8. (Sphere–Alexander Duality Theorem) Let $n \in \mathbb{N}$. Furthermore let $X \subsetneq S^n$ be a compact subspace of S^n that admits a regular neighborhood in the sense of Definition 217.2. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X; \mathbb{Z}).$$

(By the Tubular–Regular Neighborhood Theorem 217.3 this conclusion holds for any closed submanifold $X \subsetneq S^n$.)

Remark.

(1) Our isomorphism depends a priori on the choice of a regular neighborhood N of X. Only later in the Sphere–Alexander Duality Isomorphisms Proposition 218.23 will we see that the choice of N plays no role after all.
(2) In the proof of the Sphere–Alexander Duality Theorem 217.8 we can flip the roles of homology and cohomology groups and we see that (under the same hypotheses) for any $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}^i(S^n \setminus X; \mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}_{n-i-1}(X).$$

We will not make use of this observation. (Alternatively one could also use the purely algebraic Exercise 217.6 to get to this conclusion.) □

Proof. In the following let $X \subsetneq S^n$ be a compact subspace of S^n that admits a regular neighborhood N.

[2312] Note that to arrange this conclusion one needs to make use of the fact that the derivatives of the maps f_e are non-zero also at the endpoints.
[2313] Note that in [**FrH2021a, FrH2021b**] it is shown that the regular neighborhood we just constructed is also unique in an appropriate sense.



The following claim contains the part of the argument, which does not really rely on the fact that we work with the sphere S^n:

Claim 1. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i(S^n \setminus X) \;\cong\; \mathrm{H}^{n-i}(S^n, X; \mathbb{Z}).$$

Proof. We have the following isomorphisms:

by the Basics-of-Regular Neighborhood Lemma 217.4 (1) and (2), and the Topological Manifold-Induced Orientation Lemma 192.1 we know that $S^n \setminus \mathring{N}$ is a compact orientable n-dimensional topological manifolds, thus the isomorphism follows from by the Poincaré Duality Theorem 210.1 applied to $S^n \setminus \mathring{N}$

$$\mathrm{H}_i(S^n \setminus X) \;\underset{\uparrow}{\overset{\cong}{\longrightarrow}}\; \mathrm{H}_i(S^n \setminus X; \mathbb{Z}) \;\underset{\uparrow}{\overset{\cong}{\longleftarrow}}\; \mathrm{H}_i(S^n \setminus \mathring{N}; \mathbb{Z}) \;\overset{\downarrow}{\underset{\cong}{\longleftarrow}}\; \mathrm{H}^{n-i}(S^n \setminus \mathring{N}, \partial N; \mathbb{Z})$$

see page 3077

by the Basics-of-Regular Neighborhood Lemma 217.4 (3b) and the Homotopy-H_*-Proposition 144.12

$$\underset{\uparrow}{\overset{\cong}{\longleftarrow}}\; \mathrm{H}^{n-i}(S^n, N; \mathbb{Z}) \qquad \underset{\uparrow}{\overset{\cong}{\longleftarrow}}\; \mathrm{H}^{n-i}(S^n, X; \mathbb{Z}).$$

by the Manifold-Excision Theorem 148.3, which we can apply by the Codimension-Zero Submanifold Proposition 150.16

by the Homotopy Equivalence Relative-H^*-Corollary 198.14 since X is a deformation retract of N

⊞

The following claim now gives us the desired result:

Claim 2. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X; \mathbb{Z}).$$

Proof. We denote by $\iota\colon X \to S^n$ the inclusion map. We consider the long exact sequence in reduced cohomology of the pair (S^n, X) that is provided by the Reduced LES-of-Pair Proposition 198.19:[2314]

$$\underbrace{\widetilde{\mathrm{H}}^{n-i-1}(S^n; \mathbb{Z})}_{=0 \text{ for } i \neq -1} \to \widetilde{\mathrm{H}}^{n-i-1}(X; \mathbb{Z}) \xrightarrow{\delta} \underbrace{\mathrm{H}^{n-i}(S^n, X; \mathbb{Z})}_{\cong \mathrm{H}_i(S^n \setminus X), \text{ by Claim 1}} \to \underbrace{\widetilde{\mathrm{H}}^{n-i}(S^n; \mathbb{Z})}_{=0 \text{ for } i \neq 0} \xrightarrow{\iota^*} \widetilde{\mathrm{H}}^{n-i}(X; \mathbb{Z}) \to \dots$$

The case $i \neq 0$ of the claim follows from the above and the Reduced Homology Lemma 147.3 (1). The case $i = 0$ requires slightly more thought. In that case we have the following exact sequence:

$$0 \;\to\; \widetilde{\mathrm{H}}^{n-1}(X; \mathbb{Z}) \xrightarrow{\delta_{n-1}} \underbrace{\mathrm{H}^n(S^n, X; \mathbb{Z})}_{\cong \mathrm{H}_0(S^n \setminus X)} \to \underbrace{\widetilde{\mathrm{H}}^n(S^n; \mathbb{Z})}_{\cong \mathbb{Z}} \xrightarrow{\iota^*} \widetilde{\mathrm{H}}^n(X; \mathbb{Z}) \;\to\; \mathit{dots}$$

[2314] In the argument below we make use of the fact that by the calculation of $\widetilde{\mathrm{H}}^*(S^n; \mathbb{Z})$ on page 3830 we know that $\widetilde{\mathrm{H}}^i(S^n; \mathbb{Z}) = 0$ for all groups except for $i = n$ and that $\widetilde{\mathrm{H}}^i(S^n; \mathbb{Z}) \cong \mathbb{Z}$.

We make a few observations:

(1) Recall that we assume that $X \subsetneq S^n$ is a proper subset of S^n. This means that there exists a $P \in S^n \setminus X$. Thus the inclusion $\iota\colon X \to S^n$ factors through $S^n \setminus \{P\}$. In particular $\iota^*\colon \widetilde{\mathrm{H}}^n(S^n;\mathbb{Z}) \to \widetilde{\mathrm{H}}^n(X;\mathbb{Z})$ factors through $\widetilde{\mathrm{H}}^n(S^n \setminus \{P\};\mathbb{Z})$, which is zero, since $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

(2) It follows from the above long exact sequence, (1), the Free Abelian Group–Splitting Lemma 97.11 and the Splitting Lemma 145.5 that $\mathrm{H}_0(S^n \setminus X) \cong \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z}) \oplus \mathbb{Z}$.

(3) Since we assume that $S^n \setminus X \neq \varnothing$ we obtain from the Reduced Homology Lemma 147.3 that $\mathrm{H}_0(S^n \setminus X) \cong \widetilde{\mathrm{H}}_0(S^n \setminus X) \oplus \mathbb{Z}$.

(4) Finally note that it follows from (2), (3) and the Cancellation Proposition 97.26 (which we can apply by the Submanifold Complement–Invariants Lemma 217.5) that $\widetilde{\mathrm{H}}_0(S^n \setminus X) \cong \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z})$. ∎

In practice we often like to consider objects that lie in $\mathbb{R}^n$ instead of S^n. The following variation on the Sphere–Alexander Duality Theorem 217.8 deals with this situation.

Theorem 217.9. (Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem) Let $n \in \mathbb{N}_0$ and let X be a compact non-empty subspace of $\mathbb{R}^n$. If X admits a regular neighborhood, then for any $i \in \mathbb{N}_0$ we have an isomorphism[2315]

$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X) \;\cong\; \mathrm{H}^{n-i-1}(X;\mathbb{Z}).$$

(By the Tubular–Regular Neighborhood Theorem 217.3 this conclusion holds for any closed non-empty submanifold $X \subseteq \mathbb{R}^n$.)

The proof of the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9 rests on the following lemma:

Lemma 170.1. (Complement-in-$\mathbb{R}^n$-S^n-Lemma) Let $n \in \mathbb{N}_0$ and let $X \subseteq \mathbb{R}^n$ be a non-empty compact subspace. We also view X as a subspace of $\mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$. For any $i \in \mathbb{N}$ we have an isomorphism

$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X) \;\cong\; \begin{cases} \widetilde{\mathrm{H}}_i(S^n \setminus X), & \text{if } i \neq n-1, \\ \widetilde{\mathrm{H}}_i(S^n \setminus X) \oplus \mathbb{Z}, & \text{if } i = n-1. \end{cases}$$

The analogous statements also hold if we replace reduced homology by ordinary homology.

Proof of the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9. Let X be a compact non-empty subspace of $\mathbb{R}^n$ that admits a regular neighborhood N in $\mathbb{R}^n$. Recall that N is by definition compact. It follows easily that N is also a regular neighborhood for X in $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$. Now let $i \in \mathbb{N}_0$. Then

$$\mathrm{H}^{n-i-1}(X;\mathbb{Z}) \;\underset{\uparrow}{\cong}\; \begin{cases} \widetilde{\mathrm{H}}_i(S^n \setminus X), & \text{if } i \neq n-1, \\ \widetilde{\mathrm{H}}_{n-1}(S^n \setminus X) \oplus \mathbb{Z}, & \text{if } i = n-1 \end{cases} \;\underset{\uparrow}{\cong}\; \widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X).$$

follows from the Reduced Cohomology Lemma 198.17 and the Sphere–Alexander Duality Theorem 217.8 which we can apply since X admits a regular neighborhood in S^n and since $X \subseteq \mathbb{R}^n \subsetneq S^n$

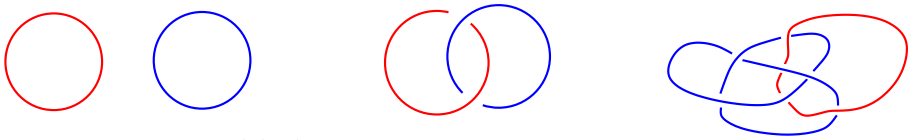
by the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1, here we use that $X \neq \varnothing$

∎

In the remainder of this section we discuss a few applications of the Sphere–Alexander Duality Theorem 217.8. First we recall Definition 108.5:

[2315] Note that the effect of replacing S^n by $\mathbb{R}^n$ is to replace reduced cohomology by unreduced cohomology.

Definition 217.10. Let $m \in \mathbb{N}_0$. An m-component link is a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1.[2316]



2-component trivial link Hopf link

The following lemma computes the homology groups of a link complement. It can be viewed as a generalization of the Knot Complement-H_*-Proposition 185.29.

Proposition 217.11. (Link Complement-H_*-Proposition) Let $m \in \mathbb{N}$. For any m-component link L in S^3 we have

$$\mathrm{H}_i(S^3 \setminus L) \cong \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus L)$ we get the same result, except for an extra $\mathbb{Z}$-summand in degree 2.

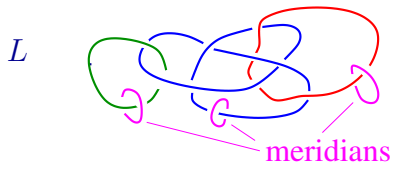
Proof. Let L be an m-component link. We calculate that

$$\widetilde{\mathrm{H}}_i(S^3 \setminus L) \underset{\text{by the Sphere–Alexander Duality Theorem 217.8, since } L \text{ is a smooth submanifold}}{\cong} \widetilde{\mathrm{H}}^{3-1-i}(L;\mathbb{Z}) \cong \widetilde{\mathrm{H}}^{2-i}(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{m\text{-copies}};\mathbb{Z}) \underset{\text{by the discussion on page 3830, the Path-Component-H}^*\text{-Lemma 198.8, and the Reduced Cohomology Lemma 198.17}}{\cong} \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The statement about the homology groups of $S^3 \setminus L$ follows from this calculation together with the Reduced Homology Lemma 147.3.

Finally note that we obtain the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus L)$ by applying the Tubular–Regular Neighborhood Theorem 217.3 together with the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9, or alternatively, by cutting out the middle man, by applying the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1 to the above calculation. ∎

Example. For the three 3-component link L shown below we see that $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^3$ and $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^2$.



$$\begin{aligned} \mathrm{H}_2(S^3 \setminus L) &\cong \mathbb{Z}^2, \\ \mathrm{H}_1(S^3 \setminus L) &\cong \mathbb{Z}^3, \\ \mathrm{H}_0(S^3 \setminus L) &\cong \mathbb{Z} \end{aligned}$$

The problem with this calculation is that we do not obtain any understanding of what the homology classes "look like". It seems reasonable to guess, along the lines of the Knot Complement-H_*-Proposition 185.29, that the first homology group is generated by the

[2316] In Definition 108.5 we also demanded that a link is equipped with a bijection from $\{1, \ldots, m\}$ to the set of components of L. We quietly drop this extra requirement.

three meridians. But (a), we do not know how to prove it, and (b), we have no idea what a generator of the second homology group looks like. □

From the previous example the following question arises:

Question 217.12. Given a link L, what are explicit bases for the homology groups of $S^3 \setminus L$?

We will return to this question in the Link Complement-H_*–Basis-Proposition 283.4. We move on to spatial graphs. It is pretty clear that every link can also be viewed as a spatial graph. The following proposition can thus be viewed as a generalization of the Link Complement-H_*-Proposition 217.11.

Proposition 217.13. (Spatial Graph Complement-H_*-Proposition) Let G be a finite abstract graph with $m \in \mathbb{N}$ components and let $\Theta\colon |G| \to \mathbb{R}^3$ be an embedding such that $\Theta(|G|)$ is a spatial graph. We also view $\Theta(|G|)$ as a subspace of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. Then

$$\mathrm{H}_i(S^3 \setminus \Theta(|G|)) \cong \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^{m-\chi(G)}, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus \Theta(|G|))$ we get the same result, except for an extra $\mathbb{Z}$-summand in degree 2.

Proof. We have the following isomorphisms:

$$\widetilde{\mathrm{H}}_i(S^3 \setminus \Theta(|G|)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{2-i}(\Theta(|G|); \mathbb{Z}) \cong \widetilde{\mathrm{H}}^{2-i}(|G|; \mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^{m-\chi(G)}, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

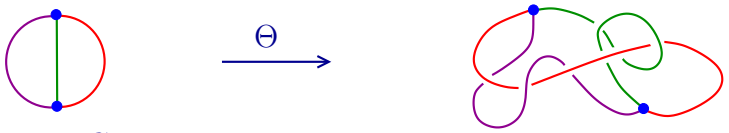
by the Spatial Graph–Regular Neighborhood Proposition 217.7 we can apply Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9

by the discussion on page 3831 together with the Path-Component-H^*-Lemma 198.8 and the Reduced Cohomology Lemma 198.17

Again the statement about the homology groups of $S^3 \setminus \Theta(|G|)$ follows from this calculation together with the Reduced Homology Lemma 147.3.

Finally note that once again we obtain the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus \Theta(|G|))$ by applying the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9, or alternatively by applying the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1. ■

Example. We consider the connected abstract graph G with two vertices and three edges that is illustrated in the figure below on the left. On the right of the figure we show a corresponding spatial graph $\Theta(|G|)$ in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ that is homeomorphic to G.



graph G

the spatial graph $\Theta(G)$ in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$

Note that $m = 1$ and $\chi(G) = -1$. From the Spatial Graph Complement-H_*-Proposition 217.13 we thus obtain that

$$\mathrm{H}_i(\mathbb{R}^3 \setminus \Theta(|G|)) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In this case it is also rather unclear what a basis of $\mathrm{H}_1(S^3 \setminus G)$ should look like. Later, on page 5457, we will find an explicit basis for $\mathrm{H}_1(S^3 \setminus G)$. □

217.4. **Immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$ II.** We can now provide a new proof of the following proposition:

Proposition 51.17. (Smooth Codimension-One Orientability Proposition) Let N be a closed n-dimensional smooth manifold. If N is non-orientable, then there is no smooth embedding of N into $\mathbb{R}^{n+1}$.

Proof. Clearly it suffices to deal with the case that N is connected. Thus let N be a closed *non-orientable* connected n-dimensional smooth manifold. Suppose that N admits a smooth embedding $N \to \mathbb{R}^{n+1}$. It follows from the Smooth Embedding Theorem 30.15 that we can view N as a submanifold of $\mathbb{R}^{n+1}$. We obtain that

$$\mathbb{Z}/2\mathbb{Z} \overset{\text{by the Manifold-Top-H}^* \text{ Theorem 201.26 (2)}}{\cong} \mathrm{H}^n(N;\mathbb{Z}) \overset{\text{by the Sphere–Alexander Duality-in-}\mathbb{R}^n\text{-Theorem 217.9 which we can apply since } N \text{ is a non-empty closed smooth submanifold of } \mathbb{R}^{n+1}}{\cong} \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus N;\mathbb{Z}) \overset{\text{by the H}_0\text{-Proposition 143.25 together with the Reduced Homology Lemma 147.3 (0)}}{\cong} \mathbb{Z}^{\text{number of path-components of } \mathbb{R}^{n+1} \setminus N \text{ minus 1}}.$$

But this is absurd. We have thus obtained the desired contradiction. ■

It is worth recalling the following almost immediately corollary to Proposition 51.17. This corollary gives in particular a negative answer to Question 30.18 (2).

Corollary 51.18. (Surface-No Smooth Embedding Corollary)

(1) Let $k \in \mathbb{N}$. The non-orientable surface of genus k does not admit a smooth embedding into $\mathbb{R}^3$.
(2) Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$.

Remark. In the Real Projective Space-Smooth Embedding Proposition 30.19 (see also Exercise 217.17) we learned that $\mathbb{R}\mathrm{P}^2$ admits a smooth embedding into $\mathbb{R}^4$. It follows from Exercise 217.15 that any non-orientable surface of some genus $k \in \mathbb{N}$ admits a smooth embedding into $\mathbb{R}^4$. □

Proof.

(1) By the Surface Non-Orientability Lemma 31.45 we know that N_k is non-orientable. Since N_k is a closed 2-dimensional smooth manifold it follows from the Smooth Codimension-One Orientability Proposition 51.17 that N_k does not admit a smooth embedding into $\mathbb{R}^3$.
(2) By Definition 9.19 the real projective plane is precisely the non-orientable surface of genus ≥ 1. Furthermore note that by Exercise 9.17 the Klein bottle is diffeomorphic to the non-orientable surface of genus two. Thus the promised statements follow from (1). ■

We recall the following related question:

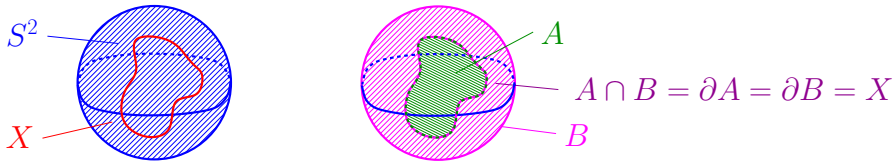
Question 33.6. Given $n \in \mathbb{N}$, what is the minimal $k \in \mathbb{N}$ such that $\mathbb{R}\mathrm{P}^n$ admits a (topological) smooth embedding into $\mathbb{R}^k$?

Recall that it follows from the Real Projective Space Orientability Lemma 31.46, or alternatively from the Orientation-Top-H_*-Theorem 193.3 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^n$ in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18, that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd. In particular it follows from the Smooth Codimension-One Orientability Proposition 51.17 that for n even it is not possible to smoothly embed $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. But this raises the following question: Given n odd, can one find a smooth embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}? For $n = 1$ we can do so: First note that $\mathbb{R}\mathrm{P}^1$ is diffeomorphic to[2317] S^1 and clearly we can find a smooth embedding of S^1 into S^2. But for odd $n \geq 3$ we can answer this question in the negative:

Proposition 217.14. ($\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition) If $n \in \mathbb{N}_{\geq 2}$, then there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.

In the proof of the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14 we will need the following lemma which is in fact basically identical to the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15.

Lemma 217.15. (Smooth Sphere-Splitting Lemma) Let $n \in \mathbb{N}_{\geq 2}$. If X is a closed orientable connected non-empty $(n-1)$-dimensional smooth submanifold of the smooth manifold S^n, then there exist two compact connected codimension-zero smooth submanifolds A and B of S^n such that $A \cap B = \partial A = \partial B = X$.



Proof. First we calculate that

$$\mathrm{H}_0(S^n \setminus X) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_0(S^n \setminus X) \oplus \mathbb{Z} \underset{\uparrow}{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}) \oplus \mathbb{Z} \underset{\uparrow}{\cong} \mathbb{Z} \oplus \mathbb{Z}.$$

by the Reduced Homology Lemma 147.3 (0b), since $X \subsetneq S^n$

by the Sphere–Alexander Duality Theorem 217.8

by the Orientation-Top Homology Theorem 193.3 and since X is closed, orientable connected, non-empty and $(n-1)$-dimensional

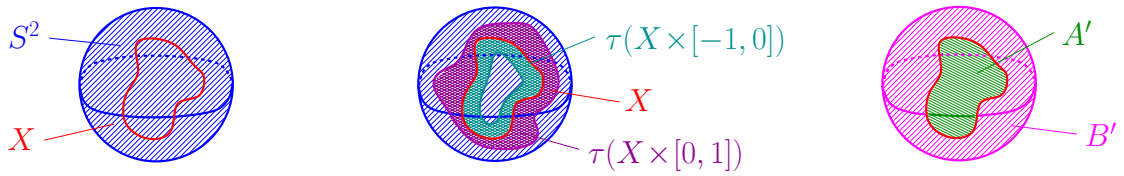
It follows from this calculation and the H_0-Proposition 143.25 that $S^n \setminus X$ has precisely two path-components A' and B'.

Claim 1. The subsets A' and B' of S^n are open.

Proof. Since $X \subseteq S^n$ is compact we know by the Compact-Closed Lemma 3.39 that X is a closed subset of S^n. Thus $S^n \setminus X$ is an open subset of S^n. Next note that it follows from the Components-are-Closed Lemma 4.49 (1) and the Manifold–Local Properties Lemma 23.8 that A' and B' are open subsets of $S^n \setminus X$. It follows from the Open-Closed Inclusion Lemma 4.13 that A' and B' are also open subsets of S^n. ⊞

[2317] In the Projective Space-Avatar Lemma 9.16 we showed that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1 and in Exercise 25.11 the reader hopefully showed that $\mathbb{R}\mathrm{P}^1$ is actually diffeomorphic to S^1.

Next note that it follows from the Product Tubular Neighborhood–Existence Theorem 58.4 (3) (or alternatively the Bicollar Neighborhood Theorem 55.2) that the submanifold X admits a tubular map $\tau\colon [-1,1]\times X \to S^n$. Also note that it follows from the Product Topology-Properties Proposition 7.9 (2) that $\tau((-1,0)\times X)$ and $\tau((0,1)\times X)$ are also path-connected.



Since $\tau((-1,0)\times X)$ is path-connected and since it is contained in $S^n \setminus X = A' \cup B'$ it follows from the Path-Component Lemma 4.44 (4b) that $\tau((-1,0)\times X)$ is contained in A' or it is contained in B'. Without loss of generality we can assume that $\tau((-1,0))\times X$ is contained in A'.

Claim 2. The subspace $\tau((0,1)\times X)$ is contained in B'.

Proof. We consider the set $A'' := A' \cup \tau((-1,1)\times X)$. Note that $S^n = A'' \cup B'$. It follows from Claim 1 and the Basics-of-Tubular Neighborhood Lemma 57.10 that both A'' and B' are open subsets of S^n. Since S^n is connected it follows that $A'' \cap B' \neq \varnothing$. Since $(A' \cup X) \cap B' = \varnothing$ we see that this implies that $\tau((0,1)\times X) \cap B' \neq \varnothing$. But since $(0,1)\times X$ is connected and contained in $S^n \setminus X = A' \cup B'$ we obtain from the Image-Preimage-Component Lemma 4.50 (2) that $\tau((0,1)\times X)$ is contained in B'. ⊞

We set $A := A' \cup X$ and $B := B' \cup X$.

Claim 3. The subspaces A and B are compact connected codimension-zero submanifolds of S^n such that $A \cap B = \partial A = \partial B = X$.

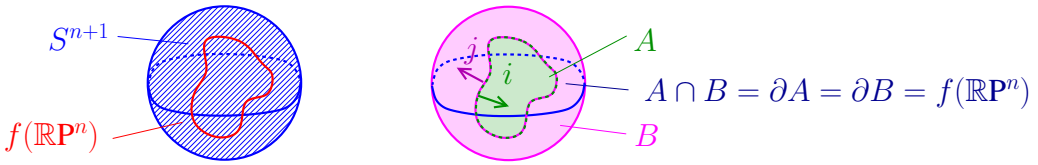
Proof. First note that it follows from the above discussion that $S^n \setminus A = B'$ and $S^n \setminus B = A'$. It follows from Claim 1 that A and B are closed subsets of S^n. It follows from the Compact-Closed Lemma 3.39 that A and B are compact. Next note that $A = A' \cup \tau((-1,0]\times X)$ and $B = B' \cup \tau([0,1)\times X)$. It follows from this observation and the Union Connected Lemma 4.39 that A and B are connected.

Finally we need to find submanifold charts for A and B. Since A' and B' are open subsets of S^n we obtain for the points in A' and B' submanifold charts of type (α). Finally, for points in X we have the tubular neighborhood $\tau((-1,1)\times X)$, which gives rise to submanifold charts of type (γ). Using these two types of charts we see that A and B are codimension-zero submanifolds with $A \cap B = \partial A = \partial B = X$. ■

Proof of Proposition 217.14. As we had discussed just before the proposition, it remains to deal with the case that $n \geq 2$ is odd. So suppose that $n \geq 3$ is odd and that there exists a smooth embedding $f\colon \mathbb{R}\mathrm{P}^n \to S^{n+1}$. First note that it follows from the Smooth Embedding Theorem 30.15 that $f(\mathbb{R}\mathrm{P}^n)$ is a submanifold of S^{n+1}. Next note that it follows from $n \geq 3$ and the discussion on page 3830 that $\mathrm{H}^1(S^n;\mathbb{F}_2) = \mathrm{H}^2(S^n;\mathbb{F}_2) = 0$.

Since n is odd we know by the Real Projective Space Orientability Lemma 31.46 that $\mathbb{R}\mathrm{P}^n$ is orientable. Thus we can apply the Smooth Sphere-Splitting Lemma 217.15 which tells us that there exist two compact connected $(n+1)$-dimensional smooth submanifolds A and B of S^{n+1} with $A \cap B = \partial A = \partial B = f(\mathbb{R}\mathrm{P}^n)$. We denote by $i\colon f(\mathbb{R}\mathrm{P}^n) \to A$ and

by $j\colon f(\mathbb{R}\mathrm{P}^n) \to B$ the inclusion maps. By the discussion on page 3832 we know that $\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2$. As usual we denote by $x \in \mathrm{H}^1(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2) \cong \mathbb{F}_2$ the unique non-zero element.



Since A and B are compact codimension-zero smooth submanifolds with $A \cap B = \partial A = \partial B$ we can apply the Mayer–Vietoris-H^*-Theorem 200.11 (3) to $S^{n+1} = A \cup B$ and we obtain the long exact sequence

$$\cdots \to \underbrace{\mathrm{H}^1(S^{n+1};\mathbb{F}_2)}_{=0 \text{ since } n\geq 3} \to \mathrm{H}^1(A;\mathbb{F}_2) \oplus \mathrm{H}^1(B;\mathbb{F}_2) \xrightarrow{i^*\oplus j^*} \underbrace{\mathrm{H}^1(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)}_{=\mathbb{F}_2\cdot x} \to \underbrace{\mathrm{H}^2(S^{n+1};\mathbb{F}_2)}_{=0 \text{ since } n\geq 3} \to \dots$$

Thus we see that the map $i^* \oplus j^*\colon \mathrm{H}^1(A;\mathbb{F}_2) \oplus \mathrm{H}^1(B;\mathbb{F}_2) \to \mathrm{H}^1(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2) = \mathbb{F}_2 \cdot x$ is an isomorphism. Without loss of generality we can assume that i^* is non-trivial which means that there exists an $a \in \mathrm{H}^1(A;\mathbb{F}_2)$ with $i^*(a) = x$. Next note that

$$i^*(a^n) \underset{\substack{\uparrow\\ \text{by the Cup Product}\\ \text{Naturality Lemma 203.9}}}{=} (i^*(a))^n = x^n \underset{\substack{\uparrow\\ \text{by the Cup Product-}\mathbb{R}\mathrm{P}^n\\ \text{Proposition 213.19}}}{\neq} 0 \quad \in \mathrm{H}^n(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2).$$

In particular the map $i^*\colon \mathrm{H}^n(A;\mathbb{F}_2) \to \mathrm{H}^n(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)$ is non-zero. Now we consider the other end of the above Mayer-Vietoris sequence and we obtain the exact sequence

$$\begin{array}{ccccccc} \mathrm{H}^n(A;\mathbb{F}_2)\oplus\mathrm{H}^n(B;\mathbb{F}_2) & \xrightarrow{i^*\oplus j^*} & \overbrace{\mathrm{H}^n(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)}^{=\mathbb{F}_2\cdot x^n} & \to & \overbrace{\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2)}^{\cong\mathbb{F}_2} & \to & \underbrace{\mathrm{H}^{n+1}(A;\mathbb{F}_2)\oplus\mathrm{H}^{n+1}(B;\mathbb{F}_2)}_{\substack{\text{both are zero by the}\\ \text{Top Cohomology Theorem 201.26 (2a),}\\ \text{since } A \text{ and } B \text{ are compact, connected}\\ (n+1)\text{-dimensional smooth manifolds}\\ \text{with non-empty boundary}}}. \\ (a^n,0) & \mapsto & x^n & & & & \end{array}$$

Thus we see that the map on the left is an epimorphism and that the group on the right is trivial. But then the fact that $\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2) \neq 0$ (which we obtained on page 3830) shows that the sequence cannot possibly be exact. We have thus obtained the desired contradiction. ∎

Remark. The above proposition does not give a complete answer to Question 33.6, we have only shown that $\mathbb{R}\mathrm{P}^n$ cannot smoothly embed into S^{n+1}. More lower bounds on the minimal dimension of some $\mathbb{R}^k$ into which we can smoothly embed $\mathbb{R}\mathrm{P}^n$ have been obtained by Michael Atiyah [**At1961a**]. □

217.5. **Alexander duality for smooth manifolds with boundary.** The following theorem generalizes the Sphere–Alexander Duality Theorem 217.8:

Theorem 217.16. (Alexander Duality–Boundary Theorem) Let $n \in \mathbb{N}$, let Σ be a compact orientable n-dimensional smooth manifold and let $\partial\Sigma = A \cup B$ be a decomposition

of the boundary $\partial\Sigma$ into unions of components. Let X be a compact subspace of Σ that admits a regular neighborhood in the sense of Definition 217.2.

(1) Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma, A) = \mathrm{H}_{j+1}(\Sigma, A) = 0$. Any choice of a regular neighborhood of $X \subseteq \Sigma$ defines an isomorphism

$$\mathrm{H}_j(\Sigma \setminus X, A \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(X, X \cap B; \mathbb{Z}).$$

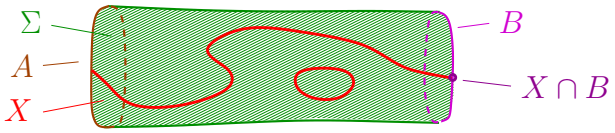
(2) We have two extra results for the "extremes" $j = 0$, $A = \varnothing$ and $j = n-1$, $B = \varnothing$:[2318]

(a) If $X \subsetneq \Sigma$ and if $\widetilde{\mathrm{H}}_0(\Sigma) = \mathrm{H}_1(\Sigma) = 0$, then we obtain an isomorphism

$$\widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X, X \cap \partial\Sigma; \mathbb{Z}).$$

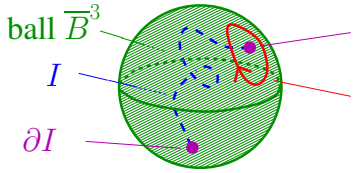
(b) If $X \subsetneq \Sigma$, if $X \neq \varnothing$, if Σ is connected and if $\mathrm{H}_{n-1}(\Sigma, \partial\Sigma) = 0$, then there exists an isomorphism

$$\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X; \mathbb{Z}).$$



we have $\mathrm{H}_*(\Sigma, A) = 0$ since A is a deformation retract of Σ

Example. Let $I \subseteq \overline{B}^3$ be a proper 1-dimensional smooth submanifold that is diffeomorphic to an interval.



generator of $\mathrm{H}_1(S^2 \setminus K)$, and thus also generator of $\mathrm{H}_1(\overline{B}^3 \setminus I)$

Note that for any $j \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_j(\overline{B}^3 \setminus I) \xrightarrow{\cong} \mathrm{H}^{3-j-1}(I, I \cap S^2; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}^{2-j}([0,1], \{0,1\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

by the Alexander Duality–Boundary Theorem 217.16 (1) (2a), applied with $A = \varnothing$ and with $B = S^2$, note that we use that the reduced homology groups of $\overline{B}^3$ are zero

since I is a properly embedded interval

this follows for example from the long exact sequence in cohomology of the pair $([0,1], \{0,1\})$

Note that these groups are isomorphic to the corresponding homology groups of $S^2 \setminus \partial I$. So what can we see about the inclusion-induced map $i_*\colon \mathrm{H}_1(S^2 \setminus \partial I) \to \mathrm{H}_1(\overline{B}^3 \setminus I)$? To answer this question we consider the long exact sequence of the pair $(\overline{B}^3 \setminus I, S^2 \setminus \partial I)$:

$$\mathrm{H}_2(\overline{B}^3 \setminus I, S^2 \setminus \partial I) \xrightarrow{\partial_2} \mathrm{H}_1(S^2 \setminus \partial I) \xrightarrow{i_*} \mathrm{H}_1(\overline{B}^3 \setminus I) \to \mathrm{H}_1(\overline{B}^3 \setminus I, S^2 \setminus \partial I).$$

[2318] These two special cases deal with the nuisance that for a compact connected orientable non-empty smooth manifold Σ we have $\mathrm{H}_0(\Sigma) \cong \mathbb{Z}$ and we have $\mathrm{H}_n(\Sigma, \partial\Sigma) \cong \mathbb{Z}$. For $j = 0$, $A = \varnothing$ and $j = n-1$ and $B = \varnothing$ this prevents us from applying (1) directly.

To understand the middle map we need to understand the groups to the left and right. We now see that for $j = 1, 2$ we have the following isomorphisms:

$$\mathrm{H}_j(\overline{B}^3 \setminus I, S^2 \setminus \partial I) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}^{3-j-1}(I;\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}^{2-j}([0,1];\mathbb{Z}) \underset{\uparrow}{=} 0.$$

by the Alexander Duality–Boundary Theorem 217.16 (1) and (2b), applied with $A = S^2$ and with $B = \varnothing$, note that we use that the homology groups of $(\overline{B}^3, S^2)$ are zero in dimensions $1, 2$

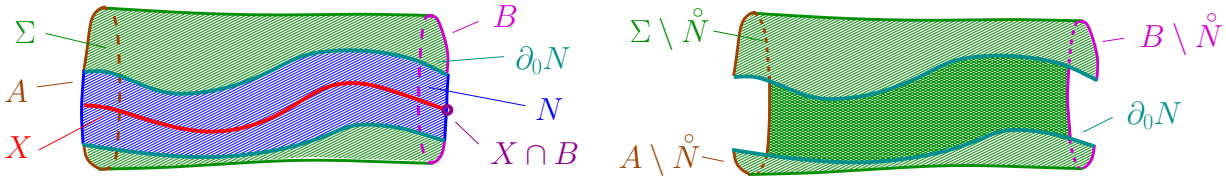
since I is a properly embedded interval

since I is contractible

It follows that in the above long exact sequence the middle map $i_*\colon \mathrm{H}_1(S^2 \setminus \partial I) \to \mathrm{H}_1(\overline{B}^3 \setminus I)$ is an isomorphism. □

The proof of the Alexander Duality–Boundary Theorem 217.16 will require the remainder of this section. The impatient reader can also jump to Appendix 217.6 where we prove a generalization of the Smooth Codimension-One Orientability Proposition 51.17, or straight to Chapter 218 where we will prove a generalization of the Sphere–Alexander Duality Theorem 217.8.

Proof of the Alexander Duality–Boundary Theorem 217.16 (1). Let $n \in \mathbb{N}$, let Σ be a compact orientable n-dimensional smooth manifold and let $\partial\Sigma = A \cup B$ be a decomposition of the boundary $\partial\Sigma$ into unions of components. Let X be a compact subspace of Σ that admits a regular neighborhood N. Finally let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma, A) = \mathrm{H}_{j+1}(\Sigma, A) = 0$.



Rather similar to the proof of the Sphere–Alexander Duality Theorem 217.8 on page 4150 we have the following isomorphisms:

$$\begin{aligned}
\mathrm{H}_j(\Sigma \setminus X, A \setminus X) &\xleftarrow[(0)]{\cong} \mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}) \xrightarrow[(1)]{\cong} \mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}; \mathbb{Z}) \\
&\xleftarrow[(2)]{\cong} \mathrm{H}^{n-j}(\Sigma \setminus \mathring{N}, \partial_0 N \cup (B \setminus \mathring{N}); \mathbb{Z}) \xleftarrow[(3)]{\cong} \mathrm{H}^{n-j}(\Sigma, N \cup B; \mathbb{Z}) \\
&\xleftarrow[(4)]{\cong} \mathrm{H}^{n-j-1}(N \cup B, B; \mathbb{Z}) \xrightarrow[(5)]{\cong} \mathrm{H}^{n-j-1}(N, N \cap B; \mathbb{Z}) \\
&\xrightarrow[(6)]{\cong} \mathrm{H}^{n-j-1}(X, X \cap B; \mathbb{Z}).
\end{aligned}$$

All that remains to do, is to explain, where the maps come from, and why they are isomorphisms. Here is the executive summary:

(0) This holds since the topological spaces are homotopy equivalent.
(1) This holds since ordinary homology is isomorphic to homology with $\mathbb{Z}$-coefficients.
(2) This follows from the Poincaré Duality Theorem 210.1.
(3) This follows from an excision argument.
(4) This follows from our hypothesis on the homology groups of (Σ, A), Poincaré Duality and a long exact sequence argument.
(5) This follows again from an excision style argument.

(6) This holds since the topological spaces are homotopy equivalent.

Here are the gory details:

(0) It follows easily from the Basics-of-Regular Neighborhood Lemma 217.4 that the inclusions $\Sigma\setminus\mathring{N} \to \Sigma\setminus X$ and $A\setminus\mathring{N} \to A\setminus X$ are homotopy equivalences. It thus follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the inclusion-induced map $\mathrm{H}_j(\Sigma\setminus\mathring{N}, A\setminus\mathring{N}) \to \mathrm{H}_j(\Sigma\setminus X, A\setminus X)$ is an isomorphism.

(1) It follows from the discussion on page 3077 of homology with $\mathbb{Z}$-coefficients that we have an isomorphism $\mathrm{H}_j(\Sigma\setminus\mathring{N}, A\setminus\mathring{N}) \to \mathrm{H}_j(\Sigma\setminus\mathring{N}, A\setminus\mathring{N};\mathbb{Z})$.

(2) By the Basics-of-Regular Neighborhood Lemma 217.4 (1) and (2), and the Topological Manifold-Induced Orientation Lemma 192.1 we know that $\Sigma\setminus\mathring{N}$ is a compact orientable n-dimensional topological manifold. Furthermore we know by the Basics-of-Regular Neighborhood Lemma 217.4 (2b) that $\partial(\Sigma\setminus\mathring{N}) = (A\setminus\mathring{N}) \cup (\partial_0 N \cup (B\setminus\mathring{N}))$ is a decomposition into two compact codimension-zero submanifolds that intersect in the common boundary. It thus follows from the Poincaré Duality Theorem 210.1 applied to $\Sigma\setminus\mathring{N}$ (together with the decomposition $\partial(\Sigma\setminus\mathring{N}) = (A\setminus\mathring{N}) \cup (\partial_0 N \cup (B\setminus\mathring{N}))$), that we have an isomorphism $\mathrm{H}^{n-j}(\Sigma\setminus\mathring{N}, \partial_0 N \cup (B\setminus\mathring{N});\mathbb{Z}) \to \mathrm{H}_j(\Sigma\setminus\mathring{N}, A\setminus\mathring{N};\mathbb{Z})$.

(3) By Definition 217.2 of a regular neighborhood we know that N is a codimension-zero submanifold with corner of N. It thus follows from a slight generalization of the Excision-in-H^*-Theorem 200.1 to codimension-zero submanifolds with corner[2319] that the inclusion-induced map

$$\mathrm{H}^{n-j}(\Sigma, N\cup B;\mathbb{Z}) \;\to\; \mathrm{H}^{n-j}(\Sigma\setminus\mathring{N}, \partial_0 N \cup (B\setminus\mathring{N});\mathbb{Z})$$

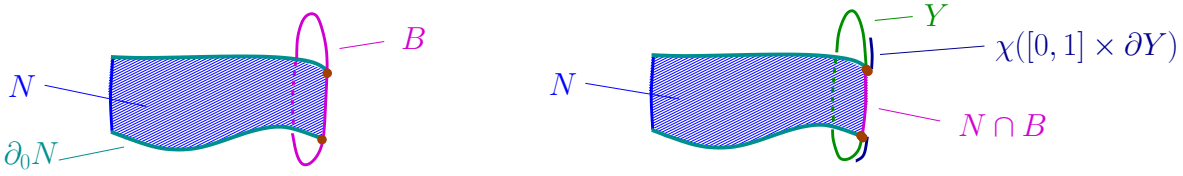
is an isomorphism.

(4) From the LES-of-Triple Proposition 198.13 we obtain the following exact sequence:

$$\underbrace{\mathrm{H}^{n-j-1}(\Sigma, B;\mathbb{Z})}_{\cong \mathrm{H}_{j+1}(\Sigma,A)=0} \to \mathrm{H}^{n-j-1}(N\cup B, B;\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^{n-j}(\Sigma, N\cup B;\mathbb{Z}) \to \underbrace{\mathrm{H}^{n-j}(\Sigma, B;\mathbb{Z})}_{\cong \mathrm{H}_j(\Sigma,A)=0} \to$$

It follows from the Poincaré Duality Theorem 210.1 and our hypothesis that the groups to the left and right are zero. Thus the homomorphism we care about is indeed an isomorphism.

(5) One can easily verify that $N\cap B$ is a compact codimension-zero submanifold of B with boundary $(\partial_0 N)\cap B$. We set $Y := B\setminus\mathring{N}$. By the Codimension-Zero Submanifold Proposition 25.49 we know that Y is a codimension-zero submanifold of B. By the Smooth Collar Neighborhood Theorem 35.6 we can pick a collar $\chi\colon [0,1]\times\partial Y \to Y$.



[2319] We proved this statement for homology groups in Exercise 148.7. (The key idea was to replace, in the proof of the Manifold–Excision Theorem 148.3, the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7 for smooth manifolds with corner.) The cohomological statement now follows from the Also-Iso-on-Cohomology Proposition 200.2. This is the one point where right now we use that Σ is a smooth manifold. With some more effort one can surely prove this result for topological manifolds as well.

For any $k \in \mathbb{N}_0$ we now consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(N \cup B, B; \mathbb{Z}) & \longrightarrow & \mathrm{H}^k(N, N \cap B; \mathbb{Z}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}^k(N \cup \chi([0,1] \times \partial Y), (N \cap B) \cup \chi([0,1] \times \partial Y); \mathbb{Z}) & \end{array}$$

We make the following clarifications and observations:

(a) All maps are induced by the inclusions of pairs of topological spaces.

(b) The diagram commutes by the functoriality of cohomology.

(c) One can easily verify that $Y \setminus \chi([0,1] \times \partial Y)$ lies in the interior of $B \subseteq N \cup B$. Thus it follows from the Excision-in-H^*-Theorem 200.1 (1) that the left diagonal map is an isomorphism.

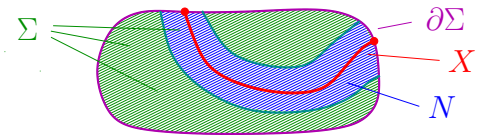
(d) A straightforward argument shows that N and $\chi([0,1] \times \partial Y)$ are closed subsets of $N \cup \chi([0,1] \times \partial Y)$. It follows from the Homotopy Combination Lemma 18.8 that the stationary homotopy and the obvious deformation retraction from $\chi([0,1] \times \partial Y)$ to ∂Y define a deformation retraction from $N \cup \chi([0,1] \times \partial Y)$ to N. It thus follows from the Homotopy Equivalence-Relative-H^*-Corollary 198.14 that the right diagonal map is an isomorphism.

Finally note that it follows from the above discussion that the top horizontal map $\mathrm{H}^k(N \cup B, B; \mathbb{Z}) \to \mathrm{H}^k(N, N \cap B; \mathbb{Z})$ is, as desired, an isomorphism.

(6) By Definition 217.2 of a regular neighborhood in Definition 217.2 we know that there exists a deformation retraction $F\colon N \times [0,1] \to N$ from N to X such that $F(\partial_1 N \times [0,1]) \subseteq \partial_1 N$. This implies that X is a deformation retract of N and that $X \cap B$ is a deformation retract of $N \cap B$. It thus follows from the Homotopy Equivalence-Relative-H^*-Corollary 198.14 that $\mathrm{H}^{n-j-1}(N, N \cap B; \mathbb{Z}) \to \mathrm{H}^{n-j-1}(X, X \cap B; \mathbb{Z})$ is an isomorphism. ■

Proof of the Alexander Duality–Boundary Theorem 217.16 (2a). Let Σ be a compact orientable n-dimensional smooth manifold such that $\widetilde{\mathrm{H}}_0(\Sigma) = \mathrm{H}_1(\Sigma) = 0$. Let $X \subsetneq \Sigma$ be a compact subspace of Σ that admits a regular neighborhood N in the sense of Definition 217.2. We need to show that there exists an isomorphism

$$\widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X, X \cap \partial\Sigma; \mathbb{Z}).$$



We start out with the following technical claim:

Claim. We have $N \neq \Sigma$.

Proof. We prove the claim by contradiction. So assume that $\Sigma = N$. Clearly we only have to deal with the case that Σ is connected. Since $X \subsetneq \Sigma$ there exists a $P \in \Sigma \setminus X$. We consider the following diagram:

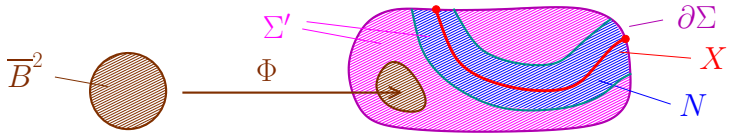
$$\begin{array}{ccc} \mathrm{H}_n(X, X\cap\partial\Sigma) & \longrightarrow & \overbrace{\mathrm{H}_n(\Sigma,\partial\Sigma)}^{\cong\mathbb{Z}} \\ & \searrow \quad \nearrow & \\ & \underbrace{\mathrm{H}_n(\Sigma\setminus\{P\},\partial\Sigma\setminus\{P\})}_{=0}. & \end{array}$$

We make the following clarifications and observations:

(1) All maps are induced by inclusions. Thus the diagram commutes.
(2) It follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8, Condition (2) of the Definition 217.2 of a regular neighborhood and the assumption that Σ is a regular neighborhood of X that the top horizontal map is an isomorphism.
(3) Since Σ is compact, orientable, connected and non-empty it follows from the Manifold-Top-H_*-Theorem 193.1 that the top right group is isomorphic to $\mathbb{Z}$.
(4) If $n = 0$, then $\Sigma\setminus\{P\} = \varnothing$ and the bottom group is zero. If $n > 0$, then one can easily show that each component of $\Sigma\setminus\{P\}$ is non-compact. Thus we obtain from the Manifold-Top-H_*-Theorem 193.1 that the bottom group is zero.

The above four statements are inconsistent. We have thus obtained the desired contradiction. ⊞

Since N is compact we obtain from the Compact-Closed Lemma 3.39 and the claim that $\Sigma\setminus N$ is an non-empty open subset of Σ. By the Topological Manifold–Boundary Proposition 23.15 we know that $\Sigma\setminus\partial\Sigma$ is an open dense subset of Σ. It follows from this discussion and the smooth analog of the discussion on page 639 that we can pick a smooth embedding $\Phi\colon \overline{B}^n \to \Sigma\setminus(N\cup\partial\Sigma)$ such that $\Phi(\overline{B}^n)$ is a codimension-zero smooth submanifold. We set $\Sigma' := \Sigma\setminus\Phi(B^n)$. It follows from the Codimension-Zero Submanifold Proposition 25.49 that Σ' is a compact orientable n-dimensional smooth manifold.



We obtain the following isomorphisms:

$$\begin{array}{ccccc} & & \text{isomorphism from the Reduced-to-Relative-H}_*\text{-Lemma 147.9} & & \text{this follows from the Homotopy Equivalence-Relative-H}_*\text{-Corollary 146.8, since } \Phi(\{0\})) \text{ is a deformation retract of } \Phi(\overline{B}^n) \\ & & \downarrow\cong & & \downarrow\cong \\ \widetilde{\mathrm{H}}_0(\Sigma\setminus X) & \xrightarrow{\cong} & \mathrm{H}_0(\Sigma\setminus X, \Phi(\{0\})) & \xrightarrow{\cong} & \mathrm{H}_0(\Sigma\setminus X, \Phi(\overline{B}^n)) \\ & \xleftarrow{\cong} & \mathrm{H}_0(\Sigma'\setminus X, \Phi(S^{n-1})) & \cong & \mathrm{H}^{n-1}(X, X\cap\partial\Sigma;\mathbb{Z}). \\ & & \uparrow & & \uparrow \\ & & \text{isomorphism by the Manifold–Excision Theorem 148.3} & & \text{follows from (1) applied to } \Sigma' \text{ with } A = \Phi(S^{n-1}) \text{ and } B = \partial\Sigma, \text{ here we use that } N \text{ is also a regular neighborhood in } \Sigma' \end{array}$$

■

Proof of the Alexander Duality–Boundary Theorem 217.16 (2b). We now assume that $X \subsetneq \Sigma$, that $X \neq \varnothing$, that Σ is connected and that $\mathrm{H}_{n-1}(\Sigma,\partial\Sigma) = 0$. We need to show that there exists an isomorphism

$$\mathrm{H}_{n-1}(\Sigma\setminus X, \partial\Sigma\setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X;\mathbb{Z}).$$

In the proof of (2a) we saw that we can pick an embedding $\Phi\colon \overline{B}^n \to \Sigma \setminus (N \cup \partial\Sigma)$ such that $\Phi(\overline{B}^n)$ is a codimension-zero submanifold. We set again $\Sigma' \coloneqq \Sigma \setminus \Phi(B^n)$. First note that

$$\mathrm{H}_{n-1}(\Sigma' \setminus X, \partial\Sigma \setminus X) \underset{\uparrow}{\cong} \mathrm{H}^0(X;\mathbb{Z}).$$

this follows from (1) applied to Σ' with $A = \partial\Sigma$ and $B = \Phi(S^{n-1})$, here we use that N is also a regular neighborhood in Σ'

Next note that we have a long exact sequence

$$\to \underbrace{\mathrm{H}_n(\Sigma \setminus X, \Sigma' \setminus X)}_{\cong \mathrm{H}_n(\overline{B}^n, S^{n-1}) \cong \mathbb{Z}} \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\Sigma' \setminus X, \partial\Sigma \setminus X) \to \mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \to \underbrace{\mathrm{H}_j(\Sigma \setminus X, \Sigma' \setminus X)}_{\cong \mathrm{H}_{n-1}(\overline{B}^n, S^{n-1}) = 0}$$

We make the following observations:

(1) For $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\Sigma \setminus X, \Sigma' \setminus X) \underset{\uparrow}{\cong} \mathrm{H}_j(\overline{B}^n, S^{n-1}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = n, \\ 0, & \text{else.} \end{cases}$$

follows from the Manifold Excision Theorem 148.3

follows from the Reduced LES-of-Pair Proposition 198.19 and the discussion on pages 3819 and 3830

(2) First recall that on page 4160 we showed that $\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X)$ is isomorphic to $\mathrm{H}_{n-1}(\Sigma \setminus \mathring{N}, \partial\Sigma \setminus \mathring{N})$. Next note that it follows from the Homology-rel-Partial Boundary Corollary 210.5 that the latter group is free abelian.

(3) It follows from the Manifold-Top-H_*-Theorem 193.1 and the fact $X \neq \varnothing$ that we have $\mathrm{H}_n(\Sigma \setminus X, \partial\Sigma \setminus X) = 0$.

(4) It follows from the above long exact sequence, (1), (2), (3), the Free Abelian Group–Splitting Lemma 97.11 and the Splitting Lemma 145.5 that there exists an isomorphism

$$\mathrm{H}_{n-1}(\Sigma' \setminus X, \partial\Sigma \setminus X) \cong \mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \oplus \mathbb{Z}.$$

(5) Since we assume that $X \neq \varnothing$ we obtain from the Reduced Cohomology Lemma 198.17 (0) that $\mathrm{H}^0(X;\mathbb{Z}) \cong \widetilde{\mathrm{H}}^0(X;\mathbb{Z}) \oplus \mathbb{Z}$.

(6) It follows from (4), (5) and the Cancellation Proposition 97.26 (which we can apply by the Submanifold Complement–Invariants Lemma 217.5) that there exists an isomorphism $\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \cong \widetilde{\mathrm{H}}^0(X;\mathbb{Z})$. ∎

217.6. **Appendix: Embeddings of non-orientable topological manifolds.** In the Smooth Codimension-One Orientability Proposition 51.17 we used the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9 to show that if M is a closed non-orientable n-dimensional smooth manifold, then there is no smooth embedding of M into $\mathbb{R}^{n+1}$.

In the next chapter we will prove the more fancy General Alexander Duality Theorem 218.2 which will allow us to prove the following strengthening of the above non-embedding result:

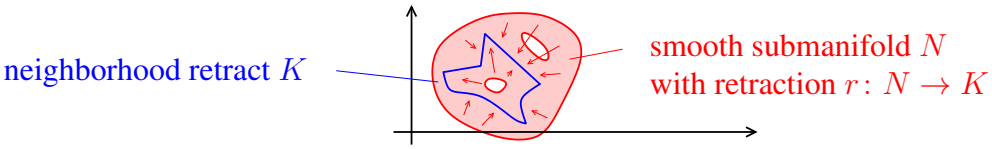
Proposition 218.9. (Topological Codimension-One Orientability Proposition) Let X be a closed n-dimensional topological manifold. If X is non-orientable, then there is no *topological* embedding of X into $\mathbb{R}^{n+1}$.

In this appendix we use a little trick to reduce, for the impatient reader, the proof of the Topological Codimension-One Orientability Proposition 218.9 to the Smooth Codimension-One Orientability Proposition 51.17. In the author's eyes, this approach is esthetically

inferior to the later, technologically more advanced proof. Thus we hide this proof in an appendix.

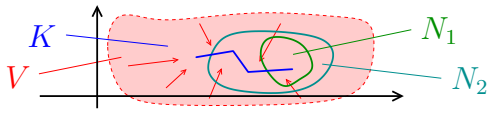
The proof of the Topological Codimension-One Orientability Proposition 218.9 which we provide in this appendix relies on the following lemma, which can be viewed as a variation on the Topological Manifold-Simplicial Complex Retract Theorem 189.13.

Lemma 217.17. (Retract-of-Smooth Manifold Lemma)[2320] Let $K \subseteq \mathbb{R}^m$ be a compact subspace. If K is a neighborhood retract, i.e. if K is the retract of an open neighborhood of $K \subseteq \mathbb{R}^m$, then K is the retract of a compact m-dimensional smooth submanifold of $\mathbb{R}^m$.



Proof. Let K be a compact subspace of $\mathbb{R}^m$ that admits an open neighborhood V together with a retraction $r\colon V \to K$. Since V is an open subset of $\mathbb{R}^m$ it is in particular an m-dimensional smooth manifold. Therefore we can apply the Compact Submanifold Exhaustion Proposition 38.2 to V and we obtain a sequence $N_1, N_2, \ldots$ of subspaces of V with the following three properties:

(a) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $N_i \subseteq N_{i+1}$.
(b) Each N_i is a compact m-dimensional smooth submanifold of V.
(c) We have $\bigcup_{i\in\mathbb{N}} \mathring{N}_i = V$ where $\mathring{N}_i := N_i \setminus \partial N_i$.



We make the following observations:

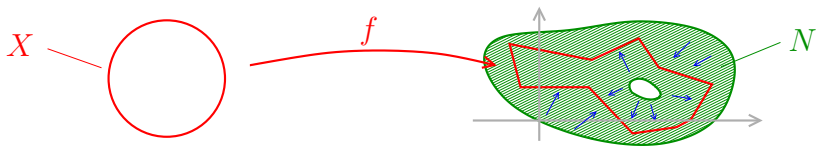
(i) By the Codimension-Zero Submanifold Proposition 25.49 we know that each $\mathring{N}_i$ is an open subset of V and thus also of $\mathbb{R}^m$.
(ii) It follows from (a), (c) and (i), the hypothesis that K is compact and the Compact Subspace Covering Lemma 3.38 that there exists an $i \in \mathbb{N}$ such that $K \subseteq N_i$.
(iii) Since V is an open subset of $\mathbb{R}^m$ every smooth submanifold of V is in fact also a smooth submanifold of $\mathbb{R}^m$.

It follows from this discussion and Property (ii) that the smooth manifold N_i together with the map $r|_{N_i}\colon N_i \to K$ has all the desired properties. ■

Proof of the Topological Codimension-One Orientability Proposition 218.9. In the following let X be a closed n-dimensional topological manifold which admits an embedding $f\colon X \to \mathbb{R}^{n+1}$. We have to show that X is orientable.

First note that it follows from Borsuk's Theorem 189.7 together with the Retract-of-Smooth Manifold Lemma 217.17 (1) that there exists a compact $(n+1)$-dimensional smooth submanifold N of $\mathbb{R}^{n+1}$ that contains $f(X)$ and such that there exists a retraction $r\colon N \to f(X)$. In the following we denote by $i\colon f(X) \to N$ the inclusion map.

[2320] The lemma is also the content of Exercise 189.7.



We now consider the following sequence of homomorphisms:

$$\mathrm{H}^n(X;\mathbb{Z}) \xleftarrow[f^*]{\cong} \mathrm{H}^n(f(X);\mathbb{Z}) \xhookrightarrow{r^*} \mathrm{H}^n(N;\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus N).$$

We make the following clarifications and observations:

(1) Since $f\colon X \to f(X)$ is a homeomorphism we see that the leftmost map is an isomorphism.
(2) Note that $r \circ i = \mathrm{id}_{f(X)}$. It follows that $i^* \circ r^* = (r \circ i)^* = (\mathrm{id}_{f(X)})^* = \mathrm{id}_{\mathrm{H}^n(f(X);\mathbb{Z})}$. Thus we see that the map $r^*\colon \mathrm{H}^n(f(X);\mathbb{Z}) \to \mathrm{H}^n(N;\mathbb{Z})$ is a monomorphism.
(3) One can easily show, using the Codimension-Zero Submanifold Proposition 25.49 and the Smooth Collar Neighborhood Theorem 35.6, that N admits a regular neighborhood (see also Exercise 217.3.) Thus it follows from the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9 that there is an isomorphism $\mathrm{H}^n(N;\mathbb{Z}) \cong \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus N)$.

Note that it follows in particular from the H_0-Proposition 143.25 together with the Reduced Homology Lemma 147.3 (0) that $\widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus N;\mathbb{Z})$ is a free abelian group. It follows from the above that $\mathrm{H}^n(X;\mathbb{Z})$ is a torsion-free abelian group. Finally note that it now follows from the Manifold-Top-H^*-Theorem 201.26 (2) that the closed n-dimensional topological manifold X is orientable. ■

217.7. **Appendix: Complements of knots in arbitrary 3-manifolds.** In the Knot Complement-H_*-Proposition 185.29 we calculated the homology groups of the complements of a knot $K \subseteq S^3$. In this appendix we will consider complements of knots in arbitrary 3-manifolds. We will not really make use of these results. But since the author tends to get confused easily at that point, it is perhaps best to write this up once and for all:

Proposition 217.18. (Manifold–Knot Complement-H_*-Proposition) Let M be a closed orientable connected 3-dimensional smooth manifold and let $K \subseteq M$ be an oriented knot in the sense of Definition 108.3, i.e. K is a closed oriented 1-dimensional smooth submanifold of M that is diffeomorphic to S^1. We denote by μ_K a meridian of K which we define exactly as in Definition 109.1.

(1) (a) If $\mathrm{H}_1(K) \to \mathrm{H}_1(M)$ is injective, then we set

$$d \;:=\; \max\{m \in \mathbb{N} \,|\, [K] = m \cdot \alpha \text{ for some } \alpha \in \mathrm{H}_1(M)\}.$$

(b) If $\mathrm{H}_1(K) \to \mathrm{H}_1(M)$ is not injective, then we set $d := 0$.

With this notation the following sequence is well-defined and exact:

$$0 \;\to\; \mathbb{Z}/d\mathbb{Z} \xrightarrow{[n] \mapsto n \cdot [\mu_K]} \mathrm{H}_1(M \setminus K) \xrightarrow{\iota_*} \mathrm{H}_1(M) \;\to\; 0.$$

(2) We have $\mathrm{H}_2(M \setminus K) \cong \mathbb{Z}^{\mathrm{rank}(\mathrm{H}_1(M \setminus K))-1}$.

As we will see shortly, even in the calculation of integral homology groups it can be convenient to have the following field-coefficient version of the above proposition at our disposal:

Proposition 217.19. (Manifold–Knot Complement-H_*-Field-Proposition) Let M be a closed orientable connected 3-dimensional smooth manifold and let $K \subseteq M$ be an oriented knot. We denote by μ_K a meridian of K. Finally let $\mathbb{F}$ be a field.

(1) (a) If $\mathrm{H}_1(K;\mathbb{F}) \to \mathrm{H}_1(M;\mathbb{F})$ is injective, then $\mathrm{H}_1(M \setminus K;\mathbb{F}) \to \mathrm{H}_1(M;\mathbb{F})$ is an isomorphism.

(b) If $\mathrm{H}_1(K;\mathbb{F}) \to \mathrm{H}_1(M;\mathbb{F})$ is not injective, then we have a short exact sequence

$$0 \to \mathbb{F} \xrightarrow{f \mapsto f \cdot [\mu_K]} \mathrm{H}_1(M \setminus K;\mathbb{F}) \xrightarrow{\iota_*} \mathrm{H}_1(M;\mathbb{F}) \to 0.$$

(2) We have $\dim_{\mathbb{F}}(\mathrm{H}_2(M \setminus K;\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_1(M \setminus K;\mathbb{F})) - 1$.

Example.

(1) Let M be a closed oriented connected 3-dimensional smooth manifold and let $K \subseteq M$ be an oriented knot that is null-homologous, i.e. $[K] = 0 \in \mathrm{H}_1(M)$. It follows from the Manifold–Knot Complement-H_*-Proposition 217.18 (1b) that we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot [\mu_K]} \mathrm{H}_1(M \setminus K) \xrightarrow{\iota_*} \mathrm{H}_1(M) \to 0.$$

If we apply this to $K = S^3$ we recover the result from the Knot Complement-H_*-Proposition 185.29.

(2) We consider $M = S^1 \times S^2$. Let $K \subseteq S^1 \times S^2$ be a knot that represents the homology class $d \cdot [S^1 \times \{*\}]$ for some $d \in \mathbb{Z}$. It follows from the Manifold–Knot Complement-H_*-Proposition 217.18 that we have a short exact sequence

$$0 \to \mathbb{Z}/d\mathbb{Z} \xrightarrow{[n] \mapsto n \cdot [\mu_K]} \mathrm{H}_1((S^1 \times S^2) \setminus K) \xrightarrow{\iota_*} \underbrace{\mathrm{H}_1(S^1 \times S^2)}_{\cong \mathbb{Z}} \to 0.$$

Since the group to the right is free abelian we obtain from the Free Abelian Group–Splitting Lemma 97.11 and the Splitting Lemma 145.5 that we have an isomorphism $\mathrm{H}_1((S^1 \times S^2) \setminus K) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$. We will illustrate this observation in Exercise 217.24.

(3) We consider $M = \mathbb{R}\mathrm{P}^3$ and a knot K that represents the unique non-zero homology class of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}/2\mathbb{Z}$.[2321] The Manifold–Knot Complement-H_*-Proposition 217.18 (1b) gives us a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{[n] \mapsto n \cdot [\mu_K]} \mathrm{H}_1(\mathbb{R}\mathrm{P}^3 \setminus K) \xrightarrow{\iota_*} \underbrace{\mathrm{H}_1(\mathbb{R}\mathrm{P}^3)}_{\cong \mathbb{Z}/2\mathbb{Z}} \to 0.$$

Unfortunately this result is not enough to pinpoint the isomorphism type of the middle homology group. More precisely, a priori there are two possibilities:

(i) $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3 \setminus K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, or

(ii) $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3 \setminus K) \cong \mathbb{Z}$.

To settle this issue we appeal to the Manifold–Knot Complement-H_*-Field-Proposition 217.19 with $\mathbb{F} = \mathbb{F}_2$. It follows from our hypothesis and the Universal Coefficient Theorems 162.16 that the inclusion induced map $\mathrm{H}_1(K;\mathbb{F}_2) \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^3;\mathbb{F}_2)$ is actually an isomorphism. Thus we obtain from the Manifold–Knot Complement-H_*-Field-Proposition 217.19 (1a) that $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3 \setminus K;\mathbb{F}_2) \cong \mathbb{F}_2$. By the Universal Coefficient Theorems 162.16 this is only consistent with the above option (ii). □

[2321] For example we could take the image of $\mathbb{R}\mathrm{P}^1$ under the standard embedding into $\mathbb{R}\mathrm{P}^3$.

Proof of the Manifold–Knot Complement-H$_*$-Proposition 217.18 (1). Let M be a closed orientable connected 3-dimensional smooth manifold and let $K \subseteq M$ be a closed oriented 1-dimensional smooth submanifold that is diffeomorphic to S^1. We start out with the following claim, which translates our desired homological sequence into a cohomological sequence:

Claim 1. There exists an isomorphism $\Theta\colon \mathrm{H}^2(M, K; \mathbb{Z}) \to \mathrm{H}_1(M \setminus K)$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\overbrace{\mathrm{H}^1(K;\mathbb{Z})}^{=\mathbb{Z}\cdot[K]^*} & \xrightarrow{\delta_1} & \mathrm{H}^2(M,K;\mathbb{Z}) & \xrightarrow{\text{я}} & \mathrm{H}^2(M;\mathbb{Z}) & \longrightarrow & 0 \\
\cong\downarrow{\scriptstyle [K]^*\mapsto -[\mu_K]} & & \cong\downarrow{\scriptstyle \Theta} & & \cong\downarrow{\scriptstyle -\cap[M]} & & \\
\mathbb{Z}\cdot[\mu_K] & \longrightarrow & \mathrm{H}_1(M\setminus K) & \xrightarrow{\iota_*} & \mathrm{H}_1(M) & \longrightarrow & 0.
\end{array}$$

Here the top sequence is an excerpt from the long exact sequence in cohomology of the pair (M, K). Note that it follows from the Poincaré Duality Theorem 210.1 that the right vertical map is an isomorphism.

Proof. Since M is orientable we know, by the Product Tubular Neighborhood–Existence Theorem 58.4 (2b), that we can pick a tubular map $\tau\colon \overline{B}^2 \times K \to M$ for $K \subseteq M$. We write $N_K := \tau(\overline{B}^2 \times K)$ and $X_K := M \setminus \tau(B^2 \times K)$. Note that it follows from the same argument as in the Meridian Proposition 109.2 that $\mu_K := \tau(S^1 \times \{*\})$ is the essentially unique meridian of K.

We consider the following more elaborate diagram:[2322]

$$\begin{array}{ccccccccc}
0 & & & & 0 & & 0 & & \\
\uparrow & & & & \uparrow & & \uparrow & & \\
\mathrm{H}_1(M) & \xrightarrow{\mathrm{id}} & \mathrm{H}_1(M) & \xleftarrow[\cong]{\cap[M]} & \mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{\mathrm{id}} & \mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{\mathrm{id}} & \mathrm{H}^2(M;\mathbb{Z}) \\
\iota_*\uparrow & & \iota_*\uparrow & & & & \text{я}\uparrow & & \text{я}\uparrow \\
\mathrm{H}_1(M\setminus K) & \xleftarrow[\cong]{\iota_*} & \mathrm{H}_1(X_K) & \xleftarrow[\cong]{\cap[X_K]} & \mathrm{H}^2(X_K,\partial X_K;\mathbb{Z}) & \xleftarrow[\iota^*]{\cong} & \mathrm{H}^2(M,N_K;\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{H}^2(M,K;\mathbb{Z}) \\
\uparrow & & \iota_*\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\mathbb{Z}\cdot[\mu_K] & \longrightarrow & \mathrm{H}_1(\partial X_K) & \xleftarrow[\cong]{\cap[\partial X_K]} & \mathrm{H}^1(\partial X_K;\mathbb{Z}) & \xleftarrow{\iota^*_{\partial X_K}} & \mathrm{H}^1(N_K;\mathbb{Z}) & \xrightarrow[\cong]{\iota_K^*} & \mathrm{H}^1(K;\mathbb{Z}). \\
 & & {\scriptstyle [\mu_K]} & & {\scriptstyle p^*([K]^*)} & & & & {\scriptstyle [K]^*}
\end{array}$$

$$-[\mu_k] \mapsfrom [K]^*$$

We make the following clarifications and observations:

(a) For lack of imagination all inclusions are called ι.

(b) The inclusion $\iota\colon K \to N_K$ is clearly a homotopy equivalence. It follows from the Homotopy Equivalence-H*-Proposition 198.12 (1) together with the Homotopy Equivalence-Relative-H*-Corollary 198.14 that the two induced maps on cohomology, to the right of the diagram, are isomorphisms.

(c) It follows from the Excision-in-H*-Theorem 200.1 (1a) that the inclusion induced map $\iota^*\colon \mathrm{H}^2(M, N_K; \mathbb{Z}) \to \mathrm{H}^2(X_K, \partial X_K; \mathbb{Z})$ is an isomorphism.

[2322]Unfortunately for space reasons we had to rotate the diagram from the claim by 90 degrees.

(d) It follows from the Poincaré Duality Theorem 210.1 that the three horizontal maps from the third to the second column are isomorphisms.
(e) It follows from the Small H_*-to-Large H^*-PD Lemma 212.8 that the two maps from $\mathrm{H}^2(X_K, \partial X_K; \mathbb{Z})$ to $\mathrm{H}_1(M)$ are the same. In other words, the big rectangle in the top center commutes.
(f) It follows from the Poincaré Duality–Boundary Proposition 211.14 that the bottom middle square (involving cap products) commutes.
(g) It follows easily from the definitions, the functoriality of cohomology and the naturality of connecting homomorphisms that all other squares also commute.
(h) We denote by $p\colon \tau(S^1 \times K) \to K$ the obvious projection. Note that the following little diagram commutes up to homotopy:

$$\begin{array}{ccc} \partial X_K & \xrightarrow{\quad p \quad} & K \\ & {\scriptstyle \iota_{\partial X_K}} \searrow \quad \swarrow {\scriptstyle \iota_K} & \\ & N_K. & \end{array}$$

By the Homotopy-H^*-Proposition 198.11 we see that the induced diagram on cohomology groups commutes. This implies that $\iota_K^*([K]^*) = \iota_{\partial X_K}^*(p^*([K]^*))$.
(i) The Fundamental Class-Product Proposition 207.2 says that $p^*([K])^* \cap [\partial X_K] = -[\mu_K]$.
(j) It follows from (h) and (i) that the bottom part of the diagram, involving the long horizontal map, commutes. More precisely, the two maps from $\mathrm{H}^1(K; \mathbb{Z})$ to $\mathrm{H}^1(\partial X_K; \mathbb{Z})$ agree. But then the maps from $\mathrm{H}^1(K; \mathbb{Z})$ to $\mathrm{H}^2(X_K, \partial X_K; \mathbb{Z})$ agree.

It follows from the above discussion that the map horizontal sequence of maps defines an isomorphism $\Theta\colon \mathrm{H}^2(M, K; \mathbb{Z}) \to \mathrm{H}_1(M \setminus K)$.[2323] Unless the author overlooked something the above discussion should show that Θ has all the desired properties. ⊞

Before we do anything else let, let us recall the following definition:

(a) If $\mathrm{H}_1(K) \to \mathrm{H}_1(M)$ is injective, then we set

$$d \;:=\; \max\{m \in \mathbb{N} \,|\, [K] = m \cdot \alpha \text{ for some } \alpha \in \mathrm{H}_1(M)\}.$$

(b) If $\mathrm{H}_1(K) \to \mathrm{H}_1(M)$ is not injective, then we set $d := 0$.

It follows from Claim 1 that it suffices to prove the following claim:

Claim 2. That the following sequence is well-defined and exact:

$$0 \;\to\; \mathbb{Z}/d\mathbb{Z} \xrightarrow{[n] \mapsto n \cdot \delta_1([K]^*)} \mathrm{H}^2(M, K; \mathbb{Z}) \xrightarrow{\text{я}} \mathrm{H}^2(M; \mathbb{Z}) \;\to\; 0.$$

Proof. First of all note that it follows from $\mathrm{H}^2(K; \mathbb{Z}) \cong \mathrm{H}^2(S^1; \mathbb{Z}) = 0$ that the following sequence is exact:

$$\mathrm{H}^1(M; \mathbb{Z}) \xrightarrow{\iota^*} \overbrace{\mathrm{H}^1(K; \mathbb{Z})}^{=\mathbb{Z}\cdot[K]^*} \xrightarrow{\delta_1} \mathrm{H}^2(M, K; \mathbb{Z}) \xrightarrow{\text{я}} \mathrm{H}^2(M; \mathbb{Z}) \;\to\; 0.$$

We see that it suffices to show that $\mathrm{im}(\iota^*\colon \mathrm{H}^1(M; \mathbb{Z}) \to \mathrm{H}^1(K; \mathbb{Z})) = d \cdot \mathbb{Z} \cdot [K]^*$. Next note that by the H^1-Hom-Proposition 201.24 we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(M; \mathbb{Z}) & \xrightarrow{\quad \iota^* \quad} & \mathrm{H}^1(K; \mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{Z}) & \xrightarrow{\quad \iota^* \quad} & \mathrm{Hom}(\mathrm{H}_1(K), \mathbb{Z}) \end{array}$$

[2323] Note that the bottom horizontal maps are *not* isomorphisms.

where the vertical isomorphisms are given by the evaluation maps and where the bottom horizontal hand map is induced by $\iota_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(M)$. We now consider the two cases separately:

(a) If $\mathrm{H}_1(K) \to \mathrm{H}_1(M)$ is injective, then it follows from an elementary argument, see Exercise 97.11, that we can write $\mathrm{H}_1(M) = \mathbb{Z} \cdot v \oplus H$ where $\iota_*(\mathrm{H}_1(K)) = d \cdot \mathbb{Z} \cdot v$. The desired statement now follows from elementary observations.

(b) If $\mathrm{H}_1(K) \to \mathrm{H}_1(M)$ is not injective, then the image of $\mathrm{H}_1(K)$ is a torsion subgroup of $\mathrm{H}_1(M)$. But this means that any homomorphism $\varphi\colon \mathrm{H}_1(M) \to \mathbb{Z}$ vanishes on $\iota_*(\mathrm{H}_1(K))$. Thus $\iota^*\colon \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_1(K), \mathbb{Z})$ is the zero map. ■

Proof of the Manifold–Knot Complement-H$_*$-Proposition 217.18 (2). We still need to show that $\mathrm{H}_2(M \setminus K) \cong \mathbb{Z}^{\mathrm{rank}(\mathrm{H}_1(M \setminus K))-1}$. We continue with the notation from (1). The desired statement follows from the following "standard" observations:

- It follows from the Basics-of-Tubular Neighborhood Lemma 57.10 (3a) together with the Homotopy Equivalence-H$_*$-Proposition 144.14 that the inclusion induces an isomorphism $\mathrm{H}_2(X_K) \to \mathrm{H}_2(M \setminus K)$.
- By the Manifold-H$_{n-1}$-Theorem 193.4 (2) we know that $\mathrm{H}_2(X_K)$ is a free abelian group.
- We have the following equalities:

$$\chi(X_K) \underset{\substack{\uparrow \\ \text{by the Odd-Dimensional} \\ \text{Manifold-}\chi\text{-Proposition 210.9}}}{=} \tfrac{1}{2} \cdot \chi(\partial X_K) = \tfrac{1}{2} \cdot \chi(\text{torus } S^1 \times S^1) = 0.$$

- We have

$$\chi(X_K) \underset{\substack{\uparrow \\ \text{by the Homology-}\chi\text{-Proposition 156.2 and} \\ \text{the Manifold-Top-H}_*\text{-Theorem 193.1}}}{=} \sum_{i=0}^{2} (-1)^i \cdot \mathrm{rank}(\mathrm{H}_i(X_K)) = \mathrm{rank}(\mathrm{H}_2(X_K)) - \mathrm{rank}(\mathrm{H}_1(X_K)) + \underbrace{\mathrm{rank}(\mathrm{H}_0(X_K))}_{=1}.$$

■

Proof of the Manifold–Knot Complement-H$_*$-Field Proposition 217.19. This proof is very similar to the proof of the Manifold–Knot Complement-H$_*$-Proposition 217.18. Throughout the proof we just need to replace the statements about $\mathbb{Z}$-coefficients by statements about $\mathbb{F}$-coefficients. For example we need to replace the H^1-Hom-Proposition 201.24 by the Universal Coefficient-H*–Field Theorem 202.2. We leave it to the reader to look up, or formulate, all the various field coefficient statements. ■

217.8. Exercises for Chapter 217 .

Exercise 217.1. Show that the subspace $K := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ does not admit a regular neighborhood.

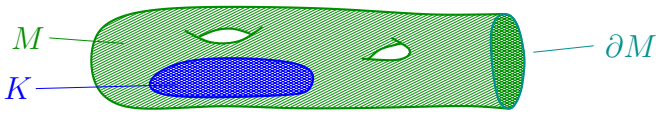
Exercise 217.2.

(a) Show that there exists a compact subspace $X \subseteq \overline{B}^2$ such that $\mathrm{H}_1(\overline{B}^2 \setminus X)$ is uncountable.

(b) Show that there exists a compact subspace $X \subseteq \overline{B}^3$ such that $\mathrm{H}_2(\overline{B}^3 \setminus X)$ is uncountable.

Exercise 217.3. Let M be a smooth manifold and let $K \subseteq M$ be a compact codimension-zero submanifold of M with $\partial K \cap \partial M = \varnothing$. Show that K admits a regular neighborhood.

Hint. Use the Codimension-Zero Submanifold Proposition 25.49 (3) and the Smooth Collar Neighborhood Theorem 35.6.



Exercise 217.4. We consider the spatial graphs $A \subseteq \mathbb{R}^3$ and $B \subseteq \mathbb{R}^3$ that are shown in the figure below.

(a) (i) Determine a presentation for $\pi_1(\mathbb{R}^3 \setminus A)$.
(ii) What is the abelianization of $\pi_1(\mathbb{R}^3 \setminus A)$?
(b) Replace A by B and again solve (i) and (ii).

Remark. This exercise is identical to Exercise 113.21.



Exercise 217.5.

(a) We say that two knots $K_0, K_1 \subseteq \mathbb{R}^3$ are *equivalent* if there exists a smooth isotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ with $F_0(S^1) = K_0$ and $F_1(S^1) = K_1$. Define a notion of what it means for two connected non-empty spatial graphs $G_0, G_1 \subseteq \mathbb{R}^3$ to be equivalent, which corresponds to the above notion of equivalence of knots.
Remark. See the Link Equivalence Theorem 126.5 for the discussion of different notions of equivalence for knots and links.
(b) Show that the two spatial graphs shown in Exercise 217.4 are not equivalent.

Remark. Most likely one has to take a few technical liberties to make things work and to preserve one's sanity.

Exercise 217.6. Let X and Y be two topological spaces and let $n \in \mathbb{N}_0$ such that the following statements hold:

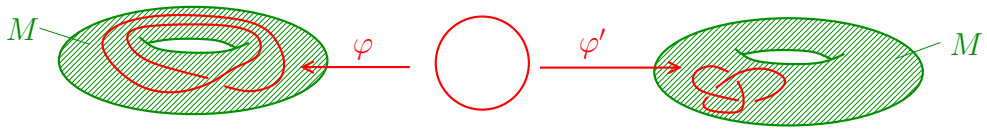
(1) All homology groups of X and Y are finitely generated.
(2) For every $j \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_j(X) \cong \mathrm{H}^{n-j}(Y;\mathbb{Z})$.

Show that for every $j \in \mathbb{N}_0$ there exists also an isomorphism $\mathrm{H}^j(X;\mathbb{Z}) \cong \mathrm{H}_{n-j}(Y)$.

Exercise 217.7. Let M be an n-dimensional smooth manifold and let $\varphi\colon S^1 \to M \setminus \partial M$ be a smooth embedding. By the Smooth Embedding Theorem 30.15 we know that $\varphi(S^1)$ is a smooth submanifold that is diffeomorphic to S^1.

(a) We suppose that $\mathrm{H}_2(M) = 0$ and that $\mathrm{H}_1(M)$ is torsion-free. (For example M could be the solid torus $\overline{B}^2 \times S^1$.) Compute the isomorphism types of the homology groups of $M \setminus \varphi(S^1)$.
(b) Under the hypothesis of (a) show that the isomorphism type of $\mathrm{H}_*(M \setminus \varphi(S^1))$ does not depend on the choice of the smooth embedding φ.

(c) Provide examples that show that in (b) the hypothesis $\mathrm{H}_2(M) = 0$ cannot be dropped and provide examples that show that the hypothesis that $\mathrm{H}_1(M)$ is torsion-free cannot be dropped.



Exercise 217.8. Let $n \in \mathbb{N}$. Prove the following two statements:

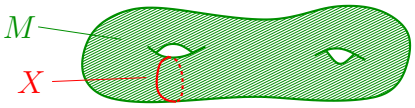
(a) There is no smooth embedding of the complex projective space $\mathbb{C}\mathrm{P}^n$ into $\mathbb{R}^{2n+1}$.

(b) There is no smooth embedding of the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ into $\mathbb{R}^{4n+1}$.

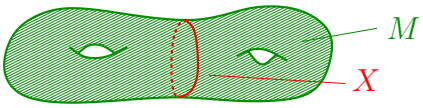
Exercise 217.9. Let M be a closed smooth manifold and let $X \subseteq M$ be an orientable connected codimension-one submanifold that is closed a smooth submanifold. Show that the following two statements are equivalent:

(1) There exists compact submanifold A of M with $\partial A = X$.

(2) The fundamental class $[X]$ represents the zero element in $\mathrm{H}_{n-1}(M)$.



Case 1: $[X] \neq 0 \in \mathrm{H}_1(M)$



Case 1: $[X] = 0 \in \mathrm{H}_1(M)$

Exercise 217.10. By the Surface-No Smooth Embedding Corollary 51.18 we know that there is no smooth embedding of the Klein bottle K into $\mathbb{R}^3$. Does there exist any orientable 3-dimensional smooth manifold M such that K admits a smooth embedding $K \to M$?

Exercise 217.11. We consider the trivial link $U \subseteq S^3$ and the Hopf link $H \subseteq S^3$ that are shown in the figure below.

(a) Determine closed oriented submanifolds of $S^3 \setminus U$ which represent a basis for the homology groups of $S^3 \setminus U$.

(b) Repeat (a) for the Hopf link.



Exercise 217.12. Let Σ be a 3-dimensional rational homology sphere. More precisely, let Σ be a 3-dimensional smooth manifold such that $\mathrm{H}_i(\Sigma;\mathbb{Q}) \cong \mathrm{H}_i(S^3;\mathbb{Q})$ for all $i \in \mathbb{N}_0$. Furthermore let $K \subseteq \Sigma$ be a knot, i.e. K is a closed connected non-empty 1-dimensional submanifold of Σ that is diffeomorphic to S^1. We denote by $i\colon K \to \Sigma$ the inclusion.

(a) Show that $\mathrm{H}_1(\Sigma \setminus K) \cong \mathbb{Z} \oplus \operatorname{coker}(i_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(\Sigma))$.
Hint. Modify the proof and statement of Claim on page 4150 and apply the Universal Coefficient-H*-Theorem 201.23.

(b) Determine the isomorphism type of the remaining homology groups of $\Sigma \setminus K$.

(c) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We consider the corresponding lens space $L(p,q)$. What are the possible homology groups of knot complements in $L(p,q)$?

Exercise 217.13. Let M be a closed oriented connected 4-dimensional smooth manifold with $\mathrm{H}_1(M) = 0$ and let $\Sigma \subseteq M$ be a closed oriented connected non-empty 2-dimensional smooth submanifold. We suppose that there exists a $d \in \mathbb{N}$ and a primitive element $\varphi \in \mathrm{H}_2(M)$ such that $[\Sigma] = d \cdot \varphi$. Show that $\mathrm{H}_1(M \setminus \Sigma)$ has a summand that is isomorphic to $\mathbb{Z}/d\mathbb{Z}$.
Hint. Use the full tool box of tubular neighborhoods, the Poincaré Duality Theorem 210.1, excision in homology and or cohomology, long exact sequences of pairs, the Universal Coefficient-H*-Theorem 201.23, and so on.

Exercise 217.14. Let $p, q \in \mathbb{N}$ be coprime with $p \geq 2$. We consider the corresponding lens space $L(p,q) = S^3/\sim$, that we originally introduced in Definition 91.4. Let t be a prime that divides p.

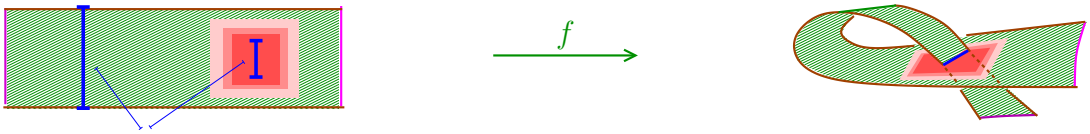
(a) Determine the cohomology ring of $L(p,q)$ with $\mathbb{Z}/t\mathbb{Z}$-coefficients.
(b) Show that there is no smooth embedding of $L(p,q)$ into S^4.

Exercise 217.15.

(a) Let M and N be compact connected non-empty k-dimensional smooth manifolds. (If both M and N are orientable we assume that they are both oriented.) Suppose that there exist smooth embeddings $f\colon M \to S^n$ and $g\colon N \to S^n$. Show that there exists a smooth embedding $h\colon M\#N \to S^n$ of the connected sum $M\#N$ into $\mathbb{R}^n$.
Remark. This exercise is identical to Exercise 54.14.
(b) We suppose that $k \leq n-2$. What can you say about $\pi_1(S^n \setminus h(M\#N))$ in terms of $\pi_1(S^n \setminus f(M))$ and $\pi_1(S^n \setminus g(N))$?

Exercise 217.16. Using a specific color, e.g. "red" we can draw 4-dimensional pictures, where "redness" indicates the fourth dimension. For example in the figure below we show a map $f\colon [0,4] \times [0,1] \to \mathbb{R}^3$ which is not injective, but if we use the "redness" function $r\colon [0,4] \times [0,1] \to \mathbb{R}_{\geq 0}$, then the map $(f,r)\colon [0,4] \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ is injective.

(a) Illustrate an embedding of the real projective plane $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^4$.
(b) Illustrate an embedding of the Klein bottle into $\mathbb{R}^4$.
(c) Let $k \in \mathbb{N}$. Illustrate an embedding of the non-orientable surface N_k of genus k into $\mathbb{R}^4$. (We refer to Definition 9.19 for the definition of N_k.)



images in $\mathbb{R}^3$ agree, but they differ in "redness" (fourth dimension)

Exercise 217.17. We denote by $\iota\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ the smooth embedding that is induced by the smooth embedding $\mathbb{R}^3 \setminus \{0\} \to \mathbb{C}^3 \setminus \{0\}$.

(a) Show that $\mathbb{C}\mathrm{P}^2 \setminus \iota(\mathbb{R}\mathrm{P}^2)$ is simply connected.
Hint. You could look at the usual CW-decomposition of $\mathbb{C}\mathrm{P}^2$ and consider in each skeleton the effect of removing $\iota(\mathbb{R}\mathrm{P}^2)$.

In the following we equip $\mathbb{C}\mathrm{P}^2$ with the involution $\rho\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ that is given by $\rho([z_0 : z_1 : z_2]) = [\overline{z}_1 : \overline{z}_2 : \overline{z}_3]$. Evidently the fixed point set of ρ is given by $\iota(\mathbb{R}\mathrm{P}^2)$. Next

we consider the quotient space $\mathbb{C}\mathrm{P}^2/P \sim \rho(P)$. The Arnold–Kuiper–Massey Theorem (see e.g. [**Hill2017, Kui1974, Mass1973, Arn1988, Sea2006**]) says that there exists a homeomorphism $f\colon \mathbb{C}\mathrm{P}^2/\sim \to S^4$ such that the map $f \circ \iota\colon \mathbb{R}\mathrm{P}^2 \to S^4$ is a smooth embedding.

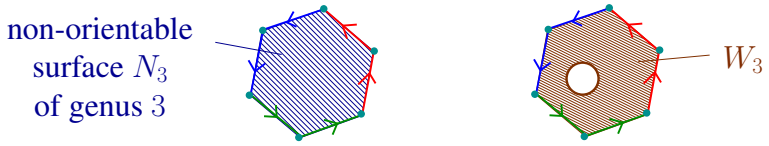
(b) Show that $\pi_1(S^4 \setminus (f \circ \iota)(\mathbb{R}\mathrm{P}^2)) \cong \mathbb{Z}/2\mathbb{Z}$.

(c) Show for any $k \in \mathbb{N}$ the non-orientable surface N_k of genus k admits a smooth embedding $f\colon N_k \to S^4$ such that $\pi_1(S^4 \setminus (f \circ \iota)(\mathbb{R}\mathrm{P}^2)) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark. The above smooth embedding $\mathbb{R}\mathrm{P}^2 \to S^4$ is called the *standard positive embedding.*. A (different?) embedding is described in the Real Projective Space-Smooth Embedding Proposition 30.19.

Exercise 217.18. For $k \in \mathbb{N}$ let N_k be the non-orientable surface of genus k. We define W_k to be N_k minus one open disk. For which $k \in \mathbb{N}$ does there exist a smooth embedding $W_k \to \mathbb{R}^3$?

Remark. Recall that $N_1 = \mathbb{R}\mathrm{P}^2$ and that N_2 is the Klein bottle.



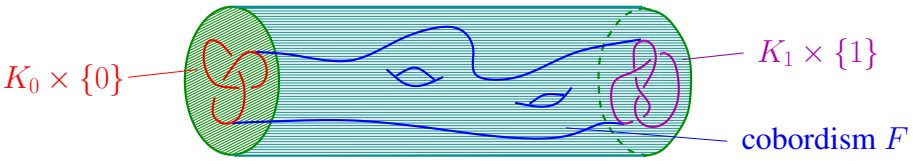
Exercise 217.19. Use the Alexander Duality–Boundary Theorem 217.16 to prove the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 217.9 without making use of the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1. So you are supposed to show that if X is a compact non-empty subspace of $\mathbb{R}^n$, and if X admits a regular neighborhood, then for any $i \in \mathbb{N}_0$ we have an isomorphism

$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X) \;\cong\; \mathrm{H}^{n-i-1}(X;\mathbb{Z}).$$

Exercise 217.20. Let $K_0 \subseteq S^3$ and $K_1 \subseteq S^3$ be knots. A *cobordism* between K_0 and K_1 is a compact oriented connected proper smooth submanifold $F \subseteq S^3 \times [0,1]$ such that $\partial F = (K_0 \times \{0\}) \cup (K_1 \times \{1\})$.

(a) Show that if F is a cobordism between K_0 and K_1, then for $i = 0, 1$ the inclusion-induced maps $\mathrm{H}_1((S^3 \setminus K_i) \times \{i\}) \to \mathrm{H}_1((S^3 \times [0,1]) \setminus F)$ are isomorphisms.

(b) Let F be a cobordism between K_0 and K_1 and let $i \in \{0,1\}$. Complete the following sentence: The inclusion-induced maps $\mathrm{H}_j((S^3 \setminus K_i) \times \{0\}) \to \mathrm{H}_j((S^3 \times [0,1]) \setminus F)$ are all isomorphisms if and only if F is ?...?



Exercise 217.21. Let M be a compact non-orientable n-dimensional smooth manifold. Show that there is no proper smooth embedding $f\colon M \to \overline{B}^{n+1}$. (Here proper just means that $f^{-1}(S^n) = \partial M$.)

Hint. Use the Alexander Duality–Boundary Theorem 217.16.

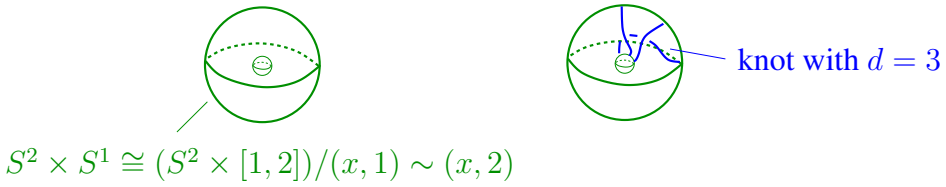
Exercise 217.22.

(a) Does there exist a compact codimension-zero submanifold M of $\mathbb{R}^4$ such that $\mathrm{H}_1(M)$ is non-zero and finite?

(b) Does there exist an open subset M of $\mathbb{R}^4$ such that $\mathrm{H}_1(M)$ is non-zero and finite?

Exercise 217.23. Let $n \in \mathbb{N}_0$ and let $k \in \{0, \ldots, n\}$. Show that $\mathrm{H}_j(\mathbb{R}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^k; \mathbb{F}_2) = 0$ for $j = n - k + 1, \ldots, n$.

Exercise 217.24. Note that $S^2 \times S^1$ is diffeomorphic to $M := (S^2 \times [1,2])/(x,1) \sim (x,2)$. In the figure below we show an oriented knot $K \subseteq M = (S^2 \times [1,2])/(x,1) \sim (x,2)$ that represents the class $d \cdot [\{*\} \times S^1]$ with $d = 3$. By the Manifold–Knot Complement-H_*-Proposition 217.18 we know that $3 \cdot [\mu_K] = 0 \in \mathrm{H}_1(M)$. Give a geometric proof that $3 \cdot [\mu_K] = 0 \in \mathrm{H}_1(M)$. More precisely, find a compact oriented surface $\Sigma \subseteq M$ such that $\partial\Sigma$ (as an oriented 1-dimensional submanifold) consists of three meridians of $K \subseteq M$.



CHAPTER 218

The General Alexander Duality Theorem

In the last chapter we proved the Sphere–Alexander Duality Theorem 217.8 which says that for any compact subspace $X \subsetneq S^n$ of S^n that admits a regular neighborhood (in the sense of Definition 217.2) and any $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X;\mathbb{Z}).$$

This result had many applications, for example we used it to show that there is no smooth embedding of $\mathbb{R}\mathrm{P}^2$ as a smooth manifold into $\mathbb{R}^3$.

In this chapter we will formulate and prove a significant generalization of the Sphere–Alexander Duality Theorem 217.8. In the proof of the more general General Alexander Duality Theorem we will not make use of regular neighborhoods, instead we will make use of the Manifold–Neighborhood Retract Theorem 189.8 (2) (which is a consequence of the tricky Borsuk Theorem 189.7) on weakly locally contractible subspaces of topological manifolds. One of the many applications of the General Alexander Duality Theorem is that there is no embedding of $\mathbb{R}\mathrm{P}^2$ as a topological space into $\mathbb{R}^3$.

This chapter is organized as follows:

- In Section 218.1 we will formulate the General Alexander Duality Theorem 218.2.
- In Section 218.2 we will prove the General Alexander Duality Theorem 218.2.
- In Section 218.3 we will state and prove the Sphere–Alexander Duality Theorem 218.7.
- In Section 218.4 we will discuss wild knots.
- In Section 218.5 we will discuss invariants of complements of subspaces.
- In Appendix 218.6 we will compare the two Sphere–Alexander Duality Theorems 217.8 and 218.7.
- In Appendix **??** we will formulate and prove the General Alexander Duality–Boundary Theorem **??**.
- In Section 218.7 we will provide exercises for Chapter 218.

In the next Chapter 219 we will discuss some "applications" of cohomology.

218.1. **The General Alexander Duality Theorem.** As we just mentioned, by the Sphere–Alexander Duality Theorem 217.8 we know that for any compact subspace $X \subsetneq S^n$ of S^n that admits a regular neighborhood and any $i \in \mathbb{N}_0$ there exists an isomorphism

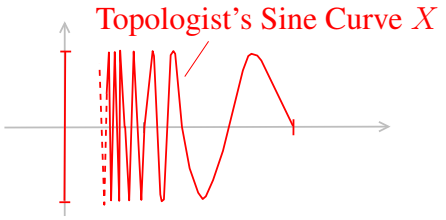
$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X;\mathbb{Z}).$$

In a bout of optimism one might hope that the statement of the Sphere–Alexander Duality Theorem 217.8 holds for any (compact) subspace of S^n. But as the following example shows, this is not the case:

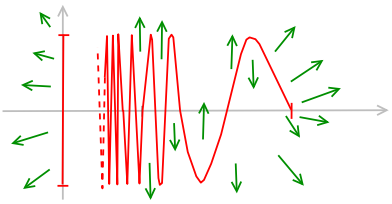
Example. We consider again the Topologist's Sine Curve

$$X \;=\; \{(0,y)\,|\,y\in[-1,1]\} \cup \{(x,\sin(\tfrac{1}{x}))\,|\,x\in(0,\tfrac{1}{\pi}]\} \;\subseteq\; \mathbb{R}^2$$

that we had initially introduced on page 195.



X is not path-connected



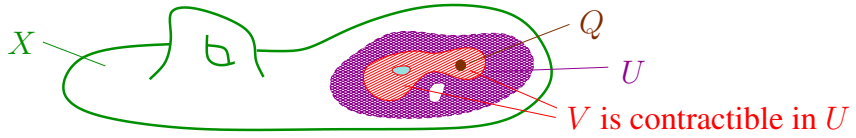
$S^2 \setminus X$ is contractible

It is fairly straightforward to see that X is a closed subset of $\mathbb{R}^2$. Evidently X is bounded. Thus the Heine-Borel Theorem 3.44 implies that X is in fact a compact subspace of $\mathbb{R}^2$ and thus of $S^2 =_{\text{i}} \mathbb{R}^2 \cup \{\infty\}$. Back on page 195 we saw that X is not path-connected, which by the H^0-Proposition 198.9 and the Path-Component-H^*-Lemma 198.8 implies that $\widetilde{\mathrm{H}}^0(X;\mathbb{Z}) \neq 0$.

On the other hand, with some effort one can show that there exists a deformation retraction from $S^2 \setminus X = (\mathbb{R}^2 \setminus X) \cup \{\infty\}$ to ∞, which implies that $S^2 \setminus X$ is contractible, which in turn shows (by the Contractible-H_*-Corollary 144.15) that $\mathrm{H}_1(S^2 \setminus X) = 0$. In summary we have shown that $\widetilde{\mathrm{H}}_1(S^2 \setminus X) \ncong \widetilde{\mathrm{H}}^{2-1-1}(X;\mathbb{Z})$. □

This example shows that we cannot hope to generalize the Sphere–Alexander Duality Theorem 217.8 to all compact subspaces of S^n. But as we will see shortly, we can extend it to compact subspaces which are weakly locally contractible. Just in case the reader has only a faint memory of the definition, let us recall the definition of a weakly locally contractible space from Item (3) (4):

Definition 218.1. Let X be a topological space. We say that X is weakly locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U such that the inclusion $V \to U$ is contractible in U, i.e. such that the inclusion is homotopic in U to a constant map.



X is weakly locally contractible

Example.

(1) As we already pointed out on page 3637, it follows from the Manifold–Local Properties Lemma 23.8 (1) and the CW-Complex–Local Properties Proposition 137.8 (2) that topological manifolds and CW-complexes are weakly locally contractible.

(2) It is an amusing little exercise to show that the Topologist's Sine Curve, which we discussed above, is not weakly locally contractible. □

Now we can formulate our main theorem of the chapter:

Theorem 218.2. (General Alexander Duality Theorem) Let Σ be a closed oriented n-dimensional topological manifold and let $X \subseteq \Sigma$ be a weakly locally contractible, compact subspace of Σ.

(1) Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma) = \mathrm{H}_{j+1}(\Sigma) = 0$. There exists a natural isomorphism

$$\mathrm{AD}_X \colon \mathrm{H}_j(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(X;\mathbb{Z}).$$

Here "natural" means two things:

- If $X \subseteq Y \subseteq \Sigma$ are two weakly locally contractible, compact subspaces of Σ, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_j(\Sigma \setminus Y) & \xrightarrow[\cong]{\mathrm{AD}_Y} & \mathrm{H}^{n-j-1}(Y;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}_j(\Sigma \setminus X) & \xrightarrow[\cong]{\mathrm{AD}_X} & \mathrm{H}^{n-j-1}(X;\mathbb{Z}), \end{array}$$

  here the left vertical map is induced by the inclusion $\Sigma \setminus Y \to \Sigma \setminus X$ and the right vertical map is induced by the inclusion $X \to Y$.
- Let $f \colon \Sigma \to \Sigma'$ be a homeomorphism between two closed oriented connected non-empty n-dimensional topological manifolds and let $X \subseteq \Sigma$ be a weakly locally contractible, compact subspace of Σ. We set $\epsilon = 1$ if f is orientation-preserving and we set $\epsilon = -1$ is f is orientation-reversing. Then the following diagram commutes up to ϵ:

$$\begin{array}{ccc} \mathrm{H}_j(\Sigma \setminus X) & \xrightarrow[\cong]{\mathrm{AD}_X} & \mathrm{H}^{n-j-1}(X;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle f^*} \\ \mathrm{H}_j(\Sigma' \setminus f(X)) & \xrightarrow[\cong]{\mathrm{AD}_{f(X)}} & \mathrm{H}^{n-j-1}(f(X);\mathbb{Z}). \end{array}$$

(2) We now assume that Σ is connected.[2324]

(a) If $\mathrm{H}_1(\Sigma) = 0$ and if $X \subsetneq \Sigma$, then there is a natural isomorphism

$$\mathrm{AD}_X \colon \widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}).$$

(b) If $\mathrm{H}_{n-1}(\Sigma) = 0$ and if $X \neq \varnothing$, then there is a natural isomorphism

$$\mathrm{AD}_X \colon \mathrm{H}_{n-1}(\Sigma \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X;\mathbb{Z}).$$

Notation 218.3. We continue with the notation from the General Alexander Duality Theorem 218.2. We refer to the isomorphism

$$\mathrm{AD}_X \colon \mathrm{H}_j(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(X;\mathbb{Z})$$

(and the reduced versions) which we will completely specify on Definition 218.6, Section 218.2 and Section 218.2, as the Alexander Duality Isomorphism.

We proceed as follows:

(1) In the remainder of this section we discuss one application of the General Alexander Duality Theorem 218.2.

[2323] It follows from the elementary discussion on page 3773 that ϵ is just the degree of f.

[2324] Note that if Σ is non-empty, then $\mathrm{H}_0(\Sigma) \neq 0$ and $\mathrm{H}_n(\Sigma) \neq 0$. Thus (1) cannot be applied for $j = 0$ and $j = n-1, j = n$. The case $j = n$ is almost trivial. But we have to modify the statements slightly to get some useful isomorphisms for $j = 0$ and $j = n-1$.

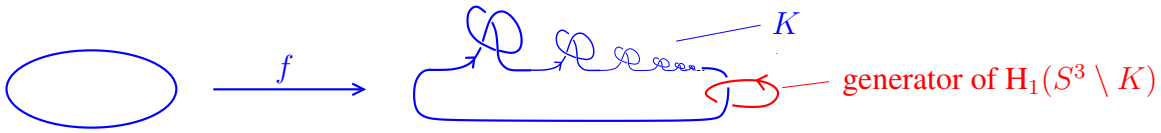
(2) In Section 218.2 we will give the explicit definition of the map AD_X and we will prove the General Alexander Duality Theorem 218.2.
(3) In Section 218.3 we will state and discuss an important special case of the General Alexander Duality Theorem 218.2, namely the Sphere–Alexander Duality Theorem 218.7.
(4) In Section 218.4 we will use the General Alexander Duality Theorem 218.2 to show that there exist "wild knots", i.e. there exist embeddings $w\colon S^1 \to S^3$ such that $w(S^1) \subseteq S^3$ is not a topological manifold.
(5) In Appendix 218.6 we will compare the two Sphere–Alexander Duality Theorems 217.8 and 218.7. In particular we will see that the latter generalizes the earlier Sphere–Alexander Duality Theorem 217.8.

For better readability we formulate our first example as a lemma:

Lemma 218.4. (Wild Knot Complement-H_*-Lemma)
(1) Let $f\colon S^1 \to S^3$ be any embedding. We set $K \coloneqq f(S^1)$. Given $j \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_j(S^3 \setminus K) \quad \cong \quad \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $K \subseteq S^3$ be the subspace that is shown in the figure below.[2325] The curve shown in the figure below represents a generator of $\mathrm{H}_1(S^3 \setminus K)$.



Proof.
(1) Since $f\colon S^1 \to S^3$ is an embedding we know that $K = f(S^1)$ is homeomorphic to S^1. This shows in particular that K is weakly locally contractible and we see that the General Alexander Duality Theorem 218.2 applies. Thus we obtain that

$$\widetilde{\mathrm{H}}_j(S^3 \setminus K) \quad \underset{\uparrow}{\overset{\cong}{\to}} \quad \widetilde{\mathrm{H}}^{2-j}(K;\mathbb{Z}) \quad \cong \quad \widetilde{\mathrm{H}}^{2-j}(S^1;\mathbb{Z}) \quad \underset{\uparrow}{\cong} \quad \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

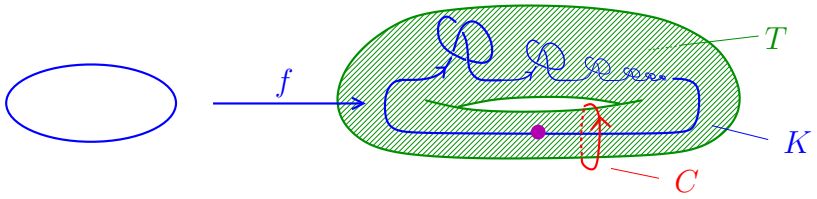
isomorphism AD_K from the General Alexander Duality Theorem 218.2

by the calculation on page 3830 together with the Reduced Cohomology Lemma 198.17

(2) Using the Compact-Hausdorff Proposition 4.23 (3) one can show that the subspace K in the above figure is indeed homeomorphic to S^1.[2326] Thus it follows from (1) that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$. But the calculation in (1) does not specify an explicit generator for $\mathrm{H}_1(S^3 \setminus K)$. As we will now see, one way to pin down a generator is to use the naturality of the Alexander Duality isomorphism AD_K.

[2325] In No Regular Neighborhood Proposition 218.21 we will show that K does not admit a regular neighborhood in the sense of Definition 217.2. In particular the Sphere–Alexander Duality Theorem 217.8 does not apply to $K \subseteq S^3$.. Furthermore, in the Wild Knot Existence Theorem 218.13 we will show the related result, that K is not a topological submanifold of S^3.
[2326] In Notation 218.15 we will give a more precise description of K, which then also makes the claim, that K is homeomorphic to S^1, more meaningful.

In the following we consider the "standard closed solid torus" $T \subseteq S^3$ that is shown in the figure below:



Note that it follows easily from the Sphere–Solid Tori-Decomposition Lemma 111.6 that $S^3 \setminus T$ is diffeomorphic to $S^1 \times B^2$, where $S^1 \times \{0\}$ corresponds to the red curve C shown in the figure. Thus C represents a generator of $\mathrm{H}_1(S^3 \setminus T)$. We want to show that C also represents a generator of $\mathrm{H}_1(S^3 \setminus K)$. We denote by $i\colon K \to T$ the inclusion map. By the above discussion it now suffices to show that the map $\mathrm{H}_1(S^3 \setminus T) \to \mathrm{H}_1(S^3 \setminus K)$ is an isomorphism.

To do so we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(S^3 \setminus T) & \xrightarrow[\cong]{\mathrm{AD}_T} & \mathrm{H}^{3-1-1}(T;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(T),\mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle (i_*)^*} \\ \mathrm{H}_1(S^3 \setminus K) & \xrightarrow[\cong]{\mathrm{AD}_K} & \mathrm{H}^{3-1-1}(K;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(K),\mathbb{Z}). \end{array}$$

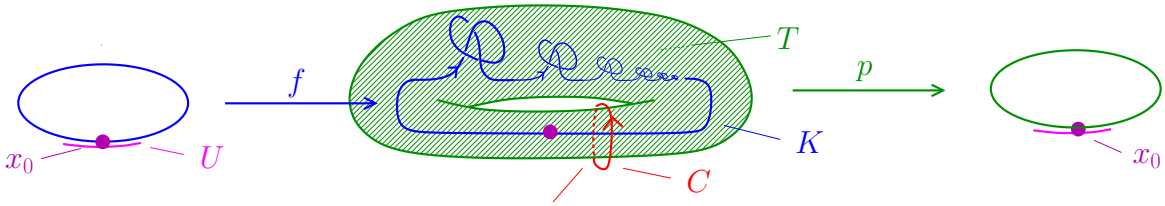
We make the following observations:

- It follows from the existence and the naturality of the isomorphism of the General Alexander Duality Theorem 218.2 that the left horizontal maps are isomorphisms and that the rectangle to the left commutes.
- The H^1-Hom-Proposition 201.24 tells us that the right horizontal maps are isomorphisms and that the rectangle to the right commutes.

We see that it now remains to prove the following claim:

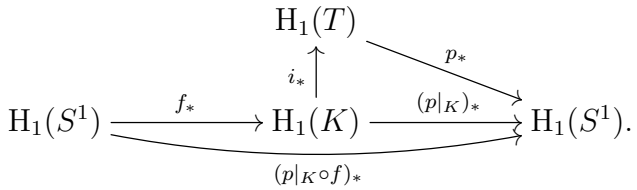
Claim. The map $i_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(T)$ is an isomorphism.

Proof. Let $f\colon S^1 \to K$ be a homeomorphism and let $p\colon T \to S^1$ be the obvious homotopy equivalence. We can pick f and p and a point $x_0 \in S^1$ such that $p \circ f$ is the identity in a neighborhood U of x_0 and such that $(p \circ f)^{-1}(x_0) = \{x_0\}$.



a generator of $\mathrm{H}_1(S^3 \setminus T)$ and therefore also a generator of $\mathrm{H}_1(S^3 \setminus K)$

We consider the following little diagram:



We make the following observations:

(a) The diagram commutes by the functoriality of homology.

(b) The diagonal map is an isomorphism since $p\colon T \to S^1$ is a homotopy equivalence.

(c) It follows from the S^n-Local-Global Degree Proposition 152.5 and the Degree-via-Local Orientations Proposition 152.11 that the composition of the bottom two horizontal maps, i.e. the map $(p|_K \circ f)_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$, is an isomorphism.

It follows from the above discussion that the vertical map $i_*\colon \mathrm{H}_*(K) \to \mathrm{H}_*(T)$ is indeed an isomorphism. ■

Remark.

(1) We leave it to the reader to determine – say using search engines provided by California based companies – whether in the setting of the General Alexander Duality Theorem 218.2 (1) there exists also a natural isomorphism

$$\mathrm{H}^j(\Sigma \setminus X; \mathbb{Z}) \;\to\; \mathrm{H}_{n-j-1}(X).$$

Here is how we can see that the groups are at least isomorphic: By the Finite-H_*-H^*-Lemma 218.19 we know that the homology groups $\mathrm{H}_i(X)$ and the cohomology groups $\mathrm{H}^j(X;\mathbb{Z})$ are all finitely generated. It follows from the General Alexander Duality Theorem 218.2 that all homology groups $\mathrm{H}_k(\Sigma \setminus X)$ are finitely generated. It follows from this discussion, the General Alexander Duality Theorem 218.2 and the purely algebraic Exercise 217.6 that the groups $\mathrm{H}^j(\Sigma \setminus X;\mathbb{Z})$ and $\mathrm{H}_{n-j-1}(X)$ are isomorphic.

(2) There exists also a version of the General Alexander Duality Theorem 218.2 for any arbitrary compact subspace of Σ, but in the general case one needs to replace ordinary cohomology by "*Čech cohomology*" of K. We refer to [**Hat2002**, p. 256], [**Dol1980**, Theorem VIII.7.2], [**Bre1993**, Corollary VI.8.7], [**Mass1981b**] or [**Pra2007**, p. 271] for more details. □

218.2. **Proof of the General Alexander Duality Theorem 218.2 .** The definition of the Alexander duality isomorphism requires some preparations. We start out with the following notation:

Notation 218.5. First we recall the following concepts from page 4011:

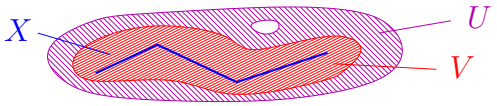
(1) Given a topological space Y we denote by $\mathcal{K}(Y)$ the set of all compact subspaces of Y. Given $K, L \in \mathcal{K}(Y)$ we write $K \leq L$ if $K \subseteq L$. The set $\mathcal{K}(Y)$ together with the relation "$\leq$" is a directed set[2327] since by the Compact–Union-Intersection Lemma 3.41 we know that the union of two compact subspaces is again compact.

We also introduce a new definition and notation:

(2) Let X be a subspace of a topological space Y. We denote by $\mathcal{N}(X)$ the set of all open neighborhoods of X in Y. Given $U, V \in \mathcal{N}(X)$ we write $U \leq V$ if $V \subseteq U$. (Note the reversal of V and U.) The set $\mathcal{N}(X)$ together with the relation "$\leq$" is a directed set since the intersection of two open neighborhoods of X is again an open neighborhood of X.[2328]

(3) Given any two subspaces $A \subseteq B$ of Σ we denote by $i_{A,B} \colon A \to B$ the inclusion map.

(4) It follows from our definition of "$\leq$" on $\mathcal{N}(X)$ that for any $m \in \mathbb{N}_0$ the cohomology groups $\{\mathrm{H}^m(U;\mathbb{Z})\}_{U\in\mathcal{N}(X)}$ together with the inclusion-induced maps form a *direct* system[2329] of abelian groups.



we have $U \leq V$ and $i^*_{V,U} \colon \widetilde{\mathrm{H}}^m(U;\mathbb{Z}) \to \widetilde{\mathrm{H}}^m(V;\mathbb{Z})$

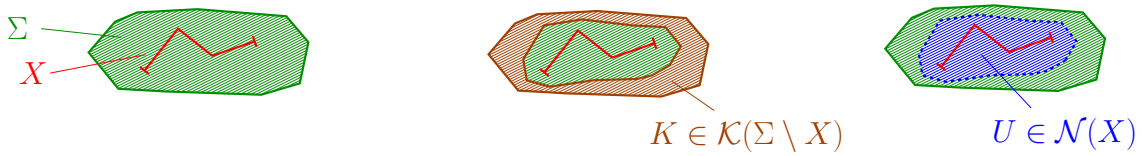
With this notation we can now finally define the Alexander duality isomorphisms:

Definition 218.6. Let $n \in \mathbb{N}$, let Σ be a closed oriented[2330] n-dimensional topological manifold and let $X \subseteq \Sigma$ be a weakly locally contractible, compact subspace of Σ. Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma) = \mathrm{H}_{j+1}(\Sigma) = 0$. Rather similar to the discussions on Section 217.3 and 4159 we consider the following sequence of homomorphisms:

$$\begin{aligned}
\mathrm{H}_j(\Sigma \setminus X) &\xrightarrow[(1)]{\cong} \mathrm{H}_j(\Sigma \setminus X;\mathbb{Z}) \xleftarrow[(2)]{\cong} \overbrace{\operatorname*{colim}_{K\in\mathcal{K}(\Sigma\setminus X)} \mathrm{H}^{n-j}(\Sigma \setminus X, \underbrace{(\Sigma \setminus X)\setminus K}_{=(\Sigma\setminus K)\setminus X};\mathbb{Z})}^{\text{naturally isomorphic to cohomology } \mathrm{H}_c^{n-j}(\Sigma\setminus X;\mathbb{Z}) \text{ with compact support of } \Sigma\setminus X} \\
&\xleftarrow[(3)]{\cong} \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j}(\Sigma \setminus X, U \setminus X;\mathbb{Z}) \\
&\xleftarrow[(4)]{\cong} \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j}(\Sigma, U;\mathbb{Z}) \xleftarrow[(5)]{\cong} \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j-1}(U;\mathbb{Z}) \xrightarrow[(6)]{\cong} \mathrm{H}^{n-j-1}(X;\mathbb{Z}).
\end{aligned}$$

Below we will see that all maps are isomorphisms. Thus it makes sense to consider the composition and we refer to this composition as the Alexander duality isomorphism

$$\mathrm{AD}_X \colon \mathrm{H}_j(\Sigma \setminus X) \to \mathrm{H}^{n-j-1}(X;\mathbb{Z}).$$



Proof of the General Alexander Duality Theorem 218.2 (1). Let $n \in \mathbb{N}$, let Σ be a closed oriented n-dimensional topological manifold and let $X \subseteq \Sigma$ be a weakly locally

[2327] Recall that according to Definition 104.1, a directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

[2328] Here the idea is that the neighborhoods "approximate" X, in particular $U \leq V$ means that V is a "better approximation of X", which just means that $V \subseteq U$.

[2329] We refer to Definition 104.2 for the definition of a direct system.

contractible, compact subspace of Σ. Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma) = \mathrm{H}_{j+1}(\Sigma) = 0$. We need to prove the following statements:

(a) All six homomorphisms in the above Definition 218.6 are isomorphisms.
(b) All six homomorphisms are natural.[2331]

The fact that the homomorphisms are natural follows, as we will see, from formal reasons. The proof that the six homomorphisms are isomorphisms can be summarized as follows:

(1) This isomorphism reflects that ordinary homology is isomorphic to homology with $\mathbb{Z}$-coefficients.
(2) This map is an isomorphism by the Poincaré Duality–Non-Compact Theorem 211.4.
(3) This map is basically an identity.
(4) It follows from excision that this map is an isomorphism.
(5) It follows from long exact sequences and our information on the homology of Σ that this map is an isomorphism.
(6) We will use Borsuk's Theorem 189.7, and our hypothesis that X is weakly locally contractible, to show that this map is an isomorphism.

We now discuss the six homomorphisms in detail:

(1) This isomorphism comes from the discussion on page 3077.
(2) (a) Since X is compact, we know from the Compact-Closed Lemma 3.39 (2) that $\Sigma \setminus X$ is an open subset of Σ. In particular $\Sigma \setminus X$ is also an n-dimensional topological manifold and via the Topological Manifold-Induced Orientation Lemma 192.1 it inherits an orientation from Σ. Also note that it follows from our hypothesis that $\partial\Sigma = \varnothing$ and Definition 23.2 that $\Sigma \setminus X$ has no boundary. After this preamble the first isomorphism is the Poincaré Duality isomorphism from the Poincaré Duality–Non-Compact Theorem 211.4 corresponding to the oriented topological manifold $\Sigma \setminus X$. [2332]
   (b) As we discussed in the (proof of) the Poincaré Duality–Open Subset Lemma 211.6, it follows from the Cap Product-Naturality Lemma 206.13 and the universal properties of direct colimits that this isomorphism is natural.
(3) (a) Since Σ is compact it follows immediately from the Compact-Closed Lemma 3.39 that a subset $U \subseteq \Sigma$ is open if and only if $\Sigma \setminus U$ is compact. This observation shows that the map

$$\begin{array}{ccc} \mathcal{N}(X) & \to & \mathcal{K}(\Sigma \setminus X) \\ U & \mapsto & \Sigma \setminus U \end{array}$$

   is an isomorphism of directed sets. From this observation we obtain an isomorphism of direct systems. It is now clear that the second homomorphism is an isomorphism.
   (b) It follows from the universal properties of direct colimits that this map is natural.

[2331] For an orientation-reserving homeomorphism $f\colon \Sigma \to \Sigma'$ the sign -1 makes its appearance in (2), due to the Fundamental Class–Opposite Orientation Lemma 194.4.
[2332] Note that we formulated the Poincaré Duality–Non-Compact Theorem 211.4 in terms of cohomology with compact support. But we really proved it in terms of the colimit and we used the notation for "cohomology with compact support" just to abbreviate the formulation.)

(4) (a) For any two open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subseteq U$, we have the diagram

$$\begin{array}{ccc} \mathrm{H}^{n-j}(\Sigma\backslash X, V\backslash X;\mathbb{Z}) & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, V;\mathbb{Z}) \\ \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} \\ \mathrm{H}^{n-j}(\Sigma\backslash X, U\backslash X;\mathbb{Z}) & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, U;\mathbb{Z}) \end{array}$$

where all the maps are induced by the obvious inclusions of pairs of topological spaces. The diagram commutes by the functoriality of cohomology. Since X is closed and since U and V are open we see that we obtain from the Excision-in-H^*-Theorem 200.1 (1a) that the horizontal maps are isomorphisms. The desired isomorphism of direct colimits is now an immediate consequence of the Cofinal-Isomorphism Lemma 209.13.

(b) It follows from the universal properties of direct colimits that this map is natural.

(5) (a) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subseteq U$, we have a commutative diagram

$$\begin{array}{ccccccc} \overbrace{\mathrm{H}^{n-j}(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_j(\Sigma)} & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, V;\mathbb{Z}) & \xleftarrow{\delta_{n-j-1}} & \mathrm{H}^{n-j-1}(V;\mathbb{Z}) & \longleftarrow & \overbrace{\mathrm{H}^{n-j-1}(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_{j+1}(\Sigma)} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow & & {\scriptstyle i^*_{V,U}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow \\ \underbrace{\mathrm{H}^{n-j}(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_j(\Sigma)} & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, U;\mathbb{Z}) & \xleftarrow{\delta_{n-j-1}} & \mathrm{H}^{n-j-1}(U;\mathbb{Z}) & \longleftarrow & \underbrace{\mathrm{H}^{n-j-1}(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_{j+1}(\Sigma)} \end{array}$$

where the vertical maps are induced by the obvious inclusions and where the horizontal maps are the connecting homomorphisms of the long exact sequences in cohomology provided by the LES-of-Triple Proposition 198.13. Since $j \geq 1$ we obtain from these long exact sequences, our hypotheses on the homology groups of Σ and the Poincaré Duality Theorem 210.1 that the middle horizontal maps are in fact isomorphisms. As in (4) we appeal to the Cofinal-Isomorphism Lemma 209.13 to obtain the desired isomorphism of direct colimits.

(b) It follows from the universal properties of direct colimits that this map is natural.

(6) (a) We need to prove the following claim:

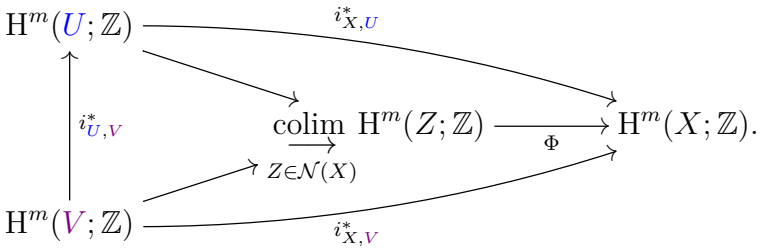
Claim. For every $m \in \mathbb{N}_0$ the map

$$\Phi\colon \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^m(U;\mathbb{Z}) \;\to\; \mathrm{H}^m(X;\mathbb{Z}),$$

that is induced by the restriction maps $i^*_{X,U}\colon \mathrm{H}^m(U;\mathbb{Z}) \to \mathrm{H}^m(X;\mathbb{Z})$, is an isomorphism.

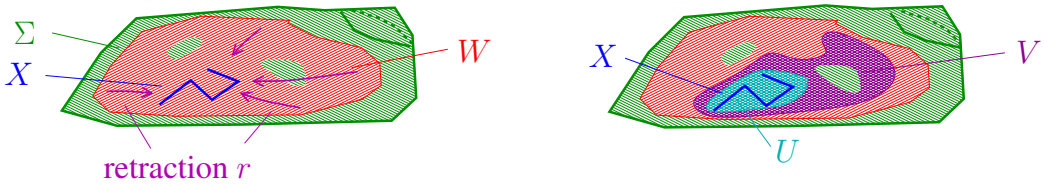
Proof. For orientation purposes we now state the commutative diagram which comes with the definition of a direct colimit. More precisely, for open neighborhoods

$X \subseteq U \subseteq V \subseteq \Sigma$ we have the following commutative diagram:

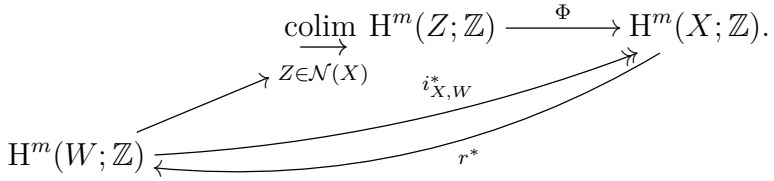


Now let us start with the actual proof of the claim. By our hypothesis we know that X is compact and weakly locally contractible. Since Σ is a topological manifold we know by the Manifold–Neighborhood Retract Theorem 189.8 (2) (which is a consequence of the tricky Borsuk Theorem 189.7) that there exists an open neighborhood $W \subseteq \Sigma$ of $X \subseteq \Sigma$ and a retraction $r\colon W \to X$ such that furthermore the following condition is satisfied:

(∗) Given any neighborhood V of X there exists an open neighborhood U with $X \subseteq U \subseteq V \cap W$ such that the inclusion map $i_{U,V}\colon U \to V$ and the retraction $r|_U\colon U \to X \subseteq V$ are homotopic maps from U to V.



Next we consider the following diagram:



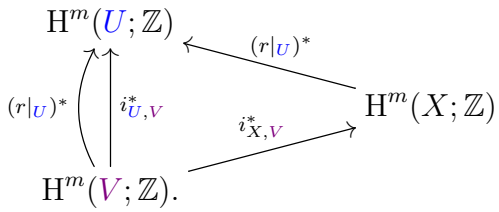
First note that the outer triangle commutes by the universal property of direct colimits. Next note that we have $i^*_{X,W} \circ r^* = (r \circ i_{X,W})^* = (\mathrm{id}_X)^* = \mathrm{id}_{\mathrm{H}^m(X;\mathbb{Z})}$. We deduce that the map $i^*_{X,W}$ is an epimorphism. It follows that the horizontal map Φ is also an epimorphism. It remains to show that Φ is a monomorphism.

By the Elements-in-Direct Colimit Lemma 104.8 (3) it remains to show that for every $V \in \mathcal{N}(X)$ there exists a $U \in \mathcal{N}(X)$ with $V \leq U$, i.e. with $U \subseteq V$, such that

$$\ker(i^*_{X,V}\colon \mathrm{H}^m(V;\mathbb{Z}) \to \mathrm{H}^m(X;\mathbb{Z})) \;\subseteq\; \ker(i^*_{U,V}\colon \mathrm{H}^m(V;\mathbb{Z}) \to \mathrm{H}^m(U;\mathbb{Z})).$$

So let $V \in \mathcal{N}(X)$. By (∗) there exists an open neighborhood U with $X \subseteq U \subseteq V \cap W$ such that the inclusion $i_{U,V}\colon U \to V$ and the retraction $r|_U\colon U \to X \subseteq V$ are homotopic maps from U to V.

We consider the following diagram:



Since the two maps $i_{U,V}\colon U \to V$ and $r|_U\colon U \to X \subseteq V$ are homotopic it follows from the Homotopy-H^*-Proposition 198.11 (1) that the two maps $i^*_{U,V}$ and r^* from $\mathrm{H}^m(V;\mathbb{Z})$ to $\mathrm{H}^m(U;\mathbb{Z})$ are actually the same. Since r is the identity on X we see that $r\colon V \to X \subseteq U$ equals the map $i_{X,V} \circ r\colon V \to X$. This shows that the diagram commutes. It follows immediately from these observations that U has the desired property. ⊞

(b) It follows from the universal properties of direct colimits that this map is natural. ■

Proof of the General Alexander Duality Theorem 218.2 (2a). Let Σ be a closed oriented n-dimensional topological manifold and let $X \subseteq \Sigma$ be a weakly locally contractible, compact subspace of Σ. We now assume that Σ is connected and that $\mathrm{H}_1(\Sigma) = 0$. As we will see, the above steps are unchanged, except for Step (5) on page 4183. To deal with our new setting we start out with the following claim:

Claim. Given any compact weakly locally contractible subspace $Y \subseteq \Sigma$ we have an exact sequence

$$\mathrm{H}^n(Y;\mathbb{Z}) \xleftarrow{i^*_{Y,\Sigma}} \mathrm{H}^n(\Sigma;\mathbb{Z}) \leftarrow \mathrm{H}_0(\Sigma \setminus Y) \leftarrow \mathrm{H}^{n-1}(Y;\mathbb{Z}) \leftarrow 0,$$

where the left hand map is induced by the inclusion $Y \to \Sigma$ and where all maps are natural in Y.

Proof. First note that, similar to the proof of (5) on page 4183, we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
\mathrm{H}^n(V;\mathbb{Z}) & \longleftarrow & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \xleftarrow{\text{я}} & \mathrm{H}^n(\Sigma,V;\mathbb{Z}) & \xleftarrow{\delta_{n-1}} & \mathrm{H}^{n-1}(V;\mathbb{Z}) & \longleftarrow & \overbrace{\mathrm{H}^{n-1}(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_1(\Sigma)=0} \\
\uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^n(U;\mathbb{Z}) & \longleftarrow & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \xleftarrow{\text{я}} & \mathrm{H}^n(\Sigma,U;\mathbb{Z}) & \xleftarrow{\delta_{n-1}} & \mathrm{H}^{n-1}(U;\mathbb{Z}) & \longleftarrow & \underbrace{\mathrm{H}^{n-1}(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_1(\Sigma)=0}.
\end{array}$$

Next note that it follows from the Direct Colimit-H_*-Lemma 154.8 that direct colimits preserve exactness. Thus we obtain the following exact sequence:

$$\underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^n(U;\mathbb{Z})}_{\substack{\text{by (6) naturally} \\ \text{isomorphic} \\ \text{to } \mathrm{H}^n(Y;\mathbb{Z})}} \leftarrow \underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^n(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}^n(\Sigma;\mathbb{Z})} \leftarrow \underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^n(\Sigma,U;\mathbb{Z})}_{\substack{\text{by (1), (2), (3), (4) naturally} \\ \text{isomorphic to } \mathrm{H}_0(\Sigma \setminus Y)}} \leftarrow \underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^{n-1}(U;\mathbb{Z})}_{\substack{\text{by (6) naturally} \\ \text{isomorphic} \\ \text{to } \mathrm{H}^{n-1}(Y;\mathbb{Z})}} \leftarrow 0$$

where all maps are natural in Y. Using the explicit natural isomorphisms (1), (2), (3), (4) and (6) from above (which we still have, since for these we did not use any hypothesis on the homology groups of Σ), we obtain the promised natural maps and exact sequence. ⊞

Now let $X \subsetneq \Sigma$ be a compact, weakly locally contractible subspace. We apply the claim to $Y = X$ and $Y = \varnothing$. By naturality we obtain the following commutative diagram of exact horizontal sequences:

$$\begin{array}{ccccccccc}
\mathrm{H}^n(X;\mathbb{Z}) & \xleftarrow{0} & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \longleftarrow & \mathrm{H}_0(\Sigma \setminus X) & \longleftarrow & \mathrm{H}^{n-1}(X;\mathbb{Z}) & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i_*} & & \downarrow & & \\
\underbrace{\mathrm{H}^n(\varnothing;\mathbb{Z})}_{=0} & \xleftarrow{0} & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \xleftarrow{\cong} & \mathrm{H}_0(\Sigma \setminus \varnothing) & \longleftarrow & \underbrace{\mathrm{H}^{n-1}(\varnothing;\mathbb{Z})}_{=0} & \longleftarrow & 0.
\end{array}$$

We make the following observations:

(i) Since $X \subsetneq \Sigma$ there exists a point $P \in \Sigma \setminus X$. Thus the horizontal maps to the top left factors through $\mathrm{H}^n(\Sigma \setminus \{P\};\mathbb{Z})$, but by the Manifold-Top-H*-Theorem 201.26 we know that that group is trivial. We thus see that the horizontal maps to the top left is the zero map.

(ii) Since $\mathrm{H}^n(\varnothing;\mathbb{Z}) = \mathrm{H}^{n-1}(\varnothing;\mathbb{Z}) = 0$ we see that in the bottom row the second map to the left is an isomorphism.

We now consider the following maps:

$$\mathrm{H}^{n-1}(X;\mathbb{Z}) \quad \underset{\uparrow}{\xrightarrow{\cong}} \quad \ker(i_*\colon \mathrm{H}_0(\Sigma \setminus X) \to \mathrm{H}_0(\Sigma)) \quad \underset{\uparrow}{\xleftarrow{\cong}} \quad \widetilde{\mathrm{H}}_0(\Sigma \setminus X).$$

it follows from the above discussion in (i) and (ii) that we have such a natural isomorphism

it follows from the Reduced Homology Lemma 147.3 (0b), and the hypotheses that $\Sigma \setminus X \neq \varnothing$ and that Σ is connected, that the natural map ж from the $\widetilde{\mathrm{H}}_n$–Functor Lemma 147.2 is an isomorphism

We have thus shown that there exists a natural isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}). \qquad \blacksquare$$

Proof of the General Alexander Duality Theorem 218.2 (2b). Let Σ be a closed oriented n-dimensional topological manifold and let $X \subseteq \Sigma$ be a weakly locally contractible, compact subspace of Σ. We now assume that Σ is connected, that $\mathrm{H}_{n-1}(\Sigma) = 0$ and that X is non-empty.

Claim. We have an exact sequence

$$0 \leftarrow \mathrm{H}_{n-1}(\Sigma \setminus X) \leftarrow \mathrm{H}^0(X;\mathbb{Z}) \leftarrow \mathrm{H}^0(\Sigma;\mathbb{Z})$$

where the right hand map is induced by the inclusion $X \to \Sigma$ and where all maps are natural in X.

Proof. Similar to the proof of (5) on page 4183 we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
\overbrace{\mathrm{H}^1(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_{n-1}(\Sigma)=0} & \xleftarrow{\text{я}} & \mathrm{H}^1(\Sigma, V;\mathbb{Z}) & \xleftarrow{\delta_0} & \mathrm{H}^0(V;\mathbb{Z}) & \longleftarrow & \mathrm{H}^0(\Sigma;\mathbb{Z}) \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\underbrace{\mathrm{H}^1(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_{n-1}(\Sigma)=0} & \xleftarrow{\text{я}} & \mathrm{H}^1(\Sigma, U;\mathbb{Z}) & \xleftarrow{\delta_0} & \mathrm{H}^0(U;\mathbb{Z}) & \longleftarrow & \mathrm{H}^0(\Sigma;\mathbb{Z})
\end{array}$$

As in the proof of (2a) above, the claim now follows from the Direct Colimit-H_*-Lemma 154.8 and the proofs of (1),(2),(3),(4) and (6). ⊞

We now consider the following maps:

$$\mathrm{H}_{n-1}(\Sigma \setminus X) \quad \underset{\uparrow}{\overset{\cong}{\longleftarrow}} \quad \operatorname{coker}(i_*\colon \mathrm{H}^0(\Sigma;\mathbb{Z}) \to \mathrm{H}^0(X;\mathbb{Z})) \quad \underset{\uparrow}{\overset{\cong}{\longrightarrow}} \quad \widetilde{\mathrm{H}}^0(X;\mathbb{Z}).$$

it follows from the above claim that we have such a natural isomorphism

it follows from the Reduced Cohomology Lemma 198.17 (0b), and the hypotheses that $X \neq \varnothing$ and that Σ is connected, that the natural map ж from the $\widetilde{\mathrm{H}}^n$–Functor Lemma 198.16 is an isomorphism

We have thus shown that there exists a natural isomorphism

$$\mathrm{AD}_X\colon \mathrm{H}_{n-1}(\Sigma \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X;\mathbb{Z}). \qquad \blacksquare$$

218.3. **The Sphere–Alexander Duality Theorems.** We start out with the following immediate consequence of the General Alexander Duality Theorem 218.2:

Theorem 218.7. (Sphere–Alexander Duality Theorem) Let $X \subsetneq S^n$ be a compact non-empty subspace of S^n. If X is weakly locally contractible, then for every $j \in \mathbb{N}_0$ the General Alexander Duality Theorem 218.2 gives us an isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_j(S^n \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z})$$

which is natural in the sense of the General Alexander Duality Theorem 218.2.

Proof. It is clear that the hypotheses of the General Alexander Duality Theorem 218.2 (1), (2a) and (2b) are satisfied. Thus we obtain the desired result for free. ■

This theorem raises the following question: What is the connection between the Sphere–Alexander Duality Theorem 218.7 and the previous Sphere–Alexander Duality Theorem 217.8? In Appendix 218.6 we will give a complete answer. More precisely: In the Regular Neighborhood–Weakly Locally Contractible Lemma 218.22 we will show that if a subspace $X \subseteq M$ admits a regular neighborhood, then X is weakly locally contractible. The lemma implies that the General Alexander Duality Theorem 218.2 generalizes the Sphere–Alexander Duality Theorem 217.8. Furthermore in the Sphere–Alexander Duality Isomorphisms Proposition 218.23 we will show that the isomorphism from Sphere–Alexander Duality Theorem 218.7 agrees with the isomorphism from the Sphere–Alexander Duality Theorem 217.8 (which a priori depended on the choice of N). This implies in particular that the choice of the regular neighborhood in the proof of the Sphere–Alexander Duality Theorem 217.8 had no impact on the isomorphism we constructed.

For most applications of Alexander Duality we will actually need the following straightforward consequence of the above theorem.

Theorem 218.8. (Sphere–Alexander Duality Theorem) Let $n \in \mathbb{N}_0$ and let X be a compact, non-empty topological manifold of dimension $< n$. If $f\colon X \to S^n$ is an injective continuous map, then for any $j \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_j(S^n \setminus f(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z})$$

which is natural in the sense of the General Alexander Duality Theorem 218.2.

Proof. Let $k < n$ and let $f\colon X \to S^n$ be an injective continuous map from a compact, non-empty k-dimensional topological manifold to S^n. It is a consequence of the Compact-Hausdorff Proposition 4.23 (3) that $f\colon X \to f(X)$ is a homeomorphism. From $k < n$ and the Topological Manifold-Dimension Proposition 150.10 it follows that $f(X) \neq S^n$. It follows from this discussion that we have the following isomorphisms:

$$\widetilde{\mathrm{H}}_j(S^n \setminus f(X)) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}^{n-j-1}(f(X);\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

we can apply the General Alexander Duality Theorem 218.2 since by the Manifold–Local Properties Lemma 23.8 we know that $f(X)$ is weakly locally contractible, and since $X \neq \varnothing$ and since $f(X) \neq S^n$

■

Examples. Let $n \in \mathbb{N}_0$.

(1) Let $k < n$ and let $f\colon S^k \to S^n$ be an injective continuous map. We obtain that

$$\widetilde{\mathrm{H}}_j(S^n \setminus f(S^k)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{n-j-1}(S^k;\mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

by the Sphere–Alexander Duality Theorem 218.8

see page 3830 and the Reduced Cohomology Lemma 198.17

This gives us a new proof for the Sphere Complement-H_*-Theorem 170.2 (2).[2333]

(2) Let X be a compact connected non-empty $(n-1)$-dimensional topological manifold with $\partial X \neq \varnothing$ and let $f\colon X \to S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ be an injective continuous map.



We see that

$$\widetilde{\mathrm{H}}_0(S^n \setminus f(X)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z}) \underset{\uparrow}{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}) \underset{\uparrow}{=} 0.$$

by the Sphere–Alexander Duality Theorem 218.8

since $n \geq 2$

by the Manifold-Top-H*-Theorem 201.26 (2a) since X is connected and since $\partial X \neq \varnothing$

It follows from this calculation together with the Reduced Homology Lemma 147.3 (0c) that $S^n \setminus f(X)$ is path-connected. □

The following proposition is the topological analog of the Smooth Codimension-One Orientability Proposition 51.17 which we proved on Section 51.4 and 4154.

[2333] Basically the same approach also gives a new proof of the Sphere Complement-H_*-Theorem 170.2 (1).

Proposition 218.9. (Topological Codimension-One Orientability Proposition) Let X be a closed n-dimensional topological manifold. If X is non-orientable, then there is no *topological* embedding of X into $\mathbb{R}^{n+1}$.

Proof. [23342335] Let X be a closed n-dimensional non-orientable topological manifold. Clearly it suffices to deal with the case that X is connected. Suppose that X admits an embedding $f\colon X \to \mathbb{R}^{n+1}$. As always we view $\mathbb{R}^{n+1}$ as a subspace of $\mathbb{R}^{n+1} \cup \{\infty\} =_{\mathrm{i}} S^{n+1}$.

We obtain that

$$\mathbb{Z}/2\mathbb{Z} \overset{\text{by the Manifold-Top-H}^*\text{-Theorem 201.26 (2), since } N \text{ is non-orientable and connected}}{\cong} \mathrm{H}^n(X;\mathbb{Z}) \overset{\text{by the Sphere–Alexander Duality Theorem 218.8}}{\cong} \widetilde{\mathrm{H}}_0(S^{n+1} \setminus X) \overset{\text{by the Reduced Homology Lemma 147.3 (0c)}}{\cong} \mathbb{Z}^{b_0(S^{n+1}\setminus X)-1}.$$

But this is absurd. We have thus obtained the desired contradiction. ■

In Question 30.18 (3) we asked whether $\mathbb{R}\mathrm{P}^2$ or the Klein bottle admit an embedding into $\mathbb{R}^3$. The following corollary, which is a strengthening of the Surface-No Smooth Embedding Corollary 51.18, gives us a negative answer to this question:

Corollary 218.10. (Surface-No Topological Embedding-Corollary)
(1) For $k \in \mathbb{N}$ the non-orientable surface of genus k does not admit an embedding into $\mathbb{R}^3$.
(2) Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit an embedding into $\mathbb{R}^3$.

Proof. Since $N_1 = \mathbb{R}\mathrm{P}^2$ and since by Exercise 9.17 we know that N_2 is homeomorphic to the Klein bottle it remains to prove Statement (1). Let $k \in \mathbb{N}$. By the Surface Non-Orientability Lemma 31.45 we know that N_k is non-orientable as a smooth manifold. It follows from the Smooth-Topological Orientation Proposition 191.25 that N_k is also non-orientable as a topological manifold. Since N_k is a closed 2-dimensional topological manifold it follows from the Topological Codimension-One Orientability Proposition 218.9 that N_k does not admit an embedding into $\mathbb{R}^3$. ■

The following proposition is a strengthening of the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14.

Proposition 218.11. ($\mathbb{R}\mathrm{P}^n$–No Topological Embedding Proposition) If $n \in \mathbb{N}_{\geq 2}$, then there is no embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.

Sketch of proof. First let n be even. In this case we know by the Real Projective Space Orientability Lemma 31.46 and the Smooth-Topological Orientation Proposition 191.25 that $\mathbb{R}\mathrm{P}^n$ is non-orientable as a topological manifold. Thus the result is an immediate consequence of the Topological Codimension-One Orientability Proposition 218.9.

Now let n be odd. This case is significantly harder and it was first proved by René Thom [**Tho1952**, p. 180] in 1952. Unfortunately we do not have the time and space to provide the argument. But it seems like the argument of Thom is somewhat reminiscent of the argument in the proof of the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 217.14. So

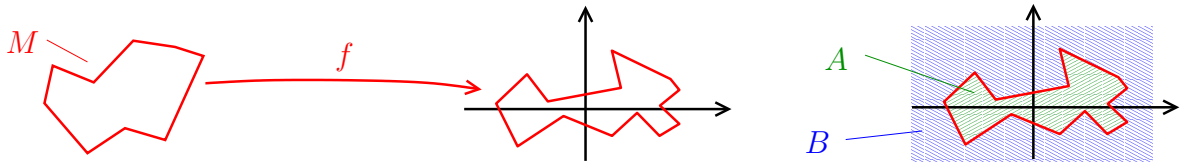
[2334] As we will see, the proof of the Topological Codimension-One Orientability Proposition 218.9 is structurally the same as the proof of the Smooth Codimension-One Orientability Proposition 51.17.
[2335] On page 4164 we gave an alternative proof of this proposition.

suppose that there exists an embedding $f\colon \mathbb{R}\mathrm{P}^n \to S^{n+1}$. It follows from the H_0-Proposition 143.25, the General Alexander Duality Theorem 218.2 and the fact that by the discussion on page 3832 we know that $\mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ (here we use that n is odd), that $S^{n+1} \setminus f(\mathbb{R}\mathrm{P}^n)$ consists of two path components. We denote the closures of the path components by A and B. The problem is that the triad (S^{n+1}, A, B) does not satisfy any of the possible hypotheses of the Mayer–Vietoris-H^*-Theorem 200.11. The idea in the proof is to replace usual singular cohomology by Alexander-Spanier cohomology for which the above triad is excisive. We refer to [**Mass1978a**, Chapter 8] or [**Mass1978b**] for an introduction to Alexander-Spanier cohomology and we refer to [**Mass1978a**, Theorem II.8.15] for the proof that in this setting the above triad is excisive. ■

We can now prove the following theorem, which generalizes the Jordan-Brouwer Separation Theorem 170.3, and which can be viewed as the topological analog of the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15 and the Smooth Sphere-Splitting Lemma 217.15.

Theorem 218.12. (Generalized Jordan-Brouwer Separation Theorem) Let $n \in \mathbb{N}_{\geq 2}$, let M be an $(n-1)$-dimensional closed connected non-empty topological manifold and let $f\colon M \to \mathbb{R}^n$ be an injective continuous map.

(1) The complement $\mathbb{R}^n \setminus f(M)$ has precisely two path-components, one of which is bounded and one which is unbounded.
(2) If A and B are these path-components, then $\partial A = \partial B = \overline{A} \cap \overline{B} = f(M)$.



Remark. The Generalized Jordan-Brouwer Separation Theorem 218.12 was first proved by James Alexander [**Ale1922**, p. 348] in 1922. □

Proof. [2336] Let $n \in \mathbb{N}_{\geq 2}$, let M be an $(n-1)$-dimensional closed connected non-empty topological manifold and let $f\colon M \to \mathbb{R}^n$ be an injective continuous map. By the Topological Codimension-One Orientability Proposition 218.9 we know that M is orientable.

(1) We use the embedding $\mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$ to view $\mathbb{R}^n$ as a subspace of S^n. Now we see that

$$\widetilde{\mathrm{H}}_0(\mathbb{R}^n \setminus f(M)) \overset{\text{by the Complement-in-}\mathbb{R}^n\text{-}S^n \text{ Lemma 170.1}}{\cong} \widetilde{\mathrm{H}}_0(S^n \setminus f(M)) \overset{\text{by the Sphere–Alexander Duality Theorem 218.8}}{\cong} \widetilde{\mathrm{H}}^{n-1}(M;\mathbb{Z}) \overset{\text{since } n \geq 2}{\cong} \mathrm{H}^{n-1}(M;\mathbb{Z}) \overset{\text{by the Top Cohomology Theorem 201.26 (2a), since } M \text{ is closed, connected, non-empty and } (n-1)\text{-dimensional}}{\cong} \mathbb{Z}.$$

[2336] The proof of the Generalized Jordan-Brouwer Separation Theorem 218.12 is very similar to the Jordan-Brouwer Separation Theorem 170.3. The biggest difference is that we need to replace the Sphere Complement-H_*-Theorem 170.2 by the Sphere–Alexander Duality Theorem 218.8. There are a few other subtle changes though.

It follows from this calculation and the Reduced Homology Lemma 147.3 (0c) that $\mathbb{R}^n \setminus f(M)$ has precisely two path-components. It remains to show that precisely one of them is unbounded.

Since M is compact there exists an $r \in \mathbb{R}$ such that $f(M) \subseteq B^n_r$. Since $n \geq 2$ it follows from an elementary argument that $\mathbb{R}^n \setminus B^n_r$ is path-connected. It follows from the Path-Component Lemma 4.44 (4b) that $\mathbb{R}^n \setminus B^n_r$ is contained in one of the two path-components. This shows that precisely one path-component is unbounded.

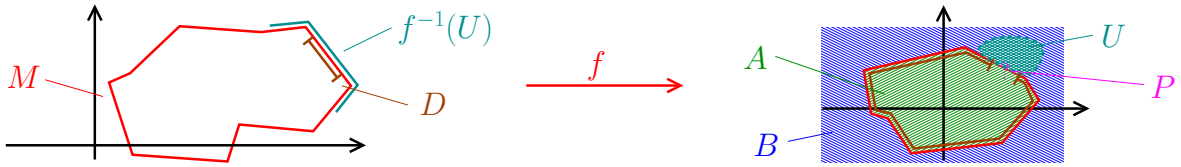
(2) We start out with the following claim:

Claim 1. A and B are open subsets of $\mathbb{R}^n$.

Proof. First note that it follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that $\mathbb{R}^n \setminus f(M)$ is open. In particular, as we remarked on page 206, $\mathbb{R}^n \setminus f(M)$ is locally path-connected. It follows from the Locally Path-Connected Lemma 4.57 that A and B are both open subsets of $\mathbb{R}^n$. ⊞

Claim 2. We have $\overline{A} = A \cup f(M)$ and $\overline{B} = B \cup f(M)$.

Proof. In the following we show that $\overline{A} = A \cup f(M)$. The proof that $\overline{B} = B \cup f(M)$ is basically identical. Since B is open we know that $A \cup f(M) = \mathbb{R}^n \setminus B$ is closed. By definition of $\overline{A}$ this implies that $\overline{A} \subseteq A \cup f(M)$. Now let $P \in A \cup f(M)$. We need to show that $P \in \overline{A}$. Suppose that this is not the case. By the Interior-Closure Lemma 3.22 we know that $\overline{A}$ is closed. Thus our assumption that $P \notin \overline{A}$ implies that there exists an open subset $U \subseteq \mathbb{R}^n$ with $P \in U \subseteq \mathbb{R}^n \setminus \overline{A}$. It follows from the discussion on page 639 that there exists a compact codimension-zero submanifold $D \subseteq M$ which is homeomorphic to $\overline{B}^{n-1}$, with $D \subseteq f^{-1}(U)$ and such that $E \coloneqq M \setminus \mathring{D}$ is also a compact $(n-1)$-dimensional topological manifold. From $n \geq 2$ it follows that E is connected with non-empty boundary.



We now obtain the following equalities:

$$\underbrace{\mathbb{R}^n \setminus f(E)}_{\substack{\text{path-connected by the discussion on page 4188 and the Complement-in-}\mathbb{R}^n\text{-}S^n\text{-Lemma 170.1 (here we use that } \partial E \neq \varnothing \text{ and that } E \text{ is connected)}}} \overset{\text{since } \mathbb{R}^{n+1} = A \sqcup B \sqcup f(M)}{\overset{\downarrow}{=}} A \sqcup B \sqcup (f(M) \setminus f(E)) \overset{\text{since } f(M) \setminus f(E) = f(M \setminus E) = f(D) \subseteq U}{\overset{\downarrow}{=}} A \sqcup (B \cup (U \cap \mathbb{R}^n \setminus f(E))).$$

$\uparrow$ disjoint union since $U \cap A = \varnothing$

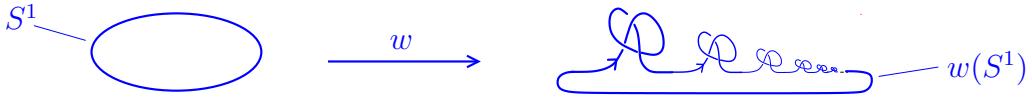
We have thus written the path-connected space $\mathbb{R}^n \setminus f(E)$ as the disjoint union of the two open non-empty sets A and $B \cup (U \cap \mathbb{R}^n \setminus f(E))$. But by the Path-Connected-Implies-Connected Corollary 4.33 this is not possible. We have therefore obtained a contradiction. ⊞

We now see that

$$\partial A \underset{\substack{\uparrow \\ \text{by Definition 3.21 of } \partial A,}}{=} \overline{A} \setminus \mathring{A} \underset{\substack{\uparrow \\ \text{by Claim 1 and 2}}}{=} (A \cup f(M)) \setminus A \underset{\substack{\uparrow \\ \text{since } A \cap f(M) = \varnothing}}{=} f(M).$$

The same way we see that $\partial B = f(M)$. This gives us the desired result. ∎

218.4. **Wild knots.** In these lecture notes we considered already on several occasions the embedding $w\colon S^1 \to \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ that is shown in the figure below.



We are aware that the image is "wild", but it is hard to pin this down. As a first step, we showed in Exercise 30.21 that $w(S^1)$ is not a smooth submanifold of $\mathbb{R}^3$. In this section we will prove the following theorem, which now makes it precise that $w(S^1)$ is indeed "wild".

Theorem 218.13. (Wild Knot Existence Theorem) There exists an embedding $S^1 \to S^3$ (e.g. the one shown above) such that the image is not a topological submanifold.

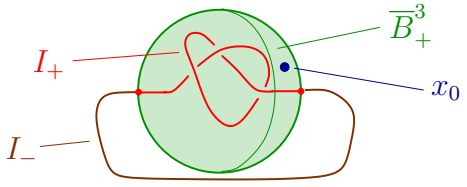
In the following we will give a very precise construction of such a map $w\colon S^1 \to S^3$. The reader who is comfortable with the above pictorial description can also jump straight to the Wild Embedding Existence Proposition 218.16 for the essential part of the proof of the Wild Knot Existence Theorem 218.13.

The precise construction of a "wild" embedding $w\colon S^1 \to S^3$ requires some preparations. We start out with the following lemma:

Lemma 218.14. (Disk Knot–Non-Abelian Lemma) There exists a proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ with the following two properties:

(1) There exists an $\epsilon > 0$ such that $\tau(t) = (2t-1, 0, 0)$ for $t \in [0, \frac{\epsilon}{2}]$ and $t \in [1 - \frac{\epsilon}{2}, 1]$ and such that $\tau^{-1}(\{x \in \overline{B}^3 \mid \|x\| > 1 - \epsilon\}) = [0, \frac{\epsilon}{2}] \cup [1 - \frac{\epsilon}{2}, 1]$.
(2) $\pi_1(\overline{B}^3 \setminus \tau([0,1]))$ is non-abelian.

Sketch of proof. We consider the smooth embedding that is illustrated above.[2337] We set $I_+ := \tau([0,1])$. We also consider the smoothly embedded interval I_- that is shown in the figure below.



Recall that in Definition 111.13 we saw that we can write $S^3 =_{\text{i}} \overline{B}^3_- \cup_{S^2} \overline{B}^3_+$. Next we pick a base point $x_0 \in S^2 \setminus (I_- \cap I_+)$. By the Manifold–Seifert-van Kampen Theorem 102.1 we

[2337] For readers who do not like "examples by picture", there is an alternative approach: One could also start out with the trefoil $K \subseteq S^3$, see Definition 111.13, then use the fact that K is a 1-dimensional submanifold to find a smooth embedding $\Phi\colon \overline{B}^3 \to S^3$ with $\Phi^{-1}(K) = \overline{B}^1 \times \{0\}$, then use the discussion on page 1229 and the Smooth Ball Isotopy Proposition 45.12 to show that there exists a smooth embedding $\Psi\colon \overline{B}^3 \to S^3$ such that $\Phi(\overline{B}^3) \cup \Psi(\overline{B}^3) = S^3$ and $\Phi(\overline{B}^3) \cap \Psi(\overline{B}^3) = \Phi(S^2) = \Psi(S^2)$. After a few other minor modifications one can arrange that the preimage of K under Ψ has the desired properties.

have the following pushout diagram:

$$\begin{array}{ccc} \pi_1(S^2 \setminus (I_- \cap I_+), x_0) & \longrightarrow & \pi_1(\overline{B}^3_+ \setminus I_+, x_0) \\ \downarrow \cong & & \downarrow \\ \pi_1(\overline{B}^3_- \setminus I_-, x_0) & \longrightarrow & \pi_1(S^3 \setminus (I_- \cup I_+)_+, x_0). \end{array}$$

We make the following observations:

(1) Note that $I_- \cup I_+$ is smoothly isotopic to the trefoil K. In the Trefoil–Non-Trivial Proposition 111.19 we showed that $\pi_1(S^3 \setminus K)$ is non-abelian. It follows from the Isotopic Link-π_1-Lemma 111.4 that $\pi_1(S^3 \setminus (I_- \cup I_+), x_0)$ is also non-abelian.

(2) Since I_- is an unknotted arc, i.e. since it is smoothly isotopic to the "standard arc" $\overline{B}^1 \times \{(0,0)\}$, we see that $S^2 \setminus (I_- \cap I_+)$ is a deformation retract of $\overline{B}^3_- \setminus I_-$. It follows from the Homotopy Equivalence-π_1-Proposition 90.9 that the left vertical map is an isomorphism.

(3) It follows from (2) and the Pushout–Opposite Lemma 6.26 (a) that the right vertical map is an isomorphism.

(4) It follows from (1) and (3) that $\pi_1(\overline{B}^3_+ \setminus I_+, x_0)$ is non-abelian. ■

Our goal is to construct an embedding $f\colon S^1 \to S^3$ such that $f(S^1)$ is not a topological submanifold of S^3. To simplify the notation we will use the homeomorphisms from the Ball Quotient-Sphere Lemma 8.10 and page 319 to make the identifications $[0,2]/0 \sim 2 =_{\mathrm{i}} S^1$ and $\overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2 =_{\mathrm{i}} S^3$.

Notation 218.15.

(1) For the remainder of this section we fix a proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ as in the Disk Knot–Non-Abelian Lemma 218.14.

(2) Given $n \in \mathbb{N}$ we consider the smooth embedding

$$\begin{array}{rcl} \varphi_n\colon \overline{B}^3 & \to & B^3 \subseteq \overline{B}^3 \subseteq S^3 \\ x & \mapsto & (-1 + \frac{3}{2^n}, 0, 0) + \frac{1}{2^n} \cdot x. \end{array}$$

(3) We consider the map

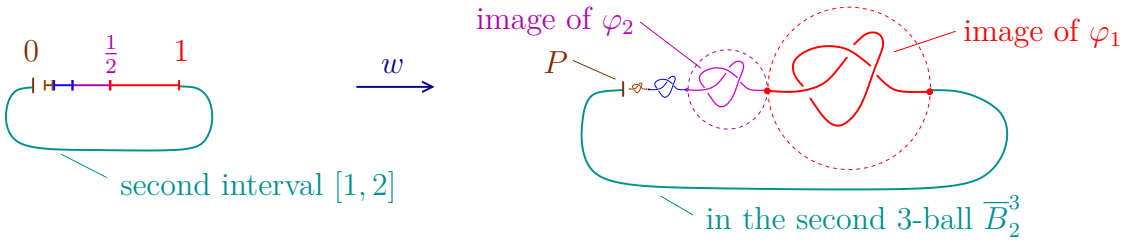
$$w\colon \overbrace{[0,2]/0 \sim 2}^{=_{\mathrm{i}} S^1} \to \overbrace{\overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2}^{=_{\mathrm{i}} S^3}$$

$$[t] \mapsto \begin{cases} (-1,0,0) \in \overline{B}^3_1, & \text{if } t = 0, \\ \varphi_n(\tau(\underbrace{2^{n-1} \cdot (t - \frac{1}{2^n})}_{\in (0,1]})) \in \overline{B}^3_1, & \text{if } t \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}] \text{ for some } n \in \mathbb{N}, \\ (3 - 2t, 0, 0) \in \overline{B}^3_2, & \text{if } t \in [1,2]. \end{cases}$$

(4) We set $W := w(S^1)$ and we write $P := w([0])$.

The following proposition immediately implies the Wild Knot Existence Theorem 218.13.

Proposition 218.16. (Wild Embedding Existence Proposition) Let $w\colon S^1 \to S^3$ be the above map.



(1) The map $w\colon S^1 \to S^3$ is an embedding.
(2) There is no topological submanifold chart, in the sense of Definition 23.18, for the subspace $W := w(S^1) \subseteq S^3$ around $P = w([0])$. In particular $W = w(S^1)$ is not a topological submanifold of S^3.

Our proof of the Wild Embedding Existence Proposition 218.16 crucially relies on the following lemma, which is a descendant of the General Alexander Duality Theorem 218.2.

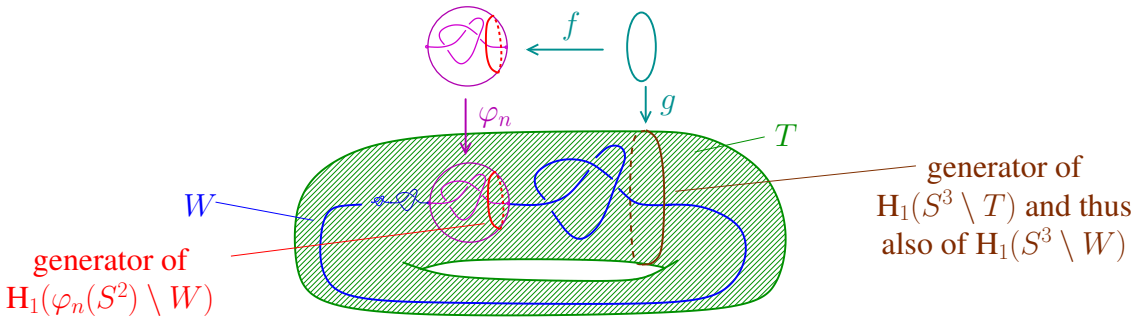
Auxiliary Lemma 218.17. For any $n \in \mathbb{N}$ the inclusion-induced map

$$\mathrm{H}_1(\varphi_n(S^2) \setminus W) \;\to\; \mathrm{H}_1(S^3 \setminus W)$$

is an isomorphism.

Proof. Let $n \in \mathbb{N}$. We make the following preparations and observations:

(1) We denote by $i\colon \varphi_n(S^2) \setminus W \to S^3 \setminus W$ the inclusion.
(2) We denote by $T =_{\mathrm{i}} \overline{B}^2 \times S^1$ the "standard solid torus" in $S^3 = \overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2$ that is given by the points $(x, y, z) \in \overline{B}_i$ in each ball with $y^2 + z^2 \leq \frac{1}{2}$. We denote by $j\colon W \to T$ the inclusion map.
(3) Note that $(\varphi_n|_{S^2})^{-1}(W) = \{(\pm 1, 0, 0)\}$.
(4) (a) Let $f\colon S^1 \to S^2 \setminus \{(\pm 1, 0, 0)\}$ be the map that is given by $(x, y) \mapsto (0, x, y)$. Note that f is an embedding and also note that $f(S^1)$ is a deformation retract of $S^2 \setminus \{(\pm 1, 0, 0)\}$. It follows that f is a homotopy equivalence.
(b) Let $g\colon S^1 \to \partial T = \partial(\overline{B}^2 \times S^1)$ be the map that is given by $z \mapsto (z, w_0)$ for some fixed $w_0 \in S^1$.
(c) The two maps $\varphi_n \circ f, g\colon S^1 \to S^3 \setminus W$ are clearly homotopic.



We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_1(S^1) & \xrightarrow{\;g_*\;} & \mathrm{H}_1(S^3 \setminus T) \\ {\scriptstyle (\varphi_n \circ f)_*}\downarrow & & \downarrow \\ \mathrm{H}_1(\varphi_n(S^2) \setminus W) & \longrightarrow & \mathrm{H}_1(S^3 \setminus W). \end{array}$$

We make a few more observations:

(5) It follows from the Wild Knot Complement-H_*-Lemma 218.4 that the right vertical map is an isomorphism.

(6) (a) It follows from (4a) and the Homotopy Equivalence-H_*-Proposition 144.14 that the left vertical map is an isomorphism.
(b) It follows from (4b) that the top horizontal map is an isomorphism.
(c) It follows from (4c) and the Homotopy-H_*-Proposition 144.12 that the diagram commutes.

It follows from (5) and (6) that the bottom horizontal map is indeed an isomorphism. ■

Our proof of the Wild Embedding Existence Proposition 218.16 relies also on the following earlier proposition.

Proposition 98.26. (Amalgamated Product-Monomorphism Proposition) Suppose we are given two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon G \to B$. We denote by $A *_G B$ the corresponding amalgamated product from Definition 98.23. If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.

Proof of Proposition 218.16. Let $w\colon S^1 \to S^3$ be the map from Notation 218.15. Recall that we need to prove the following two statements:

(1) The map $w\colon S^1 \to S^3$ is an embedding.
(2) There is no topological submanifold chart, in the sense of Definition 23.18, for the subspace $W := w(S^1) \subseteq S^3$ around $P = w([0])$. In particular $W = w(S^1)$ is not a topological submanifold of S^3.[2338]

First we provide the proof of Statement (1). Using the ϵ-δ-criterion from the Metric Continuity Proposition 4.9 one can easily show that the map $[0,1] \to S^3$, induced by w, is continuous. It is clear that the corresponding map $[1,2] \to S^3$ is continuous. It follows from the Pasting Proposition 4.7 together with the Topological-Quotient Proposition 8.2 (1b) that the given map $f\colon S^1 \to S^3$ is continuous. It now follows from the Compact-Hausdorff Proposition 4.23 (2) that $f\colon S^1 \to S^3$ is actually an embedding.

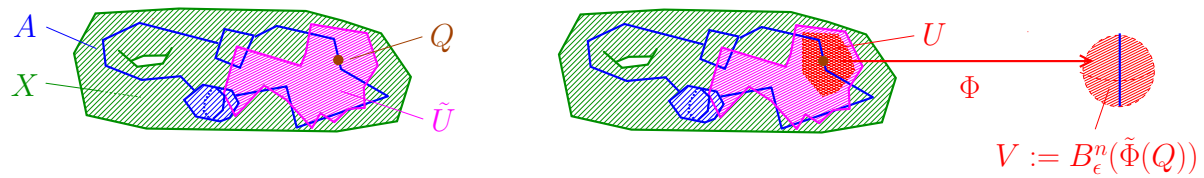
Now we turn to the proof of Statement (2). We start out with the following harmless general claim:

Claim 1. Let X be an n-dimensional topological manifold, let $A \subseteq X$ be a subspace and let $Q \in A$. If there exists an $(n-2)$-dimensional submanifold chart of type (α) for $A \subseteq X$ and $Q \in A$, then there exists an $(n-2)$-dimensional submanifold chart $\Phi\colon U \to V$ such that $U \setminus A$ is path-connected and such that for every $x_0 \in U \setminus A$ the group $\pi_1(U \setminus A, x_0)$ is isomorphic to $\mathbb{Z}$.[2339]

Proof. Let $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ be an $(n-2)$-dimensional submanifold chart of type (α) for $A \subseteq X$ and $Q \in A$. Since $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ is an $(n-2)$-dimensional submanifold chart of type (α) we know that $\widetilde{\Phi}(A \cap \widetilde{U}) = (\{0\} \times \mathbb{R}^{n-2}) \cap \widetilde{V}$.

Since $\widetilde{V} \subseteq \mathbb{R}^n$ is open there exists an $\epsilon > 0$ such that $V := B^n_\epsilon(\widetilde{\Phi}(Q)) \subseteq \widetilde{V}$. We now set $U := \widetilde{\Phi}^{-1}(V)$ and $\Phi := \widetilde{\Phi}|_U$.

[2339]Since $U \setminus A$ is path-connected we know by the Base Point Independence Lemma 110.2 that the isomorphism type of the fundamental group does not depend on the choice of the base point.



From the above it follows that $\Phi(A \cap U) = \{0\} \times B^{n-2} \subseteq B^n$. We now see that

$$\pi_1(U \setminus A, x_0) \underset{\text{since } \Phi|_U \text{ is a homeomorphism}}{\cong} \pi_1(B^n \setminus (\{0\} \times B^{n-2})) \underset{\text{since } B^n \setminus (\{0\} \times B^{n-2}) \text{ admits a deformation retraction to the circle } S^1_{\frac{1}{2}} \times \{0\}}{\cong} \pi_1(S^1) \underset{\text{by the Circle-}\pi_1 \text{ Theorem 88.22}}{\cong} \mathbb{Z}.$$

⊞

Note that S^1 and S^3 are topological manifolds with empty boundary. It follows from the Unique Type-of-Chart Proposition 150.11 that neither $\mathbb{R}$ nor $\mathbb{R}^3$ admits a chart of type (ii) in the sense of Definition 23.2. This implies that if there exists a submanifold for $P \in W \subseteq S^3$, then it has to be a chart of type (α). It follows from this discussion, together with Claim 1, that it suffices to prove the following claim:

Claim 2. For every open neighborhood U of $P \in S^3$ such that $U \setminus W$ is path-connected, the fundamental group $\pi_1(U \setminus W)$ is non-abelian.

Proof. Let U be an open neighborhood of $P \in S^3$ such that $U \setminus W$ is path-connected. Since U is a neighborhood of $P = w([0])$ there exists an $n \in \mathbb{N}$ such that $\varphi_n(\overline{B}^3) \subseteq U$. We pick an $\epsilon > 0$ as in the Disk Knot–Non-Abelian Lemma 218.14.

(a) Basically by construction we know that $\varphi_n(B^3)$ is an open subset of S^3.
(b) Note that $S^2_{1-\epsilon}$ is a submanifold of B^3 that intersects $\tau([0,1])$ transversally in two points.
(c) Note that $\overline{B}^3_{1-\epsilon}$ is a codimension-zero submanifold of B^3.
(d) It follows from (a) and (c) that $\varphi_n(\overline{B}^3_{1-\epsilon})$ is a codimension-zero submanifold of S^3.
(e) It follows from the Codimension-Zero Submanifold Proposition 150.16 and (d) that the subspace $S^3 \setminus \varphi_n(\overline{B}^3_{1-\epsilon})$ is also a codimension-zero submanifold of S^3.

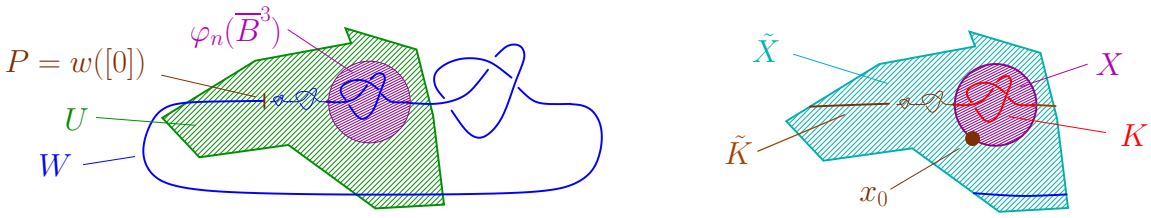
We introduce the following notation:

- We write $X \coloneqq \varphi_n(\overline{B}^3_{1-\epsilon})$, $\widetilde{X} \coloneqq U \setminus \varphi_n(B^3_{1-\epsilon})$ and $Y \coloneqq \varphi_n(S^2_{1-\epsilon})$,
- We write $K \coloneqq W \cap X$, $\widetilde{K} \coloneqq W \cap \widetilde{X}$ and $L \coloneqq K \cap Y$.

We now see that for any $x_0 \in Y \setminus L$ we have

$$\pi_1(U \setminus W, x_0) \underset{\text{follows from the Manifold Seifert-van Kampen Theorem 102.1}}{\overset{\cong}{\longleftarrow}} \underbrace{\pi_1(X \setminus K, x_0)}_{\text{non-abelian by choice of } W} *_{\pi_1(Y \setminus L, x_0)} \pi_1(\widetilde{X} \setminus \widetilde{K}, x_0).$$

We know that $\pi_1(X \setminus K)$ is non-abelian. To conclude that $\pi_1(U \setminus W)$ is non-abelian, it suffices by the Amalgamated Product-Monomorphism Proposition 98.26 to prove the following subclaim:



Subclaim. The two inclusion-induced homomorphisms $\pi_1(Y \setminus L, x_0) \to \pi_1(X \setminus K, x_0)$ and $\pi_1(Y \setminus L, x_0) \to \pi_1(\widetilde{X} \setminus \widetilde{K}, x_0)$ are monomorphisms.

Proof of subclaim. We consider the following diagram:

$$\begin{array}{ccc} \pi_1(Y \setminus L, x_0) & \longrightarrow & \pi_1(X \setminus K, x_0) \\ \downarrow & \searrow & \downarrow \\ \pi_1(\widetilde{X} \setminus \widetilde{K}, x_0) & \longrightarrow & \pi_1(S^3 \setminus W, x_0). \end{array}$$

Here all maps are induced by inclusions. By the functoriality of fundamental groups the diagram commutes. We see that it suffices to prove that the diagonal map is a monomorphism. For that we consider the following diagram:

$$\begin{array}{ccc} \pi_1(Y \setminus L, x_0) & \longrightarrow & \pi_1(S^3 \setminus W, x_0) \\ \downarrow \cong & & \downarrow \\ \pi_1(Y \setminus L, x_0)_{\mathrm{ab}} & \longrightarrow & \pi_1(S^3 \setminus W, x_0)_{\mathrm{ab}} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{H}_1(Y \setminus L) & \xrightarrow{\cong} & \mathrm{H}_1(S^3 \setminus W). \end{array}$$

We make the following observations and clarifications:

(i) The top vertical maps are the natural maps from Definition 98.15. Since the group $\pi_1(Y \setminus L, x_0) \cong \mathbb{Z}$ is abelian we know that the top left vertical map is an isomorphism.

(ii) The bottom vertical maps are are the natural Hurewicz homomorphisms from the Hurewicz Homomorphism Proposition 166.3. By the Hurewicz Theorem 166.10 we know that these are isomorphisms.

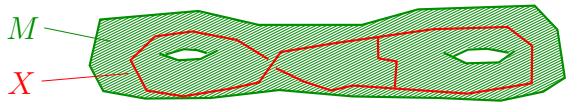
(iii) By Lemma 218.17 we know that the bottom horizontal map is an isomorphism.

By naturality the diagram commutes. It thus follows from the above discussion that the top horizontal map is, as promised, a monomorphism. ∎

218.5. **Invariants of complements.** Let us first recall a result about smooth manifolds and smooth embeddings: Let M be a compact smooth manifold, let X be a compact smooth manifold and let $f\colon X \to M$ be a *smooth* embedding. By the Complement-of-Submanifold Invariants Proposition 179.12 we know that $\pi_1(M \setminus f(X))$ is finitely presented and that the homology groups $\mathrm{H}_k(M \setminus f(X))$ are finitely generated.

We now want to consider what happens, if we consider topological manifolds instead of smooth manifolds. The following proposition shows that when it comes to homology groups, we can generalize the above statement to the more general topological setting:

Proposition 218.18. (Complement-of-Subspace-H_*-Proposition) Let M be a compact orientable topological manifold and let $X \subseteq M$ be a weakly locally contractible, compact subspace such that $X \cap \partial M$ is weakly locally contractible. Then the homology groups $H_k(M \setminus X)$ are also finitely generated.



each $H_k(M \setminus X)$ is finitely generated

The proof of the Complement-of-Subspace-H_*-Proposition 218.18 relies on the following lemma:

Lemma 218.19. (Finite-H_*-H^*-Lemma) Let M be a compact orientable topological manifold and let $X \subseteq M$ be a weakly locally contractible, compact subspace of M. The homology groups $H_i(X)$ and the cohomology groups $H^i(X;\mathbb{Z})$ of X are finitely generated.

Proof of Lemma 218.19. Since X is a compact subspace of a topological manifold and since X is weakly locally contractible we know by the Euclidean Embedding Theorem 33.2, Borsuk's Theorem 189.7 and the Simplicial Complex Retract Proposition 189.11 that X is homeomorphic to a retract of a finite simplicial complex.

By the Simplicial Complex-Retract-Invariants Lemma 189.15 (2b) we know that the homology groups of X are finitely generated. It thus follows from the Universal Coefficient-H^*-Theorem 201.23 and basic facts about Hom-groups and Ext-groups that the cohomology groups of X are also finitely generated. ■

Proof of the Complement-of-Subspace-H_*-Proposition 218.18. Let M be a compact orientable topological manifold and let $X \subseteq M$ be a weakly locally contractible, compact subspace. We first consider the case that M is actually *closed*. In this setting we consider the following maps:

$$\begin{array}{ccccccccc} & & & & & & H_k(M \setminus X) & & \\ & & & & & & \cong \downarrow \text{(1)–(4)} & & \\ \underbrace{H^{n-k}(M;\mathbb{Z})}_{\text{finitely generated}} & \leftarrow & \underset{U \in \mathcal{N}(X)}{\operatorname{colim}} H^{n-k}(M,U;\mathbb{Z}) & \xleftarrow{(5)} & \underset{U \in \mathcal{N}(X)}{\operatorname{colim}} H^{n-k-1}(U;\mathbb{Z}) & \leftarrow & H^{n-k-1}(M;\mathbb{Z}). & & \\ & & & & & & \cong \downarrow \text{(6)} & & \\ & & & & & & \underbrace{H^{n-k-1}(X;\mathbb{Z})}_{\text{finitely generated}} & & \end{array}$$

We make the following clarifications and observations:

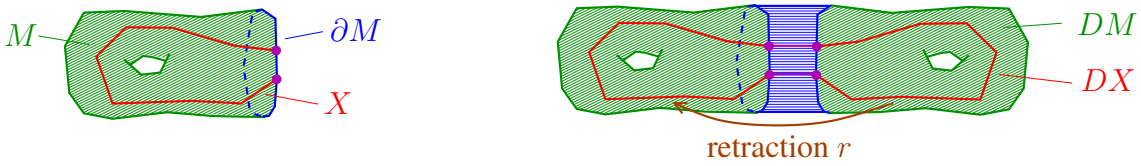
(a) In Definition 218.6 we wrote down six homomorphisms. In the proof of the General Alexander Duality Theorem 218.2 (1) we showed that, regardless of any homological information, the maps (1)–(4) and (6) are isomorphisms. These give us the vertical isomorphisms.

(b) In the proof of the General Alexander Duality Theorem 218.2 (1) we also discussed the homomorphism (5). In our setting this is not an isomorphism, but we saw that it fits into the horizontal sequence and that this sequence is exact.

(c) Since X and M are compact it follows from the Finite-H_*-H^*-Lemma 218.19 that the groups $\mathrm{H}^{n-k}(M;\mathbb{Z})$ and $\mathrm{H}^{n-k-1}(X;\mathbb{Z})$ are finitely generated.
(d) It follows from the above discussion together with the Finite Generation Lemma 97.20 that $\mathrm{H}_k(M \setminus X)$ is finitely generated.

Finally we consider the case that $\partial M \neq \varnothing$. To deal with this case we introduce the risque notation $\partial X \coloneqq X \cap \partial M$. We now assume that $X \cap \partial M$ is also weakly locally contractible. We consider[2340]

$$\begin{aligned} DM &\coloneqq M \cup_{\partial M = \partial M \times \{-1\}} (\partial M \times [-1,1]) \cup_{\partial M \times \{1\} = \partial M} M, \\ DX &\coloneqq X \cup_{\partial X = \partial X \times \{-1\}} (\partial X \times [-1,1]) \cup_{\partial X \times \{1\} = \partial X} X. \end{aligned}$$



We make the following observations:

(a) Basically the same argument as in the Doubling-Manifold Lemma 54.10 and the Double-Orientability Lemma 192.6 shows that DM is a closed oriented topological manifold.
(b) Using the hypothesis that X and ∂X are weakly locally contractible one can show, with a little bit of an effort, that DX is also weakly locally contractible.[2341]
(c) It follows from (a), (b) and the above discussion of the closed case that the homology groups $\mathrm{H}_k(DM \setminus DX)$ are finitely generated.
(d) Let $i\colon M \setminus X \to DM \setminus DX$ be the obvious embedding. Similar to the Doubling-Manifold Lemma 54.10 one sees that there exists a continuous map $r\colon DM \setminus DX \to M \setminus X$ with $r \circ i = \mathrm{id}_{M \setminus X}$.
(e) From $r \circ i = \mathrm{id}_{M \setminus X}$ we obtain that $r_*\colon \mathrm{H}_k(DM \setminus DX) \to \mathrm{H}_k(M \setminus X)$ is an epimorphism.
(f) It follows from (c) and (e) that each $\mathrm{H}_k(M \setminus X)$ is finitely generated. ■

The Complement-of-Submanifold Invariants Proposition 179.12 and the Complement-of-Subspace-H_*-Proposition 218.18 raise the question, whether we can also control the size of the *fundamental groups* of complements of weakly locally contractible, compact subspaces. But the following proposition shows that this is not the case.

Proposition 218.20. (Wild Knot-π_1-Proposition) There is an embedding $w\colon S^1 \to S^3$ such that $\pi_1(S^3 \setminus w(S^1))$ is not finitely generated.

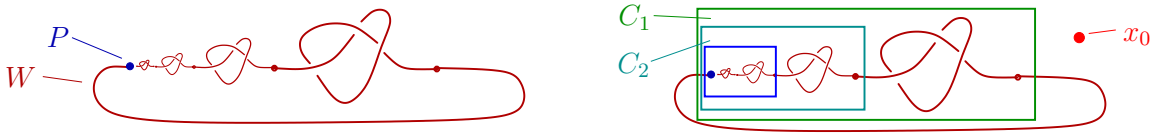
Sketch of proof. We consider the "wild knot" $W \coloneqq w(S^1) \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ from Notation 218.15. Our goal is to show that $\pi_1(S^3 \setminus W)$ is not finitely generated. We introduce the following notation:

(1) For $n \in \mathbb{N}$ we denote by $C_n \subseteq \mathbb{R}^3$ the compact cuboid that is shown in the figure below. Note that $\bigcap\limits_{n \in \mathbb{N}} C_n = \{P\}$. We set $U_n \coloneqq (S^3 \setminus C_n) \setminus W$. Note that the following statements hold:
- Each U_n is an open subset of $S^3 \setminus W$.
- For each $n \in \mathbb{N}$ we have $U_n \subseteq U_{n+1}$.

[2340] The topological space DM is almost identical to the double $\mathrm{D}M$ that we introduced in Definition 54.8.
[2341] This argument requires that we added the extra $\partial X \times [-1,1]$.

- We have $\bigcup_{n\in\mathbb{N}} U_n = S^3 \setminus W$.

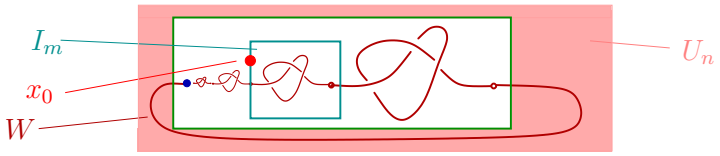
(2) We pick $x_0 \in U_1$.



We continue with the following crucial observation: If $\pi_1(S^3 \setminus W, x_0)$ was finitely generated, then it would follow from an elementary compactness argument, as in the proof of the Exhaustion-π_1-Proposition 105.2, that there exists an $n \in \mathbb{N}$ such that the inclusion-induced map $\pi_1(U_n, x_0) \to \pi_1(S^3 \setminus W, x_0)$ is an epimorphism. Thus it remains to prove the following claim:

Claim. There is no $n \in \mathbb{N}$ such that the inclusion-induced map $\pi_1(U_n, x_0) \to \pi_1(S^3 \setminus W, x_0)$ is an epimorphism.

Proof. Let $n \in \mathbb{N}$. We pick an $m > n$. We denote by I_m the m-th cube in the construction of the "wild knot" W. We consider the two inclusion maps $i\colon (\partial I_m) \setminus W \to I_m \setminus W$ and $j\colon (\partial I_m) \setminus W \to (S^3 \setminus \mathring{I}_m) \setminus W$. We pick $x_0 \in (\partial I_m) \setminus W$ and we immediately erase it from the notation in the discussion below.



Next we consider the following two diagrams:

$$\begin{array}{ccc}
\overbrace{\pi_1((\partial I_m) \setminus W)}^{\cong\mathbb{Z}} & \xrightarrow{j_*} & \pi_1((S^3 \setminus \mathring{I}_m) \setminus W) \\
\cong\downarrow & & \downarrow \\
\pi_1((\partial I_m) \setminus W)_{\mathrm{ab}} & \xrightarrow{j_*} & \pi_1((S^3 \setminus \mathring{I}_m) \setminus W)_{\mathrm{ab}} \\
\cong\downarrow & & \downarrow \\
\mathrm{H}_1((\partial I_m) \setminus W) & \xrightarrow{j_*} & \mathrm{H}_1((S^3 \setminus \mathring{I}_m) \setminus W) \\
\mathrm{id}\downarrow & & \downarrow \\
\mathrm{H}_1((\partial I_m) \setminus W) & \xrightarrow[\cong]{j_*} & \mathrm{H}_1(S^3 \setminus W).
\end{array}$$

$$\begin{array}{ccc}
 & & \pi_1(U_n) \\
 & \overset{\rho}{\swarrow} & \downarrow \\
\pi_1((\partial I_m) \setminus W) & \xrightarrow[j_*]{} & \pi_1((S^3 \setminus \mathring{I}_m) \setminus W) \\
i_*\downarrow & & \downarrow \\
\pi_1(I_m \setminus W) & \longrightarrow & \pi_1(S^3 \setminus W) \\
 & \searrow^{\mathrm{id}} & \searrow^{=:\Theta} \\
 & & \pi_1(I_m \setminus W)
\end{array}
\qquad i_* \circ \rho$$

We make the following observations and clarifications:

(1) First we consider the diagram to the left. This diagram is very similar to the diagram on page 4197. In fact we again see that the left vertical maps and the bottom horizontal maps are isomorphisms. It follows from this discussion that there exists a left-inverse $\rho\colon \pi_1((S^3 \setminus \mathring{I}_m) \setminus W) \to \pi_1((\partial I_m) \setminus W)$ to $j_*\colon \pi_1((\partial I_m) \setminus W) \to \pi_1((S^3 \setminus \mathring{I}_m) \setminus W)$.

(2) In the diagram to the right we show the pushout diagram that we obtain from the Manifold–Seifert-van Kampen Theorem 102.1. By the universal property of pushouts and

the construction of ρ we get an induced homomorphism $\Theta\colon \pi_1(S^3 \setminus W) \to \pi_1(I_m \setminus W)$ which makes the diagram commute.

(3) By construction of W in Notation 218.15 we know that $\pi_1(I_m \setminus W)$ is non-abelian.

In summary we have found a homomorphism $\Theta\colon \pi_1(S^3 \setminus W) \to \pi_1(I_m \setminus W)$ with non-abelian image but such that the image of $\pi_1((S^3 \setminus \mathring{I}_m) \setminus W)$ is abelian. From $m > n$ we obtain that $U_n \subseteq (S^3 \setminus \mathring{I}_m) \setminus W$. Thus the image of $\pi_1(U_n)$ is also abelian. It follows that the inclusion-induced map $\pi_1(U_n) \to \pi_1(S^3 \setminus W)$ is not an epimorphism. ■

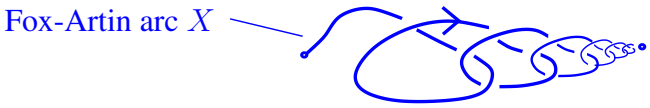
The following proposition is a straightforward consequence of the above result:

Proposition 218.21. (No Regular Neighborhood Proposition) There exists an embedding $w\colon S^1 \to S^3$ such that $w(S^1) \subseteq S^3$ does not admit a regular neighborhood.

Proof. By the Wild Knot-π_1-Proposition 218.20 there exists an embedding $w\colon S^1 \to S^3$ such that $\pi_1(S^3 \setminus w(S^1))$ is not finitely generated. It follows from this discussion and the Basics-of-Regular Neighborhood Lemma 217.4 that $w(S^1)$ does not admit a regular neighborhood. ■

Example. We conclude this discussion of "wild" subspaces with a few more examples:

(1) (a) In the following figure we consider the Fox-Artin arc X.



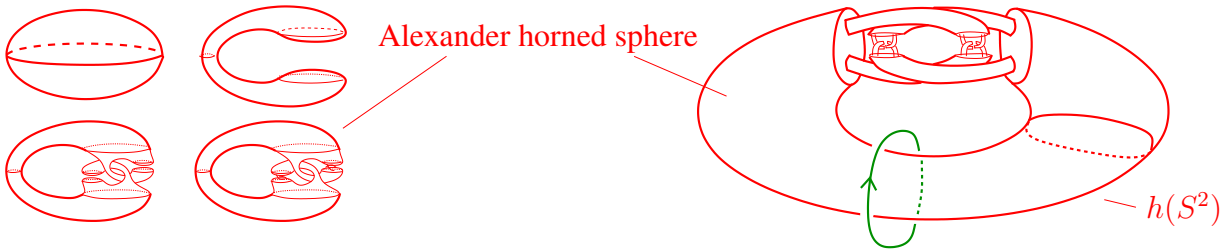
One can easily show that $\pi_1(S^3 \setminus X)$ is trivial. But in [**FA1948**] and [**DV2009**, p. 77] it is shown that X is not a topological submanifold of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$.

(b) In the figure below we show two subspaces Y and Z of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ that are homeomorphic to the compact interval $[0, 1]$.



One can easily show that $\pi_1(S^3 \setminus Y)$ is trivial. On the other hand it follows from [**DV2009**, p. 78] and [**GHM1972**, Corollary 2.6] that $\pi_1(S^3 \setminus Z)$ is not finitely generated. Furthermore note that by [**DV2009**, p. 78] we know that in fact neither $S^3 \setminus Y$ nor $S^3 \setminus Z$ is homeomorphic to the open 3-ball B^3.

(2) We now turn turn to embeddings of 2-dimensional spheres into $\mathbb{R}^3$. In the figure below we show an embedding $h\colon S^2 \to \mathbb{R}^3$ which was introduced by James Alexander [**Ale1924a**]. The image $h(S^2)$ is called the *Alexander horned sphere*.



A rigorous description is given in [**Ale1924a**] and alternatively in [**Hat2002**, p. 170] and in [**DV2009**, p. 46]. Furthermore in these references it is shown that one of the

components of $\mathbb{R}^3 \setminus h(S^2)$ is not simply connected, namely the green loop shown above is not null-homotopic in the "outside" component of $\mathbb{R}^3 \setminus h(S^2)$. □

218.6. **Appendix: Comparing the two Sphere–Alexander Duality Theorems.** Recall that we proved the following two theorems:

Theorem 217.8. (Sphere–Alexander Duality Theorem) Let $n \in \mathbb{N}$. Furthermore let $X \subsetneq S^n$ be a compact subspace of S^n that admits a regular neighborhood in the sense of Definition 217.2. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \cong \widetilde{\mathrm{H}}^{n-i-1}(X;\mathbb{Z}).$$

Theorem 218.7. (Sphere–Alexander Duality Theorem) Let $X \subsetneq S^n$ be a compact non-empty subspace of S^n. If X is weakly locally contractible, then for every $j \in \mathbb{N}_0$ the General Alexander Duality Theorem 218.2 gives us a natural isomorphism

$$\mathrm{AD}_X \colon \widetilde{\mathrm{H}}_j(S^n \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

The following question arises: What is the connection between the Sphere–Alexander Duality Theorem 218.7 and the previous Sphere–Alexander Duality Theorem 217.8?

The following lemma implies that whenever the hypotheses for the Sphere–Alexander Duality Theorem 217.8 are satisfied, then so are the hypotheses for the Sphere–Alexander Duality Theorem 218.7. In other words, the lemma implies that the General Alexander Duality Theorem 218.2 generalizes the Sphere–Alexander Duality Theorem 217.8.

Lemma 218.22. (Regular Neighborhood–Weakly Locally Contractible Lemma) Let M be a smooth manifold and let $X \subseteq M$ be a subspace. If X admits a regular neighborhood, then X is weakly locally contractible.

We will prove the Regular Neighborhood–Weakly Locally Contractible Lemma 218.22 in a second. But first let us state the following proposition which completely clarifies the relationship between the two Sphere–Alexander Duality Theorems:

Proposition 218.23. (Sphere–Alexander Duality Isomorphisms Proposition) Let $X \subsetneq S^n$ be a compact non-empty subspace of S^n. If X admits a regular neighborhood N, then X is weakly locally contractible, and the isomorphism

$$\mathrm{AD}_X \colon \widetilde{\mathrm{H}}_j(S^n \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z})$$

from the Sphere–Alexander Duality Theorem 218.7 agrees with the isomorphism from the Sphere–Alexander Duality Theorem 217.8 (which a priori depended on the choice of N).[2342]

The remainder of this appendix is dedicated to the proofs of the Regular Neighborhood–Weakly Locally Contractible Lemma 218.22 and the Sphere–Alexander Duality Isomorphisms Proposition 218.23.

Proof of Lemma 218.22. [2343] Let M be a smooth manifold and let $X \subseteq M$ be a subspace which admits a regular neighborhood N. We need to show that X is weakly locally

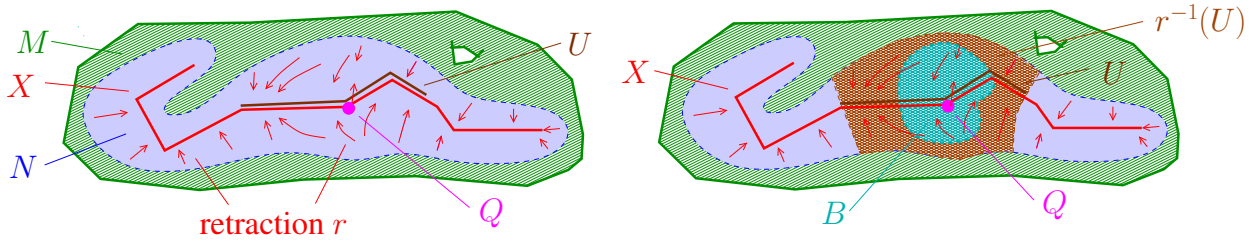
[2342] We have thus shown that the choice of the regular neighborhood in the proof of the Sphere–Alexander Duality Theorem 217.8 had no impact on the isomorphism we constructed.

[2343] As we will see, this lemma follows essential from the same argument as in the Manifold Neighborhood Retract–Contractibility Lemma 189.6.

contractible. Recall that by Definition 217.2 of a regular neighborhood in Definition 217.2 we know in particular the following:

(1) X is contained in $N \setminus \partial_0 N$.
(2) N admits a retraction $r\colon N \to X$.

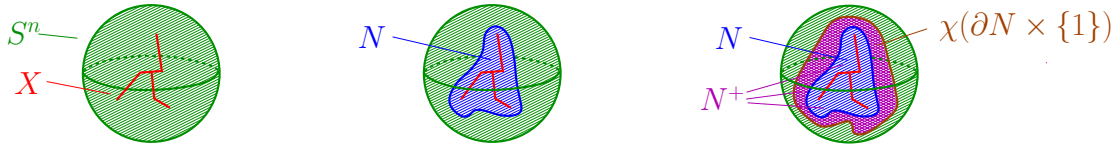
Let $Q \in X$ and let U be a neighborhood of Q in X. By the Manifold–Local Properties Lemma 23.8 we know that M is locally contractible. It follows from this information, (1) and (2) and the Contractability Criterion 19.20 that there exists an open neighborhood B of $Q \in M$ with $B \subseteq r^{-1}(U)$ and a homotopy $F\colon U \times [0,1] \to U$ from the identity to a constant map.



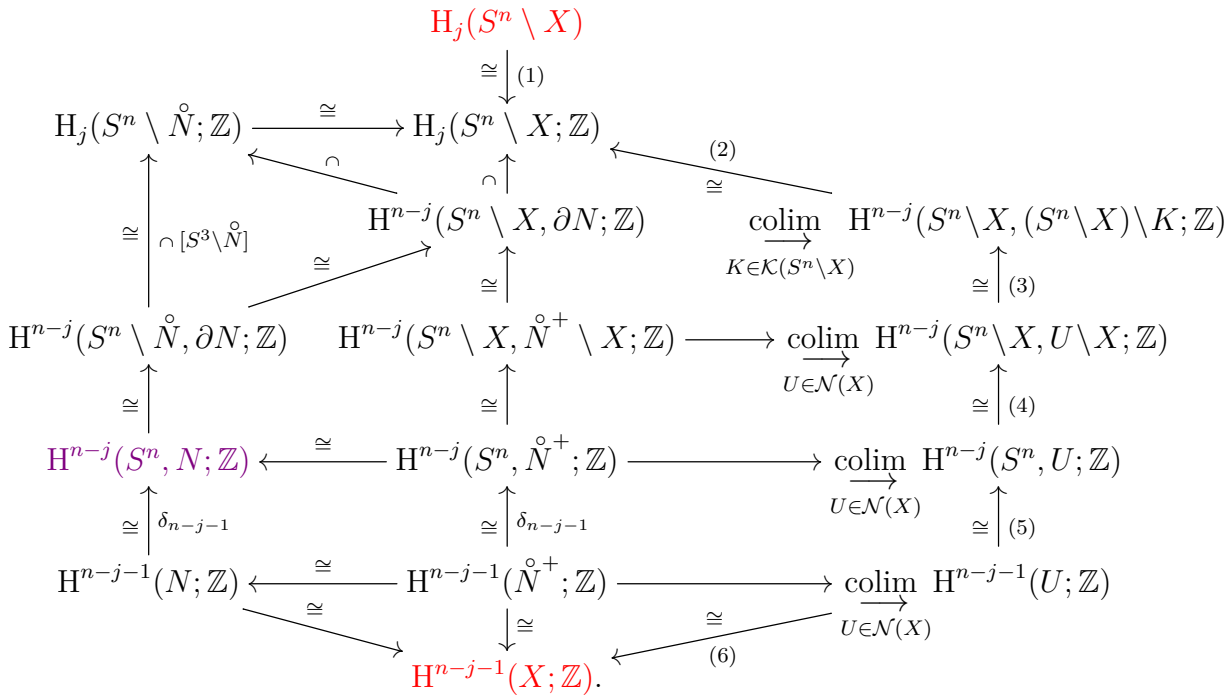
Next we consider the map
$$\begin{array}{rcl}(U \cap X) \times [0,1] & \to & U \\ (x,t) & \mapsto & r(\underbrace{F(x,t)}_{\in B \subseteq r^{-1}(U)}).\end{array}$$
Evidently this map is a homotopy from the identity to a constant map. In particular $V := U \cap X$ is contractible in U. ∎

Proof of Proposition 218.23. Let $X \subsetneq S^n$ be a compact non-empty subspace of S^n and let N be a regular neighborhood of X. We need to show that the isomorphism
$$\widetilde{\mathrm{H}}_j(S^n \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z})$$
which we gave explicitly in the proof of the Sphere–Alexander Duality Theorem 217.8 equals the Alexander duality isomorphism AD_X. In the following we deal with the case that $j \neq 0$ and $j \neq n-1$. The cases $j = 0$ and $j = n-1$ are proved almost the same way. Thus we leave it to the reader to verify these two special cases.

It follows from the Codimension-Zero Submanifold Proposition 25.49 (3) that $S^n \setminus \mathring{N}$ is a compact smooth manifold with boundary given by ∂N. We now apply the Smooth Collar Neighborhood Theorem 35.6 to this smooth manifold and we obtain a smooth embedding $\chi\colon \partial N \times [0,1] \to S^n \setminus \mathring{N}$ such that $\chi(P,0) = \chi(P)$ for all $P \in \partial N$. Next we consider $N^+ := N \cup \chi(\partial N \times [0,1])$. One can easily verify that $\mathring{N}^+ := N^+ \setminus \chi(\partial N \times \{1\})$ is an open neighborhood of X that admits a deformation retraction to X.



Next we consider the following diagram:



We make the following clarifications and observations:

(a) The left sequence of isomorphisms from $\mathrm{H}_j(S^n \setminus X)$ to $\mathrm{H}^{n-j-1}(X; \mathbb{Z})$ are essentially the isomorphisms that we considered on page 4150. The only minute difference is summarized in the following diagram:

$$\begin{array}{ccc} \mathrm{H}^{n-j}(S^n, N; \mathbb{Z}) & \longrightarrow & \mathrm{H}^{n-j}(S^n, X; \mathbb{Z}) \\ {\scriptstyle \delta_{n-j-1}}\uparrow & & \uparrow{\scriptstyle \delta_{n-j-1}} \\ \mathrm{H}^{n-j-1}(N; \mathbb{Z}) & \longrightarrow & \mathrm{H}^{n-j-1}(X; \mathbb{Z}). \end{array}$$

In the above diagram we go from the top left to bottom right via the lower left corner, whereas on page 4150 we took the route via the upper right corner. But the diagram commutes by the naturality of connecting homomorphisms. So this difference is immaterial.

(b) The right sequence of isomorphisms from $\mathrm{H}_j(S^n \setminus X)$ to $\mathrm{H}^{n-j-1}(X; \mathbb{Z})$ are the isomorphisms from the definition of the Alexander duality isomorphism AD_X which we gave in Definition 218.6.

(c)
- The maps decorated with "$\cap$" to the top left are given by capping with the image of the fundamental class $[S^n \setminus \mathring{N}]$ in $\mathrm{H}_n(S^n \setminus X, \partial N)$.
- The remaining maps are either induced by inclusions, or they are connecting homomorphisms of long exact sequences, or they come from the natural map into a direct colimit.

(d)
- It follows from the Cap Product-Naturality Lemma 206.13 that the two triangles at the top left and the very top commute and also that the pentagon to the top right commutes.

- It follows from the functoriality of cohomology groups, the naturality of connecting homomorphisms and the universal properties of direct colimits that the remaining part of the diagram commutes.

(e) It follows essentially from the Homotopy Equivalence-H*-Proposition 198.12 (1) that the maps connecting the left and the middle column are isomorphisms.

It follows from all of the above that the two maps from $\mathrm{H}_j(S^n \setminus X)$ to $\mathrm{H}^{n-j-1}(X;\mathbb{Z})$ are the same. ■

218.7. Exercises for Chapter 218 .

Exercise 218.1. Is is true that for every compact subspace of S^1 we have

$$\widetilde{\mathrm{H}}_0(S^1 \setminus X) \;\cong\; \widetilde{\mathrm{H}}^0(X;\mathbb{Z})\,?$$

Exercise 218.2.

(a) Let $(I,\leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category *AbGroup* of abelian groups. Show that if each X_i is torsion-free, then so is the direct colimit $\varinjlim X_i$.

(b) Let $U \subseteq \mathbb{R}^3$ be an open subset. Show that $\mathrm{H}_1(U)$ is a torsion-free group.
Remark. You could use Exercise 38.5.

(c) Does the conclusion of (b) also hold for open subsets $U \subseteq \mathbb{R}^2$?

(d) Does the conclusion of (c) also hold for open subsets $U \subseteq \mathbb{R}^4$?

Exercise 218.3. Let M be a closed orientable connected n-dimensional topological manifold, let X be a compact k-dimensional topological manifold with $k \leq n-2$ and let $f\colon X \to M$ be an embedding. Show that $M \setminus f(X)$ is path-connected.
Hint. Analyze the proof of the General Alexander Duality Theorem 218.2 (1).

Exercise 218.4. The General Alexander Duality Theorem 218.2 was formulated for a weakly locally contractible subspace $X \subseteq S^n$ that is *compact*. Is the adjective "compact" necessary?

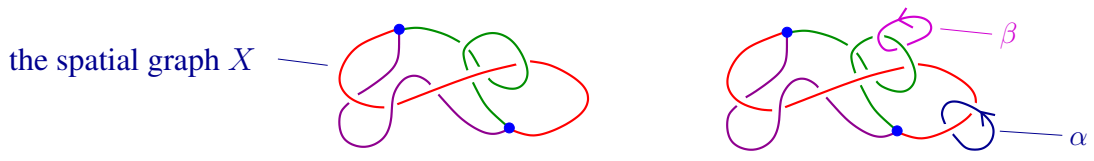
Exercise 218.5. We consider the connected spatial graph X with two vertices and three edges that is illustrated in the figure below on the left.

(a) Show that α represents a non-trivial element of $\mathrm{H}_1(\mathbb{R}^3 \setminus X)$.

(b) Show that β represents a non-trivial element of $\mathrm{H}_1(\mathbb{R}^3 \setminus X)$.

(c) Show that α and β form a generating set of $\mathrm{H}_1(\mathbb{R}^3 \setminus X)$.

Hint. Use the naturality of the Alexander duality isomorphism to get epimorphisms to $\mathbb{Z}$ respectively $\mathbb{Z}^2$ which send α, β respectively $\{\alpha,\beta\}$ to generating sets.



Exercise 218.6. We consider the three component link $L \subseteq \mathbb{R}^3$ that is illustrated in the figure below.

(a) Determine the isomorphism type of $\mathrm{H}_2(\mathbb{R}^3 \setminus L)$.

(b) Determine closed oriented 2-dimensional smooth manifolds in $\mathbb{R}^3 \setminus L$ which represent a basis for $\mathrm{H}_2(\mathbb{R}^3 \setminus L)$.

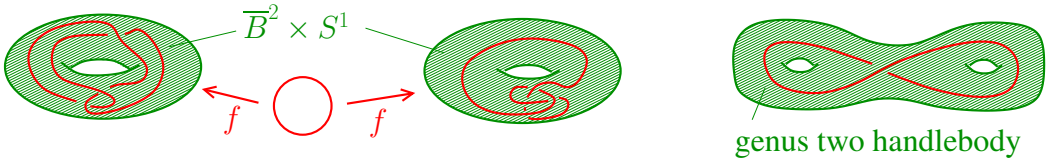


Exercise 218.7.

(a) Let $f\colon S^1 \to \overline{B}^2 \times S^1$ be a smooth embedding. Show that the isomorphism types of the homology groups $\mathrm{H}_k((\overline{B}^2 \times S^1) \setminus f(S^1))$ do not depend on the smooth embedding.

(b) Let H be a genus two handlebody, as shown in the figure below. Furthermore let $f\colon S^1 \to \overline{B}^2 \times S^1$ be a smooth embedding. Do the isomorphism types of the homology groups $\mathrm{H}_k(H \setminus f(S^1))$ depend on the smooth embedding?

(c) Let $f\colon S^1 \to S^2 \times S^1$ be a smooth embedding. Do the isomorphism types of the homology groups $\mathrm{H}_k((S^2 \times S^1) \setminus f(S^1))$ depend on the smooth embedding?

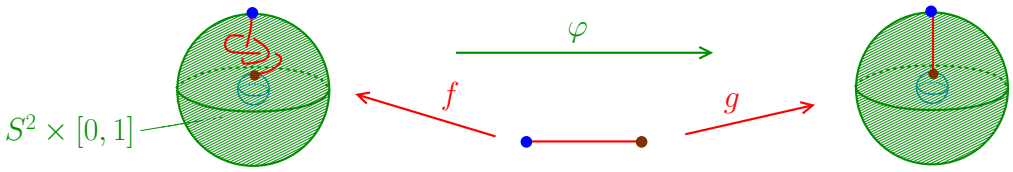


Exercise 218.8. Let $N \coloneqq (1,0,0)$ be the North Pole of S^2 and let $f\colon [0,1] \to S^2 \times [0,1]$ be a smooth embedding with $f(0) = (N,0)$ and with $f(1) = (N,1)$.

(a) Show that the smooth embedding $f\colon [0,1] \to S^2 \times [0,1]$ is smoothly isotopic rel $\{0,1\}$ to the standard embedding $g\colon [0,1] \to S^2 \times [0,1]$ that is given by $g(t) = (N,t)$.
Hint. You could for example do an induction on the number of crossings of f.

(b) Show that there exists diffeomorphism $\varphi\colon S^2 \times [0,1] \to S^2 \times [0,1]$ which is the identity on $S^2 \times \{0,1\}$ and with $\varphi \circ f = g$.
Hint. Use the Isotopy Extension Theorem 44.2.

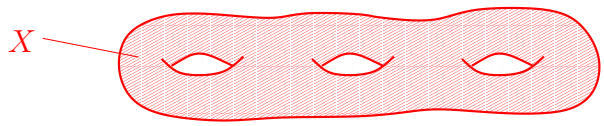
Remark. The conclusion of this exercise is sometimes referred to as the *light bulb trick*.



Exercise 218.9. Let $X \subseteq S^3$ be a closed orientable connected 2-dimensional submanifold. We view S^3 as a subspace of S^4 in the usual way.

(a) Determine the homology groups of $S^4 \setminus X$.

(b) Now let $X \subseteq S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ be the surface that is shown in the figure below. Give an example of a closed curve in $S^4 \setminus X$ that is not null-homotopic.



Exercise 218.10. This exercise is a continuation of Exercise 218.9.

(a) Let $X \subseteq S^n$ be a closed submanifold. We view S^n as a subspace of S^{n+1} in the usual way. Show that $S^{n+1} \setminus X$ is homotopy equivalent to $\Sigma(S^n \setminus X)$. (Here $\Sigma(S^n \setminus X)$ denotes the suspension of $S^n \setminus X$.)
Remark. You will need to use at some point that X is a "reasonable subspace ", e.g. that X is a closed submanifold. The statement does not hold for all subspaces of S^n.

(b) Again let $X \subseteq S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in the figure above. As above we view it as a submanifold of S^4. Determine $\pi_1(S^4 \setminus X)$.

Exercise 218.11.

(a) Let F be an orientable connected 2-dimensional smooth manifold with non-empty boundary and let $i\colon F \to \mathbb{R}^3$ be an injective continuous map. Is the complement $\mathbb{R}^3 \setminus F$ necessarily path-connected?

(b) Let F be an orientable connected surface non-compact 2-dimensional smooth manifold with empty boundary and let $i\colon F \to S^3$ be an injective continuous map. Is the complement $S^3 \setminus F$ necessarily path-connected?

Exercise 218.12. We say that an embedding $\varphi\colon [0,1] \to \mathbb{R}^3$ is *tame* if there exists a homeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(\varphi(t)) = (t,0,0)$ for all $t \in [0,1]$. In the figure below we show the images of three embeddings $\alpha, \beta, \gamma\colon [0,1] \to \mathbb{R}^3$.

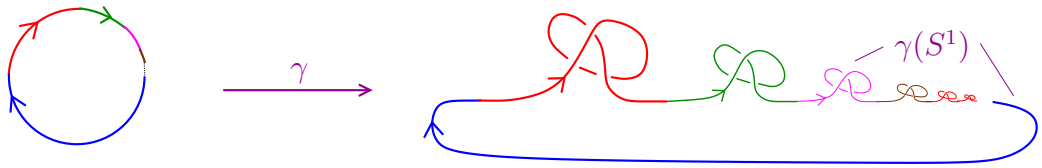
(a) Is the map α tame?
(b) Is the map β tame?
(c) Is the map γ tame?

Remark. Perhaps Exercise 218.8 can be useful.



Exercise 218.13. In the figure below we show another embedding $\gamma\colon S^1 \to \mathbb{R}^3$ which is given by infinitely many copies of the "trefail".

(a) Is $\gamma(S^1)$ a smooth submanifold of $\mathbb{R}^3$?
(b) Is $\gamma(S^1)$ a topological submanifold of $\mathbb{R}^3$?



Exercise 218.14. Let X and Y be two topological manifolds and let $f_0, f_1\colon X \to Y$ be two embeddings.

- As in Definition 18.5 we say that the two embeddings f_0, f_1 are *isotopic* if there exists a homotopy $F\colon X \times [0,1] \to Y$ such that $F_0 = f_0$, $F_1 = f_1$ and such that each $F_t\colon X \to Y$ is an embedding.
- We say that f_0, f_1 are *ambiently isotopic* if there exists a homotopy $G\colon Y \times [0,1] \to Y$ such that $G_0 = \mathrm{id}$, $G_1 \circ f_0 = f_1$ and such that each $G_t\colon Y \to Y$ is a homeomorphism.

With these definitions we can formulate the following exercises:

(a) Let $f_0, f_1 \colon S^1 \to S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ be two smooth embeddings. If these two embeddings are isotopic, does it follow that they are ambiently isotopic?

(b) Let $n \in \mathbb{N}_0$. As on pages 1815 and 1963 we consider the dihedral group

$$D_n \;=\; \mathbb{Z}/n\mathbb{Z} \underset{\uparrow}{\rtimes_\varphi} \mathbb{Z}/2\mathbb{Z} \;\cong\; \langle a, t \,|\, a^n, t^2, t \cdot a \cdot t^{-1} = a^{-1} \rangle.$$

semidirect product where $\varphi \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is given by multiplication by -1

Given coprime $p, q \in \mathbb{N}$ we consider the torus knot $T(p,q)$. Show that for odd $p, n \in \mathbb{N}$ there exists an epimorphism $\pi_1(S^3 \setminus T(p,2)) \to D_n$ if and only if $p|n$.
Remark. Recall by the Torus Knot-π_1-Proposition 111.18 we know that there exists an isomorphism $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \mid x^p = y^q \rangle$.

(c) Let K_1 and K_2 be two oriented knots and let p be a prime. By the Oriented Knot–Monoid-π_1-Proposition 111.26 we know that

$$\pi_1(S^3 \setminus (K_1 \# K_2)) \;\cong\; \pi_1(S^3 \setminus (K_1)) *_{\mu_{K_1} = \mu_{K_2}} \pi_1(S^3 \setminus K_2)$$

where μ_{K_i} is the meridian of K_i. Show that the following two statements are equivalent:
(1) There exists an epimorphism $\pi_1(S^3 \setminus (K_1 \# K_2)) \to D_p$.
(2) There exists an $i \in \{1, 2\}$ such that $\pi_1(S^3 \setminus K_i)$ admits an epimorphism onto D_p.

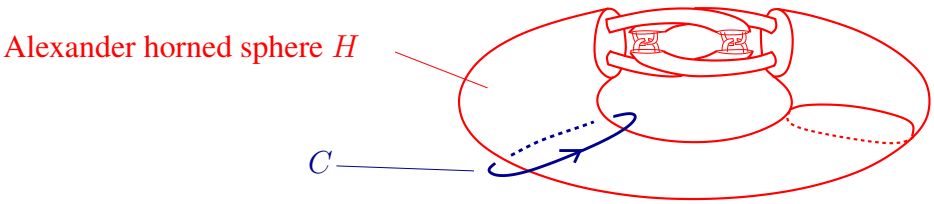
(d) Show that there are uncountably many ambient isotopy equivalence classes of embeddings $S^1 \to S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$.

Remark. In this exercise we end up showing that there exists an embedding $w \colon S^1 \to S^3$ such that $\pi_1(S^3 \setminus w(S^1))$ is not finitely generated. We proved the same type of statement in the Wild Knot-π_1-Proposition 218.20.

Exercise 218.15. In the figure below we show the Alexander horned sphere H and a curve $C \subseteq \mathbb{R}^3 \setminus H$. By [**Ale1924a**], [**Hat2002**, p. 170] and [**DV2009**, p. 46] we know that $\pi_1(\mathbb{R}^3 \setminus H)$ is non-trivial.

(a) Show that C is non-trivial in $\pi_1(\mathbb{R}^3 \setminus H)$.

(b) It follows easily from the Sphere–Alexander Duality Theorem 218.8 that $\mathrm{H}_1(\mathbb{R}^3 \setminus H) = 0$. Find a compact smoothly embedded surface $\Sigma \subseteq \mathbb{R}^3 \setminus H$ with $\partial H = C$.



CHAPTER 219

Applications of cohomology groups

In this chapter we give two applications of cohomology groups:

(1) In Section 219.1 we state and prove the Table Theorem 219.3
(2) In Section 219.3 we address two special cases of the square peg problem. For one special case we will need the Table Theorem 219.3 and for the other we will use the General Alexander Duality Theorem 218.2.

Neither of these applications can count as a "real life" application, but at least in both cases cohomology, Poincaré Duality and Alexander duality get used to prove statements which initially have absolutely no connection to cohomology theory at all.

This chapter is organized as follows:

- In Section 219.1 we will state and prove the Table Theorem 219.3.
- In Section 219.2 we will discuss trivial and non-trivial solutions to the Table Theorem 219.3.
- In Section 219.3 we will discuss the square-peg problem.
- In Section 219.4 we will provide exercises for Chapter 219.

In the next Chapter 220 we will introduce algebraic intersection numbers and we will explore their relationship to cup products.

219.1. **The Table Theorem.** In this section we will state and prove the stunning Table Theorem 219.3. As a warm-up we start out with the fairly elementary but charming Wobbly Table Theorem. To formulate the Wobbly Table Theorem we need the following definition from linear algebra:

Definition 219.1. We say that four points $P_1, P_2, P_3, P_4 \in \mathbb{R}^3$ are coplanar if there exists a 2-dimensional vector subspace $V \subseteq \mathbb{R}^3$ and a $Q \in \mathbb{R}^3$ such that $P_1, P_2, P_3, P_4 \in Q + V$.

In the vernacular the following theorem says that given a wobbly table we can rotate it around its center, by an angle of at most $\frac{\pi}{2}$, such that eventually all four legs touch the ground.[2344] Slightly more precisely, the Wobbly Table Theorem can be formulated as follows:

Theorem 219.2. (Wobbly Table Theorem) Let $f\colon \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \to \mathbb{R}$ be a function, let $x \in \mathbb{C}$ and let $z \in \mathbb{C} \setminus \{0\}$. There exists a $\theta \in [0, \frac{\pi}{2}]$ such that the four points

$$(x + e^{\mathrm{i}(\theta + k\cdot\frac{\pi}{2})} \cdot z, f(x + e^{\mathrm{i}(\theta + k\cdot\frac{\pi}{2})} \cdot z)), \qquad \text{with } k = 0, 1, 2, 3$$

are coplanar.

[2344] This might be the only theorem of these notes that I find useful in real life.

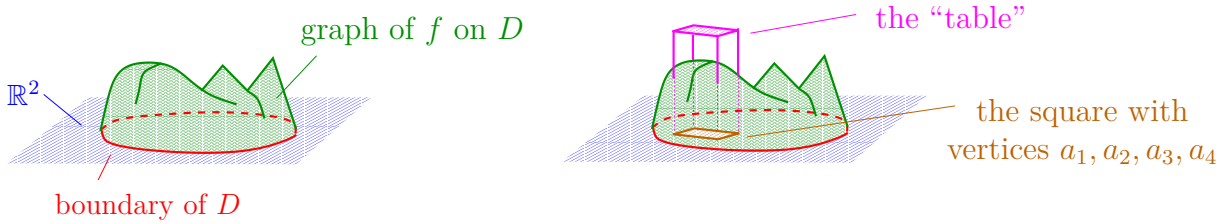
Proof. The theorem is a fairly elementary consequence of the Intermediate Value Theorem from real analysis. Filling in the details is your task in Exercise 219.1. Alternatively the reader can look at [**Loy2017**, p. 289] for a proof. ■

Our goal in this section is to prove a much more fancy theorem. More precisely we will prove the following "Table Theorem" which was first formulated and proved by Roger Fenn [**Fen1970**] in 1970.

Theorem 219.3. (Table Theorem) Let $D \subseteq \mathbb{R}^2$ be a compact convex non-empty subset and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that $f(x) \geq 0$ for all $x \in D$ and such that $f(x) = 0$ for all $x \notin D$. Given any $d \in \mathbb{R}_{>0}$ there exist four points $a_1, a_2, a_3, a_4 \in \mathbb{R}^2$ with the following properties:

(1) The points form a square of side length d.
(2) The center of the square lies in D.
(3) We have $f(a_1) = f(a_2) = f(a_3) = f(a_4)$.

In other words, the Table Theorem 219.3 says that any given square table can be placed on the ground defined by the graph of f, such that all four legs lie on the ground, such that the table is horizontal and such that the center of the table is located in D.



Remark. In [**Meye1981**] it is shown that in the formulation of the Table Theorem 219.3 one cannot drop the hypothesis that D is convex. □

The proof of the Table Theorem 219.3 will require the remainder of this section. Throughout the section we fix a compact convex non-empty subset $D \subseteq \mathbb{R}^2$ and we fix a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in D$ and such that $f(x) = 0$ for all $x \notin D$. Let $d \in \mathbb{R}_{>0}$. We make the following elementary observations and preparations:

(i) Since we can rescale D we can and will without loss of generality assume that $d = 1$.
(ii) If the interior of D is the empty set, then the function f is necessarily the zero function[2345]. Thus the conclusion of the Table Theorem 219.3 holds trivially.
(iii) It follows from (ii) that henceforth we can assume that the interior of D is non-empty. Note that the Convex-to-Ball Proposition 4.28 implies that there exists a homeomorphism of pairs $(D, \partial D) \xrightarrow{\cong} (\overline{B}^2, S^1)$.
(iv) After a translation we can and will assume that the origin $(0, 0)$ is contained in the interior of D.
(v) As always we will make the identification $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$, in particular we will work a lot with the multiplication "$\cdot\, \mathrm{i}$", i.e. with "rotation by $\frac{\pi}{2}$".
(vi) If there exists a square in $\mathbb{R}^2$ of side length $d = 1$ with center in D, such that the f-value at all of the four vertices is zero, then we have evidently found the desired table. Thus

[2345] Why is that?

we can henceforth assume that for any square of side length $d = 1$ with center in D the f-value of at least one vertex is non-zero.

In the following discussion we will at times take a few liberties with mathematical language, but these liberties will not infringe on the rigor of the argument.

We start out with introducing the following notation, which gives us a convenient way for parameterizing the set of all squares with side length $d = 1$ and whose center lies in D.

Notation 219.4.

(1) We consider the map[2346]

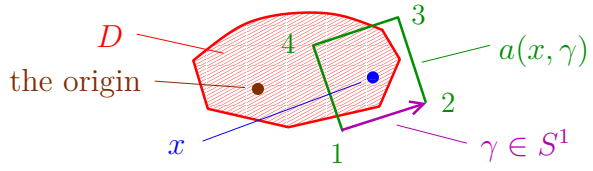
$$\begin{array}{rcl} a\colon D \times S^1 & \to & \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \\ (x, \gamma) & \mapsto & \begin{array}{c}\text{the four vertices } a_1, a_2, a_3, a_4 \text{ of the unique square} \\ \text{with center } x \text{ such that } a_2 - a_1 = \gamma \text{ and} \\ \text{such that } a_1, a_2, a_3, a_4 \text{ are ordered in the} \\ \text{counterclockwise direction.}\end{array} \end{array}$$

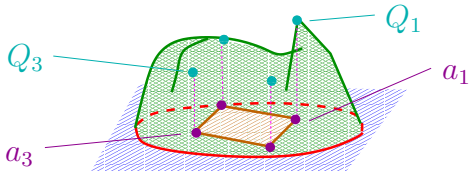
This definition is illustrated in the figure below.

(2) We also consider the closely related map

$$\begin{array}{rcl} Q\colon D \times S^1 & \to & \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \\ (x, \gamma) & \mapsto & \text{the points } Q_1, Q_2, Q_3, Q_4 \text{ on the graph corresponding to } a(x, \gamma). \end{array}$$

When $(x, \gamma) \in D \times S^1$ is understood from the context, then we drop it at times from the notation. In other words, often we write a_i instead of $a_i(x, \gamma)$ and we write Q_i instead of $Q_i(x, \gamma)$.





One of the key ideas behind the proof of the Table Theorem 219.3 is to split Condition (3) of the Table Theorem 219.3 into two separate statements. Namely we note that the condition

(3) $f(a_1) = f(a_2) = f(a_3) = f(a_4)$,

is satisfied precisely if the following two conditions are satisfied:

(3a) the points $\underbrace{(a_1, f(a_1))}_{=Q_1}, \ldots, \underbrace{(a_4, f(a_4))}_{=Q_4}$ lie on a plane,

(3b) the plane from (3a) is horizontal.

The trick now is to encode these two statements in a clever way. We will do so by introducing two maps.

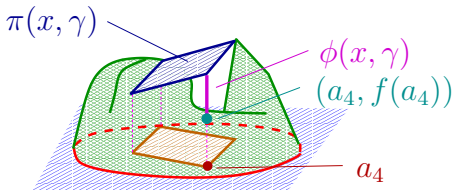
Definition 219.5.

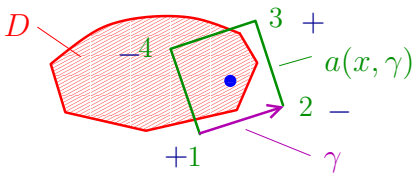
[2346] We leave the task of writing down an explicit formula to the reader. Once the formula is written down it is clear that a is actually continuous.

(1) Given $(x,\gamma) \in D^2 \times S^1$ we denote by $\pi(x,\gamma)$ the plane through the three points $Q_1(x,\gamma), Q_2(x,\gamma)$ and $Q_3(x,\gamma)$.
(2) We consider the map[2347]

$$\begin{array}{rcl} \phi\colon D \times S^1 & \to & \mathbb{R} \\ (x,\gamma) & \mapsto & \underbrace{x_3\text{-value of the plane } \pi(x,\gamma) \text{ at } a_4(x,\gamma) \text{ minus } f(a_4(x,\gamma))}_{=f(a_1)+f(a_3)-f(a_2)-f(a_4)} \end{array}$$

The definition of this map is illustrated in the figure below. Note that $\phi(x,\gamma) = 0$ if and only if the corresponding points Q_1, Q_2, Q_3, Q_4 lie on a plane.
(3) We set $A := \{(x,\gamma) \in D \times S^1 \mid \phi(x,\gamma) = 0\}$.

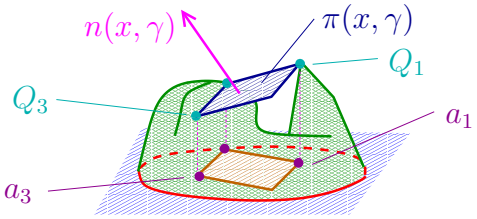


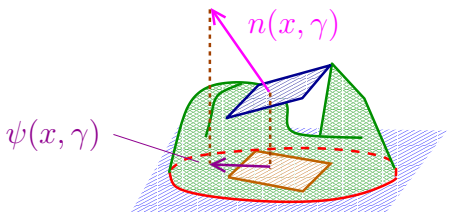


Definition 219.6.
(1) Given $(x,\gamma) \in D^2 \times S^1$ we denote by $n(x,\gamma) \in \mathbb{R}^3$ the unique vector that is normal to $\pi(x,\gamma)$, that has unit length and that has positive x_3-coordinate. This definition is illustrated in the figure below on the left.
(2) We consider the map

$$\begin{array}{rcl} \psi\colon D \times S^1 & \to & \mathbb{R}^2 \\ (x,\gamma) & \mapsto & \text{projection of } n(x,\gamma) \text{ onto the } x_1x_2\text{-plane.} \end{array}$$

The definition of this map is illustrated on the right-hand side of the figure below. Note that $\psi(x,\gamma) = 0$ if and only if the plane $\pi(x,\gamma)$ is horizontal.
(3) We set $B := \{(x,\gamma) \in D \times S^1 \mid \psi(x,\gamma) = 0\}$.





Remark. The Wobbly Table Theorem 219.2 implies that given any $x \in D$ there exist at least four $\gamma \in S^1$ with $(x,\gamma) \in A$. In particular the set A is more complicated than in our subsequent illustrations. □

We make the following elementary observation.

[2347] Note that the equality

$$a_4 = a_2 + (a_3 - a_2) + \underbrace{(a_4 - a_3)}_{=a_1-a_2} = a_2 + (a_1 - a_2) + (a_3 - a_2) = a_1 + a_3 - a_2,$$

together with the fact that the x_3-coordinate of the plane $\pi(x,\gamma)$ is an affine function on $\mathbb{R}^2$, implies that

$$x_3\text{-coordinate of the plane } \pi(x,\gamma) \text{ at } a_4 \;=\; f(a_1) + f(a_3) - f(a_2).$$

Observation 219.7. To prove the Table Theorem 219.3 it suffices to show that $A \cap B \neq \varnothing$.

In the following two lemmas we collect some properties of the two maps ϕ and ψ.

Lemma 219.8. (ϕ-Lemma) The map $\phi\colon D \times S^1 \to \mathbb{R}$ has the following properties:

(1) The map $\phi\colon D \times S^1 \to \mathbb{R}$ is continuous.
(2) For any $(x, \gamma) \in D \times S^1$ we have $\phi(x, \mathrm{i} \cdot \gamma) = -\phi(x, \gamma)$.
(3) Given $(x, \gamma) \in D \times S^1$ the following statements are equivalent:
   (a) $\phi(x, \gamma) = 0$,
   (b) $\phi(x, \mathrm{i} \cdot \gamma) = 0$,
   (c) (x, γ) lies in A,
   (d) the corresponding points Q_1, Q_2, Q_3, Q_4 lie on a plane.

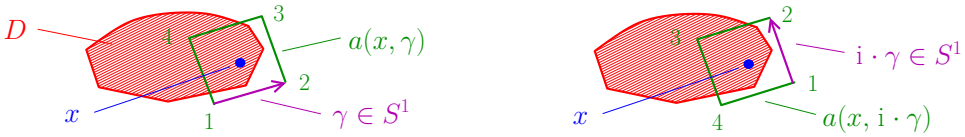
Proof.

(1) As we pointed out above, we have

$$\phi(x,\gamma) = f(\underbrace{a_1(x,\gamma)}_{\text{odd vertex}}) + f(\underbrace{a_3(x,\gamma)}_{\text{odd vertex}}) - f(\underbrace{a_2(x,\gamma)}_{\text{even vertex}}) - f(\underbrace{a_4(x,\gamma)}_{\text{even vertex}}).$$

It follows from the continuity of f and the continuity of $(x, \gamma) \mapsto a_i(x, \gamma)$ that ϕ is indeed continuous.

(2) Note that for every $(x, y) \in D \times S^1$ and every $j \in \{1, 2, 3, 4\}$ we have the equality $a_j(x, \mathrm{i} \cdot \gamma) = a_{j+1}(x, \gamma)$. In particular the odd/even vertices corresponding to $(x, \mathrm{i} \cdot \gamma)$ are precisely the even/odd vertices corresponding to (x, γ).



It follows immediately from this observation and the explicit formula for ϕ given in (1) that the equality $\phi(x, \mathrm{i} \cdot \gamma) = -\phi(x, \gamma)$ holds.

(3) This statement follows immediately from (2) and the definition of A. ∎

Next we turn to the study of the map $\psi\colon D \times S^1 \to \mathbb{R}^2$. In the formulation of the next lemma we need the following definition:

Definition 219.9. We say that $v, w \in \mathbb{R}^2$ are parallel if one of v, w is zero or if there exists an $\lambda \in \mathbb{R}_{>0}$ such that $v = \lambda \cdot w$.

Lemma 219.10. (ψ-Lemma) The map $\psi\colon D \times S^1 \to \mathbb{R}^2$ has the following properties:

(1) The map $\psi\colon D \times S^1 \to \mathbb{R}$ is continuous.
(2) If $(x, \gamma) \in A$, then $\psi(x, \gamma) = \psi(x, \mathrm{i}^k \cdot \gamma)$ for any $k \in \mathbb{Z}$.
(3) Let $x \in \partial D$ and let $\gamma \in S^1$. If $(x, \gamma) \in A$, then the vector $\psi(x, \gamma)$ is not parallel to $-x$.

Sketch of proof.

(1) Let $(x, \gamma) \in D \times S^1$. Basic linear algebra shows that

$$\psi(x,\gamma) = \begin{array}{l}\text{the vector in } \mathbb{R}^3 \text{ that is obtained as follows:}\\ \text{(a) perform the cross product } (Q_3(x,\gamma)-Q_2(x,\gamma))\times(Q_1(x,\gamma)-Q_2(x,\gamma))\\ \text{(b) normalize the vector from (a) to length one}^{2348}\\ \text{(c) project the vector from (b) onto the } xy\text{-plane.}\end{array}$$

It follows immediately from this observation that ψ is continuous.

(2) Let $(x,\gamma) \in A$. As mentioned in the ϕ-Lemma 219.8 (3), this means that the four points $Q_1(x,\gamma)$, $Q_2(x,\gamma)$, $Q_3(x,\gamma)$ and $Q_4(x,\gamma)$ lie on a plane. But this immediately implies that for any $k \in \mathbb{Z}$ we have $\pi(x,\gamma) = \pi(x, \mathrm{i}^k \cdot \gamma)$ which then gives us the equality $\psi(x,\gamma) = \psi(x, \mathrm{i}^k \cdot \gamma)$.

(3) The final statement is a little more interesting. So let $x \in \partial D$ and let $\gamma \in S^1$. We assume that $(x,\gamma) \in A$.

Claim. There exist two adjacent vertices which lie outside of $\mathring{D}$ and there are two adjacent vertices which have non-zero f-value.

Proof. We make the following observations:

(a) Since $(x,\gamma) \in A$ we have[2349] $f(a_1) + f(a_3) - f(a_2) - f(a_4) = 0$.

(b) By choice of f we know that all f-values are non-negative.

(c) By the preliminary remark (vi) on page 4211 we know that not all of the f-values are zero.

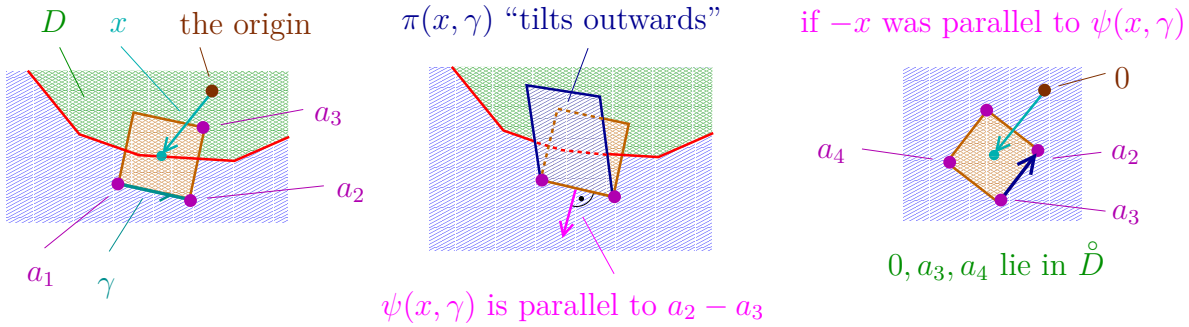
(d) Since x lies on ∂D and since D is convex we see that at most two f-values can be non-zero.[2350]

The claim follows easily from the above considerations and the elementary observation that every odd vertex is adjacent to every even vertex. ⊞

To simplify the notation let us now assume that the adjacent vertices a_1 and a_2 do *not* lie in the interior of D. Consequently $f(a_3) > 0$ and $f(a_4) > 0$. A straightforward calculation shows that

$$\psi(x,\gamma) = \begin{array}{c}\text{projection of}\\ \text{normalization of}\end{array} \begin{pmatrix} a_3 - a_2 \\ f(a_3) \end{pmatrix} \times \begin{pmatrix} a_1 - a_2 \\ 0 \end{pmatrix} \text{ into the } xy\text{-plane } = \lambda \cdot (a_2 - a_3)$$

for some $\lambda > 0$.



Now suppose that $\psi(x,\gamma)$ is parallel to $-x$. This means, by the above calculation, that $a_3 - a_2 = -\mu \cdot x$ for some $\mu > 0$. It is now straightforward to verify that x lies in the interior of the convex hull of 0, a_3 and a_4. Since 0, a_3 and a_4 lie in the interior of D we

[2348] Since $a_3(x,\gamma) - a_2(x,\gamma)$ and $a_1(x,\gamma) - a_2(x,\gamma)$ are linearly independent we see that the two vectors in the cross product are linearly independent, in particular the cross product is indeed non-zero. Thus it makes sense to normalize the vector to length one.

[2349] Recall that we write $a_i = a_i(x,\gamma)$, $i = 1,2,3,4$.

[2350] Any vertex with non-zero f-value lies in the interior of D. So if three vertices have non-zero f, then the triangle spanned by these vertices lies in the interior, in particular x would lie in the interior of D.

see that x also lies in the interior of D. But that is a contradiction to our hypothesis that $x \in \partial D$. ■

Lemma 219.11. (T-Lemma) We set

$$T \coloneqq (D \times S^1)/\sim \qquad \text{where } (x,\gamma) \sim (x, \mathrm{i}\cdot\gamma)$$

and we consider the following diagram:

$$\begin{array}{ccc} D \times S^1 & \xrightarrow{\quad\psi\quad} & \mathbb{R}^2 \\ {\scriptstyle =:p}\big\downarrow {\scriptstyle (x,\gamma)\mapsto[(x,\gamma)]} & & \big\downarrow \\ T & \xrightarrow[{[(x,\gamma)]\mapsto \frac{1}{4}\sum_{k=0}^{3}\psi(x,\mathrm{i}^k\cdot\gamma)}]{=:\psi'} & \mathbb{R}^2. \end{array}$$

The following statements hold:

(1) The map $p\colon D \times S^1 \to T$ is a 4-fold covering and the topological space T is homeomorphic to the solid torus $D^2 \times S^1$.
(2) The map ψ' is continuous.
(3) The restriction of the diagram to $A \subseteq D \times S^1$ is commutative.
(4) If we set

$$A' \coloneqq p(A) \quad \text{and} \quad B' \coloneqq \{[(x,\gamma)] \in T \mid \psi'([x,\gamma]) = 0\}$$

then we have $p(A \cap B) = A' \cap B'$.

Proof.

(1) For the experienced reader it should be clear that $p\colon D \times S^1 \to T = (D \times S^1)/\sim$ is a 4-fold covering and that $T \cong D \times (S^1/\sim) \cong D \times S^1$. On page 4210 we remarked that D is homeomorphic to $\overline{B}^2$, thus we see that T is indeed homeomorphic to the solid torus $\overline{B}^2 \times S^1$.
(2) The map $D^2 \times S^1 \to \mathbb{R}^2$ given by $(x,\gamma) \to \frac{1}{4}\sum_{k=0}^{3}\psi(x,\mathrm{i}^k\cdot\gamma)$ is continuous. It descends to our map $\psi'\colon T = (D \times S^1)/\sim \to \mathbb{R}^2$. Thus it follows from the Topological-Quotient Proposition 8.2 (1b) that ψ' is continuous.
(3) It follows immediately from the ψ-Lemma 219.10 (2) that for any $(x,\gamma) \in A$ we have the equality $\psi'([x,\gamma]) = \psi(x,\gamma)$.
(4) The equality $p(A \cap B) = A' \cap B'$ follows easily from (3). ■

From the T-Lemma 219.11 (3) and Observation 219.7 we obtain the following observation.

Observation 219.12. If we want to prove the Table Theorem 219.3, then it suffices to show that $A' \cap B' \neq \varnothing$.

After spending so much time on preparations we can now finally apply the machinery from algebraic topology to tackle the problem. We introduce the following convention.

Convention 219.13. By the T-Lemma 219.11 (1) we know that T is a solid torus. Given any $x_0 \in T$ we make an identification $\pi_1(T,x_0) =_{\mathrm{i}} \mathbb{Z}$.[2351]

The following lemma turns our data into algebraic-topological information.

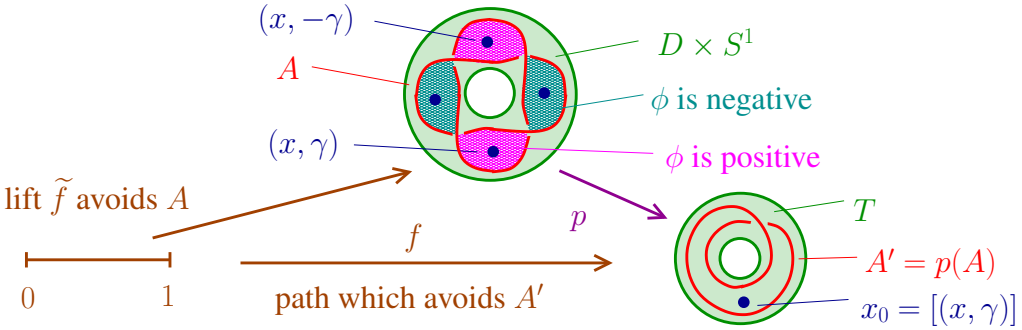
[2351] In the following we do not care about the precise isomorphism, in particular we do not care about any sign ambiguity.

Lemma 219.14. (Subgroup Lemma) We continue with the notation that we introduced above. Let $x_0 \in T \setminus A'$. The image of $\pi_1(T \setminus A', x_0)$ in $\pi_1(T, x_0) \cong \mathbb{Z}$ is contained in the subgroup $2 \cdot \mathbb{Z}$.

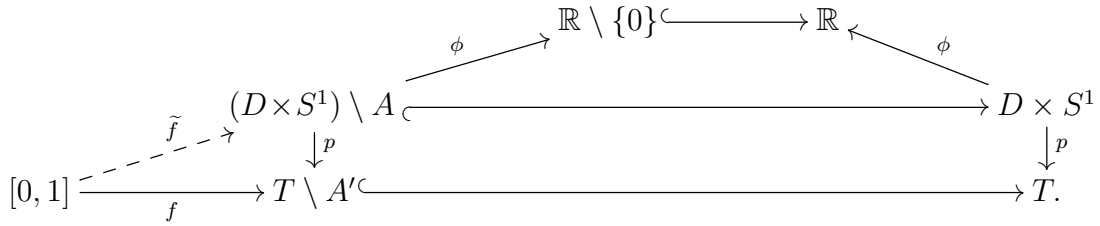
Proof. Let $x_0 = [(x, \gamma)] \in T \setminus A'$. Note that by definition of A' we have $\phi(x, \gamma) \neq 0$.

(∗) Without loss of generality we can assume that $\phi(x, \gamma) > 0$.

Now let $[f] \in \pi_1(X \setminus A', x_0)$. In other words, $f\colon [0,1] \to T \setminus A'$ is a loop in $x_0 = [(x, \gamma)]$. We need to show that $[f] \in 2 \cdot \mathbb{Z}$. By the Path-Lifting Proposition 92.6 (1) we know that there exists a unique lift $\widetilde{f}\colon [0,1] \to (D \times S^1) \setminus A$ with $\widetilde{f}(0) = (x, \gamma)$.



We consider the following diagram:



We make the following observations:

(1) By Definition 92.1 of a lift we know that $\widetilde{f}(1)$ lies in the subset
$$p^{-1}(f(1)) = p^{-1}(f(0)) = p^{-1}(x_0) = \{(x, \gamma), (x, \mathrm{i} \cdot \gamma), (x, -\gamma), (x, -\mathrm{i} \cdot \gamma)\}.$$
(2) By the ϕ-Lemma 219.8 (2) and (∗) we know that $\phi(x, \gamma) = \phi(x, -\gamma) > 0$ and we know that $\phi(x, \mathrm{i} \cdot \gamma) = \phi(x, -\mathrm{i} \cdot \gamma) < 0$.
(3) Recall that by definition $A' = p(A)$. Therefore it follows from the ϕ-Lemma 219.8 (3) (a)⇔ (b) that $A = p^{-1}(A')$. In particular we have $(D \times S^1) \setminus A = p^{-1}(T \setminus A')$.
(4) It follows from (3) and the above diagram that the image of $\phi \circ \widetilde{f}\colon [0,1] \to \mathbb{R}$ is contained in $\mathbb{R} \setminus \{0\}$.
(5) Since $[0,1]$ is connected and since $\phi(\widetilde{f}(0)) = \phi(x, \gamma) > 0$ we obtain from (4) that $\phi(\widetilde{f}(1)) > 0$.
(6) It follows from (1), (2) and (5) that $\widetilde{f}(1) = (x, \gamma)$ or $\widetilde{f}(1) = (x, -\gamma)$.
(7) It follows easily[2352] from (6) that the loop $f * f$ lifts to a loop in $(D \times S^1, (x, \gamma))$.
(8) It follows from (7) and the Loop–Lifting Proposition 92.8 that $2 \cdot [f] = [f] \cdot [f] = [f * f]$ actually lies in the subgroup $p_*(\pi_1(D \times S^1, (x, \gamma)))$.

[2352] How does this work?

(9) Since $p\colon D\times S^1 \to T$ is a 4-fold connected covering we obtain from the Covering-Degree Proposition 92.10 that $p_*(\pi_1(D\times S^1,(x,\gamma))) = 4\cdot\mathbb{Z}$.
(10) The combination of (8) and (9) shows that $[f]$ lies in $2\cdot\mathbb{Z}$. ■

The next convention is basically self-explanatory.

Convention 219.15. Any map denoted by i is given by an inclusion of the corresponding (pairs) of topological spaces.

The following lemma consists mostly of repackaging the result of the Subgroup Lemma 219.14.

Lemma 219.16. (Epimorphism Lemma) We continue with the above notation. We now suppose that we are given a compact smooth submanifold with corner $L\subseteq T$ such that $A'\subseteq \mathring{L} := L\setminus \partial_0 L$.[2353] The map

$$i_*\colon \mathrm{H}_2(L, L\cap\partial T;\mathbb{Z}/2\mathbb{Z}) \;\to\; \mathrm{H}_2(T,\partial T;\mathbb{Z}/2\mathbb{Z})$$

is an epimorphism.



Proof. Let $L\subseteq T$ be a compact smooth submanifold with corner with $A'\subseteq \mathring{L} := L\setminus\partial_0 L$. If $L = T$ then there is nothing to show. So we assume that $L\neq T$. This implies in particular that $T\setminus\mathring{L}\neq\varnothing$.

We start out with the following claim which is basically a rewriting of the Subgroup Lemma 219.14.

Claim. For any homomorphism $\varphi\colon \mathrm{H}_1(T)\to\mathbb{Z}/2\mathbb{Z}$ the composition

$$\mathrm{H}_1(T\setminus\mathring{L}) \xrightarrow{i_*} \mathrm{H}_1(T) \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z}$$

is the zero map.

Proof. It suffices to show the analogous statement for any path-component Y of $T\setminus\mathring{L}$. So let Y be a path-component of $T\setminus\mathring{L}$. We pick a point $x_0\in Y$. We consider the following diagram:

$$\begin{array}{ccccccccc}
\pi_1(Y,x_0) & \longrightarrow & \pi_1(T\setminus\mathring{L},x_0) & \longrightarrow & \pi_1(T\setminus A',x_0) & \overset{=0 \text{ by } (4)}{\longrightarrow} & \pi_1(T,x_0) & \overset{\eta}{\twoheadrightarrow} & \mathbb{Z}/2\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\cong & & \\
\mathrm{H}_1(Y) & \longrightarrow & \mathrm{H}_1(T\setminus\mathring{L}) & \longrightarrow & \mathrm{H}_1(T\setminus A') & \longrightarrow & \mathrm{H}_1(T). & &
\end{array}$$

We make the following clarifications and observations:

(1) The map $\eta\colon \pi_1(T,x_0)\to\mathbb{Z}/2\mathbb{Z}$ is the unique epimorphism.
(2) All the vertical maps are given by the corresponding Hurewicz homomorphism. The diagram commutes by the naturality of the Hurewicz homomorphisms.
(3) Since Y is path-connected it follows from the Hurewicz Theorem 166.10 that the Hurewicz homomorphism to the left is an epimorphism. Furthermore, since T is

[2353] Here we use the notation for smooth submanifolds with corner that we introduced in Notation 25.47 and the Smooth Submanifold-with-Corner Lemma 26.16.

path-connected and since $\pi_1(T, x_0) \cong \mathbb{Z}$ is abelian it follows from the Hurewicz Theorem 166.10 that the Hurewicz homomorphism to the right is an isomorphism.

(4) Note that it follows from the Subgroup Lemma 219.14 that the homomorphism $\pi_1(T \setminus A', x_0) \to \pi_1(T, x_0) \xrightarrow{\eta} \mathbb{Z}/2\mathbb{Z}$ is the trivial map.

(5) The claim follows immediately from combining (1) to (4). ⊞

Now we turn to the actual proof of the desired statement. We write $W := T \setminus \mathring{L}$. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{H}_1(W), \mathbb{Z}/2\mathbb{Z}) & \mathrm{H}^1(W; \mathbb{Z}/2\mathbb{Z}) \arrow[l, "\mathrm{ev}", "\cong"'] \arrow[r, "\cap[W]", "\cong"'] & \mathrm{H}_2(W, \partial W; \mathbb{Z}/2\mathbb{Z}) \arrow[r, "i_*", "\cong"'] & \mathrm{H}_2(T, \partial T \cup L; \mathbb{Z}/2\mathbb{Z}) \\
\mathrm{Hom}(\mathrm{H}_1(T), \mathbb{Z}/2\mathbb{Z}) \arrow[u, "i^*", "=0"'] & \mathrm{H}^1(T; \mathbb{Z}/2\mathbb{Z}) \arrow[l, "\mathrm{ev}", "\cong"'] \arrow[u, "i^*"] \arrow[rr, "\cap[T]", "\cong"'] & & \mathrm{H}_2(T, \partial T; \mathbb{Z}/2\mathbb{Z}) \arrow[u, "i_*"] \\
& & \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}/2\mathbb{Z}) \arrow[ur, "i_*"] \arrow[r, "i_*", "\cong"'] & \mathrm{H}_2(\partial T \cup L, \partial T; \mathbb{Z}/2\mathbb{Z}). \arrow[u, "i_*"]
\end{tikzcd}$$

We make the following comments and clarifications:

(1) The vertical maps are induced by the various inclusions.

(2) The square to the left commutes by the naturality of the evaluation homomorphism. Furthermore, the horizontal maps to the left are isomorphisms by the H^1-Hom-Proposition 201.24.

(3) By the Smooth Submanifold-with-Corner Lemma 26.16 we know that W is a compact 3-dimensional smooth manifold with corner. We equip it with the orientation coming from T.

(4) The horizontal maps "$\cap[W]$" and "$\cap[T]$" in the center are given by capping with the fundamental classes $[W]$ and $[T]$. These maps are isomorphisms by the Poincaré Duality Theorem 210.1.

(5) The horizontal map to the top right is an isomorphism by the Excision Theorem 212.2 (b).

(6) The vertical maps to the right are a piece of the long exact sequence of the triple $(T, \partial T \cup L, \partial T)$, see the LES-of-Triple Proposition 146.5.

(7) The rectangle to the right commutes by the Large H_*-to-Small H^*-PD Lemma 212.9, see in particular the example (a) on page 4071.

(8) By the above claim we know that the vertical map to the left is zero. Since all horizontal maps are isomorphisms and since the upper part of the diagram commutes we see that the vertical map to the top right is also the zero map.

(9) It follows from (6) and (8) that the vertical map to the bottom right is an epimorphism.

(10) The bottom triangle commutes since all maps are inclusion-induced maps.

(11) The horizontal map at the very bottom is an isomorphism by the Excision Theorem 212.2 (c).

(12) Finally we note that it follows immediately from (9), (10) and (11) that the diagonal map $i_*\colon \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_2(T, \partial T; \mathbb{Z}/2\mathbb{Z})$ is an epimorphism. Fortunately that is precisely what we wanted to show. ■

Lemma 219.17. (Map-to-S^n-Homotopy Lemma) Let X be a topological space and let $f, g\colon X \to S^n$ be continuous maps. If for every $x \in X$ we have $f(x) \neq -g(x) \in S^n$, then f and g are homotopic.[2354]

Proof. Note that for any $x \in X$ and any $t \in [0,1]$ we have $f(x)\cdot(1-t)+g(x)\cdot t \neq 0$. (For $t \neq \frac{1}{2}$ this follows from considering norms and for $t = \frac{1}{2}$ this follows from our hypothesis). Thus the map

$$\begin{array}{rcl} X \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \frac{f(x)\cdot(1-t)+g(x)\cdot t}{\|f(x)\cdot(1-t)+g(x)\cdot t\|} \end{array}$$

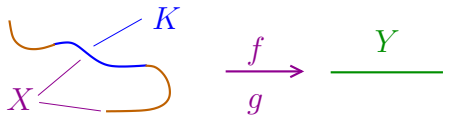
makes sense, and it is a homotopy from f to g. ∎

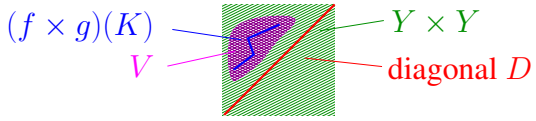
Lemma 219.18. (Maps-are-Different Lemma) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. Suppose that there exists a compact subset $K \subseteq X$ such that $f(x) \neq g(x)$ for all $x \in K$. If Y is Hausdorff, then there exists an open neighborhood U of K such that $f(x) \neq g(x)$ for all $x \in U$.

Proof. We consider the map

$$\begin{array}{rcl} f \times g\colon X & \to & Y \times Y \\ x & \mapsto & (f(x), g(x)). \end{array}$$

By the Topological-Product Proposition 7.2 (1b) we know that $f \times g$ is continuous. Since K is compact we obtain from the Compact Image Lemma 4.15 (1) that $(f \times g)(K)$ is a compact subset of $Y \times Y$. Let $D := \{(y,y) \mid y \in Y\}$ be the "diagonal" in $Y \times Y$. Since Y is Hausdorff we obtain from the Hausdorff-Diagonal Lemma 7.20 that D is a closed subset of $Y \times Y$. Now, using our hypothesis that Y is Hausdorff we obtain from Exercise 3.29 (which is a variation on the Compact Hausdorff-Normal Lemma 3.49) an open neighborhood V of $(f \times g)(K)$ with $V \cap D = \varnothing$.





We set $U := (f \times g)^{-1}(V)$. Since $f \times g$ is continuous we see that U is an open neighborhood of K. Evidently for any $x \in U$ we have $(f \times g)(x) \notin D$, i.e. we have $f(x) \neq g(x)$. ∎

Now we head towards the denouement.

Proof of the Table Theorem 219.3. First recall that by Observation 219.12 it suffices to show that $A' \cap B' \neq \varnothing$. We do a proof by contradiction. So we make the following assumption:

(∗) We assume that $A' \cap B' = \varnothing$.

In other words, we assume that ψ' never vanishes on A'. Throughout the proof the following two maps will play a major role:

$$\begin{array}{rcl} \widehat{\psi}\colon \{z \in T \mid \psi'(z) \neq 0\} & \to & S^1 \\ z & \mapsto & \frac{1}{\|\psi'(z)\|}\cdot\psi'(z) \end{array} \quad \text{and} \quad \begin{array}{rcl} \theta\colon \partial T = \partial D \times (S^1/\sim) & \to & \partial D \to S^1 \\ z & \mapsto & \frac{1}{\|z\|}\cdot z. \end{array}$$

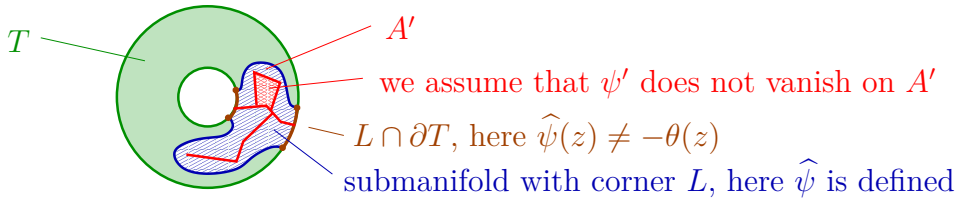
Note that A' has the following two properties:

(α) ψ' is non-zero on A' (in particular $\widehat{\psi}$ is defined on A'), and

[2354] This is the content of Exercise 18.2.

(β) By the ψ-Lemma 219.10 (3) we know that for any $z \in A' \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$. The following claim says that we can enlarge A' to a smooth submanifold with corner which still has properties (α) and (β). More precisely, we have the following claim:

Claim. The smooth manifold T admits a compact smooth submanifold with corner L with the following properties:
(i) $A' \subseteq \mathring{L}$,
(ii) ψ' is non-zero on L (in particular $\widehat{\psi}$ is defined on L), and
(iii) for any $z \in L \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$.

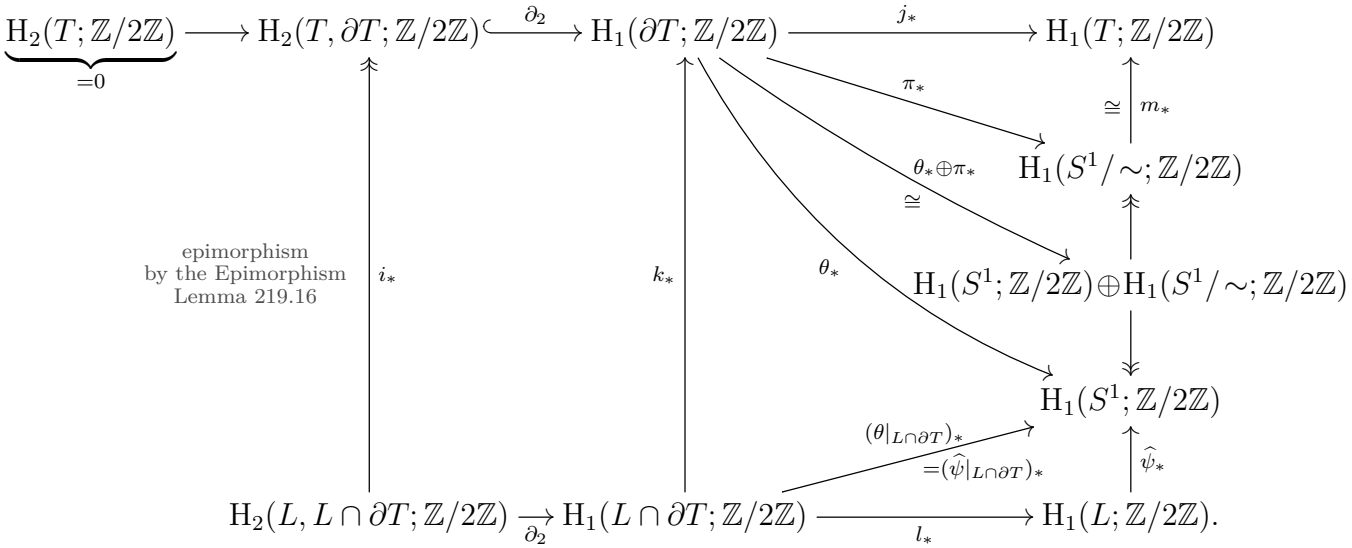


Proof. Recall that T is a solid torus, in particular T is compact. Since A is a closed subset of $D \times S^1$ we obtain from the Compact-Closed Lemma 3.39 (1) and the Compact Image Lemma 4.15 that $A' = p(A)$ is compact.

(1) It follows from (α), the fact that A' is compact and the Maps-are-Different Lemma 219.18 that there exists an open neighborhood U of A' such that ψ' vanishes nowhere on U.
(2) It follows from (β), Exercise 3.18 (which is an immediate consequence of the Compact-Closed Lemma 3.39 (1)) and the Maps-are-Different Lemma 219.18 that there exists an open neighborhood V of $A' \cap \partial T$ such that $\widehat{\psi}(z) \neq -\theta(z)$ for all $z \in V$.

Next note that it follows from the Wrapping-in-Submanifold Lemma 38.4 (2) that there exists a smooth submanifold L with corner of T such that $A' \subseteq \mathring{L} \subseteq L \subseteq U$ and such that $L \cap \partial T \subseteq V$. In particular we see that ψ' is non-zero on L and that for any $z \in L \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$. ⊞

We consider the following diagram:



We make the following comments, clarifications and observations:

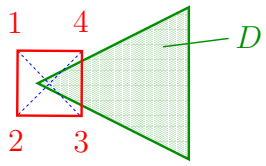
(1) The horizontal maps at the top respectively the bottom are extracts of the long exact sequence in homology of the pair $(T, \partial T)$ respectively of the pair $(L, L \cap \partial T)$.
(2) The maps i, j, k, l indicate the obvious inclusion maps.
(3) We denote by $m\colon S^1/\sim \xrightarrow{\cong} \{0\} \times S^1/\sim \to D \times (S^1/\sim) = T$ the obvious map. Since D is convex we see that m is a homotopy equivalence, hence we obtain from the Homotopy Equivalence-H_*-Proposition 144.14 (1a) that m_* is an isomorphism.
(4) The map $\pi\colon \partial T = \partial D \times (S^1/\sim) \to S^1/\sim$ is the natural projection. It is elementary to show that the maps $m \circ \pi$ and j from ∂T to T are homotopic. Thus we obtain from the Homotopy-H_*-Proposition 144.12 that they induce the same map on homology. In other words, the triangle to the top right commutes.
(5) It follows from the naturality of the connecting homomorphism, see the LES-of-Triple Proposition 146.5, that the rectangle to the left commutes. All the remaining triangles to the right commute basically by definition or by the functoriality of homology groups.
(6) By the discussion on page 190 the map $\theta\colon \partial D \to S^1$ is a homeomorphism. It follows easily from this observation, together with the fact that $\partial T = \partial D \times (S^1/\sim)$ and the Künneth Theorem 164.6 , that the map $\theta_* \oplus \pi_*\colon \mathrm{H}_1(\partial T; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_1(S^1; \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{H}_1(S^1/\sim; \mathbb{Z}/2\mathbb{Z})$ is an isomorphism.
(7) It follows from (ii) and the Map-to-S^n-Homotopy Lemma 219.17 that the restrictions of $\widehat{\psi}$ and θ to $L \cap \partial T$ are homotopic. We deduce from the Homotopy-H_*-Proposition 144.12 that the induced maps on homology are the same. Put differently, the two maps indicated at the upward-pointing diagonal arrow are actually the same.
(8) By the Epimorphism Lemma 219.16 the left vertical map is an epimorphism. Since $\mathrm{H}_2(T, \partial T; \mathbb{Z}/2\mathbb{Z}) \neq 0$ there exists a $\xi \in \mathrm{H}_1(L, L \cap \partial T; \mathbb{Z}/2\mathbb{Z})$ with $i_*(\xi) \neq 0$.
(9) It follows from the long exact sequence on top that $\partial i_*(\xi) \neq 0$ and that $j_* \partial i_*(\xi) = 0$.
(10) It follows from (3), (4) and (9) that $\pi_* \partial i_*(\xi) = 0$.
(11) By (6) we know that the map $\theta_* \oplus \pi_*$ is an isomorphism. Therefore we obtain from (8) and (10) that $\theta_* \partial i_*(\xi) \neq 0 \in \mathrm{H}_1(S^1; \mathbb{Z}/2\mathbb{Z})$.
(12) Since the diagram commutes we obtain from (11) that $l_* \partial \xi \neq 0$ in $\mathrm{H}_1(L; \mathbb{Z}/2\mathbb{Z})$. But that is absurd, since the horizontal sequence at the bottom is exact. ■

219.2. **Trivial and non-trivial solutions to the table theorem.** In this short section we want to discuss the (non-) existence of trivial solutions to the table theorem. More precisely, we have the following definition:

Definition 219.19. Let $D \subseteq \mathbb{R}^2$ be a compact convex non-empty subset of $\mathbb{R}^2$.
(1) Let $s \in \mathbb{R}_{>0}$. We say that D is s-trivial if there exists a square in $\mathbb{R}^2$ of side length s such that the vertices lie outside of $\overset{\circ}{D}$ and such that the center of the square lies in D. Otherwise we say that D is s-non trivial.
(2) We say that D is trivial if D is s-trivial for every $s \in \mathbb{R}_{>0}$.

It is basically clear that any compact convex subset with empty interior is trivial. One might initially hope that every compact convex subset with non-empty interior is non-trivial. But that is not the case.

Example. We consider the convex subset D that is the convex hull of the points $(0, 0)$, $(2, 1)$ and $(2, -1)$.

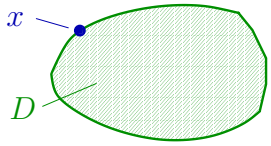


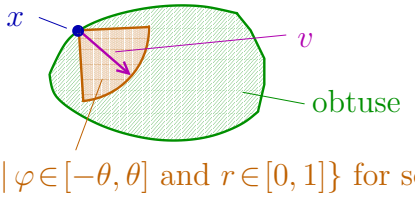
We claim that it is trivial. Indeed, let $s \in \mathbb{R}_{>0}$. We write $t = \frac{s}{2}$. The square with vertices $a_1 = (-t + \frac{t}{4}, t)$, $a_2 = (-t + \frac{t}{4}, -t)$, $a_3 = (t + \frac{t}{4}, -t)$ and $a_4 = (t + \frac{t}{4}, t)$, whose center equals $(\frac{t}{4}, 0)$, shows that D is s-trivial. □

This example leads us to the following definition:

Definition 219.20. We say that a subset D of $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ is obtuse if given any $x \in \partial D$ there exists a $v \in \mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ and a $\theta > \frac{\pi}{4}$ such that

$$\{x + r \cdot v \cdot \exp(\mathrm{i}\varphi) \,|\, \varphi \in [-\theta, \theta] \text{ and } r \in [0,1]\} \subseteq D.$$





We conclude this short section with the following lemma:

Lemma 219.21. (Obtuse–Non Trivial Lemma) Let D be a compact convex non-empty subset of $\mathbb{R}^2$. If D is obtuse, then there exists a $t > 0$ such that D is s-non trivial for every $s \in (0, t)$.

Proof. This lemma is proved in [**FI2023**]. The proof is quite elementary and the reader can try to give their own proof. ■

219.3. **The square-peg problem.** In this section we will study the "square-peg problems", which is a surprising application of the General Alexander Duality Theorem 218.2. To introduce the topic we need the following definition:

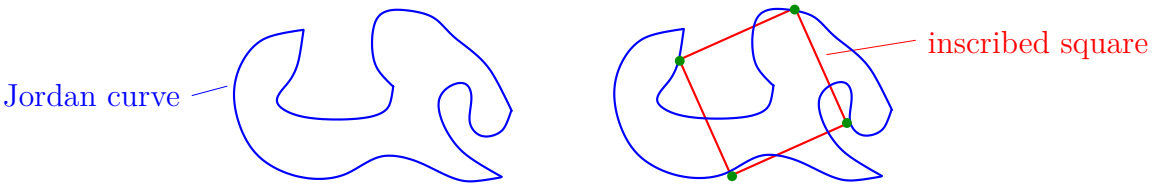
Definition 219.22.

(1) The image of an injective continuous map $S^1 \to \mathbb{R}^2$ is called a Jordan curve.[2355]
(2) In this section a rectangle is any subspace R of $\mathbb{R}^2$ for which we can find *distinct points* $P_1, P_2, P_3, P_4 \in \mathbb{R}^2$ such that R is the union of four line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$ and $\overline{P_4P_1}$. If all these line segments have the same length, then we also speak of a square.
(3) Given a Jordan curve C we say that a rectangle R is inscribed in C if all vertices of R lie on C.

The following question was asked by Otto Toeplitz [**Toep1911**] in 1911.[2356]

[2355]Note that this definition is slightly different from the definition we gave in Definition 39.19 where we defined a Jordan curve to be an injective continuous map $S^1 \to \mathbb{R}^2 \cup \{\infty\} =_{\mathrm{i}} S^2$.
[2356]Otto Toeplitz (1881-1940) was a Jewish German mathematician who was forced to emigrate in 1939. He is mainly known for his contributions to analysis.

Question 219.23. (Square Peg Problem) Does every Jordan curve admit an inscribed square?



This question is known as the "inscribed square problem" or the "square-peg problem". Lev Schnirelmann [**Schni1944**][2357] (see also [**Stromq1989**]) gave a positive answer if the closed curve is smooth, in fact he gave a proof if the closed curve is C^2, i.e. if the map $S^1 \to \mathbb{R}^2$ is twice continuously differentiable. But the general case is still open.[2358] A detailed discussion of the question and its history is given in [**Matsc2014**].



In this section we will provide two partial answers to Question 219.23.

(1) We will use the Table Theorem 219.3 to deal with the case that the Jordan curve is the boundary of a compact convex obtuse subset of $\mathbb{R}^2$.
(2) We will show that we can inscribe a rectangle into every Jordan curve.

Let us start with the first result.

Proposition 219.24. (Obtuse–Square Peg Proposition) Let J be a Jordan curve which is the boundary of a compact convex subset D. If D is obtuse, then J admits an inscribed square.

Remark. This proposition also admits proofs that do not rely on topology, see for example [**Zin1921, Chr1950**]. □

Proof. Let $D \subseteq \mathbb{R}^2$ be a compact convex obtuse subset. We need to show that there exist four points $a_1, a_2, a_3, a_4 \in \partial D$ which form a square. After a translation we can assume that the origin 0 is contained in the interior of D. Given $x \in D \setminus \{0\}$ we set

$$\rho(x) \;:=\; \sup\big\{\|r \cdot x\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot x \in D\big\} \;\in\; \mathbb{R}_{>0}.$$

It follows almost immediately from the Distance-to Boundary Continuous Lemma 4.29[2359] that the map

[2357] We had already encountered Lev Schnirelmann in the Lusternik-Schnirelmann Theorem 48.2.
[2358] The Fields medalist Terence Tao [**Tao2017**] wrote a long paper in which he outlined an approach to the square-peg problem. See also the following video:

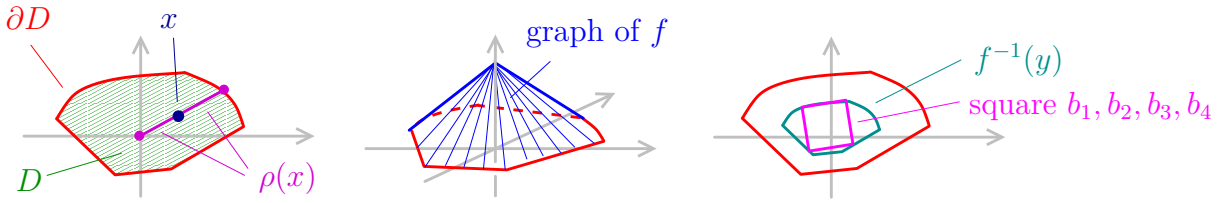
`https://www.youtube.com/watch?v=RGxnWgy5i80`

[2359] Well-trained eyes will not fail to spot that implicitly we also make use of the Interior-Closure Lemma 3.22, the Pasting Proposition 4.7 and the Compact-Closed Lemma 3.39 to show that f is indeed continuous.

$$\begin{array}{rcl} f\colon \mathbb{R}^2 & \to & [0,1] \\ x & \mapsto & \begin{cases} 1 - \|x\| \cdot \frac{1}{\rho(x)}, & \text{if } x \in D \setminus \{0\}, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x \notin D \end{cases} \end{array}$$

is continuous. Since D is obtuse we obtain from the Obtuse–Non Trivial Lemma Lemma 219.21 that there exists a $d > 0$ such that D is d-non trivial. By the Table Theorem 219.3 there exist four points $b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ with the following properties:

(1) The points form a square of side length d.
(2) The center of the square lies in D.
(3) We have $f(b_1) = f(b_2) = f(b_3) = f(b_4)$.



Since D is d-non trivial we see that at least one vertex lies inside of $\mathring{D}$. But since f is non-zero on $\mathring{D}$ we see that the common f-value, let's call it y, lies in the open interval $(0,1)$. It follows immediately from the definition of f that the four points b_1, b_2, b_3, b_4 lie on the subset $(1-y) \cdot \partial D$. In other words, if we multiply these four points by $\frac{1}{1-y}$, then we obtain the desired four points a_1, a_2, a_3, a_4 on ∂D. ■

We conclude this chapter with the second contribution to the square-peg problem.

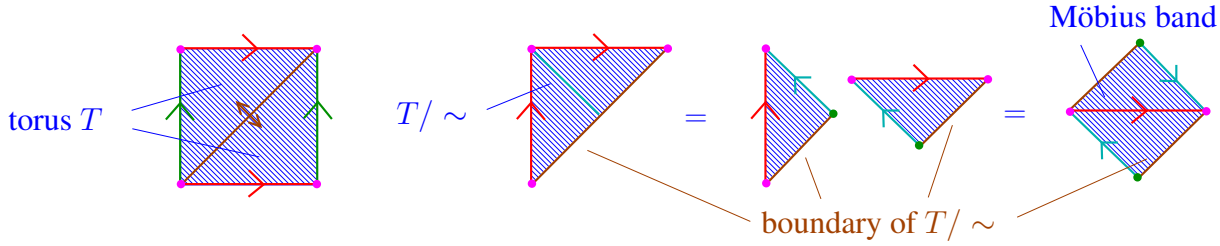
Proposition 219.25. (Inscribed Rectangle Proposition) Every Jordan curve admits an inscribed rectangle.[2360]

In preparation for the proof we need the following elementary lemma:[2361]

Lemma 219.26. (Torus Quotient–Möbius Band Lemma) Let $T = S^1 \times S^1$ be the torus. Let $\sim$ be the equivalence relation on T generated by $(z,w) \sim (w,z)$. The quotient space $T/\sim$ is homeomorphic to the Möbius band and[2362]

$$\partial(T/\sim) \;=\; \{[(z,z)] \,|\, z \in S^1\}.$$

Proof. The homeomorphism is illustrated in the figure below.



[2360] In [**Meye1982**] the subsequent ingenious proof of the Inscribed Rectangle Proposition 219.25 is attributed to Herbert Vaughan.
[2361] Note that the lemma is the content of Exercise 8.16.
[2362] Here by "$\partial(T/\sim)$" we mean the boundary of the topological manifold $T/\sim$.

We leave it to the reader to fill in the details. ■

Now we are already ready to prove the Inscribed Rectangle Proposition 219.25.

Proof of the Inscribed Rectangle Proposition 219.25. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an injective continuous map. We have to show that $\gamma(S^1)$ admits an inscribed rectangle. We consider the map

$$\begin{aligned} \Phi\colon T := S^1 \times S^1 &\to \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \\ (z,w) &\mapsto \Big(\tfrac{\gamma(z)+\gamma(w)}{2}, \|\gamma(z)-\gamma(w)\| \Big). \end{aligned}$$

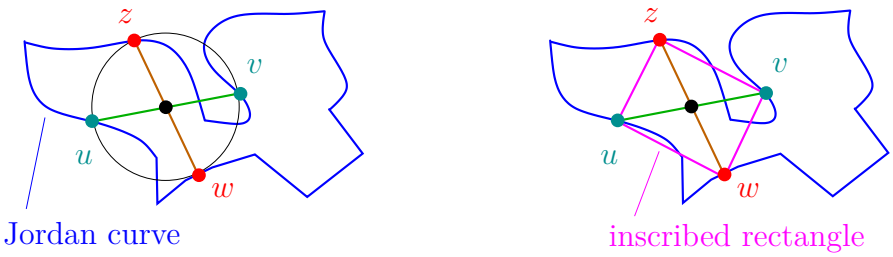
Clearly this map is continuous. Next note that for all $z, w \in S^1$ we have the equality $\Phi(z,w) = \Phi(w,z)$. Thus we obtain from the Topological-Quotient Proposition 8.2 (1b) that Φ descends to a continuous map

$$\Psi\colon T/\!\sim \to \mathbb{R}^3$$

where $\sim$ denotes of course the equivalence relation generated by $(z,w) \sim (w,z)$.

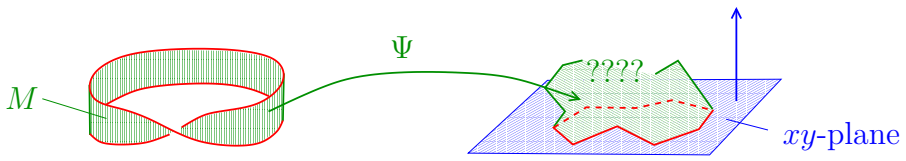
Claim. The map $\Psi\colon T/\!\sim \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is not injective.

We first show that the claim implies the proposition. So now that we know that Ψ is not injective we know that there exist distinct pairs of points $\{z,w\}$ and $\{u,v\}$ on S^1 such that $\Phi(z,w) = \Phi(u,v)$. We consider the quadrilateral given by the points z,u,v,w.



It follows from $\Phi(z,w) = \Phi(u,v)$ that the diagonals bisect each other and that the diagonals have the same length. An elementary argument (see Exercise 219.4) shows that such a quadrilateral is necessarily a rectangle.

Proof of claim. We will prove the claim by contradiction. Thus let us now assume that the map $\Psi\colon T/\!\sim \to H := \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is injective. By the Torus Quotient–Möbius Band Lemma 219.26 the quotient $M := T/\!\sim$ is homeomorphic to the Möbius band and the boundary ∂M is given by $\{[(z,z)] \,|\, z \in S^1\}$. It follows easily from the definition of Ψ and the injectivity of γ that for a point $[(z,w)] \in M = T/\!\sim$ we have $\Psi([(z,w)]) \in \partial H = \mathbb{R}\times\mathbb{R}\times\{0\}$ if and only if $[(z,w)] \in \partial M$.



We denote by M_1 and M_2 two copies of M, and as in Definition 54.8 we denote by

$$\mathrm{D}M \;=\; M_1 \cup_{\partial M_1 = \partial M_2} M_2 \;=\; (M_1 \sqcup M_2)/(P,1) \sim (P,2) \quad \text{for } P \in \partial M$$

the double of M. It follows from the proof of the Surface–Connected Sum Lemma 54.15 (1) that DM is in fact homeomorphic to the Klein bottle, in particular DM is a closed, non-orientable 2-dimensional topological manifold.
We denote by $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the xy-plane, i.e. ρ is the map that is given by $\rho(x,y,z) = (x,y,-z)$. We consider the map

$$\begin{array}{rcl} \mathrm{D}M = M_1 \cup_{\partial M_1 = \partial M_2} M_2 & \to & \mathbb{R}^3 \\ [P] & \mapsto & \begin{cases} \Psi(P), & \text{if } P \in M_1, \\ \rho(\Psi(P)), & \text{if } P \in M_2. \end{cases} \end{array}$$

It follows immediately from the above discussion that this map is well-defined and injective. Furthermore it follows from the Topological-Quotient Proposition 8.2 (1b) that the map is in fact continuous. But now we have constructed an injective continuous map from the Klein bottle to $\mathbb{R}^3$. Since the Klein bottle is compact we obtain from the Compact-Hausdorff Proposition 4.23 that such a map is in fact an embedding. But recently we proved in the Surface-No Topological Embedding-Corollary 218.10 that such an embedding cannot exist. ■

Remark.

(1) The above proof of the Inscribed Rectangle Proposition 219.25 is also explained in the following video:

`https://www.youtube.com/watch?v=AmgkSdhK4K8`

(2) There are many variations on Question 219.23. For example, given $r \neq 1$ one can ask whether every Jordan curve admits an inscribed rectangle with aspect ratio r. Even for smooth curves this had been an open problem for a long time. The first real progress since the work by Schnirelmann [**Schni1944**] happened in 2018 when Cole Hugelmeyer [**Hug2018**] showed that a deep result in knot theory, due to Josh Batson [**Bats2014**], implies that every smooth Jordan curve admits an inscribed rectangle of aspect ratio $\sqrt{3}$. The key idea of [**Hug2018**] is rather similar to the proof of the Inscribed Rectangle Proposition 219.25. Even more impressively, an affirmative answer for smooth curves and any aspect ratio was given in 2020 by Josh Greene and Andrew Lobb [**GrL2021**] (see also [**Hug2021**]). □

219.4. Exercises for Chapter 219 .

Exercise 219.1.

(a) Let $P_1, P_2, P_3, P_4 \in \mathbb{R}^3$. Show that

$$P_1, P_2, P_3, P_4 \text{ are coplanar} \iff ((P_2 - P_1) \times (P_3 - P_1)) \cdot (P_4 - P_1) = 0.$$

(b) Let $f\colon \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, let $x \in \mathbb{C}$ and let $z \in \mathbb{C} \setminus \{0\}$. Show that there exists a $\theta \in [0, \frac{\pi}{2}]$ such that the four points

$$(x + e^{\mathrm{i}(\theta + k\cdot\frac{\pi}{2})} \cdot z, f(x + e^{\mathrm{i}(\theta + k\cdot\frac{\pi}{2})} \cdot z)), \qquad \text{with } k = 0,1,2,3$$

are coplanar.

Exercise 219.2. Let X be a compact non-empty subset of $\mathbb{R}^2$ of diameter d. Does there exist a $P \in \mathbb{R}^2$ such that $X \subseteq \overline{B}^2_{\frac{d}{2}}(P)$?

Exercise 219.3. We consider the solid torus $T = \overline{B}^2 \times S^1$. Let $f\colon T \to \mathbb{R}$ be a continuous map such that for every $(x, \gamma) \in T$ we have $f(x, -\gamma) = -f(x, \gamma)$. We consider the subset $A \coloneqq \{(x, \gamma) \in T \mid f(x, \gamma) = 0\}$ of T.

(a) Show that the inclusion-induced map $\mathrm{H}_1(T \setminus A; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_1(T; \mathbb{Z}/2\mathbb{Z})$ is the zero map.

(b) Give an example of a continuous map $f\colon T \to \mathbb{R}$ as above such that the inclusion-induced map $\mathrm{H}_1(T \setminus A) \to \mathrm{H}_1(T)$ is *not* the zero map.

Exercise 219.4. Suppose that we are given a quadrilateral with four distinct vertices $A, B, C, D \in \mathbb{R}^2$ such that the diagonals have the same lengths and such that each diagonal bisects the other. Show that the quadrilateral is a rectangle.

Exercise 219.5. Show that the Möbius band is homeomorphic to $\mathrm{U}(2)/\sim$ where $A \sim B$ if A and B are conjugate in $\mathrm{U}(2)$.

Exercise 219.6. Let $\varphi\colon S^1 \to \mathbb{R}^2$ be an embedding. Show that there exist distinct three points $x, y, z \in S^1$ such that $\varphi(x), \varphi(y), \varphi(z)$ from an equilateral triangle.
Hint. Use the Generalized Jordan-Brouwer Separation Theorem 218.12 for $n = 2$.
Remark. This exercise is basically identical to Exercise 51.10.

CHAPTER 220

The cup product and algebraic intersection numbers

In Chapter 51 we introduced the algebraic intersection number $X \cdot Y$ of two oriented submanifolds X and Y of complementary dimensions of a given oriented smooth manifold M. The main technical result that we proved back then is the Intersection Number–Homotopy Invariance Theorem 51.9, which says that the algebraic intersection numbers is unchanged under smooth isotopies of $X \sqcup Y$.

In this chapter we first recall the definition of the algebraic intersection number. In the Cup Product–Intersection Number Theorem 220.10 we relate this algebraic intersection number to the cup product of suitable cohomology classes of M. The fact that algebraic intersection numbers correspond to the cup product can be used to compute cup products in a geometric and often entertaining way. Conversely this result can also be used to prove deep statements about intersections of submanifolds. For example we can give a new and very short proof of the Intersection Number–Homotopy Invariance Theorem.

This chapter is written in a way such that it can be read independently of Chapter 51. In particular we will motivate some questions and results, even though we might have solved some of the issues already in Chapter 51.

This chapter is organized as follows:

- In Section 220.1 we will recall the algebraic intersection numbers of submanifolds.
- In Section 220.2 we will formulate the Cup Product–Intersection Number Theorem 220.10 which expresses algebraic intersection numbers in terms of cup products.
- In Section 220.3 we will prove the Cup Product–Intersection Number Theorem 220.10
- In Section 220.4 we will discuss the cup product on complex manifolds.
- In Section 220.5 we will provide exercises for Chapter 220.

In the next Chapter 221 we will introduce intersection numbers for suitable cycles and we will express them in terms of cup product.

220.1. **Algebraic intersection numbers of submanifolds.** Before we (re-) introduce the algebraic intersection number of submanifolds we want to recall Definition 41.10 and we want to give a motivating question:

Definition 220.1. Let M be a smooth manifold.

(1) We say that two oriented proper[2363] submanifolds Z_0 and Z_1 of M are properly smoothly isotopic if there exists a proper smooth isotopy between Z_0 and Z_1, i.e. if there exists a smooth isotopy $F\colon Z_0 \times [0,1] \to M$ such that $F_0 = \mathrm{id}_{Z_0}$, such that each $F_t\colon Z_0 \to M$ is a proper embedding, and such that $F_1\colon Z_0 \to Z_1$ is an orientation-preserving diffeomorphism.

(2) Let X and Y be two oriented proper submanifolds of M. We say that X and Y can be made disjoint if there exist oriented proper disjoint submanifolds $\widetilde{X}$ and $\widetilde{Y}$ of M such that X and $\widetilde{X}$ and also Y and $\widetilde{Y}$ are properly smoothly isotopic.

Example. We consider the three submanifolds X, Y and Z of the torus that are illustrated in the figure below on the left.



As is shown in the same figure on the right, one can make X and Y disjoint. □

It does not look like we can make the submanifolds X and Z disjoint. But how can we prove it? We will record this question for later:

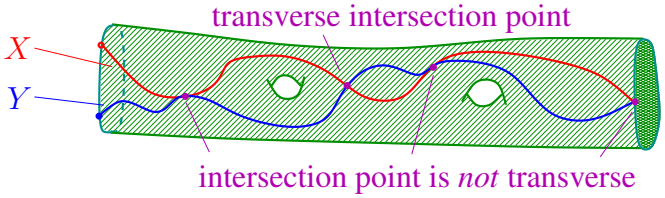
Question 220.2. Can we make the submanifolds X and Z, that are illustrated in the figure above, disjoint?

Our approach to addressing Question 220.2 is to introduce the "algebraic intersection number". Before we can introduce this notion, we need to recall Definition 50.3.

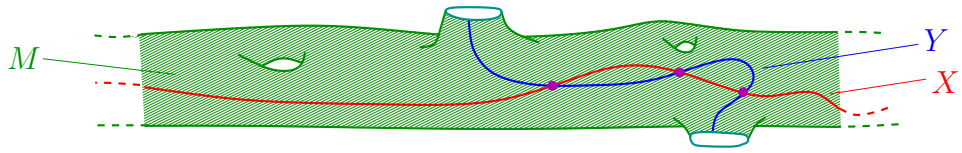
Definition 220.3. Let M be a smooth manifold and let X and Y be proper submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$.

(1) Let $P \in X \cap Y$. We say that P is a transverse intersection point if $P \notin \partial M$ and if X and Y intersect transversally in P, i.e. if the equality[2364] $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ holds.

(2) We say that X and Y intersect transversally if each intersection point is a transverse intersection point.



Lemma 220.4. (Finite Transverse Intersection Lemma) Let M be a smooth manifold and let X and Y be proper submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. If X and Y intersect transversally and if at least one of X or Y is compact, then the intersection $X \cap Y$ consists of finitely many points.



[2363] Recall that by the Proper Submanifold Proposition 25.30 (1) we know that a submanifold Z is proper if Z is a closed subset of M and if $\partial Z = Z \cap \partial M$.

[2364] Since here we assume that $\dim(X) + \dim(Y) = \dim(M)$ we see that we have $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ if and only if $\mathrm{T}_P X \oplus \mathrm{T}_P Y = \mathrm{T}_P M$.

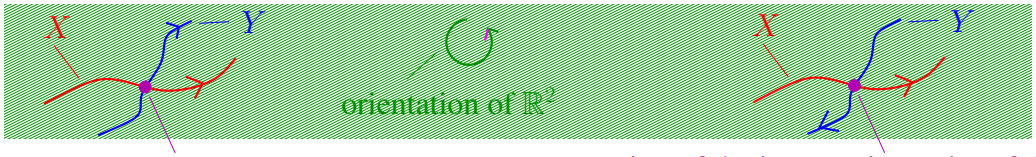
Proof. Without loss of generality we can now assume that Y is compact. Since X and Y intersect transversally it follows from the Submanifold Transversal Intersection Theorem 50.4 that $X \cap Y$ is a 0-dimensional submanifold of M. It follows easily from the definition of a 0-dimensional submanifold that $X \cap Y$ is a discrete subset of M. Since Y is compact and since X is in particular a closed subset of M we obtain from the Compact-Closed Lemma 3.39 (1) that $X \cap Y$ is also compact. It now follows from the Compact-Discrete Lemma 3.40 that $X \cap Y$ is finite. ■

Next we recall Definition 51.4:

Definition 220.5. Let $m \in \mathbb{N}$ and let $k \in \{1, \dots, m-1\}$. Furthermore let M be an m-dimensional smooth manifold, let $X \subseteq M$ be a compact proper submanifold of dimension k and let $Y \subseteq M$ be a compact proper submanifold of the complementary dimension $m-k$. Suppose that X and Y intersect transversally.

(1) We define[2365]
$$\text{geometric intersection number of } X \text{ and } Y \;:=\; \#(X \cap Y).$$

(2) We now assume that X, Y and M are oriented.
   (a) Let P be an intersection point of X and Y. We set $\operatorname{sign}(P) := 1$, if
   $$(\text{positive basis of } \mathrm{T}_P X, \text{positive basis of } \mathrm{T}_P Y) \;=\; \text{positive basis of } \mathrm{T}_P M.$$
   Otherwise we set $\operatorname{sign}(P) := -1$.[2366]
   (b) We define[2367]
   $$\text{algebraic intersection number of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \operatorname{sign}(P) \in \mathbb{Z}.$$



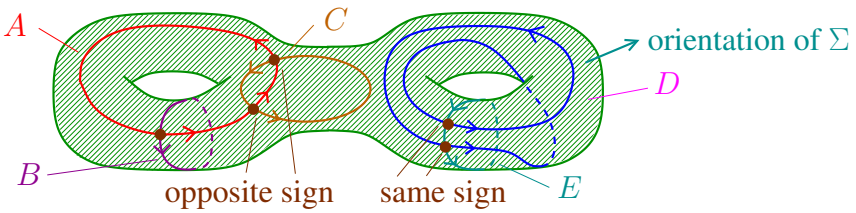
sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals $+1$

sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals -1

Examples.

(1) We consider the surface of genus 2 with the orientation where the normal vector "sticks outward". Furthermore we consider the oriented submanifolds A, B, C, D and E that are illustrated in the figure below:
For the submanifolds A and B we have
$$\text{geometric intersection number of } A \text{ and } B \;=\; 1,$$
the sign of the intersection point is negative, hence
$$\text{algebraic intersection number of } A \text{ and } B \;=\; -1.$$

[2365] It follows from the Finite Transverse Intersection Lemma 220.4 that the geometric intersection number is finite and that the algebraic intersection number is given by a finite sum.
[2366] It is a straightforward exercise in linear algebra to show that the definition of $\operatorname{sign}(P)$ does not depend on the choice of the positive bases of $\mathrm{T}_P X$ and $\mathrm{T}_P Y$.
[2367] It follows from the Intersection Numbers-via-Signs Lemma 51.5 that the above definition of the algebraic intersection number of two submanifolds agrees with Definition 51.3.



For the submanifolds A and C we have

$$\text{geometric intersection number of } A \text{ and } C \;=\; 2,$$

the intersection points have opposite signs, therefore

$$\text{algebraic intersection number of } A \text{ and } C \;=\; 0.$$

Finally for the submanifolds D and E we have

$$\text{geometric intersection number of } D \text{ and } E \;=\; 2,$$

here both intersection points have negative sign, thus

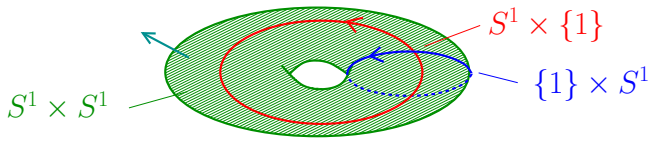
$$\text{algebraic intersection number of } D \text{ and } E \;=\; -2.$$

(2) Now suppose that M and N are oriented smooth manifolds of dimensions m respectively n. We saw in the Smooth Manifold Product Proposition 25.34 (1) and the Smooth Manifold Product Proposition 26.12 that $M \times N$ is naturally an $(m+n)$-dimensional smooth manifold (possibly with corner). By the Product Orientation Proposition 31.48 (3b) we know that for each $P \in M \setminus \partial M$ and $Q \in N \setminus \partial N$ we have a natural isomorphism

$$\mathrm{T}_P M \oplus \mathrm{T}_Q N \;\to\; \mathrm{T}_{(P,Q)}(M \times N)$$

which is induced by the obvious smooth embeddings. If $\{v_1, \ldots, v_m\}$ is a positive basis for $\mathrm{T}_P M$ and $\{w_1, \ldots, w_n\}$ is a positive basis for $\mathrm{T}_Q N$, then we endow the tangent vector space $\mathrm{T}_{(P,Q)}(M \times N)$ with the orientation that makes the above isomorphism orientation-preserving.[2368] Given $P \in M$ and $Q \in N$ it follows, basically by definition, that

$$\text{algebraic intersection number of } M \times \{Q\} \text{ and } \{P\} \times N \;=\; 1.$$



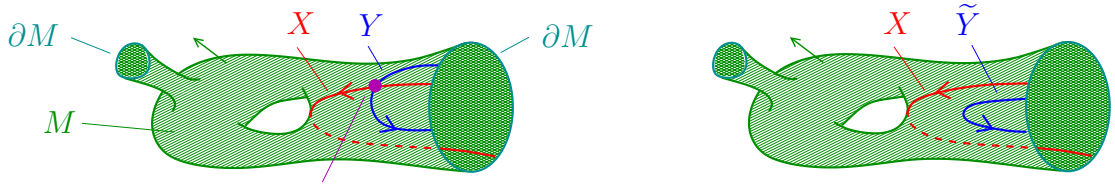
the algebraic intersection number of $S^1 \times \{1\}$ and $\{1\} \times S^1$ equals $+1$

(3) In Section 220.4.1 we will study the algebraic intersection number of complex submanifolds of a given complex manifold.

(4) The above example (1) suggests that if two submanifolds have non-zero algebraic intersection number, then they cannot be made disjoint. As we will see in the Intersection Number–Homotopy Invariance Theorem 220.12, this hunch is basically correct. But

[2368] One can easily verify that these orientations on the tangent spaces of $M \times N$ define an orientation on $M \times N$. In fact it follows fairly easily from the Product Orientation Proposition 192.7 (2) that this definition of the product orientation agrees with the definition we gave in Definition 192.8.

if we deal with submanifolds with boundary one needs to think carefully about the correct setting. For example in the figure below we show on the left a smooth manifold M and we show two proper submanifolds X and Y such that the boundaries ∂X and ∂Y lie on the same component of ∂M. The algebraic intersection number is $+1$, but as we see in the figure below on the right, these two submanifolds can be made disjoint.



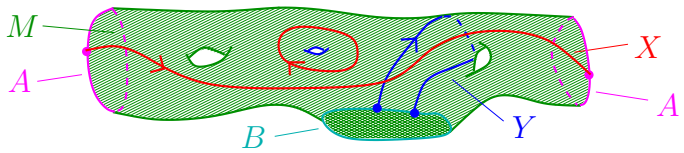
the algebraic intersection number of X and Y equals $+1$

Therefore in the following discussion we will always assume that the boundary components of X and Y lie in different boundary components of M. □

Before we continue, and in light of the last example, we introduce the following definition:

Definition 220.6. Let M be an oriented smooth manifold.

(1) A boundary decomposition for M is a decomposition $\partial M = A \sqcup B$ where A and B are disjoint unions of components of ∂M.
(2) Let X and Y be two submanifolds of M. We say that X and Y are complementary if the following five conditions are satisfied:
   (a) X and Y are proper submanifolds.
   (b) There exists a boundary decomposition $\partial M = A \sqcup B$ with $\partial X \subseteq A$ and $\partial Y \subseteq B$.
   (c) Both X and Y are oriented.
   (d) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
   (e) The submanifolds X and Y intersect transversally.



X and Y are complementary

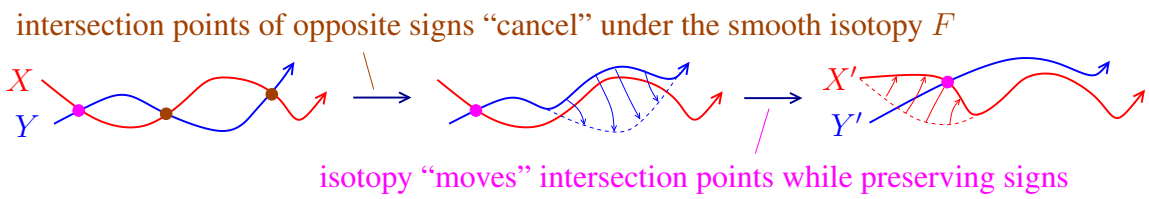
We would be able to answer Question 220.2 if we could prove the following statement:

Statement 220.7. Let M be a compact oriented smooth manifold and let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. Then the following holds:

$$\begin{array}{c} X, X' \text{ are properly smoothly isotopic and} \\ Y, Y' \text{ are properly smoothly isotopic} \end{array} \implies X \cdot Y = X' \cdot Y'.$$

But how can we prove such a statement? In the figure below we sketch an idea: We break the smooth isotopies from X to X' and from Y to Y' into "smaller isotopies" such that each smaller isotopy either does not change the type of intersection points, or two intersection points of opposite sign get created or erased. Even though the idea is fairly convincing, it is very hard to turn this idea into a proper proof.

In the Intersection Number–Homotopy Invariance Theorem 51.9 we gave a proof of the above statement, but if one goes through all the results that go into the proof of that theorem one sees that this approach is certainly not elementary.

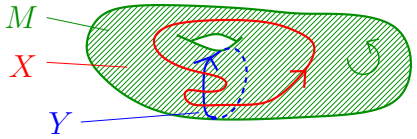


We will provide a new proof of Statement 220.7 later in this section, once we have explained the relationship between algebraic intersection numbers and the cup product.

We conclude this introduction to algebraic intersection numbers with the following elementary lemma, which is basically the same as the Intersection Number–Symmetry Lemma 51.6:

Lemma 220.8. (Intersection Number–Symmetry Lemma) Let M be a compact oriented smooth manifold. If X and Y are complementary submanifolds of M, then

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \cdot X \cdot Y.$$



$$X \cdot Y = +1 - 1 + 1 \;=\; +1$$
$$Y \cdot X = -1 + 1 - 1 \;=\; -1$$

Proof. We set $k = \dim(X)$ and $l = \dim(Y)$. It follows easily from the definitions that at each $P \in X \cap Y = Y \cap X$ the signs of the intersection points differ by the sign of the determinant of the following matrix:

$$\begin{pmatrix} 0_{l\times k} & \mathrm{id}_l \\ \mathrm{id}_k & 0_{k\times l} \end{pmatrix}.$$

An linear algebra elementary argument shows that the determinant of this matrix equals $(-1)^{k\cdot l}$. ■

220.2. **The cup product and algebraic intersection numbers.** We recall the following notation:

Notation 220.9.

(1) Given a compact oriented l-dimensional smooth manifold W we denote as usual by $[W] \in \mathrm{H}_l(W, \partial W)$ the fundamental class that we introduced on Definition 185.4 or alternatively onsee Definition 194.3. If W is a proper submanifold of some smooth manifold M, then sometimes we denote by $[W] \in \mathrm{H}_l(M, \partial M)$ the image of $[W]$ under the inclusion-induced homomorphism $\mathrm{H}_l(W, \partial W) \to \mathrm{H}_l(M, \partial M)$.

(2) Let M be a compact oriented m-dimensional smooth manifold and suppose that we are given a boundary decomposition $\partial M = A \sqcup B$. By the Poincaré Duality Theorem 210.1

we know that for each $k \in \mathbb{N}_0$ the map

$$\mathrm{H}^k(M, A; \mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{H}_{m-k}(M, B)$$

is an isomorphism. As in Definition 210.2 we denote by

$$\mathrm{PD}_M \colon \mathrm{H}_{m-k}(M, B) \longrightarrow \mathrm{H}^k(M, A; \mathbb{Z})$$

its inverse.[2369] If M is understood from the context then often we just write PD instead of PD_M.

The following theorem relates the cup product on a smooth manifold to algebraic intersection numbers. Or conversely, it expresses the algebraic intersection number of submanifolds as a cup product.

Theorem 220.10. (Cup Product–Intersection Number Theorem) Let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \sqcup B$. Let X and Y be a complementary pair of submanifolds of M with $\partial X \subseteq A$ and $\partial Y \subseteq B$. We write $k = \dim(X)$ and we denote by $i \colon (X, A) \to (M, \partial M)$ and $j \colon (Y, B) \to (M, \partial M)$ the obvious inclusion maps. Furthermore we denote by $[X] \in \mathrm{H}_k(X, \partial X)$ and $[Y] \in \mathrm{H}_{m-k}(Y, \partial Y)$ the fundamental classes of X and Y. Then[2370] [2371] [2372]

$$\underbrace{X \cdot Y}_{\text{see Definition 220.5}} = \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M,A)})}_{\in \mathrm{H}^{m-k}(M,B;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}_{\in \mathrm{H}^k(M,A;\mathbb{Z})}}^{\in \mathrm{H}^m(M,\partial M;\mathbb{Z})}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$

In particular with the abbreviation $[X] \coloneqq i_*([X])$ and $[Y] \coloneqq j_*([Y])$ we have the slightly more readable equality

$$X \cdot Y = \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle.$$

Remark.

(1) The following citation from [**Kir1989**, p. 21] encapsulates the roles of algebraic intersection numbers and cup products:

> "Think with [algebraic] intersection [numbers], prove with cup products."

More precisely, algebraic intersection numbers are very intuitive and often they can be used to guess a statement. But when it comes to proving statements, e.g. why does the intersection number only depend on the smooth isotopy type of a submanifold?, it is usually much easier to work with cup products.

[2369] If M is closed this statement is also contained in the Poincaré Duality Theorem 188.6.

[2371] It follows from the Excisive Triad-Proposition 158.20 that (M, A, B) is an excisive triad. Thus the cup product is actually defined.

[2372] One could use the right-hand side of the Cup Product–Intersection Number Theorem 220.10 as the definition of the algebraic intersection number. We will discuss this idea in Exercise 220.7.

(2) For many purposes the following asymmetric formulation of the Intersection Number-via-Cup Product Theorem 220.10 can be useful:

$$\begin{array}{ll} & \text{by the Cup-Cap Lemma 206.15} \\ X \cdot Y \overset{\text{follows from the Intersection Number via-Cup Product Theorem 220.10 and the Cap Product-Kronecker Lemma 206.12}}{\downarrow =} & \downarrow = \epsilon_M(\mathrm{PD}_M([Y]) \cap (\mathrm{PD}_M([X]) \cap [M])) \\ \epsilon_M((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]) & \\ = \epsilon_M(\mathrm{PD}_M([Y]) \cap [X]) & = \langle \mathrm{PD}_M([Y]), [X] \rangle_M . \\ \uparrow = \text{by Definition 210.2} & \uparrow = \text{by the Cap Product-Kronecker Lemma 206.12} \end{array}$$

(3) If we drop the condition that M, X and Y are orientable, then we can still define an algebraic intersection number $X \cdot_{\mathbb{F}_2} Y$ and we can formulate Poincaré Duality with $\mathbb{F}_2$-coefficients. This allows us to formulate a variation on the Cup Product–Intersection Number Theorem 220.10. To save the trees we will not spell out the details at this point, instead we refer to the more general Maps-$\mathbb{F}_2$-Cup Product–Intersection Number Theorem 221.29 for details.

(4) In Section 244.5 we will formulate and discuss the Cup Product–Intersection Manifold Theorem 244.16 which is a generalization of the Cup Product–Intersection Number Theorem 220.10 where we allow that $\dim(X \cap Y) > 0$. □

The remainder of this chapter is structured as follows:

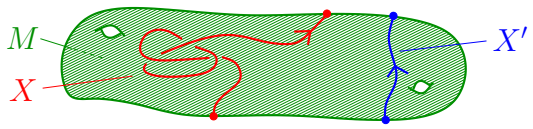
- In the following we will discuss many applications of the Cup Product–Intersection Number Theorem 220.10, to convince ourselves that proving the theorem is worth the effort.
- In Section 220.3 we will provide the lengthy proof of the Cup Product–Intersection Number Theorem 220.10.
- In Section 220.4.1 we will discuss algebraic intersection numbers of complex submanifolds.
- In Section 220.4.2 we will use Cup Product–Intersection Number Theorem 220.10 to discuss the cup product on complex projective spaces. In particular we will see that we can dispose of a sign indeterminacy, which had been bothering us.

A first major application of the Cup Product–Intersection Number Theorem 220.10 is that now we can provide a proof of a strengthened version of Statement 220.7.

To formulate this result we need to introduce the following definition:

Definition 220.11. Let M be a smooth manifold and let $X, X' \subseteq M$ be two oriented proper submanifolds. We say that X and X' are properly homotopic if there exists a homotopy $F\colon X \times [0,1] \to M$ with the following three properties:

(1) We have $F_0 = \mathrm{id}$.
(2) The map F_1 is an orientation-preserving diffeomorphism $X \to X'$.
(3) For each $t \in [0,1]$ we have $F_t(\partial X) \subseteq \partial M$.



X and X' are properly homotopic but they are not properly isotopic

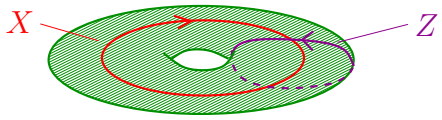
Now we can state the promised theorem, which is basically the same as the earlier Intersection Number–Homotopy Invariance Theorem 51.10:

Theorem 220.12. (Intersection Number–Homotopy Invariance Theorem) Let M be a compact oriented smooth manifold.

(1) Let $\{X, Y\}$ and $\{X', Y'\}$ be pairs of complementary submanifolds of M in the sense of Definition 220.6. If X, X' are properly homotopic and if Y, Y' are also properly homotopic, then $X \cdot Y = X' \cdot Y'$.

(2) If X and Y are two complementary submanifolds of M that can be made disjoint, then $X \cdot Y = 0$.

Example. Now we can answer Question 220.2.



It follows from the Intersection Number–Homotopy Invariance Theorem 220.12 that the submanifolds X and Z, that are illustrated above and in the figure on page 4229, cannot be made disjoint using proper homotopies (let alone through proper smooth isotopies) since their algebraic intersection number is non-zero. □

Proof. Let M be a compact oriented smooth manifold. We start out with the following claim:

Claim. Let Z and Z' be oriented proper s-dimensional submanifolds of M and let $C \subseteq \partial M$ be a union of boundary components with $\partial Z \subseteq C$ and $\partial Z' \subseteq C$. If Z and Z' are properly homotopic, then $[Z] = [Z'] \in \mathrm{H}_s(M, C)$.

Proof. We denote by $i\colon Z \to M$ and $i'\colon Z' \to M$ the inclusion maps. Since Z and Z' are properly homotopic there exists in particular an orientation-preserving diffeomorphism $f\colon Z \to Z'$ such that the maps $i, i' \circ f\colon (Z, \partial Z) \to (M, C)$ are homotopic. We obtain that

$$[Z] = i_*([Z]) \underset{\uparrow}{=} (i' \circ f)_*([Z]) = i'_*(f_*([Z])) \underset{\uparrow}{=} i'_*([Z']) = [Z'] \in \mathrm{H}_s(M, C).$$

by the Homotopy-H_* Proposition 146.11

by the Orientation Preserving–Characterisation Proposition 196.6, since f is an orientation-preserving homeomorphism

⊞

Now let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. We assume that the proper submanifolds X, X' are properly homotopic and that the smooth manifolds Y, Y' are homotopic. We obtain that

$$X \cdot Y \underset{\uparrow}{=} \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}_M([X']) \cup \mathrm{PD}_M([Y']), [M] \rangle \underset{\uparrow}{=} X' \cdot Y'.$$

by the Cup Product–Intersection Number Theorem 220.10

by the claim

by the Cup Product Intersection Number Theorem 220.10

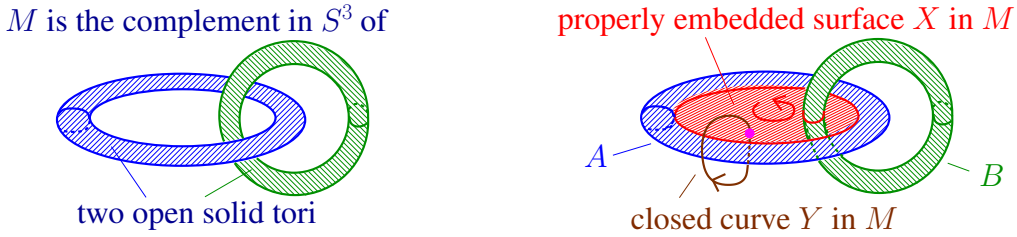
This concludes the proof of Statement (1). Statement (2) is an immediate consequence of Statement (1) and the observation that the algebraic intersection number of disjoint submanifolds is of course zero. ■

Now we will use the Cup Product–Intersection Number Theorem 220.10 "the other way", i.e. we will apply the theorem to use the algebraic intersection number to draw conclusions

on (co-) homology classes and to determine cup products using our geometric understanding of submanifolds.

Corollary 220.13. (Complementary Pair Corollary) Let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \sqcup B$. Let X be an oriented proper k-dimensional submanifold of M with $\partial X \subseteq A$. If there exists an oriented proper submanifold Y with $\partial Y \subseteq B$ such that X and Y form a complementary pair and if $X \cdot Y = \pm 1$, then $[X] \in \mathrm{H}_k(M, A)$ and $[Y] \in \mathrm{H}_{m-k}(M, B)$ generate infinite cyclic summands.

Example. Let M be the complement in S^3 of the two open 3-dimensional tori shown in the figure below on the left. On the right of that figure we see an oriented properly embedded annulus X in M and we also see a closed oriented 1-dimensional submanifold Y of M.



The two submanifolds X and Y are clearly transverse and they intersect in one point. So with whatever orientation, we have $X \cdot Y = \pm 1$. The Complementary Pair Corollary 220.13 implies that $[X] \in \mathrm{H}_2(M, \partial M)$ and $[Y] \in \mathrm{H}_1(M)$ generate infinite cyclic summands. □

Proof. We consider the homomorphism

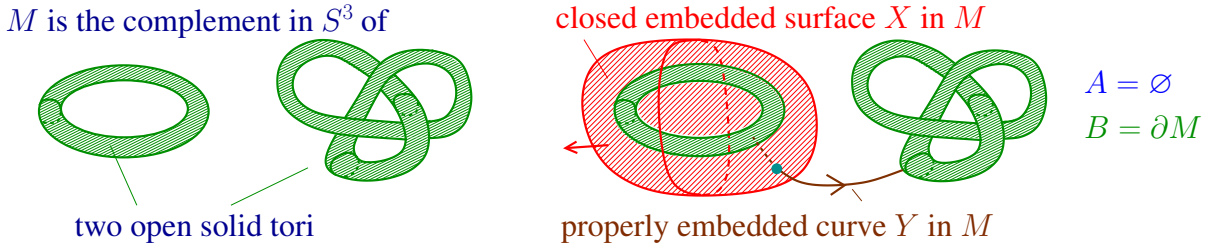
$$\begin{aligned} \Theta\colon \mathrm{H}_k(M, A) &\to \mathbb{Z} \\ \varphi &\mapsto \langle \mathrm{PD}_M(\varphi) \cup \mathrm{PD}_M([Y]), [M] \rangle. \end{aligned}$$

Note that $\Theta([X]) \underset{\uparrow}{=} \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle \underset{\uparrow}{=} X \cdot Y \underset{\uparrow}{=} \pm 1.$

by definition of Θ — by the Cup Product–Intersection Number Theorem 220.10 — by hypothesis

It now follows from the Free Abelian Group–Splitting Lemma 97.11, see also the elementary discussion on page 2782, that $[X] \in \mathrm{H}_k(M, A)$ generates an infinite cyclic summand. The same argument, with the roles of X and Y reversed, also shows that $[Y] \in \mathrm{H}_{m-k}(M, B)$ generates an infinite cyclic summand. ■

Example. This time let M be the complement in S^3 of the two open 3-dimensional tori that are shown in the figure below on the left.



We consider the oriented proper submanifolds X and Y that are shown in the figure. It follows again from the Complementary Pair Corollary 220.13 that $[X] \in \mathrm{H}_2(M)$ and $[Y] \in \mathrm{H}_1(M, \partial M)$ generate infinite cyclic summands. □

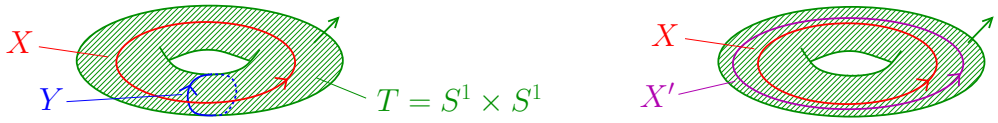
In the next example we will show how to use the Cup Product–Intersection Number Theorem 220.10 to compute cup products.

Example. We consider the torus $T = S^1 \times S^1$ together with the submanifolds $X = S^1 \times \{*\}$ and $Y = \{*\} \times S^1$ which we endow with the usual orientations. By now we have many ways[2373] for showing that $\mathrm{PD}([X])$ and $\mathrm{PD}([Y])$ form a basis for $\mathrm{H}^1(T; \mathbb{Z})$. Thus if we want to understand the cup product on the torus, it suffices to understand the cup products involving these two cohomology classes. We calculate that

$$-\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([X]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([Y]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} X \cdot Y \underset{\uparrow}{=} +1.$$

by the Cup Product Commutativity Proposition 203.20 — by the Cup Product Intersection Number Theorem 220.10 — see the figure below

Next we want to calculate the "square" $\mathrm{PD}([X]) \cup \mathrm{PD}([X])$. This requires a little trick since evidently X is not transverse to itself. To deal with this issue we consider a "parallel copy" of X, namely we consider $X' = S^1 \times \{z\}$ with $z \neq 1$.



We calculate that

$$\langle \mathrm{PD}([X]) \cup \mathrm{PD}([X]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([X']), [S^1 \times S^1] \rangle \underset{\uparrow}{=} X \cdot X' = 0.$$

by the claim in the proof of the Intersection Number Homotopy Invariance Theorem 220.12 — by the Cup Product–Intersection Number Theorem 220.10, since X and X' are disjoint, in particular transverse

and with the same argument we show that $\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]), [S^1 \times S^1] \rangle = 0$. We will now convince ourselves that these calculations match our previous calculations. To do so we denote by p_1 respectively p_2 the projection $S^1 \times S^1 \to S^1$ onto the first respectively the second factor and we denote by $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1; \mathbb{Z})$ the singular 1-cocycle that we introduced on page 3813. Now we see that

$$\begin{pmatrix} \mathrm{PD}([X]) \cup \mathrm{PD}([X]) & \mathrm{PD}([X]) \cup \mathrm{PD}([Y]) \\ \mathrm{PD}([Y]) \cup \mathrm{PD}([X]) & \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]) \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} [p_2^*\theta] \cup [p_2^*\theta] & -[p_2^*\theta] \cup [p_1^*\theta] \\ -[p_1^*\theta] \cup [p_2^*\theta] & [p_1^*\theta] \cup [p_1^*\theta] \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

on page 3955 we already showed that $[p_1^*\theta] \cap [T] = [\{1\} \times S^1] = [Y]$ and $[p_2^*\theta] \cap [T] = -[S^1 \times \{1\}] = -[X]$ — see page 3955

□

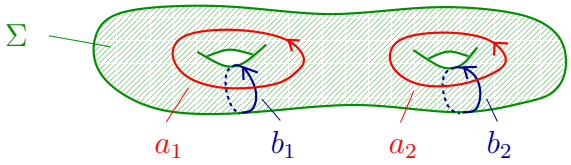
Since this was so much fun, let us consider one more example:

[2373] So, how many ways do you know?

Example. Let Σ be the surface of genus two. We consider the following bilinear pairing:[2374]

$$\begin{array}{rccl} Q\colon & \mathrm{H}_1(\Sigma)\times\mathrm{H}_1(\Sigma) & \to & \mathbb{Z} \\ & (\sigma,\tau) & \mapsto & \langle \mathrm{PD}_\Sigma(\sigma)\cup\mathrm{PD}_\Sigma(\tau),[\Sigma]\rangle. \end{array}$$

Recall that by the Surface-H_*-Proposition 155.17 we know that $\mathrm{H}_1(\Sigma)\cong\mathbb{Z}^4$. Next we consider the closed oriented curves a_1,b_1,a_2,b_2 that are shown in the figure below.



It follows easily from the Cup Product–Intersection Number Theorem 220.10 that[2375]

$$\begin{pmatrix} Q([a_1],[a_1]) & Q([a_1],[b_1]) & Q([a_1],[a_2]) & Q([a_1],[b_2]) \\ Q([b_1],[a_1]) & Q([b_1],[b_1]) & Q([b_1],[a_2]) & Q([b_1],[b_2]) \\ Q([a_2],[a_1]) & Q([a_2],[b_1]) & Q([a_2],[a_2]) & Q([a_2],[b_2]) \\ Q([b_2],[a_1]) & Q([b_2],[b_1]) & Q([b_2],[a_2]) & Q([b_2],[b_2]) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Note that the determinant of the matrix to the right is one. Since Q is bilinear we obtain from the Kronecker-Basis Lemma 199.10 (1) a new proof that $[a_1]$, $[b_1]$, $[a_2]$, $[b_2]$ is a basis for $\mathrm{H}_1(\Sigma)$.[2376] It follows from the Poincaré Duality Theorem 210.1 together with the Cup Product–Intersection Number Theorem 220.10 we have now also computed the cup product $\mathrm{H}^1(\Sigma;\mathbb{Z})\times\mathrm{H}^1(\Sigma;\mathbb{Z})\to\mathrm{H}^2(\Sigma;\mathbb{Z})$. □

We will do one more calculation of the cup product in Section 220.4.1, right after the proof of the Cup Product–Intersection Number Theorem 220.10.

220.3. **Proof of the Cup Product–Intersection Number Theorem 220.10.** [2377] In this section we will prove the Cup Product–Intersection Number Theorem 220.10. Our proof of the Cup Product–Intersection Number Theorem 220.10 relies heavily on the Small H_*-to-Large H^*-PD Lemma 212.8 and the Large H_*-to-Small H^*-PD Lemma 212.9 that relate Poincaré Duality in an oriented topological manifold M to Poincaré Duality in a compact codimension 0 submanifold W. For the reader's convenience we recall the Small H_*-to-Large H^*-PD Lemma 212.8. This lemma says that given a homology class in W we can apply Poincaré Duality in M or W and essentially we end up with the same result:

[2374] Later on, in Definition 222.3 we will refer to this map as the *intersection form of* Σ and we will introduce the intersection form of any closed oriented even-dimensional topological manifold.

[2375] For the diagonal terms we use the "little trick" from the previous example, namely we use a "parallel copy" of the closed curves a_1,b_1,a_2,b_2.

[2376] We gave a different argument on page 2970.

[2377] Alternative proofs of the Cup Product–Intersection Number Theorem 220.10 are given in [**Sam1965**, Theorem C], [**GrH1981**, Proposition 31.7], [**Bre1993**, Theorem VI.11.9] and [**DaK2001**, Theorem 10.32]. Furthermore a quick outline of a proof is given in [**GoS1999**, Proposition 1.2.5].

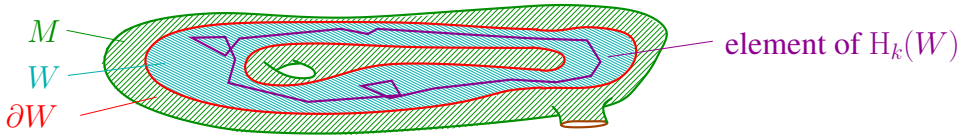
Lemma 212.8. (Small H_*-to-Large H^*-PD Lemma) Let M be a compact oriented m-dimensional topological manifold and let W be a compact codimension-zero submanifold with $W \cap \partial M = \varnothing$. We write $\mathring{W} = W \setminus \partial W$. We fix the following notation:

(1) We denote by w the inclusion map $W \to M$ and also the inclusion map of the pairs $(W, \partial W) \to (M, M \setminus \mathring{W})$.

(2) We denote by $p\colon (M, \partial M) \to (M, M \setminus \mathring{W})$ the inclusion map of pairs.

For every $k \in \mathbb{N}_0$ the following diagram commutes:[2378]

$$\begin{array}{ccccc} \mathrm{H}_k(W) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{m-k}(W, \partial W; \mathbb{Z}) & \xrightarrow[\cong]{(w^*)^{-1}} & \mathrm{H}^{m-k}(M, M \setminus \mathring{W}; \mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M) & & \xrightarrow{\qquad\qquad\mathrm{PD}_M\qquad\qquad} & & \mathrm{H}^{m-k}(M, \partial M; \mathbb{Z}). \end{array}$$



Proof. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(W) & \xleftarrow[\cong]{\cap[W]} & \mathrm{H}^{n-k}(W, \partial W; \mathbb{Z}) & \xleftarrow[w^*]{\cong} & \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z}) \\ \downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M) & & \xleftarrow[\cong]{\qquad\qquad\cap[M]\qquad\qquad} & & \mathrm{H}^{n-k}(M, \partial M; \mathbb{Z}). \end{array}$$

Now let $\beta \in \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z})$. We have the following equality in $\mathrm{H}_k(M)$:

$$w_*(w^*(\beta) \cap [W]) \underset{\text{by the Cap Product-Naturality Lemma 206.13 applied to the map } w\colon (W, \partial W, \partial W) \to (M, M \setminus \mathring{W}, \varnothing)}{=} \beta \cap w_*([W]) \underset{\text{by the Codimension Zero-Fundamental Class Lemma 212.7 we have } p_*([M]) = w_*([W]) \in \mathrm{H}_m(M, M \setminus \mathring{W})}{=} \beta \cap p_*([M]) \underset{\text{by the Cap Product-Naturality Lemma 206.13 applied to the map } p\colon (M, \partial M, \varnothing) \to (M, M \setminus \mathring{W}, \varnothing)}{=} p_*(\overbrace{p^*(\beta) \cap [M]}^{\in \mathrm{H}_k(M)}) \underset{\text{since } p\colon M \to M \text{ is id we have } p_* = \mathrm{id}_* = \mathrm{id}}{=} p^*(\beta) \cap [M].$$

We have thus shown that the above diagram does indeed commute. ∎

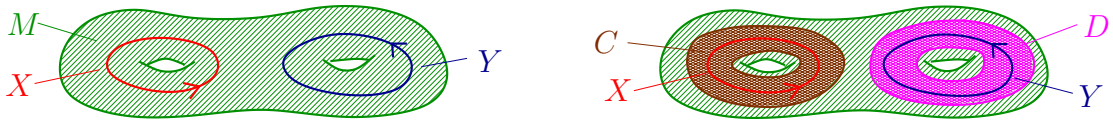
We take a short rest from more technicalities and enjoy the fact that we can already prove the Cup Product–Intersection Number Theorem 220.10 in the special case that M is closed and that the submanifolds *do not intersect*.

Proof of Theorem 220.10 for disjoint submanifolds. Let M be a closed oriented m-dimensional smooth manifold and let X and Y be a complementary pair of *disjoint* submanifolds of M. Since the submanifolds are disjoint it suffices to show that the cup product of the Poincaré duals of $[X]$ and $[Y]$ is also zero.

We write $k = \dim(X)$. We apply the Tubular Neighborhood–Existence Theorem 57.11 to the submanifold $X \cup Y$ of M. We obtain *disjoint* tubular neighborhoods C for X and D

[2378] Here we use that by the Excision-in-H^*-Theorem 200.1 we know that $w\colon (W, \partial W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on cohomology groups.

for Y. We denote by $p\colon (M,\varnothing) \to (M, M \setminus \mathring{C})$ and $p\colon (M,\varnothing) \to (M, M \setminus \mathring{D})$ the obvious inclusion maps of pairs of topological spaces.



We apply the Small H_*-to-Large H^*-PD Lemma 212.8 to the codimension-zero submanifold C and the homology class $[X] \in \mathrm{H}_k(C)$. We obtain an $\alpha \in \mathrm{H}^{m-k}(M, M \setminus \mathring{C}; \mathbb{Z})$ with $p^*(\alpha) = \mathrm{PD}_M([X])$. Similarly we see that there exists a $\beta \in \mathrm{H}^k(M, M \setminus \mathring{D}; \mathbb{Z})$ such that $p^*(\beta) = \mathrm{PD}_M([Y])$.

We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^{m-k}(M;\mathbb{Z}) \times \mathrm{H}^k(M;\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^m(M;\mathbb{Z}) \\
\uparrow{\scriptstyle p^*\times p^*} \quad (\mathrm{PD}_M([X]), \mathrm{PD}_M([Y])) \updownarrow (\alpha,\beta) & & \uparrow \\
\mathrm{H}^{m-k}(M, M\setminus\mathring{C};\mathbb{Z}) \times \mathrm{H}^k(M, M\setminus\mathring{D};\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^m(M, \underbrace{(M\setminus\mathring{C}) \cup (M\setminus\mathring{D})}_{=M,\text{ since } C\cap D=\varnothing};\mathbb{Z}).
\end{array}$$

The diagram commutes by the Cup Product-Naturality Lemma 204.8. It follows from the fact that the lower right group is zero, that the cup product of $\mathrm{PD}_M([X])$ and $\mathrm{PD}_M([Y])$ is zero. ∎

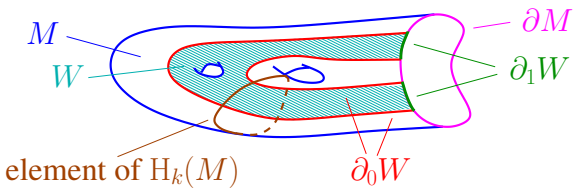
After this refreshing proof of the special case we continue with trying to prove the general case of the Cup Product–Intersection Number Theorem 220.10. We will need the Large H_*-to-Small H^*-PD Lemma 212.9. Since the statement is rather technical, we recall the statement of the special case that is discussed in example (a) on page 4071.

Lemma 212.9. (Large H_*-to-Small H^*-PD Lemma) Let M be a compact oriented m-dimensional smooth manifold and let W be a compact codimension-zero submanifold with corner. We fix the following notation:

(1) We write $\partial_1 W = W \cap \partial M$ and $\partial_0 W = \overline{\partial W \setminus \partial_1 W}$.
(2) We write $\mathring{W} = W \setminus \partial_0 W$.
(3) The inclusion maps $(W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $(W, \partial_1 W) \to (M, \partial M)$ are denoted by w.
(4) Finally we denote by $p\colon (M,\varnothing) \to (M, M \setminus \mathring{W})$ the obvious map.

For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_k(M) & & \xrightarrow{\quad\mathrm{PD}_M\quad} & & \mathrm{H}^{m-k}(M, \partial M;\mathbb{Z}) \\
\downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\
\mathrm{H}_k(M, M\setminus\mathring{W}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W, \partial_0 W) & \xrightarrow{\quad\mathrm{PD}_W\quad} & \mathrm{H}^{m-k}(W, \partial_1 W;\mathbb{Z}).
\end{array}$$



Proof. By Definition 210.2 of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M) & \xleftarrow[\cong]{\cap[M]} & & & \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M\setminus \mathring{W}) & \xleftarrow[\cong]{w_*} & \mathrm{H}_k(W,\partial_0 W) & \xleftarrow[\cong]{\cap[W]} & \mathrm{H}^{n-k}(W,\partial_1 W;\mathbb{Z}). \end{array}$$

So let $\beta \in \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z})$. We have the following equalities in $\mathrm{H}_k(M, M\setminus \mathring{W})$:

$$\begin{array}{ccccc} p_*(\beta\cap[M]) & = & \beta\cap p_*([M]) & = & \beta\cap w_*([W]) \\ & & & & = \\ & & & & w_*(w^*(\beta)\cap[W]). \end{array}$$

by the Cap Product-Naturality Lemma 206.13 applied to the map $p\colon (M,\partial M,\varnothing)\to(M,\partial M, M\setminus\mathring{W})$ of excisive triads

by the Codimension Zero-Fundamental Class Lemma 212.7 we have $p_*([M]) = w_*([W]) \in \mathrm{H}_n(M,(M\setminus\mathring{W})\cup\partial M)$

by the Cap Product-Naturality Lemma 206.13 applied to the map $w\colon (W,\partial_1 W,\partial_0 W)\to(M,\partial M, M\setminus\mathring{W})$ of excisive triads ■

In the proof of the Cup Product–Intersection Number Theorem 220.10 we will also need the following proposition:

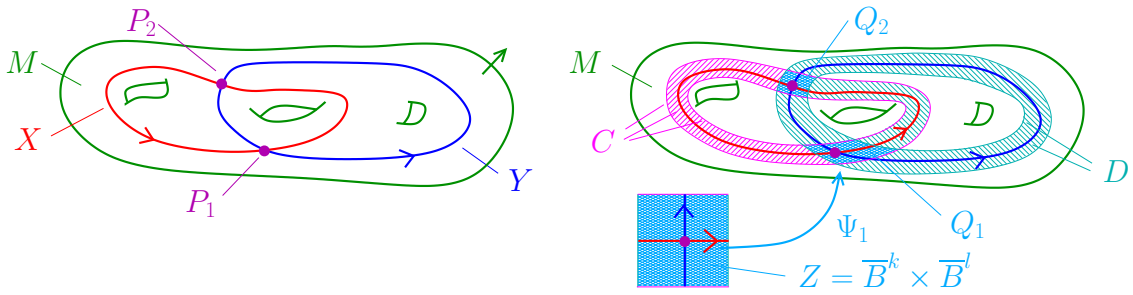
Proposition 220.14. (Transverse Neighborhood Proposition) Let M be a compact oriented smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1,\dots,P_s$ the intersection points of X and Y. Then there exist

(1) compact submanifolds with corner C and D such that C is a neighborhood for X and D is a neighborhood for Y, and
(2) embeddings $\Psi_i\colon \overline{B}^k\times\overline{B}^l\to M$, $i=1,\dots,s$

such that the following statements hold:

(a) For each $i\in\{1,\dots,s\}$ we have $\Psi_i((0,0)) = P_i$.
(b) The images $Q_i := \Psi_i\big(\overline{B}^k\times\overline{B}^l\big)$, $i=1,\dots,s$ are disjoint.
(c) We have $Q_1\cup\dots\cup Q_s = C\cap D$.
(d) For each $i\in\{1,\dots,s\}$ the following statements hold:
  (d1) The restriction of Ψ_i to $B^k\times B^l$ is a diffeomorphism onto its image.
  (d2) We have $Q_i\cap X = \Psi_i(\overline{B}^k\times\{0\})$ and the map $\Psi_i\colon \overline{B}^k\times\{0\}\to X$ is an orientation-preserving smooth embedding.
  (d3) We have $Q_i\cap Y = \Psi_i(\{0\}\times\overline{B}^l)$ and the map $\Psi_i\colon \{0\}\times\overline{B}^l\to Y$ is an orientation-preserving smooth embedding.

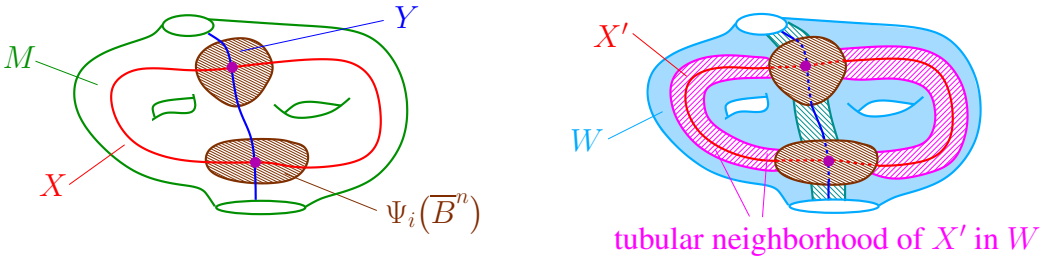
Sketch of a proof. This proposition can bed deduced from the Transverse Tubular Neighborhood Theorem 57.32. In the following we sketch a self-contained argument: Let M be a



compact oriented n-dimensional smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \dots, P_s$ the intersection points of X and Y. For $i = 1, \dots, s$ we can and will pick smooth embeddings $\Psi_i \colon \overline{B}^n \to M \setminus \partial M$ such that the following conditions are satisfied:

(1) The images are disjoint.
(2) For each $i \in \{1, \dots, s\}$ the map Ψ_i restricts to orientation-preserving diffeomorphisms $\overline{B}^k \times \{0\} \to \Psi_i(\overline{B}^n) \cap X$ and $\{0\} \times \overline{B}^l \to \Psi_i(\overline{B}^n) \cap Y$.

We consider $W \coloneqq M \setminus \bigcup_{i=1}^{s} \Psi_i(B^n)$. Note that $X' \coloneqq W \cap X$ and $Y' \coloneqq W \cap Y$ are proper submanifolds of the compact smooth manifold W.



Thus we can apply the Tubular Neighborhood Theorem 57.11 to obtain tubular neighborhoods for X' and Y' in W. Using the Tubular Neighborhood–Uniqueness Theorem 57.12 we can arrange the intersections of the tubular neighborhoods with $\Psi_i(S^{n-1})$ to be the standard tubular neighborhoods of $\Psi_i(S^{k-1} \times \{0\})$ and $\Psi_i(\{0\} \times S^{l-1})$ in $\Psi_i(S^{n-1})$. At this stage it is fairly straightforward to extend the tubular neighborhoods to the interiors of the $\Psi_i(B^n)$ and to obtain all the desired objects. We have faith in the unwavering enthusiasm of our readers for filling in the details. ■

Remark. An alternative approach to the Transverse Neighborhood Proposition 220.14 would be to use tubular neighborhoods provided by the Transverse Tubular Neighborhood–Existence Theorem 57.32. □

Given the situation of the Transverse Neighborhood Proposition 220.14 we can make one more statement.

Lemma 220.15. (Sign-Q_i-Lemma) We use the notation of the Transverse Neighborhood Proposition 220.14. Let $i \in \{1, \dots, s\}$. We equip the topological manifold Q_i, which is a codimension-zero submanifold of M, with the induced orientation coming from the Topological Manifold-Induced Orientation Lemma 192.1. Furthermore we equip the topological

manifold $\overline{B}^k \times \overline{B}^l$ with the orientation coming from the standard orientation of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$. With these choices the following statement holds:

(e) If we denote by σ_i the sign of the intersection point P_i, then

$$\Psi_{i*}([\overline{B}^k \times \overline{B}^l]) \;=\; \sigma_i \cdot [Q_i].$$

Proof. Let $i \in \{1, \ldots, s\}$. It follows from (d2) and (d3) that at the point $(0,0)$ the diffeomorphism $\Psi_i \colon B^k \times B^l \to \mathring{Q}_i$ is orientation-preserving if and only if $\sigma_i = 1$. Now the statement follows easily from the definition of σ_i, the Orientation Preserving–Characterisation Proposition 196.6 and the Orientations–Preserved Lemma 191.26. ■

The last preparation we need before we can provide the actual proof of the Cup Product–Intersection Number Theorem 220.10 is the Poincaré Duality-Products Proposition 210.3, which for the reader's convenience we recall using our present notation.

Proposition 210.3. (Poincaré Duality-Products Proposition) Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. We equip $X \times Y$ with the product orientation introduced in the Product Orientation Proposition 192.7 (1). Given any points $P \in X$ and $Q \in Y$ we have the following equality in $\mathrm{H}^{m+n}(X \times Y, \partial(X \times Y); \mathbb{Z})$:[2379]

$$\mathrm{PD}_{X\times Y}([X \times \{Q\}]) \cup \mathrm{PD}_{X\times Y}([\{P\} \times Y]) \;=\; [X \times Y]^*.$$

Proof. This proposition follows almost immediately from the Fundamental Class-Product Proposition 207.2. ■

We turn to the actual proof of the Cup Product–Intersection Number Theorem 220.10:

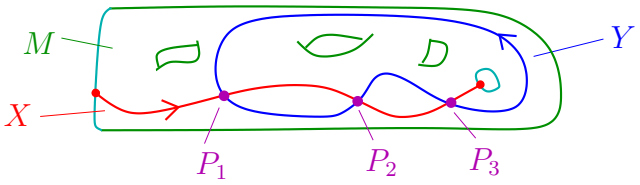
Proof of the Cup Product–Intersection Number Theorem 220.10. Let M be a compact oriented connected m-dimensional smooth manifold. To simplify the notation we assume that M is closed. The case that M has non-empty boundary is proved in a rather similar fashion.

Let X and Y be a complementary pair of submanifolds of M. We introduce the following notation:

(1) (a) We write $k = \dim(X)$ and $l = \dim(Y)$. Note that $l = m - k$.
(b) We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the natural projections.
(2) As usual we denote by $[X] \in \mathrm{H}_k(X)$ and $[Y] \in \mathrm{H}_l(Y)$ the fundamental classes of X and Y. By a slight abuse of notation we denote by $[X]$ respectively $[Y]$ also the images of these homology classes in any subspace of M that contains X respectively Y.
(3) We denote the intersection points of X and Y by $P_1, \ldots, P_s$ and we denote the corresponding signs by $\sigma_1, \ldots, \sigma_s$.
(4) As usual, given a topological space X we denote by $\epsilon_X = \epsilon \colon \mathrm{H}_0(X) \to \mathbb{Z}$ the augmentation map.

With the notation that we just introduced, and keeping in mind the Cap Product-Kronecker Lemma 206.12, the statement of the Cup Product–Intersection Number Theorem 220.10 can be reformulated as the following statement:

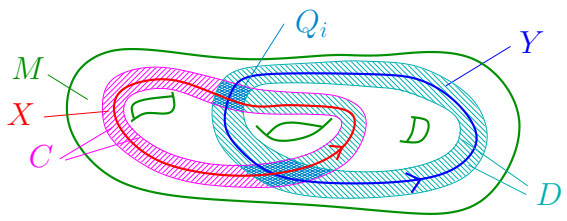
[2379] Recall that for a compact oriented connected non-empty n-dimensional topological manifold M we introduced in Definition 201.27 the dual fundamental class $[M]^* \in \mathrm{H}^n(M, \partial M; \mathbb{Z})$, which is characterized by $\langle [M]^*, [M] \rangle = 1$.

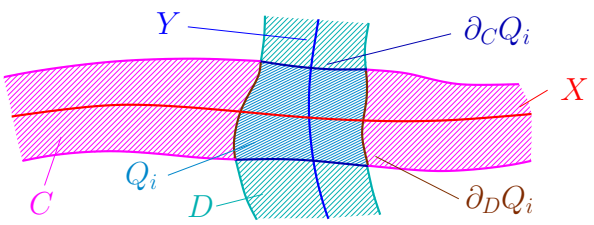


Statement. $$\epsilon_M((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]) \;=\; \sum_{i=1}^{s} \sigma_i.$$

We introduce many more objects, notations and conventions:

(5) We apply the Transverse Neighborhood Proposition 220.14 to the submanifolds X and Y. In the remainder of the proof we use the notation that we introduced in that proposition.
(6) We write $\mathring{C} = C \setminus \partial C$ and $\mathring{D} = D \setminus \partial D$.
(7) We denote all of the inclusion maps $(M, \varnothing) \to (M, M \setminus \mathring{C})$, $(M, \varnothing) \to (M, M \setminus \mathring{D})$ and $(M, \varnothing) \to (M, M \setminus (\mathring{C} \cap \mathring{D}))$ by p.
(8) We write $E \coloneqq C \cap D = Q_1 \cup \cdots \cup Q_s$. Furthermore we write $\partial_C E = \partial C \cap D$ and $\partial_D E = C \cap \partial D$. Note that $\partial E = \partial_C E \cup \partial_D E$. Also note that $(M \setminus \mathring{C}) \cup (M \setminus \mathring{D}) = M \setminus \mathring{E}$.
(9) We denote by $e\colon E \to M$ the inclusion map and we denote by c also all the corresponding inclusion maps of pairs where the first terms are E and M.
(10) Given $i \in \{1, \ldots, s\}$ we write $\partial_C Q_i \coloneqq \partial_C E \cap Q_i$ and $\partial_D Q_i \coloneqq \partial_D E \cap Q_i$. Note that $\partial Q_i = \partial_C Q_i \cup \partial_D Q_i$.
(11) Given $i \in \{1, \ldots, s\}$ we denote by $q_i\colon Q_i \to E$ the inclusion map and we denote by q_i also all the corresponding inclusion maps of pairs where the first terms are Q_i and E.
(12) It follows from the descriptions in the Transverse Neighborhood Proposition 220.14 that E and each Q_i are codimension-zero submanifolds of M. We equip these submanifolds with the corresponding orientation introduced in the Topological Manifold-Induced Orientation Lemma 192.1.





Now we consider the following diagram:[23802381]

$$\begin{array}{ccccccc}
\mathrm{H}^l(M) \times \mathrm{H}^k(M) & \xrightarrow{\cup} & \mathrm{H}^m(M) & \xrightarrow{\cap[M]} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\uparrow{\scriptstyle p^*\times p^*} & & \uparrow{\scriptstyle p^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle =} \\
\mathrm{H}^l(M, M\setminus \mathring{C}) \times \mathrm{H}^k(M, M\setminus \mathring{D}) & \xrightarrow{\cup} & \mathrm{H}^m(M, M\setminus \mathring{E}) & \xrightarrow{\cap p_*([M])} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\downarrow{\scriptstyle e^*\times e^*} & & \downarrow{\scriptstyle e^*} & & \uparrow{\scriptstyle e_*} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^l(E, \partial_C E) \times \mathrm{H}^k(E, \partial_D E) & \xrightarrow{\cup} & \mathrm{H}^m(E, \partial E) & \xrightarrow{\cap[E]} & \mathrm{H}_0(E) & \xrightarrow{\epsilon_E} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^s q_i^*\times q_i^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^s q_i^*} & & \uparrow{\scriptstyle \sum_{i=1}^s q_{i*}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^s \mathrm{H}^l(Q_i, \partial_C Q_i) \times \mathrm{H}^k(Q_i, \partial_D Q_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^s \mathrm{H}^m(Q_i, \partial Q_i) & \xrightarrow{\bigoplus_{i=1}^s \cap[Q_i]} & \bigoplus_{i=1}^s \mathrm{H}_0(Q_i) & \xrightarrow{\sum \epsilon_{Q_i}} & \mathbb{Z}.
\end{array}$$

Claim 1. The above diagram commutes.

Proof. In the following we deal with all of the nine squares of the diagram.

(a) The left squares in the first and second row commute by the Cup Product-Naturality Lemma 204.8.

(b) The left square in the third row commutes by the Cup Product–Disjoint Union Lemma 204.16 together with the aforementioned the Cup Product-Naturality Lemma 204.8.

(c) The middle square in the first row commutes by the Cap Product-Naturality Lemma 206.13.

(d) It follows from our orientation conventions and the Codimension Zero-Fundamental Class Lemma 194.10 that $e_*([E]) = p_*([M])$. It follows from the Cap Product-Naturality Lemma 206.13 that the middle square in the second row commutes.

(e) First note that it follows easily from the definitions and the Path-Component-H_*-Lemma 143.21 that $\sum_{i=1}^s q_{i*}([Q_i]) = [E]$. Now we obtain from the Cap Product-Naturality Lemma 206.13 and the Cap Product–Disjoint Union Lemma 206.14 that the middle square in the third row commutes.

(f) In the Augmentation Lemma 143.24 (3) we saw that the squares that involve the augmentation maps commute. ⊞

We introduce a few last pieces of notation:

(13) We denote by $c\colon (C, \partial C) \to (M, M\setminus \mathring{C})$ and $d\colon (D, \partial D) \to (M, M\setminus \mathring{D})$ the inclusion maps. Note that it follows from the Excision-in-H^*-Theorem 200.1 (2) that the maps c and d induce isomorphisms on cohomology.

(14) We write

$$\begin{aligned}
\alpha &= (c^*)^{-1}(\overbrace{\mathrm{PD}_C([X])}^{\in \mathrm{H}^l(C,\partial C;\mathbb{Z})}) && \in \mathrm{H}^l(M, M\setminus \mathring{C}; \mathbb{Z}) \\
\beta &= (d^*)^{-1}(\underbrace{\mathrm{PD}_D([Y])}_{\in \mathrm{H}^k(D,\partial D;\mathbb{Z})}) && \in \mathrm{H}^k(M, M\setminus \mathring{D}; \mathbb{Z}).
\end{aligned}$$

[2380] For space reasons we suppressed the $\mathbb{Z}$-coefficients from the cohomology groups.

[2381] Many of the vertical maps in the diagram are isomorphisms by Excision-in-H^*-Theorem 200.1. But this is in fact irrelevant for our purpose.

(15) Given $i \in \{1, \dots, s\}$ we write $X_i := X \cap Q_i$ and $Y_i := Y \cap Q_i$. It follows from the Transverse Neighborhood Proposition 220.14 (e) and (f) that X_i and Y_i are codimension zero submanifolds of X and Y. Since X and Y are oriented we can view X_i and Y_i also as oriented topological manifolds. As usual we denote by $[X_i] \in \mathrm{H}_k(X_i, \partial X_i)$ and $[Y_i] \in \mathrm{H}_k(Y_i, \partial Y_i)$ the corresponding fundamental classes. With the same convention as before we denote by $[X_i]$ also the image under the inclusion-induced homomorphism $\mathrm{H}_k(X_i, \partial X_i) \to \mathrm{H}_k(Q_i, \partial_D Q_i)$ and similarly we denote by $[Y_i]$ also the image under the inclusion-induced homomorphism $\mathrm{H}_l(Y_i, \partial Y_i) \to \mathrm{H}_l(Q_i, \partial_C Q_i)$.

Claim 2. We have

(a) We have $p^*(\alpha) = \mathrm{PD}_M([X]) \in \mathrm{H}^l(M; \mathbb{Z})$ and $p^*(\beta) = \mathrm{PD}_M([Y]) \in \mathrm{H}^k(M; \mathbb{Z})$.

(b) For any $i \in \{1, \dots, s\}$ we have the following two equalities:

$$\begin{array}{rcl} (q_i^* \circ e^*)(\alpha) & = & \mathrm{PD}_{Q_i}([X_i]) \in \mathrm{H}^l(Q_i, \partial_C Q_i; \mathbb{Z}) \\ (q_i^* \circ e^*)(\beta) & = & \mathrm{PD}_{Q_i}([Y_i]) \in \mathrm{H}^k(Q_i, \partial_D Q_i; \mathbb{Z}). \end{array}$$

(c) For any $i \in \{1, \dots, s\}$ we have

$$\epsilon_{Q_i}((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]) = \sigma_i.$$

Proof.

(a) By the Small H_*-to-Large H^*-PD Lemma 212.8 the following diagram commutes:

$$\begin{array}{ccccc} & & \overset{[X] \mapsto \alpha}{\frown} & & \\ \mathrm{H}_k(C) & \xrightarrow[\mathrm{PD}_C]{} & \mathrm{H}^l(C, \partial C; \mathbb{Z}) & \xrightarrow[(c^*)^{-1}]{\cong} & \mathrm{H}^l(M, M \setminus \mathring{C}; \mathbb{Z}) \\ [X] \downarrow & & & & \downarrow p^* \\ [X] & & & & \\ \mathrm{H}_k(M) & & \xrightarrow{\mathrm{PD}_M} & & \mathrm{H}^l(M; \mathbb{Z}). \end{array}$$

We apply this commutative diagram to $[X] \in \mathrm{H}_k(C)$ and we obtain immediately that $p^*(\alpha) = \mathrm{PD}_M([X])$. The same way one shows that $p^*(\beta) = \mathrm{PD}_M([Y])$.

(b) Let $i \in \{1, \dots, s\}$. We denote by $r_i \colon Q_i \to C$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccccc} \overbrace{\mathrm{H}_k(C)}^{[X]} & & \xrightarrow{\mathrm{PD}_C} & & \mathrm{H}^l(C, \partial C; \mathbb{Z}) \xleftarrow[\cong]{c^*} \overbrace{\mathrm{H}^l(M, M \setminus \mathring{C}; \mathbb{Z})}^{\alpha} \\ \downarrow & & & & \downarrow r_i^* \quad \swarrow (e \circ q_i)^* \\ \mathrm{H}_k(C, C \setminus (Q_i \setminus \partial_D Q_i)) & \xleftarrow[\cong]{r_{i*}} & \underbrace{\mathrm{H}_k(Q_i, \partial_D Q_i)}_{[X_i]} & \xrightarrow{\mathrm{PD}_{Q_i}} & \mathrm{H}^l(Q_i, \partial_C Q_i; \mathbb{Z}). \end{array}$$

We make the following observations:

(α) The triangle on the right commutes since $c \circ r_i = e \circ q_i \colon Q_i \to M$ are both the inclusion map.

(β) It follows from the Large H_*-to-Small H^*-PD Lemma 212.9, applied to the codimension-zero submanifold Q_i of C, that the rectangle commutes.[2382]

(γ) From the Codimension Zero-Fundamental Class Lemma 194.10 we obtain the equality $r_{i*}([X_i]) = [X] \in \mathrm{H}_k(C, C \setminus (Q_i \setminus \partial_D Q_i))$.

(δ) It follows from the Manifold–Excision Theorem 212.4 that the induced homomorphism $r_{i*} \colon \mathrm{H}_k(Q_i, \partial_D Q_i) \to \mathrm{H}_k(C, C \setminus (Q_i \setminus \partial_D Q_i))$ is an isomorphism.

The desired equality $(q_i^* \circ e^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i])$ follows from the above commutative diagram applied to $[X] \in \mathrm{H}_k(C)$. Furthermore, exactly the same way we can also show that $(q_i^* \circ e^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i])$.

(c) Let $i \in \{1, \dots, s\}$. We set $Z := \overline{B}^k \times \overline{B}^l$. We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^l(Q_i, \partial_C Q_i; \mathbb{Z}) \times \mathrm{H}^k(Q_i, \partial_D Q_i; \mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^m(Q_i, \partial Q_i; \mathbb{Z}) & \xrightarrow{\cap [Q_i]} & \mathrm{H}_0(Q_i) & \xrightarrow{\epsilon_{Q_i}} & \mathbb{Z} \\
\downarrow{\scriptstyle \Psi_i^* \times \Psi_i^*} & & \downarrow{\scriptstyle \Psi_i^*} & & \uparrow{\scriptstyle \Psi_i^*} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^l(Z, \overline{B}^k \times \partial \overline{B}^l; \mathbb{Z}) \times \mathrm{H}^k(Z, \partial \overline{B}^k \times \overline{B}^l; \mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^m(Z, \partial Z; \mathbb{Z}) & \xrightarrow{\cap \sigma_i \cdot [Z]} & \mathrm{H}_0(Z) & \xrightarrow{\epsilon_Z} & \mathbb{Z}.
\end{array}$$

As in the big diagram on page 4246, the squares to the left and right commute. Next note that the middle square is slightly more interesting: The Sign-Q_i-Lemma 220.15 says that $\Psi_{i*}([Z]) = \sigma_i \cdot [Q_i]$. Using this observation we obtain once again from the Cap Product-Naturality Lemma 206.13 that the middle square commutes.

We calculate that

$$\begin{aligned}
\epsilon_{Q_i}((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]) &\overset{\text{by the above commutative diagram}}{=} \sigma_i \cdot \epsilon_Z((\Psi_i^*(\mathrm{PD}_{Q_i}([X_i])) \cup \Psi_i^*(\mathrm{PD}_{Q_i}([Y_i]))) \cap [Z]) \\
&= \sigma_i \cdot \epsilon_Z((\mathrm{PD}_Z([\overline{B}^k \times \{0\}]) \cup \mathrm{PD}_Z([\{0\} \times \overline{B}^l])) \cap [Z]) \\
&\overset{\text{by the Transverse Neighborhood Proposition 220.14 (d) and since } \Psi_i \text{ is a homeomorphism}}{=} \sigma_i \cdot \epsilon_{\overline{B}^k \times \overline{B}^l}([\overline{B}^k \times \overline{B}^l]^* \cap [\overline{B}^k \times \overline{B}^l]) \\
&\overset{\text{the Poincaré Duality-Products Proposition 210.3}}{=} \sigma_i. \qquad \overset{\text{Definition 201.27 of the dual fundamental class}}{=}
\end{aligned}$$

⊞

After all these preparations we can prove the desired equality:

$$\begin{aligned}
\epsilon_M((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]) &\overset{\text{by the first row of the commutative diagram from Claim 1 and by Claim 2 (a)}}{=} \epsilon_M((\alpha \cup \beta) \cap p_*([M])) \\
&= \sum_{i=1}^{s} \epsilon_{Q_i}((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]) \\
&\overset{\text{by the second and third row of the commutative diagram from Claim 1 and by Claim 2 (b)}}{=} \sum_{i=1}^{s} \sigma_i. \qquad \overset{\text{by Claim 2 (c)}}{=}
\end{aligned}$$

This concludes the proof of the theorem. ■

[2382] One can easily convince oneself that $\partial_C Q_i = \partial_1 Q_i$ and $\partial_D Q_i = \partial_0 Q_i$ for Q_i viewed as a submanifold with corner of C.

Remark. Towards the end of the proof of the theorem we marked in red color that we used the Poincaré Duality-Products Proposition 210.3. Most of the other statements that we used in the proof were fairly straightforward lemmas regarding cup and cap products. But arguably the key ingredient in the proof is the Poincaré Duality-Products Proposition 210.3 which was a variation on the Fundamental Class-Product Proposition 207.2 and which in turn was a fairly straightforward consequence of the rather difficult Cross-Cup-Cap Theorem 207.1. □

220.4. **Cup product on complex manifolds.** This section has three subsections:

- In Subsection 220.4.1 we study intersection numbers of complex submanifolds.
- In Subsection 220.4.2 we specialize to complex projective spaces and compute their cup products without any sign ambiguities.
- In Subsection 220.4.3 we will show that the tubular neighborhood of $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$ is not a product.

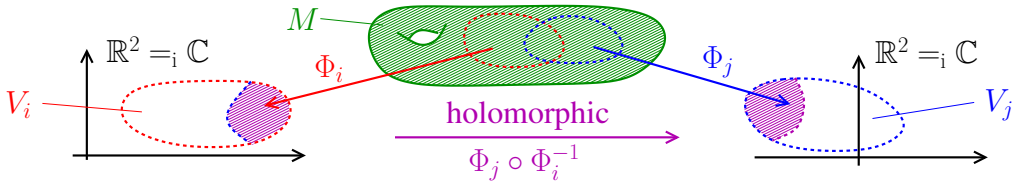
220.4.1. **Algebraic intersection numbers of complex submanifolds.** In this subsection we will study the algebraic intersection numbers of complex submanifolds of any complex manifold. It is worth recalling following definition that goes all the way back to Definition 40.2:

Definition 220.16.

(1) Let M be a $2n$-dimensional topological manifold M. An atlas $\{\Phi_i \colon U_i \to V_i\}_{i\in I}$ of charts of type (i), i.e. of charts where each V_i is an open subset of $\mathbb{R}^{2n}$, for M is called holomorphic if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1} \colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open in } \mathbb{R}^{2n}=\mathbb{C}^n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subseteq \mathbb{R}^{2n}=\mathbb{C}^n}$$

is holomorphic.



Definition 220.16.

(2) We define the maximal holomorphic atlas of a complex manifold the same way as we defined the maximal smooth atlas of a smooth manifold, see Definition 25.3.

(3) An n-dimensional complex manifold is a pair $(M, \mathcal{A})$ consisting of a $2n$-dimensional topological manifold M, together with a maximal holomorphic atlas for M.

Next we recall Definition 40.11.

Definition 220.17. Let M be a complex manifold. We say that $W \subseteq M$ is a k-dimensional complex submanifold if for every point $P \in W$ there exists a chart $\Phi \colon U \to V$ from the maximal holomorphic atlas with $P \in U$ and which satisfies $\Phi(U \cap W) \subseteq \mathbb{C}^k \times \{0\}$.

Remark.

(1) Let M be a complex manifold. As in the Submanifold Smooth Structure Lemma 25.23 we see that every complex submanifold of M is naturally complex manifold in its own right.
(2) Let V be an n-dimensional complex vector space. In the Complex Vector Space-Orientation Lemma 40.16 we showed the following: If $(v_1, \ldots, v_n)$ is a basis for V as a complex vector space, then the ordered basis $(v_1, \mathrm{i}v_1, \ldots, v_n, \mathrm{i}v_n)$ gives an orientation of the $2n$-dimensional real vector space V which does *not* depend on the choice of the basis $v_1, \ldots, v_n$. This orientation of the real vector space V is called the *natural orientation*.
(3) Let M be an n-dimensional complex manifold. In the Complex Manifold-Tangent Space Lemma 40.14 we showed that $\mathrm{T}_P M$ is naturally a complex vector space and we showed that the corresponding natural orientations from (2) define a natural orientation of M, viewed as a $2n$-dimensional smooth manifold. □

Now we can formulate the following elementary but rather useful lemma:

Lemma 220.18. (Complex Submanifolds–Intersection Number Lemma) Let M be a closed complex manifold and let X and Y be two closed complex submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. Then the following hold:
(1) The sign at every transverse intersection point of X and Y is equal to $+1$.
(2) If X and Y intersect transversally, then the algebraic intersection number equals the geometric intersection number.

Proof.

(1) We set $k = \dim(X)$ and $l = \dim(Y)$. Suppose that P is a transverse intersection point of the submanifolds X and Y. We pick a basis $(v_1, \ldots, v_k)$ for the complex vector space $\mathrm{T}_P X$ and we pick a basis $(w_1, \ldots, w_l)$ for the complex vector space $\mathrm{T}_P Y$. As in the Product Orientation Proposition 31.48 we see that the vectors $(v_1, \ldots, v_k, w_1, \ldots, w_l)$ evidently form a basis for the complex vector space $\mathrm{T}_P X \oplus \mathrm{T}_P Y =_{\mathrm{i}} \mathrm{T}_P M$. We have

$$\underbrace{v_1, \mathrm{i}v_1, \ldots, v_k, \mathrm{i}v_k}_{\substack{\text{positive basis for } \mathrm{T}_P X, \\ \text{viewed as an } \mathbb{R}\text{-vector space}}} \quad , \quad \underbrace{w_1, \mathrm{i}w_1, \ldots, w_l, \mathrm{i}w_l}_{\substack{\text{positive basis for } \mathrm{T}_P Y, \\ \text{viewed as an } \mathbb{R}\text{-vector space}}} \quad = \quad \underbrace{v_1, \mathrm{i}v_1, \ldots, v_k, \mathrm{i}v_k, w_1, \mathrm{i}w_1, \ldots, w_l, \mathrm{i}w_l}_{\substack{\text{positive basis for } \mathrm{T}_P M, \\ \text{viewed as an } \mathbb{R}\text{-vector space}}}.$$

This shows that $\mathrm{sign}(P) = +1$.
(2) Statement (2) is an immediate consequence of Statement (1). ■

Example. We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. In the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 we saw that $\mathbb{C}\mathrm{P}^n$ is a complex manifold of complex dimension n. As on page 962 we see that for any $k \leq n$ the subspace

$$X_k \;\coloneqq\; \{[z_0 : \ldots : z_k : 0 : \ldots : 0] \mid (z_0, \ldots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

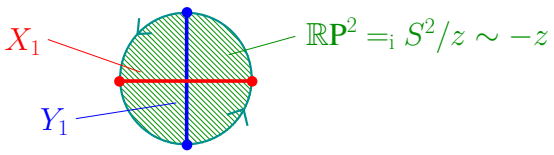
is a k-dimensional complex submanifold which is biholomorphic to $\mathbb{C}\mathrm{P}^k$. Similarly we see that for any $l \leq n$ the subspace

$$Y_l \;\coloneqq\; \{[0 : \ldots : 0 : z_{n-l} : \ldots : z_n] \mid (z_{n-l}, \ldots, z_n) \in \mathbb{C}^{l+1} \setminus \{0\}\}$$

is an l-dimensional complex submanifold which is biholomorphic to $\mathbb{C}\mathrm{P}^l$. Note that given any $k \leq n$ we have

$$X_k \cap Y_{n-k} \quad = \quad \{[0 : \ldots : 0 : \underset{\substack{\uparrow \\ (k+1)\text{-st entry}}}{1} : 0 : \ldots : 0]\}$$

consists of a single point P. We leave it to the reader to verify that X_k and Y_{n-k} intersect transversally in P. For illustration purposes we show the corresponding submanifolds in the real projective space $\mathbb{R}\mathrm{P}^2$:



Finally note that it follows from the Complex Submanifolds–Intersection Number Lemma 220.18 that $X_k \cdot Y_{n-k} = 1$ and that $Y_k \cdot X_{n-k} = 1$. □

220.4.2. **The cup product of complex projective spaces.** In this subsection we will use the results from Subsection 220.4.1 to calculate the cup product on complex projective spaces. The following notation will come in handy:

Notation 220.19. Let $n \in \mathbb{N}$ and let $m \in \{0, \ldots, n\}$.

(1) We consider the smooth embedding[2383]

$$\begin{array}{rcl} i_{m,n} \colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \ldots : z_m] & \mapsto & [z_0 : \ldots : z_m : 0 : \cdots : 0]. \end{array}$$

When m and n are understood we write i instead of $i_{m,n}$. Recall that in the proof of the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 we showed that for any $k \leq 2m$ the induced maps $i_* \colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m; \mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ and $i^* \colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$ are isomorphisms.

(2) By a slight abuse of notation we abbreviate $(i_{m,n})_*([\mathbb{C}\mathrm{P}^m])$ to $[\mathbb{C}\mathrm{P}^m] \in \mathrm{H}_{2m}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. Furthermore, again by a slight abuse of notation, we write $[\mathbb{C}\mathrm{P}^m]^* \in \mathrm{H}^{2m}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ for the unique element with $\langle [\mathbb{C}\mathrm{P}^m]^*, [\mathbb{C}\mathrm{P}^m] \rangle = 1$. Finally we write $x := [\mathbb{C}\mathrm{P}^1]^*$.

Now we can formulate the main result of this subsection:

Proposition 220.20. (Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}$. For every choice of $k \in \{0, \ldots, n\}$ we have

$$(\underbrace{[\mathbb{C}\mathrm{P}^1]^*}_{=x})^k \quad = \quad [\mathbb{C}\mathrm{P}^k]^* \quad \in \quad \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Remark. Recall that in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 we used the Poincaré Duality Theorem 210.1 to show that for given $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}$ is a generator. In other words, we had proved that $x^k = \pm[\mathbb{C}\mathrm{P}^k]^*$. The Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20 is a much better result since it allows us to nail down the cohomology class x^k, without any sign ambiguity. In particular we see that the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20 gives an affirmative answer to Question 213.9. □

The key to the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20 is the following lemma:

[2383] We showed in the Projective Space–Smooth Embedding Lemma 30.11 that this map is indeed a smooth embedding.

Lemma 220.21. (Cup Product-of-Duals-in-$\mathbb{C}\mathrm{P}^n$-Lemma) Let $n \in \mathbb{N}_0$. Given any $k \in \{0, \ldots, n\}$ we have

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}(\underbrace{[\mathbb{C}\mathrm{P}^k]}_{\in \mathrm{H}_{2k}(\mathbb{C}\mathrm{P}^n)}) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}(\underbrace{[\mathbb{C}\mathrm{P}^{n-k}]}_{\in \mathrm{H}_{2n-2k}(\mathbb{C}\mathrm{P}^n)}) = [\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$. For $k = 0$ and $k = n$ the statement follows easily from the definitions. We now consider the case that $k \in \{1, \ldots, n-1\}$. We consider the smooth embedding

$$\begin{array}{rcl} t_{k,n} \colon \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \ldots : z_k] & \mapsto & [0 : \ldots : 0 : z_0 : \ldots : z_k]. \end{array}$$

Claim. There exists a homotopy from $i_{k,n} \colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ to $t_{k,n} \colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$.

Proof. We consider the following map:

$$\begin{array}{rcl} \Phi \colon \mathbb{C}^{k+1} \times [0,1] & \to & \mathbb{C}^{n+1} \\ (z,t) & \mapsto & \underbrace{(z \cdot (1-t), 0)}_{\in \mathbb{C}^{n+1}} + \underbrace{(0, z \cdot t)}_{\in \mathbb{C}^{n+1}}. \end{array}$$

We make the following observations:

(1) For any $z \neq 0$ and any $t \in [0,1]$ we have $\Phi(z,t) \neq 0$. This can be seen as follows: If $k = n$ or if $t = 1$, then the statement is clear. If $k \neq n$ and $t \in [0,1)$, then we consider the first coordinate of z which is non-zero. We see that for $t \neq 1$ this coordinate of $\Phi(z,t)$ is also non-zero.
(2) It is clear that this map descends to a map $\mathbb{C}\mathrm{P}^k \times [0,1] \to \mathbb{C}\mathrm{P}^n$. It follows fairly easily from the Homotopy Quotient Theorem 18.11 that this map is continuous. Clearly this is now the desired homotopy. ⊞

Using this claim we can perform the following calculation:

$$\begin{array}{rcl} \langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n] \rangle & = & \langle \mathrm{PD}(i_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n] \rangle \\ & = & \langle \mathrm{PD}(t_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n] \rangle \end{array}$$

↑ by the Homotopy-H_*-Proposition 144.12, since since by the claim we know that $i_{k,n}$ and $t_{k,n}$ are homotopic

$$\begin{array}{rcl} & = & \langle \mathrm{PD}([t_{k,n}(\mathbb{C}\mathrm{P}^k)]) \cup \mathrm{PD}([i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k})]), [\mathbb{C}\mathrm{P}^n] \rangle \\ & = & t_{k,n}(\mathbb{C}\mathrm{P}^k) \cdot i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \end{array} = 1.$$

↑ by the Cup Product–Intersection Number Theorem 220.10 since the submanifolds $t_{k,n}(\mathbb{C}\mathrm{P}^k)$ and $i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k})$ are transverse — see page 4251 for details

By definition of the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$ this implies the desired result. ■

Proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20. We start out with the following claim:

Claim 1. Let $n \in \mathbb{N}$. For any $k \in \{0, \ldots, n\}$ we have

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}(\underbrace{[\mathbb{C}\mathrm{P}^k]}_{\in \mathrm{H}_{2k}(\mathbb{C}\mathrm{P}^n)}) = [\mathbb{C}\mathrm{P}^{n-k}]^* \in \mathrm{H}^{2n-2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. Then

$$\langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]), [\mathbb{C}\mathrm{P}^{n-k}] \rangle \overset{\text{by Definition 210.2 of } \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}])}{=} \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]), \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]) \cap [\mathbb{C}\mathrm{P}^n] \rangle$$
$$= \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n] \rangle \overset{\text{by the Cup Product-of-Duals-in-}\mathbb{C}\mathrm{P}^n \text{ Lemma 220.21}}{=} 1.$$

by the Cup-Cap Lemma 206.15 and the Cup Product-Commutativity Proposition 203.20

By definition of the dual fundamental class this implies the desired result. ⊞

We now start with the actual proof. Recall that we need to show that for every $n \in \mathbb{N}$ and every $k \in \{0, \dots, n\}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. We start out with the following observation that follows immediately from the discussion on page 4251 and the Cup Product-Naturality Lemma 203.9:

Observation. Let $m \in \mathbb{N}$ and let $k \in \{1, \dots, m\}$. If $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$, then for any $n \geq m$ we also have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$

By the observation it suffices to prove the following claim:

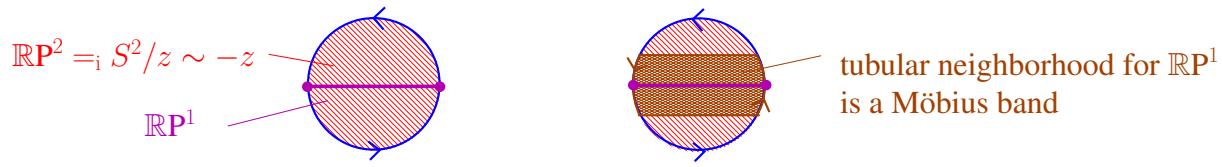
Claim 2. For any $k \in \mathbb{N}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k; \mathbb{Z})$.

Proof. We prove the claim by induction on k. For $k = 1$ the statement holds by definition. So suppose that the statement holds for some $k-1$ with $k \geq 2$. Then the following equality holds in $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k; \mathbb{Z})$:

$$([\mathbb{C}\mathrm{P}^1]^*)^k = ([\mathbb{C}\mathrm{P}^1]^*)^{k-1} \cup [\mathbb{C}\mathrm{P}^1]^* \overset{\text{by induction and by the above observation}}{=} [\mathbb{C}\mathrm{P}^{k-1}]^* \cup [\mathbb{C}\mathrm{P}^1]^* \overset{\text{by Claim 1}}{=} \mathrm{PD}([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{k-1}]) \overset{\text{by the Cup Product-of-Duals-in-}\mathbb{C}\mathrm{P}^k \text{ Lemma 220.21}}{=} [\mathbb{C}\mathrm{P}^k]^*.$$

This concludes the proof of the claim and thus of the proposition. ■

220.4.3. **Tubular neighborhoods in complex projective spaces.** We conclude this section with a short discussion of tubular neighborhoods. Recall that in the Tubular Neighborhood–Existence Theorem 57.11 we showed that every proper submanifold N of a given smooth manifold M admits a tubular neighborhood, which is, loosely speaking a neighborhood of N which is also a $\overline{B}^{\dim(M)-\dim(N)}$-bundle over N. In the Product Tubular Neighborhood–Existence Theorem 58.4 we showed that in many cases the tubular neighborhood is a product. But we also saw some examples where the tubular neighborhood is not a product. For example the figure below shows convincingly that the tubular neighborhood of $\mathbb{R}\mathrm{P}^1$ in $\mathbb{R}\mathrm{P}^2$ is a Möbius band, and thus not a product.



Similarly, if N is a non-orientable submanifold of an orientable smooth manifold, then it follows from the Product Orientation Proposition 31.48 that the tubular neighborhood of N in M cannot be a product.

The following proposition shows that, for better or worse, the tubular neighborhood of an orientable submanifold of an orientable smooth manifold is not necessarily a product either:

Proposition 220.22. (Non-Product Tubular Neighborhood Proposition) No tubular neighborhood of $\mathbb{C}\mathrm{P}^1 = i_{1,2}(\mathbb{C}\mathrm{P}^1)$ in $\mathbb{C}\mathrm{P}^2$ is a product.

Remark. In Exercise 240.8 we will give an alternative proof of the Non-Product Tubular Neighborhood Proposition 220.22. □

Proof. We now suppose that $\mathbb{C}\mathrm{P}^1$ admits a product tubular neighborhood in $\mathbb{C}\mathrm{P}^2$. By definition this would mean that there exists a smooth embedding $\tau\colon \mathbb{C}\mathrm{P}^1 \times \overline{B}^2 \to \mathbb{C}\mathrm{P}^2$ with $\tau(P,(0,0)) = P$ for all $P \in \mathbb{C}\mathrm{P}^1$. We denote by $i = i_{1,2}\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the usual smooth embedding and we denote by $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the smooth embedding that is given by $P \mapsto \tau(P,(1,0))$. Then

$$\pm 1 \underset{\uparrow}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1])) \cup \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle \underset{\uparrow}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(f_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle$$

by the Cup Product-of-Duals-in-$\mathbb{C}\mathrm{P}^n$-Lemma 220.21 or by the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7

by the Homotopy-H_*-Proposition 144.12 since $i, f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ are homotopic via the homotopy $(P,t) \mapsto \tau(P,(t,0))$

$$\underset{\uparrow}{=} i(\mathbb{C}\mathrm{P}^1) \cdot f(\mathbb{C}\mathrm{P}^1) \underset{\uparrow}{=} 0.$$

by the Cup Product–Intersection Number Theorem 220.10

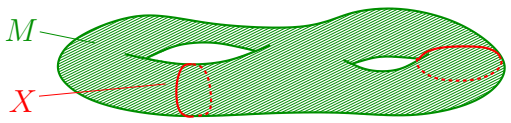
since $i(\mathbb{C}\mathrm{P}^1)$ and $f(\mathbb{C}\mathrm{P}^1)$ are disjoint

We have thus obtained a contradiction. ■

220.5. Exercises for Chapter 220 .

Exercise 220.1. Show, as a reality check, that the (anti-) commutativity of the algebraic intersection number of the Intersection Number–Symmetry Lemma 220.8 does indeed correspond precisely to the (anti-) commutativity of the cup product from the Cup Product-Commutativity Proposition 204.10.

Exercise 220.2. Let M be a compact orientable connected n-dimensional smooth manifold and let $X \subseteq M$ be an orientable proper codimension-one submanifold. Show that the fundamental class $[X] \subseteq \mathrm{H}_{n-1}(M, \partial M)$ generates an infinite cyclic summand.
Remark. The statement of this exercise is related to the statement of Exercise 194.5. But it is suggested to use very different techniques.



Exercise 220.3. We consider the torus $S^1 \times S^1$ with the usual orientation.

(a) Is it possible to find closed oriented 1-dimensional submanifolds $A, B, C \subseteq S^1 \times S^1$ with $A \cdot B = 3$, $A \cdot C = -6$ and $B \cdot C = 2$?

(b) Is it possible to find closed oriented *connected* 1-dimensional submanifolds A, B, C in $S^1 \times S^1$ with $A \cdot B = 3$, $A \cdot C = -6$ and $B \cdot C = 2$?

Exercise 220.4. Let M be a smooth manifold that admits two compact codimension-zero submanifolds A and B such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B. By the Mayer–Vietoris Theorem 153.2 (3) we obtain a long exact sequence

$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$

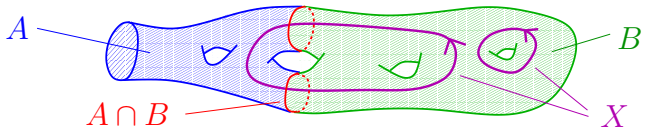
Let X be a closed oriented n-dimensional submanifold of M that intersects $A \cap B$ transversally.

(a) Show that $X \cap (A \cap B)$ is a closed $(n-1)$-dimensional submanifold of $A \cap B$.

We equip $X \cap (A \cap B)$ with the orientation coming from $X \cap (A \cap B) = \partial(X \cap A)$. We view the fundamental class $[X]$ as an element in $\mathrm{H}_n(M)$ and we view the fundamental class $[X \cap (A \cap B)]$ as an element in $\mathrm{H}_{n-1}(A \cap B)$.

(b) Show that $\partial_n([X]) = \pm[X \cap (A \cap B)] \in \mathrm{H}_{n-1}(A \cap B)$.

(c) Determine the correct sign in (b).



Exercise 220.5. Let M be an oriented connected smooth manifold and let X, Y be two closed oriented submanifolds with $\dim(X)+\dim(Y) = \dim(M)$. We suppose that $X \cdot Y = 0$. Does it follow that there exists a smooth homotopy $F\colon Y \times [0,1] \to M$ such that $F_0 = \mathrm{id}$ and $X \cap F_1(Y) = \varnothing$?

Remark. This exercise is identical to Exercise 51.4.

Exercise 220.6. Let A, B be two oriented 1-dimensional submanifolds of S^2 with $A \cdot B = 0$. Sketch a proof for the claim that A and B can be made disjoint via isotopies of A and B.

Exercise 220.7. Let M be a compact oriented m-dimensional smooth manifold. Furthermore let $k \in \{1, \dots, m-1\}$, let X be a proper oriented k-dimensional submanifold of M and let Y be a proper oriented $(m-k)$-dimensional submanifold of M. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. We could define the algebraic intersection number of X and Y via the formula in the Cup Product–Intersection Number Theorem 220.10, i.e. we could define

$$X \cdot Y \;\coloneqq\; \langle \mathrm{PD}_M(i_*([X]) \cup \mathrm{PD}_M(j_*([Y])), [M] \rangle.$$

What, if any, conditions do you need to impose on the boundaries of X and Y for this definition to make any sense?

Remark. This exercise can be viewed as a continuation of the discussion on page 4232.

Exercise 220.8. Let M be an oriented m-dimensional smooth manifold and let $X, Y \subseteq M$ be complementary submanifolds of M. Show that if X is the boundary of a compact submanifold $Z \subseteq M$, then $X \cdot Y = 0$.

Remark. This exercise can be viewed as an analog of the Intersection-Number Zero-if-Extends Proposition 51.8.

Exercise 220.9. Let M be a closed oriented connected m-dimensional smooth manifold. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering.

(a) Let X, Y be a complementary pair of submanifolds of M. We write $\widetilde{X} \coloneqq p^{-1}(X)$ and $\widetilde{Y} \coloneqq p^{-1}(Y)$. What is the relationship between $X \cdot Y$ and $\widetilde{X} \cdot \widetilde{Y}$?

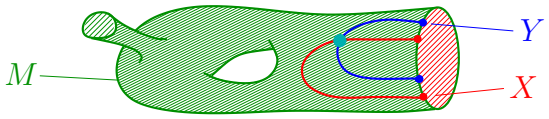
(b) Let X be a closed oriented submanifold of M. What is the relationship between $\mathrm{PD}_{\widetilde{M}}([\widetilde{X}])$ and $p^*(\mathrm{PD}_M([X])$?

(c) Reprove (a) using (b) and the Cup Product–Intersection Number Theorem 220.10.

Exercise 220.10. Let M be an oriented m-dimensional smooth manifold, let X be an oriented proper k-dimensional submanifold and let Y be an oriented proper $(m-k)$-dimensional submanifold of M. We suppose that $X \cap Y \cap \partial M = \varnothing$. What is the relationship between the algebraic intersection number $X \cdot Y$ in M and $DX \cdot DY$ in $\mathrm{D}M$?
Here, as always, given a smooth manifold N we denote by $\mathrm{D}N = N \cup_{\partial N = \partial N'} N'$ the double of N. We view $\mathrm{D}N$ as an oriented smooth manifold where the orientation is chosen such that the orientation of $\mathrm{D}N$ agrees with the orientation on N.

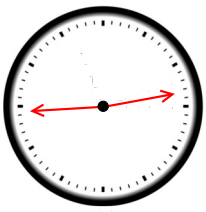
Exercise 220.11. Let $\Sigma \subseteq S^2 \times (-1,1)$ be a closed surface. Suppose that there exists a $P \in S^2$ such that Σ intersects $\{P\} \times [-1,1]$ in a single point. Show that if this intersection point is a transversal intersection point, then Σ intersects every path $\gamma\colon [0,1] \to S^2 \times [-1,1]$ from a point in $S^2 \times \{-1\}$ to a point in $S^2 \times \{1\}$.

Exercise 220.12. We consider the 2-dimensional smooth manifold M that is shown together with the two proper submanifolds X and Y that are shown in the figure below. Show that X and Y cannot be made disjoint by a smooth isotopy that keeps the boundaries of X and Y fixed.

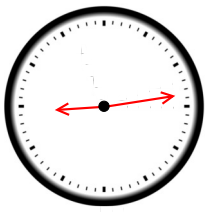


Exercise 220.13. By accident you ended up buying a clock where both hands have the same length, so you cannot distinguish the hand which shows the hour from the hand that indicates the minute. Fortunately, very often you can still tell the precise time. For example, in the configuration in the figure below it does not make sense for the hand that is indicating the hour to be just *before* 9 whereas the hand showing the minute is at 15.

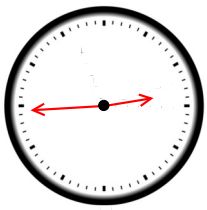
Of course there will be some configurations when it is indeed impossible to tell which hand is which. How many such confusions moments will there be in a 12 hour period?
Hint. Consider a suitable map $\gamma\colon S^1 \to S^1 \times S^1$.



clock where both hands have the same length



this makes no sense

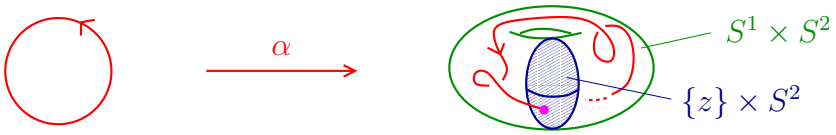


the time is 2:44

Exercise 220.14. Let C and D be smoothly embedded loops in $\mathbb{R}^2$ that intersect transversally. Give a rigorous proof that the number of intersection points between C and D is an even number.

Exercise 220.15. Let $g \in \mathbb{N}_0$ and let Σ_g be the orientable surface of genus g. We equip Σ_g with some orientation. For the purpose of this exercise, given a closed oriented 1-dimensional submanifold $C \subseteq \Sigma_g$ we define the self-intersection number $C \cdot C := 0$. What is the maximal $k \in \mathbb{N}_0$ such that there exist closed oriented 1-dimensional submanifolds $A_1, \ldots, A_k \subseteq \Sigma_g$ such that any two A_i, A_j with $i \neq j$ are transverse and such that the matrix $(A_i \cdot A_j)_{i,j=1,\ldots,k}$ has non-zero determinant?
Remark. This exercise is basically identical to Exercise 51.15.

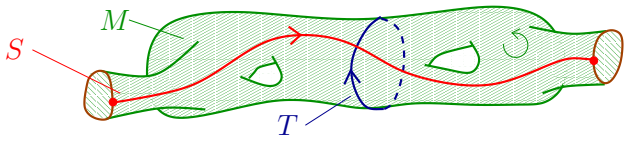
Exercise 220.16. Let $\alpha\colon S^1 \to S^1 \times S^2$ be a smooth embedding. We assume that there exists a $z \in S^1$ such that $\alpha(S^1)$ intersects the submanifold $\{z\} \times S^2$ in a single point P and we assume that the intersection point is a transversal intersection point of sign $+1$. Show that $\alpha\colon S^1 \to S^1 \times S^2$ is smoothly isotopic to the embedding $\beta\colon S^1 \to S^1 \times S^2$ that is given by $t \mapsto (t, P)$.
Remark. This statement is sometimes referred to as the *light bulb trick*.



Exercise 220.17. Let M be a compact oriented 2-dimensional smooth manifold. We consider the following construction:

$$\begin{aligned} \mathrm{H}_1(M, \partial M) \times \mathrm{H}_1(M, \partial M) &\to \mathbb{Z} \\ (\sigma, \tau) &\mapsto \begin{array}{l}\text{algebraic intersection number } S \cdot T \text{ of} \\ \text{proper oriented transverse submanifolds} \\ S \text{ and } T \text{ of } M \text{ with } [S] = \sigma \text{ and } [T] = \tau.\end{array} \end{aligned}$$

Does this give a well-defined map?



Exercise 220.18. We consider the surface Σ of genus 2 together with the four oriented curves a_1, b_1, a_2, b_2 that are shown in the figure below.

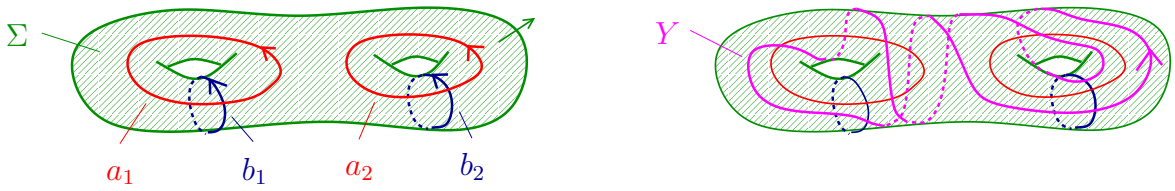
(a) Let X be a closed oriented 1-dimensional submanifold of Σ that intersects a_1, b_1, a_2, b_2 transversally. Show that

$$[X] = \underset{\uparrow}{(X \cdot b_1)} \cdot [a_1] - \underset{\uparrow}{(X \cdot a_1)} \cdot [b_1] + \underset{\uparrow}{(X \cdot b_2)} \cdot [a_2] - \underset{\uparrow}{(X \cdot a_2)} \cdot [b_2] \in \mathrm{H}_1(\Sigma).$$

these are the algebraic intersection numbers of the given oriented curves

(b) Let X and Y be two closed oriented 1-dimensional submanifolds of Σ that are both transverse to a_1, a_2, b_1, b_2. Show that if X and Y have the same algebraic intersection numbers with a_1, a_2, b_1, b_2, then $[X] = [Y] \in \mathrm{H}_1(\Sigma)$.

Remark. The above gives a practical way, for Σ and similarly for many other smooth manifolds, to determine precisely the homology class represented by a submanifold. In particular it can be used to deal with the type of problems as in Exercise 155.16.



Exercise 220.19. Let $n \in \mathbb{Z}$. In Exercise 40.8 we showed that

$$W_n \;:=\; \{[x:y:z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$$

is a closed 1-dimensional complex submanifold of the complex projective space $\mathbb{C}\mathrm{P}^2$. Show that $[W_n] = n \cdot [\mathbb{C}\mathrm{P}^1] \in \mathrm{H}_2(\mathbb{C}\mathrm{P}^2)$.
Hint. Use the Complex Submanifolds–Intersection Number Lemma 220.18.

Exercise 220.20.
(a) For which $k \in \{0, \dots, n\}$ does $\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$ admit a product tubular neighborhood?
(b) For which $k \in \{0, \dots, n\}$ does $\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$ admit a product tubular neighborhood?

Exercise 220.21. Let $m \in \mathbb{N}_0$ and let $k \in \{0, \dots, m\}$. It follows from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 together with the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20 we can write $\mathrm{H}^*(\mathbb{C}\mathrm{P}^m; \mathbb{Z}) = \mathbb{Z}[x]/(x^{m+1})$. Show that

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^m}([\mathbb{C}\mathrm{P}^k]) \;=\; x^{m-k} \;\in\; \mathrm{H}^{2(m-k)}(\mathbb{C}\mathrm{P}^m; \mathbb{Z}).$$

Remark. This exercise is closely related to Exercise 213.2.

CHAPTER 221

Intersections of cycles and the cup product

In this chapter we generalize some of the results from Chapter 220. In particular we will define the "algebraic intersection number" of two continuous maps of topological manifolds into a given topological manifold, as long as the maps intersect "nicely". This notion generalizes the algebraic intersection number of two transverse submanifolds of a given smooth manifold that we introduced in Definition 220.5.

We also introduce the algebraic intersection number of two singular cycles that intersect "nicely". Afterwards we will see in the Maps-Cup Product–Intersection Number Theorem 221.16 and 221.20 how these algebraic intersection numbers relate to the cup product. These theorems can be viewed as a generalization of the Cup Product–Intersection Number Theorem 220.10.

For many applications the results from Chapter 220 are good enough and for many readers it might make sense to move on to the next chapter. The main application of the results of this chapter will be the Lefschetz-Hopf Theorem 222.14 and the Poincaré-Hopf Theorem 223.8.

This chapter is organized as follows:

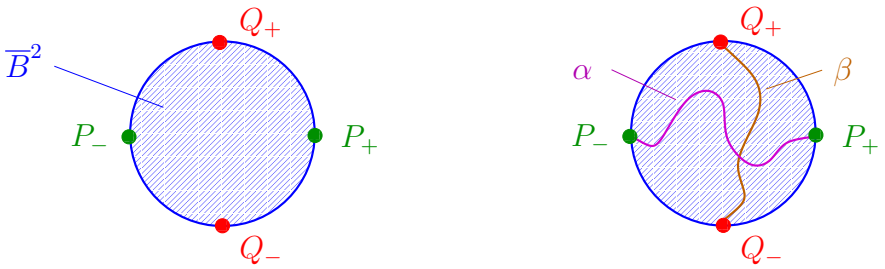
- In Section 221.1 we will introduce the algebraic intersection number of maps of topological manifolds.
- In Section 221.2 we will discuss transverse intersection points of maps.
- In Section 221.3 we will discuss intersections of cycles and we will formulate the Cycles–Cup Product-Intersection Number Theorem 221.20.
- In Section 221.4 we will provide the proof of the Cycles–Cup Product-Intersection Number Theorem 221.20.
- In Appendix 221.5 we will discuss the algebraic $\mathbb{F}_2$-intersection number.
- In Section 221.6 we will provide exercises for Chapter 221.

In the next Chapter 222 we will formulate and prove the Lefschetz-Hopf Theorem 222.9.

Since much of this chapter is quite technical we start out with the following motivating question.

Question 221.1. As shown in the figure below we consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm = (0, \pm 1)$. Do there exist disjoint paths α and β such that α connects P_- to P_+ and β connects Q_- to Q_+?

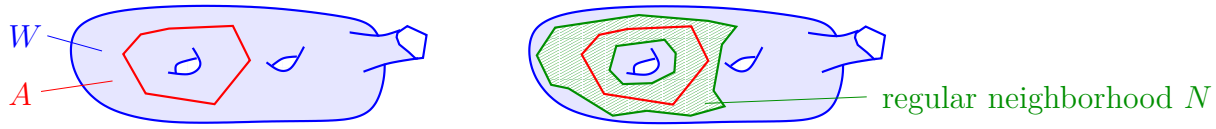
It is pretty clear that the answer should be no. In fact once we have developed all the tools we will find it quite easy to answer Question 221.1 in the negative. We refer to the Cross Intersection Lemma 221.17 for the details.



221.1. **The algebraic intersection number of maps of topological manifolds.** In this section we introduce the algebraic intersection number of two continuous maps of topological manifolds into a given topological manifold. The precise setup and the definitions require some preparations. First we recall Definition 217.2:

Definition 221.2. Let W be a topological manifold with $\partial W = \varnothing$ and let $A \subseteq W \setminus \partial W$ be a subset. A regular neighborhood of A is a compact codimension-zero submanifold N of W with the following properties:

(1) N is contained in $W \setminus \partial W$ and A is contained in the interior of N.
(2) A is a deformation retract of N.
(3) ∂N is a deformation retract of $N \setminus A$.



The following proposition can be viewed as a variation on the Poincaré Duality Theorem 210.1.

Proposition 221.3. (Neighborhood–Poincaré Duality Proposition) Let Z be a compact oriented m-dimensional topological manifold and let A be a closed submanifold of ∂Z.

(1) Let $N \subseteq \partial Z$ be a regular neighborhood of A. We denote by $j\colon A \to N$ the inclusion map. Given any $l \in \mathbb{N}_0$ the following two maps are isomorphisms:[2384]

$$\mathrm{H}^l(Z, \partial Z \setminus A; \mathbb{Z}) \xrightarrow[\cong]{\cap [Z]} \mathrm{H}_{m-l}(Z, N) \xleftarrow[\cong]{j_*} \mathrm{H}_{m-l}(Z, A).$$

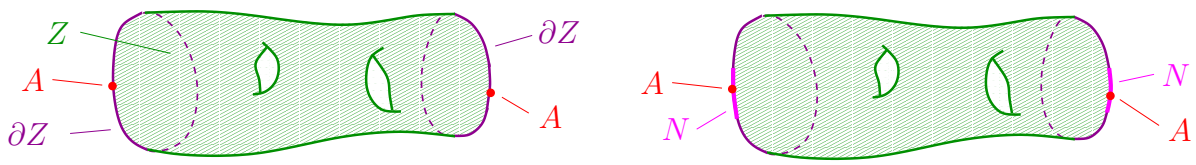
(2) We make the following technical hypothesis:
   (∗) Given any neighborhood U of $A \subseteq \partial Z$ there exists a regular neighborhood N of $A \subseteq \partial Z$ with $N \subseteq U$.

   Under this hypothesis the isomorphism

$$\mathrm{PD}_Z\colon \mathrm{H}^l(Z, \partial Z \setminus A; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_{m-l}(Z, A)$$

   given in (1) does not depend on the choice of the regular neighborhood N. We denote the inverse of this map by PD_Z as well.

[2384]Note that it follows from the Excisive Triad-Proposition 158.20 (1) that the cap product $\cap [Z]\colon \mathrm{H}^l(Z, \partial Z \setminus A; \mathbb{Z}) \to \mathrm{H}_{m-l}(Z, N)$ is actually defined. It is also worth pointing out that it is not possible to go directly from $\mathrm{H}^l(Z, \partial Z \setminus A; \mathbb{Z})$ to $\mathrm{H}_{m-l}(Z, A)$ via capping with $[Z]$, since $(\partial Z, \partial Z \setminus A, A)$ is not an excisive triad.



Proof.

(1) Let $N \subseteq \partial Z$ be a regular neighborhood of A. We denote by $i\colon \partial Z \setminus \mathring{N} \to \partial Z \setminus A$ and $j\colon A \to N$ the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^l(Z, \partial Z \setminus A; \mathbb{Z}) & \xrightarrow[\cong]{\cap[Z]} & \mathrm{H}_{m-l}(Z, N) & \xleftarrow[\cong]{j_*} & \mathrm{H}_{m-l}(Z, A). \\ & {\scriptstyle i^*}\searrow \quad \nearrow{\scriptstyle \cap[Z]}^{\cong} & & & \\ & \mathrm{H}^l(Z, \partial Z \setminus \mathring{N}; \mathbb{Z}) & & & \end{array}$$

We make the following observations:

(a) The diagonal cap product exists by the Codimension-Zero Submanifold Proposition 150.16 (3) together with the Excisive Triad-Proposition 158.20 (1).

(b) It follows from the Cap Product-Naturality Lemma 206.13 that the triangle commutes.

(c) It follows from the hypothesis that N is a regular neighborhood, together with the Basics-of-Regular Neighborhood Lemma 217.4 (1), that the maps i and j are both homotopy equivalences. Thus we obtain from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 and the Homotopy Equivalence-H^*-Proposition 198.12 (1) that the induced maps i^* and j_* are isomorphisms.

(d) It follows from the Poincaré Duality Theorem 210.1, which we can apply by the Codimension-Zero Submanifold Proposition 150.16 (3), that the right diagonal map is an isomorphism.

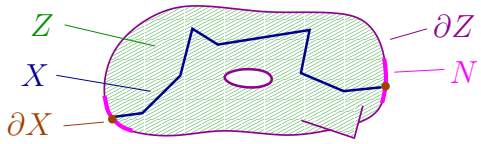
The above argument shows that the two horizontal maps are both isomorphisms.

(2) We need to show that the map in (1) does not depend on the choice of the regular neighborhood N. Thus let M and N be regular neighborhoods of $A \subseteq \partial Z$. It follows immediately from our hypothesis (∗) that it suffices to consider the case that $M \subseteq N$. We consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_{m-l}(Z, M) & & \\ & {\scriptstyle \cap[Z]}\nearrow^{\cong} & \downarrow & \nwarrow^{\cong} & \\ \mathrm{H}^l(Z, \partial Z \setminus A; \mathbb{Z}) & & & & \mathrm{H}_{m-l}(Z, A). \\ & {\scriptstyle \cap[Z]}\searrow_{\cong} & \downarrow & \swarrow_{\cong} & \\ & & \mathrm{H}_{m-l}(Z, N) & & \end{array}$$

Here all unmarked maps are induced by inclusions. The triangle to the left commutes by the Cap Product-Naturality Lemma 206.13 and the triangle to the right commutes by the functoriality of cohomology groups. But this shows that the upper and the lower path from the left to the right give exactly the same map. ■

Definition 221.4. Let Z be a topological manifold. We say that a proper submanifold X of Z is well-behaved if given any open neighborhood U of $\partial X \subseteq \partial Z$ the submanifold ∂X of ∂Z admits a regular neighborhood N with $N \subseteq U$.



Remark. Almost any proper submanifold X of some topological manifold Z that appears in nature is well-behaved. For example it follows from the Simplicial Regular Neighborhood Proposition 180.14, the Tubular Neighborhood–Existence Theorem 57.11 and the Tubular–Regular Neighborhood Theorem 217.3 that X is well-behaved if ∂X is a PL-submanifold of ∂Z or if ∂X is a submanifold of ∂Z. □

In the following we will study maps between topological manifolds. To simplify the discussion we use the following convention.

Convention 221.5. Let M be a topological manifold and let X be a submanifold of M. In our notation and our language often we do not distinguish between X as a submanifold and between the embedding $X \to M$.

Next we fix a convenient notation.

Notation 221.6. Given any continuous map $f\colon Y \to X$ between two topological spaces we write $|f| \coloneqq f(Y)$.

Now we can give the first key definition of this chapter.

Definition 221.7. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be continuous maps.

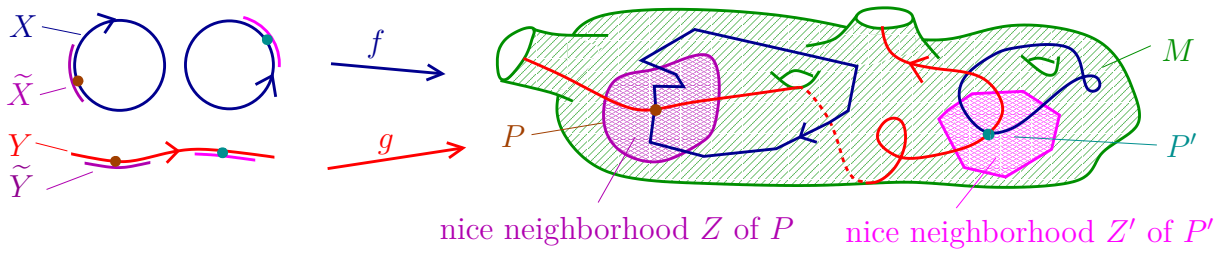
(1) We say that a point $P \in |f| \cap |g|$ is a nice intersection point of f and g if given any neighborhood U of P there exists a nice neighborhood Z, this means a submanifold $Z \subseteq M \setminus \partial M$ such that the following holds:
   (a) Z is homeomorphic to $\overline{B}^m$.
   (b) P is contained in[2385] $\mathring{Z} = Z \setminus \partial Z$ and Z is contained in U.
   (c) $|f| \cap |g| \cap Z = \{P\}$.

   Before we continue we set $\widetilde{X} \coloneqq f^{-1}(|f| \cap Z)$ and we set $\widetilde{Y} \coloneqq g^{-1}(|g| \cap Z)$. With this notation we also demand the following:
   (d) $\widetilde{X}$ is a compact codimension-zero submanifold of X and $\widetilde{Y}$ is a compact codimension-zero submanifold of Y.
   (e) The maps $f\colon \widetilde{X} \to Z$ and $g\colon \widetilde{Y} \to Z$ are both proper embeddings.
   (f) $f(\widetilde{X})$ and $g(\widetilde{Y})$ are both well-behaved submanifolds of Z.
   (g) $f(\widetilde{X})$ is homeomorphic to $\overline{B}^k$ and $g(\widetilde{Y})$ is homeomorphic to $\overline{B}^{m-k}$.

   In the figure below we illustrate the definition for two different nice intersection points.

(2) We say that f and g intersect nicely, if all points in $|f| \cap |g|$ are nice intersection points.

[2385] Note that by (a) we know that Z is a codimension-zero submanifold of M. Since $Z \subseteq M \setminus \partial M$ we know by the Codimension-Zero Submanifold Proposition 150.16 that the boundary of Z as a topological manifold agrees with the boundary of Z as a subset of M.



Remark. Note that by the above convention we now also defined the notion of a nice intersection point of two submanifolds and we defined the notion that two submanifolds intersect nicely. □

Convention 221.8. Let W be an oriented topological manifold. We equip any codimension-zero submanifold of W with the orientation coming from the Topological Manifold-Induced Orientation Lemma 192.1.

Now we are ready to give the second key definition of this chapter.

Definition 221.9. Let M be an oriented m-dimensional topological manifold, let X be an oriented k-dimensional topological manifold and let Y be an oriented $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be continuous maps and let P be a nice intersection point of f and g. We pick a nice neighborhood Z of P. We define the index of f and g at P as follows:[2386]

$$\operatorname{index}(f,g,P) = \Big\langle \overbrace{\underbrace{\mathrm{PD}_Z(\underbrace{f_*([\widetilde{X}])}_{\in \mathrm{H}_k(Z,\partial f(\widetilde{X}))})}_{\in \mathrm{H}^{m-k}(Z,\partial Z\setminus \partial f(\widetilde{X});\mathbb{Z})} \cup \underbrace{\mathrm{PD}_Z(\underbrace{g_*([\widetilde{Y}])}_{\in \mathrm{H}_{m-k}(Z,\partial(g(\widetilde{Y})))})}_{\in \mathrm{H}^{k}(Z,\partial Z\setminus \partial g(\widetilde{Y});\mathbb{Z})}}^{\in \mathrm{H}^m(Z,\partial Z;\mathbb{Z})}, \overbrace{[Z]}^{\in \mathrm{H}_m(Z,\partial Z)} \Big\rangle \in \mathbb{Z}.$$

Example. In the figure below we show, in three cases, the indices of nice intersection points.

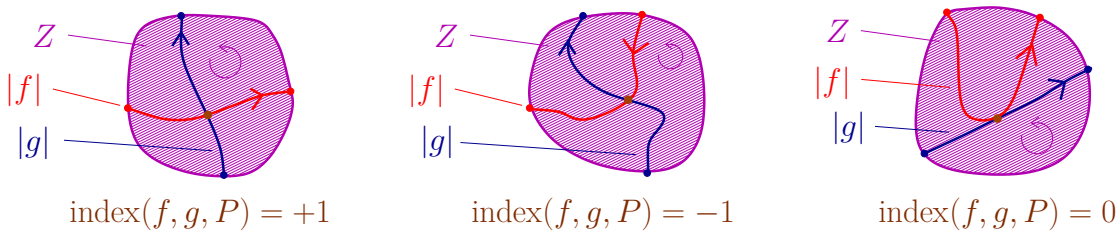
The first two indices can be determined with some effort using the Poincaré Duality-Products Proposition 210.3 together with the Product Orientation Proposition 192.7 (2). Alternatively see the Index-equals-Sign Proposition 221.14 below. Finally we outsource the calculation of the third index to Exercise 221.2. □

[2386] Here are a few clarifications:

(1) By hypothesis we know that Z, $\widetilde{X}$ and $\widetilde{Y}$ are compact codimension-zero submanifolds of M, X and Y. As we just remarked, by the Topological Manifold-Induced Orientation Lemma 192.1 they inherit an orientation from the given orientations on M, X and Y. As always we denote by $[Z]$, $[\widetilde{X}]$ and $[\widetilde{Y}]$ the corresponding fundamental classes.

(2) By hypothesis $f(\widetilde{X})$ and $g(\widetilde{Y})$ are well-behaved submanifolds of Z. Thus we obtain from the Neighborhood–Poincaré Duality Proposition 221.3 the corresponding maps PD_Z.

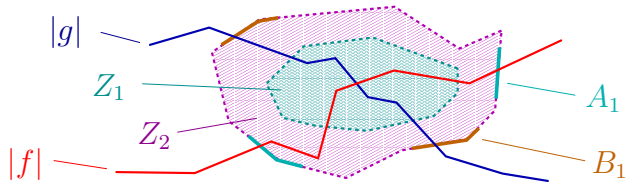
(3) By hypothesis $f(\widetilde{X})$ and $g(\widetilde{Y})$ are compact. It follows from the Topological Manifold–Boundary Proposition 23.15 (2b) that $\partial f(\widetilde{X})$ and $\partial g(\widetilde{Y})$ are compact. By the Compact-Closed Lemma 3.39 (2) this implies that $\partial Z \setminus \partial f(\widetilde{X})$ and $\partial Z \setminus \partial g(\widetilde{Y})$ are open subsets of ∂Z. Since P is the only intersection point of f and g in Z we see that these two open subsets cover all of ∂Z. It now follows from the Excisive Triad-Proposition 158.20 (1) that the cup product is actually defined.



Next we need to show that the index of a nice intersection point is well-defined.

Lemma 221.10. (Index–Well-Defined Lemma) Let M, $f\colon X \to M$ and $g\colon Y \to M$ as in the previous definition and let P be a nice intersection point of f and g. The corresponding index $\operatorname{index}(f, g, P)$ is well-defined, i.e. it is independent of the choice of the nice neighborhood Z of P.

Sketch of a proof. By the "given any neighborhood U of P" aspect of the definition of a nice intersection point it suffices to deal with nice neighborhoods Z_1 and Z_2 with $Z_1 \subseteq \mathring{Z}_2$. We define $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{Y}_1$ and $\widetilde{Y}_2$ in the obvious way. We pick regular neighborhoods A_1 of $\partial X_1 \subseteq \partial Z_1$ and B_1 of $\partial Y_1 \subseteq \partial Z_1$. Similarly we pick A_2 and B_2 with the obvious meaning.



Let $i \in \{1, 2\}$. We write $C_i \coloneqq \partial Z_i \setminus \mathring{A}_i$ and $D_i \coloneqq \partial Z_i \setminus \mathring{B}_i$. Similar to the proof of the Manifold–Excision Theorem 148.3 we use the Topological Collar Neighborhood Theorem 35.14 to find open subsets $\widetilde{A}_i$, $\widetilde{B}_i$, $\widetilde{C}_i$ and $\widetilde{D}_i$ of ∂Z_i that admit deformation retractions to A_i, B_i, C_i respectively D_i. Now we consider the following maps of triads of topological spaces:

$$\begin{array}{ccccccccc} Z_1 & & Z_1 & & Z_2 & & Z_2 & & Z_2 \\ A_1 & \to & \widetilde{A}_1 & \to & \widetilde{A}_1 \cup ((\mathring{Z}_2 \setminus Z_1) \cup \widetilde{A}_2) & \leftarrow & \widetilde{A}_2 & \leftarrow & A_2 \\ \partial Z_1 \setminus A_1 & & \partial Z_1 \setminus A_1 & & (\partial Z_1 \setminus A_1) \cup (\mathring{Z}_2 \setminus Z_1) \cup (\partial Z_2 \setminus A_2) & & \partial Z_2 \setminus A_2 & & \partial Z_1 \setminus A_2. \end{array}$$

It follows from the Excisive Triad-Proposition 158.20 (1) that all these triads are excisive. Evidently the analogous statement also holds if we replace the A's by B's. Now the lemma can be deduced from the Cup Product-Naturality Lemma 204.8 and the Cap Product-Naturality Lemma 206.13 together with the Codimension Zero-Fundamental Class Lemma 194.10. We leave it to the reader to fill in the details. ■

The following lemma is a generalization of the Finite Transverse Intersection Lemma 220.4.

Lemma 221.11. (Finite Intersection Lemma) Let M be an m-dimensional topological manifold, let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be continuous maps. If f and g intersect nicely, then $|f| \cap |g|$ is finite.

Proof. Let P be a nice intersection point of f and g. We pick a nice neighborhood Z for P. Note that this means in particular that Z is a codimension-zero submanifold of M with $Z \subseteq M \setminus \partial M$, $P \in Z \setminus \partial Z$ and with $|f| \cap |g| \cap Z = \{P\}$. By the Codimension-Zero

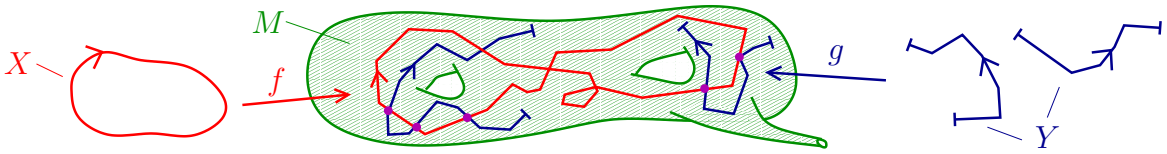
Submanifold Proposition 150.16 (2) we know that $Z \setminus \partial Z$ is an open subset of M. We have thus shown that P is an isolated point, in the sense of Definition 7.17, of the subspace $|f| \cap |g|$. Since by hypothesis every intersection point is nice we see that $|f| \cap |g|$ is a discrete subset of M. Since X and Y are compact we know by the Compact Image Lemma 4.15 together with the Compact–Union-Intersection Lemma 3.41 (2) that $|f| \cap |g|$ is also compact. Thus we see that $|f| \cap |g|$ is a discrete compact space. By the Compact-Discrete Lemma 3.40 (1) this implies that $|f| \cap |g|$ is a finite set. ■

By the Finite Intersection Lemma 221.11 it now makes sense to introduce the following definition which can be viewed as an analog of the definition on Definition 220.5.

Definition 221.12. Let M be an oriented m-dimensional topological manifold. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be continuous maps. If f and g intersect nicely, then we define the algebraic intersection number of f and g to be

$$f \cdot g \;:=\; \sum_{P \in |f| \cap |g|} \operatorname{index}(f, g, P).$$

finite sum by the Finite Intersection Lemma 221.11



Remark. Recall that by our convention on page 4262 we do not distinguish between a submanifold X of M and the actual embedding $X \to M$. In particular in the above Definition 221.12 we also introduced the algebraic intersection number $X \cdot Y$ of two suitable submanifolds of M. □

221.2. **Transverse intersection points of maps.** In this section we will see that the notion of a nice intersection points incorporates the notion of a transverse intersection point that we introduced on Definition 220.3. We will also see that the concept of the algebraic intersection number that we defined in the previous section is a generalization of the algebraic intersection number that we introduced on Definition 50.3 and Definition 220.5.

The following definition can be viewed as a generalization of the concepts we introduced on Definition 50.3 and Definition 220.3.

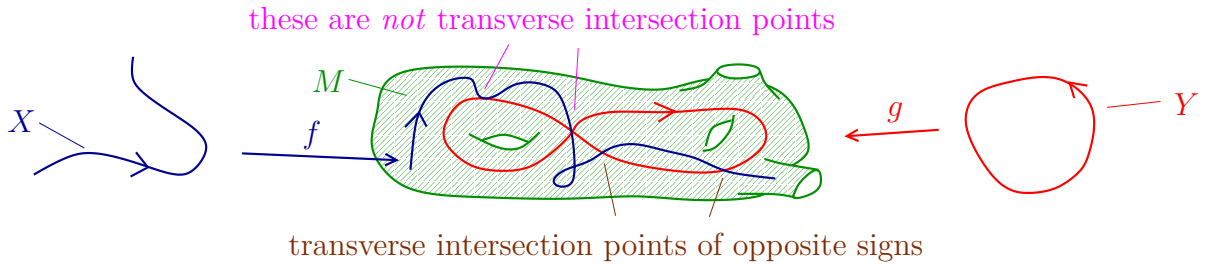
Definition 221.13. Let $m \in \mathbb{N}$, let M be an m-dimensional smooth manifold and let $k \in \{1, \ldots, m-1\}$. Furthermore let X be a k-dimensional smooth manifold and let Y be an $(m-k)$-dimensional smooth manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be smooth maps.

(1) We say that a point $P \in |f| \cap |g|$ is a transverse intersection point of f and g if $P \in M \setminus \partial M$, if $f^{-1}(P)$ contains a single point x_0, if $g^{-1}(P)$ contains a single point y_0, if $x_0 \in X \setminus \partial X$ and $y_0 \in Y \setminus \partial Y$, and if $f_*(\mathrm{T}_{x_0} X) + g_*(\mathrm{T}_{y_0} Y) = \mathrm{T}_P M$.

(2) Now we assume that M, X and Y are oriented. Let P be a transverse intersection point of f and g. We set $\operatorname{sign}(f,g,P) := +1$ if

$$(f_*(\text{positive basis of } \mathrm{T}_{x_0}X), g_*(\text{positive basis of } \mathrm{T}_{y_0}Y)) \;=\; \text{positive basis of } \mathrm{T}_P M,$$

and otherwise we set $\operatorname{sign}(f,g,P) := -1$.



Remark. If X and Y are compact proper submanifolds of M and if f and g are just the inclusion maps, then the above definitions are precisely the definitions provided in Definition 50.3 and 220.3. □

Proposition 221.14. (Index-equals-Sign Proposition) Let M be an m-dimensional smooth manifold and let $k \in \{1, \dots, m-1\}$. Furthermore let X be a k-dimensional smooth manifold and let Y be an $(m-k)$-dimensional smooth manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be smooth maps.

(1) Let $P \in |f| \cap |g|$. If P is a transverse intersection point of f and g, then P is a nice intersection point of f and g.

(2) If M, X and Y are oriented, then for any transverse intersection point P of f and g we have $\operatorname{index}(f,g,P) = \operatorname{sign}(f,g,P)$.

Proof. The alert reader will doubtlessly notice that the proof of the proposition mostly recycles ideas from the proofs of the Transverse Neighborhood Proposition 220.14 and the Sign-Q_i-Lemma 220.15.

Let $P \in |f| \cap |g|$ be a transverse intersection point. By definition this means that $f^{-1}(P)$ contains a single point x_0, that $g^{-1}(P)$ contains a single point y_0, that $x_0 \in X \setminus \partial X$ and $y_0 \in Y \setminus \partial Y$ and that $f_*(\mathrm{T}_{x_0}X) + g_*(\mathrm{T}_{y_0}Y) = \mathrm{T}_P M$. Note that the last condition, together with our information on the dimensions of the various smooth manifolds, implies that $f_*\colon \mathrm{T}_{x_0}X \to \mathrm{T}_P M$ and $g_*\colon \mathrm{T}_{y_0}Y \to \mathrm{T}_P N$ are both monomorphisms.

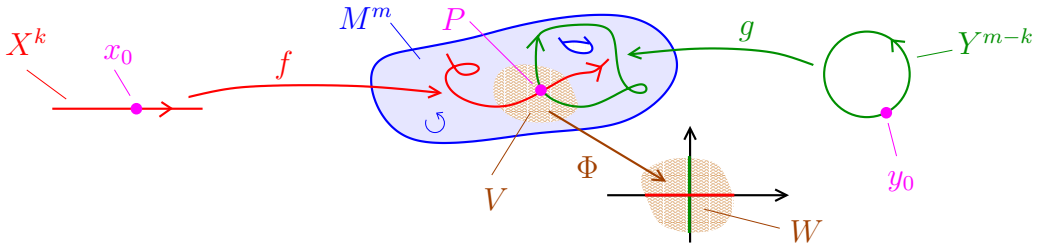
We need to show that P is a nice intersection point of f and g, in the sense of Definition 221.7. Thus let U be an open neighborhood of P. It follows from the Local Smooth Embedding Theorem 30.13 (using the fact that $f_*\colon \mathrm{T}_{x_0}X \to \mathrm{T}_P M$ is a monomorphism) that there exists an open neighborhood A of $x_0 \in X$ such that the restriction of f to $f\colon A \to M$ is a smooth embedding and such that $f(A) \subseteq U$. Furthermore, using the hypothesis that $f^{-1}(P) = \{x_0\}$ and using Exercise 23.17 we see that, after possibly shrinking A, we can assume that $f^{-1}(f(A)) = A$.

The same way we can evidently find an open neighborhood B of $y_0 \in Y$ such that $g\colon B \to M$ is a smooth embedding, such that $g(B) \subseteq U$ and such that $g^{-1}(g(B)) = B$.

By the Smooth Embedding Theorem 30.15 we know that $f(A)$ and $g(B)$ are submanifolds of M. By our hypothesis we know that P is a transverse intersection point of $f(A)$ and $g(B)$. It follows from the Submanifold Transversal Intersection Theorem 50.4 that

there exists a chart $\Phi\colon V \to W$ for M around P with

$$\Phi(V \cap f(X)) \;=\; W \cap (\mathbb{R}^k \times \{0\}) \quad \text{and} \quad \Phi(V \cap g(Y)) \;=\; W \cap (\{0\} \times \mathbb{R}^{m-k}).$$



Next note that after possibly rescaling W we can assume that $\overline{B}^k \times \overline{B}^{m-k} \subseteq W$. We set $Z := \Phi^{-1}(\overline{B}^k \times \overline{B}^{m-k})$ and $\widetilde{X} := f^{-1}(Z)$ and $\widetilde{Y} := g^{-1}(Z)$. Note that by construction the maps $\Phi \circ f\colon \widetilde{X} \to \overline{B}^k \times \{0\}$ and $\Phi \circ g\colon \widetilde{Y} \to \{0\} \times \overline{B}^{m-k}$ are diffeomorphisms. It follows easily from the construction that Z has all the properties stated in the definition of a nice intersection point. This concludes the proof of Statement (1).

We turn to the proof of Statement (2). Thus we now assume that M, X and Y are oriented. We set $\sigma := \operatorname{sign}(f, g, P)$. We make the following preparations:

(a) We equip each $\overline{B}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(b) After possibly composing $\Phi\colon Z \to \overline{B}^k \times \overline{B}^{m-k}$ with reflections in some $(x_i = 0)$-hyperplanes we can assume that the diffeomorphisms $\Phi \circ f\colon f^{-1}(Z) = \widetilde{X} \to \overline{B}^k \times \{0\}$ and $\Phi \circ g\colon g^{-1}(Z) = \widetilde{Y} \to \{0\} \times \overline{B}^{m-k}$ are both orientation-preserving.

(c) We equip the topological manifold $\overline{B}^k \times \overline{B}^{m-k}$ with the product orientation from Definition 192.8. It follows from the Product Orientation Proposition 192.7 (2) that with this orientation the inclusion map $\overline{B}^k \times \overline{B}^{m-k} \to \mathbb{R}^m$ is orientation-preserving.

(d) Elementary linear algebra shows that (b) implies that $\Phi\colon Z \to \overline{B}^k \times \overline{B}^{m-k}$ is orientation-preserving if and only if $\sigma = +1$.

(e) It follows from the Orientation Preserving–Characterisation Proposition 196.6 and the Orientations–Preserved Lemma 191.26 that $\Phi_*([Z]) = \sigma \cdot [\overline{B}^k \times \overline{B}^{m-k}]$.

Now we perform the following calculation:

$$\overbrace{\big\langle \mathrm{PD}_Z(f_*[\widetilde{X}]) \cup \mathrm{PD}_Z(g_*[\widetilde{Y}]), [Z] \big\rangle}^{=\operatorname{index}(f,g,P),\ \text{by Definition 221.9}} \;\overset{\text{by the Kronecker Pairing Natural Lemma 199.9 (2) and the Cup Product-Naturality Lemma 203.9}}{=}\; \big\langle \Phi^*(\mathrm{PD}_Z(f_*[\widetilde{X}])) \cup \Phi^*(\mathrm{PD}_Z(g_*[\widetilde{Y}])), \Phi_*([Z]) \big\rangle$$

$$= \big\langle \sigma \cdot \mathrm{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([\overline{B}^k \times 0]) \cup \sigma \cdot \mathrm{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([0 \times \overline{B}^{m-k}])), \sigma \cdot [\overline{B}^k \times \overline{B}^{m-k}] \big\rangle$$

$$\overset{\text{by (d) and the pretty elementary Exercise 210.27}}{=} \sigma^3 \cdot \underbrace{\big\langle \mathrm{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([\overline{B}^k \times 0]) \cup \mathrm{PD}_{\overline{B}^k \times \overline{B}^{m-k}}([0 \times \overline{B}^{m-k}])), [\overline{B}^k \times \overline{B}^{m-k}] \big\rangle}_{=1 \text{ by the Poincaré Duality-Products Proposition 210.3}} = \underbrace{\operatorname{sign}(f, g, P)}_{=\sigma}.$$

■

The following is an immediate corollary to the Index-equals-Sign Proposition 221.14.

Corollary 221.15. (Algebraic Intersection Numbers Agree Corollary) Let M be an oriented m-dimensional smooth manifold, let X be an oriented k-dimensional submanifold and let Y be an oriented $(m-k)$-dimensional submanifold of M. We denote by

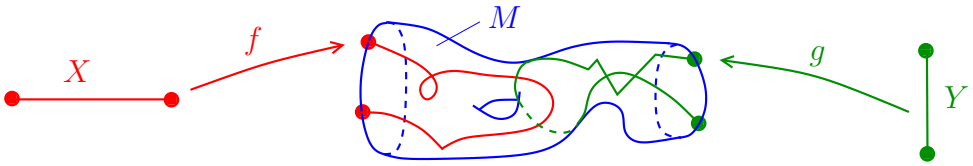
$i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. If the submanifolds are transverse in the sense of the definition on pages Definition 50.3 and Definition 220.5, then the maps $i\colon X \to M$ and $j\colon Y \to M$ are intersect nicely and the two definitions of the algebraic intersection numbers given on Definition 220.5 and Definition 221.12 agree.

By the Algebraic Intersection Numbers Agree Corollary 221.15 the following theorem is a generalization of the Cup Product–Intersection Number Theorem 220.10.

Theorem 221.16. (Maps-Cup Product–Intersection Number Theorem) Let M be a compact oriented connected m-dimensional topological manifold. Furthermore let A and B be two compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and with $\partial M = A \sqcup B$. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be two continuous maps with $f(\partial X) \subseteq A$ and $g(\partial Y) \subseteq B$. If f and g intersect nicely, then

$$f \cdot g = \Big\langle \overbrace{\mathrm{PD}_M(\underbrace{f_*([X])}_{\in \mathrm{H}_k(M,A)}) \cup \mathrm{PD}_M(\underbrace{g_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}^{\in \mathrm{H}^m(M,\partial M;\mathbb{Z})}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$

$\in \mathrm{H}^{m-k}(M,B;\mathbb{Z})$ $\quad$ $\in \mathrm{H}^k(M,A;\mathbb{Z})$



In the ∂-Cycles–Cup Product-Intersection Number Theorem 221.26 below we will give a result which takes more time to formulate and which implies the above Maps-Cup Product–Intersection Number Theorem 221.16. To keep at least the flow of the definitions and statements coherent we decided to formulate the Maps-Cup Product–Intersection Number Theorem 221.16 in this section. But rather awkwardly this means that we now provide a proof of the Maps-Cup Product–Intersection Number Theorem 221.16 that relies on the future ∂-Cycles–Cup Product-Intersection Number Theorem 221.26.

Proof of Theorem 221.16 using Theorem 221.26. It follows easily from the Chop Fundamental Class Lemma 221.24 (formulated below) that there exists a k-cycle $\sigma \in \mathrm{C}_k(M, A)$ and an $(m-k)$-cycle $\tau \in \mathrm{C}_{m-k}(M, B)$ with the following properties:

(1) We have $[\sigma] = f_*([X])$ and $[\tau] = g_*([Y])$.
(2) The cycles σ and τ are transverse.
(3) The equality $f \cdot g = \sigma \cdot \tau$ holds.

It follows that

$$f \cdot g \;=\; \sigma \cdot \tau \;\underset{\uparrow}{=}\; \langle \mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]), [M] \rangle \;\underset{\uparrow}{=}\; \langle \mathrm{PD}_M(f_*[X]) \cup \mathrm{PD}_M(g_*[Y]), [M] \rangle.$$

by the ∂-Cycles–Cup Product Intersection Number Theorem 221.26

since $[\sigma] = f_*([X])$ and since $[\tau] = g_*([Y])$

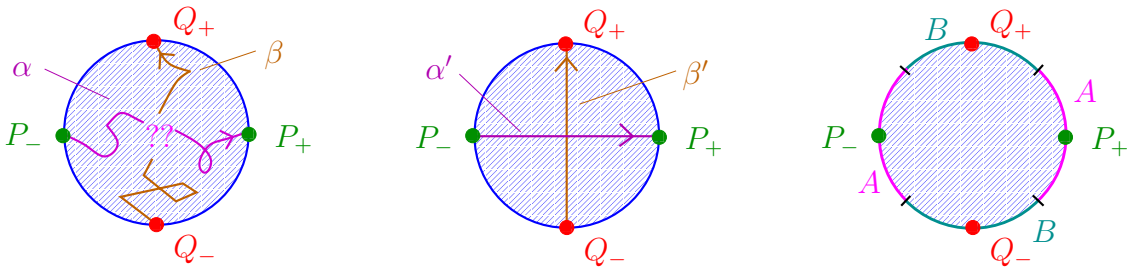
■

As promised, we now find it quite easy to answer Question 221.1 in the negative.

Lemma 221.17. (Cross Intersection Lemma) We consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm(0, \pm 1)$. If $\alpha\colon [-1,1] \to \overline{B}^2$ is a continuous map with $\alpha(\pm 1) = P_\pm$ and if $\beta\colon [-1,1] \to \overline{B}^2$ is a continuous map with $\beta(\pm 1) = Q_\pm$, then α and β intersect in at least one point.

Proof. We write $I = [-1,1]$. Suppose that there exist continuous maps $\alpha\colon I \to \overline{B}^2$ and $\beta\colon I \to \overline{B}^2$ with $\alpha(\pm 1) = P_\pm$ and with $\beta(\pm 1) = Q_\pm$ such that $|\alpha| \cap |\beta| = \varnothing$. We introduce the following notation:

(1) We consider the "straight paths" $\alpha'\colon I \to \overline{B}^2$ given by $\alpha'(t) = (t, 0)$ and $\beta'\colon I \to \overline{B}^2$ given by $\beta'(t) = (0, t)$.
(2) We set $A = \{\exp(it) \,|\, t \in [-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}]\}$ and $B = \{\exp(it) \,|\, t \in [-\frac{3\pi}{4}, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{3\pi}{4}]\}$.
(3) We denote by $\mathrm{PD}_{\overline{B}^2}$ the isomorphisms from the Poincaré Duality Theorem 210.1 corresponding to the decomposition $\partial\overline{B}^2 = A \cup B$.



We perform the following calculation:

$$\begin{array}{ccccccccc} & & \text{by the Maps-Cup Product–Intersection Number} & & & & & & \\ \text{since } |\alpha| \cap |\beta| = \varnothing & & \text{Theorem 221.16} & & & & & & \\ & \downarrow & & \downarrow & & & & & \\ 0 & = & \alpha \cdot \beta & = & \left\langle \mathrm{PD}_{\overline{B}^2}(\alpha_*([I])) \cup \mathrm{PD}_{\overline{B}^2}(\beta_*([I]), [\overline{B}^2] \right\rangle & & & & \\ & & & = & \left\langle \mathrm{PD}_{\overline{B}^2}(\alpha'_*([I])) \cup \mathrm{PD}_{\overline{B}^2}(\beta'_*([I]), [\overline{B}^2] \right\rangle & = & \alpha' \cdot \beta' & = & \pm 1. \\ & & & \uparrow & & & \uparrow & & \uparrow \\ & & \text{by considering the long exact sequences in homology} & & \text{by the Maps-Cup Product} & & \alpha', \beta' \text{ intersect} & & \\ & & \text{of the pairs } (\overline{B}^2, A) \text{ and } (\overline{B}^2, B) \text{ we see that} & & \text{Intersection Number} & & \text{nicely in one} & & \\ & & \alpha_*([I]) = \alpha'_*([I]) \text{ and } \beta_*([I]) = \beta'_*([I]) & & \text{Theorem 221.16} & & \text{point} & & \end{array}$$

We have thus obtained a contradiction. ∎

221.3. **Intersections of cycles.** Now we turn to the study of singular chains in a topological manifold. To be on the safe side in our subsequent definitions we introduce the following convention.

Convention 221.18.

(1) Let X be a topological space. When we write "let $\sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r$ be a singular m-chain in X" it is understood that the σ_i are pairwise different and that all the a_i are non-zero. We write $|\sigma| := |\sigma_1| \cup \cdots \cup |\sigma_r|$.
(2) Let M be an oriented m-dimensional topological manifold M and let $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ be singular simplices. Since each Δ^n is in particular a topological manifold it makes sense to say that the singular simplices intersect nicely. Furthermore, throughout this chapter we equip the topological manifold Δ^n with the orientation

given by the discussion on Definition 143.8 together with the Smooth-Topological Orientation Proposition 191.25. In particular, if $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ intersect nicely, then we obtain an algebraic intersection number $\sigma \cdot \tau$.

Definition 221.19. Let M be an m-dimensional topological manifold and let $\sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r$ be a singular k-chain and let $\tau = b_1 \cdot \tau_1 + \cdots + b_s \cdot \tau_s$ be a singular $(m-k)$-chain in M. We say that σ and τ intersect nicely if the following two conditions are satisfied:

(1) For all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$ the maps $\sigma_i\colon \Delta^k \to M$ and $\tau_j\colon \Delta^{m-k} \to M$ intersect nicely.

(2) For all $(i, j) \neq (i', j') \in \{1, \ldots, r\} \times \{1, \ldots, s\}$ we have

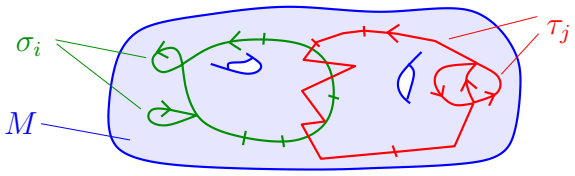
$$(|\sigma_i| \cap |\tau_j|) \cap (|\sigma_{i'}| \cap |\tau_{j'}|) \;=\; \varnothing.$$

Put differently, any intersection point is the intersection point of a unique pair of singular simplices.

If this is the case and if M is oriented, then we define the algebraic intersection number of σ and τ to be

$$\sigma \cdot \tau \;\coloneqq\; \sum_{i,j} (a_i \cdot b_j) \cdot (\sigma_i \cdot \tau_j).$$

($\sigma_i \cdot \tau_j$: algebraic intersection number of $\sigma_i\colon \Delta^k \to M$ and $\tau_j\colon \Delta^{m-k} \to M$)



Before we can state the main theorem of this chapter we recall the formulation of the Poincaré Duality theorem for (not necessarily compact) topological manifolds with empty boundary. Thus let W be an oriented m-dimensional topological manifold with empty boundary and let $K \subseteq W$ be a compact subset. By the Homology-Section Theorem 193.6 there exists a unique lift of the orientation classes of W to a class in $\mathrm{H}_m(W, W \setminus K)$. Given any $k \in \mathbb{N}_0$ we denote by

$$- \cap W\colon \mathrm{H}^k(W, W \setminus K; \mathbb{Z}) \;\to\; \mathrm{H}_{m-k}(W)$$

the map that is given by capping with this unique lift. The Cap–Non Compact Manifold Lemma 211.3 (2) says that these maps define a well-defined map

$$- \cap W\colon \mathrm{H}^k_{\mathrm{c}}(W; \mathbb{Z}) \;\to\; \mathrm{H}_{m-k}(W).$$

Furthermore the Poincaré Duality the Poincaré Duality–Non-Compact Theorem 211.4 says that this map is an isomorphism. We denote by

$$\mathrm{PD}_W\colon \mathrm{H}_{m-k}(W) \;\to\; \mathrm{H}^k_{\mathrm{c}}(W; \mathbb{Z})$$

the inverse map. Finally we remind the reader that given a topological space X we denote by $\epsilon_X = \epsilon\colon \mathrm{H}_0(X) \to \mathbb{Z}$ the augmentation map.

Now we can formulate the main theorem of this chapter.

Theorem 221.20. (Cycles–Cup Product-Intersection Number Theorem) Let M be an oriented m-dimensional topological manifold with empty boundary. Let $\sigma \in \mathrm{C}_k(M)$ and $\tau \in \mathrm{C}_{m-k}(M)$ be two cycles. If σ and τ intersect nicely, then[2387]

$$\sigma \bullet \tau \;=\; \epsilon_M\Big(\big(\overbrace{\mathrm{PD}_M([\sigma])}^{\in \mathrm{H}_{\mathrm{c}}^{m-k}(M;\mathbb{Z})} \cup \overbrace{\mathrm{PD}_M([\tau])}^{\in \mathrm{H}_{\mathrm{c}}^{k}(M;\mathbb{Z})}\big) \cap M\Big).$$
$$\in \mathrm{H}_{\mathrm{c}}^{m}(M;\mathbb{Z})$$

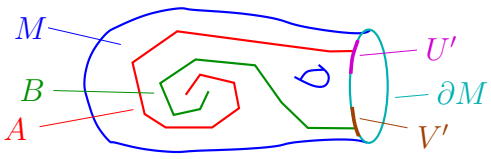
221.4. **Proof of the Cycles–Cup Product-Intersection Number Theorem 221.20 .** Before we can provide the proof of the Cycles–Cup Product-Intersection Number Theorem 221.20 we need to recall some definitions and we need to prove several lemmas. First let us recall Definition 3.48:

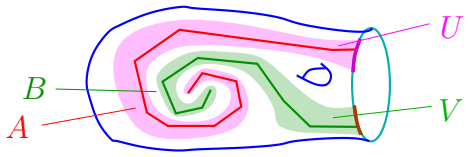
Definition 221.21. Let X be a topological space.

(1) Let A and B be disjoint subsets of X. We say that U and V are separating open neighborhoods for A and B if U and V are disjoint, if U is an open neighborhood of A and if V is an open neighborhood of B.
(2) A topological space X is called normal if any two disjoint closed subsets A and B admit separating open neighborhoods.

Proposition 221.22. (Topological Manifold–Normal Proposition)

(1) Every topological manifold is normal.
(2) Let M be a topological manifold and let A and B be two disjoint closed subsets of M. If $U' \subseteq \partial M$ and $V' \subseteq \partial M$ are separating open neighborhoods for $A \cap \partial M$ and $B \cap \partial M$, then there exist separating open neighborhoods U and V of A and B such that $U \cap \partial M = U'$ and $V \cap \partial M = V'$.





Sketch of proof.

(1) We proved this statement in the Manifold Paracompact-Normal Proposition 23.9 (2).
(2) Let M be a topological manifold and let A and B be two disjoint closed subsets of M. We pick a collar neighborhood $\partial M \times [0,1]$ as given by the Topological Collar Neighborhood Theorem 35.14.

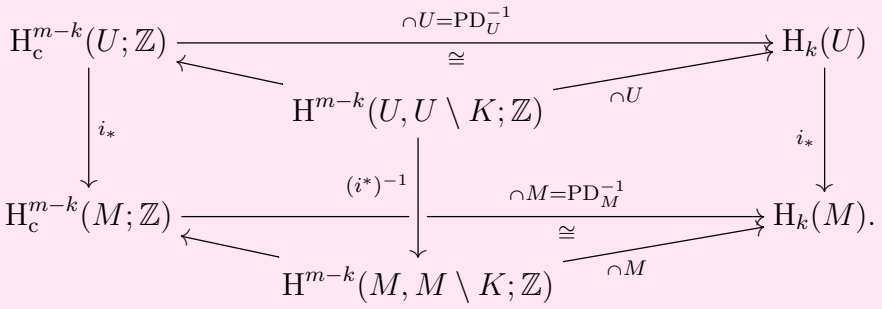
Suppose that we are given separating neighborhoods $U' \subseteq \partial M$ and $V' \subseteq \partial M$ for $A \cap \partial M$ and $B \cap \partial M$. By the Topological Manifold–Boundary Proposition 23.15 (2) the complement $M \setminus \partial M$ is again a topological manifold. By (1) it is normal. Thus there exist separating open neighborhoods $\widetilde{U}$ and $\widetilde{V}$ for the closed subsets $A \cap (M \setminus \partial M)$ and $B \cap (M \setminus \partial M)$. It follows from the Topological Manifold–Boundary Proposition 23.15 (2) that $\widetilde{U}$ and $\widetilde{V}$ are open subsets of M. It is now straightforward to verify that $U := \widetilde{U} \cup ([0,1) \times U')$ and $V := \widetilde{V} \cup ([0,1) \times V')$ have all the desired properties. ■

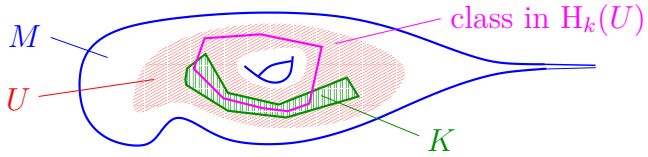
[2387] We refer to Definition 209.19 for the definition of the cup product on cohomology with compact support.

Lemma 221.23. (Open Submanifold–Poincaré Duality Lemma) Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$ and let $U \subseteq M$ be an open subset. Let $i\colon U \to M$ be the inclusion map. For any compact subset $K \subseteq U$ and any $k \in \mathbb{N}_0$ the map

$$i^*\colon \mathrm{H}^{m-k}(M, M \setminus K; \mathbb{Z}) \to \mathrm{H}^{m-k}(U, U \setminus K; \mathbb{Z})$$

is an isomorphism and the following diagram commutes:





Proof. We make the following comments and observations:

(i) It follows from the Excision-in-H*-Theorem 200.1 (1b) that the induced homomorphism $i^*\colon \mathrm{H}^{m-k}(M, M \setminus K; \mathbb{Z}) \to \mathrm{H}^{m-k}(U, U \setminus K; \mathbb{Z})$ is an isomorphism.

(ii) The triangles on the top and the bottom of the diagram commute by Definition 210.2 of PD_U and PD_M, see the Cap–Non Compact Manifold Lemma 211.3 (2).

(iii) It follows from the definition of the map $i_*\colon \mathrm{H}_c^{m-k}(U;\mathbb{Z}) \to \mathrm{H}_c^{m-k}(M)$ on page Definition 209.16 that the quadrilateral to the left commutes.

(iv) First note that it follows from the functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 193.6 that $i_*(\alpha_K^U) = \alpha_K^M$. It thus follows from the Cap Product-Naturality Lemma 206.13 that the quadrilateral to the right commutes.

(v) The rectangle in the back commutes by the Poincaré Duality–Open Subset Lemma 211.6. ■

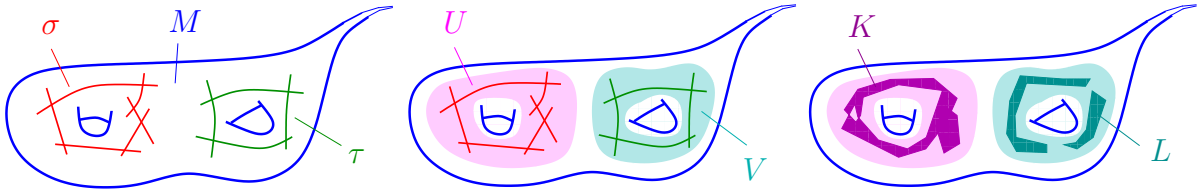
As in the case of the Cup Product–Intersection Number Theorem 220.10 it is perhaps instructive to first prove the Cycles–Cup Product-Intersection Number Theorem 221.20 in the special case that the cycles do not intersect.

Proof of Theorem 221.20 for disjoint cycles. Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$. Let $\sigma \in \mathrm{C}_k(M)$ and $\tau \in \mathrm{C}_{m-k}(M)$ be two cycles such that $|\sigma| \cap |\tau| = \varnothing$. We have to show that $\mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]) = 0$.

By the Topological Manifold–Normal Proposition 221.22 we can find separating open neighborhoods U and V for $|\sigma|$ and $|\tau|$. Given a compact subset J of M and given $i \in \mathbb{N}_0$ we denote by $\varphi_J\colon \mathrm{H}^i(M, M \setminus J; \mathbb{Z}) \to \mathrm{H}_c^i(M;\mathbb{Z})$ the natural map.

Claim.

(1) There exists a compact subset $K \subseteq U$ and a cohomology class $\alpha \in \mathrm{H}^{m-k}(M, M \setminus K; \mathbb{Z})$ such that $\varphi_K(\alpha) = \mathrm{PD}_M([\sigma])$.
(2) There exists a compact subset $L \subseteq V$ and a cohomology class $\beta \in \mathrm{H}^k(M, M \setminus L; \mathbb{Z})$ such that $\varphi_L(\beta) = \mathrm{PD}_M([\tau])$.



Proof. We consider $\mathrm{PD}_U([\sigma]) \in \mathrm{H}_c^{m-k}(U; \mathbb{Z})$. By definition of cohomology with compact support there exists a compact subset K of U such that $\mathrm{PD}_U([\sigma])$ lies in the image of the natural map $\mathrm{H}^{m-k}(U, U \setminus K; \mathbb{Z}) \to \mathrm{H}_c^{m-k}(U; \mathbb{Z})$. Statement (1) of the claim now follows from the Open Submanifold–Poincaré Duality Lemma 221.23. The proof of Statement (2) is evidently basically the same. ⊞

We consider the following diagram:

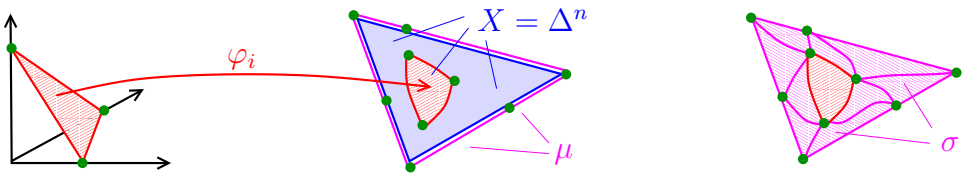
$$\begin{array}{ccc} \mathrm{H}_c^{m-k}(M;\mathbb{Z}) \times \mathrm{H}_c^{k}(M;\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}_c^{m}(M;\mathbb{Z}) \\ \uparrow{\scriptstyle \varphi_K \times \varphi_L} & & \uparrow{\scriptstyle \varphi_M} \\ \mathrm{H}^{m-k}(M, M\setminus K;\mathbb{Z}) \times \mathrm{H}^{k}(M, M\setminus L;\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{m}(M, \underbrace{(M\setminus K)\cup(M\setminus L)}_{=M};\mathbb{Z}) \end{array} \qquad (\mathrm{PD}_M(\alpha), \mathrm{PD}_M(\beta)) \updownarrow (\alpha, \beta)$$

The diagram commutes by the definition of the cup product on cohomology with compact support, see Definition 209.19. It follows from the observation that the lower right group is zero that the cup product of $\mathrm{PD}_M([\sigma])$ and $\mathrm{PD}_M([\tau])$ is zero. ■

Lemma 221.24. (Chop Fundamental Class Lemma) Let X be a compact oriented, n-dimensional topological manifold and let $\varphi_i \colon \Delta^n \to X \setminus \partial X$, $i = 1, \ldots, r$, be orientation-preserving embeddings such that the images are disjoint. In the following we write $W := X \setminus (\varphi_1(\mathring{\Delta}^n) \cup \cdots \cup \varphi_r(\mathring{\Delta}^n))$. Given any representative $\mu \in \mathrm{C}_{n-1}(\partial X)$ of the fundamental class of ∂X there exists a singular n-chain $\sigma \in \mathrm{C}_n(W)$ such that

$$\partial_n(\sigma + \varphi_1 + \cdots + \varphi_r) = \mu \quad \text{and with} \quad [\sigma + \varphi_1 + \cdots + \varphi_r] = [X].$$

Example. In the proof below of the Cycles–Cup Product-Intersection Number Theorem 221.20 we will apply the Chop Fundamental Class Lemma 221.24 to the case that $X = \Delta^n$ itself is the standard n-simplex.



This particular situation is illustrated in the figure below. □

Proof. By the Codimension-Zero Submanifold Proposition 150.16 we know that W is a compact n-dimensional topological manifold with $\partial W = \varphi_1(\partial\Delta^n) \cup \cdots \cup \varphi_r(\partial\Delta^n) \cup \partial X$. Since the φ_i are orientation-preserving we obtain from the Fundamental Class-of-∂-Proposition 194.13 that $-\partial_n(\varphi_1) - \cdots - \partial_n(\varphi_r) + \mu \in \mathrm{C}_{n-1}(\partial W)$ represents the fundamental class of ∂W. Again by the Fundamental Class-of-∂-Proposition 194.13 we know that there exists a singular n-chain $\sigma \in \mathrm{C}_n(W)$ with $\partial_n(\sigma) = -\partial_n(\varphi_1) - \cdots - \partial_n(\varphi_r) + \mu$. It follows from the Fundamental Class-Criterion 185.7 that $\sigma + \varphi_1 + \cdots + \varphi_r$ represents the fundamental class of X. ■

The formulation of the following, rather technical lemma, might initially be somewhat intimidating. We hope that the figure below helps somewhat with keeping track of the various objects. Also note that the lemma can be viewed as a cousin of first degree to the Small H_*-to-Large H^*-PD Lemma 212.8.

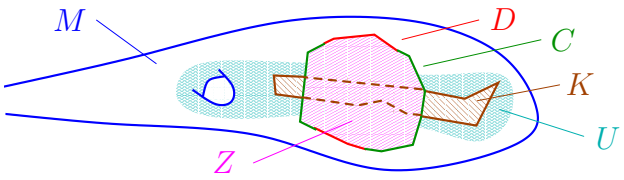
Lemma 221.25. (Subspace–Poincaré Duality Lemma) Let M be an oriented m-dimensional topological manifold and let Z be a compact codimension-zero submanifold of M. Furthermore let C and D be two compact codimension-zero submanifolds of ∂Z with $C \cup D = \partial Z$ and with $\partial C = \partial D = C \cap D$.[2388] Let $U \subseteq M$ be an open subset with $U \cap \partial Z \subseteq C$ and let $K \subseteq U$ be a compact subset. We consider the three inclusion maps $p\colon (U, \varnothing) \to (U, U \setminus \mathring{Z})$, $g\colon (Z \cap U, C \cap U) \to (Z, C)$ and $z\colon (Z \cap U, C \cap U) \to (U, U \setminus \mathring{Z})$. We note that the induced map

$$z_*\colon \mathrm{H}_k(Z \cap U, C \cap U) \;\to\; \mathrm{H}_k(U, U \setminus \mathring{Z})$$

is an isomorphism by the Manifold–Excision Theorem 148.3. With this preparation we can now formulate the conclusion of the lemma. Namely, the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathrm{H}_k(M) & \xleftarrow{\cap M} & \mathrm{H}^{m-k}(M, M\setminus K; \mathbb{Z}) \\
 & \nearrow & & \swarrow & \\
\mathrm{H}_k(U) & \xleftarrow{\cap U} & \mathrm{H}^{m-k}(U, U\setminus K; \mathbb{Z}) & & \downarrow \\
\downarrow{\scriptstyle p_*} & & & & \\
\mathrm{H}_k(U, U\setminus \mathring{Z}) & \xrightarrow{(z_*)^{-1}} & \mathrm{H}_k(Z\cap U, C\cap U) & \xrightarrow{g_*} & \mathrm{H}_k(Z, C) \xleftarrow{\cap [Z]} \mathrm{H}^{m-k}(Z, D; \mathbb{Z}).
\end{array}$$

Here all the unmarked maps are induced by the obvious inclusion maps of (pairs) of topological spaces. Also note that the bottom cap product $\cap[Z]\colon \mathrm{H}^{m-k}(Z, D; \mathbb{Z}) \to \mathrm{H}_k(Z, C)$ is defined since our hypothesis on C and D together with the Excisive Triad-Proposition 158.20 (2) imply that (Z, C, D) is an excisive triad.



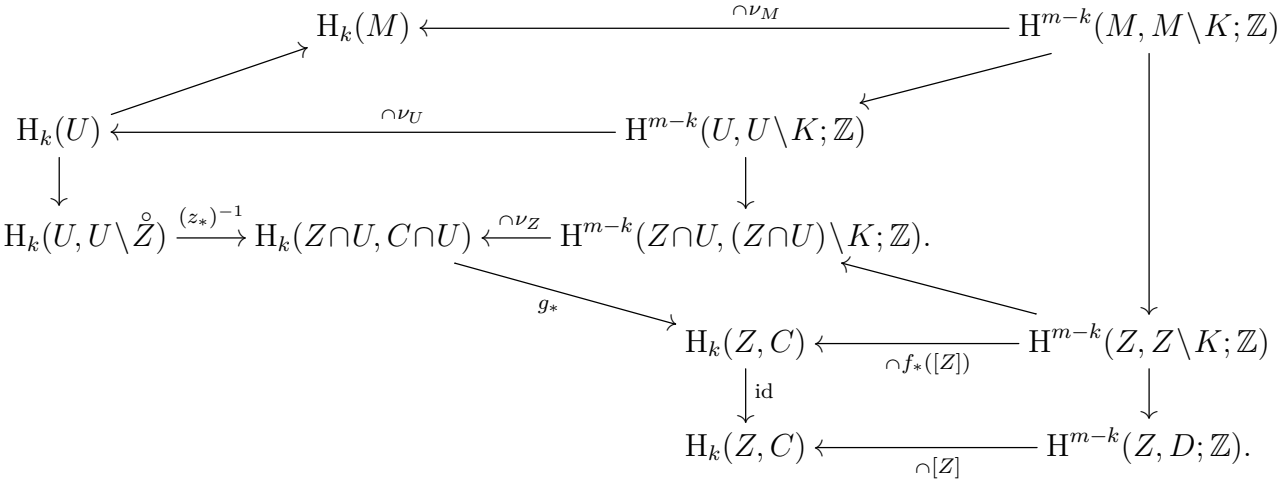
[2388] In fact in the proof we only need that we have a decomposition $\partial Z = C \cup D$ such that (Z, C, D) is an excisive triad.

Proof. We make the following preparations:

(a) We pick a representative $\nu_M \in \mathrm{H}_m(M, M \setminus K)$ for the lift of the orientation of M and similarly we pick $\nu_U \in \mathrm{H}_m(U, U \setminus K)$.

(b) We now consider the two obvious inclusion maps $f\colon (Z, \partial Z) \to (Z, \overbrace{Z \setminus (\mathring{Z} \cap K)}^{=(Z\setminus K)\cup C})$ and $\widetilde{g}\colon (Z \cap U, (Z \cap U) \setminus (\mathring{Z} \cap K)) \to (Z, Z \setminus (\mathring{Z} \cap K))$. It follows from the Excision Theorem 148.2 that the map $\widetilde{g}_*\colon \mathrm{H}_m(Z \cap U, (Z \cap U) \setminus (\mathring{Z} \cap K)) \to \mathrm{H}_m(Z, Z \setminus (\mathring{Z} \cap K))$ is an isomorphism. We write $\nu_Z \coloneqq g_*^{-1}(f_*([Z]))$.

Next we consider the following expanded version of the diagram shown in the lemma:



It suffices to show that all the five regions of the diagram commute:

(1) The parallelogram on the top of the diagram commutes by the Cap Product-Naturality Lemma 206.13 applied to the map $h\colon (U, U \setminus K, \varnothing) \to (M, M \setminus K, \varnothing)$, together with the observation that the Codimension Zero-Fundamental Class Lemma 194.10 implies that $h_*(\nu_U) = \nu_M$.

(2) The 4-gon on the right commutes since all maps are induced by inclusions.

(3) For the lower parallelogram we obtain commutativity from applying the Cap Product-Naturality Lemma 206.13 to the map[2389] $g\colon (Z \cap U, Z \cap U \setminus K, C \cap U) \to (Z, Z \setminus K, C)$ that is given by the inclusion.

(4) To show commutativity of the lower rectangle we apply the Cap Product-Naturality Lemma 206.13 to the inclusion map $f\colon (Z, D, C) \to (Z, Z \setminus K, C)$.

Finally we turn to the upper rectangle. We need to show that the following rectangle commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(U) & & \xleftarrow{\quad\cap\nu_U\quad} & & \mathrm{H}^{m-k}(U, U\setminus K;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle z^*} \\ \mathrm{H}_k(U, U\setminus \mathring{Z}) & \xleftarrow{z_*} & \mathrm{H}_k(Z\cap U, C\cap U) & \xleftarrow{\cap\nu_Z} & \mathrm{H}^{m-k}(Z\cap U, (Z\cap U)\setminus K;\mathbb{Z}). \end{array}$$

[2389] Here we use the symbol for this map of triples of topological spaces. We feel that recycling the name g is justified since this map of triples "contains" our original map $g\colon (Z \cap U, C \cap U) \to (Z, C)$ of pairs of topological spaces.

Now let $\beta \in \mathrm{H}^{m-k}(U, U \setminus K; \mathbb{Z})$. We have the following equalities in $\mathrm{H}_k(U, U \setminus \mathring{Z})$:

$$\begin{array}{ccccc} & & \text{it follows from the functoriality of homology groups and the uniqueness statement of the Boundary Homology–Section Theorem 193.20 that we have } p_*(\nu_U) = z_*(\nu_Z) \text{ in } \mathrm{H}_m(U, U\setminus K \cup U\setminus \mathring{Z}) & & \\ & & \downarrow & & \\ p_*(\beta \cap \nu_U) & = & \beta \cap p_*(\nu_U) & = & \beta \cap z_*(\nu_Z) & = & z_*(z^*(\beta) \cap \nu_Z). \\ & \uparrow & & & \uparrow & & \\ \text{by the Cap Product-Naturality Lemma 206.13 applied to the map } p\colon (U, U\setminus K, \varnothing) \to (U, U\setminus K, U\setminus \mathring{Z}) & & & & \text{by the Cap Product-Naturality Lemma 206.13 applied to the map } z\colon (Z\cap U, Z\cap U\setminus K, C\cap U) \to (U, U\setminus K, U\setminus \mathring{Z}) & & \end{array}$$

We have thus shown that the diagram commutes. ■

After all these preparations we can finally attack the Cycles–Cup Product-Intersection Number Theorem 221.20.

Proof of Theorem 221.20. In the following let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$. Furthermore let $\sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r \in \mathrm{C}_k(M)$ and $\tau = b_1 \cdot \tau_1 + \cdots + b_s \cdot \tau_s \in \mathrm{C}_{m-k}(M)$ be two singular cycles that intersect nicely.

Claim 1. We can assume that we are in the following setting: There exists an $n \in \mathbb{N}_0$, there exist distinct points $P_1, \ldots, P_n \in M$ and there exist disjoint codimension-zero submanifolds $Z_1, \ldots, Z_n$ of M such that the following hold:

(1) For $i \neq j$ we have $|\sigma_i| \cap |\tau_j| = \varnothing$.
(2) For each $i \in \{1, \ldots, n\}$ we have $|\sigma_i| \cap |\tau_i| = \{P_i\}$.
(3) For $i = n+1, \ldots, \min\{r, s\}$ we have $|\sigma_i| \cap |\tau_i| = \varnothing$.
(4) For $i \in \{1, \ldots, n\}$ the following hold:[2390]
  (a) P_i is contained in $Z_i \setminus \partial Z_i$.
  (b) Z_i is homeomorphic to $\overline{B}^m$.
  (c) The map $\sigma_i\colon \Delta^k \to M$ is a proper embedding $\sigma_i\colon \Delta^k \to Z_i$ such that $X_i := \sigma_i(\Delta^k)$ is a well-behaved submanifold of Z_i. By the definition, see Definition 221.4 this means that the submanifold $\partial X_i = \sigma_i(\partial \Delta^k)$ of ∂Z_i admits a regular neighborhood A_i.
  (d) Similarly to (c) the map $\tau_i\colon \Delta^{m-k} \to M$ is a proper embedding $\tau_i\colon \Delta^{m-k} \to Z_i$ such that $Y_i := \tau_i(\Delta^{m-k})$ is a well-behaved submanifold of Z_i. This means that the submanifold $\partial Y_i = \tau_i(\partial \Delta^k)$ of ∂Z_i admits a regular neighborhood B_i.
  (e) We have $A_i \cap B_i = \varnothing$.
  (f) For $j \neq i$ we have $|\sigma_j| \cap Z_i = \varnothing$ and $|\tau_j| \cap Z_i = \varnothing$.

Proof. The proof of the claim is built out of the following ingredients:

(i) The fact, established in Exercise 23.17, that given any continuous map $f\colon \Delta^l \to M$ and given any $P \in M \setminus f(\Delta^l)$ there exists an open neighborhood U of P with $f(\Delta^l) \cap U = \varnothing$.
(ii) The "given any neighborhood U" feature of the definition of a nice intersection point.
(iii) By the Δ^n-Homeomorphism Lemma 143.9 (2) we know that any Δ^l is homeomorphic to $\overline{B}^l$.
(iv) We use that it follows from (iii) and the definition of a nice intersection point that any $f(\widetilde{X})$ is homeomorphic to Δ^k and that any $g(\widetilde{Y})$ is homeomorphic to Δ^{m-k}. (Here we use the notation from the definition of a nice intersection point, i.e. $\widetilde{X}$ and $\widetilde{Y}$ are

[2390] These statements imply in particular that P_i is a nice intersection point of σ_i and τ_i.

introduced in Definition 221.7.) We combine this piece of information with the Chop Fundamental Class Lemma 221.24.

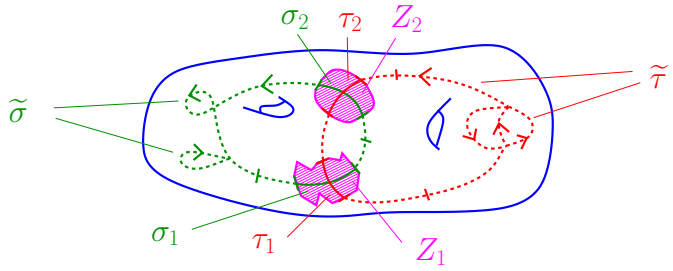
We leave it to the reader to work out the elementary albeit slightly messy details. ⊞

Next we introduce the following notation:

(5) We write $Z = Z_1 \cup \cdots \cup Z_n$.

(6) We write $\widetilde{\sigma} = a_{n+1} \cdot \sigma_{n+1} + \cdots + a_r \cdot \sigma_r$ and we denote by $|\widetilde{\sigma}|$ the union of the images of $\sigma_{n+1}, \ldots, \sigma_r$. Similarly we define $\widetilde{\tau}$ and $|\widetilde{\tau}|$.

We refer to the figure below on the left for an illustration.



Claim 2. There exist compact subsets K and L of M with the following properties:

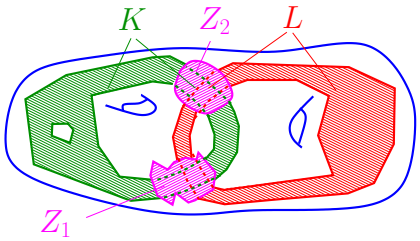
(a) There exists a cohomology class $\varphi \in \mathrm{H}^{m-k}(M, M \setminus K; \mathbb{Z})$ such that the image of φ in $\mathrm{H}_{\mathrm{c}}^{m-k}(M; \mathbb{Z})$ equals $\mathrm{PD}_M([\sigma])$.

(b) There exists a cohomology class $\psi \in \mathrm{H}^{k}(M, M \setminus L; \mathbb{Z})$ such that the image of ψ in $\mathrm{H}_{\mathrm{c}}^{k}(M; \mathbb{Z})$ equals $\mathrm{PD}_M([\tau])$.

(c) We have $K \cap L \cap (M \setminus Z) = \varnothing$.

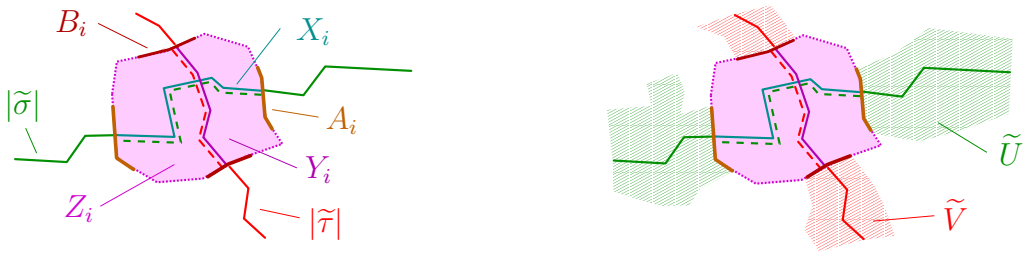
(d) For each $i \in \{1, \ldots, n\}$ we have $K \cap \partial Z_i \subseteq A_i$ and $L \cap \partial Z_i \subseteq B_i$.

We refer to the figure above on the right for an illustration.



Proof. By the Codimension-Zero Submanifold Proposition 150.16 we know that $W \coloneqq M \setminus \mathring{Z}$ is an m-dimensional topological manifold. It follows from the Topological Manifold–Normal Proposition 221.22 (2) that there exist *disjoint* open neighborhoods $\widetilde{U} \subseteq W$ of $|\widetilde{\sigma}|$ and $\widetilde{V} \subseteq W$ of $|\widetilde{\tau}|$ such that for each $i \in \{1, \ldots, n\}$ we have $\widetilde{U} \cap \partial Z_i = \mathring{A}_i$ and $\widetilde{V} \cap \partial Z_i = \mathring{B}_i$. Next we write $U = \widetilde{U} \cup \bigcup_{i=1}^{n} \mathring{Z}_i$ and $V = \widetilde{V} \cup \bigcup_{i=1}^{n} \mathring{Z}_i$. Note that it follows from the elementary Cover-Open Subset Lemma 3.20 (2) that U and V are open subsets of M. We refer to the figure below for an illustration.

We denote by $i \colon U \to M$ the inclusion map. Evidently we have $\sigma \in \mathrm{C}_k(U)$. Since U is an open subset of M we can thus consider $\mathrm{PD}_U([\sigma]) \in \mathrm{H}_{\mathrm{c}}^{m-k}(U; \mathbb{Z})$. By the Compact Support–Relative Cochain Lemma 209.7 (2b) there exists a compact subset K of U and a cohomology class $\widetilde{\varphi} \in \mathrm{H}^{m-k}(U, U \setminus K)$ such that $\mathrm{PD}_U([\sigma])$ equals the image of $\widetilde{\varphi}$ under the natural map



$\mathrm{H}^{m-k}(U, U \setminus K; \mathbb{Z}) \to \mathrm{H}_{\mathrm{c}}^{m-k}(U; \mathbb{Z})$. It follows from the Excision-in-H*-Theorem 200.1 (1b) that the map $i^*\colon \mathrm{H}^{m-k}(M, M \setminus K; \mathbb{Z}) \to \mathrm{H}^{m-k}(U, U \setminus K; \mathbb{Z})$ is an isomorphism. We set $\varphi := (i^*)^{-1}(\widetilde{\varphi})$. By the Open Submanifold–Poincaré Duality Lemma 221.23 we know that the image of φ in $\mathrm{H}_{\mathrm{c}}^{m-k}(M; \mathbb{Z})$ equals $\mathrm{PD}_M([\sigma])$.

By precisely the same argument we can also find a compact subset L of V and some cohomology class $\psi \in \mathrm{H}^k(M, M \setminus L; \mathbb{Z})$ such that the image of ψ in $\mathrm{H}_{\mathrm{c}}^k(M; \mathbb{Z})$ equals $\mathrm{PD}_M([\tau])$.

Finally note that by construction K and L, and also φ and ψ satisfy all the desired conditions set out in (a), (b), (c) and (d). ⊞

We continue with some more observations and notation.

(7) For $i = 1, \dots, n$ we write $K_i = K \cap Z_i$ and $L_i = L \cap Z_i$. Note that by construction we have $K_i \cap L_i \subseteq \mathring{Z}_i = Z_i \setminus \partial Z_i$.

(8) Given any compact subset T of M we denote by $o\colon (M, \varnothing) \to (M, M \setminus T)$ the obvious map. Also we denote by $o^*\colon \mathrm{H}^*(M, M \setminus T; \mathbb{Z}) \to \mathrm{H}_{\mathrm{c}}^*(M; \mathbb{Z})$ the obvious map.

(9) We denote by $z\colon Z \to M$ the inclusion map and we denote by z also all the inclusion maps of pairs of topological spaces where the total spaces are Z and M.

(10) Given $i \in \{1, \dots, n\}$ we denote by $z_i\colon Z_i \to Z$ the inclusion map. We adopt the same convention as in (9) for pairs of topological spaces.

(11) We view Z as an m-dimensional submanifold of M and we equip Z with the orientation coming from the Topological Manifold-Induced Orientation Lemma 192.1. Given any subset $W \subseteq Z$ with $\partial Z \subseteq W$ we denote by $[Z]$ also the image of the fundamental class of Z under the map $\mathrm{H}_m(Z, \partial Z) \to \mathrm{H}_m(Z, W)$. We adopt the same convention for the $[Z_i]$.

(12) Recall that for each $i \in \{1, \dots, n\}$ we have $K \cap \partial Z_i \subseteq A_i$ and $L \cap \partial Z_i \subseteq B_i$. This implies that we obtain the maps of pairs

$$\begin{array}{rcl} (Z_i, \partial Z_i \setminus A_i) & \to & (Z_i, Z_i \setminus K_i) \\ (Z_i, \partial Z_i \setminus B_i) & \to & (Z_i, Z_i \setminus L_i). \end{array}$$

We refer to each of these maps as the canonical map c.

(13) For each $i \in \{1, \dots, n\}$ we have maps of pairs $d\colon (Z_i, \partial Z_i \setminus A_i) \to (Z_i, \partial Z_i \setminus \partial X_i)$ and $d\colon (Z_i, \partial Z_i \setminus B_i) \to (Z_i, \partial Z_i \setminus \partial Y_i)$. Since A_i is a regular neighborhood of ∂X_i and since B_i is a regular neighborhood of ∂Y_i we obtain from the Homotopy Equivalence-H*-Proposition 198.12 (1) and the LES-of-Triple Proposition 198.13 that these maps induce isomorphisms of relative cohomology groups.

(14) By the Homology-Section Theorem 193.6 we know that there exists a unique lift $\mu \in \mathrm{H}_m(M, M \setminus (K \cap L))$ of the orientation classes of M.

With the above definitions and observations it now makes sense to consider the following diagram:[2391]

$$
\begin{array}{ccccccc}
\mathrm{H}_{\mathrm{c}}^{m-k}(M)\times\mathrm{H}_{\mathrm{c}}^{k}(M) & \xrightarrow{\cup} & \mathrm{H}_{\mathrm{c}}^{m}(M) & \xrightarrow{\cap M} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\uparrow{\scriptstyle o^*\times o^*} & & \uparrow{\scriptstyle o^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^{m-k}(M,M\backslash K)\times\mathrm{H}^{k}(M,M\backslash L) & \xrightarrow{\cup} & \mathrm{H}^{m}(M,M\backslash(K\cap L)) & \xrightarrow{\cap\mu} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\downarrow{\scriptstyle z^*\times z^*} & & \downarrow{\scriptstyle z^*} & & \uparrow{\scriptstyle z_*} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^{m-k}(Z,Z\backslash K)\times\mathrm{H}^{k}(Z,Z\backslash L) & \xrightarrow{\cup} & \mathrm{H}^{m}(Z,Z\backslash K\cap L) & \xrightarrow{\cap[Z]} & \mathrm{H}_0(Z) & \xrightarrow{\epsilon_Z} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^n z_i^*\times z_i^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^n z_i^*} & & \uparrow{\scriptstyle \sum_{i=1}^n z_{i*}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^n \mathrm{H}^{m-k}(Z_i,Z_i\backslash K_i)\times\mathrm{H}^{k}(Z_i,Z_i\backslash L_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^n \mathrm{H}^{m}(Z_i,Z_i\backslash K_i\cap L_i) & \xrightarrow{\oplus\cap[Z_i]} & \bigoplus_{i=1}^n \mathrm{H}_0(Z_i) & \xrightarrow{\sum\epsilon_{Z_i}} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^n c^*\times c^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^n c^*} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^n \mathrm{H}^{m-k}(Z_i,\partial Z_i\backslash \mathring{A}_i)\times\mathrm{H}^{k}(Z_i,\partial Z_i\backslash \mathring{B}_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^n \mathrm{H}^{m}(Z_i,\partial Z_i) & \xrightarrow{\oplus\cap[Z_i]} & \bigoplus_{i=1}^n \mathrm{H}_0(Z_i) & \xrightarrow{\sum\epsilon_{Z_i}} & \mathbb{Z} \\
\cong\uparrow{\scriptstyle \bigoplus_{i=1}^n d^*\times d^*} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^n \mathrm{H}^{m-k}(Z_i,\partial Z_i\backslash \partial X_i)\times\mathrm{H}^{k}(Z_i,\partial Z_i\backslash \partial Y_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^n \mathrm{H}^{m}(Z_i,\partial Z_i) & \xrightarrow{\oplus\cap[Z_i]} & \bigoplus_{i=1}^n \mathrm{H}_0(Z_i) & \xrightarrow{\sum\epsilon_{Z_i}} & \mathbb{Z}.
\end{array}
$$

Claim 3. The above diagram commutes.

Proof. In the following we deal with all of the fifteen squares of the diagram.

(a) The square on the top left commutes by definition of the cup product on cohomology with compact support, see Definition 209.19.

(b) The left squares in the second row and also in the fourth row commute by the Cup Product-Naturality Lemma 204.8.

(c) The left square in the third row commutes by the Cup Product–Disjoint Union Lemma 204.16 together with the aforementioned Cup Product-Naturality Lemma 204.8.

(d) The middle square in the first row commutes by definition of the map $\cap M$.

(e) It follows from functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 193.6 that $z_*([Z]) = \mu \in \mathrm{H}_m(M, M \setminus (K \cap L))$. Therefore we obtain from the Cap Product-Naturality Lemma 206.13 that the middle square in the second row commutes.

(f) It follows easily from the definitions and the Path-Component-H_*-Lemma 143.21 that $\sum_{i=1}^n z_{i*}([Z_i]) = [Z]$. Now we obtain from the Cap Product-Naturality Lemma 206.13 and the Cap Product–Disjoint Union Lemma 206.14 that the middle square in the third row commutes.

(g) It follows from the Augmentation Lemma 143.24 (3) that the squares that involve the augmentation maps commute.

(h) It follows from the Cap Product-Naturality Lemma 206.13 and our conventions regarding the meaning of $[Z_i]$ that the middle square in the fourth row commutes.

[2391] For space reasons we drop the "$\mathbb{Z}$"-coefficients from the cohomology groups.

(i) It follows from the Cup Product-Naturality Lemma 204.8 that the bottom left square commutes.

(j) The remaining squares, that we did not discuss, commute for the trivial reason that the opposite maps in the square are actually on the nose the same map. ⊞

We introduce one last piece of notation:

(15) For each $i \in \{1, \dots, n\}$ we denote by $\mathrm{PD}_{Z_i} \colon \mathrm{H}_k(Z_i, \partial X_i) \to \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \partial X_i; \mathbb{Z})$ the isomorphism from the Neighborhood–Poincaré Duality Proposition 221.3.

Claim 4. Let $i \in \{1, \dots, n\}$.

(i) For the above $\varphi \in \mathrm{H}^{m-k}(M, M \setminus K; \mathbb{Z})$ we have $(c^* \circ z_i^* \circ z^*)(\varphi) = a_i \cdot d^*(\mathrm{PD}_{Z_i}([X_i]))$.

(ii) For the above $\psi \in \mathrm{H}^k(M, M \setminus L; \mathbb{Z})$ we have $(c^* \circ z_i^* \circ z^*)(\psi) = b_i \cdot d^*(\mathrm{PD}_{Z_i}([Y_i]))$.

(iii) We have
$$\epsilon_{Z_i}\Big(\big((d^*(\mathrm{PD}_{Z_i}([X_i])) \cup d^*(\mathrm{PD}_{Z_i}([Y_i]))\big) \cap [Z_i]\Big) \;=\; \mathrm{index}(\sigma_i, \tau_i, P_i).$$

Proof.

(i) We introduce the following notation:

(α) We set $U \coloneqq (M \setminus Z) \cup \bigcup_{i=1}^{n} \mathring{Z}_i \cup \bigcup_{i=1}^{n} \mathring{A}_i = M \setminus \bigcup_{i=1}^{n} (\partial Z_i \setminus \mathring{A}_i)$. It is elementary to see that U is in fact an open subset of M.

(β) We write $C \coloneqq A_i$ and $D \coloneqq \partial Z_i \setminus \mathring{A}_i$. Note that it follows from the Codimension-Zero Submanifold Proposition 150.16 that C and D are compact codimension-zero submanifolds of ∂Z_i with $C \cup D = \partial Z_i$ and with $C \cap D = \partial C = \partial D$.

(γ) We denote by $p \colon (U, \varnothing) \to (U, U \setminus \mathring{Z}_i)$, $\widetilde{z} \colon (Z_i \cap U, C \cap U) \to (U, U \setminus \mathring{Z}_i)$ and $g \colon (Z_i \cap U, C \cap U) \to (Z_i, C)$ the obvious inclusion maps.

We perform the following calculation in $\mathrm{H}_k(Z_i, A_i)$:

$$\underbrace{(c^* \circ z_i^* \circ z^*)(\varphi)}_{\in \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \mathring{A}_i)} \cap [Z_i] \;\underset{\substack{\uparrow \\ \text{this follows from the Subspace–Poincaré Duality Lemma 221.25,} \\ \text{applied to the above choices of } C \text{ and } D\text{, and} \\ \text{the observation that } \varphi \cap M = [\sigma] \text{ with } |\sigma| \subseteq U, \\ \text{note that here we use that we have } U \cap \partial Z_i \subseteq C \text{ and } K \subseteq U}}{=}\; g_*(\widetilde{z}_*^{-1}(p_*([\sigma]))) \;\overset{\substack{\text{since } \sigma = a_1 \cdot \sigma_1 + \dots + a_r \cdot \sigma_r, \\ \text{since } p_*(\sigma_j) = 0 \text{ for } j \neq i \text{ and} \\ \text{since } [\sigma_i] = [X_i]}}{\overset{\downarrow}{=}}\; g_*(\widetilde{z}_*^{-1}(p_*(a_i \cdot [X_i]))) \;\underset{\substack{\uparrow \\ \text{by the Codimension} \\ \text{Zero Fundamental} \\ \text{Class Lemma 194.10}}}{=}\; a_i \cdot [X_i].$$

Now recall that in the proof of the Neighborhood–Poincaré Duality Proposition 221.3 we saw that the isomorphism PD_{Z_i} is given as follows:

$$\begin{tikzcd} \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \partial X_i; \mathbb{Z}) \arrow[r, "d^*", "\cong"'] & \mathrm{H}^{m-k}(Z_i, \partial Z_i \setminus \mathring{A}_i; \mathbb{Z}) \arrow[r, "\cap [Z_i]", "\cong"'] & \mathrm{H}_k(Z_i, A_i) & \mathrm{H}_k(Z_i, \partial X_i). \arrow[l, "j_*"', "\cong"] \arrow[ll, bend left, "=:\mathrm{PD}_{Z_i}"] \end{tikzcd}$$

Since the inclusion-induced map $j_* \colon \mathrm{H}_k(Z_i, \partial X_i) \to \mathrm{H}_k(Z_i, A_i)$ is an isomorphism we now see that $(c^* \circ z_i^* \circ z^*)(\varphi) = a_i \cdot d^*(\mathrm{PD}_{Z_i}([X_i]))$.

(ii) The proof of this statement is evidently the same as the proof of (2).

(iii) This statement follows immediately from the definitions together with the fact, obtained in Claim 3, that the bottom squares in the above diagram commute. ⊞

After all these preparations we can prove the desired equality:

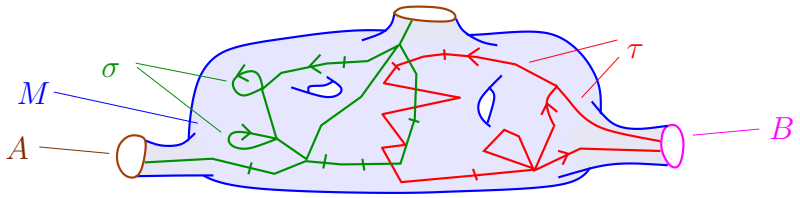
$$\begin{aligned}
\epsilon_M\big((\mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau])) \cap M\big) &\overset{\text{by construction of } \varphi \text{ and } \psi \text{ in Claim 1 and the first row of the commutative diagram from Claim 2}}{=} \epsilon_M((\varphi \cup \psi) \cap \mu) \\
&\overset{\text{by the second, third and fourth row of the commutative diagram from Claim 2}}{=} \sum_{i=1}^{n} \epsilon_{Z_i}\big(((c^* \circ z_i^* \circ z^*)(\varphi) \cup (c^* \circ z_i^* \circ z^*)(\psi)) \cap [Z_i]\big) \\
&\overset{\text{by Claim 3}}{=} \sum_{i=1}^{n} a_i \cdot b_i \cdot \mathrm{index}(\sigma_i, \tau_i, P_i) \overset{\text{by the setting from page 4276}}{=} \sigma \cdot \tau.
\end{aligned}$$

This concludes the proof of the theorem. ■

We conclude this section with the following variation on the Cycles–Cup Product-Intersection Number Theorem 221.20.

Theorem 221.26. (∂-Cycles–Cup Product-Intersection Number Theorem) Let M be a compact orientable connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and with $\partial M = A \sqcup B$. Finally let $\sigma \in \mathrm{C}_k(M, A)$ and $\tau \in \mathrm{C}_{m-k}(M, B)$ be two cycles. If σ and τ intersect nicely, then

$$\sigma \cdot \tau = \Big\langle \underbrace{\overbrace{\mathrm{PD}_M([\sigma])}^{\in \mathrm{H}^{m-k}(M,B;\mathbb{Z})} \cup \overbrace{\mathrm{PD}_M([\tau])}^{\in \mathrm{H}^{k}(M,A;\mathbb{Z})}}_{\in \mathrm{H}^{m}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle.$$



Sketch of proof. The proof of the theorem goes along the lines of the proof of the Cycles–Cup Product-Intersection Number Theorem 221.20. The key difference is that we need to use Poincaré Duality for non-compact topological manifolds with boundary. More precisely, let U be an oriented m-dimensional topological manifold with boundary. By the Boundary Homology–Section Theorem 193.20 we know that given any compact set K of U the orientation defines a natural homology class μ_K in $\mathrm{H}_m(U, (U \setminus K) \cup \partial U)$. The same construction as in the Cap–Non Compact Manifold Lemma 211.3 (2) shows that capping with these homology classes defines a well-defined map

$$\mathrm{PD}_U \colon \mathrm{H}_c^k(U; \mathbb{Z}) \to \mathrm{H}_{m-k}(U, \partial U).$$

A moderately heroic effort shows that the proof of the Poincaré Duality Theorem 210.1, as provided in Section 211.3 can be modified to show that the above map PD_U is an isomorphism. Once this is out of the way it is fairly straightforward to modify the proof of the Cycles–Cup Product-Intersection Number Theorem 221.20 to obtain a proof of the ∂-Cycles–Cup Product-Intersection Number Theorem 221.26. ■

221.5. **Appendix: The algebraic $\mathbb{F}_2$-intersection number.** In this appendix we quickly discuss the "$\mathbb{F}_2$-analog" of the results of the previous sections.

Definition 221.27. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be continuous maps. If f and g intersect nicely, then we define

$$\text{algebraic } \mathbb{F}_2\text{-intersection number of } f \text{ and } g \;:=\; f \pitchfork_{\mathbb{F}_2} g \;:=\; \#(|f| \cap |g|) \bmod 2.$$

If X and Y are in fact submanifolds and f and g are the inclusion map, then we write $X \pitchfork_{\mathbb{F}_2} Y := f \pitchfork_{\mathbb{F}_2} g$.

We recall the following notation:

Notation 221.28.

(1) It follows from the $\mathbb{F}$-Orientability Proposition 191.22 together with the Fundamental Class-Proposition 194.2 that any compact n-dimensional topological manifold W comes with its $\mathbb{F}_2$-fundamental class $[W]_{\mathbb{F}_2} \in \mathrm{H}_n(W, \partial W; \mathbb{F}_2)$. If W is in fact a proper submanifold of some other topological manifold M, then frequently we denote by $[W]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ the image of $[W]_{\mathbb{F}_2}$ under the inclusion-induced map $\mathrm{H}_n(W, \partial W; \mathbb{F}_2) \to \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$.

(2) In this section, given a compact topological manifold we denote by PD_M the Poincaré Duality isomorphisms with $\mathbb{F}_2$-coefficients provided by the Poincaré Duality Theorem 210.1.

Now we can formulate the following theorem, which is a variation on the Maps-Cup Product–Intersection Number Theorem 221.16.

Theorem 221.29. (Maps-$\mathbb{F}_2$-Cup Product–Intersection Number Theorem) Let M be a compact connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M such that $\partial A = \partial B = A \cap B$ and with $\partial M = A \sqcup B$. Let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be two continuous maps with $f(\partial X) \subseteq A$ and $g(\partial Y) \subseteq B$. If f and g intersect nicely, then

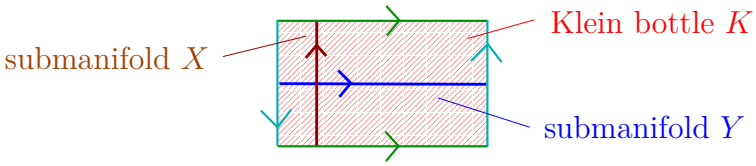
$$f \pitchfork_{\mathbb{F}_2} g \;=\; \Big\langle \overbrace{\mathrm{PD}_M(\underbrace{f_*([X]_{\mathbb{F}_2})}_{\in \mathrm{H}_k(M,A;\mathbb{F}_2)})}^{\in \mathrm{H}^{m-k}(M,B;\mathbb{F}_2)} \cup \overbrace{\mathrm{PD}_M(\underbrace{g_*([Y]_{\mathbb{F}_2})}_{\in \mathrm{H}_{m-k}(M,B;\mathbb{F}_2)})}^{\in \mathrm{H}^{k}(M,A;\mathbb{F}_2)}, \overbrace{[M]_{\mathbb{F}_2}}^{\in \mathrm{H}_m(M,\partial M;\mathbb{F}_2)} \Big\rangle \;\in\; \mathbb{F}_2^{\in \mathrm{H}^m(M,\partial M;\mathbb{F}_2)}.$$

In particular, if X and Y are already submanifolds of M, then

$$X \pitchfork_{\mathbb{F}_2} Y \;=\; \langle \mathrm{PD}_M([X]_{\mathbb{F}_2}) \cup \mathrm{PD}_M([Y]_{\mathbb{F}_2}), [M]_{\mathbb{F}_2} \rangle.$$

Proof. By the $\mathbb{F}_2$-Orientability Proposition 191.22 we know that every topological manifold is $\mathbb{F}_2$-orientable. With this observation the proof of the theorem is a variation, and basically a simplification, of the proof of the Cup Product–Intersection Number Theorem 220.10. We leave it to the skeptical reader to fill in the details. ∎

Example. Similar to the discussion on page 4237 we can use the Maps-$\mathbb{F}_2$-Cup Product–Intersection Number Theorem 221.29 to show that some homology classes are in fact non-zero. For example in the figure below we show the Klein bottle K together with two closed 1-dimensional submanifolds. These intersect in precisely one point, so their algebraic $\mathbb{F}_2$-intersection number is necessarily non-zero.



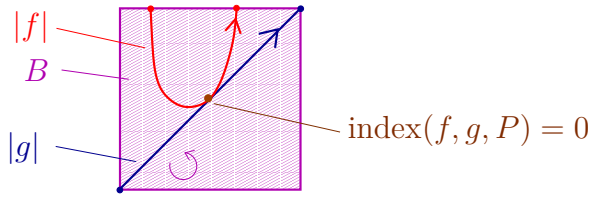
It follows from the Maps-$\mathbb{F}_2$-Cup Product–Intersection Number Theorem 221.29 that $[X]_{\mathbb{F}_2}$ and $[Y]_{\mathbb{F}_2}$ are non-zero in $\mathrm{H}_1(K;\mathbb{F}_2)$. □

Remark. A discussion of intersection numbers in topological manifolds is also given in [**Dol1980**, Chapter VIII.13]. □

221.6. **Exercises for Chapter 221 .**

Exercise 221.1. Is every topological space that is Hausdorff also normal?

Exercise 221.2. We consider the manifold $M = Z = [0,1]^2$ and the two embeddings $f, g\colon [0,1] \to Z$ that are shown in the figure below. Note that $P = (0,0)$ is the unique intersection point of $|f|$ and $|g|$. Show that $\mathrm{index}(f,g,P) = 0$.



Exercise 221.3. Can one use the General Alexander Duality Theorem 218.2 to answer Question 221.1? Or some other of the earlier theorems? If yes, how would you do it? You could also try to restrict yourselves to the case that the map α is injective.

Exercise 221.4. Let M be a compact oriented connected n-dimensional topological manifold and let $N \subseteq M$ be a compact oriented proper submanifold of codimension one. Finally let $\gamma\colon S^1 \to M$ be a continuous map that intersects N nicely. Show that

$$\langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle \;=\; \gamma \cdot N.$$

Exercise 221.5. In the figure on page 4264 we saw that $-1, 0, 1$ can be indices of nice intersection points of oriented submanifolds. Can any other numbers occur as indices of nice intersection points?

Exercise 221.6. Let $n \in \mathbb{N}$. By the discussion on page 3832 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

Use the Maps-$\mathbb{F}_2$-Cup Product–Intersection Number Theorem 221.29 to show that the obvious map

$$\mathbb{F}_2[x]/(x^{n+1}) \ \to \ (\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2),\cup)$$

is a ring isomorphism.

Remark. In the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 we used a rather different approach to calculating the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$.

CHAPTER 222

The Lefschetz-Hopf Theorem

First we recall Definition 182.13:

Definition 222.1. Let M be a compact topological manifold and let $f\colon M \to M$ be a continuous map. We define the Lefschetz number of f to be[2392]

$$\Lambda(f) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)).$$

The following is one of our main results on the Lefschetz number.

Theorems 182.16 and 189.21 (Lefschetz Fixed Point Theorem). Let M be a compact smooth or topological manifold and let $f\colon M \to M$ be a continuous map. If the Lefschetz number $\Lambda(f)$ is non-zero, then f has a fixed point.

In this chapter we will use the Cup Product–Intersection Number Theorem 220.10 to prove the Lefschetz-Hopf Theorem 222.9 and we will use the more powerful Maps-Cup Product Intersection Number 221.16 to prove the more powerful Lefschetz-Hopf Theorem 222.14. These two Lefschetz-Hopf Theorems allow us to compute the Lefschetz number $\Lambda(f)$ as the sum of "indices" of the fixed points of a given map $f\colon M \to M$. In particular, if there are no fixed points, then this sum is zero. Thus the Lefschetz-Hopf Theorems are generalizations of the above version of the Lefschetz Fixed Point Theorem.

Recall that in the Euler Characteristic–Vector Field Theorem 182.22 we used the Lefschetz Fixed Point Theorem 182.16 to show that if a closed smooth manifold has non-zero Euler characteristic, then every vector field on M has at least one zero. In the following chapter we will use the Lefschetz-Hopf Theorems 222.9 and 222.14 to prove the Poincaré-Hopf Theorem 223.8 which is a significant generalization of the Euler Characteristic–Vector Field Theorem 182.22.

This chapter is organized as follows:

- In Section 222.1 we will express the Lefschetz number as an intersection number.
- In Section 222.2 we will state and prove the Lefschetz-Hopf Theorem 222.9 for smooth manifolds.
- In Section 222.3 we will state and prove the Lefschetz-Hopf Theorem 222.14 for topological manifolds.
- In Section 222.4 we will provide exercises for Chapter 222.

In the next Chapter 223 we will state and prove the Poincaré-Hopf Theorem 223.8.

[2392]Note that we know by the Topological Manifolds-Invariants Proposition 189.14 that the homology groups of M are finitely generated and that they are zero beyond a certain degree. This shows that the Lefschetz number is indeed defined.

222.1. **The Lefschetz number as an intersection number.** We start out with recalling and introducing some notation. We will make use of this notation throughout this chapter.

Notation 222.2. Let M be an n-dimensional topological respectively smooth manifold.

(1) It follows from the Topological Manifold-Product Proposition 23.6 and the Smooth Manifold-Product Proposition 26.12 that we can view $M \times M$ as a $2n$-dimensional topological respectively smooth manifold (possibly with corner).

(2) We denote by $p_1, p_2 \colon M \times M \to M$ the projection onto the first respectively second factor.

(3) We write $\Delta_M := \{(x,x) \in M \times M \mid x \in M\}$.

(4) Let $f \colon M \to M$ be a continuous map. We denote by $\mathrm{id} \times f \colon M \to M \times M$ the map that is given by $x \mapsto (x, f(x))$. We define the graph of f as

$$\mathrm{Gr}(f) \;:=\; (\mathrm{id} \times f)(M).$$

Note that $\mathrm{Gr}(\mathrm{id}) = \Delta_M$.

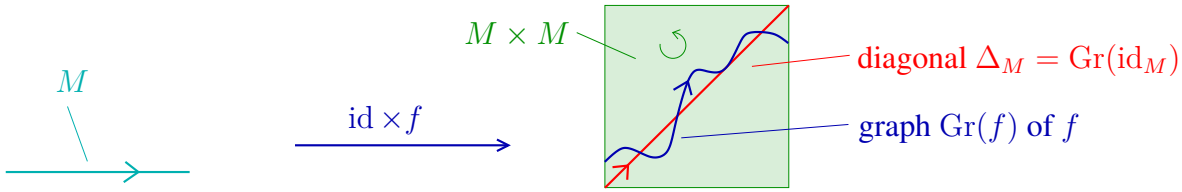
(5) The map $\mathrm{id} \times f \colon M \to M \times M$ is easily seen to be a (smooth) proper embedding. Thus we can view the graph $\mathrm{Gr}(f)$ as a proper n-dimensional submanifold of $M \times M$.

(6) Now assume that M is oriented.

(a) We equip $M \times M$ with the product orientation that we defined on pages Convention 31.49 and Definition 192.8.

(b) We equip $\mathrm{Gr}(f)$ with the unique orientation that turns $\mathrm{id} \times f$ into an orientation-preserving map.

(c) If M is furthermore closed, then we denote as usual by $[\mathrm{Gr}(f)] \in \mathrm{H}_n(\mathrm{Gr}(f))$ the fundamental class of $\mathrm{Gr}(f)$. By the usual slight abuse of notation we denote by $[\mathrm{Gr}(f)] \in \mathrm{H}_n(M \times M)$ the homology class represented by this submanifold. Note that $[\mathrm{Gr}(f)] = (\mathrm{id} \times f)_*([M]) \in \mathrm{H}_n(M \times M)$.



To formulate the main result of this section we will also need the following definition:

Definition 222.3.

(1) Given a finitely generated abelian group we write $\mathrm{F}H := H/\operatorname{Tor}(H)$, where $\operatorname{Tor}(H)$ denotes the torsion subgroup of H.

(2) Let W be a closed oriented $2n$-dimensional topological manifold. We refer to the map

$$\begin{aligned} Q_W \colon \mathrm{FH}_n(W) \times \mathrm{FH}_n(W) &\to \mathbb{Z} \\ (a,b) &\mapsto \langle \mathrm{PD}_W(a) \cup \mathrm{PD}_W(b), [W] \rangle \end{aligned}$$

as the intersection form of W.

Remark. We will discuss the intersection form in great detail in Chapter 224. □

Now we can formulate the following, potentially quite counterintuitive proposition.

Proposition 222.4. (Lefschetz-Intersection Number Proposition) Let M be a closed oriented connected n-dimensional topological manifold.
(1) We have $Q_{M\times M}([\Delta_M],[\Delta_M]) = \chi(M)$.
(2) Given any continuous map $f\colon M \to M$ we have $Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) = \Lambda(f)$.

Remark.

(1) Sometimes in the literature, in line with the notation introduced on page 4265, the statement of the Lefschetz-Intersection Number Proposition 222.4 (1) gets shortened to the potentially ambiguous statement "$M \cdot M = \chi(M)$".
(2) Let M be a closed oriented smooth manifold. In Definition 53.4 we introduced the smooth Euler characteristic $\chi^{\mathrm{sm}}(M)$. We now see that

$$\chi^{\mathrm{sm}}(M) \underset{\uparrow}{=} \Delta_M \cdot \Delta_M \underset{\uparrow}{=} Q_{M\times M}([\Delta_M],[\Delta_M]) \underset{\uparrow}{=} \chi(M).$$

by Definition 53.4; by the Cup Product–Intersection Number Theorem 220.10; by the Lefschetz-Intersection Number Proposition 222.4 □

Examples.

(1) Suppose that in the Lefschetz-Intersection Number Proposition 222.4 we actually have a smooth manifold M and a smooth map $f\colon M \to M$. In this case we can view Δ_M and $\mathrm{Gr}(f)$ as oriented n-dimensional submanifolds of $M \times M$. If these two submanifold are transverse, then it follows from the Cup Product–Intersection Number Theorem 220.10 that we can calculate $Q_{M\times M}$ geometrically, more precisely, we have the following equality:

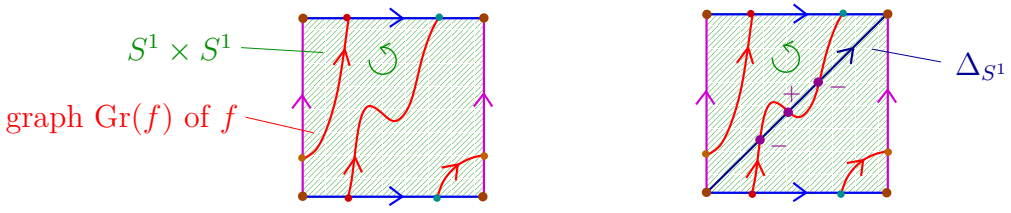
$$Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) = \begin{array}{c}\text{algebraic intersection number } \mathrm{Gr}(f)\cdot\Delta_M \text{ of the oriented}\\ \text{submanifolds } \Delta_M \text{ and } \mathrm{Gr}(f), \text{ see Definition 220.5.}\end{array}$$

For example, let us consider the map $f\colon S^1 \to S^1$ that is illustrated in the figure below. We see that

$$\mathrm{Gr}(f)\cdot\Delta_M \underset{\uparrow}{=} -1+1-1 = -1 \underset{\uparrow}{=} (-1)^0\cdot 1 + (-1)^1\cdot 2 = \Lambda(f).$$

there are three intersection points with signs -1, $+1$ and -1; since $\mathrm{tr}(f \circlearrowleft \mathrm{H}_0(S^1;\mathbb{Z})) = 1$ and $\mathrm{tr}(f \circlearrowleft \mathrm{H}_1(S^1)) = 2$

Fortunately this is consistent with the Lefschetz-Intersection Number Proposition 222.4 (2).



(2) Let $n \in \mathbb{N}$. We have

$$Q_{S^n\times S^n}([\Delta_{S^n}],[\Delta_{S^n}]) \underset{\uparrow}{=} \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ (-1)^{n^2} & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = 1+(-1)^{n^2} = \chi(S^n).$$

see the Product-of-Spheres–Cup Product Lemma 207.3

Fortunately this is again consistent with the Lefschetz-Intersection Number Proposition 222.4 (1). □

The proof of the Lefschetz-Intersection Number Proposition 222.4 requires some preparations. In particular we will need the following lemma.

Lemma 222.5. (Intersection Form–Dual Basis Lemma) Let M be a closed oriented connected n-dimensional topological manifold, let $i \in \{0,\dots,n\}$ and let $v_1,\dots,v_m$ be a basis of $\mathrm{FH}^i(M;\mathbb{Z})$. There exists a basis $\widetilde{v}_1,\dots,\widetilde{v}_m$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for every $k,l \in \{1,\dots,m\}$ we have $\langle v_k \cup \widetilde{v}_l, [M]\rangle = \delta_{kl}$.

Proof of Lemma 222.5. The statement of the lemma is precisely the content of Exercise 210.21. For the reader's convenience we nonetheless provide the proof. For $j = 1,\dots,m$ we set $\sigma_j \coloneqq v_j \cap [M] \in \mathrm{FH}_{n-i}(M)$. By the Poincaré Duality Theorem 210.1 we know that $\sigma_1,\dots,\sigma_m$ is a basis for $\mathrm{FH}_{n-i}(M)$. Next note that it follows from the Universal Coefficient-H^*-Theorem 201.23 together with the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18, the Ext-Properties Lemma 201.19 (4) and the Topological Manifolds-Invariants Proposition 189.14 (2) that the map

$$\begin{array}{ccc} \mathrm{FH}^{n-i}(M;\mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{FH}_{n-i}(M),\mathbb{Z}) \\ \varphi & \mapsto & (\sigma \mapsto \langle \varphi, \sigma\rangle) \end{array}$$

is an isomorphism. We denote by $\widetilde{v}_1,\dots,\widetilde{v}_m$ the cohomology classes with $\langle \widetilde{v}_k, \sigma_l\rangle = \delta_{kl}$. By the above $\widetilde{v}_1,\dots,\widetilde{v}_m$ are a basis for $\mathrm{FH}^{n-i}(M;\mathbb{Z})$. Now we see that for any $k,l \in \{1,\dots,m\}$ we have

$$\langle v_k \cup \widetilde{v}_l, [M]\rangle \overset{\text{follows from the Cap Product-Kronecker Lemma 206.12, under the usual identification } \mathrm{H}_0(M) =_{\mathrm{i}} \mathbb{Z}}{\overset{\downarrow}{=}} (v_k \cup \widetilde{v}_l) \cap [M] \underset{\text{by the Cup-Cap Lemma 206.15}}{\overset{\uparrow}{=}} \widetilde{v}_l \cap (v_k \cap [M]) = \widetilde{v}_l \cap \sigma_k \overset{\downarrow}{=} \langle \widetilde{v}_l, \sigma_k\rangle = \delta_{kl}.$$

■

The Intersection Form–Dual Basis Lemma 222.5 allows us to formulate our next lemma, which will be a key ingredient in the proof of the Lefschetz-Intersection Number Proposition 222.4.

Lemma 222.6. (Dual-of-Diagonal Lemma) Let M be a closed oriented connected n-dimensional topological manifold. For $i = 0,\dots,n$ we pick a basis $v_1^i,\dots,v_{m_i}^i$ of $\mathrm{FH}^i(M;\mathbb{Z})$. By the Intersection Form–Dual Basis Lemma 222.5 there exists a basis $\widetilde{v}_1^{n-i},\dots,\widetilde{v}_{m_i}^{n-i}$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for every $k,l \in \{1,\dots,m_i\}$ we have $\langle v_k^i \cup \widetilde{v}_l^{n-i}, [M]\rangle = \delta_{kl}$.

(1) We have the following equality in $\mathrm{H}^n(M\times M;\mathbb{Z})$:

$$\mathrm{PD}_{M\times M}([\Delta_M]) = \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i).$$

(2) Given any continuous map $f\colon M \to M$ we have the following equality in $\mathrm{H}^n(M\times M;\mathbb{Z})$:

$$\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) = \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{k=1}^{m_i}\sum_{l=1}^{m_i} \underbrace{\langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle}_{\in\mathbb{Z}} \cdot p_1^*(\widetilde{v}_k^{n-i}) \cup p_2^*(v_l^i).$$

Proof of the Dual-of-Diagonal Lemma 222.6. First we show that Statement (1) follows almost immediately from Statement (2). Indeed, we have

$$\mathrm{PD}_{M\times M}(\overbrace{[\mathrm{Gr}(\mathrm{id}_M)]}^{=\Delta_M}) \;=\; \sum_{i=0}^{n}(-1)^{i\cdot(n-i-1)}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}\underbrace{\langle v_r^i\cup\widetilde{v}_s^{n-i},[M]\rangle}_{=\delta_{rs}}\cdot p_1^*(\widetilde{v}_r^{n-i})\cup p_2^*(v_s^i).$$

$\uparrow$ Statement (2) applied to $f=\mathrm{id}_M$

Thus it remains to prove Statement (2) of the Dual-of-Diagonal Lemma 222.6. We start out the proof of Statement (2) with the following claim:

Claim. For $j\in\{0,\dots,n\}$ and $r,s\in\{1,\dots,m_j\}$ there exist a unique $a_{j,r,s}\in\mathbb{Z}$ such that

$$\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \;=\; \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j}a_{j,r,s}\cdot p_1^*(\widetilde{v}_r^{n-j})\cup p_2^*(v_s^j) \;\in\; \mathrm{H}^n(M\times M;\mathbb{Z}).$$

Proof. Recall that by the Topological Manifolds-Invariants Proposition 189.14 (2) we know that the homology groups of M are all finitely generated. Thus it follows from the Algebraic–Chain Complex Condition Lemma 208.3 Künneth-H*-Theorem 208.1 that there exists a short exact sequence of the following form:

$$0\to\bigoplus_{j=0}^{n}\mathrm{H}^{n-j}(M;\mathbb{Z})\otimes\mathrm{H}^j(M;\mathbb{Z})\to\mathrm{H}^n(M\times M;\mathbb{Z})\to\bigoplus_{j=0}^{n+1}\overbrace{\mathrm{Tor}(\mathrm{H}^{n+1-j}(M;\mathbb{Z}),\mathrm{H}^j(M;\mathbb{Z}))}^{\text{torsion group by the Tor-Properties Lemma 162.13}}\to 0.$$
$$\varphi\otimes\psi\;\mapsto\;p_1^*(\varphi)\cup p_2^*(\psi)$$

The Algebraic–Chain Complex Condition Lemma 208.3 and the Künneth-H*-Theorem 208.1 also imply that the short exact sequence splits. Since the group on the right is torsion it follows easily, using some of the elementary properties of tensor product that we listed in the Tensor–Direct Sum Lemma 159.15, that the map

$$\bigoplus_{j=0}^{n}\underbrace{\mathrm{FH}^{n-j}(M;\mathbb{Z})}_{=\bigoplus_r\mathbb{Z}\cdot\widetilde{v}_r^{n-j}}\otimes\underbrace{\mathrm{FH}^j(M;\mathbb{Z})}_{=\bigoplus_s\mathbb{Z}\cdot v_s^j}\xrightarrow{\varphi\otimes\psi\mapsto p_1^*(\varphi)\cup p_2^*(\psi)}\mathrm{FH}^n(M\times M;\mathbb{Z})$$

is an isomorphism. The claim follows immediately from this fact. ⊞

Next we remind the reader of a few facts on the cup, cap and cross product. To shorten the discussion we leave it to the reader to figure out what the various symbols stand for.

(a) Given $\varphi\in\mathrm{H}^i(M;\mathbb{Z})$ and $\sigma\in\mathrm{H}_i(M;)$ we have $\varphi\cap\sigma=\langle\varphi,\sigma\rangle\in\mathrm{H}_0(M;\mathbb{Z})=_{\mathrm{i}}\mathbb{Z}$.
(b) By the Cup-Cap Lemma 206.15 we have $(\psi\cup\varphi)\cap\sigma=\varphi\cap(\psi\cap\sigma)$.
(c) By the Cup Product-Commutativity Proposition 203.20 we have $\varphi\cup\psi=(-1)^{\deg(\varphi)\cdot\deg(\psi)}\cdot\psi\cup\varphi$.
(d) Given any continuous map $g\colon X\to Y$ we know by the Kronecker Pairing–Natural Lemma 199.9 (1) that we have $\langle g^*(\varphi),\sigma\rangle=\langle\varphi,g_*(\sigma)\rangle$.
(e) Given any continuous map $g\colon X\to Y$ we know by the Cup Product-Naturality Lemma 203.9 that $g^*(\varphi)\cup g^*(\psi)=g^*(\varphi\cup\psi)$.
(f) By the Fundamental Class-of-Product Proposition 194.17 we have $[M\times M]=[M]\times[M]$. Furthermore, by the Fundamental Class-Product Proposition 207.2 we know that we have the equality $p_1^*([M]^*)\cup p_2^*([M]^*)=[M\times M]^*$.

(g) By the Cross-Cup-Cap Theorem 207.1 we have

$$\big(p_1^*(\alpha) \cup p_2^*(\beta)\big) \cap (\mu \times \nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu).$$

(h) By Exercise 164.9 the cross product on H_0 corresponds to multiplication under the augmentation maps.

We will also need the following basic fact:

(i) Given any $k \in \mathbb{Z}$ we have $(-1)^k = (-1)^{k^2}$.

Given $i \in \{0, \dots, n\}$ and $k, l \in \{1, \dots, m_i\}$ we calculate that

$$\begin{aligned}
&\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle \\
&\quad= (-1)^n \cdot \langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \cap [M\times M]\rangle && \text{by (a), (b), (c) and (i)} \\
&\quad= (-1)^n \cdot \langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), [\mathrm{Gr}(f)]\rangle && \text{by Definition 210.2 of } \mathrm{PD}_{M\times M} \\
&\quad= (-1)^n \cdot \langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), (\mathrm{id}\times f)_*([M])\rangle && \text{by definition of } [\mathrm{Gr}(f)] \\
&\quad= (-1)^n \cdot \langle (\mathrm{id}\times f)^*(p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i})), [M]\rangle && \text{by (d)} \\
&\quad= (-1)^n \cdot \langle (p_1\circ(\mathrm{id}\times f))^*(v_k^i) \cup (p_2\circ(\mathrm{id}\times f))^*(\widetilde{v}_l^{n-i}), [M]\rangle && \text{by (e)} \\
&\quad= (-1)^n \cdot \langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle && \text{since } p_1\circ(\mathrm{id}\times f) = \mathrm{id} \\
&&& \text{and } p_2\circ(\mathrm{id}\times f) = f.
\end{aligned}$$

Next we perform an alternative calculation:

$$\begin{aligned}
&\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle \\
&\quad= \Big\langle (p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i})) \cup \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j} a_{j,r,s}\cdot p_1^*(\widetilde{v}_r^{n-j}) \cup p_2^*(v_s^j), [M\times M]\Big\rangle && \text{by the claim} \\
&\quad= (-1)^{(n-i)(n-j)} \cdot \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j} a_{j,r,s}\cdot \langle p_1^*(\underbrace{\overbrace{v_k^i \cup \widetilde{v}_r^{n-j}}^{\in \mathrm{H}^{i+n-j}(M)}}_{=0,\text{ for } i+n-j>n}) \cup p_2^*(\underbrace{\overbrace{\widetilde{v}_l^{n-i} \cup v_s^j}^{\in \mathrm{H}^{n-i+j}(M)}}_{=0,\text{ for } n-i+j>n}), [M\times M]\rangle && \text{by (c) and (e)} \\
&\qquad\qquad \underbrace{\qquad\qquad}_{=0,\text{ for } i\neq j} \\
&\quad= (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s}\cdot \langle p_1^*(v_k^i \cup \widetilde{v}_r^{n-i}) \cup p_2^*(\widetilde{v}_l^{n-i} \cup v_s^i), [M]\times[M]\rangle && \text{by (f) and (i)} \\
&\quad= (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s}\cdot \langle v_k^i \cup \widetilde{v}_r^{n-i}, [M]\rangle \cdot \langle \widetilde{v}_l^{n-i} \cup v_s^i, [M]\rangle && \text{by (a), (g) and (h)} \\
&\quad= (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s}\cdot (-1)^{i\cdot(n-i)} \cdot \underbrace{\langle v_k^i \cup \widetilde{v}_r^{n-i}, [M]\rangle}_{=\delta_{kr}} \cdot \underbrace{\langle v_s^i \cup \widetilde{v}_l^{n-i}, [M]\rangle}_{=\delta_{ls}} && \text{by (d)} \\
&\quad= (-1)^{n-i} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} a_{i,r,s} \cdot (-1)^{i\cdot(n-i)} \cdot \delta_{kr} \cdot \delta_{ls} \;=\; (-1)^{n-i+i\cdot(n-i)} \cdot a_{i,k,l}.
\end{aligned}$$

Comparing these two calculations of $\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle$ we see that $(-1)^{i\cdot(n-i-1)} \cdot \langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle = a_{i,k,l}$. But that is exactly what we needed to show. ∎

Now we are finally ready to provide the proof of the Lefschetz-Intersection Number Proposition 222.4 (1). Note, that as we pointed out in the Lefschetz Number–Basics Lemma 182.11, we have $\Lambda(\mathrm{id}) = \chi(M)$. Thus we see that the Lefschetz-Intersection Number Proposition 222.4 (1) is just a special case of the Lefschetz-Intersection Number Proposition 222.4 (2). Nonetheless, to reduce the notational mess we first prove the Lefschetz-Intersection Number Proposition 222.4 (1) separately.

Proof of Proposition 222.4 (1). We start out with the following claim which gives us a description of $\mathrm{PD}_{M\times M}([\Delta_M])$ that differs slightly from the one provided by the Dual-of-Diagonal Lemma 222.6 (1).

Claim.
$$\mathrm{PD}_{M\times M}([\Delta_M]) \;=\; \sum_{j=0}^{n}(-1)^j\cdot\sum_{s=1}^{m_j}p_1^*(v_s^j)\cup p_2^*(\widetilde{v}_s^{n-j}).$$

Proof. Let $j\in\{0,\dots,n\}$. Recall that we have $v_r^{n-j}\cup\widetilde{v}_s^j=\delta_{rs}$. From the Cup Product-Commutativity Proposition 203.20 we obtain that $\widetilde{v}_s^j\cup v_r^{n-j}=(-1)^{j\cdot(n-j)}\cdot\delta_{rs}$. In other words, we have $\widetilde{v}_s^j\cup(-1)^{j\cdot(n-j)}\cdot v_r^{n-j}=\delta_{rs}$. Now we see that

$$\mathrm{PD}_{M\times M}([\Delta_M]) \;\underset{\substack{\uparrow\\ \text{by the Dual-of-Diagonal Lemma 222.6 (1), applied}\\ \text{to the bases } \{\widetilde{v}_r^j\}_{r=1,\dots,m_j} \text{ and the}\\ \text{corresponding dual bases } \{(-1)^{j\cdot(n-j)}\cdot v_s^j\}_{s=1,\dots,m_j}}}{=}\; \sum_{j=0}^{n}\overbrace{(-1)^{j\cdot(n-j-1)}\cdot(-1)^{j\cdot(n-j)}}^{=(-1)^j}\cdot\sum_{s=1}^{m_j}p_1^*(v_s^j)\cup p_2^*(\widetilde{v}_s^{n-j}).$$

⊞

Now we calculate that

$$\begin{aligned}
&\mathrm{PD}_{M\times M}([\Delta_M])\cup\mathrm{PD}_{M\times M}([\Delta_M])=\\
&\underset{\substack{\uparrow\\ \text{by the Dual-of-Diagonal Lemma 222.6 (1)}\\ \text{and the claim}}}{=}\Big(\sum_{i=0}^{n}(-1)^{i\cdot(n-i-1)}\cdot\sum_{r=1}^{m_i}p_1^*(\widetilde{v}_r^{n-i})\cup p_2^*(v_r^i)\Big)\cup\Big(\sum_{j=0}^{n}(-1)^j\cdot\sum_{s=1}^{m_j}p_1^*(v_s^j)\cup p_2^*(\widetilde{v}_s^{n-j})\Big)\\
&=\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i\cdot(n-i-1)+j}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_j}p_1^*(\widetilde{v}_r^{n-i})\cup p_2^*(v_r^i)\cup p_1^*(v_s^j)\cup p_2^*(\widetilde{v}_s^{n-j})\\
&\underset{\substack{\uparrow\\ \text{by the Cup Product-Commutativity Proposition 203.20}\\ \text{and the Cup Product-Naturality Lemma 203.9}}}{=}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i\cdot(n-i-1)+j}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_j}(-1)^{i\cdot j+j\cdot(n-i)}\cdot\underbrace{p_1^*(\overbrace{\underbrace{v_s^j\cup\widetilde{v}_r^{n-i}}_{=0\text{ if }j+n-i>n}}^{\in\mathrm{H}^{j+n-i}(M)})\cup p_2^*(\overbrace{\underbrace{v_r^i\cup\widetilde{v}_s^{n-j}}_{=0\text{ if }i+n-j>n}}^{\in\mathrm{H}^{i+n-j}(M)})}_{=0,\text{ for }i\neq j}\\
&\underset{\substack{\uparrow\\ \text{since } i+i^2 \text{ is even}}}{=}\sum_{i=0}^{n}(-1)^{i\cdot(n-i-1)}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}(-1)^{i\cdot(n-i)}\cdot p_1^*(\underbrace{v_s^i\cup\widetilde{v}_r^{n-i}}_{=\delta_{sr}\cdot[M]^*})\cup p_2^*(\underbrace{v_r^i\cup\widetilde{v}_s^{n-i}}_{=\delta_{rs}\cdot[M]^*})\\
&=\sum_{i=0}^{n}(-1)^i\cdot\sum_{r=1}^{m_i}\underbrace{p_1^*([M]^*)\cup p_2^*([M]^*)}_{=[M\times M]^*\text{ by the Fundamental Class-Product Proposition 207.2}}\;=\;\chi(M)\cdot[M\times M]^*.
\end{aligned}$$

The desired statement now follows from the observation that by definition of the dual fundamental class $[M\times M]^*$ we have $\langle[M\times M]^*,[M\times M]\rangle=1$. ■

Proof of Proposition 222.4 (2). The proof of the Lefschetz-Intersection Number Proposition 222.4 (2) is a generalization of the proof of Proposition 222.4 (1). It turns out that this argument is even less readable. In the following we quickly run through the argument and we only explain the steps that differ from the proof of the Lefschetz-Intersection

Number Proposition 222.4 (2). We calculate that

$$\begin{aligned}
&\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \cup \mathrm{PD}_{M\times M}([\Delta_M]) = \\
&= \Big(\sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i) \Big) \cup \Big(\sum_{j=0}^{n} (-1)^j \cdot \sum_{t=1}^{m_j} p_1^*(v_t^j) \cup p_2^*(\widetilde{v}_t^{n-j}) \Big) \\
&\quad \uparrow \text{ by the Dual-of-Diagonal Lemma 222.6 (2) and the rewriting procedure as in the claim in the proof of Proposition 222.4 (1)} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \sum_{t=1}^{m_j} \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i) \cup p_1^*(v_t^j) \cup p_2^*(\widetilde{v}_t^{n-j}) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \sum_{t=1}^{m_j} (-1)^{i\cdot j + j\cdot(n-i)} \cdot \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot \underbrace{p_1^*(v_t^j \cup \widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i \cup \widetilde{v}_t^{n-j})}_{=0 \text{ unless } i=j} \\
&= \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)+i} \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} \sum_{t=1}^{m_i} (-1)^{i\cdot i + i\cdot(n-i)} \cdot \langle v_r^i \cup f^*(\widetilde{v}_s^{n-i}), [M]\rangle \cdot p_1^*(\underbrace{v_t^i \cup \widetilde{v}_r^{n-i}}_{=\delta_{tr}\cdot[M]^*}) \cup p_2^*(\underbrace{v_s^i \cup \widetilde{v}_t^{n-i}}_{=\delta_{st}\cdot[M]^*}) \\
&= \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \langle v_r^i \cup f^*(\widetilde{v}_r^{n-i}), [M]\rangle \cdot [M\times M]^* = \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \Big\langle v_r^i \cup \sum_{s=1}^{m_i} a_{rs}^i \cdot \widetilde{v}_s^{n-i}, [M] \Big\rangle \cdot [M\times M]^* \\
&\quad \uparrow \text{ we define } a_{rs}^i \text{ in the obvious way such that this equality holds} \\
&= \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \sum_{s=1}^{m_i} a_{rs}^i \cdot \underbrace{\langle v_r^i \cup \widetilde{v}_s^{n-i}, [M]\rangle}_{=\delta_{rs}} \cdot [M\times M]^* = \sum_{i=0}^{n} (-1)^i \cdot \underbrace{\sum_{r=1}^{m_i} a_{rr}^i}_{=\mathrm{tr}(f^*\circlearrowleft \mathrm{H}^i(M;\mathbb{Z}))} \cdot [M\times M]^* = \Lambda(f) \cdot [M\times M]^*. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \text{ by Exercise 201.12}
\end{aligned}$$

As in the proof of the Lefschetz-Intersection Number Proposition 222.4 (1) we are now done. ∎

222.2. **The Lefschetz-Hopf Theorem I.** In the Lefschetz-Intersection Number Proposition 222.4 we saw that we can express the Lefschetz number of a continuous self-map of a manifold as an intersection number. Furthermore in the Cup Product–Intersection Number Theorem 220.10 and in the more fancy Cycles–Cup Product-Intersection Number Theorem 221.20 we learned how to determine intersection numbers using signs and indices of intersection points. The combination of the results, together with some fairly standard arguments, allows us to prove two versions of the Lefschetz-Hopf Theorem. In this section we will use the more elementary Cup Product–Intersection Number Theorem 220.10 to prove the more basic version of the Lefschetz-Hopf Theorem.

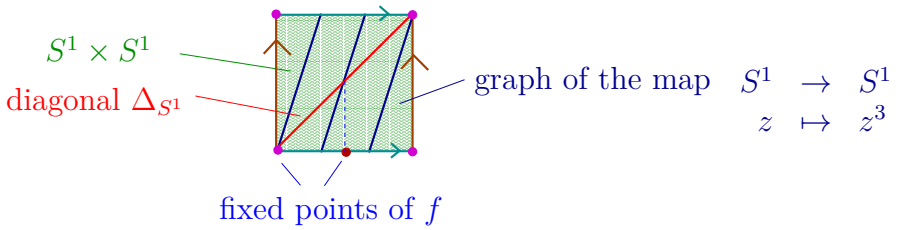
Definition 222.7.

(1) Let M be a smooth manifold and let $f\colon M \to M$ be a smooth map.
   (a) Let $x \in M$ be a fixed point of f. Note that the differential $\mathrm{D}f_x$ is an endomorphism $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_xM$. We say that the fixed point x is non-degenerate if the endomorphism $\mathrm{id}_{\mathrm{T}_xM} - \mathrm{D}f_x$ of T_xM is invertible. Otherwise we say that the fixed point is degenerate.
   (b) Given a non-degenerate fixed point $x \in M$ we define

$$\mathrm{index}(f,x) \coloneqq \mathrm{sign}(\det(\mathrm{id} - \mathrm{D}f_x)) = \frac{\det(\mathrm{id} - \mathrm{D}f_x)}{\|\det(\mathrm{id} - \mathrm{D}f_x)\|}.$$

(2) Let X be a topological space and let $f\colon X \to X$ be a continuous map. We say that a fixed point x of f is isolated if there exists a neighborhood U of x which does not contain another fixed point.

Example. Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$.



An elementary calculation shows that for each fixed point $x \in S^1$ the differential $\mathrm{D}f_x$ is given by multiplication by n.

(1) If $n = 1$, then every point on S^1 is a fixed point of f, and each fixed point is degenerate.
(2) If $n \neq 1$, then the fixed points of f are $x_j = \exp(2\pi \mathrm{i} j/(n-1))$, $j = 1, \ldots, |n-1|$. If $n \geq 1$, then the index of every fixed point is $\operatorname{sign}(1-n) = -1$, and if $n \leq 0$, then the index of every fixed point is $\operatorname{sign}(1-n) = +1$. □

Lemma 222.8. (Isolated Fixed Point Lemma)

(1) Let M be a smooth manifold and let $f\colon M \to M$ be a smooth map. Every non-degenerate fixed point is also an isolated fixed point.
(2) Let X be a topological space and let $f\colon X \to X$ be a continuous map. We assume that X is compact and Hausdorff. In this setting every fixed point of f is isolated if and only if f has only finitely many fixed points.

Proof.

(1) Using charts we see that we only need to consider the case that M is an open subset of some $\mathbb{R}^n$ and we can assume that the non-degenerate fixed point is $x = 0$. We consider the map $g\colon M \to \mathbb{R}^n$ that is given by $x \mapsto x - f(x)$. By our hypothesis we know that $\mathrm{D}g_0 = \mathrm{id}_{\mathrm{T}_0 M} - \mathrm{D}f_0$ is invertible. It follows from the Inverse Function Theorem 24.21 that the map $x \mapsto g(x) = x - f(x)$ is a local diffeomorphism. In particular there exists an open neighborhood U of 0 such that the map $x \mapsto g(x) = x - f(x)$ is injective. But since $g(0) = 0$ this implies that U does not contain any other fixed point of f.
(2) This statement is the content of Exercise 7.17. For the reader's peace of mind we provide the proof. Thus let X be a topological space that is compact and Hausdorff. Furthermore let $f\colon X \to X$ be a continuous map. We denote by $F \subseteq X$ the fixed point set of f.

If F is finite, then it follows easily from the hypothesis that X is Hausdorff that F is a discrete subset of X. Conversely, suppose that F is a discrete subset of X. Note that it follows from the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 together with the Product Topology-Properties Proposition 7.9 that Δ_X and $\mathrm{Gr}(f)$ are closed subsets of $X \times X$. Thus $\Delta_X \cap \mathrm{Gr}(f)$ itself is a closed subset of $X \times X$, in particular it is a closed subset of Δ_X. We denote by $p\colon \Delta_X \to X$ the projection onto the first factor. Evidently this map is a homeomorphism and $F = p(\Delta_X \cap \mathrm{Gr}(f))$. Thus we have shown that F is a closed subset of X and by hypothesis we know that F is discrete. Since X is compact, we obtain from the Compact-Discrete Lemma 3.40 (1) that F is indeed finite. ■

Now we can formulate the first version of the Lefschetz-Hopf Theorem.

Theorem 222.9. (Lefschetz-Hopf Theorem I) Let M be a compact orientable smooth manifold and let $f\colon M \to M$ be a smooth map. If every fixed point of f is non-degenerate and if there is no fixed point on ∂M, then[2393]

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) \;=\; \Lambda(f).$$

Remark. If a smooth map has no fixed points, then the left-hand side of the Lefschetz-Hopf Theorem 222.9 is evidently zero. Thus we see that the Lefschetz-Hopf Theorem 222.9 is a generalization of the Lefschetz Fixed Point Theorem 182.16, applied to the field $\mathbb{F} = \mathbb{Q}$. □

Our first example hopefully puts any lingering worries about signs to rest.

Example. Let $n \in \mathbb{Z} \setminus \{1\}$. As on page 4293 we consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. We perform the following calculation:

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) \;\underset{\uparrow}{=}\; \sum_{j=1}^{|n-1|} \operatorname{index}(f, e^{2\pi \mathrm{i} j/(n-1)}) \;\underset{\uparrow}{=}\; \sum_{j=1}^{|n-1|} -\operatorname{sign}(n-1) \;=\; -(n-1).$$

both equalities follow from the discussion on page 4293

Fortunately this is consistent with the Lefschetz-Hopf Theorem 222.9 and the following calculation:

$$\Lambda(f) \;=\; \underbrace{\operatorname{tr}(f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1))}_{=1\text{, by H}_0\text{-Proposition 143.25}} - \underbrace{\operatorname{tr}(f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1))}_{=n\text{, by the Degree-of-Power Lemma 151.5}} \;=\; 1-n.$$

□

Just for fun let us consider a second example.

Example. Let $n \in \mathbb{N}$. We consider $M = \mathbb{R}^n \cup \{\infty\}$. We denote by 0 the origin of $\mathbb{R}^n$. In Exercise 29.16 we showed that M is an n-dimensional smooth manifold with the atlas given by the obvious chart for $\mathbb{R}^n$ together with the chart

$$\begin{array}{rcl} M \setminus \{0\} & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} \frac{1}{\|x\|^2} \cdot x, & \text{if } x \neq \infty, \\ 0, & \text{if } x = \infty. \end{cases} \end{array}$$

Furthermore we showed that the map

$$\begin{array}{rcl} f\colon M & \to & M \\ x & \mapsto & \begin{cases} 3x, & \text{if } x \neq \infty, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is smooth, that it has two fixed points, namely the origin 0 and ∞, and that $\mathrm{D}f_0 = 3 \cdot \mathrm{id}$ and $\mathrm{D}f_\infty = \frac{1}{3} \cdot \mathrm{id}$. Thus we see that

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) = \operatorname{index}(f,0) + \operatorname{index}(f,\infty) = \operatorname{sign}(\underbrace{(1-3)^n}_{\det(\mathrm{id} - \mathrm{D}f_0)}) + \operatorname{sign}(\underbrace{(1-\tfrac{1}{3})^n}_{\det(\mathrm{id} - \mathrm{D}f_\infty)}) = (-1)^n + 1.$$

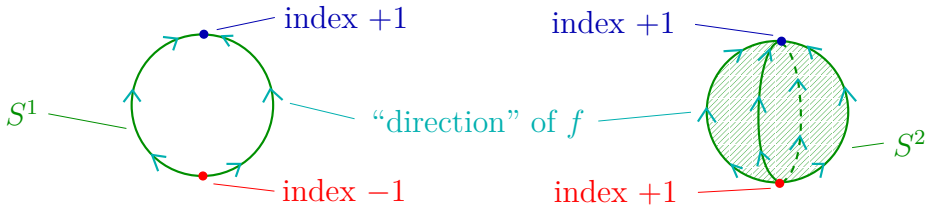
Conveniently enough this is consistent with the Lefschetz-Hopf Theorem 222.9 and the following calculation:

$$\Lambda(f) \;\underset{\uparrow}{=}\; \Lambda(\mathrm{id}) \;=\; \chi(S^n) \;=\; 1 + (-1)^n.$$

follows from the Lefschetz Number-Homotopy Lemma 182.12, since f is easily seen to be homotopic to id

[2393] Note that by the Isolated Fixed Point Lemma 222.8 we know that the left-hand side is a finite sum.

In the Stereographic Projection Lemma 4.24 we gave an explicit diffeomorphism between $\Phi\colon S^n \to M = \mathbb{R}^n \cup \{\infty\}$ that sends the South Pole to the origin and that sends the North Pole to ∞.



For $n = 1$ and $n = 2$ we illustrate the map $\Phi^{-1} \circ f \circ \Phi\colon S^n \to S^n$ in the figure above. □

Proof of the Lefschetz-Hopf Theorem 222.9. Let M be a compact orientable n-dimensional smooth manifold and let $f\colon M \to M$ be a smooth map such that every fixed point is non-degenerate. We pick an orientation. First we assume that M is closed.

We pick an orientation for M. We equip $\operatorname{Gr}(f) \subseteq M \times M$ and $\Delta_M \subseteq M \times M$ with the orientation which turns the projection onto the first factor into an orientation-preserving diffeomorphism.

Claim. Let x be a fixed point of f.[2394]

(1) The point (x, x) is a transverse intersection point of the submanifolds $\operatorname{Gr}(f)$ and Δ_M if and only if x is non-degenerate.

(2) If (x, x) is a transverse intersection point of $\operatorname{Gr}(f)$ and Δ_M, then the sign of the intersection point is given by $\operatorname{index}(f, x) = \frac{\det(\mathrm{id} - \mathrm{D}f_x)}{\|\det(\mathrm{id} - \mathrm{D}f_x)\|}$.

Proof. First recall that by the Product Orientation Proposition 31.48 (3b) we have a natural identification $\mathrm{T}_{(x,x)}(M \times M) = \mathrm{T}_x M \times \mathrm{T}_x M$. We pick a positive basis $v_1, \dots, v_n$ for $\mathrm{T}_x M$. By the definition of the product orientation on $M \times M$, see Convention 31.49, we see that the basis $(v_1, 0), \dots, (v_n, 0), (0, v_1), \dots, (0, v_n)$ is a positive basis for $\mathrm{T}_{(x,x)}(M \times M)$. We use this basis to express the automorphisms id and $\mathrm{D}f_x$ of $\mathrm{T}_x M$ by matrices. With this convention we make the following observation:

$$\underbrace{\begin{pmatrix} \mathrm{id}_n \\ \mathrm{D}f_x \end{pmatrix}}_{\substack{\text{positive basis} \\ \text{for } \mathrm{T}_{(x,x)} \operatorname{Gr}(f)}} \text{ followed by } \underbrace{\begin{pmatrix} \mathrm{id}_n \\ \mathrm{id}_n \end{pmatrix}}_{\substack{\text{positive basis} \\ \text{for } \mathrm{T}_{(x,x)} \Delta_M}} \text{ gives us the matrix } \begin{pmatrix} \mathrm{id}_n & \mathrm{id}_n \\ \mathrm{D}f_x & \mathrm{id}_n \end{pmatrix} \text{ which has the same determinant as } \begin{pmatrix} \mathrm{id}_n & 0 \\ \mathrm{id}_n & \mathrm{id}_n - \mathrm{D}f_x \end{pmatrix}.$$

The desired statements follow immediately from this observation and the definitions. ⊞

Now we perform the following calculation:

$$\Lambda(f) \underset{\substack{\uparrow \\ \text{by the Lefschetz-Intersection Number Proposition 222.4}}}{=} \operatorname{Gr}(f) \cdot \Delta_M \underset{\substack{\uparrow \\ \text{by the Cup Product–Intersection Number Theorem 220.10 and Claim (1)}}}{=} \sum_{\substack{(x,x) \text{ an intersection point} \\ \text{of } \operatorname{Gr}(f) \text{ and } \Delta_M}} \text{sign of } (x,x) \underset{\substack{\uparrow \\ \text{by Claim (2)}}}{=} \sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f, x).$$

We have thus proved the theorem in the special case that M is closed and orientable.

[2394] We refer to Definition 220.3 and Definition 220.5 for the definition of a transverse intersection point and the sign of a transverse intersection point.

Now we allow that the smooth manifold M has non-empty boundary. As in Definition 54.8 we consider the double $\mathrm{D}M := (M \times \{1\}) \sqcup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$. By the Doubling-Manifold Lemma 54.10 we know that $\mathrm{D}M$ is a closed orientable smooth manifold. We also consider the "double of f", i.e. we consider the map

$$\begin{array}{rcl} \psi\colon \mathrm{D}M & \to & \mathrm{D}M \\ [(P,\epsilon)] & \mapsto & [(f(P),\epsilon)]. \end{array}$$

A priori ψ need not be smooth, but using the Homotopic-to-Product-Near-Boundary Lemma 35.9 we can arrange that ψ is smooth. Thus we now assume that ψ is actually smooth.

Let $x_1, \ldots, x_k$ be the fixed points of $f\colon M \to M$. Recall that we demand that no fixed point of f lies on ∂M. It follows immediately that the fixed points of ψ are precisely $(x_1,1), \ldots, (x_k,1)$, $(x_1,2), \ldots, (x_k,2)$. Now we see that

$$2 \cdot \sum_{i=1}^{k} \operatorname{index}(f,x_i) \underset{\substack{\uparrow \\ \text{it is elementary to see that} \\ \operatorname{index}(f,x_i) = \operatorname{index}(\psi,(x_i,1)) = \operatorname{index}(\psi,(x_i,2))}}{=} \sum_{i=1}^{k} \operatorname{index}(\psi,(x_i,1)) + \sum_{i=1}^{k} \operatorname{index}(\psi,(x_i,2)) \underset{\substack{\uparrow \\ \text{by the above, since } \mathrm{D}M \text{ is a closed} \\ \text{orientable smooth manifold and} \\ \text{since } \psi \text{ is smooth}}}{=} \Lambda(\psi\colon \mathrm{D}M \to \mathrm{D}M)$$

$$\underset{\substack{\uparrow \\ \text{follows from Exercise 182.15}}}{=} 2 \cdot \Lambda(f\colon M \to M) + \Lambda(f\colon \partial M \to \partial M) \underset{\substack{\uparrow \\ \text{since there are no fixed points on } \partial M \text{ and since} \\ \partial M \text{ is a closed orientable smooth manifold} \\ \text{we know by the above that } \Lambda(f\colon \partial M \to \partial M) = 0}}{=} 2 \cdot \Lambda(f).$$

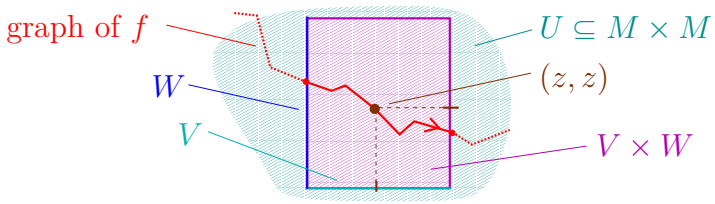
Dividing both sides by 2 gives us the desired equality. ∎

222.3. **The Lefschetz-Hopf Theorem II.** In this section we will formulate and prove an advanced version of the Lefschetz-Hopf Theorem 222.9 which puts fewer restrictions on the fixed points. The down-side is that in general we need to work with a more tricky definition of the index of a fixed point.

We start out with the following not overly exciting lemma.

Lemma 222.10. (Isolated Fixed Point–Neighborhood Lemma) Let M be an n-dimensional topological manifold and let $f\colon M \to M$ be a continuous map. Let $z \in M \setminus \partial M$ be an isolated fixed point of $f\colon M \to M$. Given any neighborhood U of $(z,z) \in M \times M$ there exists a neighborhood V of $z \in M$ and a neighborhood W of $z \in M$ with the following properties:

(1) We have $V \times W \subseteq U$.
(2) z is the only fixed point of f contained in V.
(3) V and W are both submanifolds of M and both are homeomorphic to $\overline{B}^n$.
(4) We have $V \subseteq \mathring{W}$ and $f(V) \subseteq \mathring{W}$.



Proof. This lemma follows fairly easily from Exercise 23.8. We leave it to the reader to fill in the details. ∎

The lemma allows us to introduce the following definition:

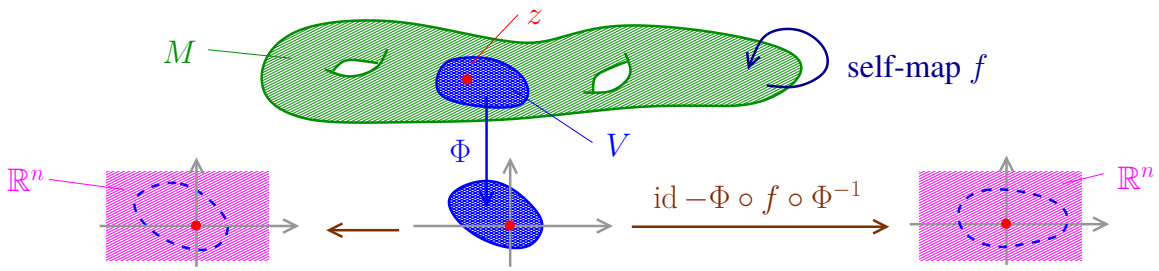
Definition 222.11. Let M be an n-dimensional topological manifold and let $f\colon M \to M$ be a continuous map. Furthermore let $z \in M \setminus \partial M$ be an isolated fixed point of f. We pick neighborhoods V and W of z that satisfy Properties (2) and (4) of the Isolated Fixed Point–Neighborhood Lemma 222.10. Furthermore we pick an embedding $\Phi\colon W \to \mathbb{R}^n$ with $\Phi(z) = 0$. We define the index $\operatorname{index}(f, z)$ of the fixed point z as the degree of the following homomorphism:

$$\overbrace{\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})}^{\cong \mathbb{Z} \text{ by the Manifold-Local Homology Lemma 150.9}} \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(\Phi(V), \Phi(V)\setminus\{0\}) \xrightarrow[\uparrow]{\mathrm{id} - (\Phi \circ f \circ \Phi^{-1})_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

this map is induced by the inclusion $\Phi(V) \subseteq \mathbb{R}^n$, it is an isomorphism by the Excision Theorem 148.2, here we also use that we obtain from the Topological Invariance-of-Domain Theorem 170.9 that $\Phi(V)$ is a neighborhood of 0

this map is defined since $f(V) \subseteq W$ and since x is the only fixed point of f contained in V

We illustrate this definition in the figure below.



Example. Let $n \in \mathbb{N}$. We consider the map

$$\begin{aligned} f\colon \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C} &\to \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \\ z &\mapsto z + z^n. \end{aligned}$$

The point $z = 0$ is evidently an isolated fixed point. Note that $f'(z) = 1 + nz^{n-1}$. It follows that for $n \geq 2$ we have $\mathrm{id} - \mathrm{D}f_0 = \mathrm{id} - \mathrm{id} = 0$. Thus we we see that for $n \geq 2$ the fixed point 0 is degenerate. Since 0 is an isolated fixed point we can nonetheless define the index using the approach above. We set $V = \overline{B}^2$ and $W = \overline{B}^2_3$ and $\Phi = \mathrm{id}$. Since

$$\mathrm{id} - (\Phi \circ f \circ \Phi^{-1}) \;=\; \mathrm{id} - f \;=\; \mathrm{id} - (\mathrm{id} + (z \mapsto z^n)) \;=\; -(z \mapsto z^n)$$

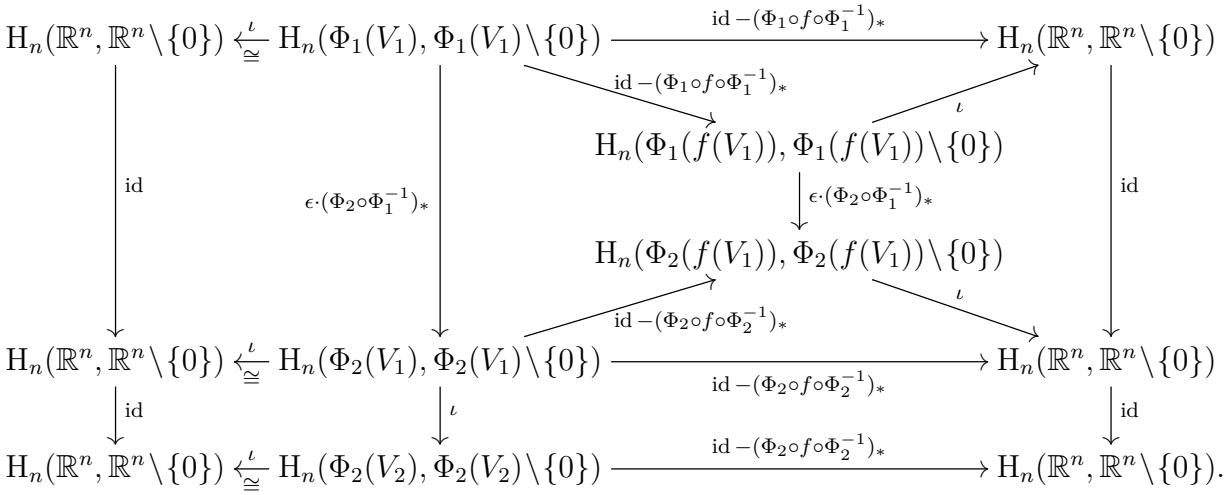
we obtain easily from the Degree-of-Power Lemma 151.5 that $\operatorname{index}(f, 0) = -n$. □

The following lemma says that the above index is well-defined.

Lemma 222.12. (Index-well-Defined Lemma) Let M be an n-dimensional topological manifold and let $f\colon M \to M$ be a continuous map. For an isolated fixed point $z \in M \setminus \partial M$ of f the above Definition 222.11 of $\operatorname{index}(f, z)$ does not depend on any choices.

Proof. Let V_1, W_1 and V_2, W_2 be neighborhoods of z that satisfy the properties stated in the Isolated Fixed Point–Neighborhood Lemma 222.10 (2) and (4). We pick embeddings $\Phi_1\colon W_1 \to \mathbb{R}^n$ and $\Phi_2\colon W_2 \to \mathbb{R}^n$ as in the definition of the index. We need to show that the choices lead to the same index.

First note that by the "given any neighborhood" feature of the Isolated Fixed Point–Neighborhood Lemma 222.10 we can assume that $V_1 \times W_1 \subseteq V_2 \times W_2$. Next we set $\epsilon := +1$ if $\Phi_2 \circ \Phi_1^{-1}$ is orientation-preserving, otherwise we set $\epsilon := -1$. Next we consider the following diagram:



We make the following clarifications and observations:

(1) The maps indicated by ι are the natural inclusion-induced maps.
(2) Since $\Phi_1 \colon W_1 \to \mathbb{R}^n$ is an embedding and since $V_1 \subseteq W_1$ we can consider the inverse $\Phi_1^{-1} \colon \Phi_1(V_1) \to V_1$. Furthermore, since $V_1 \subseteq V_2$ we can consider $\Phi_2(V_1)$ and we have $\Phi_2(V_1) \subseteq \Phi_2(V_2)$.
(3) It follows fairly immediately from the definition of ϵ that the diagram commutes.
(4) The left horizontal maps are isomorphisms by the Excision Theorem 148.2.

It follows from the above that the top and the bottom horizontal maps from left to right agree. But this means that we obtain the same degree. ■

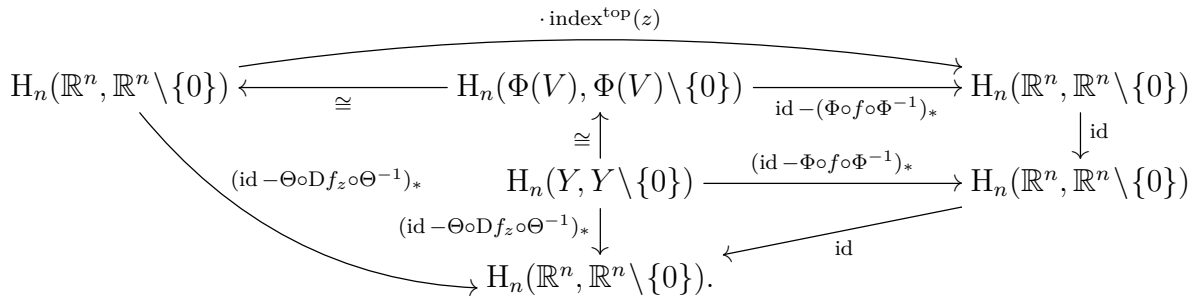
The following lemma shows that the two notions of the index of a fixed point of a smooth map, introduced on Definition 222.7 and Definition 222.11 are consistent.

Lemma 222.13. (Index-equals-Index Lemma) Let M be a smooth manifold, let $f \colon M \to M$ be a smooth map and let $z \in M \setminus \partial M$ be a non-degenerate fixed point of f. The fixed point z is isolated, and the index of z from Definition 222.7, agrees with the index of z from Definition 222.11.

Proof. Let M be a smooth manifold, let $f \colon M \to M$ be a smooth map and let $z \in M \setminus \partial M$ be a non-degenerate fixed point of f. For the purpose of this proof we denote by $\mathrm{index}^{\mathrm{sm}}(z)$ the index of z from Definition 222.7 and we denote by $\mathrm{index}^{\mathrm{top}}(z)$ the index of z from Definition 222.11.

First note that we already showed in the Isolated Fixed Point Lemma 222.8 that z is an isolated fixed point. Thus it remains to show that $\mathrm{index}^{\mathrm{top}}(z) = \mathrm{index}^{\mathrm{sm}}(z)$. Since $z \notin M$ we can pick a chart $\Phi \colon W \to \mathbb{R}^n$ around z from our smooth atlas for M with $\Phi(z) = 0$. It follows fairly easily from the hypothesis that z is a non-degenerate fixed point that there exists an open neighborhood V around z such that $V \subseteq W$, such that $\Phi(V) \subseteq W$, and such that V contains no other fixed point of f. We set $\Theta := \mathrm{D}\Phi_z \colon \mathrm{T}_z M \to \mathrm{T}_0 \mathbb{R}^n = \mathbb{R}^n$.

Now we consider the following diagram:



We make the following clarifications and observations:

(1) By construction V and W satisfy the properties of the Isolated Fixed Point–Neighborhood Lemma 222.10 (2) and (4). This implies that the composition of the top two horizontal maps is given by multiplication by $\operatorname{index}^{\mathrm{top}}(z)$.

(2) It follows immediately from the Linearization–Homotopy Lemma 152.7 that there exists an open neighborhood Y of 0, contained in $\Phi(V)$, such that the map $\mathrm{id} - \Phi \circ f \circ \Phi^{-1}$ is homotopic, as a map of pairs $(\Phi(V), \Phi(V) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, to the map $\mathrm{D}(\mathrm{id} - \Phi \circ f \circ \Phi^{-1})_0$.

(3) By the Chain Rule 24.17 we have $\mathrm{D}(\mathrm{id} - \Phi \circ f \circ \Phi^{-1})_0 = \mathrm{id} - \Theta \circ \mathrm{D}f_z \circ \Theta^{-1}$.

(4) It follows from (2) together with the Homotopy-H_*-Proposition 146.11 and some elementary considerations that the diagram commutes.

(5) It follows from the Excision Theorem 148.2 that the two maps decorated with "$\cong$" are indeed isomorphisms.

We see that

$$\begin{array}{ccc} & \text{by the above diagram and the above discussion} & \text{by the Matrix Multiplication–Degree Lemma 152.9} \\ \operatorname{index}^{\mathrm{top}}(z) & \downarrow & \downarrow \\ & = \deg((\mathrm{id}_{\mathbb{R}^n} - \Theta \circ \mathrm{D}f_z \circ \Theta^{-1})_*) & = \operatorname{sign}(\det(\mathrm{id}_{\mathbb{R}^n} - \Theta \circ \mathrm{D}f_z \circ \Theta^{-1})) \\ & = \operatorname{sign}(\det(\mathrm{id}_{\mathrm{T}_z M} - \mathrm{D}f_z)) & = \operatorname{index}^{\mathrm{sm}}(z). \\ & \uparrow & \uparrow \\ & \text{basic linear algebra} & \text{by Definition 222.7} \end{array}$$

■

It follows from the Index-equals-Index Lemma 222.13 the following theorem is a generalization of our previous Lefschetz-Hopf Theorem 222.9.

Theorem 222.14. (Lefschetz-Hopf Theorem II) Let M be a compact orientable topological manifold and let $f\colon M \to M$ be a continuous map which has no fixed points on ∂M. If f has finitely many fixed points, then

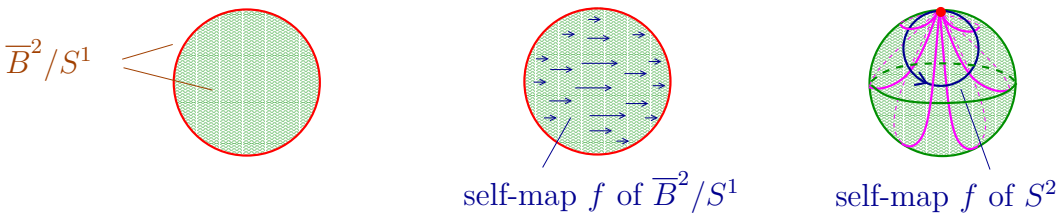
$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f, x) \;=\; \Lambda(f).$$

Remark.

(1) The Lefschetz-Hopf Theorem has its origin in the work of Heinz Hopf [**Hopf1929**, Section 8]. Various variations on the Lefschetz-Hopf Theorem 222.14 can be found in [**GD2003**, Chapter V.16] and [**JM2006**, Chapter 2.4].

(2) As on page 4294 we see that the above Lefschetz-Hopf Theorem 222.14 is a generalization of the Lefschetz Fixed Point Theorem 189.21 for compact orientable topological manifolds. □

Example. This time we consider S^2 with the avatar given by $\overline{B}^2/S^1$. We consider the map $f\colon \overline{B}^2/S^1 \to \overline{B}^2/S^1$ that is illustrated, to the best of the author's abilities, in the figure below.[2395]



It has precisely one fixed point, namely $P \coloneqq [S^1]$. We see that

$$\operatorname{index}(f,P) \underset{\substack{\uparrow\\ \text{Lefschetz-Hopf}\\ \text{Theorem 222.14}}}{=} \Lambda(f) \underset{\substack{\uparrow\\ \text{since } f \text{ is homotopic}\\ \text{to id}}}{=} \Lambda(\mathrm{id}) = \chi(S^2) = 2.$$

Note that with a little bit of care, say using the Smooth Transition Function Lemma 24.9, one can arrange that f is smooth. Also note that $\mathrm{D}f_P = \mathrm{id}$. Thus $P = [S^1]$ is a degenerate fixed point of f.[2396] □

The proof of the Lefschetz-Hopf Theorem 222.14 rests heavily on the shoulders of the following two lemmas.

Lemma 222.15. (Tilt Lemma) Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$ be two compact connected n-dimensional topological submanifolds of $\mathbb{R}^n$ that contain 0 in the interior. We assume that $V \subseteq \mathring{W}$. Let $f\colon V \to \mathbb{R}^n$ be a continuous map with $f(V) \subseteq \mathring{W}$ and which has a unique fixed point, namely 0. We set $G \coloneqq \{(v, f(v)) \,|\, v \in V\}$ and we set $D \coloneqq \{(v,v) \,|\, v \in V\}$.

(1) G and D are well-behaved proper submanifolds of $V \times W$, in the sense of Definition 221.4.

(2) The following equality holds:[2397] [2398]

$$\Big\langle \underbrace{\underbrace{\mathrm{PD}_{V\times W}([G])}_{\in \mathrm{H}^n(V\times W, \partial(V\times W)\setminus \partial G;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_{V\times W}([D])}_{\in \mathrm{H}^n(V\times W, \partial(V\times W)\setminus \partial D;\mathbb{Z})}}_{\in \mathrm{H}^{2n}(V\times W, \partial(V\times W);\mathbb{Z})}, [V\times W]\Big\rangle = \deg \begin{pmatrix} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \uparrow \cong \\ \mathrm{H}_n(V, V\setminus\{0\}) \\ \downarrow (\mathrm{id}-f)_* \\ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{pmatrix}.$$

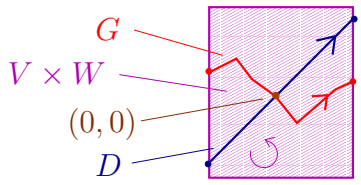
Sketch of proof. First we prove Statement (1). More precisely, we will show that G is a well-behaved proper submanifold of $V \times W$. The proof that D is also a well-behaved submanifold is basically identical. Note that by Exercise 23.11 we know that G is a proper submanifold of $V \times W$. Next note that by definition of a well-behaved submanifold we need

[2395] An alternative description of this map is given by considering $S^2 =_{\mathrm{i}} \mathbb{R}^2 \cup \{\infty\}$. In this setting our map corresponds to the map $f\colon \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ that is given by $f(x) = x + (1,0)$ for all $x \in \mathbb{R}^2$ and $f(\infty) = \infty$.

[2396] Incidentally this example gives an affirmative answer to the question asked in Exercise 4.15.

[2397] By (1) we know that G is a well-behaved proper submanifold of $V \times W$. Thus we can consider the isomorphism $\mathrm{PD}_{V\times W}\colon \mathrm{H}_n(V\times W, \partial G) \to \mathrm{H}^n(V\times W, (V\times W)\setminus \partial G;\mathbb{Z})$ that we introduced in the Neighborhood–Poincaré Duality Proposition 221.3. Similarly we consider the isomorphism $\mathrm{PD}_{V\times W}$ with G replaced by D.

[2398] Since 0 is the only fixed point of f we see that $(\mathrm{id}-f)(V\setminus\{0\}) \subseteq \mathbb{R}^n \setminus \{0\}$.



to show that given any open neighborhood U of $\partial G \subseteq \partial(V \times W) = (\partial V \times W) \cup (V \times \partial W)$ the submanifold ∂G of $\partial(V \times W)$ admits a regular neighborhood N with $N \subseteq U$. Note that using the fact that ∂V is compact and using the hypothesis that $f(V) \subseteq \mathring{W}$ it is fairly elementary to see that there exists an $\epsilon > 0$ such that for every $x \in \partial G$ we have $\{x\} \times \overline{B}^n_\epsilon(f(x)) \subseteq U$. It is straightforward to verify that given any $\nu \in (0, \epsilon)$ the set

$$N_\nu \;:=\; \{(x,y) \in \partial V \times W \,|\, x \in \partial V \text{ and } y \in \overline{B}^n_\nu(f(x))\} \;\subseteq\; \partial(V \times W)$$

is a regular neighborhood of ∂G with $N_\nu \subseteq U$. This concludes the proof of Statement (1).

Next we turn to the proof of Statement (2). The key idea is to "tilt" the situation to turn the diagonal D into a "horizontal submanifold". Afterwards we can use the Poincaré Duality-Products Proposition 210.3 to perform the desired calculation.

Now we carry out the details. To do so we introduce the following objects:

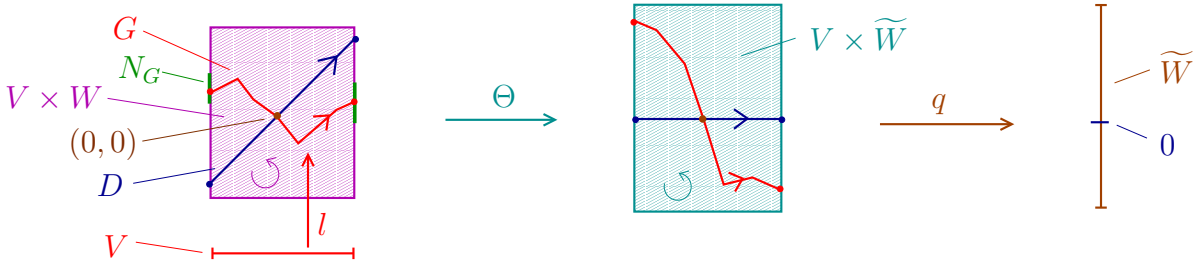
(a) We pick a compact submanifold $\widetilde{W}$ of $\mathbb{R}^n$ with $W \subseteq \widetilde{W}$ and $(\mathrm{id} - f)(W) \subseteq \widetilde{W}$.

(b) We consider the maps

$$\begin{array}{rcl} l\colon V & \to & G \\ v & \mapsto & (v, f(v)) \end{array} \qquad \begin{array}{rcl} \Theta\colon V \times W & \to & V \times \widetilde{W} \\ (v,w) & \mapsto & (v, w - f(v)) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon V \times \widetilde{W} & \to & \widetilde{W} \\ (v,w) & \mapsto & w. \end{array}$$

(c) Note that by (1) we can pick a regular neighborhood N_G for $\partial G \subseteq \partial(V \times W)$ with $N_G \subseteq \partial V \times W$ and with $N_G \cap \partial D = \varnothing$.

(d) We denote by $\mu_{\widetilde{W},0} \in \mathrm{H}_n(\widetilde{W}, \widetilde{W} \setminus \{0\}) \cong \mathbb{Z}$ the standard generator, as defined on page 3683. Furthermore we denote by $\mu^*_{\widetilde{W},0} \in \mathrm{H}^n(\widetilde{W}, \widetilde{W} \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$ the corresponding dual.



Now we can perform the desired calculation:[2399]

follows from the Cap Product-Kronecker Lemma 206.12 and the Cup-Cap Lemma 206.15, under the natural identification $\mathrm{H}_0(V\times W)=_{\mathrm{i}}\mathbb{Z}$

$=[G]\in \mathrm{H}_n(V\times W, N_G)$ by Definition 210.2 of $\mathrm{PD}_{V\times W}$

$$\begin{array}{lll}
\big\langle \mathrm{PD}_{V\times W}([G])\cup \mathrm{PD}_{V\times W}([D]), [V\times W]\big\rangle & = & \mathrm{PD}_{V\times W}([D])\cap \big(\overbrace{\mathrm{PD}_{V\times W}([G])\cap [V\times W]}\big) \\
= \mathrm{PD}_{\Theta(V\times W)}(\Theta_*([D]))\cap \Theta_*([G]) & = & \mathrm{PD}_{V\times \widetilde{W}}(\Theta_*([D]))\cap \Theta_*([G])
\end{array}$$

by the Cap Product-Naturality Lemma 206.13 and since the map $\Theta\colon V\times W\to\Theta(V\times W)\subseteq V\times\widetilde{W}$ is an *orientation-preserving* homeomorphism

follows from an argument similar to the one provided in the Small H_*-to-Large H^*-PD Lemma 212.8

$$\begin{array}{lll}
= \mathrm{PD}_{V\times\widetilde{W}}([V\times\{0\}])\cap \Theta_*(l_*([V])) & = & (-1)^n\cdot q^*(\mu^*_{\widetilde{W},0})\cap \Theta_*(l_*([V])) \\
\text{since } \Theta(D)=V\times\{0\} \text{ and } l(V)=G \text{ and since both maps are orientation-preserving} & & \text{this can be deduced fairly easily from the Poincaré Duality-Products Proposition 210.3} \\
= (-1)^n\cdot\big\langle \mu^*_{\widetilde{W},0}, q_*(\Theta_*(l_*([V])))\big\rangle & = & \big\langle \mu^*_{\widetilde{W},0}, (\mathrm{id}-f)_*([V])\big\rangle. \\
\text{by the Cap Product-Kronecker Lemma 206.12 and the Kronecker Pairing Natural Lemma 199.9 (1)} & & \text{since } q\circ\Theta\circ l = f-\mathrm{id}\colon V\to\widetilde{W} \text{ and since we obtain from the Matrix Multiplication–Degree Lemma 152.9 that } (f-\mathrm{id})_* = (-1)^n\cdot(\mathrm{id}-f)_*
\end{array}$$

$$= \deg\big(\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \xleftarrow{\cong} \mathrm{H}_n(V,V\setminus\{0\}) \xrightarrow{(\mathrm{id}-f)_*} \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\big).$$

this follows from the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & \xleftarrow{\cong} & \mathrm{H}_n(V,V\setminus\{0\}) & \xrightarrow{(\mathrm{id}-f)_*} & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \\
 & & \uparrow & & \uparrow \\
 & & \mathrm{H}_n(V,\partial V) & \xrightarrow{(\mathrm{id}-f)_*} & \mathrm{H}_n(\widetilde{W},\widetilde{W}\setminus\{0\})
\end{array}$$

together with the fact that $[V]$ and $\mu_{\widetilde{W},0}$ are both mapped to the standard generator of $\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$

Thus we have proved the desired equality. ∎

Lemma 222.16. (Index-equals-Index Lemma) Let M be a closed oriented n-dimensional topological manifold. Furthermore let $f\colon M\to M$ be a continuous map. Finally let $z\in M$ be an isolated fixed point of f. We consider the two submanifolds $\mathrm{Gr}(f)$ and Δ_M of the $2n$-dimensional topological manifold $M\times M$. The following two statements hold:

(1) The point $(z,z)\in M\times M$ is a nice intersection point of the two submanifolds $\mathrm{Gr}(f)$ and Δ_M, in the sense of Definition 221.7.

(2) We have the equality

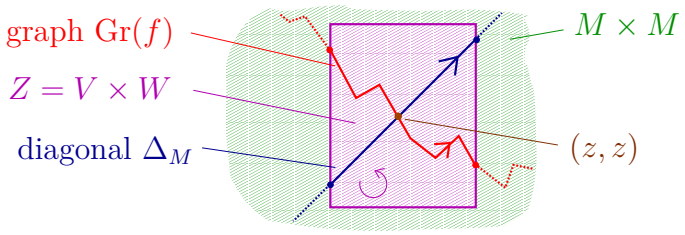
$$\mathrm{index}(f,z) = \underbrace{\mathrm{index}(\mathrm{Gr}(f),\Delta_M,(z,z))}_{\text{index of the nice intersection point } (z,z), \text{ as introduced in Definition 221.9}}.$$

[2399] Careful reading of the equalities shows that it is in fact at times difficult to determine which relative (co-) homology groups the objects lie in. In particular on several occasions we implicitly use the Cap Product-Naturality Lemma 206.13 and the Kronecker Pairing–Natural Lemma 199.9 (1) to go from one relative homology group to another. Writing down the calculation in full detail would make the proof even more unreadable that it already is.

Remark. On page 4300 we gave an example of a continuous map $f\colon S^2 \to S^2$ that has a unique fixed point P, which has index 2. It follows from the Index-equals-Index Lemma 222.16 that (P,P) is a nice intersection point of the submanifolds $\mathrm{Gr}(f)$ and Δ_{S^2} of $S^2 \times S^2$ of index 2. Note that this gives an affirmative answer to the question posed in Exercise 221.5. □

Proof.

(1) Let $z \in M$ be an isolated fixed point of $f\colon M \to M$. We need to show that the point $(z,z) \in M \times M$ is a nice intersection point of the two submanifolds $\mathrm{Gr}(f)$ and Δ_M. Thus let U be an open neighborhood of (z,z) in $M \times M$. By the Isolated Fixed Point–Neighborhood Lemma 222.10 there exists a neighborhood V of $z \in M$ and a neighborhood W of $z \in M$ with the following properties:
   (i) We have $V \times W \subseteq U$.
   (ii) z is the only fixed point of f contained in V.
   (iii) V and W are both submanifolds of M and both are homeomorphic to $\overline{B}^n$.
   (iv) We have $V \subseteq \mathring{W}$ and $f(V) \subseteq \mathring{W}$.
   By (i) it remains to show that $Z \coloneqq V \times W$ is a nice neighborhood of (z,z) in the sense of Definition 221.7. Most of the conditions are easily verified. For example by (iii) we know that $V \times W$ is homeomorphic to $\overline{B}^{2n}$. Furthermore, by (ii) we know that $\mathrm{Gr}(f) \cap \Delta_M \cap (V \times W) = \{(z,z)\}$. Finally note that it follows almost immediately from the Tilt Lemma 222.15 (1) that $\mathrm{Gr}(f)$ and Δ_M are both well-behaved proper submanifolds of $V \times W$.



(2) To simplify the notation we can and will assume that $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$, $z = 0$ and $\Phi = \mathrm{id}$. But now in this setting the desired equality is an immediate consequence of the Tilt Lemma 222.15 (2). ■

Now that we have the Index-equals-Index Lemma 222.16 under our belt, the proof of the Lefschetz-Hopf Theorem 222.14 is formally very similar to the proof of the Lefschetz-Hopf Theorem 222.9:

Proof of the Lefschetz-Hopf Theorem 222.14. Let M be a compact orientable topological manifold and let $f\colon M \to M$ be a continuous map which has only finitely many fixed points $z_1, \dots, z_k$, and all of them lie in $M \setminus \partial M$. We pick an orientation for M.

First we assume that M is actually closed. Now we easily obtain the desired equality:

$$\Lambda(f) \overset{\text{by the Lefschetz-Intersection Number Proposition 222.4}}{=} Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) \overset{\text{by the Index-equals-Index Lemma 222.16 (1) and the Maps-Cup Product Intersection Number Theorem 221.16}}{=} \mathrm{Gr}(f)\cdot\Delta_M \overset{\text{by Definition 221.12}}{=} \sum_{i=1}^{k} \mathrm{index}(\mathrm{Gr}(f),\Delta_M,(z_i,z_i)) \overset{\text{by the Index-equals-Index Lemma 222.16 (2)}}{=} \sum_{i=1}^{k} \mathrm{index}(f,z_i).$$

Now we assume that M has non-empty boundary. Once again we consider the double $\mathrm{D}M := (M \times \{1\}) \sqcup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$. By the Doubling-Manifold Lemma 54.10 and the Double-Orientability Lemma 192.6 we know that $\mathrm{D}M$ is a closed orientable topological manifold. We consider the map

$$\begin{array}{rcl} \psi\colon \mathrm{D}M & \to & \mathrm{D}M \\ [(P,\epsilon)] & \mapsto & [(f(P),\epsilon)]. \end{array}$$

Note that it follows from the Topological-Pushout Proposition 8.12 (1b) that ψ is continuous. Since all of the fixed points of f lie in $M \setminus \partial M$ we see that the fixed points of ψ are $(z_1,1),\dots,(z_k,1),(z_1,2),\dots,(z_k,2)$. Now we see that

$$2 \cdot \sum_{i=1}^{k} \operatorname{index}(f,z_i) \underset{\uparrow}{=} \sum_{i=1}^{k} \operatorname{index}(\psi,(z_i,1)) + \sum_{i=1}^{k} \operatorname{index}(\psi,(z_i,2)) \underset{\uparrow}{=} \Lambda(\psi\colon \mathrm{D}M \to \mathrm{D}M)$$

it follows immediately from the definitions that $\operatorname{index}(f,z_i) = \operatorname{index}(\psi,(z_i,1)) = \operatorname{index}(\psi,(z_i,2))$

by the above, since $\mathrm{D}M$ is a closed orientable topological manifold

$$\underset{\uparrow}{=} 2 \cdot \Lambda(f\colon M \to M) + \Lambda(f\colon \partial M \to \partial M) \underset{\uparrow}{=} 2 \cdot \Lambda(f).$$

follows from Exercise 182.15

since there are no fixed points on ∂M and since ∂M is a closed orientable topological manifold we know by the above that $\Lambda(f\colon \partial M \to \partial M) = 0$

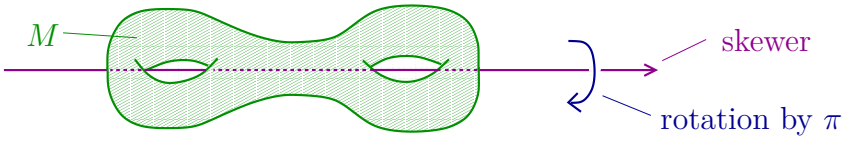
Dividing both sides by 2 yields the desired result. ∎

222.4. Exercises for Chapter 222 .

Exercise 222.1. We consider the surface $M \subseteq \mathbb{R}^3$ as shown in the figure below. Furthermore we consider the map $\varphi\colon M \to M$ which is given by "rotating M around the skewer by an angle of π".

(a) For $i = 0,1,2$ determine $\operatorname{tr}(\varphi_*\colon \mathrm{H}_i(M) \to \mathrm{H}_i(M))$.
(b) For each fixed point x of φ determine $\operatorname{index}(\varphi,x)$.

Remark. Evidently you should use the Lefschetz-Hopf Theorem 222.14 to check your calculations in (a) and (b) for consistency.



Exercise 222.2.

(a) Let $k,l \in \mathbb{Z}$. We consider the sphere $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w^2| = 1\}$ and the continuous self-map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (z^k, w^l). \end{array}$$

(b) We view S^2 as a subspace of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^2 \\ (z,t) & \mapsto & (z^2,t) \end{array}$$

(c) Let $n \in \mathbb{N}$. We consider the maps

$$h\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\} \qquad z \mapsto \begin{cases} z^n, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases}$$

and

$$k\colon \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\} \qquad z \mapsto \begin{cases} z + z^n, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty. \end{cases}$$

In each case determine all fixed points, determine whether they are non-degenerate and determine the index of each fixed point.

Exercise 222.3. For which $n \in \mathbb{N}_0$ does every continuous map $f\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ have a fixed point?

Exercise 222.4. Let $n \in \mathbb{N}_0$ and let $k \in \mathbb{N}_0$. Determine the Lefschetz number of the map

$$\begin{array}{rcl} f_k\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_n] & \mapsto & [z_0^k : \cdots : z_n^k]. \end{array}$$

Exercise 222.5. Let $g \in \mathbb{N}_0$ and let $\Xi\colon \Sigma_g \to \Sigma_g$ be an orientation-preserving diffeomorphism with $\Xi \circ \Xi = \mathrm{id}$. In other words, Ξ is an orientation-preserving involution of Σ_g. Show that if Ξ has precisely $2g+2$ fixed points, then the map $\Xi_*\colon \mathrm{H}_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g)$ equals $-\mathrm{id}$.

Hint. Use the Lefschetz-Hopf Theorem 222.14.

CHAPTER 223

The Poincaré-Hopf Theorem

Now we return to the study of vector fields on smooth manifolds.[2400] Recall that the Hairy Ball Theorem 46.18 says that there does not exist a nowhere-vanishing vector field on any even-dimensional sphere. Also recall that later on we proved a significant generalization of this result. Namely, using the Lefschetz Fixed Point Theorem 182.15 we proved the following theorem:

Theorem 182.22. (Euler Characteristic–Vector Field Theorem) Let M be a closed smooth manifold. If M admits a nowhere-vanishing vector field, then $\chi(M) = 0$.

In Chapter 221 we proved the Lefschetz-Hopf Theorems 222.9 and 222.14 which are significant generalization of the Lefschetz Fixed Point Theorem 182.16 that we had just mentioned. In this chapter we will use these recently acquired theorems to prove the Poincaré-Hopf Theorem 223.8 which is a significant generalization of the above Euler Characteristic–Vector Field Theorem 182.22.

This chapter is organized as follows:

- In Section 223.1 we will introduce the index of a zero of a vector field.
- In Section 223.2 we will state and prove the Poincaré-Hopf Theorem 223.8.
- In Section 223.3 we will give a new proof of the Gauß Theorem 53.32.
- In Section 223.4 we will provide exercises for Chapter 223.

In the next Chapter 224 we will introduce the intersection form on compact oriented even-dimensional topological manifolds.

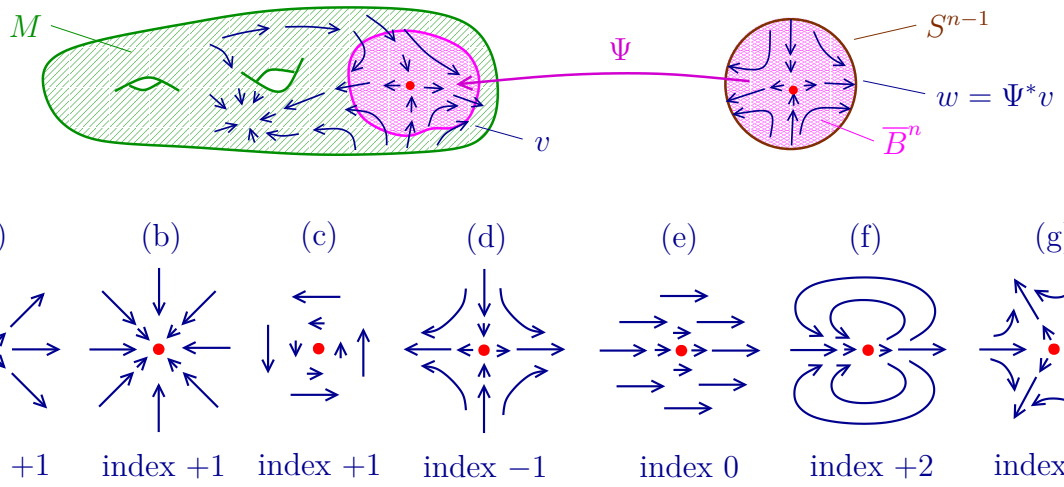
223.1. **The index of a zero of a vector field.** In the following discussion it will be helpful to study zeros of vector fields in more detail. As we will see, not all zeros are created equal, but they actually come in different flavors. More precisely, we have the following definition:

Definition 223.1. Let M be an n-dimensional smooth manifold, let v be a vector field on M and let $x \in M \setminus \partial M$ be an isolated zero of v. We pick a smooth embedding $\Psi\colon \overline{B}^n \to M \setminus \partial M$ with $\Psi(0) = x$ and such that x is the only zero of v on $\Psi(\overline{B}^n)$. We denote by $w = \Psi^*(v)$ the pullback vector field on $\overline{B}^n$ from Lemma 29.33. In other words, given $y \in \overline{B}^n$ we write $w(y) := \mathrm{D}\Psi^{-1}_{\Psi(y)}(v(\Psi(y)))$. We define the index of v at x as follows:

$$\begin{aligned} \operatorname{index}(v,x) \;:=\; \text{degree of the map}\quad \Theta_{\Psi,v}\colon S^{n-1} &\to S^{n-1} \\ y &\mapsto \frac{w(y)}{\|w(y)\|}. \end{aligned}$$

Example. We consider the vector fields on $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ with isolated zero at the origin that are shown in the figure below.

[2400] We refer to Definition 29.29 for the definition of a vector field on a given smooth manifold M.



In the following we use the identity chart $\Psi = \mathrm{id}$ to determine the various indices. In fact, using the Degree-of-Power Lemma 151.5 and the Sphere–Degree Lemma 151.4, we obtain the same table as on page 1201:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
vector field v at $(x,y)=z$	(x,y)	$(-x,-y)$	$(-y,x)$	$(x,-y)$	$(\lvert x\rvert+\lvert y\rvert,0)$	z^2	z^{-2}
degree of $\Theta_{\Psi,v}\colon S^1 \to S^1$	$+1$	$+1$	$+1$	-1	0	$+2$	-2.

□

The following lemma shows that the above Definition 223.1 makes sense:

Lemma 223.2. (Index-Well Defined Lemma) Let M be an n-dimensional smooth manifold.

(1) Let v be a vector field on M. The definition of the index $\mathrm{index}(v,x)$ of an isolated zero $x \in M \setminus \partial M$ of a vector field v does not depend on the choice of the smooth embedding $\Psi\colon \overline{B}^n \to M$.

(2) If v and w are two vector fields on M that agree on an open subset $U \subseteq M$ and if z is an isolated zero of v in U, then $\mathrm{index}(v,z) = \mathrm{index}(w,z)$.

Proof. Statement (2) is a straightforward consequence of Statement (1). Thus it remains to prove Statement (1). Let $\Psi\colon \overline{B}^n \to M$ be a smooth embedding as in the definition of $\mathrm{index}(v,x)$. We need to show that any other such smooth embedding leads to the same index. The proof is performed in two stages.

(a) Let $A \in \mathrm{O}(n)$ be an orthogonal matrix. We denote by $\rho_A\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by multiplication by the matrix A. Next we consider the smooth embedding $\Psi \circ \rho_A\colon \overline{B}^n \to M$. A straightforward calculation using the Chain Rule 24.17, shows that $\Theta_{\Psi\circ\rho_A,v} = \rho_A^{-1} \circ \Theta_{\Psi,v} \circ \rho_A\colon S^{n-1} \to S^{n-1}$. It follows from the Sphere–Degree Lemma 151.4 that $\deg(\Theta_{\Psi\circ\rho_A,v}) = \deg(\Theta_{\Psi,v})$.

(b) Now assume that we are given some other smooth embedding $\Xi\colon \overline{B}^n \to M$. We consider

$$N \;:=\; \underbrace{\{y \in M \,|\, v(y) \neq 0\}}_{\text{open subset by Exercise 182.16}} \;\cup\; \{x\}.$$

It follows from Exercise 182.16 and the fact that x is an isolated zero that N is an open subset of M. In particular we see that N is an n-dimensional submanifold of M. Since x is the only zero of v in $\Psi(\overline{B}^n)$ and $\Xi(\overline{B}^n)$ we see that $\Psi\colon \overline{B}^n \to M$ and $\Xi\colon \overline{B}^n \to M$ are tubular maps, in the sense of Definition 58.2, for the 0-dimensional submanifold $\{x\}$ of N. By the Tubular Map–Uniqueness Theorem 58.5 there exists a diffeotopy $H\colon N \times [0,1] \to N$ rel $\{x\}$ and a matrix $A \in \mathrm{O}(n)$ such that $H_0 = \mathrm{id}$ and such that $H_1 \circ \Xi = \Psi \circ \rho_A\colon \overline{B}^n \to N$. Now we see that

$$\deg(\Theta_{\Xi,v}) = \deg(\Theta_{H_0\circ\Xi,v}) \underset{\uparrow}{=} \deg(\Theta_{H_1\circ\Xi,v}) = \deg(\Theta_{\Psi\circ\rho_A,v}) \underset{\uparrow}{=} \deg(\Theta_{\Psi,v}).$$

the maps $\Theta_{H_t\circ\Xi,v}$ with $t \in [0,1]$ define a homotopy between $\Theta_{H_0\circ\Xi,v}$ and $\Theta_{H_1\circ\Xi,v}$, thus by the Sphere–Degree Lemma 151.4 (3) the maps have the same degree

by (1)

■

The next example shows that any integer can occur as the index of an isolated zero of a vector field.

Example. Let $n \in \mathbb{N}_{\geq 2}$ and let $d \in \mathbb{Z}$. Note that it follows from the Degree-of-Power Lemma 151.5 and the Suspension-Degree Lemma 151.6 that there exists a continuous self-map $\varphi\colon S^{n-1} \to S^{n-1}$ of degree d. Let $\lambda \in \mathbb{R}_{>0}$. It is straightforward to verify that

$$\begin{aligned} v(\varphi)\colon \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto \begin{cases} 0, & \text{if } x = 0, \\ \lambda \cdot x \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0 \end{cases} \end{aligned}$$

is a continuous map, in particular it defines a vector field on $\mathbb{R}^n$. Evidently it has a single isolated zero at the origin and the corresponding index is precisely $d = \deg(\varphi)$. □

The next lemma can be convenient for calculating the index of a zero of a vector field.

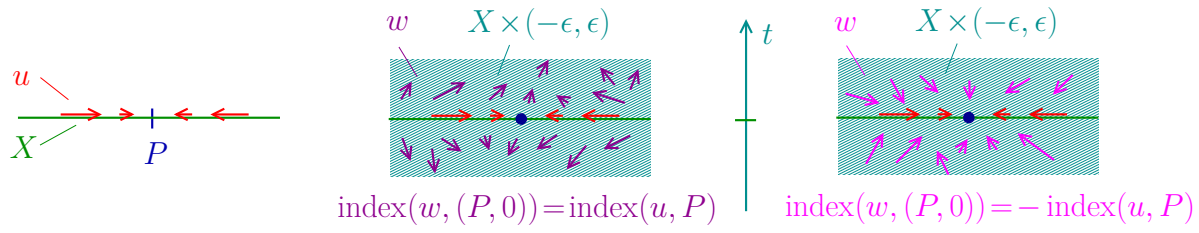
Lemma 223.3. (Index–Induction Lemma) Let $X \subseteq \mathbb{R}^n$ be an open subset and let $\epsilon > 0$. Furthermore suppose that we are given a vector field $w = (u, v)\colon X \times (-\epsilon, \epsilon) \to \mathbb{R}^n \times \mathbb{R}$ with an isolated zero at $(P, 0)$.

(1) If $v(x,t) > 0$ whenever $t > 0$ and if $v(x,t) < 0$ whenever $t < 0$, then

$$\mathrm{index}(w, (P,0)) = \mathrm{index}(u, P).$$

(2) If $v(x,t) < 0$ whenever $t > 0$ and if $v(x,t) > 0$ whenever $t < 0$, then

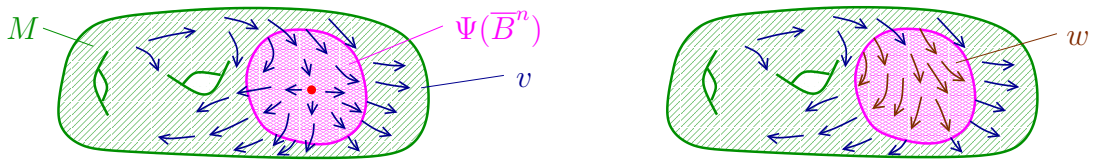
$$\mathrm{index}(w, (P,0)) = -\mathrm{index}(u, P).$$



Proof. We leave it to the reader to verify that the two statements follow easily from the definitions together with Exercise 151.10. ■

The following lemma says that isolated zeros of a vector field of index 0 can be eliminated by adjusting the vector field in the neighborhood of the zero.

Lemma 223.4. (Zero Elimination Lemma) Let M be an n-dimensional smooth manifold, let v be a vector field on M and let x be an isolated zero of v with $\text{index}(v,x) = 0$. Let $\Psi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding with $\Psi(0) = x$ and such that x is the only zero of v in $\Psi(\overline{B}^n)$. There exists a vector field w on M that agrees with v on $M \setminus \Psi(B^n)$ and which has no zeros on $\Psi(\overline{B}^n)$.



Sketch of proof. First recall that by the $\pi_n(S^n)$–Proposition 167.10 we know that the map $\deg\colon \pi_{n-1}(S^{n-1}) \to \mathbb{Z}$ is an isomorphism. In particular this implies, using our hypothesis that $\text{index}(v,x) = 0$, that there exists a homotopy $H\colon S^{n-1} \times [0,1] \to S^{n-1}$ from the map $\Theta_{\Psi,v}\colon S^{n-1} \to S^{n-1}$ to a constant map. Using the homotopy H it is now straightforward to write down an explicit vector field w on M with the following properties:

(1) On $M \setminus \Psi(B^n)$ the vector field w equals v.
(2) On $\Psi(\overline{B}^n_{\frac{1}{4}})$ the vector field w is "constant" and non-zero.
(3) On $\Psi(\overline{B}^n \setminus B_{\frac{1}{2}})$ the vector field interpolates between (1) and (2).

We leave it to the reader to fill in the details. ■

We continue with the following refinement of the Non-Degenerate Vector Field Existence Proposition 53.22.

Proposition 223.5. (Non-Degenerate Vector Field Existence Proposition II) Every closed smooth manifold M admits a smooth vector field v such that all zeros are non-degenerate and such that

$$\sum_{x \text{ a zero of } v} \text{index}(v,x) \;=\; \chi(M).$$

Remark. On page **??** we will give an alternative proof of the Non-Degenerate Vector Field Existence Proposition 223.5. □

Before we turn to the proof of the Non-Degenerate Vector Field Existence Proposition 223.5 let us first state and prove the following corollary:

Corollary 223.6. (Euler Characteristics–Equality Corollary) For every closed smooth manifold M the Euler characteristic $\chi(M)$ equals the smooth Euler characteristic $\chi^{\text{sm}}(M)$ of closed orientable smooth manifolds that we had introduced in Definition 53.4.

Proof of Corollary 223.6. [2401] Let v be a smooth vector on M as in the Non-Degenerate Vector Field Existence Proposition 223.5. Then we see that

$$\chi(M) \;\underset{\substack{\uparrow\\ \text{choice of } v}}{=}\; \sum_{x \text{ a zero of } v} \text{index}(v,x) \;\underset{\substack{\uparrow\\ \text{by the Smooth Poincaré–Hopf}\\ \text{Theorem 53.24}}}{=}\; \chi^{\text{sm}}(M).$$

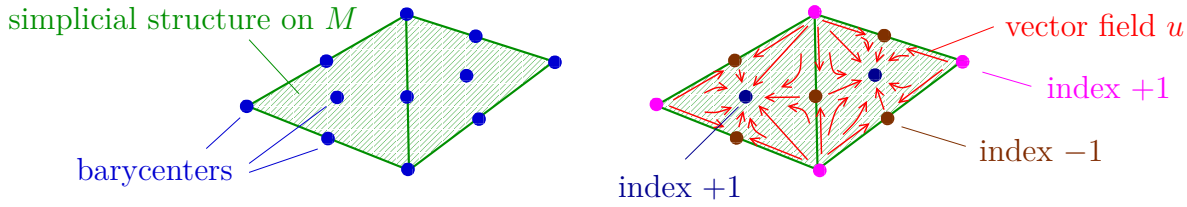
■

[2401] Note that we gave an alternative proof on page 4287.

Proof of Proposition 223.5. Let M be a closed connected smooth manifold. First note that by the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a smooth simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. Inductively we can define a smooth vector field u on M with the following properties:

(1) The vector field u is defined to be the zero vector field on the vertices of the triangulation.
(2) Suppose the vector field u is defined on the k-skeleton of M, then on every $(k+1)$-simplex s we extend the vector field u in the following way:
   (a) the vector field vanishes on the barycenter $\underline{s}$ of s,
   (b) for all other points in the open simplex the vector field "points towards" the barycenter $\underline{s}$ of s.
   This can be done explicitly.[2402]

We sketch the construction of the vector field u in the figure below.



Note that the zeros of the vector field u are precisely the barycenters of the original simplices. Furthermore, one can verify fairly easily that given a k-simplex s we have[2403] $\operatorname{index}(u, \underline{s}) = (-1)^k$. Thus we see that

$$\sum_{x \text{ a zero of } u} \operatorname{index}(u, x) \;=\; \sum_{s \text{ a } k\text{-simplex}} \operatorname{index}(u, \underline{s}) \;=\; \sum_{s \text{ a } k\text{-simplex}} (-1)^k \;=\; \chi(M).$$

by the Simplicial Complex–Topological Properties Corollary 174.52 (5)

This argument is also sketched in [**Stee1951**, p. 202]. ■

We conclude this chapter with the following converse to the Smooth-χ-Vector Field Theorem 53.29 and the Euler Characteristic–Vector Field Theorem 182.22.

[2402]For the resilient reader we provide the gory details. Suppose we are given a vector field v on $\partial\Delta^{k+1}$. We denote by $b := (\frac{1}{k+1}, \dots, \frac{1}{k+1})$ the barycenter of Δ^{k+1}. We denote by $r\colon \Delta^{k+1} \setminus \{b\} \to \partial\Delta^{k+1}$ the map that is given by sending x to the unique intersection point of the ray $b + \mathbb{R}_{\geq 0} \cdot (x - b)$ with $\partial\Delta^{k+1}$. Next we consider the map

$$\begin{array}{rcl} \Delta^{k+1} & \to & \mathbb{R}^{k+2} \\ x & \mapsto & \begin{cases} \frac{\|x-b\|}{\|r(x)-b\|} \cdot (v(r(x)) + (b-x) \cdot \|x - r(x)\|), & \text{if } x \neq b, \\ 0, & \text{if } x = b. \end{cases} \end{array}$$

It is now fairly elementary to see that this vector field, transported back to M, has the desired properties. As always, with a little more care, say using the Smooth Transition Function Lemma 24.9, one can arrange that the vector field is actually smooth.

[2403]The fact that $\operatorname{index}(u, \underline{s}) = (-1)^k$ can be seen as follows. First note that the vector field u points in k directions towards $\underline{s}$ and in $(n-k)$ directions away from $\underline{s}$. Using a suitable embedding $\Psi\colon \overline{B}^m \to M$ with $\Psi(0) = \underline{s}$ we see that the corresponding map $\Theta_{\Psi,v}\colon S^{n-1} \to S^{n-1}$ is given by the diagonal matrix with k entries equal to -1 and $(n-k)$-entries equal to $+1$. Thus it follows from the Sphere-Degree-Matrix Lemma 151.7 that the degree of this map equals $(-1)^k$.

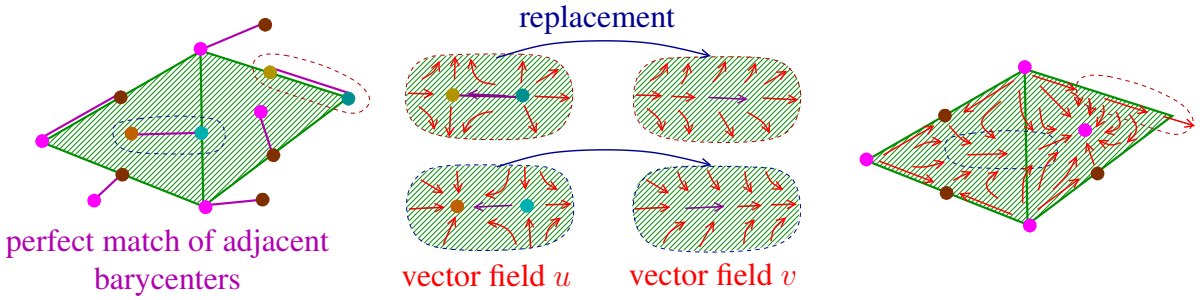
Theorem 223.7. (Nowhere Vanishing Vector Field Existence Theorem) Let M be a closed connected smooth manifold. If $\chi(M) = 0$, then M admits a nowhere-vanishing smooth vector field.

Sketch of proof. Let M be a closed connected smooth manifold with $\chi(M) = 0$. We need to show that M admits a nowhere-vanishing smooth vector field. This statement is proved in [**Bre1993**, Corollary VII.14.5], [**Stee1951**, Corollary 39.8], [**Hirs1976**, Theorem 5.2.10] and alternatively also in [**Tur1990**, Lemma 6.3.4]. In the following we provide a terse sketch of the proof given in [**Tur1990**, Lemma 6.3.4]. We continue with the discussion of the above proof of Proposition 223.5 and we make the following observations and constructions.

(1) Some thought shows that it follows from our hypothesis that M is connected and that $\chi(M) = 0$ that we can find a perfect match between adjacent barycenters. More precisely, we can find a decomposition $S = \bigsqcup_{i \in I} \{s_i, t_i\}$ of the simplex set, such that each s_i is a codimension one face of t_i.

(2) Given each pair $\{s_i, t_i\}$ we consider a small suitable neighborhood of the 1-simplex $\{\underline{s}_i, \underline{t}_i\}$ of the barycentric subdivision. This neighborhood contains two critical points. As of right now, on this 1-simplex the vector field points from $\underline{s}_i$ to $\underline{t}_i$.

We can replace the vector field u on this neighborhood by a different vector field v that agrees with u on the boundary, which on the 1-simplex now points from $\underline{t}_i$ to $\underline{s}_i$ and which has no critical points. This replacement is illustrated in the figure below.



Hopefully this outline is convincing. Evidently filling in all details is a nightmarish task. Thankfully the details are provided in [**Tur1990**, Lemma 6.3.4]. ■

223.2. **The Poincaré-Hopf Theorem.** Now we can formulate the namesake of this chapter.

Theorem 223.8. (Poincaré-Hopf Theorem) Let M be a compact smooth manifold. Let v be a vector field[2404] on M with finitely many zeros. If M has non-empty boundary, then we insist that along the boundary ∂M the vector field v is pointing outward. Then we have the formula

$$\sum_{x \text{ a zero of } v} \operatorname{index}(v, x) \;=\; \chi(M).$$

Remark.

(1) In the 2-dimensional setting the Poincaré-Hopf Theorem was first proved by Henri Poincaré [**Poi1885**] in the late 19th century. The generalization to the higher-dimensional case was first proved by Heinz Hopf [**Hopf1927b**] in 1927. There are many

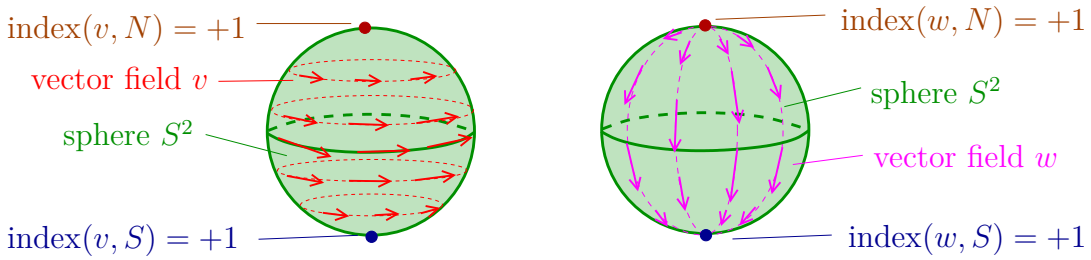
[2404]It is perhaps worth recalling that for us vector fields are assumed to be continuous. We do not make any assumptions on differentiability.

different proofs of the Poincaré-Hopf Theorem, sometimes with somewhat different hypotheses, in the literature, see e.g. [**BoT1982**, Theorem 11.25], [**Vic1994**, Theorem 7.27], [**Bre1993**, Theorem VI.12.13], [**Miln1965a**, p. 35], [**DFN1985**, Theorem 15.3.2], [**BaT2003**, Theorem 7.6.5] and [**MTo1997**, Theorem 12.1]. See also the sketch in [**Bra2007**, p. 69]. The proof we provide is partly inspired by the proofs in [**DFN1985**] and [**Vic1994**].

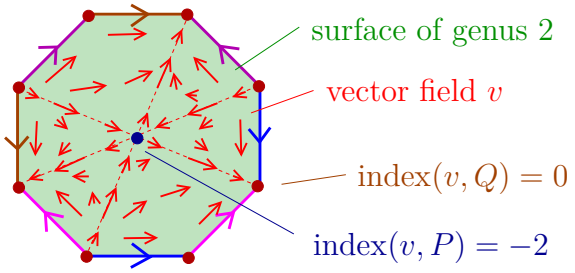
(2) The proof of the Poincaré-Hopf Theorem 223.8 relies very much on the Lefschetz-Hopf Theorem 222.14. For readers who prefer to work with the weaker Lefschetz-Hopf Theorem 222.9 we leave it as a little challenge to deduce from it the conclusion of the Poincaré-Hopf Theorem 223.8 under the stronger hypotheses that the vector field is smooth and that the indices of the zeros are ± 1. □

Examples.

(1) In the figure below we see two vector fields v and w on S^2. Each of these vector fields has precisely two zeros, namely the North Pole N and the South Pole S. Furthermore, considering the table on page 1201, it it pretty clear that for both vector fields both zeros have index $+1$. In particular we see that for both vector fields these two indices add up reassuringly to $\chi(S^2) = 2$.



(2) In the figure below we see a vector field on the surface Σ of genus two with two zeros P and Q. The table on page 1201 strongly suggests that $\text{index}(v, P) = -2$.



By the discussion on page 2987 we also know that $\chi(\Sigma) = -2$. Thus it follows from the Poincaré-Hopf Theorem 223.8 that $\text{index}(v, Q) = 0$. By the Zero Elimination Lemma 223.4 this means that we can modify the vector field v to obtain a vector field which has no zero at Q. Thus we have found a vector field on Σ with a single zero, namely at P. This gives a belated solution to Exercise 182.12. □

It is perhaps worth stating the following immediate corollary to the Poincaré-Hopf Theorem 223.8. Note that this corollary generalizes the statement of the Euler Characteristic–Vector Field Theorem 182.22 to smooth manifolds with non-empty boundary.

Corollary 223.9. (Euler Characteristic–Vector Field Corollary) Let M be a compact smooth manifold. If M admits a nowhere-vanishing vector field v that either points outward along all of ∂M or that points inward along all of ∂M, then $\chi(M) = 0$.

Proof. First note that after possibly replacing the vector field v by $-v$ we can assume that v points outward along ∂M. If v has no zeros, then the sum over the indices of the zeros is also zero. Thus it follows from the Poincaré-Hopf Theorem 223.8 that $\chi(M) = 0$. ■

In preparation for the proof of the Poincaré-Hopf Theorem 223.8 let us first let us make a little remark and prove a useful lemma.

Remark. Let M be a topological manifold. Furthermore let $U \subseteq M$ be an open subset and let $f\colon U \to M$ be a continuous map. If x is an isolated fixed point of f, then we can define $\operatorname{index}(f, x) \in \mathbb{Z}$ the same way as we did for the case $U = M$ in Definition 222.11. □

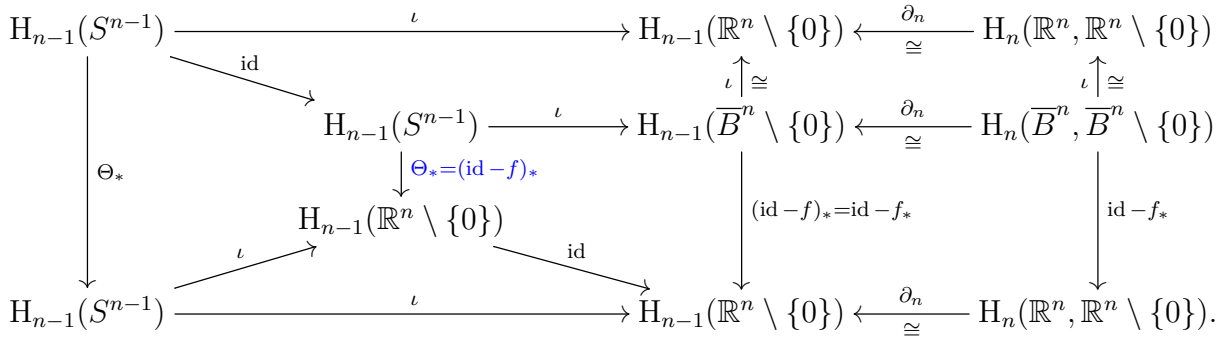
Lemma 223.10. (Index-equals-Index Lemma) Let U be an open neighborhood of the origin $0 \in \mathbb{R}^n$ which contains the closed ball $\overline{B}^n$. Furthermore let $v\colon U \to \mathbb{R}^n$ be a vector field on U which has a single zero, namely at 0. We consider the map

$$\begin{aligned} f\colon U &\to \mathbb{R}^n \\ x &\mapsto x - v(x). \end{aligned}$$

The point $x = 0$ is an isolated fixed point of f and we have the following equality:

$$\underbrace{\operatorname{index}(v, 0)}_{\substack{\text{index of a zero of a} \\ \text{vector field, see Definition 223.1}}} = \underbrace{\operatorname{index}(f, 0).}_{\substack{\text{index of a fixed} \\ \text{point, see Definition 222.11}}}$$

Proof. Since v has a single zero at $x = 0$ we see that $x = 0$ is the only fixed point of f, in particular it is an isolated fixed point. We consider the embedding $\Psi = \mathrm{id}\colon \overline{B}^n \to \mathbb{R}^n$ to compute the indices $\operatorname{index}(v, 0)$ and $\operatorname{index}(f, 0)$. We consider the map $\Theta\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto \frac{v(x)}{\|v(x)\|}$. Furthermore we consider the following diagram:



We make the following observations and clarifications:

(1) The maps ι are the various inclusion-induced maps.
(2) The map

$$\begin{aligned} H\colon S^{n-1} \times [0,1] &\to \mathbb{R}^n \setminus \{0\} \\ (x, t) &\mapsto \frac{v(x)}{(1-t)\cdot\|v(x)\|+t} \end{aligned}$$

is a homotopy between $H_0 = \Theta$ and $H_1 = v = \mathrm{id} - f$. It follows from the existence of such a homotopy together with the Homotopy-H_*-Proposition 144.12 that we have the equality $\Theta_* = (\mathrm{id} - f)_* \colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\})$.

(3) It follows from the naturality of the connecting homomorphisms, see the LES-of-Pair Proposition 146.7 that the squares to the right commute. All the remaining parts of the diagram commute by the functoriality of homology groups.

It follows from the above discussion that

$$\mathrm{index}(v,0) \;=\; \deg\begin{pmatrix} \mathrm{H}_{n-1}(S^{n-1}) \\ \downarrow \Theta_* \\ \mathrm{H}_{n-1}(S^{n-1}) \end{pmatrix} \;=\; \deg\begin{pmatrix} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ {\scriptstyle \iota} \uparrow \cong \\ \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \\ \downarrow {\scriptstyle \mathrm{id} - f_*} \\ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{pmatrix} \;=\; \mathrm{index}(f,0).$$

$\uparrow$ by Definition 223.1 — $\uparrow$ by the above discussion — $\uparrow$ see Definition 222.11 ■

Now we can provide the proof of the Poincaré-Hopf Theorem 223.8. To simplify the discussion we first deal with the special case that the smooth manifold is closed and oriented. We will deal with the general case right afterwards.

Proof of the Poincaré-Hopf Theorem for closed orientable M. Let M be a closed orientable n-dimensional smooth manifold. Let v be a vector field on M with finitely many zeros $P_1, \dots, P_k$. The idea now is similar to the idea behind the proof of the Euler Characteristic–Vector Field Theorem 182.22. Namely we intend to use the vector field v to produce a continuous self-map f of M. We will then roll out the Lefschetz-Hopf Theorem 222.9 to obtain the desired result.

As the reader surely remembers, in the Non-Zero Vector Field–No Fixed Point Proposition 182.23 we gave two different constructions of a continuous self-map f from a given vector field v. The first approach used more machinery whereas the second proof was much more hands-on. It is technically slightly tricky to adapt the first approach to our current needs. Thus in the following we present a slight refinement of the second construction of the self-map f. We refer to the proof of the Non-Zero Vector Field–No Fixed Point Proposition 182.23 for details.

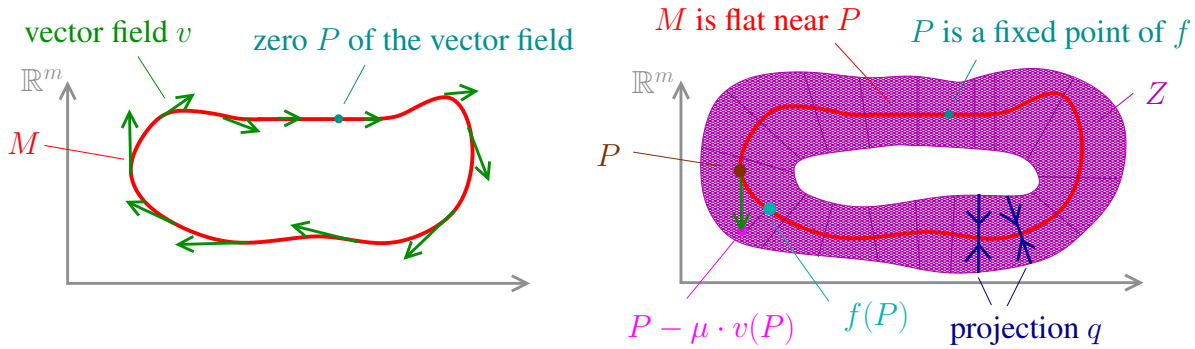
(1) By Exercise 33.6, which is a variation on the Euclidean Embedding Theorem 33.2, we can assume the following:
   (a) M is a submanifold of some $\mathbb{R}^m$.
   (b) For each zero P_i there exists an open neighborhood U_i of P_i that is "flat", in the sense that there exists a $y_i \in \mathbb{R}^{m-n}$ such that $U_i \subseteq \mathbb{R}^n \times \{y_i\}$.

(2) We consider the map
$$\begin{aligned} F \colon M \times \mathbb{R}_{\geq 0} &\to \mathbb{R}^m \\ (P,t) &\mapsto P - t \cdot v(P). \end{aligned}$$

(3) There exists an $\epsilon > 0$ such that $Z := \{P + w \mid P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon\}$ is a neighborhood of M and such that the map
$$\begin{aligned} q \colon Z &\to M \\ P + w &\mapsto P \quad \text{where } P \in M \text{ and } w \in (\widetilde{\mathrm{T}}_P M)^\perp \text{ with } \|w\| \leq \epsilon \end{aligned}$$
is well-defined and continuous.

(4) There exists a $\mu > 0$ such that $F(M \times [0,\mu]) \subseteq Z$.

(5) We set $f := q \circ F_\mu \colon M \to M$.
(6) The map $q \circ F \colon M \times [0, \mu] \to M$ defines a homotopy between the identity and f.



Claim. Let $P \in M$.
(i) P is a zero of v if and only if P is a fixed point of f.
(ii) If P is a zero of v, then $\operatorname{index}(v, P) = \operatorname{index}(f, P)$.

Proof.

(i) Evidently if P is a zero of v, then it is a fixed point of f. Now suppose that P is not a zero of v. We can write $F_\mu(P) = P - \mu \cdot v(P) =: Q + w$ with $Q \in M$ and $w \in (\widetilde{\mathrm{T}}_Q M)^\perp$ with $\|w\| \leq \epsilon$. Since $\widetilde{\mathrm{T}}_P M \cap \widetilde{\mathrm{T}}_P M^\perp = \{0\}$ and since $v(P)$ is a non-zero vector in $\widetilde{\mathrm{T}}_P M$ we see that $P \neq Q$. By the definition in (5) we have $f(P) = Q$. Thus we have shown that $f(P) \neq P$.
(ii) It follows from (1b) and our explicit construction of f that we are essentially in the setting of the Index-equals-Index Lemma 223.10, which then provides us with the desired equality. ⊞

Now we see that

$$\sum_{P \text{ a zero of } v} \operatorname{index}(v, P) \overset{\text{by the claim}}{\downarrow =} \sum_{P \text{ a fixed point of } f} \operatorname{index}(f, P) \underset{\text{by the Lefschetz-Hopf Theorem 222.14, note that here we use that } M \text{ is orientable}}{\uparrow =} \Lambda(f) \overset{\text{follows from the Lefschetz Number-Homotopy Lemma 182.12 and the fact, mentioned in (6), that } f \text{ is homotopic to id}}{\downarrow =} \Lambda(\mathrm{id}) \underset{\text{see the Lefschetz Number Basics Lemma 182.11}}{\uparrow =} \chi(M).$$

■

Next we drop the hypothesis that M needs to be closed.

Proof of the Poincaré-Hopf Theorem 223.8 for compact orientable M. In the following let M be a compact orientable n-dimensional smooth manifold and let v be a vector field on M that points outward on ∂M and that has only finitely many zeros $P_1, \ldots, P_k$.

We perform the following steps:

(1) By the Smooth Collar Neighborhood Theorem 35.6 we know that there exists a collar neighborhood $[-1, 0] \times \partial M$ for $\partial M = \{0\} \times \partial M$. Possibly after rescaling the collar neighborhood we can arrange that none of the points $P_1, \ldots, P_k$ lies in $[-1, 0] \times \partial M$.
(2) Note that given any point $(t, x) \in [-1, 0] \times \partial M$ we can use the natural isomorphism from the Smooth Manifold Product Proposition 25.34 (3) to make the identification $\mathrm{T}_{(t,x)}([-1, 0] \times \partial M) =_{\mathrm{i}} (\mathrm{T}_t \mathbb{R}) \times \mathrm{T}_x \partial M = \mathbb{R} \times \mathrm{T}_x \partial M$. In particular on $[-1, 0] \times \partial M$

we can write the vector field v as $v(t,x) = (v_1(t,x), v_2(t,x))$ with $v_1(t,x) \in \mathbb{R}$ and $v_2(t,x) \in \mathrm{T}_x\partial M$.

(3) It follows easily from the hypothesis that v points outward, see Definition 29.22 for the definition of an outward pointing tangent vector, and the fact that ∂M is compact that there exists an $\epsilon \in [-1,0)$ such that $v_1(t,x) > 0$ for every $x \in \partial M$ and every $t \in [\epsilon, 0]$.

(4) After further rescaling the collar neighborhood we might as well assume that $\epsilon = -1$.

(5) By the Non-Degenerate Vector Field Existence Proposition 223.5 we know that there exists a vector field u on ∂M with finitely many zeros $Q_1, \dots, Q_l$ and such that the sum of the indices of the zeros of u equals precisely $\chi(\partial M)$.

(6) We consider the double $\mathrm{D}M := M \sqcup_{\partial M = \partial M'} M'$ where M' is a copy of M. By the Doubling-Manifold Lemma 54.10 we know that $\mathrm{D}M$ is a closed orientable n-dimensional smooth manifold.

(7) We denote by $\rho\colon \mathrm{D}M \to \mathrm{D}M$ the natural involution swapping the two copies of M.

(8) Our next goal is to construct a vector field w on M that equals v outside of the collar neighborhood, that equals u on ∂M and that interpolates between these vector fields on the collar neighborhood without creating new zeros. In fact the following vector field has precisely those properties:

$$w(x) := \begin{cases} v(x), & \text{if } x \in M \setminus ([-1,0] \times \partial M), \\ (\underbrace{|t| \cdot v_1(y,t)}_{\in \mathbb{R}}, \underbrace{|t| \cdot v_2(y,t) + (1+t) \cdot u(y)}_{\in \mathrm{T}_y \partial M}), & \text{if } x = (t,y) \text{ with } t \in [-1,0] \ \&\ y \in \partial M. \end{cases}$$

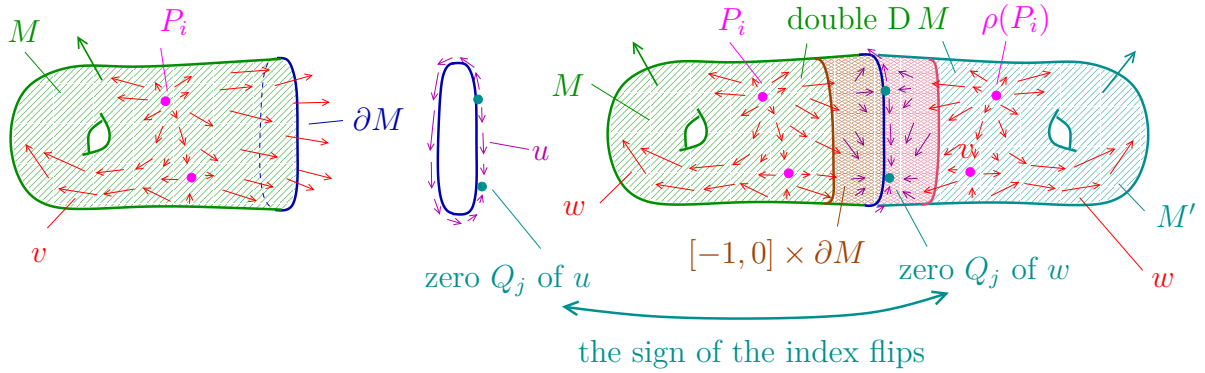
We use the natural involution $\rho\colon \mathrm{D}M \to \mathrm{D}M$ to extend this vector field to $\mathrm{D}M$. More precisely, given $x' \in M'$ we set $w(x') = \mathrm{D}\rho_{\rho(x')}(w(\rho(x))$. Note that w is indeed a vector field on $\mathrm{D}M$, i.e. there are no continuity issues along $\partial M = \partial M'$.

(9) The zeros of w are $P_1, \dots, P_k$, $\rho(P_1), \dots, \rho(P_k)$ and $Q_1, \dots, Q_l$. We make the following three observations:

(a) We have $\mathrm{index}(w, P_i) = \mathrm{index}(v, P_i)$, since w equals v in a neighborhood of P_i.

(b) We have $\mathrm{index}(w, \rho(P_i)) = \mathrm{index}(w, P_i)$, since ρ is a diffeomorphism.

(c) It follows from our hypothesis that v points *outward* together with the Index–Induction Lemma 223.3 that we have $\mathrm{index}(w, Q_j) = -\mathrm{index}(u, Q_j)$.



Now we have the following equality:

$$\sum_{i=1}^{k} \underbrace{\operatorname{index}(w,P_i)}_{\operatorname{index}(v,P_i)\text{ by (9a)}} + \sum_{i=1}^{k} \underbrace{\operatorname{index}(w,\rho(P_i))}_{=\operatorname{index}(v,P_i)\text{ by (9b) together with (9a)}} + \sum_{j=1}^{l} \underbrace{\operatorname{index}(w,Q_j)}_{=-\operatorname{index}(u,Q_j)\text{ by (9c)}} \overset{\text{since } \mathrm{D}M \text{ is a } \textit{closed} \text{ orientable smooth manifold we already know from the above that the conclusion of the Poincaré-Hopf Theorem 223.8 holds for } \mathrm{D}M}{\overset{\downarrow}{=}} \chi(\mathrm{D}M).$$

From this equality, together with (9a), (9b) and (9c) and the Union-χ-Proposition 156.10 we obtain the following equality:

$$2\cdot\sum_{i=1}^{k}\operatorname{index}(v,P_i) - \underbrace{\sum_{j=1}^{l}\operatorname{index}(u,Q_j)}_{=\chi(\partial M)\text{ by (5)}} = \underbrace{2\cdot\chi(M)-\chi(\partial M).}_{=\chi(\mathrm{D}M)\text{ by the Union-}\chi\text{-Proposition 156.10 and the Smooth Manifold CW Structure Theorem 179.9}}$$

Cleaning up the bottom equality we obtain the desired equality. ■

Finally we drop the hypothesis that M needs to be orientable.

Proof of the Poincaré-Hopf Theorem 223.8. Let M be a compact n-dimensional smooth manifold and let v be a vector field on M that points outward on ∂M and that has only finitely many zeros $P_1,\dots,P_k$. Evidently we only need to consider the case that M is connected. Furthermore, by the previous discussions it remains to consider the case that M is non-orientable.

By the Orientation Covering–Properties-Proposition 95.10 there exists a 2-fold covering $p\colon \widetilde{M}\to M$, where $\widetilde{M}$ is a compact orientable smooth manifold and p is a local diffeomorphism. We equip $\widetilde{M}$ with the vector field p^*v that is the pullback of v under p, see the Vector Field-Pullback Lemma 29.33. In other words, p^*v is the unique vector field on $\widetilde{M}$ that satisfies $\mathrm{D}p_x(p^*v(x))=v(p(x))$ for every $x\in\widetilde{M}$. We see that

$$2\cdot\sum_{i=1}^{k}\operatorname{index}(v,P_i) \underset{\text{each } P_i \text{ has two preimages and indices of the preimages agree with the index of } P_i}{\overset{\uparrow}{=}} \sum_{i=1}^{k}\sum_{Q\in p^{-1}(P_i)}\operatorname{index}(p^*v,Q) \overset{\text{by the above discussion, since } \widetilde{M} \text{ is orientable}}{\overset{\downarrow}{=}} \chi(\widetilde{M}) \underset{\text{by the Covering-}\chi\text{-Proposition 156.13 and the Smooth Manifold--CW Structure Theorem 179.9}}{\overset{\uparrow}{=}} 2\cdot\chi(M).$$

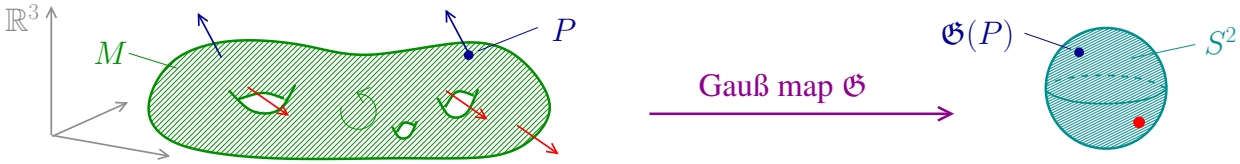
■

223.3. **The Gauß map II.** In this section we return to the study of the Gauß map which we had introduced in Section 53.7. For the reader's convenience we recall the definition:

Definition 223.11. Let $n \in \mathbb{N}_0$ and let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented smooth submanifold of codimension one. We refer to the map

$$\begin{array}{rcl} \mathfrak{G}\colon M & \to & S^n \\ P & \mapsto & \text{the unique } \mathfrak{G}(P) \in \mathbb{R}^{n+1} \text{ that has the following properties:} \\ & & \text{(1) } \mathfrak{G}(P) \text{ is normal to the physical tangent space } \widetilde{\mathrm{T}}_P M, \\ & & \text{(2) } \mathfrak{G}(P) \text{ has length one, i.e. } \mathfrak{G}(P) \in S^n, \\ & & \text{(3) } \mathfrak{G}(P) \text{ followed by a positive basis for } \widetilde{\mathrm{T}}_P M \text{ is a positive basis for } \mathbb{R}^{n+1} \end{array}$$

as the Gauß map of M.[2405] [2406]



Our next goal is to give another proof of the following theorem from Section 53.7.

Theorem 53.32. (Gauß Map-Degree Theorem) Let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. If n is even, then

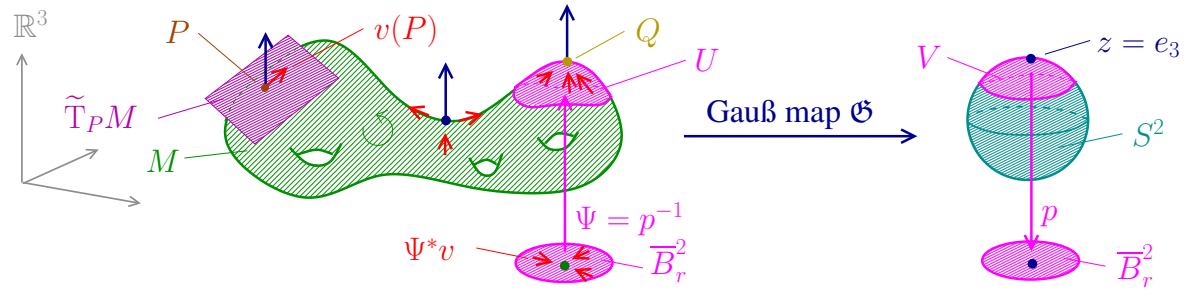
$$\deg(\text{Gauß map } \mathfrak{G}\colon M \to S^n) \;=\; \tfrac{1}{2} \cdot \chi(M).$$

Remark. We refer to Section 53.7 for a discussion of consequences and alternative proofs of the Gauß Map-Degree Theorem 53.32. □

Proof. Let $n \in \mathbb{N}_0$ and let $M \subseteq \mathbb{R}^{n+1}$ be a closed oriented connected smooth submanifold of codimension one. The proof we will now provide is a variation on the proof in Section 53.7. We refer to the earlier proof for details on the common aspects of the proofs. By Sard's Theorem 38.1 there exists some $z \in S^n$ such that z and $-z$ are both regular values of the Gauß map $\mathfrak{G}$. After a rotation, we can assume that $z = e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Next we consider the vector field

$$\begin{array}{rcl} v\colon M & \to & \mathbb{R}^{n+1} \\ Q & \mapsto & \underbrace{z - \langle z, \mathfrak{G}(Q)\rangle \cdot \mathfrak{G}(Q)}_{\text{orthogonal projection of } z \text{ to } \widetilde{\mathrm{T}}_Q M}. \end{array}$$

Clearly we have $v(Q) = 0$ if and only if $\mathfrak{G}(Q) = z$ or $\mathfrak{G}(Q) = -z$.



[2405] A variation on the Gauß map appeared also in the definition of the rotation number of an immersion $S^1 \to \mathbb{R}^2$ in Definition 49.8.

[2406] As discussed in the Orientations-via-Normal Vector Fields Proposition 31.27, this map is smooth.

Let $Q \in M$ with $\mathfrak{G}(Q) = \pm z$. By design we know that Q is a regular point of $\mathfrak{G}$. By the Stack-of-Records Lemma 32.3 (1) we know that $\mathfrak{G}\colon M \to S^n$ is a local diffeomorphism at Q, thus, as in Definition 46.1, it makes sense to introduce the following definition:

$$\deg(\mathfrak{G}, Q) \;=\; \begin{cases} +1, & \text{if } \mathfrak{G} \text{ is orientation-preserving at } Q \\ -1, & \text{if } \mathfrak{G} \text{ is orientation-reversing at } Q. \end{cases}$$

Claim. Let $Q \in M$.
(1) If $\mathfrak{G}(Q) = z$, then $\operatorname{index}(v, Q) = (-1)^n \cdot \deg(\mathfrak{G}, Q)$.
(2) If $\mathfrak{G}(Q) = -z$, then $\operatorname{index}(v, Q) = \deg(\mathfrak{G}, Q)$.

Proof. First we consider the case that $\mathfrak{G}(Q) = z$. We make the following observations and preparations.

(a) We denote by $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection onto the first n coordinates.
(b) It follows fairly easily from the Inverse Mapping Theorem 30.1 that there exists a neighborhood U of $Q \in M$, a neighborhood V of $z \in S^n$ and an $r \in \mathbb{R}_{>0}$ with the following properties:
  (i) U contains no other points x with $\mathfrak{G}(x) = z$.
  (ii) We have $\mathfrak{G}(U) \subseteq V$.
  (iii) The restriction of the projection $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ to U and V defines diffeomorphisms $p\colon U \to \overline{B}^n_r$ and $p\colon V \to \overline{B}^n_r$. It follows immediately from the orientation conventions and some elementary linear algebra that both maps are orientation-preserving if n is even and both maps are orientation-reversing if n is odd.
(c) We set $\Psi \coloneqq p^{-1}\colon \overline{B}^n_r \to U$.
(d) Recall that by definition we have $v(x) = z - \langle z, \mathfrak{G}(x)\rangle \cdot \mathfrak{G}(x)$. As we discussed on page 1217 one can easily show, using the Product Rule for partial differentials and using that $\mathfrak{G}(Q) = z$, that

$$\mathrm{D}(p \circ (x \mapsto -v(x)) \circ \Psi)_{p(Q)} \;=\; \mathrm{D}(p \circ \mathfrak{G} \circ \Psi)_{p(Q)}.$$

(e) We denote by $\rho\colon \overline{B}^n_r \to \overline{B}^n_r$ the map that is given by $\rho(x) = -x$.
(f) In Exercise 18.4 we showed that if $f\colon \overline{B}^n_r \setminus \{0\} \to \overline{B}^n_r \setminus \{0\}$ is a continuous map, then it is homotopic to the map that is given by $x \mapsto \frac{x}{\|x\|}$.

We now see that

$$\begin{aligned}
\operatorname{index}(v, Q) &= \deg\begin{pmatrix} S^{n-1}_r \to S^{n-1}_r \\ x \mapsto \frac{p(v(\Psi(x)))}{\|p(v(\Psi(x)))\|} \end{aligned}\right) \\
&= \deg\big((p \circ v \circ \Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}^n_r, \overline{B}^n_r \setminus \{0\})\big) = \\
&= \deg\big((\rho \circ p \circ \mathfrak{G} \circ \Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}^n_r, \overline{B}^n_r \setminus \{0\})\big) \\
&\to = (-1)^n \cdot \deg\big((p \circ \mathfrak{G} \circ \Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}^n_r, \overline{B}^n_r \setminus \{0\})\big) \\
&= (-1)^n \cdot \deg(\mathfrak{G}, Q).
\end{aligned}$$

definition of $\operatorname{index}(v, Q)$, see Definition 223.1, applied to our given map Ψ

follows from (f) and the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 146.7 (2)

follows from (d) together with the Local Diffeomorphism–Degree Proposition 152.6

as in the Matrix Multiplication–Degree Lemma 152.9

this follows from the definition of $\deg(\mathfrak{G}, Q)$ together with the Local Diffeomorphism-Degree Proposition 152.6 and the fact that we know that p and Ψ are either both orientation-preserving or they are both orientation-reversing

Now we consider the case that $\mathfrak{G}(Q) = -z$. The calculation is basically identical to the above. The only difference is that in this setting the map p is orientation-preserving if and

only if n is odd, whereas before p was orientation-preserving if and only if n is even. On the other hand the behavior of the orientation of Ψ does not change, i.e. Ψ is orientation-preserving if and only if n is even. Thus we obtain the same result as above, except that by the Matrix Multiplication–Degree Lemma 152.9 we pick up a sign $(-1)^n$. ⊞

Now finally we assume that n is even. We perform the following calculation:

$$\begin{aligned} 2\cdot\deg(\text{Gauß map } \mathfrak{G}\colon M\to S^n) &\overset{\text{since } z \text{ and } -z \text{ are regular values we can apply the Local-Global Degree Proposition 186.8 to } z \text{ and to } -z}{=} \sum_{\substack{Q\in M \text{ with}\\ \mathfrak{G}(Q)=z}} \deg(\mathfrak{G},Q) + \sum_{\substack{Q\in M \text{ with}\\ \mathfrak{G}(Q)=-z}} \deg(\mathfrak{G},Q) \\ &= \sum_{\substack{Q\in M \text{ with}\\ v(Q)=0}} \deg(\mathfrak{G},Q) \qquad = \sum_{\substack{Q\in M \text{ with}\\ v(Q)=0}} \operatorname{index}(v,Q) = \chi(M). \end{aligned}$$

since $v(Q)=0$ if and only if $\mathfrak{G}(Q)=-z$ or $\mathfrak{G}(Q)=z$ — by the claim and since n is even — by the Poincaré-Hopf Theorem 223.8 ■

223.4. Exercises for Chapter 223 .

Exercise 223.1. We consider the two vector fields v and w that are shown in the figure below with a single zero at P. On page 1201 we already saw that $\operatorname{index}(v,P)=+1$. What is $\operatorname{index}(w,P)$?

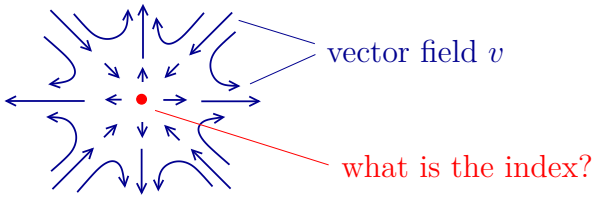


Exercise 223.2.

(a) We consider the vector field v on $\mathbb{R}^2$ shown in the figure below. What is the index of v at the unique zero?

(b) Sketch a vector field v on $\mathbb{R}^2$ with a zero at some point P, such that the corresponding index equals $+3$.

(c) Let $n\geq 2$. Show that for every $k\in\mathbb{Z}$ there exists a vector field v on $\mathbb{R}^n$ with a single zero at the origin 0 such that $\operatorname{index}(v,0)=k$.



Exercise 223.3. Let M be a closed oriented n-dimensional smooth manifold. We consider the tangent bundle

$$\mathrm{T}M \coloneqq \bigsqcup_{x\in M} (\mathrm{T}_xM\times\{x\}).$$

(a) Show that $\mathrm{T}M$ is naturally an oriented $2n$-dimensional smooth manifold such that the map $\varphi\colon M\to \mathrm{T}M$ given by $x\mapsto (0,x)\in \mathrm{T}_xM\times\{x\}$ is a smooth embedding.

(b) Show that there exists a smooth embedding $\psi\colon M \to \mathrm{T}M$, that is homotopic to φ, such that the submanifolds $\varphi(M)$ and $\psi(M)$ of $\mathrm{T}M$ intersect transversally and such that $\varphi(M)\cdot\psi(M) = \chi(M)$. Here the left-hand side is the algebraic intersection number from Definition 220.5.
Hint. Use the Non-Degenerate Vector Field Existence Proposition 223.5.
Remark. In the literature the statement we just proved sometimes gets shortened to "$M \cdot M = \chi(M)$".

Exercise 223.4. We say that two vector fields v and w on a smooth manifold M are *equivalent* if there exists a diffeotopy $H\colon M \times [0,1] \to M$ with $H_0 = \mathrm{id}$ and such that $(H_1)_*(v) = w$, i.e. such that for every $x \in M$ we have $(\mathrm{D}H_1)_x(v(x)) = w_{H_1(x)}$. Are all nowhere-vanishing smooth vector fields on the torus $S^1 \times S^1$ equivalent?

Part XXIV

The intersection form

In this part we introduce and discuss in great detail the intersection form of compact oriented even-dimensional topological manifolds. The intersection form is defined using the cup product and Poincaré duality. This part of the lecture notes is organized as follows:

- In Chapter 224 we will introduce the intersection form and we study several examples.
- In Chapter 225 we introduce two variations on the intersection form, namely the asymmetric and the reduced intersection pairings. These are only of interest for topological manifolds with non-empty boundary. This chapter can certainly be skipped at a first reading.
- In Chapter 226 we will study the algebraic classification of (anti-) symmetric forms over $\mathbb{Z}$ and $\mathbb{R}$. In particular we will introduce the all important signature of a compact oriented $4k$-dimensional topological manifold.
- In Chapter 227 we will state and prove the Novikov Additivity Theorem which says that the signature is additive under arbitrary gluings.
- In Chapter 228 we will give a longish survey on intersection forms of 4-dimensional smooth and topological manifolds. In particular we will explain some of the work by the Fields medalists Simon Donaldson and Mike Freedman.

CHAPTER 224

The intersection form: Definition and calculations

In this chapter we will introduce and study the intersection form of compact oriented even-dimensional topological manifolds. We will calculate the intersection form of various topological manifolds, e.g. of the complex projective spaces and products $S^n \times S^n$. Furthermore we will study the behavior of the intersection form under the connected sum operation.

This will lead to some interesting algebraic problems, which we will deal with in Chapter 226.

This chapter is organized as follows:

- In Section 224.1 we will introduce pairings and forms and we will see how they can be described by matrices.
- In Section 224.2 we will introduce maximal free abelian quotients.
- In Section 224.3 we will define the intersection form of topological manifolds.
- In Section 224.4 we will discuss properties of the intersection form.
- In Section 224.5 we will discuss the relationship between the intersection form and degree-one maps.
- In Section 224.6 we will calculate the intersection form of the connected sum of two topological manifolds.
- In Appendix 224.7 we will discuss the connected sum of topological manifolds.
- In Appendix 224.8 we will discuss intersection numbers on non-compact manifolds.
- In Section 224.9 we will provide exercises for Chapter 224.

In the next Chapter 225 we will introduce the asymmetric and reduced intersection pairings.

224.1. **Pairings, forms and matrices.** Before we turn to topology it is convenient to remind ourselves of a few basic notions and results in algebra. In Definition 213.1 we introduced the notion of a pairing over a commutative ring R. We recall the definition and we introduce some new notions:

Definition 224.1. Let R be a commutative ring and let V, W and S be R-modules.

(1) A pairing on $V \times W$ is a map

$$\varphi\colon V \times W \ \to \ S$$

such that for all $v, v' \in V$ and $w, w' \in W$ we have $\varphi(v + v', w) = \varphi(v, w) + \varphi(v', w)$ and $\varphi(v, w + w') = \varphi(v, w) + \varphi(v, w')$.

(2) A pairing is called R-bilinear if it is R-linear in V and if it is R-linear in W.

(3) Let $\varphi\colon V \times V \to S$ be a pairing.

- We say that φ is symmetric if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.
- We say that φ is anti-symmetric if $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$.

(4) Let (V, φ) and (W, ψ) be two R-bilinear pairings.
   (a) An isometry between (V, φ) and (W, ψ) is an isomorphism $f\colon V \to W$ of R-modules such that $\varphi(v_1, v_2) = \psi(f(v_1), f(v_2))$ for all $v_1, v_2 \in V$.
   (b) We say that (V, φ) and (W, ψ) are isometric, written $(V, \varphi) \cong (W, \psi)$, if an isometry exists.

In the discussions below we will often treat abelian groups as $\mathbb{Z}$-modules in the obvious way.

In this chapter we will mostly be interested in the following special type of an R-bilinear pairing:

Definition 224.2. Let R be a commutative ring. A form[2407] is a finitely generated free R-module V together with an R-bilinear pairing $\varphi\colon V \times V \to R$ which is either symmetric or anti-symmetric. We refer to the rank of V also as the rank of the form.[2408]

For the reader's convenience we recall Definition 213.1:

Definition 224.3. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. The form is called non-singular if the map

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(V, R) \\ v & \mapsto & \begin{pmatrix} V & \to & R \\ w & \mapsto & \varphi(v, w) \end{pmatrix} \end{array}$$

is an isomorphism.[2409] If the map is a monomorphism, then we say that the form is non-degenerate.

In the following discussion we will study the relationship between forms and matrices. We start out with the following definition:

Definition 224.4. Let R be a commutative ring.

(1) Let A be an (anti-) symmetric[2410] $(n \times n)$–matrix over R. We consider the map

$$\begin{array}{rccc} \varphi(A) \coloneqq \varphi_R(A)\colon & R^n \times R^n & \to & R \\ & (v, w) & \mapsto & v^{\mathrm{T}} A w. \end{array}$$

   It is elementary to verify that $\varphi_R(A)$ is an (anti-) symmetric form over R.[2411]

(2) Let $\varphi\colon V \times V \to R$ be a form. Given any basis $\{v_1, \ldots, v_k\}$ of the free R-module V we refer, as in Definition 2.26, to the $(k \times k)$–matrix

$$(\varphi(v_i, v_j))_{i,j=1,\ldots,k} \in \mathrm{M}(k \times k, R)$$

   as a matrix representing the form φ.

[2407] The various definitions of "form" in the literature vary wildly. It is always a good idea, reading a paper or a book, to look up the precise definition of "form" that is used by the author.

[2408] As we will see in the Commutative Ring–Invariant Basis Number Lemma 312.21, the rank of a finitely generated free module over a commutative ring is well-defined.

[2409] Note that a form is by definition (anti-) symmetric. It follows from this fact that the given condition is satisfied if and only if the condition with the two entries of φ swapped is satisfied. Thus the definition we give here is indeed consistent with Definition 213.1.

The following lemma shows in particular that isometry classes of (anti-) symmetric (non-singular) forms over a commutative ring R are in bijection to congruence classes of (anti-) symmetric (invertible) matrices over R:

Lemma 224.5. (Form–Matrix-Lemma) Let R be a commutative ring.

(1) (a) Let (V, ψ) be an (anti-) symmetric form over R of rank n. If A is a matrix that represents ψ, then $(V, \psi) \cong (R^n, \varphi_R(A))$. In particular every (anti-) symmetric form over R is isometric to $\varphi_R(A)$ for some (anti-) symmetric $(n \times n)$–matrix A over R.

(b) Let A and B be two (anti-) symmetric matrices over R. The forms $\varphi_R(A)$ and $\varphi_R(B)$ are isometric if and only if there exists a $P \in \mathrm{GL}(n, R)$ with $B = P^{\mathrm{T}} \cdot A \cdot P$.

(2) The form $\varphi_R(A)$ is non-singular, in the sense of Definition 224.3, if and only if A is invertible, i.e. if $\det(A)$ is a unit in R.

Convention 224.6. The Form–Matrix-Lemma 224.5 allows us to go back and forth between (anti-) symmetric forms and (anti-) symmetric matrices. Therefore in the following discussion we will blur the distinction between these two concepts. For example, given an (anti-) symmetric matrix A over a commutative ring we will often just refer to the (anti-) symmetric form $\varphi_R(A)$ as the form A.

Proof.

(1) (a) Let $\psi\colon V \times V \to R$ be an (anti-) symmetric form over R. Let $v_1, \dots, v_n$ be a basis of V. We denote by $A = (\psi_R(v_i, v_j))_{i,j=1,\dots,n}$ the corresponding matrix representing the form ψ. We consider the isomorphism

$$\begin{array}{rcl} \Theta\colon R^n & \to & V \\ (\lambda_1, \dots, \lambda_n) & \mapsto & \lambda_1 \cdot v_1 + \dots + \lambda_n \cdot v_n. \end{array}$$

We claim that Θ is an isometry from the form $(R^n, \varphi_R(A))$ to the form (V, ψ). Indeed, if we denote by $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ the standard basis for R^n, then for any choice of $i, j \in \{1, \dots, n\}$ we have

$$\psi_R(\Theta(\mathfrak{e}_i), \Theta(\mathfrak{e}_j)) \;=\; \psi_R(v_i, v_j) \;=\; ij\text{-entry of } A \;=\; \mathfrak{e}_i^{\mathrm{T}} A \mathfrak{e}_j \;=\; \varphi_R(A)(\mathfrak{e}_i, \mathfrak{e}_j).$$

By the bilinearity of forms we obtain the corresponding statement for all pairs of elements in R^n.

(b) Let A and B be two (anti-) symmetric matrices over R. First we prove the "if"-direction. Thus we assume that there exists a $P \in \mathrm{GL}(n, R)$ with $B = P^{\mathrm{T}} \cdot A \cdot P$. We claim that the isomorphism $v \mapsto P \cdot v$ defines an isometry from the form $(R^n, \varphi_R(P^{\mathrm{T}}AP))$ to the form $(R^n, \varphi_R(A))$. Indeed, for any $v, w \in R^n$ we have

$$\varphi_R(A)(Pv, Pw) \;=\; (Pv)^{\mathrm{T}} APw \;=\; v^{\mathrm{T}}(P^{\mathrm{T}}AP)w \;=\; \varphi_R(P^{\mathrm{T}}AP)(v, w).$$

We turn to the proof of the "only if"-direction. Thus we assume that there exists an isometry $f\colon (R^n, \varphi_R(A)) \to (R^n, \varphi_R(B))$. Let P be the matrix representing f

[2410] Recall that a matrix A is called *anti-symmetric* if $A^{\mathrm{T}} = -A$.

[2411] Indeed, for any $v, w \in R^n$ we have

$$\varphi_R(A)(w, v) \;=\; w^{\mathrm{T}} A v \;=\; (w^{\mathrm{T}} A^{\mathrm{T}} v)^{\mathrm{T}} \;=\; v^{\mathrm{T}} A^{\mathrm{T}} w \;=\; \varphi_R(A^{\mathrm{T}})(v, w) \;=\; \pm\varphi_R(A)(v, w).$$

for a (1×1)–matrix (x) we have $(x)^{\mathrm{T}} = (x)$

with respect to the standard basis of R^n. Since f is an isomorphism we know that $P \in \mathrm{GL}(n,R)$. Finally the above calculation shows that $B = P^{\mathrm{T}}AP$.

(2) This statement is basically the content of the Non-Singularity Lemma 213.2. ■

We continue with a few basic constructions of R-bilinear pairings:

Definition 224.7. Let R be a commutative ring.

(1) Given an R-bilinear pairing $\varphi\colon V \times V \to S$ we denote by $-\varphi$ the R-bilinear pairing that is given by $(-\varphi)(v,w) \coloneqq -\varphi(v,w)$ for all $v,w \in V$.

(2) Given two R-bilinear pairings $\varphi\colon V \times V \to S$ and $\psi\colon W \times W \to S$ we define the direct sum of φ and ψ to be the R-bilinear pairing

$$\begin{array}{rcl} \varphi \oplus \psi\colon (V \oplus W) \times (V \oplus W) & \to & S \\ ((v,w),(v',w')) & \mapsto & \varphi(v,v') + \psi(w,w'). \end{array}$$

(3) Given an R-bilinear pairing (V,φ) and $n \in \mathbb{Z}$ we define

$$n \cdot (V,\varphi) \coloneqq \begin{cases} \overbrace{(V,\varphi) \oplus \cdots \oplus (V,\varphi)}^{n\text{-summands}}, & \text{if } n \geq 0, \\ \underbrace{(V,-\varphi) \oplus \cdots \oplus (V,-\varphi)}_{|n|\text{-summands}}, & \text{if } n \leq 0. \end{cases}$$

Lemma 224.8. (Matrix–Form Operations Lemma) Let R be a commutative ring, let $\varphi\colon V \times V \to R$ be a form and let $\psi\colon W \times W \to R$ be a form.

(1) If A is a matrix that represents φ, then the matrix $-A$ represents the form $-\varphi$.

(2) If A is a matrix that represents φ and if B is a matrix that represents ψ, then the matrix $A \oplus B \coloneqq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ represents the form $\varphi \oplus \psi$.

(3) Let $n \in \mathbb{N}_0$. If A is a matrix that represents φ, then the matrix $n \cdot A \coloneqq \underbrace{A \oplus \cdots \oplus A}_{n\text{-times}}$ represents the form $n \cdot \varphi$.

Proof. The lemma follows immediately from the definitions. ■

224.2. **Maximal free abelian quotients.** Before we can turn to topology we still need to recall Definition 182.6:

Definition 224.9. Given a finitely generated abelian group H we refer to

$$\mathrm{F}H \coloneqq H/\operatorname{Tor}(H) \quad \text{where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$

as the maximal torsion-free quotient of H.

The following lemma summarizes everything we need to know about maximal torsion-free quotients. We will often make use of it implicitly, but to the lemma's chagrin we will never acknowledge that we are making use of it.

Lemma 224.10. (Maximal Torsion-Free Quotient Lemma)

(1) The maps

$$H \mapsto \mathrm{F}H \qquad \text{and} \qquad (\alpha\colon G \to H) \mapsto \begin{pmatrix} \alpha\colon \mathrm{F}G & \to & \mathrm{F}H \\ [g] & \mapsto & [\alpha(g)] \end{pmatrix}$$

define a covariant functor from the category of finitely generated abelian groups to the category of finitely generated free abelian groups.

(2) For every homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups the following diagram commutes:[2412]

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle a\mapsto [a]}\downarrow & & \downarrow{\scriptstyle b\mapsto [b]} \\ \mathrm{F}A & \xrightarrow{\alpha} & \mathrm{F}B. \end{array}$$

(3) Any (anti-) symmetric pairing $\varphi\colon H \times H \to \mathbb{Z}$,

where H is a finitely generated abelian group, descends to a well-defined form

$$\begin{array}{rcl} \mathrm{F}H \times \mathrm{F}H & \to & \mathbb{Z} \\ ([g],[h]) & \mapsto & \varphi(g,h). \end{array}$$

Proof. In the elementary Free Abelian Quotient Lemma 182.7 we already showed that if H is a finitely generated abelian group, then $\mathrm{F}H$ is a finitely generated free abelian group. The remaining statements of the lemma are basically elementary. ■

Convention 224.11. On many occasions we do not distinguish in the notation between an element in an abelian group H and the element it represents in the maximal torsion-free quotient $\mathrm{F}H$. Furthermore, given a homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups we often denote the induced map $\mathrm{F}A \to \mathrm{F}B$ by α as well.

224.3. **The definition of the intersection form of topological manifolds.** Let M be a compact oriented m-dimensional topological manifold. Recall that by the Poincaré Duality Theorem 210.1 we know that for any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z}) & \to & \mathrm{H}_k(M) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. As in Definition 210.2 we denote by

$$\mathrm{PD}_M\colon \mathrm{H}_k(M) \to \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z})$$

the inverse of this isomorphism. The Cup Product–Intersection Number Theorem 220.10 motivates the following definition:

Definition 224.12. Let $n \in \mathbb{N}_0$ and let M be a compact oriented topological manifold of even dimension $2n$. We refer to the map[2413] [2414]

$$\begin{array}{rcl} Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{array}$$

as the intersection form of M.

[2412] In fancy speak this means that the projections $H \to \mathrm{F}H$ define a natural transformation from the identity functor to the functor $H \mapsto \mathrm{F}H$.

Remark. Since $\mathrm{PD}_M\colon \mathrm{H}_n(M) \to \mathrm{H}^n(M,\partial M;\mathbb{Z})$ is an isomorphism and since by the Manifold-Top-H*-Theorem 201.26 we know that the cohomology group $\mathrm{H}^{2n}(M,\partial M;\mathbb{Z})$ is torsion-free it is clear that the intersection form is basically just a reformulation of the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) & \to & \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) \\ (\alpha,\beta) & \mapsto & \alpha \cup \beta \end{array}$$

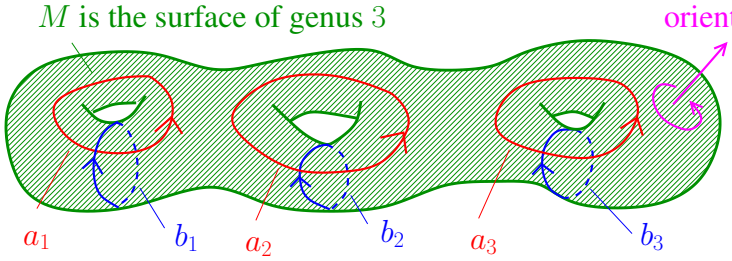
on cohomology groups in the middle dimension that we introduced in the Manifold-Non Singularity Proposition 213.3. The intersection form has two advantages:

(1) The big psychological advantage of homology is that very often one can "see homology classes". For example, in many cases homology classes are represented by submanifolds, this allows us to use the Cup Product–Intersection Number Theorem 220.10 to compute the intersection form of a smooth manifold. Sometimes we can also represent homology classes by explicit cycles and we can use the Cycles–Cup Product-Intersection Number Theorem 221.20 to compute the intersection form.

(2) The intersection form always has the target $\mathbb{Z}$, so it makes sense to say that two intersection forms are isometric. □

Examples.

(1) First we consider the surface M of genus 3. In principle we could obtain the intersection form from the calculation of the cup product that we had performed in the Surface–Cup Product Lemma 203.30. But it is much more fun and instructive to use the Cup Product–Intersection Number Theorem 220.10 instead. More precisely, we consider the surface M that is illustrated in the figure below. We give it the usual orientation. We already saw in the discussion on page 2970 that the closed oriented curves $a_1, b_1, a_2, b_2, a_3, b_3$ illustrated in the figure below represent a basis for $\mathrm{H}_1(M)$. It follows easily from the Cup Product–Intersection Number Theorem 220.10 that with respect to this basis the intersection form is represented by the matrix shown in the figure below. An analogous statement holds of course for any surface of genus $g \geq 1$.

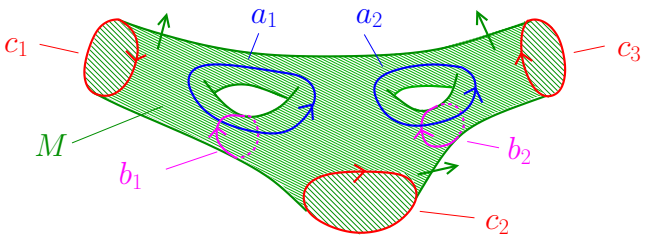


$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

(2) Next we let M be the surface of genus two minus three open disks that is shown in the figure below.

[2413] Note that the homology group $\mathrm{H}_n(M)$ in the middle dimension is not necessarily torsion-free. For example $M = S^1 \times \mathbb{R}\mathrm{P}^3$ is orientable, but it follows easily from the Topological Künneth Formula 164.6 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^3$ in the $\mathbb{R}\mathrm{P}^n$-H$_*$-Proposition 155.18 that $\mathrm{H}_2(S^1 \times \mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

[2414] As mentioned above, in the notation usually we do not distinguish between an element in $\mathrm{H}_n(M)$ and the element it represents in $\mathrm{FH}_n(M)$.



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By the discussion on page 2971 we know that a basis for $\mathrm{H}_1(M)$ is represented by the closed curves a_1, b_1, a_2, b_2 together with two out of the three boundary curves c_1, c_2, c_3. It follows from the Cup Product–Intersection Number Theorem 220.10 that with respect to this basis the intersection form is represented by the matrix given in the figure above. This calculation of course generalizes to any surface of genus g minus k open disks.

(3) Next let us consider the $2n$-dimensional smooth manifold $M = S^n \times S^n$. By the Künneth Formula 164.6 we know that a basis for $\mathrm{H}_n(S^n \times S^n)$ is given by $a = [S^n \times \{*\}]$ and $b = [\{*\} \times S^n]$. We have two approaches for determining the intersection form with respect to this basis:
   (a) We can use the Cross-Cup-Cap Theorem 207.1 and its corollaries. More precisely, we first determine the Poincaré duals of $[S^n \times \{*\}]$ and $[\{*\} \times S^n]$ as in the Fundamental Class-Product Proposition 207.2 (2) and then we can apply the Product-of-Spheres–Cup Product Lemma 207.3.
   (b) As on page 4238 we can use algebraic intersection numbers and the Cup Product–Intersection Number Theorem 220.10 to determine the intersection form.

   In both cases we see that with respect to this basis the intersection form of $S^n \times S^n$ is represented by the matrix

$$\begin{pmatrix} Q_{S^n \times S^n}(a,a) & Q_{S^n \times S^n}(a,b) \\ Q_{S^n \times S^n}(b,a) & Q_{S^n \times S^n}(b,b) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

(4) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from the Complex Manifold-Orientation Proposition 40.18. It follows from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20 that the intersection form

$$Q_{\mathbb{C}\mathrm{P}^2} \colon \mathrm{H}_2(\mathbb{C}\mathrm{P}^2) \times \mathrm{H}_2(\mathbb{C}\mathrm{P}^2) \to \mathbb{Z}$$

is represented by the (1×1)–matrix (1).

(5) Let M be a closed oriented $2n$-dimensional topological manifold. As usual we denote by $-M$ the same topological manifold with the opposite orientation. It follows immediately from the definitions together with the Orientation Preserving–Characterisation Proposition 196.6 that $Q_{-M} = -Q_M$.

(6) For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. With this notation it follows from (4) and (5) that the intersection form

$$Q_{\overline{\mathbb{C}\mathrm{P}}^2} \colon \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2) \times \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2) \to \mathbb{Z}$$

is represented by the (1×1)–matrix (-1). □

224.4. **Properties of the intersection form.** In the following two propositions we summarize some of the key properties of the intersection form:

Proposition 224.13. (Intersection Form Proposition) Let M be a compact oriented $2n$-dimensional topological manifold. Then the following hold:[2415]

(1) (a) The map $Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) \to \mathbb{Z}$ is bilinear.
(b) The group $\mathrm{FH}_n(M)$ is a finitely generated free abelian group.
(2) If n is even, then the intersection form Q_M is symmetric, otherwise it is anti-symmetric.

In particular the intersection form $Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) \to \mathbb{Z}$ is indeed a form in the sense of Definition 224.2.

Proof.

(1) (a) First note that it follows immediately from the bilinearity of the cup product that $Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) \to \mathbb{Z}$ is bilinear. It now follows from the fact that the Kronecker pairing is linear in the first entry that the map Q_M is indeed bilinear.
(b) By the Topological Manifolds-Invariants Proposition 189.14 we know that $\mathrm{H}_n(M)$ is finitely generated. This implies of course that $\mathrm{FH}_n(M)$ is also finitely generated. It follows from the Free Abelian Quotient Lemma 182.7 (or more directly from the Abelian Group-Classification Theorem 97.24) that $\mathrm{FH}_n(M)$ is a free abelian group.

(2) Given $v, w \in \mathrm{FH}_n(M)$ we have the following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}, [M] \Big\rangle}^{=Q_M(v,w)} = (-1)^{n\cdot n} \cdot \overbrace{\Big\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}, [M] \Big\rangle}^{=Q_M(w,v)}.$$

by the Cup Product Commutativity Proposition 204.10

The statement follows from the observation that $(-1)^{n\cdot n} = (-1)^n$. ∎

Proposition 224.14. (Intersection Form–Non Singularity Proposition) Let M be a closed oriented $2n$-dimensional topological manifold. The corresponding intersection form is non-singular.[2416]

Proof. Let M be a closed oriented $2n$-dimensional topological manifold. We need to show that the intersection form Q_M is non-singular. Evidently this is equivalent to showing that the form

$$\begin{array}{rcl} \mathrm{FH}^n(M;\mathbb{Z}) \times \mathrm{FH}^n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular. We dealt with the case that the cohomology groups are torsion-free in the Manifold-Non Singularity Proposition 213.3. The proof of the general case is essentially the same. For the reader's convenience we provide the argument. It follows from the Cup Product-Commutativity Proposition 203.20 that it suffices to show that the map

$$\begin{array}{rcl} \Omega\colon \mathrm{FH}^n(M;\mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}^n(M;\mathbb{Z}), \mathbb{Z}) \\ \psi & \mapsto & (\varphi \mapsto \langle \varphi \cup \psi, [M] \rangle) \end{array}$$

is an isomorphism. As on page 4082 we start out with the following claim:

[2416] This statement is a slight generalization of the Manifold-Non Singularity Proposition 213.3.

Claim. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{FH}^n(M;\mathbb{Z}) & \xrightarrow{\quad\Omega\quad} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}^n(M;\mathbb{Z}),\mathbb{Z}). \\ & {\scriptstyle \mathrm{ev}}\searrow \quad \nearrow{\scriptstyle (\cap[M])^*} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}_n(M),\mathbb{Z}) & \end{array}$$

Proof. Note that for any $\psi \in \mathrm{FH}^n(M;\mathbb{Z})$ and any $\varphi \in \mathrm{FH}^n(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:[2417]

$$\Omega(\psi)(\varphi) \underset{\text{by definition of } \Omega \text{ and by the Cap Product-Kronecker Lemma 206.12}}{=} \epsilon_M((\varphi\cup\psi)\cap[M]) \underset{\text{by the Cup-Cap Lemma 206.15}}{=} \epsilon_M(\psi\cap(\varphi\cap[M])) \underset{\text{by the Cap Product Kronecker Lemma 206.12}}{=} \langle\psi,\varphi\cap[M]\rangle = \mathrm{ev}(\psi)(\varphi\cap[M]).$$

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms:

- By the Universal Coefficient-H*-Theorem 201.23 we have a short exact sequence
$$0 \to \underbrace{\mathrm{Ext}(\mathrm{H}_{n-1}(M),\mathbb{Z})}_{\substack{\text{torsion by the Ext-Properties Lemma 201.19} \\ \text{and the } \mathbb{Z}/k\mathbb{Z}\text{-Ext-Lemma 201.18}}} \to \mathrm{H}^n(M;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(M),\mathbb{Z}) \to 0.$$
It follows from the above data that the group to the left is torsion. It follows easily that the homomorphism $\mathrm{ev}\colon \mathrm{FH}^n(M;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}_n(M),\mathbb{Z})$ is, as desired, an isomorphism.
- The diagonal map to the right is an isomorphism by the Poincaré Duality Theorem 210.1.

■

Remark. Let M be a compact oriented $2n$-dimensional topological manifold. In the Intersection Form–Non Singularity Proposition 224.14 we just saw that if M is closed, then the intersection form is non-singular. If M is not closed, then we already saw on page 4331, that this conclusion does not necessarily hold. In the Asymmetric Intersection Pairing–Non Singularity Proposition 225.4 and the Reduced Intersection Form–Non Degeneracy Proposition 225.11 we will see that we can nonetheless associate to a compact (not necessarily closed) oriented topological manifold two pairings that are non-singular or at least non-degenerate. □

In the next lemma we want to see to what degree the isometry type of the intersection form is an invariant of the homeomorphism or homotopy type of the topological manifold:

[2416]The right diagonal is the dual map corresponding to the map $\mathrm{FH}^n(M;\mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{FH}_n(M)$.
[2417]Here $\epsilon_M\colon \mathrm{H}_0(M) \to \mathbb{Z}$ is the augmentation homomorphism from Definition 143.23.

Lemma 224.15. (Intersection Form–Homeomorphism Lemma) Let M and N be two compact oriented connected non-empty $2n$-dimensional topological manifolds. Furthermore let $f\colon (M,\partial M) \to (N,\partial N)$ be a continuous map such that $f\colon M \to N$ and $f\colon \partial M \to \partial N$ are homotopy equivalences.

(1) There exists an $\varepsilon \in \{-1,1\}$ which satisfies $f_*([M]) = \varepsilon \cdot [N]$. If f is an orientation-preserving homeomorphism, then $\varepsilon = 1$, and if f is an orientation-reversing homeomorphism, then $\varepsilon = -1$.

(2) (a) If $f_*([M]) = \ [N]$, then f induces an isometry $Q_M \cong \ Q_N$.
(b) If $f_*([M]) = -[N]$, then f induces an isometry $Q_M \cong -Q_N$.

Proof. [2418] Let $f\colon (M,\partial M) \to (N,\partial N)$ be a continuous map such that $f\colon M \to N$ and $f\colon \partial M \to \partial N$ are homotopy equivalences. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 together with the Fundamental Class-Proposition 194.2 (2) that there exists an $\varepsilon \in \{-1,1\}$ which satisfies $f_*([M]) = \varepsilon \cdot [N]$. If f is a homeomorphism, then we obtain ε from the Local Degree Lemma 195.5.

Claim. For any $c \in \mathrm{H}_n(M)$ we have $\mathrm{PD}_N(f_*(c)) = \varepsilon \cdot (f^*)^{-1}(\mathrm{PD}_M(c)) \in \mathrm{H}^n(N,\partial N;\mathbb{Z})$.

Proof. So let $c \in \mathrm{H}_n(M)$. We have

$$(f^*)^{-1}(\mathrm{PD}_M(c)) \cap [N] \underset{\uparrow}{=} (f^*)^{-1}(\mathrm{PD}_M(c)) \cap \varepsilon\cdot f_*([M]) \underset{\uparrow}{=} \varepsilon\cdot f_*(\mathrm{PD}_M(c) \cap [M]) \underset{\uparrow}{=} \varepsilon\cdot f_*(c).$$

by the definition of ε — by the Cap Product Naturality Lemma 206.13 — Definition 210.2 of PD_M

It follows from the Definition 210.2 of PD_N that we get the desired equality. ⊞

Now let $a, b \in \mathrm{FH}_n(M)$. Then we see that

$$\begin{aligned} Q_N(f_*(a),f_*(b)) &\overset{\downarrow}{=} \langle \mathrm{PD}_N(f_*(a)) \cup \mathrm{PD}_N(f_*(b)), [N]\rangle \overset{\downarrow}{=} \\ &= \langle \varepsilon\cdot(f^*)^{-1}(\mathrm{PD}_M(a)) \cup \varepsilon\cdot(f^*)^{-1}(\mathrm{PD}_M(b)), \varepsilon\cdot f_*([M])\rangle \\ &= \varepsilon\cdot\langle (f^*)^{-1}(\mathrm{PD}_M(a)) \cup \mathrm{PD}_M(b)), f_*([M])\rangle \underset{\uparrow}{=} \varepsilon\cdot Q_M(a,b). \end{aligned}$$

by Definition 222.3 of Q_N — by the claim — by the Cup Product-Naturality Lemma 204.8 and since $\varepsilon^3 = \varepsilon$ — by the Kronecker Pairing–Natural Lemma 199.9 (1) and Definition 222.3 of Q_M

The lemma is now a consequence of the following observation: It follows from the hypothesis that f is a homotopy equivalence together with the Homotopy Equivalence-H_*-Proposition 144.14 (1a) that $f_*\colon \mathrm{FH}_n(M) \to \mathrm{FH}_n(N)$ is an isomorphism. ■

224.5. **The intersection form and degree-one maps.** As we remarked on page 4330, the intersection form of a compact oriented $2n$-dimensional topological manifold contains basically the same information as the cup product pairing

$$\begin{aligned} \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) &\to \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) \\ (\alpha,\beta) &\mapsto \alpha \cup \beta. \end{aligned}$$

Given a continuous map $f\colon X \to Y$ between topological spaces it follows from the Cup Product-Naturality Lemma 203.9 that the cup product behaves well under the induced

[2418] The proof is perhaps slightly longer than one might expect. In a sense the statement of the lemma is basically trivial, but we have to pay a price for using the slightly awkward Poincaré Duality isomorphism $\mathrm{PD}_M\colon \mathrm{H}_n(M) \to \mathrm{H}^{2n-n}(M,\partial M;\mathbb{Z})$.

maps f^* on cohomology groups. It stands to reason that the intersection form should also behave well with respect to induced maps on homology.

So suppose that we are given two closed oriented connected non-empty $2n$-dimensional topological manifolds and let $f\colon X \to Y$ be a continuous map of degree 1, i.e. such that $f_*([X]) = [Y]$. By the aforementioned Cup Product-Naturality Lemma 203.9 we know that for any $\alpha \in \mathrm{FH}^i(Y;\mathbb{Z})$ and $\beta \in \mathrm{FH}^j(Y;\mathbb{Z})$ we have $f^*(\alpha) \cup f^*(\beta) = f^*(\alpha \cup \beta)$. The following question arises: Given $\alpha \in \mathrm{H}_n(X)$ and $\beta \in \mathrm{H}_n(X)$, does the following equality hold:

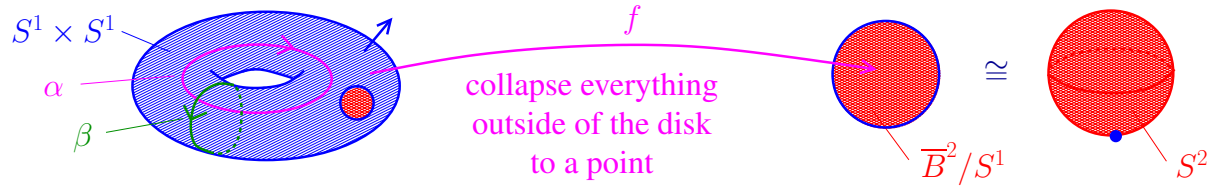
$$(*) \qquad \text{¿¿ } Q_X(\alpha,\beta) \;=\; Q_Y(f_*(\alpha), f_*(\beta)) \text{ ??}$$

In fact here is a diagram that looks awfully relevant:

$$\begin{array}{ccccccc}
\mathrm{H}_n(X)\times\mathrm{H}_n(X) & \xrightarrow[\cong]{\mathrm{PD}_X\times\mathrm{PD}_X} & \mathrm{H}^n(X;\mathbb{Z})\times\mathrm{H}^n(X;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{2n}(X;\mathbb{Z}) & \xrightarrow{\langle -,[X]\rangle} & \mathbb{Z} \\
\downarrow{\scriptstyle f_*\times f_*} & & \uparrow{\scriptstyle f^*\times f^*} & & \uparrow{\scriptstyle f^*}\cong & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}_n(Y)\times\mathrm{H}_n(Y) & \xrightarrow[\cong]{\mathrm{PD}_Y\times\mathrm{PD}_Y} & \mathrm{H}^n(Y;\mathbb{Z})\times\mathrm{H}^n(Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{2n}(Y;\mathbb{Z}) & \xrightarrow{\langle -,[Y]\rangle} & \mathbb{Z}.
\end{array}$$

If this diagram did commute, then $(*)$ would hold. One could now waste a lot of time trying to convince oneself that the diagram commutes. Or alternatively one might notice, as we do in the following example, that $(*)$ in general just does not hold.

Example. We consider the map $f\colon S^1 \times S^1 \to S^2$ that is illustrated in the figure below.



It follows easily from the Local-Global Degree Proposition 195.6 that this is a degree-one map. We consider the two homology classes α and β on the torus that are sketched in the figure below. First note that it follows easily from the Cup Product–Intersection Number Theorem 220.10 that $Q_{S^1\times S^1}(\alpha,\beta) = +1$. On the other hand note that we have $f_*(\alpha) = f_*(\beta) = 0 \in \mathrm{H}_1(S^2)$, which evidently implies that $Q_{S^2}(f_*(\alpha), f_*(\beta)) = 0$. Thus in this case the equality $(*)$ does not hold.[2419] □

Remark. With this discussion we also learned an interesting lesson:

Not every diagram commutes, not even up to sign. □

224.6. **The intersection form and the connected sum operation.** To get more examples of interesting intersection forms we want to study the intersection forms of the connected sum of smooth and topological manifolds. Recall that in Definition 54.12 we introduced the connected sum of two *smooth* manifolds. We now generalize this definition to topological manifolds:

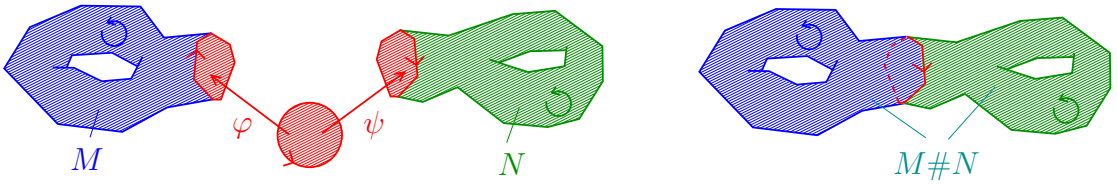
[2419] This still begs the question, in the above diagram, which of the squares does not commute?

Definition 224.16. Let $n \in \mathbb{N}_0$. Let M and N be two closed oriented connected non-empty n-dimensional topological manifolds. By the discussion on page 639 we can pick

- an orientation-preserving embedding $\varphi\colon \overline{B}^n \to M$ and
- an orientation-reversing embedding $\varphi\colon \overline{B}^n \to N$

such that the images are codimension-zero submanifolds. We define the connected sum of M and N as

$$M\#N := (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n))/\sim \qquad \text{where } \varphi(P) \sim \psi(P) \text{ for all } P \in S^{n-1}.$$



We will discuss this definition in greater generality and detail in Appendix 224.7. In particular in the appendix we will prove the following analog of the Smooth Connected Sum-Proposition 54.13:

Proposition 224.24. (Topological Connected Sum-Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented oriented connected n-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving embedding and
- let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing embedding

such that the images are codimension-zero submanifolds. The following four statements hold:[2420]

(1) The resulting connected sum $M\#N$ is an n-dimensional topological manifold.
(2) $M\#N$ is also closed and connected.
(3) $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$ are closed subsets of $M\#N$ and they are both codimension-zero submanifolds of $M\#N$.
(4) The topological manifold $M\#N$ admits a unique orientation that restricts to the given orientations of $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$.[2421]
(5) The homeomorphism type of $M\#N$ does not depend on the choice of φ and ψ.

Before we turn to the study of intersection forms of connected sums, let us first see how the homology groups and fundamental groups behave under the connected sum operation:

Proposition 224.17. (Connected Sum-Invariants Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected n-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving embedding and
- let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing embedding

[2421] The statements are mostly straightforward, except for the fact that the homeomorphism type of $M\#N$ does not depend on the choice of φ and ψ.

such that the images are codimension-zero submanifolds. Then the following two statements hold:

(1) If $n \geq 3$, then the following inclusion-induced maps are isomorphisms:

$$\pi_1(M\#N) \xleftarrow{\cong} \pi_1(M \setminus \varphi(B^n)) * \pi_1(N \setminus \psi(B^n)) \xrightarrow{\cong} \pi_1(M) * \pi_1(N).$$

(2) For any $i \neq 0, n$ the following inclusion-induced maps are isomorphisms

$$\mathrm{H}_i(M\#N) \xleftarrow{\cong} \mathrm{H}_i(M\setminus\varphi(B^n)) \oplus \mathrm{H}_i(N\setminus\psi(B^n)) \xrightarrow{\cong} \mathrm{H}_i(M) \oplus \mathrm{H}_i(N).$$

An analogous statement holds for homology groups with coefficients in any abelian group.

Proof. In the case of smooth manifolds we proved the statements in the Connected Sum-π_1-Proposition 102.4 and the Connected Sum-H_*-Proposition 185.23. The proof for topological manifolds is exactly the same. (Note that the Manifold–Seifert-van Kampen Theorem 102.1 and the Mayer–Vietoris Theorem 153.2 (3) apply to smooth *and* topological manifolds). ■

The next proposition says that the isomorphism of homology groups from the Connected Sum-Invariants Proposition 224.17 (2) induces an isometry of intersection forms:

Proposition 224.18. (Intersection Form-Additivity Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected $2n$-dimensional topological manifolds. Furthermore

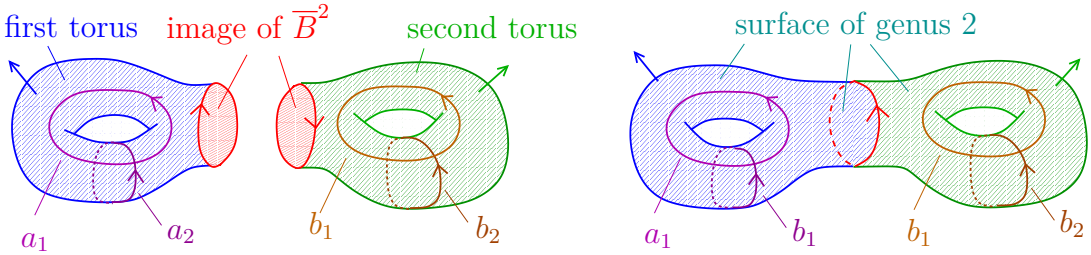
- let $\varphi\colon \overline{B}^{2n} \to M$ be an orientation-preserving embedding and
- let $\psi\colon \overline{B}^{2n} \to N$ be an orientation-reversing embedding

such that the images of both maps are codimension-zero submanifolds. The isomorphism $\mathrm{H}_n(M\#N) \xrightarrow{\cong} \mathrm{H}_n(M) \oplus \mathrm{H}_n(N)$ from the Connected Sum-Invariants Proposition 224.17 induces an isometry

$$Q_{M\#N} \xrightarrow{\cong} Q_M \oplus Q_N$$

of forms.

Example. We consider the surface Σ of genus two. In the figure below we recall that we can view Σ as the connected sum of two tori.



It thus follows from the Intersection Form-Additivity Proposition 224.18, the discussion on page 4331 and the Matrix–Form Operations Lemma 224.8 (2) that the intersection form on $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Not surprisingly this recovers the result from page 4330. □

Remark. We continue with the notation from the Intersection Form-Additivity Proposition 224.18. The figure above suggests the following approach to proving the proposition:

(1) First we pick n-dimensional cycles $a_1, \dots, a_k$ in $M \setminus \varphi(\overline{B}^{2n})$ such that $[a_1], \dots, [a_k]$ form a basis of $\mathrm{FH}_n(M)$ and which intersect nicely, in the sense of Definition 221.19.
(2) Similarly we pick transverse n-dimensional cycles $b_1, \dots, b_l$ in $N \setminus \psi(\overline{B}^{2n})$ such that $[b_1], \dots, [b_l]$ form a basis of $\mathrm{FH}_n(N)$ and which intersect nicely.
(3) Note that it follows immediately from the Connected Sum-Invariants Proposition 224.17 that $[a_1], \dots, [a_k], [b_1], \dots, [b_l]$ form a basis for $\mathrm{FH}_n(M\#N)$.
(4) In the Cycles–Cup Product-Intersection Number Theorem 221.20 we calculated the intersection form in terms of the algebraic intersection number of cycles that intersect "nicely". We use this fact together with the basis from (3) to determine the intersection form of $M\#N$. Here we make the following observations:
   (a) By the orientation conventions from the Topological Connected Sum-Proposition 224.24 (4) the algebraic intersection numbers of the a_i in M are the same as in $M\#N$.
   (b) Similarly the algebraic intersection numbers of the b_i in N are the same as in $M\#N$.
   (c) Since the a_i do not intersect the b_j we see that the algebraic intersection number $a_i \cdot b_j = 0$ for any i, j.

   Summarizing we obtain immediately from the Cycles–Cup Product-Intersection Number Theorem 221.20 that $Q_M \oplus Q_N \cong Q_{M\#N}$.

The above argument sounds very convincing. But there is a major catch: how can we find such $a_1, \dots, a_k$ and $b_1, \dots, b_l$? The problem is not that we need to avoid $\varphi(\overline{B}^{2n})$ and $\psi(\overline{B}^{2n})$. The problem is to find such cycles that are transverse. Even though we have some tools, e.g. we have the Deform-to-Embedding Theorem **??** and the Transversality Theorem 50.9, it is very hard to turn the above approach into a rigorous proof. Furthermore, any attempt to make this approach work would inevitably end up with a rather unreadable mess. □

After the discussion in the previous remark we recall the advice from Rob Kirby [**Kir1989**, p. 21], see page 4234:

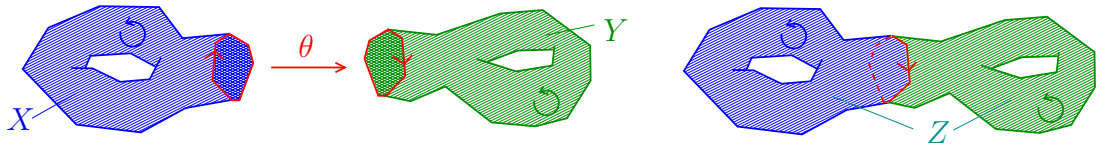
"Think with [algebraic] intersection [numbers], prove with cup products."

Thus we will use cup products to prove the Intersection Form-Additivity Proposition 224.18.

Proof of the Intersection Form-Additivity Proposition 224.18. Throughout the proof let $n \geq 1$. We start out with a lengthy preparation: Thus let X and Y be two oriented connected $2n$-dimensional topological manifolds such that ∂X and ∂Y are both homeomorphic to S^{2n-1}. We equip ∂X and ∂Y with the orientation coming from the Boundary Orientation Proposition 192.3. Let $\theta\colon \partial X \to \partial Y$ be an orientation-reversing homeomorphism. We write $Z = X \cup_\theta Y$. In the following we view X and Y as subspaces of Z. By the Gluing-Topological Manifolds-Proposition 54.3 we know that X and Y are in fact codimension-zero submanifolds of Z and that we can equip Z with an orientation such that the induced orientations on X and Y are the given orientations.

Claim. The inclusions $i\colon X \to Z$ and $j\colon Y \to Z$ induce an isometry

$$Q_X \oplus Q_Y \xrightarrow{\cong} Q_Z.$$



Proof. As in the proof of the Connected Sum-H_*-Proposition 185.23 we see that the two inclusion maps i and j induce an isomorphism $\mathrm{H}_n(X) \oplus \mathrm{H}_n(Y) \to \mathrm{H}_n(Z)$. Evidently this isomorphism descends to an isomorphism $\mathrm{FH}_n(X) \oplus \mathrm{FH}_n(Y) \to \mathrm{FH}_n(Z)$. Therefore it suffices to prove the following three statements.

(1) For $a, b \in \mathrm{FH}_n(X)$ we have $Q_X(a,b) = Q_Z(i_*(a), i_*(b))$.
(2) For $a, b \in \mathrm{FH}_n(Y)$ we have $Q_Y(a,b) = Q_Z(j_*(a), j_*(b))$.
(3) For $a \in \mathrm{FH}_n(X)$ and $b \in \mathrm{FH}_n(Y)$ we have $Q_Z(i_*(a), j_*(b)) = 0$.

We start out with the proof of Statement (1): Thus let $a, b \in \mathrm{FH}_n(X)$. In the following we write $\overset{\circ}{X} = X \setminus \partial X$. We denote by $p\colon (Z, \varnothing) \to (Z, Z \setminus \overset{\circ}{X})$ and $i\colon (X, \partial X) \to (Z, Z \setminus \overset{\circ}{X})$ the two obvious inclusion maps of pairs of topological spaces. It is a consequence of the Manifold–Excision Theorem 148.3 that $i_*\colon \mathrm{FH}_k(X, \partial X) \to \mathrm{FH}_k(Z, Z \setminus \overset{\circ}{X})$ is an isomorphism for every $k \in \mathbb{N}_0$. Then

$$Q_X(a,b) \underset{\uparrow}{=} \langle \mathrm{PD}_X(a) \cup \mathrm{PD}_X(b), [X] \rangle \underset{\uparrow}{=} \big\langle (i^*)^{-1}(\mathrm{PD}_X(a)) \cup (i^*)^{-1}(\mathrm{PD}_X(b)), i_*([X]) \big\rangle$$

by Definition 222.3 of Q_X

by the Kronecker Pairing–Natural Lemma 199.9 (1) applied to $i\colon (X, \partial X) \to (Z, Z \setminus \overset{\circ}{X})$, together with the Cup Product-Naturality Lemma 204.8

$$\underset{\uparrow}{=} \big\langle p^*((i^*)^{-1}(\mathrm{PD}_X(a))) \cup p^*((i^*)^{-1}(\mathrm{PD}_X(b))), [Z] \big\rangle$$

first note that by the Codimension Zero-Fundamental Class Lemma 194.10 we have, $i_*([X]) = p_*([Z])$, next apply the Kronecker Pairing–Natural Lemma 199.9 (1) to p, finally apply the Cup Product-Naturality Lemma 204.8 to p

$$\underset{\uparrow}{=} \big\langle \mathrm{PD}_Z(i_*(a)) \cup \mathrm{PD}_Z(i_*(b)), [Z] \big\rangle \underset{\uparrow}{=} Q_Z(i_*(a), i_*(b)).$$

by the Small H_*-to-Large H^*-PD Lemma 212.8 we know that $p^*(i^*)^{-1}(\mathrm{PD}_X(c)) = \mathrm{PD}_Z(i_*(c))$

by Definition 222.3 of Q_Z

Statement (2) is of course proved the same way. We turn to the proof of Statement (3). So let $a \in \mathrm{FH}_n(X)$ and $b \in \mathrm{FH}_n(Y)$. The above argument shows that we can compute $Q_Z(i_*(a), j_*(b))$ in terms of the cup product in Z of an element in the image $\mathrm{H}^n(Z, Z \setminus \overset{\circ}{X}; \mathbb{Z}) \to \mathrm{H}^n(Z; \mathbb{Z})$ and an element in the image of $\mathrm{H}^n(Z, Z \setminus \overset{\circ}{Y}; \mathbb{Z}) \to \mathrm{H}^n(Z; \mathbb{Z})$. But $(Z \setminus \overset{\circ}{X}) \cup (Z \setminus \overset{\circ}{Y}) = Z$. So it follows from the Cup Product-Naturality Lemma 204.8 that this cup product is zero. ⊞

After these preparations it is very easy to prove the actual statement of the proposition. Thus let M and N be closed oriented $2n$-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving embedding and
- let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing embedding

such that the images are codimension-zero submanifolds. We write $X = M \setminus \varphi(B^{2n})$ and $Y = N \setminus \psi(B^{2n})$. It follows from the Codimension-Zero Submanifold Proposition 150.16 that X and Y are compact topological manifolds. With this notation we can write $M = X \cup \overline{B}^{2n}$

and $N = Y \cup \overline{B}^{2n}$. It follows that the inclusions induce isometries

$$Q_{M\#N} \xleftarrow[\text{isometry by the claim}]{\cong} Q_X \oplus Q_Y \xrightarrow[\text{isometry by the claim since } \mathrm{H}_n(\overline{B}^{2n})=0]{\cong} Q_M \oplus Q_N.$$

We thus have proved the desired statement. ■

Now is a good moment to recall the following question:

Question 185.25. Are the topological spaces $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ and $S^2 \times S^2$ homeomorphic?

These topological spaces are hard to distinguish since we saw that they are simply connected and that their (co-) homology groups agree. Now we want to look at the intersection forms to see whether they can help us in distinguishing these three smooth manifolds.

Example. On page 4331 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the (1×1)–matrix (1) and we saw that the intersection form of $\overline{\mathbb{C}\mathrm{P}}^2$ is represented by the (1×1)–matrix (-1). From the Intersection Form-Additivity Proposition 224.18 we thus obtain:

(1) the intersection form of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, whereas

(2) the intersection form of $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Finally we recall that on page 4331 we showed that

(3) the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. □

Motivated by Question 185.25 and the Intersection Form–Homeomorphism Lemma 224.15 the following question arises:

Question 224.19. Are the forms over $\mathbb{Z}^2$ represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

isometric (possibly up to a sign reversal)?

In Chapter 226 we will develop the theory of non-singular forms to answer Question 224.19.

224.7. **Appendix: The connected sum of topological manifolds.** In Definition 224.16 we introduced the connected sum of two closed oriented connected non-empty topological manifolds. Furthermore on page 4336 we stated a preliminary version of the Topological Connected Sum-Proposition 224.24.

In this appendix we will discuss the connected sum of topological manifolds more thoroughly. The reader who is not that interested in topological manifolds can safely move on to the next chapter.

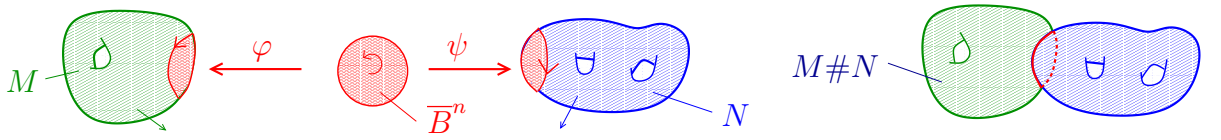
For those who are staying, let us recall the Definition 54.12 of the connected sum of two *oriented* smooth manifolds:

Definition 224.20. Let $n \in \mathbb{N}_0$ and let M and N be two oriented connected non-empty n-dimensional smooth manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M \backslash \partial M$ and $\psi\colon \overline{B}^n \to N \backslash \partial N$ be smooth embeddings (in the sense of Definition 30.9) such that φ is orientation-preserving

and such that ψ is orientation-reversing. We define the connected sum of the oriented smooth manifolds M and N as

$$M\#N := (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n))/\sim \qquad \text{where } \varphi(P)\sim\psi(P) \text{ for all } P\in S^{n-1}.$$

In the Smooth Connected Sum-Proposition 54.13 we showed that the connected sum of two oriented smooth manifolds is again a smooth manifold.



Now we intend to generalize this notion to the setting of topological manifolds. The idea is to suitably adapt the concepts used in the above Definition 224.20. Browsing through earlier chapters we see the following:

(1) In Definition 4.20 we defined an embedding to be a continuous map $f\colon X \to Y$ between two topological spaces that is a homeomorphism onto its image.
(2) In Definition 191.17 we introduced the notion of an orientation of a topological manifold and of an orientation-preserving map.

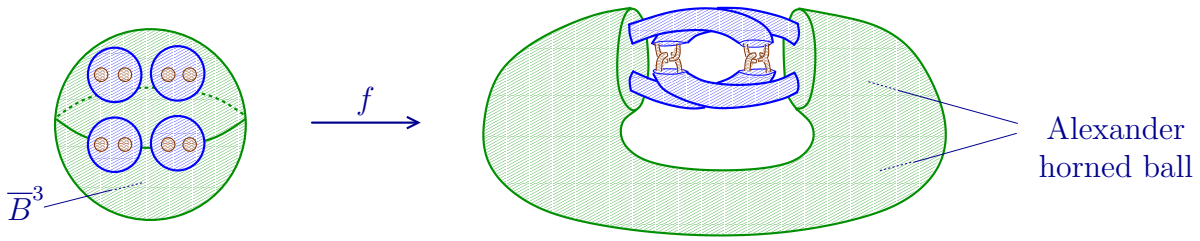
So are we done? Can we just copy-paste the above Definition 224.20?

As we had just mentioned, in the Smooth Connected Sum-Proposition 54.13 we showed that the connected sum of two oriented smooth manifolds is again a smooth manifold. Recall that the proof of the Smooth Connected Sum-Proposition 54.13 relied heavily on the Smooth Embedding Theorem 30.15 which says that the image of any smooth embedding $f\colon X \to Y \setminus \partial Y$ is a smooth submanifold of the smooth manifold Y.

The question arises, whether the analogous statement holds for topological manifolds. In other words, we have the following question:

Question 224.21. Let X be an n-dimensional topological manifold. Is the image of an embedding $f\colon \overline{B}^n \to X \setminus \partial X$ necessarily a topological submanifold of X?

Example. For better or worse the answer to Question 224.21 is no.[2422] Namely consider the embedding $f\colon \overline{B}^3 \to \mathbb{R}^3$ that is illustrated in the figure below. We refer to the image $f(\overline{B}^3)$ as the *Alexander horned ball*.[2423]



It is clear that this is a weird embedding. We leave it to the reader to make the embedding precise and to show that $f(\overline{B}^3)$ is not a topological submanifold of $\mathbb{R}^3$. But even without

[2422] That is not really a surprise, on page 786 we already saw wild knots in $\mathbb{R}^3$ and on page 3249 we saw the Alexander horned sphere. Both give examples of embeddings of topological manifolds such that the image is not a submanifold.
[2423] On page 3249 we referred to $f(S^2)$ as the Alexander horned sphere.

filling in all details this map should shatter the reader's confidence that Question 224.21 has a positive answer. □

This stupefying example leads us to the following definition:

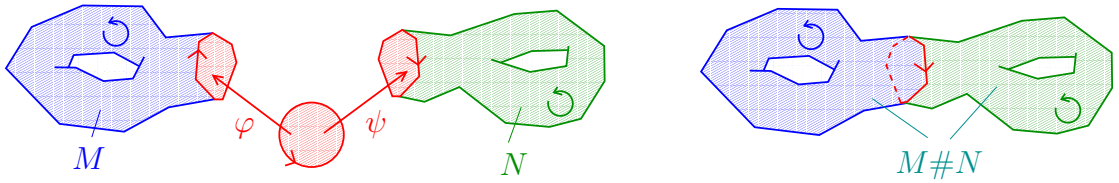
Definition 224.22. Let X and Y be topological manifolds. We say that an embedding $f\colon X \to Y$ is locally flat if $f(X)$ is a submanifold of Y.

Remark. Let X and Y be smooth manifolds. Note that it follows from the Smooth Embedding Theorem 30.15 that every smooth embedding $X \to Y$ is actually a locally flat embedding. □

With this definition we can now finally give a reasonable definition of the connected sum of two *topological* manifolds. For completeness' sake we now also define the connected sum if one or both topological manifolds involved are non-orientable.

Definition 224.23. Let $n \in \mathbb{N}$. Let M and N be two connected non-empty n-dimensional topological manifolds. (If both M and N are orientable, then we pick orientations for M and N.) By the discussion on page 639 we can pick two locally flat embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$. (If both M and N are orientable, then precisely one of φ and ψ is supposed to be orientation-preserving.) We define the connected sum of M and N as

$$M\#N := (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n))/\sim \text{ where } \varphi(P) \sim \psi(P) \text{ for all } P \in S^{n-1}.$$



The following proposition is the analog of the Smooth Connected Sum-Proposition 54.13 that we formulated in the context of smooth manifolds:

Proposition 224.24. (Topological Connected Sum-Proposition) Let $n \in \mathbb{N}$. Let M and N be two connected n-dimensional topological manifolds. (If both M and N are orientable, then we demand that M and N are oriented.) Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be two locally flat embeddings. (If M and N are orientable, then we demand that φ is orientation-preserving and that ψ is orientation-reversing.) The following five statements hold:

(0) (a) If M and N are compact, then $M\#N$ is also compact.
    (b) If $n \geq 2$, then $M\#N$ is also connected.
(1) The resulting connected sum $M\#N$ is an n-dimensional topological manifold.
(2) (a) The boundary of $M\#N$ is given by $\partial M \sqcup \partial N$.
    (b) If M and N are closed, then $M\#N$ is also closed.
(3) $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$ are closed subsets of $M\#N$ and they are both codimension-zero submanifolds of $M\#N$.
(4) If M and N are oriented, then the topological manifold $M\#N$ admits a unique orientation that restricts to the given orientations of $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$.

(5) The homeomorphism type of $M\#N$ is natural and it does not depend on the choice of φ and ψ.
(6) (a) The connected sum operation is commutative, in the sense that there exists an (orientation-preserving) homeomorphism from $M\#M'$ to $M'\#M$.
(b) If $n \geq 2$, then the connected sum operation is associative, in the sense that if we are furthermore given a connected (oriented) n-dimensional topological manifold N', then there exists an (orientation-preserving) homeomorphism from $(M\#N)\#N'$ to $M\#(N\#N')$.

Convention 224.25. We always equip the connected sum of two oriented connected n-dimensional topological manifolds with the orientation from the Topological Connected Sum-Proposition 224.24 (4).

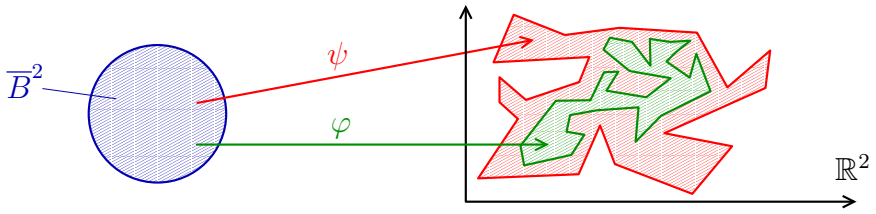
Most of the statements of the Topological Connected Sum-Proposition 224.24 are easy to prove:

Proof of Proposition 224.24 (0), (1), (2), (3), (4) and (6). The proof is basically identical to the proofs of the corresponding statements in the smooth setting that we proved in the Smooth Connected Sum-Proposition 54.13. There are just a few minor modifications that we need to make:

(i) We used the Codimension-Zero Smooth Embedding Proposition 30.21 to show that the image of a smooth embedding is a submanifold. This feature is now built into the definition of a locally flat embedding.
(ii) We need to replace the Codimension-Zero Smooth Embedding Proposition 30.21 and the Gluing-Smooth Manifolds-Proposition 54.5 by the Codimension-Zero Submanifold Proposition 150.16 and the Gluing-Topological Manifolds-Proposition 54.3. ■

The fact that the connected sum operation does not really depend on the choice of the locally flat topological embeddings is much more difficult to prove. The key to doing so is the following deep theorem.

Theorem 224.26. (Annulus Theorem) Let $n \in \mathbb{N}$ and let $\varphi, \psi\colon \overline{B}^n \to \mathbb{R}^n$ be two locally flat embeddings. If $\varphi(\overline{B}^n) \subseteq \psi(B^n)$, then $\psi(\overline{B}^n) \setminus \varphi(B^n)$ is homeomorphic to $[0,1] \times S^{n-1}$.



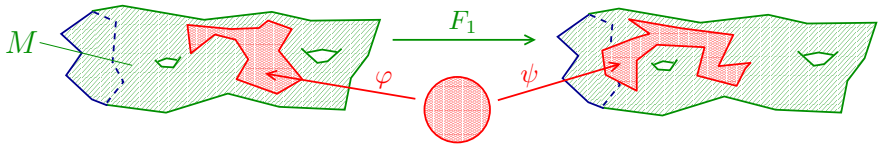
Proof. The case $n = 1$ is trivial. For $n = 2$ and $n = 3$ the Annulus Theorem follows from the work of Tibor Radó [**Rad1926**] and Edwin Moise [**Moi1952, Moi1977**]. The Annulus Theorem was proved for dimensions $n \geq 5$ by Rob Kirby [**Kir1969**] and in dimension 4 by Frank Quinn [**Quin1982**, p. 506]. We also refer to [**Edw1984**, p. 247] for more information. ■

Using the Annulus Theorem 224.26 one can now prove the following theorem which can be viewed as the topological analog of the Smooth Ball Embedding Theorem 45.10:

Theorem 224.27. (Topological Ball Embedding Theorem) Let M be an n-dimensional topological manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_m \colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_m \colon \overline{B}^n \to M \setminus \partial M$ be two sets of m locally flat embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $m = 1$, then there exists a homeotopy

$$F \colon M \times [0,1] \ \to \ M$$

rel some neighborhood of ∂M from $F_0 = \mathrm{id}_M$ to a homeomorphism $F_1 \colon M \to M$ such that for each $i \in \{1, \dots, m\}$ we have $F_1 \circ \varphi_i = \psi_i$. (Note that if M is orientable, then it follows from the Isotopy-Orientation Lemma 191.28 that the homeomorphism F_1 is in fact orientation-preserving.)



Remark. We consider $\overline{B}^3$ as a submanifold of $\mathbb{R}^3$. Since the image of the embedding $\varphi \colon \overline{B}^3 \to \mathbb{R}^3$ that is given by the Alexander horned ball is not a submanifold of the topological manifold it follows that there is no homeotopy $F \colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $F_1 \circ \varphi = \mathrm{id}$. In particular we see that in the formulation of the Topological Ball Embedding Theorem 224.27 we cannot drop the hypothesis that the embeddings are locally flat. □

Proof of the Topological Ball Embedding Theorem 224.27. This theorem follows from the Annulus Theorem 224.26 together with reasonably elementary arguments. The details are worked out in [**FNOP2025**, Proof of Theorem 5.11]. Alternatively the reader is encouraged to assemble the argument by using the obvious analogs of Exercise 95.7 and Exercise 58.6, which were both formulated in the setting of smooth manifolds, in the setting of topological manifolds. ■

Proof of the Topological Connected Sum-Proposition 224.24 (5). Thanks to our preparations the proof is basically identical to the proof of the corresponding statement in the smooth setting that we proved in the Smooth Connected Sum-Proposition 54.13. All we need to do is to replace the Smooth Ball Embedding Theorem 45.10 by the Topological Ball Embedding Theorem 224.27. ■

224.8. **Appendix form: Intersection numbers on non-compact manifolds.** In this appendix we will sketch how one can generalize some of the concepts of this chapter from the compact setting to the non-compact setting.

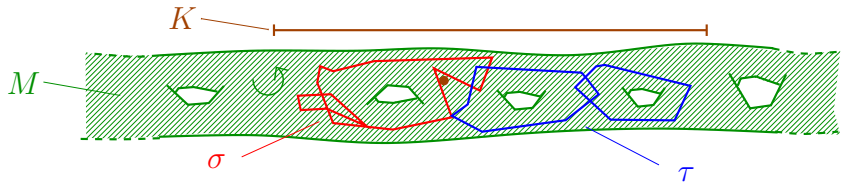
Lemma 224.28. (Non-Compact Intersection Number Lemma) Let M be a connected oriented n-dimensional topological manifold with $\partial M = \varnothing$ and let $\sigma \in \mathrm{H}_k(M)$ and let $\tau \in \mathrm{H}_{n-k}(M)$.

(1) Since $\partial M = \varnothing$ we can consider the Poincaré Duality classes $\mathrm{PD}_M(\sigma) \in \mathrm{H}_c^{n-k}(M;\mathbb{Z})$ and $\mathrm{PD}_M(\tau) \in \mathrm{H}_c^{k}(M;\mathbb{Z})$ in cohomology with compact support, as introduced in the Poincaré Duality–Non-Compact Theorem 211.4.
   (a) There exists a compact connected subset $K \subseteq M$ such that $\mathrm{PD}_M(\sigma) \in \mathrm{H}_c^{n-k}(M;\mathbb{Z})$ and $\mathrm{PD}_M(\sigma) \in \mathrm{H}_c^{k}(M;\mathbb{Z})$ both lie in the image of the natural homomorphism $\rho\colon \mathrm{H}^*(M, M\setminus K;\mathbb{Z}) \to \mathrm{H}_c^*(M;\mathbb{Z})$ from Definition 209.14.
   (b) If M is actually a smooth manifold, then we can arrange that $K \subseteq M$ is a compact codimension-zero smooth submanifold.

We pick $\varphi \in \mathrm{H}^{n-k}(M, M\setminus K;\mathbb{Z})$ with $\rho(\varphi) = \mathrm{PD}_M(\sigma)$ and we pick $\psi \in \mathrm{H}^{k}(M, M\setminus K;\mathbb{Z})$ with $\rho(\psi) = \mathrm{PD}_M(\tau)$.

(2) We have $\mathrm{H}_n(M, M\setminus K) \cong \mathbb{Z}$ and there exists a unique $\mu_K \in \mathrm{H}_n(M, M\setminus K)$ which restricts to the orientation for each $x \in K$.

(3) The integer
$$\langle \underbrace{\varphi \cup \psi}_{\in \mathrm{H}^n(M, M\setminus K;\mathbb{Z})}, \mu_K \rangle \in \mathbb{Z}$$
is independent of the choice of K.



Sketch of proof.

(1) (a) This statement follows almost immediately from the H_c^n-via-Direct Colimits Proposition 209.15, together with the Elements-in-Direct Colimit Lemma 104.8 and the Topological Manifold Exhaustion Lemma 23.12.
   (b) If M is a smooth manifold, then we can replace the Topological Manifold Exhaustion Lemma 23.12 by the Compact Submanifold Exhaustion Proposition 38.2 and we obtain the desired, more refined, statement.

(2) This statement follows by the same argument as in the proof of the Manifold-Top-H_*-Theorem on page 3734. All one really has to do is to combine the definitions with the Homology-Section Theorem 193.6 (i) and the Sections-on-Connected Proposition 191.13. The details were hopefully filled in in Exercise 193.18.

(3) To show that the given number is independent of the choice of K it suffices (by the Topological Manifold Exhaustion Lemma 23.12) to consider the case that we have two compact connected subspaces $K \subseteq L \subseteq M \setminus \partial M$ with the desired properties. At that point independence follows fairly easily from the naturality of the cup and cap product, i.e. it follows from the Cup Product-Naturality Lemma 204.8 together with the Cap Product-Naturality Lemma 206.13. We leave it to the reader to fill in the details. ■

The above lemma allows us to introduce the following definition:

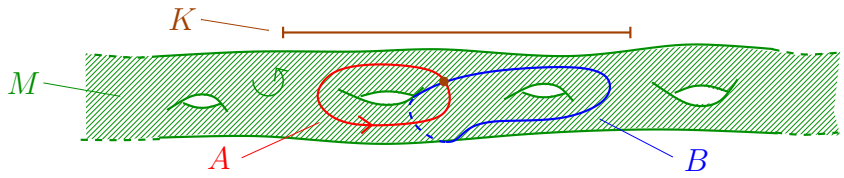
Definition 224.29. Let M be a connected oriented n-dimensional topological manifold and let $\sigma \in \mathrm{H}_k(M)$ and let $\tau \in \mathrm{H}_{n-k}(M)$. By the Manifold-minus-Boundary Lemma 193.15 together with the Homotopy Equivalence-H_*-Proposition 144.14 we know that the inclusion $\iota\colon M' := M \setminus \partial M \to M$ induces an isomorphism on homology groups. Thus we can set $\sigma' := \iota_*^{-1}(\sigma) \in \mathrm{H}_k(M)$ and $\tau' := \iota_*^{-1}(\tau) \in \mathrm{H}_{n-k}(M)$. Continuing with the (obvious variation on the) notation from the Non-Compact Intersection Number Lemma 224.28 we now set

$$Q_M(\sigma,\tau) \;:=\; \langle \varphi' \cup \psi', \mu_K \rangle \;\in\; \mathbb{Z}.$$

One can easily verify that this definition of Q_M is essentially consistent with Definition 224.12. If $\sigma = [A]$ and $\tau = [B]$ are represented by closed oriented submanifolds $A, B \subseteq M$, then we refer to

$$A \cdot B \;:=\; Q_M([A],[B])$$

as the algebraic intersection number of A and B. It follows from the Cup Product–Intersection Number Theorem 224.30 below that this definition is consistent with Definition 220.5.



Theorem 224.30. (Cup Product–Intersection Number Theorem) Let M be a connected oriented n-dimensional smooth manifold. Let $A \subseteq M$ be a closed k-dimensional oriented submanifold and let $B \subseteq M$ be a closed $(n-k)$-dimensional oriented submanifold. If A and B are transverse in the sense of Definition 220.3, then

$$A \cdot B \;=\; \sum_{P \in A \cap B} \mathrm{sign}(P)$$

where $\mathrm{sign}(P)$ is defined as in Definition 220.5.

Proof. Using the Non-Compact Intersection Number Lemma **??** (1b) one sees easily that we can reduce the desired statement to the compact case that we dealt with in the Cup Product–Intersection Number Theorem 220.10. ∎

Remark. We suppose that we are again in the setting of the Non-Compact Intersection Number Lemma 224.28. Thus let M be a connected oriented n-dimensional topological manifold with $\partial M = \varnothing$ and let $\sigma \in \mathrm{H}_k(M)$ and let $\tau \in \mathrm{H}_{n-k}(M)$. Note that by Exercise 194.18 and 194.20 we have a well-defined fundamental class $[M]$ in locally finite homology, which means that we have a well-defined fundamental class

$$[M] \;\in\; \mathrm{H}_n^{\mathrm{lf}}(M;\mathbb{Z}) \;=_{\mathrm{i}}\; \overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) \;:=\; \varprojlim_{K \in \mathcal{K}(M)} \mathrm{H}_n(M, M \setminus K;\mathbb{Z})$$

where $\mathcal{K}(M)$ is the inverse system of all compact subspaces of M. In Exercise **??** we introduced a cap product of the form

$$\cap\colon \underbrace{\varinjlim_{K \in \mathcal{K}(M)} \mathrm{H}^k(M, M \setminus K;\mathbb{Z})}_{=\mathrm{H}_c^k(M;\mathbb{Z})} \times \underbrace{\varprojlim_{K \in \mathcal{K}(M)} \mathrm{H}_l(M, M \setminus K;\mathbb{Z})}_{=\overleftarrow{\mathrm{H}}_l(M;\mathbb{Z})} \;\to\; \mathrm{H}_{l-k}(M;\mathbb{Z}).$$

It now follows easily from the definitions that

$$\langle \underbrace{\varphi \cup \psi}_{\in \mathrm{H}^n(M,M\setminus K;\mathbb{Z})}, \mu_K \rangle = \underset{\substack{\uparrow \\ \text{augmentation map} \\ \epsilon_M\colon \mathrm{H}_0(M;\mathbb{Z}) \to \mathbb{Z}}}{\epsilon_M}((\mathrm{PD}_X(\sigma) \cup \mathrm{PD}_Y(\sigma)) \cap [M]).$$

This leads to a more elegant definition of the above algebraic intersection number. □

224.9. Exercises for Chapter 224 .

Exercise 224.1. Let R be a commutative ring, let V be an R-module and let $\varphi\colon V\times V \to R$ be a symmetric R-bilinear pairing. Given a submodule $W \subseteq V$ we obtain a symmetric R-bilinear pairing $\varphi|_W\colon W \times W \to R$ by restricting φ to $W \times W$.

(a) Show that if φ is non-degenerate, then it does not necessarily follow that $\varphi|_W$ is non-degenerate.

(b) Let $\langle -,-\rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the usual Euclidean pairing. Show that for every $\mathbb{R}$-vector subspace $W \subseteq \mathbb{R}^n$ the corresponding pairing $W \times W \to \mathbb{R}$ is non-singular.

(c) Let $\langle -,-\rangle\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be the usual Euclidean pairing. Show that for a $\mathbb{Z}$-submodule $W \subseteq \mathbb{Z}^n$ the corresponding pairing $W \times W \to \mathbb{Z}$ is non-degenerate and show that in general it is not non-singular.

Exercise 224.2. We consider the intersection form $Q\colon \mathrm{H}_1(S^1 \times S^1) \times \mathrm{H}_1(S^1 \times S^1) \to \mathbb{Z}$. Express the isometry group of Q with respect to your favorite basis of $\mathrm{H}_1(S^1 \times S^1)$.

Exercise 224.3. For which $d \in \mathbb{N}$ is the isometry type of the intersection form Q_M a complete invariant of closed oriented connected non-empty 2-dimensional smooth manifolds?

Exercise 224.4. We consider the 2-dimensional torus $S^1 \times S^1$. We equip $\mathrm{H}_1(S^1 \times S^1)$ with the standard basis $\mu = [S^1 \times \{*\}]$ and $\lambda = [\{*\} \times S^1]$. Show that

$$Q_{S^1\times S^1}(a \cdot \mu + b \cdot \lambda, c \cdot \mu + d \cdot \lambda) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Remark. The exercise is really quite easy, but it does produce a neat formula.

Exercise 224.5. Let V and W be free $\mathbb{Z}$-modules of finite rank, i.e. V and W are finitely generated free abelian groups. Furthermore let $\lambda\colon V \times W \to \mathbb{Z}$ be a pairing. Recall that we say that λ is *non-singular* if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_{\mathbb{Z}}(W,\mathbb{Z}) \\ v & \mapsto & \begin{pmatrix} W & \to & \mathbb{Z} \\ w & \mapsto & \lambda(v,w) \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_{\mathbb{Z}}(V,\mathbb{Z}) \\ w & \mapsto & \begin{pmatrix} V & \to & \mathbb{Z} \\ v & \mapsto & \lambda(v,w) \end{pmatrix} \end{array}$$

are isomorphisms. We say that the pairing is *non-degenerate* if the two maps are monomorphisms.

(a) Let $v_1, \dots, v_n$ be a basis for V. Show that the following two statements are equivalent:
  (i) There exists some basis $w_1, \dots, w_n$ for W such that $\lambda(v_i, w_j) = \delta_{ij}$ for every choice of $i, j \in \{1, \dots, n\}$.[2424]
  (ii) The pairing λ is non-singular.

(b) Suppose that λ is non-degenerate. Show that V and W have the same rank.

[2424] The basis $w_1, \dots, w_n$ is in fact unique.

(c) Suppose that V and W are of rank n and suppose that we are given $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$ such that $\det(\lambda(v_i, w_j)) = \pm 1$. Show that $v_1, \dots, v_n$ is a basis for V and that $w_1, \dots, w_n$ is a basis for W.

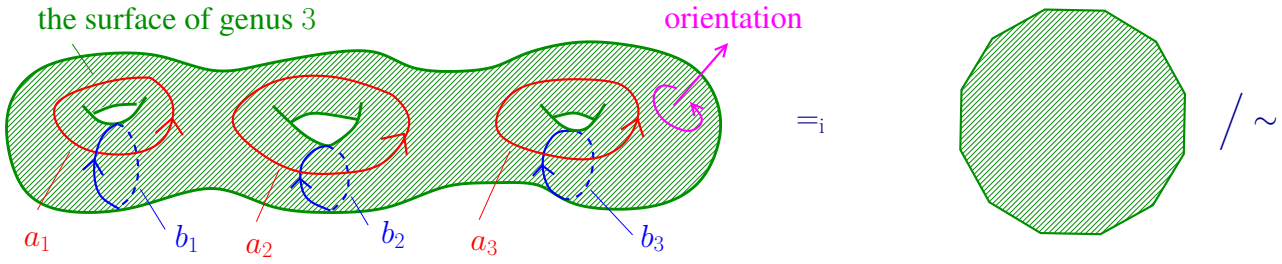
Exercise 224.6. Let V be a free $\mathbb{Z}$-module of finite rank, i.e. V is a finitely generated free abelian group. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form. Given a subgroup $U \subseteq V$ we write

$$U^\perp \;=\; \{v \in V \mid \varphi(u, v) = 0 \text{ for all } u \in U\}.$$

We suppose that the form φ is non-singular. Show that for any subgroup $U \subseteq V$ we have $\operatorname{rank}(U^\perp) = \operatorname{rank}(V) - \operatorname{rank}(U)$.

Exercise 224.7. Determine a matrix that represents the intersection form of $(S^1)^4$.

Exercise 224.8. We consider the curves a_i, b_i in the figure below. What do these closed curves a_i and b_i look like in the more precise description of the surface of genus 3 as a dodecagon with identified sides?



Exercise 224.9.

(a) Show that there is no continuous map $((S^1)^2 \times S^2)\#\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ of degree ± 1.

(b) Does there exist a finite covering of $((S^1)^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ which admits a continuous map to $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ of degree ± 1?

Exercise 224.10. Let $n \in \mathbb{N}$ and let M and M' be two closed oriented connected non-empty n-dimensional topological manifolds. Let $X \subseteq M$ and $X' \subseteq M'$ be submanifolds that are homeomorphic to $\overline{B}^n$ and let $\varphi\colon \partial X \to \partial X'$ be an orientation-reversing homeomorphism. Show that $(M \setminus \mathring{X}) \cup_{\varphi\colon \partial X \to \partial X'} (M \setminus \mathring{X}')$ is homeomorphic to the connected sum $M\#N$.

Hint. Use the Alexander trick from Exercise 8.3.

Remark. Initially it might sound like this statement holds "by definition". But in fact in Exercise 282.1 we will see that the obvious analog in the setting of smooth manifolds does *not* hold.

Exercise 224.11. Let R be a field and let $\varphi\colon V \times V \to R$ be a form. The vector subspace $R := \{v \in V \mid \varphi(v, w) = 0 \text{ for all } w \in V\}$ is called the *radical* of the form φ.

(a) Show that the map

$$\begin{array}{rcl} \overline{\varphi}\colon V/R \times V/R & \to & \mathbb{R} \\ ([v], [w]) & \mapsto & \varphi(v, w) \end{array}$$

is well-defined and non-singular.

(b) Let S be a vector subspace of V with $V = R \oplus S$. Show that $(V/R, \overline{\varphi})$ is isometric to $(S, \varphi|_S)$ where $\varphi|_S$ denotes the restriction of φ to $S \times S$.

(c) Show that (V, φ) is isometric to $(R, 0) \oplus (V/R, \overline{\varphi})$.

Remark. Recall that by our Definition 224.2 a form (V, φ) is by definition symmetric or anti-symmetric and V is a finite-dimensional free module over the ring R.

Exercise 224.12. Let H be a finite-dimensional real vector space and let $\lambda\colon H \times H \to \mathbb{R}$ be a non-singular form. Furthermore let $A_1, A_2 \subseteq H$ be two vector subspaces with $A_1^\perp = A_2$ and $A_2^\perp = A_1$. Show that $(A_1 \cap A_2)^\perp = A_1 + A_2$.

Exercise 224.13. Let M be a compact oriented 4-dimensional topological manifold and let $i\colon \partial M \to M$ be the inclusion map.

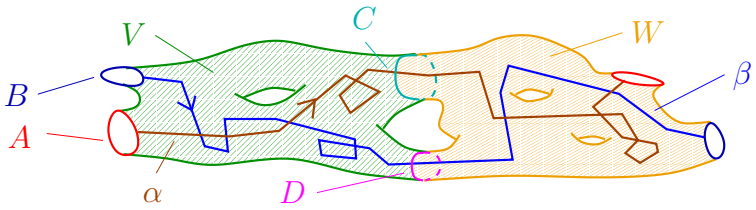
(a) Show that if $i_*\colon \mathrm{H}_1(\partial M) \to \mathrm{H}_1(M)$ is an isomorphism, then the intersection form Q_M is non-singular.

(b) Show that if $i_*\colon \mathrm{H}_1(\partial M; \mathbb{Q}) \to \mathrm{H}_1(M; \mathbb{Q})$ is a monomorphism, then the intersection form Q_M is non-degenerate.

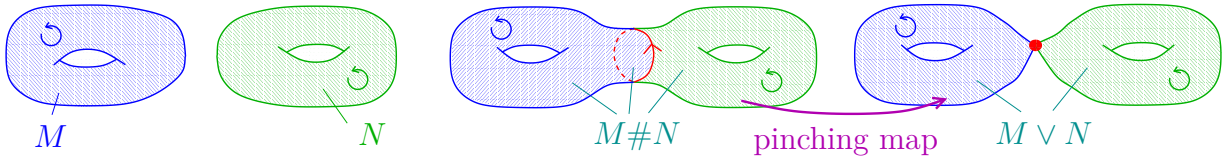
Exercise 224.14. Let M be a compact oriented n-dimensional topological manifold and let A, B be disjoint unions of components of ∂M. Furthermore let V and W be compact codimension-zero submanifolds of M as in the figure below and let $\alpha \in \mathrm{H}_k(M, A)$ and $\beta \in \mathrm{H}_{n-k}(X, B)$.

(a) Inspired by the figure below state a formula which computes $Q_M(\alpha, \beta)$ in terms of suitable intersection numbers of V and W.
Hint. What is the role of C and D in the figure below?

(b) Prove (a).



Exercise 224.15. Let $n \in \mathbb{N}$ and let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. In Exercise 185.18 we introduced the pinching map $p\colon M\#N \to M \vee N$ and studied the induced map on homology groups. Use the pinching map and the Wedge-Cup Product Lemma 204.14 to give a new proof of the Intersection Form-Additivity Proposition 224.18 for smooth manifolds. In other words, show that the intersection form $Q_{M\#N}$ is isometric to $Q_M \oplus Q_N$.



Exercise 224.16. Given $k \in \mathbb{N}$ we denote by $k \cdot (S^1 \times S^2)$ the connected sum of k copies of $S^1 \times S^2$. For which $k \in \mathbb{N}$ does there exist a continuous map $f\colon S^1 \times S^1 \times S^1 \to k \cdot (S^1 \times S^2)$ of degree $+1$?

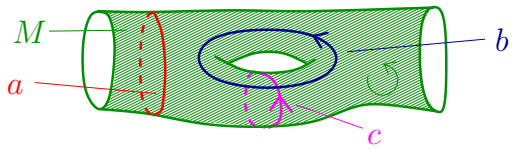
CHAPTER 225

The asymmetric and reduced intersection pairings

Let M be a compact oriented topological manifold of even dimension $2n$. In Definition 224.12 we introduced the intersection form

$$\begin{array}{rccl} Q_M\colon & \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ & (a,b) & \mapsto & \langle \mathrm{PD}_M(a) \cup \mathrm{PD}_M(b), [M] \rangle. \end{array}$$

In the Intersection Form–Non Singularity Proposition 224.14 we showed that if M is closed, then the intersection form is non-singular. If M is not closed, then this conclusion does not necessarily hold. For example let M be the torus minus two open disks. In the figure below we show a basis $\{a,b,c\}$ for $\mathrm{H}_1(M)$. Using the Cup Product–Intersection Number Theorem 220.10 one can easily see that the intersection form of M with respect to the given basis is represented by the matrix given in the figure. We see that the intersection form is anything but non-singular, in fact it is not even non-degenerate.[2425]



$$Q_M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

In this chapter we will present two approaches how this can be fixed:

- In Section 225.1 we will associate to any compact oriented topological manifold M together with a suitable boundary decomposition $\partial M = A \cup B$ the asymmetric intersection pairings
$$Q^{\mathrm{as}}_{M,A,B}\colon \mathrm{FH}_k(M,A) \times \mathrm{FH}_{n-k}(M,B) \to \mathbb{Z}$$
which are non-singular, at the price that the two groups involved are different.
- In Section 225.2 we will associate to a compact oriented $2n$-dimensional topological manifold the reduced intersection form
$$Q^{\mathrm{red}}_M\colon \mathrm{F}\widehat{\mathrm{H}}_n(M) \times \mathrm{F}\widehat{\mathrm{H}}_n(M) \to \mathbb{Z}$$
on $\widehat{\mathrm{H}}_n(M) \coloneqq \mathrm{coker}(i_*\colon \mathrm{H}_n(\partial M) \to \mathrm{H}_n(M))$, which is at least non-degenerate.

The asymmetric intersection pairing will play a role in Section 258.4 when we discuss the relationship between intersection numbers and linking numbers. The reduced intersection form will show up again in Chapter 227, in the proof of the Novikov Additivity Theorem 227.11. For most readers it is advisable to jump straight to Chapter 226 where will study the classification of (anti-) symmetric forms.

This chapter is organized as follows:

- In Section 225.1 we will introduce the asymmetric intersection pairings.
- In Section 225.2 we will introduce the reduced intersection form.
- In Section 225.3 we will provide exercises for Chapter 225.

[2425] In fact the annulus $[0,1] \times S^1$ also works as an example, but it is less fun.

In the next Chapter 226 we will discuss the classification of forms.

225.1. **The asymmetric intersection pairings.** Before we provide the definition of the asymmetric intersection pairings let us recall and introduce a new definition:

Definition 225.1. Let R be a commutative ring and let V, W and S be R-modules. Recall that according to Definition 213.1 we say that an R-bilinear pairing $\langle\ ,\ \rangle\colon V \times W \to S$ is non-singular if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \begin{pmatrix} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \begin{pmatrix} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array}$$

are isomorphisms. We now say that the R-bilinear pairing is non-degenerate if the above two maps are both monomorphisms. Otherwise we say that the R-bilinear pairing is degenerate.

Example. Let R be a commutative domain and let $A \in \mathrm{M}(n\times n; R)$. Elementary algebra, similar to the proof of the Non-Singularity Lemma 213.2 (1), shows that the form

$$\begin{array}{ccc} R^n \times R^n & \to & R \\ (v,w) & \mapsto & v^{\mathrm{T}} A w \end{array}$$

is non-singular if and only if $\det(A)$ is a unit in R and it is non-degenerate if and only if $\det(A)$ is non-zero. □

For the record let us state the following elementary lemma:

Lemma 225.2. (Non-Degeneracy–Characterisation Lemma) Let R be a commutative ring. Let V, W and S be R-modules and let

$$\langle\ ,\ \rangle\colon V \times W \to S$$

be an R-bilinear pairing. The following two statements are equivalent:

(1) The R-bilinear pairing is non-degenerate.
(2) For every $v \in V$ there exists a $w \in W$ with $\langle v,w\rangle \neq 0$ and for every $w \in W$ there exists a $v \in V$ with $\langle v,w\rangle \neq 0$.

Proof. The lemma follows almost immediately from the definitions. ■

In the Intersection Form–Non Singularity Proposition 224.14 we saw that if we are given a *closed* oriented $2n$-dimensional topological manifold M, then the intersection form

$$\begin{array}{rccl} Q_M\colon & \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ & (a,b) & \mapsto & \big\langle \underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}, [M] \big\rangle. \\ & & & \qquad\qquad \underbrace{\qquad\qquad}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})} \end{array}$$

is non-singular. On page 4330 we saw that if M is *not closed*, then this conclusion does not necessarily hold.[2426] More precisely we gave an explicit example of a compact orientable smooth manifold, namely the torus minus two open disks, such that the intersection form is degenerate, in particular it is singular.

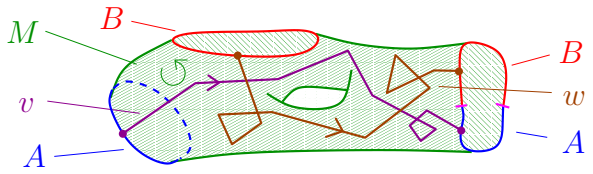
[2426] In fact perhaps the easiest example is given by $M = S^1 \times [0,1]$.

We move on to the definition of asymmetric intersection pairings on topological manifolds.

Definition 225.3. Let M be a compact oriented n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M. We suppose that $A \cap B = \partial A = \partial B$. We write $X := \partial M \setminus \mathring{A}$ and $Y := \partial M \setminus \mathring{B}$. For each $k \in \mathbb{N}_0$ we refer to the pairing[2427]

$$\begin{aligned} Q^{\mathrm{as}}_{M,A,B} \colon \mathrm{FH}_k(M,A) \times \mathrm{FH}_{n-k}(M,B) &\to \mathbb{Z} \\ (v,w) &\mapsto \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}}_{\in \mathrm{H}^{n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{aligned}$$

as an asymmetric intersection pairing of M. When A and B are clear from the context we write Q^{as}_M instead of the cumbersome $Q^{\mathrm{as}}_{M,A,B}$.



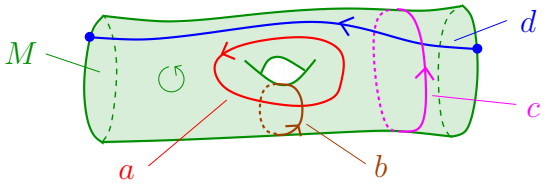
$$Q^{\mathrm{as}}_{M,A,B}(v,w) = +1 - 1 = 0$$

Remark. In the previous definition, if we take $A = B = \varnothing$, then we obtain precisely the intersection form as introduced in Definition 222.3, in other words, $Q^{\mathrm{as}}_{M,\varnothing,\varnothing} = Q_M$. □

Example. Let M be the torus minus two open disks. We are interested in the (asymmetric) intersection pairings

$$Q_M \colon \mathrm{H}_1(M) \times \mathrm{H}_1(M) \to \mathbb{Z} \quad \text{and} \quad Q^{\mathrm{as}}_{M,\varnothing,\partial M} \colon \mathrm{H}_1(M) \times \mathrm{H}_1(M,\partial M).$$

We consider the four curves a, b, c, d on M that are shown in the figure below. Note that the closed curves a, b, c represent a basis for $\mathrm{H}_1(M)$, whereas the closed curves a, b, d represent a basis for $\mathrm{H}_1(M,\partial M)$.[2428]



Using these bases and using the Cup Product–Intersection Number Theorem 220.10 we see that

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[2426] In most cases we will have $A \cup B = \partial M$. But the case $A = B = \varnothing$ can also be interesting.

[2427] Note that by the Codimension-Zero Submanifold Proposition 150.16 we know that X and Y are also compact submanifolds of ∂M with $\partial X = \partial A = A \cap X$ and $\partial Y = \partial B = B \cap Y$. Furthermore note that it follows from $A \cap B = \partial A = \partial B$ that $\partial M = X \cup Y$ and that $\partial M = (X \setminus \partial X) \cup (Y \setminus \partial Y) \cup C$ where each component of C is a component of ∂X and a component of ∂Y. Thus it follows from a slight generalization of the Excisive Triad-Proposition 158.20 (2), see also Exercise 158.8, that (M, X, Y) is an excisive triad. In particular it follows, see Definition 204.7, that the cup product is actually defined.

[2428] Why is that?

Q_M is represented by $\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$ and $Q^{\text{as}}_{M,\varnothing,\partial M}$ is represented by $\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$.

Note that the former matrix has determinant zero, whereas the latter matrix has determinant 1. This calculation, together with the Non-Singularity Lemma 213.2 (1) shows that the former pairing is degenerate whereas the latter pairing is non-singular.[2429] As we will see in the next proposition, it is not a coincidence that in this particular example the asymmetric intersection pairing is non-singular. □

The following proposition is a generalization of the Intersection Form–Non Singularity Proposition 224.14.

Proposition 225.4. (Asymmetric Intersection Pairing–Non Singularity Proposition) Let M be a compact oriented n-dimensional topological manifold. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Given any $k \in \mathbb{N}_0$ the following pairing is non-singular:[2430]

$$Q^{\text{as}}_{M,A,B}\colon \operatorname{FH}_k(M,A) \times \operatorname{FH}_{n-k}(M,B) \;\to\; \mathbb{Z}.$$

Proof. Let $k \in \mathbb{N}_0$. We need to show that the following pairing is non-singular:

$$Q^{\text{as}}_{M,A,B}\colon \operatorname{FH}_k(M,A) \times \operatorname{FH}_{n-k}(M,B) \;\to\; \mathbb{Z}.$$

Evidently this is equivalent to showing that the form

$$\begin{array}{rcl} \operatorname{FH}^{n-k}(M,B;\mathbb{Z}) \times \operatorname{FH}^k(M,A;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi,\psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular. We dealt with special cases of this statement in the Manifold-Non Singularity Proposition 213.3 and in the Intersection Form–Non Singularity Proposition 224.14. The proof of the general case is essentially the same. For the reader's convenience we provide the argument.

By the Non-Singularity Lemma 213.2 (2) it suffices to show that the map

$$\begin{array}{rcl} \Omega\colon \operatorname{FH}^{n-k}(M,B;\mathbb{Z}) & \to & \operatorname{Hom}(\operatorname{FH}^k(M,A;\mathbb{Z}),\mathbb{Z}) \\ \psi & \mapsto & (\varphi \mapsto \langle \varphi \cup \psi, [M] \rangle) \end{array}$$

is an isomorphism. As on pages 4082 and 4332 we start out with the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccc} \operatorname{FH}^{n-k}(M,B;\mathbb{Z}) & \xrightarrow[\psi \mapsto (\varphi \mapsto \langle \varphi \cup \psi, [M] \rangle)]{\Omega} & \operatorname{Hom}(\operatorname{FH}^k(M,A;\mathbb{Z}),\mathbb{Z}). \\ & {\scriptstyle \text{ev}} \searrow \quad \nearrow {\scriptstyle (\cap [M])^*} & \\ & \operatorname{Hom}_{\mathbb{Z}}(\operatorname{FH}_{n-k}(M,B),\mathbb{Z}) & \end{array}$$

Proof. Note that for any $\psi \in \operatorname{FH}^{n-k}(M,B;\mathbb{Z})$ and any $\varphi \in \operatorname{FH}_k(M,A;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:

[2429] In fact, using this observation and Exercise 224.5 (c) one can easily answer the previous footnote.

[2430] The right diagonal is the dual map corresponding to the map $\operatorname{FH}^k(M,A;\mathbb{Z}) \xrightarrow{\cap [M]} \operatorname{FH}_{n-k}(M,B)$.

$$\Omega(\psi)(\varphi) \underset{\text{by definition of } \Omega \text{ and by the Cap Product-Kronecker Lemma 206.12}}{=} \epsilon_M((\varphi\cup\psi)\cap[M]) \underset{\text{by the Cup-Cap Lemma 206.15}}{=} \epsilon_M(\psi\cap(\varphi\cap[M])) \underset{\text{by the Cap Product-Kronecker Lemma 206.12}}{=} \langle\psi,\varphi\cap[M]\rangle = \mathrm{ev}(\psi)(\varphi\cap[M]).$$

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms:

- By the Universal Coefficient-H*-Theorem 201.23 we have a short exact sequence
$$0 \to \underbrace{\mathrm{Ext}(\mathrm{H}_{n-k-1}(M,B),\mathbb{Z})}_{\substack{\text{torsion by the Ext-Properties Lemma 201.19}\\ \text{and the } \mathbb{Z}/k\mathbb{Z}\text{-Ext-Lemma 201.18}}} \to \mathrm{H}^{n-k}(M,B;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(M,B),\mathbb{Z}) \to 0.$$
It follows from the above data that the group to the left is torsion. It follows easily that the homomorphism $\mathrm{FH}^{n-k}(M,B;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}_{n-k}(M,B),\mathbb{Z})$ is, as desired, an isomorphism.
- The diagonal map to the right is an isomorphism by the Poincaré Duality Theorem 210.1.

■

We continue with some results that are specific to the new asymmetric setting:

Lemma 225.5. (Asymmetric Intersection Form–Natural Lemma) Let M be a compact oriented connected n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M with $A\cap B = \partial A = \partial B$. Suppose we are given two compact $(n-1)$-dimensional submanifolds A' and B' of ∂M such that $A'\cap B' = \partial A' = \partial B'$ and which furthermore satisfy $A\subseteq A'$ and $B\subseteq B'$. We denote by $p\colon (M,A)\to(M,A')$ and $q\colon (M,B)\to(M,B')$ the natural maps. For any $v\in \mathrm{H}_k(M,A')$ and $w\in \mathrm{H}_{n-k}(M,B')$ we have the following equality:

$$Q^{\mathrm{as}}_{M,A',B'}(p_*(v),q_*(w)) \;=\; Q^{\mathrm{as}}_{M,A,B}(v,w).$$

Proof. We write $X = \partial M \setminus \mathring{A}$ and $Y = \partial M \setminus \mathring{B}$. Not surprisingly we also write $X' = \partial M \setminus \mathring{A}'$ and $Y' = \partial M \setminus \mathring{B}'$. Note that by hypothesis we have $X'\subseteq X$ and $Y'\subseteq Y$. Let $f\colon (M,X')\to(M,X)$ and $g\colon (M,Y')\to(M,Y)$ be the natural maps. Given $v\in \mathrm{H}_k(M,A')$ and $w\in \mathrm{H}_{n-k}(M,B')$ we have the following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(p_*(v))}_{\in \mathrm{H}^{n-k}(M,X';\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(q_*(w))}_{\in \mathrm{H}^{k}(M,Y';\mathbb{Z})}, [M]\Big\rangle}^{=Q^{\mathrm{as}}_{M,A',B'}(p_*(v),q_*(w))} \underset{\text{follows from the Cap Product Naturality Lemma 206.13}}{=} \Big\langle \underbrace{f^*(\mathrm{PD}_M(v))}_{\in \mathrm{H}^{n-k}(M,X';\mathbb{Z})} \cup \underbrace{g^*(\mathrm{PD}_M(w))}_{\in \mathrm{H}^{k}(M,Y';\mathbb{Z})}, [M]\Big\rangle$$

$$\underset{\text{by the Cup Product-Naturality Lemma 204.8}}{=} \Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}, [M]\Big\rangle \;=\; Q^{\mathrm{as}}_{M,A,B}(v,w).$$

■

Proposition 225.6. (Asymmetric Intersection Pairing–Matrix Proposition) Let M be a compact oriented $2k$-dimensional topological manifold. We denote by $p_*\colon \mathrm{H}_k(M) \to \mathrm{H}_k(M,\partial M)$ the obvious map. Let $v_1,\dots,v_m$ be a basis of the free abelian group $\mathrm{FH}_k(M)$. By the Asymmetric Intersection Pairing–Non Singularity Proposition 225.4 together with Exercise 224.5 (a) there exists a unique "dual" basis $v_1^*,\dots,v_m^*$ of the free abelian group $\mathrm{FH}_k(M,\partial M)$ with $Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_i,v_j^*) = \delta_{ij}$ for all $i,j \in \{1,\dots,m\}$. With this notation the following diagram commutes:

$$\begin{array}{ccc} \mathrm{FH}_k(M) & \xrightarrow{\quad p_* \quad} & \mathrm{FH}_k(M,\partial M) \\ \cong\uparrow e_i\mapsto v_i & & e_i\mapsto v_i^*\uparrow\cong \\ \mathbb{Z}^m & \xrightarrow[\text{matrix } (Q_M(v_i,v_j))_{i,j=1,\dots,m}]{\text{multiplication by the}} & \mathbb{Z}^m. \end{array}$$

Put differently, let $(Q_M(v_i,v_j))_{i,j=1,\dots,m}$ be the $(m\times m)$–matrix that represents the intersection form Q_M with respect to the basis $v_1,\dots,v_m$. This matrix represents the map $p_*\colon \mathrm{FH}_k(M) \to \mathrm{FH}_k(M,\partial M)$ with respect to the bases $v_1,\dots,v_m$ and $v_1^*,\dots,v_m^*$.

Proof. This statement follows easily from the definitions together with the Cap Product-Naturality Lemma 206.13 and the Cup Product-Naturality Lemma 204.8. Let $i \in \{1,\dots,m\}$. We write $p_*(v_i) = \sum_{j=1}^{m} b_{ji}\cdot v_j^*$. Let $j \in \{1,\dots,m\}$. We see that

$$b_{ji} \underset{\substack{\uparrow\\ \text{since } Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_j,v_k^*)=\delta_{jk}}}{=} Q^{\mathrm{as}}_{M,\varnothing,\partial M}\Big(v_j, \sum_{k=1}^{m} b_{ki}\cdot v_k^*\Big) = Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_j,p_*(v_i)) \underset{\substack{\uparrow\\ \text{follows from the Asymmetric}\\ \text{Intersection Form–Natural}\\ \text{Lemma 225.5}}}{=} Q_M(v_j,v_i).$$

Since by definition the matrix (b_{ij}) represents the map p_* with respect to the given bases we obtain the desired result. ∎

In the remainder of this section we prove several properties of the asymmetric intersection pairings.

In the remainder of this section we prove several properties of the asymmetric intersection pairings.

Lemma 225.7. (Asymmetric Intersection Form–Exchange Lemma) Let M be a compact oriented connected n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of the boundary ∂M with $A\cap B = \partial A = \partial B$. If we are given $v \in \mathrm{H}_k(M,A)$ and $w \in \mathrm{H}_{n-k}(M,B)$, then

$$Q^{\mathrm{as}}_{M,A,B}(v,w) = (-1)^{k\cdot(n-k)}\cdot Q^{\mathrm{as}}_{M,B,A}(w,v).$$

Proof. We write $X = \partial M \setminus \mathring{A}$ and $Y = \partial M \setminus \mathring{B}$. Given $v \in \mathrm{H}_k(M,A)$ and $w \in \mathrm{H}_{n-k}(M,B)$ we have the following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}, [M]\Big\rangle}^{=Q^{\mathrm{as}}_{M,A,B}(v,w)} \underset{\substack{\uparrow\\ \text{by the Cup Product}\\ \text{Commutativity}\\ \text{Proposition 204.10}}}{=} (-1)^{(n-k)\cdot k}\cdot \overbrace{\Big\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})}, [M]\Big\rangle}^{=Q^{\mathrm{as}}_{M,B,A}(w,v)}.$$

∎

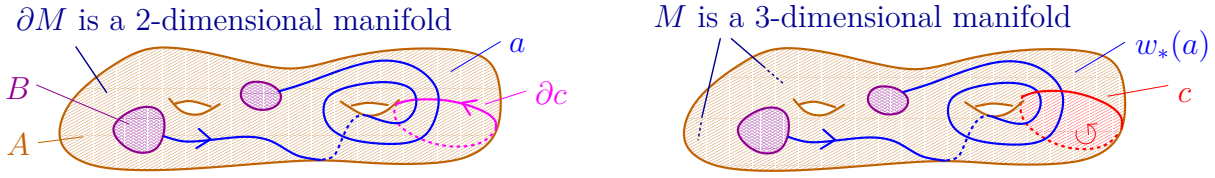
Our next result on asymmetric intersection pairing deals with the interplay between the asymmetric intersection pairings on a topological manifold and the asymmetric intersection pairings on its boundary.

Proposition 225.8. (Intersection Pairing–Boundary Proposition) Let M be a compact oriented n-dimensional topological manifold. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Now let $a \in \mathrm{H}_k(A, \partial A)$ and let $c \in \mathrm{H}_{n-k+1}(M, A)$. We denote by $w\colon (A, \partial A) \to (M, B)$ the obvious map. Furthermore we denote by $\partial_{n-k+1}\colon \mathrm{H}_{n-k+1}(M, A) \to \mathrm{H}_{n-k}(A)$ the connecting homomorphism of the long exact sequence in homology of the pair (M, A). In this setting the following equality holds:

$$Q^{\mathrm{as}}_{A,\varnothing,\partial A}(\partial_{n-k+1}(c), a) \;=\; Q^{\mathrm{as}}_{M,A,B}(c, w_*(a)).$$

In other words, the following diagram commutes in a suitable sense:

$$\begin{array}{ccc} \mathrm{H}_{n-k+1}(M, A) \times \mathrm{H}_k(M, B) & & \\ \downarrow{\scriptstyle \partial_{n-k+1}} \quad \uparrow{\scriptstyle w_*} & \xrightarrow{\;Q^{\mathrm{as}}_{M,A,B}\;} & \mathbb{Z}. \\ \mathrm{H}_{n-k}(A) \times \mathrm{H}_k(A, \partial A) & \xrightarrow{\;Q^{\mathrm{as}}_{A,\varnothing,\partial A}\;} & \end{array}$$



Proof. We consider the following diagram of homomorphisms:

$$\begin{array}{ccccc} \mathrm{H}^k(A, \partial A; \mathbb{Z}) \times \mathrm{H}^{n-k}(A; \mathbb{Z}) & \xrightarrow{\;\cup\;} & \mathrm{H}^n(A, \partial A; \mathbb{Z}) & & \\ \uparrow{\scriptstyle w^*} \quad \downarrow{\scriptstyle \delta} & & \cong\downarrow{\scriptstyle (u^*)^{-1}} & \searrow^{\langle -, [A]\rangle} & \\ & & \mathrm{H}^n(\partial M, B; \mathbb{Z}) & \xrightarrow{\;\langle -, p_*([\partial M])\rangle\;} & \mathbb{Z}. \\ & & \downarrow{\scriptstyle \delta} & \nearrow_{\langle -, [M]\rangle} & \\ \mathrm{H}^k(M, B; \mathbb{Z}) \times \mathrm{H}^{n-k+1}(M, A; \mathbb{Z}) & \xrightarrow{\;\cup\;} & \mathrm{H}^{n+1}(M, \partial M) & & \end{array}$$

We make the following clarifications and observations:

(1) By the Excisive Triad-Proposition 158.20 we know that (M, A, B) is an excisive triad.
(2) We denote by $u\colon (A, \partial A) \to (\partial M, B)$ the inclusion map. It follows from (1), the Excisive Characterization Lemma 158.19 (1)⇒(3) and the Also-Iso-on-Cohomology Proposition 200.2 that the map u induces an isomorphism on cohomology groups.
(3) We denote by $\delta\colon \mathrm{H}^n(A; \mathbb{Z}) \to \mathrm{H}^{n+1}(M, A; \mathbb{Z})$ and $\delta\colon \mathrm{H}^n(\partial M, B; \mathbb{Z}) \to \mathrm{H}^{n+1}(M, \partial M; \mathbb{Z})$ the connecting homomorphisms in cohomology of the pair (M, A) and of the triple $(M, \partial M, B)$.
(4) It follows from (1) and the Cup Product-Connecting Homomorphism Lemma 204.19 that the left part of the diagram commutes up to the sign $(-1)^k$.
(5) We denote by $p\colon (\partial M, \varnothing) \to (\partial M, B)$ the obvious map. By the Codimension Zero-Fundamental Class Lemma 194.10 we have $p_*([\partial M]) = u_*([A]) \in \mathrm{H}_n(\partial M, B)$. Together

with the Kronecker Pairing–Natural Lemma 199.9 (1) it follows that the top right triangle commutes.

(6) Next note that it follows from the Fundamental Class-of-∂-Proposition 194.13 that $\partial_n([M]) = [\partial M] \in \mathrm{H}_n(\partial M)$. Together with the Cap Product-Triple-Connecting Homomorphism Lemma 206.19 this implies that the bottom right triangle commutes.

Next we consider the following two diagrams:

$$\begin{tikzcd} \mathrm{H}_k(M,A) \arrow[r, "\partial_k"] \arrow[d, "\mathrm{PD}_M"'] & \mathrm{H}_k(A) \arrow[d, "\mathrm{PD}_A"] \\ \mathrm{H}^{n-k}(M,B) \arrow[r, "w^*"] & \mathrm{H}^{n+1-k}(A,\partial A;\mathbb{Z}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{H}_k(A,\partial A) \arrow[r, "w_*"] \arrow[d, "\mathrm{PD}_A"'] & \mathrm{H}_k(M,B) \arrow[d, "\mathrm{PD}_M"] \\ \mathrm{H}^{n-k}(A;\mathbb{Z}) \arrow[r, "\delta_{n-k}"] & \mathrm{H}^{n-k+1}(M,A;\mathbb{Z}). \end{tikzcd}$$

By the Poincaré Duality–Connecting Homomorphism Proposition 211.15 we know that the diagram to the left commutes up to the factor $(-1)^k$. The same proposition generously also tells us that the diagram to the right commutes as it is. With all these preparations we can now easily carry out the desired calculation:

$$\begin{aligned} Q_{A,\varnothing,\partial A}(\partial c, a) &\overset{\substack{\text{since the second diagram commutes}\\ \text{up to multiplication by } (-1)^k}}{\overset{\downarrow}{=}} \langle \mathrm{PD}_A(\partial c) \cup \mathrm{PD}_A(a), [A]\rangle \overset{\substack{\text{by Definition 224.12 of } Q_A}}{\overset{\downarrow}{=}} (-1)^k \cdot \langle w^*(\mathrm{PD}_M(c)) \cup \mathrm{PD}_A(a), [A]\rangle \\ &\underset{\substack{\text{since the first diagram commutes}\\ \text{up to multiplication by } (-1)^k}}{\underset{\uparrow}{=}} \langle \mathrm{PD}_M(c) \cup \delta(\mathrm{PD}_A(a)), [M]\rangle \underset{\substack{\text{since the third diagram}\\ \text{commutes}}}{\underset{\uparrow}{=}} \langle \mathrm{PD}_M(c) \cup \mathrm{PD}_M(w_*(a)), [M]\rangle \\ &= Q^{\mathrm{as}}_{M,A,B}(c, w_*(a)). \end{aligned}$$

■

The following proposition relates the asymmetric intersection pairing on a topological manifold to the asymmetric intersection pairing on a codimension-zero submanifold.

Proposition 225.9. (Intersection Form-Submanifold Proposition) Let M be a compact oriented m-dimensional topological manifold and let $W \subseteq M$ be a compact codimension-zero submanifold with corner.[2431] Let A and B be disjoint unions of components of ∂M with $\partial_1 W \subseteq A$. We denote by $i\colon (W, \partial_1 W) \to (M, A)$, $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the natural maps. For every $\alpha \in \mathrm{H}_k(W, \partial_1 W)$ and $\beta \in \mathrm{H}_{n-k}(M, B)$ we have the following equality:[2432]

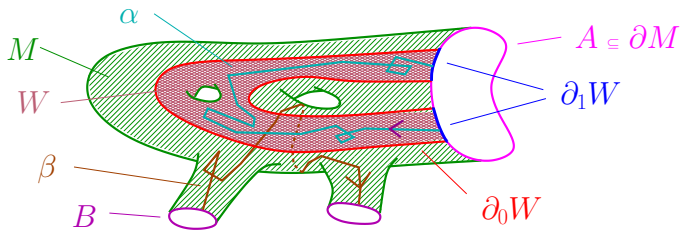
$$Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha, (w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha), \beta).$$

In other words, the following diagram commutes in a suitable sense:

$$\begin{tikzcd} \mathrm{H}_k(W,\partial_1 W) \times \mathrm{H}_{n-k}(W,\partial_0 W) \arrow[d, "i_*"] \arrow[dr, "Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}"] & \\ \mathrm{H}_{n+k}(M, M \setminus \mathring{W}) & \mathbb{Z}. \\ \mathrm{H}_k(M,A) \times \mathrm{H}_{n-k}(M,B) \arrow[u, "p_*"] \arrow[ur, "Q^{\mathrm{as}}_{M,A,B}"'] & \end{tikzcd}$$

$w_* \downarrow \cong$

The analogous statement also holds with the roles of the first and the second entry swapped.



Proof. We perform the following calculation:

$$\begin{aligned}
Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha,(w_*)^{-1}(p_*(\beta))) &= \langle \mathrm{PD}_W(\alpha)\cup \mathrm{PD}_W((w_*)^{-1}(p_*(\beta))),[W]\rangle \\
&= \langle w^*((w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup w^*(\mathrm{PD}_M(\beta)),[W]\rangle \\
&\quad \uparrow \text{ by the Large } \mathrm{H}_*\text{-to-Small } \mathrm{H}^*\text{-PD Lemma 212.9 and the Cup Product-Naturality Lemma 204.8} \\
&= \langle (w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup \mathrm{PD}_M(\beta), p_*([M])\rangle \\
&\quad \uparrow \text{ by the Codimension Zero-Fundamental Class Lemma 194.10 we have } w_*([W]) = p_*([M]) \\
&= \langle \mathrm{PD}_M(i_*(\alpha)),\mathrm{PD}_M(\beta),[M]\rangle \\
&\quad \uparrow \text{ by the Small } \mathrm{H}_*\text{-to-Large } \mathrm{H}^*\text{-PD Lemma 212.8}
\end{aligned}$$

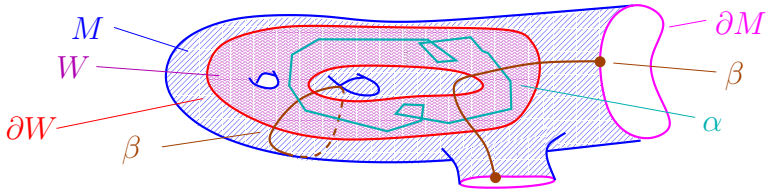
$$\begin{aligned}
\langle w^*((w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup w^*(\mathrm{PD}_M(\beta)),[W]\rangle &= \langle (w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup \mathrm{PD}_M(\beta), w_*([W])\rangle \\
&\quad \uparrow \text{ by the Kronecker Pairing Natural Lemma 199.9 (1)} \\
\langle (w^*)^{-1}(\mathrm{PD}_W(\alpha))\cup \mathrm{PD}_M(\beta), p_*([M])\rangle &= \langle p^*((w^*)^{-1}(\mathrm{PD}_W(\alpha)))\cup p^*(\mathrm{PD}_M(\beta)),[M]\rangle \\
&\quad \uparrow \text{ by the Kronecker Pairing Natural Lemma 199.9 (1)} \\
\langle \mathrm{PD}_M(i_*(\alpha)),\mathrm{PD}_M(\beta),[M]\rangle &= Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha),\beta).
\end{aligned}$$

■

The case we will use most often is the special case that $\partial_0 W = \partial W$, $\partial_1 W = \varnothing$, $A = \varnothing$ and $B = \partial M$. In this case the Intersection Form-Submanifold Proposition 225.9 specializes to the following slightly more readable proposition:

Proposition 225.10. (Intersection Form-Submanifold Proposition II) Let M be a compact oriented m-dimensional topological manifold and let $W \subseteq M \setminus \partial M$ be a compact codimension-zero submanifold. We denote by $i\colon W \to M$, $w\colon (W,\partial W) \to (M, M\setminus \mathring{W})$ and $p\colon (M,\partial M) \to (M, M\setminus \mathring{W})$ the natural maps. For every $\alpha \in \mathrm{H}_k(W)$ and every $\beta \in \mathrm{H}_{n-k}(M,\partial M)$ we have the following equality:

$$Q^{\mathrm{as}}_{W,\varnothing,\partial W}(\alpha,(w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,\varnothing,\partial M}(i_*(\alpha),\beta).$$



225.2. **The reduced intersection form.** After the exhaustive and exhausting discussion of the asymmetric intersection pairings in the previous section, let us now move on to our second variation on the intersection form.

[2431] We refer to Definition 212.2 for the definition of $\partial_0 W$ and $\partial_1 W$.

[2432] Recall that by the Manifold–Excision Theorem 212.4 the map $w\colon (W,\partial W) \to (M, M\setminus \mathring{W})$ induces an isomorphism on homology groups.

Proposition 225.11. (Reduced Intersection Form–Non Degeneracy Proposition) Let M be a compact oriented $2n$-dimensional topological manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) Let $\alpha \in \mathrm{FH}_n(M)$. The following two statements are equivalent:
   (a) $\alpha \in \mathrm{im}(i_*\colon \mathrm{FH}_n(\partial M) \to \mathrm{FH}_n(M))$.
   (b) For every $\beta \in \mathrm{FH}_n(M)$ we have $Q_M(\alpha,\beta) = 0$.

(2) We write $\widehat{\mathrm{H}}_n(M) \coloneqq \mathrm{coker}(i_*\colon \mathrm{H}_n(\partial M) \to \mathrm{H}_n(M))$. The form

$$\begin{aligned} Q_M^{\mathrm{red}}\colon \mathrm{F}\widehat{\mathrm{H}}_n(M) \times \mathrm{F}\widehat{\mathrm{H}}_n(M) &\to \mathbb{Z} \\ ([\varphi],[\psi]) &\mapsto Q_M(\varphi,\psi) \end{aligned}$$

is well-defined and non-degenerate. We refer to this form as the reduced intersection form of M.

(3) If the intersection form $Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) \to \mathbb{Z}$ is non-degenerate, then the obvious homomorphism $\mathrm{FH}_n(M) \to \mathrm{F}\widehat{\mathrm{H}}_n(M)$ is an isomorphism and it induces an isometry $Q_M \to Q_M^{\mathrm{red}}$.

Proof. We denote by $i\colon \partial M \to M$ and $j\colon (M,\varnothing) \to (M,\partial M)$ the natural maps. Before we do anything else we recall that we have a long exact sequence

$$\ldots \to \mathrm{H}_n(\partial M) \xrightarrow{i_*} \mathrm{H}_n(M) \xrightarrow{j_*} \mathrm{H}_n(M,\partial M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial M) \to \ldots$$

Now we turn to the actual proofs of the statements.

(1) Let $\alpha \in \mathrm{H}_n(M)$. First we prove "(a)⇒(b)". Thus we assume that $\alpha = i_*(\gamma)$ for some $\gamma \in \mathrm{H}_n(\partial M)$. Let $\beta \in \mathrm{H}_n(M)$. We perform the following calculation:

$$\begin{aligned} Q_M(\alpha,\beta) &\underset{\text{by Definition 222.3 of } Q_M}{=} \langle \mathrm{PD}(i_*(\gamma)) \cup \mathrm{PD}(\beta), [M]\rangle \overset{\text{this follows from the Cap Product-Kronecker Lemma 206.12 and the Cup-Cap Lemma 206.15, see also Exercise 206.5}}{=} \langle \mathrm{PD}(\beta), j_*(\underbrace{\mathrm{PD}(i_*(\gamma)) \cap [M]}_{= i_*(\gamma) \text{ by Definition 210.2 of PD}})\rangle \\ &= \langle \mathrm{PD}(\beta), j_*(i_*(\gamma))\rangle \underset{\text{since } j_* \circ i_* = 0}{=} 0. \end{aligned}$$

Now we prove "(b)⇒(a)". Thus we assume that $Q_M(\alpha,\beta) = 0$ for all $\beta \in \mathrm{FH}_n(M)$. This implies for any $\beta \in \mathrm{FH}_n(M)$ we have the following equalities:

$$Q^{\mathrm{as}}_{M,\partial M,\varnothing}(j_*(\alpha),\beta) \underset{\substack{\text{by the Asymmetric Intersection} \\ \text{Form–Natural Lemma 225.5}}}{=} Q_M(\alpha,\beta\rangle \underset{\text{by assumption}}{=} 0.$$

By the Asymmetric Intersection Pairing–Non Singularity Proposition 225.4 we know that the pairing $Q^{\mathrm{as}}_{M,\partial M,\varnothing}$ is non-singular, in particular it is non-degenerate. Therefore it follows from the above, together with the Non-Degeneracy–Characterisation Lemma 225.2, that $j_*(\alpha) = 0 \in \mathrm{FH}_n(M)$. It follows from the above long exact sequence, together with Exercise 161.4, that $\mathrm{im}(i_*\colon \mathrm{FH}_n(\partial M) \to \mathrm{FH}_n(M))$.

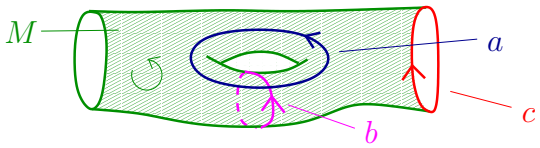
(2) It follows immediately from (1) "(a)⇒(b)" that the form Q_M^{red} is well-defined. Furthermore it follows from (1) "(b)⇒(a)" and the Non-Degeneracy–Characterisation Lemma 225.2 that Q_M^{red} is non-degenerate.

(3) This statement follows immediately from (1) and the Non-Degeneracy–Characterisation Lemma 225.2. ■

Examples.

(1) One last time we consider again our favorite example M, namely the torus minus two open disks. We use the notation from the figure on page 4360. A basis for $\mathrm{H}_1(M)$ is given by a, b, c and the image of $\mathrm{H}_1(\partial M) \to \mathrm{H}_1(M)$ is given by $\mathbb{Z} \cdot c$. With respect to these bases we see that

$$Q_M \text{ is represented by } \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \text{ and } Q_M^{\mathrm{red}} \text{ is represented by } \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}.$$



(2) In Exercise 225.2 we will provide an example of a compact orientable 4-dimensional smooth manifold M for which the reduced intersection form is not non-singular. We will obtain many more such examples later on in the Intersection Form-Realization Corollary 259.10. □

225.3. Exercises for Chapter 225 .

Exercise 225.1. Let M be a compact oriented n-dimensional topological manifold. We consider the map

$$\Psi\colon \mathrm{H}_k(M) \xrightarrow{\mathrm{PD}_M} \mathrm{H}^{n-k}(M, \partial M; \mathbb{Z}) \xrightarrow{i^*} \mathrm{H}^{n-k}(M; \mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{H}_{n-k}(M, B), \mathbb{Z}).$$

Show that for every $a \in \mathrm{H}_k(M)$ and every $b \in \mathrm{H}_{n-k}(M, B)$ we have

$$\Psi(b)(a) \;=\; \langle a \cup b, [M] \rangle.$$

Exercise 225.2. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \subseteq \partial(\overline{B}^2 \times \overline{B}^2) \\ (z, w) & \mapsto & (z, w \cdot z^n). \end{array}$$

Furthermore we consider [2433]

$$M \;:=\; (\overline{B}^2 \times \overline{B}^2) \underset{\substack{\overline{B}^2 \times S^1 \to \partial(\overline{B}^2 \times \overline{B}^2) \\ (z,w) \mapsto \varphi(z,w)}}{\cup} (\overline{B}^2 \times \overline{B}^2).$$

One can show by hand that M is a compact orientable 4-dimensional smooth manifold.[2434]

(a) Show that $\pi_1(M) = 0$, $\mathrm{H}_2(M; \mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}_2(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.

[2433] Later in Definition 246.4 we will say that M is obtained from the 4-ball $\overline{B}^2 \times \overline{B}^2$ by attaching a 2-handle.

[2434] We will carry out the details in the Handle Attachment Proposition 246.3, but this is irrelevant for the present exercise.

(b) Show that $\mathrm{H}_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{H}_2(\partial M;\mathbb{Z}) = 0$. (In fact very well-trained eyes will spot that ∂M is diffeomorphic to the lens space $L(n,1)$.)

(c) Show that if $n \geq 2$, then the intersection form Q_M is represented by one of the two (1×1)-matrices $(-n)$ or (n).
Hint. Use the Asymmetric Intersection Pairing–Matrix Proposition 225.6 (2).
Remark. Note that it follows from (b) that the intersection form equals the reduced intersection form. Thus we have shown, as promised on page 4360, that there exists a compact orientable 4-dimensional smooth manifold M for which the reduced intersection form is not non-singular.

Exercise 225.3. Is the reduced intersection form additive under the connected sum operation?

CHAPTER 226

The classification of forms

Before we do anything else, let us recall the following notation from Definition 224.4:

Definition 226.1. Let R be a commutative ring and let A be an (anti-) symmetric $(n\times n)$–matrix over R. We introduce the following (anti-) symmetric form over R:

$$\begin{aligned}\theta(A) \coloneqq \theta_R(A)\colon R^n \times R^n &\to R\\ (v,w) &\mapsto v^{\mathrm{T}}Aw.\end{aligned}$$

Sometimes, for brevity, we just say "the form A", instead of "the form $\theta(A)$". The Form–Matrix-Lemma 224.5 allows us to go back and forth between isometry classes of (anti-) symmetric forms and congruence classes of (anti-) symmetric matrices over R.

Next let us consider the following table, which we obtain from our discussions on page 3549, the Künneth Theorem 164.6, the Sphere-π_1-Proposition 88.24, the Product-π_1-Proposition 90.13, the Projective Space–π_1-Proposition 135.8 and our discussion on page 4340:

manifold	π_1	H_1	H_0	H_2	H_3	H_4	intersection form
$S^2\times S^2$	$\{e\}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$	$\{e\}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$	$\{e\}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

So if we want to distinguish $S^2\times S^2$, $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ our only hope lies in the intersection form. In fact in Question 224.19 we were wondering out loud, whether or not the forms in the last column are isometric.
In this chapter we will try our best to classify (anti-) symmetric non-singular forms over $\mathbb{Z}$ and over $\mathbb{R}$. Along the way we will in particular develop the tools to distinguish the above three intersection forms.

This chapter is organized as follows:

- In Section 226.1 we will classify non-singular anti-symmetric forms over $\mathbb{Z}$.
- In Section 226.2 we will classify symmetric forms over $\mathbb{R}$.
- In Section 226.3 we will introduce the E_8-form in a convoluted but fun way.
- In Section 226.4 we will study symmetric forms over $\mathbb{Z}$.
- In Appendix 226.5 we will discuss forms over $\mathbb{F}_2$.
- In Section 226.6 we will provide exercises for Chapter 226.

In the next Chapter 227 we will discuss the multiplicativity and additivity of signatures.

226.1. **Non-singular anti-symmetric forms.** In this section we will do the following:

- First we completely classify non-singular anti-symmetric forms over $\mathbb{R}$ and over $\mathbb{Z}$.
- We will apply this algebraic result to topology, namely we will determine which integers can arise as Euler characteristics of closed orientable topological manifolds.

226.1.1. **Non-singular anti-symmetric forms: Algebra.** First we consider the matrix

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is anti-symmetric and it has determinant 1. From Definition 224.4 and Definition 226.1 we obtain an anti-symmetric form $\theta(S)$ over any commutative ring. The following proposition shows that S is the mother of all non-singular anti-symmetric forms over $\mathbb{R}$ or over $\mathbb{Z}$:

Proposition 226.2. (Anti-Symmetric Form Proposition) Let $R = \mathbb{R}$ or $R = \mathbb{Z}$. If (V, φ) is a non-singular anti-symmetric form over R, then there exists a $k \in \mathbb{N}_0$ and an isometry[2435] $(V, \varphi) \cong k \cdot \theta_R(S)$. In other words, φ is represented by the $2k \times 2k$–matrix

$$\underbrace{\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}}_{k \text{ diagonal blocks of } 2 \times 2\text{-matrices}}.$$

In particular the rank of V is even.

Remark. The Intersection Form Proposition 224.13, the Intersection Form–Non Singularity Proposition 224.14 and the Anti-Symmetric Form Proposition 226.2 imply in particular that the isometry type of the intersection form of a surface of genus g is determined by the formal properties of an intersection form. In particular we did not need to go through the discussion on page 4330 if we just need to know the *isometry type* of the intersection form. □

Sketch of proof. First we deal with the case $R = \mathbb{R}$. Thus let (V, φ) be a non-singular anti-symmetric form over $\mathbb{R}$ of rank n. We pick an $(n \times n)$–matrix A that represents φ. By the Form–Matrix-Lemma 224.5 it suffices to prove the following claim:

Claim. The rank n is even and A is congruent to the matrix $\frac{n}{2} \cdot S$.

Proof. In the following we consider the usual three types of $(n \times n)$-elementary matrices:

$$D_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}, \lambda \neq 0, \quad P := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \quad \text{and} \quad Q_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

We make the following elementary observations:

(i) Replacing a matrix B by $D_\lambda^{\mathrm{T}} B D_\lambda$ means that first we multiply a column by λ, and then we multiply the corresponding row by λ.

(ii) Replacing a matrix B by $P^{\mathrm{T}} B P$ means that first we exchange two columns, and then we exchange the corresponding two rows.

[2435] Here, when we write $k \cdot \theta(S)$ we use Definition 224.7.

(iii) Replacing a matrix B by $Q_\lambda^{\mathrm{T}} B Q_\lambda$ means that first we add a λ-multiple of one column to another column, and then we do the same operation for the corresponding rows.

Now we can perform the following sequence of operations on A:

$$A \overset{(6)}{=} \begin{pmatrix} 0 & * & * & \dots & * \\ * & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(2)}{\to} \begin{pmatrix} 0 & 1 & * & \dots & * \\ -1 & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(3)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & * & \dots & * \\ 0 & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & * & * & * & 0 \end{pmatrix} \overset{(4)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & 0 & * & * & 0 \end{pmatrix}.$$

More precisely we perform the following steps:

(1) We write out A. Since φ is anti-symmetric we see that all terms of A on the diagonal are zero. Here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are anti-symmetric.

(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (ii) and (i) we can arrange that the $(1,2)$-entry equals 1. By the anti-symmetry we see that the $(2,1)$-entry now equals -1.

(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.

(4) Similarly, we can use operations of type (iii) to "clean up" the remaining terms on the second row and column.

At this stage we can continue with the same process, applied to the $(n-2)\times(n-2)$–matrix at the right bottom. Eventually we will have shown that n is even and we will have turned A into the matrix $\frac{n}{2} \cdot S$. ⊞

Now suppose that (V, φ) is a non-singular anti-symmetric form over $\mathbb{Z}$. In Exercise 226.14 we will modify the above proof to deal with this new situation. ■

226.1.2. **Non-singular anti-symmetric forms: Topology.** In a sense the Anti-Symmetric Form Proposition 226.2 is rather disappointing: We had hoped that intersection forms will give us new meaningful invariants. But the Anti-Symmetric Form Proposition 226.2 together with the Intersection Form Proposition 224.13 and the Intersection Form–Non Singularity Proposition 224.14 shows that intersection forms of closed oriented $(4k+2)$-dimensional topological manifolds do not give us any extra information that would allow us to distinguish manifolds.

Nonetheless, the Anti-Symmetric Form Proposition 226.2 has just enough content to answer an old question about Euler characteristics. More precisely, recall that in the Odd-Dimensional Manifold-χ-Proposition 210.9 we showed that the Euler characteristic of every closed odd-dimensional topological manifold is necessarily zero. This observation gave rise to Question 210.11: Which integers appear as Euler characteristics of closed orientable connected even-dimensional topological manifolds? Now we can provide a complete answer to this question:

Proposition 226.3. (Euler Characteristic–Realization Proposition)

(1) Let $k \in \mathbb{N}$. Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $4k$-dimensional smooth manifold with Euler characteristic n.

(2) (a) The Euler characteristic of any closed orientable connected topological manifold of dimension 2 is even and ≤ 2. Conversely, any even integer ≤ 2 is the Euler characteristic of such a smooth manifold.
(b) Let $k \in \mathbb{N}$. The Euler characteristic of any closed orientable connected topological manifold of dimension $4k+2$ is even. Conversely, any even integer is the Euler characteristic of such a smooth manifold.

Remark. In Exercise 226.1 we will consider Euler characteristics of closed, possibly non-orientable, connected smooth manifolds. Furthermore, in Exercise 246.11 we will deal with the Euler characteristics of compact connected smooth manifolds with possibly non-empty boundary. □

Proof. We start out with two claims that give us the stated restrictions on Euler characteristics of closed orientable topological manifolds.

Claim 1. Let $k \in \mathbb{N}_0$. The Euler characteristic of every closed orientable topological manifold M of dimension $4k+2$ is even.

Proof. We have

$$\chi(M) \underset{\uparrow}{=} \sum_{i=0}^{4k+2} (-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(M)) = -\operatorname{rank}(\mathrm{H}_{2k+1}(M)) + \sum_{i=0}^{2k} (-1)^i \cdot \overbrace{2 \cdot \operatorname{rank}(\mathrm{H}_i(M))}^{\text{even}}$$

by the Manifold-χ Proposition 190.24 (4)

by the Betti Number Symmetry Corollary 210.8 we have $\operatorname{rank}(\mathrm{H}_i(M)) = \operatorname{rank}(\mathrm{H}_{4k+2-i}(M))$

$$\underset{\uparrow}{\equiv} 0 \mod 2.$$

by the Intersection Form Proposition 224.13 and the Intersection Form–Non Singularity Proposition 224.14 the facts that M is closed and that $2k+1$ is odd we know that the intersection form on $\mathrm{FH}_{2k+1}(M)$ is anti-symmetric and non-singular, by the Anti-Symmetric Form Proposition 226.2 we thus know that $\operatorname{rank}(\mathrm{H}_{2k+1}(M))$ is even ⊞

Claim 2. The Euler characteristic of every every closed orientable connected 2-dimensional topological manifold M is ≤ 2.

Proof. Of course this statement follows from the Surface Classification Theorem 101.16 together with the calculation of the Euler characteristics on page 2987. But one can prove the claim directly: First note that the case that $M = \varnothing$ is trivial. Now assume that M is non-empty. In this case we have the following (in-) equalities:

$$\chi(M) \underset{\uparrow}{=} \sum_{i \in \mathbb{N}_0} (-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(M)) \underset{\uparrow}{=} \overbrace{\operatorname{rank}(\mathrm{H}_0(M))}^{=1 \text{ by the } \mathrm{H}_0\text{-Proposition 143.25}} - \operatorname{rank}(\mathrm{H}_1(M)) + \overbrace{\operatorname{rank}(\mathrm{H}_2(M))}^{=1 \text{ by the Orientation-Top Homology Theorem 193.3}} \leq 2.$$

by the Euler Characteristic-H_* Proposition 156.2

by the Manifold-Top-H_*-Theorem 193.1 we know that $\mathrm{H}_k(M) = 0$ for $k \geq 3$ ⊞

It remains to show that besides these two restrictions from Claim 1 and 2 on the possible values of Euler characteristics, all other values can be realized by closed orientable connected smooth manifolds. In other words, it remains to prove the following claim:

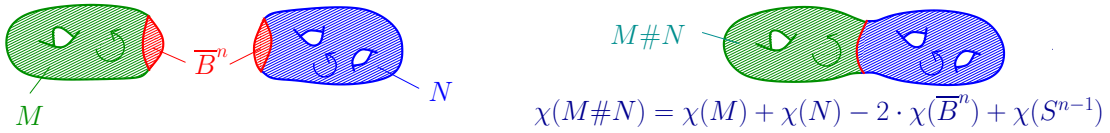
Claim 3.

(1) Let $m \in \mathbb{N}$ be even. Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $2m$-dimensional smooth manifold with Euler characteristic n.

(2) (a) Every even number ≤ 2 is the Euler characteristic of a closed orientable connected smooth manifold of dimension 2.
(b) Let $m \in \mathbb{N}_{\geq 3}$ be odd. Every even number is the Euler characteristic of a closed orientable connected smooth manifold of dimension $2m$.

Proof. We start out with the following observations:

(i) On page 2988 we saw that for every $m \in \mathbb{N}_0$ we have $\chi(\mathbb{C}\mathrm{P}^m) = m+1$.
(ii) There is any number of reasons, e.g. we could use the Product-χ-Proposition 156.12, why given any $d \in \mathbb{N}$ the Euler characteristic of the d-dimensional torus $(S^1)^d$ is zero.
(iii) The Connected Sum-χ-Lemma 156.16 says that if M and N are two compact oriented connected *even-dimensional* smooth manifolds, then $\chi(M\#N) = \chi(M) + \chi(N) - 2$.



Note that it follows from these observations that for any $m, r, s \in \mathbb{N}_0$ we have

$$(*) \quad \chi(r \cdot \mathbb{C}\mathrm{P}^m \# s \cdot (S^1)^{2m}) = r \cdot \underbrace{(m+1)}_{=\chi(\mathbb{C}\mathrm{P}^m)} - 2 \cdot \underbrace{(r+s-1)}_{\substack{\text{number of} \\ \text{connected sum} \\ \text{operations}}} = r \cdot (m-1) - 2 \cdot (s-1).$$

Now we turn to the actual proof of the claim:

(1) Let $m \in 2\mathbb{N}$ and let $n \in \mathbb{Z}$. Since m is even we see that $s := \frac{|n| \cdot (m-1) - n}{2} + 1$ is a natural number. It follows from $(*)$ with $r = |n|$ and the above s that $\chi(r \cdot \mathbb{C}\mathrm{P}^m \# s \cdot (S^1)^{2m}) = n$.
(2) (a) Let $n \in \mathbb{Z}_{\leq 2}$ be even. It follows from $(*)$ that $\chi((1 - \frac{n}{2}) \cdot (S^1 \times S^1)) = n$.[2436]
(b) Let $m \in \mathbb{N}_{\geq 3}$ be odd and let n be an even integer. We set $s := \frac{|n| \cdot (m-1) - n}{2} + 1$. Since n is even we see that s is an integer. Furthermore, since $m \geq 3$ we see that s is ≥ 0. It follows again from $(*)$ with $r = |n|$ and the above $s \in \mathbb{N}_0$ that $\chi(r \cdot \mathbb{C}\mathrm{P}^m \# s \cdot (S^1)^{2m}) = n$. ∎

226.2. **Symmetric forms over $\mathbb{R}$.** In Section 226.1 we saw that non-singular *anti-symmetric* forms are almost depressingly boring mathematical objects. In this and the following sections we will consider non-singular *symmetric* forms. These will, fortunately, turn out to be much more interesting animals. The current section is organized as follows:

- First we classify symmetric forms over $\mathbb{R}$.
- Afterwards we apply the classification to topological manifolds, in particular we introduce the signature of a compact oriented topological manifold and study some of its properties. The signature will allow us to show that $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is neither homotopy equivalent to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$ nor to $S^2 \times S^2$.

226.2.1. **Symmetric forms over $\mathbb{R}$: Algebra.** It is natural to ask whether there is some type of analog of the Anti-Symmetric Form Proposition 226.2 for symmetric forms. In fact using the methods of the proof of the Anti-Symmetric Form Proposition 226.2 one can prove the following statement:

[2436] Of course that is old news: We already saw on page 2987 that for the surface Σ_g of genus g we have $\chi(\Sigma_g) = 2 - 2g$.

Proposition 226.4. (Symmetric Forms-over-$\mathbb{R}$-Proposition) Let (V, φ) be a symmetric form over $\mathbb{R}$. There exist $n, k_+, k_- \in \mathbb{N}_0$ such that

$$(V, \varphi) \cong \underbrace{n \cdot (0) \oplus k_+ \cdot (+1) \oplus k_- \cdot (-1)}.$$

here we use the notation from the Matrix–Form Operations Lemma 224.8 i.e. we consider the diagonal matrix where n entries are zero, $k_\pm$ entries are ± 1

Remark. Recall that the Anti-Symmetric Form Proposition 226.2 dealt with anti-symmetric forms over $\mathbb{R}$ and also with anti-symmetric forms over $\mathbb{Z}$. In contrast the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 only deals with symmetric forms over $\mathbb{R}$. It is a very instructive exercise to run through the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 and to determine where it breaks down if we work over $\mathbb{Z}$. □

Proof. Let (V, φ) be a non-singular symmetric form over $\mathbb{R}$ of rank r. We pick an $(r \times r)$–matrix A that represents φ. By the Form–Matrix-Lemma 224.5 it suffices to prove the following claim:

Claim. There exist $n, k_+, k_- \in \mathbb{N}_0$ such that the matrix A is congruent to the diagonal matrix $n \cdot (0) \oplus k_+ \cdot (+1) \oplus k_- \cdot (-1)$.

Proof. In the subsequent arguments below we use the conventions and notions we introduced in the proof of the Anti-Symmetric Form Proposition 226.2 on page 4363. In particular we will also use the operations of type (i), (ii) and (iii) that we introduced in the proof of the Anti-Symmetric Form Proposition 226.2. We perform the following sequence of operations on A:

$$A \overset{(6)}{=} \begin{pmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(2a)} \begin{pmatrix} \epsilon_1 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(2b)} \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}.$$

More precisely, we perform the following steps:

(1) We write out A. Here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are symmetric.

(2)
- If the first row consists entirely of zeros, then so does (by symmetry) the first column.
- We now assume that the first row has at least one entry which is non-zero.
  - (a) After operations of type (iii) and (i) we can arrange that the $(1,1)$-entry equals some $\epsilon_1 \in \{-1, 1\}$.
  - (b) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.

At this stage we can continue with the same process, applied to the $(r-1) \times (r-1)$–matrix at the right bottom. Eventually we will have turned A into a diagonal matrix where all entries lie in $\{-1, 0, 1\}$. After operations of type (ii) we can arrange that the first diagonal entries are equal to 0, the following entries are equal to $+1$ and the final ones are equal to -1. ■

Let $n \in \mathbb{N}_0$. The Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 raises the question whether or not the symmetric forms defined by $n \cdot (0) \oplus k_+ \cdot (+1) \oplus k_+ \cdot (-1)$ are isometric for different

values of n, k_-, k_+. Put differently, the question is, can we read off n, k_-, k_+ from a given symmetric form over $\mathbb{R}$?

To answer this question it is helpful to introduce the following definition:

Definition 226.5. Let (V, φ) be a form over a field $\mathbb{F}$.

(1) The vector subspace $\operatorname{rad}(\varphi) := \{v \in V \mid \theta(v, w) = 0 \text{ for all } w \in V\}$ is called the radical of the form φ.

(2) We refer to
$$\nu(\varphi) \;:=\; \dim_{\mathbb{F}}(\operatorname{rad}(\varphi)).$$
as the nullity $\nu(\varphi)$.

Examples.

(1) In the setting of the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 we clearly have an inequality $\nu(\varphi) \geq n$. In the Signature-Eigenvalues Proposition 226.11 we will see that this inequality is in fact an equality.

(2) It follows easily from the Non-Singularity Lemma 213.2 that a symmetric form (V, φ) over $\mathbb{R}$ has nullity zero if and only if (V, φ) is non-singular. □

Definition 226.6. Let R be a subring of $\mathbb{R}$ and let (V, φ) be a symmetric form over $\mathbb{R}$.

(1) We say that φ is positive-definite if $\varphi(v, v) > 0$ for all $v \neq 0$.

(2) We say that φ is negative-definite if $\varphi(v, v) < 0$ for all $v \neq 0$.

(3) We say that φ is indefinite if φ is neither positive-definite nor negative-definite.

Examples. Let R be a subring of $\mathbb{R}$.

(1) The form $\theta_R(+1)$ is positive-definite and the form $\theta_R(-1)$ is negative-definite.

(2) We consider the matrix $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The corresponding form $\theta_R(H)$ is indefinite, indeed, we have
$$\theta_R(H)\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \;=\; (1 \;\; 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;=\; 2$$
and
$$\theta_R(H)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \;=\; (1 \;\; -1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \;=\; -2.$$
□

Notation 226.7. Let R be a commutative ring. Given a form $(V, \varphi\colon V \times V \to R)$ and given a free R-submodule U of V we denote by $\varphi|_U$ the form that is given by restricting the map $\varphi\colon V \times V \to R$ to $U \times U \to R$.

Definition 226.8. Let (V, φ) be a symmetric form over $\mathbb{R}$. We define

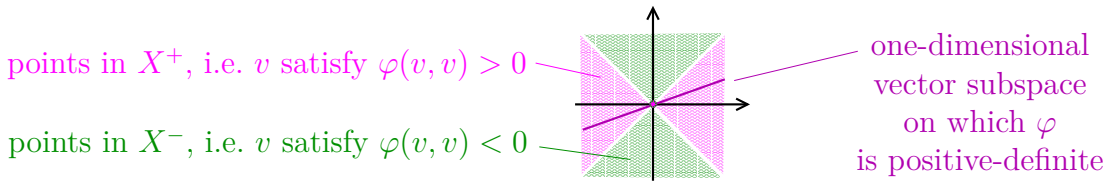
$b^+(\varphi) :=$ maximal dimension of a vector subspace U of V s.t. $\varphi|_U$ is positive-definite,
$b^-(\varphi) :=$ maximal dimension of a vector subspace U of V s.t. $\varphi|_U$ is negative-definite.

Furthermore we define the signature $\operatorname{sign}(\varphi)$ as
$$\operatorname{sign}(\varphi) \;:=\; b^+(\varphi) - b^-(\varphi).$$

Example. We consider the symmetric form
$$\begin{array}{rcl} \varphi\colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) & \mapsto & (v_1 \;\; v_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \end{array}$$

In the figure below we show the subsets

$$X^+ := \{v \in \mathbb{R}^2 \,|\, \theta(v,v) > 0\} \cup \{0\} \quad \text{and} \quad X^- := \{v \in \mathbb{R}^2 \,|\, \theta(v,v) < 0\} \cup \{0\}.$$



These subsets X^+ and X^- are evidently *not* vector subspaces of $V = \mathbb{R}^2$. The maximal dimension of a vector subspace contained in X^+ and X^- is in both cases equal to one. Thus we see that $b^+(\varphi) = b^-(\varphi) = 1$. In particular we see that $\text{sign}(\varphi) = 0$. □

In the following two lemmas we summarize very elementary statements about these invariants of symmetric forms over $\mathbb{R}$:

Lemma 226.9. (Isometry Invariants Lemma) Let (V,φ) and (W,ψ) be two symmetric forms over $\mathbb{R}$. If (V,φ) and (W,ψ) are isometric, then

$$\begin{aligned} \dim_{\mathbb{R}}(V) &= \dim_{\mathbb{R}}(W), \\ \nu(\varphi) &= \nu(\psi), \\ b^{\pm}(\varphi) &= b^{\pm}(\psi), \\ \text{sign}(\varphi) &= \text{sign}(\psi). \end{aligned}$$

Proof. The lemma is an immediate consequence of the definitions. ■

Lemma 226.10. (Signature-Sign Lemma) Given any symmetric form (V,φ) over $\mathbb{R}$ we have

$$\begin{aligned} \nu(-\varphi) &= \nu(\varphi), \\ b^{\pm}(-\varphi) &= b^{\mp}(\varphi), \\ \text{sign}(-\varphi) &= -\,\text{sign}(\varphi). \end{aligned}$$

Proof. This lemma also follows easily from the definitions. ■

In the following proposition we determine the nullity ν, the invariants $b^{\pm}$ and the signature for a symmetric form in terms of the eigenvalues of any matrix representing the form:

Proposition 226.11. (Signature-Eigenvalues Proposition) Let A be a symmetric real matrix. Then

$$\begin{aligned} \nu(\theta_{\mathbb{R}}(A)) &= \text{number of zero eigenvalues of } A\text{, counted with multiplicities,} \\ b^+(\theta_{\mathbb{R}}(A)) &= \text{number of positive eigenvalues of } A\text{, counted with multiplicities,} \\ b^-(\theta_{\mathbb{R}}(A)) &= \text{number of negative eigenvalues of } A\text{, counted with multiplicities.} \end{aligned}$$

In particular

$$\text{sign}(\theta_{\mathbb{R}}(A)) = \begin{array}{l} \text{number of positive eigenvalues of } A \\ - \text{ number of negative eigenvalues of } A. \end{array}$$

Example. The following calculation will be helpful on several occasions:

$$\text{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\uparrow}{=} \text{sign}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \text{sign}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \underset{\uparrow}{=} 0.$$

by the Forms-Matrices Lemma 224.5

Signature-Eigenvalues Proposition 226.11

□

Proof. Let A be a symmetric $(r \times r)$-matrix over $\mathbb{R}$. We write

$$\begin{aligned} n &:= \text{number of zero eigenvalues of } A\text{, counted with multiplicities,} \\ k^+ &:= \text{number of positive eigenvalues of } A\text{, counted with multiplicities,} \\ k^- &:= \text{number of negative eigenvalues of } A\text{, counted with multiplicities.} \end{aligned}$$

By the Spectral Theorem 2.35 we know that there exists an orthogonal matrix $P \in \mathrm{O}(r)$ such that $P^{-1} \cdot A \cdot P =: D$ is a diagonal matrix, such that the first n entries of D are zero, the next k_+ entries of D are positive and such that the remaining k_- entries are negative.

Since P is orthogonal we know that $P^{-1} = P^{\mathrm{T}}$. Thus it follows from the Form–Matrix-Lemma 224.5 that $\theta(A)$ is isometric to $\theta(D)$. It follows from this observation and the Isometry Invariants Lemma 226.9 that it suffices to prove the proposition for the diagonal matrix D.

We write

$$\begin{aligned} V^0 &:= \{(x_1, \dots, x_n, 0, \dots, 0) \in \mathbb{R}^r \mid x_1, \dots, x_n \in \mathbb{R}\}, \\ V^+ &:= \{(0, \dots, 0, y_1, \dots, y_{k_+}, 0, \dots, 0) \in \mathbb{R}^r \mid y_1, \dots, y_{k_+} \in \mathbb{R}\}, \\ V^- &:= \{(0, \dots, 0, z_1, \dots, z_{k_-}) \in \mathbb{R}^r \mid z_1, \dots, z_{k_-} \in \mathbb{R}\}. \end{aligned}$$

We make the following observations:

- It is clear that V^0 is contained in the radical of φ. Hence $\nu(\theta(D)) \geq n$.
- It is clear that $\theta(D)|_{V^+}$ is positive-definite. Hence $b^+(\theta(D)) \geq k_+$.
- It is clear that $\theta(D)|_{V^-}$ is negative-definite. Hence $b^-(\theta(D)) \geq k_-$.

If remains to prove the following claim:

Claim. For any symmetric form (W, ψ) over $\mathbb{R}$ we have $\nu(\psi) + b^+(\psi) + b^-(\psi) \leq \dim_{\mathbb{R}}(W)$.

Proof. We assume that we have the following:

- W^0 is vector subspace of W which is contained in the radical of φ.
- W^+ is a positive-definite vector subspace of W.
- W^- is a negative-definite vector subspace of W.

We need to show that the sum of the dimensions of these subspaces is less or equal than the dimension of W. We make the following observations:

(1) W^+ and W^- have trivial intersection: Indeed, $W^+ \cap W^-$ is a vector subspace that is positive-definite and negative-definite. So for any non-zero vector $v \in W^+ \cap W^-$ we have $\psi(v, v) > 0$ and $\psi(v, v) < 0$. This is only possible if no such v exists, i.e. we have shown that the intersection $W^+ \cap W^- = \{0\}$.

(2) W^0 and $W^+ \oplus W^-$ have trivial intersection: Let $w^+ \in W^+$ and $w^- \in W^-$ such that $w^+ + w^-$ is non-zero. We see that

$$\theta(w^+ + w^-, w^+ - w^-) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is bilinear} \\ \text{and symmetric}}}{=} \theta(w^+, w^+) - \theta(w^-, w^-) \underset{\substack{\uparrow \\ \text{since } w^+ \in W^+ \text{ and } w^- \in W^- \\ \text{and at least one is non-zero}}}{>} 0.$$

This shows that $w^+ + w^- \notin W^0$.

Now we see that

$$\dim(W^0) + \dim(W^+) + \dim(W^-) \underset{\substack{\uparrow \\ \text{this follows from elementary linear algebra} \\ \text{since } W^- \cap W^+ = \{0\} \text{ and } W^0 \cap (W^+ + W^-) = \{0\}}}{=} \dim(W^0 + W^+ + W^-) + \underset{\substack{\uparrow \\ \text{since } W^0 + W^+ + W^- \subseteq W}}{\leq} \dim(W).$$

∎

The following theorem (which gives us in particular a converse to the Isometry Invariants Lemma 226.9) says that the dimension, the nullity and the signature are a complete invariant for isometry types of symmetric forms over $\mathbb{R}$.

Theorem 226.12. (Sylvester's Theorem a. k. a. Sylvester's Law of Inertia)

(1) Let (V,φ) be a symmetric form over $\mathbb{R}$. We have an isometry[2437]

$$(V,\varphi) \;\cong\; \nu(\varphi)\cdot(0)\oplus b^+(\varphi)\cdot(+1)\oplus b^-(\varphi)\cdot(-1).$$

In particular:
$$\dim_{\mathbb{R}}(V) \;=\; \nu(\varphi)+b^+(\varphi)+b^-(\varphi).$$

(2) Let (V,φ) and (W,ψ) be two symmetric forms over $\mathbb{R}$. Then

$$(V,\varphi) \text{ and } (W,\psi) \text{ are isometric} \iff \begin{array}{rcl}\dim_{\mathbb{R}}(V) &=& \dim_{\mathbb{R}}(W),\\ \nu(\varphi) &=& \nu(\psi),\\ \operatorname{sign}(\varphi) &=& \operatorname{sign}(\psi).\end{array}$$

Proof.

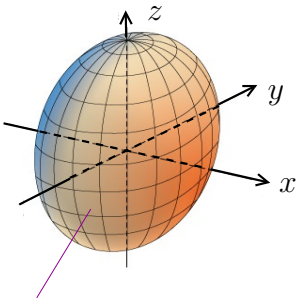
(1) By the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 and the Isometry Invariants Lemma 226.9 it suffices to prove the statement for symmetric forms that are represented by diagonal matrices. But for those forms the statement follows immediately from the Signature-Eigenvalues Proposition 226.11.

(2) The "$\Rightarrow$"-direction is precisely the content of the Isometry Invariants Lemma 226.9. It remains to prove the "$\Leftarrow$"-direction. So suppose that we are given two symmetric forms (V,φ) and (W,ψ) over $\mathbb{R}$ with same dimension, the same nullity and the same signature. It follows from our hypothesis together with (1) and the definition of the signature $\operatorname{sign}(\varphi)=b^+(\varphi)-b^-(\varphi)$ that there exist $n,k_+,k_-\in\mathbb{N}_0$ such that (V,φ) and (W,ψ) are both isometric to $n\cdot(0)\oplus k_+\cdot(+1)\oplus k_-\cdot(-1)$. But this implies that (V,φ) and (W,ψ) are themselves isometric. ∎

Example. Given a symmetric form φ on $\mathbb{R}^3$ one can visualize it by considering the "unit-sphere"

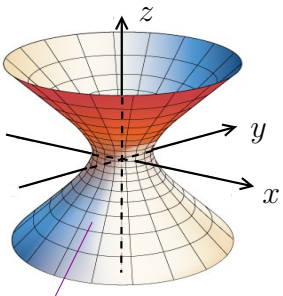
$$S(\varphi) \;:=\; \{v\in\mathbb{R}^3 \,|\, \theta(v,v)=1\}.$$

(a)



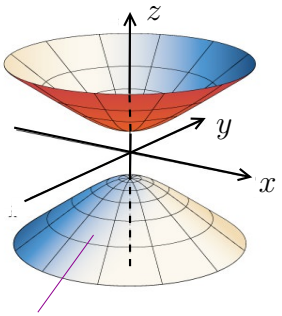
$x^2+y^2+z^2=1$

(b)



$x^2+y^2-z^2=1$

(c)



$-x^2-y^2+z^2=1$

It follows from Sylvester's Theorem 226.12, the Isometry Invariants Lemma 226.9 and the Signature-Eigenvalues Proposition 226.11 that every non-singular symmetric form on $\mathbb{R}^3$ is

[2437] If (V,φ) is non-singular, then we have $\nu(\varphi)=0$. Thus we obtain that $b^+(\varphi)+b^-(\varphi)=\dim_{\mathbb{R}}(V)$. Together with $\operatorname{sign}(\varphi)=b^+(\varphi)-b^-(\varphi)$ we see that in this case any two out of $\dim(V)$, $\operatorname{sign}(\varphi)$, $b^+(\varphi)$ and $b^-(\varphi)$ determine the other invariants.

isometric to precisely one of the following four forms:

(a) $3 \cdot (1)$, (b) $2 \cdot (1) \oplus (-1)$
(c) $2 \cdot (-1) \oplus (1)$ (d) $3 \cdot (-1)$.

In the last case the unit-sphere is the empty set. In the other cases the unit-spheres are sketched in the figure above. □

We conclude this subsection with the following proposition, which says that the signature is additive under the direct sum operation:

Proposition 226.13. (Signature Additivity Proposition) If (V, φ) and (W, ψ) are two symmetric forms over $\mathbb{R}$, then

$$\begin{aligned} \nu(\varphi \oplus \psi) &= \nu(\varphi) + \nu(\psi), \\ b^{\pm}(\varphi \oplus \psi) &= b^{\pm}(\varphi) + b^{\pm}(\psi), \\ \operatorname{sign}(\varphi \oplus \psi) &= \operatorname{sign}(\varphi) + \operatorname{sign}(\psi). \end{aligned}$$

Proof. This statement follows immediately from the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 applied to φ and ψ, the Matrix-Form Operations Lemma 224.8 (2) and the Signature-Eigenvalues Proposition 226.11. ■

We conclude this section with the following definition, which shows how we can associate to a symmetric from over $\mathbb{Z}$ a symmetric form over $\mathbb{R}$:

Definition 226.14. Let (V, φ) be a symmetric form over $\mathbb{Z}$. It follows easily from the Tensor Product–Universal Property Lemma 159.4 and the Tensor-with-Ring Lemma 159.18 that the following map is well-defined:

$$\begin{aligned} \varphi \otimes \mathbb{R} \colon V \otimes \mathbb{R} \times V \otimes \mathbb{R} &\to \mathbb{R} \\ (v \otimes r, w \otimes s) &\mapsto r \cdot \theta(v, w) \cdot s, \end{aligned}$$

and that this map defines a symmetric form over $\mathbb{R}$. This allows us to define

$$\operatorname{sign}(\varphi) := \operatorname{sign}(\varphi \otimes \mathbb{R}).$$

Remark.

(1) Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be an (anti-) symmetric matrix. Note that it follows easily from the definitions and the Free Module–Tensor-with-Ring Lemma 159.20 that the form $\theta_{\mathbb{R}}(A)$ (which we introduced in Definition 226.1) is isometric to $\theta_{\mathbb{Z}}(A) \otimes \mathbb{R}$. In particular if A is a matrix representing a form φ over $\mathbb{Z}$, then the same matrix, viewed as a matrix over $\mathbb{R}$, represents $\varphi \otimes \mathbb{R}$. We will often make use of this observation without explicitly pointing it out.

(2) Let (V, φ) be symmetric form over $\mathbb{Z}$. In Exercise 226.20 we will see that φ is positive-definite if and only if $\varphi \otimes \mathbb{R}$ is positive-definite. The analogous statement of course also holds for negative-definite and indefinite. □

226.2.2. **Symmetric forms over $\mathbb{R}$: Topology.** Now it is time to apply our recently acquired knowledge to topology:

Definition 226.15. Let $k \geq 1$ and let M be a compact oriented $4k$-dimensional topological manifold. We define the signature of M to be[2438]

$$\operatorname{sign}(M) \quad := \quad \operatorname{sign}(\underset{\substack{\uparrow \\ \text{in Definition 222.3 we introduced the intersection} \\ \text{form } Q_M \colon \mathrm{FH}_{2k}(M) \times \mathrm{FH}_{2k}(M) \to \mathbb{Z}}}{Q_M}).$$

Remark. The signature of a manifold was first introduced in 1923 by Hermann Weyl [**Wey1923**, p. 432] in a little known paper written in Spanish. As is explained in [**Eckm2006**], Hermann Weyl deliberately chose this obscurity since at this point, according to Hermann Weyl, algebraic topology was not considered "a respectable field of mathematics". □

Examples. Let $k \in \mathbb{N}$.

(a) Note that
$$\operatorname{sign}(\mathbb{C}\mathrm{P}^{2k}) \underset{\substack{\uparrow \\ \text{follows from the Cup Product}}}{=} \operatorname{sign}(1) = 1.$$

(b) Note that
$$\operatorname{sign}(S^{2k} \times S^{2k}) \underset{\substack{\uparrow \\ \text{see page 4331}}}{=} \operatorname{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\substack{\uparrow \\ \text{see page 4369}}}{=} 0.$$

□

Lemma 226.16. (Manifold–Orientation–Signature Lemma) Let $k \in \mathbb{N}$ and let M and N be two compact oriented $4k$-dimensional topological manifolds.

(1)
- If there exists an orientation-preserving homeomorphism $M \to N$, then we have $\operatorname{sign}(M) = \operatorname{sign}(N)$.
- If there exists an orientation-reversing homeomorphism $M \to N$, then we have $\operatorname{sign}(M) = -\operatorname{sign}(N)$.

(2) If there exists a homotopy equivalence $f \colon M \to N$ that restricts to a homotopy equivalence $f|_{\partial M} \colon \partial M \to \partial N$, then we have the equality $\operatorname{sign}(M) = \pm \operatorname{sign}(N)$.

Proof. These two statements are an immediate consequence of the Intersection Form–Homeomorphism Lemma 224.15, the Isometry Invariants Lemma 226.9 and the Signature-Sign Lemma 226.10. ■

Lemma 226.17. (Manifold Signature Additivity Lemma) The signature is additive under the connected sum operation and under disjoint union.

Proof. It follows easily from the Cup Product–Disjoint Union Lemma 204.16 that the signature is additive under disjoint union. If we are given compact oriented topological manifolds, then it follows immediately from the straightforward generalization of the Intersection Form-Additivity Proposition 224.18 to compact topological manifolds (with possibly non-empty boundary) and the Signature Additivity Proposition 226.13 that the signature is additive under the connected sum operation. ■

[2438] Recall that by the Intersection Form Proposition 224.13 we know that the intersection form Q_M is indeed symmetric, so it makes sense to consider its signature.

Example. Note that

$$\begin{array}{ccccccccc}
\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2)+\operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1)+\operatorname{sign}(1) & = & 1+1 & = & 2 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \text{by Manifold Signature Additivity Lemma 226.17} & & \text{see page 4373} & & & & \\
 & & \text{and the Manifold–Orientation–Signature Lemma 226.16 (1)} & & & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2)-\operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1)-\operatorname{sign}(1) & = & 1-1 & = & 0.\square
\end{array}$$

We can now give a partial answer to Question 185.25:

Proposition 226.18. ($\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$-Homotopy Type Proposition) The topological space $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is neither homotopy equivalent (in particular it is not homeomorphic and certainly not diffeomorphic) to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ nor to $S^2\times S^2$.

Proof. In the above examples we just saw that $\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2)=2$ and we also showed that $\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2)=\operatorname{sign}(S^2\times S^2)=0$. It follows from the Intersection Form–Homeomorphism Lemma 224.15 and the Manifold–Orientation–Signature Lemma 226.16 (2) that $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is neither homotopy equivalent to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ nor to $S^2\times S^2$. ■

The following corollary shows in particular that in the definition of the connected sum of two orientable (topological) smooth manifolds on Section 54.5 and Definition 224.23 we needed to work with *oriented* (topological) smooth manifolds.

Proposition 226.19. (Connected Sum–Orientation Dependence Proposition) There exist closed oriented smooth manifolds M and N such that $M\#N$ is not homotopy equivalent (let alone homeomorphic or diffeomorphic) to $M\#(-N)$.

Proof. We just saw in the $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$-Homotopy Type Proposition 226.18 that the closed oriented smooth manifolds $M=N=\mathbb{C}\mathrm{P}^2$ have this property. ■

Unfortunately we are not yet able to distinguish the two smooth manifolds $S^2\times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$. In fact Sylvester's Theorem 226.12 might suggest that we are again at a dead end. But this is not the case since Sylvester's Theorem 226.12 is a statement about symmetric forms over $\mathbb{R}$, but we are really dealing with symmetric forms defined over $\mathbb{Z}$. We will exploit this extra piece of information in Section 226.4.

226.3. **Interlude: Sphere-packing.** Before we turn to the study of symmetric forms over $\mathbb{Z}$ we take a short break, and we consider sphere packings. As we will see, the relationship with our actual goal of dealing with symmetric forms is arguably tenuous, but nonetheless fascinating.

Definition 226.20. Let $v_1,\dots,v_n$ be linearly independent vectors in $\mathbb{R}^n$. We refer to the group

$$\mathbb{Z}\cdot v_1+\cdots+\mathbb{Z}\cdot v_n \;=\; \left\{\sum_{i=1}^{n} a_i\cdot v_i \,\middle|\, a_1,\dots,a_n\in\mathbb{Z}\right\}$$

as the lattice spanned by $v_1,\dots,v_n$.[2439]

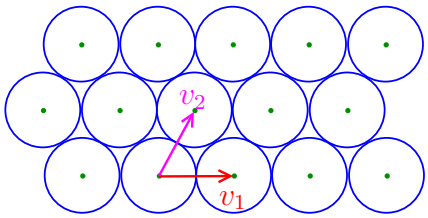
We start out with the following theorem:

Theorem 226.21. (Densest Packing Theorem)

[2439] This is perhaps not the standard definition of a lattice, but who cares?

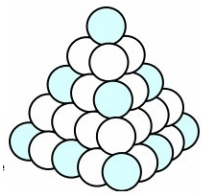
(1) A densest packing[2440] of disks of radius 1 in $\mathbb{R}^2$ is given by setting the centers on the "hexagonal" lattice spanned by $v_1 = (2, 0)$ and $v_2 = (1, \sqrt{3})$ shown in the figure below on the left. The density equals $\frac{\pi}{\sqrt{12}}$.

(2) A densest packing of balls of the same radius in $\mathbb{R}^3$ is given by the "face-centered cubic packing" shown in the figure below in the middle and on the right. The density equals $\frac{\pi}{\sqrt{18}}$.



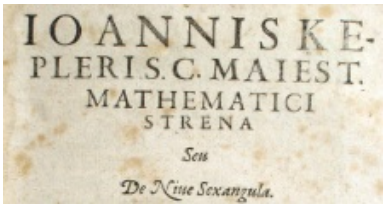
densest disk packing





ball packings of highest density

Proof. Statement (1) was proved by L. Fejes Tóth [**FT1943, FT1972**] in 1943. It is much harder to prove Statement (2). Statement (2) had been conjectured by Johannes Kepler [**Kep1611**] in 1611 and was known as the Kepler Problem.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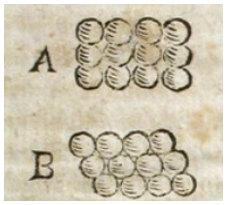
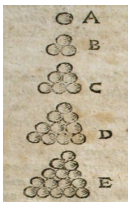




It was finally proved by Thomas Hales [**Hale2000, Hale2005, Hale2017**] in 2005. ■

For a mathematician it is natural to wonder what are the highest densities of packings in higher dimensions. Not surprisingly this is a very hard problem. Astonishingly the next case that was dealt with is the 8-dimensional setting. More precisely, the following theorem was proved by Maryna Viazovska [**Via2017**] in 2016:[2441] [2442]

Theorem 226.22. (E_8-Densest Packing Theorem) We consider the following matrix:

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

[2440] The fact that strictly speaking we did not define this term will not bother us.

[2441] In 2022 Maryna Viazovska was awarded a Fields medal for this work.

[2442] Shortly afterwards the 24-dimensional case was settled in [**CKMRV2017**], in this case a densest packing is given by the "Leech lattice".

A densest packing of balls of radius 1 in $\mathbb{R}^8$ is given by the lattice spanned by the eight column vectors of the E_8–matrix. Furthermore, this packing has density $\frac{\pi^4}{384}$.

Proof. The original proof can be found in [**Via2017**]. An exposition of the proof is also given in [**CohnH2023**]. ■

Remark. We also refer to [**Baez2018**] for other interesting appearances of the E_8–matrix, for example its relationship to the regular icosahedron. □

The E_8-matrix from the E_8-Densest Packing Theorem 226.22 looks interesting. Let us explore its properties:

Lemma 226.23. (E_8-Lemma) The form

$$\theta_{\mathbb{Z}}(E_8)\colon \mathbb{Z}^8 \times \mathbb{Z}^8 \ \to \ \mathbb{Z}$$

that is defined by the E_8-matrix is non-singular and positive-definite.

Proof. As in the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 on page 4367 we perform simultaneous row and column operations of type (iii) (which correspond to matrices in $\mathrm{GL}(8,\mathbb{R})$ of determinant $+1$) that we introduced in the proof of the Anti-Symmetric Form Proposition 226.2. This leads us to the following sequence of matrices:[2443]

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}}_{=E_8} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

This shows that there exists a matrix $P \in \mathrm{GL}(8,\mathbb{R})$ *of determinant* $+1$, such that $P^{\mathrm{T}} \cdot E_8 \cdot P$ is a diagonal matrix where the diagonal entries are all positive. In particular the determinant is $\neq 0$. It follows from the Signature-Eigenvalues Proposition 226.11, the Isometry Invariants Lemma 226.9 and the Form–Matrix-Lemma 224.5 that the form $\theta_{\mathbb{Z}}(E_8)$ is non-singular and positive-definite.[2444] ■

OK, that was fun, but now let us return to our actual problem, and let us see whether this section was of any use.

226.4. **Non-singular symmetric forms over $\mathbb{Z}$.** In this section we will turn to the study of non-singular symmetric forms over $\mathbb{Z}$. As we will see, these forms contain much more information than their real cousins.

Examples. We list some symmetric integral matrices with determinant equal to ± 1:

(1) The (1×1)-matrices $(+1)$ and (-1).
(2) By the E_8-Lemma 226.23 we know that the slightly mysterious 8×8–matrix E_8 from the E_8-Densest Packing Theorem 226.22 has determinant one and that the corresponding form is positive-definite.

[2443] The missing inbetween matrices are provided in [**Scor2005**, p. 126].

[2444] A more conceptual proof of the lemma is given in [**HGK2010**, Theorem 1.16.14], [**ErH2018**, p. 194] and [**Baro2015**, p. 140].

(3) The matrix $$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ is symmetric and it has determinant -1.

By the Form–Matrix-Lemma 224.5 (2) these matrices define in particular non-singular forms over $\mathbb{R}$ and also over $\mathbb{Z}$. □

In the following example we will see that forms over $\mathbb{Z}$ are more subtle than forms over $\mathbb{R}$:

Example. We consider the matrices
$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
It follows from Sylvester's Theorem 226.12 that the matrices give rise to *isometric forms over* $\mathbb{R}$. The question arises, whether the forms on $\mathbb{Z}^2$ defined by these matrices are isometric. To get an idea let us consider what we can say about the "squares $\theta(v,v)$" defined by these forms. Note that given any $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2$ we have

$$\theta(H)(v,v) = (a\ b)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2ab \quad \text{and} \quad \theta(H)(v,v) = (a\ b)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a^2 - b^2.$$
We see that for the form to the left we only get even integers whereas to the right we obtain odd and even integers. This already shows that the two forms are not isometric over $\mathbb{Z}$. □

This encouraging discussion leads us to the following definition:

Definition 226.24. A symmetric form (V, φ) over $\mathbb{Z}$ is called even if
$$\varphi(v,v) \equiv 0 \mod 2 \quad \text{for all } v \in V,$$
otherwise the form is called odd. We refer to the property of φ being even or odd as the parity of the form.

The following lemma summarizes some basic facts about the parity of symmetric forms:

Lemma 226.25. (Parity Lemma)
(1) Isometric symmetric forms have the same parity.
(2) For a symmetric form (V, φ) the parities of (V, φ) and $(V, -\varphi)$ agree.
(3) The direct sum of an odd form with any other symmetric form is also odd.
(4) If A is a symmetric square matrix over $\mathbb{Z}$, then
$$\theta(A) \text{ is even} \iff \text{all diagonal entries of } A \text{ are even.}$$

Example. It follows from the Parity Lemma 226.25 (4) and earlier results that we have the following table:

Form	rank	signature	parity
(± 1)	1	± 1	odd
$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2	0	even
E_8	8	8	even

□

Proof. The only statement which requires a little thought is the statement that if $A = (a_{ij})$ is a symmetric $(n \times n)$–matrix with even diagonal entries, then the form $\theta(A)$ is even. Thus

let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. We calculate that

$$\theta(A)(v,v) \;=\; v^{\mathrm{T}}Av \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} v_i \cdot v_j \cdot a_{ij} \;\underset{\text{since } A \text{ is symmetric}}{=}\; 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} v_i \cdot v_j \cdot a_{ij} + \sum_{i=1}^{n} v_i^2 \cdot a_{ii} \;\underset{\text{since the } a_{ii} \text{ are even}}{\equiv}\; 0 \bmod 2.$$

■

The above example allows us to settle the remaining case of Question 185.25:

Proposition 226.26. ($\mathbb{C}\mathbf{P}^2\#\overline{\mathbb{C}\mathbf{P}}^2$-$S^2 \times S^2$-Proposition) The topological space $S^2 \times S^2$ is not homotopy equivalent (in particular it is not homeomorphic) to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$.

Proof. In the above example we just showed that the forms

$$H \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are not isometric. The same argument also shows that H is not isometric to $-D$. Recall that it follows from the calculation of the intersection forms on page 4340 that H and D are precisely the intersection forms of $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$. Thus it follows from the Intersection Form–Homeomorphism Lemma 224.15 that $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ are not homotopy equivalent. ■

We just saw that the two matrices

$$H \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

give rise to *isometric forms over* $\mathbb{R}$ and that the two matrices give rise to *non-isometric forms over* $\mathbb{Z}$. The following question arises: How much more complicated is the classification of non-singular symmetric forms over $\mathbb{Z}$ than the classification of non-singular forms over $\mathbb{R}$? In particular the following question arises:

Question 226.27. Can we classify non-singular symmetric forms over $\mathbb{Z}$?

In the following two subsections we will see that the answer to the question depends very much on whether we study indefinite or definite forms.

226.4.1. **Indefinite forms over $\mathbb{Z}$.** The following is the key theorem in the classification of non-singular *indefinite* symmetric forms over $\mathbb{Z}$:

Theorem 226.28. (Indefinite Forms Proposition) If (V_1, θ_1) and (V_2, θ_2) are two non-singular *indefinite* symmetric forms over $\mathbb{Z}$, then the following holds:

$$\theta_1 \text{ and } \theta_2 \text{ are isometric} \iff \theta_1 \text{ and } \theta_2 \text{ have the same rank, signature and parity.}$$

Proof. The proof of the theorem is provided in [**MH1973**, Theorem II.5.3] (see also [**GoS1999**, Theorem 1.2.14] for an outline of the proof). The proof relies crucially on a significant input from algebraic number theory, namely the Hasse-Minkowski theorem: This theorem says that given $a_1, \dots, a_n \in \mathbb{Q} \setminus \{0\}$ the equation

$$x_1^2 \cdot a_1 + \cdots + x_n^2 \cdot a_n \;=\; 0$$

has a non-trivial integral solution if and only if it has a non-trivial real solution and if for each prime p the equation has also has a non-trivial solution over the p-adic numbers $\mathbb{Q}_p$. A proof of the Hasse-Minkowski theorem is given in [**Ser1973**, Chapter IV Theorem 8], [**BSh1966**, Section 7.1] or [**Neuk1999**, p. 385]. ■

Example. Using the parity we had just seen that the intersection forms of the closed oriented 4-dimensional smooth manifolds $M = S^2 \times S^2$ and $N = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, even up to sign. Now we consider the closed oriented 4-dimensional smooth manifolds that we obtain from M and N by performing the connected sum with $\mathbb{C}\mathrm{P}^2$. In other words, we are considering $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$. It follows from the Intersection Form-Additivity Proposition 224.18 that the intersection forms of these two smooth manifolds are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These two forms are isometric. This can be seen as follows:

- Either through a direct calculation, see Exercise 226.17.
- One cannot ice that both forms are non-singular, of signature $+1$ and odd. Thus it follows from the Indefinite Forms Theorem 226.28 that the forms are isometric.

Note that this discussion shows that we cannot distinguish $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ using their intersection forms. In fact, it turns out that there is a good reason why we have a hard time distinguishing them: On page **??** we will see that it follows from the Odd Form–Connected Sum Proposition **??** together with the by the Projective Space–Connected Sum Proposition 238.10 that the two smooth manifolds $(S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ are in fact diffeomorphic. □

As an immediate corollary to the Indefinite Forms Theorem 226.28 we obtain the classification of non-singular *indefinite odd* forms over $\mathbb{Z}$. More precisely, we have the following corollary:

Corollary 226.29. (Indefinite Odd Forms Corollary) If φ is a non-singular *indefinite odd* symmetric form over $\mathbb{Z}$, then

$$\varphi \cong b^+(\varphi) \cdot (1) \oplus b^-(\varphi) \cdot (-1).$$

Proof. Let φ be a non-singular indefinite odd symmetric form. It follows from the Signature-Eigenvalues Proposition 226.11 that the rank, signature and parity of the two non-singular indefinite odd forms in the statement of the corollary agree. So these two forms are isometric by the Indefinite Forms Theorem 226.28. ■

Now we turn to the study of *even* forms over $\mathbb{Z}$. We have the following, perhaps somewhat surprising proposition:

Proposition 226.30. (Even Forms Proposition) Let φ be a non-singular symmetric form over $\mathbb{Z}$. If φ is even, then $\mathrm{sign}(\varphi) \equiv 0 \bmod 8$.

Example. As a reality check, the two non-singular even symmetric forms that we are acquainted with are $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8. The signatures are 0 and 8. These calculations are consistent with the Even Forms Proposition 226.30. □

Proof. Given a non-singular symmetric form (W, ψ) over $\mathbb{Z}$ we say that an element $v \in W$ is *characteristic* if for all $w \in W$ we have

$$\psi(v, w) \equiv \psi(w, w) \mod 2.$$

We make the following observations:

(1) It follows immediately from the definitions that (W, ψ) is even if and only if 0 is a characteristic element.
(2) If $\psi = k \cdot (1) \oplus l \cdot (-1)$, then it follows from a straightforward calculation, as in the proof of the Parity Lemma 226.25, that an element $u \in \mathbb{Z}^{k+l}$ is characteristic if and only if all its entries are odd.[2445]
(3) If v is a characteristic vector for φ and v' is a characteristic element for φ', then (v, v') is a characteristic element for $\varphi \oplus \varphi'$.
(4) Isometries preserve characteristic vectors.

It follows from Observation (1) that the following claim implies in particular the desired statement of the Even Forms Proposition 226.30:

Claim. Let (V, φ) be a non-singular symmetric form over $\mathbb{Z}$. If $v \in V$ is a characteristic element, then
$$\theta(v, v) \equiv \operatorname{sign}(\varphi) \mod 8.$$

Proof. Let $v \in V$ be a characteristic element of (V, φ). It follows easily from Observation (3) that $(v, 1, 1)$ is a characteristic element of the form $(V \oplus \mathbb{Z} \oplus \mathbb{Z}, \varphi \oplus (1) \oplus (-1))$. Note that, regardless of φ, the form $\varphi \oplus (1) \oplus (-1)$ is indefinite and odd. Thus it follows from the Indefinite Odd Forms Corollary 226.29 together with the Signature Additivity Proposition 226.13 that there exists an isometry[2446]
$$\Phi\colon \varphi \oplus (1) \oplus (-1) \xrightarrow{\cong} (b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1).$$
We write $u \coloneqq \Phi(v, 1, 1)$. Since Φ is an isometry we obtain from Observation (4) that u is a characteristic element for the form on the right-hand side. The above Observation (2) tells us that each entry of u is an odd integer. Now we calculate that

$$\begin{aligned}
\theta(v, v) &\overset{\text{straightforward calculation}}{=} (\varphi \oplus (1) \oplus (-1))((v, 1, 1), (v, 1, 1)) \overset{\text{since } \Phi \text{ is an isometry}}{=} \\
&= ((b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1))(u, u) \\
&= \sum_{i=1}^{b^+(\varphi)+1} (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) \cdot (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 \\
&\equiv \sum_{i=1}^{b^+(\varphi)+1} 1 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) = (b^+(\varphi) + 1) - (b^-(\varphi) + 1) = \operatorname{sign}(\varphi) \mod 8.
\end{aligned}$$

this equality holds modulo 8, since for each odd number k we have $k^2 \equiv 1 \mod 8$[2447]

■

Now we can provide the classification of non-singular *indefinite even* symmetric forms.

Corollary 226.31. (Indefinite Even Forms Corollary) If φ is a non-singular *indefinite even* symmetric form over $\mathbb{Z}$, then
$$\varphi \cong \frac{\operatorname{sign}(\varphi)}{8} \cdot E_8 \oplus \frac{\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|}{2} \cdot H.$$

[2445] The key observation is that for $v \in \mathbb{Z}$ we have $v \cdot w \equiv \pm w \cdot w \mod 2$ for all $w \in \mathbb{Z}$.
[2446] Note that both sides are non-singular indefinite odd forms, regardless of what non-singular symmetric form φ we actually consider.
[2447] Why is it true that for each odd number k we have $k^2 \equiv 1 \mod 8$?

Proof. First let φ be a non-singular even symmetric form over $\mathbb{Z}$. Note that it follows from the Even Forms Proposition 226.30 that $\operatorname{sign}(\varphi)$ is indeed divisible by 8. Furthermore it follows from the equalities

$$\begin{array}{rcll} \operatorname{rank}(\varphi) & = & b^+(\varphi) + b^-(\varphi) & \text{(from Sylvester's Theorem 226.12 (1),} \\ \operatorname{sign}(\varphi) & = & b^+(\varphi) - b^-(\varphi) & \text{(from the definition)} \end{array}$$

that $\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|$ is divisible by 2. So it makes sense to write down the right-hand side.

Next we recall that the two non-singular symmetric forms E_8 and H are even, that by the E_8-Lemma 226.23 we know that $\operatorname{rank}(E_8) = \operatorname{sign}(E_8) = 8$ and that $\operatorname{rank}(H) = 2$ and $\operatorname{sign}(H) = 0$. Now assume that φ is furthermore indefinite. It follows from the above observation regarding E_8 and H together with the Signature Additivity Proposition 226.13 and the Parity Lemma 226.25 (3) that the rank, signature and parity of the two non-singular indefinite even forms in the statement of the corollary agree. So these two forms are isometric by the Indefinite Forms Theorem 226.28. ■

226.4.2. **Definite forms over $\mathbb{Z}$.** Now we turn to the study of non-singular *definite* symmetric forms over $\mathbb{Z}$. We start out with some more examples of such forms:

Lemma 226.32. (Γ-Forms Lemma) We denote by $\mathfrak{e}_1, \dots, \mathfrak{e}_{4m}$ the standard basis for $\mathbb{Z}^{4m}$. We denote by Γ_{4m} the submodule of $\mathbb{R}^{4m}$ spanned by the vectors $\{\mathfrak{e}_i + \mathfrak{e}_j\}_{i,j=1,\dots,4m}$ and $\frac{1}{2}(\mathfrak{e}_1 + \dots + \mathfrak{e}_{4m})$.

(1) Γ_{4m} is a free abelian group of rank $4m$.
(2) The restriction of the usual form $\langle\,,\,\rangle$ on $\mathbb{Z}^{4m}$ to Γ_{4m} is a non-singular positive-definite symmetric form.
(3) The form Γ_{4m} is odd if and only if m is odd.
(4) The form Γ_8 is isometric to E_8.
(5) The form Γ_{16} is not isometric to $E_8 \oplus E_8$.

Proof. The lemma is proved in [**MH1973**, Lemmas II.6.1 and 6.2] and [**Ser1973**, p. 51]. ■

Example. It turns out that there are many more *definite* symmetric forms than indefinite symmetric forms and that there is no known classification of such forms. For example, by [**MH1973**, p. 28] and [**Ser1973**, p. 55] we have the following table:[2448]

dimension	8	16	24	32	40
number of isometry types of non-singular positive-definite *even* symmetric forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

Here in dimension 8, up to isometry, the unique example is $\Gamma_8 = E_8$ whereas in dimension 16 the unique examples are $\Gamma_8 \oplus \Gamma_8 = E_8 \oplus E_8$ and Γ_{16}. □

Finally we turn to non-singular positive-definite odd symmetric forms:

Example. A form $\varphi\colon V \times V \to \mathbb{Z}$ is called *standard* if it can be represented by the identity matrix. In Exercise 226.23 we will show that for every $n \in \mathbb{N}$ the form $E_8 \oplus n \cdot (1)$ is not

[2448]For a non-singular positive-definite form over $\mathbb{Z}$ we have $b^+(\varphi) = \operatorname{rank}(\varphi)$ and $b^-(\varphi) = 0$, thus $\operatorname{rank}(\varphi) = \operatorname{sign}(\varphi)$. If φ is even, then it follows from the Even Forms Proposition 226.30 that this rank is divisible by 8.

standard. The non-singular positive-definite odd non-standard form of lowest rank, which does not decompose as a direct sum, is given by the following (12×12)-matrix:

$$D_{12}^{+} := \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

We refer to [**CSl1982**, Table 1], [**CSl1999**, p. 416], [**Kin2003**, Table 2] and

`http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/D12p.html`

for details and for lists of isometry classes of non-singular positive-definite *odd* symmetric forms of a given rank. □

226.5. **Appendix: Forms over $\mathbb{F}_2$.** Considering the power of the intersection form of a closed *oriented* even-dimensional topological manifold it is natural to ask, whether one can obtain similar results for closed (possibly) *non-orientable* even-dimensional topological manifolds. In fact we have the following definition:

Definition 226.33. Let M be a compact $2n$-dimensional topological manifold M (which is not necessarily orientable). Recall that it follows from the $\mathbb{F}_2$-Orientability Proposition 191.22 together with the Fundamental Class-Proposition 194.2 that we have an $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_{2n}(M, \partial M; \mathbb{F}_2)$. Given $k \in \mathbb{N}_0$ we denote by

$$\mathrm{PD}_M^{\mathbb{F}_2} \colon \mathrm{H}_k(M; \mathbb{F}_2) \to \mathrm{H}^{2n-k}(M; \mathbb{F}_2)$$

the inverse of the isomorphism given by the Poincaré Duality Theorem 210.1. Similar to the discussion in Definition 224.12 we can now define the $\mathbb{F}_2$-valued intersection form as follows:

$$\begin{aligned} Q_M^{\mathbb{F}_2} \colon \mathrm{H}_n(M; \mathbb{F}_2) \times \mathrm{H}_n(M; \mathbb{F}_2) &\to \mathbb{F}_2 \\ (a, b) &\mapsto \Big\langle \underbrace{\mathrm{PD}_M^{\mathbb{F}_2}(a) \cup \mathrm{PD}_M^{\mathbb{F}_2}(b)}_{\in \mathrm{H}^{2n}(M, \partial M; \mathbb{F}_2)}, [M]_{\mathbb{F}_2} \Big\rangle. \end{aligned}$$

The above definition is completely analogous to the definition of the $\mathbb{Z}$-valued intersection form in Definition 222.3. Not surprisingly the $\mathbb{F}_2$-valued intersection forms has many properties in common with the $\mathbb{Z}$-valued intersection form:

Proposition 226.34. ($\mathbb{F}_2$-Intersection Form Proposition)

(1) The $\mathbb{F}_2$-valued intersection form $Q_M^{\mathbb{F}_2}$ of a compact even dimensional topological manifold M is symmetric.

(2) The $\mathbb{F}_2$-valued intersection form $Q_M^{\mathbb{F}_2}$ of a *closed* even dimensional topological manifold M is non-singular.
(3) The $\mathbb{F}_2$-valued intersection forms of homotopy equivalent closed even-dimensional topological manifolds are isometric.

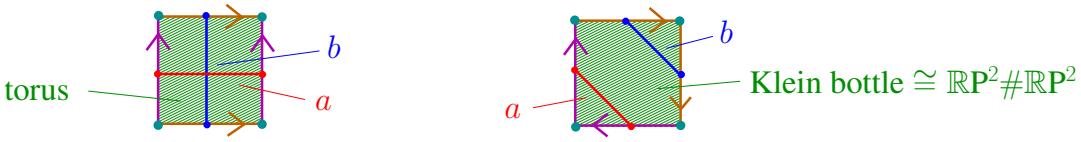
Proof. The proof is a very mild variation on the proof, of the Intersection Form Proposition 224.13 and the Intersection Form–Homeomorphism Lemma 224.15. The biggest difference is that we need to replace the Universal Coefficient-H*-Theorem 201.23 by the Universal Coefficient-H*–Field Theorem 202.2. We leave the details to the reader. ■

Proposition 226.35. ($\mathbb{F}_2$-Intersection Form–Non Singularity Proposition) The $\mathbb{F}_2$-valued intersection form $Q_M^{\mathbb{F}_2}$ of a *closed* even dimensional topological manifold M is non-singular.

Proof. The proof is almost identical to the proof of the Intersection Form–Non Singularity Proposition 224.14. Again all details are left to the reader. ■

Examples. In the figure below we show the following:

- We see the torus $T = S^1 \times S^1$ together with a basis a, b of $\mathrm{H}_1(S^1 \times S^1; \mathbb{F}_2)$.
- We see the Klein bottle K together with a basis a, b of $\mathrm{H}_1(K; \mathbb{F}_2)$.



There are many ways (e.g. one could use the $\mathbb{F}_2$-analog of the Cup Product–Intersection Number Theorem 220.10) to see that with respect to these bases we have[2449]

$$Q_T^{\mathbb{F}_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Q_K^{\mathbb{F}_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ □

The question arises, whether we can classify non-singular symmetric forms over $\mathbb{F}_2$. For better or worse it turns out that there is a straightforward classification. To state the classification we need the following definition which is inspired by our Definition 226.24:

Definition 226.36. We say that a symmetric form $\lambda\colon V \times V \to \mathbb{F}_2$ over $\mathbb{F}_2$ is even if $\lambda(v, v) = 0 \in \mathbb{F}_2$ for all $v \in V$. Otherwise we say that the symmetric form is odd. We call the property of a form to be even or to be odd the parity of the form.

Example. As we had just seen, the $\mathbb{F}_2$-valued intersection form of the torus $S^1 \times S^1$ is even whereas the $\mathbb{F}_2$-valued intersection form of the Klein bottle $K \cong \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is odd. Thus we have found a new, and particularly complicated, proof that these two 2-dimensional smooth manifolds are not homeomorphic. □

The following proposition gives a complete classification of non-singular symmetric forms over $\mathbb{F}_2$ up to isometry:

[2449] There are many other approaches to do this calculation. For example for $S^1 \times S^1$ one could adopt the discussion on page 4331 and for the Klein bottle one could use that on page 1237 we see that K is homeomorphic to $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ and the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 together with the obvious analog of the Intersection Form-Additivity Proposition 224.18.

Proposition 226.37. (Non-Singular Forms-over-$\mathbb{F}_2$-Proposition) Let (V,κ) and (W,λ) be two symmetric forms over $\mathbb{F}_2$. Then

$$(V,\kappa) \text{ and } (W,\lambda) \text{ are isometric} \iff \begin{array}{rcl} \dim_{\mathbb{F}_2}(V) & = & \dim_{\mathbb{F}_2}(W), \\ \nu(\kappa) & = & \nu(\lambda), \\ \text{parity of } \kappa & = & \text{parity of } \lambda. \end{array}$$

Sketch of proof. First note that one can show by hand (see Exercise 226.17 that the forms on $\mathbb{F}_2^3$ that are defined by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are isometric. Once one has verified this, the proof of the proposition is quite similar to the proofs of the Anti-Symmetric Form Proposition 226.2, the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 and Sylvester's Theorem 226.12. We will fill in the details in Exercise 226.28.

For peace of mind we point out that the proposition is proved in [**Hopk2015**, Theorem 2.7]. It can also be deduced from [**Szy1997**, Corollary 6.3.1]. ■

The proposition shows that, unfortunately, we extract only very little information from the $\mathbb{F}_2$-valued intersection form.

Remark. Let V be a finite-dimensional vector space over $\mathbb{F}_2$ and let $\langle\,,\,\rangle\colon V\times V\to\mathbb{F}_2$ be an even form. It follows from the Non-Singular Forms-over-$\mathbb{F}_2$-Proposition 226.37 that V admits a *symplectic basis*, i.e. there exists a basis $a_1,\dots,a_k,b_1,\dots,b_k$ for V such that the following conditions are satisfied:

(1) For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$.
(2) For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,b_j\rangle=\delta_{ij}$.

This observation will come in handy later on. □

226.6. Exercises for Chapter 226 .

Exercise 226.1. Let $n\in\mathbb{N}$ be even. What are the possible Euler characteristics of closed, possibly non-orientable, connected n-dimensional smooth manifolds?

Exercise 226.2. Let φ be a symmetric form on a finite-dimensional real vector space V. Prove the following statement: If φ is positive-definite, then φ is non-singular.

Exercise 226.3. Let R be a commutative ring. As in Definition 2.6 we denote by $R^\times$ the multiplicative units of R. Let (V,ψ) be an (anti-) symmetric form over R of rank n. We pick a basis $v_1,\dots,v_n$ of V. We refer to

$$\det(\psi) \;:=\; \det(\psi(v_i,v_j)_{i,j=1,\dots,n}) \;\in\; R^\times/\{a^2\,|\,a\in R^*\}$$

as the *determinant of the form* (V,ψ). It follows from the Form–Matrix-Lemma 224.5 (1b), (1d) and (2) that $\det(\psi)\in R^\times/\{a^2\,|\,a\in R^*\}$ is a well-defined isometry invariant.

(a) Give examples of closed oriented 4-dimensional topological manifolds M and N with isomorphic second homology groups, where the intersections are distinguished by the determinant of the intersection form.

(b) Show that for non-singular forms over $\mathbb{Z}$ the above invariant is determined by the rank and the signature.

Exercise 226.4. On page 4372 we classified the non-singular symmetric forms on $\mathbb{R}^3$ up to isometry and for each isometry type we gave a picture of the "unit-sphere"

$$S(\varphi) \;:=\; \{v \in \mathbb{R}^3 \,|\, \varphi(v,v) = 1\}.$$

Perform the same for singular symmetric forms on $\mathbb{R}^3$.

Exercise 226.5. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a symmetric form over R. We introduce the following concepts:

- We say that φ is *metabolic* if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$ and such that V' is isomorphic to V''.
- We say that φ is *hyperbolic* if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$, $\varphi|_{V''} \equiv 0$ and such that V' is isomorphic to V''.

Now we can formulate our problems:

(a) Show that not every non-singular metabolic form is also hyperbolic.
(b) Show that if φ is non-singular metabolic symmetric form over $\mathbb{R}$, then $\operatorname{sign}(\varphi) = 0$.
(c) Show that if φ is a hyperbolic symmetric form over $\mathbb{R}$, then $\operatorname{sign}(\varphi) = 0$.

Exercise 226.6. Recall that according to Definition 2.1 a *monoid* $(M,+)$ consists of a set M together with a map

$$\begin{array}{rcl} +\colon M \times M & \to & M \\ (m,n) & \mapsto & m+n \end{array}$$

which is associative and which admits a neutral element. If $(M,+)$ is abelian, then we consider

$$(M \times M)/\!\sim \qquad \begin{array}{l} \text{where } (m,n) \sim (m',n') \text{ if there exists} \\ \text{an } r \in M \text{ with } m+n'+r = m'+n+r. \end{array}$$

(a) Show that "$\sim$" defines an equivalence relation on M.
(b) Show that $((M \times M)/\!\sim, +)$ is an abelian group. This group is called the *Grothendieck group* corresponding to the monoid $(M,+)$ and this construction of turning an abelian monoid into a group is called the *Grothendieck construction*.

Exercise 226.7. Let R be a commutative domain. In this exercise we will see that non-singular symmetric forms over R give rise to a group. We introduce the following notation:

(1) Note that the set of isometry types of symmetric non-singular forms $\varphi\colon V \times V \to R$ together with the direct sum operation is a monoid $M(R)$. We consider the corresponding Grothendieck group $G(R) = M(R)/\!\sim$ as introduced in Exercise 226.6.
(2) As in Exercise 226.5 we say that a form $\varphi\colon V \times V \to R$ is *hyperbolic* if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$, $\varphi|_{V''} \equiv 0$ and such that V' is isomorphic to V''.
(3) We define the *Witt group* $W(R)$ as the quotient of the group $G(R)$ by the subgroup generated by the hyperbolic forms.

Now we can formulate the following problems:

(a) Let $\varphi \in M(R)$. Show that there exists an isometry between

$$\begin{pmatrix} \varphi & 0 & 0 \\ 0 & -\varphi & 0 \\ 0 & 0 & \varphi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \varphi & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix}.$$

(b) Let (V, φ) and (W, ψ) be two hyperbolic forms. Show that if $\operatorname{rank}(V) \leq \operatorname{rank}(W)$, then (W, ψ) is isometric to $(V, \varphi) \oplus (V', \varphi')$, where (V', φ') is also a hyperbolic form.

(c) Given $\varphi, \psi \in M(R)$ we write $\varphi \sim \psi$ if there exists a form $\theta \in M(R)$ and a hyperbolic form $\eta \in M(R)$ such that $\varphi \oplus (-\psi) \oplus \theta \cong \eta \oplus \theta$. Show that the obvious map

$$\begin{array}{rcl} M(R)/\!\sim & \to & W(R) \\ [\varphi] & \mapsto & [(\varphi, 0)] \end{array}$$

is an isomorphism of monoids.

(d) Show that the maps

$$\begin{array}{rcl} W(\mathbb{R}) & \to & \mathbb{Z} \\ \varphi & \mapsto & \operatorname{sign}(\varphi) \end{array} \quad \text{and} \quad \begin{array}{rcl} W(\mathbb{Z}) & \to & \mathbb{Z} \\ \varphi & \mapsto & \operatorname{sign}(\varphi \otimes \mathbb{R}) \end{array}$$

are isomorphisms.

(e) Show that $W(\mathbb{Q})$ admits an epimorphism onto $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^\infty$.
Hint. Consider determinants of matrices representing a given form.
Remark. The group $W(\mathbb{Q})$ is computed in [**MH1973**, Chapter IV.2] or alternatively in [**Lam2005**, Chapter VI.4].

Exercise 226.8. Show that for any non-singular symmetric form (V, φ) over $\mathbb{R}$ we have $\operatorname{sign}(\varphi) \equiv n \bmod 2$.

Exercise 226.9. Let M be a closed oriented 3-dimensional smooth manifold. Show that $\operatorname{sign}(M \times [0, 1]) = 0$.

Exercise 226.10. For which $k \in \mathbb{Z}$ does there exist a closed oriented 3-dimensional smooth manifold N with $\operatorname{sign}(S^1 \times N) = k$?

Exercise 226.11. Let M be a compact oriented $4k$-dimensional topological manifold and let W be the union of some components of ∂M. As in Definition 54.8 we refer to

$$\mathrm{D}_W M \;:=\; (M \times \{1\}) \cup_{W \times \{1\} = W \times \{2\}} ((-M) \times \{2\})$$

as the *double of M along W*. By the Doubling-Manifold Lemma 54.10 and the Double-Orientability Lemma 192.6 we know that $\mathrm{D}_W M$ is a compact orientable topological manifold. Show that the signature of $\mathrm{D}_W M$ is zero.

Exercise 226.12. Let M be a closed non-orientable $4k$-dimensional smooth manifold. Let $p\colon \widetilde{M} \to M$ be the orientation covering that we introduced in Definition 95.9. Show that $\operatorname{sign}(\widetilde{M}) = 0$.

Exercise 226.13. We consider the form

$$\begin{array}{rcl} \mathbb{Z}^2 \times \mathbb{Z}^2 & \to & \mathbb{Z} \\ (v, w) & \mapsto & v^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w. \end{array}$$

What is the set of "squares" $\varphi(v,v) \in \mathbb{Z}$?

Exercise 226.14.

(a) Explain how to modify the proof of the Anti-Symmetric Form Proposition 226.2 so that it also holds for anti-symmetric forms over $\mathbb{Z}$.

(b) Explain why the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 cannot be generalized to symmetric forms over $\mathbb{Z}$.

Exercise 226.15. Let $A \in \mathrm{GL}(n,\mathbb{R})$ be a symmetric matrix. Show that

$$b^+(\varphi(A)) \;=\; \max\left\{k \in \mathbb{N}_0 \;\middle|\; \begin{array}{c}\text{there exists a } k\text{-dimensional vector subspace } V \\ \text{of } \mathbb{R}^n \text{ with } v^{\mathrm{T}}Av > 0 \text{ for all } v \in V \cap S^{n-1}\end{array}\right\}.$$

Exercise 226.16. Show that the map

$$\begin{array}{rcl}\{A \in \mathrm{GL}(n,\mathbb{R}) \,|\, A = A^{\mathrm{T}}\} & \to & \mathbb{Z} \\ A & \mapsto & \mathrm{sign}(A)\end{array}$$

is continuous.

Hint. You can work either with the definition of the signature (or better, with the reformulation given in Exercise 226.15) or you could use the Zeros Continuity Proposition 9.22.

Exercise 226.17. Show "by hand" that the forms on $\mathbb{Z}^3$ defined by the matrices

$$A \;=\; \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad B \;=\; \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

are isometric.

Exercise 226.18. Show "by hand" that the forms on $\mathbb{Z}^2$ defined by the matrices

$$A \;=\; \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad B \;=\; \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

are isometric.

Exercise 226.19. Provide a complete classification of all non-singular symmetric forms over $\mathbb{Z}$ with signature zero.

Exercise 226.20. Let (V,φ) be a symmetric form over $\mathbb{Z}$.

(a) Show that φ is non-degenerate if and only if $\varphi \otimes \mathbb{R}$ is non-singular.

In Definition 226.6 we introduced the notion of a positive-definite (negative-definite, indefinite) form over $\mathbb{Z}$ and over $\mathbb{R}$.

(b) Show that φ is positive-definite if and only if $\varphi \otimes \mathbb{R}$ is positive-definite.

(c) Show that φ is negative-definite if and only if $\varphi \otimes \mathbb{R}$ is negative-definite.

(d) Show that φ is indefinite if and only if $\varphi \otimes \mathbb{R}$ is indefinite.

Exercise 226.21. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form over $\mathbb{Z}$. Show that there exists a non-degenerate form $\psi\colon W \times W \to \mathbb{Z}$ such that φ is isometric to the direct sum of ψ with a form that is identically zero.

Exercise 226.22. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form. As in the proof of the Even Forms Proposition 226.30 we say that an element $v \in W$ is *characteristic* if for all $w \in W$ we have

$$\psi(v,w) \;\equiv\; \psi(w,w) \mod 2.$$

Show that every non-singular indefinite form $\varphi\colon V \times V \to \mathbb{Z}$ admits a characteristic element.

Exercise 226.23. Let $n \in \mathbb{N}$. Show that the non-singular positive-definite odd symmetric forms $\varphi_1 := E_8 \oplus n \cdot (1)$ and $\varphi_2 := (8+n) \cdot (1)$ are not isometric.
Hint. Count the number of elements of $\mathbb{Z}^{n+1}$ that square to 1 with respect to the two forms.

Exercise 226.24. Let M be a compact oriented $4n$-dimensional topological manifold. Show that if M is amphichiral, i.e. if it admits an orientation-reversing homeomorphism, then $\operatorname{sign}(M) = 0$.

Exercise 226.25. Let M be a compact oriented topological manifold. Recall that we say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*. For which $k, l \in \mathbb{N}_0$ is the connected sum $k \cdot \mathbb{C}\mathrm{P}^2 \# l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ chiral? More precisely:
(a) For the values that you exclude, show that $k \cdot \mathbb{C}\mathrm{P}^2 \# l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is not chiral,
(b) For all the other values show explicitly that $k \cdot \mathbb{C}\mathrm{P}^2 \# l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is chiral.

Exercise 226.26. We consider the topological manifold $S^4 \times S^4$. Provide an example of a connected simply connected topological manifold X such that for every $k \in \mathbb{N}_0$ we have an isomorphism $\mathrm{H}_k(S^4 \times S^4) \cong \mathrm{H}_k(X)$, but such that $S^4 \times S^4$ and X are not homotopy equivalent.

Exercise 226.27. Let M and N be two closed orientable connected 4-dimensional topological manifolds such that the intersection forms Q_M and Q_N over $\mathbb{Z}$ are isomorphic. Are the intersection forms over $\mathbb{F}_2$ also isomorphic?

Exercise 226.28. Let (V, κ) and (W, λ) be two symmetric forms over $\mathbb{F}_2$. Show that

$$(V,\kappa) \text{ and } (W,\lambda) \text{ are isometric} \iff \begin{array}{rcl} \dim_{\mathbb{F}_2}(V) & = & \dim_{\mathbb{F}_2}(W), \\ \nu(\kappa) & = & \nu(\lambda), \\ \text{parity of } \kappa & = & \text{parity of } \lambda. \end{array}$$

Exercise 226.29. Let X be a closed connected non-empty 2-dimensional topological manifold. Show (without using the Surface Classification Theorem 101.16) that there exists a $g \in \mathbb{N}_0$ such that the cohomology ring $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to the cohomology ring $(\mathrm{H}^*(\Sigma_g;\mathbb{Z}), \cup)$, where Σ_g stands for the surface of genus g.
Remark. This exercise solves the algebraic analog of the Surface Classification Theorem 101.16.

Exercise 226.30.
(a) Does the $\mathbb{F}_2$-intersection form classify closed connected non-empty 2-dimensional topological manifolds up to homeomorphism?
(b) Does the $\mathbb{F}_2$-intersection form classify compact connected non-empty 2-dimensional topological manifolds up to homeomorphism?

Remark. This exercise builds on the Surface Classification Theorem 101.16.

CHAPTER 227

Signatures: Multiplicativity and additivity

In this chapter we will state and prove two results about signatures. We will see that they are multiplicative and additive.

This chapter is organized as follows:

- In Section 227.1 we will recall the definition of the signature of a topological manifold and we will summarize the results of this chapter.
- In Section 227.2 we will state and prove the Signature–Product Theorem 227.4.
- In Section 227.3 we will state the Novikov Additivity Theorem 227.11.
- In Section 227.4 we will prove the Novikov Additivity Theorem 227.11.
- In Section 227.5 we will provide exercises for Chapter 227.

In the next Chapter 228 we will study the intersection form of 4-dimensional manifolds.

227.1. **Summary of results.** Let $k \geq 1$ and let M be a compact oriented $2k$-dimensional topological manifold. Recall that in Definition 224.12 we introduced the intersection form

$$\begin{aligned} Q_M\colon \mathrm{FH}_k(M) \times \mathrm{FH}_k(M) &\to \mathbb{Z} \\ (a,b) &\mapsto \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{4k}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{aligned}$$

of M, which is $(-1)^k$-symmetric. In particular, if M is actually $4k$-dimensional, then the intersection form is symmetric. This allowed us in Definition 226.15 to introduce the signature

$$\mathrm{sign}(M) \coloneqq \mathrm{sign}(Q_M \otimes \mathbb{R}) \in \mathbb{Z}$$

of the manifold.

The signature turns out to be one of the most central invariants of compact oriented $4k$-dimensional smooth and topological manifolds. In particular, rather weirdly, it shares many properties with the Euler characteristic. In this chapter we will prove the following two theorems:

(1) In the Signature–Product Theorem 227.4 we will see that (similar to the Product-χ-Proposition 156.12) the signature is multiplicative under products.
(2) In the Novikov Additivity Theorem 227.11 we will see that (similar to the Union-χ-Proposition 156.10) signatures are additive under gluings.

Later we will also add the following theorem to our collection:

(3) In the Signature–Covering Theorem 281.16 we will see that (similar to the Covering-χ-Proposition 156.13) for *closed* smooth manifolds the signature is multiplicative under finite coverings.

By definition the signature of a compact oriented $4k$-dimensional topological manifold is defined in terms of the intersection form on homology. But as we saw above, if one strips away the Poincaré duality isomorphism, the intersection form is defined via the cup product on cohomology.

For the purpose of this chapter it is convenient to completely bypass homology (despite our emotional attachment to it) and to work with cohomology throughout. This leads us to the following notation:

Notation 227.1. Given a compact oriented $2m$-dimensional topological manifold M we consider the form

$$\begin{array}{rcl} I_M\colon \mathrm{H}^m(M,\partial M;\mathbb{R}) \times \mathrm{H}^m(M,\partial M;\mathbb{R}) & \to & \mathbb{R} \\ (\varphi,\psi) & \mapsto & \langle \varphi\cup\psi, [M]\rangle. \end{array}$$

The following lemma shows that this new form I_M is isometric to $Q_M\otimes\mathbb{R}$:

Lemma 227.2. (Intersection Form Reinterpretation Lemma) Let M be a compact oriented $2m$-dimensional topological manifold. The form $Q_M\otimes\mathbb{R}$ is isometric to the form

$$\begin{array}{rcl} I_M\colon \mathrm{H}^m(M,\partial M;\mathbb{R}) \times \mathrm{H}^m(M,\partial M;\mathbb{R}) & \to & \mathbb{R} \\ (\varphi,\psi) & \mapsto & \langle \varphi\cup\psi, [M]\rangle. \end{array}$$

Proof. Let M be a compact oriented $2m$-dimensional topological manifold. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_m(M)\otimes\mathbb{R} \times \mathrm{H}_m(M)\otimes\mathbb{R} & \xrightarrow{\quad Q_M\otimes\mathbb{R}\quad} & \mathbb{R} \\ \cong\big\downarrow{\scriptstyle \mu\times\mu} & & \\ \mathrm{H}_m(M;\mathbb{R}) \times \mathrm{H}_m(M;\mathbb{R}) & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \cong\big\downarrow{\scriptstyle \mathrm{PD}_M\times\mathrm{PD}_M} & & \\ \mathrm{H}^m(M,\partial M;\mathbb{R}) \times \mathrm{H}^m(M,\partial M;\mathbb{R}) & \xrightarrow{\quad (\varphi,\psi)\mapsto\langle\varphi\cup\psi,[M]\rangle\quad} & \mathbb{R}. \end{array}$$

We make the following clarifications and observations:

(1) The top left vertical maps are the isomorphisms μ given by the Torsion–Free Coefficients Proposition 161.8.
(2) The bottom left vertical maps are the isomorphisms given by the Poincaré Duality Theorem 210.1
(3) It follows easily from the definition of $Q_M\otimes\mathbb{R}$, the Cup Product-Coefficients Lemma 204.9 and the Kronecker Pairing–Natural Lemma 199.9 (2) that the diagram commutes.

It thus follows that the form at the top is isometric to the form at the bottom. ∎

227.2. **Multiplicativity of the signature.** Recall that we introduced the signature of a compact oriented $4k$-dimensional topological manifold. We now extend this definition to topological manifolds of other dimensions in the most unimaginative way possible:

Definition 227.3. Let X be a compact oriented topological manifold. If $\dim(X)$ is not divisible by 4, then we set $\mathrm{sign}(X) := 0$.

With this definition we can formulate the first of the two main theorems of this chapter:

Theorem 227.4. (Signature–Product Theorem) For any two compact oriented topological manifolds M and N we have

$$\operatorname{sign}(M \times N) \;=\; \operatorname{sign}(M) \cdot \operatorname{sign}(N).$$

Remark. Later, in Appendix 281.2 we will discuss the related question whether the signature is multiplicative for fiber bundles. □

Example.

(1) Note that for any $k \in \mathbb{N}$ we have

$$\operatorname{sign}(S^{2k} \times S^{2k}) \underset{\uparrow}{=} \operatorname{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\uparrow}{=} 0 = 0 \cdot 0 = \operatorname{sign}(S^{2k}) \cdot \operatorname{sign}(S^{2k}).$$

see page 4331 see page 4374

This calculation is consistent with the Signature–Product Theorem 227.4.

(2) Let $g \in \mathbb{N}$. We denote by Σ_g the surface of genus g. We consider the closed oriented 4-dimensional smooth manifolds $M = \Sigma_g \times \Sigma_g$ and $N = 2g \cdot (S^1 \times S^3) \# 4g \cdot \mathbb{C}\mathrm{P}^2$. Using the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9, the Surface-H_*-Proposition 155.17, the Connected Sum-Invariants Proposition 224.17 and the Künneth Theorem 164.6 one can easily verify that the (co-) homology groups of M and N are all isomorphic. But it follows from the Signature–Product Theorem 227.4 that $\operatorname{sign}(M) = 0$ whereas it follows from the discussion on page 4373 and the Manifold Signature Additivity Lemma 226.17 that $\operatorname{sign}(N) = 4g$. □

Before we can turn to the proof of the Signature–Product Theorem 227.4 we need to upgrade our algebraic understanding of the signature.

Definition 227.5. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a symmetric form over R.

(1) We say that φ is metabolic if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$ and such that V' is isomorphic to V''.

(2) We say that φ is hyperbolic if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$, $\varphi|_{V''} \equiv 0$ and such that V' is isomorphic to V''.

Example.

(1) Let $\mathbb{F}$ be a field and let $\varphi\colon V \times V \to \mathbb{F}$ be a symmetric form over $\mathbb{F}$. It follows immediately from the definitions that

$$\varphi \text{ is metabolic} \iff \varphi \text{ is represented by a matrix of the form } \begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & Q \end{pmatrix}$$

$$\varphi \text{ is hyperbolic} \iff \varphi \text{ is represented by a matrix of the form } \begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & 0 \end{pmatrix}$$

where P and Q are square matrices.

(2) Clearly every hyperbolic form is metabolic. But the converse does not hold. For example the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ represents a form over $\mathbb{R}$ which is clearly metabolic. But one can show easily, e.g. using the Vanishing Signature Lemma 227.6 (2), that it is not hyperbolic. □

The following lemma gives in particular a solution to Exercise 226.5:

Lemma 227.6. (Vanishing Signature Lemma) Let $\varphi\colon V \times V \to \mathbb{R}$ be a symmetric form over $\mathbb{R}$.

(1) If φ is metabolic and non-singular, then $\operatorname{sign}(\varphi) = 0$.
(2) If φ is hyperbolic, then $\operatorname{sign}(\varphi) = 0$.

Proof. Let $\varphi\colon V \times V \to \mathbb{R}$ be a symmetric form over $\mathbb{R}$.

(1) We assume that φ is metabolic and non-singular. As we mentioned above, this implies that φ can be a represented by a matrix of the form

$$\begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & Q \end{pmatrix}$$

where P and Q are square matrices and where Q is a symmetric matrix. It follows from the hypothesis that φ is non-singular, together with the Non-Singularity Lemma 213.2, that $\det(P) \neq 0$. Now we consider the following three transformations

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & Q \end{pmatrix} \begin{pmatrix} (P^{\mathrm{T}})^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix},$$
$$\begin{pmatrix} \mathrm{id} & 0 \\ -\frac{1}{2}Q & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\frac{1}{2}Q \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix}.$$

It follows from the Form–Matrix-Lemma 224.5 (1b) that the three matrices to the right all define isometric forms. In particular they are all isometric to φ. Thus it follows from the Isometry Invariants Lemma 226.9 that the signatures agree. Considering the last matrix we obtain from the Signature-Eigenvalues Proposition 226.11 that the signature is zero.

(2) We assume that φ is hyperbolic. As we mentioned above, this implies that φ can be a represented by a matrix of the form

$$\begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & 0 \end{pmatrix}$$

where P is a square matrix. Elementary linear algebraic shows that there exist invertible square matrices A and B such that $A \cdot P \cdot B =: D$ is a diagonal matrix. Now we consider the following two transformations

$$\begin{pmatrix} 0 & B^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{\mathrm{T}} \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$$

and as in (1) we see that

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2D & 0 \\ 0 & -2D \end{pmatrix}.$$

It follows from the Form–Matrix-Lemma 224.5 (1b) that the two matrices to the right define isometric forms. Considering the last matrix we obtain from the Signature-Eigenvalues Proposition 226.11 that the signature is zero. ∎

Next we recall Definition 226.5:

Definition 227.7. Let (V, φ) be a form over a field $\mathbb{F}$. The vector subspace

$$\operatorname{rad}(\varphi) \coloneqq \{v \in V \mid \varphi(v, w) = 0 \text{ for all } w \in V\}$$

is called the radical of the form φ.

The following lemma gives a solution to Exercise 224.11.

Lemma 227.8. (Radical Summand Lemma) Let $\mathbb{F}$ be a field and let $\varphi\colon V \times V \to \mathbb{F}$ be a form. There exists a non-singular form (W, ψ) over $\mathbb{F}$ and an isometry

$$(\operatorname{rad}(\varphi), 0) \oplus (W, \psi) \cong (V, \varphi).$$

Sketch of proof. We write $R \coloneqq \operatorname{rad}(\varphi)$. First note that the map

$$\begin{array}{rcl} \overline{\varphi}\colon V/R \times V/R & \to & \mathbb{F} \\ ([v], [w]) & \mapsto & \varphi(v, w) \end{array}$$

is a well-defined form. One can easily verify that this form is non-degenerate, and thus non-singular, since we work over a field. Since we are dealing with vector spaces we know that the projection $V \to V/R$ admits a right-inverse $\Theta\colon V/R \to V$. Next note that, as in the proof of the Splitting Lemma 145.5 (1)⇒(2), one can easily verify that the map

$$\begin{array}{rcl} R \oplus V/R & \to & V \\ (r, w) & \mapsto & r + \Theta(w) \end{array}$$

is an isomorphism. Finally note that it follows easily from the definitions that this map actually defines an isometry $(\operatorname{rad}(\varphi), 0) \oplus (V/R, \overline{\varphi}) \to (V, \varphi)$. ∎

Next we introduce the tensor product of two forms over $\mathbb{R}$:

Lemma 227.9. (Form–Tensor Product Lemma) Let $\varphi\colon V \times V \to \mathbb{R}$ be a μ-symmetric form over $\mathbb{R}$ and let $\psi\colon W \times W \to \mathbb{R}$ be a ν-symmetric form over $\mathbb{R}$. There exists a unique $\mathbb{R}$-bilinear map[2450]

$$\varphi \otimes \psi\colon V \otimes_{\mathbb{R}} W \times V \otimes_{\mathbb{R}} W \to \mathbb{R}$$

such that for every $v, v' \in V$ and $w, w' \in W$ we have

$$((v \otimes w), (v' \otimes w')) \mapsto \varphi(v, v') \cdot \psi(w, w').$$

Furthermore, this map has the following properties:

(1) $\varphi \otimes \psi$ is $\mu \cdot \nu$-symmetric.
(2) If φ and ψ are non-singular, then so is $\varphi \otimes \psi$.
(3) (a) If $\mu = \nu = -1$, then we have $\operatorname{sign}(\varphi \otimes \psi) = 0$.
    (b) If $\mu = \nu = 1$, then we have $\operatorname{sign}(\varphi \otimes \psi) = \operatorname{sign}(\varphi) \cdot \operatorname{sign}(\psi)$.

Sketch of proof. First note that it follows from the analog of the Tensor Product-Universal Property Lemma 159.4 for tensor products of $\mathbb{R}$-vector spaces that there does exist a unique $\mathbb{R}$-bilinear map

$$\varphi \otimes \psi\colon V \otimes_{\mathbb{R}} W \times V \otimes_{\mathbb{R}} W \to \mathbb{R}$$

with the given property. We start out with the following observations:

(i) It follows from the Radical Summand Lemma 227.8 together with the Anti-Symmetric Form Proposition 226.2 that every anti-symmetric form over $\mathbb{R}$ is the direct sum of a zero form and of copies of the form that is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[2450] Technically speaking we will introduce the tensor product over a field much later in Definition 312.3.

(ii) It follows from the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4 that every symmetric form over $\mathbb{R}$ is the direct sum of a zero form and of forms that are represented by (1×1)-matrices (ϵ_i) with $\epsilon_i \in \{\pm 1\}$.

(iii) Let (V_i, φ_i), $i = 1, \dots, k$ be μ-symmetric forms over $\mathbb{R}$ and let (W_i, ψ_i), $i = 1, \dots, l$ be ν-symmetric forms over $\mathbb{R}$ such that we have isometries $(V, \varphi) \cong \bigoplus_{i=1}^{k} (V_i, \varphi_i)$ and $(W, \psi) \cong \bigoplus_{i=1}^{l} (W_i, \psi_i)$. It follows from the real analog of the Tensor-Direct Sum Lemma 159.15 that we have a natural isomorphism

$$\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{l} V_i \otimes W_j \;\to\; V \otimes W.$$

Furthermore it follows easily from the definition of the direct sum of forms, see Definition 224.7, that this isomorphism induces an isometry

$$\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{l} \varphi_i \otimes \psi_j \;\to\; \varphi \otimes \psi.$$

(iv) It follows easily from the definitions that we have an isometry $(\epsilon) \otimes (\eta) \cong (\epsilon \cdot \eta)$ and, more interestingly, that we have the following isometry:

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \otimes \begin{array}{c} \\ b_1 \\ b_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \cong \begin{array}{c} \\ a_1\otimes b_1 \\ a_1\otimes b_2 \\ a_2\otimes b_1 \\ a_2\otimes b_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1\otimes b_1 & a_1\otimes b_2 & a_2\otimes b_1 & a_2\otimes b_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Next note that the tensor product of a zero-form with any other form is clearly again a zero-form. Finally note that one can easily verify that we have an isometry

$$(\epsilon) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}.$$

We turn to the proofs of the statements:

(1) The symmetry statement is essentially clear.

(2) If φ and ψ are non-singular, then it follows easily from (i),(iii),(iv) that $\varphi \otimes \psi$ is also non-singular.

(3) (a) We now assume that $\mu = \nu = -1$. It follows easily from (i)-(iv) together with the Vanishing Signature Lemma 227.6 that $\operatorname{sign}(\varphi \otimes \psi) = 0$.

(b) We now assume that $\mu = \nu = 1$. It follows easily from (ii)-(iv) and the Signature-Eigenvalues Proposition 226.11 that

$$\begin{aligned} b^+(\varphi \otimes \psi) &= b^+(\varphi)\cdot b^+(\psi) + b^-(\varphi)\cdot b^-(\psi), \\ b^-(\varphi \otimes \psi) &= b^+(\varphi)\cdot b^-(\psi) + b^-(\varphi)\cdot b^+(\psi). \end{aligned}$$

It follows from this discussion and the definitions that

$$\begin{aligned} \operatorname{sign}(\varphi \otimes \psi) &= b^+(\varphi \otimes \psi) - b^-(\varphi \otimes \psi) \\ &= b^+(\varphi)\cdot b^+(\psi) + b^-(\varphi)\cdot b^-(\psi) - b^+(\varphi)\cdot b^-(\psi) - b^-(\varphi)\cdot b^+(\psi) \\ &= (b^+(\varphi) - b^-(\varphi))\cdot(b^+(\psi) - b^-(\psi)) \;=\; \operatorname{sign}(\varphi)\cdot\operatorname{sign}(\psi). \end{aligned}$$

■

We also need one major topological input in the proof of the Signature–Product Theorem 227.4, namely we will need the following special case of the Cross-Cup-Cap Theorem 207.1:

Theorem 227.10. (Manifold Cross-Cup-Cap Theorem) Let M and N be topological manifolds. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the natural projections. For any $\alpha \in \mathrm{H}^k(M, \partial M; \mathbb{Z})$, $\mu \in \mathrm{H}_m(M, \partial M; \mathbb{Z})$, $\beta \in \mathrm{H}^l(N, \partial N; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(N, \partial N; \mathbb{Z})$ we have the equality[2451]

$$(p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$$

in $\mathrm{H}_{m+n-k-l}(M \times N; \mathbb{Z})$.

Proof. First note that, as we discussed on page 3970, it follows from the Topological Manifold Product Proposition 150.12 and the Excisive Triad-Proposition 158.20 (2) that the triad $(M \times N, \partial M \times N, M \times \partial N)$ is excisive. Therefore the promised equality follows directly from the Cross-Cup-Cap Theorem 207.1. ■

Now we are finally ready to prove the Signature–Product Theorem 227.4:

Proof of the Signature–Product Theorem 227.4. Let M be a compact oriented m-dimensional topological manifold and let N be a compact oriented n-dimensional topological manifold. If $\dim(M) + \dim(N)$ is not divisible by 4, then also at least one of $\dim(M)$, $\dim(N)$ is not divisible by 4. Thus in this case the statement of the theorem is trivial. So we now assume that $m + n = \dim(M) + \dim(N)$ is divisible by 4, i.e. we can assume that there exists a $d \in \mathbb{N}_0$ with $m + n = 4d$. We adopt the following convention:

> Throughout this proof we work with real coefficients. We will generously reinterpret earlier results on $\mathbb{Z}$-coefficients in terms of $\mathbb{R}$-coefficients.

Throughout the proof we will need the following notation:

- Given a compact oriented k-dimensional topological manifold W and given $r, s \in \mathbb{N}_0$ with $r + s = k$ we consider the $\mathbb{R}$-bilinear pairing

$$\begin{array}{rcl} I_W^{r,s}\colon \mathrm{H}^r(W, \partial W) \times \mathrm{H}^s(W, \partial W) & \to & \mathbb{R} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [W] \rangle \underset{\uparrow}{=} \epsilon_W((\varphi \cup \psi) \cap [W]). \end{array}$$

  by the Cap Product-Kronecker Lemma 206.12, here $\epsilon_W : \mathrm{H}_0(W; \mathbb{R}) \to \mathbb{R}$ is the augmentation map from page 3079

  Note that by the Intersection Form Reinterpretation Lemma 227.2 we know that for any $k \in \mathbb{N}$ and any compact oriented $2k$-dimensional topological manifold W the form $Q_W \otimes \mathbb{R}$ is isometric to the form

$$\begin{array}{rcl} I_W^{k,k}\colon \mathrm{H}^k(W, \partial W) \times \mathrm{H}^k(W, \partial W) & \to & \mathbb{R} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [W] \rangle = \epsilon_W((\varphi \cup \psi) \cap [W]). \end{array}$$

  Thus our goal will be to calculate $\mathrm{sign}(I_{M\times N}^{2d,2d})$.

[2451] Here "$\times$" denotes the cross product

$$\times\colon \mathrm{H}_r(M; \mathbb{Z}) \times \mathrm{H}_s(N; \mathbb{Z}) \;\to\; \mathrm{H}_{r+s}(M \times N; \mathbb{Z})$$

that we introduced in Definition 164.14.

- We denote by $p\colon M\times N\to M$ and $q\colon M\times N\to N$ the natural projections. Since we work with real coefficients we obtain by the same argument as in the Künneth-H*-Theorem 208.1[2452] and the Universal Coefficient-H*–Field Theorem 202.2 an isomorphism

$$\begin{array}{rcl}\Theta\colon \bigoplus_{k+l=2d} \mathrm{H}^k(M,\partial M)\otimes \mathrm{H}^l(N,\partial N) & \to & \mathrm{H}^{2d}(M\times N, \overbrace{(\partial M\times N)\cup(M\times\partial N)}^{=\partial(M\times N)})\\ \varphi\otimes\psi & \mapsto & p^*(\varphi)\cup q^*(\psi).\end{array}$$

Claim. Let $\alpha_i\in \mathrm{H}^{k_i}(M,\partial M)$, $i=1,2$ and let $\beta_i\in \mathrm{H}^{l_i}(N,\partial N)$, $i=1,2$ such that we have the equalities $k_1+l_1=2d$ and $k_2+l_2=2d$. If[2453] $k_1+k_2=m$, then

$$I^{2d,2d}_{M\times N}(\Theta(\alpha_1\otimes\beta_1),\Theta(\alpha_2\otimes\beta_2)) \;=\; (-1)^{k_2\cdot l_1}\cdot (I^{k_1,k_2}_M\otimes I^{l_1,l_2}_N)(\alpha_1\otimes\beta_1,\alpha_2\otimes\beta_2).$$

Otherwise

$$I^{2d,2d}_{M\times N}(\Theta(\alpha_1\otimes\beta_1),\Theta(\alpha_2\otimes\beta_2)) \;=\; 0.$$

Given $k,l\in\mathbb{N}_0$ we write $A_k\coloneqq \mathrm{H}^k(M,\partial M)$ and $B_l\coloneqq \mathrm{H}^l(N,\partial N)$. With this notation we can illustrate the statement of the claim for $\dim(M)=\dim(N)=4$ as follows: The map Θ induces an isometry

$$I^{4,4}_{M\times N}\;\cong\;\begin{array}{c|ccccc} & A_0\otimes B_4 & A_1\otimes B_3 & A_2\otimes B_2 & A_3\otimes B_1 & A_4\otimes B_1\\ \hline A_0\otimes B_4 & 0 & 0 & 0 & 0 & I^{0,4}_M\otimes I^{4,0}_N\\ A_1\otimes B_3 & 0 & 0 & 0 & -I^{1,3}_M\otimes I^{3,1}_N & 0\\ A_2\otimes B_2 & 0 & 0 & I^{2,2}_M\otimes I^{2,2}_N & 0 & 0\\ A_3\otimes B_1 & 0 & -I^{3,1}_M\otimes I^{1,3}_N & 0 & 0 & 0\\ A_4\otimes B_0 & I^{4,0}_M\otimes I^{0,4}_N & 0 & 0 & 0 & 0\end{array}.$$

Proof. Note that

$$\begin{aligned} I^{2d,2d}_{M\times N}(\Theta(\alpha_1\otimes\beta_1),\Theta(\alpha_2\otimes\beta_2)) &\overset{\substack{\text{by the definition of }\Theta\text{ and }I^{2d,2d}_{M\times N}\text{ and since}\\ \text{by the Fundamental Class-of-Product Proposition 194.17}\\ \text{we know that }[M\times N]=[M]\times[N]}}{=}\\ &= \epsilon_{M\times N}((\underbrace{p^*(\alpha_1)\cup q^*(\beta_1)}_{=\Theta(\alpha_1,\beta_1)}\cup \underbrace{p^*(\alpha_2)\cup q^*(\beta_2)}_{=\Theta(\alpha_2,\beta_2)})\cap[M\times N])\\ &\underset{\substack{\text{by the up Product-Commutativity}\\ \text{Proposition 204.10 together with the}\\ \text{Cup Product-Naturality Lemma 204.8}}}{=} (-1)^{k_2\cdot l_1}\cdot\epsilon_{M\times N}((p^*(\underbrace{\alpha_1\cup\alpha_2}_{\in \mathrm{H}^{k_1+k_2}(M,\partial M)})\cup q^*(\underbrace{\beta_1\cup\beta_2}_{\in \mathrm{H}^{l_1+l_2}(N,\partial N)}))\cap([M]\times[N])) \;=\; (*).\end{aligned}$$

Next note that it follows from the Top Cohomology Theorem 201.26 (2a) that the first cup product $p^*(\alpha_1\cup\alpha_2)\in \mathrm{H}^{k_1+k_2}(M,\partial M)$ is zero if $k_1+k_2>m$ and that the second cup product $q^*(\beta_1\cup\beta_2)\in \mathrm{H}^{l_1+l_2}(N,\partial N)$ is zero if $l_1+l_2>n$. Since $k_1+k_2+l_1+l_2=m+n$ we see that the cup product is zero, unless $k_1+k_2=m$ and $l_1+l_2=n$. Thus we henceforth assume that $k_1+k_2=m$ and that $l_1+l_2=n$. We continue with the above calculation and

[2452] It follows from the Topological Manifolds-Invariants Proposition 189.14 that the condition from the Algebraic–Chain Complex Condition Lemma 208.3 is satisfied.

[2453] Note that from $k_1+l_1=2d$ and $k_2+l_2=2d$ and $m+n=4d$ it follows that $k_1+k_2=m$ is equivalent to $l_1+l_2=n$.

we see that

$$\begin{aligned}(*) &= (-1)^{k_2\cdot l_1}\cdot\epsilon_{M\times N}((p^*(\alpha_1\cup\alpha_2)\cup q^*(\beta_1\cup\beta_2))\cap([M]\times[N]))\\ &= (-1)^{k_2\cdot l_1}\cdot\epsilon_{M\times N}(((\underbrace{\alpha_1\cup\alpha_2}_{\in \mathrm{H}^m(M,\partial M)})\cap[M])\times((\underbrace{\beta_1\cup\beta_2}_{\in \mathrm{H}^n(N,\partial N)})\cap[N]))\\ &\uparrow \text{ follows from the Manifold Cross-Cup-Cap Theorem 227.10 (note that the sign in the theorem equals } (-1)^{k_2\cdot l_1+n\cdot m}=1)\\ &= (-1)^{k_2\cdot l_1}\cdot\langle\alpha_1\cup\alpha_2,[M]\rangle\cdot\langle\beta_1\cup\beta_2,[N]\rangle\\ &\uparrow \text{ follows from the definition of } I_M^{k_1,k_2} \text{ and } I_N^{l_1,l_2}, \text{ the fact that } \epsilon_{M\times N}(\{P\}\times\{Q\})=\epsilon_M(P)\cdot\epsilon_N(Q) \text{ and the Cap Product-Kronecker Lemma 206.12}\\ &= (-1)^{k_2\cdot l_1}\cdot(I_M^{k_1,k_2}\otimes I_N^{l_1,l_2})(\alpha_1\otimes\beta_1,\alpha_2\otimes\beta_2).\\ &\uparrow \text{ by the definition of } I_M\otimes I_N \text{ in the Form–Tensor Product Lemma 227.9}\end{aligned}$$

⊞

We now see that

$$\begin{aligned}\operatorname{sign}(M\times N) &\overset{\text{by the Intersection Form Reinterpretation Lemma 227.2}}{=} \operatorname{sign}(I_{M\times N}^{2d,2d}) \overset{\text{follows from the claim together with the Signature Additivity Proposition 226.13}}{=}\\ &= \sum_{k=0}^{\max\{\lfloor\frac{m-1}{2}\rfloor,\lfloor\frac{n-1}{2}\rfloor\}} \pm\underbrace{\operatorname{sign}\begin{pmatrix}0 & I_M^{k,m-k}\otimes I_N^{2d-k,n+k-2d}\\ I_M^{m-k,k}\otimes I_N^{n+k-2d,2d-k} & 0\end{pmatrix}}_{=0 \text{ by the Vanishing Signature Lemma 227.6 (2)}}\\ &\quad+\underbrace{(-1)^{\frac14 m\cdot n}\cdot\operatorname{sign}(I_M^{\frac m2,\frac m2}\otimes I_N^{\frac n2,\frac n2})}_{\text{this summand occurs only if } m \text{ and } n \text{ are both even}}\\ &= \begin{cases}(-1)^{\frac14 m\cdot n}\cdot\operatorname{sign}(I_M^{\frac m2,\frac m2}\otimes I_N^{\frac n2,\frac n2}), & \text{if } m,n \text{ even},\\ 0, & \text{otherwise}\end{cases}\\ &= \begin{cases}\operatorname{sign}(I_M^{\frac m2,\frac m2})\cdot\operatorname{sign}(I_N^{\frac n2,\frac n2}), & \text{if } m \text{ and } n \text{ are divisible by 4},\\ 0, & \text{otherwise}\end{cases}\\ &\uparrow \text{ by the Form–Tensor Product Lemma 227.9}\\ &= \operatorname{sign}(M)\cdot\operatorname{sign}(N).\\ &\uparrow \text{ by the Intersection Form Reinterpretation Lemma 227.2}\end{aligned}$$

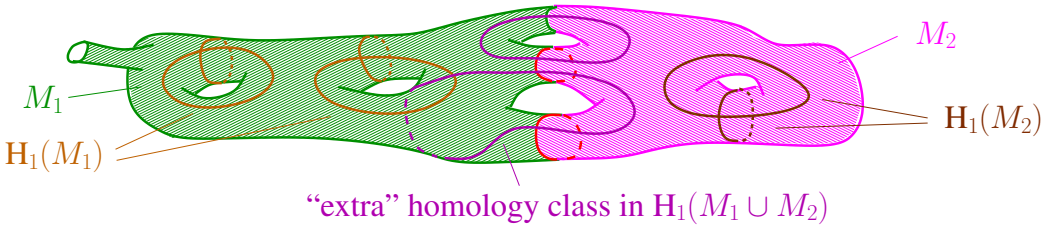
■

227.3. **Statement of the Novikov Additivity Theorem.** In the Intersection Form-Additivity Proposition 224.18 we saw that for closed topological manifolds the intersection form is additive under the connected sum operation.[2454] In the Manifold Signature Additivity Lemma 226.17 we pointed out that this implies that the signature is additive under the connected sum operation.

As we can see in the example below, if we glue two compact oriented $4k$-dimensional topological manifolds M_1 and M_2 along arbitrary boundary components, then the middle

[2454] In fact the same argument also shows that the intersection form is additive under the connected sum of compact oriented topological manifolds with non-empty boundary.

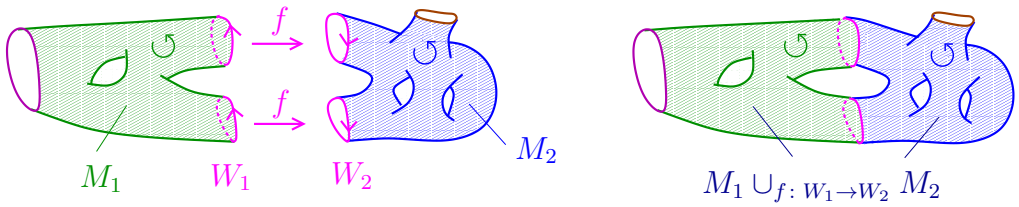
homology is in general not additive. In particular the intersection form is certainly in no way additive.



Rather surprisingly, the following theorem, which was first stated by Sergej Novikov [**Nov1970**, p. 154] in the mid 1960s, shows that the signature is additive under gluing along boundary components:

Theorem 227.11. (Novikov Additivity Theorem) Let M_1 and M_2 be two compact oriented $4k$-dimensional topological manifolds, let W_1 be a union of components of ∂M_1 and let W_2 be a union of components of ∂M_2. Finally let $f\colon W_1 \to W_2$ be an orientation-reversing homeomorphism. Then

$$\operatorname{sign}\Big(\underbrace{M_1 \cup_{f\colon W_1 \to W_2} M_2}_{\substack{\text{by the Gluing-Topological Manifolds-Proposition 54.3 this is an oriented topological manifold}}}\Big) = \operatorname{sign}(M_1) + \operatorname{sign}(M_2).$$



Example.

(1) In Exercise 227.6 we will use the Novikov Additivity Theorem 227.11 to give a new proof that the signature is additive under the connected sum operation. This gives an alternative proof of the Manifold Signature Additivity Lemma 226.17.
(2) In Exercise 227.7 we will use the Novikov Additivity Theorem 227.11 to show that the signature of the double $\mathrm{D}M$ of a compact oriented $4k$-dimensional topological manifold M is necessarily zero.
(3) Let $k \in \mathbb{N}_0$, let W be a closed oriented $(4k+3)$-dimensional smooth manifold and let $f\colon W \to W$ be an orientation-preserving diffeomorphism. We consider the corresponding mapping torus $\operatorname{Tor}(W, f) := (W \times [0,1])/(0, x) \sim (1, f(x))$ from Definition 21.19. In Exercise 227.8 we will use the Novikov Additivity Theorem 227.11 to show that $\operatorname{sign}(\operatorname{Tor}(W, f)) = 0$. □

Remark.

(1) If one glues two $4k$-manifolds along submanifolds of the boundary that are not unions of components, then in general the conclusion of the Novikov Additivity Theorem 227.11 no longer holds.

For example, by the discussion on page 4331 we know that $\text{sign}(\mathbb{C}P^2) = 1$. On the other hand the Handle Decomposition Theorem 250.7 says that every closed smooth manifold, in particular $\mathbb{C}P^2$, can be obtained iteratively by handle attachments.[2455] Each handle has vanishing second homology, in particular vanishing intersection form and thus signature zero.

(2) In this more general setting, where we glue smooth manifolds along submanifolds of the boundary, C. T. C. Wall [**Wall1969**] proved a refinement of the Novikov Additivity Theorem 227.11. This refinement is usually referred to as the Wall Non-Additivity Theorem. A detailed discussion is given in [**Brunnb2005**, Chapter 5]. □

Before we move on to the long and exhausting proof of the Novikov Additivity Theorem 227.11 let us state the following theorem for general culture. It says that the signature is essentially the only invariant of compact oriented smooth manifolds that has the additivity property stated in the Novikov Additivity Theorem 227.11.

Theorem 227.12. (Jänich Theorem) Let $k \in \mathbb{N}$ and let φ be a real valued orientation-preserving diffeomorphism[2456] invariant for compact oriented $4k$-dimensional smooth manifolds. We assume that φ satisfies the following two conditions:

(1) The map φ satisfies the additivity property stated in the Novikov Additivity Theorem 227.11.

(2) For any compact oriented $4k$-dimensional smooth manifold M we have the equality $\varphi(-M) = -\varphi(M)$.

Then there exists a $C \in \mathbb{R}$ such that for every closed oriented $4k$-dimensional smooth manifold we have $\varphi(M) = C \cdot \text{sign}(M)$.

Proof. The theorem was proved by Klaus Jänich [**Jä1968, Jä1969**] in 1968. A generalization of the theorem is provided by [**KKNO1973**, Theorem 1.3a]. ■

The remainder of this chapter is dedicated to the proof of the Novikov Additivity Theorem 227.11. Except for readers who have an insatiable appetite for technical proofs it is perhaps best to just move on to the next Chapter 228, where we discuss intersection forms of 4-dimensional smooth and topological manifolds. That chapter makes for much more pleasant and interesting reading.

227.4. **Proof of the Novikov Additivity Theorem.** The proof of the Novikov Additivity Theorem 227.11 takes place in the following three subsections:

(1) In Subsection 227.4.1 we reformulate the problem in terms of real valued cohomology and we make several preparations.

(2) In Subsection 227.4.2 we prove the Novikov Additivity Theorem 227.11 in the special case that we glue the total boundary of M_1 to the total boundary of M_2.

(3) In Subsection 227.4.3 we deal with the general case.

Before we turn to the three subsections we start out with formulating the following blanket convention that applies to the remainder of this chapter:

[2455] It follows from the Projective Space-Handle Decomposition Proposition 246.12 that $\mathbb{C}P^2$ has a handle decomposition consisting of a single 0-handle, a single 2-handle and a single 4-handle.

[2456] It is not clear to me whether the analogous statement also holds for topological manifolds.

Convention 227.13. Throughout the remainder of this chapter we will work with (co-) homology groups with real coefficients. To simplify the notation we drop the "; $\mathbb{R}$" throughout.

227.4.1. **Proof of the Novikov Additivity Theorem: Preliminaries.** First recall that in the Reduced Intersection Form–Non Degeneracy Proposition 225.11 we introduced the reduced intersection form and we showed that it is non-degenerate. The following proposition gives us the cohomological analog of the reduced intersection form.

Proposition 227.14. (Cohomology–Reduced Cup Product Proposition) Let Z be a compact oriented $4k$-dimensional topological manifold. Let $\delta\colon \mathrm{H}^{2k-1}(\partial Z) \to \mathrm{H}^{2k}(Z,\partial Z)$ be the connecting homomorphism of the long exact sequence of the pair $(Z,\partial Z)$.

(1) The following equality of vector subspaces holds:[2457]

$$\mathrm{im}(\delta\colon \mathrm{H}^{2k-1}(\partial Z) \to \mathrm{H}^{2k}(Z,\partial Z)) \;=\; \mathrm{H}^{2k}(Z,\partial Z)^{\perp}.$$

($\perp$: we take the orthogonal complement "$\perp$" with respect to the pairing I_Z)

(2) We write $\widehat{\mathrm{H}}^{2k}(Z,\partial Z) := \mathrm{coker}(\delta\colon \mathrm{H}^{2k-1}(\partial Z) \to \mathrm{H}^{2k}(Z,\partial Z))$. The form

$$\begin{array}{rcl} I_Z^{\mathrm{red}}\colon \widehat{\mathrm{H}}^{2k}(Z,\partial Z) \times \widehat{\mathrm{H}}^{2k}(Z,\partial Z) & \to & \mathbb{R} \\ ([\varphi],[\psi]) & \mapsto & \langle \varphi \cup \psi, [Z]\rangle \end{array}$$

is well-defined and non-singular.

(3) We have $\mathrm{sign}(Z) = \mathrm{sign}(I_Z) = \mathrm{sign}(I_Z^{\mathrm{red}})$.

The proof of the Cohomology–Reduced Cup Product Proposition 227.14 will require the remainder of this section. The impatient reader can also jump straight to the next Section 227.4.2.

In the proof of the Cohomology–Reduced Cup Product Proposition 227.14, and also later on in this chapter, we will need to make use of the Cup Product-Naturality Lemma 204.8. To facilitate reading the coming proofs it is perhaps convenient to have the statement of the Cup Product-Naturality Lemma 204.8 at hand:

Proposition 204.8. (Cup Product-Naturality Proposition) We suppose that we are given two excisive triads (X,A,B) and (Y,C,D) be of topological spaces. Furthermore we suppose that we are given a continuous map $f\colon (X,A,B) \to (Y,C,D)$. For all $p,q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(Y,C) \times \mathrm{H}^q(Y,D) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(Y, C\cup D) \\ \downarrow{\scriptstyle f^*\times f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X,A) \times \mathrm{H}^q(X,B) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A\cup B). \end{array}$$

Next we introduce some notation which we will use throughout this chapter.

Notation 227.15.

[2457] In the words of Definition 226.5 this means that $\mathrm{im}(\delta\colon \mathrm{H}^{2k-1}(\partial Z) \to \mathrm{H}^{2k}(Z,\partial Z))$ is the radical of the form I_Z.

(1) Given a real vector space V we denote by $\overline{V} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ the dual vector space.
(2) Given a homomorphism $f\colon V \to W$ between two real vector spaces we denote by $\overline{f}\colon \overline{W} \to \overline{V}$ the dual homomorphism.
(3) Given a compact oriented m-dimensional topological manifold Z we consider the maps

$$\begin{array}{rcl} \mathrm{D}\colon \mathrm{H}^r(Z) & \to & \overline{\mathrm{H}^{m-r}(Z, \partial Z)} \\ \varphi & \mapsto & (\psi \mapsto \langle \varphi \cup \psi, [Z] \rangle) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{D}\colon \mathrm{H}^r(Z, \partial Z) & \to & \overline{\mathrm{H}^{m-r}(Z)} \\ \varphi & \mapsto & (\psi \mapsto \langle \varphi \cup \psi, [Z] \rangle). \end{array}$$

It follows from the Manifold-Non Singularity Proposition 213.3 (2), which is a consequence of the Poincaré Duality Theorem 210.1, that these two maps are isomorphisms.[2458]
(4) Given a compact oriented m-dimensional topological manifold Z we also consider the maps

$$\begin{array}{rcl} \widetilde{\mathrm{D}}\colon \mathrm{H}^r(Z, \partial Z) & \to & \overline{\mathrm{H}^{m-r}(Z, \partial Z)} \\ \varphi & \mapsto & (\psi \mapsto \langle \varphi \cup \psi, [Z] \rangle) = I_Z(\varphi, \psi) \end{array}.$$

(5) We denote by

$$\begin{array}{rcl} i\colon \partial Z & \to & Z \\ j\colon (Z, \varnothing) & \to & (Z, \partial Z) \end{array}$$

the natural maps.

The following lemma is closely related to the Poincaré Duality–Boundary Proposition 211.14.

Lemma 227.16. (Poincaré Duality–Boundary Lemma) Let Z be a compact oriented m-dimensional topological manifold. For any $r \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathrm{H}^{r-1}(\partial Z) & \xrightarrow{\delta_{r-1}} & \mathrm{H}^r(Z, \partial Z) & \xrightarrow{j^*} & \mathrm{H}^r(Z) & \xrightarrow{i^*} & \ldots \\ & & \cong \downarrow \mathrm{D} & & \cong \downarrow \mathrm{D} & \searrow^{\widetilde{\mathrm{D}}} & \cong \downarrow \mathrm{D} & & \\ \ldots & \longrightarrow & \overline{\mathrm{H}^{m-r}(\partial Z)} & \xrightarrow{\overline{i^*}} & \overline{\mathrm{H}^{m-r}(Z)} & \xrightarrow{\overline{j^*}} & \overline{\mathrm{H}^{m-r}(Z, \partial Z)} & \xrightarrow{\overline{\delta}_{m-r}} & \ldots \end{array}$$

The following statements hold:
(1) The top and bottom horizontal sequences are exact.
(2) The diagram commutes.

Proof. The top horizontal sequence is just the long exact sequence in cohomology of the pair $(Z, \partial Z)$, hence it is exact. It follows by the same argument as in the Hom-Split-SES Lemma 197.8 that applying the $\mathrm{Hom}_{\mathbb{R}}(-, \mathbb{R})$ functor preserves exactness. This shows that the bottom horizontal sequence is also exact.

We turn to the proof that the diagram commutes. First we show that the square to the left commutes. Recall that given a topological space X we denote by $\epsilon_X\colon \mathrm{H}_0(X) \to \mathbb{R}$ the augmentation map that we introduced on Definition 143.23 and 3079. Let $\alpha \in \mathrm{H}^{r-1}(\partial Z)$

[2458] Note that this statement only holds since we work with $\mathbb{R}$-coefficients.

and let $\beta \in \mathrm{H}^{m-r}(Z)$. Then

$$\begin{array}{lllll}
(\mathrm{D}\circ\delta)(\alpha)(\beta) & \overset{\text{by definition}}{=} & \langle \delta(\alpha)\cup\beta, [Z]\rangle & \overset{\text{by the Cap Product-Kronecker Lemma 206.12}}{=} & \epsilon_Z((\delta(\alpha)\cup\beta)\cap[Z]) \\
& \overset{\text{by the Cup-Cap Lemma 206.15}}{=} & \epsilon_Z(\beta\cap(\delta(\alpha)\cap[Z])) & \overset{\text{by the Cap Product-Triple-Connecting Homomorphism Lemma 206.19 and the Fundamental Class-of-}\partial\text{-Proposition 194.13}}{=} & \epsilon_Z(\beta\cap i_*(\alpha\cap[\partial Z])) \\
& \overset{\text{by the Cap Product-Naturality Lemma 206.13 and the Augmentation Lemma 143.24 (3)}}{=} & \epsilon_{\partial Z}(i^*(\beta)\cap(\alpha\cap[\partial Z])) & \overset{\text{by the above arguments run backwards}}{=} & \langle \alpha\cup i^*(\beta), [\partial Z]\rangle \quad \overset{\text{by definition}}{=} \quad (\overline{i^*}\circ\mathrm{D})(\alpha)(\beta).
\end{array}$$

We turn to the two triangles on the right. Let $\alpha \in \mathrm{H}^r(Z, \partial Z)$ and let $\beta \in \mathrm{H}^{m-r}(Z, \partial Z)$. We perform the following calculation:

$$(\mathrm{D}\circ j^*)(\alpha)(\beta) \overset{\text{by Notation 227.15 of D}}{=} \langle j^*(\alpha)\cup\beta, [Z]\rangle \overset{=\widetilde{\mathrm{D}}(\alpha)(\beta)}{=} \overbrace{\langle \alpha\cup\beta, [Z]\rangle} \overset{\text{by the Cup Product-Naturality Lemma 204.8 applied to the map } (Z,\varnothing,\partial Z)\to(Z,\partial Z,\partial Z) \text{ and the map } (Z,\partial Z,\varnothing)\to(Z,\partial Z,\partial Z)}{=} \langle \alpha\cup j^*(\beta), [Z]\rangle \overset{\text{by Notation 227.15 of D}}{=} (\overline{j^*}\circ\mathrm{D})(\alpha)(\beta).$$

Conveniently enough this calculation takes care of both triangles. ∎

Proof of the Cohomology–Reduced Cup Product Proposition 227.14. The lemma can be deduced, with a little bit of jiu-jitsu from the Reduced Intersection Form–Non Degeneracy Proposition 225.11. But esthetically it is more appealing to give a direct proof. So let us do so:

(1) We have the equalities

$$\begin{array}{lll}
\mathrm{im}(\delta\colon \mathrm{H}^{2k-1}(\partial Z)\to \mathrm{H}^{2k}(Z,\partial Z)) & \overset{\text{by the Poincaré Duality Boundary Lemma 227.16}}{=} & \ker(\mathrm{D}\circ\overline{j^*}\colon \mathrm{H}^{2k}(Z,\partial Z)\to\overline{\mathrm{H}^{2k}(Z,\partial Z)}) \\
& \overset{\text{by the Poincaré Duality–Boundary Lemma 227.16 we have } \mathrm{D}\circ\overline{j^*}=\widetilde{\mathrm{D}}}{=} & \ker(\widetilde{\mathrm{D}}\colon \mathrm{H}^{2k}(Z,\partial Z)\to\overline{\mathrm{H}^{2k}(Z,\partial Z)}) \quad \overset{\text{by definition of D and } I_Z}{=} \quad \mathrm{H}^{2k}(Z,\partial Z)^{\perp}.
\end{array}$$

(2) It follows immediately from (1) "$\subseteq$", together with the symmetry of the cup product, that the form I_Z^{red} is well-defined. Furthermore it follows from (1b)"$\supseteq$" and some elementary algebra, see the Radical Summand Lemma 227.8, that I_Z^{red} is non-singular.

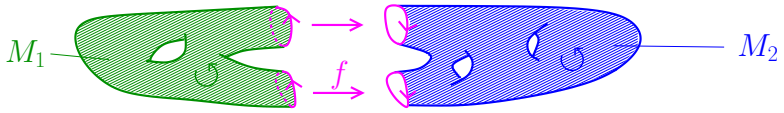
(3) We have the following equalities:

$$\mathrm{sign}(Z) \overset{\text{by the Intersection Form Reinterpretation Lemma 227.2}}{=} \mathrm{sign}(I_Z) \overset{\text{by the Radical Lemma 227.8}}{=} \mathrm{sign}\begin{pmatrix}0 & 0\\ 0 & I_Z^{\mathrm{red}}\end{pmatrix} \overset{\text{by the Signature Additivity Proposition 226.13}}{=} \mathrm{sign}(I_Z^{\mathrm{red}}).$$

∎

227.4.2. **Proof of the Novikov Additivity Theorem in a special case.** In this subsection we deal with the Novikov Additivity Theorem 227.11 in the special case that we glue the whole boundary of M_1 to the whole boundary of M_2.

Proof of Theorem 227.11 if $W_1 = \partial M_1$ and $W_2 = \partial M_2$. Let M_1 and M_2 be two compact oriented $4k$-dimensional topological manifolds and let $f\colon \partial M_1 \to \partial M_2$ be an orientation-reversing homeomorphism. We set $N := M_1 \cup_{f\colon \partial M_1 \to \partial M_2} M_2$. We will use the Gluing-Topological Manifolds-Proposition 54.3 to view M_1 and M_2 as codimension-zero submanifolds of N. We need to show that $\operatorname{sign}(N) = \operatorname{sign}(M_1) + \operatorname{sign}(M_2)$.



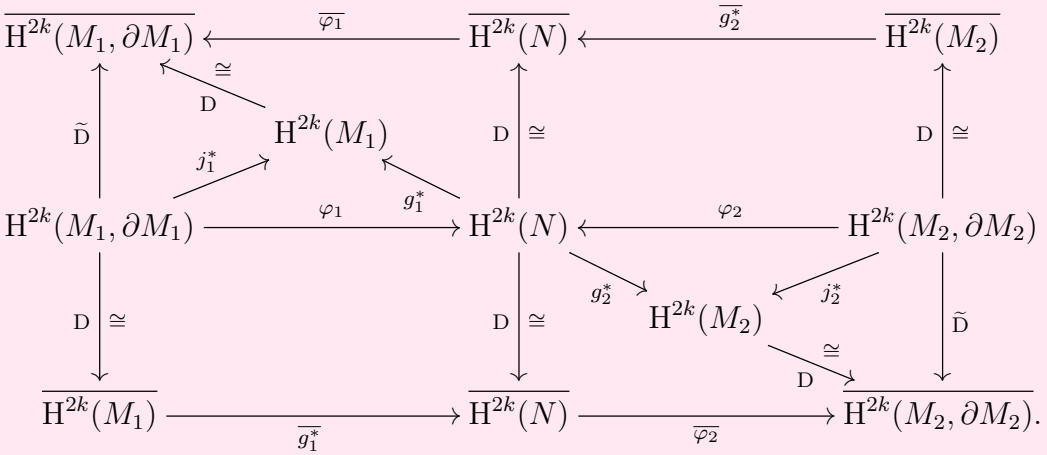
For $s = 1, 2$ we denote by

$$\begin{aligned} a_s\colon (M_s, \partial M_s) &\to (N, M_{3-s}), \\ b_s\colon (N, \varnothing) &\to (N, M_{3-s}), \\ j_s\colon (M_s, \varnothing) &\to (M_s, \partial M_s) \\ g_s\colon M_s &\to N \end{aligned}$$

the obvious maps. Note that the induced map $a_s^*\colon \mathrm{H}^{2k}(N, M_{3-s}) \to \mathrm{H}^{2k}(M_s, \partial M_s)$ is an isomorphism by the Excision-in-H*-Theorem 200.1 (2). Therefore we can define

$$\varphi_s := b_s^* \circ (a_s^*)^{-1}\colon \mathrm{H}^{2k}(M_s, \partial M_s) \to \mathrm{H}^{2k}(N).$$

Claim 1. We consider the following diagram:

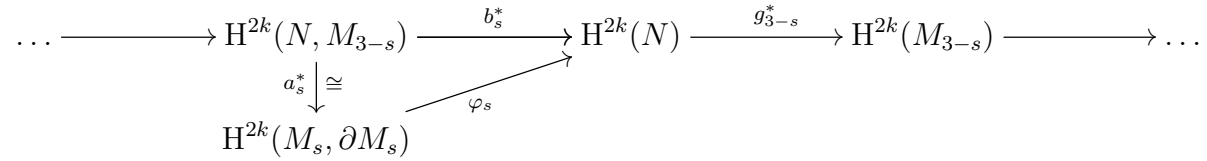


This diagram has the following two properties:

(1) The top and bottom horizontal sequences are exact.

(2) The diagram commutes.

Proof. First we show that the top and bottom horizontal sequences are exact. For $s \in \{1, 2\}$ we consider the following diagram:



The top horizontal sequence is just a segment of the long exact sequence in cohomology of the pair (N, M_{3-s}). As discussed above, the vertical map is an isomorphism by the Excision-in-H*-Theorem 200.1 (2). By definition of φ_s the diagram commutes. It follows that the sequence with the $\mathrm{H}^{2k}(M_s, \partial M_s)$-term is also exact. It follows by the same argument as in the Hom-Split-SES Lemma 197.8 that applying the $\mathrm{Hom}_{\mathbb{R}}(-,\mathbb{R})$ functor preserves exactness. This concludes the proof that the top and bottom horizontal sequences in the big diagram are exact.

We turn to the proof of the commutativity of the diagram. By symmetry it suffices to show that the left-hand side commutes.

(a) First we consider the big triangle. The argument that this triangle commutes is quite similar to the proof of the Small H$_*$-to-Large H*-PD Lemma 212.8. But for completeness' sake we carry out the details. Thus we now let $\alpha \in \mathrm{H}^{2k}(N)$ and we let $\beta \in \mathrm{H}^{2k}(M_1, \partial M_1)$. Next we set $\gamma := (a_1^*)^{-1}(\beta) \in \mathrm{H}^{2k}(N, M_2)$. We perform the following calculation:

$$\begin{array}{lllll}
\overbrace{\langle g_1^*(\alpha)\cup\beta, [M_1]\rangle}^{=(\mathrm{D}\circ g_1^*)(\alpha)(\beta)} & = \langle g_1^*(\alpha)\cup a_1^*(\gamma), [M_1]\rangle & = \langle g_1^*(\alpha\cup\gamma), [M_1]\rangle & = \langle \alpha\cup\gamma, a_{1*}([M_1])\rangle & \\
& = \langle \alpha\cup\gamma, b_{1*}([N])\rangle & = \langle b_1^*(\alpha\cup\gamma), [N]\rangle & & \\
& = \langle \alpha\cup b_1^*(\gamma), [N]\rangle & = \langle \alpha\cup(b_1\circ(a_1^*)^{-1})(\beta), [N]\rangle & = \underbrace{\langle \alpha\cup\varphi_1(\beta), [N]\rangle}_{=(\overline{\varphi_1}\circ\mathrm{D})(\alpha)(\beta)}. &
\end{array}$$

by the Cup Product-Naturality Lemma 204.8 applied to the map $(M_1, \varnothing, \partial M_1) \to (N, \varnothing, M_2)$ (↓)

by the Kronecker Pairing Natural Lemma 199.9 (1) (↓)

by the Codimension Zero Fundamental Class Lemma 194.10 (↑)

by the Kronecker Pairing Natural Lemma 199.9 (1) (↑)

by the Cup Product-Naturality Lemma 204.8 applied to the map $(N, M_1, \varnothing) \to (N, M_1, M_1)$ (↑)

(b) We showed in the Poincaré Duality–Boundary Lemma 227.16 that the small triangle to the left commutes.

(c) We turn to the small triangle for which $\mathrm{H}^{2k}(N)$ is a vertex. First note that it follows immediately from the definitions that $b_1 \circ g_1 = a_1 \circ j_1 \colon (M_1, \varnothing) \to (N, M_2)$. This implies that $g_1^* \circ b_1^* = j_1^* \circ a_1^*$. Since $\varphi_1 = b_1^* \circ (a_s^*)^{-1}$ we see that $g_1^* \circ \varphi_1 = j_1^*$.

(d) Finally we consider the square at the bottom left. In (a) we just showed that

$$\mathrm{D} \circ g_1^* = \overline{\varphi_1} \circ \mathrm{D} \colon \mathrm{H}^{2k}(N) \ \to \ \overline{\mathrm{H}^{2k}(M_1, \partial M_1)}.$$

We take the $\mathrm{Hom}(-,\mathbb{R})$ dual of this statement. Using the symmetry of the cup product, see the Cup Product-Commutativity Proposition 204.10, we see that $\overline{\mathrm{D}} = \mathrm{D}$. Thus we obtain, as desired, that

$$\overline{g_1^*} \circ \mathrm{D} = \mathrm{D} \circ \varphi_1 \colon \mathrm{H}^{2k}(M_1, \partial M_1) \ \to \ \overline{\mathrm{H}^{2k}(N)}. \qquad \boxplus$$

Claim 2. For $s = 1, 2$ we set

$$A_s \ := \ \mathrm{im}(\varphi_s \colon \mathrm{H}^{2k}(M_s, \partial M_s) \to \mathrm{H}^{2k}(N)).$$

With respect to the form $I_N \colon \mathrm{H}^{2k}(N) \times \mathrm{H}^{2k}(N) \to \mathbb{R}$ we have the following equalities:

$$(1) \qquad A_s^\perp \ = \ A_{3-s} \qquad \text{and} \qquad (2) \qquad (A_1 \cap A_2)^\perp \ = \ A_1 + A_2.$$

Proof. First we prove Equality (1). This equality follows immediately from the commutative diagram in Claim 1, the fact that the bottom sequence is exact and one minute's worth of concentration. More precisely, we consider the following diagram, which is just given by a

slight reorganization of the bottom part of the commutative diagram in Claim 1:

$$\begin{array}{ccccc} \mathrm{H}^{2k}(M_1,\partial M_1) & \xrightarrow{\varphi_1} & \mathrm{H}^{2k}(N) & \xrightarrow{g_2^*} & \mathrm{H}^{2k}(M_2) \\ \mathrm{D}\downarrow\cong & & \mathrm{D}\downarrow\cong & & \mathrm{D}\downarrow\cong \\ \overline{\mathrm{H}^{2k}(M_1)} & \xrightarrow{\overline{g_1^*}} & \overline{\mathrm{H}^{2k}(N)} & \xrightarrow{\overline{\varphi_2}} & \overline{\mathrm{H}^{2k}(M_2,\partial M_2)}. \end{array}$$

Since the diagram commutes, since the bottom sequence is exact and since the vertical maps are isomorphisms we see that

$$\underbrace{\mathrm{im}(\varphi_1\colon \mathrm{H}^{2k}(M_1,\partial M_1)\to \mathrm{H}^{2k}(N))}_{=A_1} = \underbrace{\ker(\overline{\varphi_2}\circ \mathrm{D}\colon \mathrm{H}^{2k}(N)\to \overline{\mathrm{H}^{2k}(M_2,\partial M_2)})}_{=A_2^\perp}.$$

Thus we have shown that $A_1 = A_2^\perp$. The same argument shows that $A_2 = A_1^\perp$. This concludes the proof of Equality (1).

Next note that Equality (2) follows from Equality (1) together with some elementary algebra, see Exercise 224.12. For the reader's convenience we provide the argument. First note that it follows immediately from (1) that $A_1 + A_2 \subseteq (A_1 \cap A_2)^\perp$. To show equality it suffices to show that both vector subspaces have the same dimension. As the following calculation shows, this is indeed the case:

since N is closed we know by the Manifold-Non Singularity Proposition 213.3 (2) that I_N is non-singular, hence for any vector subspace $B \subseteq \mathrm{H}^{2k}(N)$ we have $\dim(B^\perp) = \dim(\mathrm{H}^{2k}(N)) - \dim(B)$

$$\begin{array}{lll} \dim((A_1\cap A_2)^\perp) & \overset{\downarrow}{=} \dim(\mathrm{H}^{2k}(N)) - \dim(A_1\cap A_2) & \overset{\downarrow}{=} \dim(A_1)+\dim(A_1^\perp)-\dim(A_1\cap A_2) \\ & = \dim(A_1)+\dim(A_2)-\dim(A_1\cap A_2) & = \dim(A_1+A_2). \\ & \overset{\uparrow}{} & \overset{\uparrow}{} \\ & \text{by (1)} & \text{since } 0\to A_1\cap A_2\to A_1\oplus A_2\to A_1+A_2\to 0 \text{ is exact} \end{array}$$ ⊞

Claim 3. Let $s \in \{1,2\}$. As in the Cohomology–Reduced Cup Product Proposition 227.14 we write

$$\widehat{\mathrm{H}}^{2k}(M_s,\partial M_s) := \mathrm{coker}(\delta\colon \mathrm{H}^{2k-1}(\partial M_s)\to \mathrm{H}^{2k}(M_s,\partial M_s)).$$

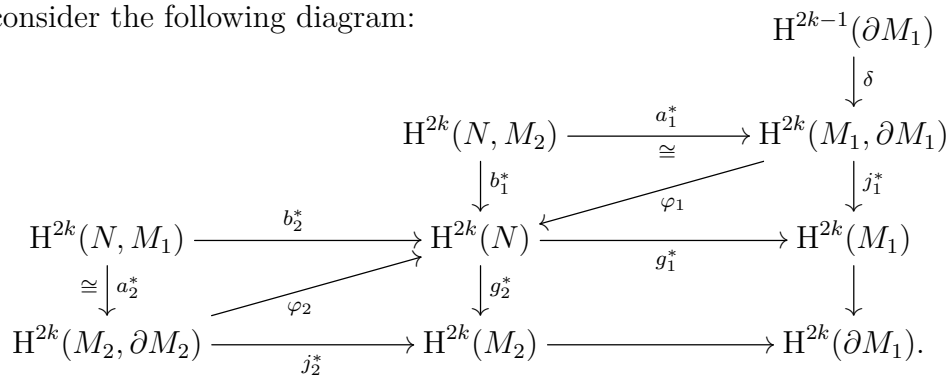
The map

$$\varphi_s\colon \mathrm{H}^{2k}(M_s,\partial M_s) \to \mathrm{H}^{2k}(N)$$

induces an isomorphism

$$\begin{array}{rcl} \widehat{\varphi}_s\colon \widehat{\mathrm{H}}^{2k}(M_s,\partial M_s) & \to & A_s/(A_1\cap A_2) \\ [\sigma] & \mapsto & [\varphi_s(\sigma)]. \end{array}$$

Proof. We consider the following diagram:



We make the following clarifications and observations:

- All undecorated maps are induced by the obvious maps.
- The middle and the right vertical sequences and also all the horizontal sequences are long exact sequences of pairs.
- All parts of the diagram commute. This basically follows from the definition of φ_s and φ_{3-s} and the fact that all the remaining maps are induced by maps of pairs of topological spaces. Note that the fact that the triangles involving φ_s, j_s^* and g_s^* commute was already discussed in Claim 1.

A little diagram chase now shows that each map $\widehat{\varphi}_s\colon \widehat{\mathrm{H}}^{2k}(M_s,\partial M_s) \to A_s/(A_1\cap A_2)$, that is given by $[\sigma]\mapsto[\varphi_s(\sigma)]$, is well-defined and an isomorphism. ⊞

Claim 4. We consider the projections

$$p_s\colon \mathrm{H}^{2k}(M_s,\partial M_s)\to\widehat{\mathrm{H}}^{2k}(M_s,\partial M_s) \quad\text{and}\quad q\colon \mathrm{H}^{2k}(N)\to\mathrm{H}^{2k}(N)/(A_1+A_2)$$

and we pick corresponding left inverses

$$t_s\colon \widehat{\mathrm{H}}^{2k}(M_s,\partial M_s)\to\mathrm{H}^{2k}(M_s,\partial M_s) \quad\text{and}\quad u\colon \mathrm{H}^{2k}(N)/(A_1+A_2)\to\mathrm{H}^{2k}(N).$$

Next we consider the following vector subspaces of $\mathrm{H}^{2k}(N)$:

$$\begin{aligned} H_1 &\coloneqq \operatorname{im}(\varphi_1\circ t_1\colon \widehat{\mathrm{H}}^{2k}(M_1,\partial M_1)\to\mathrm{H}^{2k}(N))\\ H_2 &\coloneqq \operatorname{im}(\varphi_2\circ t_2\colon \widehat{\mathrm{H}}^{2k}(M_2,\partial M_2)\to\mathrm{H}^{2k}(N))\\ D &\coloneqq A_1\cap A_2\\ E &\coloneqq \operatorname{im}(u\colon \mathrm{H}^{2k}(N)/(A_1+A_2)\to\mathrm{H}^{2k}(N)). \end{aligned}$$

The following three statements hold:

(1) We have $\dim(D)=\dim(E)$.

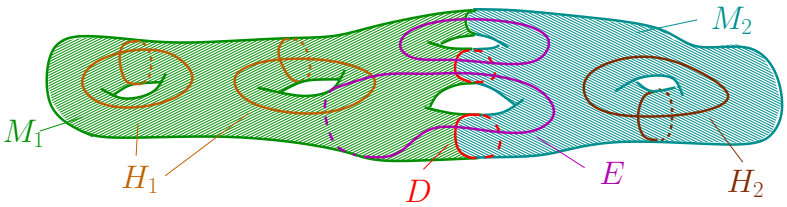
(2) The various inclusions induce an isomorphism

$$H_1\oplus H_2\oplus D\oplus E \xrightarrow{\cong} \mathrm{H}^{2k}(N).$$

(3) With respect to this decomposition of $\mathrm{H}^{2k}(N)$ given in (1) the form I_N can be written as follows:

$$I_N = \begin{array}{c} \\ H_1\\ H_2\\ D\\ E \end{array}\!\!\begin{array}{c} \begin{array}{cccc} H_1 & H_2 & D & E \end{array}\\ \begin{pmatrix} I^{\mathrm{red}}_{M_1} & 0 & 0 & * \\ 0 & I^{\mathrm{red}}_{M_2} & 0 & * \\ 0 & 0 & 0 & X \\ * & * & X^{\mathrm{T}} & * \end{pmatrix} \end{array}$$

where X is represented by an invertible square matrix.

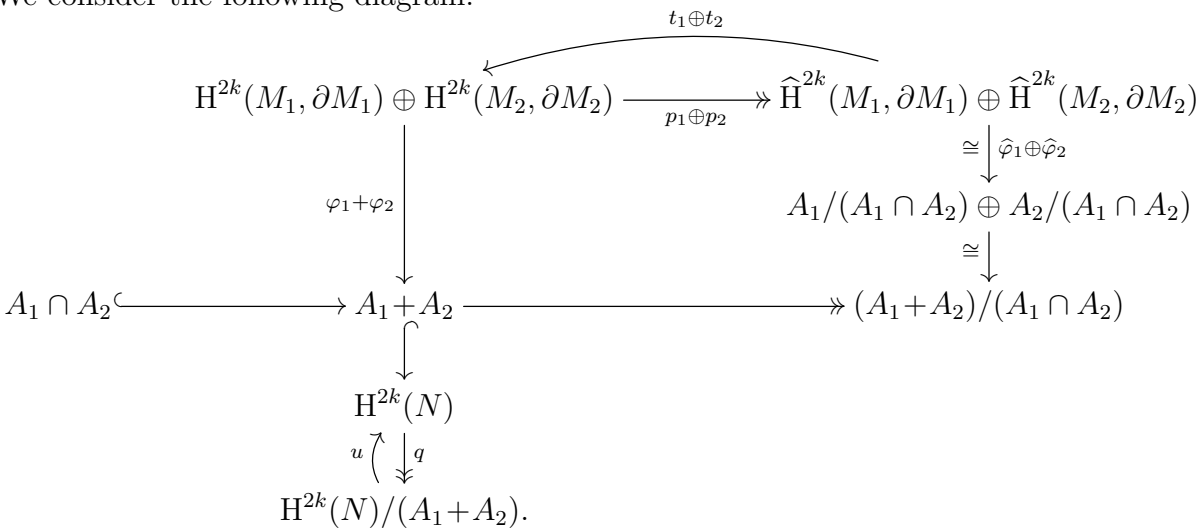


in each case the cohomology classes are the Poincaré duals of the homology classes

Proof.

(1) Recall that in the proof of Claim 2 (2) we already showed that we have the equality $\dim(A_1 \cap A_2) = \dim(N) - \dim(A_1 + A_2)$. But this means precisely that we have the promised equality $\dim(D) = \dim(E)$.

(2) We consider the following diagram:

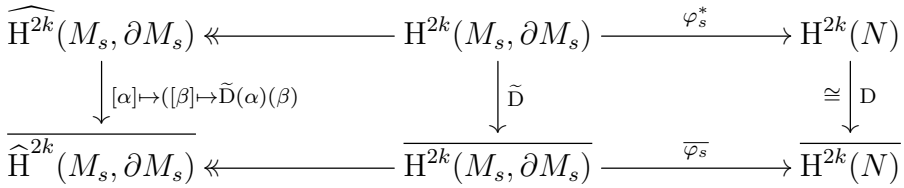


The desired statement follows easily from the following observations:

- The central horizontal sequence is exact and the vertical sequence starting with $A_1 + A_2$ are exact.
- One can easily verify that the diagram commutes.

(3) The statement follows from the following three bullet points:

- The three zeros in the "D-column" and the "D-row" follow immediately from the fact, shown in Claim 3, that $(A_1 \cap A_2)^{\perp} = A_1 + A_2$.
- We consider the following diagram:



The square to the left commutes by definition and the square to the right commutes since it equals the upper left respectively the lower right quadrant of the diagram in Claim 1. But this means precisely that the "H_s-diagonal term" is given by $I^{\mathrm{red}}_{M_s}$.

- By (1) we know that X is a square matrix. The fact that X is invertible now follows from the fact, shown in the Manifold-Non Singularity Proposition 213.3 (2), that I_N is non-singular. ⊞

Now we see that

$$\operatorname{sign}(N) \overset{\text{by the Cohomology–Reduced Cup Product Proposition 227.14 (3)}}{=} \operatorname{sign}(I_N) = \operatorname{sign}\begin{pmatrix} I^{\mathrm{red}}_{M_1} & 0 & 0 & * \\ 0 & I^{\mathrm{red}}_{M_2} & 0 & * \\ 0 & 0 & 0 & X \\ * & * & X^{\mathrm{T}} & * \end{pmatrix} \overset{\text{since } X \text{ is invertible there exists an isometry between the two forms}}{=} \operatorname{sign}\begin{pmatrix} I^{\mathrm{red}}_{M_1} & 0 & 0 & 0 \\ 0 & I^{\mathrm{red}}_{M_2} & 0 & 0 \\ 0 & 0 & 0 & X \\ 0 & 0 & X^{\mathrm{T}} & * \end{pmatrix}$$

$$= \underset{\text{by the Signature Additivity Proposition 226.13}}{\operatorname{sign}(I^{\mathrm{red}}_{M_2}) + \operatorname{sign}(I^{\mathrm{red}}_{M_2})} + \underbrace{\operatorname{sign}\begin{pmatrix} 0 & X \\ X^{\mathrm{T}} & * \end{pmatrix}}_{\substack{= 0 \text{ by the Metabolic Signature Lemma 227.6, since the matrix } X \text{ is invertible}}} \overset{\text{Proposition 227.14 (3)}}{=} \operatorname{sign}(M_1) + \operatorname{sign}(M_2).$$

■

227.4.3. **Proof of the Novikov Additivity Theorem III.** Before we formulate the next special case of Theorem 227.11 we introduce some definitions:

Definition 227.17.

(1) We say that a closed oriented n-dimensional topological manifold S is nullbordant if there exists a compact oriented $(n+1)$-dimensional topological manifold T such that $\partial T = S$ as oriented topological manifolds. Such a T is called a nullbordism for S.

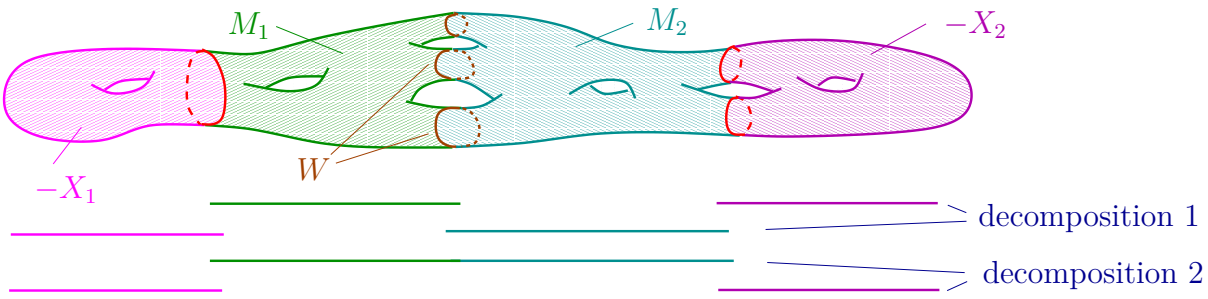
(2) In the setting of the Novikov Additivity Theorem 227.11 we refer to $\partial M_1 \setminus W_1$ as the unglued boundary of M_1 and we refer to $\partial M_2 \setminus W_2$ as the unglued boundary of M_2.

With this definition we can formulate the next special case of the Novikov Additivity Theorem 227.11 that we can deal with:

Proof of Theorem 227.11 if the unglued boundaries are nullbordant. We assume that the unglued boundaries $\partial M_1 \setminus W_1$ and $\partial M_2 \setminus W_2$ are nullbordant. By the above Definition 227.17 this means that there exist compact oriented $4k$-dimensional topological X_1 and X_2 with $\partial X_s = \partial M_s \setminus W_s$ as oriented topological manifolds. Using the homeomorphism $f\colon W_1 \to W_2$ we can write $W \coloneqq W_1 = W_2$. We perform the following calculation:

$$\operatorname{sign}(-X_1 \cup_{\partial M_1 \setminus W_1} M_1 \cup_W M_2 \cup_{\partial M_2 \setminus W_2} -X_2) \overset{\text{the special case applied to } M_1 \sqcup -X_2 \text{ glued to } M_2 \sqcup -X_1, \text{ glued along the common boundary}}{=} \operatorname{sign}(M_1 \sqcup -X_2) + \operatorname{sign}(M_2 \sqcup -X_1)$$

$$\overset{\text{by the Manifold Signature Additivity Lemma 226.17 we have } \operatorname{sign}(A \sqcup B) = \operatorname{sign}(A) + \operatorname{sign}(B) \text{ and we have } \operatorname{sign}(-A) = -\operatorname{sign}(A)}{=} \operatorname{sign}(M_1) + \operatorname{sign}(M_2) - \operatorname{sign}(X_1) - \operatorname{sign}(X_2).$$



Since this was so much fun we compute the signature of the glued manifold once again with a different decomposition:

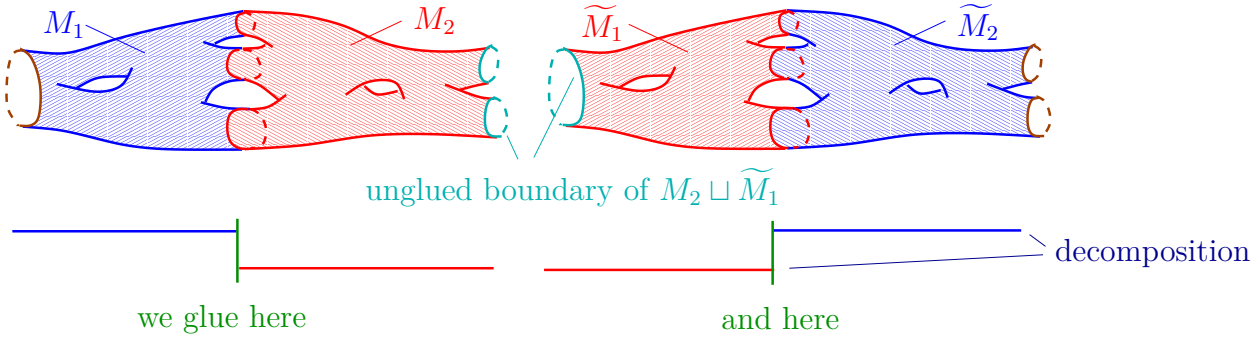
$$\begin{aligned}\operatorname{sign}(-X_1 \cup_{\partial M_1 \setminus W_1} M_1 \cup_W \cup M_2 \cup_{\partial M_2 \setminus W_2} -X_2) &\overset{\downarrow}{=} \operatorname{sign}(M_1 \cup_W M_2) + \operatorname{sign}(-X_1 \sqcup -X_2)\\ &= \operatorname{sign}(M_1 \cup_W M_2) - \operatorname{sign}(X_1) - \operatorname{sign}(X_2).\end{aligned}$$

special case applied to $M_1 \cup_W M_2$ glued to $-X_2 \sqcup -X_1$, glued along the common boundary

Comparing these two calculations we obtain the desired result. ∎

Now we can finally turn to the proof of the general case of the Novikov Additivity Theorem 227.11:

Proof of the Novikov Additivity Theorem 227.11. Let M_1 and M_2 be two compact oriented $4k$-dimensional topological manifolds, let W_1 be a union of components of ∂M_1 and let W_2 be a union of components of ∂M_2. Finally let $f\colon W_1 \to W_2$ be an orientation-reversing homeomorphism. Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be copies of M_1 and M_2. Furthermore we denote by $\widetilde{W} = \widetilde{W}_1 = \widetilde{W}_2$ a copy of $W = W_1 = W_2$.



We perform the following calculation:

$$\begin{aligned}2 \cdot \operatorname{sign}(M_1 \cup_W M_2) &= \operatorname{sign}\big((M_1 \cup_W M_2) \sqcup (\widetilde{M}_1 \cup_{\widetilde{W}} \widetilde{M}_2)\big)\\ &= \operatorname{sign}\big((M_1 \sqcup \widetilde{M}_2) \cup_{W \sqcup \widetilde{W}} (\widetilde{M}_1 \sqcup M_2)\big)\\ &= \operatorname{sign}\big(M_1 \sqcup \widetilde{M}_2\big) + \operatorname{sign}\big(\widetilde{M}_1 \sqcup M_2\big) \overset{\uparrow}{=} 2 \cdot (\operatorname{sign}(M_1) + \operatorname{sign}(M_2)).\\ &\uparrow\end{aligned}$$

by the Manifold Signature Additivity Lemma 226.17

follows from the previous special case, which we can apply for the following reason:
unglued boundary of $M_1 \sqcup \widetilde{M}_2$ is the boundary of $\widetilde{M}_1 \cup_{\widetilde{W}_1 = W_2} M_2$, in particular it is nullbordant,
unglued boundary of $\widetilde{M}_1 \sqcup M_2$ is the boundary of $M_1 \cup_{W_1 = \widetilde{W}_2} \widetilde{M}_2$, in particular it is nullbordant

Dividing both sides by 2 gives us the desired result. ∎

227.5. Exercises for Chapter 227 .

Exercise 227.1. We consider the following two matrices:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Are the corresponding two forms over $\mathbb{R}^2$ metabolic? Hyperbolic?
(b) Are the corresponding two forms over $\mathbb{Z}^2$ metabolic? Hyperbolic?

Exercise 227.2. Let (V, g) be a symmetric form over $\mathbb{R}$. Show that if $\operatorname{sign}(g) = 0$, then the form is hyperbolic.
Remark. This exercise gives us the converse to the Vanishing Signature Lemma 227.6 (2).

Exercise 227.3. Let $\varphi\colon V \times V \to \mathbb{R}$ be a μ-symmetric form over $\mathbb{R}$ and let $\psi\colon W \times W \to \mathbb{R}$ be a ν-symmetric form over $\mathbb{R}$. In the Form–Tensor Product Lemma 227.9 we introduced the $\mu \cdot \nu$-symmetric form

$$\varphi \otimes \psi\colon V \otimes_{\mathbb{R}} W \times V \otimes_{\mathbb{R}} W \ \to \ \mathbb{R}.$$

What is the relationship between the nullities $\nu(\varphi)$, $\nu(\psi)$ and $\nu(\varphi \otimes \psi)$?

Exercise 227.4. Let R be a commutative ring. Recall that in Exercise 226.7 we introduced the Witt group $W(R)$.
(a) Show that the tensor product of forms defines a ring structure on the Witt group $W(R)$. This ring is cleverly called the *Witt ring* $W(R)$ *of* R.
(b) Determine the Witt rings $W(\mathbb{R})$ and $W(\mathbb{Z})$.

Exercise 227.5. Provide a proof for the Vanishing Signature Lemma 227.6 that does not use bases.[2459] Let us elaborate. Let (V, φ) be a non-singular symmetric form over $\mathbb{R}$ and suppose that there exists a subspace W of V with $\varphi|_W = 0$ and with $\dim(W) = \frac{1}{2}\dim(V)$. Show that given any subspace A of V such that $\varphi|_A$ is positive (respectively negative) definite there exists a subspace B of V such that $\varphi|_B$ is negative (respectively positive) definite and which satisfies $\dim(A) = \dim(B)$. A proof of this statement implies that $\operatorname{sign}(\varphi) = 0$.

Exercise 227.6. Use the Novikov Additivity Theorem 227.11 to show that for compact oriented topological manifolds the signature is additive under the connected sum operation.
Remark. This gives a new proof of the Manifold Signature Additivity Lemma 226.17.

Exercise 227.7. Let M be a compact oriented $4k$-dimensional topological manifold and let W be the union of some components of ∂M. As in Definition 54.8 we refer to

$$\mathrm{D}_W M \ := \ (M \times \{1\}) \cup_{W \times \{1\} = W \times \{2\}} ((-M) \times \{2\})$$

as the *double of* M *along* W. By the Doubling-Manifold Lemma 54.10 and the Double-Orientability Lemma 192.6 we know that $\mathrm{D}_W M$ is a compact orientable topological manifold. Use the Novikov Additivity Theorem 227.11 to show that the signature of $\mathrm{D}_W M$ is zero.
Remark. This exercise is basically identical to Exercise 226.11.

Exercise 227.8. Let $k \in \mathbb{N}_0$, let W be a closed oriented $(4k+3)$-dimensional smooth manifold and let $f\colon W \to W$ be an orientation-preserving diffeomorphism. We consider the corresponding mapping torus $\operatorname{Tor}(W, f) := (W \times [0,1])/(0, x) \sim (1, f(x))$ from Definition 21.19. Use the Novikov Additivity Theorem 227.11 to show that $\operatorname{sign}(\operatorname{Tor}(W, f)) = 0$.

Exercise 227.9. Let $k \in \mathbb{N}$. Give an example of two compact oriented 4-dimensional topological manifolds M_1 and M_2 together with a homeomorphism $f\colon \partial M_1 \to \partial M_2$ such that the following statements are satisfied:
(a) $b_2(M_1) = k$ and $b_2(M_2) = k$.

[2459] As the reader surely knows, introducing bases is morally wrong and should be avoided at all costs.

(b) The inclusion-induced maps $\mathrm{H}_2(M_1) \to \mathrm{H}_2(M_1 \cup_f M_2)$ and $\mathrm{H}_2(M_2) \to \mathrm{H}_2(M_1 \cup_f M_2)$ are both the zero map.

CHAPTER 228

The intersection form of 4-dimensional manifolds

We have now introduced lots of invariants of topological spaces, namely the fundamental group, higher homotopy groups, homology groups and the cohomology groups together with the cup product. In particular, for a compact oriented $2n$-dimensional topological manifold M we introduced in Definition 224.12 the intersection form

$$\begin{array}{rcl} Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{array}$$

The question arises, whether we are done at some point, or whether this list of ever new invariants continues forever. Later in Part XXXVI we will introduce twisted (co-) homology groups and in Part XXXVII we will introduce Reidemeister torsion, which give us both extra invariants of non-simply connected topological spaces. But for simply connected topological spaces we are running out of ideas for producing new invariants.

Fortunately we have achieved some successes. For example in the Surface Classification Theorem 101.16 we saw that our invariants are strong enough to classify compact 2-dimensional topological and smooth manifolds. We also saw that the intersection form allows us to distinguish $S^2 \times S^2$ from $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$, even though the homology groups of these simply connected 4-dimensional smooth manifolds are the same.
In this chapter we want to consider closed oriented simply connected 4-dimensional (topological) smooth manifolds and their intersection forms. What follows is a long survey on some amazing results. Unfortunately the proofs of any of these statements goes well beyond what we can cover in these introductory notes.

The following question will guide us through the chapter:

Question 228.1.

(1) Is the intersection form a complete invariant of closed oriented simply connected 4-dimensional (topological) smooth manifolds?[2460]
(2) Which isometry types of non-singular symmetric forms over $\mathbb{Z}$ are realized by closed oriented simply connected 4-dimensional (topological) smooth manifolds?

[2460] First note that for a simply connected 4-dimensional topological manifold M we obtain from the Hurewicz Theorem 166.10 that $\mathrm{H}_1(M) = 0$. Next note that it follows from $\mathrm{H}_1(M) = 0$ and the Universal Coefficient-H^*-Theorem 201.23 that $\mathrm{H}^2(M;\mathbb{Z})$ is torsion-free. We now assume that M is closed and orientable. It follows from the above and the Poincaré Duality Theorem 210.1 that $\mathrm{H}_2(M) \cong \mathrm{H}^2(M;\mathbb{Z})$ is torsion-free. Furthermore it follows from the above and the Poincaré Duality Theorem 210.1 that $\mathrm{H}^3(M;\mathbb{Z}) \cong \mathrm{H}_1(M) = 0$. It follows from $\mathrm{H}^3(M;\mathbb{Z}) = 0$, the fact that $\mathrm{H}_2(M)$ is torsion-free and the Universal Coefficient-H^*-Theorem 201.23 that $\mathrm{H}_3(M) = 0$. Thus we see that for closed oriented simply

This chapter is organized as follows:

- In Section 228.1 we will introduce The K3-surface.
- In Section 228.2 we will state Rokhlin's Theorem 228.7.
- In Section 228.3 we will discuss the homotopy classification of closed simply connected 4-dimensional smooth manifolds.
- In Section 228.4 we will discuss the Kirby-Siebenmann invariant of topological manifolds.
- In Section 228.5 we will state the Freedman Classification Theorem 228.12 for closed simply connected 4-dimensional topological manifolds.
- In Section 228.6 we will state the Donaldson Diagonalization Theorem 228.18.
- In Section 228.7 we will state Furuta's 10/8-Theorem 228.19 and we will formulate the 11/8-Conjecture 228.20.
- In Section 228.8 we will discuss exotic smooth structures.
- In Section 228.9 we will see that 4-dimensional smooth manifolds that are homeomorphic become diffeomorphic after stabilizations.
- In Section 228.10 we will provide exercises for Chapter 228.

In the next Chapter 229 we will introduce inverse limits.

228.1. **1910–1958: The K3-surface.** In the following we will give a mostly chronological account of the developments regarding the two (or possibly four?) questions posed in Question 228.1.

In Theorem 226.22 we introduced the initially rather unexpected form E_8. One might wonder whether a form as strange as E_8 can ever appear "in nature". As we will now see, somewhat surprisingly this is indeed the case.

Definition 228.2. We refer to

$$\mathrm{K3} := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,\middle|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \right\}$$

as the K3-surface. Sometimes the K3-surface is also called the Kummer surface.[2461]

Remark. The odd name "K3-surface" comes about as follows:

(1) First of all, as we will see in the K3-Intersection Form Proposition 228.3, K3 is a *2-dimensional* complex manifold, i.e. it is a complex *surface*.

(2) The K3-surface was (re-) discovered[2462] by André Weil [**Weil1958**] who named it in honor of three algebraic geometers, Kummer, Kähler and Kodaira, and also the mountain K2 in Pakistan which is the second highest mountain on earth.



the mountain K2, almost as pretty as the K3-surface

connected 4-dimensional topological manifolds the intersection form contains all the (co-) homological data we could ask for. (This discussion is essentially the content of Exercise 210.13.)

[2461] There are many alternative ways to describe the K3-surface, see e.g. [**Scor2005**, Chapter 3.3].

[2462] According to [**OTL2016**] the definition of the K3-surface is already somewhat implicit in the work of Srinivasa Ramanujan in the 1910s, in particular it is connected to the famous 1729 taxi cab anecdote.

More precisely, André Weil wrote: "Dans la seconde partie de mon rapport, il s'agit des varietés kähleriennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire". □

The following proposition summarizes some of the key aspects of the K3-surface:

Proposition 228.3. (K3-Intersection Form Proposition) The K3-surface is a closed simply connected complex manifold of complex dimension 2 or equivalently of real dimension 4. Furthermore we have $H_2(K3) \cong \mathbb{Z}^{22}$ and [2463]

$$Q_{K3} \cong 2 \cdot E_8 \oplus 3 \cdot H \quad \text{where } H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular we have $\text{sign}(K3) = 16$.

Proof. Using the Complex Regular Value Theorem 40.13 it is elementary to show that the K3-surface is a closed complex manifold of complex dimension 2. Using the Lefschetz Hyperplane Theorem **??** one can show that the K3-surface is simply connected. The second homology group and the intersection form are calculated in [**GoS1999**, Theorem 1.3.8], or alternatively in [**McS2016**, p. 176] and in [**Huy2016**, Proposition 3.5].[2464]

The statement about the signature then follows from the E_8-Lemma 226.23 and the Signature Additivity Proposition 226.13. ■

Remark. The K3-surface, albeit with a somewhat different definition as ours, plays a big role in (algebraic) geometry, as is indicated by the fact that a whole book [**Huy2005**] is dedicated to the study of the K3-surface(s). Note that it follows from [**Huy2016**, Theorem 2.4.2 and 7.1.1] that the K3-surface as defined above, viewed as a smooth manifold, admits many non-biholomorphic complex structures.[2465] □

In the following we want to discuss one particular aspect of the K3-surface, namely: The calculation of the intersection form of the K3-surface, of $Q_{S^2 \times S^2} \cong H$ and the Intersection Form-Additivity Proposition 224.18 raises the question whether one can split off copies of $S^2 \times S^2$ from the K3-surface. More precisely the following question arose almost immediately after the introduction of the K3-surface:

Question 228.4.

(1) Does there exist a closed oriented 4-dimensional topological manifold M such that the K3-surface is homeomorphic to $M \# (S^2 \times S^2)$?
(2) Does there exist a closed oriented 4-dimensional smooth manifold M such that the K3-surface is diffeomorphic to $M \# (S^2 \times S^2)$?

We will address Question 228.4 in the following sections.

228.2. **1952: Rokhlin's Theorem.** We will return to Question 228.4 shortly. But before we do so, we continue with our chronological discussion. Next in line is a result proved by

[2463] Since the K3-surface is a complex manifold we know by the Complex Manifold-Orientation Proposition 40.18 that it has a natural orientation.
[2464] The calculation of the intersection form, perhaps somewhat disappointingly, makes use of the Indefinite Even Forms Corollary 226.31.
[2465] We had seen already in the Not–Biholomorphic Corollary 129.3 that there exist complex manifolds that are diffeomorphic but not biholomorphic.

Vladimir Rokhlin in 1952. To formulate Rokhlin's Theorem we need the notion of a "spin structure". Unfortunately we have to give a "black box" definition:

Definition 228.5. A smooth manifold M is called spin if its tangent bundle admits a spin structure in the sense of the definition in **??**.

At this stage the definition of a "spin structure" is of no concern to us and we will use this notion as a black box. For the discussion of this chapter it suffices to be aware of the following proposition about the (non-) existence of spin structures in our context:

Proposition 228.6. (Spin–Intersection Form Proposition) Let X be a compact oriented 4-dimensional smooth manifold.

(1) If X is spin, then the intersection form Q_X is even.
(2) If the intersection form Q_X is even and if $\mathrm{H}_1(X)$ has no 2-torsion (by the Hurewicz Theorem 166.10 this condition is satisfied if X is simply connected), then the smooth manifold X is spin.

Proof. The proposition is the content of [**GoS1999**, Corollary 5.7.6], which in turn is proved using the "Wu formula". The proposition is also discussed in [**Scor2005**, p. 163]. ■

Example. It follows from the Spin–Intersection Form Proposition 228.6 (2), the K3-Intersection Form Proposition 228.3 and familiar facts about $S^2 \times S^2$ that the K3-surface and $S^2 \times S^2$ are both spin. □

Remark. We will now see that the converse to neither of the two statements in the Spin–Intersection Form Proposition 228.6 holds:

(1) We consider the *Enriques surfaces*

$$X := \underbrace{\left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \;\middle|\; z_0^4+z_1^4+z_2^4+z_3^4=0\right\}}_{=\text{K3-surface}} \Big/ (z_0, z_1, z_2, z_3) \sim (\overline{z}_1, -\overline{z}_0, \overline{z}_3, -\overline{z}_2).$$

We make the following observations:
(a) It is elementary to show that the above equivalence relation $\sim$ on the K3-surface defines a smooth free orientation-preserving action by the group $\mathbb{Z}/2\mathbb{Z}$.
(b) It follows from (a), the Manifold–Quotient Proposition 25.39 and the Orientation-Action Proposition 31.44 that the Enriques surface X is a closed oriented connected 4-dimensional smooth manifold.
(c) Since the K3-surface is simply connected it follows from (a), together with the Free-Proper Action Lemma 25.44, the Action–Covering Proposition 91.11 and the Action–π_1-Theorem 93.1, that $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$. In particular we obtain from the Hurewicz Theorem 166.10 that $\mathrm{H}_1(X)$ has 2-torsion.
(d) As is shown in [**Hab1982**], the intersection form Q_X is isometric to $E_8 \oplus H$, in particular Q_X is even and $\mathrm{sign}(X) = 8$.

Note that it follows from (d) together with Rokhlin's Theorem 228.7 below that X cannot be spin. In particular we see that the converse to the Spin–Intersection Form Proposition 228.6 (1) does not hold. We refer to [**Hab1982**], [**FS1984**], [**GoS1999**, p. 93] and [**LaM1989**, p. 90] for more details on the Enriques surface.

(2) We consider the smooth manifold $S^1 \times \mathbb{R}\mathrm{P}^3$. It follows from the discussion on page **??** that $S^1 \times \mathbb{R}\mathrm{P}^3$ is parallelizable, i.e. its tangent bundle is a trivial bundle. By the

observation on page **??** this implies that $S^1 \times \mathbb{R}P^3$ admits a spin structure. It follows from the $\mathbb{R}P^n$-H_*-Proposition 155.18, the Sphere-H_*-Proposition 149.4 and the Künneth Theorem 164.15 that $H_2(S^1 \times \mathbb{R}P^3) = 0$ and that $H_1(S^1 \times \mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since $H_2(S^1 \times \mathbb{R}P^3) = 0$ we see that the intersection form of $S^1 \times \mathbb{R}P^3$ is even. But $H_1(S^1 \times \mathbb{R}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has 2-torsion. Thus we see that the converse to the Spin–Intersection Form Proposition 228.6 (2) does not hold. □

Let X be a closed oriented 4-dimensional smooth manifold. If X is spin, then we know by the Spin–Intersection Form Proposition 228.6 (1) that the intersection form Q_X is even. It follows from the Even Forms Proposition 226.30, for purely algebraic reasons, that the signature of X is divisible by 8. Rather mysteriously, Rokhlin's theorem from 1952 gives us a refinement of this statement:

Theorem 228.7. (Rokhlin Theorem) Let X be a closed oriented 4-dimensional smooth manifold. If X is spin, then $\operatorname{sign}(X)$ is divisible by 16.

Proof. This theorem was proved by Vladimir Rokhlin [**Rok1952**][**Rok1986a**, p. 21] in 1952. More modern proofs of Rokhlin's Theorem are given in [**FreK1978**], [**LaM1989**, Theorem II.2.13 and Corollary IV.1.2] and [**Sav2002**, Theorem 2.1]. This theorem was generalized to closed oriented $(4+8k)$-dimensional spin manifolds by Ochanine [**Oc1981**]. ■

We conclude this section with the following variation on Rokhlin's Theorem 228.7, which was historically the first result to address Question 228.1 (2):

Theorem 228.8. (Rokhlin Theorem II) Let X be a closed oriented 4-dimensional smooth manifold. If the intersection form Q_X is even and if $H_1(X)$ has no 2-torsion (by the Hurewicz Theorem 166.10 this condition is satisfied if X is simply connected), then $\operatorname{sign}(X)$ is divisible by 16.

Proof. This theorem follows immediately from the Spin–Intersection Form Proposition 228.6 together with Rokhlin's Theorem 228.7. ■

228.3. **1958: The homotopy type.** We just learned in the Rokhlin Theorem 228.8 that not every non-singular symmetric form over $\mathbb{Z}$ can appear as the intersection form of a closed oriented 4-dimensional smooth manifolds. In the opposite direction, the following theorem was the first noteworthy result towards an affirmative answer to Question 228.1 (1):

Theorem 228.9. (Homotopy Type–4D-Theorem) Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds. Then

$$X \text{ and } Y \text{ are homotopy equivalent} \iff Q_X \text{ is isometric to } \pm Q_Y.$$

Proof. The straightforward "$\Rightarrow$"-direction is the content of the Intersection Form–Homeomorphism Lemma 224.15. The much more difficult "$\Leftarrow$"-direction was proved for smooth manifolds by John Milnor [**Miln1958b**, Theorem 3] in 1958, building on work of John Whitehead [**WhdJ1949a**]. This proof is also written up in [**MH1973**, Theorem V.1.5]. Furthermore [**GoS1999**, Theorem 1.2.25] and [**Kir1989**, Theorem 2.1] sketch a proof that also works for topological manifolds. ■

228.4. **1977: The work of Rob Kirby and Larry Siebenmann.** Unfortunately we need to start out again with a "black box" definition:

Definition 228.10.
(1) Given any compact topological manifold X we denote by $\mathrm{ks}(X) \in \mathrm{H}^4(X;\mathbb{Z}/2\mathbb{Z})$ the corresponding Kirby-Siebenmann invariant which is defined in [**KSi1977**, p. 318] or alternatively in [**FQ1990**, Section 10.2B] and [**Rudy2016**, Chapter 3.4].
(2) If X is a closed oriented connected non-empty 4-dimensional topological manifold, then we know by the Manifold-Top-H^*-Theorem 201.26 (2) that we have $\mathrm{H}^4(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus in that case we think of the Kirby-Siebenmann invariant as an element of $\mathbb{Z}/2\mathbb{Z}$.

The following theorem summarizes a few key properties of the Kirby-Siebenmann invariant:

Proposition 228.11. (Kirby-Siebenmann Invariant Proposition)
(1) Let X be a compact topological manifold.
  (a) If X admits a PL-structure[2466] or a smooth structure, then $\mathrm{ks}(X) = 0$.
  (b) If $\mathrm{ks}(X) = 0$ and if $\dim(X) \geq 5$, then X admits a PL-structure.
(2) If $f\colon X \to Y$ is a local homeomorphism, then $\mathrm{ks}(X) = f^*(\mathrm{ks}(Y)) \in \mathrm{H}^4(X;\mathbb{Z}/2\mathbb{Z})$.
(3) (a) The Kirby-Siebenmann invariant of closed oriented connected non-empty 4-dimensional topological manifolds is additive under connected sums.
  (b) If X is a closed oriented simply connected non-empty 4-dimensional topological manifold with even intersection form, then[2467] [2468] $\mathrm{ks}(X) \equiv \frac{1}{8} \cdot \mathrm{sign}(X) \bmod 2$.

Proof.

(1) (a) By [**Rudy2016**, Theorem 3.4.3] the Kirby-Siebenmann invariant vanishes if X admits a PL-structure. Furthermore note that in the Smooth-implies-PL Theorem 180.16 we saw that every smooth manifold admits a PL-structure. In particular we see that the Kirby-Siebenmann invariant vanishes for smooth manifolds. We refer to [**Cer1968a**, p. IX] and [**Scor2005**, p. 220] for more details.
  (b) If $\mathrm{ks}(X) = 0$ and if $\dim(X) \geq 5$, then it follows again from [**Rudy2016**, Theorem 3.4.3] that X admits a PL-structure.
(2) This statement follows easily from the definition of the Kirby-Siebenmann invariant in [**Rudy2016**, Definition 3.4.2].
(3) These two statements are proved in [**FNOP2025**, Theorem 9.2]. ∎

A priori it is not clear whether there exist compact topological manifolds with non-zero Kirby-Siebenmann invariant. But in the next section we will see that they do exist in the 4-dimensional setup.

[2466] We refer to Definition 180.8 for the definition of a PL-structure.
[2467] It follows from the Even Forms Proposition 226.30 that $\mathrm{sign}(X)$ is divisible by 8.
[2468] For topological manifolds there is also a notion of being "spin" which will not get into. With this notion one can replace the hypothesis "simply connected and even intersection form" by "spin". We refer to [**FNOP2025**, Theorem 9.2] for details.

228.5. **1981: The work of Michael Freedman.** After the somewhat preliminary results by Vladimir Rokhlin and John Milnor in the 1950s, the early 1980s saw two major breakthroughs in the 4-dimensional setting, due to Michael Freedman and Simon Donaldson. We will explain these amazing results in the following two sections.

First, in 1981 Michael Freedman [**Free1982**][2469] gave a complete answer to Question 228.1 in the setting of topological manifolds:

Theorem 228.12. (Freedman Classification Theorem)

(1) Let Q be a non-singular symmetric form over $\mathbb{Z}$ and let $k \in \mathbb{Z}/2\mathbb{Z}$. (If Q is even, then we assume that $k \equiv \frac{1}{8} \cdot \operatorname{sign}(Q) \bmod 2$.) There exists a closed oriented simply connected 4-dimensional topological manifold X whose intersection form is isometric to Q and with $\operatorname{ks}(X) = k \in \mathbb{Z}/2\mathbb{Z}$.

(2) Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds with $\operatorname{ks}(X) = \operatorname{ks}(Y) \in \mathbb{Z}/2\mathbb{Z}$. If $\varphi\colon (\mathrm{H}_2(X), Q_X) \to (\mathrm{H}_2(Y), Q_Y)$ is an isometry of the intersection forms, then there exists an orientation-preserving homeomorphism $f\colon X \to Y$ such that $f_* = \varphi$. Furthermore f is unique up to isotopy.

Proof. The theorem is proved in [**Free1982**, Theorem 1.5] and [**FQ1990**, p. 161]. (The observation that in Statement (2) we can find an *orientation-preserving* homeomorphism is not explicitly mentioned in [**FQ1990**, p. 161], but it does follow from the proof.) Some aspects of the proof are dealt with in much greater detail in [**BKKPR2021**]. The last statement, that f is unique up to isotopy, was proved by Frank Quinn in [**Quin1986**, Theorem 1.1] (see also [**GGHKP2023**, Section 1.3]). (Finally note that many of the results were extended to the case of topological manifolds with non-empty boundary by Steve Boyer [**Boye1986, Boye1993**] and Patrick Orson and Mark Powell [**OP2023**].) ■

The Freedman Classification Theorem 228.12 says in particular the following:

- Every non-singular symmetric even form over $\mathbb{Z}$ is realized by a closed oriented simply connected 4-dimensional topological manifold that is unique up to homeomorphism.
- Every non-singular symmetric odd form over $\mathbb{Z}$ is realized by two closed oriented simply connected 4-dimensional topological manifolds that are distinguished by the Kirby-Siebenmann invariant and that are unique up to homeomorphism.

In the following we will discuss several consequences of the Freedman Classification Theorem 228.12. For the most famous application we need to recall Definition 210.19:

Definition 228.13. Let $n \in \mathbb{N}$ and let M be a closed connected non-empty n-dimensional topological manifold. If $\pi_k(M) = 0$ for $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$, then we call M a homotopy sphere.[2470]

Theorem 228.14. (Topological 4D-Poincaré Theorem) Let X be a 4-dimensional topological manifold. If X is a homotopy sphere, then X is homeomorphic to S^4.

Remark. We refer to [**Free2021**] for some history regarding the Topological 4D-Poincaré Theorem 228.14. □

[2469] Michael Freedman (*1951) is an American mathematician, in 1986 he was awarded a Fields medal for the proof of the Freedman Classification Theorem 228.12.

[2470] In the Homotopy Sphere-Characterization Propositions 210.18 and 210.18 we give several alternative characterizations of homotopy spheres.

Proof. Let X be a closed connected 4-dimensional topological manifold which is a homotopy sphere. By definition this means that we have $\pi_1(X) = 0$ and $\pi_2(X) = 0$. Since $\pi_1(X) = 0$ we obtain from the Hurewicz Theorem 167.9 that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X)$ is an isomorphism. It follows that $\mathrm{H}_2(X) = 0$. This implies in particular, by the Kirby-Siebenmann Invariant Proposition 228.11 (3b), that $\mathrm{ks}(X) = \mathrm{ks}(S^4) = 0$. It now follows from the Freedman Classification Theorem 228.12 (2) that X is homeomorphic to S^4. ■

The Freedman Classification Theorem 228.12 gives us two unusual, deeply mysterious, closed oriented 4-dimensional topological manifolds:

Definition 228.15.

(1) By the Freedman Classification Theorem 228.12 there exists a closed oriented simply connected 4-dimensional topological manifold X with $Q_X \cong E_8$, which is unique up to an orientation-preserving[2471] homeomorphism. We refer to this topological manifold as the E_8-manifold and sometimes we denote it by E_8 as well.

(2) We consider the form that is represented by the (1×1)–matrix (1). By the Freedman Classification Theorem 228.12 (2) there exist, up to homeomorphism, precisely two closed oriented simply connected 4-dimensional topological manifolds with this intersection form, and these two topological manifolds are distinguished by the Kirby-Siebenmann invariant.

- By the discussion on page 4331 and the Kirby-Siebenmann Invariant Proposition 228.11 (1) we know that the closed oriented simply connected smooth manifold $\mathbb{C}\mathrm{P}^2$ corresponds to the case of vanishing Kirby-Siebenmann invariant.
- The closed oriented simply connected 4-dimensional topological manifold with intersection form (1) and non-zero Kirby-Siebenmann invariant is usually denoted by $*\mathbb{C}\mathrm{P}^2$.

Note that the topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are not homeomorphic since the Kirby-Siebenmann invariants differ. But also note that it follows from the Homotopy Type–4D-Theorem 228.9 that the topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are homotopy equivalent.

Using the Freedman Classification Theorem 228.12 (1) and (2) we can give an affirmative answer to Question 228.4 (1):

Theorem 228.16. (K3-Topological Splitting Theorem) The K3-surface is homeomorphic to $2 \cdot E_8 \# 3 \cdot (S^2 \times S^2)$.

Proof. It follows from the Connected Sum-Invariants Proposition 224.17 and the Intersection Form-Additivity Proposition 224.18, together with the uniqueness statement of the Freedman Classification Theorem 228.12 (2), that there exists an orientation-preserving homeomorphism of the K3-surface to $2 \cdot E_8 \# 3 \cdot (S^2 \times S^2)$. ■

[2471] If X and Y are two closed oriented connected 4-dimensional topological manifolds with positive signature, then it follows from the Manifold–Orientation–Signature Lemma 226.16 that any homeomorphism $f\colon X \to Y$ that induces an isometry of the intersection forms is necessarily orientation-preserving. By the E_8-Lemma 226.23 we know that $\mathrm{sign}(E_8) > 0$. It follows from this discussion that the E_8-manifold is indeed unique up to an orientation-preserving homeomorphism.

Among many other consequences of the Freedman Classification Theorem 228.12 we obtain the following rather surprising result:

Theorem 228.17. (No Smooth Structure-4D-Theorem) There exist closed orientable 4-dimensional topological manifolds that do not admit a smooth structure.

Proof. At this point we can combine the above results to give two types of examples of such topological manifolds:

(1) It follows from Rokhlin's Theorem 228.8 (or alternatively from the Kirby-Siebenmann Invariant Proposition 228.11) that the E_8-manifold cannot admit a smooth structure.
(2) We consider $*\mathbb{C}\mathrm{P}^2$. Since by construction we have $\mathrm{ks}(*\mathbb{C}\mathrm{P}^2) = 1$ it follows from the Kirby-Siebenmann Invariant Proposition 228.11 that $*\mathbb{C}\mathrm{P}^2$ cannot admit a smooth structure. ■

We conclude this section with two remarks:

Remark.

(1) It is natural to ask to what degree the Freedman Classification Theorem 228.12 generalizes to the non-simply connected case. In [**FQ1990**, Theorem 10.7A] a generalization is proved for the case $\pi_1 = \mathbb{Z}$, where one needs to replace the ordinary intersection form by the twisted intersection pairings which we will introduce in Chapter 337. A few more (partial) classification results are summarized in [**FNOP2025**, Chapter 12]. But for most fundamental groups the classification of closed oriented 4-dimensional topological manifolds is still completely out of reach.
(2) We make a quick foray into the high-dimensional world. For $n \geq 3$ Wall [**Wall1962**] (see also [**Ran2002**, p. 269]) classified diffeomorphism classes of pairs $(M, \partial M)$ where M is an orientable $(n-1)$-connected $2n$-dimensional smooth manifold and where the boundary ∂M is a homotopy sphere. The classification goes via the intersection form and one extra invariant which lies in the homotopy group $\pi_{n-1}(\mathrm{SO}(n))$. □

228.6. **1982: The work of Simon Donaldson.** In a remarkable twist, shortly after Freedman proved his results on the intersection forms of 4-dimensional topological manifolds, Simon Donaldson[2472], proved the following result regarding intersection forms of 4-dimensional smooth manifolds.

Theorem 228.18. (Donaldson Diagonalization Theorem) Let X be a closed oriented connected 4-dimensional smooth manifold. If Q_X is positive-definite, then Q_X is represented by the identity matrix.[2473]

Proof. The simply connected case was first proved by Simon Donaldson in [**Don1983**, Theorem 1], see also [**Don1986**, Theorem A]. Later, in [**Don1987a**, Theorem 1], the general case (without any restrictions on the fundamental group) was proved. An exposition of the proof is also given in [**Law1985**]. A different proof (of the general statement),

[2472]Simon Donaldson (*1957) is an English mathematician. He proved Theorem 228.18 when he was a second year PhD student. The Fields medalist Michael Atiyah described in [**At1987**] the work of Simon Donaldson as follows: "In 1982, when he was a second-year graduate student, Simon Donaldson proved a result that stunned the mathematical world." Simon Donaldson was awarded a Fields medal in 1986.
[2473]If Q_X is negative-definite, then we apply the theorem to $-X$, and we see that Q_X is represented by $-\mathrm{id}$.

using Seiberg-Witten invariants is given in [**Nic2000**, Theorem 2.4.18] and a proof using Heegaard Floer homology is given in [**OS2003b**, Theorem 9.1]. ■

Example. We consider the positive-definite symmetric form $E_8 \oplus E_8$. It has signature 16, hence Rokhlin's Theorem 228.8 has no objections to this form being realized by a closed oriented 4-dimensional smooth manifold. But $E_8 \oplus E_8$ is an even form. It follows from the Parity Lemma 226.25 that it cannot be represented by the identity matrix. Thus the Donaldson Diagonalization Theorem 228.18 implies that $E_8 \oplus E_8$ cannot be the intersection form of a closed oriented connected 4-dimensional smooth manifold. □

To understand the significance of the Donaldson Diagonalization Theorem 228.18, often referred to as *Donaldson Theorem A*, recall that on page 4381 we saw that there are more than 10^{51} isometry classes of non-singular positive-definite symmetric forms of rank 40 over $\mathbb{Z}$. By the Freedman Classification Theorem 228.12 all of them arise as intersection forms of closed oriented simply connected 4-dimensional topological manifolds. By the Donaldson Diagonalization Theorem 228.18 the only one which appears as the intersection form of a closed oriented connected 4-dimensional smooth manifold is the one represented by the identity matrix.[2474]



Remark. It is difficult to exaggerate the importance of the Donaldson Diagonalization Theorem 228.18 in mathematics.[2475] In fact the Donaldson Diagonalization Theorem 228.18 can appear in totally unexpected situations. For example in 2017 Duncan McCoy [**McCoy2017**] used the Donaldson Diagonalization Theorem 228.18, and many other ingredients, to show that if K is an alternating knot with unknotting number one[2476], then such a crossing change can already be performed in an alternating diagram. For example, according to [**LiM**] the three alternating knots shown in the figure below have unknotting number one. By the aforementioned result we now know that we can turn these three knots into the trivial knot by changing a crossing in the alternating diagrams shown above. □

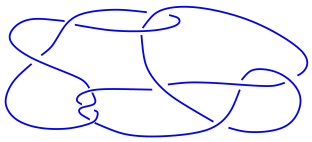
228.7. **1996: The 10/8-Theorem and the 11/8-Conjecture.** The Donaldson Diagonalization Theorem 228.18 gives us a complete answer to the question, which non-singular

[2474]The identity matrix represents, by the Intersection Form-Additivity Proposition 224.18 and the discussion on page 4331, the intersection form of the connected sum of 40 copies of $\mathbb{C}P^2$.

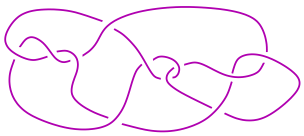
[2475]Admittedly it is also hard to underestimate the role of the Donaldson Diagonalization Theorem 228.18 outside of mathematics and perhaps physics.

[2476]We recall the following definitions from pages Exercise 113.12 and Definition 118.18:

- A knot is called *alternating* if it admits a diagram that is alternating, i.e. which has the property that each undercrossing is followed by an overcrossing and vice versa.
- The *unknotting number* of a knot K is the minimal number of crossing changes needed to turn K into the trivial knot.


the knot $11a_{12}$


the knot $11a_{41}$


the knot $11a_{56}$

positive-definite symmetric forms can arise as the intersection form of a closed oriented 4-dimensional smooth manifold: Only the forms that are represented by the identity matrix.

Now the question arises, what can we say about *indefinite* intersection forms of closed oriented 4-dimensional smooth manifolds? First recall that in the Indefinite Odd Forms Corollary 226.29 we saw that every non-singular indefinite odd symmetric form over $\mathbb{Z}$ is isometric to $m \cdot (1) \oplus n \cdot (-1)$ for some $m, n \in \mathbb{N}_0$. Next note that according to the Intersection Form-Additivity Proposition 224.18 and the discussion on page 4331 every such form is realized by the smooth manifold $m \cdot \mathbb{C}\mathrm{P}^2 \# n \cdot \overline{\mathbb{C}\mathrm{P}}^2$.

Therefore, in the following we only need to discuss the realizability of non-singular indefinite *even* symmetric forms. Recall that in the Indefinite Even Forms Corollary 226.31 we showed that every such form is isometric to $m \cdot E_8 \oplus n \cdot H$ for some $m \in \mathbb{Z}$ and some $n \in \mathbb{N}$. Which values of m and n can we realize by closed oriented connected smooth manifolds? We can certainly take connected sums of k copies of the K3-surface and l copies of $S^2 \times S^2$. It follows from the Intersection Form-Additivity Proposition 224.18, the calculation of the intersection forms of the K3-surface, see the K3-Intersection Form Proposition 228.3, and of the intersection form of $S^2 \times S^2$ that

$$Q_{k \cdot \mathrm{K3} \,\#\, l \cdot (S^2 \times S^2)} \;\cong\; 2k \cdot E_8 \oplus (l + 3k) \cdot H.$$

Recall that it follows from the E_8-Lemma 226.23 that $\operatorname{sign}(E_8) = 8$ and that on page 4369 we saw that $\operatorname{sign}(H) = 0$. Some 8th grade mathematics now shows that

$$b_2(k \cdot \mathrm{K3} \,\#\, l \cdot (S^2 \times S^2)) \;\geq\; \tfrac{11}{8} \cdot |\operatorname{sign}(k \cdot \mathrm{K3} \,\#\, l \cdot (S^2 \times S^2))|,$$

and that equality holds precisely if $l = 0$.

Note that by definition we have for any closed oriented 4-dimensional topological manifold the trivial inequality $b_2(M) \geq |\operatorname{sign}(M)|$. So the question that arises is, whether for closed oriented smooth manifolds we can do better than the inequality given by the above examples. The following theorem gives us some restrictions:

Theorem 228.19. (Furuta's 10/8-Theorem) If M is a closed oriented spin 4-dimensional smooth manifold with indefinite intersection form, then

$$b_2(M) \;\geq\; \tfrac{10}{8} \cdot |\operatorname{sign}(M)| + 2.$$

In particular $Q_M \cong m \cdot E_8 \oplus n \cdot H$ with $n \geq |m| + 1$.

Proof. Let M be a closed oriented spin 4-dimensional smooth manifold with indefinite intersection form. Mikio Furuta [**Furu2001**] (see also [**Akb1996**, Theorem (Furuta)] proved in 1996 the inequality

$$b_2(M) \;\geq\; \tfrac{10}{8} \cdot |\operatorname{sign}(M)| + 2.$$

It now follows easily from the Indefinite Even Forms Corollary 226.31 and the calculation of the rank and signature of H and E_8 that $Q_M \cong m \cdot E_8 \oplus n \cdot H$ with $n \geq |m| + 1$. ■

Furuta's 10/8 Theorem 228.19 is not yet optimal since it does not quite close the gap between the forms which we can realize by closed oriented smooth manifolds and the forms which we can exclude. The following conjecture (which was first formulated in [**Matsu1982**, p. 289]) predicts that we cannot do better than the above examples.

Conjecture 228.20. (11/8-Conjecture) If M is a closed oriented spin 4-dimensional smooth manifold with indefinite intersection form, then

$$b_2(M) \geq \frac{11}{8} \cdot |\operatorname{sign}(M)|.$$

In particular the conjecture predicts, that in this setting $Q_M \cong m \cdot E_8 \oplus n \cdot H$ with $n \geq \frac{3}{2} \cdot |m|$.[2477]

Remark. For simply connected smooth manifolds currently the best result towards the 11/8-Conjecture 228.20 is [**HLSX2022**, Corollary 1.13] which says that if M is a closed oriented spin simply connected 4-dimensional smooth manifold that is not homeomorphic to S^4, $S^2 \times S^2$ or the K3-surface and whose intersection form is indefinite, then we have the inequality $b_2(M) \geq \frac{10}{8} \cdot |\operatorname{sign}(M)| + 4$. □

Furuta's 10/8-Theorem 228.19 allows us to answer Question 228.4 (2) in the negative. More precisely, we have the following theorem in the smooth setting, which differs dramatically from the K3-Topological Splitting Theorem 228.16 in the topological setting:

Theorem 228.21. (K3-Smooth Non-Splitting Theorem) If there exist closed oriented smooth manifolds M and N such that K3 is homeomorphic[2478] to $M\#N$, then one of M or N is homeomorphic to S^4.

Proof. Suppose that we have closed oriented smooth manifolds M and N such that K3 is homeomorphic to $M\#N$. We make the following four observations:

(1) Without loss of generality we have $\operatorname{sign}(M) \geq \operatorname{sign}(N)$.

(2) (a) It follows from the K3-Intersection Form Proposition 228.3, the Parity Lemma 226.25 (3) and the Intersection Form-Additivity Proposition 224.18 that Q_M and Q_N are even.

(b) It follows from the K3-Intersection Form Proposition 228.3, the Hurewicz Theorem 166.10 and the Connected Sum-H_*-Proposition 185.23 that we have the equalities $\mathrm{H}_1(M) = \mathrm{H}_1(N) = 0$.

(3) It follows from (2) and the Spin–Intersection Form Proposition 228.6 that M and N are spin.

(4) From the Manifold Signature Additivity Lemma 226.17 and the K3-Intersection Form Proposition 228.3 we obtain that $\operatorname{sign}(M) + \operatorname{sign}(N) = 16$.

(5) By the K3-Intersection Form Proposition 228.3 and the Connected Sum-Invariants Proposition 224.17 we know that $b_2(M) + b_2(N) = 22$.

Since we assume that M is smooth it follows from these facts together with Rokhlin's Theorem 228.8 that $\operatorname{sign}(M) = 16$. Next note that Furuta's 10/8-Theorem 228.19 implies that $b_2(M) \geq \frac{10}{8} \cdot 16 + 2 = 22$. By the K3-Intersection Form Proposition 228.3 this equals $b_2(\mathrm{K3})$. Thus we obtain from the Connected Sum-Invariants Proposition 224.17 that $b_2(N) = 0$.

[2477] As before, it follows from the Indefinite Even Forms Corollary 226.31 and the calculation of the rank and signature of H and E_8 that these two statements are equivalent.

[2478] Note that "smooth manifold" and "homeomorphic" are not typos.

Next note that it follows from the Connected Sum-Invariants Proposition 224.17 and the K3-Intersection Form Proposition 228.3 that $\pi_1(N) = 0$. Recall that by the Hurewicz Theorem 167.9 we know that if $\pi_1(N) = 0$ and $\mathrm{H}_2(N) = 0$, then $\pi_2(N) = 0$. We have thus shown that N is a homotopy 4-sphere. Thus it follows from the Topological 4D-Poincaré Theorem 228.14 that N is homeomorphic to S^4.[2479] ■

228.8. **Non-diffeomorphic smooth structures.** The other question which arises in this context is the following: If two closed orientable 4-dimensional smooth manifolds are homeomorphic, are they also necessarily diffeomorphic?

The following elementary lemma let's us rephrase the question:

Lemma 228.22. (Inequivalent Smooth Structure Lemma) Let X be a topological manifold and let I be a set. The following two statements are equivalent:

(a) There exists a family $\{X_i\}_{i\in I}$ of smooth manifolds such that each X_i is homeomorphic to X and such that the X_i are pairwise non-diffeomorphic.
(b) There exists a family $\{\mathcal{A}_i\}_{i\in I}$ of smooth structures[2480] on X such that the smooth manifolds $(X, \mathcal{A}_i)$ are pairwise non-diffeomorphic.

Proof. The (b)⇒(a)-direction is obvious. We turn to the proof of the (a)⇒(b)-direction. Let $\{(X_i, \mathcal{A}_i)\}_{i\in I}$ be a family of smooth manifolds such that for each $i \in I$ there exists a homeomorphism $f\colon X \to X_i$ and such that the X_i are pairwise non-diffeomorphic. The homeomorphisms allow us to define smooth structures $f^*(\mathcal{A}_i)$ on X. These have the desired properties. ■

We continue with the following semi-mathematical definition:

Definition 228.23.

(1) Let X be a topological manifold. We say that two smooth structures $\mathcal{A}$ und $\mathcal{B}$ are non-diffeomorphic if the smooth manifolds $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are not diffeomorphic.
(2) If X is a smooth manifold that is equipped with a "standard smooth structure $\mathcal{S}$", then every smooth structure $\mathcal{A}$ which is not diffeomorphic to $\mathcal{S}$ is called exotic.

We can now answer the question, which we posed in the beginning of this section, in the negative. For example the following theorem was proved in 1998 by Ron Fintushel and Ron Stern [**FS1998**]:

Theorem 228.24. (Exotic K3-Theorem) The K3-surface admits infinitely many pairwise non-diffeomorphic smooth structures.

Similar results have been obtained for many other 4-dimensional smooth manifolds, e.g. for $\mathbb{C}\mathrm{P}^2\#k\overline{\mathbb{C}\mathrm{P}}^2$ with $k = 9$ by Simon Donaldson [**Don1987a**] in 1987, $k = 8$ by Dieter Kotschick [**Kot1989**] in 1989 and $k = 7$ by Jongil Park [**Park2005**] in 2005, and since then also for $k = 6, 5, 4, 3, 2$, see e.g. [**AP2008, AP2010, BaK2008**] for more details on the history of this problem. It is unknown whether $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ or $S^2 \times S^2$ admit an exotic smooth structure.

The following question is surely one of the most interesting problem in the theory of 4-dimensional smooth manifolds:

[2479] Since Question 228.26 is still open, we do not know whether N is diffeomorphic to S^4.
[2480] Recall that in Definition 25.5 we defined a smooth structure on a topological manifold Y as a maximal smooth atlas on Y.

Question 228.25. Does every closed oriented connected non-empty 4-dimensional smooth manifold M admit infinitely many pairwise non-diffeomorphic smooth structures?

The following question is an important special case of Question 228.25:

Question 228.26. (Smooth Poincaré 4D-Conjecture) Does S^4 admit an exotic smooth structure?[2481]

So far we discussed closed manifolds. How about non-compact manifolds? In 1961 John Stallings [**Stal1961**, p. 481] wrote:

Euclidean space $\mathbb{R}^4$ ought to have [...] a unique differentiable structure.

It turns out that this is indeed true for $n \leq 3$ by work of Christos Papakyriakopoulos [**Pap1946**], Edwin Moise [**Moi1952**, Theorem 4] and James Munkres [**Mun1960a**, Corollary 6.6]. (We refer to the 3D-Unique Smooth Structure-Theorem **??** for more details.) Furthermore John Stallings [**Stal1961**, Corollary 5.2] showed that the above statement is also true for $n \geq 5$.

It turns out that the situation is dramatically different in dimension 4. Again the Donaldson Diagonalization Theorem 228.18 together with the Freedman Classification Theorem 228.12 are at the heart of the proof of the following theorem:

Theorem 228.27. ($\mathbb{R}^4$-Smooth Structure Theorem) There are uncountably many pairwise non-diffeomorphic smooth structures on $\mathbb{R}^4$.

Proof. The first examples of exotic smooth structures on $\mathbb{R}^4$ were given by Michael Freedman (see [**Gom1983**, Chapter 2]). Later infinitely, and even uncountably many pairwise non-diffeomorphic smooth structures were found by Robert Gompf [**Gom1985**, Theorem 1.3] and Cliff Taubes [**Tau1987**, Theorem 1.1]. ■

228.9. **Stabilizations and diffeomorphisms.** We conclude the discussion of the relationship between smooth and topological aspects of 4-dimensional smooth manifolds with the following theorem, which says that smooth manifolds that are homeomorphic are in fact "stably diffeomorphic".

Theorem 228.28. (Stable Diffeomorphism Theorem) Let M and N be two compact 4-dimensional smooth manifolds that are homeomorphic.[2482]

(1) If the smooth manifolds M and N are orientable, then there exists a $k \in \mathbb{N}$ such that $M \# k \cdot (S^2 \times S^2)$ and $N \# k \cdot (S^2 \times S^2)$ are diffeomorphic.
(2) If the smooth manifolds M and N are non-orientable, then there exists a $k \in \mathbb{N}$ such that $M \# k \cdot (S^2 \tilde{\times} S^2)$ and $N \# k \cdot (S^2 \tilde{\times} S^2)$ are diffeomorphic.[2483]

Proof. The theorem was first proved by Robert Gompf [**Gom1984**, p. 116] in 1984. An alternative proof, using Matthias Kreck's modified surgery [**Kre1999**], is given in [**Tei1992**, Theorem 5.1.1] and [**FNOP2025**, Theorem 12.2]. Also note that the special

[2481] An answer to this question would lead, assuming the mathematician is under 40, straight to a Fields medal.
[2482] By the Smooth-Topological Orientation Proposition 191.25 we know that being orientable is a homeomorphism invariant.
[2483] Here $S^2 \tilde{\times} S^2$ denotes the total space of the non-trivial S^2-bundle over S^2 that we will introduce in Exercise 228.8 and in Definition 241.23.

case that the smooth manifolds are closed and simply connected was first proved by C. T. C. Wall [**Wall1964**, Theorem 3] in 1964 (see also [**Scor2005**, p. 149] and [**Quin1983**, Theorem 1.1]). ■

This raises the question, whether in the above theorem $k = 1$ suffices. More precisely the following question is still open:

Question 228.29. Let M and N be two closed oriented 4-dimensional smooth manifolds that are homeomorphic. Is $M\#(S^2 \times S^2)$ diffeomorphic to $N\#(S^2 \times S^2)$?

There are some indications, see [**JZ2018**], that the answer to the above question should be no.

228.10. Exercises for Chapter 228 .

Exercise 228.1. As we discussed in Definition 228.15, it follows from the Freedman Classification Theorem 228.12 that there exists a closed orientable simply connected 4-dimensional topological manifold $*\mathbb{C}\mathrm{P}^2$ whose intersection form is represented by the (1×1)-matrix (1) and with $\mathrm{ks}(*\mathbb{C}\mathrm{P}^2) = 1$.

Let M and N be two closed oriented simply connected 4-dimensional topological manifolds. Show that there exist $m, m', n, n' \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}$ such that the topological manifolds $M \# m \cdot \mathbb{C}\mathrm{P}^2 \# m' \cdot \overline{\mathbb{C}\mathrm{P}^2} \# \epsilon \cdot *\mathbb{C}\mathrm{P}^2$ and $N \# n \cdot \mathbb{C}\mathrm{P}^2 \# n' \cdot \overline{\mathbb{C}\mathrm{P}^2}$ are homeomorphic.
Hint. Make use of the properties of the Kirby-Siebenmann invariant discussed in the Kirby-Siebenmann Invariant Proposition 228.11. Furthermore it is strongly advisable to use the Indefinite Forms Theorem 226.28.

Exercise 228.2. Let M and N be two closed oriented simply connected 4-dimensional smooth manifolds. Show that there exist $m, m', n, n' \in \mathbb{N}_0$ such that $M \# m \cdot \mathbb{C}\mathrm{P}^2 \# m' \cdot \overline{\mathbb{C}\mathrm{P}^2}$ is diffeomorphic to $N \# n \cdot \mathbb{C}\mathrm{P}^2 \# n' \cdot \overline{\mathbb{C}\mathrm{P}^2}$.
Remark. Make use of Exercise 228.1 and use the fact from page 4379 that the two smooth manifolds $(S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$ are diffeomorphic.

Exercise 228.3. Show that given any finitely presented group π there exist infinitely many closed oriented connected 4-dimensional topological manifolds $\{X_i\}_{i\in\mathbb{N}}$ with the following properties three:
(1) For each $i \in \mathbb{N}$ we have $\pi_1(X_i) \cong \pi$.
(2) The X_i are pairwise non-homeomorphic.
(3) No X_i admits a smooth structure.
Remark. You can make use of the Manifold-π_1-Realization Proposition 102.5.

Exercise 228.4. Show that there exist closed non-orientable 4-dimensional topological manifolds that do not admit a smooth structure.

Exercise 228.5. Show that there exists a closed orientable 4-dimensional smooth manifold M and a free continuous $\mathbb{Z}/2\mathbb{Z}$-action such that $M/\mathbb{Z}/2\mathbb{Z}$ is a 4-dimensional topological manifold[2484] that does not admit a smooth structure.

[2484]It follows from the Manifold–Quotient Proposition 25.39 that $M/\mathbb{Z}/2\mathbb{Z}$ is a 4-dimensional topological manifold.

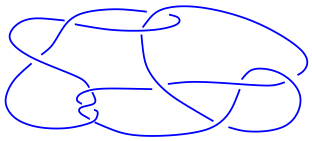
Exercise 228.6. Show that if X is a simply connected 4-dimensional smooth manifold with positive-definite, even intersection form, then X is homeomorphic to S^4.

Exercise 228.7. Let Y be a compact 3-dimensional smooth manifold which is a homology 3-sphere, i.e. we have $\mathrm{H}_*(Y) \cong \mathrm{H}_*(S^3)$. By [**FQ1990**, Theorem 9.3C] there exists a compact contractible 4-dimensional topological manifold W with $\partial W = Y$. Show that Y is homeomorphic to a topological submanifold of $\mathbb{R}^4$.
Hint. What can you say about the double DW?

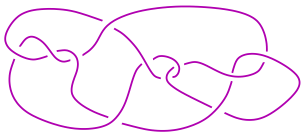
Exercise 228.8. Show that there exists an S^2-bundle $S^2 \mathbin{\widetilde{\times}} S^2 \to S^2$ such that the intersection form of $S^2 \mathbin{\widetilde{\times}} S^2$ is represented by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Hint. Construct the bundle by gluing $S^2 \times S^2_{\geq 0}$ to $S^2 \times S^2_{\leq 0}$ and use the Cup Product–Intersection Number Theorem 220.10 to calculate the intersection form.

Exercise 228.9. In each of the following alternating diagrams find a crossing which turns the given diagram into a diagram of the unknot.


the knot $11a_{12}$


the knot $11a_{41}$


the knot $11a_{56}$

Exercise 228.10. Show that if the 11/8-Conjecture 228.20 holds, then every closed oriented simply connected 4-dimensional smooth manifold is *homeomorphic* either to a connected sum of the form $m{\cdot}\mathbb{C}\mathrm{P}^2 \# n{\cdot}\overline{\mathbb{C}\mathrm{P}}^2$ or to a connected sum of the form $m{\cdot}\mathrm{K3} \# n{\cdot}(S^2 \times S^2)$.

Exercise 228.11. Show that $8 \cdot (S^2 \times S^2)$ is homeomorphic to the connected sum of two topological manifolds M and N that do not admit a smooth structure.

Exercise 228.12. Show that the K3-surface admits at least 96 orientation-preserving self-homeomorphisms that are pairwise non-homotopic.

Exercise 228.13.

(a) Show that there exists an algorithm that can determine whether or not two given non-singular *indefinite* symmetric forms φ and ψ over $\mathbb{Z}$ are isometric.

(b) Show that there exists an algorithm that, given a *positive-definite* symmetric form $\varphi\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}_{\geq 0}$ and given $C \in \mathbb{Z}$, determines the set $\{v \in \mathbb{Z}^n \mid \varphi(v,v) \leq C\}$.
Hint. Consider the minimum of the continuous function $f\colon S^{n-1} \to \mathbb{R}_{>0}$ that is given by $x \mapsto f(x) := \varphi(x,x)$.

(c) Show that there exists an algorithm that can determine whether or not two given non-singular *definite* symmetric forms φ and ψ over $\mathbb{Z}$ are isometric.

Exercise 228.14. Let N be a closed oriented simply connected 4-dimensional smooth manifold which is not spin. Show that $N \# (S^2 \times S^2)$ is homeomorphic to $N \# (S^2 \mathbin{\widetilde{\times}} S^2)$.

Exercise 228.15. Let M be a closed oriented simply connected 4-dimensional smooth manifold with $\chi(M) = 6$ and $\mathrm{sign}(M) = -2$. Is the homeomorphism type of M uniquely determined by the given data?

Exercise 228.16. For the purpose of this exercise we introduce the following three definitions:

- We say that a closed oriented connected topological manifold X is *irreducible* if the following condition holds: Whenever $X \cong Y \# Z$, i.e. whenever X is the connected sum of two closed oriented connected topological manifolds Y and Z, then at least one of Y or Z is homeomorphic to a sphere.
- A *decomposition* of a closed oriented simply connected 4-dimensional topological X manifold is a finite sequence $Y_1, \dots, Y_k$ of closed oriented irreducible topological manifolds such that X is homeomorphic to $Y_1 \# \dots \# Y_k$ and such that no Y_i is homeomorphic to S^4.
- We say that two decompositions are *equivalent* if they agree up to reordering.

Now we can formulate the problems:

(a) Show that $\mathbb{C}\mathrm{P}^2, S^2 \times S^2$ and the E_8-manifold are irreducible.

(b) Show that every closed oriented simply connected 4-dimensional topological manifold X admits a decomposition in the above sense.[2485]

(c) Let X be a closed oriented simply connected smooth manifold.

  (i) If Q_X is odd, then show that X admits two non-inequivalent decompositions if and only if $b_2(X) \geq 2$.

  (ii) If Q_X is even, then show that X admits two non-inequivalent decompositions if and only if $b_2(X) \geq 16$.

[2485]We use the convention from page 1238 which says that the connected sum of the empty family of n-dimensional topological manifolds is S^n.

Part XXV

Cohomology theories

In this part of the lecture notes we will study the question, which properties characterize the theory of real cohomology groups of smooth manifolds? In the Homotopy-H*-Proposition 198.11 we showed that singular cohomology is a homotopy invariant and in the Mayer–Vietoris-H*-Theorem 200.11 (1) we showed that singular cohomology gives rise to Mayer–Vietoris sequences.

It turns out that there is a third property one needs to consider to get a full characterization, namely one needs to consider cohomology groups of unions of open subsets of a topological space or a smooth manifold.

The formulation of that result requires the notion of an inverse limit. Thus in this part of the lecture notes we first study inverse limits, then we study cohomology groups of ascending sequences of subspaces, before we characterize cohomology on smooth manifolds. This characterization will then allow us to show that for smooth manifolds de Rham cohomology is naturally isomorphic to singular cohomology with real coefficients.

This part of the lecture notes is organized as follows:

- In Chapter 229 we will introduce the very general notion of an inverse limit of an inverse system.
- In Chapter 230 we will see that if we can write a topological space X as the ascending union of a sequence X_i of open subsets, then the cohomology groups of X agree with the inverse limit of the inverse sequence of cohomology groups of the X_i.
- In Chapter 231 we will introduce the notion of a "smooth cohomology theory". We will see that real singular cohomology and de Rham cohomology both define a "smooth cohomology theory". Furthermore we will see that any two smooth cohomology theories are isomorphic if they agree for the 0-dimensional smooth manifold given by a point and if there exists a natural transformation from one to the other. In the De Rham Isomorphism-Theorem 231.16 we will use this result to show that for smooth manifolds with empty boundary real singular cohomology is naturally isomorphic to de Rham cohomology.
- In Chapter 232 we will show in the De Rham–Ring Isomorphism Proposition 232.1 that the isomorphism from the De Rham Isomorphism-Theorem 231.16 is actually an isomorphism of rings.
- In Chapter 233 we will use the De Rham Isomorphism-Theorem 231.16 to prove the Maps-to-S^1-Proposition 233.4 which says that for a smooth manifold M homotopy classes of smooth maps $M \to S^1$ are in bijection to cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$.
- In Chapter 234 we will discuss which homology classes can be represented by submanifolds. For example, we will use the Poincaré Duality Theorem 210.1 and the Maps-to-S^1-Proposition 233.4 to prove the Codimension-One Submanifold Existence Proposition 234.6 which says that, given a compact orientable n-dimensional smooth manifold, every class in $\mathrm{H}_{n-1}(M)$ can be represented by a submanifold.

CHAPTER 229

Inverse limits

In Chapter 104 we discussed in great detail the direct colimit of a direct system. In this chapter we now study the "contravariant" analog of direct colimits, namely inverse limits. These will play an important role in the following two chapters.

This chapter is organized as follows:

- In Section 229.1 we will introduce inverse systems and inverse limits.
- In Section 229.2 we will prove the existence of inverse limits.
- In Appendix 229.3 we will discuss the inverse limit of topological spaces.
- In Appendix 229.4 we will discuss the profinite completion of a group.
- In Section 229.5 we will provide exercises for Chapter 229.

In the next Chapter 230 we will discuss the relationship between cohomology groups and limits.

229.1. **Inverse systems and inverse limits.** First we recall Definitions (1) and 104.1.

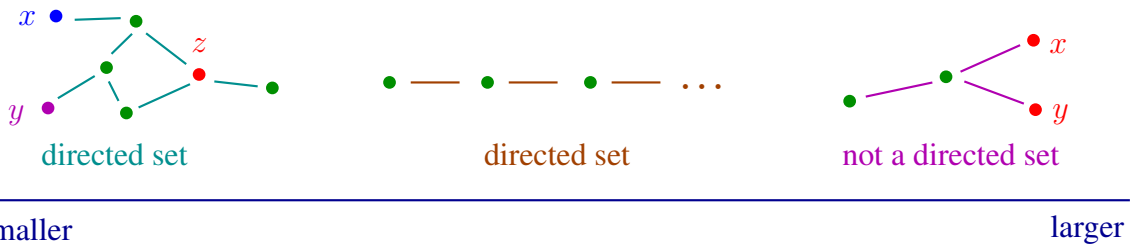
Definition 229.1.

(1) Let X be a set and let $\leq$ be a relation on X.
   (a) The relation is called reflexive if for every $x \in X$ we have $x \leq x$.
   (b) The relation is called transitive if for every $x, y, z \in X$ we have
   $$x \leq y \text{ and } y \leq z \implies x \leq z.$$
   (c) The relation is called antisymmetric if $x \leq y$ and $y \leq x$ implies $x = y$.
   (d) A relation that is reflexive, transitive and antisymmetric is called a partial order.

(2) A directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) We consider the three partial orders that are illustrated in the figure below. Here it is understood that $P \leq Q$ if P is connected to Q via a path that goes throughout from left to right. We see that the two partial orders to the left form directed sets whereas the one on the right does not form a directed set.

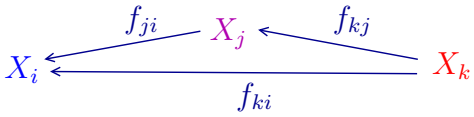


(2) Let X be a subset of $\mathbb{R}$. If we denote by $\leq$ the usual "less or equal" relation on the real numbers, then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
(3) We let $I = \mathbb{N}$, but this time we use the partial order given by divisibility, i.e. we write $k \leq l$ if and only if $k \mid l$. This turns $\mathbb{N}$ into a directed set, since given $x, y \in \mathbb{N}$ we have $x \mid xy$ and $y \mid xy$. □

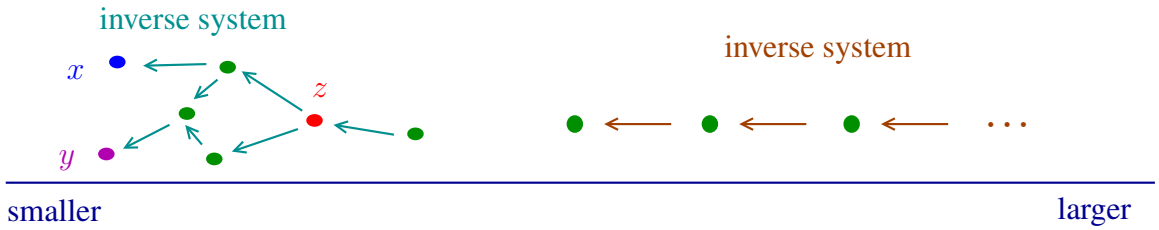
The following definition is basically the same as the definition of a direct system in Definition 104.2, except that all arrows get reversed. We will make use of this concept only rarely, mostly when we study cohomology groups.

Definition 229.2. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over I is a diagram in the opposite category $\mathcal{C}^{\text{op}}$ over the partially ordered set $(I, \leq)$ in the sense of Definition 6.10. In other words, an inverse system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[2486] such that the following two conditions are satisfied:

(1) $f_{ii} = \text{id}_{X_i}$ for all $i \in I$,
(2) $f_{ki} = f_{ji} \circ f_{kj}$ for all $i, j, k \in I$ with $i \leq j \leq k$.



Example. We consider the two directed sets from the above example.



In the figure above we illustrate corresponding inverse systems. □

Example.

(A) We let $I = \mathbb{N}$. As above we use the partial order given by divisibility, i.e. we write $k \leq l$ if and only if $k \mid l$. For each $k \in \mathbb{N}$ we define $X_k := \mathbb{Z}/k\mathbb{Z}$ and for $k \mid l$ we denote by

$$\begin{array}{rcl} f_{lk} \colon \mathbb{Z}/l\mathbb{Z} & \to & \mathbb{Z}/k\mathbb{Z} \\ n + l \cdot \mathbb{Z} & \mapsto & n + k \cdot \mathbb{Z} \end{array}$$

the natural projection. Clearly $(\{\mathbb{Z}/k\mathbb{Z}\}_{k \in \mathbb{N}}, \{f_{lk}\}_{k \mid l})$ forms an inverse system in the category *Ring* of rings. □

When we studied direct systems we saw that most direct systems we were interested in arose from "direct sequences". Similarly we will see that most inverse systems we are interested will arise from "inverse sequences". The following definition of an "inverse sequence" is almost identical to the definition of a "direct sequence" in Definition 104.3, we just need to "invert the arrows":

[2486] So here for $i \leq j$ we have a morphism $X_j \to X_i$ instead of $X_i \to X_j$ in a direct system.

Definition 229.3. Let $\mathcal{C}$ be a category and let $w \in \mathbb{Z}$. A inverse sequence in the category $\mathcal{C}$ consists of a family $\{X_i\}_{i\in\mathbb{Z}_{\geq w}}$ of objects and a family of morphisms $\{\varphi_i \colon X_i \to X_{i-1}\}_{i\in\mathbb{Z}_{\geq w+1}}$. Usually we write an inverse sequence as follows:

$$X_w \xleftarrow{\varphi_{w+1}} X_{w+1} \xleftarrow{\varphi_{w+2}} X_{w+2} \xleftarrow{\varphi_{w+3}} \dots$$

An inverse sequence as above gives rise to an inverse system over the directed set $(\mathbb{Z}_{\geq w}, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} := \varphi_{i+1} \circ \dots \circ \varphi_{j-1} \circ \varphi_j \colon X_j \to X_i$.

Examples.

(B) The following is an inverse sequence in the category *AbGroup* of abelian groups:

$$\mathbb{Z} \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z}^3 \leftarrow \dots.$$

Here all homomorphisms are the projections onto the first coordinates:

$$\begin{array}{ccc} \mathbb{Z}^n & \leftarrow & \mathbb{Z}^{n+1} \\ (x_1, \dots, x_n) & \mapsto & (x_1, \dots, x_n, x_{n+1}). \end{array}$$

(C) Let p be a prime. The following is an inverse sequence in the category *Ring* of rings:

$$\mathbb{Z}/p\cdot\mathbb{Z} \leftarrow \mathbb{Z}/p^2\cdot\mathbb{Z} \leftarrow \mathbb{Z}/p^3\cdot\mathbb{Z} \leftarrow \dots.$$

Here all homomorphisms are the obvious epimorphisms

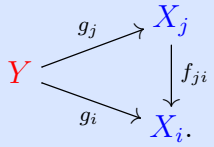
$$\begin{array}{ccc} \mathbb{Z}/p^i\cdot\mathbb{Z} & \leftarrow & \mathbb{Z}/p^{i+1}\cdot\mathbb{Z} \\ n + p^i\cdot\mathbb{Z} & \mapsto & n + p^{i+1}\cdot\mathbb{Z}. \end{array}$$

□

The following definition is the "mirror" image of the definition of the direct colimit of a direct system which we gave in Definition 104.4:

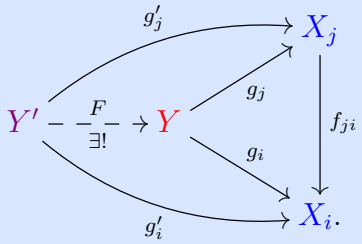
Definition 229.4. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given an inverse system $(\{X_i\}_{i\in I}, \{f_{ji} \colon X_j \to X_i\}_{i\leq j})$ in the category $\mathcal{C}$. An inverse limit[2487] of the inverse system is a limit, in the sense of Definition 6.11 of the inverse system, viewed as a diagram in the opposite category $\mathcal{C}^{\mathrm{op}}$. In other words, an inverse limit is an object Y in $\mathcal{C}$ together with morphisms $g_i \colon Y \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $f_{ji} \circ g_j = g_i \colon Y \to X_i$, i.e. for all $i \leq j$ the following diagram commutes:



(2) If we are given another object Y' and morphisms $g'_i \colon Y' \to X_i$, $i \in I$, that satisfy (1), then there exists a unique morphism $F \colon Y' \to Y$ such that for all $i \in I$ we have

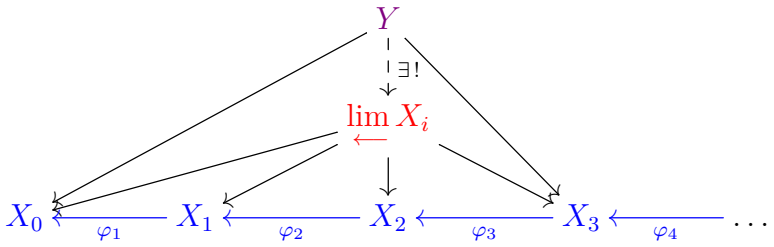
$g'_i = g_i \circ F$. In particular for all $i \leq j$ the following diagram commutes:



By the Limit Uniqueness Lemma 6.12 we know, that if an inverse limit exists, then it is unique in the usual sense. We denote the inverse limit, if it exists, by $\varprojlim X_i$ or sometimes by $\varprojlim_I X_i$.

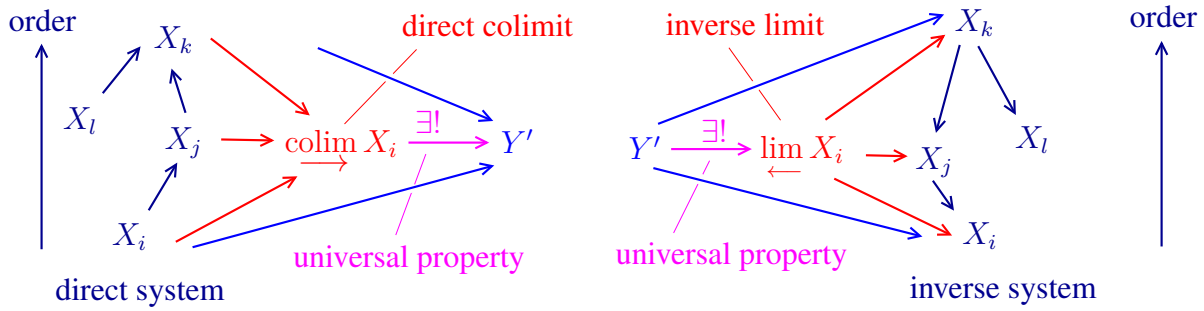
Remark.

(1) In the case of an inverse sequence $X_0 \xleftarrow{\varphi_1} X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} \dots$ of morphisms the notion of an inverse limit can be summarized in the following diagram:



More precisely, an inverse limit $\varprojlim X_i$, if it exists, comes with morphisms $\varprojlim X_i \to X_i$ which form commutative diagrams, furthermore given morphisms $Y \to X_i$ which form commutative diagrams, we get a unique morphism $Y \to \varprojlim X_i$ which makes everything commute.

(2) In the following diagram we compare schematically the definition of the direct colimit of a direct system to the inverse limit of an inverse system:

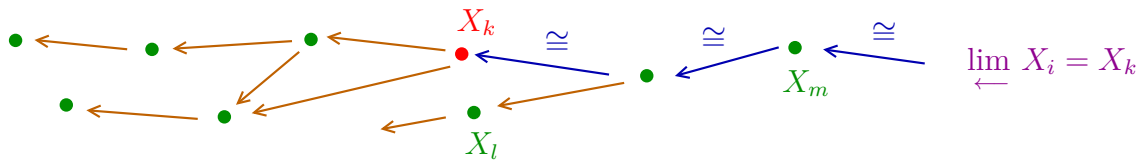


The difference between a direct colimit and an inverse limit is not the direction of the arrows between the X_i. In fact given a partial order "$\leq$" one could define a partial order "$\widetilde{\leq}$" via $a \mathrel{\widetilde{\leq}} b$ if and only if $b \leq a$, hence a direct system over $(I, \leq)$ is an inverse system over $(I, \widetilde{\leq})$. The real difference is whether the X_i map to the colimit, or whether the limit maps to the X_i. □

[2487] Somewhat confusingly an "inverse limit" is often also called an "inductive limit".

We conclude this section with the following analog of the Direct Colimit–Stabilization Lemma 104.5.

Lemma 229.5. (Inverse Limit Stabilization Lemma) Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be an inverse system in the category $\mathcal{C}$. Suppose that there exists a $k \in I$ such that for all $l \geq k$ the morphism $f_{lk}\colon X_l \to X_k$ is in fact an isomorphism. Then $\varprojlim X_i$ exists and it is naturally isomorphic to X_k. More precisely, an inverse limit is given by the object $\varprojlim X_i := X_k$ and where for each $l \in I$ the morphism $X_l \to X_k$ is given by $f_{ml} \circ f_{mk}^{-1}\colon X_l \xleftarrow{\sim} X_k$ where $m \in I$ is chosen such that $m \geq k$ and $m \geq l$.



Proof. We will provide the proof in Exercise 229.3. ■

229.2. **Existence of inverse limits.** We saw in the Sets–Direct Colimit Existence Proposition 104.6, the Algebraic–Direct Colimit Existence Proposition 104.7 and the Topological-Direct Colimit Proposition 104.10 that in categories we cared about the direct colimit could be viewed as a quotient of a coproduct. We will now see that the inverse limit can be viewed as a "subobject" of a product. Since in all categories we care about, the product has the same underlying set, we can give a uniform description of inverse limits:

Proposition 229.6. (Inverse Limit Existence Proposition) Suppose we are dealing with one of the following seven categories:

(a) the category *Set* of sets,
(b) the category *Top* of topological spaces,
(c) the category *Group* of groups,
(d) the category *AbGroup* of abelian groups,
(e) the category *Ring* of rings,
(f) the category of R-modules, where R is a commutative ring,
(g) the category *TopGr* of topological groups.

In these categories we can explicitly construct inverse limits:

(1) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system in one of the above categories. We set

$$\varprojlim X_i \;:=\; \left\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, f_{ji}(x_j) = x_i \text{ for all } i \leq j\right\}.$$

With the appropriate structures (topology, multiplication, etc.) this set, together with the obvious projections, become the inverse limit of the inverse system.

(2) Suppose we are given an inverse sequence

$$X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} X_3 \xleftarrow{\varphi_4} \dots$$

As discussed in Definition 229.3 such an inverse sequence gives rise to an inverse system. In this setting we can simplify the description of the set in (1) and we have

$$\varprojlim X_i = \{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } \varphi_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N}_{\geq 2}\}.$$

Remark. Let

$$X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} X_3 \xleftarrow{\varphi_4} \dots$$

be an inverse sequence in one of the seven categories that we considered in the Inverse Limit Existence Proposition 229.6. We can visualize the inverse limit, as described in Inverse Limit Existence Proposition 229.6 (2), as the set (topological space, group, etc.) of all infinite sequences

$$\begin{array}{ccccccccccc} X_1 & \xleftarrow{\varphi_2} & X_2 & \xleftarrow{\varphi_3} & X_3 & \xleftarrow{\varphi_4} & X_4 & \xleftarrow{\varphi_5} & \dots \\ x_1 & \longmapsto & x_2 & \longmapsto & x_3 & \longmapsto & x_4 & \longmapsto & \dots \end{array}$$

□

Proof. [2488] First we prove Statement (1). Thus let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the seven given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji} \colon X_j \to X_i\}_{i\leq j})$ be an inverse system in the category $\mathcal{C}$. We set

$$\varprojlim X_i \coloneqq \left\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, f_{ji}(x_j) = x_i \text{ for all } i \leq j\right\}.$$

Furthermore we denote by $g_i \colon \varprojlim X_i \to X_i$ the natural projections. It follows immediately from the definitions that for any $i \leq j$ we have $f_{ji} \circ g_j = g_i$. Furthermore, given an object Y' and morphisms $g_i' \colon Y' \to X_i$, $i \in I$ which satisfy the condition $f_{ji} \circ g_j' = g_i'$ for all $i \in I$ we set

$$\begin{array}{rcl} F \colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \begin{pmatrix} I & \to & \prod_{i\in I} X_i \\ i & \mapsto & g_i'(y) \end{pmatrix}. \end{array}$$

It is straightforward to verify that $F(Y')$ lies in $\varprojlim X_i$ and that F is the unique map with the desired property. This shows that the inverse limit exists in the category of sets.[2489]

For the remaining six categories we will use the same definition of the direct colimit as a set and the same maps, but we now have to verify that we have objects and morphisms in the given category.

(b) Suppose that we work in the category of topological spaces. We equip $\prod_{i\in I} X_i$ with the product topology that we defined in the Topological-Product Proposition 7.2. We equip $\varprojlim X_i \subseteq \prod_{i\in I} X_i$ with the subspace topology. It follows from the Topological-Product Proposition 7.2 (1a) and the Subspace Continuity Lemma 4.3 (2) that the maps $g_i \colon \varprojlim X_i \to X_i$ are continuous and it follows from the Topological-Product Proposition 7.2 (1b) together with the Subspace Continuity Lemma 4.3 (3) that the map $F \colon Y \to \varprojlim X_i$ is continuous.

[2488] Alternatively the statements can also be deduced easily from the Limit-via-Product+Equalizer Lemma 6.18 together with the Product-of-Sets-and-Groups Lemma 6.4 and the Topological-Product Proposition 7.2 and the discussion on page 261.

[2489] Now we will see that we can use the "same" inverse limit for all the categories. This is very different from the construction of the direct colimit for the same categories. There, in the proof of the Direct Algebraic–Limit Existence Proposition 104.7 and the Topological-Direct Colimit Proposition 104.10, we had to use several different construction.

(c) In the category *Group* of groups we view $\prod_{i\in I} X_i$ as a group via component-wise multiplication. Since the f_{ij} are group homomorphisms we see that $\varprojlim X_i$ is a subgroup of $\prod_{i\in I} X_i$. It is straightforward to verify that the above maps are all morphisms in the category *Group* of groups.

(d)–(f) These categories are dealt with the same way as (c).

(g) We equip the set $\varprojlim X_i$ with the topological structure and the group structure as in the above statements (b) and (c). It follows from the Topological Subgroup Lemma 7.33 that this turns $\varprojlim X_i$ into a topological group. One can now easily verify that this is the inverse limit of the inverse system of topological groups.

Finally note that Statement (2) is an elementary consequence of the definitions. ■

Now we return to our initial examples of inverse systems and their inverse limits. We will deal with them in a slightly modified order:

Examples.

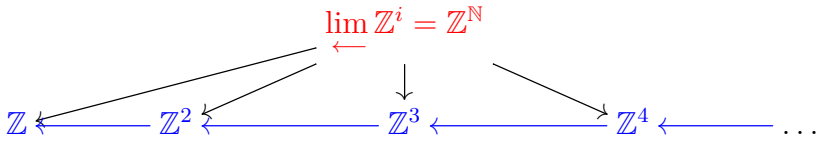
(B) As on page 4433 we consider the inverse sequence

$$\mathbb{Z} \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z}^3 \leftarrow \dots$$

of abelian groups that is given by the projections onto the first coordinates. One can show quite easily, either by hand or using the Inverse Limit Existence Proposition 229.6 (2) that[2490]

$$\varprojlim_{i\in\mathbb{N}} \mathbb{Z}^i = \{(x_1, x_2, x_3, \dots) \mid x_j \in \mathbb{Z}\} \cong \mathbb{Z}^{\mathbb{N}}$$

and where the maps from $\mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^n$ are the projections onto the first n coordinates. We can summarize the situation in the following diagram:



(C) Let p be a prime. The inverse limit $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^i \cdot \mathbb{Z}$ of the rings $\mathbb{Z}/p^i\mathbb{Z}$ is called the *ring of the p-adic numbers*.[2491] The p-adic numbers play a very prominent role in many branches of pure mathematics.

(A) The inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ of the rings $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, with the partial order on $\mathbb{N}$ defined by the divisibility relation $k \mid l$, is called the *ring of the profinite integers*. In Exercise 229.2 we will use the Chinese Remainder Theorem 97.4 to show that there exists a ring isomorphism

$$\widehat{\mathbb{Z}} \cong \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p.$$

□

Inverse limits will play a minor role in these lecture notes. They will mostly arise through the following lemma, which plays the role of the Tensor–Direct Colimit Lemma 159.16.

[2490] Note that $\mathbb{Z}^{\mathbb{N}}$ is an uncountable group.

[2491] In number theory one often uses a different convention, namely one writes explicitly $\mathbb{Z}/p\mathbb{Z}$ for the quotient ring and one denotes the p-adic numbers by $\mathbb{Z}/p\mathbb{Z}$. If the expression "$\mathbb{Z}/p\mathbb{Z}$" appears in a paper or a book it is a good idea to check the convention used by the author.

Lemma 229.7. (Hom-of-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set, let $(\{A_i\}_{i\in I}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\leq j})$ be a direct system of abelian groups and let G be an abelian group. We can consider the corresponding direct colimit $\operatorname{colim} A_i$ from Definition 104.4. The natural homomorphisms $A_j \to \operatorname{colim} A_i$ induce a natural isomorphism

$$\lim \operatorname{Hom}(A_i, G) \xrightarrow{\cong} \operatorname{Hom}\left(\operatorname{colim} A_i, G\right)$$

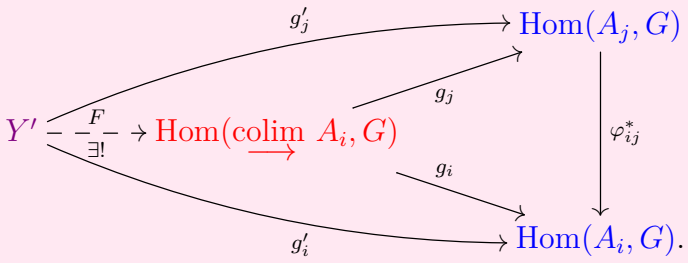
of abelian groups (or G-modules if G is a commutative ring).

Proof. Let $(I, \leq)$ be a directed set, let $(\{A_i\}_{i\in I}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\leq j})$ be a direct system of abelian groups and let G be an abelian group. It follows easily from the definitions and the elementary Hom–Contravariant Functor Lemma 197.4 that

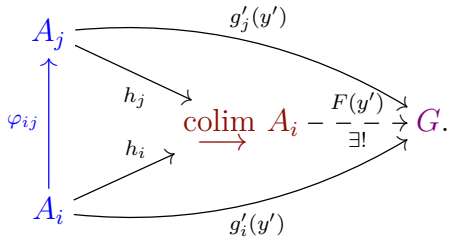
$$(\{\operatorname{Hom}(A_i, G)\}_{i\in I}, \{\varphi_{ij}^*\colon \operatorname{Hom}(A_j, G) \to \operatorname{Hom}(A_i, G)\}_{i\leq j})$$

is an inverse system of abelian groups. The natural homomorphisms $h_j\colon A_j \to \operatorname{colim} A_i$ induce homomorphisms $g_j\colon \operatorname{Hom}(\operatorname{colim} A_i, G) \to \operatorname{Hom}(A_j, G)$. By the uniqueness of inverse limits it remains to show that $\operatorname{Hom}(\operatorname{colim} A_i, G)$ together with the homomorphisms g_j has the universal property an inverse limit that we introduced in Definition 229.4. Put differently, we need to prove the following claim:

Claim. Let Y' be an abelian group and let $g_i'\colon Y' \to \operatorname{Hom}(A_i, G)$ be homomorphisms such that $\varphi_{ij}^* \circ g_j' = g_i'\colon Y' \to \operatorname{Hom}(A_i, G)$ for all $i \leq j$. Then there exists a unique homomorphism $F\colon Y' \to \operatorname{Hom}(\operatorname{colim} A_i, G)$ such that the following diagram commutes:



Proof. Let Y' be an abelian group and let $g_i'\colon Y' \to \operatorname{Hom}(A_i, G)$ be homomorphisms as above. Let $y' \in Y'$. By Definition 104.4 of a direct colimit we obtain a unique homomorphism $F(y')\colon \operatorname{colim} A_i \to G$ such that the following diagram commutes:



This discussion defines a map

$$\begin{aligned} F\colon Y' &\to \operatorname{Hom}\left(\operatorname{colim} A_i, G\right) \\ y' &\mapsto F(y'). \end{aligned}$$

Using similar arguments as above one shows that this map is a homomorphism and that it is the unique homomorphism that makes the diagram of the claim commute. ∎

229.3. **Appendix: The inverse limit of topological spaces.** In this appendix we consider the inverse limit in the category *Top* of topological spaces. Oddly enough we will not really make use of this construction, except in the second Appendix 229.4 below, on the profinite completion of a group.

The following lemma summarizes a few basic properties of inverse limits of topological spaces.[2492]

Lemma 229.8. (Inverse Limit Topology–Properties Lemma) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces.

(1) (a) If all the X_i's are Hausdorff, then $\varprojlim X_i$ is a closed subset of $\prod_{i\in I} X_i$.

(b) If all the X_i's are compact and Hausdorff, then $\varprojlim X_i$ is also compact.[2493]

(3) If all the X_i are Hausdorff, then so is the inverse limit $\varprojlim X_i$.

Example. Let p be a prime. The abelian group $\widehat{\mathbb{Z}}_p$ is the inverse limit of finite, hence compact groups. It follows from the Finite Groups–Inverse Limit Proposition 229.9 that $\widehat{\mathbb{Z}}_p$ is also a compact topological group. On the other hand $\widehat{\mathbb{Z}}_p$ is infinite,[2494] which shows that the topology on $\widehat{\mathbb{Z}}_p$ is *not* the discrete topology. □

Proof. Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces.

(1) (a) We suppose that all the X_i are Hausdorff. Given $i \in I$ we denote by $p_i\colon \prod_{j\in J} X_j \to X_i$ the projection and we denote by $D_i = \{(x, x) \mid x \in X_i\} \subseteq X_i \times X_i$ the diagonal. By the Hausdorff-Diagonal Lemma 7.20 the hypothesis that X_i is Hausdorff implies that the diagonal D_i is a closed subset of $X_i \times X_i$. By the proof of the Inverse Limit Existence Proposition 229.6 we have

$$\varprojlim X_i := \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, f_{lk}(x_l) = x_k \text{ for all } k \leq l\Big\}$$
$$= \bigcap_{k\leq l} \Big\{x \in \prod_{i\in I} X_i \,\Big|\, \underbrace{((f_{lk}\circ p_l)\times p_k)}_{\substack{(f_{lk}\circ p_l)\times p_k \text{ is a continuous map} \\ \prod_{i\in I} X_i \to X_k \times X_k}} (x) \in D_k\Big\} = \bigcap_{k\leq l} \underbrace{((f_{lk}\circ p_l)\times p_k)^{-1}(D_k)}_{\substack{\text{closed since } D_k \text{ closed and} \\ \text{since the map is continuous}}}.$$

Thus we see that $\varprojlim X_i$ is the intersection of closed subsets of $\prod_{i\in I} X_i$, hence $\varprojlim X_i$ itself is a closed subset.

(b) Now we suppose that all the X_i are Hausdorff and compact. We obtain from Tychonoff's Theorem 7.14 that $\prod_{i\in I} X_i$ is compact. Together with (1) we see that $\varprojlim X_i$ is a closed subset of a compact space. Thus it follows from the Compact-Closed Lemma 3.39 (1) that $\varprojlim X_i$ is also compact. ■

(3) We assume that all the X_i are Hausdorff. It follows from the Product Topology-Properties Proposition 7.13 (3) that the product $\prod_{i\in I} X_i$ is also Hausdorff. By the Inverse

[2492] The lemma is formatted like the Product Topology-Properties Proposition 7.13 and the Disjoint Union Topology-Properties Lemma 7.28:

[2494] In fact $\widehat{\mathbb{Z}}_p$ is uncountable (why?).

Limit Existence Proposition 229.6 we know that $\varprojlim$ can be viewed as a subspace of the product $\prod_{i\in I} X_i$. Thus it follows from the Subspace-Hausdorff Lemma 3.31 that $\varprojlim X_i$ is also Hausdorff.

229.4. **Appendix: The profinite completion of a group.** Note that we can view any group as a topological group by equipping it with the discrete topology. With this topology a group is compact if and only if it is finite. Furthermore any homomorphism of groups equipped with the discrete topology is also a morphism in the category $\mathcal{TopGr}$ of topological groups. These simple observations lead us to the following proposition:

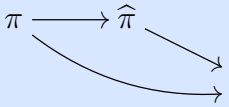
Proposition 229.9. (Finite Groups–Inverse Limit Proposition) Let $(I, \leq)$ be a directed set and let $(\{G_i\}_{i\in I}, \{f_{ji}\colon G_j \to G_i\}_{i\leq j})$ be an inverse system of finite groups. Then the inverse limit $\varprojlim G_i$ exists in the category of topological groups and it is a compact topological group.

Proof. It follows from the Inverse Limit Existence Proposition 229.6 that the inverse limit $\varprojlim G_i$ exists in the category of topological groups. Since all the G_i's are finite, in particular compact, it follows from the Inverse Limit Topology–Properties Lemma 229.8 (1b) that the inverse limit $\varprojlim G_i$ is also compact. ∎

Definition 229.10. Let π be a group. We consider the set of all finite index normal subgroups $\{\pi_i\}_{i\in I}$. We equip the index set I with the partial order defined by the convention that $i \leq j$ if and only if $\pi_j \subseteq \pi_i$.[2495] Given $i \leq j$ we denote by $\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i$ the natural projection. Then $(\{\pi/\pi_i\}_{i\in I}, \{\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i\}_{i\leq j})$ forms an inverse system. We refer to

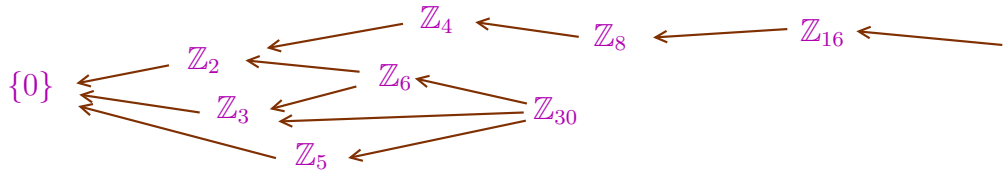
$$\widehat{\pi} \coloneqq \varprojlim \pi/\pi_i$$

as the profinite completion of π. By the Finite Groups–Inverse Limit Proposition 229.9 the profinite completion $\widehat{\pi}$ is a compact topological group. It follows immediately from the definition of an inverse limit that there exists a unique homomorphism $\pi \to \widehat{\pi}$ such that for every i the following diagram commutes:



Examples.

(1) The profinite completion $\widehat{\mathbb{Z}}$ of the group $\mathbb{Z}$ equals precisely the profinite integers that we had encountered on page 4437.



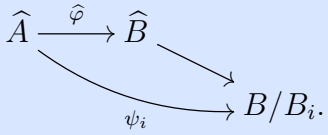
[2495] Note that we wrote $\pi_j \subseteq \pi_i$ instead of $\pi_i \subseteq \pi_j$. The mnemonic device is that $i \leq j$ corresponds to the quotient π/π_i being "smaller" than the quotient π/π_j.

(2) It follows immediately from the Inverse Limit Stabilization Lemma 229.5 that the profinite completion of a *finite* group G is the group itself, i.e. $\widehat{G} = G$.
(3) Let $G = \mathbb{Q}$. This group has no proper subgroups of finite index.[2496] It follows immediately from the definitions that $\widehat{\mathbb{Q}} = \{0\}$. □

Definition 229.11. Let $\varphi\colon A \to B$ be a group homomorphism. We denote by $\{B_i\}_{i\in I}$ the finite index normal subgroups of B. For $B_j \subseteq B_i$ we denote again by $p_{ji}\colon B/B_j \to B/B_i$ the projection. For every $i \in I$ the preimage $\varphi^{-1}(B_i)$ is a finite index normal subgroup of A.[2497] Now we define $\psi_i\colon \widehat{A} \to A/\varphi^{-1}(B_i) \xrightarrow{\varphi} B/B_i$. Note that ψ_i is continuous since the first map is continuous by definition and the second map is a map between two topological spaces that are equipped with the discrete topology. It follows immediately from the definitions that for $B_j \subseteq B_i$ we have $p_{ji} \circ \psi_j = \psi_i$. It now follows from the definition of the inverse limit in the category of topological groups that there exists a unique continuous homomorphism

$$\widehat{\varphi}\colon \widehat{A} \to \widehat{B}$$

such that for every i the following diagram commutes:



We conclude this appendix with the following cute observation.

Lemma 229.12. (Profinite Completion Functor Lemma) The map

$$A \mapsto \widehat{A}$$

together with the maps

$$\begin{aligned} \mathrm{Mor}(A,B) &\to \mathrm{Mor}(\widehat{A},\widehat{B}) \\ (\varphi\colon A \to B) &\mapsto (\widehat{\varphi}\colon \widehat{A} \to \widehat{B}) \end{aligned}$$

define a covariant functor from the category *Group* of groups to the category of compact topological groups.

Proof. The lemma follows easily from the definitions and the Finite Groups–Inverse Limit Proposition 229.9. ■

229.5. Exercises for Chapter 229 .

Exercise 229.1. We consider the profinite integers, i.e. the inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ of the rings $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, with the partial order on $\mathbb{N}$ defined by the divisibility relation $k \mid l$. Does there exist an epimorphism $\widehat{\mathbb{Z}} \to \mathbb{Z}$ of groups?

Exercise 229.2. We consider the profinite integers $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ which we introduced on page 4437 und for each prime p we consider the p-adic integers $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ which

[2496] Why not?
[2497] Why is it a finite index subgroup?

we introduced on page 4437. Use the Chinese Remainder Theorem 97.4 to show that there exists an isomorphism

$$\widehat{\mathbb{Z}} \cong \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p$$

of rings.

Exercise 229.3. Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system in the category $\mathcal{C}$. Suppose that there exists a $k \in I$ such that for all $l \geq k$ the map $f_{lk}\colon X_l \to X_k$ is an isomorphism. Show that $\operatorname{colim} X_n$ exists and show that it is naturally isomorphic to X_k.

Remark. In other words, the exercise is to prove the Direct Colimit–Stabilization Lemma 104.5.

Exercise 229.4. We consider the following inverse sequence in the category *Ring* of rings

$$\mathbb{Z}[x]/(x) \leftarrow \mathbb{Z}[x]/(x^2) \leftarrow \mathbb{Z}[x]/(x^3) \leftarrow \dots$$

where the homomorphisms are the obvious projections. Show that

$$\lim_{\leftarrow} \mathbb{Z}[x]/(x^n) \cong \mathbb{Z}[\![x]\!] := \left\{ \sum_{i=0}^{\infty} a_i \cdot x^i \,\middle|\, a_i \in \mathbb{Z} \right\}$$

is the ring of formal power series.

Exercise 229.5. We consider the directed set $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{aligned} f_{ji}\colon X_j = \mathbb{N} &\to X_i = \mathbb{N} \\ n &\mapsto n + (j - i). \end{aligned}$$

Then $(\{X_i = \mathbb{N}\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of sets. Show that the inverse limit is the empty set.

Exercise 229.6. For each $n \in \mathbb{N}$ we consider the topological space $X_n = \{1, \dots, n\}$ together with the discrete topology and for each $n \in \mathbb{N}$ we consider the map

$$\begin{aligned} \varphi_n\colon X_n &\to X_{n-1} \\ i &\mapsto \begin{cases} i, & \text{if } i \in \{1, \dots, n-1\}, \\ n-1, & \text{if } i = n. \end{cases} \end{aligned}$$

Note that these maps define an inverse sequence in the category *Top* of topological spaces. We equip $\mathbb{N} \cup \{\infty\}$ with the subspace topology coming from $\mathbb{R} \cup \{\infty\}$.

(a) We consider the inverse limit $\lim X_i$ in the category of topological spaces. Show that there exists a continuous non-constant map $\mathbb{N} \cup \{\infty\} \to \lim X_i$.

(b) Is $\mathbb{N} \cup \{\infty\}$ homeomorphic to $\lim X_i$?

Remark. This exercise is almost identical to Exercise 6.7.

Exercise 229.7. Recall that according to Definition 7.18 a topological space X is called *totally disconnected* if the only connected subsets are the empty set and the subsets consisting of a single point (e.g. X could be discrete). Show that the inverse limit of an inverse system of totally disconnected topological spaces is totally disconnected.

Exercise 229.8. Let X be a topological space. We denote by $\mathcal{K}$ the set of all compact subspaces of X. We consider $\mathcal{K}$ with the partial order given by inclusion, i.e. given $A, B \in \mathcal{K}$ we have $A \leq B$ if $A \subseteq B$. Given $A \in \mathcal{K}$ we define[2498] $X_A := \pi_0(X \setminus A)$. Furthermore given

[2498] Recall that by Definition 4.43, given a topological space Y we had denoted by $\pi_0(Y)$ the set of path-components of X.

$A \leq B$ we denote by $f_{AB}\colon \pi_0(X_B) \to \pi_0(X_A)$ the map induced by the inclusion $X_B \to X_A$. It follows immediately from the discussion on page 236 that the sets $\{\pi_0(X_A)\}_{A\in\mathcal{K}}$, together with the inclusion-induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets. The elements of $\varprojlim \pi_0(X_A)$ are called the *ends of* X.

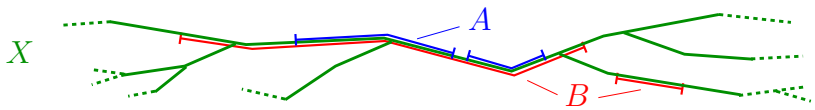
(a) Let *Top* be the category of topological spaces and let *Set* be the category of sets. Show that
$$X \;\mapsto\; \varprojlim \pi_0(X_A)$$
gives rise to a covariant functor from *Top* to *Set*.

(b) Show that for a topological space X the cardinality of $\varprojlim \pi_0(X_A)$, viewed as an element in $\mathbb{N}_0 \cup \{\infty\}$, equals the number of ends of M as defined in Exercise 4.34.

(c) Let $n \in \mathbb{N}_0$. Determine the number of ends of $X = \mathbb{R}^n$.

(d) For each $k \in \mathbb{N}_0$ give an example of a connected 1-dimensional CW-complex X with precisely k ends.

(e) Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We consider the mapping torus $\mathrm{Tor}(f\colon X \to X)$. We denote by $i\colon X \to \mathrm{Tor}(f\colon X \to X)$ the embedding that is given by $x \mapsto [(x, 0)]$. Show that i induces a surjection from the set of ends of X onto the set of ends of $\mathrm{Tor}(f\colon X \to X)$.



Exercise 229.9.

(a) We consider the inverse system $(X_i)_{i\in\mathbb{N}}$ where $X_i = \mathbb{Z}$ and where for each $i \leq j$ the map $X_j \to X_i$ is given by multiplication by 2^{i+j}. Determine $\varprojlim X_i$ in the category *AbGroup* of abelian groups.

(b) Now we work in the category of sets. Show that there exists an inverse system $(\{X_i\}_{i\in\mathbb{N}}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ of non-empty sets such that $\varprojlim X_i = \varnothing$.

Remark. In Exercise 229.11 we will see that the inverse limit of an inverse system of *finite non-empty sets* is in fact non-empty.

Exercise 229.10. Let X be a topological space and let $\{X_i\}_{i\in I}$ be a family of subspaces. We write $i \leq j$ if $X_j \subseteq X_i$. Given $i \leq j$ we denote by $\varphi_{ji}\colon X_j \to X_i$ the inclusion. Show that $\varprojlim X_i = \bigcap_{i\in I} X_i$.

Exercise 229.11. Throughout this exercise we work in the category *Top* of topological spaces. Let $(\{X_i\}_{i\in\mathbb{N}}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces that are non-empty, Hausdorff and compact.

(a) For each $j \in \mathbb{N}$ we set
$$Z_j \;\coloneqq\; \{(x_i)_{i\in\mathbb{N}} \mid \text{ for all } i \leq j \text{ we have } x_i = f_{ji}(x_j)\} \;\subseteq\; \prod_{i\in\mathbb{N}} X_i.$$
Show that Z_j is closed and non-empty.
Hint. You might want to consider the Inverse Limit Topology–Properties Lemma 229.8 (1a).

(b) Show that for any $i, j \in \mathbb{N}$ we have $Z_i \cap Z_j \neq \varnothing$.

(c) Apply Tychonoff's Theorem 7.14, together with Exercise 3.27, to conclude that we have $\varprojlim X_i \neq \varnothing$.

Remark. We can apply this statement in particular to an inverse system of finite non-empty sets, viewed as topological spaces by equipping each of the finite sets with the discrete topology. Evidently a finite discrete space is Hausdorff and compact.

Exercise 229.12. Given a topological space Y we denote by $C(Y, \mathbb{R})$ the abelian group of all real-valued continuous functions on Y. Note that for each subset $W \subseteq Y$ we have a homomorphism $C(Y, \mathbb{R}) \to C(W, \mathbb{R})$ given by $f \mapsto f|_W$, i.e. given by restricting a function on Y to a function on W. Let X be a topological space and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets such that for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$ and such that $\bigcup_{i\in\mathbb{N}} U_i = X$.

(a) Show that the groups $\{C(U_i, \mathbb{R})\}_{i\in\mathbb{N}}$ together with the restriction maps form an inverse system.

(b) Show that the restriction maps from X to the U_i gives rise to an isomorphism

$$C(X, \mathbb{R}) \ \to \varprojlim C(U_i, \mathbb{R}).$$

Exercise 229.13. Let I be directed set, let $\{f_i \colon A_i \to A_j\}_{i\leq j}$ be a direct system in the category *AbGroup* of abelian groups and let $\{g_i \colon B_i \to B_j\}_{j\leq i}$ be an inverse system in the category *AbGroup* of abelian groups. Suppose that for each $i \in I$ we are given a bilinear map $\varphi_i \colon A_i \times B_i \to \mathbb{Z}$ such that for every $i \leq j$, and $a \in A_i$ and $b \in B_j$ we have $\varphi_i(a_i, g_j(b_j)) = \varphi_j(f_i(a_i), b_j)$. Define an "interesting" bilinear map

$$\varinjlim A_i \ \times \ \varprojlim B_i \ \to \ \mathbb{Z}.$$

Exercise 229.14. Let $\{\{G_i\}_{i\in\mathbb{N}}, \{\varphi_{ij} \colon G_j \to G_i\}_{i\leq j}\}$ be an inverse system of abelian groups and let A be an abelian group. Express

$$\operatorname{Hom}\left(A, \varprojlim G_i\right)$$

in terms of the groups $\operatorname{Hom}(A, G_i)$.

Exercise 229.15. We consider the inverse sequences

$$D_* \colon \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \dots$$

and

$$E_* \colon \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 3} \mathbb{Z} \xleftarrow{\cdot 4} \mathbb{Z} \xleftarrow{\cdot 5} \mathbb{Z}$$

(a) Determine $\varprojlim D_*$.

(b) Determine $\varprojlim E_*$.

Exercise 229.16. Let A and B be two finitely generated abelian groups such that the profinite completions are isomorphic.

(a) Show that $\operatorname{rank}(A) = \operatorname{rank}(B)$.

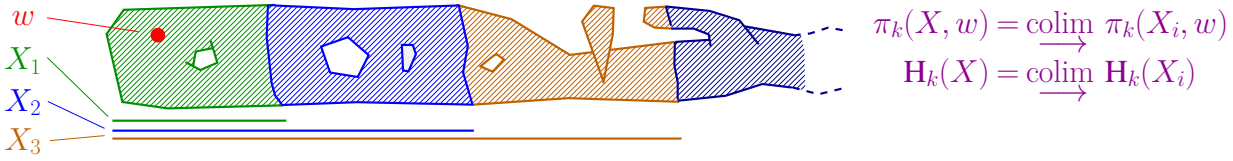
(b) Is A necessarily isomorphic to B?

Hint. Use the Abelian Group-Classification Theorem 97.24.

CHAPTER 230

Cohomology groups and limits

In the Exhaustion-π_1-Proposition 105.2, the Exhaustion-π_n-Proposition 134.21 and the Exhaustion-H_*-Proposition 154.2 we saw that in favorable situations the fundamental group, the higher homotopy groups and the homology groups of an ascending union of topological spaces are the direct colimit of the corresponding fundamental groups, homotopy groups and homology groups.



In this chapter we will try to extend these results to cohomology groups. The results of this chapter will not only be useful for computing cohomology groups, but they will also play an essential role in many proofs later on. For example the proof that de Rham cohomology is isomorphic to singular cohomology with real coefficients will build on the somewhat technical results of this chapter.

This chapter is organized as follows:

- In Section 230.1 we will study homology of ascending unions
- In Section 230.2 we will introduce the $\lim^1$-functor and the Mittag-Leffler condition.
- In Section 230.3 we will state and prove the purely algebraic Inverse Limit-H^*-Proposition 230.9 which determines the cohomology groups of an inverse limit of cochain complexes.
- In Section 230.4 we will use the Inverse Limit-H^*-Proposition 230.9 to study the cohomology group of an ascending sequence of topological spaces
- In Appendix 230.5 we will use the Inverse Limit-H^*-Proposition 230.9 to study the de Rham cohomology of non-compact smooth manifolds via exhaustions by compact smooth submanifolds.
- In Section 230.6 we will provide exercises for Chapter 230.

In the next Chapter 231 we will see that for a smooth manifold de Rham cohomology is naturally isomorphic to real-valued singular cohomology.

230.1. **Homology of ascending unions.** For the reader's convenience we recall the Exhaustion-H_*-Proposition 154.2 which we already alluded to in the above introduction:

Proposition 154.2. (Exhaustion-H_*-Proposition) Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be a sequence of subspaces such that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ and such that one of the following conditions holds:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,

then for any $n \in \mathbb{N}_0$ the inclusion-induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism[2499]

$$\operatorname*{colim}_{\longrightarrow} \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

In the following we outline the proof of the Exhaustion-H_*-Proposition 154.2. First note that singular homology of a topological space is the composition of the following two covariant functors:

$$\begin{aligned} X &\mapsto (\mathrm{C}_*(X), \partial_*) && \text{(singular chain complex functor)} \\ (C_*, \partial_*) &\mapsto \mathrm{H}_k(C_*, \partial_*) && \text{(homology functor)} \end{aligned}$$

Therefore, to prove Exhaustion-H_*-Proposition 154.2 it made sense to consider the two functors separately. We did so with the following two lemmas. The first lemma is of topological nature:

Lemma 154.1. (Exhaustion-C_*-Lemma) Let X be a topological space. Furthermore let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subspaces of X with $\bigcup_{i\in\mathbb{N}_0} X_i = X$. Then the chain complexes $\mathrm{C}_*(X_i), i \in \mathbb{N}$ together with the inclusion-induced chain maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X_j)$ for $i \leq j$ form a direct system of chain complexes. We now suppose that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Then the inclusion-induced maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \mathrm{C}_*(X_i) \xrightarrow{\cong} \mathrm{C}_*(X)$$

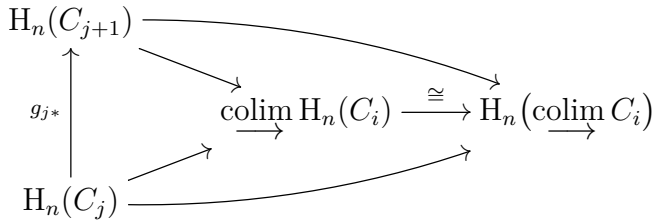
of chain complexes.

The second lemma is purely algebraic:

Lemma 154.8. (Direct Colimit-H_*-Lemma) Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct sequence of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{\mathrm{H}_n(C_i)\}_{i\in\mathbb{N}}, \{g_{i*}\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore, given any $n \in \mathbb{N}_0$ the maps $\mathrm{H}_n(C_i) \to \mathrm{H}_n\big(\operatorname*{colim}_{\longrightarrow} C_i\big)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \mathrm{H}_n(C_i) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname*{colim}_{\longrightarrow} C_i\big).$$

Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct sequence of chain complexes. The Direct Colimit-H_*-Lemma 154.8 can be summarized as follows: For every $j \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ we have the following diagram:



Finally note that the Exhaustion-H_*-Proposition 154.2 is an almost immediate consequence of the Exhaustion-C_*-Lemma 154.1 and the Direct Colimit-H_*-Lemma 154.8.

230.2. **The $\lim^1$-functor and the Mittag-Leffler condition.** Singular cohomology of a topological space with G-coefficients is the composition of three functors:

$$\begin{array}{rcll} X & \mapsto & (\mathrm{C}_*(X), \partial_*) & \text{(singular chain complex functor)} \\ (C_*, \partial_*) & \mapsto & (\mathrm{Hom}(C_*, G), \partial^*) & \text{(hom-functor)} \\ (D^*, \delta^*) & \mapsto & \mathrm{H}^k(D^*, \delta^*) & \text{(cohomology functor)} \end{array}$$

So if we want to understand cohomology of an ascending union we should again consider the three functors separately. Fortunately we already studied the first functor in the above Exhaustion-H_*-Proposition 154.2. We also already considered the second functor in the following lemma:

Lemma 229.7. (Hom-of-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set, let $(\{A_i\}_{i\in I}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\leq j})$ be a direct system of abelian groups and let G be an abelian group. We can consider the corresponding direct colimit $\varinjlim A_i$ from Definition 104.4. The natural homomorphisms $A_j \to \varinjlim A_i$ induce a natural isomorphism

$$\varprojlim \mathrm{Hom}(A_i, G) \xrightarrow{\cong} \mathrm{Hom}\left(\varinjlim A_i, G\right)$$

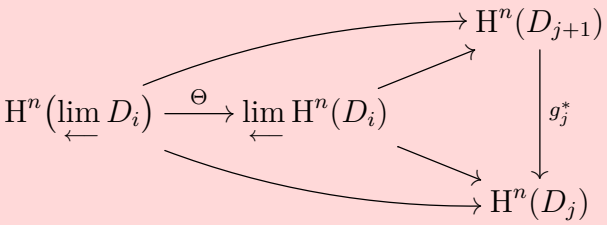
of abelian groups (or G-modules if G is a commutative ring).

The composition of the Exhaustion-C_*-Lemma 154.1 and the Hom-of-Direct Colimit Lemma 229.7 together with the Dual Cochain Map Lemma 197.20 leads us to the following question:

Question 230.1. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of cochain complexes and let $n \in \mathbb{N}_0$. Note that the natural cochain maps $\varprojlim D_i \to D_i$ induce homomorphisms $\varprojlim \mathrm{H}^n(D_i) \to \mathrm{H}^n(D_i)$ which, by the universal property of the inverse limit $\varprojlim \mathrm{H}^n(D_i)$, induce a unique homomorphism

$$\Theta\colon \mathrm{H}^n(D_i) \to \mathrm{H}^n\left(\varprojlim D_i\right)$$

such that the following diagram commutes:



The question that arises is the following: Is the above map

$$\Theta\colon \mathrm{H}^n\left(\varprojlim D_i\right) \to \varprojlim \mathrm{H}^n(D_i)$$

always an isomorphism?

It turns out that, in contrast to the Direct Colimit-H_*-Lemma 154.8, this is not necessarily the case. To formulate the relationship between the two sides of the above question it is useful to introduce a piece of notation and to rewrite the inverse limit from the Inverse Limit Existence Proposition 229.6 (1):

Notation 230.2. Given an inverse sequence $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ of abelian groups we consider the homomorphism

$$\begin{array}{rccc} \mu_{D_i}\colon & \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ & (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}}. \end{array}$$

Remark. With the above notation the Inverse Limit Existence Proposition 229.6 can be reformulated as saying that $\varprojlim D_i = \ker(\mu_{D_i})$. □

This leads us to the following definition:

Definition 230.3. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups (or of modules over a commutative ring R). Using the above notation we define[2500]

$$\varprojlim{}^1 D_i := \operatorname{coker}(\mu_{D_i}).$$

Example. We consider the inverse sequence

$$D_*\colon \quad 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow \dots.$$

In Exercise 230.1 we will see that $\varprojlim D_* = 0$ and $\varprojlim^1 D_* = 0$. We also consider the inverse sequence

$$E_*\colon \quad 0 \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 3} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{\cdot 5} \mathbb{Z} \leftarrow \dots.$$

In Exercise 230.1 we will see that $\varprojlim^1 E_* \neq 0$. Furthermore in Exercise 230.3 we will see that this inverse sequence E_* arises "in nature", i.e. we will see that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $\mathrm{H}^1(X_i;\mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the inclusion induced map $\mathrm{H}^1(X_{i+1};\mathbb{Z}) \to \mathrm{H}^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$. □

We continue with the following definition:

Definition 230.4. Let $\mathcal{C}$ be a category such that limits always exist and let $(X_i, f_i)_{i\in\mathbb{N}}$ and $(Y_i, g_i)_{i\in\mathbb{N}}$ be two inverse sequences in $\mathcal{C}$.

(1) A morphism of inverse sequences from (X_i, f_i) to (Y_i, g_i) consists of a sequence of morphisms $u_i\colon X_i \to Y_i$, $i \in \mathbb{N}$ such that for each i we have the equality $g_i \circ u_i = u_{i-1} \circ f_i$.

(2) Given a morphism of inverse sequences it follows from the definition of an inverse limit that the morphisms $\varprojlim X_i \to X_i \to Y_i$ induce a morphism

$$u_*\colon \varprojlim X_i \to \varprojlim Y_i.$$

(3) We now work in the category of abelian groups. Let $u\colon (X_i, f_i) \to (Y_i, g_i)$ be a morphism between inverse sequences. It follows easily from the definitions that the homomorphism $u\colon \prod_{i\in\mathbb{N}} X_i \to \prod_{i\in\mathbb{N}} Y_i$ restricts to a homomorphism

$$u_*\colon \varprojlim{}^1 X_i \to \varprojlim{}^1 Y_i.$$

In the following we will at times implicitly make use of the following lemma:

[2500] If R is a commutative ring and if $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ is an inverse sequence of R-modules, then $\varprojlim^1 D_i$ is also an R-module in an obvious way.

Lemma 230.5. (Inverse Limit Functor Lemma) The above notion of morphisms allows us to consider the category *InvSeq* of inverse sequences of abelian groups. The maps

$$(X_i, f_i)_{i\in\mathbb{N}} \mapsto \varprojlim (X_i, f_i) \qquad \text{and} \qquad (X_i, f_i)_{i\in\mathbb{N}} \mapsto \varprojlim{}^1 (X_i, f_i)$$

both define covariant functors from the category *InvSeq* to the category *AbGroup* of abelian groups. An analogous statement holds if, given a commutative ring R, we work with the category of R-modules instead of the category of abelian groups.

Proof. The lemma follows easily from the definitions. ■

We cannot proceed without introducing the following somewhat technical definition.

Definition 230.6. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups. We say that the inverse sequence satisfies the Mittag-Leffler condition[2501] [2502] if for every $i \in \mathbb{N}$ there exists an $n \geq i$ such that for all $s \geq n$ we have

$$\operatorname{im}(g_i \circ \cdots \circ g_{s-1}\colon D_s \to D_i) \;=\; \operatorname{im}(g_i \circ \cdots \circ g_{n-1}\colon D_n \to D_i).$$

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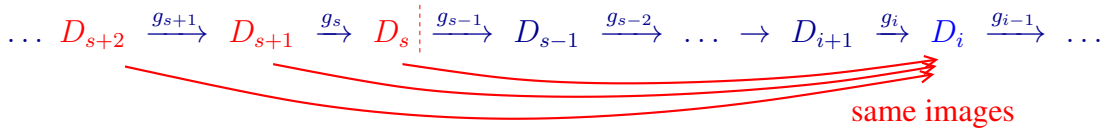


Example. It is straightforward to verify that the above inverse system D_* satisfies the Mittag-Leffler condition whereas the above inverse system E_* does not satisfy the Mittag-Leffler condition. □

In many situations one can ignore the precise definition of the Mittag-Leffler condition and instead one can appeal to the following lemma:

Lemma 230.7. (Mittag-Leffler Epimorphism Lemma) Suppose that we are given an inverse sequence $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ of abelian groups. If there exists an $m \in \mathbb{N}$ such that for all $i \geq m$ the map $g_i\colon D_{i+1} \to D_i$ is an epimorphism, then the inverse sequence satisfies the Mittag-Leffler condition.

Proof. The lemma follows almost immediately from the definitions. ■

Our main interest in the Mittag-Leffler condition arises from the fact that if an inverse sequence satisfies this condition, then the pesky $\varprojlim^1$-limit vanishes:

Lemma 230.8. (Mittag-Leffler Limit Lemma) Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups. If this inverse sequence satisfies the Mittag-Leffler condition, then

$$\varprojlim{}^1 D_i \;=\; 0.$$

[2501] The name "Mittag-Leffler condition" comes from the observation, that it is in a rather roundabout way connected to the classical Mittag-Leffler theorem from complex analysis. We refer to [**Bou1998**, Chapter 3.5] and [**Run2005**, Theorem 2.4.14] for a few more details.

[2502] For $n = i$ the map $g_i \circ \cdots \circ g_{n-1}\colon D_n \to D_i$ is a homomorphism $D_n \to D_n$ which happens to the composition of the empty set of homomorphisms. The composition of the empty set of morphisms is always understood to be the identity.

Proof. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups that satisfies the Mittag-Leffler condition. This means that given given any $i \in \mathbb{N}$ there exists an $n(i) \geq i$ such that for all $s \geq n(i)$ we have

$$\operatorname{im}(g_i \circ \cdots \circ g_{s-1}\colon D_s \to D_i) \;=\; \operatorname{im}(g_i \circ \cdots \circ g_{n(i)-1}\colon D_{n(i)} \to D_i).$$

Without loss of generality we can assume that $n(i)$ is monotonously increasing in i. We have to show that the map

$$\begin{array}{rcl} \mu_{D_i}\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}} \end{array}$$

is an epimorphism.

Given $j \geq i$ we now write $f_{ji} \coloneqq g_i \circ \cdots \circ g_{j-1}\colon D_j \to D_i$. First we consider $(a_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ with the property that for all $i \in \mathbb{N}$ we have $a_i \in \operatorname{im}(f_{n(i),i}\colon D_{n(i)} \to D_i)$. To show that $\{a_i\}_{i\in\mathbb{N}}$ lies in the image of μ_{D_i} it suffices to prove the following claim:

Claim 1. There exist $b_i \in \operatorname{im}(f_{n(i),i}\colon D_{n(i)} \to D_i)$ with $i \in \mathbb{N}$ such that for each i we have $a_i = b_i - g_i(b_{i+1})$.

Proof. We define the b_i inductively. We set $b_0 = a_0$. Suppose that $b_0, \ldots, b_k$ are already defined with the desired properties. By our hypothesis we have $a_k \in \operatorname{im}(f_{n(k),k})$ and by our choice of b_k we have $b_k \in \operatorname{im}(f_{n(k),k})$. By the Mittag-Leffler condition and since we arranged that $n(k+1) \geq n(k)$ we have $\operatorname{im}(f_{n(k+1),k}) = \operatorname{im}(f_{n(k),k})$. Thus we can pick $c \in D_{n(k+1)}$ with $f_{n(k+1),k}(c) = a_k - b_k$. Now we put

$$b_{k+1} \;\coloneqq\; -f_{n(k+1),k+1}(c).$$

Then

$$b_k - g_{k+1}(b_{k+1}) = b_k + g_{k+1}(f_{n(k+1),k+1}(c)) = b_k + \underbrace{(g_{k+1} \circ f_{n(k+1),k+1})}_{=f_{(n(k+1),k)}}(c) = b_k + (a_k - b_k) = a_k.$$

We have thus shown that $b_0, \ldots, b_{k+1}$ have the desired properties. ⊞

Now let $(b_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ be an arbitrary element. For each $i \in \mathbb{N}$ we set

$$c_i \;\coloneqq\; b_i + \sum_{j=i+1}^{n(i)} f_{j,i}(b_j).$$

Claim 2. For each $i \in \mathbb{N}$ we have $b_i - (c_i - g_i(c_{i+1})) \in \operatorname{im}(f_{n(i),i})$.

Proof. Let $i \in \mathbb{N}$. We have

$$\begin{aligned} b_i - (c_i - g_i(c_{i+1})) \;&=\; b_i - b_i - \sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + g_i\Big(b_{i+1} + \sum_{j=i+2}^{n(i+1)} f_{j,i+1}(b_j)\Big) \\ &=\; -\sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + f_{i+1,i}(b_{i+1}) + \sum_{j=i+2}^{n(i+1)} f_{j,i}(b_j) \;=\; \sum_{j=n(i)+1}^{n(i+1)} f_{j,i}(b_j). \end{aligned}$$

But by definition of $n(i)$ we know that for each $j \in \{n(i)+1, \ldots, n(i+1)\}$ we have $f_{j,i}(b_j) \in \operatorname{im}(f_{n(i),i})$. ⊞

It follows from Claim 2 and the above discussion that $(b_i - (c_i - g_i(c_{i+1})))_{i\in\mathbb{N}}$ lies in the image of μ_{D_i}. Evidently $(c_i - g_i(c_{i+1}))_{i\in\mathbb{N}}$ lies in the image of μ_{D_i}. Since μ_{D_i} is a homomorphism we see that $(b_i)_{i\in\mathbb{N}}$ also lies in the image of μ_{D_i}. ■

230.3. **Cohomology groups and inverse limits.** Now we can formulate the following proposition, which is structurally similar to the Algebraic Universal Coefficient Theorem 162.15 and the Algebraic Universal Coefficient-H*-Theorem 201.22:

Proposition 230.9. (Inverse Limit-H*-Proposition) Suppose that we are given an inverse sequence $(\{D_i^*\}_{i\in\mathbb{N}}, \{f_i\colon D_{i+1}^* \to D_i^*\}_{i\in\mathbb{N}})$ of cochain complexes. If for each $k \in \mathbb{N}_0$ the inverse sequence of cochain groups $\{D_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition[2503], then there exists a natural[2504] homomorphism

$$\tau\colon \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \;\to\; \mathrm{H}^n\big(\varprojlim D_i^*\big)$$

such that the following sequence is exact:[2505]

$$0 \;\to\; \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \;\xrightarrow{\tau}\; \mathrm{H}^n\big(\varprojlim D_i^*\big) \;\xrightarrow{\Theta}\; \varprojlim \mathrm{H}^n(D_i^*) \;\to\; 0$$

If R is a commutative ring and if the D_i^* are cochain complexes of R-modules, then the map τ is a natural R-module homomorphism.

The proof of the Inverse Limit-H*-Proposition 230.9 makes use of the following lemma:

Lemma 230.10. (Inverse Limit–Snake Lemma) Given any short exact sequence

$$0 \;\to\; (X_i, f_i) \;\xrightarrow{u_i}\; (Y_i, g_i) \;\xrightarrow{v_i}\; (Z_i, h_i) \;\to\; 0$$

of inverse sequences of abelian groups there exists a natural homomorphism

$$\delta\colon \varprojlim Z_i \;\to\; \varprojlim{}^1 X_i$$

such that the following sequence is exact:

$$0 \to \varprojlim X_i \xrightarrow{u_*} \varprojlim Y_i \xrightarrow{v_*} \varprojlim Z_i \xrightarrow{\delta} \varprojlim{}^1 X_i \xrightarrow{u_*} \varprojlim{}^1 Y_i \xrightarrow{v_*} \varprojlim{}^1 Z_i \to 0.$$

If R is a commutative ring and if all inverse sequences and all maps are in the category of R-modules, then the maps in the short exact sequence are natural R-module homomorphisms.

Proof of Lemma 230.10. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod\limits_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod\limits_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod\limits_{i\in\mathbb{N}} Z_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_{X_i}} & & \downarrow{\scriptstyle \mu_{Y_i}} & & \downarrow{\scriptstyle \mu_{Z_i}} & & \\
0 & \longrightarrow & \prod\limits_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod\limits_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod\limits_{i\in\mathbb{N}} Z_i & \longrightarrow & 0.
\end{array}$$

It follows almost immediately from the definitions that the diagram commutes and that the horizontal sequences are exact. We can view the rows as chain complexes. The existence of the desired natural homomorphism now follows from the Snake Lemma that we proved

[2503] By the Mittag-Leffler Epimorphism Lemma 230.7 this condition is satisfied if the f_i are all epimorphisms. That is the only setting we will ever make use of.

[2504] Using the category *InvSeq* of inverse sequences and using the Inverse Limit Functor Lemma 230.5 one can make precise what "natural" is actually supposed to mean.

[2505] The map to the right is given as follows: By definition an inverse limit of cochain complexes comes with cochain maps $\varprojlim D_i^* \to D_i^*$. Since cohomology of cochain complexes is covariant in cochain maps we get induced maps $\mathrm{H}^n(\varprojlim D_i) \to \mathrm{H}^n(D_i^*)$ which then, by definition of the inverse limit, give rise to a map $\mathrm{H}^n(\varprojlim D_i) \to \varprojlim \mathrm{H}^n(D_i^*)$.

in Exercise 145.24, the description of inverse limits on page 4448 and the definition of $\varprojlim^1$. ■

Proof of Proposition 230.9. Let $(\{D_i^*\}_{i\in\mathbb{N}}, \{f_i\}_{i\in\mathbb{N}})$ be an inverse sequence of cochain complexes such that for each $k \in \mathbb{N}_0$ the inverse sequence of cochain groups $\{D_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. In the following, given $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$, we denote by $B_i^m := \operatorname{im}(D_i^{m-1} \to D_i^m)$ the image of the $(m-1)$-st coboundary map and we denote by $Z_i^m := \ker(D_i^m \to D_i^{m+1})$ the kernel of the m-th coboundary map. These give rise to the following two short exact sequences of inverse sequences

$$\begin{aligned}
&\text{(i)} && 0 \to B_i^m \to Z_i^m \to \mathrm{H}^m(D_i) \to 0\\
&\text{(ii)} && 0 \to Z_i^m \to D_i^m \to B_i^{m+1} \to 0.
\end{aligned}$$

Let $m \in \mathbb{N}_0$. By our hypothesis the inverse sequences $\{D_i^m\}_{i\in\mathbb{N}}$ satisfy the Mittag-Leffler condition. It is straightforward to see that then the inverse sequence $\{B_i^m\}_{i\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition. It follows from the Mittag-Leffler Limit Lemma 230.8 that

$$\varprojlim{}^1 B_i^{m+1} = 0 \quad \text{and} \quad \varprojlim{}^1 D_i^m = 0.$$

(Here and throughout the proof all limits are taken over $i \in \mathbb{N}$.) Using this data we obtain from the Inverse Limit–Snake Lemma 230.10, applied to the two short exact sequences (i) and (ii) of inverse sequences above, the following two exact sequences:

$$0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim \mathrm{H}^m(D_i) \xrightarrow{\delta} \underbrace{\varprojlim{}^1 B_i^m}_{=0} \to \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(D_i) \to 0$$

$$0 \to \varprojlim Z_i^m \to \varprojlim D_i^m \to \varprojlim B_i^{m+1} \xrightarrow{\delta} \varprojlim{}^1 Z_i^m \to \underbrace{\varprojlim{}^1 D_i^m}_{=0} \to \varprojlim{}^1 B_i^{m+1} \to 0.$$

From these sequence we deduce that the maps

$$\begin{aligned}
&\text{(i)} && \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(D_i)\\
&\text{(ii)} && \varprojlim Z_i^m \to Z^m\big(\varprojlim D_i\big) := \ker\big(\varprojlim D_i^m \to \varprojlim B_i^{m+1}\big)
\end{aligned}$$

are isomorphisms and that the following sequences are exact

$$\begin{aligned}
&\text{(i$'$)} && 0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim \mathrm{H}^m(D_i) \to 0\\
&\text{(ii$'$)} && 0 \to \underbrace{B^m\big(\varprojlim D_i\big)}_{:=\operatorname{im}(\varprojlim D_i^{m-1} \to \varprojlim B_i^m)} \to \varprojlim B_i^m \to \varprojlim{}^1 Z_i^{m-1} \to 0.
\end{aligned}$$

From (ii$'$) and the isomorphism (ii) we obtain the inclusions

$$B^m\big(\varprojlim D_i\big) \subseteq \varprojlim B_i^m \subseteq \varprojlim Z_i^m = Z^m\big(\varprojlim D_i\big)$$

which gives rise to the short exact sequence[2506]

$$0 \to \underbrace{\frac{\varprojlim B_i^m}{B^m(\varprojlim D_i)}}_{=\varprojlim{}^1 \mathrm{H}^{m-1}(D_i) \text{ by (ii') and (ii)}} \to \underbrace{\frac{Z^m(\varprojlim D_i)}{B^m(\varprojlim D_i)}}_{=\mathrm{H}^m(\varprojlim D_i) \text{ by definition}} \to \underbrace{\frac{\varprojlim Z_i^m}{\varprojlim B_i^m}}_{=\varprojlim \mathrm{H}^m(D_i) \text{ by (i')}} \to 0.$$

■

230.4. **The cohomology group of an ascending sequence of topological spaces.** The following proposition can be viewed as the analog of the Exhaustion-H_*-Proposition 154.2 for cohomology groups:

Proposition 230.11. (Exhaustions-H^*-Proposition) Let X be a topological space. Furthermore let $X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subspaces of X with $\bigcup_{i\in\mathbb{N}_0} X_i = X$. We suppose that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Then for any $n \in \mathbb{N}_0$ and any abelian group G there exists a natural homomorphism

$$\tau\colon \varprojlim{}^1 \mathrm{H}^{n-1}(X_i;G) \to \mathrm{H}^n(X;G)$$

such that the following sequence is exact:

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(X_i;G) \xrightarrow{\tau} \mathrm{H}^n(X;G) \xrightarrow{\Theta} \varprojlim \mathrm{H}^n(X_i;G) \to 0,$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X;G) \to \mathrm{H}^n(X_i;G)$. If G is a commutative ring, then all maps are G-module homomorphisms.

Remark.

(1) In many situations the term $\varprojlim{}^1 \mathrm{H}^{n-1}(X_i;G)$ will be zero. But it can be non-zero and one should view the proposition as a warning that in general it is not true that "limits commute with everything".
(2) On page 5335 we will prove a version of the Exhaustions-H^*-Proposition 230.11 for field coefficients. □

Proof. Let X be a topological space and let $X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subspaces of X with $\bigcup_{i\in\mathbb{N}_0} X_i = X$ such that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Furthermore let G be an abelian group. For $i \in \mathbb{N}_0$ we consider the cochain complex $D_i^* := \mathrm{Hom}(\mathrm{C}_*(X_i), G)$ and for $i \in \mathbb{N}_0$ we denote by $f_i\colon D_{i+1}^* \to D_i^*$ the map that is induced by the inclusion $X_i \to X_{i+1}$.

Claim. For every choice of $k \in \mathbb{N}$ the inverse sequence of cochain groups that is given by $(\{D_i^k\}_{i\in\mathbb{N}}, \{f_i\colon D_{i+1}^k \to D_i^k\}_{i\in\mathbb{N}_0})$ satisfies the Mittag-Leffler condition.

[2506] Here we use that a filtration $P \subseteq Q \subseteq R$ of three abelian groups induces a short exact sequence $0 \to Q/P \to R/P \to R/Q \to 0$.

Proof. Let $k \in \mathbb{N}_0$. By the Mittag-Leffler Epimorphism Lemma 230.7 it suffices to show that for every $i \in \mathbb{N}_0$ the map

$$D_{i+1}^k = \operatorname{Hom}(\mathrm{C}_n(X_{i+1}); G) \xrightarrow{f_i} \operatorname{Hom}(\mathrm{C}_n(X_i); G) = D_i^k$$

is an epimorphism. Since $X_i \subseteq X_{i+1}$ we know that the set of singular chains in X_i is a subset of the set of singular chains in X_{i+1}. It follows from the universal properties of free abelian groups that the map $\operatorname{Hom}(\mathrm{C}_n(X_{i+1}); G) \to \operatorname{Hom}(\mathrm{C}_n(X_i); G)$ is indeed an epimorphism.[2507] ⊞

Let $n \in \mathbb{N}_0$. From the Inverse Limit-H*-Proposition 230.9 we obtain a natural homomorphism

$$\tau \colon \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \to \mathrm{H}^n\big(\varprojlim D_i^*\big)$$

such that the following sequence is exact:

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \xrightarrow{\tau} \mathrm{H}^n\big(\varprojlim D_i^*\big) \xrightarrow{\Theta} \varprojlim \mathrm{H}^n(D_i^*) \to 0$$

Let $i \in \mathbb{N}_0$. We make the following two observations:

(1) By definition we have $\mathrm{H}^k(D_i^*) = \mathrm{H}^k(X_i; G)$ for every $k \in \mathbb{N}_0$.
(2) We have natural isomorphisms

$$\varprojlim D_i^* \overset{\cong}{\underset{\uparrow}{\to}} \varprojlim \operatorname{Hom}(\mathrm{C}_*(X_i); G) = \underset{\uparrow}{\operatorname{Hom}}\big(\varinjlim \mathrm{C}_*(X_i), G\big) \overset{\cong}{\underset{\uparrow}{\leftarrow}} \operatorname{Hom}(\mathrm{C}_*(X), G).$$

by definition of the D_i — by the Hom-of-Direct Colimit Lemma 229.7 — by the Exhaustion-C$_*$-Lemma 154.1 and our hypothesis (1) or (2)

Combining all of the above we obtain our desired short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(X_i; G) \to \mathrm{H}^n(X; G) \to \varprojlim \mathrm{H}^n(X_i; G) \to 0.$$

If G is a commutative ring, then one can easily verify that τ and all the other maps are G-module homomorphisms. ■

We provide a few examples for the above proposition.

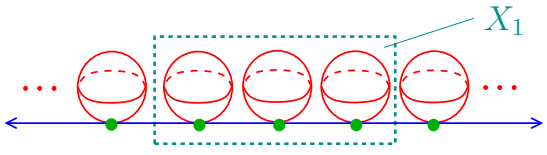
Example.

(1) As in the Infinite String-of-Spheres-H$_*$-Lemma 154.3 we consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X := (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0, 0, -1), i) \in S^2 \times \mathbb{Z}$. Furthermore, given $k \in \mathbb{N}$ we consider the open subset

$$X_k := ((-k - \tfrac{1}{2}, k + \tfrac{1}{2}) \cup (S^2 \times \{-k, \dots, k\}))/\sim .$$

$X =$ 

$$\mathrm{H}_2(X) \cong \mathbb{Z}^{(\mathbb{N})}$$
$$\mathrm{H}^2(X; \mathbb{Z}) \cong \mathbb{Z}^{\mathbb{N}}$$

[2507] Here we use the following observation: If S is a subset of a set T, then we have a natural inclusion $i \colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ and it follows from the Abelian Group-Homomorphism Lemma 197.5 (1) that the map $i^* \colon \operatorname{Hom}(\mathbb{Z}^{(T)}, G) \to \operatorname{Hom}(\mathbb{Z}^{(S)}, G)$ is an epimorphism. Indeed, we just extend the map $S \to G$ to a map $T \to G$ by assigning the trivial element in G to all elements in $T \setminus S$.

We leave it to the reader to verify that we can make the identifications[2508]

$$\mathrm{H}^n(X_k;\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{2k+1}, & \text{if } n=2,\\ 0, & \text{else.}\end{cases}$$

such that for each $k \in \mathbb{N}$ the inclusion maps $X_k \to X_{k+1}$ induce the identity map on $\mathbb{Z} = \mathrm{H}^0(X_{k+1};\mathbb{Z}) \to \mathrm{H}^0(X_k;\mathbb{Z}) = \mathbb{Z}$ and such that the inclusions induce the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+3} = \mathrm{H}^2(X_{k+1};\mathbb{Z}) & \to & \mathrm{H}^2(X_k;\mathbb{Z}) = \mathbb{Z}^{2k+1} \\ (a_1,\ldots,a_{2k+3}) & \mapsto & (a_2,\ldots,a_{2k+2}) \end{array}$$

It follows from the Mittag-Leffler Epimorphism Lemma 230.7 and the above calculations that for each $n \in \mathbb{N}_0$ the inverse sequence of cohomology groups $\{\mathrm{H}^n(X_k;\mathbb{Z})\}_{k\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. We obtain that

$$\mathrm{H}^n(X;\mathbb{Z}) \;\underset{\uparrow}{=}\; \varprojlim \mathrm{H}^n(X_k;\mathbb{Z}) \;=\; \varprojlim \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{2k+1}, & \text{if } n=2,\\ 0, & \text{else}\end{cases} \;\underset{\uparrow}{=}\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=2,\\ 0, & \text{else.}\end{cases}$$

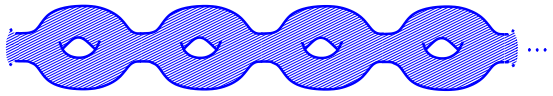
by the Exhaustions-H$_*$-Proposition 230.11 and the Mittag-Leffler Limit Lemma 230.8

see page 4437

Note that in the Infinite String-of-Spheres-H$_*$-Lemma 154.3 we saw that $\mathrm{H}_2(X) \cong \mathbb{Z}^{(\mathbb{N})}$. In this case we see that $\mathrm{H}_2(X) \ncong \mathrm{H}^2(X;\mathbb{Z})$ since the former group is countable whereas the latter group is uncountable.[2509]

(2) We consider again the surface Σ_∞ of infinite genus that we had already encountered on page 2054 and in the Surface-of-Infinite Genus–H$_*$-Lemma 155.24.

surface of "infinite genus"



Using a variation on example (1) and the proof of the Surface-of-Infinite Genus–π_1-Lemma 105.5 one can show that

$$\mathrm{H}^n(\Sigma_\infty;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=1,\\ 0, & \text{if } n\geq 2.\end{cases}$$

We leave the task of providing full details to the valiant reader. Note that this shows that the first cohomology group is uncountable. In particular we see that the cohomology groups with $\mathbb{Z}$-coefficients of a non-compact smooth manifold do not need to be countable. This is in contrast to the Smooth Manifolds-Invariants Proposition 179.10 (2a) where we showed that the homology groups with $\mathbb{Z}$-coefficients of a (possibly non-compact) smooth manifold are always countable. □

We conclude this chapter with the following technical variation on the Exhaustions-H*-Proposition 230.11. It can be viewed as the cohomological analog of the CW-Skeleton-H$_*$-Lemma 155.2 (3).

[2508] One possible approach to showing these statements is to use the Mayer–Vietoris-H*-Theorem 200.11 (1).

[2509] Of course we could have obtained the same observation from the Universal Coefficient-H*-Theorem 201.23.

Proposition 230.12. (Skeleton-H*-Proposition) Let X be a CW-complex and let $n \in \mathbb{N}$. For every $k > n$ and every abelian group G the inclusion $i\colon X^k \to X$ induces an isomorphism

$$i^*\colon \mathrm{H}^n(X;G) \;\to\; \mathrm{H}^n(X^k;G).$$

Remark. There are many other ways to prove the Skeleton-H*-Proposition 230.12. For example it can also be proved using the Also-Iso-on-Cohomology Proposition 200.2. Furthermore we will give a very different proof of this proposition in the Skeleton-H*-Theory Proposition **??**. □

Proof. By the Exhaustions-H*-Proposition 230.11 there exists a natural map τ which gives rise to a short exact sequence

$$0 \;\to\; \varprojlim_{j\in\mathbb{N}}{}^1\, \mathrm{H}^{n-1}(X_j;G) \;\xrightarrow{\tau}\; \mathrm{H}^n(X;G) \;\xrightarrow{\Theta}\; \varprojlim_{j\in\mathbb{N}} \mathrm{H}^n(X_j;G) \;\to\; 0$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X;G) \to \mathrm{H}^n(X_i;G)$. It follows immediately from the Inverse Limit Stabilization Lemma 229.5 and the Mittag-Leffler Epimorphism Lemma 230.7 that is suffices to prove the following claim:

Claim. For any $j \geq k > n$ the inclusion-induced map $\mathrm{H}^n(X^j;G) \to \mathrm{H}^n(X^k;G)$ is an isomorphism.

Proof. It follows from the CW-Skeleton-H$_*$-Lemma 155.2 (1), the Universal Coefficient-H*-Theorem 201.23 that for every $m \geq n$ we have $\mathrm{H}^n(X^{m+1}, X^m;G) = 0$. Using the long exact sequence in cohomology of the pairs $(X^{m+1}, X^m), (X^{m+2}, X^m), \ldots$ one can now easily show that for any $j \geq k > n$ the inclusion-induced map $\mathrm{H}^n(X^j;G) \to \mathrm{H}^n(X^k;G)$ is an isomorphism. ■

230.5. **Appendix: De Rham cohomology and exhaustions.** For fans of de Rham cohomology we now prove an analog of the Exhaustions-H*-Proposition 230.11:

Proposition 230.13. (Exhaustions-H$^*_{\mathrm{dR}}$-Proposition) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:

(1) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(2) for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$, and
(3) $M = \bigcup_{i\in\mathbb{N}} U_i$.

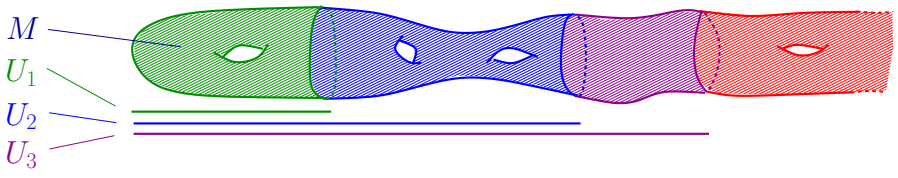
Then for every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\tau\colon \varprojlim{}^1\, \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\to\; \mathrm{H}^n_{\mathrm{dR}}(M)$$

such that the following sequence is exact:

$$0 \;\to\; \varprojlim{}^1\, \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\xrightarrow{\tau}\; \mathrm{H}^n_{\mathrm{dR}}(M) \;\xrightarrow{\Theta}\; \varprojlim \mathrm{H}^n_{\mathrm{dR}}(U_i) \;\to\; 0.$$

(Here the right-hand map Θ is induced by the obvious maps $\mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n_{\mathrm{dR}}(U_i)$.)



Proof. First recall that given a smooth manifold W we denote, as in Notation 74.5, by $\Omega^k(W)$ the real vector space of all smooth k-forms on W. In our setting we consider for each $k \in \mathbb{N}_0$ the inverse sequence

$$\Omega^k(U_1) \;\leftarrow\; \Omega^k(U_2) \;\leftarrow\; \Omega^k(U_3) \;\leftarrow\; \dots$$

where all the maps are induced by the inclusion, put differently, they are all restriction maps.

Claim. The inverse sequence satisfies the Mittag-Leffler condition from Definition 230.6.

Proof. Given any $j \geq i$ we denote by $\rho_{ji}\colon \Omega^k(U_j) \to \Omega^k(U_i)$ the restriction map. Now let $i \in \mathbb{N}$. By definition of the Mittag-Leffler condition from Definition 230.6 we have to show that there exists an $n \geq i$ such that for all $r \geq n$ we have

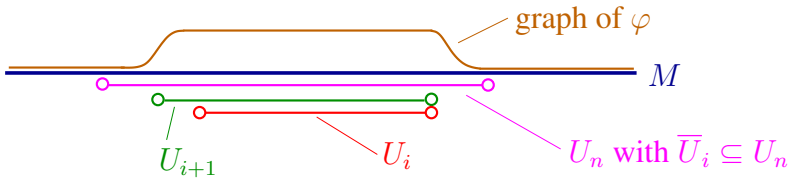
$$\operatorname{im}(\rho_{ri}\colon \Omega^k(U_r) \to \Omega^k(U_i)) \;=\; \operatorname{im}(\rho_{ni}\colon \Omega^k(U_n) \to \Omega^k(U_i)).$$

Since $\overline{U}_i$ is compact, since the U_j are nested and since $\overline{U}_i$ is contained in the union of all U_j there exists an $n \in \mathbb{N}$ such that $\overline{U}_i \subseteq U_n$.

We claim that that n has the desired property. So let $r \geq n$. We have to show that

$$\operatorname{im}(\rho_{ri}\colon \Omega^k(U_r) \to \Omega^k(U_i)) \;=\; \operatorname{im}(\rho_{ni}\colon \Omega^k(U_n) \to \Omega^k(U_i)).$$

Clearly we have the inclusion "$\subseteq$". It remains to prove the reverse inclusion. So suppose $\eta \in \Omega^k(U_i)$ is the restriction of some $\mu \in \Omega^k(U_n)$. Note that U_n and $M \setminus \overline{U}_i$ form an open cover for M. It follows immediately from the Manifold–Partitions-of-Unity Theorem 27.2 that there exists a smooth function $\varphi\colon M \to \mathbb{R}$ with $\varphi|_{\overline{U}_i} \equiv 1$ and such that $\operatorname{supp}(\varphi) \subseteq U_n$. Let $r \geq n$.



As in the Extend Smooth Forms Lemma 80.3 we denote by $\epsilon(\varphi \cdot \mu, U_r)$ the smooth k-form that is given by $\varphi \cdot \mu$ on U_r and that is extended to the zero k-form outside of U_r. We see that

$$\eta \;=\; \mu|_{U_i} \;\underset{\substack{\uparrow\\ \text{since } \varphi \equiv 1 \text{ on } U_i}}{=}\; (\varphi \cdot \mu)|_{U_i} \;=\; \epsilon(\varphi \cdot \mu, U_r)|_{U_i}.$$

We have thus shown that η is also the restriction of some smooth form on U_r. ⊞

By the claim we can apply the Inverse Limit-H*-Proposition 230.9 to the inverse sequence $(\{\Omega^*(U_i)\}_{i\in\mathbb{N}}, \{\rho_{ij}\}_{i\geq j})$ of cochain complexes over $\mathbb{N}$ and we obtain the natural homomorphism

$$\tau\colon \varprojlim{}^1 \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\to\; \mathrm{H}^n_{\mathrm{dR}}(M)$$

such that the following sequence is exact:

$$0 \to \varprojlim_{i\in\mathbb{N}}{}^1 \underbrace{\mathrm{H}^{n-1}(\Omega^*(U_i))}_{=\mathrm{H}^{n-1}_{\mathrm{dR}}(U_i)} \to \underbrace{\mathrm{H}^n\Big(\overbrace{\varprojlim_{i\in\mathbb{N}} \Omega^*(U_i)}^{=\Omega^*(M),\text{ see Exercise 229.12}}\Big)}_{=\mathrm{H}^n_{\mathrm{dR}}(M)} \to \varprojlim_{i\in\mathbb{N}} \underbrace{\mathrm{H}^n(\Omega^*(U_i))}_{=\mathrm{H}^n_{\mathrm{dR}}(U_i)} \to 0.$$

■

230.6. Exercises for Chapter 230 .

Exercise 230.1. We consider the inverse sequences

$$D_*: \quad 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow \dots.$$

and

$$E_*: \quad 0 \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 3} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{\cdot 5} \mathbb{Z} \leftarrow \dots.$$

(a) Show that $\varprojlim D_* = 0$.
(b) Show that $\varprojlim^1 D_* = 0$.
(c) Show that $\varprojlim^1 E_* \neq 0$.

Exercise 230.2. Show that the inverse sequence

$$\dots \to \mathbb{Z} \xrightarrow{\cdot 11} \mathbb{Z} \xrightarrow{\cdot 7} \mathbb{Z} \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$$

does not satisfy the Mittag-Leffler condition.

Exercise 230.3. Show that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $\mathrm{H}^1(X_i;\mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the map $\mathrm{H}^1(X_{i+1};\mathbb{Z}) \to \mathrm{H}^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$.
Put differently, we want topological spaces and continuous maps such that the inverse sequence

$$\dots \to \mathrm{H}^1(X_4;\mathbb{Z}) \to \mathrm{H}^1(X_3;\mathbb{Z}) \to \mathrm{H}^1(X_2;\mathbb{Z}) \to \mathrm{H}^1(X_1;\mathbb{Z})$$

is isomorphic to the inverse sequence E_* from Exercise 230.1.

Exercise 230.4. We consider the following short exact sequence of inverse sequences in the category *AbGroup* of abelian groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p^{i+1}\cdot\mathbb{Z} & \longrightarrow & \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p^{i+1}\cdot\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & p^{i}\cdot\mathbb{Z} & \longrightarrow & \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p^{i}\cdot\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & p^{i-1}\cdot\mathbb{Z} & \longrightarrow & \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/p^{i-1}\cdot\mathbb{Z} & \longrightarrow & 0
\end{array}$$

where we continue upward indefinitely and we continue downward up to $i = 1$. Determine for each inverse sequence the inverse limit $\varprojlim$ and also determine $\varprojlim^1$.

Exercise 230.5. Let X be a topological space and let $X_0 \subseteq X_1 \subseteq \dots$ be a sequence of subspaces such that for each $k \in \mathbb{N}_0$ the natural map

$$\mathrm{H}^k(X;\mathbb{Z}) \to \varprojlim \mathrm{H}^k(X_i;\mathbb{Z})$$

of abelian groups is an isomorphism. We now consider the cohomology rings, and in particular we consider the inverse limit in the category of rings. Is the natural map

$$(\mathrm{H}^*(X;\mathbb{Z}),\cup) \to \varprojlim (\mathrm{H}^*(X_i;\mathbb{Z}),\cup)$$

necessarily an isomorphism?

Exercise 230.6. Let X be a topological space. Furthermore let $X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be a sequence of subspaces of X with $\bigcup_{i\in\mathbb{N}_0} X_i = X$. We suppose that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Show that for any $n \in \mathbb{N}_0$ and any field $\mathbb{F}$ the natural homomorphism

$$\mathrm{H}^n(X;\mathbb{F}) \xrightarrow{\Theta} \varprojlim \mathrm{H}^n(X_i;\mathbb{F})$$

is an isomorphism.

Hint. Use the Exhaustion-H_*-Proposition 154.2, the Hom-of-Direct Colimit Lemma 229.7 and the Universal Coefficient-H^*–Field Theorem 202.2.

Exercise 230.7. Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:

(1) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(2) for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$, and
(3) $M = \bigcup_{i\in\mathbb{N}} U_i$.

In the proof of the Exhaustions-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 230.13 we showed that for each $k \in \mathbb{N}_0$ the inverse sequence

$$\Omega^k(U_1) \leftarrow \Omega^k(U_2) \leftarrow \Omega^k(U_2) \leftarrow \dots$$

satisfies the Mittag-Leffler condition. Now let $i \in \mathbb{N}$. Does there exist an $n \geq i$ such that for all $r \geq n$ the map

$$\Omega^k(U_r) \to \Omega^k(U_i)$$

is an epimorphism?

CHAPTER 231

De Rham cohomology and singular cohomology

In this chapter we will show that, given a smooth manifold M, there exists a natural isomorphism between the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(M)$ (that we introduced in Chapter 78) and the singular cohomology groups $\mathrm{H}^i(M;\mathbb{R})$ (that we introduced in Chapter 198).

This chapter is organized as follows:

- In Section 231.1 we will introduce smooth cohomology theories
- In Section 231.2 we will state and prove the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 which gives us a tool to show that two smooth cohomology theories are isomorphic.
- In Section 231.3 we will introduce smooth singular cohomology.
- In Section 231.4 we will relate the three smooth cohomology theories given by (smooth) singular cohomology and de Rham cohomology.
- In Section 231.5 we will state the Stokes-for-Simplices Proposition 231.15 and we will use it to prove the De Rham-Isomorphism Theorem 231.16 which shows that singular cohomology and de Rham cohomology are isomorphic.
- In Appendix 231.6 we will prove the Stokes-for-Simplices Proposition 231.15.
- In Section 231.7 we will provide exercises for Chapter 231.

In the next Chapter 232 we will show that the wedge product on de Rham cohomology corresponds to the cup product on singular cohomology.

231.1. **Smooth cohomology theories.** To a smooth manifold M we can associate the singular cohomology groups $\mathrm{H}^i(M;\mathbb{R})$ and also the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(M)$. Rereading the chapters on these cohomology groups one notices many common features, for example they both satisfy a Mayer–Vietoris type theorem. Now we want to put both cohomologies in a common framework. To build this common framework it is helpful to recall and introduce some notation:

Notation 231.1.

(1) As in Definition 29.7 we define

$$\mathcal{SmMfd} \;:=\; \text{category of smooth manifolds}$$

where objects are smooth manifolds and morphisms are smooth maps.

(2) Let $\mathbb{K}$ be a field.

- An $\mathbb{N}_0$-graded $\mathbb{K}$-vector space is a family $\{V_n\}_{n\in\mathbb{N}_0}$ of $\mathbb{K}$-vector spaces. We often write an $\mathbb{N}_0$-graded $\mathbb{K}$-vector space as a direct sum $\bigoplus_{n\in\mathbb{N}_0} V_n$.

- A morphism between $\mathbb{N}_0$-graded $\mathbb{K}$-vector spaces $\{V_n\}_{n\in\mathbb{N}_0}$ and $\{W_n\}_{n\in\mathbb{N}_0}$ is a family $\{\varphi_n\colon V_n \to W_n\}_{n\in\mathbb{N}_0}$ of $\mathbb{K}$-linear homomorphisms.
- We denote by $\mathcal{V}ec_{\mathbb{K}}^{\mathbb{N}_0}$ the category of $\mathbb{N}_0$-graded $\mathbb{K}$-vector spaces.

This notation lets us formulate the common framework for the different types of cohomology groups:

Definition 231.2. A smooth cohomology theory is a contravariant functor[2510]

$$\begin{array}{rccc} \mathcal{H}^*\colon & \text{category } \mathcal{S}m\mathcal{M}fd \text{ of smooth manifolds} & \to & \text{category } \mathcal{V}ec_{\mathbb{R}}^{\mathbb{N}_0} \text{ of } \mathbb{N}_0\text{-graded real vector spaces} \\ & M & \mapsto & \mathcal{H}^*(M) = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}^n(M) \\ & (f\colon M\to N) & \mapsto & \{f^* = \mathcal{H}^n(f)\colon \mathcal{H}^n(N) \to \mathcal{H}^n(M)\}_{n\in\mathbb{N}_0} \end{array}$$

together with natural homomorphisms $\delta_{\mathcal{H}}$ and $\tau_{\mathcal{H}}$ (described below) such that the following four axioms are satisfied:

(1) For every $n \in \mathbb{N}_0$ we have $\mathcal{H}^n(\varnothing) = 0$.

(2) (Smooth homotopy invariance) Let $f, g\colon M \to N$ be smooth maps between smooth manifolds. If f and g are smoothly homotopic[2511], then for all $n \in \mathbb{N}_0$ we have

$$f^* = g^*\colon \mathcal{H}^n(N) \ \to\ \mathcal{H}^n(M).$$

(3) (Mayer–Vietoris sequence) If M is a smooth manifold and if U and V are open subsets of M such that $M = U \cup V$, then there exists a family of natural homomorphisms

$$\delta_{\mathcal{H}}\colon \mathcal{H}^n(U\cap V) \ \to\ \mathcal{H}^{n+1}(M)$$

with $n \in \mathbb{N}_0$ such that the following sequence is exact:

$$\ldots \to \mathcal{H}^n(M) \xrightarrow{(j_U^*, -j_V^*)} \mathcal{H}^n(U)\oplus\mathcal{H}^n(V) \xrightarrow{i_U^*\oplus i_V^*} \mathcal{H}^n(U\cap V) \xrightarrow{\delta_{\mathcal{H}}} \mathcal{H}^{n+1}(M) \to \ldots$$

(Here $j_U\colon U \to M$, $j_V\colon V \to M$, $i_U\colon U\cap V \to U$ and $i_V\colon U\cap V \to V$ denote the inclusion maps.)

(4) (Limit property) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:

(a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,

(b) for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$, and

(c) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.

Then for every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\tau_{\mathcal{H}}\colon \varprojlim{}^1\, \mathcal{H}^{n-1}(U_i) \ \to\ \mathcal{H}^n(M)$$

such that the following sequence is exact:

$$0 \ \to\ \varprojlim{}^1\, \mathcal{H}^{n-1}(U_i) \ \xrightarrow{\tau_{\mathcal{H}}}\ \mathcal{H}^n(M) \ \xrightarrow{\Theta}\ \varprojlim \mathcal{H}^n(U_i) \ \to\ 0.$$

(Here the right-hand map Θ is induced by the maps $\mathcal{H}^n(M) \to \mathcal{H}^n(U_i)$ which in turn are induced by the inclusions $U_i \to M$.)

This time it is easy to give examples for our new definition:

Proposition 231.3. (Smooth-$\mathcal{H}^*$-Theories Proposition)

(1) Singular cohomology with real coefficients (together with the natural maps from the Mayer–Vietoris-H^*-Theorem 200.11 (1) and the Exhaustions-H^*-Proposition 230.11) is a smooth cohomology theory.

(2) De Rham cohomology (together with the natural maps from the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 and the Exhaustions-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 230.13) is a smooth cohomology theory.

Proof.

(1) It follows from the Empty Space-H^*-Lemma 198.7, the Homotopy-H^*-Proposition 198.11, the Mayer–Vietoris-H^*-Theorem 200.11 (1) and the Exhaustions-H^*-Proposition 230.11 (1) that singular cohomology with real coefficients is a smooth cohomology theory.

(2) It follows from the Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.6, the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition 79.2, the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 and the Exhaustions-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 230.13 that de Rham cohomology is a smooth cohomology theory. ■

231.2. **Isomorphisms between smooth cohomology theories.** One might now be optimistic and hope that smooth cohomology theories are necessarily isomorphic. Unfortunately, as it stands, this thought is too optimistic. For example three other smooth cohomology theories are given by

$$\begin{aligned}\mathcal{H}^n(M) &:= 0,\\ \mathcal{H}^n(M) &:= \mathrm{H}^{n+5}(M;\mathbb{R}),\\ \mathcal{H}^n(M) &:= \mathrm{H}^n(M;\mathbb{R}) \oplus \mathrm{H}^n(M;\mathbb{R}).\end{aligned}$$

This shows that if we want to prove that smooth cohomology theories are isomorphic, we have to add some extra hypotheses. For example we could demand that the 0-th cohomology groups of a point are isomorphic. This thought motivates the following notation:

Notation 231.4. In the following $\star$ denotes the topological space consisting of a single point. Evidently $\star$ is a 0-dimensional smooth manifold.

In Definition 5.27 we introduced the notion of a natural transformation between two contravariant functors. For the reader's convenience we recall the definition, since it will play an essential role in the following discussion:

Definition 231.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two contravariant functors. A natural transformation between the contravariant functors F and G assigns to each object $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each

[2510] Given a smooth map $f\colon M \to N$ the contravariant functor provides for each $n \in \mathbb{N}_0$ a morphism $\mathcal{H}^n(f)\colon \mathcal{H}^n(N) \to \mathcal{H}^n(M)$. As usual, when there is no danger of confusion we just write f^* instead of $\mathcal{H}^n(f)$.

[2511] Since we assume that $\partial M = \varnothing$ we obtain from the Replace-by-Smooth Homotopy Proposition 36.6 that two smooth maps $f, g\colon M \to N$ are smoothly homotopic if and only if they are homotopic.

morphism $\varphi\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(\varphi)} & F(X) \\ {\scriptstyle \Phi_Y}\downarrow & & \downarrow{\scriptstyle \Phi_X} \\ G(Y) & \xrightarrow{G(\varphi)} & G(X). \end{array}$$

If all the maps Φ_X are isomorphisms we also say that it is a natural isomorphism.

With this notation and this definition we can now formulate the following theorem:

Theorem 231.6. (Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem) Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories and let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation. Suppose that t has the following three properties:

(i) The natural transformation t is compatible with the connecting homomorphisms in the Mayer-Vietoris sequence, i.e. with the obvious notation we always have a commutative diagram

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \mathcal{H}^n(M) & \xrightarrow{(j_U^*, -j_V^*)} & \mathcal{H}^n(U) \oplus \mathcal{H}^n(V) & \xrightarrow{i_U^* \oplus i_V^*} & \mathcal{H}^n(U \cap V) & \xrightarrow{\delta_{\mathcal{H}}} & \mathcal{H}^{n+1}(M) & \longrightarrow \ldots \\ & \downarrow{\scriptstyle t^n} & & \downarrow{\scriptstyle t^n \oplus t^n} & & \downarrow{\scriptstyle t^n} & & \downarrow{\scriptstyle t^{n+1}} & \\ \ldots \longrightarrow & \mathcal{K}^n(M) & \xrightarrow{(j_U^*, -j_V^*)} & \mathcal{K}^n(U) \oplus \mathcal{K}^n(V) & \xrightarrow{i_U^* \oplus i_V^*} & \mathcal{K}^n(U \cap V) & \xrightarrow{\delta_{\mathcal{K}}} & \mathcal{K}^{n+1}(M) & \longrightarrow \ldots \end{array}$$

(ii) The natural transformation t is compatible with the map τ appearing in the limit property, i.e. with the obvious notation we always have a commutative diagram[2512]

$$\begin{array}{ccccccccc} 0 \longrightarrow & \varprojlim{}^1 \mathcal{H}^{n-1}(U_i) & \xrightarrow{\tau_{\mathcal{H}}} & \mathcal{H}^n(M) & \xrightarrow{\Theta} & \varprojlim \mathcal{H}^n(U_i) & \longrightarrow 0 \\ & \downarrow{\scriptstyle t^{n-1}} & & \downarrow{\scriptstyle t^n} & & \downarrow{\scriptstyle t^n} & \\ 0 \longrightarrow & \varprojlim{}^1 \mathcal{K}^{n-1}(U_i) & \xrightarrow{\tau_{\mathcal{K}}} & \mathcal{K}^n(M) & \xrightarrow{\Theta} & \varprojlim \mathcal{K}^n(U_i) & \longrightarrow 0. \end{array}$$

(iii) For all $n \in \mathbb{N}_0$ the map $t_\star^n\colon \mathcal{H}^n(\star) \to \mathcal{K}^n(\star)$ is an isomorphism.

Then for every smooth manifold M and every $n \in \mathbb{N}_0$ the map

$$t^n\colon \mathcal{H}^n(M) \ \to \ \mathcal{K}^n(M)$$

is an isomorphism.

The proof of the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 rests on the following useful little lemma, which we implicitly proved in the proof of the Second Countability Lemma 13.2 (1), and which incidentally also solves Exercise 3.28:

Lemma 231.7. (Rational Open Balls Lemma)

(1) Every open subset U of $\mathbb{R}^m$ is the union of open balls of the form $B_\epsilon^m(x)$ where ϵ is rational, $x \in \mathbb{Q}^m$ and with $\overline{B}_\epsilon^m(x) \subseteq U$.

(2) Every open subset U of $\mathbb{H}^m$ is the union of open subsets of the form $B_\epsilon^m(x) \cap \mathbb{H}^m$ where ϵ is rational, $x \in \mathbb{Q}^m$ and with $\overline{B}_\epsilon^m(x) \subseteq U$.

Proof of Lemma 231.7.

[2512]Note that the left and right vertical maps are given by Definition 230.4.

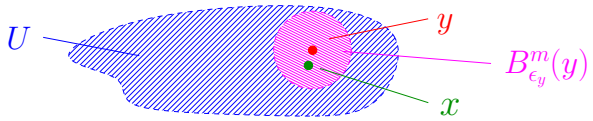
(1) Let U be an open subset of $\mathbb{R}^m$. For $x \in U$ we set

$$\mu_x \;:=\; \sup\{\mu \in [0,1] \,|\, B^m_\mu(x) \subseteq U\}.$$

The hypothesis that U is open implies that $\mu_x > 0$. We now pick a rational number $\epsilon_x \in (\frac{1}{2}\mu_x, \frac{3}{4}\mu_x)$. Note that $\overline{B^m_{\epsilon_x}(x)} \subseteq \overline{B}^m_{\frac{3}{4}\mu_x}(x) \subseteq U$. It remains to prove the following claim:

Claim. The subset U is the union of all $B^m_{\epsilon_x}(x)$ with $x \in U \cap \mathbb{Q}^m$.

Proof. Let $y \in U$. Since $\mathbb{Q}^m$ is dense in $\mathbb{R}^m$ we can pick $x \in U \cap \mathbb{Q}^m$ with $\|x-y\| < \frac{1}{4}\epsilon_y$. Note that this is equivalent to saying that $y \in B^m_{\frac{1}{4}\epsilon_y}(x)$. Thus it remains to prove that $B^m_{\frac{1}{4}\epsilon_y}(x) \subseteq B^m_{\epsilon_x}(x)$.



Recall that above we saw that $B^m_{\epsilon_y}(y) \subseteq U$. It follows from the inequality $\|x-y\| < \frac{1}{4}\epsilon_y$ and the triangle inequality that we have an inclusion $B^m_{\frac{3}{4}\epsilon_y}(x) \subseteq B^m_{\epsilon_y}(y)$. It follows that $B^m_{\frac{3}{4}\epsilon_y}(x) \subseteq U$. By definition of μ_x this implies that $\frac{3}{4}\epsilon_y \leq \mu_x$. Hence we have $\frac{1}{4}\epsilon_y \leq \frac{3}{8}\epsilon_y \leq \frac{1}{2}\mu_x \leq \epsilon_x$. But then we also have $B^m_{\frac{1}{4}\epsilon_y}(x) \subseteq B^m_{\epsilon_x}(x)$, as desired.

(2) Let U be an open subset of $\mathbb{H}^m$. By definition of the subspace topology there exists an open subset $V \subseteq \mathbb{R}^m$ with $U = \mathbb{H}^m \cap V$. We apply (1) to V and we obtain the desired statement. ∎

Proof of Theorem 231.6. Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories, which means, by Definition 231.2, in a nutshell that they both have the following properties:

(1) the invariants of the empty manifold are zero,
(2) smooth homotopy invariance,
(3) the Mayer-Vietoris sequence,
(4) the limit property.

Furthermore let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation that has the properties (i), (ii) and (iii) from the statement of the theorem on page 4463. We need to prove the following statement:

Statement. Let $m \in \mathbb{N}_0$. For every m-dimensional smooth manifold M and every $n \in \mathbb{N}_0$ the map

$$t^n_M\colon \mathcal{H}^n(M) \;\to\; \mathcal{K}^n(M)$$

is an isomorphism.

We start out with the following two observations:

(A) Suppose that M is an m-dimensional smooth manifold and that U and V are open subsets with $M = U \cup V$. Let $n \in \mathbb{N}_0$. If t^n and t^{n+1} are isomorphisms for U, V and $U \cap V$, then it follows from (3), (i) and the Five Lemma 145.11 that t^n is an isomorphism for M.

(B) Let M be an m-dimensional smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
  (a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
  (b) for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$, and

(c) $M = \bigcup_{i\in\mathbb{N}} U_i$.

Let $n \in \mathbb{N}_0$. If t^{n-1} and t^n are isomorphisms for all U_i, then it follows, as in (A), from (4) and the Five Lemma 145.11 that t^n is an isomorphism for M.

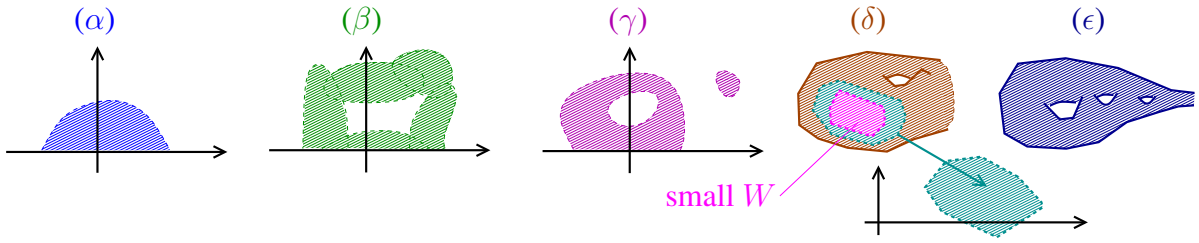
After these preparations we will prove the desired statement in five easy steps, which are reminiscent of the five steps in the proof of the Homology-Section Theorem 193.6 and the five steps in the proof of the Poincaré Duality Theorem 210.1 on page 4057:

Claim 2.

(α) Every t^n is an isomorphism for every open bounded convex subset of $\mathbb{H}^m$.
(β) Every t^n is an isomorphism for the union of k open bounded convex subsets of $\mathbb{H}^m$.
(γ) Every t^n is an isomorphism for every open subset of $\mathbb{H}^m$.

To formulate the next statement of the claim we need some extra language: We say that a subset W of a smooth manifold M is small if it is open and if W is contained in the domain of some chart for M where the target of the chart is an open subset of $\mathbb{H}^m$.

(δ) Every t^n is an isomorphism for all smooth manifolds that are covered by m small subsets.
(ϵ) Every t^n is an isomorphism for all m-dimensional smooth manifolds.



Proof.

(α) Let M be an open convex subset of $\mathbb{H}^m$. If M is empty, then the statement follows from (1). Now assume that M is non-empty. We pick an inclusion $f\colon \star \to M$. Since M is convex we have an obvious smooth deformation retraction from M to $\star$. Given $n \in \mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} \mathcal{H}^n(M) & \xrightarrow{f^*} & \mathcal{H}^n(\star) \\ {\scriptstyle t}\downarrow & & \cong\downarrow{\scriptstyle t} \\ \mathcal{K}^n(M) & \xrightarrow{f^*} & \mathcal{K}^n(\star). \end{array}$$

The diagram commutes since t is a natural transformation. It follows from (2), by the same argument as in the Homotopy Equivalence-H$_*$-Proposition 144.14, that the horizontal maps are isomorphisms. By (iii) the right vertical map is also an isomorphism. It follows that the left vertical map is, as desired, an isomorphism

[2512] In other words, in the language from Definition 4.59 we demand that W is precompact.

(β) We claim that t is an isomorphism for every subset M of $\mathbb{H}^m$ that is the union of k open convex subsets of $\mathbb{H}^m$. We prove this by induction on the number k of open convex subsets:

(a) If $k = 1$, then M itself is an open convex subset of $\mathbb{H}^m$, hence the statement follows from (α).

(b) Suppose the statement holds whenever M is the union of $k-1$ open convex subsets with $k \geq 2$. Now let M be the union of k open convex subsets $U_1, \ldots, U_k$ of $\mathbb{H}^m$. We write $V = U_1 \cup \cdots \cup U_{k-1}$. By induction the statement holds for V and also for U_k. Furthermore since the intersection of two open convex subsets is again open and convex we see that $V \cap U_k = (U_1 \cap U_k) \cup \cdots \cup (U_{k-1} \cap U_k)$ is the union of $k-1$ open convex subsets, i.e. by induction the statement holds for $V \cap U_k$. But then it follows from (A) that the statement also holds for M.

(γ) Let M be an open subset of $\mathbb{H}^m$. By the Rational Open Balls Lemma 231.7 (2) we can write M as the union of countably many open convex subsets $B_i, i \in \mathbb{N}$ of $\mathbb{H}^m$ such that each closure $\overline{B}_i \subseteq \mathbb{H}^m$ is contained in M. We obtain from this information and the elementary argument of the Compact Exhaustion Lemma 13.6 a sequence U_i of open subsets of M such that the following three conditions are satisfied:

(a) For each $i \in \mathbb{N}$ the subset U_i is the union of finitely many open balls and the closure $\overline{U}_i$ of U_i in M is compact,

(b) For each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$.

(c) $M = \bigcup_{i \in \mathbb{N}} U_i$.

By (β) the statement holds for each U_i, but then it follows from (B) that the statement holds for M.

(δ) We claim that t is an isomorphism for every m-dimensional smooth manifold M that is the union of k small subsets.

(a) If $k = 1$, then it follows from the fact that $\mathcal{H}$ and $\mathcal{K}$ are diffeomorphism invariants and from (γ) that the statement holds for M.

(b) The intersection of two small subsets is clearly small again. With this simple observation the same induction argument as in (β) shows that the statement holds whenever M is covered by finitely many small subsets.

(ϵ) Finally let M be any m-dimensional smooth manifold. Note that it follows easily from the fact that the smooth manifold M is second-countable, the existence of charts around every point and the Small Basis Lemma 3.54 that there exist small sets $W_i, i \in \mathbb{N}$ which cover M and such that each closure $\overline{W}_i$ is compact. As in (γ) we obtain from the argument of the Compact Exhaustion Lemma 13.6 a sequence U_i of open subsets of M such that the following three conditions are satisfied:

(a) For each $i \in \mathbb{N}$ the subset U_i is the union of finitely many small subsets and the closure $\overline{U}_i$ of U_i is compact,

(b) For each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$.

(c) $M = \bigcup_{i \in \mathbb{N}} U_i$.

We had just seen in (δ) that the statement holds for all the U_i. But then it follows from (B) that the statement holds for M. ■

231.3. **Smooth singular cohomology.** One might think that it follows immediately from the Smooth-$\mathcal{H}^*$-Theories Proposition 231.3 and the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 (together with the calculations of the cohomology groups of a point in the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma 78.10 and the Singleton-H^*-Lemma 198.10) that the singular real valued cohomology groups and the de Rham cohomology groups of smooth manifolds are always isomorphic.

But recall that to apply the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 we need to find a natural transformation

$$t\colon \text{de Rham cohomology} \quad \to \quad \text{real valued singular cohomology}$$

that has the property that

$$\mathrm{H}^n_{\mathrm{dR}}(\star) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(\star;\mathbb{R}) \qquad \text{for all } n \in \mathbb{N}_0.$$

Here is an idea: In Section 74.2 we had seen that given a compact oriented n-dimensional smooth manifold W and a smooth differential n-form ω on W there exists an integral

$$\int_W \omega \ \in \ \mathbb{R}$$

that has many good properties. So given some smooth manifold M (not necessarily of dimension n) we could try to define a map

$$\begin{array}{rcl} t\colon \Omega^n(M) & \to & \mathrm{Hom}(\mathrm{C}_n(M),\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_n(M) & \to & \mathbb{R} \\ \sum\limits_{i=1}^k a_i \cdot (f_i\colon \Delta^n \to M) & \mapsto & \sum\limits_{i=1}^k a_i \cdot \int\limits_{\Delta^n} f_i^*(\omega) \end{array} \right). \end{array}$$

Unfortunately, there are several problems with this "definition":

(1) The first question that arises is the following: What does it mean to integrate a form over Δ^n? After all, it is not clear in what sense Δ^n is a smooth manifold. With a little bit of luck, that should not be too much of an issue, since Δ^n is clearly pretty close to being a smooth manifold.

(2) Let us suppose that we have resolved (1). Now let $f\colon \Delta^n \to M$ be a singular n-chain. If the map f is not smooth, then $f^*(\omega)$ just does not make any sense.[2513]

(3) After fixing the previous issues, why should t be a cochain map?

Arguably the most serious issue is (2). The only way around it is, to only work with smooth singular simplices. Let us recall the following definitions:

Definition 231.8. Let $n \in \mathbb{N}_0$.

(1) As in the Δ^n-Homeomorphism Lemma 143.9 (1) we consider the planar simplex

$$\measuredangle_n \ := \ \left\{ (t_1,\ldots,t_n) \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_{i=1}^n t_i \in [0,1] \right\}.$$

[2513] Recall that if $f\colon N \to M$ is a smooth map between smooth manifolds and if ω is a smooth differential n-form on M, then in Definition 74.2 we defined $f^*(\omega)$ to be the differential n-form on N that for $v_1,\ldots,v_n \in \mathrm{T}_P N$ has the value $(f^*(\omega))_P(v_1,\ldots,v_n) = \omega_{f(P)}(\mathrm{D}f_P(v_1),\ldots,\mathrm{D}f_P(v_n))$. But if f is not differentiable, then there is no differential $\mathrm{D}f_P\colon \mathrm{T}_P N \to \mathrm{T}_{f(P)}M$.

(2) In the Δ^n-Homeomorphism Lemma 143.9 (1) we saw that the map[2514]

$$\begin{aligned} \Theta_n \colon \triangle_n &\to \Delta^n \\ (t_1, \dots, t_n) &\mapsto \Big(t_1, \dots, t_n, 1 - \sum_{i=1}^{n} t_i\Big) \end{aligned}$$

is a homeomorphism.

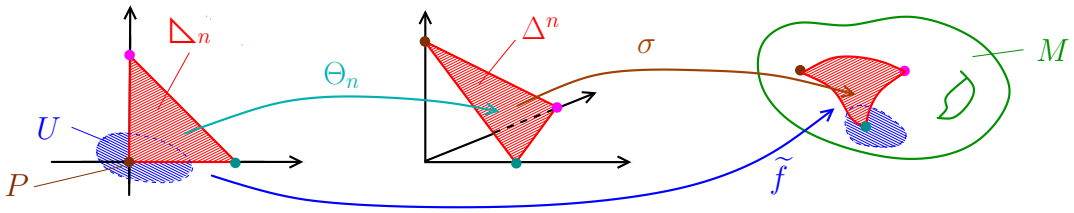


Next we turn to smooth manifolds. Continuing with the above definition we can now introduce the notion of a smooth singular simplex:

Definition 231.9. Let M be a smooth manifold.

(3) Recall that by the definition in Definition 25.20 a map $f \colon \triangle_n \to M$ is smooth if given any $P \in \triangle_n$ there exists an open neighborhood U in $\mathbb{R}^n$ of P and a smooth map $\widetilde{f} \colon U \to M$ which coincides with f on $\triangle_n \cap U$. We define the differential of f at $P \in \triangle_n$ to be the differential of $\widetilde{f}$.[2515]

(4) We say that a singular n-simplex $\sigma \colon \Delta^n \to M$ is smooth if the map $\sigma \circ \Theta_n \colon \triangle_n \to M$ is smooth.



Definition 231.10.

(5) We write

$$\mathrm{C}_n^{\mathrm{sm}}(M) \coloneqq \begin{array}{c}\text{the free abelian group generated by the} \\ \text{set of smooth singular } n\text{-simplices in } M.\end{array}$$

This is a subgroup of $\mathrm{C}_n(M)$ and it is clear that the boundary map

$$\begin{aligned} \partial_n \colon \mathrm{C}_n(M) &\to \mathrm{C}_{n-1}(M) \\ \sigma &\mapsto \sum_{j=0}^{n} (-1)^j \cdot (\sigma \circ \underset{\substack{\uparrow \\ \text{linear (thus smooth)} \\ \text{map } \Delta^{n-1} \to \Delta^n}}{i_j^n}). \end{aligned}$$

restricts to a map $\partial_n \colon \mathrm{C}_n^{\mathrm{sm}}(M) \to \mathrm{C}_{n-1}^{\mathrm{sm}}(M)$.

(6) Given an abelian group G we write

$$\mathrm{C}^n_{\mathrm{sm}}(M; G) \coloneqq \mathrm{Hom}\left(\mathrm{C}_n^{\mathrm{sm}}(M), G\right)$$

[2514] Put differently, Θ_n is the inverse of the obvious projection map $\Delta^n \to \triangle_n$.

[2515] It follows easily from the Regular Closed–∂_*-Lemma 24.5 that this definition does not depend on the choice of U and $\widetilde{f}$. This is the reason why we work with $\triangle_n$ instead of Δ^n.

and we define $\delta_n \colon \mathrm{C}^n_{\mathrm{sm}}(M;G) \to \mathrm{C}^{n+1}_{\mathrm{sm}}(M;G)$ as the map $\delta_n \coloneqq \partial^*_{n+1}$.

(7) We define the n-th smooth singular cohomology group of M with G-coefficients as

$$\mathrm{H}^n_{\mathrm{sm}}(M;G) \coloneqq \frac{\ker\left(\delta_n \colon \mathrm{C}^n_{\mathrm{sm}}(M;G) \to \mathrm{C}^{n+1}_{\mathrm{sm}}(M;G)\right)}{\mathrm{im}\left(\delta_{n-1} \colon \mathrm{C}^{n-1}_{\mathrm{sm}}(M;G) \to \mathrm{C}^n_{\mathrm{sm}}(M;G)\right)}.$$

These definitions lead us to the following unsurprising proposition:

Proposition 231.11. (Smooth Singular Cohomology Proposition) Smooth singular cohomology with real coefficients is a smooth cohomology theory.

Sketch of proof. We need to verify that smooth singular cohomology with real coefficients satisfies the properties of a smooth cohomology theory from Definition 231.2:

(0) First we introduce the induced maps: Thus let $f \colon M \to N$ be a *smooth* map between two smooth manifolds. Evidently the induced map $f_* \colon \mathrm{C}_n(M) \to \mathrm{C}_n(N)$ restricts to a chain map $f_* \colon \mathrm{C}^{\mathrm{sm}}_n(M) \to \mathrm{C}^{\mathrm{sm}}_n(N)$. In particular we get an induced chain cochain map $f^* \colon \mathrm{C}^n_{\mathrm{sm}}(N;\mathbb{R}) \to \mathrm{C}^n_{\mathrm{sm}}(M;\mathbb{R})$. Thus we obtain for each $n \in \mathbb{N}_0$ an induced map

$$f^* \colon \mathrm{H}^n_{\mathrm{sm}}(N;\mathbb{R}) \to \mathrm{H}^n_{\mathrm{sm}}(M;\mathbb{R}).$$

It is now pretty clear that smooth singular cohomology with real coefficients is a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}^{\mathbb{N}_0}_{\mathbb{K}}$ of $\mathbb{N}_0$-graded real vector spaces.

(1) It is clear that for all $n \in \mathbb{N}_0$ we have $\mathrm{H}^n_{\mathrm{sm}}(\varnothing;\mathbb{R}) = 0$.

(2) Let $f, g \colon M \to N$ be two smooth maps between two smooth manifolds. Furthermore let $F \colon M \times [0,1] \to N$ be a *smooth* homotopy between f and g. Using the Notation 144.16 we will introduce for each $n \in \mathbb{N}_0$ the singular chain

$$\Omega_n \coloneqq \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \ldots, v_j, w_j, \ldots, w_n]}_{\substack{\text{linear (in particular smooth)} \\ \text{map } \Delta^{n+1} \to \Delta^n \times [0,1]}} \in \mathrm{C}^{\mathrm{sm}}_{n+1}(\Delta^n \times [0,1]).$$

As on page 2776 we will introduce the homomorphisms

$$\begin{aligned} P_n \colon \mathrm{C}_n(M) &\to \mathrm{C}_{n+1}(N) \\ (\sigma \colon \Delta^n \to M) &\mapsto (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{aligned}$$

On page 2776 we saw that these maps form a chain homotopy $P_n \colon \mathrm{C}_n(M) \to \mathrm{C}_{n+1}(N)$. It follows immediately from the definitions that these maps restrict to a chain homotopy $P_n \colon \mathrm{C}^{\mathrm{sm}}_n(M) \to \mathrm{C}^{\mathrm{sm}}_{n+1}(N)$. Now we can once again appeal to the Dual Cochain Map Lemma 197.20 to conclude that the induced maps on smooth singular cohomology are the same.

(3) Let M be a smooth manifold and let U and V be open subsets with $M = U \cup V$. We gave two different proofs of the Mayer–Vietoris Theorem 153.2. We will explain how to modify the second proof: In Subsection 153.1.2 the key to proving the existence of the Mayer–Vietoris sequence for ordinary singular cohomology was to show that the inclusion maps

$$\underbrace{\{\alpha + \beta \,|\, \alpha \in \mathrm{C}_k(U), \beta \in \mathrm{C}_k(V)\}}_{=\mathrm{C}_k^{\{U,V\}}(M)} \to \mathrm{C}_k(M)$$

define a chain homotopy equivalence. This in turn we showed using the Comfortable Cover Proposition 148.9. As part of the proof of the Comfortable Cover Proposition 148.9 we introduced on page 2825 the subdivision maps

$$u_n \colon \mathrm{C}_n(M) \;\to\; \mathrm{C}_n(M), \quad n \in \mathbb{N}_0.$$

It follows immediately from the explicit definitions that these maps restrict to maps

$$u_n \colon \mathrm{C}_n^{\mathrm{sm}}(M) \;\to\; \mathrm{C}_n^{\mathrm{sm}}(M), \quad n \in \mathbb{N}_0.$$

Using this observation the proof of the existence of a natural connecting homomorphism in smooth singular cohomology that gives rise to a long exact sequence is otherwise identical to the proof for ordinary singular cohomology.

(4) The limit property is proved the same way as the Exhaustions-H*-Proposition 230.11 (1). We only have to replace the fact, proved in the Exhaustion-C$_*$-Lemma 154.1 that $\mathrm{C}_*(M) = \varinjlim \mathrm{C}_*(U_i)$ by the observation that the same argument as in the Exhaustion-C$_*$-Lemma 154.1 shows that $\mathrm{C}_*^{\mathrm{sm}}(M) = \varinjlim \mathrm{C}_*^{\mathrm{sm}}(U_i)$. ■

231.4. **Relating the three smooth cohomology theories.** In this section we will show that for any smooth manifold M there exist explicit natural isomorphisms

$$\mathrm{H}^n(M;\mathbb{R}) \xrightarrow[\text{Proposition 231.12}]{\cong} \mathrm{H}^n_{\mathrm{sm}}(M;\mathbb{R}) \xleftarrow[\text{Proposition 231.14}]{\cong} \mathrm{H}^n_{\mathrm{dR}}(M).$$

These two natural isomorphisms give us in particular a natural isomorphism between singular cohomology and de Rham cohomology. We will discuss this isomorphism in greater detail in the follow-up Section 231.5.

In the following proposition we provide the first of the two natural isomorphisms:

Proposition 231.12. (Smooth Singular-H*-Isomorphism Proposition) For any smooth manifold M and any $n \in \mathbb{N}_0$ we have a natural isomorphism

$$s \colon \mathrm{H}^n(M;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{sm}}(M;\mathbb{R}).$$

Proof. For any smooth manifold M we have the natural inclusion

$$\mathrm{C}_*^{\mathrm{sm}}(M) \;\to\; \mathrm{C}_*(M)$$

which evidently respects the boundary maps. These maps thus give rise to natural cochain maps

$$\mathrm{C}^*(M;\mathbb{R}) \;\to\; \mathrm{C}^*_{\mathrm{sm}}(M;\mathbb{R}).$$

Thus we obtain a natural homomorphism

$$s_M \colon \mathrm{H}^*(M;\mathbb{R}) \;\to\; \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R}).$$

Note that the proposition follows immediately from the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 and the following three observations:

(i) Since the natural transformation $s_M \colon \mathrm{H}^*(M;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R})$ is induced by the inclusion map it follows easily from the Connecting Homomorphism Lemma 145.9 (2) that the natural transformation s is compatible with the connecting homomorphism in the Mayer-Vietoris sequences.

(ii) It follows easily from the naturality of the Inverse Limit-H*-Proposition 230.9 that the natural transformation s is compatible with the maps τ appearing in the limit property.

(iii) Note that every map $\Delta^n \to \star$ is smooth. Thus there is no difference between ordinary singular cohomology and smooth singular cohomology for the smooth manifold $\star$. It follows that the map $s_\star \colon \mathrm{H}^*(\star;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{sm}}(\star;\mathbb{R})$ is an isomorphism. ■

Now we also want to relate smooth singular cohomology to de Rham cohomology. The discussion in Section 231.3 suggests that the key for doing so is the following definition:

Definition 231.13.

(1) Given $x \in \triangle_n$ we define $\widetilde{\mathrm{T}}_x \triangle_n := \mathbb{R}^n$. Using this definition it makes sense to talk of smooth n-forms on $\triangle_n$.

(2) Given a smooth n-form α on $\triangle_n$ we define[2516]

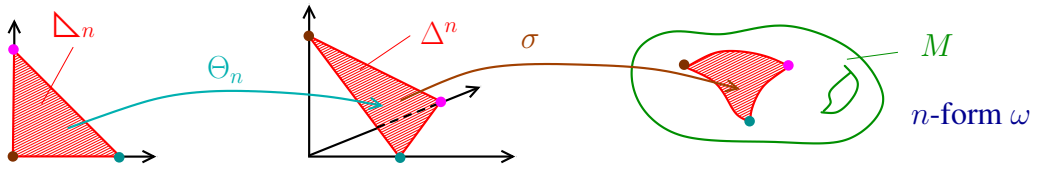
$$\int\limits_{\triangle_n} \alpha := \int\limits_{x \in \triangle_n} \alpha_x(e_1,\dots,e_n)\, d\lambda.$$

(3) Let M be a smooth manifold and let $f \colon \triangle_n \to M$ be a smooth map. Since $\triangle_n$ is regular closed, i.e. since it equals the closure of its interior, we obtain from the argument of the Regular Closed–∂_*-Lemma 24.5 that it makes sense to consider the map $\mathrm{D}f_x \colon \mathbb{R}^n \to \mathrm{T}_{f(x)}M$ for all $x \in \triangle_n$.

(4) Given a smooth manifold M, given a smooth differential n-form ω on M and given a smooth singular n-simplex $\sigma \colon \Delta^n \to M$ we use (1), (2) and (3) to define[2517]

$$\int\limits_\sigma \omega := (-1)^n \cdot \int\limits_{\triangle_n} (\sigma \circ \Theta_n)^*(\omega).$$

We extend this definition linearly to define the integral of the smooth differential n-form ω over a smooth singular n-chain $\sum\limits_{i=1}^k a_i \cdot \sigma_i \in \mathrm{C}^{\mathrm{sm}}_n(M)$.



Proposition 231.14. (De Rham–Smooth-H*-Isomorphism Theorem) For any smooth manifold M and any $k \in \mathbb{N}_0$ we have, by the above discussion, a homomorphism

$$\begin{array}{rcl} \Omega^k(M) & \to & \mathrm{C}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} \mathrm{C}^{\mathrm{sm}}_k(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int\limits_\sigma \omega \end{array} \right). \end{array}$$

This map descends to a natural isomorphism

$$t_M \colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}).$$

[2516] It follows from the Borel σ-Algebra Proposition 73.5 (1), the Lebesgue Integrability Proposition 73.21, the observation that $\triangle_n$ is a compact subset of $\mathbb{R}^n$ and the observation that the integrand is continuous that the integral of the map $x \mapsto \alpha_x(e_1,\dots,e_n)$ over $\triangle_n$ is defined.

[2517] Recall that in Definition 143.8 we equipped $\mathring{\Delta}^n$ with an orientation. The factor $(-1)^n$ is motivated by the fact that the restriction of the map $\Theta_n \colon \triangle_n \to \Delta^n$ to the interiors is orientation-preserving if and only if n is even.

The proof of the De Rham–Smooth-H*-Isomorphism Proposition 231.14 relies on the following proposition:

Proposition 231.15. (Stokes-for-Simplices Proposition) Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then[2518] [2519]

$$\int_\sigma d\omega \;=\; \int_{\partial_n(\sigma)} \omega.$$

Proof of Proposition 231.15. The proposition is evidently just a variation on Stokes' Theorem 75.11. In fact, as we will see, we can prove the proposition using a pretty elementary and direct calculation, which unfortunately turns out to be somewhat lengthy. To preserve the flow of the current discussion we outsource the proof of the proposition to Appendix 231.6. ■

Proof of Proposition 231.14. First note that it follows basically immediately from the definitions that the map

$$\begin{array}{rcl} \Omega^k(M) & \to & \mathrm{C}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ \omega & \mapsto & \begin{pmatrix} \mathrm{C}^{\mathrm{sm}}_k(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int_\sigma \omega \end{pmatrix} \end{array}$$

is natural. Next note that it is an immediate consequence of the Stokes-for-Simplices Proposition 231.15 that these maps together form a cochain map $\Omega^*(M) \to \mathrm{C}^*_{\mathrm{sm}}(M;\mathbb{R})$. Summarizing we see that we have constructed a natural map

$$t_M\colon \mathrm{H}^*_{\mathrm{dR}}(M) \;\to\; \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R}).$$

The proposition follows immediately from the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 231.6 and the following three observations:

(i) We need to prove the following claim:

Claim 1. This natural transformation $t_M\colon \mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R})$ is compatible with the connecting homomorphism in the Mayer-Vietoris sequences.

Proof. Let M be a smooth manifold and let U and V be open subsets of M such that $M = U \cup V$. We denote by $j_U\colon U \to M$, $j_V\colon V \to M$, $i_U\colon U \cap V \to U$ and $i_V\colon U \cap V \to V$ the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*(M) & \xrightarrow{(j_U^*,-j_V^*)} & \Omega^*(U)\oplus\Omega^*(V) & \xrightarrow{i_U^*\oplus i_V^*} & \Omega^*(U\cap V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t_M} & & & & & & \\ & & \mathrm{Hom}(\mathrm{C}^{\mathrm{sm}}_*(M),\mathbb{R}) & & \downarrow{\scriptstyle t_U\oplus t_V} & & \downarrow{\scriptstyle t_{U\cap V}} & & \\ & & \downarrow{\scriptstyle \rho} & & & & & & \\ 0 & \to & \mathrm{Hom}(\mathrm{C}^{\mathrm{sm}}_n(U)+\mathrm{C}^{\mathrm{sm}}_n(V),\mathbb{R}) & \xrightarrow{(j_U^*,-j_V^*)} & \mathrm{C}^*_{\mathrm{sm}}(U,\mathbb{R})\oplus\mathrm{C}^*_{\mathrm{sm}}(V,\mathbb{R}) & \xrightarrow{i_U^*\oplus i_V^*} & \mathrm{C}^*_{\mathrm{sm}}(U\cap V,\mathbb{R}) & \to & 0. \end{array}$$

[2518] Here $d\omega$ denotes the differential of the smooth n-form ω from Definition 75.5.

[2519] Note that on the left we integrate a smooth differential n-form over the smooth singular n-simplex σ and that on the right we integrate a smooth $(n-1)$-form over the smooth singular $(n-1)$-chain $\partial_n(\sigma) \in \mathrm{C}^{\mathrm{sm}}_{n-1}(M)$.

We make the following clarifications and observations:

- We denote by $\mathrm{C}_n^{\mathrm{sm}}(U) + \mathrm{C}_n^{\mathrm{sm}}(V) \subseteq \mathrm{C}_n^{\mathrm{sm}}(M)$ the vector subspace of $\mathrm{C}_n^{\mathrm{sm}}(M)$ that is spanned by $\mathrm{C}_n^{\mathrm{sm}}(U)$ and $\mathrm{C}_n^{\mathrm{sm}}(V)$.
- The vertical homomorphism ρ is defined as the restriction map that is induced by the inclusion $\mathrm{C}_n^{\mathrm{sm}}(U) + \mathrm{C}_n^{\mathrm{sm}}(V) \subseteq \mathrm{C}_n^{\mathrm{sm}}(M)$.
- It follows easily from the naturality of t and the definition of the restriction map that the diagram commutes.
- By the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem 80.2 we know that the top sequence is exact and that it gives rise to the connecting homomorphism and the long exact sequence in de Rham cohomology.
- As in the second proof of the Mayer–Vietoris-H^*-Theorem 200.11 (which builds on the Mayer–Vietoris Theorem 153.2 from Subsection 153.1.2) we see that the bottom horizontal sequence is exact and we see that, together with the natural isomorphism $\rho^*\colon \mathrm{H}^n(M;\mathbb{R}) \to \mathrm{H}^n(\mathrm{Hom}(\mathrm{C}_n^{\mathrm{sm}}(U) + \mathrm{C}_n^{\mathrm{sm}}(V), \mathbb{R}))$, it gives rise to the connecting homomorphism and the long exact sequence in smooth singular cohomology.
- It follows from the above and the Connecting Homomorphism Lemma 145.9 that t commutes with the connecting homomorphisms. ⊞

(ii) We need to prove the following claim:

Claim 2. This natural transformation $t_M\colon \mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R})$ is compatible with the map τ appearing in the limit property.

Proof. Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
(a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(b) for each $i \in \mathbb{N}$ we have $U_i \subseteq U_{i+1}$, and
(c) $M = \bigcup_{i\in\mathbb{N}} U_i$.

We consider the following maps

$$\begin{array}{ccccccc}
\Omega^k(U_1) & \longleftarrow & \Omega^k(U_2) & \longleftarrow & \Omega^k(U_3) & \longleftarrow & \dots \\
\downarrow{\scriptstyle t_{U_1}} & & \downarrow{\scriptstyle t_{U_2}} & & \downarrow{\scriptstyle t_{U_3}} & & \\
\mathrm{C}^k_{\mathrm{sm}}(U_1) & \longleftarrow & \mathrm{C}^k_{\mathrm{sm}}(U_2) & \longleftarrow & \mathrm{C}^k_{\mathrm{sm}}(U_3) & \longleftarrow & \dots
\end{array}$$

where the horizontal maps are induced by inclusions. Note that the diagram commutes by the naturality of t. The claim now follows from the Inverse Limit-H^*-Proposition 230.9. ⊞

(iii) Finally note that in the special case of $M = \star$ the map $t_\star\colon \mathrm{H}^*_{\mathrm{dR}}(\star) \to \mathrm{H}^*_{\mathrm{sm}}(\star;\mathbb{R})$ is easily seen to be an isomorphism. ■

231.5. **The isomorphism between singular and de Rham cohomology.** After the preparations in the previous sections we can now easily prove the theorem that we had been aiming for.

Theorem 231.16. (De Rham-Isomorphism Theorem) For any smooth manifold M and any $k \in \mathbb{N}_0$ we have a natural isomorphism[2520]

$$u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathrm{H}^k(M;\mathbb{R})$$

which has the property that for any closed differential k-form ω and any smooth singular k-chain $\sigma \in \mathrm{C}_k^{\mathrm{sm}}(M)$ that is a cycle we have

$$\big\langle \underbrace{u_M([\omega])}_{\in \mathrm{H}^k(M;\mathbb{R})}, \underbrace{[\sigma]}_{\in \mathrm{H}_k(M)} \big\rangle_M = \int_\sigma \omega.$$

Definition 231.17. Given a smooth manifold M we refer to the natural isomorphism

$$u_M \colon \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathrm{H}^k(M;\mathbb{R})$$

from the De Rham-Isomorphism Theorem 231.16 and its proof as the de Rham isomorphism.

Remark. A rather different proof for the De Rham-Isomorphism Theorem 231.16 is given in [**War1983**, p. 205] and [**Ram2005**, Chapte 4.6]. More precisely, in [**War1983, Ram2005**] the equivalence of de Rham cohomology and real valued singular cohomology is proved by relating both to another type of cohomology theory, namely sheaf cohomology. We also refer to [**BoT1982**, Theorems 8.9 and 15.8], [**Lee2002**, Theorem 18.14], [**Dup1978**, Theorem 1.15], [**Lau2011**, Section 6.7.6] and [**Ful1995**, Section 15C] for other proofs. □

Proof. We consider the natural isomorphisms

$$\begin{aligned} s \colon \mathrm{H}^k(M;\mathbb{R}) &\to \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ [\varphi \colon \mathrm{C}_k(M) \to \mathbb{R}] &\mapsto [\varphi|_{\mathrm{C}_k^{\mathrm{sm}}(M)} \colon \mathrm{C}_k^{\mathrm{sm}}(M) \to \mathbb{R}] \end{aligned} \quad \text{and} \quad \begin{aligned} t \colon \mathrm{H}^k_{\mathrm{dR}}(M) &\to \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ [\omega] &\mapsto \left[\begin{array}{rl} \mathrm{C}_k^{\mathrm{sm}}(M) &\to \mathbb{R} \\ \sigma &\mapsto \int_\sigma \omega \end{array} \right] \end{aligned}$$

from the Smooth Singular-H*-Isomorphism Proposition 231.12 and the De Rham–Smooth-H*-Isomorphism Proposition 231.14. It is clear that $u := s^{-1} \circ t \colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$ is a natural isomorphism. Finally note that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\langle u([\omega]), [\sigma] \rangle_M = \langle (s^{-1} \circ t)([\omega]), [\sigma] \rangle_M \underset{\substack{\uparrow \\ \text{since } \sigma \text{ is smooth}}}{=} \langle t([\omega]), [\sigma] \rangle_M \underset{\substack{\uparrow \\ \text{by definition of } t}}{=} \int_\sigma \omega.$$

■

The following proposition describes the natural isomorphism $u_M \colon \mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n(M;\mathbb{R})$ from the above de Rham-Isomorphism Theorem 231.16 in the top dimension:

Proposition 231.18. (Top–de Rham Isomorphism Proposition) Let M be a closed oriented connected non-empty n-dimensional smooth manifold. Furthermore let $\omega \in \Omega^n(M)$ be a smooth differential n-form.

(1) We have $\langle u_M([\omega]), [M] \rangle_M = \int_M \omega.$

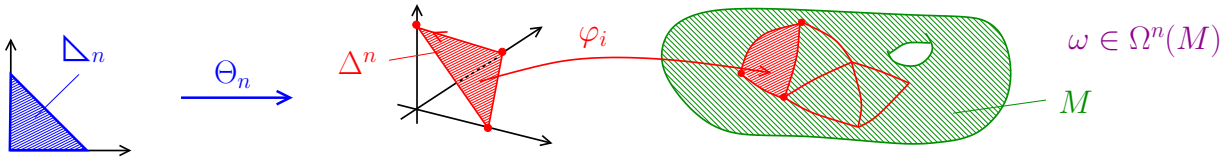
(2) (a) If $\int_M \omega = 1$, then $u_M([\omega]) = [M]^* \in \mathrm{H}^n(M;\mathbb{R})$ where $[M]^*$ denotes the dual fundamental class that we introduced in Definition 201.27.

(b) If $[M]^*_{\mathrm{dR}} \in \mathrm{H}^n_{\mathrm{dR}}(M)$ denotes the de Rham fundamental class that we introduced in Definition 84.8, then $u_M([M]^*_{\mathrm{dR}}) = [M]^*$.

Proof.

[2520] In the De Rham–Ring Isomorphism Proposition 232.1 we will see that this isomorphism is actually an isomorphism of rings.

(1) Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $\omega \in \Omega^n(M)$ be a smooth differential n-form. As we explained in the Fundamental Class Algorithm 185.5, there exist smooth singular simplices $\varphi_1, \ldots, \varphi_k \colon \Delta^n \to M$ with the following properties:
   (i) Each $\varphi_i \colon \Delta^n \to M$ is a smooth embedding in the sense of Definition 179.2.
   (ii) We have $\bigcup_{i=1}^{k} \varphi_i(\Delta^n) = M$.
   (iii) For any $i \neq j \in \{1, \ldots, k\}$ we have $\varphi_i(\Delta^n) \cap \varphi_j(\Delta^n) = \varphi_i(\partial\Delta^n) \cap \varphi_j(\partial\Delta^n)$.
   (iv) If for each $\in \{1, \ldots, k\}$ we set $\epsilon_i := +1$ if φ_i is orientation-preserving and $\epsilon_i := -1$ otherwise, then the singular chain $\sigma := \epsilon_1 \cdot \varphi_1 + \cdots + \epsilon_k \cdot \varphi_k$ is a singular cycle that represents the fundamental class $[M]$.



As above we denote by $\Theta_n \colon \Delta_n \to \Delta^n$ the inverse of the obvious projection map. We calculate that

$$\int_M \omega \;\overset{\uparrow}{=}\; \sum_{i=1}^{k} (-1)^n \cdot \epsilon_i \cdot \int_{\Delta_n} (\varphi_i \circ \Theta_n)^*(\omega) \;\overset{\uparrow}{=}\; \int_\sigma \omega \;\overset{\uparrow}{=}\; \langle u_M([\omega]), [M] \rangle_M .$$

follows from (i)–(iv), our orientation conventions the Lebesgue Integral Proposition 73.22 (2a), (3b), the Form Integration Proposition 74.15 (5) and the Measure-Zero Properties Proposition 37.3 (4)

see Definition 231.13

by the De Rham-Isomorphism Theorem 231.16

(2) (a) This statement follows immediately from (1) and the definition of the dual fundamental class $[M]^* \in \mathrm{H}^n(M; \mathbb{R})$ in Definition 201.27.
   (b) This statement follows immediately from (a) and the definition of the de Rham fundamental class $[M]^*_{\mathrm{dR}} \in \mathrm{H}^n_{\mathrm{dR}}(M)$ in Definition 84.8. ■

Next we turn to the ubiquitous notion of the degree of a map between smooth manifolds. More precisely, let $f \colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds. On numerous occasions we introduced the degree $\deg(f \colon M \to N)$ of f:

(1) In Definition 46.3 we defined the degree using regular values and local degrees.
(2) In Definition 86.2 we defined the degree using de Rham cohomology.
(3) In Definition 186.1 we defined the degree using the smooth fundamental class.
(4) In Definition 195.1 we defined the degree using the topological fundamental class.

We can now show that all the definitions agree:

Proposition 231.19. (Degrees-Agree Proposition) Let $f \colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds.

(1) The definitions of the degree given on Definition 46.3, Definition 86.2, Definition 186.1 and Definition 195.1 agree.
(2) If ω is a smooth differential n-form on N with $\int_N \omega = 1$, then we have the following equality:

$$\int_M f^*(\omega) \;=\; \deg(f \colon M \to N).$$

Proof. Let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds.

(1) It follows from the Local-Global Degree Proposition 195.6 together with the Local Degree Lemma 195.5 (3) that the definitions on pages Definition 46.3 and Definition 195.1 agree. Furthermore it follows from the Smooth-Topological Fundamental Class Lemma 194.5 that the definitions on Definition 186.1 and Definition 195.1 agree. Finally note that it follows immediately from the Dual Fundamental Class–Degree Lemma 201.28, the Top–de Rham Isomorphism Proposition 231.18 and the naturality of u that the definitions on Definition 86.2 and Definition 195.1 agree.

(2) Let ω be a smooth differential n-form on N with $\int_N \omega = 1$. We have the following equalities:[2521]

$$\begin{array}{lllll}
\int\limits_M f^*(\omega) & \overset{\text{by the Top–de Rham Isomorphism Isomorphism Proposition 231.18}}{\downarrow} & & & \\
& = \langle u_M(f^*([\omega])), [M]\rangle_M & \overset{\text{since } u \text{ is a } \textit{natural} \text{ isomorphism}}{\downarrow} & & \\
& & = \langle f^*(u_N([\omega])), [M]\rangle_M & \overset{\text{by the Kronecker Pairing Natural Lemma 199.9 (1)}}{\downarrow} & \\
& & & = \langle u_N([\omega]), f_*([M])\rangle_N & \\
& = \deg(f) \cdot \langle u_N([\omega]), [N]\rangle_N & = \deg(f) \cdot \int\limits_N \omega & = \deg(f). & \\
& \overset{\text{by Definition 195.1 of } \deg(f)}{\uparrow} & \overset{\text{by the Top–de Rham Isomorphism Proposition 231.18}}{\uparrow} & \overset{\text{by choice of } \omega}{\uparrow} &
\end{array}$$

■

The de Rham Isomorphism-Theorem 231.16 allows us to transport information from de Rham cohomology to real-valued singular cohomology and vice-versa. For example the many computational techniques that we had developed, e.g. cellular cohomology, and the results, e.g. the Manifold-Top-H*-Theorem 201.26, for singular cohomology can now be used to calculate isomorphism types of de Rham cohomology groups. In particular we can easily give a new proof of the following proposition:

Proposition 84.7 (Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition) If M is a closed orientable connected non-empty n-dimensional smooth manifold, then the homomorphism

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\eta] & \mapsto & \int\limits_M \eta \end{array}$$

from the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 is an isomorphism.

Proof. By the Top Dimension-$\mathrm{H}^n_{\mathrm{dR}}$-Integration Proposition 78.11 the map is well-defined and it is an epimorphism. On the other hand, by the Manifold-Top-H*-Theorem 201.26 (2a) and the De Rham-Isomorphism Theorem 231.16 we know that $\mathrm{H}^n_{\mathrm{dR}}(M) \cong \mathbb{R}$, which implies that the map is an isomorphism. ■

We can now also translate many earlier calculations of singular (co-) homology groups to de Rham cohomology. In particular we can now easily calculate the de Rham cohomology groups of complex projective spaces:

Corollary 231.20. ($\mathbb{C}\mathrm{P}^n$-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary) For any $n \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k \in \{0, 2, 4, \dots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

[2521] In the Degree-Integration Lemma 86.3 we gave a slightly different proof of this statement.

Proof. Given $n \in \mathbb{N}_0$ and given $k \in \mathbb{N}_0$ we have the following isomorphisms:

$$\begin{array}{ccccccc} \mathrm{H}^k_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n) & \cong & \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{R}) & \cong & \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \otimes \mathbb{R} & \cong & \begin{cases} \mathbb{R}, & \text{if } k \in \{0,2,4,\dots,2n\}, \\ 0, & \text{otherwise.} \end{cases} \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{De Rham-Isomorphism} & & \text{H}^*\text{–Change-of-Coefficients} & & \text{see page 3832} & & \\ \text{Theorem 231.16} & & \text{Theorem 202.10} & & & & \end{cases}$$

■

231.6. **Appendix: Proof of the Stokes-for-Simplices Proposition 231.15.** In this appendix we will finally provide the proof of the following technical but essential proposition:

Proposition 231.15. (Stokes-for-Simplices Proposition) Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then

$$\int\limits_{\sigma} d\omega \;=\; \int\limits_{\partial_n(\sigma)} \omega.$$

All the hard work for the proof of the Stokes-for-Simplices Proposition 231.15 is done in the following lemma:

Lemma 231.21. (Stokes-on-$\triangle_n$-Lemma) For $i = 1,\dots,n$ we consider the map

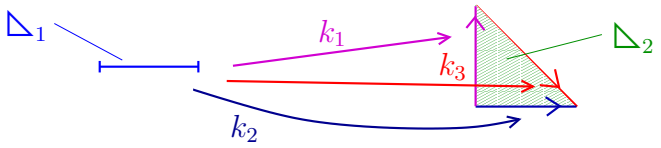
$$\begin{array}{rcl} k_i\colon \triangle_{n-1} & \to & \triangle_n \\ (x_1,\dots,x_{n-1}) & \to & (x_1,\dots,x_{i-1},0,x_i,\dots,x_{n-1}). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} k_{n+1}\colon \triangle_{n-1} & \to & \triangle_n \\ (x_1,\dots,x_{n-1}) & \to & (x_1,\dots,x_{n-1},1-x_1-\cdots-x_{n-1}). \end{array}$$

Given any smooth $(n-1)$-form α on $\triangle_n$ we have

$$\int\limits_{\triangle_n} d\alpha \;=\; \sum_{i=1}^{n} (-1)^i \cdot \int\limits_{\triangle_{n-1}} k_i^*\alpha + (-1)^{n+1} \cdot \int\limits_{\triangle_{n-1}} k_{n+1}^*\alpha.$$



Proof. Let α be a smooth $(n-1)$-form on $\triangle_n$ By the Alternating Forms-Basis Proposition 72.7 we can write

$$\alpha \;=\; \sum_{j=1}^{n} f_j \cdot dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

where $f_1,\dots,f_n\colon \triangle_n \to \mathbb{R}$ are smooth functions. First let us calculate the differential of the n-form $d\alpha$.

Claim 1. For each $x \in \triangle_n$ we have

$$(d\alpha)_x(e_1,\dots,e_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

Proof. We performed the calculation already in the proof of Stokes' Theorem 75.11. But there is no harm done in doing so again. So let us do so:

$$d\alpha \overset{\text{see Definition 75.1}}{\downarrow} = \sum_{j=1}^{n} df_j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \overset{\text{by the 1-Form-via-}dx^i\text{-Lemma 71.5}}{\downarrow} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} \cdot dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$
$$= \sum_{j=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_j}{\partial x_j} \cdot dx^1 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^n$$
$$\uparrow$$
follows from $dx^i \wedge dx^i = 0$ and from $dx^i \wedge dx^j = -dx^j \wedge dx^i$

Thus for each $x \in \triangle_n$ we have

$$(d\alpha)_x(e_1, \ldots, e_n) = \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x) \cdot \underbrace{(dx^1 \wedge \cdots \wedge dx^n)(e_1, \ldots, e_n)}_{=1} = \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

⊞

We adopt the following notation: Given $i \in \{1, \ldots, n\}$ and given $x \in \mathbb{R}^n$ we denote by $\varphi_i(x) \in \mathbb{R}^{i-1}$ the first $i-1$ coordinates and we denote by $\psi_i(x) \in \mathbb{R}^{n-i}$ the last $n-i$ coordinates. Furthermore we denote by $|x|$ the sum of the coordinates of x. Now we see that

$$\int_{\triangle_n} d\alpha = \int_{x \in \triangle_n} (d\alpha)_x(e_1, \ldots, e_n) \overset{\text{by Claim 1}}{\downarrow} = \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{x \in \triangle_n} \frac{\partial f_i}{\partial x_i}(x)\, d\lambda \overset{\text{Fubini Theorem 73.25}}{\downarrow} =$$
$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} \underbrace{\int_{x_i=0}^{x_i=1-|s|} \frac{\partial f_i}{\partial x_i}(\varphi_i(s), x_i, \psi_i(s))\, dx_i}_{\substack{\text{can be calculated using the} \\ \text{Fundamental Theorem of Calculus 73.27}}} d\lambda$$
$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} (f_i(\varphi_i(s), 1-|s|, \psi_i(s)) - f_i(\varphi_i(s), 0, \psi_i(s))\, d\lambda$$
$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} \underbrace{f_i(\varphi_i(s), 0, \psi_i(s))}_{=k_i(s)}\, d\lambda + \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} (f_i(\varphi_i(s), 1-|s|, \psi_i(s))\, d\lambda.$$

It remains to recognize the terms. We will do so in the following two claims.

Claim 2. Given any $i \in \{1, \ldots, n\}$ we have

$$\int_{s \in \triangle_{n-1}} f_i(k_i(s))\, d\lambda = \int_{\triangle_{n-1}} k_i^* \alpha.$$

Proof. We calculate that

$$\int_{\triangle_{n-1}} k_i^* \alpha = \int_{s \in \triangle_{n-1}} (k_i^* \alpha)_s(e_1, \ldots, e_{n-1})\, d\lambda \overset{\text{definition of } k_i^*}{\downarrow} = \int_{s \in \triangle_{n-1}} \alpha_{k_i(s)}(k_{i*}(e_1), \ldots, k_{i*}(e_{n-1}))\, d\lambda \overset{\text{definition of } \alpha}{\downarrow} =$$
$$= \int_{s \in \triangle_{n-1}} \sum_{j=1}^{n} f_j(k_i(s)) \cdot \underbrace{(dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n)(e_1, \ldots, \widehat{e_i}, \ldots, e_{n-1})}_{= \delta_{ij}\text{, where } \delta_{ij} \text{ denotes the Kronecker delta}} d\lambda$$
$$= \int_{s \in \triangle_{n-1}} f_i(k_i(s))\, d\lambda.$$

⊞

Claim 3. $$\sum_{i=1}^{n} \int\limits_{s\in\triangle_{n-1}} (-1)^{i-1}\cdot f_i(\varphi_i(s), 1-|s|, \psi_i(s))\,d\lambda \;=\; (-1)^{n+1}\cdot \int\limits_{\triangle_{n-1}} k_{n+1}^*\alpha.$$

Proof. We calculate that

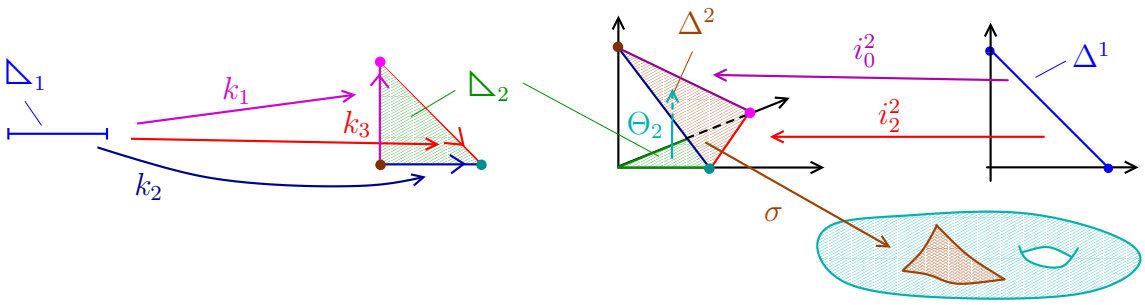
$$\begin{aligned}
\int\limits_{\triangle_{n-1}} k_{n+1}^*\alpha &= \int\limits_{s\in\triangle_{n-1}} (k_{n+1}^*\alpha)_s(e_1,\dots,e_{n-1})\,d\lambda \overset{\text{definition of } k_i^*}{=} \int\limits_{s\in\triangle_{n-1}} \alpha_{k_{n+1}(s)}(k_{n+1,*}(e_1),\dots,k_{n+1,*}(e_{n-1}))\,d\lambda \overset{\text{definition of } \alpha}{=} \\
&= \int\limits_{s\in\triangle_{n-1}} \sum_{j=1}^{n} f_j(k_{n+1}(s))\cdot \underbrace{(dx^1\wedge\dots\wedge\widehat{dx^j}\wedge\dots\wedge dx^n)(e_1-e_n,\dots,e_{n-1}-e_n)}_{=(-1)^{n-j}}\,d\lambda \\
&= \sum_{j=1}^{n} \int\limits_{s\in\triangle_{n-1}} (-1)^{n-j}\cdot f_j(k_{n+1}(s))\,d\lambda \;=\; \sum_{j=1}^{n} \int\limits_{s\in\triangle_{n-1}} (-1)^{n-j}\cdot f_j\underbrace{(\varphi_n(s), 1-|s|, \psi_n(s))}_{=k_{n+1}(s)}\,d\lambda \\
&= \sum_{j=1}^{n} \int\limits_{s\in\triangle_{n-1}} (-1)^{n-j}\cdot f_j(\varphi_j(s), 1-|s|, \psi_j(s))\,d\lambda.
\end{aligned}$$

↑ follows from the Lebesgue Transformation Theorem 73.26 applied to the diffeomorphism that swaps the j-th and the n-th coordinate

This calculation of course implies the claim. ∎

Proof of the Stokes-for-Simplices Proposition 231.15. Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. For $j\in\{0,\dots,n\}$ we consider as usual the map

$$\begin{aligned} i_j^n\colon \Delta^{n-1} &\to \Delta^n \\ (t_0,\dots,t_{n-1}) &\mapsto (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}). \end{aligned}$$



The proposition follows from the following calculation:

$$\begin{aligned}
\int_\sigma d\omega &\overset{\text{by the definition in Definition 231.13}}{=} (-1)^n \cdot \int_{\triangle_n} (\sigma\circ\Theta_n)^*(d\omega) \overset{\text{by the Differential-Natural Proposition 75.8}}{=} (-1)^n \cdot \int_{\triangle_n} d((\sigma\circ\Theta_n)^*(\omega)) \overset{\text{by the Stokes-on-}\triangle_n\text{-Lemma 231.21 applied to } \alpha=(\sigma\circ\Theta_n)^*(d\omega)}{=} \\
&= \sum_{j=1}^{n} (-1)^{n+i} \cdot \int_{\triangle_{n-1}} \underbrace{k_j^*((\sigma\circ\Theta_n)^*(\omega))}_{(\sigma\circ\Theta_n\circ k_j)^*(\omega)} + (-1)^{2n+1} \cdot \int_{\triangle_{n-1}} \underbrace{k_{n+1}^*((\sigma\circ\Theta_n)^*(\omega))}_{(\sigma\circ\Theta_n\circ k_j)^*(\omega)} \\
&= (-1)^{n-1} \cdot \sum_{l=0}^{n} (-1)^l \cdot \int_{\triangle_{n-1}} (\sigma\circ i_l^n\circ\Theta_{n-1})^*(\omega) \overset{\text{by Definition 231.13 of } \partial_n(\sigma)}{=} (-1)^{n-1} \cdot \sum_{l=0}^{n} (-1)^l \cdot \int_{\sigma\circ i_l^n} \omega \overset{\text{by definition}}{=} \int_{\partial\sigma} \omega.
\end{aligned}$$

since $\Theta_n\circ k_j = i_{j+1}^{n-1}\circ\Theta_{n-1}\colon \triangle_{n-1}\to\Delta^n$

■

231.7. Exercises for Chapter 231 .

Exercise 231.1. Does the map $M \mapsto \operatorname{Hom}(\mathrm{H}_k(M),\mathbb{R})$ give rise to a smooth cohomology theory?

Exercise 231.2. We consider the contravariant functors

$$\begin{aligned} \mathcal{H}^k\colon M &\mapsto \mathbb{R} \\ (f\colon M\to N) &\mapsto (\mathrm{id}\colon \mathbb{R}\to\mathbb{R}) \end{aligned}$$

Show that this family of functors gives rise to a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{K}}^{\mathbb{N}_0}$ of $\mathbb{N}_0$-graded $\mathbb{R}$-vector spaces that satisfies the properties (2), (3) and (4) of a smooth cohomology theory, as introduced in Definition 231.2.
Remark. Evidently this functor does not satisfy (1).

Exercise 231.3. Let $\mathcal{H}$ be a smooth cohomology theory such that $\mathcal{H}^*(\star) = 0$ for all but one k. Determine $\mathcal{H}^*(S^n)$.

Exercise 231.4. Let $\mathcal{H}^*$ be a smooth cohomology theory with $\mathcal{H}^n(\star) = 0$ for $n \geq 1$. We denote by $f\colon S^1\to S^1$ the map that is given by $(x,y)\mapsto(x,-y)$. Show, only using the axioms of a smooth cohomology theory, that $f^*\colon \mathcal{H}^1(S^1)\to\mathcal{H}^1(S^1)$ is given by multiplication by -1.

Exercise 231.5. Let M be a smooth manifold and let A be a union of components of ∂M. We define

$$\mathrm{C}^k_{\mathrm{dr}}(M,A) \coloneqq \{\varphi\in\Omega^k(M)\,|\,\text{there exists a neighborhood } U \text{ of } A \text{ such that } \varphi|_U\equiv 0\}.$$

(a) Show that the usual differential $d\colon \mathrm{C}^k_{\mathrm{dR}}(M)\to\mathrm{C}^{k+1}_{\mathrm{dR}}(M)$ restricts to a coboundary map $d\colon \mathrm{C}^k_{\mathrm{dR}}(M,A)\to\mathrm{C}^{k+1}_{\mathrm{dR}}(M,A)$. We refer to resulting cohomology groups by $\mathrm{H}^k_{\mathrm{dR}}(M,A)$.
(b) Show that there exists a natural connecting homomorphism $\delta\colon \mathrm{H}^k_{\mathrm{dR}}(A)\to\mathrm{H}^{k+1}_{\mathrm{dR}}(X,A)$ which gives rise to a long exact sequence

$$\ldots\longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M,A)\longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M)\longrightarrow \mathrm{H}^k_{\mathrm{dR}}(A)\xrightarrow{\delta} \mathrm{H}^{k+1}_{\mathrm{dR}}(M,A)\longrightarrow\ldots$$

(c) Show that there exists a natural isomorphism $u\colon \mathrm{H}^k_{\mathrm{dR}}(M, A) \to \mathrm{H}^k(M, A; \mathbb{R})$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(M, A) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(M) & \longrightarrow & \mathrm{H}^k_{\mathrm{dR}}(A) & \xrightarrow{\delta} & \mathrm{H}^{k+1}_{\mathrm{dR}}(M, A) \longrightarrow \ldots \\
 & & \cong\downarrow u & & \cong\downarrow u & & \cong\downarrow u & & \cong\downarrow u \\
\ldots & \longrightarrow & \mathrm{H}^k(M, A; \mathbb{R}) & \longrightarrow & \mathrm{H}^k(M; \mathbb{R}) & \longrightarrow & \mathrm{H}^k(A; \mathbb{R}) & \xrightarrow{\delta} & \mathrm{H}^{k+1}(M, A; \mathbb{R}) \longrightarrow \ldots
\end{array}$$

Exercise 231.6. Let $\mathcal{H}^*$ be a smooth cohomology theory.

(a) Show that for two n-dimensional smooth manifolds we have for every $k \in \mathbb{N}_0$ an isomorphism $\mathcal{H}^k(M \sqcup N) \to \mathcal{H}^k(M) \oplus \mathcal{H}^k(N)$.

(b) Let M be a smooth manifold and let $N \subseteq M$ be a smooth submanifold. Show that if there exists a smooth deformation retraction $R\colon M \times [0,1] \to M$ from M to N, then the inclusion-induced map $\mathcal{H}^i(M) \to \mathcal{H}^i(N)$ is an isomorphism.

Exercise 231.7. Let $m, n \in \mathbb{N}_0$. Compute the de Rham cohomology groups of $\mathbb{R}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$.

Project 231.1. Show that de Rham cohomology with compact support is naturally isomorphic to singular cohomology with compact support of smooth manifolds.

CHAPTER 232

The wedge product and the cup product

In the previous chapter we saw in the De Rham Isomorphism-Theorem 231.16 that for a smooth manifold M the de Rham cohomology vector spaces $\mathrm{H}^k_{\mathrm{dR}}(M)$ are naturally isomorphic to the singular cohomology vector spaces $\mathrm{H}^k(M;\mathbb{R})$. De Rham cohomology and singular cohomology each carry an extra structure which turns de Rham cohomology and singular cohomology into rings. In this fairly short chapter we will see that the above de Rham isomorphism $\mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*(M;\mathbb{R})$ is actually a ring isomorphism.

This chapter is organized as follows:

- In Section 232.1 we will discuss the wedge product of differential forms and we will state the De Rham–Ring Isomorphism Proposition 232.1 which says that the de Rham cohomology ring is isomorphic to the singular cohomology ring.
- In Section 232.2 we will prove the De Rham–Ring Isomorphism Proposition 232.1.
- In Section 232.3 we will provide exercises for Chapter 232.

In the next Chapter 233 we will see that for a smooth manifold cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$ correspond to smooth maps $M \to S^1$.

232.1. **The wedge product of differential forms.** Recall that given a smooth manifold M we introduced in the Wedge-on-de Rham Cohomology Proposition 78.12 the wedge product

$$\begin{array}{rcl} \wedge\colon \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^l_{\mathrm{dR}}(M) & \to & \mathrm{H}^{k+l}_{\mathrm{dR}}(M) \\ ([\omega],[\sigma]) & \mapsto & [\omega]\wedge[\sigma] \coloneqq [\omega\wedge\sigma]. \end{array}$$

Furthermore recall that given any topological space X and given any commutative ring R we introduced in Definition 203.8 the cup product

$$\begin{array}{rcl} \cup\colon \mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) & \to & \mathrm{H}^{k+l}(X;R) \\ ([\varphi],[\psi]) & \mapsto & [\varphi]\cup[\psi] \coloneqq [\varphi\cup\psi]. \end{array}$$

The following proposition shows that the natural isomorphism $u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$ from the De Rham Isomorphism-Theorem 231.16 preserves these structures:

Proposition 232.1. (De Rham–Ring Isomorphism Proposition) Let M be a smooth manifold and let

$$u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$$

be the de Rham isomorphism from the De Rham Isomorphism-Theorem 231.16. Given any $\varphi \in \mathrm{H}^k_{\mathrm{dR}}(M)$ and given any $\psi \in \mathrm{H}^l_{\mathrm{dR}}(M)$ we have

$$u_M(\varphi\wedge\psi) = u_M(\varphi)\cup u_M(\psi) \in \mathrm{H}^{k+l}(M;\mathbb{R}).$$

Remark. Two alternative proofs of the proposition are given in [**War1983**, Theorem 5.45] (using "sheaf cohomology") and in [**Dup1978**, Theorem 2.4] (using the "simplicial de Rham complex"). □

Proof. We postpone the proof of the De Rham–Ring Isomorphism Proposition 232.1 to Section 232.2. ∎

Example. As usual we denote by dt on $\mathbb{R}$ the 1-form that for every $v \in \mathrm{T}_P\mathbb{R} = \mathbb{R}$ is given by $dt(v) = v$. We denote by $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} =_{\mathrm{i}} S^1$ the natural projection. By the G-Invariant Form Lemma 74.22 together with the Pullback-Closed-Exact Lemma 75.10 (3) the closed 1-form dt descends to a closed 1-form on S^1, i.e. there exists a unique 1-form $\overline{dt}$ on $\mathbb{R}/\mathbb{Z} =_{\mathrm{i}} S^1$ whose pullback equals dt. Let $\sigma\colon \Delta^1 \to \mathbb{R}/\mathbb{Z} =_{\mathrm{i}} S^1$ be a smooth singular 1-simplex. Since Δ^1 is simply connected we can lift this singular 1-simplex, i.e. there exists a continuous map (which is in fact smooth) $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ with $\pi \circ \widetilde{\sigma} = \sigma$. We have

$$\int_\sigma \overline{dt} \;\underset{\uparrow}{=}\; -\int_{s=0}^{s=1} \overline{dt}(\sigma'(t,1-t))ds \;\underset{\uparrow}{=}\; -\int_{s=0}^{s=1} \widetilde{\sigma}'(t,1-t)ds \;\underset{\uparrow}{=}\; -\widetilde{\sigma}(1,0)+\widetilde{\sigma}(0,1) \;\underset{\uparrow}{=}\; \theta_{\mathbb{R}}(\sigma).$$

see Definition 231.13 — definition of $\overline{dt}$ — by the Fundamental Theorem of Calculus 73.27 — see page 3813 for for the definition of the 1-cochain $\theta_{\mathbb{R}}$

This little calculation shows that under the natural isomorphisms from the De Rham-Isomorphism Theorem 231.16 the class $[\overline{dt}] \in \mathrm{H}^1_{\mathrm{dR}}(M)$ gets sent to our favorite generator $[\theta_{\mathbb{R}}] \in \mathrm{H}^1(M;\mathbb{R})$, i.e. we have $u([\overline{dt}]) = [\theta_{\mathbb{R}}]$.

Next we consider the torus $M = S^1 \times S^1$. As before we denote by $p\colon S^1 \times S^1 \to S^1$ respectively $q\colon S^1 \times S^1 \to S^1$ the projection onto the first, respectively the second factor. On page 1541 we showed that, with the usual product orientation on the torus $M = S^1 \times S^1$ we have

$$\int_{S^1\times S^1} p^*(\overline{dt}) \wedge q^*(\overline{dt}) \;=\; +1.$$

On the other hand, by the naturality of u we have $u(p^*([\overline{dt}])) = p^*(u([\overline{dt}])) = p^*([\theta_{\mathbb{R}}])$ and similarly we also have $u(q^*([\overline{dt}]) = q^*([\theta_{\mathbb{R}}])$. The calculation of the above integral is now consistent with the calculation in the Torus–Cup Product Lemma 203.28 where we had, implicitly, shown that $\langle p^*([\theta_{\mathbb{R}}]) \cup q^*([\theta_{\mathbb{R}}]), [S^1 \times S^1]\rangle = +1$. □

We can now use the De Rham–Ring Isomorphism Proposition 232.1 together with results on singular cohomology groups to give a new proof of the following result:

Theorem 84.5 (De Rham Poincaré Duality Theorem For any closed oriented n-dimensional smooth manifold M the following $\mathbb{R}$-bilinear pairing is non-singular:

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \displaystyle\int_M \omega \wedge \eta. \end{array}$$

Proof. The De Rham–Ring Isomorphism Proposition 232.1 and the Top–de Rham Isomorphism Proposition 231.18 give us the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \xrightarrow{(\varphi,\psi)\mapsto \int_M \varphi\wedge\psi} & \mathbb{R} \\ \cong\downarrow u \quad u\downarrow\cong & & \downarrow \mathrm{id} \\ \mathrm{H}^k(M;\mathbb{R}) \times \mathrm{H}^{n-k}(M;\mathbb{R}) & \xrightarrow{(\varphi,\psi)\mapsto \langle \varphi\cup\psi,[M]\rangle} & \mathbb{R}. \end{array}$$

Since M is closed we obtain from the Manifold-Non Singularity Proposition 213.3 (2) that the $\mathbb{R}$-bilinear pairing on the bottom is non-singular. Thus the $\mathbb{R}$-bilinear pairing on the top is also non-singular. ■

The name "De Rham–Ring Isomorphism Proposition 232.1" hints at the fact, that the statement of the proposition should be phrased in terms of rings. To make this precise it is helpful to recall Definition 203.22:

Definition 232.2.

(1) A graded ring is a ring R together with a direct sum decomposition[2522] $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ into subgroups such that for every $m, n \in \mathbb{N}_0$ the multiplication map restricts to a bilinear map
$$R_m \times R_n \to R_{m+n}.$$
(2) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be two graded rings. A morphism between the two graded rings is a ring homomorphism $\varphi\colon R \to S$ such that for every $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subseteq S_n$.
(3) We define the category $\mathcal{GrRing}$ of graded rings to be the category where the objects are graded rings and the morphisms are morphisms of graded rings.

Example. De Rham cohomology and singular cohomology with real coefficients both give rise to graded rings:

(1) Let M be a smooth manifold. We write $\mathrm{H}^*_{\mathrm{dR}}(M) \coloneqq \bigoplus_{i=0}^{\infty} \mathrm{H}^i_{\mathrm{dR}}(M)$. We consider the map
$$\begin{aligned} \wedge\colon \mathrm{H}^*_{\mathrm{dR}}(M) \times \mathrm{H}^*_{\mathrm{dR}}(M) &\to \mathrm{H}^*_{\mathrm{dR}}(M) \\ \Big(\sum_{i=0}^{\infty} x_i, \sum_{i=0}^{\infty} y_i\Big) &\mapsto \sum_{i,j=0}^{\infty} \varphi_i \wedge \psi_j \end{aligned}$$
where it is understood that each x_i and y_i is homogeneous of degree i. In the $\mathrm{H}^*_{\mathrm{dR}}$-Ring Functor Proposition 78.14 we showed that the maps
$$\begin{aligned} M &\mapsto (\mathrm{H}^k_{\mathrm{dR}}(M), \cup) \\ (f\colon M \to N) &\mapsto (f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)) \end{aligned}$$
define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category of graded rings $\mathcal{GrRing}$.

(2) Let X be a topological space. We write $\mathrm{H}^*(X;\mathbb{R}) \coloneqq \bigoplus_{i=0}^{\infty} \mathrm{H}^i(X;\mathbb{R})$. We consider the map
$$\begin{aligned} \cup\colon \mathrm{H}^*(X;\mathbb{R}) \times \mathrm{H}^*(X;\mathbb{R}) &\to \mathrm{H}^*(X;\mathbb{R}) \\ \Big(\sum_{i=0}^{\infty} \varphi_i, \sum_{i=0}^{\infty} \psi_i\Big) &\mapsto \sum_{i,j=0}^{\infty} x_i \cup y_j. \end{aligned}$$
where again it is understood that each φ_i and ψ_i is homogeneous of degree i. In the Cup Product-Ring Proposition 203.25 (3) we showed that the maps
$$\begin{aligned} X &\mapsto (\mathrm{H}^k(X;\mathbb{R}), \cup) \\ (f\colon X \to Y) &\mapsto (f^*\colon \mathrm{H}^*(Y;\mathbb{R}) \to \mathrm{H}^*(X;\mathbb{R})) \end{aligned}$$
define a contravariant functor from the category $\mathcal{Top}$ of topological spaces to the category of graded rings $\mathcal{GrRing}$. □

With this language we can now formulate the following theorem:

[2522] In particular, R viewed as an abelian group is $\mathbb{N}_0$-graded.

Theorem 232.3. The natural isomorphisms u from the De Rham-Isomorphism Theorem 231.16 are in fact ring homomorphisms, i.e. they define a natural isomorphism from the functor

$$\begin{array}{rcl} \mathcal{SmMfd} & \to & \mathcal{GrRing} \\ M & \mapsto & (\mathrm{H}^*_{\mathrm{dR}}(M), \wedge) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{SmMfd} & \to & \mathcal{GrRing} \\ M & \mapsto & (\mathrm{H}^*(M;\mathbb{R}), \cup). \end{array}$$

Proof. The theorem is an immediate consequence of the De Rham-Isomorphism Theorem 231.16 together with the De Rham–Ring Isomorphism Proposition 232.1. ■

232.2. **Proof of the De Rham–Ring Isomorphism Proposition 232.1 .** In this, the second and final section of the chapter, we provide the proof of the De Rham–Ring Isomorphism Proposition 232.1. In the proof of the De Rham–Ring Isomorphism Proposition 232.1 we will need the shuffle map which we first introduced in Definition 164.23:

Definition 232.4. Let $k, l \in \mathbb{N}_0$.

(1) An (k,l)-shuffle is a pair (μ, ν) of disjoint sets of integers

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq k+l \quad \text{and} \quad 1 \leq \nu_1 < \nu_2 < \cdots < \nu_l \leq k+l.$$

(2) We denote by $S_{k,l}$ the set of all (k,l)-shuffles.

(3) We denote by $\operatorname{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_l)$.

(4) Given a sequence $1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq k+l$ we denote by $\eta^\mu \colon \mathbb{R}^{k+l+1} \to \mathbb{R}^{k+1}$ the unique linear map which satisfies $\eta^\mu(\mathfrak{v}_i) = \mathfrak{v}_j$ for $\mu_j \leq i < \mu_{j+1}$. One can easily verify that η^μ restricts to a map $\Delta^{k+l} \to \Delta^k$ which we also denote by η^μ.

(5) Given two topological spaces X and Y we consider the homomorphism

$$\begin{array}{rcl} \nabla := \nabla_{k,l} \colon \mathrm{C}_k(X) \otimes \mathrm{C}_l(Y) & \to & \mathrm{C}_{k+l}(X \times Y) \\ (\sigma \otimes \tau) & \mapsto & \displaystyle\sum_{(\mu,\nu) \in S_{k,l}} \operatorname{sign}(\mu,\nu) \cdot \Big(\underbrace{\overbrace{(\sigma \times \tau)}^{\Delta^k \times \Delta^l \to X \times Y} \circ \overbrace{(\eta^\mu, \eta^\nu)}^{\Delta^{k+l} \to \Delta^k \times \Delta^l}}_{\text{map } \Delta^{k+l} \to X \times Y} \Big). \end{array}$$

The following lemma will play an essential role in the proof of the De Rham–Ring Isomorphism Proposition 232.1:

Lemma 232.5. (Shuffle–Decomposition Lemma) Let $k, l \in \mathbb{N}_0$. The following statements hold:

(1) The images of the maps $(\eta^\mu, \eta^\nu) \colon \Delta^{k+l} \to \Delta^k \times \Delta^l$ with $(\mu, \nu) \in S_{m,n}$ cover $\Delta^k \times \Delta^l$ and the images overlap only in subsets of measure zero.

(2) For each $(\mu, \nu) \in S_{m,n}$ the map $(\eta^\mu, \eta^\nu) \colon \Delta^{k+l} \to \Delta^k \times \Delta^l$ is orientation-preserving if and only if $\operatorname{sign}(\mu, \nu) = 1$.

Sketch of proof of Lemma 232.5. The lemma follows easily from the Simplicial Complex-Product Lemma 174.38. ■

Proof of the De Rham–Ring Isomorphism Proposition 232.1. Let M be a smooth manifold. We start out with the following claim:

Claim 1. We consider the following diagram:

$$\begin{array}{ccccccc}
 & & & \overset{\wedge}{\frown} & & & \\
\mathrm{H}^k_{\mathrm{dR}}(M)\times\mathrm{H}^l_{\mathrm{dR}}(M) & & \xrightarrow{([\varphi],[\psi])\mapsto[p_1^*(\varphi)\wedge p_2^*(\psi)]} & & \mathrm{H}^{k+l}_{\mathrm{dR}}(M\times M) & \xrightarrow{d_M^*} & \mathrm{H}^{k+l}_{\mathrm{dR}}(M) \\
\downarrow{\scriptstyle s}\,{\scriptstyle\cong} & & {\scriptstyle\cong}\,\downarrow{\scriptstyle s} & & {\scriptstyle\cong}\,\downarrow{\scriptstyle s} & & {\scriptstyle\cong}\,\downarrow{\scriptstyle s} \\
\mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R})\times\mathrm{H}^l_{\mathrm{sm}}(M;\mathbb{R}) & \xrightarrow{\Xi_{k,l}} & \mathrm{H}^{k+l}(\mathrm{C}^{\mathrm{sm}}_*(M)\otimes\mathrm{C}^{\mathrm{sm}}_*(M);\mathbb{R}) & \xleftarrow[\nabla^*]{\cong} & \mathrm{H}^{k+l}_{\mathrm{sm}}(M\times M;\mathbb{R}) & \xrightarrow{d_M^*} & \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \\
\uparrow{\scriptstyle t}\,{\scriptstyle\cong} & & {\scriptstyle\cong}\,\uparrow{\scriptstyle t} & & {\scriptstyle\cong}\,\uparrow{\scriptstyle t} & & {\scriptstyle\cong}\,\uparrow{\scriptstyle t} \\
\mathrm{H}^k(M;\mathbb{R})\times\mathrm{H}^l(M;\mathbb{R}) & \xrightarrow{\Xi_{k,l}} & \mathrm{H}^{k+l}(\mathrm{C}_*(M)\otimes\mathrm{C}_*(M);\mathbb{R}) & \xleftarrow[\nabla^*]{\cong} & \mathrm{H}^{k+l}(M\times M;\mathbb{R}) & \xrightarrow{d_M^*} & \mathrm{H}^k(M;\mathbb{R}). \\
 & & & \underset{\cup}{\smile} & & &
\end{array}$$

(Some of the maps are explained in the proof of the claim.) Eventually we need to show that the diagram commutes. In this claim we prove the following statements:

(1) All parts of the diagram, except possibly the big top left rectangle, commute.
(2) All maps decorated by "$\cong$" are indeed isomorphisms.

Proof. The claim follows from the following clarifications and observations:

(a) For $i = 1, 2$ we denote by $p_i \colon M \times M \to M$ the projection onto the i-th factor.
(b) $d_M \colon M \to M \times M$ is the diagonal map.
(c) It follows from $p_1 \circ d_M = p_2 \circ d_M = \mathrm{id}_M$ and the Alternating Form-Functoriality Lemma 72.12 that the top horizontal maps define the wedge product on de Rham cohomology.
(d) We denote by $s_N \colon \mathrm{H}^j_{\mathrm{dR}}(N) \to \mathrm{H}^j_{\mathrm{sm}}(N;\mathbb{R})$ the natural isomorphisms from the Smooth Singular-H*-Isomorphism Proposition 231.12 and we denote by $t_N \colon \mathrm{H}^j(N;\mathbb{R}) \to \mathrm{H}^j_{\mathrm{sm}}(N;\mathbb{R})$ the natural isomorphisms from the De Rham–Smooth-H*-Isomorphism Theorem 231.14.
(e) We introduce the following maps:
- We denote by $\mathrm{C}_*(M)\otimes\mathrm{C}_*(M)$ the tensor product of chain complexes, as introduced in Definition 163.2. Recall that on the chain level we have by definition
$$(\mathrm{C}_*(M)\otimes\mathrm{C}_*(M))_d = \bigoplus_{r+s=d} \mathrm{C}_r(M)\otimes\mathrm{C}_s(M).$$
- We consider the map $\Pi_{k,l} \colon (\mathrm{C}_*(M)\otimes\mathrm{C}_*(M))_{k+l} \to \mathrm{C}_k(M)\otimes\mathrm{C}_l(M)$ that is given by projection onto the corresponding summand.
- We denote by $\mu_{\mathbb{R}} \colon \mathbb{R}\otimes\mathbb{R} \to \mathbb{R}$ the map that is induced by $x \otimes y \mapsto x \cdot y$.

It follows from the Cocycle Tensor Lemma 203.5 that the map
$$\begin{array}{rcl} \Xi_{k,l} \colon \mathrm{H}^k(M;\mathbb{R})\times\mathrm{H}^l(M;\mathbb{R}) & \to & \mathrm{H}^{k+l}(\mathrm{C}_*(M)\otimes\mathrm{C}_*(M),\mathbb{R}) \\ ([\varphi],[\psi]) & \mapsto & [\mu_{\mathbb{R}}\circ(\varphi\otimes\psi)\circ\Pi_{k,l}] \end{array}$$
is well-defined. Clearly this map also works for smooth simplices and we get a corresponding map on smooth (co-) chain complexes and cohomology groups.
(f) It follows from Definition 203.8 and the Shuffle Proposition 164.24 that the composition of the bottom horizontal maps gives us the cup product.
(g) Note that the right hand square commutes by the naturality of u.

(h) Given any two topological spaces X and Y we denote by

$$\begin{aligned} \nabla\colon \mathrm{C}_m(X)\otimes \mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X\times Y)\\ (\sigma\otimes\tau) &\mapsto \sum_{(\mu,\nu)\in S_{m,n}} \operatorname{sign}(\mu,\nu)\cdot((\sigma\times\tau)\circ(\eta^\mu,\eta^\nu) \end{aligned}$$

the shuffle map that we first introduced in Definition 164.23.

(i) Finally note that it follows from the Shuffle Proposition 164.24 together with the Eilenberg-Zilber Theorem 164.2 and the Cochain Homotopy-H^*-Lemma 197.14 that the map $\nabla^*\colon \mathrm{H}^{k+l}(\mathrm{C}_*(M\times M);\mathbb{R})\to \mathrm{H}^{k+l}(\mathrm{C}_*(M)\otimes \mathrm{C}_*(M);\mathbb{R})$ is an isomorphism. ⊞

It follows from Claim 1 that it remains to show that big rectangle in the statement of the claim commutes, i.e. we need to show the two maps from $\mathrm{H}^k_{\mathrm{dR}}(M)\times \mathrm{H}^l_{\mathrm{dR}}(M)$ to $\mathrm{H}^{k+l}(\mathrm{C}^{\mathrm{sm}}_*(M)\otimes \mathrm{C}^{\mathrm{sm}}_*(M);\mathbb{R})$ agree. In other words, it suffice to prove the following claim:

Claim 2. The following diagram commutes:

$$\begin{array}{ccccc} \Omega^k(M)\times\Omega^l(M) & & \xrightarrow{(\varphi,\psi)\mapsto p_1^*(\varphi)\wedge p_2^*(\psi)} & & \Omega^{k+l}(M\times M)\\ \downarrow s & \downarrow s & & & \downarrow s\\ \mathrm{C}^k_{\mathrm{sm}}(M;\mathbb{R})\times \mathrm{C}^l_{\mathrm{sm}}(M;\mathbb{R}) & \xrightarrow{\Xi_{k,l}} & \operatorname{Hom}((\mathrm{C}^{\mathrm{sm}}_*(M)\otimes \mathrm{C}^{\mathrm{sm}}_*(M))_{k+l};\mathbb{R}) & \xleftarrow{\nabla^*} & \mathrm{C}^{k+l}_{\mathrm{sm}}(M\times M;\mathbb{R}). \end{array}$$

Proof. Let $\varphi\in\Omega^k(M)$ and $\psi\in\Omega^l(M)$ be two smooth forms. We need to show that the two maps from $\Omega^k(M)\times\Omega^l(M)$ to $\operatorname{Hom}((\mathrm{C}^{\mathrm{sm}}_*(M)\otimes \mathrm{C}^{\mathrm{sm}}_*(M))_{k+l};\mathbb{R})$ send (φ,ψ) to the same homomorphism $(\mathrm{C}^{\mathrm{sm}}_*(M)\otimes \mathrm{C}^{\mathrm{sm}}_*(M))_{k+l}\to\mathbb{R}$. We only have to show that the two resulting

homomorphisms agree on a generating set of $\mathrm{C}^{\mathrm{sm}}_*(M) \otimes \mathrm{C}^{\mathrm{sm}}_*(M))_{k+l}$. Thus let $r, s \in \mathbb{N}_0$ with $r + s = k + l$ and let $\varphi\colon \Delta^r \to M$ and $\tau\colon \Delta^s \to M$ be two smooth simplices.

(1) First we assume that $r = k$ (and thus also $s = l$). Note that

$$(\nabla^* \circ s)(p_1^*(\varphi) \wedge p_2^*(\psi)))(\sigma \otimes \tau) \overset{\text{by definition of the dual map}}{=} s(p_1^*(\varphi) \wedge p_2^*(\psi))(\nabla(\sigma \otimes \tau)) \overset{\text{by definition of } s}{=} \int\limits_{\nabla(\sigma\otimes\tau)} p_1^*(\varphi) \wedge p_2^*(\psi) \overset{\text{by definition of } \nabla}{=}$$

$$= \sum_{(\mu,\nu)\in S_{k,l}} \int\limits_{(\sigma\times\tau)\circ(\eta^\mu,\eta^\nu)\colon \Delta^{k+l}\to M\times N} \operatorname{sign}(\mu,\nu) \cdot p_1^*(\varphi) \wedge p_2^*(\psi)$$

$$= \sum_{(\mu,\nu)\in S_{k,l}} \int\limits_{(\eta_\mu,\eta_\nu)\colon \Delta^{k+l}\to \Delta^k\times\Delta^l} \operatorname{sign}(\mu,\nu) \cdot ((\sigma\times\tau)^*(p_1^*(\varphi) \wedge p_2^*(\psi)))$$

$$= \sum_{(\mu,\nu)\in S_{k,l}} \int\limits_{(\eta_\mu,\eta_\nu)(\Delta^{k+l})} \operatorname{sign}(\mu,\nu) \cdot ((\sigma\times\tau)^*(p_1^*(\varphi) \wedge p_2^*(\psi)))$$

↑ by the Form Integration Proposition 74.15 (4)

$$= \int\limits_{\Delta^k\times\Delta^l} (\sigma\times\tau)^*(p_1^*(\varphi) \wedge p_2^*(\psi)) = \int\limits_{\Delta^k\times\Delta^l} p_1^*(\sigma^*(\varphi)) \wedge p_2^*(\tau^*(\psi))$$

↑ follows from the Shuffle–Decomposition Lemma 232.5 and the Lebesgue Integral Proposition 73.22 (2a), (3b)

↑ follows from $p_1 \circ (\sigma \times \tau) = \sigma$ and $p_2 \circ (\sigma \times \tau) = \tau$

$$= \int\limits_{\Delta^k} \sigma^*(\varphi) \cdot \int\limits_{\Delta^l} \tau^*(\psi) = (\mu_{\mathbb{R}} \circ (u(\varphi) \otimes u(\psi)) \circ \Pi_{k,l})(\sigma \otimes \tau)$$

↑ follows, as in the Product-Pullback-Integral Proposition 74.17 (1), from the Fubini Theorem 73.25 (note that here we use that $r = k$ and $s = l$)

↑ follows from the definitions (here we also use that $r = k$ and $s = l$)

$$= (\Xi_{k,l} \circ (s(\varphi) \times s(\psi))(\sigma \otimes \tau).$$

(2) Now we assume that $r \neq k$ (which is equivalent to $s \neq l$). The argument is almost identical to the above. Except that now instead of the product of two integrals we obtain in both cases 0. This is a consequence of a (slight variation on) the Product-Pullback-Integral Proposition 74.17 (2) and the definition of $\Pi_{k,l}$. ■

232.3. Exercises for Chapter 232 .

Exercise 232.1. Given $n \in \mathbb{N}_0$ we consider the complex projective space $\mathbb{C}\mathrm{P}^n$. In the Wedge Product-$\mathbb{C}\mathrm{P}^n$-Proposition 85.2 we gave a calculation of the wedge product on $\mathrm{H}^*_{\mathrm{dR}}(\mathbb{C}\mathrm{P}^n)$ and in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 we gave a calculation of the cup product on singular cohomology $\mathrm{H}^*(\mathbb{C}\mathrm{P}; \mathbb{Z})$. Compare the results and explain why the latter result is stronger.

Exercise 232.2. Let $n \in \mathbb{N}$. Compute the de Rham cohomology ring $\mathrm{H}^*_{\mathrm{dR}}((S^1)^n)$ of the n-dimensional torus.

Exercise 232.3. Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. Compute the de Rham cohomology ring of the connected sum $M\#N$ in terms of the de Rham cohomology rings of M and N.

Exercise 232.4. Let V be a finite-dimensional real vector space. We say a 2-form ω on V is *non-degenerate* if the map

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \\ v & \mapsto & \begin{pmatrix} V & \to & \mathbb{R} \\ w & \mapsto & \varphi(v,w) \end{pmatrix} \end{array}$$

is an isomorphism.

(a) Show that if V admits a non-degenerate 2-form, then $\dim_{\mathbb{R}}(V)$ is even.

(b) Now assume that V is $2n$-dimensional and that φ is a non-degenerate 2-form on V. Show that $\varphi^n = \underbrace{\varphi \wedge \cdots \wedge \varphi}_{n \text{ times}} \neq 0$.

A *symplectic form* on a smooth manifold is a closed non-degenerate 2-form ω.

(c) Show that is a smooth manifold $2n$-dimensional M admits a symplectic form, then M is orientable.

(d) Show that is a closed non-empty smooth manifold $2n$-dimensional M admits a symplectic form, then there exists a $\theta \in \mathrm{H}^2(M;\mathbb{R})$ with $\theta^n \neq 0 \in \mathrm{H}^{2n}(M;\mathbb{R})$.

(e) Does $S^2 \times S^4$ admit a symplectic structure?

Remark. This exercise has an overlap with Exercise 72.5.

CHAPTER 233

Realizing cohomology classes by maps to S^1

In this relatively short chapter we will use the de Rham Isomorphism-Theorem 231.16 to prove the two results:

- In Section 233.1 we will state and prove the Maps-to-S^1-Proposition 233.4 which says that for a smooth manifold M there exists a natural bijection between $\mathrm{H}^1(M;\mathbb{Z})$ and the set of homotopy classes of smooth maps $M \to S^1$.
- In Section 233.2 we will use the Maps-to-S^1-Proposition 233.4 to prove the Product-as-Mapping Torus Proposition 233.8 which says to show that for a smooth manifold M the product $M \times S^1$ can often be rewritten as non-trivial mapping tori.
- In Section 233.3 we will provide exercises for Chapter 233.

In the next Chapter 234 we will prove that in some cases homology classes can be represented by smooth submanifolds.

233.1. **Realizing cohomology classes by maps to S^1.** We start out this section with the following definition:

Definition 233.1. Let (X, x_0) and (Y, y_0) be pointed topological spaces. We say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is realized by a continuous map $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

In the following we will mostly discuss the case $(Y, y_0) = (S^1, 1)$. Before we can continue we introduce the following conventions:

Convention 233.2.

(1) By the Circle-π_1-Theorem 88.22 we have an isomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$ given by sending 1 to the equivalence class of the loop $t \mapsto \exp(2\pi\,\mathrm{i}t)$. We use this isomorphism to make the identification $\pi_1(S^1, 1) =_{\mathrm{i}} \mathbb{Z}$.

(2) Let X be a path-connected topological space and let $x_0 \in X$. As shown in the H^1-Hom-Proposition 201.24 we can use the Hurewicz homomorphism and the evaluation homomorphism to make the identifications

$$\mathrm{H}^1(X;\mathbb{Z}) \;=_{\mathrm{i}}\; \mathrm{Hom}(\mathrm{H}_1(X);\mathbb{Z}) \;=_{\mathrm{i}}\; \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}).$$

Now we can formulate the following question:

Question 233.3. Let X be a path-connected topological space that is "reasonable", e.g. that is a smooth manifold or a CW-complex. Furthermore let $x_0 \in X$ be a base point. Can every homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z} =_{\mathrm{i}} \pi_1(S^1, 1)$ be realized by a continuous map $(X, x_0) \to (S^1, 1)$?

The following proposition gives an affirmative answer if X is a smooth manifold:

Proposition 233.4. (Maps-to-S^1-Proposition) Let M be a connected smooth manifold and let $P \in M$ be a base point.

(1) (a) Every class $\phi \in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$.
   (b) Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$. If $f\colon \partial M \to S^1$ is a smooth map that realizes $\phi|_{\partial M}$, then there exists a smooth map $g\colon M \to S^1$ with $g|_{\partial M} = f$ that realizes ϕ.

(2) Let $f, g\colon M \to S^1$ be two smooth maps with $f(P) = g(P) = 1$. If the induced maps $f_* = g_*\colon \pi_1(M,P) \to \pi_1(S^1,1)$ agree, then f and g are smoothly homotopic.

Remark.

(1) We first recall some notation: As in Definition 19.11 we denote by $[X,Y]$ the set of homotopy equivalence classes of continuous maps $X \to Y$. Now let M be a smooth manifold. By applying the H^1-Hom-Proposition 201.24 to each component of M and by using the Whitney Approximation Theorem 36.1 and the Replace-by-Smooth Homotopy Proposition 36.6 we can reformulate the Maps-to-S^1-Proposition 233.4 as follows: The map

$$\begin{array}{rcl} [X,S^1] & \to & \mathrm{H}^1(M;\mathbb{Z}) \\ [f\colon X \to S^1] & \mapsto & f^*([S^1]^*) \end{array}$$

is a bijection.[2523]

(2) The Maps-to-S^1-Proposition 233.4 (2) can be viewed as a partial converse to the Homotopy-π_1-Proposition 90.8.

(3) Later on, in the Maps-to-K(π,n)-Proposition 307.18 we will prove an analog of the Maps-to-S^1-Proposition 233.4 for CW-complexes. We will use this proposition, in the Map-out-of-CW Proposition 307.24, to give a new proof of the Maps-to-S^1-Proposition 233.4. □

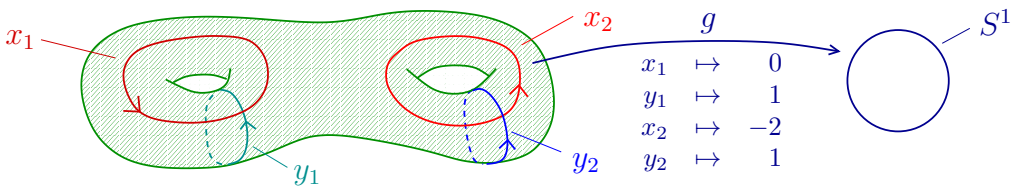
Examples.

(1) We consider the torus $T = S^1 \times S^1 =_{\mathrm{i}} \mathbb{R}^2/\mathbb{Z}^2$ and we make the usual identification $\mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2$ coming from the Künneth Theorem 164.6. We consider the cohomology classes in $\mathrm{H}^1(T;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\mathrm{H}_1(T),\mathbb{Z})$ that are given by

$$\begin{array}{rcl} \phi\colon \mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & m \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon \mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & n. \end{array}$$

These two cohomology classes are evidently realized by the projection $S^1 \times S^1 \to S^1$ onto the first, respectively the second factor. In Exercise 233.1 the goal will be to give an example of an explicit continuous map $g\colon T \to S^1$ that realizes the homomorphism $\varphi\colon \mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2 \to \mathbb{Z}$ given by $\varphi(m,n) = 6m + 4n$.

(2) In the figure below we consider the surface Σ of genus 2:



[2523] Here $[S^1]^* \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the standard dual fundamental class.

As we saw on page 2970, the oriented curves x_1, y_1, x_2, y_2 represent a basis for $\mathrm{H}_1(\Sigma)$. We consider some random epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$, e.g. we could consider the epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By the Maps-to-S^1-Proposition 233.4 we know that ϕ can be realized by some smooth map $g\colon \Sigma \to S^1$. But can one "visualize" such a map? We will return to this question on page 4512. □

We break the proof of the Maps-to-S^1-Proposition 233.4 into three parts.

Proof of the Maps-to-S^1-Proposition 233.4 (1a). Let M be a connected n-dimensional smooth manifold, let $P \in M$ and let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$. We have to show that ϕ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$. We start out with two preparations:

(1) Let ω be a smooth differential 1-form on M and let $\gamma\colon [a,b] \to M$ be a piecewise smooth path. We pick $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to each subinterval $[t_i, t_{i+1}]$ is smooth. We set

$$\int\limits_{[a,b]} \gamma^*(\omega) \;:=\; \sum_{i=0}^{n-1} \underbrace{\int\limits_{[t_i,t_{i+1}]} \gamma^*(\omega)}_{\substack{\text{defined in the}\\ \text{1-Form Integration}\\ \text{Proposition 71.17}}}.$$

It follows almost immediately from basic facts on Riemann-integrals that this definition does not depend on the choice of the subdivision.

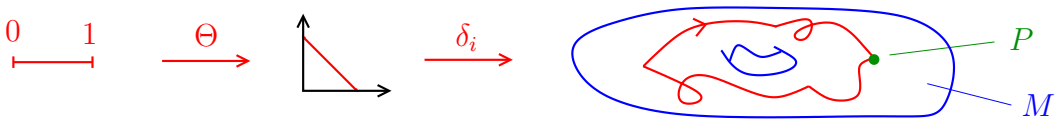
(2) We denote by $u\colon \mathrm{H}^1_{\mathrm{dR}}(M) \to \mathrm{H}^1(M;\mathbb{R})$ the natural isomorphism from the de Rham Isomorphism-Theorem 231.16.

We now start with the actual proof of the proposition. We pick a smooth closed differential 1-form ω on M with $u([\omega]) = \phi \in \mathrm{Hom}(\pi_1(M,P);\mathbb{Z}) \subseteq \mathrm{Hom}(\pi_1(M,P);\mathbb{R}) =_{\mathrm{i}} \mathrm{H}^1(M;\mathbb{R})$.

Claim 1. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. Then

$$\int\limits_{[0,1]} \delta^*(\omega) \;=\; \phi\Big(\underbrace{[\delta]}_{\in \pi_1(M,P)}\Big) \;\in\; \mathbb{R}.$$

Proof. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. We first note that one can show easily that, after possibly performing an orientation-preserving reparametrization, we can write $\delta = \delta_1 * \cdots * \delta_k$ as a concatenation of smooth paths $\delta_i\colon [0,1] \to M$, $i = 1, \ldots, k$. Let $\Theta\colon [0,1] \to \Delta^1$ be the homeomorphism that is given by $\Theta(t) = (1-t, t)$. Note that with this definition the singular chain $\delta_1 \circ \Theta^{-1} + \cdots + \delta_k \circ \Theta^{-1}$ lies in $\mathrm{C}_1^{\mathrm{sm}}(M)$.



We see that:

$$\int\limits_{[0,1]} \delta^*(\omega) \;\overset{\text{(1) and (6)}}{=}\; \int\limits_{\delta_1\circ\Theta^{-1}+\cdots+\delta_k\circ\Theta^{-1}} \omega \;\overset{\text{Theorem 231.16}}{=}\; \langle \phi, [\delta_1 \circ \Theta^{-1} + \cdots + \delta_k \circ \Theta^{-1}] \rangle \;\overset{\text{identification}}{=}\; \phi([\delta]).$$

by the definitions and by the Form Integration Proposition 74.15 (1) and (6)

by the De Rham-Isomorphism Theorem 231.16, since $u([\omega]) = \phi$

via the identification $\mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$

We define $\begin{array}{rcl} \widetilde{g}\colon M & \to & \mathbb{R}/\mathbb{Z} \\ Q & \mapsto & \left[\int\limits_{[0,1]} \gamma^*(\omega)\right] \end{array}$ and $\begin{array}{rcl} g\colon M & \to & S^1 \\ Q & \mapsto & \exp(2\pi\,\mathrm{i}\cdot\widetilde{g}(Q)). \end{array}$

where γ is a smooth path[2524] from P to Q

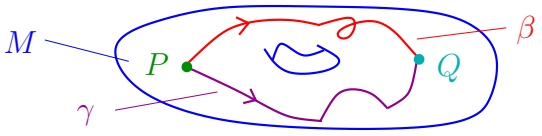
The following claim shows that the map g has all the required properties:

Claim 2.

(i) The map $\widetilde{g}$ (and thus the map g) is well-defined.
(ii) We have $g(P) = 1 \in S^1$.
(iii) The map g is smooth.
(iv) We have $g_* = \phi\colon \pi_1(M,P) \to \pi_1(S^1,1) =_{\mathrm{i}} \mathbb{Z}$.

Proof. Let us prove these four statements:

(i) Let Q in M. Suppose we are given two piecewise smooth paths β and γ from the base point P to Q.



Then $\int\limits_{[0,1]} \beta^*(\omega) - \int\limits_{[0,1]} \gamma^*(\omega) = \int\limits_{[0,1]} (\beta * \overline{\gamma})^*(\omega) = \phi([\beta * \overline{\gamma}]) \in \mathbb{Z}.$

follows from the definitions and the Form Integration Proposition 74.15 (1) and (6)

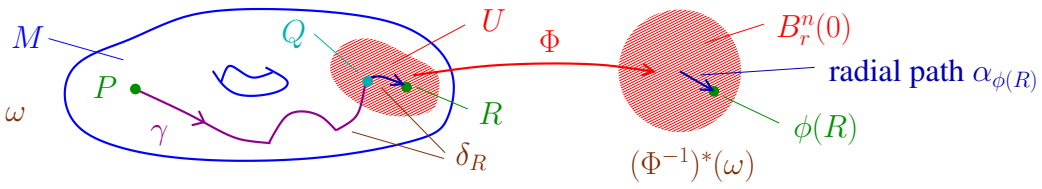
by Claim 1, since $\beta * \overline{\gamma}$ is a piecewise smooth loop

This shows that the map $g\colon M \to \mathbb{R}/\mathbb{Z}$ is well-defined.

(ii) We determine $g(P)$ by taking the constant path from P to P. It follows immediately from the definition of g that $g(P) = 1 \in S^1$.

(iii) Let $Q \in M$. We need to show that g is smooth in an open neighborhood of Q. We pick a path γ from P to Q. If $P \in M \setminus \partial M$, then we pick a chart $\Phi\colon U \to B_r^n(0)$ around Q with $\Phi(Q) = 0$. If $P \in \partial M$, then we pick a chart $\Phi\colon U \to B_r^n(0) \cap \mathbb{H}^n$ around Q with $\Phi(Q) = 0$. Either way, given a point $R \in U$ we denote by δ_R the piecewise smooth path that is given by the concatenation of the following two paths:

(a) The path γ from P to Q.
(b) The "radial path" from Q to R that corresponds, under the diffeomorphism Φ, to the radial path $\alpha_{\phi(R)}$ from 0 to $\Phi(R) \in B_r^n(0)$.



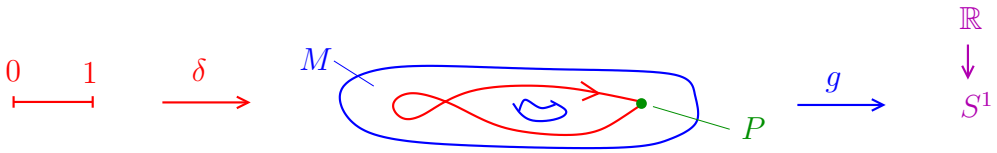
[2524]By the Smooth Path-Connectivity Proposition 25.32 (2) any two points on a connected smooth manifold are connected by a smooth path.

It now suffices to show that the map $U \to \mathbb{R}$ that is given by $R \mapsto \int\limits_{[0,1]} \delta_R^*(\omega)$ is smooth.

But this map is indeed smooth, since for any $R \in U$ we have

$$\int\limits_{[0,1]} \delta_R^*(\omega) \;=\; \int\limits_{[0,1]} \delta_Q^*(\omega) + \int\limits_{[0,1]} \alpha_{\phi(R)}^*((\Phi^{-1})^*(\omega)) \;=\; \int\limits_{[0,1]} \delta_Q^*(\omega) + \underbrace{\int\limits_{t=0}^{t=1} \overbrace{(\Phi^{-1})^*(\omega)}^{\substack{\text{smooth } n\text{-form} \\ \text{on } B_r^n(0)}}{}_{t\cdot\phi(R)}(\phi(R))}_{\text{smooth in } R}.$$

(iv) Suppose we are given an element in $\pi_1(M,P)$. By the Smooth Path-Connectivity Proposition 25.32 (1) we can represent this element by a smooth loop $\delta\colon [0,1] \to M$.



We have the following equalities:

$$\begin{aligned} g_*([\delta]) &\overset{\text{by the Action–}\pi_1\text{-Theorem 93.1}}{=} \text{endpoint of the lift of } g\circ\delta\colon[0,1]\to\mathbb{R} \text{ to the universal covering } \mathbb{R} \text{ of } S^1 \\ &= \text{endpoint of the lift of the path } t \mapsto \exp\Big(2\pi\,\mathrm{i} \int\limits_{[0,t]} \delta^*(\omega)\Big) \text{ to the universal covering } \mathbb{R} \\ &= \text{endpoint of the path } t \mapsto \int\limits_{[0,t]} \delta^*(\omega) \;=\; \text{the point } \int\limits_{[0,1]} \delta^*(\omega) \overset{\text{by Claim 1}}{=} \phi([\delta]). \end{aligned}$$

We have thus shown that $g_* = \phi\colon \pi_1(M,P) \to \pi_1(S^1,1) =_{\mathrm{i}} \mathbb{Z}$. ■

Proof of the Maps-to-S^1-Proposition 233.4 (2). Let $f, g\colon M \to S^1$ be two smooth maps with $g(P) = 1$, such that we have the equality $f_* = g_*\colon \pi_1(M,P) \to \pi_1(S^1,1)$. We need to show that f and g are homotopic. Let ω be any closed differential 1-form on S^1 such that $u_M([\omega])$ represents a generator of $\mathrm{Hom}(\pi_1(S^1,1),\mathbb{Z}) \subseteq \mathrm{Hom}(\pi_1(S^1,1),\mathbb{R}) =_{\mathrm{i}} \mathrm{H}^1(S^1;\mathbb{R})$. We set $\alpha := f^*(\omega)$ and $\beta := g^*(\omega)$.

Claim. Let $p\colon M \times [0,1] \to M$ be the natural projection. There exists a closed smooth differential 1-form ψ on $M \times [0,1]$ such that the restriction to $M \times \{0\}$ equals $p^*(\alpha)|_{M\times\{0\}}$ and such that the restriction to $M \times \{1\}$ equals $p^*(\beta)|_{M\times\{1\}}$.

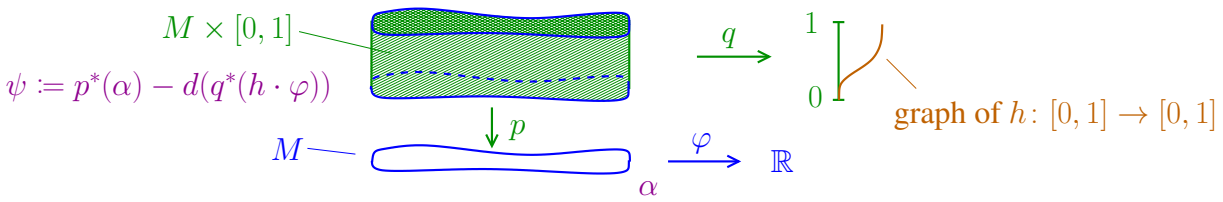
Proof. We start out with a few preparations:

- First note that it follows easily from the hypothesis that $f_* = g_*$ and from the natural isomorphisms $\mathrm{H}^1_{\mathrm{dR}}(M) = \mathrm{H}^1(M;\mathbb{R}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{R})$ that we have the equalities $[\alpha] = f^*([\omega]) = g^*([\omega]) = [\beta] \in \mathrm{H}^1_{\mathrm{dR}}(M)$. By definition of de Rham cohomology this means that there exists a smooth 0-form φ on M with $d\varphi = \alpha - \beta$.
- Let $q\colon M \times [0,1] \to [0,1]$ be the natural projection.
- It follows from the Smooth Transition Function Lemma 24.9 that there exists a smooth map $h\colon [0,1] \to [0,1]$ that is $\equiv 0$ close to 0 and that is $\equiv 1$ close to 1.

On $M \times [0,1]$ we consider the 1-form that is given by

$$\psi \;:=\; p^*(\alpha) - d(q^*(h \cdot \varphi)).$$

We claim that ψ has the desired properties.



First note that

$$d(\psi) \;=\; d(p^*(\alpha) - d(q^*(h\cdot\varphi))) \underset{\text{by the Differential-Natural Proposition 75.8}}{=} p^*(d(\alpha)) - d(d(q^*(h\cdot\varphi))) \underset{\text{by the Differentials-on-Manifolds Proposition 75.7 (4)}}{=} p^*(\overbrace{d(\alpha)}^{=0}) \;=\; 0.$$

In other words, we have just shown that ψ is closed. Finally we calculate that

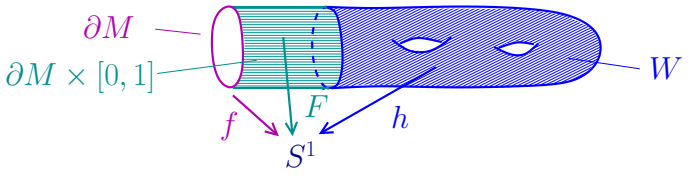
$$\begin{aligned} \psi \;=\; p^*(\alpha) - d(q^*(h\cdot\varphi)) &\overset{\text{by the Differential-Natural Proposition 75.8 and the Differentials-on-Manifold Proposition 75.7 (3)}}{=} p^*(\alpha) - (q^*(dh)\cdot\varphi + q^*(h)\cdot d\varphi) \\ &= \underbrace{p^*(\alpha) - q^*(dh)\cdot\varphi - q^*(h)\cdot d\varphi}. \end{aligned}$$

on $M\times\{0\}$ this equals $p^*(\alpha) - 0 - 0 = p^*(\alpha)$
on $M\times\{1\}$ this equals $p^*(\alpha) - 0 - (p^*(\alpha) - p^*(\beta)) = p^*(\beta)$ ⊞

Next note that, as in the proof of Statement (1), the closed 1-form ψ on $M\times[0,1]$ gives rise, by integrating the 1-form ψ along smooth paths emanating from $(P,0)$, to a smooth map $F\colon M\times[0,1]\to S^1$. We set $w := F(P,1)$. It follows fairly easily from the definitions that $F_0 = f\colon M\to S^1$ and that $F_1 = \rho_w\circ g\colon M\to S^1$ where $\rho_w\colon S^1\to S^1$ is given by multiplication by w. This shows that $F_0 = f$ and $F_1 = \rho_w\circ g$ are homotopic. But clearly ρ_w is homotopic to the identity. This implies that $\rho_w\circ g$ is homotopic to g. Since being homotopic is an equivalence relation we see that f is homotopic to g. ■

Proof of the Maps-to-S^1-Proposition 233.4 (1b). Let $P\in\partial M$ be a point in the boundary. Furthermore let $\phi\in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$, and let $f\colon \partial M\to S^1$ be a smooth map that realizes $\phi|_{\partial M}$. We need to show that there exists a smooth map $g\colon M\to S^1$ with $g|_{\partial M} = f$ that realizes ϕ.

First note that it follows from the Smooth Collar Neighborhood Theorem 35.6 that we can pick a collar neighborhood $[0,1]\times\partial M$. We set $W := M\setminus([0,1)\times\partial M)$ and we equip W with the base point $(P,1)$. By the Basics-of-Collars Lemma 35.7 (1) we know that W is an n-dimensional smooth manifold with $\partial W = \{1\}\times M$. By the Maps-to-$S^1$-Proposition 233.4 (1a) there exists a smooth map $h\colon W\to S^1$ which realizes $\phi|_W$. It is now straightforward to see that under the obvious homeomorphism $\{0\}\times M\to\{1\}\times M$ the maps $f_*\colon \mathrm{H}_1(M)\to\mathrm{H}_1(S^1)$ and $h_*\colon \mathrm{H}_1(M)\to\mathrm{H}_1(S^1)$ agree. At this point we can conclude from the Maps-to-S^1-Proposition 233.4 (2) that there exists a smooth map $F\colon [0,1]\times\partial M\to S^1$ such that $F_0 = f$ and $F_1 = h|_{\{1\}\times M}$.



Note that by definition of a collar neighborhood and by the Basics-of-Collars Lemma 35.7 we know that $[0,1]\times\partial M$ and W are closed subsets of M. Thus we obtain from the Pasting Proposition 4.7 (2) that the two maps F and g combine to a continuous map $h\colon M\to S^1$. Since h is smooth on $[0,1]\times\partial M$ we obtain from the Whitney Approximation Theorem 36.1 that h is homotopic rel $\{0\}\times\partial M$ to a smooth map $g\colon M\to S^1$. This map now has all the desired properties. ■

Of course one can ask many variations on Question 233.3. Since arbitrary topological spaces can be quite weird we restrict ourselves in the following questions to CW-complexes:

Question 233.5. Let X be a connected CW-complex.

(1) Let $g\geq 1$ and let Σ_g be the surface of genus $g\geq 1$. Can every homomorphism $\pi_1(X)\to\pi_1(\Sigma_g)$ be realized by a continuous map $X\to\Sigma_g$?

(2) Can every homomorphism $\pi_1(X)\to\pi_1\left(\bigvee_{i=1}^{k} S^1\right)=_{\mathrm{i}}\langle x_1,\dots,x_k\rangle$ be realized by a continuous map $X\to\bigvee_{i=1}^{k} S^1$?

(3) Can every homomorphism $\pi_1(X)\to\mathbb{Z}/2\mathbb{Z}=_{\mathrm{i}}\pi_1(\mathbb{R}\mathrm{P}^2)$ be realized by a continuous map $X\to\mathbb{R}\mathrm{P}^2$?

In the Map-out-of-CW Proposition 307.24 and Exercise 213.8 we will answer these three questions.

233.2. **Rewriting products $M\times S^1$.** The Maps-to-S^1-Proposition 233.4 (1) has many consequences. For example in the Codimension-One Submanifold Existence Proposition 234.6 we will use it to show that for a given compact orientable n-dimensional smooth manifold M any class in $\mathrm{H}_{n-1}(M)$ can be represented by a compact oriented proper submanifold. In this section we will give a different application, namely we will see that in many situations one can find interesting ways to rewrite a product $M\times S^1$ where M is a closed connected smooth manifold.

To formulate that application it is convenient to recall Definition 21.19:

Definition 233.6. Let X be a topological space and let $f\colon X\to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])\,/\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X,f). We refer to the continuous map

$$\begin{array}{rccc} q\colon \mathrm{Tor}(X,f)=(X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}\,t) \end{array}$$

as the natural projection onto S^1.

Throughout this section we also use the following notation:

Notation 233.7. We denote by $\Theta\colon \mathrm{H}_1(S^1)\to\mathbb{Z}$ the isomorphism given by $\Theta([S^1])=1$. (Here $[S^1]\in\mathrm{H}_1(S^1)$ is the standard generator.)

Now we can state the main result of this section:

Proposition 233.8. (Product-as-Mapping Torus Proposition) Let M be a smooth manifold and let $\phi\colon \mathrm{H}_1(M\times S^1)\to\mathbb{Z}$ be an epimorphism. We set $k=k(\phi):=\phi([\{*\}\times S^1])$.

If $k \neq 0$, then there exists a $|k|$-fold covering $p\colon \widetilde{M} \to M$ and a deck transformation $f\colon \widetilde{M} \to \widetilde{M}$ together with a diffeomorphism[2525] $\Xi\colon \mathrm{Tor}(\widetilde{M}, f) \to M \times S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(\mathrm{Tor}(\widetilde{M}, f)) & \xrightarrow{\quad \Xi_* \quad} & \mathrm{H}_1(M \times S^1) \\ & {\scriptstyle \Theta \circ q_*} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & \mathbb{Z}. & \end{array}$$

Remark.

(1) The Product-as-Mapping Torus Proposition 233.8 can be viewed as a "manifold analog" of the group-theoretic statement of Exercise 100.10.

(2) In Exercise 238.9 we will use the Ehresmann Fibration Theorem 238.6 to give an alternative proof for some of the statements of the Product-as-Mapping Torus Proposition 233.8.

(3) One mathematician (AR) once called products "slippery bastards". Indeed, as we will see in the examples below, the Product-as-Mapping Torus Proposition 233.8 shows that many products have alternative descriptions. □

Examples.

(1) Let M be a smooth manifold and let $\psi\colon \mathrm{H}_1(M) \to \mathbb{Z}$ be a homomorphism. We apply the Product-as-Mapping Torus Proposition 233.8 to the epimorphism

$$\begin{array}{rcl} \phi\colon \mathrm{H}_1(M \times S^1) \overset{\text{by the Künneth Theorem 164.15}}{\overset{\cong}{\leftarrow}} \mathrm{H}_1(M) \oplus \mathrm{H}_1(S^1) & \to & \mathbb{Z} \\ (\sigma, n \cdot [S^1]) & \mapsto & \psi(\sigma) + n. \end{array}$$

Note that we have $k = k(\phi) = 1$. This implies that the map $p\colon \widetilde{M} \to M$ is a covering of degree 1. It follows from the Covering Unique Existence Theorem 127.17 that we can now assume that $\widetilde{M} = M$ and that $p = \mathrm{id}$. Note that the only deck transformation of the 1-fold covering $\mathrm{id}\colon M \to M$ is the identity. Therefore we obtain from the Product-as-Mapping Torus Proposition 233.8 that there exists a diffeomorphism $\Xi\colon M \times S^1 \to M \times S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(M \times S^1) & \xrightarrow{\quad \Xi_* \quad} & \mathrm{H}_1(M \times S^1) \\ & {\scriptstyle \Theta \circ q_*} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & \mathbb{Z}, & \end{array}$$

where $q\colon M \times S^1 \to S^1$ denotes the natural projection. Note that if $\psi\colon \mathrm{H}_1(M) \to \mathbb{Z}$ is non-zero, then Ξ_* is not the identity. Summarizing, we have now shown that if $\mathrm{H}^1(M; \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M), \mathbb{Z}) \neq 0$, then $M \times S^1$ can be decomposed as a product in more than one way.

(2) Let Σ be the surface of some genus $g \in \mathbb{N}_{\geq 2}$ and let $\phi\colon \mathrm{H}_1(\Sigma \times S^1) \to \mathbb{Z}$ be an epimorphism such that $k = k(\phi) \neq 0$. It follows from the Product-as-Mapping Torus Proposition 233.8, together with the Subgroups-of-Surface Groups Proposition 130.7 (1)

[2525] By the Manifold-Mapping Torus Lemma 59.11 the mapping torus $\mathrm{Tor}(\widetilde{M}, f)$ naturally has the structure of a smooth manifold.

or alternatively with the Genus-of-Covering Lemma 156.18, that $\Sigma \times S^1$ is diffeomorphic to a mapping torus of the form $\mathrm{Tor}(\widetilde{\Sigma}, f)$ where $\widetilde{\Sigma}$ is a surface of genus $|k|\cdot(g-1)+1$. In particular we obtain the slightly mind boggling statement that the product $\Sigma \times S^1$ can also be viewed as a mapping torus of a self-diffeomorphism of a surface of arbitrarily high genus. □

Proof. By considering the components of M individually we can assume that M is actually connected and non-empty. In this setting we pick points $* \in M$ and $* \in S^1$. We denote by $j_1 \colon M \xrightarrow{x \mapsto (x,*)} M \times S^1$ and $j_2 \colon S^1 \xrightarrow{z \mapsto (*,z)} M \times S^1$ the corresponding embeddings. Note that by the Maps-to-S^1-Proposition 233.4 (1) there exists a smooth map $g \colon M \to S^1$ such that the map $\mathrm{H}_1(M) \xrightarrow{g_*} \mathrm{H}_1(S^1) \xrightarrow{\Theta} \mathbb{Z}$ equals $\phi \circ j_{1*} \colon \mathrm{H}_1(M) \to \mathbb{Z}$. Now let us consider the following pullback diagram:

$$\begin{array}{ccc} \widetilde{M} := \{(x,z) \in M \times S^1 \mid g(x) = z^k\} & \xrightarrow{(x,z) \mapsto z} & S^1 \\ \downarrow{\scriptstyle (x,z) \mapsto x} & & \downarrow{\scriptstyle z \mapsto z^k} \\ M & \xrightarrow{\quad g \quad} & S^1. \end{array}$$

By the Pullback-Covering Lemma 92.12 we know that $p \colon \widetilde{M} \to M$ is a k-fold covering. The map

$$\begin{array}{rcl} f \colon \widetilde{M} & \to & \widetilde{M} \\ (x,z) & \mapsto & (x, \exp(-2\pi\,\mathrm{i}/k) \cdot z) \end{array}$$

is evidently a deck transformation. Next we consider the following diagram:

$$\begin{array}{ccc} \widetilde{M} \times S^1 & \xrightarrow{((x,z),w) \mapsto ((x,z),z^{-1}\cdot w)} & \widetilde{M} \times S^1 \\ \downarrow{\scriptstyle ((x,z),\exp(2\pi\,\mathrm{i}\,t)) \mapsto [((x,z),k\cdot t \bmod \mathbb{Z})]} & & \downarrow{\scriptstyle ((x,z),w) \mapsto (x,w)} \\ \mathrm{Tor}(\widetilde{M}, f) & \xrightarrow[\Xi]{[((x,z),t)] \mapsto (x, z^{-1}\cdot \exp(2\pi\,\mathrm{i}\,t/k))} & M \times S^1. \end{array}$$

We make the following observations:

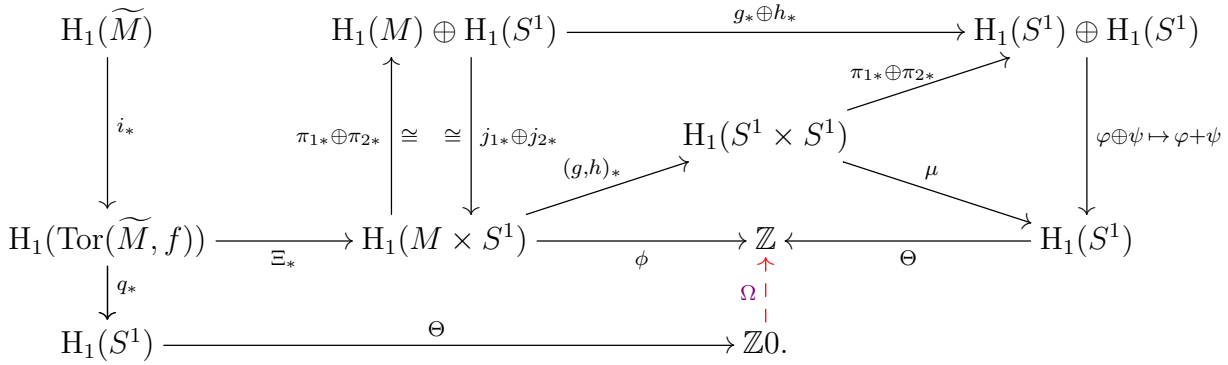
(1) One can easily verify that the bottom map is in fact well-defined and that the diagram commutes.
(2) It is pretty clear that the top map is a diffeomorphism.[2526]
(3) It is straightforward to show that the vertical maps are local diffeomorphisms and that they are degree k coverings.

It follows from the above together with Exercise 95.4 that the bottom map is also a diffeomorphism. It remains to prove the following claim:

Claim. We have $\phi \circ \Xi_* = \Theta \circ q_* \colon \mathrm{H}_1(\mathrm{Tor}(\widetilde{M}, f)) \to \mathbb{Z}$.

[2526]The reader might note that the top map is the restriction of the map $(M \times S^1) \times S^1 \to (M \times S^1) \times S^1$ given by the same formula, which is clearly a diffeomorphism, to the submanifold $\widetilde{M} \times S^1$ of $(M \times S^1) \times S^1$.

Proof. We consider the following diagram:



We make the following clarifications and observations:

(1) The map $i\colon \widetilde{M} \to \mathrm{Tor}(\widetilde{M}, f))$ is the natural embedding given by $x \mapsto [(x, 0)]$.
(2) The map $\mu\colon S^1 \times S^1 \to S^1$ is the multiplication map.
(3) In both settings we denote by π_1 and π_2 the projection onto the first respectively second factor.
(4) The map $h\colon S^1 \to S^1$ is given by $z \mapsto z^k$. Note that by the Degree-of-Power Lemma 151.5 we know that $h_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by k.
(5) It follows from the discussion on page 3142 that the vertical maps in the middle are isomorphisms and inverses of one another.
(6) The upper left triangle commutes by the functoriality of homology groups.
(7) The triangle to the right commutes by Exercise 166.2.
(8) The big rectangle to the right commutes by the choice of g and h.
(9) It follows from (5), (6), (7) and (8) that the triangle at the bottom commutes.
(10) Recall that any point $(x, z) \in \widetilde{M}$ satisfies by definition $g(x) = z^k$. It now follows immediately that the map $(g, h) \circ \mu \circ \Xi \circ i\colon \widetilde{M} \to S^1$ is the constant map.
(11) It follows from (9) and (10) that $\phi \circ \Xi_* \circ i_*\colon \mathrm{H}_1(\widetilde{M}) \to \mathbb{Z}$ is the zero map.
(12) By the Mapping Torus-H_1-Lemma 153.11 we know that the vertical sequence to the left is exact.
(13) It follows from (11) and (12) that the vertical map Ω exists.
(14) We pick $(x, z) \in \widetilde{M}$. We use the homeomorphism from the Mapping Torus-of-Homeomorphism Lemma 93.8 to make the identification

$$\mathrm{Tor}(\widetilde{M}, f) \ =_{\mathrm{i}}\ (\widetilde{M} \times \mathbb{R})/(w, t) \sim (f(w), t + 1).$$

Now we consider the singular 1-cycle

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & \mathrm{Tor}(\widetilde{M}, f) = (\widetilde{M} \times \mathbb{R})/(w, t) \sim (f(w), t + 1) \\ (1 - t, t) & \mapsto & [((x, z), kt)]. \end{array}$$

It follows immediately from the definitions that $(\phi \circ \Xi_*)([\sigma]) = k$ and $(\Theta \circ q_*)([\sigma]) = k$. Thus we have $\Omega(k) = k$. Since $k \neq 0$ we see that $\Omega = \mathrm{id}$. ∎

233.3. Exercises for Chapter 233 .

Exercise 233.1. We make the usual identification $\pi_1(\mathbb{R}^2/\mathbb{Z}^2) =_{\mathrm{i}} \mathbb{Z}^2$. We consider the homomorphism $\phi\colon \mathbb{Z}^2 = \pi_1(\mathbb{R}^2/\mathbb{Z}^2) \to \mathbb{Z}$ given by $\phi((1,0)) = 6$ and $\phi((0,1)) = 4$. Provide an *explicit* map $f\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^1$ such that $f_* = \phi$.

Exercise 233.2. Let M be a smooth manifold and let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$. Show that $\phi \cup \phi = 0$.
Remark. In the Cup Product–Square Zero Lemma 203.21 we showed that this statement holds in even greater generality.

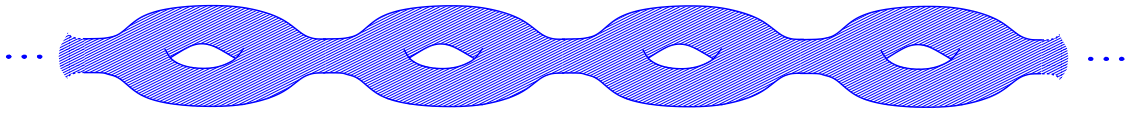
Exercise 233.3. Let $K \subseteq S^3$ be a knot. On several occasions (e.g. the Sphere Complement-H_*-Theorem 170.2) we showed that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$. Use Seifert surfaces to construct an explicit smooth map $f\colon S^3 \setminus K \to S^1$ which induces an epimorphism $f_*\colon \mathrm{H}_1(S^3 \setminus K) \to \mathbb{Z}$.

Exercise 233.4. Let $n \in \mathbb{N}$. Show that for any connected smooth manifold M, any $P \in M$ and any homomorphism $\varphi\colon \pi_1(M,P) \to \pi_1((S^1)^n, (1,\dots,1))$ there exists a smooth map $f\colon M \to (S^1)^n$ with $f(P) = (1,\dots,1)$ such that $f_* = \varphi\colon \pi_1(M,P) \to \pi_1((S^1)^n,(1,\dots,1))$.

Exercise 233.5. Let M be a smooth manifold, let A be a union of components of ∂M and furthermore let $\varphi \in \mathrm{H}^1(M,A;\mathbb{Z})$. Use Exercise 231.5 to show that there exists a smooth map $f\colon M \to S^1$ that vanishes on an open neighborhood of A and which has the property that the induced map $f_*\colon \mathrm{H}_1(M,A) \to \mathrm{H}_1(S^1) =_{\mathrm{i}} \mathbb{Z}$ equals the given cohomology class $\varphi \in \mathrm{H}^1(M,A;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M,A),\mathbb{Z})$.
Remark. This statement is the analog of the Maps-to-S^1-Proposition 233.4 (1).

Exercise 233.6. Let Σ_2 be the surface of genus two and let Σ_∞ be the surface of infinite genus that is shown in the figure below.

(a) Show that there exists a continuous map $f\colon \Sigma_2 \to S^1$ such that $f_*\colon \pi_1(\Sigma_2) \to \pi_1(S^1)$ is an epimorphism.
(b) Show that the covering corresponding to the subgroup $\ker(f_*) \subseteq \pi_1(\Sigma_2)$ is homeomorphic to Σ_∞.
(c) Show that there exists a continuous map $\varphi\colon \Sigma_\infty \to \Sigma_\infty$ such that the mapping torus $\mathrm{Tor}(\Sigma_\infty,\varphi)$ is homeomorphic to $\Sigma_2 \times \mathbb{R}$.
Hint. You could make use of Exercise 128.10.



surface Σ_∞ of "infinite genus"

Exercise 233.7. Let M be a closed smooth manifold and let $\Phi \in \mathrm{H}^1(M;\mathbb{Z})$ be a cohomology class. By the De Rham-Isomorphism Theorem 231.16 we know that Φ corresponds to a class in the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(M)$. Show that the following two statements are equivalent:

(1) There exists a smooth bundle map $p\colon M \to S^1$ (in the sense of Definition 122.1) such that $\Phi = p^*([S^1]^*)$ where $[S^1]^* \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the standard generator.
(2) We can represent Φ by a closed smooth 1-form that is non-zero everywhere.

Remark. Make use of the Ehresmann Fibration Theorem 122.6.

CHAPTER 234

Homology, cohomology and submanifolds

In this chapter, given an oriented, compact smooth manifold M we will relate the following objects:

(1) Codimension one submanifolds of M.
(2) Cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$ of dimension one.
(3) Homology classes in $\mathrm{H}_{n-1}(M,\partial M)$ of codimension one.
(4) Continuous maps $M \to S^1$.

The main input in this chapter is the Maps-to-S^1-Proposition 233.4 or alternatively the Map-out-of-CW Proposition 307.24.

This chapter is organized as follows:

- In Section 234.1 we will prove several statements about codimension-one submanifolds. For example we will prove the Non-Separating Codimension-One Submanifold Proposition 234.3 which says that a non-separating codimension-one submanifold represents a primitive element in codimension-one homology.
- In Section 234.2 we will state and prove the Codimension-One Submanifold Existence Proposition 234.6 which says that codimension-one homology classes of smooth manifolds can be represented by submanifolds.
- In Section 234.3 we will state and prove the Epimorphism-onto-Free Group Proposition 234.14 which says that a codimension-one submanifold can give rise to an epimorphism of the fundamental group onto a free group.
- In Section 234.4 we will build explicit cyclic coverings of manifolds.
- In Appendix 234.5 we will apply some of our results from this chapter to the study of curves on 2-dimensional smooth manifolds.
- In Section 234.6 we will provide exercises for Chapter 234.

234.1. **Codimension one submanifolds.** Let M be a compact oriented connected n-dimensional topological manifold. Note that every compact oriented proper submanifold $Y \subseteq M$ of *codimension* k gives rise to a homomorphism

$$\begin{array}{rcl} \mathrm{H}_k(M) & \to & \mathbb{Z} \\ \sigma & \mapsto & \langle \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^k(M;\mathbb{Z})}, \sigma \rangle. \end{array}$$

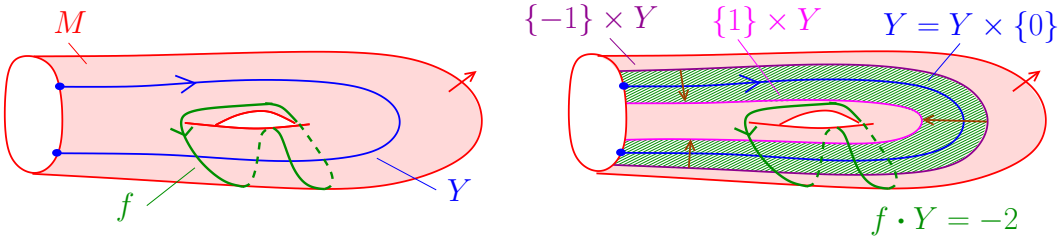
In many circumstances the next proposition gives an explicit description of this homomorphism $\mathrm{H}_k(M) \to \mathbb{Z}$:

Proposition 234.1. (Kronecker Pairing–Intersection Number Proposition) Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subseteq M$ be a

compact oriented proper submanifold of *codimension* k. Let $f\colon X \to M$ be a continuous map from a closed k-dimensional topological manifold X to M. If f is transverse to Y, then

$$\langle \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^k(M;\mathbb{Z})}, \underbrace{f_*([X])}_{\in \mathrm{H}_k(M)} \rangle \;=\; f \cdot Y.$$

Remark. Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subseteq M$ be a compact oriented proper submanifold of codimension one. In the Product Tubular Neighborhood–Existence Theorem 58.4 (3) together with the Tubular Map Lemma 58.3 we saw that Y admits a tubular neighborhood of the form $[-1,1] \times Y$ such that the product orientation of $[-1,1] \times Y$ matches the orientation of M.



If $f\colon S^1 \to M$ is a continuous map, then loosely speaking, the algebraic intersection number $f \cdot Y$ counts (with signs) how often f goes from $Y \times \{\mp 1\}$ to $Y \times \{\pm 1\}$. □

Proof. We have the following equalities:

$$f \cdot Y \underset{\uparrow}{=} \epsilon_M\big((\mathrm{PD}_M(f_*([X])) \cup \mathrm{PD}_M([Y])) \cap [M]\big) \underset{\uparrow}{=} \epsilon_M\big(\mathrm{PD}_M([Y]) \cap (\mathrm{PD}_M(f_*([X]) \cap [M]))\big)$$

by the Maps-Cup Product–Intersection Number Theorem 221.16, since f and Y are transverse

by the Cup-Cap Lemma 206.15

$$= \epsilon_M(\mathrm{PD}_M([Y]) \cap f_*([X])) \qquad = \langle \mathrm{PD}_M([Y]), f_*([X]) \rangle.$$

by the Definition 210.2 of $\mathrm{PD}_M(f_*([X]))$

by the Cap Product-Kronecker Lemma 206.12

■

For our next proposition we need the following simple-minded definition from group theory.

Definition 234.2. Let G be an abelian group. We say that an element $g \in G$ is primitive if there does not exist an $h \in G$ and an $n \geq 2$ with $n \cdot h = g$.

Now we can prove the following proposition which gives an easy criterion for showing that a codimension-one submanifold represents a non-trivial class in homology.

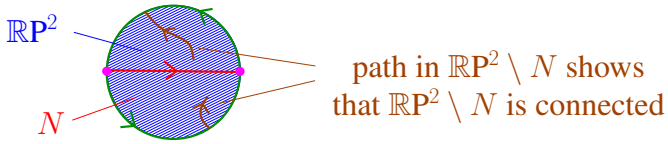
Proposition 234.3. (Non-Separating Codimension-One Submanifold Proposition) Let M be a compact oriented n-dimensional smooth manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If the complement $M \setminus N$ is connected, then $[N] \in \mathrm{H}_{n-1}(M, \partial M)$ is a primitive element of infinite order.

Example. The Non-Separating Codimension-One Submanifold Proposition 234.3 can in particular be seen as a generalization of the argument given on page 4237. □

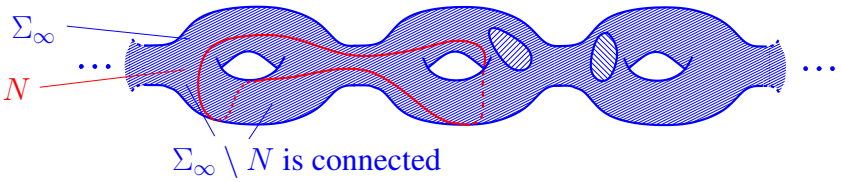
Remark. In any statement of a lemma, proposition or theorem it is always a good exercise to see whether the hypotheses stated are actually necessary. In the following we will

discuss whether the conditions in the Non-Separating Codimension-One Submanifold Proposition 234.3 that M needs to be compact and oriented are necessary.

(1) We consider the real projective plane $\mathbb{R}\mathrm{P}^2$ and the 1-dimensional submanifold N that is illustrated in the figure below to the left. We see that $\mathbb{R}\mathrm{P}^2 \setminus N$ is connected, but since $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{F}_2$ we see that N cannot possibly represent an element of infinite order. On the other hand, an "$\mathbb{F}_2$-version" of the Non-Separating Codimension-One Submanifold Proposition 234.3, which by the $\mathbb{F}_2$-Orientability Proposition 191.22 also applies to non-orientable smooth manifolds, shows that N represents a non-trivial element in $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$.



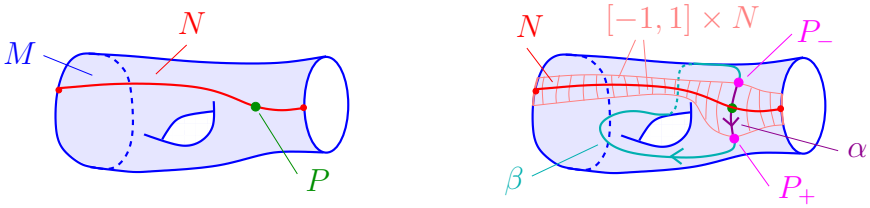
(2) Let M be a non-compact oriented n-dimensional smooth manifold and let N be a compact orientable non-empty proper submanifold of codimension one.



If ∂M is compact, then we will see in Exercise 234.1 using the Compact Submanifold Exhaustion Proposition 38.2 and the Non-Separating Codimension-One Submanifold Proposition 234.3, that the homology class $[N] \in \mathrm{H}_{n-1}(M)$ is still an element of infinite order. □

Proof. Let M be a compact oriented n-dimensional smooth manifold and let N be a compact orientable proper submanifold of codimension one such that $M \setminus N$ is connected. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can find a product tubular neighborhood $[-1,1] \times N$ in M. By hypothesis $M \setminus N$ is connected. It follows from the Basics-of-Tubular Neighborhood Lemma 57.10, the Homotopy-Equivalence π_0-Lemma 19.17 and the Manifold–Local Properties Lemma 23.8 (3) that $M \setminus (-1,1) \times N$ is path-connected.

It follows from our hypothesis that N is non-empty, together with the Interior-Non Compact Lemma 23.16, that we can pick a point $P \in N \setminus \partial N$. We write $P_\pm = P \times \{\pm 1\} \in [-1,1] \times N$. We denote by $\alpha\colon [0, \frac{1}{2}] \to [-1,1] \times N$ the "horizontal path" in $[-1,1] \times N$ from P_- to P_+. Since $M \setminus (-1,1) \times N$ is path-connected we can connect the two points P_+ to P_- via a path $\beta\colon [\frac{1}{2}, 1] \to M \setminus (-1,1) \times N$.



We consider the map $\gamma\colon S^1 = [0,1] \to M$ that is defined by α on $[0, \frac{1}{2}]$ and that is given by β on $[\frac{1}{2}, 1]$. We consider the following homomorphism

$$\begin{aligned} \Psi\colon \mathrm{H}_{n-1}(M, \partial M) &\to \mathbb{Z} \\ \sigma &\mapsto \langle \underbrace{\mathrm{PD}_M(\sigma)}_{\in \mathrm{H}^1(M;\mathbb{Z})}, \underbrace{\gamma_*([S^1])}_{\in \mathrm{H}_1(M)} \rangle. \end{aligned}$$

We want to show that $[N] \in \mathrm{H}_{n-1}(M, \partial M)$ is a primitive element of infinite order. It suffices to show that $\Psi([N]) = \pm 1 \in \mathbb{Z}$. In fact we have

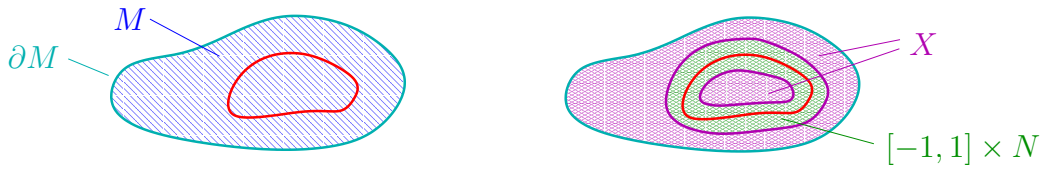
$$\Psi([N]) \overset{\text{by definition}}{=} \langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle \underset{\text{by the Kronecker Pairing–Intersection Number Proposition 234.1, since } \gamma \text{ and } N \text{ are clearly transverse}}{=} \gamma \cdot N \overset{\text{by construction of } \gamma}{=} \pm 1,$$

as desired. ∎

We conclude this section with the following lemma:

Lemma 234.4. (Codimension Zero Submanifold Bounds Lemma) Let M be a compact orientable n-dimensional smooth manifold with at most one boundary component and which furthermore satisfies $\mathrm{H}_1(M) = 0$. (For example we could take $M = \overline{B}^n$ or $M = S^n$). Then for every connected submanifold N of codimension one with $N \cap \partial M = \varnothing$ there exists a compact n-dimensional submanifold W of M with $\partial W = N$.

Proof. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can find a product tubular neighborhood $[-1,1] \times N$ in M. It follows from the argument of the proof of the Non-Separating Codimension-One Submanifold Proposition 234.3 and our assumption that M is orientable and the fact that M is compact that $X := M \setminus (-1,1) \times N$ is disconnected.



Furthermore it follows from the Codimension-Zero Submanifold Proposition 25.49 that X is a submanifold of M with $\partial X = \partial M \cup (\{-1\} \times N) \cup (\{1\} \times N)$. Since ∂M consists of at most one component, since M is connected and since X is disconnected we see that there exists a component Y of X with $\partial X = \{\epsilon\} \times N$ for some $\epsilon \in \{-1, 1\}$. By adding "half" of the tubular neighborhood to Y we obtain a submanifold$W$ with $\partial W = N$. ∎

234.2. **Representing homology classes by submanifolds II.** First we recall Definition 187.13.

Definition 234.5. Let M be a smooth manifold and let A be a union of components of ∂M.

(1) If N is a compact oriented k-dimensional submanifold of M with $\partial N \subseteq A$, then we denote by $[N]$ the image of the fundamental class $[N]$ in $\mathrm{H}_k(M, A)$ under the inclusion-induced map $\mathrm{H}_k(N, \partial N) \to \mathrm{H}_k(M, A)$.
(2) We say that a homology class $z \in \mathrm{H}_k(M, A)$ is represented by a submanifold if there exists a compact oriented proper k-dimensional submanifold N with $\partial N \subseteq A$ such that $[N] = z$.

Let M be an orientable, compact n-dimensional smooth manifold. We saw in the Cup Product–Intersection Number Theorem 220.10 that we can calculate the cup product of $\alpha \in \mathrm{H}^k(M, \partial M; \mathbb{Z})$ and $\beta \in \mathrm{H}^{n-k}(M; \mathbb{Z})$ if the Poincaré duals $\mathrm{PD}_M(\alpha) \in \mathrm{H}_{n-k}(M)$ and $\mathrm{PD}_M(\beta) \in \mathrm{H}_k(M, \partial M)$ are represented by transverse submanifolds. This rekindles our interest in the following question.

Question 187.15. Let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_k(M, A)$ be represented by submanifolds?

We already studied Question 187.15 in great detail in Section 187.4. For example in the Represent-by-1-Manifold Corollary 187.19 and the Represent-by-Connected 2-Manifold Corollary 187.22 we proved the following:

(1) If $n \geq 2$, then every class in $\mathrm{H}_1(M, A)$ can be represented by a submanifold.
(2) If $n \geq 4$, then every class in $\mathrm{H}_2(M, A)$ can be represented by a connected submanifold.

Instead of moving on to the next highest dimension, we now actually consider homology classes of *codimension one*. More precisely, we want to prove the following proposition:

Proposition 234.6. (Codimension-One Submanifold Existence Proposition) Let M be a compact orientable n-dimensional smooth manifold and let $A \subseteq \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A)$ can be represented by a submanifold.

Remark.

(1) In the Prescribed Boundary–Proposition 234.11 we will prove a refinement of the Codimension-One Submanifold Existence Proposition 234.6.
(2) In the Codimension-Two Submanifold Existence Proposition **??** we will prove the codimension-two analog of the Codimension-One Submanifold Existence Proposition 234.6. Furthermore in the $\mathbb{F}_2$-Codimension-One Submanifold Existence Proposition **??** we will prove an "$\mathbb{F}_2$-coefficient"-analog of the Codimension-One Submanifold Existence Proposition 234.6.
(3) The proof of the Tube Equivalence Theorem 291.4 shows that any two submanifolds representing a given codimension-one homology class are tube equivalent in the sense of Definition 291.3. □

The proof of the Codimension-One Submanifold Existence Proposition 234.6 rests heavily on the shoulders of the following lemma, which is also of independent interest:

Lemma 234.7. (Preimage-of-Point-in-S^1-Lemma) Let M be a compact orientable n-dimensional smooth manifold, let $f\colon M \to S^1$ be a smooth map and let $z \in S^1$ be a

regular value. We write $N \coloneqq f^{-1}(z)$. By the Regular Value Theorem 32.5 (1), (2), (3) we know that N is a compact orientable proper $(n-1)$-dimensional smooth manifold of M. In this setting we have the following equality in $\mathrm{H}^1(M;\mathbb{Z})$:

$$\mathrm{PD}_M(\underbrace{[N]}_{\in \mathrm{H}_{n-1}(M,\partial M)}) = f^*(\underbrace{[S^1]^*}_{\in \mathrm{H}^1(S^1;\mathbb{Z})}).$$

Proof of the Codimension-One Submanifold Existence Proposition 234.6. Let M be a compact orientable n-dimensional smooth manifold. In the following we deal with the case that $A = \partial M$. The general case is proved in a similar fashion, but now one needs to use Exercise 231.5 (4) instead of the Maps-to-S^1-Proposition 233.4. We leave it to the reader to work out the details.

Now let $\sigma \in \mathrm{H}_{n-1}(M, \partial M)$. By the Maps-to-$S^1$-Proposition 233.4 or alternatively by the Map-out-of-CW Proposition 307.24 there exists a smooth map $f\colon M \to S^1$ such that $\mathrm{PD}_M(\sigma) = f^*([S^1]^*)$ where $[S^1]^* \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the dual fundamental class.

By Sard's Theorem 38.1 and the Full Measure-Density Proposition 37.4 (2) there exists a regular value $z \in S^1$ of $f\colon M \to S^1$ and also of $f|_{\partial M}\colon \partial M \to S^1$. We write $N \coloneqq f^{-1}(z)$. By the Regular Value Theorem 32.5 (1), (2), (3) we know that N is a compact orientable proper $(n-1)$-dimensional smooth manifold. Finally note that

$$\mathrm{PD}_M([N]) \underset{\substack{\uparrow \\ \text{by the Preimage-of-Point-in-}S^1 \\ \text{Lemma 234.7}}}{=} f^*([S^1]^*) \underset{\substack{\uparrow \\ \text{by the above}}}{=} \mathrm{PD}_M(\sigma).$$

This implies that $[N] = \sigma \in \mathrm{H}_{n-1}(M, \partial M)$. ∎

Proof of the Preimage-of-Point-in-S^1-Lemma 234.7. Let M be a compact orientable n-dimensional smooth manifold, let $f\colon M \to S^1$ be a smooth map and let $z \in S^1$ be a regular value. We write $N \coloneqq f^{-1}(z)$. We need to show that

$$\mathrm{PD}_M([N]) = f^*([S^1]^*).$$

We give two proofs. One, which makes use of the Thom Class Proposition 244.17, and another one which proves the desired equality "by hand".

(i) Note that we have the following equalities in $\mathrm{H}^1(M;\mathbb{Z})$:

$$\mathrm{PD}_M([N]) \underset{\substack{\uparrow \\ \text{by the Submanifold Preimage} \\ \text{Thom Class Proposition 244.17}}}{=} f^*(\mathrm{PD}_{S^1}([\{1\}])) \underset{\substack{\uparrow \\ \text{see page 4029}}}{=} f^*([S^1]^*).$$

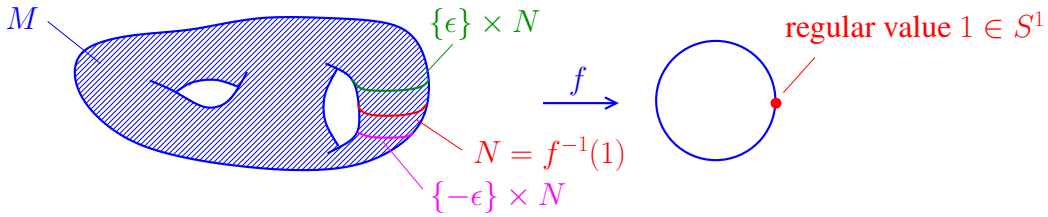
Thus $\mathrm{PD}_M([N]) = f^*([S^1]^*) = \mathrm{PD}_M(\sigma)$. Since PD_M is an isomorphism this implies that we have $[N] = \sigma \in \mathrm{H}_{n-1}(M, \partial M)$.

(ii) We turn to the "by hand" proof of the equality $\mathrm{PD}_M([N]) = f^*([S^1]^*)$. To simplify the notation a little bit we now assume that $z = 1$. Next note that we know by the Tubular-Regular Value Theorem 58.13 (4) (see also the discussion on page 1328) that there exists an $\epsilon \in (0, \pi)$ and a tubular neighborhood $[-\epsilon, \epsilon] \times N$ of the submanifold

N such that the following diagram commutes:

$$\begin{array}{ccc} [-\epsilon,\epsilon]\times N & \hookrightarrow & M \\ \downarrow{\scriptstyle (s,x)\mapsto s} & & \downarrow{\scriptstyle f} \\ [-\epsilon,\epsilon] & \xrightarrow{s\mapsto \exp(2\pi i s)} & S^1. \end{array}$$

We pick the orientation for N such that the inclusion map $[-\epsilon,\epsilon]\times N \to M$ is orientation-preserving. We need to show that $\mathrm{PD}_M([N]) = f^*([S^1]^*)$. By Definition 210.2 it suffices to show that $[N] = f^*([S^1]^*) \cap [M] \in \mathrm{H}_{n-1}(M, \partial M)$.



We introduce the following notation:

(1) We write $I = [-\epsilon,\epsilon]$ and $\mathring{I} = (-\epsilon,\epsilon)$.
(2) We denote by $i\colon I\times N \to M$ the inclusion map and we denote by i also all maps of pairs where the first map is given by the inclusion map.
(3) We denote by q all maps of pairs of topological spaces where the first map is just the identity on M.
(4) Finally we denote by $p\colon I\times N \to I$ the projection.

We consider the following diagram:[2527]

$$\begin{array}{ccccc}
\mathrm{H}^1(M;\mathbb{Z}) & \times & \mathrm{H}_n(M,\partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-1}(M,\partial M) \\
\uparrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*}\,\mathrm{id} \\
\mathrm{H}^1(M, M\setminus(\mathring{I}\times N);\mathbb{Z}) & \times & \mathrm{H}_n(M, M\setminus(\mathring{I}\times N)\cup\partial M) & \xrightarrow{\cap} & \mathrm{H}_{n-1}(M,\partial M) \\
\downarrow{\scriptstyle i^*}\,\cong & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} \\
\mathrm{H}^1(I\times N,\partial I\times N;\mathbb{Z}) & \times & \mathrm{H}_n(I\times N,\partial(I\times N)) & \xrightarrow{\cap} & \mathrm{H}_{n-1}(I\times N, I\times\partial N) \\
\uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle -\times[N]} & & \uparrow{\scriptstyle -\times[N]} \\
\mathrm{H}^1(I,\partial I;\mathbb{Z}) & \times & \mathrm{H}_1(I,\partial I) & \xrightarrow{\cap} & \mathrm{H}_0(I).
\end{array}$$

The claim follows from the following subclaim:

Subclaim.
(1) The diagram commutes.[2528]
(2) The map $i^*\colon \mathrm{H}^1(M, M\setminus(\mathring{I}\times N);\mathbb{Z}) \to \mathrm{H}^1(I\times N,\partial I\times N)$ is an isomorphism.
(3) We have $(q^*\circ(i^*)^{-1}\circ p^*)([I]^*) = f^*([S^1]^*)$.
(4) We have $i_*([I]\times[N]) = q_*([M])$.

Proof of subclaim.

[2527]For completeness' sake we recall that the relative cap product is only defined for excisive triads, see Definition 204.7. So one should think for a second about why the second cap product is in fact defined.
[2528]Note that here for the first three rows we use the convention of Convention 204.17 for what "commutativity" actually means.

(1) The top two squares commute by the Cap Product-Naturality Lemma 206.13 applied to the two maps

$$q\colon (M,\varnothing,\partial M) \;\to\; (M, M\setminus(\mathring{I}\times N),\partial M)$$

and

$$i\colon (I\times N,\partial I\times N, I\times\partial N) \;\to\; (M, M\setminus(\mathring{I}\times N),\partial M).$$

Finally regarding the bottom square, note that

$$p^*([I]^*)\cap([I]\times[N]) \underset{\uparrow}{=} p^*([I]^*)\cap([I\times N]) \underset{\uparrow}{=} [\{0\}\times N] = (\overbrace{[I]^*\cap[I]}^{=[\{0\}]})\times[N].$$

by the Fundamental Class-of-Product Proposition 194.17 — by the Fundamental Class-Product Proposition 207.2 (2)

Since $[I]^*\in \mathrm{H}^1(I,\partial I;\mathbb{Z})$ and $[I]\in \mathrm{H}_1(I,\partial I)$ are generators it follows from this calculation that the bottom square commutes.

(2) It follows from the Excision-in-H*-Theorem 200.1 (2) that i^* is indeed an isomorphism. We leave it to the reader to verify the details.

(3) We identify the interval I with its image in S^1 under the orientation-preserving map defined by $s\mapsto \exp(2\pi\,\mathrm{i}s)$. We denote by $j\colon I\to S^1$ and $j\colon (I,\partial I)\to(S^1,S^1\setminus\mathring{I})$ the inclusion maps. Furthermore we denote by $r\colon (S^1,\varnothing)\to(S^1,S^1\setminus\mathring{I})$ the obvious map of pairs of topological spaces. We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^1(S^1;\mathbb{Z}) & \xrightarrow{\quad f^*\quad} & \mathrm{H}^1(M;\mathbb{Z}) \\
{\scriptstyle r^*}\uparrow & & \uparrow{\scriptstyle q^*} \\
\mathrm{H}^1(S^1,S^1\setminus\mathring{I};\mathbb{Z}) & \xrightarrow{\quad f^*\quad} & \mathrm{H}^1(M,M\setminus(\mathring{I}\times N);\mathbb{Z}) \\
{\scriptstyle\cong}\downarrow{\scriptstyle j^*} & & {\scriptstyle i^*}\downarrow{\scriptstyle\cong} \\
\mathrm{H}^1(I,\partial I;\mathbb{Z}) & \xrightarrow{\quad f^*=p^*\quad} & \mathrm{H}^1(I\times N,\partial I\times N;\mathbb{Z}).
\end{array}$$

We make the following observations and clarifications:

- The map j^* is an isomorphism by excision for cohomology groups.
- On page 4507 we showed that $f|_{I\times N}=p$. Thus it follows that in the bottom horizontal map we have $f^*=p^*$.
- It follows from the functoriality of cohomology groups both squares commute.
- We point out that it follows immediately from the Codimension Zero-Fundamental Class Lemma 194.10 that $r^*((j^*)^{-1}([I]^*))=[S^1]^*$.

The desired equality $(q^*\circ(i^*)^{-1}\circ p^*)([I]^*)=f^*([S^1]^*)$ follows almost immediately from the above discussion.

(4) This statement follows immediately from the Codimension Zero-Fundamental Class Lemma 194.10. ■

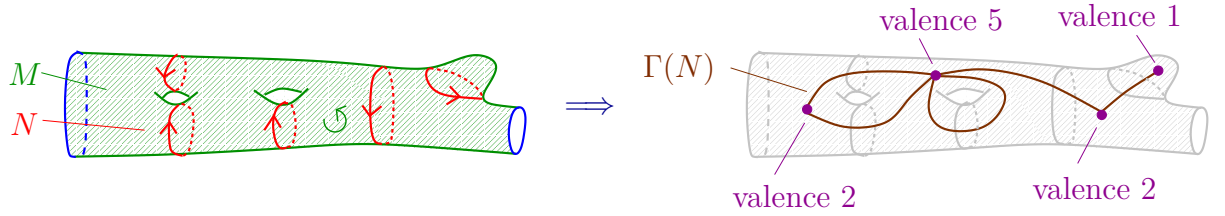
In the Represent-by-Connected Submanifold Proposition 187.21 we showed that if we can represent a homology class of codimension ≥ 2 by a submanifold, then we can also represent it by a connected submanifold. Our goal is to give a criterion when we can represent codimension-one homology classes by connected submanifolds:

Proposition 234.8. (Connected Codimension-One Representative Proposition) Let M be a compact orientable connected n-dimensional smooth manifold. Given any $\varphi \in \mathrm{H}_{n-1}(M, \partial M)$ the following two statements are equivalent:

(1) φ is primitive.
(2) We have $\varphi \neq 0$ and there exists a closed *connected* $(n-1)$-dimensional smooth submanifold $N \subseteq M$ with $[N] = \varphi$.

Sketch of proof. Let $\varphi \in \mathrm{H}_{n-1}(M, \partial M)$.

(1)⇒(2). First note that it follows from the Codimension-One Submanifold Existence Proposition 234.6 that φ is represented by some submanifold N. As in Definition 55.6 we cut M along N, i.e. we consider $M \setminus\setminus N$. We denote by $\Gamma(N)$ the graph where the vertices are the components of $M \setminus\setminus N$ and where the edges are the components of N.

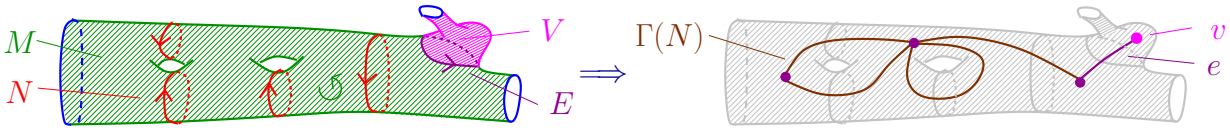


Claim 1. We can find a submanifold N of M with $[N] = \varphi$ such that $\Gamma(N)$ has only vertices of valence 2.

Proof. Given a submanifold N we write $v(N)$ for the number of vertices of valence $\neq 2$. It suffices to prove the following claim: If N is a submanifold with $[N] = \varphi$ such that $v(N) \neq 0$, then there exists a submanifold $\widetilde{N}$ with $[\widetilde{N}] = \varphi$ such that $b_0(\widetilde{N}) < b_0(N)$.

So let N be a submanifold that represents φ with $v(N) \neq 0$.

- If $\Gamma(N)$ has a vertex v of valence 1 (which corresponds to a component V of $M \setminus\setminus N$) and this vertex corresponds to a unique edge e (which in turn corresponds to a component E of N).



We consider the following simple minded diagram:

$$\begin{array}{ccc} \mathbb{Z} \cdot [E] = \mathrm{H}_{n-1}(E) & \longrightarrow & \mathrm{H}_{n-1}(M, \partial M) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}_{n-1}(V, V \cap \partial M). & \end{array}$$

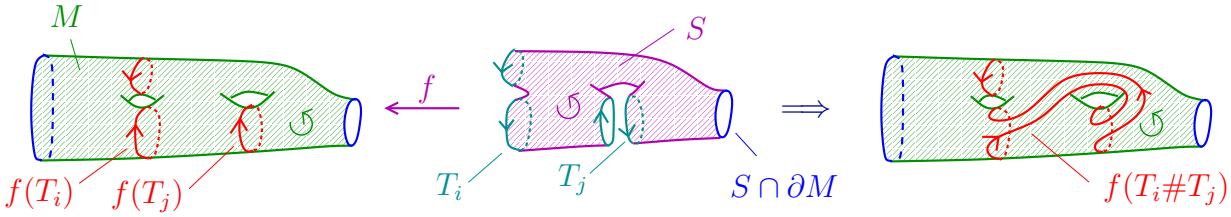
Here all maps are induced by inclusions, thus the diagram commutes. It follows from the Topological Kernel-of-Boundary Inclusion Corollary 194.14 (2) (or alternatively the Kernel-of-Boundary Inclusion Corollary 185.20 (1) together with Exercise 185.3) that the left diagonal map is the zero map. This shows that the component E represents the trivial element in $\mathrm{H}_{n-1}(M, \partial M)$. Thus we can drop this component from N and we managed to lower the number of components of N by one.

- Now suppose that $\Gamma(N)$ has a vertex of valence > 2. Let S be the corresponding component of $M \setminus\setminus N$ and let $f \colon S \to M$ be the natural map. We equip S with the

orientation coming from M. We denote by $T_1, \dots, T_l$ the components of $\partial S \setminus (S \cap \partial M)$ and we equip them with the boundary orientation that we introduced in Definition 31.37.

Note that for each $i \in \{1, \dots, l\}$ the map $f\colon T_i \to N$, with the above orientation of T_i and the given orientation of N, is either orientation-preserving or orientation-reversing. Since $l \geq 3$ we see that there are exist $i \neq j \in \{1, \dots, l\}$ such that $f\colon T_i \to N$ and $f\colon T_j \to N$ are either both orientation-preserving or orientation-reversing.

Since S is connected we can form, similar to the discussion in the Internal Connected Sum Construction 187.28, the internal connected sum $T_i \# T_j$ of the *oriented* smooth manifolds T_i and T_j.



Next note that it follows from the Recombine Fundamental Class Proposition 187.27 that $[T_i \# T_j] = [T_i] + [T_j] \in \mathrm{H}_n(S, S \cap \partial M)$. By replacing $f(T_i)$ and $f(T_j)$ by $f(T_i \# T_j)$ we have thus shown once again that we can reduce the number of components of N. ⊞

Claim 1 leads us straight to Claim 2:

Claim 2. If N is a submanifold such that $\Gamma(N)$ has only vertices of valence 2, then all components of N represent (up to sign) the same homology class in $\mathrm{H}_{n-1}(M, \partial M)$.



Proof. First note that it follows from the fact that M is connected that Γ is connected. Thus it suffices to show that if we have two components E_1 and E_2 of N such that the corresponding edges e_1 and e_2 have a vertex v in common, then $[E_1] = \pm[E_2]$. Let V be the component of $M \setminus\setminus N$ that corresponds to v.



As in Claim 1 we consider the following simple minded diagram:

$$\begin{array}{ccc} \mathbb{Z} \cdot [E_1] \oplus \mathbb{Z} \cdot [E_2] = \mathrm{H}_{n-1}(E_1 \cup E_2) & \longrightarrow & \mathrm{H}_{n-1}(M, \partial M) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}_{n-1}(V, V \cap \partial M). & \end{array}$$

It follows from the Topological Kernel-of-Boundary Inclusion Corollary 194.14 (2) (or alternatively the Kernel-of-Boundary Inclusion Corollary 185.20 (1) together with Exercise 185.3) that $[E_1 \cup E_2] = \pm[E_1] \pm [E_2]$ lies in the kernel of the left hand diagonal map. This shows that E_1 and E_2 represent, up to sign, the same homology class in $\mathrm{H}_{n-1}(M, \partial M)$. ⊞

Now we finally suppose that φ is primitive. It follows from Claim 1 and Claim 2 that we can represent φ by a submanifold with $[N] = \varphi$ such that all components $N_1, \dots, N_k$ represent the same homology class. It follows that for any component N_i of N there exists an $r_i \in \mathbb{Z}$ with $\varphi = r_i \cdot [N_i]$. But since φ is primitive we see that $r_i = \pm 1$, thus φ is represented by $\pm N_i$.

(2)⇒(1). Now we suppose that we have $\varphi \neq 0$ and that there exists a closed *connected* $(n-1)$-dimensional smooth submanifold $N \subseteq M$ with $[N] = \varphi \in \mathrm{H}_{n-1}(M, \partial M)$. It follows easily from $\varphi \neq 0$, the fact that N is connected and the Kernel-of-Boundary Inclusion Corollary 185.20 that $M \setminus N$ is connected. It follows from the Non-Separating Codimension-One Submanifold Proposition 234.3 that $[N] = \varphi$ is primitive. ■

234.3. **The basic Thom-Pontryagin Theory.** In the proof of the Codimension-One Submanifold Existence Proposition 234.6 we saw that a map from a smooth manifold N to S^1 gives rise to a codimension-one submanifold. In this section we will go in the opposite direction.

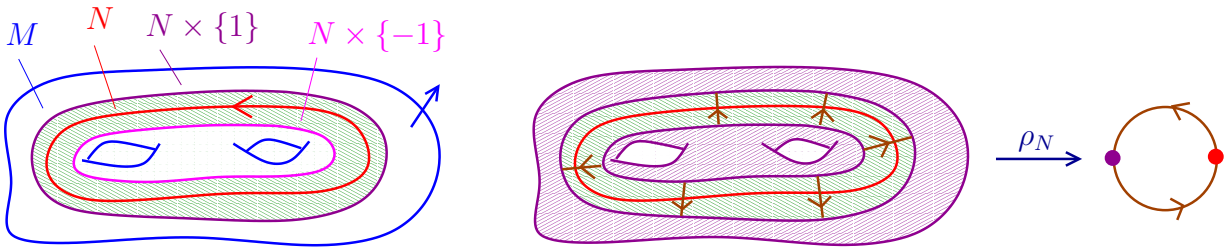
Definition 234.9. Let M be a compact oriented smooth manifold and let F be a proper oriented codimension-one submanifold of M. We introduce two objects:

- By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) together with the Tubular Map Lemma 58.3 we know that there exists an orientation-preserving tubular map $\tau\colon [-1,1] \times F \to M$.
- Using the Smooth Transition Function Lemma 24.9 one can easily find a smooth monotonously increasing map $\eta\colon [-1,1] \to [-1,1]$ with $\eta(t) = -1$ for $t \in [-1, -\frac{3}{4}]$, $\eta(t) = 2t$ for $t \in [-\frac{1}{4}, \frac{1}{4}]$ and $\eta(t) = 1$ for $t \in [\frac{3}{4}, 1]$.

We refer to the map

$$\begin{array}{rccl} \rho_F\colon & M & \to & S^1 \\ & P & \mapsto & \begin{cases} \exp(\pi \mathrm{i} \eta(t)), & \text{if } P = \tau(t,Q) \in [-1,1] \times F, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

as the Thom-Pontryagin map.[2529] Using the Smooth Pasting Proposition 25.19 one can verify that ρ_F is smooth. One can easily verify that $1 \in S^1$ is a regular value of ρ_F and that $\rho_F^{-1}(1) = F$.



Remark. The Thom-Pontryagin map ρ_N that we had just constructed depends on the choice of a tubular neighborhood. But it follows from the Tubular Neighborhood–Uniqueness Theorem 57.12, together with the Linear $\overline{B}^k$-Bundle-Isomorphism Lemma 58.6 and the

[2529] Lev Pontryagin (1908-1988) was a Soviet topologist, he made many major contributions to topology, despite having lost his eyesight in an accident when he was 14.

fact that SO(1) consists of a single point, that any other choice of a tubular neighborhood $[-1,1] \times N$ leads to a continuous map $M \to S^1$ that is homotopic to the original map. □

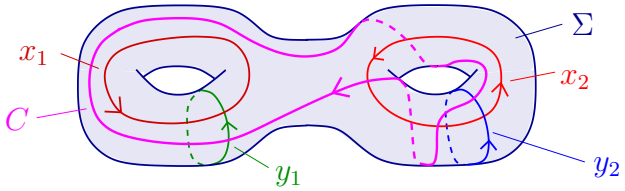
Lemma 234.10. (Codimension One–H^1-Lemma) Let M be a compact oriented n-dimensional smooth manifold and let N be a proper oriented proper $(n-1)$-dimensional submanifold. We have the following equality in $\mathrm{H}^1(M;\mathbb{Z})$:

$$\rho_N^*([S^1]^*) \;=\; \mathrm{PD}_M([N]).$$

Proof. Let M be a compact oriented n-dimensional smooth manifold and let N be a proper oriented proper $(n-1)$-dimensional submanifold. We consider the corresponding Thom-Pontryagin map $\rho_N\colon M \to S^1$. As we pointed out above, $1 \in S^1$ is a regular value of ρ_F and $\rho_F^{-1}(1) = F$. The desired conclusion thus follows from the Preimage-of-Point-in-S^1-Lemma 234.7. ■

Example. We return to the example that we had discussed on page 4492. More precisely, let Σ be the surface of genus 2. The oriented smoothly embedded closed curves x_1, y_1, x_2, y_2 shown in the figure below represent a basis for $\mathrm{H}_1(\Sigma)$. We consider the epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By the Maps-to-S^1-Proposition 233.4 we know that ϕ can be realized by some smooth map $g\colon \Sigma \to S^1$. Now we will construct such a map explicitly.

We consider the oriented smoothly embedded closed curve C that is illustrated in the figure below.



We claim that the corresponding map $\rho_C\colon \Sigma \to S^1$ has the desired property. Put differently, we claim that $\rho_{C*} = \phi \in \mathrm{Hom}(\mathrm{H}_1(N), \mathbb{Z})$. It suffices to show that the two homomorphisms agree on the basis $[x_1], [y_1], [x_2], [y_2]$ of $\mathrm{H}_1(\Sigma)$. For example we have the following equalities:

$$\begin{array}{lcccccccl} \rho_{C*}([x_2]) & = & \langle [S^1]^*, \rho_{C*}([x_2]) \rangle & \overset{\text{by the Kronecker Pairing Natural Lemma 199.9 (1)}}{=} & \langle \rho_C^*([S^1]^*), [x_2] \rangle & & & & \\ & & & \overset{\text{by the Codimension One-H}^1 \text{ Lemma 234.10}}{=} & \langle \mathrm{PD}_M([C]), [x_2] \rangle & \overset{\text{by the Kronecker Pairing Intersection Number Proposition 234.1}}{=} & x_2 \cdot C & \overset{\text{see the figure above}}{=} & -2 \;=\; \phi(x_2). \end{array}$$

It is straightforward to verify that the same equalities also hold for all the other basis elements. □

Now we have the tools to prove the following refinement of the Codimension-One Submanifold Existence Proposition 234.6 and the Codimension-Two Submanifold Existence Proposition **??**.

Proposition 234.11. (Prescribed Boundary–Proposition) Let M be a compact orientable n-dimensional smooth manifold and let $\phi \in \mathrm{H}_{n-1}(M, \partial M)$. We denote by $\partial_{n-1}(\phi) \in \mathrm{H}_{n-2}(\partial M)$ the image of ϕ under the connecting homomorphism in the long exact sequence of the pair $(M, \partial M)$. If $F \subseteq \partial M$ is a submanifold that represents $\partial\phi \in \mathrm{H}_{n-2}(A)$, then ϕ can be represented by a submanifold N of M with $\partial N = F$.

Proof. Let M be a compact oriented n-dimensional smooth manifold. Suppose we are given a homology class $\phi \in \mathrm{H}_{n-1}(M, \partial M)$ and a submanifold $F \subseteq \partial M$ that represents $\partial\phi \in \mathrm{H}_{n-2}(\partial M)$. As above we pick a product tubular neighborhood $[-1,1] \times F$ in ∂M such that the product orientation of $[-1,1] \times F$ matches the orientation of ∂M. Next we consider the corresponding Thom-Pontryagin map

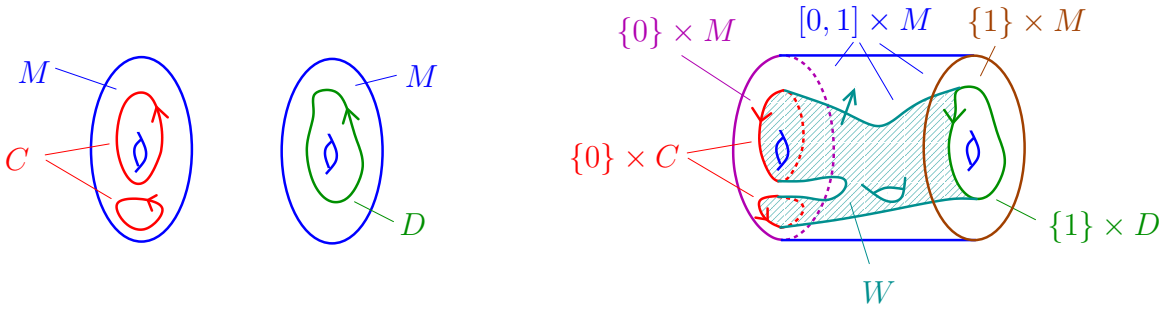
$$\begin{aligned} \rho_F \colon \partial M &\to S^1 \\ P &\mapsto \begin{cases} \exp(\pi \mathrm{i} t), & \text{if } P = (t, Q) \in [-1,1] \times F, \\ -1, & \text{otherwise} \end{cases} \end{aligned}$$

Note that ρ_F is smooth on the closed subset $[-\frac{1}{2}, \frac{1}{2}] \times F$. It follows quite easily from the Whitney Approximation Theorem 36.1 (5) that ρ_F is homotopic rel $[-\frac{1}{2}, \frac{1}{2}] \times F$ to a smooth map $f \colon \partial M \to S^1$ which is "close to ρ_F" in the sense that it has the property that for $P \notin [-\frac{1}{2}, \frac{1}{2}] \times F$ we have $f(P) \notin \{\exp(\pi \mathrm{i}) \mid t \in [-\frac{1}{4}, \frac{1}{4}]\}$. Now note that it follows from the Poincaré Duality–Boundary Proposition 211.14 together with the Maps-to-S^1-Proposition 233.4 (1b) that there exists a smooth map $g \colon M \to S^1$ that represents $\mathrm{PD}_M(\phi) \in \mathrm{H}^1(M, \partial M; \mathbb{Z})$ and that agrees with f on ∂M.

By Sard's Theorem 38.1 and the Full Measure-Density Proposition 37.4 (2) there exists an $s \in (-\frac{1}{4}, \frac{1}{4})$ such that $\exp(\pi \mathrm{i} s)$ is a regular value of g. We write $N \coloneqq g^{-1}(z)$. By the proof of the Codimension-One Submanifold Existence Proposition 234.6 (1) we know that N is a smooth proper submanifold that represents ϕ. Note that by construction we have $\partial N = N \cap \partial M = \{s\} \times F$. Using a collar neighborhood of ∂M it is straightforward to find an isotopy that arranges that $\partial N = \{0\} \times F = F$. We leave it to the reader to fill in the few remaining details. ∎

Later on we will make use of the following useful corollary.

Corollary 234.12. (Cobounding Submanifolds Corollary) Let M be a closed orientable n-dimensional smooth manifold and let C and D be two closed oriented $(n-1)$-dimensional submanifolds. If $[C] = [D] \in \mathrm{H}_{n-1}(M)$, then there exists a compact oriented proper n-dimensional submanifold $W \subseteq [0,1] \times M$ such that $\partial W = (\{0\} \times C) \cup (\{1\} \times -D)$.



Proof. We consider the following diagram:

$$\begin{array}{ccccccc} \dots \longrightarrow \mathrm{H}_n([0,1]\times M) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}((\{0\}\times M)\cup(\{1\}\times M)) & \xrightarrow{i_*} & \mathrm{H}_{n-1}([0,1]\times M) & \longrightarrow \dots \\ & & & & \Big\downarrow{\scriptstyle p_*}\cong & \\ & & & & \mathrm{H}_n(M) & \end{array}$$

Here the horizontal sequence is the long exact sequence in homology corresponding to the pair $([0,1]\times M, (\{0\}\times M)\cup(\{1\}\times M))$. Furthermore $p\colon [0,1]\times M \to M$ denotes the obvious projection. Our hypothesis that $[C]=[D]\in \mathrm{H}_n(M)$ implies immediately that $p_*([\{0\}\times C]) = p_*([\{1\}\times D])$. Since p_* is an isomorphism and since the sequence is exact we see that there exists a $\varphi\in \mathrm{H}_n([0,1]\times M)$ with $\partial_n(\varphi) = [\{0\}\times C] - [\{1\}\times D]$. Finally note that it follows immediately from the Prescribed Boundary–Proposition 234.11 that there exists in particular a compact oriented proper n-dimensional submanifold W of $[0,1]\times M$ with $[W]=\varphi$ and with $\partial W = (\{0\}\times C)\cup(\{1\}\times -D)$. ∎

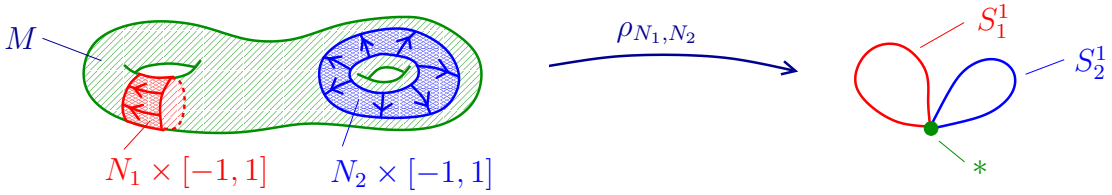
In the remainder of this chapter we study an amusing generalization of the above Thom-Pontryagin map.

Definition 234.13. Let M be a compact oriented n-dimensional smooth manifold. Furthermore let $N_1,\dots,N_k$ be disjoint proper oriented $(n-1)$-dimensional submanifolds. As in Definition 234.9 we pick a product tubular neighborhood of the form $[-1,1]\times(N_1\cup\dots\cup N_k)$. We consider the wedge $\bigvee_{j=1}^{k} S^1_j$ given by identifying the points $-1\in S^1_j$ to a single point $*$.

Again we refer to the map

$$\begin{array}{rccl} \rho_{N_1,\dots,N_k}\colon M & \to & \bigvee_{j=1}^{k} S^1_j & \\ P & \mapsto & \begin{cases} \exp(\pi\,\mathrm{i}\,t)\in S^1_j, & \text{if } P=(t,Q)\in[-1,1]\times N_j, \\ *, & \text{otherwise} \end{cases} & \end{array}$$

as the Thom-Pontryagin map.



Before we continue we recall that in the Wedge-of-Circles–π_1-Lemma 99.11 we gave an identification

$$\langle t_1,\dots,t_k\rangle \;=_{\mathrm{i}}\; \pi_1\Big(\bigvee_{j=1}^{k} S^1_j, *\Big)$$

where each t_j is represented by a loop that goes once around the j-th circle S^1_j in a positive direction.

Proposition 234.14. (Epimorphism-onto-Free Group Proposition) Let M be a compact oriented connected n-dimensional smooth manifold and let $N_1,\dots,N_k$ be disjoint proper oriented non-empty $(n-1)$-dimensional submanifolds. If $M\setminus(F_1\cup\dots\cup F_k)$ is

connected, then for any $x_0 \in M$ the map

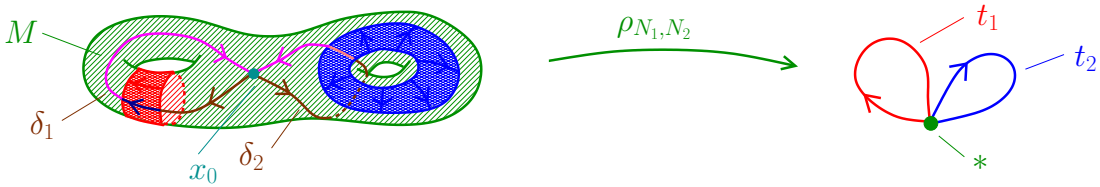
$$\rho_{N_1,\dots,N_k} \colon \pi_1(M, x_0) \;\to\; \pi_1\Big(\bigvee_{j=1}^{k} S^1_j, *\Big) \;=\; \langle t_1, \dots, t_k \rangle$$

is an epimorphism.

Proof. We write $N = N_1 \cup \dots \cup N_k$ and $\rho = \rho_{N_1,\dots,N_k}$. We pick a product tubular neighborhood of the form $[-1,1] \times N$. It follows from our hypothesis that $M \setminus N$ is connected, together with the Basics-of-Tubular Neighborhood Lemma 57.10, the Homotopy-Equivalence π_0-Lemma 19.17 and the Manifold–Local Properties Lemma 23.8 (3) that $M \setminus (-1,1) \times N$ is also connected. Since M is connected it follows from the Change-of-Base Point Proposition 90.4 that it suffices to consider a base point x_0 that lies in $M \setminus (-1,1) \times N$.

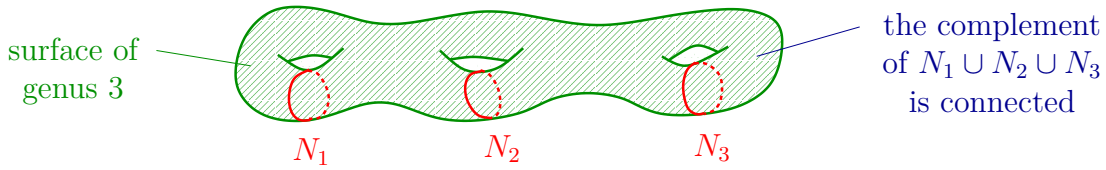
For $j = 1, \dots, k$ we do the following:

(1) We pick a point $P_i \in N_i$.
(2) Since $M \setminus (-1,1) \times N$ is connected we can pick a path α_j from x_0 to $(-1, P_j)$ and we can pick a path γ_j from $(1, P_j)$ to x_0.
(3) We denote by β_j the "horizontal" path in $[-1,1] \times \mathbb{N}_j$ from $(-1, P_j)$ to $(1, P_j)$.
(4) We write $\delta_j \coloneqq \alpha_j * \beta_j * \gamma_j$.



It follows easily from the definitions that for $j = 1, \dots, k$ we have $\rho_*([\delta_j]) = t_j$. We have thus shown that the generating set $\{t_1, \dots, t_k\}$ lies in the image of ρ_*. In other words, ρ_* is indeed an epimorphism. ■

Example. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. As should be pretty clear from considering the figure below, the surface Σ_g contains g disjoint smoothly embedded closed curves $N_1, \dots, N_g$ such that the complement is connected.



Thus it follows from the Epimorphism-onto-Free Group Proposition 234.14 that there exists an epimorphism

$$\pi_1(\Sigma_g) \;\to\; \langle x_1, \dots, x_g \rangle.$$

In fact, as we will see in Exercise 234.2, it is not difficult to show the existence of such an epimorphism on a more algebraic level. □

The previous example raises the following thought-provoking question.

Question 234.15. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.

(1) What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint smoothly embedded closed curves $F_1, \dots, F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected?

(2) What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of k generators?

In summary, as of right now we know the following:

$$\begin{array}{ccccccc} & & & & \text{by the Epimorphism-onto-Free Group Proposition 234.14} & & \\ & & & & \downarrow & & \\ g & \underset{\uparrow}{\leq} & k(g) & \leq & l(g) & \underset{\uparrow}{\leq} & 2g. \\ & \text{by the above example} & & & \text{by the Abelianization-Epimorphism Proposition 98.17} & & \end{array}$$

234.4. **Cyclic coverings of manifolds.** Our goal in this section is to give explicit constructions of cyclic coverings of manifolds. To simplify the language will use the following conventions throughout this section.

Convention 234.16.

(1) We will not distinguish between two coverings $p\colon X \to B$ and $q\colon T \to B$ that are isomorphic in the sense of Definition 127.16.

(2) We will use the Change-of-Base Point Proposition 88.14 (1) as an excuse to suppress all base points from our notation.[2530]

The following proposition combines several results on coverings that we proved a while ago.

Proposition 234.17. (Covering–Abelian Group Proposition) Let M be a connected smooth manifold and let $\varphi\colon \pi_1(M) \to G$ be an epimorphism onto an abelian group. There exists a covering $p\colon \widetilde{M} \to M$ such that $p_*(\pi_1(\widetilde{M})) = \ker(\varphi)$ and this covering is unique up to isomorphism, in the sense of Definition 127.16. Furthermore this covering has the following properties:

(1) The covering is regular and the deck transformation group is isomorphic to G.

(2) The index of the covering is precisely $|G|$.

Proof. The proposition follows immediately from the Covering Existence Theorem 127.10 and the Deck Transformation–Normal Subgroup Proposition 128.11 together with the Covering Unique Existence Theorem 127.17 and the Regular Covering-Characterization Proposition 128.13. ■

We sort-of introduced Definition 128.1.

Definition 234.18. We say that a covering $p\colon X \to Y$ is (infinite) cyclic if p is a regular covering and if the deck transformation group is an (infinite) cyclic group.

Let M be a smooth manifold. Given any primitive $\phi \in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ and given $n \in \mathbb{N}$ we know by the Covering–Abelian Group Proposition 234.17 that there exist corresponding (infinite) cyclic covers of M. Our goal is to give a very explicit constructions of these (infinite) cyclic covers for many smooth manifolds.

In the remainder of this section we will use the following notation:

[2530]More precisely, using the Change-of-Base Point Proposition 88.14 it is not difficult to show that for any topological space X, any abelian group G and any two paths $p, q\colon [0,1] \to X$ from a point x_0 to a point x_1 we have $p_* = q_*\colon \mathrm{Hom}(\pi_1(X,x_0),G) \to \mathrm{Hom}(\pi_1(X,x_1),G)$. In particular, given a path-connected topological space X and $x_0, x_1 \in X$ the groups $\mathrm{Hom}(\pi_1(X,x_0),G)$ and $\mathrm{Hom}(\pi_1(X,x_0),G)$ are canonically isomorphic.

Notation 234.19. Let M be a connected smooth manifold.

(1) We use the natural isomorphisms from the H^1-Hom-Proposition 201.24 to make the identification $\mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M),\mathbb{Z})$. It follows almost immediately from the definitions that under this identification the primitive elements in $\mathrm{H}^1(M;\mathbb{Z})$ correspond to the epimorphisms $\pi_1(M) \to \mathbb{Z}$.

(2) Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ again be a primitive element. Furthermore let $n \in \mathbb{N}$. In the following we denote by $\phi_n \colon \pi_1(M) \to \mathbb{Z}/n\mathbb{Z}$ the epimorphism that is given by the composition $\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

At this stage it is worth recalling the following definitions from Definition 55.6.

Definition 234.20. Let M be an oriented smooth manifold and let $F \subseteq M$ be a compact oriented proper submanifold of codimension one. We pick an orientation-preserving tubular map $\Phi \colon [-1,1] \times F \to M$. We define

$$M \setminus\setminus F \;:=\; (M \setminus F) \sqcup (F \times \{\pm 0\})$$

with the topology that is generated by the following sets:

(1) the usual open subsets of $M \setminus F$,

(2) the sets of the form $\Phi(U \times (0,\epsilon)) \cup (U \times \{+0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$,

(3) the sets of the form $\Phi(U \times (-\epsilon,0)) \cup (U \times \{-0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$.

We refer to $M \setminus\setminus F$ as M cut along F. This definition is illustrated in the figure below. We also introduce the following extra notation:

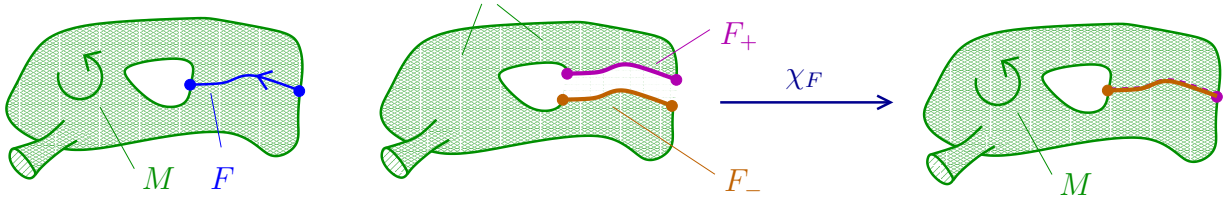
(4) We denote by $F_\pm$ the images of $F \times \{\pm 0\}$ in $M \setminus\setminus F$. Furthermore we denote by $i_\pm \colon F_\pm \to M \setminus\setminus F$ the obvious inclusion maps.

(5) We denote by $\sim_F$ the equivalence relation on the set $M \setminus\setminus F$ that is generated by $(P,+0) \sim (P,-0)$ for $P \in F$.

(6) We denote by χ_F the map

$$\begin{array}{rccl} \chi_F \colon M \setminus\setminus F & \to & M & \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M \setminus F, \\ Q, & \text{if } P = (Q, \pm 0). \end{cases} \end{array}$$

slightly incorrect picture of M cut along F



Now we can give the explicit construction of the cyclic coverings.

Proposition 234.21. (Cyclic-Covering Proposition) Let M be a compact oriented connected smooth manifold. Furthermore let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ be a

primitive element. Next let $F \subseteq M$ be a compact oriented proper codimension-one submanifold[2531] [2532] that is dual to ϕ in the sense that $\mathrm{PD}_M([F]) = \phi$. In the following we denote by $i_\pm \colon F_\pm \to M \setminus\setminus F$ the two obvious inclusion maps.

(1) We write

$$\widetilde{M}_\infty \coloneqq \Big(\bigsqcup_{k\in\mathbb{Z}} (M \setminus\setminus F) \times \{k\} \Big) / \sim \qquad \begin{array}{c} \text{where } (i_-(P),k) \sim (i_+(P),k+1) \\ \text{for } P \in F \text{ and } k \in \mathbb{Z}. \end{array}$$

The map

$$\begin{array}{rcl} p \colon \widetilde{M}_\infty & \to & M \\ [(P,k)] & \mapsto & [\chi_F(P)] \end{array}$$

is the infinite cyclic covering corresponding to ϕ.

(2) Let $n \in \mathbb{N}$. We write

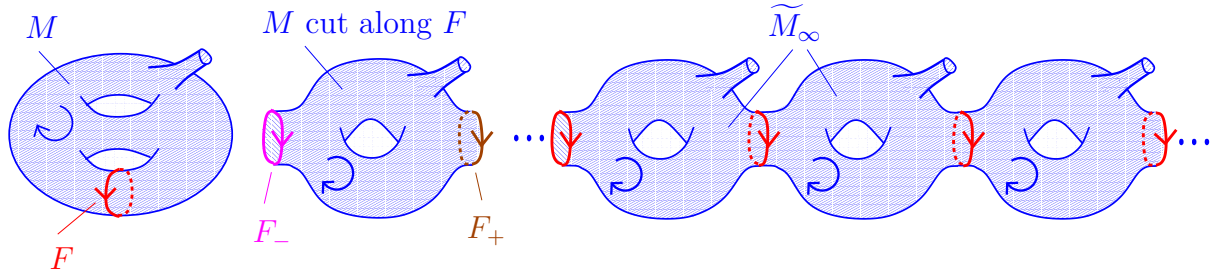
$$\widetilde{M}_n \coloneqq \Big(\bigsqcup_{k\in\mathbb{Z}/n\mathbb{Z}} (M \setminus\setminus F) \times \{k\} \Big) / \sim \qquad \begin{array}{c} \text{where } (i_-(P),k) \sim (i_+(P),k+1) \\ \text{for } P \in F \text{ and } k \in \mathbb{Z}/n\mathbb{Z}. \end{array}$$

The map

$$\begin{array}{rcl} p \colon \widetilde{M}_n & \to & M \\ [(P,k)] & \mapsto & [\chi_F(P)] \end{array}$$

is the finite cyclic covering corresponding to ϕ_n.

Example. We consider the 2-dimensional smooth manifold M together with the submanifold F that are shown in the figure below.



We illustrate the explicit construction of the infinite cyclic covering $\widetilde{M}_\infty$. The reader will not fail to spot the uncanny similarities with the Surface-of-Infinite Genus–Covering Lemma 105.6 and the figure on page 2057. □

Proof.

(1) By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we can pick a tubular map $\Phi \colon [-1,1] \times F \to M$ for F. It follows almost immediately from the definitions that the open subsets $M \setminus F$ and $\Phi((-1,1) \times F)$ are uniformly covered in the sense of Definition 91.8. Since these two open subsets cover all of M we see that $p \colon \widetilde{M}_\infty \to M$ is indeed a covering.

[2531] Recall that it follows from the Codimension-One Submanifold Existence Proposition 234.6 that such F always exists.

[2532] Note that we do not assume that F is connected.

In the following we will show that $p\colon \widetilde{M}_\infty \to M$ is indeed the covering corresponding to $\ker(\phi)$.[2533] We consider the following map:

$$\begin{array}{rcl} \alpha\colon M & \to & S^1 \\ P & \mapsto & \begin{cases} \exp(\pi\,\mathrm{i}t), & \text{if } P = \Phi(t,Q) \text{ with } t \in [-1,1] \text{ and } Q \in F, \\ -1, & \text{otherwise.} \end{cases} \end{array}$$

This is of course the Thom-Pontryagin map corresponding to the tubular map that we introduced in Definition 234.9. Let $q\colon \mathbb{R} \to S^1$ be the universal covering that is given by $t \mapsto \exp(\pi\,\mathrm{i}t)$. Furthermore let

$$\begin{array}{rcl} r\colon \alpha^*\mathbb{R} = \{(P,t) \in M \times \mathbb{R} \,|\, \alpha(P) = q(t)\} & \to & M \\ (P,t) & \mapsto & P \end{array}$$

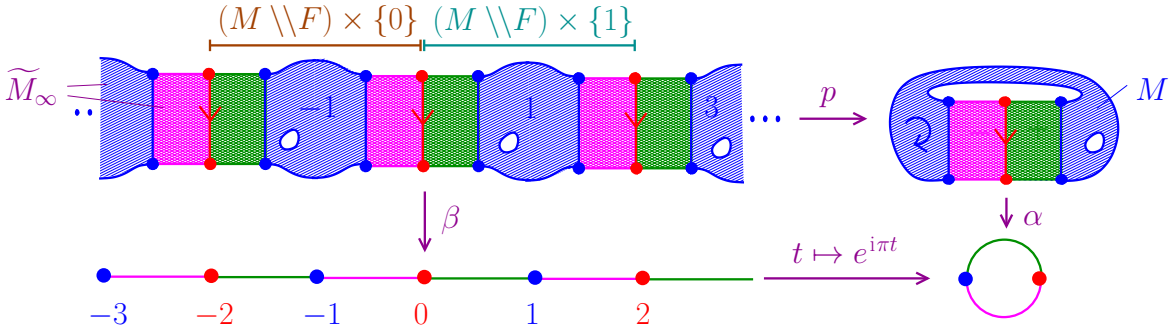
be the pullback of the covering q. Note that by the Pullback-Covering Lemma 92.12 (1) we know that $r\colon \alpha^*\mathbb{R} \to M$ is also a covering.

Claim. The covering $p\colon \widetilde{M}_\infty \to M$ is equivalent to the covering $r\colon \alpha^*\mathbb{R} \to M$.

Proof. The key to proving the claim is to consider the map

$$\begin{array}{rcl} \beta\colon \widetilde{M}_\infty & \to & \mathbb{R} \\ [P] & \mapsto & \begin{cases} 2i-1, & \text{if } P \in (M \setminus \Phi([-1,1] \times F)) \times \{i\}, \\ 2i+t, & \text{if } P \in \Phi(\{t\} \times F) \times \{i\} \text{ for some } t \in [-1,0), \\ 2i+t-2, & \text{if } P \in \Phi(\{t\} \times F) \times \{i\} \text{ for some } t \in (0,1], \\ 2i, & \text{if } P \in F_- \times \{i\} \text{ or } P \in F_+ \times \{i+1\}. \end{cases} \end{array}$$

Using the Pasting Proposition 4.7 (2) one easily verify that this map is continuous.



Next we consider the map

$$\begin{array}{rcl} \Theta\colon \widetilde{M}_\infty & \to & \alpha^*\mathbb{R} \\ P & \mapsto & (p(P), \beta(P)). \end{array}$$

It is straightforward to see that Θ is a bijection. We leave it to the reader to show that Θ is actually a homeomorphism. One can easily verify that $r \circ \Theta = p$. In summary we conclude that the two coverings q and r are indeed equivalent. ⊞

Now we see that

[2533] An alternative proof of this statement is given in [**Herr2019**, Lemma 1.31].

$$q_*(\pi_1(\widetilde{M}_\infty)) \underset{\text{since } q \text{ and } r \text{ are equivalent coverings}}{\uparrow} = r_*(\pi_1(\alpha^*\mathbb{R})) \overset{\text{by the Pullback-Covering Lemma 92.12 (3)}}{\downarrow} = \alpha_*^{-1}(\pi_1(\mathbb{R})) \underset{\text{since } \pi_1(\mathbb{R})=0}{\uparrow} = \ker(\alpha_*) \underset{\text{by the Codimension One-Lemma 234.10}}{\uparrow} = \ker(\phi).$$

We have thus shown that $q_*\colon \widetilde{M}_\infty \to M$ is indeed the covering corresponding to $\ker(\phi)$.

(2) The proof of this statement is very similar to the proof of (1). We just need to replace the covering $q\colon \mathbb{R} \to S^1$ by the covering $q\colon S^1 \to S^1$ given by $z \mapsto z^n$. We leave it to the reader to once again lessen the burden on the author by filling in the details. ■

234.5. **Appendix: Curves on 2-dimensional smooth manifolds.** We conclude this chapter with a discussion of curves on 2-dimensional smooth manifolds. The results of this section are nice-to-have, but they will not be used in the remainder of these notes. Thus it is safe to skip this section in the unlikely event that the reader is in a hurry.

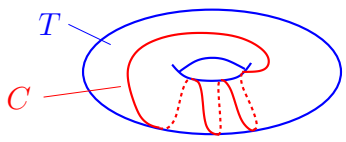
Let us now recall Definition 107.2:

Definition 234.22. A an (oriented) smoothly embedded closed curve C in a smooth manifold M is an (oriented) 1-dimensional submanifold of M that is diffeomorphic to S^1.

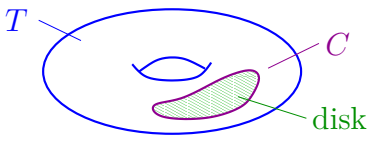
In the following our main focus will be oriented smoothly embedded closed curves on the torus. Before we state the results we recall that in Notation 185.28 we saw that an oriented smoothly embedded closed curve C in a smooth manifold M defines a homology class $[C] \in \mathrm{H}_1(M)$. We start out with the following lemma.

Lemma 234.23. (Curves-on-Torus Lemma) Let $T = S^1 \times S^1$ be the torus and let C be an oriented smoothly embedded closed curve on T.

(1) If $T \setminus C$ is connected, then $[C]$ is a primitive element of $\mathrm{H}_1(T)$.
(2) If $T \setminus C$ is disconnected, then C is the boundary of an embedded disk.



$T \setminus C$ is connected



$T \setminus C$ is disconnected

Proof. If $T \setminus C$ is connected, then it follows from the Non-Separating Codimension-One Submanifold Proposition 234.3 that $[C]$ is a primitive element of $\mathrm{H}_1(T)$. Now suppose that $T \setminus C$ is disconnected. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) there exists a tubular neighborhood $[-1,1] \times C$. From the Basics-of-Tubular Neighborhood Lemma 57.10, Homotopy-Equivalence π_0-Lemma 19.17 and the Manifold–Local Properties Lemma 23.8 (3) we obtain that $T \setminus ((-1,1) \times C)$ is also disconnected. Since C is connected this implies that $T \setminus ((-1,1) \times C)$ has two components A and B, one of which contains $\{-1\} \times C$ and one which contains $\{1\} \times C$.[2534] By the Codimension-Zero Submanifold Proposition 25.49 we know that A and B are compact 2-dimensional smooth manifolds, in

[2534] Why is that?

particular they are finite CW-complexes. We now see that

$$\chi(A)+\chi(B) \underset{\text{by the Union-}\chi\text{-Proposition 156.10}}{=} \underbrace{\chi(T)}_{=0}+\underbrace{\chi(\{-1\}\times C)}_{=0}+\underbrace{\chi(\{1\}\times C)}_{=0} = 0.$$

Since A and B have boundary it follows from the classification of compact 2-dimensional smooth manifolds and the calculation of Euler characteristics, see the Surface-χ-Lemma 156.15, that one of A or B is diffeomorphic to the disk $\overline{B}^2$. ■

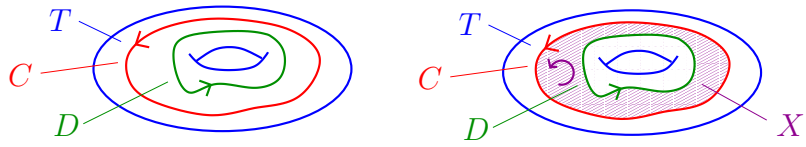
Lemma 234.24. (Disjoint Curves-on-Torus Lemma) Let $T = S^1 \times S^1$ be the torus. Let C and D be two oriented smoothly embedded closed curves on T. If C and D are disjoint and if $[C]$ and $[D]$ are both non-zero[2535] in $\mathrm{H}_1(T)$, then $[C] = \pm[D]$.

Proof. From the Cup Product–Intersection Number Theorem 220.10 together with our hypothesis that $C \cap D = \varnothing$ we obtain that $\mathrm{PD}_T([C]) \cup \mathrm{PD}_T([D]) = 0$. It follows easily from the calculation of the cup product of the torus, see the Torus–Cup Product Lemma 203.28, that $\mathrm{PD}_T([C]) = k \cdot \mathrm{PD}_T([D])$ for some $k \in \mathbb{Z}$. We obtain from the Curves-on-Torus Lemma 234.23 that $k \in \{-1, 1\}$. ■

Lemma 234.25. (Curves-on-Torus Lemma) Let $T = S^1 \times S^1$ be the torus. Let C and D be two oriented smoothly embedded closed curves on T. Suppose that $[C] = [D] \neq 0$, then the following hold:

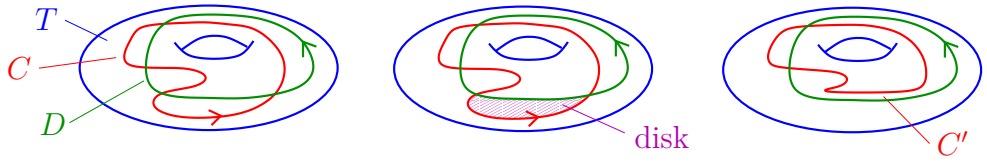
(1) The two oriented smoothly embedded closed curves are smoothly isotopic.
(2) If C and D are furthermore disjoint, then there exists an embedded annulus $X \subseteq T$ with $\partial X = C \cup -D$.

Furthermore, the same statements also hold if we replace the torus $T = S^1 \times S^1$ by the annulus $S^1 \times [-1, 1]$.



Sketch of proof. We provide a sketch of a proof for the various statements:

(1) By the Transversality Theorem 50.9 (2) we can assume that C and D are already transverse. It follows from the calculation of the cup product of the torus and the Cup Product–Intersection Number Theorem 220.10 that the algebraic intersection number is 0. The argument is now similar to the argument of the proof of the Whitney Trick-Theorem 262.5, but this time we use the Curves-on-Torus Lemma 234.23 to find the desired embedded disks. We refer to the figure below for an illustration.



[2535] Is this condition that $[C]$ and $[D]$ are both non-zero necessary?

It is a good, and not entirely trivial, mental exercise to fill in the details. The argument is sketched in more detail in [**Rolf1990**, Lemma 5 of Chapter 2.C].

(2) The proof of this statement is similar to the proof of the Curves-on-Torus Lemma 234.23 (2). We leave it to the reader to fill in the details.

The proof in the case that $T = S^1 \times [-1, 1]$ is easier. We will discuss it in Exercise 156.20. ■

For completeness sake' we state the following proposition, even though we will not make use of it.

Proposition 234.26. (Homotopy-implies-Isotopy Proposition) Let F be a 2-dimensional smooth manifold and let C and D be two oriented smoothly embedded closed curves on F.

(1) If C and D are homotopic, then C and D are smoothly isotopic.

(2) If C and D are furthermore disjoint and if they are not null-homotopic, then there exists a smoothly embedded annulus $X \subseteq F$ with $\partial X = C \cup -D$.[2536]

Proof.

(1) This statement is [**FaM2011**, Proposition 1.10]. (Closely related statements are also proved in [**Baer1928**], [**Eps1966**, Theorem 2.1].) Unfortunately all the references are somewhat cavalier when it comes to distinguishing continuous and smooth maps, but with some effort one can see that all the references provide smooth isotopies.

(2) If F is orientable, then we will provide the proof in Exercise 234.7. The proof of the non-orientable case is not much harder. ■

234.6. Exercises for Chapter 234 .

Exercise 234.1. Let M be a *non-compact* oriented n-dimensional smooth manifold with compact boundary. Furthermore let N be a compact orientable non-empty proper submanifold of codimension-one such that $M \setminus N$ is connected. Show that $[N] \in \mathrm{H}_{n-1}(M)$ is an element of infinite order.

Hint. Use the Compact Submanifold Exhaustion Proposition 38.2 and the Non-Separating Codimension-One Submanifold Proposition 234.3.

Exercise 234.2. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. By the Surface-π_1-Proposition 101.1 we know that there exists an isomorphism

$$\pi_1(\Sigma_g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle.$$

Use this isomorphism to give a purely algebraic proof that $\pi_1(\Sigma_g)$ admits an epimorphism onto a free group on g generators.

Exercise 234.3. Here is one possible approach to answering Question 234.15 (1). Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. Let $F_1, \ldots, F_k$ be disjoint smoothly embedded closed curves on Σ_g such that $\Sigma_g \setminus (F_1 \cup \cdots \cup F_k)$ is connected. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) we know that we can pick a tubular neighborhood $\bigcup_{i=1}^{k} [-1, 1] \times F_i$ for the submanifold $F_1 \cup \cdots \cup F_k \subseteq \Sigma_g$. We consider the compact orientable

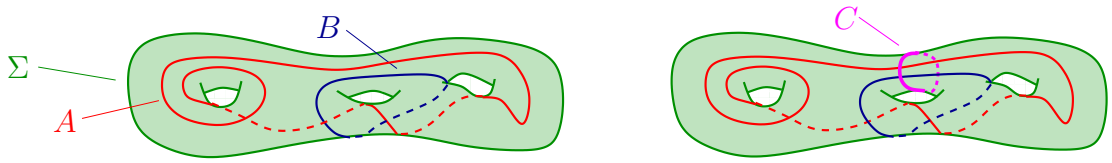
[2536] This result gives in particular a solution to Exercise 107.6.

2-dimensional smooth manifold $X \coloneqq \Sigma_g \setminus \bigcup_{i=1}^{k} (-1,1) \times F_i$. What can you say about the Euler characteristic of X? What does the classification of compact orientable 2-dimensional smooth manifolds tell you?

Exercise 234.4. Let Σ be a compact orientable connected 2-dimensional smooth manifold and let $A, B \subseteq \Sigma$ be two disjoint non-separating smoothly embedded closed curves.

(a) Show that there exists a smoothly embedded closed curve that intersects each of A and B transversally in a single point.
Remark. Exercise 155.20 might come in handy.

(b) Can one also arrange that C is connected?



Exercise 234.5. Let M be a compact oriented connected smooth manifold and let F be a compact oriented proper submanifold. We use the Definition 234.20. In particular we denote by $i_\pm \colon F \to M \setminus\setminus F$ the two inclusion maps and we denote by $\chi_F \colon M \setminus\setminus F \to M$ the natural projection. Show that there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_i(F) \xrightarrow{i_- - i_+} \mathrm{H}_i(M \setminus\setminus F) \xrightarrow{\chi_F} \mathrm{H}_i(M) \xrightarrow{\partial_i} \mathrm{H}_{i-1}(F) \to \ldots$$

Exercise 234.6. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\phi \colon \pi_1(X, x_0) \to \langle t \rangle$ be an epimorphism. We denote by $p \colon \widetilde{X}_\infty \to X$ the covering corresponding to $\ker(\phi)$. By the Deck Transformation–Normal Subgroup Proposition 128.11 we have an explicit identification of the group $\langle t \rangle$ with the deck transformation group of $p \colon \widetilde{X}_\infty \to X$. Given $\sigma \in \mathrm{C}_n(\widetilde{X}_\infty)$ and given $k \in \mathbb{Z}$ we write $t^k \cdot \sigma \coloneqq (t^k)_*(\sigma)$.

(a) Let $n \in \mathbb{N}_0$ and let $\sigma, \mu \in \mathrm{C}_n(\widetilde{X}_\infty)$. Show that $p_*(\sigma) = p_*(\mu) \in \mathrm{C}_n(X)$ if and only if there exists a $k \in \mathbb{Z}$ with $\sigma = t^k \cdot \mu$.

(b) Let $n \in \mathbb{N}_0$. Show that the following sequence is exact:

$$0 \to \mathrm{C}_n(\widetilde{X}_\infty) \xrightarrow{t-\mathrm{id}} \mathrm{C}_n(\widetilde{X}_\infty) \xrightarrow{p_*} \mathrm{C}_n(X) \to 0.$$

Remark. By the LES-Proposition 145.10 we obtain a corresponding long exact sequence

$$\ldots \to \mathrm{H}_{n+1}(X) \xrightarrow{\partial} \mathrm{H}_n(\widetilde{X}_\infty) \xrightarrow{t-\mathrm{id}} \mathrm{H}_n(\widetilde{X}_\infty) \xrightarrow{p_*} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\widetilde{X}_\infty) \to \ldots.$$

(c) As a reality check, what does the long exact sequence in (b) look like for the covering $p \colon \mathbb{R} \to S^1$?

(d) Let $r \in \mathbb{N}$. We denote by $\widetilde{X}_r$ the covering of X corresponding to the epimorphism $\ker(\phi \colon \pi_1(X, x_0) \to \langle t \rangle = \mathbb{Z} \to \mathbb{Z}/r\mathbb{Z})$. Show that there exists an exact sequence relating the homology groups of $\widetilde{X}_\infty$ and $\widetilde{X}_r$.

Exercise 234.7. Let F be an oriented 2-dimensional smooth manifold and let C and D be two *disjoint* oriented smoothly embedded closed curves on F that are not null-homotopic. Assume that C and D are homotopic. Show that there exists an embedded annulus $X \subseteq F$ with $\partial X = C \cup -D$.

Hint. Use Exercise 156.21. You might also want to use some of the ideas of the proof of the Curves-on-Torus Lemma 234.23.

Exercise 234.8. Let $T = S^1 \times S^1$ be the torus and let C and D be two closed oriented 1-dimensional submanifolds of T with $[C] = [D] \in \mathrm{H}_1(T)$. Use the results from Section 234.5 to give a down-to-earth proof of the statement that there exists a compact oriented proper 2-dimensional submanifold Ω of $T \times [0,1]$ such that

$$\partial\Omega \;=\; (\{0\}\times) \cup (\{1\} \times -D).$$

Remark. Evidently this is just a special case of the Cobounding Submanifolds Corollary 234.12. But it is fun to give a bare-hands proof in this special case.

Exercise 234.9. Let Σ be the surface of some genus $g \in \mathbb{N}_0$ and let $f\colon \Sigma \to \Sigma$ be some orientation-preserving self-diffeomorphism. We consider the corresponding mapping torus

$$\mathrm{Tor}(\Sigma, f) \;:=\; (\Sigma \times [0,1]) \,/\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in \Sigma.$$

Let F be a surface of some genus k. We now suppose that there exists a continuous map $\varphi\colon F \to \mathrm{Tor}(\Sigma, f)$ such that $\varphi_*([F]) = [\{0\} \times \Sigma] \in \mathrm{H}_2(\mathrm{Tor}(\Sigma, f))$. Show that $k \geq g$.
Hint. Make clever use of the Cyclic-Covering Proposition 234.21 and Exercise 187.3.

Exercise 234.10. Let M be a closed oriented n-dimensional smooth manifold. Furthermore let $\phi \in \mathrm{H}_{n-1}(M)$, let $k \in \mathbb{N}$ and let F be a closed oriented $(n-1)$-dimensional submanifold that represents $k \cdot \phi$. Show that we can decompose $F = F_1 \sqcup \cdots \sqcup F_k$ where each F_i represents ϕ.
Hint. Consider the Thom-Pontryagin map $\rho = \rho_F\colon M \to S^1$ that we introduced in Definition 234.9. Note that $\rho^{-1}(1) = F$. Let $p\colon S^1 \to S^1$ be the covering map that is given by $z \mapsto z^k$. Show that there exists a smooth map $\widetilde{\rho}\colon M \to S^1$ such that $\rho = p \circ \widetilde{\rho}$.

Exercise 234.11. Given a closed orientable 2-dimensional smooth manifold S with components $S_1, \ldots, S_m$ we define

$$\chi_-(S) \;:=\; \sum_{i=1}^{m} \max\{-\chi(S_i), 0\}.$$

Let M be a closed oriented 3-dimensional smooth manifold. We consider the map

$$\begin{aligned} x_M\colon \mathrm{H}_2(M) &\to \mathbb{N}_0 \\ \sigma &\mapsto \min\left\{ \chi_-(S) \,\middle|\, \begin{array}{c} S \text{ is a closed oriented 2-dimensional} \\ \text{submanifold of } M \text{ with } [S] = \sigma \end{array} \right\}. \end{aligned}$$

Now we can formulate the exercises:

(a) Show that the map x_M is a seminorm, i.e. show that it has the following two properties:
- Given any $\sigma \in \mathrm{H}_2(M)$ and $k \in \mathbb{Z}$ we have $x_M(k \cdot \sigma) = |k| \cdot x_M(\sigma)$.
  Hint. Use Exercise 234.10.
- Given $\mu, \nu \in \mathrm{H}_2(M)$ we have $x_M(\mu + \nu) \leq x_M(\mu) + x_N(\nu)$.
  Hint. Use the Transversality Theorem 50.9, the Transverse Tubular Neighborhood–Existence Theorem 57.32 and the Recombine Fundamental Class Proposition 187.27.

(b) Show that there exists a unique seminorm $x_M\colon \mathrm{H}_2(M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ such that the map $\mathrm{H}_2(M) \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1_{\mathbb{R}}]} \mathrm{H}_2(M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ agrees with the above map x_M.

Remark. This seminorm $x_M\colon \mathrm{H}_2(M) \to \mathbb{R}_{\geq 0}$ is usually called the *Thurston norm of M*. It was introduced by William Thurston [**Thu1986a**] in 1976 and, as is shown in [**Thu1986a**] and [**Gab1983b**], it has many remarkable properties.

Exercise 234.12. We continue with the definitions introduced in Exercise 234.11. Let Σ be a closed connected 2-dimensional smooth manifold with $\chi(\Sigma) \leq 0$.

(1) Let $f\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the mapping torus $\mathrm{Tor}(\Sigma, f)$. Show that $x([\Sigma]) = \chi_-(\Sigma)$.
Hint. Let F be any closed oriented 2-dimensional submanifold of $\mathrm{Tor}(\Sigma, f)$ such that $[F] = [\Sigma]$. Using the Cyclic-Covering Proposition 234.21 we obtain a map $F \to \Sigma \times \mathbb{R}$. Consider the map $F \to \Sigma \times \mathbb{R} \to \Sigma$ and use Exercise 186.4.

(2) We can to consider the Thurston norm of $N \times S^1$.
  (a) Show that $x_{\Sigma^1 \times S^1}([\Sigma \times \{*\}] = \chi_-(\Sigma)$.
  (b) Show that for any $\sigma \in \mathrm{H}_1(\Sigma)$ we have $x_{\Sigma^1 \times S^1}([S^1] \times \sigma) = 0$.
  (c) Show that for any $\phi \in \mathrm{H}_2(\Sigma \times S^1; \mathbb{R})$ we have

$$x_{\Sigma \times S^1}(\phi) \;=\; \langle \underbrace{\mathrm{PD}_{\Sigma \times S^1}}_{\in \mathrm{H}^1(M;\mathbb{R})}, \underbrace{[\{*\} \times S^1]}_{\in \mathrm{H}_1(M;\mathbb{Z})} \rangle.$$

Hint. Either use the Product-as-Mapping Torus Proposition 233.8 and (1). Alternatively use (2a) and (2b) together with Exercise 234.11 (b).

Exercise 234.13. Let M be a closed oriented n-dimensional smooth manifold. Given any $\phi \in \mathrm{H}^1(M; \mathbb{Z})$ we define

$$x_M(\sigma) \;=\; \inf\{\|N\|_{\mathbb{R}} \mid N \text{ is a closed oriented submanifold of } M \text{ with } \mathrm{PD}_M([N]) = \phi\}.$$

$\|N\|_{\mathbb{R}}$: simplicial volume as defined in Exercise 186.9

Let N be a closed oriented connected $(n-1)$-dimensional smooth manifold. Given any cohomology class $\phi \in \mathrm{H}^1(N \times S^1)$ determine $x_{N \times S^1}(\phi)$ in terms of $\|N\|_{\mathbb{R}}$ and $\phi(\{*\} \times S^1)$.
Hint. Use the Product-as-Mapping Torus Proposition 233.8 and (a).

Exercise 234.14. Let M be the 2-sphere minus four open disks. Show that there exists a class in $\mathrm{H}_1(M)$ that cannot be represented by a closed connected 1-dimensional smooth submanifold of M.

Exercise 234.15. We consider $M = S^2 \times S^2$ and the homology classes $\coloneqq [S^2 \times \{*\}]$ and $b \coloneqq [\{*\} \times S^2]$ in $\mathrm{H}_2(S^2 \times S^2)$.

(a) Show for every $n \in \mathbb{Z}$ the class $n \cdot a$ can be represented by a smoothly embedded sphere.
(b) Show that for every $m, n \in \mathbb{Z} \setminus \{0\}$ the class $m \cdot a + n \cdot b$ can be represented by a smoothly embedded surface of genus $g = (|n| - 1) \cdot (|m| - 1)$.

Part XXVI

Steenrod operations

In this rather short part of the lecture notes we will introduce and study the Steenrod operations

$$\mathrm{Sq}^i \colon \mathrm{H}^k(X;\mathbb{Z}/2\mathbb{Z}) \ \to \ \mathrm{H}^{k+i}(X;\mathbb{Z}/2\mathbb{Z})$$

on singular cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients. It is a rather non-trivial to define these operations. But we will see that they are worth the effort, since they have many interesting applications. In particular they can be used to show that many homotopy groups of spheres are non-zero. Later, in Chapter 279 the Steenrod operations get another chance to shine, when we use them to give an alternative definition of Stiefel-Whitney classes.

This part of the lecture notes is organized as follows:

- In Chapter 235 we will formulate and prove two "Acyclic Model Theorems". These can be viewed as formalizations of several of the theorems from Chapter 164 where we proved the existence and uniqueness of Eilenberg-Zilber maps.
- In Chapter 236 we will use the Acyclic-Model Theorems to introduce the Steenrod operations on the singular cohomology groups $\mathrm{H}^*(X;\mathbb{Z}/2\mathbb{Z})$ of a topological space with $\mathbb{Z}/2\mathbb{Z}$-coefficients. We will prove many essential statements about Steenrod operations. We will also prove some applications. In particular we will use them to show that all homotopy groups $\pi_{n+1}(S^n)$, $\pi_{n+3}(S^n)$ and $\pi_{n+7}(S^n)$ are non-zero.
- In Chapter 237 we will introduce the Adem relations on Steenrod operations. Unfortunately we lack the time to prove these relations. But we will explain how they can be used to prove many interesting results. For example we will use them to show that for sufficiently large $n \in \mathbb{N}$ the homotopy groups $\pi_{n+2}(S^n)$, $\pi_{n+6}(S^n)$ and $\pi_{n+14}(S^n)$ are non-zero. We will also be able to give another proof that finite-dimensional division algebras over $\mathbb{R}$ only exist in dimensions $1, 2, 4$ and 8.

CHAPTER 235

The acyclic model theorems

In Chapter 164 on the Eilenberg-Zilber maps and the Künneth Theorem we proved many statements where the proofs had a very similar structure. In this chapter we will formulate and prove several theorems which go by the name of "Acyclic Model Theorem". These theorems formalize some of the proofs of Chapter 164. The Acyclic Model Theorems will also play an essential role in the next chapter when we introduce the Steenrod operations on cohomology.

This chapter is organized as follows:

- In Section 235.1 we will state and prove the Acyclic Model Theorem 235.1.
- In Section 235.2 we will state and prove another Acyclic Model Theorem 235.7. (The author apologizes for the lack of creativity in naming the various results of this chapter.)
- In Section 235.3 we will provide exercises for Chapter 235.

In the next Chapter 236 we will use the Acyclic Model Theorems to introduce Steenrod operations on $\mathbb{F}_2$-value cohomology.

235.1. **The first Acyclic Model Theorem.** In this section we will prove two "Acyclic Model Theorems" which on many occasions will allow us to construct maps and homotopies. We start out with the following fairly friendly Acyclic Model Theorem. We will need this particular Acyclic Model Theorem later on in Chapter 236 when we construct Steenrod operations.

Theorem 235.1. (Acyclic Model Theorem) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized[2537] chain complexes. Let $G\colon \mathcal{T}op \to \mathcal{C}$ be a covariant functor. Furthermore let S be a natural transformation from the functor $X \mapsto \mathrm{C}_*(X)$ to the functor $X \mapsto G(X)$. If $\mathrm{H}_i(G(\Delta^k)) = 0$ for any $k \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$, then there exists a natural chain homotopy P from S to the zero map. In other words, for any topological space X and any $k \in \mathbb{N}$ there exists a natural map $P_X\colon \mathrm{C}_k(X) \to G_{k+1}(X)$, such that for every $k \in \mathbb{N}_0$ we have

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \;\to\; G_k(X).$$

Sketch of proof. The proof of this theorem is very similar to the "naturally chain homotopic statement" of the EZ–Natural Chain Homotopy Proposition 164.13. First recall that the fact that S is a natural transformation means that for any topological space X we get

[2537]Here "generalized" just means, see Definition 143.1, that we also allow chain groups in negative degrees.

a commutative diagram as follows:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} & & \\
\ldots & \longrightarrow & G_2(X) & \xrightarrow{\partial_2} & G_1(X) & \xrightarrow{\partial_1} & G_0(X) & \xrightarrow{\partial_0} & G_{-1}(X) & \xrightarrow{\partial_{-1}} & G_{-2}(X) & \xrightarrow{\partial_{-2}} & \ldots
\end{array}$$

Now we consider $\mathrm{id}_{\Delta^0} \in \mathrm{C}_0(\Delta^0)$. It is a cycle in the chain complex $\mathrm{C}_*(\Delta^0)$. Since the above diagram commutes we see that $S(\mathrm{id}_{\Delta^0})$ is also a cycle in $G_0(\Delta^0)$. Since we assume that $\mathrm{H}_0(G(\Delta^0)) = 0$ we see that there exists some $\mu_1 \in G_1(\Delta^0)$ with $\partial_1(\mu_1) = S(\mathrm{id}_{\Delta^0})$. Now, given any topological space X we define

$$\begin{array}{rcl}
P_0 \colon \mathrm{C}_0(X) & \to & G_1(X) \\
\sum_{i=1}^m a_i \cdot \underset{\uparrow}{f_i} & \mapsto & \sum_{i=1}^m a_i \cdot f_{i*}(\mu_1). \\
\text{map } \Delta^0 \to X & &
\end{array}$$

We iterate this procedure. Namely suppose that for $i = 0, \ldots, k-1$ we have already constructed natural maps $P_i \colon \mathrm{C}_i(X) \to G_{i+1}(X)$ such that $\partial \circ P_i + P_{i-1} \circ \partial = S_i$ for $i = 0, \ldots, k-1$. For reference we consider the following diagram:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathrm{C}_{k+1}(X) & \xrightarrow{\partial_{k+1}} & \mathrm{C}_k(X) & \xrightarrow{\partial_k} & \mathrm{C}_{k-1}(X) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle S_{k+1}} & \swarrow{\scriptstyle P_k} & \downarrow{\scriptstyle S_k} & \swarrow{\scriptstyle P_{k-1}} & \downarrow{\scriptstyle S_{k-1}} & & \\
\ldots & \longrightarrow & G_{k+1}(X) & \xrightarrow[\partial_{k+1}]{} & G_k(X) & \xrightarrow[\partial_k]{} & G_{k-1}(X) & \xrightarrow{\partial_{k-1}} & \ldots
\end{array}$$

Next we need to construct a suitable map P_k. We calculate that

$$\begin{array}{rcl}
\partial_k((S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta^k})) & = & \partial S_k(\mathrm{id}_{\Delta^k}) - \partial P_{k-1}(\partial\, \mathrm{id}_{\Delta^k}) \\
 & \underset{\uparrow}{=} & S_{k-1}(\partial\, \mathrm{id}_{\Delta^k}) - (S_{k-1} - P_{k-2}\partial)(\partial\, \mathrm{id}_{\Delta^k}) \;=\; (P_{k-2} \circ \overbrace{\partial \circ \partial}^{=0})(\mathrm{id}_{\Delta^k}) \;= 0. \\
 & \text{since } S \text{ is a chain map and} & \\
 & \text{since } \partial \circ P_{k-1} + P_{k-2} \circ \partial = S_{k-1} &
\end{array}$$

Since we assume that $\mathrm{H}_k(G(\Delta^k)) = 0$ we see that there exists some $\mu_{k+1} \in G_{k+1}(\Delta^k)$ with $\partial_{k+1}(\mu_{k+1}) = (S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta^k})$. Now, given any topological space X we define

$$\begin{array}{rcl}
P_k \colon \mathrm{C}_k(X) & \to & G_{k+1}(X) \\
\sum_{i=1}^m a_i \cdot \underset{\uparrow}{f_i} & \mapsto & \sum_{i=1}^m a_i \cdot f_{i*}(\mu_{k+1}). \\
\text{map } \Delta^k \to X & &
\end{array}$$

Using the fact that S is natural one can easily verify that P_k has the desired properties. ∎

235.2. **The second Acyclic Model Theorem.** The above version of the Acyclic Model Theorem will not work for the situations that we need to deal with in the near future. There are two problems with the above Acyclic Model Theorem:

(1) On several occasions we will be interested in natural transformations of the form $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, but for $X = \Delta^k$ the chain complex to the right does not satisfy the condition that the homology groups for $X = \Delta^k$ vanish in dimensions ≥ 0.

(2) We will also need to deal with situations where to the left we have more complicated chain complexes, e.g. we might want to study natural transformations of the type $\mathrm{C}_*(X) \times \mathrm{C}_*(X) \to \mathrm{C}_*(X \times X)$.

To formulate a version of the Acyclic Model Theorem that deals with the above two issues we need a few more definitions:

Definition 235.2.

(1) An augmented chain complex is a chain complex

$$(C_*, \partial_*) \;=\; \ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

together with an epimorphism $\epsilon\colon C_0 \to \mathbb{Z}$ such that $\epsilon \circ \partial_1\colon C_1 \to \mathbb{Z}$ is the zero map.

(2) We say that an augmented chain complex $((C_*, \partial_*), \epsilon\colon C_0 \to \mathbb{Z})$ is acyclic if the generalized chain complex

$$\ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \underset{\substack{\uparrow \\ \text{degree } -1}}{\mathbb{Z}} \to 0$$

is acyclic, i.e. if all homology groups are zero.

(3) Let $((C_*, \partial_*), \epsilon)$ and $((C'_*, \partial'_*), \epsilon')$ be two augmented chain complexes. A chain map of augmented chain complexes is a chain map $f_*\colon C_* \to C'_*$ such that the following diagram commutes:

$$\begin{array}{ccc} C_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C'_0 & \xrightarrow{\epsilon'} & \mathbb{Z}. \end{array}$$

(4) We refer to the category $\mathcal{A}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{A}) &= \text{augmented chain complexes} \\ \mathrm{Mor}((C,\epsilon),(C',\epsilon')) &= \text{set of chain maps of augmented chain complexes from } (C,\epsilon) \text{ to } (C',\epsilon') \end{aligned}$$

as the category of augmented chain complexes.

Examples.

(1) Let X be a non-empty topological space. We denote by $\epsilon_X\colon \mathrm{C}_0(X) \to \mathbb{Z}$ the augmentation map that we introduced in Definition 143.23, i.e. ϵ_X is the map that sends each singular 0-simplex to 1. It follows from the Augmentation Lemma 143.24 (1) that $(\mathrm{C}_*(X), \epsilon_X)$ is an augmented chain complex. If X is a contractible topological space, e.g. if $X = \Delta^k$, then it follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that the corresponding augmented chain complex is acyclic.

(2) If (C_*, μ) and (D_*, ν) are two augmented chain complexes, then the tensor product $C_* \otimes D_*$ of chain complexes, as defined in the Tensor–Boundary Lemma 163.3, together with the map

$$\begin{aligned} \mu \otimes \nu\colon (C_* \otimes D_*)_0 = C_0 \otimes D_0 &\to \mathbb{Z} \\ \sum_{i=1}^{k} c_i \otimes d_i &\mapsto \sum_{i=1}^{k} \mu(c_i) \cdot \nu(d_i) \end{aligned}$$

is easily seen to be an augmented chain complex.[2538] □

Definition 235.3. We say that a chain complex C_* is almost acyclic if $\mathrm{H}_i(C_*) = 0$ for $i \geq 1$ and if $\mathrm{H}_0(C_*) \cong \mathbb{Z}$.

Example. If X is a contractible topological space, then it follows from the Contractible-H_*-Corollary 144.15 that $\mathrm{C}_*(X)$ is almost acyclic. □

Proposition 235.4. (Tensor Product–Almost Acyclic Proposition) Let C_* and D_* be two chain complexes. If C_* and D_* are almost acyclic and if C_* is free, i.e. if each chain group C_n is a free abelian group, then $C_* \otimes D_*$ is also almost acyclic.

Proof. Let $n \in \mathbb{N}_0$. Since C_* is free, we can apply the Algebraic–Künneth Theorem 163.6 which gives us the following short exact sequence:

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(C_*) \otimes \mathrm{H}_q(D_*) \xrightarrow{\Theta} \mathrm{H}_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(C_*), \mathrm{H}_q(D_*)) \to 0.$$

The proposition is an immediate consequence of this short exact sequence and of some very elementary statements regarding tensor products and torsion groups, see the Tensor-with-$\mathbb{Z}$–Lemma 159.9 and the Tor-Properties Lemma 162.13. ■

The following lemma gives a convenient criterion for showing that augmented chain complexes are acyclic.

Lemma 235.5. (Augmented Chain Complex–Acyclic Lemma) Let $((C_*, \partial_*), \epsilon)$ be an augmented chain complex. If the chain complex (C_*, ∂_*) is almost acyclic, then the augmented chain complex $((C_*, \partial_*), \epsilon)$ is acyclic.

Proof. We denote by $\widetilde{C}_*$ the generalized chain complex

$$\ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

We need to show that this chain complex is acyclic. It is clear that for $i \geq 1$ we have $\mathrm{H}_i(\widetilde{C}_*) = \mathrm{H}_i(C_*) = 0$. It remains to show that $\mathrm{H}_0(\widetilde{C}_*) = 0$. Since $\mathbb{Z}$ is a free abelian group and since $\epsilon\colon C_0 \to \mathbb{Z}$ is an epimorphism, we can pick a homomorphism $s\colon \mathbb{Z} \to C_0$ with $\epsilon \circ s = \mathrm{id}_{\mathbb{Z}}$. As in the Reduced Homology Lemma 147.3 (0b) one can easily show that the map

$$\begin{array}{rcl} \mathrm{H}_0(\widetilde{C}_*) \oplus \mathbb{Z} & \to & \mathrm{H}_0(C_*) \\ [\sigma] \oplus n & \mapsto & [\sigma] + s(n) \end{array}$$

is an isomorphism. Since $\mathrm{H}_0(C_*) \cong \mathbb{Z}$ we see that $\mathrm{H}_0(\widetilde{C}_*) \oplus \mathbb{Z} \cong \mathbb{Z}$. It follows from the Cancellation Proposition 97.26 that $\mathrm{H}_0(\widetilde{C}_*) = 0$. ■

Definition 235.6. Let $\mathcal{K}$ be a category (in all applications $\mathcal{K}$ will be the category of non-empty topological spaces or the category of tuples of non-empty topological spaces)

[2538] Indeed, by definition $(C_* \otimes D_*)_1$ is generated by elements $c_1 \otimes d_0$ with $c_1 \in C_1$ and $d_0 \in D_0$ and by elements $c_0 \otimes d_1$ with $c_0 \in C_0$ and $d_1 \in D_1$. We calculate that

$$((\mu \otimes \nu) \circ \partial_1)(c_1 \otimes d_0) = (\mu \otimes \nu)(\partial c_1 \otimes d_0 + (-1)^1 \cdot c_1 \otimes \underbrace{\partial d_0}_{=0}) = (\mu \otimes \nu)(\partial c_1 \otimes d_0) = \mu(\partial c_1) \cdot \nu(d_0) = 0 \cdot \nu(d_0) = 0.$$

The same argument shows that $((\mu \otimes \nu) \circ \partial_1)(c_0 \otimes d_1) = 0$.

and let $F\colon \mathcal{K} \to \mathcal{A}$ be a covariant functor from $\mathcal{K}$ to the category $\mathcal{A}$ of augmented chain complexes. Let $\mathcal{M}$ be a collection of objects of $\mathcal{K}$.

(1) We say that the functor F is acyclic relative to $\mathcal{M}$ if given any $X \in \mathcal{M}$ the augmented chain complex $F(X)$ is acyclic.

(2) We say that the functor F is free relative to $\mathcal{M}$ if for each $k \in \mathbb{N}_0$ there exists $\mathcal{M}_k \subseteq \mathcal{M}$ and if for each $M \in \mathcal{M}_k$ there exists an element $\sigma_M \in F_k(M)$[2539] such that given any $X \in \mathcal{K}$ the following two statements hold:

(a) The set

$$\{\underbrace{F(f)}_{\substack{\uparrow \\ \text{the morphism } f\colon M \to X \text{ induces a} \\ \text{homomorphism } F(f)\colon F_k(M) \to F_k(X)}}(\sigma_M) \in F_k(X) \mid \ M \in \mathcal{M}_k \text{ and } f \in \operatorname{Mor}(M, X)\}$$

is a basis for the abelian group $F_k(X)$.[2540]

(b) The description of the set in (a) has no double elements, much more precisely, for $(M, f) \neq (M', f')$ we have $F(f)(\sigma_M) \neq F(f')(\sigma_{M'})$.

Remark. The objects in a collection $\mathcal{M}$ as in the previous definition are often called *models*. □

The definition of "free relative to $\mathcal{M}$" is arguably somewhat intimidating. The following example should convince the reader that in practice it is not that bad.

Example. Let $\mathcal{K}$ be the category of non-empty topological spaces and let F be the covariant functor that assigns to each $X \in \mathcal{K}$ the augmented singular chain complex $(C_*(X), \epsilon_X)$. We consider

$$\mathcal{M} \ := \ \{\Delta^k \mid k \in \mathbb{N}_0\} \ = \ \text{the set of all standard simplices.}$$

We claim that F is acyclic and free relative to $\mathcal{M}$:

(1) By the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 we know that for every $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_*(\Delta^k) = 0$. But by the Augmented Chain Complex–Acyclic Lemma 235.5 this just means that $\mathcal{M}$ is acyclic relative to F.

(2) Let $k \in \mathbb{N}_0$. We set $\mathcal{M}_k = \{\Delta^k\}$ and we consider $\sigma_k = \mathrm{id}_{\Delta^k} \in \mathrm{C}_k(\Delta^k)$. Now recall that given any topological space X and given any $k \in \mathbb{N}_0$ we have

$$\begin{aligned} \mathrm{C}_k(X) &\overset{\substack{\downarrow \\ \text{by Definition 143.11 of } \mathrm{C}_k(X)}}{=} \text{free abelian group generated by the set of singular } k\text{-simplices in } X \\ &= \text{free abelian group generated by the set of all continuous maps } \underbrace{f\colon \Delta^k \to X}_{=f_*(\mathrm{id}_{\Delta^k})}. \end{aligned}$$

$\uparrow$ by Definition 143.10 of a singular k-simplex

But this means that the set $\{f_*(\mathrm{id}_{\Delta^k})\}_{f\colon \Delta^k \to X}$ is a basis for $\mathrm{C}_k(X)$. □

Now we can formulate the main theorem of this section.

Theorem 235.7. (Acyclic Model Theorem) Let $\mathcal{K}$ be a category with a collection of objects $\mathcal{M}$. Furthermore let $F, G\colon \mathcal{K} \to \mathcal{A}$ be two covariant functors from $\mathcal{K}$ to the

[2539] Here $F_k(M)$ is the k-chain group of the chain complex $F(M)$.

[2540] In particular this implies that each $F_k(X)$ is a free abelian group.

category of augmented chain complexes. If F is free relative to $\mathcal{M}$ and if G is acyclic relative to $\mathcal{M}$, then the following statements hold:

(1) There exists a natural transformation T from F to G. In other words, given any object X of $\mathcal{K}$ we have a chain map $T_X\colon F(X) \to G(X)$ of augmented chain complexes such that for every morphism $f\colon X \to Y$ in $\mathcal{K}$ the following diagram commutes:[2541]

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle T_X}\downarrow & & \downarrow{\scriptstyle T_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

(2) Given two natural transformations S_X and T_X from F to G there is a natural chain homotopy P_X between them.

Proof. The proof is, ultimately very much the same as the proof of the EZ–Natural Chain Homotopy Proposition 164.13. We decline to provide a proof and we follow the lead given by Munkres [**Mun1984**, p. 185], namely like Munkres we assign the proof as an exercise to the reader. Alternatively the reader can find the proof in [**Spa1995**, p. 169], [**Vic1994**, Theorem 5.3] or [**Dol1980**, p. 175]. ■

As an example for the Acyclic Model Theorem 235.7 we will prove a statement that we will anyway need shortly. To formulate the application we need the following definition:

Definition 235.8. Let X be a topological space. A diagonal approximation is a natural chain map

$$\varphi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

such that for each $x \in X$, viewed as a singular 0-simplex, we have

$$\varphi(x) = x \otimes x \in \underbrace{\mathrm{C}_0(X) \otimes \mathrm{C}_0(X)}_{=(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_0}.$$

Proposition 235.9. (Diagonal Approximation Uniqueness Proposition) Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.

Proof. We let $\mathcal{K}$ be the category of non-empty topological spaces. As on page 4533 we consider the collection

$$\mathcal{M} := \{\Delta^k \mid k \in \mathbb{N}_0\} = \text{the set of all standard simplices.}$$

We consider the covariant functors[2542]

$$\begin{array}{rcl} F\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (\mathrm{C}_*(X), \epsilon_X) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \epsilon_X \otimes \epsilon_X). \end{array}$$

Now let $\Phi, \Psi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be two diagonal approximations. By definition these are natural transformations from F to G. The hypothesis that Φ and Ψ are diagonal approximations implies that Φ and Ψ define chain maps of augmented chain complexes.

By the Acyclic Model Theorem 235.7 (2) it remains to prove the following claim:

[2541] Note that the zero map is *not* a chain map of augmented chain complexes, since it does not respect the augmentation.

[2542] It follows from the discussion on page 4531 that both F and G are indeed functors to the category of augmented chain complexes.

Claim.

(1) The functor F is free relative to $\mathcal{M}$.
(2) The functor G is acyclic relative to $\mathcal{M}$.

Proof.

(1) Conveniently enough we showed on page 4533 that F is free relative to $\mathcal{M}$.
(2) We need to show that given any $k \in \mathbb{N}_0$ the corresponding augmented chain complex $(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k), \epsilon_X \otimes \epsilon_X)$ is acyclic. By the Augmented Chain Complex–Acyclic Lemma 235.5 it suffices to show that the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ is almost acyclic. But this is a consequence of the fact that $\mathrm{C}_*(\Delta^k)$ is almost acyclic, see page 4532, and the Tensor Product–Almost Acyclic Proposition 235.4. ■

235.3. **Exercises for Chapter 235.**

CHAPTER 236

Steenrod Operations

In this chapter we introduce the Steenrod operations. These are cohomology operations which satisfy several axioms. In the first three sections we will state the axioms and draw several interesting conclusions. Afterwards we will sketch a proof for the existence of the Steenrod operations.

This chapter is organized as follows:

- In Section 236.1 we will formulate the Steenrod Theorem 236.3 which introduces the Steenrod operations on singular cohomology.
- In Section 236.2 we will discuss the interplay between Steenrod operations and connecting homomorphisms.
- In Section 236.3 we will discuss the interplay between Steenrod operations and suspensions.
- In Section 236.4 we will start out with the proof of the Steenrod Theorem 236.3.
- In Section 236.5 we will conclude the proof of the Steenrod Theorem 236.3.
- In Section 236.6 we will provide exercises for Chapter 236.

In the next Chapter 237 we will discuss the Adem relations among Steenrod operations.

236.1. **The axioms of Steenrod operations.** We start out with the following fairly straightforward definition:

Definition 236.1.

(1) Let G and H be abelian groups and let $m, n \in \mathbb{N}_0$. Any natural transformation from the contravariant functor

$$\mathrm{H}^m(-;G)\colon \text{category of topological spaces} \to \text{category of sets}$$

to the contravariant functor

$$\mathrm{H}^n(-;H)\colon \text{category of topological spaces} \to \text{category of sets}$$

is called a cohomology operation of degree $n-m$. In other words, a cohomology operation consists for any topological space X of a map $\Phi_X\colon \mathrm{H}^m(X;G) \to \mathrm{H}^n(X;H)$ of sets such that for every continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^m(Y;G) & \xrightarrow{f^*} & \mathrm{H}^m(X;G) \\ \downarrow{\scriptstyle \Phi_Y} & & \downarrow{\scriptstyle \Phi_X} \\ \mathrm{H}^n(Y;H) & \xrightarrow{f^*} & \mathrm{H}^n(X;H). \end{array}$$

(2) A cohomology operation is called additive if each map $\Phi_X\colon \mathrm{H}^m(X;G) \to \mathrm{H}^n(X;H)$ is a homomorphism.
(3) A cohomology operation is called non-trivial if there exists a topological space X such that the map $\mathrm{H}^m(X;G) \to \mathrm{H}^n(X;H)$ is non-trivial in the sense that the image is not just the trivial element.

With the obvious modifications the same definitions also apply to relative cohomology viewed as functors from the category of pairs of topological spaces to the category of sets.

Perhaps somewhat surprisingly we have already assembled a substantial list of non-trivial cohomology operations.

Examples.

(1) Let $n \in \mathbb{N}_0$ and let G be an abelian group. Given a pair of topological spaces (X, A) we consider the connecting homomorphism $\mathrm{H}^n(X, A; G) \to \mathrm{H}^{n+1}(X; G)$. It follows from the LES-of-Triple Proposition 198.13 (3) that these homomorphisms define an additive cohomology operation of degree one.
(2) Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups and let $n \in \mathbb{N}_0$. In the Cohomology-Coefficients-Functor Lemma 197.21 we saw that given any topological space X we obtain an induced homomorphism $\varphi_*\colon \mathrm{H}^n(X;A) \to \mathrm{H}^n(X;B)$. By the Cohomology-Coefficients-Functor Lemma 197.21 we know that these maps define an additive cohomology operation of degree zero.
(3) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Next recall that in **??** we introduced the n-th Bockstein homomorphism $\beta_n\colon \mathrm{H}^n(X;C) \to \mathrm{H}^{n+1}(X;A)$. In the Bockstein–Naturality Lemma **??** we saw that the n-th Bockstein homomorphisms define an additive cohomology operation of degree one.
(4) Let $k \in \mathbb{N}$. It follows from the Cup Product-Naturality Lemma 204.8 that for any $m \in \mathbb{N}_0$ and any commutative ring R the maps[2543]
$$\begin{array}{rcl} \mathrm{H}^m(X, A; R) & \to & \mathrm{H}^{km}(X, A; R) \\ \varphi & \mapsto & \varphi^k = \underbrace{\varphi \cup \cdots \cup \varphi}_{k \text{ times}} \end{array}$$
define a cohomology operation of degree $km - m$. Note that this type of a cohomology operation is in general not additive. For example, if $k \geq 2$, then it follows easily from the calculation of the cup product of $\mathbb{C}\mathrm{P}^k$, see the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7, that the map $\mathrm{H}^2(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) \to \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ given by $\varphi \mapsto \varphi^k$ is not a homomorphism.
(5) If R is a commutative ring with $1 + 1 = 0$, then the cohomology operation $\varphi \mapsto \varphi^2$ is in fact additive. Indeed for any $\varphi, \psi \in \mathrm{H}^m(X, A; R)$ we have[2544]
$$\begin{aligned} (\varphi + \psi)^2 &= \varphi \cup \varphi + \varphi \cup \psi + \psi \cup \varphi + \psi \cup \psi \\ &\underset{\uparrow}{=} \varphi \cup \varphi + (1 + (-1)^{m^2}) \cdot \varphi \cup \psi + \psi \cup \psi \underset{\uparrow}{=} \varphi \cup \varphi + \psi \cup \psi = \varphi^2 + \psi^2. \end{aligned}$$
by the Cup Product-Commutativity Proposition 203.20 we have $\psi \cup \varphi = (-1)^{m^2} \cdot (\varphi \cup \psi)$ — since $1 - 1 = 0 = 1 + 1$ in R

□

[2543] We refer to Definition 204.7 for the definition of the relative cup product.
[2544] This can be viewed as the cohomological "freshman's dream". Perhaps it should called the incoming PhD student's dream.

Note that all of the cohomology operations that we have introduced above are of non-negative degree. This raises the following amusing question.

Question 236.2. Does there exist a non-trivial cohomology operation of negative degree?

We will come back to this question in the Cohomology Operation–Non-Negative Degree Proposition **??**.

The following theorem gives us new and interesting cohomology operations. The theorem was first proved by Norman Steenrod [**Stee1947**] in 1947.

Theorem 236.3. (Steenrod Theorem) There exists a family

$$\mathrm{Sq}^i\colon \mathrm{H}^n(X,A;\mathbb{F}_2) \ \to\ \mathrm{H}^{n+i}(X,A;\mathbb{F}_2) \quad \text{with } i\in\mathbb{N}_0 \text{ and } n\in\mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.
(2) For any $\varphi\in\mathrm{H}^i(X,A;\mathbb{F}_2)$ we have $\mathrm{Sq}^i(\varphi)=\varphi^2=\varphi\cup\varphi\in\mathrm{H}^{2i}(X,A;\mathbb{F}_2)$.
(3) For any $\varphi\in\mathrm{H}^j(X,A;\mathbb{F}_2)$ with $i>j$ we have $\mathrm{Sq}^i(\varphi)=0$.
(4) Let (X,A,B) be an excisive triad of topological spaces. For any $\varphi\in\mathrm{H}^r(X,A;\mathbb{F}_2)$ any $\psi\in\mathrm{H}^s(X,B;\mathbb{F}_2)$ and any $k\in\mathbb{N}_0$ we have[2545]

$$\mathrm{Sq}^k(\varphi\cup\psi) \ =\ \sum_{i=0}^{k} \underbrace{\mathrm{Sq}^i(\varphi)}_{\in\mathrm{H}^{r+i}(X,A;\mathbb{F}_2)} \ \cup\ \underbrace{\mathrm{Sq}^{k-i}(\psi)}_{\in\mathrm{H}^{s+k-i}(X,B;\mathbb{F}_2)} \ \in\ \mathrm{H}^{r+s+k}(X,A\cup B;\mathbb{F}_2).$$

This equality is often called the Cartan formula.

In the next two sections we will discuss some properties and applications of these cohomology operations. Only after these two sections, once we are convinced that Steenrod operations are indeed useful and interesting, will we provide a proof for the Steenrod Theorem 236.3.

Remark.

(1) The first three axioms of the Steenrod Theorem 236.3 can be summarized as follows: Given any $\varphi\in\mathrm{H}^k(X,A;\mathbb{F}_2)$ we obtain the following table

$\mathrm{Sq}^0(\varphi)$	$\mathrm{Sq}^1(\varphi)$	$\dots$	$\mathrm{Sq}^{k-1}(\varphi)$	$\mathrm{Sq}^k(\varphi)$	$\mathrm{Sq}^{k+1}(\varphi)$	$\dots$
φ	?	$\dots$	?	φ^2	0	$\dots$

(2) Throughout the remainder of this chapter we will almost invariably consider cohomology with $\mathbb{F}_2$-coefficients. We point out that by the Cup Product-Commutativity Proposition 203.20 the cup product on the $\mathbb{F}_2$-cohomology $\mathrm{H}^*(X,A;\mathbb{F}_2)$ of a given pair of topological spaces (X,A) is actually *commutative*. □

Theorem 236.4. (Steenrod Operations–Uniqueness Theorem) Any two families of additive cohomology operations that satisfy Axioms (1) to (4) are identical.

Proof. This theorem is proved in [**SE1962**, Theorem VIII.3.10]. ■

[2545]We refer to Definition 158.18 for the definition of an excisive triad. By the Excisive Triad-Proposition 158.20 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A=B$, or that at least one of A or B is the empty set. We refer again to Definition 204.7 for the definition of the relative cup product.

This theorem allows us to give a name to the above cohomology operations.

Definition 236.5. We refer to the cohomology operations from the Steenrod Theorem 236.3 as the Steenrod operations.[2546]

The following theorem shows that the first Steenrod operation Sq^1 is actually an old acquaintance of ours.

Theorem 236.6. (Steenrod Operation–Bockstein Theorem) The first Steenrod operation Sq^1 is the Bockstein homomorphism corresponding to the short exact sequence $0 \to \mathbb{F}_2 \xrightarrow{[a]\mapsto[2a]} \mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2 \to 0$.

Proof. We will not make use of this theorem, thus we decline to provide a proof. The theorem is proved in [**Bre1993**, p. 418] or alternatively in [**MTa1968**, p. 23] or [**Hat2002**, Theorem 4.L.12]. ■

Evidently our next goal is to compute Steenrod operations in specific examples. We start out with the following slightly disappointing lemma.

Lemma 236.7. (Steenrod Operation–Wedge-of-Sphere Lemma) For any wedge of finitely many spheres all Steenrod operations in degrees ≥ 1 are zero.

Proof. We will prove the lemma in Exercise 236.3. ■

In remainder of this section we intend to compute the Steenrod operations for the real projective spaces $\mathbb{R}\mathrm{P}^n$ with $n \in \mathbb{N}_0$. Before we consider the real projective spaces in detail it is convenient to reformulate the Cartan formula.
For the reformulation we need the following notation:

Notation 236.8. Let (X, A) be a pair of topological spaces. Given $\varphi \in \mathrm{H}^k(X, A; \mathbb{F}_2)$ we write

$$\mathrm{Sq}(\varphi) \;:=\; \mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi) + \cdots + \mathrm{Sq}^k(\varphi) \;\in\; \mathrm{H}^*(X, A; \mathbb{F}_2) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}^n(X, A; \mathbb{F}_2).$$

Furthermore we extend the homomorphism Sq in the obvious way to a homomorphism $\mathrm{Sq}\colon \mathrm{H}^*(X; \mathbb{F}_2) \to \mathrm{H}^*(X; \mathbb{F}_2)$.

Remark. Let (X, A) be a pair of topological spaces and let $\varphi \in \mathrm{H}^k(X, A; \mathbb{F}_2)$. Recall that by Axiom (3) given $\varphi \in \mathrm{H}^k(X, A; \mathbb{F}_2)$ we have $\mathrm{Sq}^i(\varphi) = 0$ for $i > k$. Thus we can write

$$\mathrm{Sq}(\varphi) \;=\; \mathrm{Sq}^0(\varphi) + \mathrm{Sq}^1(\varphi) + \cdots + \mathrm{Sq}^k(\varphi) \;=\; \sum_{i\in\mathbb{N}_0} \mathrm{Sq}^i(\varphi).$$ □

With this notation we can formulate the following lemma:

Lemma 236.9. (Steenrod–Cup Product Lemma) Let (X, A) be a pair of topological spaces.

(1) Given any $\varphi, \psi \in \mathrm{H}^*(X, A; \mathbb{F}_2)$ we have

$$\mathrm{Sq}(\varphi \cup \psi) \;=\; \mathrm{Sq}(\varphi) \cup \mathrm{Sq}(\psi).$$

(2) Given any $\varphi \in \mathrm{H}^*(X, A; \mathbb{F}_2)$ and given any $k \in \mathbb{N}_0$ we have $\mathrm{Sq}(\varphi^k) = \mathrm{Sq}(\varphi)^k$.

[2546] Sometimes they are also called Steenrod squares, and the term Steenrod operations is used for a more general concept.

Proof. Statement (1) follows easily from the Cartan formula. Indeed, we see that

$$\begin{aligned}\mathrm{Sq}(\varphi\cup\psi) &= \sum_{k\in\mathbb{N}_0}\mathrm{Sq}^k(\varphi\cup\psi) \overset{\text{Cartan formula, i.e. Axiom (4)}}{=} \sum_{k\in\mathbb{N}_0}\sum_{i=0}^{k}\mathrm{Sq}^i(\varphi)\cup\mathrm{Sq}^{k-i}(\psi)\\ &= \sum_{r\in\mathbb{N}_0}\sum_{s\in\mathbb{N}_0}\mathrm{Sq}^r(\varphi)\cup\mathrm{Sq}^s(\psi) = \Big(\sum_{r\in\mathbb{N}_0}\mathrm{Sq}^r(\varphi)\Big)\cup\Big(\sum_{s\in\mathbb{N}_0}\mathrm{Sq}^s(\psi)\Big) = \mathrm{Sq}(\varphi)\cup\mathrm{Sq}(\psi).\end{aligned}$$

Statement (2) is an immediate consequence of Statement (1). ■

Before we state a corollary we recall Definition 1.43.

Definition 236.10. Let $r,s\in\mathbb{N}_0$. If $0\leq r\leq s$, then we set $\binom{s}{r}=\frac{s!}{r!(s-r)!}$, otherwise we set $\binom{s}{r}=0$.

Now we can state the promised corollary.

Corollary 236.11. (Steenrod Operation–Degree 1-Corollary) Let X be a topological space. For any cohomology class $\varphi\in\mathrm{H}^1(X;\mathbb{F}_2)$ of degree *one*, any $k\in\mathbb{N}$ and any $i\in\mathbb{N}$ we have

$$\mathrm{Sq}^i(\varphi^k) \;=\; \binom{k}{i}\cdot\varphi^{k+i} \;\in\; \mathrm{H}^{k+i}(X;\mathbb{F}_2).$$

Examples.

(1) By the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 we know that given any $n\in\mathbb{N}_0$ we have a ring isomorphism $(\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2),\cup)\cong\mathbb{F}_2[x]/(x^{n+1})$, where $x\in\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$, and we have a ring isomorphism $(\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2),\cup)\cong\mathbb{F}_2[x]$, where $x\in\mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$. Since Steenrod operations are additive we see that the result of the Steenrod Operation–Degree 1-Corollary 236.11 gives us the promised calculations of all Steenrod operations for the real projective spaces.

(2) In Exercise 236.4 we will use a similar argument to calculate the Steenrod operations on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{F}_2)$. □

Proof. We calculate that

$$\begin{aligned}\mathrm{Sq}(\varphi^k) &\overset{\text{by the Steenrod–Cup Product Lemma 236.9 (2)}}{=} \mathrm{Sq}(\varphi)^k \overset{\text{by Axiom (3) since } \varphi\in\mathrm{H}^1(X;\mathbb{F}_2)}{=} (\mathrm{Sq}^0(\varphi)+\mathrm{Sq}^1(\varphi))^k \overset{\text{Axioms (1) and (2)}}{=} (\varphi+\varphi^2)^k\\ &\overset{\text{since the ring } (\mathrm{H}^*(X;\mathbb{F}_2),\cup) \text{ is commutative}}{=} \varphi^k\cup(1+\varphi)^k = \sum_{j=0}^{k}\binom{k}{j}\varphi^{k+j}.\end{aligned}$$

The desired conclusion follows from comparing the terms in degree $k+i$ on both sides. ■

236.2. **Steenrod operations and connecting homomorphisms.** This section is completely dedicated to the proof of the following proposition. In the subsequent section we will see that the proposition has powerful applications.

Proposition 236.12. (Steenrod-Coboundary Proposition) Let (X,A) be a pair of topological spaces. If $A\subseteq X$ is closed, then the Steenrod operations commute with the connecting homomorphisms in cohomology. More precisely, given any $i,n\in\mathbb{N}_0$ the

following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(A;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(X,A;\mathbb{F}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ \mathrm{H}^{n+i}(A;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+i+1}(X,A;\mathbb{F}_2). \end{array}$$

In the proof of the Steenrod-Coboundary Proposition 236.12 we will make use of the cross product on cohomology which we now introduce.

Definition 236.13. Let R be a commutative ring, let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi \in \mathrm{H}^m(X;R)$ and $\psi \in \mathrm{H}^n(Y,B;R)$. We denote by $p\colon X\times Y \to X$ and $q\colon X\times Y\to Y$ the projections. We define the (cohomological) cross product[2547] [2548]

$$\varphi\times\psi \;:=\; \underbrace{p^*(\varphi)}_{\text{lies in } \mathrm{H}^m(X\times Y;R)} \cup \underbrace{q^*(\psi)}_{\text{lies in } \mathrm{H}^n(X\times Y, X\times B;R)} \;\in\; \mathrm{H}^{m+n}(X\times Y, X\times B;R).$$

The following proposition basically says that in Axiom (4), i.e. in the Cartan formula, we can replace the cup product by the cross product.

Proposition 236.14. (Steenrod–Cross Product Proposition) Let X be a topological space and let (Y,B) be a pair of topological spaces. For any $\varphi\in\mathrm{H}^r(X;\mathbb{F}_2)$, any $\psi\in\mathrm{H}^s(Y,B;\mathbb{F}_2)$ and any $n\in\mathbb{N}_0$ the following equality holds:

$$\mathrm{Sq}^n(\varphi\times\psi) \;=\; \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^{n-i}(\psi) \;\in\; \mathrm{H}^{r+s+n}(X\times Y, X\times B;\mathbb{F}_2).$$

Proof. Let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi\in\mathrm{H}^r(X;\mathbb{F}_2)$ and $\psi\in\mathrm{H}^s(Y,B;\mathbb{F}_2)$. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projections. We start out with the following claim:

Claim. We have the following equalities:[2549]

(a) $$\begin{aligned} p^*(\varphi) &= \varphi\times 1_Y \;\in \mathrm{H}^r(X\times Y;\mathbb{F}_2), \\ q^*(\psi) &= 1_X\times\psi \;\in \mathrm{H}^s(X\times Y, X\times B;\mathbb{F}_2). \end{aligned}$$

(b) $$\varphi\times\psi \;=\; (\varphi\times 1_Y)\cup(1_X\times\psi) \;\in \mathrm{H}^{r+s}(X\times Y, X\times B;\mathbb{F}_2).$$

Proof.

(a) We perform the following easy calculation:

$$\varphi\times 1_Y \;=\; p^*(\varphi)\cup q^*(1_Y) \;\underset{\text{by definition we have } p^*(1_X)=q^*(1_Y)=1_{X\times Y}}{=}\; p^*(\varphi)\cup 1_{X\times Y} \;\underset{\text{see the Cup Product-Ring Proposition 203.25}}{=}\; p^*(\varphi).$$

[2547] Recall that in Definition 164.4, given two *homology* classes $\mu\in\mathrm{H}_m(X;\mathbb{Z})$ and $\nu\in\mathrm{H}_n(Y;\mathbb{Z})$ we defined the *(homological) cross product* $\mu\times\nu\in\mathrm{H}_{m+n}(X\times Y;\mathbb{Z})$. It should usually be clear from the context whether in a given situation we mean the cross product on homology or on cohomology.

[2548] We first introduced the cohomological cross product in Definition 203.6. It follows from the Cohomological Cross Product Proposition 203.36 that Definition 203.6 is consistent with Definition 236.13.

[2549] Recall that given a topological space W we denote by $1_Z\in\mathrm{H}^0(W;\mathbb{F}_2)$ the cohomology class that is represented by the cocycle that assigns to each singular 0-simplex the value $1\in\mathbb{F}_2$.

Evidently the same argument also shows that $q^*(\psi) = 1_X \times \psi$.

(b) This statement follows from (a) and the definition of the cross product. ⊞

Now we turn to the proof of the actual statement of the proposition. We have the following equalities in $\mathrm{H}^{r+s+n}(X \times Y, X \times B; \mathbb{F}_2)$:

$$\begin{aligned}
\mathrm{Sq}^n(\varphi \times \psi) &\overset{\text{part (b) of the claim}}{=} \mathrm{Sq}^n\left((\varphi \times 1_Y) \cup (1_X \times \psi)\right) \\
&\overset{\text{by the Cartan formula, i.e. by Axiom (4)}}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi \times 1_Y) \cup \mathrm{Sq}^{n-i}(1_X \times \psi) \\
&\overset{\text{part (a) of the claim}}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(p^*(\varphi)) \cup \mathrm{Sq}^{n-i}(q^*(\psi)) \\
&\overset{\text{since } \mathrm{Sq}^i \text{ and } \mathrm{Sq}^{n-i} \text{ are natural transformations}}{=} \sum_{i=0}^{n} p^*(\mathrm{Sq}^i(\varphi)) \cup q^*(\mathrm{Sq}^{n-i}(\psi)) \\
&\overset{\text{by Definition 236.13 of the cross product}}{=} \sum_{i=0}^{n} \mathrm{Sq}^i(\varphi) \times \mathrm{Sq}^{n-i}(\psi).
\end{aligned}$$

We have thus obtained the desired equality. ■

The proof of the Steenrod-Coboundary Proposition 236.12 rests on the following lemma:

Lemma 236.15. (Cross Product–Connecting Homomorphism Lemma) Let X be a topological space, let (Y, B) be a pair of topological spaces and let R be a commutative ring. If $\varphi \in \mathrm{H}^m(X; R)$ and $\psi \in \mathrm{H}^n(B; R)$, then

$$\underset{\text{connecting homomorphism in cohomology for the pair } (X \times Y, X \times B)}{\delta}\big(\overbrace{\varphi \times \psi}^{\in \mathrm{H}^{m+n}(X \times B; R)}\big) = (-1)^m \cdot \varphi \times \overbrace{\underset{\text{connecting homomorphism in cohomology for the pair } (Y, B)}{\delta}\psi}^{\in \mathrm{H}^{n+1}(Y, B; R)} \in \mathrm{H}^{m+n+1}(X \times Y, X \times B; R).$$

Proof of Lemma 236.15. This equality follows fairly easily from the Cup Product–Coboundary Lemma 203.12, the definition of the cross product via the cup product and the explicit description of the connecting homomorphism in Definition 197.15. We leave it to the reader to fill in the details. ■

Proof of the Steenrod-Coboundary Proposition 236.12. We start out with a warm-up case.

Claim 1. Let Z be a topological space and let $a < b$ be two real numbers. The conclusion of the proposition holds for the pair $(Z \times [a, b], Z \times \partial([a, b]))$ of topological spaces.

Proof. Let $\theta \in \mathrm{H}^n(Z \times \partial([a, b]); \mathbb{F}_2) = \mathrm{H}^n(Z \times \{a, b\}; \mathbb{F}_2)$. Recall that we need to show that we have $\mathrm{Sq}^i(\delta\theta) = \delta(\mathrm{Sq}^i(\theta))$. First note that θ can be written as $\theta = \varphi \times \psi$ for some $\varphi \in \mathrm{H}^n(Z; \mathbb{F}_2)$ and some $\psi \in \mathrm{H}^0(\partial([a, b]); \mathbb{F}_2) = \mathrm{H}^0(\{a, b\}; \mathbb{F}_2)$.[2550] Note that we have the

[2550] This statement follows quite easily from the fact that $Z \times \partial([a, b]) = (Z \times \{a\}) \sqcup (Z \times \{b\})$, the observation that $\mathrm{H}^0(\{a, b\}) = \mathbb{Z} \cdot 1_{\{a\}} \oplus \mathbb{Z} \cdot 1_{\{b\}}$, the Cup Product-Ring Proposition 203.25 and the Cup Product–Disjoint Union Lemma 204.16. We leave it to the assiduous reader to assemble these steps in the right logical order.

following equalities in $\mathrm{H}^{n+i+1}(Z \times [a,b], Z \times \partial([a,b]); \mathbb{F}_2)$:

$$\begin{aligned}
\mathrm{Sq}^i(\delta\theta) &\overset{\text{since } \theta = \varphi\times\psi}{=} \mathrm{Sq}^i(\delta(\varphi\times\psi)) \overset{\text{by the Cross Product–Connecting Homomorphism Lemma 236.15, we also use that } (-1)^n = 1 \in \mathbb{F}_2}{=} \mathrm{Sq}^i(\varphi \times \delta\psi) \\
&= \sum_{j=0}^{i} \mathrm{Sq}^{i-j}(\varphi)\times\mathrm{Sq}^j(\delta\psi) \overset{\text{since } \delta\psi \in \mathrm{H}^1([a,b],\partial([a,b]);\mathbb{F}_2) \text{ and since } \mathrm{H}^i([a,b],\partial([a,b]);\mathbb{F}_2) = 0 \text{ for every } i \geq 2 \text{ we know that } \mathrm{Sq}^j(\delta\psi) = 0 \text{ for } j \geq 1}{=} \mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^0(\delta\psi) \\
&= \mathrm{Sq}^i(\varphi) \times \delta\psi \overset{\text{Axiom (1) says that } \mathrm{Sq}^0 = \mathrm{id}}{=} \delta(\mathrm{Sq}^i(\varphi) \times \psi) \overset{\text{Cross Product–Connecting Homomorphism Lemma 236.15}}{=} \delta(\mathrm{Sq}^i(\varphi\times\psi)) \\
&\overset{\text{by the Steenrod–Cross Product Proposition 236.14, together with Axioms (1) and (3) and the fact that } \psi \in \mathrm{H}^0(\partial([a,b]);\mathbb{F}_2)}{=} \delta(\mathrm{Sq}^i(\theta)).
\end{aligned}$$

⊞

The remainder of the proof of the proposition consists in cleverly reducing the general case to the above special case. Thus let (X, A) be a pair of topological spaces where $A \subseteq X$ is a closed subset. We denote by $i\colon A \to X$ the inclusion map. As in Definition 21.1 we set

$$\mathrm{Cyl}(i) = \begin{array}{c}\text{mapping cylinder of the} \\ \text{inclusion } i\colon A \to X\end{array} \coloneqq ((A \times [0,1]) \sqcup X)/\sim \quad \text{where } (a,1) \sim a \text{ for all } a \in A.$$

Claim 2. The obvious injections $A\times[0,1] \to \mathrm{Cyl}(i)$ and $X \to \mathrm{Cyl}(i)$ are closed embeddings.

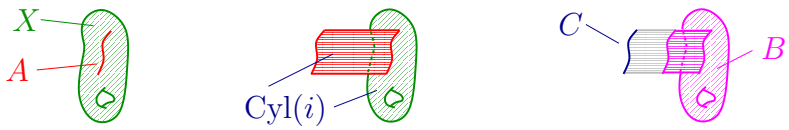
Proof. [2551] Recall that by the Mapping Cylinder–Pushout Lemma 21.3 we have the following pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{a\mapsto(a,1)} & A \times [0,1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (a,t)\mapsto[(a,t)]} \\ X & \xrightarrow{x\mapsto[x]} & \mathrm{Cyl}(f\colon A \to X). \end{array}$$

It follows from our hypothesis that $A \subseteq X$ is closed, that the inclusion $A \to X$ is a closed embedding. Furthermore the top horizontal map $X \to X \times [0,1]$ is a closed embedding. It thus follows from the Mapping Cylinder–Pushout Lemma 21.3 that the "opposite maps" $A \times [0,1] \to \mathrm{Cyl}(i)$ and $X \to \mathrm{Cyl}(f\colon A \to X)$ are closed embeddings. ⊞

It follows from Claim 2 that we can view $A \times [0,1]$ and X as subspaces of $\mathrm{Cyl}(i)$. Now we consider the following subsets of the mapping cylinder $\mathrm{Cyl}(i)$:

$$B \coloneqq (A \times [\tfrac{1}{2}, 1]) \cup X \quad \text{and} \quad C \coloneqq A \times \{0\}.$$



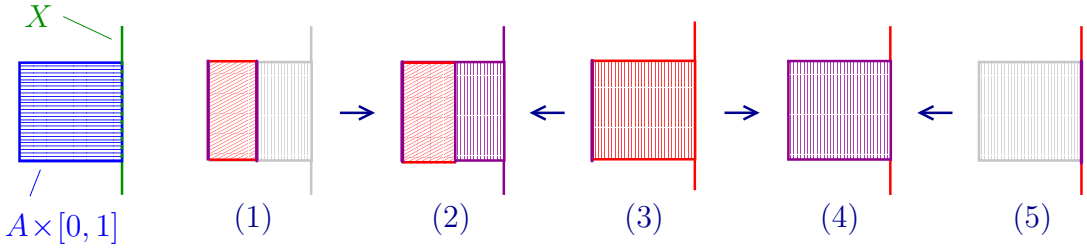
The five statements in the following claim break the proof of the proposition into five manageable steps:

Claim 3.

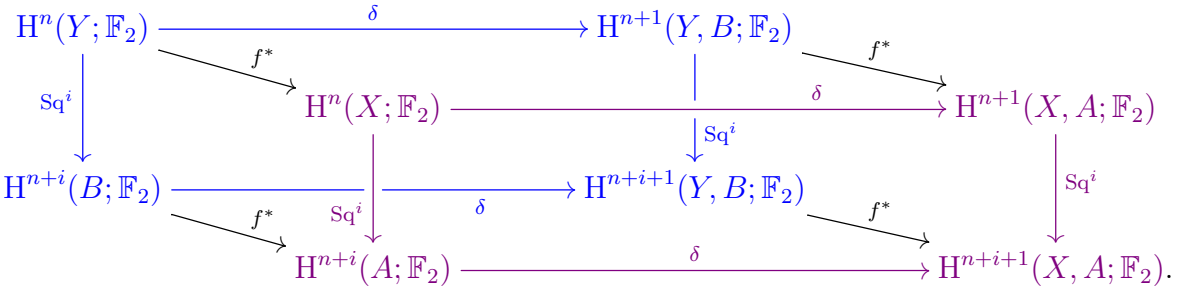
(1) The conclusion of the proposition holds for $(A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}]))$.

[2551] This claim is the content of Exercise 21.5 and the Mapping Cylinder–Embedding Lemma 21.4.

(2) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), B \cup C)$.
(3) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), C)$.
(4) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), A \times [0,1])$.
(5) The conclusion of the proposition holds for (X, A).

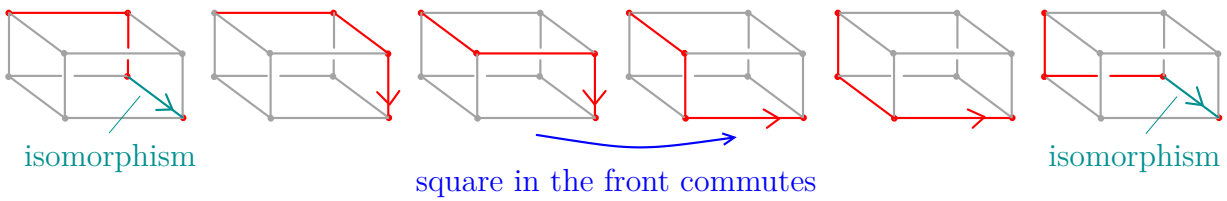


Proof. Before we work our way through the five steps we need to make two related observations. Let $f\colon (X, A) \to (Y, B)$ be a continuous map of pairs of topological spaces. We obtain the following cubical diagram:



It follows from the naturality of the Steenrod operations that the parallelograms to the left and right commute. Furthermore, it follows from the naturality of the connecting homomorphism, see the LES-of-Triple Proposition 198.13, that the top and bottom parallelograms commute. Using these facts we can make the following two observations:

(i) If the square in the front commutes and if the map f^* to the bottom right is an isomorphism, then it follows from a stroll through the cube, as indicated in the figure below, that the square in the back also commutes.
(ii) If the square in the back commutes and if the map f^* to the top left is an isomorphism, then it follows as in (i) that the square in the front also commutes.



Now we turn to the actual proof of the five steps in the claim.

(1) This statement follows immediately from the previous claim.
(2) We consider the inclusion map $f\colon (A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}])) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (1) we know that the square in the front commutes. We leave it to the reader to cleverly apply the Excision-in-H*-Theorem 200.1 to show

that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back commutes.

(3) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), C) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (2) we know that the square in the back commutes. Since on the first entry f is the identity we see that the top left map f^* is an isomorphism. Thus we obtain from (ii) that the square in the front also commutes.

(4) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), \overbrace{A \times \{0\}}^{=C}) \to (\mathrm{Cyl}(i), A \times [0,1])$. By (3) we know that the square in the front commutes. Since $A \times \{0\}$ is a deformation retract of $A \times [0,1]$ we obtain from Homotopy Equivalence-H*-Proposition 198.12 (1) and the LES-of-Triple Proposition 198.13 that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back also commutes.

(5) We consider the inclusion

$$\begin{array}{rccc} f\colon & X & \to & \mathrm{Cyl}(i) \\ & x & \mapsto & [(x,1)]. \end{array}$$

It also defines a map $f\colon (X, A) \to (\mathrm{Cyl}(i), A \times [0,1])$ of pairs of topological spaces. We consider the corresponding cube. By (4) we know that the square in the back commutes. By the Mapping Cylinder–Homotopy Lemma 21.5 (1) the map $f\colon X \to \mathrm{Cyl}(i)$ is in fact a homotopy equivalence. Thus it follows from the Homotopy Equivalence-H*-Proposition 198.12 (1) that the top left f^* is an isomorphism. Therefore we obtain from (ii) that the square in the front commutes. ∎

236.3. **Steenrod operations and suspensions.** In the Steenrod-Coboundary Proposition 236.12 we saw that Steenrod operations commute (under a mild hypothesis) with connecting homomorphisms in cohomology. In this section we will use this fact to show that Steenrod operations behave well under taking suspensions. This fact will help us to obtain new information on homology groups of spheres.

First recall that in the Suspension-H$_*$-Proposition 149.3 we saw that taking the suspension $\Sigma(X)$ of a topological space shifts the reduced homology groups "upward by one". The following proposition says that the same phenomenon occurs when we consider cohomology groups.

Proposition 236.16. (Suspension-H*-Proposition) Let X be a non-empty topological space, let $n \in \mathbb{N}^{2552}$ and let G be an abelian group. As in Definition 149.2 we denote by

$$C_-X \;:=\; \Sigma(X) \setminus \{N\} \qquad \text{and} \qquad C_+X \;:=\; \Sigma(X) \setminus \{S\}$$

the "lower and upper part" of the suspension $\Sigma(X)$. We refer to the figure below for an illustration.

(1) The following maps are isomorphisms:

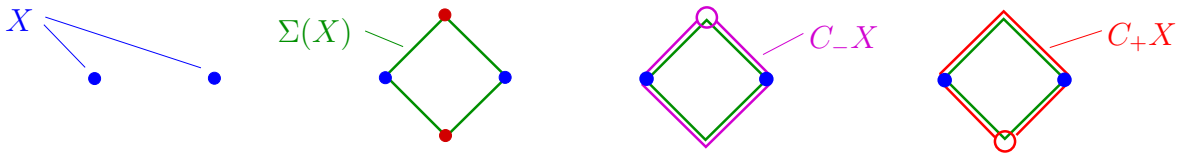
$$\mathrm{H}^n(X;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(C_+X, X; G) \leftarrow \mathrm{H}^{n+1}(\Sigma(X), C_-X; G) \to \mathrm{H}^{n+1}(\Sigma(X); G).$$

Here the map to the left is the connecting homomorphism of the pair (C_+X, X) from the LES-of-Triple Proposition 198.13 The remaining two horizontal maps are induced by the obvious inclusion maps.

(2) The isomorphism

$$\Sigma_X\colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \mathrm{H}^{n+1}(\Sigma(X);G)$$

that is given by the maps in (1) is in fact a natural isomorphism.



Proof. The proof is a slight variation on the proof of the Suspension-H_*-Proposition 149.3. We leave it to the reader to fill in the details. ■

The following proposition shows that Steenrod operations behave well under taking suspensions.

Proposition 236.17. (Steenrod-Suspension Proposition) Given any topological space X, any $n \in \mathbb{N}$ and any $i \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(X;\mathbb{F}_2) & \xrightarrow[\cong]{\Sigma_X} & \mathrm{H}^{n+1}(\Sigma(X);\mathbb{F}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ \mathrm{H}^{n+i}(X;\mathbb{F}_2) & \xrightarrow[\cong]{\Sigma_X} & \mathrm{H}^{n+i+1}(\Sigma(X);\mathbb{F}_2). \end{array}$$

Remark. The proposition says in particular that if there exists a non-trivial Steenrod operation on a topological space X, then its suspension inherits a non-trivial Steenrod operation. This is in so far interesting as we know by the Suspension-Cup Product Lemma 204.12 that the cup product vanishes on suspensions. The fact that suspensions can have non-trivial Steenrod operations will allow us to prove several hitherto inaccessible results. □

Proof. We consider the following diagram:

$$\begin{array}{ccccccc} & & & & =:\Sigma_X & & \\ \mathrm{H}^n(X;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(C_+X,X;\mathbb{F}_2) & \xleftarrow[\cong]{} & \mathrm{H}^{n+1}(\Sigma(X),C_-X;\mathbb{F}_2) & \xrightarrow[\cong]{} & \mathrm{H}^{n+1}(\Sigma(X);\mathbb{F}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ \mathrm{H}^{n+i}(X;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^{n+1+i}(C_+X,X;\mathbb{F}_2) & \xleftarrow{\cong} & \mathrm{H}^{n+1+i}(\Sigma(X),C_-X;\mathbb{F}_2) & \xrightarrow{\cong} & \mathrm{H}^{n+1+i}(\Sigma(X);\mathbb{F}_2). \\ & & & & =:\Sigma_X & & \end{array}$$

We now claim that the diagram commutes:

- Note that it follows from the Space-Cone Embedding Lemma 20.19 that the natural map $X \to C_+X$ is a closed embedding. It thus follows from the Steenrod-Coboundary Proposition 236.12 that the first square commutes.
- The two other squares commute by the naturality of the Steenrod operation Sq^i. ■

Next we recall that by Definitions 134.24 and 216.2 we have three different types of Hopf maps:

[2552]Basically the same statement works for $n = 0$, the only nuisance is that in that case we need to work with reduced cohomology groups.

Definition 236.18. We consider the following maps:

$$\underbrace{\begin{array}{ccc} H\colon S^3 & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ \underbrace{(z_1,z_2)}_{\in \mathbb{C}^2} & \mapsto & [z_1 : z_2] \end{array}}_{\text{Hopf map}} \quad \underbrace{\begin{array}{ccc} H_{\mathbb{H}}\colon S^7 & \to & S^4 = \mathbb{H}\mathrm{P}^1 \\ \underbrace{(z_1,z_2)}_{\in \mathbb{H}^2} & \mapsto & [z_1 : z_2] \end{array}}_{\text{quaternionic Hopf map}} \quad \text{and} \quad \underbrace{\begin{array}{ccc} H_{\mathbb{O}}\colon S^{15} & \to & S^8 = \mathbb{O}\mathrm{P}^1 \\ \underbrace{(z_1,z_2)}_{\in \mathbb{O}^2} & \mapsto & [z_1 : z_2]. \end{array}}_{\text{octonionic Hopf map}}$$

In the Hopf Map–Homotopy Theorem 216.6 we showed that the Hopf map $H\colon S^3 \to S^2$ is not null-homotopic. The key to doing so were the following two observations:

(1) The mapping cone $\mathrm{Cone}(H\colon S^3 \to S^2)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$.
(2) The cup product on $\mathbb{C}\mathrm{P}^2$ has the property that for a generator $\varphi \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ the square φ^2 is non-zero.

Since the Steenrod operations are a generalization of the mod 2 "cup product square" and since Steenrod operations behave well with respect to suspensions it seems reasonable that we can use Steenrod operations to show that the suspensions of the Hopf maps are not null-homotopic.

Indeed, we can prove the following theorem:

Theorem 236.19. (Steenrod Operations-π_n-Theorem)

(1) For any $n \geq 2$ the $(n-2)$-fold suspension[2553]

$$\Sigma^{n-2}(H)\colon \underbrace{\Sigma^{n-2}(S^3)}_{=S^{n+1}} \to \underbrace{\Sigma^{n-2}(S^2)}_{=S^n}$$

of the Hopf map defines a non-trivial element in $\pi_{n+1}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+1}(S^n) \neq 0$.

(2) For any $n \geq 4$ the $(n-4)$-fold suspension

$$\Sigma^{n-4}(H_{\mathbb{H}})\colon \underbrace{\Sigma^{n-4}(S^7)}_{=S^{n+3}} \to \underbrace{\Sigma^{n-4}(S^4)}_{=S^n}$$

of the quaternionic Hopf map defines a non-trivial element in $\pi_{n+3}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+3}(S^n) \neq 0$.

(3) For any $n \geq 8$ the $(n-8)$-fold suspension

$$\Sigma^{n-8}(H_{\mathbb{O}})\colon \underbrace{\Sigma^{n-8}(S^{15})}_{=S^{n+7}} \to \underbrace{\Sigma^{n-8}(S^8)}_{=S^n}$$

of the octonionic Hopf map defines a non-trivial element in $\pi_{n+7}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+7}(S^n) \neq 0$.

Before we turn to the proof of the Steenrod Operations-π_n-Theorem 236.19 we recall, for the reader's convenience, the following, frequently overlooked, proposition which says that taking suspensions and taking mapping cones commutes.

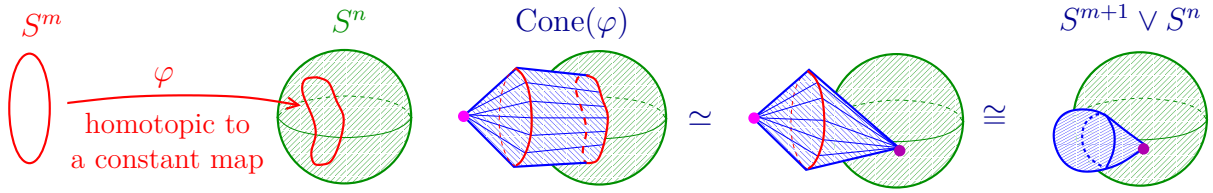
Proposition 21.26. (Mapping Cone-Suspension Proposition) If $f\colon X \to Y$ is a continuous map between topological spaces, then there exists a homeomorphism

$$g\colon \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \xrightarrow{\cong} \Sigma(\mathrm{Cone}(f\colon X \to Y)).$$

[2553] This statement gives in particular an affirmative answer to Question 134.28.

We also state the following lemma which we are already dimly aware of.

Lemma 236.20. (Mapping Cone-of-Null Homotopic Lemma) If a map $\varphi\colon S^m \to S^n$ is homotopic to a constant map, then the corresponding mapping cone $\mathrm{Cone}(\varphi\colon S^m \to S^n)$ is homotopy equivalent to $S^{m+1} \vee S^n$.



Proof. We have

$$\mathrm{Cone}(\varphi\colon S^m \to S^n) \underset{\text{by the Mapping Cone-of-Homotopy Lemma 21.17, since } \varphi \text{ is homotopic to a constant map}}{\simeq} \mathrm{Cone}(\text{constant map}\colon S^m \to S^n) \underset{\text{by the Mapping Cone-of-Constant Map Lemma 216.4}}{\cong} S^{m+1} \vee S^n.$$

■

Now we turn to the proof of the Steenrod Operations-π_n-Theorem 236.19. For clarity's sake we first provide a detailed proof of the Steenrod Operations-π_n-Theorem 236.19 (1). Afterwards we explain how to modify that proof to obtain statements (2) and (3) of the Steenrod Operations-π_n-Theorem 236.19.

Proof of the Steenrod Operations-π_n-Theorem 236.19 (1). In the following we will show that $\Sigma(H)\colon S^4 \to S^3$ is not homotopic to a constant map. The general statement regarding the $(n-2)$-fold suspension $\Sigma^{n-2}(H)$ follows easily from iterating the argument below.

For once we do a proof by contradiction. So let us assume that the map $\Sigma(H)\colon S^4 \to S^3$ is in fact homotopic to a constant map. We make the following preparations:

(1) To simplify the notation we denote the mapping cone of a continuous map $f\colon X \to Y$ by $\mathrm{C}(f)$ instead of $\mathrm{Cone}(f)$. Furthermore we write $\mathbb{F} = \mathbb{F}_2$.
(2) Note that by the Mapping Cone-of-Hopf Map Lemma 216.5 there exists a homeomorphism $f\colon \mathrm{C}(H\colon S^3 \to S^2) \xrightarrow{\cong} \mathbb{C}\mathrm{P}^2$.
(3) We denote by $g\colon \Sigma(\mathrm{C}(H)) \xrightarrow{\cong} \mathrm{C}(\Sigma(H))$ the homeomorphism from the Mapping Cone-Suspension Proposition 21.26.
(4) Since we assume that the map H is homotopic to a constant map we obtain from the Mapping Cone-of-Null Homotopic Lemma 236.20 that there exists a homotopy equivalence $h\colon \mathrm{C}(\Sigma(H)) \to S^3 \vee S^5$.

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^2(\mathrm{C}(H);\mathbb{F}) & \xrightarrow[\cong]{\Sigma} & \mathrm{H}^3(\Sigma(\mathrm{C}(H));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^3(\mathrm{C}(\overbrace{\Sigma(H)}^{S^4 \to S^3});\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^3(S^5 \vee S^3;\mathbb{F}) \\
{\scriptstyle x \mapsto x \cup x}\downarrow \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \downarrow{\scriptstyle \mathrm{Sq}^2} \\
\mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^4(C(H);\mathbb{F}) & \xrightarrow[\cong]{\Sigma} & \mathrm{H}^5(\Sigma(\mathrm{C}(H));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^5(\mathrm{C}(\Sigma(H));\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^5(S^5 \vee S^3;\mathbb{F}).
\end{array}$$

We make the following observations:

(1) The maps Σ are the suspension isomorphisms given by the Suspension-H^*-Proposition 236.16.

(2) It follows from Axiom (2) that the two vertical maps on the left are actually the same.
(3) It follows from the naturality of Sq^2 that the first, third and the fourth square commute.
(4) It follows from the Steenrod-Suspension Proposition 236.17 that the second square commutes.
(5) Since f and g are homeomorphisms we see that the induced maps f^* and g^* are isomorphisms. Furthermore, since h is a homotopy equivalence we obtain from the Homotopy Equivalence-H*-Proposition 198.12 (1) that the horizontal maps h^* are isomorphisms.
(6) The argument of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7, but with $\mathbb{Z}$-coefficients replaces by $\mathbb{F}_2$-coefficients, shows that there exists a unique non-trivial $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{F})$ and that this element satisfies $x \cup x \neq 0 \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \cong \mathbb{F}$. In particular we see that the left vertical homomorphism is an isomorphism and that it is non-trivial.
(7) By the Steenrod Operation–Wedge-of-Sphere Lemma 236.7 we know that the vertical homomorphism to the right is trivial.

In summary we have shown that the above diagram commutes, we have shown that all horizontal maps are isomorphisms, we have seen that the left vertical map is non-trivial and have pointed out that the right vertical map is trivial. But this is absurd. In other words, we have obtained a contradiction to our hypothesis that the map $\Sigma(H)\colon S^4 \to S^3$ is null-homotopic. ■

Proof of the Steenrod Operations-π_n-Theorem 236.19 (2) and (3). As we will see shortly, the proof for the quaternionic and octonionic Hopf maps is basically the same as the proof for the "classical" Hopf map.

Let us first consider the quaternionic Hopf. By the Mapping Cone-of-Hopf Map Lemma 216.5 (2) there exists a homeomorphism $f\colon \mathrm{C}(H_{\mathbb{H}}) \xrightarrow{\cong} \mathbb{H}\mathrm{P}^2$. If $\Sigma(H_{\mathbb{H}})\colon S^8 \to S^5$ was null-homotopic we would, as in the proof of (1), obtain a commutative diagram of the following form:

$$\begin{array}{ccccccccc}
\mathrm{H}^4(\mathbb{H}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^4(\mathrm{C}(H_{\mathbb{H}});\mathbb{F}) & \xrightarrow{\Sigma} & \mathrm{H}^5(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^5(\mathrm{C}(\overbrace{\Sigma(H_{\mathbb{H}})}^{S^8\to S^5});\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^5(S^9\vee S^5;\mathbb{F}) \\
{\scriptstyle x\mapsto x\cup x}\downarrow\ \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} & & \downarrow{\scriptstyle \mathrm{Sq}^4} \\
\mathrm{H}^8(\mathbb{H}\mathrm{P}^2;\mathbb{F}) & \xrightarrow[\cong]{f^*} & \mathrm{H}^8(C(H_{\mathbb{H}});\mathbb{F}) & \xrightarrow{\Sigma} & \mathrm{H}^9(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) & \xleftarrow[g^*]{\cong} & \mathrm{H}^9(\mathrm{C}(\Sigma(H_{\mathbb{H}}));\mathbb{F}) & \xleftarrow[\cong]{h^*} & \mathrm{H}^9(S^9\vee S^5;\mathbb{F}).
\end{array}$$

By the $\mathbb{F}_2$-version of the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5[2554] the cup product on the left is non-trivial. Thus the same logic as in the proof of (1) leads to a contradiction.

The argument for the octonionic Hopf map is almost identical to the argument for the quaternionic Hopf map. We just need to redo the above argument with the Mapping Cone-of-Hopf Map Lemma 216.5 (3) and with the $\mathbb{F}_2$-version of the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 215.9. ■

For the record we jot down the following proposition that we proved in the proof of the Steenrod Operations-π_n-Theorem 236.19. We will make use of it later on.

[2554]In the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5 we calculated the cohomology ring of $\mathbb{H}\mathrm{P}^n$ with $\mathbb{Z}$-coefficients. But the same proof gives the obvious analogous statement for $\mathbb{F}_2$-coefficients.

Proposition 236.21. (Steenrod-Isomorphism Proposition) Given any $n \in \mathbb{N}_0$ the maps

$$\begin{array}{lrcl} (1) & \mathrm{Sq}^2\colon \mathrm{H}^{2+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{F}_2) & \to & \mathrm{H}^{4+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{F}_2) \\ (2) & \mathrm{Sq}^4\colon \mathrm{H}^{4+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{F}_2) & \to & \mathrm{H}^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{F}_2) \\ (3) & \mathrm{Sq}^8\colon \mathrm{H}^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{F}_2) & \to & \mathrm{H}^{16+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{F}_2) \end{array}$$

are isomorphisms.

236.4. **Proof of the Steenrod Theorem 236.3 I.** In the final two sections we will give a partial proof of the Steenrod Theorem 236.3. More precisely, we will construct the Steenrod operations and we will provide the proof of some of the properties. The proof is split into two parts:

(1) The goal of this section is to construct a certain interesting sequence of natural maps $\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$, where the zeroth map Δ_0 is a diagonal approximation.
(2) In the subsequent section we will show how this sequence of maps can be used to define the Steenrod operations.

In preparation for the construction of the sequence Δ_n we recall some definitions and results from earlier on. First recall that in Definition 163.2 we introduced the tensor product $A_* \otimes B_*$ of two chain complexes A_* and B_*. We will make use of the following harmless lemma.

Lemma 207.8. (Swap Factors–Chain Map Lemma) Let A_* and B_* be chain complexes. The unique map

$$T\colon A_* \otimes B_* \to B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \mapsto (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Since we will make use of this later on, we point out that T is natural in A_* and B_* and we point out that $T^2 = \mathrm{id}$. Next we recall Definition 203.16:

Definition 236.22. Let X be a topological space. A diagonal approximation consists of a natural chain map

$$\Phi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

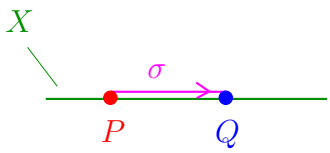
for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds in $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X)$:

$$\Phi(x) = x \otimes x.$$

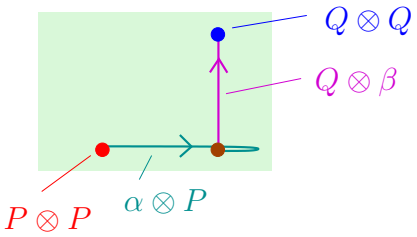
Example. We will try to rise to the challenge of finding a suitable illustration for a diagonal approximation Φ. We start out with the following observation: If $\sigma\colon \Delta^1 \to X$ is a singular 1-simplex from a point P to a point Q, then $\Phi(\sigma) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_1$ needs to satisfy

$$\partial\Phi(\sigma) \underset{\text{since } \Phi \text{ is a chain map}}{\overset{\uparrow}{=}} \Phi(\partial\sigma) = \Phi(Q-P) = \Phi(Q) - \Phi(P) \underset{\substack{\text{since } \Phi \text{ is determined} \\ \text{on singular 0-simplices}}}{\overset{\uparrow}{=}} Q \otimes Q - P \otimes P.$$

For example $\Phi(\sigma) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_1 = \mathrm{C}_1(X) \otimes \mathrm{C}_0(X) \oplus \mathrm{C}_0(X) \otimes \mathrm{C}_1(X)$ could be of the form $\alpha \otimes P + Q \otimes \beta$ where $\alpha\colon \Delta^1 \to X$ and $\beta\colon \Delta^1 \to X$ are singular simplices from P to Q.



diagonal approximation
$\Phi_0(\sigma) = \alpha \otimes P + Q \otimes \beta$



This situation is illustrated, with artistic liberties, in the figure above. □

In the following we recall, for the reader's convenience, our two main results on diagonal approximations:

Lemma 203.17 (Alexander-Whitney Lemma) There exists an explicit diagonal approximation.

Proposition 203.19. (Diagonal Approximation–Unique Existence Proposition). For any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.[2555]

The following theorem shows that any diagonal approximation can be extended to a sequence of maps.

Theorem 236.23. (Extended Diagonal Approximation Theorem)

(1) Given any topological space X there exists a sequence of natural maps

$$\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n[2556] such that the following two conditions are satisfied:

(1) The map Δ_0 is a diagonal approximation.

(2) For every $n \in \mathbb{N}_0$ the following equality holds:[25572558]

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial = (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

(2) Given two sequences Δ_n, Δ'_n, $n \in \mathbb{N}_0$, as in (1) there exists a sequence of natural maps

$$H_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n such that the following two conditions are satisfied:

(a) H_0 is the zero map.

(b) For every $n \in \mathbb{N}_0$ the following equality holds:[2559]

$$\partial \circ H_{n+1} + (-1)^n \cdot H_{n+1} \circ \partial = \Delta_n - \Delta'_n - (T + (-1)^n \cdot \mathrm{id}) \circ H_n.$$

[2555] On page 3898 we gave a down-to-earth proof of the proposition. Now we could also see it as a consequence of the Acyclic Model Theorem 235.7 (2).

In the proof of the Extended Diagonal Approximation Theorem 236.23 we will need the following innocuous variation on the definition of a chain map that we had given in Definition 143.2.

Definition 236.24.

(1) A chain map $f\colon C_* \to D_*$ of degree d between two chain complexes consists of a family $\{f_n\colon \mathrm{C}_n \to D_{n+d}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \;=\; \partial_{n+d} \circ f_n.$$

(2) An anti-chain map $f\colon C_* \to D_*$ of degree d is basically the same data as a chain map of degree d, except that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \;=\; -\partial_{n+d} \circ f_n.$$

Remark. Given an anti-chain map $\{f_n\}_{n\in\mathbb{N}_0}$ of degree d one can easily see that the maps $\{(-1)^n \cdot f_n\}_{n\in\mathbb{N}_0}$ form a chain map of degree d. This little trick makes it possible to translate all kinds of statements about chain maps to anti-chain maps. For example, as in the Chain Complex-H_*-Functor Lemma 143.4 (1) we see that an anti-chain map induces a map on homology groups. □

In the proof of the Extended Diagonal Approximation Theorem 236.23 we will need the following consequence of the Acyclic Model Theorem 235.1.

Theorem 236.25. (Acyclic Model Theorem) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized[2560] chain complexes.

(1) Let $S\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be a natural chain map of degree d. If we have $d \geq 1$, then for any topological space X and any $k \in \mathbb{N}_0$ there exists a natural map $P_X\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+d+1}$ such that for any $k \in \mathbb{N}_0$ we have

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+d}.$$

(2) For a natural anti-chain map $S\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree d the analogous data exists except that now we have

$$\partial \circ P_X - P_X \circ \partial \;=\; S_X\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+d}.$$

Proof of the Acyclic Model Theorem 236.25.

(1) Given any topological space X we consider $G(X) \coloneqq (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))[-d]$. Here we use the notation introduced in Notation 207.9, namely given a chain complex D_* we denote by $D_*[-d]$ the chain complex D_* "shifted by d to the right". This shift turns $G(X)$ into a generalized chain complex. Furthermore, this shift turns $S\colon \mathrm{C}_*(X) \to G(X)$ into a natural chain map of degree 0.

[2556] By a map of degree n between chain complexes C_* and D_* we mean a collection of homomorphisms $\mathrm{C}_k \to D_{k+n}$, $k \in \mathbb{N}_0$.

[2557] Note that both sides are maps $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree n.

[2558] The chain map $T\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is the map we introduced in the Swap Factors-Chain Map Lemma 207.8.

[2559] In other words, the map H_{n+1} is a chain homotopy between Δ_n and Δ'_n up to the "error term" $-(T + (-1)^n \cdot \mathrm{id}) \circ H_n$.

[2560] Recall that a generalized chain complex is like a chain complex except that we allow chain groups in negative degrees.

Next note that for any $k \in \mathbb{N}_0$ and any $i \geq 0$ we have

$$\begin{array}{rcccl} \mathrm{H}_i(G(\Delta^k)) & = & \mathrm{H}_i((\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))[-d]) & = & \mathrm{H}_{i+d}(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \\ & = & \bigoplus_{p+q=i+d} \mathrm{H}_p(\Delta^k) \otimes \mathrm{H}_q(\Delta^k) & = & 0. \\ & \uparrow & & & \uparrow \end{array}$$

by the Chain Complex Künneth Theorem 163.6 — by the calculation of $\mathrm{H}_*(\Delta^k)$ and since $d \geq 1$

Thus we see that all the hypotheses of the Acyclic Model Theorem 235.1 are satisfied. Therefore we see that for any topological space X and any $k \in \mathbb{N}_0$ there exists a natural map $P_X \colon \mathrm{C}_k(X) \to G_{k+1}(X)$ such that for every $k \in \mathbb{N}_0$ we have

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X \colon \mathrm{C}_k(X) \;\to\; G_k(X).$$

(2) Suppose we are given a natural anti-chain map S of degree d. As in the remark preceding the theorem we can turn S into a natural chain map T by setting $T_k = (-1)^k \cdot S_k$, $k \in \mathbb{N}_0$. We apply (1) and we obtain natural maps Q_X. Now we define $P_k := (-1)^k \cdot (Q_X)_k$, $k \in \mathbb{N}_0$. One easily verifies that the desired equality holds. ■

Now, finally, we can turn to the actual proof of the Extended Diagonal Approximation Theorem 236.23.

Proof of the Extended Diagonal Approximation Theorem 236.23.

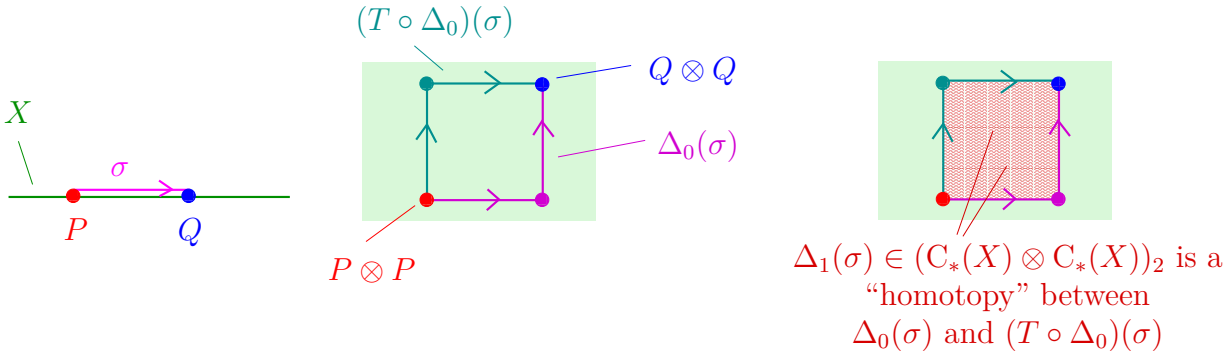
(1) We construct the maps $\Delta_n \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$ iteratively.

(i) First of all, we know by the Alexander-Whitney Lemma 203.17 that there exists a diagonal approximation $\Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$.

(ii) Note that $T \circ \Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is another diagonal approximation. It follows from the Diagonal Approximation–Uniqueness Proposition 203.19 that there exists a natural chain homotopy $\Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ between Δ_0 and $T \circ \Delta_0$. In other words, there exists a natural map $\Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree one such that

$$\partial \circ \Delta_1 + \Delta_1 \circ \partial \;=\; \underbrace{T \circ \Delta_0 - \Delta_0}_{=(T-\mathrm{id}) \circ \Delta_0}.$$

Thus we are done with the construction of the map Δ_1. This step is illustrated in the figure below.



(iii) The starting point for defining the next map Δ_2 is the following amusing little calculation:[2561]

$$0 \overset{\text{since } T^2=\mathrm{id}}{=} (T^2-\mathrm{id})\circ\Delta_0 = (T+\mathrm{id})\circ(T-\mathrm{id})\circ\Delta_0 \overset{\text{by (ii)}}{=} (T+\mathrm{id})\circ(\partial\circ\Delta_1+\Delta_1\circ\partial)$$
$$= \partial\circ(T+\mathrm{id})\circ\Delta_1 + (T+\mathrm{id})\circ\Delta_1\circ\partial.$$
$$\overset{\text{since } T \text{ and id are chain maps they commute with the boundary maps}}{=}$$

This shows that the map $(T+\mathrm{id})\circ\Delta_1\colon \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ is a natural anti-chain map of degree one. By the Acyclic Model Theorem 236.25 (2) there exists a natural map $\Delta_2\colon \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ of degree two such that

$$\partial\circ\Delta_2-\Delta_2\circ\partial = (T+\mathrm{id})\circ\Delta_1.$$

This conclude the construction of the map Δ_2.

(iv) Just for kicks we do the calculation for the next stage as well. We perform the following calculation:

$$0 \overset{\text{since } T^2=\mathrm{id}}{=} (T^2-\mathrm{id})\circ\Delta_1 = (T-\mathrm{id})\circ(T+\mathrm{id})\circ\Delta_1 \overset{\text{by (iii)}}{=} (T-\mathrm{id})\circ(\partial\circ\Delta_2-\Delta_2\circ\partial)$$
$$= \partial\circ(T-\mathrm{id})\circ\Delta_2 - (T-\mathrm{id})\circ\Delta_2\circ\partial.$$
$$\overset{\text{since } T \text{ and id are chain maps they commute with the boundary maps}}{=}$$

This shows that the map $(T-\mathrm{id})\circ\Delta_2\colon \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ is a natural chain map of degree two. By the Acyclic Model Theorem 236.25 (1) there exists a natural map $\Delta_3\colon \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ of degree three such that

$$\partial\circ\Delta_3+\Delta_3\circ\partial = (T-\mathrm{id})\circ\Delta_2.$$

This concludes the construction of the map Δ_3.

(iv) We iterate this procedure and we obtain the desired sequence of natural maps Δ_n.

(2) Given two sequences Δ_n, Δ'_n as in (1) we define the desired maps H_n, $n\in\mathbb{N}_0$ in a fashion that is quite similar to the approach taken in (1).

(i) We set $H_0=0$.

(ii) Next we point out that by definition Δ_0 and Δ'_0 are diagonal approximations. Thus it follows from the Diagonal Approximation–Uniqueness Existence Proposition 203.19 that there exist natural maps H_1 such that $\partial\circ H_1+H_1\circ\partial=\Delta_0-\Delta'_0$.

(iii) We calculate that

$$\partial\circ(\Delta_1-\Delta'_1-(T-\mathrm{id})\circ H_1)+(\Delta_1-\Delta'_1-(T-\mathrm{id})\circ H_1)\circ\partial$$
$$= (\partial\circ\Delta_1+\Delta_1\circ\partial)-(\partial\circ\Delta'_1+\Delta'_1\circ\partial)-\partial\circ(T-\mathrm{id})\circ H_1-(T-\mathrm{id})\circ H_1\circ\partial$$
$$\overset{\text{since } \partial\circ\Delta_1+\Delta_1\circ\partial=(T-\mathrm{id})\circ\Delta_0 \text{ (same for } \Delta'_0,\Delta'_1) \text{ and since } T \text{ and id are chain maps}}{=} (T-\mathrm{id})\circ(\Delta_0-\Delta'_0)-(T-\mathrm{id})\circ(\partial\circ H_1+H_1\circ\partial) \overset{\text{since } \partial\circ H_1+H_1\circ\partial=\Delta_0-\Delta'_0}{=} 0.$$

[2561] These are equalities of maps $\mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$.

This shows that $\Delta_1 - \Delta_1' - (T - \mathrm{id}) \circ H_1$ is an anti-chain map of degree one. By the Acyclic Model Theorem 236.25 (2) there exists a natural map H_2 with

$$H_2 \circ \partial - \partial \circ H_2 \;=\; -(T - \mathrm{id}) \circ H_1 + \Delta_1 - \Delta_1'.$$

(iv) Now we iterate this procedure to obtain the desired sequence of natural maps. One easily sees that we get the correct pattern for the signs involved. ■

Remark. If there did exist a diagonal approximation $\Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ that is symmetric in the sense that $T \circ \Delta_0 = \Delta_0$, then we could take $\Delta_1 = 0$ and we could also take all higher Δ_n to be zero. Later, in Chapter 236, we will see that this is in general not the case. So even though diagonal approximations are symmetric on the degree 0 level, they cannot be chosen to be symmetric in higher degrees. □

236.5. **Proof of the Steenrod Theorem 236.3 II.** In this section we will use the sequence of natural maps $\Delta_n \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$, from the Extended Diagonal Approximation Theorem 236.23 to define the Steenrod operations.

To get an inspiration for how to define the Steenrod operations we first recall the definition of the cup product. To do so we need the following lemma which is inspired by the definitions on from Definition 203.4.

Lemma 236.26. (Cochain Tensor Lemma) Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes and let R be a commutative ring. Given $p, q \in \mathbb{N}_0$ we consider the map[2562]

$$\begin{aligned} \Xi_{p,q} \colon \mathrm{Hom}(\mathcal{C}_p, R) \otimes \mathrm{Hom}(\mathcal{D}_q, R) &\to \mathrm{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\ \varphi \otimes \psi &\mapsto \left((\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\text{projection}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi \otimes \psi} \begin{array}{r} R \otimes R \to R \\ a \otimes b \mapsto a \cdot b \end{array} \right). \end{aligned}$$

The corresponding map

$$\Xi := \bigoplus_{p,q \in \mathbb{N}_0} \Xi_{p,q} \colon \mathrm{Hom}(\mathcal{C}_*, R) \otimes \mathrm{Hom}(\mathcal{D}_*, R) \;\to\; \mathrm{Hom}(\mathcal{C}_* \otimes \mathcal{D}_*, R)$$

is a cochain map.

Proof. We proved this lemma implicitly in the proof of the Cocycle Tensor Lemma 203.5. ■

The following lemma gives us a slight reformulation of the original definition of the cup product.

Lemma 236.27. (Cup Product Description Lemma) Let X be a topological space and let R be a commutative ring. Furthermore let $\Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be a diagonal approximation. We consider the cochain map

$$\mathrm{Hom}(\mathrm{C}_*(X), R) \otimes \mathrm{Hom}(\mathrm{C}_*(X), R) \xrightarrow{\Xi} \mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) \xrightarrow{\Delta_0^*} \mathrm{Hom}(\mathrm{C}_*(X), R).$$

$$=: h_0$$

The resulting map

$$\begin{aligned} \mathrm{H}^p(X; R) \times \mathrm{H}^q(X; R) &\to \mathrm{H}^{p+q}(X; R) \\ ([\varphi], [\psi]) &\mapsto [h_0(\varphi \otimes \psi)] \end{aligned}$$

[2562]The tensor product of cochain complexes is defined basically the same way as we defined the tensor product of chain complexes, see Definition 163.2.

equals the cup product as defined in Definition 203.8.

Proof. We denote by $d\colon X \to X \times X$ the diagonal map that is given by $d(x) = (x, x)$. Let $\Theta\colon \mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ denote a Eilenberg-Zilber map from the Eilenberg-Zilber Theorem 164.2. For the purpose of this proof the only thing we need to know about the Eilenberg-Zilber map Θ is that, basically by definition, the homomorphism $\Theta \circ d\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is a diagonal approximation. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{Hom}(\mathrm{C}_*(X), R) \otimes \mathrm{Hom}(\mathrm{C}_*(X), R) & \xrightarrow{\Xi} & \mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) & \xrightarrow{\Delta_0^*} & \mathrm{Hom}(\mathrm{C}_*(X), R) \\ & & \downarrow{\scriptstyle \Theta^*} & \nearrow{\scriptstyle d^*} & \\ & & \mathrm{Hom}(\mathrm{C}_*(X \times X), R). & & \end{array}$$

The lower sequence of maps is at the heart of the definition of the cup product in Definition 203.8. Since $\Theta \circ d$ is a diagonal approximation we obtain from the Diagonal Approximation–Uniqueness Proposition 203.19 that Δ_0 and $\Theta \circ d$ are chain homotopic. But by the Dual Cochain Map Lemma 197.20 (3) this means that the two maps induce the same map on cohomology. The lemma follows almost immediately from the above observations. ∎

Now we turn to the actual construction of the Steenrod operations. To do so we pick a sequence of natural maps

$$\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

which satisfies the conditions set out in the Extended Diagonal Approximation Theorem 236.23 (1). In light of the Cup Product Description Lemma 236.27 it is clear what we want to do next. Namely we want to replace Δ_0 by the "higher maps" Δ_n. This leads us straight to the following definition:

Definition 236.28. Let X be a topological space and let R be a commutative ring. Given $i \in \mathbb{N}_0$ we define

$$\mathrm{Hom}(\mathrm{C}_*(X), R) \otimes \mathrm{Hom}(\mathrm{C}_*(X), R) \xrightarrow{\Xi} \mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) \xrightarrow{\Delta_i^*} \mathrm{Hom}(\mathrm{C}_*(X), R).$$

$$=: h_i$$

Recall that Δ_i is a map of degree i. Since we dualize we see that h_i is a map of degree $-i$. For convenience we define $h_i := 0$ for $i \in \mathbb{Z}_{<0}$.

Now we can finally define the Steenrod operations.

Theorem 236.29. (Steenrod Operation–Existence Theorem) Let X be a topological space, let $A \subseteq X$ be a subset and let R be a commutative ring. If $1 + 1 = 0$ in R, then

for any $q \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{Sq}^i \colon \mathrm{H}^q(X, A; R) & \to & \mathrm{H}^{q+i}(X, A; R) \\ [\underbrace{\varphi}_{\in \mathrm{C}^q(X;R)}] & \mapsto & [h_{q-i}(\underbrace{\varphi \otimes \varphi}_{\in \mathrm{C}^q \otimes \mathrm{C}^q})] \end{array}$$

is a well-defined natural homomorphism.

Remark. The proof of the Steenrod Operation–Existence Theorem 236.29 that we give below is a long and winding calculation, with few interesting view points. Perhaps more interesting than the question, why the definition works, is the question, why does the map

$$\begin{array}{rcl} \mathrm{C}^q(X, A; R) \otimes \mathrm{C}^r(X, A; R) & \to & \mathrm{C}^{q+r+j}(X, A; R) \\ \varphi \otimes \psi & \mapsto & h_j(\underbrace{\varphi \otimes \psi}_{\in \mathrm{C}^q \otimes \mathrm{C}^r}) \end{array}$$

not descend to a map on cohomology? □

The proof of the Steenrod Operation–Existence Theorem 236.29 requires some preparations. First of all, note that the equalities in the Extended Diagonal Approximation Theorem 236.23 translate into the following lemma.

Lemma 236.30. (Coboundary-h_n-Lemma) Let X be a topological space. Furthermore let R be a commutative ring with $1_R + 1_R = 0_R$. For any $\alpha, \beta \in \mathrm{C}^*(X; R)$ we have the following equality:

$$h_{n+1}(\delta(\alpha \otimes \beta)) + \delta(h_{n+1}(\alpha \otimes \beta)) \;=\; h_n(\alpha \otimes \beta + \beta \otimes \alpha) \;\in\; \mathrm{C}^*(X; R).$$

Proof. Recall, see the Extended Diagonal Approximation Theorem 236.23, that one of the defining properties of the sequence Δ_n is that for every $n \in \mathbb{N}_0$ the following equality holds:

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial \;=\; (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

We dualize this equality, we use the equality $1 + 1 = 0$, we use the fact, shown in the Cochain Tensor Lemma 236.26 that Ξ is a cochain map. After performing all these steps one obtains the desired equality. We leave it to the reader to fill in the few remaining details. ■

For elements of the form $\alpha \otimes \alpha$ the following lemma shows how the maps h_n relate to coboundaries.

Lemma 236.31. (Square-Coboundary-h_n-Lemma) Let X be a topological space and let R be a commutative ring. If $1 + 1 = 0$ in R, then for any $\alpha \in \mathrm{C}^q(X; R)$ we have

$$h_{n+1}(\delta\alpha \otimes \delta\alpha) \;=\; \delta h_{n+1}(\alpha \otimes \delta\alpha) + \delta h_n(\alpha \otimes \alpha) \;\in\; \mathrm{C}^{2q-n+1}(X; R).$$

Proof. Since we assume that $1 + 1 = 0$ in R we can ignore conveniently enough ignore signs throughout the proof. Thus we compute

$$\begin{array}{rcl} h_{n+1}(\delta\alpha \otimes \delta\alpha) & \overset{\text{since } \delta(c\otimes d) = \delta c \otimes d + c \otimes \delta d}{\overset{\downarrow}{=}} & h_{n+1}(\delta(\alpha \otimes \delta\alpha)) \\ & = & \delta h_{n+1}(\alpha \otimes \delta\alpha) + h_n(\delta(\alpha \otimes \alpha)) \\ & \overset{\text{definition of } \delta \text{ on tensor product}}{\overset{\uparrow}{=}} & \end{array}$$

$$\begin{array}{rcl} & \overset{\text{by the Coboundary-}h_n\text{-Lemma 236.30}}{\overset{\downarrow}{=}} & \delta h_{n+1}(\alpha\otimes\delta\alpha) + h_n(\alpha\otimes\delta\alpha + \delta a\otimes\alpha) \\ & = & \delta h_{n+1}(\alpha \otimes \delta\alpha) + \delta h_n(\alpha \otimes \alpha). \\ & \overset{\text{by the Coboundary-}h_n\text{-Lemma 236.30 and since } h_{n-1}(\underbrace{\alpha\otimes\alpha + \alpha\otimes\alpha}_{=0 \text{ since } 1+1=0}) = 0}{\overset{\uparrow}{=}} & \end{array}$$

■

In the proof of the Steenrod Operation–Existence Theorem 236.29 we will need the following little lemma.

Lemma 236.32. (Binomial-h_n-Lemma) Let X be a topological space and let R be a commutative ring with $1+1=0$. For any $\alpha, \beta \in \mathrm{C}^q(X;R)$ the following equality holds:

$$h_n((\alpha+\beta)\otimes(\alpha+\beta)) = h_n(\alpha\otimes\alpha) + h_n(\beta\otimes\beta) + h_{n+1}(\delta(\alpha\otimes\beta)) + \delta(h_{n+1}(\alpha\otimes\beta)).$$

Proof of the Binomial-h_n-Lemma 236.32. This follows from "multiplying out", the linearity of h_n and the equality

$$h_n(\alpha\otimes\beta + \beta\otimes\alpha) \underset{\text{by the Coboundary-}h_n\text{-Lemma 236.30}}{\overset{\uparrow}{=}} h_{n+1}(\delta(\alpha\otimes\beta)) + \delta(h_{n+1}(\alpha\otimes\beta)).$$

■

Now, after these lengthy preparations we can provide the proof of the Steenrod Operation–Existence Theorem 236.29.

Proof of the Steenrod Operation–Existence Theorem 236.29. Let X be a topological space, let $A \subseteq X$ be a subset and let R be a commutative ring. First we consider the case that $A = \varnothing$. Let $\varphi \in \mathrm{C}^q(X;R)$ be a representative of some element in $\mathrm{H}^q(X;R)$.

(1) Note that $\varphi\otimes\varphi$ is of degree $2q$. Thus $h_{q-i}(\varphi\otimes\varphi)$ is of degree $2q-(q-i) = q+i$.

(2) Let $\varphi \in \mathrm{C}^q(X;R)$ be a cocycle. We calculate that

$$\delta h_{q-i}(\varphi\otimes\varphi) \underset{\text{by the Coboundary-}h_n\text{ Lemma 236.30}}{\overset{\uparrow}{=}} h_{q-i}(\delta(\varphi\otimes\varphi)) + h_{q-i-1}(\overbrace{\varphi\otimes\varphi+\varphi\otimes\varphi}^{=0 \text{ since } 1+1=0}) \underset{\text{definition of } \delta \text{ on tensor products}}{\overset{\uparrow}{=}} h_{q-i}(\delta\varphi\otimes\varphi + \varphi\otimes\delta\varphi) \underset{\text{since } \varphi \text{ is a cocycle}}{\overset{\uparrow}{=}} 0.$$

Thus we have shown that $h_{q-i}(\varphi\otimes\varphi)$ is a cocycle. In other words, we have shown that it defines an element in cohomology.

(3) Let $\psi \in \mathrm{C}^{q-1}(X;R)$. We have the following equality in $\mathrm{H}^{q+i}(X;R)$:

$$[h_{q-i}((\varphi+\delta\psi)\otimes(\varphi+\delta\psi))] \underset{\substack{\text{by the Binomial-}h_n\text{-Lemma 236.32, here we use}\\ \text{that } \delta(\varphi\otimes\delta\psi) = \delta\varphi\otimes\delta\psi + \varphi\otimes\delta\delta\psi = 0}}{\overset{\uparrow}{=}} [h_{q-i}(\varphi\otimes\varphi) + h_{q-i}(\delta\psi\otimes\delta\psi)] \underset{\substack{\text{by the Square-Coboundary-}h_n\text{-}\\ \text{Lemma 236.31}}}{\overset{\uparrow}{=}} [h_{q-i}(\varphi\otimes\varphi)].$$

This shows that the map Sq^i is well-defined, i.e. it does not depend on the choice of the representative of the cohomology class.

(4) It follows quite easily from the Binomial-h_n-Lemma 236.32 that Sq^i is in fact a homomorphism.

Finally we consider the case that $A \neq \varnothing$. Using the observation on page 3815 we make the identification

$$\mathrm{C}^q(X,A;R) = \{\varphi \in \mathrm{C}^q(X;R) = \mathrm{Hom}(\mathrm{C}_q(X),R) \mid \varphi \text{ vanishes on } \mathrm{C}_q(A) \subseteq \mathrm{C}_q(X)\}.$$

It follows from the naturality of h_n that if $\varphi \in \mathrm{C}^q(X)$ vanishes on $\mathrm{C}_q(A)$, then $h_{q-i}(\varphi\otimes\varphi)$ vanishes on the subgroup $\mathrm{C}_*(A)\otimes\mathrm{C}_*(X) + \mathrm{C}_*(X)\otimes\mathrm{C}_*(A)$ of $\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$. Thus we see that the map

$$\begin{array}{rcl} \mathrm{Sq}^i\colon \mathrm{H}^q(X,A;R) & \to & \mathrm{H}^{q+i}(X,A;R) \\ [\underbrace{\varphi}_{\in \mathrm{C}^q(X;R)}] & \mapsto & [h_{q-i}(\varphi\otimes\varphi)] \end{array}$$

is well-defined. ■

We continue with the following proposition:

Proposition 236.33. (Steenrod–Independence Proposition) The definition of the maps Sq^i does not depend on the choice of the natural maps Δ_n.

Proof. Suppose we are given another sequence of maps $\widetilde{\Delta}_n$ that have the properties described in the Extended Diagonal Approximation Theorem 236.23 (1). We define the maps $\widetilde{h}_i$ in the obvious way. By the Extended Diagonal Approximation Theorem 236.23 (2) there exists a sequence of natural maps

$$H_n \colon \mathrm{C}_*(X) \ \to \ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n such that $H_0 = 0$ and such that for every $n \in \mathbb{N}_0$ the following equality holds:

$$(*) \qquad \partial \circ H_{n+1} + (-1)^n \cdot H_{n+1} \circ \partial \ = \ \Delta_n - \widetilde{\Delta}_n - (T + (-1)^n \cdot \mathrm{id}) \circ H_n.$$

Given a cocycle $\varphi \in \mathrm{C}^q(X, A; \mathbb{F}_2)$ and $i \in \mathbb{N}_0$ we see that we have the following equality in $H^{2q-i}(X, A; R)$:

$$\begin{aligned}
[h_i(\varphi \otimes \varphi)] - [\widetilde{h}_i(\varphi \otimes \varphi)] &\overset{\text{definition of } h_i, \widetilde{h}_i}{=} [\Delta_i^*(\Xi(\varphi \otimes \varphi)) - \widetilde{\Delta}_i^*(\Xi(\varphi \otimes \varphi))] \\
&= [H_{i+1}^*(\delta(\Xi(\varphi \otimes \varphi)) + \underbrace{\delta(H_{i+1}^*(\Xi(\varphi \otimes \varphi)))}_{\text{coboundary}} + H_i^*((T + \mathrm{id})^*(\Xi(\varphi \otimes \varphi)))] \\
&\overset{\text{by } (*) \text{ and since } 1+1=0}{=} [H_{i+1}^*(\Xi(\underbrace{\delta(\varphi \otimes \varphi)}_{=0 \text{ since } \varphi \text{ cocycle}})) + H_i^*(\Xi(\underbrace{\varphi \otimes \varphi + \varphi \otimes \varphi}_{=0 \text{ since } 1+1=0}))] \ = \ 0.
\end{aligned}$$

since Ξ is a cochain map and since Ξ commutes with T

We have thus shown that the two definitions using Δ_n and $\widetilde{\Delta}_n$ agree. ∎

By the Steenrod–Independence Proposition 236.33 the maps Sq^i are unambiguously defined. Thus we can make the following definition:

Definition 236.34. Given a pair of topological spaces (X, A) and $i \in \mathbb{N}_0$ we refer to each map

$$\mathrm{Sq}^i \colon \mathrm{H}^q(X, A; \mathbb{F}_2) \ \to \ \mathrm{H}^{q+i}(X, A; \mathbb{F}_2)$$

as the i-th Steenrod operation.

By the Steenrod Operation–Existence Theorem 236.29 the Steenrod operations are additive cohomology operations. To complete the proof of the Steenrod Theorem 236.3 we need to verify that the maps Sq^i have all the other desired properties. In other words we need to prove the following theorem.

Theorem 236.35. (Steenrod Operation–Properties Theorem) The Steenrod operations Sq^i have the following properties:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.
(2) For any $\varphi \in \mathrm{H}^i(X, A; \mathbb{F}_2)$ we have $\mathrm{Sq}^i(\varphi) = \varphi^2$.
(3) For any $\varphi \in \mathrm{H}^j(X, A; \mathbb{F}_2)$ with $j < i$ we have $\mathrm{Sq}^i(\varphi) = 0$.
(4) For any $\varphi \in \mathrm{H}^r(X, A; \mathbb{F}_2)$ and $\psi \in \mathrm{H}^s(X, A; \mathbb{F}_2)$ we have

$$\mathrm{Sq}^k(\varphi \cup \psi) \ = \ \sum_{i=0}^{k} \mathrm{Sq}^i(\varphi) \cup \mathrm{Sq}^{k-i}(\psi).$$

Proof.

(1) The notation might suggest that this statement is surely trivial, but that is not the case. For time, space and mental energy reasons we will not give the proof, which at first glance does not seem terribly enlightening anyway. Instead we refer to [**Spa1995**, p. 274] for details.

(2) First we deal with the absolute case. Thus let $\varphi \in \mathrm{C}^i(X;R)$ be a cocycle. We calculate

$$\mathrm{Sq}^i(\varphi) \underset{\text{by definition}}{\overset{\uparrow}{=}} [h_{i-i}(\varphi \otimes \varphi)] = [h_0(\varphi \otimes \varphi)] \underset{\substack{\text{by the Cup Product} \\ \text{Description Lemma 236.27}}}{\overset{\uparrow}{=}} [\varphi] \cup [\varphi].$$

The case that we are dealing with a cocycle $\varphi \in \mathrm{C}^i(X,A;R)$ is proved basically the same way.

(3) Let $\varphi \in \mathrm{C}^j(X,A;\mathbb{F}_2)$ be a cocycle and suppose that $i > j$. We have

$$\mathrm{Sq}^i([\varphi]) \underset{\text{by definition}}{\overset{\uparrow}{=}} [h_{j-i}(\varphi)] \underset{\text{since } j-i<0 \text{ we have } h_{j-i}=0}{\overset{\uparrow}{=}} [0].$$

(4) This statement is proved in [**Spa1995**, p. 274]. ■

236.6. Exercises for Chapter 236.

Exercise 236.1. Note that it follows from the Cup Product-Naturality Lemma 204.8 that for any commutative ring R and any $m \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} \mathrm{H}^m(X;R) & \to & \mathrm{H}^{2m}(X;R) \\ \varphi & \mapsto & \varphi \cup \varphi \end{array}$$

define a cohomology operation. On page 4537 we showed that this cohomology operation is additive, if $1+1=0 \in R$.

(a) Show that if m is odd, then this cohomology operation is additive.

(b) Show for any ring R with $1+1 \neq 0 \in R$ and any even $m \in \mathbb{N}$ the above cohomology operation is not additive.

Exercise 236.2.

(a) Show that for any $k \in \mathbb{N}_{\geq 3}$ and $m \in \mathbb{N}$ the map $\mathrm{H}^m(X;\mathbb{F}_2) \to \mathrm{H}^{mk}(X;\mathbb{F}_2)$ given by $\varphi \mapsto \varphi^k$ is not an additive cohomology operation.

(b) Is the map $\mathrm{H}^1(X;\mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^3(X;\mathbb{Z}/3\mathbb{Z})$ given by $\varphi \mapsto \varphi^3$ an additive cohomology operation?

Exercise 236.3. Let $X = S^m \vee S^n$ be the wedge of two spheres. Show that for $i \geq 1$ all Steenrod operations $\mathrm{Sq}^i\colon \mathrm{H}^k(X;\mathbb{F}_2) \to \mathrm{H}^{k+i}(X;\mathbb{F}_2)$ are zero.
Hint. Reread the proof of the Wedge-of-Spheres–Cup Product Lemma 203.29.

Exercise 236.4. Find a closed formula for the i-th Steenrod operation on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{F}_2)$.
Hint. Use the obvious $\mathbb{F}_2$-version of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7.

CHAPTER 237

The Adem relations

In this chapter we will formulate and discuss the Adem Relations Theorem 237.1. This theorem establishes identities involving Steenrod operations. Unfortunately we lack the time and space to prove the theorem. But we will discuss many of the fascinating consequences of the Adem relations.

This chapter is organized as follows:

- In Section 237.1 we will formulate the Adem Relations Theorem 237.1.
- In Section 237.2 we will apply the Adem relations to the study of homotopy groups of spheres.
- In Section 237.3 we will see that some Steenrod operations can be decomposed into Steenrod operations of lower degree. We will use these results to obtain some results on the Hopf invariant.
- In Section 237.4 we will consider Steenrod operations on cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficients, where p is an odd prime.
- In Section 237.5 we will provide exercises for Chapter 237.

237.1. **The Adem relation.** The following theorem was first proved by José Ádem [**Ade1952, Ade1957**] in 1952.

Theorem 237.1. (Adem Relations Theorem) For any $0 < a < 2b$ we have the following equality of cohomology operations:[2563]

$$\mathrm{Sq}^a \circ \mathrm{Sq}^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j .$$

These equalities are called the Adem relations.

Remark. As is remarked on [**SE1962**, p. 2], the Adem relations are a consequence of the properties of Steenrod operations that were stated in the Steenrod Theorem 236.3. This observation also follows from the uniqueness result stated in the Steenrod Operations–Uniqueness Theorem 236.4. □

Proof. Since the proof would consume too much of our valuable time we will not provide it. Instead we refer to the original paper by Adem [**Ade1952, Ade1957**] or alternatively to [**BuM1982**], [**SE1962**, Theorem VIII.1.5], [**MTa1968**, p. 23], or [**Hat2002**, Theorem 4.L.13] for a proof. ■

[2563]These are equalities of cohomology operations $\mathrm{H}^n(X, A; \mathbb{F}_2) \to \mathrm{H}^{n+a+b}(X, A; \mathbb{F}_2)$. In particular, since we work over $\mathbb{F}_2$-coefficients, we only care about the binomial coefficient modulo 2. We will discuss the mod 2 reductions of binomial coefficients in the Binomial Coefficient–Mod 2-Lemma 237.3.

Even though we will not prove the Adem relations, we will shamelessly exploit them in this chapter. Afterwards the conscientious reader will surely feel compelled to read the proof of the Adem relations.

Just to get a feeling for the Adem relations we determine for small values of a what the relations really look like.

Example. For $a = 1$ and for any $b \in \mathbb{N}$ we have

$$\mathrm{Sq}^1 \circ \mathrm{Sq}^b \;=\; \underbrace{\sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{with } a=1} \;=\; \underbrace{\binom{b-1}{1} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^0}_{j=0} \;=\; (b-1) \cdot \mathrm{Sq}^{1+b} .$$

since $\mathrm{Sq}^0 = \mathrm{id}$

Furthermore, for $a = 2$ and any $b \in \mathbb{N}_{\geq 2}$ we have

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^b \;=\; \underbrace{\binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} \circ \mathrm{Sq}^0}_{j=0} + \underbrace{\binom{b-2}{0} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1}_{j=1} \;=\; \binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} + \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1 .$$

since $\mathrm{Sq}^0 = \mathrm{id}$

In particular setting $b = 2$ we see that

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^2 \;=\; \mathrm{Sq}^3 \circ \mathrm{Sq}^1 .$$

□

237.2. **Application of the Adem relations to homotopy groups of spheres.** Recall that in the Steenrod Operations-π_n-Theorem 236.19 we used Steenrod operations to show that the three Hopf maps $H\colon S^3 \to S^2$, $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ and their suspensions define non-trivial elements in the corresponding homotopy groups of spheres.

In this section we will use the Adem relations to show that the 2-fold iteration of these Hopf maps also define non-trivial elements in the homotopy groups of spheres. More precisely, we have the following theorem.

Theorem 237.2. (Double Hopf Map Theorem)

(1) For any $n \geq 2$ the map

$$\Sigma^{n-2}\Big(\underbrace{\overbrace{H}^{S^3 \to S^2} \circ \overbrace{\Sigma(H)}^{S^4 \to S^3}}_{\text{map } S^4 \to S^2}\Big)\colon S^{n+2} \;\to\; S^n$$

defines a non-trivial element in $\pi_{n+2}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+2}(S^n) \neq 0$.

(2) For any $n \geq 4$ the map

$$\Sigma^{n-4}\Big(\underbrace{\overbrace{H_{\mathbb{H}}}^{S^7 \to S^4} \circ \overbrace{\Sigma^3(H_{\mathbb{H}})}^{S^{10} \to S^7}}_{\text{map } S^{10} \to S^4}\Big)\colon S^{n+6} \;\to\; S^n$$

defines a non-trivial element in $\pi_{n+6}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+6}(S^n) \neq 0$.

(3) For any $n \geq 8$ the map

$$\Sigma^{n-8}\Big(\underbrace{\overbrace{H_{\mathbb{O}}}^{S^{15} \to S^8} \circ \overbrace{\Sigma^7(H_{\mathbb{O}})}^{S^{22} \to S^{15}}}_{\text{map } S^{22} \to S^8}\Big)\colon S^{n+14} \;\to\; S^n$$

defines a non-trivial element in $\pi_{n+14}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+14}(S^n) \neq 0$.

For the reader's convenience we recall two lemmas which will play a key role in the proof of the Double Hopf Map Theorem 237.2.

Lemma 216.7. (Mapping Cone-H*-Lemma) Let $f\colon A \to X$ be a continuous map between topological spaces. Let $j\colon X \to \mathrm{Cone}(f\colon A \to X)$ the obvious inclusion. Then there exists a natural homomorphism $\delta_{k-1}\colon \mathrm{H}^{k-1}(A;\mathbb{F}_2) \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f\colon A \to X);\mathbb{F}_2)$ such that the following sequence is exact:

$$\ldots \to \mathrm{H}^{k-1}(A;\mathbb{F}_2) \xrightarrow{\delta_{k-1}} \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f\colon A \to X);\mathbb{F}_2) \xrightarrow{j^*} \mathrm{H}^k(X;\mathbb{F}_2) \xrightarrow{f^*} \mathrm{H}^k(A;\mathbb{F}_2) \xrightarrow{\delta} \ldots$$

As an example for the Mapping Cone-H*-Lemma 216.7 let us consider the mapping cone of the iterated suspension of the Hopf map.

Example. Let $k \in \mathbb{N}_0$. We consider $\Sigma^k(H)\colon S^{k+3} \to S^{k+2}$, i.e. we consider the k-fold suspension of the Hopf map $H\colon S^3 \to S^2$. As we saw in the Mapping Cone-of-Spheres-H*-Lemma 216.9, it follows almost immediately from the long exact sequence of the Mapping Cone-H*-Lemma 216.7 that the maps

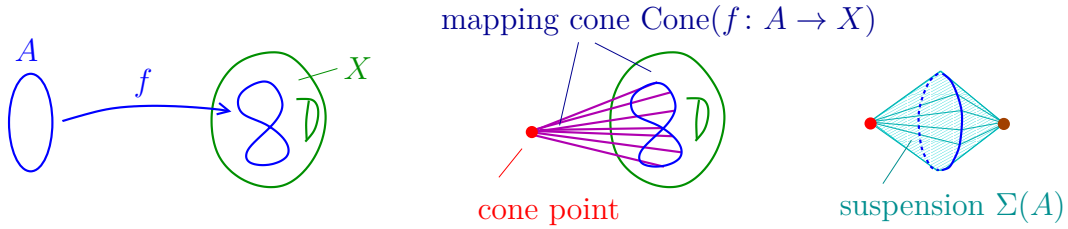
$$\widetilde{\mathrm{H}}^{k+2}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{F}_2) \xrightarrow[\cong]{j^*} \mathrm{H}^{k+2}(S^{k+2};\mathbb{F}_2)$$

and $$\mathrm{H}^{k+3}(S^{k+3};\mathbb{F}_2) \xrightarrow[\cong]{\delta} \widetilde{\mathrm{H}}^{k+4}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{F}_2)$$

are isomorphisms, and that all the other reduced cohomology groups of the mapping cone vanish. □

In the following we recall a few facts about the mapping cones that we established in the Mapping Cone–Embedding Lemma 21.11 and the in the Mapping Cone–Quotient Lemma 21.14: For any continuous map $f\colon A \to X$ the following statements hold:

- The natural injection $X \to \mathrm{Cone}(f\colon A \to X)$ is a closed embedding.
- The map
$$\begin{array}{rccc} h\colon & \mathrm{Cone}(f\colon A \to X)/X & \to & \Sigma(A) \\ & [(a,t)] & \mapsto & [(a,1-2t)] \end{array}$$
is a homeomorphism.



Now we turn to the proof of the Double Hopf Map Theorem 237.2. As in the proof of the Steenrod Operations-π_n-Theorem 236.19 we first consider the case of the classical Hopf map, i.e. we first prove Statement (1).

Proof of the Double Hopf Map Theorem 237.2 (1). Let $n \in \mathbb{N}_{\geq 2}$. We write

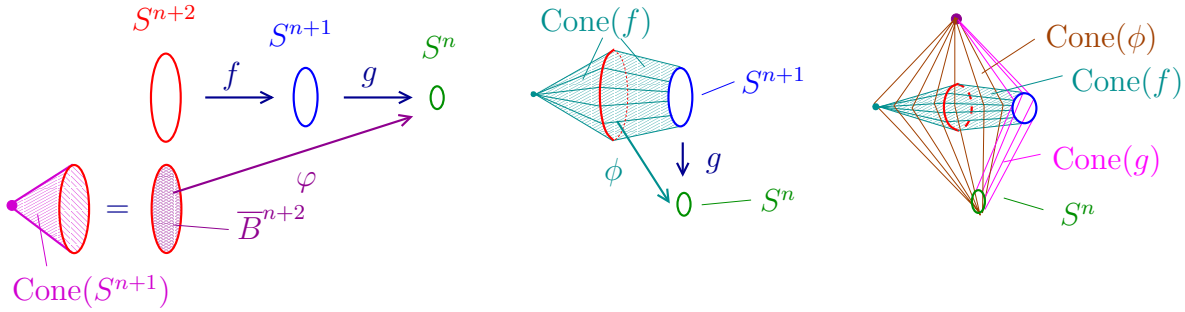
$$\Sigma^{n-2}(H \circ \Sigma(H)) \;=\; \overbrace{\underbrace{\Sigma^{n-2}(H)}_{=:g}}^{S^{n+1}\to S^n} \circ \overbrace{\underbrace{\Sigma^{n-1}(H)}_{=:f}}^{S^{n+2}\to S^{n+1}} : S^{n+2} \;\to\; S^n.$$

$\uparrow$ by the Suspension Functor Lemma 20.23 (2)

We need to show that $g \circ f \colon S^{n+2} \to S^n$ defines a non-trivial element in $\pi_{n+2}(S^n)$. As in the proof of the Steenrod Operations-π_n-Theorem 236.19 we do a proof by contradiction. Thus suppose that there exists a homotopy from $g \circ f \colon S^{n+2} \to S^n$ to a constant map. By the Trivial-in-π_n-Lemma 134.10 we can use this homotopy to construct a map[2564] $\varphi \colon \mathrm{Cone}(S^{n+2}) = \overline{B}^{n+3} \to S^n$ such that $\varphi|_{S^{n+2}} = g \circ f$. We consider the map[2565]

$$\begin{aligned} \phi \colon \overbrace{(\mathrm{Cone}(S^{n+2}) \sqcup S^{n+1})/\sim}^{=\mathrm{Cone}(f\colon S^{n+2}\to S^{n+1})} &\to S^n \\ [P] &\mapsto \begin{cases} g(P), & \text{if } P \in S^{n+1}, \\ \varphi(P), & \text{if } P \in \mathrm{Cone}(S^{n+2}) = \overline{B}^{n+3}. \end{cases} \end{aligned}$$

Our main object of interest is now, perhaps rather confusingly, the mapping cone of ϕ. This mapping cone comes with two interesting maps:



(1) Recall that it follows from the Space-Cone Embedding Lemma 20.19 that the natural map $S^{n+1} \to \mathrm{Cone}(f)$ is a closed embedding. Since $\phi|_{S^{n+1}} = g$ we thus obtain from the Mapping Cone–Closed Embedding Lemma 21.16 that the map

$$i \colon \mathrm{Cone}(g \colon S^{n+1} \to S^n) \;\to\; \mathrm{Cone}(\phi \colon \mathrm{Cone}(f) \to S^n)$$

is a closed embedding.

(2) We also consider the projection

$$p \colon \mathrm{Cone}(\phi \colon \mathrm{Cone}(f) \to S^n) \;\to\; \mathrm{Cone}(\phi \colon \mathrm{Cone}(f) \to S^n)/S^n.$$

In the following two claims we will study the maps on cohomology induced by the inclusion i and the projection p.

[2564] As we discussed on page 552, given any $m \in \mathbb{N}_0$ the map $\mathrm{Cone}(S^m) \to \overline{B}^{m+1}$ given by $[(v,t)] \mapsto v \cdot t$ is a homeomorphism. We will use this map to make the identification $\overline{B}^{m+1} = \mathrm{Cone}(S^m)$.

[2565] It follows from $\varphi|_{S^{n+2}} = g \circ f$ that this map is well-defined. Furthermore we obtain immediately from the Topological Pushout Proposition 8.12 (1b) that this map is continuous.

Claim 1.
(a) The induced map
$$i^*\colon \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{F}_2) \;\to\; \mathrm{H}^k(\mathrm{Cone}(g);\mathbb{F}_2)$$
is an isomorphism for $k \neq n+1$.
(b) We have $\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{F}_2) = 0$.

Proof. We consider the following diagram:
$$\begin{array}{ccccccccccc}
 & & \mathrm{H}^{k-2}(S^{n+2};\mathbb{F}_2) & & & & & & \mathrm{H}^{k-1}(S^{n+2};\mathbb{F}_2) & & \\
 & & \downarrow & & & & & & \downarrow & & \\
\to & \mathrm{H}^{k-1}(\mathrm{Cone}(f);\mathbb{F}_2) & \to & \mathrm{H}^k(\mathrm{Cone}(\phi\colon \mathrm{Cone}(f)\to S^n);\mathbb{F}_2) & \to & \mathrm{H}^k(S^n;\mathbb{F}_2) & \xrightarrow{\phi^*} & \mathrm{H}^k(\mathrm{Cone}(f);\mathbb{F}_2) & \to & & \\
 & \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & \\
\longrightarrow & \mathrm{H}^{k-1}(S^{n+1};\mathbb{F}_2) & \longrightarrow & \mathrm{H}^k(\mathrm{Cone}(g\colon S^{n+1}\to S^n);\mathbb{F}_2) & \longrightarrow & \mathrm{H}^k(S^n;\mathbb{F}_2) & \xrightarrow{f^*} & \mathrm{H}^k(S^{n+1};\mathbb{F}_2) & \longrightarrow & & \\
 & \downarrow{\scriptstyle f^*} & & & & & & \downarrow{\scriptstyle f^*} & & & \\
 & \mathrm{H}^{k-1}(S^{n+2};\mathbb{F}_2) & & & & & & \mathrm{H}^k(S^{n+2};\mathbb{F}_2) & & &
\end{array}$$
Here the two horizontal and the two vertical sequences are obtained from the Mapping Cone-H*-Lemma 216.7. The diagram commutes by the naturality of the long exact sequence. The two statements of the claim follow easily from the above diagram (see also the discussion on page 4563) and the calculation of the cohomology groups of spheres with $\mathbb{F}_2$-coefficients. ⊞

Claim 2. The map
$$p^*\colon \mathrm{H}^k(\mathrm{Cone}(\phi)/S^n;\mathbb{F}_2) \;\to\; \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{F}_2)$$
is an isomorphism for $k \neq n, n+1$.

Proof. We consider the following diagram:
$$\begin{array}{ccccccccc}
\dots \longrightarrow & \mathrm{H}^{k-1}(S^n;\mathbb{F}_2) & \xrightarrow{\delta} & \mathrm{H}^k(\mathrm{Cone}(\phi), S^n;\mathbb{F}_2) & \longrightarrow & \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{F}_2) & \longrightarrow & \mathrm{H}^k(S^n;\mathbb{F}_2) & \longrightarrow \dots \\
 & & & \downarrow & \nearrow_{p^*} & & & & \\
 & & & \mathrm{H}^k(\mathrm{Cone}(\phi)/S^n;\mathbb{F}_2) & & & & &
\end{array}$$
Here on top we see the long exact sequence in cohomology of the pair $(\mathrm{Cone}(\phi), S^n)$. Furthermore the vertical map is the natural map from Definition 200.5 which has in particular the property that it makes the diagram commute. Furthermore, it follows from the Cohomology-of-Quotients Proposition 200.8 that the vertical map is an isomorphism.[2566]

The desired statement follows immediately from this discussion and the calculation of the cohomology groups of S^n. ⊞

[2566]Here we use that an argument as in the Mapping Cylinder–Homotopy Lemma 21.5 (1) shows that S^n is a deformation retract of $\mathrm{Cone}(\phi) \setminus \{\text{cone point}\}$, which implies that S^n is a good subset of $\mathrm{Cone}(\phi)$.

Next we consider the following diagram:.

$$\begin{array}{ccccccc}
\mathrm{H}^n(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{F}_2) & & & & \\
\downarrow{\scriptstyle \mathrm{id}} & & \mathrm{id}\downarrow & & & & \\
\mathrm{H}^n(\mathrm{Cone}(g);\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(g);\mathbb{F}_2) & & & & \\
{\scriptstyle i^*}\uparrow{\scriptstyle \cong} & & \cong\uparrow{\scriptstyle i^*} & & & & \\
\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{F}_2) & \xrightarrow{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{F}_2) & \xrightarrow{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\phi);\mathbb{F}_2) & & \\
 & & {\scriptstyle p^*}\uparrow{\scriptstyle \cong} & & \cong\uparrow{\scriptstyle p^*} & & \\
 & & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi)/S^n;\mathbb{F}_2) & \xrightarrow{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\phi)/S^n;\mathbb{F}_2) & & \\
 & & {\scriptstyle h^*}\uparrow{\scriptstyle \cong} & & \cong\uparrow{\scriptstyle h^*} & & \\
 & & \mathrm{H}^{n+2}(\Sigma(\mathrm{Cone}(f));\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\Sigma(\mathrm{Cone}(f));\mathbb{F}_2) & & \\
 & & {\scriptstyle \theta^*}\downarrow{\scriptstyle \cong} & & \cong\downarrow{\scriptstyle \theta^*} & & \\
 & & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma(f));\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\Sigma(f));\mathbb{F}_2) & & \\
 & & \mathrm{id}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
 & & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma^n(H));\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+4}(\mathrm{Cone}(\Sigma^n(H));\mathbb{F}_2). & &
\end{array}$$

We make the following clarifications and observations:

(1) At the top and bottom we use that by definition $f = \Sigma^{n-1}(H)\colon S^{n+2} \to S^{n+1}$ and $g = \Sigma^{n-2}(H)\colon S^{n+1} \to S^n$.
(2) We obtain from the Steenrod-Isomorphism Proposition 236.21 (1) that the top and bottom horizontal maps Sq^2 are isomorphisms.
(3) It follows from the above two claims that the vertical maps i^* and p^* are isomorphisms.
(4) The map $h\colon \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n)/S^n \to \Sigma(\mathrm{Cone}(f))$ is the homeomorphism from the Mapping Cone–Quotient Lemma 21.14.
(5) The map $\theta\colon \mathrm{Cone}(\Sigma(f)\colon \Sigma(S^{n+2}) \to \Sigma(S^{n+1})) \to \Sigma(\mathrm{Cone}(f\colon S^{n+2} \to S^{n+1}))$ is the homeomorphism from the Mapping Cone-Suspension Proposition 21.26.
(6) It follows from the naturality of the Steenrod operations that all the squares commute.
(7) The combination of the above arguments implies that both of the maps Sq^2 in the third row are isomorphisms.

Finally we consider the following diagram:.

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{F}_2)}^{\neq 0} & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{F}_2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{F}_2). \\
 & {\scriptstyle \mathrm{Sq}^1}\searrow & & \nearrow{\scriptstyle \mathrm{Sq}^3} & \\
 & & \underbrace{\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{F}_2)}_{=0} & &
\end{array}$$

We make the following observations:

(1) By the discussion preceding the diagram we know that the two maps on the top are isomorphisms.
(2) As we saw on page 4562, we have the Adem relation $\mathrm{Sq}^2 \circ \mathrm{Sq}^2 = \mathrm{Sq}^3 \circ \mathrm{Sq}^1$. This implies that the diagram commutes.

(3) We have isomorphisms

$$\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{F}_2) \underset{\text{Claim 1 (a)}}{\cong} \mathrm{H}^n(\mathrm{Cone}(\overbrace{\Sigma^{n-2}(H)}^{=g});\mathbb{F}_2) \underset{\text{see the example on page 4563}}{\cong} \mathrm{H}^n(S^n;\mathbb{F}_2) \cong \mathbb{F}_2.$$

(4) In Claim 1 (b) we had shown that the group $\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{F}_2)$ is trivial.

By (1) and (3) the two horizontal maps in the diagram give us an isomorphism of a non-trivial group. On the other hand, by (2) and (4) this isomorphism factors through a trivial group. This is absurd. We have thus obtained a contradiction. ■

Now we turn to the proof of the remaining two statements of the Double Hopf Map Theorem 237.2.

Proof of the Double Hopf Map Theorem 237.2 (2) and (3). The proof for the quaternionic Hopf map $H_{\mathbb{H}}$ and the octonion Hopf map $H_{\mathbb{O}}$ is almost the same as the above proof that deals with the classical Hopf map H. We just need to replace the Steenrod-Isomorphism Proposition 236.21 (1) by the Steenrod-Isomorphism Proposition 236.21 (2) and (3) and we need to use the following Adem relations:

$$\begin{aligned} \mathrm{Sq}^4 \circ \mathrm{Sq}^4 &= \mathrm{Sq}^6 \circ \mathrm{Sq}^2 + \mathrm{Sq}^7 \circ \mathrm{Sq}^1 \\ \mathrm{Sq}^8 \circ \mathrm{Sq}^8 &= \mathrm{Sq}^{12} \circ \mathrm{Sq}^4 + \mathrm{Sq}^{14} \circ \mathrm{Sq}^2 + \mathrm{Sq}^{15} \circ \mathrm{Sq}^1 . \end{aligned}$$

These relations can be obtained easily by hand from the Adem Relation Theorem 237.1. Alternatively one can use the Adem relations calculator:

`https://math.berkeley.edu/~kruckman/adem/`

As in the proof for the Hopf maps one sees that the initial Steenrod operations on the right-hand side all end up in a cohomology group which is zero. Thus one obtains again a contradiction to the hypothesis that the maps $\Sigma^{n-4}(H_{\mathbb{H}} \circ \Sigma^3(H_{\mathbb{H}})) \colon S^{n+6} \to S^n$ and $\Sigma^{n-8}(H_{\mathbb{O}} \circ \Sigma^7(H_{\mathbb{O}})) \colon S^{n+14} \to S^n$ are null-homotopic. ■

237.3. **Decomposing Steenrod operations and the Hopf invariant revisited.** The key to extracting extra mileage out of Steenrod operations is the following innocuous lemma from elementary number theory.

Lemma 237.3. (Binomial Coefficient–Mod 2-Lemma) Let $n, k \in \mathbb{N}_0$. If we write $n = \sum_{i\in\mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum_{i\in\mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0,1\}$, then[2567]

$$\binom{n}{k} = \prod_{i\in\mathbb{N}_0} \binom{n_i}{k_i} \mod 2.$$

Remark.

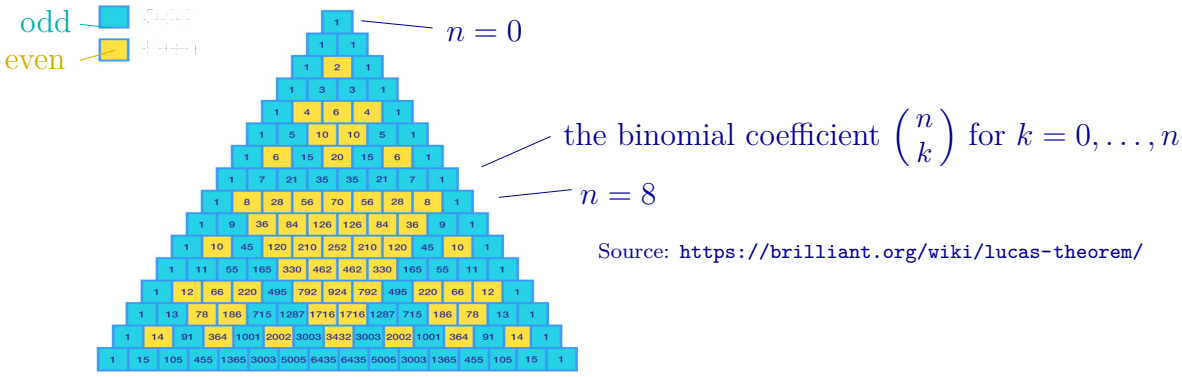
(1) Note that for $r, s \in \{0,1\}$ it follows easily from Definition 236.10 that

$$\binom{s}{r} = \begin{cases} 1, & \text{if } r \le s, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $n \in \mathbb{N}$ and let $k \in \{0,\dots,d\}$. We define n_i, k_i as above. It follows from the Binomial Coefficient–Mod 2-Lemma 237.3 and (1) that $\binom{n}{k}$ is odd if and only if for

[2567] The lemma can be seen as a generalization of the claim on page 4110.

each i we have $k_i \leq n_i$. If we draw the usual Pascal triangle, then by shading odd and even terms in different colors we obtain a Sierpinski triangle. We refer to the figure below for an illustration.[2568]



(3) In fact the obvious analog of the Binomial Coefficient–Mod 2-Lemma 237.3 holds with the prime 2 replaced by any prime p. This more general statement is known as Lucas' Theorem, a proof is for example provided in [**Fi1947**]. □

Proof. Let $n, k \in \mathbb{N}_0$. We write $n = \sum_{i \in \mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum_{i \in \mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0, 1\}$ In the polynomial ring $\mathbb{F}_2[x]$ we have the following equalities[2569]

$$\sum_{i=0}^{n} \binom{n}{i} x^i \underset{\text{by the binomial formula}}{=} (1+x)^n = (1+x)^{\sum n_i 2^i} = \prod_{i \in \mathbb{N}_0} (1+x)^{2^i n_i} \underset{\text{since in } \mathbb{F}_2[x] \text{ we have } (a+b)^2 = a^2 + b^2}{=} \prod_{i \in \mathbb{N}_0} (1+x^{2^i})^{n_i} \underset{\text{by the binomial formula}}{=} \prod_{i \in \mathbb{N}_0} \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} x^{j_i \cdot 2^i}.$$

Now we compare the x^k-coefficients, we see that[2570]

$$\binom{n}{k} = x^k\text{-coefficient of } \sum_{i=0}^{n} \binom{n}{i} x^i \underset{\text{by the above equality}}{=} x^k\text{-coefficient of } \prod_{i \in \mathbb{N}_0} \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} x^{j_i \cdot 2^i} \underset{\substack{\text{by the uniqueness of the 2-adic description} \\ \text{of a natural number we only get } x^k \\ \text{if each variable } j_i \text{ equals } k_i}}{=} \prod_{i \in \mathbb{N}_0} \binom{n_i}{k_i}.$$

We have thus proved the desired equality. ■

This lemma allows us to prove the following theorem:

Theorem 237.4. (Steenrod Indecomposability Theorem) Let $i \in \mathbb{N}_0$. If i is not a power of 2, then Sq^i is decomposable, in the sense that it can be written as the sum of compositions of Steenrod operations of smaller degree.

Example. By the discussion on page 4562 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{Sq}^{2k+1} = \mathrm{Sq}^1 \circ \mathrm{Sq}^{2k} \quad \text{and} \quad \mathrm{Sq}^{4k+2} = \mathrm{Sq}^2 \circ \mathrm{Sq}^{4k} + \mathrm{Sq}^{4k+1} \circ \mathrm{Sq}^1 .$$

[2568] A more elaborate version of this figure can be found here:

https://upload.wikimedia.org/wikipedia/commons/8/87/Sierpinski_Pascal_triangle.svg

[2569] The argument has some similarities with the proof of the claim on page 4110.

[2570] Note that the equalities also make sense for $k > n$. In this setting one sees easily that all terms are in fact zero.

In particular we see that

$$\mathrm{Sq}^3 = \mathrm{Sq}^1 \circ \mathrm{Sq}^2, \quad \mathrm{Sq}^5 = \mathrm{Sq}^1 \circ \mathrm{Sq}^4, \quad \mathrm{Sq}^6 = \mathrm{Sq}^2 \circ \mathrm{Sq}^4 + \mathrm{Sq}^5 \circ \mathrm{Sq}^1 \quad \text{and} \quad \mathrm{Sq}^7 = \mathrm{Sq}^1 \circ \mathrm{Sq}^6 .$$

□

Proof. Suppose that we are given $i \in \mathbb{N}_0$ that is not a power of two. This means that we can write $i = a + b$ where $b = 2^k$ and $a \in \{1, \dots, 2^k - 1\}$. Since $0 < a < 2b$ we obtain an Adem relation for $\mathrm{Sq}^a \circ \mathrm{Sq}^b$ from the Adem Relations Theorem 237.1. This relation can be rewritten as follows:

$$\binom{b-1}{a} \cdot \mathrm{Sq}^{a+b} = \underbrace{\mathrm{Sq}^a \circ \mathrm{Sq}^b + \sum_{j=1}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{compositions of Steenrod operations of degree} < a+b} .$$

It remains to show that $\binom{b-1}{a} = 1 \bmod 2$. We calculate

$$\binom{b-1}{a} \overset{\text{by the Binomial Coefficient–Mod 2-Lemma 237.3, where } b-1 = \sum_{i \in \mathbb{N}_0} d_i \cdot 2^i \text{ and } a = \sum_{i \in \mathbb{N}_0} c_i \cdot 2^i \text{ with } c_i, d_i \in \{0,1\}}{\downarrow} \prod_{i \in \mathbb{N}_0} \binom{d_i}{c_i} \overset{\text{since } b-1 = 2^k - 1 = 1 + 2 + \cdots + 2^{k-1} \text{ we have } d_0 = \cdots = d_{k-1} = 1 \text{ and } d_k = d_{k+1} = \cdots = 0 \text{, furthermore, since } a < 2^k - 1 \text{ we have } c_k = c_{k+1} = \cdots = 0}{=} \prod_{i=0}^{k-1} \binom{1}{c_i} \cdot \prod_{i=k}^{\infty} \binom{0}{0} = 1 \mod 2.$$

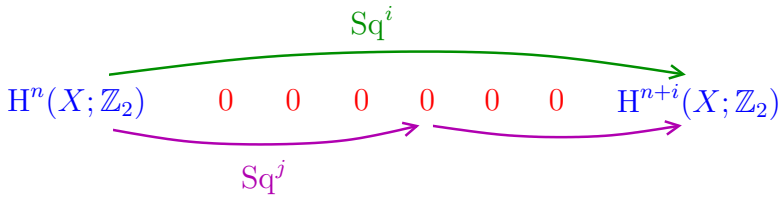
■

Corollary 237.5. (Steenrod Vanishing Corollary) Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the Steenrod operation

$$\mathrm{Sq}^i \colon \mathrm{H}^n(X; \mathbb{F}_2) \to \mathrm{H}^{n+i}(X; \mathbb{F}_2).$$

If all the $\mathbb{F}_2$-cohomology groups inbetween these two vanish, i.e. if we have the equalities $\mathrm{H}^{n+1}(X; \mathbb{F}_2) = \cdots = \mathrm{H}^{n+i-1}(X; \mathbb{F}_2) = 0$, and if i is not a power of 2, then the Steenrod operation is the zero map.

Proof. Since i is not a power of 2 we know by the Steenrod Indecomposability Theorem 237.4 that Sq^i can be written as the sum of compositions of Steenrod operations of smaller degree.



By our hypothesis for any $j \in \{1, \dots, i-1\}$ we have $\mathrm{H}^{n+j}(X; \mathbb{F}_2) = 0$. But this implies in particular that for any such j we have $\mathrm{Sq}^j \colon \mathrm{H}^n(X; \mathbb{F}_2) \to \mathrm{H}^{n+j}(X; \mathbb{F}_2) = 0$. ■

In Question 208.15 we asked whether all (reasonable) superalgebras can be realized as cohomology rings. We will now see that this is not the case. Before we can state the corresponding proposition we need to introduce the following definition:

Definition 237.6. Let $m \in \mathbb{N}$ be an even number and let $s \in \mathbb{Z}$. We define $R_m(s)$ to be the superalgebra which is given by the groups[2571]

$$R_m(s) \;=\; \begin{cases} \mathbb{Z}\cdot 1, & \text{in degree } 0, \\ \mathbb{Z}\cdot x, & \text{in degree } m, \\ \mathbb{Z}\cdot y, & \text{in degree } 2m, \\ 0, & \text{in all other degrees} \end{cases}$$

and where the multiplication is given by $x \cdot x = s \cdot y$. In other words, the multiplication table is given as follows

	1	x	y
1	1	x	y
x	x	$s\cdot y$	0
y	y	0	0.

Example.

(1) For any m the superalgebra $R_m(1)$ is evidently isomorphic to $\mathbb{Z}[x]/(x^3)$ where x is of degree m.

(2) By (1) together with the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7, the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 215.5 and the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 215.9 we have isomorphisms

$$(\mathrm{H}^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z}),\cup) \cong R_2(1),\ (\mathrm{H}^*(\mathbb{H}\mathrm{P}^2;\mathbb{Z}),\cup) \cong R_4(1) \text{ and } (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2;\mathbb{Z}),\cup) \cong R_8(1). \quad \square$$

Next we prove the following proposition which gives a negative answer to Questions 208.15 and 215.12.

Proposition 237.7. (Cohomology Ring–Non-Realization Proposition) Let m be an even number and let s be an odd number. If m is not a power of 2, then there is no topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}),\cup)$ is isomorphic to $R_m(s)$.

In the proof of the Cohomology Ring–Non-Realization Proposition 237.7 we will need the following lemma:

Lemma 237.8. (Bockstein–LES Lemma) Let $m \in \mathbb{N}$. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ the canonical map. Given any topological space X there exists a long exact sequence

$$\ldots \to \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\cdot m} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} \mathrm{H}^k(X;\mathbb{Z}/m\mathbb{Z}) \xrightarrow{\delta_k} \mathrm{H}^{k+1}(X;\mathbb{Z}) \to \ldots$$

Proof of Lemma 237.8. We denote by $\psi\colon \mathbb{Z} \to \mathbb{Z}$ the map that is given by multiplication by m. We have the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/m\mathbb{Z} \to 0.$$

By the discussion in **??** we have the corresponding Bockstein sequence in cohomology which looks as follows:

$$\ldots \xrightarrow{\delta} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\psi_*} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} \mathrm{H}^k(X;\mathbb{F}_2) \xrightarrow{\delta} \mathrm{H}^{k+1}(X;\mathbb{Z}) \to \ldots$$

By the almost, but not entirely trivial Coefficient-Multiplication Lemma **??** we know that ψ_* is indeed multiplication by m. ■

Proof of Proposition 237.7. Let m be an even number. Furthermore let X be a topological space such that $(\mathrm{H}^*(X;\mathbb{Z}),\cup) = R_m(s)$ for some odd number s. We need to show

[2571] It follows from the fact that m is even that this is indeed a superalgebra.

that m is a power of 2. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{F}_2$ the canonical map. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^m(X;\mathbb{Z}) & \xrightarrow{x\mapsto x^2=x\cup x=s\cdot y} & \mathrm{H}^{2m}(X;\mathbb{Z}) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^m(X;\mathbb{F}_2) & \xrightarrow{z\mapsto z^2=z\cup z} & \mathrm{H}^{2m}(X;\mathbb{F}_2) \\ \downarrow & & \downarrow \\ \mathrm{H}^m(X;\mathbb{F}_2) & \xrightarrow{z\mapsto \mathrm{Sq}^m(z)} & \mathrm{H}^{2m}(X;\mathbb{F}_2). \end{array}$$

We make the following clarifications and observations:

(1) It follows from the Cup Product-Coefficients Lemma 204.9 that the upper square commutes.

(2) It follows from the Bockstein–LES Lemma 237.8 and our hypothesis on the topological space X that the map $\varphi_*\colon \mathrm{H}^{2m}(X;\mathbb{Z}) \to \mathrm{H}^{2m}(X;\mathbb{F}_2)$ is an epimorphism and that $\mathrm{H}^{2m}(X;\mathbb{F}_2) \cong \mathbb{F}_2$. In particular we see that $\varphi_*(y) \in \mathrm{H}^{2m}(X;\mathbb{F}_2)$ is a non-trivial element of order 2.

(3) Since s is odd we obtain from (2) that $\varphi_*(s \cdot y) = s \cdot \varphi_*(y)$ is a non-trivial element of $\mathrm{H}^{2m}(X;\mathbb{F}_2)$.

(4) The lower square commutes by Axiom (2) of Steenrod operations, see the Steenrod Theorem 236.3 (2).

(5) By our hypothesis on X we know that the intermediate cohomology groups vanish, i.e. we have $\mathrm{H}^{m+1}(X;\mathbb{F}_2) = \cdots = \mathrm{H}^{2m-1}(X;\mathbb{F}_2) = 0$.

It follows from (3) and (4) that the bottom horizontal map has non-trivial image. But by the Steenrod Vanishing Corollary 237.5 together with (5) this is only possible if m is indeed a power of 2. ∎

As promised in the title of the section, we now return to the Hopf invariant. In the following let $f\colon S^{2n-1} \to S^n$ be a continuous map where $n \geq 2$. Recall that in Definition 216.10 we introduced the Hopf invariant $\mathrm{Hopf}(f) \in \mathbb{Z}$. The following lemma translates the Hopf invariant into the language of the above discussion. The lemma can also be viewed as an alternative definition of the Hopf invariant, at least up to sign.

Lemma 237.9. (Hopf Invariant–Cohomology Ring Lemma) Let $n \geq 2$. For every continuous map $f\colon S^{2n-1} \to S^n$ there exists an isomorphism

$$(\mathrm{H}^*(\mathrm{Cone}(f);\mathbb{Z}), \cup) \;\cong\; R_n(\mathrm{Hopf}(f))$$

of superalgebras.

Proof. This statement follows immediately from the Mapping Cone-of-Spheres-H^*-Lemma 216.9 and the definition of the Hopf invariant, see Definition 216.10. ∎

The following proposition is the summary of our previous discussion of the Hopf invariant.

Proposition 237.10. (Hopf Invariant Proposition) Let $n \geq 2$.

(1) The Hopf invariant $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism.

(2) The Hopf invariant $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ is zero for odd n.

(3) If $\mathbb{R}^n$ can be equipped with the structure of a division algebra, then there exists a continuous map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$.
(4) For $k = 1, 2, 4$ the Hopf maps $H, H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ give elements in $\pi_{4k-1}(S^{2k})$ with Hopf invariant 1.
(5) Given any $k \in \mathbb{N}$ there exists a continuous map $f\colon S^{4k-1} \to S^{2k}$ with $\mathrm{Hopf}(f) = 2$.

Proof. The proposition follows from the Hopf Invariant–Homomorphism Proposition 216.14, the Hopf Invariant One-Corollary 216.15, the Division Algebra–Bidegree-Lemma 216.19 and the Bidegree–Realization Proposition 216.22 and the Bidegree–Hopf Invariant Proposition 216.21. ■

We recall the following question that we asked a while ago.

Question 216.26. For which $n \in \mathbb{N}_{\geq 2}$ does there exist a continuous map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$?

Theorem 237.11. (Hopf Invariant–Power-of-2 Theorem) Let $n \geq 2$. If there exists a continuous map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$, then n is a power of 2.

Proof. The theorem is an immediate consequence of the Cohomology Ring–Non-Realization Proposition 237.7 and the Hopf Invariant–Cohomology Ring Lemma 237.9. ■

Theorem 214.13. (Hopf–Division Algebra Dimension Theorem) If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.

Proof. The theorem is an immediate consequence of the combination of the Hopf Invariant Proposition 237.10 (3) and the Hopf Invariant–Power-of-2 Theorem 237.11. ■

It is an amusing mental exercise, left to the reader, to compare the two proofs of the Hopf–Division Algebra Dimension Theorem 214.13. The first one relies on the calculation of the cup product of $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients and a fact in elementary number theory that is proved on page 4110. The second proof uses Steenrod operations and a different input from elementary number theory, namely the Binomial Coefficient–Mod 2-Lemma 237.3. Note that in some sense the former proof is more satisfactory since we could provide full details, whereas the latter proof relies on the Adem relations that we did not prove.

For completeness we remark that in 1960 Frank Adams proved the following strengthening of the Hopf Invariant–Power-of-2 Theorem 237.11. Note that this theorem finally gives a complete answer to Question 216.26.

Theorem 237.12. (Adams Theorem) Let $n \geq 2$. If there exists a continuous map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$, then $n = 2, 4$ or $n = 8$.

Proof. The theorem is proved in [**Adam1960**]. We refer to [**AdA1966**], [**May1999a**, Chapter 24.6] and [**Hus1993**, Theorem 15.4.3]) for alternative proofs. ■

Adams Theorem allows us to give an alternative proof of the following beautiful theorem.

Theorem 214.14. (Division Algebra–Dimensions Theorem) Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.

Proof. The theorem follows immediately from the Hopf Invariant Proposition 237.10 (3) together with the Adams Theorem 237.12. ■

237.4. **The Steenrod cyclic reduced powers.** The Steenrod operations Sq^i are cohomology operations on cohomology groups with $\mathbb{F}_2$-coefficients. It turns out that there exist also analogous cohomology operations for cohomology with $\mathbb{F}_p$-coefficients. More precisely, the following theorem holds:

Theorem 237.13. (Steenrod Theorem) Let p be an odd prime. There exists a family[2572]

$$P^i\colon \mathrm{H}^n(X,A;\mathbb{F}_p) \ \to\ \mathrm{H}^{n+2i(p-1)}(X,A;\mathbb{F}_p) \quad \text{with } i\in\mathbb{N}_0 \text{ and } n\in\mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $P^0 = \mathrm{id}$.

(2) For any $\varphi\in\mathrm{H}^{2i}(X,A;\mathbb{F}_p)$ we have $P^i(\varphi)=\varphi^p\in\mathrm{H}^{2pi}(X,A;\mathbb{F}_p)$.

(3) For any $\varphi\in\mathrm{H}^j(X,A;\mathbb{F}_p)$ with $i>j$ we have $P^i(\varphi)=0$.

(4) Let (X,A,B) be an excisive triad of topological spaces. For any $\varphi\in\mathrm{H}^r(X,A;\mathbb{F}_p)$ any $\psi\in\mathrm{H}^s(X,B;\mathbb{F}_p)$ and any $k\in\mathbb{N}_0$ we have[2573]

$$P^k(\varphi\cup\psi) \ =\ \sum_{i=0}^{k} \underbrace{P^i(\varphi)}_{\in \mathrm{H}^{r+2i(p-1)}(X,A;\mathbb{F}_p)} \ \cup\ \underbrace{P^{k-i}(\psi)}_{\in \mathrm{H}^{s+2(k-i)(p-1)}(X,B;\mathbb{F}_p)} \ \in\ \mathrm{H}^{r+s+2k(p-1)}(X,A\cup B;\mathbb{F}_p).$$

This equality is often called the Cartan formula.

Proof. This theorem is proved in [**SE1962**, Chapter VI]. A very different proof is given in [**Hat2002**, Theorem 4.L.16]. ■

The following theorem plays the role of the Adem Relations Theorem 237.1.

Theorem 237.14. (Adem Relations Theorem) Let p be an odd prime.

(1) For any $0<a<pb$ we have the following equality of cohomology operations:

$$P^a\circ P^b \ =\ \sum_{j=0}^{\left\lfloor \frac{a}{p}\right\rfloor} (-1)^{a+j}\cdot \binom{(p-1)(b-j)-1}{a-pj}\cdot P^{a+b-j}\circ P^j.$$

(2) Given $i\in\mathbb{N}_0$ we denote by $\beta\colon \mathrm{H}^i(X;\mathbb{F}_p)\to\mathrm{H}^{i+1}(X;\mathbb{F}_p)$ the Bockstein homomorphism that corresponds to the short exact sequence $0\to\mathbb{F}_p\to\mathbb{Z}/p^2\mathbb{Z}\to\mathbb{F}_p\to 0$. For any $0<a\leq pb$ the following mystifying equality holds:

$$\begin{aligned} P^a\circ\beta\circ P^b \ =\ & \sum_{j=0}^{\left\lfloor \frac{a}{p}\right\rfloor} (-1)^{a+j}\cdot \binom{(p-1)(b-j)}{a-pj}\cdot \beta\circ P^{a+b-j}\circ P^j \\ & - \sum_{j=0}^{\left\lfloor \frac{a-1}{p}\right\rfloor} (-1)^{a+j}\cdot \binom{(p-1)(b-j)-1}{a-pj-1}\cdot P^{a+b-j}\circ\beta\circ P^j. \end{aligned}$$

[2572]Note that if for argument's sake one plugs $p=2$ into the statements of the Steenrod Theorem 237.13, then one does *not* get the statements of the Steenrod Theorem 236.3 since the degree formulas are different. In other words there is no easy way to combine Theorems 236.3 and 237.13 into a single statement.

[2573]We refer to Definition 158.18 for the definition of an excisive triad. By the Excisive Triad-Proposition 158.20 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A=B$, or that at least one of A or B is the empty set. We refer again to Definition 204.7 for the definition of the relative cup product.

These equalities are called the Adem relations.

Example. Let us consider the case of the prime $p = 3$. In this case the corresponding reduced Steenrod operations are additive cohomology operations of the form

$$P^i\colon \mathrm{H}^n(X, A; \mathbb{F}_3) \ \to\ \mathrm{H}^{n+4i}(X, A; \mathbb{F}_3).$$

For $a = 1$ and $b = 3$ we obtain the Adem relation

$$P^1 \circ P^3 \underset{\uparrow}{=} \sum_{j=0}^{\left\lfloor \frac{1}{3} \right\rfloor} (-1)^{1+j} \binom{2\cdot(3-j)-1}{1-3\cdot j} \cdot P^{4-j} \circ P^j \underset{\uparrow}{=} (-1)\cdot \binom{5}{1} \cdot P^4 = -5\cdot P^4 \underset{\uparrow}{=} P^4.$$

by the Adem Relations Theorem 237.14 (1) — only one summand, namely for $j = 0$ — we work in $\mathbb{F}_3$ □

The Steenrod cyclic reduced powers can, not surprisingly, be used to give more information on homotopy groups of spheres. We refer to [**SE1962**, Chapter VI.5] and [**Hat2002**, Example 4.L.6] for details.

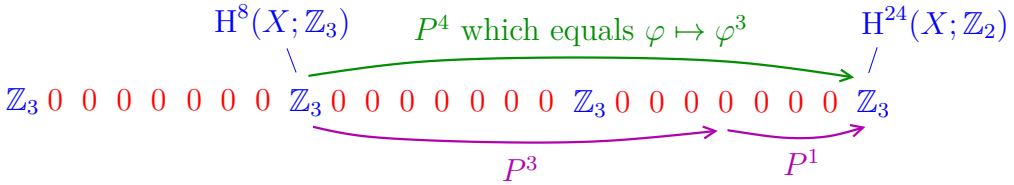
Instead of providing more results on homotopy groups of spheres we want to give a different application of the Steenrod cyclic reduced powers. To motivate the next result, recall that in Definition 215.6 we introduced the mysterious octonionic projective plane $\mathbb{O}\mathrm{P}^2$, which is a closed oriented 16-dimensional smooth manifold. In the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 215.9 we showed that there exists an isomorphism

$$\mathbb{Z}[z]/(z^3) \ \to\ (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2; \mathbb{Z}), \cup)$$

of superalgebras, where z is an element of degree 8. We were then wondering whether there exist higher dimensional octonionic spaces. But the following proposition, which gives a negative answer to Question 215.13, shows that this is not the case.

Proposition 237.15. (No-$\mathbb{O}\mathrm{P}^3$-Proposition) Let $d \in \mathbb{N}_{\geq 3}$. There is no topological space X such that $\mathrm{H}^*(X; \mathbb{F}_3)$ is isomorphic to $\mathbb{F}_3[\varphi]/(\varphi^{d+1})$ with $\varphi \in \mathrm{H}^8(X; \mathbb{F}_3)$.

Proof. We provide a proof by contradiction. So suppose that there exists a $d \in \mathbb{N}_{\geq 3}$ and topological space X such that $\mathrm{H}^*(X; \mathbb{F}_3)$ is isomorphic to $\mathbb{F}_3[\varphi]/(\varphi^{d+1})$ with $\varphi \in \mathrm{H}^8(X; \mathbb{F}_3)$.



We see that

$$0 \underset{\uparrow}{\neq} \varphi^3 \overset{\downarrow}{=} P^4(\varphi) \underset{\uparrow}{=} (P^1 \circ P^3)(\varphi) = P^1(\overbrace{P^3(\varphi)}^{\in \mathrm{H}^{8+3\cdot 4}(X;\mathbb{F}_3)}) \underset{\uparrow}{=} 0.$$

by the Steenrod Theorem 237.13 (2) — since we assume that $\mathrm{H}^*(X; \mathbb{F}_3) \cong \mathbb{F}_3[\varphi]/(\varphi^{d+1})$ for some $d \geq 3$ — since $P^4 = P^1 \circ P^3$ — since $\mathrm{H}^{20}(X; \mathbb{F}_3) = 0$

We have thus obtained a contradiction. ■

Remark. In Exercise 237.2 we will use the No-$\mathbb{O}\mathrm{P}^3$-Proposition 237.15 to give a negative answer to Question 215.11. So with most reasonable interpretations it is fair to say that there is no smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ with $n \geq 3$. □

237.5. Exercises for Chapter 237.

Exercise 237.1. Let X be a topological space and let $x \in \mathrm{H}^1(X;\mathbb{F}_2)$ be a cohomology class of degree *one*. Show that for any $k \in \mathbb{N}$ and any $i \in \mathbb{N}$ we have

$$\mathrm{Sq}^i(x^{2^k}) \;=\; \begin{cases} x^{2^k}, & \text{if } i = 0, \\ x^{2^{k+1}}, & \text{if } i = 2^k, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 237.2. Let $n \geq 3$. Show that there is no closed orientable $8n$-dimensional smooth manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \ldots, 8n$.
Remark. This exercise gives a negative answer to Question 215.11.

Part XXVII

Bundles: Basics

In this part of the lecture notes we introduce the notion of a bundle and we discuss basic properties and examples. A full-blown discussion of bundles, with all the whistles and bells, will have to wait till Part XXVII.

This part of the lecture notes is organized as follows:

- In Chapter 238 we will introduce bundles and we give many examples.
- In Chapter 239 we will see that every fiber bundle gives rise to a long exact sequence of homotopy groups. This sequence will in particular allow us to calculate some homotopy groups of spheres and matrix groups.
- In Chapter 240 we will introduce a more refined notion of bundles and we will study basic properties of such bundles.
- In Chapter 241 we will first show that if two maps are homotopic, then the pullbacks of bundles are isomorphic. This fact will allow us to show that bundles over contractible spaces are trivializable and it will allow us to classify some bundles over spheres and surfaces.
- In Chapter 242 we will consider the topology of total spaces of bundles. We will in particular see that the Euler characteristic of the total space is the product of the Euler characteristics of the base space and the fiber.
- In Chapter 243 we will discuss smooth bundles.
- In Chapter 244 we will introduce the Thom class of an (oriented) $\overline{B}^n$-bundle. We will use it to state and prove the Cup Product–Intersection Manifold Theorem 244.16 which allows us to calculate the cup product of the Poincaré duals of two transverse submanifolds of a given smooth manifold. This theorem can be viewed as a generalization of the Cup Product–Intersection Number Theorem 220.10.
- In Chapter 245 we will use the Thom class to define the Euler class of disk bundles. Using the Euler class we can state the Gysin sequence which can be used to calculate the cohomology groups of the total space of a sphere bundle.

CHAPTER 238

Bundles: Definition and examples

In this chapter we will give a short introduction to bundles, with a particular emphasis on bundles of smooth manifolds. This chapter is organized as follows:

- In Section 238.1 we will introduce bundles and we consider a few basic examples.
- In Section 238.2 we will show that the Hopf maps $S^3 \to S^2$ and $S^7 \to S^4$ are bundle maps.
- In Section 238.3 we will recall the Ehresmann Fibration Theorem 238.6, which is a convenient tool for showing that a map between smooth manifolds is a smooth bundle map.
- In Section 238.4 we will use the Ehresmann Fibration Theorem 238.6 to show that the connected sums $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)$, $\mathbb{H}\mathrm{P}^2\#(-\mathbb{H}\mathrm{P}^2)$ are total spaces of smooth bundle maps over spheres.
- In Section 238.5 we will use the Ehresmann Fibration Theorem 238.6 to show that for any $n \in \mathbb{N}$ we have an $\mathrm{O}(n)$-bundle $\mathrm{O}(n+1) \to S^n$ and that we have a $\mathrm{U}(n)$-bundle $\mathrm{U}(n+1) \to S^{2n+1}$.
- In Section 238.6 we will see that F-bundles over S^1 correspond precisely to mapping tori.
- In Section 238.7 we will provide exercises for Chapter 238.

In the next Chapter 239 we will show that fiber bundle always gives rise to a long exact sequence of homotopy groups.

238.1. **Definition of bundles.** We start out with the following definition of a bundle map:

Definition 238.1. Let F be a topological space.

(1) A continuous map $p\colon Y \to B$ between two topological spaces is called an F-bundle over B if for every $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\tau} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f)\mapsto x} & \\ & U. & \end{array}$$

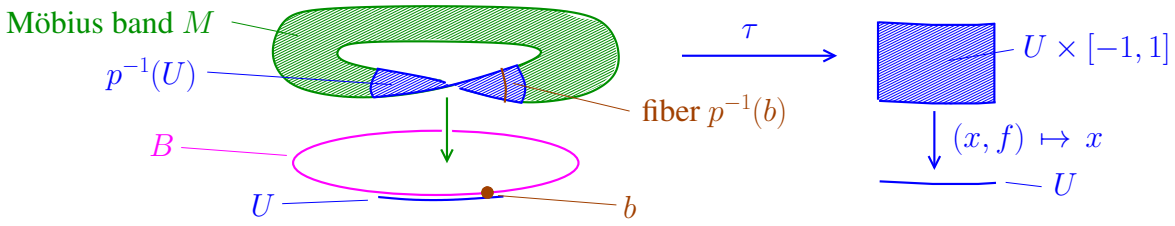
Given an F-bundle $p\colon Y \to B$ we adopt the following language:

- We refer to Y as the total space, to B as the base space, and to F as the fiber of the F-bundle.
- We refer to a homeomorphism $\tau\colon p^{-1}(U) \to U \times F$ as above as a trivialization.

(2) A continuous map $p\colon Y \to B$ between two topological spaces is called a bundle map if there exists a topological space F such that p is an F-bundle.

(3) Let $p\colon Y \to B$ be an F-bundle. For any $C \subseteq B$ we refer to $p\colon p^{-1}(C) \to C$ as the restriction of p to C.
(4) If F is a smooth manifold, then we say that a map $p\colon Y \to B$ between two smooth manifolds is a smooth F-bundle over B if in the definition in (1) we can always upgrade "homeomorphism" by "diffeomorphism".[2574] Similarly we define the notion of a smooth bundle map.[2575]

Example. In the following figure we show a map $p\colon M \to B$ from the Möbius band M to a circle B:



We also illustrate that this map is a $[-1,1]$-bundle. More formally we will see in Exercise 238.1 that the map

$$\begin{array}{rcl} M = ([0,1]\times[-1,1])/(0,y)\sim(1,-y) & \to & S^1 \\ [(t,y)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a smooth $[-1,1]$-bundle over S^1. We refer to it as the *Möbius* $[-1,1]$*-bundle*. □

Definition 238.1.

(4) For any two topological spaces B and F the projection $p\colon B \times F \to B$ onto the first factor is evidently an F-bundle. We refer to it as the trivial F-bundle over B.
(5) Let B be a topological space and let $p\colon Y \to B$ and $q\colon Z \to B$ be two F-bundles over B. An isomorphism of the two F-bundles over B is a homeomorphism $f\colon Y \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow[\cong]{f} & Z \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B. & \end{array}$$

(6) Any F-bundle over a topological space B that is isomorphic to the trivial bundle $B \times F \to B$ is called trivializable, otherwise it is called non-trivializable.

Examples.

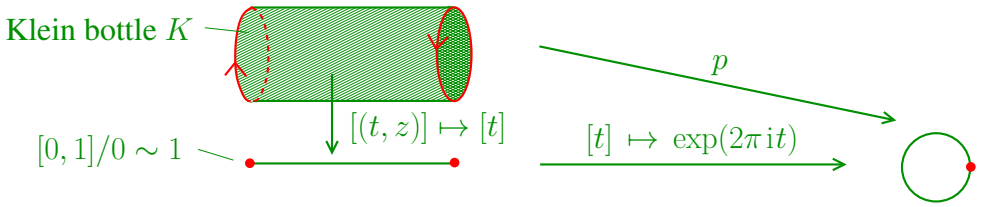
(1) Note that the Möbius band M is not homeomorphic[2576] to the annulus $S^1 \times [-1,1]$. It follows that the above bundle map $M \to S^1$ is non-trivializable.

[2574] Since the projections $U \times F \to F$ are smooth we see that for a smooth F-bundle the projection $p\colon Y \to B$ is necessarily smooth.
[2575] Note that this notion of a smooth bundle map is consistent with Definition 122.1.
[2576] The most elementary way to showing that the Möbius band M is not homeomorphic to the annulus $A = S^1 \times [-1,1]$ is by noting that the former has one boundary component and the latter has two boundary components. Note though that even here we implicitly use the not-so-elementary Unique 2D-Type-of-Chart Proposition Proposition 101.8.

(2) One can easily show that the map

$$\begin{array}{rcl} \overbrace{([0,1]\times S^1)/(0,z)\sim(1,z^{-1})}^{\text{Klein bottle}} & \to & S^1 \\ [(t,z)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a smooth S^1-bundle over S^1.



Since the Klein bottle is not diffeomorphic to $S^1\times S^1$ we see that this S^1-bundle $K\to S^1$ is not trivializable. □

The following lemma characterizes the most elementary type of a bundle, namely bundles where the fiber is a discrete topological space:

Lemma 238.2. (Covering-as-Bundle Lemma)

(1) If F is a non-empty *discrete* topological space, then any F-bundle $p\colon Y\to B$ is a covering of index $\#F$.

(2) Let $p\colon Y\to B$ be a covering over a connected space B and let $x\in B$. We equip $p^{-1}(x)$ with the discrete topology.[2577] With this topology the map p is a $p^{-1}(x)$-bundle.

Proof.

(1) Let F be a non-empty discrete topological space and let $p\colon Y\to B$ be an F-bundle. First note that it follows easily from the hypothesis that F is non-empty and the definition of an F-bundle that $p\colon Y\to B$ is surjective. Now let $x\in B$. By definition of an F-bundle there exists an open neighborhood U of $x\in B$ and a homeomorphism $\tau\colon p^{-1}(U)\to U\times F$, such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\tau} & U\times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f)\mapsto x} & \\ & U. & \end{array}$$

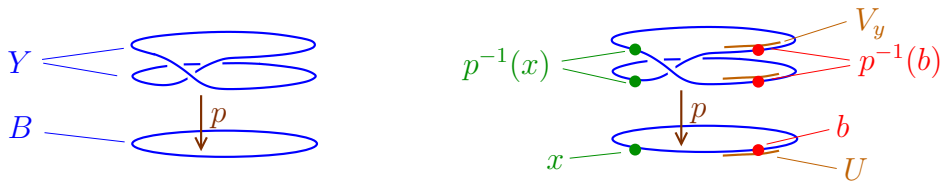
It follows now easily from the hypothesis that F is discrete (together with the elementary Exercise 7.20) that U is uniformly covered in the sense of Definition 91.8.

(2) Let $p\colon Y\to B$ be a covering map over a connected space B and let $x\in B$. We need to show that p is a $p^{-1}(x)$-bundle. Now let $b\in B$. By the definition of a covering in Definition 91.8 there exists an open neighborhood U of b such that $p^{-1}(U)$ is the disjoint union of open subsets $\{V_y\}_{y\in p^{-1}(b)}$ with the following two properties:

- For each $y\in p^{-1}(b)$ the set V_y is an open neighborhood of $y\in Y$.
- The map $p|_{V_y}\colon V_y\to U$ is a homeomorphism.

Since B is connected we know by the Covering–Preimage Lemma 91.9 that there exists a bijection $\Theta\colon p^{-1}(b)\to p^{-1}(x)$. Finally note that it follows from the Disjoint Union

[2577] In fact, we know by the Covering–Basics Lemma 91.12 that the subset topology on $p^{-1}(x)\subseteq Y$ is in fact the discrete topology.



Embedding Lemma 7.29 and the elementary Exercise 7.20 (b) that

$$\begin{array}{rcl} p^{-1}(U) = \bigsqcup_{y \in p^{-1}(b)} V_y & \to & U \times p^{-1}(x) \\ \underbrace{v}_{\in V_y} & \mapsto & (p(v), \Theta(y)) \end{array}$$

is a trivialization. We have thus verified that $p\colon Y \to B$ is a $p^{-1}(x)$-bundle. ∎

238.2. **The Hopf maps.** As in Definition 134.24 and Definition 216.2 we consider the following two Hopf maps, which build on the concept of the complex and quaternionic projective spaces which we first introduced in Definition 9.11 and Definition 215.2.

Definition 238.3. We refer to the maps

identification from page 340

$$\begin{array}{rcl} H_{\mathbb{C}}\colon S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \,|\, |z_0|^2 + |z_1|^2 = 1\} & \to & \mathbb{C}\mathrm{P}^1 \overset{\downarrow}{=}_{\mathrm{i}} S^2 \\ (z_0, z_1) & \mapsto & [z_0 : z_1] \end{array}$$

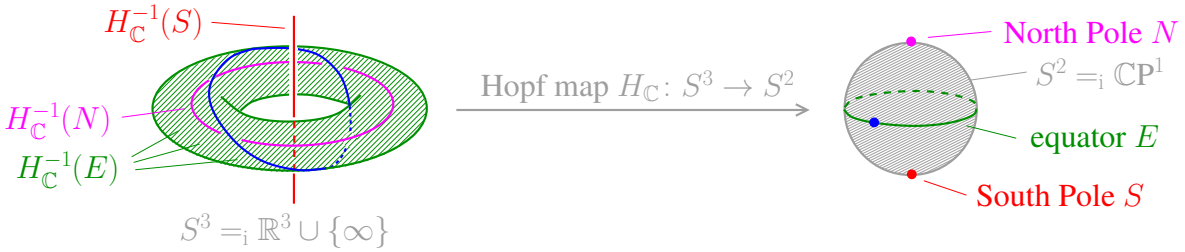
and

identification from page 4114

$$\begin{array}{rcl} H_{\mathbb{H}}\colon S^7 = \{(h_0, h_1) \in \mathbb{H}^2 \,|\, |h_0|^2 + |h_1|^2 = 1\} & \to & \mathbb{H}\mathrm{P}^1 \overset{\downarrow}{=}_{\mathrm{i}} S^4 \\ (h_0, h_1) & \mapsto & [h_0 : h_1] \end{array}$$

as Hopf maps.

With the limited artistic skills of the author we try to show in the figure below the Hopf map $H_{\mathbb{C}}\colon \mathbb{R}^3 \cup \{\infty\} \to S^2$. In particular we show, for various points on S^2, the preimages in $\mathbb{R}^3 \cup \{\infty\}$.



The following proposition shows that the above two Hopf maps are actually bundle maps:

Proposition 238.4. (Hopf-Bundle Proposition)
(1) The Hopf map $H_{\mathbb{C}}\colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ is a smooth S^1-bundle.
(2) The Hopf map $H_{\mathbb{H}}\colon S^7 \to \mathbb{H}\mathrm{P}^1 =_{\mathrm{i}} S^4$ is a smooth S^3-bundle.

Proof. We only provide the proof of (2). The proof of (1) is a straightforward modification of the proof of (2). Given $j \in \{0,1\}$ we set

$$U_j \coloneqq \{[h_0 : h_1] \in \mathbb{H}\mathrm{P}^1 \mid h_j \neq 0\}.$$

Note that it follows easily from the Definition 8.3 of the quotient topology on $\mathbb{H}\mathrm{P}^1$ that U_j is an open subset of $\mathbb{H}\mathrm{P}^1$.

Claim. Let $j \in \{0,1\}$. The map

$$\begin{aligned} \tau_j \colon H_{\mathbb{H}}^{-1}(U_j) = \{(h_0,h_1) \in \mathbb{H}^2 \mid |h_0|^2 + |h_1|^2 = 1 \text{ and } h_j \neq 0\} &\to U_j \times S^3 \\ (h_0,h_1) &\mapsto \big([h_0 : h_1], \tfrac{1}{|h_j|} \cdot h_j\big) \end{aligned}$$

is a diffeomorphism.

Proof. We consider the maps

$$\begin{array}{rcl} \Psi_0 \colon \overbrace{U_0 \times S^3}^{\subseteq \mathbb{H}\mathrm{P}^1 \times S^3} & \to & \overbrace{H_{\mathbb{H}}^{-1}(U_0)}^{\subseteq S^7} \\ ([\underbrace{h_0}_{\neq 0} : h_1], u) & \mapsto & \dfrac{(u, u \cdot h_0^{-1} \cdot h_1)}{\|(u, u \cdot h_0^{-1} \cdot h_1)\|} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi_1 \colon \overbrace{U_1 \times S^3}^{\subseteq \mathbb{H}\mathrm{P}^1 \times S^3} & \to & \overbrace{H_{\mathbb{H}}^{-1}(U_1)}^{\subseteq S^7} \\ ([h_0 : \underbrace{h_1}_{\neq 0}], u) & \mapsto & \dfrac{(u \cdot h_1^{-1} \cdot h_0, u)}{\|(u \cdot h_1^{-1} \cdot h_0, u)\|}. \end{array}$$

One can easily check that these maps Ψ_0 and Ψ_1 are well-defined.[2578] Using the Topological-Quotient Proposition 8.2 (1b) one can show that the maps Ψ_j are continuous. Furthermore, using the explicit smooth atlas on $\mathbb{H}\mathrm{P}^1 \times S^3$ and S^7 one can easily verify that the maps τ_j and Ψ_j are smooth. Finally note that it follows easily from the definitions that Ψ_j and τ_j are inverses of one another. ⊞

Finally note that it is clear that the open subsets U_j cover all of $\mathbb{H}\mathrm{P}^1$ and that the triangles

$$\begin{array}{ccc} H_{\mathbb{H}}^{-1}(U_j) & \xrightarrow[\tau_j]{(h_0,h_1) \mapsto ([h_0:h_1], \frac{1}{|h_j|} \cdot h_j)} & U_j \times S^3 \\ & {\scriptstyle (h_0,h_1) \mapsto [h_0:h_1]} \searrow \quad \overset{H_{\mathbb{H}}}{} \quad \swarrow {\scriptstyle (u,h) \mapsto u} & \\ & U_j & \end{array}$$

commute. This shows, according to Definition 238.1, that $H_{\mathbb{H}} \colon S^7 \to \mathbb{H}\mathrm{P}^1$ is a smooth S^3-bundle. ■

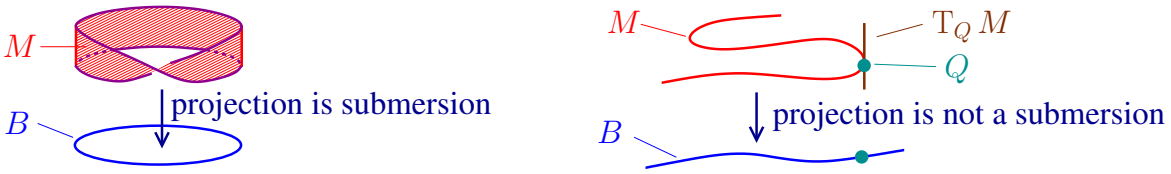
238.3. **The Ehresmann Fibration Theorem.** [2579] In Definition 30.9 we essentially called a smooth map $f \colon M \to N$ between smooth manifolds an *immersion* if for each $Q \in M$ the differential $\mathrm{D}\varphi_Q \colon \mathrm{T}_Q M \to \mathrm{T}_{\varphi(Q)} N$ is a monomorphism. It is natural to also consider the case that the differential is an epimorphism. This leads us to the following definition, which we first encountered in Definition 122.5:

[2578] Indeed, a short calculation shows that if we pick a different representative for $[h_0 : h_1]$, i.e. if we plug in $(\lambda \cdot h_0, \lambda \cdot h_1)$ for some $\lambda \in \mathbb{H} \setminus \{0\}$, then we obtain the same result. (Note though that the order of the factors in the definition of Ψ_j is important, since $\mathbb{H}$ is not commutative.)

[2579] In this section we will shamelessly copy-paste large chunks from Section 122.2.

Definition 238.5. Let M and B be smooth manifolds where $\partial B = \varnothing$. A smooth map $f\colon M \to B$ is called a submersion if the following statements hold:

(1) For every $Q \in M \setminus \partial M$ the map $\mathrm{D}f_Q\colon \mathrm{T}_Q M \to \mathrm{T}_{f(Q)}B$ is an epimorphism.
(2) For every $Q \in \partial M$ the map $\mathrm{D}f_Q\colon \mathrm{T}_Q(\partial M) \to \mathrm{T}_{f(Q)}B$ is an epimorphism.



Example. Let $p\colon M \to B$ be a smooth F-bundle where B is a smooth manifold with $\partial B = \varnothing$. We claim that p is a submersion. Let $Q \in M$. Since p is a smooth F-bundle there exists an open neighborhood U of $p(Q)$ and a diffeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\tau} & U \times F \\ {\scriptstyle p}\downarrow & \swarrow_{(x,f)\mapsto x} & \\ U. & {\scriptstyle =:q} & \end{array}$$

We also consider the inclusions $\iota\colon U \to B$ and $\widetilde{\iota}\colon p^{-1}(U) \to M$. From the above diagram and the observation that $p \circ \widetilde{\iota} = \iota \circ p$ we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{T}_Q M & \xleftarrow[\cong]{\mathrm{D}\widetilde{\iota}_Q} & \mathrm{T}_Q(p^{-1}(U)) & \xrightarrow[\cong]{\mathrm{D}\tau_Q} & \mathrm{T}_{\tau(Q)}(U \times F) \\ \downarrow{\scriptstyle \mathrm{D}p_Q} & & \downarrow{\scriptstyle \mathrm{D}p_Q} & \swarrow_{\mathrm{D}q_{\tau(Q)}} & \\ \mathrm{T}_{p(Q)}B & \xleftarrow[\cong]{\mathrm{D}\iota_{p(Q)}} & \mathrm{T}_{p(Q)}(U). & & \end{array}$$

It follows easily from the Product Orientation Proposition 31.48 (3b) that the diagonal map to the right is an epimorphism. Furthermore note that it follows from the Tangent Space-Inclusion Proposition 29.20 (3) that the left horizontal maps are isomorphisms. Since the diagram commutes we see that the left vertical map is indeed an epimorphism. □

We recall the following theorem, which we first formulated as the Ehresmann Fibration Theorem 122.6.

Theorem 238.6. (Ehresmann Fibration Theorem) Let M and B be smooth manifolds such that $\partial B = \varnothing$. Let $p\colon M \to B$ be a submersion. Given $b \in B$ we set $F_b \coloneqq p^{-1}(b)$. If M is compact and if B is connected, then the following statements hold:

(1) Each F_b is naturally a compact smooth manifold of dimension $\dim(M) - \dim(B)$.
(2) All the smooth manifolds F_b are diffeomorphic.
(3) The map $p\colon M \to B$ is a smooth bundle map.

Remark.

(1) In the literature the Ehresmann Fibration Theorem 238.6 is sometimes also called the *Submersion Theorem*.
(2) Alternative proofs of the Ehresmann Fibration Theorem 238.6 are given in [**Dun2018**, Theorem 8.5.10] or in [**Ebe2007**, Theorem 4.1].

(3) In Exercise 238.16 and the Ehresmann Fibration Theorem 250.2 we consider a variation on the Ehresmann Fibration Theorem 238.6 where we allow B to have non-empty boundary. □

To explain where this theorem comes from it is useful to recall Definition 32.1:

Definition 238.7. Let M and B be smooth manifolds and let $\varphi\colon M \to B$ be a smooth map.

(1) Let $Q \in M$. We say that Q is a regular point if $\mathrm{D}\varphi_Q\colon \mathrm{T}_Q M \to \mathrm{T}_{\varphi(Q)} B$ is an epimorphism.

(2) We say that $b \in B$ is a regular value if all points in the preimage $\varphi^{-1}(b)$ are regular.

Note that with this language a smooth map $\varphi\colon M \to B$ is a submersion precisely if every $b \in B$ is a regular value of φ.

After this remainder we can now recall the following theorem:

Theorem 56.9 (Neighborhood-Regular Value Theorem) Let $\varphi\colon M \to B$ be a smooth map between smooth manifolds M and B with $\partial B = \varnothing$. Let $b \in B$ be a regular value of φ.

(1) $F_b := \varphi^{-1}(b)$ is a proper submanifold of M of dimension $\dim(M) - \dim(B)$.

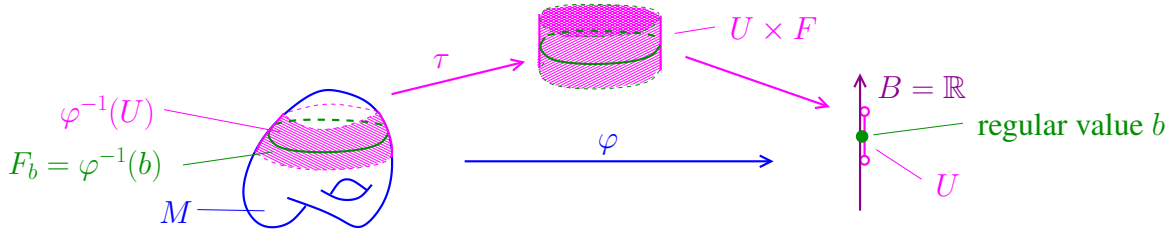
(2) If M is orientable, then F_b is also orientable.

(3) If M is compact, then F_b is also compact.

For the last statement we assume that M is compact:

(4) There exists an open neighborhood U of $b \in B$ together with a diffeomorphism $\tau\colon \varphi^{-1}(U) \to U \times F_b$ such that the following diagram commutes:

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow[\cong]{\tau} & U \times F_b \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle (y,f)\mapsto y} & \\ & U. & \end{array}$$



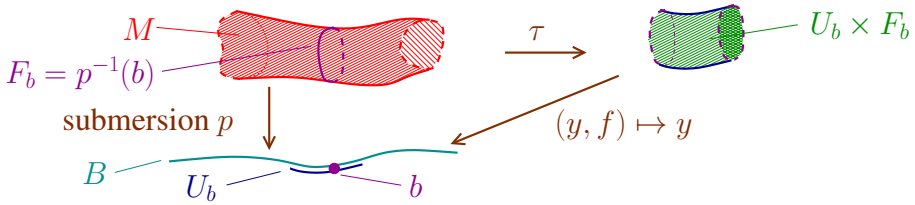
Proof of the Ehresmann Fibration Theorem 238.6. Let M and B be smooth manifolds such that $\partial B = \varnothing$, such that B is connected and such that M is compact. Furthermore let $p\colon M \to B$ be a submersion. The Neighborhood-Regular Value Theorem 56.9[2580] gives us the following two facts:

(a) $F_b := p^{-1}(b)$ is naturally a compact smooth manifold of dimension $\dim(M) - \dim(B)$.

[2580] Note that to apply this theorem we use that M is compact and that $\partial B = \varnothing$.

(b) Given any $b \in B$ there exists an open neighborhood U_b of $b \in B$ and a diffeomorphism $\tau\colon p^{-1}(U_b) \to U_b \times F_b$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow[\cong]{\tau} & U_b \times F_b \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (y,f)\mapsto y} & \\ & U_b. & \end{array}$$



We now turn to the proof of the three statements:

(1) This statement is just Statement (a).

(2) It follows (similar to the Covering–Preimage Lemma 91.9) easily from Statement (b) and the fact that B is connected, that all the smooth manifolds F_b are diffeomorphic.

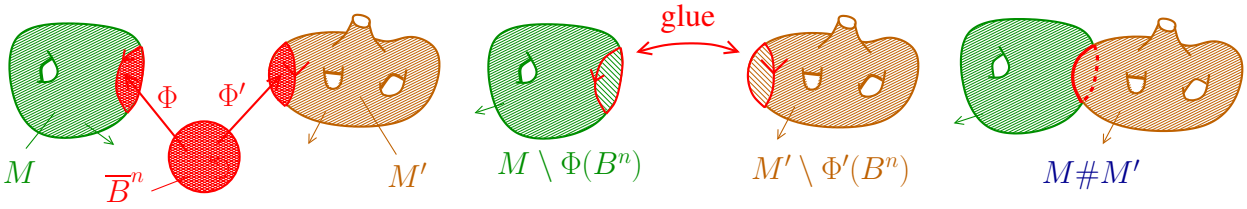
(3) It now follows easily from Statements (b) and (2) that $p\colon M \to B$ is a smooth bundle map. ■

238.4. **The connected sum of two projective spaces with opposite orientation.** First let us recall Definition 54.12:

Definition 238.8. Let $n \in \mathbb{N}_0$ and let M and M' be two connected non-empty n-dimensional smooth manifolds. (If both M and M' are orientable, then we assume that M and M' are oriented.) We pick two smooth embeddings $\Phi\colon \overline{B}^n \to M \setminus \partial M$ and $\Phi'\colon \overline{B}^n \to M' \setminus \partial M'$. (If both M and M' are orientable, then we assume that precisely one of Φ and Φ' is orientation-preserving.) We define the connected sum of M and M' as

$$M \# M' := ((M \setminus \Phi(B^n)) \sqcup (M' \setminus \Phi'(B^n)))/\sim \quad \text{where } \Phi(P) \sim \Phi'(P) \text{ for all } P \in S^{n-1}.$$

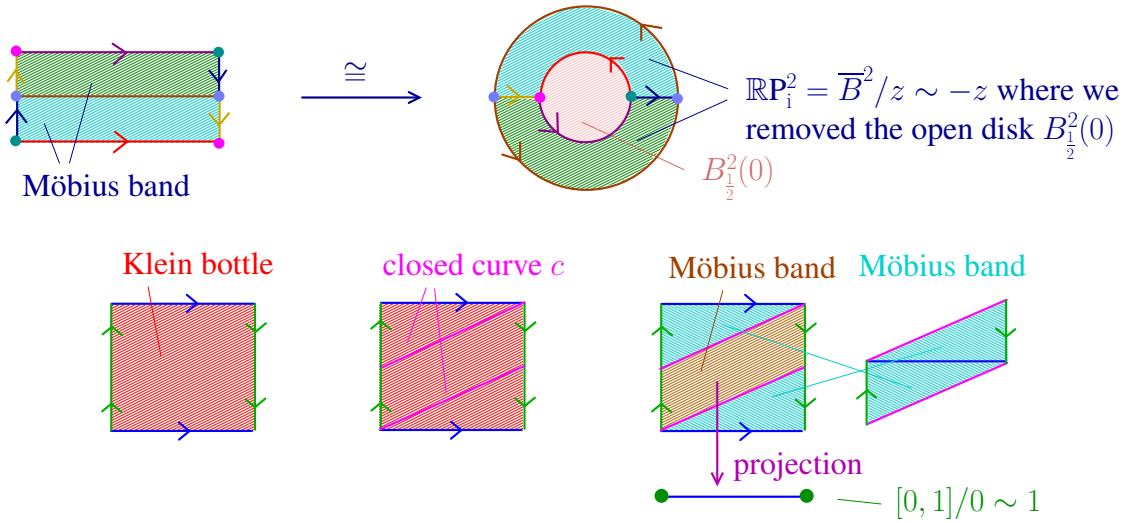
Note that we showed in the Smooth Connected Sum-Proposition 54.13 that the diffeomorphism type of $M \# M'$ is independent of any choices.



Example. In the Surface–Connected Sum Lemma 54.15 (1) we showed that the connected sum $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is diffeomorphic to the Klein bottle. For the reader's convenience we recall the argument.

(1) In the figure below we show that the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z$ minus an open disk is diffeomorphic to the Möbius band.

(2) In the figure below we show that the union of two Möbius bands, glued along the boundary, is diffeomorphic to the Klein bottle.



Furthermore recall that on page 4581 we saw that the Klein bottle is the total space of a non-trivializable smooth S^1-bundle over S^1. We have thus shown that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is the total space of a non-trivializable S^1-bundle over S^1. □

In the example we just saw that the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is the total space of a (non-trivializable) S^1-bundle over S^1. One can ask whether a similar statement also holds if we replace the real projective space $\mathbb{R}\mathrm{P}^2$ by the complex projective space $\mathbb{C}\mathrm{P}^2$ or the quaternionic projective space $\mathbb{H}\mathrm{P}^2$. The naive guess does not work: In Exercise 241.11 we will see that $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is *not* the total space of an S^2-bundle over S^2.

But we will see in a second, that if we are a little more careful about orientations, then the above example does indeed generalize. At this moment it is worth recalling the following notation:

Notation 238.9. Given an oriented smooth manifold M we denote by $-M$ the smooth manifold with the opposite orientation.

Now we can formulate the promised generalization of the above phenomenon:

Proposition 238.10. (Projective Space–Connected Sum Proposition)
(1) There exists a smooth S^1-bundle map $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2 \to S^1$.
(2) There exists a smooth S^2-bundle map $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2) \to S^2$.
(3) There exists a smooth S^4-bundle map $\mathbb{H}\mathrm{P}^2\#(-\mathbb{H}\mathrm{P}^2) \to S^4$.
All these bundles are non-trivializable.

The key to proving the Projective Space–Connected Sum Proposition 238.10 is the following lemma, which is interesting in its own right:

Lemma 238.11. (Projective Space-minus-Ball Lemma) Let $n \in \mathbb{N}_0$.
(1) The map

$$\begin{array}{rcl} \Theta\colon \overline{B}^n = \{(x_0,\dots,x_{n-1}) \in \mathbb{R}^n \,|\, |x_0|^2 + \cdots + |x_{n-1}|^2 \leq 1\} & \to & \mathbb{R}\mathrm{P}^n \\ (x_0,\dots,x_{n-1}) & \mapsto & [x_0 : \cdots : x_{n-1} : 1] \end{array}$$

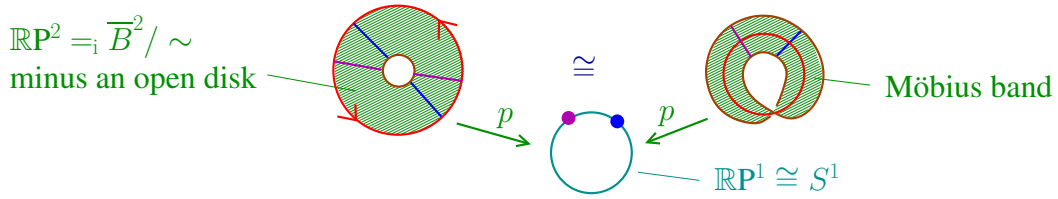
is a smooth embedding and

$$\begin{array}{rcl} p\colon \mathbb{R}\mathrm{P}^n \setminus \Theta(B^n) & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [x_0 : \cdots : x_{n-1} : x_n] & \mapsto & [x_0 : \cdots : x_{n-1}] \end{array}$$

is a smooth $\overline{B}^1$-bundle.[2581]

(2) Similar to (1) one can show that $p\colon \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) \to \mathbb{C}\mathrm{P}^{n-1}$ is a smooth $\overline{B}^2$-bundle.

(3) Similar to (1) one can show that $p\colon \mathbb{H}\mathrm{P}^n \setminus \Theta(B^{4n}) \to \mathbb{H}\mathrm{P}^{n-1}$ is a smooth $\overline{B}^4$-bundle.



To practice our techniques we will provide two different proofs of the Projective Space-minus-Ball Lemma 238.11. The first proof is more direct, whereas the second proof makes use of the Ehresmann Fibration Theorem 238.6.

First proof of the Projective Space-minus-Ball Lemma 238.11. In the following we deal with the complex projective spaces. The case of real and quaternionic projective spaces is dealt with in almost the same way.

First note that by the definition of the smooth structure on $\mathbb{C}\mathrm{P}^n$, see the Projective Space–Manifold Proposition 25.13 (2), we know that for $i = 0, \ldots, n$ the maps

$$\begin{array}{rcl} \Phi_i\colon U_i \coloneqq \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, z_i \neq 0\} & \to & \mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n} \\ [z_0 : \cdots : z_n] & \mapsto & \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \end{array}$$

form a smooth atlas for $\mathbb{C}\mathrm{P}^n$. Using this explicit atlas one can easily verify that the given map

$$\begin{array}{rcl} \Theta\colon \overline{B}^{2n} = \{(z_0, \ldots, z_{n-1}) \in \mathbb{C}^n \,|\, |z_0|^2 + \cdots + |z_{n-1}|^2 \leq 1\} & \to & \mathbb{C}\mathrm{P}^n \\ (z_0, \ldots, z_{n-1}) & \mapsto & [z_0 : \cdots : z_{n-1} : 1] \end{array}$$

is a smooth embedding. Next note that

$$\Theta(B^{2n}) \;=\; \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, |z_n|^2 > |z_0|^2 + \cdots + |z_{n-1}|^2\}.$$

In other words,

$$\mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) \;=\; \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, |z_n|^2 \leq |z_0|^2 + \cdots + |z_{n-1}|^2\}.$$

It follows from this discussion that $[0 : 0 : \cdots : 0 : 1] \notin \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n})$ which implies that the map $p\colon \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) \to \mathbb{C}\mathrm{P}^{n-1}$ is well-defined. We need to show that this map defines a smooth $\overline{B}^{2n}$-bundle.

For $i = 0, \ldots, n-1$ we consider the following open subset of $\mathbb{C}\mathrm{P}^{n-1}$:

$$V_i \;\coloneqq\; \{[z_0 : z_1 : \cdots : z_{n-1}] \in \mathbb{C}\mathrm{P}^{n-1} \,|\, z_i \neq 0\}.$$

Evidently we have $V_0 \cup \cdots \cup V_{n-1} = \mathbb{C}\mathrm{P}^{n-1}$. Since $V_0, \ldots, V_{n-1}$ is an open over of $\mathbb{C}\mathrm{P}^{n-1}$ it now remains to prove the following claim:

[2581] Note that for $\mathbb{R}\mathrm{P}^2$ we saw above that $\mathbb{R}\mathrm{P}^2 \setminus \Theta(B^2)$ is diffeomorphic to the Möbius band, which (as we saw on page 4580) is the total space of a non-trivializable $\overline{B}^1$-bundle over S^1.

Claim. For each $i \in \{0, \dots, n-1\}$ we consider the map

$$\begin{array}{rcl} \tau_i \colon p^{-1}(V_i) & \to & V_i \times \overline{B}^2 \\ [z_0 : z_1 : \cdots : z_n] & \mapsto & \left([z_0 : \cdots : z_{n-1}], \frac{z_n}{z_i} \cdot \frac{|z_i|}{\sqrt{|z_0|^2 + \cdots + |z_{n-1}|^2}}\right). \end{array}$$

(1) The map $\tau_i \colon p^{-1}(V_i) \to V_i \times \overline{B}^2$ is well-defined.
(2) The map $\tau_i \colon p^{-1}(V_i) \to V_i \times \overline{B}^2$ is a trivialization.

Proof.

(1) Let $[z_0 : \cdots : z_{n-1} : z_n] \in p^{-1}(V_i) \subseteq \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n})$. As we discussed above this implies that $|z_n|^2 \leq |z_0|^2 + \cdots + |z_{n-1}|^2$. It follows that $\left|\frac{z_n}{z_i} \cdot \frac{|z_i|}{\sqrt{|z_0|^2 + \cdots + |z_{n-1}|^2}}\right| \leq 1$. This shows that $\tau_i(p^{-1}(V_i))$ does indeed take values in $V_i \times \overline{B}^2$.
(2) We consider the map

$$\begin{array}{rcl} \sigma_i \colon V_i \times \overline{B}^2 & \to & p^{-1}(V_i) \\ ([w_0 : \cdots : w_{n-1}], w) & \mapsto & \left[w_0 : \cdots : w_{n-1} : w \cdot w_i \cdot \frac{\sqrt{|w_0|^2 + \cdots + |w_{n-1}|^2}}{|w_i|}\right] \end{array}$$

An elementary calculation shows that $\sigma_i(V_i \times \overline{B}^2)$ is indeed contained in $\mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n})$. Finally note that it follows easily from the definitions that τ_i and σ_i are inverses of one another. This shows that τ_i is indeed a trivialization. ∎

Second proof of the Projective Space-minus-Ball Lemma 238.11. Again we will only deal with the complex projective spaces. The case of real and quaternionic projective spaces is dealt with in almost the same way.

Note that in the first proof of the Projective Space-minus-Ball Lemma 238.11 on page 4588 we already showed that $\Theta \colon B^{2n} \to \mathbb{C}\mathrm{P}^n$ is a smooth embedding.

Our idea now is to use the Ehresmann Fibration Theorem 238.6 to prove that the map $p \colon \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) \to \mathbb{C}\mathrm{P}^{n-1}$ is a smooth $\overline{B}^2$-bundle. This idea leads us to the following claim:

Claim. The map $p \colon \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) \to \mathbb{C}\mathrm{P}^{n-1}$ is a submersion.

Proof. First note that using explicit charts one can easily verify that the map is smooth. Next let $Q \in \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n})$. By the definition of a submersion we need to show that the map $\mathrm{D}p_Q \colon \mathrm{T}_Q(\mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n})) \to \mathrm{T}_{p(Q)}\mathbb{C}\mathrm{P}^{n-1}$ is an epimorphism. To do so we consider the following diagram:

$$\begin{array}{ccc} \mathbb{C}^{n+1} \setminus \{(z_0, \dots, z_n) \,|\, |z_0|^2 + \cdots + |z_{n-1}|^2 < |z_n|^2\} & \xrightarrow[=:\widetilde{p}]{(z_0,\dots,z_n) \mapsto (z_0,\dots,z_{n-1})} & \mathbb{C}^n \setminus \{0\} \\ \Big\downarrow{\scriptstyle =:\pi,\ (z_0,\dots,z_n) \mapsto [z_0:\cdots:z_n]} & & \Big\downarrow{\scriptstyle =:\pi,\ (z_0,\dots,z_{n-1}) \mapsto [z_0:\cdots:z_{n-1}]} \\ \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) & \xrightarrow[p]{[z_0:\cdots:z_n] \mapsto [z_0:\cdots:z_{n-1}]} & \mathbb{C}\mathrm{P}^{n-1}. \end{array}$$

We pick $(Q_0, \dots, Q_n) \in \mathbb{C}^{n+1}$ with $\pi(Q_0, \dots, Q_n) = Q$. We consider the following maps between tangent spaces:

$$\begin{array}{ccc} \mathrm{T}_{(Q_0,\dots,Q_n)}(\mathbb{C}^{n+1} \setminus \{(z_0,\dots,z_n) \,|\, |z_0|^2+\ldots+|z_{n-1}|^2 < |z_n|^2\}) & \xrightarrow{\mathrm{D}\widetilde{p}_{(Q_0,\dots,Q_n)}} & \mathrm{T}_{(Q_0,\dots,Q_{n-1})}(\mathbb{C}^n \setminus \{0\}) \\ \downarrow{\scriptstyle \mathrm{D}\pi_{(Q_0,\dots,Q_n)}} & & \downarrow{\scriptstyle \mathrm{D}\pi_{(Q_0,\dots,Q_{n-1})}} \\ \mathrm{T}_Q(\mathbb{CP}^n \setminus \Theta(B^{2n})) & \xrightarrow{\mathrm{D}p_{[Q_0:\cdots:Q_n]}} & \mathrm{T}_{p(Q)}\mathbb{CP}^{n-1}. \end{array}$$

We make the following three observations:

(a) It follows from $\pi \circ \widetilde{p} = p \circ \pi$ and the Tangent Space-Functor Proposition 29.15 that the diagram commutes.

(b) Since on the top horizontal map we are dealing with an explicit smooth map between open subsets of $\mathbb{C}^{n+1}$ and $\mathbb{C}^n$ one can easily verify, e.g. from the Differential-equals-Differential Lemma 29.6 and the discussion on page 651, that the top horizontal map is an epimorphism.

(c) We claim that the right vertical map is an epimorphism:[2582] To see this we consider the above chart $\Phi_0 \colon U_0 \to \mathbb{C}^{n-1}$. Note that the inverse is given by the map $\Psi_0 \colon \mathbb{C}^{n-1} \to U_0$ with $\Psi_0(w) = [1 : w]$. We also consider the inclusion $\iota \colon U_0 \to \mathbb{CP}^{n-1}$ and we consider the map $f \colon \mathbb{C}^{n-1} \to \mathbb{C}^n \setminus \{0\}$ that is given by $w \mapsto (1, w)$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{T}_{Q_1}\mathbb{C}^{n-1} & \xrightarrow{\mathrm{D}f_{Q_1}} & \mathrm{T}_{(Q_0,\dots,Q_{n-1})}(\mathbb{C}^n \setminus \{0\}) \\ \cong \downarrow{\scriptstyle \mathrm{D}(\Psi_0)_{Q_1}} & & \downarrow{\scriptstyle \mathrm{D}\pi_{(Q_0,\dots,Q_{n-1})}} \\ \mathrm{T}_Q U_0 & \xrightarrow[\cong]{\mathrm{D}\iota_{[Q_0,\dots,Q_n]}} & \mathrm{T}_{p(Q)}\mathbb{CP}^{n-1}. \end{array}$$

Note that $\pi \circ f = \iota \circ \Psi_0 \colon \mathbb{C}^{n-1} \to \mathbb{CP}^{n-1}$. It follows again from the Tangent Space-Functor Proposition 29.15 that the diagram commutes. Since Ψ_0 and ι are (local) diffeomorphisms we know by the Tangent Space-Inclusion Proposition 29.20 (3) that the left vertical and the bottom horizontal maps are epimorphisms. It follows from this discussion that the right vertical map is, as promised, an epimorphism.

It follows from the above three observations that the bottom horizontal map is indeed an epimorphism. □

It follows from the claim, the above observation (b) and the Ehresmann Fibration Theorem 238.6 that[2583] the map $p \colon \mathbb{CP}^n \setminus \Theta(B^{2n}) \to \mathbb{CP}^{n-1}$ is a smooth $\overline{B}^2$-bundle. ■

Proof of the Projective Space–Connected Sum Proposition 238.10. In the following discussion we will prove that there exists a smooth S^2-bundle map $\mathbb{CP}^2 \# (-\mathbb{CP}^2) \to S^2$ and we will prove that this bundle map is non-trivializable. The proofs of the other two statements are basically the same.

First note that by the definition of the smooth structure on $\mathbb{CP}^n$, see the Projective Space–Manifold Proposition 25.13 (2), we know that for $i = 0, \dots, n$ the maps

$$\begin{array}{rcl} \Phi_i \colon U_i \coloneqq \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \,|\, z_i \neq 0\} & \to & \mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n} \\ [z_0 : \cdots : z_n] & \mapsto & \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right) \end{array}$$

[2582] This is the content of Exercise 29.8.

[2583] We also make use of the fact that $\mathbb{CP}^n \setminus \Theta(B^{2n})$ is compact and that $\mathbb{CP}^{n-1}$ has empty boundary.

form a smooth atlas for $\mathbb{C}\mathrm{P}^n$. Using this explicit atlas one can easily verify that the map

$$\begin{array}{rcl}\Theta\colon \overline{B}^4 = \{(z_0,z_1)\in\mathbb{C}^2 \,|\, |z_0|^2+|z_1|^2\leq 1\} & \to & \mathbb{C}\mathrm{P}^2 \\ (z_0,z_1) & \mapsto & [z_0:z_1:1]\end{array}$$

is a smooth embedding. We will use this map to form the connected sum $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)$. We make the following observations:

(a) The image of $\Theta\colon \overline{B}^4\to\mathbb{C}\mathrm{P}^2$ is given by $\{[w_0:w_1:w_2]\,|\,|w_0|^2+|w_1|^2\leq|w_2|^2\}$.

(b) Using (a), using the Compact-Hausdorff Proposition 4.23 (3), using the Inverse Mapping Theorem 30.1 and using the Smooth Pasting Proposition 25.19 (2) one can easily verify that for every $[z_0:z_1]\in\mathbb{C}\mathrm{P}^1$ the map

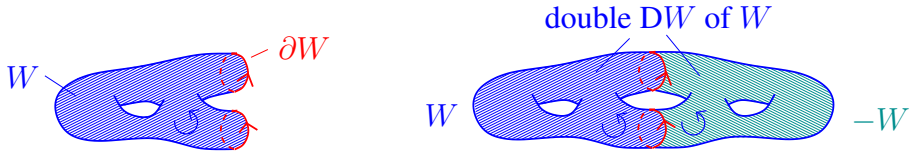
$$\begin{array}{rcl}\overline{B}^2 & \to & p^{-1}([z_0:z_1]) \\ z_2 & \mapsto & [z_0:z_1:\sqrt{|z_0|^2+|z_1|^2}\cdot z_2]\end{array}$$

defines a diffeomorphism.

Claim 1. There exists a smooth S^2-bundle map $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)\to S^2$.

Proof. First recall that for a smooth manifold W we introduced in Definition 54.8 the double

$$\mathrm{D}W \;:=\; ((W\times\{1\})\sqcup(W\times\{2\}))/\sim \qquad \text{where } (w,1)\sim(w,2) \text{ for all } w\in\partial W.$$



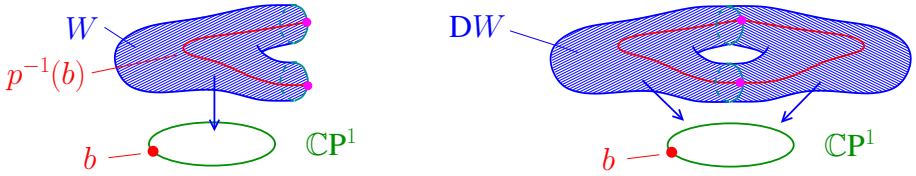
The claim now follows from the following three observations:

(1) It follows almost immediately from the definitions (and say the Doubling-Manifold Lemma 54.10 (4)) that for any oriented n-dimensional smooth manifold M and any smooth embedding $\Phi\colon\overline{B}^n\to M\setminus\partial M$ the double $\mathrm{D}(M\setminus\Phi(B^n))$ is diffeomorphic to the connected sum $M\#(-M)$.

(2) Let $p\colon\mathbb{C}\mathrm{P}^2\setminus\Phi(B^4)\to\mathbb{C}\mathrm{P}^1$ be the smooth $\overline{B}^2$-bundle from the Projective Space-minus-Ball Lemma 238.11 (2). One can easily verify that the map

$$\begin{array}{rcl}\overbrace{(((\mathbb{C}\mathrm{P}^2\setminus\Theta(B^4))\times\{1\})\cup((\mathbb{C}\mathrm{P}^2\setminus\Theta(B^4))\times\{2\}))/\sim}^{=\mathrm{D}(\mathbb{C}\mathrm{P}^2\setminus\Theta(B^4))} & \to & \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2 \\ [(x,i)] & \mapsto & p(x)\end{array}$$

is a smooth bundle map with fiber given by the double of $\overline{B}^2$.



(3) By the discussion on page 1229 the double of $\overline{B}^2$ is diffeomorphic to S^2. ⊞

It remains to prove the following claim:

Claim 2. The connected sum $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)$ is not the total space of a trivializable S^2-bundle over S^2.

Proof. We need to show that $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)$ is not homeomorphic to $S^2 \times S^2$. We will do so, as on many other occasions, by considering the intersection forms:[2584]

- It follows from the Cup Product-of-Duals-in-$\mathbb{C}\mathrm{P}^n$-Lemma 220.21 and the Intersection Form-Additivity Proposition 224.18 that the intersection form of $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- As we pointed out on page 4331, the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- Since the first form is odd and the second form is even[2585] we see that the two forms are not isometric, not even up to change of a sign.[2586]

It follows from this discussion and the Intersection Form–Homeomorphism Lemma 224.15 that $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2)$ is indeed not homeomorphic to the product $S^2 \times S^2$. ∎

238.5. **Orthogonal and unitary matrices.** In this section we will see that matrix groups also give rise to interesting bundle maps. First we recall the following proposition:

Proposition 32.10 (Matrix Group-Smooth Manifold Proposition) Let $n \in \mathbb{N}$.

(1) (a) The set
$$\mathrm{O}(n) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, A^{\mathrm{T}} \cdot A = \mathrm{id}\}$$
of orthogonal matrices is an orientable proper $\frac{1}{2}n(n-1)$-dimensional submanifold of $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$. Furthermore for every $A \in \mathrm{O}(n)$ we have
$$\mathrm{T}_A \mathrm{O}(n) \;=\; A \cdot \underbrace{\{\{B \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, B = -B^{\mathrm{T}}\}}_{\text{vector space of skew-symmetric matrices}}.$$

(b) The analogous statements also hold for $\mathrm{SO}(n)$.

(2) (a) The set
$$\mathrm{U}(n) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id}\}$$
of unitary matrices is similarly an orientable proper n^2-dimensional submanifold of $\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Furthermore for every $A \in \mathrm{U}(n)$ we have
$$\mathrm{T}_A \mathrm{U}(n) \;=\; A \cdot \underbrace{\{B \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, B = -\overline{B}^{\mathrm{T}}\}}_{\text{vector space of skew-Hermitian matrices}}.$$

(b) The set

[2584] For the reader's convenience we recall the definition of the intersection from Definition 222.3: Let W be a closed oriented $2n$-dimensional topological manifold. The intersection of W is defined as the map
$$\begin{array}{rccl} Q_W \colon & \mathrm{FH}_n(W) \times \mathrm{FH}_n(W) & \to & \mathbb{Z} \\ & (a, b) & \mapsto & \langle \mathrm{PD}_W(a) \cup \mathrm{PD}_W(b), [W] \rangle, \end{array}$$
where for an abelian group A we write $\mathrm{F}A := A/\mathrm{torsion}$.

[2585] Recall that according to Definition 226.24 a symmetric form $\varphi \colon H \times H \to \mathbb{Z}$ is called *even* if for all $h \in H$ the value $\varphi(h, h)$ is even. Otherwise we say that the form is *odd*.

[2586] Note that essentially the same argument also works for real projective spaces, i.e. it can also be used to show that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is not homeomorphic to $S^1 \times S^1$. All we need to do is to replace the $\mathbb{Z}$-intersection form by the $\mathbb{F}_2$-intersection form. Then the same argument works, since the notion of parity also makes sense over $\mathbb{F}_2$.

$$\mathrm{SU}(n) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id} \text{ and } \det(A) = 1\}$$

of special unitary matrices is an orientable proper (n^2-1)-dimensional submanifold of $\mathrm{M}(n\times n,\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Furthermore for every $A \in \mathrm{SU}(n)$ we have

$$i\mathrm{T}_A\,\mathrm{SU}(n) \;=\; A\cdot\{B \in \mathrm{M}(n\times n,\mathbb{C}) \,|\, B = -\overline{B}^{\mathrm{T}} \text{ and } \mathrm{tr}(B) = 0\}.$$

All of the above examples, viewed as smooth manifolds in their own right, are closed.

The following proposition gives us new and interesting examples of bundles:

Proposition 238.12. (Matrix Group-Fiber Proposition) Let $n \in \mathbb{N}$. The maps

$$\begin{array}{rcl} p\colon \mathrm{O}(n+1) & \to & S^n \\ (v_1 \,\ldots\, v_{n+1}) & \mapsto & v_1 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1 \,\ldots\, v_{n+1}) & \mapsto & v_1 \end{array}$$

are bundle maps with fibers given by $\mathrm{O}(n)$ and $\mathrm{U}(n)$ respectively. Furthermore, we can replace O by SO and U by SU.

Remark. Later, in the Matrix Group-Fiber Proposition 299.8, we will consider again orthogonal and unitary groups, and we will also consider the symplectic groups $\mathrm{Sp}(n)$ which we will introduce in Definition 299.3. At that point we will not give just one, but three proofs of the Matrix Group-Fiber Proposition 299.8. □

Proof. In the following we deal with the first map $p\colon \mathrm{O}(n+1) \to S^n$. As we will see in Exercise 238.15, the second map $p\colon \mathrm{U}(n+1) \to S^{2n+1}$ is dealt with in a similar fashion. We leave it to the reader to verify that we can also replace O by SO and U by SU. This can either be done by modifying the below argument or by making use of the $\mathrm{U}(n)$-Product Lemma 32.11.

So, let us consider the map $p\colon \mathrm{O}(n+1) \to S^n$. First note that

$$p^{-1}((1,0,\ldots,0)) \;=\; \{(v_1 \,\ldots\, v_{n+1}) \in \mathrm{O}(n+1) \,|\, v_1 = e_1\} \;=\; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in \mathrm{O}(n) \right\}.$$

Thus we see that the "fiber" $p^{-1}((1,0,\ldots,0))$ is diffeomorphic to $\mathrm{O}(n)$. Next note that by the Matrix Group-Smooth Manifold Proposition 32.10 we know that $\mathrm{O}(n+1)$ is a smooth submanifold of $\mathrm{M}((n+1)\times(n+1),\mathbb{R})$ and that $\mathrm{O}(n+1)$ is a closed smooth manifold. Since S^n is connected it follows from the Ehresmann Fibration Theorem 238.6 that it remains to prove the following claim:

Claim. The given map $p\colon \mathrm{O}(n+1) \to S^n$ is a submersion.

Proof. Evidently the projection $\mathrm{M}((n+1)\times(n+1),\mathbb{R}) \to \mathbb{R}^{n+1}$ onto the first column is smooth. It follows from the Submanifold Smooth Structure Lemma 25.23 that the given map $p\colon \mathrm{O}(n+1) \to S^n$ is also smooth.

Let $A \in \mathrm{O}(n+1)$. We need to show that the differential $\mathrm{D}p_A\colon \mathrm{T}_A\,\mathrm{O}(n+1) \to \mathrm{T}_{p(A)}S^n$ is an epimorphism. It follows easily from the Physical Tangent Spaces-Lemma 29.2 (3) that a basis for $\mathrm{T}_{p(A)}S^n = A\cdot(\mathrm{T}_{p(\mathrm{id})}S^n)$ is given by the vectors $A\cdot e_2,\ldots,A\cdot e_{n+1}$. It remains to show that the basis vector $A\cdot e_i$, $i = 2,\ldots,n+1$ are contained in the image of $\mathrm{D}p_A$. For

$i = 2, \dots, n+1$ we consider the curve

$$\begin{array}{rcl} \gamma_i \colon \mathbb{R} & \to & \mathrm{O}(n+1) \\ t & \mapsto & A \cdot \begin{pmatrix} \cos(t) & 0 & -\sin(t) & 0 \\ 0 & \mathrm{id}_{i-2} & 0 & 0 \\ \sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & \mathrm{id}_{n-i-1} \end{pmatrix}. \end{array}$$

Note that $\gamma_i(0) = A$ and note that

$$\mathrm{D}p_A(\gamma_i'(0)) \underset{\text{Chain Rule (from right to left)}}{=} (p \circ \gamma_i)'(0) \underset{\text{definition of } p \text{ and } \gamma_i}{=} A \cdot \begin{pmatrix} \cos(t) \\ 0_{i-2} \\ \sin(t) \\ 0 \end{pmatrix}' (0) = A \cdot \begin{pmatrix} -\sin(0) \\ 0_{i-2} \\ \cos(0) \\ 0 \end{pmatrix} = A \cdot e_i.$$

In summary we have now shown that the differential $\mathrm{D}p_A \colon \mathrm{T}_A \mathrm{O}(n+1) \to \mathrm{T}_{p(A)} S^n$ is indeed an epimorphism. ■

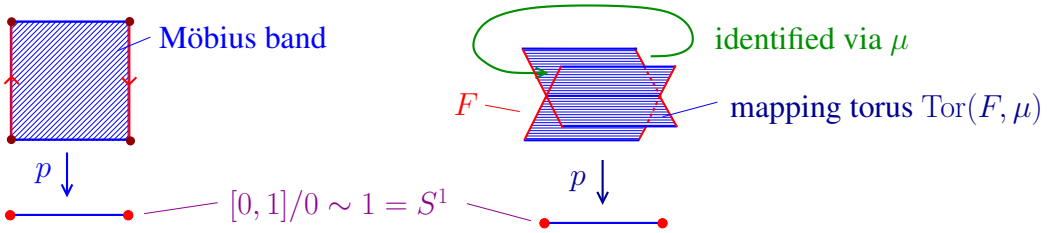
238.6. **Mapping tori as bundles.** In this final section we will mostly be interested in bundle maps over S^1. For such bundles there is a useful characterization which makes use of the following concept, which we first introduced in Definition 21.19:

Definition 238.13. Let F be a topological space and let $\mu \colon F \to F$ be a continuous map.

(1) We call
$$\mathrm{Tor}(F, \mu) \coloneqq (F \times [0,1]) / \sim \quad \text{where } (f, 0) \sim (\mu(f), 1) \text{ for all } f \in F,$$
the mapping torus of (F, μ).

(2) We refer to the map
$$\begin{array}{rcl} p \colon \mathrm{Tor}(F,\mu) = (F \times [0,1])/(f,0) \sim (\mu(f),1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi i t) \end{array}$$
as the natural projection onto S^1

Example. Basically by definition the Möbius band and the Klein bottle can be written as mapping tori.



In the figure above we also show a more exotic example, where F is a "cross" and μ is given by rotation by $\frac{\pi}{2}$. □

The following lemma can be viewed as the topological analog of the Smooth Mapping Torus–Bundle Lemma 122.3:

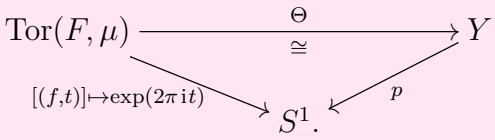
Lemma 238.14. (Mapping Torus–Bundle Lemma) Let F be a topological space.

(1) Let $\mu\colon F \to F$ be a homeomorphism. The natural projection

$$\begin{array}{rccc} q\colon & \mathrm{Tor}(F,\mu) = (F\times[0,1])/(f,0)\sim(\mu(f),1) & \to & S^1 \\ & [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is an F-bundle.

(2) For every F-bundle $q\colon Y \to S^1$ there exists a homeomorphism $\mu\colon F \to F$ and a homeomorphism $\Theta\colon \mathrm{Tor}(F,\mu) \to Y$ such that the following diagram commutes:



Remark. In the Mapping Torus–$(F,\mathcal{G})$-Bundle Lemma 241.11 we will prove a slight refinement of the Mapping Torus–Bundle Lemma 238.14. □

Proof.

(1) As we will see, this proof is almost identical to the proof of the Smooth Mapping Torus–Bundle Lemma 122.3 (1). We denote by $\pi\colon [0,1] \to [0,1]/0\sim 1$ the natural projection. We consider the two open subsets $U_1 := \pi((\frac{1}{4},\frac{3}{4}))$ and $U_2 := \pi([0,\frac{1}{3})\cup(\frac{2}{3},1])$ of $[0,1]/0\sim 1$. Furthermore we consider the maps

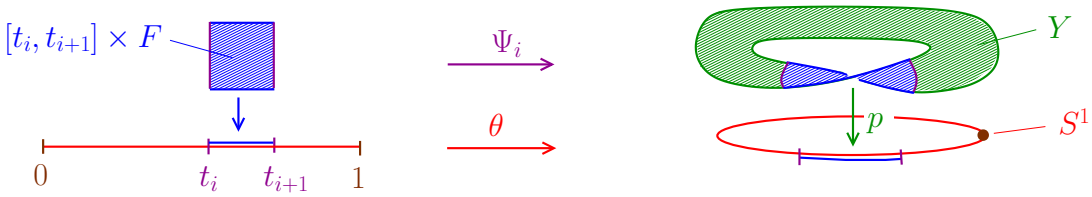
$$\begin{array}{rcl} q^{-1}(U_1) & \xrightarrow{\Phi_1} & U_1\times F \\ [(f,t)] & \mapsto & (t,f) \end{array} \quad\text{and}\quad \begin{array}{rcl} q^{-1}(U_2) & \xrightarrow{\Phi_2} & U_2\times F \\ [(f,t)] & \mapsto & \begin{cases} (t,\mu(f)), & \text{if } t\in[0,\frac{1}{3}), \\ (t,f) & \text{if } t\in(\frac{2}{3},1]. \end{cases} \end{array}$$

Note that the map on the right is indeed well-defined. Using the Topological-Quotient Proposition 8.2 (1b), the Homotopy Combination Lemma 18.8 (1) and the Topological-Product Proposition 7.2 (1) one can show quite easily see that both maps are homeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0\sim 1$, we have shown that $q\colon \mathrm{Tor}(F,\mu) \to S^1$ is an F-bundle.

(2) The proof of this statement is also very similar to the proof of the Smooth Mapping Torus–Bundle Lemma 122.3 (2). For the reader's convenience we provide the argument. Let Y be a topological space and let $p\colon Y \to S^1$ be an F-bundle. We denote by $\theta\colon [0,1] \to S^1$ the map that is given by $\theta(t) := \exp(2\pi\,\mathrm{i}t)$.

It follows easily from the definition of an F-bundle and the Lebesgue Lemma 10.9 that there exists an $n\geq 2$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for all $i = 0,\ldots,n-1$ there exists a homeomorphism

$$\Psi_i\colon [t_i,t_{i+1}]\times F \;\to\; p^{-1}(\theta([t_i,t_{i+1}])).$$



Given $s \in [t_i, t_{i+1}]$ we denote by $\Psi_i^s \colon F \to p^{-1}(\theta(s))$ the homeomorphism that is given by the composition of the obvious homeomorphism $F \to \{s\} \times F$ and the restriction of Ψ_i to $\{s\} \times F$. Furthermore given $i = 0, \dots, n-1$ we consider the homeomorphism $\nu_i \coloneqq (\Psi_{i+1}^{t_i})^{-1} \circ \Psi_i^{t_i} \colon F \to F$, where we set $\Psi_n = \Psi_0$. We set $\mu \coloneqq \nu_{n-1} \circ \cdots \circ \nu_0 \colon F \to F$. Finally we consider the map

$$\begin{array}{rccl} \Theta \colon & \operatorname{Tor}(F, \mu) & \to & Y \\ & [(f, \underset{\uparrow}{s})] & \mapsto & \Psi_i(s, (\nu_{i-1} \circ \cdots \circ \nu_0)(f)). \\ & \text{where } s \in [t_i, t_{i+1}] & & \end{array}$$

One can easily verify that this map is well-defined, that it is continuous, that it is a bijection and that it is an open map. It follows that this map is a homeomorphism. ■

238.7. Exercises for Chapter 238.

Exercise 238.1. Show that the map

$$\begin{array}{rcl} Y = \overbrace{([0,1] \times [-1,1])/(0,y) \sim (1,-y)}^{\text{Möbius band}} & \to & S^1 \\ (x,y) & \mapsto & \exp(2\pi \mathrm{i} x) \end{array}$$

is a smooth $[-1,1]$-bundle.

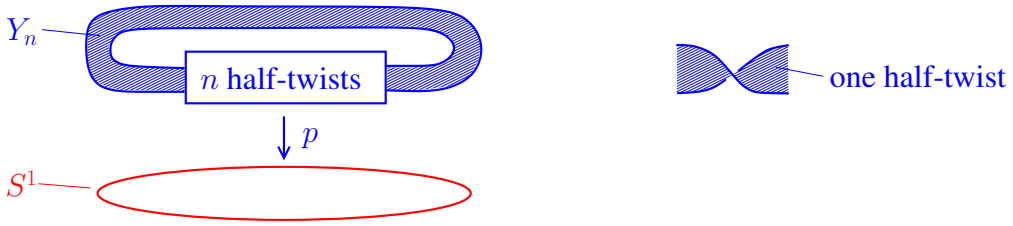
Exercise 238.2. Let $p \colon Y \to [0,1]$ be a bundle with fiber F. Show with your bare hands that p is trivial, i.e. show that p is equivalent to the trivial bundle $F \times [0,1] \to [0,1]$.

Exercise 238.3. Let $p \colon Y \to B$ be an F-bundle.

(a) Show that B and F are compact if and only if Y is compact.
(b) Show that B and F are (path-) connected if and only if Y is (path-) connected.
(c) Show that if B and F are Hausdorff if and only if Y is Hausdorff.

Remark. This exercise can be viewed as an extension of the Product Topology-Properties Proposition 7.9.

Exercise 238.4. Let $n \in \mathbb{N}_0$. In the figure below we show a $[-1,1]$-bundle $p \colon Y_n \to S^1$. For which $n \in \mathbb{N}_0$ is this bundle trivializable?



Exercise 238.5. Show that the Hopf map $H_{\mathbb{C}} \colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ descends to a bundle map $\mathbb{R}\mathrm{P}^3 \to S^2$ where the fiber is diffeomorphic to S^1..
Hint. Recall that $\mathbb{R}\mathrm{P}^1$ is diffeomorphic to S^1.

Exercise 238.6. Let F be a topological space and let $p \colon Y \to B$ be an F-bundle. Suppose that the fiber F and the base space B are path-connected. Show that the total space Y is also path-connected.

Exercise 238.7. In Definition 238.3 we introduced the Hopf maps $H\colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ and $H_{\mathbb{H}}\colon S^7 \to \mathbb{H}\mathrm{P}^1 =_{\mathrm{i}} S^4$ where the total space, the base space and the fiber are spheres. We can also consider the "real" Hopf map

$$\begin{array}{rcl} H_{\mathbb{R}}\colon S^1 = \{(x_0, x_1) \in \mathbb{R}^2 \,|\, |x_0|^2 + |x_1|^2 = 1\} & \to & \mathbb{R}\mathrm{P}^1 \\ (x_0, x_1) & \mapsto & [x_0 : x_1]. \end{array}$$

Does this also define a bundle map where the total space, the base space and the fiber are spheres?

Exercise 238.8. Let M and B be smooth manifolds and let $p\colon M \to B$ be a submersion. The Ehresmann Fibration Theorem 238.6 says that if the following conditions are satisfied:

(a) $\partial B = \varnothing$,
(b) M is compact,
(c) B is connected,

then the map $p\colon M \to B$ is a smooth bundle map. Which of these three hypotheses are necessary, and which ones are just used to simplify the proof?

Exercise 238.9. Let M be a compact connected smooth manifold. We consider the product smooth manifold $M \times S^1$. Let $\phi \in \mathrm{H}_1(M \times S^1) \to \mathbb{Z}$ be a homomorphism such that $k := \phi([\{*\} \times S^1) \neq 0$. Show that there exists a smooth bundle map $p\colon M \times S^1 \to S^1$ such that the induced map $p_*\colon \mathrm{H}_1(M \times S^1) \to \mathrm{H}_1(S^1) =_{\mathrm{i}} \mathbb{Z}$ agrees with ϕ.
Hint. Make use of the Maps-to-S^1-Proposition 233.4 which says that for any smooth manifold W and any homomorphism $\gamma\colon \mathrm{H}_1(W) \to \mathbb{Z}$ there exists a smooth map $g\colon W \to S^1$ such that the induced map $g_*\colon \mathrm{H}_1(W) \to \mathrm{H}_1(S^1) =_{\mathrm{i}} \mathbb{Z}$ agrees with γ.
Remark. In the Product-as-Mapping Torus Proposition 233.8 we gave a more precise description of the bundle map p.

Exercise 238.10. Use the Ehresmann Fibration Theorem 238.6 to show that the Hopf map

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

is a fiber bundle.
Remark. This exercise shows that, with enough effort, one can reprove the Hopf-Bundle Proposition 238.4 using the Ehresmann Fibration Theorem 238.6.

Exercise 238.11. Let $n \in \mathbb{N}$. Show that there exists a closed smooth manifold M with $\dim(M) \in \{1, \ldots, 2n\}$ together with a smooth bundle map $p\colon S^{2n+1} \to M$.

Exercise 238.12. Let $n \in \mathbb{N}$. Recall that by the Projective Space-Avatar Lemma 9.16 we have an identification $\mathbb{R}\mathrm{P}^3 =_{\mathrm{i}} \overline{B}^n/\sim$ where $x \sim -x$ for all $x \in S^{n-1}$. Show directly that $\overline{B}^n/\sim \setminus B^n_{\frac{1}{2}}$ is the total space of a $[-1, 1]$-bundle over $\mathbb{R}\mathrm{P}^{n-1} =_{\mathrm{i}} S^{n-1}/x \sim -x$.

Exercise 238.13.

(a) (i) Show that the map

$$\begin{array}{rcl} \mathrm{GL}(n, \mathbb{C}) & \to & \mathbb{C} \setminus \{0\} \\ A & \mapsto & \det(A) \end{array}$$

is an $\mathrm{SL}(n, \mathbb{C})$-bundle.
(ii) Is the bundle in (i) trivializable?

(b) (i) Show that the map

$$\begin{array}{rcl} \mathrm{U}(n) & \to & S^1 \\ A & \mapsto & \det(A) \end{array}$$

is an $\mathrm{SU}(n)$-bundle.

(ii) Is the bundle in (i) trivializable?

Exercise 238.14. Let $n \in \mathbb{N}$. Show that the connected sum $\mathbb{R}\mathrm{P}^n \# (-\mathbb{R}\mathrm{P}^n)$ is the total space of an S^1-bundle over $\mathbb{R}\mathrm{P}^{n-1}$.
Hint. Use the Projective Space-minus-Ball Lemma 238.11.

Exercise 238.15. Let $n \in \mathbb{N}$. Show that the map

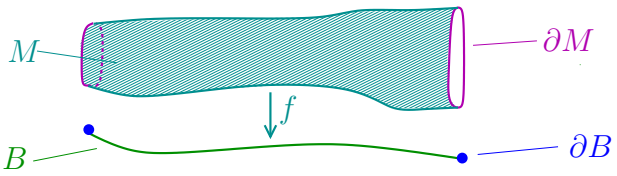
$$\begin{array}{rcl} p\colon \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1 \ \ldots \ v_{n+1}) & \mapsto & v_1 \end{array}$$

is a bundle map with fiber given by $\mathrm{U}(n)$.
Remark. The task is to modify the argument provided for $\mathrm{O}(n+1)$ which we gave in the proof of the Matrix Group-Fiber Proposition 238.12.

Exercise 238.16. A map smooth $f\colon M \to B$ between two smooth manifolds is called a *vertical submersion* if the following conditions are satisfied:[2587]

- For each $P \in M$ the map $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} B$ is an epimorphism.
- We have $f^{-1}(\partial B) = \partial M$.

Let $f\colon M \to B$ be a vertical submersion. Show that if B is connected, then f is a smooth bundle map.
Remark. This is a variation on the Ehresmann Fibration Theorem 238.6. We now allow the base manifold to have boundary, but we demand that the fiber is a closed smooth manifold.



Exercise 238.17. Let K be the Klein bottle. Do there exist smooth manifolds F and Y together with a smooth F-bundle $p\colon Y \to K$ and such that Y is an orientable 3-dimensional smooth manifold?

Exercise 238.18.

(a) Let X, Y, B be closed connected smooth manifolds and let $p\colon X \to Y$ and $q\colon Y \to B$ be two smooth bundle maps. Show that $q \circ p\colon X \to B$ is also a smooth bundle map.

(b) Let X, Y, B be topological spaces and let $p\colon X \to Y$ and $q\colon Y \to B$ be bundle maps. Is the composition $q \circ p\colon X \to B$ also necessarily a bundle map?

Exercise 238.19. Let F, Y, B be smooth manifolds. We assume that there exists a smooth F-bundle $p\colon Y \to B$.

(a) If B and F are orientable, is Y orientable?

[2587] If $\partial B = \varnothing$ and $\partial M = \varnothing$, then the notion of a vertical submersion agrees with the notion of a submersion as defined in Exercise 56.2 and in Definition 122.5 and Definition 238.5.

(b) If Y is orientable, is F orientable?
(c) If Y is orientable, is B orientable?

CHAPTER 239

Long exact sequence of homotopy groups

In this chapter we will see that an F-bundle $p\colon X \to B$ gives rise to an exact sequence involving the homotopy groups of F, X and B.

This chapter is organized as follows:

- In Section 239.1 we will recall the definition of (higher) homotopy groups and we will recall some basic facts.
- In Section 239.2 we will recall some calculations of (higher) homotopy groups.
- In Section 239.3 we will finally see that an F-bundle $p\colon X \to B$ gives rise to a long exact sequence involving the homotopy groups of F, X and B.
- In Section 239.4 we will study several applications of the long exact sequence of homotopy groups.
- In Section 239.5 we will see that the long exact sequence of homotopy groups, together with the bundle maps from Section 238.5, will allow us to calculate the fundamental groups and some homotopy groups of many matrix groups.
- In Appendix 239.6 we will prove the Polar Decomposition Theorem 239.18 from linear algebra.
- In Section 239.7 we will provide exercises for Chapter 239.

In the next Chapter 240 we will introduce bundles with structure groups.

239.1. **A summary of homotopy groups.** In this section and the following section we mostly recall definitions and results from Chapter 134. We refer to that earlier chapter for details.

As in Chapter 134 we start out by recalling Definition 19.24.

Definition 239.1. Let W and X be topological spaces and let $A \subseteq W$ and $B \subseteq X$ be subsets.

(1) A continuous map $(W, A) \to (X, B)$ of pairs of topological spaces is a continuous map $f\colon W \to X$ with $f(A) \subseteq B$.

(2) Let $f, g\colon (W, A) \to (X, B)$ be two continuous maps of pairs of topological spaces.
   (a) A homotopy between f and g is a continuous map
   $$H\colon W \times [0,1] \to X,$$
   such that $H_0 = f$ and $H_1 = g$ and such that each H_t is a map $(W, A) \to (X, B)$.
   (b) If there exists a homotopy, then we say that f and g are homotopic and we write $f \simeq g$.

(3) We set
$$[(W, A), (X, B)] := \{\text{continuous maps } (W, A) \to (X, B)\}/\simeq .$$



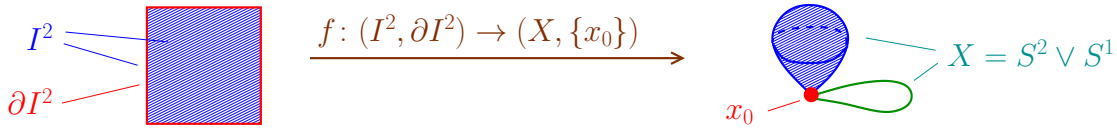
$\alpha, \beta\colon (W, A) \to (X, B)$ are homotopic but
$\alpha, \gamma\colon (W, A) \to (X, B)$ are not homotopic

Notation 239.2. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}_0$ we denote by $I^n \subseteq \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote by ∂I^n its boundary as a subspace of $\mathbb{R}^n$.

Definition 239.3. Given a pointed topological space (X, x_0) and given $n \in \mathbb{N}_0$ we define

$$\pi_n(X, x_0) \coloneqq \underbrace{[(I^n, \partial I^n), (X, \{x_0\})]}_{\substack{\text{homotopy classes of continuous} \\ \text{maps } (I^n, \partial I^n) \to (X, \{x_0\})}}.$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n-th homotopy group.



Next we recall Definition 134.4:

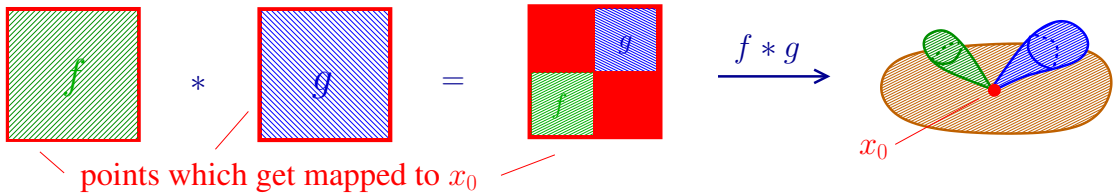
Definition 239.4. Let (X, x_0) be a pointed topological space and let $n \geq 1$.

(1) Let $f, g\colon (I^n, \partial I^n) \to (X, \{x_0\})$ be two continuous maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, \{x_0\}) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, \ldots, 2x_n), & \text{if } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \ldots, 2x_n - 1), & \text{if } (x_1, \ldots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

(2) As in Definition 134.5 we define

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] \coloneqq [f * g]. \end{array}$$



Proposition 134.6. (π_n-Group Proposition) Let (X, x_0) be a pointed topological space and let $n \geq 1$. The map

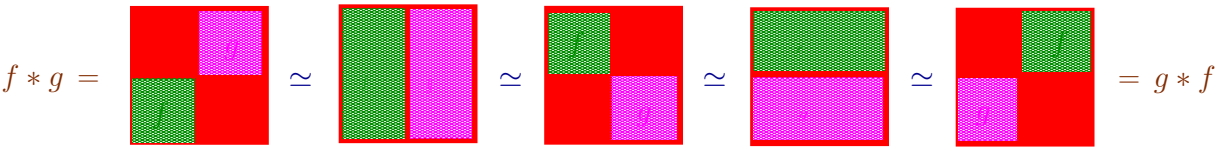
$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] \coloneqq [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

Proposition 134.11 (π_n-Abelian Proposition) For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.

The proof of the π_n-Abelian Proposition 134.11 lured this particular author into topology. Thus there is no harm done in providing it again:

Sketch of proof. Suppose we are given two continuous maps $f, g\colon (I^n, \partial I^n) \to (X, \{x_0\})$. We have to show that $f * g$ is homotopic to $g * f$.

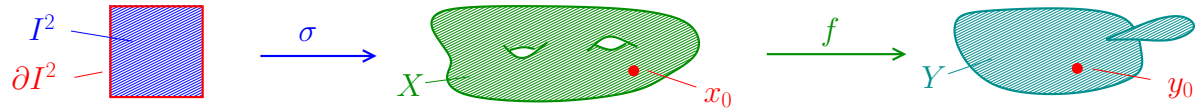


The homotopy is described in the figure above for $n = 2$. ■

Definition 239.5. Let $f\colon X \to Y$ be a continuous map between two topological spaces, let $n \in \mathbb{N}_0$ and let $x_0 \in X$. We write $y_0 = f(x_0)$. As in Definition 134.14 we consider the map

$$\begin{array}{rcl} f_*\colon \pi_n(X, x_0) & \to & \pi_n(Y, y_0) \\ [\sigma\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f \circ \sigma\colon (I^n, \partial I^n) \xrightarrow{\sigma} (X, \{x_0\}) \xrightarrow{f} (Y, \{y_0\})] \end{array}$$

In the π_n-Functor Proposition 134.15 we showed that for $n \geq 1$ these maps are group homomorphisms and note that these maps turn any π_n into a functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category of sets (for $n = 0$) or the category $\mathcal{Group}$ of groups (for $n \geq 1$).



239.2. **Calculations of higher homotopy groups.** In this section we collect several methods for calculating homotopy groups and we summarize a few calculations:

Proposition 134.17 (Homotopy Equivalence-π_n-Proposition) Let $n \geq 1$.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is an isomorphism.

(2) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proposition 134.18 (Product-π_n-Proposition) Let $n \in \mathbb{N}$ and let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j\colon \prod_{i \in I} X_i \to X_j$ the natural projection. The induced homomorphism

$$\begin{array}{rcl} \pi_n\Big(\prod_{i \in I} X_i, (x_i)_{i \in I}\Big) & \to & \prod_{i \in I} \pi_n(X_i, x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i \in I} \end{array}$$

is an isomorphism.

Proposition 134.19. (Sphere–Low-π_n-Proposition) Let $k \in \mathbb{N}$. For any $n < k$ we have

$$\pi_n(S^k) \;=\; 0.$$

Sketch of proof. There are two approaches to proving this result:

(1) We need to show that every continuous map $f\colon (S^n, x_0) \to (S^k, y_0)$ is homotopic to a constant map. First note that it follows immediately from the Whitney Approximation Theorem 36.1 that $f\colon (S^n, x_0) \to (S^k, y_0)$ is homotopic rel $\{x_0\}$ to a smooth map $g\colon (S^n, x_0) \to (S^k, y_0)$. From $n < k$ it now follows from the Measure-Zero Properties Proposition 37.3 that g misses a point P. Since $S^k \setminus \{P\}$ is homeomorphic to $\mathbb{R}^k$ it follows from the Homotopy Equivalence-π_n-Proposition 134.17 that g is homotopic rel x_0 to a constant map.

(2) As we explain on page 2723, this proposition can also be proved using the Cellular Approximation Theorem 141.2. ■

Proposition 134.22 (Covering-π_n-Isomorphism Proposition) In the following let $p\colon (Y, y_0) \to (B, b_0)$ be a covering of pointed topological spaces. For every $n \geq 2$ the map

$$p_*\colon \pi_n(Y, y_0) \;\to\; \pi_n(B, b_0)$$

is an isomorphism.

Example. For any $k \in \mathbb{N}_0$ and any $n \geq 2$ we have

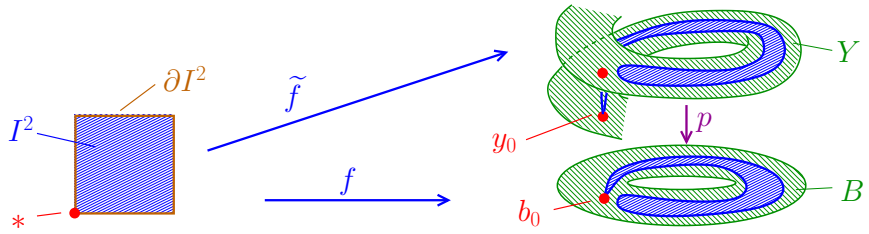
$$\pi_n((S^1)^k) \;\cong\; \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \;\xleftarrow[\uparrow]{\cong}\; \pi_n(\mathbb{R}^k) \;\underset{\uparrow}{=}\; 0.$$

by the Covering-π_n-Isomorphism Proposition 134.22, since $n \geq 2$ — by the Homotopy Equivalence π_n-Proposition 134.17 □

Proof. [2588] Let $p\colon (Y, y_0) \to (B, b_0)$ be a covering of pointed topological spaces and let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(Y, y_0) \to \pi_n(B, b_0)$.

Let $f\colon (I^n, \partial I^n) \to (B, b_0)$ be a continuous map. We set $* := (0, \ldots, 0) \in I^n$. We are in the following setting:

$$\begin{array}{ccc} & & (Y, y_0) \\ & & \downarrow p \\ (I^n, *) & \xrightarrow[f]{} & (B, b_0). \end{array} \qquad \overset{\text{by the Map Lifting Criterion 127.5, which can apply, since } \pi_1(I^n) = 0}{\rightsquigarrow} \qquad \begin{array}{ccc} & \overset{\exists!\, \widetilde{f}}{\nearrow} & (Y, y_0) \\ & & \downarrow p \\ (I^n, *) & \xrightarrow[f]{} & (B, b_0). \end{array}$$



Claim 1. We have $\widetilde{f}(\partial I^n) = \{y_0\}$.

[2588] We now provide a proof which is slightly different from our original proof on page 2587.

Proof. The claim follows from the Connected-to-Discrete Lemma 4.37 and the following three observations:

- It follows immediately from the Covering–Basics Lemma 91.12 (2) that $p^{-1}(b_0)$ is a discrete subspace of Y.
- It follows easily from $n \geq 2$ that ∂I^n is connected.
- We have $\widetilde{f}(*) = y_0$. ⊞

It follows from Claim 1 that $\widetilde{f}$ defines an element of $\pi_n(Y, y_0)$. This leads us to the following claim:

Claim 2. The map
$$\begin{array}{rccc} \lambda\colon & \pi_n(B, b_0) & \to & \pi_n(Y, y_0) \\ & [f] & \mapsto & [\widetilde{f}] \end{array}$$
is well-defined.

Proof. Suppose that we have two continuous maps $f, g\colon (I^n, \partial I^n) \to (B, \{b_0\})$ that represent the same element of $\pi_n(B, b_0)$. By definition this means that there exists a homotopy $H\colon I^n \times [0,1] \to B$ with $H_0 = f$, $H_1 = g$ and $H((\partial I^n) \times [0,1]) = \{b_0\}$. As above we can lift the map H to a continuous map $\widetilde{H}\colon I^n \times [0,1] \to Y$ with $\widetilde{H}(*, 0) = y_0$. As in Claim 1 we see that $\widetilde{H}((\partial I^n) \times [0,1]) = \{y_0\}$. This implies that $\widetilde{H}$ is a homotopy between the lifts $\widetilde{f}, \widetilde{g}\colon (I^n, \partial I^n) \to (Y, \{y_0\})$. ⊞

We have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B, b_0)$ we have
$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; [p \circ \widetilde{f}] \underset{\uparrow}{=} [f].$$
by Definition 92.1 of a lift

Finally let $[g] \in \pi_n(Y, y_0)$. We now see that
$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \underset{\uparrow}{=} [g].$$
since g is the unique lift of $p \circ g$ with $g(*) = y_0$.

■

For the last two results of this section it is helpful to switch to a slightly different point of view. Thus we recall the following identifications from page 2578:

Identification 239.6. Let $n \in \mathbb{N}$.

(1) Let
$$\begin{array}{rccc} \Phi\colon & (I^n, \partial I^n) & \to & (\overline{B}^n, S^{n-1}) \\ & x & \mapsto & \begin{cases} \frac{x}{\|x\|} \cdot \max\{|x_1|, \dots, |x_n|\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \end{array}$$
be the homeomorphism from page 189.

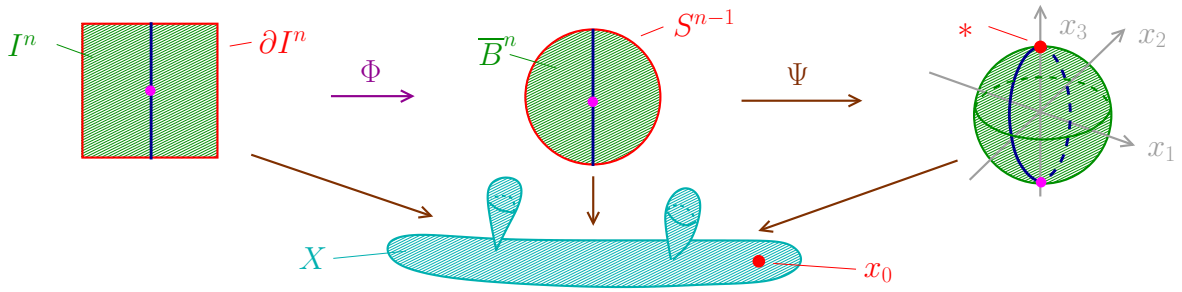
(2) We equip S^n throughout with the base point $* := (0, \dots, 0, 1) \in S^n$.

(3) We denote by $\Psi\colon (\overline{B}^n / S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 8.10

(4) Let X be a topological space and let $x_0 \in X$. Note that the above homeomorphisms Φ and Ψ define bijections

(homotopy classes) of continuous maps $(I^n, \partial I^n) \to (X, \{x_0\})$
$\iff$ (homotopy classes) of continuous maps $(\overline{B}^n, S^{n-1}) \to (X, \{x_0\})$
$\iff$ (homotopy classes) of continuous maps $(\overline{B}^n / S^{n-1}, [S^{n-1}]) \to (X, x_0)$
$\iff$ (homotopy classes) of continuous maps $(S^n, *) \to (X, x_0)$.

The three equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$ between pointed topological spaces.



Next we recall the following theorem, which is one of the main tools for calculating homotopy groups:

Theorem 167.9. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. Furthermore let $n \in \mathbb{N}_{\geq 2}$. If X is $(n-1)$-connected, i.e. if $\pi_i(X, x_0) = 0$ for $i = 0, \ldots, n-1$, then the following two statements hold:

(1) We have $\widetilde{\mathrm{H}}_k(X) = 0$ for $k = 0, \ldots, n-1$.

(2) We denote by $[S^n] \in \mathrm{H}_n(S^n)$ the standard generator. The Hurewicz homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)} \colon \pi_n(X, x_0) & \to & \mathrm{H}_n(X) \\ [f \colon (S^n, *) \to (X, x_0)] & \mapsto & f_*([S^n]) \end{array}$$

here f_* is the induced map $\mathrm{H}_n(S^n) \to \mathrm{H}_n(X)$

is an isomorphism.

We conclude this discussion with the following proposition:

Proposition 167.10. ($\pi_n(S^n)$–Proposition) For any $n \in \mathbb{N}$ we have

$$\pi_n(S^n, *) \;=\; \mathbb{Z} \cdot [\mathrm{id}_{S^n}].$$

Proof. There are several approaches to proving this result:

(1) The proposition follows from the Hurewicz Theorem 167.9 together with the calculation of the homology groups of spheres in the Sphere-H_*-Proposition 149.4.

(2) The proposition can be proved using the Hopf Degree Theorem 46.22. We refer to the proof of the earlier $\pi_n(S^n)$–Proposition 134.31 for details. ∎

239.3. **Long exact sequence of homotopy groups.** We will see in a second that an F-bundle map $p \colon X \to B$ gives rise to an exact sequence involving all $\pi_n(F, x_0)$, $\pi_n(X, x_0)$ and $\pi_n(B, b_0)$. For $n = 0$ these are not groups. So we have to think a little harder about what "exactness" might mean. This leads us to the following definition:

Definition 239.7.

(1) A pointed set is a pair (S, s_0) where S is a set and $s_0 \in S$ is a point. We call s_0 the trivial element in S.

(2) A morphism from a pointed set (S, s_0) to a pointed set (T, t_0) is a map $f \colon S \to T$ with $f(s_0) = t_0$.

(3) A morphism $f \colon (S, s_0) \to (T, t_0)$ is called trivial if $f(S) = \{t_0\}$.

(4) Given a morphism $f\colon (S,s_0)\to(T,t_0)$ we refer to $\ker(f) := f^{-1}(t_0)$ as the kernel of f.
(5) We say that a sequence

$$\ldots \xrightarrow{f_{n+2}} (S_{n+1},s_{n+1}) \xrightarrow{f_{n+1}} (S_n,s_n) \xrightarrow{f_n} (S_{n-1},s_{n-1}) \xrightarrow{f_{n-1}} \ldots$$

of morphisms of pointed sets is exact if for every $n\in\mathbb{N}_0$ we have the usual equality $\ker(f_n)=\mathrm{im}(f_{n+1})$.

Examples.

(1) Any group G gives rise to the pointed set (G,e) where $e\in G$ evidently denotes the trivial element. We will thus always view a group as a pointed set. Any group homomorphism is also a morphism of the corresponding pointed sets. Furthermore an exact sequence of homomorphisms of groups is also an exact sequence of pointed sets.
(2) Let (X,x_0) be a pointed topological space. For any $n\in\mathbb{N}_0$ the pair $(\pi_n(X,x_0),[\mathrm{const}_{x_0}])$ is a pointed set. □

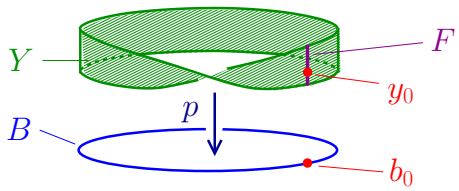
The following miraculous proposition shows that the homotopy groups involved in a bundle map fit together into a beautiful exact sequence:

Proposition 239.8. (Fiber-LES-Proposition) Let $p\colon Y\to B$ be a bundle map and let $y_0\in Y$. We set $b_0 := p(y_0)$ and we set $F := p^{-1}(b_0)$. We denote by $i\colon F=p^{-1}(b_0)\to Y$ the inclusion. There exists a sequence of natural[2589] morphisms of pointed sets

$$\partial_n\colon \pi_n(B,b_0) \to \pi_{n-1}(F,y_0), \qquad n\in\mathbb{N}$$

(which are homomorphisms for $n\geq 2$) such that the following sequence is exact:

$$\begin{aligned}\ldots \xrightarrow{\partial_3} \pi_2(F,y_0) \xrightarrow{i_*} \pi_2(Y,y_0) \xrightarrow{p_*} \pi_2(B,b_0) \xrightarrow{\partial_2}\\ \xrightarrow{\partial_2} \pi_1(F,y_0) \xrightarrow{i_*} \pi_1(Y,y_0) \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial_1}\\ \xrightarrow{\partial_1} \pi_0(F,y_0) \xrightarrow{i_*} \pi_0(Y,y_0) \xrightarrow{p_*} \pi_0(B,b_0).\end{aligned}$$



Remark.

[2589]Here "natural" means that if we are given a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

where $p\colon Y\to B$ and $p'\colon Y'\to B'$ are bundle maps, then given any base point $y_0'\in Y'$ the following diagram commutes:

$$\begin{array}{ccc} \pi_n(B',p'(y_0')) & \xrightarrow{\partial_n} & \pi_{n-1}(F',y_0') \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} \\ \pi_n(B,p(y_0)) & \xrightarrow{\partial_n} & \pi_{n-1}(F,y_0), \end{array}$$

where $y_0 := g(y_0')$, $F := p^{-1}(y_0)$ and $F' := (p')^{-1}(y_0')$.

(1) Arguably the natural home of long exact sequence of homotopy groups are fibrations, which can be viewed as a "homotopical" generalization of fiber bundles. In this setting we will prove the Fibration-LES Theorem 297.11 (1b). As we will see on page 5730, the above Fiber-LES-Proposition 239.8 can be deduced from the Fibration-LES Theorem 297.11 (1b).

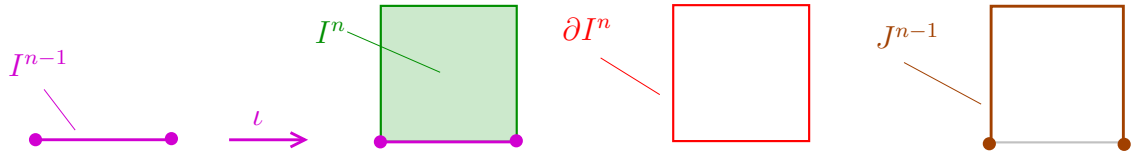
(2) In the Homological Leray-Serre Spectral Sequence 341.3 we will see that for a bundle map $p\colon Y \to B$ with fiber F there exists also a rather subtle interplay between the (twisted) homology groups of Y, B and F. □

The proof of the Fiber-LES-Proposition 239.8 will occupy the remainder of this section. In the next Section 239.4 we will study several applications of the long exact sequence of homotopy groups.

We start out with recalling and introducing some notation:

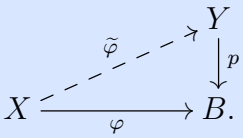
Notation 239.9. Let $n \in \mathbb{N}_0$.

- We denote by $I^n := [0,1]^n \subseteq \mathbb{R}^n$ the standard cube in $\mathbb{R}^n$.
- We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
- We denote by $\iota\colon I^{n-1} \to I^n$ the embedding that is given by $x \mapsto (x,0)$.
- We write $J^{n-1} := \overline{\partial I^n \setminus \iota(I^{n-1})}$.



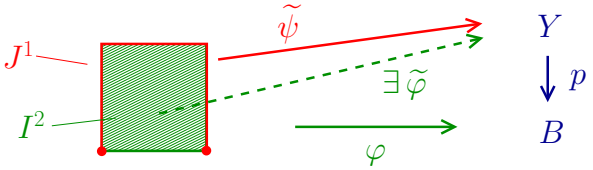
In Definition 92.1, given a covering $p\colon Y \to B$, we introduced the notion of the lift of a continuous map $f\colon X \to B$. The following definition generalizes this notion to bundle maps:

Definition 239.10. Let $p\colon Y \to B$ be a bundle map. Furthermore let X be a topological space and let $\varphi\colon X \to B$ be a continuous map. A lift of φ is a continuous map $\widetilde{\varphi}\colon X \to Y$ with $p \circ \widetilde{\varphi} = \varphi$. This can be summarized in the following commutative diagram:



The following lemma is the key technical ingredient to the proof of the Fiber-LES-Proposition 239.8:

Lemma 239.11. (Bundle–Homotopy Lifting Lemma) Let $p\colon Y \to B$ be a bundle map, let $n \in \mathbb{N}_0$, let $\varphi\colon I^n \to B$ be a continuous map and let $\widetilde{\psi}\colon J^{n-1} \to Y$ be a lift of $\psi := \varphi|_{J^{n-1}}$. Then there exists a lift $\widetilde{\varphi}\colon I^n \to Y$ of φ with $\widetilde{\varphi}|_{J^{n-1}} = \widetilde{\psi}$.



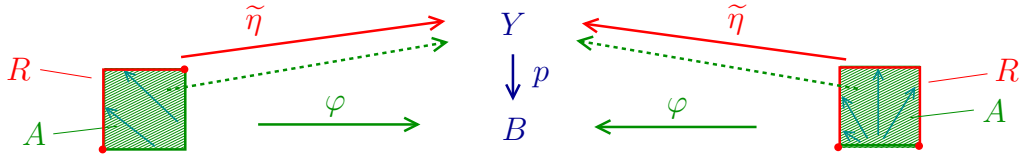
Proof of the Bundle–Homotopy Lifting Lemma 239.11. Let $p\colon Y \to B$ be a bundle map, let $n \in \mathbb{N}_0$, let $\varphi\colon I^n \to B$ be a continuous map and let $\widetilde{\psi}\colon J^{n-1} \to Y$ be a lift of $\psi := \varphi|_{J^{n-1}}$. We introduce the following language:

(a) Let $A \subseteq I^n$ be a subset. A *trivialization* for A is defined as a homeomorphism $\tau\colon p^{-1}(\varphi(A)) \to \varphi(A) \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(\varphi(A)) & \xrightarrow[\cong]{\tau} & \varphi(A) \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f)\mapsto x} & \\ & \varphi(A). & \end{array}$$

(b) We say a subset $A \subseteq I^n$ is *small* if A admits a trivialization.

Claim. Let $A \subseteq I^n$ be a small set, let $R \subseteq A$ and let $\widetilde{\eta}\colon R \to Y$ be a lift of $\varphi|_R$. If R is a retract of A, then there exists a lift $\widetilde{\varphi}\colon A \to Y$ with $\widetilde{\varphi}|_R = \widetilde{\eta}$.



Proof. Since A is small we can pick a trivialization $\tau\colon p^{-1}(\varphi(A)) \to \varphi(A) \times F$. We denote by $q_F\colon \varphi(A) \times F \to F$ the natural projection. Note that since R is a retract of A we can pick a retraction $r\colon A \to R$. One can now easily verify that

$$\begin{array}{rccl} \widetilde{\varphi}\colon & A & \to & Y \\ & a & \mapsto & \tau^{-1}(\varphi(a), q_F(\tau(\underbrace{\widetilde{\eta}(r(a))}_{\in p^{-1}(\varphi(A))}))) \end{array}$$

is a lift $\widetilde{\varphi}$ of $\varphi\colon A \to B$ with $\widetilde{\varphi}|_R = \widetilde{\eta}$. ⊞

We continue with introducing more objects:

(c) Let $k \in \mathbb{N}$. A *k-cube* is a subset of I^n of the form

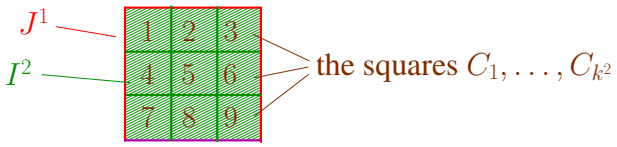
$$[a_1, a_1 + \tfrac{1}{k}] \times \cdots \times [a_n, a_n + \tfrac{1}{k}]$$

where $a_1, \ldots, a_k \in \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$.

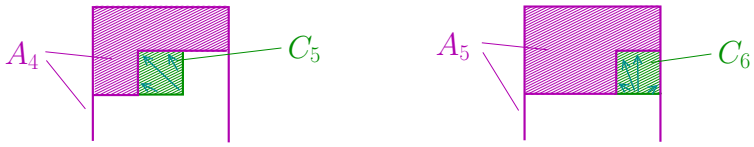
(d) It follows from the definition of an F-bundle and the continuity of φ that I^n is covered by open small subsets. It now follows from the Lebesgue Lemma 10.8 that we can find a $k \in \mathbb{N}$ such that all k-cubes are small.

In the following we consider for simplicity the case $n = 2$. The general case is essentially the same. The only difference is that the notation would become more messy.

We enumerate the k^2 squares by $C_1, \ldots, C_{k^2}$ as shown in the figure below.



Furthermore for $i = 0, \ldots, k^2$ we denote by A_i the union of J^1 with $C_1, \ldots, C_i$. On $A_0 = J^1$ we start out with the given lift $\widetilde{\psi}\colon J^1 \to Y$. For each $i = 1, \ldots, k^2$ we see that $C_i \cap A_{i-1}$ is a retract of C_i.



Thus it follows from the claim that we can iteratively extend the lift over all A_i, in particular over $A_{k^2} = I^2$. ■

Proof of the Fiber-LES-Proposition 239.8. Let $p\colon Y \to B$ be a bundle map and let $y_0 \in Y$. We set $b_0 := p(y_0)$ and we set $F := p^{-1}(b_0)$. We denote by $i\colon F = p^{-1}(b_0) \to Y$ the inclusion. We need to show that there exists a sequence of natural morphisms of pointed sets

$$\partial_n\colon \pi_n(B, b_0) \;\to\; \pi_{n-1}(F, y_0), \qquad n \in \mathbb{N}$$

(which are homomorphisms for $n \geq 2$) such that the following sequence is exact:

$$\ldots \;\to\; \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_n} \pi_{n-1}(F, y_0) \;\to\; \ldots$$

We start out with the construction of the maps ∂_n:

Claim 0. Let $[f\colon (I^n, \partial I^{n-1}) \to (B, \{b_0\})] \in \pi_n(B, b_0)$. Note that by the Bundle–Homotopy Lifting Lemma 239.11 we know that there exists a lift $\widetilde{f}\colon I^n \to Y$ of f with $\widetilde{f}(J^{n-1}) = \{y_0\}$. We set

$$\partial_n([f]) \;:=\; [\widetilde{f} \circ \iota\colon (I^{n-1}, \partial I^{n-1}) \to (F, y_0)].$$

This map has the following three properties:

(1) The map $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$ is well-defined, i.e. independent of the choice of f and the choice of the lift $\widetilde{f}$.
(2) For $n \geq 2$ the map $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$ is a homomorphism.
(3) The map $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$ is natural.

Out of desperation we provide two sketches:

- For $n = 2$ the role of I^2 and J^1 becomes clear, but it is difficult to illustrate the definition of $\widetilde{f}$:

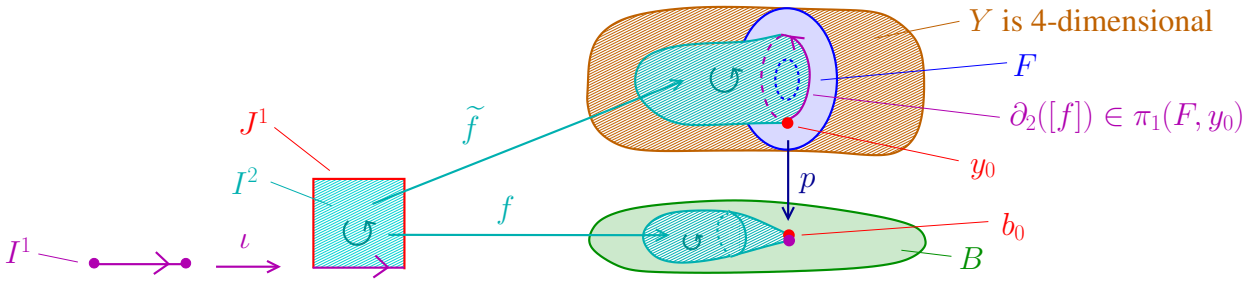


illustration for $n = 2$

- For $n = 1$ the role of I^1 and J^0 becomes confusingly trivial. But the definition of $\widetilde{f}$ can be illustrated precisely:
  In the figures below we will (rather confusingly) alternate between $n = 1$ and $n = 2$.

Proof.

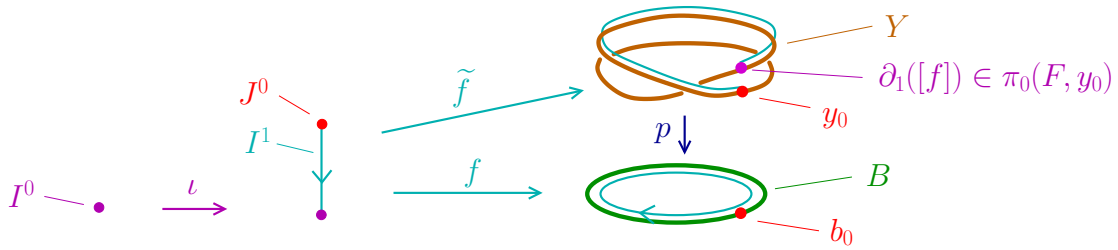


illustration for $n = 1$

(1) Let $f, g\colon (I^n, \partial I^{n-1}) \to (B, \{b_0\})$ be two maps that represent the same element in $\pi_n(B, b_0)$. This implies that there exists a continuous map $H\colon [0,1] \times I^n \to (B, \{b_0\})$ with $H(0, -) = f(-)$, $H(1, -) = g(-)$ and with $H([0,1] \times \partial I^{n-1}) = \{b_0\}$. Next let $\widetilde{f}, \widetilde{g}\colon I^n \to Y$ be lifts of f and g with $\widetilde{f}(J^{n-1}) = \widetilde{g}(J^{n-1}) = \{y_0\}$. We apply the Bundle–Homotopy Lifting Lemma 239.11 to the following data:

- We consider the map $H\colon I^{n+1} \to B$.
- We consider the lift $\widetilde{H}\colon J^{n-1} \to Y$ that is given by $\widetilde{f}$ on $\{0\} \times I^n$ and $\widetilde{g}$ on $\{1\} \times I^n$ and which assigns the value ${}_0$ to all other points on J^{n-1}.

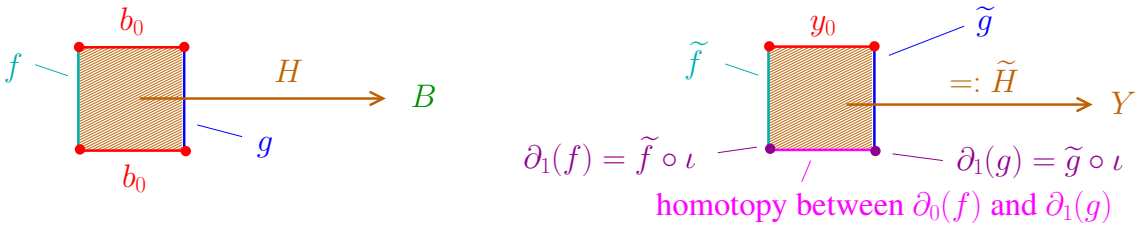


illustration for $n = 1$

We obtain a lift $\widetilde{H}\colon I^{n+1} = [0,1] \times I^n \to Y$. The map

$$\begin{array}{rcl} I^{n-1} \times [0,1] & \to & Y \\ (x,t) & \mapsto & \widetilde{H}(t,(x,0)) \end{array}$$

shows that $\widetilde{f} \circ \iota, \widetilde{g} \circ \iota\colon (I^{n-1}, \partial I^{n-1}) \to (F, \{y_0\})$ are homotopic. Thus we end up with the same element in $\pi_{n-1}(F, y_0)$.

(2) We outsource this proof to Exercise 239.2.

(3) We need to show that "∂_n" is natural. Thus suppose we are given a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\Phi} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\varphi} & B \end{array}$$

where $p\colon Y \to B$ and $p'\colon Y' \to B'$ are two bundle maps. Furthermore suppose that we are given a $y_0' \in Y'$. We set $b_0' := p'(y_0')$ and we set $F' := (p')^{-1}(b_0')$. Note that given a continuous map $f\colon (I^n, \partial I^n) \to (B', \{b_0'\})$ and given a lift $\widetilde{f}\colon I^n \to Y'$ with $\widetilde{f}(J^{n-1}) = \{y_0'\}$ the map $\Phi \circ \widetilde{f}$ is a lift of $\varphi \circ f$ with $(\Phi \circ \widetilde{f})(J^{n-1}) = \{y_0\}$. It follows easily from this observation and the definitions that

$$(\Phi|_{F'})_*(\partial_n([f])) = (\Phi|_{F'})_*([\widetilde{f} \circ \iota]) = [(\Phi \circ \widetilde{f}) \circ \iota] = \partial_n([\varphi \circ f]) = \partial_n(\varphi_*([f])). \boxplus$$

Now that we have constructed the maps $\partial_n\colon \pi_n(B,b_0)\to\pi_{n-1}(F,y_0)$ we need to show that they give rise to an exact sequence. We will do so in the following three claims:

Claim 1. The sequence is exact at $\pi_n(Y,y_0)$.

Proof.

(a) $\operatorname{im}(i_*\colon \pi_n(F,y_0)\to\pi_n(Y,y_0))\subseteq\ker(p_*\colon \pi_n(Y,y_0)\to\pi_n(B,b_0))$.
The inclusion is a consequence of the following calculation:

$$p_*\circ i_* \;=\; (p\circ i)_* \;=\; (\mathrm{const}_{y_0})_* \;=\; 0.$$

(b) $\ker(p_*\colon \pi_n(Y,y_0)\to\pi_n(B,b_0))\subseteq\operatorname{im}(i_*\colon \pi_n(F,y_0)\to\pi_n(Y,y_0))$.
Let $[\widetilde{f}\colon (I^n,\partial I^n)\to(Y,\{y_0\})]\in\ker(p_*)$. This means that there exists a homotopy $H\colon (I^n,\partial I^n)\times[0,1]\to(B,\{b_0\})$ such that $H_0=\mathrm{const}_{b_0}$ and $H_1=p\circ\widetilde{f}$. We apply the Bundle–Homotopy Lifting Lemma 239.11 to the following data:

- We consider the map $H\colon I^{n+1}\to B$.
- We consider the lift $\widetilde{H}\colon J^n\to Y$ that is given by $\widetilde{f}$ on $I^n\times\{1\}$ and which assigns the value y_0 to all other points on J^n.



illustration for $n=1$

We obtain a lift $\widetilde{H}\colon I^{n+1}=I^n\times[0,1]\to Y$. The map

$$\begin{aligned}\widetilde{H}\circ(\mathrm{id}_{[0,1]^n}\times(t\mapsto 1-t))\colon I^n\times[0,1] &\to Y\\ (x,t) &\mapsto \widetilde{H}(x,1-t)\end{aligned}$$

is a homotopy from $\widetilde{f}\colon (I^n,\partial I^n)\to(Y,\{y_0\})$ to a map $\widetilde{f}'$ that takes values in F. Thus we see that $[\widetilde{f}]=[\widetilde{f}']\in\operatorname{im}(i_*)$. ⊞

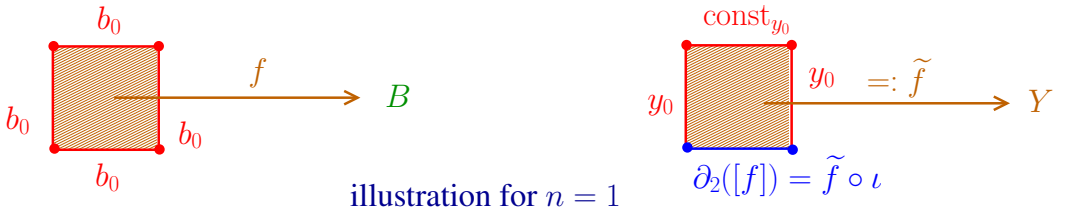
Claim 2. The sequence is exact at $\pi_n(F,y_0)$.

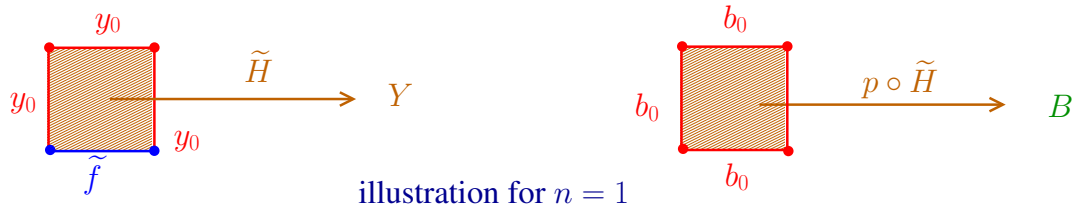
Proof.

(a) $\operatorname{im}(\partial_{n+1}\colon \pi_{n+1}(B,b_0)\to\pi_n(F,y_0))\subseteq\ker(i_*\colon \pi_n(F,y_0)\to\pi_n(Y,y_0))$.
Let $[f\colon (I^{n+1},\partial I^{n+1})\to(B,\{b_0\})]\in\pi_{n+1}(B,b_0)$. We pick a lift $\widetilde{f}\colon (I^{n+1},\partial I^{n+1})\to Y$ with $\widetilde{f}(J^n)=\{y_0\}$. We now see that

$$(i_*\circ\partial_n)([f]) \underset{\substack{\uparrow\\ \text{definition of }\partial_n}}{=} [\widetilde{f}\circ\iota\colon (I^n,\partial I^n)\to(Y,\{y_0\})] \underset{\substack{\uparrow\\ \text{since } \widetilde{f} \text{ defines a homotopy from}\\ \widetilde{f}_0=\widetilde{f}\circ\iota\colon (I^n,\partial I^n)\to(F,\{y_0\}) \text{ to}\\ \text{the constant map } (I^n,\partial I^{n-1})\to\{y_0\}}}{=} [\mathrm{const}_{y_0}] \;=\; 0 \;\in\; \pi_n(Y,y_0).$$

(b) $\ker(i_*\colon \pi_n(F,y_0)\to\pi_n(Y,y_0))\subseteq\operatorname{im}(\partial_{n+1}\colon \pi_{n+1}(B,b_0)\to\pi_n(F,y_0))$.
Let $[\widetilde{f}\colon (I^n,\partial I^n)\to(F,\{y_0\})]\in\ker(i_*)$. By definition this means that there exists a continuous map $\widetilde{H}\colon I^{n+1}\to Y$ with $\widetilde{H}_0=\widetilde{f}$ and with $\widetilde{H}(J^n)=\{y_0\}$.





Note that $p \circ \widetilde{H}$ is a map $(I^{n+1}, \partial I^{n+1}) \to (B, \{b_0\})$. This observation allows us to consider $[p \circ \widetilde{H}] \in \pi_{n+1}(B, b_0)$. Finally note that

$$\partial_n([p \circ \widetilde{H}]) \;=\; [\widetilde{H} \circ \iota] \;=\; [\widetilde{f}]. \qquad \boxplus$$

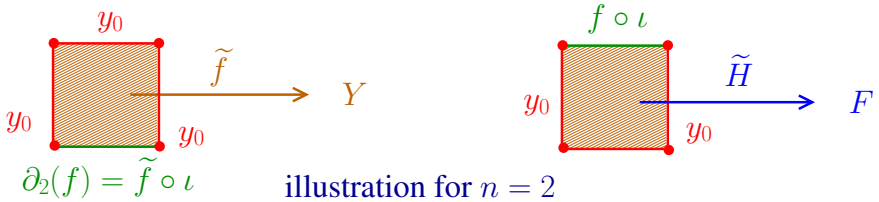
Claim 3. The sequence is exact at $\pi_n(B, b_0)$.

Proof.

(a) $\mathrm{im}(p_*\colon \pi_n(Y, y_0) \to \pi_n(B, b_0)) \subseteq \ker(\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0))$.
Let $[\widetilde{f}\colon (I^n, \partial I^n) \to (Y, \{y_0\})] \in \pi_n(Y, y_0)$. We calculate that

$$\partial_n(p_*([\widetilde{f}])) \underset{\substack{\uparrow \\ \text{definition of } p_*}}{=} \partial_n([p \circ \widetilde{f}]) \underset{\substack{\uparrow \\ \text{since } \widetilde{f} \text{ is a} \\ \text{lift of } p \circ \widetilde{f}}}{=} [\iota \circ \widetilde{f}] \underset{\substack{\uparrow \\ \text{since } \widetilde{f}(\partial I^n) = \{y_0\}}}{=} [\mathrm{const}_{y_0}] \;=\; 0.$$

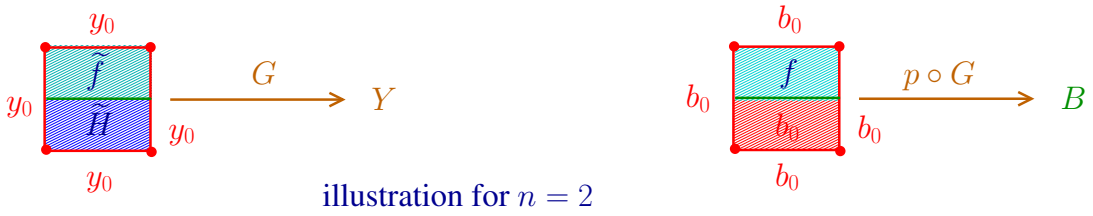
(b) $\ker(\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)) \subseteq \mathrm{im}(p_*\colon \pi_n(Y, y_0) \to \pi_n(B, b_0))$.
Let $[f\colon (I^n, \partial I^n) \to (B, \{b_0\})] \in \ker(\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0))$. We pick a lift $\widetilde{f}\colon I^n \to Y$ of f with $\widetilde{f}(J^{n-1}) = \{y_0\}$. Since $[\widetilde{f} \circ \iota] = \partial_n([f]) = 0 \in \pi_{n-1}(F, y_0)$ there exists a homotopy $\widetilde{H}\colon I^{n-1} \times [0,1] \to F$ with $\widetilde{H}_1 = \widetilde{f} \circ \iota$, $\widetilde{H}_0 = \mathrm{const}_{y_0}$ and such that $\widetilde{H}(\partial I^{n-1} \times [0,1]) = \{y_0\}$.



We consider the map

$$\begin{aligned} \widetilde{G}\colon I^{n-1} \times [0,1] &\to Y \\ (x,t) &\mapsto \begin{cases} \widetilde{H}(x, 2t), & \text{if } t \in [0, \tfrac{1}{2}], \\ \widetilde{f}(x, 2t-1), & \text{if } t \in [\tfrac{1}{2}, 1]. \end{cases} \end{aligned}$$

Note that $\widetilde{G}(\partial I^n) = \{y_0\}$. Thus $\widetilde{G}$ defines an element in $\pi_n(Y, y_0)$.



Finally we see that

$$[f] \;=\; [p \circ \widetilde{G}] \;=\; p_*([\widetilde{G}]) \;\in\; \mathrm{im}(p_*\colon \pi_n(Y, y_0) \to \pi_n(B, b_0)).$$

since $p \circ \widetilde{H} = \mathrm{const}_{b_0}$ one can easily see that $p \circ \widetilde{G}\colon (I^n, \partial I^n) \to (B, \{b_0\})$ is homotopic to $f\colon (I^n, \partial I^n) \to (B, \{b_0\})$

■

239.4. **Applications of the long exact sequence of homotopy groups.** The following theorem is one of the most memorable applications of the Fiber-LES-Proposition 239.8:

Theorem 239.12. **($\pi_n(S^3)=\pi_n(S^2)$–Theorem)**
(1) For all $n \geq 3$ we have $\pi_n(S^3) \cong \pi_n(S^2)$.
(2) We have $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_3(S^2)$ is generated by the Hopf map $H_{\mathbb{C}}\colon S^3 \to S^2$.[2590]

Proof.

(1) The Hopf-Bundle Proposition 238.4 (1) tells us that the Hopf map $H_{\mathbb{C}}\colon S^3 \to S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle. The Fiber-LES-Proposition 239.8 gives us the long exact sequence

$$\ldots \;\to\; \pi_n(S^1) \;\to\; \pi_n(S^3) \;\xrightarrow{H_{\mathbb{C}*}}\; \pi_n(S^2) \;\xrightarrow{\partial_n}\; \pi_{n-1}(S^1) \;\to\; \pi_{n-1}(S^3) \;\to\; \ldots$$

The desired statement now follows from the example on page 4603 where we saw that $\pi_k(S^1) = 0$ for $k \geq 2$.

(2) In the $\pi_n(S^n)$–Proposition 167.10 we saw that $\pi_3(S^3) = \mathbb{Z} \cdot [\mathrm{id}_{S^3}]$. By (1) the Hopf map $H_{\mathbb{C}}\colon S^3 \to S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$ induces an isomorphism $H_{\mathbb{C}*}\colon \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$. The image of $[\mathrm{id}_{S^3}]$ under $H_{\mathbb{C}*}$ is by definition precisely the class in $\pi_3(S^2)$ represented by the Hopf map. Thus we see that $\pi_3(S^2) \cong \mathbb{Z}$ and that $\pi_3(S^2)$ is generated by the Hopf map. ■

Remark. In Sections 298.4 and 298.4 we will study more Hopf bundles and this way we will obtain more results on homotopy groups of spheres. □

We will now see that the Fiber-LES-Proposition 239.8 can be used to give a new proof of the Product-π_1-Proposition 90.13 and the Product-π_n-Proposition 134.18, at least for finite products:

Proposition 134.18. **(Product-π_n-Proposition)** Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. We consider the projections

$$\begin{array}{rcl} p\colon X \times Y & \to & X \\ (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon X \times Y & \to & Y \\ (x,y) & \mapsto & y. \end{array}$$

[2590] This statement gives us an answer to Question 134.25 and to Question 216.24 (1).

For any $n \in \mathbb{N}$ the map[2591]

$$\begin{array}{rcl} \Phi\colon \pi_n(X \times Y, (x_0, y_0)) & \to & \pi_n(X, x_0) \times \pi_n(Y, y_0) \\ \psi & \to & (p_*(\psi), q_*(\psi)) \end{array}$$

is an isomorphism.

Proof. Throughout this proof we denote by $j\colon Y \to X \times Y$ the embedding that is given by $y \mapsto (x_0, y)$. Evidently the projection $p\colon X \times Y \to X$ is a bundle map with fiber $p^{-1}(x_0) = \{x_0\} \times Y = j(Y)$. Thus we obtain from the Fiber-LES-Proposition 239.8 the long exact sequence:

$$\ldots \longrightarrow \pi_n(Y, y_0) \xrightarrow[\overleftarrow{q_*}]{j_*} \pi_n(X \times Y, (x_0, y_0)) \xrightarrow{p_*} \pi_n(X, x_0) \xrightarrow{\partial_n} \pi_{n-1}(Y, y_0) \xrightarrow{j_*} \ldots.$$

Evidently we have $q \circ j = \mathrm{id}_Y$ which implies that $q_* \circ j_* = \mathrm{id}_{\pi_n(Y,y_0)}$. It follows that for every $n \in \mathbb{N}_0$ the map $j_*\colon \pi_n(Y, y_0) \to \pi_n(X \times Y, (x_0, y_0))$ is a monomorphism. Thus we obtain from the Break-into-SES Lemma 145.6 that for each $n \in \mathbb{N}$ the sequence

$$0 \;\to\; \pi_n(Y, y_0) \;\xrightarrow{j_*}\; \pi_n(X \times Y, (x_0, y_0)) \;\xrightarrow{p_*}\; \pi_n(X, x_0) \;\to\; 0$$

is exact. Once again using that $q_* \circ j_* = \mathrm{id}$ we obtain with the same argument [2592] as in the Splitting Lemma 145.5 (2)⇒(3) that the given map

$$\begin{array}{rcl} \Phi\colon \pi_n(X \times Y, (x_0, y_0)) & \to & \pi_n(X, x_0) \times \pi_n(Y, y_0) \\ \psi & \to & (p_*(\psi), q_*(\psi)) \end{array}$$

is indeed an isomorphism. ∎

239.5. **Homotopy groups of matrix groups.** In this section we want to understand the fundamental group, and various homotopy groups, of suitable matrix groups. We start out with the following proposition which shows that for the most part we can concentrate our attention on the set of orthogonal and unitary matrices:

Proposition 239.13. (Matrix Group–Homotopy Type Proposition) Let $n \in \mathbb{N}_0$.
(1) The inclusion $i\colon \mathrm{O}(n) \to \mathrm{GL}(n, \mathbb{R})$ is a homotopy equivalence.
(2) The inclusion $i\colon \mathrm{U}(n) \to \mathrm{GL}(n, \mathbb{C})$ is a homotopy equivalence.

Proof. We proved the proposition in the Matrix Groups Deformation Retract Proposition 19.4. We will give an alternative proof in Appendix 239.6 of this chapter, on page 4618. ∎

In the following our focus will be on orthogonal groups. (We outsource the study of the unitary groups $\mathrm{U}(n)$ to Exercise 239.19.) First note that $\mathrm{O}(n)$ consists of two components, each of which is homeomorphic to $\mathrm{SO}(n)$. Thus it suffices to calculate homotopy groups of $\mathrm{SO}(n)$.

[2591] Note that for $n \geq 2$ the map is a group homomorphism since in this case the group on the right-hand side is abelian. It is a priori not completely clear that the map is also a homomorphism for $n = 1$, but we will see in the proof that the map is indeed a homomorphism.

[2592] Note that for simplicity we formulated the Splitting Lemma 145.5 for abelian groups. But as we pointed out in the remark on page 2781, if one looks at the proof, one realizes that we used the hypothesis that the groups are abelian only in the proof of the (1)⇒(2) statement. The (2)⇒(3)-direction still holds with the same proof.

In very low dimensions this can be done by hand: First note that $\mathrm{SO}(1) = \{1\}$, so there is not much to say. A little bit more interesting is the fact, shown on page 202 that it follows easily from the Compact-Hausdorff Proposition 4.23 (3) that the following map is a homeomorphism:

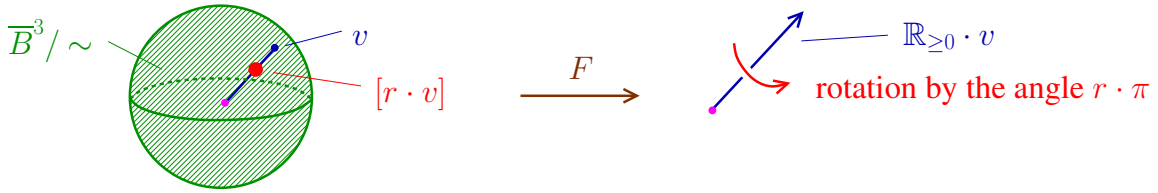
$$\begin{array}{rcl} S^1 & \to & \mathrm{SO}(2) \\ (x,y) & \mapsto & \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \end{array}$$

The next following lemma shows SO(3) is also homeomorphic to a well-known topological space:

Lemma 9.16 (Projective Space-Avatar Lemma)
(3) Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$. The map

$$\begin{array}{rcl} F\colon \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r\in[0,1] \text{ and } v\in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi\cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0}\cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. In particular SO(3) is homeomorphic to $\overline{B}^3/\sim$, to $\mathbb{R}\mathrm{P}^3$ and also to $S^3/x \sim -x$.



The following proposition allows us to move to higher dimensional orthogonal groups:

Proposition 239.14. (Matrix Group–Inclusion–π_n-Proposition) Let $n \in \mathbb{N}$. We consider the embedding

$$\begin{array}{rcl} \iota\colon \mathrm{SO}(n) & \to & \mathrm{SO}(n+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

The induced map[2593]

$$\iota_*\colon \pi_k(\mathrm{SO}(n)) \to \pi_k(\mathrm{SO}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < n-1, \\ \text{epimorphism for } k = n-1. \end{cases}$$

Proof. It follows easily from the Matrix Group-Fiber Proposition 238.12 that the map

$$\begin{array}{rcl} p\colon \mathrm{O}(n+1) & \to & S^n \\ (v_1,\ldots,v_{n+1}) & \mapsto & v_{n+1} \end{array}$$

is a bundle map. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n+1)$ is a component of $\mathrm{O}(n+1)$. It follows easily from this observation that the restriction $p\colon \mathrm{SO}(n+1) \to S^n$ is a bundle map with fiber $p^{-1}(1,0,\ldots,0) = \iota(\mathrm{SO}(n))$. From the

[2593]Recall that in the Matrix Group-π_0-Proposition 4.54 (3) we saw that $\mathrm{O}(n)$ is disconnected. Thus in principle it is not completely clear what we mean by $\pi_k(\mathrm{O}(n))$. But it follows easily from the Components-of-Topological Group Lemma 7.34 that for a topological group the isomorphism type of the homotopy groups does not depend on the choice of a base point.

Fiber-LES-Proposition 239.8 we obtain a long exact sequence of the form

$$\ldots \to \pi_{k+1}(S^n) \xrightarrow{\partial_{k+1}} \pi_k(\mathrm{SO}(n)) \xrightarrow{\iota_*} \pi_k(\mathrm{SO}(n+1)) \xrightarrow{p_*} \underbrace{\pi_k(S^n)}_{=0 \text{ for } k<n} \xrightarrow{\partial_k} \ldots$$

The desired statement now follows from the familiar fact that by the Sphere–Low-π_n-Proposition 134.19 we have $\pi_i(S^k) = 0$ for $i < k$. ∎

As a corollary to the Matrix Group–Inclusion–π_n-Proposition 299.11 (1) we can now finally answer Question 93.5. More precisely, we have the following proposition.

Proposition 239.15. (Matrix Group-π_1-Proposition)

(1) Let $n \in \mathbb{N}_{\geq 2}$. We denote by $i\colon \mathrm{SO}(n) \to \mathrm{GL}_+(n,\mathbb{R})$ the inclusion. We have isomorphisms

$$\pi_1(\mathrm{SO}(n)) \xrightarrow[i_*]{\cong} \pi_1(\mathrm{GL}_+(n,\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \geq 3. \end{cases}$$

(2) In each of the two fundamental groups of (1) a generator of the group is represented by the loop

$$\begin{array}{rcl} S^1 & \to & \mathrm{SO}(n) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Proof.

(1) First we consider the case $n = 2$. We have isomorphisms

$$\mathbb{Z} \underset{\text{by the Circle-}\pi_1\text{-Theorem 88.22}}{\overset{\uparrow}{\cong}} \pi_1(S^1,1) \overset{\text{since } \mathrm{SO}(2) \cong S^1}{\overset{\downarrow}{\cong}} \pi_1(\mathrm{SO}(2),\mathrm{id}) \underset{\substack{\text{by the Matrix Group–Homotopy Type Proposition 239.13} \\ \text{and the Homotopy-}\pi_1\text{-Proposition 90.8}}}{\overset{\uparrow}{\cong}} \pi_1(\mathrm{GL}_+(2,\mathbb{R}),\mathrm{id}).$$

Now we consider the case $n \geq 3$. We have isomorphisms

$$\mathbb{Z}/2\mathbb{Z} \underset{\text{see page 1799}}{\overset{\uparrow}{\cong}} \pi_1(\mathbb{R}\mathrm{P}^3,*) \overset{\substack{\text{since we just saw} \\ \text{that } \mathbb{R}\mathrm{P}^3 \cong \mathrm{SO}(3)}}{\overset{\downarrow}{\cong}} \pi_1(\mathrm{SO}(3),\mathrm{id}) \underset{\substack{\text{by the Matrix Group–Inclusion} \\ \pi_n\text{-Proposition 239.14}}}{\overset{\uparrow}{\cong}} \pi_1(\mathrm{SO}(n),\mathrm{id}) \overset{\substack{\text{by the Matrix Group–Homotopy Type Proposition 239.13} \\ \text{together with the Homotopy-}\pi_1\text{-Proposition 90.8}}}{\overset{\downarrow}{\cong}} \pi_1(\mathrm{GL}_+(n,\mathbb{R}),\mathrm{id}).$$

(2) We consider the loop

$$\begin{array}{rcl} \gamma\colon S^1 & \to & \mathrm{SO}(2) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \end{array}$$

It follows easily from the explicit homeomorphism $S^1 \to \mathrm{SO}(2)$ and the Circle-π_1-Theorem 88.22 that γ represents a generator of $\pi_1(\mathrm{SO}(2))$. By the Matrix Group–Inclusion–π_n-Proposition 239.14 we know that the inclusion $\mathrm{SO}(2) \to \mathrm{SO}(n)$ induces an epimorphisms of fundamental groups. It follows immediately that the loop given

[2593] Recall that we write $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$.

in (1) is indeed a generator of $\pi_1(\mathrm{SO}(n))$. Since the inclusion i induces an isomorphism of fundamental groups we see that the given loop also represents a generator of $\pi_1(\mathrm{GL}_+(n,\mathbb{R}))$. ■

We conclude this section with the following proposition:

Proposition 239.16. (Matrix Group-π_2-Proposition) For any $n \in \mathbb{N}$ we have

$$\pi_2(\mathrm{SO}(n)) \;=\; \pi_2(\mathrm{GL}_+(n,\mathbb{R})) \;=\; 0.$$

Remark. Later in the H-Space-π_n-Theorem 299.28 we will generalize this result, namely we will see that the second homotopy group of every compact Lie group vanishes. □

Proof. First note that for $n = 2$ we have

$$\pi_2(\mathrm{SO}(2)) \underset{\text{since } \mathrm{SO}(2) \cong S^1}{\overset{\uparrow}{\cong}} \pi_2(S^1) \underset{\text{see page 4603}}{\overset{\uparrow}{\cong}} 0.$$

Next we consider SO(3). Note that

$$\pi_2(\mathrm{SO}(3)) \underset{\text{since } \mathrm{SO}(3) \cong \mathbb{R}\mathrm{P}^3}{\overset{\uparrow}{\cong}} \pi_2(\mathbb{R}\mathrm{P}^3) \cong \pi_2(S^3/x \sim -x) \underset{\substack{\text{by the Covering}\\ \pi_n\text{-Isomorphism}\\ \text{Proposition 134.22}}}{\overset{\uparrow}{\cong}} \pi_2(S^3) \underset{\substack{\text{by the Sphere}\\ \text{Low-}\pi_n\\ \text{Proposition 134.19}}}{\overset{\uparrow}{=}} 0.$$

Finally note that it follows from the Matrix Group–Inclusion–π_n-Proposition 239.14 that for every $n \geq 3$ the induced map $\iota_*\colon \pi_2(\mathrm{SO}(n)) \to \pi_2(\mathrm{SO}(n+1))$ is an epimorphism. It follows that for all $n \geq 3$ we have $\pi_2(\mathrm{SO}(n)) = 0$. ■

239.6. **Appendix: The polar decomposition.** In this appendix we will prove the Polar Decomposition Theorem 239.18 from linear algebra. This amazing theorem will allow us to provide a (new) proof of the Matrix Group–Homotopy Type Proposition 239.13. We start out by introducing the following notation:

Notation 239.17. Let $n \in \mathbb{N}_0$.

(1)
- We denote by $\mathrm{Sym}(n) \subseteq \mathrm{M}(n \times n, \mathbb{R})$ the set of symmetric matrices.
- We denote by $\mathrm{Herm}(n) \subseteq \mathrm{M}(n \times n, \mathbb{C})$ the set of Hermitian matrices.

Note that these are real vector spaces of dimension $\frac{1}{2}n \cdot (n+1)$ and $2n^2$. In particular we can view the above as smooth manifolds.

(2)
- We denote by $\mathrm{Sym}^+(n) \subseteq \mathrm{Sym}(n)$ the set of positive-definite symmetric matrices.
- We denote by $\mathrm{Herm}^+(n) \subseteq \mathrm{Herm}(n)$ the set of positive-definite Hermitian matrices.

Note that it follows from the elementary Positive Definite–Open Condition Lemma 81.4 that being positive-definite is an open condition. Thus $\mathrm{Sym}^+(n) \subseteq \mathrm{Sym}(n)$ and $\mathrm{Herm}^+(n) \subseteq \mathrm{Herm}(n)$ are open subsets. It follows from the Open Subset-Submanifold Lemma 25.8 that we can view $\mathrm{Sym}^+(n)$ and $\mathrm{Herm}^+(n)$ as smooth manifolds of dimension $\frac{1}{2}n \cdot (n+1)$ and $2n^2$.

With this notation we can formulate the main result of this appendix:

Theorem 239.18. (Polar Decomposition Theorem)

(1) The map[2594]

$$\begin{array}{rcl} \mathrm{O}(n) \times \mathrm{Sym}^+(n) & \to & \mathrm{GL}(n,\mathbb{R}) \\ (A,P) & \mapsto & A \cdot P \end{array}$$

is a diffeomorphism.

(2) The map

$$\begin{array}{rcl} \mathrm{U}(n) \times \mathrm{Herm}^+(n) & \to & \mathrm{GL}(n,\mathbb{C}) \\ (A,P) & \mapsto & A \cdot P \end{array}$$

is a diffeomorphism.

Using the Polar Decomposition Theorem 239.18 we can now easily provide the promised new proof of the following proposition:

Proposition 239.13 (Matrix Group–Homotopy Type Proposition) Let $n \in \mathbb{N}_0$.

(1) The inclusion $i\colon \mathrm{O}(n) \to \mathrm{GL}(n,\mathbb{R})$ is a homotopy equivalence.
(2) The inclusion $i\colon \mathrm{U}(n) \to \mathrm{GL}(n,\mathbb{C})$ is a homotopy equivalence.

Proof. We provide the proof of Statement (1). The proof of Statement (2) is almost identical. We make the following observations:

(a) By the Polar Decomposition Theorem 239.18 (1) we know that the map

$$\begin{array}{rcl} \mathrm{O}(n) \times \mathrm{Sym}^+(n) & \to & \mathrm{GL}(n,\mathbb{R}) \\ (A,P) & \mapsto & A \cdot P \end{array}$$

is a diffeomorphism.

(b) Note that being symmetric and being positive-definite are both convex properties, i.e. they are preserved under linear combinations $A \cdot (1-t) + B \cdot t$ with $t \in [0,1]$. It follows that the set $\mathrm{Sym}^+(n)$ is a *convex* subset of $\mathrm{M}(n \times n, \mathbb{R})$. This in turn implies that $\{\mathrm{id}\} \subseteq \mathrm{Sym}^+(n)$ is a deformation retract of $\mathrm{Sym}^+(n)$.

(c) It follows from (b) that $\mathrm{O}(n) \times \{\mathrm{id}\}$ is a deformation retract of $\mathrm{O}(n) \times \mathrm{Sym}^+(n)$.

(d) It follows from (a) and (c) that $\mathrm{O}(n)$ is a deformation retract of $\mathrm{GL}(n,\mathbb{R})$.

It follows from the above together with the Deformation Retract–Homotopy Equivalence Lemma 19.10 that the inclusion $i\colon \mathrm{O}(n) \to \mathrm{GL}(n,\mathbb{R})$ is indeed a homotopy equivalence. ■

The proof of the Polar Decomposition Theorem 239.18 rests on the following powerful theorem from linear algebra:

Theorem 2.35 (Spectral Theorem)

(1) If $B \in \mathrm{M}(n \times n, \mathbb{R})$ is a symmetric matrix, then there exists an orthogonal matrix $P \in \mathrm{O}(n)$ such that $P^{\mathrm{T}} \cdot B \cdot P$ is a diagonal matrix where all diagonal entries are real numbers.
(2) If $B \in \mathrm{M}(n \times n, \mathbb{C})$ is a Hermitian matrix, then there exists a unitary matrix $P \in \mathrm{U}(n)$ such that $\overline{P}^{\mathrm{T}} \cdot B \cdot P$ is a diagonal matrix where all diagonal entries are real numbers.

If B is positive-definite, then in both cases the diagonal entries are positive.

Using the Spectral Theorem 2.35 we can now prove the following beautiful proposition:

[2594] By the Matrix Group-Smooth Manifold Proposition 32.10 we can view $\mathrm{O}(n)$ as a smooth manifold of dimension $\frac{1}{2}n \cdot (n-1)$ and we can view $\mathrm{U}(n)$ as a smooth manifold of dimension $2n^2$. Furthermore recall that it follows from the earlier discussion of $\mathrm{Sym}^+(n)$ and $\mathrm{Herm}^+(n)$ that we can view these as smooth manifolds. It now follows from the Smooth Manifold Product Proposition 25.34 that we can view the left-hand sides of (1) and (2) as smooth manifolds.

Proposition 239.19. (Matrix–Square Root Proposition) The squaring maps

(1) $\begin{array}{rcl} S\colon \mathrm{Sym}^+(n) & \to & \mathrm{Sym}^+(n) \\ A & \mapsto & A^2 \end{array}$ and (2) $\begin{array}{rcl} S\colon \mathrm{Herm}^+(n) & \to & \mathrm{Herm}^+(n) \\ A & \mapsto & A^2 \end{array}$

are diffeomorphisms.

The Matrix–Square Root Proposition 239.19 allows us to introduce the following definition.

Definition 239.20. Let $B \in \mathrm{GL}(n,\mathbb{C})$ be a positive-definite Hermitian matrix.

- By the Matrix–Square Root Proposition 239.19 there exists a unique positive-definite Hermitian matrix whose square equals B. This matrix is denoted by $B^{\frac{1}{2}}$.
- We write $B^{-\frac{1}{2}} := (B^{\frac{1}{2}})^{-1}$. An elementary argument shows that $B^{-\frac{1}{2}} = (B^{-1})^{\frac{1}{2}}$.

Proof. In the following we deal with Statement (2). The proof of Statement (1) is very similar. We leave it to the reader to make the necessary changes. We break the proof into three steps:

(a) First we show that the squaring map is surjective. Thus let $B \in \mathrm{Herm}^+(n)$. It follows immediately from the Spectral Theorem 2.35 that there exists a unitary matrix P and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ such that

$$B \;=\; P \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \cdot \overline{P}^{\mathrm{T}}.$$

One can now easily verify that the square of the matrix

$$B^{\frac{1}{2}} \;:=\; P \cdot \mathrm{diag}(\lambda_1^{\frac{1}{2}}, \ldots, \lambda_n^{\frac{1}{2}}) \cdot \overline{P}^{\mathrm{T}}$$

equals B.

(b) Next we show that the squaring map is injective. Let A, B be positive-definite Hermitian matrices with $A^2 = B^2$. We need to show that $A = B$. We consider the difference $A - B$. Evidently this is again a Hermitian matrix. It thus follows from the Spectral Theorem 2.35 that it suffices to prove the following claim:

Claim. All eigenvalues of $A - B$ are zero.

Proof. [2595] Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A - B$ and let $v \neq 0 \in \mathbb{C}^n$ be an eigenvector of the matrix $A - B$ to the eigenvalue λ. We see that

$$\begin{array}{rcl} 0 & \overset{\text{since } A^2 = B^2}{\downarrow} & \\ & = \langle v, (A^2 - B^2) \cdot v \rangle & = \langle v, (A(A-B) + (A-B)B) \cdot v \rangle \\ & = \langle v, A(A-B) \cdot v \rangle + \langle (A-B) \cdot v, B \cdot v \rangle & = \langle v, A \cdot (\lambda \cdot v) \rangle + \langle \lambda \cdot v, B \cdot v \rangle \\ & \uparrow & \uparrow \\ & \text{since } \langle -,- \rangle \text{ is sesquilinear} & \text{since } v \text{ is an eigenvalue of } A - B \\ & \text{and since } A - B \text{ is Hermitian} & \text{to the eigenvalue } \lambda \\ & = \lambda \cdot \langle v, A \cdot v \rangle + \lambda \cdot \langle v, B \cdot v \rangle & = \lambda \cdot \big(\underbrace{\langle v, A \cdot v \rangle}_{\substack{> 0 \text{ since } v \neq 0 \\ \text{and since } A \text{ is} \\ \text{positive-definite}}} + \underbrace{\langle v, B \cdot v \rangle}_{\substack{> 0 \text{ since } v \neq 0 \\ \text{and since } B \text{ is} \\ \text{positive-definite}}} \big). \\ & \uparrow & \\ & \text{since } \lambda \in \mathbb{R} & \end{array}$$

Since $v \neq 0$ and since A and B are positive-definite we now see that $\lambda = 0$. ⊞

(c) We now know that the squaring map

$$\begin{array}{rcl} S\colon \mathrm{Herm}^+(n) & \to & \mathrm{Herm}^+(n) \\ A & \mapsto & A^2 \end{array}$$

[2595] We follow the argument provided in [**KoS2006**, p. 991].

is a bijection. Clearly the map S is smooth. It remains to show that the inverse is smooth. By the Inverse Mapping Theorem 30.1 (2) it remains to prove the following claim:

Claim. For every $A \in \mathrm{Herm}^+(n)$ the differential $\mathrm{D}S_A$ is an isomorphism.

Proof. Let $A \in \mathrm{Herm}^+(n)$. First we need to calculate the differential $\mathrm{D}S_A$: For any[2596] $V \in \mathrm{Herm}(n)$ we have

$$\mathrm{D}S_A(V) \overset{\text{by the Chain Rule (which we apply going from right to left)}}{=} \frac{d}{dt} S(A+t\cdot V)|_{t=0} = \frac{d}{dt}(A+t\cdot V)^2|_{t=0}$$
$$= \frac{d}{dt}(A^2 + A\cdot tV + tV\cdot A + t^2\cdot V^2)|_{t=0} = A\cdot V + V\cdot A.$$

Thus we have shown that the differential $\mathrm{D}S_A$ is given by the map

$$\begin{array}{rcl} \mathrm{Herm}(n) & \to & \mathrm{Herm}(n) \\ V & \mapsto & A\cdot V + V\cdot A. \end{array}$$

It remains to show that this map is an isomorphism. Since $\mathrm{D}S_A$ is a homomorphism between vector spaces of the same finite-dimension it suffices to show that $\mathrm{D}S_A$ is a monomorphism. Thus suppose that we have a $V \in \mathrm{Herm}(n)$ with $A\cdot V + V\cdot A = 0$. Since $A \in \mathrm{Herm}^+(n)$ we know by the Spectral Theorem 2.35 that there exists an orthonormal basis $\{a_1, \dots, a_n\}$ of $\mathbb{R}^n$ which consists of eigenvectors of A to positive eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{>0}$. Now note that for each $i \in \{1, \dots, n\}$ we have

$$A\cdot(V\cdot a_i) \underset{\text{since } A\cdot V + V\cdot A = 0}{\overset{\uparrow}{=}} -V\cdot A\cdot a_i \underset{\text{since } a_i \text{ is an eigenvector for } \lambda_i \text{ and the matrix } A}{\overset{\uparrow}{=}} -V\cdot(\lambda_i\cdot a_i) = -\underbrace{\lambda_i}_{>0}\cdot(V\cdot a_i).$$

Since A is positive-definite we know that all eigenvalues of A are positive. The only way how this can be consistent with the above calculation is that $V\cdot a_i = 0$. Since this holds for all $i \in \{1, \dots, n\}$ we deduce that $V = 0$. ∎

Proof of the Polar Decomposition Theorem 239.18. In the following we prove Statement (2). The proof of Statement (1) is essentially identical. As a reminder, we need to show that the map

$$\begin{array}{rcl} \Xi\colon \mathrm{U}(n)\times\mathrm{Herm}^+(n) & \to & \mathrm{GL}(n,\mathbb{C}) \\ (A,P) & \mapsto & A\cdot P \end{array}$$

is a diffeomorphism. Clearly the map is smooth. Thus it suffices to construct an inverse and we need to show that the inverse is smooth.

To construct the inverse we start out with the following claim:

Claim 1. Let $B \in \mathrm{GL}(n,\mathbb{C})$.

(1) The matrix $\overline{B}^{\mathrm{T}}\cdot B$ is Hermitian and positive-definite. As discussed above, this allows us define the matrices $(\overline{B}^{\mathrm{T}}\cdot B)^{\frac{1}{2}}$ and $(\overline{B}^{\mathrm{T}}\cdot B)^{-\frac{1}{2}}$.

(2) The matrix $B\cdot(\overline{B}^{\mathrm{T}}\cdot B)^{-\frac{1}{2}}$ is unitary.

[2596] Since $\mathrm{Herm}^+(n)$ is an open subset of the vector space $\mathrm{Herm}(n)$ we obtain from the Physical Tangent Space-Inclusion Proposition 29.5 (1b) that the tangent space is given by $\mathrm{T}_A\,\mathrm{Herm}^+(n) = \mathrm{Herm}(n)$.

Proof. Let $B \in \mathrm{GL}(n,\mathbb{C})$.

(1) We now verify that the matrix $\overline{B}^{\mathrm{T}} \cdot B$ is Hermitian and positive-definite:
   (a) We have $\overline{(\overline{B}^{\mathrm{T}} \cdot B)}^{\mathrm{T}} = (\overline{\overline{B}}^{\mathrm{T}} \cdot \overline{B})^{\mathrm{T}} = \overline{B}^{\mathrm{T}} \cdot (B^{\mathrm{T}})^{\mathrm{T}} = \overline{B}^{\mathrm{T}} \cdot B$.
   (b) For every $v \in \mathbb{C}^n \setminus \{0\}$ we have $\langle v, \overline{B}^{\mathrm{T}} \cdot B \cdot v\rangle = \langle Bv, Bv\rangle > 0$.

(2) We need to show that $B \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}}$ is unitary. Note that a matrix P is unitary if and only if $P \cdot \overline{P}^{\mathrm{T}} = \mathrm{id}$. We will verify that our matrix satisfies this condition:

$$(B \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}}) \cdot \overline{(B \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}})}^{\mathrm{T}} \overset{\substack{\text{since } \overline{(C\cdot D)}^{\mathrm{T}} = \overline{D}^{\mathrm{T}} \cdot \overline{C}^{\mathrm{T}} \text{ and} \\ \text{since } (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}} \text{ is Hermitian}}}{\downarrow}$$
$$= B \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}} \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}} \cdot \overline{B}^{\mathrm{T}}$$
$$\overset{\substack{\uparrow \\ \text{by definition of } (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}}}}{=} B \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-1} \cdot \overline{B}^{\mathrm{T}} = B \cdot B^{-1} \cdot (\overline{B}^{\mathrm{T}})^{-1} \cdot \overline{B}^{\mathrm{T}} = \mathrm{id}\,. \qquad \boxplus$$

Claim 1 allows us to formulate the following claim:

Claim 2. The map

$$\begin{aligned} \Theta\colon \mathrm{GL}(n,\mathbb{C}) &\to \mathrm{U}(n) \times \mathrm{Herm}^+(n) \\ B &\mapsto (\underbrace{B \cdot (\overline{B}^{\mathrm{T}} \cdot B)^{-\frac{1}{2}}}_{\substack{\text{unitary by} \\ \text{Claim 1 (2)}}}, \underbrace{(\overline{B}^{\mathrm{T}} \cdot B)^{\frac{1}{2}}}_{\substack{\text{Hermitian and} \\ \text{positive-definite} \\ \text{by Claim 1 (1)}}}) \end{aligned}$$

is smooth and it is an inverse to Ξ.

Proof.

(1) It follows easily from the Matrix–Square Root Proposition 239.19 that Θ is smooth.

(2) We claim that Θ is an inverse to Ξ.
   - It is clear that $\Xi \circ \Theta = \mathrm{id}_{\mathrm{GL}(n,\mathbb{C})}$.
   - Let $(A,P) \in \mathrm{U}(n) \times \mathrm{Herm}^+(n)$. First note that

$$\big(\overline{\Xi(A,P)}^{\mathrm{T}} \cdot \Xi(A,P)\big)^{\frac{1}{2}} = ((\overline{A\cdot P})^{\mathrm{T}} \cdot A \cdot P)^{\frac{1}{2}} = (\underbrace{\overline{P}^{\mathrm{T}}}_{=P} \cdot \underbrace{\overline{A}^{\mathrm{T}} \cdot A}_{\substack{=\mathrm{id}, \text{ since} \\ A \in \mathrm{U}(n)}} \cdot P)^{\frac{1}{2}} = (P^2)^{\frac{1}{2}} \overset{\substack{\uparrow \\ \text{since } P \text{ is Hermitian} \\ \text{and positive-definite}}}{=} P.$$

     We now see that

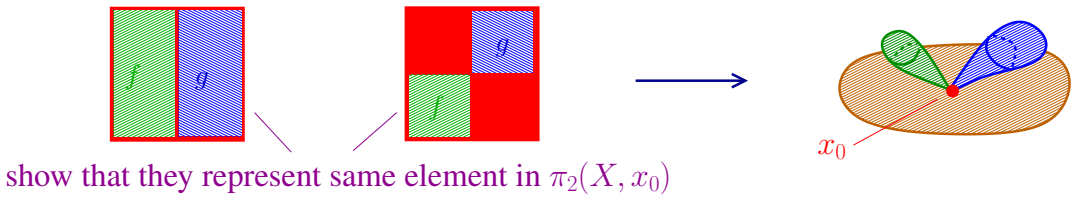
$$(\Theta \circ \Xi)(A,P) = \Theta(\Xi(A,P)) \overset{\substack{\uparrow \\ \text{this follows easily from} \\ \text{the above calculation}}}{=} (A \cdot P \cdot P^{-1}, P) = (A,P).$$

$\blacksquare$

239.7. Exercises for Chapter 239.

Exercise 239.1. Let (X,x_0) be a pointed topological space and let $f,g\colon (I^2,\partial I^2) \to (X,\{x_0\})$ be continuous maps. Show that the multiplication $[f]\cdot[g] \in \pi_2(X,x_0)$ can also be defined as shown in the figure below:

Exercise 239.2. Let $p\colon Y \to B$ be a bundle map and let $y_0 \in Y$. We set $b_0 := p(y_0)$ and we set $F := p^{-1}(b_0)$. We denote by $i\colon F = p^{-1}(b_0) \to Y$ the inclusion. In the proof of the



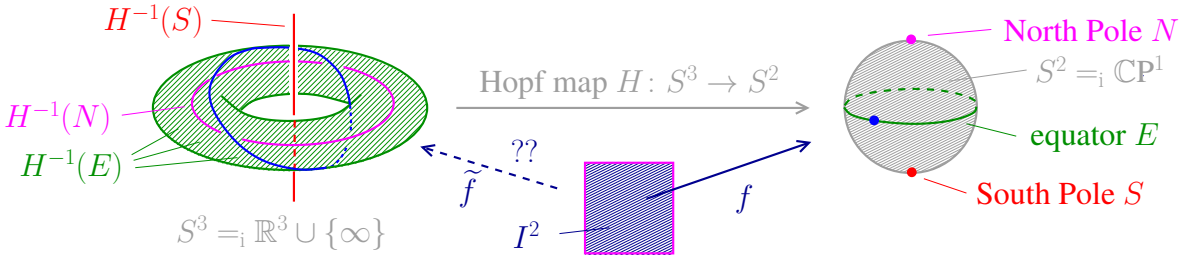
Fiber-LES-Proposition 239.8 we constructed for each $n \in \mathbb{N}$ a morphism

$$\partial_n \colon \pi_n(B, b_0) \;\to\; \pi_{n-1}(F, y_0).$$

Show that for $n \geq 2$ the maps are homomorphisms of groups.
Hint. It can be convenient to make use of Exercise 239.1.

Exercise 239.3. Are the homotopy groups of compact smooth manifolds necessarily finitely generated?

Exercise 239.4. We consider the Hopf map $H \colon S^3 \to S^2$ as illustrated below. We also show a continuous map $f \colon (I^2, \partial I^2) \to (S^2, N)$ which gives a generator of $\pi_2(S^2, N)$. Give an explicit lift $\widetilde{f} \colon I^2 \to S^3$ of f which is constant on $J^1 \subseteq I^2$.



Exercise 239.5. Let $\varphi \colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence

$$\{e\} \;\to\; G \;\xrightarrow{\varphi}\; H \;\xrightarrow{\psi}\; (S, s_0) \;\to\; \{e\}$$

where ψ is a morphism of pointed sets that is H-equivariant, i.e. for any $h, h' \in H$ we have $\psi(h \cdot h') = h \cdot \psi(h')$. Show that $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Exercise 239.6. Let $p \colon (X, x_0) \to (B, b_0)$ be a covering. What does the Fiber-LES-Proposition 239.8 have to say about such a covering?
Remark. To get the maximal output you should make use of Exercise 239.5.

Exercise 239.7. Let $n \in \mathbb{N}$. Use the Matrix Group-Fiber Proposition 238.12 together with the Fiber-LES-Proposition 239.8 to reprove the following table from the Matrix Group-π_0-Proposition 4.54:

	topological space	path-components
(1)	$\mathrm{GL}(n,\mathbb{R})$	path-components are $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$
(2)	$\mathrm{SL}(n,\mathbb{R})$	path-connected
(3)	$\mathrm{O}(n)$	path-components are $\mathrm{SO}(n)$ and $\mathrm{O}(n) \cap \mathrm{GL}_-(n,\mathbb{R})$
(4)	$\mathrm{GL}(n,\mathbb{C})$	path-connected
(5)	$\mathrm{SL}(n,\mathbb{C})$	path-connected
(6)	$\mathrm{U}(n)$	path-connected
(7)	$\mathrm{SU}(n)$	path-connected.

Exercise 239.8. Let $H_{\mathbb{C}}\colon S^3 \to S^2$ be the Hopf map. Recall that in the Hopf-Bundle Proposition 298.10 we saw that this map is an S^1-bundle. We consider the connecting homomorphism $\partial_2\colon \pi_2(S^2) \to \pi_1(S^1)$ of the long exact sequence of homotopy groups from the Fiber-LES-Proposition 239.8 (2). Show, just using the definition of the connecting homomorphism, that $\partial_2([\mathrm{id}_{S^2}]) = [\mathrm{id}_{S^1}]$.

Exercise 239.9. Let K be the Klein bottle. On page 4581 we showed that there exists an S^1-bundle $p\colon K \to S^1$. Explore the long exact sequence of homotopy groups arising from the Fiber-LES-Proposition 239.8.

Exercise 239.10. Let F be a topological space and let $\mu\colon F \to F$ be a homeomorphism. In the Mapping Torus–Bundle Lemma 238.14 we saw that we have a corresponding F-bundle

$$\begin{array}{rcl} \mathrm{Tor}(F,\mu) = (F \times [0,1])/(f,0) \sim (\mu(f),1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi i t). \end{array}$$

Explore the long exact sequence of homotopy groups arising from the Fiber-LES-Proposition 239.8.

Exercise 239.11. We consider the $\mathrm{SO}(2)$-bundle

$$\begin{array}{rcl} \mathrm{SO}(3) & \to & S^2 \\ (v_1,v_2,v_3) & \mapsto & v_3. \end{array}$$

Give an explicit description of the connecting connecting homomorphism

$$\partial_2\colon \pi_2(S^2) \to \pi_1(\mathrm{SO}(2)).$$

Exercise 239.12. Let $p\colon Y \to B$ be a bundle map and let $y_0 \in Y$. We set $b_0 := p(y_0)$ and $F := p^{-1}(b_0)$. We assume that the inclusion $F \to Y$ is null-homotopic, i.e. homotopic to a constant map.

(a) Show for each $n \in \mathbb{N}$ there exists a homomorphism

$$\sigma\colon \pi_n(B,b_0) \to \pi_n(Y,y_0)$$

with $\sigma \circ p_* = \mathrm{id}_{\pi_n(B,b_0)}$.

Hint. Let $[\gamma\colon (I^n,\partial I^n) \to (B,\{b_0\})] \in \pi_n(B,b_0)$. By the Bundle–Homotopy Lifting Lemma 239.11 we can find a lift $\widetilde{f}\colon I^n \to Y$ with $\widetilde{f}(J^{n-1}) = \{y_0\}$. Now use our hypothesis on the inclusion $F \to Y$ to modify $\widetilde{f}$.

(b) Show that for each $n \in \mathbb{N}_{\geq 2}$ we have an isomorphism

$$\pi_n(B, b_0) \;\cong\; \pi_n(Y, y_0) \times \pi_{n-1}(F, y_0).$$

Exercise 239.13. Use the Fiber-LES-Proposition 239.8 to prove the following statements:

(a) For all $n \geq 2$ we have $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$.
Remark. Make use of Exercise 239.12.

(b) For $n \in \{1, \ldots, 6\}$ we have $\pi_n(S^4) \cong \pi_{n-1}(S^3)$.

Exercise 239.14. Let $k, l \in \mathbb{N}$. Show that if there exists an S^k-bundle $p\colon S^{k+l} \to S^l$, then $l = k + 1$.

Exercise 239.15.

(a) Does there exist an F-bundle $p\colon S^4 \to B$ where F and B are closed connected smooth manifolds of dimension ≥ 1?

(b) Does there exist an F-bundle $p\colon S^5 \to B$ where F and B are closed connected smooth manifolds of dimension ≥ 1?

Exercise 239.16. Formulate and prove the "unitary" analog of the Matrix Group–Inclusion–π_n-Proposition 239.14.

Exercise 239.17. We consider the topological space $\mathbb{C}\mathrm{P}^2 \# (-\mathbb{C}\mathrm{P}^2)$. Recall that by the Projective Space–Connected Sum Proposition 238.10 we know that $\mathbb{C}\mathrm{P}^2 \# (-\mathbb{C}\mathrm{P}^2)$ admits an S^2-bundle $\mathbb{C}\mathrm{P}^2 \# (-\mathbb{C}\mathrm{P}^2) \to S^2$. Does $\mathbb{C}\mathrm{P}^2 \# (-\mathbb{C}\mathrm{P}^2)$ admit bundle maps with base space or fiber that are not homeomorphic to S^2?

Exercise 239.18. Does there exist a bundle map $S^3 \times S^1 \to \mathbb{R}\mathrm{P}^2$? If yes, what is the fiber?

Exercise 239.19.

(a) For every $n \in \mathbb{N}$ compute the fundamental group $\pi_1(\mathrm{U}(n))$.

(b) For every $n \in \mathbb{N}$ compute the second homotopy group $\pi_2(\mathrm{U}(n))$.

Remark. First recall that it follows from the SU(2)-SO(3) Covering Theorem 94.3 together with the U(n)-Product Lemma 32.11 that U(2) is homeomorphic to $S^3 \times S^1$. Secondly make use of Exercise 239.16.

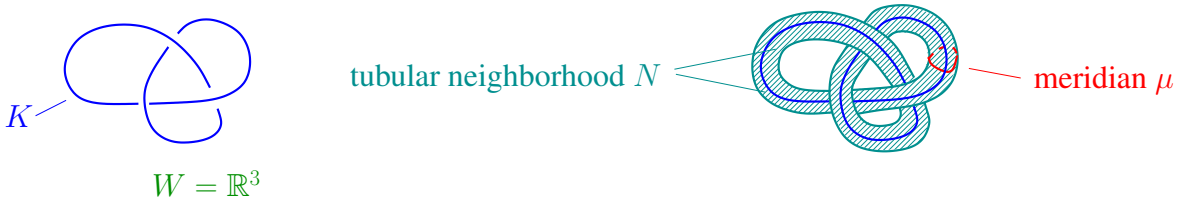
Exercise 239.20. Note that the map

$$\begin{array}{rccc} \Theta\colon & \mathrm{Sym}(2) & \to & \mathbb{R}^3 \\ & \begin{pmatrix} a & b \\ b & c \end{pmatrix} & \mapsto & (a, b, c) \end{array}$$

is an isomorphism. Can you describe $\Theta(\mathrm{Sym}^+(2)) \subseteq \mathbb{R}^3$?
Remark. Note that by the discussion in Notation 239.17 we know that $\Theta(\mathrm{Sym}^+(2))$ is an open convex subset of $\mathbb{R}^3$.

Exercise 239.21. Let $n \in \mathbb{N}_0$, let W be a simply connected n-dimensional smooth manifold and let $K \subseteq W$ be a proper oriented connected non-empty submanifold of codimension 2. Let $p\colon N \to K$ be a tubular neighborhood in the sense of Tubular Neighborhood–Existence Theorem 57.11. For any $x \in K$ we refer to $\partial p^{-1}(x)$ as a *meridian* of K. Show that $\pi_1(W \backslash K)$ is normally generated (in the sense of Definition 98.20) by any meridian of K.

Hint. Make use of Exercise 98.28.
Remark. This is a generalization of Exercise 99.20.



CHAPTER 240

Bundles with structure groups

Recall that in Definition 238.1 we introduced, given a topological space F, the notion of an F-bundle $p\colon F \to B$. In this chapter we will first refine this concept. Afterwards we will develop in this, and the following chapter, the theory of these refined bundles. One intermediate goal we have in mind is that we would like to give a suitable characterization of bundles over spheres. We will achieve this goal in the next chapter.

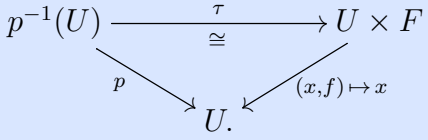
This chapter is organized as follows:

- In Section 240.1 we will recall the definition of an F-bundle and we will study the behavior of F-bundles under pullbacks.
- In Section 240.2 we will recall the compact-open topology on the set of continuous maps between two fixed topological spaces.
- In Section 240.3 we will refine the notion of an F-bundle. More precisely, given a topological space F and a subgroup $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ (which we equip with the compact-open topology) we will introduce the notion of an $(F, \mathcal{G})$-bundle. Such $(F, \mathcal{G})$-bundles are sometimes also called F-bundles with structure group $\mathcal{G}$.
- In Section 240.4 we will discuss morphisms between $(F, \mathcal{G})$-bundles.
- In Section 240.5 we will study $(F, \mathcal{G})$-bundles over a fixed base space B. In particular we will introduce the notion of a trivialization of an $(F, \mathcal{G})$-bundle over B.
- In Section 240.6 we will see that (up to isomorphism) all $(F, \mathcal{G})$-bundles can be described via transition maps.
- In Section 240.7 we will discuss pullbacks of $(F, \mathcal{G})$-bundles.
- In Section 240.8 we will provide exercises for Chapter 240.

In the next Chapter 241 we will discuss the interplay between bundles and homotopies.

240.1. **Bundles and pullbacks.** First we recall Definition 238.1.

Definition 240.1. Let F be a topological space. A continuous map $p\colon Y \to B$ between two topological spaces is called an F-bundle over B if for every $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:



Next we recall a very general construction of topological spaces and continuous maps:

Proposition 7.22 (Topological-Pullback Proposition) Let $p\colon Y \to B$ be a continuous map of topological spaces, let C be a topological space and let $h\colon C \to B$ be a

continuous map. We consider the topological space

$$h^*Y \;:=\; \{(c,y) \in C \times Y \,|\, h(c) = p(y)\}$$

and the maps

$$\begin{array}{rcl} q\colon h^*Y & \to & C \\ (c,y) & \mapsto & c \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon h^*Y & \to & Y \\ (c,y) & \mapsto & y. \end{array}$$

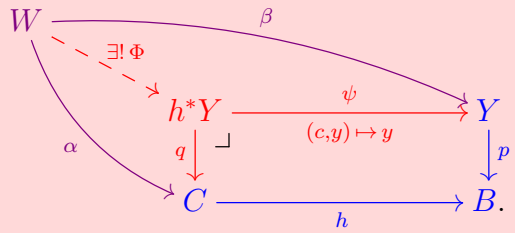
The following statements hold:

(1) (a) The maps $q\colon h^*Y \to C$ and $\psi\colon h^*Y \to Y$ are continuous.

(b) Suppose we are given a topological space W and two continuous maps $\alpha\colon W \to C$ and $\beta\colon W \to Y$ such that $h \circ \alpha = p \circ \beta\colon W \to B$. The map

$$\begin{array}{rcl} \Phi\colon W & \to & h^*Y \\ w & \mapsto & (\alpha(w), \beta(w)) \end{array}$$

is continuous. In particular we obtain the following commutative diagram:

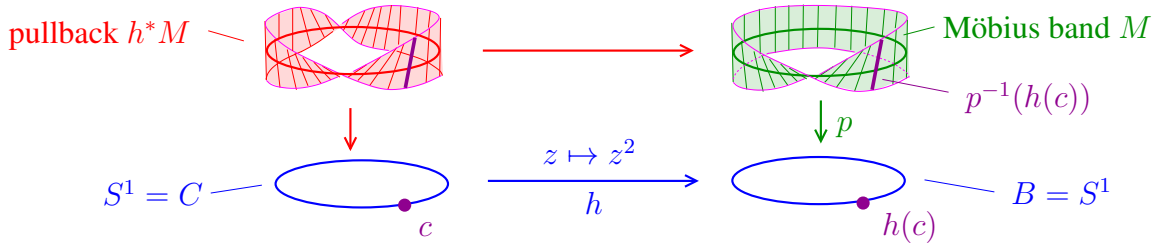


(∗) A map $\Phi\colon W \to h^*Y$ is continuous if and only if the maps $q \circ \Phi\colon W \to C$ and $\psi \circ \Phi\colon W \to Y$ are continuous.

(2) The topological space h^*Y with the natural maps $q\colon h^*Y \to C$ and $\psi\colon h^*Y \to Y$ is the pullback, in the sense of Definition 6.13, in the category $\mathcal{Top}$ of topological spaces.

Example.

(1) In the figure below we show the projection $p\colon M \to S^1$ from the Möbius band to S^1 and we see the map $h\colon C = S^1 \to S^1 = B$ that is given by $h(z) = z^2$.



We want to understand the pullback h^*M: The definition of the pullback says, loosely speaking, that for each $c \in C$ the pullback h^*M contains a "copy" of $p^{-1}(h(c))$. With this intuition it becomes perhaps reasonable that the pullback h^*M is a "doubly twisted band".

(2) Let $p\colon Y \to B$ be a continuous map of topological spaces and let $C \subseteq B$ be a subspace. We denote by $\iota\colon C \to B$ the inclusion.



It follows easily from the Topological-Product Proposition 7.2 (1) and the Subspace Continuity Lemma 4.3 (2) and (3) that the maps

$$\begin{array}{rcl} p^{-1}(C) & \to & \overbrace{\iota^*Y}^{\subseteq C\times Y} \\ y & \mapsto & (p(y),y) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \overbrace{\iota^*Y}^{\subseteq C\times Y} & \to & p^{-1}(C) \\ (c,y) & \mapsto & y \end{array}$$

are continuous and inverses of one another. Thus we see that ι^*Y is naturally homeomorphic to $p^{-1}(C)$. For fans of pullback diagrams this discussion is summarized in the following diagram:

$$\begin{array}{ccc} \iota^*Y \cong p^{-1}(C) & \hookrightarrow & Y \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xhookrightarrow{\iota} & B. \end{array}$$

(3) Let F, C, B be topological spaces, let $p\colon B\times F\to B$ be the natural projection and let $h\colon C\to B$ be a continuous map. It follows easily from the definitions (or more cleverly from the general properties of pullbacks, see Exercise 6.8) that the maps

$$\begin{array}{rcl} \overbrace{h^*(B\times F)}^{\subseteq C\times(B\times F)} & \to & C\times F \\ (c,(b,f)) & \mapsto & (c,f) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} C\times F & \to & \overbrace{h^*(B\times F)}^{\subseteq C\times(B\times F)} \\ (c,f) & \mapsto & (c,(h(c),f)) \end{array}$$

are homeomorphisms that are inverses of one another. This leads us to the following diagram:

$$\begin{array}{ccc} h^*(B\times F)\cong C\times F & \xrightarrow{(c,f)\mapsto(h(c),f)} & B\times F \\ {\scriptstyle (c,f)\mapsto c}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xhookrightarrow{h} & B. \end{array}$$

□

Proposition 240.2. (Bundle Map-Pullback Proposition) Let F be a topological space, let $p\colon Y\to B$ be an F-bundle and let $h\colon C\to B$ be a continuous map. We consider the corresponding pullback diagram

$$\begin{array}{ccc} h^*Y & \xrightarrow{(c,y)\mapsto y} & Y \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xhookrightarrow{h} & B \end{array}$$

from the Topological-Pullback Proposition 7.22.

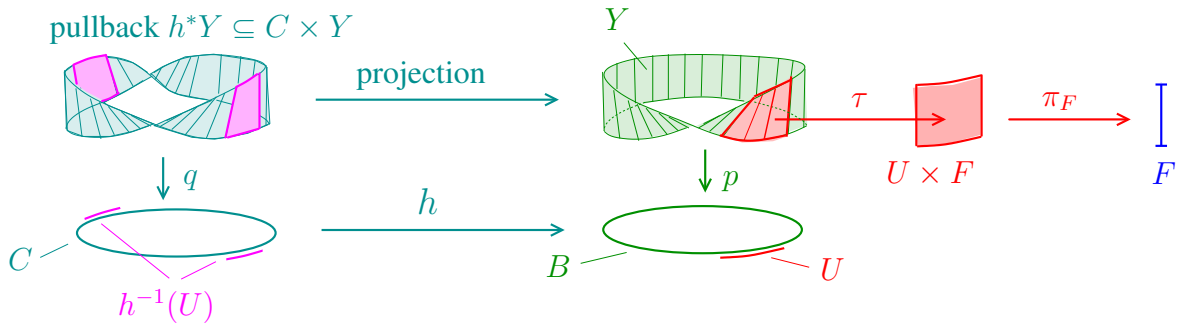
(1) Let $U\subseteq B$ be an open subset and let $\tau\colon p^{-1}(U)\to U\times F$ be a trivialization. The map

$$\begin{array}{rcl} \overbrace{q^{-1}(h^{-1}(U))}^{\subseteq h^*Y\subseteq C\times Y} & \to & \overbrace{h^{-1}(U)}^{\subseteq C}\times F \\ (c,y) & \mapsto & (c,\pi_F(\tau(y))) \end{array}$$

here $\pi_F\colon Y\times F\to F$ is the natural projection

is a local F-trivialization for the map $q\colon h^*Y\to C$.

(2) The pullback $q\colon h^*Y\to C$ is also an F-bundle.



Remark. Recall that in the Covering-as-Bundle Lemma 238.2 we showed that coverings are the same as bundle maps with discrete fiber. It thus follows from the Bundle Map-Pullback Proposition 240.2 that the pullback of a covering is again a covering. This statement is precisely the content of the Pullback-Covering Lemma 92.12. □

Example. It follows easily from the example preceding the proposition that the pullback of a trivializable F-bundle $p\colon Y \to B$ under a continuous map $h\colon C \to B$ is a trivializable F-bundle $h^*Y \to C$. □

Proof.

(1) There are two approaches to proving this statement:

(a) We consider the following two maps:

$$\begin{aligned} \Phi\colon \overbrace{q^{-1}(h^{-1}(U))}^{\subseteq h^*Y \subseteq C\times Y} &\to \overbrace{h^{-1}(U)}^{\subseteq C} \times F \\ (c,y) &\mapsto (c, \pi_F(\tau(y))) \end{aligned} \quad \text{and} \quad \begin{aligned} \Psi\colon \overbrace{h^{-1}(U)}^{\subseteq C} \times F &\to \overbrace{q^{-1}(h^{-1}(U))}^{\subseteq h^*Y \subseteq C\times Y} \\ (c,f) &\mapsto (c, \tau^{-1}(h(c), f)) \end{aligned}$$

We claim that the two maps are continuous and inverses of one another:

- Using the Subspace Continuity Lemma 4.3 (2) and (3) and the Topological-Product Proposition 7.2 (1) one can easily verify that both maps are continuous.
- The two maps are inverses of one another: For any $(c,y) \in q^{-1}(h^{-1}(U))$ we calculate that
$$\Psi(\Phi(c,y)) = \Psi(c, \pi_F(\tau(y))) = (c, \tau^{-1}(\underbrace{h(c)}_{=p(y)}, \pi_F(\tau(y))) = (c, \underbrace{\tau^{-1}(\tau(y))}_{=y}).$$
Furthermore, for any $(c,f) \in h^{-1}(U) \times F$ we have
$$\Phi(\Psi(c,f)) = \Phi(c, \tau^{-1}(h(c),f)) = (c, \pi_F(\tau(\tau^{-1}(h(c),f)))) = (c, \underbrace{\pi_F(h(c),f)}_{=f}).$$

(b) One can deduce the statement from the above observation that the pullback of a subspace is just the restriction and from the above discussion of the pullback of a projection. We leave it to the reader to make this approach work.

(2) This statement follows immediately from Statement (1). ■

240.2. **The compact-open topology.** Before we can discuss refined notions of bundles, we need to discuss the compact-open topology, which we first introduced and discussed in great detail in Chapter 16. We start out by recalling Definition 16.2:

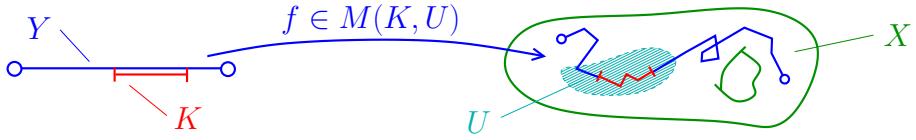
Definition 240.3. Let X and Y be topological spaces.

(1) We denote by $X^Y := C(Y,X)$ the set of all continuous maps from Y to X.

(2) The compact-open topology on X^Y is the topology generated, in the sense of Definition 3.56, by all the sets of the form

$$M(K,U) \;:=\; \{f \in X^Y \mid f(K) \subseteq U\}$$

where $K \subseteq Y$ is compact and where $U \subseteq X$ is open.



Before we can give our first example we should introduce the following convention:

Convention 240.4. Let $m,n \in \mathbb{N}$. We have the obvious injective map

$$\begin{array}{rcl} \mathrm{M}(m\times n,\mathbb{R}) & \to & C(\mathbb{R}^n,\mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n} \\ A & \mapsto & \left(\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A\cdot v \end{array}\right). \end{array}$$

We will use this map to view $\mathrm{M}(m\times n,\mathbb{R})$ as a subset of $C(\mathbb{R}^n,\mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n}$.

The following lemma gives us our most important example of the compact-open topology:

Lemma 16.8 (Matrix Topology Lemma) Let $m,n \in \mathbb{N}$. The compact-open topology on $C(\mathbb{R}^n,\mathbb{R}^m)$ induces the usual topology on $\mathrm{M}(m\times n,\mathbb{R})$ that we introduced in Identification 4.52.

We continue with a few technical statements on the compact-open topology:

Proposition 16.9. (Evaluation Map Proposition) Let X and Y be topological spaces. If Y is regionally compact, then the evaluation map[2597]

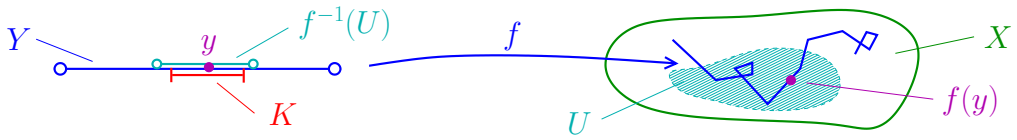
$$\begin{array}{rcl} \mathrm{ev}\colon X^Y\times Y & \to & X \\ ((f\colon Y\to X),y) & \mapsto & f(y) \end{array}$$

is continuous.

Proof. Let U be an open subset of X. We need to show that $\mathrm{ev}^{-1}(U)$ is open. By the JH-Openness Criterion 3.3 it suffices to prove the following claim:

Claim. Given any $(f,y) \in \mathrm{ev}^{-1}(U)$ there exists a neighborhood W of $(f,y) \in X^Y\times Y$ such that $W \subseteq \mathrm{ev}^{-1}(U)$.

Proof. Let $f \in X^Y$ and $y \in Y$ with $\mathrm{ev}(f,y) = f(y) \in U$. Since f is continuous we know that $f^{-1}(U)$ is an open neighborhood of y. Since Y is regionally compact there exists a compact neighborhood K of $y \in Y$ with $y \in K \subseteq f^{-1}(U)$.



[2597]Here we equip $X^Y\times Y$ with the product topology from the Topological-Product Proposition 7.2.

In other words, we have found a compact neighborhood K of y with $f(K) \subseteq U$. Therefore we have $f \in M(K,U)$ and the evaluation map ev sends $W := M(K,U) \times K$ into U. By the Definition 7.3 of the product topology the set $W = M(K,U) \times K$ is in fact a neighborhood of (f,y) in $X^Y \times Y$. ■

Before we can turn to our next proposition it is convenient to recall Definition 7.30:

Definition 240.5. A topological group is a topological space G together with a group structure, such that the two maps

$$\begin{array}{ccc} G \times G & \to & G \\ (g,h) & \mapsto & g \cdot h \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are continuous.

Example. Let $n \in \mathbb{N}$. In the Matrices–Topological Group Lemma 7.31 we showed that $\mathrm{GL}(n,\mathbb{R})$ is topological group. □

In light of the Matrix Topology Lemma 16.8 the following can be viewed as a generalization of the above example.

Proposition 16.17. (Homeomorphisms–Topological Group Proposition) Let X be a topological space. We equip the group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X with the subspace topology coming from X^X.

(1) If X is regionally compact then the map

$$\begin{array}{ccc} \mathrm{Homeo}(X) \times \mathrm{Homeo}(X) & \to & \mathrm{Homeo}(X) \\ (f,g) & \mapsto & f \circ g \end{array}$$

is continuous.

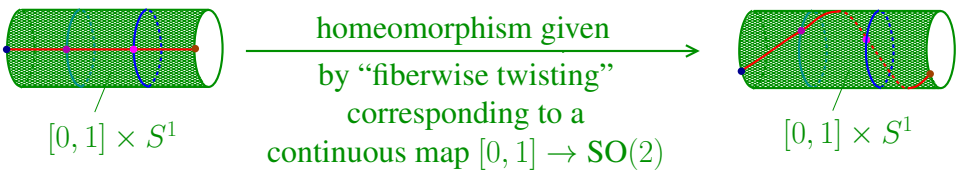
(2) If X is regionally compact, Hausdorff and locally connected, then $\mathrm{Homeo}(X)$ is a topological group.

We move on to the following proposition, which we will soon need in our discussion of bundle maps:

Proposition 240.6. (Fiber Preserving Homeo Proposition)[2598] Let B and F be topological spaces. We denote by $\pi_B \colon B \times F \to B$ the natural projection. If F is regionally compact, Hausdorff and locally connected, then the map

$$\begin{array}{ccc} \mathrm{Homeo}(F)^B & \to & \{\Phi \in \mathrm{Homeo}(B \times F) \,|\, \pi_B \circ \Phi = \pi_B\} \\ (\mu \colon B \to \mathrm{Homeo}(F)) & \mapsto & \left(\begin{array}{ccc} B \times F & \to & B \times F \\ (b,f) & \mapsto & (b, \mu(b)(f)) \end{array} \right) \end{array}$$

is well-defined and it is a bijection.



The proof of the Fiber Preserving Homeo Proposition 240.6 will require the remainder of this section. In the next Section 240.3 we will return to the study of bundles.

[2598] Note that this proposition is the content of Exercise 16.9.

Our proof of the Fiber Preserving Homeo Proposition 240.6 relies on the following proposition:

Proposition 16.10. (Compact-Open Exponential Proposition) Let Y be a topological space that is regionally compact. Furthermore let X and T be two topological spaces. Given any continuous map $H\colon Y \times T \to X$ the map

$$\begin{array}{rcl} T & \to & X^Y \\ t & \mapsto & (y \mapsto H(y,t)) \end{array}$$

is also continuous.

Proof of the Fiber Preserving Homeo Proposition 240.6. We start out with the following claim, which shows in particular that the map in the proposition is well-defined:

Claim 1. Let $\mu\colon B \to \mathrm{Homeo}(F)$ be a continuous map. We consider the map

$$\begin{array}{rcl} \Xi_\mu\colon B \times F & \to & B \times F \\ (b,f) & \mapsto & (b, \mu(b)(f)). \end{array}$$

This map has the following properties:

(0) We have $\pi_B \circ \Xi_\mu = \pi_B$.
(1) The map Ξ_μ is continuous.
(2) The map Ξ_μ is a bijection.
(3) The inverse Ξ_μ is also continuous.

Proof.

(0) It is clear that $\pi_B \circ \Xi_\mu = \pi_B$.
(1) We claim that the map $\Xi_\mu\colon B \times F \to B \times F$ is continuous. First note that it follows from the Topological-Product Proposition 7.2 (1∗) that it suffices to show that the map $B \times F \to F$ that is given by $(b,f) \mapsto \mu(b)(f)$ is continuous. This map can be written as the composition of the following two maps:

$$B \times F \xrightarrow{(\mu,\mathrm{id})} \mathrm{Homeo}(F) \times F \xrightarrow[\text{evaluation map ev}]{(\nu,f)\mapsto \nu(f)} F.$$

It follows easily from the continuity of μ and the Topological-Product Proposition 7.2 (1) that the first map is continuous. Finally note that it follows from the Evaluation Map Proposition 16.9 that the second map is continuous.
(2) Clearly an inverse to the above map Ξ_μ is given the map

$$\begin{array}{rcl} B \times F & \to & B \times F \\ (b,f) & \mapsto & (b, \mu(b)^{-1}(f)) \end{array}$$

(3) Since F is regionally compact, Hausdorff and locally contractible we know by the Homeomorphisms–Topological Group Proposition 16.17 that $\mathrm{Homeo}(F)$ is a topological group. In particular the map $\nu \mapsto \nu^{-1}$ is continuous. It now follows that the map $B \to \mathrm{Homeo}(F)$ that is given by $b \mapsto \mu(b)^{-1}$ is continuous. Thus the argument of (1) shows that the map in (2) is continuous. ⊞

We have now shown that the map

$$\begin{array}{rcl} \Omega\colon \mathrm{Homeo}(F)^B & \to & \{\Phi \in \mathrm{Homeo}(B \times F) \mid \pi_B \circ \Phi = \pi_B\} \\ (\mu\colon B \to \mathrm{Homeo}(F) & \mapsto & \begin{pmatrix} B \times F & \to & B \times F \\ (b,f) & \mapsto & (b, \mu(b)(f)) \end{pmatrix} \end{array}$$

is well-defined. Also note that it is essentially clear that the map Ω is injective. It remains to show that the map Ω is surjective. Thus let $\Phi\colon B \times F \to B \times F$ be a homeomorphism with $\pi_B \circ \Phi = \pi_B$. For each $b \in B$ we consider the map $i_b\colon F \to B \times F$ that is given by $f \mapsto (b, f)$. Furthermore we denote by $\pi_F\colon F \times B \to F$ the projection. We consider the map

$$\begin{array}{rcl} \mu\colon B & \to & \mathrm{Homeo}(F) \\ b & \mapsto & \pi_F \circ \Phi \circ i_b. \end{array}$$

As always we have to verify that a given map is continuous:

Claim 2. The above map $\mu\colon B \to \mathrm{Homeo}(F)$ is continuous.

Proof. Note that $\mu\colon B \to \mathrm{Homeo}(F)$ equals the composition of the following two maps:

$$\begin{array}{rcl} B & \to & (B \times F)^F \\ b & \mapsto & \begin{pmatrix} F \to B \times F \\ f \mapsto \Phi(b, f) \end{pmatrix} \end{array} \qquad \xrightarrow[\pi_{F*}]{g \mapsto \pi_F \circ g} \quad \mathrm{Homeo}(F)$$

Since F is regionally compact we obtain from the Compact-Open Exponential Proposition 16.10 (applied to $X = B \times F$, $Y = F$ and $T = B$) that the map to the left is continuous. Furthermore note that it follows from the Compact-Open Functoriality Lemma 16.5 (1) that the map to the right is also continuous. ⊞

The claim shows that $\mu \in \mathrm{Homeo}(F)^B$. Finally note that it follows almost immediately from the definitions that $\Omega(\mu) = \Phi$. We have thus successfully verified that Ω is surjective. ■

240.3. **Bundles with structure groups.** After this slightly painful excursion into the area of the compact-open topology we can now return to bundles.

First we recall the definition of an F-bundle, but this time we will use slightly different language:

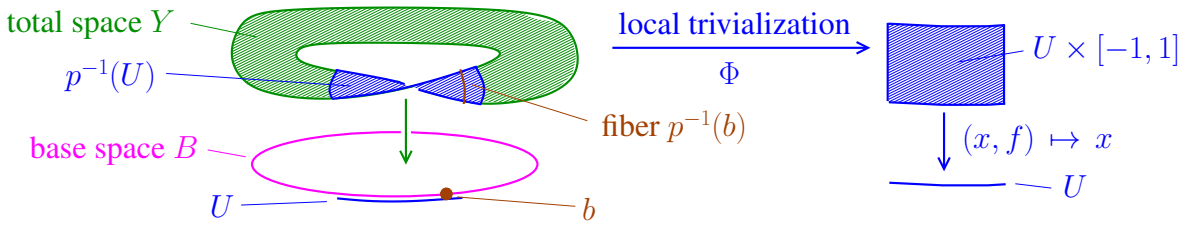
Definition 240.7. Let F be a topological space.

(1) Let C be a topological space and let W be a topological space. An F-trivialization of a given continuous map $p\colon W \to C$ is a homeomorphism $\Phi\colon W \to C \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow[\cong]{\Phi} & C \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f) \mapsto x} & \\ & C. & \end{array}$$

(2) Let $p\colon Y \to B$ be a continuous map between topological spaces.
   (a) A local F-trivialization for $p\colon Y \to B$ is an open subset $U \subseteq B$ together with a trivialization $\Phi\colon p^{-1}(U) \to U \times F$ of the map $p\colon p^{-1}(U) \to U$.
   (b) An F-atlas for $p\colon Y \to B$ is a family $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i \in I}$ of local F-trivializations for p such that $\bigcup_{i \in I} U_i = B$.
   (c) According to Definition 238.1 the map $p\colon Y \to B$ is called an F-bundle over B if p admits an F-atlas.
   (d) Let $p\colon Y \to B$ be an F-bundle. We adopt the following language:
   - For $b \in B$ we refer to $Y_b := p^{-1}(b)$ as a fiber of p.

- We refer to B as the base space and to Y as the total space.
- We refer to the map $p\colon Y \to B$ as the projection.



Example. We consider the Klein bottle in the guise of

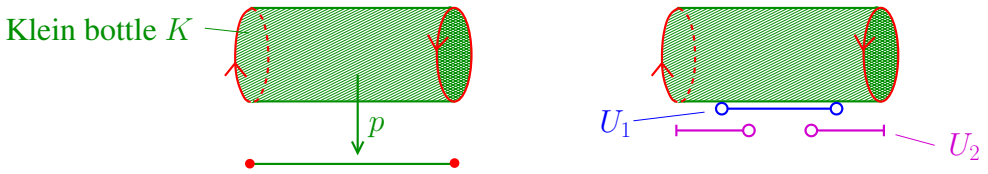
$$K \;=\; (S^1 \times [0,1])/(z,0) \sim (z^{-1},1)$$

and we consider the map

$$\begin{array}{rcl} p\colon K & \to & [0,1]/0 \sim 1 \\ [(x,t)] & \mapsto & [t]. \end{array}$$

We will now see or recall that this map is an S^1-bundle: First note that it follows from the Quotient-to-Quotient Lemma 8.6 that the map $p\colon K \to [0,1]/\sim$ is continuous. Next we consider the open subsets $U_1 := \{[x] \,|\, x \in (\frac{1}{4},\frac{3}{4})\}$ and $U_2 := \{[x] \,|\, x \in [0,\frac{1}{3}) \cup (\frac{2}{3},1])\}$ of $[0,1]/0 \sim 1$ and we consider the maps

$$\begin{array}{rcl} \Phi_1\colon p^{-1}(U_1) & \to & U_1 \times S^1 \\ [(z,t)] & \mapsto & ([t],z) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Phi_2\colon p^{-1}(U_2) & \to & U_2 \times S^1 \\ [(z,t)] & \mapsto & \begin{cases} ([t],z^{-1}), & \text{if } t \in [0,\frac{1}{3}), \\ ([t],z) & \text{if } t \in (\frac{2}{3},1]. \end{cases} \end{array}$$



Using the Pasting Proposition 4.7 (2) and the Topological-Quotient Proposition 8.2 (1b) one can show quite easily that both maps are homeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0 \sim 1$, we have shown that $\Phi_1\colon p^{-1}(U_1) \to U_1 \times S^1$ and $\Phi_2\colon p^{-1}(U_2) \to U_2 \times S^1$ form an S^1-atlas for $p\colon K \to [0,1]/\sim$. Thus we have shown that $p\colon K \to [0,1]/\sim$ is indeed an S^1-bundle. □

In the following definition we will now refine the notion of an F-bundle:

Definition 240.8. Let F be a topological space and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let $p\colon Y \to B$ be a continuous map between two topological spaces.

(1) Let $\Phi\colon p^{-1}(U) \to U \times F$ and $\Psi\colon p^{-1}(V) \to V \times F$ be two local F-trivializations for p. We say that the trivialization Φ is $\mathcal{G}$-compatible to the trivialization Ψ if there exists a *continuous* map[2599] $\Theta\colon U \cap V \to \mathcal{G}$ such that the following diagram commutes:[2600]

$$(U \cap V) \times F \xleftarrow[\cong]{\Phi} p^{-1}(U \cap V) \xrightarrow[\cong]{\Psi} (U \cap V) \times F.$$
$$(b,f) \mapsto (b, \Theta(b)(f))$$

(2) An atlas $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}$ for an F-bundle p is called $\mathcal{G}$-compatible if for all $i, j \in I$ the trivialization Φ_i is compatible to the trivialization Φ_j.
(3) An $(F, \mathcal{G})$-bundle over B is a pair $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ where $p\colon Y \to B$ is an F-bundle over B and where $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}$ is a $\mathcal{G}$-compatible F-atlas for p.[2601]
(4) Sometimes an $(F, \mathcal{G})$-bundle is also referred to as an F-bundle with structure group $\mathcal{G}$.
(5) Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$. Furthermore let Y and B be smooth manifolds. An $(F, \mathcal{G})$-bundle $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ is called smooth if all Φ_i are diffeomorphisms.[2602]

Example.

(1) Let F be a topological space and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be any subgroup. Given any topological space B the projection $B \times F \to B$ together with the identity atlas clearly defines an $(F, \mathcal{G})$-bundle. We refer to it as the trivial $(F, \mathcal{G})$-bundle over B.
(2) We consider $F = S^1$ and $\mathrm{O}(2) \subseteq \mathrm{Homeo}(S^1)$. We consider again the Klein bottle K with the S^1-atlas from page 4634. We claim that this atlas defines an $(S^1, \mathrm{O}(2))$-bundle. Indeed for the two local trivializations Φ_1 and Φ_2 the corresponding map is given by

$$\begin{array}{rcl} \Theta\colon U \cap V & \to & \mathrm{O}(2) \\ [t] & \mapsto & \begin{cases} \mathrm{id}_2, & \text{if } t \in (\frac{2}{3}, \frac{3}{4}), \\ \rho, & \text{if } t \in (\frac{1}{4}, \frac{1}{3}), \end{cases} \end{array}$$

where ρ is the matrix that corresponds to reflection in the x-axis.
(3) Let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ be an $(F, \mathcal{G})$-bundle over B and let $C \subseteq B$ be a subspace. Note that it follows immediately from the definitions that the pair $(p\colon p^{-1}(C) \to C, \{\Phi_i\colon p^{-1}(U_i \cap C) \to U_i \cap C\}_{i\in I})$ is an $(F, \mathcal{G})$-bundle over C. We refer to it as the restriction of p to C.
(4) (a) In the Hopf-Bundle Proposition 238.4 (1) we considered the Hopf map $H_{\mathbb{C}}\colon S^3 \to S^2$ and in the proof we gave an explicit atlas $\tau_j\colon (H_{\mathbb{C}})^{-1}(U_j) \to U_j \times S^1$, $j = 0, 1$. As we will see in Exercise 240.5 one can easily verify that the atlas is $\mathrm{SO}(2)$-compatible.
(b) In the Hopf-Bundle Proposition 238.4 (2) we considered the Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$ and in the proof we gave an explicit atlas $\tau_j\colon (H_{\mathbb{H}})^{-1}(U_j) \to U_j \times S^3$, $j = 0, 1$. For $j = 0, 1$ we compose τ_j with the self-diffeomorphism of $U_j \times S^3$ that is given by $(z, v) \mapsto (z, v^{-1})$. As we will see in Exercise 240.5 one can easily verify that the resulting atlas is $\mathrm{SO}(4)$-compatible.[2603]

[2599] Here and throughout this chapter we equip $\mathcal{G}$ with the compact-open topology that we introduced in Definition 240.3.
[2600] Note that this definition is asymmetric in Φ and Ψ. More precisely, if Φ is $\mathcal{G}$-compatible to Ψ, then we have a continuous map $\Theta\colon U \cap V \to \mathcal{G}$ which makes the diagram commute. To show that Ψ is $\mathcal{G}$-compatible to Φ one would like to consider the map $U \cap V \to \mathcal{G}$ that is given by $x \mapsto \Theta(x)^{-1}$. But a priori it is not clear that this map is continuous, unless we are in a nice setting, e.g. as in the Homeomorphisms–Topological Group Proposition 16.17.
[2601] Similar to many other occasions we will usually suppress this choice of atlas from the notation.
[2602] Note that it follows easily from the definition that the map $p\colon Y \to B$ is smooth.
[2603] The inversion $v \mapsto v^{-1}$ turns a right action into a left action.

(5) As we will see in Exercise 240.9, one can easily verify that the explicit atlases that we constructed in the Projective Space-minus-Ball Lemma 238.11 are compatible over the appropriate special orthogonal group. This shows:[2604]
- We can equip $\mathbb{R}\mathrm{P}^2 \setminus \Theta(B^2)$ with the structure of a $(\overline{B}^1, \mathrm{O}(1))$-bundle.
- We can equip $\mathbb{C}\mathrm{P}^2 \setminus \Theta(B^4)$ with the structure of a $(\overline{B}^2, \mathrm{SO}(2))$-bundle.
- We can equip $\mathbb{H}\mathrm{P}^2 \setminus \Theta(B^8)$ with the structure of a $(\overline{B}^4, \mathrm{SU}(2))$-bundle.

In particular these bundles are tubular neighborhoods in the sense of Definition 57.9.

(6) As we will see in Exercise 240.11, one can easily verify that the explicit atlases that we constructed in the Projective Space–Connected Sum Proposition 238.10, building on the atlases from the proof of the Projective Space-minus-Ball Lemma 238.11 on page 4588, are compatible over the appropriate special orthogonal group. This shows:
- We can equip $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ with the structure of an $(S^1, \mathrm{O}(2))$-bundle.
- We can equip $\mathbb{C}\mathrm{P}^2 \# (-\mathbb{C}\mathrm{P}^2)$ with the structure of an $(S^2, \mathrm{SO}(3))$-bundle.
- We can equip $\mathbb{H}\mathrm{P}^2 \# (-\mathbb{H}\mathrm{P}^2)$ with the structure of an $(S^4, \mathrm{SO}(5))$-bundle. □

The following lemma relates the naive notion of an F-bundle to the more refined notion of an $(F, \mathcal{G})$-bundle:

Lemma 240.9. (Bundle Definition Lemma) Let F be a topological space. Furthermore let $p\colon Y \to B$ be an F-bundle in the sense of Definition 238.1. If F is regionally compact, Hausdorff and locally connected, then any two local F-trivializations for p are $\mathrm{Homeo}(F)$-compatible. In particular the pair

$$(p\colon Y \to B, \text{family of all local } F\text{-trivializations of } p\colon Y \to B)$$

is an $(F, \mathrm{Homeo}(F))$-bundle.

Remark. Note that in all the explicit examples that we discussed in Chapter 238, and in particular in the Ehresmann Fibration Theorem 238.6, the fibers are topological manifolds. By the Manifold–Local Properties Lemma 23.8 we know that topological manifolds are regionally compact, Hausdorff and locally connected. We can and will therefore use the Bundle Definition Lemma 240.9 to view all these examples in the following also as $(F, \mathrm{Homeo}(F))$-bundles. □

Proof. Let F be a topological space and let $p\colon Y \to B$ be an F-bundle. Furthermore let $\Phi\colon p^{-1}(U) \to U \times F$ and $\Psi\colon p^{-1}(V) \to V \times F$ be two local F-trivializations for p. We need to show that they are $\mathrm{Homeo}(F)$-compatible.

First note that we have the following homeomorphisms:

$$\begin{array}{ccccc} (U \cap V) \times F & \xleftarrow[\cong]{\Phi} & p^{-1}(U \cap V) & \xrightarrow[\cong]{\Psi} & (U \cap V) \times F \\ & {\scriptstyle (b,f) \mapsto b} \searrow & \downarrow {\scriptstyle p} & \swarrow {\scriptstyle (b,f) \mapsto b} & \\ & & U \cap V. & & \end{array}$$

Since the topological space F is by hypothesis regionally compact, Hausdorff and locally connected we obtain from the Fiber Preserving Homeo Proposition 240.6 that there exists

[2604] Another way of phrasing the result below is that the structure group is given by $\{z \in \mathbb{R} \,|\, |z| = 1\} \cong S^0$, $\{z \in \mathbb{C} \,|\, |z| = 1\} \cong \mathrm{U}(1) \cong \mathrm{SO}(2) \cong S^1$ and $\{z \in \mathbb{H} \,|\, |z| = 1\} \cong \mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong S^3$.

a continuous map $\Theta\colon U\cap V\to \mathrm{Homeo}(F)$ such that the following diagram commutes:

$$(U\cap V)\times F \xleftarrow[\cong]{\Phi} p^{-1}(U\cap V) \xrightarrow[\cong]{\Psi} (U\cap V)\times F.$$
$$(b,f)\mapsto (b,\Theta(b)(f))$$

We have thus shown that Φ and Ψ are $\mathrm{Homeo}(F)$-compatible. ■

There are two special cases of $(F,\mathcal{G})$-bundles which are so important, that they deserve their own names:

Definition 240.10.

(1) We refer to any $(\mathbb{R}^n,\mathrm{GL}(n,\mathbb{R}))$-bundle as an n-dimensional real vector bundle.
(2) We refer to any $(\mathbb{C}^n,\mathrm{GL}(n,\mathbb{C}))$-bundle as an n-dimensional complex vector bundle.

We will discuss vector bundles in much greater detail later in Chapter 265. To practice the definition of vector bundles we will now prove the following lemma, even though we will not make use of it for a while:

Lemma 240.11. (Vector Bundle–Fiber Lemma) Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Furthermore let $(p\colon Y\to B,\{\Phi_i\colon p^{-1}(U_i)\to U_i\times\mathbb{K}^n\}_{i\in I})$ be a $(\mathbb{K}^n,\mathrm{GL}(n,\mathbb{K}))$-bundle. For every $b\in B$ we can define a $\mathbb{K}$-vector space structure on the fiber $p^{-1}(b)$ such that for each $i\in I$ and each $b\in U_i$ the following map is an isomorphism of $\mathbb{K}$-vector spaces:

$$p^{-1}(b) \xrightarrow{\Phi_i|_{p^{-1}(b)}} \{b\}\times\mathbb{K}^n \xrightarrow{(b,f)\mapsto f} \mathbb{K}^n.$$

Proof. Given any $b\in B$ we can pick an $i\in I$ with $b\in U_i$ and we consider the bijection

$$p^{-1}(b) \xrightarrow{\Phi_i|_{p^{-1}(b)}} \{b\}\times\mathbb{K}^n \xrightarrow{(b,f)\mapsto f} \mathbb{K}^n.$$

We equip $p^{-1}(b)$ with the unique $\mathbb{K}$-vector space structure that turns the bijection into an isomorphism of $\mathbb{K}$-vector spaces. It is clear that it remains to prove the following claim:

Claim. The above vector space structure does not depend on the choice of $i\in I$.

Proof. Let $j\in I$ with $b\in U_j$. Since the atlas is $\mathrm{GL}(n,\mathbb{K})$-compatible there exists an $A\in\mathrm{GL}(n,\mathbb{K})$ such that the following diagram commutes:

$$\mathbb{K}^n \xleftarrow{f\mapsfrom(b,f)} \{b\}\times\mathbb{K}^n \xleftarrow{\Phi_i|_{p^{-1}(b)}} p^{-1}(b) \xrightarrow{\Phi_j|_{p^{-1}(b)}} \{b\}\times\mathbb{K}^n \xrightarrow{(b,f)\mapsto f} \mathbb{K}^n.$$
$$(b,f)\mapsto(b,A\cdot f)$$

It follows easily from this diagram that the indices i and j lead to the same $\mathbb{K}$-vector space structure on $p^{-1}(b)$. ■

240.4. **Morphisms between bundles.** As always, not only do we care about objects (in our current setting $(F,\mathcal{G})$-bundles), but we also care about morphisms between objects. This leads us to the following definition:

Definition 240.12. Let F be a topological space and let $\mathcal{G}\subseteq\mathrm{Homeo}(F)$ be a subgroup. Let

$$(p\colon X\to B,\{\Phi_i\colon p^{-1}(U_i)\to U_i\times F\}_{i\in I})$$
$$(q\colon Y\to C,\{\Psi_j\colon q^{-1}(V_j)\to V_j\times F\}_{j\in J})$$

be two $(F, \mathcal{G})$-bundles. A morphism between these two $(F, \mathcal{G})$-bundles consists of two continuous maps $h\colon B \to C$ and $H\colon X \to Y$ such that the following two conditions are satisfied:

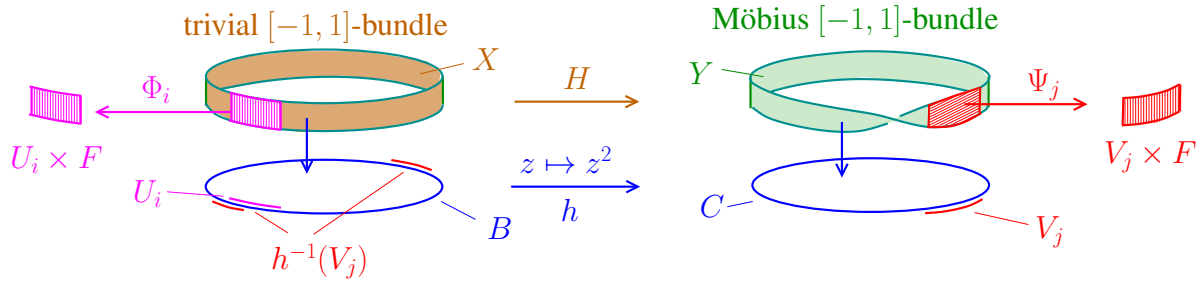
(1) The following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C. \end{array}$$

(2) For each $i \in I$ and $j \in J$ there exists a continuous map $\Theta\colon U_i \cap h^{-1}(V_j) \to \mathcal{G}$ such that the following diagram commutes:

$$(U_i \cap h^{-1}(V_j)) \times F \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap h^{-1}(V_j)) \xrightarrow{H} q^{-1}(V_j) \xrightarrow[\cong]{\Psi_j} V_j \times F.$$
$$(b,f) \mapsto (h(b), \Theta(b)(f))$$

Example. We consider $F = [-1,1]$ and the subgroup $\mathcal{G} := \{\mathrm{id}, \rho\} \subseteq \mathrm{Homeo}([-1,1])$, where $\rho\colon [-1,1] \to [-1,1]$ is the reflection given by $\rho(x) = -x$.



One can easily verify that the map

$$\begin{array}{ccc} \overbrace{S^1 \times [-1,1]}^{\substack{\text{the trivial} \\ ([-1,1],\mathcal{G})\text{-bundle} \\ \text{over } S^1}} & \to & \overbrace{(\mathbb{R} \times [-1,1])/(x,y) \sim (x+1,-y)}^{\substack{\text{this is essentially} \\ \text{the Möbius } ([-1,1],\mathcal{G})\text{-bundle} \\ \text{from page 4580}}} \\ (\exp(2\pi t\,\mathrm{i}), y) & \mapsto & [(2t,y)] \end{array}$$

is a morphism of $(F, \mathcal{G})$-bundles. □

The notion of a "morphism" suggests that there should be a category just around the corner:

Lemma 240.13. (Bundle–Category Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. We compose morphisms as indicated in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{G} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{g} & C \end{array} \quad \circ \quad \begin{array}{ccc} Y & \xrightarrow{H} & Z \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ C & \xrightarrow{h} & D \end{array} \quad := \quad \begin{array}{ccc} X & \xrightarrow{H \circ G} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} \\ B & \xrightarrow{h \circ g} & D. \end{array}$$

The class of $(F, \mathcal{G})$-bundles together with the above notion of morphisms and the above composition of such morphisms maps form a category.

Definition 240.14. Let F be a topological space that is regionally compact. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. We refer to the category that we introduced in the Bundle–Category Lemma 240.13 as the category $\mathcal{Bund}_{(F,\mathcal{G})}$ of $(F,\mathcal{G})$-bundles.

Proof. It is pretty clear that the only statement that requires just a little bit of thought is the statement that the above composition of two morphisms of $(F,\mathcal{G})$-bundles is indeed a morphism of $(F,\mathcal{G})$-bundles. Thus let

$$\begin{aligned}&(p\colon X \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}),\\ &(q\colon Y \to C, \{\Psi_j\colon q^{-1}(V_j) \to V_j \times F\}_{j\in J}),\\ &(r\colon Z \to D, \{\Xi_k\colon r^{-1}(W_k) \to W_k \times F\}_{k\in K})\end{aligned}$$

be three $(F,\mathcal{G})$-bundles. Furthermore let

$$\begin{array}{ccc} X & \xrightarrow{G} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{g} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{H} & Z \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ C & \xrightarrow{h} & D \end{array}$$

be two morphisms of $(F,\mathcal{G})$-bundles. Let $i \in I$, $j \in J$ and $k \in K$. Since (G,g) is a morphism of $(F,\mathcal{G})$-bundles we know that there exists a continuous map $\Theta\colon U_i \cap g^{-1}(V_j) \to \mathcal{G}$ such that the following diagram commutes:

$$(U_i \cap g^{-1}(V_j)) \times F \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap g^{-1}(V_j)) \xrightarrow{G} q^{-1}(V_j) \xrightarrow[\cong]{\Psi_j} V_j \times F.$$
$$(b,f) \mapsto (g(b), \Theta(b)(f))$$

Furthermore, since (H,h) is a morphism of $(F,\mathcal{G})$-bundles we also know that there exists a continuous map $\Omega\colon V_j \cap h^{-1}(W_k) \to \mathcal{G}$ such that the following diagram commutes:

$$(V_j \cap h^{-1}(W_k)) \times F \xleftarrow[\cong]{\Psi_j} q^{-1}(V_j \cap h^{-1}(W_k)) \xrightarrow{H} r^{-1}(W_k) \xrightarrow[\cong]{\Xi_k} W_k \times F.$$
$$(c,f) \mapsto (h(c), \Omega(c)(f))$$

In summary we get the following diagram:

$$(U_i \cap (h\circ g)^{-1}(W_k)) \times F \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap (h\circ g)^{-1}(W_k)) \xrightarrow{H\circ G} r^{-1}(W_k) \xrightarrow[\cong]{\Xi_k} W_k \times F.$$
$$(b,f) \mapsto ((h\circ g)(b), \Omega(g(b))(\Theta(b)(f)))$$

It remains to prove the following claim:

Claim. The map
$$\begin{aligned} U_i \cap (h\circ g)^{-1}(W_k) &\to \mathrm{Homeo}(F)\\ b &\mapsto \Omega(g(b)) \circ \Theta(b)\end{aligned}$$
is continuous.

Proof. Note that the given map can be written as the composition of the following three maps:

$$U_i \cap (h\circ g)^{-1}(W_k) \xrightarrow{(g,\iota)} V_j \times U_i \xrightarrow{\mathrm{diag}(\Omega,\Theta)} \mathrm{Homeo}(F)\times\mathrm{Homeo}(F) \xrightarrow{(\varphi,\psi)\mapsto\varphi\circ\psi} \mathrm{Homeo}(F)$$

where $\iota\colon (h\circ g)^{-1}(W_k) \to U_i$ is the inclusion. Note that it follows from the Topological-Product Proposition 7.2 and the fact that g, Θ and Ω are continuous, that the first two

maps are continuous. Finally note that it follows from our hypothesis that F is regionally compact and the Homeomorphisms–Topological Group Proposition 16.17 (1) that the third map is continuous. ■

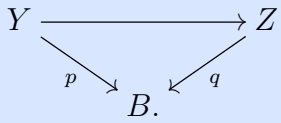
240.5. **Bundles over a fixed base space.** The notion of a morphism between $(F, \mathcal{G})$-bundles leads to two somewhat different questions:

(1) Given a fixed topological space B, can we classify in a suitable sense the $(F, \mathcal{G})$-bundles over B?
(2) Given two topological spaces B and C, when can we establish a relationship between the types of $(\mathcal{F}, G)$-bundles that can occur over B and over C?

We start out with fixing a base space. This leads us to the following definition:

Definition 240.15. Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let B be a topological space.

(1) A morphism of $(F, \mathcal{G})$-bundles over B is a morphism of $(F, \mathcal{G})$-bundles with $h = \mathrm{id}_B$, i.e. we look at a morphism of $(F, \mathcal{G})$-bundles of the form



(2) The class of $(F, \mathcal{G})$-bundles over B together with the morphism of $(F, \mathcal{G})$-bundles over B form a category written as $\mathcal{Bund}_{(F,\mathcal{G})}(B)$.

Next we introduce a few mostly self-explanatory definitions:

Definition 240.16. Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be any subgroup.

(1) Given any topological space B the projection $B \times F \to B$ together with the identity atlas clearly defines an $(F, \mathcal{G})$-bundle. As on page 4635 we refer to it as the trivial $(F, \mathcal{G})$-bundle over B.
(2) Let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle.
   (a) A trivialization of the $(F, \mathcal{G})$-bundle $p\colon Y \to B$ is an isomorphism (in the category $\mathcal{Bund}_{(F,\mathcal{G})}(B)$ of $(F, \mathcal{G})$-bundles over B) from p to the above trivial $(F, \mathcal{G})$-bundle $B \times F \to B$.
   (b) We say $p\colon Y \to B$ is trivializable if it admits a trivialization.



The following proposition shows that in general there is not a unique trivialization:

Proposition 240.17. (Bundle–Action-on-Trivializations Proposition) Let F be a topological space that is regionally compact, Hausdorff and locally connected. Furthermore

let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup, let $p\colon Y \to B$ be an $(F,\mathcal{G})$-bundle over a topological space B and let $\Phi\colon Y \to B \times F$ be a trivialization.

(1) If $\mu\colon B \to \mathcal{G}$ is a continuous map, then

$$\begin{array}{rcl} Y \xrightarrow{\Phi} B \times F & \to & B \times F \\ (b,f) & \mapsto & (b, \mu(b)(f)) \end{array}$$

is also a trivialization of the $(F,\mathcal{G})$-bundle p.

(2) Every trivialization of $p\colon Y \to B$ is of the form described in (1).

Proof. Since $\Phi\colon Y \to B \times F$ is an isomorphism of $(F,\mathcal{G})$-bundles we can essentially ignore it and we can start out with the trivial $(F,\mathcal{G})$-bundle $p\colon B \times F \to B$ and the identity trivialization $\Phi = \mathrm{id}$.

(1) By our hypothesis on F and the Homeomorphisms–Topological Group Proposition 16.17 (2) we know that the map $b \mapsto \mu(b)^{-1}$ is also continuous. Using this observation one can now easily verify that the map in (1) is an isomorphism of $(F,\mathcal{G})$-bundles.

(2) It follows easily from the definition of an isomorphism of $(F,\mathcal{G})$-bundles over B that any trivialization is of the claimed form.

We leave it to the reader to provide the few missing details. ■

240.6. **Constructing bundles via transition maps.** The starting point of this section is the following lemma which says that we can construct bundles by specifying "transition maps".

Lemma 240.18. (Bundle–Transition Map Lemma) Let F be a topological space and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let B be a topological space and let $\{U_i\}_{i\in I}$ be a family of open subsets of B such that $\bigcup_{i\in I} U_i = B$. Suppose that for every $i,j \in I$ we have a continuous map $\mu_{ij}\colon U_i \cap U_j \to \mathcal{G}$ such that the following condition is satisfied:

(∗) For every $i,j,k \in I$ and every $b \in U_i \cap U_j \cap U_k$ we have the following equality in $\mathcal{G}$:

$$\mu_{jk}(b) \circ \mu_{ij}(b) \;=\; \mu_{ik}(b).$$

The following statements hold:

(1) The map[2605]

$$\begin{array}{rcl} p\colon Y := \Big(\bigsqcup_{i\in I} (U_i \times F)\Big)/\big((b,f),i) \overset{\text{where } b\in U_i\cap U_j}{\sim} ((b,\mu_{ij}(b)(f)),j) & \to & B \\ [((b,f),i)] & \mapsto & b \end{array}$$

is well-defined and continuous.

(2) For each $i \in I$ the natural map

$$\begin{array}{rcl} \Psi_i\colon U_i \times F & \to & Y \\ (b,f) & \mapsto & [((b,f),i)] \end{array}$$

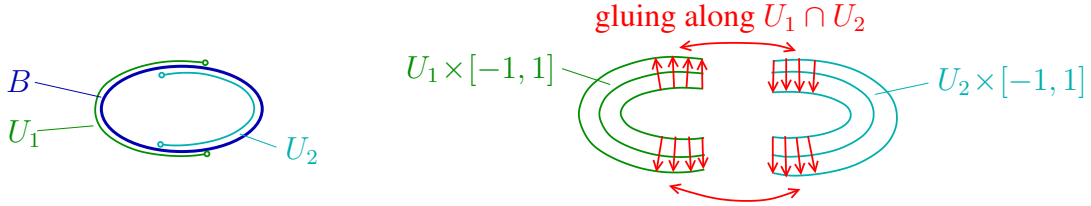
is an open embedding with image $p^{-1}(U_i)$.

(3) The maps $\{\Psi_i^{-1}\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}$ are a $\mathcal{G}$-compatible atlas for $p\colon Y \to B$.

[2605] Let $\{A_i\}_{i\in I}$ be a family of sets. Recall that in Notation 1.12 we defined the *disjoint union* as the set

$$\bigsqcup_{i\in I} A_i \;:=\; \bigcup_{i\in I} (A_i \times \{i\}).$$

Example. We consider $F := [-1,1]$ and the subgroup $\mathcal{G} := \{\mathrm{id}, \rho\} \subseteq \mathrm{Homeo}(F)$ that is given by the identity and the reflection in the origin. Furthermore we consider $B := S^1$ and two open overlapping intervals $U_1, U_2 \subseteq B = S^1$ which satisfy $U_1 \cup U_2 = S^1$.



Let $\mu_{12} = \mu_{21} \colon U_1 \cap U_2 \to \mathcal{G}$ be the map that equals id on one component of $U_1 \cap U_2$ and that equals ρ on the other component of $U_1 \cap U_2$. For $i = 1, 2$ we also set $\mu_{ii} := \mathrm{const}_{\mathrm{id}_F} \colon U_i \to \mathcal{G}$. It is fairly straightforward to see that the resulting $([-1,1], \mathcal{G})$-bundle is isomorphic to the usual Möbius $([-1,1], \mathcal{G})$-bundle. □

Proof.

(1) First note that it follows from the Quotient-Factorization Lemma 1.21 that the map

$$\begin{aligned} p \colon Y := \Big(\bigsqcup_{i \in I} (U_i \times F) \Big) / ((b,f),i) \sim ((b, \mu_{ij}(b)(f)), j) &\to B \\ [((b,f),i)] &\mapsto b \end{aligned}$$

that we are interested in, is indeed well-defined. Furthermore note that it follows from the Topological-Coproduct Proposition 7.26 (1b) and the Topological-Quotient Proposition 8.2 (1b) that this map is continuous.

(2) Let $i \in I$. We consider the natural map $\Psi_i \colon U_i \times F \to Y$. We make the following observations:

(i) It follows from the Topological-Coproduct Proposition 7.26 (1a) and the Topological-Quotient Proposition 8.2 (1a) that $\Psi_i \colon U_i \times F \to Y$ is continuous.

(ii) We now want to prove the following claim:

Claim. The map $\Psi_i \colon U_i \times F \to Y$ is injective.

Proof. Suppose that we have $\Psi_i(b,f) = \Psi_i(b',f')$. In other words, suppose that we have $((b,f),i) = ((b',f'),i)$. By definition of an equivalence relation that is generated by a relation (see Definition 1.22) this means that there exists a sequence $i = i_0, i_1, \ldots, i_m = i$ in I and elements $((b,f_j),i_j) \in (U_{i_j} \times F) \times \{i_j\}$, $j = 0, \ldots, m$ such that the following are satisfied:

- We have $((b,f_0),i_0) = ((b,f),i)$ and $((b,f_m),i_m) = ((b,f'),i)$.
- For $j = 0, \ldots, m-1$ we have $((b,f_j),i_j) = ((b, \mu_{i_j,i_{j+1}}(b)(f)), i_{j+1})$.

It follows that[2606]

$$f' = f_m = (\mu_{i_{m-1},i_m}(b) \circ \cdots \circ \mu_{i_1,i_2} \circ \mu_{i_0,i_1}(b))(f) \underset{\text{follows from } (*) \text{ applied } m-1 \text{ times}}{\uparrow =} \mu_{ii}(b)(f_0) \underset{\text{it follows from } (*) \text{ that } \mu_{ii}(b) = \mathrm{id}}{\uparrow =} f_0 = f.$$

⊞

(iii) Next we want to show that $\Psi_i \colon U_i \times F \to Y$ is an open map. Thus let $V \subseteq U_i \times F$ be an open subset. Using the fact that the U_j are open subsets of B one can easily show that the set of points in $\bigsqcup_{j \in I} (U_j \times F)$ that are equivalent to a point in V is an

[2606] This discussion gives a proof of Exercise 1.3.

open subset. It follows from this observation and the Definition 8.3 of the quotient topology on Y that the image of V under the map $\Psi_i\colon U_i \times F \to Y$ is an open subset.

(iv) It follows from (i), (ii) and (iii) and the Open-Closed–Injective Lemma 4.22 that Ψ_i is an open embedding.

(v) It remains to show that the image of Ψ_i is $p^{-1}(U_i)$. Clearly the image of Ψ_i is contained in $p^{-1}(U_i)$. Now let $[((b,f),j)] \in p^{-1}(U_i)$. Then $b \in U_i \cap U_j$ and we see that $((b,f),j) \sim (b,\mu_{ji}(b)(f),i)$. It follows that $[((b,f),j)] = [((b,\mu_{ji}(b)(f),i)]$ lies in the image of Ψ_i.

(3) Let $i,j \in I$. It follows easily from the definitions that the map

$$(U_i \cap U_j) \times F \xrightarrow[(b,f)\mapsto[((b,f),i)]]{\Psi_i} p^{-1}(U_i \cap U_j) \xrightarrow{\Psi_j^{-1}} (U_i \cap U_j) \times F$$

is given by the map $(b,f) \mapsto (b,\mu_{ij}(b)(f))$. Since $\mu_{ij}\colon U_i \cap U_j \to \mathcal{G}$ is by hypothesis continuous we see that the trivializations Ψ_i^{-1} form a $\mathcal{G}$-compatible atlas. ■

We also have the following converse to the Bundle–Transition Map Lemma 240.18:

Lemma 240.19. (Bundle-Complete Construction Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Every $(F,\mathcal{G})$-bundle $p\colon X \to B$ admits an isomorphism over B to an $(F,\mathcal{G})$-bundle of the type constructed in the Bundle–Transition Map Lemma 240.18.

Proof. Let $(p\colon X \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ be an $(F,\mathcal{G})$-bundle. By definition of a $\mathcal{G}$-compatible atlas we know that for each $i,j \in I$ there exists a continuous map $\mu_{ij}\colon U_i \cap U_j \to \mathcal{G}$ such that the following diagram commutes:

$$(U_i \cap U_j) \times F \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap U_j) \xrightarrow[\cong]{\Phi_j} (U_i \cap U_j) \times F.$$
$$\text{with } (b,f) \mapsto (b,\mu_{ij}(b)(f)) \text{ on the top arc}$$

First we verify that the maps μ_{ij} satisfy condition (∗) of the Bundle–Transition Map Lemma 240.18 is verified:

Claim 1.

(∗) For every $i,j,k \in I$ and every $b \in U_i \cap U_j \cap U_k$ we have

$$\mu_{jk}(b) \circ \mu_{ij}(b) \;=\; \mu_{ik}(b).$$

Proof. This claim follows easily from the definitions and the observation that

$$\underbrace{(\Phi_k \circ \Phi_j^{-1})}_{\text{defines } \mu_{jk}} \circ \underbrace{(\Phi_j \circ \Phi_i^{-1})}_{\text{defines } \mu_{ij}} \;=\; \underbrace{\Phi_k \circ \Phi_i^{-1}}_{\text{defines } \mu_{ik}}.$$

⊞

It follows from the claim that we can apply the Bundle–Transition Map Lemma 240.18 and we obtain the corresponding $(F,\mathcal{G})$-bundle

$$\begin{aligned} q\colon Y := \Big(\bigsqcup_{i\in I} (U_i \times F)\Big)/((b,f),i) \sim ((b,\mu_{ij}(b)(f)),j) \;&\to\; B \\ [((b,f),i)] \;&\mapsto\; b. \end{aligned}$$

It remains to prove the following claim:

Claim 2. The bundle $q\colon Y \to B$ is isomorphic to the original $(F, \mathcal{G})$-bundle $p\colon Y \to B$.

Proof. The claim follows from the following observations:

(1) The natural maps $p^{-1}(U_i) \xrightarrow{\Phi_i} U_i \times F \to Y$ coincide, by definition, on the overlaps. Thus we get a well-defined map $f\colon X \to Y$. It follows from the Pasting Proposition 4.7 that this map is continuous. It follows easily from the definitions that f is a morphism of $(F, \mathcal{G})$-bundles.

(2) It follows easily from the Topological-Coproduct Proposition 7.26 (1b) and the Topological-Quotient Proposition 8.2 (1b) that the maps $\Phi_i^{-1}\colon U_i \times F \to X$ descend to a well-defined continuous map $g\colon Y \to X$. Again one can easily verify that g is a morphism of $(F, \mathcal{G})$-bundles.

(3) Basically by definition we have $f = g^{-1}$. It now follows from this discussion that f and g are isomorphisms of $(F, \mathcal{G})$-bundles. ∎

We conclude this section with the following lemma, which says that isomorphism classes of $(F, \mathcal{G})$-bundles over a fixed topological space B do not completely get out of control:

Lemma 240.20. (Bundles–Set Lemma) Let F be a topological space that is regionally compact, let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup and let B be a topological space. The isomorphism classes of $(F, \mathcal{G})$-bundles over B form a set.

Proof. This statement follows immediately from the Bundle-Complete Construction Lemma 240.19, modulo the minor nuisance that we never said what a set is. The less this particular author says about set theory, the better. ∎

240.7. **Pullbacks of bundles with structure groups.** We start out with the following refinement of the Bundle Map-Pullback Proposition 240.2:

Proposition 240.21. (Bundle–Pullback Proposition) Let F be some topological space that is regionally compact. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Finally let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ be an $(F, \mathcal{G})$-bundle and let $h\colon C \to B$ be a continuous map between topological spaces. As in the Topological-Pullback Proposition 7.22 and in the Bundle Map-Pullback Proposition 240.2 we consider the pullback

$$\begin{array}{ccc} h^*Y & \xrightarrow{(c,y)\mapsto y} & Y \\ {\scriptstyle q(c,y):=c}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B \end{array}$$

where $h^*Y := \{(c, y) \in C \times Y \mid h(c) = p(y)\}$.

(1) Let $i \in I$. By the Bundle Map-Pullback Proposition 240.2 we know that the map

$$\begin{array}{rcl} \widetilde{\Phi}_i\colon q^{-1}(h^{-1}(U_i)) & \to & h^{-1}(U_i) \times F \\ (c, y) & \mapsto & (c, \underset{\substack{\uparrow \\ \text{here } \pi_F\colon U_i \times F \to F \text{ is the} \\ \text{natural projection}}}{\pi_F}(\Phi_i(y))) \end{array}$$

is an F-trivialization for $q\colon h^*Y \to C$. These trivializations are $\mathcal{G}$-compatible. With this $(F, \mathcal{G})$-bundle structure on $q\colon h^*Y \to C$ the above rectangle becomes a morphism of $(F, \mathcal{G})$-bundles.

(2) If we are given a morphism

$$\begin{array}{ccc} Z & \xrightarrow{H} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B \end{array}$$

of $(F, \mathcal{G})$-bundles, then the map

$$\begin{array}{rccc} T\colon & Z & \to & h^*Y \\ & z & \mapsto & (r(z), H(z)) \end{array}$$

is a natural isomorphism of $(F, \mathcal{G})$-bundles over C.

Proof.

(1) We need to show that the trivializations $\{\widetilde{\Phi}_i\}_{i\in I}$ of the map $q\colon h^*Y \to B$ are $\mathcal{G}$-compatible. Thus let $i, j \in I$. Since the trivializations Φ_i and Φ_j are by hypothesis $\mathcal{G}$-compatible we know that there exists a continuous map $\Theta\colon U_i \cap U_j \to \mathcal{G}$ such that the following diagram commutes:[2607]

$$(U_i \cap U_j) \times F \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap U_j) \xrightarrow[\cong]{\Phi_j} (U_i \cap U_j) \times F.$$
$$(b,f) \mapsto (b, \Theta(b)(f))$$

We now consider the map

$$\begin{array}{rccc} \Theta \circ h\colon & h^{-1}(U_i) \cap h^{-1}(U_j) & \to & \mathcal{G} \\ & c & \mapsto & \Theta(h(c)). \end{array}$$

One can now easily verify that the following diagram does indeed commute:

$$(h^{-1}(U_i) \cap h^{-1}(U_j)) \times F \xleftarrow[\cong]{\widetilde{\Phi}_i} p^{-1}(h^{-1}(U_i) \cap h^{-1}(U_j)) \xrightarrow[\cong]{\widetilde{\Phi}_j} (h^{-1}(U_i) \cap h^{-1}(U_j)) \times F.$$
$$(c,f) \mapsto (c, \Theta(h(c))(f))$$

(2) Let $(r\colon Z \to C, \{\Psi_j\colon r^{-1}(V_j) \to V_j \times F\}_{j\in J})$ be an $(F, \mathcal{G})$-bundle and suppose that we are given a morphism

$$\begin{array}{ccc} Z & \xrightarrow{H} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B \end{array}$$

of $(F, \mathcal{G})$-bundles.

Claim. The map

$$\begin{array}{rccc} T\colon & Z & \to & h^*Y \\ & z & \mapsto & (r(z), H(z)) \end{array}$$

is an isomorphism of $(F, \mathcal{G})$-bundles over C.

Proof.

(a) First note that it follows easily from the Topological-Product Proposition 7.2 (1) that the map $T\colon Z \to h^*Y$ is continuous.

(b) We claim that (T, id_C) is a morphism of $(F, \mathcal{G})$-bundles. By hypothesis $H\colon Z \to Y$ is a morphism of $(F, \mathcal{G})$-bundles. Recall that by Definition 240.12 this means that for each $i \in I$ and $j \in J$ there exists a continuous map $\Theta\colon V_j \cap h^{-1}(U_i) \to \mathcal{G}$ such

that the following diagram commutes:

$$(V_j \cap h^{-1}(U_i)) \times F \xleftarrow[\cong]{\Psi_j} r^{-1}(V_j \cap h^{-1}(U_i)) \xrightarrow{H} p^{-1}(U_i) \xrightarrow[\cong]{\Phi_i} U_i \times F.$$
$$(c,f) \mapsto (h(c), \Theta(c)(f))$$

Now we turn to the maps $T\colon Z \to h^*Y$. Note that for any $i \in I$ and $j \in J$ we have the following diagram:

$$(V_j \cap h^{-1}(U_i)) \times F \xleftarrow[\cong]{\Psi_j} r^{-1}(V_j \cap h^{-1}(U_i)) \xrightarrow[z \mapsto (q(z), H(z))]{T} q^{-1}(V_j \cap h^{-1}(U_i)) \xrightarrow[\cong]{\widetilde{\Phi}_i} (V_j \cap h^{-1}(U_i)) \times F.$$
$$(c,f) \mapsto (c, \Theta(c)(f))$$

We draw the following conclusions:

- One can easily verify that the diagram commutes.
- It follows from the Bundle–Action-on-Trivializations Proposition 240.17 that the bottom horizontal map is an isomorphism of $(F, \mathcal{G})$-bundles.

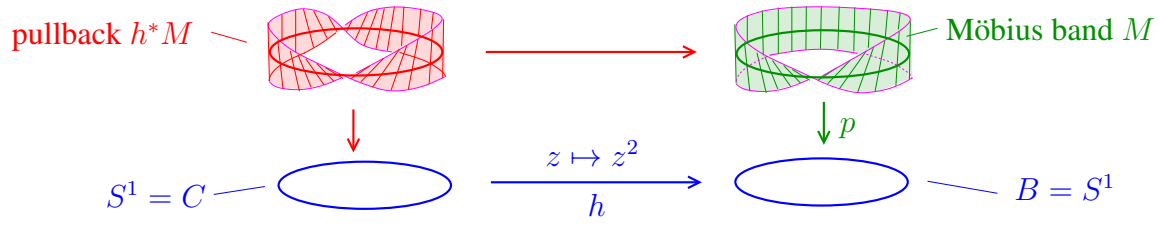
(c) The discussion in (b) shows that $T\colon Z \to h^*Y$ is an $(F, \mathcal{G})$-bundle morphism which furthermore is a local isomorphism. Since we are dealing with a morphism over id_C we see, similar to the Bijective-Local Homeomorphism Lemma 4.62, that the morphism is in fact an isomorphism of $(F, \mathcal{G})$-bundles. ■

The Bundle–Pullback Proposition 240.21 allows us to introduce the following definition:

Definition 240.22. Let F be a topological space that is regionally compact. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup, let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i \in I})$ be an $(F, \mathcal{G})$-bundle and let $h\colon C \to B$ be a continuous map between topological spaces. Using the notation from the Bundle–Pullback Proposition 240.21 we refer to the $(F, \mathcal{G})$-bundle $(q\colon h^*Y \to C, \{\widetilde{\Phi}_i\colon q^{-1}(h^{-1}(U_i)) \to h^{-1}(U_i) \times F\}_{i \in I})$ as the pullback of the $(F, \mathcal{G})$-bundle $p\colon Y \to B$ under the map $h\colon C \to B$.

We start out with a familiar explicit example:

Example. Let $p\colon M \to S^1$ be the Möbius $[-1, 1]$-bundle from page 4580. We denote by $h\colon S^1 \to S^1$ the map that is given by $h(z) = z^2$.



It follows from the Bundle–Pullback Proposition 240.21 (2) and the discussion on page 4638 that the pullback h^*M is trivializable. □

Next we consider more theoretical examples:

Example. Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) Let $p\colon Y \to B$ be an $(F,\mathcal{G})$-bundle, let $C \subseteq B$ be a subspace and let $\iota\colon C \to B$ be the inclusion. It follows from the Bundle–Pullback Proposition 240.21 (2) that the maps

$$\begin{array}{rcl} \iota^*(Y) & \to & p^{-1}(C) \\ (c,y) & \mapsto & y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p^{-1}(C) & \to & \iota^*(Y) \\ y & \mapsto & (p(y),y) \end{array}$$

give natural isomorphisms between the pullback $(F,\mathcal{G})$–bundle $\iota^*(Y)$ and the restriction $(F,\mathcal{G})$-bundle $p|_{p^{-1}(C)}\colon p^{-1}(C) \to C$, as defined on page 4635.

(2) Let B be a topological space and let $p\colon B \times F \to B$ be the trivial $(F,\mathcal{G})$-bundle. It follows from the discussion on page 4628 that the pullback of p under any continuous map $h\colon C \to B$ is naturally isomorphic to the trivial $(F,\mathcal{G})$-bundle $C \times F \to F$.

(3) Let $p\colon Y \to B$ be an $(F,\mathcal{G})$-bundle and let $\pi\colon B \times C \to B$ be the natural projection. One can easily verify that the maps

$$\begin{array}{rcl} \pi^*Y & \to & Y \times C \\ ((b,c),y) & \to & (y,c) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} Y \times C & \to & \pi^*Y \\ (y,c) & \mapsto & ((p(y),c),y) \end{array}$$

are isomorphisms of $(F,\mathcal{G})$-bundles. This discussion can be summarized in the following diagram:

$$\begin{array}{ccc} \pi^*Y \cong Y \times C & \xrightarrow{(y,c)\mapsto y} & Y \\ {\scriptstyle (y,c)\mapsto (p(y),c)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B \times C & \xrightarrow{\pi} & B. \end{array}$$

□

Lemma 240.23. (Bundle-Pullback-Functor Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. For any continuous map $h\colon C \to B$ between topological spaces the maps

$$\begin{array}{ccc} Y \\ \downarrow{\scriptstyle p} \\ B \end{array} \mapsto \begin{array}{ccc} h^*Y \\ \downarrow{\scriptstyle q} \\ C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \xrightarrow{\varphi} & Y' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array} \mapsto \begin{array}{ccc} h^*Y & \xrightarrow{(c,y)\mapsto(c,\varphi(y))} & h^*Y' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & C & \end{array}$$

define a covariant functor from the category $\mathcal{B}\mathit{und}_{(F,\mathcal{G})}(B)$ of $(F,\mathcal{G})$-bundles over B to the category $\mathcal{B}\mathit{und}_{(F,\mathcal{G})}(C)$ of $(F,\mathcal{G})$-bundles over C.

Proof. The lemma follows from the following basic observations:

- By the Bundle–Pullback Proposition 240.21 we know that $q\colon h^*Y \to C$ is an $(F,\mathcal{G})$-bundle.
- For the identity map $\mathrm{id}\colon Y \to Y$ the induced map $h^*Y \to h^*Y$ is also the identity.
- One can easily verify that if $\varphi\colon Y \to Y'$ is an $(F,\mathcal{G})$-morphism, then so is also the given induced map $h^*Y \to h^*Y'$.
- For two bundles maps $\varphi\colon Y \to Y'$ and $\varphi'\colon Y' \to Y''$ the corresponding induced map $h^*Y \to h^*Y''$ is given by $(c,y) \mapsto (c,\varphi'(\varphi(y))) = (c,(\varphi' \circ \varphi)(y))$. ■

Example. Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

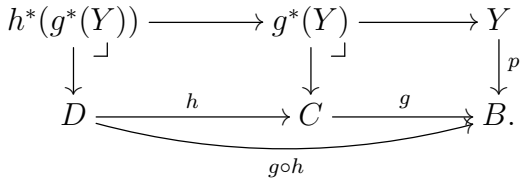
(1) It follows from the Bundle-Pullback-Functor Lemma 240.23 together with the Isomorphism Preservation Lemma 5.26 that the pullbacks of isomorphic $(F, \mathcal{G})$-bundles are again isomorphic.
(2) It follows from (1) and the fact that the pullback of a trivial bundle is naturally trivializable (see page 4647) that the pullback of a trivializable $(F, \mathcal{G})$-bundle is again a trivializable $(F, \mathcal{G})$-bundle. □

The following lemma says that the pullback construction is "contravariantly functorial", at least in a loose sense.

Lemma 240.24. (Bundle-Pullback–Functorial Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) Let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle. Then
$$\begin{array}{rcl} (\mathrm{id}_B)^*Y & \to & Y \\ (b, y) & \mapsto & y \end{array}$$
is a natural isomorphism from the pullback $(F, \mathcal{G})$-bundle $(\mathrm{id}_B)^*Y \to B$ to the original $(F, \mathcal{G})$-bundle $p\colon Y \to B$.
(2) Let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle and let $g\colon C \to B$ and $h\colon D \to C$ be continuous maps between topological spaces. The map
$$\begin{array}{rcl} h^*(g^*Y) & \to & (g \circ h)^*Y \\ (d, (c, y)) & \mapsto & (d, y) \end{array}$$
is a natural isomorphism from the double pullback $(F, \mathcal{G})$-bundle $h^*(g^*Y) \to B$ to the pullback $(F, \mathcal{G})$-bundle $(g \circ h)^*Y \to B$.

Proof. We claim that both statements follow easily from the Bundle–Pullback Proposition 240.21. We provide the details for (2). Let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle and let $g\colon C \to B$ and $h\colon D \to C$ be continuous maps between topological spaces. We consider the following diagram:



We make the following clarifications and observations:

(a) The smaller rectangles are the commutative diagrams given by the pullbacks.
(b) It follows from (a) and the Bundle–Category Lemma 240.13 that the big rectangle is a morphism of $(F, \mathcal{G})$-bundles.
(c) Since the bottom map equals $g \circ h\colon D \to B$ we obtain from the Bundle–Pullback Proposition 240.21 (2) that the $(F, \mathcal{G})$-bundle $h^*(g^*(Y)) \to D$ is naturally isomorphic to the $(F, \mathcal{G})$-bundle $(g \circ h)^*Y \to D$. In fact the natural isomorphism from the Bundle–Pullback Proposition 240.21 (2) is precisely the one we wrote down in the lemma. ■

240.8. Exercises for Chapter 240.

Exercise 240.1. Let F be a topological space and let $\mu\colon F \to F$ be a homeomorphism. We consider the mapping torus $\operatorname{Tor}(F,\mu)$ with the natural F-bundle map $\operatorname{Tor}(F,\mu) \to S^1$. Let $n \in \mathbb{Z}$ and let $h\colon S^1 \to S^1$ be the continuous map that is given by $h(z) := z^n$. Show that the total space of pullback bundle $h^*\operatorname{Tor}(F,\mu)$ is homeomorphic to a mapping torus $\operatorname{Tor}(F',\mu')$ and describe F' and μ' explicitly.

Exercise 240.2. Let X and Y be topological spaces. Suppose that X is Hausdorff. Show that X^Y is also Hausdorff.
Remark. This exercise is identical to Exercise 16.2.

Exercise 240.3. Let X and Y be non-empty topological spaces. We consider the following map:

$$\begin{array}{rcl} \varphi\colon X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

(a) Show that φ is continuous.
(b) Show that φ is an embedding.

Remark. This exercise is identical to Exercise 16.3.

Exercise 240.4. Let X and Y be two discrete topological spaces. Is the compact-open topology on X^Y the discrete topology on the set X^Y?
Remark. This exercise is identical to Exercise 16.1.

Exercise 240.5.

(a) In the Hopf-Bundle Proposition 238.4 (1) we considered the Hopf map $H_{\mathbb{C}}\colon S^3 \to S^2$ and in the proof we gave an explicit atlas $\tau_j\colon (H_{\mathbb{C}})^{-1}(U_j) \to U_j \times S^1$, $j = 0, 1$. Show that the atlas is $(S^1, \mathrm{SO}(2))$-compatible.
(b) In the Hopf-Bundle Proposition 238.4 (2) we considered the Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$ and in the proof we gave an explicit atlas $\tau_j\colon (H_{\mathbb{H}})^{-1}(U_j) \to U_j \times S^3$, $j = 0, 1$.
- Show that the atlas is not $(S^3, \mathrm{SO}(4))$-compatible.
- For $j = 0, 1$ we compose τ_j with the self-diffeomorphism of $U_j \times S^3$ that is given by $(z, v) \mapsto (z, v^{-1})$. Show that the resulting atlas is $(S^3, \mathrm{SO}(4))$-compatible.

Exercise 240.6. Let F be a topological space that is regionally compact, Hausdorff and locally connected. Let $\mathcal{G} = \{\mathrm{id}_F\}$ be the group consisting just of the identity. Show that every $(F, \mathcal{G})$-bundle is trivializable.

Exercise 240.7. Let F be a topological space that is regionally compact, Hausdorff and locally connected. Furthermore let $\mu_0, \mu_1\colon F \to F$ be homeomorphisms. Suppose that μ_0 and μ_1 are isotopic. Show that there exists a homeomorphism $\Theta\colon \operatorname{Tor}(F,\mu_0) \to \operatorname{Tor}(F,\mu_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Tor}(F,\mu_0) & \xrightarrow[\cong]{\Theta} & \operatorname{Tor}(F,\mu_1) \\ & \searrow \quad \swarrow & \\ & S^1. & \end{array}$$

Exercise 240.8. In the first proof of the Projective Space-minus-Ball Lemma 238.11 on page page 4588 we gave various atlases for bundles. Show that the following statements hold:

(a) The atlas for the $\overline{B}^1$-bundle $p\colon \mathbb{R}\mathrm{P}^n \setminus \Theta(B^n) \to \mathbb{R}\mathrm{P}^{n-1}$ is $\mathrm{O}(1)$-compatible.

(b) The atlas for the $\overline{B}^2$-bundle $p\colon \mathbb{C}\mathrm{P}^n \setminus \Theta(B^{2n}) \to \mathbb{C}\mathrm{P}^{n-1}$ is $\mathrm{SO}(2)$-compatible.
(c) The atlas for the $\overline{B}^4$-bundle $p\colon \mathbb{H}\mathrm{P}^n \setminus \Theta(B^{4n}) \to \mathbb{C}\mathrm{P}^{n-1}$ is $\mathrm{SU}(2)$-compatible.

Exercise 240.9. In the first proof of the Projective Space-minus-Ball Lemma 238.11 on page page 4588 we gave various atlases for bundles. Show that the following statements hold:

(a) Show that the $(\overline{B}^1, \mathrm{O}(1))$-bundle $p\colon \mathbb{R}\mathrm{P}^2 \setminus \Theta(B^2) \to \mathbb{R}\mathrm{P}^1 =_{\mathrm{i}} S^1$ is non-trivial.
(b) Show that the $(\overline{B}^2, \mathrm{SO}(2))$-bundle $p\colon \mathbb{C}\mathrm{P}^2 \setminus \Theta(B^4) \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ is non-trivial.
(c) Show that the $(\overline{B}^4, \mathrm{SO}(4))$-bundle $p\colon \mathbb{H}\mathrm{P}^2 \setminus \Theta(B^8) \to \mathbb{H}\mathrm{P}^1 =_{\mathrm{i}} S^4$ is non-trivial.

Hint. In each case use the explicit atlas to determine the clutching map.
Remark. This exercise can be viewed as an extension of Exercise 240.9.

Exercise 240.10. Let $n \in \mathbb{N}$. We refer to the maps

$$\begin{aligned} p\colon S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \,|\, |z_0|^2 + \dots + |z_n|^2 = 1\} &\to \mathbb{C}\mathrm{P}^n \\ (z_0, \dots, z_n) &\mapsto [z_0 : \dots : z_n] \end{aligned}$$

and[2608]

$$\begin{aligned} q\colon S^{4n+3} = \{(h_0, \dots, h_n) \in \mathbb{H}^{n+1} \,|\, |h_0|^2 + \dots + |h_n|^2 = 1\} &\to \mathbb{H}\mathrm{P}^n \\ (h_0, \dots, h_n) &\mapsto [h_0 : \dots : h_n] \end{aligned}$$

as *Hopf maps* Let $n \in \mathbb{N}$.

(1) The Hopf map $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ is an $(S^0, \mathrm{O}(1))$-bundle.
(2) The Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an $(S^1, \mathrm{SO}(2))$-bundle.
(3) The Hopf map $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an $(S^3, \mathrm{SO}(4))$-bundle.

Exercise 240.11. In the Projective Space–Connected Sum Proposition 238.10 we gave various atlas for bundles. Show that the following statements hold:

(a) The atlas for the S^1-bundle $p\colon \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \to S^1$ is $\mathrm{O}(2)$-compatible.
(b) The atlas for the S^2-bundle $p\colon \mathbb{C}\mathrm{P}^2 \# (-\mathbb{C}\mathrm{P}^2) \to S^2$ is $\mathrm{SO}(3)$-compatible.
(c) The atlas for the S^4-bundle $p\colon \mathbb{H}\mathrm{P}^2 \# (-\mathbb{H}\mathrm{P}^2) \to S^4$ is $\mathrm{SO}(5)$-compatible.

Remark. Make use of Exercise 240.9.

Exercise 240.12. Let M be a smooth manifold and let $n \in \mathbb{N}_0$.

(a) Show that the set of isomorphism classes of n-dimensional vector bundles over M is not necessarily countable.
(b) Now suppose that M is connected. Is the set of isomorphism classes of n-dimensional vector bundles over M necessarily countable?

Exercise 240.13. Let $p\colon M \to S^1$ be the Möbius $[-1, 1]$-bundle which we introduced on page 4580. Given $n \in \mathbb{Z}$ let $f\colon S^1 \to S^1$ be the map that is given by $f(z) := z^n$. For which $n \in \mathbb{Z}$ is the pullback $[-1, 1]$-bundle f^*M trivializable?

Exercise 240.14.

(a) Let $p\colon Y \to B$ be an n-dimensional real vector bundle. Show that Y is homotopy equivalent to B.
(b) Does the proof also work if $p\colon Y \to B$ is an $\mathbb{R}^n$-bundle?

Exercise 240.15. Give examples of $(F, \mathcal{G})$-bundles $p\colon X \to B$ and $q\colon Y \to B$ which are isomorphic, in the sense that there exists an isomorphism of $(F, \mathcal{G})$-bundles

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & B, \end{array}$$

but which are not isomorphic over B, i.e. we cannot find an isomorphism with $h = \mathrm{id}_B$.
Hint. You could consider bundles over product spaces $B = B_1 \times B_2$.

Exercise 240.16. Let F be a topological space that is regionally compact. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Suppose we are given the following morphism of $(F, \mathcal{G})$-bundles:

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C. \end{array}$$

Show that if h is a homeomorphism, then the above is an isomorphism of $(F, \mathcal{G})$-bundles.

Exercise 240.17. Let $p\colon Y \to B$ be a $(\overline{B}^k, \mathrm{O}(k))$-bundle.

(a) Show that there exists a continuous map $s\colon B \to Y$ with $p \circ s = \mathrm{id}_B$.
(b) Show that Y is homotopy equivalent to B.

Exercise 240.18. Show that for every odd n and every n-dimensional real vector bundle $p\colon Y \to B$ there exists an automorphism of the bundle over B which is orientation-reversing (i.e. which has negative determinant) on every fiber?
Remark. This exercise is a companion to Exercise 240.20.

Exercise 240.19. Let $n \in \mathbb{N}$. Furthermore let $(p\colon Y \to B, \{\Phi_i\colon p^{-1} \to U_i \times \mathbb{R}^n\}_{i \in I})$ be an $(\mathbb{R}^n, \mathrm{GL}_+(n, \mathbb{R}))$-bundle. Let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be a reflection in a hyperplane. For each $i \in I$ we consider the map

$$\widetilde{\Phi}_i\colon p^{-1}(U_i) \xrightarrow{\Phi_i} U_i \times \mathbb{R}^n \xrightarrow{(x,f) \mapsto (x, \rho(f))} U_i \times \mathbb{R}^n.$$

(a) Show that $(p\colon Y \to B, \{\widetilde{\Phi}_i\colon p^{-1} \to U_i \times \mathbb{R}^n\}_{i \in I})$ is an $(\mathbb{R}^n, \mathrm{GL}_+(n, \mathbb{R}))$-bundle.
(b) Show that the two $(\mathbb{R}^n, \mathrm{GL}_+(n, \mathbb{R}))$-bundles are not isomorphic.
(c) Since $\mathrm{GL}_+(n, \mathbb{R})$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ we can view both of the above bundles also as $(\mathbb{R}^n, \mathrm{GL}(n, \mathbb{R}))$-bundles. Show that the two bundles are isomorphic, if they are viewed as $(\mathbb{R}^n, \mathrm{GL}(n, \mathbb{R}))$-bundles.

Exercise 240.20. Let $A \in \mathrm{GL}(n, \mathbb{R})$. We consider the corresponding $(\mathbb{R}^n, \mathrm{GL}(n, \mathbb{R}))$-bundle $p\colon \mathrm{Tor}(\mathbb{R}^n, A) \to S^1$.

(a) Show that if there exists an automorphism of the bundle over S^1 which is orientation-reversing (i.e. which has negative determinant) on every fiber, then there exists a matrix $B \in \mathrm{GL}(n, \mathbb{R})$ with $\det(B) < 0$ and with $AB = BA$.
(b) Show that any $B \in \mathrm{GL}(n, \mathbb{R})$ with $\det(B) < 0$ has at least one real eigenvalue.
(c) Show that if $A \in \mathrm{GL}(n, \mathbb{R})$ has only complex eigenvalues, then there is no $B \in \mathrm{GL}(n, \mathbb{R})$ with $\det(B) < 0$ and with $AB = BA$.

(d) Show that for every even n there exists an n-dimensional real vector bundle $p\colon Y \to S^1$ which does not admit an automorphism of the bundle over S^1 which is orientation-reversing on every fiber.

Remark. This exercise is a companion to Exercise 240.18.

Exercise 240.21. Let F, Y and B be smooth manifolds. Furthermore suppose that we are given an $(F, \mathrm{Homeo}(F))$-bundle $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ such that all the Φ_i are diffeomorphisms. Show that the charts are $\mathrm{Diffeo}(F)$-compatible.

Exercise 240.22. Let $p\colon \widetilde{Y} \to Y$ be a regular covering of topological spaces where $\widetilde{Y}$ is connected. Furthermore let X be a topological space that is locally contractible and that is path-connected. Give a sufficient and necessary condition for a continuous map $f\colon X \to Y$ to have the property that the pullback bundle $f^*\widetilde{Y} \to X$ is trivializable.

Exercise 240.23. Let F be a topological space that is regionally compact. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup, let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle and let $S \subseteq B$ be a subspace. Can we extend every $(F, \mathcal{G})$-bundle isomorphism $Y|_S \to Y|_S$ to an $(F, \mathcal{G})$-bundle isomorphism $Y \to Y$?

Exercise 240.24. Let F be a topological space that is regionally compact. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup and let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle. We suppose that there exists an $(F, \mathcal{G})$-bundle isomorphism

$$\begin{array}{ccc} Y & \xrightarrow{H} & C \times F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle (b,f)\mapsto f} \\ B & \xrightarrow{h} & C. \end{array}$$

Does there also exist an isomorphism of $(F, \mathcal{G})$-bundles from the $(F, \mathcal{G})$-bundle Y to the trivial bundle $B \times F \to F$, such that the bottom horizontal map is the identity?

Exercise 240.25. Let B and F be oriented smooth manifolds with empty boundary. We denote by $\mathrm{Diffeo}^+(F)$ the group of orientation-preserving self-diffeomorphisms of F. Let Y be a smooth manifold and let $p\colon Y \to B$ be a smooth $(F, \mathrm{Diffeo}^+(F))$-bundle. Show that the total space Y admits a natural orientation.

Exercise 240.26.

(a) Let $n \in \mathbb{N}_0$ and let $p\colon Y \to B$ be an $(\mathbb{R}^n, \mathrm{Homeo}(\mathbb{R}^n))$-bundle. Show that there exists an $(S^n, \mathrm{Homeo}(S^n))$-bundle $q\colon Z \to B$ and an embedding $i\colon Y \to Z$ such that $p = q \circ i$.
Hint. Make use of the fact that $S^n \cong \mathbb{R}^n \cup \{\infty\}$.

(b) What happens in (a) if we start out with the "Möbius" $\mathbb{R}$-bundle

$$\begin{array}{rcl} p\colon (\mathbb{R} \times [0,1])/(x,0) \sim (-x,1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t)\,? \end{array}$$

What S^1-bundle do we obtain?

Exercise 240.27. Let $H\colon S^3 \to S^2$ be the Hopf map. By the Hopf-Bundle Proposition 238.4 we know that this is an S^1-bundle. Thus it makes sense to consider the pullback S^1-bundle $H^*S^3 \to S^3$. Is this bundle trivializable?

Exercise 240.28. We want to find out whether the analog of the Bundle–Pullback Proposition 240.21 for F-bundles holds. More precisely: Let F be a topological space, let $p\colon Y \to B$ be an F-bundle, let $q\colon Z \to C$ be an F-bundle and suppose that we have continuous maps $H\colon Z \to Y$ and $h\colon C \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{H} & Y \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B. \end{array}$$

Does it follow that Z is homeomorphic to h^*Y?

Exercise 240.29. Let F be a topological space that is regionally compact. Furthermore let B be a topological space and let $C \subseteq B$ be a subspace that is a retract of B. We denote by $\iota\colon C \to B$ be the inclusion map. Does ι induce a surjection/injection on the set of isomorphism classes of $(F, \mathcal{G})$-bundles?

Exercise 240.30. Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Given a topological space B we denote by $\mathrm{IsoBund}_{(F,\mathcal{G})}(B)$ the set[2609] of isomorphism classes of $(F, \mathcal{G})$-bundles over B. We consider the wedge $S^1_a \vee S^1_b$ be the wedge of two circles. We denote by $i_a\colon S^1_a \to S^1_a \vee S^1_b$ and $i_b\colon S^1_a \to S^1_a \vee S^1_b$ the two natural embeddings. We obtain a map

$$\begin{array}{rcl} \mathrm{IsoBund}_{(F,\mathcal{G})}(S^1_a \vee S^1_b) & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(S^1_a) \times \mathrm{IsoBund}_{(F,\mathcal{G})}(S^1_b) \\ [p\colon Y \to S^1_a \vee S^1_b] & \mapsto & ([i_a^*Y], [i_b^*Y]). \end{array}$$

Is this map injective?

Exercise 240.31. Let $n \in \mathbb{N}_{\geq 2}$. We refer to the mapping torus of the reflection

$$\begin{array}{rcl} \overline{B}^{n-1} & \to & \overline{B}^{n-1} \\ (x_1, x_2, \ldots, x_n) & \mapsto & (-x_1, x_2, \ldots, x_n) \end{array}$$

as an n-dimensional *generalized Möbius band*. Let M be an n-dimensional smooth manifold. Show that M is orientable if and only if there is no smooth embedding of the n-dimensional generalized Möbius band into M.

Hint. Use the Definition 95.18 of an orientation-reversing loop, use the Tubular Neighborhood–Existence Theorem 57.11, use the Represent-by-1-Manifold Corollary 187.19 and use the classification of $(\overline{B}^{n-1}, \mathrm{O}(n-1))$-bundles over S^1.

CHAPTER 241

Bundles and homotopies

In this chapter we will show that, under mild hypotheses, given an $(F, \mathcal{G})$-bundle $p\colon Y \to B$ and a topological space C, two *homotopic* maps $h_0, h_1\colon C \to B$ give rise to *isomorphic* pullback bundles h_0^*Y and h_1^*Y. This result is fundamental in any attempt to classify bundles and in this chapter we will indeed manage to classify some types of $(F, \mathcal{G})$-bundles.

This chapter is organized as follows:

- In Section 241.1 we will recall the notion of a paracompact topological space.
- In Section 241.2 we will see that if $p\colon Y \to B$ is an $(F, \mathcal{G})$-bundle, if $h_0, h_1\colon C \to B$ are two homotopic maps and if C is Hausdorff and paracompact, then the pullbacks h_0^*Y and h_1^*Y are isomorphic $(F, \mathcal{G})$-bundles.
- In Section 241.3 we will see that a homotopy equivalence induces a bijection of isomorphisms types of $(F, \mathcal{G})$-bundles. In particular we will see that $(F, \mathcal{G})$-bundles over contractible spaces are trivializable.
- In Section 241.4 we will give a (new) proof that $(F, \mathcal{G})$-bundles over S^1 correspond precisely to mapping tori and we will study bundles over 1-dimensional CW-complexes.
- In Section 241.5 we will give a characterization of $(F, \mathcal{G})$-bundles over spheres.
- In Section 241.6 we will use the results from Section 241.5 to prove a few general facts about bundles over certain spheres.
- In Section 241.7 we will use the results from Section 241.5 to give a classification of some types of bundles over certain spheres.
- In Section 241.8 we will study bundles over surfaces.
- In Section 241.9 we will provide exercises for Chapter 241.

In the next Chapter 242 we will prove some results about the topology of total spaces.

241.1. **Interlude: Paracompact spaces.** Before we can talk about bundles and homotopies, we have to (re-) introduce the notion of a paracompact topological space.

First we recall the following Definition 3.19:

Definition 241.1. Let X be a topological space.

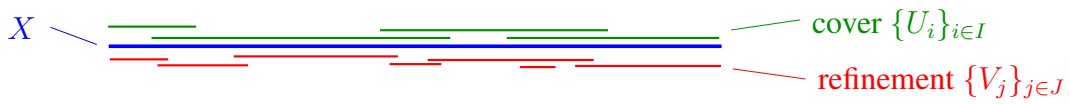
(1) A cover of X is a family $\{U_i\}_{i\in I}$ of subsets of X with $\bigcup_{i\in I} U_i = X$.
(2) We say that a cover $\{U_i\}_{i\in I}$ is open if each U_i is open.
(3) We say that a family $\{U_i\}_{i\in I}$ of subsets is locally finite if each $x \in X$ admits an open neighborhood W such that $\{i \in I \,|\, U_i \cap W \neq \varnothing\}$ is finite.

We now add one and a half definitions in the context of covers:

Definition 241.2. Let X be a topological space and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two covers of X.

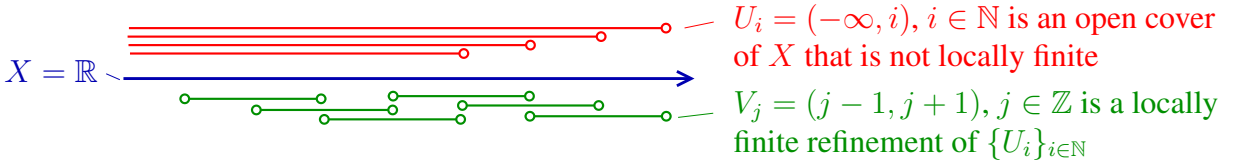
(1) We say that $\{V_j\}_{j\in J}$ is a refinement of $\{U_i\}_{i\in I}$ if each V_j is contained in some U_i.

(2) If furthermore each V_j is open, then we say that $\{V_j\}_{j\in J}$ is an open refinement of $\{U_i\}_{i\in I}$.



Now we can recall the definition of a paracompact space from Definition 14.3:

Definition 241.3. A topological space X is called paracompact if every open cover of X admits a locally finite open refinement.



Example. Evidently every compact space is paracompact. □

In the following we recall two main properties of paracompact spaces. Afterwards we will recall that many of the topological spaces that we care about are actually paracompact.

Theorem 14.15 (Paracompact-Partition of Unity Theorem) Let X be a topological space that is *Hausdorff*. The following two statements are equivalent:

(1) The topological space X is paracompact.
(3) For every open cover $\{U_i\}_{i\in I}$ of X we can find a partition of unity $\{\varphi_i\colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have $\operatorname{supp}(\varphi_i) \subseteq U_i$.[2610]

Next we recall another useful property of paracompact spaces:

Proposition 14.17 (Countable Cover Proposition) Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. If X is Hausdorff and paracompact, then there exists a *countable* locally finite open cover $\{V_j\}_{j\in J}$ where each V_j is the disjoint union of open subsets of X, each of which is contained in some U_i.

Example. We consider the topological space $X = \bigsqcup_{i\in\mathbb{R}} (-2,2)$ which is the disjoint union of $\mathbb{R}$-copies of the interval $(-2,2)$. We consider the open cover that is given by the family $U_{i-} := (-2,1) \times \{i\}$, $U_{i+} := (-1,2) \times \{i\}$ with $i \in \mathbb{R}$. The finite family consisting of the two sets $V_- := \bigsqcup_{i\in I} U_{i-}$ and $V_+ := \bigsqcup_{i\in I} U_{i+}$ has the properties of the Countable Cover Proposition 14.17. □

Fortunately we have already seen that many interesting topological spaces are Hausdorff and paracompact. In the following proposition we summarize some of our earlier results in this direction:

[2610] Recall that according to Definition 14.14 a partition of unity on a topological space X is a family of continuous maps $\{\varphi_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:

(a) For each $x \in X$ there exists an open neighborhood U of x such that φ_i vanishes on U for all but finitely many i.
(b) For every $x \in X$ we have $\sum_{i\in I} \varphi_i(x) = 1$.

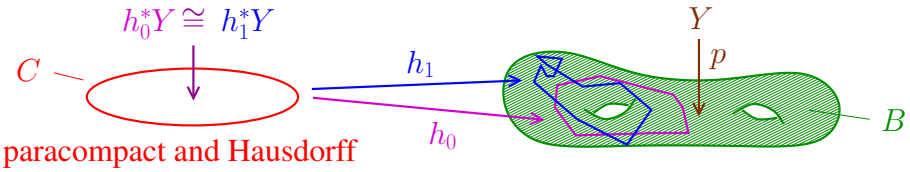
Proposition 241.4. (Manifold+CW-Complex–Paracompact Proposition)
(1) Every topological manifold is Hausdorff and paracompact.
(2) Every CW-complex is Hausdorff and paracompact.[2611]

Proof.

(1) Every topological manifold is by definition Hausdorff. In the Manifold Paracompact-Normal Proposition 23.9 (1) we showed that every topological manifold is paracompact.[2612]
(2) By the CW-Complex–Hausdorff Proposition 136.6 and the CW-Complex–Paracompact Proposition 136.33 we know that all CW-complexes are Hausdorff and paracompact. ■

241.2. **Pullbacks and homotopies.** The following proposition is one of the key statements in the theory of bundles:

Proposition 241.5. (Pullback-Homotopy Proposition) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Let $p\colon Y \to B$ be an $(F,\mathcal{G})$-bundle, let C be a topological space and finally let $h_0, h_1\colon C \to B$ be two continuous maps. If h_0 and h_1 are homotopic and if C is Hausdorff and paracompact, then the $(F,\mathcal{G})$-bundles $h_0^*Y \to C$ and $h_1^*Y \to C$ are isomorphic over C.



The key to proving the Pullback-Homotopy Proposition 241.5 is the following lemma:

Lemma 241.6. (Bundle-over-Product Lemma) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Let C be a topological space that is Hausdorff and paracompact. Given $t \in [0,1]$ we denote by

$$\begin{array}{rccc} i_t\colon & C & \to & C \times [0,1] \\ & c & \mapsto & (c,t) \end{array}$$

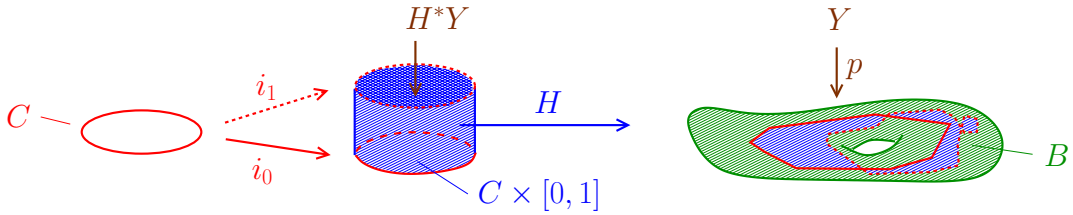
the obvious embedding. For any $(F,\mathcal{G})$-bundle $p\colon Y \to C \times [0,1]$ the pullbacks i_0^*Y and i_1^*Y are isomorphic over C.

Remark. Note that the Bundle-over-Product Lemma 241.6 does not say that the $(F,\mathcal{G})$-bundles i_0^*Y and i_1^*Y are *naturally* isomorphic. In fact, as we will see in Exercise 241.3, the isomorphism we provide depends on choices. □

Before we provide the proof of the Bundle-over-Product Lemma 241.6, let us first show that the Pullback-Homotopy Proposition 241.5 is a pretty straightforward consequence of the Bundle-over-Product Lemma 241.6:

[2611] To the author's eternal puzzlement one does not need any hypothesis (like second countable) on the CW-complex.
[2612] Note that for this proof we used for once that topological manifolds are by definition second-countable.

Proof of Proposition 241.5 assuming Lemma 241.6. Let $p\colon Y \to B$ be an $(F,\mathcal{G})$-bundle and let C be a topological space that is Hausdorff and paracompact. Furthermore let $H\colon C \times [0,1] \to B$ be a homotopy between two maps $h_0, h_1\colon C \to B$. As in the Bundle-over-Product Lemma 241.6 we denote by $i_0, i_1\colon C \to C \times [0,1]$ the two obvious embeddings.



We have the following isomorphisms of $(F,\mathcal{G})$-bundles over C:

$$\underset{\text{since } h_0 = H\circ i_0}{h_0^*Y \;=\; (H\circ i_0)^*Y} \;\xleftarrow{\cong}\; \underset{\text{by the Bundle-over-Product Lemma 241.6}}{i_0^*(H^*Y) \;\cong\; i_1^*(H^*Y)} \;\xrightarrow{\cong}\; \underset{\text{since } h_1 = H\circ i_1}{(H\circ i_1)^*Y \;=\; h_1^*Y.}$$

(natural isomorphisms from the Bundle-Pullback–Functorial Lemma 240.24 (2))

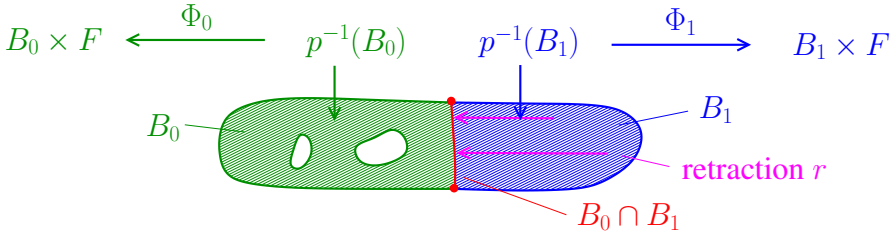
The proof of the Bundle-over-Product Lemma 241.6 will require the remainder of this section. We will continue in Section 241.3 with applications of the Pullback-Homotopy Proposition 241.5.

In preparation for the proof of the Bundle-over-Product Lemma 241.6 we need one more lemma, which might actually be interesting in its own right:

Lemma 241.7. (Trivialization Extension Lemma) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let $p\colon Y \to B$ be an $(F,\mathcal{G})$-bundle and let $B = B_0 \cup B_1$ be a decomposition of B into closed subsets. We suppose that the following two statements hold:
(a) The restrictions of p to $p^{-1}(B_0) \to B_0$ and $p^{-1}(B_1) \to B_1$ are both trivializable.
(b) $B_0 \cap B_1$ is a retract of B_1.
Then $p\colon Y \to B$ is also trivializable.

Proof. We introduce the following objects:

- By hypothesis there exist a retraction $r\colon B_1 \to B_0 \cap B_1$.
- By hypothesis there exist trivializations $\Phi_i\colon p^{-1}(B_i) \to B_i \times F$ for $i = 0, 1$.



- Note that the trivializations Φ_0 and Φ_1 both restrict to trivializations of the bundle $p\colon p^{-1}(B_0 \cap B_1) \to B_0 \cap B_1$. Since F is regionally compact, Hausdorff and locally connected it follows from the Bundle–Action-on-Trivializations Proposition 240.17 that

there exists a continuous map $\Theta\colon B_0 \cap B_1 \to \mathcal{G}$ such that the following diagram commutes:

$$(B_0 \cap B_1) \times F \xleftarrow[\cong]{\Phi_0} p^{-1}(B_0 \cap B_1) \xrightarrow[\cong]{\Phi_1} (B_0 \cap B_1) \times F.$$
$$(b,f) \mapsto (b, \Theta(b)(f))$$

Using the Pasting Proposition 4.7 one can now easily verify that

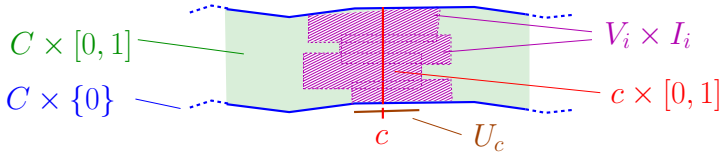
$$\begin{aligned} p^{-1}(B) &\to B \times F \\ y &\mapsto \begin{cases} \operatorname{diag}(\mathrm{id}_B, \overbrace{\Theta((r\circ p)(y))}^{\in \mathrm{Homeo}(F)})(\overbrace{\Phi_0(y)}^{\in B\times F})), & \text{if } y \in p^{-1}(B_0), \\ \Phi_1(y), & \text{if } y \in p_1^{-1}(B_1) \end{cases} \end{aligned}$$

is an isomorphism of $(F,\mathcal{G})$-bundles. ■

Proof of the Bundle-over-Product Lemma 241.6. [2613] Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let C be a topological space that is Hausdorff and paracompact. We need to show that for any $(F,\mathcal{G})$-bundle $p\colon Y \to C \times [0,1]$ the pullbacks i_0^*Y and i_1^*Y are isomorphic over C.

Claim 1. For each $c \in C$ there exists an open neighborhood U_c of c such that the restriction of the $(F,\mathcal{G})$-bundle $p\colon Y \to C \times [0,1]$ to $p^{-1}(U_c \times [0,1])$ is trivializable.

Proof. Let $c \in C$. It follows easily from the definition of an $(F,\mathcal{G})$-bundle and the compactness of $[0,1]$ that there exist open neighborhoods $V_1,\dots,V_m$ of c and intervals $I_1,\dots,I_m \subseteq [0,1]$ such the restriction of the $(F,\mathcal{G})$-bundle $p\colon Y \to C \times [0,1]$ to each $p^{-1}(V_i \times I_i)$ is trivializable and such that $I_1 \cup \dots \cup I_m = [0,1]$. We set $U_c \coloneqq V_1 \cap \dots \cap V_m$.



An elementary argument shows that we can pick $t_0 < t_1 < \dots < t_m = 1$ such that each $[t_i, t_{i+1}]$ is contained in one of $I_1,\dots,I_m$. It remains to show that p is trivializable over $U_c \times [0,t_i]$ for $i = 1,\dots,m$. For $i = 1$ this is clear. For $i = 2,\dots,m$ this follows from applying the Trivialization Extension Lemma 241.7 iteratively to $B_0 = U_c \times [0,t_{i-1}]$, $B_1 = U_c \times [t_{i-1},t_i]$, using the fact that $B_0 \cap B_1 = U_c \times \{t_{i-1}\}$ is evidently a retract of $B_1 = U_c \times [t_{i-1},t_i]$. ⊞

Claim 2. There exists a countable cover $\{V_i\}_{i\in\mathbb{N}}$ of C such that the restriction of $p\colon Y \to B$ to each $p^{-1}(V_i \times [0,1])$ is trivializable.

Proof. By Claim 1 there exists an open cover $\{U_c\}_{c\in C}$ of C such that the restriction of p to any $p^{-1}(U_c \times [0,1])$ is trivializable. Since C is Hausdorff and paracompact it follows from the Countable Cover Proposition 14.17 that there exists a *countable* locally finite open cover $\{V_i\}_{i\in\mathbb{N}}$ of C such that each V_i is the disjoint union of open subsets $\{V_{ij}\}_{j\in I_j}$ of C, each of which is contained in some U_c.

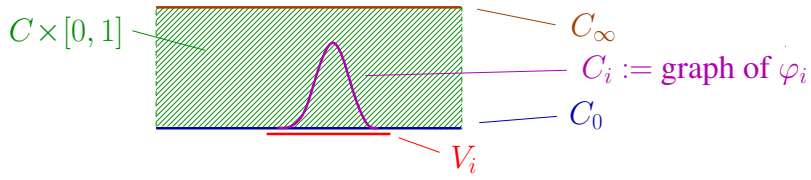
[2613] The structure of the proof is somewhat similar to the idea behind the proof of the Topological Collar Neighborhood Theorem 35.14.

Since the restriction of a trivializable $(F,\mathcal{G})$-bundle is again trivializable we see that we can pick a trivialization for each $p^{-1}(V_{ij}\times[0,1])$. Next note that it follows easily from the Disjoint Union Embedding Lemma 7.29 that each $V_i\times[0,1]$ is homeomorphic to the disjoint union of the open subsets $V_{ij}\times[0,1]$. It now follows that easily from the Topological-Coproduct Proposition 7.26 that these trivializations on $p^{-1}(V_{ij}\times[0,1])$ give rise to a trivialization on $p^{-1}(V_i\times[0,1])$. ⊞

Next note that it follows from our hypothesis that C is Hausdorff and paracompact together with the Paracompact-Partition of Unity Theorem 14.15 (1)⇒(3) that there exists a partition of unity $\{\varphi_i\colon C\to[0,1]\}_{i\in\mathbb{N}}$ such that for each $i\in\mathbb{N}$ we have $\operatorname{supp}(\varphi_i)\subseteq V_i$. Given $i\in\mathbb{N}_0$ we set $\psi_i:=\varphi_1+\cdots+\varphi_i\colon C\to[0,1]$ and we set

$$C_i \;:=\; \text{graph of } \psi_i\colon C\to[0,1] \;:=\; \{(c,\psi_i(c))\,|\,c\in C\} \;\subseteq\; C\times[0,1].$$

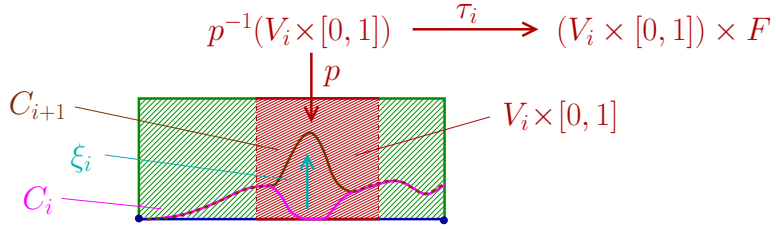
Note that $C_0=C\times\{0\}$. We also set $C_\infty:=C\times\{1\}$. Given $i\in\mathbb{N}_0\cup\{\infty\}$ we write $Y_i:=p^{-1}(C_i)$.



Claim 3. For each $i\in\mathbb{N}_0$ there exists an isomorphism

$$\begin{array}{ccc} Y_i & \xrightarrow[\cong]{\Xi_i} & Y_{i+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ C_i & \xrightarrow[=:\xi_i]{(c,\psi_i(c))\mapsto(c,\psi_i(c)+\varphi_{i+1}(c))} & C_{i+1} \end{array}$$

of $(F,\mathcal{G})$-bundles which is the identity outside of $p^{-1}(V_i\times[0,1])$.



Proof. Let $i\in\mathbb{N}_0$. Recall that by construction of the V_i we know that there exists a trivialization $\tau_i\colon p^{-1}(V_i\times[0,1])\to(V_i\times[0,1])\times F$. First we consider the map

$$\begin{array}{rccl} \widetilde{\Xi}_i\colon & ((V_i\times[0,1])\cap C_i)\times F & \to & ((V_i\times[0,1])\cap C_{i+1})\times F \\ & ((c,\psi_i(c)),f) & \mapsto & ((c,\psi_i(c)+\varphi_{i+1}(c)),f). \end{array}$$

Next we consider the map

$$\begin{array}{rcl} \Xi_i\colon Y_i & \to & Y_{i+1} \\ y & \mapsto & \begin{cases} y, & \text{if } y\notin p^{-1}(V_i\times[0,1]), \\ \tau_i^{-1}(\underbrace{\widetilde{\Xi}_i(\tau_i(y))}_{\in V_i\times[0,1]\times F}), & \text{if } y\in p^{-1}(V_i\times[0,1]). \end{cases} \end{array}$$

One can now easily verify that these maps have the desired properties. ⊞

We combine these maps to obtain the map

$$\begin{array}{rccc} \Theta\colon & Y_0 & \to & Y_\infty \\ & y & \mapsto & \Xi_{\underset{\text{we pick } k\in\mathbb{N}_0 \text{ with } \psi_k(y)=1}{\uparrow k}}(\dots(\Xi_0(y))). \end{array}$$

It is now straightforward to verify that this map is well-defined, i.e. that it does not depend on the choice of $k \in \mathbb{N}_0$ for a given y. It follows easily from this observation together with the definition of a partition of unity and the Pasting Proposition 4.7 that Θ is continuous.

Finally we consider the following diagram:

$$\begin{array}{ccccccccccccc} i_0^*Y & \longrightarrow & \iota_0^*Y & \longrightarrow & \overbrace{p^{-1}(C\times\{0\})}^{=Y_0} & \xrightarrow[\cong]{\Theta} & \overbrace{p^{-1}(C\times\{1\})}^{=Y_\infty} & \longleftarrow & \iota_1^*Y & \longleftarrow & i_1^*Y \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ C & \xrightarrow[f_0]{c\mapsto(c,0)} & C\times\{0\} & \xrightarrow{\text{id}} & C\times\{0\} & \xrightarrow{(c,0)\mapsto(c,1)} & C\times\{1\} & \xleftarrow{\text{id}} & C\times\{1\} & \xleftarrow[f_1]{(c,1)\leftarrow c} & C. \end{array}$$

We make the following clarifications and observations:

(1) It follows easily from the above discussion that the middle horizontal map Θ defines an isomorphism of $(F, \mathcal{G})$-bundles.

(2) For $j = 0, 1$ we denote by $f_j\colon C \to C \times \{j\}$ the obvious homeomorphism and we denote by $\iota_j\colon C \times \{j\} \to C \times [0, 1]$ the inclusion. Note that $i_j = \iota_j \circ f_j$.

(3) It follows from the Bundle-Pullback-Functor Lemma 240.23 (applied to the homeomorphism f_j) together with the Bundle-Pullback–Functorial Lemma 240.24 (2) (applied to $i_j = \iota_j \circ f_j$) that the first and last map are isomorphisms of $(F, \mathcal{G})$-bundles.

(4) It follows from the Bundle–Pullback Proposition 240.21 (2) that the second and the fourth map are isomorphisms of $(F, \mathcal{G})$-bundles.

It follows from these observations together with the Bundle-Pullback-Functor Lemma 240.23 that the pullbacks i_0^*Y and i_1^*Y are isomorphic over C. ■

241.3. **Bundles and homotopy equivalences.** In this section we want to pick up some fairly low-hanging fruit from the Pullback-Homotopy Proposition 241.5. To do so it is convenient to introduce the following notation:

Notation 241.8. Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Let B be a topological space. We denote by $\mathrm{IsoBund}_{(F,\mathcal{G})}(B)$ the set[2614] of isomorphism classes of $(F, \mathcal{G})$-bundles over B.

We continue with an elementary lemma that we could already have formulated in the last chapter:

[2614]Recall that we showed in the Bundles–Set Lemma 240.20 that it follows from F being regionally compact, that this is indeed a set.

Lemma 241.9. (Bundle–Isomorphism Classes Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) Given any continuous map $f\colon C \to B$ between two topological spaces the pullback of $(F,\mathcal{G})$-bundles, as defined in the Bundle–Pullback Proposition 240.21, induces a well-defined map

$$\begin{array}{rcl} f^*\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(B) & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(C) \\ [p\colon Y \to B] & \mapsto & [f^*Y \to C]. \end{array}$$

(2) The maps

$$\begin{array}{rcl} B & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(B) \\ (f\colon C \to B) & \mapsto & (f^*\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(B) \to \mathrm{IsoBund}_{(F,\mathcal{G})}(C)) \end{array}$$

define a contravariant functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{S}et$ of sets.

Proof. Statement (1) is an immediate consequence of the Bundle-Pullback-Functor Lemma 240.23, whereas Statement (2) is an immediate consequence of the Bundle-Pullback–Functorial Lemma 240.24. ■

The following proposition gives us a first classification of $(F,\mathcal{G})$-bundles over some types of topological spaces:

Proposition 241.10. (Bundles-Homotopies Proposition) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) Let B and C be two topological spaces that are both Hausdorff and paracompact. If $f\colon C \to B$ is a homotopy equivalence, then the map

$$f^*\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(B) \to \mathrm{IsoBund}_{(F,\mathcal{G})}(C),$$

that we introduced in the Bundle–Isomorphism Classes Lemma 241.9, is a bijection.

(2) Let C be a topological space that is Hausdorff and paracompact. If C is contractible, then every $(F,\mathcal{G})$-bundle over C is trivializable.

Example. Let $(F,\mathcal{G})$ as in the Bundles-Homotopies Proposition 241.10. Recall that in the Manifold+CW-Complex–Paracompact Proposition 241.4 we just mentioned that topological manifolds are Hausdorff and paracompact. It follows from this fact together with the Bundles-Homotopies Proposition 241.10 (2) that every $(F,\mathcal{G})$-bundle over the topological spaces $\overline{B}^n$, B^n and $\mathbb{R}^n$ is trivializable. □

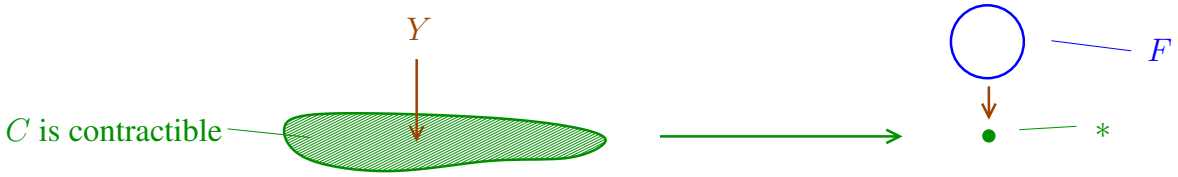
Proof.

(1) Let C and B be two topological spaces that are Hausdorff and paracompact and let $f\colon C \to B$ be a homotopy equivalence. We pick a homotopy inverse $g\colon B \to C$ to f. Recall that this means that $f \circ g$ is homotopic to id_B and that $g \circ f$ is homotopic to id_C. We consider the following induced maps $\mathrm{IsoBund}_{(F,\mathcal{G})}(B) \to \mathrm{IsoBund}_{(F,\mathcal{G})}(B)$:

$$g^* \circ f^* \overset{\text{by the Bundle–Isomorphism Classes Lemma 241.9 (2)}}{=} (f \circ g)^* \overset{\text{by the Pullback-Homotopy Proposition 241.5 (1), since } f \circ g \simeq \mathrm{id}_B}{=} \mathrm{id}_B^* = \mathrm{id}_{\mathrm{IsoBund}_{(F,\mathcal{G})}(B)}.$$

Similarly we see that $f^* \circ g^* = \mathrm{id}_{\mathrm{IsoBund}_{(F,\mathcal{G})}(C)}$.

(2) Let $Y \to C$ be an $(F,\mathcal{G})$-bundle over a topological space C that is Hausdorff, paracompact and contractible. By definition of being contractible there exists a homotopy equivalence from C to the topological space $\{*\}$ consisting of a single point.



Since every $(F,\mathcal{G})$-bundle over $\{*\}$ is trivializable it follows from (1), the discussion on page 4647 and the Bundle-Pullback-Functor Lemma 240.23 that every $(F,\mathcal{G})$-bundle over C is trivializable. ■

241.4. **Mapping tori as bundles and bundles over CW-complexes.** In Section 238.6 we already introduced mapping tori. Furthermore we saw in the Mapping Torus–Bundle Lemma 238.14 that mapping tori give rise to bundles over S^1, and that in fact every bundle over S^1 is homeomorphic to a mapping torus.

We will now slightly refine these statements and we will use the Bundles-Homotopies Proposition 241.10 (2) to give a new proof of the Mapping Torus–Bundle Lemma 238.14 (2):

Lemma 241.11. (Mapping Torus–$(F,\mathcal{G})$-Bundle Lemma) Let F be a topological space and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) (a) Let $\mu \in \mathcal{G}$. We consider the natural projection

$$\begin{array}{rccc} q\colon \mathrm{Tor}(F,\mu) = (F\times[0,1])/(f,0)\sim(\mu(f),1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

This map admits naturally the structure of an $(F,\mathcal{G})$-bundle.

(b) We now assume that F is regionally compact, Hausdorff and locally connected. Let $\mu_0,\mu_1\colon F\to F$ be two homeomorphisms in $\mathcal{G}$. The following two statements are equivalent:[2615]

(i) There exists a homeomorphism φ of X and a continuous map $H\colon[0,1]\to\mathcal{G}$ with $H(0)=\mu_0$ and $H(1)=\varphi\circ\mu_1\circ\varphi^{-1}$.

(ii) The two $(F,\mathcal{G})$-bundles $\mathrm{Tor}(F,\mu_0)\to S^1$ and $\mathrm{Tor}(F,\mu_1)\to S^1$ are isomorphic.

(2) For every $(F,\mathcal{G})$-bundle $p\colon Y\to S^1$ there exists a $\mu\in\mathcal{G}$ such that $p\colon Y\to S^1$ is isomorphic to the $(F,\mathcal{G})$-bundle $p\colon \mathrm{Tor}(F,\mu)\to S^1$ constructed in (1).

Example. Let $n\in\mathbb{N}$ and let $\rho\colon S^n\to S^n$ be the reflection in a hyperplane. We refer to $\mathrm{Tor}(S^n,\rho) := S^n\widetilde{\times}S^1$ as the twisted S^n-bundle over S^1. This simpleminded example will be used on several occasions. □

Proof.

(1) (a) In the proof of the Mapping Torus–Bundle Lemma 238.14 we constructed an explicit atlas, consisting of two trivializations, for the map $q\colon \mathrm{Tor}(f,\mu)\to S^1$. It follows almost immediately from the definitions that these charts are $\mathcal{G}$-compatible.

[2615] The statement (i)⇒(ii)-Implication is closely related to Exercise 240.7.

(b) We prove the (i)⇒(ii)-Implication: Thus let $H\colon [0,1] \to \mathcal{G}$ be a continuous map with $H(0) = \mu_0$ and $H(1) = \varphi \circ \mu_1 \circ \varphi^{-1}$. Similar to the Mapping Tori–Homeomorphic Lemma 21.25 and Exercise 21.14 we consider the map

$$\begin{array}{ccccc} \operatorname{Tor}(F,\mu_0) & \to & \operatorname{Tor}(F, \varphi \circ \mu_1 \circ \varphi^{-1}) & \to & \operatorname{Tor}(F,\mu_1) \\ [(f,t)] & \mapsto & [((H(t) \circ \mu_0^{-1})(f), t)] & & \\ & & [(f,t)] & \mapsto & [\varphi(f), t]. \end{array}$$

It is elementary to check that this map is well-defined. Furthermore, using the Homeomorphisms–Topological Group Proposition 16.17 one can easily verify that this map defines an isomorphism between the two $(F,\mathcal{G})$-bundles $\operatorname{Tor}(F,\mu_0) \to S^1$ and $\operatorname{Tor}(F,\mu_1) \to S^1$.

We outsource the proof of the (ii)⇒(i)-Implication to Exercise 241.6.

(2) This statement also follows easily from the down-to-earth argument of the Mapping Torus–Bundle Lemma 238.14. But now that we have more tools at our hands, we can give a different, more elegant argument:

Let $p\colon Y \to S^1$ be an $(F,\mathcal{G})$-bundle over S^1. We consider the following diagram:

$$\begin{array}{ccccc} [0,1] \times F & \xrightarrow[\cong]{\tau} & h^*Y & \xrightarrow[=:H]{(t,y)\mapsto y} & Y \\ \downarrow & & \downarrow & & \downarrow p \\ [0,1] & \xrightarrow{\mathrm{id}} & [0,1] & \xrightarrow[=:h]{t \mapsto \exp(2\pi \mathrm{i} t)} & S^1. \end{array}$$

We make the following clarifications:

- To the right we see a pullback diagram of $(F,\mathcal{G})$-bundles.
- Since $[0,1]$ is contractible[2616] we obtain from the Bundles-Homotopies Proposition 241.10 (2) that there exists an isomorphism $\tau\colon [0,1] \times F \to h^*Y$ of $(F,\mathcal{G})$-bundles.

We now consider the homeomorphism $\mu\colon F \to F$ that is given by

$$F \xrightarrow[\cong]{f \mapsto (f,0)} F \times \{0\} \xrightarrow[\cong]{H \circ \tau} p^{-1}(\underbrace{1}_{\in S^1}) \xleftarrow[\cong]{H \circ \tau} F \times \{1\} \xleftarrow[\cong]{(f,1) \hookleftarrow f} F.$$

Finally we consider the following diagram:

$$\begin{array}{ccc} (F \times [0,1])/(f,0) \sim (\mu(f),1) & \xrightarrow{[(f,t)] \mapsto (H\circ\tau)(f,t)} & Y \\ \downarrow{\scriptstyle [(f,t)] \mapsto \exp(2\pi \mathrm{i} t)} & & \downarrow p \\ S^1 & \xrightarrow{\mathrm{id}_{S^1}} & S^1. \end{array}$$

First note that one can easily verify that the horizontal map is well-defined. Next note that it follows from the Quotient-to-Quotient Lemma 8.6 that the top horizontal map is continuous. It is now elementary to verify that the above diagram gives us an isomorphism of $(F,\mathcal{G})$-bundles. Using the above diagram we obtain the following

[2616] Evidently we also need that $[0,1]$ is Hausdorff and paracompact. We will not bother to point this out all the time.

isomorphisms of $(F, \mathcal{G})$-bundles:

$$(F \times [0,1])/(f,0) \sim (\mu(f),1) \quad \underset{\uparrow}{\cong} \quad (\mathrm{id}_{S^1})^* Y \quad \underset{\uparrow}{\cong} \quad Y.$$

follows from the above diagram together with the Bundle Pullback Proposition 240.21 (2)

by the Pullback–Functorial Lemma 240.24 (1)

■

Let F be a topological space and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$. If $\mathcal{G}$ is path-connected, then it follows from the Mapping Torus–$(F, \mathcal{G})$-Bundle Lemma 241.11 that every $(F, \mathcal{G})$-bundle over S^1 is trivializable. We will now generalize this statement to $(F, \mathcal{G})$-bundles over connected topological graphs (or equivalently, to bundles over connected 1-dimensional CW-complexes):

Lemma 241.12. (Bundle-over-1-Complex Lemma) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. If $\mathcal{G}$ is *path-connected*, then every $(F, \mathcal{G})$-bundle over any 1-dimensional CW-complex is trivializable.

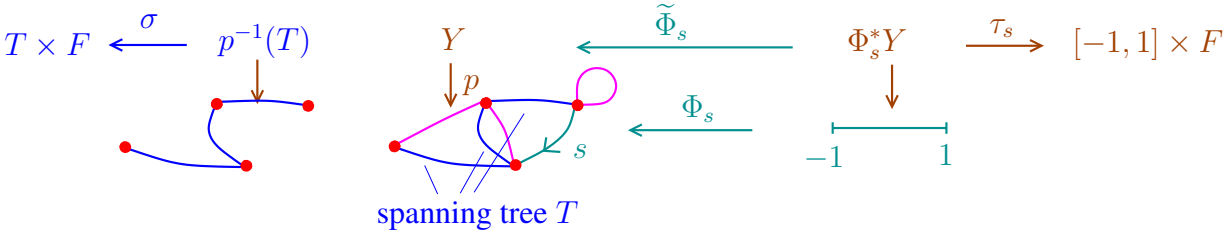
Proof. Let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be path-connected and let $p\colon Y \to B$ be an $(F, \mathcal{G})$-bundle over a 1-dimensional CW-complex B. It follows easily from the CW-Complex–Local Properties Proposition 137.8 (3) that it suffices to deal with the case that B is connected and non-empty.

We make the following preparations:

- By the 1-Complex-π_1-Theorem 137.6 (or alternatively by the Spanning Tree–Existence Proposition 11.26) we know that B admits a spanning tree T.[2617]



- Note that by the Tree–Contractible Proposition 137.4 we know that T is contractible. It follows from the Bundles-Homotopies Proposition 241.10 that there exists a trivialization $\sigma\colon p^{-1}(T) \to T \times F$.
- Let S be the set of 1-cells of B that are not contained in the spanning tree. For each $s \in S$ we pick a characteristic map $\Phi_s\colon [-1,1] \to B$. It follows again from the Bundles-Homotopies Proposition 241.10 that for each $s \in S$ there exists a trivialization $\tau_s\colon \Phi_s^* Y \to [-1,1] \times F$.



[2617]Recall that by Definition 137.5, together with the Tree–Contractible Proposition 137.4, a spanning tree is the same as a contractible subcomplex that contains all 0-cells.

Since we are dealing with trivializations of $(F, \mathcal{G})$-bundles we see that for each $s \in S$ and $\eta \in \{-1, 1\}$ there exists a $\Theta_{s,\eta} \in \mathcal{G}$ such that the following diagram commutes:

$$\{\Phi_s(\eta)\} \times F \xleftarrow{\sigma} p^{-1}(\Phi_s(\eta)) \xleftarrow{\widetilde{\Phi}_s} p_s^{-1}(\eta) \xrightarrow{\tau_s} \{\eta\} \times F.$$
$$(\Phi_s(\eta), f) \mapsto (\eta, \Theta_{s,\eta}(f))$$

Since $\mathcal{G}$ is by hypothesis path-connected there exists a continuous map $\gamma_s \colon [-1, 1] \to \mathcal{G}$ with $\gamma_s(-1) = \Theta_{c-}^{-1}$ and $\gamma_s(1) = \Theta_{c+}^{-1}$. As in the Bundle–Action-on-Trivializations Proposition 240.17 we compose the trivialization τ_c of $\Phi_c^* Y$ with the map $(t, f) \mapsto (t, \gamma(t)(f))$ to obtain a new trivialization $\widetilde{\tau}_c$.

It now suffices to prove the following claim:

Claim. The map:

$$\begin{aligned} \overbrace{\left(\left(T \sqcup \bigsqcup_{s \in S} [-1, 1]\right) / \sim\right)}^{\cong B} \times F &\to Y \\ ([P], f) &\mapsto \begin{cases} \sigma^{-1}(P, f), & \text{if } P \in T, \\ \widetilde{\tau}_s^{-1}(t, f), & \text{if } P = (t, s) \in [-1, 1] \times \{s\} \end{cases} \end{aligned}$$

is an isomorphism of $(F, \mathcal{G})$-bundles.

Proof. We start out with the following observation: Since F is regionally compact it follows from the Product-Quotient Mixing Lemma 17.8 that it suffices to show that the following, subtly different map, is an isomorphism of $(F, \mathcal{G})$-bundles:

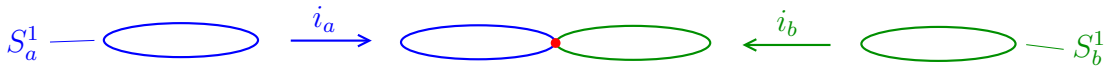
$$\begin{aligned} \left((T \times F) \sqcup \bigsqcup_{s \in S} ([-1, 1] \times F)\right) / \sim &\to Y \\ [Q] &\mapsto \begin{cases} \sigma^{-1}(Q), & \text{if } Q \in T \times F, \\ \widetilde{\tau}_s^{-1}(t, f), & \text{if } Q = ((t, f), s) \in [-1, 1] \times F \times \{s\}. \end{cases} \end{aligned}$$

Regarding this map we make the following observations:

(1) Now that we so carefully constructed the trivializations $\widetilde{\tau}_c$ one can easily verify that the map is well-defined.
(2) It is elementary to see that the map is a bijection.
(3) It follows from the Topological-Coproduct Proposition 7.26 (1b) together with the Topological-Quotient Proposition 8.2 (1b) that the map is continuous.
(4) It is pretty elementary to show that the map is open.

It follows from the above, together with the Open-Closed–Injective Lemma 4.22, that the map is an isomorphism of $(F, \mathcal{G})$-bundles. ∎

Example. Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. We consider the wedge $S_a^1 \vee S_b^1$ of two circles. We denote by $i_a \colon S_a^1 \to S_a^1 \vee S_b^1$ and $i_b \colon S_b^1 \to S_a^1 \vee S_b^1$ the two natural embeddings.

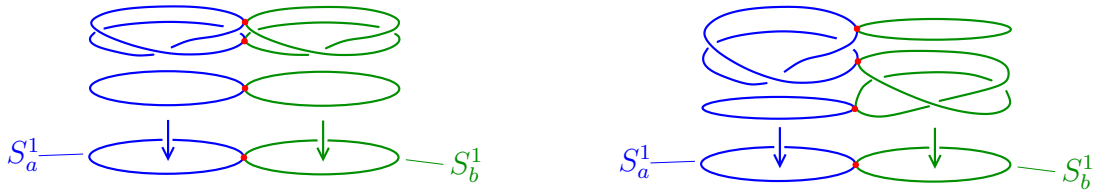


It follows from the Bundle-Pullback-Functor Lemma 240.23 that we obtain a map

$$\begin{array}{rcl} \mathrm{IsoBund}_{(F,\mathcal{G})}(S^1_a \vee S^1_b) & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(S^1_a) \times \mathrm{IsoBund}_{(F,\mathcal{G})}(S^1_b) \\ [p\colon Y \to S^1_a \vee S^1_b] & \mapsto & ([i_a^* Y], [i_b^* Y]). \end{array}$$

In Exercise 241.24 we will see that this map is surjective.

But, to the surprise of unsuspecting authors, this map is not necessarily injective. For example we consider the two bundles over $S^1_a \vee S^1_b$, with fiber $F = \{1,2,3\}$, which are shown in the figure below:



The restrictions of these two bundles to the circles are all isomorphic. But the total spaces of the two bundles are not homeomorphic. Indeed, the one to the left is disconnected, whereas the one to the right is connected.[2618] □

241.5. **The clutching construction and bundles over spheres.** In this section it is convenient to bring back the following notation which we first introduced in Definition 19.11:

Definition 241.13. Given two topological spaces X and Y we set

$$[X,Y] := \{\text{continuous maps } X \to Y\}/\sim$$

where $f \sim g$ if there exists a homotopy between f and g. We refer to $[X,Y]$ as the set of homotopy classes of continuous maps from X to Y.

We continue with a general construction of $(F,\mathcal{G})$-bundles:

Lemma 241.14. (Bundle–Clutching Map Lemma) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let B a smooth manifold and let B_0, B_1 be codimension-zero submanifolds of B such that the following conditions are satisfied:

(a) $B = B_0 \cup B_1$.
(b) $B_0 \cap B_1$ is a union of components of ∂B_0 and a union of components of ∂B_1.
(c) B_0 and B_1 are closed subsets of B.

Then the following statements hold:

(1) Let $\varphi\colon B_0 \cap B_1 \to \mathcal{G}$ be a continuous map. We set

$$Y(\varphi) := ((B_0 \times F) \sqcup (B_1 \times F))/\sim$$

where $((c,f),0) \sim ((c,\varphi(c)(f)),1)$ for all $(c,f) \in (B_0 \cap B_1) \times F$. Next we consider the map

$$\begin{array}{rcl} p(\varphi)\colon Y(\varphi) & \to & B \\ [((b,f),i)] & \mapsto & b. \end{array}$$

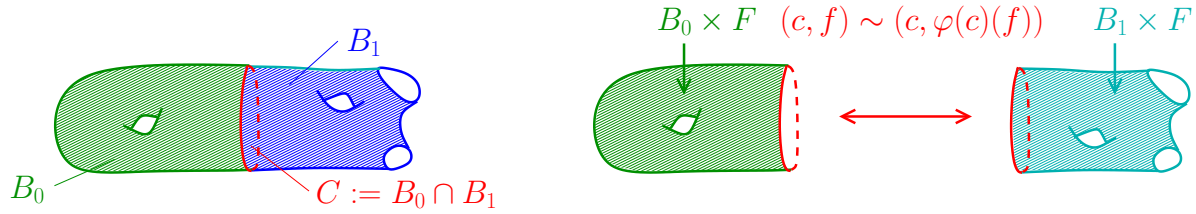
[2618] Note that this example gives a solution to Exercise 240.30.

There exists an $(F, \mathcal{G})$-bundle atlas for the map $p(\varphi)\colon Y(\varphi) \to B$ such that the restrictions to $p(\varphi)^{-1}(B_0)$ and $p(\varphi)^{-1}(B_1)$ are naturally trivializable.

(2) The map

$$\begin{array}{ccc} \overbrace{[B_0 \cap B_1, \mathcal{G}]}^{\substack{\text{homotopy classes of} \\ \text{maps } B_0 \cap B_1 \to \mathcal{G}}} & \to & \overbrace{\operatorname{IsoBund}_{(F,\mathcal{G})}(B)}^{\substack{\text{isomorphism classes of} \\ (F,\mathcal{G})\text{-bundles over } B}} \\ [\varphi\colon B_0 \cap B_1 \to \mathcal{G}] & \mapsto & [p(\varphi)\colon Y(\varphi) \to B] \end{array}$$

is well-defined.



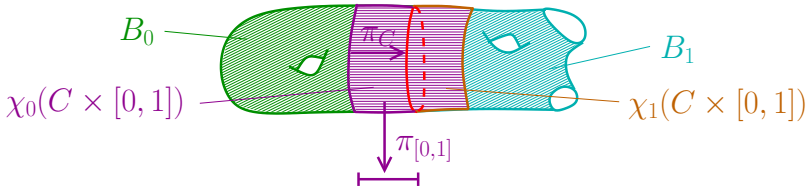
Definition 241.15. We continue with the notation from the Bundle–Clutching Map Lemma 241.14. We say that the $(F, \mathcal{G})$-bundle

$$\begin{array}{rcl} p(\varphi)\colon Y(\varphi) := ((B_0 \times F) \sqcup (B_1 \times F))/((c, f), 0) \sim ((c, \varphi(c)(f)), 1) & \to & B \\ [((b, f), i)] & \mapsto & b \end{array}$$

is obtained by the clutching construction using the clutching map $\varphi\colon B_0 \cap B_1 \to \mathcal{G}$.

Example. We continue with the notation from the Bundle–Clutching Map Lemma 241.14. Note that it follows easily from the definitions that the constant map $\varphi\colon B_0 \cap B_1 \to \mathcal{G}$ that is given by $b \mapsto \mathrm{id}_F$ gives rise to an $(F, \mathcal{G})$-bundle that is isomorphic to the trivial bundle $B \times F \to B$. □

Sketch of proof. We set $C := B_0 \cap B_1$. Note that it follows from Hypothesis (b) and the Smooth Collar Neighborhood Theorem 35.6 that for $i = 0, 1$ there exist smooth embeddings $\chi_i\colon C \times [0, 1] \to B_i$ such that $\chi_i(c, 0) = c$ for all $c \in C$.



(1) We now construct an atlas for $p(\varphi)\colon Y(\varphi) \to B$.

(a) Let $i \in \{0, 1\}$. Similar to the proof of the Bundle–Transition Map Lemma 240.18 one can show that the map

$$\begin{array}{rcl} \Psi_i\colon (B_i \setminus C) \times F & \to & Y(\varphi) \\ (b, f) & \mapsto & [((b, f), i)] \end{array}$$

is an open embedding with image given by $p(\varphi)^{-1}(B_i \setminus C)$. It follows that the inverse $\Phi_i^{-1}\colon p(\varphi)^{-1}(B_i \setminus C) \to (B_i \setminus C) \times F$ is a trivialization.

(b) We denote by $\pi_C\colon \chi_0(C\times[0,1)) \xrightarrow{\chi_0^{-1}} C\times[0,1] \to C$ the obvious projection. We consider the map

$$\begin{array}{rcl} p(\varphi)^{-1}(\overbrace{\chi_0(C\times[0,1))\cup\chi_1(C\times[0,1))}^{\text{by the Basics-of-Collars Lemma 35.7, this is an open subset of } B}) & \to & (\chi_0(C\times[0,1))\cup\chi_1(C\times[0,1)))\times F \\ [((b,f),i)] & \mapsto & \begin{cases} ((b,\varphi(\pi_C(b))(f)),i), & \text{if } i=0, \\ ((b,f),i), & \text{if } i=1. \end{cases} \end{array}$$

One can easily verify that the map is well-defined. We leave it to the reader to verify that this map is indeed a trivialization.

Finally note that it follows easily from the definitions that these three trivializations are $\mathcal{G}$-compatible.

(2) Let $\varphi_0,\varphi_1\colon C\to\mathcal{G}$ be two continuous homotopic maps. We need to show that the corresponding $(F,\mathcal{G})$-bundles are isomorphic. Let $H\colon C\times[0,1]\to\mathcal{G}$ be a homotopy between φ_0 and φ_1. We consider the map

$$\begin{array}{rcl} \widetilde{H}\colon C\times[0,1] & \to & \mathcal{G} \\ (c,t) & \mapsto & H(c,t)\circ\varphi_0(c)^{-1}. \end{array}$$

Note that it follows from our many adjectives for F and the Homeomorphisms–Topological Group Proposition 16.17 that this map is continuous.

Similar to the proof of the Mapping Torus–$(F,\mathcal{G})$-Bundle Lemma 241.11 we consider the following map:

$$\begin{array}{rcl} Y(\varphi_0) & \to & Y(\varphi_1) \\ [((b,f),i)] & \mapsto & \begin{cases} [((b,f),i)], & \text{if } i=0 \text{ and } b\notin\chi_0(C\times[0,1)), \\ [(b,\widetilde{H}(c,t)(f),i)], & \text{if } i=0 \text{ and } b=\chi_0(c,t) \text{ for } (c,t)\in C\times[0,1], \\ [((b,f),i)], & \text{if } i=1. \end{cases} \end{array}$$

It is pretty elementary to see that the map is well-defined and that it is an isomorphism of $(F,\mathcal{G})$-bundles.

We leave it to the reader to fill in the details. ∎

The following proposition shows that for the obvious decomposition of a sphere S^k into two halves the map from the Bundle–Clutching Map Lemma 241.14 (2) is a bijection:

Proposition 241.16. (Bundle-over-Sphere Proposition) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G}\subseteq\operatorname{Homeo}(F)$ be a subgroup. Furthermore let $k\in\mathbb{N}_0$. As on page 320 we use the identification $S^k =_{\mathrm{i}} \overline{B}^k_+ \cup_{S^{k-1}} \overline{B}^k_-$ (together with the obvious decomposition of the right-hand side) for the clutching construction.

(1) If $\mathcal{G}$ is path-connected, then the map

$$\begin{array}{rcl} \Theta\colon [S^{k-1},\mathcal{G}] & \to & \operatorname{IsoBund}_{(F,\mathcal{G})}(S^k) \\ [\varphi\colon S^{k-1}\to\mathcal{G}] & \mapsto & [\ \underbrace{p(\varphi)\colon Y(\varphi)\to S^k}_{\text{defined in the Bundle–Clutching Map Lemma 241.14}}\ ] \end{array}$$

is a well-defined bijection.

(2) We now assume that $k \geq 2$. We denote by $\mathcal{G}_{\mathrm{id}}$ the path-component of $\mathrm{id} \in \mathcal{G}$.

(a) The map

$$\begin{array}{rcl} \mathcal{G}/\mathcal{G}_{\mathrm{id}} \times [S^{k-1}, \mathcal{G}_{\mathrm{id}}] & \to & [S^{k-1}, \mathcal{G}_{\mathrm{id}}] \\ ([g], [\varphi\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}]) & \mapsto & \left[\begin{array}{rcl} S^{k-1} & \to & \mathcal{G}_{\mathrm{id}} \\ y & \mapsto & g \cdot \varphi(y) \cdot g^{-1} \end{array}\right] \end{array}$$

is a well-defined action of the group $\mathcal{G}/\mathcal{G}_{\mathrm{id}}$ on the set $[S^{k-1}, \mathcal{G}_{\mathrm{id}}]$.

(b) We denote by $[S^{k-1}, \mathcal{G}_{\mathrm{id}}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}}$ the set of equivalence classes corresponding to the action of $\mathcal{G}/\mathcal{G}_{\mathrm{id}}$ on $[S^{k-1}, \mathcal{G}_{\mathrm{id}}]$. The map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathcal{G}_{\mathrm{id}}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}} & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \\ [\varphi\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}] & \mapsto & [p(\varphi)\colon Y(\varphi) \to S^k] \end{array}$$

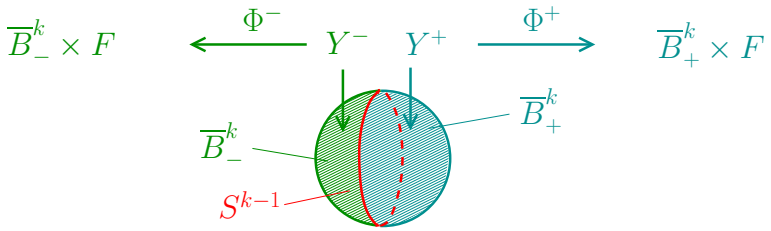
is a well-defined bijection.

The proof of the Bundle-over-Sphere Proposition 241.16 will require the remainder of this section. In the next two Sections 241.6 and 241.7 we will use the proposition to classify at least some types of bundles over spheres.

Proof of the Bundle-over-Sphere Proposition 241.16 (1). We now assume that $\mathcal{G}$ is path-connected. The idea of the proof is to construct an inverse to the map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathcal{G}] & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \\ [\varphi\colon S^{k-1} \to \mathcal{G}] & \mapsto & [p(\varphi)\colon Y(\varphi) \to S^k]. \end{array}$$

Thus let $p\colon Y \to S^k =_{\mathrm{i}} \overline{B}^k_+ \cup_{S^{k-1}} \overline{B}^k_-$ be an $(F,\mathcal{G})$-bundle. We write $Y^{\pm} := p^{-1}(\overline{B}^k_{\pm})$. Since $\overline{B}^k_+$ and $\overline{B}^k_-$ are paracompact, Hausdorff and contractible we know by the Bundles-Homotopies Proposition 241.10 (2) that the $(F,\mathcal{G})$-bundles $Y^{\pm} \to \overline{B}^k_{\pm}$ are trivializable. We pick trivializations $\Phi^{\pm}\colon Y^{\pm} \to \overline{B}^k_{\pm} \times F$.



Note that Φ^+ and Φ^- both give a trivialization of p over S^{k-1}. It follows from the Bundle–Action-on-Trivializations Proposition 240.17 that there exists a unique continuous map $\Omega_{\Phi^-,\Phi^+}\colon S^{k-1} \to \mathcal{G}$ such that the following diagram commutes:

$$S^{k-1} \times F \xleftarrow{\Phi^-} p^{-1}(S^{k-1}) \xrightarrow{\Phi^+} S^{k-1} \times F.$$
$$(b,f) \mapsto (b, \Omega_{\Phi^-,\Phi^+}(b)(f))$$

Claim 1.

(a) The class $[\Omega_{\Phi^-,\Phi^+}] \in [S^{k-1}, \mathcal{G}]$ does not depend on the choice of trivializations $\Phi^{\pm}$.

[2618] In the Mapping Torus–$(F,\mathcal{G})$-Bundle Lemma 241.11 we characterized $(F,\mathcal{G})$-bundles over the 1-dimensional sphere S^1. The overlap of these two results is minimal though, since in the current Bundle-over-Sphere Proposition 241.16 we assume that $\mathcal{G}$ is path-connected.

(b) The class $[\Omega_{\Phi^-,\Phi^+}] \in [S^{k-1}, \mathcal{G}]$ only depends on the isomorphism type of the $(F,\mathcal{G})$-bundle $p\colon Y \to B$ over S^k.

Proof.

(a) In the following we show that $[\Omega_{\Phi^-,\Phi^+}] \in [S^{k-1}, \mathcal{G}]$ does not depend on the choice of the trivialization Φ^+. The argument for Φ^- is basically the same.

So suppose that we are given another trivialization $\Psi^+\colon Y^+ \to \overline{B}^k_+ \times F$. As above it follows from the Bundle–Action-on-Trivializations Proposition 240.17 that there exists a unique continuous maps $\tau\colon \overline{B}^k_+ \to \mathcal{G}$ such that the following diagram commutes:

$$\overline{B}^k_+ \times F \xleftarrow{\Phi^+} p^{-1}(\overline{B}^k_+) \xrightarrow{\Psi^+} \overline{B}^k_\pm \times F, \qquad (b,f) \mapsto (b, \tau(b)(f))$$

Furthermore recall that we assumed that $\mathcal{G}$ is path-connected. This implies that there exists a path $\gamma\colon [0,1] \to \mathcal{G}$ with $\gamma(1) = \tau(0)$ and $\gamma(1) = \mathrm{id}_F$.

We now consider the map

$$\begin{aligned} H\colon S^{k-1} \times [0,1] &\to \mathcal{G} \\ (b,t) &\mapsto \begin{cases} \tau(\overbrace{b \cdot (1-2t)}^{\in \overline{B}^k_+}) \circ \Omega_{\Phi^-,\Phi^+}(b), & \text{if } t \in [0, \frac{1}{2}], \\ \gamma(2t-1) \circ \Omega_{\Phi^-,\Phi^+}(b), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

First note that it follows from the Homotopy Combination Lemma 18.8 together with the Homeomorphisms–Topological Group Proposition 16.17 that this map is continuous. Furthermore note that it follows easily from the definitions that $H_0 = \Omega_{\Phi^-,\Psi^+}$ and that $H_1 = \Omega_{\Phi^-,\Phi^+}$.

(b) Let $f\colon \widetilde{Y} \to Y$ be an isomorphism of $(F,\mathcal{G})$-bundles. Clearly the maps $\Phi^\pm \circ f$ are trivializations of $\widetilde{Y}$ over $\overline{B}^k_\pm$. Next note that it follows almost immediately from the definitions that $\Omega_{\Phi^- \circ f, \Phi^+ \circ f} = \Omega_{\Phi^-,\Phi^+}\colon S^{k-1} \to \mathcal{G}$. This shows that $\widetilde{Y}$ gives rise to the same class in $[S^{k-1}, \mathcal{G}]$ as Y. ⊞

It follows from Claim 1 that we have just constructed a well-defined map

$$\Omega\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \to [S^{k-1}, \mathcal{G}].$$

It remains to prove the following claim:

Claim 2. The maps $\Theta\colon [S^{k-1}, \mathcal{G}] \to \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k)$ and $\Omega\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \to [S^{k-1}, \mathcal{G}]$ are inverses of one-another.

Proof.

(a) First we claim that $\Omega \circ \Theta$ is the identity on $[S^{k-1}, \mathcal{G}]$. Thus let $\varphi\colon S^{k-1} \to \mathcal{G}$ be a continuous map. We construct the corresponding $(F,\mathcal{G})$-bundle $p(\varphi)\colon Y(\varphi) \to S^k$. This bundle has obvious trivializations $\Phi^\pm$ over $\overline{B}^k_\pm$. For these trivializations we clearly have $\Omega_{\Phi^-,\Phi^+} = \varphi$. Thus $(\Omega \circ \Theta)([\varphi]) = [\varphi]$.

(b) Now we claim that $\Theta \circ \Omega$ is the identity on $\mathrm{IsoBund}_{(F,\mathcal{G})}(S^k)$. Thus let $p\colon Y \to S^k$ be an $(F,\mathcal{G})$-bundle. We pick trivializations $\Phi^\pm$ of p over $\overline{B}^k_\pm$. One can now easily verify

that the maps $\Phi^{\pm}$ define an isomorphism

$$\begin{array}{rcl} Y & \to & Y(\varphi) \\ y & \mapsto & \begin{cases} \Phi_-(y), & \text{if } p(y) \in \overline{B}^k_-, \\ \Phi_+(y), & \text{if } p(y) \in \overline{B}^k_+. \end{cases} \end{array}$$

of $(F, \mathcal{G})$-bundles. ■

Proof of the Bundle-over-Sphere Proposition 241.16 (2). In the following we now assume that $k \geq 2$.

(a) Let $g \in \mathcal{G}$. We consider the map $\mathcal{G}_{\mathrm{id}} \to \mathcal{G}$ that is given by $h \mapsto g \cdot h \cdot g^{-1}$. Since $\mathcal{G}$ is a topological group, this map is continuous. We make two observations:

- Since the trivial element gets sent to the trivial element, and since $\mathcal{G}_{\mathrm{id}}$ is path-connected, we obtain from the Image-Preimage-Component Lemma 4.50 (2) that the image of $\mathcal{G}_{\mathrm{id}}$ is again contained in $\mathcal{G}_{\mathrm{id}}$.
- If $g \in \mathcal{G}_{\mathrm{id}}$, then this map is clearly homotopic to the identity.

It follows easily from these observations and elementary arguments that the map

$$\begin{array}{rcl} \mathcal{G}/\mathcal{G}_{\mathrm{id}} \times [S^{k-1}, \mathcal{G}_{\mathrm{id}}] & \to & [S^{k-1}, \mathcal{G}_{\mathrm{id}}] \\ ([g], [\varphi\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}]) & \mapsto & \left(\begin{array}{rcl} \varphi^g\colon S^{k-1} & \to & \mathcal{G}_{\mathrm{id}} \\ y & \mapsto & g \cdot \varphi(y) \cdot g^{-1} \end{array} \right) \end{array}$$

is a well-defined action of the group $\mathcal{G}/\mathcal{G}_{\mathrm{id}}$ on the set $[S^{k-1}, \mathcal{G}_{\mathrm{id}}]$.

(b) We start out with the following claim:

Claim 0. The map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathcal{G}_{\mathrm{id}}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}} & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \\ [\varphi\colon S^{k-1} \to \mathcal{G}] & \mapsto & [p(\varphi)\colon Y(\varphi) \to S^k] \end{array}$$

is well-defined.

Proof. The only statement that requires some thought is the verification that the action of $\mathcal{G}/\mathcal{G}_{\mathrm{id}}$ has no impact on the isomorphism type of the $(F, \mathcal{G})$-bundle. So suppose that we are given $\varphi\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}$ and let $g \in \mathcal{G}$. We form the (F, φ)-bundles $Y(\varphi) \to S^k$ and $Y(\varphi^g) \to S^k$. Note that the following diagram commutes:

$$\begin{array}{ccc} \overline{B}^k_- \times F & \xrightarrow{(b,f) \mapsto (b, \varphi(b)(f))} & \overline{B}^k_+ \times F \\ {\scriptstyle (b,f) \mapsto (b, g(f))}\downarrow & & \downarrow{\scriptstyle (b,f) \mapsto (b, g(f))} \\ \overline{B}^k_- \times F & \xrightarrow{(b,f) \mapsto (b, \varphi^g(b)(f)) = (b, (g \cdot \varphi(b) \cdot g^{-1})(f))} & \overline{B}^k_+ \times F. \end{array}$$

Note that it follows easily from this observation that an isomorphism from $Y(\varphi) \to S^k$ to $Y(\varphi^g) \to S^k$ is given by applying the map $(b, f) \mapsto (b, g(f))$ on $\overline{B}^k_- \times F$ and also on $\overline{B}^k_+ \times F$. ⊞

We still need to show that the above map

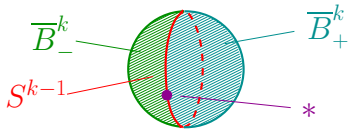
$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathcal{G}_{\mathrm{id}}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}} & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \\ [\varphi\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}] & \mapsto & [p(\varphi)\colon Y(\varphi) \to S^k] \end{array}$$

is a bijection. We will do so by constructing an inverse. We construct the inverse map as before, with one minor change: Given an $(F, \mathcal{G})$-bundle $p\colon Y \to S^k$ we again

pick trivializations $\Phi^{\pm}\colon Y^{\pm} \to \overline{B}^k_{\pm} \times F$. As before we see that there exists a unique continuous map $\Omega_{\Phi^-,\Phi^+}\colon S^{k-1} \to \mathcal{G}$ such that the following diagram commutes:

$$S^{k-1} \times F \xleftarrow{\Phi^-} p^{-1}(S^{k-1}) \xrightarrow{\Phi^+} S^{k-1} \times F.$$
$$(b,f) \mapsto (b, \Omega_{\Phi^-,\Phi^+}(b)(f))$$

Next pick $* \in S^{k-1}$.



We set $\mu := (\Omega_{\Phi^-,\Phi^+})(*)^{-1} \in \mathcal{G}$. After composing the trivialization Φ^+ by the map $(b,f) \mapsto (b, \mu(f))$ we obtain a new trivialization Ψ^+ such that $(\Omega_{\Phi^-,\Psi^+})(*) = \mathrm{id}$. Since $k \geq 2$ we know that S^{k-1} is path-connected. Thus we see that $\Omega_{\Phi^-,\Psi^+}\colon S^{k-1} \to \mathcal{G}$ takes values in $\mathcal{G}_{\mathrm{id}}$. We now set $\Omega([p\colon Y \to B]) := [\Omega_{\Phi^-,\Psi^+}\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}]$.

Claim 1.
(a) The class $[\Omega_{\Phi^-,\Phi^+}] \in [S^{k-1}, \mathcal{G}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}}$ does not depend on the choice of the two trivializations $\Phi^{\pm}$.
(b) The class $[\Omega_{\Phi^-,\Phi^+}] \in [S^{k-1}, \mathcal{G}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}}$ only depends on the isomorphism type of the $(F,\mathcal{G})$-bundle $p\colon Y \to B$ over S^k.

Proof.
(a) The proof hat the choice of Φ_+ has no impact is the same as above. Note though that a different choice of Φ_- leads to a conjugation, which means that the class is well-defined in the set $[S^{k-1}, \mathcal{G}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}}$ of equivalence classes.
(b) The proof of this statement is the same as above. ⊞

It now remains to prove the following claim:

Claim 2. The maps

$$[S^{k-1}, \mathcal{G}_{\mathrm{id}}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}} \xrightarrow{\Theta} \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k)$$
$$\mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \xrightarrow{\Omega} [S^{k-1}, \mathcal{G}_{\mathrm{id}}]_{\mathcal{G}/\mathcal{G}_{\mathrm{id}}}$$

are inverses of one another:

Proof.
(a) As before, if we use the obvious trivialization for the $(F,\mathcal{G})$-bundle $p(\varphi)$, then we see that $(\Omega \circ \Theta)([\varphi]) = \Omega([\varphi]) = [p(\varphi)]$.
(b) Now let $[p\colon Y \to B] \in \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k)$. As above we can and will pick trivializations over $\overline{B}^k_{\pm}$ which agree on $p^{-1}(*)$. We denote by $\varphi\colon S^{k-1} \to \mathcal{G}_{\mathrm{id}}$ the resulting map. As above, these two trivializations define an isomorphism $Y \to Y(\varphi)$. ■

241.6. **Classification of bundles over spheres.** The Bundle-over-Sphere Proposition 241.16 says that, in reasonable settings, isomorphism classes of $(F,\mathcal{G})$-bundles over the k-dimensional sphere S^k are in bijection to the set $[S^{k-1}, \mathcal{G}]$. This calculation gives us the curious fact, that the isomorphism types of $(F,\mathcal{G})$-bundles only depends on the *homotopy type* of the structure group $\mathcal{G}$. In particular the fiber F is largely irrelevant.

This observation leads us to the following lemma:

Lemma 241.17. (Structure Group–Homotopy Type Lemma) Let F be a topological space that is regionally compact, Hausdorff and locally connected, let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup and let $k \in \mathbb{N}$.

(1) Let $\mathcal{H} \subseteq \mathrm{Homeo}(F)$ be another subgroup with $\mathcal{G} \subseteq \mathcal{H}$. Any $(F, \mathcal{G})$-bundle also defines an $(F, \mathcal{H})$-bundle. If the inclusion $\mathcal{G} \to \mathcal{H}$ is a homotopy equivalence, then the natural map
$$\mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) \ \to \ \mathrm{IsoBund}_{(F,\mathcal{H})}(S^k)$$
is a bijection.

(2) Let $S \subseteq F$ be a subspace which is preserved by any $g \in \mathcal{G}$ and which has the property that if for some $g \in \mathcal{G}$ we have $g|_S = \mathrm{id}_S$, then $g = \mathrm{id}_F$.
   (a) We set $\mathcal{G}_S := \{g|_S \colon S \to S \,|\, g \in \mathcal{G}\}$. The obvious map $\mathcal{G} \to \mathcal{G}_S$ is a homeomorphism.
   (b) Given an $(F, \mathcal{G})$-bundle $(p \colon Y \to B, \{\Phi_i \colon p^{-1}(U_i) \to U_i \times F\}_{i \in I})$ we now set $Y_S := \bigcup_{i \in I} \Phi_i^{-1}(U_i \times S)$. The pair $(p_S := p|_{Y_S} \colon Y_S \to B, \{\Phi_i \colon p_S^{-1}(U_i) \to U_i \times S\}_{i \in I})$ is an $(S, \mathcal{G}_S)$-bundle.
   (c) The map
$$\begin{array}{ccc} \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) & \to & \mathrm{IsoBund}_{(F,\mathcal{G}_S)}(S^k) \\ [p \colon Y \to S^k] & \mapsto & [p_S \colon Y_S \to S^k] \end{array}$$
   is a bijection.

Proof. The case $k = 1$ can be dealt with easily using the Mapping Torus–$(F, \mathcal{G})$-Bundle Lemma 241.11. We leave the details to the reader. In the following we now assume that $k \in \mathbb{N}_{\geq 2}$.

(1) The explicit bijections from the Bundle-over-Sphere Proposition 241.16 (2) give us the following *commutative* diagram:
$$\begin{array}{ccc} [S^{k-1}, \mathcal{G}_{\mathrm{id}}] & \longrightarrow & [S^{k-1}, \mathcal{H}_{\mathrm{id}}] \\ \Theta \downarrow \cong & & \cong \downarrow \Theta \\ \mathrm{IsoBund}_{(F,\mathcal{G})}(S^k) & \longrightarrow & \mathrm{IsoBund}_{(F,\mathcal{H})}(S^k). \end{array}$$
We make the following observations:
- By the Bundle-over-Sphere Proposition 241.16 the vertical maps are bijections.
- We now assume that $\mathcal{G} \to \mathcal{H}$ is a homotopy equivalence. If we work in the homotopy category, then it follows easily from the Homotopy Equivalence-Reformulation Lemma 19.13 (see also Exercise 19.9) that the top horizontal map is a bijection.

It follows from this discussion that the bottom horizontal map is also a bijection.

(2) (a) It follows easily from our hypothesis on S that the map $\mathcal{G} \to \mathcal{G}_S$ is a bijection. Using the Subbasis-Continuity Proposition 4.4 and the Subbasis-Open Map Lemma 4.14 one can easily verify that the map is continuous and open. It thus follows from the Open-Closed–Injective Lemma 4.22 that the map is a homeomorphism.
   (b) This statement follows easily from the definitions.
   (c) Similar to (1) this statement follows easily from the Bundle-over-Sphere Proposition 241.16. ∎

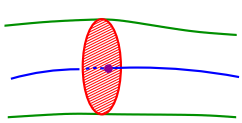
We formulate our main examples of the Structure Group–Homotopy Type Lemma 241.17 as a corollary:

Corollary 241.18. (Linear Bundle Corollary) For each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ we have bijections

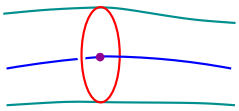
$$\underset{(\overline{B}^n,\mathrm{O}(n))}{\mathrm{IsoBund}}(S^k) \xrightarrow{\cong} \underset{(S^{n-1},\mathrm{O}(n))}{\mathrm{IsoBund}}(S^k) \xleftarrow{\cong} \underset{(\mathbb{R}^n,\mathrm{O}(n))}{\mathrm{IsoBund}}(S^k) \xrightarrow{\cong} \underset{(\mathbb{R}^n,\mathrm{GL}(n,\mathbb{R}))}{\mathrm{IsoBund}}(S^k)$$

and we have bijections

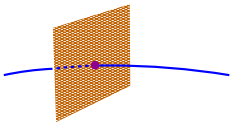
$$\underset{(\overline{B}^n,\mathrm{SO}(n))}{\mathrm{IsoBund}}(S^k) \xrightarrow{\cong} \underset{(S^{n-1},\mathrm{SO}(n))}{\mathrm{IsoBund}}(S^k) \xleftarrow{\cong} \underset{(\mathbb{R}^n,\mathrm{SO}(n))}{\mathrm{IsoBund}}(S^k) \xrightarrow{\cong} \underset{(\mathbb{R}^n,\mathrm{GL}_+(n,\mathbb{R}))}{\mathrm{IsoBund}}(S^k).$$



$(\overline{B}^2, \mathrm{O}(2))$-bundle



$(S^1, \mathrm{O}(2))$-bundle



$(\mathbb{R}^2, \mathrm{O}(2))$-bundle

Proof. The corollary follows from the following observations:

- In both cases the two bijections to the left are provided by the Structure Group–Homotopy Type Lemma 241.17 (2), applied to $F = \overline{B}^n$, $S = S^{n-1}$ and $F = \mathbb{R}^n$ and $S = S^{n-1}$.
- By the Matrix Groups Deformation Retract Proposition 19.4 we know that the inclusions $\mathrm{O}(n) \to \mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SO}(n) \to \mathrm{GL}_+(n,\mathbb{R})$ are homotopy equivalences. Thus it follows from the Structure Group–Homotopy Type Lemma 241.17 (1) that in both cases the map to the right is also a bijection. ∎

The Bundle-over-Sphere Proposition 241.16 shows that we should try to understand the set $[S^{k-1}, \mathcal{G}]$. This set looks of course very similar to the homotopy group $\pi_{k-1}(\mathcal{G}, \mathrm{id})$ of homotopy classes of continuous maps $(S^{k-1}, *) \to (\mathcal{G}, \mathrm{id})$ of *pointed* topological spaces.

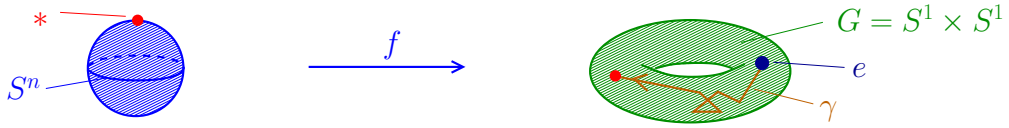
Fortunately the following proposition shows that if $\mathcal{G}$ is path-connected, then the obvious map between these two sets is in fact a bijection:

Proposition 241.19. (Sphere-to-Topological Group Proposition) For any path-connected topological group G and any $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \pi_n(G,e) & \to & [S^n, G] \\ [f\colon (S^n,*) \to (G,e)] & \mapsto & [f] \end{array}$$

is a bijection.

Proof. [2619] We first prove that the given map $\theta\colon \pi_n(G,e) \to [S^n, G]$ is surjective. Thus let $f\colon S^n \to G$ be a continuous map. Since G is by hypothesis path-connected there exists a path $\gamma\colon [0,1] \to G$ with $\gamma(0) = e$ and $\gamma(0) = f(*)$.



We consider the map

$$\begin{array}{rcl} H\colon S^n \times [0,1] & \to & G \\ (y,t) & \mapsto & f(y) \cdot \gamma(t)^{-1}. \end{array}$$

multiplication in G

[2619] Note that this proof also gives a solution to Exercise 140.3.

Next note that it follows easily from the fact that G is a topological group that the above map H is continuous. Note that $H_0 = f$ and $H_1(*) = f(*) \cdot \gamma(1)^{-1} = f(*) \cdot f(*)^{-1} = e$. This shows that the map $H_1 \colon S^n \to G$ defines an element in $\pi_n(G, e)$. We thus see that $[f] = [H_0] = [H_1] = \theta([H_1]) \in [S^n, G]$.

Now we prove (with almost the same idea) that the given map $\theta \colon \pi_n(G, e) \to [S^n, G]$ is also injective. Thus let $f_0, f_1 \colon (S^n, *) \to (G, e)$ be two continuous maps such that $\theta([f_0]) = \theta([f_1]) \in [S^n, G]$. This means that there exists a homotopy $H \colon S^n \times [0, 1] \to G$ with $H_0 = f_0$ and with $H_1 = f_1$. We consider the map

$$\begin{array}{rcl} S^n \times [0,1] & \to & G \\ (y, t) & \mapsto & H(y,t) \cdot H(*,t)^{-1}. \end{array}$$

As before it follows from the fact that G is a topological group that this map is continuous. This map is now a base-point preserving homotopy between f_0 and f_1, which implies that $[f_0] = [f_1] \in \pi_n(G, e)$. ■

241.7. **Classification of sphere bundles over spheres.** In this section we will try to classify $(S^n, \mathrm{SO}(n+1))$-bundles over S^k, at least for small values of k. By the Linear Bundle Corollary 241.18 the classification results below also classify several other types of bundles at absolutely no extra expense.

First note that it follows immediately from the Mapping Torus–$(F, \mathcal{G})$-Bundle Lemma 241.11 that every $(S^n, \mathrm{SO}(n+1))$-bundle over S^1 is trivializable and that (up to isomorphism) there exists precisely one non-trivial $(S^n, \mathrm{O}(n+1))$-bundle over S^1.

Before we move on to bundles over higher dimensional spheres let us summarize some of our earlier results:

Proposition 241.20. (Sphere Bundle-over-Sphere Proposition) Let $k, n \in \mathbb{N}$. We consider the sphere $S^k =_{\mathrm{i}} \overline{B}^k_- \cup_{S^{k-1}} \overline{B}^k_+$.

(1) The map

$$\begin{array}{rcl} \pi_{k-1}(\mathrm{SO}(n+1), \mathrm{id}) & \to & \mathrm{IsoBund}_{(S^n, \mathrm{SO}(n+1))}(S^k) \\ [\varphi \colon S^{k-1} \to \mathrm{SO}(n+1)] & \mapsto & \underbrace{(p(\varphi) \colon (\overline{B}^k_- \times S^n) \cup_{(x,v) \sim (x, \varphi(x) \cdot v)} (\overline{B}^k_+ \times S^n) \to S^k)}_{\text{defined in the Bundle–Clutching Map Lemma 241.14}} \end{array}$$

is a bijection.

(2) Let $k \in \mathbb{N}_{\geq 2}$ and let $H \in \mathrm{O}(n+1) \setminus \mathrm{SO}(n+1)$. For $\varphi \colon S^{k-1} \to \mathrm{SO}(n+1)$ we write $\varphi \sim (g \mapsto H \cdot \varphi(g) \cdot H^{-1})$. With this notation the map

$$\pi_{k-1}(\mathrm{SO}(n+1), \mathrm{id}) / \sim \to \mathrm{IsoBund}_{(S^n, \mathrm{O}(n+1))}(S^k)$$

is a bijection.

Proof. Note that by the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{O}(n+1)$ has precisely two path-components, one of which is $\mathrm{SO}(n+1)$. The proposition now follows almost immediately from the Bundle-over-Sphere Proposition 241.16 together with the Sphere-to-Topological Group Proposition 241.19. ■

We will first apply the Sphere Bundle-over-Sphere Proposition 241.20 to $(S^n, \mathrm{SO}(n+1))$-bundles over S^2. As we will see, we need to distinguish between the case of S^1-bundles over S^2 and the case of higher-dimensional sphere bundles over S^2.

Proposition 241.21. (S^1-Bundle-over-S^2-Proposition) given any $n \in \mathbb{Z}$ we consider the map

$$\begin{array}{rcl} \varphi_n \colon S^1 & \to & \mathrm{SO}(2) \\ \exp(it) & \mapsto & \begin{pmatrix} \cos(n \cdot t) & -\sin(n \cdot t) \\ \sin(n \cdot t) & \cos(n \cdot t) \end{pmatrix} \end{array}$$

and we consider the corresponding $(S^1, \mathrm{SO}(2))$-bundle $p_n(\varphi_n) \colon Y(\varphi_n) \to S^2$.

(1) (a) For every $(S^1, \mathrm{SO}(2))$-bundle $p \colon Y \to S^2$ there exists a unique $n \in \mathbb{Z}$ such that this bundle is isomorphic over S^2 to the $(S^1, \mathrm{SO}(2))$-bundle $p(\varphi_n) \colon Y(\varphi_n) \to S^2$.

(b) We have $\pi_1(Y(\varphi_n)) \cong \mathbb{Z}/n\mathbb{Z}$.

(c) The total space $Y(\varphi_n)$ admits an orientation-preserving diffeomorphism to the lens space $L(n,1)$.

(2) (a) Every $(S^1, \mathrm{O}(2))$-bundle $p \colon Y \to S^2$ is isomorphic to one of $p(\varphi_n) \colon Y(\varphi_n) \to S^2$.

(b) The $(S^1, \mathrm{SO}(2))$-bundles $p(\varphi_m) \colon Y(\varphi_m) \to S^2$ and $p(\varphi_n) \colon Y(\varphi_n) \to S^2$ are isomorphic as $(S^1, \mathrm{O}(2))$-bundles if and only if $m = \pm n$.

Example. For $n \in \mathbb{Z}$ we consider the $(S^1, \mathrm{SO}(2))$-bundle $p(\varphi_n) \colon Y(\varphi_n) \to S^2$ from the S^1-Bundle-over-S^2-Proposition 241.21. In Exercise 241.16 we will see that we have the following isomorphisms as $(S^1, \mathrm{SO}(2))$-bundles:

- For $n = 0$ we have $Y(\varphi_0) \cong S^2 \times S^1$.
- For $n = 1$ we have $Y(\varphi_1) \cong S^3$, where we view S^3 as the total space of the Hopf bundle $H_{\mathbb{C}} \colon S^3 \to S^2$, which, by the discussion on page 4635, we can view as an $(S^1, \mathrm{SO}(2))$-bundle.
- For $n = 2$ we have $Y(\varphi_2) \cong \mathbb{R}\mathrm{P}^3 \cong \mathrm{SO}(3)$, where we use the Matrix Group-Fiber Proposition 238.12 to view $\mathrm{SO}(3)$ as the total space of an $\mathrm{SO}(2) \cong S^1$-bundle over the sphere S^2.[2620] □

Proof.

(1) (a) This statement follows immediately from the Sphere Bundle-over-Sphere Proposition 241.20 (1) and the observation that by the Circle-π_1-Theorem 88.22 we know that the following map is an isomorphism:

$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_1(\mathrm{SO}(2), \mathrm{id}) \\ n & \mapsto & [\varphi_n]. \end{array}$$

(b) We introduce the following notation:

- We again work with $S^2 =_{\mathrm{i}} \overline{B}^2_+ \cup_{S^1} \overline{B}^2_-$.
- We denote by $\iota \colon S^1 \times S^1 \to \overline{B}^2_+ \times S^1$ the inclusion.
- We consider the map

$$\begin{array}{rcl} \Phi_n \colon S^1 \times S^1 & \to & \overline{B}^2_- \times S^1 \\ (c, f) & \mapsto & (c, c^n \cdot f). \end{array}$$

- We pick a base point $* \in S^1 \times S^1$.
- We make the obvious identifications $\pi_1(\overline{B}^2_\pm \times S^1, *) = \langle x_\pm \rangle$.
- We make the obvious identifications $\pi_1(S^1 \times S^1, *) = \langle a, b \,|\, [a,b] \rangle$.

[2620] We did not show that this bundle admits an $(S^1, \mathrm{SO}(2))$-bundle atlas. Some more careful work, which we will unfortunately not be able to do for time reasons, shows that it does in fact admit an $(S^1, \mathrm{SO}(2))$-bundle atlas.

It follows from the discussion on page 1760 that the induced map on fundamental groups are given by

$$\begin{array}{rcl} \iota_*\colon \pi_1(S^1\times S^1,*) & \to & \pi_1(\overline{B}^2_+\times S^1) \\ a & \mapsto & e \\ b & \mapsto & x_- \end{array} \quad \text{and} \quad \begin{array}{rcl} \Phi_{n*}\colon \pi_1(S^1\times S^1,*) & \to & \pi_1(\overline{B}^2_+\times S^1) \\ a & \mapsto & (x_+)^n \\ b & \mapsto & x_+ \end{array}$$

We now see that

$$\begin{array}{lll} & \overset{\text{by definition of } Y(\varphi_n)}{\downarrow} & \overset{\text{by the Manifold–Seifert-van Kampen Theorem 102.1}}{\downarrow} \\ \pi_1(Y(\varphi_n),*) & = \pi_1((\overline{B}^2_-\times S^1)\cup_{\Phi_n}(\overline{B}^2_+\times S^1),*) & = \pi_1(\overline{B}^2_-\times S^1,*) *_{\pi_1(S^1\times S^1,*)} \pi_1(\overline{B}^2_+\times S^1,*) \\ & = \langle x_-\rangle_{\langle a,b\,|\,[a,b]\rangle}\langle x_+\rangle & = \langle x_+, x_- \,|\, \iota_*(a)=\Phi_{n*}(a), \iota_*(b)=\Phi_{n*}(b)\rangle \\ & \overset{\text{by the above}}{\uparrow} & \overset{\text{by the Amalgamated Product Presentation Lemma 100.20}}{\uparrow} \\ & \cong \langle x_+, x_- \,|\, e=(x_+)^n, x_-=x_+\rangle & = \langle x \,|\, x^n=1\rangle. \\ & \overset{\text{by the above discussion of } \iota_* \text{ and } \Phi_{n*}}{\uparrow} & \overset{\text{Tietze transformation see Definition 100.24}}{\uparrow} \end{array}$$

(c) This statement is a special case of the Lens Space–Heegaard Splitting Proposition **??** which we will prove later, when we discuss 3-dimensional smooth manifolds in greater detail.

(2) First note that for any $n\in\mathbb{Z}$ we have

$$\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\in \mathrm{O}(2)\setminus\mathrm{SO}(2)} \cdot \underbrace{\begin{pmatrix}\cos(n\cdot t)&-\sin(n\cdot t)\\ \sin(n\cdot t)&\cos(n\cdot t)\end{pmatrix}}_{=\varphi_n(t)} \cdot \begin{pmatrix}0&1\\1&0\end{pmatrix}^{-1} = \underbrace{\begin{pmatrix}\cos(n\cdot t)&\sin(n\cdot t)\\ -\sin(n\cdot t)&\cos(n\cdot t)\end{pmatrix}}_{=\varphi_{-n}(t)}.$$

The desired statement is now, as in (1a), an immediate consequence of Sphere Bundle-over-Sphere Proposition 241.20 (2). ■

Next we turn to higher-dimensional spheres bundles over S^2:

Proposition 241.22. (S^n-Bundle-over-S^2-Proposition) Let $k\in\mathbb{N}_{\geq 2}$. We consider the map

$$\begin{array}{rcl} \varphi\colon S^1 & \to & \mathrm{SO}(n+1) \\ \exp(\mathrm{i}t) & \mapsto & \begin{pmatrix}\cos(t)&-\sin(t)&0\\ \sin(t)&\cos(t)&0\\ 0&0&\mathrm{id}_{n-1}\end{pmatrix}. \end{array}$$

(1) Every $(S^n,\mathrm{SO}(n+1))$-bundle over the sphere S^2 is either isomorphic to the trivial bundle $S^2\times S^n\to S^2$ or to the non-trivial bundle $p(\varphi)\colon Y(\varphi)\to S^2$.

(2) The statement in (1) also applies to $(S^n,\mathrm{O}(n+1))$-bundles over S^2.

Definition 241.23. Let $n\in\mathbb{N}_{\geq 2}$. Using the notation from the S^n-Bundle-over-S^2-Proposition 241.22 we write

$$S^n \mathbin{\widetilde{\times}} S^2 := Y(\varphi).$$

We refer to the $(S^n,\mathrm{SO}(n+1))$-bundle $p\colon S^n\mathbin{\widetilde{\times}} S^2\to S^2$ as the twisted $(S^n,\mathrm{SO}(n+1))$-bundle over S^2.

Example. Recall that in the Projective Space–Connected Sum Proposition 238.10 we showed that there exists a smooth bundle map $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2) \to S^2$ with fiber S^2 and which is not trivializable. Next recall that on page 4635 we showed that the explicit atlas that we constructed is in fact $\mathrm{SO}(3)$-compatible. With the above notation we can therefore write $\mathbb{C}\mathrm{P}^2\#(-\mathbb{C}\mathrm{P}^2) \cong S^2 \mathbin{\widetilde{\times}} S^2$. □

Proof. Let $n \in \mathbb{N}_{\geq 2}$. Recall that we know by the Matrix Group-π_1-Proposition 239.15 that $\pi_1(\mathrm{SO}(n+1)) \cong \mathbb{Z}/2\mathbb{Z}$ and that the unique non-trivial element is represented by the above loop $\varphi\colon S^1 \to \mathrm{SO}(n+1)$.

(1) This statement follows now from Sphere Bundle-over-Sphere Proposition 241.20 (1).
(2) As in the proof of the S^1-Bundle-over-S^2-Proposition 241.21 (2) we see that the action of $\mathrm{O}(n+1)/\mathrm{SO}(n+1)$ on $[S^1, \mathrm{SO}(2)] \cong \pi_1(\mathrm{SO}(2))$ turns $[\varphi]$ into $[g \mapsto \varphi(g)^{-1}] = [\varphi]^{-1}$. But it follows from $\pi_1(\mathrm{SO}(n+1)) \cong \mathbb{Z}/2\mathbb{Z}$ that $[g \mapsto \varphi(g^{-1})] = [\varphi]$, thus we see that the action of $\mathrm{O}(n+1)/\mathrm{SO}(n+1)$ is trivial. The promised statement follows from this observation. ■

We move on to $(S^n, \mathrm{SO}(n+1))$-bundles over S^3. Here we get a very clean and surprisingly dull statement:

Proposition 241.24. (S^n-Bundle-over-S^3-Proposition) Every $(S^n, \mathrm{SO}(n+1))$-bundle and every $(S^n, \mathrm{O}(n+1))$-bundle over S^3 is trivializable.

Proof. By the Matrix Group-π_2-Proposition 239.16 we know that $\pi_2(\mathrm{SO}(n+1)) = 0$ for all n. It follows from the Sphere Bundle-over-Sphere Proposition 241.20 that every $(S^n, \mathrm{SO}(n+1))$-bundle over S^3 is trivializable. ■

So how about $(S^n, \mathrm{SO}(n+1))$-bundles over S^4? This now requires the calculation of $\pi_3(\mathrm{SO}(n+1))$ for $n \in \mathbb{N}$. In fact we have the following isomorphisms:

- We have $\pi_3(\mathrm{SO}(2)) \cong \pi_3(S^1) = 0$.
- We note that

$$\pi_3(\mathrm{SO}(3)) \underset{\text{by the Projective Space Avatar Lemma 9.16}}{\cong} \pi_3(\mathbb{R}\mathrm{P}^3) \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_3(S^3) \underset{\text{by the } \pi_n(S^n)\text{–Proposition 167.10}}{\cong} \mathbb{Z}.$$

  So we see that there are "$\mathbb{Z}$ many" isomorphism types of $(S^2, \mathrm{SO}(3))$-bundles over S^4.
- In the later Matrix Group-Isomorphisms Proposition 299.16 we will see that we have a homeomorphism $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(\mathrm{id}, \mathrm{id})\}$. It follows that

$$\begin{array}{ccccccc} \pi_3(\mathrm{SO}(4)) & \cong & \pi_3((\mathrm{SU}(2)\times\mathrm{SU}(2))/\{\pm(\mathrm{id},\mathrm{id})\}) & \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} & \pi_3(\mathrm{SU}(2)\times\mathrm{SU}(2)) & & \\ & \underset{\text{by the Product-}\pi_n\text{ Proposition 134.18}}{\cong} & \pi_3(\mathrm{SU}(2)) \times \pi_3(\mathrm{SU}(2)) & & \underset{\text{by the SU(2) – SO(3) Covering Theorem 94.3 (1)}}{\cong} \pi_3(S^3)\times\pi_3(S^3) & \underset{\text{by the } \pi_n(S^n) \text{ Proposition 167.10}}{\cong} & \mathbb{Z}\times\mathbb{Z}. \end{array}$$

  So we see that there are "$\mathbb{Z}^2$ many" isomorphism types of $(S^3, \mathrm{SO}(4))$-bundles over S^4. In fact we know by the discussion on page 4635 that the Hopf bundle $H\colon S^7 \to S^4$ is

a non-trivial $(S^3, \mathrm{SO}(4))$-bundle over S^4. We leave it to the reader to figure out what parameter in $\mathbb{Z}^2$ this corresponds to.

- By the Homotopy Group-O-U-Sp-Proposition 299.23 together with the Homotopy Group-O-U-Sp-Lemma 299.20 we know that $\pi_3(\mathrm{SO}(n)) \cong \mathbb{Z}$ for $n \geq 5$. In fact in the Projective Space–Connected Sum Proposition 238.10 and on page 4635 we showed that there exists a smooth $(S^4, \mathrm{SO}(5))$-bundle $\mathbb{H}\mathrm{P}^2 \# (-\mathbb{H}\mathrm{P}^2) \to S^4$ which is not trivializable. Again we leave it to the reader to figure out what parameter in $\mathbb{Z}$ this corresponds to.

We will not attempt to study bundles over spheres S^k of dimension ≥ 5.

241.8. **Bundles over surfaces.** In the S^1-Bundle-over-S^2-Proposition 241.21 and the S^k-Bundle-over-S^2-Proposition 241.22 we classified some types of bundles over the simplest closed orientable non-empty surface, namely over the sphere. In this section we will obtain some partial results about bundles over arbitrary compact orientable surfaces. We start out with bundles over surfaces with *non-empty boundary*:

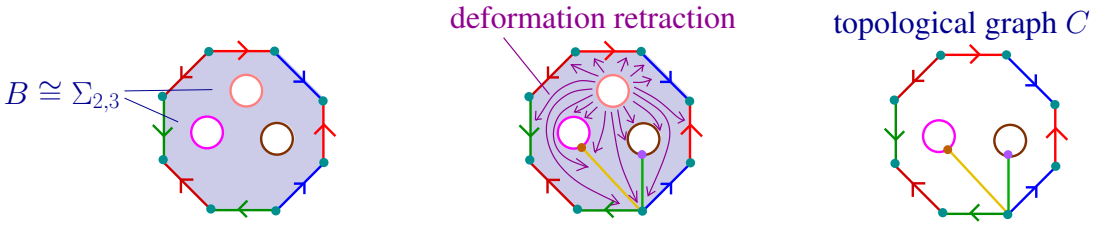
Lemma 241.25. (Bundle-over-Surface-with-Boundary Lemma) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let B be a compact connected 2-dimensional smooth manifold. If $\partial B \neq \varnothing$ and if $\mathcal{G}$ is path-connected, then every $(F, \mathcal{G})$-bundle over B is trivializable.

Proof. Let B be a compact connected 2-dimensional smooth manifold with non-empty boundary. The lemma is proved using three observations:

(1) First note that it follows from the Surface Classification Theorem 101.16 there exist $g \in \mathbb{N}_0$ and $n \geq 1$ such that B is diffeomorphic to

$$\Sigma_{g,n} \ := \ \text{the standard genus } g \text{ surface minus } n \text{ open disks.}$$

As in the proof of the Surface-with-Boundary–π_1-Lemma 101.13 one can show easily show that $\Sigma_{g,n}$ admits a deformation retraction to a topological graph C.



Alternatively, if one does not want to make use of the classification of compact smooth manifolds, one can also appeal to the No Top-Cell Proposition 183.31 to show that B is homotopy equivalent to a 1-dimensional CW-complex.

(2) Since we assume that $\mathcal{G}$ is path-connected we obtain from the Bundle-over-1-Complex Lemma 241.12 that any $(F, \mathcal{G})$-bundle over C is trivializable.

(3) It follows from (1), (2) and the Bundles-Homotopies Proposition 241.10 (1) that any $(F, \mathcal{G})$-bundle over B is also trivializable. ∎

Now we turn to bundles over closed (orientable) surfaces:

Proposition 241.26. (Bundles-over-Surface Proposition) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a *path-connected* subgroup. Furthermore let Σ be a closed connected 2-dimensional smooth manifold and let $\Sigma = B_0 \cup B_1$ be a decomposition into two compact codimension-zero submanifolds such that $C := B_0 \cap B_1 = \partial B_0 = \partial B_1$ is connected and non-empty.

(1) The map

$$\begin{array}{rccl} \Theta\colon [\overbrace{B_0 \cap B_1}^{=C}, \mathcal{G}] & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \\ [\varphi] & \mapsto & [p(\varphi)\colon Y(\varphi) \to \Sigma] \end{array}$$

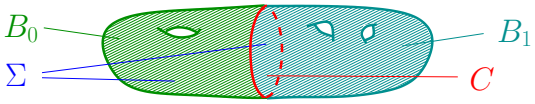
from the Bundle–Clutching Map Lemma 241.14 is a surjection.

We now assume that Σ is orientable.

(2) We pick a generator $[C]$ of the homology group $\mathrm{H}_1(C) \cong \mathbb{Z}$. There exists a map $\alpha\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \to \mathrm{H}_1(\mathcal{G})$ such that the following diagram commutes:

$$\begin{array}{ccc} [B_0 \cap B_1, \mathcal{G}] & \xrightarrow{\quad\Theta\quad} & \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \\ & {\scriptstyle \varphi \mapsto \varphi_*([C])} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & \mathrm{H}_1(\mathcal{G}). & \end{array}$$

(3) The three maps in the diagram in (2) are all bijections.



The proof of the Bundles-over-Surface Proposition 241.26 will require the remainder of this section. The proof relies on two general lemmas about induced maps on singular homology:

Lemma 241.27. (Multiplication-Homology Lemma) Let C be a path-connected topological space, let G be a path-connected topological group and let $f, g\colon C \to G$ be two continuous maps. We consider the map

$$\begin{array}{rccl} f \cdot g\colon C & \to & G \\ c & \mapsto & f(c) \cdot g(c). \end{array}$$

For any $\alpha \in \mathrm{H}_1(C)$ we have the following equality in $\mathrm{H}_1(G)$:

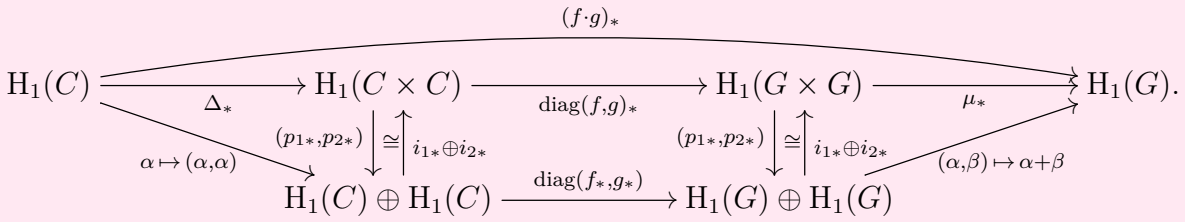
$$(f \cdot g)_*(\alpha) \;=\; f_*(\alpha) + g_*(\alpha).$$

Proof. We introduce the following notation:

(1) We denote by $\Delta\colon C \to C \times C$ the diagonal map that is given by $c \mapsto (c, c)$.
(2) We denote by $\mathrm{diag}(f, g)\colon C \times C \to G \times G$ the map that is given by $(c, c') \mapsto (f(c), g(c'))$.
(3) We denote by $\mu\colon G \times G \to G$ the multiplication map.

With this notation we can write $f \cdot g = \mu \circ \mathrm{diag}(f, g) \circ \Delta\colon C \to G$. It remains to prove the following claim:

Claim. The following diagram:

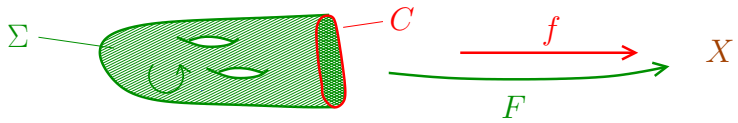


commutes. (We will explain the vertical maps in the proof.)

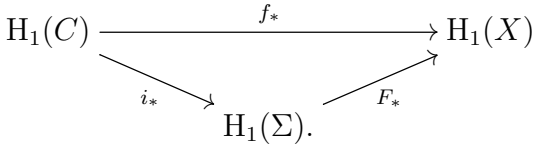
Proof. We make the following clarification and observations:

(1) We pick $c_0 \in C$. The maps $i_j\colon C \to C\times C$ and $i_j\colon G \to G\times G$ are given by the inclusions into the first or second factor (with the other entry being c_0 respectively e.)
(2) It follows from the Künneth Theorem 164.6 together with the Cross Product Lemma 164.5 (and our hypothesis that C and G are path-connected) that the upward vertical maps are isomorphisms.
(3) The maps $p_j\colon C\times C\to C$ and $p_j\colon G\times G\to G$ are the projections onto the j-th factor. The map (p_{1*},p_{2*}) is clearly a left inverse to $i_1\oplus i_2$. It follows from (2) that the maps (p_{1*},p_{2*}) are also isomorphisms.
(4) We now turn to the three parts of the diagram:
   (a) It follows from $p_i\circ\Delta=\mathrm{id}_C$ that the left triangle commutes.
   (b) It follows from $p_i\circ\mathrm{diag}(f,g)=f\circ p_i$ that the rectangle commutes.
   (c) It follows from $\mu\circ i_j=\mathrm{id}_G$ that the right triangle commutes. ∎

Lemma 241.28. (Map Extends–Homology Lemma) Let Σ be a compact orientable 2-dimensional smooth manifold with precisely one boundary component C. Let $f\colon C\to X$ be a continuous map to some topological space X. If f extends to a continuous map $F\colon\Sigma\to X$, then $f_*\colon\mathrm{H}_1(C)\to\mathrm{H}_1(X)$ is the zero map.



Proof of the Map Extends–Homology Lemma 241.28. We denote by $i\colon C\to\Sigma$ the inclusion. Let $F\colon\Sigma\to X$ be a continuous map that is an extension of $f\colon C\to X$. This means that $f=F\circ i$. By functoriality of singular homology we have the following commutative diagram:

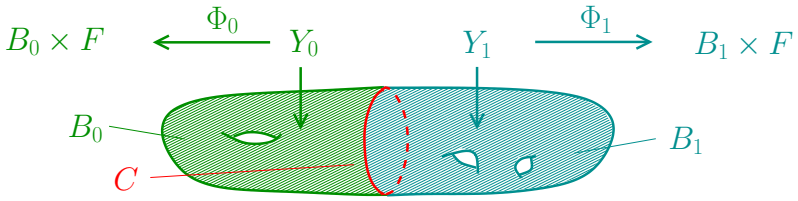


Since Σ is compact and orientable, and since C is the unique boundary component of Σ it follows from the Kernel-of-Boundary Inclusion Corollary 185.20 (or alternatively from the Kernel-of-Boundary Inclusion Corollary 194.14) that the left diagonal map is the zero map. ∎

Sketch of proof of the Bundles-over-Surface Proposition 241.26. Let F be a topological space that is regionally compact, Hausdorff and locally connected. Furthermore let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a *path-connected* subgroup.

Let Σ be a closed connected 2-dimensional smooth manifold and let $\Sigma = B_0 \cup B_1$ be a decomposition into two compact codimension-zero submanifolds such that the intersection $C := B_0 \cap B_1 = \partial B_0 = \partial B_1$ is connected and non-empty. Note that it follows easily from our hypotheses that B_0 and B_1 are both compact *connected* 2-dimensional smooth manifolds with non-empty boundary. We set $Y_0 := p^{-1}(B_0)$ and $Y_1 := p^{-1}(B_1)$.

(1) Let $p\colon Y \to \Sigma$ be an $(F, \mathcal{G})$-bundle. Since $\mathcal{G}$ is path-connected it follows from the Bundle-over-Surface-with-Boundary Lemma 241.25 that p is trivializable over B_0 and also over B_1. As in the proof of the Bundle-over-Sphere Proposition 241.16 we pick trivializations $\Phi_i\colon Y_i \to B_i \times F$.



It follows again from the Bundle–Action-on-Trivializations Proposition 240.17 that there exists a unique continuous map $\Omega := \Omega_{\Phi_0,\Phi_1}\colon C \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccccc} C \times F & \xleftarrow{\Phi_0} & p^{-1}(C) & \xrightarrow{\Phi_1} & C \times F. \\ & & \underset{(c,f)\mapsto(c,\Omega_{\Phi_0,\Phi_1}(c)(f))}{\searrow} & & \end{array}$$

As in the proof of the Bundle-over-Sphere Proposition 241.16 we see that Φ_0 and Φ_1 give rise to an isomorphism from the original $(F, \mathcal{G})$-bundle $p\colon Y \to \Sigma$ to the $(F, \mathcal{G})$-bundle $p(\Omega)\colon Y(\Omega) \to \Sigma$.

(2) We need to show that there exists a map $\alpha\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \to \mathrm{H}_1(\mathcal{G})$ such that the following diagram commutes:

$$\begin{array}{ccc} [C, \mathcal{G}] & \xrightarrow{\Theta} & \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \\ & {\scriptstyle \varphi\mapsto\varphi_*([C])} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & \mathrm{H}_1(\mathcal{G}). & \end{array}$$

Since in (1) we already established that the top horizontal map is an epimorphism it follows, by the argument of the proof of the Bundle-over-Sphere Proposition 241.16 on page 4670, that it suffices to prove the following claim:

Claim. The homology class $\Omega_{\Phi_0,\Phi_1,*}([C]) \in \mathrm{H}_1(\mathcal{G})$ does not depend on the choice of the trivializations Φ_0, Φ_1.

Proof. Since the situation is symmetric in B_0 and B_1 it suffices to show that the homology class does not depend on the choice of the trivialization Φ_1. Suppose we are given another trivialization $\Psi_1\colon Y_1 \to B_1 \times F$. As on page 4670, in the proof of the Bundle-over-Sphere Proposition 241.16, we see that there exists a continuous map

$\tau\colon B_1 \to \mathcal{G}$ such that the following diagram commutes:

$$B_1 \times F \xleftarrow{\Phi_1} p^{-1}(B_1) \xrightarrow{\Psi_1} B_1 \times F.$$
$$(b,f) \mapsto (b, \tau(b)(f))$$

It follows that for all $b \in B_1$ we have $\Omega_{\Phi_0,\Psi_1}(b) = \tau(b) \circ \Omega_{\Phi_0,\Phi_1}(b) \in \mathcal{G}$. We now see that

$$\Omega_{\Phi_0,\Psi_1,*}([C]) \underset{\uparrow}{=} (\tau|_C)_*([C]) + \Omega_{\Phi_0,\Phi_1,*}([C]) \underset{\uparrow}{=} \Omega_{\Phi_0,\Phi_1,*}([C]).$$

by the Multiplication Homology Lemma 241.27 applied to $G = \mathcal{G}$

by the Map Extends Homology Lemma 241.28, since $\tau|_C$ extends to a continuous map $\tau\colon B_1 \to \mathcal{G}$ (here we use that Σ, and thus B_1, is orientable)

⊞

(3) We consider the following diagram:

$$\begin{array}{ccc} [C,\mathcal{G}] & \xrightarrow{\Theta} & \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \\ \cong\uparrow\gamma_* & \searrow^{\varphi\mapsto\varphi_*([C])} & \nearrow_{\alpha} \\ \pi_1(\mathcal{G},\mathrm{id}) & \xrightarrow[\cong]{[f]\mapsto f_*([S^1])} & \mathrm{H}_1(\mathcal{G}). \end{array}$$

We make the following clarifications and observations:

(a) In (1) we showed that the top horizontal map is a surjection.

(b) In (2) we just showed that there exists a map $\alpha\colon \mathrm{IsoBund}_{(F,\mathcal{G})}(\Sigma) \to \mathrm{H}_1(\mathcal{G})$ such that the large triangle commutes.

(c) We pick an orientation-preserving diffeomorphism $\gamma\colon C \to S^1$. The left vertical map is the induced by the map γ together with "forgetting base points". It follows from the Sphere-to-Topological Group Proposition 241.19 that the vertical map is a bijection.

(d) The bottom horizontal map is the Hurewicz homomorphism.

(e) Since $\mathcal{G}$ is a topological group we obtain from the Topological Group-π_1-Proposition 90.16 that $\pi_1(\mathcal{G},\mathrm{id})$ is abelian.

(f) It follows easily from the definition of the Hurewicz homomorphism that the small triangle to the left commutes.

(g) It follows from the fact that $\pi_1(\mathcal{G},\mathrm{id})$ is abelian and the Hurewicz Theorem 166.10 that the bottom horizontal map is an isomorphism.

(h) It follows from (c), (f) and (g) that the left diagonal map is a bijection.

(i) It follows easily from (a), (b) and (h) that the top horizontal map and the right diagonal map are bijections. ■

241.9. Exercises for Chapter 241.

Exercise 241.1. Provide an example of a topological space X and an open cover $\{U_i\}_{i\in I}$ with the following two properties:

(1) Each $x \in X$ is contained in only finitely many U_i.

(2) The open cover is not locally finite.

Exercise 241.2. Is the set $\mathbb{R}$, equipped with the discrete topology, paracompact?

Exercise 241.3. Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Let C be a topological space that is Hausdorff and paracompact. Given $t \in [0,1]$ we denote by

$$\begin{array}{rccc} i_0 \colon & C & \to & C \times [0,1] \\ & c & \mapsto & (c,t) \end{array}$$

the natural embedding. In the Bundle-over-Product Lemma 241.6 we showed that given any $(F,\mathcal{G})$-bundle $p \colon Y \to C \times [0,1]$ the pullbacks i_0^*Y and i_1^*Y are isomorphic. Show in an example that the isomorphism that we constructed in the proof of the Bundle-over-Product Lemma 241.6 depends in general on the choices we make in the proof.

Exercise 241.4. Use the classification of self-diffeomorphisms of 1-dimensional smooth manifolds to determine which compact 2-dimensional smooth manifolds Y are the total space of a bundle map $Y \to I$ where I is a 1-dimensional smooth manifold.

Exercise 241.5. Can one use the long exact sequence of homotopy groups to determine which compact 2-dimensional smooth manifolds are the total space of a fiber bundle, where the fiber and the base space are both 1-dimensional smooth manifolds?

Exercise 241.6. Prove the (ii)⇒(i)-direction of the Mapping Torus–$(F,\mathcal{G})$-Bundle Lemma 241.11. More precisely: Let F be a topological space that is regionally compact, Hausdorff and locally connected, let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup and let $\mu_0, \mu_1 \in \mathcal{G}$. Show that if the two $(F,\mathcal{G})$-bundles $\mathrm{Tor}(F,\mu_0) \to S^1$ and $\mathrm{Tor}(F,\mu_1) \to S^1$ are isomorphic, then there exists a homeomorphism φ of X and a continuous map $H \colon [0,1] \to \mathcal{G}$ with $H(0) = \mu_0$ and $H(1) = \varphi \circ \mu_1 \circ \varphi^{-1}$.
Hint. Let $h \colon [0,1] \to S^1$ be the map that is given by $t \mapsto \exp(2\pi\,\mathrm{i}t)$. Consider the pullback $(F,\mathcal{G})$-bundles $h^*\,\mathrm{Tor}(F,\mu_0) \to [0,1]$ and $h^*\,\mathrm{Tor}(F,\mu_1) \to [0,1]$.

Exercise 241.7. For which $n \in \mathbb{N}$ does there exist a *non-orientable* n-dimensional smooth manifold that is homotopy equivalent to S^1?

Exercise 241.8. Let Σ be an oriented 2-dimensional smooth manifold and let $C \subseteq \Sigma \setminus \partial\Sigma$ be a closed oriented connected non-empty 1-dimensional smooth submanifold. By the Product Tubular Neighborhood–Existence Theorem 58.4 there exists an orientation-preserving smooth embedding $\tau \colon C \times [-1,1] \to \Sigma \setminus \partial\Sigma$. We consider the smooth map

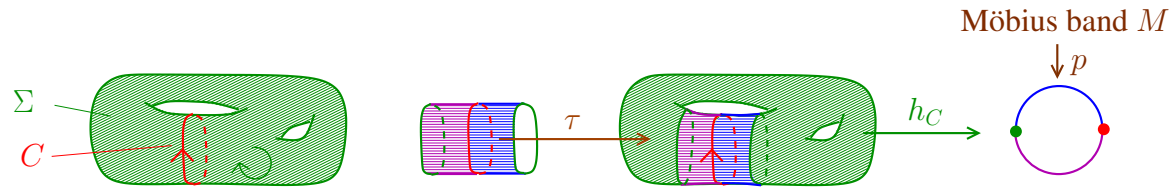
$$\begin{array}{rccl} h_C \colon & \Sigma & \to & S^1 \\ & P & \mapsto & \begin{cases} \exp(2\pi\,\mathrm{i}t), & \text{if } P = \tau(c,t) \text{ for some } c \in C, t \in [-1,1], \\ -1, & \text{otherwise.} \end{cases} \end{array}$$

We also consider the usual Möbius $[-1,1]$-bundle $p \colon M \to S^1$.

(a) Show that if C is non-separating, i.e. if $\Sigma \setminus C$ is connected, then the pullback $[-1,1]$-bundle $(h_C)^*M \to \Sigma$ is not trivializable either.
Hint. Construct an interesting map $S^1 \to \Sigma$.

(b) Show that if C is separating, then the pullback $[-1,1]$-bundle $(h_C)^*M \to \Sigma$ is trivializable.
Hint. Show that the map $\Sigma \to S^1$ is null-homotopic.

Exercise 241.9. For which natural numbers $n, k \in \mathbb{N}_0$ is it the case that the total space of every $(S^n, \mathrm{O}(n+1))$-bundle over S^k is orientable?



Exercise 241.10. Let $f\colon M \to S^2$ be a bundle over S^2 with fiber the torus $T = S^1 \times S^1$. What can we say about the homology groups? What is the possible range for the values of homology groups?

Exercise 241.11. Is the smooth manifold $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ the total space of an $(S^2, \mathrm{SO}(3))$-bundle over S^2?

Exercise 241.12. We consider the S^2-bundle $p\colon S^2 \mathbin{\widetilde{\times}} S^2 \to S^2$.

(a) Show that there exists a continuous map $f\colon S^2 \to S^2 \mathbin{\widetilde{\times}} S^2$ such that $p \circ f$ is homotopic to the identity of S^2.

(b) We fix $y_0 \in S^2$ and we set $F := p^{-1}(y_0)$. We denote by $i\colon F \to S^2 \mathbin{\widetilde{\times}} S^2$ the inclusion. Show that there is no continuous map $g\colon S^2 \mathbin{\widetilde{\times}} S^2 \to F$ such that $g \circ i$ is homotopic to the identity of F.

Hint. Consider the Fiber-LES-Proposition 239.8.

Exercise 241.13. Let $n \in \mathbb{Z}$. As in the S^1-Bundle-over-S^2-Proposition 241.21 we consider the map

$$\begin{array}{rcl} \varphi_n\colon S^1 & \to & \mathrm{SO}(2) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(n\cdot\varphi) & -\sin(n\cdot\varphi) \\ \sin(n\cdot\varphi) & \cos(n\cdot\varphi) \end{pmatrix} \end{array}$$

and we consider the corresponding $(S^1, \mathrm{SO}(2))$-bundle $p_n\colon Y(\varphi_n) \to S^2$.

(a) Compute the homology groups of $Y(\varphi_n)$.

(b) Show that there exists an orientation-reversing diffeomorphism $Y(\varphi_n) \to Y(\varphi_{-n})$.

Exercise 241.14. Let $k, l \in \mathbb{N}$.

(a) Is the total space of every $(S^k, \mathrm{SO}(k+1))$-bundle over S^l the boundary of the total space of a $(\overline{B}^{k+1}, \mathrm{SO}(k+1))$-bundle over S^l?

(b) Is the total space of every $(S^k, \mathrm{SO}(k+1))$-bundle over S^l the boundary of the total space of an $(S^k, \mathrm{SO}(k+1))$-bundle over $\overline{B}^{l+1}$?

Bonus question: What happens if we replace $\mathrm{SO}(k+1)$ by $\mathrm{Homeo}(S^k)$ and $\mathrm{Homeo}(\overline{B}^{k+1})$?

Exercise 241.15.

(a) Classify the isomorphism types of over $S^1 \vee S^1$ where the fiber is the discrete topological space consisting of two points.

(b) Repeat (a) with three instead of two points.

Exercise 241.16. For $n \in \mathbb{Z}$ we consider the $(S^1, \mathrm{SO}(2))$-bundle $p(\varphi_n)\colon Y(\varphi_n) \to S^2$ from the S^1-Bundle-over-S^2-Proposition 241.21. Show that we have the following isomorphisms as S^1-bundles:

- For $n = 0$ we have $Y(\varphi_0) \cong S^1 \times S^2$.
- For $n = 1$ we have $Y(\varphi_1) \cong S^3$, where we view S^3 as the total space of the Hopf bundle.

- For $n = 2$ we have $Y(\varphi_2) \cong \mathbb{R}\mathrm{P}^3 \cong \mathrm{SO}(3)$, where we use the Matrix Group-Fiber Proposition 238.12 to view $\mathrm{SO}(3)$ as the total space of an $\mathrm{SO}(2) \cong S^1$-bundle over S^2.

Exercise 241.17. Let $k \in \mathbb{N}_{\geq 2}$. As in Definition 241.23 we consider the corresponding twisted S^k-bundle $S^k \mathbin{\widetilde{\times}} S^2 := Y(\varphi)$.

(a) What is $\pi_1(S^k \mathbin{\widetilde{\times}} S^2)$?
(b) What are the homology groups of $S^k \mathbin{\widetilde{\times}} S^2$?

Exercise 241.18. For which $n \in \mathbb{N}$ and $k \in \mathbb{N}$ can you classify complex n-dimensional vector bundles over S^k?

Exercise 241.19. For each $n \in \mathbb{N}$ classify $(S^2, \mathrm{O}(3))$-bundles over S^n.

Exercise 241.20. Let $m < n \in \mathbb{N}$ and let $p\colon X \to S^m$ be an S^n-bundle map. Show that for $k \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}_k(X) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, m, n, m+n, \\ 0, & \text{otherwise} \end{cases}$$

such that the inclusion of the fiber induces an isomorphism on n-th homology. There are various approaches:

(a) You could use induction on m and use the Mayer–Vietoris Theorem 153.2 (3).
(b) You could also use the Fiber-LES-Proposition 239.8 and use everything that you know about homology groups of closed orientable smooth manifolds.

Hint. Why not pursue both routes, for safety?

Exercise 241.21. Which parts of the Bundles-over-Surface Proposition 241.26 can be generalized from closed *orientable* connected 2-dimensional smooth manifolds to the *non-orientable* setting?

Exercise 241.22. Let $g \in \mathbb{N}_0$. We consider a decomposition $\Sigma = B_0 \cup B_1$ of the surface of genus g as in the Bundles-over-Surface Proposition 241.26. We showed in particular that isomorphism classes of $(S^1, \mathrm{SO}(2))$-bundles are in one-to-one correspondence with homomorphisms maps $\mathrm{H}_1(C) \to \mathrm{H}_1(\mathrm{SO}(2))$. Since both groups are isomorphic to $\mathbb{Z}$ we see that such a map corresponds to multiplication by $\pm n$ for some $n \in \mathbb{N}_0$. We consider the total space $Y(\varphi_n)$. Compute the homology groups of $Y(\varphi_n)$.

Exercise 241.23. We consider the complex projective space $\mathbb{C}\mathrm{P}^2$. We view $\mathbb{C}\mathrm{P}^1$ in an obvious way as a smooth submanifold of $\mathbb{C}\mathrm{P}^2$. By the Tubular Neighborhood–Existence Theorem 57.11 we know that $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$ admits a tubular neighborhood which is, by definition, a smooth $(\overline{B}^2, \mathrm{O}(2))$-bundle over $\mathbb{C}\mathrm{P}^1$. The boundary is thus an $(S^1, \mathrm{O}(2))$-bundle over $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$. How does this bundle fit into the classification scheme of the S^1-Bundle-over-S^2-Proposition 241.21?
Hint. Consider the self-intersection number of $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$ or consider the total space of the boundary of the tubular neighborhood.

Exercise 241.24. Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let

$$B = (V, E, i\colon E \to V, t\colon E \to V)$$

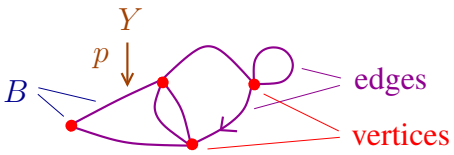
be a connected non-empty graph. [2621] By the Spanning Tree–Existence Proposition 11.26 we know that B admits a spanning tree T. Let S be the set of edges that are not contained in the spanning tree. Show that the map

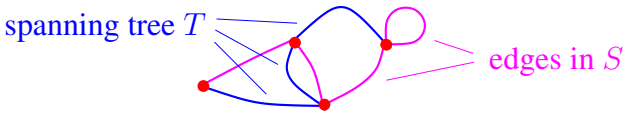
$$\begin{array}{rcl} \pi_0(\mathcal{G})^S & \to & \mathrm{IsoBund}_{(F,\mathcal{G})}(B) \\ \{[\varphi_s]\}_{s\in S} & \mapsto & \Big((T\times F)\sqcup \coprod_{s\in S}([0,1]\times F)\Big)/\sim \end{array}$$

where for every $s \in S$ and $f \in F$ we have $(i(s), f) \sim ((0, f), s)$ and $(t(s), \varphi_s(f)) \sim ((1, f), s)$

is injective.

Remark. This exercise is a companion to the Bundle-over-1-Complex Lemma 241.12.





[2621] Recall that according to Definition 11.1 a *graph* is a quadruple $(V, E, i\colon E \to V, t\colon E \to V)$ consisting of a set V of vertices, a set E of edges, an initial map $i\colon E \to V$ and a terminal map $t\colon E \to V$. As on most occasions we will use the same symbols for the corresponding objects in the topological realization.

CHAPTER 242

The topology of total spaces

In the previous chapters we discussed $(F, \mathcal{G})$-bundles in great detail. In this chapter the focus lies on the topology of the total spaces of $(F, \mathcal{G})$-bundles. In particular we will see that for an $(F, \mathcal{G})$-bundle $p\colon Y \to B$ over a finite CW-complex the Euler characteristic is multiplicative, i.e. we have the equality $\chi(Y) = \chi(B) \cdot \chi(F)$.

This chapter is organized as follows:

- In Section 242.1 we will prove the following: If we are given a morphism of $(F, \mathcal{G})$-bundles such that the map between the base spaces is a homeomorphism (homotopy equivalence), then so is the map between the total spaces.
- In Section 242.2 we will prove the aforementioned statement about Euler characteristics. As we will see, if the base space is a simplicial complex, then the statement can be proved fairly easily using the Bundles-Homotopies Proposition 241.10 and a Mayer–Vietoris sequence. We will reduce the general case to the case of simplicial complexes using the results from Section 242.1.
- In Section 242.3 we will provide exercises for Chapter 242.

In the next Chapter 243 we will introduce smooth bundles.

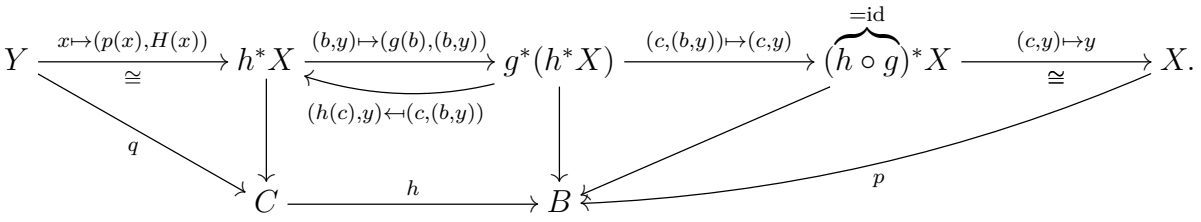
242.1. **Total spaces and homotopy equivalences.** In this section we will see that if we are given a morphism of $(F, \mathcal{G})$-bundles such that the map between the base spaces is a homeomorphism (homotopy equivalence), then so is the map between the total spaces. We start out with the "homeomorphism statement":

Proposition 242.1. (Homeomorphic Bases Proposition) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Also let $q\colon Y \to C$ and $p\colon X \to B$ be two $(F, \mathcal{G})$-bundles. Suppose we are given an $(F, \mathcal{G})$-bundle morphism of the form

$$\begin{array}{ccc} Y & \xrightarrow{H} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B. \end{array}$$

If $h\colon C \to B$ is a homeomorphism, then so is H.

Proof. We denote by $g := h^{-1}\colon B \to C$ the inverse of h. We consider the following diagram:



We make the following clarifications:

- Since we are given a morphism of $(F, \mathcal{G})$-bundles we obtain from the Bundle–Pullback Proposition 240.21 (2) that the first horizontal map is a homeomorphism.
- The second horizontal map is the explicit map provided by the Bundle–Pullback Proposition 240.21 (1). Note that it is also has an explicit inverse that we give with the curved arrow. This shows that the second horizontal map is a homeomorphism.
- The third horizontal map is the explicit isomorphism given by the Bundle-Pullback–Functorial Lemma 240.24 (2).
- The fourth horizontal map is the explicit isomorphism given by the Bundle-Pullback–Functorial Lemma 240.24 (1).

One can easily verify that the map on the top from left to right equals the map $H\colon Y \to X$. Since all four horizontal maps are in particular homeomorphisms, we see that H is indeed a homeomorphism. ■

Now we switch from homeomorphisms to homotopy equivalences:

Proposition 242.2. (Homotopy Equivalent Bases Proposition) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) Let $p\colon X \to B$ be an $(F, \mathcal{G})$-bundle and let $h\colon C \to B$ be a continuous map. We consider the corresponding pullback diagram:

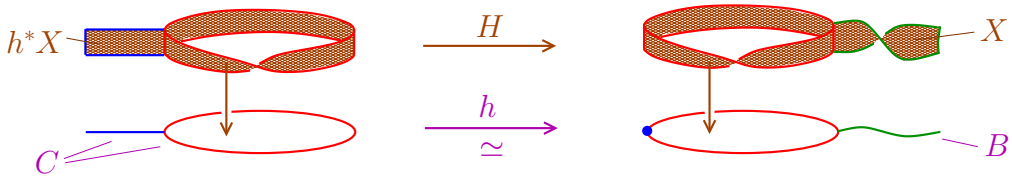
$$\begin{array}{ccc} h^*X & \xrightarrow{H} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B. \end{array}$$

If h is a homotopy equivalence and if B and C are Hausdorff and paracompact, then $H\colon h^*X \to X$ is a homotopy equivalence.

(2) Let $q\colon Y \to C$ and $p\colon X \to B$ be two $(F, \mathcal{G})$-bundles. Suppose we are given an $(F, \mathcal{G})$-bundle morphism of the form

$$\begin{array}{ccc} Y & \xrightarrow{H} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B. \end{array}$$

If $h\colon C \to B$ is a homotopy equivalence, then so is H.



The proof of the Homotopy Equivalent Bases Proposition 242.2 will occupy the remainder of this section. In the next Section 242.2 we will use this proposition to study Euler characteristics of total spaces.

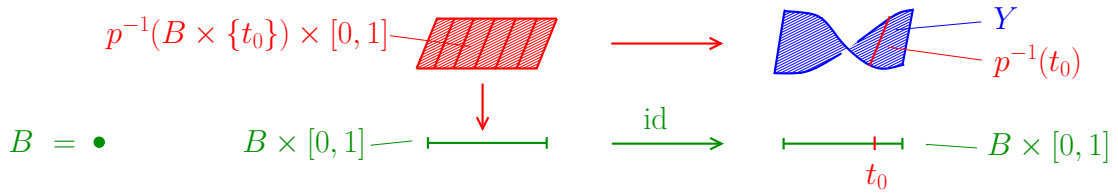
Remark. The case that $p\colon X \to B$ and $q\colon Y \to C$ are coverings is essentially the content of Exercise 127.23. □

Our proof of the Homotopy Equivalent Bases Proposition 242.2 makes use of the following lemma:

Lemma 242.3. (Bundle-over-Homotopy Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Let $p\colon Y \to B \times [0,1]$ be an $(F,\mathcal{G})$-bundle. If B is Hausdorff and paracompact, then for any $t_0 \in [0,1]$ there exists an isomorphism

$$\begin{array}{ccc} p^{-1}(B\times\{t_0\})\times[0,1] & \xrightarrow[\cong]{H} & Y \\ \downarrow{\scriptstyle (y,s)\mapsto(\pi_B(p(y)),s)} & & \downarrow{\scriptstyle p} \\ B\times[0,1] & \xrightarrow{\mathrm{id}} & B\times[0,1] \end{array}$$

of $(F,\mathcal{G})$-bundles over $B \times [0,1]$. (Here $\pi_B\colon B\times[0,1]\to B$ is the natural projection.)



Proof of the Bundle-over-Homotopy Lemma 242.3. Let $p\colon Y \to B\times[0,1]$ be an $(F,\mathcal{G})$-bundle. We introduce the following maps:

- Given any $t_0 \in [0,1]$ we consider the map
$$\begin{array}{rcl} \varphi_{t_0}\colon B\times[0,1] & \to & B\times[0,1] \\ (b,s) & \mapsto & (b,t_0). \end{array}$$
- We denote by $\pi_B\colon B\times[0,1]\to B$ the projection, we denote by $f\colon B\to B\times\{t_0\}$ the natural homeomorphism and we denote by $\iota\colon B\times\{t_0\}\to B\times[0,1]$ the inclusion. Note that $\varphi_{t_0} = \iota\circ f\circ\pi_B$.

Next we consider the following sequence of maps:

$$\begin{array}{lll} p^{-1}(B\times\{t_0\})\times[0,1] & \xrightarrow{(1)} f^*(p^{-1}(B\times\{t_0\}))\times[0,1] & \xrightarrow{(2)} f^*(\iota^*(Y))\times[0,1] \\ \xrightarrow{(3)} \pi_B^*(f^*(\iota^*(Y))) & \xrightarrow{(4)} (\iota\circ f\circ\pi_B)^*(Y) & \\ \xrightarrow{(5)} (\varphi_{t_0})^*(Y) & \xrightarrow{(6)} (\mathrm{id}_{B\times[0,1]})^*Y & \xrightarrow{(7)} Y. \end{array}$$

We will now explain all maps and we will see that all maps are isomorphisms of $(F,\mathcal{G})$-bundles over $B\times[0,1]$:

(1) Since $f\colon B\to B\times\{t_0\}$ is a homeomorphism we see that the first map is an isomorphism of $(F,\mathcal{G})$-bundles.

(2) On page 4647 we saw that the map
$$\begin{array}{rcl} \iota^*(Y) & \to & p^{-1}(B\times\{t_0\}) \\ ((b,t_0),y) & \mapsto & y \end{array}$$
is a natural isomorphism of $(F,\mathcal{G})$-bundles over $B\times[0,1]$.

(3) On page 4647, given any $(F,\mathcal{G})$-bundle $q\colon Z\to B$, we gave an explicit natural isomorphism
$$\begin{array}{rcl} \pi_B^*Z & \to & Z\times[0,1] \\ ((b,t),z) & \mapsto & (z,t) \end{array}$$

of $(F, \mathcal{G})$-bundles over $B \times [0,1]$.
(4) This isomorphism is given by the Bundle-Pullback–Functorial Lemma 240.24 (2).
(5) Since $\iota \circ f \circ \pi_B = \varphi_{t_0}$ this map is just the identity.
(6) Note that there exists an obvious homotopy H from $\mathrm{id}_{B\times[0,1]}$ to φ_{t_0} Thus we obtain from the Pullback-Homotopy Proposition 241.5 (1) that there exists an isomorphism from $\varphi_{t_0}^* Y$ to $(\mathrm{id}_{B\times[0,1]})^* Y$. (Note that here we use our hypothesis that B is Hausdorff and paracompact.)
(7) This isomorphism is provided by the Bundle-Pullback–Functorial Lemma 240.24 (1). ■

For the proof of the Homotopy Equivalent Bases Proposition 242.2 we will also need the following slight generalization of the Deformation Retract–Homotopy Equivalence Lemma 19.10:

Lemma 242.4. (Weak Deformation Retract Lemma)[2622] Let X be a topological space. Furthermore let $T \subseteq X$ be a subspace. We suppose that there exists a continuous map $H\colon X \times [0,1] \to X$ with the following properties:[2623]
(a) The map $H_0\colon X \to X$ is a homotopy equivalence.
(b) $H_1(X) \subseteq T$.
(c) We have $H(T \times [0,1]) \subseteq T$.
(d) The map $H_1\colon T \to T$ is a homotopy equivalence.
Then the inclusion $i\colon T \to X$ and the map $H_1\colon X \to T$ are both homotopy equivalences.

Proof of the Weak Deformation Retract Lemma 242.4. In this proof we work in the homotopy category $\mathcal{HomTop}$ that we introduced in Definition 19.12. Recall that the objects are topological spaces and recall that the set of morphisms between two topological spaces Y and Z is given by $[Y, Z]$, i.e. the set of homotopy classes of continuous maps $Y \to Z$. Recall that it follows from the Homotopy Equivalence-Reformulation Lemma 19.13 that a map $f\colon Y \to Z$ represents an isomorphism in the homotopy category if and only if it is a homotopy equivalences.

We denote by $i\colon T \to X$ the inclusion and we set $r := H_1\colon X \to T$. Next note that we have the following equalities in $[T, T]$:

$$[r] \circ [i] \quad = \quad [r \circ i] \quad = \quad [H_1|_T] \quad = \quad \text{isomorphism by (d)}.$$

We also have the following equalities in $[X, X]$:

$$[i] \circ [r] \quad = \quad [i \circ r] \quad = \quad [H_1] \quad = \quad [H_0] \quad = \quad \text{isomorphism by (a)}.$$

This discussion, together with the associativity of composition, shows that the morphisms $[r]$ and $[i]$ both have left and right inverses. It follows from the Inverse Uniqueness Lemma 5.10 (1) that $[r]$ and $[i]$ are both invertible in the homotopy category. By the above this means that r and i are both homotopy equivalences. ■

The following lemma gives us a frequently used special case of the Homotopy Equivalent Bases Proposition 242.2:

[2622] Note that this lemma provides a solution for Exercise 19.3.
[2623] Recall that in Definition 19.3 we introduced the notion of a deformation retraction. In contrast to a deformation retraction we now no longer assume that $H_0 = \mathrm{id}_X$ and we no longer assume that $H_t|T = \mathrm{id}_T$ for all $t \in [0,1]$.

Lemma 242.5. (Deformation Retract–Bases Lemma) Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Furthermore let $p\colon X \to T$ be an $(F, \mathcal{G})$-bundle over a topological space that is Hausdorff and paracompact. Finally let $S \subseteq T$ be a subspace. We denote by $i\colon S \to T$ the inclusion. We consider the corresponding pullback diagram:

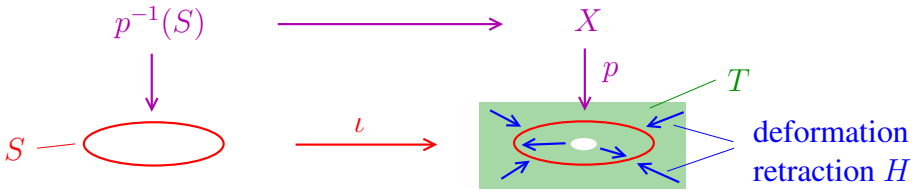
$$\begin{array}{ccc} i^*X & \xrightarrow{I} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{i} & T. \end{array}$$

If S is a deformation retract of T, then $I\colon i^*X \to X$ is a homotopy equivalence.

Proof of the Deformation Retract–Bases Lemma 242.5. Let $S \subseteq T$ be a deformation retract. Recall that this means that there exists a continuous map $H\colon T \times [0,1] \to T$ with $H_0 = \mathrm{id}$, $H_1(T) \subseteq S$ and such that $H_t(s) = s$ for all $s \in S$ and $t \in [0,1]$. We consider the following diagram:

$$\begin{array}{ccc} p^{-1}(S) & \hookrightarrow & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xhookrightarrow{i} & T. \end{array}$$

It follows from the explicit isomorphism $i^*X \to p^{-1}(S)$ from the Bundle–Pullback Proposition 240.21 (2) that it suffices to show that the inclusion $p^{-1}(S) \to X$ is a homotopy equivalence.



We consider the following diagram:

$$\begin{array}{ccccc} X \times [0,1] & & \xrightarrow{\quad =:\widetilde{H} \quad} & & X \\ & \xrightarrow{\Xi} & H^*X & \xrightarrow[((t,s),x)\mapsto x]{} & \\ {\scriptstyle p\times\mathrm{id}}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ T \times [0,1] & \xrightarrow{\mathrm{id}} & T \times [0,1] & \xrightarrow{H} & T. \end{array}$$

We make the following clarifications:

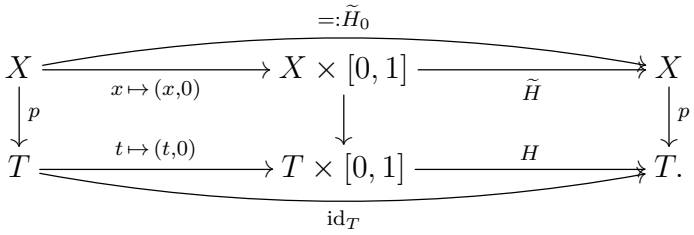
- The square to the right is just given by the pullback of the bundle $p\colon X \to T$ under the map H.
- The Bundle-over-Homotopy Lemma 242.3 provides us with a (non-natural) isomorphism $\Xi\colon X \times [0,1] \to H^*X$ of $(F, \mathcal{G})$-bundles.

By the Weak Deformation Retract Lemma 242.4 it now suffices to prove the following claim:

Claim.
(a) The map $\widetilde{H}_0\colon X \to X$ is a homeomorphism.
(b) The map $\widetilde{H}_1\colon X \to X$ restricts to a homeomorphism $p^{-1}(S) \to p^{-1}(S)$.

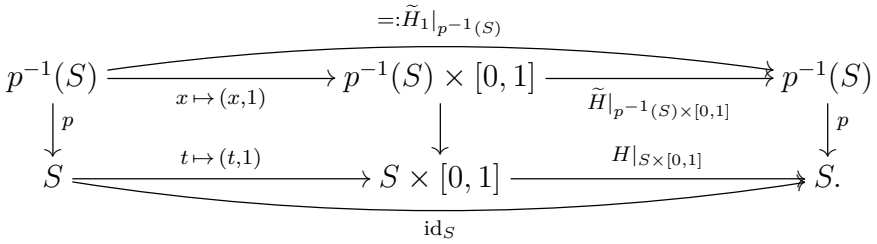
Proof.

(a) We consider the following diagram of $(F, \mathcal{G})$-bundles morphisms:



Since the bottom horizontal map is a homeomorphism we obtain from the Homeomorphic Bases Proposition 242.1 that the map $\widetilde{H}_0 \colon X \to X$ is a homeomorphism.

(b) Similar to (a) we consider the following diagram of $(F, \mathcal{G})$-bundles morphisms:



Since the bottom horizontal map is a homeomorphism it follows again from the Homeomorphic Bases Proposition 242.1 that the map $\widetilde{H}_1|_{p^{-1}(S)} \colon p^{-1}(S) \to p^{-1}(S)$ is a homeomorphism. ■

Now we are finally in a position to provide the proof of the Homotopy Equivalent Bases Proposition 242.2:

Proof of the Homotopy Equivalent Bases Proposition 242.2. Let F be a topological space that is regionally compact and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup.

(1) Let $p \colon X \to B$ be an $(F, \mathcal{G})$-bundle and let $h \colon C \to B$ be a continuous map. We consider the corresponding pullback diagram:

$$\begin{array}{ccc} h^*X & \xrightarrow{H} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B. \end{array}$$

As in Definition 21.1 we consider the mapping cylinder

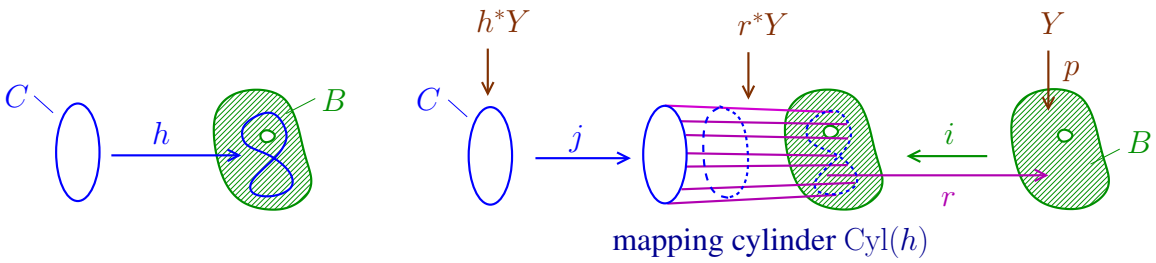
$$\mathrm{Cyl}(h) \;:=\; ((C \times [0,1]) \sqcup B)/(c,1) \sim h(c) \text{ for } c \in C$$

together with the two embeddings

$$\begin{array}{rcl} j \colon C & \to & \mathrm{Cyl}(h) \\ c & \mapsto & [(c,0)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} i \colon B & \to & \mathrm{Cyl}(h) \\ b & \mapsto & [b]. \end{array}$$

Furthermore we denote by $r \colon \mathrm{Cyl}(h) \to B$ the obvious retraction. Note that $r \circ j = h$. We recall the following basic facts about mapping cylinders:

- By the Mapping Cylinder–Homotopy Lemma 21.5 (1) we know that $i(B)$ is a deformation retract of $\mathrm{Cyl}(h)$.
- By the Homotopy Equivalent–Characterization Corollary 22.21 we know that $j(C)$ is a deformation retract of $\mathrm{Cyl}(h)$ if and only if h is a homotopy equivalence.



In the following we now assume that $h\colon C \to B$ is a homotopy equivalence and that B and C are Hausdorff and paracompact. We now obtain the following diagram:

$$\begin{array}{ccccccccccccc}
h^*Y & \xrightarrow{\mathrm{id}} & \overbrace{(r\circ j)}^{=h})^*Y & \xrightarrow{\cong} & j^*(r^*(Y)) & \xrightarrow{\simeq} & r^*Y & \xleftarrow{\simeq} & i^*(r^*Y) & \xleftarrow{\cong} & (r\circ i)^*Y & \xleftarrow{\cong} & Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow{\mathrm{id}} & C & \xrightarrow{\mathrm{id}} & C & \xrightarrow[\simeq]{j} & \mathrm{Cyl}(h) & \xleftarrow[\simeq]{i} & B & \xleftarrow{\mathrm{id}} & B & \xleftarrow{\mathrm{id}} & B.
\end{array}$$

It suffices to show that all the top horizontal maps are homeomorphisms or homotopy equivalences. We do so in the following discussion:

(a) The first top horizontal map is just the identity.

(b) The second and the fifth top horizontal maps are the natural isomorphisms from the Bundle-Pullback–Functorial Lemma 240.24 (2).

(c) As we just mentioned, $j(C)$ and $i(B)$ are deformation retracts of $\mathrm{Cyl}(h)$. Also note that it follows from our hypothesis on B and C, together with the Mapping Cylinder Topology-Properties Proposition 21.2, that $\mathrm{Cyl}(h)$ is also Hausdorff and paracompact. It thus follows from the Deformation Retract–Bases Lemma 242.5 that the third and fourth top horizontal maps are homotopy equivalences.

(d) Note that $r \circ i = \mathrm{id}_B$. The last horizontal map is thus the natural isomorphism provided by the Bundle-Pullback–Functorial Lemma 240.24 (1). ■

(2) Similar to the proof of Homeomorphic Bases Proposition 242.1 one can easily deduce Statement (2) from Statement (1). We leave it to the reader to fill in the details.

242.2. **Fiber bundles and Euler characteristics.** In this section we want to determine the Euler characteristic of the total space of a bundle. We first recall Definition 156.6:

Definition 242.6. Let X be a topological space.

(1) We define the n-th Betti number of (X) to be

$$b_n(X) \coloneqq \mathrm{rank}(\mathrm{H}_n(X)) \quad \in \mathbb{N}_0 \cup \{\infty\}.$$

(2) If $\sum\limits_{n\in\mathbb{N}_0} b_n(X)$ is finite, then we define the Euler characteristic of X to be

$$\chi(X) \coloneqq \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X).$$

In the Product-χ-Proposition 156.12 (or alternatively Exercise 164.5) we showed that the Euler characteristic is multiplicative under products, i.e. we have $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. In this section we will generalize this result from products to bundles:

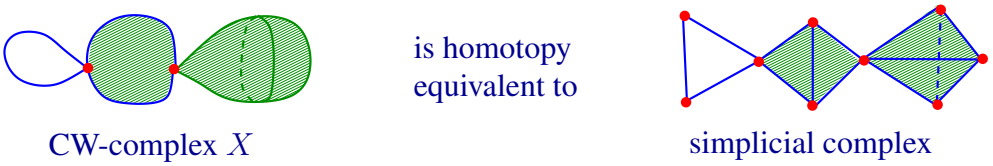
Theorem 242.7. (Fiber Bundle-χ-Theorem) Let F be a topological space that is regionally compact, Hausdorff and locally connected. Let $p\colon Y \to B$ be an F-bundle over a topological space that is Hausdorff and paracompact. If $\chi(F)$ is defined and if B is homotopy equivalent to a finite CW-complex, then

$$\chi(Y) \;=\; \chi(B) \cdot \chi(F).$$

Example. Let $n \in \mathbb{N}_0$. We consider the complex projective space $\mathbb{C}\mathrm{P}^2$. On page 2988 we saw that $\chi(\mathbb{C}\mathrm{P}^2) = 3$. As we will see in Exercise 242.3, it follows easily from the Fiber Bundle-χ-Theorem 242.7 together with a few basic observations on Euler characteristics of low-dimensional topological manifolds, that $\mathbb{C}\mathrm{P}^2$ cannot be the total space of an F-bundle $\mathbb{C}\mathrm{P}^2 \to B$ where F and B are closed orientable topological manifolds of dimension ≥ 1. □

Our proof of the Fiber Bundle-χ-Theorem 242.7 makes use of the following earlier result:

Proposition 175.21 (CW–Simplicial Homotopy Type Proposition) Every finite CW-complex is homotopy equivalent to a finite simplicial complex of the same dimension.



The following proposition deals with an important special case of the Fiber Bundle-χ-Theorem 242.7:

Proposition 242.8. (Simplicial Complex–Fiber Bundle-χ-Proposition) Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $p\colon Y \to B$ be an F-bundle. If $\chi(F)$ is defined and if B is a finite simplicial complex[2624],

$$\chi(Y) \;=\; \chi(B) \cdot \chi(F).$$

Proof of the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8. Let F be a topological space that is regionally compact, Hausdorff and locally connected and let $p\colon Y \to B$ be an F-bundle over a finite simplicial complex B. Note that it follows from our hypothesis on F and the Bundle Definition Lemma 240.9 that we can view p as an $(F, \mathrm{Homeo}(F))$-bundle.

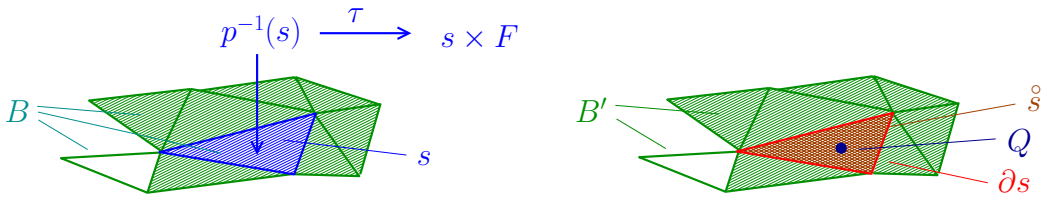
We want to show that $\chi(Y) = \chi(B) \cdot \chi(F)$. We will prove this statement by induction over the number of simplices of B. If there is no simplex, then there is nothing to show. Now let B a simplicial complex with at least one simplex and let $p\colon Y \to B$ be an F-bundle.

We make the following preparations and observations:

(1) Let s be a simplex of maximal dimension of B. Let B' be the simplicial complex that is obtained by removing s. Since s has maximal dimension, this makes sense.
(2) By the Simplex-Embedding Lemma 174.20 (3) we know that s is homeomorphic to a standard simplex. In particular we see that s is contractible.

[2624]We will not need any particular statements about simplicial complexes, except that these are CW-complexes where each characteristic map is an embedding.

(3) It follows from (2) together with the Bundles-Homotopies Proposition 241.10 that the bundle $p\colon p^{-1}(s) \to s$ is trivializable. In particular there exists a homeomorphism $\tau\colon p^{-1}(s) \to s \times F$ which commutes with the (obvious) projections to s.
(4) Let Q be a point in the interior of s.



Claim.
(a) $p^{-1}(\partial s)$ is a deformation retract of $p^{-1}(s \setminus \{Q\})$.
(b) $p^{-1}(B')$ is a deformation retract of $p^{-1}(B \setminus \{Q\})$.

Proof.

(a) First note that it should be clear that there exists a deformation retraction H from $s \setminus \{Q\}$ to ∂s. One can easily verify that

$$\begin{array}{rcl} p^{-1}(s \setminus \{Q\}) \times [0,1] & \to & p^{-1}(s \setminus \{Q\}) \\ (y,t) & \mapsto & \tau^{-1}(H(p(y),t), \pi_F(\tau(y))) \end{array}$$

$\uparrow$ $\pi_F\colon S \times F \to F$ is the projection

is a deformation retraction from $p^{-1}(s \setminus \{Q\})$ to $p^{-1}(\partial s)$.
(b) Using the definition of the topology on a simplicial complex one can easily verify that B' and $s \setminus \{Q\}$ are both closed subsets of $B \setminus \{Q\}$. It follows from the Homotopy Combination Lemma 18.8 (1) that the above deformation retraction extends to a deformation retraction from $p^{-1}(B \setminus \{Q\})$ to $p^{-1}(B')$. ⊞

Next note that it follows easily from the definition of the topology on a simplicial complex, see Definition 174.19, that the interiors of $B \setminus \{Q\}$ and of s form a cover of B. An elementary argument shows that this implies that the interiors of $p^{-1}(B \setminus \{Q\})$ and of $p^{-1}(s)$ form a cover of Y. It follows from this discussion that we can apply the Mayer–Vietoris Theorem 153.2 (1) to the cover $p^{-1}(B \setminus \{Q\})$ and $p^{-1}(s)$ of Y. We obtain an exact sequence of the following form:

$$\ldots \to \mathrm{H}_{n+1}(Y) \xrightarrow{\partial_{n+1}} \underbrace{\mathrm{H}_n(p^{-1}(s \setminus \{Q\}))}_{\cong \mathrm{H}_n(p^{-1}(\partial s)), \text{ by the claim and the Homotopy Equivalence H}_*\text{-Proposition 144.14}} \to \underbrace{\mathrm{H}_n(p^{-1}(B \setminus \{Q\}))}_{\cong \mathrm{H}_n(p^{-1}(B')), \text{ by the claim and the Homotopy Equivalence H}_*\text{-Proposition 144.14}} \oplus \mathrm{H}_n(p^{-1}(s)) \to \mathrm{H}_n(Y) \to \ldots$$

We now see that

follows from the Exact Sequence-χ–Lemma 156.5 applied to the above exact sequence

follows from applying the homeomorphism τ

$$\begin{array}{rcl}\chi(Y) & \overset{\downarrow}{=} & \chi(p^{-1}(B')) + \chi(p^{-1}(s)) - \chi(p^{-1}(\partial s)) \\ & = & \chi(B') \cdot \chi(F) + \chi(s) \cdot \chi(F) - \chi(\partial s) \cdot \chi(F) \\ & \overset{\uparrow}{=} & \chi(B) \cdot \chi(F).\end{array} \qquad \begin{array}{rcl} & \overset{\downarrow}{=} & \chi(p^{-1}(B')) + \chi(s \times F) - \chi(\partial s \times F) \\ & = & (\chi(B') + \chi(s) - \chi(\partial s)) \cdot \chi(F) \\ & & \end{array}$$

by induction and the product formula for Euler characteristics

by the Union-χ-Proposition 156.10 ■

With all these preparations we now have no problems with providing a proof of the Fiber Bundle-χ-Theorem 242.7:

Proof of the Fiber Bundle-χ-Theorem 242.7. Let F be a topological space that is regionally compact. Let $p\colon Y \to B$ be an F-bundle over a topological space B that is Hausdorff and paracompact. We assume that $\chi(F)$ is defined and that B is homotopy equivalent to a finite CW-complex.

It follows from the above CW–Simplicial Homotopy Type Proposition 175.21 together with the Homotopy-Equivalence Basics Lemma 19.14 (2) that there exists a finite simplicial complex C and a homotopy equivalence $h\colon C \to B$. We consider the $(F, \mathcal{G})$-pullback bundle $q\colon h^*Y \to C$.

We now see that

follows from the Simplicial Complex Fiber Bundle-χ-Proposition 242.8

$$\chi(Y) \underset{\uparrow}{=} \chi(h^*Y) \overset{\downarrow}{=} \chi(C) \cdot \chi(F) \underset{\uparrow}{=} \chi(B) \cdot \chi(F).$$

by the Homotopy Type-χ-Lemma 156.3 together with the Homotopy Equivalent Bases Proposition 242.2

by the Homotopy Type-χ-Lemma 156.3 since B and C are homotopy equivalent ■

Remark. Later, on page 4946, we will give a slightly different proof of the Fiber Bundle-χ-Theorem 242.7 where we will replace the CW–Simplicial Homotopy Type Proposition 175.21 by the CW-Complex–Handle Decomposition Theorem 256.12 and where we will replace the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8 by the Manifold–Fiber Bundle-χ-Theorem 250.15. □

242.3. Exercises for Chapter 242.

Exercise 242.1. Let F be a topological space that is regionally compact, Hausdorff and locally connected and such that $\chi(F)$ is defined. Use homology groups to prove that the Euler characteristic of any F-bundle $Y \to S^1$ is zero.

Exercise 242.2. Let F be a topological space that is regionally compact, Hausdorff and locally connected and such that $\chi(F)$ is defined. Give a direct proof, by induction on the dimension k, that for of any $(F, \mathcal{G})$-bundle $Y \to S^k$ we have $\chi(Y) = \chi(F) \cdot \chi(S^k)$.

Exercise 242.3.

(a) Show that $\mathbb{C}\mathrm{P}^2$ cannot be the total space of an F-bundle $\mathbb{C}\mathrm{P}^2 \to B$ where F and B are closed orientable topological manifolds of dimension ≥ 1.
Hint. Use everything you know about Euler characteristics.

(b) In (a), can we drop the hypothesis that F and B are orientable?

Exercise 242.4. Let $n \in \mathbb{N}_0$. We consider the real projective space $\mathbb{R}\mathrm{P}^n$. We recall the following facts from earlier chapters:

(1) On page 2988 we saw that $\chi(\mathbb{R}\mathrm{P}^n) = 0$ if n is odd and $\chi(\mathbb{R}\mathrm{P}^n) = 1$ if n is even.
(2) By the Odd-Dimensional Manifold-χ-Proposition 210.9 we know that the Euler characteristic of any closed odd-dimensional topological manifold is zero.
(3) By the Euler Characteristic–Realization Proposition 226.3 we know that the Euler characteristic of any closed orientable topological manifold of dimension $4k+2$ is even.

For which $n \in \mathbb{N}$ can you show that $\mathbb{R}\mathrm{P}^n$ is not the total space of an F-bundle $p\colon \mathbb{R}\mathrm{P}^n \to B$ where F and B are closed orientable topological manifolds of dimension > 0?

Exercise 242.5. Determine the Euler characteristics of the matrix groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{U}(n)$.

Exercise 242.6. Does there exist an F-bundle map $p\colon E \to X$ such that the following conditions are satisfied:

(1) F is a topological space with finitely generated homology.
(2) X is a finite CW-complex.
(3) There exists an $i \in \mathbb{N}$ such that $\mathrm{H}_i(F) \neq 0$.
(4) The induced map $p_*\colon \mathrm{H}_i(E) \to \mathrm{H}_i(X)$ is an isomorphism for each $i \in \mathbb{N}_0$.

CHAPTER 243

Smooth bundles

In this chapter we will outline the theory of smooth bundles. Many of our earlier results have smooth analogs and often we will just state the smooth analogs, without providing full proofs. It is arguably best to skip this chapter and to return to it only once its results get used later on.

This chapter is organized as follows:

- In Section 243.1 we will (re-) introduce smooth $(F, \mathcal{G})$-bundles and we will study pullbacks of smooth bundles.
- In Section 243.2 we will see that pullbacks of smooth $(F, \mathcal{G})$-bundles under homotopic maps are isomorphic. This allows us to show that smooth $(F, \mathcal{G})$-bundles over contractible smooth manifolds are trivializable.
- In Section 243.3 we will (re-) introduce tubular neighborhoods and we will use our results to prove several statements about tubular neighborhoods.
- In Section 243.4 we will provide exercises for Chapter 243.

In the next Chapter 244 we will formulate and discuss the Thom Isomorphism Theorem for disk bundles.

243.1. **Smooth bundles and pullbacks.** We start out by recalling Definition 240.8:

Definition 243.1. Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$. Furthermore let Y be a smooth manifold (potentially with corner), let B be a smooth manifold and let $p\colon Y \to B$ be a smooth map.

(1) We say a local F-trivialization[2625] $\Phi\colon p^{-1}(U) \to U \times F$ is smooth if Φ is a diffeomorphism.

(2) A $(F, \mathcal{G})$-bundle $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i \in I})$ over B is called smooth if all Φ_i are smooth.

Example. Note that in all the explicit examples that we discussed in Chapter 238, and in particular in the Ehresmann Fibration Theorem 238.6, the fibers are smooth manifolds and that the trivializations are diffeomorphisms. By the Manifold–Local Properties Lemma 23.8 we know that topological manifolds are regionally compact, Hausdorff and locally connected. We can and will therefore use the Bundle Definition Lemma 240.9 to view all these examples in the following also as smooth $(F, \mathrm{Homeo}(F))$-bundles. □

[2625] Note that $p^{-1}(U)$ is an open subset of a smooth manifold with corner, thus it is naturally a smooth manifold with corner. Furthermore note that, since U and F are smooth manifolds, we can use the Smooth Manifold Product Proposition 26.12 to equip $U \times F$ naturally with the structure of a smooth manifold with corner.

Definition 243.2. For $F = \mathbb{R}^n$, $F = S^{n-1}$ and $F = \overline{B}^n$ with $\mathcal{G} = \mathrm{O}(n)$ we refer to a (smooth) $(F, \mathrm{O}(n))$-bundle as a (smooth) linear bundle.[2626]

Lemma 243.3. (Orientable Smooth Linear Bundle Lemma) Let $p\colon Y \to B$ be a smooth $(F, \mathrm{O}(n))$-bundle. If the smooth manifold Y is orientable and if the smooth manifold B is orientable, then we can find a smooth atlas for this $(F, \mathrm{O}(n))$-bundle which is $\mathrm{SO}(n)$-compatible.

Proof. We outsource the proof to Exercise 243.3. ■

For smooth linear bundles we have the following analog of the Bundle–Action-on-Trivializations Proposition 240.17:

Proposition 243.4. (Smooth Linear Bundle–Action-on-Trivializations Proposition) Let $p\colon Y \to B$ be a smooth $(F, \mathrm{O}(n))$-bundle and let $\Phi\colon Y \to B \times F$ be a trivialization

(1) If $\mu\colon B \to \mathrm{O}(n)$ is a *smooth* map, then

$$\begin{array}{ccccc} Y & \xrightarrow{\Phi} & B \times F & \to & B \times F \\ & & (b, f) & \mapsto & (b, \mu(b)(f)) \end{array}$$

is also a trivialization of the smooth $(F, \mathrm{O}(n))$-bundle p.

(2) Every trivialization of $p\colon Y \to B$ is of the form described in (1).

Sketch of proof. The proof of the proposition is mostly identical to the proof of the Bundle–Action-on-Trivializations Proposition 240.17. The only question is the following: If we are given two smooth trivializations, we know by the Bundle–Action-on-Trivializations Proposition 240.17 that these give rise to a *continuous* map $B \to \mathrm{O}(n)$. But why is this map smooth? Fortunately smoothness follows easily from the following claim:

Claim. Let $F = \mathbb{R}^n$ or $F = S^{n-1}$ or $F = \overline{B}^n$. Let $U \subseteq \mathbb{H}^n$ be an open subset and let $A\colon U \to \mathrm{O}(n)$ be a continuous map. If the map

$$\begin{array}{ccc} U \times F & \to & U \times F \\ (u, f) & \mapsto & (u, A(u) \cdot f) \end{array}$$

is smooth, then so is $A\colon U \to \mathrm{O}(n)$.

Proof. We write $A = (a_{ij})_{i,j=1,\dots,n}$. We need to show that each map a_{ij} is smooth. Thus let $i, j \in \{1, \dots, n\}$. Note that the map $a_{ij}\colon U \to \mathbb{R}$ is the composition of the following maps:

$$U \xrightarrow{u \mapsto (u, e_j)} U \times F \xrightarrow{(u,f) \mapsto (u, A(u) \cdot f)} U \times F \xrightarrow{(u,f) \mapsto f} F \xrightarrow{(x_1, \dots, x_n) \mapsto x_i} \mathbb{R}.$$

It thus follows from our hypothesis that $a_{ij}\colon U \to \mathbb{R}$ is the composition of smooth maps. This shows that $a_{ij}\colon U \to \mathbb{R}$ is indeed a smooth map. ■

We continue with the following analog of a morphism of $(F, \mathcal{G})$-bundles that we introduced in Definition 240.12:

[2626] It follows immediately from the definitions that a linear smooth $\overline{B}^d$-bundle" from Definition 57.4 is the same as a smooth $(\overline{B}^d, \mathrm{Diffeo}(\overline{B}^d))$-bundle in the above sense.

Definition 243.5. Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$.

(1) Let
$$(p\colon X \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$$
$$(q\colon Y \to C, \{\Psi_j\colon q^{-1}(V_j) \to V_j \times F\}_{j\in J})$$
be two smooth $(F,\mathcal{G})$-bundles. A morphism between these two smooth $(F,\mathcal{G})$-bundles consists of two smooth maps $h\colon B \to C$ and $H\colon X \to Y$ which form a commutative diagram
$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C \end{array}$$
and which are a morphism of $(F,\mathcal{G})$-bundles in the sense of Definition 240.12. We say it is a morphism over B, if $B = C$ and $h = \mathrm{id}_B$.

(2) As in the Bundle–Category Lemma 240.13 we see that the class of smooth $(F,\mathcal{G})$-bundles together with the above notion of morphism and the obvious composition of such morphisms maps form a category. We refer to it as the category $\mathcal{B}\mathit{und}^{\mathrm{sm}}_{(F,\mathcal{G})}$ of smooth $(F,\mathcal{G})$-bundles.

(3) A smooth $(F,\mathcal{G})$-bundle $F\colon Y \to B$ is called trivializable if it is isomorphic over B to the trivial smooth $(F,\mathcal{G})$-bundle $p\colon B \times F \to B$.

We continue with the following analog of the Bundle–Pullback Proposition 240.21:

Proposition 243.6. (Smooth Bundle–Pullback Proposition) Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$. Furthermore let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ be a smooth $(F,\mathcal{G})$-bundle and let $h\colon C \to B$ be a smooth map between smooth manifolds.

(1) The pullback h^*Y is naturally a smooth manifold with corner.

(2) The pullback bundle $h^*Y \to B$ that we constructed in the Bundle–Pullback Proposition 240.21 is a smooth $(F,\mathcal{G})$-bundle and the diagram
$$\begin{array}{ccc} h^*Y & \xrightarrow{(c,y)\mapsto y} & Y \\ {\scriptstyle q(c,y):=c}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B \end{array}$$
describes a morphism of smooth $(F,\mathcal{G})$-bundles.

Sketch of proof. Let $i \in I$. By the Bundle Map-Pullback Proposition 240.2 we know that the map
$$\begin{array}{rcl} \widetilde{\Phi}_i\colon q^{-1}(h^{-1}(U_i)) & \to & h^{-1}(U_i) \times F \\ (c,y) & \mapsto & (c, \pi_F(\Phi_i(y))) \end{array}$$
here $\pi_F\colon Y \times F \to F$ is the natural projection

is an F-trivialization for $q\colon h^*Y \to C$. Since $h^{-1}(U_i)$ is an open subset of the smooth manifold C we can equip it with a natural smooth structure. We equip $h^{-1}(U_i) \times F$ with the product smooth structure coming from the Smooth Manifold Product Proposition 26.12. This product smooth structure defines, via $\widetilde{\Phi}_i$, also a smooth structure on $q^{-1}(h^{-1}(U_i))$. One can easily verify that these structures define a smooth structure on h^*Y. With this smooth structure the other statements of the proposition follow easily. ∎

We conclude this section with the following analog of the Homeomorphic Bases Proposition 242.1:

Proposition 243.7. (Diffeomorphic Bases Proposition) Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$. Furthermore let $q\colon Y \to C$ and $p\colon X \to B$ be two smooth $(F,\mathcal{G})$-bundles. Suppose we are given a smooth $(F,\mathcal{G})$-bundle morphism of the form

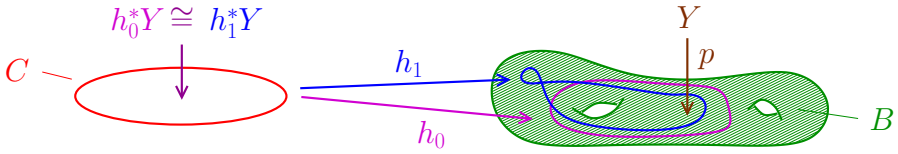
$$\begin{array}{ccc} Y & \xrightarrow{\quad H \quad} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{\quad h \quad} & B. \end{array}$$

If $h\colon C \to B$ is a diffeomorphism, then so is H.

Sketch of proof. It follows from the Homeomorphic Bases Proposition 242.1 that H is a homeomorphism, in particular it is a bijection. By definition of a smooth bundle morphism we know that H is smooth. One can easily verify that H is a local diffeomorphism. It now follows from the Smooth Pasting Proposition 25.19 that H is indeed a diffeomorphism. ■

243.2. **Smooth bundles and smooth homotopies.** We have the following smooth analog of the Pullback-Homotopy Proposition 241.5:

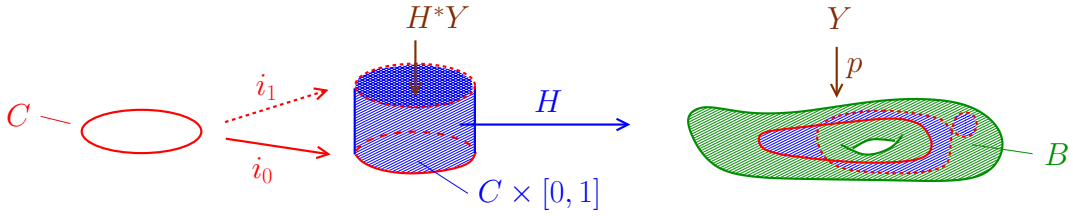
Proposition 243.8. (Smooth Pullback-Homotopy Proposition) Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$ be a subgroup. Let $p\colon Y \to B$ be a smooth $(F,\mathcal{G})$-bundle, let C be a smooth manifold and finally let $h_0, h_1\colon C \to B$ be two smooth maps. If h_0 and h_1 are homotopic, then the smooth $(F,\mathcal{G})$-bundles $h_0^*Y \to C$ and $h_1^*Y \to C$ are isomorphic over C.



Proof. We suppose that $h_0, h_1\colon C \to B$ are two smooth maps that are homotopic. It follows from the Replace-by-Smooth Homotopy Proposition 36.6 that there exists in fact a smooth homotopy $H\colon C \times [0,1] \to B$ between h_0 and h_1. As in the proof of the Pullback-Homotopy Proposition 241.5 on page 4657 the desired statement now follows easily from the following claim (which is the smooth analog of the Bundle-over-Product Lemma 241.6):

Claim. Given $t \in [0,1]$ let $\quad \begin{array}{rcl} i_t\colon C & \to & C \times [0,1] \\ c & \mapsto & (c,t) \end{array}$

be the obvious embedding. For any smooth $(F,\mathcal{G})$-bundle $p\colon Y \to C \times [0,1]$ the pullbacks i_0^*Y and i_1^*Y are isomorphic over C.



Proof. The proof of the claim is very similar to the proof of the Bundle-over-Product Lemma 241.6. The biggest change is that we one needs to use the Manifold–Partitions-of-Unity

Theorem 27.2 (2) to obtain *smooth* partitions of unity on $C \times [0,1]$. We leave it to the reader to find and fix the many details that we blissfully ignored. ■

This discussion motivates the following variation on the notation that we first introduced in Notation 241.8:

Notation 243.9. Let F be a smooth manifold, let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$ be a subgroup and let B be a smooth manifold. We denote by $\mathrm{IsoBund}^{\mathrm{sm}}_{(F,\mathcal{G})}(B)$ the set[2627] of isomorphism classes of smooth $(F,\mathcal{G})$-bundles over B.

We move on to the following analog of the Bundles-Homotopies Proposition 241.10:

Proposition 243.10. (Smooth Bundles-Homotopies Proposition) Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$ be a subgroup.

(1) Let $f\colon B \to C$ be a smooth map two smooth manifolds. If $f\colon C \to B$ is a homotopy equivalence, then the map

$$\begin{array}{rcl} f^*\colon \mathrm{IsoBund}^{\mathrm{sm}}_{(F,\mathcal{G})}(B) & \to & \mathrm{IsoBund}^{\mathrm{sm}}_{(F,\mathcal{G})}(C) \\ [p\colon Y \to B] & \mapsto & [f^*\colon f^*Y \to C] \end{array}$$

is a bijection.

(2) If B is a contractible smooth manifold, then every smooth $(F,\mathcal{G})$-bundle over B is trivializable.

Sketch of proof.

(1) The proof of this statement is very similar to the proof of the Bundles-Homotopies Proposition 241.10 (1). We just not replace the Pullback-Homotopy Proposition 241.5 by the Smooth Pullback-Homotopy Proposition 243.8 and we need to use the Whitney Approximation Theorem 36.1 which tells us that if the smooth map f has a homotopy inverse, then it also has a smooth homotopy inverse.

(2) This statement follows immediately from (1) and the Contractability Criterion 19.20 which says that the smooth manifold B is contractible if and only if for any $x_0 \in B$ the *smooth* inclusion map $\{x_0\} \to B$ is a homotopy equivalence. ■

243.3. **Tubular neighborhoods.** To practice our results from the earlier sections we now recall Definition 57.9:

Definition 243.11. Let M be an m-dimensional smooth manifold and let K be a proper k-dimensional submanifold of M. A tubular neighborhood for K is a pair $(N, p\colon N \to K)$ with the following properties:

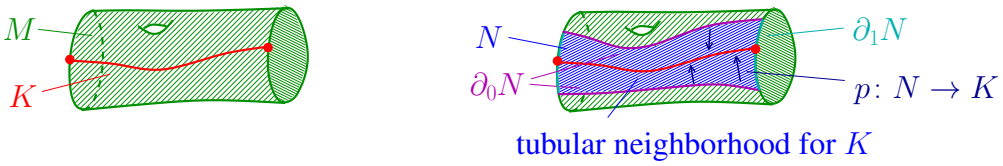
(1) N is a codimension-zero submanifold of M with corner with $\partial_1 N = p^{-1}(\partial K)$.

(2) The pair $(N, p\colon N \to K)$ is a[2628] smooth $(\overline{B}^{m-k}, \mathrm{O}(m-k))$-bundle such that K is the 0-section of p.

(3) The ∂_0-boundary of the submanifold N with corner, i.e. $\partial_0 N := (N \cap (\overline{M \setminus N})$, equals the ∂_0-boundary of the smooth $\overline{B}^{m-k}$-bundle $N \to K$.

(4) N is a closed subset of M.

[2627] Recall that we showed in the Bundles–Set Lemma 240.20 that it follows from F being regionally compact, that this is indeed a set.



Next we recall the existence and uniqueness statements for tubular neighborhoods:

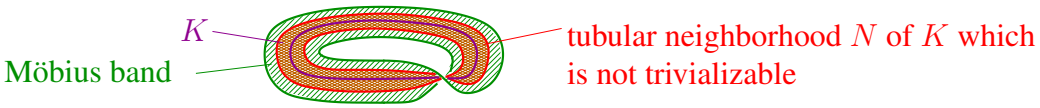
Theorem 57.11 (Tubular Neighborhood–Existence Theorem) Every compact proper submanifold of a smooth manifold admits a tubular neighborhood.

Theorem 57.12 (Tubular Neighborhood–Uniqueness Theorem) Let M be an m-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Furthermore let $p\colon A \to K$ and $q\colon B \to K$ be two tubular neighborhoods of K. There exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) F is a homotopy rel K.
(3) The diffeomorphism $\Phi = F_1\colon M \to M$ restricts to an isomorphism $\Phi\colon A \to B$ of smooth $(\overline{B}^{m-k}, \mathrm{Diffeo}(\overline{B}^{m-k}))$-bundles.
(4) The diffeotopy F has compact support.



As we already discussed on page 1316, tubular neighborhoods are in general not trivializable. For example if we take M to be the Möbius band and $K \subseteq M$ the central curve, then K does not admit a trivializable tubular neighborhood.



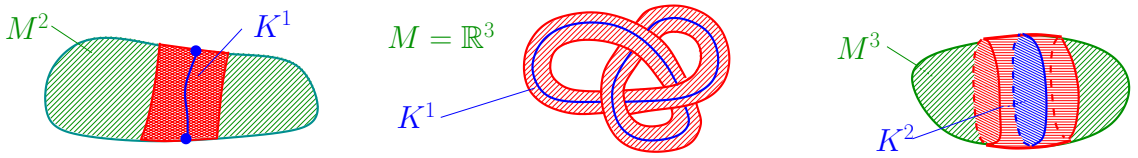
Fortunately in the Product Tubular Neighborhood–Existence Theorem 58.4 we already showed that in some settings tubular neighborhoods are trivializable. Now that we have developed more fancy tools, we can prove the following generalization of the original Product Tubular Neighborhood–Existence Theorem 58.4:

Theorem 243.12. (Product Tubular Neighborhood–Existence Theorem) Let M be a smooth manifold and let K be a compact proper submanifold. If we are in one of the following four situations:

(1) K is smoothly contractible,
(2) (a) K is one-dimensional and each component of K has non-empty boundary,
(b) K is one-dimensional and M is orientable,
(c) K is a compact orientable connected 2-dimensional smooth manifold with non-empty boundary and M is orientable.

[2628] In the original definition we wrote "linear smooth $\overline{B}^{m-k}$-bundle". As we already pointed out in Footnote 2626, if one compares Definition 57.4 with the definition in Definition 240.8 one sees that a "linear smooth $\overline{B}^{d}$-bundle" is exactly the same as a "smooth $(\overline{B}^{d}, \mathrm{O}(d+1)$-bundle.

(3) K is of codimension one and M and K are both orientable,
(4) K is a sphere of codimension two and M is orientable and if $\dim(M) \geq 5$,
then every product tubular neighborhood is a product.

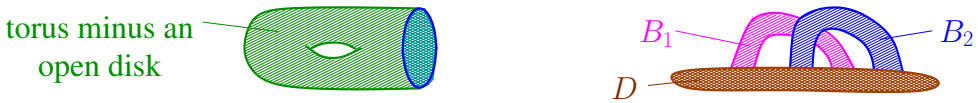


(1) K is contractible
(2a) K is 1-dimensional, $\partial K \neq \varnothing$
(2b) K is one-dimensional
(3) K is a codimension-2 sphere
(3) K has codimension 1

Sketch of proof. Let M^m be an m-dimensional smooth manifold, let K be a compact proper k-dimensional submanifold of M and let $(N, p\colon N \to K)$ be a tubular neighborhood of K. By definition p is a smooth $(\overline{B}^{m-k}, \mathrm{O}(m-k))$-bundle over K. We need to show that in the given settings this bundle is trivializable as a *smooth* $(\overline{B}^{m-k}, \mathrm{O}(m-k))$-bundle over K.

Note that it follows from our hypothesis and the Orientable Smooth Linear Bundle Lemma 243.3 that in (2b), (2c), (3) and (4) we can assume that we are actually given a smooth $(\overline{B}^{m-k}, \mathrm{SO}(m-k))$-bundle over K.

(1) This statement is an immediate consequence of the Smooth Bundles-Homotopies Proposition 243.10 (2).
(2) (a) This statement follows from (1).
(b) We already proved this statement in the original Product Tubular Neighborhood–Existence Theorem 58.4 (2a). Alternatively this statement can be proved like the Mapping Torus–$(F, \mathcal{G})$-Bundle Lemma 241.11 (3). The only extra ingredient we need is that any two matrices in $\mathrm{SO}(m-k)$ can be connected by a *smooth* path. (This follows for example from the Smooth Path-Connectivity Proposition 25.32).
(c) Let K be a connected 2-dimensional smooth manifold with non-empty boundary and let $p\colon Y \to K$ be a smooth linear $\overline{B}^{n-2}$-bundle. It follows easily from the Surface Classification Theorem 101.16 that we can decompose K as a disk D to which we attach n-squares $B_i = [0,1] \times [-1,1]$, $i = 1, \ldots, n$ along the boundary $[0,1] \times \{\pm 1\}$.



It follows from the Smooth Bundles-Homotopies Proposition 243.10 that the restriction of the smooth bundle p to $p^{-1}(A)$ and $p^{-1}(B_i)$, $i = 1, \ldots, m$ is trivializable. An argument as in the Bundle-over-1-Complex Lemma 241.12, using the Smooth Linear Bundle–Action-on-Trivializations Proposition 243.4, now shows that these trivializations give rise to a trivialization over K.
(3) We already proved this statement in the original Product Tubular Neighborhood–Existence Theorem 58.4 (3). Alternatively note that in this setting we consider the structure group $\mathrm{SO}(1) = \{1\}$. So, as we saw in Exercise 240.6, it is essentially clear that the smooth bundle is trivializable.

(4) We now assume that $m\dim(M) \geq 5$, that M is orientable and that K is a codimension-two sphere. We are thus considering a smooth $(\overline{B}^2, \mathrm{SO}(2))$-bundle over $K \cong S^{m-2}$. It follows from $m-2 \geq 3$ and $\mathrm{SO}(2) \cong S^1$ that $\pi_{m-3}(\mathrm{SO}(2)) = 0$. It thus follows immediately from the Bundle-over-Sphere Proposition 241.16 that the bundle is trivializable as a "non-smooth" bundle. To obtain a trivialization as a smooth bundle we apply the same argument as in the Bundle-over-Sphere Proposition 241.16, we just need to replace the Bundle–Action-on-Trivializations Proposition 240.17 by the Smooth Linear Bundle–Action-on-Trivializations 243.4.

In all of the above cases we leave it to the reader to work out the details. ■

243.4. Exercises for Chapter 243.

Exercise 243.1. Let $p\colon Y \to B$ be a smooth $(F, \mathrm{O}(n))$-bundle. Show that if the smooth manifold Y is orientable and if the smooth manifold B is orientable, then we can find a smooth atlas for this $(F, \mathrm{O}(n))$-bundle which is $\mathrm{SO}(n)$-compatible.

Exercise 243.2. Show that every compact non-empty 2-dimensional smooth manifold with non-empty boundary can be obtained from a disk by attaching bands to the disk.

CHAPTER 244

The Thom Isomorphism Theorem for disk bundles

Let $p\colon E \to X$ be an oriented $\overline{B}^n$-bundle over a compact topological space B with corresponding S^{n-1}-bundle $\mathrm{S}E \to X$. In this chapter we will introduce the corresponding Thom class $\Theta_{\mathbb{Z}}(E \to X) \in \mathrm{H}^n(E, \mathrm{S}E; \mathbb{Z})$ which is uniquely determined by the property that for each $x \in X$ it associates to the fundamental class of the fiber $p^{-1}(x)$ the value $+1$.

For us the main application of the Thom class is that it allows us to prove a generalization of the Cup Product–Intersection Number Theorem 220.10. More precisely, we will prove the Cup Product–Intersection Manifold Theorem 244.16 which tells us how to calculate the cup product of the Poincaré duals of two transverse submanifolds of a given smooth manifold.

This chapter is organized as follows:

- In Section 244.1 we will collect a few basic observations about disk and sphere bundles.
- In Section 244.2 we will state and discuss the Thom–Disk Bundle Isomorphism Theorem 244.5. This theorem assigns in particular to each oriented $\overline{B}^n$-bundle $p\colon E \to X$ with compact base space X the Thom class $\Theta_{\mathbb{Z}}(E \to X) \in \mathrm{H}^n(E, \mathrm{S}E; \mathbb{Z})$.
- In Section 244.3 we will provide the proof of the Thom–Disk Bundle Isomorphism Theorem 244.5.
- In Section 244.4 we will see that for an oriented smooth $\overline{B}^n$-bundle $p\colon E \to X$ over a compact smooth manifold X we can determine the Thom class via Poincaré duality.
- In Section 244.5 we will prove the main result of this chapter, namely we will use Thom classes to show that for two transverse submanifolds X and Y of a compact smooth manifold M the cup product $\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])$ is given by $\mathrm{PD}_M([X \cap Y])$.
- In Section 244.6 we will see that the Thom class behaves well under taking preimages of submanifolds.
- In Appendix 244.7 we will introduce the Thom class of sphere bundles.
- In Section 244.8 we will provide exercises for Chapter 244.

In the next Chapter 245 we will introduce the Euler class of disk and sphere bundles.

Later, in Chapter 277, after we introduced real vector bundles, we will introduce the analogous concept of the Thom class of an oriented n-dimensional real vector bundle.

244.1. **$\overline{B}^n$-bundles.** In this chapter we are mostly interested in $\overline{B}^n$-bundles and S^n-bundles. Thus we will now specialize to these situations, even though some of the concepts and results of this section would also hold in a more general context.

First of all we have the following lemma, which is a slight variation on the Boundary Fiber Bundle Lemma 57.3:

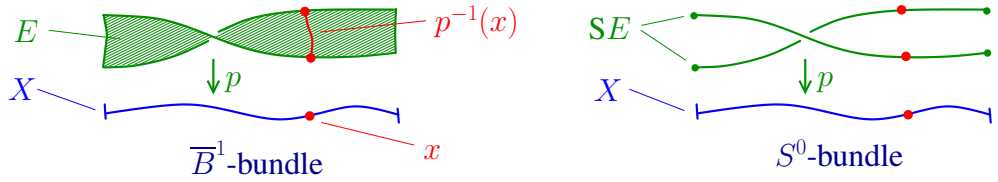
Lemma 244.1. (Boundary Fiber Bundle Lemma) Let $p\colon E \to X$ be a $\overline{B}^n$-bundle.

(1) For each $x \in X$ the preimage $p^{-1}(x)$ is homeomorphic to $\overline{B}^n$.

(2) We set

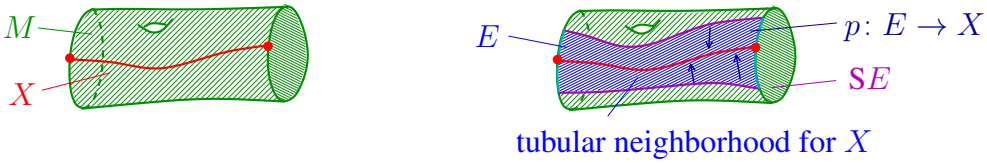
$$\mathrm{S}E \;:=\; \bigcup_{x\in X} \underbrace{\partial p^{-1}(x)}_{\substack{\uparrow \\ \text{boundary of the topological} \\ \text{manifold } p^{-1}(x)}}.$$

Then $p|_{\mathrm{S}E}\colon \mathrm{S}E \to X$ is an S^{n-1}-bundle.



Proof. The lemma follows easily from the definitions together with the Unique Type-of-Chart Proposition 150.11. We leave the details to the reader. ■

Example. Let M be an m-dimensional smooth manifold and let $X \subseteq M$ be a compact proper k-dimensional smooth manifold. In the Tubular Neighborhood–Existence Theorem 57.11 we saw that X admits a tubular neighborhood $p\colon E \to X$ which, by Definition 57.9, is a linear smooth $\overline{B}^{m-k}$-bundle over X. In particular it is a $\overline{B}^{m-k}$-bundle over X.



By Definition 57.9 of a tubular neighborhood we see that the subspace $\mathrm{S}X$ of the Boundary Fiber Bundle Lemma 244.1 agrees with ∂_0 of Notation 25.47 and the Smooth Submanifold-with-Corner Lemma 26.16. □

To state the results of this chapter we will also need to introduce the notion of an oriented $\overline{B}^n$-bundle:

Definition 244.2. Let $p\colon E \to X$ be a $\overline{B}^n$-bundle. An orientation for $p\colon E \to X$ is a map which assigns to each fiber E_x, $x \in X$ an orientation in the sense of Definition 191.17[2629] such that the following continuity condition is satisfied:

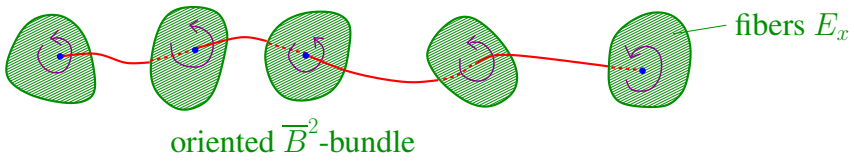
(∗) For every $\overline{B}^n$-bundle trivialization $\Phi\colon p^{-1}(U) \to U \times \overline{B}^n$ and every component U' of U the map

$$p^{-1}(y) \xrightarrow{\Phi|_{p^{-1}(y)}} \{y\} \times \overline{B}^n \xrightarrow{(y,v)\mapsto v} \overline{B}^n$$

is either orientation-preserving (in the sense of Definition 191.23) for all $y \in U'$ or it is orientation-reversing for all $y \in U'$.

We conclude this section with the appropriate notion of a morphism of *oriented* $\overline{B}^n$-bundles:

[2629]Note that each fiber E_x is homeomorphic to $\overline{B}^n$, thus it is a topological manifold, thus it makes sense to talk of an orientation in the sense of Definition 191.17.



Definition 244.3. Let $p\colon D \to X$ and $q\colon E \to Y$ be two oriented $\overline{B}^n$-bundles. A morphism of oriented $\overline{B}^n$-bundles is a morphism

$$\begin{array}{ccc} D & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{h} & Y \end{array}$$

in the sense of Definition 240.12 which has the extra property that for each $x \in X$ the homeomorphism $H|_{p^{-1}(x)}\colon D_x \to E_{h(x)}$ is orientation-preserving.

In most applications we will care about linear $\overline{B}^n$-bundles, i.e. $(\overline{B}^n, \mathrm{O}(n))$-bundles. For such bundles we will often (implicitly) use the following reformulation of an orientation:

Lemma 244.4. (Disk Bundle Definition Lemma) We have the following correspondence:

$$(\overline{B}^n, \mathrm{SO}(n))\text{-bundles} \iff \text{oriented linear } \overline{B}^n\text{-vector bundles.}$$

More precisely, let $p\colon E \to X$ be a linear $\overline{B}^n$-bundle.

(1) Suppose that we are given an atlas $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \overline{B}^n\}_{i\in I}$ that is $\mathrm{SO}(n)$-compatible in the sense of Definition 240.8. In this setting we can define for every $x \in X$ a unique orientation on the fiber $p^{-1}(x)$ such that for each $i \in I$ and each $x \in U_i$ the following map is orientation-preserving:

$$p^{-1}(b) \xrightarrow{\Phi_i|_{p^{-1}(b)}} \{b\} \times \overline{B}^n \xrightarrow{(b,f)\mapsto f} \overline{B}^n.$$

These orientations turn $p\colon E \to X$ into an oriented $\overline{B}^n$-bundle.

(2) Let $p\colon E \to X$ be an oriented $\overline{B}^n$-bundle. The family of all fiber-wise orientation-preserving trivializations turn $p\colon E \to X$ into a $(\overline{B}^n, \mathrm{SO}(n))$-bundle.

Sketch of proof. The lemma follows easily from the definitions and the following elementary observation: For $A \in \mathrm{O}(n)$ the multiplication map $\overline{B}^n \to \overline{B}^n$ that is given by $x \mapsto A\cdot x$ is orientation-preserving if and only if $A \in \mathrm{SO}(n)$. We leave it to the reader to fill in the details. ∎

244.2. **The Thom Isomorphism Theorem.** We are now in a position to formulate the main theorem of this chapter:

Theorem 244.5. (Thom–Disk Bundle Isomorphism Theorem) Let $p\colon E \to X$ be a $\overline{B}^n$-bundle over a topological space X. We assume that p admits a finite atlas.[2630]

(1) (a) There exists a unique $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E, SE; \mathbb{F}_2)$ with the property that for each $x \in X$ the restriction to $\mathrm{H}^n(E_x, SE_x; \mathbb{F}_2)$ is non-zero.[2631]

(b) For each $k \in \mathbb{Z}$ the map[2632] [2633]

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{F}_2) & \to & \mathrm{H}^{n+k}(E,\mathrm{S}E;\mathbb{F}_2) \\ \varphi & \mapsto & \Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

(2) (a) If the $\overline{B}^n$-bundle $p\colon E \to X$ is oriented in the sense of Definition 244.2, then there exists a unique $\Theta_{\mathbb{Z}}(E) \in \mathrm{H}^n(E,\mathrm{S}E;\mathbb{Z})$ with the property that for each $x \in X$ the restriction to $\mathrm{H}^n(E_x,\mathrm{S}E_x;\mathbb{Z})$ equals the dual fundamental class $[E_x]^*$ from Definition 201.27.[2634]

(b) For each $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{Z}) & \to & \mathrm{H}^{n+k}(E,\mathrm{S}E;\mathbb{Z}) \\ \varphi & \mapsto & \Theta_{\mathbb{Z}}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

Definition 244.6. Let $p\colon E \to X$ be a $\overline{B}^n$-bundle over a topological space X that admits a finite atlas. The cohomology classes $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E,\mathrm{S}E;\mathbb{Z})$ and $\Theta_{\mathbb{Z}}(E) \in \mathrm{H}^n(E,\mathrm{S}E;\mathbb{Z})$ from the Thom–Disk Bundle Isomorphism Theorem 244.5 are both called Thom class and both isomorphisms from the Thom–Disk Bundle Isomorphism Theorem 244.5 are called Thom isomorphism.

Before we turn to the proof of the Thom–Disk Bundle Isomorphism Theorem 244.5, let us state and prove a naturality property of the two types of Thom classes:

Proposition 244.7. (Thom Class–Natural Proposition)

(1) Let

$$\begin{array}{ccc} E & \xrightarrow{\quad G \quad} & F \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad g \quad} & Y \end{array}$$

be a morphism of $\overline{B}^n$-bundles which both admit a finite atlas. Then

$$\Theta_{\mathbb{F}_2}(E) \;=\; G^*(\Theta_{\mathbb{F}_2}(F)) \;\in\; \mathrm{H}^n(E,\mathrm{S}E;\mathbb{F}_2).$$

(2) For a morphism of oriented $\overline{B}^n$-bundles we have the analogous equality

$$\Theta_{\mathbb{Z}}(E) \;=\; G^*(\Theta_{\mathbb{Z}}(F)) \;\in\; \mathrm{H}^n(E,\mathrm{S}E;\mathbb{Z}).$$

Proof.

[2630] As we will discuss later in the closely related Thom–Vector Bundle Isomorphism Theorem 277.9, one can also extend this theorem to the case that $p\colon E \to X$ does not admit a finite atlas. But since currently all our examples are compact, we will at this point not thrive for maximal generality.

[2631] As the reader is surely aware, $\mathrm{H}^n(E_x,\mathrm{S}E_x;\mathbb{F}_2) \cong \mathrm{H}^n(\overline{B}^n,S^{n-1};\mathbb{F}_2) \cong \mathbb{F}_2$. Thus there is a unique non-zero element in this cohomology group.

[2632] Note that $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E,\mathrm{S}E;\mathbb{F}_2)$ and $p^*(\varphi) \in \mathrm{H}^k(E;\mathbb{F}_2)$. This implies that the relative cup product $\Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi)$ from Definition 204.7 takes values in $\mathrm{H}^n(E,\mathrm{S}E;\mathbb{F}_2)$.

[2633] For $k \in \{-n,\dots,-1\}$ the statement implies that $\mathrm{H}^{n+k}(E,\mathrm{S}E;\mathbb{F}_2) \cong \mathrm{H}^k(X;\mathbb{F}_2) = 0$.

[2634] At times we will use the observation that the Manifold-Top-H*-Theorem 201.26 (2b) tells us that the dual fundamental class $[E_x]^* \in \mathrm{H}^n(E_x,\mathrm{S}E_x;\mathbb{F}_2)$ is uniquely determined by the condition that we have $\langle [E_x]^*, [E_x]\rangle_{E_x} = 1$, where $[E_x] \in \mathrm{H}_n(E_x,\mathrm{S}E_x)$ is the fundamental class in the sense of Definition 194.3.

(1) Let $\Theta_{\mathbb{F}_2}(F) \in \mathrm{H}^n(F, \mathrm{S}F; \mathbb{F}_2)$ be the Thom class. We claim that we have the equality $\Theta_{\mathbb{F}_2}(E) = G^*(\Theta_{\mathbb{F}_2}(F)) \in \mathrm{H}^n(E, \mathrm{S}E; \mathbb{F}_2)$. By the uniqueness statement of the Thom–Disk Bundle Isomorphism Theorem 244.5 (1) it suffices to prove the following claim:

Claim. For each $x \in X$ the restriction of $G^*(\Theta_{\mathbb{F}_2}(F))$ to $\mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{F}_2)$ is non-zero.

Proof. Let $x \in X$. We set $y := g(x)$. Note that we have the following commutative diagram

$$\begin{array}{ccc} E_x & \xrightarrow[\cong]{G|_{E_x}} & F_y \\ \downarrow & & \downarrow \\ E & \xrightarrow{G} & F \end{array} \qquad \text{which induces the diagram} \qquad \begin{array}{ccc} \mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{F}_2) & \xleftarrow{\cong} & \mathrm{H}^n(F_y, \mathrm{S}F_y; \mathbb{F}_2) \\ \uparrow & & \uparrow \\ \mathrm{H}^n(E, \mathrm{S}E; \mathbb{F}_2) & \xleftarrow{G^*} & \mathrm{H}^n(F, \mathrm{S}F; \mathbb{F}_2). \\ & & \Theta_{\mathbb{F}_2}(F) \end{array}$$

By functoriality of cohomology the diagram to the right also commutes. Since the top horizontal map is an isomorphism it is now clear that the restriction of $G^*(\Theta_{\mathbb{F}_2}(F))$ to $\mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{F}_2)$ is non-zero. ⊞

(2) The proof of the statement for a morphism of oriented $\overline{B}^n$-bundles is almost the same as the above proof. We leave it to the reader to fill in the details. ■

Note that for an oriented $\overline{B}^n$-bundle $p\colon E \to B$ we now have two Thom classes, namely $\Theta_{\mathbb{F}_2}(E)$ and $\Theta_{\mathbb{Z}}(E)$. The following proposition gives us the non-surprising connection between these two cohomology class:

Proposition 244.8. (Thom Class Relationship Proposition) Let $p\colon E \to X$ be a $\overline{B}^n$-bundle. We denote by $\gamma_*\colon \mathbb{Z} \to \mathbb{F}_2$ the obvious ring homomorphism. Note that from the Cohomology-Coefficients-Functor Lemma 197.21 we obtain an induced homomorphism $\gamma_*\colon \mathrm{H}^k(E, \mathrm{S}E; \mathbb{Z}) \to \mathrm{H}^k(E, \mathrm{S}E; \mathbb{F}_2)$. If E is oriented, then we have the equality

$$\Theta_{\mathbb{F}_2}(E) \;=\; \gamma_*(\Theta_{\mathbb{Z}}(E)) \;\in\; \mathrm{H}^n(X; \mathbb{F}_2).$$

Proof. Let $\Theta_{\mathbb{Z}}(F) \in \mathrm{H}^n(E, \mathrm{S}E; \mathbb{Z})$ be the Thom class. We claim that we have the equality $\Theta_{\mathbb{F}_2}(E) = \gamma_*(\Theta_{\mathbb{Z}}(E)) \in \mathrm{H}^n(X; \mathbb{F}_2)$. By the uniqueness statement of the Thom–Disk Bundle Isomorphism Theorem 244.5 (1) it suffices to prove the following claim:

Claim. For each $x \in X$ the restriction of $\gamma_*(\Theta_{\mathbb{Z}}(E)) \in \mathrm{H}^n(X; \mathbb{F}_2)$ to $\mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{F}_2)$ is non-zero.

Proof. Let $x \in X$. We denote by $\iota_x\colon E_x \to E$ the inclusion. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^n(E, \mathrm{S}E; \mathbb{Z}) & \xrightarrow{\iota_x^*} & \mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{Z}) \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ \mathrm{H}^n(E, \mathrm{S}E; \mathbb{F}_2) & \xrightarrow{\iota_x^*} & \mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{F}_2). \end{array} \qquad \begin{array}{c} \Theta_{\mathbb{Z}}(E) \\ \\ \\ \\ \end{array} \qquad \begin{array}{c} [E_x]^* \\ \\ \\ \\ \end{array}$$

We make the following observations:

- It follows from the Cohomology-Coefficients-Functor Lemma 197.21 (2) that the diagram commutes.
- By definition of $\Theta_{\mathbb{Z}}(E)$ we know that $\iota_x^*(\Theta_{\mathbb{Z}}(E))$ is a generator of $\mathrm{H}^n(E_x, \mathrm{S}E_x; \mathbb{Z})$.
- It follows easily from the H^*–Change-of-Coefficients Theorem 202.10 together with the H^*–Coefficient Change Lemma 202.11 that the right vertical map is an epimorphism.

It is now clear that $\iota_x^*(\gamma_*(\Theta_{\mathbb{Z}}(E)))$ is the unique non-zero element. ■

244.3. **Proof of the Thom–Disk Bundle Isomorphism Theorem 244.5.** As a warm-up we will deal in the following lemma with the case that $E = X \times \overline{B}^n$ is the trivial $\overline{B}^n$-bundle over some topological space X:

Lemma 244.9. (Trivializable–Thom Lemma) Let $n \in \mathbb{N}_0$ and let X be a topological space. The statement of the Thom–Disk Bundle Isomorphism Theorem 244.5 holds for the trivial $\overline{B}^n$-bundle $X \times \overline{B}^n \to X$ where we equip each $\{x\} \times \overline{B}^n$ with the obvious orientation. In fact we can describe the Thom classes precisely: We denote by $q\colon X \times \overline{B}^n \to \overline{B}^n$ the natural projection.

(1) If we denote by $[\overline{B}^n]^*_{\mathbb{F}_2} \in \mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{F}_2)$ the unique non-zero element, then the cohomology class $\Theta := q^*([\overline{B}^n]^*_{\mathbb{F}_2}) \in \mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{F}_2)$ has the desired properties and it is the unique such element.

(2) If we denote by $[\overline{B}^n]^* \in \mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{Z})$ the dual fundamental class of the topological manifold $\overline{B}^n$ that is equipped with the standard generation, then the cohomology class $\Theta := q^*([\overline{B}^n]^*) \in \mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{Z})$ has the desired properties and it is the unique such element.

Proof. In the following we deal with $\mathbb{Z}$-coefficients. The proof for $\mathbb{F}_2$-coefficients is essentially the same. We start out with the following claim:

Claim 1. Let W be a topological space. In the following we denote by $p\colon W \times \overline{B}^n \to W$ and $q\colon W \times \overline{B}^n \to \overline{B}^n$ the natural projections. For each $k \in \mathbb{Z}$ the following map is an isomorphism[2635]

$$\begin{array}{rcl} \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^n(\overline{B}^n, S^{n-1};\mathbb{Z}) & \to & \mathrm{H}^{k+n}(W \times \overline{B}^n, W \times S^{n-1};\mathbb{Z}) \\ \varphi \otimes \psi & \mapsto & p^*(\varphi) \cup q^*(\psi). \end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^{n-1}(S^{n-1};\mathbb{Z}) & \xrightarrow{\varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} & \mathrm{H}^{k+n-1}(W \times S^{n-1};\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\delta_{n-1}} & & \downarrow{\scriptstyle \delta_{k+n-1}} \\ \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^n(\overline{B}^n, S^{n-1};\mathbb{Z}) & \xrightarrow{\varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} & \mathrm{H}^{k+n}(W \times \overline{B}^n, W \times S^{n-1};\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{я}} & & \downarrow{\scriptstyle \mathrm{я}} \\ \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^n(\overline{B}^n;\mathbb{Z}) & \xrightarrow{\varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} & \mathrm{H}^{k+n}(W \times \overline{B}^n;\mathbb{Z}) \\ \downarrow & & \downarrow \end{array}$$

We make the following clarifications and observations:

- The right vertical sequence is just given by the LES-of-Pair Proposition 146.7 corresponding to the pair $(W \times \overline{B}^n, W \times S^{n-1})$. In particular it is exact.
- The left vertical sequence is given by tensoring the exact sequence of the LES-of-Pair Proposition 146.7 corresponding to the pair $(\overline{B}^n, S^{n-1})$ with the group $\mathrm{H}^k(W;\mathbb{Z})$. It

[2635] Here we use the relative cup product from Definition 204.7.

follows from the Algebraic Universal Coefficient Theorem 162.15 that the left vertical sequence is also exact.

- It follows easily from the naturality of the connecting homomorphism in the LES-of-Triple Proposition 198.13, the Cup Product-Connecting Homomorphism Lemma 204.18 and the Cup Product-Naturality Lemma 204.8 that the diagram commutes up to a sign.
- It follows from the Künneth-H*-Theorem 208.1 together with the Tor-Properties Lemma 162.13 (3) that the top and bottom horizontal maps are isomorphisms (here we use that $\overline{B}^n$ and S^{n-1} both satisfy Condition ($\divideontimes$) from the Künneth-H*-Theorem 208.1).

It follows from the above together with the Five Lemma 145.11 that the middle horizontal map is (as promised) also an isomorphism. ⊞

Now we turn to the actual proof of the lemma. Thus let X be a topological space.

(a) Recall that we denote by $[\overline{B}^n]^* \in \mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{Z})$ the dual fundamental class of the topological manifold $\overline{B}^n$ that is equipped with the standard generation.

Claim 2. The cohomology class $\Theta := q^*([\overline{B}^n]^*) \in \mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{Z})$ has the property that for each $x \in X$ the restriction of Θ to $\mathrm{H}^n(\{x\} \times \overline{B}^n, \{x\} \times S^{n-1}; \mathbb{Z})$ equals the dual fundamental class $[\{x\} \times \overline{B}^n]^*$.

Proof. Let $x \in X$. We denote by $\iota_x \colon \{x\} \times \overline{B}^n \to X \times \overline{B}^n$ the inclusion and we denote by $f_x \colon \overline{B}^n \xrightarrow{v \mapsto (x,v)} \{x\} \times \overline{B}^n$ the obvious homeomorphism. We now see that

$$\begin{aligned} \langle \iota_x^*(\Theta), f_{x*}([\overline{B}^n]) \rangle_{\{x\} \times \overline{B}^n} &\overset{\text{since } \Theta := q^*([\overline{B}^n]^*)}{=} \langle (q \circ \iota_x)^*([\overline{B}^n]^*), f_{x*}([\overline{B}^n]) \rangle_{\{x\} \times \overline{B}^n} \\ &= \langle [\overline{B}^n]^*, \underbrace{(q \circ \iota_x \circ f_x)}_{=\mathrm{id}_{\overline{B}^n}}{}_*([\overline{B}^n]) \rangle_{\overline{B}^n} \\ &\overset{\text{by the Kronecker Pairing–Natural Lemma 199.9 (1)}}{=} \langle [\overline{B}^n]^*, [\overline{B}^n] \rangle_{\overline{B}^n} = 1. \end{aligned}$$

This shows that $\iota_x^*(\Theta)$ equals the standard generator of $\mathrm{H}^n(\{x\} \times \overline{B}^n, \{x\} \times S^{n-1}; \mathbb{Z})$. ⊞

Claim 3. The cohomology class $\Theta := q^*([\overline{B}^n]^*) \in \mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{Z})$ is the *unique* cohomology class with the property from Claim 2.

Proof. First note that it follows easily from the Path-Component-H*-Lemma 198.8 that it suffices to deal with the case that X is path-connected and non-empty. Now let $x \in X$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{Z}) & \xrightarrow{\mathrm{diag}(\iota, \mathrm{id})^*} & \mathrm{H}^n(\{x\} \times \overline{B}^n, \{x\} \times S^{n-1}; \mathbb{Z}) \\ \uparrow{\scriptstyle \varphi \otimes \psi \mapsto p^*(\varphi) \cup q^*(\psi)} & & \uparrow{\scriptstyle \varphi \otimes \psi \mapsto p^*(\varphi) \cup q^*(\psi)} \\ \mathrm{H}^0(X; \mathbb{Z}) \otimes \mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{Z}) & \xrightarrow{\iota^* \otimes \mathrm{id}} & \mathrm{H}^0(\{x\}; \mathbb{Z}) \otimes \mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{Z}) \end{array}$$

where the horizontal maps are induced by the inclusion $\iota \colon \{x\} \to X$. Clearly it suffices to show that the top horizontal map is an isomorphism. This in turn follows from the following three observations:

- It follows easily from the Cup Product-Naturality Lemma 204.8 that the diagram commutes.
- By Claim 1 we know that the vertical maps are isomorphisms.

- It follows easily from the fact that X is path-connected and non-empty, together with the H^0-Proposition 198.9 that the bottom horizontal map is an isomorphism. ⊞

(b) It follows from Claim 1 that $\Theta := q^*([\overline{B}^n]^*) \in \mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{Z})$ has the desired property (b). ■

Proof of the Thom–Disk Bundle Isomorphism Theorem 277.9. [2636] In the following we will prove Statement (1) of the theorem which deals with $\mathbb{F}_2$-coefficients. The proof of Statement (2), which deals with $\mathbb{Z}$-coefficients, is essentially the same. In fact, in the remainder of the proof we will, for space reasons, ignore coefficients, which makes the two statements almost indistinguishable.

For the purpose of this proof we introduce the following language: We say that a $\overline{B}^n$-bundle $p\colon E \to X$ satisfies (∗) if $p\colon E \to X$ satisfies the conclusion of the Thom–Disk Bundle Isomorphism Theorem 244.5, i.e. if the following two conditions hold:

(a) There exists a unique $\Theta(E) \in \mathrm{H}^n(E, \mathrm{S}E)$ with the property that for each $x \in X$ the restriction to $\mathrm{H}^n(E_x, \mathrm{S}E_x)$ is non-zero.

(b) For each $k \in \mathbb{Z}$ the map[2637]
$$\begin{array}{ccc} \mathrm{H}^k(X) & \to & \mathrm{H}^{n+k}(E, \mathrm{S}E) \\ \varphi & \mapsto & \Theta(E) \cup p^*(\varphi) \end{array}$$
is an isomorphism.

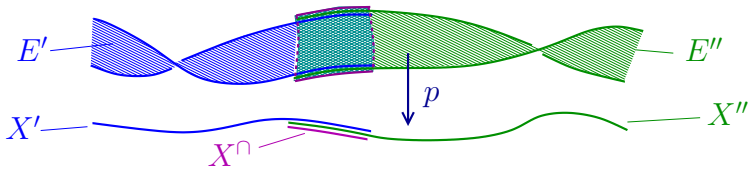
Claim 1. The statement (∗) holds if $p\colon E \to X$ is a trivializable $\overline{B}^n$-bundle.

Proof. This statement follows easily from the Trivializable–Thom Lemma 244.9 together with the Cup Product-Naturality Lemma 204.8. ⊞

Claim 2. Let $p\colon E \to X$ be a $\overline{B}^n$-bundle. We suppose that there exist two open subsets $X', X'' \subseteq X$ with $X' \cup X'' = X$ and such that the restrictions of p to X' and X'' satisfy (∗) and such that the restriction of p to $X' \cap X''$ also satisfies (∗). Then $p\colon E \to X$ also satisfies (∗).

Proof. We introduce the following notation:

- We write $X^\cap := X' \cap X''$.
- We write $E' := p^{-1}(X')$, $E'' := p^{-1}(X'')$ and $E^\cap := p^{-1}(X^\cap)$.
- We denote by $\iota'\colon X^\cap \to X'$ and $\iota''\colon X^\cap \to X''$ the inclusions. By a slight abuse of notation we denote by $\iota'\colon E^\cap \to E'$ and $\iota''\colon E^\cap \to E''$ also the inclusions.
- We denote by $\widetilde{\iota}'\colon X' \to X$ and $\widetilde{\iota}''\colon X'' \to X$ the inclusions. Similarly we denote by $\widetilde{\iota}'\colon E' \to E$ and $\widetilde{\iota}''\colon E'' \to E$ also the inclusions.



[2636] The following proof shares some features with the proof of the Orientation-Top-H_*-Theorem 193.3 and the Poincaré Duality Theorem 210.1: In each case we have to turn a local statement into a global statement.

Next note that by a variation on the Mayer–Vietoris-H*-Theorem 200.11 we have the following Mayer–Vietoris sequence

$$\to \underbrace{\mathrm{H}^{n-1}(E^\cap, \mathrm{S}E^\cap)}_{\cong \mathrm{H}^{-1}(X^\cap)=0} \xrightarrow{\delta} \mathrm{H}^n(E, \mathrm{S}E) \xrightarrow{(\widetilde{\iota}'^*, -\widetilde{\iota}''^*)} \mathrm{H}^n(E', \mathrm{S}E') \oplus \mathrm{H}^n(E'', \mathrm{S}E'') \xrightarrow{\iota'^* \oplus \iota''^*} \mathrm{H}^n(E^\cap, \mathrm{S}E^\cap) \to$$
$$\qquad \Theta' \qquad \Theta''$$

We make the following observations:

(i) By hypothesis we have unique elements $\Theta' \in \mathrm{H}^n(E', \mathrm{S}E')$ and $\Theta'' \in \mathrm{H}^n(E'', \mathrm{S}E'')$ which satisfy (a) and (b) for the vector bundles $E' \to X'$ and $E'' \to X''$. As in the Thom Class–Natural Proposition 277.11 we see that $\iota'^*(\Theta') \in \mathrm{H}^n(E^\cap, \mathrm{S}E^\cap)$ and $\iota''^*(\Theta') \in \mathrm{H}^n(E^\cap, \mathrm{S}E^\cap)$ both satisfy (a) and (b) for $p\colon E^\cap \to X^\cap$. Since we assume the uniqueness statement for $X^\cap$ we obtain that $\iota'^*(\Theta') = \iota''^*(\Theta'') \in \mathrm{H}^n(E^\cap, \mathrm{S}E^\cap)$. We denote this element by $\Theta^\cap \in \mathrm{H}^n(E^\cap, \mathrm{S}E^\cap)$.

(ii) Since $p\colon E^\cap \to X^\cap$ satisfies $(*)$ we see that $\mathrm{H}^{n-1}(E^\cap, \mathrm{S}E^\cap) \cong \mathrm{H}^{-1}(X^\cap) = 0$.

(iii) It follows from (i) and (ii) that there exists a unique cohomology class $\Theta \in \mathrm{H}^n(E, \mathrm{S}E)$ with $\widetilde{\iota}'^*(\Theta) = \Theta' \in \mathrm{H}^n(E', \mathrm{S}E')$ and with $\widetilde{\iota}''^*(\Theta) = \Theta'' \in \mathrm{H}^n(E'', \mathrm{S}E'')$.

It follows easily from the above discussion and the functoriality of cohomology that Θ is the unique class that satisfies (a) for the $\overline{B}^n$-bundle $p\colon E \to X$.

It remains to show that Θ satisfies (b). Thus let $k \in \mathbb{Z}$. We consider the following diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{H}^{k-n-1}(X^\cap) & \xrightarrow{\delta} & \mathrm{H}^{k-n}(X) & \xrightarrow{(\widetilde{\iota}'^*, -\widetilde{\iota}''^*)} & \mathrm{H}^{k-n}(X') \oplus \mathrm{H}^{k-n}(X'') & \xrightarrow{\iota'^* \oplus \iota''^*} & \mathrm{H}^{k-n}(X^\cap) & \longrightarrow \\
& \downarrow{\scriptstyle \varphi \mapsto \Theta^\cap \cup p^*(\varphi)} & & \downarrow{\scriptstyle \varphi \mapsto \Theta \cup p^*(\varphi)} & & \downarrow{\scriptstyle \mathrm{diag}(\ldots,\ldots)} & & \downarrow{\scriptstyle \varphi \mapsto \Theta^\cap \cup p^*(\varphi)} & \\
\to & \mathrm{H}^{k-1}(E^\cap, \mathrm{S}E^\cap) & \xrightarrow{\delta} & \mathrm{H}^k(E, \mathrm{S}E) & \xrightarrow{(\widetilde{\iota}'^*, -\widetilde{\iota}''^*)} & \mathrm{H}^k(E', \mathrm{S}E') \oplus \mathrm{H}^k(E'', \mathrm{S}E'') & \xrightarrow{\iota'^* \oplus \iota''^*} & \mathrm{H}^k(E^\cap, \mathrm{S}E^\cap) & \to
\end{array}$$

We make the following observations:

(a) By hypothesis we know that the vertical maps corresponding to $X^\cap$ and X', X'' are isomorphisms.

(b) We claim that the diagram commutes up to a sign:

- It follows easily from the Cup Product-Naturality Lemma 204.8 that the middle square and the square to the right both commute.
- It follows fairly easily from the fact that the connecting homomorphism in the Mayer–Vietoris sequence is defined via the connecting homomorphism in a long exact sequence, the naturality of the connecting homomorphism in the LES-of-Triple Proposition 198.13, the Cup Product-Connecting Homomorphism Lemma 204.18 and the fact that the restriction of Θ to $E^\cap$ equals $\Theta^\cap$, that the left square commutes up to a sign. We leave it to the reader to fill in the details.

It now follows from the Five Lemma 145.11 that the vertical map corresponding to X is also an isomorphism. This concludes the proof that Θ satisfies (b). ⊞

Claim 3. Let $p\colon E \to X$ be a $\overline{B}^n$-bundle. If X is covered by open subsets $X_1, \ldots, X_k$ such that the restriction to each X_i is trivializable, then $(*)$ holds for $p\colon E \to X$.

Proof. We prove the claim by induction on $k \in \mathbb{N}$. The case $k = 0$ is trivial and the case $k = 1$ is just the content of Claim 1. Now we assume that the statement holds for some $k \in \mathbb{N}$. Next let $p\colon E \to X$ be a $\overline{B}^n$-bundle such that X is covered by open subsets

$X_1, \dots, X_{k+1}$ such that the restriction to each X_i is trivializable. We need to show that (∗) holds for $p\colon E \to X$.

We set $X' := X_1 \cup \dots \cup X_k$ and $X'' = X_{k+1}$. Note that $X^\cap := X' \cap X''$ can be written as $X^\cap = (X_1 \cap X_{k+1}) \cup \dots \cup (X_k \cap X_{k+1})$. Thus $X^\cap$ is covered by the k open subsets $X_i \cap X_{k+1}$. Note that the fact that the restriction of $p\colon E \to X$ to each X_i is trivializable implies that the restriction to each open subset $X_i \cap X_{k+1}$ is also trivializable. It thus follows from our induction hypothesis that (∗) holds for X', X'' and also for $X' \cap X''$. Thus it follows from Claim 2 that (∗) holds for X. ■

244.4. **Thom class and Poincaré duality.** In this section we will consider Thom classes of oriented linear smooth $\overline{B}^n$-bundles over a compact oriented smooth manifold X. We start out by recalling the definition of a smooth bundle and of a linear bundle.

First we recall Definition 57.1:

Definition 244.10. In the following let F be a smooth manifold. A smooth F-bundle over a smooth manifold X is a smooth manifold T (possibly with corner) together with a smooth map $p\colon T \to X$ with the following property: Given any point $x \in X$ there exists a smooth F-trivialization of p around x, i.e. there exists an open neighborhood U of $x \in X$ and a diffeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\Phi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f)\mapsto x} & \\ & U. & \end{array}$$

Next we recall Definition 57.4 and Definition 243.2:

Definition 244.11. For $F = \mathbb{R}^n$, $F = S^{n-1}$ and $F = \overline{B}^n$ with $\mathcal{G} = \mathrm{O}(n)$ we refer to a (smooth) $(F, \mathrm{O}(n))$-bundle as a (smooth) linear bundle.[2638]

Next we adopt the following notation:

Notation 244.12. Let $p\colon E \to X$ be a linear smooth oriented $\overline{B}^n$-bundle over a compact oriented smooth manifold X.

(1) By the 0-Section Lemma 57.5 we know that the 0-section $X \to E$ (which is defined by sending $x \in X$ to the well-defined origin of the fiber) defines a proper smooth embedding. Thus it follows from the Smooth Embedding Theorem 30.15 that we can view X as a compact proper submanifold of E.

(2) We equip the smooth manifold E with the orientation which has the property that at each $Q \in E$ the orientations in the short exact sequence

$$0 \;\to\; \mathrm{T}_Q E_{p(Q)} \;\to\; \mathrm{T}_Q E \;\to\; \mathrm{T}_{p(Q)} X \;\to\; 0$$

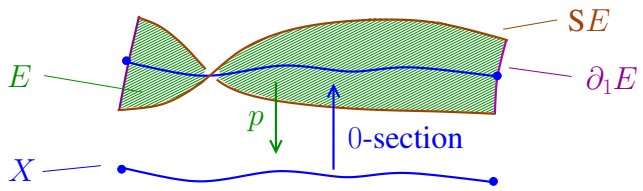
are compatible in the sense of Definition 31.11.[2639]

(3) We write $\partial_1 E := p^{-1}(\partial X)$.[2640]

(4) It follows from (2) and the Poincaré Duality Theorem 210.1 that for each $i \in \mathbb{N}_0$ the map[2641]

$$\begin{array}{ccc} \mathrm{H}^{n+k-i}(E, SE; \mathbb{Z}) & \to & \mathrm{H}_i(E, \partial_1 E) \\ \varphi & \mapsto & \varphi \cap [E]. \end{array}$$

is an isomorphism. As usual we denote by $\mathrm{PD}_E\colon \mathrm{H}_i(E, \partial_1 E) \to \mathrm{H}^{n+k-i}(E, SE; \mathbb{Z})$ the inverse of this isomorphism.



The following proposition gives us a calculation of the Thom class for linear oriented smooth $\overline{B}^n$-bundles:

Proposition 244.13. (Smooth Disk Bundle–Thom Class Proposition) In the following let $p\colon T \to X$ be a linear oriented smooth $\overline{B}^n$-bundle over a compact oriented k-dimensional smooth manifold X. As in Notation 244.12 we view X as a compact oriented proper submanifold of the compact oriented smooth manifold T. Then[2642] [2643]

$$\Theta_{\mathbb{Z}}(p\colon T \to X) \;=\; \mathrm{PD}_T(\underbrace{[X]}_{\in \mathrm{H}_k(T,\partial_1 T)}) \;\in\; \mathrm{H}^n(T, ST; \mathbb{Z}).$$

Proof. Note that it follows from the uniqueness statement of the Thom–Disk Bundle Isomorphism Theorem 244.5 (2) it suffices to prove the following claim:

Claim. For every $x \in X$ we have the equality

$$\langle \underbrace{\iota_x^* \mathrm{PD}_T([X])}_{\in \mathrm{H}^n(T_x, ST_x;\mathbb{Z})}, \underbrace{[T_x]}_{\in \mathrm{H}_n(T_x, ST_x)} \rangle_{T_x} \;=\; 1.$$

(Here $\iota_x\colon T_x \to T$ is of course the inclusion map.)

Proof. Let $x \in X$. We introduce some notation:

(a) It follows easily from the hypothesis that we are dealing with a smooth $\overline{B}^n$-bundle that we can pick a neighborhood U around x with the following two properties:
- U is a compact codimension-zero submanifold of X.
- There exists an orientation-preserving diffeomorphism $\Phi\colon p^{-1}(U) \to \overline{B}^n \times U$ of linear oriented smooth $\overline{B}^n$-bundles.[2644]

(b) We denote by $\pi\colon \overline{B}^n \times U \to \overline{B}^n$ the natural projection.

[2639] Note that we use the "fiber first"-convention for the orientation. In particular, for the trivializable $\overline{B}^n$-bundle $\overline{B}^n \times X \to X$ our convention leads us to the usual product orientation and for the trivial $\overline{B}^n$-bundle $X \times \overline{B}^n \to X$ we get possibly an orientation that differs from the product orientation. This can be very confusing.

[2640] The notation $\partial_1 X$ is inspired by Notation 25.47 and the Smooth Submanifold-with-Corner Lemma 26.16.

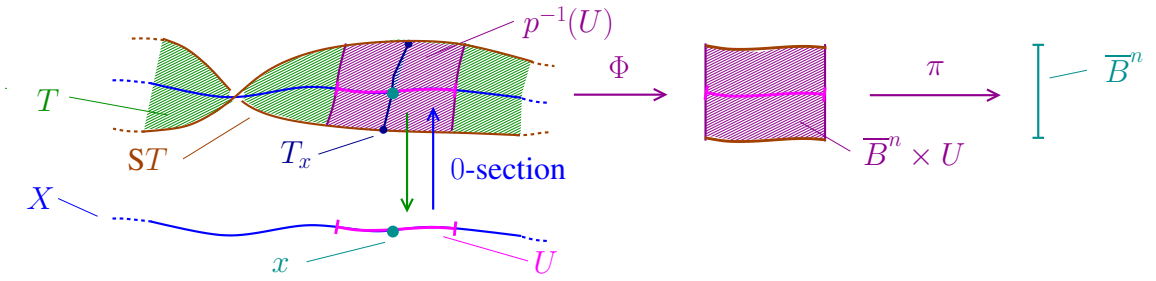
[2641] Here we implicitly use that E is a topological manifold with $\partial E = SE \cup \partial_1 E$ where SE and $\partial_1 E$ are compact codimension-zero submanifolds of ∂E with $SE \cap \partial_1 E = \partial SE = \partial\partial_1 E$. We leave the verification of these statements to the reader.

[2642] Note that in the proof of the Smooth Disk Bundle–Thom Class Proposition 244.13 we give an alternative proof for the *existence* of Thom classes, in the sense of the Thom–Disk Bundle Isomorphism Theorem 244.5 (2b). But this proposition does *not* give us the uniqueness statement for Thom classes.

[2643] There is of course also an "unoriented" $\mathbb{F}_2$-version of this statement. As ever, for this author conciseness in the exposition is an important objective, thus we will not formulate this version.

[2644] Note that Φ is essentially a trivialization, except that we wrote $\overline{B}^n \times U$ instead of $U \times \overline{B}^n$. The reason is (as in Footnote 2639) that our orientation conventions force us to pick the former.

(c) We denote by $v\colon T_x \to p^{-1}(U)$ and $w\colon p^{-1}(U) \to T$ the inclusions.

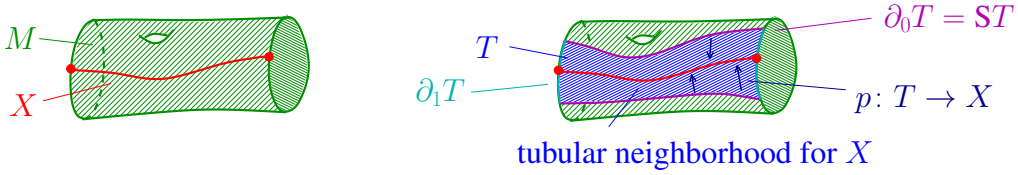


We now calculate that

$$\begin{array}{ccc}
\langle \iota_x^* \mathrm{PD}_T([X]), [T_x]\rangle_{T_x} & \overset{\text{since } \iota_x = w\circ v\colon T_x \to T}{\underset{\downarrow}{=}} & \langle v^*(w^*(\mathrm{PD}_T([X]))), [T_x]\rangle_{T_x} \\
& = & \langle \mathrm{PD}_{p^{-1}(U)}([U]), v_*([T_x])\rangle_{p^{-1}(U)} \\
& \underset{\uparrow}{=} & \\
\text{by the Large H}_*\text{-to-Small H}^*\text{-PD Lemma 212.9 together with the Codimension Zero Fundamental Class Lemma 194.10} & & \\
& = & \langle \pi^*([\overline{B}^n]), [\overline{B}^n \times \{x\}]\rangle_{\overline{B}^n \times U} \\
& \underset{\uparrow}{=} & \\
\text{by the Fundamental Class-Product Proposition 207.2 (2)} & &
\end{array}$$

$$\begin{array}{ccc}
& \overset{\text{by the Kronecker Pairing Natural Lemma 199.9 (1)}}{\underset{\downarrow}{=}} & \langle w^*(\mathrm{PD}_T([X])), v_*([T_x])\rangle_{p^{-1}(U)} \\
& = & \langle \mathrm{PD}_{\overline{B}^n \times U}([\{0\}\times U]), [\overline{B}^n \times \{x\}]\rangle_{\overline{B}^n \times U} \\
& \underset{\uparrow}{=} & \\
\text{since } \Phi \text{ is an orientation-preserving diffeomorphism and since } (\Phi\circ v)(T_x) = \overline{B}^n \times \{x\} & & \\
& = & \langle [\overline{B}^n]^*, [\overline{B}^n]\rangle_{\overline{B}^n} \;=\; 1 \\
& \underset{\uparrow}{=} & \\
\text{by the Kronecker Pairing Natural Lemma 199.9 (1) and since } \pi_*([\overline{B}^n \times \{x\}]) = [\overline{B}^n] & & \blacksquare
\end{array}$$

For us the main source of linear oriented smooth $\overline{B}^n$-bundles are tubular neighborhoods. Recall that the Tubular Neighborhood–Existence Theorem 57.11 says that for any compact m-dimensional smooth manifold X and any proper n-dimensional smooth submanifold $X \subseteq M$ we have a tubular neighborhood $p\colon T \to X$. By Definition 57.9 this is a smooth linear $\overline{B}^{m-n}$-bundle.



Furthermore, if the smooth manifold M is oriented and if the smooth submanifold X is oriented, then one can easily verify that the exact sequence from Notation 244.12 turns $p\colon T \to X$ into an oriented $\overline{B}^{m-n}$-bundle.

With this reminder the following proposition makes sense:

Proposition 244.14. (Thom Class-of-Submanifold Proposition) Let M be a compact oriented m-dimensional smooth manifold and let $\partial M = A \sqcup B$ be a decomposition into two disjoint unions of components of ∂M. Furthermore let $X \subseteq M$ be an oriented proper n-dimensional smooth submanifold with $\partial X \subseteq A$ and with tubular neighborhood $p\colon T \to X$. We introduce the following three maps:

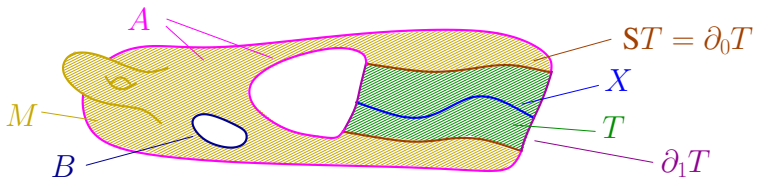
(1) The inclusion map $w\colon (T, ST) \to (M, M \setminus \mathring{T})$. Note that it follows from the Manifold–Excision Theorem 212.4 that w induces an isomorphism on cohomology groups.

(2) The natural map $\text{я}\colon \mathrm{H}^i(M, M \setminus \mathring{T}; \mathbb{Z}) \to \mathrm{H}^i(M, B; \mathbb{Z})$.

(3) We denote by $\mathrm{PD}_M\colon \mathrm{H}_n(M,A) \to \mathrm{H}^{m-n}(M,B;\mathbb{Z})$ the Poincare duality isomorphism from Definition 210.2.

With these preparations we have the following equality in $\mathrm{H}^{m-n}(M,B;\mathbb{Z})$:

$$\mathrm{PD}_M([X]) \;=\; (\text{я} \circ (w^*)^{-1})(\Theta_{\mathbb{Z}}(T \to X)).$$



Proof. Let M be a compact oriented m-dimensional smooth manifold and let $\partial M = A \sqcup B$ be a decomposition into two disjoint unions of components of ∂M. Furthermore let $X \subseteq M$ be an oriented proper n-dimensional smooth submanifold with $\partial X \subseteq A$ and with tubular neighborhood $p\colon T \to X$. We start out with two observations:

(a) By Definition 57.9 of a tubular neighborhood we see that ST of the Boundary Fiber Bundle Lemma 244.1 agrees with $\partial_0 T$ of Notation 25.47 and the Smooth Submanifold-with-Corner Lemma 26.16.

(b) It follows easily from comparing the definitions that in this case $\partial_1 T$ of Notation 244.12 agrees with $\partial_1 T$ of Notation 25.47 and the Smooth Submanifold-with-Corner Lemma 26.16.

We now see that we have the following equalities in $\mathrm{H}^{m-n}(M,B;\mathbb{Z})$:

$$\mathrm{PD}_M([X]) \underset{\text{by the Small } \mathrm{H}_*\text{-to-Large } \mathrm{H}^*\text{-PD Lemma 212.8, here implicitly we use (a) and (b)}}{\uparrow =} (\text{я} \circ (w^*)^{-1})(\mathrm{PD}_T([X])) \underset{\text{by the Smooth Disk Bundle–Thom Class Proposition 244.13}}{\uparrow =} (\text{я} \circ (w^*)^{-1})(\Theta_{\mathbb{Z}}(T \to X)).$$

■

244.5. **Intersection of submanifolds and Poincaré duality.** We start out by recalling the following theorem:

Theorem 220.10. (Cup Product–Intersection Number Theorem) Let M be a compact oriented connected smooth manifold and let $\partial M = A \sqcup B$ be a decomposition into two disjoint unions of components of ∂M. Furthermore let X be an oriented proper submanifold of M with $\partial X \subseteq A$ and let Y be an oriented proper submanifold of M with $\partial Y \subseteq B$. We make the following assumptions:

(a) X and Y intersect transversally in the sense of Definition 50.3.

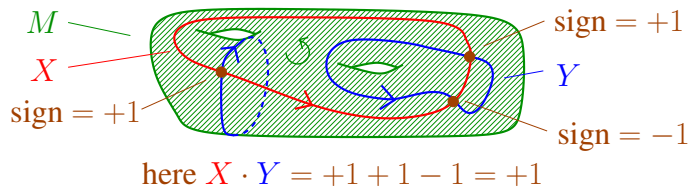
(b) $\dim(X) + \dim(Y) = \dim(M)$ (or equivalently, if $\mathrm{codim}(X) + \mathrm{codim}(Y) = \dim(M)$.)

Then we have the following equality in $\mathrm{H}^{\dim(M)}(M,\partial M;\mathbb{Z})$:

$$\underbrace{\mathrm{PD}_M(\underbrace{[X]}_{\in \mathrm{H}_{\dim(X)}(M,A)})}_{\in \mathrm{H}^{\mathrm{codim}(X)}(M,B;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(\underbrace{[Y]}_{\in \mathrm{H}_{\dim(Y)}(M,B)})}_{\in \mathrm{H}^{\mathrm{codim}(Y)}(M,A;\mathbb{Z})} \;=\; \underbrace{X \cdot Y}_{\substack{\text{algebraic intersection number, see Definition 220.5}}} \cdot [M]^*.$$

$\in \mathrm{H}^{\dim(M)}(M,\partial M;\mathbb{Z})$

In other words, the Cup Product–Intersection Number Theorem 220.10 lets us calculate the cup product $\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])$ geometrically, if the submanifolds X and Y are



transverse with $\dim(X) + \dim(Y) = \dim(M)$. In this section we want to generalize this beautiful result to arbitrary dimensions.

Before we can get to this generalization, we need to formulate the following lemma:

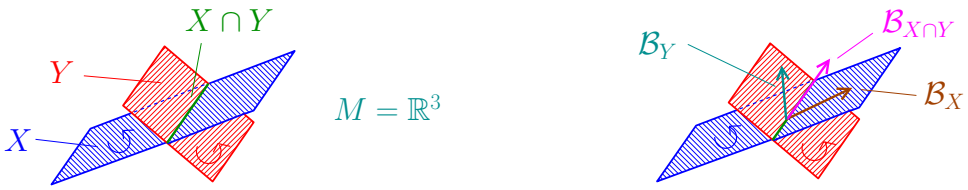
Lemma 244.15. (Submanifold Transversal Intersection Lemma) Let M be an oriented smooth manifold, let X be an oriented proper submanifold of M and let Y be an oriented proper submanifold of M. Suppose that X and Y intersect transversally. The following two statements hold:

(1) (a) The intersection $X \cap Y$ is a proper submanifold of M.
(b) We have $\operatorname{codim}(X \cap Y) = \operatorname{codim}(X) + \operatorname{codim}(Y)$.

(2) We now assume that $\dim(X) > 0$, $\dim(Y) > 0$ and $\dim(X \cap Y) > 0$.[2645] Given $P \in X \cap Y$ we have the following short exact sequence which is induced by the various inclusion maps ι:

$$0 \to \mathrm{T}_P(X \cap Y) \xrightarrow{(\iota,-\iota)} \mathrm{T}_P X \oplus \mathrm{T}_P Y \xrightarrow{\iota \oplus \iota} \mathrm{T}_P M \to 0.$$

We can pick an ordered set of vectors $\mathcal{B}_X$ in $\mathrm{T}_P X$, $\mathcal{B}_Y$ in $\mathrm{T}_P Y$ and $\mathcal{B}_{X\cap Y}$ in $\mathrm{T}_P(X \cap Y)$ such that $\mathcal{B}_X \cup \mathcal{B}_{X\cap Y}$ is a positive basis for $\mathrm{T}_P X$, such that $\mathcal{B}_{X\cap Y} \cup \mathcal{B}_Y$ is a positive basis for $\mathrm{T}_P Y$ and such that $\mathcal{B}_X \cup \mathcal{B}_{X\cap Y} \cup \mathcal{B}_Y$ is a positive basis for $\mathrm{T}_P M$. We then equip $\mathrm{T}_P(X \cap Y)$ with the orientation given by $\mathcal{B}_{X\cap Y}$. This definition is independent of the choices and it turns $X \cap Y$ into an oriented submanifold of M.



Proof.

(1) This follows from the Submanifold Transversal Intersection Theorem 50.4 (2a).

(2) This statement follows easily from the Submanifold Transversal Intersection Theorem 50.4 (2c) together with the purely algebraic Exercise 31.3. ∎

[2645] If one of these dimensions is zero, then we have still have a corresponding statement, which needs to be formulated slightly differently, since the orientation of a 0-dimensional vector space is by Definition 31.4 a sign -1 or $+1$. We leave it to the reader to fill in the details.

Example. Let $n \in \mathbb{N}$ and let $0 \leq k \leq l \leq n$. We consider the complex manifold $\mathbb{C}\mathrm{P}^n$ with the complex submanifolds

$$\begin{aligned} A &= \{[z_0 : \cdots : z_l : 0 : \cdots : 0] \,|\, (z_0, \ldots, z_l) \in \mathbb{C}^{l+1} \setminus \{0\}\} \\ B &= \{[0 : \cdots : 0 : z_k : \cdots : z_n] \,|\, (z_k, \ldots, z_n) \in \mathbb{C}^{n+1-k} \setminus \{0\}\} \end{aligned}$$

and

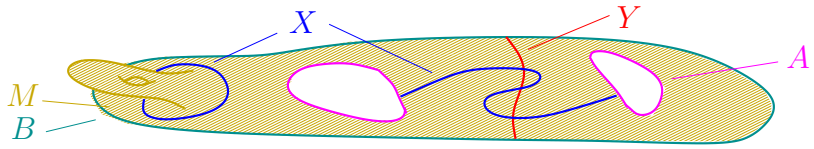
$$C = \{[0 : \cdots : 0 : z_k : \cdots : z_l : 0 : \cdots : 0] \,|\, (z_k, \ldots, z_l) \in \mathbb{C}^{l+1-k} \setminus \{0\}\}.$$

As usual we equip $\mathbb{C}\mathrm{P}^n$ and A, B, C with the natural orientations coming from the Complex Manifold-Orientation Proposition 40.18. One can easily show that A and B intersect transversely with $A \cap B = C$. Furthermore, using elementary linear algebra, see Exercise 40.3, one can easily show that the orientation of $A \cap B$ equals the orientation of C. □

Now we can formulate the main result of his section:

Theorem 244.16. (Cup Product–Intersection Manifold Theorem) Let M be a compact oriented smooth manifold and let $\partial M = A \sqcup B$ be a decomposition into two disjoint unions of components of ∂M. Furthermore let X be an oriented proper submanifold of M of codimension k with $\partial X \subseteq A$ and let Y be an oriented proper submanifold of M of codimension l with $\partial Y \subseteq B$. If X and Y intersect transversally, then[2646]

$$\underbrace{\mathrm{PD}_M([X])}_{\in \mathrm{H}^k(M,B;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^l(M,A;\mathbb{Z})} = \mathrm{PD}_M([X \cap Y]) \in \mathrm{H}^{k+l}(M, \partial M; \mathbb{Z}).$$

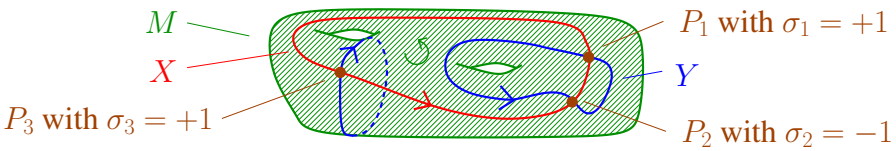


Remark. The Cup Product–Intersection Manifold Theorem 244.16 can be summarized as saying that cup products correspond to taking intersections of submanifolds. With this point of view, the symbol "$\cup$" for the cup product does not seem to be a great choice. □

Examples.

(1) Let M be a compact oriented connected smooth manifold and let $\partial M = A \sqcup B$ be a decomposition into two disjoint unions of components of ∂M. Furthermore let X be an oriented proper submanifold of M with $\partial X \subseteq A$ and let Y be an oriented proper submanifold of M with $\partial Y \subseteq B$. We make the following assumptions:
   (a) X and Y intersect transversally.
   (b) $\dim(X) + \dim(Y) = \dim(M)$.
   We denote by $P_1, \ldots, P_n$ the points in $X \cap Y$ and we denote by $\sigma_1, \ldots, \sigma_n \in \{\pm 1\}$ the corresponding signs from Definition 220.5. It follows immediately from the definitions that these signs give us precisely the orientation of the 0-dimensional submanifold $X \cap Y$.[2647]
   Now we calculate that

[2646] It is conceivable that the equality only holds up to a sign that depends on $\dim(X), \dim(Y)$ and $\dim(M)$.
[2647] Recall that on page 810 we remarked that the orientation of a 0-dimensional manifold M is the same as a map $M \to \{\pm 1\}$.

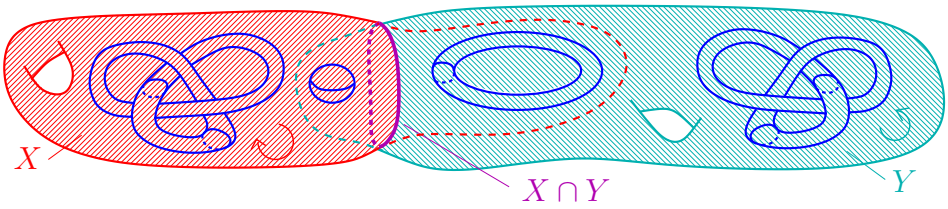


$$\begin{array}{ccccc} & & \text{by the Cup Product–Intersection Manifold Theorem 244.16} & & \\ & & \downarrow & & \\ \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) & = & \mathrm{PD}_M([X \cap Y]) & = & \sum_{i=1}^{n} \mathrm{PD}_M(\sigma_i \cdot [\{P_i\}]) \\ & = & \sum_{i=1}^{n} \sigma_i \cdot [M]^* & = & (X \cdot Y) \cdot [M]^*. \\ & & \uparrow & & \uparrow \\ & & \text{see page 4029} & & \text{see Definition 51.4 or equivalently see Definition 220.5} \end{array}$$

This shows that the Cup Product–Intersection Manifold Theorem 244.16 generalizes the Cup Product–Intersection Number Theorem 220.10.

(2) Let M be the smooth manifold that is the complement in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ of the three open solid tori and the open ball that are shown in the figure below.

M is the complement in S^3 of three open solid tori and an open ball



Furthermore let X and Y be the oriented surfaces shown in that figure. Basically the same argument as on page 4238 shows that $[X]$ and $[Y]$ are non-trivial in $\mathrm{H}_2(M)$. On the other hand the class $[X \cap Y] \in \mathrm{H}_1(M)$ is clearly trivial. Therefore we deduce from the Cup Product–Intersection Manifold Theorem 244.16 that $\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) = 0$.

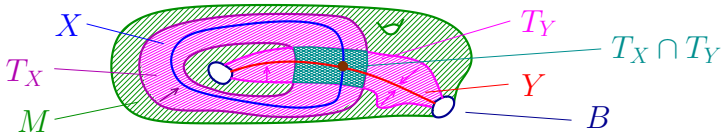
(3) Using the Cup Product–Intersection Manifold Theorem 244.16 and using the example preceding that theorem one can give fairly easily a completely geometric calculation of the cup product of $\mathbb{C}\mathrm{P}^n$ and also of the n-dimensional torus S^n. We will work out the details in Exercise 244.1 and in Exercise 244.2. □

The remainder of this section is dedicated to the proof of the Cup Product–Intersection Manifold Theorem 244.16:

Sketch of proof. Let M be a compact oriented smooth manifold and let $\partial M = A \sqcup B$ be a decomposition into two disjoint unions of components of ∂M. Furthermore let X be an oriented proper submanifold of M with $\partial X \subseteq A$ of codimension k and let Y be an oriented proper submanifold of M with $\partial Y \subseteq B$ of codimension l. Finally we assume that X and Y intersect transversally. The cases that $\dim(X) = 0$ or $\dim(Y) = 0$ are essentially trivial. So we now assume that $\dim(X) > 0$ and $\dim(Y) > 0$.

Since the two submanifolds X and Y intersect transversally we can pick tubular neighborhoods $p_X \colon T_X \to X$ of $X \subseteq M$ and $p_Y \colon T_Y \to Y$ of $Y \subseteq M$ as in the Transverse

Tubular Neighborhood–Existence Theorem 57.32. For the time being it suffices to know that $T_{X\cap Y} := T_X \cap T_Y \to X \cap Y$ is essentially a tubular neighborhood of $X \cap Y$.[2648]



Next we consider the following maps:

$$\begin{array}{ccccc}
 & \overset{\mathrm{PD}_M([X])}{\mathrm{H}^k(M,B;\mathbb{Z})} \times \overset{\mathrm{PD}_M([Y])}{\mathrm{H}^l(M,A;\mathbb{Z})} & \xrightarrow{\quad\cup\quad} & \overset{\mathrm{PD}_M([X\cap Y])}{\mathrm{H}^{k+l}(M,\partial M;\mathbb{Z})} \\
 & \uparrow{\scriptstyle я} \quad {\scriptstyle я}\uparrow & & \uparrow{\scriptstyle я} \\
 & \mathrm{H}^k(M, M\setminus \mathring{T}_X;\mathbb{Z}) \times \mathrm{H}^l(M, M\setminus \mathring{T}_Y;\mathbb{Z}) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{k+l}(M, M\setminus \mathring{T}_{X\cap Y};\mathbb{Z}) \\
 & \cong\downarrow \quad \downarrow\cong & & \downarrow\cong \\
 & \mathrm{H}^k(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_X;\mathbb{Z}) \times \mathrm{H}^l(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_{X\cap Y};\mathbb{Z}) \\
 & \cong\downarrow{\scriptstyle \widetilde{\imath}_X^*} \quad {\scriptstyle \widetilde{\imath}_Y^*}\downarrow\cong & & {\scriptstyle \widetilde{\imath}_{X\cap Y}^*}\downarrow\cong \\
 & \underset{\Theta_{\mathbb{Z}}(T_X\to X)}{\mathrm{H}^k(T_X, \mathrm{S}T_X;\mathbb{Z})} \times \underset{\Theta_{\mathbb{Z}}(T_Y\to Y)}{\mathrm{H}^l(T_Y, \mathrm{S}T_Y;\mathbb{Z})} & & \underset{\Theta_{\mathbb{Z}}(T_{X\cap Y}\to X\cap Y)}{\mathrm{H}^{k+l}(T_{X\cap Y}, \mathrm{S}T_{X\cap Y};\mathbb{Z})}
\end{array}$$

We make the following clarifications and observations:

- We set $T_{X\cup Y} := T_X \cup T_Y$.
- The maps я are the natural maps from relative to absolute cohomology.
- The downward pointing maps are all induced by inclusions. It follows from the Manifold–Excision Theorem 212.4 that they are isomorphisms.
- It follows from the Cup Product-Naturality Lemma 204.8 that the top two squares of the diagram commute.
- By the Thom Class-of-Submanifold Proposition 244.14 we know that the Thom classes of the various tubular neighborhoods get sent (by going the three vertical maps upwards) to the corresponding Poincaré duals.

This discussion shows: It thus suffices to prove that we have the following equality

$$(\widetilde{\imath}_X^*)^{-1}(\Theta_{\mathbb{Z}}(T_X \to X)) \cup (\widetilde{\imath}_Y^*)^{-1}(\Theta_{\mathbb{Z}}(T_Y \to Y)) \;=\; (\widetilde{\imath}_{X\cap Y}^*)^{-1}(\Theta_{\mathbb{Z}}(T_{X\cap Y} \to X \cap Y))$$

[2648]More precisely, $T_X \cap T_Y \to X \cap Y$ is naturally a "linear" $\overline{B}^k \times \overline{B}^l$-bundle. Such a $\overline{B}^k \times \overline{B}^l$-bundle can be treated almost the same way as a linear $\overline{B}^{k+l}$-bundle. In particular there exists a Thom class $\Theta_{\mathbb{Z}}(T_X \cap T_Y \to X \cap Y)$ which has the property that for each fiber $(T_X \cap T_Y)_P = T_{X,P} \times T_{Y,P} \cong \overline{B}^k \times \overline{B}^l$ the Thom class restricts to the dual fundamental class with respect to the product orientation. We leave it to the reader to fill in the technical details.

in the group $\mathrm{H}^{k+l}(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_{X\cap Y};\mathbb{Z})$. We now consider the following maps:

$$\begin{array}{ccc}
\mathrm{H}^k(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_X;\mathbb{Z}) \times \mathrm{H}^l(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_Y;\mathbb{Z}) & \xrightarrow{\ \cup\ } & \mathrm{H}^{k+l}(T_{X\cup Y}, T_{X\cup Y}\setminus \mathring{T}_{X\cap Y};\mathbb{Z}) \\
\cong \downarrow \tilde{\iota}_X^* \qquad \tilde{\iota}_Y^* \downarrow \cong & & \\
\underset{\Theta_{\mathbb{Z}}(T_X\to X)}{\mathrm{H}^k(T_X, \mathrm{S}T_X;\mathbb{Z})} \times \underset{\Theta_{\mathbb{Z}}(T_Y\to Y)}{\mathrm{H}^l(T_Y, \mathrm{S}T_Y;\mathbb{Z})} & & \tilde{\iota}_{X\cap Y}^* \downarrow \cong \\
\iota_X^* \downarrow \qquad \iota_Y^* \downarrow & & \\
\mathrm{H}^k(T_{X\cap Y}, \mathrm{S}T_X \cap T_Y;\mathbb{Z}) \times \mathrm{H}^l(T_{X\cap Y}, T_X \cap \mathrm{S}T_Y;\mathbb{Z}) & \xrightarrow{\ \cup\ } & \underset{\Theta_{\mathbb{Z}}(T_{X\cap Y}\to X\cap Y)}{\mathrm{H}^{k+l}(T_{X\cap Y}, \mathrm{S}T_{X\cap Y};\mathbb{Z})}
\end{array}$$

We make the following observations:

- The vertical maps are induced by inclusions.
- As pointed out above, the right vertical map is an isomorphism.
- It follows again from the Cup Product-Naturality Lemma 204.8 that the diagram commutes.

This discussion shows: It suffices to prove that

$$\iota_X^*(\Theta_{\mathbb{Z}}(T_X \to X)) \cup \iota_Y^*(\Theta_{\mathbb{Z}}(T_Y \to Y)) \;=\; \Theta_{\mathbb{Z}}(T_{X\cap Y} \to X \cap Y)$$

in the group $\mathrm{H}^{k+l}(T_{X\cap Y}, \mathrm{S}T_{X\cap Y};\mathbb{Z})$.

By the defining property of the Thom class $\Theta_{\mathbb{Z}}(T_{X\cap Y} \to X\cap Y)$ it thus suffices to prove the following claim:

Claim. For every $P \in X \cap Y$ the image of $\iota_X^*(\Theta_{\mathbb{Z}}(T_X \to X)) \cup \iota_Y^*(\Theta_{\mathbb{Z}}(T_Y \to Y))$ in $\mathrm{H}^{k+l}(T_{X\cap Y,P}, \mathrm{S}T_{X\cap Y,P};\mathbb{Z})$ equals the positive generator.

Proof. Let $P \in X \cap Y$. It follows easily from the fact that we picked tubular neighborhoods as in the Transverse Tubular Neighborhood–Existence Theorem 57.32 and our orientation convention from the Submanifold Transversal Intersection Lemma 244.15 that there exists a neighborhood U of $P \in M$, such that $\widetilde{U} := U \cap X \cap Y$ is an open subset of $X \cap Y$, and an orientation-preserving diffeomorphism

$$\Phi = (\Phi_y, \Phi_U, \Phi_x)\colon U \;\to\; \overline{B}^l \times \underbrace{((X\cap Y)\cap U)}_{=\widetilde{U}} \times \overline{B}^k$$

of smooth manifolds with corner such that the following three conditions are satisfied:

(1) (a) $\Phi(X\cap U) = \overline{B}^l \times \widetilde{U} \times \{0\}$.
(b) $\Phi(Y\cap U) = \{0\} \times \widetilde{U} \times \overline{B}^k$.
Furthermore the diffeomorphisms $\Phi\colon X\cap U \to \overline{B}^l \times \widetilde{U} \times \{0\}$, $\Phi\colon Y\cap U \to \{0\}\times\widetilde{U}\times\overline{B}^k$ and $\Phi\colon (X\cap Y)\cap U \to \{0\}\times\widetilde{U}\times\{0\}$ are orientation-preserving.

(2) (a) The map

$$\begin{array}{rcl} U & \to & (X\cap U) \times \overline{B}^k \\ Q & \mapsto & (\underbrace{\Phi^{-1}(\Phi_y(Q), \Phi_U(Q), 0)}_{\in X\cap U \text{ by (1a)}}, \underbrace{\Phi_x(Q)}_{\in \overline{B}^k}) \end{array}$$

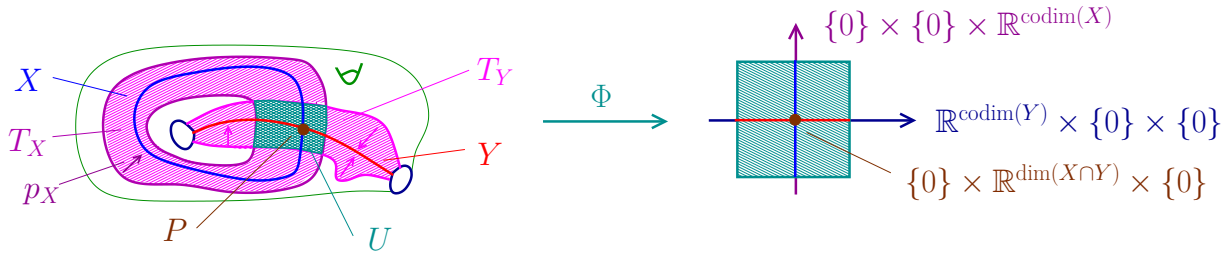
is a trivialization of the $\overline{B}^k$-bundle $p_X\colon T_X \to X$.

(b) The map

$$\begin{array}{rcl} U & \to & (Y\cap U)\times \overline{B}^l \\ Q & \mapsto & (\underbrace{\Phi^{-1}(0,\Phi_U(Q),\Phi_x(Q))}_{\in Y\cap U \text{ by (1b)}},\underbrace{\Phi_y(Q)}_{\in \overline{B}^l}) \end{array}$$

is a trivialization of the $\overline{B}^l$-bundle $p_Y\colon T_Y\to Y$.

Furthermore in (a) and (b) the trivializations are compatible with the given smooth structures. Note that in (a) and (b) we do not claim that these are orientation-preserving trivializations.



Next we consider the following diagram (we dropped the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccc}
 & & \overset{\Theta_{\mathbb{Z}}(T_X\to X)}{\mathrm{H}^k(T_X,ST_X)}\times\overset{\Theta_{\mathbb{Z}}(T_Y\to Y)}{\mathrm{H}^l(T_Y,ST_Y)} & & \\
 & & \downarrow{\scriptstyle \iota_X^*} \quad \downarrow{\scriptstyle \iota_Y^*} & & \\
\mathrm{H}^k(T_{X\cap Y},ST_X\cap T_Y)\times\mathrm{H}^l(T_{X\cap Y},T_X\cap ST_Y) & & \xrightarrow{\cup} & & \mathrm{H}^{k+l}(T_{X\cap Y},ST_{X\cap Y}) \\
 & & & \searrow & \downarrow \\
 & & \mathrm{H}^{k+l}(T_{X\cap Y,P},ST_{X\cap Y,P}) & & \\
 & & & \nearrow & \\
\mathrm{H}^k(T_{X\cap Y}\cap U,ST_X\cap T_Y\cap U)\times\mathrm{H}^l(T_{X\cap Y}\cap U,T_X\cap ST_Y\cap U) & & \xrightarrow{\cup} & & \mathrm{H}^{k+l}(T_{X\cap Y}\cap U,ST_{X\cap Y}\cap U) \\
\downarrow{\scriptstyle \Phi^*}\cong & & \cong\downarrow{\scriptstyle \Phi^*} & & \downarrow{\scriptstyle \Phi^*}\cong \\
\mathrm{H}^k(\overline{B}^l\times\widetilde{U}\times\overline{B}^k,\overline{B}^l\times\widetilde{U}\times S^{k-1})\times\mathrm{H}^l(\overline{B}^l\times\widetilde{U}\times\overline{B}^k,S^{l-1}\times\widetilde{U}\times\overline{B}^k) & & \hookrightarrow & & \mathrm{H}^{k+l}(\overline{B}^l\times\widetilde{U}\times\overline{B}^k,\partial(\overline{B}^l\times\widetilde{U}\times\overline{B}^k)) \\
\uparrow{\scriptstyle \mathrm{pr}^*_{\overline{B}^k}} & & \uparrow{\scriptstyle \mathrm{pr}^*_{\overline{B}^l}} & & \uparrow{\scriptstyle \mathrm{pr}^*_{\overline{B}^l\times\overline{B}^k}} \\
\underset{[\overline{B}^k]^*}{\mathrm{H}^k(\overline{B}^k,S^{k-1})}\times\underset{[\overline{B}^l]^*}{\mathrm{H}^l(\overline{B}^l,S^{l-1})} & & \xrightarrow[(\varphi,\psi)\mapsto \mathrm{pr}^*_{\overline{B}^l}(\psi)\cup\mathrm{pr}^*_{\overline{B}^k}(\varphi)]{} & & \underset{[\overline{B}^l\times\overline{B}^k]^*}{\mathrm{H}^{k+l}(\overline{B}^l\times\overline{B}^k,\partial(\overline{B}^l\times\overline{B}^k))}.
\end{array}$$

We make the following clarifications and observations:

(i) The unnamed maps are induced by inclusions.

(ii) It follows from the functoriality of cohomology and the Cup Product-Naturality Lemma 204.8 that the diagram commutes.

(iii) (α) We denote by $\mathrm{pr}_{\overline{B}^k}\colon \overline{B}^l \times \widetilde{U} \times \overline{B}^k \to \overline{B}^k$ the natural projection. It follows from (2a), the Thom Class–Natural Proposition 244.7 and the Trivializable–Thom Lemma 244.9 that the image of $\Theta_{\mathbb{Z}}(T_X \to X)$ (under the top three vertical maps) in $\mathrm{H}^k(\overline{B}^l \times \widetilde{U} \times \overline{B}^k, \overline{B}^l \times \widetilde{U} \times S^{k-1})$ equals $\pm\mathrm{pr}^*_{\overline{B}^k}([\overline{B}^k]^*)$.

(β) Analogous statements as in (α) also hold for Y and $X \cap Y$.

(iv) Note that it follows from the Fundamental Class-Product Proposition 207.2 (1) that $\mathrm{pr}^*_{\overline{B}^l}([\overline{B}^l]^*) \cup \mathrm{pr}^*_{\overline{B}^k}([\overline{B}^k]^*) = [\overline{B}^l \times \overline{B}^k]^*$.

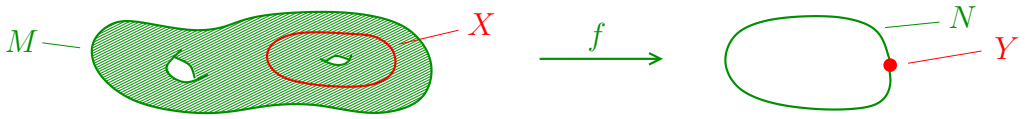
The above discussion shows that the image of $\iota_X^*(\Theta_{\mathbb{Z}}(T_X \to X)) \cup \iota_Y^*(\Theta_{\mathbb{Z}}(T_Y \to Y))$ in $\mathrm{H}^{k+l}(T_{X\cap Y} \cap U, \mathrm{S}T_{X\cap Y} \cap U; \mathbb{Z})$ equals (up to sign) the Thom class for the $(\overline{B}^l \times \overline{B}^k)$-bundle $T_{X\cap Y} \cap U \to \widetilde{U}$. But by the commutativity of the diagram we then get the desired statement also for the cohomology group corresponding to the fiber at $P \in \widetilde{U}$.

Finally we leave it to the reader to sort out the frightful mess that comes from trying to keep track of all orientation conventions. ■

244.6. **Preimages of submanifolds.** The main goal of this section is to prove the following proposition, which will play an essential role on two occasions in later chapters:

Proposition 244.17. (Submanifold Preimage–Thom Class Proposition) In the following let M be a compact oriented smooth manifold, let N be a closed oriented smooth manifold and let $f\colon M \to N$ be a smooth map. Furthermore let $Y \subseteq N$ be a closed oriented smooth submanifold. We suppose that f intersects Y transversally, in the sense of Definition 50.7. Then we know by the Transversal Preimage Theorem 50.8 that the preimage $X := f^{-1}(Y)$ is a closed oriented submanifold of M. In this setting we have the following equality in $\mathrm{H}^{\dim(M)-\dim(X)}(M, \partial M; \mathbb{Z})$:

$$f^*(\mathrm{PD}_N([Y])) \;=\; \mathrm{PD}_M([X]).$$



Remark. These are the two occasions when we will make use of the Submanifold Preimage–Thom Class Proposition 244.17:

(1) In the Codimension-One Submanifold Existence Proposition 234.6 when we consider smooth maps $f\colon M \to S^1$ and we take $Y = \{y\} \subseteq S^1$ where $y \in S^1$ is a regular value.

(2) In the Codimension-Two Submanifold Existence Proposition **??** when we consider smooth maps $f\colon M \to \mathbb{C}\mathrm{P}^k$ that are transverse to $\mathbb{C}\mathrm{P}^{k-1} \subseteq \mathbb{C}\mathrm{P}^k$. □

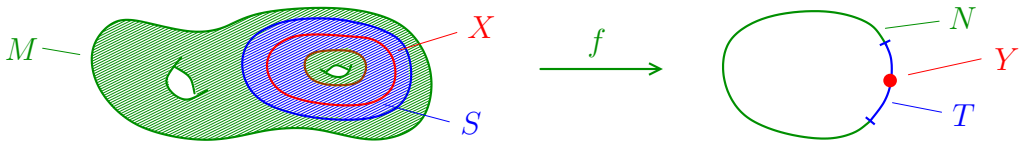
The proof of the Submanifold Preimage–Thom Class Proposition 244.17 relies on the following lemma:

Lemma 244.18. (Tubular Neighborhood Preimage Lemma) Let M be a compact oriented smooth manifold, let N be a closed oriented smooth manifold and let $f\colon M \to N$ be a smooth map. Furthermore let $Y \subseteq N$ be a closed oriented smooth submanifold of codimension k. We suppose that f intersects Y transversally. Then we know by the Transversal Preimage Theorem 50.8 that $X := f^{-1}(Y)$ is a closed oriented submanifold of M of codimension k. There exists a tubular neighborhood T for $Y \subseteq N$ such that

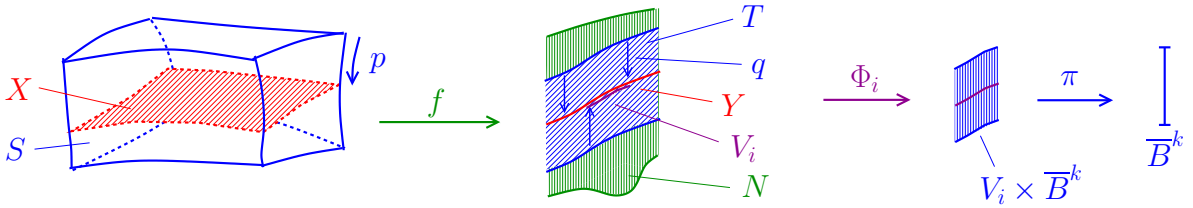
$S := f^{-1}(T)$ defines a tubular neighborhood $X \subseteq M$ and such that

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

is a morphism of oriented $\overline{B}^k$-bundles.



Sketch of proof of Lemma 244.18. It follows from the Tubular Neighborhood–Existence Theorem 57.11 that there exist tubular neighborhoods $p\colon S \to X$ and $q\colon T \to Y$. Since X is compact it follows that, after radial shrinking of the tubular neighborhood T, we can assume that $f^{-1}(T) \subseteq S$. Since Y is compact and since Y is a smooth manifold we can find a family of trivializations $\{\Phi_i\colon q^{-1}(V_i) \to V_i \times \overline{B}^k\}_{i=1,\ldots,m}$ for T such that each V_i contains a compact subset W_i such that the interiors of the W_i cover Y.



For each $i \in \{1, \ldots, m\}$ let $\pi\colon V_i \times \overline{B}^k \to \overline{B}^k$ be the natural projection. Next, for each $i \in \{1, \ldots, m\}$ we consider the map

$$\begin{array}{rcl} \Psi_i\colon f^{-1}(q^{-1}(V_i)) & \to & f^{-1}(V_i) \times \overline{B}^k \\ y & \mapsto & (\underbrace{p(y)}_{\substack{\uparrow \\ \text{defined since } f^{-1}(T) \subseteq S}}, \pi(\Phi_i(f(y)))). \end{array}$$

For each $i \in \{1, \ldots, m\}$ we make the following observations:

- Evidently $f^{-1}(V_i)$ is a closed subset of $f^{-1}(q^{-1}(V_i))$ and the restriction of Ψ_i to $f^{-1}(V_i)$ is a homeomorphism onto the closed subset $f^{-1}(V_i) \times \{0\}$.
- Using the fact that f intersects Y transversally one can easily verify that for each $y \in f^{-1}(V_i)$ the differential $\mathrm{D}(\Psi_i)_y$ is an isomorphism. It thus follows from the Inverse Mapping Theorem 30.1 that Ψ_i is a local diffeomorphism at y.

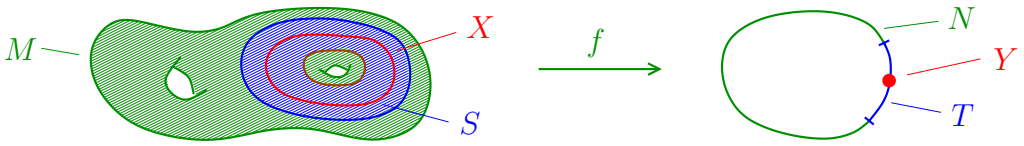
It follows from these two observations, together with the Neighborhood Smooth Embedding Theorem 27.16 that there exists an open neighborhood around $f^{-1}(V_i) \subseteq f^{-1}(q^{-1}(V_i))$ such that Ψ_i is a smooth embedding. It follows from the compactness of W_i that there exists an $r \in (0, 1]$ such that, after replacing the original tubular neighborhood of T by its "radius r-version" T_r, then $f^{-1}(T_r)$ together with the restrictions of the above map $\Psi_i\colon f^{-1}(q^{-1}(\mathring{W}_i)) \to f^{-1}(\mathring{W}_i) \times \overline{B}^k_r$ has all the desired properties. We leave it to the reader to fill in the details. ∎

Proof of Proposition 244.17. Let M be a compact oriented smooth manifold, let N be a closed oriented smooth manifold and let $f\colon M \to N$ be a smooth map. Furthermore let $Y \subseteq N$ be a closed oriented smooth submanifold of codimension k. We suppose that f intersects Y transversally. Then we know by the Transversal Preimage Theorem 50.8 that $X := f^{-1}(Y)$ is a closed oriented submanifold of M of codimension k.

It follows from the Tubular Neighborhood Preimage Lemma 244.18 that there exists a tubular neighborhood T for $Y \subseteq N$ such that $S := f^{-1}(T)$ defines a tubular neighborhood $X \subseteq M$ and such that

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

is a morphism of oriented $\overline{B}^k$-bundles.



Now we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(N;\mathbb{Z}) & \xrightarrow{f^*} & \mathrm{H}^k(M;\mathbb{Z}) \\ \uparrow{\scriptstyle \text{я}} & & \uparrow{\scriptstyle \text{я}} \\ \mathrm{H}^k(N,T;\mathbb{Z}) & \xrightarrow{f^*} & \mathrm{H}^k(M,S;\mathbb{Z}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathrm{H}^k(T,ST;\mathbb{Z}) & \xrightarrow[\Theta_{\mathbb{Z}}(T\to Y)\mapsto\Theta_{\mathbb{Z}}(S\to X)]{f^*} & \mathrm{H}^k(S,SS;\mathbb{Z}). \end{array}$$

$\mathrm{PD}_N([Y]) \updownarrow \Theta_{\mathbb{Z}}(T\to Y)$ $\qquad$ $\mathrm{PD}_M([X]) \updownarrow \Theta_{\mathbb{Z}}(S\to X)$

The proposition follows from the following observations:

- It follows from the functoriality of cohomology that the diagram commutes.
- By the Thom Class–Natural Proposition 244.7 we know that the bottom horizontal map sends $\Theta_{\mathbb{Z}}(T \to Y) \mapsto \Theta_{\mathbb{Z}}(S \to X)$.
- It follows from the Smooth Disk Bundle–Thom Class Proposition 244.13 that the left vertical map sends $\Theta_{\mathbb{Z}}(T \to Y) \mapsto \mathrm{PD}_N([Y])$. Furthermore it follows that the right vertical map sends $\Theta_{\mathbb{Z}}(S \to X) \mapsto \mathrm{PD}_M([X])$. ∎

244.7. **Appendix: The Thom class of sphere bundles.** In this appendix we will introduce the Thom class of spheres bundles. We will not make use of this concept.

Theorem 244.19. (Thom–Sphere Bundle Isomorphism Theorem) Let $p\colon E \to X$ be an S^n-bundle over a topological space X. We assume that p admits a finite atlas.

(1) (a) There exists a unique $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E;\mathbb{F}_2)$ with the property that for each $x \in X$ the restriction to $\mathrm{H}^n(E_x;\mathbb{F}_2)$ is non-zero.[2649]

(b) For each $k \in \mathbb{Z}$ the map[2650 2651]

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{F}_2) & \to & \mathrm{H}^{n+k}(E;\mathbb{F}_2) \\ \varphi & \mapsto & \Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

(2) (a) If the S^n-bundle $p\colon E \to X$ is oriented in the sense of Definition 244.2, then there exists a unique $\Theta_{\mathbb{Z}}(E) \in \mathrm{H}^n(E;\mathbb{Z})$ with the property that for each $x \in X$ the restriction to $\mathrm{H}^n(E_x;\mathbb{Z})$ equals the dual fundamental class $[E_x]^*$ from Definition 201.27.

(b) For each $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{Z}) & \to & \mathrm{H}^{n+k}(E,\mathrm{S}E;\mathbb{Z}) \\ \varphi & \mapsto & \Theta_{\mathbb{Z}}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

Both classes $\Theta_{\mathbb{F}_2}(E)$ and $\Theta_{\mathbb{Z}}(E)$ are again called Thom class and both isomorphisms are called Thom isomorphism.

Proof. The proof of this theorem is very similar to the proof of the Thom–Disk Bundle Isomorphism Theorem 244.5. We leave it to the reader to make the necessary adjustments. ■

The following proposition relates the concept of the Thom class of an $\overline{B}^n$-bundle and of an S^{n-1}-bundle.

Proposition 244.20. (Euler Class–Boundary Proposition) In the following let $p\colon E \to X$ be an $\overline{B}^n$-bundle that admits a finite atlas. For $R = \mathbb{F}_2$ and $R = \mathbb{Z}$ we denote by $\delta_{n-1}\colon \mathrm{H}^{n-1}(\mathrm{S}E;R) \to \mathrm{H}^n(E,\mathrm{S}E;R)$ the connecting homomorphism.

(1) We have the equality $e_{\mathbb{F}_2}(E \to X) = \delta_{n-1}(e_{\mathbb{F}_2}(\mathrm{S}E \to X))$.

(2) If E is oriented, then we have the equality $e_{\mathbb{Z}}(E \to X) = \delta_{n-1}(e_{\mathbb{Z}}(\mathrm{S}E \to X))$.

Proof. In the following we prove Statement (2). Thus let $x \in X$. Note that we have the following diagram:

$$\begin{array}{ccc} \mathrm{H}^{n-1}(\mathrm{S}E;\mathbb{Z}) & \xrightarrow{\delta_{n-1}} & \mathrm{H}^n(E,\mathrm{S}E;\mathbb{Z}) \\ \downarrow{\scriptstyle \iota_x^*} & & \downarrow{\scriptstyle \iota_x^*} \\ \mathrm{H}^{n-1}(\mathrm{S}E_x;\mathbb{Z}) & \xrightarrow[\cong]{\delta_{n-1}} & \mathrm{H}^n(E_x,\mathrm{S}E_x;\mathbb{Z}). \end{array}$$

Next note that it follows from the naturality of the connecting homomorphism that this diagram commutes. Also note that the bottom map is an isomorphism. The claim now follows immediately from the defining properties of $\Theta_{\mathbb{Z}}(\mathrm{S}E \to X))$ and $\Theta_{\mathbb{Z}}(E \to X)$ together with the Fundamental Class-of-∂-Proposition 194.13 and Exercise 199.12. ■

244.8. Exercises for Chapter 244.

Exercise 244.1. Let $n \in \mathbb{N}_0$. As on page 3995 we consider the ring

$$\Lambda[x_1,\dots,x_n] \;:=\; \mathbb{Z}\langle x_1,\dots,x_n\rangle/\text{ideal generated by } x_1^2,\dots,x_n^2, \{x_ix_j + x_jx_i\}_{i,j=1,\dots,n}.$$

Use the Cup Product–Intersection Manifold Theorem 244.16 to show that we have an isomorphism

$$\psi\colon \Lambda[x_1,\dots,x_n] \;\to\; \mathrm{H}^*((S^1)^n;\mathbb{Z})$$

Remark. This gives a new proof of the Torus–Cup Product Proposition 208.13.

Exercise 244.2. Use the Cup Product–Intersection Manifold Theorem 244.16 to show that we have an isomorphism $(\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}),\cup) \cong \mathbb{Z}[x]/(x^{n+1})$. You can assume that we already know that $\mathrm{H}_{2k}(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}\cdot[\iota(\mathbb{C}\mathrm{P}^k)]$ where $\iota\colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ is the standard embedding given by $[z] \mapsto [z:0]$.
Remark. This gives a new proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7.

Exercise 244.3. Let $g \in \mathbb{N}_0$ and let Σ_g be the surface of genus g. Use the Cup Product–Intersection Manifold Theorem 244.16 to calculate the cohomology ring of $S^1 \times \Sigma_g$.

Exercise 244.4. Let $p\colon E \to X$ be an $\overline{B}^n$-bundle. As usual let $\Theta_{\mathbb{F}_2}(E \to X) \in \mathrm{H}^n(E, \mathrm{S}E;\mathbb{F}_2)$ be the Thom class. By the Thom–Disk Bundle Isomorphism Theorem 244.5 (1) we know that for each $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl} \Xi\colon \mathrm{H}^k(X;\mathbb{F}_2) & \to & \mathrm{H}^{n+k}(E,\mathrm{S}E);\mathbb{F}_2) \\ \varphi & \mapsto & \Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism. This implies in particular that there exists an $e(E \to X) \in \mathrm{H}^n(X;\mathbb{F}_2)$ such that $\Xi(e) = \Theta_{\mathbb{F}_2}(E) \cup \Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^{2n}(E,\mathrm{S}E;\mathbb{F}_2)$. This cohomology class is called the *Euler class of the $\overline{B}^n$-bundle $p\colon E \to X$*.

(a) Show that the Euler class of a trivializable bundle is zero.
(b) We consider the Möbius bundle

$$\begin{array}{rcl} p\colon ([0,1]\times[-1,1])/(0,t)\sim(1,-t) & \to & S^1 \\ (x,t) & \mapsto & \exp(2\pi x\,\mathrm{i}). \end{array}$$

What is the Euler class in this case?

CHAPTER 245

The Euler class of disk and sphere bundles

Let $p\colon W \to X$ be an (oriented) $(\overline{B}^n, \mathrm{O}(n))$-bundle. In the previous Chapter 277 we proved the Thom–Disk Bundle Isomorphism Theorem 244.5 which gives us in particular the Thom class $\Theta_{\mathbb{F}_2}(p\colon W \to B)$ and (in the oriented case) the Thom class $\Theta_{\mathbb{Z}}(p\colon W \to B)$. In this chapter we will see that this allows us to define the $\mathbb{F}_2$-Euler class $e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(X;\mathbb{F}_2)$ and (in the oriented case) the Euler class $e_{\mathbb{Z}}(W) \in \mathrm{H}^n(X;\mathbb{Z})$. Furthermore we will introduce Euler classes of $(S^{n-1}, \mathrm{O}(n))$-bundles. We will use these Euler classes to formulate and prove the Gysin sequences which can be used to calculate the cohomology groups of the total space of a sphere bundle.

This chapter is organized as follows:

- In Section 245.1 we will introduce the Euler class of a $(\overline{B}^n, \mathrm{O}(n))$-bundle.
- In Section 245.2 we will discuss basic properties of the Euler class.
- In Section 245.3 we will introduce the Euler class of sphere bundles.
- In Section 245.4 we will formulate and prove the Gysin sequence.
- In Section 245.5 we will provide exercises for Chapter 245.

Later, in Chapter 278 we will introduce the closely related concept of Euler classes of vector bundles.

245.1. **Definition of the Euler class.** We can use the Thom classes from the Thom–Disk Bundle Isomorphism Theorem 244.5 to introduce the Euler classes of $(\overline{B}^n, \mathrm{O}(n))$-bundles:

Definition 245.1.

(1) Let $p\colon W \to X$ be a $(\overline{B}^n, \mathrm{O}(n))$-bundle that admits a finite atlas. We start out with some observations and some notation:
- It follows from the Radial Shrinking Lemma 57.8 (2a) that $p\colon W \to X$ is a homotopy equivalence. Thus p induces isomorphisms on cohomology groups.
- We denote by SW the associated $(S^{n-1}, \mathrm{O}(n))$-bundle from the Boundary Fiber Bundle Lemma 244.1.
- We denote by $\text{я}\colon \mathrm{H}^n(W, SW;\mathbb{F}_2) \to \mathrm{H}^n(W;\mathbb{F}_2)$ the natural map.
- We denote by $\Theta_{\mathbb{F}_2}(p\colon W \to X) \in \mathrm{H}^n(W, SW;\mathbb{F}_2)$ the Thom class from the Thom–Disk Bundle Isomorphism Theorem 244.5.

We refer to
$$e_{\mathbb{F}_2}(W) \;:=\; ((p^*)^{-1} \circ \text{я})(\underbrace{\Theta_{\mathbb{F}_2}(W)}_{\in \mathrm{H}^n(W,SW;\mathbb{F}_2)}) \;\in\; \mathrm{H}^n(X;\mathbb{F}_2)$$
as the $\mathbb{F}_2$-Euler class of $p\colon W \to X$.

(2) Let $p\colon W \to X$ be a $(\overline{B}^n, \mathrm{SO}(n))$-bundle that admits a finite atlas. We refer to

$$e_{\mathbb{Z}}(W) \;:=\; ((p^*)^{-1} \circ \text{я})(\underbrace{\Theta_{\mathbb{Z}}(W)}_{\in \mathrm{H}^n(W,SW;\mathbb{Z})}) \;\in\; \mathrm{H}^n(X;\mathbb{Z})$$

as the $\mathbb{Z}$-Euler class of $p\colon W \to X$.

Note that for a $(\overline{B}^n, \mathrm{SO}(n))$-bundle $p\colon W \to X$ we now have two Euler classes, namely $e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(X;\mathbb{F}_2)$ and $e_{\mathbb{Z}}(W) \in \mathrm{H}^n(X;\mathbb{Z})$. The following lemma gives us the non-surprising connection between these two cohomology class:

Lemma 245.2. (Euler Class Relationship Lemma) Let $p\colon W \to X$ be a $(\overline{B}^n, \mathrm{SO}(n))$-bundle with a finite atlas. We denote by $\gamma_*\colon \mathbb{Z} \to \mathbb{F}_2$ the obvious ring homomorphism. From the Cohomology-Coefficients-Functor Lemma 197.21 we obtain an induced homomorphism $\gamma_*\colon \mathrm{H}^n(X;\mathbb{Z}) \to \mathrm{H}^n(X;\mathbb{F}_2)$. With this notation we have the equality

$$e_{\mathbb{F}_2}(W) \;=\; \gamma_*(e_{\mathbb{Z}}(W)) \;\in\; \mathrm{H}^n(X;\mathbb{F}_2).$$

Proof. Let $p\colon W \to X$ be a $(\overline{B}^n, \mathrm{SO}(n))$-bundle. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^n(W,SW;\mathbb{Z}) & \xrightarrow{\text{я}} & \mathrm{H}^n(W;\mathbb{Z}) & \xleftarrow{p^*} & \mathrm{H}^n(X;\mathbb{Z}) \\
\overset{\Theta_{\mathbb{Z}}(p\colon W\to X)}{} & & & & \overset{e_{\mathbb{Z}}(p\colon W\to X)}{} \\
\downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\
\mathrm{H}^n(W,SW;\mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^n(W;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}^n(X;\mathbb{F}_2). \\
\underset{\Theta_{\mathbb{Z}}(p\colon W\to X)}{} & & & & \underset{e_{\mathbb{Z}}(p\colon W\to X)}{}
\end{array}$$

The lemma follows from Definition 245.1 and the following two observations:

(1) It follows from the Cohomology-Coefficients Functor Lemma 197.21 (2) that the diagram commutes.
(2) It follows from the Thom Class Relationship Proposition 244.8 that we have the equality $\gamma_*(\Theta_{\mathbb{Z}}(p\colon W \to X)) = \Theta_{\mathbb{F}_2}(p\colon W \to X)$. ■

245.2. Properties and calculations of the Euler class.

In this section we will discuss same basic properties and calculations of Euler classes.

Proposition 245.3. (Naturality-of-Euler-Proposition)

(1) For any morphism

$$\begin{array}{ccc} W & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{h} & X \end{array}$$

of $(\overline{B}^n, \mathrm{O}(n))$-bundles (with finite atlas) we have $h^*(e_{\mathbb{F}_2}(V)) = e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(Y;\mathbb{F}_2)$.

(2) For any morphism

$$\begin{array}{ccc} W & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{h} & X \end{array}$$

of $(\overline{B}^n, \mathrm{SO}(n))$-bundles (with finite atlas) we have $h^*(e_{\mathbb{Z}}(V)) = e_{\mathbb{Z}}(W) \in \mathrm{H}^n(Y;\mathbb{Z})$.

Proof. In the following we prove Statement (1). The proof of Statement (2) is essentially the same. Thus let

$$\begin{array}{ccc} W & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{h} & X \end{array}$$

be a morphism of $(\overline{B}^n, \mathrm{O}(n))$-bundles (with finite atlas). We consider the following diagram:

$$\begin{array}{ccccc} \overset{\Theta_{\mathbb{F}_2}(p\colon W\to Y)}{\mathrm{H}^n(W, \mathrm{S}W; \mathbb{F}_2)} & \xrightarrow{\text{я}} & \mathrm{H}^n(W; \mathbb{F}_2) & \xleftarrow{p^*} & \overset{e_{\mathbb{F}_2}(p\colon W\to Y)}{\mathrm{H}^n(Y; \mathbb{F}_2)} \\ \downarrow{\scriptstyle H^*} & & \downarrow{\scriptstyle H^*} & & \downarrow{\scriptstyle h^*} \\ \underset{\Theta_{\mathbb{F}_2}(p\colon V\to X)}{\mathrm{H}^n(V, \mathrm{S}V; \mathbb{F}_2)} & \xrightarrow{\text{я}} & \mathrm{H}^n(V; \mathbb{F}_2) & \xleftarrow{p^*} & \underset{\Theta_{\mathbb{F}_2}(p\colon V\to X)}{\mathrm{H}^n(X; \mathbb{F}_2)}. \end{array}$$

Note that the diagram commutes by the functoriality of cohomology. This proposition now easily from the Thom Class–Natural Proposition 244.7. ∎

We continue with the almost trivial calculation for trivializable bundles:

Lemma 245.4. (Trivializable-Euler Lemma) Let $n \in \mathbb{N}$.

(1) If $p\colon W \to X$ is a trivializable $(\overline{B}^n, \mathrm{O}(n))$-bundle, then $e_{\mathbb{F}_2}(X) = 0$.
(2) If $p\colon W \to X$ is a trivializable $(\overline{B}^n, \mathrm{SO}(n))$-bundle, then $e_{\mathbb{Z}}(X) = 0$.

Proof. In the following we provide a proof of Statement (2). The proof of Statement (1) is evidently almost the same. Thus let $p\colon W \to X$ be a trivializable $(\overline{B}^n, \mathrm{SO}(n))$-bundle. For fun we provide two different arguments:

(a) We consider the trivial bundle $\{x\} \times \overline{B}^n \to \{x\}$ over the singleton $\{x\}$. Let $h\colon X \to \{x\}$ be the unique map. We now see that we have the following equalities in $\mathrm{H}^n(X; \mathbb{Z})$:

$$e_{\mathbb{Z}}(p\colon W \to X) \underset{\substack{\text{since } W \text{ is trivializable} \\ \text{we see that the two} \\ \overline{B}^n\text{-bundles are isomorphic}}}{=} e_{\mathbb{Z}}(h^*(\{x\} \times \overline{B}^n)) \underset{\substack{\text{by the Naturality-of-Euler} \\ \text{Proposition 245.3}}}{=} h^*(e_{\mathbb{Z}}(\{x\} \times \overline{B}^n)) \underset{\substack{\text{since } \mathrm{H}^n(\{x\} \times \overline{B}^n; \mathbb{Z}) = 0}}{=} h^*(0) = 0.$$

(b) First note that it follows easily from the Naturality-of-Euler-Proposition 245.3 that we only have to consider the case that we are dealing with the trivial $(\overline{B}^n, \mathrm{SO}(n))$-bundle $p\colon X \times \overline{B}^n \to X$. We denote by $q\colon X \times \overline{B}^n \to \overline{B}^n$ the projection. Now we consider the following diagram:

$$\begin{array}{ccccc} \overset{\Theta_{\mathbb{Z}}(p\colon X\times\overline{B}^n\to X)}{\mathrm{H}^n(X \times \overline{B}^n, X \times S^{n-1}; \mathbb{Z})} & \xrightarrow{\text{я}} & \mathrm{H}^n(X \times \overline{B}^n; \mathbb{Z}) & \xleftarrow{p^*} & \overset{e_{\mathbb{Z}}(p\colon X\times\overline{B}^n\to X)}{\mathrm{H}^n(X; \mathbb{Z})} \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \\ \underset{[\overline{B}^n]^*}{\mathrm{H}^n(\overline{B}^n, S^{n-1}; \mathbb{Z})} & \xrightarrow{\text{я}} & \underbrace{\mathrm{H}^n(\overline{B}^n; \mathbb{Z})}_{=0}. & & \end{array}$$

The desired conclusion follows from the following observations:

- The diagram commutes by functoriality of cohomology.
- By the Trivializable–Thom Lemma 244.9 we have $\Theta_{\mathbb{Z}}(p\colon X \times \overline{B}^n \to X) = q^*([\overline{B}^n]^*)$.
- Since $n \in \mathbb{N}$ we have $\mathrm{H}^n(\overline{B}^n; \mathbb{Z}) = 0$. ∎

We conclude this section with a much more interesting calculation of an Euler class:

Proposition 245.5. (Projective Tubular Neighborhood–Euler Proposition) In the following let $n \in \mathbb{N}_0$.

(1) As usual we view $\mathbb{R}\mathrm{P}^n$ as a smooth submanifold of $\mathbb{R}\mathrm{P}^{n+1}$. Let T be the corresponding tubular neighborhood of $\mathbb{R}\mathrm{P}^n \subseteq \mathbb{R}\mathrm{P}^{n+1}$. We have

$$e_{\mathbb{F}_2}(T) = [\mathbb{R}\mathrm{P}^1]^* \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2).$$

(2) As usual we view $\mathbb{C}\mathrm{P}^n$ as a smooth submanifold of $\mathbb{C}\mathrm{P}^{n+1}$. Let T be the corresponding tubular neighborhood of $\mathbb{C}\mathrm{P}^n \subseteq \mathbb{C}\mathrm{P}^{n+1}$. We have

$$e_{\mathbb{Z}}(T) = [\mathbb{C}\mathrm{P}^1]^* \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

The proof of the Projective Tubular Neighborhood–Euler Proposition 245.5 makes use of the following lemma, which might also be of independent interest:[2652]

Lemma 245.6. (Projective Submanifold–Poincaré Duality Lemma) Let $n \in \mathbb{N}_0$ and let $k \in \{0,\ldots,n\}$.[2653]

(1) We have $\mathrm{PD}_{\mathbb{R}\mathrm{P}^n}([\mathbb{R}\mathrm{P}^k]) = x^{n-k} \in \mathrm{H}^{n-k}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ where $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$.

(2) We have $\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) = x^{n-k} \in \mathrm{H}^{2(n-k)}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ where $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$.

Proof of the Projective Submanifold–Poincaré Duality Lemma 245.6. In the following we first deal with Statement (1). We denote by $\iota\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ the standard smooth embedding. Note that it follows from the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 that the map

$$\begin{array}{ccc} \mathrm{H}^{n-k}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \to & \mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cup x^k \end{array}$$

is an isomorphism. This observation shows that it suffices to show that we have the equality $\mathrm{PD}_{\mathbb{R}\mathrm{P}^n}(\iota_*([\mathbb{R}\mathrm{P}^k])) \cup x^k = x^{n-k} \cup x^k \in \mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. In slightly different words, it suffices to show that $\langle \mathrm{PD}_{\mathbb{R}\mathrm{P}^n}(\iota_*([\mathbb{R}\mathrm{P}^k])) \cup x^k, [\mathbb{R}\mathrm{P}^n]\rangle = 1$. Indeed we have

$$\begin{array}{lll} & & \text{by the Cap Product-Kronecker Lemma 206.12} \\ & & \downarrow \\ \langle \mathrm{PD}_{\mathbb{R}\mathrm{P}^n}(\iota_*([\mathbb{R}\mathrm{P}^k])) \cup x^k, [\mathbb{R}\mathrm{P}^n]\rangle & = & \epsilon_{\mathbb{R}\mathrm{P}^n}((\mathrm{PD}_{\mathbb{R}\mathrm{P}^n}(\iota_*([\mathbb{R}\mathrm{P}^k])) \cup x^k) \cap [\mathbb{R}\mathrm{P}^n]) \\ & = & \epsilon_{\mathbb{R}\mathrm{P}^n}(x^k \cap (\mathrm{PD}_{\mathbb{R}\mathrm{P}^n}(\iota_*([\mathbb{R}\mathrm{P}^k])) \cap [\mathbb{R}\mathrm{P}^n])) \\ & & \uparrow \\ & & \text{by the Cup-Cap Lemma 206.15} \\ & = & \epsilon_{\mathbb{R}\mathrm{P}^n}(x^k \cap \iota_*([\mathbb{R}\mathrm{P}^k])) = \epsilon_{\mathbb{R}\mathrm{P}^k}(x^k \cap [\mathbb{R}\mathrm{P}^k]) = 1. \\ & & \uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow \\ & & \text{see Definition 210.2} \quad \text{by the Cap Product Naturality Lemma 206.13} \quad \text{by the Cup Product-}\mathbb{R}\mathrm{P}^n \text{ Proposition 213.19} \end{array}$$

The proof of Statement (2) is of course almost identical to the above proof. ∎

Proof of the Projective Tubular Neighborhood–Euler Proposition 245.5. In the following we deal with Statement (1). Evidently the proof of Statement (2) is essentially

[2652] Also note that the lemma is the content of Exercise 213.2.

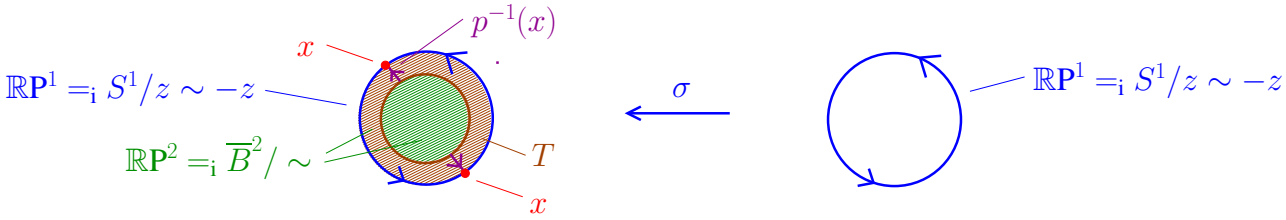
[2653] In the statements below we use the isomorphisms from the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 and the Cup Product-$\mathbb{C}\mathrm{P}^n$-Propositions 213.7 and 220.20 to make the identifications $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) =_{\mathrm{i}} \mathbb{F}_2[x]/(x^{n+1})$ and $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) =_{\mathrm{i}} \mathbb{Z}[x]/(x^{n+1})$.

the same. We consider the following diagram:

$$\begin{array}{ccccc}
 & \overset{\Theta_{\mathbb{Z}}(T)}{} & & & \\
\mathrm{H}^1(T,\partial T;\mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^1(T;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \\
\uparrow{\scriptstyle w^*} & & \uparrow{\scriptstyle \iota^*} & \overset{\sigma^*}{\nearrow} & \\
\mathrm{H}^1(\mathbb{R}\mathrm{P}^{n+1},\mathbb{R}\mathrm{P}^{n+1}\setminus\mathring{T};\mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n+1};\mathbb{F}_2). & \overset{\sigma^*\circ\iota^*=i^*}{\nearrow} & \\
 & & \underset{\mathrm{PD}_{\mathbb{R}\mathrm{P}^{n+1}}([\mathbb{R}\mathrm{P}^n])}{} & &
\end{array}$$

We make the following clarifications and observations:

(a) We denote by $w\colon (T,\partial T)\to(\mathbb{R}\mathrm{P}^{n+1},\mathbb{R}\mathrm{P}^{n+1}\setminus\mathring{T})$ the inclusion map.
(b) We denote by $\iota\colon T\to\mathbb{R}\mathrm{P}^{n+1}$ the inclusion.
(c) We denote by $\sigma\colon\mathbb{R}\mathrm{P}^n\to T$ the 0-section. Note that $\iota\circ\sigma\colon\mathbb{R}\mathrm{P}^n\to\mathbb{R}\mathrm{P}^{n+1}$ is the standard smooth embedding $i\colon\mathbb{R}\mathrm{P}^n\to\mathbb{R}\mathrm{P}^{n+1}$.
(d) Note that $p\circ\sigma=\mathrm{id}_{\mathbb{R}\mathrm{P}^n}$. By the Radial Shrinking Lemma 57.8 (2a) we know that p is a homotopy equivalence. It now follows from the Homotopy Equivalence-H*-Proposition 198.12 (1) that $\sigma^*=(p^*)^{-1}$.
(e) The diagram commutes by functoriality of cohomology.



We have the following equalities in $\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$:

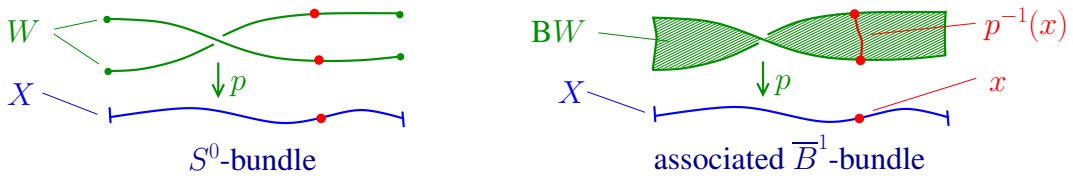
$$\begin{array}{lll}
e_{\mathbb{Z}}(T) & = & (p^*)^{-1}(\text{я}(\Theta_{\mathbb{Z}}(p\colon T\to\mathbb{R}\mathrm{P}^n))) \\
 & \uparrow & \text{by Definition 245.1} \\
 & = & (\sigma^*\circ\iota^*)((\text{я}\circ(w^*)^{-1})(\Theta_{\mathbb{Z}}(p\colon T\to\mathbb{R}\mathrm{P}^n)))) \\
 & \uparrow & \text{since the diagram commutes} \\
 & = & (\sigma^*\circ\iota^*)(x) \qquad = \ i^*(x) \\
 & \uparrow & \text{by the Projective Submanifold–Poincaré Duality Lemma 245.6}
\end{array}$$

$$\begin{array}{lll}
 & = & \sigma^*(\text{я}(\Theta_{\mathbb{Z}}(p\colon T\to\mathbb{R}\mathrm{P}^n))) \\
 & \uparrow & \text{by (d)} \\
 & = & (\sigma^*\circ\iota^*)(\text{я}(\mathrm{PD}_{\mathbb{R}\mathrm{P}^{n+1}}([\mathbb{R}\mathrm{P}^n]))) \\
 & \uparrow & \text{by the Thom Class-of-Submanifold Proposition 244.14} \\
 & = & x. \\
 & \uparrow & \text{by the } \mathbb{R}\mathrm{P}^n\text{-H}^*\text{-Lemma 213.5}
\end{array}$$

■

245.3. **The Euler class of sphere bundles.** In this section we will introduce the Euler class of $(S^{n-1},\mathrm{SO}(n))$-bundles. In the next section we will use these Euler classes to state and prove the Gysin sequence that gives us an approach to calculating the cohomology groups of the total space of a sphere bundle.

Definition 245.7. Let $n\in\mathbb{N}_0$ and let $p\colon W\to X$ be an $(S^{n-1},\mathrm{O}(n))$-bundle with atlas $\{\Phi_i\colon p^{-1}(U_i)\to U_i\times S^{n-1}\}_{i\in I}$. For $i,j\in I$ let $\Theta_{ij}\colon U_i\cap U_j\to\mathrm{O}(n)$ be the transition map from Definition 240.8. As in the Bundle–Transition Map Lemma 240.18 we use these transition maps to define a $(\overline{B}^n,\mathrm{O}(n))$-bundle $p\colon BW\to X$. We call $BW\to X$ the $(\overline{B}^n,\mathrm{O}(n))$-bundle associated to the sphere bundle.[2654] [2655]



Example.

(1) Let $p\colon W \to X$ be a $(\overline{B}^n, \mathrm{O}(n))$-bundle. In the Boundary Fiber Bundle Lemma 244.1 we defined the corresponding $(S^{n-1}, \mathrm{O}(n))$-bundle $p\colon \mathrm{S}W \to X$. It follows easily from the definitions that the corresponding $(\overline{B}^n, \mathrm{O}(n))$-bundle $p\colon \mathrm{BS}W \to X$ is naturally isomorphic to the original $(\overline{B}^n, \mathrm{O}(n))$-bundle $p\colon W \to X$.[2656]

(2) Let $n \in \mathbb{N}_0$ and let $p\colon W \to X$ be an $(S^{n-1}, \mathrm{O}(n))$-bundle. We consider the associated $(\overline{B}^n, \mathrm{O}(n))$-bundle $\mathrm{B}W \to X$ and the corresponding $(S^{n-1}, \mathrm{O}(n))$-bundle $\mathrm{SD}W \to X$ from the Boundary Fiber Bundle Lemma 244.1. It follows easily from the definitions that the $(S^{n-1}, \mathrm{O}(n))$-bundles $W \to X$ and $\mathrm{SD}W \to X$ are naturally isomorphic. □

Using Definition 245.7 we can now introduce the Euler classes of sphere bundles "for free":

Definition 245.8. Let $n \in \mathbb{N}_0$ and let $p\colon W \to X$ be an $(S^{n-1}, \mathrm{O}(n))$-bundle with a finite atlas. We consider the associated $(\overline{B}^n, \mathrm{O}(n))$-bundle $\mathrm{B}W \to X$ and we refer to

$$e_{\mathbb{F}_2}(p\colon W \to X) \;:=\; \underbrace{e_{\mathbb{F}_2}(p\colon \mathrm{B}W \to X)}_{\text{see Definition 245.1}} \;\in\; \mathrm{H}^n(X;\mathbb{F}_2)$$

as the Euler class of the $(S^{n-1}, \mathrm{O}(n))$-bundle $p\colon W \to X$. If we start out with an $(S^{n-1}, \mathrm{SO}(n))$-bundle, then we we refer to

$$e_{\mathbb{Z}}(p\colon W \to X) \;:=\; \underbrace{e_{\mathbb{Z}}(p\colon \mathrm{B}W \to X)}_{\text{see Definition 245.1}} \;\in\; \mathrm{H}^n(X;\mathbb{Z})$$

as the Euler class of the $(S^{n-1}, \mathrm{SO}(n))$-bundle $p\colon W \to X$.

Example. Let $H\colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ be the Hopf bundle. As we pointed out on page 4635, the trivializations of Hopf-Bundle Proposition 238.4 (1) turn $H\colon S^3 \to S^2$ into an $(S^1, \mathrm{SO}(2))$-bundle. It follows easily from the Projective Space-minus-Ball Lemma 238.11 that the corresponding disk bundle is given by the tubular neighborhood T of $\mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^2$. Thus we see that

$$e_{\mathbb{Z}}(H\colon S^3 \to S^2) \underset{\text{see Definition 245.8}}{\uparrow=} e_{\mathbb{Z}}(\mathrm{B}S^3 \to S^2) \underset{\substack{\text{by the above we}\\ \text{know that } \mathrm{B}S^3 \cong T}}{\uparrow=} e_{\mathbb{Z}}(T \to \mathbb{C}\mathrm{P}^1) \underset{\substack{\text{by the Euler Class-of-Projective}\\ \text{Tubular Neighborhood Proposition 245.5}}}{\uparrow=} [\mathbb{C}\mathrm{P}^1]^* = [S^2]^* \in \mathrm{H}^2(S^2;\mathbb{Z}).$$

□

[2654] Evidently, if we start out with an $(S^{n-1}, \mathrm{SO}(n))$-bundle, then this construction leads to a $(\overline{B}^n, \mathrm{SO}(n))$-bundle.

[2655] If $p\colon W \to X$ is an arbitrary S^{n-1}-bundle, then we get transition maps $\Theta_{ij}\colon U_i \cap U_j \to \mathrm{Homeo}(S^{n-1})$. We have an obvious group homomorphism $\mathrm{Homeo}(S^{n-1}) \to \mathrm{Homeo}(\overline{B}^n)$ which is given by "radial extension". This map is surely continuous and we can use these extensions to again define a $(\overline{B}^n, \mathrm{O}(n))$-bundle $\mathrm{B}X \to X$. We leave it to the reader to work out the details.

[2656] Note though that it is important that we started out with a $(\overline{B}^n, \mathrm{O}(n))$-bundle since in this case a self-homeomorphism $\overline{B}^n \to \overline{B}^n$ is completely determined by the behavior on the boundary.

245.4. **The Gysin sequence.** In this section we will use the Euler classes from the last section to state and prove the Gysin sequence that gives us an approach to calculating the cohomology groups of the total space of a sphere bundle.

Proposition 245.9. (Gysin Sequence Proposition) Let $n \in \mathbb{N}_0$ and let $p\colon W \to X$ be an $(S^{n-1}, \mathrm{SO}(n))$-bundle. For each $m \in \mathbb{N}_0$ there exists a natural homomorphism

$$\xi_m\colon \mathrm{H}^m(W;\mathbb{Z}) \;\to\; \mathrm{H}^{m-n+1}(X;\mathbb{Z})$$

such that the following sequence is exact:

$$\cdots \to \mathrm{H}^{k+n-1}(W;\mathbb{Z}) \xrightarrow{\xi_{k+n-1}} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi \mapsto e_{\mathbb{Z}}(W)\cup\varphi} \mathrm{H}^{k+n}(X;\mathbb{Z}) \xrightarrow{p^*} \mathrm{H}^{k+n}(W;\mathbb{Z}) \to \dots$$

This sequence is called the Gysin[2663] sequence. If we are dealing with an $(S^{n-1}, \mathrm{O}(n))$-bundle, then we obtain an analogous sequence with $\mathbb{F}_2$-coefficients.

Example.

(1) Let $H\colon S^3 \to S^2$ be the Hopf bundle. As we just discussed in the previous section, the Euler class $e_{\mathbb{Z}}(H\colon S^3 \to S^2)$ is the standard generator of $\mathrm{H}^2(S^2;\mathbb{Z})$. Thus (starting at $k=0$) the Gysin sequence (with $n=2$) is of the form

$$\cdots \to \mathrm{H}^1(S^3;\mathbb{Z}) \xrightarrow{\xi_1} \mathrm{H}^0(S^2;\mathbb{Z}) \xrightarrow[\cong]{\cup e_{\mathbb{Z}}(S^3)} \mathrm{H}^2(S^2;\mathbb{Z}) \xrightarrow{p^*} \mathrm{H}^2(S^3;\mathbb{Z}) \xrightarrow{\xi_2} \mathrm{H}^1(S^2;\mathbb{Z}) \to \dots$$

Fortunately this exact sequence is consistent with our calculation of the cohomology groups of spheres.

(2) In Exercises 245.6 and 245.7 we will see that one can use the Gysin Sequence Proposition 245.9 to give a new calculation of the cohomology rings $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ and $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

(3) Later, on page 6469, we will see that the Gysin Sequence Proposition 245.9 can be viewed as a special case of the Cohomological Leray-Serre Spectral Sequence 341.7.

(4) In Exercise 341.5 we will prove a homological version of the Gysin Sequence Proposition 245.9. □

Remark. Later we will use the Cohomological Leray-Serre Spectral Sequence 341.7 to prove the Gysin Sequence Proposition 341.9, which is a generalization of the above Gysin Sequence Proposition 245.9. □

Proof. Let $p\colon W \to X$ be an $(S^{n-1}, \mathrm{SO}(n))$-bundle. We denote by $p\colon D \to X$ the associated $(\overline{B}^n, \mathrm{SO}(n))$-bundle from Definition 245.7. It follows easily from the discussion on page 4736 together with the Cup Product-Naturality Lemma 204.8 that it suffices to prove the statement for the $(S^{n-1}, \mathrm{SO}(n))$-bundle $SD \to X$. Let $\Theta_{\mathbb{Z}}(D) \in \mathrm{H}^n(D, SD;\mathbb{Z})$ be the corresponding Thom class. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & \mathrm{H}^{k+n-1}(SD;\mathbb{Z}) & \xrightarrow{\delta_{k+n-1}} & \mathrm{H}^{k+n}(D,SD;\mathbb{Z}) & \xrightarrow{\text{я}} & \mathrm{H}^{k+n}(D;\mathbb{Z}) & \xrightarrow{\iota^*} & \mathrm{H}^{k+n}(SD;\mathbb{Z}) & \longrightarrow \\
 & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \cong\ \varphi\mapsto\Theta_{\mathbb{Z}}(D)\cup p^*(\varphi)} & & \uparrow{\scriptstyle \cong\ p^*} & & \uparrow{\scriptstyle \mathrm{id}} & \\
\longrightarrow & \mathrm{H}^{k+n-1}(SD;\mathbb{Z}) & \xrightarrow[=:\xi_{k+n-1}]{} & \mathrm{H}^k(X;\mathbb{Z}) & \xrightarrow[\varphi\mapsto e_{\mathbb{Z}}(SD)\cup\varphi]{} & \mathrm{H}^{k+n}(X;\mathbb{Z}) & \xrightarrow[(p|_{SD})^*]{} & \mathrm{H}^{k+n}(SD;\mathbb{Z}) & \longrightarrow
\end{array}$$

[2663]The Gysin sequence is named after Werner Gysin, who proved an early version of this the proposition [**Gys1941**, Kapitel III].

We make the following clarifications and observations:

(1) The map $\iota\colon \mathrm{S}D \to D$ is the inclusion.
(2) The top sequence is just the long exact sequence from the LES-of-Triple Proposition 198.13 applied to the pair $(D, \mathrm{S}D)$.
(3) It follows from the Radial Shrinking Lemma 57.8 (2a) that $p\colon D \to X$ is a homotopy equivalence. By the Homotopy Equivalence-H^*-Proposition 198.12 this implies that the induced map on cohomology is an isomorphism.
(4) It follows from the Thom–Disk Bundle Isomorphism Theorem 244.5 that the second vertical map is an isomorphism.
(5) We claim that the middle square commutes. In preparation we point out that it follows from the Cup Product-Naturality Lemma 204.8 that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(D;\mathbb{Z}) \times \mathrm{H}^k(D;\mathbb{Z}) & \xrightarrow{\ \cup\ } & \mathrm{H}^{k+n}(D;\mathbb{Z}) \\ \uparrow{\scriptstyle \text{я}} \quad \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \text{я}} \\ \mathrm{H}^n(D,\mathrm{S}D;\mathbb{Z}) \times \mathrm{H}^k(D;\mathbb{Z}) & \xrightarrow{\ \cup\ } & \mathrm{H}^{k+n}(D,\mathrm{S}D;\mathbb{Z}). \\ \quad\ \ {\scriptstyle p^*(\varphi)} & & \end{array}$$

Now let $\varphi \in \mathrm{H}^k(X;\mathbb{Z})$. We see that

$$p^*(e_{\mathbb{Z}}(\mathrm{S}D) \cup \varphi) \overset{\text{by Definition 245.1}}{=} p^*(((p^*)^{-1} \circ \text{я})(\Theta_{\mathbb{Z}}(D))) \cup p^*(\varphi) \overset{\text{since } p^* \text{ and } (p^*)^{-1} \text{ cancel}}{=} \text{я}(\Theta_{\mathbb{Z}}(D)) \cup p^*(\varphi) = \text{я}(\Theta_{\mathbb{Z}}(D) \cup p^*(\varphi)).$$

by the above commutative diagram

It follows from the above discussion that the bottom sequence is exact. Finally note that it follows from the naturality of the connecting homomorphism of the LES-of-Triple Proposition 198.13 that the homomorphisms ξ_m are natural. ■

245.5. Exercises for Chapter 245.

Exercise 245.1. Let $p\colon W \to X$ be a $(\overline{B}^n, \mathrm{O}(n))$-bundle that admits a finite atlas. Recall that it follows from the Thom–Disk Bundle Isomorphism Theorem 244.5 (1) that there exists a unique cohomology class $\alpha \in \mathrm{H}^n(X;\mathbb{F}_2)$ with

$$\Theta_{\mathbb{F}_2}(W) \cup p^*(\alpha) \ = \ \Theta_{\mathbb{F}_2}(W) \cup \Theta_{\mathbb{F}_2}(W) \ \in \ \mathrm{H}^{2n}(W, \mathrm{S}W;\mathbb{F}_2).$$

Show that $\alpha = e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(X;\mathbb{F}_2)$.
Remark. Evidently we have an analogous statement for $(\overline{B}^n, \mathrm{SO}(n))$-bundles and $\mathbb{Z}$-coefficients.

Exercise 245.2. Let $p\colon W \to B$ be a $(\overline{B}^n, \mathrm{SO}(n))$-bundle that admits a finite atlas. Show that if n is odd, then $2 \cdot e_{\mathbb{Z}}(W) = 0$.
Hint. Consider the self-map of W that is given by $w \mapsto -w$ on each fiber.

Exercise 245.3. Explore the statement of the Gysin Sequence Proposition 245.9 for the $(S^1, \mathrm{O}(2))$-bundle

$$\begin{array}{rcl} (S^1 \times [0,1])/((x,y),0) \sim ((x,-y),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}) \end{array}$$

which describes the Klein bottle as the total space of an S^1-bundle over S^1.

Exercise 245.4. Explore the statement of the Gysin Sequence Proposition 245.9 for the trivial S^k-bundle $X \times S^k \to X$.

Exercise 245.5.

(a) Show that the Hopf map $H_{\mathbb{C}}\colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ descends to a bundle map $\mathbb{R}\mathrm{P}^3 \to S^2$ where the fiber is diffeomorphic to S^1.
(b) Determine the Euler class $e_{\mathbb{Z}}(\mathbb{R}\mathrm{P}^3) \in \mathrm{H}^2(S^2;\mathbb{Z})$ up to sign.
Hint. Use the Gysin Sequence Proposition 245.9.

Exercise 245.6. We consider the projection $S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$. By Exercise 240.10 this map can be viewed as an $(S^0, \mathrm{O}(1))$-bundle. Use the Gysin Sequence Proposition 245.9 and the calculation of the cohomology groups $\mathrm{H}^k(S^n;\mathbb{F}_2)$ from page 3830 to prove the following facts:

(a) We have isomorphisms

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{for } k \in \{0,\dots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $e_{\mathbb{F}_2}(S^n \to \mathbb{R}\mathrm{P}^n) \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is non-trivial.
(c) The cohomology ring $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[x]/(x^{n+1})$.

Remark. This exercise gives a new proof of most of the results of the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 213.17 and of the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19.

Exercise 245.7. We consider the projection $S^{2n+1} \to \mathbb{C}\mathrm{P}^n =_{\mathrm{i}} S^{2n+1}/x \sim \lambda \cdot x$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. By Exercise 240.10 this map can be viewed as an $(S^1, \mathrm{SO}(2))$-bundle. Use the Gysin Sequence Proposition 245.9 and the calculation of the cohomology groups $\mathrm{H}^k(S^{2n+1};\mathbb{Z})$ from page 3830 to prove the following facts:

(a) We have isomorphisms

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } k \in \{0,2,4,\dots,2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $e_{\mathbb{Z}}(S^{2n+1} \to \mathbb{C}\mathrm{P}^n) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator.
(c) The cohomology ring $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$.

Remark. This exercise gives a new proof of most of the results of the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 and of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7.

Exercise 245.8. Let X be a compact connected non-empty topological manifold, let $x_0 \in X$ and let $\varphi\colon \pi_1(X,x_0) \to \mathbb{Z}/2\mathbb{Z}$ be an epimorphism. We denote by $p\colon \widetilde{X} \to X$ the covering corresponding to $\ker(\varphi)$ which we obtain from the Covering Unique Existence Theorem 127.17.

(a) Show that $p\colon \widetilde{X} \to X$ admits a finite atlas as an $(S^0, \mathrm{O}(1))$-bundle.
(b) What is the relationship between the Euler class $e_{\mathbb{F}_2}(p\colon \widetilde{X} \to X) \in \mathrm{H}^1(X;\mathbb{F}_2)$ and the homomorphism $\varphi\colon \pi_1(X,x_0) \to \mathbb{Z}/2\mathbb{Z}$?
(c) Explore the Gysin Sequence Proposition 245.9 in this context.

Exercise 245.9. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering of a topological space X. Let

$$\begin{array}{rcl} \tau\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(\widetilde{X}) \\ (\sigma\colon \Delta^n \to X) & \mapsto & \text{sum of the two lifts of } \sigma \text{ to } \Delta^n \to \widetilde{X} \end{array}$$

be the transfer map from Definition 165.1. Note that we showed in the Transfer Map Lemma 165.2 that τ is a chain map.

(a) Show that we have a long exact sequence of the form

$$\cdots \to \mathrm{H}^n(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}^n(\widetilde{X};\mathbb{F}_2) \xrightarrow{\tau^*} \mathrm{H}^n(X;\mathbb{F}_2) \to \mathrm{H}^{n+1}(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}^{n+1}(X;\mathbb{F}_2) \to \dots$$

Remark. This part of the exercise is identical Exercise 198.12. Also note that the above exact sequence is an analog of the Double Covering-LES-Proposition 165.5.

(b) Show that the sequence in (a) is precisely the long exact sequence from the Gysin Sequence Proposition 245.9 applied to the $(S^0, \mathrm{O}(1))$-bundle $\widetilde{X} \to X$.

Exercise 245.10. Show that if there exists a bundle map $S^m \to S^k$ with fiber S^n, then $m = 2k - 1$.

Hint. Use the Gysin Sequence Proposition 245.9.

Part XXVIII

Handle Structures and Morse Theory

In this part of the lecture notes we will develop Morse theory. This theory shows that every smooth manifold admits a "handle decomposition". Such handle decomposition have many uses, in particular we will use them to finally conclude the classification of compact 2-dimensional smooth manifolds.

This part of the lecture notes is organized as follows:

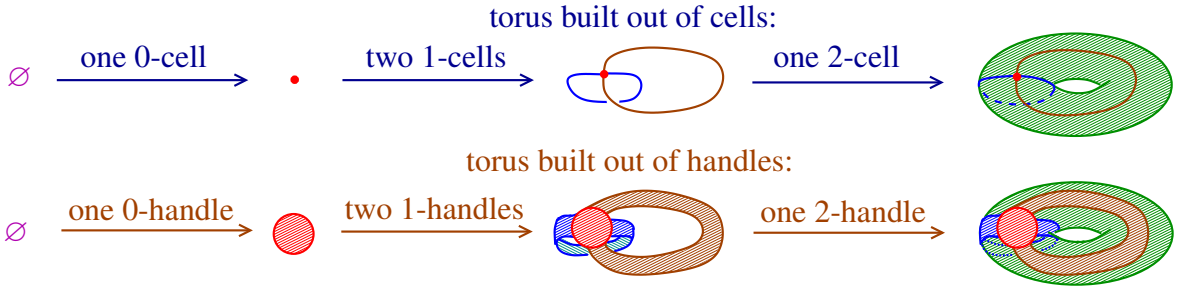
- In Chapter 246 we will introduce the notion of adding a handle a smooth manifold. We will study the effects of a handle addition on various invariants.
- In Chapter 247 we will introduce handle decompositions of a smooth manifold M rel a union A of components of ∂M. In particular we will see that if a handle decomposition exists, then one can always arrange that the indices are of non-decreasing order.
- In Chapter 248 we will see that to a handle decomposition rel A we can also associate a dual handle decomposition rel $\partial M \setminus A$.
- In Chapter 249 we will introduce the notion of a Morse functions $M \to \mathbb{R}$ on a smooth manifold and we will show that every smooth manifold admits a Morse function.
- In Chapter 250 we will use Morse functions to show that every smooth manifold M admits a handle decomposition rel any union A of components of ∂M.
- In Chapter 251 we will see that handle decompositions give rise to Morse functions. The results of this chapter are thus complementary to the results of Chapter 250.
- In Chapter 252 we will associate to a handle decomposition of a smooth manifold the corresponding handle homology and we will show that handle homology is naturally isomorphic to singular homology. We will use handle homology to prove the Morse inequalities, which say, that the "size" of singular homology gives a lower bound on the number of singular points of a Morse function.
- In Chapter 253 we will extend some of the earlier results to "cobordisms with vertical boundary".
- In Chapter 254 we will see that a handle decomposition gives rise to a CW-complex.
- In Chapter 255 we will use handle decompositions to conclude the long overdue classification of compact 2-dimensional smooth manifolds.
- In Chapter 256 we will use handles to construct interesting compact and closed smooth manifolds out of finite presentations of groups. Furthermore we will show that every finite CW-complex is homotopy equivalent to a compact smooth manifold.
- In Chapter 257 we will use handle decompositions to study fundamental groups of 3-dimensional smooth manifolds. In particular we will see that "most" finitely generated abelian groups are *not* isomorphic to the fundamental group of a compact 3-dimensional smooth manifold.

CHAPTER 246

Handle attachments

As we have seen, CW-complexes and simplicial complexes have turned out to be useful tools in topology, since often they make it possible to break a given topological space into smaller, more manageable pieces. In the Smooth Manifold-Simplicial Structure Theorem 179.5 we showed that every compact smooth manifold M admits a finite simplicial structure. By the Smooth Manifold-CW-Theorem 179.9 this implies in particular that M admits a finite CW-structure. Thus in principle we can study a smooth manifold by starting with a 0-dimensional CW-complex and attaching cells of dimensions $1, 2, \ldots$. The catch is that the inbetween objects are usually not smooth manifolds.

In this chapter we will introduce handle decompositions of smooth manifolds. These are in many ways quite similar to CW-structures, except that now all intermediate objects are smooth manifolds. In Chapter 250 we will use Morse theory to show that every compact smooth manifold admits a handle decomposition.



This chapter is organized as follows:

- In Section 246.1 we will recall some conventions and facts about smooth manifolds with corner.
- In Section 246.2 we will define what it means to attach a handle to a smooth manifold.
- In Section 246.3 we will see how handle attachments affect connectedness and orientability.
- In Section 246.4 we will consider several instances of attaching a single handle to a closed ball.
- In Section 246.5 we will introduce the notion of a handle decomposition of a smooth manifold and we will provide some basic examples. Furthermore we will discuss handle decompositions of projective spaces.
- In Section 246.6 we will see how handle attachments affect the fundamental group and homology groups.
- In Section 246.7 we will provide exercises for Chapter 246.

In the next Chapter 247 we will discuss handle decompositions of smooth manifolds.

246.1. **Conventions about smooth manifolds with corner.** In the following we recall some statements and conventions that we introduced in Part VII:

(1) Let $k \in \mathbb{N}_0$. We equip $\overline{B}^k$ with the standard collar that is given by

$$\begin{array}{rcl} \chi\colon S^{k-1} \times [0, \frac{1}{2}] & \to & \overline{B}^k \\ (x,t) & \mapsto & x \cdot (1-t). \end{array}$$

(2) Let M and N be smooth manifolds. We consider the smooth manifold $M \times N$ with corner $\angle(M \times N) = \partial M \times \partial N$. Let $\chi\colon \partial M \times [0,\mu] \to M$ and $\lambda\colon \partial N \times [0,\nu] \to N$ be two collars.

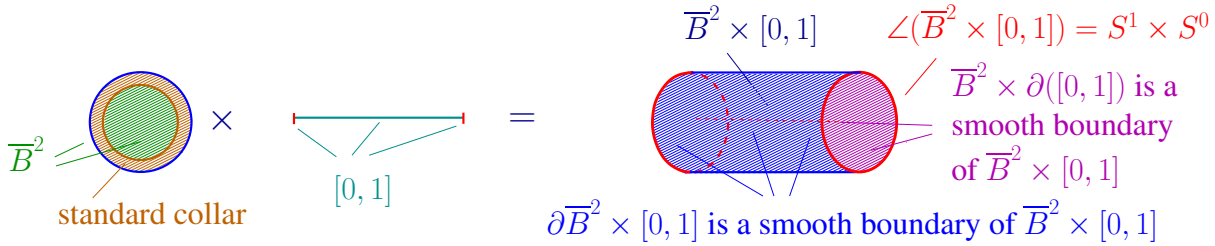
(a) As on page 1352 we equip $M \times N$ with the corner-collar

$$\begin{array}{rcl} \angle(M \times N) \times [0,\mu] \times [0,\nu] & \to & M \times N \\ ((x,y),s,t) & \mapsto & (\chi(x,s), \lambda(y,t)). \end{array}$$

(b) We equip the smooth boundary $M \times \partial N$ of $M \times N$ with the collar

$$\begin{array}{rcl} (M \times \partial N) \times [0,\nu] & \to & M \times N \\ ((x,y),t) & \mapsto & (x, \lambda(y,t)). \end{array}$$

Similarly we get a collar for the smooth boundary $\partial M \times N$ of $M \times N$.



(3) Let M be a smooth manifold and let $W \subseteq \partial M$ be a two-sided codimension-one submanifold of ∂M. As in the Introducing Corner Proposition 61.4 we denote by $M^{\angle = W}$ the smooth manifold that we obtain by introducing a corner at W.



Note that this smooth manifold depends on the choice of a double-collar.

246.2. **Handle attachments.** Throughout the following chapters it is convenient to work with the following notation which we first introduced in Definition 135.1:

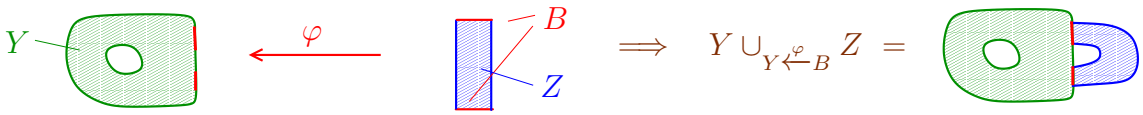
Definition 246.1. Let Y and Z be topological spaces, let $B \subseteq Z$ be a subspace and let $\varphi\colon B \to Y$ be a continuous map. We write[2664]

$$Y \cup_\varphi Z \coloneqq Y \cup_{Y \xleftarrow{\varphi} B} Z \coloneqq (Y \sqcup Z)/\sim$$

where $\sim$ is the equivalence relation generated by $\varphi(b) \sim b$ for $b \in B$. We call $Y \cup_\varphi Z$ the topological space obtained by gluing Y to Z via the map φ. Note that according to the

Topological Pushout Proposition 8.12 we obtain a pushout diagram

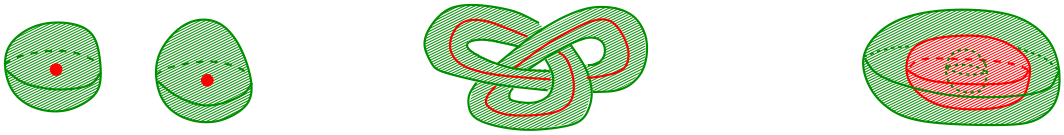
$$\begin{array}{ccc} B & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \cup_\varphi Z. \end{array}$$



We move on to a new notion:

Definition 246.2. Let W be an m-dimensional smooth manifold. A thickened l-sphere in W is a smooth embedding $\overline{B}^{m-l} \times S^l \to W \setminus \partial W$.

thickened 0-sphere in $\mathbb{R}^3$ thickened 1-sphere in $\mathbb{R}^3$ thickened 2-sphere in $\mathbb{R}^3$



Our first proposition, which is formatted in analogy to the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8, uses the above two definitions to introduce the notion of a handle attachment:

Proposition 246.3. (Handle Attachment Proposition) Let M be an n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a smooth embedding into the *boundary* of M, i.e. φ is a thickened $(k-1)$-sphere in ∂M. We refer to

$$M \cup_\varphi h^k \coloneqq M \cup_{\partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \left(\overline{B}^{n-k} \times \overline{B}^k\right)$$

as the result of a k-handle attachment to M along φ. The following statements hold:

(0) (a) If M is compact, then $M \cup_\varphi h^k$ is compact.

(b) (We will deal with connectivity in the Handle Attachment-π_0-Lemma 246.5.)

(1) The topological space $M \cup_\varphi h^k$ is an n-dimensional topological manifold. Furthermore it also admits a smooth structure:

(a) We can pick a double-collar for $\varphi(S^{n-k-1} \times S^{k-1}) \subseteq \partial M \subseteq M$ and we can pick a collar for the smooth boundary $\varphi(\overline{B}^{n-k} \times S^{k-1}) \subseteq M^{\angle = \varphi(S^{n-k-1} \times S^{k-1})}$. Every such choice equips

$$M \cup_\varphi h^k = M^{\angle = \varphi(S^{n-k-1} \times S^{k-1})} \cup_{\partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \left(\overline{B}^{n-k} \times \overline{B}^k\right)$$

canonically with the structure of an n-dimensional smooth manifold.

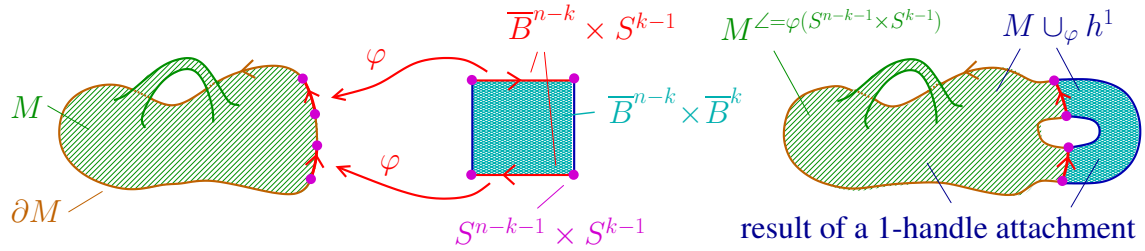
(b) Let $\mathcal{A}$ and $\mathcal{A}'$ be two smooth atlases on $M \cup_\varphi h^k$ that arise by the construction in (a). Given any neighborhood U of the image of $\varphi(\overline{B}^{n-k} \times S^{k-1})$ in $M \cup_\varphi h^k$ there exists a diffeomorphism from $(M \cup_\varphi h^k, \mathcal{A})$ to $(M \cup_\varphi h^k, \mathcal{A}')$ which is the identity outside of U.

(c) Any two choices as in (a) give rise to diffeomorphic smooth structures on $M \cup_\varphi h^k$.

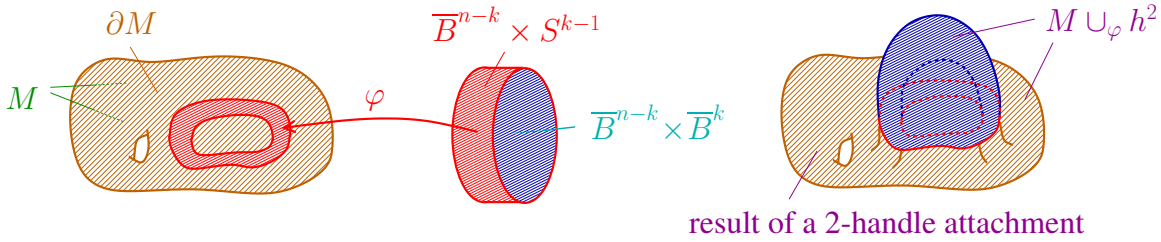
[2665]The fact that in the notation the map φ goes from right to left is a little awkward, but all other conventions lead sooner or later also to slightly unnatural results.

(2) The boundary of the smooth manifold $M \cup_\varphi h^k$ is given by[2666]
$$\partial(M \cup_\varphi h^k) \;=\; (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup_{\varphi|_{S^{n-k-1}\times S^{k-1}}} (S^{n-k-1} \times \overline{B}^k).$$
(3) The images of M and $\overline{B}^{n-k} \times \overline{B}^k$ in $M \cup_\varphi h^k$ are both submanifolds with corner, where the corner set in both cases is given by $\varphi(S^{n-k-1} \times S^{k-1})$.
(4) (We will discuss orientability in the Handles-Orientability Proposition 246.6.)
(5) If $\varphi, \psi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ are two smooth embeddings that are smoothly isotopic, then $M \cup_\varphi h^k$ and $M \cup_\psi h^k$ are diffeomorphic. Furthermore, given any neighborhood U of $\partial M \subseteq M$ one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.

Example. In the figure below we consider the case $n = 2$ and $k = 1$, i.e. we show a 2-dimensional smooth manifold together with a 1-handle attachment:



In the figure below we consider the case $n = 3$ and $k = 2$, i.e. we show a 3-dimensional smooth manifold together with a 2-handle attachment:



More examples of handle attachments will be given after the proof of the proposition. □

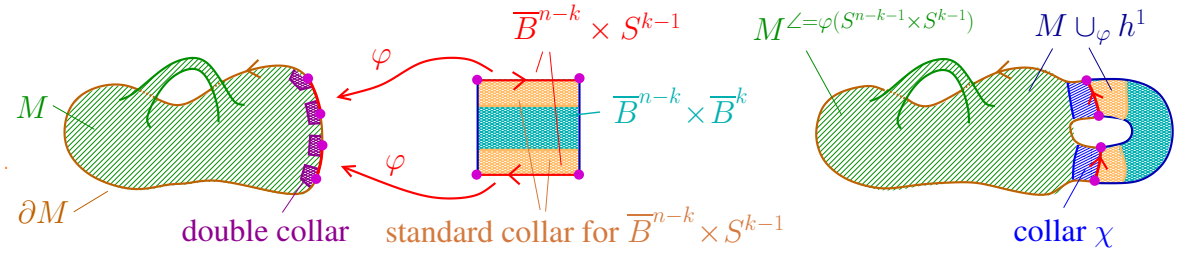
Proof.

(0) This follows from the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (0), or more directly, from the Pushout-Properties Proposition 8.14 (1).
(1) (a) We start out with two observations:
- Recall that on page 4744 we equipped $\overline{B}^{n-k} \times \overline{B}^k$ with a standard corner collar and we equipped the smooth boundary $\overline{B}^{n-k} \times S^{k-1}$ of $\overline{B}^{n-k} \times \overline{B}^k$ with a standard collar.
- Since $\varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is the smooth embedding of a codimension-zero submanifold we obtain from the discussion on page 1345 that $\varphi(S^{n-k-1} \times S^{k-1})$ is two-sided. It now follows from the Double Collar Theorem 61.2 that we can find a double-collar δ for $\varphi(S^{n-k-1} \times S^{k-1}) \subseteq M$. The Introducing Corner Proposition 61.4 lets us introduce a corner at $\varphi(S^{n-k-1} \times S^{k-1})$. As we discussed on page 1367, we know that $\varphi(\overline{B}^{n-k} \times S^{k-1})$ is a smooth boundary

[2666] This modification of the boundary is called a *surgery along* $\varphi(\{0\} \times S^{k-1}) \subseteq \partial M$. We will study surgeries in greater detail in **??**.

of $M^{\angle=\varphi(S^{n-k-1}\times S^{k-1})}$. It follows from the Smooth Collar Neighborhood Theorem 59.7 that a collar χ for $\varphi(\overline{B}^{n-k}\times S^{k-1})$ in $M^{\angle=\varphi(S^{n-k-1}\times S^{k-1})}$ exists.

Using these two observations we can now appeal to the Gluing-Smooth Manifolds-Proposition 59.9 to show that $M^{\angle=\varphi(S^{n-k-1}\times S^{k-1})}\cup_\varphi(\overline{B}^{n-k}\times\overline{B}^{k})$ admits the structure of a smooth manifold.



(b) This statement is proved as in the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (1). Slightly more precisely, this statement follows from (slight refinements of) the Double Collar Uniqueness Proposition 61.3 and the Smooth Collar-Uniqueness Proposition 59.8. We leave it to the reader to provide the details.

(c) This statement is just a reader-friendly consequence of (b).

(2) This statement follows from the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (2) and the observation that $\partial(\overline{B}^{n-k}\times\overline{B}^{k})\setminus(B^{n-k}\times S^{k-1}) = S^{n-k-1}\times\overline{B}^{k}$.

(3) This follows from the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (3).

(5) This follows from the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (5). ■

The Handle Attachment Proposition 246.3 motivates the following definition:

Definition 246.4. Let M be an n-dimensional smooth manifold. Suppose we are given a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial M$.

(1) We say that

$$M\cup_\varphi h^k \;:=\; M\cup_{\partial M\overset{\varphi}{\leftarrow}\overline{B}^{n-k}\times S^{k-1}}\left(\overline{B}^{n-k}\times\overline{B}^{k}\right)$$

is obtained from M by attaching the k-handle $\overline{B}^{n-k}\times\overline{B}^{k}$ along the attaching map φ. In particular we refer to the image of $\overline{B}^{n-k}\times\overline{B}^{k}$ in $M\cup_\varphi h^k$ as a k-handle or as a handle of index k. [2667]

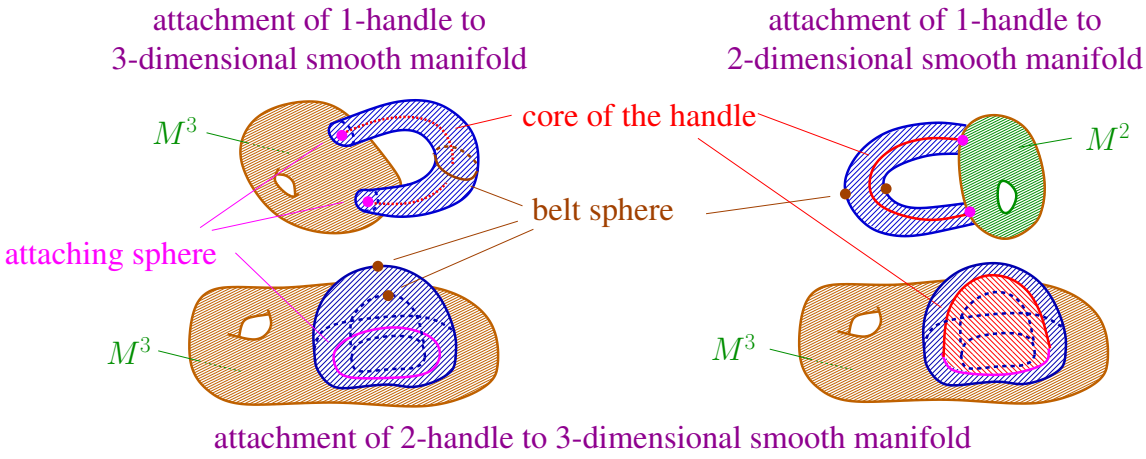
(2) Let W be a smooth manifold with submanifold M. We say W is obtained through a k-handle attachment of M if there exists an attaching map $\varphi\colon\overline{B}^{n-k}\times S^{k-1}\to\partial M$ and a diffeomorphism $M\cup_\varphi h^k\to W$ which is the identity on M.

(3) (a) We refer to the image of $\{0\}\times S^{k-1}$ in ∂M as the attaching sphere of the handle.

(b) We refer to the image of $\{0\}\times\overline{B}^{k}$ in $M\cup_\varphi h^k$ as the core of the handle.

(c) We refer to the image of $S^{n-k-1}\times\{0\}$ in $\partial(M\cup_\varphi h^k)$ as the belt sphere of the handle.

[2667] The name "handle" was introduced (?) by Felix Klein, see [**Kle1882b**] and[**Kle2018**, p. 25].

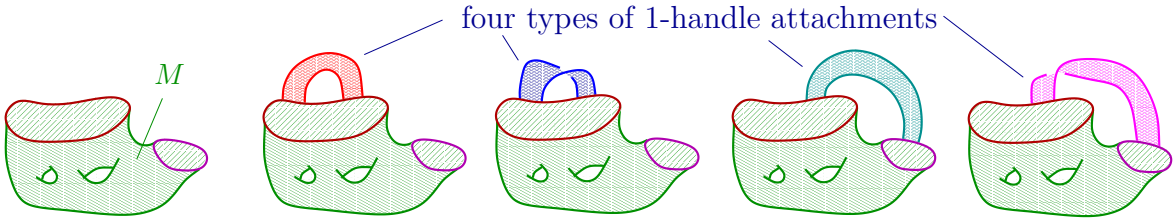


Example. Let M be a compact n-dimensional smooth manifold. Ideally one would like to classify the possible handle attachments. We make a few preliminary observations:

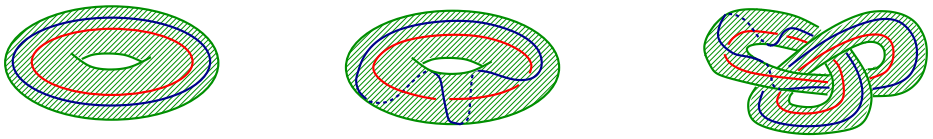
(0) We consider the extreme case of attaching a 0-handle. It follows basically from the definitions and the fact that $S^{-1} = \varnothing$ that attaching a 0-handle is the same as taking the disjoint union with $\overline{B}^n$.



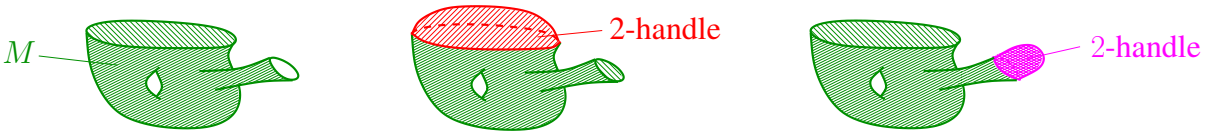
(1) We turn to attachments of 1-handles. The figure below illustrates the fact that in general the effect of attaching a 1-handle depends on the choice of the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$. The argument of the 1-Handle Attachment Lemma 246.8 below shows that for a connected smooth manifold the four examples of 1-handle attachments that we see in the figure below exhaust in a suitable sense all possibilities for 1-handle attachments.



(2) In general there is no straightforward classification of all possibilities to attach a k-handle with $k \geq 2$. For example we consider the smooth manifold $M = \overline{B}^4$ with boundary $\partial\overline{B}^4 = S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. In the figure below we show three different thickened 1-spheres $\overline{B}^2 \times S^1 \to \mathbb{R}^3 \subseteq S^3$. More precisely, we show the images in green, and we show in red the image of $\{(0,0)\} \times S^1$ and we show in blue the image of $\{(1,0)\} \times S^1$. Note that the two maps to the left have the same image, but the maps are different. Any such thickened sphere describes a 2-handle attachment to $\overline{B}^4$. There is no reason to believe that all these 2-handle attachments can be classified easily.



(*n*) Finally we turn to the other extreme, namely to attaching n-handles to n-dimensional smooth manifolds. It follows immediately from the Codimension-Zero Smooth Embedding Proposition 30.21 (3) that any smooth embedding $\varphi\colon \overline{B}^0 \times S^{n-1} \to \partial M$ is a diffeomorphism onto a component of ∂M. Thus an n-handle attachment is essentially the same as gluing an n-ball to a boundary component of M that is diffeomorphic to S^{n-1} via the diffeomorphism φ.



We will discuss the good, the bad and the ugly regarding this setting a little later, in the Top-Index Handle Attachment Proposition **??**. □

246.3. **Handle attachments: Connectedness and orientability.** In this section we will deal with the missing statements of the Handle Attachment Proposition 246.3, namely we will see how handle attachments affect connectedness and orientability. Later in Section 246.6 we will see how handle attachments affect various algebraic invariants.

But first let us deal with the basic concept of connectedness. This gets taken care of by the following analog of the Cell Attachment-π_0-Lemma 135.4:

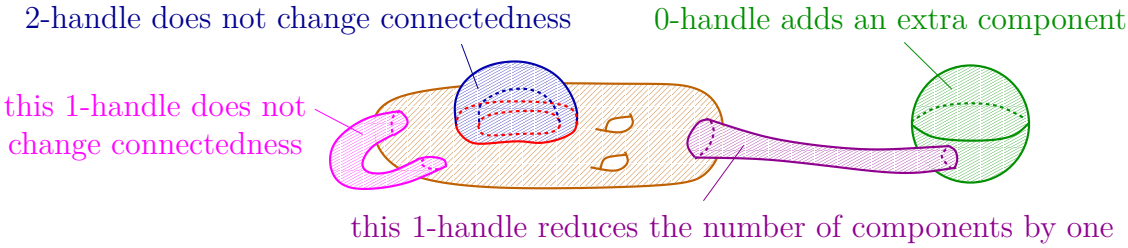
Lemma 246.5. (Handle Attachment-π_0-Lemma) Let M be an n-dimensional smooth manifold. Suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose that we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. The following statements hold:

($k \geq 2$) The inclusion-induced map $\pi_0(M) \to \pi_0(M \cup_\varphi h^k)$ is a bijection.

($k=1$) (a) If the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ takes values in a single component of M, then the inclusion-induced map $\pi_0(M) \to \pi_0(M \cup_\varphi h^k)$ is a bijection.

(b) If the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ takes values in two components of M, then the number of components goes down by one.

($k=0$) $M \cup_\varphi h^0$ has one extra component, namely the 0-handle.



Proof. The statements for $k \geq 2$ and $k = 1$ follow easily from the Union Connected Lemma 4.39. The statement for $k = 0$ follows from the fact that adding a 0-handle is, as we pointed out on page 4748, the same as taking the disjoint union with the 0-handle. ■

Next we study the slightly subtle question how attaching a handle affects orientability.

Proposition 246.6. (Handles-Orientability Proposition) Let M be an *oriented* n-dimensional smooth manifold. Suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose that we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. The following statements hold:

(i) Let $\tau\colon S^{k-1} \to S^{k-1}$ be a reflection in some hyperplane of $\mathbb{R}^k$. The corresponding map $\varphi \circ (\mathrm{id}_{\overline{B}^{n-k}} \times \tau)\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is also a thickened $(k-1)$-sphere and the smooth manifold $M \cup_{\varphi\circ(\mathrm{id}\times\tau)} h^k$ is diffeomorphic to $M \cup_\varphi h^k$.

(ii) If $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is an orientation-reversing (orientation-preserving) smooth embedding, then there exists an orientation-preserving (orientation-reversing) smooth embedding $\psi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ such that $M \cup_\psi h^k$ is diffeomorphic to $M \cup_\varphi h^k$.

Now we consider the various dimensions separately:

($k \geq 2$) $M \cup_\varphi h^k$ admits an orientation which agrees with the given orientation on $M \setminus \partial M$.

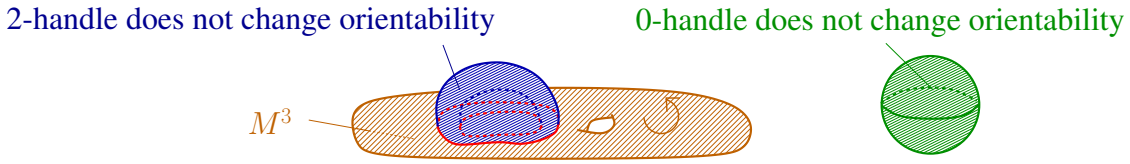
($k = 1$) (a) If[2668] $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ is orientation-preserving or orientation-reversing, then $M \cup_\varphi h^k$ admits an orientation that agrees with the given orientation on $M \setminus \partial M$.

(b) We suppose that the image of φ lies in a single component of M. If (a) is not satisfied, then $M \cup_\varphi h^k$ is non-orientable.

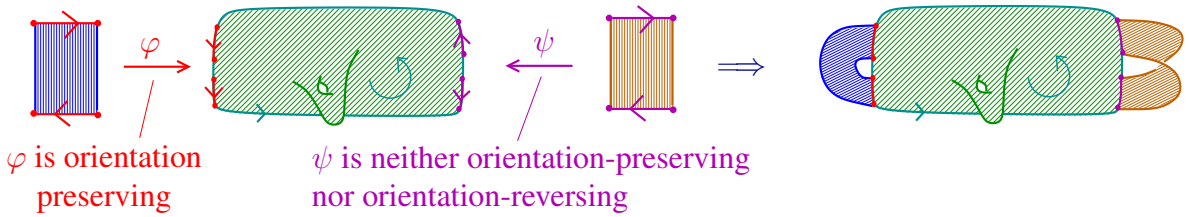
(c) If the image of φ lies in two different components of M, then $M \cup_\varphi h^k$ is orientable.

($k = 0$) $M \cup_\varphi h^0 = M \sqcup \overline{B}^n$ is again orientable.

Example. The figure below illustrates the fact that attaching a 0-handle or a handle of index ≥ 2 does not affect orientability.



In the figure below we see two instances of attaching a 1-handle:



We see that for the left handle orientability is preserved, whereas attaching the right handle destroys orientability. □

[2668] Recall that the orientation on S^0 is given by $-\{-1\} \cup +\{1\}$. By definition of the product orientation, see page 832, the obvious map $\overline{B}^{n-1} \times S^0 \to -\overline{B}^{n-1} \sqcup \overline{B}^{n-1}$ is an orientation-preserving diffeomorphism.

Proof. Let M be an oriented n-dimensional smooth manifold.

(i) Note that the reflection $\tau\colon S^{k-1} \to S^{k-1}$ extends in an obvious way to a diffeomorphism $\tau\colon \overline{B}^k \to \overline{B}^k$. The diffeomorphism from $M \cup_{\varphi\rho(\mathrm{diag}(\mathrm{id},\tau))} h^k$ to $M \cup_\varphi h^k$ is given by the identity on M and by $\mathrm{id}_{\overline{B}^{n-k}} \times \tau$ on $\overline{B}^{n-k} \times \overline{B}^k$.

(ii) This statement follows immediately from (i).

After these preparations we turn to the proofs of the main statements of the proposition.

($k \geq 2$) This statement follows immediately from (ii) together with the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (4).

($k = 1$) (a) This statement follows immediately from (ii) together with the Gluing-via Embedding-Smooth Manifolds-Proposition 61.8 (4).

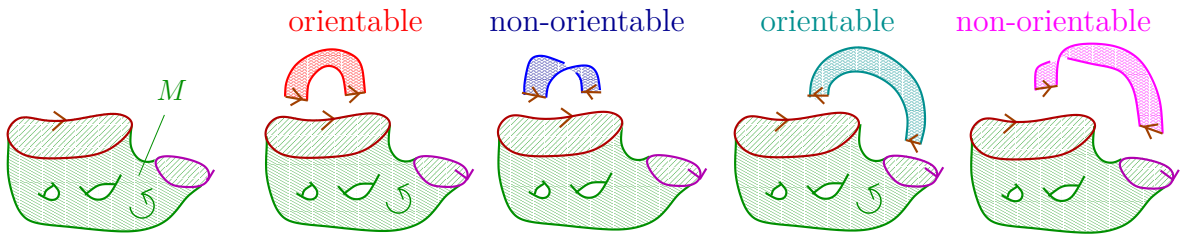
(b) This statement is basically a special case of Exercise 54.12.

(c) If the image of φ lies in two different components A and B of M, then, after possibly swapping the orientation of B, we see that we are in the setting of (a), and we see that $M \cup_\varphi h^k$ is orientable.

($k = 0$) This case is trivial. ■

The Handles-Orientability Proposition 246.6 (3) motivates the following definition:

Definition 246.7. Let M be an oriented n-dimensional smooth manifold. We say that a thickened 0-sphere $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ is orientable if φ is orientation-preserving or if it is orientation-reversing. Otherwise we say that the thickened 0-sphere is non-orientable.



The following lemma basically shows that there are not that many different possibilities for attaching 1-handles:

Lemma 246.8. (1-Handle Attachment Lemma) Let M be a smooth manifold, let F be a component of ∂M and let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres.

(1) If F is non-orientable, then $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.

(2) We now assume that F is orientable. We pick an orientation. If φ and ψ are both orientation-preserving or if they are both orientation-reversing, then $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.

Proof. Let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres.

(1) Since F is non-orientable it follows immediately from the Smooth Ball Embedding Theorem 45.10 that the two maps φ and ψ are smoothly isotopic.[2669] Thus it follows from the Handle Attachment Proposition 246.3 (5) that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.

[2669]Strictly speaking we also use the Smooth 1-Manifold Classification Theorem 28.8 which implies in particular that 1-dimensional smooth manifolds are orientable.

(2) As in the Handles-Orientability Proposition 246.6 (1) let $\tau\colon S^0 \to S^0$ be the map that swaps the two points ± 1. By the Handles-Orientability Proposition 246.6 (i) we can, if convenient, replace φ by $\varphi \circ \operatorname{diag}(\mathrm{id}, \tau)$ and we can replace ψ by $\psi \circ \operatorname{diag}(\mathrm{id}, \tau)$. It follows from this observation that we can arrange that the maps $\varphi\colon \overline{B}^{n-1} \times \{1\} \to F$ and $\psi\colon \overline{B}^{n-1} \times \{1\} \to F$ are both orientation-preserving. It follows from our orientation hypothesis on φ and ψ that $\varphi\colon \overline{B}^{n-1} \times \{-1\} \to F$ and $\psi\colon \overline{B}^{n-1} \times \{-1\} \to F$ are both orientation-preserving or that they are both orientation-reversing.



Finally note that after these preparations we can finally again apply the Smooth Ball Embedding Theorem 45.10 which implies that the two maps φ and ψ are smoothly isotopic. Thus it follows again from the Handle Attachment Proposition 246.3 (5) that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic. ■

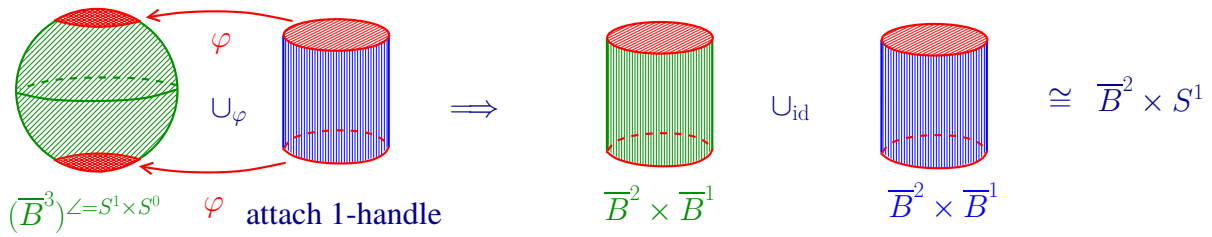
246.4. **Attaching a handle to a ball.** In this section we will discuss some of the most important examples of handle attachments, namely we will study the result of attaching a single handle to a ball:

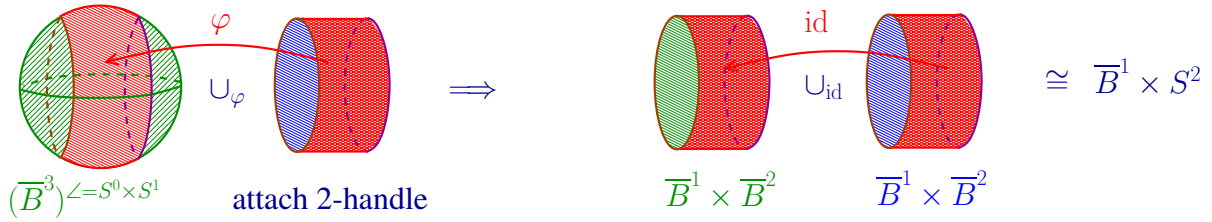
Lemma 246.9. (Ball–Handle Attachment Lemma) Let $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$.

(1) We consider the thickened $(k-1)$-sphere

$$\begin{aligned} \varphi\colon \overline{B}^{n-k} \times S^{k-1} &\to S^{n-1} = \partial \overline{B}^n \\ (x, y) &\mapsto \frac{(x, y)}{\|(x, y)\|} \end{aligned}$$

Then $\overline{B}^n \cup_\varphi h^k$ is diffeomorphic to $\overline{B}^{n-k} \times S^k$.





Before we can formulate Statement (2) of the Ball–Handle Attachment Lemma 246.9 we should remind the reader of the clutching construction from the Bundle–Clutching Map Lemma 241.14:

Definition 246.10. Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup. Let $\varphi\colon S^{k-1} \to \mathcal{G}$ be a continuous map. We set

$$Y(\varphi) \;:=\; ((\overline{B}^k_{\geq 0} \times F) \sqcup (\overline{B}^k_{\leq 0} \times F))/\sim$$

where $((c,f),0) \sim ((c,\varphi(c)(f)),1)$ for all $(c,f) \in S^{k-1} \times F$. In the Bundle–Clutching Map Lemma 241.14 we showed that the map

$$\begin{array}{rcl} p(\varphi)\colon Y(\varphi) & \to & S^k =_{\mathrm{i}} \overline{B}^k_{\geq 0} \cup \overline{B}^k_{\leq 0} \\ [((b,f),i)] & \mapsto & b \end{array}$$

is an $(F,\mathcal{G})$-bundle.

Lemma 246.9 (Ball–Handle Attachment Lemma)

(2) Let $\gamma\colon S^{k-1} \to \mathrm{O}(n-k)$ be a smooth map. We consider the thickened $(k-1)$-sphere[2670]

$$\begin{array}{rcccl} \varphi(\gamma)\colon \overline{B}^{n-k} \times S^{k-1} & \to & \overline{B}^{n-k} \times S^{k-1} & \xrightarrow{\varphi} & S^{n-1} = \partial \overline{B}^n \\ (x,y) & \mapsto & (\gamma(y)\cdot x, y) & & \\ & & (x,y) & \mapsto & \frac{(x,y)}{\|(x,y)\|}. \end{array}$$

Then $\overline{B}^n \cup_{\varphi(\gamma)} h^k$ is diffeomorphic to the total space $Y(\varphi(\gamma))$ of the $(\overline{B}^{n-k}, \mathrm{O}(n-k))$-bundle that we introduced above.

Proof.

(1) Let $n \in \mathbb{N}_0$ and $k \in \{0,\ldots,n-1\}$. First note that we saw on page 1369 that it follows from the Product-of-Balls-Proposition 60.11 and the Recornering–Rerounding Proposition 61.6 that there exists a diffeomorphism

$$\Theta\colon \overline{B}^{n-k} \times \overline{B}^k \;\to\; (\overline{B}^n)^{\angle = \varphi(S^{n-k-1} \times S^{k-1})}$$

which on $\overline{B}^{n-k} \times S^{k-1}$ is given by

$$\Theta(x,y) \;=\; \tfrac{(x,y)}{\|(x,y)\|}.$$

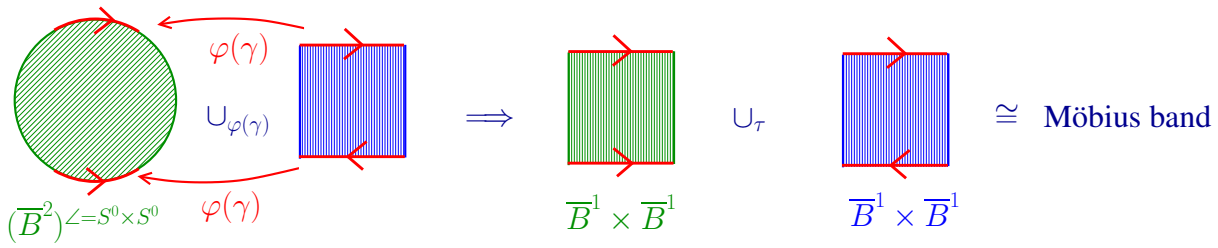
Now we see that

$$\begin{array}{rcl} \overline{B}^n \cup_\varphi h^k & \overset{\text{by Definition 246.4}}{=} & (\overline{B}^n)^{\angle = \varphi(S^{n-k-1} \times S^{k-1})} \cup_{S^{n-1} \overset{\varphi}{\leftarrow} \overline{B}^{n-k} \times S^{k-1}} (\overline{B}^{n-k} \times \overline{B}^k) \\ & \cong & (\overline{B}^{n-k} \times \overline{B}^k) \cup_{\overline{B}^{n-k} \times S^{k-1} \overset{\Theta^{-1} \circ \varphi}{\longleftarrow} \overline{B}^{n-k} \times S^{k-1}} (\overline{B}^{n-k} \times \overline{B}^k) \\ & = & (\overline{B}^{n-k} \times \overline{B}^k) \cup_{\mathrm{id}_{\overline{B}^{n-k} \times S^{k-1}}} (\overline{B}^{n-k} \times \overline{B}^k) \\ & \overset{\substack{\text{follows from the above discussion} \\ \text{that } \Theta^{-1} \circ \varphi \text{ is the identity on } \overline{B}^{n-k} \times S^{k-1}}}{=} & \overline{B}^{n-k} \times (\overline{B}^k_- \cup_{\mathrm{id}_{S^{k-1}}} \overline{B}^k_+) \;\cong\; \overline{B}^{n-k} \times S^k. \end{array}$$

(2) The proof of this statement is almost the same as the proof of Statement (1). We leave it to the reader to fill in the details. ∎

Example. Since we did not provide a detailed proof of the Ball–Handle Attachment Lemma 246.9 (2), we should at least consider one more example with $n = 2$ and $k = 1$. We consider the map $\gamma\colon S^0 \to \mathrm{O}(1)$ that is given by $\gamma(\epsilon) = (\epsilon)$.

[2670] Note that Statement (1) is the case where $\gamma \equiv \mathrm{const}_{\mathrm{id}}$.



Under the above diffeomorphism $\Theta\colon \overline{B}^1 \times \overline{B}^1 \to (\overline{B}^2)^{\angle=(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})}$ we replace the attaching map $\varphi(\gamma)$ by the map $\tau\colon \overline{B}^1 \times S^0 \to \overline{B}^1 \times S^0$ that is given by $(x,\epsilon) \mapsto (\epsilon \cdot x, \epsilon)$. The resulting smooth manifold is easily seen to be diffeomorphic to the Möbius band. □

246.5. **Handle decompositions.** The following is a good example of a self-explanatory definition:

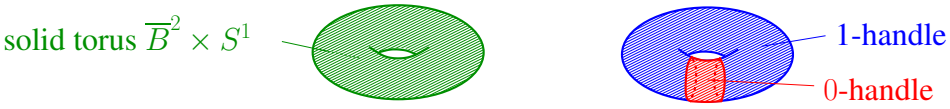
Definition 246.11. Let M be a compact n-dimensional smooth manifold. A handle decomposition for M is a diffeomorphism

$$\Theta\colon M \xrightarrow{\cong} \varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

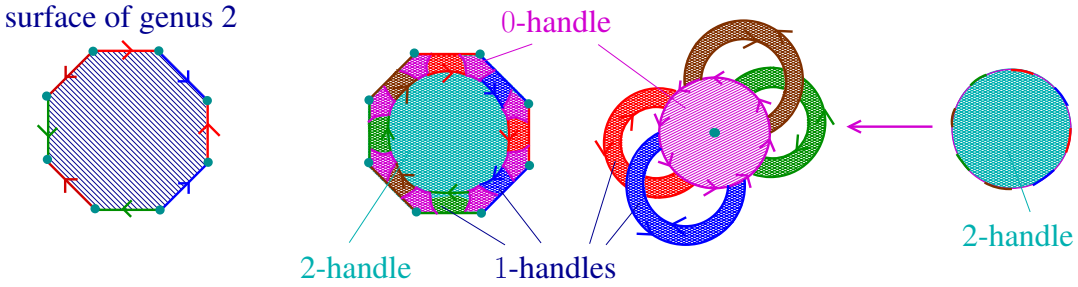
where for every $i = 1, \ldots, m$ the map φ_i is a thickened $(k_i - 1)$-sphere in the boundary of the smooth manifold $\varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}}$. Usually we suppress the name of the diffeomorphism, in this case Θ, from the notation.

Examples.

(1) The Ball–Handle Attachment Lemma 246.9 shows that $\overline{B}^n \times S^k$ admits a handle decomposition consisting of a single 0-handle and a single k-handle.

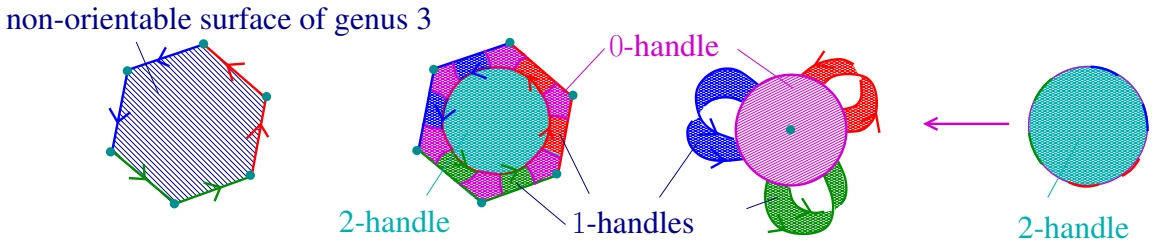


(2) In the figure below we give two different descriptions of a handle decomposition for the surface of genus $g = 2$. These two handle decompositions consist both of one 0-handle, $2g$ 1-handles and one 2-handle.



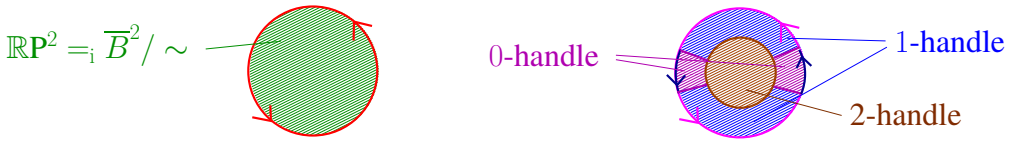
(3) In the figure below we give two different descriptions of a handle decomposition for the non-orientable surface of genus $k = 3$.
Both handle decompositions consist of one 0-handle, k 1-handles and one 2-handle. □

We will formulate our next examples as a proposition:



Proposition 246.12. (Projective Space-Handle Decomposition Proposition) Given any $n \in \mathbb{N}_0$ the projective spaces $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ admit handle decompositions with precisely $n+1$ handles. The indices are given as follows:

(1) $\mathbb{R}\mathrm{P}^n$ admits a handle decomposition with one handle of index $0, 1, 2, \ldots, n$.
(2) $\mathbb{C}\mathrm{P}^n$ admits a handle decomposition with one handle of index $0, 2, 4, \ldots, 2n$.
(3) $\mathbb{H}\mathrm{P}^n$ admits a handle decomposition with one handle of index $0, 4, 8, \ldots, 4n$.



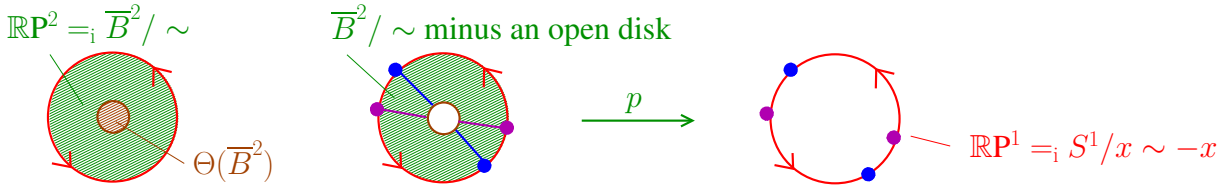
Sketch of proof. [2671] In the following we will prove the case of real projective spaces. The proofs for the complex and quaternionic projective spaces are essentially the same.

Our next goal is to prove that $\mathbb{R}\mathrm{P}^n$ admits a handle decomposition with precisely one handle of index $0, 1, 2, \ldots, n$. We prove the statement by induction on n. The case $n = 0$ is clear, since $\mathbb{R}\mathrm{P}^0$ consists of a single point. Now suppose that the statement holds for $n - 1 \in \mathbb{N}_0$.

Claim 1. There exists a smooth embedding $\Theta\colon \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ such that the map

$$\begin{array}{rcl} p\colon \mathbb{R}\mathrm{P}^n \setminus \Theta(B^n) & \to & \mathbb{R}\mathrm{P}^{n-1} \\ [x_0 : x_1 : \cdots : x_{n-1} : x_n] & \mapsto & [x_0 : x_1 : \cdots : x_{n-1}] \end{array}$$

is well-defined and it is a smooth $(\overline{B}^1, \mathrm{O}(1))$-bundle.[2672]



Proof. As we pointed out on page 4635, the explicit atlas of the Projective Space-minus-Ball Lemma 238.11 is indeed a smooth $(\overline{B}^1, \mathrm{O}(1))$-bundle atlas. ⊞
The claim shows that we can write $\mathbb{R}\mathrm{P}^n = W \cup_\varphi h^n$ where $W := \mathbb{R}\mathrm{P}^n \setminus \Theta(B^n)$ is the total space of a smooth $(\overline{B}^1, \mathrm{O}(1))$-bundle $p\colon W \to \mathbb{R}\mathrm{P}^{n-1}$. It follows from this observation and our induction hypothesis that it remains to prove the following claim:

[2671] Later, on page 4850, we will give a different proof of the proposition. An alternative argument is also sketched in [**GoS1999**, p. 106].
[2672] For $\mathbb{C}\mathrm{P}^n$ this would be a $(\overline{B}^2, \mathrm{SO}(2))$-bundle and for $\mathbb{H}\mathrm{P}^n$ this would be a $(\overline{B}^4, \mathrm{SO}(4))$-bundle.

Claim 2. Let $p\colon W \to N$ be a smooth $(\overline{B}^m, \mathrm{O}(m))$-bundle over N.

(1) (a) W is naturally a smooth manifold with corner.
(b) A collar for ∂N together with the standard collar for $S^{m-1} \subseteq \overline{B}^m$ gives rise to a corner-collar for the corner of W.

(2) We suppose that $N = M \cup_\varphi h^k$ where $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is a smooth embedding. Then

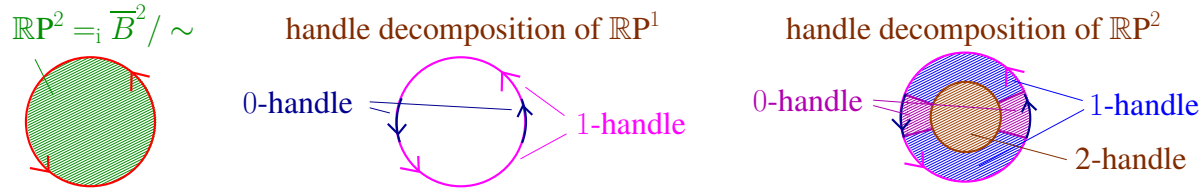
$$W \;=\; p^{-1}(M) \cup_{\overline{B}^{n-k} \times S^{k-1} \times \overline{B}^m} (\overline{B}^{n-k} \times \overline{B}^k) \times \overline{B}^m.$$

As a smooth manifold we have

$$W^{\llcorner} \;=\; ((p^{-1}(M))^{\angle = \varphi(S^{n-k-1} \times S^{k-1})})^{\llcorner X} \cup h^k.$$

Here X is the corner coming from the boundary of the $\overline{B}^m$-bundle over the smooth boundary $\partial M \setminus \varphi(B^{n-k} \times S^{k-1})$ and $\llcorner X$ means that we only round this higher degree corner.

(3) If N is a closed smooth manifold that admits a handle decomposition, then W admits a handle decomposition with the same number of handles and the same indices as N.



Proof.

(1) (a) This statement follows immediately from the fact that the boundary of a fiber is preserved under any self-diffeomorphism.
(b) This statement follows easily from the fact that we are dealing with a $(\overline{B}^m, \mathrm{O}(m))$-bundle.

(2) We write $N = M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)$ where $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is a smooth embedding. We make two preparations:

- Note that $\overline{B}^{n-k} \times \overline{B}^k$ is evidently contractible. Therefore we obtain from the Smooth Bundles-Homotopies Proposition 243.10 that there exists a diffeomorphism $\Theta\colon p^{-1}(\overline{B}^{n-k} \times \overline{B}^k) \to (\overline{B}^{n-k} \times \overline{B}^k) \times \overline{B}^m$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(\overline{B}^{n-k} \times \overline{B}^k) & \xrightarrow[\cong]{\Theta} & (\overline{B}^{n-k} \times \overline{B}^k) \times \overline{B}^m \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle (x,y,z) \mapsto (x,y)} & \\ & \overline{B}^{n-k} \times \overline{B}^k. & \end{array}$$

Note that this diagram shows that the horizontal map Θ restricts to a diffeomorphism $p^{-1}(\overline{B}^{n-k} \times S^{k-1}) \to (\overline{B}^{n-k} \times S^{k-1}) \times \overline{B}^m$.
- We denote by $j\colon p^{-1}(\overline{B}^{n-k} \times \overline{B}^k) \to W$ the inclusion.

We now see that

$$\begin{aligned} W \;=\; p^{-1}(N) \;&=\; p^{-1}(M) \cup_j p^{-1}(\overline{B}^{n-k} \times \overline{B}^k) \\ &\cong\; p^{-1}(M) \cup_{\partial M \xleftarrow{i \circ \Theta^{-1}} \overline{B}^{n-k} \times S^{k-1} \times \overline{B}^m} (\overline{B}^{n-k} \times \overline{B}^k) \times \overline{B}^m \end{aligned}$$

We leave it to the reader to make sense of the smooth statement and to prove it.

(3) This statement follows from iteratively applying (2) to the handle decomposition of N. ∎

In the above we have now seen that some of our favorite smooth manifolds admit a handle decomposition. In Chapter 250 we will prove the Handle Decomposition Theorem 250.7 which states that in fact every compact smooth manifold admits a handle decomposition. In the remainder of this chapter we perform the less exciting task of proving several basic facts about handle attachments and handle decompositions.

246.6. **Handle attachments: Algebraic invariants.** In this section we will discuss how a handle attachment alters some of our favorite algebraic invariants of topological spaces, namely the fundamental group and the homology groups. The statements below are basically the same as for cell attachments that we gave in the Cell Attachment-π_1-Proposition 135.5 and the Cell Attachment-H_*-Proposition 153.12. First we have the following analog of the Cell Attachment-π_1-Proposition 135.5:

Proposition 246.13. (Handle Attachment-π_1-Proposition) Let M be an n-dimensional smooth manifold, let $k \in \mathbb{N}$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We pick $(x_0, y_0) \in \overline{B}^{n-k} \times S^{k-1}$ and we set $P := \varphi(x_0, y_0) \in \partial M$. The following statements hold:

($k \geq 3$) The inclusion-induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^k, P)$ is an isomorphism.

($k = 2$) Let $\beta\colon S^1 \to M$ be the loop that is given by $z \mapsto \varphi(x_0, z)$. The inclusion-induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^2, P)$ descends to an isomorphism

$$\pi_1(M, P)/\langle\langle [\beta] \rangle\rangle \xrightarrow{\cong} \pi_1(M \cup_\varphi h^2, P).$$

($k = 1$) We set $P' := \varphi(x_0, -y_0) \in \partial M$.

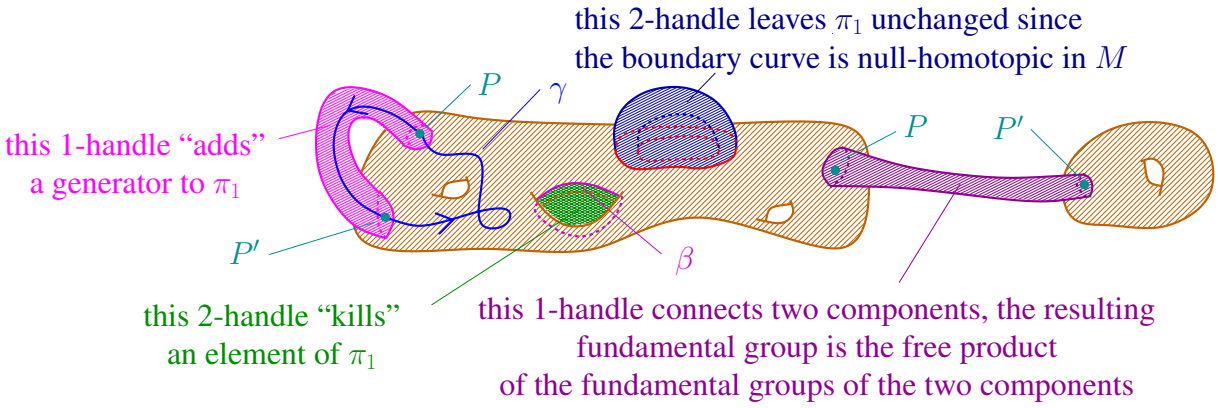
(a) Suppose that P and P' lie in the same component of M. Let γ be a loop in $M \cup_\varphi h^1$ that is given by the concatenation of the obvious path in the 1-handle from P to P' and some path in M from P' to P. With this notation the map

$$\pi_1(M, P) * \langle t \rangle \to \pi_1(M \cup_\varphi h^1, P)$$

that is given by the inclusion-induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^1, P)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that P and P' lie in two different components of M. Then we get an isomorphism

$$\pi_1(M, P) * \pi_1(M, P') \to \pi_1(M \cup_\varphi h^1, P).$$



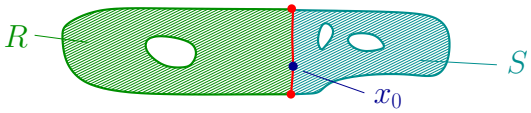
Example. In the Projective Space-Handle Decomposition Proposition 246.12 we showed that the complex and quaternionic projective spaces $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ admit handle decompositions without 1-handles. It thus follows easily from the Handle Attachment-π_1-Proposition 246.13 that $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$ and $\pi_1(\mathbb{H}\mathrm{P}^n) = 0$. □

Sketch of proof. Overall the proof of the Handle Attachment-π_1-Proposition 246.13 is very similar to the proof of the Cell Attachment-π_1-Proposition 135.5. Thus we will now give only an outline of the argument. In the proof of the proposition we will need the following general statement:

Claim. Let N be an m-dimensional smooth and let $R, S \subseteq N$ be two codimension-zero submanifolds with corner of N such that $R \cap S$ is a smooth boundary of R and a smooth boundary of S. Then for every base point $x_0 \in R \cap S$ the inclusion-induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \;\to\; \pi_1(N, x_0)$$

is an isomorphism.



Proof. This is essentially the statement of the Manifold–Seifert-van Kampen Theorem 102.1. The only difference is that in the claim we now allow submanifolds *with corner*. As we explain in Footnote 1060, in the Manifold–Seifert-van Kampen Theorem 102.1 we need to replace the Topological Collar Neighborhood Theorem 35.14 by the Smooth Collar Neighborhood Theorem 59.7 to get the promised result. ⊞

Now we turn to the proofs of the statements:

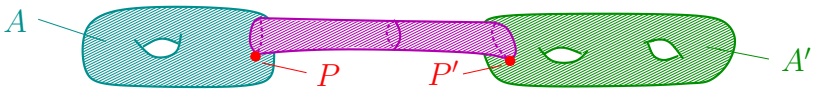
($k \geq 3$) Recall that by the Handle Attachment Proposition 246.3 we can write $M \cup_\varphi h^k$ as the union of two submanifolds with corner given by M and $h^k = \overline{B}^{n-k} \times \overline{B}^k$, which intersect in $\overline{B}^{n-k} \times S^{k-1}$. The statement follows from this observation, the claim, the observation that $\pi_1(\overline{B}^{n-k} \times S^{k-1}) = 0$ and the fact that $\pi_1(\overline{B}^{n-k} \times \overline{B}^k) = 0$.

($k = 2$) Let $\gamma\colon S^1 \to \overline{B}^{n-k} \times S^1$ be the map that is given by $z \mapsto (0, z)$. As above the statement follows from the claim and the observation that $\pi_1(\overline{B}^{n-2} \times S^1) = \langle\gamma\rangle$ and $\pi_1(\overline{B}^{n-2} \times \overline{B}^2) = 0$.

($k = 1$) We set $P' \coloneqq \varphi(x_0, -y_0) \in \partial M$.

(a) We suppose that P and P' lie in the same components of M. This statement follows easily from the HNN-Seifert–van Kampen Theorem 106.7 (2). To be precise, as in the claim we need to work with a slight generalization of the HNN-Seifert–van Kampen Theorem 106.7 (2) (see Footnote 1098) since we are dealing with submanifold with corners.

(b) We suppose that P and P' lie in two different components A and A' of M.



The desired statement now follows from the claim applied twice: First we attach $\overline{B}^{k-1} \times \overline{B}^1$ to A, and then we attach A' to $A \cup (\overline{B}^{k-1} \times \overline{B}^1)$.

We leave it to the reader to fill in all the details. ■

Next we study the effect of a handle attachment to homology groups. The following lemma is an analog of the Cell Attachment-H_*-Proposition 153.12:

Lemma 246.14. (Handle Attachment-H_*-Lemma) Let M be an n-dimensional smooth manifold, let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We denote by $i\colon M \to M \cup_\varphi h^k$ the natural embedding. There exists a natural homomorphism

$$\partial_k\colon \mathrm{H}_k(M \cup_\varphi h^k) \to \widetilde{\mathrm{H}}_{k-1}(S^{k-1})$$

such that the following sequence is exact:[2673]

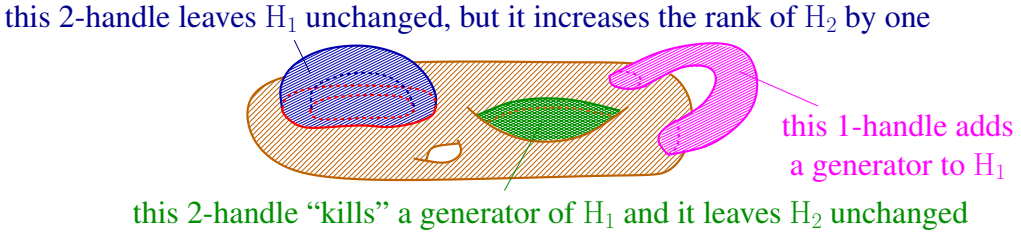
induced by the map $y \mapsto \varphi(0,y)$, together with the natural homomorphism $\text{ж}\colon \widetilde{\mathrm{H}}_{k-1}(M) \to \mathrm{H}_{k-1}(M)$

$$0 \to \mathrm{H}_k(M) \xrightarrow{i_*} \mathrm{H}_k(M \cup_\varphi h^k) \xrightarrow{\partial_k} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{k-1})}_{\cong \mathbb{Z}} \xrightarrow{\downarrow} \mathrm{H}_{k-1}(M) \xrightarrow{i_*} \mathrm{H}_{k-1}(M \cup_\varphi h^k) \to 0.$$

Furthermore, for any $l \neq k-1, k$ the inclusion-induced map

$$i_*\colon \mathrm{H}_l(M) \xrightarrow{\cong} \mathrm{H}_l(M \cup_\varphi h^k)$$

is an isomorphism. Analogous statements also hold if we consider (reduced) homology with coefficients given by an abelian group.



Example.

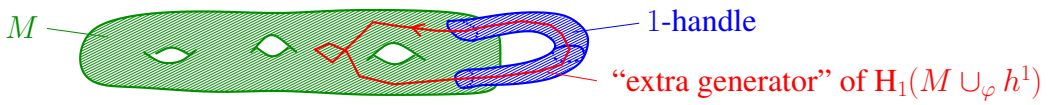
(1) In the Projective Space-Handle Decomposition Proposition 246.12 we showed that the complex space $\mathbb{C}\mathrm{P}^n$ admits a handle decomposition with precisely one handle of index $0, 2, \dots, 2n$ and no other handles. As we will see in Exercise 246.20 it follows easily from this observation together with the Handle Attachment-H_*-Lemma 246.14 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

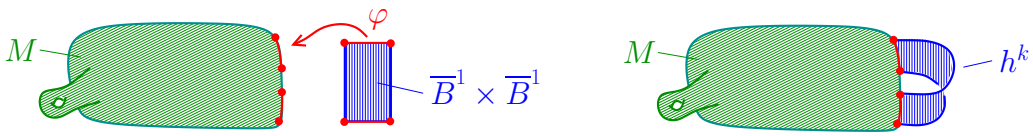
Similarly one can easily calculate the homology groups of the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$.

(2) Let M be a connected n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ be a thickened 0-sphere. We consider $M \cup_\varphi h^1$. Since M is connected it follows from the Reduced Homology Lemma 147.3 (0c) that $\widetilde{\mathrm{H}}_0(M) = 0$. This means that the map $\widetilde{\mathrm{H}}_0(S^0) \to \mathrm{H}_0(M)$ in the Handle Attachment-H_*-Lemma 246.14 is the zero map. It thus follows from the Splitting Lemma 145.5 and the Free Abelian Group–Splitting Lemma 97.11 that $\mathrm{H}_1(M \cup_\varphi h^1) \to \mathrm{H}_1(M) \oplus \mathbb{Z}$.
We also see that for all $i \neq 1$ the inclusion induced map $\mathrm{H}_1(M) \cong \mathrm{H}_1(M \cup_\varphi h^1)$ is an isomorphism. □

[2673] Note that this statement also holds for $k = 0$, we just need to remember that by the Reduced Homology Lemma 147.3 we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.



Proof. Let M be an n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere.



We consider the following diagram:

$$\widetilde{\mathrm{H}}_j(M \cup_\varphi h^k) \xrightarrow{\partial_j} \widetilde{\mathrm{H}}_{j-1}(\varphi(\overline{B}^{n-k} \times S^{k-1})) \to \widetilde{\mathrm{H}}_{j-1}(M) \oplus \overbrace{\widetilde{\mathrm{H}}_{j-1}(\overline{B}^{n-k} \times \overline{B}^k)}^{=0} \to \widetilde{\mathrm{H}}_{j-1}(M \cup_\varphi h^k) \to$$
$$\cong \uparrow$$
$$\widetilde{\mathrm{H}}_{j-1}(S^{k-1})$$

We make the following clarifications and observations:

(1) By the Handle Attachment Proposition 246.3 (3) we can view M and $\overline{B}^{n-k} \times \overline{B}^k$ as submanifolds with corner of $M \cup_\varphi h^k$. The upper sequence is now the long exact sequence coming from the Mayer-Vietoris Theorem 153.2 (3). To be a little more precise, we need a slight generalization on the Mayer-Vietoris Theorem 153.2 (3) which is explained in Footnote 1578.

(2) The two maps emanating from below are the maps induced by the continuous maps that are given by $x \mapsto \varphi(0, x)$.

(3) It follows from the Homotopy Equivalence-H_*-Proposition 144.14 (1a) that the vertical map is an isomorphism.

(4) By the Sphere-H_*-Proposition 149.4 we know that $\widetilde{\mathrm{H}}_m(S^m) \cong \mathbb{Z}$ and that $\widetilde{\mathrm{H}}_r(S^m) = 0$ for $r \neq m$.

The statements of the lemma follow from staring at this diagram for a few short minutes and applying the Reduced Homology Lemma 147.3 and $\widetilde{\mathrm{H}}_n$-Functor Lemma 147.2 which explain the relationship between reduced and ordinary homology.

Finally, if we work with (reduced) homology with G-coefficients, then using the statements of Chapter 160 one can easily adapt the above arguments. ∎

We conclude this discussion with the most basic of all topological invariants, namely the Euler characteristic:

Lemma 246.15. (Handle Attachment-χ-Lemma) Let M be a compact n-dimensional smooth manifold.

(1) Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Then

$$\chi(M \cup_\varphi h^k) \;=\; \chi(M) + (-1)^k.$$

(2) If M has a handle decomposition, then

$$\chi(M) \;=\; \sum_{k=0}^{n} (-1)^k \cdot \#k\text{-handles}.$$

Proof.

(1) This statement follows easily from the Handle Attachment-H_*-Lemma 246.14, together with the Exact Sequence-χ-Lemma 156.5 and the Homology-χ-Proposition 156.2.

(2) This statement follows easily from (1). ■

246.7. Exercises for Chapter 246.

Exercise 246.1.

(a) How many different diffeomorphism types can we obtain by attaching a single 1-handle to the annulus $A := [0,1] \times S^1$?

(b) How many different diffeomorphism types can we obtain by attaching a single 1-handle to the Möbius band M?

(c) How many different diffeomorphism types can we obtain by attaching a single 1-handle to the disjoint union $A \sqcup M$?

Exercise 246.2. We consider $M := S^2 \times [0,1]$ and we attach a 1-handle that connects the two boundary components such that $M \cup_\varphi h^1$ is orientable.

(a) Determine the diffeomorphism type of $\partial(M \cup_\varphi h^1)$.

(b) Determine $\pi_1(M \cup_\varphi h^1)$ and determine the homology groups of $M \cup_\varphi h^1$.

(c) Is $M \cup_\varphi h^1$ diffeomorphic to a smooth submanifold of $\mathbb{R}^3$?

Exercise 246.3. We consider $M := S^1 \times S^1 \times [0,1]$ and we attach a 1-handle that connects the two boundary components such that $M \cup_\varphi h^1$ is orientable.

(a) Determine the diffeomorphism type of $\partial(M \cup_\varphi h^1)$.

(b) Determine $\pi_1(M \cup_\varphi h^1)$ and determine the homology groups of $M \cup_\varphi h^1$.

(c) Is $M \cup_\varphi h^1$ diffeomorphic to a handlebody, i.e. is it diffeomorphic to the result of attaching 1-handles to $\overline{B}^3$?

Exercise 246.4. Does there exist a handle decomposition of S^2 with ...

(a) ... two 0-handles, one 1-handle and one 2-handle?

(b) ... one 0-handle, two 1-handles and one 2-handle?

(c) ... one 0-handle, one 1-handle and two 2-handles?

Exercise 246.5. We denote by $\rho\colon S^2 \to S^2$ and $\rho\colon \overline{B}^2 \to \overline{B}^2$ the reflections in a hyperplane. We set $S^1 \widetilde{\times} S^2 := \mathrm{Tor}(S^2, \rho)$.[2674]

(a) Show that $S^1 \widetilde{\times} S^2$ is diffeomorphic to the double of the solid Klein bottle $\mathrm{Tor}(\overline{B}^2, \rho)$.

(b) Let $k \in \mathbb{N}_{\geq 2}$. Show that the connected sum of k copies of $S^1 \widetilde{\times} S^2$ is diffeomorphic to the connected sum of $S^1 \widetilde{\times} S^2$ with $k-1$ copies of $S^1 \times S^2$.
Hint. Make use of the 1-Handle Attachment Lemma 246.8 and use the following observation: If M and N are diffeomorphic smooth manifolds, then the doubles $\mathrm{D}M$ and $\mathrm{D}N$ are also diffeomorphic.

[2674] Note that $S^1 \times \times S^2$ is the unique non-orientable S^2-bundle over S^1.

Exercise 246.6. Let M be an oriented n-dimensional smooth manifold with connected boundary and let $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ be a thickened 0-sphere.

(a) We assume that $M \cup_\varphi h^1$ is orientable. Show that $\partial(M \cup_\varphi h^1)$ is diffeomorphic to the connected sum of ∂M and $S^{n-2} \times S^1$.

(b) We assume that $M \cup_\varphi h^1$ is orientable. Show that $\partial(M \cup_\varphi h^1)$ is diffeomorphic to the connected sum of ∂M and the "twisted product" $S^{n-2} \widetilde{\times} S^1$, which is defined as the mapping torus $\operatorname{Tor}(S^{n-2}, \rho)$ where $\rho\colon S^{n-2} \to S^{n-2}$ is the reflection in a hyperplane.

Hint. Make use of the 1-Handle Attachment Lemma 246.8 and the Ball–Handle Attachment Lemma 246.9.

Exercise 246.7. Does every handle decomposition of a non-empty smooth manifold contain at least one 0-handle?

Exercise 246.8. Use the Handle Attachment-H_*-Lemma 246.14 to give a new proof of the Handle Attachment-π_0-Lemma 246.5.

Exercise 246.9.

(a) Let $M = g \cup h_1 \cup \cdots \cup h_m$ be a standard handle decomposition for an oriented smooth manifold M consisting of a single 0-handle g and k 1-handles $h_1, \dots, h_m$. We pick $x_0 \in g$. The Handle Attachment-π_1-Proposition 246.13 allows us to make a natural identification

$$\Theta\colon \langle h_1, \dots, h_m \rangle \xrightarrow{\cong} \pi_1(M, x_0).$$

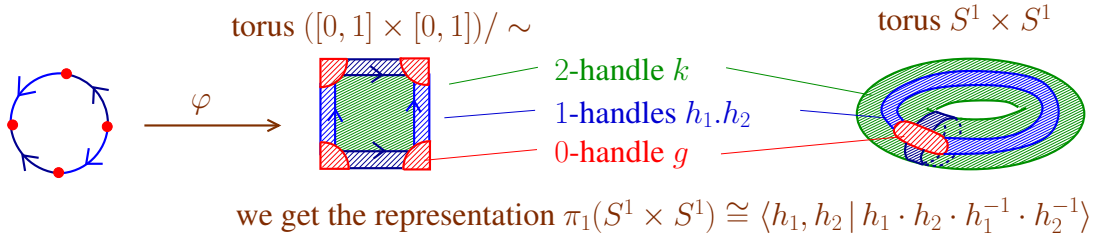
Now let $\varphi\colon S^1 \to \partial M$ be a smooth embedding that is transverse to all the belt spheres of the 1-handles. Let $i_1, \dots, i_n \in \{1, \dots, m\}$ and $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ such that the map $[0,1] \xrightarrow{t \mapsto \exp(2\pi \mathrm{i} t)} S^1 \xrightarrow{\varphi} M$ successively intersects the belt sphere of the 1-handle h_{i_j} with sign $\epsilon_j \in \{-1, 1\}$ for $j = 1, \dots, n$. Show that under the above isomorphism we have

$$\Theta(h_{i_1}^{\epsilon_1} \cdot \ldots \cdot h_{i_n}^{\epsilon_n}) \;=\; [\varphi].$$

Remark. By the Handle Attachment-π_1-Proposition 246.13 this implies that we have an isomorphism

$$\langle h_1, \dots, h_m \,|\, h_{i_1}^{\epsilon_1} \cdot \ldots \cdot h_{i_n}^{\epsilon_n} \rangle \;\to\; \pi_1(M \cup_\varphi 2\text{-handle } k).$$

(b) Use (a) to give a new proof that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$.
Hint. Make use of the suggestive figure below.



Exercise 246.10. State and prove the cohomological analog of the Handle Attachment-H_*-Lemma 246.14.

Exercise 246.11.

(a) Let M be a compact orientable connected 2-dimensional topological manifold. Show that $\chi(M) \leq 2$.

Remark. Evidently this follows from the Surface Classification Theorem 101.16. But it would be more elegant to have a proof that does not rely on this result.

(b) Show that given any $k \in \mathbb{Z}_{\leq -2}$ there exists a compact orientable connected 2-dimensional smooth manifold with $\chi(M) = k$.

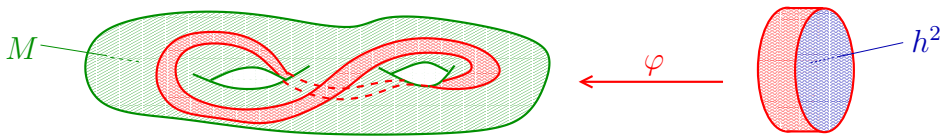
(c) Let $n \in \mathbb{N}_{\geq 3}$. Show that given any $k \in \mathbb{Z}$ there exists a compact orientable connected n-dimensional smooth manifold with $\chi(M) = k$.

Exercise 246.12. On page 4754 we saw that the surface of genus 2 admits a handle decomposition with four 1-handles. Does it also admit a handle decomposition with less than four 1-handles?

Exercise 246.13. Let M be a compact smooth manifold. Show that the minimal number of generators of $\mathrm{H}_1(M)$ is a lower bound on the number of 1-handles in any handle decomposition of M.
Hint. Use the Handle Attachment-H_*-Lemma 246.14.

Exercise 246.14. Let $K \subseteq S^3$ be a knot. By the Product Tubular Neighborhood–Existence Theorem 58.4 we can pick a tubular neighborhood $\overline{B}^2 \times K$. As in Definition 108.30 we consider the knot exterior $X_K := S^3 \setminus (B^2 \times K)$. Can we add handles to X_K to obtain a smooth manifold that is diffeomorphic to S^3?

Exercise 246.15. We consider the 3-dimensional smooth manifold M shown in the figure below. Furthermore we consider the thickened 1-sphere $\varphi\colon \overline{B}^1 \times S^1 \to \partial M$ shown in the figure below. Can you recognize the diffeomorphism type of $M \cup_\varphi h^2$? Can you prove that it is what you think it is?



Exercise 246.16. Let M be a connected n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ be a thickened 0-sphere in ∂M. Let $\tau\colon \overline{B}^{n-1} \to \overline{B}^{n-1}$ be the reflection in a hyperplane. We consider the thickened 0-sphere

$$\begin{array}{rcl} \overline{B}^n \times S^0 & \to & \partial M \\ (P, \epsilon) & \mapsto & \begin{cases} \varphi(\tau(P), -1), & \text{if } \epsilon = -1, \\ \varphi(P, 1), & \text{if } \epsilon = 1. \end{cases} \end{array}$$

Is it possible that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic?



Exercise 246.17. Let M be a compact oriented n-dimensional smooth manifold, let F be a component of ∂M and let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres. Show that if F is non-orientable, then $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.
Remark. This exercise is a variation on the 1-Handle Attachment Lemma 246.8.

Exercise 246.18. Let M be an orientable n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened k-sphere.

(a) Show that for $k = 1$ the smooth manifold $M \cup_\varphi h^1$ is orientable if and only if $\partial(M \cup_\varphi h^1)$ is orientable.
Hint. Use the Gluing-Smooth Manifolds-Proposition 54.5 (4).

(b) Does (a) also hold for all other k?

Exercise 246.19. Recall that by the Projective Space-Avatar Lemma 9.16 we have an identification $\mathbb{R}\mathrm{P}^3 =_{\mathrm{i}} \overline{B}^3/\sim$ where $x \sim -x$ for all $x \in S^2$. Illustrate the handle decomposition from the Projective Space-Handle Decomposition Proposition 246.12.

Exercise 246.20. Let $n \in \mathbb{N}_0$.

(a) Use the Projective Space-Handle Decomposition Proposition 246.12 together with the Handle Attachment-H_*-Lemma 246.14 to show that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Determine the homology groups of the quaternionic projective space $\mathbb{H}\mathrm{P}^n$.

Exercise 246.21. Let M and N be connected non-empty smooth manifolds with a handle decomposition.

(a) Show that the connected sum $M \# N$ admits a handle decomposition.

(b) Use (a) to determine $\chi(M \# N)$ in terms of $\chi(M)$ and $\chi(N)$.

Exercise 246.22. Does there exist a compact n-dimensional smooth manifold M and a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ such that for each $i \in \mathbb{N}_0$ we have

$$\text{minimal number of generators of } \mathrm{H}_i(M) \;=\; \text{minimal number of generators of } \mathrm{H}_i(M \cup_\varphi h^k)?$$

Exercise 246.23. Let

$$K \;=\; ([0,1] \times [0,1])/\sim \;\text{ where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y)$$

be the Klein bottle. We consider the smooth manifold $K \times [0,1]$.

(a) Does there exist an attaching map $\varphi\colon \overline{B}^1 \times S^1 \to K \times \{0\}$ of a 1-handle such that $\varphi(\{0\} \times S^1) = \{[(\frac{1}{2}, t)] \,|\, t \in [0,1]\}$? If yes, determine $\mathrm{H}_*((K \times [0,1]) \cup_\varphi h^1)$.

(b) Does there exist an attaching map $\varphi\colon \overline{B}^1 \times S^1 \to K \times \{0\}$ of a 1-handle such that $\varphi(\{0\} \times S^1) = \{[(t, \frac{1}{2})] \,|\, t \in [0,1]\}$? If yes, determine $\mathrm{H}_*((K \times [0,1]) \cup_\varphi h^1)$.

Exercise 246.24. Show that there exists a handle attachment such that the homomorphism $\widetilde{\mathrm{H}}_{k-1}(S^{k-1}) \to \mathrm{H}_{k-1}(M)$ from the Handle Attachment-H_*-Lemma 246.14 is non-trivial but not injective.

Exercise 246.25. Show that there exist infinitely many attaching maps $\varphi_n\colon \overline{B}^2 \times S^1 \to \overline{B}^4$, $n \in \mathbb{N}$, such that the smooth manifolds $\overline{B}^4 \cup_{\varphi_n} h^2$ are pairwise non-diffeomorphic.
Hint. You can try to distinguish smooth manifolds by considering their boundaries.

Exercise 246.26. Does $\mathbb{C}\mathrm{P}^n$ admit a handle decomposition with fewer than $n+1$ handles?

Exercise 246.27. Let M and N be two closed smooth manifolds that are equipped with a handle decomposition.

(a) Show that $M \times N$ admits a handle decomposition.

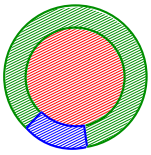
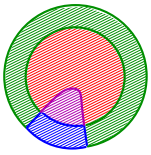
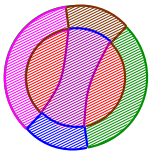
(b) Use handle decompositions to show that $\chi(M \times N) = \chi(M) \cdot \chi(N)$.

Remark. To preserve sanity it is best to ignore rounding corners.

Exercise 246.28. In the figure below we show three different decompositions of the closed disk $\overline{B}^2$. In each case do the following:

(a) Determine whether this decomposition comes from a handle decomposition.

(b) If it does come from a handle decomposition, in what order where they attached? Is there a unique order?

(1)  (2)  (3) 

Exercise 246.29. Let $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(a) Give a handle decomposition of $\Sigma_{g,n}$.

(b) What is a handle decomposition of $\Sigma_{g,n}$ with a minimal number of handles?

Exercise 246.30. Let $\widetilde{M}$ and M be smooth manifolds and let $p\colon \widetilde{M} \to M$ be a covering of degree d such that p is local diffeomorphism. We suppose that M admits a handle decomposition.

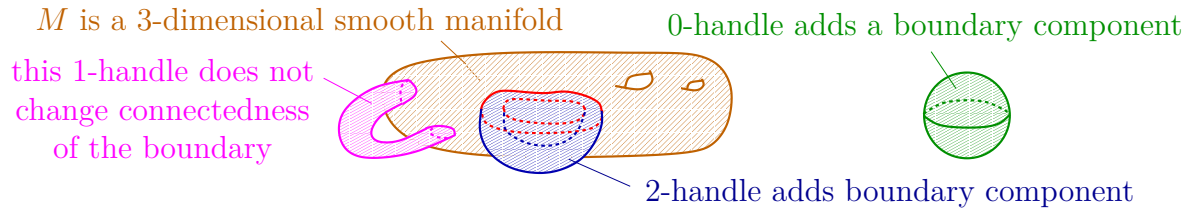
(a) Show that $\widetilde{M}$ admits a handle decomposition such that

$$\text{number of } k\text{-handles of } \widetilde{M} \;=\; d \cdot \text{number of } k\text{-handles of } M.$$

(b) Show that $\chi(\widetilde{M}) = d \cdot \chi(M)$.

(c) We equip $\mathbb{R}\mathrm{P}^2$ with the handle decomposition that we constructed in the Projective Space-Handle Decomposition Proposition 246.12. We consider the usual 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} S^2/x \sim -x$. Draw the induced handle decomposition of S^2.

Exercise 246.31. Let M be an n-dimensional smooth manifold such that ∂M is connected. Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Show that if $k \in \{1, \dots, n-2\}$, then $\partial(M \cup_\varphi h^k)$ is also connected.



Exercise 246.32. Let M be a compact n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened k-sphere. Calculate $\chi(\partial(M \cup_\varphi h^k))$ in terms of $\chi(\partial M)$ and k.

Hint. Use the Manifold-χ-Proposition 190.24.

CHAPTER 247

Handle decompositions of smooth manifolds

In this chapter we will first introduce the notion of a handle decomposition of a smooth manifold M rel a union of components of ∂M. Then we will see that if a handle decomposition exists, then one can also find a handle decompositions that is "standard". This means in particular the handles are arranged such that the indices are non-decreasing. Furthermore we will give a simple criterion for a 0-handle and a 1-handle to "cancel". Finally, to a handle decomposition we will also associate a dual handle decomposition.

This chapter is organized as follows:

- In Section 247.1 we will introduce relative handle decompositions.
- In Section 247.2 we will state and prove the Handle Swapping Proposition 247.4 which allows us to swap two successive handles to ensure that the one of lower index happens first.
- In Section 247.3 we will introduce "standard" handle decompositions. These have in particular the property that the indices are in non-decreasing order. We will use the Handle Swapping Proposition 247.4 to show that if we are given a handle decomposition, then we can make it standard.
- In Section 247.4 we will relate the boundary connected sum operation to handle attachments and decompositions. As we will see, earlier results on the boundary connected sum imply that often we can cancel a 0-handle and a successive 1-handle. We will also show that if a handle decomposition exists, then we can reduce the number of 0-handles to the bare minimum.
- In Section 247.5 we will (re-) introduce handlebodies and we will use the results of Section 247.4 to give several characterizations of handlebodies.
- In Section 247.6 we will provide exercises for Chapter 247.

In the next Chapter 248 we will introduce dual handle decomposition.

247.1. **Relative Handle Decompositions.** For smooth manifolds with non-empty boundary it will be convenient to study relative handle decompositions. This concept generalizes the notion of a handle decomposition that we introduced in Definition 246.11:

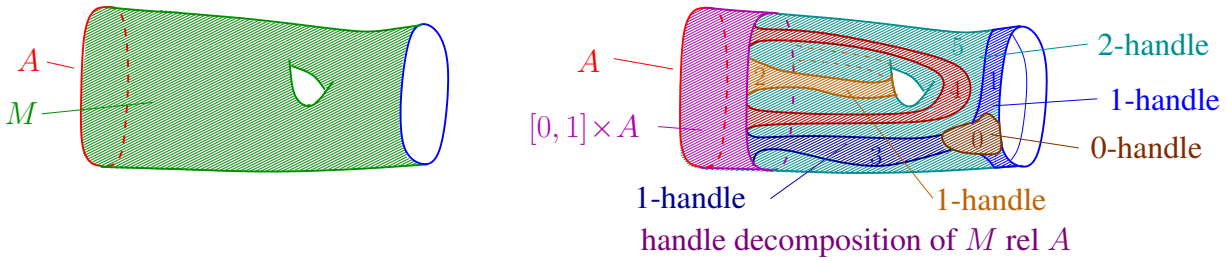
Definition 247.1. Let M be a compact[2675] n-dimensional smooth manifold and let A be a union of components of ∂M.[2676] A handle decomposition for M rel A is a diffeomorphism

$$\Theta\colon M \xrightarrow{\cong} ([0,1]\times A)\cup_{\varphi_0} h^{k_0}\cup\cdots\cup_{\varphi_m} h^{k_m}$$

such that for each $P\in A$ we have $\Theta(P)=[(0,P)]$ and such that for $i=0,\dots,m$ the map φ_i is a thickened (k_i-1)-sphere that takes values in[2677]

$$\partial(([0,1]\times A)\cup_{\varphi_0} h^{k_0}\cup\cdots\cup_{\varphi_{i-1}} h^{k_{i-1}})\setminus(\{0\}\times A).$$

In the following we will mostly suppress Θ from the notation. Furthermore, usually we will use Θ to identify A with $\{0\} \times A$. Sometimes a handle decomposition rel $\varnothing$ is just called a handle decomposition.



The following proposition says that we can "stack" relative handle decompositions.

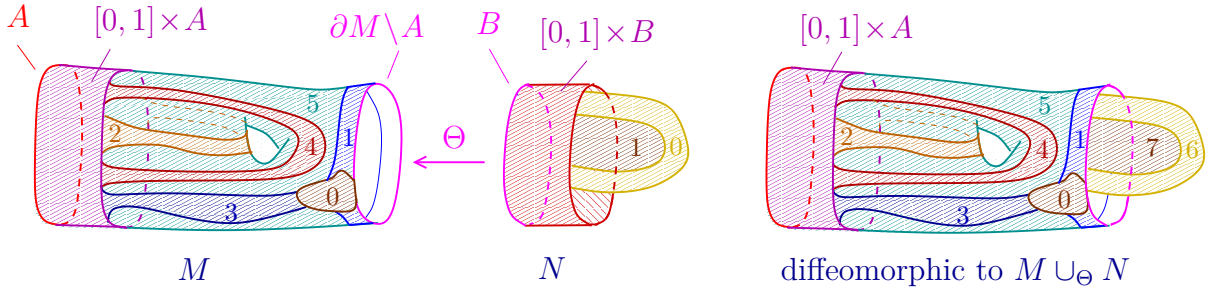
Proposition 247.2. (Handle Decomposition–Stacking Proposition) Let M be a compact n-dimensional smooth manifold and let N be a compact n-dimensional smooth manifold. Furthermore let A be a union of components of ∂M and let B be a union of components of ∂N. Next let

$$\begin{aligned} M &= ([0,1] \times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \\ \text{and} \qquad N &= ([0,1] \times B) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \end{aligned}$$

be handle decompositions rel A respectively rel B. Finally let $\Theta\colon B \to \partial M \setminus A$ be a diffeomorphism. We consider the smooth manifold $M \cup_\Theta N$ which we obtain from the Gluing-Smooth Manifolds-Proposition 54.5. There exists a handle decomposition of the form

$$M \cup_\Theta N \xrightarrow{\cong} ([0,1] \times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s}$$

rel A, where the attaching maps $\widetilde{\psi}_0, \ldots, \widetilde{\psi}_s$ are obtained in a hopefully obvious way from the attaching maps $\psi_0, \ldots, \psi_s$.



In the proof of the Handle Decomposition–Stacking Proposition 247.2 we will make use of the following lemma, which might be of independent interest:[2677]

Lemma 247.3. (External Collar Lemma) Let M be a compact smooth manifold with corner and let C be a smooth boundary of M in the sense of Exercise 59.4. Let $\varphi\colon C \to C$

[2676] In Definition 332.4 we will introduce the notion of handle-structure, which then also makes sense for non-compact smooth manifolds.

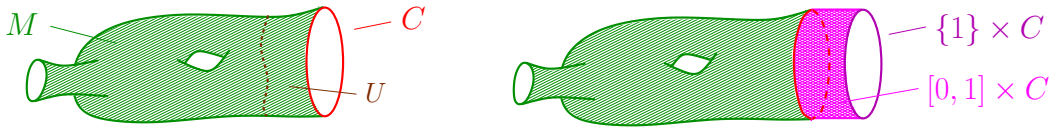
[2676] For $A = \varnothing$ we obtain precisely the notion from see Definition 246.11.

[2677] Note that this lemma also doubles as Exercise 59.4.

be a diffeomorphism. There exists a diffeomorphism

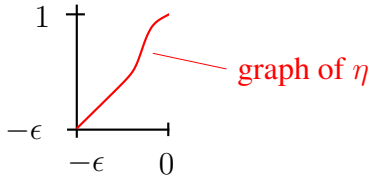
$$M \;\to\; \underbrace{(M \sqcup ([0,1]\times C))/\!\sim}_{\substack{\text{smooth manifold by the}\\ \text{Gluing-Smooth Manifolds}\\ \text{Proposition 59.9}}} \qquad \text{where } c \sim (\varphi(c),0) \text{ for } c \in C$$

In fact, given any neighborhood U of $C \subseteq M$ there exists a diffeomorphism that is the identity on the common subset $M \setminus U$ and that restricts to the obvious diffeomorphism from C to $\{1\}\times C$.



Remark. In the setting of the External Collar Lemma 247.3, we sometimes refer to the "extra space" $[0,1]\times C$ as an *external collar*. □

Proof of the External Collar Lemma 247.3. Since C is a smooth boundary of M we obtain from the Smooth Collar Neighborhood Theorem 59.7 a collar $\chi\colon [-1,0]\times C \to M$. Using this collar (and using the obvious collar for $\{0\}\times C \subseteq \partial([0,1]\times C)$) we obtain from the Gluing-Smooth Manifolds-Proposition 59.9 a smooth structure on $(M \sqcup ([0,1]\times C))/\!\sim$. Since U is a neighborhood of C and since C is compact it follows from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that $\chi([-\epsilon,0]\times C) \subseteq U$. Using the Smooth Transition Function Lemma 24.9 one can easily show that there exists a smooth function $\eta\colon [-\epsilon,0] \to [-\epsilon,1]$ with $\eta(t) = t$ for $t \in [-\epsilon, -\frac{\epsilon}{2}]$, $\eta(0) = 1$ and such that $\eta'(t) > 0$ for all t.



Finally we consider the map $M \to (M \sqcup ([0,1]\times C))/\!\sim$ that is given by the identity on $M \setminus \chi([-\epsilon,0]\times C)$ and that is given by stretching $\chi([-\epsilon,0]\times C)$ to $\chi([-\epsilon,1]\times C)$ via the map $\chi(t,P) \mapsto \chi(\eta(t),P)$. Using the Smooth Pasting Proposition 25.19 one can easily verify that this map is a diffeomorphism. ■

Proof of the Handle Decomposition–Stacking Proposition 247.2. Let

$$\begin{aligned} M &= ([0,1]\times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \\ N &= ([0,1]\times B) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \end{aligned}$$

be two handle decompositions and let $\Theta\colon B \to \partial M \setminus A$ be a diffeomorphism. We have the following diffeomorphisms:

$$\begin{aligned} M \cup_\Theta N &= M \cup_\Theta (([0,1]\times B) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s}) = (M \cup_\Theta ([0,1]\times B)) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \\ &\cong M \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s} = (([0,1]\times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r}) \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s}. \end{aligned}$$

↑ by the External Collar Lemma 247.3

■

247.2. **Handle rearrangements.** The following proposition shows that we can "slide" an l-handle off a k-handle, as long as $l \leq k$:

Proposition 247.4. (Handle Swapping Proposition) Let M be an n-dimensional smooth manifold and let

$$\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M.$$

be a thickened $(k-1)$-sphere. Furthermore let

$$\psi\colon \overline{B}^{n-l} \times S^{l-1} \to \partial(M \cup_\varphi h^k)$$

be a thickened $(l-1)$-sphere. We suppose that $l \leq k$.

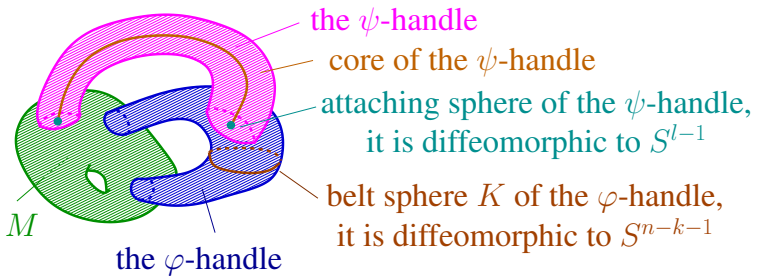
(1) There exists a smooth isotopy from ψ to a smooth embedding

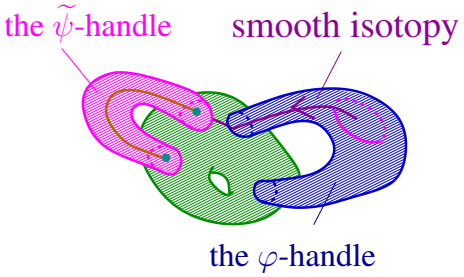
$$\widetilde{\psi}\colon \overline{B}^{n-l} \times S^{l-1} \to (\partial M) \setminus \varphi(\overline{B}^{n-k} \times S^{k-1}).$$

(2) There exists a diffeomorphism[2678]

$$M \cup_\varphi h^k \cup_\psi h^l \xrightarrow{\cong} M \cup_{\widetilde{\psi}} h^l \cup_\varphi h^k.$$

Furthermore, given any neighborhood U of ∂M one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.





Remark. Colloquially, replacing the ψ-handle by the $\widetilde{\psi}$-handle in (the proof of) the Handle Swapping Proposition 247.4 is called a *handle slide* □

The proof of the Handle Swapping Proposition 247.4 will require the remainder of this section. In the next Section 247.3 we will introduce "standard" handle decompositions.

Our proof of the Handle Swapping Proposition 247.4 relies on the following lemma which might be of independent interest:[2679].

Lemma 247.5. (Push Off-Handle Lemma) Let M be an n-dimensional smooth manifold, let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere and let W be an m-dimensional smooth manifold.

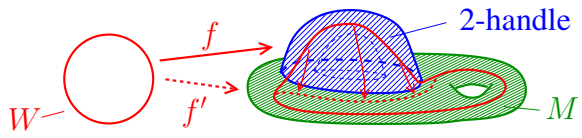
(1) If $m \leq k-1$, then every proper smooth embedding $f\colon W \to M \cup_\varphi h^k$ is smoothly isotopic to a smooth embedding $f'\colon W \to M \cup_\varphi h^k$ with $f'(W) \subseteq M$.

(2) If $m \leq k-1$, then every proper smooth embedding $f\colon W \to \partial(M \cup_\varphi h^k)$ is smoothly isotopic to a smooth embedding $f'\colon W \to \partial(M \cup_\varphi h^k)$ such that we have the inclusion $f'(W) \subseteq \partial M \setminus (\varphi(\overline{B}^{n-k} \times S^{k-1}))$.

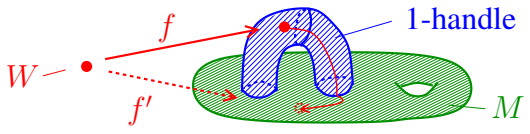
[2678]Note that it follows from the fact that $\widetilde{\psi}$ takes values in $(\partial M) \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$ that the map φ can also be used to attach a handle to $M \cup_{\widetilde{\psi}} h^l$.

[2679]The following lemma can be viewed as a smooth analog of the Ball-avoid-Cell Lemma 141.6

In both statements we can also simultaneously replace "smooth embedding" and "smooth isotopy" by "smooth map" and "smoothly homotopic".



$n = 3$, $m = 1$ and $k = 2$



$n = 3$, $m = 0$ and $k = 1$

Proof of the Push Off-Handle Lemma 247.5. In the following we will prove Statement (2), which is the only statement we need. We leave it to the reader to make the required change to obtain Statement (1). Also we leave it to the reader to verify that the "smooth map" and "smoothly homotopic" statements hold.

Let M be an n-dimensional smooth manifold, let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be an attaching map, let W be an m-dimensional smooth manifold and let $f\colon W \to \partial(M \cup_\varphi h^k)$ be a smooth embedding. We assume that $m \leq k-1$. We denote by $K := S^{n-k-1} \times \{0\} \subseteq \partial(M \cup_\varphi h^k)$ the belt sphere of the φ-handle. First we show that we can push W off the belt sphere:

Claim 1. There exists a smooth isotopy $F\colon W \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $F_0 = f$ and such that $F_1(W) \subseteq \partial(M \cup_\varphi h^k) \setminus K$.

Proof. Note that it is a consequence of the Smooth Embedding Theorem 30.15 and the Handle Attachment Proposition 246.3 (3) that $f(W)$ and K are both submanifolds of $\partial(M \cup_\varphi h^k)$. By the Transversality Theorem 50.9 there exists a smooth isotopy $F\colon W \times [0,1] \to \partial(M \cup_\varphi h^k)$ with $F_0 = f$ such that $F_1(W)$ is transverse to the belt sphere K. Recall that, according to Definition 50.3, "transverse" means that for all $P \in F_1(W) \cap K$ we have

$$\underbrace{\mathrm{T}_P(F_1(W))}_{\dim = m} + \underbrace{\mathrm{T}_P K}_{\dim = n-k-1} = \underbrace{\mathrm{T}_K \partial(M \cup_\varphi h^k)}_{\dim = n-1}.$$

Next note that

$$\dim(F_1(W)) + \dim(K) = m + n - k - 1 \underset{\substack{\uparrow \\ \text{by hypothesis} \\ \text{we have } m \leq k-1}}{\leq} n - 2 < n - 1 = \dim(\partial(M \cup_\varphi h^k)).$$

It follows immediately from the above definition of transversality and this inequality of dimensions that $F_1(W)$ is actually disjoint from the belt sphere K. ⊞

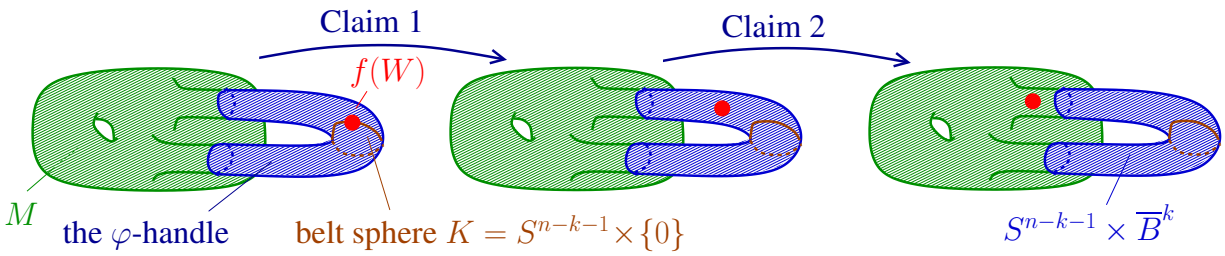
Now we can show that we can push K off the boundary of the φ-handle. In other words, the following claim gives us the desired statement:

Claim 2. There exists a smooth isotopy $F\colon W \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $F_0 = f$ and such that $F_1(W) \subseteq \partial M \setminus \varphi(S^{n-k-1} \times \overline{B}^k)$.

Proof. The key to the proof of Claim 2 is the following subclaim:

Subclaim. There exists a smooth isotopy

$$G\colon (\partial(M \cup_\varphi h^k) \setminus K) \times [0,1] \to \partial(M \cup_\varphi h^k) \setminus K$$



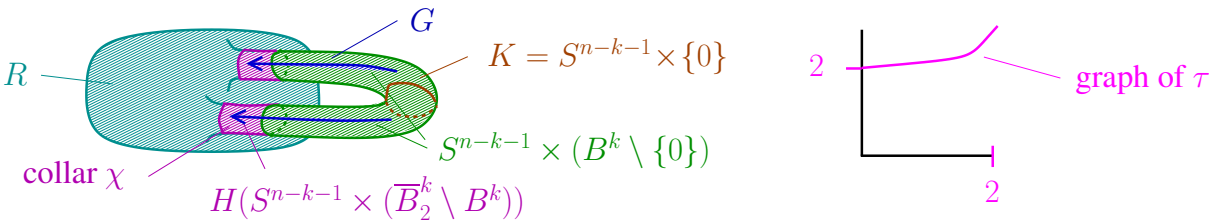
such that $G_0 = \mathrm{id}$ and such that

$$G_1(\partial(M \cup_\varphi h^k) \setminus K) \subseteq \partial(M \cup_\varphi h^k) \setminus (S^{n-k-1} \times \overline{B}^k) = \partial M \setminus \varphi(\overline{B}^{n-k} \times S^{k-1}).$$

by the Handle Attachment Proposition 246.3 (2)

Proof of subclaim. First we make a few preparations:

- Recall that in the definition of the smooth structure on $M \cup_\varphi h^k$ we picked a collar for $\varphi(\overline{B}^{n-k} \times S^{k-1}) \subseteq M^{\angle = \varphi(S^{n-k-1} \times S^{k-1})}$. Next note that it follows fairly easily from the explicit description of the gluing that this collar gives us a smooth embedding $\chi\colon \varphi(S^{n-k-1} \times S^{k-1}) \times [1,2] \to \partial M^{\angle = \varphi(S^{n-k-1} \times S^{k-1})}$ such that the map
$$\begin{array}{rcl} H\colon S^{n-k-1} \times (\overline{B}^k_2 \setminus \{0\}) & \to & \partial(M \cup_\varphi h^k) \setminus K \\ (x,y) & \mapsto & \begin{cases} [(x,y)], & \text{if } \|y\| \in (0,1], \\ [\chi(\varphi(x, \frac{y}{\|y\|}), \|y\|)], & \text{if } \|y\| \in [1,2] \end{cases} \end{array}$$
is a smooth embedding.
- We set $R := \partial M \setminus \chi(\varphi(S^{n-k-1} \times S^{k-1}) \times [0,1])$.
- One can easily show, using the Smooth Transition Function Lemma 24.9, that there exists a diffeomorphism $\tau\colon (0,2] \to (\frac{3}{2}, 2]$ with $\tau(t) = t$ for all $t \in [\frac{7}{4}, 2]$.



One can now easily verify that the desired smooth isotopy is given by

$$\begin{array}{rcl} G\colon (\partial(M \cup_\varphi h^k) \setminus K) \times [0,1] & \to & \partial(M \cup_\varphi h^k) \setminus K \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \in R, \\ H(Q, s \cdot t + (1-t) \cdot \tau(s)), & \text{if } P = H(Q,s). \end{cases} \end{array}$$ ⊞

Now we turn to the actual proof of Claim 2. First note that it follows from Claim 1 and the Smooth Isotopy-Equivalence Relation Proposition 41.5 that we can assume that $f(W) \subseteq \partial(M \cup_\varphi h^k) \setminus K$. But now the desired smooth isotopy is given by

$$\begin{array}{rcl} W \times [0,1] & \to & \partial(M \cup_\varphi h^k) \setminus K \\ (w,t) & \mapsto & G(f(w),t). \end{array}$$ ■

Proof of the Handle Swapping Proposition 247.4. Suppose that we are given a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ and suppose that we are given a thickened $(l-1)$-sphere $\psi\colon \overline{B}^{n-l} \times S^{l-1} \to \partial(M \cup_\varphi h^k)$ with $l \leq k$.

(1) Recall that we need to prove the following claim:

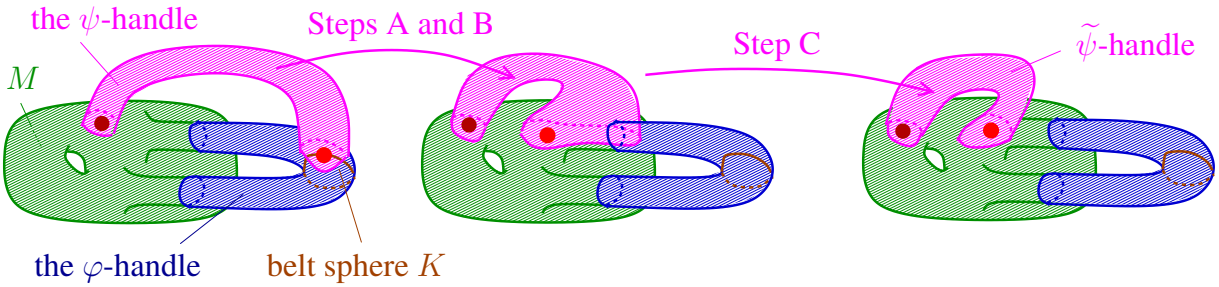
Claim. There exists a smooth isotopy
$$\Psi\colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] \;\to\; \partial(M \cup_\varphi h^k)$$
with $\Psi_0 = \psi$ and such that $\Psi_1(\overline{B}^{n-l} \times S^{l-1}) \subseteq (\partial M) \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$.

Proof. We define $f\colon S^{l-1} \to \partial(M \cup_\varphi h^k)$ via $f(x) = \psi(x,0)$. Note that $f(S^{l-1})$ is precisely the attaching sphere of the ψ-handle.

(A) First note that it follows from $l \leq k$ that we can apply the Push Off-Handle Lemma 247.5 to the smooth embedding $f\colon S^{l-1} \to \partial(M \cup_\varphi h^k)$. Therefore we obtain a smooth isotopy $F\colon S^{l-1} \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $F_0 = f$ and such that $F_1(S^{l-1}) \subseteq \partial M \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$.

(B) Note that using the Isotopy Extension Theorem 44.2 we can extend F to a smooth isotopy $G\colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] \to \partial(M \cup_\varphi h^k)$.

(C) Unfortunately a priori we do not know yet that we have the desired inclusion $G_1(\psi(\overline{B}^{n-l} \times S^{l-1})) \subseteq \partial M \setminus \varphi(\overline{B}^{k} \times S^{n-k-1})$. To fix this we observe the following: It follows immediately from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that $G_1(\psi(\overline{B}^{n-l}_\epsilon \times S^{l-1})) \subseteq \partial M \setminus \varphi(S^{n-k-1} \times \overline{B}^{k})$. One can now easily verify that the map
$$\begin{aligned} \Psi\colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] &\to \partial(M \cup_\varphi h^k) \\ ((v,P),t) &\mapsto G(\psi(\underbrace{((1-t)+\epsilon\cdot t)\cdot v, P)}_{\in \overline{B}^{n-l} \times S^{l-1}}, t) \end{aligned}$$
has all the desired properties.



These three steps are illustrated above. ⊞

(2) We set $\widetilde{\psi} := \Psi_1$. We see that
$$M \cup_\varphi h^k \cup_\psi h^l \;\underset{\text{by the Handle Attachment Proposition 246.3 (5)}}{\cong}\; M \cup_\varphi h^k \cup_{\widetilde{\psi}} h^l \;\underset{\text{since } \varphi \text{ and } \psi \text{ have disjoint images in } \partial M}{\cong}\; M \cup_{\widetilde{\psi}} h^l \cup_\varphi h^k.$$

The extra statement regarding an open neighborhood U of ∂M is fortunately enough taken care of by the Handle Attachment Proposition 246.3 (5). ■

247.3. **Standard handle decompositions.** The Handle Swapping Proposition 247.4 leads us to the following definition:

Definition 247.6. Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M.

(1) (a) We say that a handle decomposition

$$\Theta\colon M \xrightarrow{\cong} ([0,1]\times A)\cup_{\varphi_1} h^{k_1}\cup\cdots\cup_{\varphi_m} h^{k_m}$$

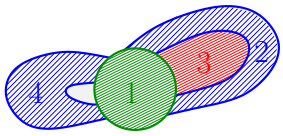
of M rel A is non-decreasing if for every $i\in\{1,\dots,m-1\}$ we have $k_i\le k_{i+1}$.

(b) Given a non-decreasing handle decomposition we define the i-skeleton M^i of the relative handle decomposition as the union of $[0,1]\times A$ together with all handles of index $\le i$.

(2) We say that a handle decomposition of M rel A is standard if the following two conditions are satisfied:

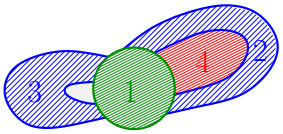
(a) The handle decomposition is non-decreasing.

(b) For each i-handle the attaching map takes values in $(\partial W^{i-1})\setminus(\{0\}\times A)$, i.e. it takes values in the part of the boundary of the $(i-1)$-skeleton of the handle decomposition which does not belong to $\{0\}\times S$.

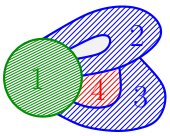


non-standard handle decomposition

violates (a)



standard handle decomposition



non-standard handle decomposition

violates (b)

Proposition 247.7. (Standard Handle Decomposition Proposition) Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M. If M admits a handle decomposition rel A, then it also admits a *standard* handle decomposition rel A with the same number of handles of each index.

In the proof of the Standard Handle Decomposition Proposition 247.7 we will need the following elementary lemma:

Lemma 247.8. (Modify Handle Attachment Region Lemma) Suppose we are given a handle decomposition

$$\Theta\colon W \xrightarrow{\cong} M\cup_{\varphi_1} h^{k_1}\cdots\cup_{\varphi_m} h^{k_m}$$

[2680]Here we use the following observation: If $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial M$ and $\psi\colon \overline{B}^{n-l}\times S^{l-1}\to\partial M$ are two attaching maps with disjoint images in ∂M, then it does not matter in what order we attach these two handles. This can be seen as follows: First of all we apply the Double Collar Theorem 61.2 to $\varphi(S^{n-k-1}\times S^{k-1})\sqcup\psi(S^{n-l-1}\times S^{l-1})$ and secondly we apply the Smooth Collar Neighborhood Theorem 59.7 to $\varphi(\overline{B}^{n-k}\times S^{k-1})\sqcup\psi(\overline{B}^{n-l}\times S^{l-1})$ in $M^{\angle=\varphi(S^{n-k-1}\times S^{k-1})\sqcup\psi(S^{n-l-1}\times S^{l-1})}$. If we use the restrictions of these collars to the two individual pieces it becomes clear that the order of the handle attachment is essentially irrelevant.

and suppose that we are given a diffeomorphism $f\colon M \to M'$. Then there exists also a handle decomposition

$$\Theta'\colon W \xrightarrow{\cong} M' \cup_{\widetilde{\varphi}_1} h^{k_1} \cdots \cup_{\widetilde{\varphi}_m} h^{k_m}.$$

Proof of the Modify Handle Attachment Region Lemma 247.8. We just use the diffeomorphism $f\colon M \to M'$ to turn the old attaching maps φ_i into attaching maps $\widetilde{\varphi}_i$. ■

Proof of the Standard Handle Decomposition Proposition 247.7. In the following we will explain how we can turn any handle decomposition into a standard decomposition, i.e. into a handle decomposition that satisfies the above Conditions (a) and (b). We will do so in the following two claims:

Claim A. There exists a handle decomposition rel A that is non-decreasing.

Proof. Given a handle decomposition

$$\Theta\colon M \xrightarrow{\cong} ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

we define its *deficiency* as $d(\Theta) = \#\{(i,j) \in \{1,\dots,m\} \,|\, i < j \text{ but } k_i > k_j\}$. Our goal is to find a handle decomposition of deficiency zero. It suffices to show that for each handle decomposition of deficiency $d > 0$ one can find a handle decomposition of deficiency $\leq d-1$. Suppose we are given a handle decomposition (with the same notation as above), of deficiency $d > 0$. An elementary argument shows there must exist an $i \in \{1,\dots,m-1\}$ with $k_i > k_{i+1}$. Then

$$\begin{aligned} M &\cong ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}} \cup_{\varphi_i} h^{k_i} \cup_{\varphi_{i+1}} h^{k_{i+1}} \cup_{\varphi_{i+2}} h^{k_{i+2}} \cup \cdots \cup_{\varphi_m} h^{k_m} \\ &\underset{\text{by the Handle Swapping Proposition 247.4 together with the Modify Handle Attachment Region Lemma 247.8}}{\cong} ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}} \cup_{\widetilde{\varphi}_{i+1}} h^{k_{i+1}} \cup_{\varphi_i} h^{k_i} \cup_{\widetilde{\varphi}_{i+2}} h^{k_{i+2}} \cup \cdots \cup_{\widetilde{\varphi}_m} h^{k_m}. \end{aligned}$$

Elementary logic shows that the deficiency of the new presentation is less than before. ⊞

Claim B. There exists a handle decomposition rel A that is standard.

Proof. By Claim 1 we can start out with a non-decreasing handle decomposition. We need to show that we can arrange the handle decomposition such that at the end for each index i and each i-handle the attaching map takes values in ∂M^{i-1}.

Suppose that this desired outcome already holds for all indices $i < k$. Now it remains to rearrange the indices of the same degree k. We can write

$$M = M^{k-1} \cup_{\varphi_1} h^k_1 \cdots \cup_{\varphi_m} h^k_m \cup \text{handles of higher index.}$$

We equip $\{1,\dots,m\}^2$ with the lexicographic ordering. For each subset $S \subseteq \{1,\dots,m\}^2$ we denote by $\min(S) \in \{1,\dots,m\}^2 \cup \{\infty\}$ its minimum. Here, as in Notation 1.35, we write $\min(S) = \infty$ precisely if $S = \varnothing$.

We consider the set

$$S := \{(i,j) \in \{1,\dots,m\}^2 \,|\, i < j \text{ and } \varphi_j(\overline{B}^{n-k} \times S^{k-1}) \cap h^k_i \neq \varnothing\}.$$

If $S = \varnothing$, then there is nothing to show. So it remains to show that if $S \neq \varnothing$, then we can rearrange the handles such that $\min(S)$ increases.

Let $(i,j) = \min(S)$. We consider

$$W := M^{k-1} \setminus \bigcup_{r=1,\dots,i-1,i+1,\dots,j-1} \varphi_r(\overline{B}^{n-k} \times S^{k-1}).$$

Note that this is an open subset of M^{k-1}, so W is an n-dimensional smooth manifold. By choice of (i,j) we know that φ_i, φ_j take values in ∂W. We can thus apply the Handle Swapping Proposition 247.4 to W, φ_i and φ_j.



We obtain that there exists a smooth isotopy $H\colon \overline{B}^{n-k} \times S^{k-1} \times [0,1] \to \partial W$ with $H_0 = \varphi_j$ and such that $H_1(\overline{B}^{n-k} \times S^{k-1}) \subseteq \partial W \setminus \varphi_i(\overline{B}^{n-k} \times S^{k-1})$. One can now easily verify (using the Modify Handle Attachment Region Lemma 247.8) that if we replace φ_j by $\widetilde{\varphi}_j := H_1$, then we obtain a handle decomposition of the same diffeomorphism type, but where $\min(S)$ went up. ■

The Standard Handle Decomposition Proposition 247.7 allows us to give a straightforward proof of the following lemma:

Lemma 247.9. (1-Handle–π_1–Inequality Lemma) Let M be a connected non-empty smooth manifold. For every handle decomposition of M we have the inequality:

$$1 + \#\text{1-handles} - \#\text{0-handles} \;\geq\; \text{minimal size of a generating set of } \pi_1(M).$$

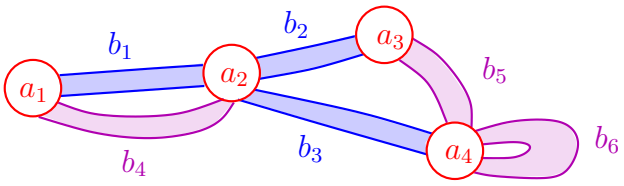
Proof. Let M be a compact connected non-empty smooth manifold that is equipped with a handle decomposition. We denote by r the number of 0-handles and we denote by s the number of 1-handles.

First note that by the Standard Handle Decomposition Proposition 247.7 we can assume that the handle decomposition is standard. Given $k \in \mathbb{N}_0$ we denote by M^k the k-skeleton of M.

We pick a base point P in one of the 0-handles. By the definition of a generating set of a group, see Definition 98.29, it suffices to prove that there exists an epimorphism from a free group of rank $1+s-r$ onto $\pi_1(M,P)$. This desired statement is a consequence of the following two claims:

Claim 1. The fundamental group $\pi_1(M^1, P)$ is a free group on $1+s-r$ generators.

Proof. First note that it follows fairly easily from our hypothesis that M is connected, together with the Handle Attachment-π_0-Lemma 246.5, that we can order the 0-handles as $a_1, \ldots, a_r$ with $P \in a_1$ and that we can find $r-1$ 1-handles $b_1, \ldots, b_{r-1}$ such that the attaching map of each b_i takes values in $(a_1 \cup \cdots \cup a_i) \cup (b_1 \cup \cdots \cup b_{i-1})$ and also in a_{i+1}. It follows from iteratively applying the Handle Attachment-π_0-Lemma 246.5 and the Handle Attachment-π_1-Proposition 246.13 that each $(a_1 \cup \cdots \cup a_i) \cup (b_1 \cup \cdots \cup b_{i-1})$ is simply connected. We denote by $b_r, \ldots, b_s$ the remaining 1-handles.



Now note that

$$\begin{aligned}\pi_1(M^1,P) &= \pi_1((a_1\cup\cdots\cup a_r)\cup(b_1\cup\cdots\cup b_{r-1})\cup(b_r\cup\cdots\cup b_s),P)\\ &\cong \underbrace{\pi_1((a_1\cup\cdots\cup a_r)\cup(b_1\cup\cdots\cup b_{r-1}))}_{=\{e\}} * \langle x_r\rangle * \ldots * \langle x_s\rangle = \underbrace{\langle x_r,\ldots,x_s\rangle}_{\text{free group of rank } 1+s-r}.\end{aligned}$$

$\uparrow$ by the Handle Attachment π_1-Proposition 246.13

⊞

Claim 2. The inclusion induced map $\pi_1(M^1,P)\to\pi_1(M,P)$ is an epimorphism.

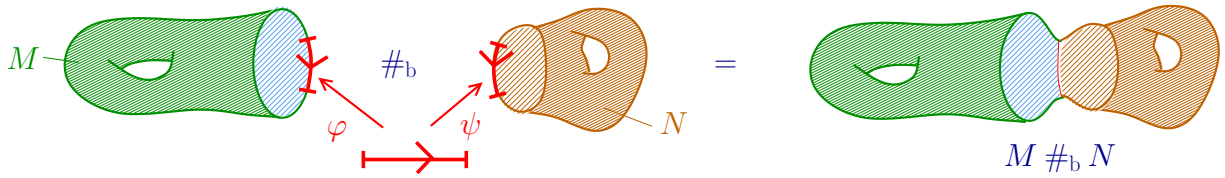
Proof. Note that we obtain M from M^1 by attaching handles of index ≥ 2. The claim follows from this observation together with the Handle Attachment-π_1-Proposition 246.13. ■

247.4. **The boundary connected sum operation.** We start out by recalling Definition 61.10:

Definition 247.10. Let $n\in\mathbb{N}$, let M and N be two n-dimensional smooth manifolds and let $\varphi\colon \overline{B}^{n-1}\to\partial M$ and $\psi\colon \overline{B}^{n-1}\to\partial N$ be two smooth embeddings.[2681] We refer to

$$M\#_{\mathrm{b}}N := (M^{\angle=\varphi(S^{n-2})}\sqcup N^{\angle=\psi(S^{n-2})})/\sim \quad \text{where } \varphi(P)\sim\psi(P) \text{ for all } P\in\overline{B}^{n-1}$$

as a boundary connected sum of M and N.



The following proposition shows that we can express boundary connected sums through handle attachments:

Proposition 247.11. (Boundary Connected Sum-via-Handle Proposition) Let M and N be n-dimensional smooth manifolds and let $\varphi\colon \overline{B}^{n-1}\to\partial M$ and $\psi\colon \overline{B}^{n-1}\to\partial N$ be smooth embeddings. We use these to define the boundary connected sum $M\#_{\mathrm{b}}N$. Next we consider the thickened 0-sphere

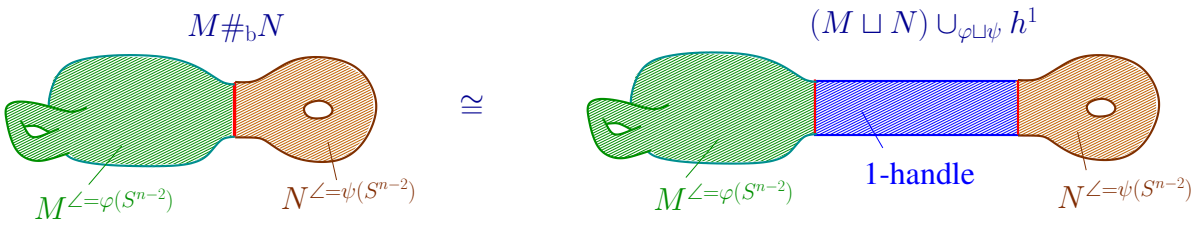
$$\begin{aligned}\varphi\sqcup\psi\colon \overline{B}^{n-1}\times S^0 &\to \partial(M\sqcup N)\\ (x,\epsilon) &\mapsto \begin{cases}\varphi(x), & \text{if } \epsilon=-1,\\ \psi(x), & \text{if } \epsilon=1.\end{cases}\end{aligned}$$

There exists a diffeomorphism

$$M\#_{\mathrm{b}}N \to (M\sqcup N)\cup_{\varphi\sqcup\psi} h^1.$$

In fact, given any open neighborhood U of $\partial(M\sqcup N)$ there exists a diffeomorphism that is the identity on the complement of U.

[2681] In contrast to the original Definition 61.10 we do not ask for any specific orientation-behavior of φ and ψ. This extra data was originally necessary to get a notion of boundary connected sum that is independent of the choices of φ and ψ. But in our current setting we do not care about this slight ambiguity.



Proof. We have the following identities and diffeomorphisms:

$$\begin{array}{rcl} & & \text{by Definition 247.10 of the boundary connected sum} \\ & \downarrow & \\ M\#_{\mathrm{b}}N & = & M^{\angle=\varphi(S^{n-2})} \cup_{\varphi(x)=\psi(x)} N^{\angle=\psi(S^{n-2})} \\ & \cong & M^{\angle=\varphi(S^{n-2})} \cup_{\varphi(x)=(x,-1)} (\overline{B}^{n-1} \times [-1,1]) \cup_{(x,1)=\psi(x)} N^{\angle=\psi(S^{n-2})} \\ & \uparrow & \\ & & \text{by the External Collar Lemma 247.3} \\ & = & (M \sqcup N) \cup_{\partial(M\sqcup N) \xleftarrow{\varphi\sqcup\psi} \overline{B}^{n-1}\times S^0} \underbrace{(\overline{B}^{n-1} \times \overline{B}^1)}_{=h^1}. \\ & \uparrow & \\ & & \text{by definition} \end{array}$$

■

The Boundary Connected Sum-via-Handle Proposition 247.11, together with earlier results on boundary connected sums, allows us to prove the following useful proposition:

Proposition 247.12. (0-Handle Cancellation Proposition) Let M be an n-dimensional smooth manifold. We do the following:

- First we attach a 0-handle to obtain $M \cup h^0 = M \sqcup \overline{B}^n$.
- Afterwards we attach a 1-handle along an attaching map of the form

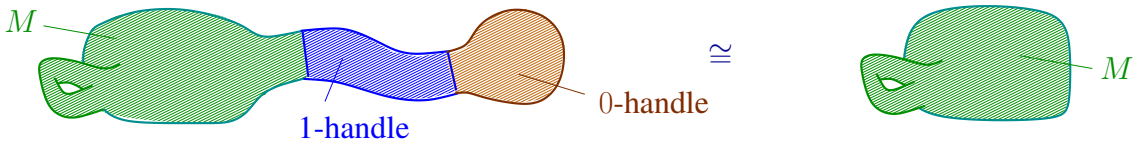
$$\begin{array}{rcl} \varphi \sqcup \psi \colon \overline{B}^{n-1} \times S^0 & \to & \partial(M \sqcup N) \\ (x,\epsilon) & \mapsto & \begin{cases} \varphi(x), & \text{if } \epsilon = -1, \\ \psi(x), & \text{if } \epsilon = 1 \end{cases} \end{array}$$

  where $\varphi \colon \overline{B}^{n-1} \to \partial M$ and $\psi \colon \overline{B}^{n-1} \to \partial \overline{B}^n$ are smooth embeddings.

Then there exists a diffeomorphism

$$(M \cup h^0) \cup_{\varphi\sqcup\psi} h^1 \xrightarrow{\cong} M.$$

In fact, given any open neighborhood U of ∂M there exists a diffeomorphism that is the identity on the complement of U.



Remark. In the 0-Handle Cancellation Proposition 247.12 we just gave a criterion for a 0-handle and a 1-handle to "cancel each other out". In the Handle Cancellation Theorem 260.1 we will give a criterion for a k-handle and a successive $(k+1)$-handle to cancel each other out. □

Proof. We have the following diffeomorphisms:

$$(M \cup h^0) \cup_{\varphi \sqcup \psi} h^1 \underset{\text{by Definition 246.4}}{=} (M \sqcup \overline{B}^n) \cup_{\varphi \sqcup \psi} h^1 \underset{\text{follows from the Boundary Connected Sum via-Handle Proposition 247.11}}{\cong} M \#_{\mathrm{b}} \overline{B}^n \underset{\text{by the Boundary Connected Sum-with Ball Proposition 61.13}}{\cong} M.$$

■

The 0-Handle Cancellation Proposition 247.12 gives us the tools for proving the following proposition, which lets us reduce the number of 0-handles in a relative handle decomposition to the absolute minimum:

Proposition 247.13. (Minimal 0-Handle Proposition) Let M be a compact connected non-empty n-dimensional smooth manifold.

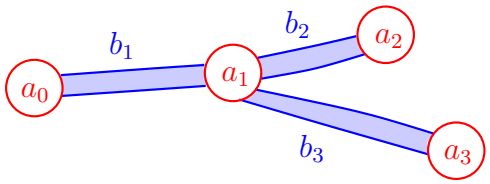
(1) Given any handle decomposition of M with d_i i-handles there exists a handle decomposition of M with the following properties:
- The number of 0-handles equals one.
- The number of 1-handles equals $d_1 - d_0 + 1$.
- The number of handles of index $\neq 0, 1$ is unchanged.

(2) Let A be a non-empty union of components of ∂M. Given any handle decomposition of M rel A with d_i i-handles there exists a handle decomposition of M rel A with the following properties:
- The number of 0-handles equals zero.
- The number of 1-handles equals $d_1 - d_0$.
- The number of handles of index $\neq 0, 1$ is unchanged.

Proof. Let M be a compact connected non-empty n-dimensional smooth manifold.

(1) Suppose we are given a handle decomposition of M. It follows from the Standard Handle Decomposition Proposition 247.7 that we might as well assume that the handle decomposition is standard. Note that the only handle we can attach to the empty set is the 0-handle. Since M is non-empty, we see that M has at least one 0-handle.

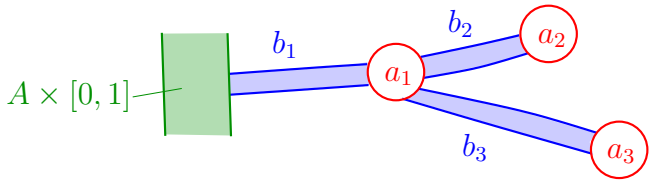
Since the smooth manifold M is connected we see, as in the proof of the 1-Handle–π_1–Inequality Lemma 247.9, that we can order the 0-handles as $a_0, \ldots, a_r$ and that we can find r 1-handles $b_1, \ldots, b_r$ such that the attaching map of each b_i takes values in $(a_0 \cup \cdots \cup a_{i-1}) \cup (b_1 \cup \cdots \cup b_{i-1})$ and also in a_i.



Now we see that we can apply the 0-Handle Cancellation Proposition 247.12 $r-1$-times to reduce the number of 0-handles to one. In the same process we reduced the number of 1-handles by $r - 1$. We have now seen that the 0-skeleton is diffeomorphic to $\overline{B}^n$. The desired statement follows from this observation together with the Modify Handle Attachment Region Lemma 247.8.

(2) Let A be a non-empty union of components of ∂M. Suppose we are given a handle decomposition of M rel A. Similar to (1) one can order the 0-handles as $a_1, \ldots, a_r$ and we can find r 1-handles $b_1, \ldots, b_r$ such that the attaching map of each b_1 takes

values in $A \times \{1\}$ and b_1 and for each $i \in \{2, \ldots, r-1\}$ the map b_i takes values in $(a_0 \cup \cdots \cup a_{i-1}) \cup (b_1 \cup \cdots \cup b_{i-1})$ and also in a_i.



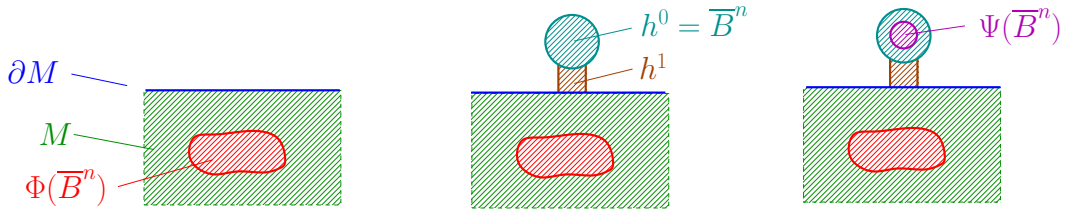
By the argument of (1) we can now dispose of all 0-handles $a_1, \ldots, a_m$ and we can reduce the number of 1-handles by r. But this time we thus disposed of all 0-handles. ■

Finally, for the record we also mention the following slightly curious consequence of the 0-Handle Cancellation Proposition 247.12:

Proposition 247.14. (Manifold-with-Boundary-minus-Ball Proposition) Let M be a connected n-dimensional smooth manifold with non-empty boundary. Furthermore let $\Phi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding. Then the following two statements hold:
(1) The smooth manifold $M \setminus \Phi(B^n)$ is diffeomorphic to $M \#_{\mathrm{b}} (\overline{B}^1 \times S^{n-1})$.
(2) The smooth manifold $M \setminus \Phi(B^n)$ is homotopy equivalent to $M \vee S^{n-1}$.

Proof. Let M be a connected n-dimensional smooth manifold with non-empty boundary. We start out with a two preparations:

(a) It follows from the 0-Handle Cancellation Proposition 247.12 that we have a diffeomorphism $M \cong (M \sqcup \overline{B}^n) \cup h^1$ which is the identity outside of any given neighborhood of ∂M.
(b) Let $\Psi\colon \overline{B}^n \to \overline{B}^n_{\frac{1}{2}} \subseteq (M \sqcup \overline{B}^n) \cup h^1$ be the obvious smooth embedding.



We now see that we have the following diffeomorphisms and homotopy equivalences:

$$M \setminus \Phi(B^n) \underset{\uparrow}{\cong} ((M \sqcup \overline{B}^n) \cup h^1) \setminus \Phi(B^n) \underset{\uparrow}{\cong} ((M \sqcup \overline{B}^n) \cup h^1) \setminus \Psi(B^n)$$

by the 0-Handle Cancellation Proposition 247.12 / follows from the Smooth Ball Embedding Theorem 45.10

$$\underset{\uparrow}{\cong} (M \sqcup (\overline{B}^n \setminus B^n_{\frac{1}{2}})) \cup h^1 \qquad \underset{\uparrow}{\cong} (M \sqcup (\overline{B}^1 \times S^{n-1})) \cup h^1$$

definition of Ψ / since $\overline{B}^n \setminus B^n_{\frac{1}{2}}$ is diffeomorphic to $\overline{B}^1 \times S^{n-1}$

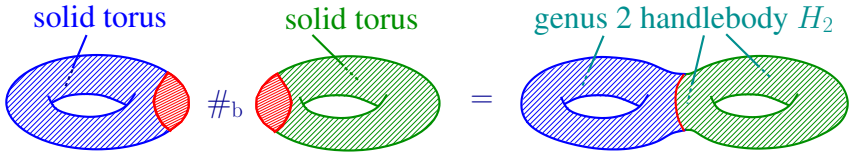
$$\underset{\uparrow}{\cong} M \#_{\mathrm{b}} (\overline{B}^1 \times S^{n-1}) \qquad \underset{\uparrow}{\simeq} M \vee (\overline{B}^1 \times S^{n-1}) \qquad \cong M \vee S^{n-1}.$$

by the Boundary Connected Sum-via-Handle Proposition 247.11 / Exercise 61.3

■

247.5. **Handlebodies.** We start out by recalling the following definition from page 1374:

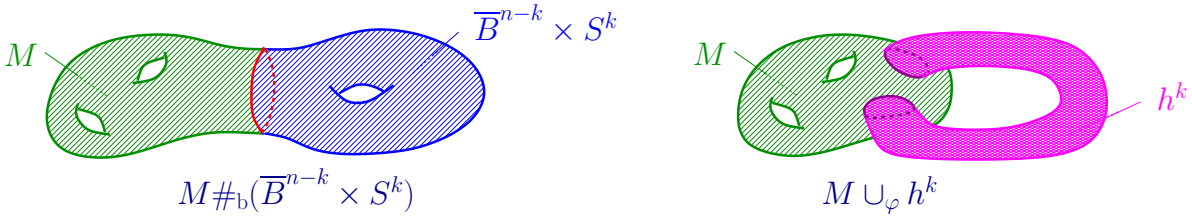
Definition 247.15. Any 3-dimensional smooth manifold that is diffeomorphic to the boundary connected sum of g copies of $\overline{B}^2 \times S^1$ is called a genus g-handlebody.



The name "handlebody", and also the figure, suggest that this notion of a handlebody can be expressed in terms of handles. The key to showing this is the following proposition:

Proposition 247.16. (Boundary Connected Sum-with-Solid Torus Proposition) Let M be an n-dimensional smooth manifold. There exists a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ and a diffeomorphism

$$M \#_b (\overline{B}^{n-k} \times S^k) \xrightarrow{\cong} M \cup_\varphi h^k.$$



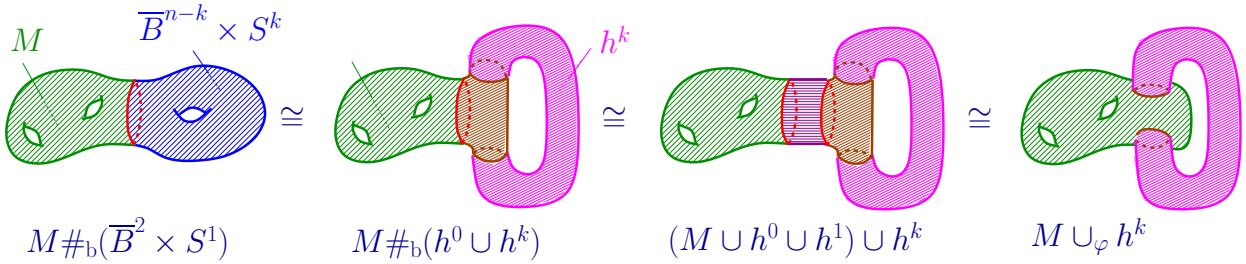
Proof. We have the following diffeomorphisms:

$$M\#_b(\overline{B}^{n-k} \times S^k) \underset{\uparrow}{\cong} M\#_b(h^0 \cup h^k) \underset{\uparrow}{\cong} \overbrace{(M \cup h^0 \cup h^1) \cup h^k}^{\cong (M \cup h^0) \cup h^k) \cup h^1} \underset{\uparrow}{\cong} M \cup h^k.$$

by the Ball-Handle Attachment Lemma 246.9, note that we can arrange that the gluing map takes values in h^0

follows from the Boundary Connected Sum-via-Handle Proposition 247.11

by the 0-Handle Cancellation Proposition 247.12

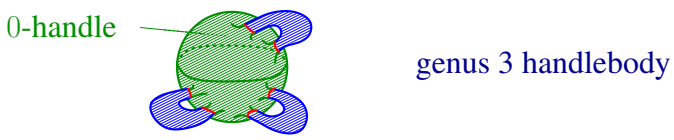


In the figure above we give a "cartoon version" of the proof. ■

We conclude this section with the following definition, which gets used frequently in topology:

Definition 247.17. Let $k \in \mathbb{N}_{\geq 2}$. Any k-dimensional smooth manifold that is diffeomorphic to the result of attaching g 1-handles to $\overline{B}^k$ is called a genus g k-dimensional handlebody.

Example. It follows easily from the Minimal 0-Handle Proposition 247.13 that every connected non-empty smooth manifold that admits a handle decomposition consisting only of



r 0-handles, s 1-handles, and no handles of higher index, is in fact a handlebody of genus $s - r + 1$. □

Lemma 247.18. (Orientable Handlebody Lemma) Let $k \in \mathbb{N}_{\geq 2}$ and let $g \in \mathbb{N}_0$.

(1) Any two orientable genus g k-dimensional handlebodies are diffeomorphic.
(2) Any orientable genus g k-dimensional handlebody is diffeomorphic to the iterated boundary connected sum of g copies of $\overline{B}^{k-1} \times S^1$.[2682]
(3) The boundary of an orientable k-dimensional genus g-handlebody is diffeomorphic to the connected sum of g copies of $S^{k-2} \times S^1$.[2683]

Proof.

(1) This statement follows easily from the Handles-Orientability Proposition 246.6, the 1-Handle Attachment Lemma 246.8 and the Handle Swapping Proposition 247.4.
(2) This statement follows easily from Statement (1) and the Boundary Connected Sum-via-Handle Proposition 247.11.
(3) It follows from the Boundary Connected Sum Proposition 61.11 (2) and Statement (2) that the boundary of an orientable k-dimensional genus g-handlebody is diffeomorphic to the connected sum of g copies of $S^{k-2} \times S^1$. ■

247.6. Exercises for Chapter 247.

Exercise 247.1. We consider $M := S^1 \times S^1 \times [0,1]$.

(a) Provide a handle decomposition of M rel $\varnothing$.
(b) Provide a handle decomposition of M rel $S^1 \times S^1 \times \{0\}$.
(c) Provide a handle decomposition of M rel ∂M.

Exercise 247.2. Let M be a closed n-dimensional smooth manifold that is equipped with a handle decomposition such that for each $k \in \mathbb{N}_0$ we have d_k k-handles. Show that there exists a handle decomposition of $M \times [0,1]$ rel $\partial(M \times [0,1])$ such that for each $k \in \mathbb{N}_0$ we have d_k $(k+1)$-handles.

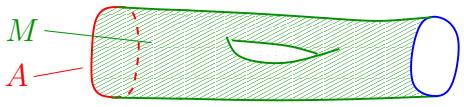
Exercise 247.3. Let M be a compact smooth manifold and let A be a union of components of ∂M. We assume that M admits a handle decomposition rel A such that all handles are of index $\leq k$. Prove the following two statements:

(a) For each $i > k$ we have $\mathrm{H}_i(M, A) = 0$.
(b) $\mathrm{H}_k(M, A)$ is a free abelian group.

[2682] This shows that our new definition of a handlebody is consistent with the earlier definition from page 1374.
[2683] For $k = 3$ it follows from this observation and the Surface–Connected Sum Lemma 54.15 that the boundary of an orientable genus g-handlebody is diffeomorphic to the surface of genus g. This is of course clear from the pictures. But it is not that easy to provide a convincing argument for this fact.

Exercise 247.4. Let M be a non-empty smooth manifold that admits a handle decomposition. Show that it also admits infinitely many handle decompositions.

Exercise 247.5. In the figure below we show once again M, the torus with two disks removed. Let A be one of the two boundary components. In the figure on page 4767 we showed that M admits a handle decomposition rel A with one 0-handle, three 1-handles and one 2-handle. Does there exist a handle decomposition of M rel A with fewer handles?



Exercise 247.6. Let M be a connected smooth manifold that is equipped with a standard handle decomposition. For $i \in \mathbb{N}_0$ we denote by M^i its i-skeleton, i.e. the union of all handles of index $\leq i$. Next let x_0 be a point in the 0-skeleton of M and let $j \in \mathbb{N}_0$.
(a) Show that the inclusion-induced map $\pi_j(M^j, x_0) \to \pi_j(M, x_0)$ is an epimorphism.
(b) Show that for any $k \geq j+1$ the inclusion-induced map $\pi_j(M^k, x_0) \to \pi_j(M, x_0)$ is an isomorphism.

Hint. Use the Whitney Approximation Theorem 36.1, the Replace-by-Smooth Homotopy Proposition 36.6 and Transversality Theorem 50.9 applied to a suitable choice of proper submanifolds.
Remark. This exercise can be viewed as a handle analog of the corresponding statement for CW-complexes that we proved in the CW-Skeleton-π_1-Proposition 136.22 and the CW-Skeleton-π_n-Proposition 141.3.

Exercise 247.7. Let Y be a k-dimensional smooth manifold. Show that Y is a genus g handlebody if and only if there exist g properly smoothly embedded k-balls $B_1, \ldots, B_k \subseteq Y$ such that $Y \setminus\setminus (B_1 \sqcup \cdots \sqcup B_k)$ is diffeomorphic to a ball.

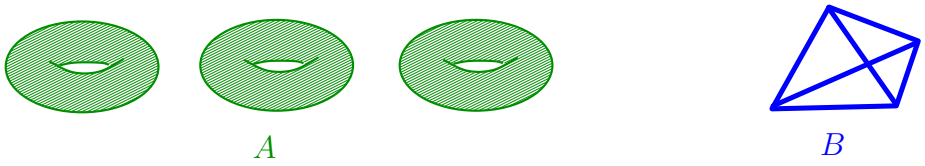
Exercise 247.8. Let M be a smooth manifold that is equipped with a standard handle decomposition. For $j \in \mathbb{N}_0$ we denote by M^j its j-skeleton, i.e. the union of all handles of index $\leq j$. Determine the homology groups $\mathrm{H}_*(M^i, M^{i-1})$.

Exercise 247.9. Let $k \in \mathbb{N}_{\geq 2}$ and let $g \in \mathbb{N}_0$. Determine the homology groups of any genus g k-dimensional handlebody.

Exercise 247.10.
(a) Are any two orientable genus g 2-dimensional handlebodies diffeomorphic?
(b) Are any two orientable genus g 3-dimensional handlebodies diffeomorphic?

Exercise 247.11.
(a) Are any two non-orientable genus g 2-dimensional handlebodies diffeomorphic?
(b) Are any two non-orientable genus g 3-dimensional handlebodies diffeomorphic?

Exercise 247.12. In the figure below we show two open subsets of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$, namely we show A, the union of three open solid tori and we show B, a suitable open neighborhood of a graph in $\mathbb{R}^3$.
(a) Describe a handle decomposition of $S^3 \setminus A$ rel $\varnothing$ and rel $\partial(S^3 \setminus A)$.

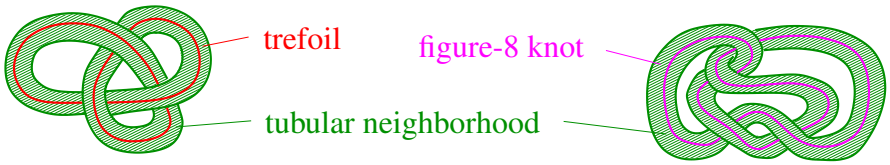


(b) Describe a handle decomposition of $S^3 \setminus B$ rel $\varnothing$ and rel $\partial(S^3 \setminus B)$.

Exercise 247.13. Let Σ be a closed connected orientable 2-dimensional smooth manifold and let $\Phi\colon \overline{B}^3 \to \Sigma \times (0,1)$ be an orientation-preserving smooth embedding. We consider the smooth manifold $M := (\Sigma \times [0,1]) \setminus \Phi(B^3)$.
(a) Provide an explicit handle decomposition for M rel $\Sigma \times \{0\}$.
(b) Provide an explicit handle decomposition for M rel $(\Sigma \times \{0\} \cup \Phi(S^2)$.

Remark. Note that it follows from the Smooth Ball Embedding Theorem 45.10 that all such smooth embeddings of the 3-ball are smoothly isotopic.

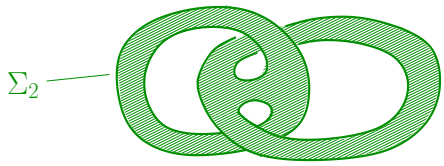
Exercise 247.14. Given any knot K we can consider, as in Definition 108.30, the knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$ where $\tau\colon \overline{B}^2 \times K \to S^3$ is a tubular map for K.
(a) Find an explicit handle decomposition for the exterior of the trefoil.
(b) Find an explicit handle decomposition for the exterior of the figure-8 knot.



Exercise 247.15. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We consider the torus knot $T := T(p,q)$ from Definition 111.13. Recall that it has the feature that it lies on the standard solid torus in $\mathbb{R}^3$. We pick a tubular map $\tau\colon \overline{B}^2 \times T(p,q) \to S^3$ and we consider the corresponding knot exterior $X_T := S^3 \setminus \tau(B^2 \times T(p,q))$. Show that the exterior X_K admits a handle decomposition with one 0-handle, one 1-handle and one 2-handle.
Hint. Take a look at the proof of the Torus Knot-π_1-Proposition 111.18.



Exercise 247.16. We consider the surface $\Sigma_2 \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ that is shown below. The surface Σ_2 splits S^3 into two compact codimension-zero submanifolds A and B with $A \cap B = \Sigma_2$. Are both A and B handlebodies?


Σ_2

CHAPTER 248

Dual handle decomposition

Let M be an n-dimensional smooth manifold with a decomposition $\partial M = A \sqcup B$ of its boundary into unions of components. In this chapter we will see that we can associate to each handle decomposition of M rel A a "dual" handle decomposition of M rel B. The dual handle decomposition reverses the "fortunes" of handles: Whatever was an i-handle in the original handle decomposition becomes an $(n-i)$-handle in the dual handle decomposition.

The dual handle decomposition has many uses. It can in particular be used to show that if a closed connected non-empty n-dimensional smooth manifold admits a handle decomposition, then it also admits a handle decomposition with a single 0-handle and a single n-handle.

This chapter is organized as follows:

- In Section 248.1 we will introduce the notion of a dual handle decomposition.
- In Section 248.2 we will discuss examples and applications of dual handle decompositions. In particular we will see that we can minimize the number of 0-handles and of n-handles at the same time.
- In Section 248.3 we will discuss splittings of smooth manifolds into codimension-zero submanifolds built out of low-index handles.
- In Section 248.4 we will provide exercises for Chapter 248.

In the next Chapter 249 we will introduce the notion of a Morse function on a smooth manifold.

248.1. **Dual handle decomposition.** In this section we will see that we can associate to each handle decomposition a "dual" handle decomposition. More precisely, we have the following proposition:

Proposition 248.1. (Dual Handle Proposition) Let M be a compact n-dimensional smooth manifold. Furthermore let $\partial M = A \sqcup B$ be a decomposition of ∂M where A and B are unions of components of ∂M. Finally let

$$\Theta\colon M \xrightarrow{\cong} ([0,1] \times A) \cup_{\varphi_0} \underbrace{h^{k_0}}_{\text{0-th handle}} \cup \cdots \cup_{\varphi_m} \underbrace{h^{k_m}}_{m\text{-th handle}}$$

be a handle decomposition of M rel A. There exists a natural handle decomposition[2684]

$$\Theta^\dagger\colon M \xrightarrow{\cong} ([0,1] \times B) \cup_{\psi_m} \underbrace{\widetilde{h}^{n-k_m}}_{\text{0-th dual handle}} \cup \cdots \cup_{\psi_0} \underbrace{\widetilde{h}^{n-k_0}}_{m\text{-th dual handle}}$$

of M rel B, called the dual handle decomposition, with the following properties:

(1) The indices of the handles are reversed, in the sense that

$$\begin{array}{c}\text{index of the } i\text{-th handle of}\\ \text{the dual handle decomposition}\end{array} = n - \begin{array}{c}\text{index of the } (n-i)\text{-th handle of}\\ \text{the original handle decomposition.}\end{array}$$

In particular the number of i-handles in the dual handle decomposition equals the number of $(n-i)$-handles in the original decomposition.

(2) If the original handle decomposition is standard, then so is the dual handle decomposition.

(3) The dual of the dual handle decomposition is the original handle decomposition.

(4) We consider the map

$$([0,1]\times A)\underset{\varphi_0}{\cup} h_0^{k_0}\cup\ldots\underset{\varphi_m}{\cup} h_m^{k_m}\cup([0,1]\times B)\to([0,1]\times B)\underset{\psi_m}{\cup}\widetilde{h}_0^{n-k_m}\cup\ldots\underset{\psi_0}{\cup}\widetilde{h}_m^{n-k_0}\cup([0,1]\times A)$$

that is given by the maps $\overline{B}^{n-k_i}\times\overline{B}^{k_i}=h_i^{k_i}\to\widetilde{h}_{m-i}^{n-k_i}=\overline{B}^{k_i}\times\overline{B}^{n-k_i}$ exchanging the factors, has the following properties:

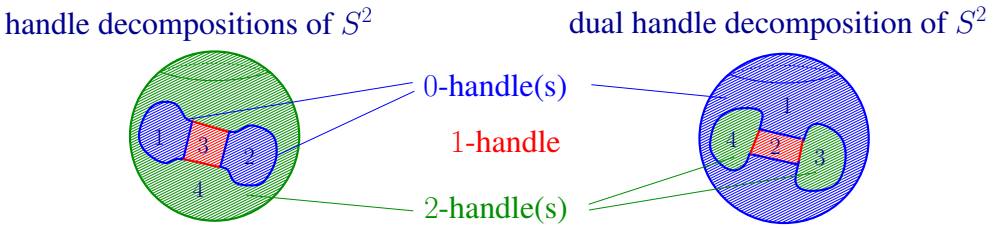
- It is a homeomorphism.
- It is homeotopic to a diffeomorphism.
- Under this diffeomorphism the roles of attaching spheres and belt spheres is reversed in the sense that

$$\begin{array}{c}\text{belt sphere of the } i\text{-th handle of}\\ \text{the dual handle decomposition}\end{array} = \begin{array}{c}\text{attaching sphere of the } (n-i)\text{-th handle}\\ \text{of the original handle decomposition.}\end{array}$$

Furthermore the same statement, with the roles of attaching and belt spheres reversed, also holds.

Example.

(1) In the figure below we show to the left a handle decomposition of S^2 and to the right we see the corresponding dual handle decomposition of S^2.

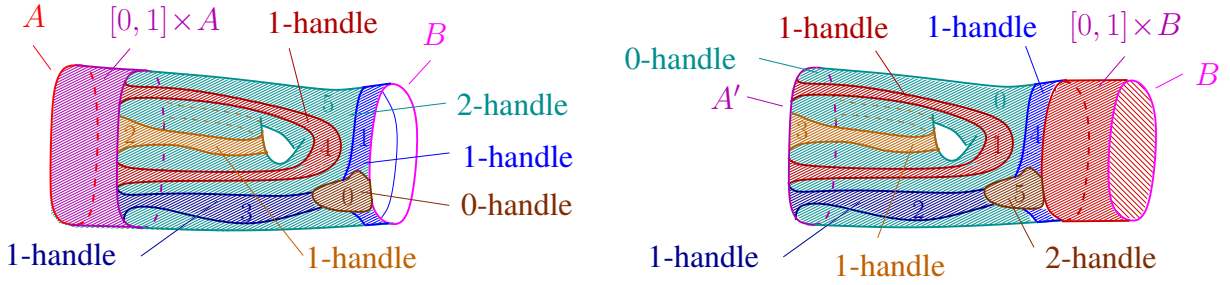


Note that we have an obvious bijection between the i-handles of the original handle decomposition and the $(n-i)$-handles of the dual decomposition. Note though that the i-handles to the left are not exactly the $(n-i)$-handles to the right. For example the 2-handle to the left is (like any n-handle of an n-dimensional smooth manifold) a smooth submanifold but the corresponding 0-handle to the right is (like any 0-handle of a smooth manifold to which a handle got attached) a smooth submanifold *with non-empty corner*.

This shows why in Statement (4) of the Dual Handle Proposition 248.1 we claim that the obvious map is just a homeomorphism.

[2684] We write $\widetilde{h}^{n-k_j}$ instead of h^{n-k_j} to distinguish these handles from their very close relatives in the original handle decomposition of M.

(2) In the figure below we show M, the torus minus two open disks, with two boundary components A and B.



standard handle decomposition of M rel A dual handle decomposition of M rel B

To the left we show the above handle decomposition of M rel A. To the right we see the corresponding dual handle decomposition of M rel B. Note that here, with non-empty A and B, the handles do not quite match. The handles only match up, if we have external collars $[0,1] \times A$ and $[0,1] \times B$ in both settings. □

The proof of the Dual Handle Proposition 248.1 will require the remainder of this section. In the next Section 248.2 we will discuss more examples and applications of dual handle decompositions.

The following lemma is the key ingredient in the proof of the Dual Handle Proposition 248.1:

Lemma 248.2. (Handle Flipping Lemma) Let X be an n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial X$ be a thickened $(k-1)$-sphere. Let

$$X \cup_\varphi h^k \;:=\; X \cup_{\partial X \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} (\overline{B}^{n-k} \times \overline{B}^k)$$

be the corresponding k-handle attachment. Furthermore let Y be another n-dimensional smooth manifold and let $\gamma\colon \partial Y \to \partial(X \cup_\varphi h^k)$ be a diffeomorphism. We denote by $\tau\colon \overline{B}^{n-k} \times \overline{B}^k \to \overline{B}^k \times \overline{B}^{n-k}$ the obvious map exchanging the two factors. We consider the maps

$$\psi\colon \overline{B}^k \times S^{n-k-1} \to \partial Y,\quad (x,y) \mapsto \gamma^{-1}(\underbrace{\tau(x,y)}_{\in \overline{S}^{n-k-1} \times \overline{B}^k \subseteq \partial h^k})$$

&

$$\delta\colon \partial X \to \partial(Y \cup_\psi h^{n-k}),\quad P \mapsto \begin{cases} \tau(\varphi^{-1}(P)), & \text{if } P \in \varphi(B^{n-k} \times S^{k-1}), \\ \gamma^{-1}(P), & \text{otherwise.} \end{cases}$$

We consider the map[2685]

$$\underbrace{\big(X \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)\big)}_{=X \cup_\varphi h^k} \cup_\gamma Y \;\to\; \underbrace{\big(Y \cup_\psi (\overline{B}^k \times \overline{B}^{n-k})\big)}_{=Y \cup_\psi h^{n-k}} \cup_\delta X$$

that is induced by id_X, τ and id_Y. This map is a homeomorphism. Furthermore there exists a homeotopy

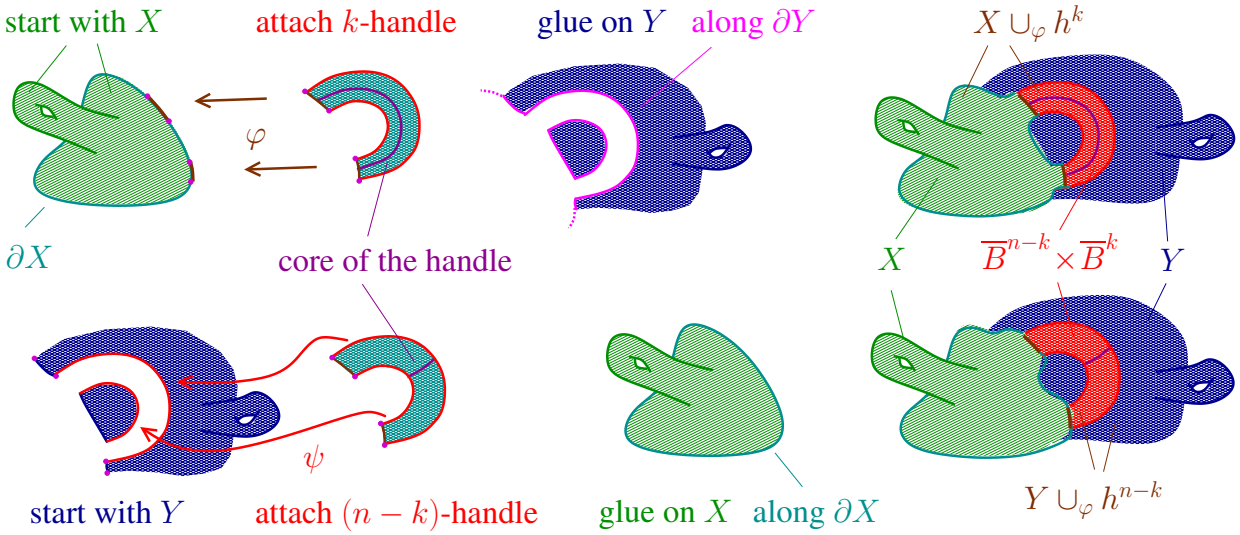
$$H\colon ((X \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)) \cup_\gamma Y) \times [0,1] \;\to\; (Y \cup_\psi (\overline{B}^k \times \overline{B}^{n-k})) \cup_\delta X$$

with the following properties:

(1) H_0 is the above obvious map induced by id_X, τ and id_Y.

(2) H_1 is a diffeomorphism.
Furthermore, given any open neighborhood of $\varphi(\overline{B}^{n-k} \times S^{k-1}) \subseteq X$ and given any open neighborhood of $\psi(\overline{B}^{k} \times S^{n-k-1}) \subseteq Y$ we can find H_t which is stationary outside of the neighborhoods.

The Handle Flipping Lemma 248.2 can be summarized as follows: If a smooth manifold W is obtained from X by attaching a k-handle and then gluing on Y, then we can also obtain W from Y by attaching an $(n-k)$-handle and then gluing on X.



Sketch of proof of the Handle Flipping Lemma 248.2. First note that the map

$$\Theta\colon (X \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)) \cup_\gamma Y \;\to\; (Y \cup_\psi ((\overline{B}^k \times \overline{B}^{n-k})) \cup_\delta X,$$

that is induced by id_X, τ and id_Y, is well-defined. Using the Topological Pushout Proposition 8.12 (1b) it is straightforward to show that the above map is continuous. The map has an obvious inverse, which for the same reasons as above is continuous. This shows that the map is a homeomorphism.

Next we turn to the more tricky issue of the smooth structures. Recall that these depend on various choices. We pick a double-collar $\alpha\colon S^{n-k-1} \times S^{k-1} \times [-\frac{1}{2}, \frac{1}{2}] \times [0,1] \to X$ and we pick a double-collar $\beta\colon S^{n-k-1} \times S^{k-1} \times [-\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}] \to Y$. Furthermore we consider the standard corner collar $\alpha\colon S^{n-k-1} \times S^{k-1} \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \to \overline{B}^{n-k} \times \overline{B}^k$. This data defines all collars and all smooth structures that we will need.

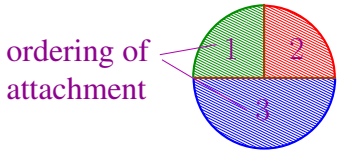
If one goes through all the definitions one will see that the map Θ is a local diffeomorphism outside of $\varphi(S^{n-k-1} \times S^{k-1})$. Furthermore one will see that there exists an open neighborhood of $\varphi(S^{n-1} \times S^{k-1})$ where Θ is of the form

$$\begin{array}{rcl} \Theta\colon S^{n-k-1} \times S^{k-1} \times \overline{B}^2 & \to & S^{k-1} \times S^{n-k-1} \times \overline{B}^2 \\ ((x,y),z) & \mapsto & ((y,x), f(z)) \end{array}$$

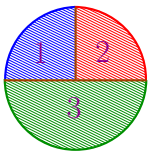
[2685] In greater detail, the topological spaces and the map are of the form

$$\underbrace{(X \cup_{\partial X \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \overline{B}^{n-k} \times \overline{B}^k)}_{= X \cup_\varphi h^k} \cup_{\partial(X \cup_\varphi h^k) \xleftarrow{\gamma} \partial Y} Y \to \underbrace{(Y \cup_{\partial Y \xleftarrow{\psi} \overline{B}^k \times S^{n-k-1}} \overline{B}^k \times \overline{B}^{n-k})}_{= Y \cup_\psi h^{n-k}} \cup_{\partial(Y \cup_\psi h^{n-k}) \xleftarrow{\delta} \partial X} X$$

where $f\colon \overline{B}^2 \to \overline{B}^2$ is a smooth map which is of the form $f(r\cdot v) = r\cdot \widetilde{f}(v)$ for some orientation-preserving diffeomorphism $\widetilde{f}\colon S^1 \to S^1$.



neighborhood of $(x,y)\times S^{k-1}\times S^{n-k-1}$
in $(X \cup_\varphi (\overline{B}^{n-k}\times \overline{B}^k)) \cup_\gamma Y$



neighborhood of $(y,x)\times S^{n-k-1}\times S^{k-1}$
in $(Y \cup_\varphi (\overline{B}^{k}\times \overline{B}^{n-k})) \cup_\delta X$

Claim. We can find a diffeomorphism $g\colon \overline{B}^2 \to \overline{B}^2$ which agrees with f outside of $\overline{B}^2_{\frac{1}{2}}$.

Proof. By the Diffeotopies-of-S^1-Proposition 42.8 we know that $f|_{S^1}\colon S^1 \to S^1$ is diffeotopic to the identity. This means that there exists a diffeotopy $H\colon S^1\times[0,1]\to S^1$ with $H_0 = f$ and $H_1 = \mathrm{id}$. Next note that, using the Smooth Transition Function Lemma 24.9, we can find a smooth map $\tau\colon [0,1]\to[0,1]$ with $\tau(t)=1$ for $t\in[0,\frac{1}{4}]$ and $\tau(t)=0$ for $t\in[\frac{1}{2},1]$. One can now easily verify that the map

$$\begin{array}{rcl} g\colon \overline{B}^2 & \to & \overline{B}^2 \\ P & \mapsto & \begin{cases} H(\frac{P}{\|P\|}, \tau(\|P\|)), & \text{if } P\neq 0, \\ P, & \text{if } \|P\| < \frac{1}{4} \end{cases} \end{array}$$

is well-defined and that it is a diffeomorphism. ⊞

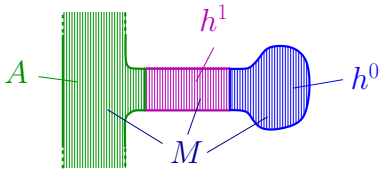
If in the above definition of Θ, we replace the original map $f\colon \overline{B}^2\to\overline{B}^2$ by the diffeomorphism $g\colon \overline{B}^2\to\overline{B}^2$ we obtain the desired diffeomorphism. ■

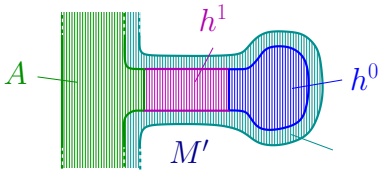
Proof of the Dual Handle Proposition 248.1. The summary of the proof is: Apply the Handle Flipping Lemma 248.2 many times.

For readers who want to see more details, we now explain in greater detail how to make this idea work. Let M be a compact n-dimensional smooth manifold and let $\partial M = A \sqcup B$ be a decomposition of ∂M, where A and B are unions of components of ∂M. Furthermore let

$$M = ([0,1]\times A)\cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

be a handle decomposition of M rel A.





We introduce the following new objects:

- For $i = m,\dots,0$ we consider the map ψ_i that is iteratively defined by

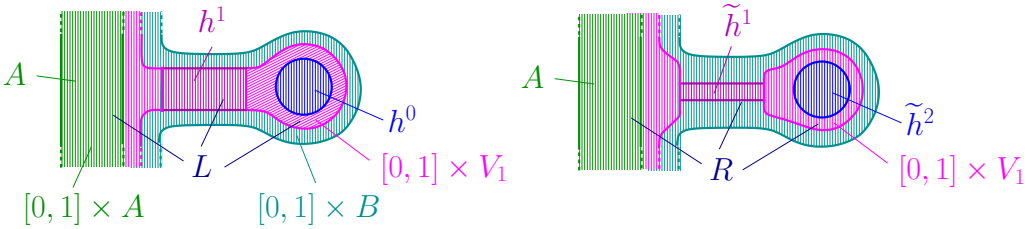
$$\begin{array}{rcl} \psi_i\colon \overline{B}^{k_i}\times S^{n-k_i-1} & \to & \partial(([0,1]\times B)\cup_{\psi_m} \widetilde{h}^{n-k_m} \cup\cdots\cup_{\psi_{i+1}} \widetilde{h}^{n-k_{i+1}}) \\ (x,y) & \mapsto & [\ \underbrace{(y,x)}_{\in h^{k_i}=\overline{B}^{m-k_i}\times\overline{B}^{k_i}}\ ]. \end{array}$$

- We set $V_0 := A$. For $i=1,\dots,m+1$ we set $V_i := (\partial([0,1]\times A)\cup_{\varphi_0} h^{k_0}\cup\cdots\cup_{\varphi_{i-1}} h^{k_{i-1}})\setminus A$. Note that $V_{m+1} = B$.

- We consider[2686]

$$\begin{aligned} L &:= \overbrace{([0,1]\times V_0)}^{=[0,1]\times A} \cup_{\varphi_0} h^{k_0} \cup ([0,1]\times V_0) \quad \cup_{\varphi_1} h^{k_1} \cup ([0,1]\times V_1) \cup \ldots \quad \cup_{\varphi_m} h^{k_m} \cup \overbrace{([0,1]\times V_{m+1})}^{=[0,1]\times B} \\ R &:= \underbrace{([0,1]\times V_0)}_{=[0,1]\times A} \cup \widetilde{h}^{k_0} \cup_{\psi_0^{-1}} ([0,1]\times V_0) \cup \widetilde{h}^{k_1} \cup_{\psi_1^{-1}} ([0,1]\times V_1) \cup \ldots \cup \widetilde{h}^{k_m} \cup_{\psi_m^{-1}} \underbrace{([0,1]\times V_{m+1})}_{=[0,1]\times B}. \end{aligned}$$

  We make the following clarifications:
  - ▷ We have $\widetilde{h}^{k_i} := \overline{B}^{k_i} \times \overline{B}^{n-k_i}$.
  - ▷ The gluing maps always go from right to left.
  - ▷ For $i \geq 1$ the maps φ_i on top are strictly speaking given by $(1, \varphi_i(x)) \hookleftarrow x$.
  - ▷ The maps on top that we did not specify are the obvious identification $x \hookleftarrow (x, 0)$.
  - ▷ The maps ψ_i^{-1} at the bottom are the maps $\psi_i(\overline{B}^{k_i} \times S^{n-k_i-1}) \to \overline{B}^{k_i} \times S^{n-k_i-1}$.
  - ▷ The maps on the bottom that we did not specify are the obvious identification maps $(x, 1) \hookleftarrow x$.



The proposition follows easily from the above explicit descriptions and the following claim:

Claim.

(1) (a) L is diffeomorphic to M.
  (b) R is diffeomorphic to

$$([0,1] \times B) \cup_{\psi_m} \widetilde{h}^{n-k_m} \cup \cdots \cup_{\psi_0} \widetilde{h}^{n-k_0}.$$

(2) L and R are diffeomorphic.

Proof.

(1) Both statements follow from the External Collar Lemma 247.3.
(2) For $i = 0, \ldots, m$ we apply the Handle Flipping Lemma 248.2 to $X_i = V_i \times (0, 1]$ and $Y_i = V_{i+1} \times [0, 1)$ and the open neighborhoods $V_i \times (\frac{3}{4}, 1]$ and $V_{i+1} \times [0, \frac{1}{2})$. We get diffeomorphisms which, by our careful choice of our neighborhoods and the Smooth Pasting Proposition 25.19, fit together to give us a diffeomorphism. ■

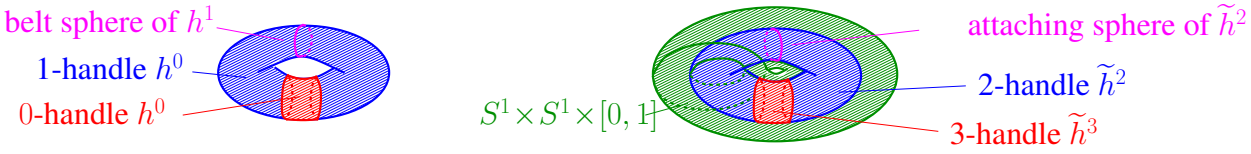
248.2. **Examples and applications of dual handle decompositions.** First we provide a few more examples of dual handle decompositions:

Examples.

(1) We consider $M = \overline{B}^2 \times S^1$. By the Ball–Handle Attachment Lemma 246.9 we know that $M = \overline{B}^2 \times S^1 = h^0 \cup h^1$ admits a handle decomposition into a single 0-handle and a single 1-handle. It follows from the Dual Handle Proposition 248.1 that $M = \overline{B}^2 \times S^1$

[2686] The mnemonic is that we think of L as building a smooth manifold by starting from the left whereas we think of R as building a smooth manifold by starting on the right.

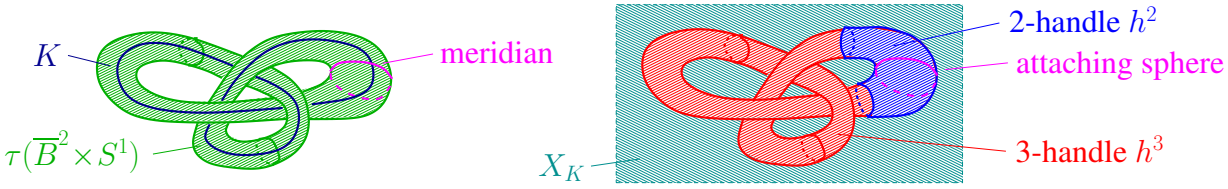
admits a handle decomposition rel $\partial M = S^1 \times S^1$ consisting of a single 2-handle $\widetilde{h}^2$ and a single 3-handle $\widetilde{h}^3$, and such that the attaching sphere of the dual 2-handle equals the belt sphere of the original 1-handle, i.e. it is given by $S^1 \times \{*\} = \partial \overline{B}^2 \times \{*\}$.



(2) Let $K \subseteq S^3$ be a knot. By the Link Tubular Map Theorem 108.29 we know that there exists a smooth embedding $\tau\colon \overline{B}^2 \times S^1 \to S^3$ such that $\tau(\{0\} \times S^1) = K$. As in Definition 108.30 we consider the knot exterior $X_K := S^3 \setminus \tau(B^2 \times S^1)$. Note that

$$\begin{aligned} S^3 \;=\; X_K \cup \tau(\overline{B}^2 \times S^1) \;&\overset{\text{by the discussion in (1)}}{\cong}\; X_K \cup \tau((S^1 \times S^1 \times [0,1]) \cup_{\varphi_2} h^2 \cup_{\varphi_3} h^3) \\ &\cong\; X_K \cup \tau(S^1 \times S^1 \times [0,1]) \cup_{\tau\circ\varphi_2} h^2 \cup_{\tau\circ\varphi_3} h^3 \;\underset{\text{by the External Collar Lemma 247.3 together with the Modify Handle Attachment Region Lemma 247.8}}{\cong}\; X_K \cup h^2 \cup h^3. \end{aligned}$$

This shows that one can recover S^3 from X_K by gluing in a 2-handle and a 3-handle. If one keeps track of the descriptions of the various maps one sees that the attaching sphere of the 2-handle is given by the meridian of K.



(3) Let M be a compact smooth manifold with a handle decomposition

$$M \;=\; \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r}.$$

We can view this as a handle decomposition rel $\varnothing$. By the Dual Handle Proposition 248.1 we can equip M with the dual handle decomposition, which is a handle decomposition rel ∂M. If we glue two copies of M along the common boundary, i.e. if we consider the double $\mathrm{D}M$, then it follows from the Handle Decomposition–Stacking Proposition 247.2 that we can stack the original handle decomposition and the dual decompositions. In summary we obtain a handle decomposition for the double $\mathrm{D}M$ of the form

$$\mathrm{D}M \;=\; \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \;\cup_{\widetilde{\psi}_r} h^{n-k_r} \cup \cdots \cup_{\widetilde{\psi}_0} h^{n-k_0}. \qquad \square$$

We continue with the following variation on the 1-Handle–π_1–Inequality Lemma 247.9:

Lemma 248.3. $(n-1)$–π_1–**Handle Inequality Corollary)** Let M be a *closed* connected non-empty n-dimensional smooth manifold. For every handle decomposition of M we have the inequality:

$$1 + \#(n-1)\text{-handles} - \#n\text{-handles} \;\geq\; \text{minimal size of a generating set of } \pi_1(M).$$

Proof. Let M be a closed connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition of M. We have the following (in-) equalities:

$$\overbrace{\begin{array}{l}1+\#(n-1)\text{-handles}\\ \quad -\#n\text{-handles}\end{array}}^{\text{original handle decomposition}} \underset{\text{by the Dual Handle Proposition 248.1}}{\overset{=}{\uparrow}} \overbrace{\begin{array}{l}1+\#1\text{-handles}\\ \quad -\#0\text{-handles}\end{array}}^{\text{dual handle decomposition}} \underset{\text{by the 1-Handle–}\pi_1\text{–Inequality Lemma 247.9}}{\overset{\geq}{\uparrow}} \begin{array}{c}\text{minimal size of a}\\ \text{generating set of } \pi_1(M).\end{array}$$

■

We move on to the following analog of the Minimal 0-Handle Proposition 247.13:

Proposition 248.4. (Minimal n-Handle Proposition) Let M be a compact connected non-empty n-dimensional smooth manifold.

(1) Given any handle decomposition of M rel ∂M with d_i i-handles there exists a handle decomposition of M with the following properties:
- The number of n-handles equals one.
- The number of $(n-1)$-handles equals $d_{n-1} - d_n + 1$.
- The number of handles of index $\neq n-1, n$ is unchanged.

(2) Let B be a union of components of ∂M with $B \neq \partial M$. Given any handle decomposition of M rel B with d_i i-handles there exists a handle decomposition of M rel B with the following properties:
- The number of n-handles equals zero.
- The number of $(n-1)$-handles equals $d_{n-1} - d_n$.
- The number of handles of index $\neq n-1, n$ is unchanged.

Proof. Let M be a compact connected non-empty n-dimensional smooth manifold and let B be a union of components of ∂M. We write $A := \partial M \setminus B$. We do the following:

(1) Given a handle decomposition of M rel B we consider the dual handle decomposition of M rel A, which we defined in the Dual Handle Proposition 248.1.

(2) We apply the Minimal 0-Handle Proposition 247.13 to the dual handle decomposition to reduce the number of 0-handles in the dual decomposition to 1 if $A = \varnothing$ (i.e. if $B = \partial M$) or to 0 if $A \neq \varnothing$ (i.e. if $B \neq \partial M$).

(3) We take the dual of the modified dual handle decomposition from (2). Since the number of k-handles in a dual handle decomposition equals the number of $(n-k)$-handles in the original handle decomposition we have proved the desired result. ■

We conclude this section with the following consequence:

Proposition 248.5. (Minimal Handle Proposition) Let M be a *closed* connected non-empty n-dimensional smooth manifold. Given any handle decomposition of M with d_i i-handles there exists another handle decomposition with the following properties:

(1) The new handle decomposition is standard.

(2) The new handle decomposition has precisely one 0-handle and precisely one n-handle.

(3) The number of 1-handles equals $d_1 - d_0 + 1$ and the number of $(n-1)$-handles equals $d_{n-1} - d_n + 1$.

(4) The number of handles of index $\neq 0, 1, n-1, n$ is unchanged.

Remark. In the Single Zero-and-Top Cell Proposition 190.13 we proved a topological analog of the Minimal Handle Proposition 248.5. □

Proof. For $n = 0$ the statement is clear. So now let $n \in \mathbb{N}$. First we apply the Minimal 0-Handle Proposition 247.13 to obtain a handle decomposition with a single 0-handle. If $n = 1$, then we see that there is only one way to attach a 1-handle, so we already have found a handle decomposition with a single 0-handle and a single n-handle.

For $n \geq 2$ we apply the Minimal n-Handle Proposition 248.4. Since $n \geq 2$ we see that the number of 0-handles is unchanged while applying the Minimal n-Handle Proposition 248.4. Thus we have found a handle decomposition with a single 0-handle and a single n-handle.

Finally we apply the Standard Handle Decomposition Proposition 247.7 to obtain a standard handle decomposition, with the same number of handles. ■

248.3. **Splittings of smooth manifolds.** On numerous occasions the following proposition allows us to decompose a given smooth manifold into two smooth manifolds with handle decompositions of low indices:

Proposition 248.6. (Manifold Handle Decomposition–Splitting Proposition) Let M be a compact n-dimensional smooth manifold and let $\partial M = A \sqcup B$ be a decomposition of ∂M where A and B are unions of components of ∂M. Furthermore let

$$\Theta\colon M \xrightarrow{\cong} ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

be a handle decomposition of M rel A. Finally let $i \in \{1, \ldots, m\}$. There exists a diffeomorphism

$$M \cong X \cup_f Y$$

where X, Y and f are given as follows:

(1) We set $X := ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_i} h^{k_i}$.

(2) We set $Y := ([0,1] \times B) \cup_{\psi_m} h^{n-k_m} \cup \cdots \cup_{\psi_{i+1}} h^{n-k_{i+1}}$

for suitable attaching maps $\psi_{i+1}, \ldots, \psi_m$.

(3) $f\colon \partial Y \setminus B \to \partial X \setminus A$ is a diffeomorphism.

Proof. We consider

$$M = \underbrace{([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_i} h^{k_i}}_{=:X} \cup_{\varphi_{i+1}} h^{k_{i+1}} \cup \cdots \cup_{\varphi_m} h^{k_m}.$$

We set $\Sigma := \partial X \setminus A$. It follows from the External Collar Lemma 247.3 and the Modify Handle Attachment Region Lemma 247.8 that we have a diffeomorphism

$$M \cong \underbrace{([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_i} h^{k_i}}_{=X} \cup_{\substack{x = (0,x) \\ \text{for } x \in \Sigma}} \underbrace{([0,1] \times \Sigma) \cup_{\widetilde{\varphi}_{i+1}} h^{k_{i+1}} \cup \cdots \cup_{\widetilde{\varphi}_m} h^{k_m}}_{=:Y}.$$

Note that $\partial Y = \Sigma \cup B$. We now apply the Dual Handle Proposition 248.1 to Y and the given handle decomposition. We see that there exist attaching maps $\psi_{i+1}, \ldots, \psi_m$ and a diffeomorphism

$$Y \cong ([0,1] \times B) \cup_{\psi_m} \widetilde{h}^{n-k_m} \cup \cdots \cup_{\psi_{i+1}} \widetilde{h}^{n-k_{i+1}}$$

which is the obvious map on B. One can easily verify that this diffeomorphism naturally gives rise to a diffeomorphism $\partial Y \setminus \Sigma = B$ to $\partial X \setminus A = \Sigma$ such that $M \cong X \cup_f Y$. ■

248.4. Exercises for Chapter 248.

Exercise 248.1. Let M be a compact n-dimensional smooth manifold with connected boundary. We assume that M has a handle decomposition with no handles of index $n-1$ and n. Show that the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is an epimorphism.

Exercise 248.2. Let M be a closed connected non-empty n-dimensional smooth manifold. Show that every handle decomposition of M admits at least one n-handle.

Exercise 248.3. Does there exist a non-empty non-orientable n-dimensional smooth manifold with no $(n-1)$-handles?

Exercise 248.4. Does there exist a handle decomposition for S^3 with a single 0-handle and a single 1-handle?

Exercise 248.5. Let M be the submanifold of $\mathbb{R}^3$ that is shown in the figure below.

(a) Equip M with a handle decomposition consisting of a single 0-handle and two 1-handles and show the handle decomposition in the figure.

(b) We consider the double $\mathrm{D}M$. We equip the double $\mathrm{D}M$ with the handle decomposition that we described on page 4791. Illustrate this handle decomposition.



Exercise 248.6. Let $n \in \mathbb{N}$, let $k \in \mathbb{N}$ and let $\varphi_1, \dots, \varphi_m \colon \overline{B}^{n-k} \times S^{k-1} \to S^{n-1}$ be orientation-preserving thickened $(k-1)$-spheres with disjoint images. We set

$$M := \overline{B}^n \cup_{\varphi_1} h^k \cup \dots \cup_{\varphi_m} h^k.$$

(a) Show that if $k \in \{3, \dots, n-3\}$, then ∂M is simply connected.

(b) Show that if $2k < n$, then the homology groups of ∂M depend only on m. Compute the isomorphism types of these groups.

(c) Show that if $2k = n$, then in general the homology groups of ∂M depend on the choice of the thickened $(k-1)$-spheres.

Exercise 248.7. Let $k \in \mathbb{N}$ and let M be a compact orientable $2k$-dimensional smooth manifold which admits a handle decomposition such that all handles have index $\leq k$. Is the inclusion-induced map $\mathrm{H}_k(\partial M) \to \mathrm{H}_k(M)$ necessarily an epimorphism?

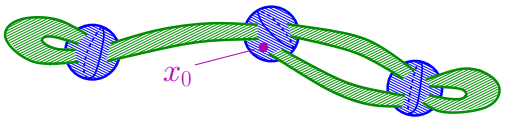
Exercise 248.8. Let $n \in \mathbb{N}_{\geq 4}$ and let M be a connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition. Let $x_0 \in \partial M$.

(a) Show that if all handles have index $\leq n-3$, then the boundary ∂M is connected and the inclusion-induced map $\pi_1(\partial M, x_0) \to \pi_1(M, x_0)$ is an isomorphism.

(b) Show that if all handles have index $\leq n-2$, then the boundary ∂M is connected and the inclusion-induced map $\pi_1(\partial M, x_0) \to \pi_1(M, x_0)$ is an epimorphism.

Exercise 248.9. Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M.

(a) Show that if M admits a handle decomposition rel A, then it also admits a handle decomposition where the number of 0-handles equals $b_0(M, A) := \operatorname{rank}(\mathrm{H}_0(M, A))$.



3-dimensional smooth manifold built out of handles of index ≤ 1

$\pi_1(\partial M), x_0) \to \pi_1(M, x_0)$ is an epimorphism

(b) Show that if M admits a handle decomposition rel A, then it also admits a handle decomposition where the number of n-handles equals $b_0(M, \partial M \setminus A)$.

Exercise 248.10. Let M be a compact 3-dimensional smooth manifold. The *hiding number* $h(M)$ *of* M is defined as the minimal number of handle attachments needed to turn M into a smooth manifold that is diffeomorphic to S^3. We set $h(M) := \infty$ if M if no such sequence of handle attachments exists.

(a) Determine the hiding number of the exterior of an m-component link.
(b) Determine the hiding number of the genus g handlebody.
(c) Determine the hiding number of $\mathbb{R}\mathrm{P}^3 \setminus \Theta(B^3)$ where $\Theta(\overline{B}^3) \to \mathbb{R}\mathrm{P}^3$ is a smooth embedding.
(d) Does there exist a compact orientable connected 3-dimensional smooth manifold M such that $\partial M \cong S^1 \times S^1$ but such that the hiding number of M is infinite?

Exercise 248.11. Let M be a closed smooth manifold and let $\varphi\colon M \to M$ be a self-diffeomorphism. Suppose that we have a handle decomposition of M such that for each $k \in \mathbb{N}_0$ we have d_k handles of index k. Show that the mapping torus

$$\mathrm{Tor}(M, \varphi) \;:=\; (M \times [0,1])/\,(x,0) \sim (\varphi(x), 1)$$

admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of handles of index k equals $d_k + d_{k-1}$.
Hint. Make use of Exercise 246.27 and break the mapping torus along $M \times \{\frac{1}{2}\}$ into two halves.

Exercise 248.12. Let M be a closed connected non-empty 3-dimensional smooth manifold. Show that if M admits a handle decomposition, then M can be obtained by gluing two handlebodies along their boundary.

Exercise 248.13. Let M be a closed connected odd-dimensional smooth manifold. Show that

$$\sum_{k \in \mathbb{N}_0} (-1)^k \cdot \#k\text{-handles} \;=\; 0.$$

Remark. The idea is to prove this equality only using the Handle Attachment-χ-Lemma 246.15 and handle theory.

Exercise 248.14. Let M be a closed connected n-dimensional smooth manifold. Show that the rank of $\mathrm{H}_n(M)$ is at most one.

Exercise 248.15.

(a) Show that there exists a thickened 1-sphere $\varphi\colon \overline{B}^2 \times S^1 \to S^3$ such that the double $\mathrm{D}(\overline{B}^4 \cup_\varphi h^2)$ is diffeomorphic to $S^2 \times S^2$.
(b) Show that there exists a thickened 1-sphere $\varphi\colon \overline{B}^2 \times S^1 \to S^3$ such that the double $\mathrm{D}(\overline{B}^4 \cup_\varphi h^2)$ is diffeomorphic to $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$.
Hint. Consider the proof of the Projective Space–Connected Sum Proposition 238.10.

CHAPTER 249

Morse functions

In Chapter 246 we introduced handle decompositions and we showed explicitly that some of our favorite compact smooth manifolds admit handle decompositions. Our next goal is to show that every compact smooth manifold admits a handle decomposition. We will do so in the next chapter, where we will show that every "Morse function" on a smooth manifold gives rise to a handle decomposition.

In this chapter we will introduce Morse functions, we will give several explicit examples and we will show that each compact smooth manifold admits a Morse function.

This chapter is organized as follows:

- In Section 249.1, we will define, given a critical point P of a smooth function $f\colon M \to \mathbb{R}$, the Hessian form $\mathrm{Hess}(f)_P\colon \mathrm{T}_P M \times \mathrm{T}_P M \to \mathbb{R}$.
- In Section 249.2 we will use the Hessian form to introduce the notion of a non-degenerate critical point of a smooth function $f\colon M \to \mathbb{R}$ and we will introduce the corresponding index.
- In Section 249.3 we will state and prove the Morse Lemma 249.11 which describes the local behavior of a function at a non-degenerate critical point in terms of the index.
- In Section 249.4 we will introduce Morse functions and we will provide many explicit examples of Morse functions, in particular we will construct explicit Morse functions for projective spaces.
- In Section 249.5 we will prove the Distance-Squared Function Proposition 249.21 which says that given a smooth submanifold $M \subseteq \mathbb{R}^d$ "most" functions $\delta_P\colon M \to \mathbb{R}$ of the form $x \mapsto \|x - P\|^2$ have no non-degenerate critical points.
- In Section 249.6 we will use the Distance-Squared Function Proposition 249.21 to show that Morse functions always exist.
- In Section 249.7 we will prove that if a Morse function exists, then there exists also a Morse function such that the values of the critical points are pairwise distinct.
- In Section 249.8 we will provide exercises for Chapter 249.

In the next Chapter 250 we will see that Morse functions give rise to handle decompositions.

249.1. **The Hessian form.** First we recall Definitions 29.11 and 29.16:

Definition 249.1. Let M be a smooth manifold and let $P \in M$.

(1) (a) A derivation at P is an $\mathbb{R}$-linear map $w\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M,\mathbb{R})$ we have

$$w(f \cdot g) \;=\; f(P) \cdot w(g) + w(f) \cdot g(P) \qquad \text{"Leibniz Rule".}$$

Given a derivation w at P and given $f \in C^\infty(M,\mathbb{R})$ we sometimes call $w(f)$ the derivative of f at P in the w-direction.

(b) The set of all derivations at P is called the tangent space $\mathrm{T}_P M$ at P. We refer to elements of $\mathrm{T}_P M$, i.e. we refer to derivations at P, as tangent vectors.

(2) Given a smooth map $f\colon M \to N$ between two smooth manifolds and given a point $P \in M$ we refer to the map

$$\begin{array}{rcl} \mathrm{D}f_P\colon \mathrm{T}_P M & \to & \mathrm{T}_{f(P)} N \\ v & \mapsto & ((g\colon N \to \mathbb{R}) \mapsto v(g \circ f\colon M \to \mathbb{R})) \end{array}$$

from the Tangent Space-Functor Proposition 29.15 (1) as the differential of f at P.

Example. Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and let $P \in U$. On page 762 we used the isomorphism

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathrm{T}_P U \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(U,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & \frac{d}{dt}\, g(P + t \cdot v)|_{t=0} \end{array} \right) \end{array}$$

from the Tangent Space-Isomorphism Proposition 29.24 to make the standard identification $\mathbb{R}^n =_{\mathrm{i}} \mathrm{T}_P U$. As we discussed in the Open Subset-of-$\mathbb{R}^n$-Tangent Space Proposition 29.14, we can give an explicit inverse of the above isomorphism as follows: For $i = 1, \ldots, n$ we denote by $\pi_i\colon U \to \mathbb{R}$ the projection onto the i-th coordinate. With this notation we saw that the inverse of the above isomorphism is given by

$$\begin{array}{rcl} \mathrm{T}_P U & \to & \mathbb{R}^n \\ w & \mapsto & (w(\pi_1), \ldots, w(\pi_n)). \end{array}$$

□

We continue by recalling Definition 32.1.

Definition 249.2. Let $f\colon M \to N$ be a smooth map between smooth manifolds. We say $P \in M$ is a regular point of f if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an epimorphism.

Example. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth map. Since $\mathbb{R}$ is 1-dimensional we see that a point $P \in M$ is a regular point if and only if the differential $\mathrm{D}f_P$ is non-trivial. □

In our previous discussions we were trying hard to avoid critical points. In the following we will see that the study of critical points can actually be quite fruitful. In particular, as we will see shortly, critical points come in different flavors which correlate with the topology of the smooth manifold.

The reader might remember the following strategy from real analysis: If $U \subseteq \mathbb{R}^n$ is an open subset and if $P \in U$ is a critical point of a smooth map $f\colon U \to \mathbb{R}$, then one obtains information on the local behavior of f in the neighborhood of P by studying the Hessian matrix

$$\mathrm{Hess}(f)_P \coloneqq \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|_P \right)_{i,j=1,\ldots,n} \in \mathrm{M}(n \times n, \mathbb{R})$$

which we first introduced in Definition 81.5.

To define an analog of the Hessian matrix for a critical point on a smooth manifold we will need the following lemma:

Lemma 249.3. (Vector Field–Derivative Lemma) Let M be a smooth manifold and let X be a smooth vector field on M. The map

$$\begin{array}{rcl} X\colon C^\infty(M,\mathbb{R}) & \to & C^\infty(M,\mathbb{R}) \\ (f\colon M\to\mathbb{R}) & \mapsto & \left(\begin{array}{ccc} M & \to & \mathbb{R} \\ P & \mapsto & \underset{\substack{\uparrow \\ \text{derivative of } f \text{ in} \\ \text{the } X(P)\text{-direction}}}{X_P(f)} \end{array}\right) \end{array}$$

is well-defined, in the sense that $P \mapsto X_P(f)$ is indeed smooth.

Proof. Let M be a smooth manifold and let X be a smooth vector field on M. We start out with an observation, which we will also use on a later occasion:

Observation. Given any smooth function $f\colon M\to\mathbb{R}$ and given any $P\in M$ we have the equality $X_P(f) = \mathrm{D}f_P(X_P) \in \mathbb{R} =_{\mathrm{i}} \mathrm{T}_P\mathbb{R}$.

Proof. The observation follows immediately from looking up the various conventions and identifications. For peace of mind let us do this carefully: First of all, on page 4797 we mentioned that for each $Q\in\mathbb{R}$ we use the isomorphism

$$\begin{array}{rcl} \Theta\colon \mathrm{T}_Q\mathbb{R} & \to & \mathbb{R} \\ w & \mapsto & w(\underbrace{x\mapsto x}_{=\mathrm{id}_{\mathbb{R}}}) \end{array}$$

to make the identification $\mathrm{T}_Q\mathbb{R} =_{\mathrm{i}} \mathbb{R}$. We have the following equality in $\mathbb{R}$:

$$\Theta(\mathrm{D}f_P(X_P)) \underset{\substack{\uparrow \\ \text{see Definition 249.1} \\ \text{of } \mathrm{D}f_P}}{=} \Theta((g\colon\mathbb{R}\to\mathbb{R})\mapsto X_P(g\circ f\colon M\to\mathbb{R})) \underset{\substack{\uparrow \\ \text{see the above} \\ \text{definition of } \Theta}}{=} X_P(g\circ f)(g=\mathrm{id}_{\mathbb{R}}) = X_P(f)$$

⊞

Now we turn to the actual proof of the claim. Note that it follows from the Vector Field Definitions–Agree Lemma 29.30, and using charts, that it suffices to prove the statement for open subsets U of $\mathbb{R}^n$ or $\mathbb{H}^n$, where the smooth vector field is given by a smooth function $X=(X_1,\ldots,X_n)\colon U\to\mathbb{R}^n$ and where we consider a smooth function $f\colon U\to\mathbb{R}$. In this setting the resulting function is given by

$$\begin{array}{rcl} X(f)\colon U & \to & \mathbb{R} \\ P & \mapsto & X_P(f) \underset{\substack{\uparrow \\ \text{by the above} \\ \text{observation}}}{=} \mathrm{D}f_P(X_P) \underset{\substack{\uparrow \\ \text{by the Tangent Space-Isomorphism Proposition 29.24} \\ \text{and the Differential-equals-Differential Lemma 29.6}}}{=} \sum_{i=1}^n \frac{\partial f}{\partial x_i}(P)\cdot X_i(P). \end{array}$$

But this map is clearly smooth. ■

The following proposition gives us an analog of the Hessian matrix for critical points of smooth functions on smooth manifolds:

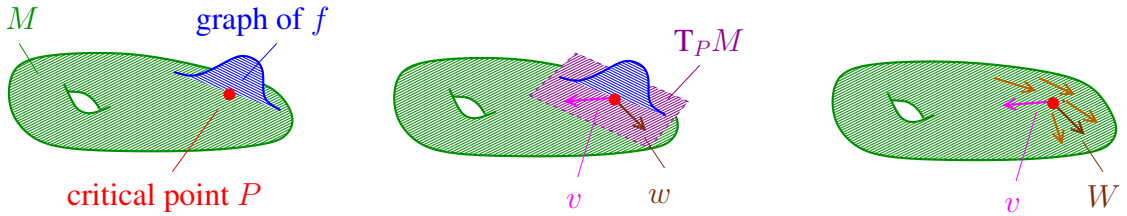
Proposition 249.4. (Hessian Form Proposition) Let M be a smooth manifold and let $f\colon M\to\mathbb{R}$ be a smooth function. Let $P\in M$ be a *critical point*[2687] of f. Let $v,w\in\mathrm{T}_PM$. Using charts and the Smooth Transition Function Lemma 24.9 we can pick smooth vector

fields V, W on M with $V(P) = v$ and $W(P) = w$. We define

$$\mathrm{Hess}(f)_P(v,w) \;:=\; V_P(\underbrace{W(f)}_{\in C^\infty(M,\mathbb{R})}).$$

(1) This definition is independent of the choice of the extensions V and W.

(2) The map $\mathrm{Hess}(f)_P\colon \mathrm{T}_PM \times \mathrm{T}_PM \to \mathbb{R}$, $(v,w) \mapsto V_P(W(f))$ is $\mathbb{R}$-bilinear and symmetric. This form is called the Hessian form of f at the critical point P.



Before we turn to the proof of the Hessian Form Proposition 249.4, let us formulate the following lemma, which tells us how one can compute the Hessian form through charts:

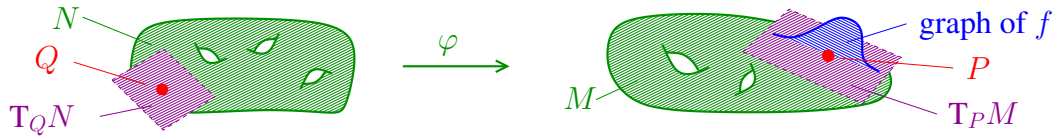
Lemma 249.5. (Hessian Form-Calculation Lemma)

(1) Let $\varphi\colon N \to M$ be a local diffeomorphism between two smooth manifolds and let $Q \in N$ with $\varphi(Q) = P$. The point $Q \in N$ is a critical point of $f \circ \varphi\colon N \to \mathbb{R}$ if and only if $P \in M$ is a critical point of f. Furthermore, if one point (and thus both) are critical points, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}_QN \times \mathrm{T}_QN & \xrightarrow{\mathrm{Hess}(f\circ\varphi)_Q} & \mathbb{R} \\ {\scriptstyle \mathrm{D}\varphi_Q}\downarrow \quad \downarrow{\scriptstyle \mathrm{D}\varphi_Q} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{T}_PM \times \mathrm{T}_PM & \xrightarrow{\mathrm{Hess}(f)_P} & \mathbb{R}. \end{array}$$

(2) If M is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then the form $\mathrm{Hess}(f)_P$ from the Hessian Form Proposition 249.4 is represented, with respect to the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ of the tangent space $\mathrm{T}_PM =_{\mathrm{i}} \mathbb{R}^n$ by the Hessian matrix

$$\mathrm{Hess}(f)_P = \left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\Big|_P\right)_{i,j=1,\ldots,n}.$$



The proof of the Hessian Form Proposition 249.4 and the Hessian Form-Calculation Lemma 249.5 will require the remainder of this section. In the next Section 249.2 we will use the Hessian Form Proposition 249.4 to introduce the notion of a *non-degenerate* critical point.

[2687]Note that it is essential that P is a critical point. The proof that the definition of the Hessian $\mathrm{Hess}(f)_P(v,w)$ is independent of the choices does make use of the fact that we are considering a critical point.

Our proof of the Hessian Form Proposition 249.4 makes use of the following lemma, which is interesting in its own right:

Lemma 249.6. (Lie Bracket Lemma) Let M be a smooth manifold.

(1) Let X be a smooth vector field on M.
   (a) The map $X\colon C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ from the Vector Field–Derivative Lemma 249.3 is a derivation, i.e. for any two smooth functions $f, g\colon M \to \mathbb{R}$ we have
   $$X(f\cdot g) \;=\; f\cdot X(g) + X(f)\cdot g.$$
   (b) Every derivation $C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ comes from a smooth vector field.

(2) Let X, Y be two smooth vector fields on M. The map
$$\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & C^\infty(M,\mathbb{R}) \\ f & \mapsto & \underbrace{(X\circ Y - Y\circ X)(f)}_{X(Y(f))-Y(X(f))} \end{array}$$
is a derivation. In particular by (1) it defines a vector field on M, written $[X,Y]$, which is called the Lie bracket of X and Y.

Remark. In Exercise 249.2 we will discuss more properties of the Lie bracket. □

Proof of the Lie Bracket Lemma 249.6.

(1) Both statements follow easily from the definition of the tangent space via derivations, see Definition 249.1.

(2) Let X, Y be two smooth vector fields on M. We need to show that $[X,Y] = X\circ Y - Y\circ X$ is a derivation. Thus let $f, g\colon M \to \mathbb{R}$ be smooth functions. We see that we indeed have
$$\begin{aligned} (X\circ Y - Y\circ X)(f\cdot g) &= X(Y(f\cdot g)) - Y(X(f\cdot g)) \\ &= X(f\cdot Y(g) + Y(f)\cdot g) - Y(f\cdot X(g) + X(f)\cdot g) \\ &= \quad f\cdot X(Y(g)) + X(f)\cdot Y(g) + Y(f)\cdot X(g) + X(Y(f))\cdot g \\ &\quad - f\cdot Y(X(g)) - Y(f)\cdot X(g) - X(f)\cdot Y(g) - Y(X(f))\cdot g \\ &= f\cdot (X(Y(g)) - Y(X(g))) + (X(Y(f)) - Y(X(f)))\cdot g \\ &= f\cdot (X\circ Y - Y\circ X)(g) - (X\circ Y - Y\circ X)(f)\cdot g. \end{aligned}$$ ■

Now we can finally provide the proofs of the Hessian Form Proposition 249.4 and the Hessian Form-Calculation Lemma 249.5:

Proof of the Hessian Form Proposition 249.4. Let M be a smooth manifold. First note that for any smooth two vector fields V and W on M and any $P \in M$ we have
$$V_P(W(f)) - W_P(V(f)) \underset{\uparrow}{=} [V,W]_P(f) \underset{\uparrow}{=} \mathrm{D}f_P(\overbrace{[V,W]_P}^{\in \mathrm{T}_P M}) \underset{\uparrow}{=} 0.$$
by definition of the vector field $[V,W]$ in the Lie Bracket Lemma 249.6; see the observation in the proof of the Vector Field Derivative Lemma 249.3; since P is a critical point of f

We now turn to the proofs of the two statements:

(1) In the above calculation we just showed that $V_P(W(f)) = W_P(V(f))$. Evidently the first expression only depends on the value of V at P and the second expression only depends on the value of W at P. This shows that the choice of extensions of V and W

in the definition of $\quad \mathrm{Hess}(f)_P(v,w) \;:=\; V_P(W(f)).$

is irrelevant.

(2) It follows immediately from the above calculation that the map

$$\begin{array}{rcl} \mathrm{Hess}(f)_P\colon \mathrm{T}_PM \times \mathrm{T}_PM & \to & \mathbb{R} \\ (v,w) & \mapsto & V_P(W(f)) \end{array}$$

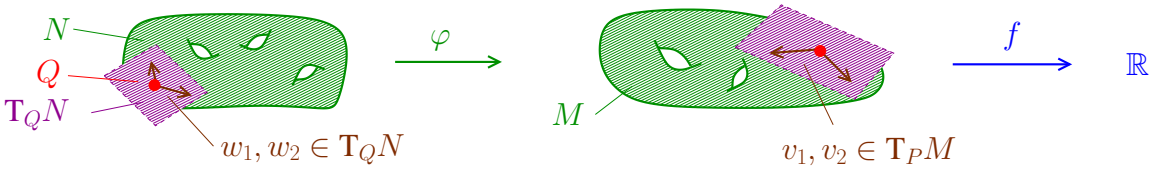
is symmetric. Finally note that it is clear that this form is $\mathbb{R}$-bilinear. ■

Proof of the Hessian Form-Calculation Lemma 249.5.

(1) Let $\varphi\colon N \to M$ be a local diffeomorphism map and let $Q \in N$ with $\varphi(Q) = P$. Note that

$$\mathrm{D}(f\circ\varphi)_Q \underset{\text{by the Chain Rule}}{=} \mathrm{D}f_{\varphi(Q)} \circ \mathrm{D}\varphi_Q \underset{\text{since } \varphi(Q)=P}{=} \mathrm{D}f_P \circ \mathrm{D}\varphi_Q.$$

Since φ is a local diffeomorphism we know by the Tangent Space-Inclusion Proposition 29.20 (3) that $\mathrm{D}\varphi_Q$ is an isomorphism. It thus follows from the above calculation that $Q \in N$ is a critical point of $f\circ\varphi\colon N \to \mathbb{R}$ if and only if P is a critical point of $f\colon M \to \mathbb{R}$.



Now let $w_1, w_2 \in \mathrm{T}_QN$. We set $v_i := \mathrm{D}\varphi_Q(w_i) \in \mathrm{T}_PM$, $i = 1,2$. We pick smooth vector fields V_1, V_2 on M with $V_i(P) = v_i$, $i = 1,2$. Next, given any vector field V on M we consider the vector field φ^*V on N that is given by $(\varphi^*V)(y) := (\mathrm{D}\varphi_y)^{-1}(V(\varphi(y))$. We make two observations:

(a) It follows from the Vector Field-Pullback Lemma 29.33 that if V is smooth, then φ^*V is also a smooth vector field on N.

(b) Given any smooth function $g\colon M \to \mathbb{R}$ and given any $y \in N$ we have the following equalities in $\mathbb{R}$:

$$\begin{array}{rcccl} V_{\varphi(y)}g & \underset{\text{by the claim in the proof of the Vector Field–Derivative Lemma 249.3}}{=} & \mathrm{D}g_{\varphi(y)}(V(\varphi(y))) & = & (\mathrm{D}g_{\varphi(y)}\circ\mathrm{D}\varphi_y)((\mathrm{D}\varphi_y)^{-1}(V(\varphi(y)))) \\ & \underset{\text{by the Chain Rule and the definition of } \varphi^*V}{=} & \mathrm{D}(g\circ\varphi)_y((\varphi^*V)(y)) & \underset{\text{by the claim in the proof of the Vector Field–Derivative Lemma 249.3}}{=} & (\varphi^*V)_y(g\circ\varphi). \end{array}$$

In summary we have shown that $V(g)\circ\varphi = (\varphi^*V)(g\circ\varphi)\colon N \to \mathbb{R}$.

Now we see that

$$\begin{array}{rcccl} \mathrm{Hess}(f)_P(\mathrm{D}\varphi_Q(w_1),\mathrm{D}\varphi_Q(w_2)) & \underset{\text{definition of } v_i}{=} & \mathrm{Hess}(f)_P(v_1,v_2) & \underset{\text{by the definition of Hess}(f)_P \text{ and since } \varphi(Q)=P}{=} & (V_1(V_2(f))\circ\varphi)(Q) \\ & \underset{\text{Observation (b) applied to } V_1 \text{ and to } g = V_2(f)}{=} & ((\varphi^*V_1)(V_2(f)\circ\varphi))(Q) & \underset{\text{Observation (b) applied to } V_2 \text{ and to } g = f}{=} & (\varphi^*V_1)((\varphi^*V_2)(f\circ\varphi)))(Q) \\ & = & (\varphi^*V_1)_Q((\varphi^*V_2)(f\circ\varphi)) & = & \mathrm{Hess}(f\circ\varphi)_Q(w_1,w_2). \end{array}$$

(2) Let M be an open subset of $\mathbb{R}^n$ and let $i, j \in \{1, \ldots, n\}$. In the following we denote by $E_i \colon M \to \mathbb{R}^n$ and $E_j \colon M \to \mathbb{R}^n$ the constant vector fields which assign to each $x \in M$ the standard basis vectors $\mathfrak{e}_i$ respectively $\mathfrak{e}_j$. Now we see that

$$\underset{\text{by definition}}{\uparrow} \quad \underset{\text{see page 4797}}{\uparrow} \quad \underbrace{\frac{d}{dt}(x \mapsto f(x + t \cdot \mathfrak{e}_j))}_{=\frac{\partial f}{\partial x_j}} \quad \underset{\text{same argument again}}{\uparrow}$$

$$\operatorname{Hess}(f)_P(\mathfrak{e}_i, \mathfrak{e}_j) = E_{iP}(E_j(f)) = E_{iP}\Big(\frac{d}{dt}(x \mapsto f(x + t \cdot \mathfrak{e}_j))\Big) = \frac{\partial^2 f}{\partial x_i \, \partial x_j}\Big|_P .$$

■

249.2. **Non-degenerate critical points.** Let M be a smooth manifold, let $f \colon M \to \mathbb{R}$ be a smooth function and let $P \in M$ be a critical point of f. In the Hessian Form Proposition 249.4 we just introduced the symmetric $\mathbb{R}$-bilinear form

$$\operatorname{Hess}(f)_P \colon \mathrm{T}_P M \times \mathrm{T}_P M \to \mathbb{R}.$$

At this stage it is worth recalling some concepts from Definition 224.3, Definition 226.6 and Definition 226.8 about symmetric forms over $\mathbb{R}$:

Definition 249.7. Let V be a finite-dimensional real vector space and let $\varphi \colon V \times V \to \mathbb{R}$ be a symmetric form over $\mathbb{R}$.

- The form is called non-singular if the map

$$\begin{array}{ccc} V & \to & \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ v & \mapsto & \left(\begin{array}{ccc} V & \to & \mathbb{R} \\ w & \mapsto & \varphi(v, w) \end{array} \right) \end{array}$$

  is an isomorphism.
- We say that φ is negative-definite if $\varphi(v, v) < 0$ for all $v \neq 0$.
- We set

$$b^-(\varphi) \coloneqq \max\{\dim_{\mathbb{R}}(U) \,|\, U \subseteq V \text{ vector subspace such that } \varphi|_U \text{ is negative-definite}\}.$$

The following theorem shows why we are interested in these two algebraic concepts:

Theorem 249.8. (Sylvester's Theorem) Let (V, φ) and (W, ψ) be two symmetric non-singular forms over finite-dimensional real vector spaces. Then

$$(V, \varphi) \text{ and } (W, \psi) \text{ are isometric} \iff \begin{array}{rcl} \dim_{\mathbb{R}}(V) & = & \dim_{\mathbb{R}}(W), \\ b^-(\varphi) & = & b^-(\psi). \end{array}$$

Proof. The theorem follows easily from the more general Sylvester's Theorem 226.12. ■

These purely algebraic definitions allow us to introduce new concepts regarding critical points:

Definition 249.9. Let M be a smooth manifold, let $f \colon M \to \mathbb{R}$ be a smooth function and let $P \in M$ be a critical point of f.

(1) We say that the critical point P of $f \colon M \to \mathbb{R}$ is non-degenerate if the Hessian form $\operatorname{Hess}(f)_P \colon \mathrm{T}_P M \times \mathrm{T}_P M \to \mathbb{R}$ is non-singular. Otherwise we call the critical point degenerate.

(2) We refer to

$$\operatorname{index}(f, P) \coloneqq b^-(\operatorname{Hess}(f)_P \colon \mathrm{T}_P M \times \mathrm{T}_P M \to \mathbb{R})$$

as the index of the critical point P.

In practice the following lemma is a very convenient tool for determining whether a Hessian form is non-degenerate, and to determine the index of a critical point:

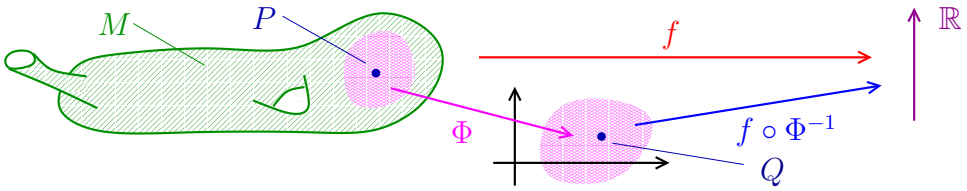
Lemma 249.10. (Critical Point-via-Matrix Lemma) Let M be an n-dimensional smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M$ be a critical point. Furthermore let $\Phi\colon U \to V$ be a chart around P. We set $Q := \Phi(P)$.

(0) $$P \text{ is a critical point of } f \iff \frac{\partial(f\circ\Phi^{-1})}{\partial x_i}(Q) = 0 \text{ for } i = 1,\dots,n.$$

(1) $$P \text{ is a non-degenerate critical point} \iff \det(\underbrace{\mathrm{Hess}(f\circ\Phi^{-1})_Q}_{=\left(\frac{\partial^2(f\circ\Phi^{-1})}{\partial x_i \partial x_j}\big|_Q\right)}) \neq 0.$$

(2) Let P be a critical point. Then

$$\mathrm{index}(f,P) = \text{number of negative eigenvalues of the Hessian matrix } \mathrm{Hess}(f\circ\Phi^{-1})_Q.$$



Remark. In principle one could also use the statement of the Critical Point-via-Matrix Lemma 249.10 to define, via charts, the notion of a non-degenerate critical point and to define the index of a critical point. If one wants to go down that route, then one needs to show that the notions do not depend on the choice of the chart. The, slightly tricky, details are worked out in [**Gaul1982**] and [**Cou**]. □

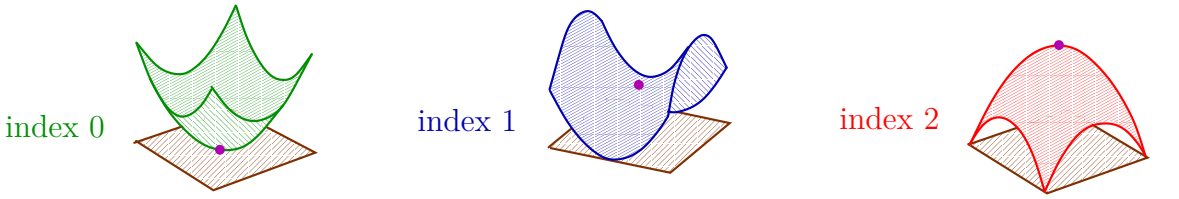
Proof. The lemma follows easily from the fact that the inclusion $U \to M$ is a local diffeomorphism, that Φ is a diffeomorphism, the Hessian Form-Calculation Lemma 249.5 (1) and (2), the Isometry Invariants Lemma 226.9, the Form–Matrix-Lemma 224.5 (2) and the Signature-Eigenvalues Proposition 226.11. ■

Examples.

(1) The "mother of all examples" are the functions

$$\begin{aligned} f\colon \mathbb{R}^n &\to \mathbb{R} \\ (x_1,\dots,x_n) &\mapsto c + \sum_{i=1}^n a_i \cdot x_i^2 \end{aligned}$$

with $c, a_1, \dots, a_n \in \mathbb{R}$. Evidently the origin 0 is a critical point of f. With respect to the identity chart the Hessian $\mathrm{Hess}(f)_0 \in \mathrm{M}(n\times n, \mathbb{R})$ is the diagonal matrix with diagonal entries $2a_1, \dots, 2a_n$.

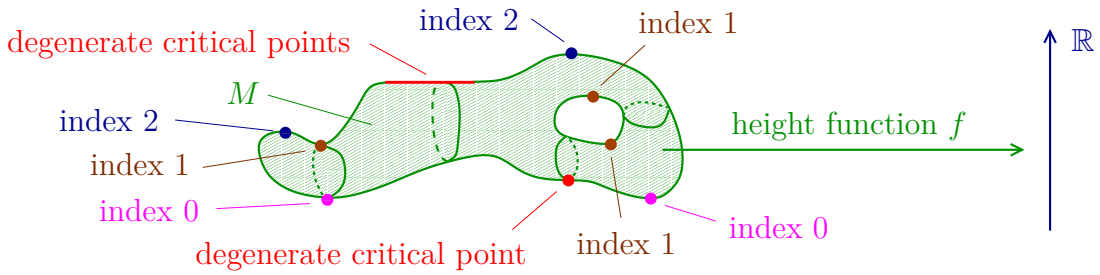


graph of $x_1^2 + x_2^2$ graph of $x_1^2 - x_2^2$ graph of $-x_1^2 - x_2^2$

It follows from the Critical Point-via-Matrix Lemma 249.10 (1) that the origin is a non-degenerate critical point if and only if all a_i are non-zero. Finally it follows from the Critical Point-via-Matrix Lemma 249.10 (2) that we have the following equality:

$$\text{index}(f, 0) \;=\; \text{number of negative } a_i.$$

(2) We consider the smooth manifold $M \subseteq \mathbb{R}^3$ that is shown in the figure below together with the "height function" $f\colon M \to \mathbb{R}$ that is defined by $f(x, y, z) = z$.

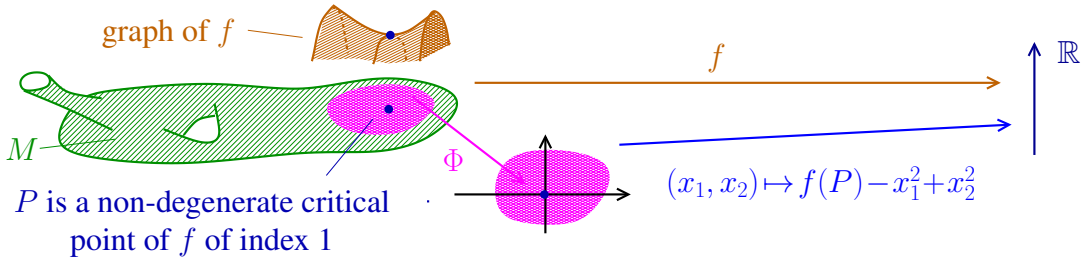


In the figure we indicate all critical points of f and we show their indices. □

249.3. **The Morse Lemma.** The following result says that, up to taking an appropriate chart, the above mother of all examples is the only example of a non-degenerate critical point. The result was first proved by Marston Morse [**MorsM1934**] in 1934.

Theorem 249.11. (Morse Lemma) Let M be an n-dimensional smooth manifold. Furthermore let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M \setminus \partial M$ be a critical point of index $k = \text{index}(f, P)$. If P is non-degenerate, then there exists a chart $\Phi\colon U \to V$ around P with $U \subseteq M \setminus \partial M$ and with $\Phi(P) = 0$ such that for every $(x_1, \ldots, x_n) \in V$ we have

$$(f \circ \Phi^{-1})(x_1, \ldots, x_n) \;=\; f(P) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$



Remark.

(1) Recall that the Inverse Mapping Theorem 30.1 says that if the differential $\mathrm{D}f_P$ of a given map $f\colon M \to N$ between two smooth manifolds at a point $P \in M \setminus \partial M$ is invertible, then the map is locally a diffeomorphism. The Morse lemma is a similar statement: at a non-degenerate critical point the local behavior of a smooth map $f\colon M \to \mathbb{R}$ is determined by the second partial derivatives.

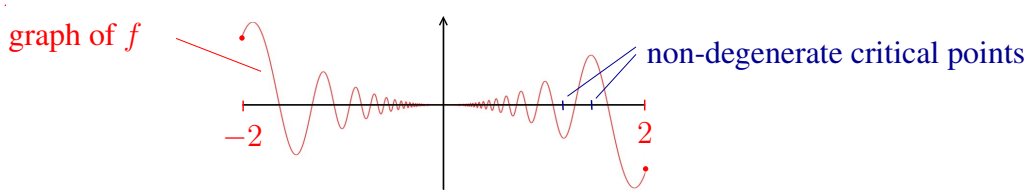
(2) Our proof of the Morse Lemma 249.11 is based on the proof in [**Miln1963a**, Lemma 2.2]. Somewhat different proofs are given in [**Nic2011**, Theorem 1.12] and [**Lau2011**, Section 3.3.2]. Finally more general proofs, that also work in the infinite-dimensional setting, are given by Richard Palais [**Pal1963**, p. 307] and [**Pal1969**, p. 968].

(3) Let M be an n-dimensional smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M \setminus \partial M$ be a non-degenerate critical point. It follows easily from the Morse Lemma 249.11 that
   (a) P is a local maximum of f if and only if $\operatorname{index}(f, P) = n$.
   (b) P is a local minimum of f if and only if $\operatorname{index}(f, P) = 0$. □

Before we turn to the proof of the Morse Lemma 249.11, let us formulate the following fairly straightforward corollary:

Corollary 249.12. (Morse Function–Finiteness Corollary) Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function which has no critical points on ∂M. If all critical points of f are non-degenerate, then there exist only finitely many non-degenerate critical points of f.

Remark. In the figure below we show the graph of a smooth function $f\colon [-2, 2] \to \mathbb{R}$ where not all critical points are non-degenerate.



In this example the set of non-degenerate critical points of f is infinite. This shows in particular that in the Morse Function–Finiteness Corollary 249.12 we really need to assume that all critical points are non-degenerate. □

Proof of Corollary 249.12 assuming the Morse Lemma 249.11. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function which has no critical points on ∂M. We start out with the following claim:

Claim. The set of non-degenerate critical points of f is a discrete subset of M.

Proof. Since f has no critical points on ∂M we see that it follows immediately from the Morse Lemma 249.11 together with the Critical Point-via-Matrix Lemma 249.10 that every non-degenerate critical point admits an open neighborhood which contains no other critical point. But this means, by definition, that the non-degenerate critical points of f form a discrete subset of M. ⊞

Next note that in the Regular Value-Open Subset Lemma 32.2 we showed that the set of critical points of f is a closed subset of M.

We now assume that M is compact and that all critical points are non-degenerate. It follows from the claim and the above discussion that this implies that the set of non-degenerate critical points is a discrete and closed subset of the compact topological space M. It now follows from the Compact-Discrete Lemma 3.40 (2) that the set of non-degenerate critical points is indeed finite. ■

The proof of the Morse Lemma 249.11 will occupy the remainder of this section. In the next Section 249.4 we will introduce the notion of a "Morse function".

One of the key ingredients to the proof of the Morse Lemma 249.11 is the following earlier proposition:

Proposition 24.23. (Map-as-Matrix Multiplication Proposition) Let $U \subseteq \mathbb{R}^n$ be an open convex subset that contains the origin and let $f\colon U \to \mathbb{R}$ be a smooth function with $f(0) = 0$. There exist smooth maps $a_1, \ldots, a_n\colon U \to \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) \;=\; \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \cdot x_i \quad \text{for all } x = (x_1, \ldots, x_n) \in U$$

and such that for $i = 1, \ldots, n$ we have $a_i(0) = \frac{\partial f}{\partial x_i}(0)$.

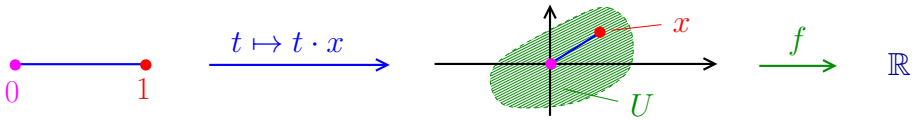
Proof. For the reader's convenience we repeat the proof of the proposition. First observe that given any $x = (x_1, \ldots, x_n) \in U$ we have

$$f(x) \;=\; f(x) - \underbrace{f(0)}_{=0} \;=\; \int_{t=0}^{t=1} \frac{d}{dt} f(\underbrace{t \cdot x}_{\in U})\, dt \;=\; \int_{t=0}^{t=1} \sum_{i=1}^{n} \big(\frac{\partial f}{\partial x_i}\big)(t \cdot x) \cdot x_i \, dt$$

by the Fundamental Theorem of Calculus 73.27

by the Chain Rule 24.17

$$= \sum_{i=1}^{n} \underbrace{\int_{t=0}^{t=1} \big(\frac{\partial f}{\partial x_i}\big)(t \cdot x)\, dt}_{=:a_i(x)} \cdot x_i.$$



Note that it follows from standard methods in analysis, e.g. the Leibniz Integral Rule 73.28, that each $a_i\colon U \to \mathbb{R}$ is smooth. Finally note that it follows from the above definition of a_i and the Product Rule that we have the equality $a_i(0) = \frac{\partial f}{\partial x_i}(0)$. ∎

The Map-as-Matrix Multiplication Proposition 24.23 gives us the following corollary, which is actually the content of Exercise 24.3.

Corollary 249.13. (Function-near-Critical Point Corollary) Let $U \subseteq \mathbb{R}^n$ be an open convex subset that contains the origin and let $f\colon U \to \mathbb{R}$ be a smooth function with $f(0) = 0$. If 0 is a critical point of f, then there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$, $i, j = 1, \ldots, n$ such that $h_{ij} = h_{ji}$ and such that

$$f(x) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j \quad \text{for all } x = (x_1, \ldots, x_n) \in U.$$

Proof of Corollary 249.13. First note that by the Map-as-Matrix Multiplication Proposition 24.23 there exist smooth maps $a_1, \ldots, a_n\colon U \to \mathbb{R}$ such that

$$f(x) \;=\; \sum_{i=1}^{n} a_i(x) \cdot x_i \quad \text{for all } x = (x_1, \ldots, x_n) \in U$$

and such that for $i = 1, \ldots, n$ we have $a_i(0) = \frac{\partial f}{\partial x_i}(0)$. Since 0 is a critical point of f, we know that $a_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$ for $i = 1, \ldots, n$. Thus, given any $i \in \{1, \ldots, n\}$ we can once again apply the ever-useful Map-as-Matrix Multiplication Proposition 24.23 to the smooth map a_i and we obtain smooth maps $b_{i1}, \ldots, b_{in}\colon U \to \mathbb{R}$ with $b_{ij}(0) = (\frac{\partial}{\partial x_j} a_i)(0) = (\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f)(0)$

and such that

$$a_i(x) \;=\; \sum_{j=1}^{n} b_{ij}(x)\cdot x_j \qquad \text{for all } x=(x_1,\dots,x_n)\in U.$$

It now clear that the maps $h_{ij} := \frac{1}{2}(b_{ij}+b_{ji})$ have the desired properties. ■

Proof of the Morse Lemma 249.11. The statement of the Morse Lemma 249.11 is a local statement which, by the Hessian Form-Calculation Lemma 249.5 (1), does not change under an application of a local diffeomorphism. It follows that it suffices to deal with the case that M is an open convex neighborhood of $0\in\mathbb{R}^n$ and that $P=0$ is the non-degenerate critical point of $f\colon M\to\mathbb{R}$. We set $k := \operatorname{index}(f,0)$. Finally it is clear that we might as well assume that $f(0)=0$.

Note that it follows from the Function-near-Critical Point Corollary 249.13 that there exists a smooth map $H\colon M\to \operatorname{Sym}(n\times n,\mathbb{R})$ such that for every $x\in M$ the map $f\colon M\to\mathbb{R}$ is given by

$$\begin{array}{rccc} f\colon M & \to & \mathbb{R} \\ x & \mapsto & x^{\mathrm{T}}\cdot H(x)\cdot x. \end{array}$$

We set $J := \operatorname{diag}(\epsilon_1,\dots,\epsilon_n)$ where the first k entries are equal to -1 and the remaining entries are equal to 1. We need to show that there exists a chart $\Phi\colon U\to V$ around 0 with $U\subseteq M$ and with $\Phi(0)=0$ such that the map $f\circ\Phi^{-1}\colon V\to\mathbb{R}$ is given by

$$\begin{array}{rccc} f\circ\Phi^{-1}\colon V & \to & \mathbb{R} \\ y & \mapsto & y^{\mathrm{T}}\cdot J\cdot y. \end{array}$$

Claim 1. There exists an open neighborhood W of $0\subseteq M$ and there exists a smooth map $Q\colon W\to \mathrm{GL}(n,\mathbb{R})$ such that for every $x\in W$ we have

$$Q(x)^{\mathrm{T}}\cdot H(x)\cdot Q(x) \;=\; J.$$

Proof. A straightforward calculation shows that $H(0)=\operatorname{Hess}(f)_0$. By hypothesis we know that $b^-(H(0))=k$. It follows from the above formulation of Sylvester's Theorem 249.8 together with the Signature-Eigenvalues Proposition 226.11 and the Form–Matrix-Lemma 224.5 (1b) that there exists a $Q\in\mathrm{GL}(n,\mathbb{R})$ with $Q^{\mathrm{T}}\cdot H(0)\cdot Q=J$. So we might as well assume that $H(0)=J$.

Next we turn the claim into a statement that lends itself to an inductive proof: We claim that for each $s\in\{0,\dots,d\}$ there exists an open neighborhood W of 0 and a smooth map $Q_s\colon W\to\mathrm{GL}(n,\mathbb{R})$ with $Q_s(0)=\mathrm{id}$ such that for every $x\in W$ we have

$$Q_s(x)^{\mathrm{T}}\cdot H(x)\cdot Q_s(x) \;=\; \begin{pmatrix} \operatorname{diag}(\epsilon_1,\dots,\epsilon_s) & 0 \\ 0 & * \end{pmatrix}.$$

Note that for $s=0$ the statement holds for trivial reasons and note that our goal is to prove the statement for $s=n$. Now suppose that we know that the statement for some $s\in\{0,\dots,n-1\}$.

We set $H'(x) := Q_s(x)^{\mathrm{T}}\cdot H(x)\cdot Q_s(x)$. Note that

$$H'(0) \;=\; \underbrace{Q_s(0)^{\mathrm{T}}}_{=\mathrm{id}}\cdot H(0)\cdot \underbrace{Q_s(0)}_{=\mathrm{id}} \;=\; H(0) \;=\; J.$$

It follows that, after possibly replacing the open neighborhood W by a smaller open neighborhood of 0, we can assume that the $(s+1,s+1)$-entry of $H'(x)=Q_s(x)^{\mathrm{T}}\cdot H(x)\cdot Q_s(x)$

is non-zero for all $x \in W$. We now set

$$Q_{s+1}(x) := \begin{pmatrix} \mathrm{id}_s & 0 & 0 \\ 0 & \frac{1}{\sqrt{|H_{s+1,s+1}(x)|}} & \left(-\frac{H_{s+1,i}(x)}{H_{s+1,s+1}(x)}\right)_{i=s+2,\dots,n} \\ 0 & 0 & \mathrm{id}_{n-s-1} \end{pmatrix}$$

One can easily verify that this map $Q_{s+1}\colon W \to \mathrm{GL}(n,\mathbb{R})$ has the desired properties. ⊞

We now consider the map

$$\begin{aligned} \Phi\colon W &\to \mathbb{R}^n \\ x &\mapsto Q(x)^{-1} \cdot x. \end{aligned}$$

It follows easily from the Product Rule that $\mathrm{D}\Phi_0 = Q(0)^{-1}$. Since this matrix is invertible we obtain from the Inverse Function Theorem 24.21 that there exists an open neighborhood U of $0 \in W$ and an open neighborhood V of $\Phi(0) = 0 \in \mathbb{R}^n$ such that $\Phi\colon U \to V$ is a diffeomorphism. We claim that Φ is the desired diffeomorphism:

Claim 2. For every $y \in V$ we have $(f \circ \Phi^{-1})(y) = y^{\mathrm{T}} \cdot J \cdot y$.

Proof. Let $y \in V$. We set $x := \Phi^{-1}(y)$. Note that this means that $y = \Phi(x) = Q^{-1}(x) \cdot x$. We see that

$$(f \circ \Phi^{-1})(y) \underset{\substack{\uparrow \\ \text{definition of } x}}{=} f(x) \underset{\substack{\uparrow \\ \text{choice of } H}}{=} x^{\mathrm{T}} \cdot H(x) \cdot x \underset{\substack{\uparrow \\ \text{choice of } Q}}{=} \overbrace{x^{\mathrm{T}} \cdot (Q(x)^{\mathrm{T}})^{-1}}^{=(Q^{-1}(x)\cdot x)^{\mathrm{T}} = y^{\mathrm{T}}} \cdot J \cdot \overbrace{Q(x)^{-1} \cdot x}^{=y} \underset{\substack{\uparrow \\ \text{definition of } x}}{=} y^{\mathrm{T}} \cdot J \cdot y.$$

■

249.4. **Morse functions.** In this section we will introduce Morse functions on smooth manifolds. To simplify the discussion we do so in two steps:

(1) First we introduce Morse functions on closed smooth manifolds.
(2) Then we introduce Morse functions for compact smooth manifolds rel a union of boundary components.

249.4.1. **Morse functions on closed smooth manifolds.** Here is the key definition of this subsection:

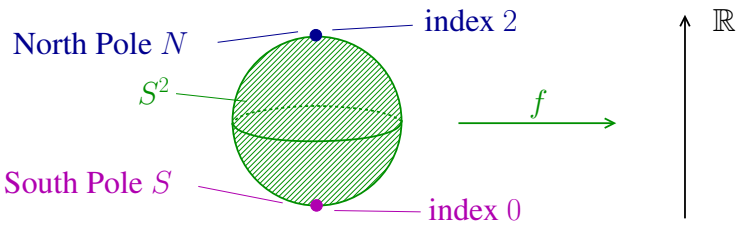
Definition 249.14. Let M be a closed smooth manifold. A smooth function $f\colon M \to \mathbb{R}$ is called a Morse function if all critical points of f are non-degenerate.

Example.

(1) Let $n \in \mathbb{N}$. We claim that

$$\begin{aligned} f\colon S^n &\to \mathbb{R} \\ (x_1,\dots,x_{n+1}) &\mapsto x_{n+1} \end{aligned}$$

is a Morse function on the sphere S^n.



Indeed: First note that one can easily verify that the only critical points of f are the South Pole $S = (0,\dots,-1)$ and the North Pole $N = (0,\dots,0,1)$. We consider the

charts

$$\begin{array}{rcl}\Phi_S\colon S^n_{<0} & \to & B^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n)\end{array} \quad \text{and} \quad \begin{array}{rcl}\Phi_N\colon S^n_{>0} & \to & B^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n).\end{array}$$

Note that

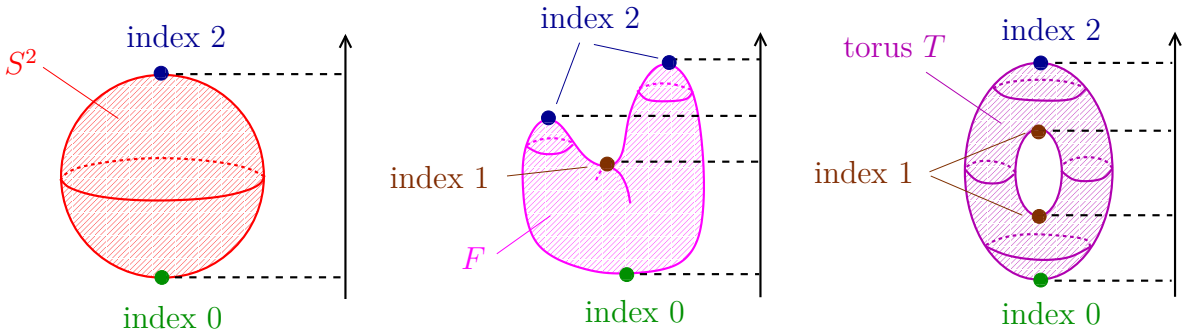
$$\begin{aligned}(f\circ(\Phi_S)^{-1})(x_1,\dots,x_n) &= -\sqrt{1-x_1^2-\cdots-x_n^2} \\ (f\circ(\Phi_N)^{-1})(x_1,\dots,x_n) &= \sqrt{1-x_1^2-\cdots-x_n^2}.\end{aligned}$$

A completely elementary calculation shows that $\mathrm{Hess}(f\circ(\Phi_S)^{-1})_0 = \mathrm{id}_n$ and that $\mathrm{Hess}(f\circ(\Phi_N)^{-1})_0 = -\mathrm{id}_n$. It thus follows from the Critical Point-via-Matrix Lemma 249.10 that both critical points are non-degenerate and that $\mathrm{index}(f,S) = 0$ and $\mathrm{index}(f,N) = n$.

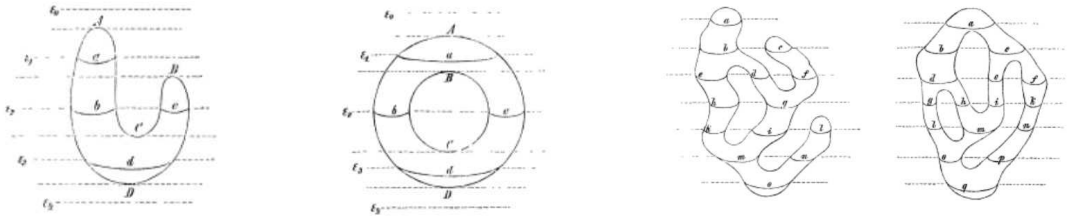
(2) In the figure below we show three surfaces S^2, F and T in $\mathbb{R}^3$. In each case the "height function"

$$\begin{array}{rcl}h\colon \mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & z\end{array}$$

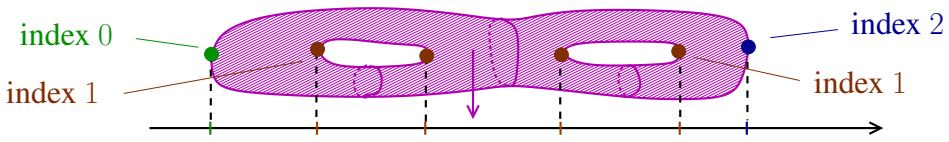
is a Morse function. Evidently S^2 and F are diffeomorphic, but the numbers of critical points differ.



Just for fun we point out that in 1863 August Möbius [**Möb1863**] drew basically the same pictures, see the figure below.



(3) The figure below strongly suggests that for every $g \in \mathbb{N}_0$ the surface of genus g admits a Morse function with one critical point of index 0, $2g$ critical points of index 1 and one critical point of index two.



A more formal proof of that statement will be given in Exercise 249.17.

(4) Let $n \in \mathbb{N}$. In the Matrix Group-Smooth Manifold Proposition 32.10 we showed that $\mathrm{SO}(n)$ is a closed smooth manifold of dimension $\frac{1}{2}n(n-1)$. Let $1 < c_1 < \cdots < c_n$ be

real numbers. In [**Mata2002**, Lemma 3.12] it is shown that the map

$$\begin{array}{rcl} \mathrm{SO}(n) & \to & \mathbb{R} \\ A = (a_{ij}) & \mapsto & \sum_{j=1}^{n} c_j \cdot a_{jj} \end{array}$$

is a Morse function. Furthermore it is shown that the critical points are precisely the diagonal matrices in $\mathrm{SO}(n)$. Finally in [**Mata2002**, p. 95] it is shown that for a critical point $\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ the index is given as follows: Let $i_1 < \cdots < i_s$ be the entries where $\epsilon = 1$. The index of $\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ equals $(i_1 - 1) + \cdots + (i_s - 1)$. With very similar approaches one can also obtain explicit Morse functions on the smooth manifolds $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. We refer to [**Mata2002**, Examples 3.13 and 3.14] for details. □

Next we want to give explicit Morse functions on projective spaces. We have the following proposition, which is suspiciously similar to the Projective Space-Handle Decomposition Proposition 246.12.[2688]

Proposition 249.15. (Projective Space–Morse Function Proposition) Let $n \in \mathbb{N}_0$. The projective spaces $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ admit Morse functions with precisely $n + 1$ critical points. The indices are given as follows:

(1) The Morse function on $\mathbb{R}\mathrm{P}^n$ has one critical point of index $0, 1, 2, \ldots, n$.
(2) The Morse function on $\mathbb{C}\mathrm{P}^n$ has one critical point of index $0, 2, 4, \ldots, 2n$.
(3) The Morse function on $\mathbb{H}\mathrm{P}^n$ has one critical point of index $0, 4, 8, \ldots, 4n$.

Proof. Let $n \in \mathbb{N}_0$. In the following we will deal with the complex projective space $\mathbb{C}\mathrm{P}^n$. (The other two types of projective spaces are dealt with in a very similar way.) We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the function:

$$\begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \dfrac{c_0 \cdot |z_0|^2 + \cdots + c_n \cdot |z_n|^2}{|z_0|^2 + \cdots + |z_n|^2}. \end{array}$$

One can easily verify that this map is well-defined. We claim that this map is a Morse function, with precisely $n + 1$ critical points and that these are of index $0, 2, \ldots, 2n$.

For $i = 0, \ldots, n$ we consider the chart

$$\begin{array}{rcl} \Phi_i\colon U_i := \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\} & \to & \mathbb{C}^n \\ [z_0 : \cdots : z_n] & \mapsto & \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \end{array}$$

from the Projective Space–Manifold Proposition 25.13. Furthermore, as on page 652 we consider the diffeomorphism

$$\begin{array}{rcl} \Xi\colon \mathbb{C}^n & \to & B^{2n} \\ z & \mapsto & \dfrac{z}{\sqrt{1 + \|z\|^2}}. \end{array}$$

We will now work with the charts $\Xi \circ \Phi_i\colon U_i \to B^{2n}$, $i = 0, \ldots, n$.

[2688] Also note that the proposition is just a reformulation of Exercise 53.5.

Claim. Let $i \in \{0, \dots, n\}$. The map $g \circ (\Xi \circ \Phi_i)^{-1} \colon B^{2n} \to \mathbb{R}$ is given by

$$\begin{aligned} g \circ (\Xi \circ \Phi_i)^{-1} \colon B^{2n} &\to \mathbb{R} \\ \underbrace{(z_1, \dots, z_n)}_{\in B^{2n} \subseteq \mathbb{R}^{2n} =_{\mathrm{i}} \mathbb{C}^n} &\mapsto \sum_{j=1}^{i} (c_{j-1} - c_i) \cdot |z_j|^2 + c_i + \sum_{j=i+1}^{n} (c_j - c_i) \cdot |z_j|^2. \end{aligned}$$

Proof. First note that the inverse of the chart $\Phi_i \colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}^n$ is given by the map

$$\begin{aligned} \Phi_i^{-1} \colon \mathbb{R}^{2n} =_{\mathrm{i}} \mathbb{C}^n &\to U_i \\ (z_1, \dots, z_n) &\mapsto [z_1 : \cdots : z_i : 1 : z_{i+1} : \cdots : z_n]. \end{aligned}$$

Furthermore recall that we saw on page 652 that the inverse to $\Xi \colon \mathbb{C}^n \to B^{2n}$ is given by the map

$$\begin{aligned} \Xi^{-1} \colon B^{2n} &\to \mathbb{C}^n \\ z &\mapsto \frac{z}{\sqrt{1 - \|z\|^2}}. \end{aligned}$$

Now we see that for any $(z_1, \dots, z_n) \in B^{2n}$ we have

$$\begin{aligned} (g \circ \Phi_i^{-1} \circ \Xi^{-1})(z_1, \dots, z_n) &\overset{\text{by the above calculation of } \Xi^{-1}}{=} (g \circ \Phi_i^{-1})\Big(\frac{z}{\sqrt{1-\|z\|^2}}\Big) \overset{\text{by the above description of } \Phi_i^{-1} \text{ and since } [\frac{z_1}{\lambda} : 1 : \frac{z_n}{\lambda}] = [z_1 : \lambda : z_n]}{=} \\ &= g([z_1 : \cdots : z_i : \sqrt{1 - \|z\|^2} : z_{i+1} : \cdots : z_n]) \\ &= \sum_{j=1}^{i} c_{j-1} \cdot |z_j|^2 + c_i \cdot (1 - \|z\|^2) + \sum_{j=i+1}^{n} c_j \cdot |z_j|^2 \\ &\overset{\text{definition of } g}{=} \sum_{j=1}^{i} (c_{j-1} - c_i) \cdot |z_j|^2 + c_i + \sum_{j=i+1}^{n} (c_j - c_i) \cdot |z_j|^2. \\ &\overset{\text{since } \|z\|^2 = |z_1|^2 + \ldots + |z_n|^2}{} \end{aligned}$$

⊞

In the following let $i \in \{0, \dots, n\}$. The claim, rewritten in real coordinates, tells us that the map $f \circ (\Xi \circ \Phi_i)^{-1} \colon B^{2n} \to \mathbb{R}$ is given by

$$\begin{aligned} g \circ (\Xi \circ \Phi_i)^{-1} \colon B^{2n} &\to \mathbb{R} \\ \underbrace{(x_1, y_1, \dots, x_n, y_n)}_{=(x_1 + \mathrm{i} y_1, \dots, x_n + \mathrm{i} y_n)} &\mapsto \sum_{j=1}^{i} \underbrace{(c_{j-1} - c_i)}_{<0} \cdot \underbrace{(x_j^2 + y_j^2)}_{=|x_j + \mathrm{i} y_j|^2} + c_i + \sum_{j=i+1}^{n} \underbrace{(c_j - c_i)}_{>0} \cdot \underbrace{(x_j^2 + y_j^2)}_{=|x_j + \mathrm{i} y_j|^2}. \end{aligned}$$

It follows easily from this calculation together with the Critical Point-via-Matrix Lemma 249.10 and $c_0 < c_1 < \cdots < c_n$ that $g \colon \mathbb{C}\mathrm{P}^n \to \mathbb{R}$ has precisely one critical point in U_i, namely $P_i := [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ (where the 1 is in the i-th entry), that this critical point is non-degenerate and that its index is given by $2i$.

Since P_i is not contained in any U_j, $j \neq i$, and since $U_0, \dots, U_n$ cover all of $\mathbb{C}\mathrm{P}^n$ we obtain from the above calculation the desired result. ■

249.4.2. **Morse functions on arbitrary compact smooth manifolds.** Even though it is not entirely clear yet what Morse functions are good for, we will now extend Definition 249.14 from closed smooth manifolds to smooth manifolds with non-empty boundary:

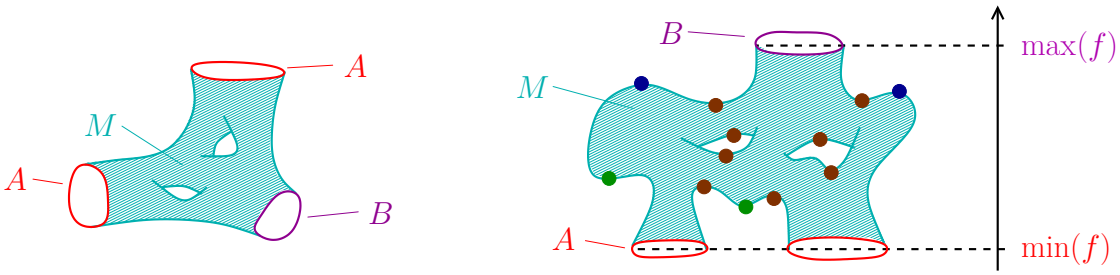
Definition 249.16. Let M be a compact smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. A smooth function $f\colon M \to \mathbb{R}$ is called a Morse function rel A if the following statements holds:

(1) All critical points of f are non-degenerate.
(2) There is no critical point on ∂M.
(3) • If $A \neq \varnothing$, then $A = f^{-1}(\min(f))$.
• If $B \neq \varnothing$, then $B = f^{-1}(\max(f))$.

Note that conditions (2) and (3) are automatically satisfied if $\partial M = \varnothing$. In particular, if M is closed, then a Morse function rel $\varnothing$ is exactly the same as a Morse function as introduced in Definition 249.14.

Example.

(1) In the figure below we show to the left a surface M and a union A of components of ∂M.

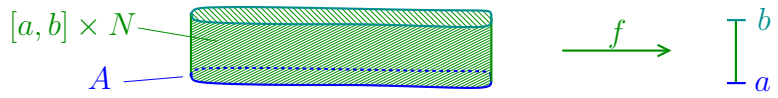


Furthermore we show to the right a smooth embedding of M into $\mathbb{R}^3$ such that the height function $(x, y, z) \mapsto z$ is a Morse function rel A.

(2) Let N be a closed smooth manifold and let $a < b$ be two real numbers. We consider the compact smooth manifold $[a, b] \times N$. Not surprisingly we set $A := \{a\} \times N$. Note that $B = \{b\} \times N$. We consider the map

$$\begin{array}{rcl} f\colon [a,b] \times N & \to & \mathbb{R} \\ (t, x) & \mapsto & t. \end{array}$$

Note that the following two statements hold:
(a) All points of $[a, b] \times N$ are regular points of f.
(b) The function f is a Morse function rel A.

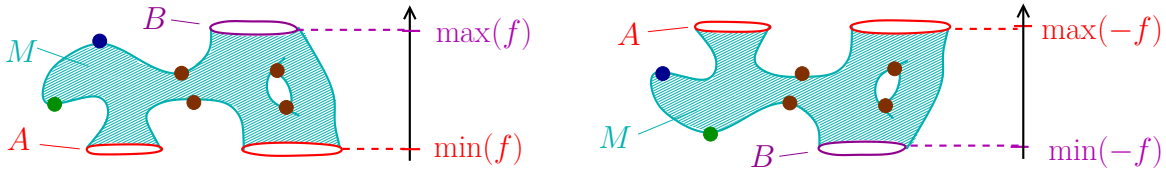


Later, in the Smooth Manifold-Product Theorem 250.1, we will prove a converse to the above. More precisely we will see that it follows from the Smooth Manifold-Product Theorem 250.1 that if a smooth manifold M admits a Morse function $f\colon M \to \mathbb{R}$ rel some A with Properties (a) and (b), then M is in fact diffeomorphic to a product $[a, b] \times A$.

(3) Let M be a compact smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Let $f\colon M \to \mathbb{R} \subseteq \mathbb{R}$ be a Morse function rel A. It follows easily from the definitions that the function $-f\colon M \to \mathbb{R}$ is a Morse function rel B. Furthermore one can easily verify that f and $-f$ have the same critical points, with

the key difference that for a critical point x of f we have

$$\operatorname{index}(-f,x) \;=\; n-\operatorname{index}(f,x).$$



(4) Let $n\in\mathbb{N}$. We consider $M=\overline{B}^n$ with $A=\varnothing$. One can easily verify that the map

$$\begin{array}{rcl} f\colon \overline{B}^n & \to & \mathbb{R} \\ (x_1,\ldots,x_n) & \mapsto & x_1^2+\cdots+x_n^2 \end{array}$$

is a Morse function of $M=\overline{B}^n$ rel $\varnothing$. It follows from (3) that $-f$ is a Morse function of $\overline{B}^n$ rel S^{n-1}. □

We continue with a general construction of Morse functions:

Proposition 249.17. (Product–Morse Function Proposition) Let M and N be two closed smooth manifolds and let $f\colon M\to\mathbb{R}$ and $g\colon N\to\mathbb{R}$ be Morse functions rel $\varnothing$. The map[2689]

$$\begin{array}{rcl} \Xi\colon M\times N & \to & \mathbb{R} \\ (x,y) & \mapsto & f(x)+g(y) \end{array}$$

is a Morse function rel $\varnothing$ with the following properties:

(1) The critical points of Ξ are precisely the points (x_0,y_0) where $x_0\in M$ is a critical point of $f\colon M\to\mathbb{R}$ and $y_0\in N$ is a critical point of $g\colon N\to\mathbb{R}$.

(2) (a) A critical point (x_0,y_0) of Ξ is non-degenerate if and only if x_0 is a non-degenerate critical point of f and if y_0 is a non-degenerate critical point of g.

(b) Given any critical point (x_0,y_0) of Ξ we have

$$\operatorname{index}(\Xi,(x_0,y_0)) \;=\; \operatorname{index}(f,x_0)+\operatorname{index}(g,y_0).$$

Example. We consider the torus $S^1\times S^1$. It follows from the Product–Morse Function Proposition 249.17 and the discussion of the "height Morse function" on S^1 from page 4808 that the map

$$\begin{array}{rcl} S^1\times S^1 & \to & \mathbb{R} \\ ((x_1,y_1),(x_2,y_2)) & \mapsto & y_1+y_2 \end{array}$$

is a Morse function with the following types of critical points:

- one critical point of index 0,
- two critical points of index 1,
- one critical point of index 2.

Thus this Morse function has the same features as the Morse function on $S^1\times S^1$ from page 4809. □

[2689]Recall that by the Smooth Manifold Product Proposition 25.34 we know that the product $M\times N$ of two smooth manifolds M and N, where at most one has non-empty boundary, is naturally a smooth manifold.

Proof. It follows from the Hessian Form Proposition 249.4 that, after using charts, it suffices to deal with the case that M is an open subset of $\mathbb{R}^m$ and that N is an open subset of $\mathbb{R}^n$.

(1) Note that for any $x_0 \in M$ and $y_0 \in N$ we have

$$\mathrm{D}\Xi_{(x_0,y_0)} \;=\; \mathrm{D}(\underbrace{(x,y) \mapsto f(x)+g(y)}_{=\Xi(x,y)})_{(x_0,y_0)} \;=\; \begin{pmatrix}\mathrm{D}f_{x_0} & \mathrm{D}g_{y_0}\end{pmatrix}.$$

It is clear that the matrix $\mathrm{D}\Xi_{(x_0,y_0)}$ is zero if and only if both $\mathrm{D}f_{x_0}$ and $\mathrm{D}g_{y_0}$ are zero. This shows that (x_0, y_0) is a critical point of $\Xi(x, y) = f(x) + g(y)$ if and only if x_0 is a critical point of f and y_0 is a critical point of g.

(2) Now let (x_0, y_0) be a critical point of $\Xi(x, y) = f(x) + g(y)$. We calculate that

$$\mathrm{Hess}(\overbrace{(x,y) \mapsto f(x)+g(y)}^{=\Xi(x,y)})_{(x_0,y_0)} \;=\; \begin{pmatrix}\mathrm{Hess}(f)_{x_0} & 0\\ 0 & \mathrm{Hess}(g)_{y_0}\end{pmatrix}$$

$\uparrow$ follows easily from the above calculation of $\mathrm{D}\Xi_{(x_0,y_0)}$

It is now clear that $\det(\mathrm{Hess}\,\Xi_{(x_0,y_0)}) \neq 0$ if and only if $\det(\mathrm{Hess}\,f_{x_0}) \neq 0$ and also $\det(\mathrm{Hess}\,g_{y_0}) \neq 0$. Furthermore it follows from elementary linear algebra that the number of negative eigenvalues of $\mathrm{Hess}\,\Xi_{(x_0,y_0)}$ equals the sum of the number of negative eigenvalues of $\mathrm{Hess}(f)_{x_0}$ and the number of negative eigenvalues of $\mathrm{Hess}(g)_{x_0}$. ∎

In the following proposition we give an alternative approach to constructing Morse functions on products of smooth manifolds. The alternative approach is slightly more complicated to formulate, but it has the big advantage that it also works if one of the two factors has non-empty boundary:

Proposition 249.18. (Product–Morse Function Proposition) Let M and N be smooth manifolds where M is closed and N is compact (possibly with non-empty boundary). Let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions rel $\varnothing$. Let $a \in \mathbb{R}$ such that $a + f(x) > 0$ for all $x \in M$ and let $b \in \mathbb{R}$ such that $b + g(y) < 0$ for all $y \in N \setminus \partial N$ and such that $b + g(y) = 0$ for all $y \in \partial N$. The map

$$\begin{array}{rcl}\Xi\colon M \times N & \to & \mathbb{R}\\ (x,y) & \mapsto & (a+f(x))\cdot(b+g(y))\end{array}$$

is a Morse function rel $\varnothing$ with the following properties:

(1) The critical points of Ξ are precisely the points (x_0, y_0) where $x_0 \in M$ is a critical point of $f\colon M \to \mathbb{R}$ and $y_0 \in N$ is a critical point of $g\colon N \to \mathbb{R}$.

(2) Let (x_0, y_0) be a critical point of Ξ.

(a) The critical point (x_0, y_0) is non-degenerate if and only if x_0 and y_0 are non-degenerate critical points.

(b) We have the following equality:

$$\mathrm{index}(\Xi, (x_0, y_0)) \;=\; \mathrm{index}(f, x_0) + \mathrm{index}(g, y_0).$$

Proof. It follows again from the Hessian Form Proposition 249.4 that, after using charts, it suffices to deal with the case that M is an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$ and that N is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$.

(1) Note that for any $x_0 \in M$ and $y_0 \in N$ we have

$$\mathrm{D}\Xi_{(x_0,y_0)} = \mathrm{D}(\underbrace{(a+f(x))\cdot(b+g(y))}_{=\Xi(x,y)})_{(x_0,y_0)} \underset{\text{by the Product Rule for partial derivatives}}{=} \begin{pmatrix}\mathrm{D}f_{x_0}\cdot(b+g(y_0)) & (a+f(x_0))\cdot\mathrm{D}g_{y_0}\end{pmatrix}.$$

We first consider the case that $y_0 \in N \setminus \partial N$. By our choice of a and b we know that $b + g(y_0) < 0$ and $a + f(x_0) > 0$. Thus we see that this matrix $\mathrm{D}\Xi_{(x_0,y_0)}$ is zero if and only if both $\mathrm{D}f_{x_0}$ and $\mathrm{D}g_{y_0}$ are zero. This shows that (x_0, y_0) is a critical point of $\Xi = (a + f(x)) \cdot (b + g(y))$ if and only if x_0 is a critical point of f and y_0 is a critical point of g.

For $y_0 \in \partial N$ we know that $\mathrm{D}g_{y_0}$ is non-zero. It follows from the above calculation, that $\mathrm{D}\Xi_{(x_0,y_0)}$ is non-zero. This shows that Ξ has no critical points on the boundary $\partial(M \times N) = M \times \partial N$.

(2) Now let (x_0, y_0) be a critical point of $\Xi = ((a + f(x)) \cdot (b + g(y)))$. We calculate that

$$\begin{aligned}\mathrm{Hess}(\overbrace{(a+f(x))\cdot(b+g(y))}^{=\Xi(x,y)})_{(x_0,y_0)} &\underset{\text{follows from the above calculation of } \mathrm{D}\Xi_{(x_0,y_0)} \text{ and the Product Rule}}{\downarrow} \\ &= \begin{pmatrix}\mathrm{Hess}(f)_{x_0}\cdot(b+g(y_0)) & (\mathrm{D}f_{x_0})^{\mathrm{T}}\cdot\mathrm{D}g_{y_0} \\ (\mathrm{D}g_{y_0})^{\mathrm{T}}\cdot\mathrm{D}f_{x_0} & (a+f(x_0))\cdot\mathrm{Hess}(g)_{y_0}\end{pmatrix} \\ &\underset{\text{since by (1) we know that } \mathrm{D}f_{x_0}=0 \text{ and } \mathrm{D}g_{y_0}=0}{=} \begin{pmatrix}\mathrm{Hess}(f)_{x_0}\cdot(b+g(y_0)) & 0 \\ 0 & (a+f(x_0))\cdot\mathrm{Hess}(g)_{y_0}\end{pmatrix}\end{aligned}$$

(a) Since $b + g(y_0) > 0$ and $a + f(x_0) > 0$ we see that $\det(\mathrm{Hess}(\Xi)_{(x_0,y_0)}) \neq 0$ if and only if $\det(\mathrm{Hess}(f)_{x_0}) \neq 0$ and $\det(\mathrm{Hess}(g)_{y_0}) \neq 0$. In other words, (x_0, y_0) is a non-degenerate critical point if and only if x_0 and y_0 are non-degenerate critical points.

(b) Since $b + g(y_0) > 0$ and $a + f(x_0) > 0$ we also see that the number of negative eigenvalues of $\mathrm{Hess}\,\Xi_{(x_0,y_0)}$ equals the sum of the number of negative eigenvalues of $\mathrm{Hess}(f)_{x_0}$ and the number of negative eigenvalues of $\mathrm{Hess}(g)_{x_0}$.

Finally, if $\partial(M \times N) \neq \varnothing$ we need to show that $\partial(M \times N) = \Xi^{-1}(\max(\Xi))$. This follows from the following elementary consideration: By construction we have $\Xi(x, y) < 0$ for all $(x, y) \in M \times (N \setminus \partial N)$ and $\Xi(x, y) = 0$ for all $(x, y) \in \partial(M \times N) = M \times \partial N$. ∎

We conclude this list of explicit constructions of Morse functions with the following proposition:

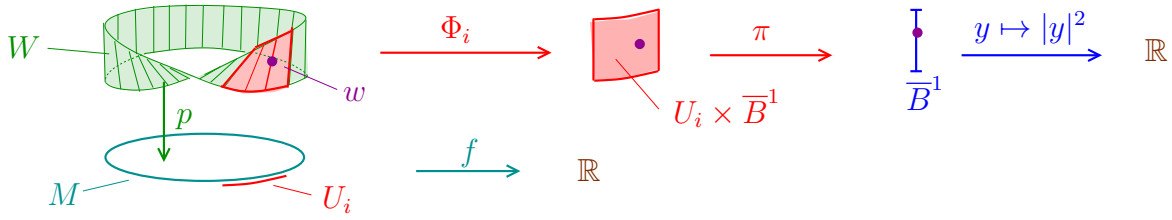
Proposition 249.19. (Linear Bundle–Morse Function Proposition) In the following let $p\colon W \to M$ be a smooth $(\overline{B}^n, \mathrm{O}(n))$-bundle[2690] where M is a closed smooth manifold. Given any Morse function $f\colon M \to \mathbb{R}$ there exists a Morse function $g\colon W \to \mathbb{R}$ with the same number of critical points as f and the same indices.

[2690] We refer to Definition 240.8 for the definition of a smooth $(\overline{B}^n, \mathrm{O}(n))$-bundle.

Sketch of proof. Let $(p\colon W \to M, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times \overline{B}^n\}_{i\in I})$ be a smooth $(\overline{B}^n, \mathrm{O}(n))$-bundle. We first construct a "radial distance squared" function $\rho\colon W \to \mathbb{R}_{\geq 0}$ as follows: Given $w \in W$ we pick $i \in I$ with $w \in p^{-1}(U_i)$. We denote by $\pi\colon U_i \times \overline{B}^n \to \overline{B}^n$ the natural projection. We now set

$$\rho(w) \;:=\; \|\pi(\Phi_i(w))\|^2.$$

As in the Radial Distance Lemma 57.6 one can show, using the fact that we started out with an $\mathrm{O}(n)$-compatible atlas, that $\rho(w)$ is independent of the choice of $i \in I$ and that $\rho\colon W \to \mathbb{R}_{\geq 0}$ is a smooth function.



Since M is in particular compact we know that there exists an $m \in \mathbb{R}$ such that $m+f(x) > 0$ for all $x \in M$. Inspired by the Product–Morse Function Proposition 249.18 and the example on page 4813 we consider the function

$$\begin{aligned} g\colon W &\to \mathbb{R} \\ w &\mapsto (m + (f \circ p)(w)) \cdot (-1 + \rho(w)). \end{aligned}$$

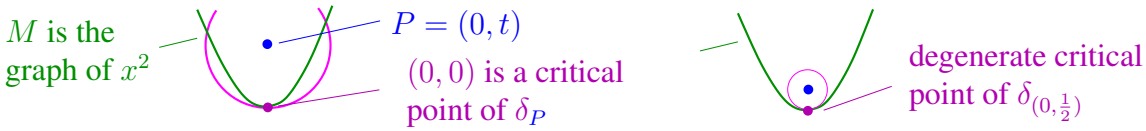
Similar to the Product–Morse Function Proposition 249.18 one can easily verify that this function has all the desired properties. We leave it to the reader to fill in the details. ■

249.5. **The distance-squared functions.** Our next goal is to show that given any compact smooth manifold M and given any union A of components of ∂M there exists a Morse function on M rel A. The idea is to turn the following "distance-squared" function into Morse functions:

Notation 249.20. Given $P \in \mathbb{R}^d$ we consider the map

$$\begin{aligned} \delta_P\colon \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto \|x - P\|^2. \end{aligned}$$

Example. We consider the smooth manifold $M := \{(x, x^2) \,|\, x \in \mathbb{R}\} \subseteq \mathbb{R}$ and we consider a point $P = (0, t)$ with $t \in \mathbb{R}$.



In Exercise 249.18 we will see $(0, 0)$ is always a critical point of $\delta_{(0,t)}$ and we will see that the critical point $(0, 0)$ is degenerate if and only if $t = \frac{1}{2}$. □

Our main goal of this section is to prove the following proposition:

Proposition 249.21. (Distance-Squared Function Proposition) Let $M \subseteq \mathbb{R}^d$ be a smooth submanifold. The set of $P \in \mathbb{R}^d$ for which the function

$$\delta_P\colon M \to \mathbb{R}$$

has no degenerate critical point is a set of full measure[2691] in $\mathbb{R}^d$.

The reader will sense that the Distance-Squared Function Proposition 249.21 is an extremely useful tool for showing that any smooth manifold M admits Morse functions rel a given union of boundary components. We will work out the details in the next Section 249.6.

The proof of the Distance-Squared Function Proposition 249.21 occupies the remainder of this section. The statement of the Distance-Squared Function Proposition 249.21 smells as if it should be a consequence of Sard's Theorem 38.1, which we now recall for the reader's convenience:

Theorem 38.1. (Sard's Theorem) If $\varphi\colon W \to N$ is a smooth map between smooth manifolds, then the set of regular values of φ has full measure in N.

We will indeed prove the Distance-Squared Function Proposition 249.21 using Sard's Theorem 38.1. The trick is to reformulate the statement about non-degenerate critical points of δ_P as a statement about the non-existence of critical points of a suitable map.

To make this idea work we need to make a few preparations:

Notation 249.22. Let M be a smooth submanifold of some $\mathbb{R}^d$. We consider

$$\mathrm{N}M := \{(Q,v) \in \mathbb{R}^d \times \mathbb{R}^d \,|\, Q \in M \text{ and } v \in (\mathrm{T}_Q M)^\perp\}.$$

The proof of the Normal Vectors Lemma 35.3 shows that $\mathrm{N}M$ is a d-dimensional submanifold of $\mathbb{R}^{2d}$.[2692]

Definition 249.23. Let M be a smooth submanifold of some $\mathbb{R}^d$. We consider the "endpoint map"

$$\begin{aligned} E\colon \mathrm{N}M &\to \mathbb{R}^d \\ (Q,v) &\mapsto Q+v. \end{aligned}$$

This map is evidently smooth.

(1) Let $Q \in M$. We say that $P \in \mathbb{R}^d$ is a focal point of (M,Q) if $P = E(Q,v)$ for some $v \in (\mathrm{T}_Q M)^\perp$ and if (Q,v) is a critical point of E.

(2) We say $P \in \mathbb{R}^d$ is a focal point of M if P is a focal point for some (M,Q). Note that focal points are exactly the same as critical values of E.

Example.

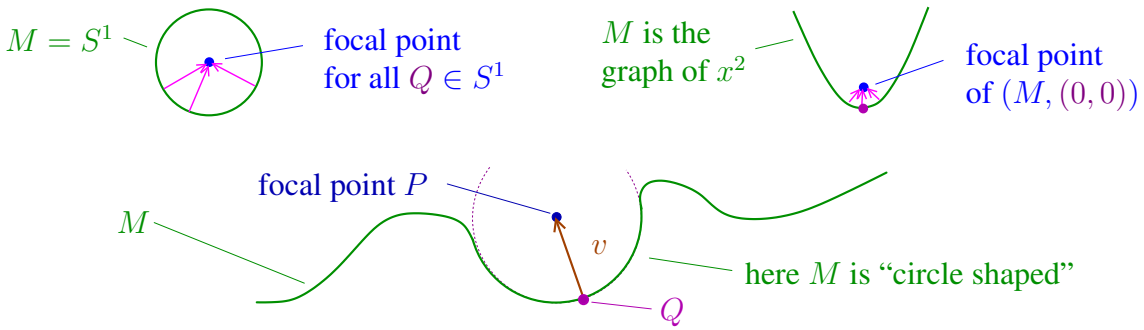
(1) (a) Let $n \in \mathbb{N}$. We consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$. In Exercise 249.19 we will see that $0 \in \mathbb{R}^{n+1}$ is a focal point for every $Q \in S^n$ and that 0 is the only focal point of S^n.

(b) We consider the smooth manifold $M := \{(x,x^2) \,|\, x \in \mathbb{R}\}$ and we consider the point $Q = (0,0) \in M$. In Exercise 249.20 we will see that $P = (0,\frac{1}{2})$ is the only focal point of $(M,(0,0))$.

(2) Let M be a smooth submanifold of $\mathbb{R}^d$. Note that $P \in \mathbb{R}^d$ being a focal point of (M,Q) only depends on the local behavior of M around Q.

[2691] We refer to Definition 37.1 for the definition of a subset of full measure. For the current purpose it suffices to know that the Full Measure-Density Proposition 37.4 (2) says that every subset of full measure in $\mathbb{R}^d$ is in particular dense.

[2692] The Normal Vectors Lemma 35.3 says that if M is a *proper* smooth submanifold of $\mathbb{H}^d$, then $\mathrm{N}M$ is a *proper* submanifold of $\mathbb{H}^d \times \mathbb{R}^d$. If we drop "proper" in the hypothesis, then we also need to drop "proper" in the conclusion. Furthermore we can replace $\mathbb{H}^d$ by $\mathbb{R}^d$. Fortunately the proof of the modified statement is almost identical.



For example, if $M \subseteq \mathbb{R}^n$ is "sphere shaped" in a neighborhood of $Q \in M$, then it follows from the above discussion and (1a) that the center of that sphere is a focal point of (M, Q). □

The following proposition allows us to characterize non-degenerate critical points of the map $\delta_P \colon M \to \mathbb{R}$ in terms of focal points of M, i.e. in terms of critical points of the endpoint map $E \colon \mathrm{N}M \to \mathbb{R}^d$:

Proposition 249.24. (Focal Point Proposition) Let M be a smooth submanifold of some $\mathbb{R}^d$, let $Q \in M$ and let $P \in \mathbb{R}^d$. The following two statements are equivalent:

(1) The point $Q \in M$ is a degenerate critical point of $\delta_P \colon M \to \mathbb{R}$.
(2) The point $P \in \mathbb{R}^d$ is a focal point of (M, Q).

Example. We consider again the smooth manifold $M = \{(x, x^2) \,|\, x \in \mathbb{R}\}$.

- On page 4816 we mentioned that $Q = (0,0)$ is a non-degenerate critical point of $\delta_{(0,\frac{1}{2})}$.
- Above we mentioned that $P = (0, \frac{1}{2})$ is a focal point of $(M, (0,0))$. □

We first show that the Focal Point Proposition 249.24 implies the Distance-Squared Function Proposition 249.21:

Proof of Proposition 249.21 assuming Proposition 249.24. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold. Let X be the set of $P \in \mathbb{R}^d$ for which the function

$$\delta_P \colon M \to \mathbb{R}$$

has no degenerate critical point. By the Focal Point Proposition 249.24 we know that X is precisely the set of critical values of $E \colon \mathrm{N}M \to \mathbb{R}^d$. It now follows from Sard's Theorem 38.1 that X is a set of full measure in $\mathbb{R}^d$. ■

The proof of the Focal Point Proposition 249.24 will require the remainder of this section. In the next Section 249.6 we will use the Focal Point Proposition 249.24 to prove that Morse functions always exist.

In the proof of the Focal Point Proposition 249.24 we will need the following elementary lemma from linear algebra:

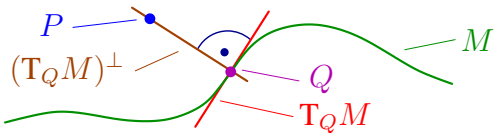
Lemma 249.25. (Basis Criterion) Let $a_1, \ldots, a_d \in \mathbb{R}^d$ be vectors and let $b_1, \ldots, b_d \in \mathbb{R}^d$ be a basis. Then

$$\operatorname{rank} \underbrace{(a_1 \ldots a_d)}_{(d \times d)\text{-matrix}} = \operatorname{rank} \underbrace{((a_i \cdot b_j)_{i,j=1,\ldots,d})}_{(d \times d)\text{-matrix}}.$$

Proof. We write $A := (a_1 \dots a_d)$ and $B := (b_1 \dots b_d)$. The lemma follows from the observation that $(a_i \cdot b_j)_{i,j=1,\dots,d} = A^{\mathrm{T}} \cdot B$. ∎

Proof of the Focal Point Proposition 249.24. Let M be an n-dimensional smooth submanifold of some $\mathbb{R}^d$. We start out with the following straightforward claim:

Claim 0. Let $P \in \mathbb{R}^d$. A point $Q \in M$ is a critical point of $\delta_P \colon M \to \mathbb{R}$, if and only if $P - Q \in (\mathrm{T}_Q M)^\perp$.



Proof. Note that for any $v \in \mathrm{T}_P M$ we have

$$\mathrm{D}(\delta_P)_Q(v) \underset{\substack{\uparrow \\ \text{since } \delta_P(x) = \|x-P\|^2 = (x-P)\cdot(x-P)}}{=} \mathrm{D}(x \mapsto (x-P)\cdot(x-P))|_{x=Q}(v) \underset{\substack{\uparrow \\ \text{by the Product Rule}}}{=} 2(Q-P)^{\mathrm{T}} \cdot v = 2(Q-P)\cdot v.$$

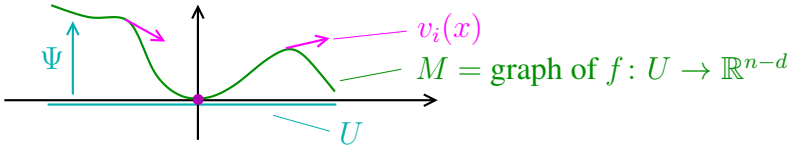
Since $\mathbb{R}$ is 1-dimensional we see that Q is a critical point of $\delta_P \colon M \to \mathbb{R}$ if and only if $\mathrm{D}(\delta_P)_Q \colon \mathrm{T}_Q M \to \mathbb{R}$ is the zero map. It follows from these observations that Q is a critical point of $\delta_P \colon M \to \mathbb{R}$ if and only if $P - Q \in (\mathrm{T}_Q M)^\perp$. ⊞

After this preamble we make a few preparations:

- First note that the desired statements are not affected by application of an isometry to $\mathbb{R}^d$. By the Orthogonal Action Lemma 2.37 we know that there exists an orthogonal matrix that turns $\mathrm{T}_Q M$ into $\mathbb{R}^n \times \{0\}$. This shows that, after applying a translation and after applying an orthogonal matrix, we can assume that $Q = (0,0) \in \mathbb{R}^n \times \mathbb{R}^{d-n}$ and that $\mathrm{T}_{(0,0)} M = \mathbb{R}^n \times \{0\}$.
- It follows easily from $\mathrm{T}_{(0,0)} M = \mathbb{R}^n \times \{0\}$ and the Local Graph Lemma 30.5 that M is locally the graph of a smooth function $f \colon U \to \mathbb{R}^{d-n}$, where $U \subseteq \mathbb{R}^n$ is an open neighborhood of 0.
- Since we are dealing with a local statement we can assume that M is the graph of the map $f \colon U \to \mathbb{R}^{d-n}$.
- We consider the diffeomorphism $\Psi \colon U \to M$ that is given by $x \mapsto (x, f(x))$. For $i = 1, \dots, n$ and $x \in U$ we consider the tangent vector

$$v_i(x) := \mathrm{D}\Psi(x) \cdot \mathfrak{e}_i = \frac{\partial \Psi}{\partial x_i}(x) = \big(\mathfrak{e}_i, \frac{\partial f}{\partial x_i}(x)\big).$$

  It follows from $\mathrm{T}_{(0,0)} M = \mathbb{R}^n \times \{0\}$ that $v_i(0) = \mathfrak{e}_i$. Also note that it follows from the fact that Ψ is a diffeomorphism that $v_1(x), \dots, v_n(x)$ form a basis for $\mathrm{T}_{\Psi(x)} M$.



Now recall what we need to prove: For every $P \in \mathbb{R}^d$ the following statements are equivalent:

(1) $Q = (0,0)$ is a degenerate critical point of $\delta_P \colon M \to \mathbb{R}$.
(2) P is a focal point of $(M, Q) = (M, (0,0))$.

The idea for the proof of the proposition is to characterize, for the given $Q = (0,0)$, the relevant two types of P, and to compare the results. More precisely we will do the following:

(1) In Claim 1 we will characterize for which $P \in \mathbb{R}^d$ the point $Q = (0,0)$ is a non-degenerate critical point of $\delta_P \colon M \to \mathbb{R}$.
(2) In Claim 2 we will characterize the $P \in \mathbb{R}^d$ which are the focal points of (M, Q).

Claim 1. Let $P \in \mathbb{R}^n$. The point $Q = (0,0)$ is a critical point of $\delta_P \colon M \to \mathbb{R}$ if and only if $P \in \{0\} \times \mathbb{R}^{d-n}$. Furthermore, for $P \in \{0\} \times \mathbb{R}^{d-n}$ the critical point $Q = (0,0)$ of $\delta_P \colon M \to \mathbb{R}$ is degenerate it and only if

$$\det\left(\mathrm{id}_n - \left(P \cdot \frac{\partial^2 \Psi}{\partial x_j \, \partial x_i}\Big|_0 \right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \right) \;=\; 0.$$

Proof. First note that it follows from $\mathrm{T}_{(0,0)} M = \mathbb{R}^n \times \{0\}$ and Claim 0 that the point $Q = (0,0)$ is a critical point of δ_P if and only if $P \in \{0\} \times \mathbb{R}^{d-n}$. Now let $P \in \{0\} \times \mathbb{R}^{d-n}$. Note that for any $j \in \{1, \dots, n\}$ and any $x \in U$ we have

$$\frac{\partial}{\partial x_j}(\delta_P \circ \Psi)_x \;\underset{\text{by the chain rule}}{=}\; (\mathrm{D}\delta_P)_{\Psi(x)} \cdot \frac{\partial \Psi}{\partial x_j}\Big|_x \;\underset{\text{as in the proof of Claim 0}}{=}\; 2(\Psi(x) - P) \cdot \frac{\partial \Psi}{\partial x_j}\Big|_x.$$

It follows that for any $i, j \in \{1, \dots, n\}$ we have

$$\begin{aligned} \frac{\partial^2 (\delta_P \circ \Psi)}{\partial x_i \, \partial x_j}\Big|_0 \;=\; \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(\delta_P \circ \Psi) \right)_0 &\overset{\text{by the above calculation}}{=} \frac{\partial}{\partial x_i}\left(2(\Psi(x) - P) \cdot \frac{\partial \Psi}{\partial x_j}\Big|_0 \right)\Big|_0 \overset{\text{by the Product Rule}}{=} \\ &= 2\Big(\underbrace{\frac{\partial \Psi}{\partial x_i}(0)}_{=\mathfrak{e}_i} \cdot \underbrace{\frac{\partial \Psi}{\partial x_j}(0)}_{=\mathfrak{e}_j} + \underbrace{(\Psi(0)}_{=Q=(0,0)} - P) \cdot \frac{\partial^2 \Psi}{\partial x_i \, \partial x_j}\Big|_0 \Big). \end{aligned}$$

The claim follows from this calculation and the Hessian Form Proposition 249.4. ⊞

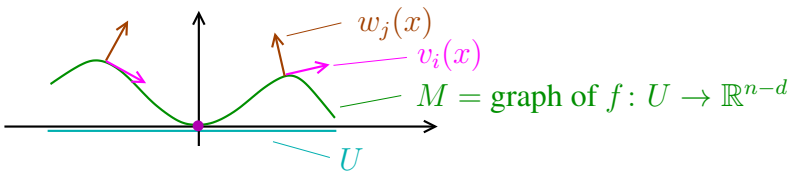
In the previous claim we answered the question, for which $P \in \mathbb{R}^n$ is $(0,0)$ a non-degenerate critical point of δ_P. Now we consider the question, for which $P \in \mathbb{R}^d$ is P a focal point of $(M, (0,0))$. Fortunately we get the same answer:

Claim 2. A point $P \in \mathbb{R}^d$ is a focal point of $(M, (0,0))$ if and only if $P \in \{0\} \times \mathbb{R}^{d-n}$ and if

$$\det\left(\mathrm{id}_n - \left(P \cdot \frac{\partial^2 \Psi}{\partial x_j \, \partial x_i}\Big|_0 \right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \right) \;=\; 0.$$

Proof. We start out with the popular concept of a subclaim:

Subclaim 1. There exist smooth maps $w_{n+1}, \dots, w_d \colon U \to \mathbb{R}^d$ such that for each $x \in U$ the following condition is satisfied:
(1) $w_{n+1}(x), \dots, w_d(x)$ is an orthonormal basis of $(\mathrm{T}_{\Psi(x)} M)^\perp$.
Note that this implies two other statements:
(2) $v_1(x), \dots, v_n(x), w_{n+1}(x), \dots, w_d(x)$ form a basis for $\mathbb{R}^d$.
(3) Each $v_i(x)$ is orthogonal to each $w_j(x)$.



Proof of subclaim. First note that it follows easily from the fact that M is a graph that for each $x \in U$ the vectors $v_1(x), \dots, v_n(x), \mathfrak{e}_{n+1}, \dots, \mathfrak{e}_d$ form a basis for $\mathbb{R}^d$. For each $x \in U$ let $w_{n+1}(x), \dots, w_d(x)$ be the last $d-n$ vectors that we obtain from applying the Gram-Schmidt Orthogonalization Process 2.32 to the vectors $v_1(x), \dots, v_n(x), \mathfrak{e}_{n+1}, \dots, \mathfrak{e}_d$. Since the Gram-Schmidt Orthogonalization Process 2.32 is given by an explicit expression we see that the resulting maps $w_{n+1}, \dots, w_d\colon U \to \mathbb{R}^d$ are smooth. Furthermore it follows from the properties stated in the Gram-Schmidt Orthogonalization Process 2.32 that the maps $w_{n+1}(x), \dots, w_d(x)$ do indeed have the desired properties. □

Using the maps from the subclaim we obtain a map

$$\begin{array}{rcl} \Xi\colon U \times \mathbb{R}^{d-n} & \to & \mathrm{N}M \\ (x, (t_{n+1}, \dots, t_d)) & \mapsto & \Big(\Psi(x), \sum\limits_{r=n+1}^{d} t_r \cdot w_r(x)\Big). \end{array}$$

Using the Inverse Mapping Theorem 30.1 one can easily verify that $\Xi\colon U \times \mathbb{R}^{d-n} \to \mathrm{N}M$ is in fact a diffeomorphism. In the following let $E\colon \mathrm{N}M \to \mathbb{R}^d$ be the endpoint map that is given by $(x, v) \mapsto x + v$. We are interested in the critical points of E. But since Ξ is a diffeomorphism this is equivalent to the study of the critical points of $E \circ \Xi\colon U \to \mathbb{R}^d$. This leads us to the following claim:

Subclaim 2. The point $(x, t_{n+1}, \dots, t_d) \in U \times \mathbb{R}^{d-n}$ is a critical point of the function $E \circ \Xi\colon U \times \mathbb{R}^{d-n} \to \mathbb{R}^d$ if and only if

$$\det\left(\left(v_i(x) \cdot v_j(x) - \sum_{r=n+1}^{d} t_r \cdot w_r(x) \cdot \frac{\partial^2 \Psi}{\partial x_j\, \partial x_i}\Big|_x\right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}}\right) = 0.$$

Proof of subclaim. First note that $E \circ \Xi$ is of the form

$$\begin{array}{rcl} E \circ \Xi\colon U \times \mathbb{R}^{d-n} & \to & \mathbb{R}^d \\ (x, (t_{n+1}, \dots, t_d)) & \mapsto & \Psi(x) + \sum\limits_{r=n+1}^{d} t_r \cdot w_r(x) \end{array}$$

Next note that an elementary calculation shows that the differential $\mathrm{D}(E \circ \Xi)_{(x,t)}$ is of the form

$$\Big(\underbrace{\Big(v_i(x) + \sum_{r=n+1}^{d} t_r \cdot \frac{\partial w_r}{\partial x_i}\big|_x\Big)_{i=1,\dots,n}}_{(d \times n)\text{-matrix}} \quad \underbrace{w_{n+1}(x) \dots w_d(x)}_{(d \times (d-n))\text{-matrix}}\Big).$$

By Subclaim 1 (2) we can apply the Basis Criterion 249.25 to the columns of this matrix and the vectors $v_1(x), \dots, v_n(x), w_{n+1}(x), \dots, w_d(x)$. Using the properties of $w_{n+1}(x), \dots, w_d(x)$

stated in Subclaim 1 we see that for $(x,t) = (x,(t_{n+1},\dots,t_d))$ we have

$$\operatorname{rank}(\mathrm{D}(E\circ\Xi)_{(x,t)}) = \operatorname{rank}\begin{pmatrix} \left(v_i(x)\cdot v_j(x) + \sum_{r=n+1}^{d} t_r\cdot \frac{\partial w_r}{\partial x_i}\Big|_x \cdot v_j(x)\right)_{\substack{i=1,\dots,n\\ j=1,\dots,n}} & 0 \\ \left(\sum_{r=n+1}^{d} t_r\cdot \frac{\partial w_r}{\partial x_i}\Big|_x \cdot w_j(x)\right)_{\substack{i=1,\dots,n\\ j=d+1,\dots n}} & \mathrm{id}_{(d-n)\times(d-n)} \end{pmatrix}.$$

It follows that the rank of $\mathrm{D}(E\circ\Xi)_{(x,t)}$ is precisely the rank of the $(n\times n)$-matrix in the upper left corner.

We want to rewrite the matrix in the upper left corner. Thus let $i\in\{1,\dots,n\}$ and let $r\in\{n+1,\dots,d\}$. Note that

$$0 \underset{\substack{\uparrow\\ \text{since } w_r\cdot v_j\equiv 0 \text{ as}\\ \text{functions on } M}}{=} \frac{\partial}{\partial x_i}(w_r\cdot v_j) = \frac{\partial w_r}{\partial x_i}\cdot v_j + w_r\cdot\frac{\partial v_j}{\partial x_i} \underset{\substack{\uparrow\\ \text{by the definition of } v_i}}{=} \frac{\partial w_r}{\partial x_i}\cdot v_j + w_r\cdot\frac{\partial^2\Psi}{\partial x_j\,\partial x_i}.$$

It follows that the $(n\times n)$-matrix in the upper left corner equals

$$\left(v_i(x)\cdot v_j(x) - \sum_{r=n+1}^{d} t_r\cdot w_r\cdot\frac{\partial^2\Psi}{\partial x_j\,\partial x_i}\Big|_x\right)_{\substack{i=1,\dots,n\\ j=1,\dots,n}}.$$

Finally note that $v_i(0)\cdot v_j(0) = \mathfrak{e}_i\cdot\mathfrak{e}_j = \delta_{ij}$. These results put together give us a proof of Subclaim 2. □

Now let $P\in\mathbb{R}^d$. For P to be a focal point of $(M,Q) = (M,(0,0))$ we need, by definition of the smooth manifold $\mathrm{N}M$ and the evaluation map E, that $P-Q = P-(0,0)\in(\mathrm{T}_Q M)^\perp$. Now let $P\in(\mathrm{T}_Q M)^\perp = (\mathbb{R}^n\times\{0\})^\perp = \{0\}\times\mathbb{R}^{d-n}$. We can write $P = \sum_{r=n+1}^{d} t_r\cdot w_r(0)$.

Now we see that

$$\begin{aligned} P \text{ is a focal point of } (M,(0,0)) &\iff \left(0,\sum_{r=n+1}^{d} t_r\cdot w_r(0)\right) \text{ is a critical point of } E\\ &\iff ((0,0),(t_{n+1},\dots,t_d)) \text{ is a critical point of } E\circ\Xi\\ &\overset{\text{Subclaim 2}}{\iff} \det\left(\mathrm{id}_n - \left(\sum_{r=n+1}^{d} r_i\cdot w_r(0)\cdot\frac{\partial^2\Psi}{\partial x_j\,\partial x_i}\Big|_0\right)_{i,j=1,\dots,n}\right) = 0\\ &\iff \det\left(\mathrm{id}_n - \left(P\cdot\frac{\partial^2\Psi}{\partial x_j\,\partial x_i}\Big|_0\right)_{i,j=1,\dots,n}\right) = 0. \end{aligned}$$

■

249.6. **Existence of Morse functions.** We can now finally prove that Morse functions always exist:

Theorem 249.26. (Morse Function-Existence Theorem) Let M be a compact smooth manifold. Given any union A of components of ∂M there exists a Morse function $f\colon M\to\mathbb{R}$ rel A.

Remark.

(1) The literature contains many alternative proofs for the Morse Function-Existence Theorem 249.26, see e.g. [**Miln1965b**, Theorem 2.5]. The literature also contains many refinements of the Morse Function-Existence Theorem 249.26 in the literature. For

example, by [**Mata2002**, Theorem 2.20], [**Knu2015**, Theorem 2.9], [**AuD2014**, Proposition 1.2.4] or [**Miln1963a**, Corollary 6.8] we know that every smooth function on a closed smooth manifold can be approximated by Morse functions.

(2) In these lecture notes we will prove two refinements of the Morse Function-Existence Theorem 249.26:
   (a) In the Distinct Critical Values Lemma 249.30 we will see that there exists a Morse function $f\colon M \to [0,1]$ rel A such that the values of f at the critical points are pairwise distinct.
   (b) In the Self-Indexing Proposition 251.9 we will see that one can always find "self-indexing" Morse functions. □

In the remainder of this section we will provide the proof of the Morse Function-Existence Theorem 249.26. In the final Section 249.7 we will state and prove the Distinct Critical Values Lemma 249.30 which we had just mentioned.

First we will consider the case that M is closed; as we will see in a minute, in this setting we are essentially done. Afterwards we will consider the case that M has non-empty boundary. This case will require some more work.

Proof of the Morse Function-Existence Theorem 249.26 if $\partial M = \varnothing$. Let M be a closed smooth manifold. It follows from the Euclidean Embedding Theorem 33.2 together with the Smooth Embedding Theorem 30.15 that we can assume that M is a submanifold of some $\mathbb{R}^d$.

By the Distance-Squared Function Proposition 249.21 we know that the set of $P \in \mathbb{R}^d$ for which the function

$$\begin{array}{rccc} \delta_P\colon & M & \to & \mathbb{R} \\ & x & \mapsto & \|x-P\|^2 \end{array}$$

has no degenerate critical point, is a set of full measure in $\mathbb{R}^d$. The theorem now follows from the Full Measure-Density Proposition 37.4 which says that any set of full measure of $\mathbb{R}^d$ is dense, in particular non-empty. ■

It remains to prove the Morse Function-Existence Theorem 249.26 for smooth manifolds with non-empty boundary. The problem with the current proof is that there is no reason, why the functions δ_P should be constant on A and B.

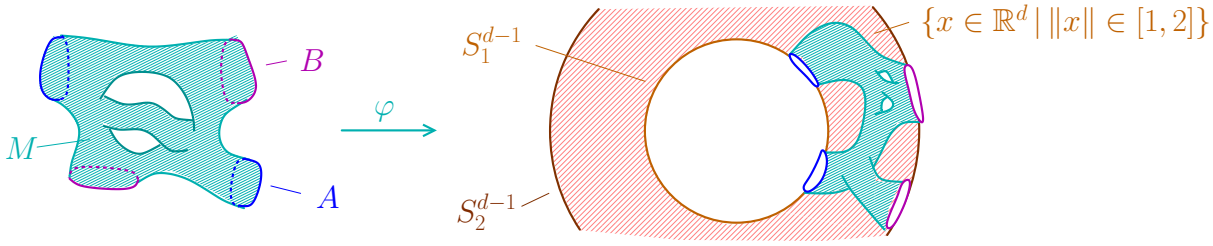
To fix this issue we will need the following lemma, which might even be of independent interest:[2693]

Lemma 249.27. (Cylinder Embedding Lemma) Let M be a compact smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. There exists a $d \in \mathbb{N}$ and a proper smooth embedding

$$\varphi\colon M \to \underbrace{\{x \in \mathbb{R}^d \,|\, \|x\| \in [1,2]\}}_{\substack{\text{by the Preimage-of-Interval} \\ \text{Lemma 38.3 this is a submanifold of } \mathbb{R}^d \\ \text{with boundary } S_1^{d-1} \sqcup S_2^{d-1}}}$$

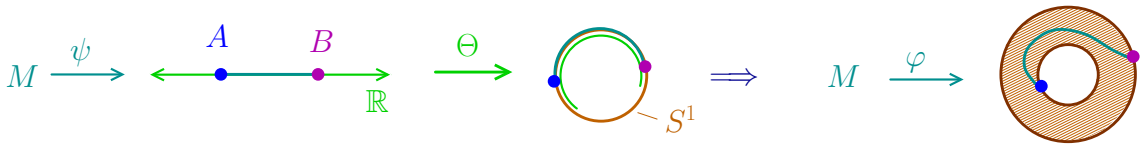
such that $\varphi^{-1}(S_1^{d-1}) = A$ and such that $\varphi^{-1}(S_2^{d-1}) = B$.

[2693] Also note that the lemma is similar to the task set out in Exercise 33.3.



Proof of the Cylinder Embedding Lemma 249.27. In preparation we need to pick and fix several maps:

(1) It follows from the Euclidean Embedding Theorem 33.2 that there exists a $d \in \mathbb{N}$ and a smooth embedding $\psi\colon M \to \mathbb{R}^{d-1}$.
(2) We pick a smooth embedding $\Theta\colon \mathbb{R}^{d-1} \to S^{d-1}$.
(3) By the Smooth Collar Neighborhood Theorem 35.6 we know that there exists a smooth embedding $\chi\colon [0,1] \times \partial M \to M$ such that $\chi(0,P) = P$ for all $P \in \partial M$.
(4) By the Smooth Transition Function Lemma 24.9 we can find smooth functions of the following type:
   - $\alpha\colon [0,1] \to [1,\frac{3}{2}]$ with $\alpha(t) = 1+t$ for all $t \in [0,\frac{1}{4}]$ and with $\alpha(t) = \frac{3}{2}$ for all $t \in [\frac{3}{4},1]$.
   - $\beta\colon [0,1] \to [1,\frac{3}{2}]$ with $\beta(t) = 2-t$ for all $t \in [0,\frac{1}{4}]$ and with $\beta(t) = \frac{3}{2}$ for all $t \in [\frac{3}{4},1]$.



We now consider the map

$$\begin{array}{rcl} \varphi\colon M & \to & \mathbb{R}^d \\ P & \mapsto & \begin{cases} \frac{3}{2} \cdot (\Theta \circ \psi)(P), & \text{if } P \notin \chi([0,1) \times \partial M), \\ \alpha(t) \cdot (\Theta \circ \psi)(P), & \text{if } P = \chi(t,a) \text{ for some } t \in [0,1] \text{ and } a \in A, \\ \beta(t) \cdot (\Theta \circ \psi)(P), & \text{if } P = \chi(t,b) \text{ for some } t \in [0,1] \text{ and } b \in B. \end{cases} \end{array}$$

Note that it follows from the Smooth Pasting Proposition 25.19 that this map is smooth. It is now straightforward to verify that φ has all the desired properties. ∎

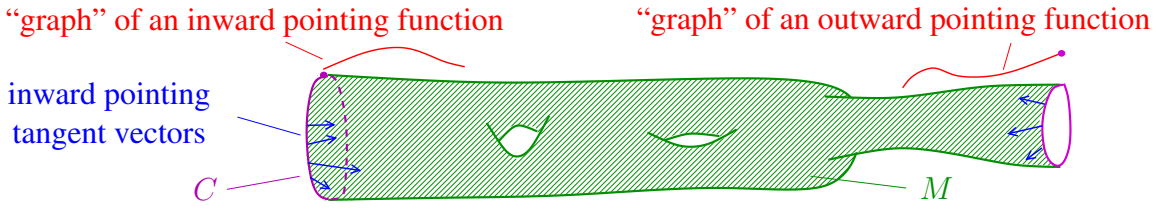
We continue with the notation from the Cylinder Embedding Lemma 249.27: If the map

$$\begin{array}{rcl} \delta_0\colon M & \to & \mathbb{R} \\ x & \mapsto & \|x-0\|^2 = \|x\|^2 \end{array}$$

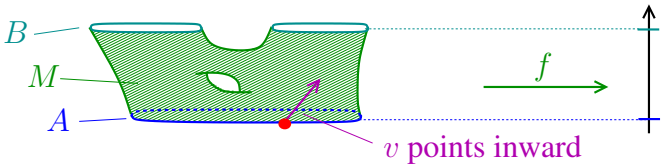
has no non-degenerate critical points, then we are done, since $\delta_0|_A \equiv 1$ and $\delta_0|_B \equiv B$. But we only know that for "most" P in any neighborhood of 0 the function δ_P has no critical points. For such P the function δ_P is "close" to being constant on A and B, but it will not be "completely constant". To get around this issue we have to work quite a bit harder. The idea is, that for a P "close" to 0 the function $\delta_P\colon M \to \mathbb{R}$ can be modified such that it is constant on A and on B, without introducing new critical points. This can be done, but, as we will see, it requires some effort.

First of all we need the following somewhat technical definition:

Definition 249.28. Let M be a smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let C be a union of components of ∂M. We say that f is pointing inward on C if for every $P \in C$ and every inward pointing tangent vector[2694] $P \in \mathrm{T}_P M$ we have $\mathrm{D}f_P(v) > 0$. Similarly we are defining that f is pointing outward on C.



Example. Let M be a smooth manifold, let A be a union of components of ∂M and let $f\colon M \to \mathbb{R}$ be a Morse function rel A. We claim that f points inward on A and outward on $B := \partial M \setminus A$. This can be seen as follows: Let $P \in A$ and let $v \in \mathrm{T}_P M$ be a vector that points inward.



Note that $\mathrm{T}_P M = \mathrm{T}_P(\partial M) \oplus \mathbb{R} \cdot v = \mathrm{T}_P A \oplus \mathbb{R} \cdot v$. Since f is a Morse function we know that $f|_A$ is constant, which implies that the restriction of $\mathrm{D}f_P$ to $\mathrm{T}_P A$ is zero. Since f is a Morse function we also know that f has no critical point on ∂M, which implies that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}$ is non-zero. The above discussion implies that $\mathrm{D}f_P(v) \neq 0$. Since f assumes the global minimum on A and since v is inward pointing it follows from the definition of an inward pointing tangent vector in Definition 29.22 that $\mathrm{D}f_P(v) > 0$. Thus we have shown that f points inward on A. Almost the same argument shows that f points outward on B. □

Now we can formulate the missing link in the proof of the Morse Function-Existence Theorem 249.26:

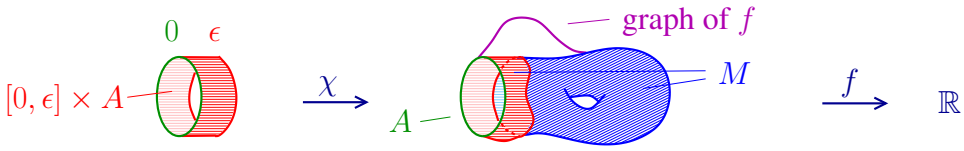
Lemma 249.29. (Standard-near-Boundary Lemma) Let M be a smooth manifold, let A be a union of components of ∂M, let $\chi\colon [0,\epsilon] \times A \to M$ be a collar and let $f\colon M \to \mathbb{R}$ be a function that points inward on A. We denote by m the minimum of f on $\chi([0,\epsilon] \times A)$. If f points inward on A, then given any $c < m$ there exists an $\eta \in (0, \frac{\epsilon}{2})$ and a smooth function $g\colon M \to \mathbb{R}$ with the following properties:

(1) The map g agrees with f outside of $\chi([0, \frac{3}{2}\eta] \times A)$.
(2) The map g has no critical points on $\chi([0, 2\eta] \times A)$.

[2694] We refer to Definition 29.22 for the definition of an inward pointing tangent vector. For the reader's convenience we give a characterization of such vectors: It follows from the Tangent Space-Isomorphism Proposition 29.24 (2) that for $P \in \partial M$ a vector $v \in \mathrm{T}_P M$ is inward pointing precisely if for a chart $\Phi\colon U \to V$ around P, with V an open subset of $\mathbb{H}^n$, the vector $\mathrm{D}\Phi_P(v) = (w_1, \ldots, w_n)$ satisfies $w_n \geq 0$.

(3) For any $a \in A$ and any $t \in [0, \eta]$ we have $g(\chi(t, a)) = c+t$. (In particular the restriction of g to A is const $\equiv c$.)

An analogous statement also holds for *outward* pointing functions, in fact it follows from the above, applied to the function $-f$.



Example. Let M be a smooth manifold, let A be a union of components of ∂M, let $\chi\colon [0, \epsilon] \times A \to M$ be a collar and let $f\colon M \to \mathbb{R}$ be a Morse function rel A. It follows from the above example and the Standard-near-Boundary Lemma 249.29 that there exists a $c \in \mathbb{R}$, an $\eta \in (0, \frac{\epsilon}{2})$ and a Morse function $g\colon M \to \mathbb{R}$ with the following properties:

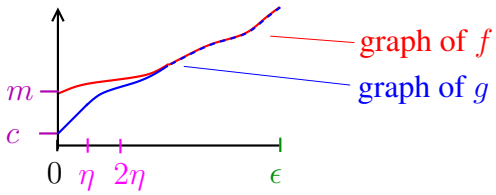
(1) The map g agrees with f outside of $\chi([0, \frac{3}{2}\eta] \times A)$.
(2) The map g has no critical points on $\chi([0, 2\eta] \times A)$.
(3) For any $a \in A$ and any $t \in [0, \eta]$ we have $g(\chi(t, a)) = c + t$.

This shows that we can always arrange Morse functions to be "standard" on collars. □

Proof of the Standard-near-Boundary Lemma 249.29. Let M be a smooth manifold, let A be a union of components of ∂M. Evidently the statement of the lemma is all about the collar neighborhood. In fact it follows easily from the Smooth Pasting Proposition 25.19 that it suffices to prove the following claim:

Claim. Let A be a closed smooth manifold and let $f\colon [0, \epsilon] \times A \to \mathbb{R}$ be a smooth function that is pointing inward on $\{0\} \times A$. We denote by m the minimum of f. Given any $c < m$ there exists an $\eta \in (0, \frac{\epsilon}{2})$ and a smooth function $g\colon [0, \epsilon] \times A \to \mathbb{R}$ with the following properties:

(1) The map g agrees with f outside of $[0, \frac{3}{2}\eta] \times A$.
(2) The map g has no critical points on $[0, 2\eta] \times A$.
(3) For any $a \in A$ and any $t \in [0, \eta]$ we have $g(t, a) = c + t$.



Proof. We denote by m the minimum of $f\colon [0, \epsilon] \times A \to \mathbb{R}$. Let $c < m$. For each $(t, a) \in [0, \epsilon] \times A$ we denote by $e \in \mathrm{T}_{(t,a)}([0, \epsilon] \times A)$ the horizontal tangent vector pointing into the $[0, \epsilon]$-direction, as defined on page 764. Note that our hypothesis that f points inward on $\{0\} \times A$ implies that for each $a \in A$ we have $(\mathrm{D}f)_{(0,a)}(e) > 0$. It follows easily from the compactness of A and the Tube Lemma 7.12 that there exists a $\rho \in (0, \epsilon]$ such that $(\mathrm{D}f)_{(t,a)}(e) > 0$ for all $(t, a) \in [0, \rho] \times A$.

We set $\eta = \min\{\frac{1}{2}(m - c), \frac{\epsilon}{4}, \frac{1}{2}\rho\} \in (0, \frac{\epsilon}{2})$. By the Smooth Transition Function Lemma 24.9 we know that there exists a smooth function $\tau\colon [0, \epsilon] \to [0, 1]$ such that $\tau(s) = 0$

for all $s \in [0, \eta]$ and such that $\tau(s) = 1$ for all $s \in [\frac{3}{2}\eta, \epsilon]$. We consider the function

$$\begin{array}{rccl} g\colon [0,\epsilon] \times A & \to & \mathbb{R} \\ (s,a) & \mapsto & (1-\tau(s)) \cdot (c+s) + \tau(s) \cdot f(s,a). \end{array}$$

By design this map has properties (1) and (3). It remains to show that the differential $\mathrm{D}g_{(t,a)}$ is non-zero for all $(s,a) \in [0, 2\eta] \times A$. We do so by applying the differential to the horizontal tangent vectors e. For $(s_0, a_0) \in [0, 2\eta] \times A$ we calculate:

$$\begin{aligned} \mathrm{D}g_{(s_0,a_0)}(e) &= \mathrm{D}((s,a) \mapsto (1-\tau(s)) \cdot (c+s) + \tau(s) \cdot f(s,a))_{(s_0,a_0)}(e) \\ &= (-\tau'(s_0)) \cdot (c+s_0) + (1-\tau(s_0)) + \tau'(s_0) \cdot f(s_0,a_0) + \tau(s_0) \cdot \mathrm{D}f_{(s_0,a_0)}(e) \\ &\quad \uparrow \\ &\quad \text{by the Product Rule} \\ &\quad \text{and the definition of } e \\ &= \tau'(s_0) \cdot (\underbrace{f(s_0,a_0) - c - s_0}_{\geq 0, \text{ since } s_0 \leq 2\eta \leq m-c}) + 1 - \tau(s_0) + \tau(s_0) \cdot \underbrace{\mathrm{D}f_{(s_0,a_0)}(e)}_{> 0, \text{ since } s_0 \leq 2\eta \leq \rho} > 0. \end{aligned}$$

This shows that the differential $\mathrm{D}g$ is indeed non-zero on $[0, 2\eta] \times A$. Thus we have finally verified that g also satisfies Condition (2). ■

Finally we can provide the proof of the Morse Function-Existence Theorem 249.26, even if the boundary is non-empty:

Proof of the Morse Function-Existence Theorem 249.26. Let M be a compact smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. It follows from the Cylinder Embedding Lemma 249.27 together with the Smooth Embedding Theorem 30.15 that we can view M as a proper submanifold of $\{x \in \mathbb{R}^d \mid \|x\| \in [1,2]\}$ such that $\varphi(A) \subseteq S_1^{d-1}$ and such that $\varphi(B) \subseteq S_2^{d-1}$.

Claim. There exists a $P \in \mathbb{R}^d$ with the following two properties:

(1) The function
$$\begin{array}{rccl} \delta_P\colon M & \to & \mathbb{R} \\ x & \mapsto & \|x-P\|^2 \end{array}$$
points inward on A and it points outward on B.

(2) Every critical point of δ_P is non-degenerate.

Proof. We start out with the following observations:

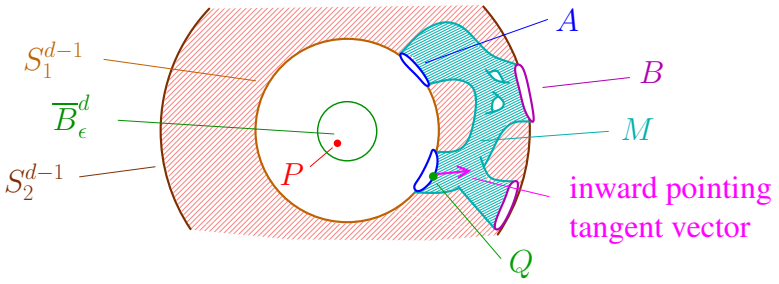
(a) It follows from the Tangent Space-Isomorphism Proposition 29.24 (1b) that for any $Q \in S_1^{d-1}$ a tangent vector $v \in \mathrm{T}_Q M$ is inward pointing if and only if $v \cdot Q > 0$.

(b) Let $P \in \mathbb{R}^d$, let $Q \in M$ and let $v \in \mathrm{T}_Q M \subseteq \mathbb{R}^d$. On page 4819 we showed that $\mathrm{D}(\delta_P)_Q(v) = 2(Q-P) \cdot v$.

It follows from these two observations that δ_0 points inward on A and it points outward on B. It follows easily from the compactness of M and the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that for any $P \in \overline{B}_\epsilon^d$ the function $\delta_P\colon M \to \mathbb{R}$ is pointing inward on A and that it is pointing outward on B.

Finally note that it follows from the Distance-Squared Function Proposition 249.21 together with the Full Measure-Density Proposition 37.4 that there exists a $P \in \overline{B}_\epsilon^d$ such that every critical point of δ_P is non-degenerate. ⊞

By the claim we can pick a point $P \in \mathbb{R}^d$ with Properties (1) and (2). Next recall that by the Smooth Collar Neighborhood Theorem 35.6 we know that the smooth manifold M



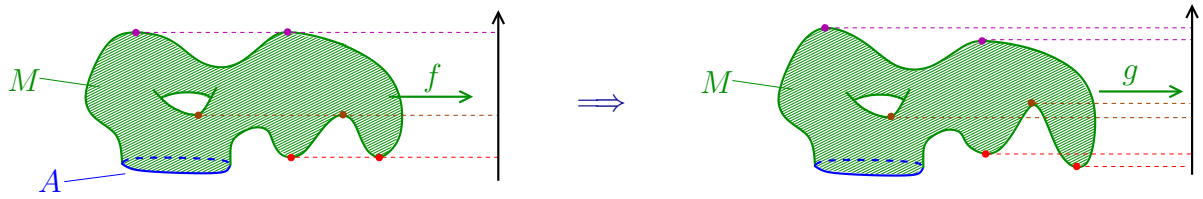
admits a collar $\chi\colon [0,1] \times \partial M \to M$. We can thus apply the Standard-near-Boundary Lemma 249.29 to the collars $\chi|_{[0,1]\times A}$ and $\chi|_{[0,1]\times B}$ and we can modify δ_P to obtain a Morse function on M rel A. ■

249.7. **Morse functions with distinct critical values.** In the Morse Function-Existence Theorem 249.26 we just showed that Morse functions always exist. For some purposes it is useful to find Morse functions, such that the values of the critical points are pairwise distinct.

The following lemma shows that this can always be achieved:

Lemma 249.30. (Distinct Critical Values Lemma) Let M be a compact smooth manifold and let A be a union of components of ∂M. Given any Morse function $f\colon M \to \mathbb{R}$ rel A there exists a Morse function $g\colon M \to \mathbb{R}$ rel A with the following properties:

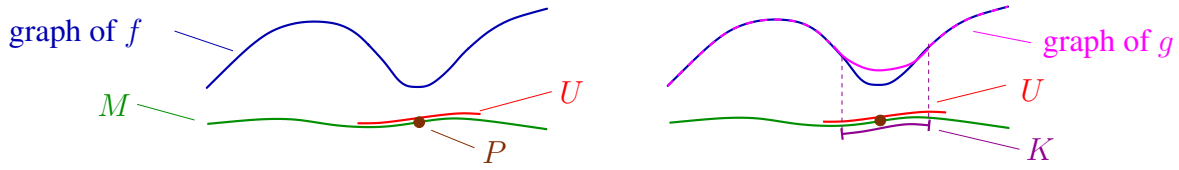
(1) The Morse functions f and g have the same number of critical points, with the same indices.
(2) The values of g at the critical points are pairwise distinct.



Proof. Let M be a compact n-dimensional smooth manifold, let A be a union of components of ∂M and let $f\colon M \to \mathbb{R}$ be a Morse function rel A. Recall that by the Morse Function–Finiteness Corollary 249.12 we know that f has only finitely many critical points. The lemma thus follows easily from iteratively applying the following claim:

Claim. Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $P \in M \setminus \partial M$ be a non-degenerate critical point of f. There exists an open neighborhood U of $P \in M \setminus \partial M$ and an $\epsilon > 0$ such that for every $\mu \in (-\epsilon, \epsilon)$ there exists a smooth function $g\colon M \to \mathbb{R}$ and a compact neighborhood $K \subseteq U$ of P with the following four properties:

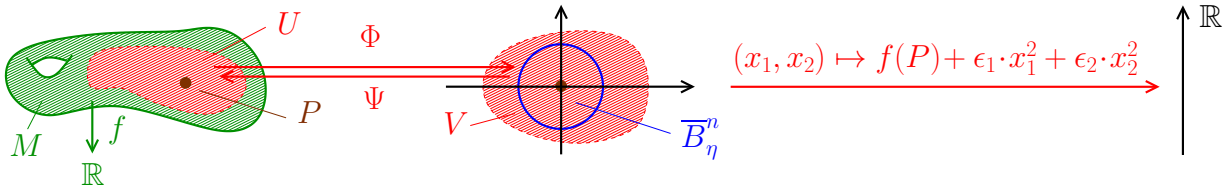
(1) The function g agrees with f outside of K.[2695]
(2) P is a critical point of g with $\operatorname{index}(g, P) = \operatorname{index}(f, P)$.
(3) $g(P) = f(P) + \mu$.
(4) P is the only critical point of g in U.



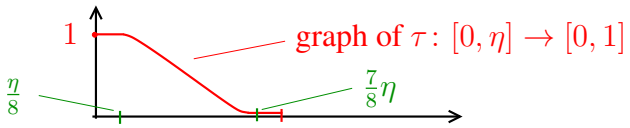
Proof. Let k be the index of the critical point P. By the Morse Lemma 249.11 there exists an open neighborhood U of P, an open neighborhood V of $0 \in \mathbb{R}^n$ and a chart $\Phi\colon U \to V$ around P with $U \subseteq M \setminus \partial M$, with $\Phi(P) = 0$ and such that there exist $a_1, \ldots, a_n \in \{-1, 1\}$ such that for every $(x_1, \ldots, x_n) \in V$ we have

$$(f \circ \Phi^{-1})(x_1, \ldots, x_n) \;=\; f(P) + \sum_{i=1}^{n} a_i \cdot x_i^2.$$

We set $\Psi := \Phi^{-1}\colon V \to U$. Since V is an open neighborhood of $0 \in \mathbb{R}^n$ there exists an $\eta > 0$ such that $\overline{B}_\eta^n \subseteq V$. We claim that $\epsilon := \frac{\eta^2}{8}$ has the desired properties.



Using the Smooth Transition Function Lemma 24.9 one can show, see Exercise 249.22, that there exists a smooth function $\tau\colon [0, \eta] \to [0, 1]$ with $\tau(t) = 1$ for $t \in [0, \frac{\eta}{8}]$, $\tau(t) = 0$ for $t \in [\frac{7\eta}{8}, \eta]$ and such that $\tau'(t) \geq -\frac{2}{\eta}$ for all $t \in [0, \eta]$.



Next, given any $\mu \in \mathbb{R}$ we consider the function

$$\begin{array}{rcl} \rho_\mu\colon [0, \eta] & \to & \mathbb{R} \\ s & \mapsto & \mu \cdot \tau(s) + s^2. \end{array}$$

Note that $\rho(0) = \mu$ and that $\rho(s) = s^2$ for $s \in [\frac{7\eta}{8}, \eta]$. Next we claim that if $\mu < \epsilon = \frac{\eta^2}{8}$, then $\rho'_\mu(s) > 0$ for all $s \in (0, \eta]$. Indeed, this can be seen as follows: For $s \in [0, \frac{\eta}{8}]$ this follows immediately from the fact that for such s we have $\rho_\mu(s) = \mu + s^2$. Now we consider $s \in [\frac{\eta}{8}, \eta]$. In this setting we see that

$$\rho'_\mu(s) \underset{\substack{\uparrow \\ \text{definition of } \rho_\mu}}{=} (\mu \cdot \tau(s) + s^2)' \;=\; \mu \cdot \underbrace{\tau'(s)}_{\geq -\frac{2}{\eta}} + \underbrace{2s}_{\geq 2 \cdot \frac{\eta}{8}} \geq -\mu \cdot \tfrac{2}{\eta} + \tfrac{\eta}{4} \underset{\substack{\uparrow \\ \text{since } \mu < \frac{\eta^2}{8}}}{>} -\tfrac{\eta^2}{8} \cdot \tfrac{2}{\eta} + \tfrac{\eta}{4} \;=\; 0.$$

[2695] In Exercise 249.21 we will see why it does not suffice to demand that g agrees with f outside of U.

Recall that we set $\epsilon := \frac{\eta^2}{8}$. Now let $\mu \in (-\epsilon, \epsilon)$. We consider the function

$$g\colon M \to \mathbb{R}$$
$$Q \mapsto \begin{cases} f(Q), & \text{if } Q \notin \Psi(\overline{B}^n_\eta), \\ f(P) + a_1 \cdot \rho_{\mu \cdot a_1}(x_1) + \sum_{i=2}^{n} a_i \cdot x_i^2, & \text{if } Q = \Psi(x_1, \dots, x_n) \text{ with } (x_1, \dots, x_n) \in \overline{B}^n_\eta. \end{cases}$$

Since $|\mu \cdot a_1| = |\mu| < \frac{\eta^2}{8}$ it follows easily from the above properties of ρ that g has all the desired properties. ■

249.8. Exercises for Chapter 249.

Exercise 249.1.

(a) Do there exist smooth vector fields X and Y on $M = \mathbb{R}$ with Lie bracket $[X, Y] \neq 0$?
(b) Do there exist smooth vector fields X and Y on $M = \mathbb{R}^2$ with Lie bracket $[X, Y] \neq 0$?

Exercise 249.2. Let M be a smooth manifold and let X, Y, Z be smooth vector fields on M.

(a) Show that for any two smooth functions $f, g\colon M \to \mathbb{R}$ we have

$$[f \cdot X, g \cdot Y] \;=\; f \cdot g \cdot [X, Y] + f \cdot (Xg) \cdot Y - g \cdot (Yf) \cdot X.$$

(Recall that by the Vector Field–Derivative Lemma 249.3 we know that Xg and Yf are smooth functions on M.)

(b) Show that the following identity holds:

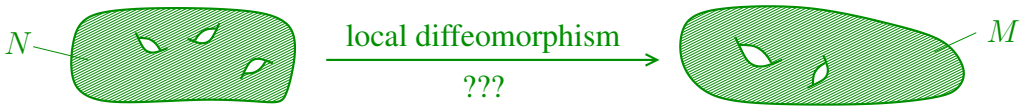
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \;=\; 0.$$

(c) Let $\varphi\colon N \to M$ be a local diffeomorphism. Given any vector field V on M we consider the vector field $\varphi^* V$ on N that is given by $(\varphi^* V)(y) := (\mathrm{D}\varphi_y)^{-1}(V(\varphi(y))$. Recall that we showed in the Vector Field-Pullback Lemma 29.33 that $\varphi^* V$ is a smooth vector field on N. Show that

$$\varphi^*([X, Y]) \;=\; [\varphi^* X, \varphi^* Y].$$

Hint. It might be helpful to look at the proof of the Hessian Form-Calculation Lemma 249.5 (1).

Exercise 249.3. In the figure on page 4801 we seem to suggest that there exists a local diffeomorphism φ from the surface N of genus 3 to the surface M of genus 2. Is this indeed the case, does there indeed exist such a local diffeomorphism?



Exercise 249.4. We consider the following two functions:

$$\begin{array}{rcl} \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 & \to & \mathbb{R} \\ z & \mapsto & \mathrm{Re}(z^2) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 & \to & \mathbb{R} \\ z & \mapsto & \mathrm{Im}(z^2). \end{array}$$

In both cases determine the index of the critical point 0.

Exercise 249.5. Let $f\colon \mathbb{C} =_{\mathrm{i}} \mathbb{R}^2 \to \mathbb{C}$ be a holomorphic function and let $z_0 \in \mathbb{C}$. We consider the real-valued functions

$$\begin{array}{rcl} u\colon \mathbb{C} & \to & \mathbb{R} \\ z & \mapsto & u(z) := \mathrm{Re}(f(z)) \end{array} \quad \text{and} \quad \begin{array}{rcl} v\colon \mathbb{C} & \to & \mathbb{R} \\ z & \mapsto & v(z) := \mathrm{Im}(f(z)) \end{array}$$

(a) Show that z_0 is a critical point of u if and only if it is a critical point of v.
(b) Show that z_0 is a non-degenerate critical point of u if and only if it is a non-degenerate critical point of v.
(c) Show that if z_0 is a non-degenerate critical point, then $\mathrm{index}(u, z_0) = \mathrm{index}(v, z_0) = 1$.

Hint. Make use of the Smooth-Complex Differentiability Proposition 39.5.

Exercise 249.6. Let M be a closed non-empty n-dimensional smooth manifold of dimension $n \geq 1$. Show that every Morse function $f\colon M \to \mathbb{R}$ admits at least two critical points.

Exercise 249.7. Show that every Morse function is a height function. More precisely, let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Show that there exists a smooth embedding $\varphi\colon M \to \mathbb{R}^n$ such that $f = h \circ \varphi$ where $h\colon \mathbb{R}^n \to \mathbb{R}$ is the height function that is given by $(x_1, \ldots, x_n) \mapsto x_n$.

Exercise 249.8.

(a) Provide an explicit Morse function $f\colon S^1 \times [0,1] \to \mathbb{R}$ rel $\varnothing$.
(b) Provide an explicit Morse function $f\colon S^1 \times [0,1] \to \mathbb{R}$ rel $S^1 \times \{0\}$.
(c) Provide an explicit Morse function $f\colon S^1 \times [0,1] \to \mathbb{R}$ rel $S^1 \times \{0,1\}$.

Exercise 249.9.

(a) Does there exist a smooth function $f\colon S^1 \to \mathbb{R}$ with precisely three critical points?
(b) Does there exist a Morse function $f\colon S^1 \to \mathbb{R}$ with precisely three critical points?

Exercise 249.10. Show that the map

$$\begin{array}{rcl} \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & \cos(2\pi x) + \cos(2\pi y) \end{array}$$

is a Morse function and determine all critical points together with the corresponding indices.

Exercise 249.11.

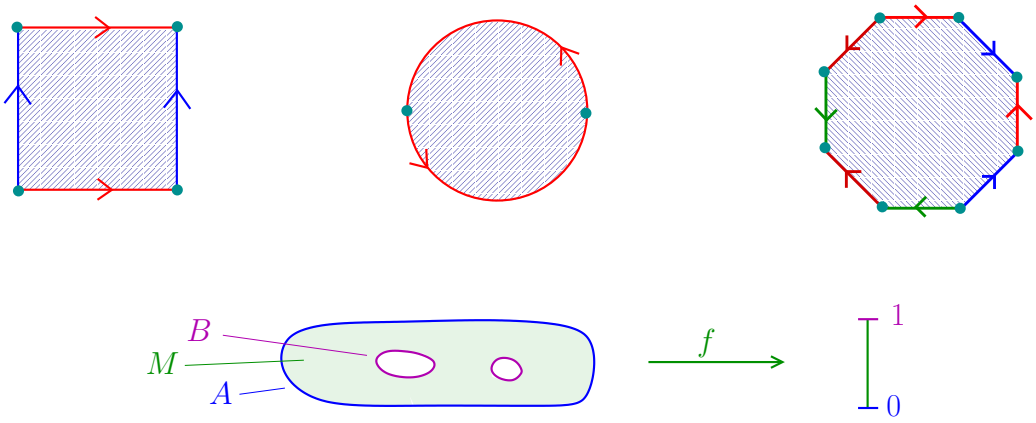
(a) We consider the function

$$\begin{array}{rcl} S^2 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & y^2 + 2z^2. \end{array}$$

Find all critical points of f and show that f is a Morse function.
(b) Show that f descends to a Morse function on $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} S^2/x \sim -x$. How many critical points does this Morse function on $\mathbb{R}\mathrm{P}^2$ have?

Exercise 249.12. Sketch Morse functions on the torus $T = [0,1]^2/\sim$, the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/x \sim -x$ and the surface of genus two $E_8/\sim$ such that each Morse function has a single local maximum and a single local minimum.

Exercise 249.13. We consider the smooth manifold M that is shown in the figure below with the boundary component A. Illustrate a Morse function $f\colon M \to [0,1]$ rel A with a single critical point by showing the level sets $f^{-1}(t)$ for several regular values and for the critical value.



Exercise 249.14. Let M be a closed smooth manifold and let $\varphi\colon M \to M$ be a self-diffeomorphism. Suppose that $f\colon M \to \mathbb{R}$ is a Morse function such that for each $k \in \mathbb{N}_0$ we have d_k critical points of index k. Show that the mapping torus

$$\operatorname{Tor}(M,\varphi) \;:=\; (M \times [0,1])/\,(x,0) \sim (\varphi(x),1)$$

admits a Morse function where for each $k \in \mathbb{N}_0$ the number of critical points of index k equals $d_k + d_{k-1}$.

Exercise 249.15. Let M be a closed non-empty n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function.

(a) Show that there exists a smooth embedding $\Phi\colon \overline{B}^n \to M$ such that $M \setminus \Phi(B^n)$ admits a Morse function rel $\Phi(S^{n-1})$ with the following properties:
- The number of critical points of index 0 is reduced by one.
- The number of critical points of index > 0 is unchanged.

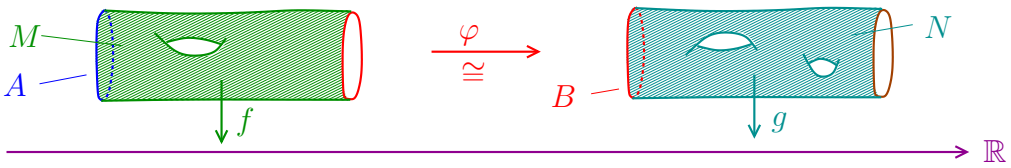
Hint. Use the Morse Lemma 249.11.

(b) Does the conclusion also hold for every smooth embedding $\Phi\colon \overline{B}^n \to M$?

Exercise 249.16. Let M be a compact n-dimensional smooth manifold and let $A \subseteq \partial M$ be a union of components of ∂M. Furthermore let N be another compact n-dimensional smooth manifold and let $B \subseteq \partial N$ be a union of components of ∂N. Let $f\colon M \to \mathbb{R}$ be a Morse function rel A and let $g\colon N \to \mathbb{R}$ be a Morse function rel B. Finally let $\varphi\colon \partial M \setminus A \to B$ be a diffeomorphism. As we explained in the Gluing-Smooth Manifolds-Proposition 54.5, we can form a smooth manifold $M \cup_\varphi N$ by gluing $\partial M \setminus A$ to B via φ. Show that the smooth manifold $M \cup_\varphi N$ admits a Morse function $h\colon M \cup_\varphi N \to \mathbb{R}$ rel A such that for any index $k \in \mathbb{N}_0$ the number of critical points of h equals the sum of the number of critical points of f and of g.

Remark. Explain the smooth structure on $M \cup_\varphi N$ and explain why the map h you construct is smooth.

Hint. Make use of the discussion on page 4826.



Exercise 249.17. Let $n \in \mathbb{N}$, let M and N be two closed n-dimensional smooth manifolds and let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions.

(a) Show that the connected sum $M\#N$ admits a Morse function h with the following properties:
- For the indices 0 and n the number of critical points of h equals the sum of the numbers for f and g minus one.
- The number of critical points of index $\geq 0, n$ is the sum of the numbers for f and g.

Hint. Use Exercises 249.15 and 249.16.

(b) Use (a) to show that the surface of genus g admits a Morse function with one critical point of index 0, $2g$ critical points of index 1 and one critical point of index 2.

(c) Use (a) to show that the non-orientable surface of genus k admits a Morse function with one critical point of index 0, k critical points of index 1 and one critical point of index 2.

Exercise 249.18. Let $a_1, \ldots, a_n \in \mathbb{R}$. We consider the smooth manifold

$$M \;:=\; \{(x_1, \ldots, x_n, a_1 \cdot x_1^2 + \cdots + a_n \cdot x_n^2) \,|\, x_1, \ldots, x_n \in \mathbb{R}\} \;\subseteq\; \mathbb{R}^{n+1}.$$

Furthermore we consider the point $(0,t) \in \mathbb{R}^n \times \mathbb{R}$ with $t \in \mathbb{R}$.

(a) Show that $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ is a critical point point of any $\delta_{(0,t)}$.

(b) For which t is $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ a non-degenerate critical point of $\delta_{(0,t)}$?

Exercise 249.19. Let $n \in \mathbb{N}$. We consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$ and, as in Notation 249.22, we consider the $(n+1)$-dimensional smooth manifold

$$\mathrm{N}S^n \;=\; \{(P,v) \,|\, P \in S^n \text{ and } v \in (\mathrm{T}_P S^n)^\perp \underset{\uparrow}{=} \mathbb{R} \cdot P\} \;\subseteq\; \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$

follows from the Physical Tangent Spaces–Lemma 29.2 (3)

(a) Given $(P, \lambda \cdot P) \in \mathrm{N}S^n$ show that the tangent space $\mathrm{T}_{(P,\lambda\cdot P)}\mathrm{N}S^n$ is spanned by the vectors $\{(v, \lambda \cdot v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,|\, v \in \mathrm{T}_P S^n\}$ and by $(0,P) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

(b) As in Definition 249.23 we consider the endpoint map

$$\begin{aligned} E\colon \mathrm{N}S^n &\to \mathbb{R}^{n+1} \\ (P,v) &\mapsto P + v. \end{aligned}$$

Show that $(P, \lambda \cdot P)$ is a critical point of E if and only if $\lambda = -1$ and show that $0 \in \mathbb{R}^{n+1}$ is the only critical value of E.

Remark. This exercise is identical to Exercise 35.1.

Exercise 249.20. We consider the smooth manifold $M := \{(x, x^2) \,|\, x \in \mathbb{R}\}$ and, as in Notation 249.22, we consider the 2-dimensional smooth manifold

$$\mathrm{N}M \;=\; \{((x,x^2), v) \,|\, (x,x^2) \in M \text{ and } v \in (\mathrm{T}_{(x,x^2)}M)^\perp\} \;\subseteq\; \mathbb{R}^4.$$

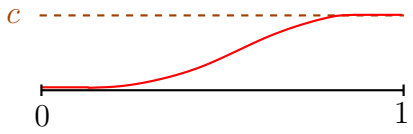
(a) Given $(x, x^2) \in M$ find a non-zero vector in $(\mathrm{T}_{(x,x^2)}M)^\perp \subseteq \mathbb{R}^2$.
(b) Given $((x, x^2), v) \in \mathrm{N}M$ find a basis for the tangent space $\mathrm{T}_{((x,x^2),v)}\mathrm{N}M \subseteq \mathbb{R}^4$.
(c) As in Definition 249.23 we consider the endpoint map

$$\begin{array}{rcl} E\colon \mathrm{N}M & \to & \mathbb{R}^2 \\ (P, v) & \mapsto & P + v. \end{array}$$

Show that $((0, 0), v)$ is a critical point of E if and only if $v = (0, \frac{1}{2})$.

Exercise 249.21. Let M be an n-dimensional smooth manifold and let $f, g\colon M \to \mathbb{R}$ be smooth functions. We suppose that there exists an open subset $U \subseteq M$ with the following two properties:
(1) The function g agrees with f outside of U.
(2) The functions f and g have no critical point on U.
Does it follow that f and g have the same number of critical points?

Exercise 249.22. Let $c > 0$. Show that there exists a smooth function $\tau\colon [0, 1] \to [0, c]$ with $\tau(t) = 0$ for $t \in [0, \frac{1}{8}]$, $\tau(t) = c$ for $t \in [\frac{7}{8}, 1]$ and $\tau'(t) \in [0, c)$ for all $t \in [0, 1]$.
Hint. Use the Smooth Transition Function Lemma 24.9 to construct τ' and use the Fundamental Theorem 73.27 of Calculus.
Remark. This exercise is identical to Exercise 24.6.



CHAPTER 250

From Morse functions to handle decompositions

In the last chapter we showed that every compact smooth manifold admits Morse functions. In this and the next chapter we will explore the connection between Morse functions and handle decompositions. In this chapter we will show that each Morse function gives rise to a handle decomposition, with precisely one handle for each critical point. Together with the existence result from the last chapter this will give us many interesting results on smooth manifolds.

This chapter is organized as follows:

- In Section 250.1 we will state and prove the Smooth Manifold-Product Theorem 250.1 which says that if a smooth manifold M admits a Morse function rel some A without critical points, then M is diffeomorphic to a product on A.
- In Section 250.2 we will state and prove the Critical Point–Handle Addition Theorem 250.5 which says that if a smooth manifold M admits a Morse function rel A with a single critical point of index k, then M is obtained from $[0,1] \times A$ by attaching a single k-handle.
- In Section 250.3 we will use the Smooth Manifold-Product Theorem 250.1 and the Critical Point–Handle Addition Theorem 250.5 to prove the Morse Function–Handle Decomposition Theorem 250.6 which says that any Morse function gives rise to a handle decomposition.
- In Section 250.4 we will apply the Morse Function–Handle Decomposition Theorem 250.6 to various types and examples of Morse functions. In particular we will prove the Handle Decomposition Theorem 250.7 which says that handle decompositions always exist.
- In Section 250.5 we will present several consequences of the Morse Function–Handle Decomposition Theorem 250.6.
- In Section 250.6 we will state and prove Reeb's Theorem 250.10.
- In Section 250.7 we will use handle decompositions to show that the Euler characteristic of bundles over compact smooth manifolds is multiplicative.
- In Appendix 250.8 we will discuss handle decompositions of topological manifolds.
- In Section 250.9 we will provide exercises for Chapter 250.

In the next Chapter 251 we will go backwards, namely we will see that handle decompositions give rise to Morse functions.

250.1. **The Smooth Manifold-Product Theorem.** We start out by reminding the reader of a simple example: Let N be a closed smooth manifold and let $a < b$ be two real numbers. As on page 4812 we consider the product $[a,b] \times N$, we set $A := \{a\} \times N$ and we consider the map

$$\begin{array}{rccc} f\colon [a,b] \times N & \to & [a,b] \\ (t,x) & \mapsto & t. \end{array}$$

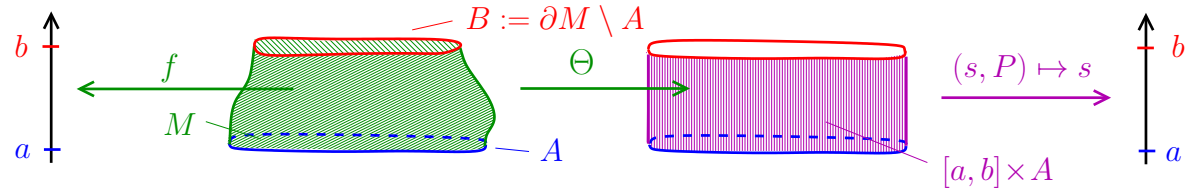
Evidently the following two statements hold:

(a) All points are regular.
(b) The function f is a Morse function rel A.

We will now see that a converse to this statement holds: If a smooth manifold M admits a Morse function $f\colon M \to \mathbb{R}$ rel A with properties (a) and (b), then the smooth manifold is a product on A. More precisely we have the following theorem:

Theorem 250.1. (Smooth Manifold-Product Theorem) Let M be a compact smooth manifold, let A be a union of boundary components of M and let $f\colon M \to [a,b]$ be a Morse function rel A with $f^{-1}(a) = A$ and with $f^{-1}(b) = \partial M \setminus A$. If all points in M are regular points of f, then there exists a diffeomorphism $\Theta\colon M \to [a,b] \times A$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow[\cong]{\Theta} & [a,b] \times A \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle (s,P) \mapsto s} & \\ & [a,b]. & \end{array}$$



In the following we will provide two very different proofs of the Smooth Manifold-Product Theorem 250.1:

(1) The first proof makes use of a variation on the Ehresmann Fibration Theorem 238.6
(2) The second proof is quite different, it makes use of some facts from the theory of differential equations, along the lines of the Fundamental Theorem 43.15 for Autonomous ODEs.

In the first proof we need the following variation on the Ehresmann Fibration Theorem 238.6:

Theorem 250.2. (Ehresmann Fibration Theorem) Let M and B be smooth manifolds and let $f\colon M \to B$ be a smooth map with the following properties: [2696]

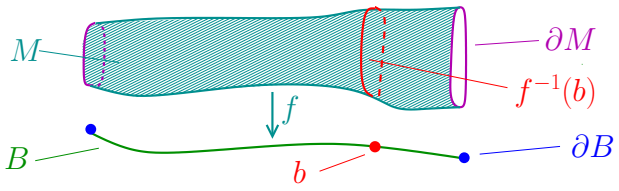
- For every $x \in M$ the differential $\mathrm{D}f_x\colon \mathrm{T}_x M \to \mathrm{T}_{f(x)} B$ is an epimorphism.
- We have $f^{-1}(\partial B) = \partial M$.

Given $b \in B$ we set $F_b := f^{-1}(b)$. If M is compact and if B is connected, then the following statements hold: [2697]

(1) Each $f^{-1}(b)$ is naturally a closed smooth manifold of dimension $\dim(M) - \dim(B)$.
(2) All the smooth manifolds $f^{-1}(b)$ are diffeomorphic.
(3) The map $f\colon M \to B$ is a smooth bundle map.

[2696] If $\partial M = \partial B = \varnothing$, then these conditions just say that p is a submersion in the sense of Definitions 122.5 and Definition 238.5.

[2697] The difference to the original Ehresmann Fibration Theorems 238.6 is as follows: In the earlier Ehresmann Fibration Theorem 238.6 we demanded that B has no boundary and we allowed that the fiber $p^{-1}(b)$



Remark. One could of course combine the Ehresmann Fibration Theorems 238.6 and 250.2 to a single theorem. But this would involve smooth manifolds with corner, which usually authors and readers want to avoid. □

Proof of the Ehresmann Fibration Theorem 250.2. As we saw in Exercise 238.16, the proof of the current Ehresmann Fibration Theorem 250.2 is, not surprisingly, very similar to the proof of the original Ehresmann Fibration Theorem 238.6. We leave it again to the reader to fill in the details. ■

First proof of Theorem 250.1. Let M be a compact smooth manifold, let A be a union of boundary components of M and let $f\colon M \to [a,b]$ be a Morse function rel A with $f^{-1}(a) = A$, with $f^{-1}(b) = \partial M \setminus A$ and such that all points in M are regular points of f.

The hypothesis that all points of f are regular means that $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathbb{R}$ is an epimorphism for all $P \in M$. Since $f^{-1}(\partial[a,b]) = \partial M$ it follows from the above Ehresmann Fibration Theorem 250.2 that $f\colon M \to [a,b]$ is a smooth bundle map where the fiber is given by $A = f^{-1}(a)$. Since $[a,b]$ is contractible it follows from the Smooth Bundles-Homotopies Proposition 243.10 (or from an elementary argument as in the proof of the Smooth Mapping Torus–Bundle Lemma 122.3 (2)) that the smooth bundle is trivializable, i.e. there exists a diffeomorphism $\Theta\colon M \to [a,b] \times A$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow[\cong]{\Theta} & [a,b] \times A \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle (s,P)\mapsto s} & \\ & [a,b]. & \end{array}$$

■

Sketch of second proof of Theorem 250.1. Note that using the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can assume that M is a submanifold of some $\mathbb{R}^n$. Given $P \in M$ we denote by $\operatorname{grad} f_P \in \mathrm{T}_P M$ the gradient of f at P. Recall that the gradient $\operatorname{grad} f_P \in \mathrm{T}_P M$ of f at $P \in M$ is defined as the unique vector in $\mathrm{T}_P M$ which satisfies $\langle w, \operatorname{grad} f_P\rangle = \mathrm{D}f_P(w) \in \mathbb{R}$ for all $w \in \mathrm{T}_P M$.[2698] Note that our hypothesis on f just says that given any $P \in M$ the gradient $\operatorname{grad} f_P$ is non-zero. Thus given any $P \in M$ we can define

$$v(P) \coloneqq \frac{\operatorname{grad} f_P}{\|\operatorname{grad} f_P\|^2} \in \mathrm{T}_P M.$$

Claim 1. If $\gamma\colon [a,b] \to M$ is a solution to the differential equation $\gamma'(t) = v(\gamma(t))$, then $(f \circ \gamma)' \equiv 1$.

---

is a smooth manifold with boundary. We now have the opposite setup: We allow that B has boundary but we demand that $p^{-1}(\partial B) = \partial M$, in particular we demand that the fiber $p^{-1}(b)$ is a smooth manifold with empty boundary.

[2698] In the Gradient Vector Field Proposition **??** we will show that $P \mapsto \operatorname{grad} f_P$ is a smooth map $M \to \mathbb{R}^n$.

Proof. Let $t \in [a,b]$. We perform the following calculation:

$$\begin{array}{lcl} (f\circ\gamma)'(t) & = & \mathrm{D}(f\circ\gamma)_t \\ & & \downarrow \text{Chain Rule 24.17} \\ & \stackrel{\downarrow}{=} & \mathrm{D}f_{\gamma(t)}(\gamma'(t)) \\ & = & \langle v(\gamma(t)), \operatorname{grad} f_P\rangle \\ & & \uparrow \text{by hypothesis we have } \gamma'(t)=v(\gamma(t)) \end{array} \quad \begin{array}{lcl} & & \text{by the definition of a gradient, here we set } P:=\gamma(t) \\ & \stackrel{\downarrow}{=} & \langle \gamma'(t), \operatorname{grad} f_P\rangle \\ & = & \left\langle \frac{1}{\|\operatorname{grad} f_P\|^2}\cdot \operatorname{grad} f_P, \operatorname{grad} f_P\right\rangle \\ & & \uparrow \text{since } P=\gamma(t) \text{ and by definition of } v \end{array} \quad \begin{array}{lcl} & & \\ & & \\ & = & 1. \\ & & \uparrow \text{since } \langle v,v\rangle = \|v\|^2 \end{array}$$

⊞

It remains to prove the following claim:

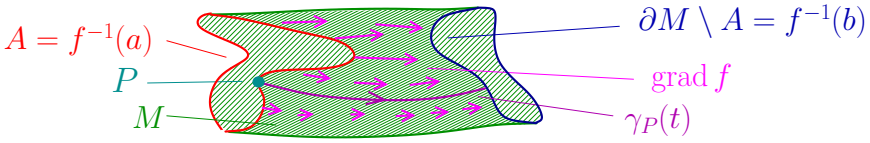
Claim 2.

(1) Given any $P \in A$ there exists a unique smooth map $\gamma_P\colon [a,b] \to M$ with $\gamma_P(a) = P$ and such that for each $t \in [a,b]$ we have $\gamma_P'(t) = v(\gamma_P(t))$.
(2) For any $P \in A$ and any $t \in [a,b]$ we have $f(\gamma_P(t)) = t$.
(3) The map

$$\begin{array}{rcl} \Theta\colon [a,b]\times A & \to & M \\ (t,P) & \mapsto & \gamma_P(t) \end{array}$$

is a diffeomorphism such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow[\cong]{\Theta} & [a,b]\times A \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle (s,P)\mapsto s} & \\ & [a,b]. & \end{array}$$



Proof.

(1) This statement follows from the previous claim and standard theory of differential equations, see the Fundamental Theorem 43.15 for Autonomous ODEs.
(2) This statement follows immediately from Claim 1 and $f(\gamma(a)) = f(P) = a$.
(3) It follows from (1) that the map $\Theta\colon [a,b]\times A \to M$ is well-defined. Furthermore, using the Fundamental Theorem 43.15 for Autonomous ODEs one can show that the map is smooth and that for each $t \in [a,b]$ the map $A \to M$ given by $P \mapsto \gamma_P(t)$ is a smooth embedding with image $f^{-1}(t)$. It follows from these observation, the previous claim and the Inverse Mapping Theorem 30.1 (2) that the map $\Theta\colon [a,b]\times A \to M$ is actually a diffeomorphism. Finally note that it follows from (2) that the diagram commutes. ⊞

We leave it to the reader to dig out their favorite books on differential equations to fill in the details. Very similar statements are also proved in [**Miln1963a**, p. 10] and [**Lee2002**, Theorem 9.12, Corollary 9.17]. ■

250.2. **The Critical Point–Handle Addition Theorem.** In the last section we saw that if we are given a Morse function without a single critical point, then the smooth manifold is diffeomorphic to a product. In this section we will see what happens if we have a single critical point.

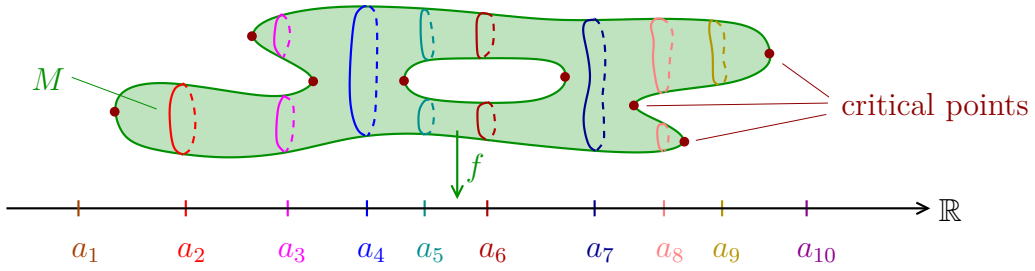
We start out by introducing the following notation:

Notation 250.3. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Given any $a \in \mathbb{R}$ we write

$$M_a \;:=\; f^{-1}((-\infty, a]) \;=\; \{x \in M \mid f(x) \leq a\}.$$

Note that if a is a regular value of f and if $\partial M \cap f^{-1}(a) = \varnothing$, then it follows from the Regular Value Theorem 32.5 that $f^{-1}(a)$ is a closed submanifold of M and it follows from the Preimage-of-Interval Lemma 38.3 that M_a is a submanifold of M with boundary $f^{-1}(a) \cup (\partial M \cap f^{-1}((-\infty, a]))$.

Example. In the figure below we show a closed smooth manifold M together with the "length" function $f\colon M \to \mathbb{R}$ given by $(x, y, z) \mapsto x$.



For various $a_i \in \mathbb{R}$ we show the corresponding preimage $f^{-1}(a_i)$. The submanifold M_{a_i} is of course everything "to the $\leq$-left" of $f^{-1}(a_i)$. The figure suggests two statements:

- If we are given $a < b$ such that $f^{-1}([a, b])$ contains no critical point, then M_a is diffeomorphic to M_b.
- If we are given $a < b$ such that $f^{-1}([a, b])$ contains a single (non-degenerate) critical point, then the topology of M_b is different from the topology of M_a, and the change in topology seems to be related to the index of the critical point.

In the following we will see that the two educated guess are indeed correct. □

The following theorem shows what happens if in an interval $[a, b]$ there are no critical values:

Theorem 250.4. (No Critical Point-Diffeomorphism Theorem) Let M be a compact smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$. If $f^{-1}([a, b])$ does not contain a critical point of f and if $f^{-1}([a, b]) \cap \partial M = \varnothing$, then there exists a diffeomorphism $M_a \to M_b$. Furthermore, given any $\epsilon > 0$ one can find a diffeomorphism $M_a \to M_b$ that is the identity on the common subset $M_{a-\epsilon}$.

Example. In the figure above there is no critical point in $f^{-1}([a_5, a_6])$. Thus we obtain from the No Critical Point-Diffeomorphism Theorem 250.4 that the smooth manifolds M_{a_5} and M_{a_6} are diffeomorphic. Furthermore, we can find a diffeomorphism that is the identity on "most of M_{a_5}". □

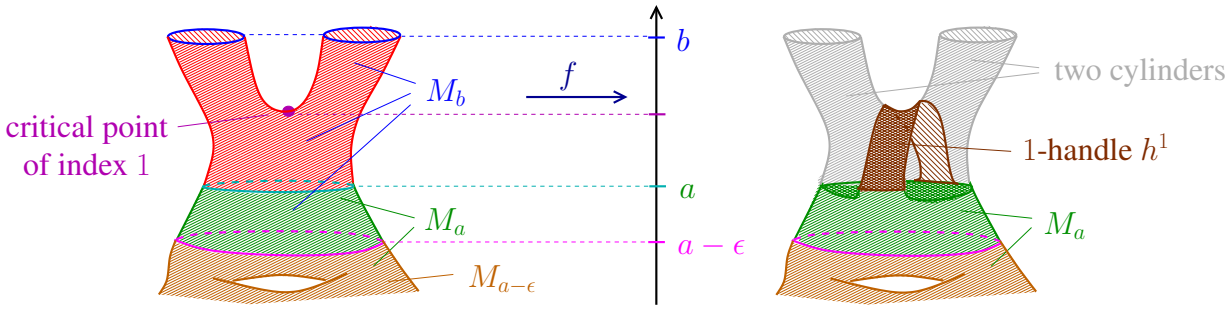
Proof of the No Critical Point-Diffeomorphism Theorem 250.4. Let M be a compact smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$. We assume that the preimage $f^{-1}([a, b])$ does not contain a critical point of f and we assume that $f^{-1}([a, b]) \cap \partial M = \varnothing$.

We set $A := f^{-1}(a)$. By the Smooth Manifold-Product Theorem 250.1 we know that $f^{-1}([a, b])$ is diffeomorphic to the product $[a, b] \times A$. The Argument of the External Collar

Lemma 247.3 shows that $M_b = M_a \cup f^{-1}([a,b])$ is diffeomorphic to M_a. Finally: If we are given some $\epsilon > 0$, then it follows from the "in fact statement" of the External Collar Lemma 247.3 that we can find a diffeomorphism that is the identity on $M_{a-\epsilon}$. ■

Again let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $a < b \in \mathbb{R}$ such that a and b are regular values and such that we have $f^{-1}([a,b]) \cap \partial M = \varnothing$. We just learned that if $f^{-1}([a,b])$ does not contain a critical point of f, then M_b is diffeomorphic to M_a. But what happens if $f^{-1}((a,b))$ contains a critical point? We consider an example:

Example. On the left of the figure below we show a 2-dimensional smooth manifold M together with a smooth function $f\colon M \to \mathbb{R}$ such that there exists a unique critical point of index $k = 1$ in $f^{-1}((a,b))$. On the right of the figure below we show the result of attaching a handle of index $k = 1$ to M_a.



With good eyes one see that $M_b \setminus (M_a \cup h^1)$ is a product, which implies that M_b is diffeomorphic to $M_a \cup h^1$. □

The following all-important theorem shows that a suitable generalization of the above example holds in all reasonable settings:

Theorem 250.5. (Critical Point–Handle Addition Theorem) Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $a < b \in \mathbb{R}$ such that a and b are regular values of f and such that $f^{-1}([a,b]) \cap \partial M = \varnothing$. We suppose that there exists a single critical point $P \in f^{-1}([a,b])$ and we suppose that P is non-degenerate. We set $k := \operatorname{index}(f,P)$. Then there exists a thickened $(k-1)$-sphere φ in ∂M_a and a diffeomorphism[2699]

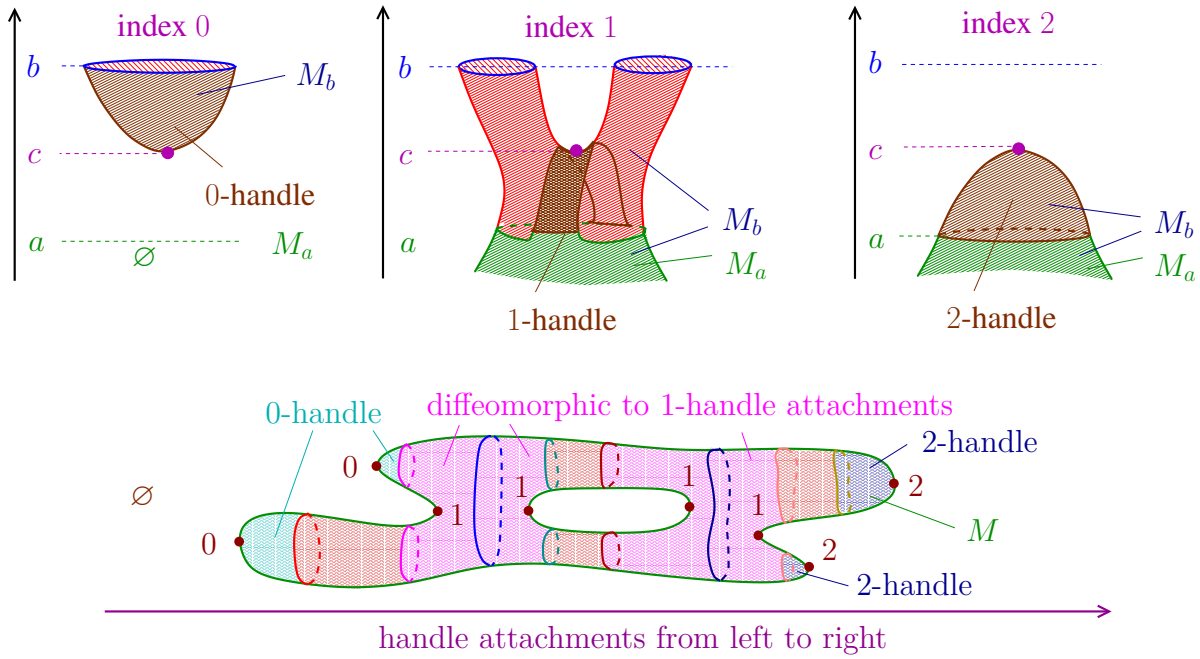
$$M_a \cup_\varphi h^k \xrightarrow{\cong} M_b.$$

Furthermore, given any $\epsilon > 0$ there exists a diffeomorphism as above which is the identity on the common subset $M_{a-\epsilon}$.

Remark. Alternative proofs of the Handle Addition Theorem 250.5 are given in [**Nic2011**, Theorem 2.7 and Remark 2.9], [**Miln1963a**, Theorem 3.2, Remark on p. 17, Remark 3.3] and [**Kos1993**, Chapter VII.2]. □

Example. In the figure below we return to the setting of the figure on page 4839. We show all the critical points and we indicate their indices.

[2699] As we see in the center of the figure below, M_b is diffeomorphic to $M_a \cup_\varphi h^k$, but we *not* arrange that in any way we have $M_b = M_a \cup_\varphi h^k$ as submanifolds of M.



Reading the figure from left to right we see the following: By applying the Critical Point–Handle Addition Theorem 250.5 several times we see how M gets built out of the empty manifold by various handle attachments.[2700] □

The proof of the Handle Addition Theorem 250.5 will require the remainder of this section. In the next Section 250.4 we will use the theorem to show that any Morse function gives rise to a handle decomposition.

Proof. Let M be a compact n-dimensional smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$ be regular values of f such that $f^{-1}([a,b]) \cap \partial M = \varnothing$. We suppose that there exists a single critical point $P \in f^{-1}([a,b])$ and we suppose that P is non-degenerate. We set $c := f(P)$ and we set $k := \operatorname{index}(f,P)$. Note that $c \in (a,b)$ since a and b are regular values of f. We need to show that M_b is diffeomorphic to the result of attaching a k-handle to M_a.

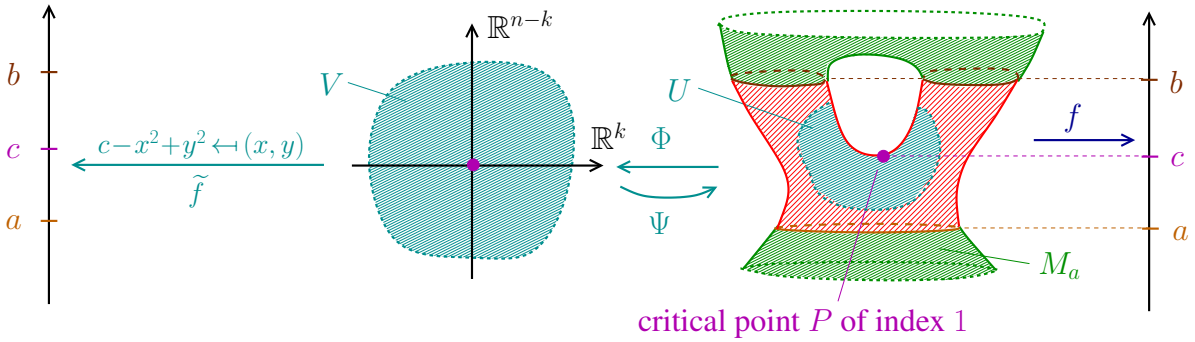
We start out with the following "standard" preparation:

(1) Note that by the Morse Lemma 249.11 there exists an open neighborhood U of $P \in M$, an open neighborhood V of $(0,0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ and a diffeomorphism $\Phi\colon U \to V$ such that $\Phi(P) = (0,0)$ and such that for every $(x,y) \in V$ we have the equality $f(\Phi^{-1}(x,y)) = c - \|x\|^2 + \|y\|^2$. We write $\Psi := \Phi^{-1}\colon V \to U$ and we consider the smooth function

$$\begin{aligned} \widetilde{f} := f \circ \Psi\colon V &\to \mathbb{R} \\ (x,y) &\mapsto c - \|x\|^2 + \|y\|^2. \end{aligned}$$

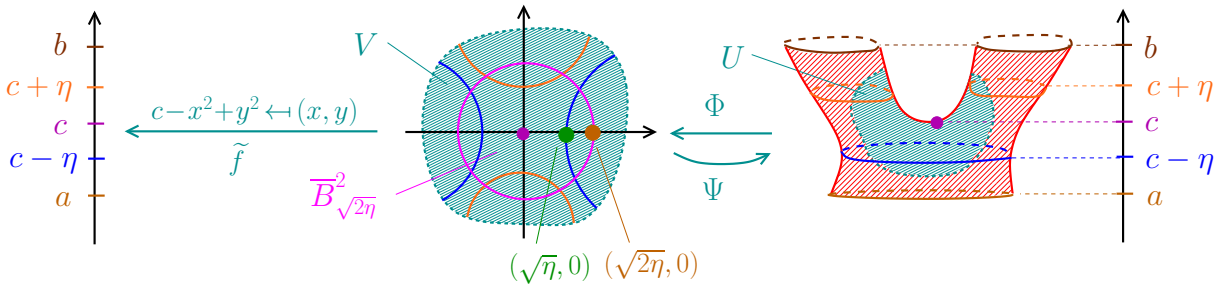
Note that only in the neighborhood U of $P \in M$ do we have complete control over f. Unfortunately a priori the difference $M_b \setminus \mathring{M}_a$ is not contained in the neighborhood U. To

[2700] To simplify our pictures we do not distinguish between a smooth manifold that is the result of attaching a k-handle and a smooth manifold that is diffeomorphic to the result of attaching a k-handle.



deal with this issue we have to make several fancy dance moves. First of all we will restrict ourselves to a cleverly chosen subset:

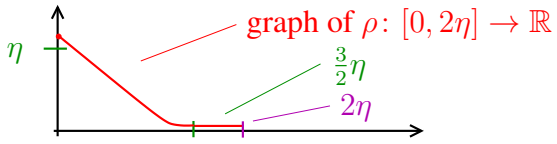
(2) Since $c \in (a,b)$ and since V is an open neighborhood of $(0,0)$ we can pick an $\eta > 0$ such that $a \leq c-\eta < c+\eta \leq b$ and such that $\overline{B}^n_{\sqrt{2\eta}} \subseteq V$. Note that by our hypothesis there are no critical points in $f^{-1}([a,c-\eta])$ and there are no critical points in $f^{-1}([c+\eta,b])$. Thus it follows from the No Critical Point-Diffeomorphism Theorem 250.4 together with the Modify Handle Attachment Region Lemma 247.8 that it suffices to show that $M_{c+\eta}$ is diffeomorphic to the result of attaching a k-handle to $M_{c-\eta}$.



There is still no reason why the difference $M_{c+\eta} \setminus \mathring{M}_{c-\eta}$ should be contained in the neighborhood U. To deal with this issue we will introduce an auxiliary function $F\colon M \to \mathbb{R}$. This requires some preparations:

(3) Using the Smooth Transition Function Lemma 24.9 one can show, similar to Exercise 249.22, that there exists a smooth map $\rho\colon [0,2\eta] \to \mathbb{R}_{\geq 0}$ with the following properties:

- $\rho(0) > \eta$,
- $\rho(t) = 0$ for $t \geq \frac{3}{2}\eta$, and
- $\rho'(t) \in (-1,0]$ for all $t \in [0,2\eta]$.



(4) Given $s \in [0, 2\eta]$ we consider the ellipsoid

$$\overline{E}^n_s \;:=\; \{(x,y) \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \mid \|x\|^2 + 2\|y\|^2 \le s\} \;\underset{\substack{\uparrow \\ \text{this inclusion follows from} \\ \|x\|^2+\|y\|^2 \le \|x\|^2+2\|y\|^2 \le s}}{\subseteq}\; \overline{B}^n_{\sqrt{s}} \;\underset{\substack{\uparrow \\ \text{since } s \le 2\eta}}{\subseteq}\; \overline{B}^n_{\sqrt{2\eta}}.$$

Furthermore we denote by E^n_s the corresponding open ellipsoid.

(5) Finally we consider the functions

$$\begin{aligned} \widetilde{F}\colon \overline{E}^n_{2\eta} &\to \mathbb{R} \\ (x,y) &\mapsto \underbrace{c - \|x\|^2 + \|y\|^2}_{=\widetilde{f}(x,y)} - \rho(\|x\|^2 + 2\|y\|^2) \end{aligned}$$

and

$$\begin{aligned} F\colon M &\to \mathbb{R} \\ Q &\mapsto \begin{cases} \widetilde{f}(\Phi(Q)), & \text{if } Q \in \Psi(\overline{E}^n_{2\eta}), \\ f(Q), & \text{otherwise.} \end{cases} \end{aligned}$$

The following claim summarizes some of the key properties of F:

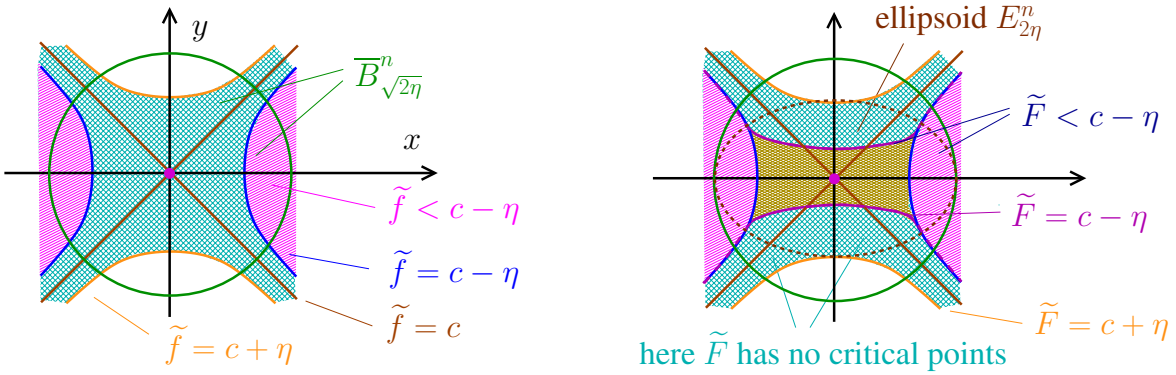
Claim 1.

(1) The map $F\colon M \to \mathbb{R}$ agrees with the map $f\colon M \to \mathbb{R}$ outside of $\Psi(\overline{E}^n_{\frac{3}{2}\eta})$.
(2) The map F is smooth.
(3) We have $f^{-1}((-\infty, c-\eta]) \subseteq F^{-1}((-\infty, c-\eta])$.
(4) The smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to $F^{-1}((-\infty, c+\eta])$.
(5) We have $F^{-1}((-\infty, c+\eta]) = f^{-1}((-\infty, c+\eta])$.

In particular we have

$$M_{c-\eta} \;=\; f^{-1}((-\infty, c-\eta]) \;\underset{\substack{\uparrow \\ \text{by (1) the difference is} \\ \text{contained in } \Psi(\overline{E}^n_{\frac{3}{2}\eta}) \subseteq \Psi(\overline{B}^n_{\sqrt{2\eta}})}}{\overset{(3)}{\subseteq}}\; F^{-1}((-\infty, c-\eta]) \;\underset{\downarrow \text{ diffeomorphic by (4)}}{\cong}\; F^{-1}((-\infty, c+\eta]) \;\overset{(5)}{=}\; f^{-1}((-\infty, c+\eta]) \;=\; M_{c+\eta}.$$

In the figure below we illustrate the statements of the claim, but translated via the chart Φ to the Euclidean setting:



Proof. We prove the statements in a somewhat different order.

(1) Note that for $(x,y) \in \overline{E}^n_{2\eta}$ we have $\rho(x^2 + 2y^2) = 0$, unless $(x,y) \in E_{\frac{3}{2}\eta}$. It follows that F agrees with the map f outside of $\Psi(E^n_{\frac{3}{2}\eta})$.

(2) It follows from (1), the smoothness of ρ and f and the Smooth Pasting Proposition 25.19 (applied to $\Psi(E^n_{2\eta})$ and $M \setminus \Psi(\overline{E}^n_{\frac{3}{2}\eta})$ that F is smooth.

(3) Since ρ takes values in $\mathbb{R}_{\geq 0}$ we see that $F(P) \leq f(P)$ for all $P \in M$. But this implies that for any $r \in \mathbb{R}$ we have $f^{-1}((-\infty, r]) \subseteq F^{-1}((-\infty, r])$.

(5) We need to show that $F^{-1}((-\infty, c+\eta]) = f^{-1}((-\infty, c+\eta])$. It follows from (1) that it remains to show that the intersections of the two sets with $\Psi(\overline{E}^n_{\frac{3}{2}\eta})$ are the same. But for any $(x, y) \in \overline{E}^n_{\frac{3}{2}\eta}$ we have

$$F(\Psi(x,y)) \underset{\substack{\uparrow \\ \text{since } \rho(\|x\|^2+2\|y\|^2)\geq 0}}{\leq} f(\Psi(x,y)) \underset{\substack{\uparrow \\ \text{since } \overline{E}^n_{\frac{3}{2}\eta}\subseteq \overline{B}^n_{\sqrt{2\eta}}\subseteq V}}{=} c - \|x\|^2 + \|y\|^2 \leq c + \tfrac{1}{2}\|x\|^2 + \|y\|^2 \underset{\substack{\uparrow \\ \text{since } (x,y)\in \overline{E}^n_{\frac{3}{2}\eta} \text{ means} \\ \text{that } \|x\|^2+2\|y\|^2 \leq 2\eta}}{\leq} c + \eta.$$

Thus we have shown that $\overline{E}^n_{\frac{3}{2}\eta}$ is contained in $F^{-1}((-\infty, c+\eta])$ and it is also contained in $f^{-1}((-\infty, c+\eta])$.

(4) We need to show that the smooth manifold $F^{-1}((-\infty, c+\eta])$ is diffeomorphic to $F^{-1}((-\infty, c-\eta])$. By the No Critical Point-Diffeomorphism Theorem 250.4 it suffices to prove the following subclaim:

Subclaim.

(a) The critical points of F are precisely the critical points of f.

(b) $F^{-1}([c-\eta, c+\eta])$ does not contain a critical point of f.

Proof of subclaim.

(a) First note that it follows from (1) that we only need to consider the points $\Psi(x, y)$ where $(x, y) \in E^n_{2\eta}$. By design f has a single critical point, namely $P = \Psi(0, 0)$ in $\Psi(E^n_{2\eta})$.

Since the map $\Psi\colon E^n_{2\eta} \to U$ is a smooth embedding it suffices to show that the function $\widetilde{F} = F \circ \Psi\colon E^n_{2\eta} \to \mathbb{R}$ has a unique critical point at $(0, 0)$. For any point $(x_0, y_0) \in E^n_{2\eta}$ we compute that:

$$\begin{aligned} \mathrm{D}\widetilde{F}_{(x_0,y_0)} &= \mathrm{D}((x,y)\mapsto c - \|x\|^2 + \|y\|^2 - \rho(\|x\|^2+2\|y\|^2))_{(x_0,y_0)} \\ &= (2x_0 \cdot \overbrace{(-1 - \underbrace{\rho'(\|x_0\|^2+2\|y_0\|^2)}_{\substack{>-1 \text{ since} \\ \rho'(t)\in(-1,0]}})}^{<0}, 2y_0 \cdot \overbrace{(1 - 2\cdot\underbrace{\rho'(\|x_0\|^2+2\|y_0\|^2)}_{\substack{\leq 0 \text{ since} \\ \rho'(t)\in(-1,0]}})}^{\geq 1}). \\ &\uparrow \\ &\text{by the Chain Rule 24.17} \end{aligned}$$

Clearly this differential is zero if and only if $(x_0, y_0) = (0, 0)$.

(b) We need to show that the preimage $F^{-1}([c-\eta, c+\eta])$ does not contain a critical point of f. This statement follows from the following inclusions and statements:

$$\begin{aligned} \overbrace{F^{-1}([c-\eta,c+\eta])}^{\substack{\text{by construction we have } \rho(0)>\eta \\ \text{thus } F(P)=c-\rho(0)<c-\eta, \\ \text{thus this set does not contain } P}} &= F^{-1}((-\infty,c+\eta]) \setminus F^{-1}((-\infty,c-\eta)) \\ &\subseteq f^{-1}((-\infty,c+\eta]) \setminus f^{-1}((-\infty,c-\eta)) = \underbrace{f^{-1}([c-\eta,c+\eta])}_{\substack{\text{by hypothesis this contains} \\ \text{a single critical point of } f, \\ \text{namely } P}}. \\ &\qquad\uparrow \\ &\qquad\text{follows from (3) and (5)} \end{aligned}$$

⊞

By Claim 1 it suffices to prove that the smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to the result of attaching a k-handle to $M_{c-\eta} = f^{-1}((-\infty, c-\eta])$. This step is performed in the following claim:

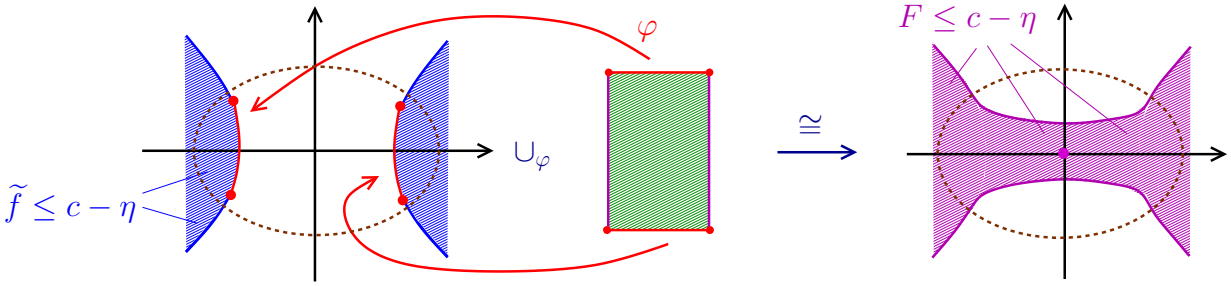
Claim 2. We consider the thickened $(k-1)$-sphere

$$\begin{aligned} \varphi\colon \overline{B}^{n-k} \times S^{k-1} &\to \widetilde{f}^{-1}(c-\eta) \subseteq E^n_{2\eta} \\ (x,y) &\mapsto \big(\sqrt{\tfrac{3}{2}\eta \cdot x^2 + \eta} \cdot y, \sqrt{\tfrac{3}{2}\eta} \cdot x\big). \end{aligned}$$

There exists a diffeomorphism

$$\underbrace{f^{-1}((-\infty, c-\eta]) \cup_{\Psi\circ\varphi} h^k}_{=M_{c-\eta}} \xrightarrow{\cong} F^{-1}((-\infty, c-\eta]).$$

In the figure below we illustrate the claim, but again translated via the chart Φ to the Euclidean setting:



Proof. We start out with two preliminary observations:

(1) It follows from Claim 1 that we can restrict ourselves to $E^n_{2\eta}$.
(2) Note that for $(x,y) \in E^n_{2\eta}$ with $\|x\| \ge 2\sqrt{\frac{\eta}{3}}$ we have

$$\widetilde{f}(x,y) \;=\; c - \|x\|^2 + \|y\|^2 \;\underset{\substack{\uparrow \\ \text{for } (x,y)\in E^n_{2\eta} \text{ we have} \\ \|x\|^2 + 2\|y\|^2 < 2\eta}}{<}\; c - \|x\|^2 + \eta - \tfrac{1}{2}\|x\|^2 \;\underset{\substack{\uparrow \\ \text{since } \|x\| \ge 2\sqrt{\frac{\eta}{3}}}}{\le}\; c + \eta - \tfrac{3}{2}(2\sqrt{\tfrac{\eta}{3}})^2 \;\le c - \eta.$$

This also implies that $\widetilde{F}(x,y) \le \widetilde{f}(x,y) \le c-\eta$. These two observations show that we can restrict ourselves to the "partial ellipsoid"

$$\widetilde{E}^n_{2\eta} \;=\; \big\{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \,\big|\, \|x\| < 2\sqrt{\tfrac{\eta}{3}} \text{ and } \|x\|^2 + 2\|y\|^2 < 2\eta\big\}.$$

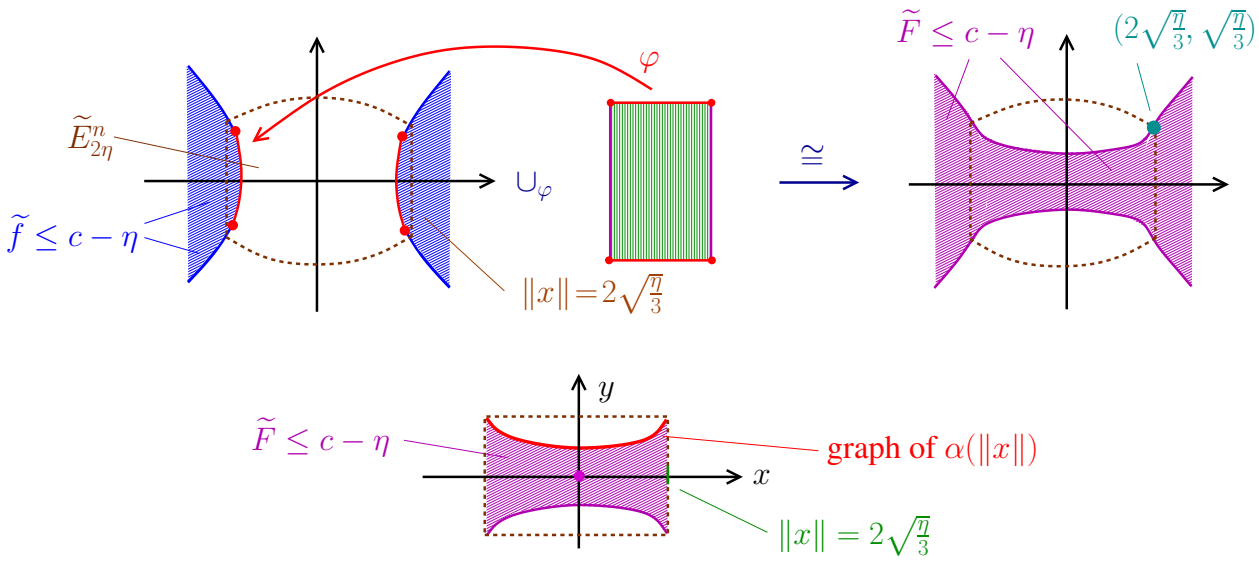
More precisely, it follows from Claim 1 and the above discussion that it suffices to show that there exists a diffeomorphism

$$\{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \widetilde{f}(x,y) \le c-\eta\} \cup_\varphi h^k \;\xrightarrow{\cong}\; \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \widetilde{F}(x,y) \le c-\eta\}$$

which is the identity if $\|x\| \in [\sqrt{\frac{7\eta}{6}}, 2\sqrt{\frac{\eta}{3}}]$.
This goal will be achieved in the following three subclaims:

Subclaim 1. There exists a smooth function $\alpha\colon [0, 2\sqrt{\frac{\eta}{3}}] \to (0, \sqrt{\frac{\eta}{3}}]$ such that

$$\{(x,y) \in E^n_{2\eta} \,|\, \widetilde{F}(x,y) \le c-\eta\} \;=\; \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \|y\| \le \alpha(\|x\|)\}.$$



Proof of subclaim. First we fix $s \in \left[0, 2\sqrt{\frac{\eta}{3}}\right]$. We consider the following smooth function

$$\begin{array}{rccl} \gamma_s(t)\colon [0, \sqrt{\frac{\eta}{3}}\,] & \to & \mathbb{R} \\ t & \mapsto & \underbrace{c - s^2 + t^2 - \rho(s^2 + 2t^2)}_{\text{note that } \widetilde{F}(x,y) = \gamma_{\|x\|}(\|y\|)}. \end{array}$$

We make three observations:

(a) First we claim that $\gamma_s(0) < c - \eta$. This can be seen as follows: By definition we have $\gamma_s(0) = c - s^2 - \rho(s^2)$. Thus we consider the function $g(w) := c - w - \rho(w)$. Recall that we know that $\rho(0) > \eta$ and we know that $\rho'(w) \in (-1, 0]$ for all $w \in [0, 2\eta]$. It follows that $g(0) < c - \eta$ and that $g'(w) < 0$ for all $s \in [0, \frac{4}{3}\eta]$. Elementary real analysis tells us that $g(w) < c - \eta$ for all $w \in [0, \frac{4\eta}{3}]$. But this implies that $\gamma_s(0) = g(s^2) < c - \eta$.

(b) Next we consider the derivative of the function γ_s. Note that

$$(\gamma_s)'(t) \;=\; 2t - \underbrace{\rho'(s^2 + 2t^2)}_{\in(-1,0]} \cdot 4t \;=\; 2t \cdot \underbrace{(1 - 2 \cdot \rho'(s^2 + 2t^2))}_{> 0, \text{ since } \rho' \in (-1,0]}.$$

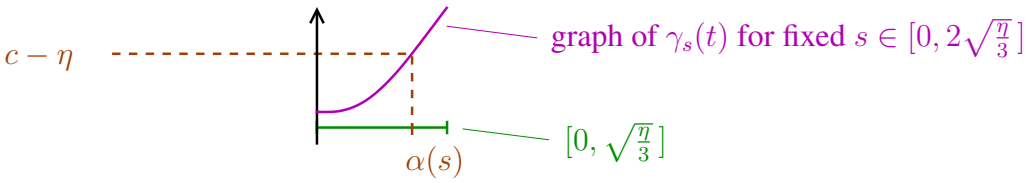
This shows that $(\gamma_s)'(t) > 0$ for all $t \in (0, \sqrt{\frac{\eta}{3}}]$.

(c) We claim that $\gamma_s(\sqrt{\frac{\eta}{3}}) \geq c - \eta$. This can be seen as follows: First note that by definition we have $\gamma_s(\sqrt{\frac{\eta}{3}}) = c - s^2 + \frac{\eta}{3} - \rho(s^2 + \frac{2\eta}{3})$. Similar to (a) we consider the function $g(w) := c - w + \frac{\eta}{3} - \rho(w + \frac{2\eta}{3})$. It follows from $\rho(2\eta) = 0$ that $g(\frac{4\eta}{3}) = c - \eta$. As in (a) we see that $g'(w) \leq 0$. It follows that $g(w) \geq c - \eta$ for all $w \in [0, \frac{4\eta}{3}]$. But this implies that $\gamma_s(\sqrt{\frac{\eta}{3}}) = g(s^2) \geq c - \eta$.

It follows from these three observations and elementary real analysis that there exists a unique $\alpha(s) \in (0, \sqrt{\frac{\eta}{3}}\,]$ such that $\gamma_s(\alpha(s)) = c - \eta$.
In other words, there exists a unique $\alpha(s) \in (0, \sqrt{\frac{\eta}{3}}]$ such that

$$\{(x,y) \in \widetilde{E}_{2\eta} \,|\, \|x\| = s \text{ and } \widetilde{F}(x,y) \leq c - \eta\} \;=\; \{(x,y) \in \widetilde{E}_{2\eta} \,|\, \|x\| = s \text{ and } \|y\| \leq \alpha(x)\}.$$

Finally note that it follows from the Implicit Function Theorem [**Lee2002**, Theorem C.40] that the map α is actually smooth. □

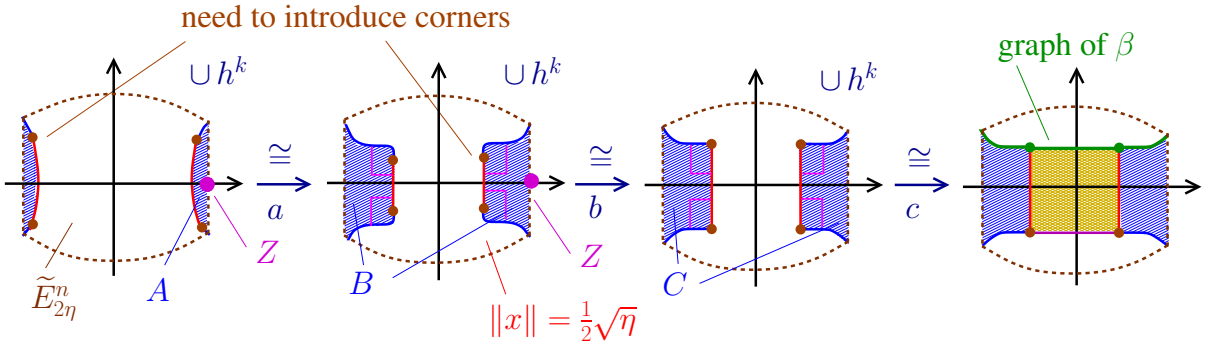


Subclaim 2. There exists a smooth function $\beta\colon [0, 2\sqrt{\frac{\eta}{3}}] \to (0, \sqrt{\frac{\eta}{3}}]$ such that there exists a diffeomorphism

$$\{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, f(x,y) \le c-\eta\} \cup_\varphi h^k \xrightarrow{\cong} \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \|y\| \le \beta(\|x\|)\}$$

and such that β agrees with α on $[\sqrt{\frac{7\eta}{6}}, 2\sqrt{\frac{\eta}{3}}]$.

Proof of subclaim. We only provide an outline of a proof. The diffeomorphism is obtained as a composition of three diffeomorphisms that are sketched in the figure below:



We now provide at least some details:

(a) We set

$$\begin{aligned} A &:= \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \overbrace{c - \|x\|^2 + \|y\|^2}^{\tilde{f}(x,y)} \le c-\eta\}, \\ B &:= \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \xi(\|x\|, \|y\|) \le c-\eta\}, \end{aligned}$$

where

$$\xi\colon D := \{(s,t) \in \mathbb{R}^2 \,|\, s \in [0, 2\sqrt{\tfrac{\eta}{3}}] \text{ and } s^2 + 2t^2 \le 2\eta\} \to \mathbb{R}$$

is a smooth function with the following properties:

- We have $\xi(s,t) = c - s^2 + t^2$ for all $t \in [\sqrt{\frac{7\eta}{6}}, 2\sqrt{\frac{\eta}{3}}]$.
- $\{(s,t) \in D \,|\, \xi(s,t) \le c-\eta\}$ is a submanifold of D.
- The set $\{(s,t) \in F \,|\, \xi(s,t) = c-\eta\}$ has a horizontal segment and a vertical segment at $s = \frac{1}{2}\sqrt{\eta}$.
- The set $\{(s,t) \in D \,|\, \xi(s,t) \le c-\eta\}$ is star-shaped with respect to $(2\sqrt{\frac{\eta}{3}}, 0)$.

Since A and B are both star-shaped with respect to the same point one can show that there exists a diffeomorphism $a\colon A \to B$ which is the identity for all $(x,y) \in A$ with $\|x\| \in [\sqrt{\frac{7\eta}{6}}, 2\sqrt{\frac{\eta}{3}}]$.

(b) As in the Single Corner-to-Convex Lemma 60.9 we see that there exists a diffeomorphism $b\colon B^{\angle=(a\circ\varphi)(S^{n-k-1}\times S^{k-1})} \to C$ where $C \subseteq \widetilde{E}^n_{2\eta}$ is a submanifold with corner and where a smooth boundary is given by a vertical copy of $\overline{B}^{n-k} \times S^{k-1}$ which has a "horizontal" product tubular neighborhood.

(c) We glue $\overline{B}^{n-k} \times \overline{B}^k$ to C along $b \circ a \circ \varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial C$. The resulting smooth manifold $C \cup h^k$ admits an obvious smooth embedding into $\widetilde{E}^n_{2\eta}$. Finally note that the resulting smooth submanifold of $\widetilde{E}^n_{2\eta}$ is of the form $\{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \|y\| \leq \beta(\|x\|)\}$ for some smooth function $\beta \colon [0, 2\sqrt{\frac{\eta}{3}}\,] \to (0, \sqrt{\frac{\eta}{3}}]$.

The above discussion just gives some of the key ideas, but leaves out many details. We leave it to the reader to fill in the details. □

Subclaim 3. The map

$$\begin{array}{rcl} \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \|y\| \leq \alpha(\|x\|)\} & \to & \{(x,y) \in \widetilde{E}^n_{2\eta} \,|\, \|y\| \leq \beta(\|x\|)\} \\ (x,y) & \mapsto & \left(x, y \cdot \frac{\beta(\|x\|)}{\alpha(\|x\|)}\right) \end{array}$$

is a diffeomorphism.

Proof of subclaim. This statement is clear, since α and β are smooth functions which are non-zero everywhere, and we thus have an obvious smooth inverse to the given map. ⊞

For the reader's convenience we summarize the argument of the theorem: We have now shown that

$$\begin{array}{ccccccccc} M_a \cup h^k & \cong & M_{c-\eta} \cup h^k & \cong & F^{-1}((-\infty, c-\eta]) & \cong & M_{c+\eta} & \cong & M_b. \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } f \text{ has no critical} & & \text{by Claim 2} & & \text{by Claim 1} & & \text{since } f \text{ has no critical} & \\ & \text{value in } [a, c-\eta] & & & & (5) \text{ and } (6) & & \text{value in } [c+\eta, b] & \end{array}$$

Finally suppose that we are given an $\epsilon > 0$. We need to show that there exists a diffeomorphism as above which is the identity on the common subset $M_{a-\epsilon}$. As the reader will surely have noticed, this is an immediate consequence of our construction together with the "furthermore" statement of the No Critical Point-Diffeomorphism Theorem 250.4. ■

250.3. **Morse functions give rise to handle decompositions.** With the results from the previous two sections it is now pretty straightforward to prove the following result:

Theorem 250.6. (Morse Function–Handle Decomposition Theorem) Let M be a compact smooth manifold and let A be a union of components of ∂M. If $f \colon M \to \mathbb{R}$ is a Morse function rel A, then there exists a standard handle decomposition of M rel A such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k.

Remark. In the Handle Decomposition–Morse Function Theorem 251.2 we will prove a converse to the Morse Function–Handle Decomposition Theorem 250.6, namely we will see that handle decompositions give rise to Morse functions. □

The proof of the Morse Function–Handle Decomposition Theorem 250.6 will require the remainder of this section. In the next Section 250.4 we will apply the Morse Function–Handle Decomposition Theorem 250.6 to various types and examples of Morse functions.

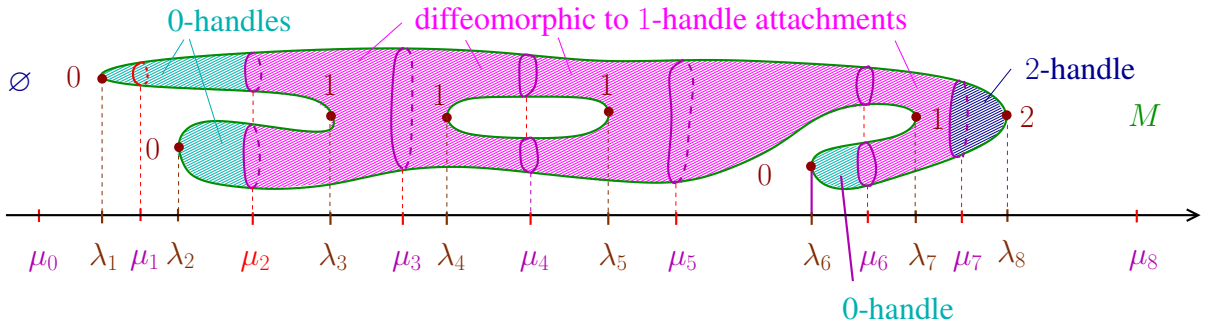
Proof. Let M be a compact smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Furthermore let $f \colon M \to \mathbb{R}$ be a Morse function rel A. Recall that by the Morse Function–Finiteness Corollary 249.12 we know that f has only finitely many critical points. Let s be the number of critical points of f. Using the Distinct Critical Values Lemma 249.30 we can arrange that the f-values of the s critical points are pairwise distinct.

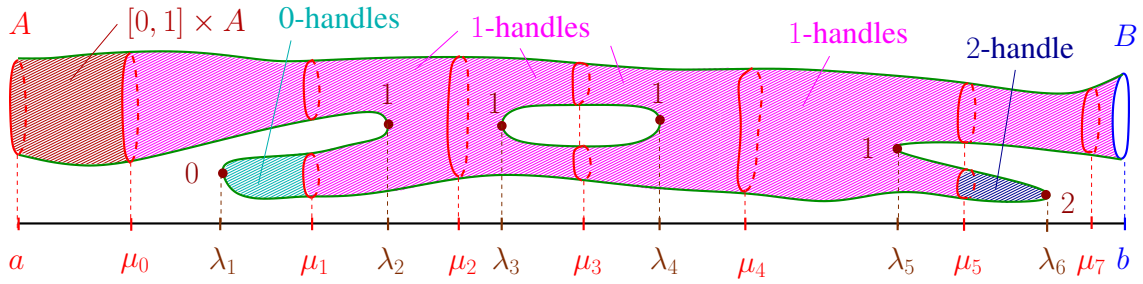
We order the f-values of the s critical points of the function f by $\lambda_1 < \cdots < \lambda_s$. Next note that we can and will pick real numbers $\mu_0, \ldots, \mu_s$ which satisfy the inequalities $\mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{s-1} < \lambda_s < \mu_s$. Since f has, by definition of a Morse function, no critical points on ∂M we can make the following two refinements:

- If $A \neq \varnothing$, then we pick in fact μ_0 such that $a := \min(f) < \mu_0 < \lambda_1$.
- If $B \neq \varnothing$, then we pick in fact μ_s such that $\lambda_s < \mu_s < \max(f) =: b$.

Note that each μ_i is a regular value of f such that $f^{-1}(\mu_i) \cap \partial M = \varnothing$. Thus, as in Notation 250.3, we consider the smooth manifold

$$M_{\mu_i} := f^{-1}((-\infty, \mu_i]).$$





We will now show that M admits a handle decomposition rel A:

(1) It follows from the Smooth Manifold-Product Theorem 250.1 that M_{μ_0} is diffeomorphic to $[0,1] \times A$.

(2) Next note that it follows from the Critical Point–Handle Addition Theorem 250.5 that for $i = 1, \ldots, s-1$ the smooth manifold M_{μ_i} is diffeomorphic to the result of attaching a single handle to $M_{\mu_{i-1}}$, where the index of the handle is given by the index of the corresponding critical point.

If $B = \varnothing$, then $M_{\mu_s} = M$ and we are done. It remains to deal with the case $B \neq \varnothing$, which requires one extra step:

(3) As we discussed in Notation 250.3 we know that M_{μ_s} is a codimension-zero submanifold of M with $\partial M_{\mu_s} = A \cup f^{-1}(\mu_s)$. It follows from this discussion and the Codimension-Zero Submanifold Proposition 25.49 that $f^{-1}([\mu_s, b])$ is a codimension-zero submanifold of M with boundary $f^{-1}(\mu_s) \cup B$. Since μ_s is a regular value and since f has no critical points on B we obtain from the Smooth Manifold-Product Theorem 250.1 that $f^{-1}([\mu_s, b])$ is diffeomorphic to $f^{-1}([\mu_s, b]) \times [\mu_s, b]$. It follows from this observation together with the External Collar Lemma 247.3 that M_b is diffeomorphic to M_{μ_s}. ■

250.4. **The Handle Decomposition Theorem.** In the last section we proved the Morse Function–Handle Decomposition Theorem 250.6 which says that Morse functions give rise to handle decomposition. We now apply this proposition to two types of Morse functions that we constructed in the previous Chapter 249.

First we have the following theorem, which finally shows that handle decompositions always exist:

Theorem 250.7. (Handle Decomposition Theorem) Let M be a compact smooth manifold and let A be a union of components of ∂M. There exists a standard handle decomposition of M rel A.

Proof. Let M be a compact smooth manifold and let A be a union of components of ∂M. By the Morse Function-Existence Theorem 249.26 we know that there exists a Morse function $f\colon M \to \mathbb{R}$ rel A. It follows from the Morse Function–Handle Decomposition Theorem 250.6 together with the Standard Handle Decomposition Proposition 247.7 that M admits a standard handle decomposition rel A. ■

We can now also give a new proof of the following proposition:

Proposition 246.12 (Projective Space-Handle Decomposition Proposition) Given any $n \in \mathbb{N}_0$ the projective spaces $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ admit handle decompositions with precisely $n+1$ handles. The indices of the handles are given as follows:

(1) $\mathbb{R}\mathrm{P}^n$ admits a handle decomposition with one handle of index $0, 1, 2, \ldots, n$.
(2) $\mathbb{C}\mathrm{P}^n$ admits a handle decomposition with one handle of index $0, 2, 4, \ldots, 2n$.
(3) $\mathbb{H}\mathrm{P}^n$ admits a handle decomposition with one handle of index $0, 4, 8, \ldots, 4n$.

Proof. In the Projective Space–Morse Function Proposition 249.15 we showed that the projective spaces $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ admit Morse functions with precisely $n+1$ critical points and that the indices are given as follows:

(1) The Morse function on $\mathbb{R}\mathrm{P}^n$ has one critical point of index $0, 1, 2, \ldots, n$.
(2) The Morse function on $\mathbb{C}\mathrm{P}^n$ has one critical point of index $0, 2, 4, \ldots, 2n$.
(3) The Morse function on $\mathbb{H}\mathrm{P}^n$ has one critical point of index $0, 4, 8, \ldots, 4n$.

The current proposition follows from the above together with the Morse Function–Handle Decomposition Theorem 250.7. ■

250.5. **Consequences of the Morse Function–Handle Decomposition Theorem 250.6.** We can now give a new proof for some of the statements of the Smooth Manifolds-Invariants Proposition 179.10.

Proposition 179.10. (Smooth Manifolds-Invariants Proposition) Let M be a compact n-dimensional smooth manifold.

(1) Given any $P \in M$ the fundamental group $\pi_1(M, P)$ is finitely presented.
(2) All homology groups $\mathrm{H}_i(M)$ of M are finitely generated abelian groups.
(3) For every $k > n$ and every abelian group G we have $\mathrm{H}_k(M; G) = 0$.

Remark.

(1) As we saw in the proof of the Smooth Manifolds-Invariants Proposition 179.10 (2a), one can use the above version of the Smooth Manifolds-Invariants Proposition 179.10

to show that homology groups and fundamental groups of arbitrary, not necessarily compact, smooth manifolds are countable.

(2) In the Morse Inequalities Proposition 252.13 and the Critical Points–Morse Inequalities Theorem 252.14 we will give explicit upper bounds on the "size" of the homology groups of a closed smooth manifold in terms of the number of critical points of a Morse function respectively the number of handles in a handle decomposition. □

Proof. Let M be a compact n-dimensional smooth manifold. By the Handle Decomposition Theorem 250.7 we know that M admits a standard handle decomposition.

(1) It follows from a straightforward induction argument, using the Handle Attachment-π_1-Proposition 246.13, that for any $P \in M$ the fundamental group $\pi_1(M,P)$ is finitely presented.

(2) As in (1) it follows from a straightforward induction argument, using the Handle Attachment-H_*-Lemma 246.14 and using the Finite Generation Lemma 97.20 (3), that all homology groups $\mathrm{H}_i(M)$ are finitely generated abelian groups.

(3) As in (2) it follows from another straightforward induction argument, using the Handle Attachment-H_*-Lemma 246.14, that for every $k > n$ and every abelian group G we have $\mathrm{H}_k(M;G) = 0$. ■

We continue with another pleasing consequence:

Proposition 250.8. (Critical Points-χ-Proposition) Let M be a compact smooth manifold and let A be a union of components of ∂M. For any Morse function $f\colon M \to \mathbb{R}$ rel A we have

$$\chi(M,A) \;=\; \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \#\text{critical points of } f \text{ of index } k.$$

Remark.

(1) For $M = S^2$ the statement of the Critical Points-χ-Proposition 250.8 appears implicitly already in a paper written by Clerk Maxwell [**Max1870**] in 1870 with the beautiful title "on hills and dales".

(2) In Exercise 250.17 we will see that the Critical Points-χ-Proposition 250.8 can also be deduced from the Poincaré-Hopf Theorem 223.8. □

Proof. Let $f\colon M \to \mathbb{R}$ be a Morse function rel A. It follows from the Morse Function–Handle Decomposition Theorem 250.6 that M admits a handle decomposition rel A such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of index k. We now see that

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \#\,\text{critical points of } f \text{ of index } k \;&=\; \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \#\text{handles of index } k\\ &=\; \chi(M,A).\end{aligned}$$

$=$ ↑ by essentially the same argument as in proof of the Handle Attachment χ-Lemma 246.15 (2)

■

We conclude this section with another pretty result:[2701]

Proposition 250.9. (Odd-Dimensional Manifold-χ-Proposition)

[2701] As we saw in Exercise 248.13 one can also prove this result using the Dual Handle Proposition 248.1.

(1) The Euler characteristic of every closed odd-dimensional smooth manifold is zero.
(2) For every compact odd-dimensional smooth manifold we have
$$\chi(M) \;=\; \tfrac{1}{2}\cdot\chi(\partial M).$$

Remark.

(1) In the Odd-Dimensional Manifold-χ-Proposition 53.6 we showed that the smooth Euler characteristic of a closed *orientable* odd-dimensional smooth manifold is zero.
(2) In the Odd-Dimensional Manifold-χ-Proposition 210.9 we used Poincaré Duality to prove a generalization of the Odd-Dimensional Manifold-χ-Proposition 210.9 to compact odd-dimensional *topological* manifolds. □

Proof. Evidently Statement (1) is a special case of Statement (2). Thus it suffices prove Statement (2). Let M be a compact $(2n+1)$-dimensional smooth manifold. By the Morse Function-Existence Theorem 249.26 we know that M admits a Morse function $f\colon M\to\mathbb{R}$ rel $\varnothing$. Now we see that

$$\begin{array}{rcl}
\chi(M) & \overset{\text{by the Critical Points-}\chi\text{ Proposition 250.8}}{=} & \sum_{k=0}^{2n+1}(-1)^k\cdot\left(\begin{array}{c}\text{number of critical}\\ \text{points of index } k \text{ of}\\ \text{the Morse function } f\end{array}\right)\\
& = & -\sum_{m=0}^{2n+1}(-1)^m\cdot\left(\begin{array}{c}\text{number of critical}\\ \text{points of index } m \text{ of}\\ \text{the Morse function } -f\end{array}\right)\\
& \overset{\text{by the substitution } m=2n+1-k}{=} & \sum_{k=0}^{2n+1}(-1)^k\cdot\left(\begin{array}{c}\text{number of critical points}\\ \text{of index } 2n+1-k \text{ of}\\ \text{the Morse function } -f\end{array}\right)\\
& \overset{\text{see the discussion on page 4813}}{=} & -\chi(M,\partial M)\\
& \overset{\text{by the Critical Points }\chi\text{-Proposition 250.8}}{=} & -\chi(M)+\chi(\partial M).
\end{array}$$

Thus we see that $2\chi(M)=\chi(\partial M)$. ■

250.6. **Reeb's Theorem.** The main goal of this section is to prove the following theorem, which was first proved by Georges Reeb [**Reeb1946**] in 1946.

Theorem 250.10. (Reeb's Theorem) Let $n\in\mathbb{N}_0$, let M be a closed n-dimensional smooth manifold. If M admits a smooth function $f\colon M\to\mathbb{R}$ with precisely two critical points which are both non-degenerate, then M is *homeomorphic* to S^n.

Remark. For the record we mention that by [**Miln1964b**, Theorem 1'] the conclusion of Reeb's Theorem 250.10 also holds if the critical points are degenerate. □

As we will see in a moment, Reeb's Theorem 250.10 follows easily from the Handle Addition Theorem 250.5 together with the following proposition:

Proposition 250.11. (Ball-Glued-to-Ball Proposition) Let $n\in\mathbb{N}$.
(1) If $\varphi\colon S^{n-1}\to S^{n-1}$ is a homeomorphism, then $\overline{B}^n\cup_\varphi\overline{B}^n$ is homeomorphic to S^n.
(2) Let M be a closed n-dimensional smooth manifold. If M admits a handle decomposition consisting of a single 0-handle and a single n-handle, then M is *homeomorphic* to S^n.

Proof of Reeb's Theorem 250.10 assuming Proposition 250.11. Let M be a closed n-dimensional smooth manifold and let $f\colon M\to\mathbb{R}$ be a smooth function with precisely

two critical points which are both non-degenerate. If $n = 0$, then it is clear that M is homeomorphic to S^0. So let us now assume that $n \geq 1$.

Claim. The function f is a Morse function with one critical point of index 0 and one critical point of index n.

Proof. Since M is compact we know that f assumes necessarily a global minimum at a point x_0 and a global maximum at a point y_0. It follows from $n \geq 1$ and the hypothesis that we only have two critical points, that f is not constant. This implies that $x_0 \neq y_0$.

Next, since M has no boundary we know, by elementary real analysis, that any point where f assumes a global extremum is a critical point. Since we have only two critical points we now see that x_0 and y_0 are precisely the critical points of f. Finally note that it follows from the discussion on page 4805 that the critical point x_0 has index 0 and that the critical point y_0 has index n. ⊞

It follows from the claim and the Handle Addition Theorem 250.5 that M has a handle decomposition consisting of a 0-handle together with an n-handle. It follows from the Ball-Glued-to-Ball Proposition 250.11 that M is homeomorphic to S^n. ■

Proof of the Ball-Glued-to-Ball Proposition 250.11.

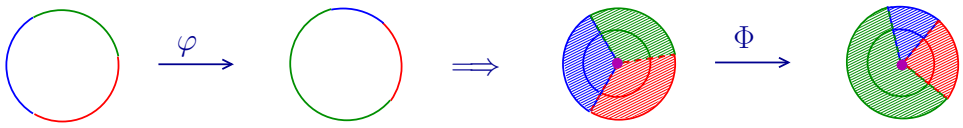
(1) Let $\varphi\colon S^{n-1} \to S^{n-1}$ be a homeomorphism.

Claim. The homeomorphism $S^{n-1} \xrightarrow{\varphi} S^{n-1}$ extends to a homeomorphism $\overline{B}^n \xrightarrow{\Phi} \overline{B}^n$.

Proof. We apply the Alexander trick to get an extension of $\varphi\colon S^{n-1} \to S^{n-1}$ to the ball $\overline{B}^n$. More precisely we consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

It is elementary to show, see e.g. Exercise 8.3, that Φ is continuous. Furthermore it is straightforward to verify that Φ is a bijection.

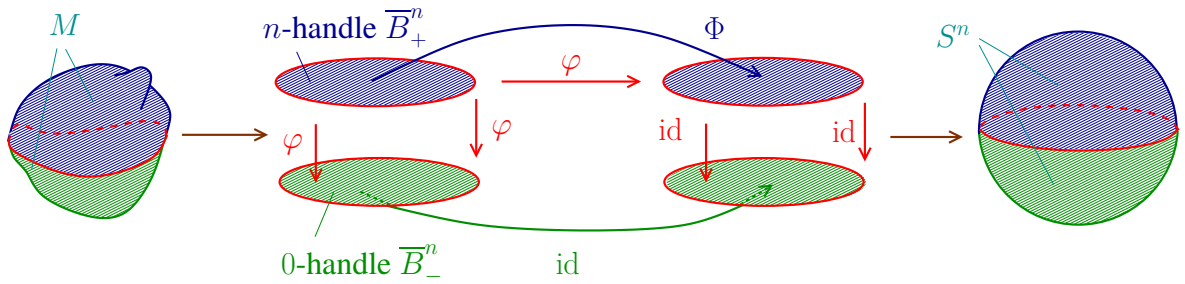


It now follows from the Compact-Hausdorff Proposition 4.23 (3) that Φ is a homeomorphism. ⊞

Finally we consider the following sequence of homeomorphisms:

$$\begin{array}{rcl} \overline{B}^n_- \cup_{S^{n-1} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n_+ & \xrightarrow{\cong} & \overline{B}^n_- \cup_{S^{n-1} \xleftarrow{\mathrm{id}} S^{n-1}} \overline{B}^n_+ \xrightarrow[\text{from page 319}]{\text{homeomorphism}} S^n \\ [P] & \mapsto & \begin{cases} [\Phi(P)], & \text{if } P \in \overline{B}^n_+, \\ [P], & \text{if } P \in \overline{B}^n_-. \end{cases} \end{array}$$

(2) Let M be a closed n-dimensional smooth manifold that admits a handle decomposition consisting of a single 0-handle and a single n-handle.



As we already discussed on page 4749, this means that there exists a diffeomorphism $\varphi\colon S^{n-1} \to S^{n-1}$ such that M is diffeomorphic to[2702] $\overline{B}^n \cup_\varphi \overline{B}^n$. It now follows from (1) that M is homeomorphic to S^n. ■

Reeb's Theorem 250.10 and the Ball-Glued-to-Ball Proposition 250.11 raise several fantastic questions. First of all, it is natural to ask whether the analog of the Ball-Glued-to-Ball Proposition 250.11 (1) holds if we replace "homeomorphism" by "diffeomorphism". Put differently, we have the following question.

Question 250.12. Let $\varphi\colon S^{n-1} \to S^{n-1}$ be a diffeomorphism. Is the smooth manifold $\overline{B}^n \cup_\varphi \overline{B}^n$ *diffeomorphic* to S^n?

Next let us reflect whether we can upgrade the composition of maps towards the end of the proof of the Ball-Glued-to-Ball Proposition 250.11 (1) into a diffeomorphism. The good news is that, by the discussion on page 1229, we can replace the map on the right-hand side by a diffeomorphism. The bad news is that, as we will see in Exercise 250.7, the map $\Phi\colon \overline{B}^n \to \overline{B}^n$ given by the Alexander trick is in general not smooth at the origin, in particular Φ is not a diffeomorphism. But perhaps we can pick Φ more cleverly? This leads us to our second question.

Question 250.13. Let $n \in \mathbb{N}$. Does every diffeomorphism $\varphi\colon S^n \to S^n$ extend to a diffeomorphism of $\overline{B}^{n+1}$?

In the Diffeomorphism–Non Extension Proposition 282.11 and in the Ball Gluing–Non-Uniqueness Proposition 282.9 we will provide answers to Questions 250.13 and 250.12.

250.7. **Multiplicativity of the Euler characteristic.** In the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8 we showed that the Euler characteristic is multiplicative for bundles over finite simplicial complexes. In this section we will prove an analogous statement for bundles over compact smooth manifolds.

First let us remind ourselves of the definition of the Euler characteristic from Definition 156.6:

Definition 250.14. Let X be a topological space.

(1) We say X has finitely generated homology if all homology groups $\mathrm{H}_k(X)$ are finitely generated and if all but finitely many of these homology groups are zero.

[2702] Note that by the Gluing-Smooth Manifolds-Proposition 54.5 we can equip $\overline{B}^n \cup_\varphi \overline{B}^n$ with a canonical smooth structure.

(2) If X has finitely generated homology, then we define its Euler characteristic as[2703]

$$\chi(X) := \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)).$$

The following theorem is now the promised analog of the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8:

Theorem 250.15. (Manifold–Fiber Bundle-χ-Theorem) Let $p\colon E \to M$ be an F-bundle in the sense of Definition 238.1. If F has finitely generated homology and if M is a compact smooth manifold, then

$$\chi(E) = \chi(M) \cdot \chi(F).$$

The idea is to imitate the proof of the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8. So it is worth recalling that the proof of the earlier Simplicial Complex–Fiber Bundle-χ-Proposition 242.8 relied on the following inputs and ideas:

(1) The Euler characteristic is multiplicative under products. More precisely, by the Product-χ-Proposition 156.12 (or alternatively Exercise 164.5) we know the following: If X and Y are two topological spaces X and Y with finitely generated homology, then we have $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.
(2) Since simplices are embedded contractible subspaces we obtain from the Bundles-Homotopies Proposition 241.10 (2) that the restriction of a bundle over a simplex is a product.
(3) Given a bundle over a finite simplicial complex we can perform an induction over the number of simplices in the simplicial complex. More precisely: Using (1), (2) and using a Mayer–Vietoris argument we can control the Euler characteristic of the base space and the total space when we add a simplex.

Not surprisingly we will prove the Manifold–Fiber Bundle-χ-Theorem 250.15 by a very similar approach. The key difference is that we will need to replace simplices by handles, and we will use that we now know that every compact smooth manifold admits a handle decomposition.

Proof. Let F be a topological space with finitely generated homology, let M be a compact smooth manifold and let $p\colon E \to M$ be an F-bundle. Recall that we need to show that $\chi(E) = \chi(M) \cdot \chi(F)$.

By the Handle Decomposition Theorem 250.7 we know that M admits a handle decomposition. Thus it suffices to prove the following statement:

Statement. For every $s \in \mathbb{N}_0$ the following holds: If M is a compact smooth manifold that admits a handle decomposition with s handles, then for every F-bundle $p\colon E \to M$ we have

$$\chi(E) = \chi(M) \cdot \chi(F).$$

Proof. Evidently we prove the statement by induction on s. In the case $s = 0$ we have $M = \varnothing$ and $E = \varnothing$, so the equality holds trivially. Now we assume that the statement holds for some $s \in \mathbb{N}_0$. Let N be a compact n-dimensional smooth manifold that admits a handle decomposition with $s + 1$ handles and let $p\colon E \to N$ be an F-bundle.

We can write

[2703]Note that it follows from the Homology-χ-Proposition 156.2 that this definition is consistent with Definition 156.1.

$$N \;=\; M \cup_\varphi h^k \;:=\; M \cup_{\partial M \overset{\varphi}{\leftarrow} \overline{B}^{n-k} \times S^{k-1}} \left(\overline{B}^{n-k} \times \overline{B}^k\right)$$

where $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ is a smooth embedding and M is a compact smooth manifold that admits a handle decomposition consisting of s handles.

We adopt the following notation:

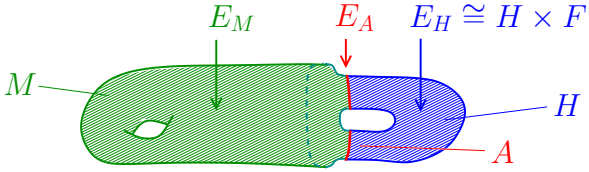
- We write $H := \overline{B}^{n-k} \times \overline{B}^k \subseteq N$ and $E_H := p^{-1}(H)$.
- We write $A := \overline{B}^{n-k} \times S^{k-1} \subseteq N$ and $E_A := p^{-1}(A)$.

We continue with the following observations:

(a) If X is a topological space with finitely generated homology, then it follows from the Product-χ-Proposition 156.12 that $\chi(X \times F) = \chi(X) \cdot \chi(F)$.

(b) Since $\overline{B}^{n-k} \times \overline{B}^k$ is contractible we obtain from the Bundles-Homotopies Proposition 241.10 (2) that the bundle $p\colon E_H \to H$ is trivializable. This means that there exists a homeomorphism $\Theta\colon E_H \to H \times F$ which commutes with the projections to H.

(c) Note that the homeomorphism Θ from (b) restricts to a homeomorphism $E_A \overset{\cong}{\to} A \times F$.



It follows from the Handle Attachment-χ-Lemma 246.15 that

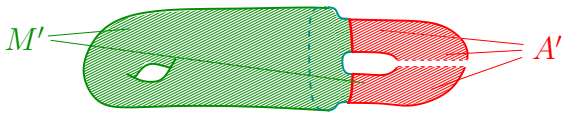
$$\chi(\underbrace{M \cup_\varphi h^k}_{=N}) \;=\; \chi(M) + \chi(H) - \chi(A).$$

It follows from this discussion and our induction hypothesis that it suffices to prove the following claim:

Claim. $$\chi(E) \;=\; \underbrace{\chi(E_M)}_{\substack{=\chi(M)\cdot\chi(F)\\ \text{by induction}}} + \underbrace{\chi(E_H)}_{\substack{=\chi(H)\cdot\chi(F)\\ \text{by (a) and (b)}}} - \underbrace{\chi(E_A)}_{\substack{=\chi(A)\cdot\chi(F)\\ \text{by (a) and (c)}}} .$$

Proof. The obvious idea, as in the proof of the Handle Attachment-χ-Lemma 246.15, is to apply a Mayer-Vietoris sequence to the decomposition $E_N = E_M \cup_{E_A} E_H$. The technical issue is that these topological spaces are not necessarily smooth (sub-) manifolds.[2704] So we cannot apply the "manifold version" of the Mayer-Vietoris Theorem 153.2, instead we have to work with the "open covering version" of the Mayer-Vietoris Theorem 153.2. To do so we introduce the following objects:

- We set $A' := \overline{B}^{n-k} \times (\overline{B}^k \setminus \{0\}) \subseteq N$ and we set $E_{A'} := p^{-1}(A')$.
- We set $M' := M \cup (\overline{B}^{n-k} \times (\overline{B}^k \setminus \{0\})) \subseteq N$ and we set $E_{M'} := p^{-1}(M')$.



[2704] In fact, since F is just assumed to be a topological space, there is no reason to expect that E is a smooth manifold.

We make the following observations:

(d) Note that $A = \overline{B}^{n-k} \times S^{k-1}$ is a deformation retract of $A' = \overline{B}^{n-k} \times (\overline{B}^k \setminus \{0\})$. Since we have a homeomorphism $\Theta\colon E_H \to H \times F$ we also see that $E_A = \Theta(A \times F)$ is a deformation retract of $E_{A'} = \Theta(A' \times F)$.
(e) It follows from (d) and the Homotopy Combination Lemma 18.8 (1) that E_M is a deformation retract of $E_{M'}$.
(f) The interiors of M' and H cover N and we have $M' \cap H = A'$.
(g) It follows from (f) that $E_{M'}$ and E_H cover E and that $E_{M'} \cap E_H = E_{A'}$.

It follows from (g) that we can apply the Mayer-Vietoris Theorem 153.2 (1) to the decomposition $E = E_{M'} \cup E_H$. Finally we see that

$$\chi(E) \underset{\uparrow}{=} \chi(E_{M'}) + \chi(E_H) - \chi(E_{A'}) \underset{\uparrow}{=} \chi(E_M) + \chi(E_H) - \chi(E_A).$$

follows from the Mayer-Vietoris Theorem 153.2 (1) together with the Sequence-χ-Lemma 156.5

by the Homotopy Type-χ-Lemma 156.3 applied to (d) and (e)

■

250.8. **Appendix: Handle decompositions of topological manifolds.** The notion of a handle decomposition also makes sense, with a few suitable modifications in the definition, for compact topological manifolds. So the question arises, does every compact topological manifold admit a handle decomposition? It turns out that this is a subtle question. In the following we summarize what is known:

(1) By the 2D-Smooth Structure-Theorem **??** and the 3D-Unique Smooth Structure-Theorem **??** every compact topological manifold of dimension ≤ 3 admits a smooth structure. Thus it follows from the Handle Decomposition Theorem 250.7 that every such topological manifold admits a handle decomposition.
(2) Rob Kirby and Larry Siebenmann [**KSi1977**, Essay III.2, Theorem 2.1] (see also [**Kupe2017b**, Theorem 1.22]) showed that every compact topological manifold of dimension ≥ 6 admits a topological handle decomposition. Later Frank Quinn [**Quin1982**, Theorem 2.3.1] extended this result to the 5-dimensional setting.
(3) It follows from the Isotopic-to-Smooth Embedding-3D-Proposition **??** that every closed 4-dimensional topological manifold that admits a topological handle decomposition also admits a smooth structure. But in the No Smooth Structure-4D-Theorem 228.17 we saw that there exist closed 4-dimensional topological manifolds that do not admit a smooth structure. In particular there exist closed 4-dimensional topological manifolds that do not admit a handle decomposition.

250.9. **Exercises for Chapter 250.**

Exercise 250.1. Use handle decompositions to show that every closed n-dimensional smooth manifold admits an atlas with $n+1$ charts.
Remark. This exercise gives a new proof for some of the statements of the Small Atlas Proposition 33.10.

Exercise 250.2. Recall that in Definition 205.4 we introduced for every topological space X the Lusternik-Schnirelmann category $\operatorname{cat}(X) \in \mathbb{Z}_{\geq -1} \cup \{\infty\}$ as follows:

- A *Lusternik-Schnirelmann cover of* X is an open cover $\{U_i\}_{i \in I}$ of X such that each inclusion map $U_i \to X$ is homotopic to a constant map.
- If X admits a finite Lusternik-Schnirelmann cover, then we define the *Lusternik-Schnirelmann category* $\operatorname{cat}(X) \in \mathbb{Z}_{\geq -1}$ as

$$\operatorname{cat}(X) \;:=\; \min\left\{ n \in \mathbb{Z}_{\geq -1} \;\middle|\; \begin{array}{c} \text{there exists a Lusternik-Schnirelmann} \\ \text{cover with } n+1 \text{ sets} \end{array} \right\}.$$

  Otherwise we set $\operatorname{cat}(X) := \infty$.

Use handle decompositions to show that for every closed connected n-dimensional smooth manifold M we have $\operatorname{cat}(M) \leq n$.
Remark. This exercise gives a new proof for the LS Category–Upper Bound Proposition 205.7 (3) for closed connected smooth manifolds.

Exercise 250.3. For which $n \in \mathbb{N}$ does $\mathrm{SO}(n)$ admit a handle decomposition without 2-handles?

Exercise 250.4.

(a) Does there exist a compact topological space X such that $\mathrm{H}_0(X)$ is uncountable?
(b) Does there exist a compact topological space X such that $\mathrm{H}_0(X)$ is uncountable?

Exercise 250.5. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function such that all critical points are non-degenerate and of even degree. Show that there is no smooth function $M \to \mathbb{R}$ such that all critical points are non-degenerate and such that the number of critical points of g is less than the number of critical points of f.

Exercise 250.6. Let M be a closed n-dimensional smooth manifold that admits a Morse function $f\colon M \to \mathbb{R}$ with precisely two critical points. Reeb's Theorem 250.10 says that M is homeomorphic to S^n. Give a direct proof of this theorem, just using the Morse Lemma 249.11 and the Smooth Manifold-Product Theorem 250.1, but without using the Handle Addition Theorem 250.5.

Exercise 250.7. For which $n \in \mathbb{N}$ does there exist a diffeomorphism $\varphi\colon S^n \to S^n$ such that the extension

$$\begin{array}{rcl} \Phi\colon \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise} \end{cases} \end{array}$$

is not a diffeomorphism?

Exercise 250.8. Let M be a closed n-dimensional smooth manifold. We say M admits a *decomposition into small pieces* if there exist compact codimension zero submanifolds A, B of M with the following properties:

(1) $M = A \cup B$.
(2) $A \cap B = \partial A = \partial B$.
(3) A and B are both homotopy equivalent to smooth manifold of dimension $\leq n-2$.

For which $n \in \mathbb{N}$ does every closed n-dimensional smooth manifold admit a decomposition into small pieces?

Exercise 250.9. Let M be a 3-dimensional smooth manifold such that the boundary is a surface of genus g. Suppose that there exists a Morse function $f\colon M \to [0,1]$ rel $\varnothing$ with a single critical point of index 0 and no critical point of index ≥ 2. How many critical points of index 1 does f have?



Exercise 250.10.

(a) For which $g \in \mathbb{N}_0$ is the surface $\Sigma_g = g \cdot (S^1 \times S^1)$ of genus g the total space of a bundle over a 1-dimensional smooth manifold?
(b) For which $k \in \mathbb{N}$ is the non-orientable surface $k \cdot \mathbb{R}\mathrm{P}^2$ of genus k the total space of a bundle over a 1-dimensional smooth manifold?

Exercise 250.11. Find infinitely many diffeomorphism types of closed connected 4-dimensional smooth manifolds which are not the total space of a fiber bundle where the fiber and the base are smooth manifolds of dimension > 0.

Exercise 250.12. Let $p\colon E \to S^1$ be an F-bundle over S^1 where F has finitely generated homology. Give a direct proof, only using Mayer–Vietoris sequences, that $\chi(E) = 0$.

Exercise 250.13. Does there exist a smooth F-bundle map $p\colon E \to M$ such that the following conditions are satisfied:

(1) F is a connected smooth manifold with finitely generated homology.
(2) M is a closed smooth manifold.
(3) There exists an $i \in \mathbb{N}$ such that $\mathrm{H}_i(F) \neq 0$.
(4) The induced map $p_*\colon \mathrm{H}_i(E) \to \mathrm{H}_i(M)$ is an isomorphism for each $i \in \mathbb{N}_0$.

Exercise 250.14. Let $f\colon \widetilde{M} \to M$ be a d-fold covering of smooth manifolds such that f is a local diffeomorphism. Use the Critical Points-χ-Proposition 250.8 to show that

$$\chi(\widetilde{M}) \;=\; d \cdot \chi(M).$$

Exercise 250.15. Let M and N be closed smooth manifolds. Use the Product–Morse Function Proposition 249.17 and the Critical Points-χ-Proposition 250.8 to show that

$$\chi(M \times N) \;=\; \chi(M) \cdot \chi(N).$$

Exercise 250.16. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function with finitely many critical points. We do not assume that they are non-degenerate. Note though that the definition of the index also makes sense for a degenerate critical point. Does the equality

$$\sum_{k \in \mathbb{N}_0} (-1)^k \cdot \#\text{critical points of } f \text{ of index } k \;=\; \chi(M)$$

necessarily hold in this context?

Remark. In other words, the task is to find out whether we can generalize the Critical Points-χ-Proposition 250.8.

Exercise 250.17. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth map. Furthermore let g be a Riemannian structure on M, as introduced in Definition 131.2. Given $P \in M$ we define the *gradient of f at P* to be the unique vector $\operatorname{grad} f_P \in \mathrm{T}_P M$ that satisfies

$$g_P(\operatorname{grad} f_P, w) \;=\; \mathrm{D}f_P(w) \;\in\; \mathrm{T}_{f(P)}\mathbb{R} = \mathbb{R} \qquad \text{for all } w \in \mathrm{T}_P M.$$

One can easily show that $P \mapsto \operatorname{grad} f_P$ defines a vector field on M in the sense of Definition 29.27.

(a) We suppose that f is a Morse function. Show that every zero of the vector field $\operatorname{grad} f$ is isolated.

(b) Let P be a critical point of f. Show that

$$\underbrace{\operatorname{index}(\operatorname{grad} f, P)}_{\text{see Definition 223.1 for the definition}} \;=\; (-1)^{\operatorname{index}(f,P)}.$$

Remark. This exercise shows that the Critical Points-χ-Proposition 250.8 can also be deduced from the Poincaré-Hopf Theorem 223.8.

Exercise 250.18. Let $n \in \mathbb{N}_{\geq 2}$. We refer to the mapping torus of the reflection

$$\begin{array}{ccc} \overline{B}^{n-1} & \to & \overline{B}^{n-1} \\ (x_1, x_2, \ldots, x_n) & \mapsto & (-x_1, x_2, \ldots, x_n) \end{array}$$

as an n-dimensional *generalized Möbius band*. Let M be a compact n-dimensional smooth manifold. Show that M is orientable if and only if there is no smooth embedding of the n-dimensional generalized Möbius band into M.

Hint. Use the Handles-Orientability Proposition 246.6, the Ball–Handle Attachment Lemma 246.9 and the Handle Decomposition Theorem 250.7.

Remark. In Exercise 240.31 we gave a very different proof of that statement.

CHAPTER 251

From handle decompositions to Morse functions

In the previous Chapter 250 we proved the Morse Function–Handle Decomposition Theorem 250.6 which says that any Morse function gives rise to a handle decomposition, with precisely one handle for each critical point. In this chapter we will see that we can also go backwards, namely we will show that every handle decomposition gives rise to a Morse function, with precisely one critical point for each handle.

This chapter is organized as follows:

- In Section 251.1 we will state and prove the Handle Decomposition–Morse Function Theorem 251.2 which tells us how one can construct Morse functions from handle decompositions.
- In Section 251.2 we will exploit the symmetry between handles and Morse functions to prove several results on handle decompositions.
- In Section 251.3 we will apply the Handle Decomposition–Morse Function Theorem 251.2 and our knowledge of handles to construct Morse functions with particularly nice features. In particular we will see that we can always find "self-indexing" Morse functions.
- In Section 251.4 we will provide exercises for Chapter 251.

In the next Chapter 252 we will introduce handle homology and we will state and prove the Morse inequalities.

251.1. **Constructing Morse functions from handle decompositions.** In this section we will prove the following converse to the Critical Point–Handle Addition Theorem 250.5:

Theorem 251.1. (Handle Addition–Critical Point Theorem) Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. Furthermore let $f\colon M \to [a,b]$ be a Morse function for M rel A with $f^{-1}(b) = B$. Finally let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to B$ be a thickened $(k-1)$-sphere and let U be an open neighborhood of B. Then there exists a Morse function $g\colon M \cup_\varphi h^k \to [a,c]$ with the following properties:

(1) The Morse function g agrees with f outside of U.

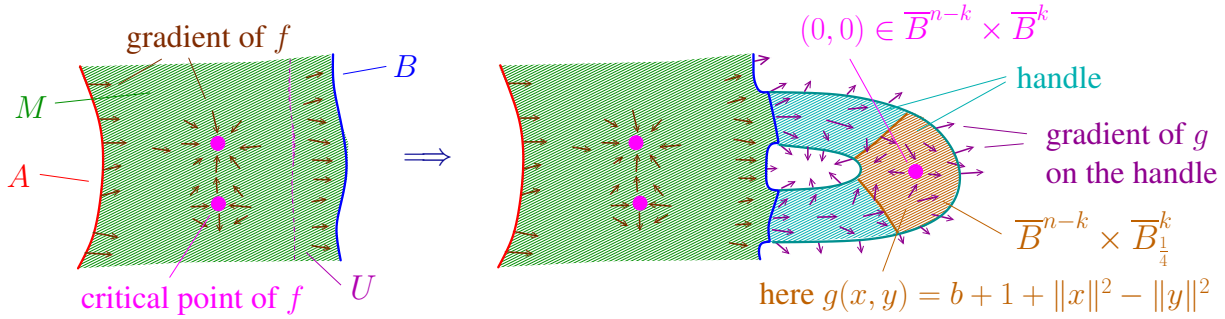
(2) The function $g\colon M \cup_\varphi h^k \to \mathbb{R}$ restricts to the map

$$\begin{array}{rcl} \overbrace{\overline{B}^{n-k} \times \overline{B}^k_{\frac{1}{4}}}^{\subseteq h^k = \overline{B}^{n-k} \times \overline{B}^k} & \to & \mathbb{R} \\ (x,y) & \mapsto & b+1+\|x\|^2 - \|y\|^2. \end{array}$$

In particular $(0,0) \in \overline{B}^{n-k} \times \overline{B}^k$ is a non-degenerate critical point of g of index k.

(3) The critical points of g are precisely the critical points of f together with the extra critical point $(0,0)$.
(4) We have $c > b+1$ and $g^{-1}(c) = \partial(M \cup_\varphi h^k) \setminus A$.

Example. In the figure below we show a smooth submanifold M of some $\mathbb{R}^n$ to which we attached a 1-handle.



We also show a Morse function $f\colon M \to \mathbb{R}$ rel A. To be more precise, since smooth functions are hard to illustrate, we show instead the gradient vectors $\operatorname{grad} f_P$. The figure illustrates that one can extend the Morse function f to a Morse function g on $M \cup_\varphi h^1$ with one extra critical point of index 1. □

Before we turn to the proof of the Handle Addition–Critical Point Theorem 251.1, let us discuss the following theorem, which can be viewed as a converse to the Handle Decomposition–Morse Function Theorem 251.2:

Theorem 251.2. (Handle Decomposition–Morse Function Theorem) Let M be a compact smooth manifold and let A be a union of components of ∂M. If M is equipped with a handle decomposition $M = ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$ rel A, then there exists a Morse function $f\colon M \to \mathbb{R}$ rel A with the following properties:

(1) On $[0,1] \times A$ the map g equals the projection onto $[0,1]$.
(2) The map f has precisely m critical points $P_1, \ldots, P_m$, each of which is non-degenerate, and such that for $i = 1, \ldots, m$ the index of the critical point P_i equals k_i, i.e. it equals the index of the corresponding handle.

Proof of the Theorem 251.2 assuming Theorem 251.1. On $[0,1] \times A$ we consider the natural projection to $[0,1]$. Now the desired conclusion follows from applying the Handle Addition–Critical Point Theorem 251.1 iteratively altogether m times. ■

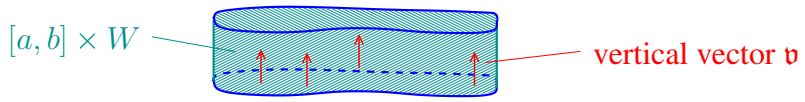
The proof of the Handle Addition–Critical Point Theorem 251.1 will require the remainder of this section. In the next Section 251.2 we will study several applications of this theorem.

The proof of the Handle Addition–Critical Point Theorem 251.1 requires a very good understanding of the smooth structure on handle attachments. To achieve this it is convenient to formulate a slightly more general proposition, which might be of independent interest. The formulation of the proposition makes use of the following simple minded definition:

Definition 251.3. Let W be a smooth manifold.

- For $(t,w) \in [a,b] \times W$ let $\mathfrak{v} \in \mathrm{T}_{(t,w)}([a,b] \times W)$ be the "vertical tangent vector" corresponding to the positive $[a,b]$-direction, as defined on page 764.

- We say a smooth function $\xi\colon [a,b]\times W \to \mathbb{R}$ points upward at a point $(t,w)\in[a,b]\times W$ if $\mathrm{D}\xi_{(t,w)}(\mathfrak{v}) > 0 \in \mathbb{R}$.



Now we can formulate the promised proposition:

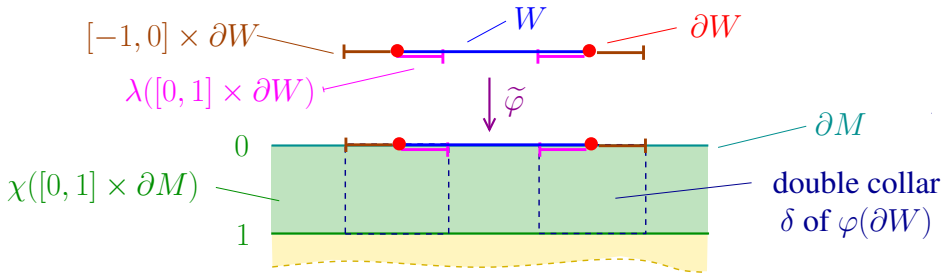
Proposition 251.4. (Codimension-Zero-in-Boundary Proposition) Let M be an n-dimensional smooth manifold, let W be a compact $(n-1)$-dimensional smooth manifold and let $\lambda\colon [0,1]\times\partial W \to W$ be a collar for W. We use this collar and the standard collar on $[-1,0]\times\partial W$ to obtain the smooth manifold $W\cup([-1,0]\times\partial W)$, as defined in the Gluing-Smooth Manifolds-Proposition 54.5, and let $\varphi\colon W\to\partial M$ be a smooth embedding. The following statements hold:

(1) There exists a smooth embedding $\widetilde{\varphi}\colon W\cup([-1,0]\times\partial W)\to\partial M$ that agrees with the given smooth embedding φ on W.

Next let $\chi\colon [0,1]\times\partial M\to M$ be a collar.

(2) The map
$$\begin{array}{rcl}\delta\colon \varphi(\partial W)\times[-1,1]\times[0,1] & \to & M\\ ((\varphi(x),s,t)) & \mapsto & \begin{cases}\chi(\varphi(\lambda(s,x)),t), & \text{if } s\in[0,1],\\ \chi(\widetilde{\varphi}([(s,x)]),t), & \text{if } s\in[-1,0]\end{cases}\end{array}$$
is a double-collar[2705] for $\varphi(\partial W)\subseteq\partial M$. We will use it to define $M^{\angle=\varphi(\partial W)}$, where, following the Introducing Corner Proposition 61.4, we introduce a corner at $\varphi(\partial W)$.

(3) $\varphi(W)$ is a smooth boundary of $M^{\angle=\varphi(\partial W)}$.



Proposition 251.4. (Codimension-Zero-in-Boundary Proposition))

(4) There exists a collar $\gamma\colon[0,1]\times\varphi(W)\to M^{\angle=\varphi(\partial W)}$ with the following properties:

(a) $\gamma([0,1]\times\varphi(W))\subseteq\chi([0,1]\times\partial M)$.

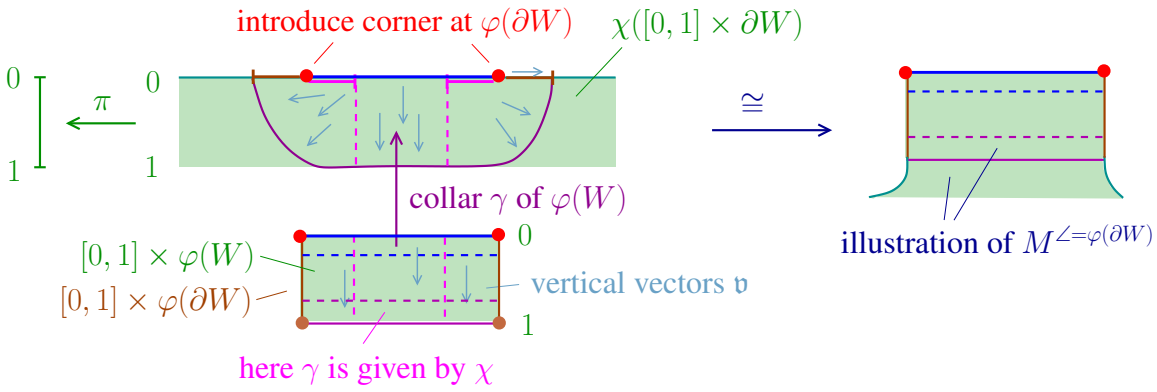
(b) We denote by $\pi\colon\chi([0,1]\times\partial M)\to[0,1]$ the obvious map. The map[2706]
$$\pi\circ\gamma\colon[0,1]\times\varphi(W)\to[0,1]$$
points upward on $(0,1]\times\varphi(W\setminus\partial W)$.

(c) The map
$$\pi\circ\gamma\colon[0,1]\times\varphi(W)\to[0,1]$$
points outward[2707] on $(0,1]\times\varphi(\partial W)$.

(d) For $w\in W\setminus\lambda([0,1]\times\partial W)$ and any $t\in[0,1]$ we have $\gamma(t,\varphi(w))=\chi(t,\varphi(w))$.

[2704]We refer to Definition 61.1 for the definition of a double-collar.

[2705]We refer to Definition 59.4 for the definition of a smooth boundary of a smooth manifold with corner.



Proof of the Codimension-Zero-in-Boundary Proposition 251.4.

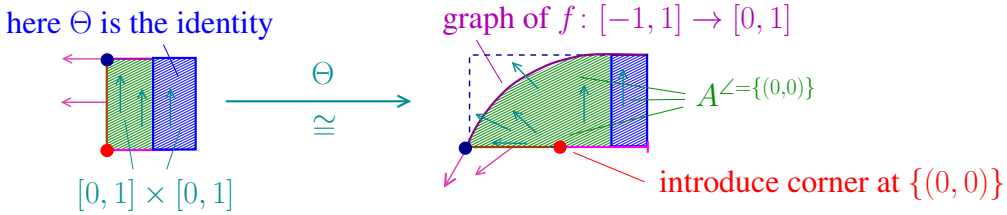
(1) This statement is just a special case of the Embedding Extension Theorem 54.17.
(2) This statement follows easily from the definitions.
(3) This statement also follows easily from the definitions.
(4) It follows easily from the Smooth Transition Function Lemma 24.9 that we can pick a smooth function $f\colon [-1,1]\to[0,1]$ such that $f(-1)=0$, such that $f(t)=1$ for $t\in[\frac{1}{2},1]$ and such that $f'(t)\geq 0$ for all $t\in[-1,1]$. We set

$$A \;:=\; \{(x,y)\,|\,x\in[-1,1] \text{ and } y\in[0,f(x)]\}.$$

Using the Single Corner-to-Convex Lemma 60.9 one can show, with a little bit of effort, that there exists a diffeomorphism

$$\Theta\colon [0,1]\times[0,1] \;\to\; A^{\angle=(0,0)}$$

which is the identity on $[\frac{1}{2},1]\times[0,1]$ and which has the following two properties (which will correspond to Properties (b) and (c) of γ):
(b) For any point $(x,y)\notin[0,\frac{1}{2}]\times\{0\}$ the vector $\mathrm{D}\Theta_{(x,y)}(0,1)$ has positive y-coordinate.
(c) For any point $(x,y)\in\{0\}\times(0,1]$ the vector $\mathrm{D}\Theta_{(x,y)}(1,0)$ has negative y-coordinate.



Now we consider the following map:

$$\begin{aligned}\gamma\colon [0,1]\times\varphi(W) &\to M^{\angle=\varphi(\partial W)}\\ (t,\varphi(w)) &\mapsto \begin{cases}\chi(t,\varphi(w)), & \text{if } w\in W\setminus\lambda([0,1]\times\partial W),\\ \delta(\varphi(w),\Theta(t,s)), & \text{if } w=\lambda(s,u) \text{ with } s\in[0,1] \text{ and } u\in\partial W.\end{cases}\end{aligned}$$

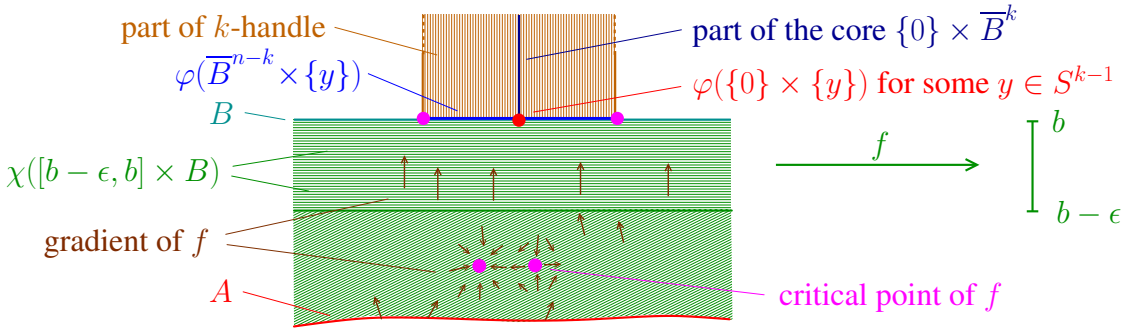
here $\delta\colon \varphi(\partial W)\times[0,1]\times[0,1]\to M$ is the double collar from (1)

[2706] It follows from (a) that $\pi\circ\gamma$ is actually defined.
[2707] We refer to Definition 249.28 for the defining of a smooth function "pointing outward".

We leave it to the reader to verify that this map $\gamma\colon [0,1] \times \varphi(W) \to M^{\angle=\varphi(\partial W)}$ does have all the desired properties. ■

Proof of the Handle Addition–Critical Point Theorem 251.1. Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M. As usual we set $B := \partial M \setminus A$. Furthermore let $f\colon M \to [a,b]$ be a Morse function for M rel A with $f^{-1}(b) = B$. Finally let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to B$ be a thickened $(k-1)$-sphere and let U be an open neighborhood of B that contains all critical points of f. Our goal is to show that there exists a Morse function $g\colon M \cup_\varphi h^k \to [a,c]$ with several interesting properties.

First note that it follows from the Smooth Collar Neighborhood Theorem 35.6 that we can pick a collar $\chi\colon [b-\epsilon, b] \times B \to M$ such that the image is contained in U and such that $\chi(b,x) = x$ for all $x \in B$. As we discussed on page 4826, it follows from the Standard-near-Boundary Lemma 249.29 that we can arrange the Morse function f and the collar χ to have the extra property that $f \circ \chi\colon [b-\epsilon, b] \times B \to \mathbb{R}$ is precisely the projection onto $[b-\epsilon, b]$.

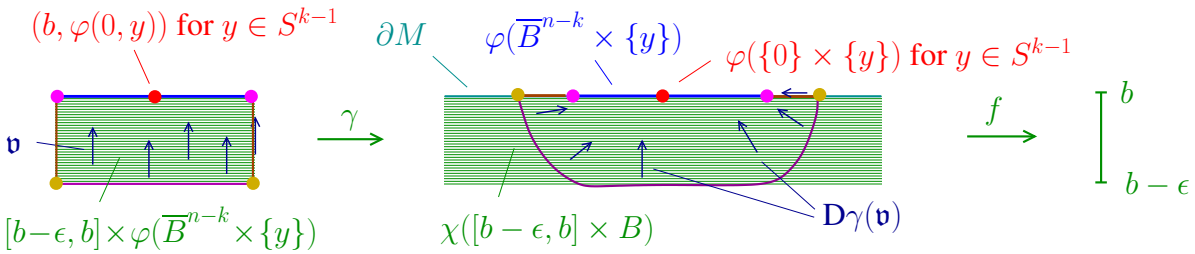


Next note that it follows immediately from the Embedding Extension Theorem 54.17 that we can extend the given smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to B$ to a smooth embedding $\varphi\colon \overline{B}_2^{n-k} \times S^{k-1} \to B$. As in the Codimension-Zero-in-Boundary Proposition 251.4 we use the map φ and the collar χ to define a bicollar for $\varphi(S^{n-k-1} \times S^{k-1}) \subseteq M$, which in turn we use to introduce a corner at $\varphi(S^{n-k-1} \times S^{k-1}) \subseteq M$, and we use φ to construct a collar $\gamma\colon [b-\epsilon, b] \times \varphi(\overline{B}^{n-k} \times S^{k-1}) \to M$ for the smooth boundary $\varphi(\overline{B}^{n-k} \times S^{k-1}) \subseteq M^{\angle=\varphi(S^{n-k-1} \times S^{k-1})}$ with the following three properties:

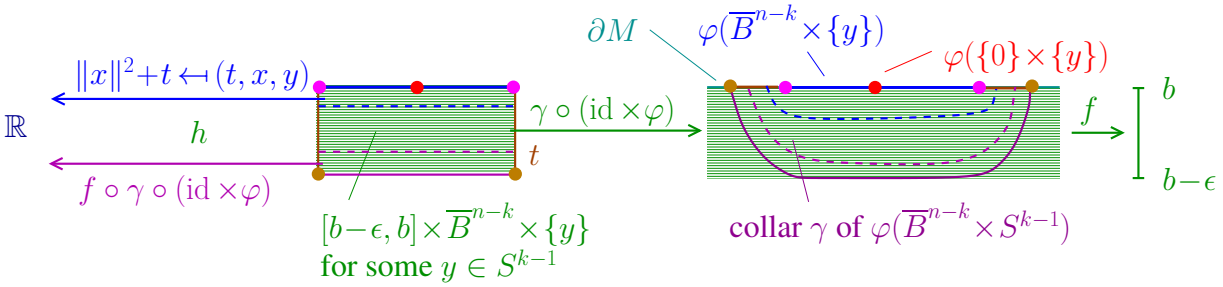
(a) $\gamma([b-\epsilon, b] \times \varphi(\overline{B}^{n-k} \times S^{k-1})) \subseteq \chi([b-\epsilon, b] \times \partial M)$.

(b) The map
$$f \circ \gamma\colon [b-\epsilon, b] \times \varphi(\overline{B}^{n-k} \times S^{k-1}) \to [b-\epsilon, b]$$
points upward on $[b-\epsilon, b) \times \varphi(B^{n-k} \times S^{k-1})$.

(c) The map
$$f \circ \gamma\colon [b-\epsilon, b] \times \varphi(\overline{B}^{n-k} \times S^{k-1}) \to [b-\epsilon, b]$$
points outward on $[b-\epsilon, b) \times \varphi(S^{n-k-1} \times S^{k-1})$.



Claim 1. There exists a smooth function $h\colon [b-\epsilon,b]\times \overline{B}^{n-k}\times S^{k-1}\to[b-\epsilon,b]$ with the following properties:
(a) On $[b-\epsilon,b-\frac{3}{4}\epsilon]\times\overline{B}^{n-k}\times S^{k-1}$ the map h equals $f\circ\gamma\circ(\mathrm{id}_{[b-\epsilon,b]}\times\varphi)$.
(b) On $[b-\frac{1}{4}\epsilon,b]\times\overline{B}^{n-k}\times S^{k-1}$ the map h is given by $h(t,x,y)=\|x\|^2+t$.
(c) h points upward on $[b-\epsilon,b]\times B^{n-k}\times S^{k-1}$.
(d) h points outward on $[b-\epsilon,b]\times S^{n-k-1}\times S^{k-1}$.
(e) h has no critical points.



Proof. We consider the functions

$$\alpha\colon [b-\epsilon,b]\times\overline{B}^{n-k}\times S^{k-1}\to\mathbb{R},\quad (t,x,y)\mapsto (f\circ\gamma)(t,\varphi(x,y)) \qquad \& \qquad \beta\colon [b-\epsilon,b]\times\overline{B}^{n-k}\times S^{k-1}\to\mathbb{R},\quad (t,x,y)\mapsto \|x\|^2+t.$$

We make the following observations:

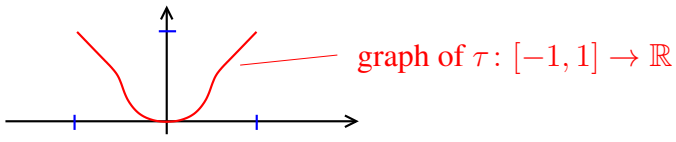
(α) It follows from Property (b) of γ that α points upward on $[b-\epsilon,b)\times B^{n-k}\times S^{k-1}$ and it follows from Property (c) of γ points that α points outward on $[b-\epsilon,b)\times S^{n-k}\times S^{k-1}$.
(β) It follows easily from the explicit definition of the map β that β points upward on $[b-\epsilon,b)\times B^{n-k}\times S^{k-1}$ and that β points outward on $[b-\epsilon,b)\times S^{n-k}\times S^{k-1}$.

By the Smooth Transition Function Lemma 24.9 we know that there exists a smooth function $\sigma\colon[b-\epsilon,b]\to[0,1]$ such that $\sigma(t)=0$ for $t\in[b-\epsilon,b-\frac{3}{4}\epsilon]$, such that $\sigma(t)=1$ for $t\in[b-\frac{1}{4}\epsilon,b]$ and such that $\sigma'(t)\geq 0$ for every $t\in[b-\epsilon,b]$. We now consider the map

$$\begin{aligned} h\colon [b-\epsilon,b]\times\overline{B}^{n-k}\times S^{k-1} &\to \mathbb{R}\\ (t,x,y) &\mapsto \underbrace{(f\circ\gamma)(t,\varphi(x,y))}_{=\alpha(t,x,y)}\cdot(1-\sigma(t))+\underbrace{(\|x\|^2+t)}_{=\beta(t,x,y)}\cdot\sigma(t). \end{aligned}$$

By design this function has Properties (a) and (b). Using the observation that "pointing upward" and "pointing outward" are convex conditions one can easily verify that Properties (c) and (d) are verified. Finally note that Property (e) is a consequence of Property (c) and (d) and of the observation that $\overline{B}^{n-k}=B^{n-k}\cup S^{n-k-1}$. ⊞

Next, using the ever-helpful Smooth Transition Function Lemma 24.9 we can easily find a smooth function $\tau\colon[-1,1]\to[0,1]$ such that $\tau(t)=|t|$ for $|t|\geq\frac{1}{2}$, such that $\tau(t)=t^2$ for $|t|\leq\frac{1}{4}$ and such that τ has no critical point except for $t=0$.



Claim 2. We consider the following function:

$$M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k) \xrightarrow{g} \mathbb{R}$$
$$P \mapsto \begin{cases} f(P), & \text{if } P \in M \setminus \gamma([b-\epsilon, b] \times \varphi(\overline{B}^{n-k} \times S^{k-1})), \\ h(t,x,y), & \text{if } P = \gamma(t, \varphi(x,y)), \\ b+1+\|x\|^2 - \tau(\|y\|), & \text{if } P = [(x,y)] \text{ with } (x,y) \in \overline{B}^{n-k} \times \overline{B}^k. \end{cases}$$

This function is smooth and it has the following properties:

(1) g equals f outside of $\gamma([b-\frac{3}{4}\epsilon, b] \times \varphi(\overline{B}_2^{n-k} \times S^{k-1}))$.

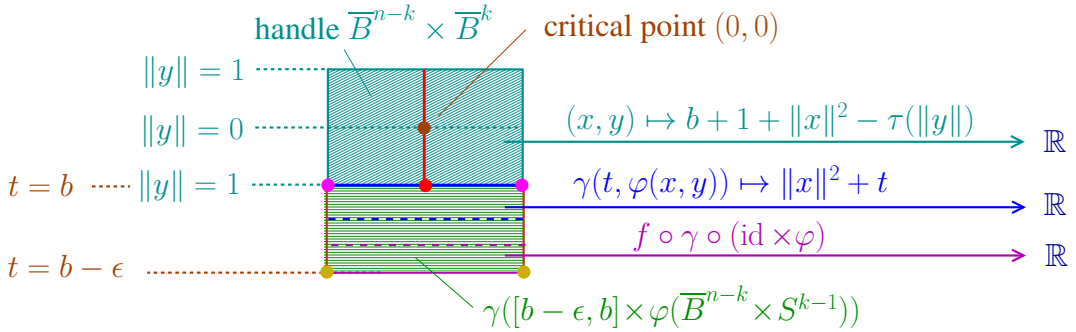
(2) The function $g\colon M \cup_\varphi h^k \to \mathbb{R}$ restricts to the map

$$\begin{array}{rcl} \overbrace{\overline{B}^{n-k} \times \overline{B}^k_{\frac{1}{4}}}^{\subseteq h^k = \overline{B}^{n-k} \times \overline{B}^k} & \to & \mathbb{R} \\ (x,y) & \mapsto & b+1+\|x\|^2 - \|y\|^2. \end{array}$$

In particular $(0,0) \in \overline{B}^{n-k} \times \overline{B}^k$ is a non-degenerate critical point of g of index k.

(3) The critical points of g are precisely the critical points of f together with the extra critical point $(0,0)$.

(4) g points outward on $\partial((M \cup_\varphi h^k) \setminus A)$.



Proof. First we point out that it follows from the definition of the smooth structure on $M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)$, via the collar γ and the standard collar $\overline{B}^{n-k} \times S^{k-1} \subseteq \overline{B}^{n-k} \times S^{k-1}$, together with our careful choice of τ and f, and the Smooth Pasting Proposition 25.19, that this map g is actually smooth. The remaining properties of g follow immediately from the properties of h and the fact that we started out with a Morse function f rel B. ⊞

Note that the function $g\colon M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k) \to \mathbb{R}$ from Claim 2 has all the desired properties, except that Property (4) is weaker than the desired condition. But this issue can be dealt with by appealing to the Standard-near-Boundary 249.29. ■

251.2. **Applications of the symmetry between handles and Morse functions.** The Handle Addition–Critical Point Theorem 251.1 and the Handle Decomposition–Morse Function Theorem 251.2 allow us to go back and forth between Morse functions and handle decompositions. In the following two sections we will see that this can be quite helpful.

In this section we will exploit the fact, that it is often much easier to modify functions, to construct handle decompositions. First of all, we have the following proposition, which gives a new solution to Exercise 246.27:

Proposition 251.5. (Product–Handle Decomposition Proposition) Let M and N be two compact smooth manifolds. We assume that $\partial M = \varnothing$. We suppose that M and N are each equipped with a handle decomposition rel $\varnothing$. Given $k \in \mathbb{N}_0$ we denote by r_k the number of k-handles of M and we denote by s_k the number of k-handles of N. The product $M \times N$ admits a handle decomposition rel $\varnothing$ such that

$$\text{number of } k\text{-handles of } M \times N \;=\; \sum_{i=0}^{k} r_i \cdot s_{k-i}.$$

Proof. The promised conclusion can be proved in three easy steps:

(1) It follows from the Handle Decomposition–Morse Function Theorem 251.2 that there exists a Morse function $f\colon M \to \mathbb{R}$ such that the number of critical points of index k equals r_k and that there exists a Morse function $g\colon M \to \mathbb{R}$ such that the number of critical points of index k equals s_k.

(2) It follows from the Product–Morse Function Proposition 249.17 that the map

$$\begin{aligned} \Xi\colon M \times N &\to \mathbb{R} \\ (x,y) &\mapsto f(x) + g(y) \end{aligned}$$

is a Morse function with the following properties:

- The critical points of Ξ are precisely the points (x_0, y_0) where $x_0 \in M$ is a critical point of $f\colon M \to \mathbb{R}$ and $y_0 \in N$ is a critical point of $g\colon N \to \mathbb{R}$.
- Given any critical point (x_0, y_0) of Ξ we have

$$\operatorname{index}(\Xi, (x_0, y_0)) \;=\; \operatorname{index}(f, x_0) + \operatorname{index}(g, y_0).$$

Elementary arguments now show that the number of critical points of index k of Ξ equals $\sum_{i=0}^{k} r_i \cdot s_{k-i}$.

(3) The desired statement follows immediately from (2) together with the Morse Function–Handle Decomposition Theorem 250.7. ■

The following proposition gives us a new proof of Claim 2 on page 4756.

Proposition 251.6. (Ball Bundle–Handle Decomposition Proposition) Let M be a closed smooth manifold, let $n \in \mathbb{N}_0$ and let $p\colon W \to M$ be a smooth $(\overline{B}^n, \mathrm{O}(n))$-bundle. For any handle decomposition of M there exists a handle decomposition of W with the same number of handles and same indices.

Proof. Suppose we are given a handle decomposition of M. For $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles. The promised conclusion can again be proved in three easy steps:

(1) From the Handle Decomposition–Morse Function Theorem 251.2 we obtain a Morse function $f\colon M \to \mathbb{R}$ such that the number of critical points of index k equals d_k.

(2) We apply the Linear Bundle–Morse Function Proposition 249.19 and we obtain from the Morse function $f\colon M \to \mathbb{R}$ a Morse function $g\colon W \to \mathbb{R}$ such that the number of critical points of index k equals d_k.

(3) From (2) and the Morse Function–Handle Decomposition Theorem 250.7 we obtain a handle decomposition of W such that the number of k-handles equals d_k. ■

We can now also give a short proof of some statements of the Dual Handle Proposition 248.1:

Proposition 248.1 (Dual Handle Proposition) Let M be a compact n-dimensional smooth manifold and let $\partial M = A \sqcup B$ be a decomposition of ∂M where A and B are unions of components of ∂M. Given any handle decomposition of M rel A there exists a handle decomposition of M rel B such that the number of k-handles of the new handle decomposition rel B equals the number of $(n-k)$-handles of the original handle decomposition rel A.

Proof. Suppose we are given a handle decomposition of M rel A. For $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles. The promised conclusion follows again from the by now familiar three-step process:

(1) From the Handle Decomposition–Morse Function Theorem 251.2 we obtain a Morse function $f\colon M \to \mathbb{R}$ rel A such that the number of critical points of index k equals d_k.
(2) On page 4813 we pointed out that it follows easily from the definitions that the function $-f\colon M \to \mathbb{R}$ is a Morse function rel B. Furthermore we pointed out that f and $-f$ have the same critical points, and that for each critical point x of f we have

$$\operatorname{index}(-f, x) \;=\; n - \operatorname{index}(f, x).$$

In particular we see that the number of critical points of $-f$ of index k equals $n - d_k$.
(3) From the Morse Function–Handle Decomposition Theorem 250.7, applied to the Morse function $-f$ rel B, we obtain a handle decomposition of M rel B such that the number of k-handles equals $n - d_k$. ■

251.3. **Self-indexing Morse functions.** In this section we will use some of our results on handle decompositions, together with the Handle Decomposition–Morse Function Theorem 251.2, to construct Morse functions with nice properties.

First of all we have the following result:

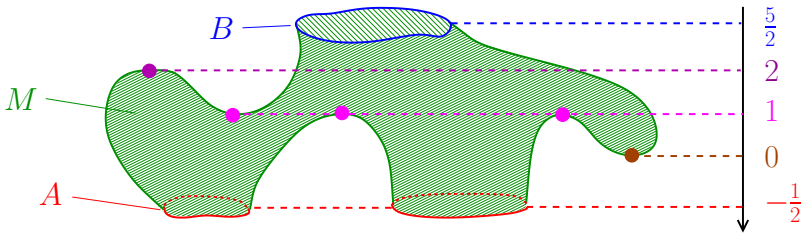
Proposition 251.7. (Single Minimum-Single Maximum Proposition) Let M be a closed connected non-empty n-dimensional smooth manifold. There exists a smooth map $f\colon M \to \mathbb{R}$ with a single local minimum and a single local maximum.

Proof. Since M is a closed connected non-empty n-dimensional smooth manifold we obtain from the Handle Decomposition Theorem 250.7 together with the Minimal Handle Proposition 248.5 that M admits a handle decomposition with a single 0-handle and a single n-handle. The desired statement now follows from the Handle Decomposition–Morse Function Theorem 251.2 and the elementary discussion on page 4805. ■

We conclude this section with a neat refinement of the Morse Function-Existence Theorem 249.26. To formulate the result we introduce the following definition:

Definition 251.8. Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M. We set $B := \partial M \setminus A$. A Morse function $f\colon M \to \mathbb{R}$ rel A is called self-indexing, if the following conditions are satisfied.
(1) We have $f|_A \equiv -\frac{1}{2}$ and $f|_B \equiv n + \frac{1}{2}$.
(2) If P is a critical point of f, then $f(P) = \operatorname{index}(f, P)$.

Proposition 251.9. (Self-Indexing Proposition) Let M be a compact smooth manifold and let A be a union of components of ∂M. There exists a self-indexing Morse function $f\colon M \to \mathbb{R}$ rel A.



Remark. The proposition was originally proved by Stephen Smale [**Sma1961b**, Theorem C]. Alternative proofs of the proposition are given in [**Miln1965b**, p. 45] and [**Nic2011**, Chapter 2.4]. □

Proof. Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M. By the Handle Decomposition Theorem 250.7 there exists a standard handle decomposition

$$M \;=\; ([0,1]\times A)\underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

rel A. Given $k \in \{-1,\ldots,n\}$ we denote by M^k the k-skeleton of the handle decomposition, i.e. M^k is the union of $[0,1]\times A$ and all the handles up to index $\leq k$. We set $B_k := \partial M^k \setminus A$. It suffices to prove the following claim:

Claim. For $k = -1,\ldots,n$ there exists a Morse function $f_k\colon M^k \to \mathbb{R}$ rel A with the following properties:
(1) We have $f|_A \equiv -\frac{1}{2}$ and $f|_{B_k} \equiv k+\frac{1}{2}$.
(2) If P is a critical point of f_k, then $f_k(P) = \operatorname{index}(f_k, P)$.

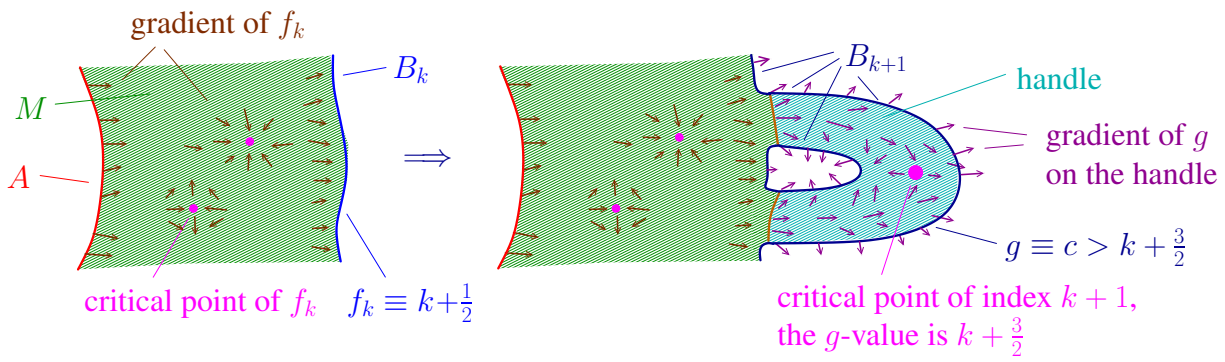
Proof. We prove the claim by induction on k. For $k=-1$ we have $M^k = [0,1]\times A$ and we choose the map that is given by $(t,a) \mapsto t - \frac{1}{2}$. Clearly this map has all the desired properties.

Now suppose the claim holds for some $k \in \{-1,\ldots,n-1\}$. So we are given Morse function $f_k\colon M^k \to \mathbb{R}$ rel A with the desired properties. It follows from an elementary extension of the Handle Addition–Critical Point Theorem 251.1 to multiple simultaneous k-handle attachments that there exists a Morse function $g\colon M^{k+1} \to \mathbb{R}$ with the following properties:

(1) The critical points of f_k are also critical points of g, with the same indices.
(2) g has precisely r_{k+1} extra critical points.
(3) Each of these extra critical points has index $k+1$.
(4) The value of g at each of the new critical points is $k+\frac{1}{2}+1 = k+\frac{3}{2}$.
(5) There exists a $c > k+\frac{1}{2}+1 = k+\frac{3}{2}$ such that $g|_{B_{k+1}} \equiv c$.

It follows easily from the Smooth Transition Function Lemma 24.9 that there exists a diffeomorphism $\tau\colon \mathbb{R} \to \mathbb{R}$ with the following three properties:

(1) We have $\tau(x) = x$ for $x < k+\frac{1}{2}$.
(2) We have $\tau(k+\frac{3}{2}) = k+1$.
(3) We have $\tau(c) = k+\frac{3}{2}$.



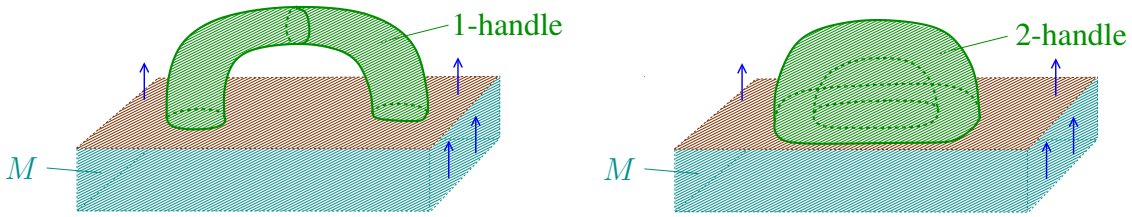
An elementary argument, see Exercise 251.1, shows that the composition of a Morse function with an orientation-preserving diffeomorphism of $\mathbb{R}$ is again a Morse function with exactly the same critical points and exactly the same indices. It follows from this discussion that $f_{k+1} := \tau \circ g \colon M_{k+1} \to \mathbb{R}$ has all of the desired properties. ■

251.4. Exercises for Chapter 251.

Exercise 251.1. Let M be a smooth manifold and let $f \colon M \to \mathbb{R}$ be a Morse function. Furthermore let $\lambda \colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\lambda'(x) > 0$ for all $x \in \mathbb{R}$. Show that $\lambda \circ f \colon M \to \mathbb{R}$ is also a Morse function and that the critical points, and the corresponding indices, of $\lambda \circ f$ agree with the critical points, and the corresponding indices, of f.

Exercise 251.2. We consider the smooth manifold $M := \mathbb{R}^2 \times [0,1]$ with the Morse function that is given by $f(x,y,z) = z$. In the figure below we show M, the gradient of f and the attachment of a 1-handle and of a 2-handle to $\mathbb{R}^2 \times \{1\}$.

(a) Indicate the gradient of a Morse function on $(M \times [0,1]) \cup h^1$.
(b) Indicate the gradient of a Morse function on $(M \times [0,1]) \cup h^2$.



Exercise 251.3. Let M be a smooth manifold and let A be a union of components of ∂M. Let $f \colon M \to \mathbb{R}$ be a Morse function rel A. Show that there exists a self-indexing Morse function rel A with the same number of critical points of each given index.

Exercise 251.4. Let $\widetilde{M}$ and M be closed smooth manifold and let $p \colon \widetilde{M} \to M$ be a covering of degree d such that p is local diffeomorphism. Show that if M is equipped with a handle decomposition, then $\widetilde{M}$ admits a handle decomposition such that the number of handles is d-times the number of handles of M.
Remark. This exercise is closely related to Exercise 246.30.

Exercise 251.5. Let M be a closed smooth manifold and let $\varphi\colon M \to M$ be a self-diffeomorphism. Suppose that we have a handle decomposition of M such that for each $k \in \mathbb{N}_0$ we have d_k handles of index k. Show that the mapping torus

$$\operatorname{Tor}(M,\varphi) \;:=\; (M \times [0,1])/\,(x,0) \sim (\varphi(x),1)$$

admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of handles of index k equals $d_k + d_{k-1}$.
Hint. You could make use of Exercise 249.14.

Exercise 251.6. Give an explicit self-indexing Morse function for the following three smooth manifolds:
(a) The torus $S^1 \times S^1$.
(b) The Klein bottle.
(c) The Möbius band rel $\varnothing$.

Exercise 251.7. Can you find a smooth embedding $\varphi\colon S^1 \times S^1 \to \mathbb{R}^3$ such that the usual height function, that is given by $h(x,y,z) \mapsto z$, is a self-indexing Morse function for the torus $\varphi(S^1 \times S^1)$?

CHAPTER 252

Handle homology and the Morse inequalities

Let M be a compact smooth manifold and let A be a union of components of ∂M. In this chapter we will see, imitating the definition of cellular homology, that a handle decomposition of M rel A gives rise to a chain complex, namely the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(M, A)$. We will see that the corresponding homology groups $\mathrm{H}_k^{\mathrm{han}}(M, A)$, called handle homology groups, are naturally isomorphic to the usual relative singular homology groups $\mathrm{H}_k(M, A)$. We will use this isomorphism to prove the Morse inequalities, which show that the "size" of the singular homology groups $\mathrm{H}_k(M, A)$ gives us a lower bound on the number of singularities of a Morse function on M rel A.

This chapter is organized as follows:

- In Section 252.1 we will introduce the handle homology groups $\mathrm{H}_k^{\mathrm{han}}(M, A)$ of a compact smooth manifold M that is equipped with a handle decomposition rel some A. We will also state and prove the Singular-Handle-H_*-Proposition 252.4 which says that handle homology is naturally isomorphic to singular homology.
- In Section 252.2 we will see that we can calculate the boundary maps in the handle chain complex via algebraic intersection numbers.
- In Section 252.3 we will formulate and prove the Algebraic Morse Inequalities Proposition 252.10 which gives us lower bounds on the ranks of chain groups in terms of the "size" of the homology groups.
- In Section 252.4 we will apply the Algebraic Morse Inequalities Proposition 252.10 to the handle chain complex. This way we will obtain in the Morse Inequalities Proposition 252.13 lower bounds on the number of handles in terms of the "size" of the singular homology groups. Similarly, we will obtain in the Critical Points–Morse Inequalities Theorem 252.14 lower bounds on the number of critical points of a Morse function.
- In Section 252.5 we will introduce the notion of a "perfect" Morse function. Loosely speaking a Morse function is perfect, if the number of critical points matches the lower bounds obtained from the Critical Points–Morse Inequalities Theorem 252.14. We will summarize some results in the literature about perfect Morse functions.
- In Section 252.6 we will provide exercises for Chapter 252.

In the next Chapter 253 we will study handle decompositions of cobordisms with vertical boundary.

252.1. **Handle homology.** In Chapter 155 we introduced the cellular chain complex of a CW-complex X. Recall that this chain complex has the following two features:

(1) The chain groups are defined to be $\mathrm{C}_k^{\mathrm{CW}}(X) \coloneqq \mathrm{H}_k(X^k, X^{k-1})$, where X^i denotes the i-skeleton of X. By the CW-Skeleton-H_*-Lemma 155.2 (1) this chain group can be viewed as the free abelian groups on the set of k-cells.

(2) In the Singular-Cellular-H_*-Proposition 155.8 we showed that the corresponding cellular homology groups are naturally isomorphic to the singular homology groups of X.

In this section we will see how one can perform something very similar in the setting of smooth manifolds with a given handle decomposition. We start out with the following analog of the CW-Skeleton-H_*-Lemma 155.2:

Lemma 252.1. (Handle-Skeleton-H_*-Lemma) Let M be a compact n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let[2708]

$$M \;=\; ([0,1]\times A)\underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

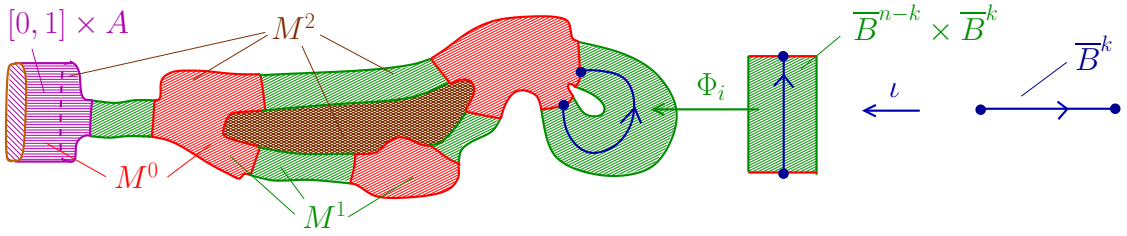
be a standard handle decomposition of M rel A. Let $k \in \mathbb{N}_0$. As in Definition 247.6 we denote by M^k the k-skeleton of the handle decomposition, i.e. M^k is the union of all handles of index $\leq k$.[2709]

(1) Let $k \in \{0,\ldots,n\}$. We denote by $\iota\colon \overline{B}^k \to \overline{B}^{n-k}\times\overline{B}^k$ the embedding that is given by $x \mapsto (0,x)$. Furthermore, given $i \in \{1,\ldots,r_i\}$ we denote by $\Phi_i\colon \overline{B}^{n-k}\times\overline{B}^k \to M^k$ the embedding that is given by the i-th k-handle. For every $j \in \mathbb{N}_0$ the following map is an isomorphism:[2710]

$$\bigoplus_{i=1}^{r_i} (\Phi_i\circ\iota)_*\colon \bigoplus_{i=1}^{r_i} \underbrace{\mathrm{H}_j(\overline{B}^k,\partial\overline{B}^k)}_{\substack{=0, \quad \text{if } j\neq k,\\ \cong\mathbb{Z}, \quad \text{if } j=k}} \;\longrightarrow\; \mathrm{H}_j(M^k,M^{k-1}).$$

In particular the following holds:
(a) For $j \neq k$ we have $\mathrm{H}_j(M^k,M^{k-1}) = 0$.
(b) The group $\mathrm{H}_k(M^k,M^{k-1})$ is a free abelian group where the rank equals the number of k-handles.[2711]

(2) For $i > k$ we have $\mathrm{H}_i(M^k,A) = 0$.
(3) For $i < k$ the inclusion $\iota\colon M^k \to M$ induces an isomorphism $\mathrm{H}_i(M^k,A) \xrightarrow{\cong} \mathrm{H}_i(M,A)$.
(4) The inclusion $\iota\colon M^k \to M$ induces an epimorphism $\mathrm{H}_k(M^k,A) \twoheadrightarrow \mathrm{H}_k(M,A)$.



Remark. The statement of the Handle-Skeleton-H_*-Lemma 252.1 (3) and (4) is very similar in spirit to the content of Exercise 247.6. □

[2708] As we discussed in Definition 247.1, we will suppress in our notation the choice of a diffeomorphism from M to the right-hand side.
[2709] Note that this definition makes sense since the handle decomposition is standard.
[2710] Note that by definition of a standard handle decomposition, see Definition 247.6, the restriction of $\Phi_i\circ\iota$ to $\partial\overline{B}^k$ does indeed take values in M^{k-1}.
[2711] Sometimes one summarizes this statement as saying that "the cores of the handles form a basis of $\mathrm{H}_k(M^k,M^{k-1})$".

Proof. Not surprisingly the proof of the lemma is very similar to the proof of the CW-Skeleton-H_*-Lemma 155.2. But for completeness sake let us go through the motions.

(1) For $j \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccc}
\bigoplus_{i=1}^{r_i} \mathrm{H}_j(\overline{B}^k, \partial\overline{B}^k) & \xrightarrow{\quad\oplus(\Phi_i\circ\iota)_*\quad} & \mathrm{H}_j(M^k, M^{k-1}) \\
\Big\downarrow \oplus\iota_* & & \Big\uparrow \\
 & & \mathrm{H}_j\Big(\bigsqcup_{i=1}^{r_i} \Phi(\overline{B}^{n-k}\times\overline{B}^k), \bigsqcup_{i=1}^{r_i} \Phi(\overline{B}^{n-k}\times\partial\overline{B}^k)\Big) \\
 & & \Big\uparrow \\
\bigoplus_{i=1}^{r_i} \mathrm{H}_j(\overline{B}^{n-k}\times\overline{B}^k, \overline{B}^{n-k}\times\partial\overline{B}^k) & \xrightarrow{\quad\mathrm{diag}(\Phi_{i*})\quad} & \bigoplus_{i=1}^{r_i} \mathrm{H}_j(\Phi_i(\overline{B}^{n-k}\times\overline{B}^k), \Phi_i(\overline{B}^{n-k}\times\partial\overline{B}^k)).
\end{array}$$

We make the following clarifications and observations:

- The left vertical map is induced by the embeddings $\iota\colon \overline{B}^k \to \overline{B}^{n-k}\times\overline{B}^k$. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the left vertical map is an isomorphism.
- The bottom horizontal map is induced by homeomorphisms, thus it is also an isomorphism.
- The right bottom vertical map is induced by the natural inclusions. By the obvious relative analog of the Path-Component–H_*–Lemma 143.21 we know that this map is an isomorphism.
- The right top vertical map is induced by the inclusion. It follows easily from the Handle Attachment Proposition 246.3 (3) that $\bigsqcup_{i=1}^{r_i} \Phi_i(\overline{B}^{n-k}\times\overline{B}^k)$ is a codimension-zero submanifold with corner of M^k. It follows almost immediately from the Manifold–Excision Theorem 212.4 that the right top vertical map is an isomorphism.

One can easily verify that the diagram commutes. Hence we get that the top horizontal map is indeed an isomorphism. Finally the "in particular" statements now follow immediately from the Ball-rel-Sphere-H_*-Lemma 149.7.

(2) We fix $i \in \mathbb{N}_0$. We need to prove the following claim:

Claim. For $k = -1, \dots, i-1$ we have $\mathrm{H}_i(M^k, A) = 0$.

Proof. We prove the claim by induction on k. First of all for $k = -1$ we have

$$\mathrm{H}_i(M^{-1}, A) \;=\; \mathrm{H}_i([0,1]\times A, A) \;\underset{\uparrow}{=}\; 0.$$

follows from the Deformation Retract Relative H_*-Corollary 146.9

Now assume that we already know that $\mathrm{H}_i(M^k, A) = 0$ for some $k < i-1$. We want to show that $\mathrm{H}_i(M^{k+1}, A) = 0$. We consider the long exact sequence of the triple (M^{k+1}, M^k, A) that is given to us by the LES-of-Triple Proposition 146.5:

$$\dots \to \mathrm{H}_{i+1}(M^{k+1}, M^k) \to \underbrace{\mathrm{H}_i(M^k, A)}_{=0} \to \mathrm{H}_i(M^{k+1}, A) \to \underbrace{\mathrm{H}_i(M^{k+1}, M^k)}_{=0 \text{ by (1) since } i>k+1} \to \dots$$

Thus we obtain that indeed $\mathrm{H}_i(M^{k+1}, A) = 0$. ⊞

(3) We fix $i \in \mathbb{N}_0$. Let $k > i$. This time we consider the following excerpt from the long exact sequence of the triple (M^{k+1}, M^k, A):

$$\ldots \to \underbrace{\mathrm{H}_{i+1}(M^{k+1}, M^k)}_{=0 \text{ by (1) since } i<k} \to \mathrm{H}_i(M^k, A) \to \mathrm{H}_i(M^{k+1}, A) \to \underbrace{\mathrm{H}_i(M^{k+1}, M^k)}_{=0 \text{ by (1) since } i<k} \to \ldots$$

This discussion implies in particular that we have inclusion-induced isomorphisms $\mathrm{H}_i(M^k, A) \xrightarrow{\cong} \mathrm{H}_i(M^{k+1}, A) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathrm{H}_i(M^n, A) = \mathrm{H}_i(M, A)$.

(4) In (3) we showed that the inclusion-induced map $\mathrm{H}_k(M^{k+1}, A) \to \mathrm{H}_k(M, A)$ is an isomorphism. It follows from this observation that it suffices to show that the inclusion-induced map $\mathrm{H}_k(M^k, A) \to \mathrm{H}_k(M^{k+1}, A)$ is an epimorphism. To show this we consider the following excerpt from the long exact sequence of the triple (M^{k+1}, M^k, A):

$$\ldots \to \mathrm{H}_{k+1}(M^{k+1}, M^k) \to \mathrm{H}_k(M^k, A) \to \mathrm{H}_k(M^{k+1}, A) \to \underbrace{\mathrm{H}_k(M^{k+1}, M^k)}_{=0 \text{ by (1)}} \to \ldots$$

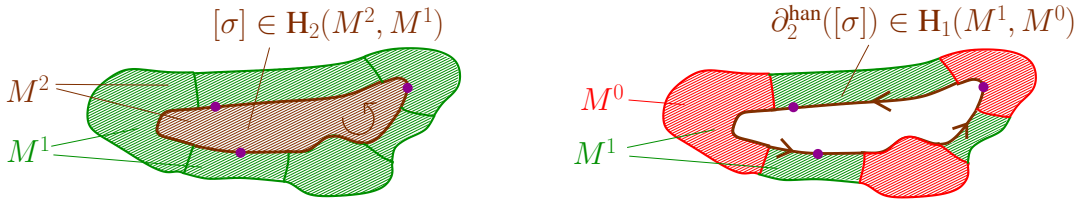
We see that the inclusion-induced map $\mathrm{H}_k(M^k, A) \to \mathrm{H}_k(M^{k+1}, A)$ is indeed an epimorphism. ■

As for CW-complexes the idea now is to turn the groups $\mathrm{H}_k(M^k, M^{k-1})$, $k \in \mathbb{N}_0$ into a chain complex. The key to doing so is the following lemma, which is an analog of the Cellular Boundary Map Lemma 155.4:

Lemma 252.2. (Handle Boundary Map Lemma) Let M be a compact smooth manifold and let A be a union of components of ∂M. We suppose that M is equipped with a standard handle decomposition rel A. Given $k \in \mathbb{N}_0$ we denote by $\partial_k^{\mathrm{han}}$ the map[2712]

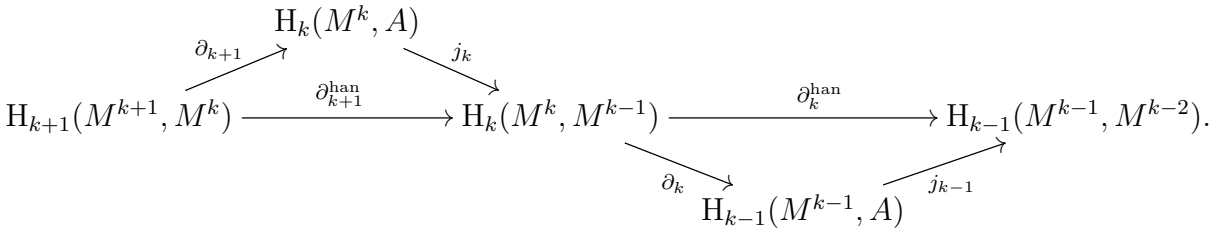
$$\mathrm{H}_k(M^k, M^{k-1}) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(M^{k-1}, A) \xrightarrow{j_{k-1}} \mathrm{H}_{k-1}(M^{k-1}, M^{k-2})$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{=:\partial_k^{\mathrm{han}}}$$

where ∂_k is the connecting map in the long exact sequence of the triple (M^k, M^{k-1}, A) and where j_{k-1} is the map induced by the natural map $(M^{k-1}, A) \to (M^{k-1}, M^{k-2})$ of pairs of topological spaces. These maps $\partial_k^{\mathrm{han}}$ have the property that for every $k \in \mathbb{N}_0$ we have $\partial_k^{\mathrm{han}} \circ \partial_{k+1}^{\mathrm{han}} = 0$.



[2712]The argument of Footnote 1610 shows that $\partial_k^{\mathrm{han}}$ can also be viewed as the connecting homomorphism in the long exact sequence of the triple (M^k, M^{k-1}, M^{k-2}). But for the purpose of this proof and later for the proof of the Singular-Handle-H_*-Proposition 252.4 it is better to adopt our two-step definition.

Proof. This proof is easy: We just need to copy-paste the proof of the Cellular Boundary Map Lemma 155.4. Thus we consider the following commutative diagram of maps



The map $\partial_k \circ j_k \colon \mathrm{H}_k(M^k, A) \to \mathrm{H}_{k-1}(M^{k-1}, A)$ is the composition of two successive maps in the long exact sequence of the triple (M^k, M^{k-1}, A), i.e. the map $\partial_k \circ j_k$ is the zero map. Since the diagram commutes we obtain that $\partial^{\mathrm{han}}_k \circ \partial^{\mathrm{han}}_{k+1} = 0$. ■

The Handle Boundary Map Lemma 252.2 allows us to introduce the following definition:

Definition 252.3. Let M be a compact smooth manifold and let A be a union of components of ∂M. We suppose that M is equipped with a standard handle decomposition rel A.

(1) We write $\mathrm{C}^{\mathrm{han}}_k(M, A) := \mathrm{H}_k(M^k, M^{k-1})$.
(2) We refer to the map $\partial^{\mathrm{han}}_k \colon \mathrm{C}^{\mathrm{han}}_k(M, A) \to \mathrm{C}^{\mathrm{han}}_{k-1}(M, A)$ from the Handle Boundary Map Lemma 252.2 as the handle boundary map.
(3) We refer to $(\mathrm{C}^{\mathrm{han}}_*(M, A), \partial^{\mathrm{han}}_*)$ as the handle chain complex.
(4) We denote the homology groups of the handle chain complex by $\mathrm{H}^{\mathrm{han}}_k(M, A)$ and we refer to these groups as the handle homology groups of the smooth manifold M rel A with the given handle decomposition.

In the Singular-Cellular-H_*-Proposition 155.8 we showed that for a CW-complex the singular and cellular homology groups are naturally isomorphic. The following proposition proves the analogous statement in our present setting:

Proposition 252.4. (Singular-Handle-H_*-Proposition) Let M be a compact smooth manifold and let A be a union of components of ∂M. We suppose that M is equipped with a standard handle decomposition rel A.

(1) Given any $k \in \mathbb{N}_0$ there exists a uniquely determined isomorphism

$$\Theta_M \colon \mathrm{H}_k(M, A) \xrightarrow{\cong} \mathrm{H}^{\mathrm{han}}_k(M, A)$$

with the property that the following diagram commutes:[2713]

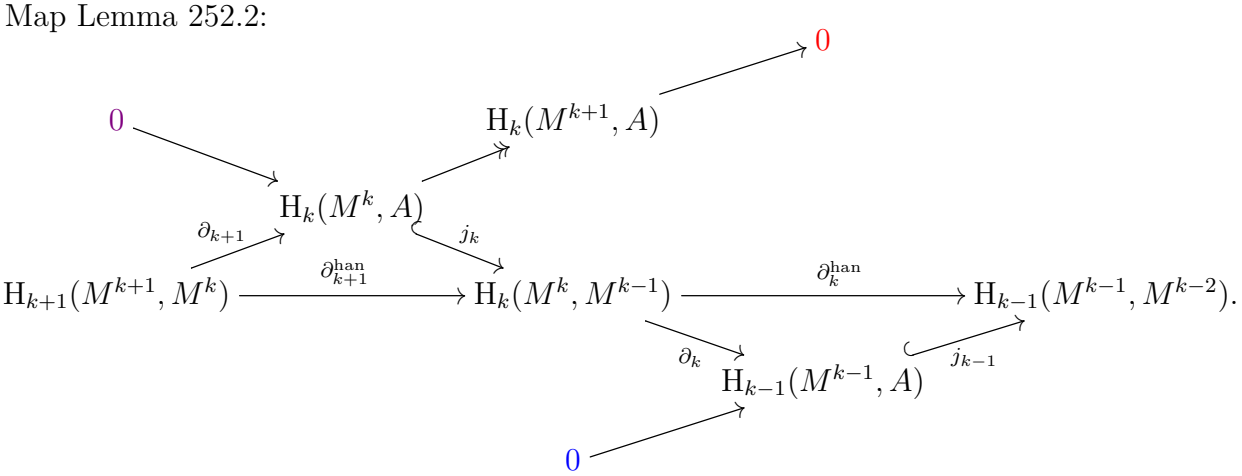
$$\begin{array}{ccc} \mathrm{H}_k(M^k, A) & \longrightarrow & \mathrm{H}_k(M, A) \\ {\scriptstyle j_k}\downarrow & & \downarrow{\scriptstyle \Theta_M} \\ \ker(\partial^{\mathrm{han}}_k) & \twoheadrightarrow & \mathrm{H}^{\mathrm{han}}_k(M, A). \end{array}$$

(2) The homomorphism from (1) is natural in the following sense: For any smooth map $f \colon M \to N$ between two smooth manifolds, that are equipped with relative handle decompositions rel $A \subseteq \partial M$ and rel $B \subseteq \partial N$, such that $f(M^k) \subseteq N^k$ for all $k \in \mathbb{N}_0$,

the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(M,A) & \xrightarrow{f_*} & \mathrm{H}_n(N,B) \\ {\scriptstyle\Theta_M}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Theta_N} \\ \mathrm{H}_n^{\mathrm{han}}(M,A) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{han}}(N,B). \end{array}$$

Proof. The proof of the Singular-Handle-H_*-Proposition 252.4 is almost embarrassingly similar to the proof of the Singular-Cellular-H_*-Proposition 155.8 (1). For the reader's convenience we nonetheless provide an abbreviated version of the argument. As in the proof of the Singular-Cellular-H_*-Proposition 155.8 (1) we add a couple of maps to the commutative diagram that we had already encountered in the proof of the Handle Boundary Map Lemma 252.2:



By the Handle-Skeleton-H_*-Lemma 252.1 (1) and (2) we know that $\mathrm{H}_k(M^{k+1},M^k)=0$, $\mathrm{H}_k(M^{k-1},A)=0$ and $\mathrm{H}_{k-1}(M^{k-2},A)=0$. It follows almost immediately from these facts that the diagonal sequences in the above diagram are exact. Next we consider the diagram:

$$\begin{array}{ccccc} & & \text{isomorphism since the left diagonal sequence is exact} & & \text{isomorphism by the Handle-Skeleton-H}_*\text{-Lemma 252.1 (3)} \\ & & \downarrow & & \downarrow \\ \mathrm{H}_k(M^k,A)/\operatorname{im}(\partial_{k+1}) & \to & \mathrm{H}_k(M^{k+1},A) & \to & \mathrm{H}_k(M,A). \\ \text{a diagram chase shows that } j_k \text{ induces an isomorphism}\ \downarrow\cong & & & & \downarrow =:\Theta_M \\ \ker(\partial^{\mathrm{han}}_k)/\operatorname{im}(\partial^{\mathrm{han}}_{k+1}) & & \xrightarrow[\mathrm{id}]{} & & \mathrm{H}^{\mathrm{han}}_k(M,A). \end{array}$$

Next note that by construction the isomorphism Θ_M has the property that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(M^k,A) & \longrightarrow & \mathrm{H}_k(M,A) \\ {\scriptstyle j_k}\downarrow & & \downarrow{\scriptstyle\Theta_M} \\ \ker(\partial^{\mathrm{han}}_k) & \twoheadrightarrow & \mathrm{H}^{\mathrm{han}}_k(M,A). \end{array}$$

[2713]Here we again denote by $j_k\colon \mathrm{H}_k(M^k,A)\to\mathrm{H}_k(M^k,M^{k-1})$ the obvious map. In the long exact sequence of the triple (M^k,M^{k-1},A) we see that the image of $\mathrm{H}_k(M^k,A)$ in $\mathrm{H}_k(M^k,M^{k-1})$ lies in the kernel of the connecting homomorphism ∂_k. This implies that the map $j_k\colon \mathrm{H}_k(M^k,A)\to\mathrm{H}_k(M^k,M^{k-1})$ takes values in $\ker(\partial^{\mathrm{han}}_k)$.

Note that the top horizontal map is an epimorphism by the Handle-Skeleton-H_*-Lemma 252.1 (4). This implies immediately that Θ_M is uniquely determined.

Finally, the proof that the isomorphisms Θ_M are natural is very similar to the proof of the corresponding statement in the Singular-Cellular-H_*-Proposition 155.8. We leave it to the reader to fill in the details. ■

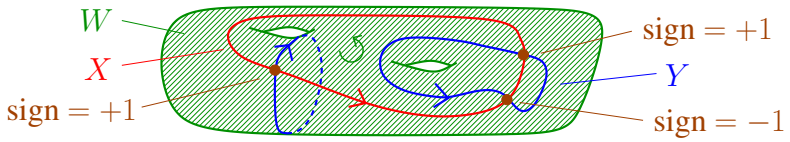
252.2. **The boundary maps in the handle chain complex.** In the Cellular Boundary–Degree Proposition 155.15 we learned about a fairly practical way to calculate the boundary maps in the cellular chain complex. In this section we will devise a tool for calculating the boundary maps in a given handle chain complex.

Definition 252.5. Let $m \in \mathbb{N}_0$, let W be a closed oriented m-dimensional smooth manifold and let $l \in \{0, \dots, m\}$. Furthermore let X be a closed oriented l-dimensional submanifold of W and let Y be a closed oriented $(m-l)$-dimensional submanifold of W. As usual we denote by $[X] \in \mathrm{H}_l(W)$ and $[Y] \in \mathrm{H}_{m-l}(W)$ the images of the fundamental classes of X and Y in the homology groups of W. We define the algebraic intersection $X \cdot Y$ number of X and Y as[2714]

$$X \cdot Y = \Big\langle \overbrace{\mathrm{PD}_W(\underbrace{[X]}_{\in \mathrm{H}_l(W)}) \cup \mathrm{PD}_W(\underbrace{[Y]}_{\in \mathrm{H}_{m-l}(W)})}^{\in \mathrm{H}^m(W;\mathbb{Z})}, \overbrace{[W]}^{\in \mathrm{H}_m(W)} \Big\rangle \in \mathbb{Z}.$$

$\in \mathrm{H}^{m-l}(W;\mathbb{Z})$ $\quad$ $\in \mathrm{H}^l(W;\mathbb{Z})$

Example. We continue with the notation from the definition. If X and Y intersect transversally, then the Cup Product–Intersection Number Theorem 220.10 tells us that we can calculate the algebraic intersection number $X \cdot Y$ by assigning signs to the intersection points and adding up the signs.



$$X \cdot Y = +1 - 1 + 1 = +1$$

We will not repeat the definition of the signs of the intersection points, but we hope that the above example will remind the reader of the definitions from Definition 220.5. □

Now we can formulate the main result of this section, which tells us that we can calculate the boundary maps in handle chain complexes through algebraic intersection numbers:

Proposition 252.6. (Handle Chain Complex-Boundary Map Proposition) Let M be an oriented n-dimensional smooth manifold, let A be a union of components of ∂M and let

$$M = ([0,1] \times A) \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \dots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

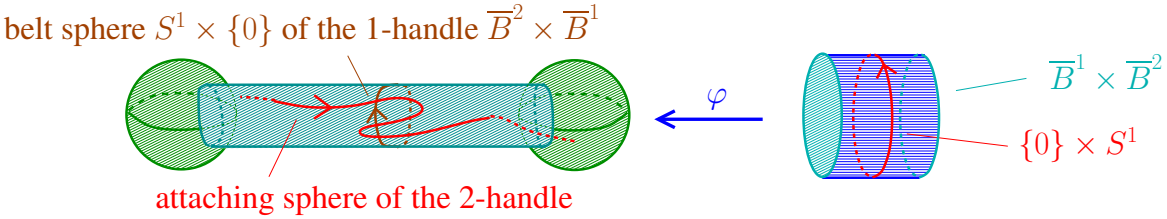
[2714] By the Cup Product–Intersection Number Theorem 220.10 this definition is consistent with the original Definition 220.5.

be a standard handle decomposition of M rel A. We assume that the inclusion maps of all the handles are orientation-preserving. Given $k \in \mathbb{N}_0$ and given $i \in \{1, \dots, r_i\}$ we denote by $\Phi_{k,i} \colon \{0\} \times \overline{B}_i^k \to M^k$ the map that is given by the inclusion map of the core of the handle. By the Handle-Skeleton-H_*-Lemma 252.1 (1) we know that

$$\mathcal{B}_k \;:=\; \{\Phi_{k,1*}([\{0\} \times \overline{B}^k]), \dots, \Phi_{k,r_k*}([\{0\} \times \overline{B}^k])\}$$

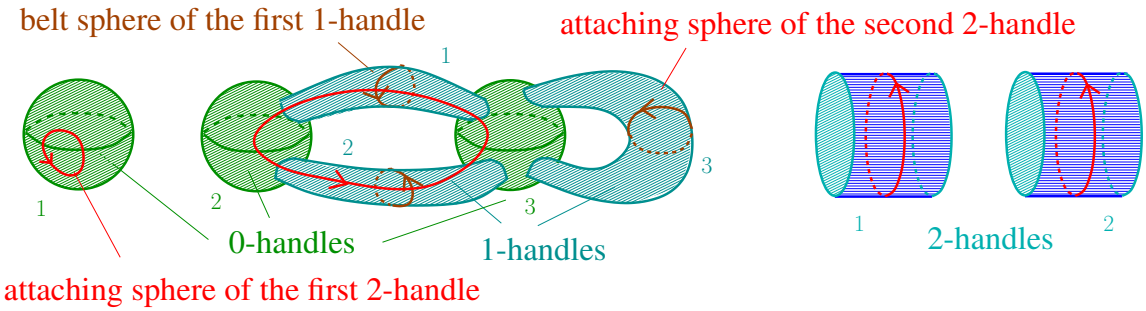
is a basis of $\mathrm{C}_k^{\mathrm{han}}(M, A) := \mathrm{H}_k(M^k, M^{k-1})$. Given $j \in \{1, \dots, r_{k+1}\}$ and $i \in \{1, \dots, r_k\}$ we have the following equality:[2715]

(i,j)-entry of the matrix representing $\partial_{k+1}^{\mathrm{han}} \colon \mathrm{C}_{k+1}^{\mathrm{han}}(M, A) \to \mathrm{C}_k^{\mathrm{han}}(M, A)$ with respect to the bases $\mathcal{B}_{k+1}$ and $\mathcal{B}_k$ = algebraic intersection number in ∂M^k of the attaching sphere of the $(k+1, j)$-handle with the belt sphere of the (k, i)-handle.



We will provide the proof of the Handle Chain Complex-Boundary Map Proposition 252.6 towards the end of the section. First we want to discuss this proposition.

Example. We consider the 3-dimensional smooth manifold M which is shown in the figure below.



It has three 0-handles, three 1-handles and two 2-handles. Using the Handle Chain Complex-Boundary Map Proposition 252.6 one sees that the corresponding handle chain complex is isomorphic to the following chain complex:

$$0 \to \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}0 & 1\\ 0 & 1\\ 0 & 2\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix}0 & 0 & 0\\ -1 & 1 & 0\\ 1 & -1 & 0\end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\end{matrix} \to 0. \qquad \text{Thus we obtain that} \quad \mathrm{H}_k^{\mathrm{han}}(M, A) \cong \begin{cases} 0, & \text{if } k \geq 3,\\ \mathbb{Z}, & \text{if } k = 2,\\ \mathbb{Z}, & \text{if } k = 1,\\ \mathbb{Z}^2, & \text{if } k = 0. \end{cases}$$

By the Ball–Handle Attachment Lemma 246.9 we know that the component of M that is given by gluing the first 2-handle to the first 0-handle is diffeomorphic to $\overline{B}^1 \times S^2$. But

[2715] For the right-hand side note that ∂M^k is a compact oriented $(n-1)$-dimensional smooth manifold, the attaching sphere of the $(k+1, j)$-handle is an oriented k-dimensional sphere in ∂M^k and the belt sphere of the (k, i)-handle is an oriented $(n-k-1)$-dimensional sphere in ∂M^k. So, according to the above definition it makes sense to consider in ∂M^k the algebraic intersection number of the two spheres.

what can we say about the other component of M? We will answer this question later on page 5026. □

Let M be a compact oriented smooth manifold that is equipped with a standard handle decomposition rel a union of components of ∂M. If the various attaching spheres and the belt spheres intersect transversally, then we know by the Cup Product–Intersection Number Theorem 220.10 that the algebraic intersection number of the attaching spheres and the belt spheres can be determined in a pleasingly geometric way. We will now see that we can always put ourselves into such a pleasant situation.

Definition 252.7. Let M be an oriented n-dimensional smooth manifold and let A be a union of components of ∂M. A handle decomposition of M rel A is called transverse if for any $k \in \mathbb{N}$ the intersection of the attaching sphere of any $(k+1)$-handle with the belt sphere of any k-handle is transverse.

Proposition 252.8. (Transversal Handle Decomposition Proposition) Let M be an oriented n-dimensional smooth manifold and let A be a union of components of ∂M. For any standard handle decomposition of M rel A there exists a *transverse* standard handle decomposition of M rel A with the same number of handles.

Proof of the Transversal Handle Decomposition Proposition 252.8. The proposition follows easily from iteratively applying the Transversality Theorem 50.10, the Handle Attachment Proposition 246.3 (5) and the Modify Handle Attachment Region Lemma 247.8. We leave it to the reader to write down the corresponding lean code. ■

Remark. Let M be a compact oriented smooth manifold that is equipped with a transverse standard handle decomposition rel a union of components of ∂M. It follows from the Handle Chain Complex-Boundary Map Proposition 252.6 together with the Cup Product–Intersection Number Theorem 220.10 that in this case the boundary maps in the handle chain complex can be determined geometrically. In [**DGK2019**] this approach is used to define handle homology without referring to singular homology. Furthermore in [**DGK2019**] it is shown, using the Cerf Theorem 260.15, but again without making use of singular homology groups, that the isomorphism types of these geometrically defined handle homology groups only depend on the diffeomorphism type of the underlying smooth manifold. An outline of the argument is also given in [**GoS1999**, Chapter 4.2].

In fact this approach, together with the Handle Flipping Lemma 248.2, can be used to give an alternative proof of some of the statements of the Poincaré Duality Theorem 210.1. We refer to the Twisted Poincaré Duality Theorem **??** (or alternatively to [**Kos1993**, Theorem VII.5.1] and [**DGK2019**]) for details. □

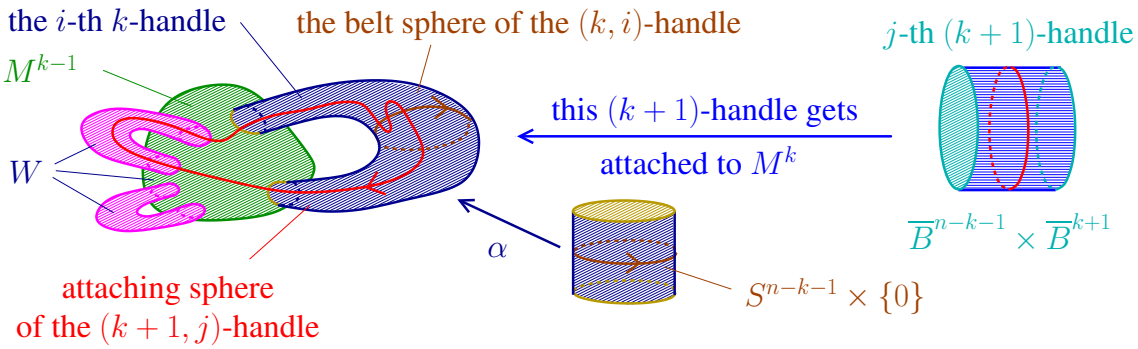
Now we will finally provide the proof of the Handle Chain Complex-Boundary Map Proposition 252.6:

Proof of the Handle Chain Complex-Boundary Map Proposition 252.6. towards the end of the section. First we want to discuss this proposition. Let M be an oriented n-dimensional smooth manifold, let A be a union of components of ∂M and let

$$M \;=\; ([0,1]\times A)\underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

be a standard handle decomposition rel A which has the property that the inclusion maps of all the handles are orientation-preserving. Let $k \in \mathbb{N}_0$, let $j \in \{1, \dots, r_{k+1}\}$ and let $i \in \{1, \dots, r_k\}$. We introduce the following notation:

(1) We denote by W the result of attaching to M^{k-1} all k-handles *except* for the i-th k-handle.
(2) We denote by $\alpha\colon \overline{B}^{n-k} \times \overline{B}^k \to M^k$ the embedding corresponding to the (k,i)-handle. Since the handle decomposition is standard we see that $\alpha(\overline{B}^{n-k} \times S^{k-1}) \subseteq M^{k-1} \subseteq W$. Furthermore note that $\alpha(S^{n-k-1} \times \{0\})$ is the belt sphere of the (k,i)-handle.
(3) Let $\beta\colon \overline{B}^{n-k-1} \times \overline{B}^{k+1} \to M^{k+1}$ be the embedding corresponding to the $(k+1,j)$-handle. Note that $\beta(\{0\} \times S^k)$ is the attaching sphere of the $(k+1,j)$-handle. Since we are dealing with a standard handle decomposition we see that $\beta(\{0\} \times S^k) \subseteq \partial M^k$.



Next we consider the following diagram:

$$
\begin{array}{ccccc}
\mathrm{H}_{k+1}(\{0\}\times\overline{B}^{k+1}, \{0\}\times S^k) & \xrightarrow{\beta_*} & \mathrm{H}_{k+1}(M^{k+1}, M^k) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(M^k, A) \\
\downarrow{\scriptstyle \partial_{k+1}} & & & & \downarrow \\
\mathrm{H}_k(\{0\}\times S^k) & \xrightarrow{\beta_*} & \mathrm{H}_k(M^k, W) & \longleftarrow & \mathrm{H}_k(M^k, M^{k-1}) \\
\downarrow{\scriptstyle \beta_*} & \nearrow & \uparrow{\scriptstyle h_*}\cong \downarrow{\scriptstyle h_*^{-1}} & & \\
\mathrm{H}_k(\partial M^k, \partial M^k \setminus \alpha(S^{k-1}\times B^{n-k})) & & & & \\
\uparrow{\scriptstyle h_*}\cong & & & & \\
\mathrm{H}_k(\alpha(S^{n-k-1}\times\overline{B}^k), \alpha(S^{n-k-1}\times S^{k-1})) & \longrightarrow & \mathrm{H}_k(\alpha(\overline{B}^{n-k}\times\overline{B}^k), \alpha(\overline{B}^{n-k}\times S^{k-1})) & & \\
\uparrow{\scriptstyle \alpha_*}\cong & & \downarrow{\scriptstyle \alpha_*^{-1}} & & \\
\mathrm{H}_k(S^{n-k-1}\times\overline{B}^k, S^{n-k-1}\times S^{k-1}) & \longrightarrow & \mathrm{H}_k(\overline{B}^{n-k}\times\overline{B}^k, \overline{B}^{n-k}\times S^{k-1}) & & \\
\downarrow{\scriptstyle \sigma\mapsto Q([S^{n-k-1}\times\{0\}],\sigma)} & & \downarrow{\scriptstyle \sigma\mapsto \widetilde{Q}([\overline{B}^{n-k}\times\{0\}],\sigma)} & \searrow{\scriptstyle n\cdot[\{0\}\times\overline{B}^k]\mapsto n} & \\
\mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Z}.
\end{array}
$$

We make the following clarifications and observations:

(1) By a slight abuse of notation we denote all of the following inclusion maps by h:

$$
\begin{aligned}
h\colon \alpha(\overline{B}^{n-k} \times \overline{B}^k) &\to M^k \\
h\colon (\alpha(\overline{B}^{n-k} \times \overline{B}^k), \alpha(\overline{B}^{n-k} \times S^{k-1})) &\to (M^k, W) \\
h\colon (\alpha(S^{n-k-1} \times \overline{B}^k), \alpha(S^{n-k-1} \times S^{k-1})) &\to (\partial M^k, \partial M^k \setminus \alpha(S^{k-1} \times B^{n-k})).
\end{aligned}
$$

(2) As in the proof of Handle-Skeleton-H_*-Lemma 252.1 (1) it follows from the Manifold–Excision Theorem 212.4 (which we can apply by the Handle Attachment Proposition 246.3 (3)) that the maps

$$\begin{aligned} h_*\colon \mathrm{H}_k(\alpha(\overline{B}^{n-k}\times\overline{B}^k),\alpha(\overline{B}^{n-k}\times S^{k-1})) &\to \mathrm{H}_k(M^k,W) \quad \text{and}\\ h_*\colon \mathrm{H}_k(\alpha(S^{n-k-1}\times\overline{B}^k),\alpha(S^{n-k-1}\times S^{k-1})) &\to \mathrm{H}_k(\partial M^k,\partial M^k\backslash\alpha(S^{k-1}\times B^{n-k})) \end{aligned}$$

are isomorphisms.

(3) We consider the two asymmetric intersection pairings

$$\begin{aligned} Q\colon \mathrm{FH}_{n-k-1}(S^{n-k-1}\times\overline{B}^k)\times\mathrm{FH}_k(S^{n-k-1}\times\overline{B}^k,S^{n-k-1}\times S^{k-1}) &\to \mathbb{Z}\\ \widetilde{Q}\colon \mathrm{FH}_{n-k}(\overline{B}^{n-k}\times\overline{B}^k,S^{n-k-1}\times\overline{B}^k)\times\mathrm{FH}_k(\overline{B}^{n-k}\times\overline{B}^k,\overline{B}^{n-k}\times S^{k-1}) &\to \mathbb{Z} \end{aligned}$$

from Definition 225.3.

(4) The undecorated maps are the obvious maps induced by inclusions.

(5) It follows from the naturality of the connecting homomorphism that the top rectangle in the diagram commutes.

(6) The regions between the second and the fifth row commute by the functoriality of relative homology groups.

(7) It follows from the Intersection Pairing–Boundary Proposition 225.8 that the bottom left square commutes.

(8) Finally note that it follows from the Poincaré Duality-Products Proposition 210.3 that $\widetilde{Q}([\overline{B}^{n-k}\times\{0\}],[\{0\}\times\overline{B}^k])=1$. It follows from this observation that the bottom right triangle commutes.

We denote by $\Xi\colon \mathrm{H}_{k+1}(\{0\}\times\overline{B}^{k+1},\{0\}\times S^k)\to\mathbb{Z}$ the composition of the blue maps, i.e. the composition of the maps on right-hand path. We can now prove the desired equality:

$$\begin{array}{ll} (i,j)\text{-entry of the matrix representing } \partial^{\mathrm{han}}_{k+1} & \overset{\text{this equality holds basically by definition}}{=} \Xi([\{0\}\times\overline{B}^{k+1}]) \\ \text{intersection form of } S^{n-k-1}\times\overline{B}^k & \text{intersection form of } \alpha(S^{n-k-1}\times\overline{B}^k) \\ \overset{\text{by the commutative diagram and since by the Fundamental Class-of-}\partial\text{-Proposition 194.13 we have } \partial([\{0\}\times\overline{B}^{k+1}])=[\{0\}\times S^k]}{=} Q([S^{n-k-1}\times\{0\}],\alpha_*^{-1}(h_*^{-1}([\beta(\{0\}\times S^k)]))) & \overset{\text{since } \alpha \text{ is an orientation-preserving smooth embedding}}{=} Q([\alpha(S^{n-k-1}\times\{0\})],h_*^{-1}([\beta(\{0\}\times S^k)])) \\ \overset{\text{by the Intersection Form Submanifold Proposition 225.9}}{=} Q_{\partial M^k}([\underbrace{\alpha(S^{n-k-1}\times\{0\})}_{\text{belt sphere}}],[\underbrace{\beta(\{0\}\times S^k)}_{\text{attaching sphere}}]) & = \begin{array}{l}\text{algebraic intersection number of the}\\ \text{belt sphere } \alpha(S^{n-k-1}\times\{0\}) \text{ and}\\ \text{the attaching sphere } \beta(\{0\}\times S^k).\end{array} \end{array}$$

This is precisely the equality that we had intended to show. ∎

252.3. **The algebraic Morse inequalities.** Given a topological space X or a smooth manifold M we will show in this section that the "sizes" of the homology groups give rise to lower bounds on the number of cells of a CW-structure for X and they give rise to a lower bounds on the number of handles in a handle decomposition of M. Furthermore

we will see that the "sizes" of the homology groups give lower bounds on the number of non-degenerate critical points of a Morse function on M.

The key to proving these results is some charming algebra. To formulate the main algebraic statement we need to introduce the following notation:

Notation 252.9. Given a chain complex C_* and given $j \in \mathbb{N}_0$ we introduce the following objects in $\mathbb{N}_0 \cup \{\infty\}$:

$$\begin{array}{rcl} b_j(C_*) & := & \operatorname{rank}(\mathrm{H}_j(C_*)), \\ q_j(C_*) & := & \text{minimal number of generators of the torsion subgroup } \operatorname{Tor} \mathrm{H}_j(C_*). \end{array}$$

Proposition 252.10. (Algebraic Morse Inequalities Proposition) Let C_* be a chain complex[2716] of finitely generated free abelian groups.

(a) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(C_{k-j}) \ \geq \ q_k(C_*) + \sum_{j=0}^{k} (-1)^j \cdot b_{k-j}(C).$$

(b) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\operatorname{rank}(C_k) \ \geq \ q_k(C_*) + q_{k-1}(C_*) + b_k(C_*).$$

Remark.

- (i) Let $k \in \mathbb{N}_0$. If we add Inequality (a) for k to Inequality (a) for $k-1$, then we obtain precisely Inequality (b). In other words, Algebraic Morse Inequalities Proposition 252.10 (b) is a consequence of Algebraic Morse Inequalities Proposition 252.10 (a). In Exercise 252.5 we will see that the converse is not true. Put differently, in general the inequalities one obtains from (a) are stronger than the inequalities from (b).
  - (ii) By adding and subtracting equations one can also show easily that the following two statements are equivalent:
    - (a) For every $k \in \mathbb{N}_0$ the inequality in (a) is an equality.
    - (b) For every $k \in \mathbb{N}_0$ the inequality in (b) is an equality.
- We refer to [**Far2004**, Chapter 1] for an alternative proof of the Algebraic Morse Inequalities Proposition 252.10 and for a generalization to chain complexes over more general rings. □

The proof of the Algebraic Morse Inequalities Proposition 252.10 relies on the following elementary lemma:

Lemma 252.11. (Torsion–Estimate Lemma) Let $F \xrightarrow{\varphi} G \to H \to 0$ be an exact sequence of abelian groups. If F and G are finitely generated free abelian groups, then

$$\operatorname{rank}(F) \ \geq \ \text{minimal number of generators of the torsion subgroup of } H.$$

Proof of Lemma 252.11. We write $m = \operatorname{rank}(F)$ and $n = \operatorname{rank}(G)$. Furthermore we set $k := \min\{m, n\}$. It follows from the Smith Normal Form Theorem 97.22 that we can find bases for F and G such that the only entries of the corresponding $(n \times m)$-matrix A representing φ are on the diagonal. We denote by $a_1, \ldots, a_k$ these diagonal entries. Then

[2716] Note that according to our convention in Definition 143.1 a chain complex "starts" at degree 0.

we see that

$$\text{torsion subgroup of } H \;\cong\; \text{torsion subgroup of } \mathbb{Z}^n/A\mathbb{Z}^m \;\cong\; \bigoplus_{a_i\neq 0} \mathbb{Z}/a_i\mathbb{Z}.$$

Thus we see that the torsion subgroup of H admits a generating set with at most k elements. Since $k \leq m$ we see that the minimal number of generators of the torsion subgroup of H is $\leq \operatorname{rank}(F)$. ■

Proof of the Algebraic Morse Inequalities Proposition 252.10.

(a) Let (C_*, ∂_*) be a chain complex of finitely generated free abelian groups. For each $k \in \mathbb{N}_0$ we write

$$\begin{aligned} Z_k &:= \ker(\partial_k \colon C_k \to C_{k-1}),\\ B_k &:= \operatorname{im}(\partial_{k+1} \colon C_{k+1} \to C_k),\\ H_k &:= Z_k/B_k = \mathrm{H}_k(C_*). \end{aligned}$$

Basically by definition we have for each $k \in \mathbb{N}_0$ the short exact sequences

(i) $0 \to B_k \to Z_k \to H_k \to 0$ and (ii) $0 \to Z_k \to C_k \xrightarrow{\partial_k} B_{k-1} \to 0$.

Given $k \in \mathbb{N}_0$ we perform the following calculation:

$$\begin{aligned} \sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(C_{k-j}) &\overset{\text{follows from the Rank Additivity Proposition 97.18 applied to the above two short exact sequences}}{=} \sum_{j=0}^{k} (-1)^j \cdot (\operatorname{rank}(H_{k-j}) + \operatorname{rank}(B_{k-j}) + \operatorname{rank}(B_{k-j-1}))\\ &= \underset{\substack{\uparrow\\ \text{all the other terms cancel}}}{\operatorname{rank}(B_k) + \sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(H_{k-j})} \;\underset{\substack{\uparrow\\ \text{by the Torsion-Estimate Lemma 252.11 applied to the exact sequence (i), this gives that } \operatorname{rank}(B_k) \geq q_k(C_*)}}{\geq}\; q_k(C_*) + \sum_{j=0}^{k} (-1)^j \cdot \overbrace{\operatorname{rank}(H_{k-j})}^{=b_{k-j}(C)}. \end{aligned}$$

(b) As mentioned above, this inequality follows immediately from adding the inequality (a) for k and $k-1$. Indeed, one sees that almost all of the terms cancel and thus one obtains the promised inequality. ■

252.4. **The topological Morse inequalities.** Evidently the idea now is to apply the Algebraic Morse Inequalities Proposition 252.10 to topology. To do so it is convenient to introduce the following definition:

Notation 252.12. Given a topological space X, given a subspace $A \subseteq X$ and given $j \in \mathbb{N}_0$ we write[2717]

$$b_j(X,A) := b_j(\mathrm{C}_*(X,A)) = \operatorname{rank}(\mathrm{H}_j(X,A)) \quad \text{and} \quad q_j(X,A) := q_j(\mathrm{C}_*(X,A)).$$

In other words, $q_j(X,A)$ is the minimal number of generators of the torsion subgroup $\operatorname{Tor}\mathrm{H}_j(X,A)$.

Proposition 252.13. (Morse Inequalities Proposition)

[2717] In Definition 156.6 we called $b_j(X,A)$ the j-th Betti number of (X,A).

(1) Let M be a compact smooth manifold and let A be a union of components of ∂M. We suppose that M is equipped with a handle decomposition rel A. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in the handle decomposition.
   (a) Given any $k \in \mathbb{N}_0$ we have the inequality
$$\sum_{j=0}^{k} (-1)^j \cdot d_{k-j} \;\geq\; q_k(M, A) + \sum_{j=0}^{k} (-1)^j \cdot b_{k-j}(M, A).$$
   (b) Given any $k \in \mathbb{N}_0$ we have the inequality
$$d_k \;\geq\; q_k(M, A) + q_{k-1}(M, A) + b_k(M, A).$$
(2) Let X be a CW-complex with finitely many cells in each dimension. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-cells. The same inequalities as in (1) hold.

Example. Let $n \in \mathbb{N}_0$. By the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we know that given any $j \in \mathbb{N}_0$ we have
$$\mathrm{H}_j(\mathbb{R}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } j = 0 \text{ or if } j = n \text{ and } n \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } j \text{ is odd and } j < n, \\ 0, & \text{if } j \text{ is even and } j \neq 0 \text{ or if } j > n. \end{cases}$$
It follows almost immediately from this calculation that for any $j \in \{0, \ldots, n\}$ we have $q_j(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) + b_j(\mathbb{R}\mathrm{P}^n) = 1$. By the Morse Inequalities Proposition 252.13 (1b) this implies that any handle decomposition of $\mathbb{R}\mathrm{P}^n$ has at least one handle of each index $0, \ldots, n$. This shows that the handle decomposition of $\mathbb{R}\mathrm{P}^n$ from the Projective Space-Handle Decomposition Proposition 246.12 is minimal in a suitable sense.

Furthermore by the Morse Inequalities Proposition 252.13 (2b) this calculation implies that any CW-decomposition of $\mathbb{R}\mathrm{P}^n$ has at least one cell of each index $0, \ldots, n$. In particular this shows that the CW-structure of $\mathbb{R}\mathrm{P}^n$ that we introduced on page 2628 is "minimal". This observation also gives a solution to Exercise 155.10. □

Proof.

(1) Let M be a compact smooth manifold and let A be a union of components of ∂M. We suppose that M is equipped with a handle decomposition rel A. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in the handle decomposition. It follows from the Standard Handle Decomposition Proposition 247.7 that without loss of generality we can assume that the handle decomposition is standard. This allows us to consider the

handle chain complex $\mathrm{C}_*^{\mathrm{han}}(M,A)$. Now we can prove Inequality (a):

$$\begin{aligned}
\sum_{j=0}^{k}(-1)^j\cdot d_{k-j} &\overset{\text{by the Handle-Skeleton-H}_*\text{-Lemma 252.1 (1) we have } d_i=\mathrm{rank}(\mathrm{C}_i^{\mathrm{han}}(M,A))}{\downarrow\ =}\ \sum_{j=0}^{k}(-1)^j\cdot\mathrm{rank}(\mathrm{C}_{k-j}^{\mathrm{han}}(M,A))\\
&\overset{\text{by the Morse Inequalities Proposition 252.13 (a)}}{\uparrow\ \geq}\ q_k(\mathrm{C}_*^{\mathrm{han}}(M,A))+\sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(\mathrm{C}_*^{\mathrm{han}}(M,A))\\
&\overset{\text{by the Singular-Handle-H}_*\text{-Proposition 252.4 we know that } \mathrm{H}_j(\mathrm{C}_*^{\mathrm{han}}(M,A))\cong\mathrm{H}_j(M,A)}{\uparrow\ =}\ q_k(M,A)+\sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(M,A).
\end{aligned}$$

Finally, as we pointed out on page 4884, Inequality (b) is an immediate consequence of Inequality (a). Alternatively one can just replace Morse Inequalities Proposition 252.13 (a) by the Morse Inequalities Proposition 252.13 (b).

(2) Let X be a CW-complex with finitely many cells in each dimension. The proof in this setting is almost identical to the proof in (1), we just need to work with the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and we make use of the fact that by the CW-Skeleton-H$_*$-Lemma 155.2 (1) we know that $d_i=\mathrm{rank}(\mathrm{C}_i^{\mathrm{CW}}(X))$ and we use that by the Singular-Cellular-H$_*$-Proposition 155.8 we know that $\mathrm{H}_j(\mathrm{C}_*^{\mathrm{CW}}(X))\cong\mathrm{H}_j(X)$. ■

Now we can formulate what is arguably the most interesting result of this section, since it connects two objects, which initially look like they are rather far removed in the mathematical universe.

Theorem 252.14. (Critical Points–Morse Inequalities Theorem) Let M be a compact smooth manifold, let A be a union of components of ∂M and $f\colon M\to\mathbb{R}$ be a Morse function rel A. Given $k\in\mathbb{N}_0$ we write

$$d_k\ :=\ \text{number of critical points of index } k.$$

The following two inequalities hold:

(a) Given any $k\in\mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k}(-1)^j\cdot d_{k-j}\ \geq\ q_k(M,A)+\sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(M,A).$$

(b) Given any $k\in\mathbb{N}_0$ we have the inequality

$$d_k\ \geq\ q_k(M,A)+q_{k-1}(M,A)+b_k(M,A).$$

Proof. Let M be a compact smooth manifold, let A be a union of components of ∂M and $f\colon M\to\mathbb{R}$ be a Morse function rel A. It follows from the Morse Function–Handle Decomposition Theorem 250.6 that M admits a handle decomposition rel A such that for each $k\in\mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k. The inequalities now follow from the inequalities of the Morse Inequalities Proposition 252.13 (1). ■

Examples. Let $n \in \mathbb{N}_0$.

(1) Recall that on page 4886 we just showed that for every $j \in \{0, \dots, n\}$ we have the equality $q_j(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) + b_j(\mathbb{R}\mathrm{P}^n) = 1$. By the Critical Points–Morse Inequalities Theorem 252.14 (b) this implies that any Morse function on $\mathbb{R}\mathrm{P}^n$ has at least one critical point of each index $0, \dots, n$. In particular the Morse function on $\mathbb{R}\mathrm{P}^n$ that we gave in the Projective Space–Morse Function Proposition 249.15 has the minimal number of critical points.

(2) In the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 we showed that given any $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } j = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

As in (1) it follows that the Morse function on $\mathbb{C}\mathrm{P}^n$ that we gave in the Projective Space–Morse Function Proposition 249.15 has the minimal number of critical points.

(3) Let M be a closed connected non-empty n-dimensional smooth manifold. By the Manifold-Top-H_*-Theorem 183.18, or alternatively by the Orientation-Top Homology-Theorem 193.3 and the Manifold-H_{n-1}-Theorem 193.4 we know the following:
(a) If M is orientable, then $\mathrm{H}_n(M) \cong \mathbb{Z}$ and $\mathrm{Tor}(\mathrm{H}_{n-1}(M)) = 0$.
(b) If M is non-orientable, then $\mathrm{H}_n(M) = 0$ and $\mathrm{Tor}(\mathrm{H}_{n-1}(M)) \cong \mathbb{Z}/2\mathbb{Z}$.
Rather charmingly, in either case we see that $q_n(M) + q_{n-1}(M) + b_n(M) = 1$, which matches the fact that any Morse function has to have at least one global maximum, i.e. one critical point of index n. □

We conclude this section with a de Rham cohomological variation on the previous results:

Proposition 252.15. (De Rham-Morse Inequalities Proposition) Let M be a closed smooth manifold. Given $k \in \mathbb{N}_0$ we denote by $\mathrm{H}^k_{\mathrm{dR}}(M)$ the k-th de Rham cohomology group of M from Definition 78.2. Let $f\colon M \to \mathbb{R}$ be a Morse function. Given $j \in \mathbb{N}_0$ we denote by d_j the number of critical points of f of index j.

(a) Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k} (-1)^j \cdot d_{k-j} \;\geq\; \sum_{j=0}^{k} (-1)^j \cdot \dim_{\mathbb{R}}(\mathrm{H}^{k-j}_{\mathrm{dR}}(M)).$$

(b) Given any $k \in \mathbb{N}_0$ we have the inequality

$$d_k \;\geq\; \dim_R(\mathrm{H}^k_{\mathrm{dR}}(M)).$$

Remark. This inequality can also proved directly using purely analytic methods, we refer to [**Chan1993**, Appendix] and [**Wit1982**] for details. □

Proof. First note that we obtain Inequality (b) by adding Inequality (a) for k and $k-1$. Thus it remains to prove Inequality (a). Given $k \in \mathbb{N}_0$ we see that

$$\begin{aligned}
\sum_{j=0}^{k} (-1)^j \cdot d_{k-j} &\geq \sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(\mathrm{H}_{k-j}(M)) \\
&\quad \uparrow \text{ by the Morse Inequalities Theorem 252.14} \\
&= \sum_{j=0}^{k} (-1)^j \cdot \dim_{\mathbb{R}}(\mathrm{H}_{k-j}(M;\mathbb{R})) \\
&\quad \uparrow \text{ by the Torsion–Free Coefficients Proposition 161.8} \\
&= \sum_{j=0}^{k} (-1)^j \cdot \dim_{\mathbb{R}}(\mathrm{H}^{k-j}(M;\mathbb{R})) \\
&\quad \uparrow \text{ follows from the Universal Coefficient-H}^*\text{–Field Theorem 202.2} \\
&= \sum_{j=0}^{k} (-1)^j \cdot \dim_{\mathbb{R}}(\mathrm{Hom}(\mathrm{H}_{k-j}(M;\mathbb{R}),\mathbb{R})) \\
&\quad \uparrow \text{ since by the Smooth Manifolds-Invariants Proposition 179.10 we know that } \mathrm{H}_{k-j}(M;\mathbb{R}) \text{ is finite-dimensional} \\
&= \sum_{j=0}^{k} (-1)^j \cdot \dim_{\mathbb{R}}(\mathrm{H}_{\mathrm{dR}}^{k-j}(M)). \\
&\quad \uparrow \text{ by the de Rham Isomorphism Theorem 231.16}
\end{aligned}$$

■

252.5. **Perfect Morse functions.** The question arises how "sharp" the Morse inequalities are. This leads us to the following definition:

Definition 252.16. Let M be a closed smooth manifold.

(1) A Morse function $f\colon M \to \mathbb{R}$ is called perfect if for every $k \in \mathbb{N}_0$ we have

$$\text{number of critical points of index } k \;=\; q_k(M) + q_{k-1}(M) + b_k(M).$$

(2) A handle decomposition is called minimal if for every $k \in \mathbb{N}_0$ we have

$$\text{number of handles of index } k \;=\; q_k(M) + q_{k-1}(M) + b_k(M).$$

Remark. Let M be a closed smooth manifold.

(1) Recall that by the Morse Function–Handle Decomposition Theorem 250.6 and the Handle Decomposition–Morse Function Theorem 251.2 we can go back and forth between Morse functions and handle decompositions. In particular we see that M admits a perfect Morse function if and only if M admits a minimal handle decomposition.

(2) Note that by the remark on page 4884 we know that if a Morse function is perfect, then the inequalities stated in the Critical Points–Morse Inequalities Theorem 252.14 (a) are also equalities. The same logic applies to the inequalities on the number of handles stated in the Morse Inequalities Proposition 252.13 (1a). □

It is now natural to ask which closed smooth manifolds admit perfect Morse functions. It follows from the 1-Handle–π_1–Inequality Lemma 247.9 and the $(n-1)$–π_1–Handle Inequality Lemma 248.3 that the fundamental group gives additional lower bounds on the number of critical points of a Morse function. Thus in the following discussion we restrict ourselves to the case of simply connected smooth manifolds. So now we have a reasonable question:

Question 252.17. Does every closed simply connected smooth manifold admit a perfect Morse function?

In fact we have the following theorem which gives an affirmative answer to Question 252.17 for simply connected high-dimensional smooth manifolds.

Theorem 252.18. (Perfect Morse Function-Existence Theorem) Let M be a closed smooth manifold. If M is simply connected and if $\dim(M) \geq 6$, then M admits a perfect Morse function.

Remark.

(1) Note that the Perfect Morse Function-Existence Theorem 252.18, together with the Ball-Glued-to-Ball Proposition 250.11, implies that every homotopy sphere of dimension ≥ 6 is homeomorphic to S^n.
(2) In [**Shark1993**, Chapter VII] the above Perfect Morse Function-Existence Theorem 252.18 gets generalized, in a suitable sense, to certain closed non-simply connected smooth manifolds. □

Proof. This theorem was first proved by Steven Smale [**Sma1962**, Theorem 6.1] in 1962. In the Morse Inequalities–Realization Proposition 263.22 we will formulate a closely related result. In Exercise 263.5 we will see how one can get from the Morse Inequalities–Realization Proposition 263.22 to the Perfect Morse Function-Existence Theorem 252.18. ■

Thus let us turn our gaze towards the low-dimensional case. It follows from the Surface Classification Theorem 101.16, and the resolution of the Poincaré Conjecture in dimension 3, see the Poincaré 3D-Conjecture Theorem **??**, that Question 252.17 has an affirmative answer in dimensions 2 and 3 for the silly reason that the spheres are the only closed simply connected smooth manifolds in those dimensions. Furthermore for 5-dimensional smooth manifolds it follows from the classification of closed simply connected 5-dimensional smooth manifolds given by Denis Barden in 1965, see [**Bard1965**, Corollary 2.2.2], that Question 252.17 has an affirmative answer.

Finally we turn to the 4-dimensional setting. Since low-dimensional topologists usually prefer to think about handles we switch from Morse functions to handles. The following lemma gives a condition for a handle decomposition of a closed orientable *simply connected* 4-dimensional smooth manifold to be minimal.

Lemma 252.19. (Minimal 4D-Handle Decomposition Lemma) Let M be a closed orientable simply connected 4-dimensional smooth manifold. A handle decomposition is minimal if and only if it has no 1-handles and no 3-handles.

Proof. Let M be a closed orientable simply connected 4-dimensional smooth manifold. Note that in Exercise 210.13 we saw that it follows easily from the Hurewicz Theorem 166.10, the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16, the Poincaré Duality Theorem 210.1 and the Universal Coefficient-H*-Theorem 201.23 that $\mathrm{H}_1(M) = 0$, that $\mathrm{H}_3(M) = 0$ and that $\mathrm{H}_2(M)$ is torsion-free. In particular we see that $q_k(M) + q_{k-1}(M) + b_k(M) = 0$ for $k = 1$ and $k = 3$.

Now we turn to the actual proof of the lemma. First note that the above calculation shows that a minimal handle decomposition has no 1-handles and no 3-handles. Conversely, suppose that we have a handle decomposition with no 1-handles and no 3-handles. It follows easily from the Handle Attachment-π_0-Lemma 246.5 that the handle decomposition contains precisely one 0-handle and, say by considering the dual handle decomposition, we

see that the handle decomposition contains precisely one 4-handle. Finally note that

$$\#2\text{-handles} \underset{\substack{\uparrow \\ \text{by the Handle Attachment} \\ \chi\text{-Lemma 246.15}}}{=} \chi(M) - 2 \underset{\substack{\uparrow \\ \text{since } \mathrm{H}_1(M)=\mathrm{H}_3(M)=0 \\ \text{and since } \mathrm{H}_0(M)=\mathrm{H}_4(M)\cong\mathbb{Z}}}{=} b_2(M) \underset{\substack{\uparrow \\ \text{since } q_1(M)=q_2(M)=0}}{=} q_2(M) + q_1(M) + b_2(M).$$

We have thus shown that the handle decomposition is minimal. ■

This leads us to the following open problem, see [**Kir1997**, Propblem 4.18].

Question 252.20.

(1) Does every closed simply connected 4-dimensional smooth manifold admit a handle decomposition without 1-handles?

(2) Does every closed simply connected 4-dimensional smooth manifold admit a handle decomposition without 1- and 3-handles?

Note that a positive answer to Question 252.20 (2) would imply, by Reeb's Theorem 250.10 that every closed simply connected 4-dimensional smooth manifold M with $\mathrm{H}_2(M) = 0$ is homeomorphic to S^4. By the Topological 4D-Poincaré Theorem 228.14 this statement is correct. But this indicates that in all likelihood it will be very difficult to show that Question 252.20 (2) has a positive answer.

252.6. Exercises for Chapter 252.

Exercise 252.1. Let M be a compact smooth manifold and let A be a union of components of ∂M. We suppose that M is equipped with a standard handle decomposition rel A. Given $k \in \mathbb{N}_0$ consider the map

$$\begin{tikzcd} \mathrm{H}_k(M^k, M^{k-1}) \arrow[r, "\partial_k"] \arrow[rr, bend right, "=:\widetilde{d}"'] & \mathrm{H}_{k-1}(M^{k-1}, \varnothing) \arrow[r, "j_{k-1}"] & \mathrm{H}_{k-1}(M^{k-1}, M^{k-2}) \end{tikzcd}$$

where ∂_k is the connecting map in the long exact sequence of the triple $(M^k, M^{k-1}, \varnothing)$ and where j_{k-1} is the map induced by the natural map $(M^{k-1}, \varnothing) \to (M^{k-1}, M^{k-2})$ of pairs of topological spaces.

(a) Does this map $\widetilde{d}$ agree with the map $\partial_k^{\mathrm{han}}$ defined in the Handle Boundary Map Lemma 252.2, where instead of $\varnothing$ we wrote A?

(b) If the answer in (a) is yes, why did we pick the definition in the Handle Boundary Map Lemma 252.2?

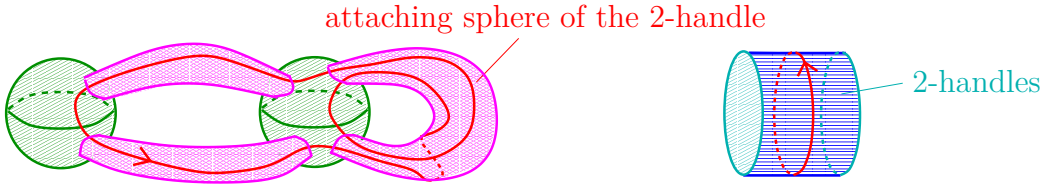
(c) If the answer in (a) is no, does there exist an example, such that the homology groups are different?

Exercise 252.2. Let M be a closed smooth manifold that is equipped with a handle decomposition. Let $n \in \mathbb{N}_0$. We consider $\overline{B}^n \times M$ which we equip with the product handle decomposition (where we equip $\overline{B}^n$ with the handle decomposition consisting of a single n-handle). Show that the handle chain complexes of M and $\overline{B}^n \times M$ are isomorphic.

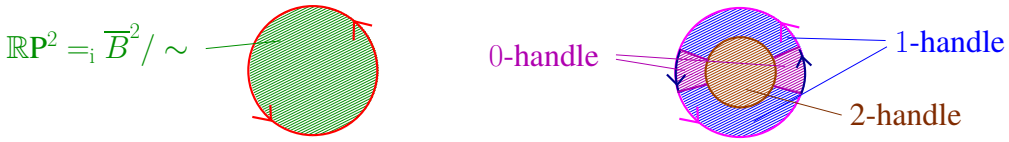
Exercise 252.3. Consider the 3-dimensional smooth manifold that is sketched in the figure below.

(a) Determine $\pi_1(M)$.

(b) Make an educated guess what well-known smooth manifold M is diffeomorphic to.



Exercise 252.4. In the figure below we consider a handle decomposition of the real projective space $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ with the handle decomposition. Determine the corresponding handle chain complex.
Remark. A priori the Handle Chain Complex-Boundary Map Proposition 252.6 does not apply since $\mathbb{R}\mathrm{P}^2$ is not orientable. A slight modification of the Handle Chain Complex-Boundary Map Proposition 252.6 says that if the belt and attaching spheres intersect transversally, then it suffices to orient the handles and the statement still makes sense.



Exercise 252.5. Give an example of a chain complex (C_*, ∂_*) of finitely generated free abelian groups where the inequalities of the Algebraic Morse Inequalities Proposition 252.10 (a) give a stronger lower bound on the rank of the groups C_*, than the inequalities of the Algebraic Morse Inequalities Proposition 252.10 (b).

Exercise 252.6. Given a closed connected non-empty smooth manifold M and given a Morse function $f\colon M \to \mathbb{R}$ we denote by $d_k(M, f)$ the number of critical points of f of index k. From the Critical Points–Morse Inequalities Theorem 252.14 and from the 1-Handle–π_1–Inequality Lemma 247.9 and the $(n-1)$–π_1–Handle Inequality Lemma 248.3 we obtain lower bounds on the number of critical points $d_k(M, f)$ of a Morse function f in terms of the homology groups and the fundamental group of M. Show, given any $k \in \mathbb{N}$, that these lower bounds are not optimal, i.e. show that there exists a closed connected non-empty smooth manifold M that does not admit a Morse function $f\colon M \to \mathbb{R}$ such that these lower bounds for k are equalities.
Hint. Apply the Critical Points–Morse Inequalities Theorem 252.14 to finite covers of a suitable M. For example you could take M to be the mapping torus $\mathrm{Tor}((S^1)^m, \varphi)$ for a suitable diffeomorphism of a high-dimensional torus $(S^1)^m$.

Exercise 252.7. Let M be a smooth homology 3-sphere, i.e. let M be a 3-dimensional smooth manifold such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(S^3;\mathbb{Z})$. Show that if $\pi_1(M)$ is non-abelian, then every Morse function on M has at least six critical points.
Remark. In the Poincaré Homology Sphere Proposition 183.26 we saw that the Poincaré Homology Sphere (which we introduced in Definition 183.27) is an example of such M.

Exercise 252.8. Let M be a closed smooth homology 5-sphere, i.e. let M be a 5-dimensional smooth manifold such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(S^5;\mathbb{Z})$. Show that if $\pi_1(M)$ is non-abelian, then every Morse function on M has at least ten critical

points.

Remark. In Exercise 252.7 we dealt with the case of smooth homology 3-spheres. Furthermore note that in Exercise 183.17 and in Homology Sphere-Existence Proposition 256.9 we give examples of such M.

Exercise 252.9. Let M be a compact non-empty n-dimensional smooth manifold. Show that given any $k \in \mathbb{N}_0$ there exists a handle decomposition for M such that for each $i \in \{0, \dots, n\}$ the number of i-handles is at least k.

Exercise 252.10. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \qquad \begin{array}{c}\text{and we consider the orientation-}\\ \text{reversing diffeomorphism}\end{array} \qquad \begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

Let $\theta\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$ be the usual diffeomorphism that is given by $[(x,y)] \mapsto (e^{2\pi \mathrm{i} x}, e^{2\pi \mathrm{i} y})$. We set

$$M \coloneqq ((\overline{B}^2 \times S^1) \sqcup (\overline{B}^2 \times S^1))/\sim \qquad \text{where } (x,y) \sim (\theta \circ \varphi \circ \theta^{-1})(x,y) \text{ for } x, y \in S^1 \times S^1.$$

By the Gluing-Smooth Manifolds-Proposition 54.5 we know that M is a closed 3-dimensional smooth manifold, let A be a union of components of ∂M.

(a) Show that M admits a handle decomposition with a unique handle in dimensions 0,1,2 and 3.

(b) Show that for every $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{han}}(M) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n = 1. \end{cases}$$

Remark. In the Lens Space–Heegaard Splitting Proposition **??** we will see that M is diffeomorphic to the lens space $L(p,q)$. Thus this exercise together with the Singular-Handle-H_*-Proposition 252.4 gives us a new calculation of the singular homology groups of lens spaces.

Exercise 252.11. Let $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. We consider the diffeomorphism

$$\begin{array}{rcl} \varphi\colon S^1 \times S^1 & \to & S^1 \times S^1 \\ (x,y) & \mapsto & (x^a \cdot y^b, x^c \cdot y^d). \end{array}$$

Show that for the smooth manifold $M_\varphi \coloneqq (\overline{B}^2 \times S^1) \cup_\varphi (\overline{B}^2 \to S^1)$ there exists a Morse function $f\colon M_\varphi \to \mathbb{R}$ such that the inequalities from the Critical Points–Morse Inequalities Theorem 252.14 are in fact equalities.

Exercise 252.12. Let $g, n \in \mathbb{N}_0$. As usual we denote by $\Sigma_{g,n}$ the surface of genus g minus n open disks. What is the minimal number of handles in a handle decomposition of $\Sigma_{g,n}$?

Exercise 252.13. Given a finitely generated abelian group A we denote by $d(A)$ the minimal size of a generating set of A. Show that for every finitely generated abelian group A we have the equality

$$d(A) = d(\mathrm{Tor}(A)) + d(A/\mathrm{Tor}(A)).$$

Exercise 252.14. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function such that all critical points are of even index. Show that there is no Morse function with fewer critical points.

Exercise 252.15. Does there exist a smooth function $S^1 \times S^1 \to \mathbb{R}$ with three critical points?

Exercise 252.16. Let $n \in \mathbb{N}$. What is the minimal number k of critical points of a Morse function on the n-dimensional torus $(S^1)^n$? You need to show that there exists a Morse function such that the number of critical points equals k and you need to show that every Morse function has at least k critical points.

Exercise 252.17. Let $n \in \mathbb{N}$. In the Matrix Group-Smooth Manifold Proposition 32.10 we showed that $\mathrm{SO}(n)$ is a closed smooth manifold of dimension $\frac{1}{2}n \cdot (n-1)$. In the following let $1 < c_1 < \cdots < c_n$ be real numbers. On page 4810 we mentioned that the map

$$\begin{aligned} \mathrm{SO}(n) &\to \mathbb{R} \\ A = (a_{ij}) &\mapsto \sum_{j=1}^{n} c_j \cdot a_{jj} \end{aligned}$$

is a Morse function and that the critical points are precisely the diagonal matrices in $\mathrm{SO}(n)$.

(a) Show that $\mathrm{H}_*(\mathrm{SO}(n))$ is generated by at most 2^{n-1} elements.

(b) For which $n \in \mathbb{N}$ is 2^{n-1} the minimal size of a generating set?
Hint. Consider $\pi_1(\mathrm{SO}(n))$.

Exercise 252.18. Let M and N be closed smooth manifolds. Furthermore let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions that are perfect in the sense of Definition 252.16. By the Product–Morse Function Proposition 249.17 we know that the map

$$\begin{aligned} \Xi\colon M \times N &\to \mathbb{R} \\ (x, y) &\mapsto f(x) + g(y) \end{aligned}$$

is a Morse function. Is this Morse function perfect?

Exercise 252.19. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Given $k \in \mathbb{N}_0$ we denote by d_k the number of critical points of index k. The Morse function is called *lacunary* if for every $k \in \mathbb{N}_0$ at least one of d_k and d_{k+1} is zero. Show that every lacunary Morse function is perfect.
Remark. This statement is sometimes called the *Morse lacunary principle*.

CHAPTER 253

Cobordisms with vertical boundary

In this chapter we will extend some of the results from the earlier chapters to "smooth manifolds with vertical boundary", these are smooth manifolds with corner, where the boundary has a suitable decomposition $\partial M = \partial_- M \cup \partial_+ M \cup \partial_{\mathrm{v}} M$ such that the "vertical boundary" $\partial_{\mathrm{v}} M$ is in fact a product. This extension is a little technical, but the need for such an extension arises quite naturally in Chapter 291 in the context of knot theory. At a first reading it might be best to give this chapter at most a cursory look.

This chapter is organized as follows:

- In Section 253.1 we will introduce the notion of a cobordism with vertical boundary.
- In Section 253.2 we will introduce the notion of a handle decompositions of a cobordism with vertical boundary.
- In Section 253.3 we will introduce Morse functions on smooth manifolds with vertical boundary and we will show that such Morse functions always exist.
- In Section 253.4 we will use Morse functions to show that every cobordism with vertical boundary admits a handle decomposition.
- In Section 253.5 we will provide exercises for Chapter 253.

In the next Chapter 254 we will see that handle decompositions give rise to CW-complexes.

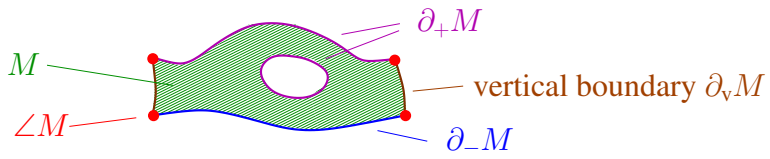
253.1. **Definition of a cobordism with vertical boundary.** In this chapter we will return to the popular topic of a smooth manifold M with corner that we introduced in Definition 26.9. Recall that in Definition 26.4 we introduced the boundary ∂M and the corner $\angle M$ of a smooth manifold with corner. As our first task we have to introduce the main concept of this chapter:

Definition 253.1. An n-dimensional cobordism with vertical boundary is a smooth manifold M with corner together with a decomposition $\partial M = \partial_- M \cup \partial_+ M \cup \partial_{\mathrm{v}} M$ such that the following statements hold:

(1) Each of $\partial_- M$, $\partial_+ M$ and $\partial_{\mathrm{v}} M$ is a smooth boundary in the sense of Definition 59.4.
(2) $\partial_- M \cap \partial_+ M = \varnothing$.
(3) $\angle M = (\partial_- M \cap \partial_{\mathrm{v}} M) \sqcup (\partial_+ M \cap \partial_{\mathrm{v}} M)$.
(4) There exists an $(n-2)$-dimensional smooth manifold C with empty boundary and a diffeomorphism $\Xi\colon [0,1] \times C \to \partial_{\mathrm{v}} M$ such that $\Xi_0(C) = \partial(\partial_-(M))$ and such that $\Xi_1(C) = \partial(\partial_+(M))$.

We refer to $\partial_{\mathrm{v}} M$ as the vertical boundary and we refer to $\partial_{\mathrm{h}} M := \partial_- M \cup \partial_+ M$ as the horizontal boundary of M.

Example.



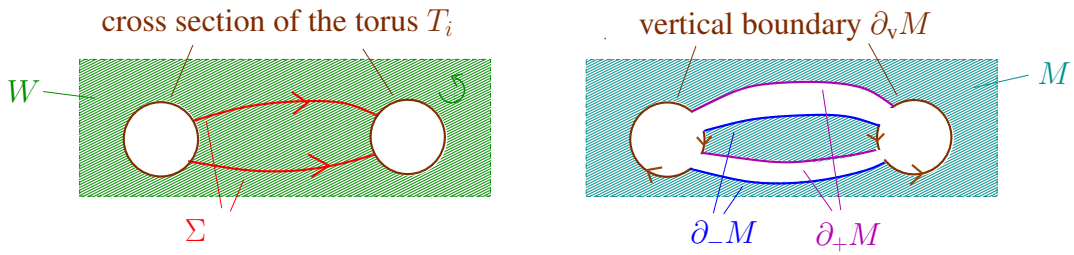
(1) Let Σ be an $(n-1)$-dimensional smooth manifold. Clearly $M := [0,1] \times \Sigma$ with $\partial_- M := \{0\} \times \Sigma$, $\partial_+ M := \{1\} \times \Sigma$, and $\partial_\mathrm{v} M := [0,1] \times \partial\Sigma$ is an n-dimensional cobordism with vertical boundary.
(2) Let M be a smooth manifold with $\angle M = \varnothing$ and let A_- be a union of components of ∂M. We set $A_+ = \partial M \setminus A_-$. Clearly M with the decomposition $\partial M = A_- \cup A_+ \cup \varnothing$ is a cobordism with vertical boundary $\partial_\mathrm{v} M = \varnothing$. □

The following lemma will be our main source of "interesting" smooth manifolds with vertical boundary:

Lemma 253.2. (Cut Along Surface–Vertical Boundary Lemma) Let W be a compact oriented 3-dimensional smooth manifold such that $\partial W = T_1 \cup \cdots \cup T_k$ is a union of tori. Let $\Sigma \subseteq W$ be an oriented proper codimension-one submanifold such that for each $i \in \{1, \dots, k\}$ the following holds:

(∗) The intersection $\partial\Sigma \cap T_i$ consists of a union of curves $C_1, \dots, C_{m_i}$ which (if we equip them with the boundary orientation coming from Σ) represent the same non-zero homology class in $\mathrm{H}_1(T_i)$.

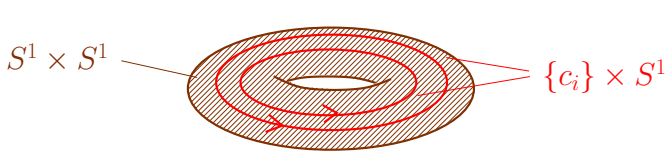
We consider $M := W \setminus\setminus \Sigma$, i.e. we cut M along Σ as in Definition 59.18. We also consider $\partial_- M := M_-$, $\partial_+ M := M_+$ and $\partial_\mathrm{v} M := \partial_* M$ from Definition 59.18. This is a 3-dimensional cobordism with vertical boundary.



Sketch of proof. First note that it follows easily from our hypothesis, the Curves-on-Surfaces-Classification Theorem 107.7 and the Curves-on-Torus Lemma 234.25 (1) (see also Exercise 107.6) that for each $i \in \{1, \dots, k\}$ there exists a diffeomorphism $\Phi_i \colon S^1 \times S^1 \to T_i$ and $c_1, \dots, c_{m_i} \in S^1$ such that $\Sigma \cap T_i = \Phi(\{c_1\} \times S^1) \cup \cdots \cup \Phi(\{c_{m_i}\} \times S^1)$ as *oriented* curves.

[2716] It follows from the orientation hypothesis and the observation on page 1344 that $\Sigma \subseteq W$ is two-sided. Thus $W \setminus\setminus \Sigma$ is indeed defined.

[2717] By the Bicollar Neighborhood Theorem 59.16 there exists a bicollar $\beta \colon [-1,1] \times \Sigma \to W$ in the sense of Definition 59.15. In Exercise 59.15 we showed that M is diffeomorphic (as a cobordism with vertical boundary) to $M' := W \setminus \beta((-1,1) \times \Sigma)$ with $\partial_- M' := \beta(\{-1\} \times \Sigma)$, $\partial_+ M' := \beta(\{1\} \times \Sigma)$ and with $\partial_\mathrm{v} M :' = \partial W \setminus \beta((-1,1) \times \Sigma)$. In illustrations we usually show $W \setminus \beta((-1,1) \times \Sigma)$ since this is easier to draw.



Now the lemma can be proved easily from the definitions, a suitable choice of the bicollar on the boundary, Exercise 41.3 and the Cutting Manifold Proposition 59.19. We leave it to the reader to connect these dots. ■

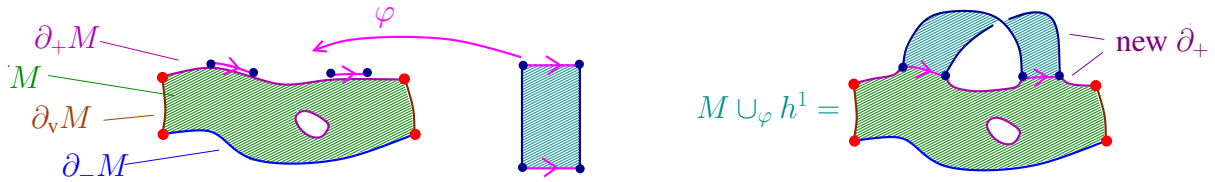
253.2. **Handle decomposition of a cobordism with vertical boundary.** Our main goal of this chapter is to show that cobordisms with vertical boundary admit a handle decomposition. To make sense of this statement we first need to explain what it means to attach a handle to a cobordism with vertical boundary:

Definition 253.3. Let M be an n-dimensional cobordism with vertical boundary. Suppose we are given a thickened $(k-1)$-sphere[2718] $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial_+ M \setminus \angle M$. As in the Handle Attachment Proposition 246.3 and as in Definition 246.4 we set

$$M \cup_\varphi h^k \;:=\; M \cup_{\partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \big(\overline{B}^{n-k} \times \overline{B}^k\big).$$

One can easily verify that we can view $M \cup_\varphi h^k$ as a cobordism with vertical boundary where

$$\begin{aligned} \partial_-(M \cup_\varphi h^k) &:= \partial_- M, \\ \partial_+(M \cup_\varphi h^k) &:= \partial_+ M \setminus \varphi(B^{n-k} \times S^{k-1}) \cup (S^{n-k-1} \times \overline{B}^k), \text{[2719]} \\ \partial_{\mathrm{v}}(M \cup_\varphi h^k) &:= \partial_{\mathrm{v}} M. \end{aligned}$$



The following definition is a variation on Definition 246.11 and Definition 247.1:

Definition 253.4. Let M be a compact n-dimensional cobordism with vertical boundary. A handle decomposition for M[2720] is a diffeomorphism

$$\Theta\colon M \xrightarrow{\cong} ([0,1] \times \partial_- M) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

such that for each $P \in \partial_- M$ we have $\Theta(P) = [(0, P)]$ and such that for $i = 0, \ldots, m$ the map φ_i is a thickened $(k_i - 1)$-sphere that takes values in the interior of

$$\underbrace{\partial_+(([0,1] \times \partial_- M) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}})}_{\text{defined iteratively in Definition 253.3}}.$$

[2718] By the Smooth Manifold–Boundary Proposition 59.5 we know that $\partial_+ M$ is naturally a smooth manifold with boundary $\partial \partial_+ M = \angle M$. Thus it makes sense to say that a map $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial_+ M \setminus \angle M$ is a smooth embedding, i.e. it makes sense to talk of a thickened sphere as in Definition 246.2.

[2719] In the language of **??**, $\partial_+(M \cup_\varphi h^k)$ is obtain from $\partial_+ M$ by surgery along φ.

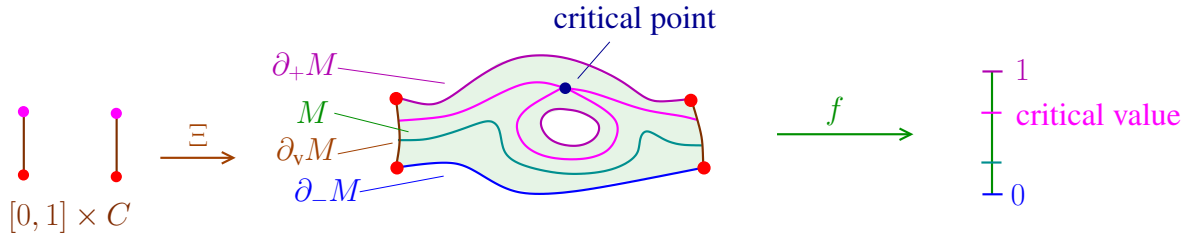
[2720] If $\partial_{\mathrm{v}} = \varnothing$, then this is the same as a handle decomposition rel $\partial_- M$ in the sense of Definition 247.1.

Our goal is to show that such handle decompositions always exist. As before the key to showing the existence of handle decomposition is given by Morse functions. In our current context these will be introduced in the following section.

253.3. **Morse functions on smooth manifolds with vertical boundary.** The following definition is a variation on Definition 249.14 and Definition 249.16:

Definition 253.5. Let M be an n-dimensional combordism with vertical boundary where the decomposition is given by $\partial M = \partial_- M \cup \partial_+ M \cup \partial_{\mathrm{v}} M$. A smooth function $f\colon M \to [0,1]$ is called a Morse function for the combordism with vertical boundary if the following statements holds:

(1) All critical points of f are non-degenerate.
(2) There is no critical point on ∂M.
(3) • We have $\partial_- M = f^{-1}(0)$.
    • We have $\partial_+ M = f^{-1}(1)$.
(4) There exists a compact $(n-2)$-dimensional smooth manifold C and a diffeomorphism $\Xi\colon [0,1] \times C \to \partial_{\mathrm{v}} M$ as in Definition 253.1 such that for all $(t,c) \in [0,1] \times C$ we have $f(\Xi(t,c)) = c$.



The following is a generalization of the Morse Function-Existence Theorem 249.26:

Theorem 253.6. (Morse Function-Existence Theorem) Every compact combordism with vertical boundary admits a Morse function in the sense of Definition 253.5.

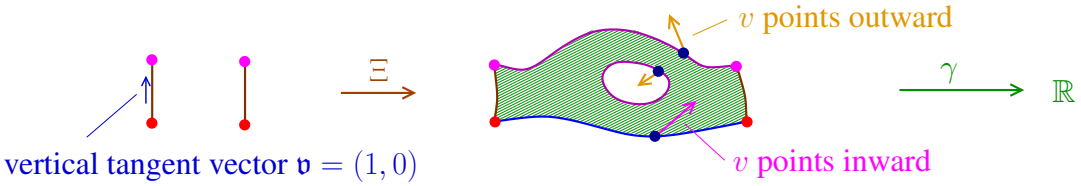
Remark. An alternative proof for this theorem is given in [**Wall2016**, Lemma 5.1.1]. □

Sketch of proof. [2721] Let M be a compact cobordism with vertical boundary, let C be a closed $(n-2)$-dimensional smooth manifold and let $\Xi\colon [0,1]\times C \to \partial_{\mathrm{v}} M$ be a diffeomorphism such that $\Xi_0(C) = \partial(\partial_-(M))$ and such that $\Xi_1(C) = \partial(\partial_+(M))$. We start out with a few preparations:

(a) By the Smooth Manifold–Boundary Proposition 59.5 together with the Smooth Collar Neighborhood Theorem 59.7 we know that the cobordism with vertical boundary admits a smooth collar $\chi\colon [0,1] \times \partial_{\mathrm{h}} M \to M$ for the horizontal boundary $\partial_{\mathrm{h}} M = \partial_- M \cup \partial_+$ and also a smooth collar $\lambda\colon [0,1] \times \partial_{\mathrm{v}} M \to M$ for the vertical boundary, both in the sense of Definition 59.6.
(b) Let $\gamma\colon M \to \mathbb{R}$ be a smooth function.
    • As in Definition 249.28 we say γ points inward on $\partial_- M$ if for every $P \in M_0$ and every inward pointing vector $v \in \mathrm{T}_P M$ we have $\mathrm{D}\gamma_P(v) > 0$.
    • Similarly we define the notion γ points outward on $\partial_+ M$.

[2721] In the following follow the logic of the proof of the Morse Function-Existence Theorem 249.26, we will just out a few steps where one needs to fine-tune the argument.

- We say γ points along $\partial_{\mathrm{v}} M$ if for every $(t, P) \in [0,1] \times C$ and for the "vertical tangent vector" $\mathfrak{v} = (1,0) \in \mathrm{T}_{(t,P)}([0,1] \times C) =_{\mathrm{i}} \mathbb{R} \oplus \mathrm{T}_P C$ we have $\mathrm{D}(\gamma \circ \Xi)_{(t,P)}(\mathfrak{v}) > 0$.



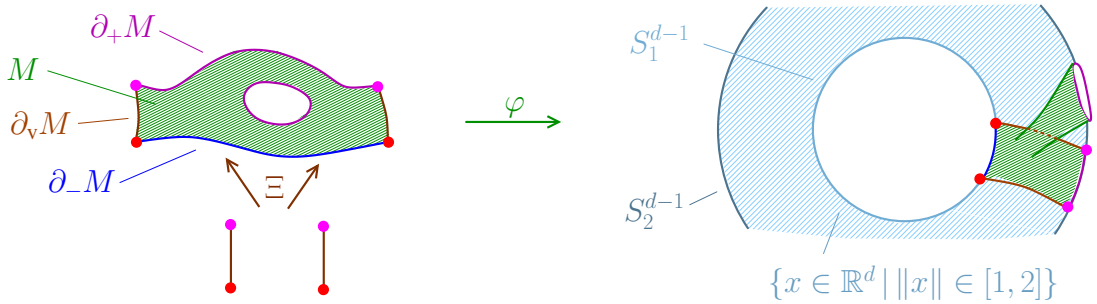
We continue with the following claim, which is a slight generalization of the Cylinder Embedding Lemma 249.27:

Claim 1. There exists a $d \in \mathbb{N}$ and a proper smooth embedding

$$\varphi \colon M \to \{x \in \mathbb{R}^d \mid \|x\| \in [1,2]\}$$

with the following two properties:

- We have $\varphi^{-1}(S_1^{d-1}) = \partial_- M$ and $\varphi^{-1}(S_2^{d-1}) = \partial_+ M$.
- For each $(t, c) \in \Xi([0,1] \times C)$ we have $\|\varphi(\Xi(t,c))\| = t + 1$.



Proof. [2722] In preparation we need to pick and fix several maps:

(1) It follows from the Euclidean Embedding Theorem 33.2 that there exists a $d \in \mathbb{N}$ and a smooth embedding $\psi \colon M \to \mathbb{R}^{d-1}$.
(2) We pick a smooth embedding $\Theta \colon \mathbb{R}^{d-1} \to S^{d-1}$.
(3) Using the two collars χ and λ and using the Smooth Transition Function Lemma 24.9 we can easily find a smooth function $\rho \colon M \to [1,2]$ with the following properties:
   - For each $(t, c) \in \Xi([0,1] \times C)$ we have $\rho(\Xi(t,c)) = t + 1$.
   - We have $\partial_- M = \rho^{-1}(1)$ and the smooth function ρ points inward on $\partial_- M$.
   - We have $\partial_+ M = \rho^{-1}(2)$ and the smooth function ρ points outward on $\partial_+ M$.

   We delegate the details to Exercise 253.3.

We now consider the map

$$\begin{aligned} \varphi \colon M &\to \mathbb{R}^d \\ P &\mapsto \rho(P) \cdot \underbrace{(\Theta \circ \psi)(P)}_{\in S^{d-1}}. \end{aligned}$$

It is straightforward to verify that φ has all the desired properties. ⊞

[2722] As we will see, the proof of the claim is almost identical to the proof of the Cylinder Embedding Lemma 249.27.

In the following we will use φ to view M as a proper submanifold of $\{x \in \mathbb{R}^d \mid \|x\| \in [1,2]\}$ as in the above claim. The next claim is modelled on the claim on page 4827:

Claim 2. For each $\epsilon > 0$ there exists a $P \in B^d_\epsilon(0) \subseteq \mathbb{R}^d$ with the following two properties:
(a) The function
$$\begin{array}{rcl} \delta_P \colon M & \to & \mathbb{R} \\ x & \mapsto & \|x - P\|^2 \end{array}$$
points inward on $\partial_- M$, it points outward on $\partial_+ M$ and it points along $\partial_{\mathrm{v}} M$.
(b) Every critical point of δ_P is non-degenerate.

Proof. Similar to the earlier discussion one can prove this claim quite easily, using the Distance-Squared Function Proposition 249.21 together with the Full Measure-Density Proposition 37.4. We leave the details to the reader. ⊞

Claim 3. There exists a smooth function $\gamma \colon M \to [0,1]$ with the following properties:
(a) We have $\partial_- M = \gamma^{-1}(0)$ and $\partial_+ M = \gamma^{-1}(1)$.
(b) The function points along $\partial_{\mathrm{v}} M$.
(c) There is no critical point on ∂M.
(d) Every critical point of δ_P is non-degenerate.

Proof. The proof of this claim is almost identical to the proof of the Standard-near-Boundary Lemma 249.29, except that this time we use the collar χ for the smooth boundaries $\partial_- M$ and $\partial_+ M$. Again the details are left to the reader. ⊞

The following claim now gives us the desired statement:

Claim 4. There exists a smooth function $f \colon M \to [0,1]$ with the following properties:
(a) We have $\partial_- M = f^{-1}(0)$ and $\partial_+ M = f^{-1}(1)$.
(b′) For all $(t,c) \in [0,1] \times C$ we have $f(\Xi(t,c)) = c$.
(b) There is no critical point on ∂M.
(c) Every critical point of δ_P is non-degenerate.

Proof. Let $\gamma \colon M \to [0,1]$ be a Morse function as in Claim 3. We just need to upgrade (b) to (b′). As we will see, this can be done quite easily using our collar $\lambda \colon [0,1] \times \partial_{\mathrm{v}} M \to M$. First note that it follows easily from the hypothesis that M is compact (which implies that the closed subset $\partial_{\mathrm{v}} M$ is compact) that there exists an $\epsilon > 0$ such that for each $s \in [0,\epsilon]$ and each $(t,c) \in [0,1] \times C$ we have $\mathrm{D}_{(t,c)}(\underbrace{\gamma \circ \lambda_s \circ \Xi}_{\text{map } [0,1] \times C \to \mathbb{R}})(\mathfrak{v}) > 0$. It follows almost immediately from the Smooth Transition Function Lemma 24.9 that there exists a smooth function $\eta \colon [0,\epsilon] \to [0,1]$ such that $\eta(0) = 1$ and such that $\eta(t) = 0$ for $t \in [\frac{\epsilon}{2}, \epsilon]$. Now we consider the function
$$\begin{array}{rcl} f \colon M & \to & [0,1] \\ Q & \mapsto & \begin{cases} \gamma(Q), & \text{if } Q \notin \lambda(\Xi([0,1] \times C) \times [0,\epsilon]), \\ \eta(s) \cdot t + (1 - \eta(s)) \cdot \gamma(Q) & \text{if } Q = \lambda(\Xi(t,c),s) \text{ for } (t,c,s) \in [0,1] \times C \times [0,\epsilon]. \end{cases} \end{array}$$
Note that it follows from our choice of η and the Smooth Pasting Proposition 25.19 that f is smooth. Finally, using the observation that "pointing along" is a convex property one can easily verify that $f \colon M \to [0,1]$ has all the desired properties. ■

253.4. **Existence of handle decompositions of a cobordism with vertical boundary.** In the previous section we introduced Morse functions on cobordisms with vertical

boundary. For our purposes their only raison d'être is the following theorem, which is a generalization of the Morse Function–Handle Decomposition Theorem 250.6:

Theorem 253.7. (Morse Function–Handle Decomposition Theorem) Let M be a compact cobordism with vertical boundary. If $f\colon M \to \mathbb{R}$ is a Morse function in the sense of Definition 253.5, then there exists a standard handle decomposition of M in the sense of Definition 253.4 such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k.

Sketch of proof. Let M be a compact cobordism with vertical boundary. We pick an $(n-2)$-dimensional smooth manifold C and a diffeomorphism $\Xi\colon [0,1] \times C \to \partial_{\mathrm{v}} M$ such that $\Xi_0(C) = \partial(\partial_- (M))$ and $\Xi_1(C) = \partial(\partial_+(M))$. Next let $f\colon M \to \mathbb{R}$ be a Morse function in the sense of Definition 253.5. We make several initial preparations and observations:

- It follows from a straightforward generalization of the Distinct Critical Values Lemma 249.30 that we can assume that the critical values are distinct.
- It follows from the compactness of M and the fact that there are no critical points on ∂M that there exists an $\epsilon > 0$ such that all values in $[0, \epsilon]$ are regular.
- Let $a \in \mathbb{R}$. As in Notation 250.3 we write
$$M_a \;:=\; f^{-1}((-\infty, a]) \;=\; \{x \in M \mid f(x) \leq a\}.$$
Note that if $a \in (0,1]$ is a regular value of f, then it follows easily from the Preimage-of-Interval Lemma 38.3 that M_a is a submanifold of M with corner, and that it is a cobordism with vertical boundary where $\partial_- M_a = \partial_- M$, where $\partial_+ M_a = f^{-1}(a)$ and where the vertical boundary is given by $\partial_{\mathrm{v}} M_a = f^{-1}([0,a]) \cap \partial_{\mathrm{v}} M = \Xi([0,a] \times C)$.

The following claim contains the key innovative feature of this proof:

Claim 1. There exists a smooth embedding $\gamma\colon M \to \mathbb{R}^d$ such that for any $P \in \partial_{\mathrm{v}} \gamma(M)$ we have $\operatorname{grad}(f \circ \gamma^{-1})|_P \in \mathrm{T}_P \gamma(M)$.

Proof. We introduce a few maps:

- First note that it follows from the Euclidean Embedding Theorem 33.2 that we can pick a smooth embedding $\alpha\colon M \setminus \partial M \to \mathbb{R}^k$ and that we can pick a smooth embedding $\beta\colon C \to \mathbb{R}^l$.
- By the Smooth Manifold–Boundary Proposition 59.5 together with the Smooth Collar Neighborhood Theorem 59.7 we can find a smooth collar $\chi\colon [0,1] \times \partial_{\mathrm{v}} M \to M$ in the sense of Definition 59.6.
- It follows immediately from the Smooth Transition Function Lemma 24.9 that there exists a smooth function $\eta\colon [0,1] \to [0,1]$ with $\eta \equiv 0$ on $[0, \frac{1}{4}]$ and $\eta \equiv 1$ on $[\frac{3}{4}, 1]$.

We now consider the map

$$\begin{aligned} \gamma\colon M &\to \mathbb{R}^{k+l+2} \\ P &\mapsto \begin{cases} \alpha(P), & \text{if } P \notin \chi([0, \frac{7}{8}] \times \partial_{\mathrm{v}} M), \\ (0, \beta(c), s, t), & \text{if } P = \chi(\Xi(t,c), s) \text{ for } (t,c) \in [0,1] \times C \text{ and } s \in [0,1). \end{cases} \end{aligned}$$

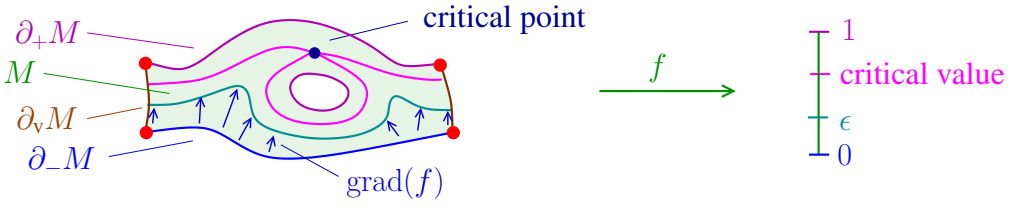
One can easily verify that this is well-defined that this is indeed a smooth embedding. Next recall that by Condition (4) of Definition 253.5 we know that for all $(t,c) \in [0,1] \times C$ we have $f(\Xi(t,c)) = c$. In our setting this implies that the map $f \circ \Xi\colon [0,1] \times C \to \mathbb{R}$ is given by $\mathrm{pr}_{k+l+2} \circ \gamma\colon C \to \mathbb{R}$ where $\mathrm{pr}_{k+l+2}\colon \mathbb{R}^{k+l+2} \to \mathbb{R}$ is just the projection onto the last

coordinate. The gradient of $f \circ \Xi \circ \gamma^{-1} \colon \gamma(M) \to \mathbb{R}$ is thus given by $\mathfrak{e}_{k+l+2}$. By our choice of the smooth embedding one sees easily that this gradient vector lies in the tangent space $\mathrm{T}_{\gamma(P)}\gamma(M)$. ⊞

We use the smooth embedding γ from Claim 1 to view M as a smooth submanifold of some $\mathbb{R}^d$ such that for any $P \in \partial_{\mathrm{v}} M$ we have $\operatorname{grad} f_P \in \mathrm{T}_P M$.

Claim 2. There exists a diffeomorphism $[0, \epsilon] \times \partial_- M \to f^{-1}([0, \epsilon])$ with $(0, x) \mapsto x$.

Proof. The claim follows from a mild generalization of the Smooth Manifold-Product Theorem 250.1.



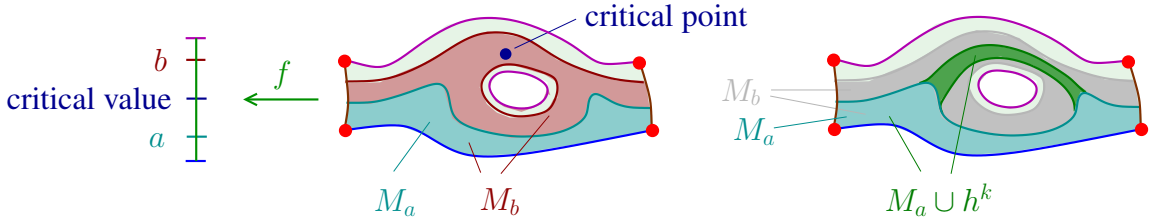
The key extra ingredient is that we use that in Claim (1) we arranged that the gradient vector field $\operatorname{grad}(f)$ is tangent to $\partial_{\mathrm{v}} M$. Thus the flow corresponding to $\operatorname{grad}(f)$ preserves $\partial_{\mathrm{v}} M$. We leave the details to the reader. ⊞

Claim 3. Let $a < b \in \mathbb{R}$ such that a and b are regular values of f. We suppose that there exists a single critical point $P \in f^{-1}([a, b])$. We set $k := \operatorname{index}(f, P)$. Then there exists a thickened $(k-1)$-sphere φ in $\partial_+ M_a$ and a diffeomorphism

$$M_a \cup_\varphi h^k \xrightarrow{\cong} M_b$$

of cobordisms with vertical boundary.

Proof. This time the proof of the claim is a mild generalization of the proof of the Critical Point–Handle Addition Theorem 250.5.



Again we need to make use the fact that our gradient vector field $\operatorname{grad}(f)$ is tangent to $\partial_{\mathrm{v}} M$. Once again leave the details to the reader. ⊞

As in the proof of the Morse Function–Handle Decomposition Theorem 250.6 the desired statement now follows easily from the above claims together with the trivial observation that $[0, \epsilon] \times \partial_- M$ is diffeomorphic to $[0, 1] \times \partial_- M$. ■

We conclude this chapter with an utterly predictable generalization of the Handle Decomposition Theorem 250.7:

Theorem 253.8. (Handle Decomposition Theorem) Let M be a compact cobordism with vertical boundary. There exists a standard handle decomposition of M in the sense of Definition 253.5 and Definition 247.6.

Proof. Let M be a compact cobordism with vertical boundary. By the Morse Function-Existence Theorem 253.7 we know that there exists a Morse function $f\colon M \to \mathbb{R}$. It follows from the Morse Function–Handle Decomposition Theorem 253.7 together with a very mild generalization of the Standard Handle Decomposition Proposition 247.7 that M admits a standard handle decomposition. ■

253.5. Exercises for Chapter 253.

Exercise 253.1. Show that Condition (3) in Definition 253.1 is implied by all the other conditions.
Remark. In other words we just added Condition (3) for clarity.

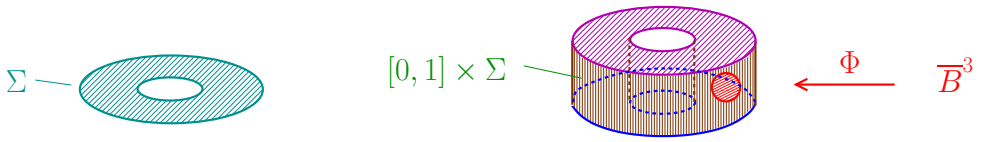
Exercise 253.2. In the statement of the Cut Along Surface–Vertical Boundary Lemma 253.2 we wrote " such that for each $i \in \{1, \dots, k\}$ the intersection $\partial\Sigma \cap T_i$ consists of a union of curves $C_1, \dots, C_m$ which [...] represent the same non-zero homology class in $\mathrm{H}_1(T_i)$".
(a) What goes wrong if we drop the word "same"?
(b) What goes wrong if we drop the word "non-zero"?

Exercise 253.3. Let M be a compact cobordism with vertical boundary. Furthermore let C be a closed $(n-2)$-dimensional smooth manifold and let $\Xi\colon [0,1] \times C \to \partial_{\mathrm{v}} M$ be a diffeomorphism such that $\Xi_0(C) = \partial(\partial_- (M))$ and such that $\Xi_1(C) = \partial(\partial_+(M))$. Show that there exists a smooth function $\rho\colon M \to [0,1]$ with the following properties:

- For each $(t,c) \in \Xi([0,1] \times C)$ we have $\rho(\Xi(t,c)) = t$.
- We have $\partial_- M = \rho^{-1}(0)$ and the smooth function ρ points inward on $\partial_- M$.
- We have $\partial_+ M = \rho^{-1}(1)$ and the smooth function ρ points outward on $\partial_+ M$.

Remark. The goal of the exercise is to fill in the details in the argument on page 4899.

Exercise 253.4. Let Σ be a compact connected orientable 2-dimensional smooth manifold and let $\Phi\colon \overline{B}^3 \to (\Sigma \setminus \partial\Sigma) \times (0,1)$ be an orientation-preserving smooth embedding. We consider the smooth manifold $M := (\Sigma \times [0,1]) \setminus \Phi(B^3)$ with corner $\angle M = \partial\Sigma \times \{0,1\}$ and we view M as a cobordism with vertical boundary where the decomposition is given by $\partial_- M := \Sigma \times \{0\}$, $\partial_+ M := (\Sigma \times \{1\}) \cup \Phi(S^2)$ and $\partial_{\mathrm{v}} M := \partial\Sigma \times [0,1]$. Provide an explicit handle decomposition for M.
Remark. Note that it follows from the Smooth Ball Embedding Theorem 45.10 that all such smooth embeddings of the 3-ball are smoothly isotopic.
Remark. This exercise is very similar to Exercise 247.13.



CHAPTER 254

Handle decompositions and CW-complexes

In this chapter we will show that a handle decomposition of a smooth manifold M gives rise to a finite CW-complex that is *homotopy equivalent* to the smooth manifold M. As we will see, this result can be used to deduce some results about smooth manifolds from our current and future expertise on CW-complexes.

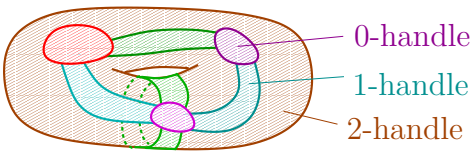
This chapter is organized as follows:

- In Section 254.1 we will formulate the Handle Decomposition–CW-Complex Theorem 254.1 and the Handle–Cell Attachment Proposition 254.2 which show that handle attachments correspond to cell attachments, and that handle decompositions give rise to CW-complexes.
- In Section 254.2 we will provide two proofs of the Handle–Cell Attachment Proposition 254.2.
- In Section 254.3 we will discuss several variations on the Handle Decomposition–CW-Complex Theorem 254.1.
- In Section 254.4 we will discuss several relatively straightforward consequences of the Handle Decomposition–CW-Complex Theorem 254.1 and of its siblings.
- In Section 254.5 we will provide exercises for Chapter 254.

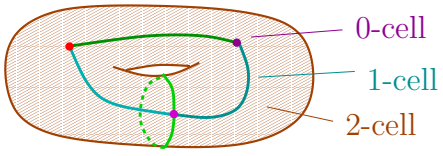
In the next Chapter 255 we will finally complete the classification of compact 2-dimensional smooth manifolds.

254.1. **Handle decompositions and CW-structures.** The following theorem is the main result of this chapter:

Theorem 254.1. (Handle Decomposition–CW-Complex Theorem) Every compact smooth manifold M that is equipped with a handle decomposition is homotopy equivalent to a CW-complex X such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-cells of X.



handle decomposition of the torus



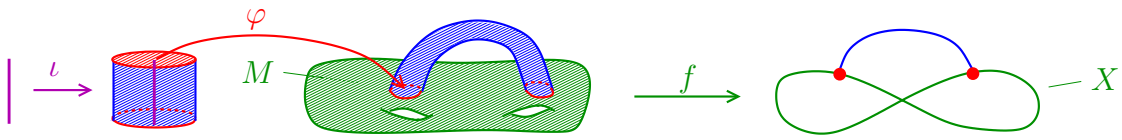
CW-complex

Not surprisingly we will prove the Handle Decomposition–CW-Complex Theorem 254.1 by considering the handle attachments iteratively. This leads us to the following proposition:

Proposition 254.2. (Handle–Cell Attachment Proposition) Let M be an n-dimensional smooth manifold and let $f\colon M \to X$ be a homotopy equivalence to some topological

space X. Furthermore let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We denote by $\iota\colon \overline{B}^k \to \overline{B}^{n-k} \times \overline{B}^k$ the embedding that is given by $x \mapsto (0, x)$. For any continuous map $\psi\colon S^{k-1} \to X$ that is homotopic to $f \circ \varphi \circ \iota\colon S^{k-1} \to X$ there exists a homotopy equivalence

$$M \cup_{M \leftarrow \overline{B}^{n-k} \times S^{k-1} : \varphi} (\overline{B}^{n-k} \times \overline{B}^k) \;\to\; X \cup_{X \leftarrow S^{k-1} : \psi} \overline{B}^k.$$



We will provide the proof of the Handle–Cell Attachment Proposition 254.2 in the next section. First we will see that we can easily deduce the Handle Decomposition–CW-Complex Theorem 254.1 from the Handle–Cell Attachment Proposition 254.2:

Proof of Theorem 254.1 assuming Proposition 254.2. Let M be a compact smooth manifold that is equipped with a handle decomposition

$$M \;=\; \varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}.$$

Given $i = 0, \ldots, m$ we denote by M_i the result of the first i handle attachments. Clearly it now suffices to prove the following claim:

Claim. For $i = 0, \ldots, m$ there exists a CW-complex X_i, where the number of k-cells equals the number of k-handles in M_i, which admits a homotopy equivalence $M_i \to X_i$.

Proof. For $i = 0$ we have $M_0 = \varnothing$ and we set $X_0 := \varnothing$. Now we suppose that the statement holds for some $i \in \{0, \ldots, m-1\}$. So we have a CW-complex X_i, with the correct number and types of cells, together with a homotopy equivalence $f_i\colon M_i \to X_i$.

We consider the map $f_i \circ \varphi_{i+1} \circ \iota\colon S^{k_{i+1}-1} \to X_i$. By the Cellular Approximation Theorem 141.2 we know that there exists a homotopy from $f_i \circ \varphi_{i+1} \circ \iota\colon S^{k_{i+1}-1} \to X_i$ to a cellular map[2723] $\psi\colon S^{k_{i+1}-1} \to X_i$.

Since ψ is cellular it follows from the CW-Gluing Proposition 139.9 that $X_i \cup_\psi \overline{B}^{k_{i+1}}$ admits a CW-structure such that X_i is a subcomplex and such that we have one extra cell, namely a k_{i+1}-cell. Finally note that it follow from the Handle–Cell Attachment Proposition 254.2 that there exists homotopy equivalence

$$\underbrace{M_i \cup_{\varphi_{i+1}} (\overline{B}^{n-k_{i+1}} \times \overline{B}^{k_{i+1}})}_{=M_{i+1}} \;\to\; \underbrace{X_i \cup_\psi \overline{B}^{k_{i+1}}}_{=:X_{i+1}}.$$

■

254.2. **Proof of the Handle–Cell Attachment Proposition 254.2.** We will provide two slightly different proofs for the Handles-Cells Proposition 254.2. Both will build on some earlier results.

254.2.1. **First proof of the Handle–Cell Attachment Proposition 254.2.** The first proof of the Handle–Cell Attachment Proposition 254.2 builds on the following two more general results:

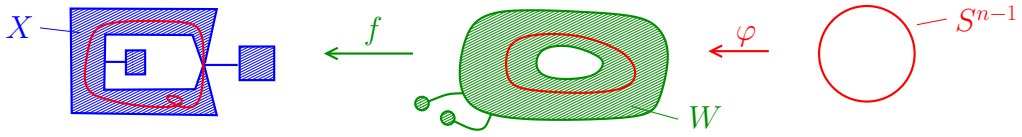
[2723] We equip any sphere S^d with the standard CW-structure from page 2625 which has one 0-cell and one d-cell.

Lemma 135.12 (Homotopic Attaching Map Lemma) Let X be a topological space and let $\varphi_0, \varphi_1 \colon S^{n-1} \to X$ be two continuous maps. If φ_0 and φ_1 are homotopic, then there exists a homotopy equivalence $X \cup_{X \leftarrow S^{k-1}:\varphi_0} \overline{B}^k \to X \cup_{X \leftarrow S^{k-1}:\varphi_1} \overline{B}^k$ which is the identity on X.

Proposition 135.13 (Homotopy Equivalent Cell Attachment Proposition) In the following let $f \colon W \to X$ be a continuous map between two topological spaces and let $\varphi \colon S^{k-1} \to W$ be an attaching map. If f is a homotopy equivalence, then the map

$$F \colon W \cup_{W \leftarrow S^{k-1}:\varphi} \overline{B}^k \xrightarrow{\simeq} X \cup_{X \leftarrow S^{k-1}:f\circ\varphi} \overline{B}^k$$

which is given by f on W and by id on $\overline{B}^k$ is a homotopy equivalence.



Example. Let $\varphi \colon S^1 \to \mathbb{R}^2$ be the inclusion and let $f \colon \mathbb{R}^2 \to \{*\}$ be the retraction to the origin. In the figure below we show the corresponding cell attachments.



It follows from the Homotopy Equivalent Cell Attachment Proposition 135.13 that these results are homotopy equivalent □

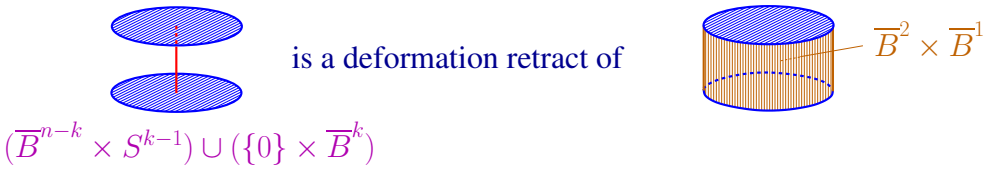
We will also need the following lemma:

Lemma 254.3. (Core Deformation Retract Lemma) Let M be an n-dimensional smooth manifold and let $\varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. With this notation $M \cup_{\varphi|_{\{0\}\times S^{k-1}}} (\{0\} \times \overline{B}^k)$ is a deformation retract of $M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)$.



Sketch of proof. We start out with the following claim:

Claim. The subset $(\overline{B}^{n-k} \times S^{k-1}) \cup (\{0\} \times \overline{B}^k)$ is a deformation retract of $\overline{B}^{n-k} \times \overline{B}^k$.



Proof. We set $Z := (\overline{B}^{n-k} \times S^{k-1}) \cup (\{0\} \times \overline{B}^k)$. Given $(x, y) \in (\overline{B}^{n-k} \setminus \{0\}) \times \overline{B}^k$ we consider the ray starting at $(2\frac{x}{\|x\|}, 0)$ through (x, y). An elementary argument shows

that this ray will intersect Z in a unique point $f(x,y)$. Together with $f(0,y) = (0,y)$ for $(0,y) \in \{0\} \times \overline{B}^k$ this define a retraction from $\overline{B}^{n-k} \times \overline{B}^k$ to Z. The desired deformation retraction is now given by the map

$$\begin{array}{rcl} R\colon (\overline{B}^{n-k} \times \overline{B}^k) \times [0,1] & \to & \overline{B}^{n-k} \times \overline{B}^k \\ (P,t) & \mapsto & P \cdot (1-t) + f(P) \cdot t. \end{array}$$

We leave it to the reader to fill in the details. ⊞

The lemma now follows from the Homotopy Quotient Theorem 18.11 which implies that we can extend the above deformation retraction R to a deformation retraction from $M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)$ to $M \cup_\varphi (\{0\} \times \overline{B}^k)$. ■

Proof of the Handle–Cell Attachment Proposition 254.2. Let M be an n-dimensional smooth manifold, let $f\colon M \to X$ be a homotopy equivalence to some topological space X and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We denote by the $\iota\colon \overline{B}^k \to \overline{B}^{n-k} \times \overline{B}^k$ the embedding that is given by $\iota(x) = (0,x)$. Finally let $\psi\colon S^{k-1} \to X$ be a continuous map that is homotopic to $f \circ \varphi \circ \iota\colon S^{k-1} \to X$. We now obtain the following homotopy equivalences:

$$\begin{array}{ccccc} M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k) & \xleftarrow{\simeq} & M \cup_{\varphi|_{\{0\}\times S^{k-1}}} (\{0\} \times \overline{B}^k) & \cong & M \cup_{\varphi\circ\iota} \overline{B}^k \\ \uparrow & & \uparrow & & \uparrow \\ \text{by the Core Deformation Retract Lemma 254.3} & & & & \text{given by the obvious homeomorphism} \\ \xrightarrow{\simeq} & X \cup_{f\circ\varphi\circ\iota} \overline{B}^k & & \xrightarrow{\simeq} & X \cup_\psi \overline{B}^k. \\ \uparrow & & & & \uparrow \\ \text{by the Homotopy Equivalent Cell Attachment Proposition 135.13} & & & & \text{by the Homotopic Attaching Map Lemma 135.12} \end{array}$$

It follows from this discussion and the Homotopy-Equivalence Basics Lemma 19.14 (2) that $M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k)$ is indeed homotopy equivalent to $X \cup_\psi \overline{B}^k$. ■

254.2.2. **Second proof of the Handle–Cell Attachment Proposition 254.2.** The key to our second proof of the Handle–Cell Attachment Proposition 254.2 is the following theorem whose easy formulation belies the difficulty of the proof.

Theorem 22.24. (Homotopy Pushout Theorem) Suppose we are given the following commutative diagram of continuous maps between topological spaces:

$$\begin{array}{ccccc} Y & \xleftarrow{f} & A & \xhookrightarrow{i} & X \\ \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_X} \\ Y' & \xleftarrow{f'} & A' & \xhookrightarrow{i'} & X'. \end{array}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the right are closed cofibrations[2724], then the induced map

$$Y \cup_A X \to Y' \cup_{A'} X'$$

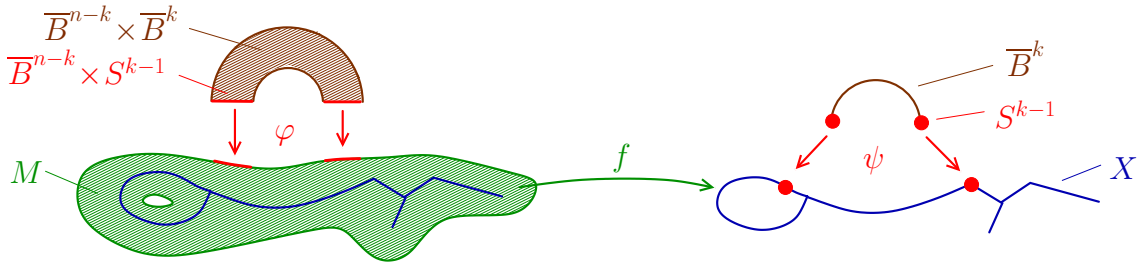
between the pushouts is a homotopy equivalence.

Proof of the Handle–Cell Attachment Proposition 254.2. Let M be an n-dimensional smooth manifold and let $f\colon M \to X$ be a homotopy equivalence to some topological

[2724]We refer to Definition 22.3 for the definition of a closed cofibration. To get the hang of the argument it is not that important though to recall the definition of a closed cofibration.

space X. Furthermore let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We denote by $\iota\colon \overline{B}^k \to \overline{B}^{n-k} \times \overline{B}^k$ the embedding that is given by $x \mapsto (0, x)$. Finally let $\psi\colon S^{k-1} \to X$ be a continuous map that is homotopic to $f \circ \varphi \circ \iota\colon S^{k-1} \to X$. We need to show that there exists a homotopy equivalence

$$M \cup_{M \leftarrow \overline{B}^{n-k} \times S^{k-1} : \varphi} (\overline{B}^{n-k} \times \overline{B}^k) \;\to\; X \cup_{X \leftarrow S^{k-1} : \psi} \overline{B}^k.$$



By hypothesis there exists a homotopy $F\colon S^{k-1} \times [0,1] \to X$ from $F_0 = f \circ \varphi \circ \iota\colon S^{k-1} \to X$ to the map $F_1 = \psi\colon S^{k-1} \to X$. We obtain the following commutative diagram:

$$\begin{array}{ccccc}
M & \hookleftarrow & \partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1} & \hookrightarrow & \overline{B}^{n-k} \times \overline{B}^k \\
{\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \iota}\ {\scriptstyle x \mapsto (0,x)} & & \uparrow{\scriptstyle \iota}\ {\scriptstyle x \mapsto (0,x)} \\
M & \xleftarrow[\varphi \circ \iota]{x \mapsto \varphi(0,x)} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xleftarrow{f \circ \varphi \circ \iota} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \iota}\ {\scriptstyle x \mapsto (x,0)} & & \downarrow{\scriptstyle \iota}\ {\scriptstyle x \mapsto (x,0)} \\
X & \xleftarrow{F} & S^{k-1} \times [0,1] & \hookrightarrow & \overline{B}^k \times [0,1] \\
{\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle x \mapsto (x,1)} & & \uparrow{\scriptstyle x \mapsto (x,1)} \\
X & \xleftarrow{\psi} & S^{k-1} & \hookrightarrow & \overline{B}^k.
\end{array}$$

It is clear that all of the vertical maps are homotopy equivalences. Furthermore note that it follows immediately from the Manifold Boundary-Cofibration Proposition 22.11 and the Product-Cofibration Proposition 22.15 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 22.24, applied altogether four times, that the induced maps between the various pushouts have homotopy inverses. Combining these four homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence

$$\underbrace{M \cup_{\overline{B}^{n-k} \times S^{k-1}} (\overline{B}^{n-k} \times \overline{B}^k)}_{=(M \sqcup (\overline{B}^{n-k} \times \overline{B}^k))/\sim} \;\to\; \underbrace{X \cup_{S^{k-1}} \overline{B}^k}_{=(X \sqcup \overline{B}^k)/\sim}.$$

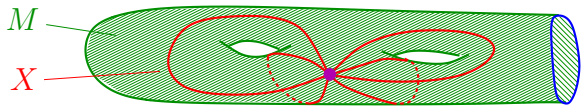
■

254.3. **Smooth manifolds, CW-complexes and simplicial complexes.** In this section we present several straightforward consequences of the Handle Decomposition–CW-Complex Theorem 254.1. We start out with the following low hanging fruit:

Proposition 254.4. (Smooth Manifold-CW-Homotopy Type Proposition)
(1) Every compact smooth manifold is homotopy equivalent to a finite CW-complex.

(2) (a) Every closed connected non-empty n-dimensional smooth manifold is homotopy equivalent to a finite CW-complex with a single 0-cell and a single n-cell.
(b) Every compact n-dimensional smooth manifold with non-empty boundary is homotopy equivalent to a finite CW-complex with a single 0-cell and no n-cell.

Example. In the figure below we show a compact 2-dimensional smooth manifold M with non-empty boundary. The Smooth Manifold-CW-Homotopy Type Proposition 254.4 says that M is homotopy equivalent to 1-dimensional finite CW-complex with a single 0-cell.



In fact in the figure we show a 1-dimensional CW-complex X contained in M, with a single 0-cell, which is deformation retract of M. □

Proof.

(1) This statement follows immediately from the Handle Decomposition Theorem 250.7 together with the Handle Decomposition–CW-Complex Theorem 254.1.
(2) This statement follows from the logic of Statement (1), except that we now require Minimal n-Handle Proposition 248.4 as an extra input. ■

The next proposition is also a straightforward consequence of our earlier results:

Proposition 254.5. (Morse Functions-Cells Proposition) Let M be a compact smooth manifold. If f is a Morse function for M, then M is homotopy equivalent to a CW-complex where the number of k-cells equals precisely the number of critical points of index k.

Proof. This statement follows immediately from the Morse Function–Handle Decomposition Theorem 250.6 together with the Handle Decomposition–CW-Complex Theorem 254.1. ■

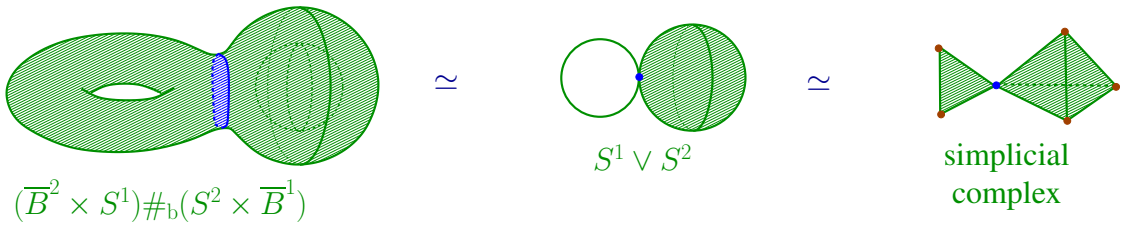
Example. Let $n \in \mathbb{N}_0$. Recall that in the Projective Space–Morse Function Proposition 249.15 we showed that there exists a Morse function $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ with precisely one critical point for each index $0, \ldots, n$. It follows from the Morse Functions-Cells Proposition 254.5 that $\mathbb{R}\mathrm{P}^n$ is homotopy equivalent to a CW-complex with precisely one cell of dimension $0, \ldots, n$ and no other cells. In the same vein the above results show that $\mathbb{C}\mathrm{P}^n$ is homotopy equivalent to a CW-complex with precisely one cell of degree $0, 2, \ldots, 2n$ and no other cells. Of course on page 2628 we showed that $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ have CW-structures with this number of cells. □

Here is one more result that we now get for "free":

Proposition 254.6. (Smooth Manifold–Simplicial Complex–Homotopy Type Proposition) Every compact smooth manifold is homotopy equivalent to a finite simplicial complex.[2725]

Proof. Recall that the CW–Simplicial Homotopy Type Proposition 175.21 says that every finite CW-complex is homotopy equivalent to a finite simplicial complex. Thus the current

[2725]Note though that in this setup we cannot control the number of k-simplices in terms of the number of k-handles of a given handle decomposition.



proposition is an immediate consequence of the Smooth Manifold-CW-Homotopy Type Proposition 254.4 (1). ■

Remark. The Smooth Manifold-CW-Homotopy Type Proposition 254.4 and the Smooth Manifold–Simplicial Complex–Homotopy Type Proposition 254.6 are related to several earlier results:

(1)
- In the Smooth Manifold-Simplicial Structure Theorem 179.5 we showed that every compact smooth manifold M admits a finite simplicial structure. By the Smooth Manifold-CW-Theorem 179.9 this implies in particular that M admits a finite CW-structure.
- In the Smooth Manifold–Weak Homotopy Type Corollary 178.10 we used the Nerve Theorem 178.5 to show that every compact smooth manifold is weakly homotopy equivalent to a finite CW-complex.
- Finally in the Manifold–Simplicial Complex Homotopy Type Corollary 190.7 we used Borsuk's Theorem 189.7 to show that every topological manifold is homotopy equivalent to a CW-complex.

(2) In the Single Top-Cell Proposition 183.34 and the Single Zero-and-Top Cell Proposition 190.13 we proved and will prove two variations on the Smooth Manifold-CW-Homotopy Type Proposition 254.4 (2). □

We conclude this section with a look at the literature:

Remark.

(1) There are various other proofs that Morse functions give rise to CW-complexes that are homotopy equivalent to M, see e.g. [**BaH2004**, Theorem 3.28], [**Miln1965b**, Chapter 3] and [**Mata2002**, Theorem 4.18]. Terse argument are also given in [**Nic2011**, Theorem 2.10] and [**Wall2016**, Proposition 5.3.1]. Furthermore an even terser argument is given in [**Hirs1976**, Theorem 6.4.1].

(2) Given a closed smooth manifold there is a notion of a more refined version of a Morse function, namely of a Morse-Smale function. We will not provide the definition, instead we refer to [**BaH2004**, p. 7] for the definition. But we try to pique the reader's interest with the following two statements:
   (a) The Kupka-Smale Theorem, proved independently by Ivan Kupka [**Kupk1963**] and Steven Smale [**Sma1963**], says that every closed smooth manifold admits a Morse-Smale function. Alternatively see [**BaH2004**, Theorem 6.6] for a proof.
   (b) In [**Lau1992**, p. 222] (see also [**BaH2004**, p. 179]) it is shown that a Morse-Smale function gives rise to a CW-complex that is actually *homeomorphic* to M and such that for each $k \in \mathbb{N}_0$ the critical points of index k are in one-to-one correspondence to the k-cells. □

254.4. **Applications.** At this stage it is perhaps not entirely clear, what the Smooth Manifold-CW-Homotopy Type Proposition 254.4, and all of its siblings from the previous section, are good for, except for satisfying one's need for ever more results in topology.

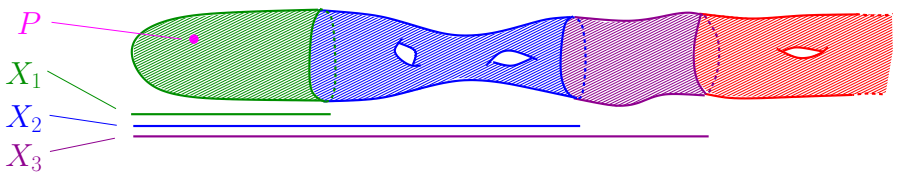
One answer is, that the results allow us to apply some of the rich theory of CW-complexes and simplicial complexes to smooth manifolds. The following proposition gives us one such application:

Proposition 254.7. (Manifold–π_n-Countable Proposition) Let M be a connected non-empty smooth manifold. For every $n \in \mathbb{N}$ the group $\pi_n(M)$ is countable.

Proof. If M is compact, then this statement follows immediately from the Smooth Manifold–Simplicial Complex–Homotopy Type Proposition 254.6 together with the Simplicial Complex–π_n-Countable Proposition 175.16.

We now turn to the case that M is non-compact. By the Push-Away-from-Boundary Corollary 35.8 we know that M is homotopy equivalent to $M \setminus \partial M$. It follows from this fact, together with the Homotopy Equivalence-π_n-Proposition 134.17, that we might as well assume that M has no boundary. Since M has no boundary we know by the Compact Submanifold Exhaustion Proposition 38.2 that there exists a sequence $X_1, X_2, \ldots$ of codimension-zero smooth submanifolds of M with the following three properties:

- For each $i \in \mathbb{N}$ we have $X_i \subseteq \overset{\circ}{X}_{i+1}$.
- Each X_i is compact, path-connected and non-empty.
- We have $\bigcup_{i\in\mathbb{N}} \overset{\circ}{X}_i = M$.



First we consider the case $n \in \mathbb{N}_{\geq 2}$. Given any $P \in X_1$ we have the following isomorphisms:

$$\pi_n(M,P) \underset{\uparrow}{\cong} \operatorname*{colim}_{\longrightarrow} \pi_n(X_i,P) \underset{\uparrow}{\cong} \Big(\bigoplus_{i\in\mathbb{N}} \pi_n(X_i,P)\Big) / \{\varphi_{ij*}(g) - g \mid \ i \leq j \text{ and } g \in \pi_n(X_i,P)\}.$$

by the Exhaustion-π_n Proposition 134.21

by the Algebraic Direct Colimit Existence Proposition 104.7 (1b)

Since each X_i is compact it follows from the discussion in the beginning of the proof that each $\pi_n(X_i,P)$ is countable. It follows from these observations, together with the Countability Lemma 1.37, that $\pi_n(M,P)$ is countable.

The case $n = 1$ is almost identical. We just need to replace the Algebraic Direct Colim Existence Proposition 104.7 (1b) by the Algebraic Direct Colimit Existence Proposition 104.7 (1a). In other words, we just need to replace the direct sum of abelian groups by the free product of groups. ∎

Our results also tell us that the following (future) theorem applies to compact smooth manifolds:

Theorem 305.1. (Whitehead Theorem) Let $f\colon X \to Y$ be a continuous map between two connected non-empty topological spaces. If $f\colon X \to Y$ is a weak homotopy

equivalence, i.e. if for some $P \in X$ all induced maps

$$f_*\colon \pi_n(X,P) \;\to\; \pi_n(Y,P)$$

are isomorphisms, *and if both X and Y are homotopy equivalent to CW-complexes*, then f is a homotopy equivalence.

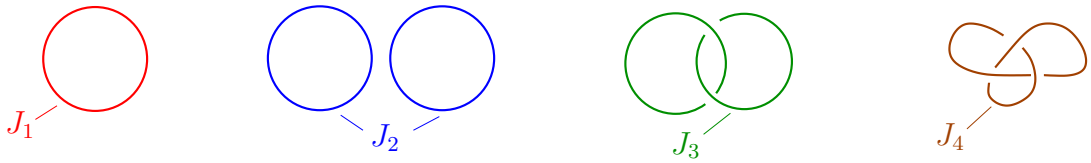
254.5. Exercises for Chapter 254.

Exercise 254.1. For every $n \in \mathbb{N}$ give an example of a compact connected smooth manifold M such that $\pi_n(M)$ is infinitely generated.

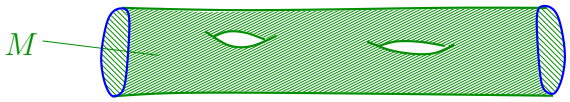
Exercise 254.2. Let M and N be smooth manifolds. Show that if M is homotopy equivalent to a CW-complex X and if N is homotopy equivalent to a CW-complex Y, then any boundary connected sum $M\#_{\mathrm{b}}N$ is homotopy equivalent to a wedge $X \vee Y$.

Exercise 254.3. Let $L \subseteq \mathbb{R}^3$ be a link, i.e. L is a closed 1-dimensional smooth submanifold of $\mathbb{R}^3$. Show that $\mathbb{R}^3 \setminus L$ is homotopy equivalent to a 2-dimensional CW-complex.

Exercise 254.4. We consider the four knots and links J_i, $i = 1, 2, 3, 4$, shown in the figure below. In each case find a 2-dimensional CW-complex X_i in $\mathbb{R}^3 \setminus J_i$ such that X_i is a deformation retract of $\mathbb{R}^3 \setminus J_i$.



Exercise 254.5. In the figure below we show a compact 2-dimensional smooth manifold M with *two* boundary components. The Smooth Manifold-CW-Homotopy Type Proposition 254.4 says that M is homotopy equivalent to 1-dimensional finite CW-complex with a single 0-cell. Show that there exists a 1-dimensional CW-complex X contained in M, with a single 0-cell, which is deformation retract of M.



CHAPTER 255

The classification of compact 2-dimensional smooth manifolds II

One of the dream results in mathematics is to classify interesting classes $\mathcal{S}$ of mathematical objects in a suitable category $\mathcal{C}$. A classification of $\mathcal{S}$ usually consists of a family of objects $\{S_i\}_{i\in I}$ with the following two properties:

(1) No two objects in the family are isomorphic.

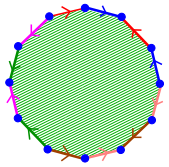
(2) Every $S \in \mathcal{S}$ is isomorphic to some S_i. Furthermore, in an ideal situation one would like an algorithm[2726] which takes as input S (given in a suitable form) and outputs the corresponding S_i.

For example, given a field $\mathbb{F}$ we could consider the class of finite-dimensional $\mathbb{F}$-vector spaces. Elementary linear algebra shows that the family $\{\mathbb{F}^n\}_{n\in\mathbb{N}_0}$ has the above properties (1) and (2).

In this chapter we will classify the class of closed (orientable or non-orientable) connected non-empty 2-dimensional smooth manifolds. We will see the following:

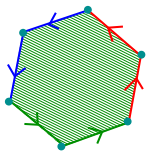
- In the orientable case the family will be given by $\{\Sigma_g\}_{g\in\mathbb{N}_0}$ the surfaces of genus g.
- In the non-orientable case the family will be given by $\{N_k\}_{k\in\mathbb{N}}$ the non-orientable surfaces of genus k.

Recall that in Chapter 101 we showed, using the calculation of fundamental groups, that in both settings the surfaces are pairwise non-homeomorphic. In this chapter we will see that any compact connected (non-) orientable non-empty 2-dimensional smooth manifold is diffeomorphic to one of the above standard surfaces.



$\cong \quad g \cdot (S^1 \times S^1)$

surface of genus g



$\cong \quad k \cdot \mathbb{R}\mathrm{P}^2$

non-orientable surface N_3 of genus 3

This chapter is organized as follows:

- In Section 255.1 we will, as a warm up, use handle decompositions to give a new proof that every compact connected non-empty 1-dimensional smooth manifold is diffeomorphic to S^1 or $[0, 1]$.
- In Section 255.2 we will discuss several features and instances of handle decompositions of 2-dimensional smooth manifolds.
- In Section 255.3 we will see how one can modify handle decompositions of surfaces and we will study handle decompositions of connected sums of surfaces.

[2726] In this discussion we will work with a very naive notion of an algorithm. We will not make precise what we mean by an algorithm.

- In Section 255.4 we will use the results from Sections 255.2 and 255.3 to complete the classification of compact 2-dimensional smooth manifolds.
- In Section 255.5 we will provide exercises for Chapter 255.

In the next Chapter 256 we will see how one can construct interesting smooth manifolds using handles.

255.1. **Classification of 1-dimensional smooth manifolds.** Before we turn to 2-dimensional smooth manifolds, let us practice our skills in the 1-dimensional setting. The following proposition covers an important special case of the Smooth 1-Manifold Classification Theorem 28.8.

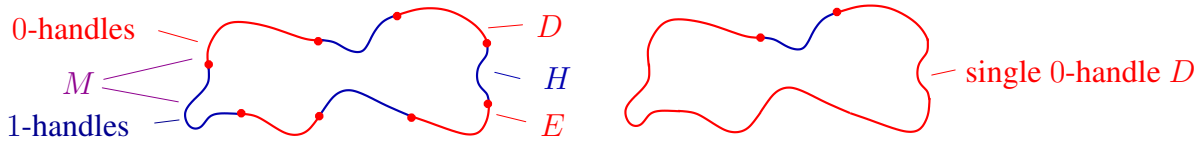
Proposition 255.1. (Compact Smooth 1-Manifold Classification Proposition) Every compact connected non-empty 1-dimensional smooth manifold is diffeomorphic either to S^1 or to $[0,1]$.

Remark. In Exercise 255.3 the reader is invited to show, using the Compact Submanifold Exhaustion Proposition 38.2 and the Compact Smooth 1-Manifold Classification Proposition 255.1, that every non-compact connected 1-dimensional smooth manifold is diffeomorphic either to $[0,1)$ or to $\mathbb{R}$. This completes the reproof of the Smooth 1-dimensional Manifold Classification Theorem 28.8. □

Proof. Let M be a compact connected non-empty 1-dimensional smooth manifold. By the Handle Decomposition Theorem 250.7 we can equip M with a standard handle decomposition.

Claim. There exists a standard handle decomposition for M with a single 0-handle.

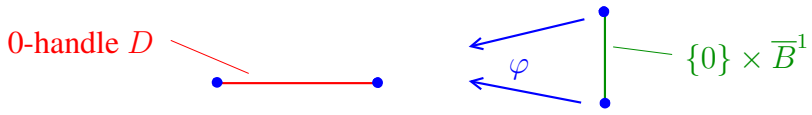
Proof. This claim is just a special case of the Minimal 0-Handle Proposition 247.13. For the reader's convenience we give an elementary self-contained argument: Since M is non-empty we know that M has at least one 0-handle. Now suppose that our handle decomposition has more than one 0-handle. Since M is connected and since the handle decomposition is standard it follows from the Handle Attachment-π_0-Lemma 246.5 that there exists a 1-handle H that gets glued to two different 0-handles D and E.



Since we are in the 1-dimensional setting it is elementary to see that $D \cup E \cup H$ is diffeomorphic to $[-1,1]$. Thus we can combine $D \cup E \cup H$ to a single 0-handle. Note that the resulting handle decomposition is still standard. We now proceed iteratively. ⊞

We now fix a handle decomposition for M with a single 0-handle D. If the handle decomposition does not contain a 1-handle, then $M = D$, in particular M is diffeomorphic to $[0,1]$.

Now suppose that the handle decomposition contains at least one 1-handle. Since the handle decomposition is standard we know that all attaching maps of the 1-handles take values in ∂D. Now let $\varphi\colon \{0\} \times S^0 \to \partial D$ be an attaching map of a 1-handle.
Note that the map is injective. Since both sets consist of precisely two elements we see that the attaching map is a bijection. In particular we see that $D \cup_\varphi (\{0\} \times \overline{B}^1)$ is a



closed 1-dimensional smooth manifold. It is elementary to see that this smooth manifold is diffeomorphic to S^1. Since this smooth manifold is closed we see that there are no other 1-handles. Thus we have shown that M is diffeomorphic to S^1.

Finally note that S^1 and $[0,1]$ are not diffeomorphic, since one has empty boundary and the other one has non-empty boundary. So M is diffeomorphic to either S^1 or $[0,1]$. ■

255.2. **Handle decompositions of 2-dimensional smooth manifolds.** After this warm-up we now turn to 2-dimensional smooth manifolds. Our goal is to show that every closed connected non-empty 2-dimensional smooth manifold M is diffeomorphic to one of the standard surfaces Σ_g or N_k. The idea will be to use handle decompositions. More precisely, we will show that we can always find a handle decomposition for M which is equivalent to a handle decomposition of some Σ_g or some N_k.

In the following we summarize what we already know about handle decompositions of 2-dimensional smooth manifolds:

Lemma 255.2. (Surface-Handle Decomposition Lemma) Every closed connected non-empty 2-dimensional smooth manifold admits a handle decomposition with the following properties:

(1) There exists a single 0-handle.
(2) The attaching maps of all of the 1-handles take values in the boundary of the 0-handle.[2727]
(3) There exists a single 2-handle.

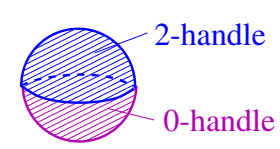
Proof. Let M be a closed connected non-empty 2-dimensional smooth manifold. By the Handle Decomposition Theorem 250.7 we know that M admits a handle decomposition. It follows from the Minimal 0-Handle Proposition 247.13 and the Minimal n-Handle Proposition 248.4 that we can arrange that we have a single 0-handle and a single n-handle. Finally, using the Standard Handle Decomposition Proposition 247.7 we can turn this handle decomposition of M into a standard handle decomposition. Since this process does not affect the number of handles of any given index we see that we have now obtained the desired handle decomposition. ■

The Surface-Handle Decomposition Lemma 255.2 motivates the following definition:

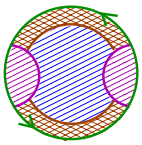
Definition 255.3. We say that a handle decomposition of a 2-dimensional smooth manifold is completely standard, if it is of the form described in the Surface-Handle Decomposition Lemma 255.2.

Examples. In the figure below we show completely standard handle decompositions for several of our favorite surfaces.
We will see more examples shortly. □

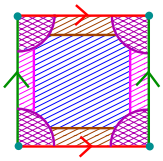
[2727] In fact this Statement (2) is in this context equivalent to saying that the handle decomposition is standard in the sense of Definition 247.6.



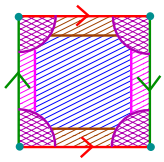
sphere S^2



real projective plane
$\mathbb{R}\,\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$



torus $[0,1]^2/\sim$

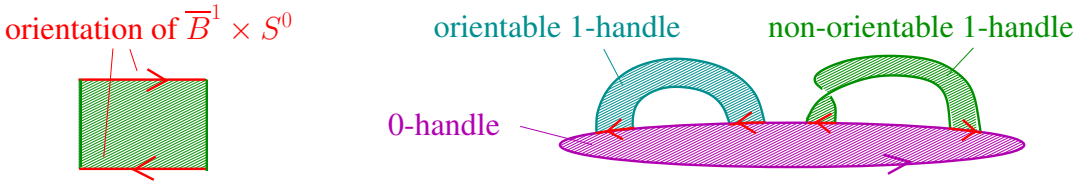


Klein bottle $[0,1]^2/\sim$

We now want to gain a better understanding of completely standard handle decompositions. Evidently there is not much to say about the single 0-handle. Thus let us turn to the 1-handle attachments. Let us recall Definition 246.7:

Definition 255.4. We say that a thickened 0-sphere $\varphi\colon \overline{B}^n \times S^0 \to \partial\overline{B}^2$ is orientable if φ is orientation-preserving or if it is orientation-reversing. Otherwise we say that the thickened 0-sphere is non-orientable. We call the corresponding 1-handle in $\overline{B}^2 \cup_\varphi h^1$ also orientable (respectively non-orientable).

Example. In the figure below we show an orientable and a non-orientable 1-handle.



The illustration to the right of figure (and the many figures below) should be interpreted as follows:

- The figure should *not* be viewed as a subspace of $\mathbb{R}^3$, but the figure should give a mnemonic device for an orientable (respectively) non-orientable handle, and the resulting smooth manifold $\overline{B}^2 \cup_\varphi h^1$ should be viewed as an abstract smooth manifold.
- To simplify the pictures we do not worry about introducing and rounding corners.
- By the 1-Handle Attachment Lemma 246.8 we know that 1-handle attachments are essentially determined by their orientability. So we do not need to specify the handle attachment precisely, it suffices to indicate whether it is orientable or non-orientable. □

We continue with the following elementary lemma:

Lemma 255.5. (Orientability-via-Handles Lemma) Let M be a 2-dimensional smooth manifold M with a completely standard handle decomposition. The following two statements are equivalent:
(1) M is orientable.
(2) All 1-handles are orientable.

Proof. This lemma follows immediately from the Handles-Orientability Proposition 246.6. ■

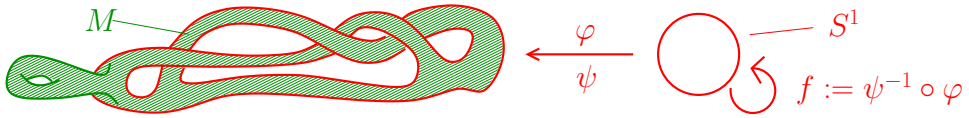
Convention 255.6. Let M be a 2-dimensional smooth manifold M with a single 0-handle $\overline{B}^2$. If M is orientable, then we equip M with the orientation such that the natural map $\overline{B}^2 \to M$ given by the 0-handle is orientation-preserving.

Now we turn to the final 2-handle. The following lemma shows that in the 2-dimensional setting the attaching map for 2-handles is essentially irrelevant:

Lemma 255.7. (Glue-in-Disk Lemma) Let M be a 2-dimensional smooth manifold, let C be a boundary component of M and let $\varphi\colon S^1 \to C$ be a diffeomorphism. The diffeomorphism type of $M \cup_\varphi h^2$ does not depend on the choice of φ.

Remark. The proof of the Glue-in-Disk Lemma 255.7 rests on the fact, proved in the Diffeomorphisms-of-S^1-Extension Corollary 42.9, that for $n = 2$ every diffeomorphism of S^{n-1} extends to a diffeomorphism of $\overline{B}^n$. Note that by the Munkres–Smale-Theorem **??** and the Cerf Theorem **??** the analogous statement also holds for $n = 3$ and $n = 4$. In particular the obvious analog of the Glue-in-Disk Lemma 255.7 also holds in those dimensions. But we will see in the Ball Gluing–Non-Uniqueness Proposition 282.9 that the analog of the Glue-in-Disk Lemma 255.7 does *not* hold for $n = 7$. □

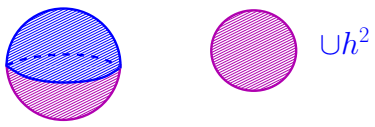
Proof. [2728] Let M be a 2-dimensional smooth manifold, let C be a boundary component of M and let $\varphi, \psi\colon S^1 \to C$ be two diffeomorphisms. We consider the self-diffeomorphism $f := \psi^{-1} \circ \varphi\colon S^1 \to S^1$.



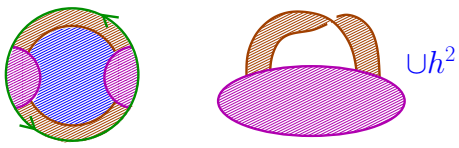
Note that by the Diffeotopies-of-S^1-Proposition 42.8 we know that there exists a diffeotopy $F\colon S^1 \times [0,1] \to S^1$ from $F_0 = f$ to the identity $F_1 = \mathrm{id}$ or to the reflection $F_1 = \rho$. As we saw in the proof of the Diffeomorphisms-of-S^1-Extension Corollary 42.9 this implies that the self-diffeomorphism f of S^1 extends to a self-diffeomorphism F of the disk $\overline{B}^2$. Finally note that we now obtain from the Smooth Manifold Pushout-Proposition 55.12 together with the Regluing Proposition 55.10 that the maps id_M and $F\colon \overline{B}^2 \to \overline{B}^2$ give rise to a diffeomorphism from $M \cup_\varphi h^2 = M \cup_{C \xleftarrow{\varphi} S^1} \overline{B}^2$ to $M \cup_\psi h^2 = M \cup_{C \xleftarrow{\psi} S^1} \overline{B}^2$. ■

Convention 255.8. Let M be a closed 2-dimensional smooth manifold. If we are given a handle decomposition for M with a single 2-handle, then in the figures we show the 0-handles and the 1-handle attachments and we indicate with a discrete $\cup h^2$ the 2-handle. Note that by the Glue-in-Disk Lemma 255.7 the 2-handle attachment is basically unambiguous.

Examples. In the figure below we use the above notation to redraw the above completely standard handle decompositions:



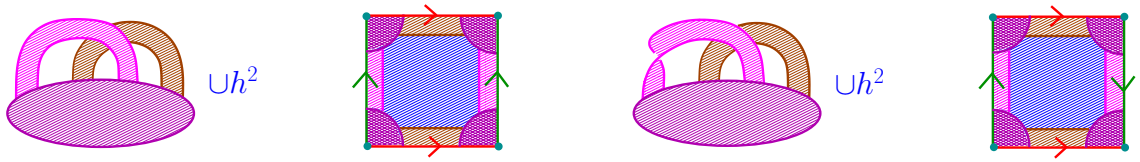
handle decomposition for the sphere



handle decomposition for $\mathbb{R}\mathrm{P}^2$

In the new notation we start out with the "same" 0-handle, we attach the 1-handles, and then we just need to indicate by "$\cup h^2$" the remaining 2-handle. □

[2728] We gave the argument already on page 1233.

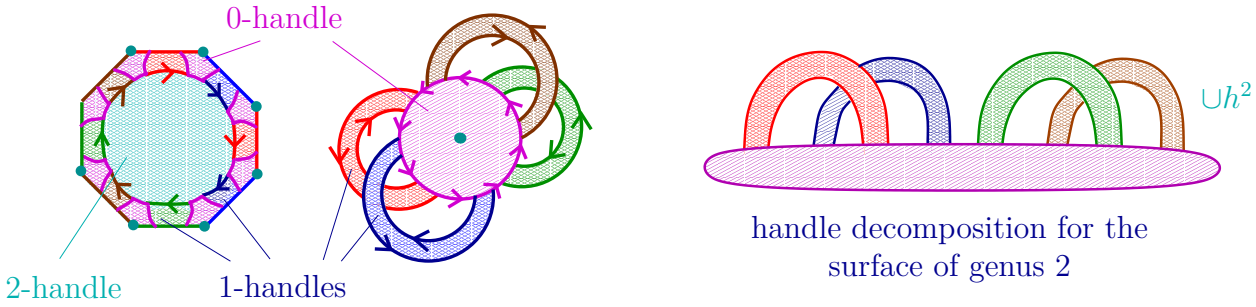


handle decomposition for the torus — handle decomposition for the Klein bottle

We conclude this section with more examples:

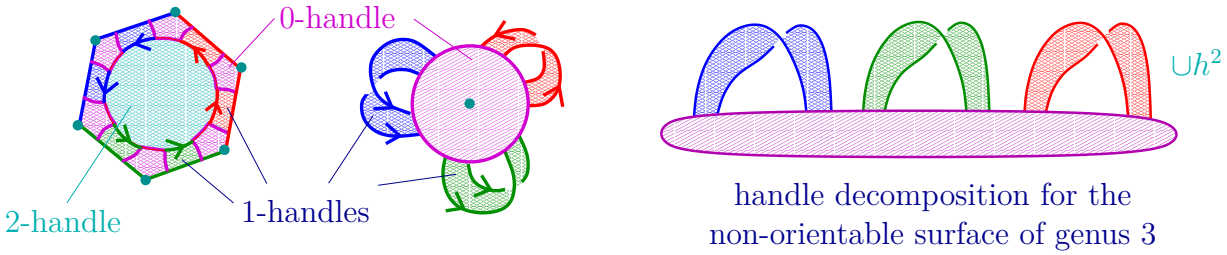
Examples.

(1) In the figure below we show a completely standard handle decomposition for the surface of genus two. This handle decomposition has obvious generalizations to higher genera.



handle decomposition for the surface of genus 2

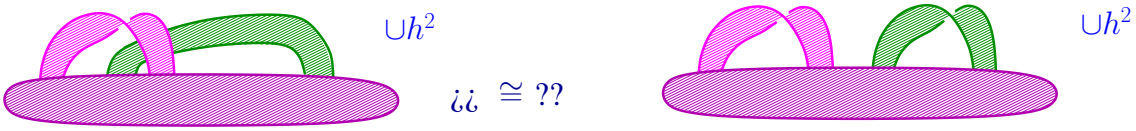
(2) In the figure below we show a completely standard handle decomposition for the non-orientable surface of genus 3, as introduced in Definition 9.19.



handle decomposition for the non-orientable surface of genus 3

Again this example has obvious generalizations to higher genera. □

255.3. **Modifying handle decompositions of 2-dimensional smooth manifolds.** In the previous section we saw the following handle decomposition of the Klein bottle and of the non-orientable surface of genus 2:



Klein bottle — non-orientable surface of genus 2

Both are non-orientable surfaces with completely standard handle decompositions which have precisely two 1-handles. The question arises, whether these two surfaces are actually diffeomorphic. To answer this question it is helpful to have a criterion for surfaces to be diffeomorphic. This leads us to the following lemma:

Lemma 255.9. (Handle Slide Lemma)

(1) Let M be a 2-dimensional smooth manifold and let $\varphi, \psi \colon \overline{B}^1 \times S^0 \to \partial M$ be two attaching maps. If φ, ψ are smoothly isotopic, then there exists a diffeomorphism $M \cup_\varphi h^1 \to M \cup_\psi h^1$. Any such replacement of the 1-handle given by φ by the 1-handle given by ψ is called a handle slide.

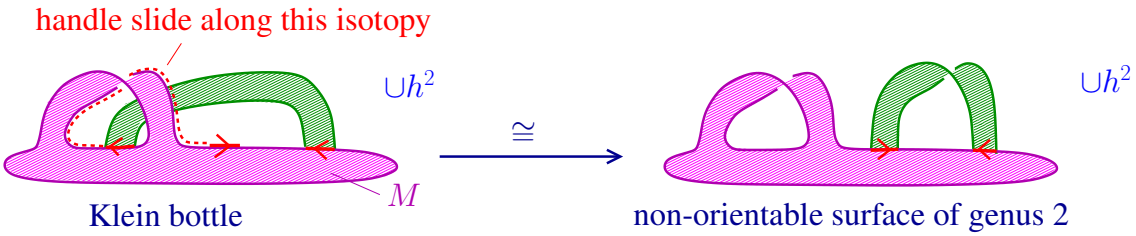
(2) Let M and N be two closed 2-dimensional smooth manifolds that are equipped with completely standard handle decompositions. If the 1-handle attachments for the handle decomposition of M can be turned into the 1-handle attachments for the handle decomposition of N via handle slides, then M and N are diffeomorphic.

Proof.

(1) This statement is just a convenient special case of the Handle Attachment Proposition 246.3 (5).

(2) This statement follows from (1), the Modify Handle Attachment Region Lemma 247.8 together with the Glue-in-Disk Lemma 255.7. ■

Example. In the figure below we show again the Klein bottle K and the non-orientable surface N_2 of genus 2.

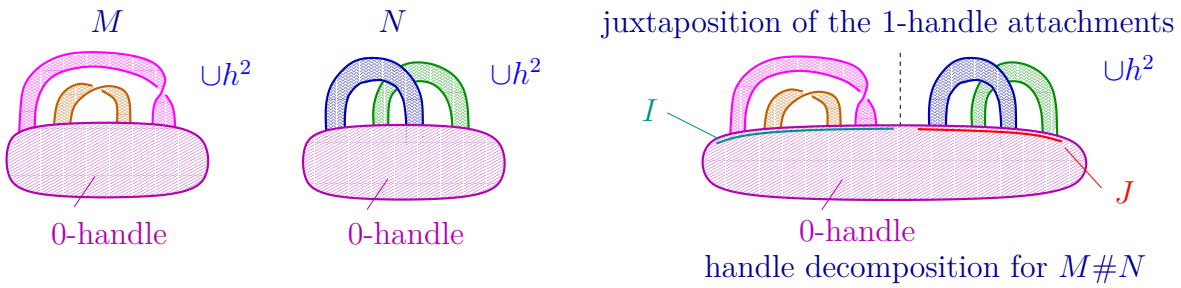


More precisely, we consider $M = \overline{B}^2 \cup h^1$ and we write $K = M \cup_\varphi h^1 \cup h^2$ and we write $N_2 = M \cup_\psi h^1 \cup h^2$. In the figure we see that the attaching maps $\varphi, \psi \colon \overline{B}^1 \times S^0 \to \partial M$ are smoothly isotopic. It thus follows from the Handle Slide Lemma 255.9 (2) that the Klein bottle K is diffeomorphic to N_2. □

Next we study the connected sum of two surfaces that are equipped with a completely standard handle decomposition:

Lemma 255.10. (Surface Connected Sum–Handles Lemma) Let M and N be two closed 2-dimensional smooth manifolds. We assume that both are equipped with a completely standard handle decomposition. Then $M \# N$ has a completely standard handle decomposition such that the 1-handles are given by "juxtaposing" the 1-handle attachments for M and the 1-handle attachments for N. We will not give a formal definition of "juxtaposing", instead we refer to the figure below for a hopefully unambiguous "definition by illustration".

Proof. For the proof we introduce the following essentially self-explanatory notation: We write $\overline{B}^2_{\geq 0} := \{(x,y) \in \overline{B}^2 \,|\, y \geq 0\}$, we write $\overline{B}^2_{=0} := \{(x,y) \in \overline{B}^2 \,|\, y = 0\}$ and finally we write $B^2_{\geq 0} := \{(x,y) \in B^2 \,|\, y \geq 0\}$. The proof of the lemma makes use of the following technical claim:



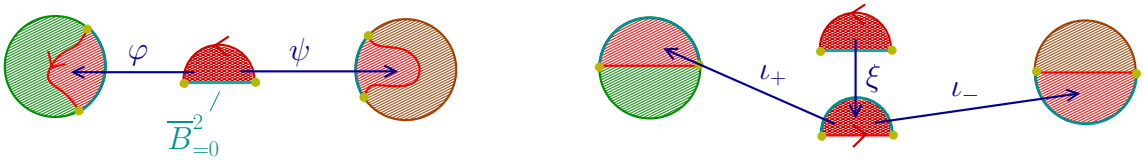
Claim. We assume we are given the following:

- Let $\varphi\colon \overline{B}^2_{\geq 0} \to \overline{B}^2$ be an orientation-preserving smooth embedding with $\varphi(\overline{B}^2_{=0}) \subseteq S^1$.
- Let $\psi\colon \overline{B}^2_{\geq 0} \to \overline{B}^2$ be an orientation-reversing smooth embedding with $\psi(\overline{B}^2_{=0}) \subseteq S^1$.

Then
$$((\overline{B}^2 \setminus \varphi(B^2_{\geq 0})) \sqcup (\overline{B}^2 \setminus \psi(B^2_{\geq 0})))/\varphi(x,y) \sim \psi(x,y) \quad \text{for } (x,y) \in \overline{B}^2_{\geq 0} \cap S^1$$
is diffeomorphic to $\overline{B}^2$.

Proof. We start out with the following observations and preparations:

(1) As in the Smooth Ball Embedding Theorem 45.10 one can show that all such smooth embeddings are smoothly isotopic.
(2) As in the proof of the Smooth Connected Sum-Proposition 54.13 one sees that smoothly isotopic embeddings lead to diffeomorphic smooth manifolds.
(3) As in the Half Ball-Proposition 60.12 one can show that there exists an orientation-preserving diffeomorphism $\xi\colon \overline{B}^2_{\geq 0} \to \overline{B}^2_{\geq 0}$ which flips $\overline{B}^2_{=0}$ and $\overline{B}^2_{\geq 0} \cap S^1$.
(4) We denote by $\iota_-\colon \overline{B}^2_{\geq 0} \to \overline{B}^2$ and $\iota_+\colon \overline{B}^2_{\leq 0} \to \overline{B}^2$ the inclusions into the upper and lower hemisphere.



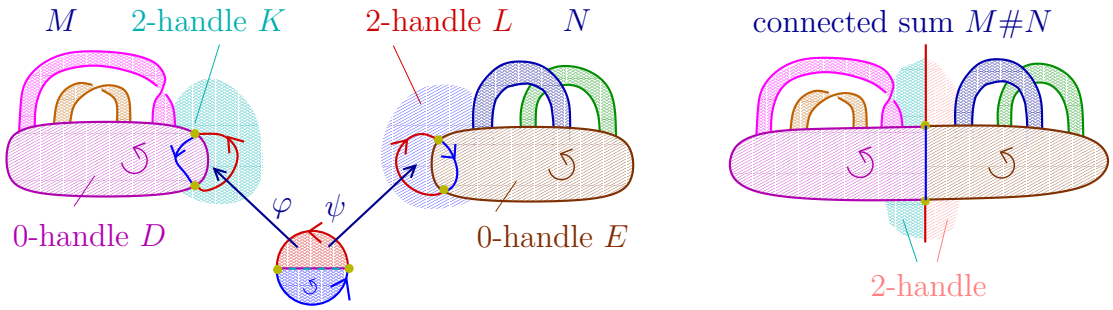
We now see that
$$((\overline{B}^2 \setminus \varphi(B^2_{\geq 0})) \sqcup (\overline{B}^2 \setminus \psi(B^2_{\geq 0})))/\varphi(x,y) \sim \psi(x,y) \;=\; (\overline{B}^2_{\leq 0} \sqcup \overline{B}^2_{\geq 0})/(x,0) \sim (x,0) \;=\; \overline{B}^2.$$
$\uparrow$ follows from (1), (2), (3) and the observation that $(\iota_+ \circ \xi)(B^2_{\geq 0}) = \overline{B}^2_{>0}$ and $(\iota_- \circ \xi)(B^2_{\geq 0}) = \overline{B}^2_{<0}$ ⊞

After these preliminary discussions we turn to the actual proof. We denote by $D = \overline{B}^2$ the unique 0-handle of M and we denote by K the unique 2-handle of M. Furthermore we denote by $E = \overline{B}^2$ the unique 0-handle of N and we denote by L the unique 2-handle of N. Note that it follows almost immediately from Exercise 248.2 that we can pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^2 \to M$ that restricts to a smooth embedding $\varphi\colon \overline{B}^2_{\geq 0} \to D$ and to a smooth embedding $\varphi\colon \overline{B}^2_{\leq 0} \to K$. Similarly we pick an orientation-reversing

smooth embedding $\varphi\colon \overline{B}^2 \to N$ with analogous properties. Now we see that

$$M\#N \overset{\text{by Definition 54.12 of the connected sum, note that if } M \text{ and } N \text{ are oriented, then the orientation conventions work out}}{\overset{\downarrow}{=}} (M \setminus \varphi(B^2)) \sqcup (N \setminus \psi(B^2))/\sim \quad \text{where } \varphi(x) \sim \psi(x) \text{ for } x \in S^1$$

$$= \overbrace{(D \setminus \varphi(B^2_{\leq 0})) \cup (E \setminus \psi(B^2_{\leq 0}))}^{\text{by the claim: diffeomorphic to } \overline{B}^2} \cup \text{ 1-handles of } M \cup \text{ 1-handles of } N \cup \underbrace{(K \setminus \varphi(B^2_{\geq 0})) \cup (L \setminus \psi(B^2_{\geq 0}))}_{\text{by the claim: diffeomorphic to } \overline{B}^2}.$$

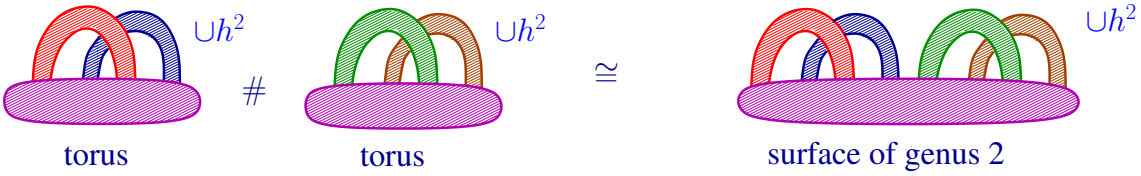


Thus we have obtained a handle decomposition for $M\#N$ with a single 0-handle, a single 2-handle, and such that the handles of M and N get attached to disjoint semicircles in the boundary of the 0-handle. In other words, the 1-handle attachments are juxtaposed. ■

The following examples give us in particular a new proof of the Surface–Connected Sum Lemma 54.15:

Examples.

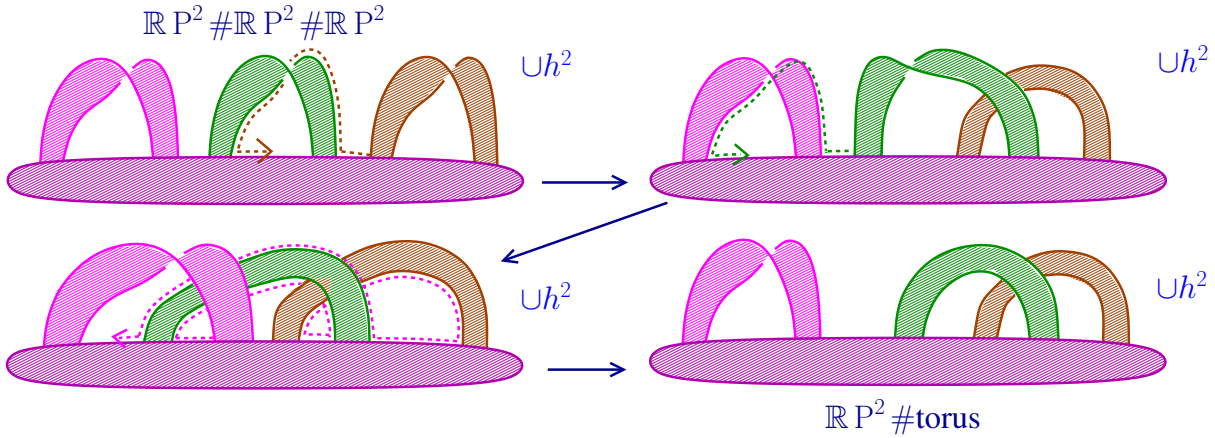
(1) It follows immediately from the completely standard handle decomposition of the surface of genus g from page 4918, together with the Surface Connected Sum–Handles Lemma 255.10, that the connected sum of the surface of genus g and the surface of genus h is diffeomorphic to the surface of genus $g + h$. Iteratively we see that the connected sum of g tori is diffeomorphic to the surface of genus g.



(2) As in (1) it follows immediately from the completely standard handle decomposition of the non-orientable surface of genus g from page 4918, together with the Surface Connected Sum–Handles Lemma 255.10, that the connected sum of the non-orientable surface of genus k and the non-orientable surface of genus l is diffeomorphic to the non-orientable surface of genus $k + l$. Iteratively we see that the connected sum of k real projective places is diffeomorphic to the non-orientable surface of genus k.



(3) (a) In the figure below we show, using altogether three handle slides, that the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the connected sum of $\mathbb{R}\mathrm{P}^2$ with the torus.



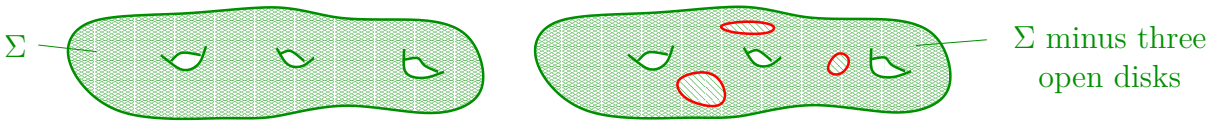
(b) Now let $g \in \mathbb{N}_0$. We consider the connected sum of $2g+1$ copies of $\mathbb{R}\mathrm{P}^2$ as given by attaching $2g+1$ non-orientable 1-handles to a single 0-handle. We iteratively apply the procedure in the figure above to the three right-most non-orientable 1-handles. We turn the two right-most non-orientable 1-handles into a pair of interlocking orientable 1-handles. Eventually we see that there exists a diffeomorphism

$$(2g+1)\cdot\mathbb{R}\mathrm{P}^2 \;\to\; \mathbb{R}\mathrm{P}^2\#\Sigma_g.$$

□

255.4. **The classification of compact 2-dimensional smooth manifolds.** In this section we will state and prove the classification theorem for compact connected non-empty 2-dimensional smooth manifolds. To do so state the theorem we first need to introduce the standard models for the non-closed case. We remind the reader of Definition 101.10:

Definition 255.11. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks.



We recall the following lemma, which captures some of the main features of the above definition.

Lemma 101.11 (Surface-Minus-Open Disks Lemma) Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.

(1) Σ minus n open disks is a compact submanifold of the smooth manifold Σ and it has n boundary components.
(2) The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.
(3) Σ minus n open disks is orientable if and only if Σ is orientable.

Next we recall the following notation that we first introduced in Notation 101.12.

Notation 255.12. Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} \;:=\; \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \;:=\; \text{the non-orientable surface of genus } g \text{ minus } n \text{ open disks.}$$

By now we have more than enough tools to show that these smooth manifolds are pairwise non-homeomorphic, let alone diffeomorphic. More precisely, in the following proposition[2729] we collect many of our invariants which we have calculated throughout these notes. Note that the first three rows are enough to distinguish all of the above examples up to homeomorphism:[2730]

Proposition 255.13. (Surface–Invariants Proposition) Given a smooth manifold X we denote by $\widehat{X}$ the topological space that we obtain by identifying all points that lie on the same boundary component. For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:

	X	Σ_g	$\Sigma_{g,m}$ with $m \geq 1$	N_k	$N_{k,n}$ with $n \geq 1$
(1)	$\pi_1(X)_{\mathrm{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{k+n-1}$
(2)	$b_0(\partial X)$	0	m	0	n
(3)	$\pi_1(\widehat{X})_{\mathrm{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$
(4)	$\mathrm{H}_1(X)$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{k-1}$
(5)	$\mathrm{H}_1(X, \partial X)$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{k+n-1}$
(6)	$\chi(X)$	$2-2g$	$2-2g-m$	$1-k$	$1-k-n$
(7)	orientable	yes	yes	no	no.

Proof. Rows (1), (2) and (3) already appeared in the earlier Surface–Full Invariants Proposition 101.20. Let us look at the remaining rows:

(4) This row was calculated in the Surface-H_*-Proposition 155.17.
(5) The quickest way to get these results is perhaps as follows: We have isomorphisms

$$\mathrm{H}_1(X, \partial X) \;\underset{\text{by the Poincaré Duality Theorem 210.1}}{\overset{\cong}{\uparrow}}\; \mathrm{H}^1(X; \mathbb{Z}) \;\underset{\text{by the Universal Coefficient H}^*\text{-Theorem 201.23}}{\overset{\cong}{\uparrow}}\; \mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z})$$

[2729] The proposition is an expanded version of the Surface–Full Invariants Proposition 101.20.
[2730] Alternatively, in the Surfaces-Non Diffeomorphic Corollary 53.28 we used the smooth Euler characteristic to prove that the surfaces Σ_g are pairwise non-diffeomorphic.

and thus the results follow from the calculation in Row (4). Alternative approaches to calculating these relative homology groups are explored in Exercise 155.22 and in Exercise 193.2.

(6) This row can be obtained from the Surface-H_*-Proposition 155.17 together with the Homology-χ-Proposition 156.2. Alternatively one can get the values from the discussion on page 2987 together with the Manifold-minus-Ball-χ-Lemma 156.14.

(7) This row follows from the discussions on page 822, the Manifold–Orientation Basics Lemma 31.31 (6b), the Surface Non-Orientability Lemma 31.45 and the Gluing-Smooth Manifolds-Proposition 54.5 (4). ■

In this section we will prove that any compact connected non-empty 2-dimensional smooth manifold is diffeomorphic to one of the above surfaces. This finally completes the classification of compact surfaces that we started in Chapter 101.

Here is the precise theorem we intend to prove:

Theorem 255.14. (Diffeomorphism-to-Standard Surfaces Theorem) Let M be a compact connected non-empty 2-dimensional smooth manifold.

(1) If M is orientable, then M is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.

(2) If M is non-orientable, then M is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Remark.

(1) Our proof of the Diffeomorphism-to-Standard Surfaces Theorem 255.14 is inspired by the ambitious [**GoS1999**, Exercise 4.2.4]. The key idea was communicated to us by Robert Gompf. The proof we provide is also outlined in [**Mart2016**, Theorem 6.1.6].

(2) In Exercise **??** we will also sketch an alternative approach to proving the Diffeomorphism-to-Standard Surfaces Theorem 255.14.

(3) One can prove the Diffeomorphism-to-Standard Surfaces Theorem 255.14 through Morse theory, without using handles. This approach is taken in [**Mata2002**, Chapter 5.2] and [**Hirs1976**, Chapter 3].

(4) Recall that on page 1992 we discussed several other approaches to proving the analog of the Diffeomorphism-to-Standard Surfaces Theorem 255.14 in the setting of topological manifolds. □

To increase readability we split the proof of the Diffeomorphism-to-Standard Surfaces Theorem 255.14 into two parts. First we treat closed manifolds, afterwards we will prove the general case.

Proof of Theorem 255.14 for closed manifolds. Let M be a closed connected non-empty 2-dimensional smooth manifold. We need to show that M is diffeomorphic to some Σ_g or some N_k. By the Surface-Handle Decomposition Lemma 255.2 we know that M admits a completely standard handle decomposition, i.e. we can pick a handle decomposition with the following properties:

(1) There exists a single 0-handle D.

(2) The attaching maps of all of the 1-handles take values in the boundary of the 0-handle.

(3) There exists a single 2-handle.

We prove this statement by induction on the number of 1-handles. If we have zero 1-handles, then M is given by gluing a 2-handle to a 0-handle via some attaching map $\varphi\colon S^1 \to S^1$.

Now we see that we have the following diffeomorphisms:

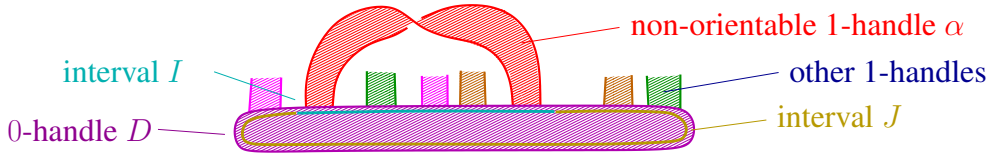
$$M = \overline{B}^2 \cup_\varphi h^2 = \overline{B}^2 \cup_{S^1 \xleftarrow{\varphi} S^1} \overline{B}^2 \underset{\uparrow}{\cong} \overline{B}^2 \cup_{S^1 \xleftarrow{\mathrm{id}} S^1} \overline{B}^2 \underset{\uparrow}{\cong} S^2.$$

by the Glue-in-Disk Lemma 255.7 all attaching maps give the same diffeomorphism type

by the Ball–Handle Attachment Lemma 246.9

Next suppose that the claim holds whenever the number of 1-handles is $\leq k$. Thus let M be a smooth manifold as above with $k+1$ 1-handles.

Perhaps slightly surprisingly the case that the smooth manifold is non-orientable is a little bit easier. So we first deal with this case. Since M is now assumed to be non-orientable we obtain from the Orientability-via-Handles Lemma 255.5 that there exists a non-orientable 1-handle $\cup_\alpha h^1$. Let $I \subseteq \partial D$ be a compact interval that contains the image of the attaching map of the chosen 1-handle α and let $J \subseteq \partial D$ be the compact interval with $\partial D = I \cup J$ and $I \cap J = \partial I = \partial J$.



standard handle decomposition for the non-orientable M with $k+1$ 1-handles

Finally we denote by $\psi_1, \dots, \psi_k$ the attaching maps of the remaining 1-handles. Here comes the key observation:

> *Since the 1-handle is non-orientable we see that $\partial(D \cup_\alpha h^1)$ is connected. Thus we can iteratively slide all handle attachments $\psi_1, \dots, \psi_k$ that end up on I across the chosen 1-handle so that they take values in the interval J.*

We denote the new handle attachment maps by $\widetilde{\psi}_1, \dots, \widetilde{\psi}_k$. We now see that

follows from the Handle Slide Lemma 255.9 (2)

$$M \cong \overline{B}^2 \cup_\alpha h^1 \cup_{\psi_1} h^1 \cdots \cup_{\psi_k} h^1 \cup h^2 \underset{\downarrow}{\cong} \overline{B}^2 \cup_\alpha h^1 \cup_{\widetilde{\psi}_1} h^1 \cdots \cup_{\widetilde{\psi}_k} h^1 \cup h^2$$

$$\cong \underbrace{(\overline{B}^2 \cup_\alpha h^1 \cup h^2)}_{\cong \mathbb{RP}^2,\ \text{see the figure on page 4917}} \# \underbrace{(\overline{B}^2 \cup_{\widetilde{\psi}_1} h^1 \dots \cup_{\widetilde{\psi}_k} h^1 \cup h^2)}_{\text{by induction} \cong k \cdot \mathbb{RP}^2 \text{ or} \cong \Sigma_{\frac{k}{2}}} \cong (k+1) \cdot \mathbb{RP}^2 \cong \text{non-orientable surface of genus } k+1.$$

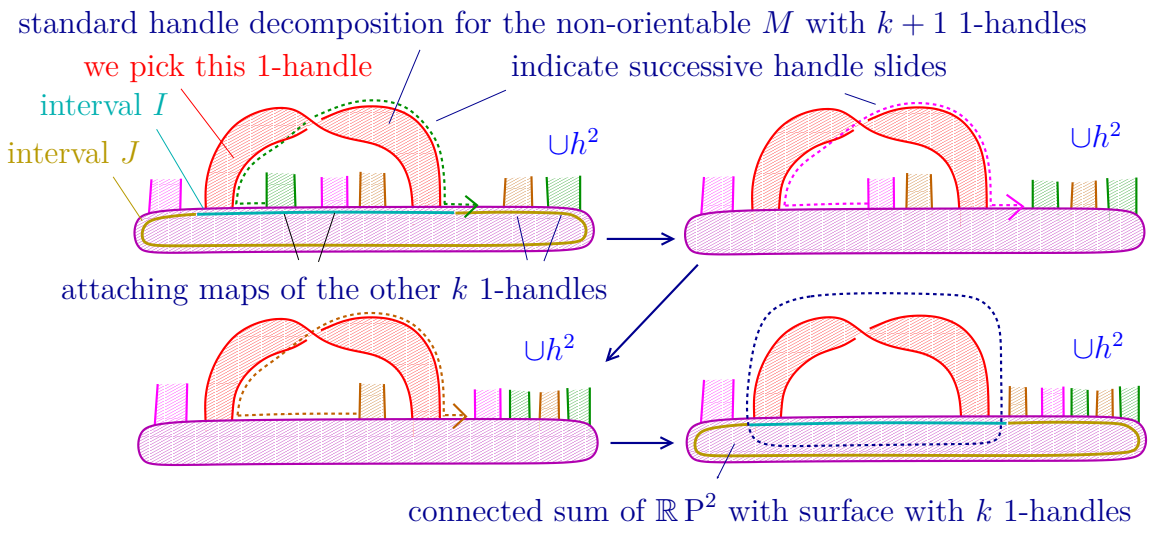
by the Surface–Connected Sum–Handles Lemma 255.10

on page 4922 we showed that $\mathbb{RP}^2 \# \Sigma_l \cong (2l+1) \cdot \mathbb{RP}^2$

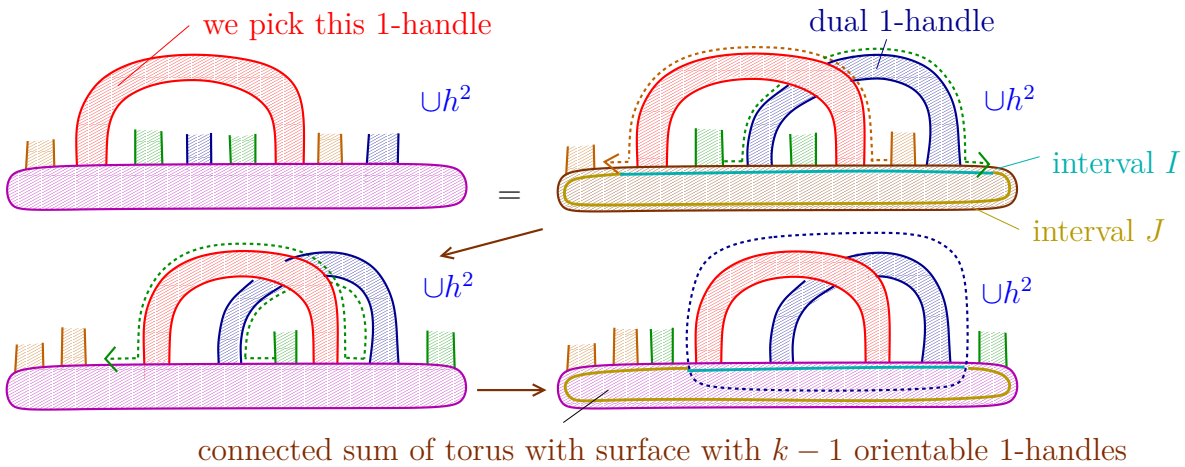
by the discussion on page 4921

We have thus proved the induction step if M is non-orientable.

Now we turn to the case that M is orientable. We pick any 1-handle $\cup_\alpha h^1$. Since M is orientable the chosen 1-handle is orientable. Note that this implies that the boundary of $\overline{B}^2 \cup_\alpha h^1$ is disconnected. But since we work with a completely standard handle decomposition we know that, once we attached all 1-handles, the boundary is connected. This implies that there exists at least one other 1-handle that connects the two components of the boundary of $\overline{B}^2 \cup_\alpha h^1$. We pick such a 1-handle $\cup_\beta h^1$. We denote by $\psi_1, \dots, \psi_{k-1}$ the attaching maps of the remaining 1-handles. Let $I \subseteq \partial D$ be a compact interval that



contains the image of the attaching maps α and β and let $J \subseteq \partial D$ be the compact interval with $\partial D = I \cup J$ and $I \cap J = \partial I = \partial J$.



Here comes the key observation:

It is straightforward to see that $\partial(\overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1)$ is connected. Thus we can iteratively slide all handle attachments $\psi_1, \ldots, \psi_k$ that end up on I across the two 1-handles so that they take values in the interval J.

We denote the new handle attachment maps by $\widetilde{\psi}_1, \dots, \widetilde{\psi}_{k-1}$. We now see that

$$M \cong \overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup_{\psi_1} h^1 \cdots \cup_{\psi_{k-1}} h^1 \cup h^2 \overset{\text{follows from the Handle Slide Lemma 255.9 (2)}}{\cong} \overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup_{\widetilde{\psi}_1} h^1 \cdots \cup_{\widetilde{\psi}_{k-1}} h^1 \cup h^2$$

$$\cong \underbrace{(\overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup h^2)}_{\cong \text{ torus, see the figure on page 4918}} \ \# \ \underbrace{(\overline{B}^2 \cup_{\widetilde{\psi}_1} h^1 \dots \cup_{\widetilde{\psi}_{k-1}} h^1 \cup h^2)}_{\text{since all handles are orientable we know that this is an orientable surface, thus by induction we know that it is diffeomorphic to } \Sigma_{\frac{k-1}{2}}} \overset{\text{follows from the Surface–Connected Sum Lemma 54.15 (1) or alternatively the discussion on page 4921}}{\cong} \text{surface of genus } \tfrac{k+1}{2}.$$

by the Surface Connected Sum–Handles Lemma 255.10

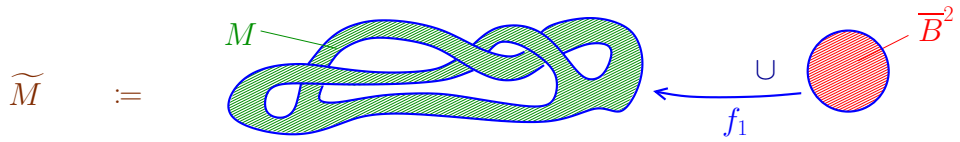
We have thus now proved the induction step also in the case that M is orientable. ■

In the following we will finally deal with 2-dimensional smooth manifolds with possibly non-empty boundary.

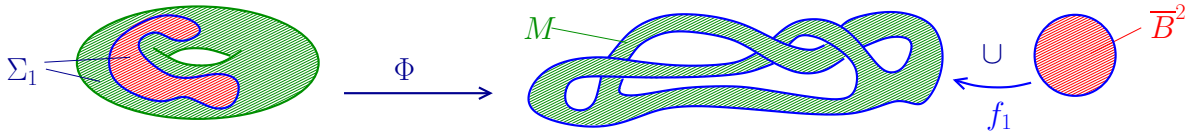
Proof of Theorem 255.14 for all manifolds. Let M be a compact connected non-empty 2-dimensional smooth manifold. By the Smooth Manifold–Boundary Proposition 25.27 (1) and (2) the boundary ∂M is a closed 1-dimensional smooth manifold. Since ∂M is compact we know by the Manifold–Local Properties Lemma 23.8 (3b) that it has finitely many components. We denotes these by $C_1, \dots, C_n$. (Note that we allow the case that $n = 0$.) Each component C_i is compact and has no boundary. Thus it follows from the Compact Smooth 1-Manifold Classification Proposition 255.1 that for each $i = 1, \dots, n$ there exists a diffeomorphism $f_i \colon S^1 \to C_i$.

We denote by $\widehat{M}$ the result of gluing a disk to each boundary component. More precisely, we consider

$$\widehat{M} \coloneqq \Big(M \sqcup \bigsqcup_{i=1}^{n} (\overline{B}^2 \times \{i\}) \Big) / \sim \quad \text{where } f_i(P) \sim (P, i) \text{ for } P \in S^1 \text{ and } i \in \{1, \dots, n\}.$$



By the Gluing-Smooth Manifolds-Proposition 54.5 we know that $\widehat{M}$ is a closed connected non-empty 2-dimensional smooth manifold that contains M as a smooth submanifold and which has the property that the obvious maps $\overline{B}^2 \times \{i\} \to \widehat{M}$ are all smooth embeddings. Furthermore it follows from the Gluing-Smooth Manifolds-Proposition 54.5 (5) and the observation that $\overline{B}^2$ admits an orientation-reversing diffeomorphism, that M is orientable if and only if $\widehat{M}$ is orientable.



Now suppose that M is orientable. By the above we know that $\widehat{M}$ is also orientable. Since $\widehat{M}$ is closed orientable connected and non-empty we know by the above that there exists a

diffeomorphism $\Phi\colon \Sigma_g \to \widehat{M}$ from the surface of some genus g to $\widehat{M}$. Then Φ restricts to a diffeomorphism

$$\Phi\colon \underbrace{\Sigma_g \setminus \bigcup_{i=1}^{n} \Phi^{-1}\big(B^2 \times \{i\}\big)}_{= \Sigma_{g,n} \text{ since the maps } \overline{B}\times\{i\}\to\Sigma_g \text{ are all smooth embeddings}} \to M.$$

If M is non-orientable, then $\widehat{M}$ is closed non-orientable connected and non-empty, thus we know that it is diffeomorphic to the non-orientable surface N_g for some g. As above we see that M is diffeomorphic to $N_{g,n}$. ■

255.5. **Exercises for Chapter 255.** Note that many of the exercises which we formulated in Chapter 101, are also relevant for this chapter. We refer to Section 101.7 for these exercises.

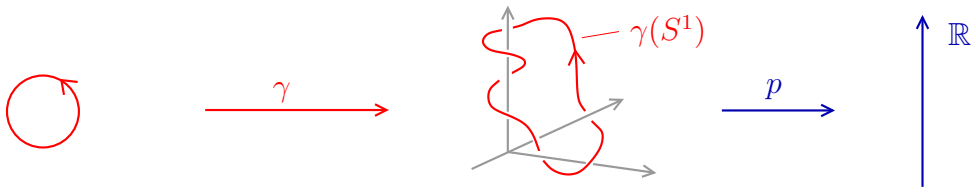
Exercise 255.1. An n-dimensional *non-Hausdorff-manifold* is a topological space that is locally homeomorphic to an open subset of $\mathbb{R}^n$. Note that the notion of a smooth structure and of an orientation makes sense for non-Hausdorff-manifolds.

(a) In the figure we show the topological space M which is the "circle with two East Poles". Give a mathematical precise definition of M and show that it is a 1-dimensional smooth non-Hausdorff-manifold.
(b) Are there infinitely many pairwise non-diffeomorphic compact connected smooth non-Hausdorff-manifolds?
(c) Are there uncountable many pairwise non-diffeomorphic compact connected 1-dimensional smooth non-Hausdorff-manifolds?
(d) Show that there exists a non-orientable 1-dimensional smooth non-Hausdorff-manifold.

Remark. This exercise is very similar to Exercises 28.1, 31.23 and 191.5.



Exercise 255.2. We denote by $p\colon \mathbb{R}^3 \to \mathbb{R}$ the projection onto the third coordinate. Let $\gamma\colon S^1 \to \mathbb{R}^3$ be a smooth embedding such that the map $p \circ \gamma\colon S^1 \to \mathbb{R}$ has precisely two critical points. Show that $\gamma(S^1)$, viewed as a submanifold of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ is smoothly isotopic to the trivial knot.



Exercise 255.3. Show that every non-compact connected 1-dimensional smooth manifold is diffeomorphic either to $[0, 1)$ or to $\mathbb{R}$.

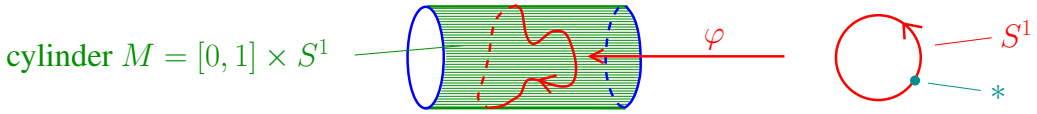
Hint. Use the Compact Submanifold Exhaustion Proposition 38.2 and the Compact Smooth 1-Manifold Classification Proposition 255.1.

Exercise 255.4. Let F, G be compact connected 2-dimensional smooth manifolds and let M be some closed non-empty smooth manifold of arbitrary dimension. We assume that $F \times M$ is diffeomorphic to $G \times M$. Does it follow that F is diffeomorphic to G?

Exercise 255.5. Let $k, n \in \mathbb{N}$. We consider the non-orientable surface $N_{k,n}$ minus n open disks and we consider the double $W := \mathrm{D}N_{k,n}$. To what "standard" surface is W diffeomorphic to?

Exercise 255.6. We consider $\Sigma = \Sigma_{2,1}$, i.e. the surface of genus 2 minus one open disk.
(a) What are the possible diffeomorphism types of connected 2-fold coverings $\widetilde{\Sigma} \to \Sigma$?
(b) What are the possible diffeomorphism types of connected 3-fold coverings $\widetilde{\Sigma} \to \Sigma$?

Exercise 255.7. We consider the cylinder $M = [0,1] \times S^1$. Let $\varphi\colon S^1 \to M \setminus \partial M$ be a smooth embedding such that $\varphi_*\colon \pi_1(S^1, *) \to \pi_1(M, \varphi(*))$ is a monomorphism. Show that there exists a smooth isotopy from φ to the map $x \mapsto (0, x)$.



Exercise 255.8. Let $p\colon M \to B$ be a smooth bundle map where M is a closed simply connected 4-dimensional smooth manifold and where B is a 2-dimensional smooth manifold. Show that B and the fiber both are diffeomorphic to S^2.
Hint. Make use of the Fiber-LES-Proposition 239.8.

CHAPTER 256

Constructing smooth manifolds using handles

In this chapter we will do the following:

(1) We will use handles to construct interesting compact and closed smooth manifolds out of finite presentations of groups. In particular we will show that given any finitely presented group π and given any $n \geq 4$ there exists a closed orientable connected n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.
(2) We will show that every finite CW-complex is homotopy equivalent to a compact smooth manifold.

This chapter is organized as follows:

- In Section 256.1 we will see that we can associate to any finite presentation a smooth manifold together with a handle decomposition.
- In Section 256.2 we will construct homology spheres (with non-trivial fundamental groups) in dimension ≥ 4.
- In Section 256.3 we will see that every finite CW-complex is homotopy equivalent to a compact smooth manifold.
- In Section 256.4 we will discuss the following question: Under what circumstances is a given finite CW-complex homotopy equivalent to a *closed* topological manifold?
- In Section 256.5 we will use the results from Section 256.3 and some earlier results to give a new proof of the Fiber Bundle-χ-Theorem 242.7 which says that the Euler characteristic of bundles over finite CW-complexes is multiplicative.
- In Section 256.6 we will provide exercises for Chapter 256.

In the next Chapter 257 we will discuss fundamental groups of 3-dimensional smooth manifolds.

256.1. **Finite presentations of groups and handle decompositions.** The following proposition tells us that presentations of groups give rise to smooth manifolds:

Proposition 256.1. (Presentation–Handle Realization Proposition) Let

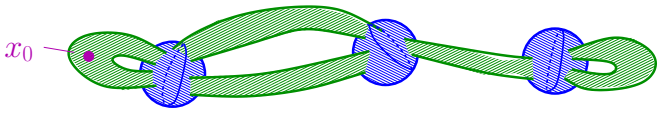
$$\pi \;=\; \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle$$

be a finite presentation. Given any $n \geq 4$ there exists a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a standard handle decomposition such that:
   - number of 0-handles = 1,
   - number of 1-handles = number of generators,
   - number of 2-handles = number of relators,
   - there are no handles of index ≥ 3.

In the proof of the Presentation–Handle Realization Proposition 256.1 we will make use of the following lemma, which we will also use a little later, on a second occasion:

Lemma 256.2. (Boundary–Handle–π_1-Lemma) Let $n \in \mathbb{N}$ and let M be a connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition. Let $x_0 \in \partial M$. If all handles have index $\leq n-2$, then the boundary ∂M is connected and the inclusion-induced map $\pi_1(\partial M, x_0) \to \pi_1(M, x_0)$ is an epimorphism.[2731]



3-dimensional smooth manifold built out of handles of index ≤ 1

$\pi_1(\partial M, x_0) \to \pi_1(M, x_0)$ is an epimorphism

Proof of the Boundary–Handle–π_1-Lemma 256.2. We equip M with the dual handle decomposition rel ∂M from the Dual Handle Proposition 248.1. Recall that the i-handles of the original handle decomposition are in one-to-one correspondence with $(n-i)$-handles of the relative handle decomposition.

Since we assume that all handles of the original handle decomposition have index $\leq n-2$ we see that we obtain M from $[0,1] \times \partial M$ by attaching handles of index ≥ 2. It follows from the Handle Attachment-π_0-Lemma 246.5 that $\pi_0(\partial M) \to \pi_0(M)$ is a bijection. Since M is connected we see that ∂M is also connected.

Furthermore it follows from the Handle Attachment-π_1-Proposition 246.13 that the inclusion induced map $\pi_1(\partial M, x_0) \to \pi_1(M, x_0)$ is an epimorphism. ■

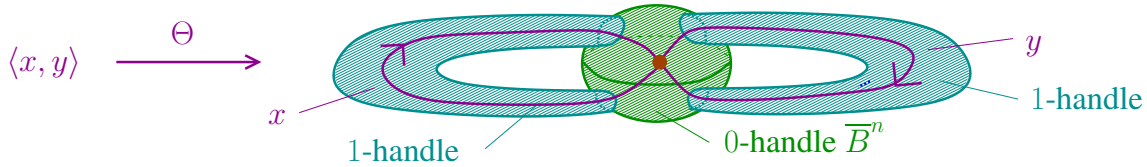
Proof of the Presentation–Handle Realization Proposition 256.1. Let

$$\pi \;=\; \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle$$

be a finite presentation and let $n \geq 4$. We start with the 0-handle $\overline{B}^n$. Since $n \geq 2$ we can pick k *orientation-preserving* thickened 0-spheres $\varphi_1, \ldots, \varphi_k \colon \overline{B}^{n-1} \times S^0 \to \partial \overline{B}^n = S^{n-1}$ with disjoint images. We denote by W the result of attaching the k 1-handles to the 0-handle $\overline{B}^n$ along these thickened 0-spheres. Note that it follows from the Handles-Orientability Proposition 246.6 that W, and thus also ∂W, are orientable.

Next we pick a base point $P \in \partial W$. It follows from the Handle Attachment-π_1-Proposition 246.13 that there exists an isomorphism

$$\Theta \colon \langle g_1, \ldots, g_k \rangle \;\xrightarrow{\cong}\; \pi_1(W, P).$$



The idea now is to attach l 2-handles to W which correspond to the relators $r_1, \ldots, r_l$. We find the corresponding thickened 1-spheres in several steps:

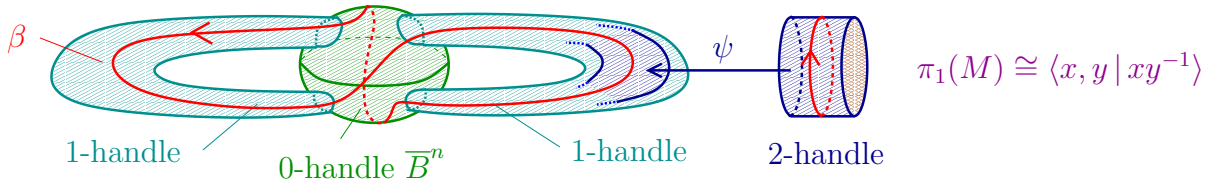
(a) Since $n \geq 3$ we know by the Boundary–Handle–π_1-Lemma 256.2 that the inclusion-induced map $\pi_1(\partial W, P) \to \pi_1(W, P)$ is an epimorphism.

[2731] This lemma gives a solution to Exercise 248.8.

(b) By (a) we can pick loops $\alpha_1, \dots, \alpha_l \colon (S^1, *) \to (\partial W, P)$ that represent the relators $r_1, \dots, r_l \in \pi_1(W, P) = \langle g_1, \dots, g_k \rangle$.

(c) By hypothesis we have $\dim(\partial W) = n - 1 \geq 3$. Thus we can apply the Curve–Smooth Embedding Theorem **??** and we obtain that the map $\alpha_1 \sqcup \dots \sqcup \alpha_l \colon S^1 \sqcup \dots \sqcup S^1 \to \partial W$ is in fact homotopic to a smooth embedding $\beta_1 \sqcup \dots \sqcup \beta_l \colon S^1 \sqcup \dots \sqcup S^1 \to \partial W$.

(d) It follows from the Smooth Embedding Theorem 30.15 together with the Product Tubular Neighborhood–Existence Theorem 58.4 (2)[2732] that we can find smooth embeddings $\psi_1, \dots, \psi_l \colon \overline{B}^{n-2} \times S^1 \to \partial W$ with disjoint images such that for each $i \in \{1, \dots, l\}$ and each $x \in S^1$ we have $\psi_i(0, x) = \beta_i(x)$.



Finally we consider

$$M \coloneqq W \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2 = \varnothing \cup \overline{B}^n \cup_{\varphi_1} h^1 \cup \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2.$$

The proposition follows from the following calculation:

it follows from the Handle Attachment-π_1-Proposition 246.13, together with the Homotopies-of-Loops Proposition 88.29 that there exist $x_1, \dots, x_l \in \langle g_1, \dots, g_k \rangle$ such that we have an isomorphism of this form

$$\begin{array}{lcccl} \pi_1(M, P) & \overset{\downarrow}{\cong} & \langle g_1, \dots, g_k \rangle / \langle\langle x_1 r_1 x_1^{-1}, \dots, x_l r_l x_l^{-1} \rangle\rangle & & \\ & \overset{\uparrow}{\cong} & \langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_l \rangle\rangle & \overset{\uparrow}{=} & \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle. \end{array}$$

by the Normal Generator Lemma 100.9 (3) — by Definition 100.10

■

Now we can provide a new proof of the following proposition:

Proposition 102.5. (Manifold-π_1-Realization Proposition) Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$$

be a finite presentation. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$.

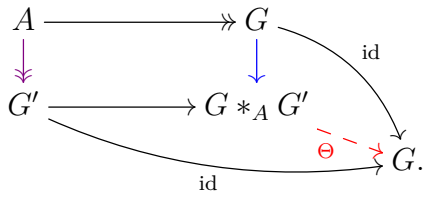
Our new proof of the Manifold-π_1-Realization Proposition 102.5 rests on the following group-theoretic lemma, which is precisely the content of Exercise 98.30.

Lemma 256.3. (Group Double Epimorphism Lemma) Let A and G be groups and let $\varphi \colon A \to G$ be an epimorphism. Furthermore let G' be a copy of G. We consider the amalgamated product $G *_A G'$ given by using twice the epimorphism $\varphi \colon A \to G$ and $\varphi \colon A \to G' = G$. In this setting the natural homomorphism $G \to G *_A G'$ is an isomorphism.

Proof of the Group Double Epimorphism Lemma 256.3. By the Group Pushout Proposition 98.22 (1b) there exists a unique homomorphism $\Theta \colon G *_A G' \to G'$ that makes the

[2732] Note that we can apply the Product Tubular Neighborhood–Existence Theorem 58.4 since ∂W is orientable.

following diagram commute:



In particular we see that the right vertical map is a monomorphism. The left vertical map is by hypothesis an epimorphism. It follows from the Amalgamated Product Lemma 98.24 (2b) that the right vertical map is also an epimorphism. Thus we have shown that the right vertical map is indeed an isomorphism. ∎

Proof of the Manifold-π_1-Realization Proposition 102.5. Let

$$\pi \;=\; \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$$

be a finite presentation and let $n \geq 4$. By the Presentation–Handle Realization Proposition 256.1 there exists a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a handle decomposition $M = \varnothing \cup h^0 \cup_{\varphi_1} h^1 \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2$.

By the Doubling-Manifold Lemma 54.10 the double

$$W \;:=\; \mathrm{D}M \;:=\; (M \times \{1\}) \cup_{\partial M \times \{1\} = \partial M \times \{2\}} (M \times \{2\})$$

is a closed orientable connected non-empty n-dimensional smooth manifold. Thus it suffices to prove the following claim:

Claim. There exists an isomorphism $\pi_1(\mathrm{D}M) \cong \pi$.

Proof. For fun we provide two different proofs.

(1) First we point out that it follows from the Boundary–Handle–π_1-Lemma 256.2 that the boundary ∂M is connected. Therefore we can apply the Manifold–Seifert-van Kampen Theorem 102.1 which tells us that for any $P \in \partial M$ we have an isomorphism $\pi_1(\mathrm{D}M, P) \cong \pi_1(M, P) *_{\pi_1(\partial M, P)} \pi_1(M, P)$. Since $n \geq 4$ and since M has only handles of index ≤ 2 we obtain from the Boundary–Handle–π_1-Lemma 256.2 that the inclusion-induced map $\pi_1(\partial M, P) \to \pi_1(M, P)$ is an epimorphism. It follows from the Group Double Epimorphism Lemma 256.3 that the map $\pi_1(M, P) \to \pi_1(\mathrm{D}M, P)$ is actually an isomorphism.
(2) On page 4791 we pointed out that it is a consequence of the Dual Handle Proposition 248.1 and the Handle Decomposition–Stacking Proposition 247.2 that the double admits a handle decomposition of the following form:

$$\mathrm{D}M = \underbrace{\varnothing \cup h^0 \cup_{\varphi_1} h^1 \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cup_{\psi_l} h^2}_{=M} \cup_{\alpha_l} h^{n-2} \cup_{\alpha_1} h^{n-2} \cup_{\beta_k} h^{n-1} \cup_{\beta_1} h^{n-1} \cup h^n,$$

where the $(n-2)$-handles corresponding to $\alpha_l, \dots, \alpha_1$ are dual to the 2-handles given by $\psi_l, \dots, \psi_1$ and the $(n-1)$-handles corresponding to $\beta_k, \dots, \beta_1$ are dual to the 1-handles given by $\varphi_k, \dots, \varphi_1$.

(a) First let us deal with the case that $n \geq 5$. In this case $\mathrm{D}M$ is obtained from M by attaching handles of index ≥ 3. But by the Handle Attachment-π_1-Proposition 246.13 we know that handle attachments of index ≥ 3 do not affect the fundamental group. Thus we see that $\pi_1(\mathrm{D}M) \cong \pi_1(M) \cong \pi$.

(b) The case $n = 4$ is a little bit more interesting. We have

$$\begin{array}{rcl} \pi_1(\mathrm{D}M) & \overset{\text{by the Handle Attachment-}\pi_1 \text{ Proposition 246.13, since } n=4}{\downarrow} & \\ & = \pi_1(M)/\langle\langle \text{attaching circles of the dual 2-handles } \alpha_l, \dots, \alpha_1 \rangle\rangle & \\ & = \pi_1(M)/\langle\langle \text{belt circles of the 2-handles } \psi_l, \dots, \psi_1 \rangle\rangle \overset{\text{since belt circles are null-homotopic}}{\cong} \pi_1(M) \cong \pi. \\ & \uparrow \text{by the Dual Handle Proposition 248.1 (1c)} & \end{array}$$

■

We have no shown that for every $n \geq 4$, every finitely presented group occurs as the fundamental group of a closed n-dimensional smooth manifold. In contrast, it is clear that for very low dimensions this statement does not hold. For example it follows from the Diffeomorphism-to-Standard Surfaces Theorem 255.14 that only "few" groups occur as fundamental groups of closed 2-dimensional smooth manifolds.

But what about the 3-dimensional case? This is a good moment to reminder the reader of the following question:

Question 102.8. Is every finitely presented group the fundamental group of a closed (or at least compact) connected 3-dimensional smooth manifold?

We will answer Question 102.8 in the 3-Manifold–π_1–Abelian Theorem 257.7.

256.2. **Integral homology spheres and homotopy spheres.** We remind the reader of the Definitions 183.22 and 210.19.

Definition 256.4. Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold.

(1) If for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M) \cong \mathrm{H}_k(S^n)$, then we refer to M as a smooth homology n-sphere.

(2) If M is a homology sphere and if M is simply connected, then we refer to M as a smooth homotopy n-sphere.[2733]

Our goal in this section is to apply the construction used in the Manifold-π_1-Realization Proposition 102.5 to construct smooth homology spheres, with non-trivial fundamental group, and to construct potentially interesting smooth homotopy spheres.

The key to constructing such manifolds is the following refinement of the Manifold-π_1-Realization Proposition 102.5.

Proposition 256.5. (Small Manifold-π_1-Realization Proposition) Let

$$\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$$

[2733] In the Homotopy Sphere-Characterization Proposition 210.18 and **??** we give several alternative characterizations of homotopy spheres.

be a finite presentation.

(1) Given any $n \geq 5$ there exists a closed orientable connected non-empty n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$ and such that[2734]

$$\mathrm{H}_i(W) \cong \begin{cases} \pi_{\mathrm{ab}}, & \text{if } i = 1, \\ \mathbb{Z}^{l-k+\mathrm{rank}(\pi_{\mathrm{ab}})}, & \text{if } i = 2, \\ \mathbb{Z}^{l-k+\mathrm{rank}(\pi_{\mathrm{ab}})} \oplus \mathrm{Tor}(\pi_{\mathrm{ab}}), & \text{if } i = n-2, \\ \mathbb{Z}^{\mathrm{rank}(\pi_{\mathrm{ab}})}, & \text{if } i = n-1, \\ \mathbb{Z}, & \text{if } i = 0 \text{ or if } i = n, \\ 0, & \text{else.} \end{cases}$$

(2) There exists a closed orientable connected non-empty smooth manifold W of dimension $n = 4$ with $\pi_1(W) \cong \pi$ and such that the homology groups are as in (1), except that for $i = 2$ we take the direct sum of the terms that we wrote down for $i = 2$ and $i = n - 2$.

The proof of the Small Manifold-π_1-Realization Proposition 256.5 makes use of the following lemma, which the reader might recognize as Exercise 210.29:

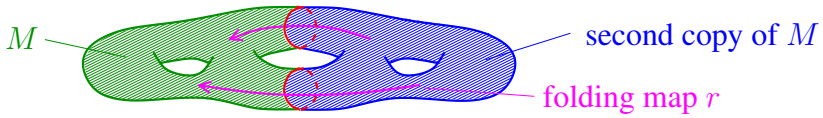
Lemma 256.6. (Homology-of-Double Lemma) Let M be a compact orientable n-dimensional topological manifold. For every $j \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_j(\mathrm{D}M) \cong \mathrm{H}_j(M) \oplus \mathrm{Hom}(\mathrm{H}_{n-j}(M), \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{n-j-1}(M)).$$

Proof of the Homology-of-Double Lemma 256.6. Let $i\colon M \to \mathrm{D}M$ be the natural inclusion and let

$$\begin{array}{rcl} r\colon \mathrm{D}M & \to & M \\ [(P,k)] & \mapsto & P \end{array}$$

be the folding map that we introduced in Definition 54.9. Note that by definition we have $r \circ i = \mathrm{id}_M$.



We consider the long exact sequence in homology of the pair $(\mathrm{D}M, M)$:

$$\ldots \to \mathrm{H}_j(M) \xrightarrow{i_*} \mathrm{H}_j(\mathrm{D}M) \to \mathrm{H}_j(\mathrm{D}M, M) \to \ldots$$

Now we see that we have the following isomorphisms:

$$\begin{array}{ccccc} & & & & \text{by the Manifold–Excision Theorem 148.3, here the blue } M \text{ stands for the second copy of } M \text{ in the double } \mathrm{D}M \\ & & & & \downarrow \cong \\ \mathrm{H}_j(\mathrm{D}M) & \underset{\uparrow}{\cong} & \mathrm{H}_j(M) \oplus \mathrm{H}_j(\mathrm{D}M, M) & & \cong \mathrm{H}_j(M) \oplus \mathrm{H}_j(M, \partial M) \end{array}$$

since $r \circ i = \mathrm{id}_M$ we see that the above long exact sequence breaks up into short exact sequences, furthermore it follows from the Splitting Lemma 145.5 (2)⇒(3) that the middle term is isomorphic to the direct sum of the two other terms

$$\underset{\uparrow}{\cong} \mathrm{H}_j(M) \oplus \mathrm{H}^{n-j}(M;\mathbb{Z}) \quad \underset{\uparrow}{\cong} \mathrm{H}_j(M) \oplus \mathrm{Hom}(\mathrm{H}_{n-j}(M), \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{n-j-1}(M)).$$

by the Poincaré Duality Theorem 210.1

by the Universal Coefficient H*-Theorem 201.23 and the discussion on page 3859

■

[2734] We denote by π_{ab} the abelianization of π from Definition 98.15.

Proof of the Small Manifold-π_1-Realization Proposition 256.5. As in the proof of the Manifold-π_1-Realization Proposition 102.5 that we just gave on page 4933 we use the Presentation–Handle Realization Proposition 256.1 to obtain a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a handle decomposition with precisely one 0-handle, k 1-handles, l 2-handles and no handles of index > 2.

In the above proof of the Manifold-π_1-Realization Proposition 102.5 we saw that the fundamental group of the double $W \coloneqq \mathrm{D}M$ is isomorphic to π and that W is a closed orientable connected non-empty n-dimensional smooth manifold. It remains to show that the homology groups of $W = \mathrm{D}M$ are as stated. In fact it follows from the Homology-of-Double Lemma 256.6 that it suffices to prove the following claim:

Claim.

$$\mathrm{H}_0(M) \cong \mathbb{Z}, \quad \mathrm{H}_1(M) \cong \pi_{\mathrm{ab}}, \quad \mathrm{H}_2(M) \cong \mathbb{Z}^{l-k+\mathrm{rank}(\pi_{\mathrm{ab}})} \quad \text{and} \quad \mathrm{H}_i(M) = 0 \text{ for } i \geq 3.$$

Proof. First note that the statements for $\mathrm{H}_0(M)$ and $\mathrm{H}_1(M)$ follow immediately from the H_0-Proposition 143.25, the fact that $\pi_1(M) \cong \pi$ and the Hurewicz Theorem 166.10. Next note that

$$\mathrm{H}_i(M) \underset{\substack{\uparrow \\ \text{by the Singular-Handle-H}_* \\ \text{Proposition 252.4}}}{\cong} \mathrm{H}_i^{\mathrm{han}}(M) \underset{\substack{\uparrow \\ \text{since } M \text{ has no handles} \\ \text{of index} \geq 3}}{=} \begin{cases} 0, & \text{if } i > 2, \\ \ker(\mathrm{C}_2^{han}(M) \to \mathrm{C}_1^{\mathrm{han}}(M)), & \text{if } i = 2. \end{cases}$$

Note that this observation for $i = 2$, together with the Free Abelian Subgroup Theorem 97.14, shows that $\mathrm{H}_2(M)$ is a free abelian group. Finally note that

$$\mathrm{rank}(\mathrm{H}_2(M)) \underset{\substack{\uparrow \\ \text{by definition of } \chi(M)}}{=} \chi(M) + \sum_{i \neq 2} (-1)^{i+1} \cdot \mathrm{rank}(\mathrm{H}_i(M)) \underset{\substack{\uparrow \\ \text{by the Handle Attachment-}\chi\text{-Lemma 246.15} \\ \text{and the above calculations}}}{=} (1 - k + l) - 1 + \mathrm{rank}(\pi_{\mathrm{ab}})$$
$$= l - k + \mathrm{rank}(\pi_{\mathrm{ab}}). \qquad \blacksquare$$

As we mentioned, our goal is to construct interesting smooth homology spheres and smooth homotopy spheres. The Small Manifold-π_1-Realization Proposition 256.5 leads us to the following group theoretic definitions:

Definition 256.7.

(1) The deficiency of a finite presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ is defined to be $k - l$.
(2) A finite presentation is called balanced, if its deficiency equals zero.
(3) A group π is called perfect if its abelianization π_{ab} is trivial.

The following proposition gives us some interesting examples of (presentations of) groups.

Proposition 256.8. (Balanced Presentations Proposition)

(1) For every closed connected non-empty 3-dimensional smooth manifold M the fundamental group $\pi_1(M)$ admits a balanced presentation.
(2) The binary icosahedral group, that we introduced in Definition 183.25, is a perfect group of order 120 that admits a balanced presentation.

(3) For every odd prime p the group $\mathrm{SL}(2,\mathbb{F}_p)$ is perfect and it admits a balanced presentation.

Proof.

(1) We will prove this statement in Exercise 256.2. Alternatively the statement is proved in the 3-Manifold–Deficiency Theorem 257.4.
(2) Let π be the binary icosahedral group. In the Poincaré Homology Sphere Proposition 183.26 we showed that π is a perfect group of order 120. It remains to show that π admits a balanced presentation. There are two possible approaches:
   (a) In [**KSc1979**, Chapter II] it is shown that the binary icosahedral group is isomorphic to $\langle x,y \,|\, x^3=y^5, x^3=(xy)^2\rangle$.
   (b) Recall that by definition $\pi \subseteq \mathrm{SU}(2)$. Thus it makes sense to consider the quotient $M := S^3/\pi$, which we called the Poincaré Homology Sphere. In the Poincaré Homology Sphere Proposition 183.26 we showed that M is a closed orientable 3-dimensional smooth manifold. Thus it follows from (1) that $\pi \cong \pi_1(M)$ admits a balanced presentation.
(3) In Exercise 256.3 we will see that $\mathrm{SL}(2,\mathbb{F}_p)$ is perfect. The much more subtle fact that $\mathrm{SL}(2,\mathbb{F}_p)$ admits a balanced presentation is shown in [**CR1980**]. ■

In the Poincaré Homology Sphere Proposition 183.24 we showed that there exist smooth homology 3-spheres with non-trivial fundamental groups. We now obtain a higher-dimensional analog:

Proposition 256.9. (Homology Sphere-Existence Proposition) Given any $n \geq 4$ there exists a smooth homology n-sphere with non-trivial fundamental group.

Remark.

(1) In Exercise 183.17 we used a different construction to obtain high-dimensional smooth homology spheres with non-trivial fundamental groups.
(2) Michel Kervaire [**Kerv1969**, Theorem 1] completely determined which finitely presented groups can occur as fundamental groups of smooth homology n-spheres for $n \geq 4$. □

Proof. Let $n \geq 4$. It follows from the Balanced Presentations Proposition 256.8 together with the Small Manifold-π_1-Realization Proposition 256.5 that there exists a smooth homology n-sphere with non-trivial fundamental group. ■

The construction that we used in the proof of the Presentation–Handle Realization Proposition 256.1 and that we used in the above proof of the Manifold-π_1-Realization Proposition 102.5 is a never-ending source of joy, since even for presentations of the trivial group one obtains interesting examples of smooth manifolds:

Example. Let $k \in \mathbb{Z}$. We consider the presentation

$$\langle x,y \,|\, \underbrace{xyx=yxy}_{=(1)}, \underbrace{x^{k+1}=y^k}_{=(2)}\rangle.$$

We will now see that this is a presentation of the trivial group. Indeed, we have

(a) $y = x^{-1}y^{-1}yxy = (yx)^{-1}yxy \underset{\text{by (1)}}{=} (yx)^{-1}x(yx)$

(b) $y^{k+1} \underset{\substack{\text{follows from taking the} \\ (k+1)\text{-st power of (a)}}}{=} (yx)^{-1}x^{k+1}(yx) \underset{\text{by (2)}}{=} (yx)^{-1}y^k(yx) = x^{-1}y^kx \underset{\text{by (2)}}{=} x^{-1}x^{k+1}x = x^{k+1} \underset{\text{by (2)}}{=} y^k.$

Thus $y^{k+1} = y^k$ which implies that $y = 1$. From (1) it follows that also $x = 1$. We have thus shown that this is indeed a presentation of the trivial group.

Let $n \geq 4$. We apply the construction of the proof of the Manifold-π_1-Realization Proposition 102.5 and the Presentation–Handle Realization Proposition 256.1 to the above balanced presentation of the trivial group. By the Small Manifold-π_1-Realization Proposition 256.5 we obtain a smooth homotopy n-sphere. □

The following question arises:

Question 256.10. Are the smooth homotopy n-spheres that we just constructed[2735] diffeomorphic to S^n?

For $n = 4$ and some special cases, Question 256.10 was answered in the affirmative by Robert Gompf [**Gom1991**]. In general though, at least for $n = 4$, Question 256.10 seems to be wide open. We refer to [**GoS1999**, p. 149] and [**Kir1989**, p. 18] for more information.

We have now seen that finite presentations of groups give rise to interesting smooth manifolds. The question arises, to what degree different presentations give rise to different smooth manifolds. It is worth recalling the following theorem.

Theorem 100.25. (Tietze Theorem) Any two finite presentations for some given group are related by a finite sequence of Tietze transformations, i.e. by a sequence of the following two transformations:

(a) $\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle \quad \rightsquigarrow \quad \langle x_1, \dots, x_k \mid r_1, \dots, r_l, s \rangle$

where s is a word in $r_1, \dots, r_l$, and

(b) $\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle \quad \rightsquigarrow \quad \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, s \cdot x \cdot t \rangle$

where $x \neq x_1, \dots, x_k$ and where s, t are words in $x_1, \dots, x_k$.

In our context, where we study balanced presentations of the trivial group, we would like to go from one presentation to another while keeping the deficiency fixed. This leads us to the following definition:

[2735] Evidently there are various choices involved in these constructions.

Definition 256.11. Let $\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$ be a presentation. An Andrews-Curtis transformation is one of the following ways to obtain a new presentation:

(a) $$\langle x_1, \dots, x_k \mid r_1, \dots, r_i, \dots, r_l \rangle \leftrightsquigarrow \langle x_1, \dots, x_k \mid r_1, \dots, x_j^{-\epsilon} \cdot r_i \cdot x_j^{\epsilon}, \dots r_l \rangle$$

where $j \in \{1, \dots, k\}$ and $\epsilon \in \{\pm 1\}$, or

(b) $$\langle x_1, \dots, x_k \mid r_1, \dots, r_i, \dots, r_l \rangle \leftrightsquigarrow \langle x_1, \dots, x_k, \mid r_1, \dots, r_i \cdot r_j, \dots, r_l \rangle$$

where $i \neq j \in \{1, \dots, l\}$, or

(c) $$\langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle \leftrightsquigarrow \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, x \rangle$$

where $x \neq x_1, \dots, x_k$.

The following is now a slight reformulation of the Andrews-Curtis Conjecture which we first encountered in Section 100.6.

Conjecture 100.34. (Andrews-Curtis Conjecture) Any balanced presentation of the trivial group can be turned into the empty presentation by a finite sequence of Andrews-Curtis transformations.

In the Handle Cancellation Theorem 260.1 and the Handle Slide Theorem 260.13 we will see that the Andrews-Curtis transformations have analogs for handle decompositions. In particular, as is argued on [**GoS1999**, p. 149], an affirmative answer to the Andrews-Curtis Conjecture 100.34 would imply that Question 256.10 has a positive answer.

For better or worse the evidence towards the Andrews-Curtis Conjecture 100.34 is rather shaky. For example, let $k \in \mathbb{Z}$. As above we consider the following balanced presentation of the trivial group: $$\langle x, y \mid xyx = yxy, x^{k+1} = y^k \rangle.$$

For most k it is unknown whether the presentation can be turned into the empty presentation by a finite sequence of Andrews-Curtis transformations. In fact the consensus seems to be that the Andrews-Curtis Conjecture 100.34 is most likely false.

Remark.

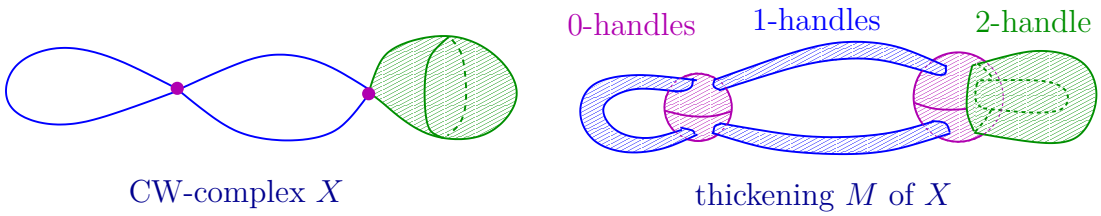
(1) More information and some partial results regarding the Andrews-Curtis Conjecture 100.34 can be found in [**Barm2011**, Chapter 11].

(2) Sergey Matveev [**Matv1987**, p. 917] (see also [**Kupe2017a**, Proposition 4.2] and [**Barm2011**, Chapter 11]) showed that the Andrews-Curtis Conjecture 100.34 is equivalent to a special case of the Zeeman Conjecture 305.23. □

256.3. **Handle decompositions and CW-structures.** Recall that in the Handle Decomposition–CW-Complex Theorem 254.1 we saw that every compact smooth manifold M that is equipped with a handle decomposition is homotopy equivalent to a CW-complex X such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-cells of X. In this section we will prove the following partial converse to this statement.

Theorem 256.12. (CW-Complex–Handle Decomposition Theorem) Given any finite CW-complex X there exists a compact orientable smooth manifold M that is homotopy equivalent to X and which admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-cells of X equals the number of k-handles of M.

Remark.



(1) In the CW-Complex–Handle Decomposition Theorem 256.12 we do not make any claims about the dimension of M. We leave it to the reader to figure out what is the lowest dimension for M that one can achieve with our proof.
(2) Let X be a finite CW-complex. Any compact smooth manifold M that is homotopy equivalent to X is called a *thickening of* X. The CW-Complex–Handle Decomposition Theorem 256.12 thus says in particular that every finite CW-complex admits a thickening. The theory of thickenings of finite CW-complexes is developed in detail in [**Maz1963**] and [**Wall1966b**]. In particular given a finite k-dimensional CW-complex it follows from [**Wall1966b**, p. 76] (see also [**BCFHLN2019**, Theorem 4.1] that the following statements hold:
   (a) Given any $n \geq 2k$ there exists an n-dimensional thickening of X.
   (b) If $n \geq 2k+1$ and if $n \geq 6$, then any two n-dimensional thickenings of X are actually diffeomorphic.

   We refer to [**BCFHLN2019**, Chapter 4] for more information on thickenings.
(3) The topic of finding smooth manifolds that "look" like a given topological space is also discussed in [**BuS2010**, Chapter 2]. □

The proof of the CW-Complex–Handle Decomposition Theorem 256.12 rests on the following two lemmas:

Lemma 256.13. (Thicken–High Codimension Sphere Lemma) If $N \geq 2r+2$, then for every smooth embedding $\varphi\colon S^r \to \mathbb{R}^N$ there exists an orientation-preserving smooth embedding $\Phi\colon \overline{B}^{N-r} \times S^r \to \mathbb{R}^N$ such that $\Phi(0,P) = \varphi(P)$ for all $P \in S^r$.

Proof. First we consider the standard smooth embedding $\theta\colon S^r \to \mathbb{R}^{r+1} \times \mathbb{R}^{N-r-1} = \mathbb{R}^N$ that is given by $x \mapsto (x,0)$. This smooth embedding can be "thickened" up. For example we could consider the smooth embedding

$$\begin{array}{rcl} \Theta\colon \overline{B}^{N-r} \times S^r & \to & \mathbb{R}^{r+1} \times \mathbb{R}^{N-r-1} = \mathbb{R}^N \\ (\underbrace{(x,y)}_{\substack{\uparrow \\ \text{with } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^{N-r-1}}}, P) & \mapsto & ((1+\frac{x}{2}) \cdot P, y). \end{array}$$

After possibly composing the map $\Theta\colon \overline{B}^{N-r} \times S^r \to \mathbb{R}^N$ with a reflection in the last coordinate we see that Θ is orientation-preserving. By hypothesis we have $N \geq 2r+2$. Thus we obtain from the Embeddings-become-Isotopic Criterion **??** that there exists a smooth isotopy $F\colon S^r \times [0,1] \to \mathbb{R}^N$ with $F_0 = \theta$ and $F_1 = \varphi$. By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $G\colon \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ with $G_1 \circ \theta = \varphi$. The map $\Phi := G_1 \circ \Theta\colon \overline{B}^{N-r} \times S^r \to \mathbb{R}^N$ has all the desired properties. ■

The heart of the proof of the CW-Complex–Handle Decomposition Theorem 256.12 is contained in the following lemma, which can be viewed as a partial converse to the Handle–Cell Attachment Proposition 254.2:

Proposition 256.14. (Cell–Handle Attachment Proposition) Let X be a finite CW-complex and let $\alpha\colon S^{k-1} \to X$ be a continuous map. If X is homotopy equivalent to a compact orientable smooth manifold M, then there exists a compact orientable smooth manifold N together with an orientation-preserving thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{\dim(N)-k} \times S^{k-1} \to \partial N$ such that the following are satisfied:

(1) N is also homotopy equivalent to X.
(2) $N \cup_\varphi h^k$ is homotopy equivalent to $X \cup_\alpha \overline{B}^k$.
(3) If M is equipped with a handle decomposition, then we can equip N with a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-handles of N.



Proof of Theorem 256.12 assuming Proposition 256.14. Let X be a finite CW-complex. We start out with two observations:

(1) It follows from the Iterative Cell-Attachment Lemma 135.10 that we can obtain X from X by iteratively attaching cells.
(2) In the setting of the Cell–Handle Attachment Proposition 256.14 we know by the Handle Attachment Proposition 246.3 and the Handles-Orientability Proposition 246.6 that $N \cup_\varphi h^k$ is again a compact orientable smooth manifold with one extra k-handle.

Starting with the empty manifold we now see, by an elementary iterative argument, that there exists a compact orientable smooth manifold M that is homotopy equivalent to X and which admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-cells of X equals the number of k-handles of M. ∎

Proof of Proposition 256.14. Let $f\colon X \to M$ be a homotopy equivalence between a finite CW-complex X and a compact orientable smooth manifold M. Furthermore let $\alpha\colon S^{k-1} \to X$ be a continuous map.

We pick some $r \geq \max\{k, 2k - \dim(M)\}$. We write $\widetilde{M} \coloneqq M \times \overline{B}^r$ and we equip the product $\widetilde{M}$ with the smooth atlas provided by the Smooth Manifold Rounded Product Proposition 60.7, i.e. we consider the smooth manifold $\widetilde{M} = (M \times \overline{B}^r)^\backsim$. Note that the map $i\colon M \to \widetilde{M}$ given by $x \mapsto (x, 0)$ is a homotopy equivalence.

Claim 1. We write $\widetilde{f} \coloneqq i \circ f\colon X \to \widetilde{M}$. There exists a homotopy $F\colon S^{k-1} \times [0,1] \to \widetilde{M}$ from the map $\widetilde{f} \circ \alpha\colon S^{k-1} \to \widetilde{M} = (M \times \overline{B}^r)^\backsim$ to a smooth embedding $\theta\colon S^{k-1} \to \partial\widetilde{M}$.

Proof. By the combination of the Whitney Approximation Theorem 36.1 and the Transversality Theorem 50.9 we know that the map $\widetilde{f} \circ \alpha\colon S^{k-1} \to \widetilde{M}$ is homotopic to a smooth

map that is transverse to the proper submanifold[2736] $M \times \{0\}$ of $\widetilde{M} = M \times \overline{B}^r$. Note that

$$\dim(S^{k-1}) + \dim(M \times \{0\}) \;=\; k-1+\dim(M) \underset{\substack{\uparrow \\ \text{since } r \geq k}}{<} r+\dim(M) \;=\; \dim(M \times \overline{B}^r).$$

Thus the image of a map $S^{k-1} \to \widetilde{M}$, which is transverse to $M \times \{0\}$, actually misses $M \times \{0\}$. Therefore we now know that $\widetilde{f} \circ \alpha \colon S^{k-1} \to \widetilde{M}$ is homotopic to a map that misses $M \times \{0\}$. Next, using the deformation retraction from $\overline{B}^k \setminus \{0\}$ to S^{k-1} we see that $\widetilde{f} \circ \alpha \colon S^{k-1} \to \widetilde{M}$ is actually homotopic to a map $\varphi \colon S^{k-1} \to M \times S^{k-1} \subseteq \partial\widetilde{M}$. Finally, since

$$2(k-1) \underset{\substack{\uparrow \\ \text{since } r \geq 2k-\dim(M)}}{<} \dim(M)+r-1 \;=\; \dim(\partial\widetilde{M})$$

we obtain from the Whitney Approximation Theorem 36.1 together with the Deform-to-Embedding Theorem **??** (1) that $\varphi \colon S^{k-1} \to \partial\widetilde{M}$ is homotopic to a smooth embedding $\theta \colon S^{k-1} \to \partial\widetilde{M}$. ⊞

We set $m := \dim(\widetilde{M}) = \dim(M)+r$. Note that it follows from the Euclidean Embedding Theorem 33.2 there exists an $n \geq 2m+1$ such that there exists a proper smooth embedding $\widetilde{M} \to \mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,\big|\, x_n \geq 0\}$. In particular, by the Smooth Embedding Theorem 30.15 we can assume that $\widetilde{M}$ is a proper smooth submanifold of $\mathbb{H}^n$ for some $n \geq 2m+1$.

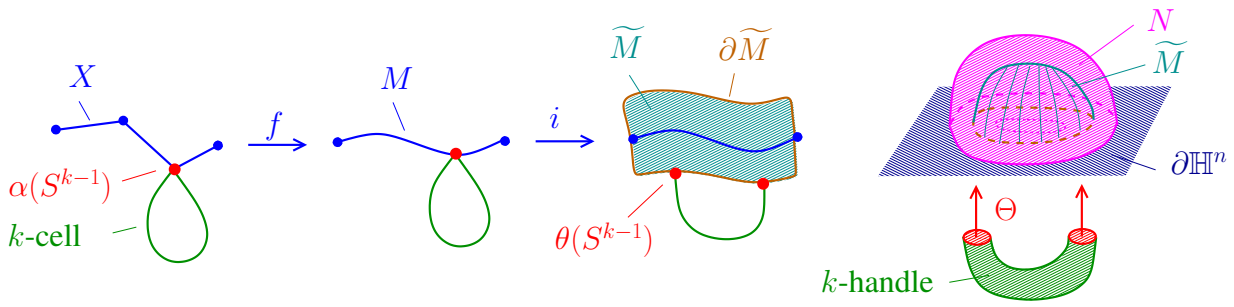
By the Tubular Neighborhood–Existence Theorem 57.11 the proper submanifold $\widetilde{M}$ of $\mathbb{H}^n$ admits a tubular neighborhood $N \subseteq \mathbb{H}^n$. By definition of a tubular neighborhood, see Definition 57.9, we know that N is a compact codimension-zero submanifold of $\mathbb{H}^n$ with corner and which is a $(\overline{B}^{n-m}, \mathrm{O}(n-m))$-bundle over M. Furthermore, by the Basics-of-Tubular Neighborhood Lemma 57.10 we know that the inclusion $j \colon \widetilde{M} \to N$ is a homotopy equivalence. By the Smooth Submanifold-with-Corner Lemma 26.16 (1) we know that N is a smooth manifold with corner. It follows from the Smooth Manifold Rounded Product Proposition 60.7 that we can view N as an n-dimensional smooth manifold.

The map $j \circ \theta \colon S^{k-1} \to \partial\widetilde{M} \subseteq \partial N$ takes values in $\partial\,\mathbb{H}^n$. Since $N \cap \partial\,\mathbb{H}^n$ is a codimension-zero submanifold of $\partial\,\mathbb{H}^n$ we see that there exists an open neighborhood U of $\theta(S^{k-1})$ in $\partial\,\mathbb{H}^n$ that is contained in N. Since $\partial\,\mathbb{H}^n$ is diffeomorphic to $\mathbb{R}^{n-1}$ and since, by our choice of n, we have $n-1 \geq 2(k-1)+2$, we obtain from the Thicken–High Codimension Lemma 256.13 that there exists an orientation-preserving smooth embedding $\Theta \colon \overline{B}^{n-k} \times S^{k-1} \to \partial\,\mathbb{H}^n$ such that $\Theta(0,P) = \theta(P)$ for all $P \in S^{k-1}$. It follows from the Tube Lemma 7.12 that there exists an $r \in (0,1)$ such that $\Theta(\overline{B}^{n-k}_r \times S^{k-1}) \subseteq U$. Thus, after possibly precomposing Θ by the obvious diffeomorphism $\overline{B}^{n-k} \times S^{k-1} \to \overline{B}^{n-k}_r \times S^{k-1}$, we can assume that the image of Θ actually lies in $U \subseteq N \cap \partial\,\mathbb{H}^n$.
Statement (2) of the lemma is a consequence of the following claim:

Claim 2. The topological space $X \cup_\alpha \overline{B}^k$ is homotopy equivalent to the smooth manifold $N \cup_\Theta h^k$.

Proof. The claim can be proved easily using the Homotopic Attaching Map Lemma 135.12, the Homotopy Equivalent Cell Attachment Proposition 135.13 and the Core Deformation Retract Lemma 254.3. We leave it to the reader to fill in the details.

[2736] It follows easily from the Smooth Manifold Rounded Product Proposition 60.7 that $M \times \{0\}$ is indeed a smooth submanifold of $(M \times \overline{B}^r)^\circlearrowleft$.



The experienced reader will also doubtlessly have spotted that alternatively this is another good moment to roll out the Homotopy Pushout Theorem 22.24. To apply this great theorem let us consider the following commutative diagram:

$$\begin{array}{ccccc}
X & \xleftarrow{\alpha} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
M & \xleftarrow{f\circ\alpha} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\widetilde{M} & \xleftarrow{\tilde{f}\circ\alpha} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto(x,0)} & & \downarrow{\scriptstyle x\mapsto(x,0)} \\
\widetilde{M} & \xleftarrow{F} & S^{k-1}\times[0,1] & \hookrightarrow & \overline{B}^k\times[0,1] \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle x\mapsto(x,1)} & & \uparrow{\scriptstyle x\mapsto(x,1)} \\
\widetilde{M} & \xleftarrow{\theta} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
N & \xleftarrow{j\circ\theta} & S^{k-1} & \hookrightarrow & \overline{B}^k \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto(0,x)} & & \downarrow{\scriptstyle x\mapsto(0,x)} \\
N & \xleftarrow{\Theta} & \overline{B}^{n-k}\times S^{k-1} & \hookrightarrow & \overline{B}^{n-k}\times\overline{B}^k.
\end{array}$$

All of the vertical maps are clearly homotopy equivalences. Furthermore note that it follows immediately from the Manifold Boundary-Cofibration Proposition 22.11 and the Product-Cofibration Proposition 22.15 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 22.24, applied altogether six times, that the induced maps between the various pushouts have homotopy inverses. Combining these six homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence from $X \cup_\alpha \overline{B}^k$ to $N \cup_\Theta h^k$. ⊞

Finally we have to worry about handle decompositions. At this stage it remains to prove the following claim:

Claim 3. The smooth manifold N has a handle-structure with exactly the same number of handles of each index as M.

Proof. Recall that N is obtained from M by a two-stage process:

(1) First we take a product with a closed ball (and we round corners).
(2) Then we consider a closed-ball bundle (and we round corners).

Since Operation (1) is just a special case of Operation (2) it suffices to show that if we perform Operation (2), then we get an appropriate handle decomposition out of the original handle decomposition. So suppose we are given a compact smooth manifold W with a handle decomposition and let $Z \to W$ be a smooth $(\overline{B}^d, \mathrm{O}(d))$-bundle. We denote by s_k the number of k-handles. We perform the following steps to obtain a handle decomposition on $Z^{\llcorner}$.

(a) We pick a collar $\chi\colon [0,1] \times \partial W \to W$. It follows from the Handle Decomposition–Morse Function Theorem 251.2 together with the Standard-near-Boundary Lemma 249.29 that there exists a Morse function $f\colon W \to \mathbb{R}$ such that the number of critical points of index k equals s_k and such that on the collar the Morse function f is given by $\chi(s,x) \mapsto -t$.

(b) Next we employ the Linear Bundle–Morse Function Proposition 249.19 to obtain an explicit Morse function $g\colon Z \to \mathbb{R}$ which again has the property that the number of critical points of index k equals s_k.

(c)
- We use the collar χ together with the standard collar $[0, \frac{1}{2} \times S^{d-1} \to \overline{B}^d$ for $\overline{B}^d$ to obtain a double-collar $\delta\colon [0,1] \times [0,\frac{1}{2}] \times \angle N \to N$ for the corner of N. We use this double-collar to define $N^{\llcorner}$.
- If one goes through the explicit definitions of our various Morse functions one sees that the function $g \circ \delta\colon [0,1] \times [0,\frac{1}{2}] \times \angle N \to \mathbb{R}$ is of the form $(s,t,x) \mapsto -s \cdot t^2$. Using the Single Corner-to-Convex Lemma 60.9 one can now easily redefine this Morse function to obtain a Morse function on $N^{\llcorner}$, with the same number of critical points of a given index.

(d) We apply the Morse Function–Handle Decomposition Theorem 250.6 to the Morse function from (c) and we finally have the desired type of handle decomposition. ■

The next proposition explains what type of closed smooth manifolds one can obtain by the ever-popular doubling procedure applied to the smooth manifolds constructed in the proof of the CW-Complex–Handle Decomposition Theorem 256.12.

Proposition 256.15. (CW Complex–Manifold Proposition) Let X be a finite k-dimensional CW-complex. There exists an $N \geq 2k+1$ such that for every $n \geq N$ there exists a closed orientable n-dimensional smooth manifold M which admits a standard handle decomposition and which admits a homotopy equivalence

$$\varphi\colon X \ \to \ M^{n-k-1}$$

from X to the $(n-k-1)$-skeleton of M.

Remark. Note that in the setting of the CW Complex–Manifold Proposition 256.15 we obtain from Exercise 247.6 and the Handle-Skeleton-H_*-Lemma 252.1 (3) and (4), together with the Homotopy Equivalence-π_n-Proposition 134.17 (1) and the Homotopy Equivalence-H_*-Proposition 144.14 (1a), that the map $X \to M^{n-k-1} \to M$ induces an isomorphism of homotopy groups and homology groups up to dimension $n-k-2$. □

Proof. Let X be a finite k-dimensional CW-complex. By the CW-Complex–Handle Decomposition Theorem 256.12 there exists a homotopy equivalence $\psi\colon X \to V$ from X to a compact smooth manifold V which has the nice property that it admits a handle decomposition such that all handles are of index $\leq k$. By the Standard Handle Decomposition

Proposition 247.7 we can arrange that this handle decomposition is standard. After possibly replacing V by some[2737] $(\overline{B}^s \times V)^\backprime$ we can arrange that $\dim(V) \geq 2k+1$. We set $N := \dim(V)$. Now let $n \geq N$. We set $W := (\overline{B}^{n-N} \times V)^\backprime$ and we set $M := \mathrm{D}W$. We equip the double $M = \mathrm{D}W$ with the handle decomposition that we described on page 4791, namely which is given by the handle decomposition on M and the dual handle decomposition on the second copy of M. In particular we see that we can equip M with a handle decomposition such that the $(n-k-1)$-skeleton equals W. The map $\varphi\colon X \to M^{n-k-1} = (\overline{B}^{n-N} \times V)^\backprime$ given by $P \mapsto (\psi(P), 0)$ is now the promised homotopy equivalence from X to the $(n-k-1)$-skeleton of M. ■

256.4. **Homotopy equivalences to closed smooth manifolds.** In the CW-Complex–Handle Decomposition Theorem 256.12 we just showed that every finite CW-complex is homotopy equivalent to a compact smooth manifold. The following question, which implicitly we had already asked on page **??**, arises:

Question 256.16. Given a finite CW-complex X, what extra conditions do we need to impose on X to ensure that X is homotopy equivalent to a *closed* smooth or topological manifold?

We make a couple of initial observations. Let X be a CW-complex.

(1) If X is homotopy equivalent to a closed orientable n-dimensional topological manifold, then it follows from the Poincaré Duality Theorem 210.1, together with the Homotopy-H_*-Proposition 144.12, the Homotopy Equivalence-H^*-Proposition 198.12 (1) and the Cap Product-Naturality Lemma 206.13, that there exists a class $\sigma \in \mathrm{H}_n(X)$ such that for each $k \in \mathbb{N}_0$ the map
$$\begin{array}{rcl} \cap\sigma\colon \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X) \\ \psi & \mapsto & \psi \cap \sigma \end{array}$$
is an isomorphism.

(2) If X is homotopy equivalent to a closed *non-orientable* topological manifold, then it follows from the Orientation Covering–Properties-Proposition 191.29 that X admits a 2-fold covering $\widetilde{X}$ which is homotopy equivalent to a closed orientable topological manifold.

(3) If X is homotopy equivalent to a closed (orientable) n-dimensional topological manifold, then every finite covering of X is also homotopy equivalent to a closed (orientable) n-dimensional topological manifold. In particular given any finite covering $\widetilde{X}$ the properties stated in (1) and (2) need to hold.

(4) These observations in (1), (2) and (3) already give lots of restrictions. We will generalize these conditions in **??**. There we will show that any CW-complex X which is homotopy equivalent to a closed orientable n-dimensional topological manifold, is in fact a "Poincaré Duality space".

The extra conditions stated in (2), (3) and (4) vanish, if X is a simply connected CW-complex. Thus we have arrived at the following subtle variation on Question 256.16:

[2737]Here we use the observation from page 4944 which says that if W has a handle decomposition, then any $(\overline{B}^t \times W)^\backprime$ has a handle decomposition with the same number of handles.

Question 256.17. Let X be a finite simply connected CW-complex X which admits a class $\sigma \in \mathrm{H}_n(X)$ as in (1), i.e. such that the map

$$\begin{array}{rcl} \cap\sigma\colon \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X) \\ \psi & \mapsto & \psi \cap \sigma \end{array}$$

is an isomorphism. Is X homotopy equivalent to a closed n-dimensional smooth or topological manifold?

It turns out that the answer to Question 256.17 is no. Namely, there are other, more subtle bundle-theoretic obstructions to being homotopy equivalent to a closed smooth or topological manifold. Slightly more precisely, the obstruction is given by showing that the "Spivak Normal Fibration" of a given finite CW-complex has the structure of a bundle. In fact in [**DaP2015**, p. 21] (see also [**MadM1979**, p. 32]) it is shown that there exists a finite simply connected 5-dimensional CW-complex X such that the cohomology ring is isomorphic to the cohomology ring of $S^2 \times S^3$ but such that X is not homotopy equivalent to a closed topological manifold. Similar examples of such simply connected CW-complexes are given in [**GiS1965**, Theorem 5.2]. We also refer to [**LM2024**] and to [**CW2021**, Chapter 1.1.2] for details.

256.5. **Euler characteristics of bundles.** As an application of the CW-Complex–Handle Decomposition Theorem 256.12 we can give a new proof of the following theorem:

Theorem 242.7 (Fiber Bundle-χ-Theorem) Let F be a topological space that is regionally compact, Hausdorff and locally connected. Let $p\colon Y \to B$ be an F-bundle over a topological space that is Hausdorff and paracompact. If F has finitely generated homology and if B is homotopy equivalent to a finite CW-complex, then

$$\chi(Y) \;=\; \chi(B) \cdot \chi(F).$$

Proof. [2738] Let F be a topological space that is regionally compact. Let $p\colon Y \to B$ be an F-bundle over a topological space B that is Hausdorff and paracompact. We assume that $\chi(F)$ is defined and that B is homotopy equivalent to a finite CW-complex.

It follows from the CW-Complex–Handle Decomposition Theorem 256.12 together with the Homotopy-Equivalence Basics Lemma 19.14 (2) that there exists a compact smooth manifold C and a homotopy equivalence $h\colon C \to B$. We consider the $(F, \mathcal{G})$-pullback bundle $q\colon h^*Y \to C$.

We now see that

follows from the Manifold Fiber Bundle-χ-Theorem 250.15

$$\chi(Y) \underset{\uparrow}{=} \chi(h^*Y) \overset{\downarrow}{=} \chi(C) \cdot \chi(F) \underset{\uparrow}{=} \chi(B) \cdot \chi(F).$$

by the Homotopy Type-χ-Lemma 156.3 together with the Homotopy Equivalent Bases Proposition 242.2

by the Homotopy Type-χ-Lemma 156.3 since B and C are homotopy equivalent

■

[2738] The proof we now give is almost identical to the original proof on page 4697, the only difference is in the first proof we used the CW–Simplicial Homotopy Type Proposition 175.21 and the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8, whereas now we are using the CW-Complex–Handle Decomposition Theorem 256.12 together with the Manifold–Fiber Bundle-χ-Theorem 250.15.

256.6. Exercises for Chapter 256.

Exercise 256.1. In the proof of the Manifold-π_1-Realization Proposition 102.5 that we gave on page 4933 we used the following statement: Let M be a compact connected non-empty smooth manifold that admits a handle decomposition such that all handles have index ≤ 2. If $\dim(M) \geq 4$, then the inclusion-induced map $\pi_1(M) \to \pi_1(\mathrm{D}M)$ is an isomorphism. Does this statement also hold if $\dim(M) = 3$?

Exercise 256.2.

(a) Let M be a closed connected 3-dimensional smooth manifold. Show that $\pi_1(M)$ admits a balanced presentation.
Hint. By the Handle Decomposition Theorem 250.7 we know that M admits a handle decomposition. By the Handle Attachment-χ-Lemma 246.15 and the Odd-Dimensional Manifold-χ-Proposition 250.9 we know that

$$\sum_{k=0}^{3}(-1)^k \cdot \text{number of } k\text{-handles} \;=\; 0.$$

You could first consider the case that the handle decomposition contains precisely one 0-handle and one 3-handle. Afterwards try to tackle the general case.

(b) Let M be a compact connected 3-dimensional smooth manifold such that ∂M is a non-empty union of tori. Show that $\pi_1(M)$ admits a presentation of deficiency one.
Remark. If M is the exterior of a non-empty link $L \subseteq S^3$, then this conclusion also follows from the Wirtinger Presentation Proposition 113.6.

Exercise 256.3. Let $\mathbb{F}$ be a field with at least three elements. Show that $\mathrm{SL}(2,\mathbb{F})$ is perfect, i.e. show that its abelianization is the trivial group.
Hint. Use that for any non-zero $d \in \mathbb{F}$ and $a \in \mathbb{F}$ we have

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \;=\; \begin{pmatrix} 1 & a\cdot d^2 \\ 0 & 1 \end{pmatrix}.$$

Exercise 256.4. Give an example of a finite CW-complex X which is not homotopy equivalent to a closed topological manifold but which admits a class $\sigma \in \mathrm{H}_n(X)$ such that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \cap\sigma\colon \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X) \\ \psi & \mapsto & \psi \cap \sigma \end{array}$$

is an isomorphism.
Hint. You could consider the wedge of a sphere with a suitable CW-complex with non-trivial fundamental group.

Exercise 256.5. Let $n \in \mathbb{N}_{\geq 4}$ and let $\pi = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle$ be a finite presentation. Provide a new proof of the Manifold-π_1-Realization Proposition 102.5 by the following approach: Start out with $\overline{B}^{n+1}$, attach k 1-handles and l 2-handles and then take the boundary.

CHAPTER 257

Fundamental groups of 3-dimensional smooth manifolds

Let $n \geq 4$. In the Manifold-π_1-Realization Proposition 102.5 we saw that given any finitely presented group π there exists a closed orientable connected n-dimensional smooth manifold with fundamental group isomorphic to π. In this chapter we will see that this statement dramatically fails for $n = 3$.

This chapter is organized as follows:

- In Section 257.1 we will introduce the deficiency of a finitely presented group.
- In Section 257.2 we will use our knowledge of handle decompositions to provide lower bounds on the deficiency of fundamental groups of compact connected 3-dimensional smooth manifolds.
- In Section 257.3 we will state the Deficiency–Abelian Group Theorem 257.6 which gives upper bounds on the deficiency of abelian groups. This will allow us to show that only "few" finitely generated abelian groups occur as the fundamental group of a compact connected 3-dimensional smooth manifold.
- In Section 257.4 we will give a terse introduction to homology of groups. More precisely, given a (possibly non-abelian) group π we will introduce the corresponding homology groups $\mathrm{H}_*(\pi)$.
- In Section 257.5 we will see that the first and second homology groups of a given group π give rise to an upper bound on the deficiency of π. This fact will then allow us to prove the Deficiency–Abelian Group Theorem 257.6.
- In Appendix 257.6 we will give a completely group theoretic formulation of the 3-dimensional Poincaré Conjecture.
- In Section 257.7 we will provide exercises for Chapter 257.

In the next Chapter 258 we will study high-dimensional knots and linking numbers.

257.1. **The deficiency of a group.** We start out with the following definition from group theory:

Definition 257.1. The deficiency of a finite presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ is defined to be $k - l$, i.e. the deficiency is the number of generators minus the number of relators.

Examples.

(1) Let F be a free group on k generators. A presentation for F is given by $\langle x_1, \ldots, x_k \rangle$. Evidently this presentation has deficiency k.

(2) Let $m \in \mathbb{N}$. In the Free Abelian Group-Presentation Lemma 100.14 we saw that there exists an isomorphism[2739]

$$\langle x_1, \ldots, x_m \mid [x_i, x_j] = e \text{ for all } i < j \in \{1, \ldots, m\} \rangle \xrightarrow{\cong} \mathbb{Z}^m.$$

[2739] As usual, given a group G and $x, y \in G$ we denote by $[x, y] := xyx^{-1}y^{-1}$ the commutator of x and y.

This shows that $\mathbb{Z}^m$ admits a presentation of deficiency $m - \frac{m(m-1)}{2}$.

(3) Let $r \in \mathbb{N}$ and $d_1, \ldots, d_s \in \mathbb{N}$. It follows basically immediately from the above isomorphism in (2) that there exists an isomorphism

$$\left\langle x_1, \ldots, x_{r+s} \;\middle|\; \begin{array}{c} [x_i, x_j] = e \text{ for all } i < j \in \{1, \ldots, r+s\} \\ \text{and } x_1^{d_1} = \cdots = x_s^{d_s} = e \end{array} \right\rangle \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z}.$$

This shows that $\mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z}$ admits a presentation of deficiency $r - \frac{(r+s)(r+s-1)}{2}$.

(4) Let $g \in \mathbb{N}_0$ and let Σ_g be the surface of g. By the Surface-π_1-Proposition 101.1 we know that

$$\pi_1(\Sigma_g) \;\cong\; \langle x_1, \ldots, x_{2g} \,|\, [x_1, x_2] \cdot \ldots \cdot [x_{2g-1}, x_{2g}] \rangle.$$

Thus we see that $\pi_1(\Sigma_g)$ has a presentation of deficiency $2g - 1$. □

Given a finite presentation for a group π it is straightforward to find a finite presentation of smaller deficiency: We just add trivial relations to the given presentation. The name of the game is to find presentations with large deficiency, i.e. a presentation where the number of relators is as small as possible compared to the number of generators.

This leads us to the following definition:

Definition 257.2. Let π be a finitely presented group. We define the deficiency $\mathrm{def}(\pi)$ of π as the maximal deficiency of a finite presentation for π.

Remark. Since the author tends to get confused about deficiencies, we point out the obvious: The deficiency of a group π is negative if and only if in each finite presentation the number of relators exceeds the number of generators. The Deficiency-upper Bound-Lemma 257.3 below gives us a straightforward non-negative upper bound on the deficiency of a group. Finding negative lower bounds usually requires more sophisticated tools. Fortunately we will develop such tools in the latter part of this chapter. □

The following lemma shows, in particular, that the deficiency is in fact a finite natural number.

Lemma 257.3. (Deficiency-upper Bound-Lemma) For any finitely presented group π we have the inequality[2740]

$$\mathrm{def}(\pi) \;\leq\; \mathrm{rank}(\pi_{\mathrm{ab}}).$$

Example. Let F be a free group on k generators. A presentation for F is of course given by $\langle x_1, \ldots, x_k \rangle$. Evidently the deficiency of F is at least k. By the Abelianization Coproduct Proposition 98.18 (2) we know that $F_{\mathrm{ab}} \cong \mathbb{Z}^k$. Thus it follows from the Deficiency-upper Bound-Lemma 257.3 that the deficiency of F is at most k. In summary, we have seen that the deficiency of a free group on k generators equals k. □

Proof. Let π be a finitely presented group and let $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finite presentation of π. It follows from the Abelianization-of-Presentation Proposition 100.17 that there exists a short exact sequence of the form

$$0 \;\to\; A \;\to\; \mathbb{Z}^k \;\to\; \pi_{\mathrm{ab}} \;\to\; 0$$

[2740]Here we denote by π_{ab} the abelianization of π that we introduced in Definition 98.15. Furthermore, given an abelian group H we denote by $\mathrm{rank}(H)$ the rank that we introduced on Definition 97.15

where A is a subgroup of $\mathbb{Z}^k$ that is generated by l elements of $\mathbb{Z}^k$. We obtain that

$$k-l \quad \leq \quad k-\operatorname{rank}(A) \quad \underset{\uparrow}{=} \quad \operatorname{rank}(\pi_{\mathrm{ab}}).$$

by the Rank Additivity Proposition 97.18

It follows that $\operatorname{def}(\pi) \leq \operatorname{rank}(\pi_{\mathrm{ab}})$. ■

257.2. **The deficiency of fundamental groups of 3-manifolds.** We now turn to the study of fundamental groups of compact connected 3-dimensional smooth manifolds. It is worth collecting some of the examples of such fundamental groups that we have calculated so far. First we list examples where the fundamental group is abelian:[2741]

M	$\pi_1(M)$	presentation	deficiency of presentation
S^3	$\{e\}$	$\langle \mid \rangle$	0
$\mathbb{R}\mathrm{P}^3$	$\mathbb{Z}/2\mathbb{Z}$	$\langle t \mid t^2 \rangle$	0
$L(p,q)$	$\mathbb{Z}/p\mathbb{Z}$	$\langle t \mid t^p \rangle$	0
$(S^1)^3$	$\mathbb{Z}^3$	$\langle x_1, x_2, x_3 \mid \{[x_i, x_j]\}_{i<j} \rangle$	0
$\mathbb{R}\mathrm{P}^2 \times S^1$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$	$\langle s, t \mid s^2, [s,t] \rangle$	0
$S^2 \times S^1$	$\mathbb{Z}$	$\langle t \mid \rangle$	1
$[0,1] \times S^1 \times S^1$	$\mathbb{Z}^2$	$\langle s, t \mid [s,t] \rangle$	1.

Now we turn to other examples and calculations we have performed so far:[2742]

M	$\pi_1(M)$	presentation	deficiency of presentation
$[0,1] \times \Sigma_g$	$\pi_1(\Sigma_g)$	see above	$2g-1$
$S^1 \times \Sigma_g$	$\langle t \rangle \times \pi_1(\Sigma_g)$	see Exercise 257.1	0
$\operatorname{Tor}(\Sigma_g, \varphi)$	$\pi_1(\Sigma_g) \rtimes_\varphi \langle t \rangle$	see Exercise 257.2	0
$S^1 \times \Sigma_{g,n}, n \geq 1$	$\langle t \rangle \times F_{2g+n-1}$	see Exercise 257.1	1
$S^3 \setminus \text{link}$		Wirtinger presentation	1
$k \cdot (S^2 \times S^1)$	$\langle x_1, \ldots, x_k \rangle$	$\langle x_1, \ldots, x_k \rangle$	k
$M \# M'$	$\pi_1(M) * \pi_1(M')$		additive
$M \#_{\mathrm{b}} M'$	$\pi_1(M) * \pi_1(M')$		additive.

We see that in each case we found a presentation of non-negative deficiency. The following theorem shows that this is not a coincidence:

Theorem 257.4. (3-Manifold–Deficiency Theorem) Let M be a compact connected non-empty 3-dimensional smooth manifold.

(1) If M is closed, then $\operatorname{def}(\pi_1(M)) \geq 0$.

(2) If $\partial M \neq \varnothing$, then

$$\operatorname{def}(\pi_1(M)) \;\geq\; 1-\chi(M) \;\underset{\uparrow}{=}\; 1-\tfrac{1}{2}\chi(\partial M).$$

by the Odd-Dimensional Manifold-χ-Proposition 250.9

[2741] We refer to the discussion on page 1799, the Lens Space-π_1-Proposition 93.2, the Torus-π_1-Proposition 90.14 and the Product-π_1-Proposition 90.13 for details.

[2742] We refer to the Mapping Torus-π_1-Proposition 93.13, the Wirtinger–Presentation Proposition 113.6, the Connected Sum-π_1-Proposition 102.4 and Exercises 99.18, 257.1 and 257.2 for details.

Remark. Let M be a compact connected non-empty 3-dimensional smooth manifold such that the boundary is a non-empty union of tori. It follows from the 3-Manifold–Deficiency Theorem 257.4 (2) that $\pi_1(M)$ admits a presentation of deficiency one. This implies in particular that, according to Definition 116.14, we can define an Alexander function $\Delta_{\pi_1(M),\Phi} \in \mathbb{Q}(H)$ for any epimorphism $\phi\colon \pi_1(M) \to H$ onto a non-trivial free abelian group H. □

Proof. In the proof we start out with the case of non-empty boundary, and then we will deal with the closed case:

(2) Let M be a compact connected 3-dimensional smooth manifold *with non-empty boundary*. It follows from the Handle Decomposition Theorem 250.7, the Minimal 0-Handle Proposition 247.13 and the Minimal n-Handle Proposition 248.4 that M admits a handle decomposition with a single 0-handle and no 3-handle. We denote by g the number of 1-handles and we denote by h the number of 1-handles. We now see that

$$\pi_1(M) \underset{\uparrow}{\cong} \pi_1(\text{1-skeleton of } M, P)/\langle r_1, \dots, r_h\rangle \underset{\uparrow}{\cong} \langle x_1, \dots, x_g \,|\, r_1, \dots, r_h\rangle$$

by the Handle Attachment-π_1-Proposition 246.13 applied to the 2-handle attachments

by the Handle Attachment-π_1-Proposition 246.13 applied to the 1-handle attachments

$$\underset{\uparrow}{\cong} \langle x_1, \dots, x_g \,|\, r_1, \dots, r_{-1+g+\chi(M)}\rangle.$$

it follows from the Handle Attachment χ-Lemma 246.15 that $\chi(M) = 1 - g + h$

It follows that $\operatorname{def}(\pi_1(M)) \geq g - (-1 + g + \chi(M)) = 1 - \chi(M)$.

(1) Let M be a *closed* connected non-empty 3-dimensional smooth manifold. We give two slightly different arguments why $\operatorname{def}(\pi_1(M)) \geq 0$:

- If in the proof of (2) we appeal to the Minimal Handle Proposition 248.5, then it follows from a slight modification of the argument in (1) that $\operatorname{def}(\pi_1(M)) \geq 0$.
- Since M is non-empty we can pick a smooth embedding $\varphi\colon \overline{B}^3 \to M$. We now see that

$$\operatorname{def}(\pi_1(M)) \underset{\uparrow}{=} \pi_1(M \setminus \varphi(B^3)) \underset{\uparrow}{\geq} 1 - \tfrac{1}{2}\chi(\partial(M \setminus \varphi(B^3))) \underset{\uparrow}{=} 1 - \tfrac{1}{2}\chi(S^2) = 0.$$

this equality follows from $\pi_1(M) = \pi_1(M \setminus \varphi(B^3)) * \pi_1(\overline{B}^3)$

by (2)

since $\partial(M \setminus \varphi(B^2)) \cong S^2$

■

If one is optimistically inclined, then one might hope that the two inequalities of the 3-Manifold–Deficiency Theorem 257.4 are in fact equalities. It turns that this is not always the case: For example for the closed 3-dimensional smooth manifold $S^2 \times S^1$ we have $\pi_1(S^2 \times S^1) \cong \langle t\rangle$, where $\operatorname{def}(\pi_1(M)) = \operatorname{def}(\langle t\rangle) = 1$, which is larger than the bound 0 from the 3-Manifold–Deficiency Theorem 257.4 (1). Rather surprisingly the following theorem give a straightforward criterion for the inequalities of the 3-Manifold–Deficiency Theorem 257.4 to be in fact equalities:

Theorem 257.5. (3-Manifold–Deficiency Equality Theorem) Let M be a compact orientable connected non-empty 3-dimensional smooth manifold such that $\pi_1(M)$ is infinite and such that $\pi_2(M) = 0$.

(1) If M is closed, then $\operatorname{def}(\pi_1(M)) = 0$.

(2) If M has non-empty boundary, then $\operatorname{def}(\pi_1(M)) = 1 - \chi(M)$.

Proof. We will not make use of the theorem. Thus we refer to [**Eps1961a**, Lemma 3.1] for a proof. ■

257.3. **The deficiency of abelian groups.** In general it is rather tricky to show that the deficiency of a given group is negative. Fortunately the following theorem, which was proved by David Epstein [**Eps1961a**] in 1961, gives us some non-trivial results for abelian groups:

Theorem 257.6. (Deficiency–Abelian Group Theorem) The deficiency of any finitely generated abelian group is at most one. Furthermore:

(1) The only finitely generated abelian groups of deficiency 0 (up to isomorphism) are $\mathbb{Z}^3$, $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $n \neq -1, 0, 1$.
(2) The only finitely generated abelian groups of deficiency 1 (up to isomorphism) are $\mathbb{Z}^2$ and $\mathbb{Z}$.

Proof. We will sketch a proof of the Deficiency–Abelian Group Theorem 257.6 in the next two sections. More precisely we will do the following:

- In Section 257.4 we will introduce, given a group π, the homology groups $\mathrm{H}_*(\pi)$ of the group π.
- In Section 257.5 we will see that the first and second homology groups of a given group π give rise to an upper bound on the deficiency of π. This fact will then allow us to prove the Deficiency–Abelian Group Theorem 257.6. ■

Now we have all the tools to state the following theorem, which comes quite close to classifying the abelian groups that can occur as fundamental groups of compact 3-dimensional smooth manifolds. Also note that the theorem gives in particular a negative answer to Question 102.8.

Theorem 257.7. (3-Manifold–π_1–Abelian Theorem) Let M be a compact connected non-empty 3-dimensional smooth manifold such that $\pi_1(M)$ is abelian.

(1) If M is closed, then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^3$, or to $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}_0$, or to $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}_0$.
(2) If $\partial M \neq \varnothing$, and if no component of ∂M is diffeomorphic to S^2 or to $\mathbb{R}\mathrm{P}^2$, then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^2$ or to $\mathbb{Z}$.

Remark.

(1) If we compare the abelian groups which are allowed by the 3-Manifold–π_1–Abelian Theorem 257.7 with the groups we found on page 4950 we unfortunately find some discrepancies: It is not clear whether $\mathbb{Z}^2$ or $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}_{\geq 3}$ can occur as the fundamental group of a closed 3-dimensional smooth manifold. Much later, once we have developed more tools in the topology of 3-dimensional smooth manifolds, we will exclude these types of groups in the 3-Manifold–π_1–Abelian Theorem **??**.
(2) For closed connected 3-dimensional smooth manifolds the theorem was first proved by Kurt Reidemeister [**Rei1936**, p. 28]. The general case was dealt with by David Epstein [**Eps1961a**, Theorem 3.3]. In fact, both authors could also handle the cases $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, $n \geq 3$ and $\mathbb{Z}^2$. □

Proof. The theorem follows immediately from the combination of the 3-Manifold–Deficiency Theorem 257.4, the Deficiency–Abelian Group Theorem 257.6 and the observation that the Euler characteristic of a closed non-empty 2-dimensional smooth manifold, where no component is diffeomorphic to S^2 or $\mathbb{R}\mathrm{P}^2$, is negative. ■

This discussion should have convinced the reader that fundamental groups of compact connected 3-dimensional smooth manifolds form an interesting class of finitely presented groups: The class is large enough to contain many different types of groups, but it is small enough so that one might be able to prove non-trivial classification results. Fortunately this class of groups is studied in great detail in [**AFW2015**].

257.4. **Interlude: Group homology.** Our intermediate goal is to sketch a proof of the Deficiency–Abelian Group Theorem 257.6. The proof builds on the concept of the homology groups of a group. In this section we will give a short introduction to group homology, namely we will sketch the definition and we will provide a few basic calculations of group homology. In the next Section 257.5 we will see that the first and second homology groups of a given group π give rise to an upper bound on the deficiency of π. This fact will then allow us to prove the Deficiency–Abelian Group Theorem 257.6.

We start out by recalling Definition 126.11:

Definition 257.8. A path-connected non-empty topological space X is called aspherical if $\pi_i(X) = 0$ for all $i \in \mathbb{N}_{\geq 2}$.

This definition allows us to introduce the following more interesting definition:

Definition 257.9. Let π be a group and let X be a path-connected non-empty topological space. We say that X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$ if the following three conditions hold:[2743]

(a) $\pi_1(X) \cong \pi$.
(b) X is aspherical.
(c) X admits a CW-structure.

The following lemma gives us the two types of Eilenberg-Maclane spaces that we care about in this chapter:

Lemma 257.10. (Eilenberg–Maclane Example Lemma)[2744]

(1) An Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^m, 1)$ is given by $(S^1)^m$.
(2) An Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$ is given by $\mathbb{R}\mathrm{P}^\infty = \varinjlim \mathbb{R}\mathrm{P}^i$.

Proof.

(1) Let $m \in \mathbb{N}_0$. It follows from the Torus-π_1-Proposition 90.14, the discussion on page 2587, the CW-Category–Product Proposition 139.16 and the CW-Complex Product Proposition 139.18 that the m-dimensional torus $(S^1)^m$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^m, 1)$.
(2) First recall that on page 2628 we showed that $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure. Furthermore recall that on page 2643 we showed, using the CW-Direct Colimit-π_n-Proposition 136.25, that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \cong \varinjlim \pi_1(\mathbb{R}\mathrm{P}^i) \cong \mathbb{Z}/2\mathbb{Z}$ and we saw that for every $n \geq 2$ we have $\pi_n(\mathbb{R}\mathrm{P}^\infty) \cong \varinjlim \pi_n(\mathbb{R}\mathrm{P}^i) \cong \varinjlim \pi_n(S^i) = 0$. This shows that $\mathbb{R}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$. ■

[2743] In the Eilenberg-Maclane Existence Theorem 307.10 we showed that such an Eilenberg-Maclane space exists for every group π.
[2744] In the Eilenberg–Maclane Example Lemma 307.3 we will provide an extended version of this lemma.

Our next goal is to show that Eilenberg-Maclane spaces always exist. A first step towards this goal is provided by the following proposition:

Proposition 257.11. (π_1-Realization Proposition)

(1) Let $\pi = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finitely presented group. There exists a connected 2-dimensional CW-complex with one 0-cell, k 1-cells and l 2-cells, such that $\pi_1(X)$ is isomorphic to π.

(2) Let π be any group. There exists a canonical triple $(X, x_0, \varphi\colon \pi_1(X, x_0) \to \pi)$ where X is a connected 2-dimensional CW-complex with a single 0-cell $\{x_0\}$ and where $\varphi\colon \pi_1(X, x_0) \to \pi$ is an isomorphism.

Proof.

(1) We proved this statement in the Presentation–CW Lemma 138.5.

(2) This statement is just a special case of the Homotopy Group-Realization Proposition 307.12. ■

The next proposition shows that we can upgrade the CW-complexes from the π_1-Realization Proposition 257.11 to Eilenberg-Maclane spaces:

Proposition 257.12. (Embed-in-Aspherical CW-Complex Proposition) Let $n \in \mathbb{N}_0$. Given any connected non-empty n-dimensional CW-complex X and given any $x_0 \in X$ there exists a canonical[2745] CW-complex $\widetilde{X}$ with the following three properties:

(a) The n-skeleton of $\widetilde{X}$ equals X.

(b) We have $\pi_i(\widetilde{X}) = 0$ for $i \geq n$.

(c) For $i = 0, \ldots, n-1$ the inclusion-induced map $\pi_i(X, x_0) \to \pi_i(\widetilde{X}, x_0)$ is an isomorphism.

Proof. This proposition is just a special case of the Kill-All Higher-π_n-Proposition 307.9. ■

The following theorem says that any two Eilenberg-Maclane spaces of the same type are essentially the same:

Theorem 257.13. (Eilenberg-Maclane Uniqueness Theorem) Let π be a group. Any two Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$ are homotopy equivalent.

Proof. Let π be a group and let (X, x_0) and (Y, y_0) be two pointed Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. Since $\pi_1(X, x_0) \cong \pi$ and since $\pi_1(Y, y_0) \cong \pi$ we see that there exists an isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Since X is a connected CW-complex and since Y is aspherical it follows from the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 307.18 that there exists a continuous map $f\colon (X, x_0) \to (Y, y_0)$ with $f_* = \varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Since X and Y are aspherical we see that the induced maps on all higher homotopy groups are also isomorphisms. Since X and Y are CW-complexes it follows from the Whitehead Theorem 305.1 that f is a homotopy equivalence. ■

The above results allow us to introduce the following definition:

[2744] Here "canonical" means that we provide an explicit construction, which does not involve any choices.

[2745] Here "canonical" again means that we provide an explicit construction, which does not involve any choices.

Definition 257.14. Let π be a group. Let X be the canonical connected non-empty CW-complex X with $\pi_1(X) \cong \pi$ that we obtain from the π_1-Realization Proposition 257.11 (2) together with the Embed-in-Aspherical CW-Complex Proposition 257.12. We set[2746]

$$\mathrm{H}_i(\pi) \;:=\; \mathrm{H}_i(X).$$

For actual calculations of group homology one usually appeals to the following proposition:

Proposition 257.15. (Group H_*-via-$\mathrm{K}(\pi,1)$-Proposition) Let π be a group. For any Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$ and any $n \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_n(\pi) \;\cong\; \mathrm{H}_n(X).$$

Proof. This proposition is an immediate consequence of the Eilenberg-Maclane Uniqueness Theorem 257.13 and the Homotopy Equivalence-H_*-Proposition 144.14 (2). ■

Examples.

(1) It follows from the H_0-Proposition 143.25 that for any group π we have $\mathrm{H}_0(\pi) \cong \mathbb{Z}$.

(2) Let π be a group and let X be the corresponding canonical Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$. We see that

$$\mathrm{H}_1(\pi) \;:=\; \mathrm{H}_1(X) \underset{\text{by the Hurewicz Theorem 166.10}}{\overset{\uparrow}{\cong}} \pi_1(X)_{\mathrm{ab}} \underset{\text{since } \pi_1(X) \cong \pi}{\overset{\uparrow}{\cong}} \pi_{\mathrm{ab}}.$$

(3) For any $m \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$ we have

$$\mathrm{H}_i(\mathbb{Z}^m) \underset{\text{by the Group } \mathrm{H}_*\text{-via-}\mathrm{K}(\pi,1)\text{-Proposition 257.15, and since } (S^1)^m \text{ is a } \mathrm{K}(\mathbb{Z}^m,1)}{\overset{\uparrow}{\cong}} \mathrm{H}_i((S^1)^m) \underset{\text{by the Torus-}\mathrm{H}_*\text{-Proposition 153.9}}{\overset{\uparrow}{\cong}} \begin{cases} \mathbb{Z}^{\binom{m}{i}}, & \text{if } i \in \{0,\dots,m\}, \\ 0, & \text{if } i > m \end{cases}$$

(4) For any $i \in \mathbb{N}_0$ we have

$$\mathrm{H}_i(\mathbb{Z}/2\mathbb{Z}) \underset{\text{since } \mathbb{RP}^\infty \text{ is a } \mathrm{K}(\mathbb{Z}/2\mathbb{Z},1)}{\uparrow\cong} \mathrm{H}_i(\mathbb{RP}^\infty) \underset{\text{by the } \mathbb{RP}^n\text{-}\mathrm{H}_*\text{-Proposition 155.18}}{\uparrow\cong} \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } i \text{ odd}, \\ 0, & \text{if } i \text{ is even and } i > 0. \end{cases}$$

(5) The homology groups of $\mathbb{Z}/d\mathbb{Z}$ are analogous to the homology groups for $\mathbb{Z}/2\mathbb{Z}$: We just need to replace $\mathbb{Z}/2\mathbb{Z}$ for odd dimensions by $\mathbb{Z}/d\mathbb{Z}$ for odd-dimensions. This can be shown by constructing an explicit model of $\mathrm{K}(\mathbb{Z}/d\mathbb{Z},1)$ by considering a suitable action of $\mathbb{Z}/d\mathbb{Z}$ on the odd-dimensional spheres S^{2n+1}. We will not work out the details. Instead we refer to the Group-H_*-of-$\mathbb{Z}/k\mathbb{Z}$-Lemma 317.9 for an alternative, more algebraic, approach to calculating these homology groups. In this chapter we will not make use of these homology groups. □

Next we would like to calculate homology groups of arbitrary finitely generated abelian groups. By the Abelian Group-Classification Theorem 97.24 we know that any finitely

[2746]The constructions in the proofs of the π_1-Realization Proposition 257.11 (2) and the Embed-in-Aspherical CW-Complex Proposition 257.12 do not involve choices. So X is well-defined and thus each $\mathrm{H}_i(\pi)$ is well-defined.

generated abelian group is the direct product of finitely many cyclic groups. This observation explains why we are interested in the following theorem:

Theorem 257.16. (Group-H_*-Künneth Theorem) For any two groups A and B and any $n \in \mathbb{N}_0$ there exists a short exact sequence of the following form

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(A) \otimes \mathrm{H}_q(B) \to \mathrm{H}_n(A \times B) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(A), \mathrm{H}_q(B)) \to 0$$

and which has the property that it splits.

Example. We now calculate the second homology groups of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$\mathrm{H}_2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \overset{\text{by the Group-H}_*\text{-Künneth Theorem 257.16}}{\cong} \underbrace{\bigoplus_{p+q=2} \mathrm{H}_p(\mathbb{Z}/2\mathbb{Z}) \otimes \mathrm{H}_q(\mathbb{Z}/2\mathbb{Z})}_{\cong \mathbb{Z}/2\mathbb{Z}, \text{ by the above calculation of } \mathrm{H}_*(\mathbb{Z}/2\mathbb{Z}) \text{ and the Cyclic-Tensor-Cyclic Lemma 159.5}} \oplus \underbrace{\bigoplus_{p+q=1} \mathrm{Tor}(\mathrm{H}_p(\mathbb{Z}/2\mathbb{Z}), \mathrm{H}_q(\mathbb{Z}/2\mathbb{Z}))}_{= 0, \text{ by the Tor-Properties Lemma 162.13 (3) and (4)}} \cong \mathbb{Z}/2\mathbb{Z}.$$

□

Proof. Let X be an aspherical CW-complex with $\pi_1(X) \cong A$ and let Y be an aspherical CW-complex with $\pi_1(Y) \cong B$. Recall that there are two natural topologies on the set $X \times Y := \{(x, y) \,|\, x \in X, y \in Y\}$:

(1) We have the usual product topology $\mathcal{TP}$ from the Topological-Product Proposition 7.2.
(2) We have the CW-product topology $\mathcal{CWP}$ on $X \times Y$ that we introduced in Definition 139.15.

Next we consider the following maps:

$$\begin{array}{ccccccccc} & & & & \overbrace{\mathrm{H}_n((X \times Y, \mathcal{CWP}))}^{\cong \mathrm{H}_n(A \times B)} & & & & \\ & & & & \cong \big\downarrow f_* & & & & \\ 0 & \to & \bigoplus_{p+q=n} \underbrace{\mathrm{H}_p(X)}_{\cong \mathrm{H}_p(A)} \otimes \underbrace{\mathrm{H}_q(Y)}_{\cong \mathrm{H}_q(B)} & \to & \mathrm{H}_n((X \times Y, \mathcal{TP})) & \to & \bigoplus_{p+q=n-1} \mathrm{Tor}(\underbrace{\mathrm{H}_p(X)}_{\cong \mathrm{H}_p(A)}, \underbrace{\mathrm{H}_q(Y)}_{\cong \mathrm{H}_q(B)}) & \to & 0. \end{array}$$

The theorem follows from the following clarifications and observations:

(1) (a) By the CW-Category–Product Proposition 139.16 we know that $(X \times Y, \mathcal{CWP})$ admits the structure of a CW-complex.
(b) By the CW-Complex Product Proposition 139.18 we know that the identity map

$$f \colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$$

is continuous. By the Product-not-Product Lemma 139.19 we know that in general this map is *not* a homeomorphism. But the argument of the CW-Products–Invariants Proposition 139.20 shows that the induced map

$$f_* \colon \pi_i(X \times Y, \mathcal{CWP}) \to \pi_i(X \times Y, \mathcal{TP})$$

is an isomorphism for each $i \in \mathbb{N}_0$.

It follows from this discussion together with the Product-π_n-Proposition 134.18 that the CW-complex $(X \times Y, \mathcal{CWP})$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, 1)$.

(2) The argument of the CW-Products–Invariants Proposition 139.20 also implies that the induced map
$$f_*\colon \mathrm{H}_n((X\times Y,\mathcal{CWP}))\ \to\ \mathrm{H}_n((X\times Y,\mathcal{TP}))$$
is an isomorphism for each $i\in\mathbb{N}_0$.[2747]

(3) The bottom sequence is given by the Künneth Theorem 164.6. Note that by the Künneth Theorem 164.6 the sequence is exact and is known to split. ■

We have now already come to an end of our short introduction to group homology. As the reader will surely sense, there is much more to day. Thus we will return to this topic in Chapter 317.

257.5. **Group homology give upper bounds on the deficiency.** We start out by introducing a new notation:

Notation 257.17. Let A be an abelian group. Inspired by the Notation 252.9 we write
$$q(A)\ :=\ \text{minimal number of generators of the torsion subgroup Tor}\,A.$$

Now we can formulate the result which explains why we introduced homology groups of groups:

Theorem 257.18. (Deficiency-upper-Bound Theorem) For every finitely presented group π we have the inequality
$$\mathrm{def}(\pi)\ \leq\ \mathrm{rank}(\mathrm{H}_1(\pi))-\mathrm{rank}(\mathrm{H}_2(\pi))-q(\mathrm{H}_2(\pi)).$$

Example.

(1) Let π be a group. On page 4955 we showed that $\mathrm{H}_1(\pi)\cong\pi_{\mathrm{ab}}$. It follows that the inequality of the Deficiency-upper-Bound Theorem 257.18 implies the simple minded inequality from the Deficiency-upper Bound-Lemma 257.3.

(2) Let $m\in\mathbb{N}_0$. On page 4949 we showed that $\mathbb{Z}^m$ admits a presentation of deficiency $m-\frac{m\cdot(m-1)}{2}$. On the other hand we also have
$$\mathrm{def}(\mathbb{Z}^m)\ \underset{\substack{\uparrow\\ \text{by the Deficiency-upper-Bound}\\ \text{Theorem 257.18}}}{\leq}\ \mathrm{rank}(\overbrace{\mathrm{H}_1(\mathbb{Z}^m)}^{\cong\mathbb{Z}^m})-\mathrm{rank}(\overbrace{\mathrm{H}_2(\mathbb{Z}^m)}^{\cong\mathbb{Z}^{\frac{m\cdot(m-1)}{2}}})-q(\overbrace{\mathrm{H}_2(\mathbb{Z}^m)}^{\cong\mathbb{Z}^{\frac{m\cdot(m-1)}{2}}})\ =\ m-\frac{m\cdot(m-1)}{2}.$$
We have thus shown that the deficiency of $\mathbb{Z}^m$ equals $m-\frac{m\cdot(m-1)}{2}$.

(3) We consider the group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. On page 4949 we showed that $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ admits a presentation of deficiency -1. Now we see that
$$-1\ \leq\ \mathrm{def}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$$
$$\underset{\substack{\uparrow\\ \text{by the Deficiency-upper-Bound}\\ \text{Theorem 257.18}}}{\leq}\ \mathrm{rank}(\overbrace{\mathrm{H}_1(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})}^{\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}})-\mathrm{rank}(\overbrace{\mathrm{H}_2(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})}^{\cong\mathbb{Z}/2\mathbb{Z}})-q(\overbrace{\mathrm{H}_2(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})}^{\cong\mathbb{Z}/2\mathbb{Z}})\ =\ -1.$$
We have thus shown that the deficiency of $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ equals -1. □

[2747] We trust that the reader worked out the details in Exercise 164.20.

The proof of the Deficiency-upper-Bound Theorem 257.18 rests very much on the following earlier proposition:

Proposition 252.10. (Algebraic Morse Inequalities Proposition) Let C_* be a chain complex of finitely generated free abelian groups. Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(C_{k-j}) \;\geq\; q(\mathrm{H}_k(C_*)) + \sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(\mathrm{H}_{k-j}(C_*)).$$

Proof of the Deficiency-upper-Bound Theorem 257.18. Let π be a finitely presented group. It follows from the Deficiency-upper Bound-Lemma 257.3 that we can pick a presentation $\pi = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ with $\operatorname{def}(\pi) = k - l$. By the π_1-Realization Proposition 257.11 (1) we know that there exists a connected CW-complex X with a single 0-cell, k 1-cells and l 2-cells such that $\pi_1(X) \cong \pi$. By the Embed-in-Aspherical CW-Complex Proposition 257.12 we can in fact find an aspherical CW-complex X with a single 0-cell, k 1-cells and l 2-cells such that $\pi_1(X) \cong \pi$.

We see that

$$\begin{aligned}
\operatorname{def}(\pi) \;=\; k - l \;&\overset{\downarrow}{=}\; 1 - \sum_{j=0}^{2} (-1)^j \cdot \operatorname{rank}(\mathrm{C}^{\mathrm{CW}}_{2-j}(X)) \\
&\leq\; 1 - q(\mathrm{H}_2(X)) - \underbrace{\operatorname{rank}(\mathrm{H}_0(X))}_{=1} + \operatorname{rank}(\mathrm{H}_1(X)) - \operatorname{rank}(\mathrm{H}_2(X)) \\
&\overset{\uparrow}{=}\; \operatorname{rank}(\mathrm{H}_1(\pi)) - \operatorname{rank}(\mathrm{H}_2(\pi)) - q(\mathrm{H}_2(\pi)).
\end{aligned}$$

it follows from the Skeleton-H_*-Lemma 155.2 that for the cellular chain complex $\mathrm{C}^{\mathrm{CW}}_*(X)$ we have $\operatorname{rank}(\mathrm{C}^{\mathrm{CW}}_0(X)) = \#0\text{-cells} = 1$, $\operatorname{rank}(\mathrm{C}^{\mathrm{CW}}_1(X)) = \#1\text{-cells} = k$, $\operatorname{rank}(\mathrm{C}^{\mathrm{CW}}_2(X)) = \#2\text{-cells} = l$

by the Algebraic Morse Inequalities Proposition 252.10, together with the Singular-Cellular-H_*-Proposition 155.8

by Definition 257.14 of $\mathrm{H}_i(\pi)$ ■

We can now provide the proof of the Deficiency–Abelian Group Theorem 257.6. For the reader's convenience we recall the statement:

Theorem 257.6. (Deficiency–Abelian Group Theorem) The deficiency of any finitely generated abelian group is at most one. Furthermore:

(1) The only finitely generated abelian groups of deficiency 0 (up to isomorphism) are $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $n \neq -1, 0, 1$ and $\mathbb{Z}^3$.
(2) The only finitely generated abelian groups of deficiency 1 (up to isomorphism) are $\mathbb{Z}$ and $\mathbb{Z}^2$.

Proof. On page 4957 we showed that the deficiency of a free abelian group of rank m is given by $m - \frac{m \cdot (m-1)}{2}$. This calculation easily implies the desired statement for finitely generated *free* abelian groups.

Next note that it follows easily from the discussion on page 4949 that $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $n \neq -1, 0, 1$ admit presentations of deficiency 0.

Thus it remains to prove the following claim:

Claim. Let π be a finitely generated abelian group, that is not torsion-free.
(1) The deficiency of π is less or equal than zero.
(2) If the deficiency of π is 0, then π is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$ or to $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $n \neq -1, 0, 1$.

Proof. Let π be a finitely generated abelian group, that is not torsion-free. It follows fairly easily from the Abelian Group-Classification Theorem 97.24 that there exists an $m \in \mathbb{N}_0$ and a non-zero finite abelian group G such that π is isomorphic to $\mathbb{Z}^m \times G$.

Next note that we have the following isomorphisms:

$$\mathrm{H}_2(\mathbb{Z}^m \times G) \overset{\text{by the Group-H}_*\text{-Künneth Theorem 257.16}}{\cong} \bigoplus_{p+q=2} \mathrm{H}_p(\mathbb{Z}^m) \otimes \mathrm{H}_q(G) \oplus \bigoplus_{p+q=1} \underbrace{\mathrm{Tor}(\mathrm{H}_p(\mathbb{Z}^m), \mathrm{H}_q(G))}_{=0, \text{ by the Tor-Properties Lemma 162.13 (4)}}$$
$$\cong \mathbb{Z} \otimes \mathrm{H}_2(G) \oplus \mathbb{Z}^m \otimes \underbrace{\mathrm{H}_1(G)}_{=G} \oplus \mathbb{Z}^{\frac{m\cdot(m-1)}{2}} \otimes \mathrm{H}_2(G).$$

We now see that

$$\mathrm{def}(\mathbb{Z}^m \times G) \overset{\text{by the Deficiency-upper-Bound Theorem 257.18}}{\leq} m - \tfrac{m\cdot(m-1)}{2} - q(\mathrm{H}_2(\mathbb{Z}^m \times G))$$
$$\overset{\text{since } \mathbb{Z}^m \otimes G \text{ is a torsion group and since we just just showed that } \mathbb{Z} \otimes G \text{ is a quotient of the torsion subgroup of } \mathrm{H}_2(\mathbb{Z}^m \times G)}{\leq} m - \tfrac{m\cdot(m-1)}{2} - q(\mathbb{Z}^m \otimes G) \leq m - \tfrac{m\cdot(m-1)}{2} - m \cdot q(G).$$

Since G is non-zero we know that $q(G) \geq 1$. It is now straightforward to verify that the only options are that $m = 0, 1$ and that $q(G) \leq 1$. But this means that G is cyclic. ■

257.6. **Appendix: A group theoretic formulation of the Poincaré Conjecture.** In this appendix we will discuss a particularly fascinating aspect of handle decompositions of closed 3-dimensional smooth manifolds: They lead to a purely group theoretic formulation of the Poincaré Conjecture in dimension three.

To understand where this formulation comes from it helpful to present a slightly different proof of the following theorem:

Theorem 257.4. (3-Manifold–Deficiency Theorem) Let M be a closed connected 3-dimensional smooth manifold. We have $\mathrm{def}(\pi_1(M)) \geq 0$.

Proof. Let M be a closed orientable connected 3-dimensional smooth manifold. We start out with the following claim:

Claim. There exists a $g \in \mathbb{N}_0$ and two compact codimension-zero submanifolds X and Y of M with the following properties:
(1) X and Y both admit a handle decomposition with a single 0-handle, g 1-handles and no other handles.
(2) $\partial X = \partial Y = X \cap Y$ is a surface of genus g.

Proof. It follows from the Handle Decomposition Theorem 250.7 together with the Standard Handle Decomposition Proposition 247.7 and the Minimal Handle Proposition 248.5 that M admits a standard handle decomposition

$$M \;=\; h^0 \cup_{\varphi_1} h^1 \cup \cdots \cup_{\varphi_g} h^1 \cup_{\psi_1} h^2 \cup \cdots \cup_{\psi_h} h^2 \cup h^3$$

with a single 0-handle and a single 3-handle. We denote by g the number of 1-handles and we denote by h the number of 2-handles.

We denote by X the 1-skeleton of M. Note that it follows from the Manifold Handle Decomposition–Splitting Proposition 248.6 that there exists a smooth submanifold Y that we obtain from a 0-handle by attaching h 1-handles (namely the duals of the original 2-handles) such that $M = X \cup Y$ and such that $\partial X = \partial Y = X \cap Y$.

It remains to show that $g = h$. This follows from the following two observations:

- Since M is closed it follows from the Odd-Dimensional Manifold-χ-Proposition 250.9 that $\chi(M) = 0$.
- By the Handle Attachment-χ-Lemma 246.15 we know that $\chi(M)$ equals the alternating sum of the number of k-handles. In our case this implies that $\chi(M) = 1 - g + h - 1$.⊞

We set $F := X \cap Y$ and we pick a base point $x_0 \in F = X \cap Y$. By the Handlebody-Invariants Lemma **??** and the Surface-π_1-Proposition 101.1 we can make the following identifications:

- $\pi_1(X, x_0) =_{\mathrm{i}} \langle x_1, \ldots, x_g \rangle$,
- $\pi_1(Y, x_0) =_{\mathrm{i}} \langle y_1, \ldots, y_g \rangle$,
- $\pi_1(F, x_0) =_{\mathrm{i}} \langle a_1, \ldots, a_{2g} \,|\, [a_1, a_2] \cdot \ldots \cdot [a_{2g-1}, a_{2g}] \rangle$.

We denote by $\varphi\colon \pi_1(F, x_0) \to \pi_1(X, x_0)$ and $\psi\colon \pi_1(F, x_0) \to \pi_1(Y, x_0)$ the inclusion-induced maps. Now we see that

$$\begin{array}{rcl} \pi_1(M, x_0) & \overset{\text{by the Manifold–Seifert-van Kampen Theorem 102.1}}{\cong} & \pi_1(X, x_0) *_{\pi_1(F, x_0)} \pi_1(Y, x_0) \\ & = & \langle x_1, \ldots, x_g, y_1, \ldots, y_g \,|\, \varphi(a_1) = \psi(a_1), \ldots, \varphi(a_{2g}) = \psi(a_{2g}) \rangle. \\ & \underset{\text{by the Amalgamated Product–Presentation Lemma 100.20}}{\uparrow} & \end{array}$$

We have thus again shown that $\pi_1(M)$ admits a presentation of deficiency zero. ■

The proof of the above theorem explains hopefully somewhat the setup for the following theorem:

Theorem 257.19. (Poincaré Conjecture Reformulation Theorem) We introduce the following notation and language:

(a) Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g and we denote by F_g the free group of rank g.

(b) We say that a homomorphism $\beta\colon G \to A * B$ from a group G to the free product of two groups A and B is *essential*, if there is no $h \in A * B$ such that $h \cdot \beta(G) \cdot h^{-1}$ is contained in A or B.

The following two statements are equivalent:

(1) Every 3-dimensional homotopy sphere[2748] is diffeomorphic to S^3.

(2) For every $g \in \mathbb{N}_{\geq 2}$ every epimorphism $\alpha\colon \pi_1(\Sigma_g) \to F_g \times F_g$ factors through an essential monomorphism $\beta\colon \pi_1(\Sigma_g) \to A * B$ from $\pi_1(\Sigma_g)$ to the free product of two groups A and B.

Proof. The theorem was proved by John Stallings [**Stal1965a**, Chapter 3] and William Jaco [**Jaco1969**, Theorem 6.3]. ■

Put differently, by Perelman's proof of the Poincaré 3D-Conjecture Theorem **??** we now know that Statement (1) of the Poincaré Conjecture Reformulation Theorem 257.19 holds, thus Statement (2) also holds. This is the only known proof of that purely group theoretic statement. Conversely, any proof of Statement (2) of the Poincaré Conjecture Reformulation Theorem 257.19 would immediately reprove the Poincaré 3D-Conjecture Theorem **??**. Surely that is a quick and easy way to mathematical glory.

257.7. Exercises for Chapter 257.

Exercise 257.1.

(a) Let $A = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_m \rangle$ and $B = \langle y_1, \dots, y_l \,|\, s_1, \dots, s_n \rangle$ be two presentations. Show that

$$\langle x_1, \dots, x_k, y_1, \dots, y_l \,|\, r_1, \dots, r_m, s_1, \dots, s_n, \{[r_i, s_j]\}_{i=1,\dots,k, j=1,\dots,l} \rangle$$

is a presentation for $A \times B$.

(b) Let $g \in \mathbb{N}_0$. Prove directly, without making use of handle decompositions, that the group $\pi_1(\Sigma_g \times S^1)$ admits a presentation of deficiency.

Exercise 257.2. Let Σ be a compact connected orientable 2-dimensional smooth manifold and let $\varphi\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the corresponding mapping torus $\mathrm{Tor}(\Sigma, \varphi)$.

(a) Show directly, without making use of handle decompositions, that if Σ is closed, then $\pi_1(\mathrm{Tor}(\Sigma, \varphi))$ admits a presentation of deficiency zero.

(b) Show directly, without making use of handle decompositions, that if Σ has non-empty boundary, then $\pi_1(\mathrm{Tor}(\Sigma, \varphi))$ admits a presentation of deficiency one.

Hint. Use the Mapping Torus-π_1-Proposition 93.13 and the Semidirect Product-Presentation Lemma 100.21.

Exercise 257.3. Let Σ be a closed connected orientable 2-dimensional smooth manifold of genus $g \geq 1$ and let $\varphi\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the corresponding mapping torus $\mathrm{Tor}(\Sigma, \varphi)$.

(a) Show that $\mathrm{Tor}(\Sigma, \varphi))$ is aspherical.

(b) Compute the homology groups of $\pi_1(\mathrm{Tor}(\Sigma, \varphi))$ in terms of the induced homomorphism $\varphi_*\colon \mathrm{H}_1(\Sigma) \to \mathrm{H}_1(\Sigma)$.

(c) Show that the deficiency of $\pi_1(\mathrm{Tor}(\Sigma, \varphi))$ equals zero.

[2748] We introduced homotopy spheres in Definition 210.19.

Exercise 257.4. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p,q)$ from Definition 111.13.

(a) Show that $S^3 \setminus T(P,q)$ is aspherical.
Hint. Use the description of $S^3 \setminus T(p,q)$ that we gave in the proof of the Torus Knot-π_1-Proposition 111.18, use the Covering-π_n-Isomorphism Proposition 134.22 and use the Hurewicz Theorem 167.9

(b) Compute the homology groups of $\pi_1(S^3 \setminus T(p,q))$.

Exercise 257.5. In Section 257.2 we provided two tables with fundamental groups of 3-manifolds. Which groups appear in both tables?

Exercise 257.6.

(a) Show that for any $m \in \mathbb{N}_0$ there exists a closed connected non-empty 3-dimensional smooth manifold with $\mathrm{def}(\pi_1(M)) \geq m$.

(b) Show that for any $m \in \mathbb{N}_0$ there exists a compact connected non-empty 3-dimensional smooth manifold with $\mathrm{def}(\pi_1(M)) \geq 1 - \chi(M) + m$.

Exercise 257.7. Let $r \in \mathbb{N}$ and $d_1, \ldots, d_s \in \mathbb{N}_{\geq 2}$. As we discussed on page 4949 there exists an isomorphism

$$\left\langle x_1, \ldots, x_{r+s} \,\middle|\, \begin{array}{c} [x_i, x_j] = e \text{ for all } i < j \in \{1, \ldots, r+s\} \\ \text{and } x_1^{d_1} = \cdots = x_s^{d_s} = e \end{array} \right\rangle \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z}.$$

Is the presentation on the right-hand side a presentation of maximal deficiency?

Exercise 257.8. Show that there exist at least seven pairwise non-diffeomorphic compact connected non-empty 3-dimensional smooth manifolds with the following two properties:

(1) The fundamental group is abelian and infinite.

(2) The boundary has no spherical boundary components.

Exercise 257.9. Show that there exist at least five pairwise non-diffeomorphic compact connected non-empty 3-dimensional smooth manifolds with the following two properties:

(1) The fundamental group is abelian.

(2) The boundary is non-empty but it has no spherical boundary components.

Exercise 257.10. Let $d \in \mathbb{N}$. We consider the lens space $L(d,1) = S^3/\mathbb{Z}/d\mathbb{Z}$. Build an Eilenberg-Maclane space out of $L(\mathbb{Z}/d\mathbb{Z}, 1)$ and show that $\mathrm{H}_2(\mathbb{Z}/d\mathbb{Z}) = 0$.

Exercise 257.11. For which $g, h \in \mathbb{N}_0$ does $\pi_1(\Sigma_g \times \Sigma_h)$ have non-negative deficiency?

Exercise 257.12. Show that $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ admits a finite covering by $S^1 \times S^2$.
Hint. What is $\mathbb{R}\mathrm{P}^3$ minus an open 3-ball?

Exercise 257.13. Are there infinitely many pairwise non-diffeomorphic closed dimensional 3-dimensional smooth manifolds such that the fundamental group is solvable?

Exercise 257.14. Let M be a compact 3-dimensional smooth manifold with two boundary components each of which is a torus. Is the deficiency necessarily equal to one?

Exercise 257.15. Let π be a finite non-trivial group. Show that there is no finite CW-complex X that is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$.
Hint. If there was such a finite CW-complex, what could you say about the Euler characteristic of X?

Exercise 257.16. Let M be a closed connected non-empty 3-dimensional smooth manifold. We now also assume that M is aspherical.

(a) Show that $\pi_1(M)$ cannot be isomorphic to $\mathbb{Z}^2$.

(b) Show that $\pi_1(M)$ cannot be isomorphic to $\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ for some $d \in \mathbb{N}_{\geq 3}$.

Hint. Since every smooth manifold admits a CW-structure we obtain from our hypothesis that M is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$.

CHAPTER 258

High-dimensional knots and linking numbers

In the next chapter we want to calculate the intersection form of a smooth manifold that we obtain by attaching k-handles to the $2k$-ball $\overline{B}^{2k}$. As we will see, the calculation will be in terms of "linking numbers" of the attaching spheres.

To prepare the ground for the next chapter we need to (re-) introduce linking numbers of high-dimensional links in spheres and we need to prove a few general results on their relationship to intersection forms.

This chapter is organized as follows:

- In Section 258.1 we will introduce the notion of a link in any sphere and we will calculate the homology groups of link complements.
- In Section 258.2 we will (re-) introduce linking numbers of two k-dimensional disjoint knots in S^{2k+1}.
- In Section 258.3 we will prove a realization result for linking numbers namely we will see that every symmetric matrix is the linking matrix of a (high-) dimensional link.
- In Section 258.4 we will state the Linking #–Sphere Intersection #-Proposition 258.13 and the Linking #–Ball Intersection #-Proposition 258.14 which relate the linking numbers to intersection numbers in spheres and balls.
- In Section 258.5 we will introduce the homological longitude of a knot.
- In Section 258.6 we will use the notion of a homological longitude to provide the proofs of the Linking #–Sphere Intersection #-Proposition 258.13 and of the Linking #–Ball Intersection #-Proposition 258.14.
- In Appendix 258.7 we will state Haefliger's Unknotting Theorem 111.35 and we will use it to prove some statements on tubular neighborhoods of high-dimensional knots. We will not make use of these results.
- In Appendix 258.8 we will relate the different definitions of linking numbers to one another.
- In Section 258.9 we will provide exercises for Chapter 258.

In the next Chapter 259 we will discuss the interaction between handle attachments, intersection forms and linking numbers.

258.1. **High-dimensional links.** By now the reader is surely familiar with knots and links in S^3. For our current purposes it is useful to generalize these concepts to all dimensions.

This goal leads us to the following definition that recalls and generalizes some of the notions that we introduced in Definitions 108.3, 108.30, 111.27 and 108.5:

Definition 258.1. Let $k, m, n \in \mathbb{N}_0$.

(1) (a) An m-component k-dimensional link is a submanifold of S^n that is diffeomorphic to the disjoint union of m copies of S^k together with an ordering of the components, i.e. together with a bijection from $\{1, \dots, m\}$ to the set of components of L. A 1-component link is of course also called a knot.

(b) Given an m-component link $L \subseteq S^n$ and given $i \in \{1, \dots, m\}$ we denote by L_i the i-th component of L.

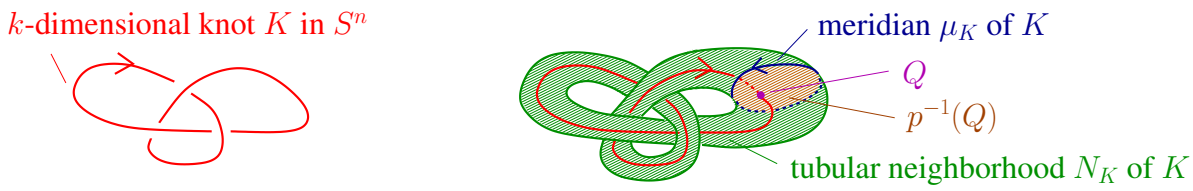
For $k = 1$ and $n = 3$ we say sometimes that a link is a classical link. For $n \geq 4$ we speak at times of high-dimensional links.

(2) We say that a link is oriented if it is oriented as a k-dimensional smooth manifold.

(3) Let $L \subseteq S^n$ be a k-dimensional knot. By the Tubular Neighborhood–Existence Theorem 57.11 we know that there exists a tubular neighborhood $p\colon N_L \to L$ in the sense of Definition 57.9, i.e. N_L is compact codimension-zero submanifold of S^n and $p\colon N_L \to L$ is a linear smooth $\overline{B}^{n-k}$-bundle in the sense of Definition 57.4.

(a) We refer to
$$X_L \;:=\; S^n \setminus \mathring{N}_L$$
as the exterior of L.

(b) Let $i \in \{1, \dots, m\}$. We pick $Q \in L_i$. By definition of a tubular neighborhood we know that $p^{-1}(Q)$ is a smoothly embedded ball of dimension $n-k$ that intersects L_i transversally in a single point, namely Q. We give $p^{-1}(Q)$ the orientation such that the sign of the intersection of $p^{-1}(Q)$ and L_i is $+1$. We refer to the oriented $(n-k-1)$-dimensional smooth manifold $\partial p^{-1}(Q)$ as a meridian μ_{L_i} of L_i.[2749][2750]



The following definition, which is essentially a generalization of Definition 108.6, doubles as an example of a high-dimensional link:

Definition 258.2. Let $n, k \in \mathbb{N}$ with $k < n$ and let $m \in \mathbb{N}_0$.

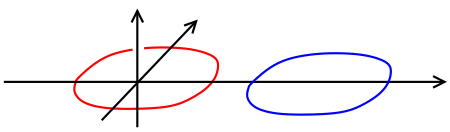
(1) The link
$$\bigcup_{i=1}^{m} \{(\underbrace{x + 3(i-1) \cdot \mathfrak{e}_1}_{\in \mathbb{R}^{k+1}}, 0) \in \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n \,|\, x \in S^k\}$$
is called the standard trivial m-component link.

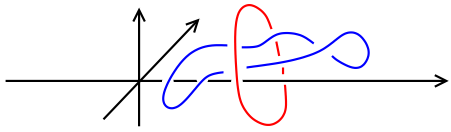
(2) We say that an m-component k-dimensional link in S^n is trivial if it is smoothly isotopic (in the sense of Definition 41.10) to the above standard trivial link.

Remark. In the Non-Trivial High-Dimensional Knot Existence Corollary 111.34 we showed that given any $n \in \mathbb{N}_{\geq 3}$ there exists a non-trivial $(n-2)$-dimensional knot in S^n. □

[2749] In the following we mostly care about the case of a k-dimensional link L in the sphere S^{2k+1} of dimension $2k+1$. In this case the meridian is also k-dimensional.

[2750] By the Meridian Proposition 109.2 we know that for $k = 1$ and $n = 3$ this definition is consistent with Definition 109.1.


standard trivial link


trivial link

The following lemma generalizes some of the statements of the Knot Exterior Lemma 185.27 to the higher-dimensional setting:

Lemma 258.3. (High-Dimensional Link Exterior Lemma) Let $n \in \mathbb{N}_0$, $k \in \{0, \dots, n\}$ and let $m \in \mathbb{N}_0$. Furthermore let $L \subseteq S^n$ be an m-component k-dimensional link. We pick a tubular neighborhood N_L and we write $X_L := S^n \setminus \mathring{N}_L$.

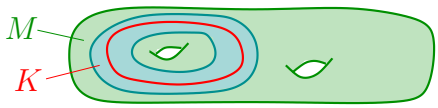
(1) The exterior X_L of the link L is a compact n-dimensional smooth manifold which has a unique orientation coming from S^n.
(2) The diffeomorphism type of X_L as an oriented smooth manifold does not depend on the choice of the tubular neighborhood.
(3) The exterior X_L is a deformation retract of the link complement $S^n \setminus L$.

Proof.

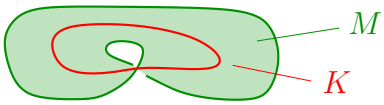
(1) This is a consequence of the Codimension-Zero Smooth Embedding Proposition 30.21.
(2) This is an immediate consequence of the Tubular Map–Uniqueness Theorem 58.5.
(3) This is a special case of the Basics-of-Tubular Neighborhood Lemma 57.10 (3a). ■

For technical purposes it is often much easier to deal with links for which the tubular neighborhood is trivializable. This desire leads us to the following definition:

Definition 258.4. Let M be a smooth manifold and let $K \subseteq M$ be a closed submanifold. We say K is frameable[2751] if the tubular neighborhood is trivializable, i.e. if K admits a tubular map $\tau\colon \overline{B}^{\dim(M)-\dim(K)} \times K \to M \setminus \partial M$ in the sense of Definition 58.2. This means that τ is a smooth embedding such that for every $P \in K$ we have $\tau(0, P) = P$.



$K \subseteq M$ is frameable



$K \subseteq M$ is not frameable

Example.

(1) By the Product Tubular Neighborhood–Existence Theorem 58.4 we know that every 1-dimensional link in S^3 is frameable.
(2) It was shown by André Haefliger (see the remark on p. 174 of [**HSL1965**]) and Jerry Levine [**Levine1965b**, Table 7.2] that there exists an 11-dimensional knot $K \subseteq S^{17}$ such that the "normal vector bundle is non-trivial".[2752] By the Frameable Submanifold

[2751] A *framing* of a submanifold is a trivialization of the "normal vector bundle". In the Frameable Submanifold Proposition **??** we will see that the tubular neighborhood is trivializable, if and only if the normal vector bundle is trivializable, in other words, if the submanifold is frameable. The name "frameable" was suggested by ChatGPT and it seems as good as any name the author could come up with.

[2752] We refer to

`https://mathoverflow.net/questions/237384/is-the-normal-bundle-of-a-torus-trivial`

Proposition **??** this implies that K does not admit a tubular map. This shows that not every high-dimensional knot is necessarily frameable. □

The following lemma proves the almost elementary fact, that every trivial knot is frameable:

Lemma 258.5. (Trivial Knot–Tubular Neighborhood Lemma) Let $K \subseteq S^n$ be a k-dimensional knot. If K is trivial, then K is frameable, i.e. there exists a tubular map $\tau\colon \overline{B}^{n-k} \times K \to S^n$.

Example. In the Trivial Knot–Tubular Neighborhood Lemma 258.5 we just proved that every trivial knot is frameable. This observation has two consequences:

(1) It follows easily from the Thicken–High Codimension Sphere Lemma 256.13 that for $n \geq 2k+2$ every k-dimensional knot in S^n is frameable.
(2) The Sphere-in-Sphere Tubular Neighborhood–Existence Theorem 258.18, which we will not really prove, says that the conclusion of (a) also holds for $n \geq 2k+1$.

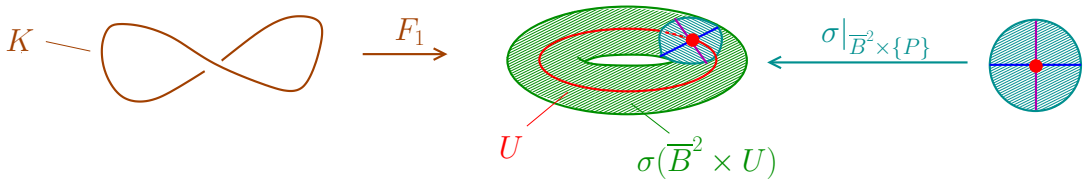
We will not make use of these two facts. □

Proof. Let $n \in \mathbb{N}_0$ and let $k \in \{0, \dots, n\}$. The case $k = n$ is trivial. Thus we now assume that $k \in \{0, \dots, n-1\}$. First we consider the k-dimensional standard trivial knot $U := \{(x,0) \in \mathbb{R}^n \cup \{\infty\} \underset{\text{i}}{=} S^n \,|\, x \in S^k\}$ in S^n. In this setting we consider the map

$$\begin{array}{rcl} \sigma\colon \overline{B}^{n-k} \times U & \to & \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1} \subseteq \mathbb{R}^n \cup \{\infty\} =_{\text{i}} S^n \\ (\underbrace{(x,y)}_{\text{with } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^{n-k-1}}, P) & \mapsto & \big(\big(1+\tfrac{x}{2}\big) \cdot P, y\big). \end{array}$$

One can easily verify that this map is a tubular map for U.

Next let $K \subseteq S^n$ be a k-dimensional trivial knot. By Definitions 258.2 and 41.10 there exists a smooth isotopy $F\colon S^k \times [0,1] \to S^n$ with $F_0 = \mathrm{id}$ and such that $F_1(K) = U$.



By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $G\colon S^n \times [0,1] \to S^n$ with $G \circ (F_0 \times \mathrm{id}_{[0,1]}) = F\colon S^k \times [0,1] \to S^n$. It now follows easily from the definitions that the map

$$\tau := G_1^{-1} \circ \sigma \circ (\mathrm{id}_{\overline{B}^{n-k}} \times F_1)\colon \overline{B}^{n-k} \times K \;\to\; \mathbb{R}^n \cup \{\infty\} =_{\text{i}} S^n$$

is a tubular map for K. ■

The following lemma is a generalization of the Knot Complement-H_*-Proposition 185.29, and also of Exercises 185.7 and 194.6, and it can be viewed as an analog of the Knot Complement-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 85.10.

for more guidance to the confusing literature.

Lemma 258.6. (Link Complement-H_*-Lemma) Let $k, n \in \mathbb{N}$ with $k < n-1$ and let $m \in \mathbb{N}$.

(1) For any k-dimensional m-component link $L = L_1 \sqcup \cdots \sqcup L_m \subseteq S^n$ and any $j \in \mathbb{N}_0$ we have the following isomorphisms:

$$\mathrm{H}_j(S^n \setminus L) \xleftarrow[\uparrow]{\cong} \mathrm{H}_j(X_L) \cong \begin{cases} 0, & \text{if } j \neq 0, n-k-1, n-1, \\ \mathbb{Z}^{m-1}, & \text{if } j = n-1, \\ \mathbb{Z}^m, & \text{if } j = n-k-1, \\ \mathbb{Z}, & \text{if } j = 0. \end{cases}$$

induced by the inclusion $X_L \to S^n \setminus L$

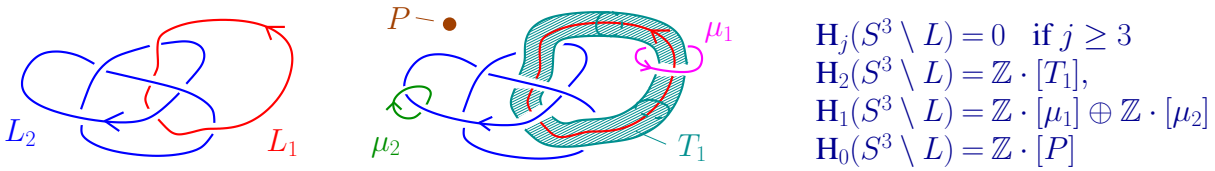
If $L \subseteq \mathbb{R}^n \subseteq \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$, then the inclusion induced map $\mathrm{H}_i(\mathbb{R}^n \setminus L) \to \mathrm{H}_i(S^n \setminus L)$ is an isomorphism for all $i < n-1$.

(2) We now assume that L is frameable and we assume that L is oriented. In this setting the following more precise statements hold:

(a) The boundary ∂X_L is diffeomorphic to the disjoint union of m smooth manifolds $T_1, \ldots, T_m$, each of which is diffeomorphic to $S^k \times S^{n-k-1}$.

(b) A basis for $\mathrm{H}_{n-k-1}(X_L)$ is given by the fundamental classes $[\mu_{L_1}], \ldots, [\mu_{L_m}]$ of the meridians $\mu_{L_1}, \ldots, \mu_{L_m}$ of the components of L.

(c) For any $i \in \{1, \ldots, m\}$ a basis for $\mathrm{H}_{n-1}(X_L)$ is given by the fundamental classes $[T_1], \ldots, \widehat{[T_i]}, \ldots, [T_m]$, i.e. by considering the fundamental classes of all but one boundary component.



$\mathrm{H}_j(S^3 \setminus L) = 0 \quad \text{if } j \geq 3$
$\mathrm{H}_2(S^3 \setminus L) = \mathbb{Z} \cdot [T_1],$
$\mathrm{H}_1(S^3 \setminus L) = \mathbb{Z} \cdot [\mu_1] \oplus \mathbb{Z} \cdot [\mu_2]$
$\mathrm{H}_0(S^3 \setminus L) = \mathbb{Z} \cdot [P]$

Proof.

(1) The statements follow from the following three results:

(a) For each $j \in \mathbb{N}_0$ we have an isomorphism

$$\widetilde{\mathrm{H}}_j(S^n \setminus L) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{n-j-1}(L; \mathbb{Z}) \cong \widetilde{\mathrm{H}}^{n-j-1}\Big(\bigsqcup_{i=1}^m S^k; \mathbb{Z}\Big) \underset{\uparrow}{\cong} \begin{cases} 0, & \text{if } n-j-1 \neq 0, k, \\ \mathbb{Z}^{m-1}, & \text{if } n-j-1 = 0, \\ \mathbb{Z}^m, & \text{if } n-j-1 = k. \end{cases}$$

by the Sphere–Alexander Duality Theorem 217.8

follows from the Path-Component-H_*-Lemma 198.8, the discussion on page 3830, and the Reduced Cohomology Lemma 198.17

Together with the Reduced Homology Lemma 147.3 we now obtain the promised isomorphism types of the homology groups of $S^n \setminus L$.[2753]

[2752] It is not clear to the author whether one needs the hypothesis that L is frameable.

[2753] The problem with the Sphere–Alexander Duality Theorem 217.8 is, that it is rather non-committal when it comes to finding explicit generators of the homology groups. One could try to pin down the generators using the General Alexander Duality Theorem 218.2, but we will not go down that route.

(b) By the High-Dimensional Link Exterior Lemma 258.3 (3) we know that the inclusion $X_L \to S^n \setminus L$ is a homotopy equivalence. It follows from the Homotopy Equivalence-H_*-Proposition 144.14 (1b) that the inclusion $X_L \to S^3 \setminus L$ induces an isomorphism on all homology groups.

(c) By the Complement-in-$\mathbb{R}^n$-S^n-Lemma 170.1 we know that for any compact non-empty subspace $X \subseteq \mathbb{R}^n$ and any $i \in \{0, \dots, n-2\}$ the inclusion induced homomorphism $\mathrm{H}_i(\mathbb{R}^n \setminus X) \to \mathrm{H}_i(S^n \setminus X)$ is an isomorphism.

(2) We now assume that L is frameable and oriented. In the following we follow the approach of the Knot Complement-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 85.10 and of the Knot Complement-H_*-Proposition 185.29 to compute (again) the homology groups of X_L. This new calculation also gives us the promised generators. It follows from the hypothesis that L is frameable and the Tubular Map Lemma 58.3 (2) that we can pick an orientation-preserving tubular map $\tau\colon \overline{B}^{n-k} \times L \to S^n$. By the Codimension-Zero Submanifold Proposition 25.49 (3) this implies that the boundary ∂X_L is diffeomorphic to the disjoint union of m smooth manifolds $T_1, \dots, T_m$, each of which is diffeomorphic to $S^k \times S^{n-k-1}$.

Next we consider the decomposition

$$S^n = \underbrace{(S^n \setminus \tau(B^{n-k} \times L))}_{=X_L} \cup_{\tau(S^{n-k-1} \times L)} \tau(\overline{B}^{n-k} \times L)$$

of S^n. It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that we can apply the Mayer-Vietoris Theorem 153.2 (3) to the above decomposition. We now consider the most interesting excerpts of the corresponding long exact sequence:

First of all, "at the upper end" we obtain the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}_n(S^n) & \xrightarrow{\partial_n} & \overbrace{\mathrm{H}_{n-1}(\partial X_L)}^{=\mathbb{Z}\cdot[T_1]\oplus\dots\oplus\mathbb{Z}\cdot[T_m]} & \xrightarrow{(i_*,-i_*)} & \mathrm{H}_{n-1}(X_L) \oplus \underbrace{\mathrm{H}_{n-1}(\tau(\overline{B}^{n-k} \times L))}_{=0,\text{ since } n-1>k} & \to & \underbrace{\mathrm{H}_{n-1}(S^n)}_{=0} \\ & & [S^n] & \mapsto & \underbrace{[T_1]+\dots+[T_m]}_{=[\partial X_L]}. & & & & \end{array}$$

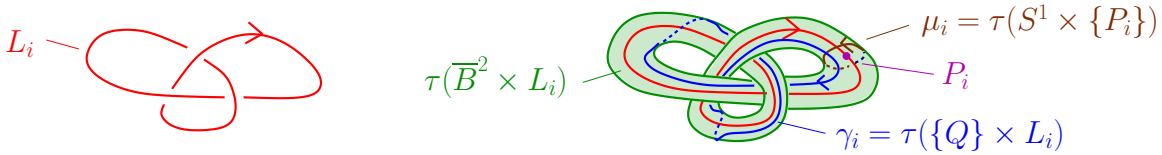
We make the following observations:

- By the discussion on page 3751 we know that $[\partial X_L] = [T_1] + \dots + [T_m]$.
- It follows from the Manifold-Top-H_*-Theorem 193.1 and the Path-Component-H_*-Lemma 143.21 that $\mathrm{H}_{n-1}(\partial X_L) = \mathbb{Z} \cdot [T_1] \oplus \dots \oplus \mathbb{Z} \cdot [T_m]$.
- It follows from the Manifold–MV-Connecting Homomorphism Corollary 194.16 that the connecting homomorphism sends $[S^n]$ to $[\partial X_L] \in \mathrm{H}_{n-1}(\partial X_L)$.

Note that it follows easily from the above discussion that we have an isomorphism $\mathrm{H}_{n-1}(X_L) \cong (\mathbb{Z} \cdot [T_1] \oplus \dots \oplus \mathbb{Z} \cdot [T_m])/\mathbb{Z} \cdot ([T_1] + \dots + [T_m])$. This implies that we have an isomorphism $\mathrm{H}_{n-1}(X_L) \cong \mathbb{Z}^{m-1}$ and that for any $i \in \{1, \dots, m\}$ a basis for $\mathrm{H}_{n-1}(X_L)$ is given by the fundamental classes $[T_1], \dots, \widehat{[T_i]}, \dots, [T_m]$.

Next we have to deal with the cases $j = k$ and $j = n-k-1$. The argument differs slightly depending on whether $k = n-k-1$ or whether $k \neq n-k-1$. In the following we assume that $k = n-k-1$, since that is what we are really interested in. We leave the case $k \neq n-k-1$ to the reader.

So we now assume $k = n-k-1$. In other words, we assume that $n = 2k+1$. For each $i \in \{1,\dots,m\}$ we pick $P_i \in L_i$ and we consider the corresponding meridian $\mu_i := \tau(S^k \times \{P_i\})$. We also pick $Q \in S^k$ and for $i = 1,\dots,m$ we write $\gamma_i := \tau(\{Q\} \times L_i)$.



Note that it follows from the Künneth Theorem 164.6 that for each $i \in \{1,\dots,m\}$ a basis for $\mathrm{H}_k(\tau(S^k \times L_i))$ is given by $[\gamma_i], [\mu_i]$ and that a basis for $\mathrm{H}_k(\tau(\overline{B}^{k+1} \times L_i))$ is given by $[\gamma_i]$. With this notation we obtain the following exact sequence

$$\underbrace{\mathrm{H}_{k+1}(S^{2k+1})}_{=0} \to \underbrace{\mathrm{H}_k(\tau(S^k \times L))}_{\substack{= \mathbb{Z}\cdot[\gamma_1] \oplus \cdots \oplus \mathbb{Z}\cdot[\gamma_m] \\ \oplus \mathbb{Z}\cdot[\mu_1] \oplus \cdots \oplus \mathbb{Z}\cdot[\mu_m]}} \to \mathrm{H}_k(X_L) \oplus \underbrace{\mathrm{H}_k(\tau(\overline{B}^{n-k} \times L))}_{=\mathbb{Z}\cdot[\gamma_1]\oplus\cdots\oplus\mathbb{Z}\cdot[\gamma_m]} \to \underbrace{\mathrm{H}_k(S^{2k+1})}_{=0}.$$

It follows from staring at this data for a few seconds that $\mathrm{H}_k(X_L)$ is a free abelian group on the meridians $[\mu_1],\dots,[\mu_m]$.

Finally it follows easily from the Mayer–Vietoris sequence that $\mathrm{H}_0(X_L) \cong \mathbb{Z}$ and that all other homology groups of X_L are zero. ∎

258.2. **Linking numbers.** On several occasions we introduced the notion of a linking number of two oriented disjoint submanifolds of a sphere:

(1) In Definition 52.2 we used the notion of a degree to introduce the linking number of two disjoint closed oriented smooth submanifolds of $\mathbb{R}^n$ with $\dim(K)+\dim(L) = n-1$.
(2) In Definition 86.10 we used the notion of a degree in de Rham cohomology to introduce the linking number of two disjoint oriented knots in $\mathbb{R}^3$.
(3) In Definition 114.1 we used the fundamental group to introduce the linking number of two disjoint oriented knots in S^3.

In this section we will give a new definition of the linking number of two oriented k-dimensional knots in S^{2k+1}. More precisely, the Link Complement-H_*-Lemma 258.6 allows us to make the following definition:

Definition 258.7. Let $k \in \mathbb{N}$.

(1) Let K be an oriented frameable k-dimensional knot in S^{2k+1}. We denote by Φ_K the isomorphisms

$$\begin{array}{rcl} \Phi_K\colon \mathrm{H}_k(X_K) & \xrightarrow{\cong} & \mathbb{Z} \\ [\mu_K] & \to & 1 \end{array} \quad \text{and} \quad \begin{array}{rcl} \Phi_K\colon \mathrm{H}_k(S^{2k+1}\setminus K) \xleftarrow{\cong} \mathrm{H}_k(X_K) & \xrightarrow{\cong} & \mathbb{Z} \\ [\mu_K] & \to & 1 \end{array}$$

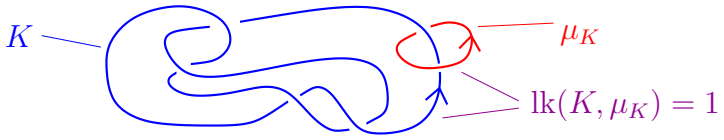
that are given by the Link Complement-H_*-Lemma 258.6 (2).

(2) Let K and J be two disjoint oriented frameable k-dimensional knots in S^{2k+1}. We define the linking number of K and J to be

$$\mathrm{lk}(K,J) \;:=\; \Phi_K([J])$$

where $[J] \in \mathrm{H}_k(S^{2k+1}\setminus K)$ denotes the image of the fundamental class of the closed oriented smooth manifold J under the inclusion $J \to S^{2k+1}\setminus K$.

Example. Let K be an oriented frameable k-dimensional knot in S^{2k+1}.



We denote by μ_K the meridian of the oriented knot K. It follows immediately from Definition 258.7 that $\mathrm{lk}(K,\mu_K)=1$. □

Later in this chapter we want to perform some explicit calculations of linking numbers. We thus remind the reader that for classical knots in S^3 we developed in Section 114.2 a very practical tool for calculating linking numbers for oriented knots that are given by a diagram in the sense of Definition 112.6. Before we state this tool, let us recall the following definition:

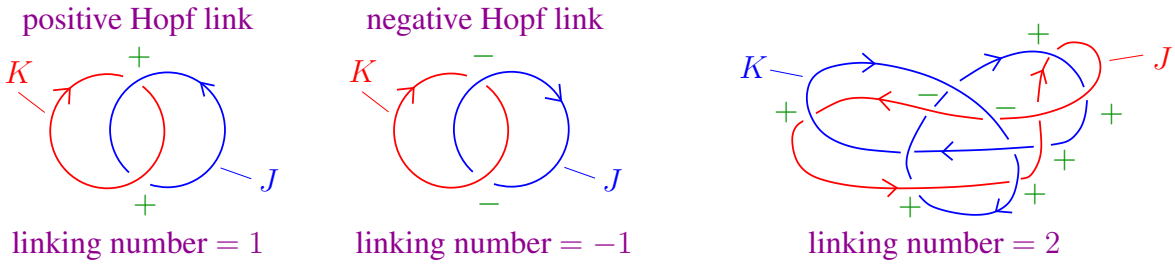
Definition 114.7. A crossing in a link diagram is called positive (respectively negative) if the direction of the upper strand followed by the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$.



Now we can recall the pretty formula that expresses the linking number as a signed count of crossings:

Proposition 114.9. (Linking Number-via-Crossings Proposition) Let K and J be two disjoint oriented 1-dimensional knots in S^3. If the link $K \cup J \subseteq S^3$ is associated to a link diagram, then

$$\mathrm{lk}(K,J) = \begin{array}{l} \frac{1}{2}\cdot \text{number of positive crossings between } K \text{ and } J \\ - \ \frac{1}{2}\cdot \text{number of negative crossings between } K \text{ and } J. \end{array}$$



Proof. It follows easily from the Hurewicz Theorem 166.10 that in the classical setting the two definitions of the linking number provided in Definition 114.1 and 258.7 agree. Thus the proposition does indeed hold with our current definition of the linking number. ■

In Definition 258.7 of the linking number the roles of K and J are very very different. On the other hand in the classical setting the Linking Number-via-Crossings Proposition 114.9 is completely symmetric in K and J. So one wonders, whether in the general setting there is some symmetry after all. It turns out that this is indeed the case:

Proposition 258.8. (Linking Number-Symmetry Proposition) Let $k \in \mathbb{N}$. Given any two disjoint oriented k-dimensional knots K and J in S^{2k+1} we have

$$\mathrm{lk}(J,K) \;=\; (-1)^{k+1} \cdot \mathrm{lk}(K,J).$$

Sketch of proof. We will not really make use of the proposition. Thus we just provide a sketch of a proof. In fact there are various approaches to proving this proposition:

(a) For classical links, i.e. for the case $k = 1$, one can use the Linking Number-via-Crossings Proposition 114.9 to show that the linking number is symmetric. We worked out the details in the Linking Number-Symmetry Propositions 114.10.

(b) The equality follows easily from the Linking Number–Symmetry Proposition 52.6, and the Linking Number Reformulation Proposition 258.19, where we show that Definitions 258.7 and 52.2 of linking numbers are consistent.

(c) For $k = 1$ we will prove this statement, with the above definition of the linking number, in the Linking Number-Symmetry Propositions 284.9. We leave it to the reader to generalize the proof of the Linking Number-Symmetry Proposition 284.9 to the higher-dimensional setting, in particular we leave it to the reader to figure out where the $(-1)^{k+1}$ comes from.

(d) The equality follows from the Linking #–Ball Intersection #-Proposition 258.14 (which we will prove shortly) together with the Cup Product-Commutativity Proposition 204.10. We delegate the details to Exercise 258.7. ■

In the remainder of this chapter we will prove various results about linking numbers. For the purpose of drawing explicit examples of these upcoming results it is convenient to introduce the linking number of disjoint oriented 0-dimensional knots in S^1. Since this definition is mostly for the purpose of illustration, we will take a few minor liberties in our language. Here we go:

Definition 258.9.

(1) An oriented 0-dimensional knot in S^1 is an oriented submanifold of S^1 of the form $\{-P, Q\}$ for two distinct points in S^1.[2754]

(2) Given an oriented 0-dimensional knot $K = \{-P, Q\}$ we define its fundamental class as $[K] := [-P + Q] \in \widetilde{\mathrm{H}}_0(Z) \cong \mathbb{Z}$.

(3) Given an oriented 0-dimensional knot $K = \{-P, Q\} \subseteq S^1$ we consider the isomorphism[2755]

$$\begin{aligned} \Phi_K \colon \widetilde{\mathrm{H}}_0(S^1 \setminus \{-P,Q\}) &\to \mathbb{Z} \\ \Big[\sum_{i=1}^{k} n_i \cdot \{z_i\}\Big] &\mapsto \sum_{i=1}^{k} n_i \cdot \epsilon_i \end{aligned}$$

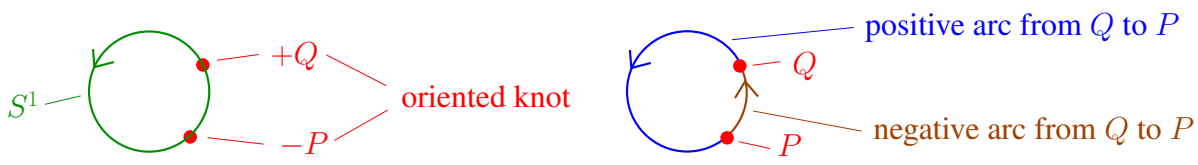
where $\epsilon_i = +1$ if z_i lies in the positive oriented arc between Q and P and 0 otherwise.[2756]

[2753] As we discussed on page 810, an orientation of a 0-dimensional smooth manifold M is essentially the same as a map $\sigma \colon M \to \{-1, 1\}$. Given $P \in M$ we write $+P$ (or just P) if $\sigma(P) = 1$ and we write $-P$ if $\sigma(P) = -1$.

[2754] Recall that on page 817 we equipped the 0-dimensional sphere $S^0 = \{\pm 1\}$ with the obvious orientation that is given by $\pm 1 \mapsto \pm 1$.

[2755] It follows easily from the H_0-Proposition 143.25 and the Reduced Homology Lemma 147.3 that this map is an isomorphism.

[2756] If one goes carefully through the definitions, then one sees that also in this setting we have $\Phi_K([\mu_K]) = 1$. Thus this definition of the linking number is consistent with Definition 258.7.

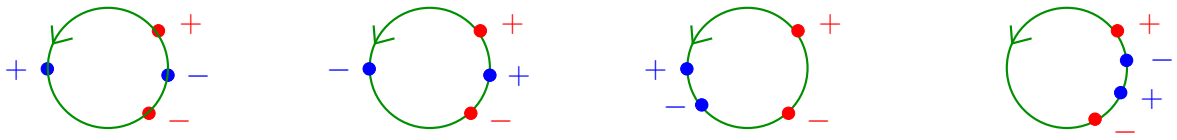


Definition 258.9 allows us to extend, almost verbatim, the above Definition 258.7 of a linking number to oriented 0-dimensional knots in S^1:

Definition 258.10. Let K and J be two disjoint oriented 0-dimensional knots in S^1. We define the linking number of K and J to be

$$\mathrm{lk}(K,J) \;:=\; \Phi_K([J]),$$

where $[J] \in \widetilde{\mathrm{H}}_0(S^n \setminus K)$ denotes the image of the fundamental class of the oriented smooth manifold J under the inclusion $J \to S^1 \setminus K$. Note that it follows easily from the Isotopy Extension Theorem 44.2 that the linking number of the link $K \sqcup J$ depends only on the smooth isotopy of the link $K \sqcup J$.



In the discussions below we will on occasion not treat the case $k = 0$ explicitly, and we will leave it to the reader to adapt the statements and the arguments from the case $k > 0$ to the case $k = 0$.

258.3. **Realization results for linking numbers.** Let $k \in \mathbb{N}$ be odd. Given an oriented m-component k-dimensional link $L_1 \sqcup \cdots \sqcup L_m \subseteq S^{2k+1}$ we can consider the "partial matrix" of linking numbers $\mathrm{lk}(L_i, L_j)$ for $i \neq j$. It is natural to wonder what values can occur. By the Linking Number-Symmetry Proposition 258.8 we know that the linking numbers are symmetric, i.e. the "partial matrix" $\{\mathrm{lk}(L_i, L_j)\}_{i \neq j}$ is symmetric. The following proposition shows that this is the only restriction on the "partial matrix":

Proposition 258.11. (Linking Number–Realization Proposition) Let $m \in \mathbb{N}$ and let $(a_{ij})_{i \neq j = 1,\ldots,m}$ by a symmetric partial matrix of integers. Given any odd $k \in \mathbb{N}$ there exists an oriented m-component k-dimensional link $L_1 \sqcup \cdots \sqcup L_m$ in S^{2k+1} such that the following two conditions are satisfied:

(1) The knots $L_1, \ldots, L_m$ are trivial.
(2) For any $i \neq j \in \{1, \ldots, m\}$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$.

The proof of the Linking Number–Realization Proposition 258.11 will require the remainder of this section. In the next Section 258.4 we will study the relationship between linking numbers and intersection numbers.

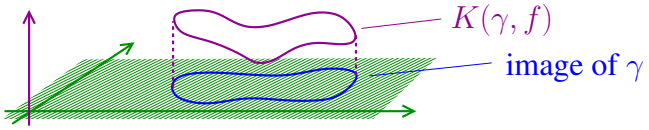
We break the proof of the Linking Number–Realization Proposition 258.11 into two parts. First we deal with the case $k = 1$. Afterwards we will see how the links constructed in this low-dimensional setting give rise to the desired links for arbitrary odd k.

Sketch of proof of Proposition 258.11 for $k = 1$. We start out by providing a method to construct trivial knots:

Claim. For every smooth embedding $\gamma\colon S^1 \to \mathbb{R}^2$ and for every smooth map $f\colon S^1 \to \mathbb{R}$ the knot

$$K(\gamma, f) \;\coloneqq\; \{(\gamma(z), f(z)) \in \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3 \,|\, z \in S^1\}$$

is trivial.



Proof. The knot $K(\gamma, f)$ is evidently smoothly isotopic to the knot $K(\gamma, 0)$ that is defined by γ together with the zero function $S^1 \to \mathbb{R}$. In particular $K(\gamma, f)$ is smoothly isotopic to a knot J that is contained in a plane $H \subseteq \mathbb{R}^3$. It follows from the Fill-in-Disk Proposition 107.15 that J bounds a smoothly embedded disk in H. It now follows from the Genus-Zero Knots Proposition 121.10 that J, and thus also $K(\gamma, f)$, is trivial. ⊞

Let $(a_{ij})_{i\neq j=1,\dots,m}$ be a symmetric partial matrix of integers. We denote by $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ the projection onto the first two coordinates. We construct iteratively an oriented m-component link $L_1 \sqcup \cdots \sqcup L_m$ as follows:

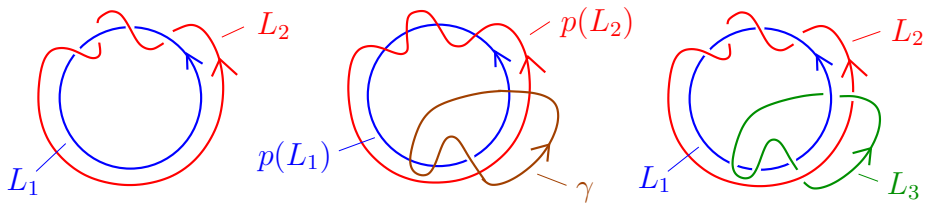
(1) We set L_1 to be the standard trivial knot in $\mathbb{R}^2 \times \{1\}$, which we equip with the counterclockwise orientation.
(2) Suppose we have constructed $L_1, \dots, L_k$ with $1 \leq k < m$.
   (a) Some thought shows that we can pick a smooth embedding $\gamma\colon S^1 \to \mathbb{R}^2$ such that the following hold:
      - For each $i \in \{1, \dots, k\}$ the closed curves $p(L_i)$ and $\gamma(S^1)$ intersect transversally and there are at least $2 \cdot |a_{i,k+1}|$ intersection points.
      - For any $i \neq j \in \{1, \dots, k\}$ we have $p(L_i) \cap p(L_j) \cap \gamma(S^1) = \varnothing$.
   (b) We pick a smooth map $f\colon S^1 \to \mathbb{R}$ which for the most time equals $k + 1$, except that for each $i \in \{1, \dots, k\}$ we do the following:
      - If $a_{ij} > 0$, then we pick $|a_{ij}|$ intersection points of the projections of L_i and γ to $\mathbb{R}^2$ where the crossing is positive (with respect to the orientation of L_i and the orientation coming from S^1) and at these points the function f dips to the value $i - \frac{1}{2}$.
      - If $a_{ij} < 0$, then we pick $|a_{ij}|$ intersection points of the projections of L_i and γ to $\mathbb{R}^2$ where the crossing is negative and at these points the function f dips to the value $i - \frac{1}{2}$.

   We set $L_{k+1} \coloneqq K(\gamma, f)$. We equip $L_{k+1} \coloneqq K(\gamma, f)$ with the orientation coming from the usual orientation of S^1 and the smooth embedding γ.

By the claim we know that the knots $L_1, \dots, L_m$ are trivial. Furthermore using the Linking Number-via-Crossings Proposition 114.9 one can easily verify that for any $i \neq j$ in $\{1, \dots, m\}$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$.
Evidently throughout the above proof we did not provide all details. We leave it to the reader to fill in the details. ■

Sketch of proof Proposition 258.11 for odd $k > 1$. Let $k = 2s+1$ be an odd number with $s \in \mathbb{N}$. Furthermore let $(a_{ij})_{i\neq j=1,\dots,m}$ be a symmetric partial matrix of integers.

$$A = \begin{pmatrix} * & 2 & 1 \\ 2 & * & -1 \\ 1 & -1 & * \end{pmatrix}$$



By the discussion above of the case $k = 1$ we know that there exists an oriented m-component 1-dimensional link $J_1 \sqcup \cdots \sqcup J_m$ in $\mathbb{R}^3 \subseteq S^3$ such that the following two conditions are satisfied:

(1) The knots $J_1, \ldots, J_m$ are trivial.
(2) For any $i \neq j \in \{1, \ldots, m\}$ we have $\mathrm{lk}(J_i, J_j) = a_{ij}$.

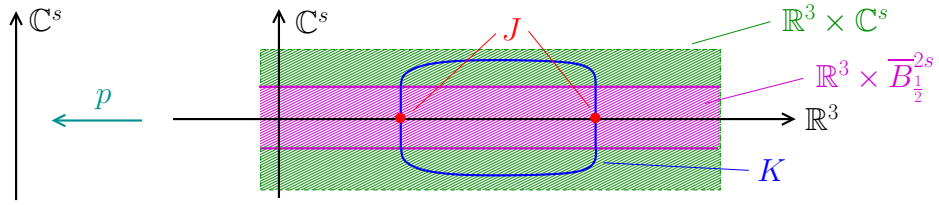
The hope now is to build the desired k-dimensional link out of the link $J_1 \sqcup \cdots \sqcup J_m$. We start out with the following fairly straightforward claim.

Claim 1. Let $\iota\colon \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{C}^s$ be the obvious linear embedding given by $x \mapsto (x, 0)$ and let $p\colon \mathbb{R}^3 \times \mathbb{C}^s \to \mathbb{C}^s$ be the natural projection. Given any trivial 1-dimensional knot $J \subseteq \mathbb{R}^3$ there exists an oriented trivial $(2s+1)$-dimensional knot $K \subseteq \mathbb{R}^3 \times \mathbb{C}^s$ with the following properties:

(1) $K \cap (\mathbb{R}^3 \times \overline{B}^{2s}_{\frac{1}{2}}) = J \times \overline{B}^{2s}_{\frac{1}{2}}$.
(2) K intersects $\iota(\mathbb{R}^3)$ transversally.
(3) $K \cap \iota(\mathbb{R}^3) = \iota(J)$.
(4) For any $P \in J$ the orientations of J at P, of U at $(P, 0)$ and of $\mathbb{C}^s$ are compatible. Let us explain what this means. Given $P \in J$ it follows from (1) that we have a short exact sequence

$$0 \;\to\; \mathrm{T}_P J \;\xrightarrow{\mathrm{D}\iota_P}\; \mathrm{T}_{(P,0)} K \;\xrightarrow{p_*}\; \mathbb{C}^s \;\to\; 0.$$

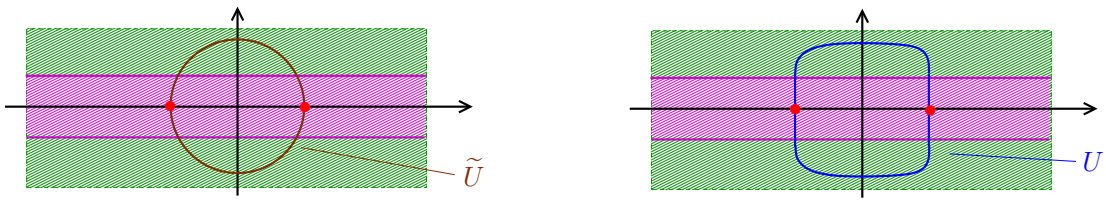
We now demand that the orientations of $\mathrm{T}_P J$, $\mathrm{T}_{(P,0)} K$ and of $\mathbb{C}^s$ are compatible in the sense of Definition 31.11.



Proof. If J is the standard trivial knot $\{(0, x) \in \mathbb{R}\times\mathbb{R}^2 = \mathbb{R}^3 \mid x \in \mathbb{R}^2 \text{ with } \|x\| = 1\}$ with the usual orientation, then we set $\widetilde{U} := \{(0, y) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{C}^s = \mathbb{R}^{3+2s} \mid y \in \mathbb{R}^{2s+1} \text{ with } \|x\| = 1\}$, which we equip with the usual orientation. One can easily verify that $\widetilde{U}$ satisfies conditions (2), (3) and (4). It is elementary to verify that $\widetilde{U}$ is smoothly isotopic to a knot U that on top also satisfies condition (1).[2757]

Now let $\widetilde{J}$ be any other oriented trivial knot in $\mathbb{R}^3$. By Exercise 41.9 together with the Isotopy Extension Theorem 44.2 there exists a diffeotopy $G\colon \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ with $G_0 = \mathrm{id}$ and such that G_1 restricts to an orientation-preserving diffeomorphism $J \to \widetilde{J}$. Next we consider the diffeotopy $H\colon (\mathbb{R}^3 \times \mathbb{R}^{k-1}) \times [0, 1] \to \mathbb{R}^{k+2}$ given by $H(x, y, t) = (H(x, t), y)$.

[2757]Details are provided in the author's office.

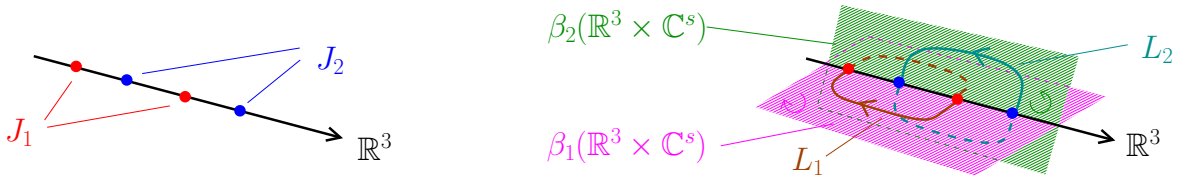


By definition $H_1(U)$ is a trivial knot in $\mathbb{R}^{k+2}$. Again one can easily verify that the oriented knot $H_1(U)$ has all the required properties. ⊞

We pick length-preserving complex linear monomorphisms $\alpha_1, \dots, \alpha_m \colon \mathbb{C}^s \to \mathbb{C}^{2s}$ such that for every $i \neq j \in \{1, \dots, m\}$ we have $\operatorname{im}(\alpha_i) \cap \operatorname{im}(\alpha_j) = \{0\}$. For $i = 1, \dots, m$ we then consider the map

$$\begin{array}{rccl} \beta_i \colon & \mathbb{R}^3 \times \mathbb{C}^s & \to & \mathbb{R}^3 \times \mathbb{C}^{2s} = \mathbb{R}^{2k+1} \\ & (v, w) & \mapsto & (v, \alpha_i(w)). \end{array}$$

Furthermore, for $i = 1, \dots, m$ we apply the previous claim to the oriented trivial 1-dimensional knot $J_i \subseteq \mathbb{R}^3$ and we obtain a corresponding oriented trivial k-dimensional knot $K_i \subseteq \mathbb{R}^3 \times \mathbb{C}^s$. We set $L_i := \beta_i(K_i)$.

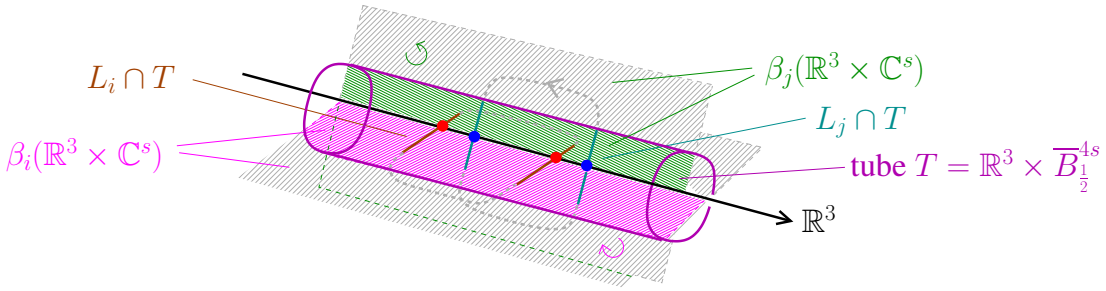


Since the images of the α_i intersect trivially we see that the L_i are disjoint. This means that $L_1 \sqcup \dots \sqcup L_m$ is indeed an oriented k-dimensional link and it is essentially clear that each L_i is a trivial knot. It now remains to prove the following claim:

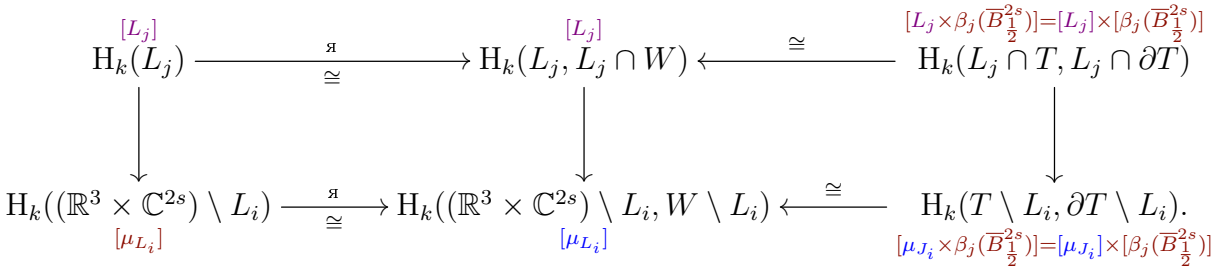
Claim 2. For any $i \neq j$ we have $\operatorname{lk}(L_i, L_j) = \operatorname{lk}(J_i, J_j)$.

Proof. Let $i \neq j$ in $\{1, \dots, m\}$. By construction we know that $\operatorname{lk}(J_i, J_j) = a_{ij}$. It follows easily from Definition 258.7 and the "If $L \subseteq \mathbb{R}^n$"-statement of the Link Complement-H_*-Lemma 258.6 that $[J_j] = a_{ij} \cdot [\mu_{J_i}] \in \mathrm{H}_1(\mathbb{R}^3 \setminus J_i)$.

We need to show that $[L_j] = a_{ij} \cdot [\mu_{L_i}] \in \mathrm{H}_{s+1}((\mathbb{R}^3 \times \mathbb{C}^{2s}) \setminus L_i)$. To do so we consider the "tube" $T := \mathbb{R}^3 \times \overline{B}^{4s}_{\frac{1}{2}} \subseteq \mathbb{R}^3 \times \mathbb{C}^{2s}$. We set $W := (\mathbb{R}^3 \times \mathbb{C}^{2s}) \setminus \mathring{T}$.



Next we consider the following diagram:



We make the following clarifications and observations:

(1) The left horizontal maps are the natural maps from absolute to relative homology. Using the LES-Proposition 145.10 one can easily verify that these maps are isomorphisms.
(2) It follows from the Manifold–Excision Theorem 148.3 that the right horizontal maps are isomorphisms.
(3) Using the Complex Vector Space-Orientation Lemma 40.16 one can verify that the image of the meridian $[\mu_{L_i}]$ in $\mathrm{H}_k((\mathbb{R}^3 \times \mathbb{C}^{2s}) \setminus L_i, W \setminus L_i)$ agrees with the image of $[\mu_{J_i} \times \beta_j(\overline{B}^{2s}_{\frac{1}{2}})]$.[2758]
(4) It follows from the Fundamental Class-of-Product Proposition 194.17 that we have the equalities $[L_j \times \beta_j(\overline{B}^{2s}_{\frac{1}{2}})] = [L_j] \times [\beta_j(\overline{B}^{2s}_{\frac{1}{2}})]$ and $[\mu_{J_i} \times \beta_j(\overline{B}^{2s}_{\frac{1}{2}})] = [\mu_{J_i}] \times [\beta_j(\overline{B}^{2s}_{\frac{1}{2}})]$, where on the right hand side we consider the cross product that we introduced in Definition 164.14.
(5) It follows from (4) and the choice of J_i, J_j that the vertical map to the right sends $[L_j \times \beta_j(\overline{B}^{2s}_{\frac{1}{2}})] = [L_j] \times [\beta_j(\overline{B}^{2s}_{\frac{1}{2}})]$ to a_{ij}-times $[\mu_{J_i} \times \beta_j(\overline{B}^{2s}_{\frac{1}{2}})] = [\mu_{J_i}] \times [\beta_j(\overline{B}^{2s}_{\frac{1}{2}})]$.
(6) Clearly the diagram commutes. It thus follows from the above discussion that the left vertical map sends $[L_j]$ to a_{ij}-times $[\mu_{L_i}]$. That is what we needed to show. ■

258.4. **Linking numbers and intersection numbers.** In this section we will reinterpret the linking number in terms of intersection numbers. Before we can state the main results we need to introduce and recall some definitions.

First we introduce some new objects:

Definition 258.12. Let $k \in \mathbb{N}_0$ and let $K \subseteq S^{2k+1}$ be an oriented k-dimensional knot.

(1) (a) First assume that $k \neq 0$. We use the long exact sequences in homology of the pair (S^{2k+1}, K) to define a homology class $\Theta_K \in \mathrm{H}_{k+1}(S^{2k+1}, K)$ as follows:

$$\underbrace{\mathrm{H}_{k+1}(S^{2k+1})}_{=0} \to \mathrm{H}_{k+1}(S^{2k+1}, K) \xrightarrow{\partial_{k+1}} \mathrm{H}_k(K) \to \underbrace{\mathrm{H}_k(S^{2k+1})}_{=0}$$
$$\Theta_K \quad \mapsto \quad [K]$$

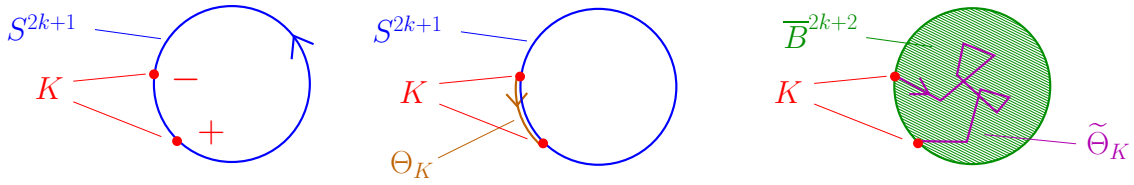
(b) For $k = 0$ and an oriented 0-dimensional knot $K = \{-P, Q\}$ we replace in (1) the sphere S^1 by the counterclockwise arc connecting P to Q and we replace ordinary homology by reduced homology.

[2758] That is perhaps the step that requires the most extra thought.

(2) We use the long exact sequence in reduced homology of the pair $(\overline{B}^{2k+2}, K)$ to define a homology class $\widetilde{\Theta}_K \in \mathrm{H}_{k+1}(\overline{B}^{2k+2}, K)$ as follows:

$$\begin{array}{ccccccc} \underbrace{\widetilde{\mathrm{H}}_{k+1}(\overline{B}^{2k+2})}_{=0} & \to & \mathrm{H}_{k+1}(\overline{B}^{2k+2}, K) & \xrightarrow{\partial_{k+1}} & \widetilde{\mathrm{H}}_k(K) & \to & \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^{2k+2})}_{=0} \\ & & \widetilde{\Theta}_K & \mapsto & [K]. & & \end{array}$$

Given any (e.g. tubular) neighborhood of K in S^{2k+1} or $\overline{B}^{2k+2}$ we denote by Θ_K and $\widetilde{\Theta}_K$ the images in the corresponding relative homology groups.

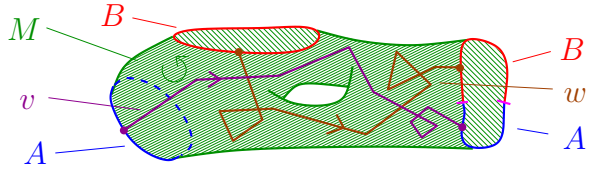


Next we recall Definition 225.3:

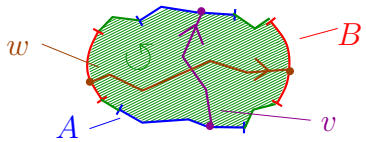
Definition 225.3. Let M be a compact oriented connected n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M such that $\mathring{A} \cap \mathring{B} = \varnothing$. For each $k \in \mathbb{N}_0$ we refer to the pairing

$$\begin{array}{rccl} Q^{\mathrm{as}}_{M,A,B}\colon & \mathrm{FH}_k(M,A) \times \mathrm{FH}_k(M,B) & \to & \mathbb{Z} \\ & (a,b) & \mapsto & \big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^k(M,\partial M \setminus \mathring{A};\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^k(M,\partial M \setminus \mathring{B};\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \big\rangle \end{array}$$

as an asymmetric intersection pairing of M.



$Q^{\mathrm{as}}_{M,A,B}(v,w) = +1 - 1 = 0$



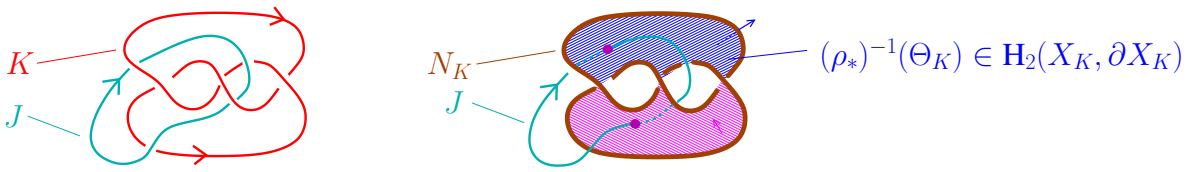
$Q^{\mathrm{as}}_{M,A,B}(v,w) = +1$

The goal of this section is to state and discuss two propositions, which let us calculate linking numbers in terms of intersection numbers in the sphere, respectively the bounding ball. We start out with intersection numbers in spheres:

Proposition 258.13. (Linking #–Sphere Intersection #-Proposition) Let $k \in \mathbb{N}$ odd and let $K, J \subseteq S^{2k+1}$ be two disjoint oriented frameable k-dimensional knots. We pick a tubular neighborhood N_K for K and we denote by $X_K = S^{2k+1} \setminus \mathring{N}_K$ the corresponding exterior of K. We consider the class $\Theta_K \in \mathrm{H}_{k+1}(S^{2k+1}, N_K)$ from Definition 258.12. Let $\rho\colon (X_K, \partial X_K) \to (S^{2k+1}, N_K)$ be the obvious map. Note that by the Manifold–Excision Theorem 148.3 we know that the induced map $\mathrm{H}_{k+1}(X_K, \partial X_K) \to \mathrm{H}_{k+1}(S^{2k+1}, N_K)$ is an isomorphism. We have the following equality:

$$\mathrm{lk}(K,J) \;=\; Q^{\mathrm{as}}_{X_K,\partial X_K,\varnothing}((\rho_*)^{-1}(\Theta_K), [J]).$$

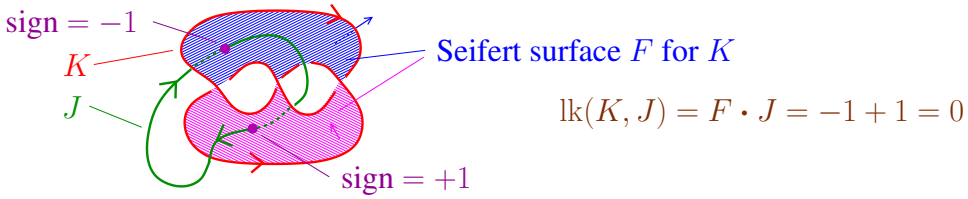
Proof. We postpone the proof of the proposition to Section 258.6. ∎



Example. Let K and J be two disjoint oriented classical knots in S^3 and let F be a Seifert surface for K that is transverse to J.[2759] Using the Linking #–Sphere Intersection #-Proposition 258.13 together with the Cup Product–Intersection Number Theorem 220.10 one can show that

$$\mathrm{lk}(K,J) \;=\; F \cdot J$$

where $F \cdot J$ denotes the algebraic intersection number of F and J as introduced in Definitions 51.4 and 220.5.

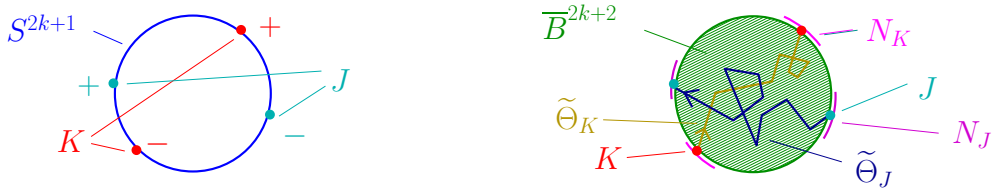


We will work out the details for the argument in the Linking #-via-Intersection #-Proposition 285.5. □

Let us hop on to the next, closely related, proposition, which expresses linking numbers in terms of intersection numbers in a ball:

Proposition 258.14. (Linking #–Ball Intersection #-Proposition) Let $k \in \mathbb{N}$ odd and let $K, J \subseteq S^{2k+1}$ be two disjoint oriented frameable k-dimensional knots. Given any disjoint tubular neighborhoods N_K and N_J of K and J we have the following equality

$$\mathrm{lk}(K,J) \;=\; Q^{\mathrm{as}}_{\overline{B}^{2k+2},N_K,N_J}(\widetilde{\Theta}_K,\widetilde{\Theta}_J).$$



Proof. We postpone the proof of the proposition to Section 258.6. ■

Example. Let K and J be two disjoint oriented classical knots in S^3. Furthermore let $F_K, F_J \subseteq \overline{B}^4$ be compact oriented proper 2-dimensional smooth submanifolds such that $\partial F_K = K$ and $\partial F_J = J$ as oriented smooth manifolds. If F_K and F_J intersect transversally, then it follows easily from the Linking #–Ball Intersection #-Proposition 258.14 together with the Cup Product–Intersection Number Theorem 220.10 that

$$\mathrm{lk}(K,J) \;=\; F_K \cdot F_J.$$ □

[2759]Recall that according to Definition 121.1 a Seifert surface for K is a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented smooth manifolds. By the Seifert Surface Existence Theorem 285.2 we know that such a surface always exists.

258.5. **The homological longitude.** We still owe the reader the proofs of the Linking #–Sphere Intersection #-Proposition 258.13 and of the Linking #–Ball Intersection #-Proposition 258.14. The proofs of the propositions will rest on the concept of a "homological longitude". This might ring a bell: For a classical knot $K \subseteq S^3$ we saw in Chapter 115 that the meridian is rarely seen without its best buddy, the longitude. In Definition 258.1 we generalized the notion of a meridian to knots of arbitrary dimension. Somewhat similarly we will see in this section that all oriented knots (in any dimension) have a *homological* longitude.

We start out with the following lemma, which can be viewed as an addendum to the Link Complement-H_*-Lemma 258.6:

Lemma 258.15. (Homological Longitude Lemma) Let $k, n \in \mathbb{N}$ with $k < n$ and let $K \subseteq S^n$ be an oriented frameable k-dimensional knot. Let N_K be a tubular neighborhood for K. We consider the corresponding knot exterior $X_K := S^n \setminus \mathring{N}_K$.

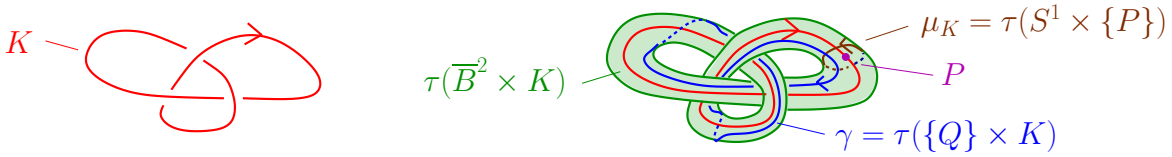
(1) There exists a unique homology class[2760] $[\lambda_K] \in \mathrm{H}_k(\partial X_K) = \mathrm{H}_k(\partial N_K)$ with the following two properties:
   (a) The image of $[\lambda_K]$ in $\mathrm{H}_k(X_K)$ is zero.
   (b) The image of $[\lambda_K]$ in $\mathrm{H}_k(N_K)$ equals $[K]$.

(2) We now assume that $n = 2k+1$. We denote by $\mu_K \subseteq \partial X_K = \partial N_K$ a meridian for K.
   (a) $[\mu_K]$ and $[\lambda_K]$ form a basis for $\mathrm{H}_k(\partial N_K)$.
   (b) We now assume that k is odd. The intersection form of ∂N_K with respect to the basis $[\mu_K], [\lambda_K]$ is given by the matrix

$$\begin{pmatrix} Q_{\partial N_K}([\mu_K],[\mu_K]) & Q_{\partial N_K}([\mu_K],[\lambda_K]) \\ Q_{\partial N_K}([\lambda_K],[\mu_K]) & Q_{\partial N_K}([\lambda_K],[\lambda_K]) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Again we only deal with the case $n = 2k+1$, since that is the most interesting case. Thus let $K \subseteq S^n$ be an oriented frameable k-dimensional knot with tubular neighborhood N_K. Since K is frameable we know that there exists a tubular map $\tau\colon \overline{B}^{n-k} \times K \to S^n$ with $N_K := \tau(\overline{B}^{n-k} \times K)$ and such that $\mu_K = \tau(S^k \times \{P\})$ for some $P \in K$.

(1) We pick $Q \in S^k$ and we write $\gamma := \tau(\{Q\} \times K)$. We set $l := \Phi_K([\gamma])$.



As in the proof of the Link Complement-H_*-Lemma 258.6 on page 4970 we obtain the following exact sequence:

$$\overbrace{\mathrm{H}_{k+1}(S^{2k+1})}^{=0} \to \overbrace{\mathrm{H}_k(\tau(S^k \times K))}^{=\mathbb{Z}\cdot[\gamma]\oplus\mathbb{Z}\cdot[\mu_K]} \to \overbrace{\mathrm{H}_k(X_K)}^{=\mathbb{Z}\cdot[\mu_K]} \oplus \overbrace{\mathrm{H}_k(\tau(\overline{B}^{k+1} \times K))}^{=\mathbb{Z}\cdot[\gamma]} \to \overbrace{\mathrm{H}_k(S^{2k+1})}^{=0}.$$

$$\begin{array}{rcl} [\mu_K] & \mapsto & ([\mu_K], 0) \\ [\gamma] & \mapsto & (l \cdot [\mu_K], [\gamma]). \end{array}$$

[2760] The notation $[\lambda_K]$ is rather risqué, since it suggests that there is in general a longitude "λ_K". We will see that this is frequently the case, but we do not claim that this is always the case.

It follows easily from this data that $[\lambda_K] := [\gamma] - l \cdot [\mu_K] \in \mathrm{H}_k(\partial N_K) = \mathrm{H}_k(\tau(S^k \times S^k))$ is the unique homology class that has all the desired homological properties.

(2) Note that the discussion in (1) shows that $[\mu_K]$ and $[\lambda_K]$ form a basis for $\mathrm{H}_k(\partial N_K)$. It remains to calculate the intersection form of ∂N_K with respect to this basis. Recall that on page 4331 we determined that for $\partial N_K = \tau(S^k \times K)$ and the two homology classes $[\mu_K] = [\tau(S^k \times \{P\})]$ and $[\gamma] = [\tau(\{Q\} \times K)]$ we have

$$\begin{pmatrix} Q_{\partial N_K}([\mu_K],[\mu_K]) & Q_{\partial N_K}([\mu_K],[\gamma]) \\ Q_{\partial N_K}([\gamma],[\mu_K]) & Q_{\partial N_K}([\gamma],[\gamma]) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

It follows from $[\lambda_K] = [\mu_K] - l \cdot [\gamma]$, the above calculation and some elementary linear algebra (as in the Form–Matrix-Lemma 224.5 (1b)) that with respect to our new basis the intersection form is given by

$$\underbrace{\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}}_{=P^{\mathrm{T}}} \cdot \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}}_{=:P} = \begin{pmatrix} l + (-1)^k \cdot l & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

$\uparrow$ elementary calculation

Since we assume that k is odd we get the desired result. ∎

The Homological Longitude Lemma 258.15 motivates the following definition:

Definition 258.16. Let $k, n \in \mathbb{N}$ with $k < n$, let $K \subseteq S^n$ be an oriented frameable k-dimensional knot and let N_K be a tubular neighborhood for K.

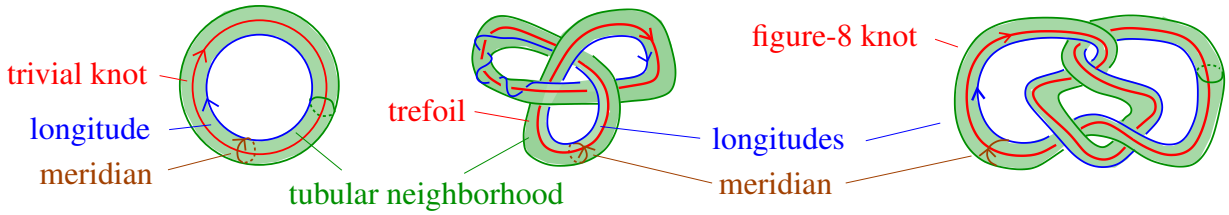
(1) We refer to the homology class $[\lambda_K] \in \mathrm{H}_{n-k}(\partial N_K)$ from the Homological Longitude Lemma 258.15 as the homological longitude of K.

(2) Any oriented submanifold of ∂N_K that is diffeomorphic to S^k and that represents the homology class $[\lambda_K]$ is called a longitude of K. A longitude is usually denoted by λ_K.[2761]

Example.

(1) Given an oriented knot $K \subseteq S^3$ we introduced in Definition 115.1 the notion of a longitude. Note that it follows from the Longitude Proposition 115.2 that for any such "classical" longitude λ_K of K there exists a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ such that λ_K is contained in $\tau(S^1 \times K)$ and such that $[\lambda_K]$ satisfies the conditions set out in the Homological Longitude Lemma 258.15. This shows that λ_K is also a longitude in the sense of Definition 258.16.

(2) In the figure below we consider again the ever popular trio of the trivial knot, the trefoil and the figure-8 knot.



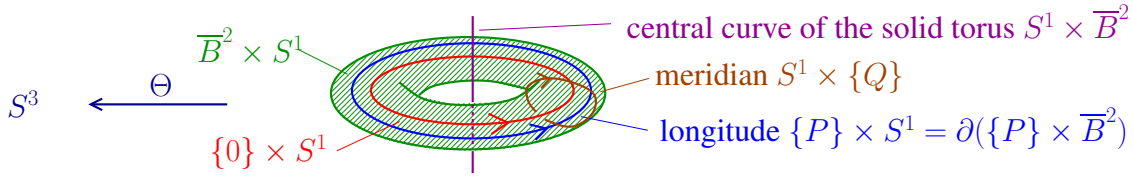
It is a hilarious exercise to verify, using the Linking Number-via-Crossings Proposition 114.9, that the curves shown in the figure are indeed longitudes.

[2761] Note that in contrast to the classical setting, we neither claim existence nor uniqueness of longitudes.

(3) Let $k, n \in \mathbb{N}$ with $k < n$. In the Sphere–Solid Tori-Decomposition Lemma 111.6 we gave an explicit diffeomorphism

$$\Theta\colon S^n \;\to\; (S^{n-k-1} \times \overline{B}^{k+1}) \cup_{S^{n-k-1}\times S^k} (\overline{B}^{n-k} \times S^k)$$

which has the property that the image of the standard trivial knot is precisely $\{0\} \times S^k$.

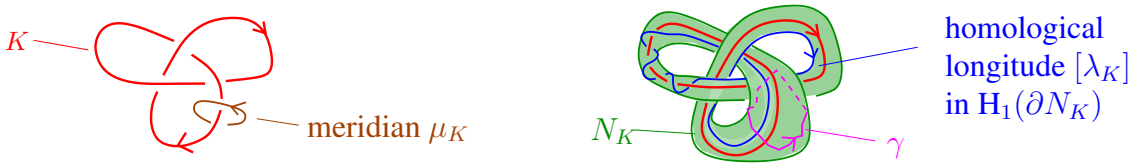


One can now easily verify that for any $P \in S^{n-k-1}$ the submanifold $\Theta^{-1}(\{P\} \times S^k)$ is a longitude for the standard trivial knot. □

We conclude this discussion of homological longitudes with a little lemma which is our first step towards interpreting linking numbers in terms of intersection numbers. The lemma will play an essential role in the next section.

Lemma 258.17. (Φ_K-via-Q-Lemma) Let $k \in \mathbb{N}$ be odd and let $K \subseteq S^{2k+1}$ be an oriented frameable k-dimensional knot. As usual we denote by N_K a tubular neighborhood. Let $[\lambda_K] \in \mathrm{H}_k(\partial N_K)$ be a homological longitude for K. Given any $\gamma \in \mathrm{H}_k(\partial N_K)$ we have

$$Q_{\partial N_K}(\gamma, [\lambda_K]) \;=\; \Phi_K(\gamma).$$



Proof. We need to prove the following claim:

Claim. Given any $\gamma \in \mathrm{H}_k(\partial N_K)$ we have

$$Q_{\partial N_K}(\gamma, [\lambda_K]) \;=\; \Phi_K(\gamma).$$

Proof. Since K is frameable we can pick a tubular map $\tau\colon \overline{B}^{k+1} \times K \to S^{2k+1}$ with $N_K = \tau(\overline{B}^{k+1} \times K)$. We consider a corresponding meridian $\mu_K := \tau(S^k \times \{P\})$. By the Homological Longitude Lemma 258.15 we know that $[\mu_K]$ and $[\lambda_K]$ form a basis for $\mathrm{H}_k(\tau(S^k \times K))$. Thus we can write $\gamma = a \cdot [\mu_K] + b \cdot [\lambda_K]$ for some $a, b \in \mathbb{Z}$.

Next note by the Homological Longitude Lemma 258.15 we know that

$$\begin{pmatrix} Q_{\partial N_K}([\mu_K],[\mu_K]) & Q_{\partial N_K}([\mu_K],[\lambda_K]) \\ Q_{\partial N_K}([\lambda_K],[\mu_K]) & Q_{\partial N_K}([\lambda_K],[\lambda_K]) \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We now see that

$$Q_{\partial N_K}(\overbrace{a \cdot [\mu_K] + b \cdot [\lambda_K]}^{\gamma}, [\lambda_K]) \;\underset{\text{follows from the above matrix}}{\overset{\uparrow}{=}}\; a \;\underset{\text{by the Homological Longitude Lemma 258.15 and by Definition 258.7 of } \Phi_K}{\overset{\uparrow}{=}}\; a \cdot \overbrace{\Phi_K([\mu_K])}^{=1} + b \cdot \overbrace{\Phi_K([\lambda_K])}^{=0}$$

$$= \Phi_K(\underbrace{a \cdot [\mu_K] + b \cdot [\lambda_K]}_{=\gamma}) \;=\; \Phi_K(\gamma).$$

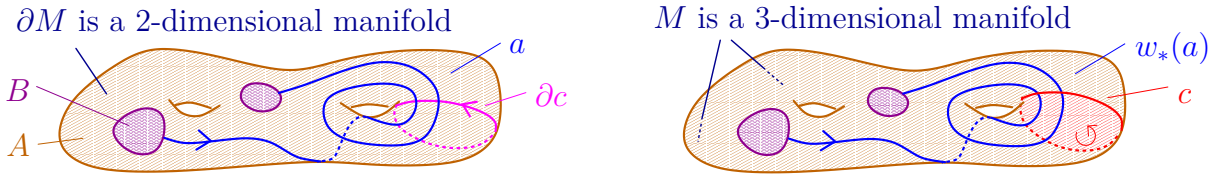
■

258.6. **Proofs of Proposition 258.13 and of Proposition 258.14.** The sole purpose of this section is to provide the proofs of the Linking #–Sphere Intersection #-Proposition 258.13 and of the Linking #–Ball Intersection #-Proposition 258.14.

For the proofs we will need the following two propositions from earlier on:

Proposition 225.8. (Intersection Pairing–Boundary Proposition) Let M be a compact oriented connected n-dimensional topological manifold. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Let $a \in \mathrm{H}_k(A, \partial A)$ and let $c \in \mathrm{H}_{n-k+1}(M, A)$. We denote by $w\colon (A, \partial A) \to (M, B)$ the obvious map and we denote by $\partial\colon \mathrm{H}_{n-k+1}(M, A) \to \mathrm{H}_{n-k}(A)$ the connecting homomorphism of the long exact sequence in homology of the pair $(M, \partial M)$. In this setting the following equality holds:

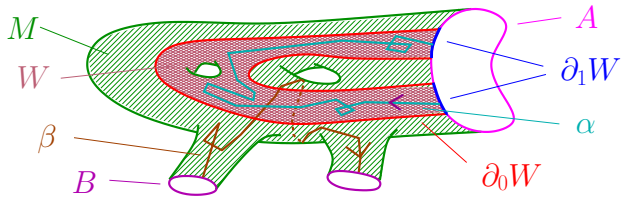
$$Q^{\mathrm{as}}_{A,\varnothing,\partial A}(\partial_{n-k+1} c, a) \;=\; (-1)^k \cdot Q^{\mathrm{as}}_{M,A,B}(c, w_*(a)).$$



Proposition 225.9. (Intersection Form-Submanifold Proposition) Let M be a compact oriented n-dimensional topological manifold and let $W \subseteq M$ be a compact n-dimensional submanifold with corner. Let A and B be disjoint unions of components of ∂M with $\partial_1 W \subseteq A$.[2762] We denote by $i\colon (W, \partial_1 W) \to (M, A)$, $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious maps. For every $\alpha \in \mathrm{H}_k(W, \partial_1 W)$ and $\beta \in \mathrm{H}_{n-k}(M, B)$ we have the following equality:[2763]

$$Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha, (w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha), \beta).$$

The analogous statement also holds with the roles of the first and the second entry swapped.



Proof of the Linking #–Sphere Intersection #-Proposition 258.13. Let $k \in \mathbb{N}_0$ and let $K, J \subseteq S^{2k+1}$ be two disjoint oriented frameable k-dimensional knots. We introduce the following notation:

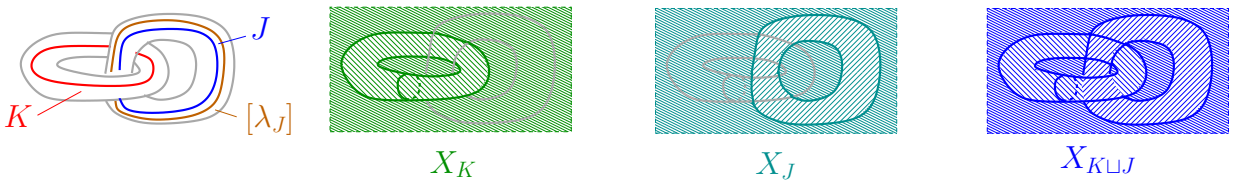
(a) • Since K and J are both frameable we can pick tubular maps $\tau\colon \overline{B}^{k+1} \times K \to S^{2k+1}$ and $\sigma\colon \overline{B}^{k+1} \times J \to S^{2k+1}$ with disjoint images.

[2762] As in Definition 212.2 and as in the Topological Submanifold-with-Corner Lemma 212.3 (0) we write $\partial_0 W := W \cap \overline{M \setminus W}$ and we write $\partial_1 W := W \cap \partial M$.

[2763] It follows from the Manifold–Excision Theorem 212.4 that the map $w\colon (W, \partial W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on homology groups.

- We denote by N_K and N_J the corresponding tubular neighborhoods for K and J and we denote by $X_K = S^{2k+1} \setminus \mathring{N}_K$ the corresponding exterior of K.
- We pick a homological longitude $[\lambda_J] \in \mathrm{H}_k(\partial X_J)$ for J.

(b) We write $X_{K \sqcup J} \coloneqq S^{2k+1} \setminus (\mathring{N}_K \cup \mathring{N}_J) = X_K \setminus \mathring{N}_J$.



(c) Let

$$\begin{aligned} \rho\colon (X_K, \partial X_K) &\to (S^{2k+1}, N_K), \\ p\colon (X_K, \partial X_K) &\to (X_K, X_{K \sqcup J}), \\ w\colon (N_J, \partial N_J) &\to (X_K, X_{K \sqcup J}) \end{aligned}$$

be the obvious maps.

(d) We denote all inclusions maps with target X_J by i.

(e) Note that it follows almost immediately from the Universal Coefficient-H^*-Theorem 201.23 and the Link Complement-H_*-Lemma 258.6 that the evaluation homomorphism $\mathrm{ev}\colon \mathrm{H}^k(X_J; \mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_k(X_J), \mathbb{Z})$ is an isomorphism. By a slight abuse of notation we denote the cohomology class in $\mathrm{H}^k(X_J; \mathbb{Z})$ that corresponds to the homomorphism $\Phi_J\colon \mathrm{H}_k(X_J) \to \mathbb{Z}$ by Φ_J as well.

Recall that we want to show that

$$\mathrm{lk}(K, J) \;=\; Q^{\mathrm{as}}_{X_K, \partial X_K, \varnothing}((\rho_*)^{-1}(\Theta_K), [J]).$$

Before we turn to the actual proof of the equality, let us mention that it follows from the naturality of the connecting homomorphisms and the functoriality of homology groups that the following diagram commutes:

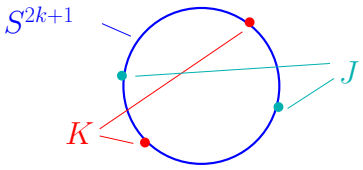
$$\begin{array}{ccccccc}
\mathrm{H}_{k+1}(N_J, \partial N_J) & \xrightarrow[\cong]{w_*} & \mathrm{H}_{k+1}(X_K, X_{K \sqcup J}) & \xleftarrow{p_*} & \mathrm{H}_{k+1}(X_K, \partial X_K) & \xrightarrow[\cong]{\rho_*} & \mathrm{H}_{k+1}(S^{2k+1}, N_K) \\
\downarrow{\scriptstyle \partial_{k+1}} & & \downarrow{\scriptstyle \partial_{k+1}} & & \downarrow{\scriptstyle \partial_{k+1}} & & \downarrow{\scriptstyle \partial_{k+1}}\ {\scriptstyle \Theta_K \mapsto [K]} \\
\mathrm{H}_k(\partial N_J) & \longrightarrow & \mathrm{H}_k(X_{K \sqcup J}) & \longleftarrow & \mathrm{H}_k(\partial X_K) & \longrightarrow & \mathrm{H}_k(N_K) \\
& {\scriptstyle i_*}\searrow & & {\scriptstyle i_*}\searrow & \downarrow{\scriptstyle i_*} & \swarrow{\scriptstyle i_*} & \\
& & & & \mathrm{H}_k(X_J). & &
\end{array}$$

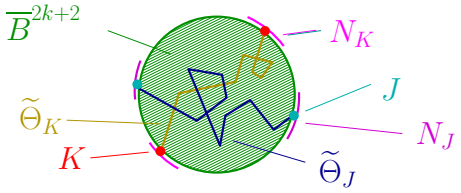
Also note that we know by the Manifold–Excision Theorem 148.3 that ρ_* and w_* are isomorphisms. Now we perform the following calculation:

$$
\begin{aligned}
Q^{\mathrm{as}}_{X_K,\partial X_K,\varnothing}((\rho_*)^{-1}(\Theta_K),[J]) &\overset{\substack{\text{by the Intersection Form-Submanifold Proposition 225.9,}\\ \text{applied to } M=X_K \text{ and } W=X_{K\sqcup J}\text{, we also use that}\\ \text{by Definition 258.16 of a homological longitude we have } [J]=[\lambda_J]\in \mathrm{H}_k(\partial N_J)\\ \text{which implies that } [J]=[\lambda_J]\in \mathrm{H}_k(\partial X_K)}}{\downarrow}{=} Q^{\mathrm{as}}_{X_{K\sqcup J},\partial X_{K\sqcup J},\varnothing}(((w_*)^{-1}\circ p_*\circ(\rho_*)^{-1})(\Theta_K)),[\lambda_J])\\
&\underset{\substack{\text{by the Intersection Pairing–Boundary Proposition 225.8, note that}\\ \text{we only need to consider the component } \partial X_J \text{ since } [\lambda_J]\in \mathrm{H}_k(\partial X_J)}}{=} Q^{\mathrm{as}}_{\partial X_J,\varnothing,\varnothing}(\partial_{k+1}(((w_*)^{-1}\circ p_*\circ(\rho_*)^{-1})(\Theta_K)),[\lambda_J])\\
&\underset{\substack{\text{by the } \Phi_K\text{-via-}Q\text{-Lemma 258.17}}}{=} \langle i^*\Phi_J,\partial_{k+1}((w_*)^{-1}\circ p_*\circ(\rho_*)^{-1})(\Theta_K)\rangle_{X_J}\\
&\underset{\substack{\text{by the Kronecker Pairing}\\ \text{Natural Lemma 199.9 (1)}}}{=} \langle \Phi_J, i_*(\partial_{k+1}((w_*)^{-1}\circ p_*\circ(\rho_*)^{-1}))(\Theta_K)\rangle_{X_J} \underset{\substack{\text{by the above commutative}\\ \text{diagram and Definition 258.12}\\ \text{of } \Theta_K}}{=} \langle \Phi_J,[K]\rangle_{X_J} \underset{\substack{\text{by Definition 258.7}\\ \text{of } \mathrm{lk}(J,K)}}{=} \mathrm{lk}(J,K).
\end{aligned}
$$

■

Proof of the Linking #–Ball Intersection #-Proposition 258.14. Let $k\in\mathbb{N}_0$ and let $K,J\subseteq S^{2k+1}$ be two disjoint oriented frameable k-dimensional knots. We pick disjoint tubular neighborhoods N_K and N_J and we denote by $X_K=S^{2k+1}\backslash\mathring{N}_K$ and $X_J=S^{2k+1}\backslash\mathring{N}_J$ the corresponding interiors.





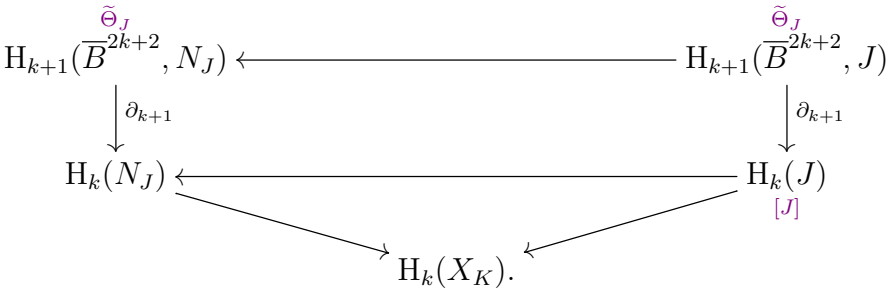
Let

$$
\begin{aligned}
\rho\colon (X_K,\partial X_K) &\to (S^{2k+1},N_K),\\
w\colon (X_K,\partial X_K) &\to (\overline{B}^4,N_K),\\
i\colon (S^{2k+1},N_K) &\to (\overline{B}^{2k+2},N_K)
\end{aligned}
$$

be the obvious maps. It follows from the naturality of connecting homomorphisms and from $i\circ p=w$ that we have the following commutative diagrams:

$$
\begin{tikzcd}
\mathrm{H}_{k+1}(S^{2k+1},N_K) \arrow[rr, bend left, "i_*"] \arrow[dr, "\partial_{k+1}"'] & \mathrm{H}_{k+1}(X_K,\partial X_K) \arrow[l, "\rho_*"', "\cong"] \arrow[r, "w_*"] & \mathrm{H}_{k+1}(\overline{B}^{2k+2},N_K) \arrow[dl, "\partial_{k+1}"] \\
& \mathrm{H}_k(N_K). &
\end{tikzcd}
$$

Furthermore it follows from again from the naturality of connecting homomorphisms that we have the following commutative diagram:



We perform the following calculation:

$$
\begin{array}{lll}
 & \text{by the Linking \# Sphere Intersection \#-Proposition 258.13} & \text{by the Asymmetric Intersection Form-Exchange Lemma 225.7, here we use that } (-1)^{k(k+1)} = 1 \\
\mathrm{lk}(K,J) & \overset{\downarrow}{=} Q^{\mathrm{as}}_{X_K,\partial X_K,\varnothing}(\rho_*^{-1}(\Theta_K),[J]) & \overset{\downarrow}{=} Q^{\mathrm{as}}_{X_K,\varnothing,\partial X_K}([J],\rho_*^{-1}(\Theta_K)) \\
 & = Q^{\mathrm{as}}_{X_K,\varnothing,\partial X_K}(\partial_{k+1}(\widetilde{\Theta}_J),\rho_*^{-1}(\Theta_K)) & = (-1)^{k+1}\cdot Q^{\mathrm{as}}_{\overline{B}^{2k+2},X_K,N_K}(\widetilde{\Theta}_J, w_*(\rho_*^{-1}(\Theta_K))) \\
 & \quad\uparrow & \quad\uparrow \\
 & \text{by Definition 258.12 of } \widetilde{\Theta}_J \text{ together with the second diagram} & \text{by the Intersection Pairing–Boundary Proposition 225.8 applied to } A = X_K \text{ and } B = N_K \\
 & = Q^{\mathrm{as}}_{\overline{B}^{2k+2},N_K,X_K}(w_*(\rho_*^{-1}(\Theta_K)),\widetilde{\Theta}_J) & = Q^{\mathrm{as}}_{\overline{B}^{2k+2},N_K,X_K}(\widetilde{\Theta}_K,\widetilde{\Theta}_J) \\
 & \quad\uparrow & \quad\uparrow \\
 & \text{by the Asymmetric Intersection Form–Natural Lemma 225.5} & \text{by the first commutative diagram and Definition 258.12 of } \Theta_K \text{ and } \widetilde{\Theta}_K \\
 & = Q^{\mathrm{as}}_{\overline{B}^{2k+2},N_K,N_J}(\widetilde{\Theta}_K,\widetilde{\Theta}_J). & \\
 & \quad\uparrow & \\
 & \text{by the Asymmetric Intersection Form Natural Lemma 225.5, since } N_J \subseteq X_K &
\end{array}
$$

■

258.7. **Appendix: Tubular neighborhoods of high-dimensional knots.** In the previous sections we were mostly interested in k-dimensional knots in S^{2k+1}. To make things work, e.g. to define the linking number, we had to add the hypothesis that our knots are frameable. In this section we will point out that we did not really have to impose this condition, since all such knots are actually frameable. More precisely, we have the following theorem:[2764]

Theorem 258.18. (Sphere-in-Sphere Tubular Neighborhood–Existence Theorem) Let $K \subseteq S^n$ be a k-dimensional knot. If $k < \frac{m}{2}$, then K admits a tubular map $F\colon \overline{B}^{m-n} \times K \to M$.

Remark. It is not entirely clear in what range of dimensions this statement holds. For example it follows from [**Kerv1959**, Theorem 8.2] that the conclusion holds if $3k < 2n-1$ and it follows from [**MW2017**, Proposition 2.10] that the conclusion holds if $2k+1 \leq n$. On the other hand, as we already pointed out on page 4966, it was shown by André Haefliger [**HSL1965**, p. 174] and Jerry Levine [**Levine1965b**, Table 7.2] that there exist 11-dimensional knots in S^{17} that are not frameable. □

[2764]Note that this theorem is a generalization of the earlier Thicken–High Codimension Sphere Lemma 256.13.

The proof of the Sphere-in-Sphere Tubular Neighborhood–Existence Theorem 258.18 relies on Haefliger's Unknotting Theorem 111.35, whose proof is unfortunately out of the scope of these modest lecture notes:

Theorem 111.35. (Haefliger's Unknotting Theorem) Let $n, k \in \mathbb{N}$. If $2n > 3(k+1)$, then every k-dimensional knot in S^n is trivial.

Example. Let $k \in \mathbb{N}_{\geq 2}$. It follows immediately from Haefliger's Unknotting Theorem that every k-dimensional knot in S^{2k+1} is trivial. □

Proof. First note that the case $k = 1$ is dealt with in the Product Tubular Neighborhood–Existence Theorem 58.4. Thus we now assume that $k \geq 2$. Since $2 \leq k < \frac{n}{2}$ we obtain from Haefliger's Unknotting Theorem 111.35 that K is trivial. Now the statement follows of course from the Trivial Knot–Tubular Neighborhood Lemma 258.5. ■

258.8. **Appendix: Definitions of the linking number.** In this appendix we will see that the various definitions of a linking number are coherent:

Proposition 258.19. (Linking Number Reformulation Proposition) Let K and J be two disjoint oriented k-dimensional knots in S^{2k+1}. The following numbers agree:

(1) The linking number from Definition 52.2 where we used the notion of a degree.
(2) The linking number from Definition 258.7 where we used the homology groups of the complement of K to define the linking number.

For $k = 1$ the above linking numbers also agree with the following:

(3) The linking number from Definition 86.10 where we used the notion of a degree in de Rham cohomology.
(4) The linking number from Definition 114.1 where we used the fundamental group of the fundamental group of the complement of K to define the linking number.

Remark. In the setting of classical knots we will see in the Linking Number Reformulation Proposition 285.8 that there are even more ways to think about the linking number. □

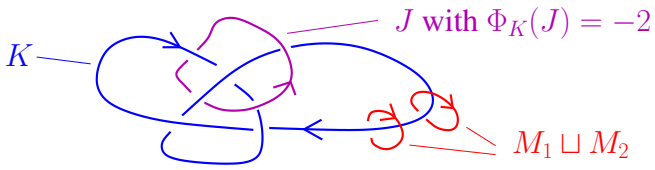
Sketch of proof. First note that it follows immediately from the Degrees Agree Proposition 86.7 that for $k = 1$ Definitions (1) and (3) are consistent. Furthermore it follows easily from the Hurewicz Theorem 166.10 that for $k = 1$ Definitions (2) and (4) are also consistent. It remains to prove that Definitions (1) and (3) are consistent.[2765]

In the remainder of the proof we denote by $\mathrm{lk}_d(K, J)$ the linking number from Definition 52.2 where we used the notion of a degree and we denote by $\mathrm{lk}_h(K, J)$ the linking number from Definition 258.7 where we used homology groups to define the linking number.

Let $k \in \mathbb{N}$ and let K and J be two disjoint oriented k-dimensional knots in S^{2k+1}. We set $l := \mathrm{lk}_h(K, J) := \Phi_K([J])$. We denote by $M = M_1 \cup \cdots \cup M_{|l|} \subseteq X_K$ a link that is given by $|l|$ disjoint meridians $M_1, \ldots, M_{|l|}$ of K. We give M the usual orientation of a meridian if $l \geq 0$ and we give M the opposite orientation if $l < 0$.

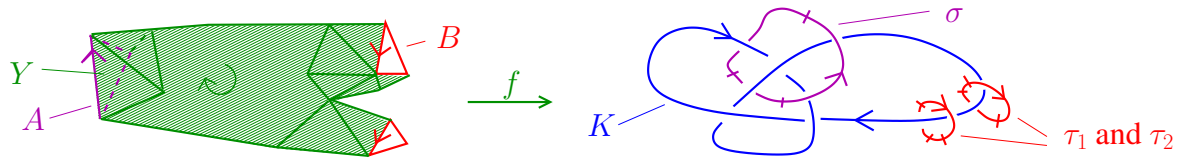
Claim. There exists a pseudomanifold Y, in the sense of Definition 183.6, with a decomposition $\partial Y = A \sqcup B$ into two sets of components and a continuous map $f\colon Y \to S^{2k+1} \setminus K$

[2765] In the case $k = 1$ we will give in the Linking Number Reformulation Proposition 285.8 an alternative definition for this statement.



such that $f\colon A \to J$ and $f\colon B \to M_1 \cup \cdots \cup M_{|l|}$ are orientation-preserving homeomorphisms.

Proof. We pick a cycle σ in X_K that represents $[J] \in \mathrm{H}_k(X_K)$ and we pick cycles $\tau_1, \ldots, \tau_{|l|}$ in X_K that represent $[M_1], \ldots, [M_{|l|}] \in \mathrm{H}_k(X_K)$. Since $\Phi_K\colon \mathrm{H}_k(X_K) \to \mathbb{Z}$ is an isomorphism we know that $[\sigma] = [\tau] \in \mathrm{H}_k(X_K)$. This implies that there exists a $(k+1)$-dimensional singular chain $\Sigma = \sum_{i=1}^{n} a_i \cdot \xi_i \in \mathrm{C}_{k+1}(X_K)$ such that $\partial\Sigma = \sigma - \tau_1 - \cdots - \tau_{|l|}$. As in the proof of H_2-via-Surface Proposition 187.4 we can build an abstract simplicial complex Y out of Σ with n $(k+1)$-simplices together with a continuous map $f\colon Y \to S^{2k+1} \setminus K$ and a decomposition $\partial Y = A \sqcup B$ such that $f\colon A \to J$ and $f\colon B \to M_1 \cup \cdots \cup M_{|l|}$ are orientation-preserving homeomorphisms.



We leave the joyful task of filling in the details to the reader. ⊞

Finally note that we have the following equalities:

$$\begin{aligned}
\mathrm{lk}_d(K,J) &\overset{\substack{\text{follows by the same argument as in the Linking Number Zero-Criterion 52.9, here we use}\\ \text{that it follows from the argument of the Pseudomanifold Homology Theorem 183.12}\\ \text{that we can equip } Y \text{ with a fundamental class } [Y] \in \mathrm{H}_{k+1}(Y,\partial Y)}}{\downarrow}{=}\ \mathrm{lk}_d(K, M_1 \cup \cdots \cup M_{|l|}) &&= \sum_{i=1}^{|l|} \mathrm{lk}_d(K, M_i)\\
&\underset{\substack{\uparrow\\ \text{by the Linking Number–Disk}\\ \text{Intersection Lemma 52.10}}}{=} \sum_{i=1}^{|l|} \begin{cases} +1, & \text{if } M \text{ is a “positive” meridian,}\\ -1, & \text{if } M \text{ is a “negative” meridian}\end{cases} &&= l = \mathrm{lk}_h(K,J).
\end{aligned}$$

■

258.9. Exercises for Chapter 258.

Exercise 258.1. Let $L \subseteq S^n$ be an m-component k-dimensional frameable link. We pick a tubular neighborhood N_L. Use the following ingredients to give an alternative calculation of the homology groups of the link exterior $X_L = S^n \setminus \mathring{N}_L$:

(a) Use a long exact of the pair (S^n, K).
(b) Consider the pairs (S^n, K), (S^n, N_L) and $(X_L, \partial X_L)$.
(c) Use whatever tricks you might know otherwise.

Exercise 258.2. Let $m \in \mathbb{N}$, let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, n-1\}$. Furthermore let $L \subseteq \mathbb{R}^n \subseteq \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$ be an m-component k-dimensional link.

(a) Show that for $i < n-1$ the inclusion induced map

$$\mathrm{H}_i(\mathbb{R}^n \setminus L) \;\to\; \mathrm{H}_i(S^n \setminus L)$$

is an isomorphism.

(b) Determine $\mathrm{H}_{n-1}(\mathbb{R}^n \setminus L)$.

Exercise 258.3.

(a) We consider the Hopf map

$$\begin{array}{ccc} H_{\mathbb{C}} \colon S^3 & \to & \mathbb{C}\mathrm{P}^2 =_{\mathrm{i}} S^2 \\ (z_0, z_1) & \mapsto & [z_0 : z_1] \end{array}$$

from Definition 134.24. Let $P, Q \in S^2$ be two distinct points. Determine the linking number between $H_{\mathbb{C}}^{-1}(P)$ and $H_{\mathbb{C}}^{-1}(Q)$.

(b) We consider the quaternionic Hopf map $H_{\mathbb{H}} \colon S^7 \to S^4$ from Definition 216.2. Once again let $P, Q \in S^4$ be two distinct points. Determine the linking number between $H_{\mathbb{H}}^{-1}(P)$ and $H_{\mathbb{H}}^{-1}(Q)$.

Hint. In both cases you might first want to show that the linking does not depend on the points P and Q and afterwards you do the calculation for $P = [0:1]$ and $Q = [1:0]$.

Exercise 258.4. Let $k \in \mathbb{N}$ and let $K, J \subseteq S^{2k+1}$ be two disjoint oriented k-dimensional knots.

(a) Let $-K$ be the same knot but with the opposite orientation. What is the relationship between $\mathrm{lk}(K, J)$ and $\mathrm{lk}(-K, J)$?

(b) Let $\rho \colon S^{2k+1} \to S^{2k+1}$ be an orientation-reversing diffeomorphism. What is the relationship between $\mathrm{lk}(K, J)$ and $\mathrm{lk}(\rho(K), \rho(J))$?

Exercise 258.5.

(a) Does there exist a 2-component 1-dimensional link L in S^3 such that the two boundary components of X_L are smoothly isotopic in X_L?

(b) Does there exist a $k \geq 2$ and a 2-component k-dimensional link L in S^{2k+1} such that the two boundary components of X_L are smoothly isotopic in X_L?

Exercise 258.6. Let $k < n$ and let $m \in \mathbb{N}$. Let U be the trivial m-component k-dimensional link in S^n. Determine $\pi_i(S^n \setminus U)$ for as many $i \in \mathbb{N}_0$ as possible.

Exercise 258.7. Let $k \in \mathbb{N}$ and let K, J be two disjoint oriented k-dimensional knots in S^{2k+1}. Use the Linking #–Ball Intersection #-Proposition 258.14 together with the Cup Product-Commutativity Proposition 204.10 to show that

$$\mathrm{lk}(J, K) \;=\; (-1)^{k+1} \cdot \mathrm{lk}(K, J).$$

Remark. This provides a proof of the Linking Number-Symmetry Proposition 258.8.

Exercise 258.8. We consider the symmetric partial matrix

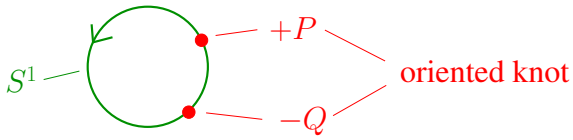
$$A \;=\; (a_{ij})_{i \neq j} \;=\; \begin{pmatrix} * & -2 & 3 \\ -2 & * & 1 \\ 3 & 1 & * \end{pmatrix}.$$

Provide an explicit oriented 3-component link in S^3 such that each component is trivial and such that for any $i \neq j \in \{1,2,3\}$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$.

Exercise 258.9. Does there exist a closed smooth manifold M and a smooth embedding $f\colon S^2 \to M$ such that $f(S^2) \subseteq M$ is not frameable?

Exercise 258.10. We consider the 0-dimensional oriented knot K in S^1.

(a) Determine a meridian μ_K for K.
(b) Show that $\Phi([\mu_K]) = 1$.



Exercise 258.11. Let $p, q \in \mathbb{N}$ be coprime. As in Definition 111.13 we consider the torus knot

$$T(p,q) := \{\exp(2\pi \mathrm{i} t \cdot p), \exp(2\pi \mathrm{i} t \cdot q) \mid t \in \mathbb{R}\}$$

where we think of the torus $S^1 \times S^1$ as the "standard torus" in the 3-dimensional sphere $S^3_2 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$ of radius 2. Let T' be a knot on $S^1 \times S^1$ that runs parallel to $T(p,q)$ on $S^1 \times S^1$. To be more precise, T' is smoothly isotopic on $S^1 \times S^1$ to $T(p,q)$. Determine the linking number between $T(p,q)$ and T'.
Remark. This exercise is identical to Exercise 114.9.

Exercise 258.12. We say that an m-component link $L = L_1 \sqcup \cdots \sqcup L_m$ in S^3 is a *boundary link* if there exists a collection of m disjoint Seifert surfaces $F_1, \ldots, F_m \subseteq S^3$ such that for $i = 1, \ldots, m$ we have $\partial F_i = L_i$.

(a) Show that the linking numbers of an oriented boundary link are zero.
(b) Give an example of a boundary link that is not "split", i.e. that does not contain an "obvious" 2-sphere embedded in the complement.

Exercise 258.13. Two oriented m-component links L and $\widetilde{L}$ in S^3 are called *smoothly concordant* if there exist disjoint properly smoothly embedded annuli $A_1, \ldots, A_m \subseteq [0,1] \times S^3$ such that for each $i \in \{1, \ldots, m\}$ we have $\partial A_i = (\{0\} \times (-L_i)) \cup (\{1\} \times \widetilde{L}_i)$. Show that if L and $\widetilde{L}$ are smoothly concordant, then for all $i \neq j \in \{1, \ldots, m\}$ we have $\mathrm{lk}(L_i, L_j) = \mathrm{lk}(\widetilde{L}_i, \widetilde{L}_j)$.

Exercise 258.14.

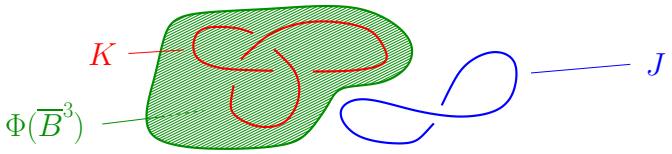
(a) We consider the two-component link $L = L_1 \sqcup L_2$ that is shown in the figure below to the left. Are there compact orientable proper 2-dimensional smooth submanifolds F_1 and F_2 in $\overline{B}^4$ such that $\partial F_1 = L_1$, $\partial F_2 = L_2$ and such that F_1 and F_2 are *disjoint*?
(b) Do the same exercise with the two-component link $\widetilde{L} = \widetilde{L}_1 \sqcup \widetilde{L}_2$ that is shown in the figure below to the right.



Exercise 258.15. Let $k \in \mathbb{N}$ and let $L = K \sqcup J$ be a k-dimensional knot in S^{2k+1}. We say that L is *splittable* if there exists a smooth embedding $\Phi\colon \overline{B}^{2k+1} \to S^{2k+1}$ such that $K \subseteq \Phi(B^{2k+1})$ and $J \subseteq S^{2k+1} \setminus \Phi(\overline{B}^{2k+1})$.[2766]

(a) Show that if $L = K \sqcup J$ is an oriented splittable link, then $\operatorname{lk}(K, J) = 0$.
Remark. This exercise can be viewed as a generalization of the Linking Number-of-Split Link Lemma 114.6.

(b) Show that if L is splittable, then $\pi_{2k}(S^{2k+1} \setminus (K \cup J)) \neq 0$.
Hint. Show that there exists a retraction from $S^{2k+1} \setminus (K \cup J)$ to a $2k$-dimensional sphere.



[2766]Using the Smooth Ball Embedding Theorem 45.10 one can easily show that for $k = 1$ this definition is consistent with Definition 109.5.

CHAPTER 259

Handle attachments, intersection forms and linking numbers

Let $k \in \mathbb{N}_0$ and let $\varphi_1, \ldots, \varphi_m \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. From this data we obtain, by the Handle Attachment Proposition 246.3, the $(2k+2)$-dimensional smooth manifold $M := \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1}$. Our previous results let us determine many facts about M. For example it follows from the Handles-Orientability Proposition 246.6 and the Handle Attachment-H_*-Lemma 246.14 that M is a compact orientable $(2k+2)$-dimensional smooth manifold such that all reduced homology groups vanish except for the middle-dimensional homology group $\mathrm{H}_{k+1}(M) \cong \mathbb{Z}^m$. If $k \geq 1$, then we know furthermore by the Handle Attachment-π_1-Proposition 246.13 that M is simply connected.

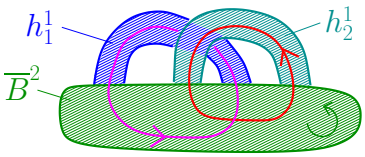
In the current setting we have one extra invariant, which we have not accounted for yet, namely the intersection form. For the reader's convenience we recall the general Definition 222.3 of the intersection form:

Definition 224.12. Let M be a compact oriented $2n$-dimensional topological manifold. We refer to the map [2767]

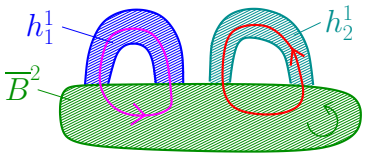
$$\begin{array}{rcl} Q_M \colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{array}$$

as the intersection form of M.

The following question arises: Can we determine the intersection form of the manifold $M = \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1}$ directly from the attaching spheres?



$$Q_M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$$Q_M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In this chapter we will see that we can indeed calculate the intersection form using the linking numbers that we introduced in the previous Chapter 258. We will use this calculation, and some realization results from the previous chapter, to construct interesting new smooth manifolds of dimension four and higher.

[2767] Here we adopt the following notation from page 243: Given an abelian group A we denote by

$$\mathrm{Tor}(A) := \{g \in A \,|\, \text{there exists an } n \in \mathbb{N} \text{ with } n \cdot g = 0\}$$

its torsion subgroup and we denote by $\mathrm{F}A := A/\mathrm{Tor}(A)$ the maximal torsion-free quotient.

This chapter is organized as follows:

- In Section 259.1 we will calculate the intersection form of a smooth manifold of the form $\overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1}$ in terms of linking numbers.
- In Section 259.2 we will study thickenings of knots.
- In Section 259.3 we will use the realization results for linking numbers from Section 258.3 and the results on thickenings of knots to prove realization results for intersection forms. In particular we will see that every symmetric form occurs as the intersection form of a compact (but not necessarily closed) oriented 4-dimensional smooth manifold.
- In Section 259.4 we will consider the boundary of the smooth manifolds from Section 259.1.
- In Section 259.5 we will recall the definition of the E_8-matrix and we will use it to construct an interesting 4-dimensional homology sphere.
- In Section 259.6 we will use the E_8-matrix to construct (potentially) interesting smooth homotopy spheres in dimension $4k-1$ with $k \in \mathbb{N}_{\geq 2}$.
- In Section 259.7 we will provide exercises for Chapter 259.

In the next **??** we will introduce the notion of a surgery along a thickened sphere.

259.1. **Handle decompositions and intersection forms.** As we mentioned in the beginning of the chapter, our goal is to calculate the intersection form of a smooth manifold that we obtain from $\overline{B}^{k+2}$ by attaching $(k+1)$-handles. The first goal is to find explicit generators of the middle homology group. This leads us to the following definition:

Definition 259.1. Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be a thickened k-sphere in S^{2k+1}. We denote by $C(\varphi) \in \mathrm{H}_{k+1}(\overline{B}^{2k+2} \cup_\varphi h^{k+1})$ the element that is the image of $[\{0\} \times \overline{B}^{k+1}]$ under the following sequence of maps:

$$\overbrace{\mathrm{H}_{k+1}(\overline{B}^{k+1} \times \overline{B}^{k+1}, \overline{B}^{k+1} \times S^k)}^{=\mathbb{Z}\cdot[\{0\}\times\overline{B}^{k+1}]} \xrightarrow[\uparrow]{\cong} \mathrm{H}_k(\overline{B}^{2k+2} \cup_\varphi h^{k+1}, \overline{B}^{2k+2}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_{k+1}(\overline{B}^{2k+2} \cup_\varphi h^{k+1}).$$

induced by the inclusion, it follows from the Manifold–Excision Theorem 212.4 that this map is an isomorphism

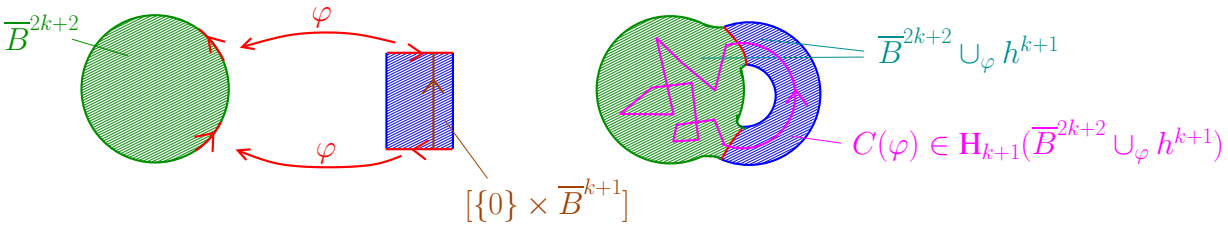
isomorphism by the long exact sequence of the pair in reduced homology

If $\overline{B}^{2k+2} \cup_\varphi h^{k+1}$ is embedded in some other manifold W, then we denote by $C(\varphi)$ also the corresponding homology class in $\mathrm{H}_{k+1}(W)$.

We will see shortly that the classes $C(\varphi_i)$ give rise to a basis of the middle homology of $M = \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cdots \cup_{\varphi_m} h^{k+1}$. The reader will furthermore most likely already sense that the intersection number between $C(\varphi_i) \in \mathrm{H}_{k+1}(M)$ and $C(\varphi_j) \in \mathrm{H}_{k+1}(M)$ for $i \neq j$ is given by a linking number.

But to express the intersection form of M on the diagonal we still need to introduce the following definition:

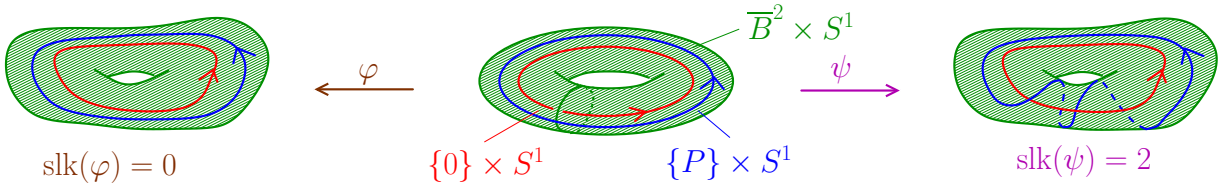
[2767] Here we equip $\{0\} \times \overline{B}^{k+1}$ with the standard orientation which we introduced in the Standard Euclidean Orientation Lemma 191.19. Next note that $[\{0\} \times \overline{B}^{k+1}]$ stands first of all for the fundamental class $[\{0\} \times \overline{B}^{k+1}] \in \mathrm{H}_{k+1}(\{0\} \times \overline{B}^{k+1}, \{0\} \times S^k)$ of the oriented topological manifold $\{0\} \times \overline{B}^{k+1}$ from Definition 194.3. As usual we also denote by $[\{0\} \times \overline{B}^{k+1}]$ its image under the inclusion induced homomorphism $\mathrm{H}_{k+1}(\{0\} \times \overline{B}^{k+1}, \{0\} \times S^k) \to \mathrm{H}_{k+1}(\overline{B}^{k+1} \times \overline{B}^{k+1}, \overline{B}^{k+1} \times S^k)$.



Definition 259.2. Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be a thickened k-sphere in S^{2k+1}. We define the self-linking number of $\varphi\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ to be

$$\mathrm{slk}(\varphi) \;:=\; \underbrace{\mathrm{lk}(\varphi(\{0\} \times S^k), \varphi(\{P\} \times S^k))}_{\substack{\text{linking number of the disjoint}\\ \text{oriented } k\text{-dimensional knots}\\ \varphi(\{0\} \times S^k) \text{ and } \varphi(\{P\} \times S^k) \text{ in } S^{2k+1}}}$$

where $P = (0, \ldots, 0, 1) \in S^k$.[2768]



To formulate the next proposition we will also need the following almost self-explanatory definition:

Definition 259.3. Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ and $\psi\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be thickened k-spheres in S^{2k+1} with disjoint images. We define their linking number to be

$$\mathrm{lk}(\varphi, \psi) \;:=\; \underbrace{\mathrm{lk}(\varphi(\{0\} \times S^k), \psi(\{0\} \times S^k))}_{\substack{\text{linking number of the disjoint}\\ \text{oriented } k\text{-dimensional knots}\\ \varphi(\{0\} \times S^k) \text{ and } \psi(\{0\} \times S^k) \text{ in } S^{2k+1}}} .$$

Now we can formulate the main result of this section:

Proposition 259.4. (Middle Dimension Handles Proposition) Let $k \in \mathbb{N}_0$ and let $\varphi_1, \ldots, \varphi_m\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. We denote by

$$M \;:=\; \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1}$$

[2768] In fact, if $k > 0$, then we can take any point P in $\overline{B}^{k+1} \setminus \{0\}$ and we get the same self-linking number. This can be seen as follows: Let $P, P' \in \overline{B}^{k+1} \setminus \{0\}$. Since $k > 0$ there exists a path $\gamma\colon [0,1] \to \overline{B}^{k+1} \setminus \{0\}$ with $\gamma(0) = P$ and $\gamma(1) = P'$. This defines a homotopy $F\colon S^k \times [0,1] \to S^{2k+1} \setminus (\varphi(\{0\} \times S^k)$ from $\varphi(\{P\} \times S^k)$ to $\varphi(\{P'\} \times S^k)$. It now follows that

$$\begin{aligned} \mathrm{lk}(\varphi(\{0\} \times S^k), \varphi(\{P\} \times S^k) &= \Phi_{\varphi(\{0\} \times S^k)}([\{P\} \times S^k]) \\ &= \Phi_{\varphi(\{0\} \times S^k)}([\{P'\} \times S^k]) \;=\; \mathrm{lk}(\varphi(\{0\} \times S^k), \varphi(\{P'\} \times S^k). \end{aligned}$$

$\uparrow$ by the Homotopy-H_*-Proposition 144.12

the result of attaching the $(k+1)$-handles corresponding to the m thickened k-spheres.

(1) M is a compact $(2k+2)$-dimensional smooth manifold that admits a unique orientation such that the natural embedding $\overline{B}^{2k+2} \to M$ is orientation-preserving.

(2) (a) For $j \neq 0$ the map

$$\bigoplus_{i=1}^{m} \underbrace{\mathrm{H}_j(\overline{B}^{2k+2} \cup_{\varphi_i} h^{k+1})}_{\substack{= 0, & \text{if } j \neq 0, k+1, \\ = \mathbb{Z} \cdot C(\varphi_i), & \text{if } j = k+1}} \to \mathrm{H}_j(M),$$

which is induced by the natural embeddings $\overline{B}^{2k+2} \cup_{\varphi_i} h^{k+1} \to M$, is an isomorphism.

(b) We have $\mathrm{H}_j(M) = 0$ for $j \neq 0, k+1$.

(c) The homology classes $C(\varphi_1), \dots, C(\varphi_m)$ form a basis of $\mathrm{H}_{k+1}(M)$.

(d) If $k \geq 1$, then M is simply connected.

(3) With respect to the basis $C(\varphi_1), \dots, C(\varphi_m)$ of $\mathrm{H}_{k+1}(M)$ the intersection form Q_M is represented by the following matrix

$$\begin{pmatrix} \mathrm{slk}(\varphi_1) & \mathrm{lk}(\varphi_1, \varphi_2) & \dots & \mathrm{lk}(\varphi_1, \varphi_m) \\ \mathrm{lk}(\varphi_2, \varphi_1) & \mathrm{slk}(\varphi_2) & & \vdots \\ \vdots & & \ddots & \mathrm{lk}(\varphi_{m-1}, \varphi_m) \\ \mathrm{lk}(\varphi_m, \varphi_1) & \dots & \mathrm{lk}(\varphi_m, \varphi_{m-1}) & \mathrm{slk}(\varphi_m) \end{pmatrix}.$$

If we do *not* demand that the thickened k-spheres are orientation-preserving, then the resulting smooth manifold M is not necessarily orientable, but the above statements still hold, under the proviso that we replace $\mathbb{Z}$-coefficients throughout by $\mathbb{F}_2$-coefficients.

The Middle Dimension Handles Proposition 259.4 (3) motivates the following definition:

Definition 259.5. Let $k \in \mathbb{N}_0$ and let $\varphi_1, \dots, \varphi_m \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. We refer to the matrix

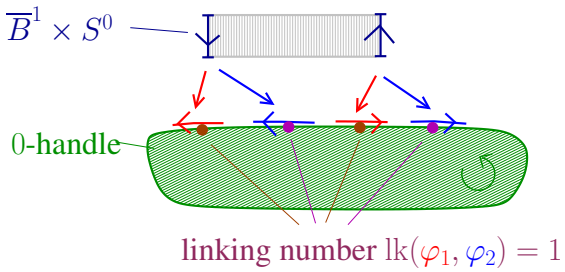
$$\mathrm{lk}(\varphi_1, \dots, \varphi_m) \coloneqq \begin{pmatrix} \mathrm{slk}(\varphi_1) & \mathrm{lk}(\varphi_1, \varphi_2) & \dots & \mathrm{lk}(\varphi_1, \varphi_m) \\ \mathrm{lk}(\varphi_2, \varphi_1) & \mathrm{slk}(\varphi_2) & & \vdots \\ \vdots & & \ddots & \mathrm{lk}(\varphi_{m-1}, \varphi_m) \\ \mathrm{lk}(\varphi_m, \varphi_1) & \dots & \mathrm{lk}(\varphi_m, \varphi_{m-1}) & \mathrm{slk}(\varphi_m) \end{pmatrix}$$

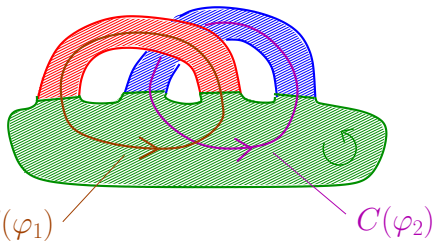
as the linking matrix of $\varphi_1, \dots, \varphi_m$.

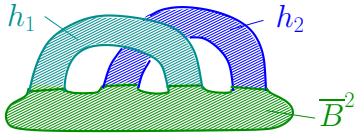
Example.

(1) We consider the disk $\overline{B}^2$ to which we attach two orientation-preserving 1-handles as shown in the figure below.
It follows from the Middle Dimension Handles Proposition 259.4 that the intersection form of $\overline{B}^2 \cup_{\varphi_1} h^1 \cup_{\varphi_2} h^1$ is represented by the linking matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of φ_1, φ_2.

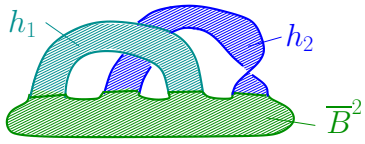
(2) We again consider the 2-dimensional setting, but this time with thickened 0-spheres that are not necessarily orientation-preserving.







$$Q_M^{\mathbb{F}_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$Q_M^{\mathbb{F}_2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

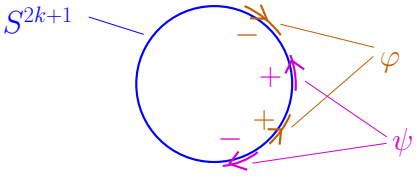
The Middle Dimension Handles Proposition 259.4 provides us with two potentially different intersection forms. In Exercise 259.3 we will discuss the question, whether these two forms are isometric. □

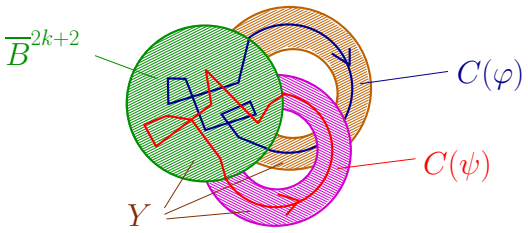
The proof of the Middle Dimension Handles Proposition 259.4 will require the remainder of this section. In the next Section 259.2 we will discuss which matrices can occur as linking matrices of a family of thickened spheres.

The key to proving the Middle Dimension Handles Proposition 259.4 (2) is the following lemma:

Lemma 259.6. (Attach-two-Handles Lemma) Let $k \in \mathbb{N}_0$. Suppose that we are given two orientation-preserving thickened k-spheres $\varphi, \psi \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ with disjoint images. We set $Y := \overline{B}^{2k+2} \cup_\varphi h^{k+1} \cup_\psi h^{k+1}$. We have the following equality:

$$\mathrm{lk}(\varphi, \psi) \;=\; Q_Y(C(\varphi), C(\psi)).$$

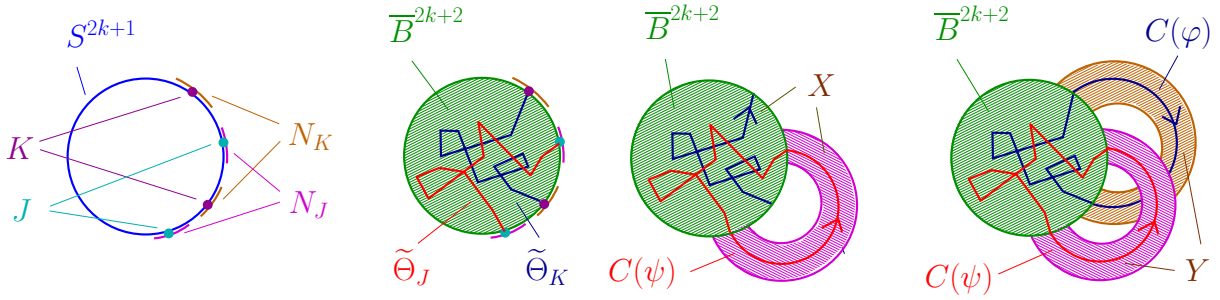




Proof of the Attach-two-Handles Lemma 259.6. Let $\varphi, \psi \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. We set

$$\begin{aligned} K &:= \varphi(\{0\} \times S^k), & N_K &:= \varphi(\overline{B}^{k+1} \times S^k), \\ J &:= \psi(\{0\} \times S^k), & N_J &:= \psi(\overline{B}^{k+1} \times S^k). \end{aligned}$$

We also set $X := \overline{B}^{2k+2} \cup_\psi h^{k+1}$ and $Y := \overline{B}^{2k+2} \cup_\varphi h^{k+1} \cup_\psi h^{k+1}$.
We now see that



$$\begin{array}{ccccccc}
 & \text{by Definition 259.3 of } \mathrm{lk}(\varphi,\psi) & & \text{by the Linking \# Ball Intersection \# Proposition 258.14} & & \text{by the Intersection Form Submanifold Proposition 225.9 applied to } M=X \text{ and } W=\overline{B}^{2k+2} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{lk}(\varphi,\psi) & = & \mathrm{lk}(K,J) & = & Q^{\mathrm{as}}_{\overline{B}^{2k+2},N_K,N_J}(\widetilde{\Theta}_K,\widetilde{\Theta}_J) & = & Q^{\mathrm{as}}_{X,\partial X,\varnothing}(\widetilde{\Theta}_K,C(\psi)) \\
 & = & Q^{\mathrm{as}}_{X,N_K,\varnothing}(\widetilde{\Theta}_K,C(\psi)) & = & Q^{\mathrm{as}}_{Y,\varnothing,\varnothing}(C(\varphi),C(\psi)) & = & Q_Y(C(\varphi),C(\psi)). \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{by the Asymmetric Intersection Form Lemma 225.5, and since } N_K\subseteq\partial X & & \text{by the Intersection Form Submanifold Proposition 225.9 applied to } M=Y \text{ and } W=X & & \text{by Definitions 224.12 and 225.3} &
\end{array}$$

■

Now we turn to the actual proof of the Middle Dimension Handles Proposition 259.4:

Proof of the Middle Dimension Handles Proposition 259.4. Let $k\in\mathbb{N}_0$. Furthermore let $\varphi_1,\ldots,\varphi_m\colon \overline{B}^{k+1}\times S^k\to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. We set

$$M \;:=\; \overline{B}^{2k+2}\cup_{\varphi_1} h^{k+1}\cup\cdots\cup_{\varphi_m} h^{k+1}.$$

(1) It follows from the Handle Attachment Proposition 246.3 and the Handles-Orientability Proposition 246.6 that M is a compact $(2k+2)$-dimensional smooth manifold that admits a unique orientation such that the natural embedding $\overline{B}^{2k+2}\to M$ is orientation-preserving.

(2) First note that it follows fairly easily from iteratively applying the Handle Attachment-π_1-Proposition 246.13 altogether m times that, under the hypothesis that $k\geq 1$, the manifold M is simply connected.

Next we turn to the calculation of homology groups:

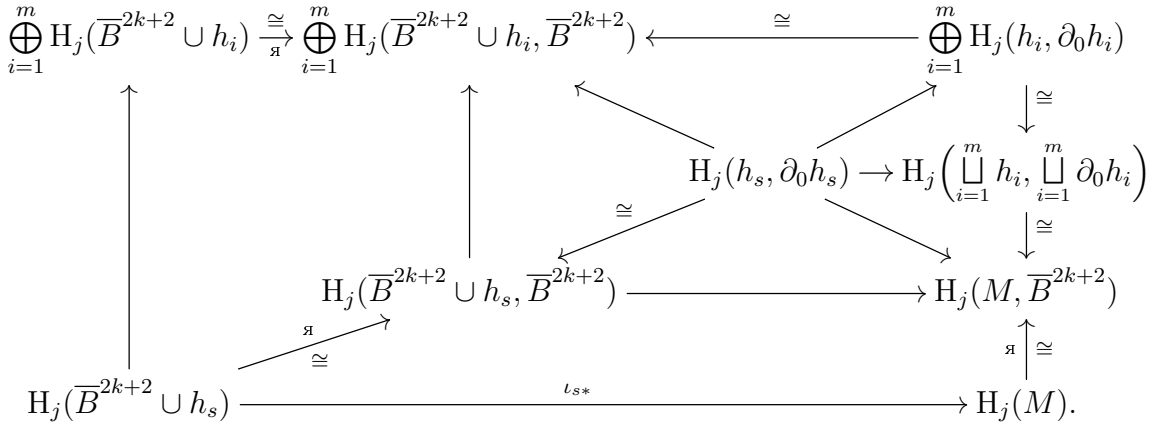
Claim. For $i\in\{1,\ldots,m\}$ we denote by $\iota_i\colon \overline{B}^{2k+2}\cup_{\varphi_i} h^{k+1}\to M$ the inclusion. For each $j\in\mathbb{N}$ the map

$$\bigoplus_{i=1}^m \iota_{i*}\colon \bigoplus_{i=1}^m \mathrm{H}_j(\overline{B}^{2k+2}\cup_{\varphi_i} h^{k+1}) \;\to\; \mathrm{H}_j(M)$$

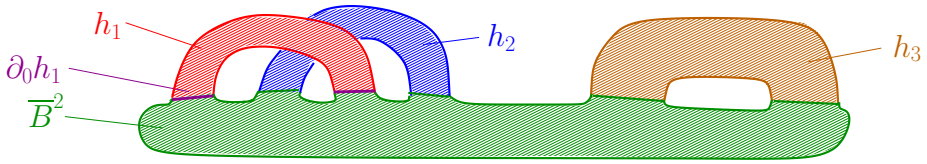
is an isomorphism.

Proof. Given $i\in\{1,\ldots,m\}$ we denote by $h_i\subseteq M$ the i-th handle and as in Notation 25.47 we write $\partial_0 h_i := h_i\cap\overline{B}^{2k+2}$. For each $s\in\{1,\ldots,m\}$ we consider the

following diagram:



Here all maps are the obvious maps induced by inclusions or going from absolute to relative homology.



It follows easily from the Reduced LES-of-Pair Proposition 147.8 (together with the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6), the Manifold–Excision Theorem 212.4 (which we can apply by the Handle Attachment Proposition 246.3 (3)) and the Path-Component-H_*-Lemma 143.21 that the top horizontal maps and the right vertical maps are all isomorphisms.

Furthermore one can easily verify that the diagram commutes. But this implies precisely that the map from the top left to the bottom right is the map that is induced by the inclusions $\iota_1, \dots, \iota_m$. ⊞

Finally note that it follows from the discussion in Definition 259.1, or the arguments in the proof of the claim, that $\widetilde{\mathrm{H}}_j(\overline{B}^{2k+2} \cup_{\varphi_i} h^{k+1}) = 0$ for $j \neq k+1$ and that $\mathrm{H}_{k+1}(\overline{B}^{2k+2} \cup_{\varphi_i} h^{k+1}) = \mathbb{Z} \cdot [C(\varphi_i)]$.

(3) We still need to calculate the intersection form of M. In the following two claims we first calculate the off-diagonal entries and afterwards we calculate the diagonal entries:

Claim 1. For any $i, j \in \{1, \dots, m\}$ with $i \neq j$ we have $Q_M(C(\varphi_i), C(\varphi_j)) = \mathrm{lk}(\varphi_i, \varphi_j)$.

Proof. We consider the smooth manifold $Y \coloneqq \overline{B}^{2k+2} \cup_{\varphi_i} h^{k+1} \cup_{\varphi_j} h^{k+1}$. We calculate

$$\begin{aligned} Q_M(C(\varphi_i), C(\varphi_j)) &= Q^{\mathrm{as}}_{M,\varnothing,\varnothing}(C(\varphi_i), C(\varphi_j)) && \text{follows from comparing Definitions 224.12 and 225.3} \\ &= Q^{\mathrm{as}}_{Y,\partial Y,\varnothing}(C(\varphi_i), C(\varphi_j)) && \text{by the Intersection Form Submanifold Proposition 225.9 applied to } M = M \text{ and } W = Y \\ &= Q^{\mathrm{as}}_{Y,\varnothing,\varnothing}(C(\varphi_i), C(\varphi_j)) && \text{by the Asymmetric Intersection Form–Natural Lemma 225.5} \\ &= \mathrm{lk}(\varphi_i, \varphi_j). && \text{by the Attach-two-Handles Lemma 259.6} \end{aligned}$$

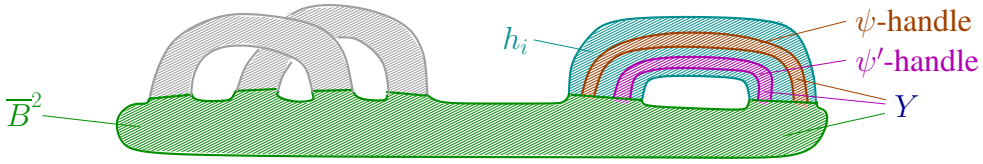
⊞

Claim 2. Given any $i \in \{1, \dots, m\}$ we have $Q_M(C(\varphi_i), C(\varphi_i)) = \mathrm{slk}(\varphi_i)$.

Proof. We pick $P \in B^{k+1} \setminus \{0\}$ and we pick an $\epsilon > 0$ such that $\overline{B}^{k+1}_\epsilon(0) \cap \overline{B}^{k+1}_\epsilon(P) = \varnothing$ and such that $\overline{B}^{k+1}_\epsilon(P) \subseteq \overline{B}^{k+1}$. We consider the thickened k-spheres

$$\begin{aligned} \psi\colon \overline{B}^{k+1} \times S^k &\xrightarrow{\cong} \overline{B}^{k+1}_\epsilon(0) \times S^k \xrightarrow{\varphi_i} S^{2k+1} \\ \psi'\colon \overline{B}^{k+1} \times S^k &\xrightarrow{\cong} \overline{B}^{k+1}_\epsilon(P) \times S^k \xrightarrow{\varphi_i} S^{2k+1} \end{aligned}$$

where in both cases the map to the left is the obvious diffeomorphism given by contracting and possibly translating. Next we set $Y \coloneqq \overline{B}^{2k+2} \cup_\psi h^{k+1} \cup_{\psi'} h^{k+1}$.



Note that we can view Y as a codimension-zero submanifold of $\overline{B}^{2k+2} \cup_{\varphi_i} h^{k+1}$, in particular we can view Y as a codimension-zero submanifold of M. We denote by $\iota\colon Y \to M$ the inclusion. Finally we perform the following calculation:

$$\begin{array}{lll} Q_M(C(\varphi_i), C(\varphi_i)) & = & Q^{\mathrm{as}}_{M,\varnothing,\varnothing}(C(\varphi_i), C(\varphi_i)) \\ & \uparrow & \\ & \text{by Definitions 224.12 and 225.3} & \end{array} \quad \begin{array}{lll} = & Q^{\mathrm{as}}_{M,\varnothing,\varnothing}(\iota_*(C(\psi)), \iota_*(C(\psi'))) \\ \uparrow & \\ \text{since the maps defining the homology classes are homotopic} & \end{array}$$

$$\begin{array}{lll} = & Q^{\mathrm{as}}_{Y,\partial Y,\varnothing}(C(\psi), C(\psi')) \\ \uparrow & \\ \text{by the Intersection Form Submanifold Proposition 225.9 applied to } M = M \text{ and } W = Y & \end{array} \quad \begin{array}{lll} = & Q^{\mathrm{as}}_{Y,\varnothing,\varnothing}(C(\psi), C(\psi')) \\ \uparrow & \\ \text{by the Asymmetric Intersection Form Natural Lemma 225.5} & \end{array}$$

$$\begin{array}{lll} = & \mathrm{lk}(\psi, \psi') \\ \uparrow & \\ \text{by the Attach-two-Handles Lemma 259.6} & \end{array} \quad \begin{array}{lll} = & \mathrm{slk}(\varphi_i). \\ \uparrow & \\ \text{by Definitions 259.2 and 259.3 and Footnote 2768} & \end{array}$$

⊞

Finally note, that essentially exactly the same arguments also apply to $\mathbb{F}_2$-coefficients, regardless of whether the thickened k-spheres are orientation-preserving or not. ■

259.2. **Thickenings of knots.** The Middle Dimension Handles Proposition 259.4 raises the question, what (symmetric) matrices can we realize as linking matrices of families of thickened spheres? By the Linking Number–Realization Proposition 258.11 we know that we can realize all "off-diagonal" entries. It remains the question, whether we can also realize all diagonal values, i.e. whether we can realize any integer as the self-linking numbers of a given knot.

In the following discussion it is convenient to introduce the following definition:

Definition 259.7. Let $K \subseteq S^n$ be an oriented k-dimensional knot in S^n. A thickening of K is an orientation-preserving tubular map for K, i.e. it is an orientation-preserving smooth embedding $F\colon \overline{B}^{n-k} \times K \to S^n$ such that $F(0, P) = P$ for all $P \in K$.

With this definition we can formulate the following proposition:

Proposition 259.8. (Knot Thickening Proposition) Let $k \in \mathbb{N}$ and let K be an oriented k-dimensional knot in S^{2k+1}.

(1) If $k = 1$ or $k = 3$ and if k is frameable, then given any $n \in \mathbb{Z}$ there exists a thickening of K with self-linking number n.
(2) If k is odd and if K is trivial, then given any even $n \in \mathbb{Z}$ there exists a thickening of K with self-linking number n.

Proof.

(1) The cases $k = 1$ and $k = 3$ are dealt with in almost the same way, except that for $k = 1$ we use the complex numbers $\mathbb{C}$ and for $k = 3$ we use the quaternions $\mathbb{H}$. In the following we will write down the argument for $k = 3$. We leave it to the reader to "dumb down" the argument to $k = 1$. Alternatively the case $k = 1$ will be dealt with in the Knot Thickening–Self Linking Lemma 284.11.

Now we assume that $k = 3$. Thus let $K \subseteq S^7$ be a 3-dimensional frameable knot and let $n \in \mathbb{Z}$. Since K is frameable there exists a thickening $\Theta\colon \overline{B}^4 \times K \to S^7$ of K of unknown self-linking number. The idea is to show that given any $n \in \mathbb{Z}$ there exists a suitable self-map $\overline{B}^4 \times K \to \overline{B}^4 \times K$ such that the resulting map $\overline{B}^4 \times K \to \overline{B}^4 \times K \to S^7$ is a thickening of self-linking number $\mathrm{slk}(\Theta) + n$.

We start out with the following claim:

Claim 1. The map

$$\begin{array}{rccl} \mu\colon & \overline{B}^4 \times S^3 & \to & \overline{B}^4 \\ & (x, y) & \mapsto & x \cdot_{\mathbb{H}} y^n \\ & & & \uparrow \\ & & & \text{multiplication in } \mathbb{H} \end{array}$$

has the following properties:

(a) The map μ is smooth and for each $y \in S^3$ the map $\overline{B}^4 \to \overline{B}^4$ given by $x \mapsto \mu(x, y)$ is an orientation-preserving diffeomorphism.
(b) The maps μ restricts to a map $S^3 \times S^3 \to S^3$ and this map has bidegree[2769] $(1, n)$.

Proof.

(a) It follows from the Quaternion Unit Proposition 171.5 (2) that multiplication by a quaternion in S^3 defines an orientation-preserving diffeomorphism of $\overline{B}^4$.
(b) Next note that by Exercise 151.17 we know that the map $S^3 \times S^3 \to S^3$ given by quaternion multiplication has bidegree $(1, 1)$. Furthermore by Exercise 214.7 or alternatively by Exercise 259.1 we know that the map $S^3 \to S^3$ given by $y \mapsto y^n$ has degree n. It follows from these observations and the multiplicativity of the bidegree, see the discussion on page 4135, that the map $S^3 \times S^3 \to S^3$ has bidegree $(1, n)$. ⊞

It remains to prove the following claim:

[2769] We recall Definition 216.18 of the bidegree: Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a continuous map. We pick $x_0, y_0 \in S^n$. We define the *bidegree* of f as

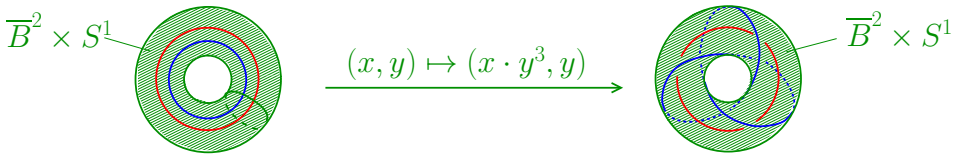
$$\mathrm{bideg}(f) := (p, q) \text{ where } p := \deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & f(x, y_0) \end{pmatrix} \text{ and } q := \deg\begin{pmatrix} S^n & \to & S^n \\ y & \mapsto & f(x_0, y) \end{pmatrix}.$$

Alternatively one can easily verify that the bidegree is precisely (m, n) if the induced homomorphism $f_*\colon \mathrm{H}_n(S^n \times S^n) \to \mathrm{H}_n(S^n)$ with respect to the bases $\{[S^n \times \{y_0\}], [\{x_0\} \times S^n]\}$ and $\{[S^n]\}$ is given by the matrix $(m \;\; n)$.

Claim 2. Let $\Theta\colon \overline{B}^4 \times K \to S^7$ be a thickening of the knot K. Let $n \in \mathbb{Z}$ and let $\mu\colon \overline{B}^4 \times S^3 \to \overline{B}^4$ be the map from Claim 1 such that the restriction to $S^k \times S^k \to S^k$ has bidegree $(1, n)$. We pick an orientation-preserving diffeomorphism $\gamma\colon S^3 \to K$. We consider the map

$$\begin{array}{rccccc} \Xi\colon \overline{B}^4 \times K & \xrightarrow{\operatorname{diag}(\mathrm{id},\gamma^{-1})} & \overline{B}^4 \times S^3 & \to & \overline{B}^4 \times S^3 & \xrightarrow{\operatorname{diag}(\mathrm{id},\gamma)} \overline{B}^4 \times K \\ & & (x,y) & \mapsto & (\underbrace{\mu(x,y)}_{=x\cdot_{\mathbb{H}} y^n}, y). & \end{array}$$

The map $\Theta \circ \Xi\colon \overline{B}^4 \times K \to S^7$ is a thickening of K with self-linking number $\operatorname{slk}(\Theta)+n$.



Proof. It follows easily from Properties (a) and (b) stated in Claim 1 that the map $\Theta \circ \Xi$ is indeed a thickening of K. Thus it remains to determine the self-linking number of $\Theta \circ \Xi$. We pick $P := (0, \dots, 0, 1) \in S^3$, $Q \in K$ and we write $\lambda_K := \Theta(\{P\} \times K)$ and $\mu_K := \Theta(S^3 \times \{Q\})$. Next we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_3(S^3 \times K) & \xrightarrow{\Xi_*} & \mathrm{H}_3(S^3 \times K) \\ {\scriptstyle \operatorname{diag}(\mathrm{id},\gamma)_*}\downarrow \cong & & \cong \downarrow {\scriptstyle \operatorname{diag}(\mathrm{id},\gamma)_*} \\ \mathrm{H}_3(S^3 \times S^3) & \xrightarrow{\substack{\text{induced by the map} \\ (x,y) \,\mapsto\, (\mu(x,y),y)}} & \mathrm{H}_3(S^3 \times S^3). \end{array}$$

We now calculate that

$$\begin{array}{rcl} \text{self-linking number of } \Theta \circ \Xi & \overset{\text{by Definition 259.2}}{=} & \Phi_K(\Theta_*(\Xi_*([\{P\} \times K]))) \\ & = & \Phi_K(\Theta_*([\lambda_K] + n \cdot [\mu_K])) \\ & \overset{\substack{\text{since } \Phi_K(\mu_K) = 1 \text{ and} \\ \text{by definition of } \operatorname{slk}(\Theta)}}{=} & \operatorname{slk}(\Theta) + n. \end{array}$$

(follows easily from Claim1 and the $(S^k \times S^l)$-H_*-Lemma 164.7)

(2) Now suppose that k is any odd number and let $n = 2m \in \mathbb{Z}$ be even. Furthermore let K be a k-dimensional *trivial* knot in S^{2k+1}.

Claim 0. There exists a thickening of self-linking number zero.

Proof. By the Isotopy Extension Theorem 44.2 it suffices to prove the statement for the standard trivial knot U. For the standard trivial knot U we gave in the proof of the Trivial Knot–Tubular Neighborhood Lemma 258.5 an explicit tubular map, which clearly has self-linking number zero.

Alternatively note that by the Sphere–Solid Tori-Decomposition Lemma 111.6 we have an explicit orientation-preserving diffeomorphism

$$\Theta\colon (\overline{B}^{k+1} \times S^k) \cup_{S^k \times S^k} (S^k \times \overline{B}^{k+1}) \to S^{2k+1}$$

which sends $\{0\} \times S^k$ to the standard trivial knot U in S^{2k+1}. The smooth embedding $\Theta\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ is a thickening of U. Note that with this thickening we have

the exterior $X_U = \Theta(S^k \times \overline{B}^{k+1})$. We now see that

$$\begin{array}{ccccc} \operatorname{slk}(\Theta) & = & \operatorname{lk}(U, \Theta(\{P\} \times S^k) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{by Definition 259.2,} & & \text{since } \Theta(\{P\} \times S^k) = \partial\Theta(\{P\} \times \overline{B}^{k+1}) \text{ which} & \\ & \text{for some } P \in S^k & & \text{implies that } \Theta_*([\{P\} \times S^k]) = 0 \in \mathrm{H}_k(X_U) & \end{array}$$

⊞

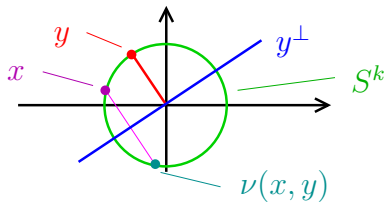
The following claim is the analog of Claim 1 in Statement (1):

Claim 1. There exists a continuous map $\mu\colon \overline{B}^{k+1} \times S^k \to \overline{B}^{k+1}$ that has the following properties:
(a) The map μ is smooth and for each $y \in S^k$ the map $\overline{B}^{k+1} \to \overline{B}^{k+1}$ given by $x \mapsto \mu(x, y)$ is an orientation-preserving diffeomorphism.
(b) The maps μ restricts to a map $S^k \times S^k \to S^k$ and this map has bidegree $(1, 2)$.
Similarly we can find such μ such that the restriction to a map $S^k \times S^k \to S^k$ has bidegree $(1, -2)$.

Proof. Rather luckily we already encountered a map with very similar properties in the Bidegree–Realization Proposition 216.22. More precisely we consider the following map:

$$\begin{array}{rccl} \nu\colon \overline{B}^{k+1} \times S^k & \to & S^k & \\ (x, y) & \mapsto & x - 2 \cdot \langle x, y \rangle_{\mathbb{R}^n} \cdot y & = \text{ reflection of } x \text{ in the hyperplane } y^\perp. \end{array}$$

Recall that in the Bidegree–Realization Proposition 216.22 we showed that the restriction $\nu\colon S^k \times S^k \to S^k$ has bidegree $(-1, 2)$.



Now let $\rho\colon \overline{B}^{k+1} \to \overline{B}^{k+1}$ be the reflection in a fixed hyperplane. It follows easily from the multiplicativity of the bidegree, see page 4135, the Sphere–Degree Lemma 151.4 (4) and the Sphere–Degree–Matrix Lemma 151.7 that the maps

$$\begin{array}{rccl} \overline{B}^{k+1} \times S^k & \to & S^k & \\ (x, y) & \mapsto & \nu(\rho(x), y) & \end{array} \quad \text{and} \quad \begin{array}{rccl} \overline{B}^{k+1} \times S^k & \to & S^k & \\ (x, y) & \mapsto & \rho(\nu(x, y)) & \end{array}$$

have all the desired properties, where the bidegree for the map to the left is given by $(1, 2)$ and the bidegree for the map to the right is given by $(1, -2)$. ⊞

By Claim 0 it remains to prove the following claim:

Claim 2. Let $\Theta\colon \overline{B}^{k+1} \times K \to S^{2k+1}$ be a thickening of K. Let $\mu\colon \overline{B}^{k+1} \times S^k \to \overline{B}^{k+1}$ be a map as in Claim 1 such that the restriction to $S^k \times S^k \to S^k$ has bidegree $(1, \pm 2)$. We pick an orientation-preserving diffeomorphism $\gamma\colon S^k \to K$. The map

$$\begin{array}{ccccccccc} \overline{B}^{k+1} \times K & \xrightarrow{\operatorname{diag}(\mathrm{id}, \gamma^{-1})} & \overline{B}^{k+1} \times S^k & \to & \overline{B}^{k+1} \times S^k & \xrightarrow{\operatorname{diag}(\mathrm{id}, \gamma)} & \overline{B}^{k+1} \times K & \xrightarrow{\Theta} & S^{2k+1} \\ & & (x, y) & \mapsto & (\mu(x, y), y) & & & & \end{array}$$

is a thickening of K with self-linking number $\operatorname{slk}(\Theta) \pm 2$.

Proof. The proof of the claim is essentially the same as the proof of Claim 2 in Statement (1). ■

259.3. **Realization results for intersection forms.** The reader will be aware that the Knot Thickening Proposition 259.8 together with the Linking Number–Realization Proposition 258.11 now gives us finally a realization result for a "total" linking matrix:

Proposition 259.9. (Linking Matrix-Realization Proposition) Let $k \in \mathbb{N}$ be odd and let $A \in \mathrm{M}(m \times m, \mathbb{Z})$ be a symmetric matrix. If $k \geq 5$, then we assume that every diagonal entry of the matrix A is even. There exist orientation-preserving thickened k-spheres $\varphi_1, \ldots, \varphi_m \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ with disjoint images such that the following two conditions are satisfied:

(1) The knots $\varphi_i(\{0\} \times S^k)$ are trivial.
(2) The linking matrix $\mathrm{lk}(\varphi_1, \ldots, \varphi_m)$ equals A.

Proof. Let $k \in \mathbb{N}$ be odd and let $A \in \mathrm{M}(m \times m, \mathbb{Z})$ be a symmetric matrix. If $k \geq 5$, then we assume that every diagonal entry is even. Since A is symmetric it follows from the Linking Number–Realization Proposition 258.11 that there exists an oriented m-component k-dimensional link $L_1 \sqcup \cdots \sqcup L_m$ in S^{2k+1} such that the following two conditions are satisfied:

(1) The knots $L_1, \ldots, L_m$ are trivial.
(2) For any $i \neq j \in \{1, \ldots, m\}$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$.

Since the knots L_i are all trivial, and since we assume that the diagonal entries are even if $k \geq 5$, we obtain from the Knot Thickening Proposition 259.8 that we can provide each L_i with a thickening with the self-linking number a_{ii}. After possibly doing some radial shrinking we can assume that the thickenings are disjoint. ■

By now we get the following corollary almost for free:

Corollary 259.10. (Intersection Form-Realization Corollary) Let $k \in \mathbb{N}$ and let $A \in \mathrm{M}(m \times m, \mathbb{Z})$ be a symmetric matrix. If $k \geq 2$, then we assume that every diagonal entry is even. There exists a compact oriented simply connected $4k$-dimensional smooth manifold M such that the intersection form of M is isometric to the form

$$\begin{array}{rcl} \mathbb{Z}^m \times \mathbb{Z}^m & \to & \mathbb{Z} \\ (v, w) & \mapsto & v^{\mathrm{T}} \cdot A \cdot w. \end{array}$$

Proof. The corollary is an immediate consequence of the Linking Matrix-Realization Proposition 259.9 and the Middle Dimension Handles Proposition 259.4. ■

The Intersection Form-Realization Corollary 259.10 implies in particular that every symmetric form over $\mathbb{Z}$ is isometric to the intersection form of a compact oriented simply connected 4-dimensional smooth manifold. The well-versed reader might remember that the Rokhlin Theorem 228.8, the Donaldson Diagonalization Theorem 228.18 and Furuta's 10/8–Theorem 228.19 give us restrictions on the intersection forms of closed oriented simply connected 4-dimensional smooth manifolds. This motivates the study of the boundaries of the smooth manifolds that we just constructed. For example one can ask, whether boundaries are spheres, so that we can cap them off with balls.

We will study the boundaries in the next section.

259.4. **The boundary of handle attachments.** As we will see, the boundaries of the smooth manifolds that we constructed in the Middle Dimension Handles Proposition 259.4 are particularly interesting. So let us summarize what we can say about the boundaries at this moment:

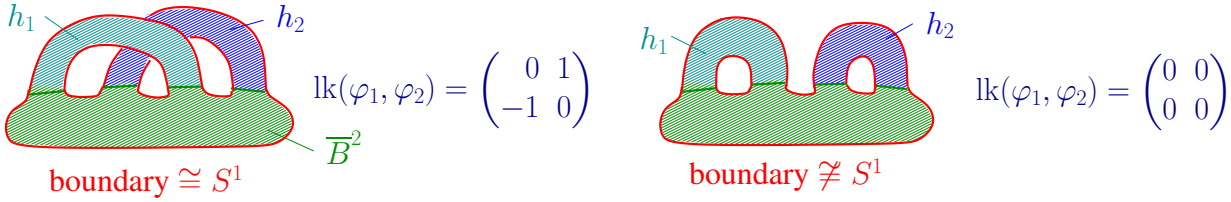
Proposition 259.11. (Boundary Invariants Proposition) Let $k \in \mathbb{N}_0$. Furthermore let $\varphi_1, \dots, \varphi_m \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. We set

$$M := \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1}.$$

(1) The boundary ∂M is a closed oriented $(2k+1)$-dimensional smooth manifold.
(2) (a) We have $\mathrm{H}_i(\partial M) = 0$ for $i \neq 0, k, k+1, 2k+1$.
(b) There exists an exact sequence of the following form

$$0 \to \mathrm{H}_{k+1}(\partial M) \to \mathbb{Z}^m \xrightarrow{v \mapsto \mathrm{lk}(\varphi_1,\dots,\varphi_m)\cdot v} \mathbb{Z}^m \to \mathrm{H}_k(\partial M) \to 0$$

where $\mathrm{lk}(\varphi_1, \dots, \varphi_m)$ is the linking matrix from Definition 259.5.
(3) If $k \geq 2$, then ∂M is simply connected.
(4) We now assume that the linking matrix $\mathrm{lk}(\varphi_1, \dots, \varphi_m)$ is invertible.
(a) The boundary ∂M is a $(2k+1)$-dimensional homology sphere.[2770]
(b) If $k \geq 2$, then the boundary ∂M is a $(2k+1)$-dimensional homotopy sphere[2771].



For the record less us point out that the Boundary Invariants Proposition 259.11 (4) raises the following very interesting question:

Question 259.12. Let $k \in \mathbb{N}_{\geq 2}$ be odd and let $\varphi_1, \dots, \varphi_m \colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images such that the linking matrix $\mathrm{lk}(\varphi_1, \dots, \varphi_m) \in \mathrm{M}(m \times m, \mathbb{Z})$ is invertible.
(1) Is $\partial(\overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1})$ necessarily homeomorphic to S^{2k+1}?
(2) Is $\partial(\overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \cdots \cup_{\varphi_m} h^{k+1})$ necessarily diffeomorphic to S^{2k+1}?

We will answer Question 259.12 (1) on page 5105 and we will answer Question 259.12 (2) in the Exotic 7-Sphere Theorem 282.6.

[2770] Recall that in Definition 183.22 we said that an n-dimensional topological manifold X is a *topological homology sphere* if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$.
[2770] By the Homotopy Sphere-Characterization Proposition 210.18 there are several equivalent definitions of a homotopy sphere. Currently we use the following definition: A closed connected non-empty n-dimensional topological manifold X is called a *homotopy sphere* if $\pi_1(X)$ is trivial and if $\mathrm{H}_i(X) = 0$ for $i = 2, \dots, n-1$. Later, in the Homotopy Sphere-Characterization Proposition **??** we will see that a closed connected n-dimensional topological manifold is a homotopy sphere if and only if it is homotopy equivalent to S^n.

The proof of the Boundary Invariants Proposition 259.11 will require the remainder of this section. In the next Section 259.5 we will consider examples of 4-dimensional homology spheres.

Almost all of the hard work in the proof of the Boundary Invariants Proposition 259.11 is performed in the following lemma:

Lemma 259.13. (Boundary-H_*-Lemma) Let $n \in \mathbb{N}$ and let M be a compact oriented $2n$-dimensional topological manifold such that $H_n(M)$ is a free abelian group, and such that all other reduced homology groups $\widetilde{H}_j(M)$, $j \neq n$, vanish.

(1) The boundary ∂M is a closed oriented $(2n-1)$-dimensional topological manifold.
(2) (a) We have $H_i(\partial M) = 0$ for $i \neq 0, n-1, n, 2n-1$.
(b) Let $v_1, \dots, v_m$ be a basis of $H_n(M)$. There exists an exact sequence of the following form

$$0 \to H_n(\partial M) \to \mathbb{Z}^m \xrightarrow{v \mapsto Q_M(v_i, v_j)\cdot v} \mathbb{Z}^m \to H_{n-1}(\partial M) \to 0.$$

Example. It follows easily from the Boundary-H_*-Lemma 259.13, together with the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 95.16, the Hurewicz Theorem 166.10 and the Manifold-Top-H_*-Theorem 193.1 that the boundary of a $2n$-dimensional homology ball is an $(2n-1)$-dimensional homology sphere. □

Proof of the Boundary-H_*-Lemma 259.13. Let $n \in \mathbb{N}$ and let M be a compact oriented $2n$-dimensional topological manifold such that $H_n(M) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}_0$ and such that all other reduced homology groups $\widetilde{H}_j(M)$ vanish.

(1) It follows immediately from the Boundary Orientation Proposition 192.3 that ∂M is a closed oriented $(2n-1)$-dimensional topological manifold.
(2) We will calculate the homology groups of the boundary by "squeezing" them into the long exact sequence of the pair $(M, \partial M)$, which allows us to express the homology groups of ∂M in terms of the homology groups of M and $(M, \partial M)$.

We are given the homology groups of M. To get to the homology groups of ∂M we make use of the Poincaré Duality Theorem 210.1 which says in our setting that for each $j \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} H^j(M;\mathbb{Z}) & \to & H_{2n-j}(M, \partial M) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. As in Definition 210.2 we denote the inverse of this homomorphism by $PD_M \colon H_j(M, \partial M) \to H^j(M;\mathbb{Z})$.

Note that for any $i \in \mathbb{N}_0$ we have

$$\begin{array}{ccccc} H_i(M, \partial M) & \cong & H^{2n-i}(M;\mathbb{Z}) & \cong & \begin{array}{c}\mathrm{Hom}(H_{2n-i}(M), \mathbb{Z}) \\ \oplus \\ \mathrm{Ext}(H_{2n-i-1}(M), \mathbb{Z})\end{array} \\ \uparrow & & \uparrow & & \uparrow \\ \text{by the above Poincaré Duality Theorem 210.1} & & \text{by the Universal Coefficient } H^*\text{-Theorem 201.23} & & \text{by the hypotheses on the groups } H_*(M) \text{ together with the Ext-Properties Lemma 201.19 (4)} \end{array} \cong \begin{cases} \mathbb{Z}, & \text{if } i = 2n, \\ \mathbb{Z}^m, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

Now we consider the long exact sequence

$$\cdots \to H_{j+1}(M) \xrightarrow{\text{я}} \underbrace{H_{j+1}(M, \partial M)}_{=0 \text{ for } j+1 \neq n, 2n} \xrightarrow{\partial_j} H_j(\partial M) \to \underbrace{H_j(M)}_{=0 \text{ for } j \neq 0, n} \to \dots$$

in homology of the pair $(M, \partial M)$. It follows immediately from the above calculations of the homology groups $\mathrm{H}_*(M)$ and $\mathrm{H}_*(M, \partial M)$ that $\mathrm{H}_j(\partial M) = 0$ for $i \neq 0, n-1, n, 2n-1$. This concludes the proof of (2a).

It remains to consider the middle part of the long exact sequence of the pair $(M, \partial M)$:

$$\underbrace{\mathrm{H}_{n+1}(M, \partial M)}_{=0} \to \mathrm{H}_n(\partial M) \to \mathrm{H}_n(M) \xrightarrow{\text{я}} \mathrm{H}_n(M, \partial M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M).$$

In fact by the above discussion it remains to prove the following claim which can be viewed as the Asymmetric Intersection Pairing–Matrix Proposition 225.6 (2) in disguise:

Claim. Let $v_1, \dots, v_m$ be a basis of the free abelian group $\mathrm{H}_n(M)$.
- We denote by $v_1^*, \dots, v_m^*$ the dual basis of the free abelian group $\mathrm{Hom}(\mathrm{H}_n(M), \mathbb{Z})$.
- By the Universal Coefficient Duality Theorem 210.1 the map
$$\begin{array}{ccc} \mathrm{H}^n(M; \mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{H}_n(M), \mathbb{Z}) \\ \varphi & \mapsto & (\sigma \mapsto \langle \varphi, \sigma \rangle) \end{array}$$
is an isomorphism. By a slight abuse of notation we denote by $v_1^*, \dots, v_m^*$ also the corresponding cohomology classes $\mathrm{H}^n(M; \mathbb{Z})$.
- By the Poincaré Duality Theorem 210.1 we know that $v_1^* \cap [M], \dots, v_m^* \cap [M]$ is a basis of the free abelian group $\mathrm{H}_n(M, \partial M)$.

With respect to these bases the natural homomorphism

$$\underbrace{\mathrm{H}_n(M)}_{\text{basis } v_1, \dots, v_m} \xrightarrow{\text{я}} \underbrace{\mathrm{H}_n(M, \partial M)}_{\text{basis } v_1^* \cap [M], \dots, v_m^* \cap [M]}$$

is represented by the matrix $Q_M(v_i, v_j)$.

Proof. Let $j \in \{1, \dots, m\}$. We write $\text{я}(v_j) = \sum_{i=1}^m b_{ij} \cdot (v_i^* \cap [M]) \in \mathrm{H}_n(M, \partial M)$. Given any $i \in \{1, \dots, m\}$ we now see that

$$\begin{array}{rcccl} b_{ij} & = & \sum_{r=1}^m b_{rj} \cdot \overbrace{\langle v_r^*, v_i \rangle}^{=\delta_{ri}} & & \\ & = & \langle \mathrm{PD}(\text{я}(v_j)), (\mathrm{PD}(v_i) \cap [M]) \rangle & = & \langle \mathrm{PD}(v_i) \cup \mathrm{PD}(\text{я}(v_j)), [M] \rangle \\ & & \uparrow & & \uparrow \\ & & \text{Definition 210.2} & & \text{follows from the Cap Product-Kronecker} \\ & & & & \text{Lemma 206.12 and the Cup-Cap Lemma 206.15} \\ & = & Q_M(v_i, v_j). & & \\ & & \uparrow & & \\ & & \text{by the Asymmetric Intersection} & & \\ & & \text{Form Natural Lemma 225.5} & & \end{array}$$

$$\begin{array}{rcl} & \text{bilinearity of } \langle -, - \rangle & \text{follows from } \text{я}(v_j) = \sum_{i=1}^m b_{ij} \cdot (v_i^* \cap [M]) \\ & \downarrow & \text{and the definition of } \mathrm{PD}_M \\ & \left\langle \sum_{r=1}^m b_{rj} \cdot v_r^*, v_i \right\rangle & \downarrow \\ = & & \langle \mathrm{PD}(\text{я}(v_j)), v_i \rangle \\ = & & Q^{\mathrm{as}}_{M, \varnothing, \partial M}(v_i, \text{я}(v_j)) \\ & & \uparrow \\ & & \text{by Definition 225.3} \end{array}$$

It follows from Definition 2.11 and this calculation that the matrix $\{Q_M(v_i, v_j)\}_{i,j=1,\dots,m}$ represents the homomorphism $\text{я}\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, \partial M)$. ∎

Proof of the Boundary Invariants Proposition 259.11. Let $k \in \mathbb{N}_0$. Furthermore let $\varphi_1, \dots, \varphi_m\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images. We set

$$M := \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \dots \cup_{\varphi_m} h^{k+1}.$$

(1) It follows immediately from the Middle Dimension Handles Proposition 259.4 (1) together with the Boundary Orientation Lemma 31.38 that ∂M is a closed oriented $(2k+1)$-dimensional smooth manifold.

(2) One could try to calculate the homology groups of ∂M "by hand", by investigating directly, using the Handle Attachment Proposition 246.3 (2), the effect of a handle attachment on the homology groups of the boundary. But around the middle dimension this calculation will turn out to be quite tricky.

We will thus use the indirect approach to calculating the homology groups of ∂M that we pioneered in the Boundary-H_*-Lemma 259.13.

First recall that by the Middle Dimension Handles Proposition 259.4 we know that $\mathrm{H}_i(M) = 0$ for $i \neq 0, k+1$ and we know that $\mathrm{H}_{k+1}(M) \cong \mathbb{Z}^m$ where a basis is given by the homology classes $C(\varphi_1), \dots, C(\varphi_m)$.

We thus see that the hypothesis of the Boundary-H_*-Lemma 259.13 are satisfied. We thus get all the desired statements directly from the Boundary-H_*-Lemma 259.13 together with the observation that in the Middle Dimension Handles Proposition 259.4 we showed that the matrix $\{Q_M(C(\varphi_i), C(\varphi_j))\}_{i,j=1,\dots,m}$ is precisely the linking matrix $\mathrm{lk}(\varphi_1, \dots, \varphi_m)$.

(3) We need to prove the following claim:

Claim. If $k \geq 2$, then $\pi_1(\partial M)$ is trivial.

Proof. We now assume that $k \in \mathbb{N}_{\geq 2}$. The smooth manifold

$$M \coloneqq \overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \dots \cup_{\varphi_m} h^{k+1}$$

has an obvious handle decomposition consisting of one 0-handle and m $(k+1)$-handles. We consider the corresponding dual handle decomposition rel ∂M provided by the Dual Handle Proposition 248.1. Recall that the dual of a j-handle is a handle of "opposite" index $(2k+2) - j$. Thus we see that the dual handle decomposition is of the form

$$(*) \qquad M \cong ([0,1] \times \partial M) \cup_{\psi_m} h^{k+1} \cdots \cup_{\psi_1} h^{k+1} \cup h^{2k+2}.$$

This shows that we obtain M from $[0,1] \times \partial M$ by attaching m handles of index $k+1$ and one handle of index $2k+2$. If $k \geq 2$, then we see that[2771]

$$\begin{array}{ccccc} \pi_1(\partial M) & \cong & \pi_1([0,1]\times\partial M) & \overset{\text{by the Handle Attachment } \pi_1\text{-Proposition 246.13 since } k+1\geq 3}{\cong} & \pi_1(([0,1]\times\partial M) \cup_{\psi_m} h^{k+1} \cdots \cup_{\psi_1} h^{k+1} \cup h^{2k+2}) \\ & \cong & \pi_1(M) & \cong & \pi_1(\overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \dots \cup_{\varphi_m} h^{k+1}) \cong \pi_1(\overline{B}^{2k+2}) = \{e\}. \\ & & \uparrow & & \uparrow \\ & & \text{by } (*) & & \text{by definition of } M \end{array}$$

by the Handle Attachment-π_1 Proposition 246.13, since $k+1 \geq 2$ ⊞

(4) We now suppose that the linking matrix $\mathrm{lk}(\varphi_1, \dots, \varphi_m)$ is invertible. It follows from (3) that ∂M is a $(2k+1)$-dimensional homology sphere. If furthermore $k \geq 2$, then it follows from (2) and (3) and the definition in Footnote 2770 that ∂M is a $(2k+1)$-dimensional homotopy sphere. ■

[2771] This calculation gives the solution to Exercise 248.6.

259.5. **Examples of 4-dimensional homology spheres.** In this section we will recall the definition of the E_8-matrix and we will see that the E_8-matrix gives rise to interesting smooth manifolds.

Notation 259.14. We set

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

In the following lemma we summarize a few basic properties of the E_8-matrix and the corresponding form:

Lemma 259.15. (E_8-Lemma) The matrix E_8 has determinant one, all eigenvalues are positive and the diagonal entries are even. In other words, the form

$$\begin{array}{rcl} \mathbb{Z}^8 \times \mathbb{Z}^8 & \to & \mathbb{Z} \\ (v, w) & \mapsto & v^{\mathrm{T}} \cdot E_8 \cdot w \end{array}$$

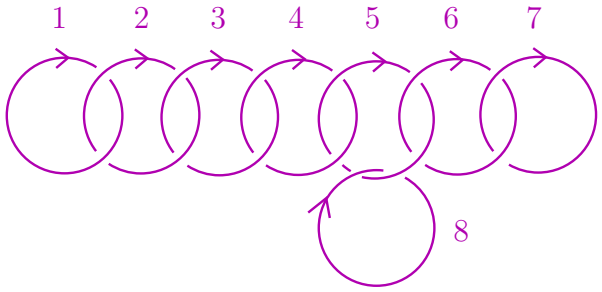
that is defined by the E_8-matrix is non-singular, positive-definite and even.

Proof. This lemma follows from the earlier E_8-Lemma 226.23, the Signature-Eigenvalues Proposition 226.11, the Form–Matrix-Lemma 224.5 and the Parity Lemma 226.25. ■

Definition 259.16. In the figure below this definition we show an oriented 8-component link $L = L_1 \sqcup \cdots \sqcup L_8 \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ where each component is unknotted and which has the property that the linking numbers $\mathrm{lk}(L_i, L_j)$ with $i \neq j$ are precisely the off-diagonal entries of E_8.[2772] By the Knot Thickening Proposition 259.8 we can pick orientation-preserving thickened 1-spheres $\varphi_1, \ldots, \varphi_8 \colon \overline{B}^2 \times S^1 \to S^3$ with disjoint images such that for each $i \in \{1, \ldots, 8\}$ we have $\varphi_i(\{0\} \times S^1) = L_i$ and such that the self-linking number of φ_i equals 2.[2773] By construction we have $\mathrm{lk}(\varphi_1, \ldots, \varphi_8) = E_8$. We set

$$M_4(E_8) := \overline{B}^4 \cup_{\varphi_1} h^2 \cup \cdots \cup_{\varphi_8} h^2.$$

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$



[2772] Using the Tubular Map–Uniqueness Theorem 58.5 and the Smooth Linear Bundle–Action-on-Trivializations Proposition 243.4 one can show that the φ_i are essentially unique. We will not make use of this observation.

[2773] That the linking matrix is E_8 follows either directly from the definition of linking numbers or one can deduce it from the Linking Number-via-Crossings Proposition 114.9.

By the E_8-Lemma 259.15 we know that $\det(E_8) = 1$. It thus follows from the Boundary Invariants Proposition 259.11 (4) that $\partial M_4(E_8)$ is a 3-dimensional homology sphere. Rather amazingly it is a homology sphere that we had already encountered before:

Proposition 259.17. (Poincaré-$\mathbb{Z}\mathbf{H}S^3$–E_8-Manifold Proposition) The 3-dimensional smooth manifold $\partial M_4(E_8)$ is diffeomorphic to the Poincaré homology sphere that we first encountered on in Definition 183.27, where we defined it is a quotient of S^3 by a group of order 120, namely the binary icosahedral group that we introduced in Definition 183.25.

Proof. Unfortunately we do not have the time to prove the proposition. We refer instead to [**KSc1979**] for a proof. In fact [**KSc1979**] contains many other ways how to think about the Poincaré Homology Sphere. ■

It follows from the argument of the Boundary-H_*-Lemma 259.13 that the boundary of a $\mathbb{Z}$-homology ball is a $\mathbb{Z}$-homology sphere. One can ask whether the converse holds: Is every $\mathbb{Z}$-homology sphere the boundary of a $\mathbb{Z}$-homology ball? In particular one can ask, whether the Poincaré Homology Sphere is the boundary of a 4-dimensional $\mathbb{Z}$-homology ball. It turns out, that the answer to this question depends on whether one considers smooth or topological manifolds. We consider these two types of manifolds in the following two subsections.

259.5.1. **The Poincaré Homology Sphere and smooth homology balls.** The following theorem answers the question whether the Poincaré Homology Sphere is the boundary of a 4-dimensional smooth $\mathbb{Z}$-homology ball:

Theorem 259.18. (Poincaré-$\mathbb{Z}\mathbf{H}S^3$–Nonbounding Theorem) The Poincaré Homology Sphere is not the boundary of a 4-dimensional smooth $\mathbb{Z}$-homology ball.

The key ingredient in the proof of the Poincaré-$\mathbb{Z}\mathrm{H}S^3$–Nonbounding Theorem 259.18 is Rokhlin's Theorem which we recall for the readers convenience:

Theorem 228.7. (Rokhlin Theorem) Let Z be a closed oriented 4-dimensional smooth manifold. If Z is spin, then[2774] $\mathrm{sign}(Z)$ is divisible by 16.

We also recall the following (partial) characterization of spin manifolds in dimension four:

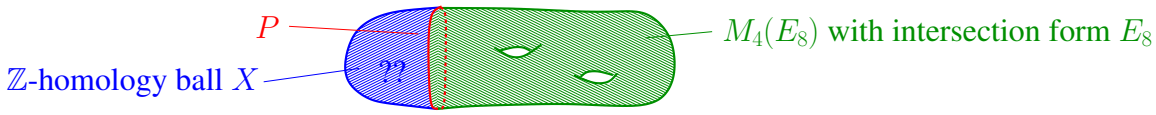
Proposition 228.6. (Spin–Intersection Form Proposition) Let Z be a compact oriented 4-dimensional smooth manifold.

(1) If Z is spin, then the intersection form Q_Z is even.
(2) If the intersection form Q_Z is even and if $\mathrm{H}_1(Z)$ has no 2-torsion, then the smooth manifold Z is spin.

Proof of the Poincaré-$\mathbb{Z}\mathbf{H}S^3$–Nonbounding Theorem 259.18. Let P be the Poincaré Homology Sphere. By the Poincaré Homology Sphere–E_8-Manifold Proposition 259.17 we know that P is the boundary of the compact oriented 4-dimensional smooth manifold $M := M_4(E_8)$ which satisfies $Q_M \cong E_8$.

Now assume that P was the boundary of a 4-dimensional smooth $\mathbb{Z}$-homology ball X. By the Gluing-Smooth Manifolds-Proposition 54.5 we can form the closed orientable 4-dimensional smooth manifold $Z := X \cup_P M$, which we can give an orientation such that the natural embedding $M \to Z = X \cup_P M$ is orientation-preserving.

[2774] In Definition 226.15 we defined the signature of Z as the signature of its intersection form.



We summarize a few properties of $Z = X \cup_P M$.

(1) By the Mayer–Vietoris Theorem 153.2 (3) for reduced homology groups we have the following exact sequence:

$$\dots \to \underbrace{\mathrm{H}_1(X)}_{=0} \oplus \underbrace{\mathrm{H}_1(M)}_{=0} \to \mathrm{H}_1(X \cup_P M) \xrightarrow{\partial_1} \underbrace{\widetilde{\mathrm{H}}_0(P)}_{=0} \to \dots$$

It follows that $\mathrm{H}_1(Z) = \mathrm{H}_1(X \cup_P M) \cong 0$.

(2) It follows again from the Mayer–Vietoris Theorem 153.2 (3) that we have the following exact sequence:

$$\dots \to \underbrace{\mathrm{H}_2(P)}_{=0} \to \underbrace{\mathrm{H}_2(X)}_{=0} \oplus \underbrace{\mathrm{H}_2(M)}_{\cong \mathbb{Z}^8} \to \mathrm{H}_2(X \cup_P M) \xrightarrow{\partial_1} \underbrace{\mathrm{H}_1(P)}_{=0} \to \dots$$

It follows that the inclusion induced map $\mathrm{H}_2(M) \to \mathrm{H}_2(Z)$ is an isomorphism. Exactly the same argument as in the proof of the Intersection Form-Additivity Proposition 224.18 shows that the isomorphism $\mathrm{H}_2(M) \to \mathrm{H}_2(Z)$ induces an isomorphism of intersection forms. It follows that the intersection form of $Z = X \cup_P M$ is represented by the matrix E_8.

(3) It follows from (2) and the E_8-Lemma 259.15 that the intersection form of $Z = X \cup_P M$ is even and that its signature equals 8.

(4) It follows from (1) and (3), together with the Spin–Intersection Form Proposition 228.6, that $Z = X \cup_P M$ is spin.

In summary we see that $X \cup_P M$ would be a closed oriented spin 4-dimensional smooth manifold with intersection form E_8. But that violates the Rokhlin Theorem 228.7. ■

259.5.2. **The Poincaré Homology Sphere and topological homology balls.** The following theorem shows that in the topological setting the situation is dramatically different from the smooth setting. Not only does the Poincaré Homology Sphere P bound a topological manifold that is a homology ball, it even bounds a *contractible* topological manifold:

Theorem 259.19. (Topological-$\mathbb{Z}\mathrm{H}S^3$–Bounding Theorem) Every 3-dimensional integral homology sphere bounds a *contractible* 4-dimensional topological manifold.

Proof. This theorem was proved by Michael Freedman [**Free1982**, Theorem 1.40], see also [**FQ1990**, Corollary 9.3C]. The methods that Michael Freedman used in the proof of the Topological-$\mathbb{Z}\mathrm{H}S^3$–Bounding Theorem 259.19 are closely related to the methods in Freedman's proof of the topological Poincaré Conjecture in dimension 4, see the Topological 4D-Poincaré Theorem 228.14. ■

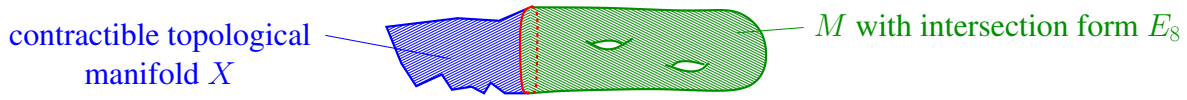
Remark. The Topological-$\mathbb{Z}\mathrm{H}S^3$–Bounding Theorem 259.19 gives us the slightly surprising statement that the boundary of a contractible 4-dimensional topological manifold is not necessarily homeomorphic to S^3. In Appendix 260.6 we will show in the Mazur Manifold Proposition 260.18 that there exists a 3-dimensional $\mathbb{Z}$-homology sphere, namely the Mazur

manifold M, with non-trivial fundamental group but which actually is the boundary of a compact contractible 4-dimensional smooth manifold.[2775] □

Using the Topological-$\mathbb{Z}$HS^3–Bounding Theorem 259.19 we can prove the following proposition which gives us some of the statements of the Freedman Classification Theorem 228.12.

Proposition 259.20. (Intersection Form-Realization Proposition) Given any symmetric non-singular form $\lambda\colon H \times H \to \mathbb{Z}$ over a free abelian group H there exists a closed oriented simply connected topological manifold such that the intersection form of Z is isometric to λ.

Proof. Let $\lambda\colon H \times H \to \mathbb{Z}$ be a symmetric non-singular over a free abelian group H of rank m. We pick a basis $v_1, \dots, v_m$ of H and we consider the corresponding matrix $A = (\lambda(v_i, v_j))_{i,j=1,\dots,m}$. Since λ is symmetric and non-singular we obtain from the Form–Matrix-Lemma 224.5 that A is symmetric and that $\det(A) \pm 1$. It follows from the Linking Matrix-Realization Proposition 259.9 that there exists a compact oriented simply connected 4-dimensional smooth manifold such that Q_M is represented by A. Since $\det(A) = \pm 1$ we obtain from the Boundary Invariants Proposition 259.11 (4) that ∂M is a $\mathbb{Z}$-homology sphere. By the Topological-$\mathbb{Z}$HS^3–Bounding Theorem 259.19 we know that there exists a contractible 4-dimensional topological manifold X with $\partial X = \partial M$.



As in the proof of the Poincaré-$\mathbb{Z}$HS^3–Nonbounding Theorem 259.18 one can easily verify that $Z := X \cup_{\partial X = \partial M} M$ is a closed oriented simply connected topological manifold such that the intersection form of Z is represented by A. ■

259.6. **Examples of high-dimensional homotopy spheres.** Finally let us consider high-dimensional analogs of the 4-dimensional smooth manifold $M_4(E_8)$ that we constructed in Definition 259.16:

Definition 259.21. Let $k \in \mathbb{N}_{\geq 2}$. We consider the matrix E_8 that we introduced in Theorem 226.22 and that we reintroduced in Definition 259.16. Since E_8 is an even matrix we obtain from the Linking Matrix-Realization Proposition 259.9 thickened $(2k-1)$-spheres $\varphi_1, \dots, \varphi_8\colon \overline{B}^{2k} \times S^{2k-1} \to S^{4k-1}$ with disjoint images such that the linking matrix is E_8. We refer to

$$M := \overline{B}^{4k} \cup_{\varphi_1} h^{2k} \cup \dots \cup_{\varphi_8} h^{2k}$$

as a $4k$-dimensional E_8-manifold.[2776]

By the Boundary Invariants Proposition 259.11 (4) we know that for $k \geq 2$ the boundary ∂M of a $4k$-dimensional E_8-manifold is a $(4k-1)$-dimensional smooth homotopy sphere. This raises the following questions, which can be viewed as more specific version of Question 259.12:

Question 259.22. Let $k \in \mathbb{N}_{\geq 2}$.

(1) Is the boundary of every $4k$-dimensional E_8-manifold homeomorphic to S^{4k-1}?

[2775] Note that Mazur Manifold Proposition 260.18 appears in a later chapter, but the reader already has all the tools to follow the argument.

[2776] Note that we do not make any claims regarding uniqueness of such a smooth manifold.

(2) Is the boundary of every $4k$-dimensional E_8-manifold diffeomorphic to S^{4k-1}?

259.7. Exercises for Chapter 259.

Exercise 259.1. Let $n \in \mathbb{Z}$. Show that the degree of the map

$$\begin{array}{rcl} S^3 = \{v \in \mathbb{H} \,|\, \|v\| = 1\} & \to & S^3 \\ z & \mapsto & z^n \end{array}$$

equals n.
Hint. You could use the Degree-via-Local Orientations Proposition 152.11.
Remark. This exercise is identical to Exercise 259.1.

Exercise 259.2. Let M be a compact oriented simply connected 4-dimensional topological manifold. Show that the intersection $Q_M \colon \mathrm{H}_2(M) \times \mathrm{H}_2(M) \to \mathbb{Z}$ is non-singular if and only if each boundary component of M is a $\mathbb{Z}$-homology sphere.

Exercise 259.3. Inspired by the examples on page 4995 we consider forms over $\mathbb{F}_2$ that are represented by the following two matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Are these forms isometric over $\mathbb{F}_2$?

Exercise 259.4. Let M be a compact oriented connected non-empty smooth manifold of dimension $n \geq 3$ such that $\pi_1(M)$ admits an epimorphism onto a finite non-trivial group G. Given $k \in \mathbb{N}$ we denote by $k \cdot M$ the connected sum of k copies of M. Show that the smooth manifolds $k \cdot M$, $k \in \mathbb{N}$ are pairwise non-diffeomorphic.
Hint. Consider the set of epimorphisms of a fundamental group onto G.
Remark. This exercise is identical to Exercise 259.4.

Exercise 259.5. Show that there exist infinitely many diffeomorphism types of 3-dimensional $\mathbb{Z}$-homology spheres that are not the boundary of a smooth $\mathbb{Z}$-homology ball.
Hint. You could use Exercise 102.4.

Exercise 259.6. Show that given any finitely generated abelian group A there exists a closed orientable 7-dimensional smooth manifold W such that $\mathrm{H}_3(W) \cong A$ and such that $\mathrm{H}_1(W) = \mathrm{H}_2(W) = 0$.
Hint. First construct a suitable 8-dimensional smooth manifold.
Remark. We will deal with the higher dimensional case in Exercise 259.7.

Exercise 259.7.
(a) Let $m \in \mathbb{N}$. We consider the $(m-1) \times (m-1)$-matrix

$$A_m := \begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & \vdots \\ 1 & 1 & 2 & \dots & \vdots \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 2 \end{pmatrix}$$

Show that $\mathbb{Z}^{m-1}/A_m \cdot \mathbb{Z}^{m-1}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.

(b) Let $k \in \mathbb{N}$. Show that given any finitely generated abelian group A there exists a closed orientable $(4k-1)$-dimensional smooth manifold W such that $\mathrm{H}_{2k-1}(W) \cong A$ and such that $\mathrm{H}_1(W) = \cdots = \mathrm{H}_{2k-2}(W) = 0$.
Hint. First construct a suitable $4k$-dimensional smooth manifold.

Exercise 259.8. Let $K \subseteq S^3$ be the trivial knot. In the following we consider a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ of self-thickening number 1.

(a) Show that $\partial(\overline{B}^4 \cup_\varphi h^2)$ is diffeomorphic to S^3.

(b) Show that $\overline{B}^4 \cup_\varphi h^2 \cup h^4$ admits an orientation-preserving diffeomorphism to $\mathbb{C}\mathrm{P}^2$.

Exercise 259.9. Let $k \in \mathbb{N}_0$. For which invertible antisymmetric $(m \times m)$-matrices over $\mathbb{Z}$ does there exist a compact orientable smooth manifold of dimension $4k+2$ such that the intersection form is represented by the matrix A?
Hint. What can you can say about antisymmetric forms over $\mathbb{Z}$ that can be represented by invertible matrices? Can one hope to classify such forms?

Exercise 259.10. Let $K, J \subseteq S^3$ be two knots.

(a) How can you build a $\mathbb{Z}$-homology sphere out of the knot exteriors X_K and X_J?

(b) Can you make an educated guess under what hypothesis the fundamental group of the $\mathbb{Z}$-homology sphere you constructed is non-trivial?

Part XXIX

The h-Cobordism Theorem

In this part of the lecture notes we will use handle decompositions to prove the h-Cobordism Theorem 263.11 which says, roughly speaking, that in dimension ≥ 6 a simply connected smooth manifold that looks like a product, is indeed *diffeomorphic* to a product. The h-Cobordism Theorem 263.11 has many consequences. In particular it allows us to prove the High-Dimensional Poincaré Conjecture 264.7 which says that in dimension ≥ 6 any smooth homotopy sphere is *homeomorphic* to the standard sphere.

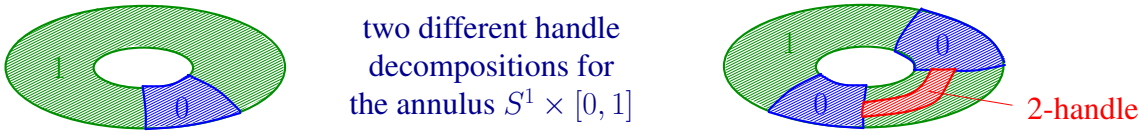
This part of the lecture notes is organized as follows:

- In Chapter 260 we will state and prove the Handle Cancellation Theorem.
- In Chapter 261 we will do a little detour and we will introduce and study Stiefel manifolds. These will play at one point an essential role in the proof of the Whitney trick. they cannot be made disjoint.
- In Chapter 262 we will prove the Whitney Trick-Theorem 262.5 which shows that if two submanifolds have zero algebraic intersection number then, under suitable hypotheses, they can actually be made disjoint.
- In Chapter 263 we will state and prove the h-Cobordism Theorem 263.11.
- In Chapter 264 we will consider several applications of the h-Cobordism Theorem 263.11. In particular we will prove the Poincaré Conjecture in dimensions ≥ 6, see the High-Dimensional Poincaré Conjecture 264.7.

CHAPTER 260

Handle cancellations and handle slides

By the Handle Decomposition Theorem 250.7 we now know that every closed smooth manifold admits a handle decomposition. As we see in the figure below, it is easy to come up with examples of handle decompositions of a fixed smooth manifold M with different number of handles.



We also saw in the Morse Inequalities Proposition 252.13 that the homology groups of a smooth manifold M give rise to a lower bound on the number of handles in any handle decomposition. For example one can easily verify that the handle decomposition of the annulus, that is shown to the left, has the minimal number of handles.

It is natural to ask whether, given an explicit handle decomposition, it is possible to reduce the number of handles. Furthermore it is reasonable to ask, whether it is possible to relate any two handle decompositions of a given smooth manifold.

To address these issues we do the following:

- We will state and prove the Handle Cancellation Theorem 260.1 which gives us a criterion for two handles to "cancel".
- We will state and prove the Handle Slide Theorem 260.13 which says that often we can "slide" one k-handle over another k-handle. Sliding handles can at times be used to modify a handle decomposition to obtain a handle decomposition to which the Handle Cancellation Theorem 260.1 applies.
- We will formulate and discuss the Cerf Theorem 260.15 which says that any two handle decompositions of a given compact smooth manifold are related by a sequence of handle cancellations and handle slides. We will not make really use of the Cerf Theorem 260.15, hence we will not prove it.

Later the Handle Cancellation Theorem 260.1 and the Handle Slide Theorem 260.13 will turn out to be essential tools in the proof of the h-Cobordism Theorem 263.11.

This chapter is organized as follows:

- In Section 260.1 we will state and prove the Handle Cancellation Theorem 260.1.
- In Section 260.2 we will discuss some consequences of the Handle Cancellation Theorem 260.1.
- In Section 260.3 we will see that in some settings handle cancellations become possible after smooth isotopies.
- In Section 260.4 we will state and prove the Handle Slide Theorem 260.13.
- In Section 260.5 we will formulate and discuss the Cerf Theorem 260.15.

- In Appendix 260.6 we will construct the Mazur manifold which is a compact 4-dimensional smooth manifold that is contractible but such that the boundary is not homotopy equivalent to a sphere.
- In Section 260.7 we will provide exercises for Chapter 260.

In the next Chapter 261 we will introduce Stiefel manifolds.

260.1. **The Handle Cancellation Theorem.** The next theorem says the following for a given smooth manifold M: If we first attach a k-handle to M and if afterwards we attach a $(k+1)$-handle such that the attaching sphere of the $(k+1)$-handle intersects the belt sphere of the k-handle transversally in a single point, then these two handles can be "canceled", i.e. $M \cup h^k \cup h^{k+1}$ is diffeomorphic to M. More precisely the following theorem holds.

Theorem 260.1. (Handle Cancellation Theorem) Let $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$. Furthermore let M be an n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere and let

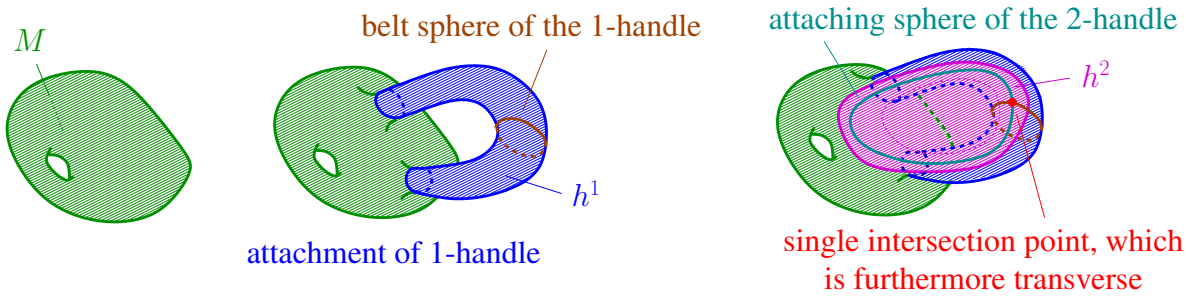
$$\psi\colon \overline{B}^{n-k-1} \times S^k \;\to\; \partial(M \cup_\varphi h^k) \;=\; \partial(M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k))$$

be a thickened k-sphere. If the attaching sphere $\psi(\{0\} \times S^k)$ of the $(k+1)$-handle intersects the belt sphere $S^{n-k-1} \times \{0\}$ of the original k-handle transversally in a single point, then $M \cup_\varphi h^k \cup_\psi h^{k+1}$ is diffeomorphic[2777] to M.

Remark. Alternative proofs for the Handle Cancellation Theorem 260.1 are provided in [**Wall2016**, Theorem 5.4.3], in [**Kos1993**, Theorem VI.7.4] and in [**Miln1965b**, Theorem 5.4]. □

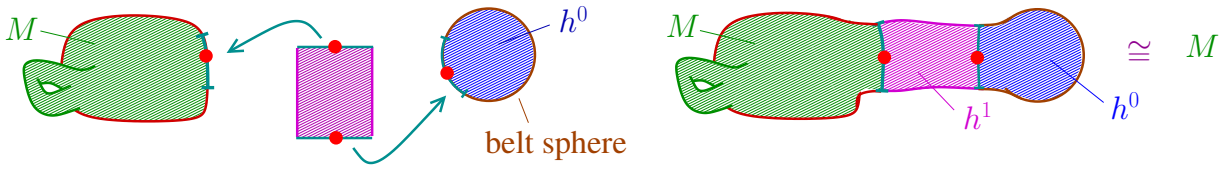
Example.

(1) In the figure below we show a 3-dimensional smooth manifold M together with a 1-handle and a 2-handle. The attaching sphere of the 2-handle intersects the belt sphere of the 1-handle transversally, in a single point. Thus we obtain from the Handle Cancellation Theorem 260.1 that $M \cup h^1 \cup h^2$ is diffeomorphic to M.



(2) It is worth considering the case $k = 0$. Thus let M be an n-dimensional smooth manifold. As we discussed on page 4748, the result of a 0-handle attachment is given by $M \cup h^0 = M \sqcup \overline{B}^n$. Note that $S^{n-1} = \partial \overline{B}^n$ is the belt sphere of the 0-handle. Next let $\psi\colon \overline{B}^{n-1} \times S^0 \to \partial(M \sqcup \overline{B}^n) = \partial M \sqcup S^{n-1}$ be a thickened 0-sphere.

[2777] In fact the proof shows that $M \cup_\varphi h^k \cup_\psi h^{k+1}$ admits a diffeomorphism to M *which is the identity outside of any given neighborhood of the common subset* $\varphi(\overline{B}^{n-k} \times S^{k-1}) \subseteq M$. Such further specifications are often used implicitly in applications of this and other theorems.



Note that it follows easily from the definitions that the following two statements are equivalent:

- The attaching sphere $\psi(\{0\} \times S^0)$ intersects the belt sphere S^{n-1} of the 0-handle transversally in a single point.
- One of the two points of $\psi(\{0\} \times S^0)$ lies in ∂M and one lies in S^{n-1}.

This discussion shows that the case $k = 0$ of the Handle Cancellation Theorem 260.1 is equivalent to the 0-Handle Cancellation Proposition 247.12. □

The proof of the Handle Cancellation Theorem 260.1 will require the remainder of this section. In the next Section 260.2 we will discuss some consequences of the Handle Cancellation Theorem 260.1.

We start out with the following lemma, which will be a key ingredient in the proof of the Handle Cancellation Theorem 260.1:

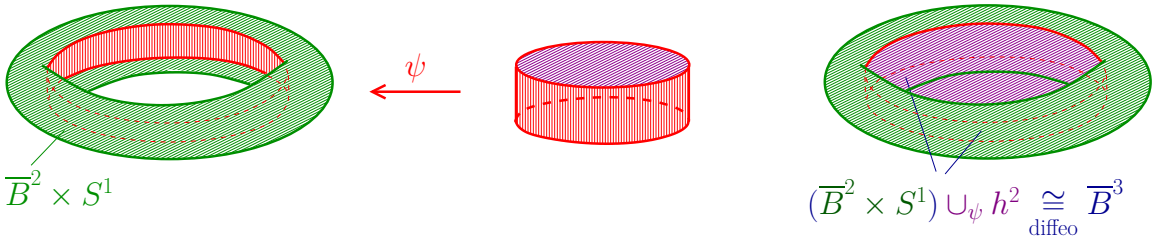
Lemma 260.2. (Basic Handle Cancellation Lemma) Let $n \in \mathbb{N}$. Furthermore let $k \in \{0, \ldots, n-1\}$. We consider the "solid torus" $\overline{B}^{n-k} \times S^k$ and we consider the thickened k-sphere

$$\begin{aligned} \psi\colon \overline{B}^{n-k-1} \times S^k &\to S^{n-k-1} \times S^k = \partial(\overline{B}^{n-k} \times S^k) \\ (x, y) &\mapsto \left(\frac{(-1, x)}{\|(-1, x)\|}, y\right). \end{aligned}$$

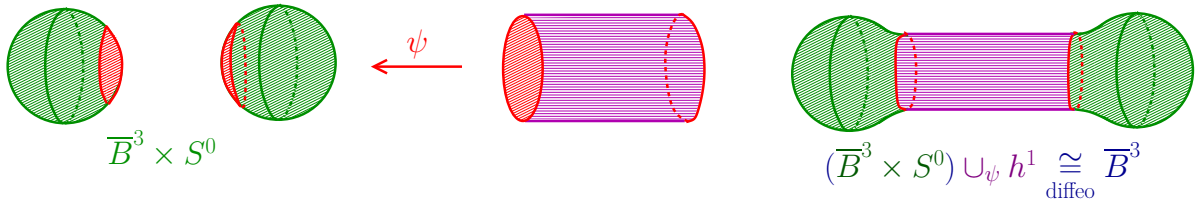
The smooth manifold $(\overline{B}^{n-k} \times S^k) \cup_\psi h^{k+1}$ is diffeomorphic to $\overline{B}^n$.

Example.

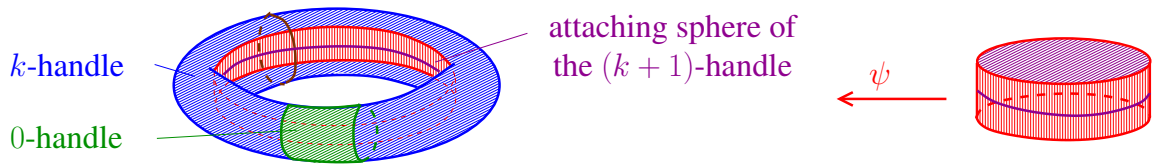
(1) In the figure below we illustrate the case $n = 3$ and $k = 1$ of the Basic Handle Cancellation Lemma 260.2.



(2) In the figure below we illustrate the case $n = 3$ and $k = 0$.



(3) In the Ball–Handle Attachment Lemma 246.9 we gave an explicit diffeomorphism from $\overline{B}^{n-k} \times S^k$ to $\overline{B}^n \cup_\varphi h^k$ for an explicit thickened $(k-1)$-sphere φ.



One can easily verify that under this diffeomorphism the attaching sphere of the h^{k+1}-handle of the Basic Handle Cancellation Lemma 260.2 intersects the belt sphere of the k-handle transversally in a single point. This discussion shows that the Basic Handle Cancellation Lemma 260.2 can be viewed as a special case of the Handle Cancellation Theorem 260.1. □

Proof of Lemma 260.2. [2778] Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, n-1\}$. One could prove the lemma, as in the proof of the Product-of-Balls-Proposition 60.11, using the Single Corner-to-Disk Proposition 60.8 and the Smooth Convex-to-Ball Proposition 30.22. But as the author found out the hard way, this becomes something of a mess.

Thus we will pursue a more indirect approach. We start out with a few observations:

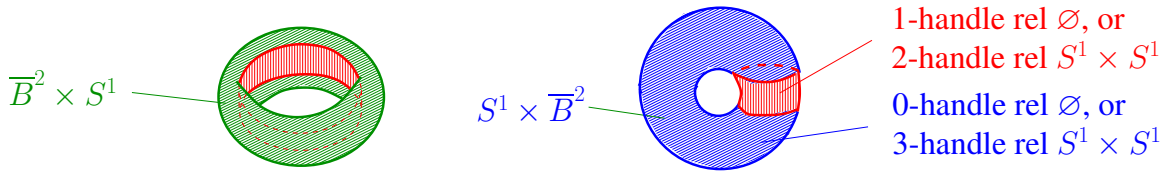
(1) By the Sphere–Solid Tori-Decomposition Lemma 111.6 we have an explicit diffeomorphism
$$(\overline{B}^{n-k} \times S^k) \cup (S^{n-k-1} \times \overline{B}^{k+1}) \xrightarrow{\cong} S^n.$$

(2) In the Ball–Handle Attachment Lemma 246.9 we showed that there exists a diffeomorphism from $S^{n-k-1} \times \overline{B}^{k+1}$ to $\overline{B}^n \cup_\theta h^{n-k-1}$ where we consider the attaching map
$$\begin{aligned} \theta \colon \overline{B}^{k-1} \times S^{n-k-2} &\to S^{n-1} = \partial \overline{B}^n \\ (x, y) &\mapsto \frac{(x, y)}{\|(x, y)\|}. \end{aligned}$$

(3) By the Dual Handle Proposition 248.1 the handle decomposition $\overline{B}^n \cup_\theta h^{n-k-1}$ has a dual handle decomposition $((S^{n-k-1} \times S^k) \times [0,1]) \cup_\psi h^{k+1} \cup h^n$.

In summary we see that we have diffeomorphisms
$$S^n \xleftarrow[\text{by (1)}]{\cong} (\overline{B}^{n-k} \times S^k) \cup (S^{n-k-1} \times \overline{B}^{k+1}) \xrightarrow[\substack{\text{by (2) and (3) and by the Handle} \\ \text{Decomposition Stacking Proposition 247.2}}]{\cong} (\overline{B}^{n-k} \times S^k) \cup_\psi h^{k+1} \cup h^n.$$

Looking at the explicit attaching map for the dual $(k+1)$-handle we see that it is given by our map ψ.



[2778] An alternative proof for the Basic Handle Cancellation Lemma 260.2 is given in [**Wall2016**, Lemma 5.4.2].

We now see that we have the following diffeomorphisms

$$(\overline{B}^{n-k} \times S^k) \cup_\psi h^{k+1} \quad \underset{\text{by the above}}{\cong} \quad S^n \setminus \operatorname{int}(h^n) \quad \underset{\substack{\text{by the Ball-in-Sphere Corollary 45.11,}\\ \text{which is a consequence of the}\\ \text{Smooth Ball Embedding Theorem 45.10}}}{\cong} \quad \overline{B}^n.$$

■

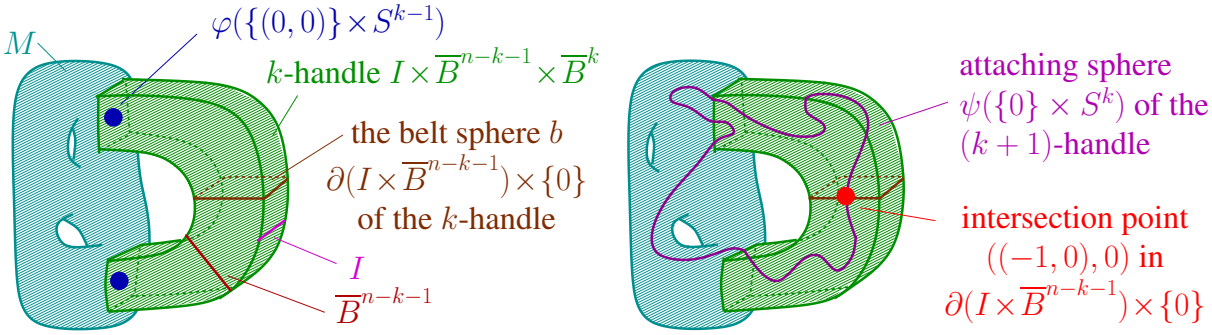
Sketch of a proof of the Handle Cancellation Theorem 260.1. [2779] Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n-1\}$. Before we do anything else, let us adopt the following notation:

- We write $I := [-1, 1]$.
- We use the diffeomorphism from the Product-of-Balls-Proposition 60.11 to make the identification $(I \times \overline{B}^{n-k-1})^\backprime =_{\text{i}} \overline{B}^{n-k}$.
- We use the diffeomorphisms from pages 319 and 1229 to make throughout the proof the identification $S^k =_{\text{i}} \overline{B}^k_+ \cup \overline{B}^k_-$.

Now we start with the actual proof. Let M be an n-dimensional smooth manifold.

- Let $\varphi\colon \overbrace{I \times \overline{B}^{n-k-1}}^{=_{\text{i}} \overline{B}^{n-k}} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We consider the handle attachment $M \cup_\varphi h^k = M \cup_\varphi ((I \times \overline{B}^{n-k-1}) \times \overline{B}^k)$. We denote by $b := \partial(I \times \overline{B}^{n-k-1}) \times \{0\}$ the belt sphere of the k-handle, see Definition 246.4.
- Let $\psi\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ be a smooth embedding such that the attaching sphere $\psi(\{0\} \times S^k)$ of the $(k+1)$-handle intersects the belt sphere b of the k-handle transversally in a single point. After a smooth isotopy of ψ we can assume that this point is given by $((-1, 0), 0) \in \partial(I \times \overline{B}^{n-k-1}) \times \{0\}$.

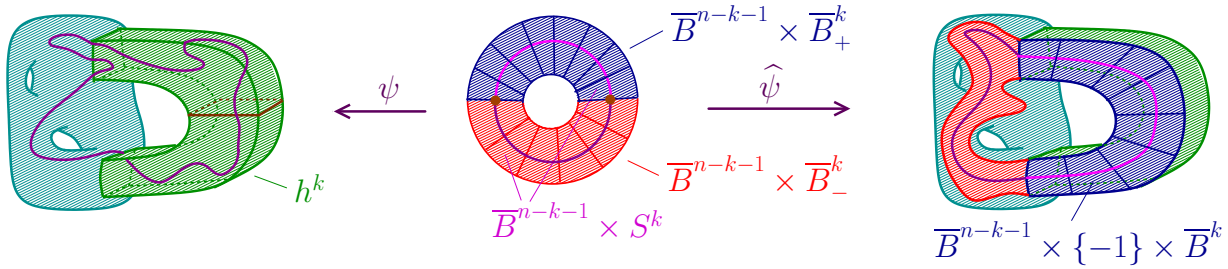
  Furthermore, after possibly using a reflection on the $\overline{B}^{n-k-1}$ factor, we can assume that on ∂h^k the map $\psi\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ is orientation-preserving.



Recall that we need to show that $(M \cup_\varphi h^k) \cup_\psi h^{k+1}$ is diffeomorphic to M. In the following claim we first show that we can arrange the attaching sphere for the ψ-handle to be as nice as possible on the boundary of the k-handle h^k:

Claim. The smooth embedding $\psi\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ is smoothly isotopic to a smooth embedding $\widehat{\psi}\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ which has the following two properties:
(a) On $\overline{B}^{n-k-1} \times \overline{B}^k_+$ it is given by the obvious map $\overline{B}^{n-k-1} \times \overline{B}^k_+ \to \overline{B}^{n-k-1} \times \{-1\} \times \overline{B}^k$.
(b) $\widehat{\psi}(\overline{B}^{n-k} \times B^k_-) \cap h^k = \varnothing$.

[2779] In this proof, in an attempt to preserve sanity, we will resolutely swipe arguments about rounding and introducing corners under the carpet.



Proof. The proof of the claim proceeds in two steps:

(1) First we want to arrange that $\psi(\{0\} \times \overline{B}^k_+) = \{(0,-1)\} \times \overline{B}^k$. We start out with the construction of two tubular neighborhoods of the belt sphere:

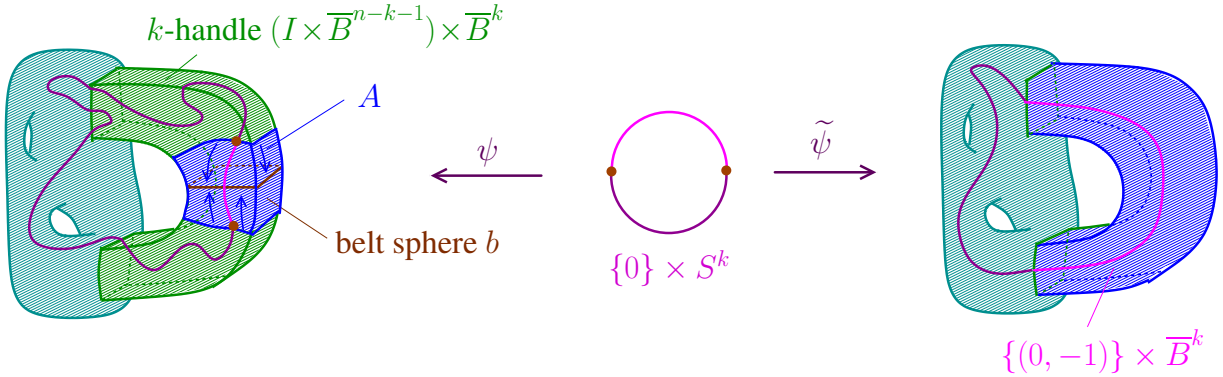
- First note that it follows from the hypothesis that the attaching sphere intersects the belt sphere transversally in the single point $((-1,0),0)$, together with the Transverse Tubular Neighborhood–Existence Theorem 57.32, that there exists a tubular neighborhood $p\colon A \to b$ for the belt sphere b such that $\psi(\{0\} \times S^k) \cap A = p^{-1}(((-1,0),0))$.
- Next note that it follows easily from the Handle Attachment Proposition 246.3 (3) that another tubular map for the belt sphere $b = \partial(I \times \overline{B}^{n-k-1}) \times \{0\}$ is given by the obvious map $\beta\colon \partial(I \times \overline{B}^{n-k-1}) \times \overline{B}^k \to \partial(M \cup_\varphi h^k)$.

By the Tubular Neighborhood–Uniqueness Theorem 57.12 there exists a diffeotopy F of $\partial(M \cup_\varphi h^k)$ with $F_0 = \mathrm{id}$ and such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad F_1 \quad} & \partial(I \times \overline{B}^{n-k-1}) \times \overline{B}^k \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle ((t,x),y) \mapsto ((t,x),0)} & \\ & \underbrace{\partial(I \times \overline{B}^{n-k-1}) \times \{0\}}_{=b}. & \end{array}$$

We set $\widetilde{\psi} \coloneqq F_1 \circ \psi$. Evidently $\widetilde{\psi}$ is smoothly isotopic to ψ and by the above discussion we now have $\widetilde{\psi}(\{0\} \times S^k) \cap \partial(I \times \overline{B}^{n-k-1}) \times \overline{B}^k = \{(-1,0)\} \times \overline{B}^k$.

Next note that $\overline{B}^k \xrightarrow{x \mapsto ((-1,0),x)} \{(-1,0)\} \times \overline{B}^k \xrightarrow{\widetilde{\psi}^{-1}} S^k \times \{0\} \xrightarrow{(x,0) \mapsto x} S^k$ is a smooth embedding. It follows from the Smooth Ball Embedding Theorem 45.10 that, after possibly applying a diffeotopy to S^k, we can assume that the above map $\overline{B}^k \to S^k$ is the obvious map from $\overline{B}^k \to \overline{B}^k_+ \subseteq S^k$.



(2) The map $\widetilde{\psi}\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ that we just constructed behaves correctly on $\{0\} \times S^k$. Next we will have to find the right "thickening" of this map. To achieve this goal first note that the smooth submanifold $\widetilde{\psi}(\{0\} \times S^k) \subseteq \partial(M \cup_\varphi h^k)$ comes with two tubular maps:

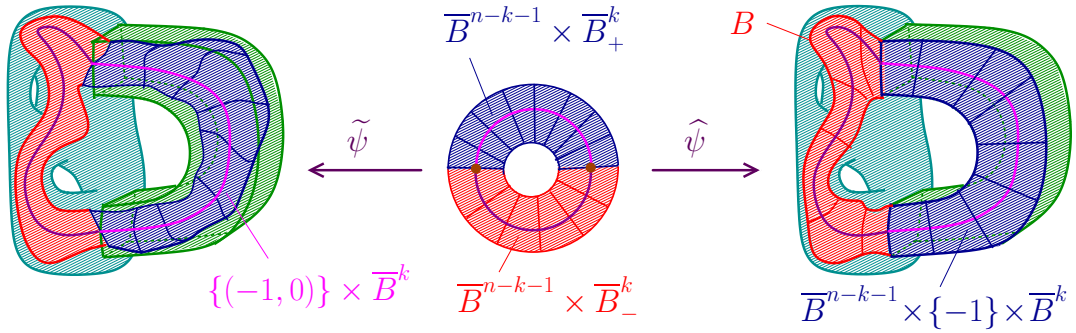
- Evidently the map $\widetilde{\psi}\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ is a tubular map for the submanifold $\widetilde{\psi}(\{0\} \times S^k) \subseteq \partial(M \cup_\varphi h^k)$.
- Note that the obvious map $p\colon \overline{B}^{n-k-1} \times \{-1\} \times \overline{B}^k \to \{(-1,0)\} \times \overline{B}^k$ is a tubular map for $\widetilde{\psi}(\{0\} \times \overline{B}^k_+) = \{(-1,0)\} \times \overline{B}^k \subseteq \partial(I \times \overline{B}^{n-k-1}) \times \overline{B}^k$. It follows from the Tubular Neighborhood–Submanifold Extension Theorem 57.30 (which we can apply by the Handle Attachment Proposition 246.3 (3)) that there exists a tubular neighborhood $q\colon B \to \widetilde{\psi}(\{0\} \times \overline{B}^k_-) \subseteq (\partial M) \setminus \varphi((-1,1) \times B^{n-k-1} \times S^k)$ that, together with p, defines a tubular map $\widetilde{\psi}'\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ for $\widetilde{\psi}(\{0\} \times S^k) \subseteq \partial(M \cup_\varphi h^k)$.

It remains to prove the following subclaim:

Subclaim. There exists a smooth map $\beta\colon S^k \to \mathrm{SO}(n-k-1)$ which is constant $\equiv \mathrm{id}$ on B^k_+ such that the smooth embeddings $\widetilde{\psi}\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ and

$$\begin{aligned} \widehat{\psi}\colon \overline{B}^{n-k-1} \times S^k &\to \partial(M \cup_\varphi h^k) \\ (v,x) &\mapsto \widehat{\psi}(v,x) := \widetilde{\psi}'(\beta(x) \cdot v, x) \end{aligned}$$

are smoothly isotopic.



Proof of subclaim. First note that it follows from the Tubular Map–Uniqueness Theorem 58.5 that there exists a smooth isotopy $G\colon \overline{B}^{n-k-1} \times S^k \times [0,1] \to \partial(M \cup_\varphi h^k)$ and a smooth map $\alpha\colon S^k \to \mathrm{O}(n-k-1)$ with the following properties:

- $G_0 = \widetilde{\psi}$
- G is a homotopy rel $\{0\} \times S^k$.
- The map G_1 is given by $(v,x) \mapsto \widetilde{\psi}'(\alpha(x) \cdot v, x)$.

Next recall that we arranged that $\widetilde{\psi}\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ is orientation-preserving on ∂h^k. Furthermore the map $\widetilde{\psi}'\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ is by design orientation-preserving. This implies that the above map $\alpha\colon S^k \to \mathrm{O}(n-k-1)$ takes values in the connected component $\mathrm{SO}(n-k-1)$ of $\mathrm{O}(n-k-1)$.

Now note that it follows from an elementary argument, see Exercise 260.4, that there exists a smooth homotopy $\Gamma\colon S^k \times [0,1] \to \mathrm{SO}(n-k-1)$ from α to a map β that takes the constant value id on the smoothly embedded ball $\overline{B}^k_+ \subseteq S^k$.

We use this map β to define the smooth embedding $\widehat{\psi}$ as in the statement of the subclaim. Since $\widetilde{\psi}$ is smoothly isotopic to G_1 it remains to prove that G_1 is smoothly

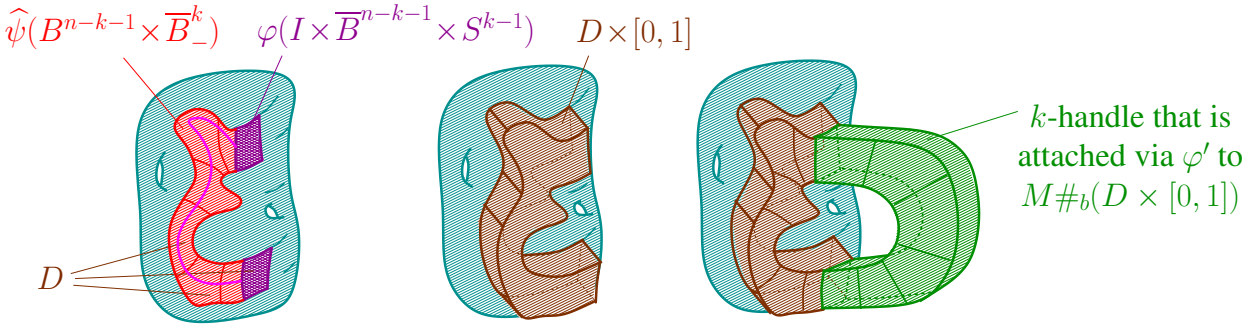
isotopic to $\widehat{\psi}$. In fact such a smooth isotopy is given by

$$\begin{aligned} H\colon \overline{B}^{n-k-1} \times S^k \times [0,1] &\to \partial(M \cup_\varphi h^k) \\ ((v,x),t) &\mapsto \widetilde{\psi}'(\Gamma(x,t)\cdot v, x). \end{aligned}$$ ⊞

We introduce a few more players:

(1) We set $D \coloneqq \widehat{\psi}(B^{n-k-1} \times \overline{B}^k_-) \cup \varphi(I \times \overline{B}^{n-k-1} \times S^{k-1})$. Since $\overline{B}^k \cup ([-1,1] \times S^{k-1})$ is diffeomorphic to $\overline{B}^k$ we see that D is diffeomorphic to $\overline{B}^{n-1}$.

(2) We denote by $M \#_{\mathrm{b}} (D \times [0,1])$ the boundary connected sum of M and $D \times [0,1]$ which in this case is defined by gluing $D \subseteq \partial M$ to $D \times \{0\}$ in the obvious way.

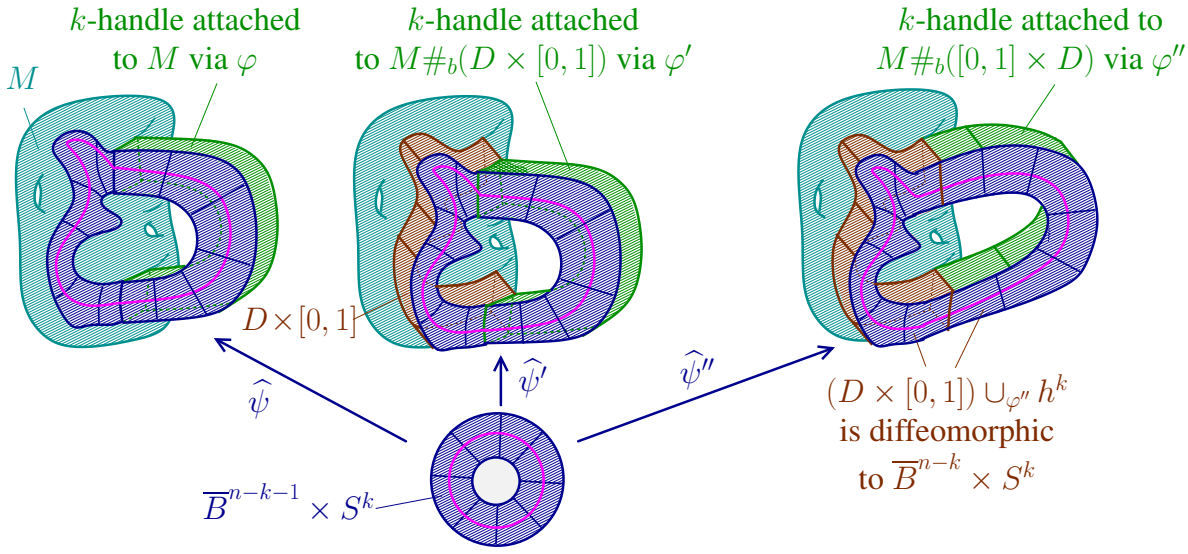


(3) Let $\varphi'\colon I \times \overline{B}^{n-k-1} \times S^{k-1} \to \partial(M \#_{\mathrm{b}} (D \times [0,1]))$ be the smooth embedding that is given by $\varphi'(P) = (\varphi(P), 1)$.

(4) We consider the smooth embedding

$$\begin{aligned} \varphi''\colon I \times \overline{B}^{n-k-1} \times S^{k-1} &\to \partial(M \#_{\mathrm{b}} (D \times [0,1])) \\ (t,x,y) &\mapsto (\varphi(1,x,y), \underbrace{\tfrac{1}{2}(1-t)}_{\in[0,1]}). \end{aligned}$$

Note that φ'' is smoothly isotopic to φ'.



Now we see that

$$
\begin{array}{ccc}
(M \cup_\varphi h^k) \cup_\psi h^{k+1} & \cong & (M \cup_\varphi h^k) \cup_{\widehat{\psi}} h^{k+1} \\
\text{by the Handle Attachment Proposition 246.3 (5)} & \downarrow & \text{by the Boundary Connected Sum-with-Ball Proposition 61.13 there exists a diffeomorphism } M \cong M \#_\mathrm{b} (D \times [0,1]), \text{ thus we get this diffeomorphism for the above } \varphi' \text{ and suitable } \widehat{\psi}' \\
M \#_\mathrm{b} (((D \times [0,1]) \cup_{\varphi''} h^k) \cup_{\widetilde{\psi}''} h^{k+1}) & \cong & ((M \#_\mathrm{b} (D \times [0,1])) \cup_{\varphi'} h^k) \cup_{\widehat{\psi}'} h^{k+1} \\
\text{by the Handle Attachment Proposition 246.3 (5)} & \uparrow & \text{by the Ball–Handle Attachment Lemma 246.9 we know that } (D \times [0,1]) \cup_{\varphi''} h^k \text{ is diffeomorphic to } \overline{B}^{n-k} \times S^k \\
M \#_\mathrm{b} \overline{B}^n & \cong & M \#_\mathrm{b} (\overline{B}^{n-k} \times S^k) \cup_{\widetilde{\psi}''} h^{k+1}) \\
\text{by the Basic Handle Cancellation Lemma 260.2} & \uparrow & \text{by the Boundary Connected Sum-with-Ball Proposition 61.13} \\
M & \cong & M.
\end{array}
$$

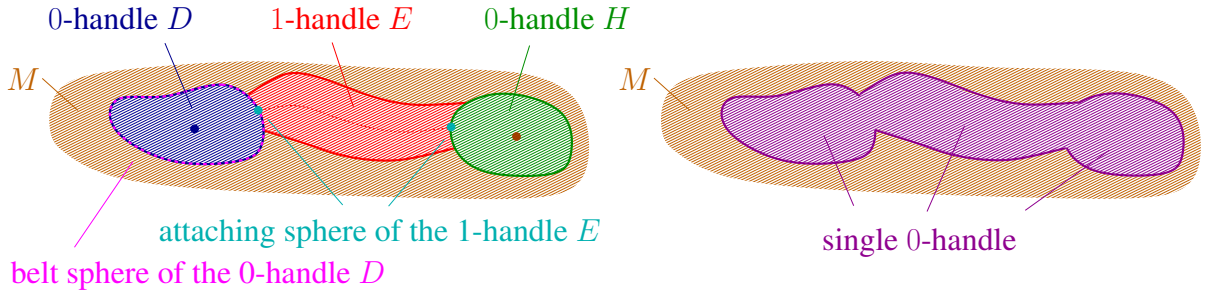
The reader will surely have noticed that various details were swiped under the carpet. We leave it to the reader to look under the carpet and to clean up properly. ■

260.2. **Consequences of the Handle Cancellation Theorem 260.1.** The Handle Cancellation Theorem 260.1 motivates the following definition:

Definition 260.3. Let $n \in \mathbb{N}_0$ and let M be an n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^{n-k-2} \times S^{k-1} \to \partial M$ and $\psi\colon \overline{B}^{n-k-3} \times S^k \to \partial(M \cup_\varphi h^k)$ be two thickened spheres. If the attaching sphere $\psi(\{0\} \times S^k)$ of the $(k+1)$-handle intersects the belt sphere $S^{n-k-1} \times \{0\}$ of the k-handle transversally in a single point, then we say that (φ, ψ) is a canceling pair of handles.

In many applications the following corollary to the Handle Cancellation Theorem 260.1 gets used:

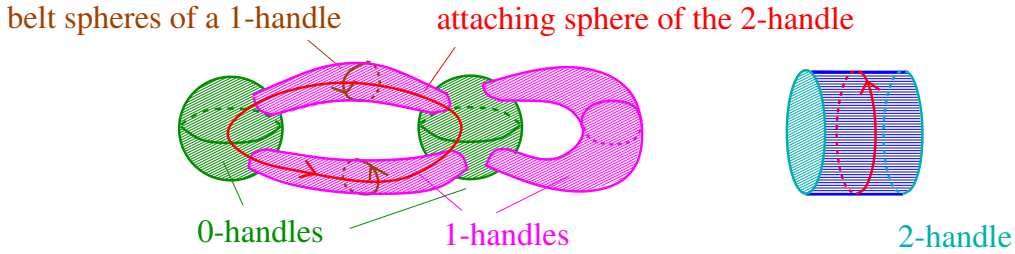
Theorem 260.4. (Inbetween–Handle Cancellation Theorem) Let M be a smooth manifold with a standard handle decomposition. If in the given handle decomposition there exists a k-handle D and a $(k+1)$-handle E which form a canceling pair of handles, i.e. such that the attaching sphere of E intersects the belt sphere of D transversally in a single point, then M admits a standard handle decomposition with one handle less in dimensions k and $k+1$, and such that the number of handles in all other dimensions stays unchanged.



Proof. By the Handle Swapping Proposition 247.4 we can assume that D is the last k-handle that gets attached and that E is the first $(k+1)$-handle that gets attached. We denote by N the union of all handles of dimension $< k$ together with all k-handles that are not equal to D. From our hypothesis and the Handle Cancellation Theorem 260.1 we obtain a diffeomorphism $\Theta\colon (N \cup D) \cup E \to N$. As we discussed in the Modify Handle

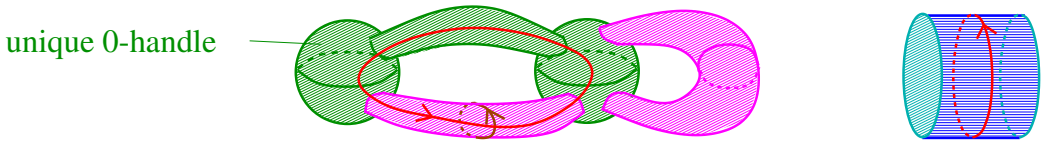
Attachment Region Lemma 247.8, using the diffeomorphism Θ, we can attach all the other handles of $(N \cup D) \cup E$ to N and we obtain the desired handle decomposition. ■

Example. We consider the 3-dimensional smooth manifold M which is shown in the figure below. It has two 0-handles, three 1-handles and one 2-handle.

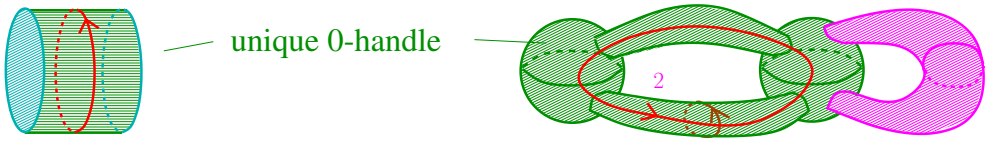


We simplify the handle decomposition as follows:

(1) Using the Inbetween–Handle Cancellation Theorem 260.4 we can cancel one 0-handle and one 1-handle, i.e. we can combine two 0-handles and one 1-handle to a single 0-handle.



(2) Afterwards we use the Inbetween–Handle Cancellation Theorem 260.4 to cancel the 2-handle with one 1-handle. So are left with one-handle and one 1-handle.

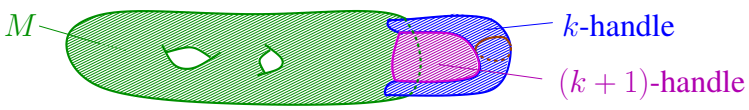


(3) Note that it follows easily from the Handles-Orientability Proposition 246.6 that M is orientable. We now know that M is an orientable 3-dimensional smooth manifold that is built out of a single 0-handle and a single 1-handle. It follows from the 1-Handle Attachment Lemma 246.8 and the Ball–Handle Attachment Lemma 246.9 that M is diffeomorphic to $\overline{B}^2 \times S^1$.

Note that this answers the question that we had asked on page 4881. □

If we find a canceling pair of handles, then we can essentially just remove it from the handle decomposition, and we have simplified the handle decomposition. On a few occasions the opposite construction is also useful, namely it can be convenient to introduce a canceling pair of handles. Fortunately this can always be done:

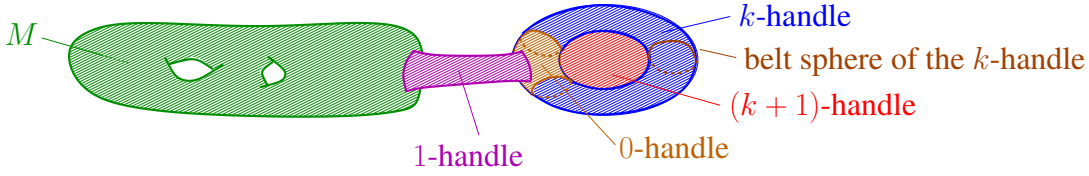
Lemma 260.5. (Canceling Pair–Creation Lemma) Let M be an n-dimensional smooth manifold and let F be a boundary component of M. For any $k \in \{0, \ldots, n-1\}$ there exists a canceling pair $\varphi\colon \overline{B}^{n-k-2} \times S^{k-1} \to F \subseteq \partial M$ and $\psi\colon \overline{B}^{n-k-1} \times S^k \to \partial(M \cup_\varphi h^k)$ of handles.



Sketch of proof. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold, let F be a boundary component of M and let $k \in \{0, \dots, n-1\}$. We have the following diffeomorphisms:

$$
\begin{array}{lll}
\text{by the Boundary Connected Sum-with-Ball Proposition 61.13} & \text{by the Basic Handle Cancellation Lemma 260.2} & \text{by the Ball–Handle Attachment Lemma 246.9} \\
\downarrow & \downarrow & \downarrow \\
M \cong M \#_{\mathrm{b}} \overline{B}^n & \cong M \#_{\mathrm{b}} ((\overline{B}^{n-k} \times S^k) \cup h^{k+1}) & \cong M \#_{\mathrm{b}} ((h^0 \cup h^k) \cup h^{k+1}) \\
\cong M \cup ((h^0 \cup h^k) \cup h^{k+1}) \cup h^1 & \cong (M \cup h^0 \cup h^1) \cup h^k \cup h^{k+1} & \cong M \cup h^k \cup h^{k+1} \\
\uparrow & \uparrow & \uparrow \\
\text{by the Boundary Connected Sum-via-Handle Proposition 247.11} & \text{since we perform the boundary connected sum along } h^0 & \text{by the Inbetween–Handle Cancellation Theorem 260.4}
\end{array}
$$

Here the boundary connected sum is of course performed along F, and such that it glues M to a disk in h^0.



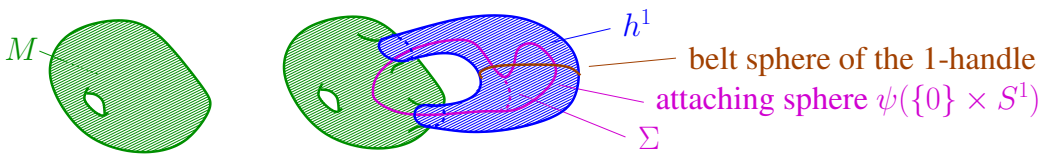
If one goes through the fairly explicit descriptions of the handles one sees that the k-handle and the $(k+1)$-handle do indeed form a canceling pair of handles. ∎

260.3. **Smooth isotopies and handle cancellations.** The following slight generalization of the Handle Cancellation Theorem 260.1 can be quite useful:

Theorem 260.6. (Isotopy–Handle Cancellation Theorem) Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold. Also let $k \in \{0, \dots, n-1\}$, let $\varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere and let

$$\psi \colon \overline{B}^{n-k-1} \times S^k \;\to\; \partial(M \cup_\varphi h^k) \;=\; \partial(M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k))$$

be a thickened k-sphere. If the attaching sphere $\psi(\{0\} \times S^k)$ is smoothly isotopic to a k-dimensional sphere Σ that intersects the belt sphere $S^{n-k-1} \times \{0\}$ of the original k-handle transversally in a single point, then $(M \cup_\varphi h^k) \cup_\psi h^{k+1}$ is diffeomorphic to M.



Proof. By our hypothesis there exists a smooth isotopy $F \colon S^k \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $F_0(x) = \psi(0,x)$ for all $x \in S^k$ and such that $F_1(S^k) = \Sigma$. Note that it follows from the ever-useful Isotopy Extension Theorem 44.2 that there exists a smooth isotopy $G \colon (\overline{B}^{n-k-1} \times S^k) \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $G_0 = \psi$ and such that the attaching

sphere $G_1(\{0\} \times S^k)$ equals Σ. We now see that we have the following diffeomorphisms:

$$(M \cup_\varphi h^k) \cup_\psi h^{k+1} \;\underset{\text{by the Handle Attachment Proposition 246.3 (5)}}{\cong}\; (M \cup_\varphi h^k) \cup_{G_1} h^{k+1} \;\underset{\text{by the Handle Cancellation Theorem 260.1, since the attaching sphere } G_1(\{0\} \times S^k) = \Sigma \text{ intersects the belt sphere of the } k\text{-handle transversally in a single point}}{\cong}\; M.$$

■

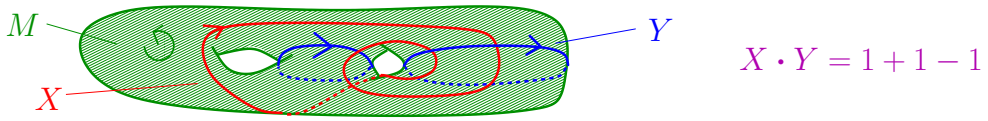
The Isotopy–Handle Cancellation Theorem 260.6 gives us a potentially useful tool for simplifying a handle decomposition. In this context the following question arises: How can we tell whether an attaching sphere is smoothly isotopic to a sphere Σ that intersects a given sphere (in this case the belt sphere), transversally in a single point?

To address this question it is useful to introduce the following definition:

Definition 260.7. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold. Furthermore let $k \in \{1, \dots, n-1\}$, let X be a closed oriented k-dimensional submanifold of M and let Y be a closed oriented proper $(m-k)$-dimensional submanifold of M. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. Then we refer to

$$X \cdot Y \;:=\; \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M)})}_{\in \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M)})}_{\in \mathrm{H}^{k}(M,\partial M;\mathbb{Z})}}^{\in \mathrm{H}^m(M,\partial M;\mathbb{Z})}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \;\in\; \mathbb{Z}$$

as the algebraic intersection number of X and Y. By the Cup Product–Intersection Number Theorem 220.10 this definition is consistent with the earlier Definition 220.5.



The following lemma now gives us a *necessary* criterion for the hypothesis of the Isotopy–Handle Cancellation Theorem 260.6 to be satisfied:

Lemma 260.8. (Isotopy–Handle Cancellation Requirement Lemma) Let M be an n-dimensional smooth manifold. Let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a smooth embedding and let

$$\psi\colon \overline{B}^{n-k-1} \times S^k \;\to\; \partial(M \cup_\varphi h^k) \;=\; \partial(M \cup_\varphi (\overline{B}^{n-k} \times \overline{B}^k))$$

be a smooth embedding. If the attaching sphere $\psi(\{0\} \times S^k)$ is smoothly isotopic to a k-dimensional sphere Σ that intersects the belt sphere $S^{n-k-1} \times \{0\}$ of the original k-handle transversally in a single point, then the algebraic intersection number of the attaching sphere $\psi(\{0\} \times S^k)$ and the belt sphere $S^{n-k-1} \times \{0\}$ equals ± 1, i.e. we have

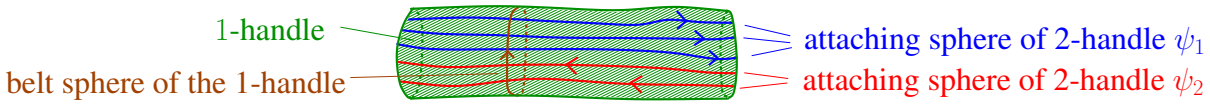
$$\psi(\{0\} \times S^k) \cdot (S^{n-k-1} \times \{0\}) \;=\; \pm 1.$$

Proof. We have the following equalities:

$$\psi(\{0\} \times S^k) \cdot (S^{n-k-1} \times \{0\}) \;\underset{\text{follows from the Homotopy } \mathrm{H}_*\text{-Proposition 144.12}}{=}\; \Sigma \cdot (S^{n-k-1} \times \{0\}) \;\underset{\text{by the Cup Product–Intersection Number Theorem 220.10 and our hypothesis}}{=}\; \pm 1.$$

■

In the following, very schematic, picture we show a 1-handle and we show two attaching spheres $\psi_1(\{0\} \times S^1)$ and $\psi_2(\{0\} \times S^1)$ for 2-handles such that the algebraic intersection numbers with the belt sphere b are 3 and -2. It follows from the Isotopy–Handle Cancellation Requirement Lemma 260.8 that we will never be able to cancel the 1-handle with one of the two 2-handles, even after applying smooth isotopies to the attaching spheres.



But now a very vague idea emerges: The sum of the algebraic intersection numbers is $3 - 2 = 1$. Perhaps we can build out of the two 2-handle attachments another 2-handle attachment ψ such that $[\psi(\{0\} \times S^k)] = [\psi_1(\{0\} \times S^k)] + [\psi_2(\{0\} \times S^k)]$. This would imply that

$$\psi(\{0\} \times S^k) \cdot b \;=\; \psi_1(\{0\} \times S^k) \cdot b - \psi_2(\{0\} \times S^k) \cdot b \;=\; 3 - 2 \;=\; 1.$$

This new 2-handle then has at least a chance of forming a canceling pair with the 1-handle.

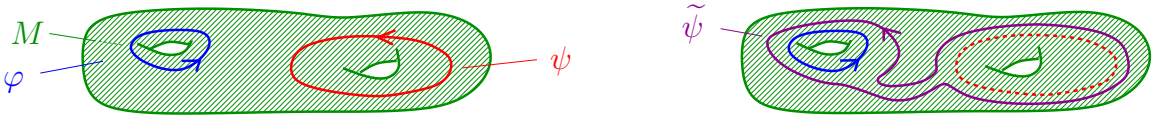
In the next section we will see that in many situations this procedure of finding new handles can indeed be done by a procedure called "handle slide".

260.4. **The Handle Slide Theorem.** In this section we will introduce the notion of a handle slide. To formulate this notion succinctly let us introduce the following notation:

Notation 260.9. Let W be an m-dimensional smooth manifold and let $\gamma\colon \overline{B}^{m-l} \times S^l \to W$ be a smooth embedding. We write $[\gamma] := \gamma_*([\{0\} \times S^l]) \in \mathrm{H}_l(W)$.

With this notation we can formulate the goal of this section: The idea (or hope) is that given two thickened $(k-1)$-spheres φ and ψ one can "slide the ψ-sphere over the φ-handle to obtain a new thickened $(k-1)$-sphere $\widetilde{\psi}$ with the following two properties:

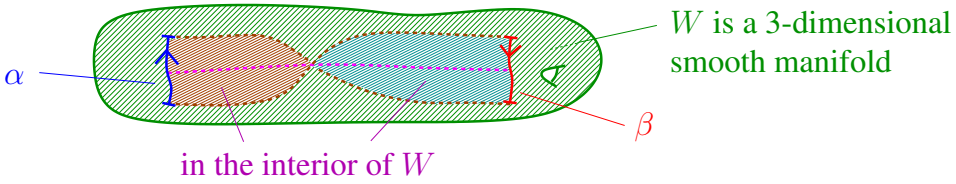
(1) The smooth manifolds $M \cup_\varphi h^k \cup_\psi h^k$ and $M \cup_\varphi h^k \cup_{\widetilde{\psi}} h^k$ are diffeomorphic.
(2) With Notation 260.9 we have $[\widetilde{\psi}] = [\psi] \pm [\varphi] \in \mathrm{H}_{k-1}(\partial M)$.



One of the keys to creating a new handle attachment out of two given handle attachments is the following, rather general, definition:

Definition 260.10. Let W be an m-dimensional smooth manifold, let $l \in \{0, \dots, m-1\}$ and $\alpha, \beta\colon \overline{B}^l \to \partial W$ be smooth embeddings. An internal band from α to β is a smooth embedding $\Xi\colon \overline{B}^l \times [0,1] \to W$ with the following properties:

(1) $\Xi_0 = \alpha$ and $\Xi_1 = \beta$.
(2) $\Xi(\overline{B}^l \times [0,1])$ is a submanifold with corner such that $\Xi(\overline{B}^l \times [0,1]) \cap \partial W = \Xi(\overline{B}^l \times \{0,1\})$.



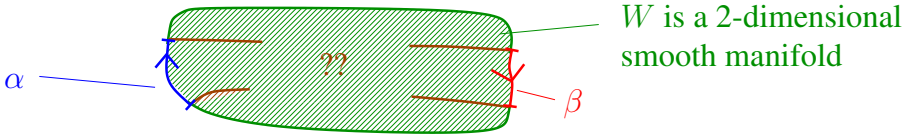
We will soon see that it is desirable to have internal bands. Fortunately the following proposition shows that in most settings we can indeed find such internal bands.[2780]

Proposition 260.11. (Internal Band Proposition) Let W be an m-dimensional smooth manifold, let $l \in \{0, \dots, m-1\}$ and $\alpha, \beta \colon \overline{B}^l \to \partial W$ be smooth embeddings with disjoint images.

(1) If $l \leq m-2$, then we can find an internal band from α to β.

(2) If $l = m-1$, then we can find an internal band from α to β or from α to $\beta \circ \rho$ where $\rho \colon \overline{B}^l \to \overline{B}^l$ is the reflection in a hyperplane.

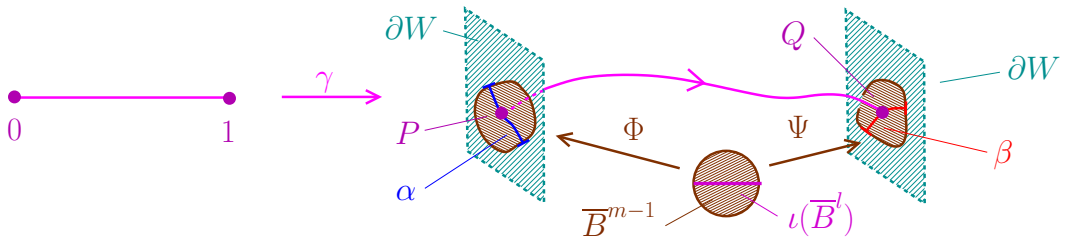
Example. The example below shows that for $l = m-1$ we run into troubles if α and β "point into opposite directions".



This explains why in the Internal Band Proposition 260.11 we need a little bit of extra flexibility if $l = m-1$. □

Proof. [2781] Let W be an m-dimensional smooth manifold. In the following we consider the case that $l \in \{0, \dots, m-2\}$. We leave it to the reader to modify the proof for the case $l = m-1$. Let $\alpha, \beta \colon \overline{B}^l \to \partial W$ be smooth embeddings with disjoint images. We set $P := \alpha(0)$ and $Q := \beta(0)$. We make a few preparations:

- Since W is path-connected it follows from the Whitney Approximation Theorem 36.1 together with the Curve–Smooth Embedding Theorem **??** that there exists a proper smooth embedding $\gamma \colon [0,1] \to W$ with $\gamma(0) = P$ and $\gamma(1) = Q$.
- We denote by $\iota \colon \overline{B}^l \to \overline{B}^{m-1}$ the smooth embedding that is given by $x \mapsto (x, 0)$. It follows easily from the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 that there exists a smooth embedding $\Phi \colon \overline{B}^{m-1} \to \partial W$ with $\Phi \circ \iota = \alpha$ and that there exists a smooth embedding $\Psi \colon \overline{B}^{m-1} \to \partial W$ with $\Psi \circ \iota = \beta$.



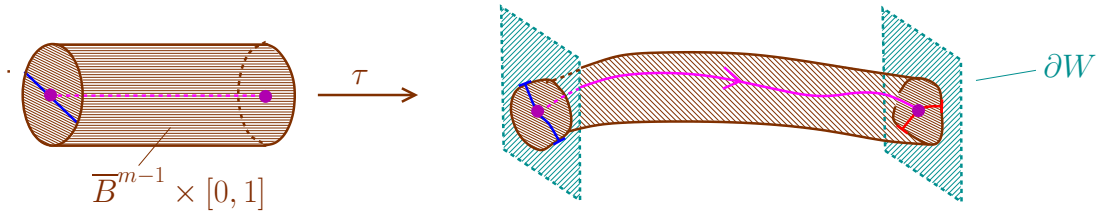
Next note that it follows easily from the Tubular Neighborhood–Boundary Extension Theorem 57.29 together with the Smooth Bundles-Homotopies Proposition 243.10 (2) that there exists a smooth embedding $\tau \colon \overline{B}^{m-1} \times [0,1] \to W$ with the following properties:

(1) For each $t \in [0,1]$ we have $\tau(0,t) = \gamma(t)$.

[2780] Note that the following proposition is related to the results in Section 187.7 where we discussed the internal connected sum of submanifolds.

[2781] Alternative proofs for this proposition are given in [**Muk2015**, Theorem 10.7.1] and [**Wall1962**, Theorem 5.4.5].

(2) We have $\tau_0 = \Phi\colon \overline{B}^{m-1} \to \partial W$.
(3) There exists an $A \in \mathrm{O}(m-1)$ such that for every $x \in \overline{B}^{m-1}$ we have $\tau_1(x) = \Psi(A \cdot x)$.



Claim. There exists a $P \in \mathrm{SO}(m-1)$ such that for all $x \in \overline{B}^{l}$ we have $A \cdot P \cdot (x,0) = (x,0)$.

Proof. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_{m-1}$ be the standard basis for $\mathbb{R}^{m-1}$. For $i = 1, \dots, l$ we consider the vector $v_i := A^{-1} \cdot \mathfrak{e}_i$. Since $l < m-1$ we obtain from elementary linear algebra that there exist vectors $v_{l+1}, \dots, v_{m-1} \in \mathbb{R}^{m-1}$ such that $P := (v_1 \ \dots \ v_{m-1})$ is an orthogonal matrix of determinant one (for the last condition we use that $l < m-1$), in other words, such that $P := (v_1 \ \dots \ v_{m-1}) \in \mathrm{SO}(m-1)$. By construction we have $P \cdot \mathfrak{e}_i = A^{-1} \cdot \mathfrak{e}_i$ for $i = 1, \dots, l$. This implies that $P \cdot (x,0) = A^{-1} \cdot (x,0)$ for all $x \in \overline{B}^{l}$, which gives us the desired result. ⊞

By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(m-1)$ is path-connected. It follows from the Whitney Approximation Theorem 36.1 that there exists a smooth path $\delta\colon [0,1] \to \mathrm{SO}(m-1)$ with $\delta(0) = \mathrm{id}$ and $\delta(1) = P$. One can now easily verify that the map

$$\begin{array}{rccc} \Xi\colon & \overline{B}^{l} \times [0,1] & \to & W \\ & (x,t) & \mapsto & \tau_t(\delta(t) \cdot (x,0)) \end{array}$$

has all the right properties. ■

Next we will see that we can use such internal bands to slide one handle over another. Before we can formulate this result we need to introduce some more notation:

Notation 260.12. Let $m \in \mathbb{N}_0$.
(1) As usual we write $S^m_{\geq 0} := \{(x_1, \dots, x_{m+1}) \in S^m \mid x_{m+1} \geq 0\}$ and we use obvious variations on this notation.
(2) As on page 1229 let $\iota_+\colon \overline{B}^m \to S^m_{\geq 0}$ and $\iota_-\colon \overline{B}^m \to S^m_{\leq 0}$ be the diffeomorphisms that are given by the inverses of stereographic projection from the South Pole S respectively the North Pole N.
(3) For $i = 0, 1$ we write $\overline{B}^m_i := \{i\} \times \overline{B}^m$ and we consider the map

$$\overline{B}^m_0 \cup ([0,1] \times S^{m-1}) \cup \overline{B}^m_1 \ \to \ S^m$$

which is given by:
- A diffeomorphism $\overline{B}^m_0 \to S^m_{\leq -\frac{1}{2}}$ as in (2).
- A diffeomorphism $\overline{B}^m_1 \to S^m_{\geq \frac{1}{2}}$ as in (2).
- The obvious diffeomorphism $[0,1] \times S^{m-1} \to S^m_{[-\frac{1}{2}, \frac{1}{2}]}$.

As we saw in Exercise 55.8, using the Bicollar Neighborhood Theorem 55.2 one can easily show that for appropriate collars the above map $\overline{B}^m_0 \cup ([0,1] \times S^{m-1}) \cup \overline{B}^m_1 \to S^m$ is a diffeomorphism. In the following we will make use this diffeomorphism to make an identification $S^m =_{\mathrm{i}} \overline{B}^m_0 \cup ([0,1] \times S^{m-1}) \cup \overline{B}^m_1$.

(4) Given any map $\varphi\colon X \times S^m \to Y$ and given $x \in X$ we consider the maps

$$\begin{array}{rcl} \varphi_{x,\pm}\colon \overline{B}^m & \to & Y \\ z & \mapsto & \varphi(x, \underbrace{\iota_\pm(z)}_{\in S^m}). \end{array}$$



Now we can finally provide the promised theorem:

Theorem 260.13. (Handle Slide Theorem) We start out with the following data:

- Let M be an n-dimensional smooth manifold.
- Let $\varphi, \psi\colon \overline{B}^{n-k}_2 \times S^{k-1} \to \partial M$ be two thickened $(k-1)$-spheres (where we use balls of radius 2) with disjoint images. We set[2782] $W := \partial M \backslash (\varphi(B^{n-k} \times S^{k-1}) \cup \psi(B^{n-k} \times S^{k-1}))$.
- We pick $P, Q \in \partial \overline{B}^{n-k}_2 = S^{n-k-1}_2$.
- Let $\rho\colon \overline{B}^{k-1} \to \overline{B}^{k-1}$ be either the identity or the reflection in a hyperplane.
- Finally let $\Xi\colon [0,1] \times \overline{B}^{k-1} \to W$ be an internal band from the smooth embedding $\beta := \psi_{Q,+}\colon \overline{B}^{k-1} \to \partial W$ to the smooth embedding $\alpha := \varphi_{P,+} \circ \rho\colon \overline{B}^{k-1} \to \partial W$.

With this data the following statements hold:

(1) We consider the map[2783]

$$G\colon \underbrace{\overline{B}^{k-1}_0 \cup ([0,1] \times S^{k-2}) \cup \overline{B}^{k-1}_1}_{=_{\mathrm{i}}\, S^{k-1},\ \text{see Notation 260.12}} \to \partial M$$

that is given as follows:

(a) $G|_{\overline{B}^k_0}\colon \overline{B}^k_0 \to \partial W$ is given by $\beta\colon \overline{B}^k \to \partial W$.

(b) $G|_{\overline{B}^k_1}\colon \overline{B}^k_1 \to \partial W$ is given by $\alpha\colon \overline{B}^k \to \partial W$.

(c) $G|_{[0,1]\times S^{k-1}}$ is given by $\Xi|_{[0,1]\times \partial \overline{B}^{k-1}}$.

This map is (after rounding corners which we will ignore) a smooth embedding.

Theorem 260.13. (Handle Slide Theorem)

(2) The smooth embeddings $G\colon S^{k-1} \to \partial(M \cup_\varphi h^k)$ and $S^{k-1} \xrightarrow{x \mapsto \psi(0,x)} \partial(M \cup_\varphi h^k)$ are smoothly isotopic.

(3) There exists a thickened $(k-1)$-sphere $\widetilde{\psi}\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ with the following three properties:

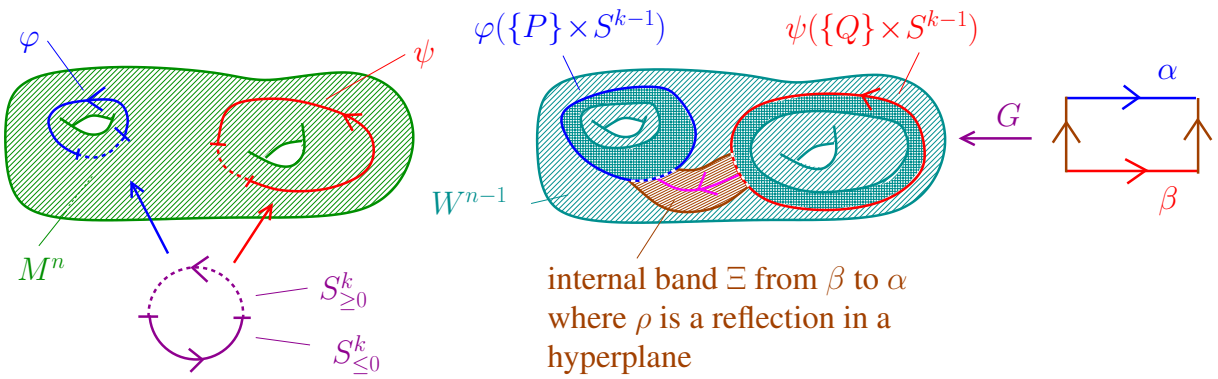
(a) For each $x \in S^{k-1}$ we have $\widetilde{\psi}(0,x) = G(x)$.

(b) In $\partial(M \cup_\varphi h^k)$ the smooth embedding $\widetilde{\psi}$ is smoothly isotopic to ψ.

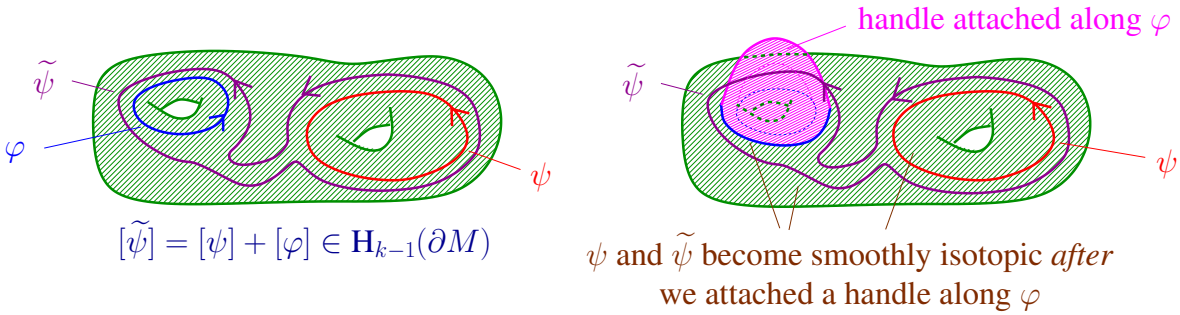
(c) The image of $\widetilde{\psi}$ is disjoint from the image of φ.

[2782] It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that W is an $(n-1)$-dimensional smooth manifold where $\varphi(S^{n-k-1}_2 \times S^{k-1})$ and $\psi(S^{n-k-1}_2 \times S^{k-1})$ are boundary components.

[2783] In the Internal Connected Sum Construction 187.28 we will refer to the map G as an "internal connected sum" of the spheres $\varphi(\{P\} \times S^{k-1})$ and $\psi(\{Q\} \times S^{k-1})$.



(4) The smooth manifolds $M \cup_\varphi h^k \cup_\psi h^k$ and $M \cup_\varphi h^k \cup_{\widetilde{\psi}} h^k$ are diffeomorphic.
(5) (a) If $\overline{B}^{k-1} \xrightarrow{\rho} \overline{B}^{k-1}$ is a reflection in a hyperplane, then $[\widetilde{\psi}] = [\psi] + [\varphi] \in \mathrm{H}_{k-1}(\partial M)$.
(b) If $\rho = \mathrm{id}_{\overline{B}^{k-1}}$, then $[\widetilde{\psi}] = [\psi] - [\varphi] \in \mathrm{H}_{k-1}(\partial M)$.[2784]



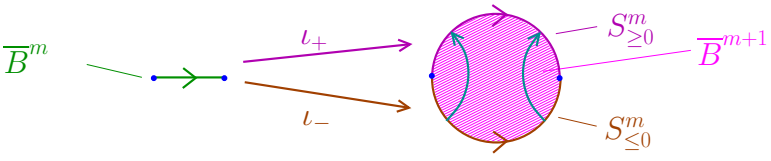
Slang 260.14. In the setting of the Handle Slide Theorem 260.13 we say sometimes that we slide the ψ-handle over the φ-handle.

Sketch of proof.

(1) One can easily verify that the map $G\colon \overline{B}_0^{k-1} \cup ([0,1] \times S^{k-2}) \cup \overline{B}_1^{k-1} \to \partial M$ is well-defined and that it is an embedding. By construction the map is furthermore a smooth embedding on three pieces $\overline{B}_0^{k-1}$, $[0,1] \times S^{k-1}$ and $\overline{B}_1^{n-k}$. We leave it to the reader to make sense of the statement that after rounding corners this map G is a smooth embedding.

(2) We start out with the following claim:

Claim. Let $m \in \mathbb{N}_0$. As in Notation 260.12 we consider the two smooth embeddings $\iota_-\colon \overline{B}^m \to S^m$ and $\iota_+\colon \overline{B}^m \to S^m$ into the lower (respectively upper) hemisphere from that are given by the inverse of the stereographic projection from the North (respectively South) Pole. If we consider these as maps to $\overline{B}^{m+1}$, then they become smoothly isotopic.



[2784] Here we use Notation 260.9.

Proof. For $m = 1$ one can write down an explicit smooth isotopy. For the general case one can take the smooth isotopy for $m = 1$ and "rotate it around the x_1-axis". We will not attempt to write down an explicit smooth isotopy. ⊞

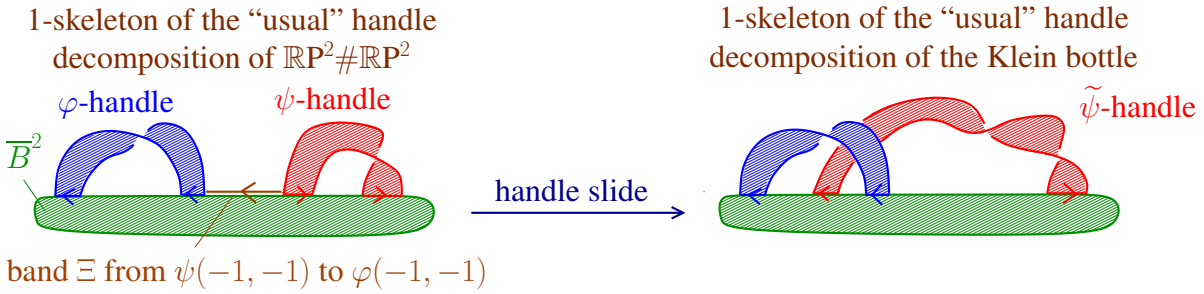
Using this smooth isotopy it is hopefully fairly straightforward to show that a smooth isotopy between ψ and $\widetilde{\psi}$ is given by "by sliding ψ along the band Ξ and by sliding over the boundary of k-handle". We leave it to the reader to fill in the details.

(3) It follows from (2) and the Isotopy Extension Theorem 44.2 that there exists a smooth embedding $\widetilde{\psi}\colon \overline{B}^{m-k} \times S^{k-1} \to \partial(M \cup_\varphi h^k)$ that is smoothly isotopic to ψ and which has the property that for any $x \in S^{k-1}$ we have $\widetilde{\psi}(0, x) = G(x)$. Since the image of G is disjoint from the image of φ one can easily modify $\widetilde{\psi}$ such that its image is also disjoint from the image of φ.

(4) It follows immediately from (3) together with the Handle Attachment Proposition 246.3 (5) that the smooth manifolds $M \cup_\varphi h^k \cup_\psi h^k$ and $M \cup_\varphi h^k \cup_{\widetilde{\psi}} h^k$ are diffeomorphic.

(5) This statement follows easily from the Recombine Fundamental Class Proposition 187.27. We will sketch an alternative proof in Exercise 260.8. ■

Example. In the figure below we see the disk together with 0-dimensional attaching spheres $\varphi\colon \overline{B}^1 \times S^0 \to \partial\overline{B}^2$ and $\psi\colon \overline{B}^1 \times S^0 \to \partial\overline{B}^2$ for two 1-handles.



We also show a band Ξ (in this case a path) from $\psi(-1, -1)$ to $\varphi(-1, -1)$. The reader will undoubtedly notice that the resulting handle slide is essentially the same as the handle slide that we discuss on page 4919 where we showed that the non-orientable surface of genus 2 is diffeomorphic to the Klein bottle. □

260.5. **The Cerf Theorem.** We have now encountered several ways how one can modify a handle decomposition:

- By the Handle Attachment Proposition 246.3 (5) we can change thickened spheres up to smooth isotopy.
- By the Handle Cancellation Theorem 260.1 we can, with some luck, cancel two handles.
- By the Canceling Pair–Creation Lemma 260.5 we can always create a pair of canceling handles.
- By the Handle Slide Theorem 260.13 we can often slide one handle over another.

It turns out that we now have a complete list of moves needed to relate the handle decompositions of a fixed smooth manifold:

Theorem 260.15. (Cerf Theorem) Let M be a compact smooth manifold and let A be a union of components of ∂M. Any two handle decompositions of M rel A are related by a finite sequence of

- diffeomorphisms that are diffeotopic to the identity,
- creation of a canceling pair of handles of index k and $k+1$ (in the sense of Definition 260.3),
- cancellation of pairs of handles of index k and $k+1$,
- handle slides (in the sense of the Handle Slide Theorem 260.13 and Slang 260.14).

If we start out with two standard handle decompositions, then we can relate them by a finite sequence as above such that each intermediate handle decomposition is also standard.

Example. It follows immediately from the Cerf Theorem 260.15 that for a compact smooth manifold M with a handle decomposition the Euler characteristic

$$\chi(M) \ := \ \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \text{number of } k\text{-handles of } M$$

is a diffeomorphism invariant. On a more sophisticated level one can use handles and algebraic intersection numbers to define homology groups of smooth manifolds and one can use the Cerf Theorem 260.15 to show that the isomorphism types of these homology groups are well-defined, without making use of singular homology. The details for this approach are worked out in [**DGK2019**]. □

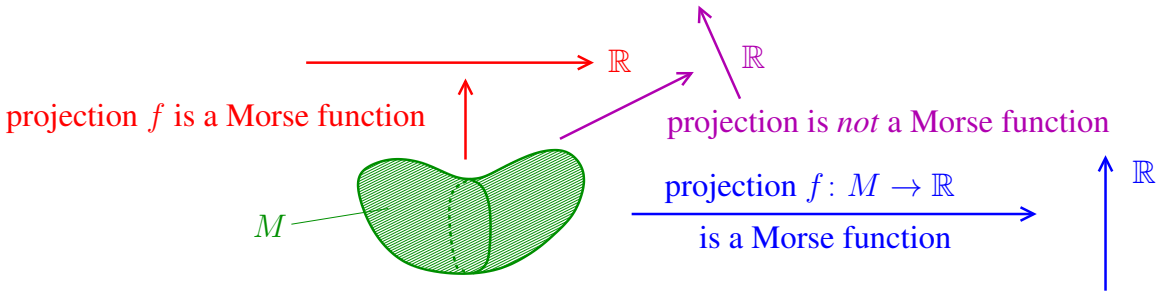
Proof. It seems to be well-known, see [**GoS1999**, p. 110] and [**Lic1997b**, p. 375] that this theorem follows from the work of Jean Cerf [**Cer1970**] (see also [**Cer1968b**] and [**HatW1973**].) Unfortunately the authors forgot to point to the precise statement in Cerf's paper which gives us the desired result. At least in the closed case the theorem should also follow from [**Lau2014**, Theorem 1.6].

The very very rough idea is as follows: Suppose we are given two (standard) handle decompositions for M rel A. In the Handle Decomposition–Morse Function Theorem 251.2 we showed that these give rise to two Morse functions $f, g\colon M \to \mathbb{R}$ rel A. Evidently the Morse functions are smoothly homotopic rel A. Thus we can find a smooth map $H\colon M \times [0,1] \to \mathbb{R}$ with $H_0 = f$ and $H_1 = g$. In a suitable sense a generic smooth function is a Morse function. One can now show that one can find a smooth map H as above such that H_t is a Morse function for all but finitely many $t_1, \ldots, t_m \in (0,1)$, and such that the handle decompositions "before and after" such a t_i differ by the cancellation or creation of a pair of handles. A slightly less rough sketch of the proof is given in [**GoS1999**, p. 110]. ■

Example. We consider the 2-dimensional "skin of a potato" that is shown in the figure below. We also show the following two maps:

- The projection $f\colon M \to \mathbb{R}$ onto the z-axis. This a Morse function with one critical point of index 0, one critical point of index 1 and two critical points of index 2.
- The projection $g\colon M \to \mathbb{R}$ onto the x-axis. This a Morse function with one critical point of index 0 and one critical point of index 2.

There is an obvious way to "deform" the Morse function f to the Morse function g, namely we just rotate the projection axis. In the figure above we show an inbetween projection that is *not* a Morse function. □



For the time being we will not make use of the Cerf Theorem 260.15. The main application much later will be in the proof of the Torsion–Diffeomorphism Theorem **??** when we show that "Reidemeister torsion" of a smooth manifold does not depend on the choice of a handle decomposition.

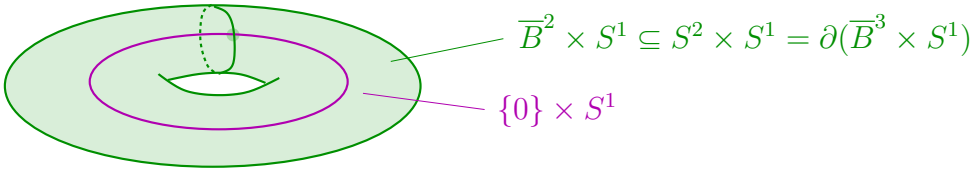
260.6. **Appendix: The Mazur manifold.** The goal of this appendix is to show that there exists a compact 4-dimensional smooth manifold that is contractible but such that the boundary is not homotopy equivalent to a sphere. The construction of the Mazur manifold is a special case of the following more general construction:

Definition 260.16. Given any oriented 1-dimensional smooth submanifold $\varphi \subseteq \overline{B}^2 \times S^1$ we construct a smooth manifold $M(\varphi)$ as follows:

(1) We start out with $\overline{B}^3 \times S^1$.
(2) We use stereographic projections to identify $\overline{B}^2$ with the upper hemisphere $S^2_{\geq 0}$. We also use this identification to consider the solid torus $\overline{B}^2 \times S^1$ as a smooth submanifold of $\partial(\overline{B}^3 \times S^1)$.
(3) We also consider the solid torus $\overline{B}^2 \times S^1$ as a submanifold of $\mathbb{R}^3$ via the standard smooth embedding that we introduced in the Torus Embedding Lemma 7.11.
(4) By the Knot Thickening Proposition 259.8 and the Knot Thickening–Self Linking Lemma 284.11 there exists an (essentially unique) thickening $\Phi\colon \overline{B}^2 \times S^1 \to \overline{B}^2 \times S^1 \subseteq \mathbb{R}^3$ of the knot $\varphi \subseteq \overline{B}^2 \times S^1 \subseteq S^3$ with self-linking number 0.

We set $M(\varphi) := (\overline{B}^3 \times S^1) \cup_\Phi h^2$.

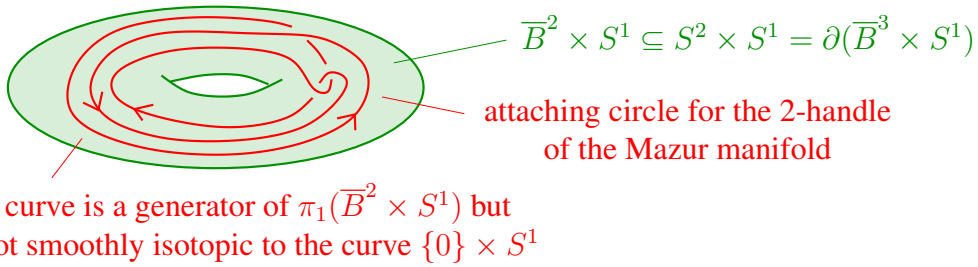
Example. We first consider the case that $\varphi\colon S^1 \to \overline{B}^2 \times S^1$ is the obvious smooth embedding $z \mapsto (0, z)$. Note that, as in Definition 260.16, the map φ gives rise to a smooth embedding $S^1 \to \overline{B}^2 \times S^1 \to S^2_{\geq 0} \times S^1 \to \partial(\overline{B}^3 \times S^1)$.



It follows immediately from the Basic Handle Cancellation Lemma 260.2 that the resulting smooth manifold $M(\varphi)$ is diffeomorphic to $\overline{B}^4$. □

Now we move on to a much more interesting example, where we take a smooth embedding that is homotopic but not smoothly isotopic to the above "trivial" smooth embedding $S^1 \to \overline{B}^2 \times S^1$:

Definition 260.17. We consider the smooth embedding $\varphi\colon S^1 \to \overline{B}^2 \times S^1 \subseteq \partial(\overline{B}^3 \times S^1)$ that is shown in the figure below. We refer to the compact 4-dimensional smooth manifold $M(\varphi) := (\overline{B}^3 \times S^1) \cup_\Phi h^2$ as the Mazur manifold.[2785]



The following proposition summarizes some properties of the Mazur manifold:

Proposition 260.18. (Mazur Manifold Proposition) The Mazur manifold M has the following properties:

(1) M is a compact orientable connected 4-dimensional smooth manifold.
(2) $\pi_1(M) = \{e\}$ and $\mathrm{H}_i(M) = 0$ for $i \geq 1$.
(3) M is contractible.
(4) ∂M is a homology sphere but $\pi_1(\partial M)$ is non-trivial. In particular ∂M is not homeomorphic to S^3.

Remark. Generalizations of the Mazur manifold are studied in detail in [**AK1979**]. □

Proof. Let M be the Mazur manifold.

(1) It follows immediately from the general theory of handle attachments, see the Handle Attachment Proposition 246.3, the Handle Attachment-π_0-Lemma 246.5 and the Handles-Orientability Proposition 246.6 that M is a compact orientable connected 4-dimensional smooth manifold.

In the following let $\Phi\colon \overline{B}^2 \times S^1 \to \overline{B}^2 \times S^1 \to \partial(\overline{B}^3 \times S^1)$ be the thickened 1-sphere in the definition of the Mazur manifold M.

(2) We pick $x \in S^2$ and we denote by $\alpha\colon S^1 \to \partial(\overline{B}^3 \times S^1)$ the map that is given by $\alpha(z) = (x, z)$. Next we perform the following calculation:

$$\pi_1(M) \;=\; \pi_1((\overline{B}^3 \times S^1) \cup_\Phi h^2) \;\underset{\substack{\uparrow\\ \text{by the Handle Attachment-}\pi_1\\ \text{Proposition 246.13}}}{\cong}\; \pi_1(\overline{B}^3 \times S^1)/\langle\langle \alpha \rangle\rangle \;\underset{\substack{\uparrow\\ \text{since } \alpha \text{ is a generator}\\ \text{of } \pi_1(\overline{B}^3 \times S^1) = \langle t \rangle}}{=}\; \langle t \rangle / \langle\langle [\alpha] \rangle\rangle \;=\; \{e\}.$$

Furthermore, using the fact that $\alpha\colon S^1 \to \overline{B}^3 \times S^1$ is a homology equivalence we obtain basically immediately from the Handle Attachment-H_*-Lemma 246.14 that $\mathrm{H}_i(M) = 0$ for $i \geq 1$.

(3) Let $P \in M$ be a point. By (2) we know that the inclusion $\{P\} \to M$ is homology equivalence. By (2) also know that M is simply connected. Thus it follows from the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 (which is a consequence of the Relative Hurewicz Theorem 296.25) that the inclusion-induced homomorphism $\pi_i(\{P\}, P) \to \pi_i(M, P)$ is an isomorphism for every $i \in \mathbb{N}$. Furthermore by the

[2785] The Mazur manifold was introduced by Barry Mazur [**Maz1961**].

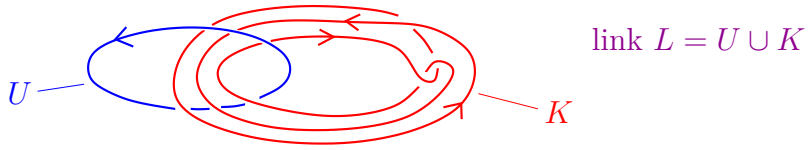
Smooth Manifold-CW-Theorem 179.9 we know that every compact smooth manifold admits a CW-structure. Thus we can apply the Whitehead Theorem 305.1 which tells us that the map $\{P\} \to M$ is a homotopy equivalence. In particular M is contractible.

(4) (a) As we discussed on page 5005 it follows easily from the Boundary-H_*-Lemma 259.13 (which in turn is a consequence of the Poincaré Duality Theorem 210.1) that the boundary of an n-dimensional homology ball is an $(n-1)$-dimensional homology sphere.[2786] This fact, together with (2) implies that ∂M is a homology sphere.

(b) We need to show that $\pi_1(\partial M)$ is non-trivial. In the following we will only sketch the argument that is given in [**Maz1961**, p. 227]. Note that

$$\partial M \overset{\text{by the Handle Attachment Proposition 246.3 (2) and by the decomposition } S^2 = \overline{B}^2_- \cup \overline{B}^2_+}{=} \underbrace{((\overline{B}^2_- \times S^1) \setminus \Phi(B^2 \times S^1))}_{\text{exterior } X_L \text{ of the link } L \subseteq S^3} \cup \underbrace{(S^1 \times \overline{B}^2)}_{\text{attached via } \Phi} \cup \underbrace{(\overline{B}^2_+ \times S^1)}_{\text{attached to } \overline{B}^2_- \times S^1} .$$

It follows easily from the Trivial Knot–Complement Proposition 111.11 that the solid torus $\overline{B}^2_- \times S^1$ is diffeomorphic to the exterior of the trivial knot U. It follows from this observation that $((\overline{B}^2_- \times S^1) \setminus \Phi(B^2 \times S^1))$ is diffeomorphic to the exterior X_L of the link $L = U \cup K$ that is shown below.



Furthermore one sees that the disks of the solid tori are glued in along the longitudes of U and K. Thus one sees that

$$\pi_1(\partial M) \overset{\text{by the above together with the Manifold–Seifert-van Kampen Theorem 102.1}}{\cong} \pi_1(X_L)/\langle\langle \text{longitudes of } U \text{ and } K \rangle\rangle \overset{\pi_1(S^3 \setminus L) \text{ can be computed using the Wirtinger algorithm that we introduced in the Wirtinger Presentation Proposition 113.6}}{\cong} \langle x, y \,|\, y^5 = x^7, y^4 = x^2 y x^2 \rangle.$$

Next one considers the epimorphism[2787]

$$\begin{array}{rcl} \langle x, y \,|\, y^5 = x^7, y^4 = x^2 y x^2 \rangle & \to & \langle y, z \,|\, z^7, y^5, (yz)^2 \rangle \\ x & \mapsto & z^4 \\ y & \mapsto & y. \end{array}$$

By the Van Dyck Group-Presentation Lemma 132.31 the group to the right is isomorphic to the $(2,5,7)$-von Dyck group that we had introduced in Definition 132.30. Furthermore, since $\frac{1}{2} + \frac{1}{5} + \frac{1}{7} < 1$ we obtain from the Van Dyck Group-Cardinality Corollary 132.32 that the group to the right is infinite. ■

260.7. **Exercises for Chapter 260.**

[2786] Alternatively see Exercise 210.22 for a proof of this statement.

[2787] A short calculation shows that the assignment $x \mapsto z^4$ and $y \mapsto y$ actually does descend to a homomorphism and that it is an epimorphism.

Exercise 260.1. Illustrate the Basic Handle Cancellation Lemma 260.2 with $n = 3$ and $k = 2$.

Exercise 260.2. Let $n \in \mathbb{N}$. Furthermore let $k \in \{0, \dots, n-1\}$. We consider the "solid torus" $\overline{B}^{n-k} \times S^k$ and we consider the thickened k-sphere

$$\begin{array}{rcl} \psi\colon \overline{B}^{n-k-1} \times S^k & \to & S^{n-k-1} \times S^k = \partial(\overline{B}^{n-k} \times S^k) \\ (x, y) & \mapsto & \left(\frac{(-1, x)}{\|(-1, x)\|}, y\right). \end{array}$$

Recall that in the Basic Handle Cancellation Lemma 260.2 we showed that the smooth manifold $(\overline{B}^{n-k} \times S^k) \cup_\psi h^{k+1}$ is diffeomorphic to $\overline{B}^n$. Now let $\gamma\colon S^k \to \mathrm{O}(n-k-1)$ be a smooth map. We consider the diffeomorphism

$$\begin{array}{rcl} \Xi\colon \overline{B}^{n-k-1} \times S^k & \to & \overline{B}^{n-k-1} \times S^k \\ (z, w) & \mapsto & (\gamma(w) \cdot z, w). \end{array}$$

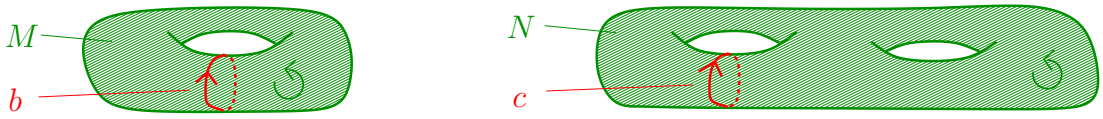
Is the smooth manifold $(\overline{B}^{n-k} \times S^k) \cup_\psi h^{k+1}$ also diffeomorphic to $\overline{B}^n$?

Exercise 260.3. We consider the two 2-dimensional smooth manifolds M (which is a torus) and N (which is a surface of genus two) that are shown below. We also consider the oriented 1-dimensional smooth submanifolds $b \subseteq M$ and $c \subseteq N$ that are shown below.

(a) Let $k \in \mathbb{Z}$. Can you find smooth embeddings $\varphi\colon S^1 \to M$ and $\psi\colon S^1 \to M$ with *disjoint images* such that $\varphi(S^1) \cdot b = k$ and $\psi(S^1) \cdot b = 1 - k$?

(b) Let $k \in \mathbb{Z}$. Can you find smooth embeddings $\varphi\colon S^1 \to N$ and $\psi\colon S^1 \to N$ with *disjoint images* such that $\varphi(S^1) \cdot c = k$ and $\psi(S^1) \cdot c = 1 - k$?

Remark. For sanity it might be best to stick to $k = 1$ and perhaps $k = 2$.



Exercise 260.4. Let M be an n-dimensional smooth manifold, let $\Phi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding and let $f\colon M \to N$ be a smooth map to some connected smooth manifold N. Finally let $Q \in N$. Show that f is smoothly homotopic to a smooth map $g\colon M \to N$ such that $(g \circ \Phi)(x) = Q$ for all $x \in \overline{B}^n$.

Hint. Note that it follows easily from the Codimension-Zero Smooth Embedding Proposition 30.21 together with the Smooth Collar Neighborhood Theorem 35.6 that we can extend Φ to a smooth embedding $\overline{B}^n_2 \to M \setminus \partial M$.

Exercise 260.5. Let M be a compact connected 3-dimensional smooth manifold with non-empty boundary. A *k-component tunnel system* for M consists of k disjoint submanifolds $T_1, \dots, T_k$ with corner of M with the following properties:

- Each T_i admits a diffeomorphism $\Phi_i\colon \overline{B}^2 \times \overline{B}^1 \to T_i$ such that $\Phi_i(\overline{B}^2 \times S^0) = T_i \cap M$.
- $M \setminus (\mathring{T}_1 \cup \dots \cup \mathring{T}_k)$ is, after rounding corners, a handlebody.

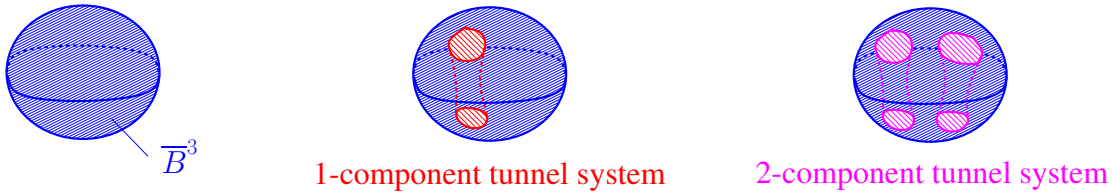
Now we can formulate our exercises:

(a) Show that M admits a tunnel system.

(b) Show that M admits tunnel systems with an arbitrarily large number of components.

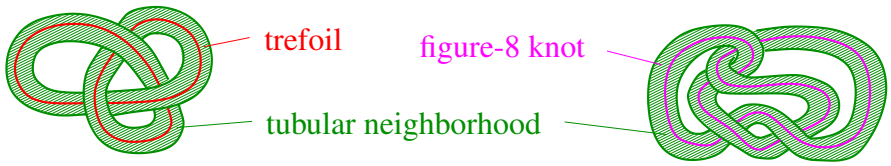
(c) The *tunnel number number of* M is defined as the minimal number of components of a tunnel system. Provide an example of a compact connected non-empty 3-dimensional smooth manifold of tunnel number 1.
(d) Let $\Phi\colon \overline{B}^3 \to \mathbb{R}\mathrm{P}^3$ be a smooth embedding. Determine the tunnel number of the smooth manifold $\mathbb{R}\mathrm{P}^3 \setminus \Phi(B^3)$.
(e) Show that there exist compact connected non-empty 3-dimensional smooth manifolds with non-empty boundary of arbitrarily large tunnel number.

Remark. We refer to Definition 26.14 for the definition of a submanifold with corner and we refer to the Smooth Submanifold-with-Corner Lemma 26.16 for a few facts about submanifolds with corner.



Exercise 260.6. Given any knot K we can consider, as in Definition 108.30, the knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$ where $\tau\colon \overline{B}^2 \times K \to S^3$ is a tubular map for K. We define the *tunnel number of* K to be the tunnel number of X_K, as defined in Exercise 260.5.

(a) Find upper bounds on the tunnel number of the trefoil.
(b) Find upper bounds on the tunnel number of the figure-8 knot.



Exercise 260.7. Given any knot K we can consider, as in Definition 108.30, the knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$ where $\tau\colon \overline{B}^2 \times K \to S^3$ is a tubular map for K.

(1) Show that any diagram of K gives rise to a handle decomposition.
(2) We can change a diagram by the Reidemeister moves from the Reidemeister Moves Theorem 112.10. Show that the resulting handle decompositions are related by a sequence of handle cancellations and handle slides.

Remark. This exercise is on purpose a little vague. It is meant as food for thought, and not for completely rigorous proofs.

Exercise 260.8. Let $k \in \mathbb{N}$. As in Notation 260.12 let $\iota_+\colon \overline{B}^k \to S^k_{\geq 0}$ be a homeomorphism to the upper hemisphere and let

$$\Theta\colon (\overline{B}^k \times \{-1\}) \cup (S^{k-1} \times [-1,1]) \cup (\overline{B}^k \times \{1\}) \;\to\; S^k$$

be the diffeomorphism from Notation 260.12.

(a) We consider the disjoint union $(S^k \times \{-1\}) \sqcup (S^k \times \{1\})$ and we consider the topological space

$$X \;:=\; (S^k \times \{-1\}) \sqcup (\overline{B}^k \times [-1,1]) \sqcup (S^k \times \{1\}) / \sim$$

where given (x,i) with $x \in \overline{B}^k$ and $i \in \{\pm 1\}$ we identify $(x,i) \in \overline{B}^k \times \{i\}$ with $(\iota_+(x),i) \in S^k \times \{i\}$. For $i = -1, 1$ we denote by $\xi_i \colon S^k \to X$ the obvious map and we denote by $\varphi \colon (\overline{B}^k \times \{-1\}) \cup (S^{k-1} \times [-1,1]) \cup (\overline{B}^k \times \{1\}) \to X$ the map that is induced by the homeomorphism $\iota_- \colon \overline{B}^k \to S^k_{\leq 0}$ Show that

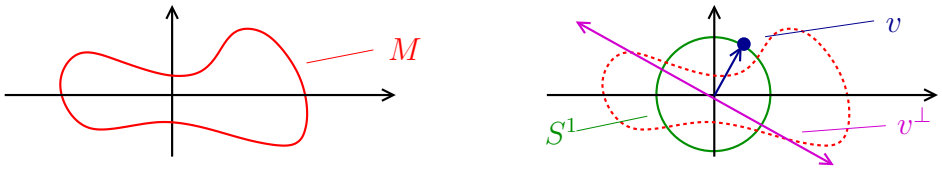
$$\xi_{-1,*}([S^k]) + \xi_{1,*}([S^k]) \;=\; (\varphi \circ \Theta^{-1})_*([S^k]).$$

(b) Use (a) to prove Statement (5) of the Handle Slide Theorem 260.13.



Exercise 260.9. We consider the (fairly) explicit closed 1-dimensional smooth submanifold $M \subseteq \mathbb{R}^2$ that is shown in the figure below. For each $v \in S^1$ we consider the map $\pi_v \colon M \to \mathbb{R}$ that is given by orthogonal projection onto $v^\perp \cong \mathbb{R}$.

(a) For how many $v \in S^1$ is π_v *not* a Morse function?
(b) What is the maximal number of critical points of index 1 for a Morse function π_v?
(c) What is the minimal number of critical points of index 1 for a Morse function π_v?

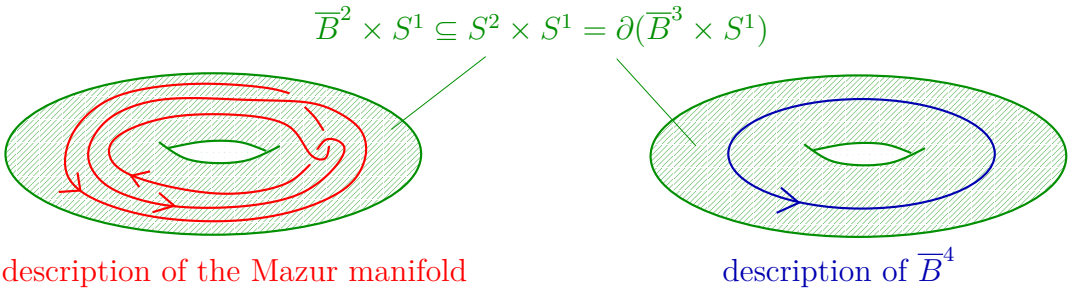


Exercise 260.10. Given a closed 1-dimensional smooth submanifold $M \subseteq \mathbb{R}^2$ and given $v \in S^1$ we consider, as in Exercise 260.9, the map $\pi_v \colon M \to \mathbb{R}$ that is given by orthogonal projection onto $v^\perp \cong \mathbb{R}$. We set $c_M := \{v \in S^1 \,|\, \pi_v \colon M \to \mathbb{R} \text{ is not a Morse function}\}$.

(a) Is the set c_M necessarily finite?
(b) Does there exist a closed 1-dimensional smooth submanifold $M \subseteq \mathbb{R}^2$ and an isolated $v \in c_M$ such that the number of critical points "before and after" v is the same?

Exercise 260.11. In the figure below we show to the left the description of the Mazur manifold M.

(a) We interpret the picture in the figure below to the right the same way as we interpret the picture to the left. Show that the smooth manifold it describes is diffeomorphic to $\overline{B}^4$.
(b) Show that $\overline{B}^1 \times M$ is diffeomorphic to $\overline{B}^1 \times \overline{B}^4 = \overline{B}^5$.
Hint. Make use of the Handle Attachment Proposition 246.3 (5).


$\overline{B}^2 \times S^1 \subseteq S^2 \times S^1 = \partial(\overline{B}^3 \times S^1)$
description of the Mazur manifold
description of $\overline{B}^4$

CHAPTER 261

Stiefel manifolds

In the previous chapter we saw that it would be desirable if one could achieve the following: If two submanifolds have *algebraic* intersection number ± 1, then we would like to find a smooth isotopy which assures that even the *geometric* intersection number is $= 1$. In the next chapter we will state and prove the Whitney Trick-Theorem 262.5 which says that in some settings this dream can be achieved.

In preparation for the proof of the Whitney Trick-Theorem 262.5 we will need to switch gears for a minute namely we will need to introduce Stiefel manifolds. These are interesting in their own right. But for us the only role they will play, is that will allow us to prove the Extend Frame-to-Ball Proposition 261.11, which will be a tool in the proof of the Whitney Trick-Theorem 262.5.

This chapter is organized as follows:

- In Section 261.1 we will provide the definition of Stiefel manifolds and we will show that they are indeed smooth manifolds.
- In Section 261.2 we will calculate some homotopy groups of Stiefel manifolds.
- In Section 261.3 we will use the calculation of homotopy groups of Stiefel manifolds to prove the Extend Frame-to-Ball Proposition 261.11.
- In Section 261.4 we will provide exercises for Chapter 261.

In the next Chapter 262 we will formulate and prove the Whitney trick.

261.1. **Definition of Stiefel manifolds.** We start out with the following definition which arises from linear algebra:

Definition 261.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let V be a $\mathbb{K}$-vector space and let $n \in \mathbb{N}_0$. An n-frame in V is an n-tuple $(v_1, \ldots, v_n)$ of linearly independent vectors in V.

Definition 261.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let V be a finite-dimensional $\mathbb{K}$-vector space and let $n \in \mathbb{N}_0$. We consider[2788]

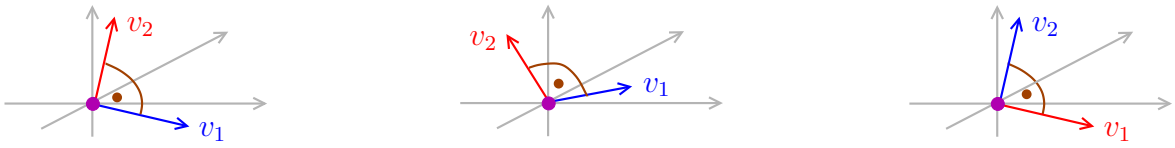
$$\mathrm{GL}_n(V) \;:=\; \text{the set of } n\text{-frames in } V$$

which is a subset of V^n. Since V is a finite-dimensional $\mathbb{K}$-vector space we know by the discussion on page 183 that it comes with a standard topology. We equip V^n with the product topology and we equip $\mathrm{GL}_n(V)$ with the subspace topology.

Examples. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $m \in \mathbb{N}$. Note that by definition we have $\mathrm{GL}_1(\mathbb{K}^m) = \mathbb{K}^m \setminus \{0\}$ and we have $\mathrm{GL}_m(\mathbb{K}^m) = \mathrm{GL}(m, \mathbb{K})$. □

[2788]For $V = \mathbb{K}^m$ we obtain the topological space $G_n(\mathbb{K}^m)$ which we already encountered in the Matrix Groups Deformation Retract Proposition 19.4 under the name $\mathrm{GL}(m, n)$.

Definition 261.3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $(V, \langle -, - \rangle)$ be a unitary $\mathbb{K}$-vector space. An n-frame $(v_1, \ldots, v_n)$ in V is called orthonormal if for all $i, j \in \{1, \ldots, n\}$ we have $\langle v_i, v_j \rangle = \delta_{ij}$.



three different orthonormal 2-frames in $V = \mathbb{R}^3$

Definition 261.4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given natural numbers $m, n \in \mathbb{N}_0$ we define the Stiefel manifold[2789] [2790]

$$V_n(\mathbb{K}^m) \quad := \quad \text{the set of orthonormal } n\text{-frames in } \mathbb{K}^m$$

which is a subset of $(\mathbb{K}^m)^n$. We equip $V_n(\mathbb{K}^m)$ with the subspace topology.

Examples. Let $m \in \mathbb{N}$.

(1) By definition we have $V_1(\mathbb{R}^m) = S^{m-1}$ and $V_m(\mathbb{R}^m) = \mathrm{O}(m)$. Furthermore we will see in Exercise 261.5 that the map

$$\begin{array}{rcl} \mathrm{SO}(m) & \to & V_{m-1}(\mathbb{R}^m) \\ (v_1 \ \ldots \ v_{m-1} \, v_m) & \mapsto & (v_1, \ldots, v_{m-1}) \end{array}$$

is a homeomorphism.

(2) By definition we have $V_1(\mathbb{C}^m) = S^{2m-1}$ and $V_m(\mathbb{C}^m) = \mathrm{U}(m)$. □

The following proposition, which is basically a slight generalization of the Matrix Group-Smooth Manifold Proposition 32.10, justifies the name Stiefel *manifold*.

Proposition 261.5. (Stiefel Manifold-Smooth Manifold Proposition) In the following let $m, n \in \mathbb{N}_0$ with $m \geq n$.

(1) $V_n(\mathbb{R}^m)$ is a closed smooth manifold of dimension $mn - \frac{1}{2}n(n+1)$.
(2) $V_n(\mathbb{C}^m)$ is a closed smooth manifold of dimension $2mn - n^2$.

Proof. Note that the case $m = n$ is essentially the content of the Matrix Group-Smooth Manifold Proposition 32.10. Not surprisingly the proof of the current proposition is just a mild generalization of the proof of the Matrix Group-Smooth Manifold Proposition 32.10. In the following we outline the proof of Statement (1). The proof of Statement (2) is evidently essentially the same.

We start out with a harmless claim which allows us to describe $V_n(\mathbb{R}^m)$ in a way which is more suitable for this proof:

Claim 1. We have

$$V_n(\mathbb{R}^m) \quad = \quad \{A \in \mathrm{M}(m \times n; \mathbb{R}) \,|\, A^{\mathrm{T}} \cdot A = \mathrm{id} \in \mathrm{M}(n \times n, \mathbb{R})\}.$$

Proof. This claim is of course just an elementary observation in linear algebra. For readers who want to practice their linear algebra skills we assign the proof of this claim as Exercise 261.2. ⊞

[2788] We equip $\mathbb{R}^m$ and $\mathbb{C}^m$ with the standard inner product from 122 and 123.

[2789] In the Matrix Groups Deformation Retract Proposition 19.4 we wrote $\mathrm{O}(m, n)$ instead of $V_n(\mathbb{R}^m)$.

Claim 2.

(i) The set

$$\mathrm{Sym}(n,\mathbb{R}) \;=\; \{B \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, B^{\mathrm{T}} = B\} \;=\; \{(b_{ij}) \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, b_{ij} = b_{ji}\}$$

is a submanifold of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n+1)$.

(ii) The map

$$\begin{array}{rccc} f\colon & \mathrm{M}(m\times n,\mathbb{R}) & \to & \mathrm{Sym}(n,\mathbb{R}) \\ & A & \mapsto & A^{\mathrm{T}}\cdot A \end{array}$$

is smooth and id is a regular value.

Proof. Statement (i) is clear. We turn to the proof of Statement (ii). It should be clear that f is smooth and that f does indeed take values in $\mathrm{Sym}(n,\mathbb{R})$. So it remains to show that id is a regular value.

Recall that it follows from Physical Tangent Spaces–Lemma 29.2 (2) that the physical tangent spaces of the submanifolds $\mathrm{Sym}(n,\mathbb{R})$ and $\mathrm{M}(m\times n,\mathbb{R})$ are just given by these vector subspaces.

Now let $A \in \mathrm{M}(m\times n,\mathbb{R})$ with $A^{\mathrm{T}}\cdot A = \mathrm{id}$. We need to show that A is a regular point of f. Thus let $C \in \mathrm{T}_{\mathrm{id}}\,\mathrm{Sym}(n,\mathbb{R}) = \mathrm{Sym}(n,\mathbb{R})$. Note that we need to show that there exists a B in $\mathrm{T}_A\,\mathrm{M}(m\times n,\mathbb{R}) = \mathrm{M}(m\times n,\mathbb{R})$ with $\mathrm{D}f_A(B) = C$.

We start out with the following little calculation: Given $B \in \mathrm{M}(m\times n,\mathbb{R})$ we have

$$\begin{array}{rcl} \mathrm{D}f_A(b) & \overset{\text{by the Chain Rule 24.17}}{\overset{\downarrow}{=}} & \frac{d}{dt} f(A+tB)\big|_{t=0} \\ & = & \frac{d}{dt}(A^{\mathrm{T}}+tB^{\mathrm{T}})\cdot(A+tB)\big|_{t=0} \\ & = & \frac{d}{dt}\big(A^{\mathrm{T}}\cdot A + tA^{\mathrm{T}}\cdot B + tB^{\mathrm{T}}\cdot A + t^2B^{\mathrm{T}}\cdot B\big)\big|_{t=0} \;=\; A^{\mathrm{T}}\cdot B + B^{\mathrm{T}}\cdot A. \end{array}$$

Using that $A^{\mathrm{T}}\cdot A = \mathrm{id}_n$ and $C = C^{\mathrm{T}}$ it is now easy to see that[2790] $B \coloneqq \frac{1}{2}A\cdot C$ satisfies $\mathrm{D}f_A(B) = C$. ⊞

First note that by Claim 1 we have $V_n(\mathbb{R}^m) = f^{-1}(\mathrm{id})$. Thus it follows from Claim 2, together with the Regular Value Theorem 32.5, that $V_n(\mathbb{R}^m)$ is an orientable proper submanifold of $\mathrm{M}(m\times n,\mathbb{R}) = \mathbb{R}^{m\cdot n}$ of dimension $m\cdot n - \frac{1}{2}n(n+1)$. Since $\mathbb{R}^{m\cdot n}$ has no boundary and since $V_n(\mathbb{R}^m)$ is a proper submanifold we see that the boundary of $V_n(\mathbb{R}^m)$ is also empty. Finally note that, as in the proof of the Matrix Group Lemma 4.53 (3), it follows easily from the definitions and the Heine-Borel Theorem 3.44 that $V_n(\mathbb{R}^m)$ is compact. These last two observations imply that $V_n(\mathbb{R}^m)$ is in fact a closed smooth manifold. ■

261.2. **Homotopy groups of Stiefel manifolds.** In this section we want to compile a few basic facts about homotopy groups of the spaces $\mathrm{GL}_n(\mathbb{K}^m)$ and the Stiefel manifolds $V_n(\mathbb{K}^m)$. We start out with the following proposition which shows that the topological spaces $\mathrm{GL}_n(\mathbb{K}^m)$ and the Stiefel manifolds $V_n(\mathbb{K}^m)$ are "homotopically indistinguishable":

Proposition 261.6. (Stiefel Manifold–Deformation Retract Proposition) In the following let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For every choice of $m, n \in \mathbb{N}_0$ the Stiefel manifold $V_n(\mathbb{K}^m)$ is a deformation retract of $\mathrm{GL}_n(\mathbb{K}^m)$.

[2790] Note that the sizes of A and C match, so that it does indeed make sense to consider the product $A\cdot C$.

Proof. As in the proof of the Matrix Groups Deformation Retract Proposition 19.4 one sees that it follows easily from the Gram-Schmidt Orthonormalization Process 2.32 that $V_n(\mathbb{K}^m)$ is a deformation retract of $\mathrm{GL}_n(\mathbb{K}^m)$. ■

It follows from the Stiefel Manifold–Deformation Retract Proposition 261.6 and the Homotopy-π_n-Proposition 134.16 that the inclusions $V_n(\mathbb{K}^m) \to \mathrm{GL}_n(\mathbb{K}^m)$ induce isomorphisms of homotopy groups. Thus it suffices to consider the homotopy groups of Stiefel manifolds.

In the following proposition we summarize some basic calculations of homotopy groups of Stiefel manifolds:

Proposition 261.7. (Stiefel Manifold-π_*-Proposition) Let $m, n \in \mathbb{N}_0$ with $m \geq n$. Given $i \in \mathbb{N}_0$ we have an isomorphism

$$\pi_i(V_n(\mathbb{R}^m)) \cong \begin{cases} 0, & \text{if } i \in \{0, \ldots, m-n-1\}, \\ \mathbb{Z}, & \text{if } i = m-n, \text{ and if } i \text{ even or if } n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } i = m-n, \text{ and if } i \text{ is odd, and if } n \geq 2. \end{cases}$$

Remark.

(1) More information on homotopy groups of Stiefel manifolds can be found in [**NRG2004**, p. 179] and [**Stee1951**, Chapter 25.6].
(2) The homology groups of the Stiefel manifolds are calculated in [**Hat2002**, p. 302] and the cohomology rings of the Stiefel manifolds are calculated in [**Jam1976**, Proposition (3.8)] and [**SE1962**, Chapter IV]. □

Example. Let $m \in \mathbb{N}_{\geq 3}$. We see that

$$\pi_1(\mathrm{SO}(m)) \underset{\text{see page 5044}}{\cong} \pi_1(V_{m-1}(\mathbb{R}^m)) \underset{\substack{\text{by the Stiefel Manifold} \\ \pi_*\text{-Proposition 261.7,} \\ \text{since } m-1 \geq 2}}{\cong} \mathbb{Z}/2\mathbb{Z}.$$

This recovers the calculation from the Matrix Group-π_1-Proposition 239.15. □

The proof of the Stiefel Manifold-π_*-Proposition 261.7 will require the remainder of this section. In the next Section 261.3 we will discuss applications to frames in vector bundles.

The key to proving the Stiefel Manifold-π_*-Proposition 261.7 is the following proposition, which is a generalization of the Matrix Group-Fiber Proposition 238.12:

Proposition 261.8. (Stiefel Manifold-Fiber Proposition) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} p\colon V_n(\mathbb{R}^m) & \to & S^{m-1} \\ (v_1, \ldots, v_n) & \mapsto & v_n \end{array}$$

is a bundle map with fiber $V_{n-1}(\mathbb{R}^{m-1})$.

Example. The special case $m = n$ of the Stiefel Manifold-Fiber Proposition 261.8 says that the given map $p\colon \mathrm{O}(m) \to S^{m-1}$ is bundle map with fiber $\mathrm{O}(m-1)$. This shows that the Stiefel Manifold-Fiber Proposition 261.8 is a generalization of the earlier Matrix Group-Fiber Proposition 238.12. □

Sketch of proof. First recall that in the Stiefel Manifold-Smooth Manifold Proposition 261.5 we showed that $V_n(\mathbb{R}^m)$ is a closed smooth manifold. As in the case of the Matrix Group-Fiber Proposition 238.12 one can easily show that the given map $p\colon V_n(\mathbb{R}^m) \to S^{m-1}$ is a

submersion in the sense of Definition 35.1, i.e. the map is smooth and the differential at any point is an epimorphism.

Since $V_n(\mathbb{R}^m)$ is a closed smooth manifold it now follows from the Ehresmann Fibration Theorem 238.6 that the map $p\colon V_n(\mathbb{R}^m) \to S^{m-1}$ is a bundle map. It is essentially clear that for $e = (0,\dots,0,1) \in S^{m-1}$ the fiber $p^{-1}(e)$ is diffeomorphic to $V_{n-1}(\mathbb{R}^{m-1})$. ■

In the following we first prove the Stiefel Manifold-π_*-Proposition 261.7 for the case $i \in \{0,\dots,m-n-1\}$. Afterwards we will deal with the case $i = m-n$.

Proof of the Stiefel Manifold-π_*-Proposition 261.7 for $i \leq m-n-1$. Let $n \in \mathbb{N}_0$ and let $s \in \mathbb{N}_0$. We need to show that for any $i \in \{0,\dots,s-1\}$ we have $\pi_i(V_n(\mathbb{R}^{n+s})) = 0$.

Claim. Let $k \in \mathbb{N}_0$. As in the Stiefel Manifold-Fiber Proposition 261.8 we consider the fiber inclusion map $V_{k-1}(\mathbb{R}^{k+s-1}) \to V_k(\mathbb{R}^{k+s})$. If $i \leq k+s-2$, then the induced homomorphism $\pi_i(V_{k-1}(\mathbb{R}^{k+s-1})) \to \pi_i(V_k(\mathbb{R}^{k+s}))$ is an epimorphism.

Proof. First recall that by the Stiefel Manifold-Fiber Proposition 261.8 we have a bundle map $p\colon V_k(\mathbb{R}^{k+s}) \to S^{k+s-1}$ with fiber $V_{k-1}(\mathbb{R}^{k+s-1})$. Thus it follows from the Fiber-LES-Proposition 239.8 that we have a long exact sequence of the form

$$\dots \to \pi_i(V_{k-1}(\mathbb{R}^{k+s-1})) \to \pi_i(V_k(\mathbb{R}^{k+s})) \xrightarrow{p_*} \underbrace{\pi_i(S^{k+s-1})}_{\substack{=0 \text{ if } i \leq k+s-2 \\ \text{see the Sphere–Low-}\pi_n \\ \text{Proposition 134.19}}} \to \dots$$

Since $i \leq k+s-2$ it follows from the Sphere–Low-π_n-Proposition 134.19 that the group to the right is zero. Thus the middle map is an epimorphism. ⊞
For $i \in \{0,\dots,s-1\}$ we see that

$$\begin{array}{ccccccccc} 0 & \cong & \pi_i(S^s) & \cong & \pi_i(V_1(\mathbb{R}^{s+1})) & \twoheadrightarrow & \pi_i(V_2(\mathbb{R}^{s+2}) & \xrightarrow{\cong} \dots \xrightarrow{\cong} & \pi_i(V_n(\mathbb{R}^{n+s})). \\ & \uparrow & & \uparrow & & \uparrow & \uparrow & & \uparrow \\ & \text{since } i<s & & \text{see the example} & & \text{by the claim applied to } k=n,n-1,\dots,1, \text{ note that} & & & \\ & & & \text{on page 5044} & & \text{the claim applies since } i<s \text{ and } k \geq 1 & & & \end{array}$$

■

For the case $i = m-n$ of the Stiefel Manifold-π_*-Proposition 261.7 we will also need the following lemma:

Lemma 261.9. (Stiefel Manifold-to-Sphere Lemma) Let $s \in \mathbb{N}$ and $k \in \{0,\dots,s\}$.
(1) If s is even, then $p_*\colon \pi_{s+1}(V_2(\mathbb{R}^{s+2})) \to \pi_{s+1}(S^{s+1})$ is an epimorphism.
(2) If s is odd, then the image of $p_*\colon \pi_{s+1}(V_2(\mathbb{R}^{s+1})) \to \pi_{s+1}(S^{s+1})$ is isomorphic to $2\cdot\mathbb{Z}$.??

Proof of Lemma 261.9. First recall that in the Physical Tangent Spaces–Lemma 29.2 (3) we showed that for any $P \in S^m$ we have

$$\mathrm{T}_P S^m = \{w \in \mathbb{R}^{m+1} \mid \langle P, w\rangle = 0\}.$$

It follows from this observation that the bundle $p\colon V_2(\mathbb{R}^{s+2}) \to S^{s+1}$ is isomorphic to the bundle $q\colon \mathrm{ST}(S^{s-1}) \to S^{s-1}$ of tangent vectors of S^{s-1} of length one.

(1) First we assume that s is even. On page 763 we saw that

$$\begin{array}{rcl} v\colon S^{s+1} & \to & \mathbb{R}^{s+2} \\ (x_1,x_2,\dots,x_{s+1},x_{s+2}) & \mapsto & (x_2,x_1,\dots,x_{s+2},x_{s+1}) \end{array}$$

defines a vector bundle over the sphere S^{s+1}. Note that this shows that the bundle $q\colon \mathrm{ST}(S^{s-1}) \to S^{s-1}$ t has a section $\sigma\colon S^{s-1} \to V_2(\mathbb{R}^l)$, i.e. a continuous map with $p\circ\sigma =$

$\mathrm{id}_{S^{s-1}}$. It follows from the functoriality of homotopy groups that $p_*\colon \pi_{s+1}(V_2(\mathbb{R}^{s+2})) \to \pi_{s+1}(S^{s+1})$ is an epimorphism.

(2)

■

Proof of the Stiefel Manifold-π_*-Proposition 261.7 for $i = m - n$. Let $n \in \mathbb{N}_0$ and let $s \in \mathbb{N}_0$. We need to show that we have isomorphisms

$$\pi_s(V_n(\mathbb{R}^{n+s})) \cong \begin{cases} \mathbb{Z}, & \text{if } s \text{ is even or if } n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } s \text{ is odd and if } n > 1. \end{cases}$$

We start out with the following claim, which is an extension of the claim on page 5047:

Claim. Let $k \in \mathbb{N}_0$.
(a) If $i < k + s - 2$, then $\pi_i(V_{k-1}(\mathbb{R}^{k+s-1})) \to \pi_i(V_k(\mathbb{R}^{k+s}))$ is an isomorphism.
(b) If $i = k + s - 2$, then $\pi_i(V_{k-1}(\mathbb{R}^{k+s-1}) \to \pi_i(V_k(\mathbb{R}^{k+s}))$ is an epimorphism.

Proof. First recall that by the Stiefel Manifold-Fiber Proposition 261.8 we have a bundle map $p\colon V_k(\mathbb{R}^{k+s}) \to S^{k+s-1}$ with fiber $V_{k-1}(\mathbb{R}^{k+s-1})$. Thus it follows from the Fiber-LES-Proposition 239.8 that we have a long exact sequence of the form

$$\ldots \to \underbrace{\pi_{i+1}(S^{k+s-1})}_{\substack{= 0 \text{ if } i < k+s-2 \\ \text{see Proposition 134.19}}} \to \pi_i(V_{k-1}(\mathbb{R}^{k+s-1})) \to \pi_i(V_k(\mathbb{R}^{k+s})) \xrightarrow{p_*} \underbrace{\pi_i(S^{k+s-1})}_{\substack{= 0 \text{ if } i < k+s-2 \\ \text{see Proposition 134.19}}} \to \ldots$$

We make the following observations:

(a) If $i < k + s - 2$, then it follows from the Sphere–Low-π_n-Proposition 134.19 that the groups to the left and right are both zero, thus the middle map is an isomorphism.
(b) We now assume that $i = k + s - 2$. In this case the above sequence becomes

$$\ldots \to \pi_{k+s}(S^{k+s-1}) \to \pi_{k+s-1}(V_{k-1}(\mathbb{R}^{k+s-1})) \to \pi_{k+s-1}(V_k(\mathbb{R}^{k+s})) \to 0.$$

Thus the map to the right is an epimorphism. ⊞

Claim 2. Let $i \in \mathbb{N}_0$.
(a) If s is even, $\pi_s(V_1(\mathbb{R}^{s+1})) \to \pi_s(V_2(\mathbb{R}^{s+2}))$ is an isomorphism.
(b) If s is odd, then $\pi_s(V_1(\mathbb{R}^{s+1}) \to \pi_s(V_2(\mathbb{R}^{s+2}))$ is an epimorphism.

Proof. Similar to the proof of Claim 1 we consider the following long exact sequence:

$$\ldots \to \pi_{s+1}(V_2(\mathbb{R}^{s+2})) \xrightarrow{p_*} \pi_{s+1}(S^{s+1}) \to \pi_s(V_1(\mathbb{R}^{s+1})) \xrightarrow{i_*} \pi_s(V_2(\mathbb{R}^{s+2})) \to 0.$$

(a) If s is even, then it follows from the Stiefel Manifold-to-Sphere Lemma 261.9 that the map p_* to the left is an epimorphism. Thus i_* is an isomorphism.
(b) If s is odd, then ??

⊞

We return to the proof of the actual statement.

- First we suppose that s is even or that $n = 1$. We obtain that

$$\begin{array}{ccccccccc} \mathbb{Z} & \cong & \pi_s(S^s) & \underset{\uparrow}{\cong} & \pi_s(V_1(\mathbb{R}^{s+1})) & \overset{\cong}{\underset{\downarrow}{\to}} & \pi_s(V_2(\mathbb{R}^{s+2}) & \overset{\cong}{\underset{\uparrow}{\to}} \ \dots \ & \overset{\cong}{\underset{\uparrow}{\to}} \ \pi_s(V_n(\mathbb{R}^{s+n})). \end{array}$$

by the example on page 5044 we have $S^s = V_1(\mathbb{R}^{s+1})$

by Claim (2a), since s is even

by Claim (1a) applied to $k = n, n-1, \dots, 2$

- Now we suppose that s is odd and that $n > 1$. We obtain that

$$\begin{array}{ccccccccc} \mathbb{Z} & \cong & \pi_s(S^s) & \underset{\uparrow}{\cong} & \pi_s(V_1(\mathbb{R}^{s+1})) & \overset{\cong}{\underset{\downarrow}{\to}} & \pi_s(V_2(\mathbb{R}^{s+2}) & \overset{\cong}{\underset{\uparrow}{\to}} \ \dots \ & \overset{\cong}{\underset{\uparrow}{\to}} \ \pi_s(V_n(\mathbb{R}^{s+n})). \end{array}$$

see the example on page 5044

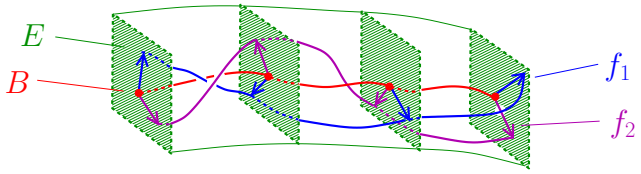
by Claim (2b), since s is even

by Claim (2a) applied to $k = n, n-1, \dots, 2$

■

261.3. **Frames in vector bundles.** In this section we will formulate an application of the Stiefel Manifold-π_*-Proposition 261.7 to the theory of frames in vector bundles. To make sense of this we first need to define what we mean by a frame in a vector bundle.

Definition 261.10. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p\colon E \to B$ be a $\mathbb{K}$-vector bundle and let $n \in \mathbb{N}_0$. An n-frame in $p\colon E \to B$ is an n-tuple $(f_1, \dots, f_n)$ of continuous maps $f_i\colon B \to E$ such that for each $b \in B$ the n-tuple $(f_1(b), \dots, f_n(b))$ forms an n-frame in the vector space $p^{-1}(b)$. In other words, we demand that for each $b \in B$ the vectors $f_1(b), \dots, f_n(b)$ of the vector space $p^{-1}(b)$ are linearly independent.



Example.

(1) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle. It follows easily from the definitions that E admits an n-frame if and only if $p\colon E \to B$ is trivializable, i.e. if it is isomorphic to the trivial n-dimensional $\mathbb{K}$-vector bundle.

(2) Clearly n-frames in the trivial m-dimensional $\mathbb{R}$-vector bundle $X \times \mathbb{R}^m \to X$ over some topological space naturally correspond to continuous maps $X \to \mathrm{GL}_n(\mathbb{R}^m)$.

(3) Let M be a smooth manifold. It follows again easily from the definitions that a 1-frame in the tangent bundle of M is the same as a nowhere vanishing vector field on M in the sense of Definition 29.29. □

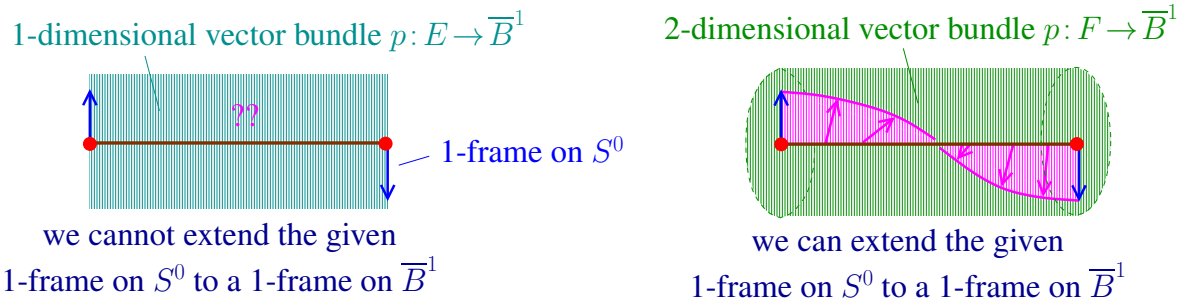
In the proof of the Whitney Trick-Theorem 262.5 in the next chapter we will make use of the following proposition:

Proposition 261.11. (Extend Frame-to-Ball Proposition) Let $E \to \overline{B}^d$ be a trivializable m-dimensional $\mathbb{R}$-vector bundle over $\overline{B}^d$. For $n \leq m - d$ every (smooth) n-frame in E that is defined over S^{d-1} can be extended to a (smooth) n-frame over $\overline{B}^d$.

Example.

(1) Note that in the figure below we see to the left the trivial 1-dimensional real vector bundle $p\colon E = \overline{B}^1 \times \mathbb{R} \to \overline{B}^1$. Furthermore we see a 1-frame on S^0. It follows say from the Intermediate Value Theorem that this 1-frame cannot be extended to a 1-frame on $\overline{B}^1$.

(2) In the figure below we see to the right the trivial 2-dimensional real vector bundle $p\colon F = \overline{B}^1 \times \mathbb{R}^2 \to \overline{B}^1$.



Furthermore we see "the same" 1-frame on S^0 as in (1). This time we can use the "extra space" provided by the extra dimension to extend this 1-frame on S^0 to a 1-frame on $\overline{B}^1$.[2791] □

Proof. Let $E \to \overline{B}^d$ be a trivializable m-dimensional $\mathbb{R}$-vector bundle over $\overline{B}^d$. In fact, without loss of generality we can assume that $E = \overline{B}^d \times \mathbb{R}^m$. It follows from the above example that (smooth) n-frames in the trivial m-dimensional $\mathbb{R}$-vector bundle $X \times \overline{B}^d \to \overline{B}^d$ naturally correspond to continuous (smooth) maps $X \to \mathrm{GL}_n(\mathbb{R}^m)$ or $X \to V_n(\mathbb{R}^m)$.

Now suppose that $n \leq m - d$. Note that it follows from the Stiefel Manifold-π_*-Proposition 261.7 together with the Stiefel Manifold–Deformation Retract Proposition 261.6 that $\pi_i(\mathrm{GL}_n(\mathbb{R}^m)) = 0$. But by the above, together with the Trivial-in-π_n-Lemma 134.10, this implies that every n-frame over S^{d-1} extends to an n-frame on $\overline{B}^d$.

Finally note that it follows from the Whitney Approximation Theorem 36.1 that if we are given a *smooth* n-frame over S^{d-1}, then it extends to a *smooth* n-frame on $\overline{B}^d$. ■

261.4. Exercises for Chapter 261.

Exercise 261.1. Let W be a finite-dimensional real vector space and let $n \in \mathbb{N}$. Given any positive-definite symmetric form $\varphi\colon W \times W \to \mathbb{R}$ we consider

$$V_n(W, \varphi) \;:=\; \text{set of all } n\text{-frames of } W \text{ that are orthonormal with respect to } \varphi$$

Now let $\varphi\colon W \times W \to \mathbb{R}$ and $\psi\colon W \times W \to \mathbb{R}$ be two positive-definite symmetric forms. Show that $V_n(W, \varphi)$ and $V_n(W, \psi)$ are homeomorphic.

Exercise 261.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $m, n \in \mathbb{N}_0$.

(a) Show that $\mathrm{GL}_n(\mathbb{K}^m) \;=\; \{A \in \mathrm{M}(m \times n; \mathbb{K}) \mid \det(\overline{A}^{\mathrm{T}} \cdot A) \neq 0\}$.

(b) Show that $V_n(\mathbb{K}^m) \;=\; \{A \in \mathrm{M}(m \times n; \mathbb{K}) \mid \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id} \in \mathrm{M}(n \times n, \mathbb{K})\}$.

[2791] This discussion is somewhat similar to the example on page 5030.

Exercise 261.3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let V be a $\mathbb{K}$-vector space and let $n \in \mathbb{N}_0$. Show that $\mathrm{GL}(n, \mathbb{K})$ admits a free action on $\mathrm{GL}_n(V)$.

Exercise 261.4. Let $m, n \in \mathbb{N}_0$. Show that $\mathrm{O}(n)$ admits a free action on $V_n(\mathbb{R}^m)$.

Exercise 261.5. Let $n \in \mathbb{N}_0$.

(a) Show that the map

$$\begin{array}{rcl} \mathrm{SO}(n) & \to & V_{n-1}(\mathbb{R}^n) \\ (v_1 \ \ldots \ v_{n-1}\, v_n) & \mapsto & (v_1, \ldots, v_{n-1}) \end{array}$$

is a homeomorphism.

(b) Show that the map

$$\begin{array}{rcl} \mathrm{SU}(n) & \to & V_{n-1}(\mathbb{C}^n) \\ (v_1 \ \ldots \ v_{n-1}\, v_n) & \mapsto & (v_1, \ldots, v_{n-1}) \end{array}$$

is a homeomorphism.

Exercise 261.6. Let $n \in \mathbb{N}_0$ and let $k \in \{0, \ldots, n\}$.

(a) Show that the map

$$\begin{array}{rcl} \mathrm{O}(n) & \to & V_k(\mathbb{R}^n) \\ (v_1 \ \ldots \ v_n) & \mapsto & (v_1, \ldots, v_k) \end{array}$$

is a smooth bundle map and determine its fiber.

(b) Show that the map

$$\begin{array}{rcl} \mathrm{U}(n) & \to & V_k(\mathbb{C}^n) \\ (v_1 \ \ldots \ v_n) & \mapsto & (v_1, \ldots, v_k) \end{array}$$

is a smooth bundle map and determine its fiber.

CHAPTER 262

The Whitney trick

In Chapter 260, when we discussed handle cancellations, the following question arose: Suppose that we are given a compact oriented smooth manifold M and two closed oriented smooth submanifolds X and Y with $\dim(X) + \dim(Y) = \dim(M)$. If the algebraic intersection $X \cdot Y = \pm 1$, under what conditions can we find a smooth isotopy of Y such that X and Y intersect transversally in a single point?

In this chapter we will state and prove the Whitney Trick-Theorem 262.5 that gives us some conditions which allow us to obtain the desired result. The Whitney Trick-Theorem 262.5 will be a key input in the proof of the h-Cobordism Theorem 263.11 in the next chapter.

This chapter is organized as follows:

- In Section 262.1 we will first introduce and discuss the algebraic intersection number of submanifolds. We will then formulate the Whitney Trick-Theorem 262.5.
- In Section 262.2 we will provide an extended sketch of a proof of the Whitney Trick-Theorem 262.5.
- In Section 262.3 we will discuss the Strong Whitney Embedding Theorem 262.11 which was the original raison d'être for the Whitney trick.
- In Appendix 262.4 we will discuss the question whether analogs of the Whitney trick also hold in the remaining dimensions.
- In Section 262.5 we will provide exercises for Chapter 262.

In the next Chapter 263 we will state and prove the h-Cobordism Theorem 263.11.

262.1. **The Whitney trick.** For the reader's convenience let us recall Definition 220.5 of the algebraic intersection number:

Definition 262.1. Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold and let $Y \subseteq M$ be a compact oriented proper submanifold. We suppose that the following conditions are satisfied:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) We have $\partial X \cap \partial Y = \varnothing$.
(c) X and Y intersect transversally, i.e. for each $P \in X \cap Y$ we have $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$.

We introduce the following notions:

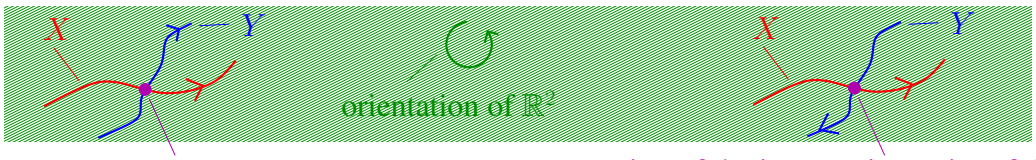
(1) Let P be an intersection point of X and Y. We set $\mathrm{sign}(P) \coloneqq 1$, if

$$(\text{positive basis of } \mathrm{T}_P X, \text{positive basis of } \mathrm{T}_P Y) \;=\; \text{positive basis of } \mathrm{T}_P M.$$

Otherwise we set $\mathrm{sign}(P) \coloneqq -1$.

(2) We define[2792]

$$\text{algebraic intersection number of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \operatorname{sign}(P) \in \mathbb{Z}.$$



sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals $+1$

sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals -1

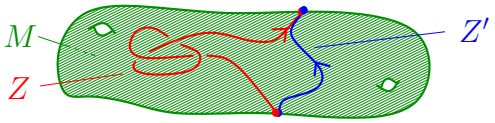
We will soon see that the algebraic intersection number is invariant under suitable "deformations". To make this notion precise we will need the following definition:

Definition 262.2. Let M be a smooth manifold and let $Z, Z' \subseteq M$ be two proper submanifolds.

(1) We say that the two smooth submanifolds Z and Z' are properly homotopic if there exists a homotopy $F\colon Z \times [0,1] \to M$ with the following three properties:
   (a) We have $F_0 = \mathrm{id}$.
   (b) The map F_1 is a diffeomorphism $Z \to Z'$.
   (c) For each $t \in [0,1]$ we have $F_t(\partial Z) \subseteq \partial M$.
   If Z and Z' are both oriented, then we demand that F_1 is an orientation-preserving diffeomorphism $Z \to Z'$.

(2) We say Z and Z' are properly smoothly isotopic if we can find a homotopy F as above with the following extra properties:
   (d) The map $F\colon Z \times [0,1] \to M$ is smooth.
   (e) Each $F_t\colon Z \to M$ is a proper smooth embedding.

(3) In both settings we say "rel boundary" if the restriction of each F_t to ∂Z is the identity.



Z and Z' are properly homotopic rel boundary, but they are not smoothly properly isotopic rel boundary

Theorem 262.3. (Intersection Number–Homotopy Invariance Theorem) Let M be an oriented m-dimensional smooth manifold, let $X \subseteq M$ be a compact oriented proper submanifold and let $Y, Y' \subseteq M$ be two compact oriented proper submanifolds. We suppose that the pair X, Y and also the pair X, Y' satisfy Conditions (a), (b), (c) from Definition 262.1. If Y and Y' are properly homotopic rel boundary, then $X \cdot Y = X \cdot Y'$.

Proof. The theorem can be proved in two ways:

(1) The theorem follows easily from a slight extension of the Intersection Number–Homotopy Invariance Theorem 51.9.

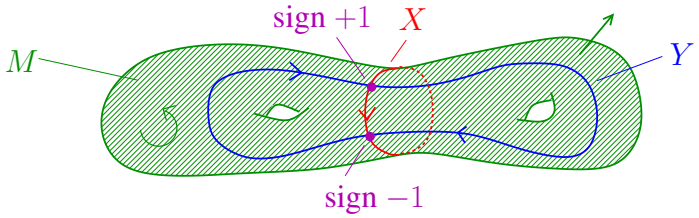
(2) We now suppose that M is compact. As in Exercise 38.3 one can show that we can find a decomposition $\partial M = A \cup B$ where A and B are compact codimension-zero submanifolds of ∂M with $\partial A = \partial B = A \cap B$ and where $\partial X \subseteq A \setminus \partial A$ and $\partial Y, \partial Y' \subseteq B \setminus \partial B$.

[2792] It follows from the Cup Product–Intersection Number Theorem 220.10 that this definition is consistent with Definition 260.7.

The desired statement now follows from the Cup Product–Intersection Number Theorem 220.10. ■

Let M be an oriented smooth manifold and let X and Y be two compact oriented proper submanifolds of M that satisfy Conditions (a), (b) and (c) from Definition 262.1. In the Intersection Number–Homotopy Invariance Theorem 262.3 we just saw that the algebraic intersection number is invariant under proper smooth isotopies of Y rel boundary. In particular we saw the following: If we can push Y off X via a proper smooth isotopy rel boundary, then $X \cdot Y = 0$. It is natural to ask whether the converse to this statement holds. Playing around with curves on a surface shows that in general this is unfortunately wrong:

Example. We consider the closed oriented 1-dimensional submanifolds X and Y on the surface M of genus 2 that are illustrated in the figure below. Their algebraic intersection number is zero.



There are several ways to see that the closed curves cannot be made disjoint:

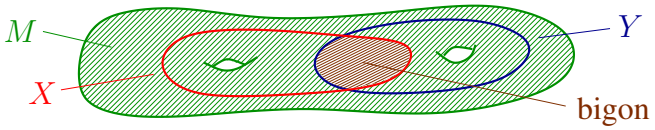
(1) In Exercise 262.2 we will use covering spaces to see that the closed 1-dimensional submanifolds X and Y cannot be made disjoint via a smooth isotopy.

(2) In Exercise 262.3 we consider $[Y]$ in a suitable relative homology group to see that the closed 1-dimensional submanifolds X and Y cannot be made disjoint via a smooth isotopy.

(3) The so-called *bigon criterion* says the following:[2793] Suppose that M is a closed 2-dimensional smooth manifold and suppose that X and Y are closed oriented 1-dimensional submanifolds on M that intersect transversally. Then the following two statements are equivalent:[2794]

   (a) X and Y have the minimal number of geometric intersection points within their smooth isotopy classes.

   (b) X and Y do not admit a bigon. Here we use the following notion: A bigon for X and Y is a disk $D \subseteq M$ such that $\partial D \subseteq X \cup Y$ and such that $D \cap X = \partial D \cap X$ is an interval and such that $D \cap Y = \partial D \cap Y$ is also an interval.



[2793] We refer to [**FaM2011**, Proposition 1.7] for details on the bigon criterion and for a proof of the bigon criterion.

[2794] The bigon criterion might appear obvious. The (a)⇒(b)-direction is indeed essentially trivial. The (b)⇒(a)-direction requires a proof since a priori one might have a smooth isotopy that in between introduces extra intersection points, before reducing the number of intersection points.

For instance, in the case considered in the beginning X and Y clearly do not admit a bigon. Thus it follows from the bigon criterion that we cannot dispose of the two intersection points. □

If one thinks for a while about the previous example, then one realizes that the issue in the example is that the smooth manifold M is not simply connected. So it is more reasonable to demand some condition on the fundamental groups. Furthermore in Exercise 262.7 we will see that it is best to restrict to submanifolds X and Y which are connected. This leads us to the following question:

Question 262.4. Let M be a compact oriented smooth manifold and X and Y be two compact oriented connected proper submanifolds of M, such that the following three conditions hold:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) We have $\partial X \cap \partial Y = \varnothing$.
(c) X and Y intersect transversally.

Suppose that the algebraic intersection number $X \cdot Y = 0$ and that M is simply connected, does it follow that X and Y can be made disjoint using a proper smooth isotopy of Y? In other words, does there exist a proper smooth *isotopy* from Y to Y' such that X and Y' are disjoint?

The following theorem says, that under the hypothesis that the (co-) dimensions are high enough, we obtain an affirmative answer to Question 262.4:

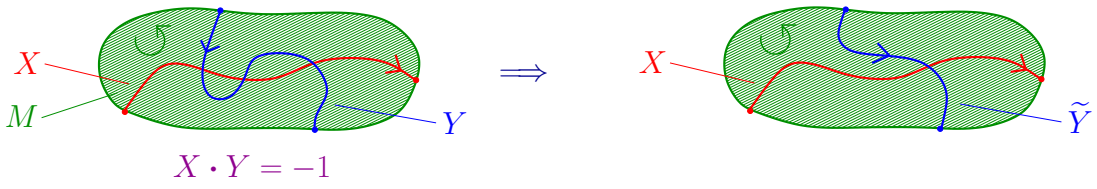
Theorem 262.5. (Whitney Trick-Theorem) Let M be a compact oriented smooth manifold and let X and Y be two compact oriented connected proper submanifolds of M such that the following three conditions hold:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) We have $\partial X \cap \partial Y = \varnothing$.
(c) X and Y intersect transversally.

Furthermore we suppose that one of the following two conditions holds:

(I) $\dim(X) \geq 3$, $\dim(Y) \geq 3$ and $\pi_1(M) = 0$,
(II) $\dim(X) \geq 3$, $\dim(Y) = 2$ and $\pi_1(M \setminus X) = 0$.[2795]

Then there exists a proper smooth isotopy rel ∂Y from Y to a smooth submanifold $\widetilde{Y}$ such that X and $\widetilde{Y}$ intersect transversally in precisely $|X \cdot Y|$ points.



Remark.

[2795]Note that the codimension of X in M is $\dim(Y)$. If this codimension is ≥ 3, then it follows from the Transversality Theorem 50.9 that $\pi_1(M \setminus X) \cong \pi_1(M)$. We refer to Exercise 134.8 for details. (An alternative proof is also given in Exercise 262.4.) Thus we see that the hypothesis $\pi_1(M) = 0$ from (I) implies that $\pi_1(M \setminus X)$.

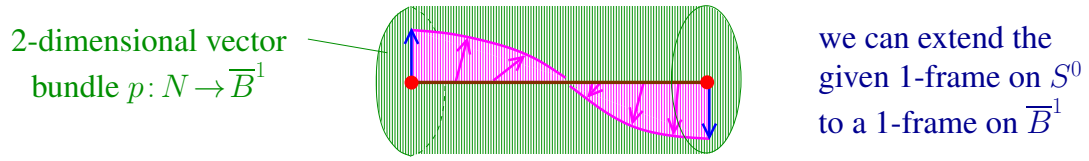
(1) For dimensions ≥ 5 the theorem was first proved by Hassler Whitney [**Why1944a**, Theorem 4] in 1944.
(2) The authoritative version of the proof was provided by John Milnor in [**Miln1965b**, p. 73-84]. As the page range shows, the proof is quite long. One of the main difficulties is to prove rigorously that everything can be done smoothly.
(3) An extended exposition of the proof is given in [**Scor2005**, Chapters 1.5 and 1.7]. Furthermore summaries of the proof are also given in [**Kir2013**], [**Scor2005**, Chapter 1.5], [**GoS1999**, Theorem 9.2.7] and [**RS1972**, Theorem 5.12]. □

262.2. **Proof of the Whitney Trick-Theorem 262.5.** In this section we will provide an extended sketch of a proof of the Whitney Trick-Theorem 262.5. We start out with two remarks and caveats:

(1) We do not claim to provide a complete proof of the Whitney Trick-Theorem 262.5. For example we will ignore various issues with corners. But we hope that our argument will make it clear how the dimension hypotheses enter and what role the algebraic intersection number plays.
(2) In the proof of the Whitney Trick-Theorem 262.5 we will work a lot with real vector bundles. We introduced the notion of a real vector bundle in Definition 240.10, but we did not really develop the theory of real vector bundles in detail, we will do so only later in Chapter 265. In the following discussion we will use several notions, like an orientation of a vector bundle, that strictly speaking we did not introduce yet. We hope that intuitively it is clear what is meant. Otherwise details can be found in Chapter 265

After these remarks we continue with some preparations for our proof of the Whitney Trick-Theorem 262.5. First of all we will need to make use of the following result from the previous chapter, which is worth restating, since it can be difficult to keep track of the various dimensions involved:

Proposition 261.11. (Extend Frame-to-Ball Proposition) Let $N \to \overline{B}^d$ be a trivializable m-dimensional $\mathbb{R}$-vector bundle over $\overline{B}^d$.[2796] For $n \leq m - d$ every smooth n-frame in N that is defined over S^{d-1} can be extended to a smooth n-frame over $\overline{B}^d$.



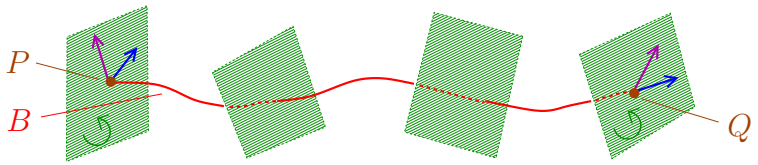
We will also need the following elementary extension property for trivializable vector bundles:

Lemma 262.6. (Trivialization–Interval Extension Lemma) Let $n \in \mathbb{N}$. Furthermore let $p\colon E \to B$ be a trivializable n-dimensional $\mathbb{R}$-vector bundle over a 1-dimensional smooth manifold B that is diffeomorphic to $\overline{B}^1 = [-1, 1]$. Let P and Q be the endpoints of B. Suppose we are given bases $v_1, \ldots, v_n \in p^{-1}(P)$ and $w_1, \ldots, w_n \in p^{-1}(Q)$. The following two statements are equivalent:

(1) There exists an n-frame of E that restricts to the given n-frames on the endpoints.

[2796] In the proof of the Whitney Trick-Theorem 262.5 we will require the statement for $d = 2$.

(2) $v_1, \dots, v_n \in p^{-1}(P)$ and $w_1, \dots, w_n \in p^{-1}(Q)$ are either both positive or both negative bases.



Proof. First note that the (1)⇒(2)-implication is a straightforward consequence of the elementary Vector Bundle–Orientation Basics Lemma 265.29 (1). Now we turn to the proof of the (2)⇒(1)-implication. Without loss of generality we can assume that $B = [-1, 1]$, $P = -1$ and that $Q = 1$. Furthermore we can assume that $v_1, \dots, v_n \in p^{-1}(-1)$ and $w_1, \dots, w_n \in p^{-1}(1)$ are both positive bases. Since p is trivializable we know that the bundle map $p\colon E \to \overline{B}^1$ admits a trivialization $\tau\colon E \to \overline{B}^1 \times \mathbb{R}^n$ which on each fiber is orientation-preserving. It follows from the definitions that there exists a map $\varphi\colon \{\pm 1\} \to \mathrm{GL}(n, \mathbb{R})$ such that $\tau(v_i) = (-1, \varphi(-1) \cdot e_i)$, $i = 1, \dots, n$ and such that $\tau(w_i) = (1, \varphi(1) \cdot e_i)$, $i = 1, \dots, n$. Since we assume that $v_1, \dots, v_n \in p^{-1}(-1)$ and $w_1, \dots, w_n \in p^{-1}(1)$ are both positive bases we obtain from the Vector Bundle–Orientation Basics Lemma 265.29 (1) that $\det(\varphi(-1)) > 0$ and $\det(\varphi(1)) > 0$. By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{GL}_+(n, \mathbb{R})$ is path-connected. This implies that there exists a continuous map $\varphi\colon [-1, 1] \to \mathrm{GL}_+(n, \mathbb{R})$ that extends φ. But then the maps

$$\begin{array}{rccc} u_i\colon & \overline{B}^1 & \to & E \\ & t & \mapsto & \tau^{-1}(\varphi(t) \cdot e_i) \end{array}$$

with $i = 1, \dots, n$ form an n-frame on E with the desired property. ∎

Finally we will also need the following proposition which will eventually bear the main responsibility for the dimension restriction in the statement of the Whitney Trick-Theorem 262.5:

Proposition ??. (Embedded Sphere–Embedded Ball Proposition) Let M be an n-dimensional smooth manifold and let $\varphi\colon S^k \to M \setminus \partial M$ be a smooth embedding. If φ is null-homotopic and if $n \geq 2(k+1) + 1$, then we can extend φ to a smooth embedding $\Phi\colon \overline{B}^{k+1} \to M \setminus \partial M$.

Example. In the proof of the Whitney Trick-Theorem 262.5 we will require the statement of the Embedded Sphere–Embedded Ball Proposition **??** for $k = 1$. So let us consider this case in greater detail. In the figure below we show a smooth embedding $\varphi\colon S^1 \to \mathbb{R}^3$ such that the image is given by the popular trefoil knot.



We make the following two observations:

(1) It follows from the Trefoil–Non-Trivial Proposition 111.19 together with the Genus-Zero Knots Proposition 121.10 that we *cannot* extend $\varphi\colon S^1 \to \mathbb{R}^3$ to a smooth embedding $\Phi\colon \overline{B}^2 \to \mathbb{R}^3$.

(2) Let $\iota\colon \mathbb{R}^3 \to \mathbb{R}^4$ be the smooth embedding that is given by $\iota(x) := (x, 0)$. In Exercise 262.6 we will provide a hands-on proof that the smooth embedding $\iota \circ \varphi\colon S^1 \to \mathbb{R}^4$ admits an extension to a smooth embedding $\overline{B}^2 \to \mathbb{R}^4$.[2797] □

In the following we will first give an extended sketch of a proof of Statement (I) of the Whitney Trick-Theorem 262.5. Afterwards we will deal with Statement (II).

Extended sketch of the proof of the Whitney Trick-Theorem 262.5 (I). Let M be a compact oriented *simply connected* smooth manifold and let X and Y be two compact oriented connected proper submanifolds of M such that the following three conditions hold:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) We have $\partial X \cap \partial Y = \varnothing$.
(c) X and Y intersect transversally.

Furthermore we suppose that the following condition is satisfied:

(I) $\dim(X) \geq 3$ and $\dim(Y) \geq 3$.

Clearly it suffices to show that if $\#(X \cap Y) > |X \cdot Y|$ then Y is properly smoothly isotopic rel ∂Y to a smooth manifold $\widetilde{Y}$ that is transverse to X and which satisfies $\#(X \cap \widetilde{Y}) < \#(X \cap Y)$.

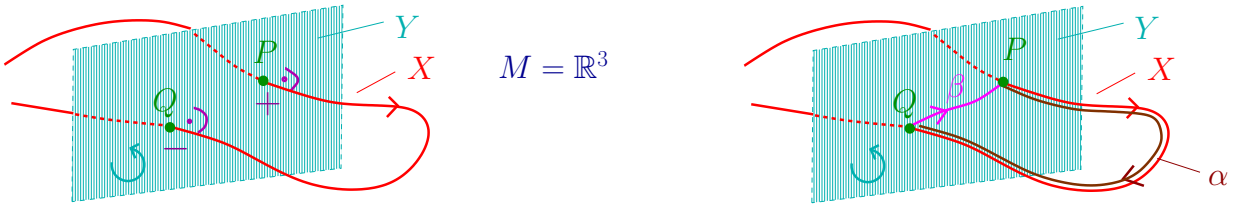
We start out with the following two observations:

(1) By the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can assume that M is a submanifold of some $\mathbb{R}^n$.
(2) As in the proof of the Submanifold Transversal Intersection Theorem 50.4 we can arrange, after possibly applying a smooth isotopy to Y, that at each intersection point the tangent spaces of X and Y are orthogonal.

Since we assume that $\#(X \cap Y) > |X \cdot Y|$ and since $\partial X \cap \partial Y = \varnothing$ we obtain from Definition 220.5 that there exist two intersection points P and Q in $M \setminus \partial M$ of opposite signs. We can assume that $\operatorname{sign}(P) = +1$ and that $\operatorname{sign}(Q) = -1$.

We make the following observations:

- Since X is connected we obtain from the Embedded Path Corollary 44.7 a smooth embedding $\alpha\colon [0,1] \to X \setminus \partial X$ with $\alpha(0) = P$ and $\alpha(1) = Q$.
- Since Y is connected we obtain from the Embedded Path Corollary 44.7 a smooth embedding $\beta\colon [0,1] \to Y \setminus \partial Y$ with $\beta(0) = Q$ and $\beta(1) = P$.



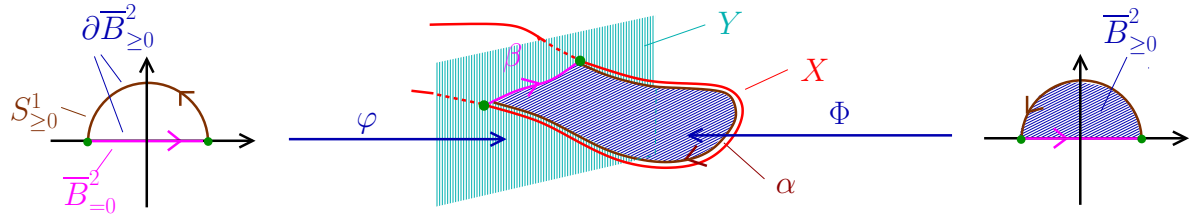
In the following we will work a lot with the upper half-ball $\overline{B}^2_{\geq 0} = \{(x, y) \in \overline{B}^2 \mid y \geq 0\}$ and its boundary $S^1_{\geq 0}$ that we can write as the union of $S^1_{\geq 0}$ and $\overline{B}^2_{=0}$. In particular we will

[2797] Note that the Embedded Sphere–Embedded Ball Proposition **??** says in this setting that for every $n \geq 5$ every smooth embedding $S^1 \to \mathbb{R}^n$ extends to a smooth embedding $\overline{B}^2 \to \mathbb{R}^n$.

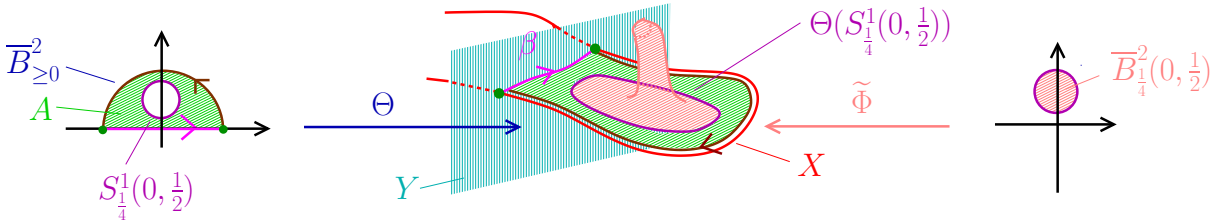
now consider the loop

$$\begin{array}{rcl} \varphi\colon S^1_{\geq 0} & \to & M \\ z & \mapsto & \begin{cases} \alpha(t), & \text{if } z = \exp(\pi i t) \text{ with } t \in [0,1], \\ \beta(\frac{1}{2}(x+1)), & \text{if } z = (x,0) \in \overline{B}^2_{=0}. \end{cases} \end{array}$$

Claim 1. There exists a smooth embedding $\Phi\colon \overline{B}^2_{\geq 0} \to M$ that extends the smooth embedding $\varphi\colon S^1_{\geq 0} \to M$ and such that $\Phi(\overline{B}^2_{\geq 0}) \cap (X \cup Y) = \varphi(S^1_{\geq 0})$.



Proof. First note that throughout the proof of the claim we will make use of the "annulus" $A := \overline{B}^2_{\geq 0} \setminus B^2_{\frac{1}{4}}(0, \frac{1}{2})$. Next note that one can show fairly easily, say using the Extend Frame-to-Ball Proposition 261.11 that there exists a vector field along $\varphi(S^1_{\geq 0})$ that is orthogonal to $\alpha([0,1])$ and also to $\beta([0,1])$. Note that we can use this vector field to extend the smooth embedding $\varphi\colon S^1_{\geq 0} \to M$ to a smooth embedding $\Theta\colon A \to M$ such that $\Theta|_{S^1_{\geq 0}} = \varphi$ and such that $\Theta(A) \cap (X \cup Y) = \varphi(S^1_{\geq 0})$.



Next we consider the loop $\Theta\colon S^1_{\frac{1}{4}}(0, \frac{1}{2}) \to M$. Since M is by hypothesis simply connected we know that the loop $\Theta\colon S^1_{\frac{1}{4}}(0, \frac{1}{2}) \to M$ is null-homotopic. Since

$$\dim(M) = \dim(X) + \dim(Y) \underset{\substack{\uparrow \\ \text{follows from } \dim(X) \geq 3 \\ \text{and } \dim(Y) \geq 3}}{\geq} 3+3 = 6 \geq 5 = 2 \cdot \dim(S^1_{\frac{1}{4}}(0, \frac{1}{2})) + 3$$

we obtain from the above Embedded Sphere–Embedded Ball Proposition **??** that there exists a smooth embedding $\widetilde{\Phi}\colon \overline{B}^2_{\frac{1}{4}}(0, \frac{1}{2}) \to M$ that extends the given smooth embedding $\Theta\colon S^1_{\frac{1}{4}}(0, \frac{1}{2}) \to M$.

Next note that we have

$$\dim(\widetilde{\Phi}(\overline{B}^2_{\frac{1}{4}}(0, \tfrac{1}{2})) + \dim(X) = 2 + \dim(M) - \dim(Y) \underset{\substack{\uparrow \\ \text{since } \dim(Y) \geq 3}}{\leq} 2 + \dim(M) - 3 < \dim(M).$$

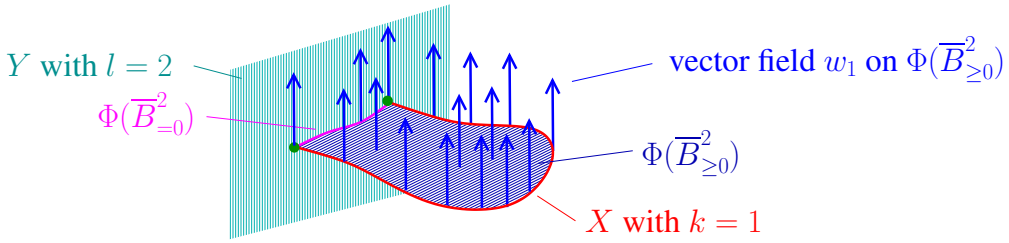
It follows from the Transversality Theorem 50.10 that we can find a smooth isotopy of $\widetilde{\Phi}$ rel boundary such that $\widetilde{\Phi}(\overline{B}^2_{\frac{1}{4}}(0, \frac{1}{2}))$ is disjoint from X.

Similarly it follows from $\dim(X) \geq 3$ that we can also arrange that $\widetilde{\Phi}(\overline{B}^2_{\frac{1}{4}}(0, \frac{1}{2}))$ does not intersect Y either. The map Θ together with the map $\widetilde{\Phi}$ defines the promised smooth embedding $\Phi\colon \overline{B}^2_{\geq 0} \to M$. ⊞

Claim 2. We set $k := \dim(X)$ and $l := \dim(Y)$. There exist smooth vector fields $v_1, \dots, v_{k-1}$ and $w_1, \dots, w_{l-1}$ on $\Phi(\overline{B}^2_{\geq 0})$ with the following properties:

(1) For each $P \in \Phi(\overline{B}^2_{\geq 0})$ the vectors $v_1, \dots, v_{k-1}, w_1, \dots, w_{l-1}$ together with $\mathrm{T}_P\Phi(\overline{B}^2_{\geq 0})$ span the vector space $\mathrm{T}_P M$.

(2) On $\Phi(\overline{B}^2_{=0})$ the vectors $w_1, \dots, w_{l-1}$ are tangent to Y.



Proof. [2798] First we construct the smooth vector fields $w_1, \dots, w_{l-1}$. We will do so in several steps. Here is the short outline for the upcoming steps:

(c) We will define the vector fields on the point Q.
(d) We will define the vector fields on $\beta([0,1])$.
(g) We will define the vector fields on $\beta([0,1]) \cup \alpha([0,1]) = \varphi(S^1_{\geq 0})$.
(h) We will define the vector fields on $\Phi(\overline{B}^2_{\geq 0})$.

Here is the extended version:

(a) Let τ be the unit length tangent vector field along $\beta([0,1])$ that points into the direction of β.

(b) Let $\pi_Y\colon F \to \beta([0,1])$ be the $(l-1)$-dimensional vector bundle that for $y \in \beta([0,1])$ is given by all tangent vectors in $\mathrm{T}_y M$ that are *tangent* to Y and that are *orthogonal* to[2799] $\mathrm{T}_y\beta([0,1])$.

(c) We pick a basis $w_1(Q), \dots, w_{l-1}(Q)$ of the vector space $F_Q = \pi_Y^{-1}(Q)$ such that the vectors $\tau(Q), w_1(Q), \dots, w_{l-1}(Q)$ form a *positive* basis for $\mathrm{T}_Q Y$.

(d) Since $\Phi(\overline{B}^2_{=0}) = \beta([0,1])$ is contractible it follows from the Smooth Bundles-Homotopies Proposition 243.10 (2) that the vector bundle $\pi_Y\colon F \to \beta([0,1])$ is trivializable. This implies that we can extend the $(l-1)$-frame $w_1(Q), \dots, w_{l-1}(Q)$ to a smooth $(l-1)$-frame $w_1, \dots, w_{l-1}$ in the vector bundle $\pi_Y\colon F \to \beta([0,1])$.

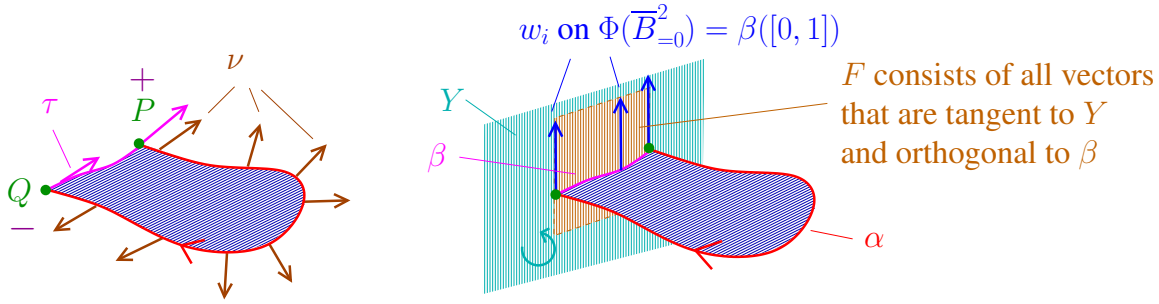
(e) Next let ν be the vector field along $\alpha([0,1])$ that is given by the vectors of length one that are orthogonal to $\alpha([0,1])$, that are tangent to $\Phi(\overline{B}^2_{\geq 0})$ and that point out of $\Phi(\overline{B}^2_{\geq 0})$.

(f) Note that $\tau(P) = \nu(P)$ and $\tau(Q) = -\nu(Q)$.

[2797] Note that this implies that at each $P \in M$ the vectors $v_1(P), \dots, v_{k-1}(P), w_1(P), \dots, w_{l-1}(P)$ span a vector subspace of dimension $\dim(M) - 2 = k + l - 2$. Thus these vectors are linearly independent.

[2798] An alternative proof of the claim is given in [**Miln1965b**, Lemma 6.13].

[2799] Recall that we view M as a submanifold of some $\mathbb{R}^n$. Thus the tangent vectors can be viewed as vectors in $\mathbb{R}^n$. Therefore it makes sense to say that tangent vectors are orthogonal.



(g) Over $\Phi(S^1_{\geq 0}) = \alpha([0,1])$ we consider the vector bundle $\pi_X \colon E \to \alpha([0,1])$ that for any $x \in \alpha([0,1])$ is given by all tangent vectors in $\mathrm{T}_x M$ that are *orthogonal* to $\mathrm{T}_x X$ and that are orthogonal to $\mathrm{T}_x \Phi(\overline{B}^2_{\geq 0})$. Note that this is an $(l-1)$-dimensional vector bundle. We make the following observations:

(i) It follows easily from the fact that X and Y meet orthogonally that the vector bundles $\pi_X \colon E \to \alpha([0,1])$ and $\pi_Y \colon F \to \beta([0,1])$ agree over P and Q.

(ii) We will now use the orientations of X and M to equip the above vector bundle $\pi_X \colon E \to \alpha([0,1])$ with an orientation. More precisely we do the following: Given any $x \in \alpha([0,1])$ a basis $\mathcal{B}$ of the fiber $E_x = \pi_X^{-1}(x)$ is now called *positive* if the ordered basis

$$\text{positive basis of } \mathrm{T}_x X, \nu(x), \text{ the given basis } \mathcal{B} \text{ of } E_x$$

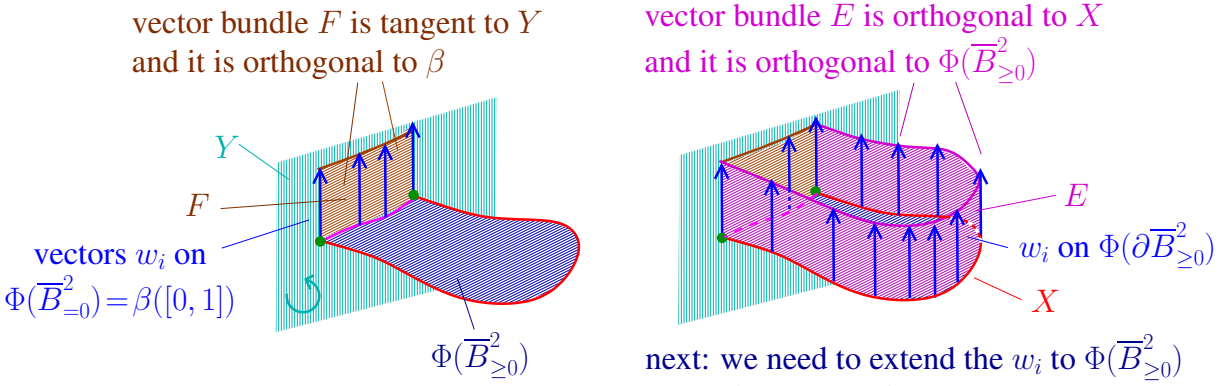
is a positive basis for $\mathrm{T}_x M$.

(iii) It follows from $\mathrm{sign}(P) = +1$ and the fact that $\tau(P) = \nu(P)$ that the ordered set of vectors $w_1(P), \ldots, w_{l-1}(P)$ is a positive basis for $E_P = \pi_X^{-1}(P)$.

(iii′) It follows from $\mathrm{sign}(Q) = -1$ and the fact that $\tau(Q) = -\nu(Q)$ that the ordered set of vectors $w_1(Q), \ldots, w_{l-1}(Q)$ is a positive basis for $E_Q = \pi_X^{-1}(Q)$.

(iv) Since $\Phi(S^1_{\geq 0}) = \alpha([0,1])$ is contractible it follows again from the Smooth Bundles-Homotopies Proposition 243.10 (2) that this vector bundle $\pi_X \colon E \to \alpha([0,1])$ is trivializable.

(v) It follows from (i), (ii), (iii), (iii′) and (iv), together with the Trivialization–Interval Extension Lemma 262.6, that we can extend the smooth $(l-1)$-frame $w_1, \ldots, w_{l-1}$ from $\{P, Q\}$ to a smooth $(l-1)$-frame $w_1, \ldots, w_{l-1}$ across $\Phi(S^1_{\geq 0}) = \alpha([0,1])$.



(h) We consider the normal vector bundle $N \to \Phi(\overline{B}^2_{\geq 0})$ of $\Phi(\overline{B}^2_{\geq 0}) \subseteq M$. Note that this is a $(k+l-2)$-dimensional vector bundle. First note that it follows easily from the

definitions that $\pi_X\colon E \to \alpha([0,1])$ can be viewed as a "subbundle" of the restriction of $N \to \Phi(\overline{B}^2_{\geq 0})$ to $\alpha([0,1])$ and that $\pi_Y\colon F \to \beta([0,1])$ can be viewed as a "subbundle" of the restriction of $N \to \Phi(\overline{B}^2_{\geq 0})$ to $\beta([0,1])$.

Since $\Phi(\overline{B}^2_{\geq 0})$ is contractible it follows from the Smooth Bundles-Homotopies Proposition 243.10 (2) that the vector bundle $N \to B$ is smoothly trivializable. Note that in (d) and (g) we just constructed a smooth $(l-1)$-frame $w_1, \dots, w_{l-1}$ in this vector bundle on $\partial\Phi(\overline{B}^2_{\geq 0})$.

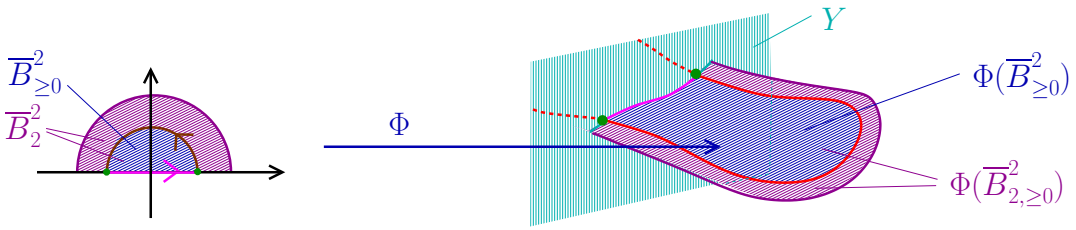
Next note that we have

$$l-1 \underset{\substack{\uparrow \\ \text{since } k = \dim(X) \geq 3}}{\leq} k+l-4 \;=\; (k+l-2)-2.$$

It thus follows from the Extend Frame-to-Ball Proposition 261.11[2800] that we can extend this smooth $(l-1)$-frame of the $(k+l-2)$-dimensional vector bundle from $\partial\Phi(\overline{B}^2_{\geq 0})$ to a smooth $(l-1)$-frame $w_1, \dots, w_{l-1}$ on the 2-dimensional disk $\Phi(\overline{B}^2_{\geq 0})$.

Note that now we have found the desired smooth vector fields $w_1, \dots, w_{l-1}$ on $\Phi(\overline{B}^2_{\geq 0})$ with suitable properties. Fortunately finding $v_1, \dots, v_{k-1}$ is going to be straightforward. To do so we consider the $(k-1)$-dimensional vector bundle $W \to \Phi(\overline{B}^2_{\geq 0})$ of all vectors that are orthogonal to $w_1, \dots, w_{l-1}$ and to $\Phi(\overline{B}^2_{\geq 0})$. Since $\Phi(\overline{B}^2_{\geq 0})$ is contractible we obtain from the Smooth Bundles-Homotopies Proposition 243.10 (2) that this vector bundle is smoothly trivializable. It follows from the elementary discussion on page 5049 that we can find a smooth $(k-1)$-frame $v_1, \dots, v_{k-1}$ in this vector bundle. This smooth frame has all the desired properties. ⊞

Next we consider $\overline{B}^2_{2,\geq 0} := \{(x,y) \in \overline{B}^2_2 \,|\, y \geq 0\}$. Note that it follows from the Embedding Extension Theorem 54.17 that we can extend the smooth embedding $\Phi\colon \overline{B}^2_{\geq 0} \to M$ to a suitable smooth embedding $\Phi\colon \overline{B}^2_{2,\geq 0} \to M$ such that $\Phi(\overline{B}^2_{2,=0}) \subseteq Y$.



Note that $\Phi(\overline{B}^2_{2,\geq 0}) = \Phi(\overline{B}^2_{\geq 0}) \cup \Phi(S^1_{\geq 0} \times [1,2])$. Using this observation it is fairly elementary to show that we can extend the smooth frames $v_1, \dots, v_{k-1}$ and $w_1, \dots, w_{l-1}$ from Claim 2 from $\Phi(\overline{B}^2_{\geq 0})$ to $\Phi(\overline{B}^2_{2,\geq 0})$ with the same properties as before.

Claim 3. There exists a smooth embedding $\Theta\colon \overline{B}^2_2 \times \overline{B}^{k-1} \times \overline{B}^{l-1} \to M$ with the following properties:

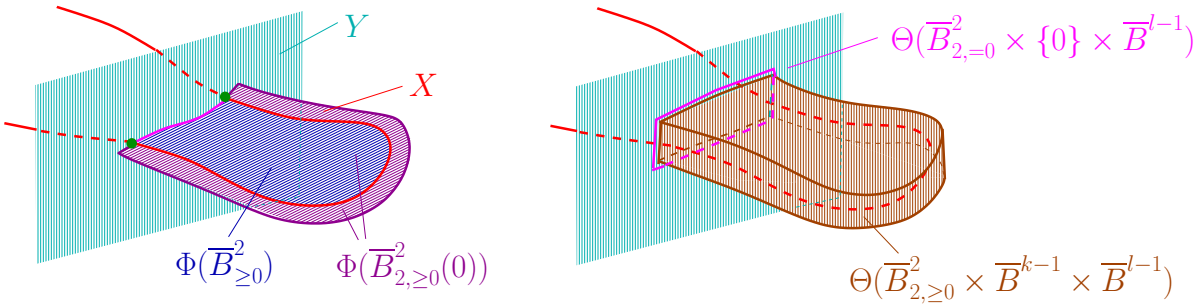
(1) For each $\xi \in \overline{B}^2_{2,\geq 0}$ we have $\Theta(\xi, 0, 0) = \Phi(\xi)$.

(2) We have $\Theta(\overline{B}^2_{2,=0} \times \{0\} \times \overline{B}^{l-1}) \subseteq Y$.

Proof. We start out with the following technical observation:

(∗) Using the Lower-Dimensional Smooth Ball Embedding Theorem 45.13, applied to the interval $\beta([0,1]) \subseteq Y$ and an interval $I \subseteq Y$, that is contained in a chart, one can first

[2800] It is worth noting that on this occasion we also make use of one of our dimension hypotheses.



of all show that there exists a smooth embedding $\xi\colon \overline{B}^l \to Y \subseteq \mathbb{R}^n$ of an l-ball into Y that contains $\beta([0,1])$ in the interior. By applying the Lower-Dimensional Smooth Ball Embedding Theorem 45.13 to the above smooth embedding ξ and a smooth embedding $\zeta\colon \overline{B}^l \to \mathbb{R}^n$ into an l-dimensional vector subspace of $\mathbb{R}^n$ we see that without loss of generality we can assume that there exists an open neighborhood of $\beta([0,1]) \subseteq Y$ that is contained in an l-dimensional vector subspace of $\mathbb{R}^n$.

Now we start with the actual proof. Let $v_1, \dots, v_{k-1}, w_1, \dots, w_{l-1}$ be the $(k-l-2)$-frame on $\Phi(\overline{B}^2_{2,\geq 0})$ that we constructed above. Now let $\epsilon \in \mathbb{R}_{>0}$. We consider the map

$$\begin{aligned} \Psi_\epsilon\colon \overline{B}^2_{2,\geq 0} \times \overline{B}^{k-1} \times \overline{B}^{l-1} &\to \mathbb{R}^n \\ (\xi, s_1, \dots, s_{k-1}, t_1, \dots, t_{l-1}) &\mapsto \Phi(\xi) + \epsilon \cdot \Big(\sum_{i=1}^{k-1} s_i \cdot v_i(\xi) + \sum_{i=1}^{l-1} t_i \cdot w_i(\xi) \Big). \end{aligned}$$

Recall that by the Neighborhood Projection Theorem 35.2 (or more fancy, by the Tubular Neighborhood–Existence Theorem 57.11) we know that there exists an open neighborhood U of the smooth submanifold $M \subseteq \mathbb{R}^n$ together with a submersion $\pi\colon U \to M$ that is the identity on M.

Since U is an open neighborhood of $M \subseteq \mathbb{R}^n$ we obtain from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that the image of Ψ_ϵ is contained in U. Finally, as in the proof of the Tubular Neighborhood–Existence Theorem 57.11, it follows from the Neighborhood Smooth Embedding Theorem 27.16 that for a (possibly smaller $\epsilon > 0$) the map $\Theta := \pi \circ \Psi_\epsilon$ is a smooth embedding. It is clear that this map satisfies Condition (1).

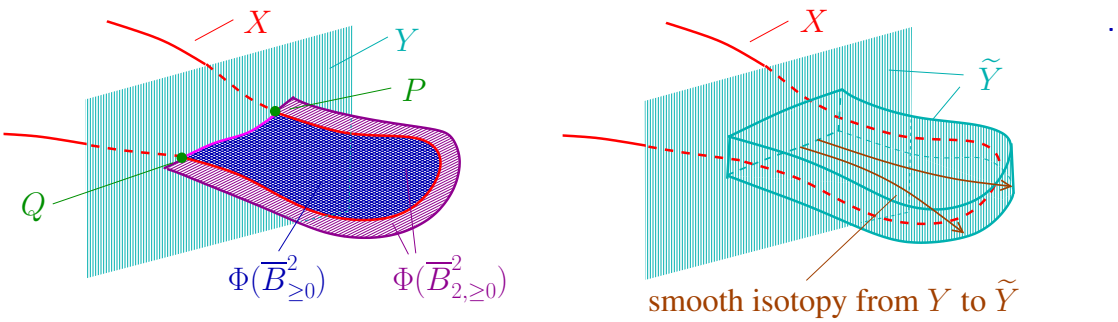
Finally recall that we arranged that on $\Phi(\overline{B}^2_{2,=0})$ the vectors $w_1, \dots, w_{l-1}$ are tangent to Y. It follows from (∗) that (after possibly shortening the vectors w_i) we obtain that $\Theta(\overline{B}^2_{2,=0} \times \{0\} \times \overline{B}^{l-1}) \subseteq Y$. ⊞

We now set

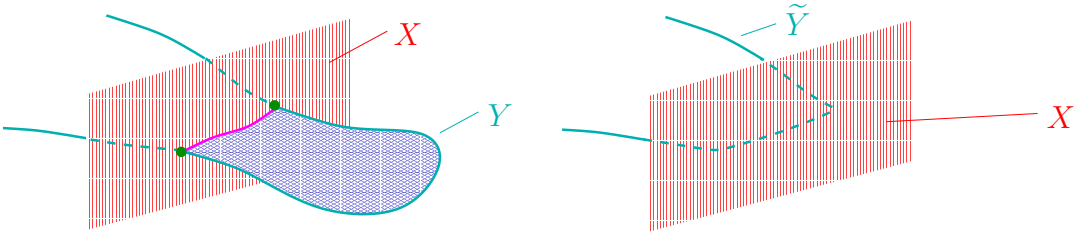
$$\widetilde{Y} \;:=\; Y \setminus \Theta(\overline{B}^2_{2,=0} \times \{0\} \times B^{l-1}) \cup \Theta(S^1_{2,\geq 0} \times \{0\} \times \overline{B}^{l-1}).$$

Using Θ and using the obvious smooth isotopy from the bottom interval $\overline{B}^2_{2,=0}$ to the upper half-circle $S^1_{2,\geq 0}$ we see that Y is smoothly isotopic to $\widetilde{Y}$. By construction we removed the intersection points P and Q from the tally.

Evidently we just rode roughshod over several technical issues, in particular we did not bother very much about smoothness and we forgot to round corners. Fortunately all these issues are dealt with carefully and to everybody's intellectual satisfaction in John Milnor's proof [**Miln1965b**, p. 74-84]. ■



Remark. In the above discussion we illustrated all arguments for the case $\dim(X) = k = 1$ and $\dim(Y) = l = 2$. Since the argument is not symmetric in X and Y it is fun to draw illustrations for $\dim(X) = k = 2$ and $\dim(Y) = l = 1$.



We leave it to the reader to verify that in this case we end up with the above cartoon. □

Sketch of a proof of the Whitney Trick-Theorem 262.5 (II). Let M be a compact oriented smooth manifold and let X and Y be two compact oriented connected proper submanifolds of M such that the following three conditions hold:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) We have $\partial X \cap \partial Y = \varnothing$.
(c) X and Y intersect transversally.

Furthermore we now suppose that the following condition holds:

(II) $\dim(X) \geq 3$, $\pi_1(M \setminus X) = 0$ and $\dim(Y) = 2$.

The proof is quite similar to the argument provided in (I). So we again suppose that we have $\#(X \cap Y) > |X \cdot Y|$. This means that we have two intersection points P and Q of opposite signs. As before we pick smooth embeddings α on X and β on Y.

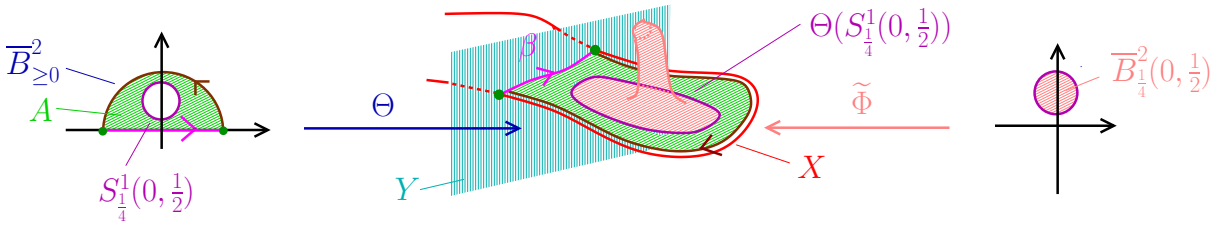
Again we consider the loop

$$\begin{array}{rcl} \varphi\colon S^1_{\geq 0} & \to & M \\ z & \mapsto & \begin{cases} \alpha(t), & \text{if } z = \exp(\pi\,\mathrm{i}t) \text{ with } t \in [0,1], \\ \beta(\frac{1}{2}(x+1)), & \text{if } z = (x,0) \in \overline{B}^2_{=0}. \end{cases} \end{array}$$

As on page 5059 we start out with the following claim:

Claim 1. There exists a smooth embedding $\Phi\colon \overline{B}^2_{\geq 0} \to M$ that extends the smooth embedding $\varphi\colon S^1_{\geq 0} \to M$ and such that $\Phi(\overline{B}^2_{\geq 0}) \cap (X \cup Y) = \varphi(S^1_{\geq 0})$.

Proof. As on page 5059 we will make use of the "annulus" $A := \overline{B}^2_{\geq 0} \setminus B^2_{\frac{1}{4}}(0, \frac{1}{2})$ and again we see that we can extend the smooth embedding $\varphi\colon S^1_{\geq 0} \to M$ to a smooth embedding $\Theta\colon A \to M$ such that $\Theta|_{S^1_{\geq 0}} = \varphi$ and such that $\Theta(A) \cap (X \cup Y) = \varphi(S^1_{\geq 0})$.



Next we consider the loop $\Theta\colon S^1_{\frac{1}{4}}(0,\frac{1}{2}) \to M \setminus X$. Since $M \setminus X$ is by hypothesis simply connected we know that the loop $\Theta\colon S^1_{\frac{1}{4}}(0,\frac{1}{2}) \to M \setminus X$ is null-homotopic. Since

$$\dim(M \setminus X) = \dim(X) + \dim(Y) \underset{\substack{\uparrow \\ \text{follows from } \dim(X) \geq 3 \\ \text{and } \dim(Y) \geq 2}}{\geq} 3 + 2 = 5 = 2 \cdot \dim(S^1_{\frac{1}{4}}(0,\tfrac{1}{2})) + 3$$

we obtain from the above Embedded Sphere–Embedded Ball Proposition **??** that there exists a smooth embedding $\widetilde{\Phi}\colon \overline{B}^2_{\frac{1}{4}}(0,\frac{1}{2}) \to M \setminus X$ that extends the given smooth embedding $\Theta\colon S^1_{\frac{1}{4}}(0,\frac{1}{2}) \to M \setminus X$. Next note that we have

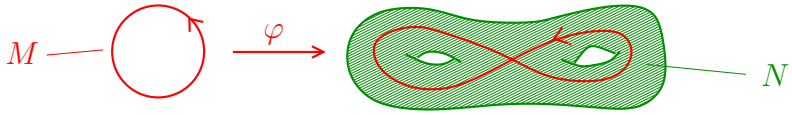
$$\dim(\widetilde{\Phi}(\overline{B}^2_{\frac{1}{4}}(0,\tfrac{1}{2}))) + \dim(Y) = 2 + \dim(M \setminus X) - \dim(X) \underset{\substack{\uparrow \\ \text{since } \dim(X) \geq 3}}{\leq} 2 + \dim(M \setminus X) - 3 < \dim(M \setminus X).$$

It follows from the Transversality Theorem 50.10 that in $M \setminus X$ we can find a smooth isotopy of $\widetilde{\Phi}$ rel boundary such that $\widetilde{\Phi}(\overline{B}^2_{\frac{1}{4}}(0,\frac{1}{2}))$ is disjoint from Y. The map Θ together with the map $\widetilde{\Phi}$ defines the promised smooth embedding $\Phi\colon \overline{B}^2_{\geq 0} \to M$. ⊞
At this point the argument proceeds as in (I). One can easily verify that for the other steps in the proof of (I) it was enough to assume that $\dim(Y) \geq 2$. ■

262.3. **The Strong Whitney Embedding Theorem.** In this section we want to discuss the Strong Whitney Embedding Theorem 262.11 which was the original motivation for proving the Whitney Trick-Theorem 262.5.

Before we talk about the Strong Whitney Embedding Theorem 262.11 let us recall Definition 30.9:

Definition 262.7. Let M, N be two smooth manifolds with $\partial N = \varnothing$. A map $\varphi\colon M \to N$ is called an immersion if the following conditions are satisfied:
(1) The map φ is smooth.
(2) For each $P \in M$ the differential $\mathrm{D}\varphi_P\colon \mathrm{T}_P M \to \mathrm{T}_{\varphi(P)} N$ is a monomorphism.
An immersion that also satisfies the following extra condition
(3) $\varphi\colon M \to N$ is an embedding, i.e. $\varphi\colon M \to N$ is a homeomorphism onto its image,
is called a smooth embedding.



immersion but not a smooth embedding

A long time ago, in the Euclidean Embedding Theorem 33.2 we saw that every m-dimensional smooth manifold M can be properly smoothly embedded into some $\mathbb{R}^n$. The question arises: What is the lowest possible dimension n that works for any closed m-dimensional smooth manifold?

We will deal with this question in the following two subsections:

- In Subsection 262.3.1 we will (re-) prove the Weak Whitney Embedding Theorem 262.8 which says that every closed m-dimensional smooth manifold M admits a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$.
- In Subsection 262.3.2 we will use the Whitney Trick-Theorem 262.5 to upgrade the Weak Whitney Embedding Theorem 262.8 to the Strong Whitney Embedding Theorem 262.11 which says that every closed m-dimensional smooth manifold M admits a smooth embedding $f\colon M \to \mathbb{R}^{2m}$.

262.3.1. **The Weak Whitney Embedding Theorem.** The following theorem is just a special case of the Weak Whitney Embedding Theorem 37.9:

Theorem 262.8. (Weak Whitney Embedding Theorem) Every closed m-dimensional smooth manifold M admits a smooth embedding $f\colon M \to \mathbb{R}^{2m+1}$.

For immersions we can do "one dimension better":

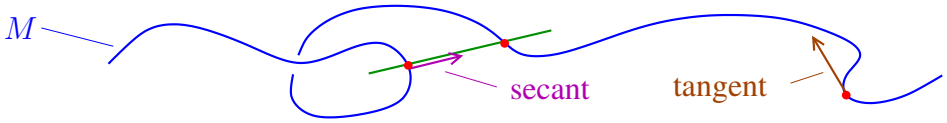
Theorem 262.9. (Weak Whitney Immersion Theorem) Every closed m-dimensional smooth manifold M admits an immersion $f\colon M \to \mathbb{R}^{2m}$.

Below we will essentially repeat the proof of the Weak Whitney Embedding Theorem 37.9. We will see that a careful rereading of the earlier proof allows us to prove the Weak Whitney Immersion Theorem 262.9 along the way.

The proofs of the Weak Whitney Embedding Theorem 262.8 and of the Weak Whitney Immersion Theorem 262.9 require the following lemma, which is a slight variation on the Generic Vector Lemma 37.10:

Lemma 262.10. (Generic Vector Lemma) Let M be a closed m-dimensional smooth submanifold of $\mathbb{R}^n$.

(1) We consider the set of vectors $v \in S^{n-1}$ that lie in any tangent space $\mathrm{T}_P M$. If $n > 2m$, then the set of such vectors has measure zero (in the sense of Definition 37.1) in S^{n-1}.
(2) We say a vector $v \in \mathbb{R}^n$ is *secant to* M if there exist two distinct points $x, y \in M$ such that the difference $x - y$ is parallel to v. If $n > 2m + 1$, then the set of vectors secant to M has measure zero in S^{n-1}.



the set of tangent vectors of length 1 is a 1-dimensional smooth manifold
the set of secant vectors of length 1 is a 2-dimensional smooth manifold

Proof of the Generic Vector Lemma 262.10. Let M be a closed m-dimensional submanifold of $\mathbb{R}^n$.

(1) We start out with the following claim:

Claim. The subset
$$X \;:=\; \{(P,w)\,|\,P\in M, w\in \mathrm{T}_PM \text{ with } \|w\|=1\} \;\subseteq\; \mathbb{R}^n\times\mathbb{R}^n$$
is a smooth submanifold of $\mathbb{R}^n\times\mathbb{R}^n$ of dimension $2m-1$.

Proof. First we consider the subset
$$\widetilde{X} \;:=\; \{(P,w)\,|\,P\in M, w\in \mathrm{T}_PM\}\} \;\subseteq\; \mathbb{R}^n\times\mathbb{R}^n.$$
In the proof of the Generic Vector Lemma 37.10 we showed that $\widetilde{X}$ is a $2m$-dimensional smooth submanifold of $\mathbb{R}^{2n}$. For the reader's convenience we recall the argument: Thus let $(P,w)\in\widetilde{X}$. We pick a submanifold chart $\Phi\colon U\to V$ around $P\in M$. The map
$$\begin{array}{rcl}\Psi\colon U\times(\mathbb{R}^n\setminus\{0\}) & \to & V\times\mathbb{R}^n)\\ (x,v) & \mapsto & (\Phi(x),\mathrm{D}\Phi_x(v))\end{array}$$
is easily seen to be smooth and, using Φ^{-1}, one can easily write down a smooth inverse. Thus we see that Ψ is a diffeomorphism. Up to a permutation of coordinates the map Ψ is a $2m$-dimensional smooth submanifold chart for $(P,w)\in\widetilde{X}$.

Next now that
$$X \;=\; \widetilde{X}\cap(\mathbb{R}^n\times S^{n-1}).$$
One can easily verify that the two submanifolds $\widetilde{X}$ and $\mathbb{R}^n\times S^{n-1}$ intersect transversally in the sense of Definition 50.3. It thus follows from the Submanifold Transversal Intersection Theorem 50.4 that X is a smooth submanifold of $\mathbb{R}^n\times\mathbb{R}^n$ of dimension $2m-1$. ⊞

In the following we suppose that $n>2m$. We consider the map
$$\begin{array}{rcl}f\colon X \;=\; \{(P,w)\,|\,P\in M, w\in \mathrm{T}_PM \text{ with } \|w\|=1\} & \to & S^{n-1}\\ (P,w) & \mapsto & w.\end{array}$$
As we just showed in the claim, the domain X of the above map is a smooth submanifold of $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$ of dimension $2m-1$. It follows easily from the definitions of the smooth structures that the map f is smooth. Since by hypothesis $2m-1<n-1$ we obtain from the Measure-Zero Properties Proposition 37.3 (4) that the image of f has measure zero. But the image of f is by definition precisely the set we are interested in.

(2) This time we start out with the following claim:

Claim. The subset
$$Y \;:=\; \{(x,y)\in M\times M\,|\,x\neq y\}$$
is a smooth submanifold of $\mathbb{R}^n\times\mathbb{R}^n$ of dimension $2m$.

Proof. Using products of submanifold charts one can easily show that the product of two closed smooth submanifolds of $\mathbb{R}^n$, is a smooth submanifold of $\mathbb{R}^n\times\mathbb{R}^n$ of the obvious dimension. By Exercise 7.9 we know that the diagonal $\{(x,x)\in M\times M\,|\,x\in M\}$ is a closed subset of $M\times M$. It follows that the subset Y is an open subset of the $2m$-dimensional smooth submanifold $M\times M$. Thus Y itself is a $2m$-dimensional smooth submanifold. ⊞

In the following we suppose that $n>2m+1$. We consider the map
$$\begin{array}{rcl}g\colon Y \;=\; \{(x,y)\in M\times M\,|\,x\neq y\} & \to & S^{n-1}\\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|}.\end{array}$$
As we just showed in the claim, the domain Y of the above map is a smooth submanifold of $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$ of dimension $2m$. Clearly g is smooth. Since $2m<n-1$ we obtain

again from the Measure-Zero Properties Proposition 37.3 (4) that the image of g has measure zero.

Finally note that it follows easily from the definitions that the set we are interested in is given by $\{\pm g(y) \mid y \in Y\}$. It follows from the above together with the Measure-Zero Properties Proposition 37.3 (2a) that this set has measure zero. ■

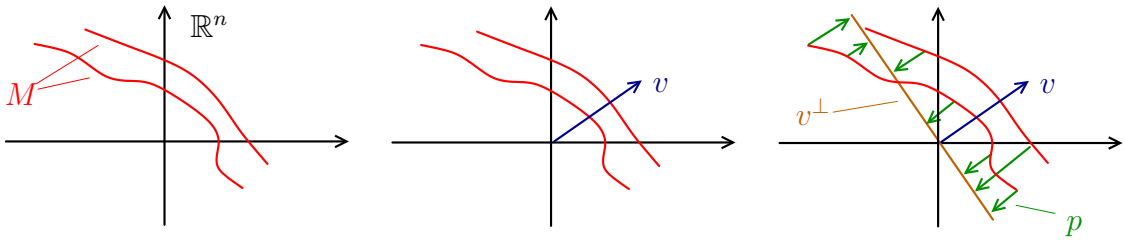
In the following we will provide the proofs of the Weak Whitney Embedding Theorem 262.8 and of the Weak Whitney Immersion Theorem 262.9 in reverse order:

Proof of the Weak Whitney Immersion Theorem 262.9. [2801] Let M be a closed m-dimensional smooth manifold. We claim that we can find an immersion of M to $\mathbb{R}^{2m}$. By the Euclidean Embedding Theorem 33.2 together with the Smooth Embedding Theorem 30.15 we can assume that M is already a submanifold of some $\mathbb{R}^n$. If $n \leq 2m$ there is nothing to prove, so we may assume that $n > 2m$. It then follows from the Generic Vector Lemma 262.10 (1) together with the Full Measure-Density Proposition 37.4 (1) that there exists a $v \in S^{n-1}$ which does not lie in any tangent space $\mathrm{T}_P M$.

We consider the projection of $\mathbb{R}^n$ onto the orthogonal complement of v, i.e. we consider the projection

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & v^\perp := \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\} \\ u & \mapsto & u - \langle u, v\rangle \cdot \frac{v}{\|v\|^2}. \end{array}$$

Since v does not lie in any tangent space $\mathrm{T}_P M$ we obtain, through elementary considerations and the discussion on page 651, that the map $p\colon M \to v^\perp$ is an immersion.



In summary we have shown that the restriction of the projection p to M is an immersion to $\mathbb{R}^{n-1}$. Iterating this process we get an immersion of M to $\mathbb{R}^{2m}$. ■

Proof of the Weak Whitney Embedding Theorem 262.8. Let M be a closed m-dimensional smooth manifold. We claim that we can smoothly embed M into $\mathbb{R}^{2m+1}$. By the Euclidean Embedding Theorem 33.2 together with the Smooth Embedding Theorem 30.15 we can assume that M is already a submanifold of some $\mathbb{R}^n$. If $n \leq 2m+1$ there is nothing to prove, so we may assume that $n > 2m+1$. It then follows from the Generic Vector Lemma 262.10 (1) and (2) together with the Full Measure-Density Proposition 37.4 (1) and the Measure-Zero Properties Proposition 37.3 (2) that there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space $\mathrm{T}_P M$ and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

As above we consider the projection of $\mathbb{R}^n$ onto the orthogonal complement of v, i.e. we consider the projection

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & v^\perp = \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\} \\ u & \mapsto & u - \langle u, v\rangle \cdot \frac{v}{\|v\|^2}. \end{array}$$

[2801] This proof gives a solution to Exercise 37.5.

Since v does not lie in any tangent space $\mathrm{T}_P M$ we see, as in the proof of the Weak Whitney Immersion Theorem 262.9, that $p\colon M \to v^\perp$ is an immersion. Furthermore, since v is not secant to M we obtain via an elementary argument that $p\colon M \to v^\perp$ is injective. Since we assume that M is compact we obtain from the Compact-Hausdorff Proposition 4.23 (3) that $p\colon M \to v^\perp$ is actually an embedding.

In summary we have shown that the restriction of the projection p to M is a smooth embedding. Thus we have embedded M into a real vector space of dimension $n-1$. Iterating this process we get a smooth embedding of M into $\mathbb{R}^{2m+1}$. ■

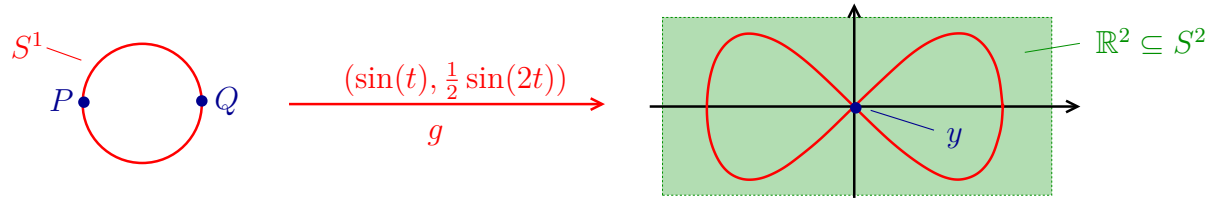
262.3.2. **The Strong Whitney Embedding Theorem.** In this subsection we will see that in the Weak Whitney Embedding Theorem 262.8 and also in the Weak Whitney Immersion Theorem 262.9 we can lower the dimension of the target by one.

More precisely, we first of all have the following strengthening of the Weak Whitney Embedding Theorem 262.8:

Theorem 262.11. (Strong Whitney Embedding Theorem) Every closed n-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2n}$.

In the proof of the Strong Whitney Embedding Theorem 262.11 we will need the following lemma:

Lemma 262.12. (Sphere–Double Point Immersion Lemma) Let $n \in \mathbb{N}$. There exists an immersion $g\colon S^n \to S^{2n}$ such that there exists a single point $y \in S^{2n}$ with more than one preimage. Furthermore for the two points $P, Q \in g^{-1}(y)$ the intersection is transverse in the sense of Definition 50.3, i.e. $\mathrm{D}g_P(\mathrm{T}_P S^n) + \mathrm{D}g_Q(\mathrm{T}_Q S^n) = \mathrm{T}_y S^{2n}$.



Sketch of proof of the Sphere–Double Point Immersion Lemma 262.12. First we consider the case $n = 1$. In this case one can easily verify that the map

$$\begin{array}{rcl} g\colon S^1 & \to & \mathbb{R}^2 \subseteq S^2 \\ (\sin(t), \cos(t)) & \mapsto & (\sin(t), \frac{1}{2}\sin(2t)) \end{array}$$

has the desired properties with $y = (0,0)$ and $P = (0,0)$, $Q = (0,\pi)$. For $n \geq 2$ we first consider the map

$$\begin{array}{rcl} f\colon S^{n-1} \times [0,\pi] & \to & \mathbb{R}^{2n} \subseteq S^{2n} \\ (x,t) & \mapsto & (x \cdot \sin(t), \frac{1}{2} \cdot x \cdot \sin(2t)). \end{array}$$

Note that all points in $S^{n-1} \times \{0\}$ and $S^{n-1} \times \{\pi\}$ get sent to the origin. Thus f can be written as $g \circ p$ where $p\colon S^{n-1} \times [0,\pi] \to S^n$ is given by $p(x,t) = (x \cdot \sin(t), \cos(t))$ and $g\colon S^n \to \mathbb{R}^{2n}$ is a map. It is elementary to verify that g is smooth, that g is an immersion and that g has all the desired properties. The details are worked out in [**Wall2016**, Proposition 6.3.3]. An alternative proof of the lemma is sketched in [**Juh2023**, Lemma 2.3]. ■

Sketch of proof of the Strong Whitney Embedding Theorem 262.11. This result was first proved by Hassler Whitney [**Why1944a**, p. 236] in 1944. For closed smooth manifolds more recent accounts of the proof can be found in [**Adac1993**, Theorem II.2.11], [**Pra2007**, Chapter 6.3] or [**Wall2016**, Theorem 6.3.4].

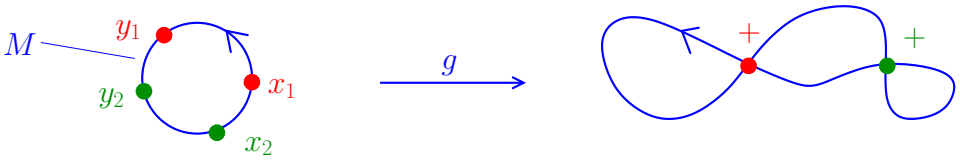
In the following we sketch an argument if M is a closed orientable n-dimensional smooth manifold. We need to show that M admits a smooth embedding into $\mathbb{R}^{2n}$. If $n = 1$ or $n = 2$, then the statement follows basically from the Classification Theorems 28.8 and 101.16.

Now we assume that $n \geq 3$. Clearly it suffices to deal with the case that M is connected. We proceed in several steps:

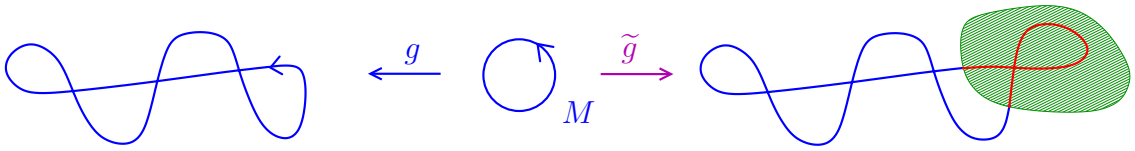
(1) By the Weak Whitney Immersion Theorem 262.9] we have an immersion $f\colon M \to \mathbb{R}^{2n}$.

(2) It follows from the Self-Transversality Theorem **??** that we can in fact find an immersion $g\colon M \to \mathbb{R}^{2n}$ that has the following two extra properties:
- The map g is self-transverse in the sense of **??**. This means that for every $P, Q \in M$ with $g(P) = g(Q)$ we have $\mathrm{D}g_P(\mathrm{T}_P M) + \mathrm{D}g_Q(\mathrm{T}_Q M) = \mathrm{T}_{g(P)}\mathbb{R}^{2n}$.
- The map g has no triple points, in the sense of **??**. This means that the preimage of any $P \in \mathbb{R}^{2n}$ consists of at most two points.

(3) It follows from (2) and the compactness of M that there exist only finitely many $z_1, \dots, z_k \in \mathbb{R}^{2n}$ such that the preimage of each z_i consists of two points. For each z_i we pick an ordering (x_i, y_i) of the preimages in M. For each $i \in \{1, \dots, k\}$ we can now consider the corresponding sign ϵ_i of the intersection.[2802]



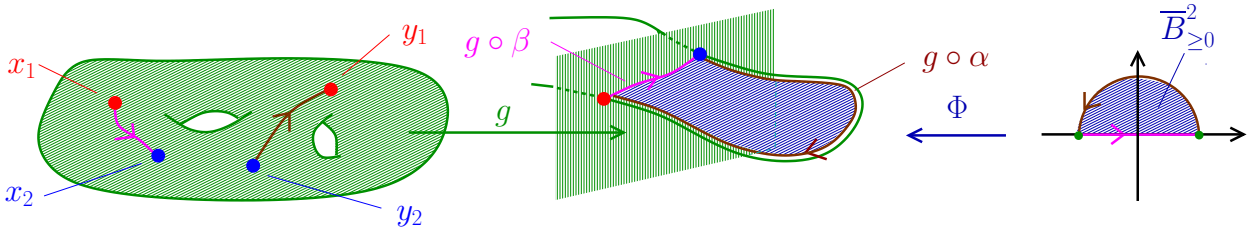
(4) Using the Sphere–Double Point Immersion Lemma 262.12 and using the connected sum operation of submanifolds we can arrange in (3) that the sum of the signs of the intersection points is zero.



(5) We will now eliminate double points of opposite sign. For argument's sake let us suppose that $\epsilon_1 = -\epsilon_2$. Since M is connected we can pick, as in the proof of the Whitney Trick-Theorem 262.5, a smooth embedding $\alpha\colon [0,1] \to M$ from x_1 to x_2 and we can pick a smooth embedding $\beta\colon [0,1] \to M$ from y_2 to y_1. Since $n \geq 2$ we can find such smooth embeddings with disjoint images and such that the images of α and β do not contain pairs of double points. The latter conditions implies that $g \circ \alpha$ and $g \circ \beta$ are also smooth embeddings

Now we proceed as in the proof of the Whitney Trick-Theorem 262.5, namely, using the fact that $2n \geq 6$ and using that $\mathbb{R}^{2n}$ is simply connected we show that there exists

[2802]Note that if we swap the order of x_i and y_i and if n, then it follows from the Intersection Number–Symmetry Lemma 51.6 that we flip the sign of the intersection if n is odd, and the is unchanged if n is even.



a smooth embedding $\Phi\colon \overline{B}^2_{\geq 0} \to \mathbb{R}^{2n}$ that bounds the loop given by $g \circ \alpha$ and $g \circ \beta$. We use this disk to change the immersion of M locally around $\beta([0,1])$ to remove the two intersection points y_1 and y_2. Iterating this procedure gives us the desired result. ■

Remark.

(1) Under some extra conditions one can obtain better results. For example we saw in Exercise 32.4 that every closed orientable 2-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^3$. Furthermore Hirsch [**Hirs1961**] and Wall [**Wall1965b**] showed that every 3-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^5$. Finally for $n > 4$ Haefliger-Hirsch [**HiH1963**, Theorem 1.2] showed that every closed orientable n-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2n-1}$.

(2) As we pointed out on page 866, it follows easily from the Topological Space–Euclidean Embedding Theorem 15.16 that every n-dimensional topological manifold admits an embedding into $\mathbb{R}^{2n+1}$. At least for compact topological manifolds one can actually find an embedding into $\mathbb{R}^{2n}$. We refer to [**MeS2006**, Corollary 4.14] and [**BryM1999**] for details.

(3) If M and N are two smooth manifolds that can be embedded (respectively immersed) into $\mathbb{R}^n$, then the construction in [**Hae1961a**, Chapter 1] [**Hae1967**, Chapter 3] shows that the connected sum $M \# N$ can also be embedded in $\mathbb{R}^n$.

(4) Let M be a compact connected smooth manifold and let N be a simply connected smooth manifold. If $\dim(N) \geq 2 \cdot \dim(M)$ and $\dim(M) \geq 3$, then [**Wall2016**, Theorem 6.3.4] says that any continuous map $M \to N$ is homotopic to a smooth embedding. As is explained on [**Wall2016**, p. 182] the various hypothesis on M, N and the dimensions are necessary. □

We conclude this chapter with the following strengthening of the Weak Whitney Immersion Theorem 262.9:

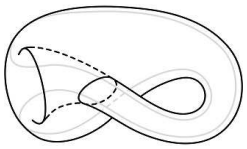
Theorem 262.13. (Whitney Immersion Theorem) Every smooth manifold of dimension $n \geq 2$ admits an immersion into $\mathbb{R}^{2n-1}$.

Proof. This theorem is proved in [**Why1944b**, p. 247] and [**Wall2016**, Theorem 6.3.6]. ■

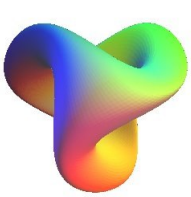
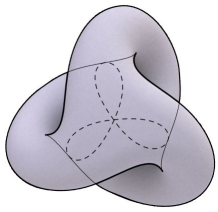
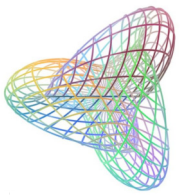
Example.

(1) The Whitney Immersion Theorem 262.13 says in particular that every closed 2-dimensional smooth manifold admits an immersion to $\mathbb{R}^3$. We will illustrate this fact for the Klein bottle and the real projective plane:

- In the figure below we show the image of an immersion of the Klein bottle into $\mathbb{R}^3$.




- As we discussed on page 788 there exists an explicit immersion of the real projective plane $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$.


Boy's surface at Oberwolfach

The first example of such an immersion was found by Werner Boy [**Boy1903**]. The image of Boy's immersion is shown in the figure above.

(2) As always one can ask whether one can do better than the Whitney Immersion Theorem 262.13. It is it pretty clear that one cannot do better for $n = 2$. But more interestingly we will see on page 5308 that for any $r \in \mathbb{N}$ the 2^r-dimensional real projective space $\mathbb{R}\mathrm{P}^{2^r}$ does not admit an immersion into $\mathbb{R}^{2\cdot 2^r-2}$. □

262.4. **Appendix: Making submanifolds disjoint in low dimensions.** In this appendix we will see that Question 262.4 has, with suitable refinements, also an affirmative answer in very low dimensions. More precisely the following variation on the Whitney Trick-Theorem 262.5 holds:

Theorem 262.14. (Low-Dimensional Whitney Trick Theorem) Let M be a compact oriented smooth manifold and let X and Y be two compact oriented connected proper submanifolds of M such that the following three conditions hold:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) We have $\partial X \cap \partial Y = \varnothing$.
(c) X and Y intersect transversally.

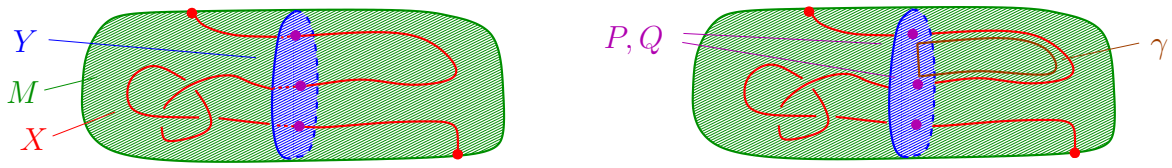
Furthermore we suppose that the following condition holds:

(∗) $\dim(M) = \dim(X) + \dim(Y) \in \{2, 3\}$.

We assume that there exist intersection points of opposite signs P and Q and we assume that there exist the following:

- A smooth embedding $\alpha\colon [0,1] \to X \setminus \partial X$ with $\alpha(0) = P$ and $\alpha(1) = Q$,
- a smooth embedding $\beta\colon [0,1] \to Y \setminus \partial Y$ with $\beta(0) = Q$ and $\beta(1) = P$,
- a smooth embedding curve $\gamma\colon S^1 \to M \setminus (X \cup Y)$ such that $\gamma(S^1)$ that cobounds in M an annulus with $\alpha * \beta$ and such that $[\gamma] = e \in \pi_1(M \setminus (X \cup Y))$.

Then there exists a proper smooth isotopy rel ∂Y from Y to a smooth submanifold $\widetilde{Y}$ such that X and $\widetilde{Y}$ intersect transversally in precisely $\#(X \cap Y) - 2$ points.



Proof. [2803] For 2-dimensional smooth manifolds the statement can be proved, using the fact, proved in the Surface Classification Theorem 101.16, that every simply connected 2-dimensional smooth manifold is diffeomorphic to S^2 or to $\overline{B}^2$. We leave it to the reader to fill in the details.

For 3-dimensional smooth manifolds the proof is similar to the proof in dimensions ≥ 5 except that one also needs to make use of the Loop Theorem **??** to find the embedded disk needed in the proof of the Whitney Trick-Theorem 262.5. Once again we leave it as a slightly challenging exercise to fill in the details in the 3-dimensional setting. ■

So we are still missing 4-dimensional smooth manifolds. Here the situation becomes tricky. It was known by 1961, see [**KeM1961**, Corollary 1], that the above approach of the proof of the Whitney Trick-Theorem 262.5 of making submanifolds disjoint does not work for 4-dimensional smooth manifolds. Later, see [**Lacke1996**], it was also shown that the above approach does not work for 4-dimensional topological manifolds either.

In the 4-dimensional setting it is not only the case that the Whitney trick does not work. But more fundamentally, Question 262.4 has in fact a negative answer. For example it is known that there exist closed simply connected 4-dimensional smooth manifolds and embedded spheres X, Y with zero intersection number that cannot be made disjoint by a smooth isotopy, in fact they cannot even be made disjoint by an isotopy that is not necessarily smooth, see e.g. [**As1998**, Theorem 1.2] for details.[2804] As is pointed out in [**As1998**, Proposition 3.9], it is a consequence of the Freedman disk smooth embedding theorem [**FQ1990**, Theorem 5.1.A] that under even stronger hypotheses the algebraic intersection number is in fact a complete invariant in the setting of 4-dimensional *topological* manifolds.

In **??** we already saw that 4-dimensional smooth manifolds exhibit many peculiar properties. Arguably one of the main reasons why 4-dimensional smooth manifolds differ dramatically from higher-dimensional smooth manifolds is the failure of the Whitney trick.

262.5. Exercises for Chapter 262.

Exercise 262.1. Determine whether there exist proper 1-dimensional smooth submanifold $X, X' \subseteq \overline{B}^3$ with $\partial X = \partial X'$ and the following two properties:

(a) X and X' are properly smoothly isotopic in the sense of Definition 262.2.
(b) X and X' are *not* properly smoothly isotopic *rel boundary* in the sense of Definition 262.2.

[2803] The reader should *not* assume that this theorem is a 100% correct until the reader has managed to fill in all the details. For such a statement the devil is in the detail.

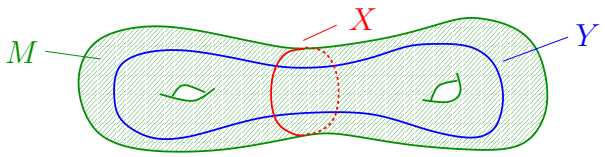
[2804] More examples for the failure of the Whitney trick in dimension 4, i.e. for fact that the conclusion of the Whitney Trick-Theorem 262.5 does not hold in dimension 4, are given by Saŝo Strle [**Strl2004**, Section 15.1] and Patrick Gilmer [**Gil1981**].

Note that here we do *not* view X and X' as oriented submanifolds.

Exercise 262.2.

(a) Let M be a smooth manifold, let $p\colon \widetilde{M} \to M$ be a finite covering and let X and Y be two closed submanifolds of M. Show that if X and Y can be made disjoint through a smooth isotopy of Y, then any *component* of $p^{-1}(X)$ can be made disjoint from any *component* of $p^{-1}(Y)$ via a smooth isotopy of the latter.

(b) We consider the surface M of genus two together with the two submanifolds X and Y that are shown in the figure below. Use (a) to show that X and Y cannot be made disjoint through a smooth isotopy of Y.



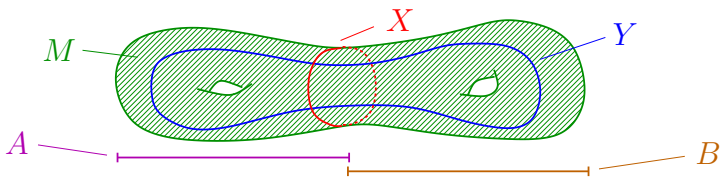
Exercise 262.3. As in Exercise 262.2 we consider the surface M of genus two together with the two submanifolds X and Y that are shown in the figure below. We also consider the compact codimension-zero smooth submanifolds A and B of M that are shown in the figure below.

(a) Show that for any closed oriented 1-dimensional submanifold Z of M the homology classes $[Z] \in \mathrm{H}_1(M, A)$ and $[Z] \in \mathrm{H}_1(M, B)$ are unchanged under homotopies of Z.

(b) Show that $[Y] \in \mathrm{H}_1(M, A)$ and $[Y] \in \mathrm{H}_1(M, B)$ are both non-zero.
Remark. Note that by the Manifold–Excision Theorem 148.3 the inclusion induced maps $\mathrm{H}_1(A, X) \to \mathrm{H}_1(M, B)$ and $\mathrm{H}_1(B, X) \to \mathrm{H}_1(M, A)$ are isomorphisms.

(c) Use (a) and (b) to show that there is no homotopy of Y to a smooth submanifold that is disjoint from X.

Remark. This is known as the DZ-trick.



Exercise 262.4. Let M be an n-dimensional smooth manifold and let $Y \subseteq M$ be a compact proper k-dimensional submanifold with tubular neighborhood $\tau\colon N \to Y$. (Recall that $p\colon N \to Y$ is in particular a linear $\overline{B}^{n-k}$-bundle.) Let $P \in \partial N$. We set $X := M \setminus \mathring{N}$. Use the Manifold–Seifert-van Kampen Theorem 102.1, applied to $M = X \cup_{\partial N} N$, together with the Fiber-LES-Proposition 239.8 to prove the following two statements:

(a) If $\dim(M) > \dim(Y) + 2$, then the map $\pi_1(X, P) \to \pi_1(M, P)$ is an isomorphism.

(b) If $\dim(M) > \dim(Y) + 1$, then the map $\pi_1(X, P) \to \pi_1(M, P)$ is an epimorphism.

Remark. In Exercise 134.8 we used the Transversality Theorem 50.9 to give an alternative proof of this statement.



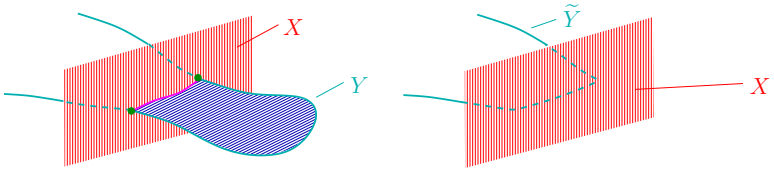
Exercise 262.5. Does there exist a closed 1-dimensional smooth submanifold $M \subseteq \mathbb{R}^3$ such that every vector in $\mathbb{R}^3$ is secant to M, i.e. such that for every $v \in \mathbb{R}^3$ there exist $x, y \in M$ with $v \in \mathbb{R} \cdot (x - y)$?
Remark. This exercise is identical to Exercise 37.7.

Exercise 262.6. In the figure below we see a smooth embedding $\varphi\colon S^1 \to \mathbb{R}^3$. If we compose it with the smooth embedding $\iota\colon \mathbb{R}^3 \xrightarrow{x \mapsto (x,0)} \mathbb{R}^3 \times \{0\}$, then we obtain a smooth embedding $\iota \circ \varphi\colon S^1 \to \mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^4$. Show "by hand" that $\iota \circ \varphi$ admits an extension to a smooth embedding $\Phi\colon \overline{B}^2 \to \mathbb{R}^4$.
Hint. First show that you can turn the trefoil into an unknot using the "fourth dimension".
Hint. Much later, in the Trefoil-not-Slice Theorem **??** we will see that the trefoil does not bound a smooth disk in $\mathbb{R}^3 \times [0, \infty)$. So you will need to go into both positive and negative x_4-coordinates.



Exercise 262.7. Give an example of a closed oriented simply connected smooth manifold of dimension ≥ 6 and two closed orientable disconnected submanifolds X and Y of dimension ≥ 3 with $X \cdot Y = 0$ such that X and Y cannot be made disjoint via a smooth isotopy.
Remark. This exercise shows that in the formulation of the Whitney Trick-Theorem 262.5 we cannot drop the hypothesis that X and Y are connected.

Exercise 262.8. Go through the proof of the Whitney Trick-Theorem 262.5 and draw cartoons for the case $\dim(X) = k = 2$ and $\dim(Y) = l = 1$. You should see that the end result looks like the figure shown below.



Exercise 262.9. Find a "naive" argument why every compact 2-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^4$.

Exercise 262.10.
(a) Show that every compact 2-dimensional smooth manifold admits an immersion into $\mathbb{R}^3$.
(b) Show that every compact orientable connected 2-dimensional smooth manifold with non-empty boundary admits an immersion into $\mathbb{R}^2$.

CHAPTER 263

The h-Cobordism Theorem

In this chapter we will, not surprisingly, state and prove the h-Cobordism Theorem 263.11. This theorem shows that if in dimension ≥ 6 a simply connected smooth manifold "looks like a product" in terms of invariants, then it is actually *diffeomorphic* to a product. This theorem has many consequences. The most famous one is surely the High-Dimensional Poincaré Conjecture 264.7 which says that in dimension ≥ 6 any smooth homotopy n-sphere is *homeomorphic* to S^n.[2805]

In the proof of the h-Cobordism Theorem 263.11 we will make use of the Handle Cancellation Theorem 260.1, the Handle Slide Theorem 260.13 and the Whitney Trick-Theorem 262.5 which we established in the previous chapters.

This chapter is organized as follows:

- In Section 263.1 we will introduce the notion of a cobordism and of a homology cobordism.
- In Section 263.2 we will introduce the notion of a smooth h-cobordism.
- In Section 263.3 we will state and discuss the h-Cobordism Theorem 263.11.
- In Section 263.4 we will prove the h-Cobordism Theorem 263.11.
- In Appendix 263.5 we will discuss a generalization of the h-Cobordism Theorem 263.11. This generalization says that under suitable hypothesis one can find handle decompositions that turn the inequalities provided by the Morse Inequalities Proposition 252.13 into equalities.
- In Section 263.6 we will provide exercises for Chapter 263.

In the next Chapter 264 we will discuss several applications of the h-Cobordism Theorem 263.11. In particular we will use it to prove the famous High-Dimensional Poincaré Conjecture 264.7.

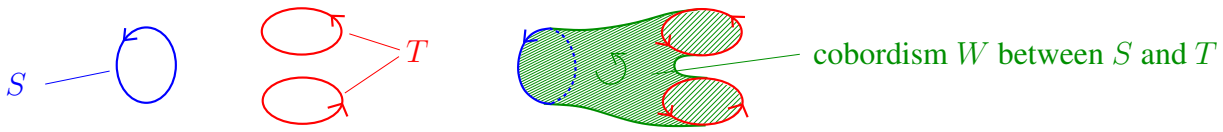
263.1. **Homology cobordisms.** We start out with a very general definition, which is a variation on Definition 212.11:

Definition 263.1. Let $n \in \mathbb{N}_0$ and let S and T be two closed oriented n-dimensional smooth manifolds.[2806]

(1)
- A cobordism between S and T is a pair $(W, \Theta\colon \partial W \to S \sqcup -T)$ consisting of a compact oriented $(n+1)$-dimensional smooth manifold W together with an orientation-preserving diffeomorphism[2807] $\Theta\colon \partial W \xrightarrow{\sim} S \sqcup -T$.[2808]
- We refer to the maps $S \to S \sqcup -T \xrightarrow{\Theta^{-1}} \partial W$ and $T \to S \sqcup -T \xrightarrow{\Theta^{-1}} \partial W$ as the natural smooth embeddings.

(2) We say that S and T are cobordant if there exists a cobordism between S and T.

[2805]Here we following the weird mathematical convention, that we say "the Poincaré *Conjecture*" instead of "the Poincaré Theorem".



Example.

(1) Let S be a closed oriented n-dimensional smooth manifold. It follows from the Smooth Manifold Product Proposition 25.34 (4) that $\partial([0,1]\times S) = -(\{0\}\times S)\cup(\{1\}\times S)$. This observation shows that $W := [0,1]\times S$, together with the obvious diffeomorphism $-(\{0\}\times S)\cup(\{1\}\times S)\to S\sqcup -S$, is a cobordism between S and itself.

(2) It follows easily from the Smooth 1-Manifold Classification Theorem 28.8 that any two closed oriented 1-dimensional smooth manifolds are cobordant.

(3) Similar to (2) it follows easily from the Surface Classification Theorem 101.16 that any two closed oriented 2-dimensional smooth manifolds are cobordant.

(4) We will discuss cobordisms in greater detail in Chapter 280. In particular we will study the non-trivial question whether or not all closed oriented smooth manifolds of a fixed dimension are cobordant. □

Definition 263.2. Let $n\in\mathbb{N}_0$ and let S and T be two closed oriented n-dimensional smooth manifolds. A cobordism $(W,\Theta\colon \partial W\to S\sqcup -T)$ from S to T is called a product cobordism if there exists a diffeomorphism $f\colon [0,1]\times S\to W$ such that the map

$$\underbrace{\{0,1\}\times S}_{=\partial([0,1]\times S)} \xrightarrow{f} \partial W \xrightarrow{\Theta} S\sqcup -T$$

restricts to a diffeomorphism $\{0\}\times S\to S$ and to a diffeomorphism $\{1\}\times S\to T$.



Note that if for two closed oriented smooth manifolds S and T there exists a product cobordism, then S and T are diffeomorphic. This gives us the following approach to showing that two smooth manifolds S and T are diffeomorphic:

- First try to find a cobordism between S and T.
- Then try to "upgrade" the cobordism to a product cobordism.

The following question arises: If we are given a cobordism, how can we tell whether it is a product cobordism? The main story of this chapter will be to give a good answer to this question.

In our first attempt to inch towards an answer to this question we will make use of the following definition:

[2806] It is worth remembering that the empty set is a closed oriented n-dimensional smooth manifold.

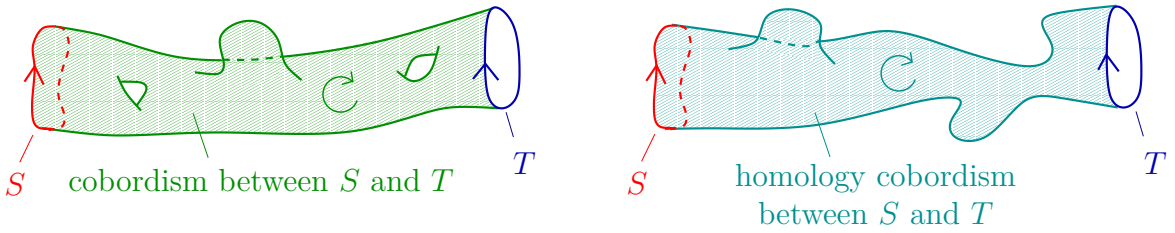
[2807] Recall that it follows from the Boundary Orientation Lemma 31.38 that the boundary ∂W inherits a natural orientation from the orientation of W.

[2808] In practice the orientation-preserving diffeomorphism Θ often gets suppressed from the notation: Often we just say that if W is a cobordism between S and T, then $\partial W = S\cup -T$, even though strictly speaking we only have a diffeomorphism $\partial W\to S\cup -T$.

Definition 263.3. A continuous map $f\colon X \to Y$ between two topological spaces is called a homology equivalence if for each $k \in \mathbb{N}_0$ the map $f_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ is an isomorphism.

Definition 263.4. Let $n \in \mathbb{N}_0$ and let S and T be two closed oriented n-dimensional smooth manifolds.

(1) A cobordism W between S and T is called a homology cobordism if the inclusion maps $S \to W$ and $T \to W$ are both homology equivalences.
(2) We say that S and T are homology cobordant if there exists a homology cobordism between S and T.



Example.

(1) It follows easily from the Homotopy Equivalence-H_*-Proposition 144.14 that any product cobordism is also a homology cobordism. One can ask whether the converse holds. In other words, is every homology cobordism already a product cobordism? We will answer this question shortly.
(2) Let W be a 2-dimensional cobordism from S to T. It follows fairly easily from the Surface Classification Theorem 101.16 that W is a homology cobordism between S and T if and only if W is a product cobordism between S and T. This shows that the question posed in (1) is not a complete flight of the fancy. □

The following (awfully convenient!) proposition cuts the work of showing that a cobordism is a homology cobordism in half:

Proposition 263.5. (Homology Cobordism-on-One Side Proposition) Let $n \in \mathbb{N}_0$ and let W be a cobordism between two closed oriented n-dimensional smooth manifolds S and T. The following statements are equivalent:

(1) The inclusion map S to W is a homology equivalence.
(2) The inclusion map T to W is a homology equivalence.
(3) W is a homology cobordism.

Proof. Clearly we have just have to show (1) and (2) are equivalent. By symmetry it suffices to show that (1) implies (2). We proceed in several easy steps:

(a) We now assume that the inclusion $S \to W$ is a homology equivalence. In other words, we assume that the inclusion-induced maps $\mathrm{H}_k(S) \to \mathrm{H}_k(W)$ are all isomorphisms.
(b) It follows from (a) and the LES-of-Pair Proposition 146.7 applied to the pair (W, S) that the relative homology groups $\mathrm{H}_k(W, S)$ all vanish.
(c) Since W is compact and oriented we obtain from the Poincaré Duality Theorem 210.1 that for each $k \in \mathbb{N}_0$ we have an isomorphism $\mathrm{H}_k(W, S) \to \mathrm{H}^{n+1-k}(W, T; \mathbb{Z})$. It thus follows from (b) that all the relative cohomology groups $\mathrm{H}^l(W, T; \mathbb{Z})$ vanish.
(d) We make the following claim:

Claim. Let C_* be a chain complex such that all homology groups $\mathrm{H}_k(C_*)$ are finitely generated abelian groups. If all the cohomology groups $\mathrm{H}^*(C_*;\mathbb{Z})$ are zero, then so are the homology groups $\mathrm{H}_*(C_*)$.

Proof. As we explain in the Vanishing-of-(Co-) Homology Proposition 201.29 (1) the claim follows easily from the Abelian Group-Classification Theorem 97.24 together with the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18 and the Algebraic Universal Coefficient-H^*-Theorem 201.22. ⊞

(e) It follows from (c) and (d) (applied to the chain complex $C_* = \mathrm{C}_*(W,T)$) that all the relative homology groups $\mathrm{H}_l(W,T)$ vanish.

(f) It follows from (e) and the LES-of-Pair Proposition 146.7 that the inclusion-induced maps $\mathrm{H}_k(T) \to \mathrm{H}_k(W)$ are all isomorphisms. In other words, the inclusion $T \to W$ is a homology equivalence. ■

We will soon give some hopefully interesting examples of homology cobordisms. To construct these examples it is convenient to (re-) introduce the following notions that we introduced in Definition 183.22 and in Exercise 210.22:

Definition 263.6. Let $n \in \mathbb{N}_0$.

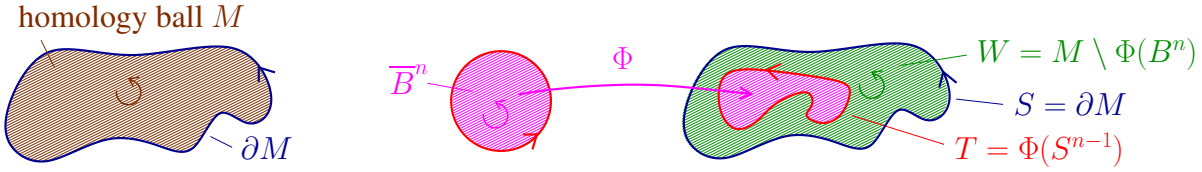
- A smooth homology n-ball is a compact n-dimensional smooth manifold X such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(\overline{B}^n)$.
- A smooth homology n-sphere is a closed n-dimensional smooth manifold X such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$.

The following lemma is a source for potentially interesting homology cobordisms:

Lemma 263.7. (Homology Ball-minus-Ball Lemma) Let M be an oriented n-dimensional smooth homology ball and let $\Phi\colon \overline{B}^n \to M \setminus \partial M$ be an orientation-preserving smooth embedding. We equip $W := M \setminus \Phi(B^n)$ with the orientation coming from M, we equip $S := \partial M$ with the boundary orientation coming from M and we equip $T := \Phi(S^{n-1})$ with the orientation arising from the standard orientation of S^{n-1}.

(1) The smooth manifold $W = M \setminus \Phi(B^n)$ is a homology cobordism between $S = \partial M$ and $T = \Phi(S^{n-1})$.

(2) If $n \geq 3$, then $\pi_1(M \setminus \Phi(B^n)) \cong \pi_1(M)$.



Proof.

(1) First note that it follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that $M \setminus \Phi(B^n)$ is a compact n-dimensional smooth submanifold of M. Next note that it follows from the Mayer-Vietoris Theorem 153.2 (3), applied to the decomposition $M = M \setminus \Phi(B^n) \cup \Phi(\overline{B}^n)$ that we have a long exact sequence

$$\to \widetilde{\mathrm{H}}_k(\Phi(S^{n-1})) \xrightarrow{(i_*,-i_*)} \widetilde{\mathrm{H}}_k(M \setminus \Phi(B^n)) \oplus \underbrace{\widetilde{\mathrm{H}}_k(\Phi(\overline{B}^n))}_{=0} \to \underbrace{\widetilde{\mathrm{H}}_k(M)}_{=0} \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\Phi(S^{n-1})) \to$$

where i always stands for the corresponding inclusion map. We make the following two observations regarding the above homology groups:

- It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that $\widetilde{\mathrm{H}}_*(\Phi(\overline{B}^n)) = 0$.
- It follows from the hypothesis that M is a homology ball and the Reduced Homology Lemma 147.3 that $\widetilde{\mathrm{H}}_*(M) = 0$.

Note that it follows from the above discussion that the inclusion induced homomorphisms $\widetilde{\mathrm{H}}_k(\Phi(S^{n-1})) \to \widetilde{\mathrm{H}}_k(M \setminus \Phi(B^n))$ are all isomorphisms. But then it follows from the $\widetilde{\mathrm{H}}_n$-Functor Lemma 147.2 (3) together with the Reduced Homology Lemma 147.3 that the inclusion $\partial M \to M \setminus \Phi(B^n)$ is a homology equivalence.

Next note that it follows from the above and the Homology Cobordism-on-One Side Proposition 263.5 that the inclusion $\Phi(S^{n-1}) \to M \setminus \Phi(B^n)$ is also a homology equivalence.

Finally note that it follows easily from our orientation conventions and the fact that Φ is orientation-preserving together with the Opposite Side–Opposite Orientation Lemma 31.40 that $\partial(M \setminus \Phi(B^n)) = \partial M \cup -(\Phi(S^{n-1}))$. In summary we have shown that $M \setminus \Phi(B^n)$ is indeed a homology cobordism between ∂M and $\Phi(S^{n-1})$.

(2) We now suppose that $n \geq 3$. We have the following isomorphisms:

$$\pi_1(M) \underset{\text{by the Manifold–Seifert-van Kampen Theorem 102.1}}{\uparrow\cong} \pi_1(M \setminus \Phi(B^n)) *_{\pi_1(\Phi(S^{n-1}))} \pi_1(\Phi(\overline{B}^n)) \underset{\text{since } n \geq 3 \text{ we know by the Sphere–Low-}\pi_n\text{-Proposition 134.19 that } \pi_1(S^{n-1}) = \{e\}}{\uparrow\cong} \pi_1(M \setminus \Phi(B^n)) * \pi_1(\Phi(\overline{B}^n))$$

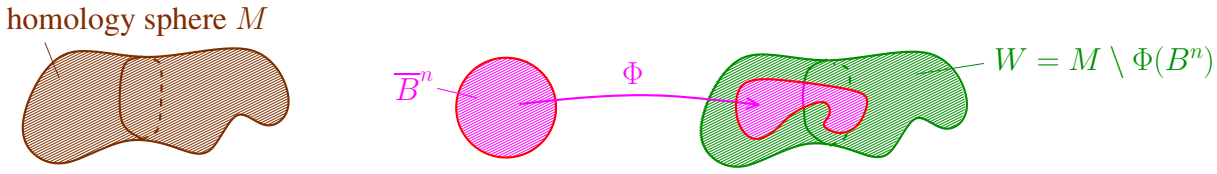
$$\cong \pi_1(M \setminus \Phi(B^n)).$$

■

The Homology Ball-minus-Ball Lemma 263.7 shows that homology balls give rise to homology cobordisms. The following lemma shows that homology spheres give rise to homology balls:

Lemma 263.8. (Homology Sphere-minus-Ball Lemma) Let M be an n-dimensional smooth homology sphere and let $\Phi\colon \overline{B}^n \to M$ be a smooth embedding. We consider the smooth manifold $W := M \setminus \Phi(B^n)$.

(1) The smooth manifold $W := M \setminus \Phi(B^n)$ is a homology ball.
(2) If $n \geq 3$, then $\pi_1(M \setminus \Phi(B^n)) \cong \pi_1(M)$.



Proof.

(1) First note that it follows again from the Codimension-Zero Smooth Embedding Proposition 30.21 that $W := M \setminus \Phi(B^n)$ is a compact n-dimensional smooth submanifold of M. We need to show that $M \setminus \Phi(B^n)$ is a homology ball. Just for fun we give two different arguments:

(i) Again we consider the long exact sequence from the Mayer-Vietoris Theorem 153.2 (3) that arises from the decomposition $M = M \setminus \Phi(B^n) \cup \Phi(\overline{B}^n)$. In the following we show the most interesting segment of this long exact sequence, namely the one

where the non-zero reduced homology group of the homology sphere M raises its head:

$$\to \underbrace{\widetilde{\mathrm{H}}_n(\Phi(S^{n-1}))}_{=0} \to \widetilde{\mathrm{H}}_n(M \setminus \Phi(B^n)) \oplus \underbrace{\widetilde{\mathrm{H}}_n(\Phi(\overline{B}^n))}_{=0} \to \underbrace{\widetilde{\mathrm{H}}_n(M)}_{=\mathbb{Z}\cdot[M]} \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(\Phi(S^{n-1}))}_{=\mathbb{Z}\cdot[\Phi(S^{n-1})]} \to$$

We make the following three observations regarding the above sequence:

- It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 147.6 that $\widetilde{\mathrm{H}}_*(\Phi(\overline{B}^n)) = 0$.
- It follows from the hypothesis that M is a homology sphere and the Reduced Homology Lemma 147.3 that $\widetilde{\mathrm{H}}_i(M) = 0$ for $i < n$ and by Definition 194.3 we have $\widetilde{\mathrm{H}}_n(M) = \mathbb{Z} \cdot [M]$.
- By the Manifold–MV-Connecting Homomorphism Corollary 194.16 we know that $\partial_n([M]) = [\Phi(S^{n-1})]$. In other words we know that the connecting homomorphism $\partial_n \colon \widetilde{\mathrm{H}}_n(M) \to \widetilde{\mathrm{H}}_{n-1}(\Phi(S^{n-1}))$ is an isomorphism.

It follows easily from the above discussion that all reduced homology groups of $M \setminus \Phi(B^n)$ are zero. It now follows from the Reduced Homology Lemma 147.3 that $M \setminus \Phi(B^n)$ is a homology ball.

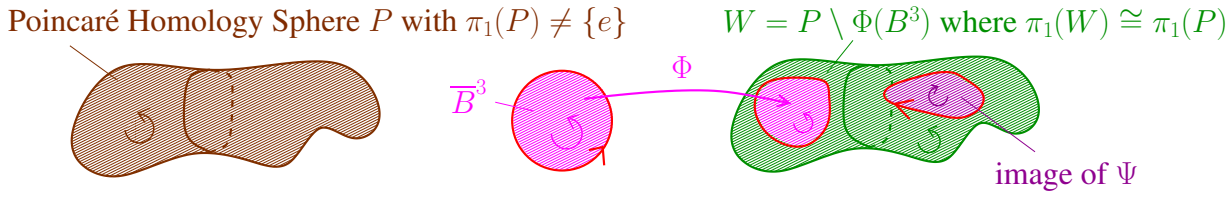
(ii) We turn to our second argument why $W := M \setminus \Phi(B^n)$ is a homology ball. Note that for $i \in \mathbb{N}$ we have the following isomorphisms:

$$\begin{array}{ccccc}
 & \text{by the Poincaré Duality Theorem 210.1} & & \text{by the Excision-in-H}^* \text{ Theorem 200.1 (2)} & \text{by the long exact sequence of the pair } (M, \Phi(\overline{B}^n)) \\
 & \downarrow & & \downarrow & \downarrow \\
\mathrm{H}_i(W) & \cong & \mathrm{H}^{n-i}(W, \partial W; \mathbb{Z}) & \cong & \mathrm{H}^{n-i}(M, \Phi(\overline{B}^n); \mathbb{Z}) \quad \cong \quad \widetilde{\mathrm{H}}^{n-i}(M; \mathbb{Z}) \\
 & \cong & \mathrm{Hom}(\widetilde{\mathrm{H}}_{n-i}(M), \mathbb{Z}) \oplus \mathrm{Ext}((\widetilde{\mathrm{H}}_{n-i-1}(M), \mathbb{Z}) & & \quad = \quad 0. \\
 & \uparrow & & & \uparrow \\
 & \text{by the Universal Coefficient-H}^* \text{ Theorem 201.23} & & & \text{since } M \text{ is a homology sphere and since } i > 0
\end{array}$$

In a very similar fashion one can show that $\mathrm{H}_0(W) \cong \mathbb{Z}$. We have thus shown that W is a homology ball.

(2) The proof of this statement is identical to the proof of the Homology Ball-minus-Ball Lemma 263.7 (2). ■

Example. Let P be the Poincaré Homology Sphere that we introduced in the Poincaré Homology Sphere Proposition 183.24 and that we encountered in the Poincaré-$\mathbb{Z}$HS^3–E_8-Manifold Proposition 259.17. Recall that P is a 3-dimensional smooth homology sphere, but such that $\pi_1(P)$ is a non-trivial group. We equip P with an orientation. Let $\Phi \colon \overline{B}^3 \to P$ be an orientation-preserving smooth embedding and let $\Psi \colon \overline{B}^3 \to P$ be an orientation-reversing smooth embedding such that Φ and Ψ have disjoint images. We now consider the smooth manifold $W := P \setminus (\Phi(B^3) \cup \Psi(B^3))$.



It follows from the Homology Sphere-minus-Ball Lemma 263.8 together with the Homology Ball-minus-Ball Lemma 263.7 that W is a homology cobordism between $\Phi(S^2)$ and $\Psi(S^2)$

with $\pi_1(W) \cong \pi_1(P)$. Since $\pi_1(P)$ is non-trivial we see that W is not a product cobordism. In summary we have found a homology cobordism that is not a product cobordism. We have thus finally obtained a negative answer to the question we asked on page 5078. □

263.2. **Smooth h-cobordisms.** In the previous section we introduced product cobordisms and homology cobordisms and we saw that in general a homology cobordism is not a product cobordism. In this section we will introduce smooth h-cobordisms, which are a much stricter version of homology cobordisms:

Definition 263.9. Let $n \in \mathbb{N}_0$ and let S and T be two closed oriented n-dimensional smooth manifolds.

(1) A cobordism W between S and T is called a smooth h-cobordism if the inclusion maps $S \to W$ and $T \to W$ are both homotopy equivalences.
(2) We say that S and T are smoothly h-cobordant if there exists a smooth h-cobordism between S and T.[2809]

Examples.

(1) Let S be a closed oriented smooth manifold. Similar to the discussion on page 5077 we see that $[0,1] \times S$ is a smooth h-cobordism from S to itself.
(2) It follows from the Homotopy Equivalence-H$_*$-Proposition 144.14 that any smooth h-cobordism is also a homology cobordism. □

Before we consider more examples of smooth h-cobordisms let us mention the following proposition which can be a convenient tool for showing that a given map between smooth manifolds is a homotopy equivalence:

Proposition 263.10. (Homotopy Equivalence–Characterization Proposition) Let $f\colon M \to N$ be a continuous map between compact connected non-empty smooth manifolds. The following statements are equivalent:

(1) The map f is a homotopy equivalence.
(2) There exists a $P \in M$ such that for every $i \in \mathbb{N}_0$ the map $f_*\colon \pi_i(M,P) \to \pi_i(N, f(P))$ is an isomorphism.

If M and N are both simply connected, then the above two statements are equivalent to the following statement:

(3) The map $M \to N$ is a homology equivalence.

Remark. In general it is certainly not true that a homology equivalence is a homotopy equivalence. For example let Σ be a homology sphere of dimension ≥ 3 with non-trivial fundamental group.[2810] By the Degree Realization Proposition 46.19 there exists a degree-one map $f\colon \Sigma \to S^n$. This map is a homology equivalence, but certainly not a homotopy equivalence, since the fundamental groups of Σ and π are not isomorphic. □

[2809] The notion of two manifolds being h-cobordant was first introduced by René Thom in [**Tho1958**, p. 58] under the name "J-equivalent".

[2810] For example Σ could be the 3-dimensional Poincaré Homology Sphere that we had introduced in Section 183.4 or it could be any of the homology spheres that we constructed in the Homology Sphere-Existence Proposition 256.9.

Proof. It follows from the fairly straightforward Homotopy Equivalence-π_n-Proposition 134.17 (1) that (1) implies (2). Next note that by the Smooth Manifold-CW-Theorem 179.9 we know that M and N admit CW-structures.[2811] Using this fact we see that it is a consequence of the Whitehead Theorem 305.1 that (2) implies (1).

If M and N are simply connected, then it follows from the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 that (2) and (3) are equivalent.[2812] ■

Example. Let M be the Mazur manifold that we introduced and studied in the Mazur Manifold Proposition 260.18. We recall its main features:

- The Mazur manifold M is a homology 4-ball with trivial fundamental group.
- The boundary ∂M is a homology sphere with non-trivial fundamental group.

We pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^4 \to M \setminus \partial M$ and we write $W := M \setminus \Phi(B^4)$. By the Homology Ball-minus-Ball Lemma 263.7 we know that W is a simply connected homology cobordism between ∂M and $\Phi(S^3)$.

(1) It follows from the above together with the fact that $\pi_1(S^3) = 0$ and the Homotopy Equivalence–Characterization Proposition 263.10 (3) that the inclusion $\Phi(S^3) \to W$ is a homotopy equivalence.
(2) Since $\pi_1(\partial M)$ is non-trivial we see that the inclusion $\partial M \to W$ is not a homotopy equivalence.

These two observations show that there is no analog of the Homology Cobordism-on-One Side Proposition 263.5 in the setting of smooth h-cobordisms. □

263.3. **The statement of the h-Cobordism Theorem.** The following major theorem says that in dimensions ≥ 6 every simply connected smooth h-cobordism is in fact a product cobordism:

Theorem 263.11. (h-Cobordism Theorem) Let W be an n-dimensional smooth h-cobordism between two closed oriented smooth manifolds S and T. If $n \geq 6$ and if W is simply connected, then W is a product cobordism, i.e. there exists a diffeomorphism $\Theta\colon [0,1] \times S \to W$ such that $\Theta(0, P) = P$ for all $P \in S$.

The remainder of this chapter is organized as follows:

- In the remainder of this section we discuss a few aspects of the h-Cobordism Theorem 263.11.
- In Section 264.2 we will use the h-Cobordism Theorem 263.11 to prove the High-Dimensional Poincaré Conjecture 264.7.
- In Section 263.4 we will prove the h-Cobordism Theorem 263.11.

Remark.

(1) The h-Cobordism Theorem 263.11 was first proved in 1962 by Steven Smale [**Sma1962**, Theorem 1.1]. As we will see in the next chapter, the High-Dimensional Poincaré

[2811] Alternatively we could also use the Manifold–Simplicial Complex Homotopy Type Corollary 190.7 or the Smooth Manifold-CW-Homotopy Type Proposition 254.4 or the Manifold–Simplicial Complex Homotopy Type Corollary 190.7 to argue that M and N are homotopy equivalent to CW-complexes which would be good enough.
[2812] The author would like to apologize for making use of two future results in the current proof.

Conjecture 264.7 is a relatively straightforward consequence of the h-Cobordism Theorem 263.11. For this reason in 1966 Steven Smale was awarded a Fields medal for the proof of the h-Cobordism Theorem 263.11. The authoritative proof of the h-Cobordism Theorem 263.11 is arguably given in [**Miln1965b**]. Note that Milnor works with Morse functions instead of handles. This has the advantage that one does not need to worry about corners and other undesirable mathematical topics. This approach has arguably the (dis?) advantage of being less pictorial. The proof we will give is close in spirit to the proof given in [**Der1966**] and [**Juh2023**, Chapter 2.3]. A slightly different proof using handles is given in [**Wall2016**, Chapter 5.5].

(2) It is natural to ask whether the conclusion of the h-Cobordism Theorem 263.11 holds in lower dimensions. We will discuss this question in the Low-Dimensional h-Cobordism Theorem 264.14. □

In the h-Cobordism Theorem 263.11 we demand that the cobordism is simply connected. In general this condition cannot be dropped. More concretely the following theorem holds:

Theorem ??. (h-Cobordism–Failure Theorem) Given coprime $p, q \in \mathbb{Z}$ with $p \neq 0$ we denote by $L(p,q) = S^3/\mathbb{Z}/p\mathbb{Z}$ the lens space from Definition 91.4. The closed oriented 7-dimensional smooth manifolds $L(7,1) \times S^4$ and $L(7,2) \times S^4$ are smoothly h-cobordant, but they are not diffeomorphic.[2813]

Proof. We will later see, using Reidemeister torsion, that $L(7,1) \times S^4$ and $L(7,2) \times S^4$ are not diffeomorphic. The proof that they are smoothly h-cobordant was given by John Milnor [**Miln1961**, Theorem 4] in 1961. ■

Remark.

(1) In the h-Cobordism–Failure Theorem **??** we just saw that in the h-Cobordism Theorem 263.11 we cannot drop the hypothesis that W is simply connected. Fortunately we will see in the s-Cobordism Theorem **??** that there does exist an analog of the h-Cobordism Theorem 263.11 in the non-simply connected setting. In this general setting we have to replace the hypothesis that we have an "smooth h-cobordism" by the hypothesis that we have an "s-cobordism". Here an "s-cobordism" is a concept that we will introduce in **??** once we introduced the notion of Whitehead torsion.

(2) A variation on the h-Cobordism Theorem 263.11 where the cobordism is 3-dimensional (and not necessarily simply connected) will be discussed in the 3D-s-Cobordism Theorem **??**. □

263.4. **Proof of the h-Cobordism Theorem 263.11.** In this section we will prove the h-Cobordism Theorem 263.11. Throughout this section we will adopt the following notation:

Notation 263.12. If W is a cobordism between S and T, then we write $\partial_- W := S$ and $\partial_+ W := T$.

For the reader's convenience we formulate the h-Cobordism Theorem 263.11 with the above notation:

[2813] On page **??** we will use Reidemeister torsion to prove that the lens spaces $L(7,1)$ and $L(7,2)$ are not diffeomorphic.

Theorem 263.11. (h-Cobordism Theorem) Let W be an n-dimensional smooth h-cobordism. If $n \geq 6$ and if W is simply connected, then W is a product cobordism, i.e. there exists a diffeomorphism $\Theta\colon [0,1] \times \partial_- W \to W$ such that $\Theta(0,P) = P$ for all $P \in \partial_- W$.

The idea will be to study handle decompositions of smooth h-cobordisms and to try to simplify them as far as possible. Before we get to this point we recall the contents of Definitions 247.1 and 247.6 and we throw in a few new concepts:

Definition 263.13. Let W be a smooth manifold and let S be a union of components of ∂W.

(1) A handle decomposition for W rel S is a diffeomorphism
$$\Theta\colon W \xrightarrow{\cong} ([0,1] \times S) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_m} h^{k_m}$$
such that for each $P \in S$ we have $\Theta(P) = [(0,P)]$ and such that for $i = 0, \ldots, m$ the map φ_i is a thickened $(k_i - 1)$-sphere that takes values in
$$\partial(([0,1] \times S) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}}) \setminus (\{0\} \times S).$$
Note that if the handle decomposition has no handles at all, then W is diffeomorphic to the product $[0,1] \times S$.

(2) Given such a handle decomposition we write $\partial_- W := \{0\} \times S$ and furthermore we write $\partial_+ W := (\partial W) \setminus \partial_- W$. Note that with this notation W is a cobordism between $\partial_- W$ and $\partial_+ W$.

(3) (a) We say that a handle decomposition
$$\Theta\colon W \xrightarrow{\cong} ([0,1] \times S) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$$
of W rel S is non-decreasing if for every $i \in \{1, \ldots, m-1\}$ we have $k_i \leq k_{i+1}$.

(b) Given a non-decreasing handle decomposition we define the i-skeleton W^i of the relative handle decomposition as the union of $[0,1] \times S$ together with all handles of index $\leq i$.

(4) We say that a handle decomposition of W rel S is standard if the following two conditions are satisfied:

(a) The handle decomposition is non-decreasing.

(b) For each i-handle the attaching map takes values in $\partial_+ W^{i-1}$, i.e. it takes values in the part of the boundary of the $(i-1)$-skeleton of the handle decomposition which does not belong to $\{0\} \times S$.

By the Handle Decomposition Theorem 250.7 together with the Standard Handle Decomposition Proposition 247.7 we know that any cobordism W admits a standard handle decomposition rel $\partial_- W$. The h-Cobordism Theorem 263.11 will thus follow if we manage to achieve the following goal:

Goal 263.14. Let W be an n-dimensional smooth h-cobordism. We assume that W is equipped with a handle decomposition rel $\partial_- W$ with $m \in \mathbb{N}$ handles. If $n \geq 6$ and if W is simply connected, then our goal is to find a handle decomposition of W rel $\partial_- W$ with fewer than m handles.

The idea is to start out with an arbitrary standard handle decomposition and to use our hypotheses to simplify the handle decomposition: First note that for an h-cobordism

$(W, \partial_- W, \partial_+)$ we know that each component of W has non-empty ∂_- and non-empty ∂_+. It thus follows from the Minimal 0-Handle Proposition 247.13 and the Minimal n-Handle Proposition 248.4, applied to each component of W, that we can already assume that we have no 0-handles and no n-handles. Getting rid of all other handles will require more of an effort. We will proceed as follows:

- In Subsection 263.4.2 we will see that we can use the Whitney Trick-Theorem 262.5 and the Handle Cancellation Theorem 260.1 to remove handles of index $2, \dots, n-4$ via handle cancellations. The restrictions in the indices comes from the dimension restrictions of the Whitney Trick-Theorem 262.5.
- To get around the restrictions of the Whitney Trick-Theorem 262.5 we will first describe in Subsection 263.4.1 a separate argument which shows that we can "trade" 1-handles for 3-handles.
- In Subsection 263.4.3 we will use the above results, applied to the handle decomposition and the corresponding dual handle decomposition, to show that we can reduce the number of handles to naught. In other words, we will complete the proof of the h-Cobordism Theorem 263.11.

263.4.1. **Proof of the h-Cobordism Theorem 263.11: 1-handles.** In this section we will see that one can dispose of 1-handles, admittedly at the price of adding 3-handles. More precisely, we have the following proposition:

Proposition 263.15. (Remove 1-Handles Proposition) Let W be a smooth cobordism such that $\partial_- W$ is connected. Suppose that W is equipped with a standard handle decomposition with no 0-handle. If W is simply connected[2814] and if $\dim(W) \geq 5$ then there exists a standard handle decomposition of W rel $\partial_- W$ with the following properties:

(1) There is no 1-handle.
(2) The number of handles of index $\neq 1, 3$ is unchanged.

Remark.

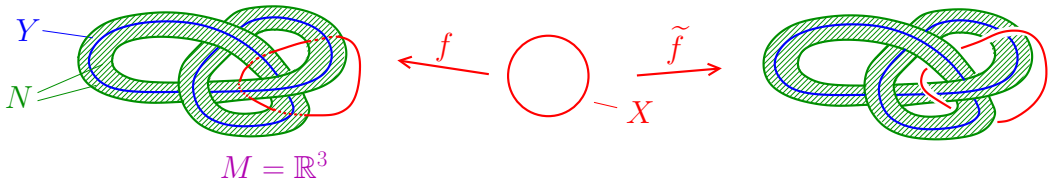
(1) In the proof of the Remove 1-Handles Proposition 263.15 we will see that, under the given hypotheses, we can trade a 1-handle for a 3-handle. This procedure is sometimes called *handle trading*.
(2) By applying the Remove 1-Handles Proposition 263.15 to the dual handle decomposition from the Dual Handle Proposition 248.1 we can also trade $(n-1)$-handles for $(n-3)$-handles.
(3) Under suitable hypothesis, see e.g. [**Wall2016**, Theorem 5.5.2 and Proposition 5.5.3], one can also generalize the Remove 1-Handles Proposition 263.15 to handles of higher index. But since the number of handles is unchanged this does not really help us with our goal. Therefore in the next section we will develop a procedure which allows to eliminate handles of index $2, \dots, n-2$. □

In the proof of the Remove 1-Handles Proposition 263.15 we will need the following proposition:

[2814]Note that according to Definition 88.16 this means that W is non-empty, path-connected and with trivial fundamental group. In the following discussion we will implicitly use on several occasions that W is non-empty and path-connected.

Proposition 263.16. (Avoid-Tubular Neighborhoods Proposition) Let M be a smooth manifold and let Y be a proper submanifold of M. Furthermore let X be a compact smooth manifold and let $f\colon X \to M$ be a continuous map. If $\dim(X)+\dim(Y) < \dim(M)$, then the following statements hold:

(1) Given any tubular neighborhood N of Y there exists a homotopy H from f to a smooth map that misses N.
(2) If f is smooth, then we can arrange H to be smooth.
(3) If f is a smooth embedding, then we can arrange H to be a smooth isotopy.
(4) If we are given a closed subset $C \subseteq M \setminus N$, then we can find a smooth isotopy rel $f^{-1}(C)$ and such that $H^{-1}(C \times [0,1]) = f^{-1}(C)$.



Proof of Proposition 263.16. Let $f\colon X \to M$ be a (smooth) map (embedding). It follows immediately from the Whitney Approximation Theorem 36.1 together with the Transversality Theorem 50.9 that there exists a (smooth) homotopy (isotopy) from f to a smooth map (embedding) g that is transverse to Y. Since $\dim(X) + \dim(Y) < \dim(M)$ this just means that $g(X) \cap Y = \varnothing$.

By the Manifold Paracompact-Normal Proposition 23.9 (2) we know that M is normal in the sense of Definition 3.48. Since Y and $f(X)$ are in particular closed subsets of M this implies that there exists an open neighborhood V of Y with $g(X) \cap V = \varnothing$.

By the Tubular Neighborhood–Existence Theorem 57.11 (applied to $Y \subseteq V$) there exists a tubular neighborhood $\widetilde{N}$ of Y with $\widetilde{N} \subseteq V$, in particular with $\widetilde{N} \cap f(X) = \varnothing$. We have thus found a (smooth) homotopy (isotopy) from f to a smooth map (embedding) g that misses the tubular neighborhood $\widetilde{N}$.

It follows from the Tubular Neighborhood–Uniqueness Theorem 57.12 that, after possibly applying a diffeotopy of M, we can find a (smooth) homotopy (isotopy) from f to a smooth map (embedding) $\widetilde{g}$ that misses N.

Finally, the statement regarding C follows from the refined versions of the theorems cited above. ∎

Proof of Proposition 263.15. Let $n \geq 5$ and let W be a simply connected n-dimensional smooth cobordism such that $\partial_- W$ is connected. We suppose that W is equipped with a standard handle decomposition with no 0-handle. It follows from the Transversal Handle Decomposition Proposition 252.8 that we can assume that the handle decomposition is transverse in the sense of Definition 252.7.

By the Standard Handle Decomposition Proposition 247.7 it suffices to show the following statement:

Statement. If W has $k \geq 1$ handles of index 1, then there exists a handle decomposition rel $\partial_- W$ with one 1-handle less and such that the number of handles of index $\neq 1, 3$ is unchanged.

So in the following we assume that we have at least one 1-handle $\widetilde{h}^1 = \overline{B}^{n-1} \times \overline{B}^1$. The idea is as follows:

> *Eventually we would like be able to cancel the 1-handle $\widetilde{h}^1$ with a 2-handle. The Handle Cancellation Theorem 260.1 shows that we can do so if we can find a 2-handle such that the corresponding 1-dimensional attaching sphere intersects the belt sphere $S^{n-2} \times \{0\}$ of $\widetilde{h}^1 = \overline{B}^{n-1} \times \overline{B}^1$ transversally in a single point. So we need to try to find or create such a 2-handle.*

The following claim shows that we can find a 1-dimensional sphere in $\partial_+ W^1$ that intersects the belt sphere transversally in a single point:

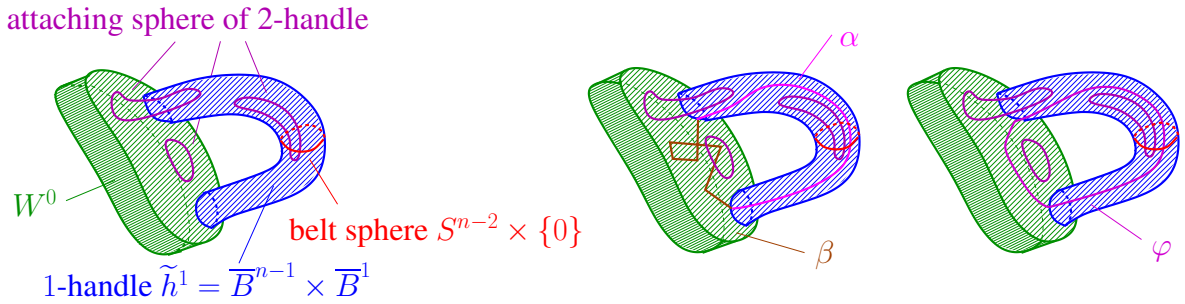
Claim 1. There exists a smooth embedding $\varphi\colon S^1 \to \partial_+ W^1$ with the following two properties:

(a) φ intersects the belt sphere $S^{n-2} \times \{0\}$ of $\widetilde{h}^1 = \overline{B}^{n-1} \times \overline{B}^1$ transversally in a single point.

(b) φ does not intersect the attaching maps of any of the 2-handles.

Proof. Using the fact that the attaching spheres of the 2-handles are transverse to the belt $S^{n-2} \times \{0\}$ one can show (with a little bit of an effort) that there exists a point $(P, 0)$ on the belt sphere $S^{n-2} \times \{0\}$ that does not lie in the image of any of the attaching maps of the 2-handles. Now let α be the smooth path on the boundary $S^{n-2} \times \overline{B}^1$ of the 1-handle $\widetilde{h}^1$ that corresponds to $\{P\} \times \overline{B}^1$. Note that this intersects the belt sphere $S^{n-2} \times \{0\}$ transversally in the single point $(P, 0)$ and that the endpoints lie on $\partial_+ W^0$.

Next note that an elementary argument (along the lines of Claim 2 below) shows that $\partial_+ W^1$ is path-connected. This implies that we can connect the two endpoints of α by a path β in $\partial_- W^1$. Using the Whitney Approximation Theorem 36.1 one can easily show that we can turn the resulting loop $\alpha * \beta\colon S^1 \to \partial_+ W^1$ into a smooth map $\gamma\colon S^1 \to \partial_+ W^1$ such that the image intersects the belt sphere transversally in the single point $(P, 0)$.



Since $\dim(\partial_+ W) = n - 1 \geq 4 = 2 \cdot 1 + 2$ we can use the Avoid-Tubular Neighborhoods Proposition 263.16 to turn γ, after possibly another smooth isotopy, into a smooth embedding $\varphi\colon S^1 \to \partial_+ W^1$ that not only satisfies (a) but also satisfies (b). ⊞

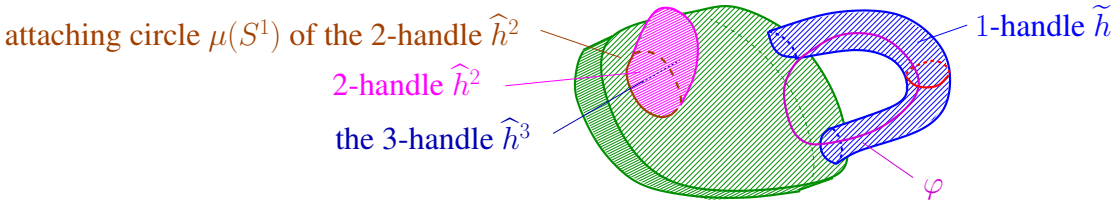
Note that Property (b), together with the Handle Attachment Proposition 246.3 (2), implies that the loop φ actually takes values in $\partial_+ W^2$, i.e. we can view it as a smooth embedding $\varphi\colon S^1 \to \partial_+ W^2$.

We continue with the above simple minded discussion:

> *If $\varphi(S^1)$ was the attaching sphere of a 2-handle, then it would follow from (a) and the Handle Cancellation Theorem 260.1 that we could cancel the 1-handle $\widetilde{h}^1$ with the*

putative 2-handle. Evidently there is no reason why such a 2-handle should exist, in fact it is not even clear why any 2-handle should exist at all. But we can try to create such a 2-handle ex nihilo at the price of also adding a 3-handle.

Note that by the Canceling Pair–Creation Lemma 260.5 we can attach a canceling pair of handles $\widehat{h}^2$ and $\widehat{h}^3$ of index 2 and 3 to W^2, along the boundary component $\partial_+ W^2$, with attaching maps $\mu\colon \overline{B}^{n-2} \times S^1 \to \partial_+ W^2$ and $\nu\colon \overline{B}^{n-3} \times S^2 \to \partial_+(W^2 \cup_\mu \widehat{h}^2)$.



The trick now is to show that we can find a *smooth isotopy* of μ such that the attaching sphere $\mu(\{0\} \times S^1)$ becomes the above sphere $\varphi(S^1)$ from Claim 1. Note that the following claim implies that we can at least find a *homotopy* from the map $\xi \mapsto \mu(0,\xi)$ to the map $\varphi\colon S^1 \to \partial_+ W^2$:

Claim 2. The topological space $\partial_+ W^2$ is path-connected and we have $\pi_1(\partial_+ W^2) = \{e\}$.

Proof. We equip $W^2 = ([0,1] \times \partial_- W) \cup h_1^1 \cup \dots \cup h_r^1 \cup h_1^2 \cdots \cup h_s^2$ with the dual decomposition from the Dual Handle Proposition 248.1. Note that it is of the form

$$W^2 = ([0,1] \times \partial_+ W^2) \cup h_1^{n-2} \cup \dots \cup h_s^{n-2} \cup h_1^{n-1} \cup \dots \cup h_r^{n-1}.$$

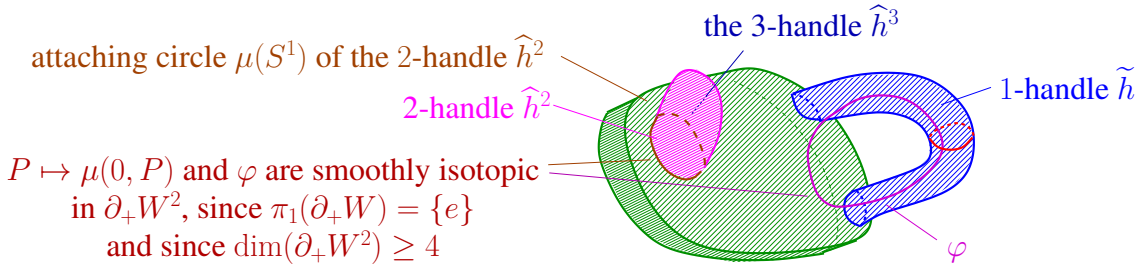
First note that it follows easily from this description, the connectedness of W and the Handle Attachment-π_0-Lemma 246.5 that $\partial_+ W^2$ is connected. Next note that[2815]

$$\begin{array}{ccccc} \pi_1(\partial_+ W^2) & \overset{\text{by the Handle Attachment-}\pi_1\text{-Proposition 246.13, note that here we use that } n-2 \geq 3}{\cong} & \pi_1(([0,1] \times \partial_+ W^2) \cup h_1^{n-2} \cup \dots \cup h_s^{n-2} \cup h_1^{n-1} \cup \dots \cup h_r^{n-1}) & = & \pi_1(W^2) \\ & \underset{\text{by the Handle Attachment-}\pi_1\text{-Proposition 246.13 since we add handles of index} \geq 3 \text{ to get from } W^2 \text{ to } W}{\cong} & \pi_1(W) & \underset{\text{by hypothesis}}{=} & \{e\}. \end{array}$$

Next we consider the two smooth embeddings $S^1 \to \partial_+ W^2$ that are given by the maps $\xi \mapsto \mu(0,\xi)$ and φ. We just saw that $\pi_1(\partial_+ W^2)$ is trivial. This implies that the two maps are homotopic. Since $\dim(\partial_+ W^2) \geq 4 = 2 \cdot \dim(S^1) + 2$ we obtain from the Upgrade-Homotopic-to-Isotopic Proposition **??** that the two smooth embeddings $\xi \mapsto \mu(0,\xi)$ and φ are even smoothly isotopic.

It thus follows from the Isotopy Extension Theorem 44.2 that there exists a diffeotopy $\Theta\colon \partial_+ W^2 \times [0,1] \to \partial_+ W^2$ with $\Theta_0 = \mathrm{id}$ and such that $(\Theta_1 \circ \mu)(0,\xi) = \varphi(\xi)$ for all $\xi \in S^1$.

[2815] This calculation is very similar to the argument on page 5007 and it is very similar to Exercise 248.6.



Now we see that

$$\begin{aligned} W^2 &\cong W^0 \cup (\text{1-handles other than } \widetilde{h}^1) \cup \widetilde{h}^1 \cup \text{2-handles} \\ &\cong W^0 \cup (\text{1-handles other than } \widetilde{h}^1) \cup \widetilde{h}^1 \cup_\mu \widehat{h}^2 \cup_\nu \widehat{h}^3 \cup \text{2-handles} \\ &\quad \uparrow \\ &\quad \text{by the Handle Cancellation Theorem 260.1 applied to } \widehat{h}^2 \text{ and } \widehat{h}^3 \\ &\cong W^0 \cup (\text{1-handles other than } \widetilde{h}^1) \cup \widetilde{h}^1 \cup_{\Theta_1 \circ \mu} \widehat{h}^2 \cup_{\widetilde{\nu}} \widehat{h}^3 \cup \text{2-handles} \\ &\quad \uparrow \\ &\quad \text{follows from the above diffeotopy } \Theta \text{ and the Handle Attachment Proposition 246.3 (5),} \\ &\quad \text{together with the Modify Handle Attachment Region Lemma 247.8} \\ &\cong W^0 \cup (\text{1-handles other than } \widetilde{h}^1) \cup_\nu \widehat{h}^3 \cup \text{2-handles} \\ &\quad \uparrow \\ &\quad \text{by the Handle Cancellation Theorem 260.1 applied to } \widetilde{h}^1 \text{ and } h^2\text{, here we use that} \\ &\quad \text{the attaching sphere } (\Theta_1 \circ \mu)(S^1 \times \{0\}) = \varphi(S^1) \text{ of the 2-handle } \widehat{h}^2 \text{ intersects} \\ &\quad \text{the belt sphere } S^{n-2} \times \{0\} \text{ of the 1-handle } \widetilde{h}^1 \text{ transversally in a single point} \end{aligned}$$

Note that in this construction the number of 1-handles was reduced by one, the number of 2-handles stays unchanged and we gained a 3-handle. We obtain the desired handle decomposition of W from the above discussion together with the Modify Handle Attachment Region Lemma 247.8. ■

263.4.2. **Proof of the h-Cobordism Theorem 263.11: Removing intermediate handles.** The Remove 1-Handles Proposition 263.15 allows us to trade 1-handles for 3-handles. If we feel like it, then we can also apply this approach to the dual handle decomposition, which allows us to trade $(n-1)$-handles for $(n-3)$-handles. But the total number of handles is unaffected by these "trades".

The following proposition finally allows us to eliminate handles from the tally:

Proposition 263.17. (Zero Relative Homology-No Handle Proposition) Let W be a smooth cobordism. We suppose that the following conditions are satisfied:

(1) W is equipped with a standard handle decomposition rel $\partial_- W$.
(2) We are given a $k \in \mathbb{N}$ such that there are no handles of index $\leq k-1$.
(3) $\mathrm{H}_k(W, \partial_- W) = 0$.

If $n \geq 6$, if $2 \leq k \leq n-4$ and if W is simply connected, then we can equip W with a standard handle decomposition that has no handles of index $\leq k$ and such that the number of handles of index $\geq k+2$ is unchanged.

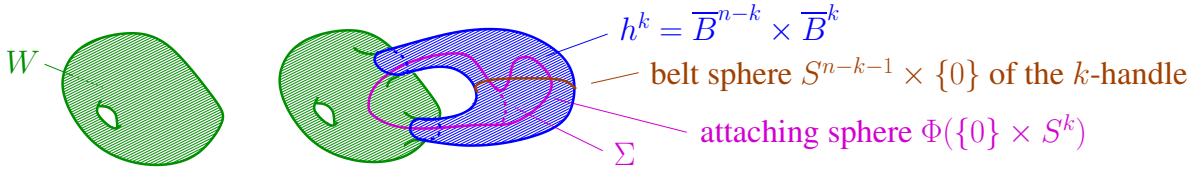
The proof of the Zero Relative Homology-No Handle Proposition 263.17 will require the remainder of this subsection. In the next Subsection 263.4.3 we will combine all our little tricks to prove the h-Cobordism Theorem 263.11.

The idea for the proof of the Zero Relative Homology-No Handle Proposition 263.17 is that we want to cancel a k-handle with a $(k+1)$-handle. This cancellation of two handles of successive indices will come from an application of the following theorem:

Theorem 260.6. (Isotopy–Handle Cancellation Theorem) Let $n \in \mathbb{N}$ and let W be an n-dimensional smooth manifold. Suppose that we attached a k-handle to get $W \cup h^k$ and suppose we are given a thickened k-sphere

$$\Phi\colon \overline{B}^{n-k-1} \times S^k \;\to\; \partial(W \cup h^k) \;=\; \partial(W \cup (\overline{B}^{n-k} \times \overline{B}^k)).$$

If the attaching sphere $\Phi(\{0\} \times S^k)$ is smoothly isotopic to a k-dimensional sphere Σ that intersects the belt sphere $S^{n-k-1} \times \{0\}$ of the original k-handle transversally in a single point, then $(W \cup h^k) \cup_\Phi h^{k+1}$ is diffeomorphic to W.



Next we give a slightly modified version of the Whitney Trick-Theorem 262.5:

Theorem 263.18. (Whitney Trick-Theorem) Let M be a closed oriented smooth manifold and let X and Y be two closed oriented connected submanifolds of M such that the following two conditions hold:

(a) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
(b) X and Y intersect transversally.

Furthermore we suppose that one of the following two conditions holds:

(I) $\dim(X) \geq 3$, $\dim(Y) \geq 3$ and $\pi_1(M) = 0$,
(II) $\dim(X) \geq 3$, $\dim(Y) = 2$ and $\pi_1(M \setminus X) = 0$.

Then there exists a smooth isotopy from X to a smooth submanifold $\widetilde{X}$ such that $\widetilde{X}$ and Y intersect transversally in precisely $|X \cdot Y|$ points.

Proof. It follows from our hypothesis and the original formulation of the Whitney Trick-Theorem 262.5 that there exists a smooth isotopy $F\colon Y \times [0,1] \to M$ with $F_0 = \mathrm{id}$ to a smooth submanifold $\widetilde{Y} := F_1(Y)$ such that X and $\widetilde{Y}$ intersect transversally in precisely $|X \cdot Y|$ points.

By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $G\colon M \times [0,1] \to M$ from $G_0 = \mathrm{id}$ to a diffeomorphism $G_1\colon M \to M$ such that for each $y \in Y$ we have $G(y,t) = F(y,t)$. We now claim that $\widetilde{X} := G_1^{-1}(X)$ has the desired properties. To see this first note that

$$\widetilde{X} \cap Y \;=\; G_1^{-1}(X) \cap Y \;\xrightarrow[G_1]{\cong}\; X \cap G_1(Y) \;=\; X \cap \widetilde{Y}.$$

Thus we see that $\widetilde{X}$ and Y intersect transversally in precisely $|X \cdot Y|$ points. Finally note that the smooth isotopy

$$\begin{array}{rcl} X \times [0,1] & \to & M \\ (x,t) & \mapsto & (G_t)^{-1}(x) \end{array}$$

witnesses that $\widetilde{X} = G_1^{-1}(X)$ is smoothly isotopic to X. ∎

Note that the Whitney Trick-Theorem 262.5 implies that, under suitable circumstances, the verification of the hypothesis of the Isotopy–Handle Cancellation Theorem 260.6 reduces to the verification that certain algebraic intersection numbers are ± 1.

To relate algebraic intersection numbers to our setting of a smooth h-cobordism we need to make use of handle homology. For the reader's convenience we recall the main concepts of handle homology:

Definition 252.3. Let W be a compact smooth manifold and let S be a union of components of ∂W. We suppose that W is equipped with a standard handle decomposition rel S.

(0) Given $k \in \mathbb{N}_0$ we denote by W^k the k-skeleton as introduced in Definition 247.6.
(1) We write $\mathrm{C}_k^{\mathrm{han}}(W,S) \coloneqq \mathrm{H}_k(W^k, W^{k-1})$. Note that by the Handle-Skeleton-H_*-Lemma 252.1 we know that the k-handles give rise to a basis for $\mathrm{C}_k^{\mathrm{han}}(W,S)$.
(2) We refer to the map $d_k \colon \mathrm{C}_k^{\mathrm{han}}(W,S) \to \mathrm{C}_{k-1}^{\mathrm{han}}(W,S)$ from the Handle Boundary Map Lemma 252.2 as the handle boundary map.
(3) We refer to $(\mathrm{C}_*^{\mathrm{han}}(W,S), d_*)$ as the handle chain complex.
(4) We denote the homology groups of the handle chain complex by $\mathrm{H}_k^{\mathrm{han}}(W,S)$ and we refer to these groups as the handle homology groups of the smooth manifold W rel S with the given handle decomposition.

To understand the handle chain complex we also need a good understanding of the boundary maps. Fortunately the following proposition gives a very geometric description of the boundary maps. Note that the following proposition also finally gives us a link between homology groups and algebraic intersection numbers:

Proposition 252.6. (Handle Chain Complex-Boundary Map Proposition) Let W be an oriented n-dimensional smooth manifold and let S be a union of components of ∂W. In the following we suppose that W is equipped with a standard handle decomposition rel S. We introduce the following notation:

- Let $B_1, \dots, B_r \subseteq \partial_+ W^k$ be the belt spheres of the k-handles.
- Let $\Phi_1, \dots, \Phi_s \colon \overline{B}^{n-k-1} \times S^k \to \partial_+ W^k$ be the thickened k-spheres corresponding to the $(k+1)$-handles.

For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$ we have

$$\begin{array}{c}(i,j)\text{-entry of the matrix representing} \\ d_{k+1} \colon \mathrm{C}_{k+1}^{\mathrm{han}}(W,S) \to \mathrm{C}_k^{\mathrm{han}}(W,S) \\ \text{with respect to the bases given} \\ \text{by the handles}\end{array} = \begin{array}{c}\text{algebraic intersection number in } \partial_+ W^k \\ \text{of the attaching sphere } \Phi_j(\{0\} \times S^k) \\ \text{with the belt sphere } B_i \text{ of the } i\text{-th } k\text{-handle.}\end{array}$$

In the proof of the Zero Relative Homology-No Handle Proposition 263.17 we will want to describe the boundary homomorphism d_{k+1} in a convenient way. To do so we will need the following two basic lemmas from linear algebra:

Lemma 263.19. (Matrix–Right Simplification Lemma) Let $r, s \in \mathbb{N}_0$ with $s \geq r$. Given any matrix $A \in \mathrm{M}(r \times s, \mathbb{Z})$ there exists a matrix $Q \in \mathrm{GL}(s, \mathbb{Z})$ such that $A \cdot Q$ is of the form

$$A \cdot Q \;=\; \begin{pmatrix} c_1 & 0 & 0 & \dots & 0 \\ * & \ddots & 0 & \dots & 0 \\ * & * & c_r & \dots & 0 \end{pmatrix}.$$

Proof. The proof of this lemma is very similar to the proof of the Smith Normal Form Theorem 97.22 that we gave in Appendix 97.6. We leave the task of providing all details to Exercise 263.6. ■

Lemma 263.20. (Matrix Group-Generating Set Lemma) Every matrix in $\mathrm{GL}(s,\mathbb{Z})$ can be written as the product of elementary matrices, i.e. of matrices of the following form:

$$\underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(I)}}, \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(II)}} \text{ and } \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(III)}}, \quad \text{with } r \in \{-1, 1\}.$$

Proof. This lemma is essentially the same as the Matrix Group-Generating Set Lemma 100.5 except that right now we have a slightly more restrictive type of matrices of type (III). We leave it to the reader to verify that the above lemma is a consequence of the Matrix Group-Generating Set Lemma 100.5. ■

Now, after so many preparations, we are finally ready to prove the Zero Relative Homology-No Handle Proposition 263.17:

Proof of the Zero Relative Homology-No Handle Proposition 263.17. First note that it follows from elementary logic together with the Standard Handle Decomposition Proposition 247.7 that it suffices to prove the following statement:

Statement. Let $n \in \mathbb{N}_{\geq 6}$ and let W be an n-dimensional smooth cobordism. We suppose that the following conditions are satisfied:

(1) W is equipped with a standard handle decomposition rel $\partial_- W$.
(2) We are given a $k \in \{2, \dots, n-4\}$ such that there are no handles of index $\leq k-1$.
(3) $\mathrm{H}_k(W, \partial_- W) = 0$.
(4) W is simply connected.

If W has $r \in \mathbb{N}$ handles of index k, then W also admits a handle decomposition rel $\partial_- W$ with $r-1$ handles of index k, such that the number of handles of index $\neq k, k+1$ is unchanged.

So let us prove this statement. First note that it follows from the Transversal Handle Decomposition Proposition 252.8 that we can assume that the handle decomposition is transverse in the sense of Definition 252.7. This implies that the attaching spheres of the $(k+1)$-handles intersect the belt spheres of the k-handles transversally.

Next let us consider the handle chain complex of $(W, \partial_- W)$. Note that it is of the form

$$\dots \xrightarrow{d_{k+2}} \mathrm{C}^{\mathrm{han}}_{k+1}(W, \partial_- W) \xrightarrow{d_{k+1}} \mathrm{C}^{\mathrm{han}}_{k}(W, \partial_- W) \xrightarrow{d_k} \underbrace{\mathrm{C}^{\mathrm{han}}_{k-1}(W, \partial_- W)}_{\substack{= 0, \text{ since there is no} \\ \text{handle of index } k-1}} \to \dots$$

We see that

$$\operatorname{coker}\left(\mathrm{C}_{k+1}^{\mathrm{han}}(W,\partial_- W) \xrightarrow{d_{k+1}} \mathrm{C}_k^{\mathrm{han}}(W,\partial_- W)\right) \underset{\text{by the above discussion}}{\cong} \mathrm{H}_k^{\mathrm{han}}(W,\partial_- W) \underset{\text{by the Singular-Handle Homology Proposition 252.4}}{\cong} \mathrm{H}_k(W,\partial_- W) \underset{\text{by our hypotheses}}{=} 0.$$

We have thus shown that the map $d_{k+1}\colon \mathrm{C}_{k+1}^{\mathrm{han}}(W,\partial_- W) \to \mathrm{C}_k^{\mathrm{han}}(W,\partial_- W)$ is an epimorphism.

We now consider this map d_{k+1} in greater detail. Recall that we assume that we have r k-handles. We denote by s the number of $(k+1)$-handles. Recall that the handles give rise to a basis for the relative handle chain groups $\mathrm{C}_k^{\mathrm{han}}(W,\partial_- W)$ and $\mathrm{C}_{k+1}^{\mathrm{han}}(W,\partial_- W)$. Next let $A \in \mathrm{M}(r\times s,\mathbb{Z})$ be the matrix that describes $d_{k+1}\colon \mathrm{C}_{k+1}^{\mathrm{han}}(W,\partial_- W) \to \mathrm{C}_k^{\mathrm{han}}(W,\partial_- W)$ with respect to the above bases. Since d_{k+1} is an epimorphism we see that $s \geq r$.

It follows from the Matrix–Right Simplification Lemma 263.19 that there exists a matrix $Q \in \mathrm{GL}(s,\mathbb{Z})$ such that $A\cdot Q$ is of the form

$$A\cdot Q = \begin{pmatrix} c_1 & 0 & 0 & \dots & 0 \\ * & \ddots & 0 & \dots & 0 \\ * & * & c_r & \dots & 0 \end{pmatrix}.$$

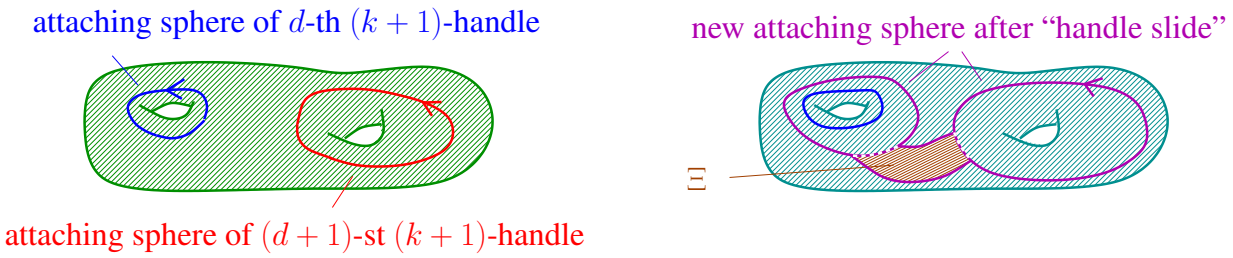
Claim 1. There exists a handle decomposition for W rel $\partial_- W$, with the same number of handles as before, such that the matrix representing the handle boundary map d_{k+1} is exactly the above matrix.

Proof. By the Matrix Group-Generating Set Lemma 263.20 we can write Q as a product of elementary matrices of the types (I), (II) and (III). Recall that the effect of these matrices can be summarized as follows:

(I) We multiply the d-th basis vector by -1.
(II) We swap the basis vectors d and e.
(III) We add the r-multiple of the d-th basis vector to the $(d+1)$-st basis vector.

The amazing observation is that all these types of multiplications by elementary matrices can be realized geometrically by changing the attaching maps for the $(k+1)$-handles:

(I) We precompose the attaching map $\varphi\colon \overline{B}^{n-k-1}\times S^k \to \partial_+ W^k$ of the d-th $(k+1)$-handle by an orientation-reversing self-diffeomorphism of S^k, e.g. by a reflection in a hyperplane. It follows easily from the Sphere–Degree–Matrix Lemma 151.7 that this corresponds to multiplying the d-th basis vector by -1.
(II) We swap the ordering of the $(k+1)$-handles, more precisely, we swap the d-th and the e-th $(k+1)$-handle.
(III)
- Note that our hypothesis that $k \in \{2,\dots,n-4\}$ implies in particular that we have $k \leq n-3$. It thus follows from the Internal Band Proposition 260.11 that we can find an internal band $\Xi\colon [0,1]\times \overline{B}^k \to \partial_+(W^k \cup h_1^{k+1} \cup \dots \cup h_{d-1}^{k+1})$ from the attaching sphere for the d-th $(k+1)$-handle to the attaching sphere of the $(d+1)$-st $(k+1)$-handle. If $r=-1$ we throw in a reflection in a hyperplane.
- We apply the construction of the Handle Slide Theorem 260.13 which "slides the $(d+1)$-st handle over the d-th handle". It follows from the Handle Slide Theorem 260.13 (4) that we obtain a new handle decomposition of W rel $\partial_- W$ and it



follows from the Handle Slide Theorem 260.13 (5) that this new handle decomposition has the desired boundary map.

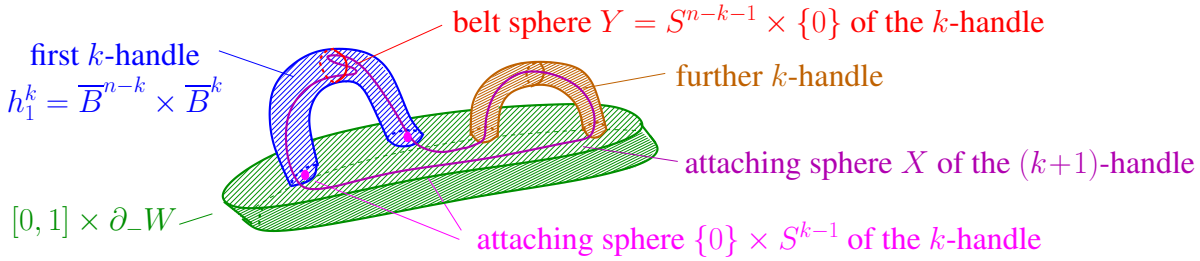
In summary, after applying these moves to the $(k+1)$-handle attachments we can assume that the boundary map in the handle chain complex is given by the above matrix. ⊞

Note that it follows from Claim 1 that we can now assume that the boundary homomorphism $d_{k+1}\colon \mathrm{C}^{\mathrm{han}}_{k+1}(W,\partial_- W) \to \mathrm{C}^{\mathrm{han}}_{k}(W,\partial_- W)$ is represented (with respect to the basis coming from the handle decomposition) by the following matrix

$$\begin{pmatrix} c_1 & 0 & 0 & \dots & 0 \\ * & \ddots & 0 & \dots & 0 \\ * & * & c_r & \dots & 0 \end{pmatrix}.$$

Next note that it follows from the fact that A corresponds to an epimorphism, using some elementary linear algebra, that $c_1,\dots,c_r \in \{-1,1\}$. We will now see that we can cancel the first k-handle and the first $(k+1)$-handle. To do so we introduce the following notation:

- For the first k-handle we now write $h^k = \overline{B}^{n-k} \times \overline{B}^k$.
- Let $Y^{n-k-1} = S^{n-k-1} \times \{0\} \subseteq \partial_+ W^k$ be the belt sphere of the first k-handle and let $X^k := \Phi(\{0\} \times S^k)$ be the attaching k-sphere corresponding to the first $(k+1)$-handle.



Next note that it follows from the above Handle Chain Complex-Boundary Map Proposition 252.6 that in $\partial_+ W^k = \partial_+((([0,1] \times \partial_- W) \cup h^k_1 \cup \dots \cup h^k_r)$ we have

$$X \cdot Y \;=\; c_1 \;= \pm 1.$$

By the Isotopy–Handle Cancellation Theorem 260.6, it remains to prove the following claim:

Claim 2. In the $(n-1)$-dimensional smooth manifold $M := \partial_+ W^k$ we can find a smooth isotopy of X to a sphere $\widetilde{X}$ such that Y and $\widetilde{X}$ intersect transversally in a single point.

Proof. We need to distinguish two cases:
Case 1. $k \in 3,\dots,n-4$.
Note that this implies that $\dim(X^k) = k \geq 3$ and that $\dim(Y^{n-k-1}) = n-k-1 \geq 3$. Thus we see that this case follows immediately from the Whitney Trick-Theorem 263.18 (I).

Case 2. $k = 2$ and $k \leq n-4$.
Note that this implies that $\dim(X^2) = 2$ and that $\dim(Y^{n-2-1}) = n-3 \geq 6-3 = 3$. Now our goal is to apply the Whitney Trick-Theorem 263.18 (II) to X and Y. Thus we need to show that $\pi_1(\partial_+ W^2 \setminus Y) = 0$. First recall that we assume that there are no handles of index $\leq k-1 = 1$. Thus we see that $W^2 = ([0,1] \times \partial_- W) \cup h_1^2 \cup \cdots \cup h_r^2$. We now see that

$$\begin{array}{rcl} \pi_1(\partial_+ W^2 \setminus Y) & \overset{\text{by definition of } Y}{\downarrow} & \\ & = & \pi_1(\partial_+ W^2 \setminus (S^{n-3} \times \{0\})) \\ & \overset{\text{since } S^{n-3}\times S^1 \text{ is a deformation retract of } S^{n-3}\times(\overline{B}^2 \setminus \{0\})}{\downarrow} & \\ & \cong & \pi_1(\partial_+ W^2 \setminus (S^{n-3} \times B^2)) \\ & = & \pi_1(\partial_+(([0,1] \times \partial_- W) \cup h_2^2 \cup \cdots \cup h_r^2) \setminus (B^{n-2} \times S^1)) \\ & \overset{\text{by the Handle Attachment Proposition 246.3 (2)}}{\uparrow} & \\ & \cong & \pi_1(\partial_+(([0,1] \times \partial_- W) \cup h_2^2 \cup \cdots \cup h_r^2)) \\ & \overset{\text{follows from the Manifold–Seifert-van Kampen-Theorem 102.1 and the fact that } n-1 \geq 4}{\uparrow} & \\ & = & \{e\}. \\ & \overset{\text{see Claim 2 on page 5089}}{\uparrow} & \end{array}$$

■

263.4.3. **Proof of the h-Cobordism Theorem 263.11: Conclusion.** At this stage we have no troubles with proving the h-Cobordism Theorem 263.11.

Proof of the h-Cobordism Theorem 263.11. Let W be an n-dimensional smooth h-cobordism between two closed oriented smooth manifolds S and T. We suppose that $n \geq 6$ and that W is simply connected. Recall that we need to show that W is a product cobordism. We proceed as follows:

(∗) Note that by the Handle Decomposition Theorem 250.7 there exists a standard handle decomposition for W rel $\partial_- W = S$. For each $i \in \mathbb{N}_0$ we denote by l_i the number of i-handles. Note that

$$\begin{array}{ccccccc} \sum_{i=0}^{n} (-1)^i \cdot l_i & = & \chi(W,S) & = & \chi(W) - \chi(S) & = & 0. \\ & \overset{\text{by same argument as in the Handle Attachment-}\chi\text{-Lemma 246.15}}{\uparrow} & & \overset{\text{by the Relative } \chi\text{-Lemma 156.8}}{\uparrow} & & \overset{\text{by the Homotopy Type-}\chi\text{-Lemma 156.3 since } S \text{ and } W \text{ are homotopy equivalent}}{\uparrow} & \end{array}$$

We now simplify this handle decomposition as follows:

(0) Since W is a smooth h-cobordism we obtain in particular that $\partial_- W$ is non-empty. Thus it follows from the Minimal 0-Handle Proposition 247.13 that we can arrange that we have no 0-handles without changing the number of handles of index $\neq 0, 1$.

(1) Since W is simply connected and since $\dim(W) \geq 5$ we obtain from the Remove 1-Handles Proposition 263.15 that we can arrange that we have no handles of index 1 without changing the number of handles of index $\neq 1, 3$.

(2) It follows from (0) and (1), together with our hypothesis that W is a smooth h-cobordism of dimension ≥ 6, that we can iteratively apply the Zero Relative Homology-No Handle Proposition 263.17 to $k = 2, \ldots, n-4$. This allows to eliminate all handles of index $2, \ldots, n-4$ without changing the number of handles of index $\neq 2, \ldots, n-3$.

(3) As in the proof of the Minimal n-Handle Proposition 248.4 we switch to the dual handle decomposition, we apply the Minimal 0-Handle Proposition 247.13, and immediately switch to the dual of the modified dual handle decomposition. The effect is that we canceled all n-handles with $(n-1)$-handles, without changing the number of handles of index $\neq n-1, n$.

(4) Similar to (3) we switch to the dual handle decomposition, we apply the Remove 1-Handles Proposition 263.15, and immediately switch to the dual of the modified dual handle decomposition. The effect is that we got rid of all $(n-1)$-handles at the expense of adding $(n-3)$-handles, without changing the number of handles of index $\neq n-1, n-3$.

(5) Similar to (3) we switch to the dual handle decomposition, we apply the Zero Relative Homology-No Handle Proposition 263.17 and immediately switch to the dual of the modified dual handle decomposition. The effect is that we got rid of all handles of index $4, \dots, n-2$ without changing the number of handles of index $\neq 3, \dots, n-2$.

So are we done? The above discussion shows that we managed to dispose of all handles of index $\leq n-4$ and of index ≥ 4. Thus, if $n-4 \geq 3$, then we are done. If $n=6$, then, at first glance, the above discussion does not seem to take care of the number of 3-handles. But since the alternating sum of the number of handles is, by the above, zero, and since we do not have any handles of index $\neq 3$, we see that there is also no 3-handle.

In summary we have shown that W admits a handle decomposition rel $S = \partial_- W$ without handles, but this means that W is in fact a product cobordism.

For the reader's convenience we illustrate the progression of the number of handles for the case $n=6$ in the table below:

	(∗)	(0)	(1)	(2)	(3)	(4)	(5)
# 0-handles	l_0	0	0	0	0	0	0
# 1-handles	l_1	l_1-d	0	0	0	0	0
# 2-handles	l_2	l_2	$l_2-l_1+l_0$	0	0	0	0
# 3-handles	l_3	l_3	l_3	$l_3-l_2-l_0+l_1$	...	$l_3-l_2-l_0+l_1+l_5-l_6$	$\chi(W,S)$
# 4-handles	l_4	l_4	l_4	l_4	l_4	l_4	0
# 5-handles	l_5	l_5	l_5	l_5	l_5-l_6	0	0
# 6-handles	l_6	l_6	l_6	l_6	0	0	0

Since $\chi(W,S)=0$ we see that in the last column all numbers are indeed zero. ■

Remark. Note that there is an alternative way to proving the h-Cobordism Theorem 263.11: As in [**Wall2016**, Theorem 5.5.2 and Proposition 5.5.3] one could first prove a generalization of the Remove 1-Handles Proposition 263.15 to trade handles upwards and downwards till at the end one has no handles of index $\neq 2, 3$ and then one could use the Zero Relative Homology-No Handle Proposition 263.17 to dispose of the resulting handles of index $2, 3$. □

263.5. **Appendix: Handle decompositions realizing the Morse inequalities.** We recall the following notation that we first introduced in Notations 252.9 and 252.12:

Notation 263.21. Given a topological space X, given a subspace $S \subseteq X$ and given $k \in \mathbb{N}_0$ we write

$b_j(X,S) \coloneqq \operatorname{rank}(\mathrm{H}_k(X,S))$,

$q_k(X,S) \coloneqq$ minimal number of generators of the torsion subgroup $\operatorname{Tor} \mathrm{H}_k(X,S)$.

Next we recall the following proposition:

Proposition 252.13. (Morse Inequalities Proposition) Let M be a compact smooth manifold and let S be a union of components of ∂M. We suppose that M is equipped with a handle decomposition rel S. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in

the handle decomposition. Given any $k \in \mathbb{N}_0$ we have the inequality

$$d_k \geq q_k(M,S) + q_{k-1}(M,S) + b_k(M,S).$$

If M is not simply connected, then it follows from the Handle Attachment-π_1-Proposition 246.13 that there are no reasons why one should expect the inequalities of the Morse Inequalities Proposition 252.13 to be achievable by a handle decomposition. But if M is simply connected, then the following proposition shows that in many cases we can find handle decompositions which turn the above inequalities into equalities:

Proposition 263.22. (Morse Inequalities–Realization Proposition) Let M be a compact n-dimensional smooth manifold and let S be a union of components of ∂M. If $n \geq 6$ and if M, S and $\partial M \setminus S$ are simply connected[2816], then there exists a handle decomposition of M rel S such that for every $k \in \mathbb{N}_0$ we have the equality

$$d_k = q_k(M,S) + q_{k-1}(M,S) + b_k(M,S).$$

Proof. Note that the case that M is a homology cobordism is exactly the statement of the h-Cobordism Theorem 263.11. In fact the proof of the above proposition is very similar to the proof of the h-Cobordism Theorem 263.11, except that one needs somewhat more elementary linear algebra. More details are provided in [**Wall2016**, Theorem 5.6.9]. (See also [**Kos1993**, Theorem VIII.4.1] for a related result.) ■

263.6. Exercises for Chapter 263.

Exercise 263.1. Let S and T be two closed oriented 2-dimensional smooth manifolds. Does there necessarily exist a *connected* cobordism between S and T?

Exercise 263.2. Let $n \in \mathbb{N}_{\geq 2}$ and let X be a compact n-dimensional topological manifold such that for each $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^{n-1})$.

(a) Show that ∂X has two boundary components.
(b) Show that X is a homology cobordism between these two boundary components.

Remark. This exercise is identical to Exercise 210.35.

Exercise 263.3. Let $n \in \mathbb{N}$ and let S and T be two closed n-dimensional smooth manifolds. (Note that we no longer assume that the smooth manifolds S and T are orientable, let alone oriented.) A *cobordism between the unoriented smooth manifolds S and T* is defined as a pair $(W, \Theta\colon \partial W \to S \sqcup T)$ consisting of a compact $(n+1)$-dimensional smooth manifold W together with a diffeomorphism $\Theta\colon \partial W \to S \sqcup T$.

(a) Show that $\mathbb{R}\mathrm{P}^2$ is *not* cobordant to the empty manifold.
Hint. Consider the Euler characteristic and homology with $\mathbb{F}_2$-coefficients.
(b) Is the Klein bottle cobordant to the empty manifold?

Exercise 263.4. We consider the two oriented submanifolds S and T of $\mathbb{R}^3$ that are shown in the figure below. Does there exist a connected smoothly embedded smooth h-cobordism in $\mathbb{R}^3$?

[2816]Recall that by Definition 88.16 we say that a topological space is *simply connected* if it is non-empty, path-connected and if the fundamental group is trivial.



Exercise 263.5. Let $k \in \mathbb{N}_{\geq 2}$. Give an example a closed simply smooth manifold M such that for any handle decomposition of M we have

$$\text{number of handles of index } k \;>\; q_k(M) + q_{k-1}(M) + b_k(M).$$

Hint. Construct a non-simply connected manifold for which the bound from the Morse Inequalities Proposition 252.13, applied to a finite covering of M, is stronger than the bound for M.
Remark. This exercise is essentially the same as Exercise 252.6.

Exercise 263.6. Let $r, s \in \mathbb{N}_0$ with $s \geq r$. Let $A \in \mathrm{M}(r \times s, \mathbb{Z})$. Show that there exists a matrix $Q \in \mathrm{GL}(s, \mathbb{Z})$ such that $A \cdot Q$ is of the form

$$A \cdot Q \;=\; \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ \vdots & \ddots & 0 & \dots & 0 \\ a_{r1} & \dots & a_{rr} & \dots & 0 \end{pmatrix}.$$

Remark. In other words, your task is to prove the Matrix–Right Simplification Lemma 263.19.

Exercise 263.7. Let M be a simply connected closed smooth manifold of dimension $\dim(M) \geq 6$. Show that M admits a Morse function $f\colon M \to \mathbb{R}$ that is perfect in the sense of Definition 252.16.
Remark. Make use of the Morse Inequalities–Realization Proposition 263.22.
Remark. This is the content of the Perfect Morse Function-Existence Theorem 252.18.

Exercise 263.8. Let $n \in \mathbb{N}_{\geq 6}$ and let Σ be an $(n-1)$-dimensional smooth homotopy sphere that admits a smooth embedding into S^n. Show that Σ is diffeomorphic to S^{n-1}.

Exercise 263.9. Let $n \in \mathbb{N}_{\geq 6}$ and let Σ be an n-dimensional smooth homotopy sphere.
(a) Show that $([0,1] \times S^{n-1}) \# \Sigma$ is diffeomorphic to $[0,1] \times S^{n-1}$.
(b) Show that $(S^1 \times S^{n-1}) \# \Sigma$ is diffeomorphic to a mapping torus.

CHAPTER 264

The High-Dimensional Poincaré Conjecture

In the last chapter we proved the h-Cobordism Theorem 263.11. In this chapter we harvest the fruits, namely we will do the following:

- We will use the h-Cobordism Theorem 263.11 to prove the Homotopy Ball-Diffeomorphism Theorem 264.2 which says that in dimension ≥ 6 every smooth homotopy ball with simply connected boundary is actually diffeomorphic to the standard ball.
- We will use the Homotopy Ball-Diffeomorphism Theorem 264.2 to prove the High-Dimensional Poincaré Conjecture 264.7 which says that in dimension ≥ 6 every smooth homotopy sphere is *homeomorphic* to the standard sphere.
- We will use the Homotopy Ball-Diffeomorphism Theorem 264.2 to prove the High-Dimensional Smooth Schoenflies Theorem 264.10 which says in dimension ≥ 6 the image of every smooth embedding $S^{n-1} \to S^n$ is "standard", i.e. up to a diffeomorphism it is given by the equatorial sphere $S^{n-1} \times \{0\} \subseteq S^n$.

With these results we conclude this part of the lecture notes.

This chapter is organized as follows:

- In Section 264.1 we will prove the Homotopy Ball-Diffeomorphism Theorem 264.2.
- In Section 264.2 we will prove the High-Dimensional Poincaré Conjecture 264.7.
- In Section 264.3 we will prove the High-Dimensional Smooth Schoenflies Theorem 264.10.
- In Section 264.4 we will discuss to what degree the results in the above three sections (and the h-Cobordism Theorem) also hold in dimensions ≤ 5.
- In Section 264.5 we will provide exercises for Chapter 264.

There are many more applications of the h-Cobordism Theorem 263.11 that unfortunately we cannot discuss in these notes. We refer to [**Kos1993**, Chapter VIII] and [**Wall2016**, Chapter 5.6] for a discussion of more consequences of the h-Cobordism Theorem 263.11.

264.1. **High-dimensional homotopy balls.** To formulate our first application of the h-Cobordism Theorem 263.11 we need to (re-) introduce the following notion that we introduced on page 4040:

Definition 264.1. Let $n \in \mathbb{N}_0$.

(1) A topological homotopy n-ball is a compact n-dimensional topological manifold X such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(\overline{B}^n)$ and such that $\pi_1(X) = \{e\}$.[2817]

(2) A smooth homotopy n-ball is a compact n-dimensional smooth manifold that is also a topological homotopy ball.

[2817] It follows from the Hurewicz Corollary 167.13 that this definition of a homotopy ball is equivalent to the earlier definition on page 4040. Using the Whitehead Theorem 305.1 one can easily show that a compact n-dimensional topological manifold M is a topological homotopy n-ball if and only if M is contractible, i.e. if it is homotopy equivalent to a ball.

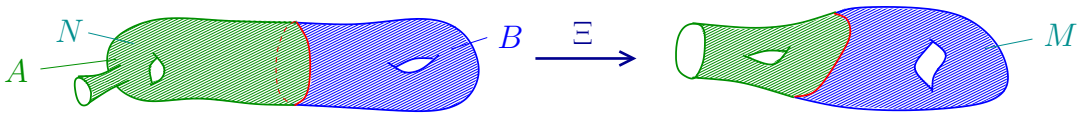
Now we are ready to formulate the first major consequence of the h-Cobordism Theorem 263.11:

Theorem 264.2. (Homotopy Ball-Diffeomorphism Theorem) Let M be a smooth homotopy n-ball. If $n \geq 6$ and if ∂M is simply connected, then M is diffeomorphic to $\overline{B}^n$.

Remark. We will discuss the case $n \leq 5$ in the Low-Dimensional Homotopy Ball Theorem 264.12. □

In the proof of the Homotopy Ball-Diffeomorphism Theorem 264.2 we will need the following proposition:

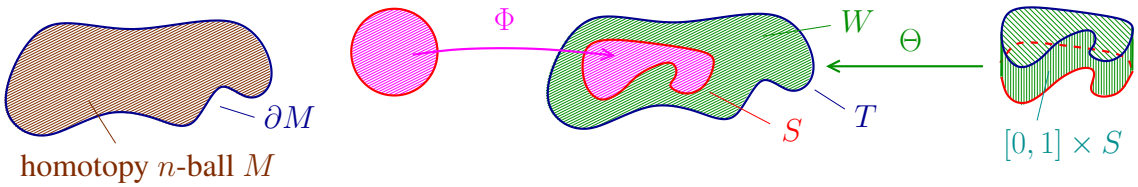
Proposition 55.12. (Smooth Manifold Pushout-Proposition) Let N be a compact orientable smooth manifold and let A and B be two compact codimension-zero submanifolds of N such that $A \cup B = N$ and such that $A \cap B$ is a union of components of ∂A and a union of components of ∂B. Let M be a smooth manifold and let $\Xi\colon N \to M$ be a homeomorphism. If the two restrictions $\Xi|_A\colon A \to M$ and $\Xi|_B\colon B \to M$ are smooth embeddings, then there exists a homeotopy rel A from Ξ to a diffeomorphism $N \to M$.



Proof of the Homotopy Ball-Diffeomorphism Theorem 264.2. Let $n \geq 6$ and let M be a smooth homotopy n-ball with simply connected boundary. We pick a smooth embedding $\Phi\colon \overline{B}^n \to \mathring{M} = M \setminus \partial M$. We write $W := M \setminus \Phi(B^n)$. Furthermore we write $S := \Phi(S^{n-1})$ and $T := \partial M$.

By the Homology Ball-minus-Ball Lemma 263.7 we know that W is a simply connected homology cobordism between S and T. Since we assume that $\pi_1(\partial M)$ is trivial and since $\pi_1(S^{n-1})$ is trivial it follows from the Homotopy Equivalence–Characterization Proposition 263.10 that W is an smooth h-cobordism between $S = \Phi(S^{n-1})$ and $T = \partial M$.

Since $n \geq 6$ we obtain from the h-Cobordism Theorem 263.11 that W is a product cobordism in the sense of Definition 263.2. Recall that this means that there exists a diffeomorphism $\Theta\colon [0,1] \times S \to W$ with $\Theta(0, P) = P$ for all $P \in S$.



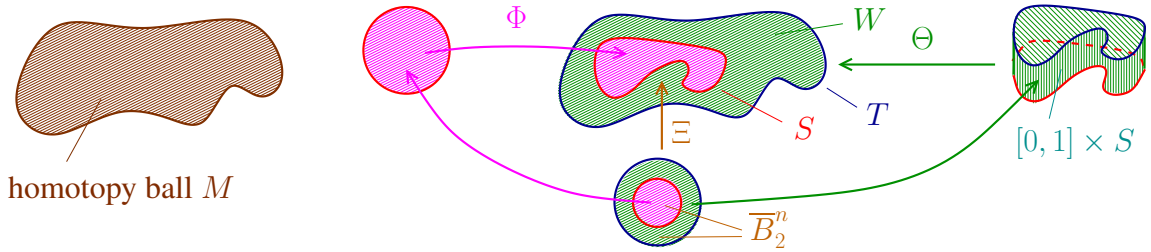
Note that the ball $\overline{B}^n_2$ of radius 2 is diffeomorphic to the standard ball $\overline{B}^n$. It now follows from the above Smooth Manifold Pushout-Proposition 55.12 together with the Regluing Proposition 55.10 that it remains to prove the following claim:

Claim. There exists a homeomorphism $\Xi\colon \overline{B}^n_2 \to M$ such that the restrictions of Ξ to $\overline{B}^n$ and $\overline{B}^n_2 \setminus B^n$ are both smooth embeddings.

Proof. We consider the map

$$\begin{array}{rcl} \Xi\colon \overline{B}_2^n & \to & M \\ x & \mapsto & \begin{cases} \Phi(x), & \text{if } \|x\| \leq 1, \\ \Theta((\|x\|-1), \underbrace{\Phi(\frac{x}{\|x\|})}_{\in S^{n-1}}), & \text{if } \|x\| \geq 1. \end{cases} \end{array}$$

First note that it follows from the Pasting Proposition 4.7 (2) that this map $\Xi\colon \overline{B}_2^n \to M$ is continuous. Using the Compact-Hausdorff Proposition 4.23 (3) one can then verify that this map $\Xi\colon \overline{B}_2^n \to M$ is a homeomorphism.



Next note that the restriction of Ξ to $\overline{B}^n$ is of course the smooth embedding Φ. Finally note that the restriction of Ξ to $\overline{B}_2^n \setminus B^n$ is the composition of the diffeomorphism

$$\begin{array}{rcl} \overline{B}_2^n \setminus B^n & \to & [0,1] \times S^{n-1} \\ x & \mapsto & ((\|x\|-1), \frac{x}{\|x\|}) \end{array}$$

and the smooth embedding $[0,1] \times S^{n-1} \to M$ that is given by $\Theta \circ \mathrm{diag}(\mathrm{id}_{[0,1]}, \Phi)$. Thus we have shown that the restriction of Ξ to $\overline{B}_2^n \setminus B^n$ is indeed a smooth embedding. ∎

264.2. **The high-dimensional Poincaré Conjecture.** In this section we will reach the climax of this part of the lecture notes, namely we will prove the High-Dimensional Poincaré Conjecture 264.7. To maximize the fun we will take our time and develop this story step-by-step.

First we need to (re-) introduce the following notion that we first introduced in Definition 210.19:

Definition 264.3. Let $n \in \mathbb{N}$.

(1) A topological homotopy n-sphere is a closed n-dimensional topological manifold X such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$. If $n \geq 2$, then we also demand that $\pi_1(X) = \{e\}$.[2818]

(2) A smooth homotopy n-sphere is a closed n-dimensional smooth manifold that is also a topological homotopy n-sphere.

Next we study handle decompositions of smooth homotopy spheres:

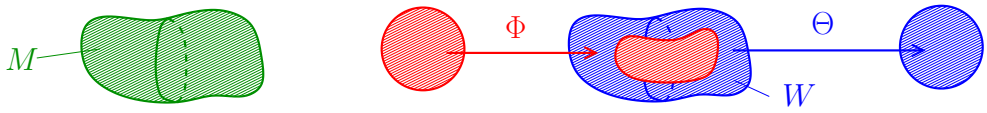
Theorem 264.4. (Homotopy Sphere–Handle Decomposition Theorem) Let $n \in \mathbb{N}$. We assume that the following statement holds:

(∗) Every n-dimensional homotopy ball with simply connected boundary is diffeomorphic to $\overline{B}^n$.

[2818] Recall that in the Homotopy Sphere-Characterization Propositions 210.18 and **??** we gave and we will give several alternative characterizations of topological homotopy spheres.

In this setting every n-dimensional smooth homotopy sphere admits a handle decomposition consisting of a single 0-handle and a single n-handle. In other words there exists a diffeomorphism $f\colon S^{n-1} \to S^{n-1}$ such that M is diffeomorphic to $\overline{B}^n \cup_f \overline{B}^n$. In fact we can arrange that f is orientation-preserving.

Proof. Let $n \in \mathbb{N}$ and let M be a smooth homotopy n-sphere. We pick a smooth embedding $\Phi\colon \overline{B}^n \to M$. By the Homology Sphere-minus-Ball Lemma 263.8 we know that the smooth manifold $W := M \setminus \Phi(B^n)$ is a simply connected homology n-ball.



Under our hypothesis (∗) we know that there exists a diffeomorphism $\Theta\colon W \to \overline{B}^n$. One can now easily verify that $f := (\Theta|_{\partial W})^{-1} \circ \Phi|_{S^{n-1}}\colon S^{n-1} \to S^{n-1}$ has the desired properties. Finally note that, after possibly composing Θ with a reflection in a hyperplane, we can arrange that $f\colon S^{n-1} \to S^{n-1}$ is actually orientation-preserving. ∎

The last theorem raises the following question: What can we say about closed smooth manifolds that admit a handle decomposition with precisely two handles? The next proposition gives some answers:

Proposition 264.5. (Two Handles Manifold Proposition) Let $n \in \mathbb{N}$ and let M be a closed n-dimensional smooth manifold that admits a handle decomposition consisting of precisely two handles.

(S) If every diffeomorphism of S^{n-1} extends to a diffeomorphism of $\overline{B}^n$, then M is diffeomorphic to S^n.

(T) If every homeomorphism of S^{n-1} extends to a homeomorphism of $\overline{B}^n$, then M is homeomorphic to S^n.[2819]

Proof. Let $n \in \mathbb{N}$ and let M be a closed n-dimensional smooth manifold that admits a handle decomposition consisting of precisely two handles. Essentially by definition of a handle decomposition this means that there exists a diffeomorphism $f\colon S^{n-1} \to S^{n-1}$ such that M is diffeomorphic to

$$(\overline{B}^n \times \{0\}) \cup (\overline{B}^n \times \{1\})/\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in S^{n-1}.$$

Now let $F\colon \overline{B}^n \to \overline{B}^n$ be an extension of $f\colon S^{n-1} \to S^{n-1}$. We consider the map

$$\begin{array}{ccc} \overbrace{(\overline{B}^n \times \{0\}) \cup (\overline{B}^n \times \{1\})/(x,0) \sim (f(x),1)}^{\text{diffeomorphic to } M} & \xrightarrow{\Theta} & \overbrace{(\overline{B}^n \times \{0\}) \cup (\overline{B}^n \times \{1\})/(x,0) \sim (x,1)}^{\text{diffeomorphic to } S^n, \text{ see page 1229}} \\ [(x,\epsilon)] & \mapsto & \begin{cases} (x,0), & \text{if } \epsilon = 0, \\ (F(x),1), & \text{if } \epsilon = 1. \end{cases} \end{array}$$

One can easily verify that the above map Θ is well-defined. The proposition now follows from the above discussion and the following observations:

(S) We now suppose that F is a diffeomorphism. It follows from the Smooth Manifold Pushout-Proposition 55.12 together with the Regluing Proposition 55.10 that Θ is homeotopic to a diffeomorphism.

[2819] This statement is essentially the same as Statement (1) of the Ball-Glued-to-Ball Proposition 250.11.

(T) We now suppose that F is a homeomorphism. It follows from the Pasting Proposition 4.7 (2) that Θ is continuous and it follows easily from the Compact-Hausdorff Proposition 4.23 (3) that Θ is a homeomorphism. ■

So the question is, when do diffeomorphisms (homeomorphisms) of spheres extend to diffeomorphisms (homeomorphisms) of balls? Let us first consider diffeomorphisms:

(1) By the elementary Diffeotopies-of-S^1-Proposition 42.8 every diffeomorphism of S^1 is diffeotopic to the identity.
(2) By the tricky Munkres–Smale-Theorem **??** every diffeomorphism of S^2 is diffeotopic to the identity.
(3) By the very tricky Cerf Theorem **??** every diffeomorphism of S^3 is diffeotopic to the identity.

It follows from the above discussion and the elementary Exercise 264.3 that for $m = 1, 2, 3$ every diffeomorphism of S^m extends to a diffeomorphism of $\overline{B}^m$. Finally note that the case $m = 4$ is still unknown.

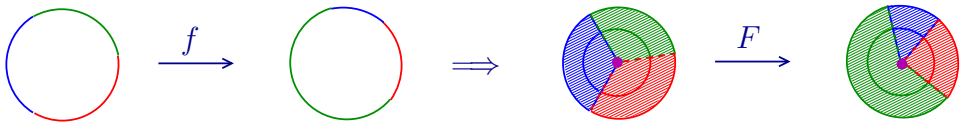
Now let us turn to homeomorphisms: It turns out that when it comes to homeomorphisms it is actually quite easy to show that every self-homeomorphism of a sphere extends to a homeomorphism of the ball:

Proposition 264.6. (Alexander Trick Proposition) Let $n \in \mathbb{N}_0$. Every homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a self-homeomorphism $F\colon \overline{B}^n \to \overline{B}^n$.

Proof. [2820] Let $f\colon S^{n-1} \to S^{n-1}$ be a homeomorphism. We consider the following map:

$$\begin{array}{rcl} F\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} \|x\| \cdot f(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

It is elementary to show, see e.g. Exercise 264.2, that F is continuous. Furthermore it is straightforward to verify that F is a bijection.



It now follows from the Compact-Hausdorff Proposition 4.23 (3) that F is a homeomorphism. ■

Now can now prove the most famous consequence of the h-Cobordism Theorem 263.11:

Theorem 264.7. (High-Dimensional Poincaré Conjecture) Let M be a smooth homotopy n-sphere. If $n \geq 6$, then M is homeomorphic to S^n.

Remark. We will discuss the case $n \leq 5$ in the Low-Dimensional Poincaré Conjecture 264.11 below. □

Example.

(1) Let $k \in \mathbb{N}_{\geq 2}$ be odd and let $\varphi_1, \ldots, \varphi_m\colon \overline{B}^{k+1} \times S^k \to S^{2k+1}$ be orientation-preserving thickened k-spheres with disjoint images such that the corresponding linking matrix

[2820] This proposition is essentially the content of Exercise 8.3 and we already gave a proof on page 4853. For the reader's convenience we recall the proof.

$\mathrm{lk}(\varphi_1,\dots,\varphi_m) \in \mathrm{M}(m\times m,\mathbb{Z})$ is invertible. By the Boundary Invariants Proposition 259.11 (4b) we know that the boundary $\partial(\overline{B}^{2k+2} \cup_{\varphi_1} h^{k+1} \cup \dots \cup_{\varphi_m} h^{k+1})$ is a $(2k+1)$-dimensional homotopy sphere. By the High-Dimensional Poincaré Conjecture 264.7 we now know that this homotopy sphere is homeomorphic to S^{2k+1}. This answers Question 259.12 (1) in the affirmative.

(2) Let $l \in \mathbb{N}_{\geq 2}$. In Definition 259.21 we saw that the E_8-matrix gives rise to $4l$-dimensional smooth manifolds such that the boundary is a homotopy sphere. It now follows from the High-Dimensional Poincaré Conjecture 264.7 that all these homotopy spheres are homeomorphic to the standard sphere. This gives an affirmative answer to Question 259.22 (1). It is not clear though whether these smooth manifolds are also diffeomorphic to the standard sphere. We will return to this question at a later moment. □

Proof. Let $n \geq 6$ and let M be a smooth homotopy n-sphere. Note that it follows from the Homotopy Sphere–Handle Decomposition Theorem 264.4 that there exists a diffeomorphism $f\colon S^{n-1} \to S^{n-1}$ such that M is diffeomorphic to $\overline{B}^n \cup_f \overline{B}^n$. Next note that it follows from the Alexander Trick Proposition 264.6 that there exists a *homeomorphism* $F\colon \overline{B}^n \to \overline{B}^n$ with $F|_{S^{n-1}} = f$. It now follows from the Two Handles Manifold Proposition 264.5 (T) that M is homeomorphic to S^n. ■

It is a little disturbing that the High-Dimensional Poincaré Conjecture 264.7 starts out with a *smooth* homotopy sphere but that it only concludes that it is homeomorphic to the standard sphere. This raises the following obvious question, which we will answer in the Exotic 7-Sphere Theorem 282.6.

Question 264.8. Let M be a smooth manifold of dimension $n \geq 6$. If M is homeomorphic to S^n, is it also diffeomorphic to S^n?

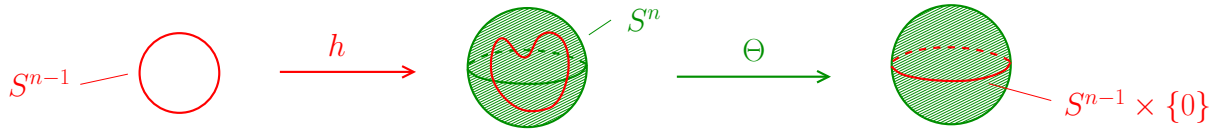
We will discuss Question 264.8 in greater detail in the Smooth Poincaré Conjecture **??** and the Homotopy Sphere Group–Calculation Theorem **??** (1).

264.3. **High-dimensional smooth Schoenflies Theorem.** In this section we discuss another theorem that is closed related to the Homotopy Ball-Diffeomorphism Theorem 264.2:

Theorem 264.9. (Codimension-One Sphere Theorem) Let $n \in \mathbb{N}$. We assume that the following statement holds:

(∗) Every n-dimensional homotopy ball with simply connected boundary is diffeomorphic to $\overline{B}^n$.

In this case the following conclusion holds: For every smooth embedding $h\colon S^{n-1} \to S^n$ there exists a diffeomorphism $\Theta\colon S^n \to S^n$ such that $(\Theta \circ h)(S^{n-1})$ equals the equatorial sphere $S^{n-1} \times \{0\}$.



Before we provide the proof of the Codimension-One Sphere Theorem 264.9, let us mention the obvious consequence:

Theorem 264.10. (High-Dimensional Smooth Schoenflies Theorem) Let $n \in \mathbb{N}_{\geq 6}$ and let $h\colon S^{n-1} \to S^n$ be a smooth embedding. There exists a diffeomorphism $\Theta\colon S^n \to S^n$ such that $(\Theta \circ h)(S^{n-1})$ equals the equatorial sphere $S^{n-1} \times \{0\}$.

Remark. We will discuss the case $n \leq 5$ in the Low Dimensional Smooth Schoenflies Theorem 264.16. □

Proof of Theorem 264.10 assuming Theorem 264.9. Let $n \geq 6$. In the Homotopy Ball-Diffeomorphism Theorem 264.2 we just showed that the hypothesis (∗) of the Codimension-One Sphere Theorem 264.9 is satisfied. Thus we get the desired result from the Codimension-One Sphere Theorem 264.9. ■

Proof of the Codimension-One Sphere Theorem 264.9. Let $n \in \mathbb{N}$. We assume that the following statement holds:

(∗) Every n-dimensional homotopy ball with simply connected boundary is diffeomorphic to $\overline{B}^n$.

Let $h\colon S^{n-1} \to S^n$ be a smooth embedding. Note that it follows easily from the Smooth Embedding Theorem 30.15 and the Generalized Smooth Jordan-Brouwer Separation Theorem 51.15 that there exist two smooth codimension-zero submanifolds B and C of S^n such that $\partial B = \partial C = B \cap C = h(S^{n-1})$.



Claim. Both B and C are smooth homotopy balls.

Proof. We leave the cases $n = 1, 2$ to the reader and we henceforth assume that $n \geq 3$. First note that we have

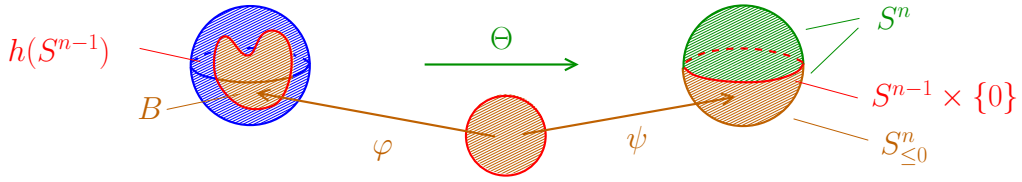
$$\pi_1(B) * \pi_1(C) \underset{\substack{\uparrow \\ \text{since } B\cap C \cong S^{n-1} \text{ and} \\ \text{since } n \geq 3}}{\cong} \pi_1(B) *_{\pi_1(B\cap C)} \pi_1(C) \underset{\substack{\uparrow \\ \text{by the Manifold–Seifert-van Kampen} \\ \text{Theorem 102.1}}}{\cong} \pi_1(S^n) = \{e\}.$$

It follows that B and C are simply connected. It remains to show that B and C are both homology balls. This follows easily from the Sphere–Alexander Duality Theorem 217.8. As an alternative we now provide, for the reader's convenience, a more down-to-earth argument using the Mayer-Vietoris Theorem 153.2 (3). We consider the long exact sequence from the Mayer-Vietoris Theorem 153.2 (3) that arises from the decomposition $M = B \cup C$. In the following we show the most interesting segment of this long exact sequence, namely the one where the non-zero reduced homology group of the sphere S^n makes its appearance:

$$\to \underbrace{\widetilde{\mathrm{H}}_n(B \cap C)}_{=0} \to \widetilde{\mathrm{H}}_n(B) \oplus \widetilde{\mathrm{H}}_n(C) \to \underbrace{\widetilde{\mathrm{H}}_n(S^n)}_{=\mathbb{Z}\cdot[S^n]} \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(B \cap C)}_{=\mathbb{Z}\cdot[B\cap C]} \to \dots$$

Note that by the Manifold–MV-Connecting Homomorphism Corollary 194.16 we know that $\partial_n([S^n]) = [B \cap C]$. It now follows easily from $B \cap C \cong S^{n-1}$ and the Sphere-H_*-Proposition 149.4 that the reduced homology groups of B and C are all zero. It follows from the Reduced Homology Lemma 147.3 that B and C are homology balls. ⊞

Note that it follows from the claim and our hypothesis (∗) that there exists a diffeomorphism $\varphi\colon \overline{B}^n \to B \subseteq S^n$. Let $\psi\colon \overline{B}^n \to S^n_{\leq 0} \subseteq S^n$ be the diffeomorphism between $\overline{B}^n$ and the lower hemisphere $S^n_{\leq 0}$ that is given by stereographic projection. It now follows from the Smooth Ball Embedding Theorem 45.10 that there exists a diffeomorphism $\Theta\colon S^n \to S^n$ such that $\Theta \circ \varphi = \psi$.



Clearly this diffeomorphism Θ has the desired properties. ■

264.4. **Results in low dimensions.** It is natural to ask to what degree the statements of the h-Cobordism Theorem 263.11, the Homotopy Ball-Diffeomorphism Theorem 264.2, the High-Dimensional Poincaré Conjecture 264.7 and the High-Dimensional Smooth Schoenflies Theorem 264.10 hold in dimensions ≤ 5. We will discuss this question in the following four subsections:

- In Subsection 264.4.1 we will consider the Poincaré Conjecture in low dimensions.
- In Subsection 264.4.2 we will consider smooth homotopy balls in low dimensions.
- In Subsection 264.4.3 we will consider smooth h-cobordisms in low dimensions.
- In Subsection 264.4.4 we will consider the Schoenflies problem in low dimensions.

264.4.1. **The Poincaré Conjecture in low dimensions.** We start out by summarizing the status of the Poincaré Conjecture in low dimensions:

Theorem 264.11. (Low-Dimensional Poincaré Conjecture)
(1) Every smooth homotopy 1-sphere is diffeomorphic to S^1.
(2) Every smooth homotopy 2-sphere is diffeomorphic to S^2.
(3) Every smooth homotopy 3-sphere is diffeomorphic to S^3.
(4) (S) It is not known whether every smooth homotopy 4-sphere is diffeomorphic to S^4.
(T) Every topological homotopy 4-sphere is homeomorphic to S^4.
(5) Every smooth homotopy 5-sphere is diffeomorphic to S^5.

Proof.

(1) This statement follows from the Smooth 1-Manifold Classification Theorem 28.8.
(2) This statement follows from the Surface Classification Theorem 101.16.
(3) This statement is dealt with by the Poincaré 3D-Conjecture Theorem **??** which was first proved by Grigori Perelman [**Per2002, Per2003a, Per2003b, MTi2007**].
(4) (T) This statement was proved by Michael Freedman [**Free1982**], see the Topological 4D-Poincaré Theorem 228.14 for a few more details.
(5) Let M be a smooth homotopy 5-sphere. It was shown by Stephen Smale [**Sma1961a**, p. 373] (see also [**Kos1993**, Corollary VIII.4.6] or [**Miln1965b**, p. 107]) that M is homeomorphic to S^5. In [**Miln1965b**, p. 107] it is shown that M is actually diffeomorphic to S^5. (Alternatively see the Homotopy Sphere Group–Calculation Theorem **??** (1).) ■

We already mentioned the big gap that becomes apparent in the above formulation of the Low-Dimensional Poincaré Conjecture 264.11 (4S). Let us record it again:

Question 228.26. (Smooth Poincaré 4D-Conjecture) If M is a 4-dimensional smooth homotopy sphere, is it diffeomorphic to S^4?

264.4.2. **Smooth homotopy balls in low dimensions.** In this subsection we will summarize what we know about smooth homotopy balls of low dimensions:

Theorem 264.12. (Low-Dimensional Homotopy Ball Theorem)

(1) Every smooth homotopy 1-ball is diffeomorphic to $\overline{B}^1$.
(2) Every smooth homotopy 2-ball is diffeomorphic to $\overline{B}^2$.
(3) Every smooth homotopy 3-ball is diffeomorphic to $\overline{B}^3$.
(4) (S) It is not known whether every smooth homotopy 4-ball with simply connected boundary is diffeomorphic to S^4. More precisely, this statement is equivalent to the Smooth Poincaré 4D-Conjecture 228.26.
    (T) Every topological homotopy 4-ball with simply connected boundary is homeomorphic to $\overline{B}^4$.
(5) Every smooth homotopy 5-ball with simply connected boundary is homeomorphic to $\overline{B}^5$.

In the proof of some of the statements of the Low-Dimensional Homotopy Ball Theorem 264.12 we will make use of the following proposition:

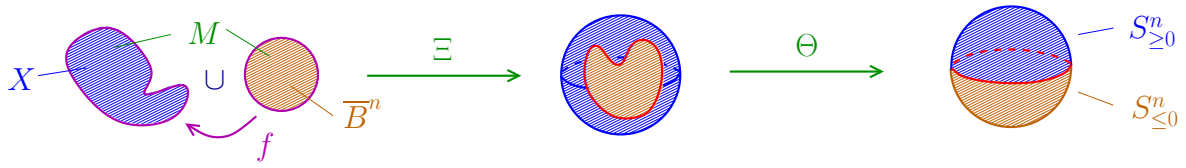
Proposition 264.13. (Homotopy Sphere–Homotopy Ball Proposition) Let $n \in \mathbb{N}_0$.

(S) We assume that every smooth homotopy n-sphere is diffeomorphic to S^n. Then given any smooth homotopy n-ball M, whose boundary ∂M is diffeomorphic to S^{n-1}, there exists a diffeomorphism $M \to \overline{B}^n$.
(T) If every topological homotopy n-sphere is homeomorphic to S^n, then every topological homotopy n-ball whose boundary is homeomorphic to S^{n-1} is homeomorphic[2821] to $\overline{B}^n$.

Proof of the Homotopy Sphere–Homotopy Ball Proposition 264.13. Let $n \in \mathbb{N}_0$.

(S) We assume that we are in a dimension such that every smooth homotopy n-sphere is diffeomorphic to S^n. Now let X be a smooth homotopy n-ball for which there exists a diffeomorphism $f\colon S^{n-1} \to \partial X$. We consider the smooth manifold $M := X \cup_f \overline{B}^n$. Using the Manifold–Seifert-van Kampen Theorem 102.1 and using the Mayer-Vietoris Theorem 153.2 (3) one can easily verify that $M = X \cup_f \overline{B}^n$ is a smooth homotopy n-sphere.
By hypothesis we know that there exists a diffeomorphism $\Xi\colon X \cup_f \overline{B}^n \to S^n$. As in the proof of the Codimension-One Sphere Theorem 264.9 we see, using the Smooth Ball Embedding Theorem 45.10, that there exists a diffeomorphism $\Theta\colon S^n \to S^n$ such that $\Theta(\Xi(\overline{B}^n)) = S^n_{\leq 0}$. The diffeomorphism $\Theta \circ \Xi$ then gives us a diffeomorphism

[2820] Note that it follows from the Topological 4D-Poincaré Theorem 228.14 that this question is equivalent to the following question: If a 4-dimensional smooth manifold is homeomorphic to S^4, is it also diffeomorphic to S^4?
[2821] The author does not know whether one can upgrade "homeomorphic" to "diffeomorphic.



$X \to S^n_{\geq 0}$. But the latter is diffeomorphic to $\overline{B}^n$. In summary we see that X is indeed diffeomorphic to $\overline{B}^n$.

(T) The proof in the topological setting is essentially identical to the proof in the smooth setting, we just need to replace the Smooth Ball Embedding Theorem 45.10 by the (much harder) Topological Ball Embedding Theorem 224.27. An alternative proof of this statement is sketched in [**Miln1965b**, p. 111]. ■

Proof of the Low-Dimensional Homotopy Ball Theorem 264.12.

(1) This statement follows from the Smooth 1-Manifold Classification Theorem 28.8.

(2) This statement follows from the Surface Classification Theorem 101.16.

(3) This statement follows from the Low-Dimensional Poincaré Conjecture 264.11 (2) and (3) together with the Homotopy Sphere–Homotopy Ball Proposition 264.13 (1).

(4) (S) We first assume that the Smooth Poincaré 4D-Conjecture 228.26 is answered in the affirmative. It would then follow from the Low-Dimensional Poincaré Conjecture 264.11 (3) together with the Homotopy Sphere–Homotopy Ball Proposition 264.13 (S) that every smooth homotopy 4-ball with simply connected boundary is diffeomorphic to S^4.

Conversely, we now assume that every smooth homotopy 4-ball with simply connected boundary is diffeomorphic to S^4. Now let M be a smooth homotopy 4-sphere. It follows from our hypothesis together with the Homotopy Sphere–Handle Decomposition Theorem 264.4 that there exists a diffeomorphism $f\colon S^3 \to S^3$ such that M is diffeomorphic to $\overline{B}^4 \cup_f \overline{B}^4$. By the Cerf Theorem **??**, which we mentioned above, we know that the self-diffeomorphism f of S^3 extends to a diffeomorphism of $\overline{B}^4$. It thus follows from the Two Handles Manifold Proposition 264.5 (S) that M is diffeomorphic to S^4.

(T) This statement follows from the Low-Dimensional Poincaré Conjecture 264.11 (3) and (4T) together with the Homotopy Sphere–Homotopy Ball Proposition 264.13 (T).

(5) This statement follows from the Low-Dimensional Poincaré Conjecture 264.11 (4T) and (5) together with the Homotopy Sphere–Homotopy Ball Proposition 264.13 (T). ■

264.4.3. **Smooth h-cobordisms in low dimensions.** Next we consider the low-dimensional analogs of the h-Cobordism Theorem 263.11 holds in lower dimensions. Here is what we know:

Theorem 264.14. (Low-Dimensional h-Cobordism Theorem)

(1) A simply connected 1-dimensional smooth h-cobordism is diffeomorphic to a product.

(2) A simply connected 2-dimensional smooth h-cobordism is diffeomorphic to a product.

(3) A simply connected 3-dimensional smooth h-cobordism is diffeomorphic to a product.

(4) (S) It is not known whether every simply connected 4-dimensional smooth h-cobordism is diffeomorphic to S^4. More precisely, this statement is equivalent to the Smooth Poincaré 4D-Conjecture 228.26.
(T) Every simply connected topological 4-dimensional h-cobordism is homeomorphic to a product.
(5) (S) There exist simply connected 5-dimensional smooth h-cobordisms that are *not* diffeomorphic to a product.[2822]
(T) Every 5-dimensional topological h-cobordism is homeomorphic to a product.

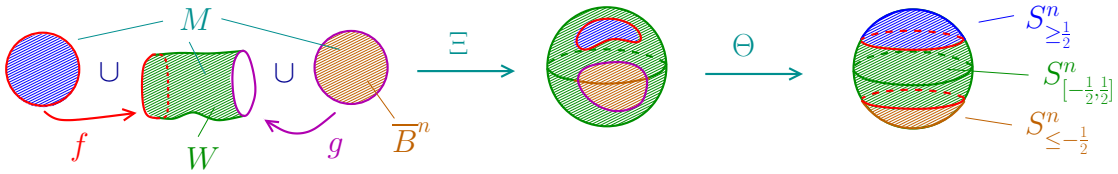
The following proposition allows us to connect many of the statements of the Low-Dimensional h-Cobordism Theorem 264.14 to the Low-Dimensional Poincaré Conjecture 264.11:

Proposition 264.15. (Homotopy Sphere–h-Cobordism Proposition) Let $n \in \mathbb{N}_0$.
(S) We assume that every smooth homotopy n-sphere is diffeomorphic to S^n. Then every n-dimensional smooth h-cobordism between two copies of S^{n-1} is diffeomorphic to a product.
(T) We assume that every topological homotopy n-sphere is homeomorphic to S^n. Then every n-dimensional smooth h-cobordism between two copies of S^{n-1} is homeomorphic to a product.

Proof of the Homotopy Sphere–h-Cobordism Proposition 264.15.

(S) We assume that every smooth homotopy n-sphere is diffeomorphic to S^n. Let W be a smooth h-cobordism between S and T where both S and T are diffeomorphic to S^{n-1}. We pick diffeomorphisms $f\colon S^{n-1} \to S$ and $g\colon S^{n-1} \to T$. We consider the smooth manifold $M := W \cup_f \overline{B}^n \cup_g \overline{B}^n$.

Using the Manifold–Seifert-van Kampen Theorem 102.1 and using the Mayer-Vietoris Theorem 153.2 (3) one can easily verify that $M = W \cup_f \overline{B}^n \cup_g \overline{B}^n$ is a smooth homotopy n-sphere.



By hypothesis there exists a diffeomorphism $\Xi\colon W \cup_f \overline{B}^n \cup_g \overline{B}^n \to S^n$. Next let $\varphi\colon \overline{B}^n \to S^n_{\geq\frac{1}{2}}$ and $\psi\colon \overline{B}^n \to S^n_{\leq-\frac{1}{2}}$ be the smooth embeddings given by stereographic projection. It follows from the Smooth Ball Embedding Theorem 45.10 that there exists a diffeomorphism $\Theta\colon S^n \to S^n$ such that $(\Theta \circ \Xi)(\overline{B}^n) = S^n_{\geq\frac{1}{2}}$ and such that $(\Theta \circ \Xi)(\overline{B}^n) = S^n_{\leq-\frac{1}{2}}$. It follows that $\Theta \circ \Xi$ defines a diffeomorphism $\overline{W} \to S^n_{[-\frac{1}{2},\frac{1}{2}]}$. but the latter is clearly diffeomorphic to the product $S^{n-1} \times [-1,1]$.

(T) The proof in the topological setting is essentially identical to the proof in the smooth setting, we just need to replace the Smooth Ball Embedding Theorem 45.10 by the Topological Ball Embedding Theorem 224.27. ■

Proof of the Low-Dimensional h-Cobordism Theorem 264.14.

(1) This statement follows easily the Smooth 1-Manifold Classification Theorem 28.8.
(2) It follows easily from the Surface Classification Theorem 101.16 that there exists no simply connected 2-dimensional smooth h-cobordism. So here there is nothing to prove.

(3) This statement follows easily from the Low-Dimensional Poincaré Conjecture 264.11 (2) and (3) together with the Homotopy Sphere–h-Cobordism Proposition 264.15 (S).

(4) (S) We first assume that the Smooth Poincaré 4D-Conjecture 228.26 is answered in the affirmative. It would then follow from the Low-Dimensional Poincaré Conjecture 264.11 (3) together with the Homotopy Sphere–h-Cobordism Proposition 264.15 (S) that every 4-dimensional smooth h-cobordism is diffeomorphic to a product. Conversely suppose that every 4-dimensional smooth h-cobordism is diffeomorphic to a product. It would then follow from the argument of the Homotopy Ball-Diffeomorphism Theorem 264.2 that every smooth homotopy 4-ball with simply connected boundary is diffeomorphic to $\overline{B}^4$. By the Low-Dimensional Homotopy Ball Theorem 264.12 (4S) this would give an affirmative answer to the Smooth Poincaré 4D-Conjecture 228.26.

(T) This statement follows easily from the Low-Dimensional Poincaré Conjecture 264.11 (3) and (4T) together with the Homotopy Sphere–h-Cobordism Proposition 264.15 (T).

(5) (S) For $n = 5$ it is known that the analog of the h-Cobordism Theorem 263.11 does *not* hold. This can be seen as follows: By [**Wall1964**, Theorem 2] (see also [**GoS1999**, Theorem 9.2.13]) any two closed oriented simply connected 4-dimensional smooth manifolds with isometric intersection forms are smoothly h-cobordant via a 5-dimensional smooth h-cobordism. If the 5-dimensional analog of the h-Cobordism Theorem 263.11 did hold, then any two closed oriented simply connected 4-dimensional smooth manifolds with isometric intersection forms would be diffeomorphic. But we saw in the Exotic K3-Theorem 228.24 that this is not the case.[2823]

(T) This statement is proved in [**FQ1990**, Theorem 7.1.A]. ■

264.4.4. **Schoenflies in low dimensions.** We conclude this chapter with the following low-dimensional analog of the High-Dimensional Smooth Schoenflies Theorem 264.10:

Theorem 264.16. (Low Dimensional Smooth Schoenflies Theorem)[2824] Let $n \in \mathbb{N}$ and let $h\colon S^{n-1} \to S^n$ be a smooth embedding.

(1) If $n = 1$, then there exists a diffeomorphism $\Theta\colon S^1 \to S^1$ with $(\Theta \circ h)(S^0) = S^0 \times \{0\}$.
(2) If $n = 2$, then there exists a diffeomorphism $\Theta\colon S^2 \to S^2$ with $(\Theta \circ h)(S^1) = S^1 \times \{0\}$.
(3) If $n = 3$, then there exists a diffeomorphism $\Theta\colon S^3 \to S^3$ with $(\Theta \circ h)(S^2) = S^2 \times \{0\}$.
(4) The statement for $n = 4$ is not known.
(5) If $n = 5$, then there exists a diffeomorphism $\Theta\colon S^5 \to S^5$ with $(\Theta \circ h)(S^4) = S^4 \times \{0\}$.

Proof. [2825]

(1) The case $n = 1$ is essentially trivial.

(2) For $n = 2$ the statement follows immediately from the Low-Dimensional Homotopy Ball Theorem 264.12 (2) together with the Codimension-One Sphere Theorem 264.9 (2). Alternatively it can be deduced from the Fill-in-Disk Proposition 107.15.

[2823] The first proof of the fact that there is no 5-dimensional smooth h-cobordism theorem was given by Larry Siebenmann [**Sie1971**].

[2824] We mentioned the contents of this theorem already in the earlier Generalized Smooth Schoenflies Theorem 170.7.

[2825] This theorem and its proof has some overlap with the proof of the Generalized Smooth Schoenflies Theorem 170.7.

(3) For $n = 3$ this statement was first proved by James Alexander [**Ale1924b**]. (See also [**Hat3**, Chapter 1] and [**Schul2014**, Theorem 3.2.5] for an exposition of the proof.) Alternatively the statement follows immediately from the Low-Dimensional Homotopy Ball Theorem 264.12 (3) together with the Codimension-One Sphere Theorem 264.9 (3).
(4) We recorded this statement already as Question 170.8. To the best of the author's knowledge nobody has managed to answer the question since we first wrote it down.
(5) For $n \geq 5$ this statement was first independently proved by Barry Mazur [**Maz1959**] in 1959 and by Morton Brown [**BrownM1960**] in 1960. Alternatively note that the statement follows immediately from the Low-Dimensional Homotopy Ball Theorem 264.12 (5) together with the Codimension-One Sphere Theorem 264.9 (5). ■

264.5. Exercises for Chapter 264.

Exercise 264.1. Let $n \in \mathbb{N}_{\geq 2}$. Let M be a compact n-dimensional smooth manifold that is simply connected.
(a) Suppose that for some $k \in \mathbb{N}$ we have $\mathrm{H}_i(M) \cong \mathrm{H}_i(S^n)$ for all $i = 0, \ldots, k$. What is the minimal k for which you can confidently conclude that M is a homotopy sphere?
(b) Suppose that for some $k \in \mathbb{N}$ we have $\mathrm{H}_i(M) \cong \mathrm{H}_i(\overline{B}^n)$ for all $i = 0, \ldots, k$. What is the minimal k for which you can confidently conclude that M is a homotopy ball?

Exercise 264.2. Let $n \in \mathbb{N}$ and let $\varphi\colon S^n \to S^n$ be a homeomorphism. We consider the map

$$\begin{array}{rcl} \Phi\colon \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

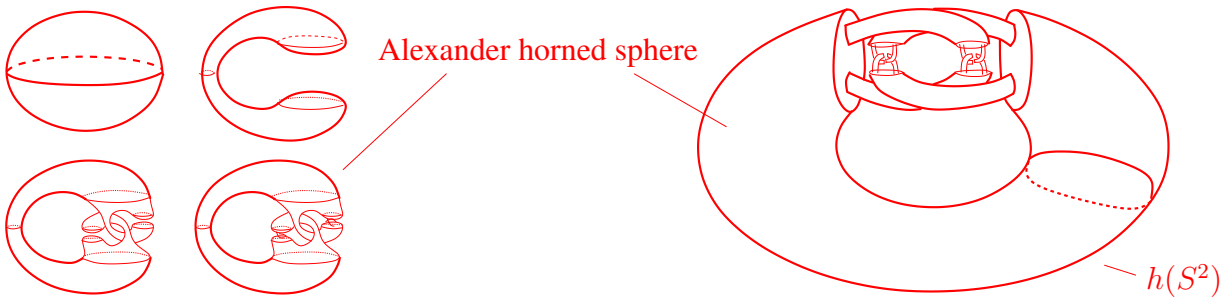
(a) Show that Φ is continuous.
(b) Give any example of a *diffeomorphism* $\varphi\colon S^n \to S^n$ such that the corresponding map $\Phi\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ is *not* a diffeomorphism.

Hint. You might want to use the Cylinder Quotient-Ball Lemma 8.9.

Exercise 264.3. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a diffeomorphism that is diffeotopic to id_{S^n}. Show that there exists a diffeomorphism $F\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ with $F|_{S^n} = f$.

Exercise 264.4. Let $n \in \mathbb{N}_0$ and let M be an n-dimensional smooth manifold. We suppose that there exists a $P \in M$ such that $M \setminus \{P\}$ is contractible. Show that M is homeomorphic to S^n.

Exercise 264.5. In the figure below we show a certain embedding $h\colon S^2 \to \mathbb{R}^3 \subseteq S^3$. The image is called the *Alexander horned sphere*.
(a) Draw the Alexander horned sphere on your own.
(b) Convince yourself that one of the two components of $S^3 \setminus h(S^2)$ is not simply connected.
(c) Show that there exists an embedding $k\colon S^2 \to S^3$ such that neither of the two components of the complement of $k(S^2)$ is simply connected.


Alexander horned sphere
$h(S^2)$

Part XXX

Vector bundles and characteristic classes

In this part of the lecture notes we will (re-) introduce vector bundles over topological spaces. There are many other references for vector bundles and characteristic classes, the most famous one is surely the classic book by John Milnor and James Stasheff [**MiS1974**]. Other references are [**Hat2, Hus1993, OsbH1982**]. A detailed discussion of the various ways to define characteristic classes is given in [**OsbH1982**, Chapter V.7].

This part of the lecture notes contains the following chapters:

- In Chapter 265 we will introduce real and complex vector bundles and we will give many examples.
- In Chapter 266 we will prove that the two definitions of vector bundles, namely in Definition 240.10 and from Definition 265.1 are essentially the same.
- In Chapter 267 we will show that pullbacks of vector bundles under homotopic maps are isomorphic. This is chapter is very close to the earlier Chapter 241 where we studied similar statements for more general bundles.
- In Chapter 268 we will take various constructions from linear algebra and generalize them "fiberwise" to vector bundles.
- In Chapter 269 we will introduce inner products on vector bundles. We will see that in most reasonable settings such inner products exist and that they are actually quite useful.
- In Chapter 270 we will introduce the Grassmannians $G_m(\mathbb{R}^\infty)$ and $G_m(\mathbb{C}^\infty)$ and the corresponding tautological vector bundles. We will see that essentially all vector bundles are pullbacks of the tautological vector bundles.
- In Chapter 271 we will calculate the (co-) homology groups of the Grassmannians $G_m(\mathbb{K}^d)$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $d \in \mathbb{N}_0 \cup \{\infty\}$.
- In Chapter 272 we will calculate the cohomology rings of the infinite-dimensional Grassmannians $G_m(\mathbb{R}^\infty)$ and $G_m(\mathbb{C}^\infty)$.
- In Chapter 273 we will discuss the splitting principle and we will discuss the effect of certain maps on Grassmannians on the cohomology groups. This chapter is very short, but it is essential for proving the key property of Chern classes and Stiefel-Whitney classes.
- In Chapter 274, given a complex vector bundle $p\colon E \to B$ we will introduce the corresponding Chern classes $c_i(E) \in \mathrm{H}^{2i}(B;\mathbb{Z})$ and given a real vector bundle $p\colon E \to B$ we will introduce the corresponding Stiefel-Whitney classes $w_i(E) \in \mathrm{H}^i(B;\mathbb{F}_2)$.
- In Chapter 275 we will discuss applications of Stiefel-Whitney classes. In particular, given $n \in \mathbb{N}_0$, we will give lower bounds on the dimensions m for which we can find an immersion $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^m$.
- In Chapter 276 we will use the Chern classes from Chapter 274 to introduce the Pontryagin classes $p_i(E) \in \mathrm{H}^{4i}(B;\mathbb{Z})$ of a real vector bundle $p\colon E \to B$.
- In Chapter 277 we will prove the Thom Isomorphism Theorem for an (oriented) n-dimensional real vector bundle $p\colon E \to B$. This allows us to define the Thom class $\Theta_{\mathbb{F}_2}(E)$ (and the Thom class $\Theta_{\mathbb{F}_2}(E)$).
- In Chapter 278 we will use Thom classes to define the Euler classes of an (oriented) real vector bundle. These are characteristic classes of vector bundles that are related to Stiefel-Whitney and Chern classes, but we will see that in general they contain new information.

- In Chapter 279 we will use the Euler classes $e_{\mathbb{F}_2}(E \to B)$ and the Steenrod operations to give an alternative definition of Stiefel-Whitney classes.

CHAPTER 265

Vector Bundles

In this chapter we will (re-) introduce vector bundles and we will study some basic examples, like the tautological bundles over projective spaces and the tangent bundles of smooth manifolds. We will also discuss the orientability of real vector bundles and we will discuss general constructions of vector bundles, like pullbacks. Many definitions and results of this chapter are just special cases of the earlier more general theory of $(F, \mathcal{G})$-bundles. So the reader, who is familiar with the earlier chapters on bundles, will find this chapter particularly easy to read.

This chapter is organized as follows:

- In Section 265.1 we will introduce real and complex vector bundles.
- In Section 265.2 we will introduce the notion of a smooth vector bundle over a smooth manifold.
- In Section 265.3 we will introduce the tangent bundle $\mathrm{T}M$ of a smooth manifold M and we will introduce the normal vector bundle $\mathrm{N}M$ of a smooth submanifold $M \subseteq \mathbb{R}^k$. These two types of vector bundles will be some of the main sources of smooth vector bundles.
- In Section 265.4 we will study sections of vector bundles. As we will see, these can be a convenient tool for showing that a vector bundle is (not) trivializable.
- In Section 265.5 we will discuss orientations of vector bundles.
- In Section 265.6 we will use sections and orientations of vector bundles to introduce the Euler number of a smooth vector bundle. We will use the Euler number to show that various smooth manifolds are not parallelizable.
- In Section 265.7 we will introduce and study the pullback of vector bundles.
- In Section 265.8 we will provide exercises for Chapter 265.

In the next Chapter 266 we will give some alternative formulations of the concept of a vector bundle.

265.1. **Vector bundles.** In this section we will (re-) introduce vector bundles and we will give several constructions and examples. Loosely speaking, a vector bundle over a topological space B consists of a vector space for each $b \in B$, where the "vector spaces vary continuously with b". The following definition makes this idea precise.

Definition 265.1. Let $n \in \mathbb{N}_0$.

(1) An n-dimensional real vector bundle[2826] consists of the following two pieces of data:[2827]
   (a) A continuous map $p\colon E \to B$ between topological spaces,
   (b) a real vector space structure on each $E_b := p^{-1}(b)$, $b \in B$,

such that the following condition is satisfied: For every $b \in B$ there exists an open neighborhood U of $b \in B$ and a homeomorphism $\Phi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ such that the following two conditions are satisfied:

- The following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{R}^n \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (y,v)\mapsto y} & \\ & U. & \end{array}$$

- For each $y \in U$ the map

$$p^{-1}(y) \xrightarrow{\Phi|_{p^{-1}(y)}} \{y\} \times \mathbb{R}^n \xrightarrow{(y,v)\mapsto v} \mathbb{R}^n$$

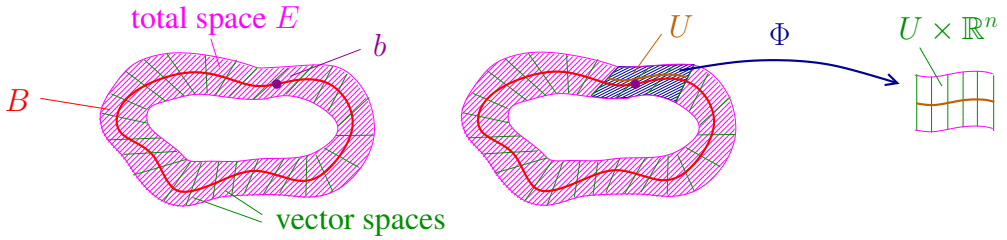
is an isomorphism of real vector spaces.

We introduce the following language:

- Any map $\Phi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ as above is called a local trivialization of the vector bundle.
- We refer to the topological space E as the total space of the vector bundle.
- We refer to B as the base space of the vector bundle.
- We refer to the map $p\colon E \to B$ as the projection.
- A 1-dimensional real vector bundle is often called a real line bundle.

(2) Exactly the same way as in (1) we also define the notion of an n-dimensional complex vector bundle.

When we are agnostic about whether we consider real or complex vector bundles, then sometimes we just say vector bundle.



Examples.

(1) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For each topological space B and each $n \in \mathbb{N}_0$ the product $B \times \mathbb{R}^n$ together with the projection $p\colon B \times \mathbb{K}^n \to B$, and together with the obvious $\mathbb{K}$-vector space structure on each $p^{-1}(b) = \{b\} \times \mathbb{K}^n$, is evidently an n-dimensional $\mathbb{K}$-vector bundle over B. We refer to this vector bundle as the trivial n-dimensional $\mathbb{K}$-vector bundle over B and sometimes we denote it by $\mathrm{Tr}_B(\mathbb{K}^n)$.

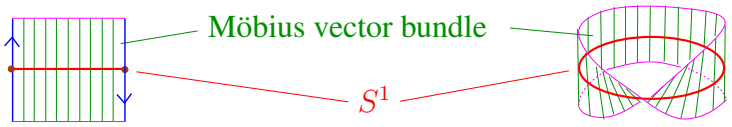
(2) Similar to the discussion on page 4580 we consider the following map:

$$\begin{array}{rcl} ([0,1] \times \mathbb{R})/(0,v) \sim (1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

[2826] In Definition 240.10 we gave a slightly different definition of a vector bundle. In Chapter 266 we will see that the two notions of vector bundles are essentially equivalent.

[2827] We follow the common convention of suppressing the choice of the vector space structure from the notation. In most cases the vector space structure is clear from the context.

In Exercise 265.1 we will see that this a real line bundle over S^1. As in the Möbius Band-as-Quotient Lemma 9.6 we see that the total space is homeomorphic to the open Möbius band $([0,1]\times(0,1))/(0,y)\sim(1,1-y)$. Therefore we refer to this bundle as the Möbius vector bundle.



(3) Given any vector bundle $p\colon E\to B$ and given any subspace $W\subseteq B$ the restricted map $p\colon p^{-1}(W)\to W$ is evidently also a vector bundle. We refer to it as the restriction of p to W. □

Before we introduce more examples of vector bundles, let us give the following definition of a morphism between vector bundles:

Definition 265.2. [2828] Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let $p\colon E\to B$ and $q\colon F\to C$ be two n-dimensional $\mathbb{K}$-vector bundles. A morphism between these $\mathbb{K}$-vector bundles consists of two continuous maps $h\colon B\to C$ and $H\colon E\to F$ such that the following two conditions are satisfied:

(1) The following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C. \end{array}$$

(2) For each $b\in B$ the map $H|_{E_b}\colon E_b\to F_{h(b)}$ is an isomorphism[2829] of $\mathbb{K}$-vector spaces.

If $B=C$ and $h=\mathrm{id}_B$, then we call it a morphism over B.

The notion of a "morphism" suggests that there should be a category lurking in the background. The following lemma, which is an analog of the Bundle–Category Lemma 240.13, shows that this is indeed the case:

Lemma 265.3. (Vector Bundle–Category Lemma) Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ and let $n\in\mathbb{N}_0$. In the following diagram we show how to compose two morphisms of n-dimensional $\mathbb{K}$-vector bundles:

$$\begin{array}{ccc} X & \xrightarrow{G} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{g} & C \end{array} \quad\circ\quad \begin{array}{ccc} Y & \xrightarrow{H} & Z \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ C & \xrightarrow{h} & D \end{array} \quad:=\quad \begin{array}{ccc} X & \xrightarrow{H\circ G} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} \\ B & \xrightarrow{h\circ g} & D. \end{array}$$

The class of n-dimensional $\mathbb{K}$-vector bundles together with the above notion of morphisms and the above composition of such morphisms maps form a category.

Proof. This lemma follows almost immediately from the definitions. Surely there is no need to elaborate on the arguments. ■

[2828] The definition is inspired by Definition 240.12.

[2829] It would also make perfect sense to ask for this map to just be a homomorphism of $\mathbb{K}$-vector spaces. We will use this more restrictive notion of a morphism of $\mathbb{K}$-vector bundles. For other purposes, and in many other books, the looser notion of a morphism gets used.

The following definition is an analog of Definitions 240.14 and 240.15:

Definition 265.4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$.

(1) We refer to the category that we introduced in the Vector Bundle–Category Lemma 265.3 as the category $\mathcal{VectBund}_{\mathbb{K}^n}$ of n-dimensional $\mathbb{K}$-vector bundles.

(2) For a topological space B we denote by category $\mathcal{VectBund}_{\mathbb{K}^n}(B)$ of n-dimensional $\mathbb{K}$-vector bundles over B.

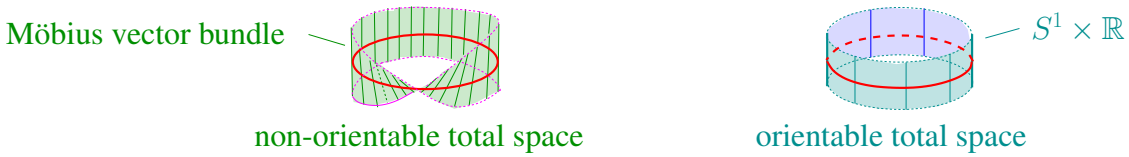
Before we discuss a few more examples of vector bundles, let us introduce the following definition, which mirrors Definitions 238.1 (6) and 240.16:

Definition 265.5. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle.

(1) We say that p is trivializable if p is isomorphic over B to the trivial n-dimensional $\mathbb{K}$-vector bundle $B \times \mathbb{K}^n \to B$.

(2) Any choice of an isomorphism $E \to B \times \mathbb{K}^n$ is called a trivialization of the vector bundle p.

Example. Now that we have a category it makes sense to talk of isomorphisms of vector bundles. One of the main challenges will be to determine whether or not two vector bundles are isomorphic. We start out with an observation: Assume that two vector bundles $p\colon E \to B$ and $q\colon F \to C$ are isomorphic. It follows immediately from the definitions that the total spaces E and F are in particular homeomorphic. Now consider the Möbius vector bundle and the trivial vector bundle $S^1 \times \mathbb{R} \to S^1$. First note that it follows from the Surface Non-Orientability Lemma 31.45 together with the Smooth-Topological Orientation Proposition 191.25 that the total space of the Möbius vector bundle is a non-orientable 2-dimensional topological manifold. On the other hand it follows from the Product Orientation Proposition 192.7 (1) that the product $S^1 \times \mathbb{R}$ is an orientable 2-dimensional topological manifold.



Note that this shows that the Möbius vector bundle is not isomorphic to the trivial 1-dimensional real vector bundle over S^1. In other words, the Möbius vector bundle is not trivializable. □

We formulate our next example as a lemma:

Lemma 265.6. (Tautological Line Bundle Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $n \in \mathbb{N}_0$. We consider the projective space $\mathbb{K}\mathrm{P}^n := (\mathbb{K}^{n+1} \setminus \{0\})/\sim$ where $\sim$ denotes the equivalence relation on $\mathbb{K}^{n+1} \setminus \{0\}$ that is given by multiplication by an element in $\mathbb{K} \setminus \{0\}$. The map[2830]

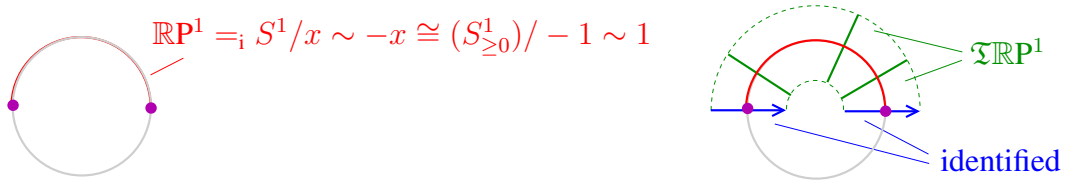
$$\begin{array}{rcl} p\colon \mathfrak{T}\mathbb{K}\mathrm{P}^n := \{([v], \lambda \cdot v) \in \mathbb{K}\mathrm{P}^n \times \mathbb{K}^{n+1} \mid v \in \mathbb{K}^{n+1} \setminus \{0\}, \lambda \in \mathbb{K}\} & \to & \mathbb{K}\mathrm{P}^n \\ ([v], w) & \mapsto & [v] \end{array}$$

defines a one-dimensional smooth $\mathbb{K}$-vector bundle over $\mathbb{K}\mathrm{P}^n$. We refer to it as the tautological $\mathbb{K}$-line bundle over $\mathbb{K}\mathrm{P}^n$.

(2) In the Infinite Projective Space-Sphere Quotient Lemma 104.22 we gave an explicit homeomorphism[2831]

$$\mathbb{K}\mathrm{P}^\infty := \operatorname{colim} \mathbb{K}\mathrm{P}^n \xrightarrow{\cong} (\mathbb{K}^\infty \setminus \{0\})/(\mathbb{K} \setminus \{0\})$$

which we use to identify these two topological spaces. The same way as in (1) we can now define a $\mathbb{K}$-line bundle $\mathfrak{T}\mathbb{K}\mathrm{P}^\infty \to \mathbb{K}\mathrm{P}^\infty$ which is called the tautological $\mathbb{K}$-line bundle over $\mathbb{K}\mathrm{P}^\infty$.



Proof.

(1) Let $n \in \mathbb{N}_0$. We need to show that $p\colon \mathfrak{T}\mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^n$ is a one-dimensional $\mathbb{K}$-vector bundle. First, given any point $b \in \mathbb{K}^{n+1} \setminus \{0\}$ we equip for $[b] \in \mathbb{K}\mathrm{P}^n$ the preimage $p^{-1}([b]) = \{[b]\} \times \mathbb{K} \cdot b$ with the obvious $\mathbb{K}$-vector space structure.

Next let $i \in \{0, \dots, n\}$. As in the proof of the Projective Space–Manifold Proposition 25.13 we consider the open subset $U_i := \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\} \subseteq \mathbb{K}\mathrm{P}^n$. Next we consider the maps

$$\begin{array}{rcl} \Phi_i\colon p^{-1}(U_i) & \to & U_i \times \mathbb{K} \\ ([z_0 : \cdots : z_n], (v_0, \dots, v_n)) & \mapsto & ([z_0 : \cdots : z_n], v_i) \end{array}$$

and

$$\begin{array}{rcl} \Psi_i\colon U_i \times \mathbb{K} & \to & p^{-1}(U_i) \\ ([z_0 : \cdots : z_n], w) & \mapsto & \big([z_0 : \cdots : z_n], w \cdot (\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i})\big). \end{array}$$

One can easily verify that both maps are well-defined, that they are continuous and that they are inverses of one another. It follows that the maps Φ_i are local trivializations of a one-dimensional $\mathbb{K}$-vector bundle. Since $U_0 \cup \cdots \cup U_n = \mathbb{K}\mathrm{P}^n$ we have shown that we have a $\mathbb{K}$-line bundle.

(2) The proof that $\mathfrak{T}\mathbb{K}\mathrm{P}^\infty \to \mathbb{K}\mathrm{P}^\infty$ is a $\mathbb{K}$-line bundle is essentially the same as the proof in (1), except that now we have a local trivialization for each $i \in \mathbb{N}$. We leave it to the reader to provide full details. ∎

Example. By now we have two potentially interesting real line bundles, namely the Möbius vector bundle and the tautological line bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$. For the reader's convenience we recall the definitions:

$$\begin{array}{rcl} ([0,1] \times \mathbb{R})/(0,\lambda) \sim (1,-\lambda) & \to & S^1 \\ [(t,\lambda)] & \mapsto & \exp(2\pi \mathrm{i} t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathfrak{T}\mathbb{R}\mathrm{P}^1 & \to & \mathbb{R}\mathrm{P}^1 \\ ([v], \lambda \cdot v) & \mapsto & [v]. \end{array}$$

[2831] The notation $\mathfrak{T}\mathbb{R}\mathrm{P}^n$ for the tautological real line bundle comes out of desperation: The prettier notation $\mathrm{T}\mathbb{R}\mathrm{P}^n$ will stand for the tangent bundle of $\mathbb{R}\mathrm{P}^n$ which we will introduce in Definition 265.11. The notation $\tau\mathbb{R}\mathrm{P}^n$, which sometimes gets used in the literature, is not ideal either, since τ has too little optical punch, so it is easy to overlook τ.

Recall that by the Projective Space-Avatar Lemma 9.16 we know that S^1 is homeomorphic to $\mathbb{R}\mathrm{P}^1$. The following diagram shows that this homeomorphism induces an isomorphism

$$\begin{array}{ccc} ([0,1]\times\mathbb{R})/(0,\lambda)\sim(1,-\lambda) & \xrightarrow{[(t,\lambda)]\mapsto([\cos(\pi t):\sin(\pi t)],\lambda\cdot(\cos(\pi t),\sin(\pi t)))} & \mathfrak{T}\mathbb{R}\mathrm{P}^1 \\ \downarrow{\scriptstyle [(t,\lambda)]\mapsto\exp(2\pi\,\mathrm{i}\,t)} & & \downarrow{\scriptstyle ([v],\lambda\cdot v)\mapsto[v]} \\ S^1 & \xrightarrow{(\cos(t),\sin(t))\mapsto[\cos(2t):\sin(2t)]} & \mathbb{R}\mathrm{P}^1 \end{array}$$

of real line bundles. □

265.2. **Smooth vector bundles.** In the following we will often consider vector bundles over smooth manifolds. For such vector bundles it is often useful to add some smoothness conditions to the recipe. This desire leads us to the following definition which combines aspects of Definitions 240.8, 243.1 and 265.1:

Definition 265.7. Let $n\in\mathbb{N}_0$.

(1) A smooth n-dimensional real vector bundle consists of the following two pieces of data:
   (a) A smooth map $p\colon E\to B$ between smooth manifolds,
   (b) a real vector space structure on each $p^{-1}(b)$, $b\in B$,
   such that the following condition is satisfied: For every $b\in B$ there exists an open neighborhood U of $b\in B$ and a diffeomorphism[2832] $\Phi\colon p^{-1}(U)\to U\times\mathbb{R}^n$ such that the following two conditions are satisfied:
   - The following diagram commutes:
   $$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & U\times\mathbb{R}^n \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (y,v)\mapsto y} & \\ & U. & \end{array}$$
   - For each $y\in U$ the map
   $$p^{-1}(y) \xrightarrow{\Phi|_{p^{-1}(y)}} \{y\}\times\mathbb{R}^n \xrightarrow{(y,v)\mapsto v} \mathbb{R}^n$$
   is an isomorphism of real vector spaces.

(2) Exactly the same way as in (1) we also define the notion of a smooth n-dimensional complex vector bundle.

Example. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ and let $n\in\mathbb{N}_0$. We consider the map

$$\begin{array}{rcl} p\colon \mathfrak{T}\mathbb{K}\mathrm{P}^n := \{([v],\lambda\cdot v)\in\mathbb{K}\mathrm{P}^n\times\mathbb{K}^{n+1}\mid v\in\mathbb{K}^{n+1}\setminus\{0\},\lambda\in\mathbb{K}\} & \to & \mathbb{K}\mathrm{P}^n \\ ([v],w) & \mapsto & [v] \end{array}$$

from the Tautological Line Bundle Lemma 265.6. The proof of that lemma implicitly equips $\mathfrak{T}\mathbb{K}\mathrm{P}^n$ with the structure of a smooth manifold and the proof implicitly shows that the map $p\colon\mathfrak{T}\mathbb{K}\mathrm{P}^n\to\mathbb{K}\mathrm{P}^n$ is in fact a *smooth* $\mathbb{K}$-line bundle. □

We continue with the following obvious variation on Definition 265.2:

Definition 265.8. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let $p\colon E\to B$ and $q\colon F\to C$ be two *smooth* $\mathbb{K}$-vector bundles. A morphism between these smooth $\mathbb{K}$-vector bundles consists of two

[2832] Here we use the Smooth Manifold Product Proposition 25.34 to view $U\times\mathbb{R}^n$ as a smooth manifold.

smooth maps $h\colon B \to C$ and $H\colon E \to F$ such that the following two conditions are satisfied:

(1) The following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C. \end{array}$$

(2) For each $b \in B$ the map $H|_{E_b}\colon E_b \to F_{h(b)}$ is an isomorphism of $\mathbb{K}$-vector spaces.

If $B = C$ and $h = \mathrm{id}_B$, then we call it a morphism over B.

Evidently one can show, as in the Vector Bundle–Category Lemma 265.3, that smooth vector bundles of a fixed dimension form a category. This leads us to the following definition:

Definition 265.9. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$.

(1) The class of smooth n-dimensional $\mathbb{K}$-vector bundles form a subcategory of the category $\mathcal{VectBund}_{\mathbb{K}^n}$ of n-dimensional $\mathbb{K}$-vector bundles. We denote this subcategory by $\mathcal{VectBund}^{\mathrm{sm}}_{\mathbb{K}^n}$.

(2) For a smooth manifold B we denote by $\mathcal{VectBund}_{\mathbb{K}^n}(B)$ the category of smooth n-dimensional $\mathbb{K}$-vector bundles over B.

265.3. **The tangent and the normal vector bundle.** In the following two subsections we will discuss two main examples of smooth vector bundles. More precisely, we will associate to a smooth manifold M the tangent bundle $\mathrm{T}M$ and to a smooth submanifold M of some $\mathbb{R}^k$ we will associate the normal vector bundle $\mathrm{N}M$.

265.3.1. **The tangent bundle.** The following proposition shows that to any abstract smooth manifold we can associate a vector bundle:

Proposition 265.10. (Tangent Bundle Proposition)

(1) Let M be an m-dimensional submanifold of $\mathbb{R}^k$. The subset[2833]

$$\mathrm{T}M \coloneqq \{(P, v) \in M \times \mathbb{R}^k \mid P \in M \text{ and } v \in \mathrm{T}_P M\} \subseteq \mathbb{R}^k \times \mathbb{R}^k$$

is a $2m$-dimensional submanifold of $\mathbb{R}^k \times \mathbb{R}^k$ and the map

$$\begin{array}{rcl} p\colon \mathrm{T}M \coloneqq \{(P, v) \in M \times \mathbb{R}^k \mid P \in M \text{ and } v \in \mathrm{T}_P M\} & \to & M \\ (P, v) & \mapsto & P \end{array}$$

is a smooth m-dimensional real vector bundle over M.

(2) Let M be an m-dimensional manifold. The set[2834]

$$\mathrm{T}M \coloneqq \bigsqcup_{P \in M} \mathrm{T}_P M$$

has a natural structure of a $2m$-dimensional smooth manifold such that the map

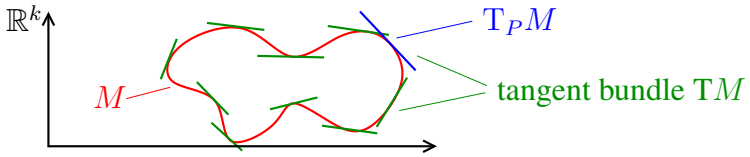
$$\begin{array}{rcl} p\colon \mathrm{T}M \coloneqq \bigsqcup_{P \in M} \mathrm{T}_P M & \to & M \\ (P, v) & \mapsto & P \end{array}$$

is a smooth m-dimensional real vector bundle over M.[2835]

(3) Given a submanifold $M \subseteq \mathbb{R}^k$ the Tangent Space-Isomorphism Proposition 29.24 gives us a natural isomorphism from the tangent space $\mathrm{T}_P M \subseteq \mathbb{R}^k$ to the "abstract" tangent

space $\mathrm{T}_P M$. These natural isomorphisms define an isomorphism from the smooth vector bundle in (1) to the smooth vector bundle in (2).

If we start out with a complex manifold, in the sense of Definition 40.2, then the above give rise to smooth complex vector bundles.



Definition 265.11. Let M be a smooth manifold. In the following we refer to the vector bundle[2837] $p\colon \mathrm{T}M \to M$ from the Tangent Bundle Proposition 265.10 (2) as the tangent bundle of M. If M is a submanifold of some $\mathbb{R}^k$, then we use the Tangent Bundle Proposition 265.10 (3) to identify the tangent bundle with the more hands-on vector bundle from the Tangent Bundle Proposition 265.10 (1).

Remark.

(1) To the complex projective plane $\mathbb{C}\mathrm{P}^1$ we can now associate two potentially interesting 1-dimensional complex vector bundles, namely the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ and the tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. In the Chern Class Characterization Proposition 274.7 we will discuss the question whether these two complex vector bundles are isomorphic.

(2) In the Tangent Bundle Proposition 265.10 we just showed that if a smooth manifold admits in fact a holomorphic atlas, then the tangent bundle admits the structure of a complex vector bundle. This motivates the following definition: If M is a smooth manifold, then a smooth $\mathbb{C}$-*vector bundle* structure on the tangent space $\mathrm{T}M$ of a smooth manifold M is called an *almost complex structure*. The Newlander–Nirenberg Theorem, see [**NN1957**] and [**Hör1990b**, Section 5.7], gives a complete criterion for whether or not an almost complex vector bundle arises from a complex manifold structure on the smooth manifold M. □

Sketch of proof.

[2834]Here we denote by $\mathrm{T}_P M$ the "physical tangent space", which we introduced Definition 29.1. Back in the day we denoted this physical vector space by $\widetilde{\mathrm{T}}_P M$ to distinguish it from the "abstract tangent space" which we introduced in Definition 29.11. In the Tangent Space-Isomorphism Proposition 29.24 we showed that these two tangent spaces are naturally isomorphic. We will use this fact as an excuse to use the notation $\mathrm{T}_P M$ for both tangent spaces. We leave it to the reader to figure out any given moment what interpretation we might mean.

[2835]Given a family $\{A_i\}_{i\in I}$ of sets we defined in Notation 1.12 the disjoint union as

$$\bigsqcup_{i\in I} A_i \coloneqq \bigcup_{i\in I} (\{i\} \times A_i).$$

[2836]Note that Definition 265.1 (1a) together with Notation 1.12 leads to the hilarious equality $\mathrm{T}M_P = \{P\} \times \mathrm{T}_P M$.

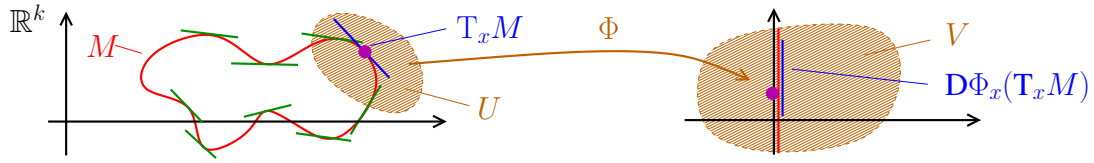
[2837]We introduced the topological space $\mathrm{T}M$ in Notation 53.16 where we referred to it as the *total tangent space of* M.

(1) Let M be an m-dimensional submanifold of $\mathbb{R}^k$.

(a) We already showed in the Physical Tangent Bundle Lemma 53.17 that $\mathrm{T}M$ is an m-dimensional smooth submanifold of $\mathbb{R}^k \times \mathbb{R}^k$. For the reader's convenience we now recall the elementary argument:
Let $(y, w) \in \mathrm{T}M$. We pick a submanifold chart $\Phi\colon U \to V$ for $y \in M$. One can easily verify that the map

$$\begin{aligned} \widetilde{\Phi}\colon U \times \mathbb{R}^k &\to V \times \mathbb{R}^k \\ (x, v) &\mapsto (\Phi(x), \mathrm{D}\Phi_x(v)) \end{aligned}$$

is, after a permutation of coordinates, a submanifold chart for $(y, w) \in \mathrm{T}M$. Thus we have shown that $\mathrm{T}M$ is a $2k$-dimensional smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.



(b) We equip each $p^{-1}(P) = \{P\} \times \mathrm{T}_P M$ with the obvious real vector space structure. We need to show that $p\colon \mathrm{T}M \to M$ is a smooth m-dimensional real vector bundle. Thus let $P \in M$. We pick a submanifold chart $\Phi\colon U \to V$ for $P \in M$. We denote by $\pi\colon \mathbb{R}^k \to \mathbb{R}^m$ the projection that is given by $(x, y) \mapsto x$. One can easily verify that the map

$$\begin{aligned} \widetilde{\Phi}\colon p^{-1}(U) &\to U \times \mathbb{R}^m \\ (x, v) &\mapsto (x, \pi(\underbrace{\mathrm{D}\Phi_x(v)}_{\in \mathbb{R}^k})) \end{aligned}$$

is a diffeomorphism and a trivialization of the vector bundle.

(2) Let M be an m-dimensional smooth manifold with smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i \in I}$.

(a) For each $i \in I$ we consider the map

$$\begin{aligned} \widetilde{\Phi}_i\colon p^{-1}(U_i) = \bigsqcup_{P \in U_i} (\{P\} \times \mathrm{T}_P M) &\to \overbrace{V_i \times \mathbb{R}^m}^{\subseteq \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}} \\ (P, v) &\mapsto (\Phi_i(P), \underbrace{\mathrm{D}(\Phi_i)_P(v)}_{\in \mathrm{T}_{\Phi_i(P)} V_i = \mathbb{R}^m}). \end{aligned}$$

We equip $\mathrm{T}M$ with the topology that is generated by all preimages of the open subsets of $V_i \times \mathbb{R}^m$ under the map $\widetilde{\Phi}_i$. It is elementary (but tedious) to verify that this topology turns $\mathrm{T}M$ into a second countable Hausdorff space such that all the maps $\widetilde{\Phi}_i\colon p^{-1}(U_i) = \bigsqcup_{P \in U_i} (\{P\} \times \mathrm{T}_P M) \to V_i \times \mathbb{R}^m$ are homeomorphisms and such that all transition maps between the charts $\widetilde{\Phi}_i$ are smooth. This shows that $\{\widetilde{\Phi}_i\colon p^{-1}(U_i) = V_i \times \mathbb{R}^m\}_{i \in I}$ is a smooth atlas for $\mathrm{T}M$. This equips $\mathrm{T}M$ with the structure of a smooth manifold.

(b) Again we equip each $p^{-1}(P) = \{P\} \times \mathrm{T}_P M$ with the obvious real vector space structure. We would like to show that $p\colon \mathrm{T}M \to M$ is an m-dimensional real vector bundle. One can easily verify that with respect to the above vector space

structures the maps

$$\begin{aligned} p^{-1}(U_i) = \bigsqcup_{P\in U_i} (\{P\} \times \mathrm{T}_P M) &\to U_i \times \mathbb{R}^m \\ (P, v) &\mapsto (P, \underbrace{\mathrm{D}(\Phi_i)_P(v)}_{\in \mathrm{T}_{\Phi_i(P)} V_i = {}_{\mathrm{i}}\mathbb{R}^m}). \end{aligned}$$

become smooth trivializations.

(3) Now let M again be an m-dimensional submanifold of $\mathbb{R}^k$. Using the natural isomorphism between the two flavors of the tangent space, provided by the Tangent Space-Isomorphism Proposition 29.24, one can show fairly easily that the two tangent bundles that we defined above are isomorphic. In practice we will not distinguish between these two notions.

Finally assume that we are given an n-dimensional complex manifold. In the Complex Manifold-Tangent Space Lemma 40.14 we equipped each tangent space with a natural complex structure. Using the Smooth-Complex Differentiability Proposition 39.22 one can easily verify that these complex vector space structures turn, in both settings, $\mathrm{T}M$ into a smooth n-dimensional complex vector bundle. ■

The following lemma shows that the tangent bundle is in a suitable sense functorial:

Lemma 265.12. (Tangent Bundle–Induced Morphism Lemma)

(1) Let $f\colon M \to N$ be a smooth map between smooth manifolds.
   (a) The map
   $$\begin{aligned} f_*\colon \mathrm{T}M &\to \mathrm{T}N \\ (P, v) &\mapsto (f(P), \mathrm{D}f_P(v)) \end{aligned}$$
   is a smooth map between smooth manifolds.
   (b) If $f\colon M \to N$ has the property, that for each $x \in M$ the corresponding differential $\mathrm{D}f_x\colon \mathrm{T}_x M \to \mathrm{T}_{f(x)} N$ is an isomorphism, then the maps
   $$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{(P,v)\mapsto (f(P),\mathrm{D}f_P(v))} & \mathrm{T}N \\ \downarrow{\scriptstyle (P,v)\mapsto P} & & \downarrow{\scriptstyle (Q,w)\mapsto Q} \\ M & \xrightarrow{f} & N \end{array}$$
   form a morphism of smooth vector bundles.

(2) Let $f\colon M \to N$ be a local diffeomorphism between two m-dimensional smooth manifolds. The maps from (1), duly reinterpreted in the abstract context, form a morphism of smooth vector bundles.

(3) The isomorphisms from the Tangent Bundle Proposition 265.10 (3) are natural.

Sketch of proof.

(1) Let $f\colon M \to N$ be a smooth map between smooth manifolds.
   (a) Using charts one can easily show (similar to the proof of the Manifold–Orientation Basics Lemma 31.31 (6)) that the map $f_*\colon \mathrm{T}M \to \mathrm{T}N$ is smooth.

(b) We now assume that for each $x \in M$ the differential $\mathrm{D}f_x \colon \mathrm{T}_x M \to \mathrm{T}_{f(x)} N$ is an isomorphism. We consider the diagram

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow[f_*]{(P,v)\mapsto (f(P),\mathrm{D}f_P(v))} & \mathrm{T}N \\ \downarrow{\scriptstyle (P,v)\mapsto P} & & \downarrow{\scriptstyle (Q,w)\mapsto Q} \\ M & \xrightarrow{f} & N. \end{array}$$

It is clear that the diagram commutes. By (a) we know that the top horizontal map $f_* \colon \mathrm{T}M \to \mathrm{T}N$ is smooth. It follows immediately from our hypothesis that the maps on the fibers are isomorphisms. It follows easily from these observations that the diagram defines a morphism of smooth vector bundles.

(2) The proof of this statement is almost identical to the proof of Statement (1). We just need to make a few minor changes, for example we need to replace the Physical Tangent Space-Inclusion Proposition 29.5 (3) by the Tangent Space-Inclusion Proposition 29.20 (3). We leave it to the reader to fill in the details.

(3) This statement follows easily from the definitions. ■

Tangent bundles are one of the main sources of vector bundles. This explains why some concepts for vector bundles have separate names in the context of tangent bundles. For example we will make use of the following standard definition:

Definition 265.13. We say that a smooth manifold M is parallelizable if its tangent bundle $\mathrm{T}M$ is trivializable.

Example.

(1) Let U be an open subset of $\mathbb{R}^n$ or of $\mathbb{H}^n$. In the setting of the Tangent Bundle Proposition 265.10 it follows from the elementary Physical Tangent Spaces–Lemma 29.2 (1) that the tangent bundle $\mathrm{T}U$ is isomorphic to the trivial vector bundle $U \times \mathbb{R}^n$. In other words, U is parallelizable.

(2) The circle S^1 is parallelizable. Indeed, using the Physical Tangent Spaces–Lemma 29.2 one can easily verify that the map

$$\begin{array}{rcl} S^1 \times \mathbb{R} & \to & \mathrm{T}S^1 \\ ((x,y),\lambda) & \mapsto & ((x,y),\lambda\cdot(-y,x)) \end{array}$$

is an isomorphism of smooth real vector bundles.

(3) For any $n \in \mathbb{N}$ the n-dimensional torus $(S^1)^n =_{\mathrm{i}} \mathbb{R}^n/\mathbb{Z}^n$ is parallelizable: In fact if we denote by $q \colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection, then an isomorphism of smooth vector bundles over $\mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\begin{array}{rcl} \varphi \colon (\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^n & \to & \mathrm{T}(\mathbb{R}^n/\mathbb{Z}^n) \\ (q(b),v) & \mapsto & (q(b), \underset{\uparrow}{\mathrm{D}q_b(v)}) \\ & & \mathrm{D}q_b \colon \mathbb{R}^n = \mathrm{T}_b\mathbb{R}^n \to \mathrm{T}_{q(b)}(\mathbb{R}^n/\mathbb{Z}^n) \end{array}$$

□

In fact we can easily generalize the above example: Recall that according to Definition 34.2 a *Lie group* is a smooth manifold M together with a group structure such that the two maps

$$\begin{array}{rcl} M \times M & \to & M \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} M & \to & M \\ x & \mapsto & x^{-1} \end{array}$$

are smooth. We recall some examples:

- On page 873 we pointed out that the tori $(S^1)^n$ and $\mathbb{R}^n/\mathbb{Z}^n$ have obvious Lie group structures.
- More interestingly we showed in the Matrix–Lie Group Lemma 34.3 that the matrix groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are Lie groups.
- As we saw in Exercise 171.3 quaternion multiplications turns S^3 into a Lie group.

Lemma 265.14. (Lie Group–Parallelizability Lemma) Every Lie group is parallelizable.

Proof. Let M be an m-dimensional Lie group with trivial element e. Given $g \in M$ we consider the map

$$\begin{array}{rcl} \lambda_g \colon M & \to & M \\ x & \mapsto & g \cdot x \end{array}$$

that is given by left multiplication. We consider the map

$$\begin{array}{rcl} M \times \mathrm{T}_e M & \to & \mathrm{T}M \\ (g, v) & \mapsto & (g, (\mathrm{D}\lambda_g)_e(v)). \end{array}$$

Using the fact that the Lie group multiplication is smooth one can show, basically by the argument of the Lie Group–Orientable Lemma 34.4, that this map is smooth. In particular it is an isomorphism of smooth vector bundles. Since the left hand side is isomorphic to the trivial vector bundle $M \times \mathbb{R}^m$ we have shown that the tangent bundle $\mathrm{T}M$ is trivializable. ■

265.3.2. **The normal vector bundle.** In the following proposition we will see that to a smooth submanifold of some $\mathbb{R}^k$ we cannot only associate the tangent bundle $\mathrm{T}M$ but also the normal vector bundle $\mathrm{N}M$:

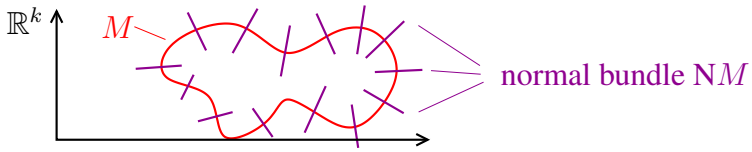
Proposition 265.15. (Normal Vector Bundle Proposition) Let M be an m-dimensional submanifold of $\mathbb{R}^k$. The subset

$$\mathrm{N}M \ := \ \{(P, v) \in M \times \mathbb{R}^k \,|\, P \in M \text{ and } v \in (\mathrm{T}_P M)^\perp\}$$

is a k-dimensional submanifold of $\mathbb{R}^k \times \mathbb{R}^k$ and the map

$$\begin{array}{rcl} \mathrm{N}M := \{(P, v) \in M \times \mathbb{R}^k \,|\, P \in M \text{ and } v \in (\mathrm{T}_P M)^\perp\} & \to & M \\ (P, v) & \mapsto & P \end{array}$$

is a smooth $(k-m)$-dimensional real vector bundle over M.



Definition 265.16. Let M be a smooth submanifold of $\mathbb{R}^k$. We refer to the real vector bundle $p \colon \mathrm{N}M \to M$ from the Normal Vector Bundle Proposition 265.15 as the normal vector bundle of $M \subseteq \mathbb{R}^k$.

Proof. [2838] Let M be an m-dimensional smooth submanifold of $\mathbb{R}^k$.

(a) First we show that $\mathrm{N}M$ is a k-dimensional submanifold of $\mathbb{R}^k \times \mathbb{R}^k$. Let $\Phi\colon U \to V$ be a submanifold chart of $M \subseteq \mathbb{R}^k$. One would be tempted to consider, as in the proof of the Tangent Bundle Proposition 265.10, the map

$$\begin{aligned} \widetilde{\Phi}\colon U \times \mathbb{R}^k &\to V \times \mathbb{R}^k \\ (x, v) &\mapsto (\Phi(x), \mathrm{D}\Phi_x(v)). \end{aligned}$$

The problem with this approach is, that the differential $\mathrm{D}\Phi_x(v)$ does not necessarily preserve orthogonality. Thus we need to work somewhat harder to make this approach work.

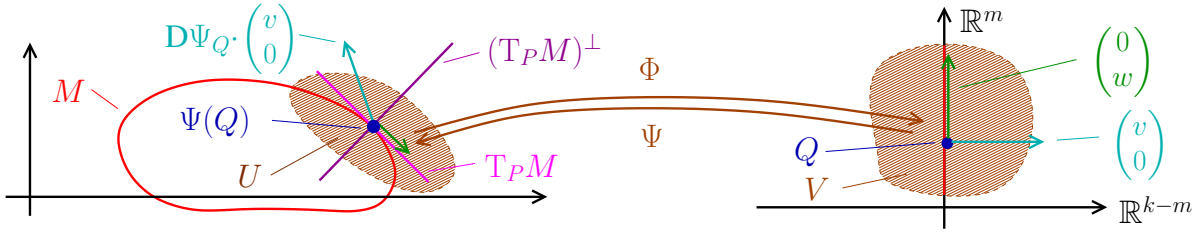
In the following we write $\Psi := \Phi^{-1}\colon V \to U$. As in the Matrix Groups Deformation Retract Proposition 19.4 we denote by $\mathrm{GL}(k,m)$ the set of m-tuples of linearly independent vectors in $\mathbb{R}^k$ and we denote by $\mathrm{O}(k,m)$ the set of m-tuples of orthonormal linearly independent vectors in $\mathbb{R}^k$. The Gram-Schmidt Orthogonalization Process 2.32 gives us a smooth retraction $R\colon \mathrm{GL}(k,m) \to \mathrm{O}(k,m)$. Next we introduce the map $\Theta\colon \mathrm{GL}(k,m) \times \mathbb{R}^k \to \mathbb{R}^k$ which is defined as the composition of the following two maps:[2839]

$$\begin{array}{rcccl} \mathrm{GL}(k,m) \times \mathbb{R}^k & \to & \mathrm{O}(k,m) \times \mathbb{R}^k & \to & \mathbb{R}^k \\ ((v_1,\dots,v_m), w) & \mapsto & (R(v_1,\dots,v_m), w) & & \\ & & (v_1,\dots,v_m, w) & \mapsto & \underbrace{w - \sum_{i=1}^{m} \langle w, v_i\rangle \cdot \frac{v_i}{\|v_i\|^2}}_{\text{orthogonal to } v_1,\dots,v_m} \end{array}$$

Next we consider the map

$$\begin{array}{rcl} V \times \mathbb{R}^{m-k} \times \mathbb{R}^k & \xrightarrow{G_\Psi} & U \times \mathbb{R}^k \\ (Q, v, w) & \mapsto & \Big(\Psi(Q), \underbrace{\Theta\Big(\underbrace{\big(\mathrm{D}\Psi_Q(\mathfrak{e}_{m-k+1}),\dots,\mathrm{D}\Psi_Q(\mathfrak{e}_n)\big)}_{\text{basis of } \mathrm{T}_{\psi(Q)}M}\Big)}_{\in (\mathrm{T}_{\psi(Q)}M)^\perp}, \mathrm{D}\Psi_Q \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}, \mathrm{D}\Psi_Q \cdot \begin{pmatrix} 0 \\ w \end{pmatrix}\Big). \end{array}$$

One can easily show that the map G_Ψ is a bijection and smooth. Furthermore using the Smooth Pasting Proposition 25.19 (2) one can show fairly easily that G_Ψ is in fact a diffeomorphism. Finally, once one has digested the various definitions, it is also quite clear that the $G_\Psi^{-1}\colon U \times \mathbb{R}^k \to V \times \mathbb{R}^k$ defines (after a suitable permutation of the coordinates) a submanifold chart for $\mathrm{N}M$. All the charts G_Ψ^{-1} show that $\mathrm{N}M$ is a k-dimensional submanifold of $\mathbb{R}^k \times \mathbb{R}^k$.



[2838] The following proof is mostly copy-pasted from the proof of the Normal Vectors Lemma 35.3.
[2839] Here $\langle\ ,\ \rangle\colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ denotes the usual Euclidean bilinear symmetric form on $\mathbb{R}^k$.

(b) Note that for each $P \in M$ we have $p^{-1}(P) = \{P\} \times (\mathrm{T}_P M)^\perp$. We give each such fiber the obvious vector space structure. A slight modification of the maps in (a) now shows that $p\colon \mathrm{N}M \to M$ is a smooth $(k-m)$-dimensional real vector bundle over M. ■

As for tangent bundles we have a special name for submanifolds of $\mathbb{R}^k$ with trivializable normal vector bundle:

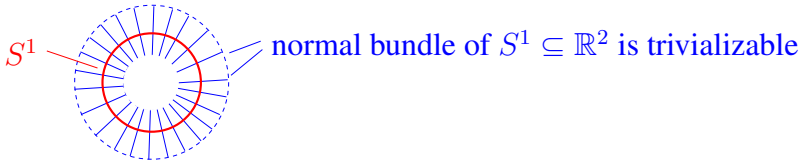
Definition 265.17. Let M be a smooth submanifold of $\mathbb{R}^k$. We say $M \subseteq \mathbb{R}^k$ is frameable if the normal vector bundle is trivializable.

Example.

(1) Let $n \in \mathbb{N}$. Using the Physical Tangent Spaces–Lemma 29.2, one can easily verify that the map

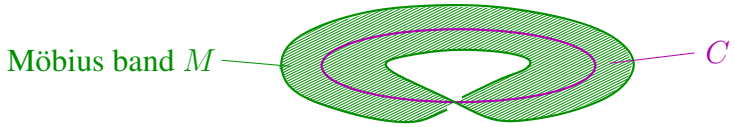
$$\begin{array}{rcl} S^n \times \mathbb{R} & \to & \mathrm{N}S^n \\ (P, \lambda) & \mapsto & (P, \lambda \cdot P) \end{array}$$

is an isomorphism of smooth vector bundles.



This shows that the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is frameable.

(2) We consider the Möbius band $M \subseteq \mathbb{R}^3$ as shown in the figure below.



We claim that the Möbius band is not frameable. This can be seen as follows: We consider the central curve $C \subseteq M$. One can easily show that the restriction of the normal vector bundle $p\colon \mathrm{N}M \to M$ to $p^{-1}(C) \to C$ is isomorphic to the Möbius vector bundle from page 5119. On page 5121 we saw that the Möbius vector bundle is not trivializable. It follows that the normal vector bundle $\mathrm{N}M \to M$ is not trivializable either. In other words, the Möbius band is not frameable. □

265.4. **Sections in vector bundles.** Now that we have discovered a never ending source of real vector bundles, let us return to developing the theory of vector bundles. As a next step we introduce the following versatile definition:

Definition 265.18.

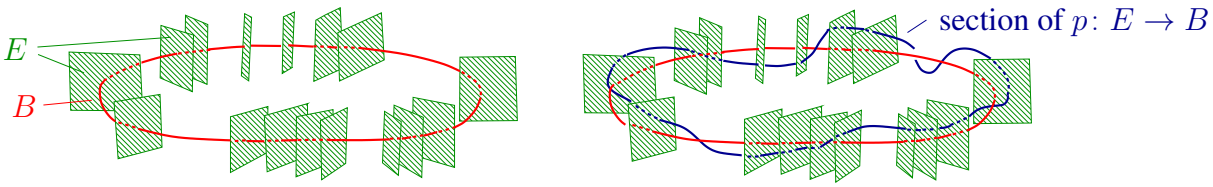
(1) A section of a vector bundle $p\colon E \to B$ is a map $s\colon B \to E$ such that $p \circ s = \mathrm{id}_B$.

(2) Given a (smooth) vector bundle $p\colon E \to B$ we write

$$\begin{array}{rcl} \Gamma(p\colon E \to B) & := & \text{set of continuous sections of } p\colon E \to B, \\ \Gamma^{\mathrm{sm}}(p\colon E \to B) & := & \text{set of smooth sections of a smooth vector bundle } p\colon E \to B. \end{array}$$

Example. Let M be a smooth manifold. We consider its tangent bundle $\mathrm{T}M \to M$. A (smooth/continuous) section

$$\begin{array}{rcl} M & \to & \mathrm{T}M \\ P & \mapsto & (P, s(P)) \end{array}$$



assigns to each $P \in M$ a tangent vector $s(P) \in \mathrm{T}_P M$. Thus we refer to such a section as a (smooth/continuous) vector field on M. A little bit of thought shows that this notion of a (smooth/continuous) vector field is essentially the same notion that we introduced in Definition 29.29. Furthermore given a submanifold M of $\mathbb{R}^n$, this notion of a (smooth/continuous) vector field corresponds in a natural way to the notion that we introduced in Definition 29.27. □

Lemma 265.19. (Vector Bundle-Section Lemma) Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth/continuous) $\mathbb{K}$-vector bundle.

(1) (a) Every continuous section $s\colon B \to E$ is a closed embedding.
   (b) Every smooth section $s\colon B \to E$ is a proper smooth embedding and the image $s(E)$ is a proper submanifold of E.[2840]

(2) (a) The map $B \to E$ that sends each $b \in B$ to the unique zero element of the $\mathbb{K}$-vector space $p^{-1}(b)$ is a (smooth/continuous) section. This section is called the 0-section of the vector bundle.[2841]
   (b) If $s, t\colon B \to E$ are two (smooth/continuous) sections, then the map $b \mapsto s(b) + t(b)$ is also a (smooth/continuous) section.
   (c) If $s\colon B \to E$ is a (smooth/continuous) section and $\lambda \in \mathbb{K}$, then the map $b \mapsto \lambda \cdot s(b)$ is also a (smooth/continuous) section.

We thus see that the set $\Gamma(p\colon E \to B)$ (respectively $\Gamma^{\mathrm{sm}}(p\colon E \to B)$) of (smooth/continuous) sections of the $\mathbb{K}$-vector bundle form a $\mathbb{K}$-vector space.

Sketch of proof.

(1) (a) First note that the restriction of the projection $p\colon E \to B$ to $s(B)$ is by definition a left inverse to the section $s\colon B \to E$. Since the projection is continuous we see that the section is an embedding. Furthermore, using the Cover-Open Subset Lemma 3.20 (1b) one can easily verify that the image of a section is a closed subset.
   (b) The logic of Statement (a) shows that smooth sections are smooth embeddings. Since $p^{-1}(\partial B) = \partial E$ we now see that $s\colon B \to E$ is a proper smooth embedding. Finally note that it follows from the Smooth Embedding Theorem 30.15 that the image $s(B)$ of the section is a proper smooth submanifold of E.

(2) We need to show that the given maps $B \to E$ are continuous (respectively smooth). By the Pasting Proposition 4.7 and the Smooth Pasting Proposition 25.19 we have to verify continuity (and smoothness) only locally. But locally the statement follows easily from the definition of a vector bundle. ■

[2840] In Definition 30.7 we introduced the notion of a proper map between smooth manifolds.

[2841] Sometimes we will use the image of the 0-section to view B as a subspace (or submanifold) of E.

Lemma 265.20. (Vector Bundle-Section Functorial Lemma) Given any morphism

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C \end{array}$$

of vector bundles we get an induced homomorphism

$$\begin{array}{rcl} h^*\colon \Gamma(F \to C) & \to & \Gamma(E \to B) \\ (t\colon C \to F) & \mapsto & \left(\begin{array}{ccc} B & \to & E \\ b & \mapsto & (H_{h(b)})^{-1}(t(h(b))) \end{array} \right). \end{array}$$

This map shows that the map that assigns to each vector bundle the corresponding vector space of sections is a contravariant functor from the category $\mathcal{VectBund}_{\mathbb{K}^n}$ of n-dimensional $\mathbb{K}$-vector bundles to the category $\mathcal{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces. An analogous statement holds of course for smooth sections of smooth vector bundles and smooth maps $h\colon B \to C$.

Proof. So suppose that we are given a morphism of n-dimensional $\mathbb{K}$-vector bundles as above. The only aspect one really needs to verify is the following claim (and its smooth analog):

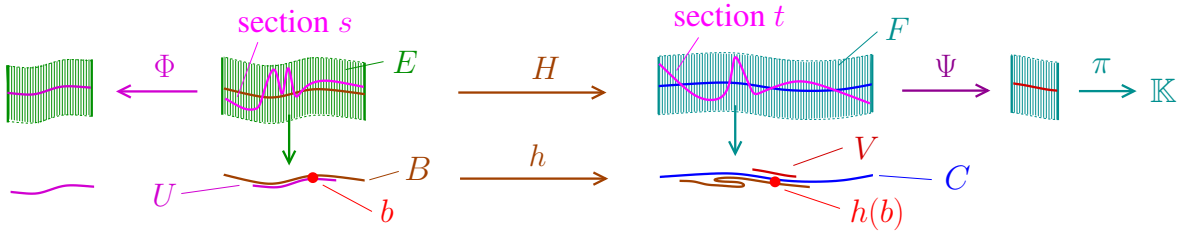
Claim. If $t\colon C \to F$ is a section, then the induced map

$$\begin{array}{rcl} s\colon B & \to & E \\ b & \mapsto & (H_{h(b)})^{-1}(t(h(b))) \end{array}$$

is continuous.

Proof. By the Pasting Proposition 4.7 it suffices to show that s is locally continuous. So let $b \in B$. We make a few preparations:

- By definition of a vector bundle there exists an open neighborhood U of b and a trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{K}^n$.
- By definition of a vector bundle there exists an open neighborhood V of $h(b)$ and a trivialization $\Psi\colon q^{-1}(V) \to V \times \mathbb{K}^n$.
- Note that $h^{-1}(V)$ is an open neighborhood of b. After replacing U by $U \cap h^{-1}(V)$ we can assume that $h(U) \subseteq V$.
- Let $\pi\colon V \times \mathbb{K}^n \to \mathbb{K}^n$ be the projection.



It follows easily from the definitions that the restriction of s to U is given by

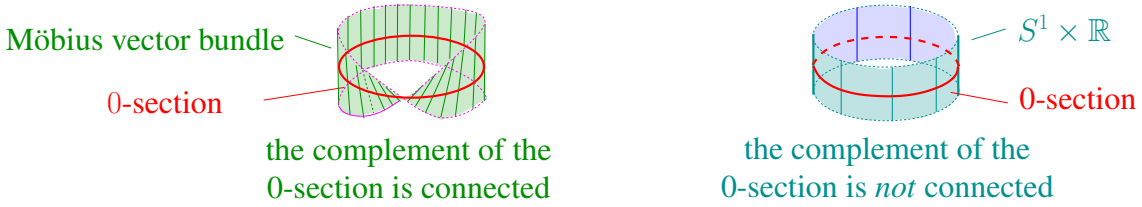
$$\begin{array}{rcl} s|_U\colon U & \to & p^{-1}(U) \\ x & \mapsto & \Phi^{-1}(x, \underbrace{\pi(\Psi(t(h(x))))}_{\in \mathbb{K}^n}). \end{array}$$

It is now clear that $s|_U$ is continuous. ∎

Example.

(1) We start out with a little observation: If two vector bundles are isomorphic, then it follows easily from the Vector Bundle-Section Functorial Lemma 265.20 that the complements of the 0-sections are homeomorphic. In particular if an m-dimensional $\mathbb{K}$-vector bundle $p\colon E \to B$ is trivializable, then the complement of the 0-section is homeomorphic to $(B \times \mathbb{K}^m) \setminus (B \times \{0\}) = B \times (\mathbb{K}^m \setminus \{0\})$. This observation can be a useful approach to showing that a vector bundle is *not* trivializable.

(2) We now apply the logic from (1) to the Möbius vector $p\colon M \to S^1$ bundle from page 5119.



Since the complement of the 0-section in the Möbius vector bundle $p\colon M \to S^1$ is connected, whereas $S^1 \times (\mathbb{R} \setminus \{0\})$ is not connected we see, once again, that the Möbius vector bundle is not trivializable. □

We move on to the next, almost self-explanatory definition:

Definition 265.21. We say that a section of a vector bundle $p\colon E \to B$ is nowhere vanishing if $s(b) \neq 0$ for all $b \in B$.

Lemma 265.22. (Trivializable–Vector Bundle Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Given any n-dimensional $\mathbb{K}$-vector bundle $p\colon E \to B$ the following two statements are equivalent:
   (a) The $\mathbb{K}$-vector bundle is trivializable.
   (b) There exist n sections $s_1, \ldots, s_n\colon B \to E$ such that for every $b \in B$ the vectors $s_1(b), \ldots, s_n(b) \in E_b := p^{-1}(b)$ form a basis of the $\mathbb{K}$-vector space $E_b = p^{-1}(b)$.

(2) A line bundle is trivializable if and only if it admits a nowhere vanishing section.

The statements also have obvious smooth analogs.

Sketch of proof.

(1) Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle. Suppose that there exists an isomorphism $\Phi\colon B \times \mathbb{K}^n \to E$ of $\mathbb{K}$-vector bundles over B. It is essentially clear that in this case the maps $s_i\colon b \mapsto \Phi(b, \mathfrak{e}_i)$, $i = 1, \ldots, n$, define sections such that for each $b \in B$ the vectors $s_1(b), \ldots, s_n(b) \in E_b = p^{-1}(b)$ form a basis of $E_b = p^{-1}(b)$. Conversely, if we are given such sections, then the map

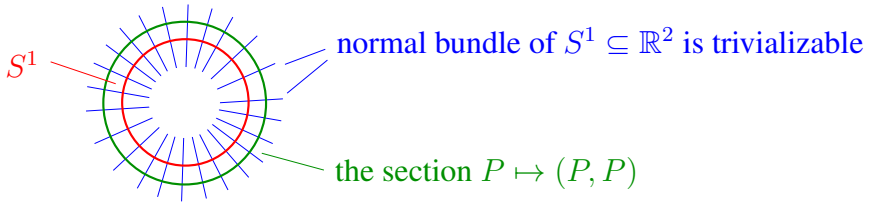
$$\begin{array}{rcl} B \times \mathbb{K}^n & \to & E \\ (b, (\lambda_1, \ldots, \lambda_n)) & \mapsto & \sum\limits_{i=1}^{n} s_i(b) \cdot \lambda_i \end{array}$$

is easily seen to be an isomorphism of $\mathbb{K}$-vector bundles over B.

(2) This statement is an immediate consequence of (1). ■

Examples.

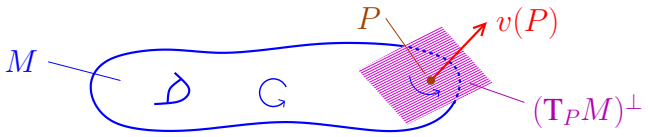
(1) We consider again the normal vector bundle of $S^n \subseteq \mathbb{R}^{n+1}$. This bundle admits the nowhere vanishing smooth section $P \mapsto (P, P)$. It follows from the Trivializable–Vector Bundle Lemma 265.22 (2) that the normal vector bundle of $S^n \subseteq \mathbb{R}^{n+1}$ is trivializable.



(2) We can easily generalize the previous example. More precisely, let $n \in \mathbb{N}$ and let M be an *oriented* submanifold of $\mathbb{R}^n$ of codimension one. It is elementary, albeit a little tedious, to show that the map

$$\begin{array}{rcl} v\colon M & \to & \mathrm{N}M \\ & & \text{the unique vector } v(P) \in \mathbb{R}^n \text{ of length one that is normal to } \mathrm{T}_P M \\ P & \mapsto & \text{and which, followed by a positive basis for } \mathrm{T}_P M, \\ & & \text{forms a positive basis for } \mathbb{R}^n \end{array}$$

is a nowhere vanishing smooth section of the normal vector bundle. Since the normal vector bundle is one-dimensional we obtain from the Trivializable–Vector Bundle Lemma 265.22 (2) that the normal vector bundle of $M \subseteq \mathbb{R}^n$ is trivializable.



(3) Recall that it follows from the Hairy Ball Theorem 46.18 that even-dimensional spheres do not admit nowhere vanishing vector fields. It follows from this observation and the Trivializable–Vector Bundle Lemma 265.22 (1) that even-dimensional spheres of dimension > 0 are not parallelizable. □

265.5. **Orientations of vector bundles.** Recall that in Definition 31.3 and Definition 31.4 we introduced the notion of an orientation of a finite-dimensional real vector space. Like most other concepts from linear algebra we can adapt this notion for real vector bundles:

Definition 265.23. [2842]

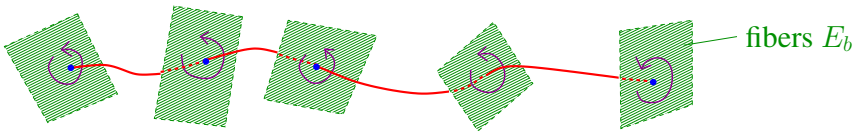
(1) Let $p\colon E \to B$ be an n-dimensional real vector bundle. An orientation for $p\colon E \to B$ is a map which assigns to each fiber E_b, $b \in B$, an orientation such that the following continuity condition is satisfied:

($*$) For every real vector bundle trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{R}^n$ and every component U' of U the map

$$p^{-1}(y) \xrightarrow{\Phi|_{p^{-1}(y)}} \{y\} \times \mathbb{R}^n \xrightarrow{(y,v) \mapsto v} \mathbb{R}^n$$

is either orientation-preserving for all $x \in U'$ or it is orientation-reversing for all $x \in U'$.

(2) A real vector bundle that admits an orientation is called orientable, otherwise we say that the real vector bundle is non-orientable.



Examples. Let $n \in \mathbb{N}_0$.

(1) Any trivial vector bundle $p\colon B \times \mathbb{R}^n \to B$ has an obvious natural orientation.
(2) In Exercise 267.2 we will see that the Möbius vector bundle $p\colon M \to S^1$ from page 5119 is not orientable.
(3) If we are given a real vector bundle $p\colon E \to B$ with an orientation, then by taking the opposite orientation on each fiber we get again an orientation of the real vector bundle, which is cleverly called the *opposite orientation*. □

The following definition is an analog of Definition 244.3:

Definition 265.24. Let $p\colon E \to B$ and $q\colon F \to C$ be oriented real vector bundles. A morphism oriented real vector bundles is a morphism

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C \end{array}$$

of real vector bundles such that for each $b \in B$ the isomorphism $H|_{p^{-1}(b)}\colon E_b \to F_{h(b)}$ is orientation-preserving. (Sometimes, for clarity we also say that the morphism is orientation-preserving.)

The next lemma shows that complex vector bundles are a rich source of orientable real vector bundles:

Lemma 265.25. (Complex Vector Bundle–Orientable Lemma)

(1) Let $p\colon E \to B$ be an n-dimensional complex vector bundle. If we view $p\colon E \to B$ as a $2n$-dimensional real vector bundle, then it comes with a natural orientation.
(2) Any morphism of complex vector bundles is a morphism of oriented real vector bundles.

Sketch of proof.

(1) We start out with the following observation: Let V be an n-dimensional complex vector space. If $(v_1, \dots, v_n)$ is a basis of V viewed as a complex vector space, then the set of vectors $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ is a basis of V viewed as a real vector space. The Complex Vector Space-Orientation Lemma 40.16 (1) says that the orientation defined by this basis is independent of the choice of the complex basis $(v_1, \dots, v_n)$. This shows that any complex vector space comes with a natural orientation.

Now let $p\colon E \to B$ be an n-dimensional complex vector bundle. We equip each fiber with the above orientation. It follows easily from the above and the definitions that this defines an orientation of $p\colon E \to B$ viewed as a $2n$-dimensional real vector bundle.
(2) This statement follows quite easily from the Complex Vector Space-Orientation Lemma 40.16 (1). ■

[2842]Note that this definition is consistent with the Definition 244.2 of an orientation of a $\overline{B}^n$-bundle.

Before we state the next lemma, let us introduce the following notation:

Notation 265.26. For a real vector bundle $p\colon E \to B$ we denote by $\mathrm{Or}(E)$ the set of orientations on E.

The following lemma can be viewed as an analog of the Vector Bundle-Section Functorial Lemma 265.20:

Lemma 265.27. (Vector Bundle-Orientation Functorial Lemma)

(1) Let

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C \end{array}$$

be a morphism of real vector bundles. Given any orientation on $q\colon F \to C$ there exists a unique orientation on $p\colon E \to B$ that turns the above morphism into a morphism of oriented real vector bundles.

(2) The maps

$$\begin{array}{rcl} (p\colon E \to F) & \mapsto & \mathrm{Or}(E) \\ (h\colon B \to C, H\colon E \to F) & \mapsto & h^*\colon \mathrm{Or}(F \to C) \to \mathrm{Or}(E \to B) \end{array}$$

from (1) and Notation 265.26 form a contravariant functor from the category $\mathcal{VectBund}_{\mathbb{R}^n}$ of n-dimensional real vector bundles to the category $\mathcal{Set}$ of sets.

Sketch of proof.

(1) Given an orientation of $F \to C$ we equip each fiber $E_b = p^{-1}(b)$ with the unique orientation that has the property that the isomorphism $H|_{E_b}\colon E_b \to F_{h(b)}$ is orientation-preserving. Similar to the proof of the Vector Bundle-Section Functorial Lemma 265.20 one can show that this defines an orientation of $E \to B$.

(2) This statement follows easily from the definitions. ■

Lemma 265.28. (Manifold Orientation-via-Tangent Bundle Lemma) Let M be a smooth manifold. Let $p\colon \mathrm{T}M \to M$ be the tangent bundle. In Definition 31.19 we introduced the notion of an orientation of M. Note that for each $x \in M$ be have a natural isomorphism $p^{-1}(x) = \{x\} \times \mathrm{T}_x M \to \mathrm{T}_x M$. These natural isomorphisms define a natural bijection between the set of orientations of the tangent bundle $\mathrm{T}M$ and the set of orientations of the smooth manifold M.

Proof. The two notions of an orientation are almost identical. So surely there is no need to provide any details. ■

Example. Let M be a complex manifold. By the Tangent Bundle Proposition 265.10 we know that the tangent bundle $\mathrm{T}M \to M$ is a complex vector bundle. By the Manifold Orientation-via-Tangent Bundle Lemma 265.28 we can equip $\mathrm{T}M$ with a natural orientation which, by the Manifold Orientation-via-Tangent Bundle Lemma 265.28, endows M with a natural orientation. It follows easily from the definitions that this orientation is the same as in the Complex Manifold-Orientation Proposition 40.18. □

By the Manifold Orientation-via-Tangent Bundle Lemma 265.28 we can view the following lemma as a mildly interesting generalization of the Manifold–Orientation Basics Lemma 31.31:

Lemma 265.29. (Vector Bundle–Orientation Basics Lemma) Let $p\colon E \to B$ be a real vector bundle.

(1) If B is connected, then two orientations agree if and only if they agree at a point.
(2) If B is connected and non-empty, then E is either non-orientable or it admits precisely two orientations.

Sketch of proof.

(1) Suppose we are given two orientations that agree at a $b \in B$. Let U be the set of all $x \in B$ where the orientations agree. As in the proof of the Manifold–Orientation Basics Lemma 31.31 (1) it follows easily from the definitions that U and $B \setminus U$ are open. Since by hypothesis $b \in U$ and since B is connected we see that $U = B$.
(2) The statement follows from (1) together with the following observation: If we are given an orientation, then the opposite orientation (which we introduced above on page 5136), gives us a second orientation. ■

We conclude this section with the following lemma, which we will use in the next section:

Lemma 265.30. (Orientable Total Space Lemma) Let $p\colon E \to B$ be a smooth real vector bundle. We consider the following:

(a) The base manifold B.
(b) The total manifold E.
(c) The vector bundle $p\colon E \to B$.

Orientations of any two out of three endow the third with a natural orientation.

Sketch of proof. We start out with a little observation: Let U and V be two finite-dimensional real vector spaces. We consider the real vector spaces $U, V, U \times V$. Elementary linear algebra shows that the orientations of any two out of these three vector spaces endow the third with a natural orientation. The lemma follows easily from this observation. We leave it to the reader to fill in the details. ■

265.6. **The Euler number of a smooth vector bundle.** The last section was arguably a little dull. The current section is much more interesting since we will get to know an interesting way to distinguish at least some smooth vector bundles via the Euler number.

To get to this point we first need to recall Definition 53.1:

Definition 265.31. Let W be an oriented smooth manifold.

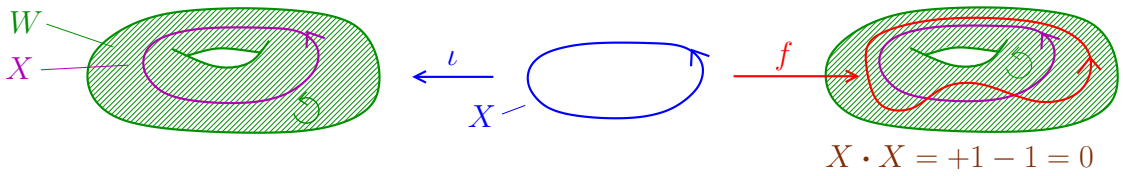
(1) Let X and Y be two closed oriented submanifolds of W of complementary dimension, i.e. we have $\dim(X) + \dim(Y) = \dim(W)$.
   (a) If X and Y are transverse, then we define the algebraic intersection number $X \cdot Y$ as in Definitions 51.3 and 262.1 by adding up signs of intersection points.
   (b) In the general case we do the following: Let $\iota\colon Y \to W$ be the inclusion map. By the Transversality Theorem 50.10 the map ι is smoothly isotopic to a smooth embedding $f\colon Y \to W$ such that X is transverse to $f(Y)$. It follows from the Intersection Number–Homotopy Invariance Theorem 51.9 that

$$X \cdot Y \;:=\; X \cdot f(Y) \;\in\; \mathbb{Z}$$

does not depend on the choice of f. We refer to $X \cdot Y$ as the algebraic intersection number of X and Y.

(2) If W is even-dimensional and if X is a closed oriented submanifold of W such that $2 \cdot \dim(X) = \dim(W)$, then in (1) we can take $Y = X$ and we refer to $X \cdot X \in \mathbb{Z}$ as the algebraic self-intersection number of X.

In the absence of orientations we obtain from Definition 51.12 well-defined algebraic (self-) intersection numbers $X \cdot_{\mathbb{Z}/2\mathbb{Z}} Y \in \mathbb{Z}/2\mathbb{Z}$ and $X \cdot_{\mathbb{Z}/2\mathbb{Z}} X \in \mathbb{Z}/2\mathbb{Z}$.



The above definition allows us to introduce the following interesting definition:

Definition 265.32. Let M be a closed oriented m-dimensional smooth manifold and let $p\colon E \to M$ be an oriented smooth m-dimensional real vector bundle over M. By the Vector Bundle-Section Lemma 265.19 the 0-section $s\colon M \to E$ is a smooth embedding. We use this fact to view M as a closed oriented smooth submanifold of E. As in the Orientable Total Space Lemma 265.30 we see that the orientations of M and of the real vector bundle $p\colon E \to M$ endow the total manifold E with an orientation.[2843] It thus makes sense to consider the self-intersection number $M \cdot M$ of $M \subseteq E$. We refer to

$$\chi(E) \;:=\; M \cdot M \;\in\; \mathbb{Z}$$

as the Euler number of the oriented smooth vector bundle $p\colon E \to M$. In the absence of orientations we still get a $\mathbb{Z}/2\mathbb{Z}$-valued Euler number $\chi_{\mathbb{Z}/2\mathbb{Z}}(E) := M \cdot_{\mathbb{Z}/2\mathbb{Z}} M \in \mathbb{Z}/2\mathbb{Z}$.

Before we consider examples, let us point out two basic facts about Euler numbers:

Lemma 265.33. (Euler Number Basics Lemma)

(1) Let M be a closed oriented m-dimensional smooth manifold and let $p\colon E \to M$ be an oriented smooth m-dimensional real vector bundle over M. If we equip E with the opposite orientation, or if we flip the orientation of M, then the sign of the Euler number flips.

(2) Let $q\colon F \to N$ be another oriented smooth m-dimensional real vector bundle over a closed oriented m-dimensional smooth manifold. If there exists an orientation-preserving isomorphism between these vector bundles, which also preserves the orientations of M and N, then $\chi(p\colon E \to M) = \chi(q\colon F \to N)$. An analogous statement also holds for the $\mathbb{Z}/2\mathbb{Z}$-Euler number.

Proof. Both statements follow easily from the definitions. We leave it to the reader to fill in the details. ∎

Example.

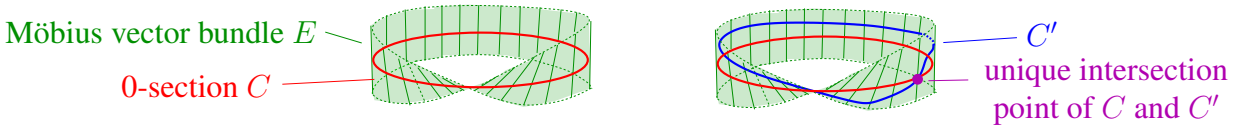
[2843] More precisely: We can and will pick the convention that we equip the smooth manifold E with the orientation which turns a trivialization $p^{-1}(U) \to U \times E$ (which is orientation-preserving on the fibers) into an orientation-preserving diffeomorphism.

(1) Let $m \in \mathbb{N}$ and let $p\colon E \to M$ be a trivializable oriented m-dimensional real vector bundle over a closed oriented m-dimensional smooth manifold M. Since $m > 0$ we obtain from the Trivializable–Vector Bundle Lemma 265.22 that there exists a nowhere vanishing smooth section $s\colon M \to E$. We see that

$$\chi(p\colon E \to M) \underset{\uparrow}{=} M \cdot s(M) \underset{\uparrow}{=} 0.$$

since s is nowhere vanishing we see that $s(M)$ is transverse to M with algebraic intersection number $= 0$

(2) We consider again the Möbius vector bundle $p\colon E \to C$ from page 5119.



We see that

$$\chi_{\mathbb{Z}/2\mathbb{Z}}(E \to C) \underset{\uparrow}{=} C \cdot_{\mathbb{Z}/2\mathbb{Z}} C \underset{\uparrow}{=} C \cdot_{\mathbb{Z}/2\mathbb{Z}} C' \underset{\uparrow}{=} 1 \in \mathbb{Z}/2\mathbb{Z}.$$

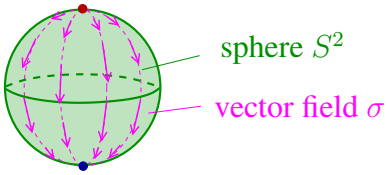
by Definition 265.32 — by Definition 265.31 — see the figure

This result together with (1) and the Euler Number Basics Lemma 265.33 (2) gives another proof that the Möbius vector bundle is not parallelizable.

(3) On the sphere S^n we consider the smooth vector field

$$\begin{aligned} \sigma\colon S^n &\to \mathbb{R}^{n+1} \\ x &\mapsto \underbrace{\mathfrak{e}_{n+1} - \langle x, \mathfrak{e}_{n+1}\rangle \cdot x}_{\text{orthogonal projection of } \mathfrak{e}_{n+1} \text{ to } \mathrm{T}_x S^n}. \end{aligned}$$

As we discussed on page 5132,courseNote that such a map defines a smooth section $\sigma\colon M \to \mathrm{T}M$.



In Exercise 265.24 we will see that $\sigma(S^n) \subseteq \mathrm{T}S^n$ is transverse to $S^n \subseteq \mathrm{T}S^n$ and we will see that there exist precisely two intersection points of $S^n, \sigma(S^n) \subseteq \mathrm{T}S^n$, and we will see that these have signs 1 and $(-1)^n$. Thus we see that

$$\chi(\mathrm{T}S^n) \underset{\uparrow}{=} S^n \cdot S^n \underset{\uparrow}{=} S^n \cdot \sigma(S^n) \underset{\uparrow}{=} 1 + (-1)^n.$$

by Definition 265.32 — by Definition 265.31 — by the above

This result together with (1) and the Euler Number Basics Lemma 265.33 (2) gives another proof that even-dimensional spheres are not parallelizable.

(4) Let $g \in \mathbb{N}_0$. In Exercise 265.25 we will see that on the surface Σ_g there exists a smooth vector field v such that for the corresponding section s we obtain (somewhat similar to (3)) that $\chi(\mathrm{T}\Sigma_g) = 2 - 2g$. As in (3) this shows that for $g \neq 1$ the surface Σ_g is not parallelizable. □

We continue with the following proposition, which in particular explains the name *Euler* number:

Proposition 265.34. (Euler Number-of-Tangent Bundle Proposition) Given any closed oriented m-dimensional smooth manifold we have

$$\chi(\mathrm{T}M) \;=\; \chi(M).$$

Proof. We start with a few preparations:

(1) We denote by $\Delta_M := \{(x,x) \in M \times M \,|\, x \in M\}$ the diagonal of M.
(2) In the Total Tangent Space-Diagonal Embedding Proposition 53.23 we showed that there exists an orientation-preserving smooth embedding $f\colon \mathrm{T}M \to M$ such that f restricts to an orientation-preserving diffeomorphism $M \to \Delta_M$.
(3) In the Lefschetz-Intersection Number Proposition 222.4 we showed, using a heroic calculation, that $Q_{M\times M}([\Delta_M],[\Delta_M]) = \chi(M)$.

We now see that

$$\chi(\mathrm{T}M) \underset{\text{by Definition 265.32}}{\overset{\uparrow}{=}} M \underset{\mathrm{T}M}{\cdot} M \underset{\substack{\text{by the Total Tangent Space-Diagonal}\\ \text{Embedding Proposition 53.23}}}{\overset{\uparrow}{=}} \Delta_M \underset{M\times M}{\cdot} \Delta_M \overset{\substack{\text{by the Cup Product–Intersection}\\ \text{Number Theorem 220.10}}}{\overset{\downarrow}{=}} Q_{M\times M}([\Delta_M],[\Delta_M]) \underset{\substack{\text{by the Lefschetz-Intersection}\\ \text{Number Proposition 222.4}}}{\overset{\uparrow}{=}} \chi(M).$$

■

Example. We now consider the parallelizability of compact connected 2-dimensional smooth manifolds:

(1) By the discussion on page 5128 we know that the torus $S^1 \times S^1$ is parallelizable.
(2) We now have many arguments why for $g \neq 1$ the surface Σ_g of genus g is not parallelizable:
   - On page 5135 we saw that S^2 is not parallelizable.
   - In the previous example we just showed that for $g \neq 1$ the surface Σ_g is not parallelizable.
   - On page 2987 we calculated that the Euler characteristic of the surface of genus g equals $\chi(\Sigma_g) = 2 - 2g$. It thus follows from the Euler Number-of-Tangent Bundle Proposition 265.34 that for any $g \neq 1$ the surface of genus g is non-parallelizable..
   - If $g \neq 1$, then we obtain from the Euler Characteristic–Vector Field Theorem 182.22, which is a generalization of the Hairy Ball Theorem 46.18, that Σ_g does *not* admit a nowhere vanishing vector field. By the Trivializable–Vector Bundle Lemma 265.22 this implies that Σ_g is not parallelizable.
(3) It follows from the Vector Bundle-Orientation Functorial Lemma 265.27 and the Manifold Orientation-via-Tangent Bundle Lemma 265.28 that non-orientable smooth manifolds are not parallelizable.
(4) In Exercise 265.21 we will show that compact connected 2-dimensional smooth manifolds with non-empty boundary are parallelizable. □

We conclude this section with a much more fancy theorem, which unfortunately we will not really be able to prove:

Theorem 265.35. (Sphere–Parallelizability Theorem) Let $n \in \mathbb{N}_0$. The sphere S^n is parallelizable if and only if $n = 0, 1, 3, 7$.

Proof. As we discussed above on page 5135 and 5140, it follows from the Euler Number-of-Tangent Bundle Proposition 265.34 that even-dimensional spheres of dimension > 0 are not parallelizable. In contrast note that it follows almost immediately from the definitions that S^0 is actually parallelizable.

We now turn to odd-dimensional spheres. Complex and quaternion multiplication turns S^1 and S^3 into Lie groups. It thus follows from the Lie Group–Parallelizability Lemma 265.14 that S^1 and S^3 are parallelizable. A little later, in the Parallelizability-of-$\mathbb{R}P^n$-S^n-Proposition 275.3 we will show that the octonions $\mathbb{O} = \mathbb{R}^8$, which we introduced in the Octonion Lemma 214.12, can be used to show that S^7 is also parallelizable.

By far the hardest aspect is to prove that for for n odd, only S^1, S^3 and S^7 are parallelizable. This statement was first proved by Michel Kervaire [**Kerv1958**, p. 280] and John Milnor [**Miln1958a**, Theorem 2], building on the Bott Periodicity Theorem 299.22 proved by Raoul Bott [**Bot1957, Bot1959**]. Full details for a proof can also be found in [**AtH1961a**] and [**Hat2**, Theorem 2.16].[2844] ■

265.7. **The pullback of vector bundles.** In this last section of this chapter we will introduce the notion of a pullback of a vector bundle. Most of the concepts and statements are very similar to what we covered in Sections 240.1 and 240.7.

The following lemma, which is closely related to the Bundle Map-Pullback Proposition 240.2 and the Bundle–Pullback Proposition 240.21 introduces the pullback of vector bundles:

Proposition 265.36. (Vector Bundle–Pullback Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle and let $h\colon C \to B$ be a continuous map. We consider the corresponding pullback diagram

$$\begin{array}{ccc} h^*E & \xrightarrow[=:H]{(c,y)\mapsto y} & E \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xrightarrow{\quad h \quad} & B \end{array}$$

from the Topological-Pullback Proposition 7.22.

(1) The map

$$\begin{array}{rccc} q\colon h^*E & := & \{(c,v)\,|\,c \in C \text{ and } v \in p^{-1}(h(c))\} & \to & C \\ & & (c,v) & \mapsto & c \end{array}$$

is also an n-dimensional $\mathbb{K}$-vector bundle where for each $c \in C$ the $\mathbb{K}$-vector space structure on $q^{-1}(c) = \{c\} \times p^{-1}(h(c)) \subseteq C \times E$ is the obvious $\mathbb{K}$-vector space structure coming from $p^{-1}(h(c))$. Furthermore the maps h and H define a morphism of $\mathbb{K}$-vector bundles.

(2) If we are given a morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B \end{array}$$

[2844] The fact that only S^0, S^1, S^3 and S^7 are parallelizable is related to the statement of the Division Algebra–Dimensions Theorem 214.14 which says that only in dimensions $n = 1, 2, 4$ and 8 do we have a division algebra on R^n.

of $\mathbb{K}$-vector bundles, then the map

$$\begin{aligned} T\colon F &\to h^*E \\ z &\mapsto (r(z), H(z)) \end{aligned}$$

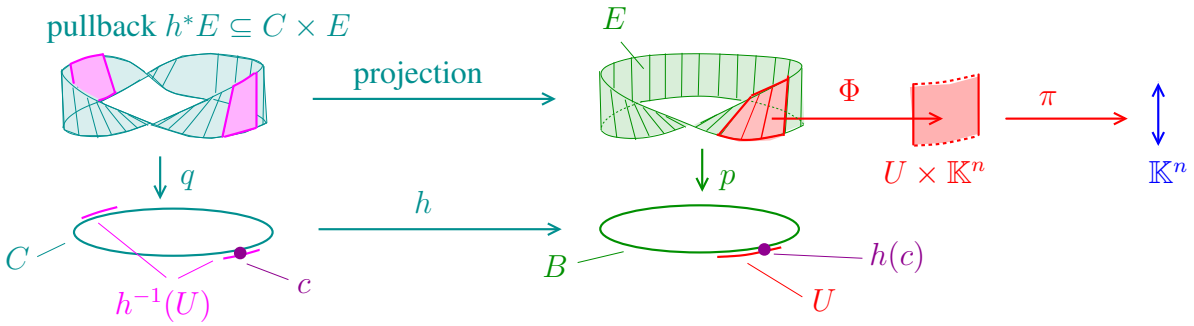
is a natural isomorphism of $\mathbb{K}$-vector bundles over C.

If we start out with a smooth vector bundle $p\colon E \to B$ and with a smooth map $h\colon C \to B$, then the resulting vector bundle $h^*E \to C$ is also smooth and we obtain a morphism of smooth vector bundles.

Definition 265.37. Using the notation from the Vector Bundle–Pullback Proposition 265.36 we refer to $h^*E \to C$ as the pullback of the $\mathbb{K}$-vector bundle $p\colon E \to B$ under the continuous (smooth) map $h\colon C \to B$.

Sketch of proof.

(1) Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle and let $h\colon C \to B$ be a continuous map. We need to show that $q\colon h^*E \to C$ is an n-dimensional $\mathbb{K}$-vector bundle. Thus let $c \in C$. By definition there exists a neighborhood $U \subseteq B$ of $h(c)$ and a local trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{K}^n$. We denote by $\pi\colon U \times \mathbb{K}^n \to \mathbb{K}^n$ the projection.



Next we consider the open neighborhood $V \coloneqq h^{-1}(U) \subseteq C$ of c and we consider the map

$$\begin{aligned} \Psi\colon q^{-1}(V) &\to V \times \mathbb{K}^n \\ (c, v) &\mapsto (c, \pi(\Phi(v))). \end{aligned}$$

As in the proof of the Bundle Map-Pullback Proposition 240.2 one can easily verify that this is a trivialization of the restriction of $q\colon h^*E \to C$ to V. This shows that the map $q\colon h^*E \to C$ is a $\mathbb{K}$-vector bundle. It is clear that the maps h and H define a morphism of $\mathbb{K}$-vector bundles.

(2) Suppose we are given a morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{h} & B \end{array}$$

of $\mathbb{K}$-vector bundles. We need to show that the map

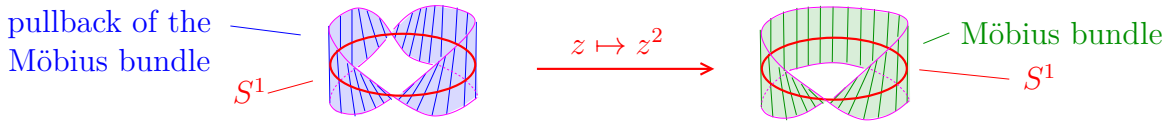
$$\begin{aligned} T\colon F &\to h^*E \\ z &\mapsto (r(z), H(z)) \end{aligned}$$

is a natural isomorphism of $\mathbb{K}$-vector bundles over C. The proof of this statement is very similar to the proof on page 4645 of the corresponding statement of the Bundle–Pullback Proposition 240.21. We leave it to the reader to make the necessary adjustments.

Finally we consider the case that we start out with a *smooth* vector bundle $p\colon E \to B$: First we need to equip h^*E with the structure of a smooth manifold. Since we already know that h^*E is a vector bundle over a smooth manifold we obtain from the elementary Exercise 265.2 that h^*E is Hausdorff and second-countable. Next note that one can easily verify that the trivializations in the proof of Statement (1) form a smooth atlas for h^*E. Finally note that this smooth structure on h^*E is chosen in such a way that the above trivializations of the vector bundle $h^*E \to C$ are in fact smooth. ■

Example.

(1) Let $p\colon M \to S^1$ be the Möbius vector bundle from page 5119. We denote by $h\colon S^1 \to S^1$ the map that is given by $h(z) = z^2$. In Exercise 265.20 we will see that the pullback of the Möbius vector bundle under h is a trivializable vector bundle.



(2) Let $q\colon E \to B$ be a $\mathbb{K}$-vector bundle, let $C \subseteq B$ be a subspace and let $\iota\colon C \to B$ be the inclusion. As on page 4647 one can easily verify that the maps

$$\begin{array}{ccc} \iota^*(E) & \to & q^{-1}(C) \\ (c,e) & \mapsto & e \end{array} \qquad \text{and} \qquad \begin{array}{ccc} q^{-1}(C) & \to & \iota^*(E) \\ e & \mapsto & (q(e),e) \end{array}$$

give natural isomorphisms between the pullback $\mathbb{K}$-vector bundle $\iota^*(E)$ and the restriction $\mathbb{K}$-vector bundle $q\colon q^{-1}(C) \to C$, as defined on page 5120.

(3) Let B be a topological space, let $p\colon B \times \mathbb{K}^n \to B$ be a trivial vector bundle and let $h\colon C \to B$ be a continuous map. As on page 4647 we see that the maps

$$\begin{array}{ccc} \overbrace{h^*(B\times\mathbb{K}^n)}^{\subseteq C\times(B\times\mathbb{K}^n)} & \to & C\times\mathbb{K}^n \\ (c,(b,f)) & \mapsto & (c,f) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C\times\mathbb{K}^n & \to & \overbrace{h^*(B\times\mathbb{K}^n)}^{\subseteq C\times(B\times\mathbb{K}^n)} \\ (c,f) & \mapsto & (c,(h(c),f)) \end{array}$$

define a natural isomorphism from the pullback bundle $h^*(B\times\mathbb{K}^n)$ to the trivial vector bundle $C \times \mathbb{K}^n$. □

The following lemma is an analog of the Bundle-Pullback-Functor Lemma 240.23:

Lemma 265.38. (Vector Bundle-Pullback-Functor Lemma) Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$. Furthermore let $n \in \mathbb{N}_0$. For any continuous map $h\colon C \to B$ between topological spaces the maps

$$\begin{array}{ccc} E & & h^*E \\ \downarrow{\scriptstyle p} & \mapsto & \downarrow{\scriptstyle q} \\ B & & C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array} \quad \mapsto \quad \begin{array}{ccc} h^*E & \xrightarrow{(c,y)\mapsto(c,\varphi(y))} & h^*E' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & C & \end{array}$$

define a covariant functor from the category $\mathcal{VectBund}_{\mathbb{K}^n}(B)$ of n-dimensional $\mathbb{K}$-vector bundles over B to the category of n-dimensional $\mathcal{VectBund}_{\mathbb{K}^n}(C)$ of $\mathbb{K}$-vector bundles over C.

Proof. As in the case of the Bundle-Pullback-Functor Lemma 240.23 the lemma follows easily from the definitions. In fact, in the proof of the Bundle-Pullback-Functor Lemma 240.23 we just need to replace the Bundle–Pullback Proposition 240.21 by the Vector Bundle–Pullback Proposition 265.36. ■

Example. Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$.

(1) It follows from the Vector Bundle-Pullback-Functor Lemma 265.38 together with the Isomorphism Preservation Lemma 5.26 that the pullbacks of isomorphic $\mathbb{K}$-vector bundles are again isomorphic.
(2) It follows from (1) and the fact that the pullback of a trivial vector bundle is naturally trivializable (see the discussion above) that the pullback of a trivializable $\mathbb{K}$-vector bundle is again a trivializable $\mathbb{K}$-vector bundle. □

The following lemma, which is basically just a slight variation on the Bundle-Pullback–Functorial Lemma 240.24, shows that the pullback of vector bundles is covariantly functorial in a suitable sense.

Lemma 265.39. (Vector Bundle-Pullback–Functorial Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $p\colon E \to B$ be a $\mathbb{K}$-vector bundle The map
$$\begin{array}{rcl} (\mathrm{id}_B)^*E & \to & E \\ (b, e) & \mapsto & e \end{array}$$
is a natural isomorphism from the pullback $\mathbb{K}$-vector bundle $(\mathrm{id}_B)^*E \to B$ to the original $\mathbb{K}$-vector bundle $p\colon E \to B$.
(2) Let $p\colon E \to B$ be a $\mathbb{K}$-vector bundle and let $g\colon C \to B$ and $h\colon D \to C$ be continuous maps between topological spaces. The map
$$\begin{array}{rcl} h^*(g^*E) & \to & (g \circ h)^*E \\ (d, (c, e)) & \mapsto & (d, e) \end{array}$$
is a natural isomorphism from the double pullback $\mathbb{K}$-vector bundle $h^*(g^*E) \to B$ to the pullback $\mathbb{K}$-vector bundle $(g \circ h)^*E \to B$.

Proof. The proof of this lemma is basically identical to the not very substantial proof of the Bundle-Pullback–Functorial Lemma 240.24. We just need to replace the Bundle–Pullback Proposition 240.21 (2) by the Vector Bundle–Pullback Proposition 265.36 (2). ■

The following lemma is a very convenient tool for showing that a morphism of vector bundles is in fact an isomorphism of vector bundles.

Lemma 265.40. (Vector Bundle Isomorphism Criterion)[2845] Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$. Suppose we are given the following morphism of n-dimensional $\mathbb{K}$-vector bundles:
$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C. \end{array}$$
If h is a homeomorphism (e.g. if $h = \mathrm{id}_B$), then the above is an isomorphism of n-dimensional $\mathbb{K}$-vector bundles. An analogous statement also holds for smooth vector bundles.

[2845] The content of this lemma is very similar to Exercise 240.16.

Proof. Suppose we are given the following morphism of n-dimensional $\mathbb{K}$-vector bundles:

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C. \end{array}$$

Next we consider the following expanded diagram:

$$\begin{array}{ccccc} & & \overset{(b,f)\mapsto f}{\frown} & & \\ h^*F & \overset{(p(z),H(z))\leftarrow\!\shortmid z}{\longleftarrow} & E & \xrightarrow{H} & F \\ {\scriptstyle (b,f)\mapsto b}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xleftarrow{\mathrm{id}} & B & \xrightarrow{h} & C. \\ & & \underset{h}{\smile} & & \end{array}$$

By the Vector Bundle–Pullback Proposition 265.36 we know that the left square is an isomorphism of $\mathbb{K}$-vector bundles. Since h is a homeomorphism we obtain from the Vector Bundle-Pullback–Functorial Lemma 265.39 that the outer rectangle defines an isomorphism of $\mathbb{K}$-vector bundles. It follows from the Vector Bundle–Category Lemma 265.3 that the right square is, as promised, an isomorphism of $\mathbb{K}$-vector bundles. ■

We conclude this discussion of pullbacks of vector bundles with a lemma, which is a little more interesting than the above rather formal statements, since it is very specific to the setting of vector bundles:

Lemma 265.41. (Tangent Bundle–Pullback Lemma) Let M and N be smooth manifolds and let $f\colon M \to N$ be a smooth map that has the property that for each $x \in M$ the differential $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_{f(x)}N$ is an isomorphism. (e.g. M could be an immersion, a local diffeomorphism, the inclusion of a codimension-zero submanifold, or a covering in the sense of Definition 95.3). Then the map

$$\begin{array}{ccc} \mathrm{T}M & \to & f^*\mathrm{T}N \\ (b,v) & \mapsto & (b, \underbrace{(f(b), \mathrm{D}f_b(v))}_{\in \mathrm{T}N}) \end{array}$$

is a natural isomorphism of smooth real vector bundles.

Proof. Let M and N be smooth manifolds and let $f\colon M \to N$ be a smooth map that has the property that for each $x \in M$ the differential $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_{f(x)}N$ is an isomorphism. It follows from the Tangent Bundle–Induced Morphism Lemma 265.12 (1) that the maps

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{(P,v)\mapsto (f(P),\mathrm{D}f_P(v))} & \mathrm{T}N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

form a morphism of vector bundles. The desired statement now follows from the Vector Bundle–Pullback Proposition 265.36 (2). ■

265.8. Exercises for Chapter 265.

Exercise 265.1. Show that the map

$$\begin{array}{rcl} ([0,1]\times\mathbb{R})/(0,v)\sim(1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a smooth 1-dimensional real vector bundle.

Exercise 265.2. Let $p\colon F\to B$ be a vector bundle.

(a) Show that if the base space B is Hausdorff, then so is the total space F.
(b) Show that if the base space B is connected, then so is the total space F.
(c) Show that if the base space B is second countable, then so is total space F.

Remark. This exercise is a variation on Exercise 238.3.

Exercise 265.3. We consider the real projective line $\mathbb{R}\mathrm{P}^1$. Is the corresponding tautological vector bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^1\to\mathbb{R}\mathrm{P}^1$ isomorphic to the tangent bundle $\mathrm{T}\mathbb{R}\mathrm{P}^1$?

Exercise 265.4. Let M be an m-dimensional smooth submanifold of $\mathbb{R}^k$. Show that the tangent bundle $\mathrm{T}M$ is orientable if and only if the normal vector bundle $\mathrm{N}M$ is orientable.

Exercise 265.5. Let $n\in\mathbb{N}$.

(a) We consider the tautological $\mathbb{R}$-line bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^n\to\mathbb{R}\mathrm{P}^n$. Show that the complement of the zero-section is homotopy equivalent to a closed n-dimensional smooth manifold.
(b) Show that the tautological $\mathbb{R}$-line bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^n\to\mathbb{R}\mathrm{P}^n$ is not trivializable.

Exercise 265.6. Let $n\in\mathbb{N}$.

(a) We consider the tautological $\mathbb{C}$-line bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^n\to\mathbb{C}\mathrm{P}^n$. Show that the complement of the zero-section is homotopy equivalent to a closed $(2n+1)$-dimensional smooth manifold.
(b) Show that the tautological $\mathbb{C}$-line bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^n\to\mathbb{C}\mathrm{P}^n$ is not trivializable.

Exercise 265.7. Let M be an m-dimensional smooth manifold. We consider the total space $\mathrm{T}M$ of the tangent bundle. By the Tangent Bundle Proposition 265.10 we can view $\mathrm{T}M$ as a $2m$-dimensional smooth manifold. Show that $\mathrm{T}M$ is orientable, regardless of whether or not M itself is orientable.

Exercise 265.8. Let $p\colon E\to X$ be a vector bundle and let $s\colon X\to E$ be a section. Show that the zero set $\{x\in X\mid s(x)=0\}$ is a closed subset of X.
Hint. Use the Cover-Open Subset Lemma 3.20 (2b).

Exercise 265.9.

(a) Let $p\colon\mathfrak{T}\mathbb{R}\mathrm{P}^1\to\mathbb{R}\mathrm{P}^1$ be the vector bundle that is provided by the Tautological Line Bundle Lemma 265.6. Show that there exists a closed 2-dimensional smooth manifold M such that $\mathfrak{T}$ is diffeomorphic to $M\setminus\partial M$.
(b) Prove an analog of (a) for the complex setting.

Exercise 265.10. For which $n\in\mathbb{N}$ and which choice of $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ is the n-dimensional $\mathbb{K}$-vector bundle

$$\begin{array}{rcl} ([0,1]\times\mathbb{K}^n)/(0,v)\sim(1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

trivializable?

Exercise 265.11. Does there exist a 2-dimensional real vector bundle $p\colon E \to S^2$ such that the complement of the zero-section is homotopy equivalent to S^3?

Exercise 265.12. Let $\pi\colon S^n \to \mathbb{R}\mathrm{P}^n$ be the natural projection and let $\mathfrak{T}\mathbb{R}\mathrm{P}^n$ be the tautological real line bundle on $\mathbb{R}\mathrm{P}^n$. Is the pullback $\pi^*\mathfrak{T}\mathbb{R}\mathrm{P}^n$ trivializable?

Exercise 265.13. Does there exist a smooth vector field on S^2 with a single zero?

Exercise 265.14. Show by hand that the Möbius vector bundle, as defined on page 5119, does not admit a section which is nowhere-vanishing.
Remark. Note that this exercise, together with the Trivializable–Vector Bundle Lemma 265.22 (2), gives one more proof that the Möbius vector bundle is non-trivial.

Exercise 265.15. Let $g \geq 1$ and let Σ be the surface of genus g. Does there exist a real line bundle $p\colon E \to \Sigma$ that is not trivializable?

Exercise 265.16. For which $n \in \{1, 2, 3, 4\}$ is $\mathbb{R}\mathrm{P}^n$ parallelizable?

Exercise 265.17. Let $p\colon E \to B$ be a 1-dimensional real vector bundle. Show that if E is orientable, then it is trivializable.

Exercise 265.18. Show that the tangent bundle $\mathrm{T}M$ of any m-dimensional smooth manifold M, viewed as a $2m$-dimensional smooth manifold, is orientable.

Exercise 265.19. Let M be an m-dimensional smooth manifold. We consider

$$\mathrm{ST}M \;:=\; (\mathrm{T}M \setminus M)/\sim$$

where $(x, v) \sim (x, w)$ if there exists a $\lambda > 0$ with $w = \lambda \cdot v$. This topological space is sometimes called the *unit tangent bundle of* M.

(a) Show that $\mathrm{ST}M$ is naturally a smooth manifold of dimension $2m - 1$ such that the projections $\mathrm{T}M \setminus M \to \mathrm{ST}M$ and $\mathrm{ST}M \to M$ are both smooth.
(b) Show that $\mathrm{SO}(3)$ is diffeomorphic to $\mathrm{ST}S^2$.

Remark. This exercise is closely related to Exercise 29.12.

Exercise 265.20. Let $p\colon M \to S^1$ be the Möbius vector bundle from page 5119. Given $n \in \mathbb{Z}$ let $g\colon S^1 \to S^1$ be the map that is given by $g(z) = z^n$. For which $n \in \mathbb{Z}$ is the pullback of the Möbius vector bundle under g a trivializable vector bundle?

Exercise 265.21.

(a) Let M be an m-dimensional smooth manifold. Show that if M admits an immersion into $\mathbb{R}^m$, then M is parallelizable.
(b) Show that every compact connected 2-dimensional smooth manifold with non-empty boundary admits an immersion into $\mathbb{R}^2$.

Remark. This exercise is related to Exercise 30.25.

Exercise 265.22. Let $p\colon E \to M$ be an oriented m-dimensional real vector bundle over a closed oriented m-dimensional smooth manifold. Furthermore let $f\colon N \to M$ be a finite covering of M. Express the Euler number $\chi(f^*E)$ in terms of the Euler number $\chi(E)$.

Exercise 265.23. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R} \setminus \{0\}$. On the smooth manifold $\mathbb{R}^n$ we consider the vector field

$$\begin{array}{rcl} v\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (v_1, \dots, v_n) & \mapsto & (\lambda_1 \cdot v_1, \dots, \lambda_n \cdot v_n). \end{array}$$

This vector field defines a section $s\colon \mathbb{R}^n \to \mathrm{T}\mathbb{R}^n$.

(a) Show that $s(\mathbb{R}^n)$ is transverse to $\mathbb{R}^n \times \{0\} \subseteq \mathrm{T}\mathbb{R}^n = \mathbb{R}^{2n}$.

(b) Show that the index of the intersection point $(0,0)$ of the two oriented smooth manifolds $\mathbb{R}^n \times \{0\}$ and $s(\mathbb{R}^n) = \{(x, v(x)) \,|\, x \in \mathbb{R}\}$ in $\mathrm{T}\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is given by the sign of $\lambda_1 \cdot \ldots \cdot \lambda_n$.

Exercise 265.24. On the sphere S^n we consider the smooth vector field

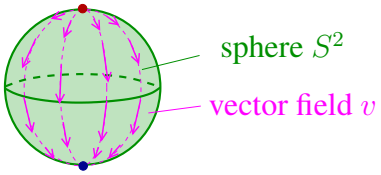
$$\begin{array}{rcl} v\colon S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & \underbrace{\mathfrak{e}_{n+1} - \langle x, \mathfrak{e}_{n+1} \rangle \cdot x}_{\substack{\text{orthogonal projection} \\ \text{of } \mathfrak{e}_{n+1} \text{ to } \mathrm{T}_x S^n}}. \end{array}$$

Note that this map defines a section $\sigma\colon M \to \mathrm{T}M$.

(a) Show that $\sigma(S^n) \subseteq \mathrm{T}S^n$ is transverse to $S^n \subseteq \mathrm{T}S^n$.

(b) Show that $\sigma(S^n) \cap S^n$ consists of precisely two intersection points, which have signs 1 and $(-1)^n$.

Hint. Use Exercise 265.23.



Exercise 265.25. Let $g \in \mathbb{N}$. Find a smooth vector field v on Σ_g such that for the corresponding section $s\colon \Sigma_g \to \mathrm{T}\Sigma_g$ the following two statements hold:

(a) The image $s(\Sigma_g) \subseteq \mathrm{T}\Sigma_g$ is transverse to $\Sigma_g \subseteq \mathrm{T}\Sigma_g$.

(b) The intersection $\sigma(\Sigma_g) \cap \Sigma_g$ consists of precisely $2g-2$ intersection points which have sign -1.

Hint. Use Exercise 265.23.

Exercise 265.26. Determine the Euler number of the tautological bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$, viewed as an oriented 2-dimensional real vector bundle.

Exercise 265.27. Let $p\colon E \to B$ be an $\mathbb{R}^n$-vector bundle. Show that there exists a 2-fold covering $f\colon \widetilde{B} \to B$ such that $p^*\widetilde{E}$ is an orientable $\mathbb{R}^n$-vector bundle.

Exercise 265.28. For which compact 2-dimensional smooth manifolds M does there exist a covering $p\colon \widetilde{M} \to M$ such that the pullback $p^*(\mathrm{T}M)$ is trivializable?

Exercise 265.29. Show that the tautological $\mathbb{K}$-line bundles from the Tautological Line Bundle Lemma 265.6 are consistent. More precisely, show that for any $0 \leq m \leq n \leq \infty$ the pullback of $\mathfrak{T}\mathbb{K}\mathrm{P}^n$ under the usual embedding $\mathbb{K}\mathrm{P}^m \to \mathbb{K}\mathrm{P}^n$ is isomorphic to $\mathfrak{T}\mathbb{K}\mathrm{P}^m$.

CHAPTER 266

Alternative views of vector bundles

The alert reader will have noticed that as of right now we have two, a priori different, definitions of vector bundles in these notes. First of all we have the Definition 240.10 where we used the language of $(F, \mathcal{G})$-bundles. On the other hand we have the more ad hoc Definition 265.1. In this chapter we will see that these two definitions are essentially equivalent. Furthermore we will also show that every vector bundle arises from a construction via transition maps and we will use this fact to show that the isomorphism classes of vector bundles over a fixed topological space B form a set.

This chapter can be viewed as an overflow from the last chapter. The reader who already has some experience with (vector) bundles might find the content rather unsurprising and might be best served by jumping straight to the next Chapter 267.

This chapter is organized as follows:

- In Section 266.1 we will study trivializations of vector bundles.
- In Section 266.2 we will see that the notion of a vector bundle from Section 265.1 is essentially equivalent to the notion of a vector bundle from Definition 240.10.
- In Section 266.3 we will see that we can construct and describe vector bundles via transition maps.
- In Section 266.4 we will consider the isomorphism classes of vector bundles over a fixed topological space and we will see that these form a set.
- In Section 266.5 we will provide exercises for Chapter 266.

In the next Chapter 267 we will study the interplay between vector bundles and homotopies.

266.1. **Trivializations of vector bundles.** The only goal of this section is to prove the following slightly subtle proposition, which underpins what is happening in the next two sections:[2846]

Proposition 266.1. (Vector Bundle–Action-on-Trivializations Proposition) In the following let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p\colon E \to W$ be an n-dimensional $\mathbb{K}$-vector bundle over a topological space W and let $\Phi\colon E \to W \times \mathbb{K}^n$ be a trivialization. We denote by

[2846]The following proposition can be viewed as an analog of the Bundle–Action-on-Trivializations Proposition 240.17.

$\pi\colon W \times \mathbb{K}^n \to \mathbb{K}^n$ the projection. We equip $\mathrm{GL}(m,\mathbb{K}) \subseteq \mathrm{M}(n\times n,\mathbb{K}) = \mathbb{K}^{n^2}$ with the usual topology that we introduced on page 202.

(1) If $f\colon W \to \mathrm{GL}(n,\mathbb{K})$ is a continuous map, then the map

$$\begin{array}{ccc} E & \to & W\times\mathbb{K}^n \\ e & \mapsto & (p(e), \underbrace{f(p(e))}_{\in \mathrm{GL}(n,\mathbb{K})} \cdot \underbrace{\pi(\Phi(e))}_{\in\mathbb{K}^n}) \end{array}$$

is also a trivialization.

(2) Every trivialization of $p\colon E \to W$ is of the form described in (1).

If we start out with a smooth vector bundle, then the analogous statements hold if we replace "continuous" by "smooth".

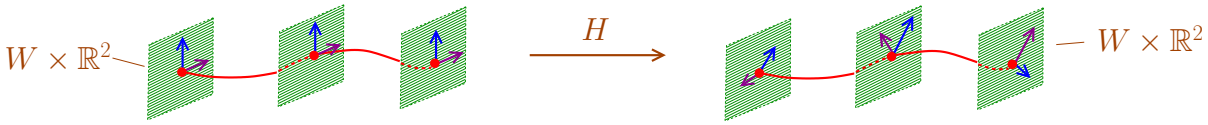
Proof.

(1) Statement (1) is basically clear.

(2) We start out with the following elementary claim:

Claim 1. If $H\colon W\times\mathbb{K}^n \to W\times\mathbb{K}^n$ is an isomorphism of $\mathbb{K}$-vector bundles over W, then there exists a unique map $f\colon W\to \mathrm{GL}(n,\mathbb{K})$ such that for every $(w,v)\in W\times\mathbb{K}^n$ we have $H(w,v) = (w, f(w)\cdot v)$.

Proof. It follows immediately from the definition of an isomorphism of $\mathbb{K}$-vector bundles that for any $w\in W$ the restriction of H to $\{w\}\times\mathbb{K}^n \to \{w\}\times\mathbb{K}^n$ is an isomorphism of $\mathbb{K}$-vector spaces. Thus it is of the form $(w,v)\mapsto(w, f(w)(v))$ for a unique matrix $f(w)\in\mathrm{GL}(n,\mathbb{K})$. ⊞



It remains to show that the map H from Claim 1 is continuous, respectively smooth. First we consider the "non-smooth" case.

Claim 2. If $H\colon W\times\mathbb{K}^n \to W\times\mathbb{K}^n$ is an isomorphism of $\mathbb{K}$-vector bundles over W, then the map $f\colon W\to\mathrm{GL}(n,\mathbb{K})$ from Claim 1 is continuous.

Proof. Since math is so much fun, we provide two separate arguments:

(a) Given $i,j\in\{1,\dots,n\}$ we denote by $f_{ij}\colon W\to\mathbb{K}$ the (i,j)-entry of the map $f\colon W\to\mathrm{GL}(n,\mathbb{K})$. We need to show that each map f_{ij} is continuous. Thus let $i,j\in\{1,\dots,n\}$. Note that the map $f_{ij}\colon W\to\mathbb{K}$ is the composition of the following maps:

$$W \xrightarrow{w\mapsto(w,e_j)} W\times\mathbb{K}^n \xrightarrow{(w,v)\mapsto(w,f(w)\cdot v)} W\times\mathbb{K}^n \xrightarrow{(w,v)\mapsto v} \mathbb{K}^n \xrightarrow{(x_1,\dots,x_n)\mapsto x_i} \mathbb{K}.$$

It thus follows from our hypothesis that $f_{ij}\colon W\to\mathbb{K}$ is the composition of continuous maps. This shows that $f_{ij}\colon W\to\mathbb{K}$ is indeed a continuous map.[2847]

(b) Since $\mathbb{K}^n$ is regionally compact, Hausdorff and locally connected we obtain from the Fiber Preserving Homeo Proposition 240.6 that the map $f\colon W\to\mathrm{GL}(n,\mathbb{K})$ is continuous with respect to the compact-open topology. But by the Matrix Topology

[2847] This argument is essentially the same argument as in the proof of the Smooth Linear Bundle–Action-on-Trivializations 243.4.

Lemma 16.8 the compact-open topology on $\mathrm{GL}(n,\mathbb{K})$ agrees with the subspace topology coming from $\mathrm{GL}(n,\mathbb{K}) \subseteq \mathbb{K}^{n^2}$. ■

Now we turn to the smooth case. After applying charts we see that we need to prove the following claim:

Claim 3. Let $U \subseteq$ be an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$. If we are given an isomorphism

$$\begin{array}{rcl} H\colon U \times \mathbb{K}^n & \to & U \times \mathbb{K}^n \\ (u,v) & \mapsto & (u, f(u)\cdot v) \end{array}$$

of smooth n-dimensional $\mathbb{K}$-vector bundles over U, then the map $f\colon U \to \mathrm{GL}(n,\mathbb{K})$ from Claim 1 is smooth.

Proof. Exactly as in the proof (a) of Claim 2 we see that each $f_{ij}\colon U \to \mathbb{K}$ is a composition of smooth maps, hence it is smooth.

266.2. **Two equivalent notions of vector bundles.** In Definition 240.10 we gave our first definition of a vector bundle. This definition made use of the following more general concepts:

Definition 240.8. Let F be a topological space and let $\mathcal{G} \subseteq \mathrm{Homeo}(F)$ be a subgroup of the homeomorphism group of F. Furthermore let $p\colon Y \to B$ be a continuous map between two topological spaces.

(1) Let $\Phi\colon p^{-1}(U) \to U \times F$ and $\Psi\colon p^{-1}(V) \to V \times F$ be two local F-trivializations for p. We say that the trivialization Φ is $\mathcal{G}$-compatible to the trivialization Ψ if there exists a *continuous* map $\Theta\colon U \cap V \to \mathcal{G}$ such that the following diagram commutes:

$$(U\cap V)\times F \xleftarrow[\cong]{\Phi} p^{-1}(U\cap V) \xrightarrow[\cong]{\Psi} (U\cap V)\times F.$$
$$(b,f)\mapsto(b,\Theta(b)(f))$$

(2) An atlas $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}$ for an F-bundle p is called $\mathcal{G}$-compatible if for all $i,j \in I$ the trivialization Φ_i is compatible to the trivialization Φ_j.

(3) An $(F,\mathcal{G})$-bundle over B is a pair $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ where $p\colon Y \to B$ is an F-bundle over B and where $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I}$ is a $\mathcal{G}$-compatible F-atlas for p.

(4) Let F be a smooth manifold and let $\mathcal{G} \subseteq \mathrm{Diffeo}(F)$. Furthermore let Y and B be smooth manifolds. An $(F,\mathcal{G})$-bundle $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times F\}_{i\in I})$ is called smooth if all Φ_i are diffeomorphisms.

The above definition allows us to recall our first definition of a vector bundle:

Definition 240.10.

(1) We refer to any $(\mathbb{R}^n, \mathrm{GL}(n,\mathbb{R}))$-bundle as an n-dimensional real vector bundle.

(2) We refer to any $(\mathbb{C}^n, \mathrm{GL}(n,\mathbb{C}))$-bundle as an n-dimensional complex vector bundle.

Since $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{Diffeo}(\mathbb{R}^n)$ and since $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathrm{Diffeo}(\mathbb{C}^n)$ we also get the notion of smooth vector bundles.

On the other hand in Definition 265.1 we introduced a somewhat different notion of an n-dimensional $\mathbb{K}$-vector bundle. The following lemma says that the two notions are essentially the same:

Lemma 266.2. (Vector Bundle Definition Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n\}_{i\in I})$ be a $(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))$-bundle. For every $b \in B$ we can define a unique $\mathbb{K}$-vector space structure on the fiber $p^{-1}(b)$ such that for each $i \in I$ and each $b \in U_i$ the following map is an isomorphism of $\mathbb{K}$-vector spaces:

$$p^{-1}(b) \xrightarrow{\Phi_i|_{p^{-1}(b)}} \{b\} \times \mathbb{K}^n \xrightarrow{(b,f)\mapsto f} \mathbb{K}^n.$$

These vector space structures turn $p\colon Y \to B$ into an n-dimensional $\mathbb{K}$-vector bundle in the sense of Definition 265.1.

(2) Let $p\colon Y \to B$ be an n-dimensional $\mathbb{K}$-vector bundle in the sense of Definition 265.1. The family of all $\mathbb{K}$-vector bundle trivializations of $p\colon Y \to B$ is a $(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))$-bundle.

These two maps define a weak equivalence of categories (in the sense of Definition 5.29) between the category $\mathcal{Bund}_{(\mathbb{K}^n,\mathrm{GL}(n,\mathbb{K}))}$ of $(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))$-bundles and the category $\mathcal{VectBund}_{\mathbb{K}^n}$ of n-dimensional $\mathbb{K}$-vector bundles. Furthermore analogous statements hold for smooth bundles and for bundles over a fixed topological space B.

Remark. We will use the Vector Bundle Definition Lemma 266.2 to freely go back and forth between the two notions of a vector bundle. In most cases we will not explicitly point out that we are making use of the Vector Bundle Definition Lemma 266.2. Both notions have some advantages:

(1) Definition 265.1 from the previous chapter has the advantage that for certain examples, like tangent bundles, it is much easier to describe.
(2) The earlier Definition 240.10 has the advantage that it fits into the general theory of $(F, \mathcal{G})$-bundles. So this approach allows us to use all of the theory that we have developed so far. □

Sketch of proof.

(1) Let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n\}_{i\in I})$ be a $(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))$-bundle. As we saw in the proof of the Vector Bundle–Fiber Lemma 240.11, it follows easily from the fact that the trivializations Φ_i are $\mathrm{GL}(n,\mathbb{K})$-compatible that the definition of the $\mathbb{K}$-vector space structure on $p^{-1}(b)$ is independent of the choice of $i \in I$. It is now clear that the maps Φ_i define local trivializations of the $\mathbb{K}$-vector bundle.
(2) Let $p\colon Y \to B$ be an n-dimensional $\mathbb{K}$-vector bundle. We need to show that the family of all $\mathbb{K}$-vector bundle trivializations is $\mathrm{GL}(n,\mathbb{K})$-compatible. Fortunately this is precisely the content of the Vector Bundle–Action-on-Trivializations Proposition 266.1.

It follows easily from the definitions that the maps in (1) define a covariant functor F from the category $\mathcal{Bund}_{(\mathbb{K}^n,\mathrm{GL}(n,\mathbb{K}))}$ to the category $\mathcal{VectBund}_{\mathbb{K}^n}$ and that the maps in (2) define a covariant functor G from the category $\mathcal{VectBund}_{\mathbb{K}^n}$ to the category $\mathcal{Bund}_{(\mathbb{K}^n,\mathrm{GL}(n,\mathbb{K}))}$. It is elementary to write down a natural isomorphism between $F \circ G$ and the identity functor $\mathrm{id}_{\mathcal{VectBund}_{\mathbb{K}^n}}$ and to write down a natural isomorphism between $G \circ F$ and the identity functor $\mathrm{id}_{\mathcal{Bund}_{(\mathbb{K}^n,\mathrm{GL}(n,\mathbb{K}))}}$. This shows that F and G define a weak equivalence of categories in the sense of Definition 5.29.

One can easily verify that in the smooth setting the analogous statements hold and that the analogous statements for vector bundles over a fixed topological space B hold. ■

266.3. **Constructing vector bundles via transition maps.** The starting point of this section is the following lemma, which is closely related to the Bundle–Transition Map Lemma 240.18, and which says that we can construct vector bundles by specifying "transition maps".

Lemma 266.3. (Vector Bundle–Transition Map Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let B be a topological space and let $\{U_i\}_{i\in I}$ be family of open subsets of B such that $\bigcup_{i\in I} U_i = B$. Furthermore let $n \in \mathbb{N}_0$. Suppose that for every $i, j \in I$ we have a continuous map $\mu_{ij}\colon U_i \cap U_j \to \mathrm{GL}(n, \mathbb{K})$ such that the following conditions are satisfied:

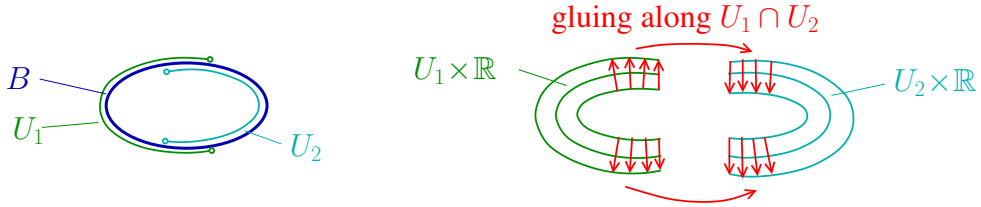
(1) For every $i, j, k \in I$ we have $\mu_{jk} \circ \mu_{ij} = \mu_{ik}\colon U_i \cap U_j \cap U_k \to \mathrm{GL}(n, \mathbb{K})$.
(2) For every $i, j \in I$ and any $b \in U_i \cap U_j$ we have $\mu_{ij}(b) = \mu_{ji}(b)^{-1}$.

The map

$$p\colon \Big(\bigsqcup_{i\in I} (U_i \times \mathbb{K}^n)\Big)/\overset{\text{where } b\in U_i\cap U_j}{\overset{\downarrow}{(b,v)}} \sim (b, \mu_{ij}(b)\cdot v) \;\to\; B$$
$$[(b,v)] \;\mapsto\; b,$$

together with the obvious $\mathbb{K}$-vector space structure on each $p^{-1}(b)$, is an n-dimensional $\mathbb{K}$-vector bundle over B. An analogous statement also holds in the smooth setting.

Example. Similar to the example on page 4642 we consider the circle $B = S^1$ and two open overlapping intervals $U_1, U_2 \subseteq B = S^1$. Let $\varphi\colon U_1 \cap U_2 \to \mathrm{GL}(1, \mathbb{R})$ be the smooth map that equals $(1) \in \mathrm{GL}(1, \mathbb{R})$ on one component of $U_1 \cap U_2$ and that equals $(-1) \in \mathrm{GL}(1, \mathbb{R})$ on the other component of $U_1 \cap U_2$.



It is fairly straightforward to see that the resulting smooth real vector bundle is isomorphic to the Möbius vector bundle which we introduced on page 5119. □

Sketch of proof. Using the isomorphisms from the Vector Bundle Definition Lemma 266.2 one sees that the lemma can be viewed as a special case of the Bundle–Transition Map Lemma 240.18. Surely there is no need to repeat the elementary, but slightly lengthy argument from the Bundle–Transition Map Lemma 240.18. We just point out that it follows fairly easily from the hypotheses (1) and (2) that for any $i \in I$ the map

$$\begin{tikzcd} U_i \times \mathbb{K}^n \arrow[rr, "(b,v)\mapsto[(b,v)]"] \arrow[dr, "(b,v)\mapsto v"'] & & p^{-1}(U_i) \arrow[dl, "p"] \\ & U_i & \end{tikzcd}$$

is a well-defined map, which is a trivialization of $\mathbb{K}$-vector bundles over U_i. Since the U_i form an open cover of B we see that p is indeed an n-dimensional $\mathbb{K}$-vector bundle over B. The smooth analog follows from the same type of argument. ■

We also have the following converse to the Vector Bundle–Transition Map Lemma 266.3:

Lemma 266.4. (Vector Bundle-Complete Construction Lemma) Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$. Every (smooth) $\mathbb{K}$-vector bundle $p\colon E \to B$ admits an isomorphism to a $\mathbb{K}$-vector bundle of the type constructed in the Vector Bundle–Transition Map Lemma 266.3.

Sketch of proof. Using the isomorphisms from the Vector Bundle Definition Lemma 266.2 one sees that the lemma can be viewed as a special case of the Bundle-Complete Construction Lemma 240.19. Thus in the following we will just outline the proof "without the boring bits".

Thus let $p\colon E \to B$ be a (smooth) n-dimensional $\mathbb{K}$-vector bundle. Note that by definition of a vector bundle we know that there exists a family $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n\}_{i\in I}$ of local trivializations such that the U_i form an open cover of B. It follows easily from the definition of a $\mathbb{K}$-vector bundle that for each $i, j \in I$ there exists a unique map $\mu_{ij}\colon U_i \cap U_j \to \mathrm{GL}(n, \mathbb{K})$ such that the following diagram commutes:

$$(U_i \cap U_j) \times \mathbb{K}^n \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap U_j) \xrightarrow[\cong]{\Phi_j} (U_i \cap U_j) \times \mathbb{K}^n.$$
$$(b,f) \mapsto (b, \mu_{ij}(b)(f))$$

Next note that it follows from the Vector Bundle–Action-on-Trivializations Proposition 266.1 that each map $\mu_{ij}\colon U_i \cap U_j \to \mathrm{GL}(n, \mathbb{K})$ is continuous (respectively smooth). It is elementary to verify that these maps satisfy Conditions (1) and (2) from the Vector Bundle–Transition Map Lemma 266.3 and that the map

$$\begin{aligned} p\colon \Big(\bigsqcup_{i\in I} (U_i \times \mathbb{K}^n)\Big)/(b, v) \sim (b, \mu_{ij}(b) \cdot v) &\to E \\ [(b, v)] &\mapsto \Phi_i^{-1}(b, v) \quad \text{if } b \in U_i \end{aligned}$$

is an isomorphism of (smooth) $\mathbb{K}$-vector bundles over B. ∎

266.4. **Vector bundles over a fixed topological space.** The following Lemma can be viewed as a special case of the Bundles–Set Lemma 240.20:

Lemma 266.5. (Vector Bundles–Set Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $n \in \mathbb{N}_0$ and let B be a topological space. The isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over B form a set.

Proof. This statement follows easily from the Vector Bundle-Complete Construction Lemma 266.4, at least, once one has settled what a set is meant to be. ∎

The Vector Bundles–Set Lemma 266.5 leads us to the following notation, which is the vector bundle analog of Notation 241.8:

Notation 266.6. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let B be a topological space. Given $n \in \mathbb{N}_0$ we denote by $\mathrm{IsoVectBund}_{\mathbb{K}^n}(B)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over B.

The following lemma is an analog of the Bundle–Isomorphism Classes Lemma 241.9:

Lemma 266.7. (Vector Bundle–Isomorphism Classes Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Given any continuous map $h\colon C \to B$ between two topological spaces the pullback of $\mathbb{K}$-vector bundles, as defined in the Vector Bundle–Pullback Proposition 265.36,

induces a well-defined map

$$\begin{array}{rcl} h^*\colon \operatorname{IsoVectBund}_{\mathbb{K}^n}(B) & \to & \operatorname{IsoVectBund}_{\mathbb{K}^n}(C) \\ [p\colon E \to B] & \mapsto & [h^*E \to C]. \end{array}$$

(2) The maps

$$\begin{array}{rcl} B & \to & \operatorname{IsoVectBund}_{\mathbb{K}^n}(B) \\ (h\colon C \to B) & \to & (h^*\colon \operatorname{IsoVectBund}_{\mathbb{K}^n}(B) \to \operatorname{IsoVectBund}_{\mathbb{K}^n}(C)) \end{array}$$

define a contravariant functor from the category *Top* of topological spaces to the category *Set* of sets.

Remark. By using "linear algebra of vector bundles" we will see in Chapter 268 that $\bigcup_{n\in\mathbb{N}_0} \operatorname{IsoVectBund}_{\mathbb{K}^n}(B)$ is "more than just a set", namely we will see that we can endow this set with two monoid structures corresponding to the direct sum and the tensor product of vector bundles. Thus we will upgrade the Vector Bundle–Isomorphism Classes Lemma 266.7 to a covariant functor from the category of topological spaces to an algebraic category. □

Proof. Statement (1) is an immediate consequence of the Vector Bundle-Pullback-Functor Lemma 265.38, whereas Statement (2) is an immediate consequence of the Vector Bundle-Pullback–Functorial Lemma 265.39. ■

266.5. Exercises for Chapter 266.

Exercise 266.1. Show that there exists an equivalence of categories between the category of oriented real vector bundles and the category of $(\mathbb{R}^n, \mathrm{GL}_+(n,\mathbb{R}))$-bundles.

Exercise 266.2. Let $\mathrm{GL}_{\pm 1}(n,\mathbb{R})$ be the subgroup of $\mathrm{GL}(n,\mathbb{R})$ that is given by matrices A with $|\det(A)| = 1$. We consider the category of $(\mathbb{R}^n, \mathrm{GL}_{\pm 1}(n,\mathbb{R}))$-bundles. How can we describe the corresponding subcategory of $\mathcal{VectBund}_{\mathbb{R}^n}$? What extra structures do such vector bundles have?

Exercise 266.3. We consider the category of $(\mathbb{R}^n, \mathrm{O}(n))$-bundles. How can we describe the corresponding subcategory of $\mathcal{VectBund}_{\mathbb{R}^n}$?

Exercise 266.4. Let $D \subseteq \mathrm{GL}(n,\mathbb{R})$ be the subgroup of diagonal matrices with entries in $\mathbb{R}_{>0}$. We consider the category of $(\mathbb{R}^n, D)$-bundles. How can we describe the corresponding subcategory of $\mathcal{VectBund}_{\mathbb{R}^n}$?

CHAPTER 267

Vector bundles and homotopies

In Chapter 241 we proved several results on pullbacks of bundles under homotopies. In this chapter we will discuss the corresponding statements for vector bundles. Most of the results of this chapter can be deduced from the corresponding results in Chapter 241. To keep this part of the lecture notes on vector bundles somewhat self-contained we will nonetheless provide sketches of proofs.

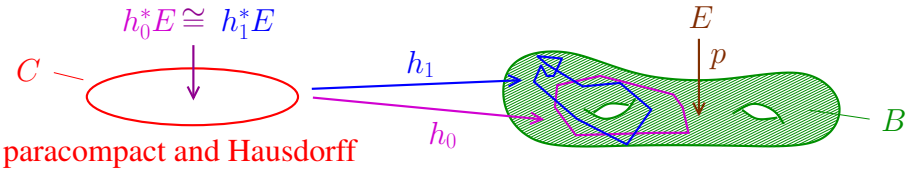
This chapter is organized as follows:

- In Section 267.1 we will see that given two homotopic maps the corresponding pullbacks of vector bundles are isomorphic.
- In Section 267.2 we will discuss some consequences of the Vector Bundle-Pullback-Homotopy Proposition 267.1.
- In Section 267.3 we will study vector bundles over spheres.
- In Section 267.4 we will provide exercises for Chapter 267.

In the next Chapter 268 we will introduce several linear algebraic constructions on vector bundles.

267.1. **Pullbacks and homotopies.** We start out with the following analog of the Pullback-Homotopy Proposition 241.5:

Proposition 267.1. (Vector Bundle–Pullback-Homotopy Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ be a $\mathbb{K}$-vector bundle, let C be a topological space and finally let $h_0, h_1\colon C \to B$ be two continuous maps. If h_0 and h_1 are homotopic and if C is Hausdorff and paracompact, then the $\mathbb{K}$-vector bundles $h_0^*E \to C$ and $h_1^*E \to C$ are isomorphic over C.



Remark. In the statement of the Vector Bundle–Pullback-Homotopy Proposition 267.1 and throughout this chapter we will use on several occasions the notion of a "paracompact" topological space that we introduced in Definition 14.3. We will use the explicit definition of a paracompact topological space only in the proof of the Vector Bundle-over-Product Lemma 267.2, which is a key ingredient in the proof of Vector Bundle–Pullback-Homotopy Proposition 267.1. Otherwise it is enough to know that most topological spaces that we are interested in are paracompact. For example we established in the Manifold+CW-Complex–Paracompact Proposition 241.4 that all topological manifolds and all CW-complexes are Hausdorff and paracompact. □

The key to proving the Vector Bundle–Pullback-Homotopy Proposition 267.1 is the following lemma, which is an analog of the Bundle-over-Product Lemma 241.6:

Lemma 267.2. (Vector Bundle-over-Product Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let C be a topological space that is Hausdorff and paracompact. Given any $t \in [0,1]$ we denote by

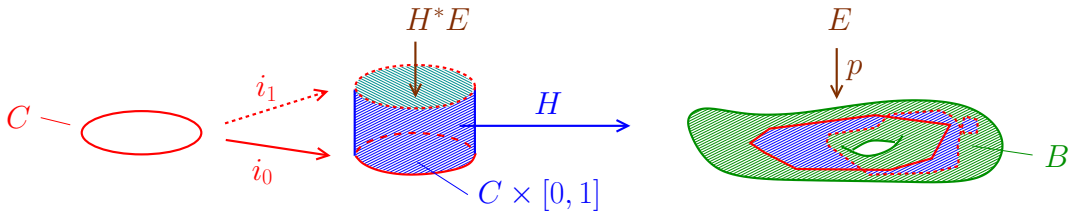
$$\begin{array}{rcl} i_t \colon C & \to & C \times [0,1] \\ c & \mapsto & (c,t) \end{array}$$

the obvious embedding. For any $\mathbb{K}$-vector bundle $p \colon E \to C \times [0,1]$ the pullbacks $i_0^* E$ and $i_1^* E$ are isomorphic over C.

Remark. Note that the Vector Bundle-over-Product Lemma 267.2 does not say that the vector bundles $i_0^* E$ and $i_1^* E$ are *naturally* isomorphic. In fact, as we will see in Exercise 267.1, the isomorphism we provide depends on choices. □

Before we provide the proof of the Vector Bundle-over-Product Lemma 267.2, let us first show that the Vector Bundle-Pullback-Homotopy Proposition 267.1 is a pretty straightforward consequence of the Vector Bundle-over-Product Lemma 267.2:

Proof of Proposition 267.1 assuming Lemma 267.2. Let $p \colon E \to B$ be a $\mathbb{K}$-vector bundle and let C be a topological space that is Hausdorff and paracompact. Furthermore let $H \colon C \times [0,1] \to B$ be a homotopy between two maps $h_0, h_1 \colon C \to B$. As in the Vector Bundle-over-Product Lemma 267.2 we denote by $i_0, i_1 \colon C \to C \times [0,1]$ the two obvious embeddings.



We have the following isomorphisms of $\mathbb{K}$-vector bundles over C:

$$\begin{array}{ccccccccc} h_0^* E & = & (H \circ i_0)^* E & \xleftarrow{\cong} & i_0^*(H^* E) & \cong & i_1^*(H^* E) & \xrightarrow{\cong} & (H \circ i_1)^* E & = & h_1^* E. \\ \uparrow & & & & & & & & & & \uparrow \\ \text{since } h_0 = H \circ i_0 & & & & & & & & & & \text{since } h_1 = H \circ i_1 \end{array}$$

natural isomorphisms from the Vector Bundle-Pullback–Functorial Lemma 265.39 (2)

by the Vector Bundle-over-Product Lemma 267.2

■

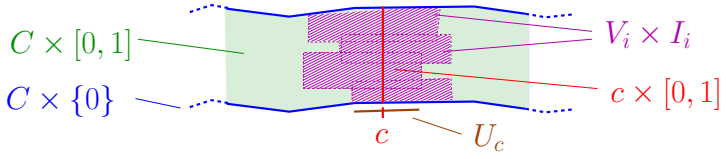
The proof of the Vector Bundle-over-Product Lemma 267.2 will require the remainder of this section. We will continue in Section 267.2 with applications of the Vector Bundle-Pullback-Homotopy Proposition 267.1.

Proof of the Vector Bundle-over-Product Lemma 267.2. This lemma follows almost immediately from the more general Bundle-over-Product Lemma 241.6 together with the Vector Bundle Definition Lemma 266.2. For the reader's convenience we outline the argument again.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let C be a topological space that is Hausdorff and paracompact. We need to show that for any n-dimensional $\mathbb{K}$-vector bundle $p \colon E \to C \times [0,1]$ the pullbacks $i_0^* E$ and $i_1^* E$ are isomorphic over C.

Claim 1. For each $c \in C$ there exists an open neighborhood U_c of c such that the restriction of the $\mathbb{K}$-vector bundle $p\colon E \to C \times [0,1]$ to $p^{-1}(U_c \times [0,1])$ is trivializable.

Proof. Let $c \in C$. It follows easily from the definition of a $\mathbb{K}$-vector bundle and the compactness of $[0,1]$ that there exist open neighborhoods $V_1, \dots, V_m$ of c and intervals $I_1, \dots, I_m \subseteq [0,1]$ such the restriction of the $\mathbb{K}$-vector bundle $p\colon E \to C \times [0,1]$ to each $p^{-1}(V_i \times I_i)$ is trivializable and such that $I_1 \cup \cdots \cup I_m = [0,1]$. We set $U_c := V_1 \cap \cdots \cap V_m$.



An elementary argument shows that we can pick $t_0 < t_1 < \cdots < t_m = 1$ such that each $[t_i, t_{i+1}]$ is contained in one of $I_1, \dots, I_m$. This implies that p is trivializable over each $U_c \times [t_i, t_{i+1}]$. It follows from the elementary Trivialization Extension Lemma 241.7 that p is trivializable over $U_c \times [0,1]$. ⊞

Claim 2. There exists a countable cover $\{V_i\}_{i\in\mathbb{N}}$ of C such that the restriction of $p\colon E \to B$ to each $p^{-1}(V_i \times [0,1])$ is trivializable.

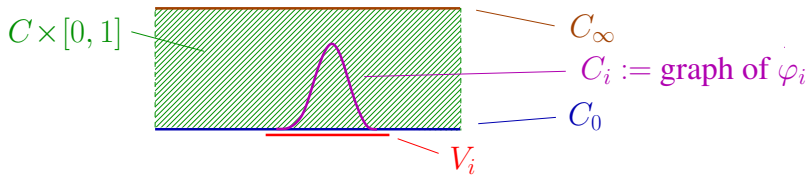
Proof. By Claim 1 there exists an open cover $\{U_c\}_{c\in C}$ of C such that the restriction of p to any $p^{-1}(U_c \times [0,1])$ is trivializable. Since C is Hausdorff and paracompact it follows from the Countable Cover Proposition 14.17 that there exists a *countable* locally finite open cover $\{V_i\}_{i\in\mathbb{N}}$ of C such that each V_i is the disjoint union of open subsets $\{V_{ij}\}_{j\in I_j}$ of C, each of which is contained in some U_c.

Since the restriction of a trivializable $\mathbb{K}$-vector bundle is again trivializable we see that we can pick a trivialization for each $p^{-1}(V_{ij} \times [0,1])$. Next note that it follows easily from the Disjoint Union Embedding Lemma 7.29 that each $V_i \times [0,1]$ is homeomorphic to the disjoint union of the open subsets $V_{ij} \times [0,1]$. It now follows that easily from the Topological-Coproduct Proposition 7.26 that these trivializations on $p^{-1}(V_{ij} \times [0,1])$ give rise to a trivialization on $p^{-1}(V_i \times [0,1])$. ⊞

Next note that it follows from our hypothesis that C is Hausdorff and paracompact together with the Paracompact-Partition of Unity Theorem 14.15 (1)⇒(3) that there exists a partition of unity $\{\varphi_i\colon C \to [0,1]\}_{i\in\mathbb{N}}$ such that for each $i \in \mathbb{N}$ we have $\operatorname{supp}(\varphi_i) \subseteq V_i$. Given $i \in \mathbb{N}_0$ we set $\psi_i := \varphi_1 + \cdots + \varphi_i\colon C \to [0,1]$ and we set

$$C_i \;:=\; \text{graph of } \psi_i\colon C \to [0,1] \;:=\; \{(c, \psi_i(c)) \,|\, c \in C\} \;\subseteq\; C \times [0,1].$$

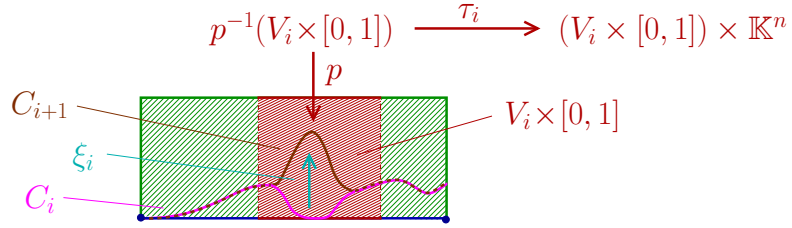
Note that $C_0 = C \times \{0\}$. We also set $C_\infty := C \times \{1\}$. Given $i \in \mathbb{N}_0 \cup \{\infty\}$ we write $E_i := p^{-1}(C_i)$.



Claim 3. For each $i \in \mathbb{N}_0$ there exists an isomorphism

$$\begin{array}{ccc} E_i & \xrightarrow[\cong]{\Xi_i} & E_{i+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ C_i & \xrightarrow[=:\xi_i]{(c,\psi_i(c))\mapsto(c,\psi_i(c)+\varphi_{i+1}(c))} & C_{i+1} \end{array}$$

of $\mathbb{K}$-vector bundles which is the identity outside of $p^{-1}(V_i \times [0,1])$.



Proof. Let $i \in \mathbb{N}_0$. Recall that by construction of the V_i we know that there exists a trivialization $\tau_i \colon p^{-1}(V_i \times [0,1]) \to (V_i \times [0,1]) \times \mathbb{K}^n$. First we consider the map

$$\begin{array}{rccl} \widetilde{\Xi}_i \colon & ((V_i \times [0,1]) \cap C_i) \times \mathbb{K}^n & \to & ((V_i \times [0,1]) \cap C_{i+1}) \times \mathbb{K}^n \\ & ((c, \psi_i(c)), f) & \mapsto & ((c, \psi_i(c) + \varphi_{i+1}(c)), f). \end{array}$$

Next we consider the map

$$\begin{array}{rccl} \Xi_i \colon & E_i & \to & E_{i+1} \\ & y & \mapsto & \begin{cases} y, & \text{if } y \notin p^{-1}(V_i \times [0,1]), \\ \tau_i^{-1}(\underbrace{\widetilde{\Xi}_i(\tau_i(y))}_{\in V_i \times [0,1] \times \mathbb{K}^n}), & \text{if } y \in p^{-1}(V_i \times [0,1]). \end{cases} \end{array}$$

One can now easily verify that these maps have the desired properties. ⊞

We combine these maps to obtain the map

$$\begin{array}{rccl} \Theta \colon & E_0 & \to & E_\infty \\ & y & \mapsto & \Xi_k(\ldots(\Xi_0(y))). \\ & & & \uparrow \\ & & & \text{we pick } k \in \mathbb{N}_0 \text{ with } \psi_k(y) = 1 \end{array}$$

It is now straightforward to verify that this map is well-defined, i.e. that it does not depend on the choice of $k \in \mathbb{N}_0$ for a given y. It follows easily from this observation together with the definition of a partition of unity and the Pasting Proposition 4.7 that Θ is continuous.

Finally we consider the following diagram:

$$\begin{array}{ccccccccccc} i_0^*E & \longrightarrow & \iota_0^*E & \longrightarrow & \overbrace{p^{-1}(C\times\{0\})}^{=E_0} & \xrightarrow[\cong]{\Theta} & \overbrace{p^{-1}(C\times\{1\})}^{=E_\infty} & \longleftarrow & \iota_1^*E & \longleftarrow & i_1^*E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ C & \xrightarrow[f_0]{c\mapsto(c,0)} & C\times\{0\} & \xrightarrow{\text{id}} & C\times\{0\} & \xrightarrow{(c,0)\mapsto(c,1)} & C\times\{1\} & \xleftarrow{\text{id}} & C\times\{1\} & \xleftarrow[f_1]{(c,1)\leftarrow c} & C. \end{array}$$

We make the following clarifications and observations:

(1) It follows easily from the above discussion that the middle horizontal map Θ defines an isomorphism of $\mathbb{K}$-vector bundles.

(2) For $j = 0, 1$ we denote by $f_j \colon C \to C \times \{j\}$ the obvious homeomorphism and we denote by $\iota_j \colon C \times \{j\} \to C \times [0,1]$ the inclusion. Note that $i_j = \iota_j \circ f_j$.
(3) It follows from the Vector Bundle-Pullback-Functor Lemma 265.38 (applied to the homeomorphism f_j) together with the Vector Bundle-Pullback–Functorial Lemma 265.39 (2) (applied to $i_j = \iota_j \circ f_j$) that the first and last map are isomorphisms of $\mathbb{K}$-vector bundles.
(4) It follows from the Vector Bundle–Pullback Proposition 265.36 (2) that the second and the fourth map are isomorphisms of $\mathbb{K}$-vector bundles.

It follows from these observations together with the Vector Bundle-Pullback-Functor Lemma 265.38 that the pullbacks i_0^*E and i_1^*E are isomorphic over C. ∎

267.2. **Vector bundles and homotopy equivalences.** The only goal of this section is to prove the following proposition, which is the vector bundle analog of the Bundles-Homotopies Proposition 241.10:

Proposition 267.3. (Vector Bundles-Homotopies Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n \in \mathbb{N}_0$.

(1) Let B and C be two topological spaces that are both Hausdorff and paracompact. If $f \colon C \to B$ is a homotopy equivalence, then the map[2848]
$$f^* \colon \operatorname{IsoVectBund}_{\mathbb{K}^n}(B) \to \operatorname{IsoVectBund}_{\mathbb{K}^n}(C),$$
that we introduced in the Vector Bundle–Isomorphism Classes Lemma 266.7, is a bijection.
(2) Let C be a topological space that is Hausdorff and paracompact. If C is contractible, then every $\mathbb{K}$-vector bundle over C is trivializable.

Proof.

(1) Let C and B be two topological spaces that are Hausdorff and paracompact and let $f \colon C \to B$ be a homotopy equivalence. We pick a homotopy inverse $g \colon B \to C$ to f. Recall that this means that $f \circ g$ is homotopic to id_B and that $g \circ f$ is homotopic to id_C. We consider the following induced maps $\operatorname{IsoVectBund}_{\mathbb{K}^n}(B) \to \operatorname{IsoVectBund}_{\mathbb{K}^n}(B)$:
$$g^* \circ f^* \overset{\downarrow}{=} (f \circ g)^* \underset{\uparrow}{=} \mathrm{id}_B^* \overset{\downarrow}{=} \mathrm{id}_{\operatorname{IsoVectBund}_{\mathbb{K}^n}(B)}.$$
by the Vector Bundle–Isomorphism Classes Lemma 266.7 (2)
follows from the Vector Bundle-Pullback-Homotopy Proposition 267.1 (1), since $f \circ g \simeq \mathrm{id}_B$
Similarly we see that $f^* \circ g^* = \mathrm{id}_{\operatorname{IsoVectBund}_{\mathbb{K}^n}(C)}$.
(2) Let $E \to C$ be a $\mathbb{K}$-vector bundle over a topological space C that is Hausdorff, paracompact and contractible. By definition of being contractible there exists a homotopy equivalence from the topological space $\{*\}$ consisting of a single point to the topological space C.
Since every $\mathbb{K}$-vector bundle over $\{*\}$ is trivializable it follows from (1), the discussion on page 5144 and the Vector Bundle-Pullback-Functor Lemma 265.38 that every $\mathbb{K}$-vector bundle over C is trivializable. ∎

[2848]Recall that according to Notation 266.6 we we denote by $\operatorname{IsoVectBund}_{\mathbb{K}^n}(B)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over B.



267.3. **Vector bundles over spheres.** We conclude this chapter with a discussion of vector bundles over spheres. We start out by reminding the reader of the following notation which we already introduced in Definitions 19.11 and 241.13:

Definition 267.4. Given two topological spaces X and Y we set

$$[X,Y] := \{\text{continuous maps } X \to Y\}/\sim$$

where $f \sim g$ if there exists a homotopy between f and g. We refer to $[X,Y]$ as the set of homotopy classes of continuous maps from X to Y.

We can formulate the following variation on the Bundle–Clutching Map Lemma 241.14:

Lemma 267.5. (Vector Bundle-Clutching Map Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $k \in \mathbb{N}$ and let $n \in \mathbb{N}$.

(1) Let $\varphi\colon S^{k-1} \to \mathrm{GL}(n,\mathbb{K})$ be a continuous map. The map

$$\begin{array}{rcl} p(\varphi)\colon ((S^k_{\geq 0} \times \mathbb{K}^n) \sqcup (S^k_{\leq 0} \times \mathbb{K}^n))/(x,v) \sim (x, \varphi(x)\cdot v) & \to & S^k \\ [(x,v)] & \mapsto & x \end{array}$$

is an n-dimensional $\mathbb{K}$-vector bundle over S^k. (The map $\varphi\colon S^{k-1} \to \mathrm{GL}(n,\mathbb{K})$ is called the clutching map and this construction of $\mathbb{K}$-vector bundles is called the clutching construction.)

(2) The map

$$\begin{array}{rcl} [S^{k-1}, \mathrm{GL}(n,\mathbb{K})] & \to & \mathrm{IsoVectBund}_{\mathbb{K}^n}(S^k) \\ [\varphi] & \mapsto & [p(\varphi)] \end{array}$$

is well-defined.

Example. Let $k = 1$. We consider the map $\varphi\colon S^0 \to \mathrm{GL}(1,\mathbb{R})$ which is conveniently given by $\pm 1 \to (\pm 1)$. It is not difficult to see that the resulting real 1-dimensional vector bundle over S^1 is isomorphic to the Möbius vector bundle. □

Sketch of proof. For Statement (1) we just note that, using collar neighborhoods of S^{k-1} in $S^k_{\geq 0}$ and $S^k_{\leq 0}$ it is straightforward to verify by hand that the clutching construction gives rise to a $\mathbb{K}$-vector bundle. We leave the details to the reader.

We turn to the proof of Statement (2). Let $f_0, f_1\colon S^{k-1} \to \mathrm{GL}(n,\mathbb{K})$ be two continuous maps that are homotopic. One can easily see that this implies that there exists a homotopy $F\colon S^{k-1} \times [0,1] \to \mathrm{GL}(n,\mathbb{K})$ with $F_0(x) = f_1(x) \cdot f_0^{-1}(x)$ and $F_1 = \mathrm{id}$. By picking a suitable collar neighborhood of $S^{k-1} = \partial S^k_{\geq 0}$ we obtain an identification $S^k_{\geq 0} = (S^{k-1} \times [0,1]) \cup S^k_{\geq \frac{1}{2}}$. One can now easily verify that the following map is an isomorphism of $\mathbb{K}$-vector bundles over S^k:

$$\begin{array}{rcl} ((S^k_{\geq 0} \times \mathbb{K}^n) \sqcup (S^k_{\leq 0} \times \mathbb{K}^n))/(x,v) \sim (x, f_0(x)\cdot v) & \to & ((S^k_{\geq 0} \times \mathbb{K}^n) \sqcup (S^k_{\leq 0} \times \mathbb{K}^n))/(x,v) \sim (x, f_1(x)\cdot v) \\ (x,v) & \mapsto & \begin{cases} (x,v), & \text{if } x \in S^k_{\leq 0} \text{ or if } x \in S^k_{\geq \frac{1}{2}}, \\ (x, F(y,t)\cdot v), & \text{if } x = (y,t) \in S^{k-1} \times [0,1]. \end{cases} \end{array}$$

This concludes the proof of Statement (2). ■

In the following two subsections we consider complex vector bundles and oriented real vector bundles over spheres in greater detail.

267.3.1. **Complex vector bundles over spheres.** The following proposition, which is a variation on the Sphere Bundle-over-Sphere Proposition 241.20, gives a convenient description of $\mathrm{IsoVectBund}_{\mathbb{C}^n}(S^k)$:

Proposition 267.6. (Complex Vector Bundles-over-Sphere Proposition) The map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathrm{GL}(n,\mathbb{C})] & \to & \mathrm{IsoVectBund}_{\mathbb{C}^n}(S^k) \\ [\varphi] & \mapsto & [p(\varphi)] \end{array}$$

is a well-defined bijection.

Proof. Recall that we know by the Matrix Group-π_0-Proposition 4.54 that $\mathrm{GL}(n,\mathbb{C})$ is path-connected. We now see that the Complex Vector Bundles-over-Sphere Proposition 267.6 follows from the Sphere Bundle-over-Sphere Proposition 241.20 together with the Vector Bundle Definition Lemma 266.2. For the reader's convenience we outline the argument again.

Let $\overline{B}^k_\pm$ be two copies of $\overline{B}^k$. In the following we make the hopefully obvious identification $S^k = S^k_{\geq 0} \cup_{S^{k-1}} S^k_{\leq 0} = \overline{B}^k_+ \cup_{S^{k-1}} \overline{B}^k_-$ and henceforth we work with the topological space to the right.

Now let $p\colon E \to S^k = \overline{B}^k_+ \cup_{S^{k-1}} \overline{B}^k_-$ be an n-dimensional complex vector bundle. We write $E^\pm \coloneqq p^{-1}(\overline{B}^k_\pm)$. Since $\overline{B}^k_+$ and $\overline{B}^k_-$ are paracompact and contractible we know by the Vector Bundles-Homotopies Proposition 267.3 (2) that the vector bundles $E^\pm \to \overline{B}^k_\pm$ are trivializable. Clearly this implies that there exist continuous maps $\Phi^\pm\colon E^\pm \to \mathbb{C}^n$ such that $p \times \Phi^\pm\colon E^\pm \to \overline{B}^k_\pm \times \mathbb{C}^n$ are trivializations of $E^\pm \to \overline{B}^k_\pm$. Given any point $x \in \overline{B}^n_\pm$ we write $\Phi^\pm_b \coloneqq \Phi^\pm|_{E_b}\colon E_b \xrightarrow{\cong} \mathbb{C}^n$. Similar to the proof of the Vector Bundle-Complete Construction Lemma 266.4 (or the Vector Bundle–Action-on-Trivializations Proposition 266.1) we consider the map

$$\begin{array}{rcl} \Omega(E)\colon S^{k-1} & \to & \mathrm{GL}(n,\mathbb{C}) = \mathrm{Aut}(\mathbb{C}^n) \\ x & \mapsto & \Phi^-_b \circ (\Phi^+_b)^{-1} = \left(\mathbb{C}^n \xrightarrow{(\Phi^+_b)^{-1}} E_b \xrightarrow{\Phi^-_b} \mathbb{C}^n\right) \end{array}$$

Claim. The homotopy class $[\Omega(E)] \in [S^{k-1}, \mathrm{GL}(n,\mathbb{C})]$ does not depend on the choice of trivializations $\Phi^\pm$.

Proof. Suppose we are given two other maps $\Psi^\pm\colon E^\pm \to \mathbb{C}^n$ such that the maps $p \times \Psi^\pm\colon E^\pm \to \overline{B}^k_\pm \times \mathbb{C}^n$ are trivializations of $E^\pm \to \overline{B}^k_\pm$. By the Matrix Group-$\pi_0$-Proposition 4.54 we know that $\mathrm{GL}(n,\mathbb{C})$ is path-connected. This implies that there exists a path $\gamma\colon [1,2] \to \mathrm{GL}(n,\mathbb{C})$ with $\gamma(1) = \Phi^-_0 \circ (\Phi^+_0)^{-1}$ and $\gamma(2) = \Psi^-_0 \circ (\Psi^+_0)^{-1}$. The map

$$\begin{array}{rcl} S^{k-1} \times [0,3] & \to & \mathrm{GL}(n,\mathbb{C}) \\ (x,t) & \mapsto & \begin{cases} \Phi^-_{x\cdot(1-t)} \circ (\Phi^+_{x\cdot(1-t)})^{-1}, & \text{if } t \in [0,1], \\ \gamma(t), & \text{if } t \in [1,2], \\ \Psi^-_{x\cdot(t-2)} \circ (\Psi^+_{x\cdot(t-2)})^{-1}, & \text{if } t \in [2,3] \end{cases} \end{array}$$

is now the desired homotopy. ⊞

We leave it to the reader (or to the proof of the Sphere Bundle-over-Sphere Proposition 241.20) to verify that the maps

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathrm{GL}(n,\mathbb{C})] & \to & \mathrm{IsoVectBund}_{\mathbb{C}^n}(S^k) \\ \Omega\colon \mathrm{IsoVectBund}_{\mathbb{C}^n}(S^k) & \to & [S^{k-1}, \mathrm{GL}(n,\mathbb{C})] \end{array}$$

are actually inverses of one-another. ■

In the following lemma we consider complex line bundles over S^2 in greater detail:

Lemma 267.7. (Complex Line Bundle-over-S^2-Lemma) Given $n \in \mathbb{Z}$ we consider the map $S^1 \to \mathrm{U}(1)$ that is given by $z \mapsto (z^n)$. We consider the corresponding $\mathbb{C}$-vector bundle $p(\varphi_n)\colon p(\varphi_n) \to S^2$ from the Vector Bundle-Clutching Map Lemma 267.5. We also recall that by the Projective Space-Avatar Lemma 9.16 we have a diffeomorphism $S^2 \xrightarrow{\cong} \mathbb{C}\mathrm{P}^1$ which we use to identify these two topological spaces.

(1) For $n = 1$ there exist an isomorphism $p(\varphi_1) \to \mathfrak{T}\mathbb{C}\mathrm{P}^1$ from the $\mathbb{C}$-line bundle $p(\varphi_1)$ to the tautological complex line bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1$ over $\mathbb{C}\mathrm{P}^1$.
(2) For $n = 2$ the vector bundle $p(\varphi_2)$ is isomorphic to the complex tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^1$ from Definition 265.11.
(3) Given $n \in \mathbb{Z}$ let $f_n\colon S^2 \to S^2$ be the usual map of degree n. There exists an isomorphism $(f_n)^*(\mathfrak{T}\mathbb{C}\mathrm{P}^1) \to p(\varphi_n)$ over S^2 of complex line bundles.

Proof.

(1) In the Tautological Line Bundle Lemma 265.6 we gave an explicit atlas for the vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. Under the identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ the two sets U_0, U_1 correspond to $S^2 \setminus \{S\}$ and $S^2 \setminus \{N\}$. Comparing the two trivializations along S^1 we easily obtain the desired result.
(2) We will illustrate this statement below on page 5165. We will provide the details in Exercise 267.3.
(3) Let $n \in \mathbb{Z}$. We consider $p(\varphi_n) = (B_- \times \mathbb{C}) \cup (B_+ \times \mathbb{C})$. We consider S^2 as a subspace of $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$. We consider the map

$$\begin{array}{rcl} f_n\colon S^2 & \to & S^2 \\ (z,t) & \mapsto & (z^n, t). \end{array}$$

One can easily verify that the map

$$\begin{array}{rcl} f_n^* p(\varphi_1) & \to & p(\varphi_n) \\ [(z_\pm, (z_\pm^n, w))] & \mapsto & [(z_\pm, w)] \end{array}$$

is well-defined and that it is an isomorphism over S^2 of complex line bundles. ■

267.3.2. **Oriented real vector bundles over spheres.** The reader might wonder why we formulated the Complex Vector Bundles-over-Sphere Proposition 267.6 for complex vector bundles instead of real vector bundles. The reason is that in the proof of the Complex Vector Bundles-over-Sphere Proposition 267.6 we used that $\mathrm{GL}(n,\mathbb{C})$ is path-connected. In the Matrix Group-π_0-Proposition 4.54 we saw that in contrast $\mathrm{GL}(n,\mathbb{R})$ has two path-components, one of which is $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$. For formulate a "real" analog of the Complex Vector Bundles-over-Sphere Proposition 267.6 it is best to introduce the following notation.

Notation 267.8. Let X be a topological space.

(1) We say that two oriented vector bundles $p\colon E \to X$ and $q\colon F \to X$ are isomorphic if there exists an isomorphism $\Phi\colon E \to F$ of vector bundles as defined in Definition 265.2, such that for each $x \in X$ the map $E_b \to F_b$ is an orientation-preserving isomorphism.

(2) Let $n \in \mathbb{N}_0$. We denote by $\operatorname{IsoVectBund}^+_{\mathbb{R}^n}(X)$ the set of isomorphism classes of oriented n-dimensional$\mathbb{R}$-vector bundles over X.

Now we can formulate the following analog of the Complex Vector Bundles-over-Sphere Proposition 267.6.

Proposition 267.9. (Real Vector Bundle-over-Sphere Proposition) The map

$$\begin{array}{rcl} \Theta\colon [S^{k-1}, \mathrm{GL}_+(n,\mathbb{R})] & \to & \operatorname{IsoVectBund}^+_{\mathbb{R}^n}(S^k) \\ [\varphi] & \mapsto & [\Theta(\varphi)] \end{array}$$

is a well-defined bijection.

Proof. The proof if is quite similar to the proof of the Complex Vector Bundles-over-Sphere Proposition 267.6. We leave it to the reader to verify that the statement regarding the orientations works out. ■

Remark. Similar to the above we have the following bijections:

$$\operatorname{IsoVectBund}^{2,+}_{\mathbb{R}}(S^2) \xleftarrow[\uparrow]{\cong} [S^1, \mathrm{GL}_+(2,\mathbb{R})] \xleftarrow[\uparrow]{\cong} [S^1, O(2)] = [S^1, S^1] \xrightarrow[\uparrow]{[f]\mapsto \deg(f)} \mathbb{Z}.$$

bijection by the Matrix Groups Deformation Retract Proposition 19.4 the Deformation Retract–Homotopy Equivalence Lemma 19.10, the Homotopy Equivalence-Reformulation Lemma 19.13 and the discussion on page 240

bijection given by the Real Vector Bundle-over-Sphere Proposition 267.9

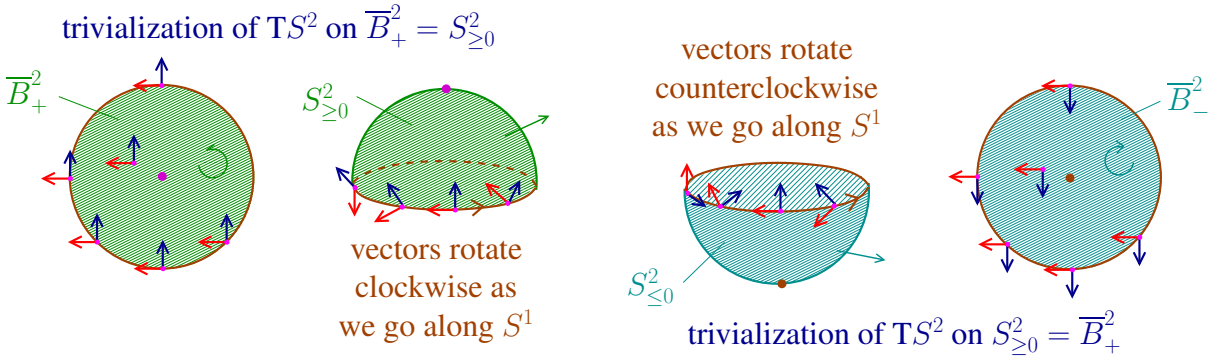
group isomorphism by the Degree Isomorphism Proposition 169.2

Given an oriented 2-dimensional real vector bundle $p\colon E \to S^2$ we refer to the corresponding integer on the right as the *Euler number of the vector bundle* $p\colon E \to S^2$. In Exercise 267.4 we will see that this definition is consistent with the notion of an Euler number from Definition 265.32. □

Example. We just introduced the Euler number of an oriented 2-dimensional real vector bundle over S^2. In this final example we will try to understand the Euler number of the tangent bundle $\mathrm{T}S^2$ of S^2. In the figure below we try our best to illustrate the situation. We show two trivializations of the $\mathrm{T}S^2$ over $\overline{B}^2_+ = S^2_{\geq 0}$ and $S^2_{\leq 0} = \overline{B}^2_-$. We can restrict the two trivializations of $\mathrm{T}S^2$ to S^1.[2849] As we go along S^1 we see that at $z \in S^1$ we have to rotate the trivialization coming from $S^2_{\geq 0}$ by z^2 to obtain the trivialization coming from $S^2_{\leq 0}$. This implies that the Euler number of $\mathrm{T}S^2$ equals $+2$.
The author is full aware that the figure below is only mildly helpful and we encourage the readers to draw pictures themselves. □

Remark. The Complex Vector Bundles-over-Sphere Proposition 267.6 and the Real Vector Bundle-over-Sphere Proposition 267.9, together with the discussion in Section 140.3, tell us that the key to understanding vector bundles over spheres lies in the understanding of the

[2849] Perhaps it is worth stressing, we consider the restriction of $\mathrm{T}S^2$ to S^1, we do not consider $\mathrm{T}S^1$.



homotopy groups of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}_+(n,\mathbb{R})$. It turns out that this is a difficult problem. We will obtain some partial calculations in Chapter 299. □

267.4. Exercises for Chapter 267.

Exercise 267.1. Let X be a topological space that is Hausdorff and paracompact. Furthermore let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by $i_0 \colon X \xrightarrow{x \mapsto (x,0)} X \times [0,1]$ and $i_1 \colon X \xrightarrow{x \mapsto (x,1)} X \times [0,1]$ the natural embeddings. In the Vector Bundle-over-Product Lemma 267.2 we showed that given any $\mathbb{K}$-vector bundle $p \colon E \to X \times [0,1]$ the pullbacks $i_0^* E$ and $i_1^* E$ are isomorphic. Show in an example that the isomorphism that we constructed in the proof of the Vector Bundle-over-Product Lemma 267.2 depends in general on the choices we make in the proof.

Exercise 267.2. As on page 5119 we consider the Möbius vector bundle:

$$\begin{array}{rcl} M := ([0,1] \times \mathbb{R})/(0,v) \sim (1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

In this exercise we want to give another proof that the Möbius vector bundle $p \colon M \to S^1$ is not orientable. This can be seen as follows: We consider the map $f \colon [0,1] \to S^1$ that is given by $f(t) = \exp(2\pi\,\mathrm{i}t)$. By the Vector Bundles-Homotopies Proposition 267.3 the pullback f^*M is a trivializable bundle. It follows from the Vector Bundle-Orientation Functorial Lemma 265.27 that if M was orientable, then we could equip the pullback with a corresponding orientation such that the corresponding vector bundle morphism is orientation-preserving. How can this be lead to a contradiction?
Remark. On page 5136 we gave an alternative proof of this statement.

Exercise 267.3. Given $n \in \mathbb{Z}$ we consider the map $S^1 \to \mathrm{U}(1)$ that is given by $z \mapsto (z^n)$. We consider the corresponding $\mathbb{C}$-vector bundle $p(\varphi_n) \colon p(\varphi_n) \to S^2$ from the Vector Bundle-Clutching Map Lemma 267.5. We also recall that by the Projective Space-Avatar Lemma 9.16 we have a diffeomorphism $S^2 \xrightarrow{\cong} \mathbb{C}\mathrm{P}^1$. Show that the $\mathbb{C}$-line bundle $p(\varphi_2)$ is isomorphic to the complex tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^1$ from Definition 265.11.

Exercise 267.4. Let $p \colon E \to S^2$ be an oriented 2-dimensional real vector bundle. Show that the Euler number from page 5165 agrees with the Euler number from Definition 265.32.

Exercise 267.5. Show that there exists an n-dimensional smooth manifold such that for the tangent bundle $p \colon \mathrm{T}M \to M$ the pullback under the map $p \colon \mathrm{T}M \to M$ (which is now an n-dimensional real vector bundle over $\mathrm{T}M$) is not trivializable.
Remark. This is in contrast to Exercise 240.27.

Exercise 267.6. We consider the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ and the Hopf map $H\colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$. Is the pullback $H^*\mathfrak{T}\mathbb{C}\mathrm{P}^1$ trivializable or not? If yes, provide an explicit trivialization.

Exercise 267.7. Does there exist a vector bundle over S^3 that is not trivializable?

CHAPTER 268

Linear algebraic constructions on vector bundles

In this chapter we will use the tools from linear algebra to construct many new vector bundles out of given vector bundles. In particular we will introduce the direct sum, the tensor product and the dual of vector bundles. The most important construction will be the direct sum (also called Whitney sum), which we will use very frequently in the coming chapters.

This chapter is organized as follows:

- In Section 268.1 we will give the straightforward definitions of the product and of the Whitney sum of two vector bundles.
- In Section 268.2 we will recall several basic constructions in linear algebra.
- In Section 268.3 we will introduce the notion of a continuous (or smooth) functor in linear algebra. We will see that many constructions, e.g. direct sums, tensor products, dual space can be viewed as smooth functors.
- In Section 268.4 we will see that smooth functors on vector spaces lead to functors on vector bundles.
- In Section 268.5 we will apply the general results from Section 268.4 to specific smooth functors. This will allow us to define, among others, the direct sum and the tensor product of two vector bundles.
- In Section 268.6 we will study the relationship between the tangent bundle and the tautological vector bundle on projective spaces.
- In Section 268.7 we will provide exercises for Chapter 268.

In the next Chapter 269 we will study inner products on vector bundles.

268.1. **The product and the Whitney sum of vector bundles.** In this section we start out with two rather simply minded construction of new vector bundles out of given vector bundles, namely the product and the Whitney sum. Let us begin with the product of vector bundles, which is defined in the most obvious way:

Definition 268.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) m-dimensional $\mathbb{K}$-vector bundle and let $q\colon F \to C$ be a (smooth) n-dimensional $\mathbb{K}$-vector bundle. If we are dealing with smooth bundles, then we assume that B and C are smooth manifolds such that $\partial B = \varnothing$ or such that $\partial C = \varnothing$.[2850] The map

$$\begin{array}{rccc} \operatorname{diag}(p,q)\colon & E \times F & \to & B \times C \\ & (e,f) & \mapsto & (p(e), q(f)) \end{array}$$

is easily seen to be a (smooth) $(m+n)$-dimensional $\mathbb{K}$-vector bundle. It is called the product of the $\mathbb{K}$-vector bundles $p\colon E \to B$ and $q\colon F \to C$.

We consider one interesting instance of a product of vector bundles:

Proposition 268.2. (Tangent Bundle-of-Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. To avoid smooth manifolds with corner we assume that $\partial M = \varnothing$ or $\partial N = \varnothing$. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the natural projections. The map[2851]

$$\begin{array}{rcl} \mathrm{T}(M\times N) & \to & \mathrm{T}M \times \mathrm{T}N \\ ((x,y),v) & \mapsto & ((x, \underbrace{\mathrm{D}p_{(x,y)}(v)}_{\in \mathrm{T}_xM}), (y, \underbrace{\mathrm{D}q_{(x,y)}(v)}_{\in \mathrm{T}_yN})) \\ \uparrow & & \\ \text{where } v \in \mathrm{T}_{(x,y)}(M\times N) & & \end{array}$$

is an isomorphism of smooth $(m+n)$-dimensional real vector bundles over $M \times N$. A similar results holds for the tangent bundles of complex manifolds.

Proof. We need to show that the map

$$\begin{array}{rcl} \Xi\colon \mathrm{T}(M\times N) & \to & \mathrm{T}M \times \mathrm{T}N \\ ((x,y),v) & \mapsto & ((x, \mathrm{D}p_{(x,y)}(v)), (y, \mathrm{D}q_{(x,y)}(v))) \end{array}$$

is an isomorphism of smooth $(m+n)$-dimensional real vector bundles over $M \times N$. By the Vector Bundle Isomorphism Criterion 265.40 it suffices to show that the map Ξ is a morphism of smooth real vector bundles in the sense of Definition 265.8, i.e. we need to prove the following:

(1) We need to show that the map Ξ is smooth.
(2) We need to show that the map Ξ induces an isomorphism on each fiber.

We now take care of these two statements:

(1) By the Smooth Manifold Product Proposition 25.34 (1b) it suffices to show that the maps $p_*\colon \mathrm{T}(M\times N) \to \mathrm{T}M$ and $q_*\colon \mathrm{T}(M\times N) \to \mathrm{T}N$ are smooth. But this follows from the Tangent Bundle–Induced Morphism Lemma 265.12 and the fact that by the Smooth Manifold Product Proposition 25.34 (1a) we know that the projections $p\colon M \times N \to M$ and $q\colon M \times N \to N$ are smooth.
(2) Let $x_0 \in M$ and let $y_0 \in N$. It follows easily from the definitions that it suffices to show that the map

$$(\mathrm{D}p_{(x_0,y_0)}, \mathrm{D}q_{(x_0,y_0)})\colon \mathrm{T}_{(x_0,y_0)}(M\times N) \to \mathrm{T}_{x_0}M \oplus \mathrm{T}_{y_0}N$$

is an isomorphism. To do so we consider the two smooth embeddings

$$\begin{array}{rcl} i_{y_0}\colon M & \to & M\times N \\ x & \mapsto & (x,y_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j_{x_0}\colon N & \to & M\times N \\ y & \mapsto & (x_0,y). \end{array}$$

As in the proof of the Product Orientation Proposition 31.48 we see that it follows easily from the chain rule that the map

$$(\mathrm{D}i_{y_0})_{x_0} \oplus (\mathrm{D}i_{x_0})_{y_0}\colon \mathrm{T}_{x_0}M \oplus \mathrm{T}_{y_0}N \to \mathrm{T}_{(x_0,y_0)}(M\times N)$$

is an inverse to $(\mathrm{D}p_{(x_0,y_0)}, \mathrm{D}q_{(x_0,y_0)})\colon \mathrm{T}_{(x_0,y_0)}(M\times N) \to \mathrm{T}_{x_0}M \times \mathrm{T}_{y_0}N$. ∎

[2850]This hypothesis allows us, by the Smooth Manifold Product Proposition 25.34, to view $B \times C$ again as a smooth manifold. If B and C both had non-empty boundary we would have to walk into the realm of smooth manifolds with corner, which we want to avoid.
[2851]We equip $M \times N$ with the smooth manifold structure coming from the Smooth Manifold Product Proposition 25.34.

In the following we are mostly interested in vector bundles over a fixed topological space or a fixed smooth manifold. This leads us to the following variation on the product of two vector bundles:

Definition 268.3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) m-dimensional $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be a (smooth) n-dimensional $\mathbb{K}$-vector bundle over the same topological space B. We will see in Exercise 268.1 that the map

$$\begin{array}{rcl} E \oplus F \coloneqq \{(e,f) \in E \times F \,|\, p(e) = q(f)\} & \to & B \\ (e,f) & \mapsto & p(e) \end{array}$$

is a (smooth) $(m+n)$-dimensional $\mathbb{K}$-vector bundle. It is called the Whitney sum of the $\mathbb{K}$-vector bundles E and F.[2852]

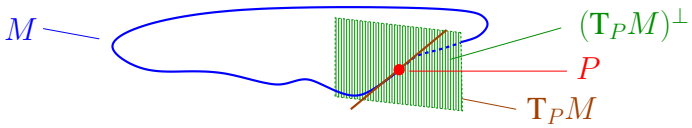
Examples.

(1) It follows easily from the definitions that the Whitney sum of two trivializable vector bundles is again trivializable.

(2) Let $M \subseteq \mathbb{R}^n$ be a submanifold. We consider the following map:

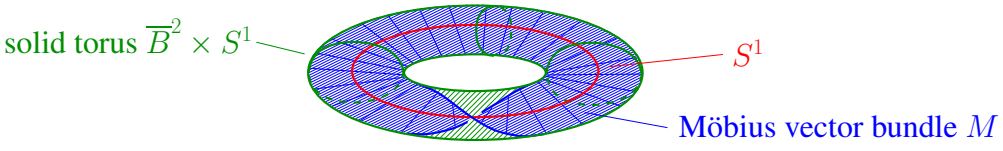
$$\begin{array}{rcl} \Xi\colon \mathrm{T}M \oplus \mathrm{N}M & \to & M \times \mathbb{R}^n \\ ((P,v),(P,w)) & \mapsto & (P, v+w). \end{array}$$

Using the Topological-Product Proposition 7.2 it is elementary to verify that this map is continuous. Note that elementary linear algebra shows that for any point $P \in M$ we have the equality $\mathrm{T}_P M \oplus (\mathrm{T}_P M)^\perp = \mathbb{R}^n$. It follows from this observation that the above map Ξ is a morphism of real vector bundles. Since Ξ is a morphism over id_M we obtain from the Vector Bundle Isomorphism Criterion 265.40 that Ξ is in fact an isomorphism of real vector bundles. Put differently, the Whitney sum of the tangent bundle and the normal vector bundle is a trivializable real vector bundle.



(3) We consider the sphere $S^2 \subseteq \mathbb{R}^3$. On page 5135 we saw that the tangent bundle $\mathrm{T}S^2$ is non-trivial and on page 5135 we saw that the normal vector bundle $\mathrm{N}S^2$ is trivializable. Together with (2) this shows that the Whitney sum of a non-trivializable real vector bundle with a trivializable real vector bundle can be trivial.

(4) Let $p\colon M \to S^1$ be the Möbius vector bundle from page 5119. We claim that the Whitney sum of two copies of the Möbius vector bundle is trivializable. We will now try to visualize this statement.



[2852] In Definition 268.26 we will construct the direct sum of two $\mathbb{K}$-vector bundles. Furthermore in the Whitney Sum–Direct Sum Lemma 268.27 we will see that the direct sum is naturally isomorphic to the *Whitney sum*. This perhaps justifies the notation $E \oplus F$.

We think of one copy of the Möbius vector bundle M as sitting in the solid torus $\overline{B}^2 \times S^1$. If we take the normal vector bundle of $M \subseteq \mathbb{R}^3$, restricted to $S^1 \subseteq M$, then we get another 1-dimensional real vector bundle $N \to S^1$. It is elementary to see that this vector bundle is isomorphic to the Möbius bundle. The Whitney sum of M and N gives us the normal vector bundle of $S^1 \subseteq \overline{B}^2 \times S^1$, which is clearly trivializable. The above argument is hopefully very intuitive, but it is admittedly not very precise. Fortunately it is not hard to write down a completely explicit trivialization of $M \oplus M$. We will do so in Exercise 268.2. □

In the following two lemmas we will see that given two vector bundles the "product construction" and the "Whitney sum construction" are the two sides of the same coin:

Lemma 268.4. (Whitney Sum-via-Diagonal Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) m-dimensional $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be a (smooth) n-dimensional $\mathbb{K}$-vector bundle. As in Definition 268.1 we consider the product bundle

$$\begin{array}{rcl} \operatorname{diag}(p,q)\colon E \times F & \to & B \times B \\ (e,f) & \mapsto & (p(e),q(f)). \end{array}$$

Recall that this is an $(m+n)$-dimensional $\mathbb{K}$-vector bundle. We denote by $d\colon B \to B \times B$ the diagonal map that is given by $d(b) = (b,b)$. In this setting the map

$$\begin{array}{rcl} E \oplus F & \xrightarrow{\cong} & d^*(E \times F) \\ (e,f) & \mapsto & (b,(e,f)) \end{array}$$

is an isomorphism of (smooth) $(m+n)$-dimensional $\mathbb{K}$-vector bundle over B.

Proof. This lemma follows easily from the Vector Bundle–Pullback Proposition 265.36. ■

Lemma 268.5. (Bundle Product-as-Whitney Sum Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be (smooth) $\mathbb{K}$-vector bundle and let $q\colon F \to C$ be a (smooth) $\mathbb{K}$-vector bundle. We denote by $\operatorname{pr}_B\colon B \times C \to B$ and $\operatorname{pr}_C\colon B \times C \to C$ the natural isomorphisms. We have natural isomorphisms

$$\begin{array}{rcl} \operatorname{pr}_B^*E \oplus \operatorname{pr}_C^*F & \to & E \times F \\ (((b,c),e),((b,c),f)) & \mapsto & (e,f) \end{array} \quad \& \quad \begin{array}{rcl} E \times F & \to & \operatorname{pr}_B^*E \oplus \operatorname{pr}_C^*F \\ (e,f) & \mapsto & (((p(e),q(c)),q(f)),((p(e),q(f)),f)) \end{array}$$

of (smooth) $\mathbb{K}$-vector bundles over $B \times C$, which are inverses of one another.

Proof. It follows easily from the Topological-Product Proposition 7.2 and the Smooth Manifold Product Proposition 25.34 that the given map $\operatorname{pr}_B^*E \oplus \operatorname{pr}_B^*F \to E \times F$ is continuous (smooth). Furthermore one can easily verify that the given map is fiberwise an isomorphism. It thus follows from the Vector Bundle Isomorphism Criterion 265.40 that the given map to the left is vector bundle isomorphism. Finally note that it is clear that the maps are inverses of one another. In particular the map to the right is also a vector bundle isomorphism. ■

We continue with the following variation on the Tangent Bundle-of-Product Proposition 268.2, which we will need later on:

Proposition 268.6. (Tangent Bundle-of-Product Proposition) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. To avoid

smooth manifolds with corner we assume that $\partial M = \varnothing$ or $\partial N = \varnothing$. We denote by $\mathrm{pr}_M \colon M \times N \to M$ and $\mathrm{pr}_N \colon M \times N \to N$ the natural projections. The map

$$\begin{array}{rcl} \mathrm{T}(M \times N) & \to & \mathrm{pr}_M^*(\mathrm{T}M) \oplus \mathrm{pr}_Q^*(\mathrm{T}N) \\ ((x,y),v) & \mapsto & \left(((x,y),(x,\mathrm{D}(\mathrm{pr}_M)_{(x,y)}(v))) \,,\, ((x,y),(y,\mathrm{D}(\mathrm{pr}_N)_{(x,y)}(v)))\right) \end{array}$$

is an isomorphism of smooth $(m+n)$-dimensional real vector bundles over $M \times N$.

Example. Recall that according to Definition 265.13 a smooth manifold W is called *parallelizable* if its tangent bundle $\mathrm{T}W$ is trivializable. Now let M and N be two closed smooth manifolds. If M and N are both parallelizable, then it follows from the Tangent Bundle-of-Product Proposition 268.6 and the discussion on page 5145 and the discussion on page 5170 that the product smooth manifold $M \times N$ is also parallelizable. □

Proof. Recall that in the original Tangent Bundle-of-Product Proposition 268.2 we showed that the map

$$\begin{array}{rcl} \mathrm{T}(M \times N) & \to & \mathrm{T}M \times \mathrm{T}N \\ ((x,y),v) & \mapsto & ((x,\mathrm{D}(\mathrm{pr}_M)_{(x,y)}(v)),(y,\mathrm{D}(\mathrm{pr}_N)_{(x,y)}(v))) \end{array}$$

is an isomorphism of $(m+n)$-dimensional real vector bundles over $M \times N$. The proposition now follows from the Bundle Product-as-Whitney Sum Lemma 268.5. ■

268.2. **Interlude: Constructions in linear algebra.** Our next goal is to introduce constructions of vector bundles which are very specific to the setting of vector bundles. For example we would like to introduce the tensor product of two vector bundles. These linear algebraic constructions cannot be achieved by the "general machine" that we used in the previous section. As we will see, it will take a little while to explain these new constructions.

In this purely algebraic section we will remind the reader of some constructions in linear algebra. We start out recalling and introducing the following notation:

Notation 268.7. Let $\mathbb{K}$ be a field.

(1) As in Definition 5.4 we denote by $\mathcal{V}ec_{\mathbb{K}}$ the category of $\mathbb{K}$-vector spaces. As on page 231 we can also consider the subcategory $\mathcal{V}ec_{\mathbb{K}}^{\cong}$ where the only morphisms we are interested in are isomorphisms.

(2) We denote by $\mathcal{FDV}ec_{\mathbb{K}}$ the full subcategory of $\mathcal{V}ec_{\mathbb{K}}$ given by the finite-dimensional $\mathbb{K}$-vector spaces and similar to (1) we also consider the subcategory $\mathcal{FDV}ec_{\mathbb{K}}^{\cong}$ given by the finite-dimensional $\mathbb{K}$-vector spaces where the only morphisms we are interested in are isomorphisms.

Next we recall Definition 5.7:

Definition 268.8. Let $\mathcal{C}_1, \ldots, \mathcal{C}_d$ be categories.

(1) As in Definition 5.7 the product category $\mathcal{C}_1 \times \cdots \times \mathcal{C}_d$ is defined as the category where

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}_1 \times \ldots \mathcal{C}_d) & \coloneqq & \{C_1 \times \cdots \times C_d \,|\, C_i \in \mathrm{Ob}(\mathcal{C}_i)\} \\ \mathrm{Mor}_{\mathcal{C}_1 \times \cdots \times \mathcal{C}_d}(C_1 \times \cdots \times C_d, D_1 \times \cdots \times D_d) & \coloneqq & \mathrm{Mor}_{\mathcal{C}_1}(C_1,D_1) \times \ldots \mathrm{Mor}_{\mathcal{C}_d}(C_d,D_d) \end{array}$$

with the composition given by $(f_1,\ldots,f_d) \circ (g_1,\ldots,g_d) \coloneqq (f_1 \circ g_1, \ldots, f_d \circ g_d)$.

(2) Given a category $\mathcal{C}$ and given $n \in \mathbb{N}$ we write $\mathcal{C}^n \coloneqq \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n \text{ times}}$.

Now we turn to various constructions in linear algebra.

268.2.1. **The direct sum of vector spaces.** First we recall Notation 6.9 of the direct sum construction of two vector spaces:

Notation 268.9. Let $\mathbb{K}$ be a field and let V_1, V_2 be two $\mathbb{K}$-vector spaces. We define the direct sum

$$V_1 \oplus V_2 \;:=\; \{(v_1, v_2) \,|\, v_1 \in V_1, v_2 \in V_2\}.$$

Lemma 268.10. (Direct Sum-Functor-Lemma) Let $\mathbb{K}$ be a field and let $d \in \mathbb{N}$. The maps

$$\begin{array}{rcl} (V_1, V_2) & \mapsto & V_1 \oplus V_2 \\ (f_1 \colon V_1 \to W_1, f_2 \colon V_2 \to W_2) & \mapsto & \left(\begin{array}{rcl} \operatorname{diag}(f_1, f_2) \colon V_1 \oplus V_2 & \to & W_1 \oplus W_2 \\ (v_1, v_2) & \mapsto & (f_1(v_1), f_2(v_2)) \end{array} \right) \end{array}$$

define a covariant functor from the product category $(\mathcal{V}ec_{\mathbb{K}})^2$ of $\mathbb{K}$-vector spaces to the category $\mathcal{V}ec_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces. We can also replace $\mathcal{V}ec_{\mathbb{K}}$ by $\mathcal{FDV}ec_{\mathbb{K}}$ and $\mathcal{FDV}ec_{\mathbb{K}}^{\cong}$.

Proof. One can either argue that the lemma follows formally from the Product-of-Sets-and-Groups Lemma 6.4 and the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8, or one can say, that the lemma follows trivially from the definitions. ■

268.2.2. **Tensor products of vector spaces.** In Chapter 159 we introduced the tensor product of abelian groups. Not surprisingly we also have a corresponding notion of the tensor product of vector spaces over a fixed field:[2853]

Definition 268.11. Let $\mathbb{K}$ be a field and let V and W be two $\mathbb{K}$-vector spaces. We define the tensor product $V \otimes_{\mathbb{K}} W$ to be the following $\mathbb{K}$-vector space:

$$V \otimes_{\mathbb{K}} W \;:=\; \left\{ \begin{array}{l} \mathbb{K}\text{-vector spaces that is generated by all} \\ \text{symbols of the form } v \otimes w \text{ with } v \in V,\, w \in W \end{array} \right\} \Big/ N(V, W)$$

where $N(V, W)$ is the linear subspace that is generated by the elements of the form

$$(\mu v + \mu' v') \otimes \nu w - (\mu v \otimes \nu w + \mu' v' \otimes \nu w) \quad \text{and} \quad \mu v \otimes (\nu w + \nu' w') - (\mu v \otimes \nu w + \mu v \otimes \nu' w')$$

with $v, v' \in V$, $w, w' \in W$ and with $\mu, \mu', \nu, \nu' \in \mathbb{K}$. For $v \in V$ and $w \in W$ we denote, by a slight abuse of notation, the image of $v \otimes w$ in $V \otimes W$ again by $v \otimes w$.

Remark. Note that the tensor product of vector spaces satisfies the obvious analogs of the tensor product of abelian groups, see the Tensor Product–Universal Property Lemma 159.4, the Tensor Commutative-Associative Lemma 159.11 and the Tensor–Direct Sum Lemma 159.15. Surely there is no need to spell these statements out in detail. □

Lemma 268.12. (Tensor Product-Functor-Lemma) Let $\mathbb{K}$ be a field. The maps

$$\begin{array}{rcl} (V_1, V_2) & \mapsto & V_1 \otimes V_2 \\ (f_1 \colon V_1 \to W_1, f_2 \colon V_2 \to W_2) & \mapsto & \left(\begin{array}{rcl} f_1 \otimes f_2 \colon V_1 \otimes V_2 & \to & W_1 \otimes W_2 \\ v_1 \otimes v_2 & \mapsto & f_1(v_1) \otimes f_2(v_2) \end{array} \right) \end{array}$$

define a covariant functor from the product category $(\mathcal{V}ec_{\mathbb{K}})^2$ of $\mathbb{K}$-vector spaces to the category $\mathcal{V}ec_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces. We can also replace $\mathcal{V}ec_{\mathbb{K}}$ by $\mathcal{FDV}ec_{\mathbb{K}}$ and $\mathcal{FDV}ec_{\mathbb{K}}^{\cong}$.

Proof. As in the Tensor–Map Composition Lemma 159.7 this lemma follows easily from the definitions. ■

[2853] Later, in Section 312.2 we will introduce the more general tensor product of modules over a ring.

The following lemma gives us a neat application of tensor products:

Lemma 268.13. (Tensor-with-$\mathbb{C}$-Lemma)

(1) Let V be a real vector space. We consider the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$. The map

$$\begin{array}{rcl} V \otimes_{\mathbb{R}} \mathbb{C} \times \mathbb{C} & \to & V \otimes_{\mathbb{R}} \mathbb{C} \\ \left(\sum\limits_{k=1}^{m} v_i \otimes z_i, w \right) & \mapsto & \sum\limits_{k=1}^{m} v_i \otimes (z_i \cdot_{\mathbb{C}} w) \end{array}$$

defines a $\mathbb{C}$-vector space structure on $V \otimes_{\mathbb{R}} \mathbb{C}$. We refer to $V \otimes_{\mathbb{R}} \mathbb{C}$ as the complexification of V.

(2) The maps

$$\begin{array}{rcl} V & \mapsto & V \otimes_{\mathbb{R}} \mathbb{C} \\ (f \colon V \to W) & \mapsto & (f \otimes \mathrm{id}_{\mathbb{C}} \colon V \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}) \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces to the category $\mathcal{V}ec_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces. This functor restricts to the subcategories given by finite-dimensional vector spaces and where we only consider isomorphisms.

Proof. Once again we do not feel like working out the details of this elementary proof. Thus we leave it to the reader to provide a proof. ∎

268.2.3. **Homomorphisms of vector spaces.** We continue with the following well-known notation, which is inspired by Definition 71.1 and the discussion on page 1503:

Notation 268.14. Let $\mathbb{K}$ be a field. Given any $\mathbb{K}$-vector space V we consider the dual $\mathbb{K}$-vector space

$$V^* \ := \ \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}).$$

We all know that the dual space is *contravariantly* functorial in V. In the later discussion we will only be interested in the category $\mathcal{FDV}ec^{\cong}$. Rather amusingly, with respect to this category we can treat the dual space as *covariantly* functorial:

Lemma 268.15. (Dual Space–Functor Lemma) Let $\mathbb{K}$ be a field. The maps

$$\begin{array}{rcl} V & \mapsto & V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) \\ (f \colon V \xrightarrow{\cong} W) & \mapsto & \left(\begin{array}{rcl} V^* & \to & W^* \\ (\varphi \colon V \to \mathbb{K}) & \mapsto & (\varphi \circ (f^{-1}) \colon W \to \mathbb{K}) \end{array} \right) \end{array}$$

form a *covariant* functor from $\mathcal{FDV}ec_{\mathbb{K}}^{\cong}$ to $\mathcal{FDV}ec_{\mathbb{K}}^{\cong}$.

Proof. The proof is verbatim the same as the proof of the Tensor-with-$\mathbb{C}$-Lemma 268.13. ∎

Next we recall Notation 72.6:

Notation 268.16. Let $\mathbb{K}$ be a field, let V be a $\mathbb{K}$-vector space and let $d \in \mathbb{N}_0$. As in Definition 72.2 we say that a map $\omega \colon V^d \to \mathbb{K}$ is an alternating d-form on V if it is multilinear and alternating in the sense of Definition 72.1. We set

$$\textstyle\bigwedge^d V^* \ := \ \text{the real vector space of all alternating } d\text{-forms on } V.$$

Note that by definition $\bigwedge^1 V^* = V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$. Furthermore, as we discussed on page 1503, we have a natural isomorphism $\bigwedge^0 V^* \cong \mathbb{K}$.

The following lemma can be viewed as a generalization of the Dual Space–Functor Lemma 268.15:

Lemma 268.17. (Alternating Form–Functor Lemma) Let $\mathbb{K}$ be a field and let $d \in \mathbb{N}_0$. The maps

$$\begin{array}{rcl} V & \mapsto & \wedge^d V^* \\ (f\colon V \xrightarrow{\cong} W) & \mapsto & \left(\begin{array}{ccc} \wedge^d V^* & \to & \wedge^d W^* \\ (\omega\colon V^d \to \mathbb{K}) & \mapsto & (\varphi \circ \operatorname{diag}(f^{-1}, \ldots, f^{-1})\colon W^d \to \mathbb{K}) \end{array} \right) \end{array}$$

form a *covariant* functor from the category $\mathcal{FDVec}_{\mathbb{K}}^{\cong}$ of $\mathbb{K}$-vector spaces to the category $\mathcal{FDVec}_{\mathbb{K}}^{\cong}$ of $\mathbb{K}$-vector spaces.

Proof. As in the proof of the Alternating Form-Functoriality Lemma 72.12 this statement follows easily from the definitions. ■

For completeness' sake we also recall the following standard notation:

Notation 268.18. Let $\mathbb{K}$ be a field. Given any two $\mathbb{K}$-vector spaces V_1, V_2 we consider the $\mathbb{K}$-vector space $\operatorname{Hom}_{\mathbb{K}}(V_1, V_2)$ of homomorphisms $V_1 \to V_2$.

Lemma 268.19. (Homomorphism–Functor Lemma) Let $\mathbb{K}$ be a field. The maps

$$\begin{array}{rcl} (V_1, V_2) & \mapsto & \operatorname{Hom}_{\mathbb{K}}(V_1, V_2) \\ (f_1\colon V_1 \xrightarrow{\cong} W_1, f_2\colon V_2 \xrightarrow{\cong} W_2) & \mapsto & \left(\begin{array}{ccc} \operatorname{Hom}_{\mathbb{K}}(V_1, V_2) & \to & \operatorname{Hom}_{\mathbb{K}}(W_1, W_2) \\ (\varphi\colon V_1 \to V_2) & \mapsto & (f_2 \circ \varphi \circ f_1^{-1}\colon W_1 \to W_2) \end{array} \right) \end{array}$$

define a covariant functor from the product category $(\mathcal{FDVec}_{\mathbb{K}}^{\cong})^2$ of $\mathbb{K}$-vector spaces to the category $\mathcal{FDVec}_{\mathbb{K}}^{\cong}$ of $\mathbb{K}$-vector spaces.

Proof. As above the lemma follows easily from the definitions. ■

268.3. **Smooth functors of vector spaces.** In the next section we will present a very general way to do "fiberwise" linear algebraic constructions. To make sense of this construction we will need to introduce the concept of a "continuous functor" or a "smooth functor". Before we can introduce this concept we should better recall the following convention which was introduced on page 183 and 667:

Convention 268.20. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let V be a finite-dimensional $\mathbb{K}$-vector space. There exists an isomorphism $\varphi\colon V \to \mathbb{K}^n$ for some $n \in \mathbb{N}_0$. We equip V with the unique topology that turns φ into a homeomorphism and we equip V with the unique smooth structure that turns φ into a diffeomorphism. Since any two such isomorphisms to $\mathbb{K}^n$ differ by a matrix in $\mathrm{GL}(n, \mathbb{K})$, and since such matrices correspond to homeomorphisms and diffeomorphisms we see that this definition (of the topology and the smooth structure on V) does not depend on the choice of φ.

This convention allows us to introduce the following concept:

Definition 268.21. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $d \in \mathbb{N}_0$. A covariant functor

$$F\colon (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^d \to \mathcal{FDVec}^{\cong}$$

is called continuous (respectively smooth) if for any choice of $\mathbb{K}$-vector spaces $V_1, \ldots, V_d$ and $W_1, \ldots, W_d$ the corresponding map[2854]

$$\begin{array}{ccc} \overbrace{\mathrm{Isom}_{\mathbb{K}}(V_1, W_1) \times \cdots \times \mathrm{Isom}_{\mathbb{K}}(V_d, W_d)}^{=\mathrm{Mor}_{(\mathcal{FDVec}_{\mathbb{K}}^{\cong})^d}((V_1,\ldots,V_d),(W_1,\ldots,W_d))} & \to & \overbrace{\mathrm{Isom}_{\mathbb{K}}(F(V_1, \ldots, V_d), F(W_1, \ldots, W_d))}^{=\mathrm{Mor}_{\mathcal{FDVec}_{\mathbb{K}}^{\cong}}(F(V_1,\ldots,V_d),F(W_1,\ldots,W_d))} \\ (\varphi_1, \ldots, \varphi_d) & \mapsto & F(\varphi_1, \ldots, \varphi_d) \end{array}$$

is continuous (respectively smooth).

Not surprisingly we will now see that all the functors that we have encountered above are in fact smooth:

Lemma 268.22. (Smooth Vector Space Functor Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $d \in \mathbb{N}_0$. We consider the covariant functors

$$\begin{array}{rcl} (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^2 & \to & \mathcal{FDVec}_{\mathbb{K}}^{\cong} \\ (V_1, V_2) & \mapsto & V_1 \oplus V_2 \end{array} \qquad \begin{array}{rcl} (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^2 & \to & \mathcal{FDVec}_{\mathbb{K}}^{\cong} \\ (V_1, V_2) & \mapsto & V_1 \otimes_{\mathbb{K}} V_2 \end{array} \qquad \begin{array}{rcl} \mathcal{FDVec}_{\mathbb{R}}^{\cong} & \to & \mathcal{FDVec}_{\mathbb{C}}^{\cong} \\ V & \mapsto & V \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

$$\begin{array}{rcl} \mathcal{Vec}_{\mathbb{K}}^{\cong} & \to & \mathcal{Vec}_{\mathbb{K}}^{\cong} \\ V & \mapsto & V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) \end{array} \qquad \begin{array}{rcl} \mathcal{Vec}_{\mathbb{K}}^{\cong} & \to & \mathcal{Vec}_{\mathbb{K}}^{\cong} \\ V & \mapsto & \wedge^d V^* \end{array} \qquad \begin{array}{rcl} (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^2 & \to & \mathcal{FDVec}_{\mathbb{K}}^{\cong} \\ (V_1, V_2) & \mapsto & \mathrm{Hom}_{\mathbb{K}}(V_1, V_2) \end{array}$$

from the Direct Sum-Functor-Lemma 268.10, the Tensor Product-Functor-Lemma 268.12, the Tensor-with-$\mathbb{C}$-Lemma 268.13, the Dual Space–Functor Lemma 268.15, the Alternating Form–Functor Lemma 268.17 and the Homomorphism–Functor Lemma 268.19. All these functors are smooth.

Proof. In the following we prove the statement for the tensor product, i.e. we will prove the following claim:

Claim. The following functor is smooth:

$$\begin{array}{rcl} (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^2 & \to & \mathcal{FDVec}_{\mathbb{K}}^{\cong} \\ (V_1, V_2) & \mapsto & V_1 \otimes_{\mathbb{K}} V_2. \end{array}$$

The proofs of the other statements of the lemma are quite similar and they are left to the reader.

Proof. Let V_1, V_2 and W_1, W_2 be finite-dimensional $\mathbb{K}$-vector spaces. By Definition 268.21 we need to show that the map

$$\begin{array}{rcl} \mathrm{Isom}_{\mathbb{K}}(V_1, W_1) \times \mathrm{Isom}_{\mathbb{K}}(V_2, W_2) & \to & \mathrm{Isom}_{\mathbb{K}}(V_1 \otimes V_2, W_1 \otimes W_2) \\ (f_1, f_2) & \mapsto & f_1 \otimes f_2 \end{array}$$

is smooth. If $\dim(V_1) \neq \dim(W_1)$ or $\dim(V_2) \neq \dim(W_2)$, then the left-hand is the empty set and there is nothing to show. So in the following discussion we can now assume that $\dim(V_1) = \dim(W_1) =: n_1$ and $\dim(V_2) = \dim(W_2) =: n_2$. We pick isomorphisms

[2854]First note $\mathrm{Hom}_{\mathbb{K}}(V_1, W_1), \ldots, \mathrm{Hom}_{\mathbb{K}}(V_d, W_d)$ and $\mathrm{Hom}_{\mathbb{K}}(F(V_1, \ldots, V_d), F(W_1, \ldots, W_d))$ are finite-dimensional vector spaces. Thus we obtain from Convention 268.20 that we can view both sides as topological spaces and smooth manifolds. In the current definition we consider open subsets of these topological spaces and smooth manifolds, so these are topological spaces and smooth manifolds in their own right.

$\Phi_1\colon \mathbb{K}^{n_1} \to V_1$, $\Psi_1\colon \mathbb{K}^{n_1} \to W_1$, $\Phi_2\colon \mathbb{K}^{n_2} \to V_2$ and $\Psi_2\colon \mathbb{K}^{n_2} \to W_2$. We obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathbb{K}}(V_1,W_1)\times\mathrm{Isom}_{\mathbb{K}}(V_2,W_2) & \xrightarrow{(f_1,f_2)\mapsto f_1\otimes f_2} & \mathrm{Isom}(V_1\otimes_{\mathbb{K}} V_2, W_1\otimes_{\mathbb{K}} W_2) \\
\cong\uparrow & & \uparrow\cong \\
\mathrm{Isom}_{\mathbb{K}}(\mathbb{K}^{n_1},\mathbb{K}^{n_1})\times\mathrm{Isom}_{\mathbb{K}}(\mathbb{K}^{n_2},\mathbb{K}^{n_2}) & \xrightarrow{(f_1,f_2)\mapsto f_1\otimes f_2} & \mathrm{Isom}(\mathbb{K}^{n_1}\otimes_{\mathbb{K}}\mathbb{K}^{n_2},\mathbb{K}^{n_1}\otimes_{\mathbb{K}}\mathbb{K}^{n_2}) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{GL}(n_1,\mathbb{K})\times\mathrm{GL}(n_2,\mathbb{K}) & \longrightarrow & \mathrm{GL}(n_1\cdot n_2,\mathbb{K}) \\
\downarrow & & \downarrow \\
\mathbb{K}^{\{1,\dots,n_1\}^2}\times\mathbb{K}^{\{1,\dots,n_2\}^2} & \xrightarrow[\substack{\text{under this map the pair}\\ (\varphi_1\colon\{1,\dots,n_1\}^2\to\mathbb{K},\varphi_2\colon\{1,\dots,n_2\}^2\to\mathbb{K})\text{ gets}\\ \text{sent to the map }(\{1,\dots,n_1\}\times\{1,\dots,n_2\})^2\to\mathbb{K}\text{ that}\\ \text{is given by }((i_1,i_2),(j_1,j_2)\mapsto\varphi_1(i_1,j_1)\cdot\varphi_2(i_2,j_2))}}{} & \mathbb{K}^{(\{1,\dots,n_1\}\times\{1,\dots,n_2\})^2}.
\end{array}$$

The claim follows from the following clarifications and observations:

(1) The top vertical maps are induced by the isomorphisms Φ_i and Ψ_i.

(2) (a) It follows by the same argument as in the Tensor–Direct Sum Lemma 159.15 and the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that $\{\mathfrak{e}_{i_1}\otimes\mathfrak{e}_{i_2}\}_{i_1=1,\dots,n_1,i_2=1,\dots,n_2}$ is a basis for the $\mathbb{K}$-vector space $\mathbb{K}^{n_1}\otimes_{\mathbb{K}}\mathbb{K}^{n_2}$.

(b) The middle vertical bijections are given by the usual reinterpretation of homomorphisms as matrices, in this case with respect to the bases $\mathfrak{e}_1,\dots,\mathfrak{e}_{n_1}$, $\mathfrak{e}_1,\dots,\mathfrak{e}_{n_2}$ and $\{\mathfrak{e}_{i_1}\otimes\mathfrak{e}_{i_2}\}_{i_1=1,\dots,n_1,i_2=1,\dots,n_2}$.

(3) The bottom vertical maps are just the inclusions.

(4) The bottom horizontal map is clearly smooth.

(5) One can easily verify that the diagram commutes. ∎

268.4. **Linear algebraic constructions on vector bundles.** The following proposition tells us that continuous (smooth) functors can be applied "fiberwise" to obtain new vector bundles out of given ones:

Proposition 268.23. (Vector Bundle Construction Proposition) Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, let $d\in\mathbb{N}_0$ and let $F\colon(\mathcal{FDVec}_{\mathbb{K}}^{\cong})^d\to\mathcal{FDVec}_{\mathbb{K}}^{\cong}$ be a continuous (smooth) covariant functor.

(1) Given d (smooth) $\mathbb{K}$-vector bundles $E_1\xrightarrow{p_1}B,\dots,E_d\xrightarrow{p_d}B$ we consider the map[2855]

$$\begin{array}{rcl} q\colon F(E_1,\dots,E_d):=\bigsqcup_{b\in B}F(E_{1,b},\dots,E_{d,b}) & \to & B \\ (b,v) & \mapsto & b. \end{array}$$

In the proof we will equip $F(E_1,\dots,E_d)$ with a topology (smooth manifold) structure such that $q\colon F(E_1,\dots,E_d)\to B$ becomes a (smooth) $\mathbb{K}$-vector bundle.

(2) Let $n_1,\dots,n_d\in\mathbb{N}_0$. We set $m:=\dim(F(\mathbb{K}^{n_1},\dots,\mathbb{K}^{n_d}))$. The vector bundles from (1) define a functor

$$\begin{array}{rcl} F\colon \mathcal{VectBund}_{\mathbb{K}^{n_1}}\times\cdots\times\mathcal{VectBund}_{\mathbb{K}^{n_d}} & \to & \mathcal{VectBund}_{\mathbb{K}^m} \\ (E_1\xrightarrow{p_1}B,\dots,E_d\xrightarrow{p_d}B) & \mapsto & (q\colon F(E_1,\dots,E_d)\to B) \end{array}$$

from the product category $\mathcal{VectBund}_{\mathbb{K}^{n_1}} \times \cdots \times \mathcal{VectBund}_{\mathbb{K}^{n_d}}$ of (smooth) $\mathbb{K}$-vector bundles to the category $\mathcal{VectBund}_{\mathbb{K}^m}$ of (smooth) m-dimensional $\mathbb{K}$-vector bundles. This functor has the property that we have a natural isomorphism

$$F(B \times \mathbb{K}^{n_1}, \dots, B \times \mathbb{K}^{n_d}) \;\cong\; B \times F(\mathbb{K}^{n_1}, \dots, \mathbb{K}^{n_d}).$$

Proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the following discussion, to simplify the notation, we deal with the case $d = 1$. Furthermore we will only consider the continuous case. The other cases are left to the reader.

Thus let $F\colon \mathcal{FDVec}_{\mathbb{K}}^{\cong} \to \mathcal{FDVec}_{\mathbb{K}}^{\cong}$ be a continuous functor.

(1) In the following let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle. Throughout the proof let $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n\}_{i\in I}$ be the family of all trivializations of the $\mathbb{K}$-vector bundle $p\colon E \to B$. Next we consider the map

$$\begin{array}{rccc} q\colon F(E) := \bigsqcup\limits_{b\in B} F(E_b) & \to & B \\ (b, v) & \mapsto & b. \end{array}$$

First we need to equip the set $F(E)$ with a suitable topology. To do so let $i \in I$.

(a) For $b \in B$ we denote by $\Phi_i|_{p^{-1}(b)}\colon p^{-1}(b) \to \{b\} \times \mathbb{K}^n$ the restriction of Φ_i.

(b) For each $b \in B$ we denote by $\pi\colon \{b\} \times \mathbb{K}^n \to \mathbb{K}^n$ the natural projection.

(c) Note that $\pi \circ \Phi_i|_{p^{-1}(b)}\colon E_b \to \mathbb{K}^n$ is an isomorphism of $\mathbb{K}$-vector spaces.

(d) Since F is a functor we see that $F(\pi \circ \Phi_i|_{p^{-1}(b)})\colon F(E_b) \to F(\mathbb{K}^n)$ is also an isomorphism of $\mathbb{K}$-vector spaces.

It follows from (d) that the following map is a bijection

$$\begin{array}{rccc} \widetilde{\Phi}_i\colon q^{-1}(U_i) & \to & U_i \times F(\mathbb{K}^n) \\ (b, v) & \mapsto & (b, F(\pi \circ \Phi_i|_{p^{-1}(b)})(v)). \\ \uparrow & & \\ \text{where } v \in F(E_b) & & \end{array}$$

Claim. For any $i, j \in I$ the map

$$\widetilde{\Phi}_j \circ (\widetilde{\Phi}_i|_{p^{-1}(U_i\cap U_j)})^{-1}\colon (U_i \cap U_j) \times F(\mathbb{K}^n) \;\to\; (U_i \cap U_j) \times F(\mathbb{K}^n)$$

is a homeomorphism.

Proof. Let $i, j \in I$. Essentially by definition of an n-dimensional $\mathbb{K}$-vector bundle we know that there exists a unique map $\Theta\colon U_i \cap U_j \to \mathrm{Aut}(\mathbb{K}^n)$ such that the following diagram commutes:

$$(U_i \cap U_j) \times \mathbb{K}^n \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap U_j) \xrightarrow[\cong]{\Phi_j} (U_i \cap U_j) \times \mathbb{K}^n.$$
$$(b,f) \mapsto (b, \Theta(b)(f))$$

[2855] Recall that given a family $\{A_b\}_{b\in B}$ of sets we defined in Notation 1.12 the disjoint union as

$$\bigsqcup_{b\in B} A_b \;:=\; \bigcup_{b\in B} (\{b\} \times A_b).$$

Next note that for any $(b,v) \in (U_i \cap U_j) \times F(\mathbb{K}^n)$ we have

$$\begin{array}{rcl}
(\widetilde{\Phi}_j \circ (\widetilde{\Phi}_i|_{p^{-1}(U_i \cap U_j)})^{-1})(b,v) & \overset{\text{definition of } \widetilde{\Phi}_j \text{ and } \widetilde{\Phi}_i}{\downarrow} & \\
& = & (b, (F(\pi \circ \Phi_j|_{p^{-1}(b)}) \circ F(\pi \circ \Phi_i|_{p^{-1}(b)})^{-1})(v)) \\
& = & (b, (F(\pi \circ \Phi_j|_{p^{-1}(b)} \circ (\pi \circ \Phi_i|_{p^{-1}(b)})^{-1}))(v)) \\
& \overset{\text{since } F \text{ is a covariant functor}}{\uparrow} & \\
& = & (b, F(\Theta(b))(v)). \\
& \overset{\text{by definition of } \Theta}{\uparrow} &
\end{array}$$

Thus we see that the map that we are interested in is given by

$$\begin{array}{rcl}
\widetilde{\Phi}_j \circ (\widetilde{\Phi}_i|_{p^{-1}(U_i \cap U_j)})^{-1} \colon (U_i \cap U_j) \times F(\mathbb{K}^n) & \to & (U_i \cap U_j) \times F(\mathbb{K}^n) \\
(b,v) & \mapsto & (b, F(\Theta(b))(v)).
\end{array}$$

Next note that by the Vector Bundle–Action-on-Trivializations Proposition 266.1 we know that the map $\Theta \colon U_i \cap U_j \to \mathrm{Aut}(\mathbb{K}^n)$ is continuous. Since F is a continuous functor we know that the map $F \colon \mathrm{Aut}(\mathbb{K}^n) \to \mathrm{Aut}(F(\mathbb{K}^n))$ is also continuous. In summary we see that the map

$$\begin{array}{rcl}
U_i \cap U_j & \to & \mathrm{Aut}(F(\mathbb{K}^n)) \\
b & \mapsto & F(\Theta(b))
\end{array}$$

is also continuous. This implies that the map $\widetilde{\Phi}_j \circ (\widetilde{\Phi}_i|_{p^{-1}(U_i \cap U_j)})^{-1}$ is continuous. It follows from the functoriality of F that the map, with the roles of i and j reversed, is an inverse. Since the inverse is also continuous we have shown that our map is indeed a homeomorphism. ⊞

We now equip the set $F(E)$ with the topology that is generated the preimages of all open subsets of $U_i \times F(\mathbb{K}^n)$ under the above map $\widetilde{\Phi}_i \colon q^{-1}(U_i) \to U_i \times F(\mathbb{K}^n)$, where we consider all $i \in I$. Using the claim one can now easily verify that each $\widetilde{\Phi}_i$ is a trivialization.

(2) This statement follows easily from the definitions. ■

The following lemma shows that we turn trivializable vector bundles again into trivializable vector bundles:

Lemma 268.24. (Functor-on-Trivializable Vector Bundle) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $d \in \mathbb{N}_0$ and let $F \colon (\mathcal{FDVec}_{\mathbb{K}})^d \to \mathcal{FDVec}_{\mathbb{K}}$ be a continuous (smooth) functor. Furthermore let $E_1 \xrightarrow{p_1} B, \ldots, E_d \xrightarrow{p_d} B$ be $\mathbb{K}$-vector bundles. If all of them are trivializable then the resulting $\mathbb{K}$-vector bundle $F(E_1, \ldots, E_d) \to B$ is also trivializable.

Proof. By hypothesis we know that for $i = 1, \ldots, d$ the $\mathbb{K}$-vector bundle $p_i \colon E_i \to B$ is isomorphic to the trivial $\mathbb{K}$-vector bundle $B \times \mathbb{K}^{n_i} \to B$. We now see that we have the following isomorphisms of $\mathbb{K}$-vector bundles over B:

$$\begin{array}{ccccccc}
F(E_1, \ldots, E_d) & \cong & F(B \times \mathbb{K}^{n_1}, \ldots, B \times \mathbb{K}^{n_d}) & \cong & B \times F(\mathbb{K}^n) & \cong & B \times \mathbb{K}^{\dim(F(\mathbb{K}^{n_1}, \ldots, \mathbb{K}^{n_d}))}. \\
& \overset{\text{by the Vector Bundle Construction Proposition 268.23 (2)}}{\uparrow} & & & & &
\end{array}$$

■

We conclude this section with the following lemma which says that natural isomorphisms between continuous (smooth) functors give rise to natural isomorphisms between the corresponding (smooth) vector bundles:

Lemma 268.25. (Smooth Natural Isomorphism–Vector Bundle Lemma) In the following let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $d \in \mathbb{N}_0$ and let

$$\begin{aligned} F\colon (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^d &\to \mathcal{FDVec}^{\cong} \\ G\colon (\mathcal{FDVec}_{\mathbb{K}}^{\cong})^d &\to \mathcal{FDVec}^{\cong} \end{aligned}$$

be two continuous (smooth) covariant functors. Furthermore let $n_1, \ldots, n_d \in \mathbb{N}_0$. In the following we set $m := \dim_{\mathbb{K}}(F(\mathbb{K}^{n_1}, \ldots, \mathbb{K}^{n_d}))$. Any natural isomorphism from F to G defines a natural isomorphism from

$$\begin{aligned} F\colon \mathcal{VectBund}_{\mathbb{K}^{n_1}} \times \cdots \times \mathcal{VectBund}_{\mathbb{K}^{n_d}} &\to \mathcal{VectBund}_{\mathbb{K}^m} \\ (E_1 \xrightarrow{p_1} B, \ldots, E_d \xrightarrow{p_d} B) &\mapsto (q\colon F(E_1, \ldots, E_d) \to B) \end{aligned}$$

to

$$\begin{aligned} G\colon \mathcal{VectBund}_{\mathbb{K}^{n_1}} \times \cdots \times \mathcal{VectBund}_{\mathbb{K}^{n_d}} &\to \mathcal{VectBund}_{\mathbb{K}^m} \\ (E_1 \xrightarrow{p_1} B, \ldots, E_d \xrightarrow{p_d} B) &\mapsto (q\colon G(E_1, \ldots, E_d) \to B) \end{aligned}$$

Sketch of proof. Suppose that we are given a natural isomorphism from F to G. Recall that by Definition 5.27 this means that for any $\mathbb{K}$-vector spaces $V_1, \ldots, V_d$ we have an isomorphism[2856]

$$\Phi_{(V_1,\ldots,V_d)}\colon F(V_1, \ldots, V_d) \to G(V_1, \ldots, V_d)$$

of $\mathbb{K}$-vector spaces such that for any choice of isomorphisms $\Theta_i\colon V_i \to W_i$, $i = 1, \ldots, d$ the following diagram commutes:

$$\begin{array}{ccc} F(V_1, \ldots, V_d) & \xrightarrow{\Phi_{(V_1,\ldots,V_d)}} & F(V_1, \ldots, V_d) \\ \downarrow{\scriptstyle F(\Theta_1,\ldots,\Theta_d)} & & \downarrow{\scriptstyle G(\Theta_1,\ldots,\Theta_d)} \\ G(W_1, \ldots, W_d) & \xrightarrow{\Phi_{(W_1,\ldots,W_d)}} & G(W_1, \ldots, W_d). \end{array}$$

Now let $E_1 \xrightarrow{p_1} B, \ldots, E_d \xrightarrow{p_d} B$ be $\mathbb{K}$-vector bundles. It follows easily from the definitions that the induced map

$$\begin{aligned} \overbrace{\bigsqcup_{b\in B} F(E_{1,b}, \ldots, E_{d,b})}^{=F(E_1,\ldots,E_d)} &\to \overbrace{\bigsqcup_{b\in B} G(E_{1,b}, \ldots, E_{d,b})}^{=G(E_1,\ldots,E_d)} \\ (b, v) &\mapsto (b, \Phi_{E_{1,b},\ldots,E_{d,b}}(v)) \end{aligned}$$

is a natural isomorphism of (smooth) $\mathbb{K}$-vector bundles. ∎

268.5. **Examples of vector bundle constructions.** In this section we will apply the Vector Bundle Construction Proposition 268.23 to our favorite covariant smooth functors.

268.5.1. **The direct sum of vector bundles.** The Vector Bundle Construction Proposition 268.23 applied to the direct sum functor $(V, W) \mapsto V \oplus W$ leads us to the following definition:

Definition 268.26. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an m-dimensional (smooth) $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be an n-dimensional (smooth) $\mathbb{K}$-vector bundle with

[2856] Note that any isomorphism between finite-dimensional $\mathbb{K}$-vector spaces is (by definition) smooth. Thus there is no need to impose any continuity or smoothness conditions on the natural isomorphism.

the same base space B. We consider the set

$$E \oplus F \coloneqq \bigsqcup_{b \in B} \underbrace{E_b \oplus F_b}_{\substack{(m+n)\text{-dimensional} \\ \mathbb{K}\text{-vector space}}} \qquad \text{with the obvious map } r\colon E \oplus F \to B.$$

By the Vector Bundle Construction Proposition 268.23 we can equip this set $E \oplus F$ with a natural $(m+n)$-dimensional (smooth) $\mathbb{K}$-vector bundle structure. It is called the direct sum product of the $\mathbb{K}$-vector bundles E and F.

Remark. Note that the usual isomorphisms between direct sums give rise to isomorphisms of vector bundles. For examples it follows from the Smooth Natural Isomorphism–Vector Bundle Lemma 268.25 that we have natural isomorphisms $E \oplus F \to F \oplus E$ of $\mathbb{K}$-vector bundles and that we have natural isomorphisms $E \oplus (F \oplus G) \to (E \oplus F) \oplus G$ of $\mathbb{K}$-vector bundles. As usual, see also the remark on page 115, this implies that the isomorphism type of the direct sum of finitely many vector bundles is well-defined. □

As the notation suggests, the direct sum of vector bundles is essentially the same as the Whitney sum of vector bundles:

Lemma 268.27. (Whitney Sum–Direct Sum Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be a (smooth) $\mathbb{K}$-vector bundle with the same base space B. The maps

$$\begin{array}{ccc} \overbrace{E \oplus F}^{\text{Whitney sum}} & \to & \overbrace{E \oplus F}^{\text{direct sum}} \\ (e,f) & \mapsto & (p(e),(e,f)) \end{array}$$

form a natural isomorphism of (smooth) $\mathbb{K}$-vector bundles between the Whitney sum from Definition 268.3 and the direct sum from Definition 268.26.

Proof. Using the Vector Bundle Isomorphism Criterion 265.40 one can easily verify that the given map is an isomorphism. ■

A slightly annoying problem with essentially all constructions of vector bundles is that they do not commute with pullbacks "on the nose", but only up to a natural isomorphism. For example we have the following lemma:

Lemma 268.28. (Pullback–Direct Sum-Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the following let $p\colon E \to B$ and $q\colon F \to B$ be (smooth) $\mathbb{K}$-vector bundles and let $h\colon C \to B$ be a continuous (smooth) map. The map[2857]

$$\begin{array}{ccc} h^*E \oplus h^*F & \to & h^*(E \oplus F) \\ (c,((c,e),(c,f))) & \mapsto & (c,(h(c),(e,f))) \end{array}$$

is a natural isomorphism of (smooth) $\mathbb{K}$-vector bundles over C.

[2857]The hardest part about these lemmas is to figure out, what the underlying set is at each given moment.

Proof. It is elementary to verify that the two pullback $\mathbb{K}$-vector bundles for $p\colon E \to B$ under $h\colon C \to B$, and for $q\colon F \to B$ under $h\colon C \to B$ give rise to a morphism

$$\begin{array}{ccc} h^*E \oplus h^*F & \xrightarrow{(c,((c,e),(c,f))) \mapsto ((h(c),(e,f)))} & E \oplus F \\ \downarrow & & \downarrow \\ C & \xrightarrow{\quad h \quad} & B \end{array}$$

of (smooth) $\mathbb{K}$-vector bundles. The desired statement now follows from the Vector Bundle–Pullback Proposition 265.36. ■

268.5.2. **Tensor products of vector bundles.** Next we define the tensor product of vector bundles:

Definition 268.29. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an m-dimensional (smooth) $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be an n-dimensional (smooth) $\mathbb{K}$-vector bundle with the same base space B. We consider the set

$$E \otimes_{\mathbb{K}} F := \bigsqcup_{b \in B} \underbrace{E_b \otimes_{\mathbb{K}} F_b}_{\substack{m \cdot n\text{-dimensional} \\ \mathbb{K}\text{-vector space}}} \qquad \text{with the obvious map } r\colon E \otimes_{\mathbb{K}} F \to B.$$

By the Vector Bundle Construction Proposition 268.23 we can equip this set $E \otimes_{\mathbb{K}} F$ with a natural $m \cdot n$-dimensional (smooth) $\mathbb{K}$-vector bundle structure. It is called the tensor product of the $\mathbb{K}$-vector bundles E and F.

The following lemma describes what is for us the most important construction involving the tensor product:

Lemma 268.30. (Bundle–Tensor-with-$\mathbb{C}$-Lemma)

(1) Let $p\colon E \to B$ be a (smooth) real vector bundle as in Definition 268.29. The fiberwise multiplication given by the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.13 turns $E \otimes_{\mathbb{R}} \mathbb{C}$ into a (smooth) complex vector bundle.

(2) (a) Given a morphism

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow p & & \downarrow q \\ B & \xrightarrow{h} & C \end{array}$$

of real vector bundles we obtain an obvious induced morphisms of complex vector bundles.

(b) The map $E \mapsto E \otimes_{\mathbb{R}} \mathbb{C}$ defines a covariant functor from $\mathcal{VectBund}_{\mathbb{R}}$ to $\mathcal{VectBund}_{\mathbb{C}}$.

Proof.

(1) It follows easily from the construction in the proof of the Vector Bundle Construction Proposition 268.23 and the Tensor-with-$\mathbb{C}$-Lemma 268.13 that we can equip this set $E \otimes_{\mathbb{R}} \mathbb{C}$ with a natural m-dimensional (smooth) $\mathbb{C}$-vector bundle structure.

(2) This statement follows immediately from the definitions. ■

This lemma motivates the following definition:

Definition 268.31. Let $p\colon E \to B$ be an m-dimensional (smooth) $\mathbb{R}$-vector bundle. We refer to the $\mathbb{C}$-vector bundle

$$E \otimes_{\mathbb{R}} \mathbb{C} \coloneqq \bigsqcup_{b\in B} \underbrace{E_b \otimes_{\mathbb{R}} \mathbb{C}}_{\substack{m\text{-dimensional} \\ \mathbb{C}\text{-vector space}}} \qquad \text{with the obvious map } r\colon E \otimes_{\mathbb{R}} \mathbb{C} \to B$$

from the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.30 as the complexification of the $\mathbb{R}$-vector bundle E.

We will discuss this construction in much greater detail in Section 276.1.

While we are already discussing complex vector bundles, let us introduce the following definition:

Definition 268.32. Let V be a $\mathbb{C}$-vector space. We denote by $\overline{V}$ the $\mathbb{C}$-vector space where the underlying group is the same as for V, but where for $v \in V$ and $z \in \mathbb{C}$ the multiplication is given by $v \cdot_{\overline{V}} z := v \cdot_V \overline{z}$. Clearly this defines a smooth covariant functor from the category of $\mathbb{C}$-vector spaces to itself. We refer to $\overline{V}$ as the complex conjugate of V.

This linear algebraic notion leads us, as always, straight to a corresponding definition for vector bundles:

Definition 268.33. Let $p\colon E \to B$ be a complex vector bundle. We denote by $p\colon \overline{E} \to B$ the complex vector bundle which has the same underlying topological space and where on each fiber E_b we now consider the complex conjugate structure from Definition 268.32. We refer to $\overline{E}$ as the complex conjugate of E.

268.5.3. **Homomorphisms of vector bundles.** In this subsection we give further constructions of vector bundles:

Definition 268.34. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an m-dimensional (smooth) $\mathbb{K}$-vector bundle. We consider the set

$$E^* \coloneqq \bigsqcup_{b\in B} \underbrace{E_b^*}_{\substack{m\text{-dimensional} \\ \mathbb{K}\text{-vector space}}} \qquad \text{with the obvious map } r\colon E^* \to B.$$

By the Vector Bundle Construction Proposition 268.23 we can equip this set E^* with a natural m-dimensional (smooth) $\mathbb{K}$-vector bundle structure. We refer to $E^* \to B$ as the dual vector bundle of $p\colon E \to B$.

For completeness' sake we also introduce the following concept, even though we will not really make use of it, except for Appendix 269.7:

Definition 268.35. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an m-dimensional (smooth) $\mathbb{K}$-vector bundle and let $d \in \mathbb{N}_0$. We consider the set

$$\wedge^d E^* \coloneqq \bigsqcup_{b\in B} \underbrace{\wedge^d E_b^*}_{\mathbb{K}\text{-vector space}} \qquad \text{with the obvious map } r\colon \wedge^d E^* \to B.$$

By the Vector Bundle Construction Proposition 268.23 we can equip this set $\wedge^d E^*$ with a natural (smooth) $\mathbb{K}$-vector bundle structure. By the Alternating Forms-Basis Proposition 72.7 we know that this $\mathbb{K}$-vector bundle has dimension $\binom{m}{d}$.

Example. Let M be a smooth manifold and let $d \in \mathbb{N}_0$. Recall that in Definitions 74.1 and 74.3 we introduced the notion of a smooth d-form on M and in Notation 74.5 we introduced the notation:

$$\Omega^d(M) \;:=\; \text{the real vector space of all smooth } d\text{-forms on } M.$$

Alternatively, using Definitions 265.18 and 268.35, we can also consider

$$\Gamma^{\mathrm{sm}}(\wedge^d \mathrm{T}M^*) \;:=\; \text{the real vector space of all smooth sections in } \wedge^d \mathrm{T}M^*.$$

Once one has untangled the definitions one sees that there exists an obvious natural isomorphism

$$\Omega^d(M) \;\xrightarrow{\cong}\; \Gamma^{\mathrm{sm}}(\wedge^d \mathrm{T}M^*)$$

of real vector spaces. □

We turn to our last construction of new vector bundles:

Definition 268.36. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an m-dimensional (smooth) $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be an n-dimensional (smooth) $\mathbb{K}$-vector bundle with the same base space B. We consider the set

$$\mathrm{Hom}_{\mathbb{K}}(E,F) := \bigsqcup_{b\in B} \underbrace{\mathrm{Hom}_{\mathbb{K}}(E_b,F_b)}_{\substack{m\cdot n\text{-dimensional}\\ \mathbb{K}\text{-vector space}}} \quad \text{with the obvious map } r\colon \mathrm{Hom}_{\mathbb{K}}(E,F) \to B.$$

By the Vector Bundle Construction Proposition 268.23 we can equip this set $\mathrm{Hom}_{\mathbb{K}}(E,F)$ with a natural $m \cdot n$-dimensional (smooth) $\mathbb{K}$-vector bundle structure.

Example. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(1) Let $p\colon E \to B$ be a $\mathbb{K}$-vector bundle. As on page 5119 we denote by $\mathrm{Tr}_B(\mathbb{K})$ the trivial 1-dimensional vector bundle $B \times \mathbb{K} \to B$. It follows easily from the definitions that the map

$$\begin{array}{rcl} E^* & \to & \mathrm{Hom}_{\mathbb{K}}(E, \mathrm{Tr}_B\,\mathbb{K}) \\ (b, \varphi\colon E_b \to \mathbb{K}) & \mapsto & (b, E_b \xrightarrow{v\mapsto(b,\varphi(v))} \{b\}\times\mathbb{K}) \end{array}$$

is a natural isomorphism of 1-dimensional $\mathbb{K}$-vector bundles over B.

(2) Let E be a 1-dimensional $\mathbb{K}$-vector bundle. We claim that the corresponding 1-dimensional $\mathbb{K}$-vector bundle $\mathrm{Hom}_{\mathbb{K}}(E,E)$ is trivializable. This can be seen in several ways:

(a) Note that the vector bundle $\mathrm{Hom}_{\mathbb{K}}(E,E) \to B$ has the section $b \mapsto \mathrm{id}_{E_b}$. It follows from the Trivializable–Vector Bundle Lemma 265.22 that $\mathrm{Hom}_{\mathbb{K}}(E,E)$ is a trivializable $\mathbb{K}$-vector bundle.

(b) Note that for any 1-dimensional $\mathbb{K}$-vector space V the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{K}}(V,V) & \to & \mathbb{K} \\ (\varphi, v) & \mapsto & \varphi(v) \end{array}$$

defines a natural isomorphism from the covariant functor

$$\begin{array}{rcl} F\colon V & \mapsto & \mathrm{Hom}_{\mathbb{K}}(V,V) \\ (f\colon V\to W) & \mapsto & f^{-1}\otimes f \end{array} \qquad \text{to the functor} \qquad \begin{array}{rcl} G\colon V & \mapsto & \mathbb{K} \\ (f\colon V\to W) & \mapsto & (\mathrm{id}_{\mathbb{K}}\colon \mathbb{K}\to\mathbb{K}). \end{array}$$

Thus we see that for any 1-dimensional $\mathbb{K}$-vector bundle $E \to B$ we have

$$\mathrm{Hom}_{\mathbb{K}}(E,E) \;=\; F(E) \;\underset{\uparrow}{\cong}\; G(E) \;\cong\; B \times \mathbb{K}.$$

by the Smooth Natural Isomorphism Vector Bundle Lemma 268.25

(3) From the Smooth Natural Isomorphism–Vector Bundle Lemma 268.25 we get many natural isomorphisms: For example, given $\mathbb{K}$-vector bundles E, F, G over B we get a natural isomorphism

$$\mathrm{Hom}_{\mathbb{K}}(E,F) \oplus \mathrm{Hom}_{\mathbb{K}}(E,G) \;\xrightarrow{\cong}\; \mathrm{Hom}_{\mathbb{K}}(E, F \oplus G).$$

Similarly we get a natural isomorphism

$$E^* \otimes F \;\to\; \mathrm{Hom}_{\mathbb{K}}(E,F). \qquad \square$$

268.5.4. **Tensor product of line bundles.** We conclude this discussion of linear algebraic constructions on vector bundles with the following, rather puzzling application of tensor products:

Proposition 268.37. (Line Bundle–Group Proposition) Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$.

(1) Given any topological space B we denote, as in Notation 266.6, by $\mathrm{IsoVectBund}_{\mathbb{K}}(B)$ the set of isomorphism classes of 1-dimensional $\mathbb{K}$-vector bundles over B. The tensor product from Definition 268.29 defines an abelian group structure on this set. Here, in the group $(\mathrm{IsoVectBund}_{\mathbb{K}}(B), \otimes)$ the neutral element is given by the trivial 1-dimensional $\mathbb{K}$-vector bundle and the inverse of $[E \to B]$ is given by $[E^* \to B]$.

(2) The maps

$$\begin{array}{rcl} B & \mapsto & (\mathrm{IsoVectBund}_{\mathbb{K}}(B), \otimes) \\ (f\colon B \to C) & \mapsto & \left(\begin{array}{rcl} f^*\colon \mathrm{IsoVectBund}_{\mathbb{K}}(C) & \to & \mathrm{IsoVectBund}_{\mathbb{K}}(B) \\ E & \mapsto & f^*E \end{array} \right) \end{array}$$

define a contravariant functor from the category *Top* of topological space to the category *AbGroup* of abelian groups.

Proof.

(1) Let B be a topological space. First of all, note that by the Vector Bundles–Set Lemma 266.5 we know that $\mathrm{IsoVectBund}_{\mathbb{K}}(B)$ is indeed a set. Next note that it follows easily from the Smooth Natural Isomorphism–Vector Bundle Lemma 268.25 that "$\otimes$" defines an associative and commutative product on $\mathrm{IsoVectBund}_{\mathbb{K}}(B)$ with neutral element given by the trivial 1-dimensional vector bundle. It remains to prove the following claim:

Claim. For any a 1-dimensional $\mathbb{K}$-vector bundle E the vector bundle $E^* \otimes_{\mathbb{K}} E$ is trivializable.

Proof. For any 1-dimensional $\mathbb{K}$-vector bundle $E \to B$ we have

$$E^* \otimes_{\mathbb{K}} E \;\underset{\uparrow}{=}\; \mathrm{Hom}_{\mathbb{F}}(E,E) \;\underset{\uparrow}{\cong}\; B \times \mathbb{K}.$$

see page 5185 see page 5185 $\boxplus$

(2) This statement follows almost immediately from the Vector Bundle Construction Proposition 268.23 (2). ■

The proposition raises the following question:

Question 268.38. Is the contravariant functor $B \mapsto \mathrm{IsoVectBund}_{\mathbb{K}}(B)$ a "new" contravariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups, or is it a well-known functor in slight disguise?

We will give a complete answer to Question 268.38 in Theorem **??**. Now we provide two more definitions.

268.6. **Tangent bundles of projective spaces.** To practice our concepts we will prove in this section some subtle statements about vector bundles on real projective spaces and complex projective spaces. These will be useful later on.

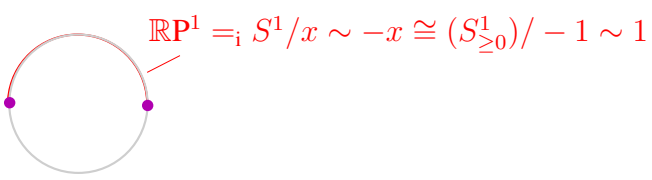
First we recall the following notation:

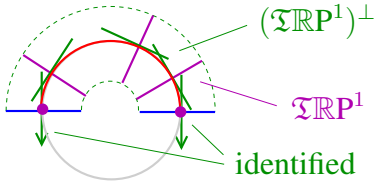
Notation 268.39. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$. On the projective space $\mathbb{K}\mathrm{P}^n$ we have the following $\mathbb{K}$-vector bundles:

- We have the tautological $\mathbb{K}$-line bundle
$$\begin{array}{rcl} p\colon \mathfrak{T}\mathbb{K}\mathrm{P}^n := \{([P], v) \in \mathbb{K}\mathrm{P}^n \times \mathbb{K}^{n+1} \,|\, P \in \mathbb{K}^{n+1} \setminus \{0\}, v \in \mathbb{K} \cdot P\} & \to & \mathbb{K}\mathrm{P}^n \\ ([v], w) & \mapsto & [v] \end{array}$$
that we introduced in the Tautological Line Bundle Lemma 265.6.
- We have the tangent bundle $\mathrm{T}\mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^n$ from the Tangent Bundle Proposition 265.10 which is an n-dimensional $\mathbb{K}$-vector bundle.
- Let $m \in \mathbb{N}_0$. As on page 5119 we denote by $\mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K}^m)$ the trivial m-dimensional $\mathbb{K}$-vector bundle $\mathbb{K}\mathrm{P}^n \times \mathbb{K}^m \to \mathbb{K}\mathrm{P}^n$.

We now add one extra vector bundle to the list:

Definition 268.40. Let $n \in \mathbb{N}_0$. We consider[2858]
$$(\mathfrak{T}\mathbb{K}\mathrm{P}^n)^\perp := \{([P], v) \,|\, P \in \mathbb{K}^{n+1} \setminus \{0\} \text{ and } v \in (\mathbb{K} \cdot P)^\perp\}$$
together with the obvious map $(\mathfrak{T}\mathbb{K}\mathrm{P}^n)^\perp \to \mathbb{K}\mathrm{P}^n$.





In the following lemma we will relate these $\mathbb{K}$-vector bundles over $\mathbb{K}\mathrm{P}^n$ in mysterious ways:

Lemma 268.41. (Tangent Bundle–$\mathbb{K}\mathrm{P}^n$-Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$.

(1) $(\mathfrak{T}\mathbb{K}\mathrm{P}^n)^\perp$ is a smooth n-dimensional $\mathbb{K}$-vector bundle over $\mathbb{K}\mathrm{P}^n$.

(2) The $\mathbb{K}$-vector bundle $\mathfrak{T}\mathbb{K}\mathrm{P}^n \oplus (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^\perp$ is trivializable, i.e. it is isomorphic to the trivial $\mathbb{K}$-vector bundle $\mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K}^{n+1})$.

[2858]Given a vector subspace $V \subseteq \mathbb{K}^{n+1}$ we denote by $V^\perp$ the orthogonal complement with respect to the standard inner product $\langle -, - \rangle_{\mathbb{K}^n}$ that we introduced on pages 122 and 123.

(3) There exists an isomorphism

$$\mathrm{T}\mathbb{K}\mathrm{P}^n \;\to\; \underbrace{\mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^{\perp})}_{\substack{n\text{-dimensional } \mathbb{K}\text{-vector bundle} \\ \text{by Definition 268.36}}}$$

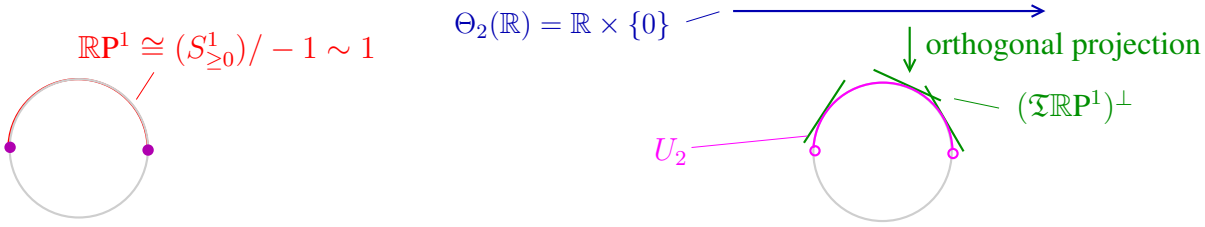
of smooth $\mathbb{K}$-vector bundles over $\mathbb{K}\mathrm{P}^n$.

Remark. Statement (3) of the Tangent Bundle–$\mathbb{K}\mathrm{P}^n$-Lemma 268.41 will play an important role in the proof of the Division Algebra–Parallelizable $\mathbb{K}\mathrm{P}^n$-Proposition 275.2, when we return to the following time-honored question: For which $n \in \mathbb{N}$ does $\mathbb{R}^n$ admit the structure of a division algebra? □

Example. In Exercise 268.9 we will see that the vector bundle $(\mathfrak{T}\mathbb{R}\mathrm{P}^1)^{\perp} \to \mathbb{R}\mathrm{P}^1$ is isomorphic to the Möbius vector bundle $p\colon M \to S^1$ from page 5119. On page 5122 we saw that the tautological vector bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ is also isomorphic to the Möbius vector bundle. Thus we see that the Tangent Bundle–$\mathbb{K}\mathrm{P}^n$-Lemma 268.41 (2) gives us the statement that the Whitney sum of two copies of the Möbius vector bundle is trivializable. This recovers the result from page 5170 □

Proof. [2859]

(1) Given $i \in \{0, \ldots, n\}$ we consider, as in the proof of the Projective Space–Manifold Proposition 25.13, the open subset $U_i := \{[z_0 : \cdots : z_n] \in \mathbb{K}\mathrm{P}^n \,|\, z_i \neq 0\}$ of $\mathbb{K}\mathrm{P}^n$. Furthermore let $\Theta_i\colon \mathbb{K}^n \to \{(v_0, \ldots, v_n) \in \mathbb{K}^{n+1} \,|\, v_i = 0\}$ be the obvious isomorphism of $\mathbb{K}$-vector spaces.



Next we consider the map

$$\begin{array}{rcl} U_i \times \mathbb{K}^n & \to & (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^{\perp} \\ ([P], v) & \to & ([P], \underbrace{\text{orthogonal projection of } \Theta_i(v) \text{ to } (\mathbb{K} \cdot P)^{\perp}}_{\text{see the Orthogonal Projection Lemma 2.33}}) \end{array}$$

Note that for each point $P = (P_0, \ldots, P_n) \in \mathbb{K}^{n+1}$ with $P_i \neq 0$ we have the equality $\Theta_i(\mathbb{K}^n) \oplus ((\mathbb{K} \cdot P)^{\perp})^{\perp} = \Theta_i(\mathbb{K}^n) \oplus \mathbb{K} \cdot P = \mathbb{K}^{n+1}$. It follows easily from this observation together with the Orthogonal Projection Lemma 2.33 (3) that the above map is fiberwise an isomorphism of $\mathbb{K}$-vector spaces. Furthermore, given the explicit description of the orthogonal projection, see the Orthogonal Projection Lemma 2.33 (2), and using the Gram-Schmidt Orthogonalization Process 2.32 we see that the above map is a diffeomorphism. In summary, it is an isomorphism of $\mathbb{K}$-vector bundles. We leave it to the reader to fill in the details.

[2859] We refer to [**MiS1974**, Lemma 4.4 and p. 174] for a different write up of the same proof.

(2) Similar to the discussion on page 5170 we consider the map

$$\begin{array}{rcl}\Xi\colon \mathfrak{T}\mathbb{K}\mathrm{P}^n \oplus (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^\perp & \to & \mathbb{K}\mathrm{P}^n \times \mathbb{K}^{n+1} \\ (([P],v),([P],w)) & \mapsto & ([P],v+w).\end{array}$$

Using the Vector Bundle Isomorphism Criterion 265.40 one can easily verify that this map is an isomorphism of $\mathbb{K}$-vector bundles.

(3) We prove this statement for $\mathbb{K}=\mathbb{C}$. The case $\mathbb{K}=\mathbb{R}$ follows from the same argument. We leave it to the reader to adjust the argument.

Thus we need to show that there exists an isomorphism

$$\mathrm{T}\mathbb{C}\mathrm{P}^n \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{T}\mathbb{C}\mathrm{P}^n, (\mathfrak{T}\mathbb{C}\mathrm{P}^n)^\perp)$$

of $\mathbb{C}$-vector bundles over $\mathbb{C}\mathrm{P}^n$. The idea is to first define an analogous map on spheres, and then to project it to the projective spaces. To make this work, we first introduce some notation:

- Given $Q \in S^{2n+1}$ we introduce the map

$$\begin{array}{rcl}\widetilde{\Xi}_Q\colon (\mathbb{C}\cdot Q)^\perp & \to & \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\cdot Q, (\mathbb{C}\cdot Q)^\perp) \\ v & \mapsto & \begin{pmatrix} \mathbb{C}\cdot Q & \to & (\mathbb{C}\cdot Q)^\perp \\ \mu\cdot Q & \mapsto & \mu\cdot v \end{pmatrix}.\end{array}$$

  Note that this map is clearly an isomorphism of vector spaces.
- We denote by $\pi\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n =_{\mathrm{i}} S^{2n+1}/S^1$ the natural projection. Recall that by the Physical Tangent Spaces–Lemma 29.2 we know that $\mathrm{T}_Q S^{2n+1} = (\mathbb{R}\cdot Q)^\perp$. Using the fact that the obvious map $S^{2n+1}/S^1 \to \mathbb{C}\mathrm{P}^n$ is a diffeomorphism one can show that the map $\mathrm{D}\pi_Q\colon \mathrm{T}_Q S^{2n+1} \to \mathrm{T}_{\pi(Q)}\mathbb{C}\mathrm{P}^n$ restricts to an isomorphism $\mathrm{D}\pi_Q\colon (\mathbb{C}\cdot Q)^\perp \to \mathrm{T}_{\pi(Q)}\mathbb{C}\mathrm{P}^n$.
- Given $Q \in S^{2n+1}$ we also consider the isomorphisms

$$\begin{array}{rcl}\alpha_Q\colon \mathbb{C}\cdot Q & \to & \mathfrak{T}\mathbb{C}\mathrm{P}^n_{\pi(Q)} \\ v & \mapsto & (\pi(Q),v)\end{array} \quad \text{and} \quad \begin{array}{rcl}\beta_Q\colon (\mathbb{C}\cdot Q)^\perp & \to & (\mathfrak{T}\mathbb{C}\mathrm{P}^n)^\perp_{\pi(Q)} \\ w & \mapsto & (\pi(Q),w).\end{array}$$

Now let $\pi(P) \in \mathbb{C}\mathrm{P}^n$ and let $u \in \mathrm{T}\mathbb{C}\mathrm{P}^n_{\pi(P)}$. Let $v \in (\mathbb{C}\cdot P)^\perp$ be the unique vector with $\mathrm{D}\pi_P(v) = u$. We define $\Xi(u)\colon (\mathfrak{T}\mathbb{C}\mathrm{P}^n)_{\pi(P)} \to (\mathfrak{T}\mathbb{C}\mathrm{P}^n)^\perp_{\pi(P)}$ to be the unique homomorphism that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{C}\cdot P & \xrightarrow[\widetilde{\Xi}_P(v)]{\mu\cdot P \mapsto \mu\cdot v} & (\mathbb{C}\cdot P)^\perp \\ {\scriptstyle\alpha_P}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\beta_P} \\ (\mathfrak{T}\mathbb{C}\mathrm{P}^n)_{\pi(P)} & \underset{=:\Xi(u)}{\dashrightarrow} & (\mathfrak{T}\mathbb{C}\mathrm{P}^n)^\perp_{\pi(P)} \end{array}$$

Claim. This definition of $\Xi(u)$ does not depend on the choice of P.

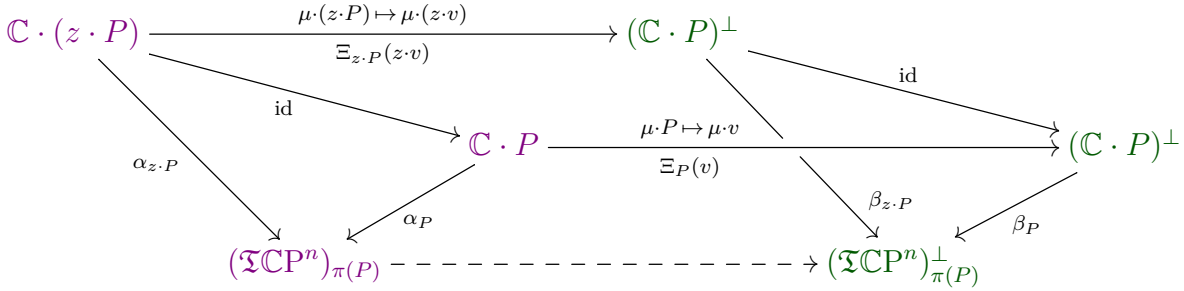
Proof. Let $z \in S^1$. We need to show that P and $z\cdot P$ lead to the same result. In the following we argue, that if we work with $z\cdot P$ instead of P, then we need to take the vector $z\cdot v$ instead of the vector v. This follows from the following two observations:

- Since multiplication by z is an isometry we see that $z\cdot v \in (\mathbb{C}\cdot(z\cdot P))^\perp$.

- We denote by ρ the self-map of $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ that is given by multiplication. We see that

$$\mathrm{D}\pi_{z\cdot P}(z\cdot v) \quad = \quad \mathrm{D}\pi_{\rho(P)}(\rho(v)) \underset{\text{by the Chain rule}}{=} \mathrm{D}(\pi\circ\rho)(v) \underset{\text{since } \pi\circ\rho=\pi}{=} \mathrm{D}\pi(v) \underset{\text{by choice of } v}{=} u.$$

Next we consider the following diagram:



Once one has overcome all confusions it is clear that the top quadrilateral commutes and that the triangles to the left and right commute. This shows that the diagram commutes. It now follows that P and $z\cdot P$ define the same bottom horizontal homomorphism. ⊞

In summary we have defined a map $\Xi\colon \mathfrak{T}\mathbb{K}\mathrm{P}^n \oplus (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^{\perp} \to \mathbb{K}\mathrm{P}^n \times \mathbb{C}^{n+1}$ which is an isomorphism on the fibers. We leave it to the reader to verify that Ξ is smooth. It then follows from the Vector Bundle Isomorphism Criterion 265.40 that the map Ξ is an isomorphism of smooth $\mathbb{C}$-vector bundles. ■

We conclude this section with another subtle isomorphism of vector bundles:

Proposition 268.42. (Stable Tangent Bundle–$\mathbb{K}\mathrm{P}^n$-Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$. There exists an isomorphism

$$\mathrm{T}\mathbb{K}\mathrm{P}^n \oplus \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K}) \xrightarrow{\cong} \underbrace{\mathfrak{T}(\mathbb{K}\mathrm{P}^n)^* \oplus \cdots \oplus \mathfrak{T}(\mathbb{K}\mathrm{P}^n)^*}_{n+1 \text{ copies}}$$

of smooth $\mathbb{K}$-vector bundles over $\mathbb{K}\mathrm{P}^n$.

Remark. The slightly obscure looking Stable Tangent Bundle–$\mathbb{K}\mathrm{P}^n$-Proposition 268.42 will be an essential ingredient in the proof of the $\mathbb{C}\mathrm{P}^n$-Chern-Proposition 274.9 where we calculate the "Chern classes" of $\mathbb{C}\mathrm{P}^n$ and in the proof of the $\mathbb{R}\mathrm{P}^n$-SW-Proposition 274.21 where we calculate the "Stiefel-Whitney classes" of $\mathbb{R}\mathrm{P}^n$. □

Proof. We have the following isomorphisms of smooth $\mathbb{K}$-vector bundles over $\mathbb{K}\mathrm{P}^n$:

$$\begin{aligned}
\mathrm{T}\mathbb{K}\mathrm{P}^n \oplus \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K}) &\cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^{\perp}) \oplus \overbrace{\mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathfrak{T}\mathbb{K}\mathrm{P}^n)}^{\text{trivializable}} \\
&\uparrow \\
&\text{by the Tangent Bundle–}\mathbb{K}\mathrm{P}^n\text{-Lemma 268.41 (3) and since by the} \\
&\text{discussion on page 5184 we know that } \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathfrak{T}\mathbb{K}\mathrm{P}^n) \text{ is trivializable} \\
&\cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, (\mathfrak{T}\mathbb{K}\mathrm{P}^n)^{\perp} \oplus \mathfrak{T}\mathbb{K}\mathrm{P}^n) \quad \cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K}^{n+1})) \\
&\uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\
&\text{as on page 5185 this follows from the} \quad \text{by the Tangent Bundle–}\mathbb{K}\mathrm{P}^n \\
&\text{Smooth Natural Isomorphism–Vector} \quad \text{Lemma 268.41 (2)} \\
&\text{Bundle Lemma 268.25} \\
&\cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K}) \oplus \cdots \oplus \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K})) \\
&\uparrow \\
&\text{by the Vector Bundle Construction Proposition 268.23 (2)} \\
&\cong \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K})) \oplus \cdots \oplus \mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathrm{Tr}_{\mathbb{K}\mathrm{P}^n}(\mathbb{K})) \\
&\uparrow \\
&\text{as on page 5185 this follows from the} \\
&\text{Smooth Natural Isomorphism–Vector Bundle Lemma 268.25} \\
&\cong \mathfrak{T}(\mathbb{K}\mathrm{P}^n)^* \oplus \cdots \oplus \mathfrak{T}(\mathbb{K}\mathrm{P}^n)^*. \\
&\uparrow \\
&\text{see page 5184}
\end{aligned}$$

■

268.7. Exercises for Chapter 268.

Exercise 268.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) m-dimensional $\mathbb{K}$-vector bundle and let $q\colon F \to B$ be a (smooth) n-dimensional $\mathbb{K}$-vector bundle over the same topological space B. Show that the map

$$\begin{aligned} E \oplus F \coloneqq \{(e,f) \in E \times F \,|\, p(e) = q(f)\} &\to B \\ (e,f) &\mapsto p(e) \end{aligned}$$

is a (smooth) $(m+n)$-dimensional $\mathbb{K}$-vector bundle.

Exercise 268.2. Let

$$\begin{aligned} p\colon M \coloneqq ([0,1] \times \mathbb{R})/(0,v) \sim (1,-v) &\to S^1 \\ [(t,v)] &\mapsto \exp(2\pi \mathrm{i} t) \end{aligned}$$

be the Möbius vector bundle that we introduced on page 5119. We consider the Whitney (or direct sum) $M \oplus M \to S^1$ of two copies of the Möbius vector bundle. Show that this 2-dimensional real vector bundle is trivializable by providing a completely explicit trivialization $M \oplus M \to \mathbb{R}^2 \times S^1$.

Exercise 268.3. Let $p\colon E \to B$ and $q\colon F \to B$ be two *non-orientable* vector bundles over the same connected topological space B. Is the Whitney sum $E \oplus F$ necessarily *orientable*?

Exercise 268.4. Let $p\colon E \to B$ be a real vector bundle. Show that the Whitney sum $E \oplus E \to B$ is always orientable.

Exercise 268.5. By the Tangent Bundle Proposition 265.10 we can view $\mathrm{T}M$ as a $2m$-dimensional smooth manifold.

(a) Let $p\colon \mathrm{T}M \to M$ be the natural projection. Show that there exists a natural isomorphism $\mathrm{TT}M \to p^*\mathrm{T}M \oplus p^*\mathrm{T}M$ of real vector bundles over $\mathrm{T}M$.

(b) Show that the tangent bundle $\mathrm{T}M$ of any m-dimensional smooth manifold M, viewed as a $2m$-dimensional smooth manifold, is orientable.
Hint. Use Exercise 268.4.
Remark. Note that this gives a new solution to Exercise 265.18.

Exercise 268.6. Let $n \in \mathbb{N}$. We consider the n-dimensional $\mathbb{R}$-vector bundle

$$\begin{array}{rccc} p_n \colon E_n := ([0,1] \times \mathbb{R}^n)/(0,v) \sim (1,-v) & \to & S^1 \\ [(t,v)] & \mapsto & \exp(2\pi \mathrm{i} t). \end{array}$$

What is the largest $k \in \{0, \ldots, n\}$ such that E_n is isomorphic to the Whitney sum of a k-dimensional trivializable real vector bundle and some other real vector bundle?
Remark. This exercise can be viewed as an addendum to Exercise 265.10.

Exercise 268.7. Let $p \colon E \to B$ and $q \colon E \to B$ be two real vector bundles. We consider the three vector bundles given by E, F and the direct sum $E \oplus F$. Show that if two out of the three vector bundles E, F and $E \oplus F$ are orientable, then so is the third.
Remark. This exercise is closely related to Exercise 265.4.

Exercise 268.8. A smooth manifold W is called *stably parallelizable* if there exists a trivializable $\mathbb{R}$-vector bundle E such that the Whitney sum $\mathrm{T}W \oplus E$ is trivializable.

(a) Give an example of a smooth manifold that is not parallelizable, but that is stable trivializable.
(b) Let M and N be smooth manifolds with empty boundary. Show that if M and N are stably parallelizable, then $M \times N$ is also stably parallelizable.

Exercise 268.9. In Definition 268.40 we introduced the 1-dimensional real vector bundle $(\mathfrak{T}\mathbb{R}\mathrm{P}^1)^\perp \to \mathbb{R}\mathrm{P}^1$. Show that this vector bundle is isomorphic to the Möbius vector bundle $p \colon M \to S^1$ from page 5119.

Exercise 268.10. Let $p \colon E \to M$ be an oriented smooth m-dimensional vector bundle over a closed m-dimensional smooth manifold M and let $q \colon F \to N$ be an oriented smooth n-dimensional vector bundle over a closed n-dimensional smooth manifold N.

(a) On the $2m$-dimensional smooth manifold $\mathrm{T}(M \times N) = \mathrm{T}M \times \mathrm{T}$ there are two orientations: We can consider the oriented smooth manifold $M \times N$ and endow the smooth manifold $\mathrm{T}(M \times N)$ with the orientation coming from Definition 265.32. Alternatively we first use the orientations of M and N to equip the total spaces $\mathrm{T}M$ and $\mathrm{T}N$ with orientations, and then consider the product orientation. What is the relationship between these orientations?
(b) Can you express the Euler number $\chi(E \times F \to M \times N)$ in terms of the Euler numbers $\chi(E \to M)$ and $\chi(F \to N)$?
Remark. Recall that we introduced the Euler number in Definition 265.32.

Exercise 268.11. First we recall the following notation: Given two topological spaces X and Y we denote by $[X, Y]$ the set of homotopy classes of continuous maps from X to Y. Let $k \in \mathbb{N}$.

(a) Note that the group structure on $\mathrm{GL}(n, \mathbb{C})$ endows $[S^{k-1}, \mathrm{GL}(n, \mathbb{C})]$ with a group structure. Furthermore note that by the Line Bundle–Group Proposition 268.37 we know

that $\mathrm{IsoVectBund}_{\mathbb{C}}(S^k)$ comes with a group structure. We consider the map

$$\begin{array}{rcl}\Theta\colon [S^{k-1}, \mathrm{GL}(1,\mathbb{C})] & \to & \mathrm{IsoVectBund}_{\mathbb{C}}(S^k) \\ [\varphi] & \mapsto & [p(\varphi)]\end{array}$$

from the Vector Bundle-Clutching Map Lemma 267.5. By the Complex Vector Bundles-over-Sphere Proposition 267.6 we know that this map is a bijection. Show that this map is a group homomorphism. In other words, show that given $\varphi, \psi\colon S^{k-1} \to \mathrm{GL}(1,\mathbb{C})$ we have the equality $[p(\varphi\cdot\psi)] = [p(\varphi)] \otimes [p(\psi)]$.

(b) Determine the isomorphism type of the group $(\mathrm{IsoVectBund}_{\mathbb{C}}(S^k), \otimes)$.

Remark. Later in Theorem **??** we will generalize this calculation to give a complete calculation of $\mathrm{IsoVectBund}^1_{\mathbb{C}}(X)$ for any CW-complex X.

Exercise 268.12. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n \in \mathbb{N}_0$ and let $d \in \mathbb{Z}$. We consider

$$E_d \;:=\; ((\mathbb{K}^{n+1} \setminus \{0\}) \times \mathbb{K})/\!\sim$$

where $(z_0, \ldots, z_n, w) \sim (\lambda\cdot z_0, \ldots, \lambda\cdot z_n, \lambda^d \cdot z)$ for every $\lambda \in \mathbb{K}\setminus\{0\}$.

(a) Show that the map

$$\begin{array}{rcl} E_d & \to & \mathbb{K}\mathrm{P}^n \\ [(z_0, \ldots, z_n, w)] & \mapsto & [z_0 : \cdots : z_n]\end{array}$$

is a 1-dimensional $\mathbb{K}$-vector bundle.

(b) For which $d \in \mathbb{Z}$ do we get a trivializable $\mathbb{K}$-vector bundle?

(c) Show that E_{-1} is isomorphic to the tautological $\mathbb{K}$-vector bundle $\mathfrak{T}\mathbb{K}\mathrm{P}^n$.

(d) Show that E_1 is isomorphic to the $\mathbb{K}$-vector bundle $\mathrm{Hom}_{\mathbb{K}}(\mathfrak{T}\mathbb{K}\mathrm{P}^n, \mathbb{K})$.

Exercise 268.13. In the Vector Bundle Definition Lemma 266.2 we saw that there exists a weak equivalence of categories (in the sense of Definition 5.29) between the category $\mathcal{B}\mathit{und}_{(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))}$ of $(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))$-bundles and the category $\mathcal{V}\mathit{ectBund}_{\mathbb{K}^n}$ of n-dimensional $\mathbb{K}$-vector bundles. How can the statement, that an n-dimensional $\mathbb{K}$-vector bundle can be written as a Whitney sum of two $\mathbb{K}$-vector bundles of dimension k and $n-k$, be expressed in the language of $(\mathbb{K}^n, \mathrm{GL}(n,\mathbb{K}))$-bundles?

Exercise 268.14. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering from Definition 127.21. Furthermore let $\varphi\colon \pi \to \mathrm{GL}(n,\mathbb{R})$ be a representation. Note that φ defines a right-action of π on the vector space $\mathbb{R}^{1\times n}$ of row vectors. We consider

$$\mathbb{R}^{1\times n} \times_\pi \widetilde{X}_{x_0} \;:=\; (\mathbb{R}^{1\times d} \times \widetilde{X}_{x_0})/\!\sim$$

where $(v\cdot\varphi(g), x) \sim (v, g\cdot x)$ for all $v \in \mathbb{R}^{1\times d}$, $g \in \pi$ and $x \in \widetilde{X}_{x_0}$. We equip $\mathbb{R}^{1\times n}$ with the usual topology, we equip $\mathbb{R}^{1\times n} \times \widetilde{X}_{x_0}$ with the product topology and finally we equip $\mathbb{R}^{1\times n} \times_\pi \widetilde{X}_{x_0} = (\mathbb{R}^{1\times d} \times \widetilde{X}_{x_0})/\!\sim$ with the quotient topology.

(a) Show that map

$$\begin{array}{rcl} q\colon \mathbb{R}^{1\times n} \times_\pi \widetilde{X}_{x_0} & \to & X \\ [(v,x)] & \mapsto & p(x)\end{array}$$

defines an n-dimensional real vector bundle.

(b) Is every n-dimensional real vector bundle isomorphic to a vector bundle as in (a)?

(c) How does the construction in (a) relate to the Whitney sum, the dual and the tensor product f vector bundles?

Exercise 268.15. Let M be a connected smooth manifold and let $x_0 \in M$. In the following we denote by $w_1 \colon \pi_1(M, x_0) \to \{\pm 1\} \subseteq \mathrm{GL}(1, \mathbb{R})$ the orientation character from Definition 95.15. Furthermore let $\wedge^n \mathrm{T}M^*$ be the orientation bundle from Definition 269.27. It follows from the construction in Exercise 268.14 that the orientation character defines a 1-dimensional real vector bundle ${}_{w_1}\mathbb{R}$ over M. Show that this vector bundle is isomorphic to the orientation bundle.

CHAPTER 269

Inner products on vector bundles

In the last chapter we used the tools from linear algebra to construct many new vector bundles out of given vector bundles. In this chapter we will consider another linear algebraic structure, namely inner products. We will see that vector bundles over "most reasonable" topological spaces admit an inner product. Inner products have various applications, for example they can be used to show that in most reasonable settings the dual E^* of a real vector bundle E is in fact isomorphic to the original real vector bundle E. They can also be used to introduce the normal vector bundle of a vector subbundle.

This chapter is organized as follows:

- In the purely algebraic Section 269.1 we will discuss inner products on vector spaces.
- In Section 269.2 we will show that vector bundles (over most reasonable topological spaces) admit an inner product.
- In Section 269.3 we will use inner products to relate a vector bundle to its dual vector bundle.
- In Section 269.4 we will introduce vector subbundles and normal vector bundles.
- In Section 269.5 we will study several instances and applications of subbundles and normal vector bundles.
- In Appendix 269.6 we will introduce the stable normal vector bundle of a smooth manifold. We will not make use of this mind-bending concept.
- In Appendix 269.7 we will discuss the orientation bundle of a real vector bundle.
- In Section 269.8 we will provide exercises for Chapter 269.

In the next Chapter 270 we will introduce the Grassmannians and we will see that they are classifying spaces for vector bundles.

269.1. **Bilinear pairings and inner products.** In this section we recall some basic concepts from linear algebra.

269.1.1. **Bilinear pairings and tensor products.** We start out by recalling some linear algebraic concepts from Definition 2.24, Definition 2.25 and Definition 125.16:

Definition 269.1. Let $\mathbb{K}$ be a field with a (possibly trivial) involution $k \mapsto \overline{k}$ and let V be a $\mathbb{K}$-vector space.

(1) A pairing on $V \times V \to \mathbb{K}$ is a map such that for all $v, v' \in V$ and for all $w, w' \in W$ we have the equalities $\varphi(v+v', w) = \varphi(v, w)+\varphi(v', w)$ and $\varphi(v, w+w') = \varphi(v, w)+\varphi(v, w')$.
(2) A pairing $\varphi\colon V \times V \to \mathbb{K}$ is called $\mathbb{K}$-sesquilinear if for all $\lambda \in \mathbb{K}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \overline{\lambda}$.[2860]
(3) We denote by $\mathrm{Sesqu}(V \times V \to K)$ the set of all $\mathbb{K}$-sesquilinear pairings $V \times V \to \mathbb{K}$.
(4) A pairing $\varphi\colon V \times V \to \mathbb{K}$ is called Hermitian if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

If the involution is trivial, then we also say $\mathbb{K}$-bilinear and symmetric.

To formulate the only lemma of this section we will also need the following definition:

Definition 269.2. Let $\mathbb{K}$ be a field with a (possibly trivial) involution and let V be a $\mathbb{K}$-vector space. We denote by $\overline{V}$ the $\mathbb{K}$-vector space where the underlying group is the same as for V, but where for $v \in V$ and $k \in \mathbb{K}$ the multiplication is given by $v \cdot_{\overline{V}} k := v \cdot_V \overline{k}$.

The following lemma gives us an alternative view of $\mathbb{K}$-sesquilinear pairings:

Lemma 269.3. (Bilinear Pairing–Tensor Lemma) Let $\mathbb{K}$ be a field with a (possibly trivial) involution and let V be a $\mathbb{K}$-vector space. The map

$$\begin{array}{rcl} \operatorname{Sesqu}(V \times V \to \mathbb{K}) & \to & \operatorname{Hom}_{\mathbb{K}}(V \otimes_{\mathbb{K}} \overline{V}, \mathbb{K}) \\ (\varphi\colon V \times V \to \mathbb{K}) & \mapsto & \left(\begin{array}{ccc} V \otimes_{\mathbb{K}} V & \to & \mathbb{K} \\ \sum_{i=1}^{m} v_i \otimes w_i & \mapsto & \sum_{i=1}^{m} \varphi(v_i, w_i) \end{array} \right) \end{array}$$

is a natural bijection.

Remark. In the following we will use the Bilinear Pairing–Tensor Lemma 269.3 as an excuse to go freely back and forth between $\operatorname{Sesqu}(V \times V \to \mathbb{K})$ and $\operatorname{Hom}_{\mathbb{K}}(V \otimes_{\mathbb{K}} \overline{V}, \mathbb{K})$. □

Proof. The lemma follows from the analog of the Tensor Product–Universal Property Lemma 159.4 for tensor products of vector spaces of $\mathbb{R}$-vector spaces. Alternatively it is straightforward to verify that the map

$$\begin{array}{rcl} \operatorname{Hom}_{\mathbb{K}}(V \otimes_{\mathbb{K}} \overline{V}, \mathbb{K}) & \to & \operatorname{Sesqu}(V \times V \to \mathbb{K}) \\ (\varphi\colon V \otimes_{\mathbb{K}} V \to \mathbb{K}) & \mapsto & \left(\begin{array}{ccc} V \times V & \to & \mathbb{K} \\ (v, w) & \mapsto & \varphi(v \otimes w) \end{array} \right) \end{array}$$

is an inverse to the map given in the lemma. We leave it to the reader (or their linear algebra books) to fill in the details. ■

269.1.2. **Inner products on vector spaces.** In this subsection we recall some more definitions and concepts from linear algebra. First of all we have the following definition that summarizes Definition 2.27:

Definition 269.4. Let V be a real vector space.

(1) We say that a symmetric $\mathbb{R}$-bilinear pairing $\varphi\colon V \times V \to \mathbb{R}$ is positive-definite if $\varphi(v, v) \in \mathbb{R}_{>0}$ for all $v \in V \setminus \{0\}$.
(2) An inner product on V is a positive-definite, symmetric $\mathbb{R}$-bilinear pairing $V \times V \to \mathbb{R}$.

Example.

(1) Let $n \in \mathbb{N}_0$. As on page 122 we define the standard inner product on $\mathbb{R}^n$ to be

$$\begin{array}{rcl} \langle -, - \rangle_{\mathbb{R}^n}\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ (v, w) & \mapsto & \langle v, w \rangle := \sum_{i=1}^{n} v_i \cdot w_i. \end{array}$$

[2859] We will only care about the case $\mathbb{K} = \mathbb{R}$ with the trivial involution and the case $\mathbb{K} = \mathbb{C}$ with the usual involution given by complex conjugation.
[2860] In other words a pairing is "sesquilinear" if it is $\mathbb{K}$-linear in the first entry and $\mathbb{K}$-antilinear in the second entry.

(2) Let $\varphi\colon V \times V \to \mathbb{R}$ be an inner product on a real vector subspace V. It follows immediately from the definitions that for any vector subspace $W \subseteq V$ the restriction $\varphi|_{W\times W}\colon W \times W \to \mathbb{R}$ is also an inner product. □

Next we recall Definition 2.29, which is just the complex analog of the above Definition 269.4:

Definition 269.5. Let V be a complex vector space.

(1) We say that a Hermitian $\mathbb{C}$-sesquilinear pairing $\varphi\colon V\times V \to \mathbb{C}$ is positive-definite if[2861] $\varphi(v,v) \in \mathbb{R}_{>0}$ for all $v \in V \setminus \{0\}$.

(2) An inner product on V is a positive-definite, Hermitian $\mathbb{C}$-sesquilinear pairing.

Example. Let $n \in \mathbb{N}_0$. As on page 123 we define the standard inner product on $\mathbb{C}^n$ to be

$$\begin{aligned} \langle -,-\rangle_{\mathbb{C}^n}\colon \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{R} \\ (v,w) &\mapsto \langle v,w\rangle := \sum_{i=1}^{n} v_i \cdot \overline{w}_i. \end{aligned}$$

□

To describe objects in linear algebra explicitly it is usually convenient to express them in terms of matrices. As this point it is worth recalling Definitions 81.3 and 239.17:

Notation 269.6. Let $n \in \mathbb{N}_0$.

(1)
- We denote by $\mathrm{Sym}(n) \subseteq \mathrm{M}(n\times n,\mathbb{R})$ the set of symmetric matrices.
- We denote by $\mathrm{Herm}(n) \subseteq \mathrm{M}(n\times n,\mathbb{C})$ the set of Hermitian matrices.

Note that these are real vector spaces of dimension $\frac{1}{2}n\cdot(n+1)$ and $2n^2$. In particular we can view the above as smooth manifolds.

(2)
- We say $H \in \mathrm{M}(n\times n,\mathbb{R})$ is positive-definite if $v^{\mathrm{T}}\cdot H\cdot v > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$.
- We say $H \in \mathrm{M}(n\times n,\mathbb{C})$ is positive-definite if $v^{\mathrm{T}}\cdot H\cdot \overline{v} \in \mathbb{R}_{>0}$ for all $v \in \mathbb{C}^n \setminus \{0\}$.

(3)
- We denote by $\mathrm{Sym}^+(n) \subseteq \mathrm{Sym}(n)$ the set of positive-definite symmetric matrices.
- We denote by $\mathrm{Herm}^+(n) \subseteq \mathrm{Herm}(n)$ the set of positive-definite Hermitian matrices.

In the following lemma we summarize a few basic facts about $\mathrm{Sym}^+(n)$ and $\mathrm{Herm}^+(n)$:

Lemma 269.7. (Positive Definite-Matrices Lemma) Given any $n \in \mathbb{N}_0$ the following statements hold:

(1) $\mathrm{Sym}^+(n) \subseteq \mathrm{Sym}(n)$ is a convex open subset.

(2) $\mathrm{Herm}^+(n) \subseteq \mathrm{Herm}(n)$ is a convex open subset.

It follows from the Open Subset-Submanifold Lemma 25.8 that we can view $\mathrm{Sym}^+(n)$ and $\mathrm{Herm}^+(n)$ as smooth manifolds of dimension $\frac{1}{2}n\cdot(n+1)$ and $2n^2$.

Proof.

(1) Note that it follows from the elementary Positive Definite–Open Condition Lemma 81.4 that being positive-definite is an open condition. Thus $\mathrm{Sym}^+(n)$ is an open subset of $\mathrm{Sym}(n)$. It remains to show that it is a convex subset. So let $\varphi,\psi \in \mathrm{Sym}^+(n)$ and let $t \in [0,1]$. For any $v \in \mathbb{R}^n \setminus \{0\}$ we have

$$((1-t)\cdot\varphi + t\cdot\psi)(v,v) \;=\; (1-t)\cdot\underbrace{\varphi(v,v)}_{>0} + t\cdot\underbrace{\psi(v,v)}_{>0} \;>\; 0.$$

[2861] Note that it follows from the fact that φ is Hermitian that for any $v \in V$ we have $\varphi(v,v) = \overline{\varphi(v,v)}$, in other words, for any $v \in V$ we have $\varphi(v,v) \in \mathbb{R}$.

(2) The proof of this statement is almost the same as in (1). ■

269.2. **Inner products on vector bundles.** As in the previous chapter we want to transport concepts about vector spaces to vector bundles. In particular we would like to discuss inner products on vector bundles.

Clearly an inner product on a vector bundle $p\colon E \to B$ is supposed to be an inner product on each fiber E_b such that "the inner product is continuous in $b \in B$". Evidently we need to make the phrase in the quotes precise. Fortunately we already developed the tools and the language for doing so:

Definition 269.8. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) $\mathbb{K}$-vector bundle. A (smooth) inner product on p is a (smooth) section[2862] $\varphi\colon B \to \mathrm{Hom}_{\mathbb{K}}(E \otimes_{\mathbb{K}} \overline{E}, \mathbb{K})$ of the $\mathbb{K}$-vector bundle $\mathrm{Hom}_{\mathbb{K}}(E \otimes_{\mathbb{K}} \overline{E}, \mathbb{K}) \to B$ such that for each $b \in B$ we get (under the bijection from the Bilinear Pairing–Tensor Lemma 269.3) an inner product $\varphi(b)\colon E_b \times E_b \to \mathbb{K}$ on the fiber E_b.[2863]

Example.

(1) An inner product on the trivial vector bundle $B \times \mathbb{R}^n$ is essentially the same as a continuous map $B \to \mathrm{Sym}^+(n)$. Similarly an inner product on the trivial vector bundle $B \times \mathbb{C}^n$ corresponds to a continuous map $B \to \mathrm{Herm}^+(n)$.

(2) Let $M \subseteq \mathbb{R}^n$ be a submanifold. For each $P \in M$ we consider the restriction of the standard inner product $\langle -, - \rangle_{\mathbb{R}^n}$ to the tangent space $\mathrm{T}_P M \subseteq \mathbb{R}^n$. It follows easily from the Vector Bundle-Section Functorial Lemma 265.20 and the discussion on page 5196 that these restrictions define an inner product on the tangent bundle $\mathrm{T}M$.

(3) Let M be a smooth bundle. An inner product on the tangent bundle $\mathrm{T}M$ is usually called a Riemannian structure. We introduced this notion, in a somewhat naive form, earlier in Definition 131.2. □

Consciously and subconsciously we will frequently use the following lemma:

Lemma 269.9. (Inner Product–Section Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p\colon E \to B$ be a (smooth) $\mathbb{K}$-vector bundle and let φ be a (smooth) inner product on p. For any two (smooth) sections $s, t\colon B \to E$ the map

$$\begin{array}{ccc} B & \to & \mathbb{R} \\ b & \mapsto & \varphi(s(b), t(b)) \end{array}$$

is continuous (smooth).

Sketch of proof. By the Pasting Proposition 4.7 (respectively the Smooth Pasting Proposition 25.19) it suffices to check continuity (smoothness) locally. Using (smooth) trivializations we only have to deal with trivial vector bundles $U \times \mathbb{K}^n \to U$, continuous (smooth) maps $\varphi\colon U \to \mathrm{Herm}^+(\mathbb{K}^n)$ and $s, t\colon U \to \mathbb{K}^n$. But at that point it is clear that the map is indeed continuous (smooth). ■

Our goal is to show that vector bundles admit inner products. It turns out that for all reasonable base spaces this is possible:

[2863] For $\mathbb{K} = \mathbb{R}$ we define $\overline{E} = E$ and for $\mathbb{K} = \mathbb{C}$ we denote by $\overline{E}$ the vector bundle that we introduced in Definition 268.33. Note that it follows from Definitions 268.31, 268.33 and 268.36 that $\mathrm{Hom}_{\mathbb{K}}(E \otimes_{\mathbb{K}} \overline{E}, \mathbb{K})$ is indeed a (smooth) $\mathbb{K}$-vector bundle.

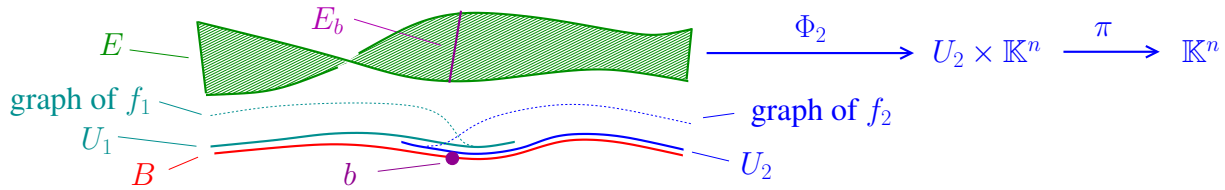
Proposition 269.10. (Inner Product-on-Vector Bundle Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be a (smooth) $\mathbb{K}$-vector bundle. If B is Hausdorff and paracompact, then there exists a (smooth) inner product on the vector bundle.

Proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ be a (smooth) n-dimensional $\mathbb{K}$-vector bundle over a topological space that is Hausdorff and paracompact. We pick a family $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n\}_{i\in I}$ of (smooth) trivializations of the vector bundle. Given any $i \in I$ we denote by $\pi\colon U_i \times \mathbb{K}^n \to \mathbb{K}^n$ the natural projection.

Since B is paracompact and Hausdorff we obtain from the Paracompact-Partition of Unity Theorem 14.15 (3) (respectively the Manifold–Partitions-of-Unity Theorem 27.2) that there exists a family of continuous (smooth) functions $\{f_i\colon B \to [0,1]\}_{i\in I}$ with the following properties:

(1) For every $i \in I$ we have $\operatorname{supp}(f_i) \subseteq U_i$.[2864]
(2) For each $b \in B$ there exists an open neighborhood U of $b \in B$ such that f_i vanishes on U for all but finitely many i.
(3) For every $b \in B$ we have $\displaystyle\sum_{i\in I} f_i(b) \;=\; 1.$

Now let $b \in B$. By (2) there exists an open neighborhood U of b and a finite subset $J \subseteq I$ such that for all $i \in I \setminus J$ we have $f_i|_U \equiv 0$.



Given $v, w \in E_b$ we now set

$$\varphi(b)(v \otimes w) \;:=\; \sum_{j\in J} f_j(b) \cdot \langle \underbrace{(\pi \circ \Phi_j)(v)}_{\in \mathbb{K}^n}, \underbrace{(\pi \circ \Phi_j)(w)}_{\in \mathbb{K}^n} \rangle_{\mathbb{K}^n}.$$

Claim. The map
$$\begin{array}{rcl} B & \to & \operatorname{Hom}_{\mathbb{K}}(E \otimes_{\mathbb{K}} \overline{E}, \mathbb{K}) \\ b & \mapsto & \varphi(b) \end{array}$$
is an inner product on E.

Proof. Note that for each $b \in B$ the map $\varphi(b)$ is a finite linear combination of inner products (with non-negative scalars). It follows from the same argument as in the Positive Definite-Matrices Lemma 269.7 that $\varphi(b)$ is also an inner product.

It remains to show that φ is continuous (smooth). By the Pasting Proposition 4.7 (respectively the Smooth Pasting Proposition 25.19) it suffices to check continuity (smoothness) locally. But by the above definition we know that locally φ is given as a sum of finitely many continuous maps, so φ is locally continuous. ∎

269.3. **The dual vector bundle revisited.** As a first application of the Inner Product-on-Vector Bundle Proposition 269.10 we will show in this section that dual vector bundles

[2864] Recall that according to Definition 14.14 the *support* of a map $f\colon X \to \mathbb{R}$ is defined as
$$\operatorname{supp}(f) \;:=\; \overline{\{x \in X \mid f(b) \neq 0\}}.$$

(over reasonable topological spaces) are essentially well-known vector bundles in slight disguise:

Proposition 269.11. (Dual Vector Bundle Isomorphism Lemma)
(1) Let $E \to B$ be a (smooth) real vector bundle. Any choice of a (smooth) inner product on E gives rises to an isomorphism $E \to E^* = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ of (smooth) real vector bundles over B.
(2) Let $E \to B$ be a (smooth) complex vector bundle. Any choice of a (smooth) inner product on E gives rises to an isomorphism $\overline{E} \to E^* = \mathrm{Hom}_{\mathbb{R}}(\overline{E}, \mathbb{R})$ of (smooth) complex vector bundles over B.

Example. For any $n \in \mathbb{N}_0$ we have the following isomorphisms of smooth $\mathbb{R}$-vector bundles over $\mathbb{RP}^n$:

$$\mathrm{T}\mathbb{RP}^n \oplus \mathrm{Tr}_{\mathbb{RP}^n}(\mathbb{K}) \underset{\substack{\uparrow \\ \text{by the Stable Tangent Bundle–}\mathbb{KP}^n\text{-Proposition 268.42}}}{\cong} \overbrace{\mathfrak{T}(\mathbb{KP}^n) \oplus \cdots \oplus \mathfrak{T}(\mathbb{KP}^n)}^{n+1 \text{ copies}} \underset{\substack{\uparrow \\ \text{by the Dual Vector Bundle Isomorphism Proposition 269.11}}}{\cong} \overbrace{\mathfrak{T}(\mathbb{KP}^n) \oplus \cdots \oplus \mathfrak{T}(\mathbb{KP}^n)}^{n+1 \text{ copies}}.$$

This statement will play a major role later in the $\mathbb{RP}^n$-SW-Proposition 274.21. In fact this calculation is (together with its complex analog) for us the main application of the Dual Vector Bundle Isomorphism Proposition 269.11. □

Sketch of proof. In the following we first give a proof of Statement (2), since that statement is slightly more subtle than Statement (1).

(2) In the following discussion we deal with the continuous statement, the smooth statement is proved essentially the same way. Thus let $E \to B$ be a complex vector bundle and let $\{\varphi_b \colon E_b \otimes_{\mathbb{K}} \overline{E}_b \to \mathbb{C}\}_{b \in B}$ be an inner product on E. We denote by $q \colon E^* \to B$ the dual vector bundle. Our goal is to show that the map

$$\begin{array}{rcl} \Xi \colon \overline{E} & \to & E^* \\ v & \mapsto & \left(p(v), \left(\begin{array}{rcl} E_{p(v)} & \to & \mathbb{C} \\ w & \mapsto & \varphi_{p(v)}(w, v) \end{array}\right)\right) \end{array}$$

is an isomorphism of complex vector bundles over B. By the Vector Bundle Isomorphism Criterion 265.40 it remains to prove the following following two statements:
(A) The map Ξ is fiberwise well-defined and an isomorphism of $\mathbb{C}$-vector spaces.
(B) The map Ξ is continuous.
We will take care of these issues in the following two claims:

Claim A. For each $b \in B$ the map $\Xi \colon \overline{E}_b \to E_b^* = \{b\} \times \mathrm{Hom}_{\mathbb{C}}(E_b, \mathbb{C})$ is fiberwise well-defined and it is an isomorphism of $\mathbb{C}$-vector spaces.

Proof. Let $b \in B$. Since $\varphi_{p(v)}$ is sesquilinear we see that for any $v \in E_b$ the map

$$\begin{array}{rcl} E_b & \to & \mathbb{C} \\ w & \mapsto & \varphi_{p(v)}(w, v) \end{array}$$

is indeed $\mathbb{C}$-linear. The sesquilinearity also implies that the map $\overline{E}_b \to \mathrm{Hom}_{\mathbb{C}}(E_b, \mathbb{C})$ is a $\mathbb{C}$-linear homomorphism.

We will need to show that it is actually an isomorphism of $\mathbb{C}$-vector spaces. We start out with the following observation: Since $\varphi_{p(v)}$ is positive-definite we see that for

any $v \in E_b$ we have $\varphi_{p(v)}(v,v) > 0$. This shows that the map $\overline{E}_b \to \mathrm{Hom}_{\mathbb{C}}(E_b, \mathbb{C})$ is a monomorphism. But since both vector spaces $\overline{E}_b$ and $\mathrm{Hom}_{\mathbb{C}}(E_b, \mathbb{C})$ have the same (finite!) dimension we see that the homomorphism $\overline{E}_b \to \mathrm{Hom}_{\mathbb{C}}(E_b, \mathbb{C})$ is actually an isomorphism. ⊞

Claim B. The above map $\Xi\colon \overline{E} \to E^*$ is continuous.

Proof. By the Pasting Proposition 4.7 it suffices to check continuity locally. Let $\Phi\colon p^{-1}(U) \to U \times \mathbb{C}^n$ be a local trivialization of the $\mathbb{C}$-vector bundle $p\colon E \to B$. We denote by $\pi\colon U \times \mathbb{C}^n \to \mathbb{C}^n$ the natural projection. It follows from the observation on page 5197 that there exists a continuous map $A\colon U \to \mathrm{Herm}^+(n)$ such that the following diagram commutes:

$$\begin{array}{ccc} \overbrace{p^{-1}(U)}^{\subseteq \overline{E}} & \xrightarrow{\quad\Xi\quad} & \overbrace{q^{-1}(U)}^{\subseteq E^*} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi^*} \\ U \times \mathbb{C}^n & \xrightarrow{(b,v) \mapsto (b, w \mapsto \langle v, A(b)\cdot w\rangle_{\mathbb{C}^n})} & U \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}). \end{array}$$

It remains to show that the bottom map is continuous. To do so we consider the following commutative diagram:

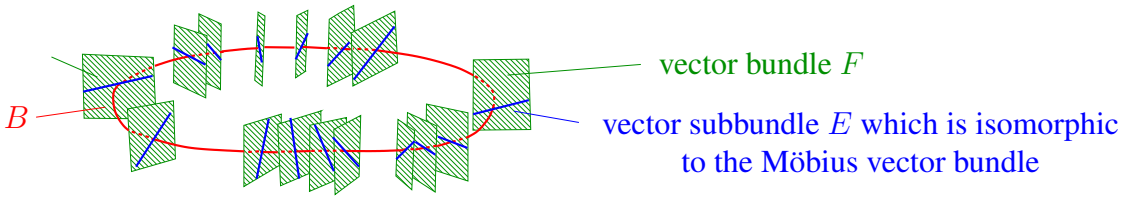
$$\begin{array}{ccc} U \times \mathbb{C}^n & \xrightarrow{(b,v) \mapsto (b, w \mapsto \langle v, A(b)\cdot w\rangle_{\mathbb{C}^n})} & U \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (b,\theta) \mapsto (b, \theta(\mathfrak{e}_1, \ldots, \mathfrak{e}_n))} \\ U \times \mathbb{C}^n & \xrightarrow{(b,v) \mapsto (b, (\langle v, A(b)\cdot \mathfrak{e}_1\rangle_{\mathbb{C}^n}, \ldots, \langle v, A(b)\cdot \mathfrak{e}_n\rangle_{\mathbb{C}^n}))} & U \times \mathbb{C}^n. \end{array}$$

The bottom horizontal map is of course continuous. The vertical maps are homeomorphisms (essentially by definition). It follows that the top horizontal map is also continuous. ⊞

(1) The proof of this statement is almost the same as the proof of Statement (2). ■

269.4. **Subbundles and normal vector bundles.** We start out by introducing the notion of a vector subbundle:

Definition 269.12. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon F \to B$ be an n-dimensional (smooth) $\mathbb{K}$-vector bundle. We say a subset $E \subseteq F$ is an m-dimensional (smooth) $\mathbb{K}$-vector subbundle if for each $b \in B$ there exists an open neighborhood U of b and a (smooth) trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{K}^n$ such that $\Phi(p^{-1}(U) \cap E) = U \times \mathbb{K}^m \times \{0\}$.



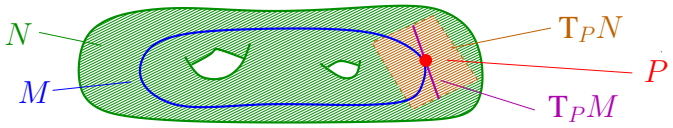
Example.

(1) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ and $q\colon F \to B$ be two (smooth) $\mathbb{K}$-vector bundles. We form the Whitney sum $E \oplus F$. One can easily verify that the maps

$$\begin{array}{ccc} E & \to & E \oplus F \\ e & \mapsto & (e, 0) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E & \to & E \oplus F \\ f & \mapsto & (0, f). \end{array}$$

are (smooth) embeddings and that the images are $\mathbb{K}$-vector subbundles of $E \oplus F$.

(2) Let N be an n-dimensional smooth manifold and let $M \subseteq N$ be an m-dimensional smooth submanifold. It is not hard to see that $\mathrm{T}M$ can be viewed as a subbundle of $\mathrm{T}N|_M$.



(3) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ be a (smooth) $\mathbb{K}$-vector bundle. Furthermore let $s_1, \dots, s_m\colon B \to E$ be (smooth) sections such that for each $b \in B$ the vectors $s_1(b), \dots, s_m(b)$ are linearly independent. In Exercise 269.1 we will see that $s_1, \dots, s_m$ span a trivializable (smooth) subbundle. More precisely we will see that there exists a unique trivializable m-dimensional $\mathbb{K}$-vector subbundle $E \subseteq F$ such that for each $b \in B$ we have $E_b = \operatorname{span}\{s_1(b), \dots, s_m(b)\}$. □

The following proposition shows that essentially all vector subbundles of a vector bundle are subsummands of Whitney sum:

Proposition 269.13. (Normal Vector Bundle Proposition) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $p\colon F \to B$ be a (smooth) $\mathbb{K}$-vector bundle, let φ be a (smooth) inner product on F and let $E \subseteq F$ be a (smooth) $\mathbb{K}$-vector subbundle.

(1) We set $$E^\perp := \{v \in F \,|\, \varphi_{p(v)}(v, e) = 0 \text{ for all } e \in E_{p(v)}\}.$$ This is a (smooth) $\mathbb{K}$-vector subbundle of F.

(2) The inclusions $E \to F$ and $E^\perp \to F$ induce a natural isomorphism $E \oplus E^\perp \to F$ of (smooth) $\mathbb{K}$-vector bundles.

(3) If A is a $\mathbb{K}$-vector subbundle of F such that the inclusions $E \to F$ and $A \to F$ induce an isomorphism $E \oplus A \to F$, then there exists a (smooth) isomorphism $\Theta\colon F \to F$ of $\mathbb{K}$-vector bundles with $\Theta|_E = \mathrm{id}_E$ and with $\Theta(A) = E^\perp$. In particular the vector bundle A is isomorphic to the vector bundle $E^\perp$.

Definition 269.14. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p\colon F \to B$ be a $\mathbb{K}$-vector bundle over a topological space B that is Hausdorff and paracompact. Furthermore let E be a $\mathbb{K}$-vector subbundle of F. By the Inner Product-on-Vector Bundle Proposition 269.10 we know that F admits an inner product. We refer to the $\mathbb{K}$-vector subbundle $E^\perp$ from the Normal Vector Bundle Proposition 265.15 as the normal vector bundle of $E \subseteq F$.[2865] It follows from the Normal Vector Bundle Proposition 269.13 (3) that $E^\perp$ is essentially unique.

[2864] This says in particular that the isomorphism type of the normal $\mathbb{K}$-vector subbundle $E^\perp$ does not depend on the choice of the inner product φ.

[2865] In the above setting one could in principle define a corresponding "quotient vector bundle" $F/E \to B$ and one could show that it is isomorphic to $E^\perp$. This would give a natural "candidate" for a normal vector bundle that does not depend on the choice of an inner product. But we will not go down that route.

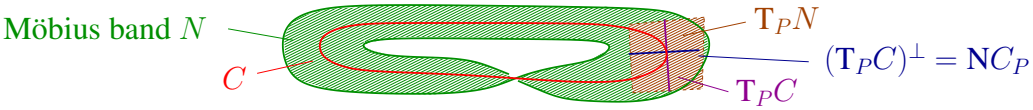
We want to highlight a special case:

Definition 269.15. Let N be an n-dimensional smooth manifold and let $M \subseteq N$ be an m-dimensional smooth submanifold. On page 5201 we pointed out that $\mathrm{T}M$ can be viewed as a subbundle of $\mathrm{T}N|_M$. We refer to the normal vector bundle of $\mathrm{T}M \subseteq \mathrm{T}N|_M$ as the normal vector bundle of $M \subseteq N$. If N is understood, then we refer to it as $\mathrm{N}M$.

Example.

(1) Let M be a smooth submanifold of some $\mathbb{R}^n$. It follows easily from the discussion on page 5197 that the normal vector bundle from Definition 269.14 is isomorphic to the normal vector bundle $\mathrm{N}M$ provided by the earlier Normal Vector Bundle Proposition 265.15.

(2) We consider the Möbius band N and the submanifold C that is given by the central circle.



It is pretty clear that the normal vector bundle of $C \subseteq N$ is isomorphic to the Möbius vector bundle from page 5119. □

Proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p\colon F \to B$ be an n-dimensional (smooth) $\mathbb{K}$-vector bundle, let φ be a (smooth) inner product on F and let $E \subseteq F$ be an m-dimensional (smooth) $\mathbb{K}$-vector subbundle.

(1) We need to prove the following claim:

Claim. The set $\quad E^\perp := \{v \in F \,|\, \varphi_{p(v)}(v, e) = 0 \text{ for all } e \in E_{p(v)}\}.$

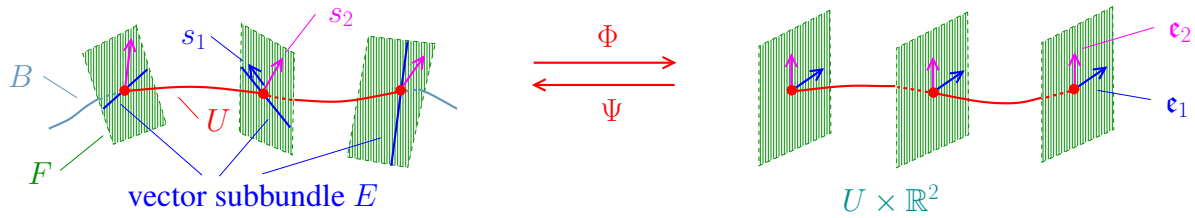
is a (smooth) $\mathbb{K}$-vector subbundle of F.

Proof. Let $b_0 \in B$. Recall that by Definition 269.12 we know that there exists an open neighborhood U of $b_0 \in B$ and a (smooth) trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{K}^n$ such that $\Phi(p^{-1}(U) \cap E) = U \times \mathbb{K}^m \times \{0\}$. We denote by $\Psi\colon U \times \mathbb{K}^n \to p^{-1}(U)$ the inverse.

Given $i = 1, \ldots, n$ we consider the (smooth) map

$$\begin{aligned} s_i\colon U &\to p^{-1}(U) \\ b &\mapsto \Psi(b, \mathfrak{e}_i). \end{aligned}$$

Note that for each $b \in U$ the vectors $s_1(b), \ldots, s_n(b)$ form a basis of F_b and that the vectors $s_1(b), \ldots, s_m(b)$ form a basis of E_b.



For each $b \in B$ we now apply the Gram-Schmidt Orthogonalization Process 2.32 to the vectors $s_1(b), \dots, s_n(b) \in E_b$. We get new vectors $\widetilde{s}_1(b), \dots, \widetilde{s}_n(b) \in E_b$ with the following properties:

- $\widetilde{s}_1(b), \dots, \widetilde{s}_n(b)$ form an orthonormal basis of F_b.
- $\widetilde{s}_1(b), \dots, \widetilde{s}_m(b)$ form a basis of E_b.

Furthermore note that it follows from the explicit formulas of the Gram-Schmidt Orthogonalization Process 2.32 (1), (4), (5a) and the Inner Product–Section Lemma 269.9 that the resulting maps $\widetilde{s}_1, \dots, \widetilde{s}_n \colon B \to F$ are continuous (smooth). We now consider the map

$$\begin{aligned} \widetilde{\Psi} \colon U \times \mathbb{K}^n &\to p^{-1}(U) \\ (b, (\lambda_1, \dots, \lambda_n)) &\mapsto \sum_{i=1}^{n} \lambda_i(b) \cdot \widetilde{s}_i(b). \end{aligned}$$

It follows easily from the Vector Bundle Isomorphism Criterion 265.40 that this is an isomorphism of (smooth) vector bundles. Clearly the inverse of $\widetilde{\Psi}$ shows us that $E^\perp$ is a (smooth) $\mathbb{K}$-vector subbundle. ⊞

(2) We consider the map

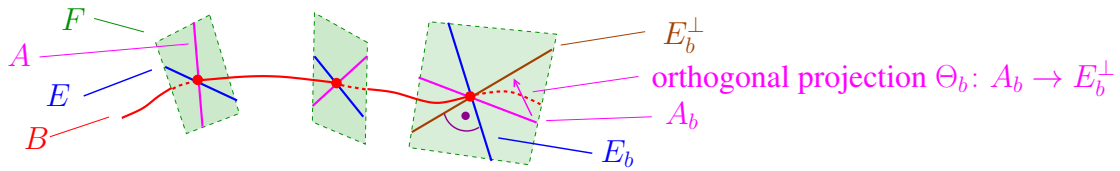
$$\begin{aligned} E \oplus E^\perp &\to F \\ (v, w) &\mapsto v + w \end{aligned}$$

which is induced by the natural inclusions $E \to F$ and $E^\perp \to F$. Implicitly we showed in (1) that this map is locally (and thus also globally) an isomorphism. Alternatively note that it follows also easily from the Vector Bundle Isomorphism Criterion 265.40 that this map is an isomorphism of $\mathbb{K}$-vector bundles.

(3) We need to prove the following claim:

Claim. If A is a $\mathbb{K}$-vector subbundle of the vector bundle F such that the inclusions $E \to F$ and $A \to F$ induce an isomorphism $E \oplus A \to F$, then there exists a (smooth) isomorphism $\Xi \colon F \to F$ of $\mathbb{K}$-vector bundles with $\Xi|_E = \mathrm{id}_E$ and with $\Xi(A) = E^\perp$.

Proof. For each $b \in B$ we consider the orthogonal projection $\Theta_b \colon A_b \to E_b^\perp$ that is given by the inner product φ_b and the Orthogonal Projection Lemma 2.33. Since $A_b \cap (E_b^\perp)^\perp = A_b \cap E_b = \{0\}$ we obtain from the Orthogonal Projection Lemma 2.33 (3b) that the map $\Theta_b \colon A_b \to E_b^\perp$ is in fact an isomorphism of $\mathbb{K}$-vector bundles. The maps $\{\Theta_b\}_{b \in B}$ define a map $\Theta \colon A \to E^\perp$.



We claim that this map $\Theta \colon A \to E^\perp$ is continuous (smooth). It suffices to do this locally. Locally this statement follows from the explicit description of the orthogonal projection, see the Orthogonal Projection Lemma 2.33 (2), and since (as in (1)) we can find locally orthonormal sections in $E^\perp$.

It now follows easily from the Vector Bundle Isomorphism Criterion 265.40 that the map $\Theta \colon A \to E^\perp$ together with the identity on E define the desired (smooth) isomorphism $\Xi \colon F \to F$ of $\mathbb{K}$-vector bundles with $\Xi|_E = \mathrm{id}_E$ and with $\Xi(A) = E^\perp$. ■

269.5. **Applications of subbundles and normal vector bundles.** As a warm-up, and as an illustration and application of our results, we prove the following little lemma:

Lemma 269.16. (Vector Bundle-Trivializable Criterion) Let $p\colon E \to B$ be a 2-dimensional real vector bundle over a topological space that is Hausdorff and paracompact. If E is orientable and if E admits a nowhere vanishing section, then E is trivializable.

Proof. Let $p\colon E \to B$ be an orientable 2-dimensional real vector bundle and let $s\colon B \to E$ be a nowhere vanishing section. As we discussed on page 5201, the nowhere vanishing section s spans a 1-dimensional trivializable subbundle S of E.

Since B is Hausdorff and paracompact we obtain from the Normal Vector Bundle Proposition 269.13 that there exists a 1-dimensional vector bundle $S^\perp$ of E such that $S \oplus S^\perp$ is isomorphic to E. Since S is trivializable we know that it is orientable. By hypothesis E is also orientable. By Exercise 268.7 we know that this implies that $S^\perp$ is also orientable. Thus $S^\perp$ is an orientable 1-dimensional real vector bundle. By Exercise 265.17 this implies that it is actually trivializable. It now follows that E is isomorphic to the Whitney sum of two trivializable vector bundles. As we discussed on page 5170, this implies that E is also trivializable. ■

We continue with a more interesting lemma:

Lemma 269.17. (Immersion–Tangent Bundle Lemma) Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold with $\partial M = \partial N = \varnothing$. Furthermore let $f\colon M \to N$ be an immersion.[2866]

(1) The image of the map

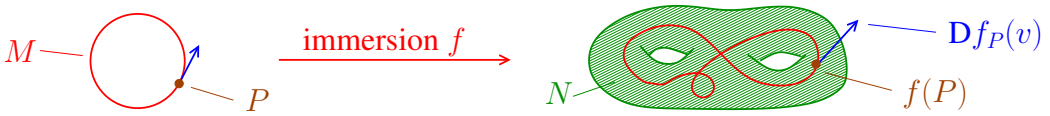
$$\begin{array}{rcl} \Theta\colon \mathrm{TM} & \to & f^*\mathrm{T}N \\ (P,v) & \mapsto & (P, \underbrace{\mathrm{D}f_P(v)}_{\in \mathrm{T}_{f(P)}N}) \end{array}$$

is a subbundle of $f^*\mathrm{T}N$.

(2) There exists an $(n-m)$-dimensional smooth vector bundle $p\colon \mathrm{N}_f M \to M$ (which is unique up to isomorphism) such that there exists an isomorphism

$$\mathrm{T}M \oplus \mathrm{N}_f M \xrightarrow{\cong} f^*\mathrm{T}N$$

of smooth vector bundles.



Definition 269.18. Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold with $\partial M = \partial N = \varnothing$ and let $f\colon M \to N$ be an immersion. We refer to vector bundle $\mathrm{N}_f M$ from the Immersion–Tangent Bundle Lemma 269.17 (which is well-defined up to isomorphism) as the normal vector bundle of the immersion. If $M \subseteq N$ is a submanifold and if we just consider the inclusion, then sometimes (as in Definition 269.15) we just write $\mathrm{N}M$.

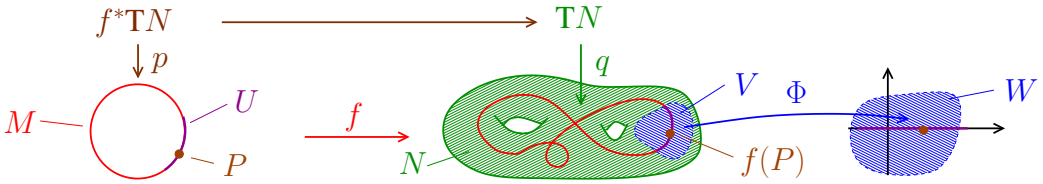
[2866] Since we assume that $\partial M = \partial N = \varnothing$ we know, see Definition 30.9, that f is an immersion precisely if for each $P \in M$ the differential $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_P N$ is a monomorphism.

Proof. Let M be an m-dimensional smooth manifold and let N be an n-dimensional smooth manifold with $\partial M = \partial N = \varnothing$. Furthermore let $f\colon M \to N$ be an immersion. We consider the tangent bundle $q\colon \mathrm{T}N \to N$ and the pullback bundle $p\colon f^*\mathrm{T}N \to M$.

(1) Our goal is to show that the image of the map

$$\begin{aligned} \Theta\colon \mathrm{T}M &\to f^*\mathrm{T}N \\ (P, v) &\mapsto (P, \underbrace{\mathrm{D}f_P(v)}_{\in \mathrm{T}_{f(P)}N}) \end{aligned}$$

is a subbundle of $f^*\mathrm{T}N$. Thus let $P \in M$. Since $\mathrm{D}f_P$ is a monomorphism and since $\partial M = \partial N = \varnothing$ we obtain from the Local Smooth Embedding Theorem 30.13 together with the Smooth Embedding Theorem 30.15 that there exists an open neighborhood U of $P \in M$, an open neighborhood V of $f(P) \in N$ and a diffeomorphism $\Phi\colon V \to W$ to some open subset $W \subseteq \mathbb{R}^n$ such that $\Phi(f(U)) = W \cap (\{0\} \times \mathbb{R}^{n-m})$.



It is now elementary to verify that the map

$$\begin{aligned} \Xi\colon f^*\mathrm{T}N &\to U \times \mathbb{R}^n \\ (P, w) &\mapsto (P, \mathrm{D}\Phi_{f(P)} w) \end{aligned}$$

is a local trivialization of $f^*\mathrm{T}N|_U$ such that $\Xi(p^{-1}(U) \cap \Theta(\mathrm{T}M)) = U \times \mathbb{R}^m \times \{0\}$.

(2) We consider the subbundle $\Theta(\mathrm{T}M) \subseteq f^*\mathrm{T}N$. It follows from the fact that smooth manifolds are Hausdorff and paracompact, the Inner Product-on-Vector Bundle Proposition 269.10 and the Normal Vector Bundle Proposition 265.15 that there exists a smooth $(n-m)$-dimensional vector bundle $p\colon E \to M$ such that $\Theta(\mathrm{T}M) \oplus E$ is isomorphic to $f^*\mathrm{T}N$. Since $\Theta\colon \mathrm{T}M \to \Theta(\mathrm{T}M)$ is an isomorphism of smooth vector bundles we also see that $\mathrm{T}M \oplus E$ is isomorphic to $f^*\mathrm{T}N$. ■

Later, once we have developed the theory of characteristic classes, we will use the following corollary to restrict the dimensions for which there exists an immersion $\mathbb{R}\mathrm{P}^m \to \mathbb{R}^n$:

Corollary 269.19. (Euclidean Immersion–Tangent Bundle Corollary) In the following let M be an m-dimensional smooth manifold with $\partial M = \varnothing$. If there exists an immersion $f\colon M \to \mathbb{R}^n$, then there exists an $(m-n)$-dimensional smooth real vector bundle $E \to M$ such that the vector bundle $\mathrm{T}M \oplus E$ is trivializable.

Proof. In the following let $f\colon M \to \mathbb{R}^n$ be an immersion. By the Immersion–Tangent Bundle Lemma 269.17 we know that there exists an $(n-m)$-dimensional smooth vector bundle $p\colon E \to M$ such that the smooth vector bundle $\mathrm{T}M \oplus E$ is isomorphic to the smooth

vector bundle $f^*\mathrm{T}\mathbb{R}^n$. We now see that

$$\begin{array}{rcl} \mathrm{T}M \oplus E & \cong & f^*\mathrm{T}\mathbb{R}^n \\ & & \overset{\text{see page 5128}}{\downarrow} \\ & & = \text{ pullback of a trivializable } n\text{-dimensional vector bundle} \\ & & = \text{ trivializable } n\text{-dimensional vector bundle} \\ & & \overset{\text{see page 5145}}{\uparrow} \end{array}$$

■

We conclude this section with the following proposition, which will be useful later on:

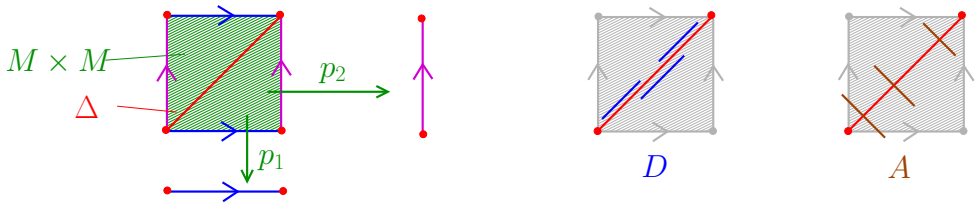
Proposition 269.20. (Tangent Bundle–Normal-to-Diagonal Proposition) Let M be a smooth manifold with empty boundary. We consider the diagonal

$$\Delta := \{(x,x) \in M \times M \,|\, x \in M\}.$$

As we saw in Exercise 25.30, one can easily verify that this is a proper smooth submanifold of $M \times M$. The normal vector bundle $\mathrm{N}\Delta$ of $\Delta \subseteq M \times M$ is isomorphic to the tangent bundle $\mathrm{T}\Delta$.

Sketch of proof. [2867] For $i = 1, 2$ we consider the projections $p_i \colon M \times M \to M$ onto the i-th factor. Furthermore we consider the following subsets of $\mathrm{T}(M \times M)|_\Delta$:

$$\begin{array}{rcl} A & := & \{((x,x),v) \,|\, x \in M, v \in \mathrm{T}_{(x,x)}(M \times M) \text{ such that } \mathrm{D}p_{1x}(v) = \mathrm{D}p_{2x}(v)\}, \\ B & := & \{((x,x),v) \,|\, x \in M, v \in \mathrm{T}_{(x,x)}(M \times M) \text{ such that } \mathrm{D}p_{1x}(v) = -\mathrm{D}p_{2x}(v)\}. \end{array}$$



It follows from Definition 269.15 and the Normal Vector Bundle Proposition 269.13 that it remains to prove the following claim:

Claim.

- As on page 5201 we view $\mathrm{T}\Delta$ as a subbundle of $\mathrm{T}(M \times M)|_\Delta$. With this identification we have an equality $A = \mathrm{T}\Delta$.
- B is also a subbundle of $\mathrm{T}(M \times M)|_\Delta$.
- The vector bundles A and B are isomorphic.
- The inclusions $A, B \to \mathrm{T}(M \times M)|_\Delta$ induce an isomorphism $A \oplus B \to \mathrm{T}(M \times M)|_\Delta$ of real vector bundles.

Proof.

(1) Let $\delta \colon M \times M \to M \times M$ be the diffeomorphism that is given by $\delta(x,y) = \delta(y,x)$. Clearly we have $\delta|_\Delta = \mathrm{id}_\Delta$ and we have $p_1 \circ \delta = p_2$. It now follows easily from the chain rule that $\mathrm{T}\Delta \subseteq A$. Since both are n-dimensional we get the desired equality.
(2) We leave the straightforward verification of this statement to the reader.

[2867] The proposition is somewhat related to the Euler Number-of-Tangent Bundle Proposition 265.34 and to the Total Tangent Space-Diagonal Embedding Proposition 53.23.

(3) Given $x \in M$ we consider the smooth embeddings

$$\begin{array}{rcl} \iota_{1x}\colon M & \to & M \times M \\ y & \mapsto & (y,x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \iota_{2x}\colon M & \to & M \times M \\ y & \mapsto & (x,y). \end{array}$$

For each $(x,x) \in M \times M$ we now consider the map

$$A_{(x,x)} \xrightarrow{\mathrm{D}p_{1x}} \mathrm{T}_x M \xrightarrow{v \mapsto (\mathrm{D}\iota_{1x}(v), \mathrm{D}\iota_{2x}(-v))} B_{(x,x)}.$$

We leave it to the reader to verify that this defines an isomorphism of vector bundles.

(4) This statement follows easily from the observation that for any real vector space V the map

$$\begin{array}{rcl} V \times V & \to & V \times V \\ (a,b) & \mapsto & (a+b, a-b) \end{array}$$

is an isomorphism.

We leave it to the reader to fill in the various little gaps in the arguments. ■

269.6. **Appendix: The stable normal vector bundle.** Let M be a smooth submanifold of $\mathbb{R}^n$. In Definition 265.16 we associated to M the normal vector bundle $p\colon \mathrm{N}M \to M$. A priori this notion makes of course no sense for a general abstract smooth manifold. For example, what should the normal vector bundle of $\mathbb{R}^n/\mathbb{Z}^n$ be?

But rather surprisingly we will see in a moment that a smooth manifold has a well-defined "stable normal vector bundle". To make sense of this notion we first need to introduce what we mean by "stable":

Definition 269.21. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let B be a topological space.

(1) We say that two $\mathbb{K}$-vector bundles $p\colon E \to B$ and $q\colon F \to B$ are stably isomorphic if there exist trivializable $\mathbb{K}$-vector bundles $S \to B$ and $T \to B$ such that $E \oplus S$ is isomorphic to $F \oplus T$.

(2) A $\mathbb{K}$-vector bundle $p\colon E \to B$ is called stably trivializable if it is stably isomorphic to a trivializable $\mathbb{K}$-vector bundle.

Now we can formulate the one and only result of this appendix:

Proposition 269.22. (Stable Normal Vector Bundle Proposition) Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}^m$ and $g\colon M \to \mathbb{R}^n$ be two smooth embeddings. By Definition 269.18 these smooth embeddings give rise to normal vector bundles $\mathrm{N}_f M$ and $\mathrm{N}_g M$. These normal vector bundles are stably isomorphic over M.

Definition 269.23. Let M be a closed smooth manifold. By the Weak Whitney Embedding Theorem 37.9 we know that there exists an $n \in \mathbb{N}_0$ and a smooth embedding $f\colon M \to \mathbb{R}^n$. It follows from the Stable Normal Vector Bundle Proposition 269.22 that the stable isomorphism type of $\mathrm{N}_f M$ is well-defined. The stable isomorphism type of $\mathrm{N}_f M$ is called the stable normal vector bundle of M.

Proof. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}^m$ and $g\colon M \to \mathbb{R}^n$ be two smooth embeddings. We need to show that normal vector bundles $\mathrm{N}_f M$ and $\mathrm{N}_g M$ are stably isomorphic over M. We start out with a few preparations:

(1) Let $i\colon \mathbb{R}^m \to \mathbb{R}^{m+n}$ be the smooth embedding that is given by $x \mapsto (x,0)$.

(2) Let $j\colon \mathbb{R}^n \to \mathbb{R}^{m+n}$ be the smooth embedding that is given by $x \mapsto (0,x)$.

(3) As in the proof of the Weak Unique Embedding Proposition **??** we consider the map

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & \mathbb{R}^{m+n} \\ (x,t) & \mapsto & (\underbrace{f(x)\cdot(1-t)\cdot t}_{\in\mathbb{R}^m},\underbrace{g(x)\cdot t}_{\in\mathbb{R}^n}). \end{array}$$

Note that F is a smooth homotopy from $i\circ f\colon M\to\mathbb{R}^{m+n}$ to $j\circ f\colon M\to\mathbb{R}^{m+n}$. Also note that each F_t is a smooth embedding. In other words, F is a smooth isotopy.

(4) It follows from (3) and the Isotopy Extension Theorem 44.2 that there exists diffeomorphism $H\colon\mathbb{R}^{m+n}\to\mathbb{R}^{m+n}$ with $H\circ F_0=F_1$, i.e. with $H\circ(i\circ f)=j\circ g$.

We now see that

$$\begin{array}{rcl} \mathrm{N}_f M\oplus\mathrm{Tr}_M(\mathbb{R}^n) & \cong & f^*(\mathrm{N}(f(M)\subseteq\mathbb{R}^m))\oplus\mathrm{Tr}_M(\mathbb{R}^n) \\ & \cong & f^*(\mathrm{N}(f(M)\subseteq\mathbb{R}^m)\oplus\mathrm{Tr}_{\mathbb{R}^{m+n}}(\mathbb{R}^n)) \\ & \cong & (i\circ f)^*(\mathrm{N}((i\circ f)(M)\subseteq\mathbb{R}^{m+n})) \\ & \cong & (H\circ i\circ f)^*(\mathrm{N}((H\circ i\circ f)(M)\subseteq\mathbb{R}^{m+n})) \\ & \cong & (j\circ f)^*(\mathrm{N}(g(M)\subseteq\mathbb{R}^{m+n}))\ \cong\ \ldots\ \cong\ \mathrm{N}_g M\oplus\mathrm{Tr}_M(\mathbb{R}^m). \end{array}$$ ■

We will not make use of the stable normal vector bundle of a smooth manifold. But we figured that this concept is an interesting construction, and it does play a role if one goes further into some aspects of topology, like surgery theory.

269.7. **Appendix: Orientation bundles.** In Chapter 268 we studied various linear algebraic constructions on vector bundles. In this appendix we concentrate on one of the constructions, namely the hitherto little studied vector bundle $\wedge^k E^*$. As we will see, this type of vector bundle will allow us to introduce the orientation bundle corresponding to a real vector bundle. Since we will not make use of this concept we will keep proofs short and we will leave the details to the reader.

We start out by recalling Notations 72.6 and 268.16:

Notation 269.24. Let V be a real vector space and let $k\in\mathbb{N}_0$. As in Definition 72.2 we say that a map $\omega\colon V^k\to\mathbb{R}$ is an alternating k-form on V if it is multilinear and alternating in the sense of Definition 72.1. We set

$$\wedge^k V^* \ :=\ \text{the real vector space of all alternating } k\text{-forms on } V.$$

By the Alternating Forms-Basis Proposition 72.7 we know that this real vector space has dimension $\binom{n}{k}$.

Next we recall Definition 268.35:

Definition 269.25. Let $p\colon E\to B$ be an n-dimensional (smooth) real vector bundle and let $k\in\mathbb{N}_0$. We consider the set

$$\wedge^k E^* \ :=\ \bigsqcup_{b\in B}\wedge^k E_b^* \quad \text{with the obvious map } r\colon \wedge^k E^*\to B.$$

By the Vector Bundle Construction Proposition 268.23 we can equip this set $\wedge^k E^*$ with a natural (smooth) real vector bundle structure. By the above discussion we know that this real vector bundle has dimension $\binom{n}{k}$.

Note that $\wedge^{\dim(E)} E^*$ is a 1-dimensional real vector bundles. The following proposition shows that this 1-dimensional real vector bundle contains interesting information:

Proposition 269.26. (Orientation Bundle Proposition) Let $p\colon E \to B$ be an n-dimensional real vector bundle.

(1) If the vector bundle $\wedge^n E^*$ is trivializable, then E is orientable.
(2) If E is orientable, and if B is paracompact and Hausdorff, then $\wedge^n E^*$ is trivializable.

The Orientation Bundle Proposition 269.26 motivates the following definition:

Definition 269.27. Given an n-dimensional real vector bundle $p\colon E \to B$ we refer to the vector bundle $\wedge^n E^*$ as the corresponding orientation bundle.

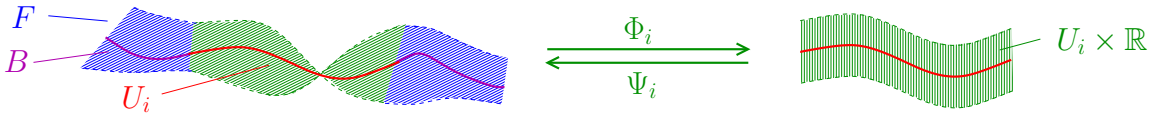
In the proof of the Orientation Bundle Proposition 269.26 we will need the following lemma:

Lemma 269.28. (1-Dimensional Trivializable Criterion) Let $p\colon F \to B$ be a 1-dimensional real vector bundle. In the following we assume that there exists a family of local trivializations $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{R}\}_{i\in I}$ which cover all of B and such that the transition maps take values in $\mathrm{GL}_+(1,\mathbb{R}) = \mathbb{R}_{>0}$. If B is paracompact and Hausdorff, then the vector bundle $p\colon F \to B$ is trivializable.

Proof of the 1-Dimensional Trivializable Criterion 269.28. First we introduce some notation: For each $i \in I$ we denote by $\Psi_i\colon U_i \times \mathbb{R} \to p^{-1}(U_i)$ the inverse of Φ_i. Next note that it follows from the Inner Product-on-Vector Bundle Proposition 269.10 that we can pick a smooth inner product $\varphi\colon B \to \mathrm{Hom}_{\mathbb{R}}(F \otimes_{\mathbb{R}} F, \mathbb{R})$ on the vector bundle F. For $b \in B$ and $v \in F_b$ we write $\|v\| := \sqrt{\varphi_b(v \otimes v)}$. For each $i \in I$ we now consider the map

$$\begin{array}{rccl} \widetilde{\Psi}_i\colon & U_i \times \mathbb{R} & \to & p^{-1}(U_i) \\ & (c,t) & \mapsto & \dfrac{1}{\sqrt{\|\Psi_i(c,1)\|}} \cdot \Psi_i(c,t). \end{array}$$

We leave it to the reader to verify that these maps are isomorphisms of real vector bundles, and that the inverses $\widetilde{\Phi}_i\colon p^{-1}(U_i) \to U_i \times \mathbb{R}$ form local trivializations of the vector bundle $p\colon F \to B$ which have the property that the transition maps are $\equiv 1$.



But this means that the maps $\widetilde{\Phi}_i$ can be combined to give a trivialization of F. ∎

Sketch of proof of the Orientation Bundle Proposition 269.26. Let $p\colon E \to B$ be an n-dimensional real vector bundle.

(1) First we assume that $\wedge^n E^*$ is trivializable. Thus let $\Phi\colon B \times \mathbb{R} \to \wedge^n E^*$ be an isomorphism of real vector bundles. Let $b \in B$. Note that we can write $\Phi(b,1) = (b, \Theta_b)$ where $\Theta_b \in \wedge^n E_b^*$. Next let $(v_1,\ldots,v_n)$ be an ordered basis of E_b. We say that $(v_1,\ldots,v_n)$ is a positive basis if $\Theta_b(v_1,\ldots,v_n) > 0$. We leave it to the reader to verify that this defines an orientation of the given real vector bundle $p\colon E \to B$.
(2) Now we assume that the real vector bundle E admits an orientation. We need to show that the real vector bundle $\pi\colon \wedge^n E^* \to B$ is trivializable. First note that it follows easily from the definitions that for each $b \in B$ we can pick an open neighborhood U_b of $b \in b$ and a trivialization $\Phi_b\colon p^{-1}(U_b) \to U_b \times \mathbb{R}^n$ such that for each $c \in U_b$ the map $p^{-1}(c) \xrightarrow{\Phi_b} \{c\} \times \mathbb{R}^n \xrightarrow{(c,v) \mapsto v} \mathbb{R}^n$ is an orientation-preserving isomorphism of oriented

n-dimensional real vector spaces. It follows easily from Definitions 31.17 and 265.23 that the transition maps between these trivializations take values in $\mathrm{GL}_+(n,\mathbb{R})$.

Now we consider the maps

$$\begin{array}{rccc} \widetilde{\Phi}_b\colon \pi^{-1}(U_b) & \to & U_b \times \wedge^n (\mathbb{R}^n)^* & \to & U_b \times \mathbb{R} \\ & & (b,\varphi) & \mapsto & (b, \varphi(\mathfrak{e}_1,\ldots,\mathfrak{e}_n)). \end{array}$$

Here the map to the left is induced by the original chart Φ_b and the Vector Bundle Construction Proposition 268.23. It follows from the above fact about transition maps and the discussion on page 1509 that the transition maps between these trivializations $\widetilde{\Phi}_b$ take values in $\mathrm{GL}_+(1,\mathbb{R}) = \mathbb{R}_{>0}$. It thus follows from the 1-Dimensional Trivializable Criterion 269.28 that we obtain a trivialization of the vector bundle $p\colon \wedge^n E^* \to B$. ■

Next we consider the orientation bundle of the tangent bundles of smooth manifolds:

Proposition 269.29. (Orientation Bundle-of-Manifold Proposition) Let M be an n-dimensional smooth manifold. The orientation bundle $\wedge^n \mathrm{T}M^*$ is an *orientable* $(n+1)$-dimensional smooth manifold.

Remark. Let M be an n-dimensional smooth manifold. We consider the orientation bundle $\wedge^n \mathrm{T}M^*$. With this data there are two ways to define a 2-fold covering $\widetilde{M} \to M$:

(1) Let $i\colon M \to \wedge^n \mathrm{T}M^*$ be the 0-section from Vector Bundle-Section Lemma 265.19 (2a). Note that we have an obvious action of $\mathbb{R}_{>0}$ on $(\wedge^n \mathrm{T}M^*) \setminus i(M)$. We leave it to the reader to verify that this action is free, smooth and proper. Note that it now follows from the Quotient-by-Lie Group Action Proposition **??** that $(\wedge^n \mathrm{T}M^* \setminus i(M))/\mathbb{R}_{>0}$ is an n-dimensional smooth manifold.

(2) We pick a smooth inner product on the 1-dimensional real vector bundle $\wedge^n \mathrm{T}M^*$ and we denote by $\widetilde{M}$ the set of all vectors of length 1.

It is pretty elementary to show that these two smooth manifolds are naturally diffeomorphic to the orientation covering introduced in Definition 95.9. Since we will not make use of this observation we have no qualms with suppressing all details. □

Sketch of proof. First note that it follows from the Vector Bundle Construction Proposition 268.23 that the orientation bundle $\wedge^n \mathrm{T}M^*$ is an $(n+1)$-dimensional smooth manifold. We still need to show that $\wedge^n \mathrm{T}M^*$ is *orientable*.

Let $P \in M$. We need to equip the tangent space $\mathrm{T}_P(\wedge^n \mathrm{T}M^*)$ with an orientation. First note that it follows easily from the definition of $\wedge^n \mathrm{T}M^*$ that for each $P \in M$ we have a short exact sequence

$$0 \to \wedge^n(\mathrm{T}_P M)^* \to \mathrm{T}_P(\wedge^n \mathrm{T}M^*) \to \mathrm{T}_P M \to 0.$$

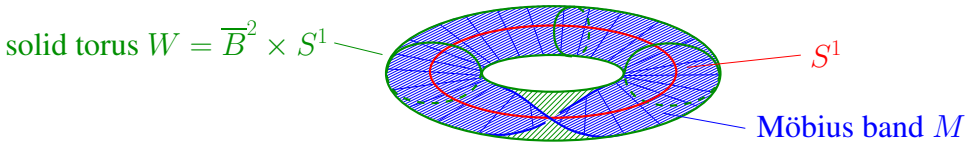
If $(v_1,\ldots,v_n)$ is an ordered basis of $\mathrm{T}_P M$, then we equip the 1-dimensional real vector space $\wedge^n \mathrm{T}_P M$ with the basis given by $v_1 \wedge \cdots \wedge v_n$. Finally we equip $\mathrm{T}_P(\wedge^n \mathrm{T}M^*)$ with the orientation that makes the orientations on the three vector spaces compatible in the sense of Definition 31.11. An elementary argument, using the Compatible Orientations Lemma 31.12 and the Alternating Forms-Basis Proposition 72.7, shows that this definition of the orientation of $\mathrm{T}_P(\wedge^n \mathrm{T}M^*)$ does not depend on the choice of the basis $v_1,\ldots,v_n$.

Now it is fairly straightforward to verify that these orientations of the tangent spaces define an orientation, in the sense of Definition 31.19, on the smooth manifold $\wedge^n \mathrm{T}M^*$. ■

We conclude this appendix with the following pleasant corollary to the Orientation Bundle-of-Manifold Proposition 269.29:

Corollary 269.30. (Embed-in-Orientable Manifold Corollary) Let M be a non-orientable n-dimensional smooth manifold. There exists an orientable $(n+1)$-dimensional smooth manifold W and a proper smooth embedding $M \to W$.[2868]

Example. Let M be the Möbius band.



The figure above illustrates the fact that the Möbius band M admits a proper smooth embedding into the orientable smooth manifold $W = \overline{B}^2 \times S^1$. □

Proof. Let $W = \wedge^n \mathrm{T}_P M$ be the orientation bundle. By the Orientation Bundle-of-Manifold Proposition 269.29 we know that W is an orientable $(n+1)$-dimensional smooth manifold. We denote by $i\colon M \to W$ the zero-section from the Vector Bundle-Section Lemma 265.19 (2a). By the Vector Bundle-Section Lemma 265.19 (1b) we know that $i\colon M \to W$ is a proper smooth embedding. ■

269.8. Exercises for Chapter 269.

Exercise 269.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ be a (smooth) $\mathbb{K}$-vector bundle. Furthermore let $s_1, \ldots, s_m\colon B \to E$ be (smooth) sections such that for each $b \in B$ the vectors $s_1(b), \ldots, s_m(b)$ are linearly independent. Show that $s_1, \ldots, s_m$ that there exists a unique trivializable m-dimensional $\mathbb{K}$-vector subbundle $E \subseteq F$ such that for each $b \in B$ we have $E_b = \mathrm{span}\{s_1(b), \ldots, s_m(b)\}$.

Exercise 269.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $p\colon E \to B$ be a $\mathbb{K}$-vector bundle over a topological space B that is Hausdorff and paracompact and let $n \in \mathbb{N}_0$. According to Definition 261.10 an *n-frame in $p\colon E \to B$* is an n-tuple $(f_1, \ldots, f_n)$ of continuous maps $f_i\colon B \to E$ such that for each $b \in B$ the n-tuple $(f_1(b), \ldots, f_n(b))$ form an n-frame in the vector space $p^{-1}(b)$. In other words, we demand that for each $b \in B$ the vectors $f_1(b), \ldots, f_n(b)$ of the vector space $p^{-1}(b)$ are linearly independent. Show that $p\colon E \to B$ admits an n-frame if and only if p splits of an n-dimensional trivializable vector bundle.
Remark. Make use of Exercise 269.1.

Exercise 269.3. Use the Vector Bundle-Trivializable Criterion 269.16 to show that every compact orientable connected 2-dimensional smooth manifold with non-empty boundary is parallelizable.

[2868] It thus follows from the Smooth Embedding Theorem 30.15 that we can view M as a proper smooth submanifold of W.

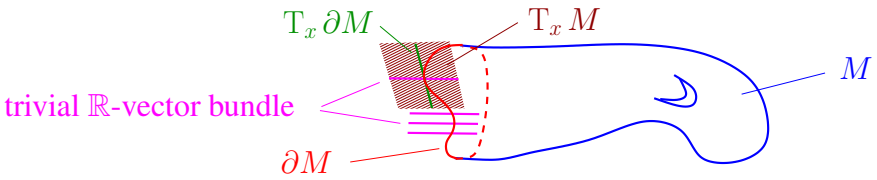
Exercise 269.4. Let M be a smooth manifold. Show that there exists an isomorphism

$$i^*(\mathrm{T}M) \;\cong\; \mathrm{T}\partial M \oplus \mathrm{Tr}_{\partial M}\mathbb{R}$$

($\mathrm{Tr}_{\partial M}\mathbb{R}$: the trivial one-dimensional vector bundle over ∂M)

of vector bundles over ∂M.

Remark. You could make use of the Smooth Collar Neighborhood Theorem 35.6.



Exercise 269.5. Let $p\colon E \to B$ be an oriented 2-dimensional real vector bundle over a topological space that is Hausdorff and paracompact. We want to show that E can be equipped with the structure of a 1-dimensional complex vector bundle. First note that it follows from the Inner Product-on-Vector Bundle Proposition 269.10 that we can equip E with an inner product. For each $b \in B$ we denote by $\varphi_b\colon E_b \times E_b \to \mathbb{R}$ the corresponding inner product on the fiber.

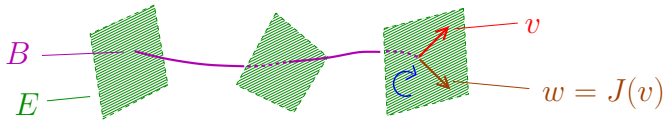
(a) Given $b \in B$ we consider the map

$$\begin{aligned} J\colon E_b \setminus \{0\} &\to E_b \setminus \{0\} \\ v &\mapsto \text{the unique vector } w \in E_b \text{ with } \varphi_b(v,w)_b = 0,\ \varphi_b(w,w) = \varphi(v,v) \\ &\qquad \text{and such that } v, w \text{ is a positive basis for } E_b. \end{aligned}$$

Show that $J^2 = -\,\mathrm{id}_{E_b}$.

(b) Show that E can be equipped with the structure of a 1-dimensional complex vector bundle.

(c) Show that any two inner products on E lead to isomorphic complex vector bundle structures.



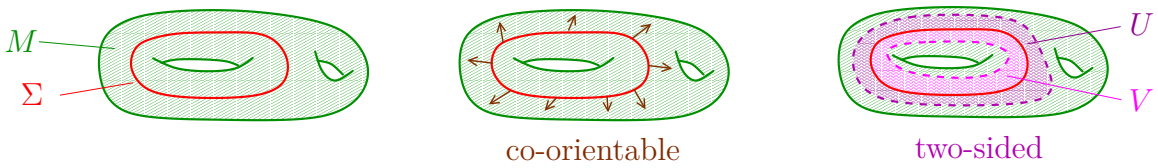
Exercise 269.6. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a closed codimension-one submanifold of M. We introduce the following two notions:

- We say that Σ is *co-orientable* if the normal vector bundle $\mathrm{N}\Sigma$ of $\Sigma \subseteq M$ is orientable.
- We say that Σ is *two-sided* if there exist disjoint subsets U and V of M with $U \cap V = \Sigma$ and such that $U \cup V$ is an open neighborhood of Σ.

Show that the following three statements are equivalent:

(1) Σ is co-orientable,
(2) Σ is two-sided,
(3) Σ admits a bicollar in the sense of Definition 55.1.

Hint. Make use of the Tubular Neighborhood–Existence Theorem 57.11.



Exercise 269.7. Let B be a topological space. In the Line Bundle–Group Proposition 268.37 we showed that $(\mathrm{IsoVectBund}_{\mathbb{R}}(B), \otimes)$ forms a group. Show that if B is Hausdorff and paracompact, then this group is 2-torsion, i.e. the square of any element of the group is trivial.

Exercise 269.8. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(a) Define the notion of a short exact sequence $0 \to U \to V \to W \to 0$ of $\mathbb{K}$-vector bundles over a topological space B.

(b) Show that in the setting of (a) the vector bundle $V \to B$ is isomorphic to the direct sum $U \oplus W \to X$.

Exercise 269.9. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We view $\mathbb{K}\mathrm{P}^n$ as a submanifold of $\mathbb{K}\mathrm{P}^{n+1}$ in the usual way. Show that the corresponding normal vector bundle $\mathrm{N}\mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^n$ is isomorphic to the tautological vector bundle $\mathfrak{T}\mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^n$.

Exercise 269.10. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p\colon E \to B$ and $q\colon F \to B$ be two (smooth) $\mathbb{K}$-vector bundles over a topological space that is Hausdorff and paracompact. We form the Whitney sum $E \oplus F$. Show that there exists an inner product on $E \oplus F$ such that the subbundle F of $E \oplus F$ *equals* the subbundle $E^\perp$ of $E \oplus F$.

Exercise 269.11. Let $p\colon F \to B$ be a real vector bundle and let $E, E' \subseteq F$ be two vector subbundles of F. If E and E' are isomorphic real vector bundles, does it follow that there exists an isomorphism $\varphi\colon F \to F$ with $\varphi(E) = E'$?

Exercise 269.12. Is the smooth manifold $\mathbb{R}^n/\mathbb{Z}^n$ stably trivializable?

Exercise 269.13. Let $f\colon M \to N$ be a smooth map between smooth manifolds with empty boundary. Let $Q \in N$ be a regular value of f. We consider the submanifold $W \subseteq M$. Show that the normal vector bundle of $W \subseteq M$ is trivializable.

Exercise 269.14. Does there exist a smooth map $f\colon \mathbb{R}^4 \to \mathbb{R}^2$ and a regular value $Q \in \mathbb{R}^2$ such that $f^{-1}(Q)$ is diffeomorphic to $\mathbb{R}\mathrm{P}^2$?
Hint. Use Exercise 269.13.

Exercise 269.15. Let M be a closed smooth manifold. We consider the corresponding diagonal $\Delta_M \subseteq M \times M$. Show that the corresponding normal vector bundle $\mathrm{N}\Delta_M$ is isomorphic to the tangent bundle $\mathrm{T}M$.
Remark. In the Total Tangent Space-Diagonal Embedding Proposition 53.23 we showed that there exists a smooth embedding $f\colon \mathrm{T}M \to M$ such that f restricts to a diffeomorphism $M \to \Delta_M$. This of course implies the statement of the proposition. But it is not very hard to give a direct proof of the statement.

Exercise 269.16. Let $p\colon E \to B$ and $q\colon F \to B$ be real vector bundles.

(a) If $E \oplus F$ is trivializable, are E and F also trivializable?

(b) If $E \otimes_{\mathbb{R}} F$ is trivializable, are E and F also trivializable?
(c) If $\mathrm{Hom}_{\mathbb{R}}(E, F)$ is trivializable, are E and F also trivializable?
(d) If E^* is trivializable, is E also trivializable?

Exercise 269.17. A pairing $\varphi\colon V \times V \to \mathbb{R}$ on a real vector space is called *skew-symmetric* if for all $v, w \in V$ we have $\varphi(v, w) = -\varphi(w, v)$ and it is called *non-degenerate* if the induced map $V \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ given by $v \mapsto (w \mapsto \varphi(v, w))$ is an isomorphism. Show that there exists a 2-dimensional real vector bundle which does *not* admit a non-degenerate skew-symmetric $\mathbb{R}$-bilinear pairing.

Exercise 269.18. Let M be an n-dimensional smooth manifold. We consider the orientation bundle $\wedge^n \mathrm{T}M^*$. With this data there are two ways to define a new topological space:

(1) First note that via the map $P \mapsto (P, 0)$ we can view M as a smooth submanifold of the orientation bundle. Next note that we have an obvious action of $\mathbb{R}_{>0}$ on $(\wedge^n \mathrm{T}M^*) \setminus M$. We leave it to the reader to verify that this action is free, smooth and proper. It thus follows from the Quotient-by-Lie Group Action Proposition **??** that $(\wedge^n \mathrm{T}M^* \setminus M)/\mathbb{R}_{>0}$ is an n-dimensional smooth manifold.
(2) We pick a smooth inner product on the 1-dimensional real vector bundle $\wedge^n \mathrm{T}M^*$ and we denote by $\widetilde{M}$ the set of all vectors of length 1.

Show that the resulting topological spaces are homeomorphic, that they define orientable smooth manifolds, and that we have a 2-fold covering $\widetilde{M} \to M$.

Exercise 269.19. Show that every non-orientable n-dimensional smooth manifold M admits a smooth embedding into the total space of the orientation bundle $\wedge^n \mathrm{T}M^*$.
Remark 1. It follows from the exercise together with the Orientation-via-Orientation Bundle Proposition 269.26 that every non-orientable n-dimensional smooth manifold admits a smooth embedding into an orientable $(n+1)$-dimensional smooth manifold.
Remark 2. This exercise is closely related to Exercise 95.15.

Exercise 269.20. Show that every compact non-orientable n-dimensional smooth manifold M is homotopy equivalent to a compact orientable $(n+1)$-dimensional smooth manifold.
Hint. Make use of the Orientation-via-Orientation Bundle Proposition 269.26.

Exercise 269.21. Show that every closed non-orientable n-dimensional smooth manifold M admits a 2-fold covering $p\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is the boundary of a compact smooth manifold.
Hint. Make use of the Orientation-via-Orientation Bundle Proposition 269.26. It might also be convenient to revisit the solution to Exercise 269.19.

Exercise 269.22. Let M be a non-orientable n-dimensional smooth manifold.

(a) If M is closed, does there exist a compact orientable $(n+1)$-dimensional smooth manifold W and a proper smooth embedding $M \to W$?
(b) If M is closed, does there exist a closed orientable $(n+1)$-dimensional smooth manifold W and a proper smooth embedding $M \to W$?

Exercise 269.23. Let M be an n-dimensional smooth manifold. Show that there exists a smooth S^1-bundle $p\colon W \to M$ such that the total space W is an orientable $(n+1)$-dimensional smooth manifold.
Hint. Make use of the orientation covering $p\colon \widetilde{M} \to M$.

CHAPTER 270

Grassmannians and classifying spaces for vector bundles

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Throughout these notes the projective spaces $\mathbb{K}\mathrm{P}^d$ and $\mathbb{K}\mathrm{P}^\infty$ have turned out to be very pretty and extremely interesting smooth manifolds and topological spaces. There are many equivalent ways to describe these projective spaces. One way is to say that they are the space of all 1-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^{m+1}$. With this point of view it makes sense to consider the space of of all n-dimensional $\mathbb{K}$-vector subspaces of some $\mathbb{K}^d$. This thought leads us to the Grassmannians $G_n(\mathbb{K}^d)$ and $G_n(\mathbb{K}^\infty)$ that we will introduce in this chapter.

Similar to case of the projective spaces, Grassmannians come with "obvious" tautological vector bundles $\mathfrak{T}G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^d)$. Amazingly we will see in this chapter that the tautological vector bundles $\mathfrak{T}G_n(\mathbb{K}^\infty)$ are "universal" in the sense that given any n-dimensional $\mathbb{K}$-vector bundle $p\colon E \to B$ over a base space B (that is Hausdorff and paracompact) there exists a continuous map $f\colon B \to G_n(\mathbb{K}^\infty)$ (unique up to homotopy) such that E is isomorphic to the pullback bundle $f^*\mathfrak{T}G_n(\mathbb{K}^\infty)$.

This chapter is organized as follows:

- In Section 270.1 we will introduce the Grassmannians $G_n(\mathbb{K}^d)$ and $G_n(\mathbb{K}^\infty)$.
- In Section 270.2 we will introduce the tautological vector bundles over Grassmannians.
- In Section 270.3 we will see that the tautological vector bundles over Grassmannians are "universal" in the above sense.
- In Section 270.4 we will provide exercises for Chapter 270.

In the next Chapter 271 we will calculate the (co-) homology groups of Grassmannians.

270.1. **The Grassmannians.** In this section we will introduce the finite-dimensional Grassmannians $G_n(\mathbb{K}^d)$ with $d \in \mathbb{N}_0$ and the infinite-dimensional Grassmannians $G_n(\mathbb{K}^\infty)$. We will do so in two separate subsections.

270.1.1. **The finite-dimensional Grassmannians.** The following definition recalls objects that we introduced in Definitions 261.1, 261.2, 261.3 and 261.4:

Definition 270.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n, d \in \mathbb{N}_0$.

(1) (a) An n-frame in $\mathbb{K}^d$ is an n-tuple $(v_1, \ldots, v_n)$ of linearly independent vectors in $\mathbb{K}^d$.
(b) We say that an n-frame $(v_1, \ldots, v_n)$ in $\mathbb{K}^d$ is orthonormal if $\langle v_i, v_j \rangle_{\mathbb{K}^d} = \delta_{ij}$.

(2) (a) We write
$$\mathrm{GL}_n(\mathbb{K}^d) \coloneqq \text{the set of } n\text{-frames in } \mathbb{K}^d$$
which is a subset of $(\mathbb{K}^d)^n$. We equip $(\mathbb{K}^d)^n$ with the usual topology and we equip $\mathrm{GL}_n(\mathbb{K}^d)$ with the subspace topology.
(b) We also define the Stiefel manifold[2869]
$$V_n(\mathbb{K}^d) \coloneqq \text{the set of orthonormal } n\text{-frames in } \mathbb{K}^d$$
which is a subset of $\mathrm{GL}_n(\mathbb{K}^d)$. We equip $V_n(\mathbb{K}^d)$ with the subspace topology.

Now we can introduce some new objects:

Definition 270.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n, d \in \mathbb{N}_0$. On $\mathrm{GL}_n(\mathbb{K}^d)$ we introduce the equivalence relation

$$(v_1, \dots, v_n) \sim (w_1, \dots, w_n) \quad :\Longleftrightarrow \quad \mathrm{span}\{v_1, \dots, v_n\} = \mathrm{span}\{w_1, \dots, w_n\}.$$

We set[2870]

$$G_n(\mathbb{K}^d) \coloneqq \mathrm{GL}_n(\mathbb{K}^d)/\!\sim$$

which we equip with the quotient topology. We refer to $G_n(\mathbb{K}^d)$ as a $\mathbb{K}$-Grassmannian.[2871]

Example. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We consider a few "extreme" cases for the Grassmannians $G_n(\mathbb{K}^d)$:

(1) If $n > d$, then $G_n(\mathbb{K}^d) = \varnothing$.
(2) $n = d$, then $G_n(\mathbb{K}^d)$ consists of a single point.
(3) If one compares Definition 9.11 with Definition 270.2, then we see that for any $d \in \mathbb{N}_0$ we have the honest-to-G∗d equality $G_1(\mathbb{K}^{d+1}) = (\mathbb{K}^{d+1} \setminus \{0\})/(\mathbb{K} \setminus \{0\}) = \mathbb{K}\mathrm{P}^d$. □

Remark. There are several alternative, but equivalent, ways to describe Grassmannians:

(1) There is an obvious bijection from $G_n(\mathbb{K}^d)$ to the set of n-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^d$. Psychologically and visually this can be a helpful point of view. But for technical purposes it is better to work with our definition.
(2) The group $\mathrm{GL}(n, \mathbb{K}) \subseteq \mathrm{M}(n \times n, \mathbb{K})$ acts on $\mathrm{GL}_n(\mathbb{K}^d) \subseteq \mathrm{M}(d \times n, \mathbb{K})$ by right-multiplication. As we will see in Exercise 270.1, it follows from elementary linear algebra that this action defines exactly the same equivalence relation on $\mathrm{GL}_n(\mathbb{K}^d)$ as in the above Definition 270.2. In particular we have $G_n(\mathbb{K}^d) = \mathrm{GL}_n(\mathbb{K}^d)/\mathrm{GL}(n, \mathbb{K})$.
(3) It follows easily from the Gram-Schmidt Orthogonalization Process 2.32 that the map $V_n(\mathbb{K}^d) \to \mathrm{GL}_n(\mathbb{K}^d) \to G_n(\mathbb{K}^d)$ is surjective. This implies that the induced map $V_n(\mathbb{K}^d)/\!\sim \to G_n(\mathbb{K}^d)$ is a homeomorphism. This alternative point of view can at times be helpful. □

The following proposition is a generalization of the Projective Space–Manifold Proposition 25.13 and the $\mathbb{C}\mathrm{P}^n$-Complex Manifold Proposition 40.4:

Proposition 270.3. (Grassmannian–Manifold Proposition) Let $n, d \in \mathbb{N}_0$ with $d \geq n$.

(1) We can equip $G_n(\mathbb{R}^d)$ with a smooth atlas such that $G_n(\mathbb{R}^d)$ becomes a closed $n(d-n)$-dimensional smooth manifold.
(2) We can equip $G_n(\mathbb{C}^d)$ with a smooth atlas such that $G_n(\mathbb{C}^d)$ becomes a closed $n(d-n)$-dimensional[2872] complex manifold.

[2869] In the Stiefel Manifold-Smooth Manifold Proposition 261.5 we showed that $V_n(\mathbb{K}^d)$ is a closed smooth manifold. More precisely, we showed that $V_n(\mathbb{R}^d)$ is a closed smooth manifold of dimension $dn - \frac{1}{2}n(n+1)$ and we showed that $V_n(\mathbb{C}^d)$ is a closed smooth manifold of dimension $2dn - n^2$. This explain why these topological spaces are called Stiefel *manifold*.

[2871] Grassmannians are named for the German mathematician Hermann Graßmann (1809-1877).

[2872] This implies that $G_n(\mathbb{C}^d)$ is in particular a $2n(d-n)$-dimensional smooth manifold.

Convention 270.4. Let $n, d \in \mathbb{N}_0$ with $d \geq n$.

(1) We refer to the smooth structure on $G_n(\mathbb{R}^d)$ that we just introduced as the standard smooth structure on $G_n(\mathbb{R}^d)$.
(2) We refer to the complex structure on $G_n(\mathbb{C}^d)$ that we just introduced as the standard complex structure on $G_n(\mathbb{C}^d)$. We equip the complex manifold $G_n(\mathbb{C}^d)$ with the orientation coming from the Complex Manifold-Orientation Proposition 40.18.

Proof. [2873] Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by

$$\pi\colon \mathrm{GL}_n(\mathbb{K}^d) \;\to\; G_n(\mathbb{K}^d) = \mathrm{GL}_n(\mathbb{K}^d)/\!\sim$$

the natural projection.

Claim 1. The topological space $G_n(\mathbb{K}^d)$ is compact.

Proof. As we discussed on page 5217 it follows from the Gram-Schmidt Orthogonalization Process 2.32 that the continuous map $V_n(\mathbb{K}^d) \to \mathrm{GL}_n(\mathbb{K}^d) \xrightarrow{\pi} G_n(\mathbb{K}^d)$ is surjective. By the Stiefel Manifold-Smooth Manifold Proposition 261.5 we know that $V_n(\mathbb{K}^d)$ is compact. Thus it follows from the Compact Image Lemma 4.15 that $G_n(\mathbb{K}^d)$ is also compact. ⊞

Claim 2. The topological space $G_n(\mathbb{K}^d)$ is Hausdorff.

Proof. We start out with a preparation:

Subclaim. Given any point $v \in \mathbb{K}^d$ and given a non-empty subset $S \subseteq \mathbb{K}^d$ we denote by $d(v, S) \coloneqq \inf\{\|v - s\| \mid s \in S\}$ the distance as introduced in Definition 10.3. The map

$$\begin{array}{rcl} d_v\colon G_n(\mathbb{K}^d) & \to & \mathbb{R}_{\geq 0} \\ [(u_1, \ldots, u_n)] & \mapsto & d(v, \mathrm{span}\{u_1, \ldots, u_n\}) \end{array}$$

is continuous.

Proof of subclaim. We consider the following diagram:

$$\begin{array}{ccccc} V_n(\mathbb{K}^d) & \xrightarrow{\pi} & V_n(\mathbb{K}^d)/\!\sim & \xrightarrow{\cong} & G_n(\mathbb{K}^d) \\ & \searrow & \downarrow & \swarrow_{d_v} & \\ & & \mathbb{R} & & \end{array}$$

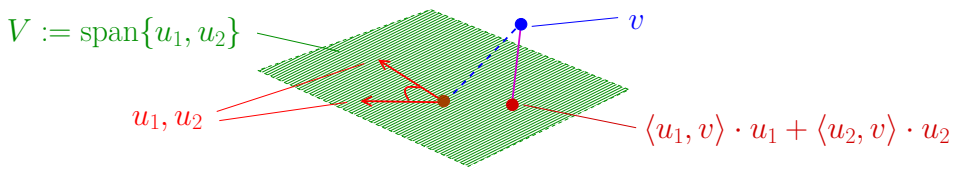
where each map to $\mathbb{R}$ is given by taking the distance to v. We make the following observation:

(a) The diagram commutes, essentially by definition.
(b) Elementary linear algebra shows that for orthonormal vectors $u_1, \ldots, u_n \in \mathbb{K}^d$ we have

$$d(v, \mathrm{span}\{u_1, \ldots, u_n\}) \;=\; \Big\| v - \underbrace{\sum_{i=1}^{n} \langle u_i, v\rangle_{\mathbb{K}^d} \cdot u_i}_{\substack{\text{orthogonal projection} \\ \text{of } v \text{ to } \mathrm{span}\{u_1, \ldots, u_n\}}} \Big\|.$$

This shows that the left diagonal map $d_v\colon V_n(\mathbb{K}^d) \to \mathbb{R}_{\geq 0}$ is continuous.
(c) It follows from (a), (b) and the Topological-Quotient Proposition 8.2 (1b) that the middle vertical map is also continuous.

[2873] Alternative proofs of the proposition are given in [**MiS1974**, p. 56], [**Wel2008**, Example 1.8] and [**Hat2**, p. 28 and 34].



(d) As we discussed on page 5217, the inclusion $V_n(\mathbb{K}^d) \to \mathrm{GL}_n(\mathbb{K}^d)$ induces a homeomorphism $V_n(\mathbb{K}^d)/\!\sim \to \mathrm{GL}_n(\mathbb{K}^d)/\!\sim$. Thus we see that the right horizontal map is a homeomorphism.

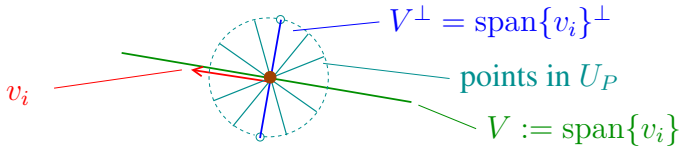
It follows from the above discussion that the map in the statement of the subclaim is indeed continuous. □

Now let $V, W \in G_n(\mathbb{K}^d)$ be two distinct points. We pick $v \in \mathrm{span}(V) \setminus \mathrm{span}(W)$. Clearly $d_v(V) = 0$. Furthermore it follows from the calculation in the subclaim (or the Distance-to-Subspace Lemma 10.4 (2)), that $d_v(W) \neq 0$. Since d_v is continuous we can now find (by taking preimages of two suitable open intervals under the map d_v) two disjoint open neighborhoods of V and W. ⊞

Claim 3. Any point $P \in G_n(\mathbb{K}^d)$ admits an open neighborhood U_P and a homeomorphism $\Phi_P \colon U_P \to \mathbb{K}^{n\cdot(d-n)}$.

Proof. Let $P := [(v_1, \ldots, v_n)] \in G_n(\mathbb{K}^d)$ where $v_1, \ldots, v_n \in \mathbb{K}^d$ are n linearly independent vectors. We consider the vector space $V := \mathrm{span}\{v_1, \ldots, v_n\}$ and its orthogonal complement $V^\perp \subseteq \mathbb{K}^d$. We set

$$U_P := \{[(w_1, \ldots, w_n)] \,|\, (w_1, \ldots, w_n) \in \mathrm{GL}_n(\mathbb{K}^d) \text{ such that } \mathrm{span}\{w_1, \ldots, w_n\} \oplus V^\perp = \mathbb{K}^d\}.$$



We consider the map

$$\begin{array}{rcl} \Theta \colon \overbrace{\mathrm{Hom}_{\mathbb{K}}(V, V^\perp)}^{\cong \mathbb{K}^{n\cdot(d-n)}} & \to & \mathrm{GL}_n(\mathbb{K}^d) \\ (\varphi \colon V \to V^\perp) & \mapsto & \underbrace{(v_1 + \varphi(v_1), \ldots, v_n + \varphi(v_n))}_{\text{linearly independent}}. \end{array}$$

Elementary linear algebra shows that the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(V, V^\perp)$ is of dimension $n \cdot (d-n)$ and it shows that $\pi \circ \Theta \colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) \to U_P$ is a bijection. It is clear that the map $\pi \circ \Theta \colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) \to U_P$ is continuous. By the Open-Closed–Injective Lemma 4.22 it remains to show that the map $\pi \circ \Theta \colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) \to G_n(\mathbb{K}^d)$ is open. To do so we consider the map

$$\begin{array}{rcl} \widetilde{\Theta} \colon \overbrace{\mathrm{Aut}_{\mathbb{K}}(V)}^{\text{of dimension } n^2} \times \overbrace{\mathrm{Hom}_{\mathbb{K}}(V, V^\perp)}^{\text{of dimension } n\cdot(d-n)} & \to & \overbrace{\mathrm{GL}_n(\mathbb{K}^d)}^{\text{of dimension } n\cdot d} \\ (\xi \colon V \to V, \varphi \colon V \to V^\perp) & \mapsto & [(\xi(v_1) + \varphi(\xi(v_1)), \ldots, \xi(v_n) + \varphi(\xi(v_n)))]. \end{array}$$

We make the following observations:

- Using the Inverse Mapping Theorem 30.1 one can verify that the map $\widetilde{\Theta}$ is an open map.
- By definition of the quotient topology a subset of $G_n(\mathbb{K}^d)$ is open if and only if its preimage under the projection $\pi\colon \mathrm{GL}_n(\mathbb{K}^d) \to G_n(\mathbb{K}^d)$ is open.
- One can easily verify that for any subset $W \subseteq \mathrm{Hom}_{\mathbb{K}}(V, V^\perp)$ we have the equality $\pi^{-1}((\pi \circ \Theta)(W)) = \widetilde{\Theta}(\mathrm{Aut}_{\mathbb{K}}(V) \times W)$.

Note that it follows from these three observations that $\pi \circ \Theta\colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) \to G_n(\mathbb{K}^d)$ is indeed an open map. ⊞

Claim 4. The topological space $G_n(\mathbb{K}^d)$ is second-countable.

Proof. In Claim 3 we just showed that $G_n(\mathbb{K}^d)$ admits an atlas. By Claim 1 we know that $G_n(\mathbb{K}^d)$ is compact. This implies that there exists a finite atlas. It now follows from the Countable Atlas-Second Countable Lemma 23.4 that $G_n(\mathbb{K}^d)$ is second-countable. ⊞

Claim 5. The atlas $\{\Phi_P\colon U_P \to \mathbb{K}^{n\cdot(d-n)}\}_{P\in G_n(\mathbb{K}^d)}$ from Claim 3 is smooth (holomorphic).

Proof. Let $P, Q \in G_n(\mathbb{K}^d)$. Using the explicit formulas of the Gram-Schmidt Orthogonalization Process 2.32 and of the Orthogonal Projection Lemma 2.33 one can write down explicit transition maps between the charts $\Phi_P\colon U_P \to \mathbb{K}^{n\cdot(d-n)}$ and $\Phi_Q\colon U_Q \to \mathbb{K}^{n\cdot(d-n)}$. Once one has down so one will observe, with great pleasure, that these transition maps are clearly smooth (respectively holomorphic). Evidently we have no desire to rob the reader of this pleasure. ■

270.1.2. **The infinite-dimensional Grassmannians.** In this subsection we will introduce the infinite Grassmannians. The idea is to imitate the definition of the infinite-dimensional projective spaces as the direct colimit of finite-dimensional projective spaces.

Recall that on page 338 we pointed out that the obvious maps $\mathbb{R}\mathrm{P}^d \to \mathbb{R}\mathrm{P}^{d+1}$ and $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^{d+1}$ are embeddings. The following lemma generalizes this observation to Grassmannians:

Lemma 270.5. (Grassmannian-Embedding Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n, d \in \mathbb{N}_0$. The maps $\mathbb{K}^d \xrightarrow{v \mapsto (v,0)} \mathbb{K}^{d+1}$ induce the embeddings

$$\begin{array}{rcl} \iota_d\colon G_n(\mathbb{K}^d) & \to & G_n(\mathbb{K}^{d+1}) \\ [(v_1, \dots, v_n)] & \mapsto & [((v_1, 0), \dots, (v_n, 0))] \end{array}$$

Proof. First note that it follows easily from the Quotient-to-Quotient Lemma 8.6 that the map $G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^{d+1})$ is continuous. Using the fact that $\mathbb{K}^d \to \mathbb{K}^{d+1}$ is a closed embedding one can also easily show that the map $G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^{d+1})$ is a closed map. Evidently the map $G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^{d+1})$ is injective. It now follows from the Open-Closed–Injective Lemma 4.22 (1) that the map $G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^{d+1})$ is an embedding.[2874] ■

As in the setting of projective spaces, the Grassmannian-Embedding Lemma 270.5 together with the Topological-Direct Colimit Proposition 104.10 leads us to the following definition:

[2874] Alternatively one could have argued using the Grassmannian–Manifold Proposition 270.3 and the good ol' Compact-Hausdorff Proposition 4.23.

Definition 270.6. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n \in \mathbb{N}_0$. We refer to the direct colimit

$$G_n(\mathbb{K}^\infty) \;:=\; \operatorname*{colim}_{\longrightarrow} G_n(\mathbb{K}^d) \;:=\; \Big(\bigsqcup_{d\in\mathbb{N}} G_n(\mathbb{K}^d)\Big)/(d,x) \sim (d, \iota_d(x))$$

as the infinite-dimensional $\mathbb{K}$-Grassmannian $G_n(\mathbb{K}^\infty)$.

Example. Once again, when we compare Definition 104.21 with Definition 270.6 we see that for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we have the equality $G_1(\mathbb{K}^\infty) = \mathbb{K}\mathrm{P}^\infty$. □

For practical and psychological purposes it can be convenient to have a "smaller" model of the infinite $\mathbb{K}$-Grassmannian $G_n(\mathbb{K}^\infty)$. This is provided by the following variation on the Infinite Projective Space-Sphere Quotient Lemma 104.22:

Lemma 270.7. (Infinite Grassmannian Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n \in \mathbb{N}_0$. On $\mathrm{GL}_n(\mathbb{K}^\infty)$ we use the "same" equivalence relation as in Definition 270.2, i.e. we work with the equivalence relation

$$(v_1, v_2, \dots) \sim (w_1, w_2, \dots) \quad :\Longleftrightarrow \quad \operatorname{span}\{v_1, v_2, \dots\} = \operatorname{span}\{w_1, w_2, \dots\}.$$

The maps $\mathbb{K}^d \to \mathbb{K}^\infty$ induce a homeomorphism

$$\underbrace{\operatorname*{colim}_{\longrightarrow} \overbrace{G_n(\mathbb{K}^d)}^{\mathrm{GL}_n(\mathbb{K}^d)/\sim}}_{=G_n(\mathbb{K}^\infty)} \xrightarrow{\cong} \mathrm{GL}_n(\mathbb{K}^\infty)/\sim$$

Here we equip $\mathbb{K}^\infty$ and $(\mathbb{K}^\infty)^n$ with the direct colimit topology from Definition 104.15. Furthermore we equip $\mathrm{GL}_n(\mathbb{K}^\infty) \subseteq (\mathbb{K}^\infty)^n$ with the subspace topology and $\mathrm{GL}_n(\mathbb{K}^\infty)/\sim$ with the quotient topology.

Proof. Let $d \in \mathbb{N}_0 \cup \{\infty\}$. Note that the group $\mathrm{GL}(n,\mathbb{K}) \subseteq \mathrm{M}(n \times n, \mathbb{K})$ acts on the set $\mathrm{GL}_n(\mathbb{K}^d) \subseteq \mathrm{M}(d \times n, \mathbb{K})$ by right-multiplication. As we discussed on page 5217 it follows follows from elementary linear algebra that this action defines exactly the same equivalence relation on $\mathrm{GL}_n(\mathbb{K}^d)$ as in Definition 270.2 and as in the lemma. In particular we have $G_n(\mathbb{K}^d) = \mathrm{GL}_n(\mathbb{K}^d)/\mathrm{GL}(n,\mathbb{K})$. Thus the desired statement follows from the Direct Colimit–Action Lemma 104.20 which says that direct colimits commute with group actions. ■

270.2. **Tautological vector bundles over Grassmannians.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the Tautological Line Bundle Lemma 265.6 we saw that the projective spaces $\mathbb{K}\mathrm{P}^m$ come with the tautological line bundle

$$\begin{aligned} p\colon \mathfrak{T}\mathbb{K}\mathrm{P}^m := \{([v], \lambda\cdot v) \in \mathbb{K}\mathrm{P}^m \times \mathbb{K}^{m+1} \mid v \in \mathbb{K}^{m+1}\setminus\{0\}, \lambda \in \mathbb{K}\} &\to \mathbb{K}\mathrm{P}^m \\ ([v], w) &\mapsto [v]. \end{aligned}$$

The projective space $\mathbb{K}\mathrm{P}^m$ admits an obvious bijection with the set of all 1-dimensional $\mathbb{K}$-vector subspaces of $\mathbb{K}^{m+1}$. With this view point the fiber of each point in $\mathbb{K}\mathrm{P}^m$ is precisely this 1-dimensional vector subspace. In the following we generalize this construction from projective spaces to Grassmannians:

Lemma 270.8. (Tautological Bundle Lemma) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$.

(1) For every $d \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathfrak{T}G_n(\mathbb{K}^d) := \overbrace{\{([V], w) \,|\, V = (v_1, \ldots, v_n) \in \mathrm{GL}_n(\mathbb{K}^d), w \in \mathrm{span}\{v_1, \ldots, v_n\}\}}^{\subseteq G_n(\mathbb{K}^d) \times \mathbb{K}^d} & \to & G_n(\mathbb{K}^d) \\ (V, w) & \mapsto & V \end{array}$$

is an n-dimensional smooth $\mathbb{K}$-vector bundle.

(2) In the Infinite Grassmannian Lemma 270.7 we gave an explicit homeomorphism

$$G_n(\mathbb{K}^\infty) := \operatorname{colim}_{\longrightarrow} G_n(\mathbb{K}^d) \xrightarrow{\cong} \mathrm{GL}_n(\mathbb{K}^\infty)/\!\sim$$

which we use to identify these two topological spaces. The same way as in (1) we can now define an n-dimensional $\mathbb{K}$-vector bundle $\mathfrak{T}G_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty) =_{\mathrm{i}} \mathrm{GL}_n(\mathbb{K}^\infty)/\!\sim$.

(3) The tautological $\mathbb{K}$-vector bundles that we defined in (1) and (2) are consistent in the following sense: For any $0 \leq d \leq e \leq \infty$ the pullback of $\mathfrak{T}G_n(\mathbb{K}^e)$ under the usual embedding $G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^e)$ is isomorphic to $\mathfrak{T}G_n(\mathbb{K}^d)$.[2875]

Sketch of proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$.

(1) Let $d \in \mathbb{N}_0$. We need to show that the given map $p\colon \mathfrak{T}G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^d)$ is an n-dimensional smooth $\mathbb{K}$-vector bundle. Thus let $V = [(v_1, \ldots, v_n)] \in G_n(\mathbb{K}^d)$. Recall that this means that $v_1, \ldots, v_n \in \mathbb{K}^d$ are n linearly independent vectors. By a slight abuse of notation we also write $V := \mathrm{span}\{v_1, \ldots, v_n\}$. We consider its orthogonal complement $V^\perp$. As in the proof of the Grassmannian–Manifold Proposition 270.3 we set

$$U_V := \{[(w_1, \ldots, w_n)] \,|\, (w_1, \ldots, w_n) \in \mathrm{GL}_n(\mathbb{K}^d) \text{ with } \mathrm{span}\{w_1, \ldots, w_n\} \oplus V^\perp = \mathbb{K}^d\}.$$

In the proof of the Grassmannian–Manifold Proposition 270.3 we showed that U_V is an open neighborhood of $V \in G_n(\mathbb{K}^d)$ and we showed that the map

$$\begin{array}{rcl} \Theta\colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) & \to & U_V \\ (\varphi\colon V \to V^\perp) & \mapsto & [(v_1 + \varphi(v_1), \ldots, v_n + \varphi(v_n))] \end{array}$$

is a homeomorphism. Next we consider the map

$$\begin{array}{rcl} \Psi\colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) \times \mathbb{K}^n & \to & p^{-1}(U_V) \\ (\varphi\colon V \to V^\perp, (\lambda_1, \ldots, \lambda_n)) & \mapsto & \Big([(v_1 + \varphi(v_1), \ldots, v_n + \varphi(v_n)]), \sum_{i=1}^n \lambda_i \cdot (v_i + \varphi(v_i))\Big). \end{array}$$

One can easily verify (say using the Inverse Mapping Theorem 30.1 (2)) that this map is a diffeomorphism and that it is fiberwise an isomorphism of $\mathbb{K}$-vector spaces. Thus we see that the inverse $\Psi^{-1}\colon \mathrm{Hom}_{\mathbb{K}}(V, V^\perp) \times \mathbb{K}^n \to p^{-1}(U_V)$ is a local trivialization of the map $p\colon \mathfrak{T}G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^d)$ as an n-dimensional smooth $\mathbb{K}$-vector bundle.

(2) We leave it to the reader to verify that the argument in (1) also works, with minor modifications, in the infinite-dimensional context.

(3) This statement follows easily from the Vector Bundle Isomorphism Criterion 265.40. ■

The Tautological Bundle Lemma 270.8 leads us to the following definition:

Definition 270.9. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $n \in \mathbb{N}_0$. We refer to the n-dimensional $\mathbb{K}$-vector bundles

[2875] This statement is a generalization of Exercise 265.29.

$$\begin{array}{rcl} p\colon \mathfrak{T}G_n(\mathbb{K}^d) \coloneqq \{([V],w)\,|\,V=(v_1,\ldots,v_n)\in \mathrm{GL}_n(\mathbb{K}^d), w\in \mathrm{span}\{v_1,\ldots,v_n\}\} & \to & G_n(\mathbb{K}^d) \\ (V,w) & \mapsto & V \end{array}$$

and $\mathfrak{T}G_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$ from the Tautological Bundle Lemma 270.8 as the tautological $\mathbb{K}$-vector bundle over the $\mathbb{K}$-Grassmannian $G_n(\mathbb{K}^d)$ and over the $\mathbb{K}$-Grassmannian $G_n(\mathbb{K}^\infty)$.

Example. We already mentioned on page 5217 and 5221 that for $d \in \mathbb{N}_0 \cup \{\infty\}$ we have the equality $\mathbb{K}\mathrm{P}^d = G_1(\mathbb{K}^{d+1})$. If one compares the definition of the tautological $\mathbb{K}$-line bundle $\mathfrak{T}\mathbb{K}\mathrm{P}^d$ from the Tautological Line Bundle Lemma 265.6 with the definition of the tautological 1-dimensional $\mathbb{K}$-vector bundle $\mathfrak{T}G_1(\mathbb{K}^{d+1})$ one sees that these are also exactly the same objects. □

270.3. **Grassmannians as universal classifying spaces.** Initially these tautological vector bundles might appear only mildly interesting. But rather stunningly they can be viewed as the "mother of all $\mathbb{K}$-vector bundles". More precisely, the following theorem says that "most" n-dimensional $\mathbb{K}$-vector bundles are the pullback of the tautological vector bundle over the infinite Grassmannian $G_n(\mathbb{K}^\infty)$ in an "essentially unique way":

Theorem 270.10. (Grassmannian–Universal Bundle Theorem) Let $\mathbb{K} = \mathbb{R}$ or let $\mathbb{K} = \mathbb{C}$. We denote by $\mathfrak{T}G_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$ the tautological n-dimensional $\mathbb{K}$-vector bundle over the infinite Grassmannian $G_n(\mathbb{K}^\infty)$ that we introduced in Definition 270.9. Let B be a topological space that is Hausdorff and paracompact. The following two statements hold:

(1) Given any n-dimensional $\mathbb{K}$-vector bundle $p\colon E \to B$ over B there exists a continuous map $g\colon B \to G_n(\mathbb{K}^\infty)$ such that the pullback $g^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ is isomorphic over B to E.

(2) If $g_0, g_1\colon B \to G_n(\mathbb{K}^\infty)$ are two continuous maps such that the two $\mathbb{K}$-vector bundles $g_0^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ and $g_1^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ over B are isomorphic, then g_0 and g_1 are homotopic.

We can paraphrase Statements (1) and (2) more succinctly as follows: The two statements say that the map[2876] [2877]

$$\begin{array}{rcl} [B, G_n(\mathbb{K}^\infty)] & \to & \mathrm{IsoVectBund}_{\mathbb{K}^n}(B) \\ [g\colon B \to G_n(\mathbb{K}^\infty)] & \mapsto & g^*\mathfrak{T}G_n(\mathbb{K}^\infty) \end{array}$$

is a bijection.

Example.

(1) Let $p\colon E \to B$ be a trivializable n-dimensional $\mathbb{K}$-vector bundle. We make the following observations:
- By the discussion on page 5145 we know that the pullback of a trivializable vector bundle is again trivializable.
- Any vector bundle over a point is trivializable.

It follows easily from these two observations, together with the Vector Bundle-Pullback–Functorial Lemma 265.39, that for any constant map $g\colon B \to G_n(\mathbb{K}^\infty)$ the pullback $g^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ of the tautological vector bundle is isomorphic to E.

[2875] Given two topological spaces X and Y we denote by $[X,Y]$, as in Definitions 19.11 and 241.13, by $[X,Y]$ the set of homotopy classes of continuous maps $X \to Y$.

[2876] Recall that we denote by $\mathrm{IsoVectBund}_{\mathbb{K}^d}(B)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over B.

[2877] It follows from the Vector Bundle–Pullback-Homotopy Proposition 267.1 that this map is well-defined.

(2) Let $d \in \mathbb{N}_0$. By the Tautological Bundle Lemma 270.8 (3) the tautological $\mathbb{K}$-vector bundle $\mathfrak{T}G_n(\mathbb{K}^d)$ is isomorphic to $g^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ where $g\colon G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^\infty)$ is the obvious embedding. □

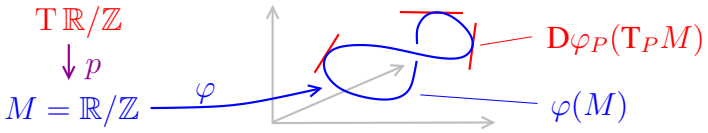
Proof of the Grassmannian–Universal Bundle Theorem 270.10 (1). In the following let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a topological space B that is Hausdorff and paracompact.

Claim 1. There exists a continuous map $\Theta\colon E \to \mathbb{K}^\infty$ such that for every $b \in B$ the restriction to the fiber E_b is a monomorphism of $\mathbb{K}$-vector spaces.

Proof.

(1) First, just for fun, we consider the important special case that we are dealing with the tangent bundle $p\colon \mathrm{T}M \to M$ of an n-dimensional smooth manifold M. By the Euclidean Embedding Theorem 33.2 we know that there exists a $d \in \mathbb{N}$ and a smooth embedding $\varphi\colon M \to \mathbb{R}^d$.[2878] The following map has the desired properties:

$$\begin{array}{rccc} g\colon \mathrm{T}M & \to & \mathbb{R}^d & \xrightarrow{v \mapsto (v,0)} \mathbb{R}^\infty \\ (P,v) & \mapsto & \mathrm{D}\varphi_P(v). & \end{array}$$



(2) Now we consider the general case, i.e. let $p\colon E \to B$ be any n-dimensional $\mathbb{K}$-vector bundle over a topological space B that is Hausdorff and paracompact space.[2879]

We start out with many preparations:

(a) By definition of a vector bundle there exists an open cover $\{U_i\}_{i\in I}$ of B such that the restriction of p to each U_i is trivializable.

(b) Recall that we assume that B is Hausdorff and paracompact. It thus follows from y the Countable Cover Proposition 14.17 that there exists a *countable* locally finite open cover $\{V_j\}_{j\in J}$ such that each V_j is the disjoint union of open subsets of B, each of which is contained in some U_i.

(c) It follows easily from the Disjoint Union Embedding Lemma 7.29 that the restriction of p to each V_j is also trivial. It follows that for each $j \in \mathbb{N}$ we can pick a trivialization $\Phi_j\colon p^{-1}(V_j) \to V_j \times \mathbb{K}^n$.

(d) We denote by $\pi\colon V_j \times \mathbb{K}^n \to \mathbb{K}^n$ the natural projection.

(e) Since B is Hausdorff and paracompact we know by the Paracompact-Partition of Unity Theorem 14.15 that there exists a family of continuous maps $\{f_j\colon B \to [0,1]\}_{j\in\mathbb{N}}$ with the following properties:

(i) For each $j \in \mathbb{N}$ we have $\mathrm{supp}(f_j) \subseteq V_j$.

(ii) For each $b \in B$ there exists an open neighborhood W of b such that f_j vanishes on W for all but finitely many j.

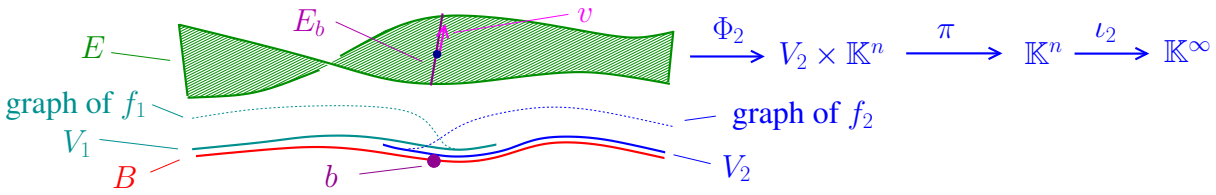
(iii) For every $b \in B$ we have

[2878] Note that in the proof of that proposition we used implicitly that smooth manifolds are paracompact.

[2879] The following argument shares some features with the proof of the Euclidean Embedding Theorem 33.2 and of the Inner Product-on-Vector Bundle Proposition 269.10.

$$\sum_{j\in\mathbb{N}} f_j(b) \;=\; 1.$$

(f) Given $j \in \mathbb{N}$ we denote by $\iota_j\colon \mathbb{K}^n \to \mathbb{K}^\infty = (\mathbb{K}^n)^\infty$ the isomorphism to the j-th copy of $\mathbb{K}^n$.



Subclaim. The map

$$\begin{array}{rcl} \Theta\colon E & \to & \mathbb{K}^\infty = (\mathbb{K}^n)^\infty \\ v & \mapsto & \displaystyle\sum_{\substack{j\in J \text{ with} \\ f_j(p(v))\neq 0}} f_j(p(v)) \cdot \underbrace{(\iota_j \circ \pi \circ \Phi_j)(v)}_{\in \mathbb{K}^\infty} \end{array}$$

has all the desired properties.

Proof. First we need to show that the map is continuous. By the Pasting Proposition 4.7 it suffices to prove continuity locally. It follows from (2e-ii) that locally the map Θ takes values in some $\mathbb{K}^{d\cdot m}$. Next note that it follows easily from (2e-i) that locally the map to $\mathbb{K}^{d\cdot m}$ is continuous. Since the injection $\mathbb{K}^{d\cdot m} \xrightarrow{v \,\mapsto\, (v;0)} \mathbb{K}^\infty$ is continuous we obtain that the map in the claim is continuous.

It remains to show that for each $b \in B$ the map $\Theta\colon E_b \to \mathbb{K}^\infty$ is a monomorphism of $\mathbb{K}$-vector spaces. So let $b \in B$. First of all note that the map $g\colon E_b \to \mathbb{K}^\infty$ is clearly a homomorphism. By (2e-iii) there exists a $j \in \mathbb{N}$ with $f_j(b) \neq 0$. It follows immediately from the definition of Θ that the composition of $\Theta\colon E_b \to \mathbb{K}^\infty$ with the projection onto the j-th $\mathbb{K}^n$-factor is an isomorphism. This implies that Θ is a monomorphism. ⊞

Claim 2. The map

$$\begin{array}{rcl} g\colon B & \to & G_n(\mathbb{K}^\infty) \\ b & \mapsto & [\ \underbrace{\Theta(E_b)}_{\substack{n\text{-dimensional} \\ \text{vector subspace of } \mathbb{K}^\infty}}\ ] \end{array}$$

is continuous.

Proof. As always it follows from the Pasting Proposition 4.7 that it suffices to show that the map g is locally continuous. So let $b \in B$. We pick an open neighborhood U of $b \in B$ and a local trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{K}^n$ of $p\colon E \to B$. This allows us to define continuous sections $s_1, \ldots, s_n\colon U \to p^{-1}(U)$ such that for each $c \in U$ the vectors $s_1(c), \ldots, s_n(c)$ form a basis of E_c. Next we consider the map

$$\begin{array}{rcl} \widetilde{g}\colon U & \to & \mathrm{GL}_n(\mathbb{K}^\infty) \\ c & \mapsto & (\Theta(s_1(c)), \ldots, \Theta(s_n(c))). \end{array}$$

It is clear that this map is continuous. The composition with the projection to $G_n(\mathbb{K}^\infty)$ is clearly also continuous. But that is precisely the restriction of g to U. ⊞

We consider the following maps:

$$\begin{array}{ccc} E & \xrightarrow{v\mapsto(g(p(v)),\Theta(v))} & \mathfrak{T}G_n(\mathbb{K}^\infty) \\ \downarrow{\scriptstyle p} & & \downarrow \\ B & \xrightarrow{g} & G_n(\mathbb{K}^\infty). \end{array}$$

It follows from Claim 1 and Claim 2 that the top and bottom map are continuous. Clearly the top map restricts to isomorphisms of fibers. Thus we see that the maps form a morphism of $\mathbb{K}$-vector bundles. Finally note that it now follows immediately from the Vector Bundle–Pullback Proposition 265.36 (2) that the $\mathbb{K}$-vector bundles $g^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ and E over B are isomorphic. ■

Proof of the Grassmannian–Universal Bundle Theorem 270.10 (2). Let B be a topological space.[2880] In the following let $g_0, g_1\colon B \to G_n(\mathbb{K}^\infty)$ be two continuous maps such that the $\mathbb{K}$-vector bundles $g_0^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ and $g_1^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ over B are isomorphic. We need to show g_0 and g_1 are homotopic.

We first introduce several pieces of notation:

- By hypothesis there exists an isomorphism
$$\Theta\colon g_0^*\mathfrak{T}G_n(\mathbb{K}^\infty) \;\to\; g_1^*\mathfrak{T}G_n(\mathbb{K}^\infty)$$
of $\mathbb{K}$-vector bundles over B.
- Given $V = [(v_1, \dots, v_n)] \in G_n(\mathbb{K}^\infty)$ we denote, by a slight abuse of notation, by V also the vector subspace $\operatorname{span}\{v_1, \dots, v_n\}$.
- Given any $b \in B$ there exists by definition an isomorphism $\Theta_b\colon g_0(b) \to g_1(b)$ such that the following diagram commutes:
$$\begin{array}{ccc} g_0(b) & \xrightarrow{\Theta_b} & g_1(b) \\ {\scriptstyle v\mapsto(b,v)}\downarrow & & \downarrow{\scriptstyle v\mapsto(b,v)} \\ (g_0^*\mathfrak{T}G_n(\mathbb{K}^\infty))_b & \xrightarrow{\Theta} & (g_1^*\mathfrak{T}G_n(\mathbb{K}^\infty))_b. \end{array}$$
- Given any *monomorphism* $\varphi\colon V \to W$ between two $\mathbb{K}$-vector spaces we obtain a well-defined induced map
$$\begin{array}{rcl} \varphi_*\colon \mathrm{GL}_n(V) & \to & \mathrm{GL}_n(W) \\ [(v_1, \dots, v_n)] & \mapsto & [(\varphi(v_1), \dots, \varphi(v_n))]. \end{array}$$

Now we can formulate our first claim:

Claim 1. We suppose that for any $(b, v) \in g_0^*\mathfrak{T}G_n(\mathbb{K}^\infty)$ and any $\lambda \in \mathbb{K}$ we have $\Theta_b(v) \neq \lambda \cdot v$. In this setting the maps g_0 and g_1 are homotopic, in fact a homotopy is given by

$$\begin{array}{rcl} H\colon B \times [0,1] & \to & G_n(\mathbb{K}^\infty) \\ (b,t) & \mapsto & (\ \underbrace{(1-t)\cdot \mathrm{id} + t\cdot\Theta_b}_{\substack{\text{monomorphism } g_0(b)\to\mathbb{K}^\infty \\ \text{since for any } v\in g_0(b) \text{ and} \\ \text{any } \lambda\in\mathbb{K} \text{ we have by} \\ \text{hypothesis } \Theta_b(v)\neq\lambda v}}\ )_*(g_0(b)). \end{array}$$

[2880] For this part we actually do not need to assume that B is Hausdorff and paracompact.

Proof. It remains to show that the given map H is actually continuous. By the Pasting Proposition 4.7 it suffices to show that H is locally continuous.

So let $b \in B$. We denote by $p\colon \mathfrak{T}G_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$ the projection in the tautological $\mathbb{K}$-vector bundle. We pick an open neighborhood U of $g_0(b) \in G_n(\mathbb{K}^\infty)$ which admits a trivialization $\Phi\colon p^{-1}(U) \to U \times \mathbb{K}^n$. We claim that H is continuous on $g_0^{-1}(U) \times [0,1]$. We introduce some notation:

- For $i = 1, \dots, n$ we consider the section
$$\begin{array}{rcl} s_i\colon U & \to & \mathfrak{T}G_n(\mathbb{K}^\infty) \\ c & \mapsto & \Phi^{-1}(c, \mathfrak{e}_i). \end{array}$$
- Let $\pi\colon \mathfrak{T}G_n(\mathbb{K}^\infty) \to \mathbb{K}^\infty$ be the projection.

Using this notation we can consider the map

$$\begin{array}{rcl} \widetilde{H}\colon g_0^{-1}(U) \times [0,1] & \to & \mathrm{GL}_n(\mathbb{K}^\infty) \\ (c,t) & \mapsto & ((1-t)\cdot \mathrm{id} + t \cdot \Theta_b)_* \underbrace{(\pi(s_1(g_0(c))), \dots, \pi(s_n(g_0(c))))}_{\substack{\text{basis of the vector subspace } g_0(c) \subseteq \mathbb{K}^\infty \\ \text{this basis is continuous in } c}}. \end{array}$$

This map is clearly continuous. But the map $H\colon g_0^{-1}(U) \times [0,1] \to G_n(\mathbb{K}^\infty)$ we are interested in is the map $\widetilde{H}$ followed by the obvious projection $\mathrm{GL}_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$. Thus this map is also continuous. ⊞

As we will now see, the general case can be reduced fairly easily to the above special case from Claim 1. But this needs some preparation. First of all we consider the continuous maps

$$\begin{array}{rcl} h_{\mathrm{even}}\colon \mathbb{K}^\infty & \to & \mathbb{K}^\infty \\ (t_1, t_2, \dots) & \mapsto & (0, t_1, 0, t_2, 0, \dots) \end{array} \quad \text{and} \quad \begin{array}{rcl} h_{\mathrm{odd}}\colon \mathbb{K}^\infty & \to & \mathbb{K}^\infty \\ (t_1, t_2, \dots) & \mapsto & (t_1, 0, t_2, 0, \dots) \end{array}$$

Claim 2. The induced maps $h_{\mathrm{even}\,*}\colon G_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$ and $h_{\mathrm{odd}\,*}\colon G_n(\mathbb{K}^\infty) \to G_n(\mathbb{K}^\infty)$ are homotopic to the identity.

Proof. First we consider the following map:

$$\begin{array}{rcl} \widetilde{H}\colon \mathbb{K}^\infty \times [0,1] & \to & \mathbb{K}^\infty \\ (v,t) & \mapsto & g_{\mathrm{even}}(v) \cdot (1-t) + v \cdot t. \end{array}$$

Note that it follows from the Direct Colimit–Product Topology Proposition 104.24 that this map is continuous.

As in the proof of the S^∞–Contractibility Lemma 104.25 one can use the CW×Interval Proposition 137.1 (2) to show that this map is actually continuous.

Next note that each $\widetilde{H}_t\colon \mathbb{K}^\infty \to \mathbb{K}^\infty$ is a monomorphism. Thus we see that $\widetilde{H}$ defines a continuous map

$$\widetilde{H}\colon \mathrm{GL}_n(\mathbb{K}^\infty) \times [0,1] \to \mathrm{GL}_n(\mathbb{K}^\infty).$$

This map in turn descends to a map

$$H\colon G_n(\mathbb{K}^\infty) \times [0,1] \to G_n(\mathbb{K}^\infty).$$

Note that it follows from the Homotopy Quotient Theorem 18.11 that this map is also continuous. This map is now clearly our desired homotopy from $h_{\mathrm{even}\,*}$ to id. The argument for $h_{\mathrm{odd}\,*}$ is almost the same. ⊞

Claim 3. The vector bundles $(h_{\text{even}\,*} \circ g_0)^* E$ and $(h_{\text{odd}\,*} \circ g_1)^* E$ are isomorphic.

Proof. We have the following isomorphisms

$$(h_{\text{even}\,*} \circ g_0)^* E \underset{\text{by the Vector Bundle-Pullback Functorial Lemma 265.39}}{\cong} g_0^*(h_{\text{even}\,*}^* E) \overset{\text{by Claim 2 and the Vector Bundle Pullback Homotopy Proposition 267.1}}{\cong} g_0^* E \underset{\text{by hypothesis}}{\cong} g_1^* E \cong \ldots \underset{\text{same argument backwards}}{\cong} (h_{\text{odd}\,*} \circ g_1)^* E.$$

Finally we obtain the following homotopies of continuous maps $B \to G_n(\mathbb{K}^\infty)$:

$$g_0 \underset{\text{since } h_{\text{even}\,*} \text{ and } h_{\text{odd}\,*} \text{ are homotopic to the identity}}{\sim} h_{\text{even}\,*} \circ g_0 \overset{\text{now the special case from Claim 1 applies}}{\sim} h_{\text{odd}\,*} \circ g_1 \underset{}{\sim} g_1.$$

This shows that g_0 is homotopic to g_1. ∎

270.4. Exercises for Chapter 270.

Exercise 270.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n, d \in \mathbb{N}_0$.

(a) Show that the group $\mathrm{GL}(n, \mathbb{K}) \subseteq \mathrm{M}(n \times n, \mathbb{K})$ acts on $\mathrm{GL}_n(\mathbb{K}^d) \subseteq \mathrm{M}(d \times n, \mathbb{K})$ by right-multiplication.

(b) Show that this action defines, via the Group Action-Equivalence Relation Lemma 2.38, exactly the same equivalence relation on $\mathrm{GL}_n(\mathbb{K}^d)$ as in Definition 270.2.

Remark. This shows we can write $G_n(\mathbb{K}^d) = \mathrm{GL}_n(\mathbb{K}^d)/\mathrm{GL}(n, \mathbb{K})$.

Exercise 270.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $n, d \in \mathbb{N}_0$. What is the number of path-components of $G_n(\mathbb{K}^d)$?

Exercise 270.3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $n, d \in \mathbb{N}_0$. Show that there exists a $\mathbb{K}$-vector bundle $p\colon E \to G_n(\mathbb{K}^d)$ such that $\mathfrak{T}G_n(\mathbb{K}^d) \oplus E$ is trivializable.

Exercise 270.4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a topological space B which is Hausdorff and compact (in particular paracompact). Show that there exists a $d \in \mathbb{N}_0$ and a continuous map $g\colon B \to G_n(\mathbb{K}^d)$ such that E is isomorphic to the pullback $g^*\mathfrak{T}G_n(\mathbb{K}^d)$.

Exercise 270.5. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a compact topological space B which is Hausdorff. Show that there exists a $\mathbb{K}$-vector bundle $q\colon F \to B$ such that $E \oplus F$ is trivializable.

Hint. Make use of Exercise 270.3 and Exercise 270.4.

Remark. In Exercise 271.9 we will see that we cannot drop the hypothesis that B is compact.

Exercise 270.6. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a topological space that is Hausdorff and paracompact. Show that the following two statements are equivalent:

(1) There exists a $\mathbb{K}$-vector bundle $q\colon F \to B$ such that $E \oplus F$ is trivializable.

(2) There exists an $n \in \mathbb{N}_0$ and a continuous map $f\colon B \to G_n(\mathbb{K}^d)$ such that E is isomorphic to the vector bundle $f^*\mathfrak{T}G_n(\mathbb{K}^d)$.

Exercise 270.7. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Furthermore let $m, n \in \mathbb{N}_0$. Show that there exists a homeomorphism $G_m(\mathbb{K}^{m+n}) \to G_n(\mathbb{K}^{m+n})$.

Exercise 270.8. Show that given any n-dimensional smooth manifold M there exist vector fields $s_1, \dots, s_{2m+1}$ on M such that for any $x \in M$ the vector $s_1(x), \dots, s_{2m+1}(x)$ span $\mathrm{T}_x M$.

Exercise 270.9. Let $p\colon F \to B$ and $q\colon G \to B$ be two complex vector bundles of dimensions m respectively n. Suppose that have continuous maps $f\colon B \to G_m(\mathbb{C}^\infty)$ and $g\colon B \to G_n(\mathbb{C}^\infty)$ such that F is isomorphic to $f^*\mathfrak{T}G_m(\mathbb{C}^\infty)$ and such that G is isomorphic to $g^*\mathfrak{T}G_n(\mathbb{C}^\infty)$. Construct out of this data a continuous map $h\colon B \to G_{m+n}(\mathbb{C}^\infty)$ such that the Whitney sum $F \oplus G$ is isomorphic to $h^*\mathfrak{T}G_{m+n}(\mathbb{C}^\infty)$.

Exercise 270.10. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Show that every $\mathbb{K}$-vector bundle $p\colon E \to B$ is isomorphic to a smooth $\mathbb{K}$-vector bundle $q\colon F \to B$.

Exercise 270.11. Fortunately we already know $G_1(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^\infty$ quite well. Use known facts about $\mathbb{C}\mathrm{P}^\infty$ to deal with the following problems:

(a) Let $p\colon E \to B$ be a 1-dimensional $\mathbb{C}$-vector bundle over a CW-complex of dimension ≤ 2. Show that there exists a continuous map $g\colon B \to \mathbb{C}\mathrm{P}^1$ such that E is isomorphic to $g^*\mathfrak{T}\mathbb{C}\mathrm{P}^1$.

(b) Is the map $B \to \mathbb{C}\mathrm{P}^1$ in (a) well-defined up to homotopy?

Exercise 270.12. Let $x := [\mathbb{C}\mathrm{P}^1]^* \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ be the dual fundamental class of the complex manifold $\mathbb{C}\mathrm{P}^1$. Given $d \in \mathbb{N} \cup \{\infty\}$ we denote by $\iota\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^d$ the obvious injection. Recall that in the $\mathbb{C}\mathrm{P}^n$-H*-Lemma 213.5 we showed that the induced homomorphism $\mathrm{H}^2(\mathbb{C}\mathrm{P}^d;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ is an isomorphism.

Now let $p\colon E \to B$ be a $\mathbb{C}$-line bundle over a topological space B that is paracompact and Hausdorff. By the Grassmannian–Universal Bundle Theorem 270.10 there exists a continuous map $g\colon B \to \mathbb{C}\mathrm{P}^\infty$ such that E is isomorphic to $g^*\mathfrak{T}\mathbb{C}\mathrm{P}^\infty$. We refer to

$$c_1(E) \;:=\; g^*((\iota^*)^{-1}(x)) \;\in\; \mathrm{H}^2(B;\mathbb{Z})$$

as the *first Chern class of the* $\mathbb{C}$*-line bundle*.

(a) Show that $c_1(E)$ is well-defined.

(b) Show that if $p\colon E \to B$ is a trivializable 1-dimensional complex vector bundle over a topological space B that is paracompact and Hausdorff, then $c_1(E) = 0 \in \mathrm{H}^2(B;\mathbb{Z})$.

(c) Determine c_1 of the tangent bundle of the 2-dimensional torus $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \mathrm{i})$, viewed as a 1-dimensional complex manifold.

(d) For $n \in \mathbb{N} \cup \{\infty\}$ determine c_1 of the tautological $\mathbb{C}$-line bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^n$ over $\mathbb{C}\mathrm{P}^n$.

(e) Let $p\colon E \to B$ be a 1-dimensional complex vector bundle over over a topological space B that is paracompact and Hausdorff. We denote by $\overline{E}$ the complex vector bundle that is given by Definition 268.33. What is the relationship between $c_1(E)$ and $c_1(\overline{E})$?

(f) Show that all cohomology classes in $\mathrm{H}^2(S^2;\mathbb{Z})$ occur as the first Chern class of a $\mathbb{C}$-line bundle $p\colon E \to S^2$.

Exercise 270.13.

(a) What are the homotopy groups of $G_1(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^\infty$?

(b) What does (a) say about the classification of 1-dimensional $\mathbb{C}$-vector bundles over spheres?
(c) Is the result in (b) consistent with the Complex Vector Bundles-over-Sphere Proposition 267.6?

Exercise 270.14. Let $p\colon M \to S^1$ be the Möbius vector bundle from page 4580. Can you provide an explicit map $g\colon S^1 \to G_1(\mathbb{R}^\infty)$ such that the vector bundle M is isomorphic to $g^*\mathfrak{T}G_1(\mathbb{R}^\infty)$.

Exercise 270.15. Is the following statement correct: Every 1-dimensional real vector bundle over a simply connected manifold is trivializable?

Exercise 270.16. Let $n \in \mathbb{N}$. Show that the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ is not trivializable.

Exercise 270.17. Let $n \in \mathbb{N}_0$.
(a) Show that for each $k \in \mathbb{N}_0$ we have $\pi_k(G_n(\mathbb{C}^\infty)) = 0$.
Hint. For each d consider the map $G_n(\mathbb{C}^d) \to \mathbb{C}\setminus\{0\}$ that is given by $(v_1, \ldots, v_n) \to v_1$. Shows that this map is fiber bundle with fiber $G_{n-1}(\mathbb{C}^{d-1})$.
(b) Show that the map $\mathrm{GL}_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ is a fiber bundle with fiber $\mathrm{GL}(n,\mathbb{C})$.
(c) Show that the long exact sequence from the Fiber-LES-Proposition 239.8 gives for each $k \in \mathbb{N}_0$ an isomorphism

$$\partial_k\colon \pi_k(G_n(\mathbb{C}^\infty)) \;\to\; \pi_{k-1}(\mathrm{GL}(n,\mathbb{C}))$$

is an isomorphism.

Remark. This exercise is consistent with the fact that the set of isomorphisms classes of n-dimensional complex vector bundles over S^k is classified by $[S^{k-1}, \mathrm{GL}(n,\mathbb{C})]$ (see the Bundle-over-Sphere Proposition 241.16) and also by $[S^k, G_n(\mathbb{C}^\infty)$. ++++ still need to worry about base points +++

Exercise 270.18. Let $n \in \mathbb{N}$.
(a) Use the Bundle-over-Sphere Proposition 241.16 to show that there exists a non-trivial n-dimensional complex vector bundle over S^2.
(b) Show that given any complex vector bundle $p\colon E \to S^2$ there exists a continuous map $f\colon S^2 \to \mathbb{C}\mathrm{P}^1$ such that $f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1$ is isomorphic to E.
(c) Show that the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1$ is non-trivializable.
(d) Show that for any $d \in \mathbb{N}$ the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^d$ is non-trivializable.

Exercise 270.19. Let $n \in \mathbb{N}$ and let $d \in \mathbb{N}_0 \cup \{\infty\}$. We consider the corresponding tautological complex vector bundle $\mathfrak{T}G_n(\mathbb{C}^d) \to G_n(\mathbb{C}^d)$ and we consider the complex vector bundle $\overline{\mathfrak{T}G_n(\mathbb{C}^d)}$ from Definition 268.33. Give an example of a continuous map $h\colon G_n(\mathbb{C}^d) \to G_n(\mathbb{C}^\infty)$ such that $h^*\mathfrak{T}G_n(\mathbb{C}^\infty) = \overline{\mathfrak{T}G_n(\mathbb{C}^d)}$.

CHAPTER 271

The (co-) homology groups of Grassmannians

In the previous chapter we got to know several new topological spaces, namely the Grassmannians $G_n(\mathbb{K}^d)$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and with $d \in \mathbb{N}_0 \cup \{\infty\}$. We also introduced the corresponding tautological n-dimensional $\mathbb{K}$-vector bundles $\mathfrak{T}G_n(\mathbb{K}^d) \to G_n(\mathbb{K}^d)$ and we proved that these tautological vector bundles $\mathfrak{T}G_n(\mathbb{K}^\infty)$ are "universal". This shows that Grassmannians are extremely interesting and worthy of study. In this chapter we will first find a CW-structure on Grassmannians and afterwards we will calculate the (co-) homology groups of Grassmannians. We will also make first steps towards calculating the cup product on Grassmannians.

In this chapter it is notationally quite tricky to deal with the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ in parallel. For the most part we will discuss the case $\mathbb{K} = \mathbb{C}$. At the end of the chapter we will outline what changes one needs to perform for the case $\mathbb{K} = \mathbb{R}$.

This chapter is organized as follows:

- In Section 271.1 we will introduce Schubert symbols, Schubert varieties and Schubert open cells.
- In Section 271.2 we will use the Schubert open cells to equip the Grassmannians with CW-structures.
- In Section 271.3 we will use cellular homology to calculate the homology groups of Grassmannians. We will see that the Schubert varieties give rise to a basis of the homology groups of Grassmannians.
- In Section 271.4 we will calculate the intersection numbers of the homology classes corresponding to Schubert varieties.
- In Section 271.5 we will summarize the corresponding results for *real* Grassmannians.
- In Section 271.6 we will provide exercises for Chapter 271.

In the next Chapter 272 we will use the results of this chapter to calculate the cohomology ring of infinite Grassmannians.

271.1. **Schubert symbols, Schubert open cells and Schubert varieties.** As we already pointed out on page 5217, for $d < n$ we have $G_n(\mathbb{C}^d) = \varnothing$. So we only need to understand $G_n(\mathbb{C}^d)$ for $d \geq n$. Throughout this chapter we will take the equivalent point of view that we want to understand $G_n(\mathbb{C}^{n+r})$ for some $r \in \mathbb{N}_0$. It might be helpful to note that with this notation the Grassmannian–Manifold Proposition 270.3 says that $G_n(\mathbb{C}^{n+r})$ is a complex manifold of dimension $n \cdot r$.

271.1.1. **Schubert symbols and Schubert open cells.** In the following discussion we will need the following notation:

Notation 271.1. Let $n, r \in \mathbb{N}_0$.
(1) Given $i \in \{0, \ldots, n+r\}$ we write $\mathfrak{F}_i := \mathbb{C}^i \times \{0\} \subseteq \mathbb{C}^{n+r}$.[2881]
(2) Given $W = [(w_1, \ldots, w_n)] \in G_n(\mathbb{C}^{n+r})$ we denote, by abuse of notation, by W also the vector subspace of $\mathbb{C}^{n+r}$ that is spanned by $\{w_1, \ldots, w_n\}$.

Our next major goal is to equip the complex Grassmannians $G_n(\mathbb{C}^{n+r})$ with a CW-structure. For this we need to decompose Grassmannians into open cells. This desire leads us to the following two definitions:

Definition 271.2. Let $n, r \in \mathbb{N}_0$. An $(n \times r)$-Schubert[2888] symbol λ is a sequence $(\lambda_1, \ldots, \lambda_n)$ in $\{0, \ldots, r\}$ such that we have the inequalities $r \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$.[2889] Sometimes we extend this sequence to $\lambda_0 := r$ and $\lambda_{n+1} := 0$.

Definition 271.3. Let $n, r \in \mathbb{N}_0$. Given an $(n \times r)$-Schubert symbol $(\lambda_1, \ldots, \lambda_n)$ we consider the corresponding Schubert open cell[2890] [2891]

$$\Omega_\lambda := \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_j) = i \text{ if } r + i - \lambda_i \leq j \leq r + i - \lambda_{i+1}\}.$$

(3×7)-Schubert symbol
$\lambda = (\lambda_1, \lambda_2, \lambda_3) = (5, 3, 1)$



this defines the
Schubert open cell $\Omega_\lambda \subseteq G_7(\mathbb{C}^{7+3})$

Example.

(1) Note that for the $(n \times r)$-Schubert symbol $(r, \ldots, r)$ we have

$$\Omega_{(r,\ldots,r)} \underset{\text{see Definition 271.3}}{=} \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_i) = i \text{ for } i = 1, \ldots, n\} = \underbrace{[(\mathfrak{e}_1, \ldots, \mathfrak{e}_n)]}_{\text{single point that corresponds to } \mathfrak{F}_n}.$$

(2) We consider the special case $n = 1$. Let $r \in \mathbb{N}_0$. Let $k \in \{0, \ldots, r\}$. We consider the $(1 \times r)$-Schubert symbol $(\lambda_1) = (k)$. Recall that we set $\lambda_0 := r$ and $\lambda_2 := 0$. We see

[2881] In fact for the whole subsequent discussion it is only really relevant that $\mathfrak{F}_0, \ldots, \mathfrak{F}_{n+r}$ form a sequence of vector subspaces of $\mathbb{C}^{n+r}$ such that the dimension goes up by one at each step. Such a sequence of vector spaces is sometimes called a *flag*, which explains our notation $\mathfrak{F}_i$.
[2888] Schubert symbols and Schubert varieties are named after the German mathematician Hermann Cäsar Hannibal Schubert (1848–1911).
[2889] There are basically two conventions for Schubert symbols: Either one demands that one has a decreasing sequence $r \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$ or one has a *strictly* decreasing sequence $n + r \geq \lambda_1 > \cdots > \lambda_n \geq 0$. The former is for example used in [**Mani2001**, Chapter 3.2], whereas the latter is used in [**MiS1974**, Chapter 6]. We will follow the former convention. This can cause some confusion if one tries to compare the results of this chapter with the results in [**MiS1974**, Chapter 6].
[2890] The expression $r + i - \lambda_i$ is at this stage very confusing. The reason for this initially rather awkward choice will become apparent later in this chapter.
[2891] What we call *Schubert open cell* is usually called a *Schubert cell*. The reason why we picked the non-standard name is that in the Grassmannian–CW-Structure Proposition 271.10 we will see that the Schubert open cells are the *open* cells of a CW-structure on Grassmannians.

that we obtain the following subset of $G_1(\mathbb{C}^{1+r}) = \mathbb{C}\mathrm{P}^r$:

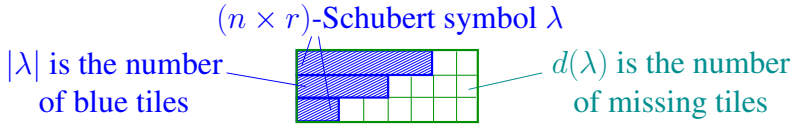
$$\begin{aligned}\Omega_{(k)} &= \{W \in G_1(\mathbb{C}^{1+r}) \,|\, \dim_{\mathbb{C}}(W \cap \mathfrak{F}_j) = i \text{ if } r+i-\lambda_i \leq j \leq r+i-\lambda_{i+1}\} \\ &= \left\{W \in G_1(\mathbb{C}^{1+r}) \,|\, \dim_{\mathbb{C}}(W \cap \mathfrak{F}_j) \begin{array}{ll} = 0 & \text{if } 0 \leq j \leq r-k, \\ = 1 & \text{if } r+1-k \leq j \leq r+1 \end{array}\right\} \\ &= \{W \in G_1(\mathbb{C}^{1+r}) \,|\, W \subseteq \mathfrak{F}_{r+1-k} \setminus \mathfrak{F}_{r-k}\} \underset{\substack{\uparrow \\ \text{where } \iota\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^r \\ \text{is induced by } [x] \mapsto [(x,0)]}}{=} \iota(\mathbb{C}\mathrm{P}^{r-k}) \setminus \iota(\mathbb{C}\mathrm{P}^{r-k-1}).\end{aligned}$$

□

We need to introduce two more concepts before we can describe the topology of the Schubert open cells:

Definition 271.4. Let $n, r \in \mathbb{N}_0$ and let $(\lambda_1, \ldots \lambda_n)$ be an $(n \times r)$-Schubert symbol. We write

$$\begin{aligned} |\lambda| &:= \lambda_1 + \cdots + \lambda_n \quad \text{(weight of } \lambda) \\ d(\lambda) &:= n \cdot r - |\lambda| \quad \text{(defect of } \lambda). \end{aligned}$$



Proposition 271.5. (Schubert Open Cell Proposition) Let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol.

(1) The Schubert open cell Ω_λ is a complex submanifold of dimension $d(\lambda)$ of $G_n(\mathbb{C}^{n+r})$[2892].

(2) There exists a diffeomorphism $\Theta_\lambda\colon \mathbb{C}^{d(\lambda)} \to \Omega_\lambda$ of complex manifolds.

We equip the complex submanifold Ω_λ with the orientation coming from the Complex Manifold-Orientation Proposition 40.18.

Sketch of proof. First we consider the map

$$\widetilde{\Theta}_\lambda\colon \mathbb{C}^{r-\lambda_1} \times \cdots \times \mathbb{C}^{r-\lambda_n} \to \mathrm{GL}_n(\mathbb{C}^{n+r})$$

which assign to complex numbers $a_{i,1}, \ldots, a_{i,r-\lambda_i}$, $i = 1, \ldots, n$ the following matrix:

$$\widetilde{\Theta}_\lambda(a_{ij}) := \begin{pmatrix} & & & r+1-\lambda_1 & & & r+2-\lambda_2 & & & r+n-\lambda_n & & \\ a_{1,1} & \ldots & a_{1,r-\lambda_1} & 1 & 0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{2,1} & \ldots & a_{2,r-\lambda_1} & 0 & a_{2,r-\lambda_1+1} & \ldots & 1 & 0 & \ldots & 0 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n,1} & \ldots & a_{n,r-\lambda_1} & 0 & a_{n,r-\lambda_1+1} & \ldots & 0 & \ldots & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix}$$

where in the i-th row the asterisks lie in the columns $1, \ldots, r+i-1-\lambda_i$ and the 1 lies in column $r+i-\lambda_i$. Note that this defines a smooth map

$$\widetilde{\Theta}_\lambda\colon \mathbb{C}^{r-\lambda_1} \times \cdots \times \mathbb{C}^{r-\lambda_n} \to \mathrm{GL}_n(\mathbb{C}^{n+r}).$$

As usual we denote by $\pi\colon \mathrm{GL}_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r})$ the natural projection. Elementary linear algebra shows that the map

$$\Theta_\lambda := \pi \circ \widetilde{\Theta}_\lambda\colon \underbrace{\mathbb{C}^{r-\lambda_1} \times \cdots \times \mathbb{C}^{r-\lambda_n}}_{=\mathbb{C}^{r-\lambda_1+\cdots+r-\lambda_n}=\mathbb{C}^{d(\lambda)}} \to G_n(\mathbb{C}^{n+r})$$

[2892]Recall that in the proof of the Grassmannian–Manifold Proposition 270.3 we equipped $G_n(\mathbb{C}^{n+r})$ with the structure of a $n \cdot r$-dimensional complex manifold.

is injective and that the image equals Ω_λ. We leave it to the reader to verify that this map is a smooth embedding such that the image is in fact a complex submanifold of $G_n(\mathbb{C}^{n+r})$. ■

271.1.2. **Schubert varieties.** In the previous subsection we introduced the notion of a Schubert symbol λ and we associated to it the Schubert open cell Ω_λ. Now we will associate to a Schubert symbol the closely related concept of the Schubert variety:

Definition 271.6. Let $n, r \in \mathbb{N}_0$. Given an $(n \times r)$-Schubert symbol $(\lambda_1, \dots, \lambda_n)$ we consider the Schubert variety

$$X_\lambda := \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \underbrace{\mathfrak{F}_{r+i-\lambda_i}}_{=\mathbb{C}^{r+i-\lambda_i}\times\{0\}}) \geq i \text{ for } i = 1, \dots, n\}.$$

Example.

(1) Let $n, r \in \mathbb{N}_0$. Note that in the Definition 271.3 of a Schubert open cell the restrictions for $j = r + i - \lambda_i$ give us precisely the conditions for the Schubert variety X_λ. This implies that $\Omega_\lambda \subseteq X_\lambda$.

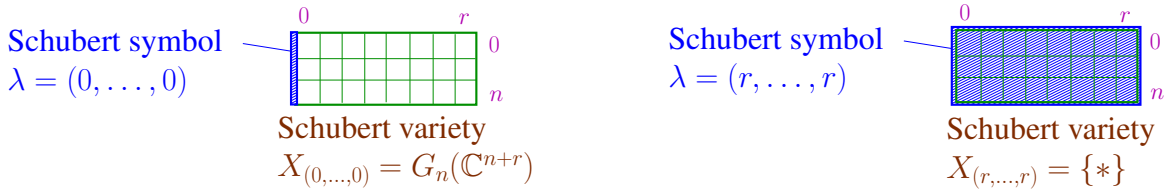
(2) Let $n, r \in \mathbb{N}_0$. In Exercise 271.1 we will see that:

(a) For the $(n \times r)$-Schubert symbol $(0, \dots, 0)$ we have

$$X_{(0,\dots,0)} = \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_{r+i}) \geq i \text{ for } i = 1, \dots, n\} = G_n(\mathbb{C}^{n+r}).$$

(b) For the $(n \times r)$-Schubert symbol $(r, \dots, r)$ we have

$$X_{(r,\dots,r)} = \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_i) \geq i \text{ for } i = 1, \dots, n\} = \underbrace{[(\mathfrak{e}_1, \dots, \mathfrak{e}_n)]}_{\text{single point that corresponds to } \mathfrak{F}_n}.$$



(3) Let $n = 1$ and let $r \in \mathbb{N}_0$. Given $k \in \{0, \dots, r\}$ we consider the $(1 \times r)$-Schubert symbol (k). We see that the corresponding Schubert variety in $G_1(\mathbb{C}^{1+r}) = \mathbb{CP}^r$ is given by

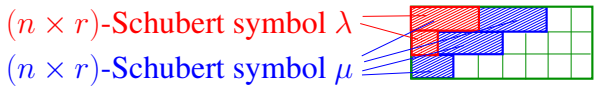
$$\begin{aligned} X_{(k)} &= \{W \in G_1(\mathbb{C}^{1+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_{r+1-k}) \geq 1\} \\ &= \{W \in G_1(\mathbb{C}^{1+r}) \mid W \subseteq \underbrace{\mathfrak{F}_{r+1-k}}_{=\mathbb{C}^{r+1-k}\times\{0\}}\} \\ &= \iota(\mathbb{CP}^{r+1-k}). \end{aligned}$$

(since $\dim_{\mathbb{C}}(W) = 1$; where $\iota\colon \mathbb{CP}^{r+1-k} \to \mathbb{CP}^r$ is induced by $[x] \mapsto [(x, 0)]$)

(4) In the Schubert Open Cell Proposition 271.5 we saw that the Schubert open cells Ω_λ are submanifolds of $G_n(\mathbb{C}^{n+r})$. Furthermore in the previous examples all the Schubert varieties $X_\lambda \subseteq G_n(\mathbb{C}^{n+r})$ we considered are submanifolds of the Grassmannian. Note though it is shown in [**Mani2001**, Chapter 3.4] that in general this is *not* the case, i.e. in general Schubert varieties are *not* submanifolds. In Exercise 271.3 we will discuss why the name Schubert *variety* is appropriate. □

To describe the precise relationship between Schubert varieties and Schubert open cells it is convenient to introduce the following concept:

Definition 271.7. Let $n, r \in \mathbb{N}_0$. Given two $(n \times r)$-Schubert symbols $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ we write $\mu \geq \lambda$ if for each $i \in \{1, \ldots, n\}$ we have $\mu_i \geq \lambda_i$. We write $\mu > \lambda$ if $\mu \geq \lambda$ and $\mu \neq \lambda$.



Remark. Let $n, r \in \mathbb{N}_0$ and let λ, μ be two $(n \times r)$-Schubert symbols. If $\mu \geq \lambda$, then it follows easily from Definition 271.6 that $X_\lambda \subseteq X_\mu$. □

Lemma 271.8. (Schubert Variety–Schubert Cell Lemma) Let $n, r \in \mathbb{N}_0$. For each $(n \times r)$-Schubert symbol λ we have

$$X_\lambda \;=\; \bigsqcup_{\mu \geq \lambda} \Omega_\mu.$$

Proof. It follows immediately from Definition 271.3 that the Schubert open cells are disjoint. Elementary logic gives us the decomposition of the Schubert variety X_λ into the Schubert open cells Ω_μ with $\mu \geq \lambda$. ■

The following proposition is a key step towards showing that Grassmannians are CW-complexes:

Proposition 271.9. (Schubert Characteristic Map Proposition) Let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol. There exists a continuous map

$$\Phi_\lambda \colon \overline{B}^{2d(\lambda)} \;\to\; G_n(\mathbb{C}^{n+r})$$

with the following properties:

(1) The image of Φ_λ equals the Schubert variety X_λ.
(2) The restriction of Φ_λ to $B^{2d(\lambda)}$ is an orientation-preserving[2893] homeomorphism to the Schubert open cell Ω_λ.
(3) The restriction of Φ_λ to $\partial\overline{B}^{2d(\lambda)} = S^{2d(\lambda)-1}$ takes values in $\bigcup_{\mu > \lambda} \Omega_\lambda$.

Proof. [2894] Let $n, r \in \mathbb{N}_0$. We start out with a preamble: Let $(\lambda_1, \ldots, \lambda_n)$ be an $(n \times r)$-Schubert symbol. We consider complex $(n \times (n+r))$-matrices of the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} & & & & & & r+1-\lambda_1 & & r+2-\lambda_2 & & & r+n-\lambda_n & & \end{array} \\
\begin{pmatrix}
* & \ldots & * & * & 0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
* & \ldots & * & * & * & \ldots & * & 0 & \ldots & 0 & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots \\
* & \ldots & * & * & * & \ldots & * & \ldots & \ldots & * & 0 & \ldots & 0
\end{pmatrix}.
\end{array}$$

[2893] Here we equip Ω_λ with the orientation coming from the Schubert Open Cell Proposition 271.5.
[2894] Recall that in the proof of the Schubert Open Cell Proposition 271.5 we gave an explicit diffeomorphism $\mathbb{C}^{d(\lambda)} \to \Omega_\lambda$. Next note that, similar to the discussion on page 652, one sees that for each m we have a diffeomorphism $B^{2m} \to \mathbb{C}^m$. Thus we get a diffeomorphism $B^{2d(\lambda)} \to \mathbb{C}^{d(\lambda)} \to \Omega_\lambda$. The catch with this map is that it is not clear why it should extend to a continuous map on $\overline{B}^{2d(\lambda)}$. Thus we will need to go a somewhat different route to define our map.

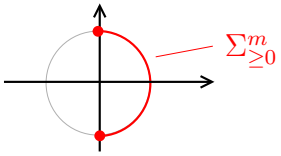
We make two observations:

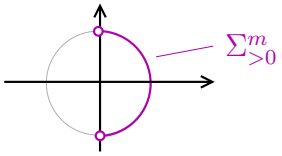
(a) If the entries that we indicate by "$*$" are all non-zero, then the rows of such a matrix define a point in Ω_λ.

(b) If the last non-zero entry for the rows $1, \dots, n$ occurs in increasing columns, then the rows of such a matrix are linearly independent, thus the rows of such a matrix define a point in $\Omega_\mu \subseteq X_\lambda$ for some $\mu \geq \lambda$. Conversely any point in X_λ is represented by such a matrix.

After all these initial observations we want to turn to the actual construction of the map $\Phi_\lambda \colon \overline{B}^{2d(\lambda)} \to G_n(\mathbb{C}^{n+r})$. To do so we need to introduce the following notation:

- We denote by $\pi \colon \mathrm{GL}_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r})$ the natural projection.
- Given $m \in \mathbb{N}_0$ we write
$$\begin{aligned} \Sigma^m_{\geq 0} &:= \{(z_1, \dots, z_m) \in \mathbb{C}^m \,|\, |z_1|^2 + \dots + |z_m|^2 = 1 \text{ and } z_m \in \mathbb{R}_{\geq 0}\}, \\ \Sigma^m_{>0} &:= \{(z_1, \dots, z_m) \in \mathbb{C}^m \,|\, |z_1|^2 + \dots + |z_m|^2 = 1 \text{ and } z_m \in \mathbb{R}_{>0}\}. \end{aligned}$$
Note that $\Sigma^m_{\geq 0}$ is homeomorphic to the upper half-sphere $S^{2m-2}_{\geq 0}$, in particular it is homoemorphic to $\overline{B}^{2(m-1)}$.





- As in Definition 261.4 we consider the complex Grassmannian
$$V_n(\mathbb{C}^{n+r}) := \{(v_1, \dots, v_n) \in \mathrm{GL}_n(\mathbb{C}^{n+r}) \,|\, \text{the } v_i \text{ are orthonormal}\}.$$
- We write
$$\begin{aligned} E(\lambda)_{\geq 0} &:= \{(v_1, \dots, v_n) \in V_n(\mathbb{C}^{n+r}) \,|\, \text{ for } i = 1, \dots, n \text{ we have } v_i \in \Sigma^{r+i-\lambda_i}_{\geq 0}\} \\ E(\lambda)_{>0} &:= \{(v_1, \dots, v_n) \in V_n(\mathbb{C}^{n+r}) \,|\, \text{ for } i = 1, \dots, n \text{ we have } v_i \in \Sigma^{r+i-\lambda_i}_{>0}\}. \end{aligned}$$
In other words $E(\lambda)_{\geq 0}$ corresponds to $(n \times (n+r))$-matrices as above where the rows are orthonormal, the red entries lie in the columns $r + i - \lambda_i$ and the red entries lie in $\mathbb{R}_{\geq 0}$. An analogous statement holds for $E(\lambda)_{>0}$.

It follows easily from the above observations (a) and (b) and elementary linear algebra that the map $\pi \colon E(\lambda)_{>0} \to \Omega_\lambda$ is a bijection and that $\pi(E(\lambda)_{\geq 0}) = X_\lambda$.

Claim. There exists a homeomorphism $\widetilde{\Phi}_\lambda \colon \overline{B}^{2d(\lambda)} \to E(\lambda)_{\geq 0}$.

Proof. We prove the statement by induction on n. For $n = 1$ this follows from the above discussion since in the case we have $E(\lambda)_{\geq 0} \cong \Sigma^{r+1-\lambda_1} \cong \overline{B}^{2(r-\lambda_1)}$.

Now we assume that the statement holds for $n - 1$. We consider the map
$$\begin{aligned} p \colon E(\lambda)_{\geq 0} &\to \Sigma^{r+1-\lambda_1}_{\geq 0} \\ (v_1, v_2, \dots, v_n) &\mapsto v_1. \end{aligned}$$
As in the Stiefel Manifold-Fiber Proposition 261.8 one can show that this is a fiber bundle where the fiber is homeomorphic to $E(\lambda_2, \dots, \lambda_n)_{\geq 0}$[2895]. Since $\Sigma^{r+1-\lambda_1}$ is contractible

[2895] A more direct argument is given in [**MiS1974**, p. 76] and in [**Hat2**, p. 33].

we obtain from the Bundles-Homotopies Proposition 241.10 (2) that the fiber bundle is trivializable. Thus we see that

$$E(\lambda)_{\geq 0} \underset{\text{by the Bundles-Homotopies Proposition 241.10 (2)}}{\cong} \Sigma^{r+1-\lambda_1}_{\geq 0} \times E(\lambda_2, \ldots, \lambda_n)_{\geq 0} \underset{\text{by induction}}{\cong} \overline{B}^{2(r-\lambda_1)} \times \overline{B}^{2d(\lambda_2,\ldots,\lambda_n)} \cong \overline{B}^{2d(\lambda)}.$$

⊞

We now consider the map $\Phi_\lambda := \pi \circ \widetilde{\Phi}_\lambda \colon \overline{B}^{2d(\lambda)} \to G_n(\mathbb{C}^{n+r})$. We leave it to the reader to verify that the homeomorphism Φ_λ restricts to a homeomorphism $E(\lambda)_{>0} \to B^{2d(\lambda)}$ and that the restriction of Φ_λ to $S^{2d(\lambda)-1}$ takes values in $\bigsqcup_{\mu > \lambda} \Omega_\lambda$. Finally note that, possibly after precomposing the map Φ_λ by a reflection in a hyperplane, we can assume that the map $\Phi_\lambda \colon B^{2d(\lambda)} \to \Omega_\lambda$ is orientation-preserving. ■

271.2. **CW-structures on Grassmannians.** Now we can show that all complex Grassmannians admit CW-structures:

Proposition 271.10. (Grassmannian–CW-Structure Proposition) Let $n, r \in \mathbb{N}_0$.

(1) Given $k \in \mathbb{N}_0$ we set

$$G_n(\mathbb{C}^{n+r})_k := \bigsqcup_{\substack{(n\times r)\text{-Schubert} \\ \text{symbol } \lambda \\ \text{with } 2d(\lambda) \leq k}} \Omega_\lambda \underset{\text{by the Schubert Variety–Schubert Cell Lemma 271.8}}{=} \bigsqcup_{\substack{(n\times r)\text{-Schubert} \\ \text{symbol } \lambda \\ \text{with } 2d(\lambda) \leq k}} X_\lambda.$$

The filtration

$$G_n(\mathbb{C}^{n+r})_0 \subseteq G_n(\mathbb{C}^{n+r})_1 \subseteq \ldots \subseteq G_n(\mathbb{C}^{n+r})_{2n\cdot r}$$

is a CW-structure, in the sense of Definition 136.1, on $G_n(\mathbb{C}^{n+r})$. This CW-structure has the following properties:

- The open cells are precisely the Schubert open cells Ω_λ corresponding to the $(n \times r)$-Schubert symbols λ. Furthermore the maps from the Schubert Characteristic Map Proposition 271.9 are characteristic maps of the cells and the cells are the Schubert varieties.
- For every $(n \times r)$-Schubert symbol λ the corresponding Schubert variety X_λ is a subcomplex.

(2) The embedding $G_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r+1})$ from the Grassmannian-Embedding Lemma 270.5 is cellular in the sense of Definition 139.2. Furthermore under this map the Schubert open cell $\Omega_{(\lambda_1,\ldots,\lambda_n)}$ gets sent to the Schubert open cell $\Omega_{(\lambda_1+1,\ldots,\lambda_n+1)}$ and a similar statement holds for Schubert varieties.

(3) The CW-structures on $G_n(\mathbb{C}^{n+r})$, $r \in \mathbb{N}_0$ define a CW-structure on the infinite-dimensional Grassmannian

$$G_n(\mathbb{C}^\infty) = \operatorname{colim}_{\longrightarrow} G_n(\mathbb{C}^{n+r})$$

such that every $G_n(\mathbb{C}^{n+r})$ is a subcomplex.

Example.

(1) Let us once again consider the case $G_1(\mathbb{C}^{1+r})$ in greater detail. In this case the $(1 \times r)$-Schubert symbols are $(0), (1), \ldots, (r)$. It follows from the discussion on page 5234 that

we get the following table:

Schubert symbol (k)	(0)	(1)	(2)	$\dots$	$(r-1)$	(r)
Schubert variety $X_{(k)}$	$\mathbb{C}\mathrm{P}^r$	$\iota(\mathbb{C}\mathrm{P}^{r-1})$	$\iota(\mathbb{C}\mathrm{P}^{r-1})$	$\dots$	$\iota(\mathbb{C}\mathrm{P}^1)$	$\iota(\mathbb{C}\mathrm{P}^0)$.

Thus we see that the CW-structure for $G_1(\mathbb{C}^{1+r}) = \mathbb{C}\mathrm{P}^r$ that we construct in the Schubert Variety Proposition 271.10 is precisely the usual CW-structure that we constructed on page 2628.

(2) Similarly the CW-structure on $G_n(\mathbb{C}^\infty)$ is the same as the CW-structure on $\mathbb{C}\mathrm{P}^\infty$ that we constructed on page 2628. □

Proof. Let $n, r \in \mathbb{N}_0$.

(1) We start out by the following claim:

Claim. We have $G_n(\mathbb{C}^{n+r}) = \bigsqcup_{\substack{(n\times r)\text{-Schubert}\\ \text{symbol } \lambda}} \Omega_\lambda$

where we run over all $(n \times r)$-Schubert symbols λ.

Proof. We have the following equalities:

$$G_n(\mathbb{C}^{n+r}) \overset{\text{see page 5234}}{=} X_{(0,\dots,0)} \overset{\text{by the Schubert Variety Schubert Cell Lemma 271.8}}{=} \bigsqcup_{\substack{(n\times r)\text{-Schubert}\\ \text{symbol } \lambda}} \Omega_\lambda.$$

⊞

Next note that it follows from the Grassmannian–Manifold Proposition 270.3 that the Grassmannian $G_n(\mathbb{C}^{n+r})$ is Hausdorff. It now follows from the Schubert Characteristic Map Proposition 271.9, the Topological-Pushout Proposition 8.12 (1b), the Pushout-Properties Proposition 8.14 (1) and the Compact-Hausdorff Proposition 4.23 that the given filtration is a CW-structure where the open cells are precisely the Schubert open cells and where the maps from the Schubert Characteristic Map Proposition 271.9 are characteristic maps.

Next note that it follows from the Schubert Characteristic Map Proposition 271.9 together with the Cell-Compact Proposition 136.8 that the cells of the CW-decomposition are precisely the Schubert varieties. Finally note that it follows from the Schubert Variety–Schubert Cell Lemma 271.8 and the Subcomplex–Characterization Proposition 136.12 that the Schubert varieties X_λ are subcomplexes of $G_n(\mathbb{C}^{n+r})$.

(2) Let $(\lambda_1, \dots, \lambda_n)$ be an $(n \times r)$-Schubert symbol. We consider the $(n \times (r+1))$-Schubert symbol $(\lambda_1+1, \dots, \lambda_n+1)$. It follows immediately from Definition 271.3 that the image of the Schubert open cell $\Omega_{(\lambda_1,\dots,\lambda_n)}$ under the usual map

$$\begin{array}{rcl} G_n(\mathbb{C}^{n+r}) & \to & G_n(\mathbb{C}^{n+r+1}) \\ [(v_1, \dots, v_n)] & \mapsto & [((v_1, 0), \dots, (v_n, 0))] \end{array}$$

is precisely the Schubert open cell $\Omega_{(\lambda_1+1,\dots,\lambda_n+1)}$. It follows from our definition of the CW-structure that the map $G_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r+1})$ is cellular in the sense of Definition 139.2. The same way we see the corresponding statement for Schubert varieties.

(3) It follows from (2) together with the Direct Colimit–CW Lemma 139.13 that the direct colimit $G_n(\mathbb{C}^\infty) = \operatorname{colim}_{\longrightarrow} G_n(\mathbb{C}^{n+r})$ inherits a CW-structure such that the $G_n(\mathbb{C}^{n+r})$ are all subcomplexes. ■

271.3. **Homology groups of Schubert varieties and Grassmannians.** The main goal of this section is to calculate the homology groups of complex Grassmannians. In preparation for the calculation we first need to prove a result about the homology groups of Schubert varieties.

As we mentioned before, in general there is no reason why one should expect the Schubert varieties X_λ to be manifolds. But the following lemma shows that homologically they look a little bit like a closed orientable manifold in the sense that the top-dimensional homology is isomorphic to $\mathbb{Z}$:

Lemma 271.11. (Schubert Variety-H$_*$-Lemma) In the following let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol. Let $\Phi_\lambda \colon \overline{B}^{2d(\lambda)} \to X_\lambda$ be the continuous map from the Schubert Characteristic Map Proposition 271.9. The following maps are isomorphisms:

$$\mathrm{H}_{2d(\lambda)}(\overline{B}^{2d(\lambda)}, S^{2d(\lambda)-1}) \xrightarrow{\Phi_{\lambda *}} \mathrm{H}_{2d(\lambda)}(X_\lambda, \underbrace{X_\lambda^{2d(\lambda)-1}}_{\substack{\uparrow \\ \text{the } (2d(\lambda)-1)\text{-skeleton} \\ \text{of the Schubert variety } X_\lambda}}) \xleftarrow{\text{я}} \mathrm{H}_{2d(\lambda)}(X_\lambda).$$

In particular we have an isomorphism $\mathrm{H}_{2d(\lambda)}(X_\lambda) \cong \mathbb{Z}$.

Notation 271.12. Let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol λ.

(1) In the following we denote by $[X_\lambda] \in \mathrm{H}_{2d(\lambda)}(X_\lambda)$ the image of the standard generator $[\overline{B}^{2d(\lambda)}] \in \mathrm{H}_{2d(\lambda)}(\overline{B}^{2d(\lambda)}, S^{2d(\lambda)-1})$ under the isomorphism from the Schubert Variety-H$_*$-Lemma 271.11. We refer to $[X_\lambda]$ as the fundamental class of X_λ.

(2) Sometimes we will also denote by $[X_\lambda] \in \mathrm{H}_{2d(\lambda)}(G_n(\mathbb{C}^{n+r}))$ the image of $[X_\lambda]$ under the inclusion induced map $\mathrm{H}_{2d(\lambda)}(X_\lambda) \to \mathrm{H}_{2d(\lambda)}(G_n(\mathbb{C}^{n+r}))$.

Proof. Let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol. Let $\Phi_\lambda \colon \overline{B}^{2d(\lambda)} \to X_\lambda$ be the continuous map from the Schubert Characteristic Map Proposition 271.9. We need to show that the following two maps are isomorphisms:

$$\mathrm{H}_{2d(\lambda)}(\overline{B}^{2d(\lambda)}, S^{2d(\lambda)-1}) \xrightarrow{\Phi_{\lambda *}} \mathrm{H}_{2d(\lambda)}(X_\lambda, X_\lambda^{2d(\lambda)-1}) \xleftarrow{\text{я}} \mathrm{H}_{2d(\lambda)}(X_\lambda).$$

By the Grassmannian–CW-Structure Proposition 271.10 we know that X_λ is a subcomplex of $G_n(\mathbb{C}^{n+r})$. Note that this provides us with a CW-structure on X_λ with a single $2d(\lambda)$-cell and no cell of dimension $2d(\lambda) - 1$. Since the map $\Phi_\lambda \colon \overline{B}^{2d(\lambda)} \to X_\lambda$ from the Schubert Characteristic Map Proposition 271.9 is a characteristic map of this cell we obtain from the CW-Skeleton-H$_*$-Lemma 155.2 (1b) that the map to the left is an isomorphism.

Since the $(2d(\lambda)-1)$-skeleton $X_\lambda^{2d(\lambda)-1}$ has no cells in dimension $\geq 2d(\lambda)-1$ we obtain from the CW-Skeleton-H$_*$-Lemma 155.2 (2) that the homology groups of $X_\lambda^{2d(\lambda)-1}$ vanish for $j \geq 2d(\lambda) - 1$. It now follows from the long exact sequence of the pair $(X_\lambda, X_\lambda^{2d(\lambda)-1})$ that the right map is also an isomorphism. ■

Now we can finally compute the homology groups of Grassmannians. More precisely, we have the following proposition which is a generalization of the $\mathbb{C}\mathrm{P}^n$-H$_*$-Proposition 155.9:

Proposition 271.13. **($G_n(\mathbb{C}^{n+r})$-H_*-Proposition)** Let $n, r \in \mathbb{N}_0$.

(0) The odd-dimension-homology groups of $G_n(\mathbb{C}^{n+r})$ and of $G_n(\mathbb{C}^\infty)$ are all zero.

(1) (a) For each $k \in \mathbb{N}_0$ the group $\mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r}))$ is a free abelian group where the rank is given by the number of $(n \times r)$-Schubert symbols with $d(\lambda) = k$.

(b) The homology classes $[X_\lambda]$, where λ runs over all $(n \times r)$-Schubert symbols with $d(\lambda) = k$ form a basis for the free abelian group $\mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r}))$.

(2) (a) The obvious map $G_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r+1})$ induces a monomorphism of homology groups. This map sends each $[X_{(\lambda_1,\dots,\lambda_n)}]$ to $[X_{(\lambda_1+1,\dots,\lambda_n+1)}]$.

(b) If $k \leq r$, then the induced map $\mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r})) \to \mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r+1}))$ is an isomorphism.

(3) If $k \leq r$, then the induced map $\mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r})) \to \mathrm{H}_{2k}(G_n(\mathbb{C}^\infty))$ is an isomorphism.

Proof. We equip $G_n(\mathbb{C}^{n+r})$ with the CW-structure from the Grassmannian–CW-Structure Proposition 271.10. The obvious idea is to determine the homology groups of $G_n(\mathbb{C}^{n+r})$ via the cellular chain complex. We make the following observations:

Claim.

(a) The odd-dimensional cellular chain groups are all zero.

(b) The cellular chain groups are isomorphic to the singular chain groups.

Proof.

(a) This statement follows from the CW-Skeleton-H_*-Lemma 155.2 (1b) and the simple observation that in the Grassmannian–CW-Structure Proposition 271.10 we cleverly equipped $G_n(\mathbb{C}^{n+r})$ and $G_n(\mathbb{C}^\infty)$ with a CW-structure without odd-dimensional cells.

(b) It follows from (a) that the cellular boundary maps in $\mathrm{C}_*^{\mathrm{CW}}(G_n(\mathbb{C}^{n+r}))$ are all zero. Thus the cellular homology groups are isomorphic to the groups in the cellular chain complex. The desired statement now follows from the Singular-Cellular-H_*-Proposition 155.8. ⊞

Now we turn to the actual proofs:

(0) It follows immediately from the claim that the odd-dimensional homology groups of $G_n(\mathbb{C}^{n+r})$ and of $G_n(\mathbb{C}^\infty)$ are all zero.

(1) (a) This statement also follows immediately from the claim, the CW-Skeleton-H_*-Lemma 155.2 (1b) and the Grassmannian–CW-Structure Proposition 271.10.

(b) Let $k \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{array}{ccccc}
\bigoplus_\lambda \mathrm{H}_{2k}(\overline{B}^{2k}, S^{2k-1}) & \xrightarrow{\oplus \Phi_{\lambda *}} & \bigoplus_\lambda \mathrm{H}_{2k}(X_\lambda, X_\lambda^{2k-1}) & \xleftarrow[\cong]{\text{я}} & \bigoplus_\lambda \mathrm{H}_{2k}(X_\lambda) \\
 & \searrow^{\oplus \Phi_{\lambda *}} & \downarrow & & \downarrow \\
 & & \mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r})^{2k}, G_n(\mathbb{C}^{n+r})^{2k-1}) & \xleftarrow{\text{я}} & \mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r})^{2k}) \\
 & & & & \downarrow \cong \\
 & & & & \mathrm{H}_{2k}(G_n(\mathbb{C}^{n+r})).
\end{array}$$

We make the following observations and clarifications:

- We take the direct sum over all $(n \times r)$-Schubert symbols λ with $d(\lambda) = k$.
- The vertical maps are induced by the inclusions $X_\lambda \to G_n(\mathbb{C}^{n+r})^{2k}$.

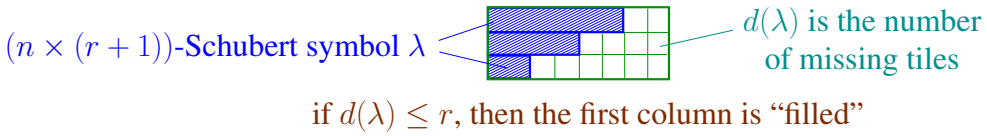
- Clearly the diagram commutes.
- It follows from the CW-Skeleton-H_*-Lemma 155.2 (1) that the diagonal map is an isomorphism.
- By the Schubert Variety-H_*-Lemma 271.11 we know that the top horizontal maps are isomorphisms.
- By the Grassmannian–CW-Structure Proposition 271.10 we know that $G_n(\mathbb{C}^{n+r})$ has no cells in dimension $2k \pm 1$. Thus, as in the proof of the Schubert Variety-H_*-Lemma 271.11, we see that the middle horizontal map is an isomorphism and that the bottom vertical map is an isomorphism.

In summary we see that the bent right vertical map is an isomorphism. This gives us the desired result.

(2) Let $k \in \mathbb{N}_0$.

(a) First of all note that we see, as in the Grassmannian–CW-Structure Proposition 271.10, that the induced map $H_{2k}(G_n(\mathbb{C}^{n+r})) \to H_{2k}(G_n(\mathbb{C}^{n+r+1}))$ clearly has the property that it sends $[X_{(\lambda_1,\dots,\lambda_n)}]$ to $[X_{(\lambda_1+1,\dots,\lambda_n+1)}]$. By (1a) this implies that the map $H_{2k}(G_n(\mathbb{C}^{n+r})) \to H_{2k}(G_n(\mathbb{C}^{n+r+1}))$ is in fact a monomorphism of groups.

(b) We now assume that $r \geq k$. Note that this implies that for any $(n \times (r+1))$-Schubert symbol $(\lambda_1, \dots, \lambda_n)$ with $d(\lambda) = k$ we have $\lambda_n > 0$. Thus $\lambda_1 - 1, \dots, \lambda_n - 1$ is an $(n \times r)$-Schubert symbol with the same defect.



It follows from (1a) that all generators of $H_k(G_n(\mathbb{C}^{n+r+1}))$ lie in the image of $H_k(G_n(\mathbb{C}^{n+r}))$. This shows that the map $H_{2k}(G_n(\mathbb{C}^{n+r})) \to H_{2k}(G_n(\mathbb{C}^{n+r+1}))$ is an epimorphism. Together with (a) this implies that this map is in fact an isomorphism.

(3) This statement follows from (2b), the Exhaustions-H_*-Proposition 154.2 (2)[2896] and the Direct Colimit–Stabilization Lemma 104.5. ∎

271.4. **The cohomology groups and rings on Grassmannians.** Our goal in this section is to determine the isomorphism type of the cohomology groups of the finite-dimensional Grassmannians $G_n(\mathbb{C}^{n+r})$ and to obtain some first results on their cup products.

271.4.1. **The cohomology groups of Grassmannians.** Now that we have calculated the homology groups of the finite-dimensional Grassmannians $G_n(\mathbb{C}^{n+r})$ it is straightforward, using Poincaré Duality, to determine the cohomology groups. To formulate the result it is convenient to introduce the following notation:

Notation 271.14. Let $n, r \in \mathbb{N}_0$. Given an $(n \times r)$-Schubert symbol λ we denote by

$$\sigma_\lambda := \mathrm{PD}_{G_n(\mathbb{C}^{n+r})}([X_\lambda]) \in H^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$$

[2896] Note that here we use that by the Grassmannian–CW-Structure Proposition 271.10 we know that the infinite-dimensional Grassmannian $G_n(\mathbb{C}^\infty) := \varinjlim G_n(\mathbb{C}^{n+r})$ has a CW-structure such that the $G_n(\mathbb{C}^{n+r})$ are all subcomplexes.

the Poincaré dual[2897] [2898] [2899] of the fundamental class $[X_\lambda] \in \mathrm{H}_{2d(\lambda)}(G_n(\mathbb{C}^{n+r}))$ that we introduced in Notation 271.12.

The following lemma is a generalization of the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5:

Lemma 271.15. ($G_n(\mathbb{C}^{n+r})$-H^*-Lemma) Let $n, r \in \mathbb{N}_0$.

(1) The odd-dimensional cohomology groups of $G_n(\mathbb{C}^{n+r})$ are zero.

(2) Let $k \in \mathbb{N}$. We consider all $(n \times r)$-Schubert symbols λ with $|\lambda| = k$. The corresponding cohomology classes σ_λ form a basis for $\mathrm{H}^{2k}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$.

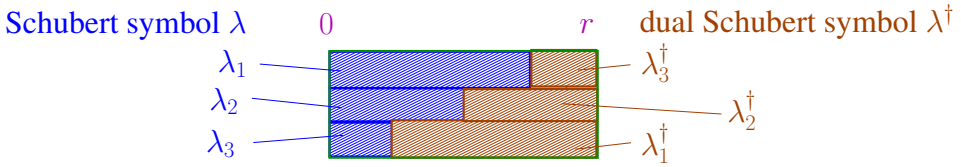
Proof. The lemma follows immediately from the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13, the Poincaré Duality Theorem 210.1 and the fact that $|\lambda| = n \cdot r - d(\lambda)$. ■

271.4.2. **The cup product on Grassmannians.** In this subsection we will calculate some cup products of Grassmannians. To formulate the result we will need the following definition:

Definition 271.16. Let $n, r \in \mathbb{N}_0$. Given an $(n \times r)$-Schubert symbol $\lambda = (\lambda_1, \ldots, \lambda_n)$ we refer to the $(n \times r)$-Schubert symbol

$$\lambda^\dagger := (r - \lambda_n, \ldots, r - \lambda_1)$$

as the dual Schubert symbol. Note that $|\lambda^\dagger| = n \cdot r - |\lambda| = d(\lambda)$.



The following proposition gives us a calculation of some cup product on the cohomology of finite-dimensional Grassmannians:

Proposition 271.17. (Schubert Variety Intersection Proposition) Let $n, r \in \mathbb{N}_0$ and let λ, μ be two $(n \times r)$-Schubert symbols with $|\lambda| + |\mu| = n \cdot r$. Then

$$\sigma_\lambda \cup \sigma_\mu = \underset{\substack{\uparrow \\ \text{Kronecker symbol}}}{\delta_{\lambda,\mu^\dagger}} \cdot [G_n(\mathbb{C}^{n+r})]^*.$$

Here $[G_n(\mathbb{C}^{n+r})]^* \in \mathrm{H}^{2n \cdot r}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$ is the dual fundamental class of the oriented smooth manifold $G_n(\mathbb{C}^{n+r})$ from Definition 201.27.

Remark. The Schubert Variety Intersection Proposition 271.17 feels like unfinished business. Even though we just calculated the cup product on a basis of the cohomology groups (at least for the opposite dimensions k and $2n \cdot r - k$) we do not really have a good sense of what the cohomology ring of the Grassmannian $G_n(\mathbb{C}^{n+r})$ looks like. We will deal with this issue in the next Chapter 272. □

[2897] Here, for a closed oriented m-dimensional smooth manifold M we denote, as in Definition 210.2, by $\mathrm{PD}_M \colon \mathrm{H}_k(M) \to \mathrm{H}^{m-k}(M; \mathbb{Z})$ the isomorphism that is provided by the Poincaré Duality Theorem 210.1.

[2898] Recall that in the Grassmannian–Manifold Proposition 270.3 we equipped $G_n(\mathbb{C}^{n+r})$ with the structure of a closed $n \cdot r$-dimensional complex manifold. By the Complex Manifold-Orientation Proposition 40.18 we can thus view $G_n(\mathbb{C}^{n+r})$ as a closed oriented $2n \cdot r$-dimensional smooth manifold.

[2899] We also use that $2|\lambda| = 2n \cdot r - 2d(\lambda)$.

Sketch of proof. [2900] We start out with some general observations: Let $(\nu_1, \dots, \nu_n)$ be an $(n \times r)$-Schubert symbol. In Definitions 271.3 and 271.6 we introduced the corresponding Schubert open cell

$$\Omega_\nu \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap (\underbrace{\mathbb{C}^j \times \{0\}}_{=\mathfrak{F}^j})) = i \text{ if } r+i-\nu_i \leq j \leq r+i-\nu_{i+1}\}$$

and we introduced the Schubert variety

$$X_\nu \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap (\mathbb{C}^{r+i-\nu_i} \times \{0\})) \geq i \text{ for } i = 1, \dots, n\}.$$

We now introduce a variation on these definitions, where we use the "opposite flag". More precisely, similar to Definitions 271.3 and 271.6 we set

$$\widetilde{\Omega}_\nu \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap (\{0\} \times \mathbb{C}^j) = i \text{ if } r+i-\nu_i \leq j \leq r+i-\nu_{i+1}\}$$

and we set

$$\widetilde{X}_\nu \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap (\{0\} \times \mathbb{C}^{r+i-\nu_i})) \geq i \text{ for } i = 1, \dots, n\}.$$

Note that, as in the Schubert Variety-H_*-Lemma 271.11, we obtain for $\widetilde{X}_\nu$ a "fundamental class" $[\widetilde{X}_\nu] \in \mathrm{H}_{2d(\nu)}(\widetilde{X}_\nu)$ which then also defines a homology class $[\widetilde{X}_\nu] \in \mathrm{H}_{2d(\nu)}(G_n(\mathbb{C}^{n+r}))$.

Claim 1. We have $$[X_\nu] \;=\; [\widetilde{X}_\nu] \;\in\; \mathrm{H}_{2d(\nu)}(G_n(\mathbb{C}^{n+r})).$$

Proof. We consider the $(n+r) \times (n+r)$-matrix

$$J \;:=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{GL}(n+r, \mathbb{C})$ is path-connected. Thus there exists a continuous map $\gamma\colon [0,1] \to \mathrm{GL}(n+r, \mathbb{C})$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = J$. It follows easily from the Homotopy Quotient Theorem 18.11 that the map

$$\begin{array}{rcl} F\colon G_n(\mathbb{C}^{n+r}) \times [0,1] & \to & G_n(\mathbb{C}^{n+r}) \\ ([(w_1, \dots, w_n)], s) & \mapsto & [\gamma(s) \cdot (w_1, \dots, w_n)] \end{array}$$

is continuous. Clearly the map $F_1\colon G_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r})$ restricts to an orientation-preserving diffeomorphism $F_1\colon \Omega_\nu \to \widetilde{\Omega}_\nu$ and to a homeomorphism $F_1\colon X_\nu \to \widetilde{X}_\nu$. It follows from the Homotopy-H_*-Proposition 144.12 that we have the following equalities in $\mathrm{H}_{2d(\nu)}(G_n(\mathbb{C}^{n+r}))$:

$$[\widetilde{X}_\nu] \;=\; F_{1*}([X_\nu]) \;=\; F_{0*}([X_\nu]) \;=\; [X_\nu]. \qquad \boxplus$$

After this initial observation we turn to the actual proof of the proposition: Let $n, r \in \mathbb{N}_0$ and let λ, μ be two $(n \times r)$-Schubert symbols with $|\lambda| + |\mu| = n \cdot r$. Note that this is equivalent to $d(\lambda) + d(\mu) = n \cdot r$.

Claim 2.

(1) If $\lambda \neq \mu^\dagger$, then $X_\lambda \cap \widetilde{X}_\mu = \varnothing$.
(2) If $\lambda = \mu^\dagger$, then $X_\lambda \cap \widetilde{X}_\mu$ consists of a single point P, it lies in $\Omega_\lambda \cap \widetilde{\Omega}_\mu$ and the oriented smooth submanifolds Ω_λ and $\widetilde{\Omega}_\mu$ intersect transversally in P with a positive sign.

[2900] For the case $\mathbb{C}\mathrm{P}^r = G_1(\mathbb{C}^{1+r})$ the proof of this lemma is essentially the same as the proof of the Cup Product-of-Duals-in-$\mathbb{C}\mathrm{P}^n$-Lemma 220.21.

Proof. As in the proof of the Schubert Characteristic Map Proposition 271.9 we consider complex $(n \times (n+r))$-matrices of the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \overset{}{} \\ \begin{pmatrix} * & \dots & * & * & 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & * & * & \dots & * & 0 & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & * & * & \dots & * & \dots & \dots & * & 0 & \dots & 0 \end{pmatrix} \\ \overset{}{} \end{array} \begin{array}{c} \\ r+1-\lambda_1 \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \\ r+2-\lambda_2 \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \\ r+n-\lambda_n \\ \\ \\ \\ \end{array}.$$

Recall that on page 5236 we made two observations:

(a) If the entries that we indicate by "*" are all non-zero, then the rows of such a matrix define a point in Ω_λ.
(b) If the last non-zero entry for the rows $1, \dots, n$ occur in increasing columns, then the rows of such a matrix are linearly independent, thus the rows of such a matrix define a point in $\Omega_\mu \subseteq X_\lambda$ for some $\mu \geq \lambda$. Conversely any point in X_λ is represented by such a matrix.

Now we consider complex $(n \times (n+r))$-matrices of the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \overset{}{} \\ \begin{pmatrix} 0 & \dots & 0 & * & * & \dots & * & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & * & * & \dots & * & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \dots & * & * & \dots & * \end{pmatrix} \\ \overset{}{} \end{array} \begin{array}{c} \\ 1+\mu_n \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \\ 2+\mu_{n-1} \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \\ n+\mu_1 \\ \\ \\ \\ \end{array}.$$

Similarly to the above we can make two observations: We make two observations:

(a) If the entries that we indicate by "*" are all non-zero, then the rows of such a matrix define a point in $\widetilde{\Omega}_\mu$.
(b) If the first non-zero entry for the rows $n, \dots, 1$ occurs in decreasing columns, then the rows of such a matrix are linearly independent, thus the rows of such a matrix define a point in $\widetilde{\Omega}_\nu \subseteq \widetilde{X}_\mu$ for some $\nu \geq \mu$. Conversely any point in $\widetilde{X}_\mu$ is represented by such a matrix.

Using our hypothesis $d(\lambda) + d(\mu) = n \cdot r$ and using the above discussion one can show (without too much effort) the following statements:

(1) If $\lambda \neq \mu^\dagger$, then $X_\lambda \cap \widetilde{X}_\mu = \varnothing$.
(2) If $\lambda = \mu^\dagger$, then $X_\lambda \cap \widetilde{X}_\mu$ consists of a single point P, namely the point which is represented by the matrix where all the red entries are one. This intersection point lies in $\Omega_\lambda \cap \widetilde{\Omega}_\mu$, i.e. it is an intersection point of the smooth submanifolds Ω_λ and $\widetilde{\Omega}_\mu$.

    We leave it to the reader to verify that this is a transverse intersection point. Since we take the intersection point of two complex submanifolds we see (say using the Complex Vector Space-Orientation Lemma 40.16) that the intersection point has sign $+1$. ⊞

We now see that

$$\begin{aligned}\sigma_\lambda \cup \sigma_\mu &\overset{\text{see Notation 271.14}}{=} \langle \mathrm{PD}([X^\lambda]) \cup \mathrm{PD}([X^\mu]), [G_n(\mathbb{C}^{n+r}]\rangle \\ &\underset{\text{by Claim 1}}{=} \langle \mathrm{PD}([X^\lambda]) \cup \mathrm{PD}([\widetilde{X}^\mu]), [G_n(\mathbb{C}^{n+r}]\rangle \underset{\substack{\text{follows from Claim 2 and a variation on the}\\ \text{Cup Product–Intersection Number Theorem 220.10,}\\ \text{alternatively one can alternatively employ the}\\ \text{Cycles–Cup Product-Intersection Number Theorem 221.20}}}{=} \delta_{\delta,\mu^\dagger}\end{aligned}$$

■

271.5. **Real Grassmannians.** In the previous sections we considered the complex Grassmannians $G_n(\mathbb{C}^{n+r})$ in great detail. In this section we will outline what the analogous notions and results are for real Grassmannians $G_n(\mathbb{R}^{n+r})$.

- Given an $(n \times r)$-Schubert symbol the notion of a Schubert variety $X_\lambda \subseteq G_n(\mathbb{R}^{n+r})$ and of a Schubert open cell $\Omega_\lambda \subseteq G_n(\mathbb{R}^{n+r})$ are defined the same way as in Definitions 271.3 and 271.6, we really just need to replace $\mathbb{C}$ by $\mathbb{R}$. Note though that the Schubert open cells now no longer carry a preferred orientation.
- We have the obvious analog of the Schubert Characteristic Map Proposition 271.9.
- Basically the same argument as in the proof of Grassmannian–CW-Structure Proposition 271.10 shows that $G_n(\mathbb{R}^{n+r})$ has a CW-structure where the k-cells are in one-to-one correspondence to $(n \times r)$-Schubert open cells with $d(\lambda) = k$.

Next we summarize the results on the (co-) homology groups of real Grassmannians. First of all we have the following analog of the Schubert Variety-H_*-Lemma 271.11:

Lemma 271.18. (Schubert Variety-$\mathbb{F}_2$-H_*-Lemma) Let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol. Let $\Phi_\lambda \colon \overline{B}^{d(\lambda)} \to X_\lambda$ be the continuous map from the analog of the Schubert Characteristic Map Proposition 271.9. The following maps are isomorphisms:

$$\mathrm{H}_{d(\lambda)}(\overline{B}^{d(\lambda)}, S^{d(\lambda)-1}; \mathbb{F}_2) \xrightarrow{\Phi_{\lambda *}} \mathrm{H}_{d(\lambda)}(X_\lambda, X_\lambda^{d(\lambda)-1}; \mathbb{F}_2) \xleftarrow{\text{я}} \mathrm{H}_{d(\lambda)}(X_\lambda; \mathbb{F}_2).$$

We denote the unique non-zero element of $\mathrm{H}_{d(\lambda)}(X_\lambda; \mathbb{F}_2)$ by $[X_\lambda]_{\mathbb{F}_2}$.

Proof. The proof of this proposition is very similar to the proof of the Schubert Variety-H_*-Lemma 271.11. The biggest difference is that now we do have odd-dimensional cells. We leave it to the reader to show that nonetheless the cellular boundary maps with $\mathbb{F}_2$-coefficients are all zero. Once one has verified this statement, all other statements follow easily. ■

Not surprisingly we continue with the following analog of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 and of the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13:

Proposition 271.19. ($G_n(\mathbb{R}^{n+r})$-H_*-Proposition) Let $n, r \in \mathbb{N}_0$.

(1) (a) For each $k \in \mathbb{N}_0$ the group $\mathrm{H}_k(G_n(\mathbb{R}^{n+r}); \mathbb{F}_2)$ is an $\mathbb{F}_2$-vector space where the dimension is given by the number of $(n \times r)$-Schubert symbols with $d(\lambda) = k$.
(b) The homology classes $[X_\lambda]_{\mathbb{F}_2}$, where λ runs over all $(n \times r)$-Schubert symbols with $d(\lambda) = k$ form a basis for the $\mathbb{F}_2$-vector space $\mathrm{H}_k(G_n(\mathbb{R}^{n+r}); \mathbb{F}_2)$.

(2) (a) The obvious map $G_n(\mathbb{R}^{n+r}) \to G_n(\mathbb{R}^{n+r+1})$ induces a monomorphism of homology groups with $\mathbb{F}_2$-coefficients. This map sends each fundamental class $[X_{(\lambda_1,\ldots,\lambda_n)}]_{\mathbb{F}_2}$ to the corresponding fundamental class $[X_{(\lambda_1+1,\ldots,\lambda_n+1)}]_{\mathbb{F}_2}$.

(b) If $k \leq r$, then the induced map $\mathrm{H}_k(G_n(\mathbb{R}^{n+r});\mathbb{F}_2) \to \mathrm{H}_k(G_n(\mathbb{R}^{n+r+1});\mathbb{F}_2)$ is an isomorphism.

(3) If $k \leq r$, then the induced map $\mathrm{H}_k(G_n(\mathbb{R}^{n+r});\mathbb{F}_2) \to \mathrm{H}_k(G_n(\mathbb{R}^{\infty});\mathbb{F}_2)$ is an isomorphism.

Proof. The proof of this proposition is very similar to the proof of the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13. The biggest difference is that now we do have odd-dimensional cells. As in the proof of the Schubert Variety-$\mathbb{F}_2$-H_*-Lemma 271.18 we leave it to the reader to show that nonetheless the cellular boundary maps with $\mathbb{F}_2$-coefficients are all zero. Once one has verified this statement, all other statements follow easily. ■

We continue with the analog of Notation 271.14:

Notation 271.20. Let $n, r \in \mathbb{N}_0$. Given an $(n \times r)$-Schubert symbol λ we denote by

$$\sigma_\lambda \ := \ \mathrm{PD}_{G_n(\mathbb{R}^{n+r})}([X_\lambda]_{\mathbb{F}_2}) \ \in \ \mathrm{H}^{|\lambda|}(G_n(\mathbb{R}^{n+r});\mathbb{F}_2)$$

the Poincaré dual[2901] of the fundamental class $[X_\lambda]_{\mathbb{F}_2} \in \mathrm{H}_{d(\lambda)}(G_n(\mathbb{R}^{n+r});\mathbb{F}_2)$.

The following lemma is an analog of the $G_n(\mathbb{C}^{n+r})$-H^*-Lemma 271.15 and a generalization of the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 213.17:

Lemma 271.21. ($G_n(\mathbb{R}^{n+r})$-H^*-Lemma) Let $n, r \in \mathbb{N}_0$. Let $k \in \mathbb{N}$. We consider all $(n \times r)$-Schubert symbols λ with $|\lambda| = k$. The corresponding cohomology classes σ_λ form a basis for $\mathrm{H}^k(G_n(\mathbb{R}^{n+r});\mathbb{F}_2)$.

Proof. The lemma follows immediately from the $G_n(\mathbb{R}^{n+r})$-H_*-Proposition 271.19, the Poincaré Duality Theorem 210.1 and the fact that $|\lambda| = n \cdot r - d(\lambda)$. ■

We conclude this section with the following analog of the Schubert Variety Intersection Proposition 271.17:

Proposition 271.22. (Schubert Variety $\mathbb{F}_2$-Intersection Proposition) Let $n, r \in \mathbb{N}_0$ and let λ, μ be two $(n \times r)$-Schubert symbols with $|\lambda| + |\mu| = n \cdot r$. Then

$$\sigma_\lambda \cup \sigma_\mu \ = \ \delta_{\lambda,\mu^\dagger} \cdot [G_n(\mathbb{R}^{n+r})]^*.$$

Here $[G_n(\mathbb{R}^{n+r})]^* \in \mathrm{H}^{n \cdot r}(G_n(\mathbb{R}^{n+r});\mathbb{F}_2)$ is the dual fundamental class of the closed smooth manifold $G_n(\mathbb{R}^{n+r})$ from Definition 201.27.

Proof. Evidently the proof is very similar to the proof of the Schubert Variety Intersection Proposition 271.17. We just need to replace the Cup Product–Intersection Number Theorem 220.10 by the Maps-$\mathbb{F}_2$-Cup Product–Intersection Number Theorem 221.29. ■

271.6. Exercises for Chapter 271.

Exercise 271.1. Let $n, r \in \mathbb{N}_0$.

(a) Show that for the $(n \times r)$-Schubert symbol $(0, \ldots, 0)$ we have

$$X_{(0,\ldots,0)} \ = \ \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_{r+i}) \geq i \text{ for } i = 1, \ldots, n\} \ = \ G_n(\mathbb{C}^{n+r}).$$

[2901] It follows from the $\mathbb{F}_2$-Orientability Proposition 191.22 that we have a Poincaré Duality isomorphism with $\mathbb{F}_2$-coefficients.

(b) Show that for the $(n \times r)$-Schubert symbol $(r, \dots, r)$ we have

$$X_{(r,\dots,r)} = \{W \in G_n(\mathbb{C}^{n+r}) \mid \dim_{\mathbb{C}}(W \cap \mathfrak{F}_i) \geq i \text{ for } i = 1, \dots, n\} = [(\mathfrak{e}_1, \dots, \mathfrak{e}_n)].$$

Exercise 271.2. Let $n, r \in \mathbb{N}_0$ and let $k \in \{0, \dots, r\}$. Show that for the $(n \times r)$-Schubert symbol $(k, 0, \dots, 0)$ we have

$$X_{(k,\dots,0)} = \{W \in G_n(\mathbb{C}^{n+r}) \mid W \cap \mathfrak{F}_{r+1} \neq \{0\}\}.$$

Exercise 271.3. In what sense is a Schubert variety X_λ a variety, i.e. the zero set of a finite set of multivariable polynomials?
Hint. The determinant can be viewed as a polynomial map $\mathrm{M}(k \times k, \mathbb{C}) \to \mathbb{C}$.

Exercise 271.4. Determine the rank of $\mathrm{H}_2(G_3(\mathbb{C}^{3+r}))$ for $r = 0, 1, 2, 3, 4, 5$.

Exercise 271.5. Let $n, r \in \mathbb{N}_0$.

(a) Show that the number of $(n \times r)$-Schubert symbols equals $\binom{n+r}{n}$.

(b) Determine the Euler characteristic of $G_n(\mathbb{C}^{n+r})$.

Exercise 271.6. Let $n, r \in \mathbb{N}_0$ and let $s \in \{0, \dots, r\}$. For which $(n \times r)$-Schubert symbol λ do we get $X_\lambda = G_n(\mathbb{C}^{n+s}) \times \{0\} \subseteq G_n(\mathbb{C}^{n+r})$?

Exercise 271.7.

(a) Determine the intersection pairing of $G_2(\mathbb{C}^4)$.

(b) Determine the signature of $G_2(\mathbb{C}^4)$.

Exercise 271.8. Let $n \in \mathbb{N}$ and let $r \in \mathbb{N}_0$.

(a) Show that the inclusion induced map $\mathrm{H}^j(G_n(\mathbb{C}^\infty); \mathbb{Z}) \to \mathrm{H}^j(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$ is non-zero for some $j \in \mathbb{N}$.

(b) Show that the tautological vector bundle $\mathfrak{T}G_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r})$ is non-trivial.
Remark. This gives a new solution to Exercise 270.18.

Exercise 271.9. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a compact topological space B which is Hausdorff. In Exercise 270.5 we showed that there exists a $\mathbb{K}$-vector bundle $q\colon F \to B$ such that $E \oplus F$ is trivializable. Show that in general this conclusion does not hold if B non-compact.
Hint. Make use of Exercise 270.6

CHAPTER 272

The cohomology ring of infinite Grassmannians

In the previous chapter we determined the (co-) homology groups of the Grassmannians $G_n(\mathbb{K}^d)$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and for any $d \in \mathbb{N}_0 \cup \{\infty\}$. In particular we showed that Schubert varieties give rise to a basis of the homology and cohomology groups. Furthermore in the Schubert Variety Intersection Proposition 271.17 we calculated the cup product of the cohomology classes (in opposite dimensions) corresponding to Schubert varieties. This calculation though is not completely satisfactory, since it is not clear at all, what the cohomology ring of say $G_n(\mathbb{K}^\infty)$ really looks like.

In this chapter, which contains lots of clever algebra, we will deal with this issue. At the end of the chapter we will have found a very satisfactory description of the cohomology ring of the infinite-dimensional Grassmannians $G_n(\mathbb{K}^\infty)$. These calculations are the key to introducing the characteristic classes of vector bundles in Chapter 274.

As in the previous chapter we will discuss for the most time the case $\mathbb{K} = \mathbb{C}$. At the end of the chapter we will outline what changes one needs to perform for the case $\mathbb{K} = \mathbb{R}$.

This chapter is organized as follows:

- In Section 272.1 we will discuss the isomorphism type of the cohomology groups of the infinite-dimensional complex Grassmannians $G_n(\mathbb{C}^\infty)$.
- In Section 272.2 we will leave topology for a minute and we immerse ourselves into the algebraic theory of symmetric polynomials.
- In Section 272.3 we will study the Pieri formula for Schur functions and for the cohomology ring of complex Grassmannians.
- In Section 272.4 we will use the algebraic tools from Section 272.2 and the calculations from Section 272.3 to calculate the cohomology rings of the complex Grassmannians $G_n(\mathbb{C}^\infty)$.
- In Section 272.5 we will summarize the results for the infinite-dimensional *real* Grassmannians $G_n(\mathbb{R}^\infty)$.
- In Section 272.6 we will provide exercises for Chapter 272.

In the next Chapter 273 we will discuss the splitting principle and Whitney sums.

272.1. **The cohomology groups of infinite-dimensional Grassmannians.** In this section we want to determine a basis for the cohomology groups of infinite-dimensional Grassmannians. Evidently it is worth to first recall the calculation of the cohomology groups for finite-dimensional Grassmannians. To do so, let us recall the following notation:

Notation 271.14 Let $n, r \in \mathbb{N}_0$. Given an $(n \times r)$-Schubert symbol λ we denote by

$$\sigma_\lambda \ := \ \mathrm{PD}_{G_n(\mathbb{C}^{n+r})}([X_\lambda]) \ \in \ \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$$

the Poincaré dual of the fundamental class $[X_\lambda] \in \mathrm{H}_{2d(\lambda)}(G_n(\mathbb{C}^{n+r}))$ that we introduced in Notation 271.12.

Next we recall the following lemma:

Lemma 271.15. ($G_n(\mathbb{C}^{n+r})$-H^*-Lemma) Let $n, r \in \mathbb{N}_0$.
(1) The odd-dimensional cohomology groups of $G_n(\mathbb{C}^{n+r})$ are zero.
(2) Let $k \in \mathbb{N}$. We consider all $(n \times r)$-Schubert symbols λ with $|\lambda| = k$. The corresponding cohomology classes σ_λ form a basis for $\mathrm{H}^{2k}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$.

The question arises, how do these cohomology classes behave under inclusion induced maps? The following lemma, which is a generalization of the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5, gives us a very satisfactory answer:

Lemma 272.1. ($G_n(\mathbb{C}^{n+r})$-H^*-Maps-Lemma) Let $n, r \in \mathbb{N}_0$. In the following we denote by $\iota\colon G_n(\mathbb{C}^{n+r}) \to G_n(\mathbb{C}^{n+r+1})$ the usual embedding. For any $(n \times r)$-Schubert symbol λ we have

$$\iota^*(\sigma_\lambda) \;=\; \sigma_\lambda \;\in\; \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z}).$$

Here to the left we consider the cohomology class $\sigma_\lambda \in \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r+1}); \mathbb{Z})$ and to the right we consider the cohomology class $\sigma_\lambda \in \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$.

Proof. Let $n, r \in \mathbb{N}_0$ and let λ be an $(n \times r)$-Schubert symbol. We need to show that for the cohomology class $\sigma_\lambda \in \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r+1}); \mathbb{Z})$ we have the equality

$$\iota^*(\sigma_\lambda) \;=\; \sigma_\lambda \;\in\; \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z}).$$

Note that it follows from the Universal Coefficient-H^*-Theorem 201.23 together with the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13 that it suffices to prove the following claim:

Claim. For every $(n \times r)$-Schubert symbol μ with $d(\mu) = |\lambda|$ we have

$$\langle \iota^*(\sigma_\lambda), [X_\mu] \rangle_{G_n(\mathbb{C}^{n+r})} \;=\; \langle \sigma_\lambda, [X_\mu] \rangle_{G_n(\mathbb{C}^{n+r})}.$$

Proof. We start out by reminding the reader of a general statement: Let M be a closed oriented m-dimensional smooth manifold, let $\Theta \in \mathrm{H}^k(M; \mathbb{Z})$ and let $X \subseteq M$ be a closed oriented k-dimensional smooth submanifold. As on page 4082 we see that

$$(*) \qquad \langle \Theta, [X] \rangle_M \underset{\text{by Definition 210.2 and by the Cap Product-Kronecker Lemma 206.12}}{=} \Theta \cap (\mathrm{PD}_M([X] \cap [M])) \underset{\text{by the Cup-Cap Lemma 206.15, the Cup Product-Commutativity Proposition 203.20 and the Cap Product-Kronecker Lemma 206.12}}{=} \langle \Theta \cup \mathrm{PD}_M([X]), [M] \rangle_M.$$

Now we start with the actual proof of the claim. In the following let $\mu = (\mu_1, \ldots, \mu_n)$ be an $(n \times r)$-Schubert symbol with $d(\mu) = |\lambda|$. We consider the $(n \times (r+1))$-Schubert symbol

$\widetilde{\mu} = (\mu_1 + 1, \ldots, \mu_n + 1)$. We see that

$$\begin{array}{lll}
\langle \iota^*(\sigma_\lambda), [X_\mu] \rangle_{G_n(\mathbb{C}^{n+r})} & & \\
\quad \overset{\text{by the Kronecker Pairing–Natural Lemma 199.9}}{\downarrow} & & \overset{\text{by the } G_n(\mathbb{C}^{n+r})\text{-H}_* \text{ Proposition 271.13 (2a)}}{\downarrow} \\
\quad = \langle \sigma_\lambda, \iota_*([X_\mu]) \rangle_{G_n(\mathbb{C}^{n+r+1})} & & = \langle \sigma_\lambda, [X_{\widetilde{\mu}}] \rangle_{G_n(\mathbb{C}^{n+r+1})} \\
\quad = \langle \sigma_\lambda \cup \sigma_{\widetilde{\mu}}, G_n(\mathbb{C}^{n+r+1}) \rangle & & = \delta_{\lambda, \widetilde{\mu}^\dagger} \text{ with } \widetilde{\mu}^\dagger \text{ defined in } (n \times (r+1)) \\
\quad \underset{\text{by (*) and Notation 271.14}}{\uparrow} & & \underset{\text{by the Schubert Variety Intersection Proposition 271.17}}{\uparrow} \\
\quad = \delta_{\lambda, \mu^\dagger} \text{ with } \mu^\dagger \text{ defined in } (n \times (r)) & & = \langle \sigma_\lambda \cup \sigma_\mu, G_n(\mathbb{C}^{n+r}) \rangle \\
\quad \underset{\text{follows easily from Definition 271.16}}{\uparrow} & & \underset{\text{by the Schubert Variety Intersection Proposition 271.17}}{\uparrow} \\
\quad = \langle \sigma_\lambda, [X_\mu] \rangle_{G_n(\mathbb{C}^{n+r})} & & \\
\quad \underset{\text{by (*) and Notation 271.14}}{\uparrow} & & \blacksquare
\end{array}$$

To describe the cohomology of the infinite-dimensional Grassmannians $G_n(\mathbb{C}^\infty)$ it is convenient to introduce the following variation on the Definition 271.2 of an $(n \times r)$-Schubert symbol:

Definition 272.2. Let $n \in \mathbb{N}_0$.

(1) An $(n \times \infty)$-Schubert[2908] symbol λ is a sequence $(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{N}_0$ such that we have the inequalities $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.[2909]

(2) As in Definition 271.4 we call $|\lambda| := \lambda_1 + \cdots + \lambda_n$ the weight of λ.

Evidently every $(n \times r)$-Schubert symbol is also an $(n \times \infty)$-Schubert symbol.

Now we can describe the cohomology groups of the infinite Grassmannians $G_n(\mathbb{C}^\infty)$:[2910]

Lemma 272.3. ($G_n(\mathbb{C}^\infty)$-H*-Lemma) Let $n \in \mathbb{N}_0$.

(1) The odd-dimensional cohomology groups of $G_n(\mathbb{C}^\infty)$ are zero.

(2) (a) For each $k \leq r$ the inclusion induced map $\mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z}) \to \mathrm{H}^{2k}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$ is an isomorphism.

(b) For every $(n \times \infty)$-Schubert symbol $\lambda = (\lambda_1, \ldots, \lambda_n)$ there exists a unique cohomology class $\sigma_\lambda \in \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ with the property that for each $r \geq \lambda_1$ we have the equality[2911] $\iota^*(\sigma_\lambda) = \sigma_\lambda \in \mathrm{H}^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$.

(c) Let $k \in \mathbb{N}$. The cohomology classes σ_λ from (b), where λ runs over all $(n \times \infty)$-Schubert symbols of weight k, form a basis for $\mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z})$.

Proof. Let $n \in \mathbb{N}_0$.

(1) Recall that in the Grassmannian–CW-Structure Proposition 271.10 we equipped $G_n(\mathbb{C}^\infty)$ with a CW-structure with no odd-dimensional cells. Thus the statement follows from the Singular-Cellular-H*-Isomorphism Proposition 199.5.

[2909] The notion of an $(n \times \infty)$-Schubert symbol seems to be non-standard. The literature, e.g. [**Mani2001**], often works with the closely related notion of a partition of length $\leq n$.

[2910] Note that the following lemma can be viewed as a generalization of the $\mathbb{C}\mathrm{P}^n$-H*-Lemma 213.5.

[2911] Note that $r \geq \lambda_1$ implies that λ is also an $(n \times r)$-Schubert symbol.

[2911] For many of the statements of the lemma there are also alternative proofs. For example for the proofs of some statements one could also use the Exhaustions-H*-Proposition 230.11 together with the Mittag-Leffler Limit Lemma 230.8.

(2) (a) This statement follows from the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13 together with the Universal Coefficient-H^*-Theorem 201.23.

(b) Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an $(n \times \infty)$-Schubert symbol. We pick an $r \in \mathbb{N}_0$ with $r \geq \lambda_1$. Note that this implies in particular that λ is an $(n \times r)$-Schubert symbol. We consider $\sigma_\lambda \in H^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$ as in Notation 271.14. By (2a) there exists a unique $\sigma_\lambda \in H^{2|\lambda|}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ such that the restriction to $H^{2|\lambda|}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$ equals σ_λ. It follows easily from the $G_n(\mathbb{C}^{n+r})$-H^*-Maps-Lemma 272.1 and the functoriality of cohomology that this definition does not depend on the choice of r and that this cohomology class $\sigma_\lambda \in H^{2|\lambda|}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ has the promised properties.

(c) This statement follows from (2a) together with the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13.

■

272.2. **Interlude: Symmetric polynomials.** In the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.3 we just calculated the cohomology groups of the complex Grassmannian $G_n(\mathbb{C}^\infty)$. It is not really clear though what the cup product structure on $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z})$ should be. In the Cup Product-$\mathbb{C}P^n$-Proposition 213.7 we calculated the cup product on the projective space $\mathbb{C}P^\infty$ and we saw that the cohomology ring is given by the polynomial ring $\mathbb{Z}[x]$ where x is a generator of second cohomology. It is thus not unreasonable to expect that the cohomology ring of the infinite-dimensional Grassmannians $G_n(\mathbb{C}^\infty)$ is also some type of polynomial ring.

In this purely algebraic section we will collect some facts about the ring of (symmetric) polynomials, which will come in handy in the next sections.[2912]

Definition 272.4. Let $n \in \mathbb{N}_0$. We refer to

$$\mathbb{Z}[x_1, \dots, x_n]^{\text{sym}} := \{p \in \mathbb{Z}[x_1, \dots, x_n] \,|\, p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = p(x_1, \dots, x_n) \text{ for all } \sigma \in S_n\}$$

as the ring of symmetric polynomials in n variables.

Definition 272.5. Let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}$ we define

$$\text{elementary symmetric polynomial } e_k := \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdot \ldots \cdot x_{i_k} \in \mathbb{Z}[x_1, \dots, x_n]^{\text{sym}}$$

$$\text{complete symmetric polynomial } h_k := \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \cdot \ldots \cdot x_{i_k} \in \mathbb{Z}[x_1, \dots, x_n]^{\text{sym}}.$$

Furthermore we set $e_0 := 1$ and $h_0 := 1$. Note that $e_k = 0$ for $k > n$. Sometimes, to highlight the polynomial ring we are in, we write $e_k(x_1, \dots, x_n)$ and $h_k(x_1, \dots, x_n)$.

[2912] For the reader who in the following acquires a taste for this type of algebra, we refer to [**Mani2001**, Chapter 1], [**Pro2007**] and [**MacD1998**, Chapter 1] for more details.

Example. For $n = 3$ we obtain the following table of elementary and complete symmetric polynomials:

	e_k	h_k
$k = 1$	$x_1 + x_2 + x_3$	$x_1 + x_2 + x_3$
$k = 2$	$x_1x_2 + x_1x_3 + x_2x_3$	$x_1x_2 + x_1x_3 + x_2x_3 + x_1^2 + x_2^2 + x_3^2$
$k = 3$	$x_1x_2x_3$	$x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + \ldots + x_1x_3^2 + x_2x_3^2 + x_1^3 + x_2^3 + x_3^3$
$k = 4$	0	$x_1^2x_2x_3 + \cdots + x_3^4$
$\vdots$	$\vdots$	$\vdots$

□

The following theorem is most likely well-known from an earlier algebra course:

Theorem 272.6. (Fundamental Theorem-of-Symmetric Polynomials) Let $n \in \mathbb{N}_0$. The unique ring homomorphism

$$\begin{array}{rcl} \mathbb{Z}[e_1, \ldots, e_n] & \to & \mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{sym}} \\ e_i & \mapsto & e_i(x_1, \ldots, x_n) \end{array}$$

is an isomorphism of rings.

Proof. This proposition is proved in every self-respecting book or course on algebra. For example a proof is given in [**Lang2002b**, Chapter IV.6]. ■

We will also need the following lesser known variation on the Fundamental Theorem-of-Symmetric Polynomials 272.6:

Proposition 272.7. (Complete Symmetric Function Proposition) Let $n \in \mathbb{N}_0$. The unique ring homomorphism

$$\begin{array}{rcl} \mathbb{Z}[h_1, \ldots, h_n] & \to & \mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{sym}} \\ h_i & \mapsto & h_i(x_1, \ldots, x_n) \end{array}$$

is an isomorphism of rings.

Sketch of proof. In the following we provide the argument that is given in [**Mani2001**, p. 9] and [**Stanl2001**, Theorem 7.6.1]. By the Fundamental Theorem-of-Symmetric Polynomials 272.6 we have a ring homomorphism

$$\begin{array}{rcl} \Xi\colon \overbrace{\mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{sym}}}^{=\mathbb{Z}[e_1,\ldots,e_n]} & \to & \mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{sym}} \\ e_i & \mapsto & h_i. \end{array}$$

We claim that this ring homomorphism Ξ is an involution, i.e. we claim that $\Xi^2 = \mathrm{id}$. We start out with a few preparations:

(1) Given any commutative ring R we introduce the Laurent series ring

$$R[\![t]\!] := \left\{ \sum_{i=0}^{\infty} a_i \cdot t^i \,\middle|\, a_0, a_1, \cdots \in R \right\}.$$

Note that in this ring we have for any $r \in R$ the equality

$$(1 - r \cdot t) \cdot \sum_{k \in \mathbb{N}_0} r^i \cdot t^i = 1, \quad \text{in other words:} \quad (1 - r \cdot t)^{-1} = \sum_{k \in \mathbb{N}_0} r^i \cdot t^i.$$

(2) It is elementary to obtain from Definition 272.5 that we have the following two identities in the ring $\mathbb{Z}[x_1,\ldots,x_n][t]$:

$$e(t) := \sum_{k\in\mathbb{N}_0} e_k\cdot t^k = \prod_{i=1}^n (1+x_i\cdot t) \quad \text{and} \quad h(t) := \sum_{k\in\mathbb{N}_0} h_k\cdot t^k = \prod_{i=1}^n \underbrace{(1-x_i\cdot t)^{-1}}_{=1+x_i\cdot t+x_i^2\cdot t^2+\ldots}.$$

(3) Note that in the ring $\mathbb{Z}[x_1,\ldots,x_n][t]$ we have the identity

$$e(t)\cdot h(-t) = \prod_{i=1}^n (1+x_i\cdot t)\cdot\prod_{i=1}^n (1-x_i\cdot(-t))^{-1} = 1.$$

(4) Let $k\in\mathbb{N}$. By considering the t^k-coefficient of the equality $e(t)\cdot h(-t)=1$ in the ring $\mathbb{Z}[x_1,\ldots,x_n][t]$ we obtain that

$$\sum_{i+j=k} (-1)^j\cdot e_i\cdot h_j = 0.$$

Note that this relation is symmetric with respect to e and h. It follows from an elementary induction argument that if $h_k = P_k(e_1,\ldots,e_n)$ for a polynomial $P_k\in\mathbb{Z}[y_1,\ldots,y_n]$, then we also have $e_k = P(h_1,\ldots,h_n) = \Xi(h_k)$.

It follows from this discussion that the ring homomorphism

$$\begin{array}{rcl}\Xi\colon \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}} & \to & \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}}\\ e_i & \mapsto & h_i\end{array}$$

is an involution. But this implies in particular that Ξ is an isomorphism of rings. ■

To understand the ring of symmetric polynomials better we need to introduce the following notation:

Definition 272.8.

(1) Given $(\lambda_1,\ldots,\lambda_n)\in\mathbb{N}_0^n$ we write

$$A_\lambda = \left(x_i^{\lambda_j+n-j}\right)_{i,j=1,\ldots n} = \begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \ldots & x_1^{\lambda_n}\\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \ldots & x_2^{\lambda_n}\\ \vdots & \vdots & & \vdots\\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \ldots & x_n^{\lambda_n}\end{pmatrix}.$$

Note that $A_{(0,\ldots,0)}$ is the popular Vandermonde matrix, i.e. we have

$$A_{(0,\ldots,0)} = \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \ldots & 1\\ x_2^{n-1} & x_2^{n-2} & \ldots & 1\\ \vdots & \vdots & & \vdots\\ x_n^{n-1} & x_n^{n-2} & \ldots & 1\end{pmatrix}.$$

(2) Given $(\lambda_1,\ldots,\lambda_n)\in\mathbb{N}_0^n$ we consider the Schur function

$$s_\lambda := \frac{s_\lambda}{s_{(0,\ldots,0)}} = \frac{\det(A_\lambda)}{\det(A_{(0,\ldots,0)})}.$$

Note that a priori s_λ lies in the quotient field $\mathbb{Q}(x_1,\ldots,x_n)$.

Example. We consider the case $n = 2$. Given $k \in \mathbb{N}_0$ we consider the $(2 \times \infty)$-Schubert symbol $\lambda = (k, 0)$. We see that the Schur function is given by

$$s_\lambda \;=\; s_{(k,0)} \;=\; \frac{\det\begin{pmatrix} x_1^{k+1} & 1 \\ x_2^{k+1} & 1 \end{pmatrix}}{\det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix}} \;=\; \frac{x_1^{k+1} - x_2^{k_1}}{x_1 - x_2} \;=\; x_1^k + x_1^{k-1} \cdot x_2 + \cdots + x_2^k \;=\; h_k(x_1, x_2).$$

elementary algebra — see Definition 272.5 □

The next proposition generalizes the above example:

Proposition 272.9. (Schur Function–Complete Symmetric Proposition) Let $k \in \mathbb{N}$. For the $(n \times \infty)$-Schubert symbol $\lambda = (k, 0, \ldots, 0)$ the corresponding Schur function equals the complete symmetric function $h_k(x_1, \ldots, x_n)$, i.e. we have

$$s_{(k,0,\ldots,0)} \;=\; h_k(x_1, \ldots, x_n).$$

Sketch of proof. There does not seem to be a completely elementary proof of the proposition. One approach is to observe that the proposition can be obtained from the more general Jacobi-Trudi formula, see [**Bum2013**, Theorem 35.1] and [**Mani2001**, Formulas 1.2.13]. Alternatively the identity can be seen as a consequence of the Cauchy identity, see [**Stanl2001**, Theorem 7.21.1], [**LLMSSZ2014**, Chapter 6.1] and [**Pro2007**, p. 280]. ■

The following proposition contains some basic facts about Schur functions:

Proposition 272.10. (Schur Function Basis Proposition)

(1) For each $(n \times \infty)$-Schubert symbol λ the corresponding Schur function s_λ lies in the subring $\mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}}$ of $\mathbb{Q}(x_1, \ldots, x_n)$.
(2) The Schur functions s_λ, corresponding to all $(n \times \infty)$-Schubert symbols form a basis of the *free abelian group* $\mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}}$.

Sketch of proof. We start out by recalling the following concepts and facts:

(a) As we mentioned above, $A_{(0,\ldots,0)}$ is just the Vandermonde matrix. It is well-known, see e.g. [**Came2008**, Proposition 4.27], that its determinant equals the Vandermonde polynomial
$$V(x_1, \ldots, x_n) \;:=\; \prod_{1 \leq i < j \leq n} (x_i - x_j).$$
(b) We set $\delta = (n-1, \ldots, 2, 1, 0)$.
(c) Given $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$ we consider, the polynomial
$$a_\nu \;:=\; \det(x_i^{\nu_j})_{i,j=1,\ldots,n} \;=\; \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot x_1^{\nu_{\sigma(1)}} \cdot \ldots \cdot x_n^{\nu_{\sigma(n)}}.$$
Note that it follows immediately from the definitions that for any $\mu \in \mathbb{N}_0^n$ we have the equality $\det(A_\mu) = a_{\nu+\delta}$.
(d) A polynomial $p(x_1, \ldots, x_n)$ is called *alternating* if for every permutation $\sigma \in S_n$ we have $p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \operatorname{sign}(\sigma) \cdot p(x_1, \ldots, x_n)$ where $\operatorname{sign}(\sigma)$ is the sign of the permutation σ. Clearly the Vandermonde polynomial $V(x_1, \ldots, x_n)$ is alternating. Furthermore note that it follows immediately from the multilinearity of the determinant that any a_λ is alternating.
(e) We denote by $\mathbb{Z}[x_1, \ldots, x_n]^{\text{alt}}$ the ring of alternating polynomials.

(f) As we will see in Exercise 272.1, it is pretty elementary to show that the map

$$\begin{array}{rcl} \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}} & \to & \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{alt}} \\ p(x_1,\ldots,x_n) & \mapsto & p(x_1,\ldots,x_n)\cdot \underbrace{V(x_1,\ldots,x_n)}_{=\det(A_{(0,\ldots,0)})} \end{array}$$

defines an isomorphism between the abelian group $\mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}}$ of symmetric polynomials and the abelian group $\mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{alt}}$ of alternating polynomials. Alternatively see [**Pro2007**, p. 28] for more details.

(g) A sequence $\mu=(\mu_1,\ldots,\mu_n)$ is called strictly decreasing if we have $\mu_1>\cdots>\mu_n\geq 0$. As we will see in Exercise 272.2, it is elementary to see that the alternating polynomials $a_\mu=\det(A_\mu)$ corresponding to the strictly decreasing sequences form a basis of the abelian group $\mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{alt}}$ of alternating polynomials. Alternatively see [**Pro2007**, p. 29] for more details.

(h) The assignment

$$\underbrace{(\mu_1,\ldots,\mu_n)}_{=\mu} \mapsto \underbrace{(\mu_1+n-1,\mu_2+n-2,\ldots,\mu_n)}_{\mu+\delta}$$

defines a bijection between the set of decreasing sequences and the set of strictly decreasing sequences of length n in $\mathbb{N}_0$.



Now we turn to proofs of our two statements:

(1) This statement follows easily from (a)–(f).

(2) This statement follows easily from (a)–(h).

The statements are also proved, with more authority (but not necessarily with more details), in [**Mani2001**, p. 10], [**MacD1998**, p. 8] and [**Pro2007**, Chapter 2.3.2]. ■

If we compare the Schur Function Basis Proposition 272.10 (2) with the earlier $G_n(\mathbb{C}^\infty)$-H*-Lemma 272.3, then we are lead to the following lemma:

Lemma 272.11. ($G_n(\mathbb{C}^\infty)$-H*-Lemma) For any $n\in\mathbb{N}_0$ the map

$$\begin{array}{rccl} \Theta\colon & \overbrace{\mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}}}^{=\bigoplus_\lambda \mathbb{Z}\cdot s_\lambda} & \to & \mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \\ & s_\lambda & \mapsto & \sigma_\lambda \\ & \uparrow & & \\ & \lambda \text{ is an } (n\times\infty)\text{-Schubert symbol} & & \end{array}$$

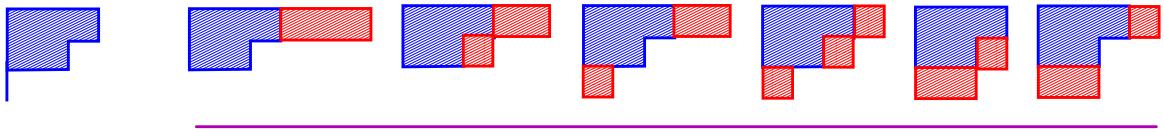
is an isomorphism of *abelian groups*.

Proof. This lemma follows immediately from the original $G_n(\mathbb{C}^\infty)$-H*-Lemma 272.3 together with the Schur Function Basis Proposition 272.10 (2). ■

Note that in the $G_n(\mathbb{C}^\infty)$-H*-Lemma 272.11 we consider two rings, namely $\mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}}$ and $\mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z})$, and an isomorphism of *abelian groups* between them. This begs the question, whether this map is actually an isomorphism of *rings*. In the next section we will acquire the tools to answer this question.

272.3. **The Pieri formula.** [2913] We adopt the following notation from [**Mani2001**, p. 11]:

Notation 272.12. Let $n \in \mathbb{N}$. Given an $(n \times \infty)$-Schubert symbol λ and given $k \in \mathbb{N}_0$ we denote by $\lambda \otimes k$ the set of all $(n \times \infty)$-Schubert symbols which we obtain from λ by adding at most one box per column.



Schubert symbol λ Schubert symbols in $\lambda \otimes k$ with $k = 3$

Proposition 272.13. (Polynomial Pieri Formula) Let $n \in \mathbb{N}$. Given any $(n \times \infty)$-Schubert symbol λ and given any $k \in \mathbb{N}_0$ we have the following equality in the ring $\mathbb{Z}[x_1, \ldots, x_n]^{\text{sym}}$ of symmetric polynomials:

$$s_\lambda \cdot \underbrace{h_k}_{=s_{(k,0,\ldots,0)}} = \sum_{\nu \in \lambda \otimes k} s_\nu.$$

Proof. In the following we provide, with more details and less confidence, the proof given in [**Mani2001**, p. 11]. Let $n \in \mathbb{N}$, let λ be an $(n \times \infty)$-Schubert symbol and let $k \in \mathbb{N}_0$. We recall some notation:

- Given $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$ we consider, as on page 5254, the alternating polynomial
$$a_\nu := \det(x_i^{\nu_j})_{i,j=1,\ldots,n} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot x_1^{\nu_{\sigma(1)}} \cdot \ldots \cdot x_n^{\nu_{\sigma(n)}}.$$
- We set $\delta := (n-1, \ldots, 2, 1, 0)$.

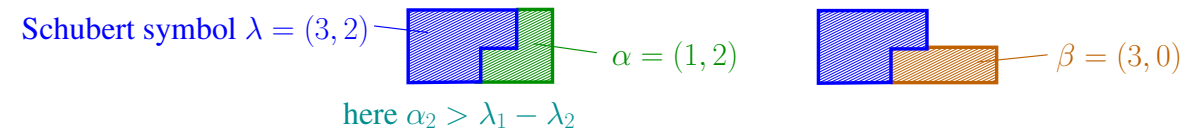
Note that by Definition 272.8 we have $s_\nu = \frac{a_{\nu+\delta}}{a_\delta}$. Thus it suffices to prove the following claim:

Claim. $$a_{\lambda+\delta} \cdot h_k = \sum_{\nu \in \lambda \otimes k} a_{\nu+\delta}.$$

Proof. First note that we have the identity

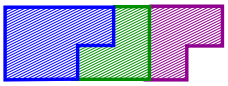
$$a_{\lambda+\delta} \cdot h_k \overset{\text{follows easily from Definition 272.5}}{\downarrow} = a_{\lambda+\delta} \cdot \sum_{\substack{(\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n \\ \alpha_1+\cdots+\alpha_n = k}} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \overset{\text{follows from the multilinearity of the determinant}}{\downarrow} = \sum_{\substack{(\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n \\ \alpha_1+\cdots+\alpha_n = k}} a_{\lambda+\alpha+\delta}.$$

We need to show that the "excess summands" cancel. Thus suppose that we have an $\alpha = (\alpha_1, \ldots, \alpha_n)$ as above such that $\lambda + \alpha$ does not lie in $\nu \otimes k$. It follows easily from the definitions that this implies there exists an $i \in \mathbb{N}_0$ with $\alpha_{i+1} > \lambda_i - \lambda_{i+1}$. We define a new sequence $\beta \in \mathbb{N}_0^n$ by setting $\beta_i = \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1)$, $\beta_{i+1} = \alpha_i + (\lambda_i - \lambda_{i+1} + 1)$ and $\beta_j = \alpha_j$ for $j \neq i, i+1$.

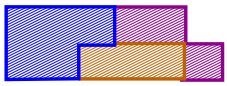


[2913] That's the section where some arguments are still a little vague.

Next we consider the sequences $\lambda + \alpha + \delta$ and $\lambda + \beta + \delta$. Note that these are the same, except that the i and the $i+1$-entry are swapped. It follows from the multilinearity of the determinant that $a_{\lambda+\alpha+\delta} = -a_{\lambda+\beta+\delta}$.



$\lambda + \alpha + \delta$



$\lambda + \beta + \delta$

Finally note that an elementary argument shows that $\lambda + \beta$ does not lie in $\nu \otimes k$ either. Thus the contributions of α and β cancel. ■

The Polynomial Pieri Formula 272.13 raises the question, whether the Polynomial Pieri Formula 272.13 also holds in the context of the cohomology groups of Grassmannians. The following proposition shows that this is indeed the case:

Proposition 272.14. (Grassmann Pieri Formula) Let $n \in \mathbb{N}_0$. Given any $(n \times \infty)$-Schubert symbol λ and given any $k \in \mathbb{N}_0$ we have the following equality in the cohomology group $\mathrm{H}^{2(|\lambda|+k)}(G_n(\mathbb{C}^\infty); \mathbb{Z})$:

$$\sigma_\lambda \cup \sigma_{(k,0,\ldots,0)} = \sum_{\nu \in \lambda \otimes k} \sigma_\nu.$$

Proof. [2914] We set $r := |\lambda| + k$. By the $G_n(\mathbb{C}^\infty)$-H*-Lemma 272.3 (2) we know that the inclusion induced map

$$\mathrm{H}^{2(|\lambda|+k)}(G_n(\mathbb{C}^\infty); \mathbb{Z}) \to \mathrm{H}^{2(|\lambda|+k)}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$$

is an isomorphism. Thus it suffices to verify the promised equality

$$\sigma_\lambda \cup \sigma_{(k,0,\ldots,0)} = \sum_{\nu \in \lambda \otimes k} \sigma_\nu$$

in $\mathrm{H}^{2(|\lambda|+k)}(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$.

By the Schubert Variety Intersection Proposition 271.17 together with the $G_n(\mathbb{C}^{n+r})$-H$_*$-Proposition 271.13 and the Intersection Form–Non Singularity Proposition 224.14 it suffices to show that the cup products of the left and right-hand side with all σ_μ where $|\mu| = n \cdot r - |\lambda| - k$ are the same.

First we consider the cup products with the right hand side:

Claim 1. Let μ be an $(n \times r)$-Schubert symbol with $|\mu| = n \cdot r - |\lambda| - k$. We consider the condition

$$(*) \qquad r - \lambda_n \geq \mu_1 \geq r - \lambda_{n-1} \geq \mu_2 \geq \cdots \geq r - \lambda_1 \geq \mu_n.$$

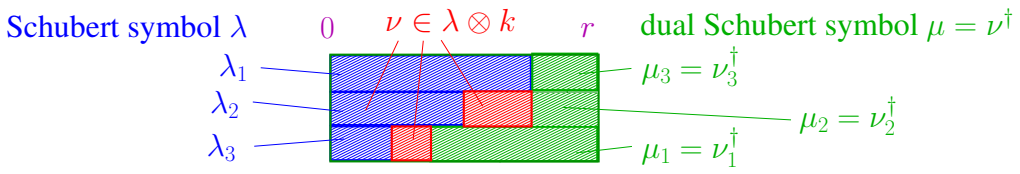
Then

$$\sum_{\nu \in \lambda \otimes k} \sigma_\nu \cup \sigma_\mu = \begin{cases} 1, & \text{if } (*) \text{ is satisfied,} \\ 0, & \text{if } (*) \text{ is not satisfied.} \end{cases}$$

Proof. First we recall some notation: Given any $(n \times r)$-Schubert symbol $\nu = (\nu_1, \ldots, \nu_n)$ we consider, as in Definition 271.16, the dual Schubert symbol $\nu^\dagger := (r - \nu_n, \ldots, r - \nu_1)$. Next note that elementary logic shows that given any $(n \times r)$-Schubert symbol μ with $|\mu| = n \cdot r - |\lambda| - k$ the following two statements are equivalent:

$(*)$ μ satisfies $(*)$.
$(\star)$ There exists a $\nu \in \lambda \otimes k$ such that $\nu^\dagger = \mu$.

[2914] In the following we try our best to follow the argument in [**Mani2001**, p. 109].



We now see that

$$\sum_{\nu\in\lambda\otimes k} \sigma_\nu \cup \sigma_\mu \;\underset{\uparrow}{=}\; \begin{cases} 1, & \text{if } (\star) \text{ is satisfied,} \\ 0, & \text{if } (\star) \text{ is not satisfied.} \end{cases} \;\underset{\uparrow}{=}\; \begin{cases} 1, & \text{if } (*) \text{ is satisfied,} \\ 0, & \text{if } (*) \text{ is not satisfied.} \end{cases}$$

follows easily from the Schubert Variety Intersection Proposition 271.17 — by the above observation

⊞

By Claim 1 together with the Cup Product-Commutativity Proposition 203.20 it suffices to prove the following claim:

Claim 2. Let μ be an $(n \times r)$-Schubert symbol with $|\mu| = n \cdot r - |\lambda| - k$. As in Claim 1 we consider the condition

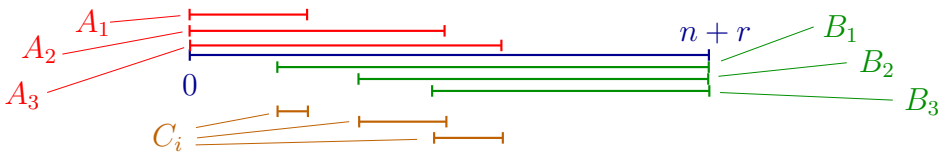
$$(*) \qquad r - \lambda_n \geq \mu_1 \geq r - \lambda_{n-1} \geq \mu_2 \geq \cdots \geq r - \lambda_1 \geq \mu_n.$$

Then

$$\sigma_\lambda \cup \underbrace{\sigma_\mu \cup \sigma_{(k,0,\ldots,0)}}_{=\sigma_{(k,0,\ldots,0)}\cup\sigma_\mu} = \begin{cases} 1, & \text{if } (*) \text{ is satisfied,} \\ 0, & \text{if } (*) \text{ is not satisfied.} \end{cases}$$

Proof. For $i = 1, \ldots n$ we consider the following vector subspaces of $\mathbb{C}^{n+r}$:

$$\begin{array}{rcll} A_i & = & \operatorname{span}\{\mathfrak{e}_1, \ldots, \mathfrak{e}_{r+i-\lambda_i}\} & = \overbrace{\mathbb{C}^{r+i-\lambda_i} \times \{0\}}^{=\delta_{r+i-\lambda_i}}, \\ B_i & = & \operatorname{span}\{\mathfrak{e}_{\mu_{n+1-i}+i}, \ldots, \mathfrak{e}_{n+r}\} & = \{0\} \times \mathbb{C}^{n+r+1-i-\mu_{n+1-i}}, \\ C_i & := & \operatorname{span}\{\mathfrak{e}_{\mu_{n+1-i}+i}, \ldots, \mathfrak{e}_{r+i-\lambda_i}\} & = A_i \cap B_i. \end{array}$$



Subclaim 1. We consider the internal sum $C := C_1 + \cdots + C_n \subseteq \mathbb{C}^{n+r}$. The following two statements hold:
(1) We have $\dim_{\mathbb{C}}(C) \leq n + k$.
(2) The condition $(*)$ is satisfied if and only if $\dim_{\mathbb{C}}(C) = n + k$.

Proof of subclaim. First we assume that for all i we have $\lambda_i + \mu_{n+1-i} \leq r$. It follows that

$$\dim_{\mathbb{C}}(C_i) \;=\; \#\{\mathfrak{e}_{\mu_{n+1-i}+i}, \ldots, \mathfrak{e}_{r+i-\lambda_i}\} \;=\; 1 + r - (\lambda_i + \mu_{n+1-i}) \geq 1.$$

This calculation implies that

$$\sum_{i=1}^{n} \dim_{\mathbb{C}}(C_i) \;=\; \sum_{i=1}^{n} (1 + r - \lambda_i - \mu_{n+1-i}) \;=\; n + \underbrace{n \cdot r - |\lambda| - |\mu|}_{=k} \;=\; n + k.$$

This calculation already implies that $\dim_{\mathbb{C}}(C) \leq n + k$.

Next note that it follows easily from the definitions, elementary logic, the fact that $\lambda_i + \mu_{n+1-i} \leq r$ and the above calculation that the following six statements are equivalent:

(a) $(*)$ is satisfied,
(b) for all i we have $r - \lambda_i \leq \mu_{n-i}$,
(c) for all i we have $r + i - \lambda_i \leq \mu_{n-i} + i$,
(d) for all i we have $r + i - \lambda_i < \mu_{n-i} + i + 1$,
(e) for all i we have $A_i \cap B_{i+1} = \{0\}$,
(f) the vector subspaces $C_1, \ldots, C_n$ are direct summands,
(g) the internal sum $C := C_1 + \cdots + C_n$ has dimension $n + k$.

It remains to deal with the case that for some i we have $\lambda_i + \mu_{n+1-i} > r$. Note that this implies that $(*)$ is not satisfied. A slight variation on the above argument also shows that $\dim_{\mathbb{C}}(C) < n + k$. □

Next, we consider the Schubert variety[2915]

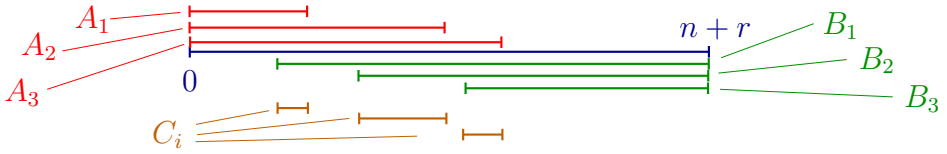
$$X_\lambda \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \,|\, \dim_{\mathbb{C}}(W \cap (\mathbb{C}^{r+i-\lambda_i} \times \{0\})) \geq i \text{ for } i = 1, \ldots, n\}.$$

from Definition 271.6 and as in the proof of the Schubert Variety Intersection Proposition 271.17 we consider the "Schubert-like variety"

$$\widetilde{X}_\mu \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \,|\, \dim_{\mathbb{C}}(W \cap (\{0\} \times \mathbb{C}^{r+i-\mu_i})) \geq i \text{ for } i = 1, \ldots, n\}.$$

Subclaim 2. If $W \in X_\lambda \cap \widetilde{X}_\mu \subseteq G_n(\mathbb{C}^{n+r})$ and if $(*)$ is satisfied, then $W \subseteq C$.+++ used how ??

Proof. Let $W \in X_\lambda \cap \widetilde{X}_\mu$. By definition of X_λ this implies that for all $i \in \{1, \ldots, n\}$ we have $\dim_{\mathbb{C}}(W \cap A_i) \geq i$. Furthermore, by definition of $\widetilde{X}_\mu$ this implies that for all $i \in \{1, \ldots, n\}$ we have $\dim_{\mathbb{C}}(W \cap B_i) \geq n - i + 1$. We claim that this implies that for all $i \in \{1, \ldots, n\}$ we have $W \subseteq A_i + B_{i+1}$. As we saw in Claim 1, the Condition $(*)$ implies that $A_i \cap B_{i+1} = \{0\}$. Thus $\dim_{\mathbb{C}}(W \cap (A_i + B_{i+1})) = \dim_{\mathbb{C}}(W \cap A_i) + \dim_{\mathbb{C}}(W \cap B_{i+1}) \geq i + (n - i) = n$. This implies that $W \subseteq A_i + B_{i+1}$. Thus we have now shown that $W \subseteq A_i + B_{i+1}$.



here $(*)$ is satisfied

The subclaim now follows from the observation that Condition $(*)$ implies that

$$C \;\subseteq\; \bigcap_{i=1}^{n} (A_i + B_{i+1}).$$

□

Next we introduce some notation: For any $(r + 1 - k)$-dimensional vector subspace L of $\mathbb{C}^{n+r}$ we consider the following subset:

$$X_k(L) \;:=\; \{W \in G_n(\mathbb{C}^{n+r}) \,|\, W \cap L \neq 0\}.$$

Subclaim 3. We have $[X_k(L)] = [X_{(k,0,\ldots,0)}] \in \mathrm{H}_{2(nr-k))}(G_n(\mathbb{C}^{n+r}))$.

[2915]Here we use the usual convention that for $W = [(w_1, \ldots, w_n)] \in G_n(\mathbb{C}^{n+r})$ we denote by W also the vector subspace spanned by $\{w_1, \ldots, w_n\}$.

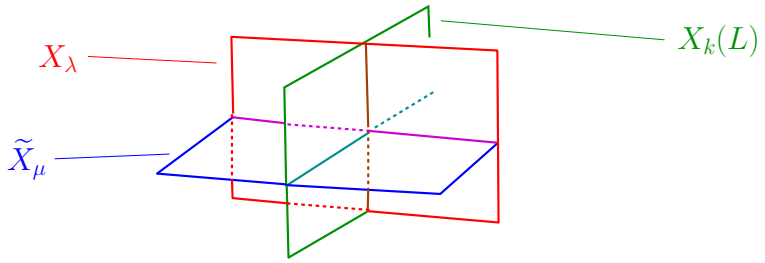
Proof of subclaim. First note for $L = \mathbb{C}^{r+1-k} \times \{0\}$ we obtain easily from the definitions (see also Exercise 271.2) that $X_k(L) = X_{(k,0,\dots,0)}$. Now let L be any $(r+1-k)$-dimensional vector subspace of $\mathbb{C}^{n+r}$ As in the Schubert Variety Intersection Proposition 271.17 we see that there exists a diffeotopy $F\colon G_n(\mathbb{C}^{n+r}) \times [0,1] \to G_n(\mathbb{C}^{n+r})$ with $F_0 = \mathrm{id}$ and such that F_1 restricts to an orientation preserving homeomorphism $X_k(L) \to X_{(k,0,\dots,0)}$. In particular we see that $[X_k(L)] = [X_{(k,0,\dots,0)}] \in \mathrm{H}_{2(nr-k))}(G_n(\mathbb{C}^{n+r}))$. □

Now we can finally perform the desired calculation:
If (∗) is not satisfied, then this implies by Subclaim 1 that $\dim_{\mathbb{C}}(C) \leq n+k-1$. It follows that we can find an $(r+1-k)$-dimensional vector subspace L of $\mathbb{C}^{n+r}$ that intersects C trivially. This implies that $X_\lambda \cap \widetilde{X}_\mu \cap X_k(L) = \varnothing$. Since $[X_k(\lambda)] = [X_{(k,0,\dots,0)}]$ we see that $\sigma_\lambda \cup \sigma_\mu \cup \sigma_{(k,0,\dots,0)} = 0$.

Finally we assume that (∗) is satisfied. Note that Subclaim 1 tells us that $\dim_{\mathbb{C}}(C) = n+k$. This implies that there exists an $(r+1-k)$-dimensional vector subspace L of $\mathbb{C}^{n+r}$ such that the intersection $C \cap L$ is 1-dimensional, i.e. it is of the form $\mathbb{C} \cdot c$ for some non-zero $c \in C$. In the proof of Claim 1 we also saw that our hypothesis (∗) implies that $C = C_1 \oplus \cdots \oplus C_n$. Thus we can write $c = c_1 + \cdots + c_n$ for unique $c_i \in C_i$. Let W be the span of $c_1, \dots, c_n$. This is an n-dimensional vector subspace (??) which is contained in $X_\lambda \cap \widetilde{X}_\mu$. The triple intersection $X_\lambda \cap \widetilde{X}_\mu \cap X_k(L)$ thus consists of the point W.

We have

$$X_\lambda \cap \widetilde{X}_\mu \cap X_k(L) \;=\; \{W \in X_\lambda \cap \widetilde{X}_\mu \,|\, W \subseteq X_k(L)\} \;=\; \{W \in X_\lambda \cap \widetilde{X}_\mu \cap C \,|\, W \subseteq X_k(L)\}$$



It is elementary to verify that the intersection is transverse. It follows that $\sigma_\lambda \cup \sigma_\mu \cup \sigma_{(k,0,\dots,0)} = 1$. ■

272.4. **The cohomology ring of Grassmannians.** Now we are finally ready to calculate the cohomology ring of the infinite-dimensional complex Grassmannians $G_n(\mathbb{C}^\infty)$:

Theorem 272.15. (Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem) Let $n \in \mathbb{N}_0$. For $k = 1, \dots n$ we consider the $(n \times \infty)$-Schubert symbol $(k, 0, \dots, 0)$ and we consider the corresponding cohomology class $\sigma_{(k,0,\dots,0)} \in \mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ that we introduced in the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.3. The ring homomorphism[2916]

$$\begin{array}{rcl} \Xi_n\colon \mathbb{Z}[c_1, \dots, c_n] & \to & \mathrm{H}^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) \\ c_k & \mapsto & \sigma_{(k,0,\dots,0)} \end{array}$$

is a ring isomorphism. In particular we see that the graded ring $\mathrm{H}^*(G_n(\mathbb{C}^\infty); \mathbb{Z})$ is isomorphic to a graded polynomial ring on n generators, $c_1, \dots, c_n$, with $\deg(c_k) = 2k$ for $k = 1, \dots, n$.

[2916]The reason why we call the variables $c_1, \dots, c_n$ will become apparent in Definition 274.1.

Remark.

(1) The cohomology ring $\mathrm{H}^*(G_n(\mathbb{C}^{n+r});\mathbb{Z})$ of a finite-dimensional complex Grassmannian can also be expressed algebraically, albeit in a somewhat less elegant way. We refer to [**Mani2001**, Exercise 3.2.13] for details.
(2) We refer to [**Mani2001**, Chapter 3.2], [**Ful1997**, Chapter 9.4] and [**GrH1994**, p. 203] for more background and somewhat different treatments of the proof of the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15.
(3) Note that [**MiS1974**, Theorem 14.5] and [**Hat2**, Theorem 3.9] give completely different proofs of the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15. □

Example. On page 5221 we pointed out that $G_1(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^\infty$. Thus the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15 shows in particular that we have a ring isomorphism

$$\begin{array}{rcl} \Xi_1\colon \mathbb{Z}[c_1] & \to & \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \\ c_1 & \mapsto & \sigma_{(1)}. \end{array}$$

Note that if one compares Notation 213.6 with Notation 271.14, then one sees that $\Xi(c_1) = \sigma_{(1)} \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ equals the usual generator $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ that we used in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7. □

Proof. We recall some facts:

(a) As we pointed out in the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.11, it follows from the Schur Function Basis Proposition 272.10 together with the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.3 that the map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}} & \to & \mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \\ s_\lambda & \mapsto & \sigma_\lambda \\ \uparrow & & \\ \lambda \text{ is an } (n\times\infty)\text{-Schubert symbol} & & \end{array}$$

is an isomorphism of *abelian groups*.
(b) By the Complete Symmetric Function Proposition 272.7 we know that the map

$$\begin{array}{rcl} \widetilde{\Xi}\colon \mathbb{Z}[c_1,\ldots,c_n] & \to & \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}} \\ c_k & \mapsto & h_k(x_1,\ldots,x_n) \end{array}$$

is an isomorphism of *rings*.
(c) By the Schur Function–Complete Symmetric Proposition 272.9 we know that we have the equality $s_{(k,0,\ldots,0)} = h_k$. Thus $\Theta(h_k) = (\Theta\circ\widetilde{\Xi})(c_k) = \Theta(h_k) = \Theta(s_{(k,0,\ldots,0)}) = \sigma_{(k,0,\ldots,0)}$.

It follows from the above three facts that it remains to prove the following claim:

Claim. The map $\Xi := \Theta\circ\widetilde{\Xi}\colon \mathbb{Z}[c_1,\ldots,c_n] \to \mathrm{H}^*(G_n(\mathbb{C}^{n+r});\mathbb{Z})$ is an isomorphism of rings.

Proof. By the above discussion it remains to show that this map is a ring homomorphism. Since Ξ is a group homomorphism and since rings are associative it suffices to show that for any $f \in \mathbb{Z}[c_1,\ldots,c_n]$ and any $k \in \{1,\ldots,n\}$ we have $\Xi(f\cdot c_k) = \Xi(f)\cup\Xi(c_k)$. As we will now see, this is indeed the case: First of all note that by the Complete Symmetric

Function Proposition 272.7 we can write

$$\widetilde{\Xi}(f) \;=\; \sum_{r=1}^{m} a_r \cdot s_{\lambda_r}$$

for some $a_1, \dots, a_m \in \mathbb{Z}$ and some $(n \times \infty)$-Schubert symbols $\lambda_1, \dots, \lambda_m$. We then see that

$$\begin{array}{rcll}
\Xi(f \cdot c_k) & \underset{\text{by definition of } \Xi}{=} & (\Theta \circ \widetilde{\Xi})(f \cdot c_k) & \underset{\text{by definition}}{=} \Theta(\widetilde{\Xi}(f \cdot c_k)) \quad \underset{\text{since } \widetilde{\Xi} \text{ is a ring homomorphism}}{=} \Theta(\widetilde{\Xi}(f) \cdot \widetilde{\Xi}(c_k)) \\
& = & \Theta\Big(\sum_{r=1}^{m} a_r \cdot s_{\lambda_r} \cdot h_k \Big) & = \sum_{r=1}^{m} a_r \cdot \Theta(s_{\lambda_r} \cdot h_k) \quad \underset{\text{this follows from comparing the Polynomial Pieri Formula 272.13 with the Grassmann Pieri Formula 272.14}}{=} \sum_{r=1}^{m} a_r \cdot \Theta(s_{\lambda_r}) \cup \Theta(h_k) \\
& \underset{\text{by definition of } \widetilde{\Xi}}{=} & \Xi(f) \cdot \Xi(c_k). & \\
& \underset{\text{follows, as above, easily from the definitions}}{=} & &
\end{array}$$

■

272.5. **Cohomology of the real Grassmannians.** In the previous sections we only dealt with complex Grassmannians. In this section we will summarize the corresponding results for real Grassmannians.

First of all we have the following analog of the $G_n(\mathbb{C}^\infty)$-H*-Lemma 272.3:

Lemma 272.16. ($G_n(\mathbb{R}^\infty)$-H*-Lemma) Let $n \in \mathbb{N}_0$.

(a) For each $k \leq r$ the inclusion induced map $\mathrm{H}^k(G_n(\mathbb{R}^\infty); \mathbb{F}_2) \to \mathrm{H}^k(G_n(\mathbb{R}^{n+r}); \mathbb{F}_2)$ is an isomorphism.

(b) For every $(n \times \infty)$-Schubert symbol $\lambda = (\lambda_1, \dots, \lambda_n)$ there exists a unique $\mathbb{F}_2$-cohomology class $\sigma_\lambda \in \mathrm{H}^{|\lambda|}(G_n(\mathbb{R}^\infty); \mathbb{F}_2)$ with the property that for each $r \geq \lambda_1$ we have the equality $\iota^*(\sigma_\lambda) = \sigma_\lambda \in \mathrm{H}^{|\lambda|}(G_n(\mathbb{R}^{n+r}); \mathbb{F}_2)$, where $\iota\colon G_n(\mathbb{R}^{n+r}) \to G_n(\mathbb{R}^\infty)$ is the usual embedding.

(c) Let $k \in \mathbb{N}$. The cohomology classes σ_λ, where λ runs over all $(n \times \infty)$-Schubert symbols of weight k, form a basis for $\mathrm{H}^k(G_n(\mathbb{R}^\infty); \mathbb{F}_2)$.

Proof. The proof is very similar to the proof of the $G_n(\mathbb{C}^\infty)$-H*-Lemma 272.3. Evidently we need to make all the obvious changes, in particular we need to replace the Universal Coefficient-H*-Theorem 201.23 by the Universal Coefficient-H*–Field Theorem 202.2. We leave it to the reader to make things work. ■

We conclude this section with the following theorem, which is the "real analog" of the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15:[2917]

Theorem 272.17. (Cup Product-$G_n(\mathbb{R}^\infty)$-Theorem) Let $n \in \mathbb{N}_0$. We consider the Grassmannian $G_n(\mathbb{R}^\infty)$. For $k = 1, \dots n$ we consider the $(n \times \infty)$-Schubert symbol $(k, 0, \dots, 0)$ and we consider the corresponding cohomology class $\sigma_{(k,0,\dots,0)} \in \mathrm{H}^k(G_n(\mathbb{C}^\infty); \mathbb{F}_2)$ that we introduced in the $G_n(\mathbb{R}^\infty)$-H*-Lemma 272.16. The ring homomorphism[2918]

$$\begin{array}{rcl}
\Xi_n\colon \mathbb{Z}[w_1, \dots, w_n] & \to & \mathrm{H}^*(G_n(\mathbb{R}^\infty); \mathbb{F}_2) \\
w_k & \mapsto & \sigma_{(k,0,\dots,0)}
\end{array}$$

[2917] Note that this theorem can be viewed as a generalization of the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19.

is a ring isomorphism. In particular we see that the graded ring $\mathrm{H}^*(G_n(\mathbb{R}^\infty);\mathbb{F}_2)$ is isomorphic to a graded polynomial ring (with $\mathbb{F}_2$-coefficients) on n generators $w_1,\ldots,w_n$ of degree $\deg(w_k)=k$ for $k=1,\ldots,n$.

Remark.

(1) Note that [**MiS1974**, Theorem 7.1] and [**Hat2**, Theorem 3.9] give completely different proofs of the Cup Product-$G_n(\mathbb{R}^\infty)$-Theorem 272.17.
(2) The calculation of the cohomology ring $\mathrm{H}^*(G_n(\mathbb{R}^\infty);\mathbb{Z})$ with $\mathbb{Z}$-coefficients is somewhat more involved, see [**Hat2**, Theorem 3.9] for more information. □

Proof. The proof of this theorem is very similar to the proof of the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15. Evidently we need to replace the Schubert Variety Intersection Proposition 271.17 by the Schubert Variety $\mathbb{F}_2$-Intersection Proposition 271.22 and we need to replace the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.3 by the $G_n(\mathbb{R}^\infty)$-H^*-Lemma 272.16. We leave it to the reader to adapt the nifty algebra of the previous sections to obtain the desired result. ■

272.6. Exercises for Chapter 272.

Exercise 272.1. Let $n\in\mathbb{N}_0$. Recall that a polynomial $p(x_1,\ldots,x_n)\in\mathbb{Z}[x_1,\ldots,x_n]$ is called *alternating* if for every permutation $\sigma\in S_n$ we have the equality

$$p(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \;=\; \mathrm{sign}(\sigma)\cdot p(x_1,\ldots,x_n)$$

where $\mathrm{sign}(\sigma)$ is the sign of the permutation σ. We consider the Vandermonde polynomial $V(x_1,\ldots,x_n) := \prod_{1\leq i<j\leq n}(x_i-x_j)$. Show that a polynomial $p(x_1,\ldots,x_n)$ is alternating if and only if there exists a symmetric polynomial $f(x_1,\ldots,x_n)$ with

$$p(x_1,\ldots,x_n) \;=\; f(x_1,\ldots,x_n)\cdot V(x_1,\ldots,x_n).$$

Remark. We need to show that for any $i<j$ we have $(x_i-x_j)|p(x_1,\ldots,x_n)$. You can show this by performing a suitable substitution.

Exercise 272.2. Let $n\in\mathbb{N}_0$. Let $\mu=(\mu_1,\ldots,\mu_n)$ be a strictly decreasing sequence of natural numbers, i.e. we have $\mu_1>\cdots>\mu_n\geq 0$. As in Definition refdef:schur-function we set $a_\mu := \det(A_\mu)$. Show that these alternating polynomials a_μ form a basis of the abelian group $\mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{alt}}$ of alternating polynomials in $x_1,\ldots,x_n$.
Hint. You could consider the map $\mathbb{Z}[x_1,\ldots,x_n]\to\mathbb{Z}[x_1,\ldots,x_n]$ which is the identity on each monomial $x_1^{r_1}\cdot\ldots\cdot x_n^{r_n}$ with $r_1>r_2>\cdots>r_n\geq 0$ and which is zero on all other monomials.

Exercise 272.3. Let $n\in\mathbb{N}_0$. By the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15 we know that there exist cohomology classes $c_1,\ldots,c_n$ of degree $\deg(c_k)=2k$ for $k=1,\ldots,n$ such that the following map is an isomorphism:

$$\begin{array}{ccc}\mathbb{Z}[c_1,\ldots,c_n] & \to & \mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z})\\ c_k & \mapsto & c_k\end{array}$$

Are the $c_1,\ldots,c_n$ unique up to sign?

[2918] The reason why we call the variables $w_1,\ldots,w_n$ will become apparent in Definition 274.14.

Exercise 272.4. Let $n, r \in \mathbb{N}_0$. Show that the cohomology ring $\mathrm{H}^*(G_n(\mathbb{C}^{n+r}); \mathbb{Z})$ is the quotient of a polynomial ring $\mathbb{Z}[y_1, \ldots, y_n]$.

CHAPTER 273

The splitting principle and Whitney sums

In the last chapter we proved the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15 and we proved the Cup Product-$G_n(\mathbb{R}^\infty)$-Theorem 272.17. As we will also see in the next chapter, these two theorems are all we need to define the Chern classes of complex vector bundles and to define the Stiefel-Whitney classes of real vector bundles.

As we will see in the next chapter, the key property of Chern classes and Stiefel-Whitney classes is that they satisfy a multiplicativity formula. To prove this formula we will have to do some extra work on the cohomology of Grassmannians, which we will do in this chapter. +++++ this argument still contains a gap ++++

This chapter is organized as follows:

- In Section 273.1 we will introduce the splitting principle for vector bundles.
- In Section 273.2 we will discuss the interaction between complex Grassmannians and the Whitney sum.
- In Section 273.3 we will discuss the analog of the results of Section 273.2 in the context of real vector bundles.
- In Section 273.4 we will provide exercises for Chapter 273.

In the next Chapter 274 we will introduce the Chern classes of complex vector bundles and the Stiefel-Whitney classes of real vector bundles.

273.1. **The splitting principle.** We start out by introducing some notation:

Notation 273.1. Suppose that we fixed some $s \in \mathbb{N}$. For $j = 1, \dots, s$ we consider the monomorphism

$$\begin{array}{rcl} \iota_j \colon \mathbb{C}^\infty & \to & \mathbb{C}^\infty \\ (v_1, v_2, \dots) & \mapsto & (\underbrace{0, \dots, 0}_{j-1 \text{ zeros}}, v_1, \underbrace{0, \dots, 0}_{s-1 \text{ zeros}}, v_2, \underbrace{0, \dots, 0}_{s-1 \text{ zeros}}, v_3, \dots) \end{array}$$

Note that the map $\iota_1 \oplus \cdots \oplus \iota_s \colon \mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty \to \mathbb{C}^\infty$ is also a monomorphism.

This notation allows us to introduce an interesting map between Grassmannians:

Proposition 273.2. (Grassmann Product Proposition) Let $s \in \mathbb{N}_0$. Furthermore let $n_1, \dots, n_s \in \mathbb{N}_0$. The map[2919]

$$\begin{array}{rcl} \mu \colon G_{n_1}(\mathbb{C}^\infty) \times \cdots \times G_{n_s}(\mathbb{C}^\infty) & \to & G_{n_1+\cdots+n_s}(\mathbb{C}^\infty) \\ (V_1, \dots, V_s) & \mapsto & \iota_1(V_1) + \cdots + \iota_s(V_s) \end{array}$$

is well-defined and continuous.

.

[2919]As usual, for $V = [(v_1, \dots, v_m)] \in G_m(\mathbb{C}^\infty)$ we denote by V also the vector subspace of $\mathbb{C}^\infty$ that is spanned by $v_1, \dots, v_m$. Converse, we view any m-dimensional vector subspace of $\mathbb{C}^\infty$ as an element of $G_m(\mathbb{C}^\infty)$

Proof. As we pointed out in Notation 273.1, the map $\iota_1 \oplus \cdots \oplus \iota_s \colon \mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty \to \mathbb{C}^\infty$ is a monomorphism. It follows easily from this observation that, using the above notation, $\iota_1(V_1) + \cdots + \iota_s(V_s)$ is indeed a vector subspace of dimension $n_1 + \cdots + n_s$. This shows that the map μ is well-defined. Next note that the map

$$\begin{array}{rcl} \mathrm{GL}_{n_1}(\mathbb{C}^\infty) \times \cdots \times \mathrm{GL}_{n_s}(\mathbb{C}^\infty) & \to & \mathrm{GL}_{n_1+\cdots+n_s}(\mathbb{C}^\infty) \\ ((v_{1,1}, \ldots, v_{1,n_1}), \ldots, (v_{s,1}, \ldots, v_{s,n_s})) & \mapsto & ((v_{1,1}, \ldots, v_{1,n_1}, \ldots, v_{s,1}, \ldots, v_{s,n_s})) \end{array}$$

is clearly continuous. It follows from the Quotient-to-Quotient Lemma 8.6 that our current map μ is also continuous. ∎

When we see a continuous map to an (infinite-dimensional) Grassmannian the following question arises immediately: What is the pullback of the tautological bundle? In the setup of the Grassmann Product Proposition 273.2 we get a nice answer:

Proposition 273.3. (Grassmann Product Pullback Proposition) Let $s \in \mathbb{N}_0$ and let $n_1, \ldots, n_s \in \mathbb{N}_0$. We consider the map

$$\begin{array}{rcl} \mu \colon G_{n_1}(\mathbb{C}^\infty) \times \cdots \times G_{n_s}(\mathbb{C}^\infty) & \to & G_{n_1+\cdots+n_s}(\mathbb{C}^\infty) \\ (V_1, \ldots, V_s) & \mapsto & \iota_1(V_1) + \cdots + \iota_s(V_s) \end{array}$$

from the Grassmann Product Proposition 273.2. The map

$$\begin{array}{rcl} \overbrace{\mathfrak{T}G_{n_1}(\mathbb{C}^\infty) \times \cdots \times \mathfrak{T}G_{n_s}(\mathbb{C}^\infty)}^{\text{product of vector bundles, see Definition 268.1}} & \to & \mu^*(\mathfrak{T}G_{n_1+\cdots+n_s}(\mathbb{C}^\infty)) \\ (V_1, v_1), \ldots, (V_s, v_s) & \mapsto & ((V_1, \ldots, V_s), (\mu(V_1, \ldots, V_s), \iota_1(v_1) + \cdots + \iota_s(v_s))) \end{array}$$

is an isomorphism of $(n_1 + \cdots + n_s)$-dimensional complex vector bundles over the base space $G_{n_1}(\mathbb{C}^\infty) \times \cdots \times G_{n_s}(\mathbb{C}^\infty)$.

Proof. It follows almost immediately from the Vector Bundle–Pullback Proposition 265.36 that the given map is an isomorphism of $\mathbb{C}$-vector bundles. ∎

Next, in the context of the Grassmann Product Proposition 273.2 we want to understand the induced map

$$\mu^* \colon \mathrm{H}^*(G_{n_1+\cdots+n_s}(\mathbb{C}^\infty); \mathbb{Z}) \to \mathrm{H}^*(G_{n_1}(\mathbb{C}^\infty) \times \cdots \times G_{n_s}(\mathbb{C}^\infty); \mathbb{Z})$$

on cohomology. To describe this map we need to understand the cohomology ring of a product of infinite-dimensional Grassmannians. First we recall the following theorem which gives us the cohomology ring of a single infinite-dimensional Grassmannian:

Theorem 272.15. (Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem) Let $n \in \mathbb{N}_0$. For $k = 1, \ldots n$ we consider the $(n \times \infty)$-Schubert symbol $(k, 0, \ldots, 0)$ and we consider the corresponding cohomology class $\sigma_{(k,0,\ldots,0)} \in \mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ that we introduced in the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.3. The ring homomorphism

$$\begin{array}{rcl} \Xi_n \colon \mathbb{Z}[c_1, \ldots, c_n] & \to & \mathrm{H}^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) \\ c_k & \mapsto & \sigma_{(k,0,\ldots,0)} \end{array}$$

is a ring isomorphism.

Notation 273.4.

(1) Let $n \in \mathbb{N}$. For $k \in \{1, \ldots, n\}$ we use the notation from the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15 to write
$$c_k \ := \ \Xi(c_k) \ = \ \sigma_{(k,0,\ldots,0)} \in \mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z}).$$
Furthermore we set $c_0 := 0$ and we set $c_k := 0$ for $k \geq n+1$.

(2) For $n = 1$, i.e. for the Grassmannian $G_1(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^\infty$ we denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ the standard generator from Notation 213.6.

As we pointed out on page 5261, we have $x = c_1 \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) = \mathrm{H}^2(G_1(\mathbb{C}^\infty); \mathbb{Z})$.

Next we need to understand the cohomology ring of a product of Grassmannians. Before we can do so, we need to recall a few concepts:

- In Definition 203.23 we introduced the notion of a superalgebra. For the purpose of the current discussion it suffices to know that superalgebras are $\mathbb{N}_0$-graded rings with a certain commutativity rule. In particular polynomials ring (with all generators in even degrees) are superalgebras and it follows from the Cup Product-Ring Proposition 203.25 that cohomology rings are superalgebras.
- On page 3996 we introduced the notion of the tensor product of two superalgebras. For example, on page 3996 we saw the following: If we set $\deg(x) = 2$ and $\deg(x_i) = 2$, then the map
$$\begin{array}{rcl} \mathbb{Z}[x] \otimes \cdots \otimes \mathbb{Z}[x] & \to & \mathbb{Z}[x_1, \ldots, x_n] \\ x^{r_1} \otimes \cdots \otimes x^{r_n} & \mapsto & x_1^{r_1} \cdot \ldots \cdot x_n^{r_n}. \end{array}$$
is an isomorphism of superalgebras. We use this isomorphism to make the identification $\mathbb{Z}[x] \otimes \cdots \otimes \mathbb{Z}[x] =_{\mathrm{i}} \mathbb{Z}[x_1, \ldots, x_n]$.

The following proposition is a generalization of the Cup Product-Multiple-$\mathbb{C}\mathrm{P}^n$-Proposition 213.8:

Proposition 273.5. (Grassmannian-Product-H*-Proposition) Let $n_1, \ldots, n_s \in \mathbb{N}_0$. For each $i \in \{1, \ldots, s\}$ we denote by $p_i \colon G_{n_1}(\mathbb{C}^\infty) \times \cdots \times G_{n_s}(\mathbb{C}^\infty) \to G_{n_i}(\mathbb{C}^\infty)$ the natural projections. The map
$$\begin{array}{rcl} \Phi \colon \mathrm{H}^*(G_{n_1}(\mathbb{C}^\infty); \mathbb{Z}) \otimes \cdots \otimes \mathrm{H}^*(G_{n_s}(\mathbb{C}^\infty); \mathbb{Z}) & \to & \mathrm{H}^*(G_{n_1}(\mathbb{C}^\infty) \times \cdots \times G_{n_s}(\mathbb{C}^\infty); \mathbb{Z}) \\ \varphi_1 \otimes \cdots \otimes \varphi_s & \mapsto & p_1^*(\varphi_1) \cup \cdots \cup p_s^*(\varphi_s) \end{array}$$
is a ring isomorphism.

Example. For $n_1 = \cdots = n_s = 1$ we obtain from the Grassmannian-Product-H*-Proposition 273.5 an isomorphism
$$\underbrace{\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})}_{=\mathbb{Z}[x]} \otimes \ldots \otimes \underbrace{\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})}_{=\mathbb{Z}[x]} \ \to \ \mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^n; \mathbb{Z}).$$
As we discussed on page 5261 the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15 (or alternatively the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7) give us an explicit isomorphism $\mathbb{Z}[x] \cong \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$. Thus we obtain an isomorphism
$$\underbrace{\mathbb{Z}[x] \otimes \cdots \otimes \mathbb{Z}[x]}_{=_{\mathrm{i}}\mathbb{Z}[x_1,\ldots,x_n]} \ \xrightarrow{\cong} \ \mathrm{H}^*(G_1(\mathbb{C}\mathrm{P}^\infty)^n; \mathbb{Z}).$$
□

Proof. First recall that we know by the Grassmannian–CW-Structure Proposition 271.10 that each $G_k(\mathbb{C}^\infty)$ has a CW-structure with finitely many cells in each dimension and recall that we know by the $G_n(\mathbb{C}^{n+r})$-H_*-Proposition 271.13 (1a) and (3) that all homology

groups of $G_k(\mathbb{C}^\infty)$ are finitely generated free abelian groups. These observations, together with the CW–Chain Complex Condition Lemma 208.2, tell us, that the Cohomology Ring-of-Product Proposition 208.12 (2) applies in our context, and that it gives us the desired result. ■

The following theorem is technically the most important result of this chapter:

Theorem 273.6. (Splitting Theorem) Let $n \in \mathbb{N}_0$. As in the Grassmann Product Proposition 273.2 we consider the map

$$\begin{array}{rcl} \mu_n\colon (\mathbb{C}\mathrm{P}^\infty)^n & \to & G_n(\mathbb{C}^\infty) \\ (V_1,\dots,V_n) & \mapsto & (\iota_1(V_1)+\cdots+\iota_n(V_n)). \end{array}$$

We consider the induced continuous map

$$\mu^*\colon \underbrace{\mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z})}_{=\mathbb{Z}[c_1,\dots,c_n]} \to \underbrace{\mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^n;\mathbb{Z})}_{=\mathbb{Z}[x_1,\dots,x_n]}$$

on cohomology. This map is given by

$$c_k \mapsto e_k(x_1,\dots,x_n)$$

where $e_k(x_1,\dots,x_n)$ is the k-th elementary symmetric polynomial from Definition 272.5.

Remark. An (alternative) proof can be found in [**Koc1996**, Proposition 2.3.3]. That proof makes use of the Serre spectral sequence. □

Remark. The Grassmann Product Pullback Proposition 273.3 together with the Splitting Theorem 273.6 says in particular that the map

$$\begin{array}{rcl} \mu\colon (\mathbb{C}\mathrm{P}^\infty)^n & \to & G_n(\mathbb{C}^\infty) \\ (V_1,\dots,V_n) & \mapsto & (\iota_1(V_1)+\cdots+\iota_n(V_n)) \end{array}$$

has the following two features:

- The pullback of the tautological bundle $\mathfrak{T}G_n(\mathbb{C}^\infty)$ under the map μ^* is a product of 1-dimensional complex vector bundles.
- The induced map μ^* on cohomology is a monomorphism.

The fact that we can find a continuous map with these two properties is sometimes called the *splitting principle*. □

Proof. This proof is unfortunately still missing. ■

273.2. **Grassmannians and the Whitney sum.** Before we state our next result, let us introduce the following suggestive variation on Notation 273.1:

Notation 273.7. We consider the monomorphisms

$$\begin{array}{rcl} \iota_{\mathrm{odd}}\colon \mathbb{C}^\infty & \to & \mathbb{C}^\infty \\ (v_1,v_2,\dots) & \mapsto & (v_1,0,v_2,0,\dots) \end{array} \quad \text{and} \quad \begin{array}{rcl} \iota_{\mathrm{even}}\colon \mathbb{C}^\infty & \to & \mathbb{C}^\infty \\ (v_1,v_2,\dots) & \mapsto & (0,v_1,0,v_2,0,\dots). \end{array}$$

Note that in Notation 273.1 these two monomorphisms corresponded to the case $s = 2$ and the monomorphisms ι_0, ι_1.

Proposition 273.8. (Complex Grassmannian-Whitney Sum-H*-Proposition) Let $m, n \in \mathbb{N}_0$. As in the Grassmann Product Proposition 273.2 we consider the continuous

map
$$\begin{array}{rcl}\mu\colon G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty) & \to & G_{m+n}(\mathbb{C}^\infty)\\ (V,W) & \mapsto & \iota_{\mathrm{odd}}(V)+\iota_{\mathrm{even}}(W)\end{array}$$
and we consider the induced maps
$$\mu^*\colon \mathrm{H}^*(G_{m+n}(\mathbb{C}^\infty);\mathbb{Z}) \to \mathrm{H}^*(G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty);\mathbb{Z}).$$
We denote by $p\colon G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty)\to G_m(\mathbb{C}^\infty)$ and $q\colon G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty)\to G_n(\mathbb{C}^\infty)$ the natural projections. By the Grassmannian-Product-H^*-Proposition 273.5 we know that the map
$$\begin{array}{rcl}\Phi\colon \mathrm{H}^*(G_m(\mathbb{C}^\infty);\mathbb{Z})\otimes \mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) & \to & \mathrm{H}^*(G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty);\mathbb{Z})\\ \varphi\otimes\psi & \mapsto & p^*(\varphi)\cup q^*(\psi)\end{array}$$
is a ring isomorphism. Finally we consider the map
$$\Phi^{-1}\circ\mu^*\colon \underbrace{\mathrm{H}^*(G_{m+n}(\mathbb{C}^\infty);\mathbb{Z})}_{=\mathbb{Z}[c_1,\dots,c_{m+n}]} \to \underbrace{\mathrm{H}^*(G_m(\mathbb{C}^\infty);\mathbb{Z})}_{=\mathbb{Z}[c_1,\dots,c_m]}\otimes\underbrace{\mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z})}_{=\mathbb{Z}[c_1,\dots,c_n]}.$$
For any $k\in\{0,\dots,m+n\}$ we have
$$c_k \mapsto \sum_{i=0}^{k} c_i\otimes c_{k-i}.$$

Proof. For each $s\in\mathbb{N}_0$ we also consider the continuous map $\mu_s\colon(\mathbb{C}\mathrm{P}^\infty)^s\to G_s(\mathbb{C}^\infty)$ from the Grassmann Product Proposition 273.2.

Claim. We consider the following diagram:
$$\begin{tikzcd}
\overset{e_i(x_1,\dots,x_m)}{\mathbb{Z}[x_1,..,x_m]}\otimes\overset{e_i(y_1,\dots,y_n)}{\mathbb{Z}[y_1,..,y_n]} \arrow[r, "\cong"] \arrow[dd] & \mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^m)\otimes\mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^n) \arrow[d, "\Phi", "\cong"'] & \overset{c_i}{\mathrm{H}^*(G_m(\mathbb{C}^\infty))}\otimes\overset{c_{k-i}}{\mathrm{H}^*(G_n(\mathbb{C}^\infty))} \arrow[l, hook', "\mu_m^*\otimes\mu_n^*"'] \arrow[d, "\Phi", "\cong"'] \\
 & \mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^m\times(\mathbb{C}\mathrm{P}^\infty)^n) & \mathrm{H}^*(G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty)) \arrow[l, hook', "\mathrm{diag}(\mu_m,\mu_n)^*"'] \\
\underset{e_k(x_1,\dots,x_m,y_1,\dots,y_n)}{\mathbb{Z}[x_1,\dots,x_m,y_1,\dots,y_n]} \arrow[r, "\cong"] & \mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^{m+n}) \arrow[u, "\mathrm{id}"] & \underset{c_k}{\mathrm{H}^*(G_{m+n}(\mathbb{C}^\infty))}. \arrow[l, hook', "\mu_{m+n}^*"'] \arrow[u, "\mu^*"']
\end{tikzcd}$$
Here we left out, for space reasons, the $\mathbb{Z}$-coefficients in cohomology. This diagram has the following properties:

(1) The diagram commutes.
(2) The horizontal maps to the right are monomorphisms.

Proof. We now prove the two statements:

(1) First we consider the following diagram:
$$\begin{tikzcd}
(\mathbb{C}\mathrm{P}^\infty)^m\times(\mathbb{C}\mathrm{P}^\infty)^n \arrow[r, "\mathrm{diag}(\mu_m,\mu_n)"] \arrow[d, "\mathrm{id}"] & G_m(\mathbb{C}^\infty)\times G_n(\mathbb{C}^\infty) \arrow[d, "\mu"] \\
(\mathbb{C}\mathrm{P}^\infty)^{m+n} \arrow[r, "\mu_{m+n}"] & G_{m+n}(\mathbb{C}^\infty).
\end{tikzcd}$$

It follows immediately from the definitions of the maps μ_s and μ that the diagram commutes up to permutation of the coordinates. Since in $\mathrm{GL}(\infty,\mathbb{C})$ a permutation

is homotopic to the identity we see that the diagram commutes up to homotopy. It thus follows from the Homotopy-H*-Proposition 198.11 that the bottom right corner in cohomology commutes.

Since the isomorphisms of the Grassmannian-Product-H*-Proposition 273.5 are natural we see that the top right corner commutes.

(2) It follows from the Splitting Theorem 273.6 that the μ^*-maps are all injective. Since the cohomology groups are finitely generated free abelian groups we see that the top right map is also a monomorphism. ⊞

Note that the horizontal maps μ^* are precisely the maps from the Splitting Theorem 273.6. Recall that we showed that these are monomorphisms which send c_j to an elementary symmetric polynomial e_j in the appropriate polynomial ring. The proposition now follows from the elementary, purely algebraic, observation that we have the following equality:

$$e_k(x_1,\dots,x_m,y_1,\dots,y_n) \;=\; \sum_{i=0}^{k} e_i(x_1,\dots,x_m)\cdot e_{k-i}(y_1,\dots,y_n).$$

We will prove this equality in Exercise 273.1. ■

273.3. **Whitney sum of real vector bundles.** In this last section we consider again real vector bundles. Later we will need the following proposition, which is the analog of the Complex Grassmannian-Whitney Sum-H*-Proposition 273.8:

Proposition 273.9. (Real Grassmannian-Whitney Sum-H*-Proposition) In the following let $m,n\in\mathbb{N}_0$. As in the Grassmann Product Proposition 273.2 we consider the continuous map

$$\begin{array}{rcl}\mu\colon G_m(\mathbb{R}^\infty)\times G_n(\mathbb{R}^\infty) & \to & G_{m+n}(\mathbb{R}^\infty)\\ (V,W) & \mapsto & \iota_{\text{odd}}(V)+\iota_{\text{even}}(W)\end{array}$$

and we consider the induced maps

$$\mu^*\colon \mathrm{H}^*(G_{m+n}(\mathbb{R}^\infty);\mathbb{F}_2) \;\to\; \mathrm{H}^*(G_m(\mathbb{R}^\infty)\times G_n(\mathbb{R}^\infty);\mathbb{F}_2).$$

We denote by $p\colon G_m(\mathbb{R}^\infty)\times G_n(\mathbb{R}^\infty)\to G_m(\mathbb{R}^\infty)$ and $q\colon G_m(\mathbb{R}^\infty)\times G_n(\mathbb{R}^\infty)\to G_n(\mathbb{R}^\infty)$ the natural projections. By the same argument as in the Grassmannian-Product-H*-Proposition 273.5 we see that the map

$$\begin{array}{rcl}\Phi\colon \mathrm{H}^*(G_m(\mathbb{R}^\infty);\mathbb{F}_2)\otimes \mathrm{H}^*(G_n(\mathbb{R}^\infty);\mathbb{F}_2) & \to & \mathrm{H}^*(G_m(\mathbb{R}^\infty)\times G_n(\mathbb{R}^\infty);\mathbb{F}_2)\\ \varphi\otimes\psi & \mapsto & p^*(\varphi)\cup q^*(\psi)\end{array}$$

is a ring isomorphism. Finally we consider the map

$$\Phi^{-1}\circ\mu^*\colon \underbrace{\mathrm{H}^*(G_{m+n}(\mathbb{R}^\infty);\mathbb{F}_2)}_{=\mathbb{Z}[w_1,\dots,w_{m+n}]} \;\to\; \underbrace{\mathrm{H}^*(G_m(\mathbb{R}^\infty);\mathbb{F}_2)}_{=\mathbb{Z}[w_1,\dots,w_m]}\otimes\underbrace{\mathrm{H}^*(G_n(\mathbb{R}^\infty);\mathbb{F}_2)}_{=\mathbb{Z}[w_1,\dots,w_n]}.$$

For any $k\in\{0,\dots,m+n\}$ we have

$$w_k \;\mapsto\; \sum_{i=0}^{k} w_i\otimes w_{k-i}.$$

Proof. The proof of the proposition is essentially the same as the proof of the Complex Grassmannian-Whitney Sum-H*-Proposition 273.8. We leave it to the reader to make the necessary adjustments. ■

273.4. Exercises for Chapter 273.

Exercise 273.1. As usual we denote by $e_j \in \mathbb{Z}[z_1, \ldots, z_n]$ the symmetric polynomials from Definition 272.5. Show that

$$e_k(x_1, \ldots, x_m, y_1, \ldots, y_n) \;=\; \sum_{i=0}^{k} e_i(x_1, \ldots, x_m) \cdot e_{k-i}(y_1, \ldots, y_n).$$

Exercise 273.2. Let $m, n \in \mathbb{N}_0$. Furthermore let $p\colon E \to B$ be an m-dimensional complex vector bundle and let $q\colon F \to B$ be an n-dimensional complex vector bundle. Furthermore let $e\colon B \to G_m(\mathbb{C}^\infty)$ be a continuous map such that $E \cong e^*\mathfrak{T}G_m(\mathbb{C}^\infty)$ and finally let $f\colon B \to G_n(\mathbb{C}^\infty)$ be a continuous map such that $F \cong f^*\mathfrak{T}G_n(\mathbb{C}^\infty)$. As in the Grassmann Product Proposition 273.2 we consider the continuous map

$$\begin{array}{rcl} \mu\colon G_m(\mathbb{C}^\infty) \times G_n(\mathbb{C}^\infty) & \to & G_{m+n}(\mathbb{C}^\infty) \\ (V, W) & \mapsto & \iota_{\mathrm{odd}}(V) + \iota_{\mathrm{even}}(W). \end{array}$$

Show that $E \oplus F \cong (\mu \circ (f, g))^*\mathfrak{T}G_{m+n}(\mathbb{C}^\infty)$.

CHAPTER 274

Chern classes and Stiefel-Whitney classes

Loosely speaking, a *characteristic class* is a map that associates to a (real or complex) vector bundle $p\colon E \to B$ a cohomology class in the base space B. In this chapter we will get to know several interesting characteristic classes. More precisely, given a complex vector bundle $p\colon E \to B$ we will introduce the corresponding Chern classes $c_k(E) \in \mathrm{H}^{2k}(B;\mathbb{Z})$, $k \in \mathbb{N}_0$ and given a real vector bundle $p\colon E \to B$ we will introduce the corresponding Stiefel-Whitney classes $w_k(E) \in \mathrm{H}^k(B;\mathbb{F}_2)$, $k \in \mathbb{N}_0$.

This chapter is organized as follows:

- In Section 274.1 we will introduce Chern classes of complex vector bundles and we will study their basic properties. We will also perform some calculations, for example we will calculate the Chern classes of the tangent bundle of complex projective spaces.
- In Section 274.2 we will introduce Stiefel-Whitney classes of real vector bundles. These are the "real" analog of the "complex" Chern classes. As we will see, the definition and the discussion of Stiefel-Whitney classes is very similar to the setting of Chern classes.
- In Appendix 274.3 we will explain a relationship between Chern classes and Stiefel-Whitney classes.
- In Section 274.4 we will provide exercises for Chapter 274.

In the next Chapter 275 we will study several applications of Stiefel-Whitney classes.

274.1. **Chern classes.** This section is organized as follows:

- In Subsection 274.1.1 we will introduce Chern classes.
- In Subsection 274.1.2 we will prove the multiplicativity theorem for Chern classes.
- In Subsection 274.1.3 we will give a characterization of Chern classes.
- In Subsection 274.1.4 we will calculate the Chern classes of the tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^n$ of the complex projective spaces $\mathbb{C}\mathrm{P}^n$.
- In Subsection 274.1.5 we will calculate the first Chern class of surfaces.

274.1.1. **Definition of Chern classes.** First we recall the following theorem:

Theorem 272.15 (Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem) Let $n \in \mathbb{N}_0$. For $k = 1, \dots n$ we consider the $(n \times \infty)$-Schubert symbol $(k, 0, \dots, 0)$ and we consider the corresponding cohomology class $\sigma_{(k,0,\dots,0)} \in \mathrm{H}^{2k}(G_n(\mathbb{C}^\infty);\mathbb{Z})$ that we introduced in the $G_n(\mathbb{C}^\infty)$-H^*-Lemma 272.3. The ring homomorphism

$$\begin{array}{rcl} \Xi_n\colon \mathbb{Z}[c_1, \dots, c_n] & \to & \mathrm{H}^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \\ c_k & \mapsto & \sigma_{(k,0,\dots,0)} \end{array}$$

is a ring isomorphism.

Next we recall Notation 273.4:

Notation 273.4. Using the notation from the Cup Product-$G_n(\mathbb{C}^\infty)$-Theorem 272.15 we define for $k \in \{1, \dots, n\}$ the following cohomology class:

$$c_k \ := \ \Xi_n(c_k) \ = \ \sigma_{(k,0,\dots,0)}) \ \in \ \mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z}).$$

The above theorem allows us to introduce the Chern classes of complex vector bundles:

Definition 274.1. Let $p\colon E \to B$ be an n-dimensional complex vector bundle over a topological space B that is Hausdorff and paracompact.[2920]

(1) By the Grassmannian–Universal Bundle Theorem 270.10 (1) there exists a continuous map $f\colon B \to G_n(\mathbb{C}^\infty)$ such that the pullback $f^*\mathfrak{T}G_n(\mathbb{C}^\infty)$ of the tautological vector bundle is isomorphic to the vector bundle $p\colon E \to B$. Given $k \in \{1, \dots, n\}$ we refer to

$$c_k(E) \ := \ f^*(c_k) \ \in \ \mathrm{H}^{2k}(B; \mathbb{Z})$$

(c_k: element of $\mathrm{H}^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ from Notation 273.4)

as the k-th Chern[2927] class of the vector bundle. Next we point out that it follows from the Grassmannian–Universal Bundle Theorem 270.10 (1) that the continuous map $f\colon B \to G_n(\mathbb{C}^\infty)$ is unique up to homotopy. It thus follows from the Homotopy-H^*-Proposition 198.11 that the definition of $c_k(E)$ does not depend on the choice of the map f.

(2) For notational convenience we also set
   (a) $c_0(E) := 1 \in \mathrm{H}^0(B; \mathbb{Z})$.
   (b) $c_k(E) := 0 \in \mathrm{H}^{2k}(B; \mathbb{Z})$ for $k \in \mathbb{N}_{>n}$.

In the following we will discuss some properties of Chern classes:

Proposition 274.2. (Naturality-of-Chern-Proposition) Let $k \in \mathbb{N}_0$. Furthermore let B and C be topological spaces that are Hausdorff and paracompact.

(1) For any morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of complex vector bundles we have $h^*(c_k(E)) = c_k(F) \in \mathrm{H}^{2k}(C; \mathbb{Z})$.

(2) (a) If $p\colon E \to B$ and $p'\colon E' \to B$ are two complex vector bundles that are isomorphic over B, then $c_k(E) = c_k(E') \in \mathrm{H}^{2k}(B; \mathbb{Z})$.
   (b) Let $p\colon E \to B$ be a complex vector bundle. For any continuous map $h\colon C \to B$ between the topological spaces C and B we have the following equality:[2928]

$$c_k(h^*E) \ = \ h^*(c_k(E)) \ \in \ \mathrm{H}^{2k}(C; \mathbb{Z}).$$

[2927] Note that we now defined Chern classes only for vector bundles over topological spaces that are Hausdorff and paracompact. In these notes we mostly care about vector bundles over CW-complexes and topological spaces. It it thus reassuring to know that by the CW-Complex–Paracompact Proposition 136.33 and the CW-Complex–Hausdorff Proposition 136.6 we know that every CW-complex is Hausdorff and paracompact. (Also note that in the Grassmannian–CW-Structure Proposition 271.10 we showed that the Grassmannians are CW-complexes.) Furthermore recall that topological manifolds are by definition Hausdorff and recall that in the Manifold Paracompact-Normal Proposition 23.9 we showed that topological spaces are paracompact.

[2927] Chern classes were introduced by Shiing-Shen Chern [**Chern1946**] in 1946.

Proof. Slightly confusingly we will prove the two statements in the opposite order:

(2) (a) Let $p\colon E \to B$ and $p'\colon E' \to B$ be complex vector bundles that are isomorphic over B. It follows from the Grassmannian–Universal Bundle Theorem 270.10 (1) that there exist continuous maps $f\colon B \to G_n(\mathbb{C}^\infty)$ and $f'\colon B \to G_n(\mathbb{C}^\infty)$ such that E is isomorphic to $f^*\mathfrak{T}G_n(\mathbb{C}^\infty)$ and such that E' is isomorphic to $(f')^*\mathfrak{T}G_n(\mathbb{C}^\infty)$. Since E and E' are isomorphic it follows from the Grassmannian–Universal Bundle Theorem 270.10 (2) that f and f' are homotopic. It follows that

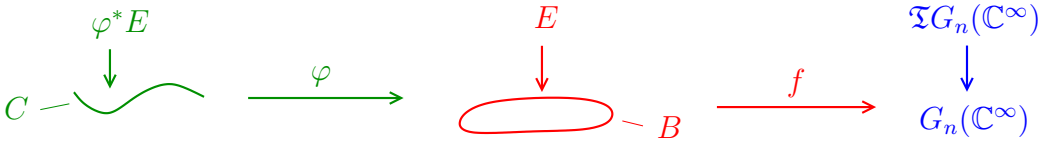
$$\underset{\text{Definition 274.1}}{c_k(E) \overset{\uparrow}{=}} f^*(c_k) \underset{\text{by the Homotopy H}^*\text{-Proposition 198.11}}{\overset{\uparrow}{=}} (f')^*(c_k) \underset{\text{Definition 274.1}}{\overset{\uparrow}{=}} c_k(E').$$

(b) Let $p\colon E \to B$ be an n-dimensional complex vector bundle, let C be a topological space and let $h\colon C \to B$ be a continuous map. We need to show that for every $k \in \mathbb{N}_0$ we have the equality

$$c_k(h^*E) \;=\; h^*(c_k(E)) \;\in\; \mathrm{H}^{2k}(C;\mathbb{Z}).$$

For $k = 0$ and $k > n$ this follows immediately from the rather simple minded Definition 274.1 (2) of the Chern classes in this range.

Now assume that $k \in \{1,\dots,n\}$. By the Grassmannian–Universal Bundle Theorem 270.10 (1) there exists a continuous map $f\colon B \to G_n(\mathbb{C}^\infty)$ such that $f^*\mathfrak{T}G_n(\mathbb{C}^\infty)$ is isomorphic to the vector bundle $p\colon E \to B$.



By the Vector Bundle-Pullback–Functorial Lemma 265.39 we know that the complex vector bundle $(f\circ h)^*\mathfrak{T}G_n(\mathbb{C}^\infty)$ is isomorphic to the complex vector bundle $h^*(f^*\mathfrak{T}G_n(\mathbb{C}^\infty)))$ which in turn is isomorphic to $h^*(E)$. We now see that

$$\underset{\substack{\text{by Definition 274.1 of } c_k(h^*E)\\ \text{and since we saw that } h^*(E)\\ \text{is isomorphic to } (f\circ h)^*\mathfrak{T}G_n(\mathbb{C}^\infty)))}}{c_k(h^*E) \overset{\uparrow}{=}} (f\circ h)^*(c_k) \underset{\substack{\text{since cohomology}\\ \text{is contravariant}}}{\overset{\uparrow}{=}} h^*(f^*(c_k)) \underset{\substack{\text{by Definition 274.1}\\ \text{of } c_k(E)}}{\overset{\uparrow}{=}} h^*(c_k(E)).$$

[2928] We can rephrase the content of the Naturality-of-Chern-Proposition 274.2 in a more categorical language:

- Let $\mathcal{Top}_{hp}$ be the category of topological spaces that are Hausdorff and paracompact and let $\mathcal{Set}$ be the category of sets.
- As in Notation 266.6 we denote by $\mathrm{IsoVectBund}_{\mathbb{C}^n}(B)$ the set of isomorphism classes of n-dimensional complex vector bundles over B.
- For each $k \in \mathbb{N}_0$ we obtain from the H*–Functor Lemma 198.3 and the Vector Bundle-Pullback-Functor Lemma 265.38 the following two contravariant functors:

$$\begin{array}{rcl} \mathcal{Top}_{hp} & \to & \mathcal{Set} \\ C & \mapsto & \mathrm{H}^{2k}(C;\mathbb{Z}) \\ (C \xrightarrow{h} B) & \mapsto & (\mathrm{H}^{2k}(B;\mathbb{Z}) \xrightarrow{h^*} \mathrm{H}^{2k}(C;\mathbb{Z})) \end{array} \quad \& \quad \begin{array}{rcl} \mathcal{Top}_{hp} & \to & \mathcal{Set} \\ C & \mapsto & \mathrm{IsoVectBund}_{\mathbb{C}}(C) \\ (C \xrightarrow{h} B) & \mapsto & \left(\begin{array}{rcl} \mathrm{IsoVectBund}_{\mathbb{C}}(B) & \to & \mathrm{IsoVectBund}_{\mathbb{C}}(C) \\ [E \xrightarrow{p} B] & \mapsto & [h^*E \to C] \end{array}\right). \end{array}$$

- In this language we can say that Chern classes are a natural transformation between these two contravariant functors.

(1) This statement follows from (2) and the Vector Bundle–Pullback Proposition 265.36 (2).[2929] ■

We continue with two basic calculations of Chern classes:

Lemma 274.3. (Trivializable-Chern-Lemma) Let $p\colon E \to B$ be a trivializable complex vector bundle over a topological space that is Hausdorff and paracompact. For every $k \in \mathbb{N}$ we have $c_k(E) = 0$.

Proof. Let $p\colon E \to B$ be a trivializable n-dimensional complex vector bundle over a topological space that is Hausdorff and paracompact. We need to show that for every $k \in \mathbb{N}$ we have $c_k(E) = 0$. For $k > n$ this holds by Definition 274.1 (2b). Now let $k \in \{1, \ldots, n\}$. Let $f\colon B \to G_n(\mathbb{C}^\infty)$ be a constant map. As we mentioned on page 5223, since f is constant map we know that $f^*(\mathfrak{T}G_n(\mathbb{C}^\infty))$ is trivializable, in particular it is isomorphic to E. We now see that for $k \in \{1, \ldots, n\}$ we have $c_k(E) = f^*(c_k)$. Since f is a constant map we see that f^* is the zero map on $2k$-th cohomology. Thus we see that $c_k(E) = 0$. ■

Lemma 274.4. (Tautological Bundle-Chern-Lemma) Let $n \in \mathbb{N}$. Furthermore let $p\colon \mathfrak{T}\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be the tautological complex line bundle from the Tautological Line Bundle Lemma 265.6. Then

$$c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^n) \;=\; x \;\in\; \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}),$$

where $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is the standard generator that we introduced in Notation 213.6.

Proof. Let $n \in \mathbb{N}$ and let $p\colon \mathfrak{T}\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be the tautological complex line bundle. We denote by $\iota\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty = G_1(\mathbb{C}^\infty)$ the usual embedding. We have the following equalities in $\mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$:

$$c_1(\overbrace{\mathfrak{T}G_1(\mathbb{C}^{n+1})}^{=\mathbb{C}\mathrm{P}^n}) \underset{\text{by the Tautological Bundle Lemma 270.8 (3)}}{=} c_1(\iota^*\mathfrak{T}G_1(\mathbb{C}^\infty)) \underset{\text{by Definition 274.1}}{=} \iota^*(c_1) \underset{\text{by the discussion on page 3832 the map } \iota^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \text{ is an isomorphism}}{=} x,$$

where $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is the standard generator. ■

Example. Given any $n \in \mathbb{Z}$ we consider the continuous map $S^1 \to \mathrm{U}(1)$ that is given by $z \mapsto (z^n)$. We consider the corresponding $\mathbb{C}$-vector bundle $p(\varphi_n)\colon Y(\varphi_n) \to S^2$ from the Vector Bundle-Clutching Map Lemma 267.5. Recall that in the Complex Line Bundle-over-S^2-Lemma 267.7 we showed that there exists a continuous map $f\colon S^2 \to S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$ of degree n such that $f^*(\mathfrak{T}\mathbb{C}\mathrm{P}^1) = Y(\varphi_n)$. It follows that we have the following equality in $\mathrm{H}^2(S^2;\mathbb{Z}) = \mathbb{Z} \cdot [S^2]^*$:

$$c_1(Y(\varphi_n)) \underset{\text{by the above}}{=} c_1(f^*(\mathfrak{T}\mathbb{C}\mathrm{P}^1)) \underset{\text{by the Naturality of-Chern-Proposition 274.2}}{=} f^*(c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1)) \overset{\text{by the Tautological Bundle Chern-Lemma 274.4}}{=} f^*(x) \underset{\text{by the Dual Fundamental Class Degree Lemma 201.28 and since } f \text{ has degree } n}{=} n \cdot [S^2]^8.$$

□

[2929] In fact one can easily show that Statements (1) and (2) of the proposition are logically equivalent.

274.1.2. **The multiplicativity theorem for Chern classes.** In this subsection we will formulate and discuss the multiplicativity theorem for Chern classes. To formulate the theorem it is helpful to introduce the following definition:

Definition 274.5. Given a complex vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact we refer to[2930 2931]

$$c(E) \coloneqq (c_0(E), 0, c_1(E), 0, c_2(E), 0, \dots) \in \mathrm{H}^*(B;\mathbb{Z}) := \bigoplus_{i=0}^{\infty} \mathrm{H}^i(B;\mathbb{Z})$$

as the total Chern class of $p\colon E \to B$.

With this notation we can now formulate the multiplicativity theorem:

Theorem 274.6. (Chern-Multiplicativity Theorem) The total Chern class is multiplicative with respect to the direct sum of complex vector bundles. More precisely, if $e\colon E \to B$ and $f\colon F \to B$ are two complex vector bundles over a topological space that is Hausdorff and paracompact, then

$$c(E \oplus F) = c(E) \cup c(F) \in \mathrm{H}^*(B;\mathbb{Z}).$$

In other words, for each $k \in \mathbb{N}_0$ we have

$$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) \cup c_{k-i}(F) \in \mathrm{H}^{2k}(B;\mathbb{Z}).$$

Example. Let B be a topological space that is Hausdorff and paracompact, let $E \to B$ be a complex vector bundle and let $F \to B$ be a trivializable complex vector bundle. We see that we have the following equality in $\mathrm{H}^*(B;\mathbb{Z})$:

$$c(E \oplus F) \underset{\text{by the Chern-Multiplicativity Theorem 274.6}}{=} c(E) \cup c(F) \underset{\text{by the Trivializable Chern-Lemma 274.3 together with Definition 274.1 (2a)}}{=} c(E) \cup 1 = c(E).$$

□

Proof. Evidently the equality $c(E \oplus F) = c(E) \cup c(F)$ is an immediate consequence of the statements regarding $c_k(E \oplus F)$. Thus it remains to prove the claim regarding $c_k(E \oplus F)$. To simplify the notation we write throughout the proof $G_k := G_k(\mathbb{C}^\infty)$ and $\mathfrak{T}_k := \mathfrak{T}G_k(\mathbb{C}^\infty)$.

Let $m, n \in \mathbb{N}_0$. We consider the map

$$\begin{aligned} \mu\colon G_m \times G_n &\to G_{m+n} \\ (V, W) &\mapsto (\iota_{\mathrm{odd}}(V) + \iota_{\mathrm{even}}(W)) \end{aligned}$$

from Notation 273.7 and the Grassmann Product Proposition 273.2. Recall that by the Grassmannian–Universal Bundle Theorem 270.10 we can pick continuous maps $e\colon B \to G_m$ and $f\colon B \to G_n$ such that $E \cong e^*\mathfrak{T}_n$ and such that $F \cong f^*\mathfrak{T}_n$. Next we consider the diagonal map $\Delta\colon B \to B \times B$ that is given by $b \mapsto (b, b)$ and we consider the map

$$h := \mu \circ \mathrm{diag}(f, g) \circ \Delta\colon B \to G_{m+n}.$$

[2930] Recall that for a topological space X and a commutative ring R we introduced in Definition 203.26 the cohomology ring $\bigoplus_{k\in\mathbb{N}_0} \mathrm{H}^k(X;R)$.

[2931] It follows from Definition 274.1 (2b) that $c(E)$ is indeed an element in the direct *sum* of cohomology groups.

Claim 1. The Whitney sum $E \oplus F$ is isomorphic to $h^*(\mathfrak{T}_{m+n})$.

Proof. We consider the following sequence of maps

$$\begin{array}{ccccccccc}
E \oplus F & \longrightarrow & E \times F & \longrightarrow & \mathfrak{T}_m \times \mathfrak{T}_n & \xrightarrow{\Theta} & \mu^*(\mathfrak{T}_{m+n}) & \longrightarrow & \mathfrak{T}_{m+n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{\Delta} & B \times B & \xrightarrow{\operatorname{diag}(f,g)} & G_m \times G_n & \xrightarrow{\mathrm{id}} & G_m \times G_n & \xrightarrow{\mu} & G_{m+n}.
\end{array}$$

$$B \xrightarrow{\quad =:h \quad} G_{m+n}.$$

We make the following clarifications and observations:

(0) The vertical maps are always the obvious projections.
(1) The first square is a morphism of $\mathbb{C}$-vector bundles by the Whitney Sum-via-Diagonal Lemma 268.4.
(2) As in Definition 268.1 it is clear that the second square is a morphism of $\mathbb{C}$-vector bundles.
(3) The map $\Theta\colon \mathfrak{T}_m \times \mathfrak{T}_n \to \mu^*(\mathfrak{T}_{m+n})$ is the isomorphism of $\mathbb{C}$-vector bundles over $G_m \times G_n$ from the Grassmann Product Pullback Proposition 273.3. This shows that the third square is an isomorphism of $\mathbb{C}$-vector bundles.
(4) The fourth square is the usual morphism of $\mathbb{C}$-vector bundles given by a pullback.

It follows from the above and the Vector Bundle–Pullback Proposition 265.36 (2) that the Whitney sum $E \oplus F$ is indeed isomorphic to $h^*(\mathfrak{T}_{m+n})$. ⊞

Note that it follows from Claim 1 and Definition 274.1 that for every $k \in \mathbb{N}_0$ we have the equality $c_k(E \oplus F) = h^*(c_k)$. Thus it remains to prove the following claim:

Claim 2. Given any $k \in \mathbb{N}_0$ we have

$$h^*(c_k) \;=\; \sum_{i=0}^{k} c_i(E) \cup c_{k-i}(F).$$

Proof. We denote by $p\colon G_m \times G_n \to G_n$ and $q\colon G_m \times G_n \to G_n$ the natural projections. Next we consider the following maps in cohomology:

$$\begin{array}{ccccccc}
 & & & \overset{h^*}{\curvearrowleft} & & & \\
\mathrm{H}^{2k}(B;\mathbb{Z}) & \xleftarrow{\Delta^*} & \mathrm{H}^{2k}(B\times B;\mathbb{Z}) & \xleftarrow{\operatorname{diag}(f,g)^*} & \mathrm{H}^{2k}(G_m\times G_n;\mathbb{Z}) & \xleftarrow{\mu^*} & \mathrm{H}^{2k}(G_{m+n};\mathbb{Z}). \\
\uparrow{\scriptstyle a\otimes b\mapsto a\cup b} & \overset{\Theta}{\nearrow} & \uparrow{\scriptstyle \varphi\otimes\psi\mapsto p^*(\varphi)\cup q^*(\psi)} & & \uparrow{\scriptstyle \varphi\otimes\psi\mapsto p^*(\varphi)\cup q^*(\psi)} & \overset{=:\Phi}{\nearrow} & \overset{\Phi\circ\mu^*}{\nwarrow} \\
\mathrm{H}^*(B;\mathbb{Z})\otimes\mathrm{H}^*(B;\mathbb{Z}) & & \xleftarrow[f^*\otimes g^*]{} & & \mathrm{H}^*(G_m;\mathbb{Z})\otimes\mathrm{H}^*(G_n;\mathbb{Z}) & & \overset{c_k}{\underset{c_k\mapsto\sum_{i=0}^{k} c_i\otimes c_{k-i}}{}}
\end{array}$$

We make the following clarifications and observations:

(1) As we discussed in the Complex Grassmannian-Whitney Sum-H*-Proposition 273.8 the right vertical map Φ is an isomorphism.
(2) Note that it follows easily from the Cup Product-Naturality Lemma 203.9 and the fact that $p \circ \Delta = q \circ \Delta = \mathrm{id}_B$ that the triangle to the left and the middle square of the diagram commute. The triangle to the right commutes by definition of the right diagonal map.

We now see that

$$\begin{array}{rcl} h^*(c_k) & \overset{\substack{\text{by the definition of } h\\ \downarrow}}{=} & (\Delta^* \circ \operatorname{diag}(f,g)^* \circ \mu^*)(c_k) \\ & \underset{\substack{\uparrow\\ \text{by the Complex Grassmannian-Whitney}\\ \text{Sum-H}^*\text{-Proposition 273.8}}}{=} & (\Theta \circ (f^* \otimes g^*))\Big(\sum_{i=0}^{k} c_i \otimes c_{k-i}\Big) \end{array} \quad \begin{array}{rcl} \overset{\substack{\text{since the diagram commutes}\\ \downarrow}}{=} & & (\Theta \circ (f^* \otimes g^*))(\Phi^{-1} \circ \mu^*)(c_k) \\ = & & \sum_{i=0}^{k} \underbrace{f^*(c_i)}_{=c_i(E)} \cup \underbrace{g^*(c_{k-i})}_{=c_{k-i}(E)}. \end{array}$$

■

274.1.3. **Characterization of Chern classes.** The following proposition shows that the combination of

(1) the dimension conditions on the Chern classes from Definition 274.1,
(2) the Naturality-of-Chern-Proposition 274.2,
(3) the Chern-Multiplicativity Theorem 274.6 and
(4) the Tautological Bundle-Chern-Lemma 274.4

completely characterizes the Chern classes:

Proposition 274.7. (Chern Class Characterization Proposition) Suppose we have a sequence of maps $\{\widetilde{c}_k\}_{k\in\mathbb{N}_0}$ such that $\widetilde{c}_k$ associates to any complex vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact a class $\widetilde{c}_k(E) \in \mathrm{H}^{2k}(B;\mathbb{Z})$ such that the following statements hold:[2932]

(1) For any n-dimensional complex vector bundle $E \to B$ we have $\widetilde{c}_0(E) = 1$ and we have $\widetilde{c}_k(E) = 0$ for all $k > n$.
(2) Each $\widetilde{c}_k$ is natural in the sense that for every morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of complex vector bundles we have $h^*(\widetilde{c}_k(E)) = \widetilde{c}_k(F) \in \mathrm{H}^{2k}(C;\mathbb{Z})$.
(3) The total class
$$\widetilde{c}(E) \;:=\; (\widetilde{c}_0(E), 0, \widetilde{c}_1(E), 0, \dots)$$
satisfies the Chern-Multiplicativity Theorem 274.6.
(4) For the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ we have the equality
$$\widetilde{c}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) \;=\; [\mathbb{C}\mathrm{P}^1]^* \;\in\; \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}).$$

Then $\widetilde{c}_k = c_k$ for all $k \in \mathbb{N}_0$.

Proof. Suppose we have a sequence of maps $\{\widetilde{c}_k\}_{k\in\mathbb{N}_0}$ that satisfy the above Conditions (1)–(4). As we mentioned above, the Chern classes $\{c_k\}_{k\in\mathbb{N}_0}$ also satisfy Conditions (1)–(4).

We need to show that for every complex vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact we have $\widetilde{c}_k(E) = c_k(E)$ for all $k \in \mathbb{N}_0$. By (4) we know that the equality holds for the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. In the following claims we will expand the class of complex vector bundles where equality holds:

Claim 1. We have $\widetilde{c}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) = c(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

[2932]These four statements correspond to Axioms 1–4 for Stiefel-Whitney classes as formulated in [**MiS1974**, p. 37] and they correspond to Axioms I–IV of [**Hirz1956**, Kapitel I.4][**Hirz1966**, Chapter I.4.].

Proof. First note that it follows from the Tautological Bundle Lemma 270.8 and the Vector Bundle–Pullback Proposition 265.36 (2) that the pullback of $\mathfrak{T}\mathbb{C}\mathrm{P}^\infty$ under the usual embedding $\iota\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ is isomorphic to $\mathfrak{T}\mathbb{C}\mathrm{P}^1$. Thus we see that

$$\begin{array}{ccccccc} \iota^*(\widetilde{c}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) & = & \widetilde{c}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) & = & c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) & = & \iota^*(c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the above discussion} & & \text{since } \widetilde{c}_1 \text{ and } c_1 & & \text{by the above discussion} & \\ & \text{and since } \widetilde{c}_1 \text{ satisfies (2)} & & \text{both satisfy (4)} & & \text{and since } c_1 \text{ satisfies (2)} & \end{array}$$

By the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 we know that the induced map $\iota^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ is an isomorphism. Thus we obtain from the above calculation that $\widetilde{c}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) = c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)$. Finally note that it follows from the fact that $\widetilde{c}_k$ and c_k satisfy (1) that we have the equality $\widetilde{c}_k(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) = c_k(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)$ for all $k \in \mathbb{N}_0$. ⊞

Claim 2. For every $n \in \mathbb{N}$ we have $\widetilde{c}(\mathfrak{T}G_n(\mathbb{C}^\infty)) = c(\mathfrak{T}G_n(\mathbb{C}^\infty)) \in \mathrm{H}^n(G_n(\mathbb{C}^\infty);\mathbb{Z})$.

Proof. Let $n \in \mathbb{N}$. We start out with a few observations:

(a) It follows from the Grassmann Product Pullback Proposition 273.3 together with the Splitting Theorem 273.6 that there exists a continuous map $\mu\colon (\mathbb{C}\mathrm{P}^\infty)^n \to G_n(\mathbb{C}^\infty)$ with the following two properties:
  (i) We have an isomorphism $\mathfrak{T}\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathfrak{T}\mathbb{C}\mathrm{P}^\infty \xrightarrow{\cong} \mu^*(\mathfrak{T}G_n(\mathbb{C}^\infty))$ of complex vector bundles.
  (ii) The induced map $\mu^*\colon \mathrm{H}^*(G_n(\mathbb{C}\mathrm{P}^\infty);\mathbb{Z}) \to \mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^n;\mathbb{Z})$ is a monomorphism.
(b) Let $\pi_1, \ldots, \pi_n\colon (\mathbb{C}\mathrm{P}^\infty)^n \to \mathbb{C}\mathrm{P}^\infty$ be the n natural projections. Recall that by the Bundle Product-as-Whitney Sum Lemma 268.5 we know that $\mathfrak{T}(\mathbb{C}\mathrm{P}^\infty) \times \cdots \times \mathfrak{T}\mathbb{C}\mathrm{P}^\infty$ is isomorphic to the Whitney sum $\pi_1^*(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \oplus \cdots \oplus \pi_n^*(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)$.

We now see that we have the following equality in $\mathrm{H}^*(G_n(\mathbb{C}\mathrm{P}^\infty);\mathbb{Z})$:

$$\begin{array}{lll} & \text{since } \widetilde{c} \text{ satisfies (2)} & \text{by (a-i)} \\ & \downarrow & \downarrow \\ \mu^*(\widetilde{c}(\mathfrak{T}G_n(\mathbb{C}^\infty))) & = \widetilde{c}(\mu^*(\mathfrak{T}G_n(\mathbb{C}^\infty))) & = \widetilde{c}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \\ & = \widetilde{c}(\pi_{1*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \oplus \cdots \oplus \pi_{n*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) & = \widetilde{c}(\pi_{1*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \cup \cdots \cup \widetilde{c}(\pi_{n*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \\ & \uparrow & \uparrow \\ & \text{by (b)} & \text{since } \widetilde{c} \text{ satisfies (3)} \\ & = \pi_{1*}(\widetilde{c}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \cup \cdots \cup \pi_{n*}(\widetilde{c}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) & = \pi_{1*}(c(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \cup \cdots \cup \pi_{n*}(c(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \\ & \uparrow & \uparrow \\ & \text{since } \widetilde{c} \text{ satisfies (2)} & \text{by Claim 1} \\ & = \ldots = \mu^*(c(\mathfrak{T}G_n(\mathbb{C}^\infty))). & \\ & \uparrow & \\ & \text{same argument backwards} & \end{array}$$

Finally note that it follows from this calculation together with the injectivity of the homomorphism μ^* from (a-ii) that we obtain the desired equality $\widetilde{c}(\mathfrak{T}G_n(\mathbb{C}^\infty)) = c(\mathfrak{T}G_n(\mathbb{C}^\infty))$ in $\mathrm{H}^n(G_n(\mathbb{C}^\infty);\mathbb{Z})$. ⊞

The following claim gives us finally the desired statement:

Claim 3. For every complex vector bundle $p\colon E \to B$ over a topological space B that is Hausdorff and paracompact we have $\widetilde{c}_k(E) = c_k(E)$ for all $k \in \mathbb{N}_0$.

Proof. Let $p\colon E \to B$ be an n-dimensional complex vector bundle over a topological space B that is Hausdorff and paracompact. By the Grassmannian–Universal Bundle Theorem 270.10 there exists a continuous map $f\colon B \to G_n(\mathbb{C}^\infty)$ such that E is isomorphic to

$f^*\mathfrak{T}G_n(\mathbb{C}^\infty)$. We now see that

$$\widetilde{c}_k(E) \underset{\text{by hypothesis (2)}}{=} f^*(\widetilde{c}_k(\mathfrak{T}G_n(\mathbb{C}^\infty))) \underset{\text{by Claim 2}}{=} f^*(c_k(\mathfrak{T}G_n(\mathbb{C}^\infty))) \underset{\text{since } c_k \text{ satisfies hypothesis (2)}}{=} c_k(E).$$

■

We conclude this subsection with an application of the Chern Class Characterization Proposition 274.7: Let $p\colon E \to B$ be a complex vector bundle. Recall that in Definition 268.33 we introduced the corresponding complex conjugate bundle $p\colon \overline{E} \to B$. This is a complex vector bundle which is almost the same as the original complex vector bundle, except that complex multiplication on each fiber differs by complex conjugation. Using the Chern Class Characterization Proposition 274.7 we will now be able to relate the Chern classes of E and $\overline{E}$:

Lemma 274.8. (Complex Conjugate-Chern-Lemma) Let $p\colon E \to B$ be a complex vector bundle over a topological space that is Hausdorff and paracompact. For every $k \in \mathbb{N}_0$ we have $c_k(\overline{E}) = (-1)^k \cdot c_k(E) \in \mathrm{H}^k(B;\mathbb{Z})$.

Proof. [2933] Given a complex vector bundle $p\colon E \to B$ we define $\widetilde{c}_k(E) := (-1)^k \cdot c_k(\overline{E})$. By the Chern Class Characterization Proposition 274.7 it suffices to prove the following claim:

Claim. These cohomology classes $\widetilde{c}_k$ satisfies the four properties set out in the Chern Class Characterization Proposition 274.7.

Proof. In the following we verify that the cohomology classes $\widetilde{c}_k(E) = (-1)^k \cdot c_k(\overline{E})$ have these four properties:

(1) Clearly for any n-dimensional complex vector bundle $E \to B$ we have $\widetilde{c}_0(E) = 1$ and we have $\widetilde{c}_k(E) = 0$ for all $k > n$.

(2) Clearly each $\widetilde{c}_k$ is natural in the sense of the Naturality-of-Chern-Proposition 274.2 (1).

(3) Let $E \to B$ and $F \to B$ be two complex vector bundles over a topological space that is Hausdorff and paracompact. We see that for each $k \in \mathbb{N}_0$ we have

$$\begin{aligned}
\widetilde{c}_k(E \oplus F) &\underset{\text{by definition of } \widetilde{c}_k}{=} (-1)^k \cdot c_k(\overline{E \oplus F}) \underset{\text{since } \overline{E \oplus F} = \overline{E} \oplus \overline{F}}{=} (-1)^k \cdot c_k(\overline{E} \oplus \overline{F}) \\
&\underset{\text{by the Chern-Multiplicativity Theorem 274.6}}{=} (-1)^k \cdot \sum_{i=0}^{k} c_i(\overline{E}) \cup c_{k-i}(\overline{F}) \\
&\underset{\text{eighth grade algebra}}{=} \sum_{i=0}^{k} (-1)^i \cdot c_i(\overline{E}) \cup (-1)^{k-i} \cdot c_{k-i}(\overline{F}) \\
&\underset{\text{by definition of } \widetilde{c}_j}{=} \sum_{i=0}^{k} \widetilde{c}_i(E) \cup \widetilde{c}_{k-i}(F).
\end{aligned}$$

(4) We consider the tautological vector bundle

$$\mathfrak{T}\mathbb{C}\mathrm{P}^1 := \{[v], w) \,|\, v \in \mathbb{C}^2 \setminus \{0\} \text{ and } w \in \mathbb{C} \cdot v\}.$$

We also consider the following continuous maps:

$$\begin{array}{ccc}
\overline{\mathfrak{T}\mathbb{C}\mathrm{P}^1} & \xrightarrow{([v],w) \mapsto ([\overline{v}],\overline{w})} & \mathfrak{T}\mathbb{C}\mathrm{P}^1 \\
\downarrow{\scriptstyle ([v],w) \mapsto w} & & \downarrow{\scriptstyle ([v],w) \mapsto w} \\
\mathbb{C}\mathrm{P}^1 & \xrightarrow[=:f]{[v] \mapsto [\overline{v}]} & \mathbb{C}\mathrm{P}^1.
\end{array}$$

[2933] We refer to [**MiS1974**, Lemma 14.9] for a very different proof of the lemma.

Note that, despite first appearances, it follows easily from the definitions that this diagram constitutes a morphism of complex vector bundles. Thus we see that

$$c_1(\overline{\mathfrak{T}\mathbb{C}\mathrm{P}^1}) \underset{\substack{\uparrow \\ \text{follows from the above} \\ \text{and Definition 274.1}}}{=} f^*(x) \underset{\substack{\uparrow \\ \text{follows from the Sphere–Degree} \\ \text{Matrix Lemma 151.7}}}{=} -x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}).$$

This implies that $\widetilde{c}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) = -c_1(\overline{\mathfrak{T}\mathbb{C}\mathrm{P}^1}) = -(-x) = x$. ■

274.1.4. **Chern classes of the tangent bundle of complex projective spaces.** Let $n \in \mathbb{N}_0$. We consider the complex projective space $\mathbb{C}\mathrm{P}^n$ and we consider the corresponding tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ which, by the Tangent Bundle Proposition 265.10, is an n-dimensional complex vector bundle. It is natural to ask what the Chern classes of this complex vector bundle are.

The following proposition gives us an elegant, but rather surprising answer to this question:

Proposition 274.9. ($\mathbb{C}\mathrm{P}^n$-Chern-Proposition) Let $n \in \mathbb{N}$. In the following we denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ the standard generator as introduced in Notation 213.6 and in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20. Then

$$c(\mathrm{T}\mathbb{C}\mathrm{P}^n) = (1-x)^{n+1} \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}).$$

In other words, for $k = 0, \ldots, n$ we have

$$c_k(\mathrm{T}\mathbb{C}\mathrm{P}^n) = (-1)^k \cdot \binom{n+1}{k} \cdot x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z} \cdot x^k.$$

Example. It follows from the $\mathbb{C}\mathrm{P}^n$-Chern-Proposition 274.9 that

$$c(\mathbb{C}\mathrm{P}^1) = (1-x)^2 = 1 - 2x + x^2 = 1 - 2x \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) = \mathbb{Z}[x]/(x^2).$$

Note that this calculation, together with the Tautological Bundle-Chern-Lemma 274.4 and the Naturality-of-Chern-Proposition 274.2 shows in particular that the tangent bundle of $\mathbb{C}\mathrm{P}^1$ is not isomorphic to the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1$. □

Proof. First we recall two pieces of notation:

- Let $m \in \mathbb{N}_0$. As on page 5119 we denote by $\mathrm{Tr}_{\mathbb{C}\mathrm{P}^n}(\mathbb{C}^m)$ the trivial m-dimensional $\mathbb{C}$-vector bundle $\mathbb{C}\mathrm{P}^n \times \mathbb{C}^m \to \mathbb{C}\mathrm{P}^n$.
- As in Definition 268.34 we denote by $\mathfrak{T}(\mathbb{C}\mathrm{P}^n)^*$ the dual vector bundle of the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^n$.

We perform the following calculation:

$$
\begin{array}{lll}
c(\mathrm{T}\mathbb{C}\mathrm{P}^n) & \overset{\text{see the discussion on page 5276}}{=} & c(\mathrm{T}\mathbb{C}\mathrm{P}^n \oplus \mathrm{Tr}_{\mathbb{C}\mathrm{P}^n}(\mathbb{C})) \\
& \overset{\text{by the Stable Tangent Bundle–}\mathbb{K}\mathrm{P}^n\text{-Proposition 268.42}}{=} & c(\underbrace{\mathfrak{T}(\mathbb{C}\mathrm{P}^n)^* \oplus \cdots \oplus \mathfrak{T}(\mathbb{C}\mathrm{P}^n)^*}_{n+1 \text{ copies}}) \\
& \overset{\text{by the Chern-Multiplicativity Theorem 274.6}}{=} & c(\mathfrak{T}(\mathbb{C}\mathrm{P}^n)^*)^{n+1} \\
& \overset{\text{by the Dual Vector Bundle Isomorphism Proposition 269.11}}{=} & c(\overline{\mathfrak{T}(\mathbb{C}\mathrm{P}^n)})^{n+1} \\
& \overset{\text{since } \overline{\mathfrak{T}(\mathbb{C}\mathrm{P}^n)} \text{ is a 1-dimensional vector bundle}}{=} & (1 + c_1(\overline{\mathfrak{T}(\mathbb{C}\mathrm{P}^n)}))^{n+1} \\
& \overset{\text{by the Complex Conjugate Chern-Lemma 274.8}}{=} & (1 - c_1(\mathfrak{T}(\mathbb{C}\mathrm{P}^n))^{n+1} \\
& \overset{\text{by the Tautological Line Bundle-Chern Lemma 274.4}}{=} & (1-x)^{n+1}.
\end{array}
$$

■

-

Remark. Let $p\colon E \to M$ be a smooth complex vector bundle over a smooth manifold M. We can consider the Chern classes $c_k(E) \in \mathrm{H}^{2k}(M;\mathbb{Z})$. By the Cohomology-Coefficients-Functor Lemma 197.21 the group homomorphism $\iota\colon \mathbb{Z} \to \mathbb{R}$ gives us an induced homomorphism $\iota_*\colon \mathrm{H}^{2k}(M;\mathbb{Z}) \to \mathrm{H}^{2k}(M;\mathbb{R})$. Thus we obtain the "real" Chern classes in $\iota_*(c_k) \in \mathrm{H}^{2k}(M;\mathbb{R})$. By the De Rham-Isomorphism Theorem 231.16 we have a natural isomorphism $\mathrm{H}^{2k}(M;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}^{2k}_{\mathrm{dR}}(M)$ from singular cohomology to de Rham cohomology. Thus we see that the real Chern classes can be viewed as objects in de Rham cohomology. It turn out that these real Chern classes in de Rham cohomology can be defined directly via de Rham cohomology, completely bypassing singular cohomology. We refer to [**Chern1979**], [**MiS1974**, Appendix C] and [**Hus1993**, Chapter 19] for details. □

274.1.5. **The first Chern class of surfaces.** In this subsection we want to calculate the first Chern classes of surfaces. To make sense of this goal we first have to explain in what sense the tangent bundle of a surface is actually a complex vector bundle. This will be consequence of the following more general construction:[2934]

Lemma 274.10. (Oriented 2-Dimensional Vector Bundle) Let $p\colon E \to B$ be an oriented 2-dimensional real vector bundle over a topological space B that is Hausdorff and paracompact. It follows from the Inner Product-on-Vector Bundle Proposition 269.10 that we can equip E with an inner product. For each $b \in B$ we denote by $\varphi_b\colon E_b \times E_b \to \mathbb{R}$ the corresponding inner product.

(1) Given $b \in B$ we consider the map

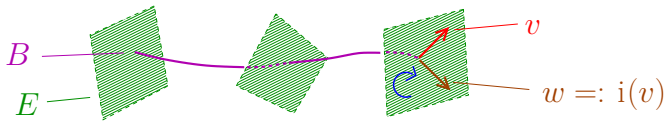
$$
\begin{array}{rcl}
\mathrm{i}\colon E_b & \to & E_b \\
& & \text{the unique vector } w \in E_b \text{ with the following properties:} \\
v & \mapsto & \text{(a) } \varphi_b(v,w)_b = 0 \text{ and } \varphi_b(w,w) = \varphi(v,v), \\
& & \text{(b) if } v \neq 0 \text{ we also demand that } (v,w) \text{ is a positive basis for } E_b.
\end{array}
$$

These maps equip E with the structure of a 1-dimensional complex vector bundle.

(2) Any two inner products on E lead to isomorphic complex vector bundle structures.

Sketch of proof.

[2934]Note that the next lemma is essentially the content of Exercise 269.5.



(1) We start out with the following claim:

Claim. For each $b \in B$ we have $\mathrm{i}^2 = -\mathrm{id}_{E_b}$.

Proof. This claim follows from elementary logic and the observation that if (v, w) is a positive basis for E_b, then so is $(w, -v)$. ⊞

Using this claim it is now elementary to verify that for each $b \in B$ the map

$$\begin{array}{rcl} \mathbb{C} \times E_b & \to & E_b \\ ((x + \mathrm{i}y), v) & \mapsto & x \cdot v + y \cdot \mathrm{i}(v) \end{array}$$

defines a complex vector space structure on E_b. We leave it to the reader to show that these complex vector space structures on the fibers now turn $p\colon E \to B$ into a complex vector bundle.

(2) Using the fact, that the set of inner products is convex one can show that any two inner products on E lead to isomorphic complex vector bundle structures. We leave it to the reader to turn this slogan into a proper proof of Statement (2). ■

Convention 274.11. Let Σ_g be an oriented surface of genus g. We use the Oriented 2-Dimensional Vector Bundle 274.10[2935] to view $\mathrm{T}\Sigma_g$ as a complex vector bundle.[2936]

Now it makes sense to talk of the first Chern class of an oriented surface:

Proposition 274.12. (Chern-of-Surface Proposition) Let Σ_g be an oriented surface of genus g. We have the following equality in $\mathrm{H}^2(\Sigma_g; \mathbb{Z}) = \mathbb{Z} \cdot [\Sigma_g]^*$:

$$c_1(\mathrm{T}\Sigma_g) \;=\; (2 - 2g) \cdot [\Sigma_g]^*.$$

Proof. The idea for the proof is to describe an explicit continuous map $f\colon \Sigma_g \to G_1(\mathbb{C}^\infty)$ such that $f^*\mathfrak{T}G_1(\mathbb{C}^\infty)$ is isomorphic to $\mathrm{T}\Sigma_g$ and to then calculate $f^*(c_1)$.

The construction of the map f will make use of the following claim:

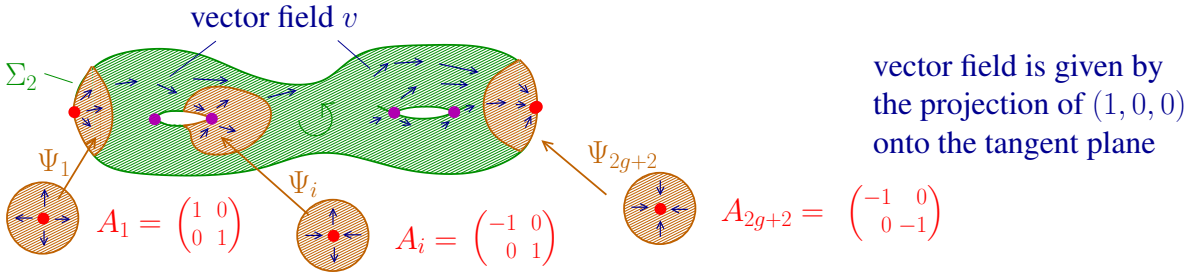
Claim 1. There exists a smooth vector field v on Σ_g, *holomorphic* embeddings $\Psi_i\colon \overline{B}^2 \to \Sigma_g$, $i = 1, \ldots, 2g+2$ with disjoint images and matrices $A_1, \ldots, A_{2g+2} \in \mathrm{O}(2)$ with the following properties:

- The vector field v is non-zero outside of $\Psi_1(\overline{B}^2) \cup \cdots \cup \Psi_{2g+2}(\overline{B}^2)$.
- The vector field $\Psi_i^* v$ on $\overline{B}^2$ is given by $(\Psi_i^* v)(P) = A_i \cdot \overline{P}$ where $\overline{P}$ is the complex conjugate of $P \in \overline{B}^2 \subseteq \mathbb{C}$.
- We have

$$\sum_{i=1}^{2g+2} \det(A_i) \;=\; 2 - 2g.$$

[2936] Alternatively we could also use the Surface–Möbius Structure Proposition 132.4 to view Σ_g as a 1-dimensional complex manifold. It follows from the Tangent Bundle Proposition 265.10 that $\mathrm{T}\Sigma_g$ is naturally a complex vector bundle. As we will see in Exercise 274.4, one easily show that this complex vector bundle structure on $\mathrm{T}\Sigma_g$ corresponds to the one that we introduced in the Oriented 2-Dimensional Vector Bundle 274.10.

Proof. We illustrate such a vector field in the figure below. More precisely, we show Σ_g as a submanifold of $\mathbb{R}^3$ in the "usual way". The corresponding tangent vector at $P \in \Sigma_g$ is then given by projecting $(1,0,0)$ onto $\mathrm{T}_P\Sigma_g$.



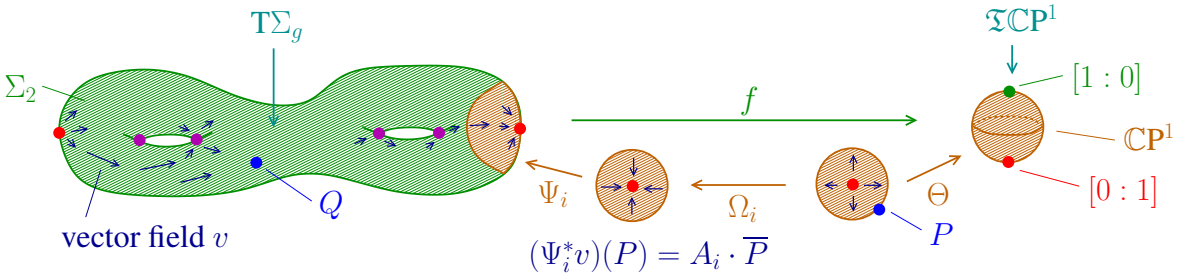
We will not attempt to give a completely rigorous description of such a vector field. Non-Degenerate Vector Field Existence Proposition 53.22 Smooth Poincaré-Hopf Theorem 53.24 ⊞

Next we introduce the following notation:

- We set $M := \Sigma_g \setminus (\Psi_1(B^2) \cup \cdots \cup \Psi_{2g+2}(B^2))$.
- For $i \in \{1, \ldots, n\}$ let $\Omega_i \colon \overline{B}^2 \to \overline{B}^2$ be the map that is given by multiplication by A_i^{-1}.
- We consider the continuous map

$$\begin{aligned} \Theta \colon \overline{B}^2 &\to \mathbb{C}\mathrm{P}^1 \\ z &\mapsto [z : \sqrt{1 - |z|^2}\,]. \end{aligned}$$

  Note that at $z = 0$ the map Θ is an orientation-preserving local diffeomorphism.[2937] Also note that every point on S^1 gets sent to $[1:0] \in \mathbb{C}\mathrm{P}^1$.
- Finally we consider the map $f \colon \Sigma_g \to \mathbb{C}\mathrm{P}^1$ that is given by $x \mapsto [1:0]$ on M and that is given by $\Theta \circ \Omega_i \circ \Psi_i^{-1}$ on each $\Psi_i(\overline{B}^2)$. It follows easily from the Pasting Proposition 4.7 that this map is continuous.



Claim 2. There exists an isomorphism $f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 \xrightarrow{\cong} \mathrm{T}\Sigma_g$ of complex vector bundles over Σ_g.

Proof. First note that by Definition 265.37 we have

$$f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 \;=\; \{(Q, (f(Q), w)) \,|\, Q \in \Sigma_g \text{ and } w \in \mathbb{C} \cdot f(Q) \subseteq \mathbb{C}^2\}.$$

[2937] The orientation-behavior can be seen as follows: We consider the chart $\Psi \colon \mathbb{C}\mathrm{P}^1 \setminus \{[1:0] \to \mathbb{C}\}$ from the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4 that is given by $[z:w] \mapsto z \cdot w^{-1}$. By definition this chart is orientation-preserving. One can easily verify that the composition of $\Theta \colon B^2 \to \mathbb{C}\mathrm{P}^1$ with Ψ is orientation-preserving. Thus Θ is orientation-preserving.

We need to construct an isomorphism $f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 \xrightarrow{\cong} \mathrm{T}\Sigma_g$ of complex vector bundles. First of all we consider the map

$$\begin{array}{rcl} G\colon f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1|_M & \to & \mathrm{T}\Sigma_g|_M \\ (\underbrace{Q}_{\in M}, (\underbrace{f(Q)}_{=[1:0]}, \underbrace{(z,0)}_{\in\mathbb{C}\cdot(1,0)})) & \mapsto & (Q, \underbrace{z}_{\in\mathbb{C}} \cdot \underbrace{v(Q)}_{\in \mathrm{T}_Q\Sigma_g}). \end{array}$$

Since v is non-zero on M we see that this map is an isomorphism of complex vector bundles over M. Our goal is to extend this map from M to all of Σ_g. To do so we have to understand what G does on the boundary curves:

Subclaim 1. Let $i \in \{1, \dots, 2g+2\}$. The following diagram commutes:

$$\begin{array}{ccc} (f\circ\Psi_i\circ\Omega_i|_{S^1})^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 & \xrightarrow{(P,(\overbrace{(f\circ\Psi_i\circ\Omega_i)(z)}^{=[1:0]},(z,0)))\ \mapsto\ (P, z\cdot P^{-1})} & \mathrm{T}\overline{B}^2|_{S^1} \\ \downarrow{\scriptstyle (\Psi_i\circ\Omega_i)_*} & & \downarrow{\scriptstyle (\Psi_i\circ\Omega_i)_*\ (P,w)\mapsto((\Psi_i\circ\Omega_i)(P),\mathrm{D}(\Psi_i\circ\Omega_i)_P(w))} \\ f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1|_{\Psi_i(S^1)} & \xrightarrow{\qquad G \qquad} & \mathrm{T}\Sigma_g|_{\Psi_i(S^1)}. \end{array}$$

Here the left vertical map is induced by $\Psi_i \circ \Omega_i$, as in the Vector Bundle–Pullback Proposition 265.36 and the right vertical map is induced as in the Tangent Bundle–Pullback Lemma 265.41.

Proof of subclaim. Let $(P, ([1:0], (z,0))) \in (f\circ\Psi_i\circ\Omega_i|_{S^1})^*\mathfrak{T}\mathbb{C}\mathrm{P}^1$. We show that "down-right" leads to the same result as "right-down":

$$\begin{array}{rcl} (G\circ(\Psi_i\circ\Omega_i)_*)(\underbrace{P}_{\in S^1}, ([1:0],(z,0))) & = & G((\Psi_i\circ\Omega_i)(P), ([1:0],(z,0))) \\ & = & ((\Psi_i\circ\Omega_i)(P), z\cdot v((\Psi_i\circ\Omega_i)(P))) \\ & = & ((\Psi_i\circ\Omega_i)(P), z\cdot \mathrm{D}(\Psi_i)_{\Omega_i(P)}((\Psi_i^*v)(\Omega_i(P)))) \\ & & \uparrow \\ & & \text{see the Vector Field-Pullback Lemma 29.33 for} \\ & & \text{the definition of } \Psi_i^*v \\ & = & ((\Psi_i\circ\Omega_i)(P), z\cdot \mathrm{D}(\Psi_i)_{\Omega_i(P)}(A_i\cdot A_i^{-1}\cdot P^{-1})) \\ & & \uparrow \\ & & \text{since } (\Psi_i^*v)(Q) = A_i\cdot Q \text{ and} \\ & & \text{since } \Omega_i(P) = A_i^{-1}\cdot\overline{P} = A_i^{-1}\cdot P^{-1} \\ & = & ((\Psi_i\circ\Omega_i)(P), \mathrm{D}(\Psi_i)_{\Omega_i(P)}(z\cdot P^{-1})) \\ & & \uparrow \\ & & \text{since } \Psi_i \text{ is holomorphic} \\ & = & \cdots = (\Psi_i\circ\Omega_i)(P), \mathrm{D})\Psi_i)_{\Omega_i(P)}\mathrm{D}\Omega_i(z\cdot P^{-1}). \end{array}$$

□

Subclaim 2. There exists an isomorphism

$$\Xi\colon \Theta^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathrm{T}\overline{B}^2$$

of complex vector bundles over $\overline{B}^2$ such that for each $P \in S^1$ the map is given by

$$(P, (\underbrace{\Theta(P)}_{=[1:0]}, (z,0))) \mapsto (P, z\cdot P^{-1}).$$

Proof of subclaim. Claim that the map

$$\begin{array}{rcl} & & \Theta^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 \ \to \ \mathrm{T}\overline{B}^2 \\ (P, \underbrace{[P:\sqrt{1-|P|^2}]}_{=\Theta(P)}, \underbrace{\lambda\cdot(P,\sqrt{1-|P|^2})}_{\in\mathbb{C}\cdot\Theta(P)}) & \mapsto & (P,\lambda) \end{array}$$

has the desired properties. Indeed, for $P \in S^1$ it is given by $(P,[P:0],\lambda\cdot(P,0)) \mapsto (P,\lambda)$, which can be rewritten as $(P,[P:0],(Q,0)) \mapsto (P, Q\cdot P^{-1})$. □

For $i = 1,\dots,2g+2$ we now consider the map which is defined by

$$\begin{array}{ccc} (\overbrace{f\circ\Psi_i\circ\Omega_i}^{=\Theta})^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 & \xrightarrow{\quad\Xi\quad} & \mathrm{T}\overline{B}^2|_{S^1} \\ \downarrow{\scriptstyle (\Psi_i\circ\Omega_i)_*} & & \downarrow{\scriptstyle (\Psi_i\circ\Omega_i)_* \ (P,w)\mapsto((\Psi_i\circ\Omega_i)(P),\mathrm{D}(\Psi_i\circ\Omega_i)_P(w))} \\ f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1|_{\Psi_i(S^1)} & \xrightarrow{\quad=:G_i\quad} & \mathrm{T}\Sigma_g|_{\Psi_i(S^1)}. \end{array}$$

Note that it follows from Subclaim 1 and 2 that these maps, together with the map G, define an isomorphism $f^*\mathfrak{T}\mathbb{C}\mathrm{P}^1 \xrightarrow{\cong} \mathrm{T}\Sigma_g$ of complex vector bundles over Σ_g. ⊞

It follows from Claim 2 and Definition 274.1 that it remains to prove the following claim:

Claim 3. We have $c_1(\mathrm{T}\Sigma_g) = (2-2g)\cdot[\Sigma_g]^*$ (up to sign).

Proof. For $i = 1,\dots,2g+2$ we set $P_i := \Psi_i(0)$, $i = 1,\dots,2fg+2$. Note that $[1:0]$ is a regular value of f with preimages $P_1,\dots,P_{2g+2}$. We now see that

$$\begin{array}{rcccl} & \text{by Claim 2 and the Naturality of-Chern-Proposition 274.2} & \text{by the Tautological Bundle Chern-Lemma 274.4} & \text{by the Dual Fundamental Class Degree Lemma 201.28} \\ & \downarrow & \downarrow & \downarrow \\ c_1(\mathrm{T}\Sigma_g) & = & f^*(c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1)) & = & f^*([\mathbb{C}\mathrm{P}^1]^*) & = & \deg(f)\cdot[\Sigma_g]^* \\ & = & [\Sigma_g]^*\cdot\sum_{i=1}^{2g+2}\deg(f,P_i) & = & [\Sigma_g]^*\cdot\sum_{i=1}^{2g+2}\det(A_i) & = & [\Sigma_g]^*\cdot(2-2g). \\ & \uparrow & & \uparrow & & & \\ & \text{by the Local-Global Degree Proposition 195.6} & & \text{by the Local Degree Lemma 195.5} & & & \end{array}$$

■

274.2. **Stiefel-Whitney classes.** This section is organized as follows:

- In Subsection 274.2.1 we will introduce Stiefel-Whitney classes.
- In Subsection 274.2.2 we will prove and discuss the multiplicativity theorem for Stiefel-Whitney classes.
- In Subsection 274.2.3 we will provide a characterization of Stiefel-Whitney classes.
- In Subsection 274.2.4 we will see that the first Stiefel-Whitney class of a real vector bundle is zero if and only if the real vector bundle is orientable.

Throughout this section the reader will have a pronounced sense of déja vu, since most of the concepts and results are very similar to the corresponding concepts and results for Chern classes. To get from the theory of Chern classes to the theory of Stiefel-Whitney classes we just need, for the most part, to replace "complex" by "real", $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients and the degrees $2k$ by k.

274.2.1. **Definition of Stiefel-Whitney classes.** This time we start out by recalling the following theorem:

Theorem 272.17. (Cup Product-$G_n(\mathbb{R}^\infty)$-Theorem) Let $n \in \mathbb{N}_0$. For $k = 1, \dots n$ we consider the $(n \times \infty)$-Schubert symbol $(k, 0, \dots, 0)$ and we consider the corresponding cohomology class $\sigma_{(k,0,\dots,0)} \in \mathrm{H}^k(G_n(\mathbb{R}^\infty); \mathbb{F}_2)$ that we introduced in the $G_n(\mathbb{R}^\infty)$-H^*-Lemma 272.16. The ring homomorphism

$$\begin{array}{rccc} \Xi_n \colon \mathbb{Z}[w_1, \dots, w_n] & \to & \mathrm{H}^*(G_n(\mathbb{R}^\infty); \mathbb{F}_2) \\ w_k & \mapsto & \sigma_{(k,0,\dots,0)} \end{array}$$

is a ring isomorphism.

Similar to Notation 273.4 we introduce the following notation:

Notation 274.13. Using the notation from the Cup Product-$G_n(\mathbb{R}^\infty)$-Theorem 272.17 we define for $k \in \{1, \dots, n\}$ the following cohomology class:

$$w_k \ := \ \Xi_n(w_k) \ = \ \sigma_{(k,0,\dots,0)}) \ \in \ \mathrm{H}^k(G_n(\mathbb{R}^\infty); \mathbb{F}_2).$$

The above theorem allows us, similar to Definition 274.1, to introduce the Stiefel-Whitney classes of real vector bundles:

Definition 274.14. Let $p \colon E \to B$ be an n-dimensional real vector bundle over a topological space B that is Hausdorff and paracompact.

(1) By the Grassmannian–Universal Bundle Theorem 270.10 (1) there exists a continuous map $f \colon B \to G_n(\mathbb{R}^\infty)$ such that the pullback $f^*\mathfrak{T}G_n(\mathbb{R}^\infty)$ is isomorphic to the vector bundle $p \colon E \to B$. Given $k \in \{1, \dots, n\}$ we refer to

$$w_k(E) \ := \ f^*(\underset{\substack{\uparrow \\ \text{element of } \mathrm{H}^k(G_n(\mathbb{R}^\infty); \mathbb{F}_2) \text{ defined in the} \\ \text{Cup Product-}G_n(\mathbb{R}^\infty)\text{-Theorem 272.17}}}{w_k}) \ \in \ \mathrm{H}^k(B; \mathbb{F}_2)$$

as the k-th Stiefel-Whitney class[2938] of the vector bundle. Note that by the Grassmannian–Universal Bundle Theorem 270.10 (1) we know that the continuous map f is unique up to homotopy. It thus follows from the Homotopy-H^*-Proposition 198.11 that the definition of $w_k(E)$ does not depend on the choice of f.

(2) For notational convenience we also set
   (a) $w_0(E) := 1 \in \mathrm{H}^0(B; \mathbb{F}_2)$.
   (b) $w_k(E) := 0 \in \mathrm{H}^k(B; \mathbb{F}_2)$ for $k \in \mathbb{N}_{>n}$.

In the following we will see that the properties of Stiefel-Whitney classes are very similar to the properties of Chern classes. For example the following proposition is an analog of the Naturality-of-Chern-Proposition 274.2:

Proposition 274.15. (Naturality-of-SW-Proposition) Let $k \in \mathbb{N}_0$ and let B and C be topological spaces that are Hausdorff and paracompact.

(1) For any morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

[2938] These cohomology classes were first introduced (in a somewhat different mathematical language) by Eduard Stiefel [**Stie1935**] and Hassler Whitney [**Why1935**].

of real vector bundles we have $h^*(w_k(E)) = w_k(F) \in \mathrm{H}^k(C;\mathbb{F}_2)$.

(2) (a) If $p\colon E \to B$ and $p'\colon E' \to B$ are real vector bundles that are isomorphic over B, then $w_k(E) = w_k(E') \in \mathrm{H}^k(B;\mathbb{F}_2)$.

(b) Let $p\colon E \to B$ be a real vector bundle. For any continuous map $h\colon C \to B$ between the topological spaces C and B we have the following equality:

$$w_k(h^*E) \;=\; h^*(w_k(E)) \;\in\; \mathrm{H}^k(C;\mathbb{F}_2).$$

Proof. The proof is very similar to the proof of the Naturality-of-Chern-Proposition 274.2. ■

We continue with the obvious analog of the Trivializable-Chern-Lemma 274.3:

Lemma 274.16. (Trivializable-SW-Lemma) Let $p\colon E \to B$ be a trivializable real vector bundle over a topological space that is Hausdorff and paracompact. For every $k \in \mathbb{N}$ we have $w_k(E) = 0$.

Proof. The proof is almost identical to the proof of the Trivializable-Chern-Lemma 274.3. ■

We conclude this subsection with the analog of the Tautological Bundle-Chern-Lemma 274.4:

Lemma 274.17. (Tautological Bundle-SW-Lemma) Let $n \in \mathbb{N}$. Furthermore let $p\colon \mathfrak{T}\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ be the tautological real line bundle that we introduced in the Tautological Line Bundle Lemma 265.6. In this case we have

$$w_1(\mathfrak{T}\mathbb{R}\mathrm{P}^n) \;=\; x \;\in\; \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$$

where $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is the unique non-trivial element.

Proof. Let $n \in \mathbb{N}$ and let $p\colon \mathfrak{T}\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ be the tautological real line bundle. We denote by $\iota\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty = G_1(\mathbb{R}^\infty)$ the usual embedding. We have the following equalities in $\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$:

$$w_1(\overbrace{\mathfrak{T}G_1(\mathbb{R}^{n+1})}^{=\mathbb{R}\mathrm{P}^n}) \underset{\text{by the Tautological Bundle Lemma 270.8 (3)}}{=} w_1(\iota^*\mathfrak{T}G_1(\mathbb{R}^\infty)) \underset{\text{by Definition 274.14}}{=} \iota^*(w_1) \underset{\text{by the discussion on page 3832 the map } \iota^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \text{ is an isomorphism}}{=} x.$$

where $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is the unique non-trivial element. ■

274.2.2. **The multiplicativity theorem for Stiefel-Whitney classes.** We start out this subsection with the following analog of Definition 274.5:

Definition 274.18. Given a real vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact we refer to

$$w(E) \;:=\; (w_0(E), w_1(E), w_2(E), \dots) \;\in\; \mathrm{H}^*(B;\mathbb{F}_2) \;:=\; \bigoplus_{i=0}^{\infty} \mathrm{H}^i(B;\mathbb{F}_2)$$

as the total Stiefel-Whitney class.

The following theorem is the utterly predictable analog of the Chern-Multiplicativity Theorem 274.6:

Theorem 274.19. (SW-Multiplicativity Theorem) The total Stiefel-Whitney class is multiplicative with respect to the direct sum of real vector bundles. More precisely, if $e\colon E \to B$ and $f\colon F \to B$ are two real vector bundles over a topological space that is Hausdorff and paracompact, then for each $k \in \mathbb{N}_0$ we have

$$w_k(E \oplus F) \;=\; \sum_{i=0}^{k} w_i(E) \cup w_{k-i}(F) \;\in\; \mathrm{H}^k(B;\mathbb{F}_2).$$

In other words, we have

$$w(E \oplus F) \;=\; w(E) \cup w(F) \;\in\; \mathrm{H}^*(B;\mathbb{F}_2).$$

Proof. Again the proof is very similar to the proof of the Chern-Multiplicativity Theorem 274.6. For the most part we just need to replace the Complex Grassmannian-Whitney Sum-H*-Proposition 273.8 by the Real Grassmannian-Whitney Sum-H*-Proposition 273.9. We leave it to the reader to fill in the details. ■

Example.

(1) Let B be a topological space that is Hausdorff and paracompact, let $E \to B$ be a real vector bundle and let $F \to B$ be a *trivializable* real vector bundle. As on page 5276 we see that we have the following equalities in $\mathrm{H}^*(B;\mathbb{F}_2)$:

$$w(E \oplus F) \underset{\substack{\uparrow\\ \text{by the SW-Multiplicativity}\\ \text{Theorem 274.19}}}{=} w(E) \cup w(F) \underset{\substack{\uparrow\\ \text{by the Trivializable}\\ \text{SW-Lemma 274.16 and}\\ \text{Definition 274.14 (2b)}}}{=} w(E) \cup 1 = w(E).$$

(2) Let $p\colon F \to B$ be a real vector bundle over a topological space that is Hausdorff and paracompact. Let $E \subseteq F$ be a vector subbundle. We consider the normal vector bundle $E^\perp$ from Definition 269.14. We now see that we have the following equalities in $\mathrm{H}^*(B;\mathbb{F}_2)$:

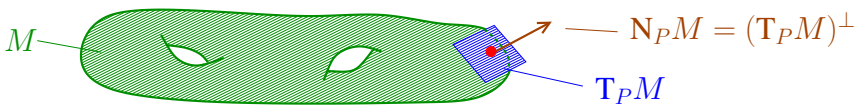
$$w(E) \cup w(E^\perp) \underset{\substack{\uparrow\\ \text{by the SW-Multiplicativity}\\ \text{Theorem 274.19}}}{=} w(E \oplus E^\perp) \underset{\substack{\uparrow\\ \text{by the Normal Vector Bundle}\\ \text{Proposition 269.13 (1)}}}{=} w(F).$$

(3) We calculate that we have the following equalities in $\mathrm{H}^1(\mathbb{RP}^1;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^2)$:

$$w(\mathfrak{T}\mathbb{RP}^1 \oplus \mathfrak{T}\mathbb{RP}^1) \underset{\substack{\uparrow\\ \text{by the SW-Multiplicativity}\\ \text{Theorem 274.19}}}{=} w(\mathfrak{T}\mathbb{RP}^1) \cup w(\mathfrak{T}\mathbb{RP}^1) \underset{\substack{\uparrow\\ \text{by the Tautological}\\ \text{Bundle-SW-Lemma 274.17}}}{=} (1+x) \cup (1+x) \;=\; 1 + 2x + x^2 \;=\; 1.$$

Thus we have shown that the total Stiefel-Whitney class of $\mathfrak{T}\mathbb{RP}^1 \oplus \mathfrak{T}\mathbb{RP}^1$ is trivial. In fact this is not surprising, since by the discussion on pages 5122 and 5171 we know that the Whitney sum of two copies of $\mathfrak{T}\mathbb{RP}^1$ is actually a trivializable vector bundle.

(4) Let $n \in \mathbb{N}$. Let M be a closed orientable submanifold of $\mathbb{R}^n$ of codimension one. For example we could take $M = S^{n-1} \subseteq \mathbb{R}^n$ or M could be the surface of some genus g which, by the discussion in Exercise 32.4, we can view as a submanifold of $\mathbb{R}^3$.



We want to determine the total Stiefel-Whitney class of the tangent bundle $\mathrm{T}M$. As before we denote by $\mathrm{N}M$ the normal vector bundle of $M \subseteq \mathbb{R}^n$. Then[2939]

$$w(\mathrm{T}M) \underset{\uparrow}{=} w(\mathrm{T}M) \cup w(\mathrm{N}M) \underset{\uparrow}{=} w(\mathrm{T}M \oplus \mathrm{N}M)$$

by the Trivializable-SW-Lemma 274.16 since the normal vector bundle $\mathrm{N}M$ of any orientation codimension-one submanifold is trivializable, see page 5135

by the SW-Multiplicativity Theorem 274.19

$$\underset{\uparrow}{=} w(\text{trivial vector bundle } M \times \mathbb{R}^n) \underset{\uparrow}{=} 1.$$

see page 5170

by the Trivializable SW-Lemma 276.6

(5) In (4) we just showed that the Stiefel-Whitney classes of the tangent bundles of all spheres are trivial. On the other hand we know by the Hairy Ball Theorem 46.18 that the tangent bundles of even-dimensional spheres are non-trivializable. This shows that, unfortunately, SW-classes are not full invariants of real vector bundles. □

We record one special case of the Stiefel-Whitney-Multiplicativity Theorem 274.19 as a lemma:

Lemma 274.20. (Manifold Product-SW-Lemma) Let M and N be two smooth manifolds. We assume that $\partial M = \varnothing$ or $\partial N = \varnothing$. We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the natural projections. We have the following equality in $\mathrm{H}^*(M \times N; \mathbb{F}_2)$:

$$w(\mathrm{T}(M \times N)) = p^*(w(\mathrm{T}M)) \cup q^*(w(\mathrm{T}N)).$$

Proof. We have the following equalities in $\mathrm{H}^*(M \times N; \mathbb{F}_2)$:

$$\begin{array}{lll} & \text{by the Tangent Bundle-of-Product Proposition 268.2} & \text{by the Bundle Product-as-Whitney Sum Lemma 268.5} \\ w(\mathrm{T}(M \times N)) \underset{\downarrow}{=} & w(\mathrm{T}M \times \mathrm{T}N) & \underset{\downarrow}{=} w(p^*\mathrm{T}M \oplus q^*\mathrm{T}N) \\ \underset{\uparrow}{=} & w(p^*\mathrm{T}M) \cup w(q^*\mathrm{T}N) & \underset{\uparrow}{=} p^*w(\mathrm{T}M) \cup q^*w(\mathrm{T}N). \\ & \text{by the Stiefel-Whitney-Multiplicativity Theorem 274.19} & \text{by the Naturality-of-SW Proposition 274.15} \end{array}$$

■

The following proposition is the analog of the $\mathbb{C}\mathrm{P}^n$-Chern-Proposition 274.9:

Proposition 274.21. ($\mathbb{R}\mathrm{P}^n$-SW-Proposition) Let $n \in \mathbb{N}_0$. As usual we denote by $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$ the unique non-trivial element. Then

$$w(\mathrm{T}\mathbb{R}\mathrm{P}^n) = (1+x)^{n+1} \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \underset{\uparrow}{=} \mathbb{F}_2[x]/(x^{n+1}).$$

by the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19

Example. We compute the total Stiefel-Whitney class $w(\mathrm{T}\mathbb{R}\mathrm{P}^n)$ for small values of n. More precisely, for $n = 1, 2, 3$ we have the following equalities in $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$:

$$\begin{array}{llll} w(\mathrm{T}\mathbb{R}\mathrm{P}^1) = (1+x)^2 = 1+2x+x^2 & = 1 & \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^1; \mathbb{F}_2) & = \mathbb{F}_2[x]/(x^2), \\ w(\mathrm{T}\mathbb{R}\mathrm{P}^2) = (1+x)^3 = 1+3x+3x^2+x^3 & = 1+x+x^2 & \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) & = \mathbb{F}_2[x]/(x^3), \\ w(\mathrm{T}\mathbb{R}\mathrm{P}^3) = (1+x)^4 = 1+4x+6x^2+4x^3+x^4 & = 1 & \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^3; \mathbb{F}_2) & = \mathbb{F}_2[x]/(x^4). \end{array}$$

We see that $w(\mathbb{R}\mathrm{P}^1) = w(\mathbb{R}\mathrm{P}^3) = 1$. But this is not very surprising: By the Projective Space-Avatar Lemma 9.16 we know that $\mathbb{R}\mathrm{P}^1$ is diffeomorphic to the Lie group S^1 and

[2939] In Exercise 274.12 we will give an alternative proof that $w_n(\mathrm{T}S^n) = 0$.

that $\mathbb{R}\mathrm{P}^3$ is diffeomorphic to the Lie group $\mathrm{SO}(3)$. It thus follows from the Lie Group–Parallelizability Lemma 265.14 that both are parallelizable. □

Remark. Later, in the Parallelizability-of-$\mathbb{R}\mathrm{P}^n$-Proposition 275.4 we will use the $\mathbb{R}\mathrm{P}^n$-SW-Proposition 274.21 and some elementary to show that if $\mathbb{R}\mathrm{P}^n$ is parallelizable, then $n+1$ is a power of 2. □

Proof. First we recall two pieces of notation:

- Let $m \in \mathbb{N}_0$. As on page 5119 we denote by $\mathrm{Tr}_{\mathbb{R}\mathrm{P}^n}(\mathbb{R}^m)$ the trivial m-dimensional $\mathbb{R}$-vector bundle $\mathbb{R}\mathrm{P}^n \times \mathbb{R}^m \to \mathbb{R}\mathrm{P}^n$.
- As in Definition 268.34 we denote by $\mathfrak{T}(\mathbb{R}\mathrm{P}^n)^*$ the dual vector bundle of the tautological vector bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^n$.

We perform the following calculation:

$$\begin{array}{ccccc} & \overset{\text{see the discussion on page 5289}}{\downarrow =} & & \overset{\text{by the Stable Tangent Bundle } \mathbb{K}\mathrm{P}^n\text{-Proposition 268.42}}{\downarrow =} & \\ w(\mathrm{T}\mathbb{R}\mathrm{P}^n) & w(\mathrm{T}\mathbb{R}\mathrm{P}^n \oplus \mathrm{Tr}_{\mathbb{R}\mathrm{P}^n}(\mathbb{R})) & & \underbrace{w(\mathfrak{T}(\mathbb{R}\mathrm{P}^n)^* \oplus \cdots \oplus \mathfrak{T}(\mathbb{R}\mathrm{P}^n)^*)}_{n+1 \text{ copies}} & \\ & = \uparrow & & = \uparrow & = \uparrow \\ & w(\mathfrak{T}(\mathbb{R}\mathrm{P}^n)^*)^{n+1} & & w(\mathfrak{T}(\mathbb{R}\mathrm{P}^n))^{n+1} & (1+x)^{n+1}. \\ & \text{by the SW-Multiplicativity Theorem 274.19} & & \text{by the Dual Vector Bundle Isomorphism Proposition 269.11} & \text{by the Tautological Bundle-SW-Lemma 274.17} \end{array}$$

■

274.2.3. **Characterization of Stiefel-Whitney classes.** The following proposition, which is the obvious analog of the Chern Class Characterization Proposition 274.7, shows that the combination of

(1) the dimension conditions on the Stiefel-Whitney classes from Definition 274.14,
(2) the Naturality-of-SW-Proposition 274.15,
(3) the Stiefel-Whitney-Multiplicativity Theorem 274.19 and
(4) the Tautological Bundle-SW-Lemma 274.17

completely characterizes the Stiefel-Whitney classes:

Proposition 274.22. (Stiefel-Whitney Class Characterization Proposition) Suppose that we have a sequence of maps $\{\widetilde{w}_k\}_{k\in\mathbb{N}_0}$ such that $\widetilde{w}_k$ associates to any real vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact a class $\widetilde{w}_k(E) \in \mathrm{H}^k(B;\mathbb{F}_2)$ such that the following statements hold:

(1) For any n-dimensional real vector bundle $E \to B$ we have $\widetilde{w}_0(E) = 1$ and we have $\widetilde{w}_k(E) = 0$ for all $k > n$.
(2) Each $\widetilde{w}_k$ is natural in the sense that for every morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of real vector bundles we have $h^*(\widetilde{w}_k(E)) = \widetilde{w}_k(F) \in \mathrm{H}^k(C;\mathbb{F}_2)$.
(3) The total class $\widetilde{w}(E) := (\widetilde{w}_0(E), \widetilde{w}_1(E), \dots)$ satisfies the SW-Multiplicativity Theorem 274.19.

(4) For the tautological vector bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ we have $\widetilde{w}_1(\mathfrak{T}\mathbb{R}\mathrm{P}^1) = x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^1;\mathbb{F}_2)$. Then $\widetilde{w}_k = w_k$ for all $k \in \mathbb{N}_0$.

Proof. The proof is very similar to the proof of the Chern Class Characterization Proposition 274.7. We leave it to the reader to make adequate changes. ■

274.2.4. **The first Stiefel-Whitney class and orientability.** We conclude this discussion of Stiefel-Whitney classes with a subsection, that for once does not have an analog in the setting of Chern classes. More precisely, we have the following proposition which gives us a very pleasant characterization of the first Stiefel-Whitney classes:

Proposition 274.23. (Bundle Orientation-w_1-Proposition) Let $p\colon E \to B$ be a real vector bundle over a topological space that is Hausdorff and orientable. If B is locally path-connected, then the following statements are equivalent:
(1) The vector bundle $p\colon E \to B$ is orientable (in the sense of Definition 265.23).
(2) $w_1(E) = 0$.

Example. Let $n \in \mathbb{N}_0$. We consider the real projective space $\mathbb{R}\mathrm{P}^n$. Note that we have the following equalities in $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2$:

$$w_1(\mathrm{T}\mathbb{R}\mathrm{P}^n) \underset{\substack{\uparrow\\ \text{by the } \mathbb{R}\mathrm{P}^n\text{-SW-Proposition 274.21}}}{=} x\text{-coefficient of } \underbrace{(1+x)^{n+1}}_{=1+(n+1)\cdot x+\dots} = n+1 = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

This is consistent with the fact, shown on page 822, that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd. □

Sketch of proof. [2940] We will only provide a sketch of a proof. In particular we will swipe all questions about continuity under a spacious carpet. Let $d \in \mathbb{N}_0$. As in Definition 270.1 we consider

$$\mathrm{GL}_n(\mathbb{R}^d) = \{(v_1,\dots,v_n) \in (\mathbb{R}^d)^n \mid v_1,\dots,v_n \text{ are linearly independent}\}.$$

Next, similar to Definitions 270.2 and 270.6 we introduce the following two equivalence relations on $\mathrm{GL}_n(\mathbb{R}^d)$:

$$(v_1,\dots,v_n) \sim (w_1,\dots,w_n) \quad :\Longleftrightarrow \quad \operatorname{span}\{v_1,\dots,v_n\} = \operatorname{span}\{w_1,\dots,w_n\}$$
$$(v_1,\dots,v_n) \sim_{\mathrm{or}} (w_1,\dots,w_n) \quad :\Longleftrightarrow \quad \begin{array}{l}\operatorname{span}\{v_1,\dots,v_n\} = \operatorname{span}\{w_1,\dots,w_n\} \text{ and the} \\ \text{base change matrix has determinant } > 0\end{array}$$

and we set

$$G_n(\mathbb{R}^d) := \mathrm{GL}_n(\mathbb{R}^d)/\!\sim$$
$$G_n^{\mathrm{or}}(\mathbb{R}^d) := \mathrm{GL}_n(\mathbb{R}^d)/\!\sim_{\mathrm{or}},$$

both of which we equip with the quotient topology. Finally we set

$$G_n(\mathbb{R}^\infty) := \operatorname*{colim}_{\longrightarrow} G_n(\mathbb{R}^d)$$
$$G_n^{\mathrm{or}}(\mathbb{R}^\infty) := \operatorname*{colim}_{\longrightarrow} G_n^{\mathrm{or}}(\mathbb{R}^d).$$

[2939] These four statements are precisely Axioms 1–4 for Stiefel-Whitney classes as formulated in [**MiS1974**, p. 37].
[2940] Alternative proofs are sketched in [**MiS1974**, Problem 12-A], [**Hus1993**, p. 260], [**OsbH1982**, Proposition V.3.2] and [**Bre1993**, Proposition VI.17.2].

Claim 1.
(1) $G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ is path-connected.
(2) The obvious map

$$\begin{array}{rcl} \mathrm{pr}\colon G_n^{\mathrm{or}}(\mathbb{R}^\infty) & \to & G_n(\mathbb{R}^\infty) \\ [(v_1,\dots,v_n)] & \mapsto & [(v_1,\dots,v_n)] \end{array}$$

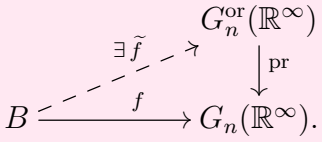
is a 2-fold covering.

Proof.

(1) In Exercise 274.23 we will show that $G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ is path-connected.
(2) First note that it follows easily from the explicit description of colimits (see the Topological-Direct Colimit Proposition 104.10) that the map $\mathrm{pr}\colon G_n^{\mathrm{or}}(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ is surjective. Next we consider the same open subsets of $G_n(\mathbb{R}^\infty)$ as in the proof of the Tautological Bundle Lemma 270.8. One can easily verify that these are uniformly covered in the sense of Definition 91.8. Thus we have shown that the map is a covering. Finally, since the preimage of each point in $G_n(\mathbb{R}^\infty)$ consists of two points we see that the covering has index 2. ⊞

Claim 2. Let B be a topological space and let $f\colon B \to G_n(\mathbb{R}^\infty)$ be a continuous map. The pullback bundle $f^*\mathfrak{T}G_n(\mathbb{R}^\infty)$ is orientable if and only if the map $f\colon B \to G_n(\mathbb{R}^\infty)$ lifts to a continuous map $\widetilde{f}\colon B \to G_n^{\mathrm{or}}(\mathbb{R}^\infty)$. In other words:

the pullback bundle $f^*\mathfrak{T}G_n(\mathbb{R}^\infty)$ is orientable $\iff$



Proof. Similar to the Tautological Bundle Lemma 270.8 we see that there exists a tautological *oriented* n-dimensional smooth $\mathbb{R}$-vector bundle $\mathfrak{T}G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ which has the property that $\mathrm{pr}^*\mathfrak{T}G_n(\mathbb{R}^\infty) = \mathfrak{T}G_n^{\mathrm{or}}(\mathbb{R}^\infty)$. The "$\Leftarrow$"-direction now follows easily from the Vector Bundle-Orientation Functorial Lemma 265.27 which says that the pullback of an oriented vector bundle inherits an orientation.

Next note that any n-dimensional vector subspace $V \subseteq \mathbb{R}^\infty$ defines a point in $G_n(\mathbb{R}^\infty)$, and that any choice of an orientation $\mathcal{O}$ for V defines a point in $G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ which we call $(V,\mathcal{O})$. The "$\Rightarrow$"-direction follows easily from the observation that an orientation $\mathcal{O}_b$ on each $(b, f(b))$ defines a lift $b \mapsto (f(b), \mathcal{O}_b)$ to a map $\widetilde{f}\colon B \to G_n^{\mathrm{or}}(\mathbb{R}^\infty)$. ⊞

Now it remains to prove the following claim:

Claim 3. Let B be a topological space that is locally path-connected. Furthermore let $f\colon B \to G_n(\mathbb{R}^\infty)$ be a continuous map. The map $f\colon B \to G_n(\mathbb{R}^\infty)$ lifts to a continuous map $\widetilde{f}\colon B \to G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ if and only if $f^*(w_1) = 0$.

Proof. One can easily verify that it suffices to deal with the case that B is path-connected and non-empty. We pick $b_0 \in B$ and we set $x_0 := f(b_0)$. We pick $y_0 \in G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ with

$\mathrm{pr}(y_0) = x_0$. We consider the following diagram:

$$\begin{array}{ccccccc}
\pi_1(B,b_0) & \twoheadrightarrow & \mathrm{H}_1(B) & \xrightarrow{\sigma\mapsto\sigma\otimes 1} & \mathrm{H}_1(B)\otimes\mathbb{F}_2 & \xrightarrow{\cong} & \mathrm{H}_1(B;\mathbb{F}_2) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\pi_1(G_n(\mathbb{R}^\infty),x_0) & \twoheadrightarrow & \mathrm{H}_1(G_n(\mathbb{R}^\infty)) & \xrightarrow{\sigma\mapsto\sigma\otimes 1} & \mathrm{H}_1(G_n(\mathbb{R}^\infty))\otimes\mathbb{F}_2 & \xrightarrow{\cong} & \mathrm{H}_1(G_n(\mathbb{R}^\infty);\mathbb{F}_2) \\
\downarrow{\scriptstyle =:\alpha} & & \downarrow{\scriptstyle \beta} & & & & \cong\downarrow{\scriptstyle \gamma} \\
\dfrac{\pi_1(G_n(\mathbb{R}^\infty),x_0)}{\mathrm{pr}_*(\pi_1(G_n^{\mathrm{or}}(\mathbb{R}^\infty),y_0))} & \xrightarrow[\theta]{\cong} & \mathbb{F}_2 & & \xrightarrow{\quad\mathrm{id}\quad} & & \mathbb{F}_2.
\end{array}$$

We make the following clarifications and observations:

- It follows from Claim 1 together with the Covering-Degree Proposition 92.10 and the Index 2–Subgroup Lemma 95.12 (2) that the bottom left group admits an isomorphism θ to $\mathbb{F}_2$.
- The top left horizontal map and the middle left horizontal map are the epimorphisms given by the Hurewicz Theorem 166.10.
- The top right horizontal map and the middle right horizontal map are the isomorphisms given by the Universal Coefficient Theorem 162.16.[2941]
- It follows easily from the Hurewicz Theorem 166.10 that there exists a homomorphism $\beta\colon \mathrm{H}_1(G_n(\mathbb{R}^\infty)) \to \mathbb{F}_2$ that makes the bottom left square commute.
- The argument of the $G_n(\mathbb{R}^{n+r})$-H_*-Proposition 271.19 shows that there exists an isomorphism $\mathrm{H}_1(G_n(\mathbb{R}^\infty);\mathbb{F}_2) \xrightarrow{\cong} \mathbb{F}_2$. Since $\mathbb{F}_2$ has only one automorphism we see that this isomorphism is unique. This implies that the bottom right square commutes.

We now see that

f lifts to a continuous map $\widetilde{f}\colon B \to G_n^{\mathrm{or}}(\mathbb{R}^\infty)$

$\Big\downarrow$ by the Map Lifting Criterion 127.5, here we use that B is locally path-connected

$\iff$ $\theta\circ\alpha\circ f_*\colon \pi_1(B,b_0)\to\mathbb{F}_2$ is the trivial map

$\iff$ $\beta\circ f_*\colon \mathrm{H}_1(B)\to\mathbb{F}_2$ is the trivial map

$\Big\uparrow$ by the above commutative diagram

$\iff$ $\gamma\circ f_*\colon \mathrm{H}_1(B;\mathbb{F}_2)\to\mathbb{F}_2$ is the trivial map

$\iff$ $f^*(w_1) = 0$.

$\Big\uparrow$ by the Universal Coefficient-H^*–Field Theorem 202.2, since w_1 is the unique non-zero element of $\mathrm{H}^1(G_n(\mathbb{R}^\infty);\mathbb{F}_2)$ ■

Remark. Let M be a smooth manifold and let $x_0 \in M$. In the lecture notes we introduced the following two objects:

(1) In Definition 95.9 we introduced the orientation covering of M. The orientation covering is a natural 2-fold covering $\widetilde{M} \to M$ such that $\widetilde{M}$ admits a natural orientation. In Definition 95.15 we used this natural covering to define the orientation character $w_1^{x_0}\colon \pi_1(M,x_0)\to\mathbb{Z}/2\mathbb{Z}$.

(2) In Definition 274.14 we just introduced the Stiefel-Whitney class $w_1(\mathrm{T}M) \in \mathrm{H}^1(M;\mathbb{F}_2)$.

[2941] In summary: The top and middle horizontal maps from the left to the right are the natural isomorphisms from Exercises 166.12 and 162.12.

It follows from the Hurewicz Theorem 166.10 and the Universal Coefficient-H^*-Theorem 201.23 that we have a natural isomorphism $\mathrm{H}^1(M;\mathbb{F}_2) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(M,x_0),\mathbb{F}_2)$. We leave it to the reader to show, using the ideas from the proof of the Bundle Orientation-w_1-Proposition 274.23 and using the Orientation Covering–Properties-Proposition 191.29 (4), that under this natural isomorphism the first Stiefel-Whitney class corresponds to the orientation character. □

274.3. **Appendix: The relationship between Chern classes and Stiefel-Whitney classes.** A complex vector bundle $p\colon E \to B$ can of course also be viewed as a real vector bundle, thus we obtain Chern classes $c_k(E) \in \mathrm{H}^{2k}(B;\mathbb{Z})$ and also Stiefel-Whitney classes $w_l(E) \in \mathrm{H}^l(B;\mathbb{F}_2)$. The following proposition explains the relationship between these characteristic classes:

Proposition 274.24. (Chern–Stiefel Whitney-Relationship Proposition) We denote by $\gamma_*\colon \mathbb{Z} \to \mathbb{F}_2$ the obvious ring homomorphism. By the Cohomology-Coefficients-Functor Lemma 197.21 we obtain an induced homomorphism $\gamma_*\colon \mathrm{H}^k(X;\mathbb{Z}) \to \mathrm{H}^k(X;\mathbb{F}_2)$. For every complex vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact and any $k \in \mathbb{N}_0$ we have the following equality in $\mathrm{H}^{2k}(B;\mathbb{F}_2)$:

$$w_k(E) \;=\; \begin{cases} \gamma_*(c_l(E)), & \text{if } k = 2l \text{ is even,} \\ 0, & \text{else.} \end{cases}$$

Proof. [2942] We start out with the following claim which is a variation on the Chern Class Characterization Proposition 274.7 and the Stiefel-Whitney Class Characterization Proposition 274.22:

Claim 1. Suppose we have a sequence of maps $\{\widetilde{g}_k\}_{k\in\mathbb{N}_0}$ such that $\widetilde{g}_k$ associates to any complex vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact a class $\widetilde{g}_k(E) \in \mathrm{H}^k(B;\mathbb{F}_2)$ such that the following statements hold:[2943]

(1) For any n-dimensional complex vector bundle $E \to B$ we have $\widetilde{g}_0(E) = 1$ and we have $\widetilde{g}_k(E) = 0$ for all $k > 2n$.

(2) Each $\widetilde{g}_k$ is natural in the sense that for every morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of complex vector bundles we have $h^*(\widetilde{g}_k(E)) = \widetilde{g}_k(F) \in \mathrm{H}^{2k}(C;\mathbb{F}_2)$.

(3) The total class

$$\widetilde{g}(E) \;:=\; (\widetilde{g}_0(E), \widetilde{g}_1(E), \dots)$$

is multiplicative.

(4) For the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ we have $\widetilde{g}_2(\mathfrak{T}\mathbb{C}\mathrm{P}^1) = x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{F}_2)$.

If we are given two such families $\widetilde{g}_k$ and g_k with these properties, then $\widetilde{g}_k = g_k$ for all $k \in \mathbb{N}_0$.

Proof. First note that it follows from the argument of the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 that $\mathrm{H}^1(\mathbb{C}\mathrm{P}^1;\mathbb{F}_2) = 0$. It follows from this observation that $\widetilde{g}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) = 0 = g_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1)$. It follows from this observation that the total characteristic class of $\mathfrak{T}\mathbb{C}\mathrm{P}^1$ are determined by

[2942] An alternative proof of the proposition is given in [**Hat2**, p. 87].

(1) and (4). With this observation the proof of the claim is very similar to the proof of the Chern Class Characterization Proposition 274.7. We leave it to the reader to make the necessary changes. ⊞

By Claim 1 it remains to prove that the left and the right hand side of the proposition satisfy Properties (1)-(4). We will do so in the following two claims:

Claim 2. The cohomology classes

$$\begin{cases} \gamma_*(c_l(E)), & \text{if } k = 2l \text{ is even,} \\ 0, & \text{else} \end{cases}$$

satisfy Properties (1)-(4).

Proof. The claim follows easily from Definition 274.1 together with the Naturality-of-Chern-Proposition 274.2, the Chern-Multiplicativity Theorem 274.6, the Tautological Bundle-Chern-Lemma 274.4, the Cup Product-Coefficients Lemma 204.9 and the H^*–Change-of-Coefficients Theorem 202.10. ⊞

Claim 3. The Stiefel-Whitney classes $w_k(E)$ satisfy Properties (1)-(4).

Proof. First note that it follows immediately from Definition 274.14, with the Naturality-of-SW-Proposition 274.15, the Stiefel-Whitney-Multiplicativity Theorem 274.19 and the Cup Product-Coefficients Lemma 204.9 that the Stiefel-Whitney classes $w_k(E)$ satisfy Properties (1)-(3).

It remains to prove that $w_2(\mathfrak{T}\mathbb{C}\mathrm{P}^1) = x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1; \mathbb{F}_2)$. This can be done directly, see Exercise 274.21. Alternatively one can see it as follows:

$$w_2(\mathfrak{T}\mathbb{C}\mathrm{P}^1) \underset{\substack{\uparrow \\ \text{by the Euler–Stiefel-Whitney Class Proposition 278.14 together with the Euler Class Relationship Lemma 278.2}}}{=} \gamma_*(e(\mathfrak{T}\mathbb{C}\mathrm{P}^1)) \underset{\substack{\uparrow \\ \text{by the Euler–Chern Class Proposition 278.14}}}{=} \gamma_*(c_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1)) \underset{\substack{\uparrow \\ \text{by Claim 2}}}{=} x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1; \mathbb{F}_2).$$

■

274.4. Exercises for Chapter 274.

Exercise 274.1.

(a) Let $n \in \mathbb{N}_0$. We consider the complex projective space $\mathbb{C}\mathrm{P}^n$ and we consider the corresponding tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$. Show that the dual tangent bundle $\mathrm{T}\mathbb{C}\mathrm{P}^n$ is never isomorphic to the tangent bundle.

(b) Does the analog of (a) also hold in the real setting?

Exercise 274.2. Let $n \in \mathbb{N}$. Show that given any $m \in \mathbb{N}$ there exists a closed n-dimensional complex manifold M and an n-dimensional complex vector bundle $p\colon E \to M$ such that $\langle c_n(E), [M] \rangle \geq m$.

Exercise 274.3. Let $n \in \mathbb{N}_0$ and let $k \in \{0, \dots, n\}$. As usual we view $\mathbb{C}\mathrm{P}^k$ as a submanifold of $\mathbb{C}\mathrm{P}^n$ via the smooth embedding $[P] \mapsto [(P, 0)]$. We consider the normal vector bundle $\mathrm{N}\mathbb{C}\mathrm{P}^k$ of $\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$ from Definition 265.16. Determine $c(\mathrm{N}\mathbb{C}\mathrm{P}^k) \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^k; \mathbb{Z})$.

Exercise 274.4. Let $p\colon E_{\mathbb{C}} \to B$ be a 1-dimensional complex vector bundle. Using the Complex Vector Bundle–Orientable Lemma 265.25 we can view $p\colon E_{\mathbb{C}} \to B$ as an oriented

2-dimensional real vector bundle $p\colon E_{\mathbb{R}} \to B$. We assume that B is Hausdorff and paracompact. Show that there exists an inner product on E such that the construction of the Oriented 2-Dimensional Vector Bundle 274.10 turns the real $p\colon E_{\mathbb{R}} \to B$ into the original complex vector bundle. $p\colon E_{\mathbb{C}} \to B$.

Exercise 274.5. Let M be a closed n-dimensional smooth manifold and let $i_1, \dots, i_r \in \mathbb{N}_0$ such that $i_1 + \cdots + i_r = n$. We consider

$$\langle \underbrace{w_{i_1}(M) \cup \cdots \cup w_{i_r}(M)}_{\in \mathrm{H}^n(M;\mathbb{F}_2)}, \underbrace{[M]_{\mathbb{F}_2}}_{\in \mathrm{H}_n(M;\mathbb{F}_2)} \rangle \in \mathbb{F}_2.$$

Show that if M is the boundary of a compact $(n+1)$-dimensional smooth manifold W, then $\langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M), [M]_{\mathbb{F}_2} \rangle = 0$.
Hint. What is the restriction of the $(n+1)$-dimensional real vector bundle $\mathrm{T}W$ to M?

Exercise 274.6. Let Σ be a closed oriented connected non-empty 2-dimensional smooth manifold. Show that given any $n \in \mathbb{Z}$ there exists a 1-dimensional complex vector bundle $p\colon E \to B$ with $c_1(E) = n \cdot [\Sigma]^* \in \mathrm{H}^2(\Sigma;\mathbb{Z}) = \mathbb{Z} \cdot [\Sigma]^*$.

Exercise 274.7. Let $k \in \mathbb{N}$.

(a) Show that $w_{2k}(\mathbb{R}\mathrm{P}^{2k}) \neq 0$.

(b) Use (a) to show that every vector field on $\mathbb{R}\mathrm{P}^{2k}$ has at least one zero.
Remark. This statement also follows from the Hairy Ball Theorem. But that is right now not the point.

Exercise 274.8. We consider the real projective space $\mathbb{R}\mathrm{P}^7$.

(a) Compute $w(\mathbb{R}\mathrm{P}^7) \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^7;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^8)$.

(b) Show that $w(\mathbb{R}\mathrm{P}^7)$ has an inverse in the ring $w(\mathbb{R}\mathrm{P}^7) \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^7;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^8)$ and compute the inverse $w(\mathbb{R}\mathrm{P}^7)^{-1}$ explicitly.

Exercise 274.9. Compute the Stiefel-Whitney classes of $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^3$.
Remark. The $\mathbb{F}_2$-Cohomology Ring-of-Product Proposition 208.16 together with the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 allows you to make the identifications

$$\begin{aligned} \mathrm{H}^*(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^3;\mathbb{F}_2) &= \mathrm{H}^*(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \otimes \mathrm{H}^*(\mathbb{R}\mathrm{P}^3;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^3) \otimes \mathbb{F}_2[y]/(y^4) \\ &= \mathbb{F}_2[x,y]/(x^3, y^4). \end{aligned}$$

Exercise 274.10. Let M and N be two closed smooth manifolds. Show that if $w_k(\mathrm{T}M) \neq 0$ and $w_l(\mathrm{T}N) \neq 0$, then $w_{k+l}(\mathrm{T}(M \times N)) \neq 0$.
Remark. The $\mathbb{F}_2$-Cohomology Ring-of-Product Proposition 208.16 allows you to make the identification $\mathrm{H}^*(M \times N;\mathbb{F}_2) = \mathrm{H}^*(M;\mathbb{F}_2) \otimes \mathrm{H}^*(N;\mathbb{F}_2)$.

Exercise 274.11. Let $p\colon E \to B$ and $q\colon F \to C$ be real vector bundles. Show that for each $k \in \mathbb{N}_0$ we have the following equality in $\mathrm{H}^k(B \times C;\mathbb{F}_2)$:

$$w_k(E \times F) = \sum_{i=0}^{k} \underbrace{w_i(E) \times w_{k-i}(F)}_{\substack{\text{cross product from} \\ \text{Definition 236.13}}}.$$

Exercise 274.12. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ be the natural projection.

(a) Show that the induced map $p^*\colon \mathrm{H}^n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \mathrm{H}^n(S^n;\mathbb{F}_2)$ is the zero map.

(b) Use (a) to show that $w_n(\mathrm{T}S^n) = 0$.
Remark. On page 5290 we gave an alternative proof for this statement.

Exercise 274.13. Let $p\colon \widetilde{M} \to M$ be the orientation-covering of a closed non-orientable connected n-dimensional smooth manifold.

(a) Show that the induced map $p^*\colon \mathrm{H}^n(M;\mathbb{F}_2) \to \mathrm{H}^n(\widetilde{M};\mathbb{F}_2)$ is the zero map.
(b) Show that $w_n(\widetilde{M}) = 0$.
(c) Which closed orientable connected 2-dimensional smooth manifolds are diffeomorphic to the orientation-covering of a non-orientable smooth manifolds?

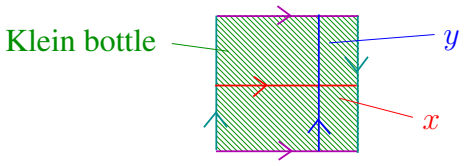
Remark. This exercise is a generalization of Exercise 274.12.

Exercise 274.14. We say that a smooth manifold M admits a *field of tangent k-planes* if its tangent bundle admits a vector subbundle of dimension k.

(a) Show that $\mathbb{R}\mathrm{P}^4$ and $\mathbb{R}\mathrm{P}^6$ do not admit fields of tangent 2-planes.
(b) Show that $\mathbb{R}\mathrm{P}^n$ admits a field of tangent 1-planes if and only if n is odd.

Exercise 274.15. We consider the Klein bottle K as shown in the figure below together with two oriented 1-dimensional smooth submanifolds x and y.

(a) Show that x and y define elements of $\mathrm{H}_1(K;\mathbb{F}_2)$ and show that they form a basis for $\mathrm{H}_1(K;\mathbb{F}_2)$.
(b) Recall that by the Universal Coefficient-H^*–Field Theorem 202.2 the evaluation map $\mathrm{H}^1(K;\mathbb{F}_2) \to \mathrm{Hom}(\mathrm{H}_1(K;\mathbb{F}_2),\mathbb{F}_2)$ is an isomorphism. Use this isomorphism to express $w_1(K)$ as an element of $\mathrm{Hom}(\mathbb{F}_2 \cdot [x] \oplus \mathbb{F}_2 \cdot [y];\mathbb{F}_2),\mathbb{F}_2)$.



Exercise 274.16. Let $(M,+)$ be an abelian monoid. We consider $M \times M$ with the relation

$$(m_1,m_2) \sim (n_1,n_2) \iff \text{there exists } k \in M \text{ with } m_1+n_2+k = n_1+m_2+k.$$

(a) Show that $\sim$ is an equivalence relation on $M \times M$.
(b) We set $G := (M \times M)/\!\sim$. Show that $[(m_1,m_2)] + [(n_1,n_2)] := [(m_1+n_1, m_2+n_2)]$ is a well-defined operation on G and show that this defines a group structure on G.

Remark. This construction of a group out of a monoid is called the *Grothendieck construction*. Note that this exercise is essentially the same as Exercise 226.6.

Exercise 274.17. Let X be a topological space. We consider the monoid that is given by isomorphism classes of complex vector bundles over X with addition given by the Whitney sum. The group which we obtain from this abelian monoid by the Grothendieck construction from Exercise 274.16 is called the *K-group* $\mathrm{K}(X)$. Show that $\mathrm{K}(S^2)$ admits an epimorphism onto $\mathbb{Z}$.

Exercise 274.18. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a compact topological space B which is Hausdorff. In Exercise 270.4 we showed that there exists a $d \in \mathbb{N}_0$ and a continuous map $g\colon B \to G_n(\mathbb{K}^d)$ such that E is isomorphic

to the pullback $g^*\mathfrak{T}G_d(\mathbb{K}^d)$. Show that in general this conclusion does not hold if B non-compact.

Exercise 274.19. Let B be a topological space that is Hausdorff, paracompact and which satisfies $\mathrm{H}_1(B) \cong \mathbb{Z}$. Furthermore let $p_i\colon E_i \to B$, $i = 1, \ldots, m$ be real vector bundles. Let s be the number of non-orientable vector bundles in this list. Show that the Whitney sum $E_1 \oplus \cdots \oplus E_m$ is orientable if and only if s is even.

Exercise 274.20. Let M be a smooth manifold.

(a) Show that if there exists a $k \in \mathbb{N}$ with $w_k(\mathrm{T}M) \neq 0$, then for any non-empty smooth manifold N the product $M \times N$ is non-parallelizable.
(b) Does the conclusion of (a) also hold, if we just assume that M is not parallelizable?

Exercise 274.21.

(a) We consider the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^1 & \to & G_2(\mathbb{R}^4) \\ [(v_0, v_1)] & \mapsto & \mathbb{C} \cdot (v_0, v_1). \end{array}$$

Show that $f^*\mathfrak{T}G_2(\mathbb{R}^4)$ is isomorphic to $\mathfrak{T}\mathbb{C}\mathrm{P}^1$.
(b) Show that $f^*\colon \mathrm{H}^2(G_2(\mathbb{R}^4); \mathbb{F}_2) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^1; \mathbb{F}_2)$ is a non-zero map.
Suggestion. You could study cellular cohomology.
(c) Show that $\mathrm{H}^2(G_2(\mathbb{R}^4); \mathbb{F}_2) = w_1^2 \cdot \mathbb{F}_2 \oplus w_2 \cdot \mathbb{F}_2$.
(d) Show that $w_2(\mathfrak{T}\mathbb{C}\mathrm{P}^1) \neq 0 \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1; \mathbb{F}_2)$.
Hint. What is $f^*(w_1)$?

Exercise 274.22. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $p\colon E \to B$ be an n-dimensional $\mathbb{K}$-vector bundle over a compact topological space B which is Hausdorff. In Exercise 270.5 we showed that there exists a $\mathbb{K}$-vector bundle $q\colon F \to B$ such that $E \oplus F$ is trivializable. Show that in general this conclusion does not hold if B non-compact.
Remark. This exercise is identical to Exercise 274.22, but not it is perhaps easier to solve the exercise.

Exercise 274.23. We introduce the following two equivalence relations on $\mathrm{GL}_n(\mathbb{R}^d)$:

$$(v_1, \ldots, v_n) \sim_{\mathrm{or}} (w_1, \ldots, w_n) \quad :\Longleftrightarrow \quad \begin{array}{l} \mathrm{span}\{v_1, \ldots, v_n\} = \mathrm{span}\{w_1, \ldots, w_n\} \text{ and the} \\ \text{base change matrix has determinant} > 0 \end{array}$$

and we set

$$G_n^{\mathrm{or}}(\mathbb{R}^d) \coloneqq \mathrm{GL}_n(\mathbb{R}^d)/\sim_{\mathrm{or}}.$$

Show that $G_n^{\mathrm{or}}(\mathbb{R}^\infty)$ is path-connected.

Exercise 274.24. Let $p\colon E \to B$ be a real vector bundle over a topological space that is Hausdorff and paracompact. Use Stiefel-Whitney classes to show that the Whitney sum $E \oplus E \to B$ is always orientable.
Remark. This gives a new solution to Exercise 268.4.

Exercise 274.25. Let M be an m-dimensional smooth manifold. By the Tangent Bundle Proposition 265.10 we can view $\mathrm{T}M$ as a $2m$-dimensional smooth manifold.

(a) Let $p\colon \mathrm{T}M \to M$ be the natural projection. Show that there exists a natural isomorphism $\mathrm{TT}M \to p^*\mathrm{T}M \oplus p^*\mathrm{T}M$ of real vector bundles over $\mathrm{T}M$.
Remark. This is the content of Exercise 265.18.

(b) Show that the tangent bundle $\mathrm{T}M$ of any m-dimensional smooth manifold M, viewed as a $2m$-dimensional smooth manifold, is orientable.
Hint. Use Exercise 274.24.
Remark. Note that this gives a new solution to Exercise 268.5.

(c) Provide an example where the tangent bundle of $\mathrm{T}M$, viewed as a smooth manifold, is not parallelizable.

CHAPTER 275

Applications of Stiefel-Whitney classes

Given a real vector bundle $p\colon E \to B$ we introduced in the previous Chapter 274 the Stiefel-Whitney classes $w_k(E) \in \mathrm{H}^k(B;\mathbb{F}_2)$ and we calculated these Stiefel-Whitney classes for the tangent bundles of the real projective spaces $\mathbb{R}\mathrm{P}^n$. In this chapter we will use these calculations to obtain several results which a priori seem to have little to do with vector bundles. More precisely, we will give a new proof that division algebras can only exist in dimensions 2^n and, given $n \in \mathbb{N}_0$, we will give lower bounds on the dimensions m for which we can find an immersion $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^m$.

This chapter is organized as follows:

- In Section 275.1 we will discuss the relationship between the existence of division algebras and the parallelizability of real projective spaces.
- In Section 275.2 we will use Stiefel-Whitney classes to show that "most" real projective spaces are not parallelizable and we will use the results from Section 275.1 to show that "most" $\mathbb{R}^n$ do not admit a division algebra structure.
- In Section 275.3 we will give an application of Stiefel-Whitney classes to the nonexistence of immersions of smooth manifolds into a given $\mathbb{R}^m$.
- In Appendix 275.4 we will state and discuss Massey's Vanishing Theorem 275.8 and Cohen's Immersion Theorem 275.9.
- In Section 275.5 we will provide exercises for Chapter 275.

In the next Chapter 276 we will introduce the Pontryagin classes of real vector bundles.

275.1. **Division algebras and parallelizability of real projective spaces.** We start out by recalling the following definition:

Definition 265.13. We say that a smooth manifold M is parallelizable if its tangent bundle $\mathrm{T}M$ is trivializable.

In this and the next section we will study which real projective spaces are parallelizable. Some certainly are: Clearly $\mathbb{R}\mathrm{P}^0 = \{1\}$ is parallelizable. Next, by the Projective Space-Avatar Lemma 9.16 we know that $\mathbb{R}\mathrm{P}^1$ is diffeomorphic to the Lie group S^1 and that $\mathbb{R}\mathrm{P}^3$ is diffeomorphic to the Lie group $\mathrm{SO}(3)$. It now follows from the Lie Group–Parallelizability Lemma 265.14 that $\mathbb{R}\mathrm{P}^1$ and $\mathbb{R}\mathrm{P}^3$ are both parallelizable.

In the following we will make this discussion a little more systematic by relating the parallelizability of real projective spaces to the existence of division algebra structures.

This is a good moment to recall Definition 171.1:

Definition 275.1.

(1) An $\mathbb{R}$-algebra is an $\mathbb{R}$-vector space A together with an $\mathbb{R}$-bilinear map

$$\begin{array}{rcl} A \times A & \to & A \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

(2) A division algebra is an algebra A such that for all $a \neq 0$ and all $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.[2944]

Example.

(1) Evidently we can equip $\mathbb{R}$ and $\mathbb{C} = \mathbb{R}^2$ with the structure of a division algebra.
(2) In the Quaternion Lemma 171.3 we saw that we can equip $\mathbb{H} = \mathbb{R}^4$ with the structure of a division algebra, called the quaternions.
(3) In the Octonion Lemma 214.12 we saw that we can equip $\mathbb{O} = \mathbb{R}^8$ with the structure of a division algebra, called the octonions. □

The following proposition now gives us the promised connection between the parallelizability of real projective spaces and the existence of division algebra structures:

Proposition 275.2. (Division Algebra–Parallelizable $\mathbb{R}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}$. If $\mathbb{R}^n$ admits the structure of a division algebra, then $\mathbb{R}\mathrm{P}^{n-1}$ is parallelizable.

Proof. Let $n \in \mathbb{N}$. As in Definition 268.40 we consider the $(n-1)$-dimensional real vector bundle

$$(\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^\perp := \{([P], v) \,|\, P \in \mathbb{R}^n \setminus \{0\} \text{ and } v \in (\mathbb{R} \cdot P)^\perp\}$$

over $\mathbb{R}\mathrm{P}^{n-1}$. By the Tangent Bundle–$\mathbb{K}\mathrm{P}^{n-1}$-Lemma 268.41 we know that there exists an isomorphism

$$\mathrm{T}\mathbb{R}\mathrm{P}^{n-1} \to \underbrace{\mathrm{Hom}_{\mathbb{R}}(\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1}, (\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^\perp)}_{\text{vector bundle by Definition 268.36}}$$

of real vector bundles over $\mathbb{R}\mathrm{P}^{n-1}$. It follows from this discussion that it suffices to prove the following statement:

Statement. If $\mathbb{R}^n$ admits the structure of a division algebra, then the real vector bundle $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1}, (\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^\perp)$ over $\mathbb{R}\mathrm{P}^{n-1}$ is parallelizable.

To prove the statement we introduce the following notation:

(1) As in Definition 2.10 let $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ be the standard basis of $\mathbb{R}^n$.
(2) Given a vector subspace $V \subseteq \mathbb{R}^n$ we denote by $\pi_V \colon \mathbb{R}^n \to V$ the orthogonal projection onto V which we introduced in the Orthogonal Projection Lemma 2.33.

Claim 1. For each $i \in \{1, \ldots, n-1\}$ the map

$$\begin{array}{rcl} s_i \colon \mathbb{R}\mathrm{P}^{n-1} & \to & \mathrm{Hom}_{\mathbb{R}}(\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1}, (\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^\perp) \\ [P] & \mapsto & \left(\begin{array}{rcl} \mathfrak{T}\mathbb{R}\mathrm{P}^{n-1} & \to & (\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^\perp \\ ([P], w) & \mapsto & ([P], \pi_{(\mathbb{R}\cdot P)^\perp}(\mathfrak{e}_i \cdot \mathfrak{e}_n^{-1} \cdot w)) \end{array} \right) \end{array}$$

(the $\cdot$ in $\mathfrak{e}_i \cdot \mathfrak{e}_n^{-1} \cdot w$: multiplication in the division algebra)

is continuous.

Proof. Recall that for any $k \in \{1, \ldots, n\}$ we gave in the Tautological Line Bundle Lemma 265.6 an explicit trivialization of the tautological line bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ over

[2944] Note that we do *not* demand that there exists a multiplicatively neutral element, nor do we demand that the multiplication is associative let alone commutative.

the open subset $U_k := \{[x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^{n-1} \,|\, x_k \neq 0\}$. By the Pasting Proposition 4.7 it suffices to show that s_i is continuous on each U_k. But this follows easily from the explicit trivialization, the Gram-Schmidt Orthogonalization Process 2.32, the explicit description of the Orthogonal Projection Lemma 2.33, and the observation (which we first pointed out on page 3262) that multiplication in a division algebra is continuous. ⊞

Claim 2. For every $[P] \in \mathbb{R}\mathrm{P}^{n-1}$ the homomorphisms $s_1([P]), \ldots, s_{n-1}([P])$ from the vector space $\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1}_{[P]} = \{[P]\} \times \mathbb{R} \cdot P$ to the vector space $(\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^{\perp}_{[P]} = \{[P]\} \times (\mathbb{R} \cdot P)^{\perp}$ are linearly independent.

Proof. Let $[P] \in \mathbb{R}\mathrm{P}^{n-1}$. The claim follows from the following sequence of observations:

(a) The n vectors $\mathfrak{e}_1, \ldots, \mathfrak{e}_{n-1}, \mathfrak{e}_n$ are evidently linearly independent in $\mathbb{R}^n$.
(b) Since we are dealing with a division algebra and since $\mathfrak{e}_n^{-1} \cdot P \neq 0$ we obtain from the No Zero Divisor Lemma 171.7 that right multiplication by $\mathfrak{e}_n^{-1} \cdot P$ is an isomorphism of $\mathbb{R}^n$.
(c) It follows from (a) and (b) that the n vector $\mathfrak{e}_1 \cdot \mathfrak{e}_n^{-1} \cdot P, \ldots, \mathfrak{e}_{n-1} \cdot \mathfrak{e}_n^{-1} \cdot P, \mathfrak{e}_n \cdot \mathfrak{e}_n^{-1} \cdot P$ are linearly independent in $\mathbb{R}^n$.
(d) We now apply the projection $\pi_{(\mathbb{R}\cdot P)^{\perp}}$ to the vectors from (c). Evidently we have $\pi_{(\mathbb{R}\cdot P)^{\perp}}(\mathfrak{e}_n \cdot \mathfrak{e}_n^{-1} \cdot P) = \pi_{(\mathbb{R}\cdot P)^{\perp}}(P) = 0$. It follows easily from this observation together with (c) and the Orthogonal Projection Lemma 2.33 (3b) that the projections of the first $n-1$ vectors are still linearly independent. In other words, the $n-1$ vectors $\pi_{(\mathbb{R}\cdot P)^{\perp}}(\mathfrak{e}_1 \cdot \mathfrak{e}_n^{-1} \cdot P), \ldots, \pi_{(\mathbb{R}\cdot P)^{\perp}}(\mathfrak{e}_{n-1} \cdot \mathfrak{e}_n^{-1} \cdot P)$ are linearly independent in $(\mathbb{R} \cdot P)^{\perp}$.
(e) For $i = 1, \ldots, n-1$ we consider the homomorphism

$$\begin{array}{rcl} \varphi_i \colon \mathbb{R} \cdot P & \to & (\mathbb{R} \cdot P)^{\perp} \\ w & \mapsto & \pi_{(\mathbb{R}\cdot P)^{\perp}}(\mathfrak{e}_i \cdot \mathfrak{e}_n^{-1} \cdot w). \end{array}$$

Since $\mathbb{R} \cdot P$ is one-dimensional it follows from (d) that $\varphi_1, \ldots, \varphi_{n-1}$ are linearly independent in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R} \cdot P, (\mathbb{R} \cdot P)^{\perp})$.
(f) The homomorphisms $\varphi_1, \ldots, \varphi_{n-1}$ are essentially the same as the homomorphisms $s_1([P]), \ldots, s_{n-1}([P])$. Thus the claim follows from (e). ⊞

Finally note that it follows from Claims 1 and 2, together with the Trivializable–Vector Bundle Lemma 265.22, that the vector bundle $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1}, (\mathfrak{T}\mathbb{R}\mathrm{P}^{n-1})^{\perp})$ is parallelizable. ■

The Division Algebra–Parallelizable $\mathbb{R}\mathrm{P}^n$-Proposition 275.2 cuts both ways. It can be used to show that some projective spaces are parallelizable and it can be used to show that many $\mathbb{R}^n$ do not admit a division algebra structure. We start out by stating the result on the parallelizability of real projective spaces and spheres:

Proposition 275.3. (Parallelizability-of-$\mathbb{R}\mathrm{P}^n$-S^n-Proposition)
(1) The real projective spaces $\mathbb{R}\mathrm{P}^0, \mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^7$ are parallelizable.
(2) The spheres S^0, S^1, S^3 and S^7 are parallelizable.

Proof.

(1) This statement follows immediately from the Division Algebra–Parallelizable $\mathbb{R}\mathrm{P}^n$-Proposition 275.2 and the observation that $\mathbb{R}$ and $\mathbb{C} = \mathbb{R}^2$, $\mathbb{H} = \mathbb{R}^4$ and $\mathbb{O} = \mathbb{R}^8$ are division algebras.[2945]

(2) This statement follows from (1) and the following general claim:

Claim. If $\mathbb{R}\mathrm{P}^n$ is parallelizable, then so is S^n.

Proof. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ be the natural projection. By the discussion on page 1842 we know that p is a local diffeomorphism. It thus follows from the Tangent Bundle–Pullback Lemma 265.41 that $\mathrm{T}S^n$ is isomorphic to $p^*\mathrm{T}\mathbb{R}\mathrm{P}^n$. The claim follows from the observation that the pullback of a trivializable vector bundle is again trivializable. ■

275.2. **Stiefel-Whitney classes and the parallelizability of real projective spaces.** In the Parallelizability-of-$\mathbb{R}\mathrm{P}^n$-S^n-Proposition 275.3 we just showed that the real projective spaces $\mathbb{R}\mathrm{P}^0, \mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^7$ are parallelizable. Considering the fact we used some very clever tools to show that these real projective spaces are parallelizable it is perhaps not surprising that the following proposition shows that "most" real projective spaces are in fact *not* parallelizable:

Proposition 275.4. (Parallelizability-of-$\mathbb{R}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}_0$. If $\mathbb{R}\mathrm{P}^n$ is parallelizable, then $n+1$ is a power of 2.

Remark. So what about the remaining real projective spaces? In the proof of the Parallelizability-of-$\mathbb{R}\mathrm{P}^n$-S^n-Proposition 275.3 we just showed that if $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ is parallelizable, then S^n is also parallelizable. But by the Sphere–Parallelizability Theorem 265.35 (which unfortunately we did not prove) we know that S^n is parallelizable if and only if $n = 0, 1, 3, 7$. This shows that the only parallelizable real projective spaces are $\mathbb{R}\mathrm{P}^0, \mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^7$. □

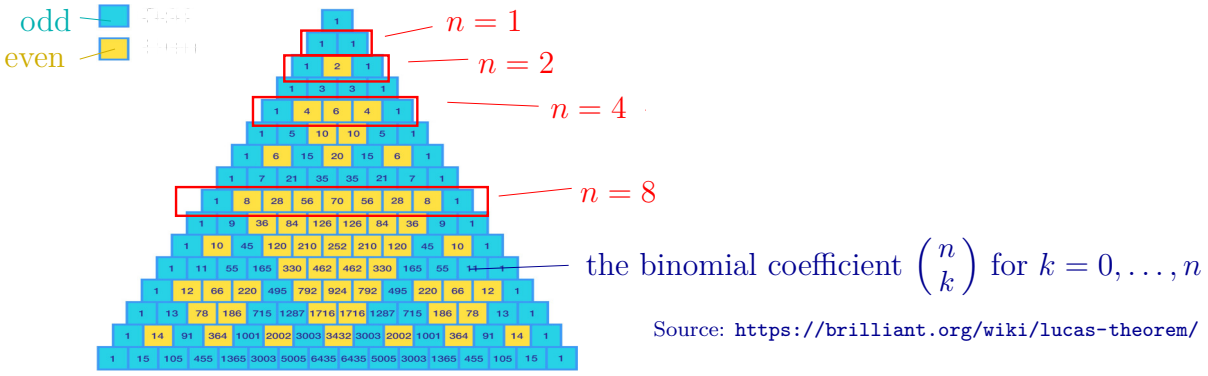
Proof. Let $n \in \mathbb{N}_0$ such that $\mathbb{R}\mathrm{P}^n$ is parallelizable. We see that we have the following equality in $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$:

$$\begin{array}{ccccccc} 1 & = & w(\mathrm{T}\mathbb{R}\mathrm{P}^n) & = & (1+x)^{n+1} & = & \displaystyle\sum_{k=0}^{n} \binom{n+1}{k} \cdot x^k. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the Trivializable SW-Lemma 274.16 and our hypothesis} & & \text{by the } \mathbb{R}\mathrm{P}^n\text{-SW Proposition 274.21} & & \text{by the binomial formula} & \end{array}$$

This discussion shows that $\binom{n+1}{k} = 0 \bmod 2$ for all $k \in \{1, \dots, n\}$. Thus the proposition follows from the following purely algebraic claim:

Claim. Let $m \in \mathbb{N}_0$. If $\binom{m}{k} \equiv 0 \bmod 2$ for $k = 1, \dots, m-1$, then m is a power of 2.

[2945] We refer to the Quaternion Lemma 171.3 and the Octonion Lemma 214.12 for details regarding $\mathbb{H}$ and $\mathbb{O}$.



Proof. [2946] We can and will write m as a sum $m = m_1 + \cdots + m_r$ where each m_i is a power of 2 and with $m_1 < \cdots < m_r$. In the polynomial ring $\mathbb{F}_2[y]$ we obtain that

$$1 + y^m \overset{\text{follows from our hypothesis}}{\underset{\downarrow}{=}} \sum_{k=0}^{m} \binom{m}{k} \cdot y^k \overset{\text{by the generalized binomial theorem}}{\underset{\downarrow}{=}} (1+y)^m \overset{\text{since } m = m_1 + \cdots + m_r}{\underset{\downarrow}{=}} (1+y)^{m_1} \cdot \ldots \cdot (1+y)^{m_r}$$
$$\overset{\text{since each } m_i \text{ is a power of 2 and since in } \mathbb{F}_2[y] \text{ we have } (a+b)^2 = a^2 + b^2}{\underset{\uparrow}{=}} (1+y^{m_1}) \cdot \ldots \cdot (1+y^{m_r}) \overset{\text{since } m_1 < \cdots < m_r}{\underset{\uparrow}{=}} 1 + y^{m_1} + \text{higher powers.}$$

But this is only possible if $m = m_1$, i.e. if m itself is a power of 2. ∎

In this section we want to apply the results from the previous section to the time-honored question, which $\mathbb{R}^n$ support a division algebra structure. We now have the tools to give a new proof of the following theorem:

Theorem 214.13. (Hopf–Division Algebra Dimension Theorem) If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.

Remark.

(1) Our first proof of the Hopf–Division Algebra Dimension Theorem 214.13, which we gave on page 4107, made use of the calculation of the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2)$. On page 4572 we gave an alternative proof using Steenrod operations.

(2) As we mentioned on page 5304, it follows from Sphere–Parallelizability Theorem 265.35 (which we did not prove) that S^n is parallelizable only if $n = 0, 1, 3, 7$. This implies that division algebras only exist in dimensions 1,2,4 and 8. □

Proof. The theorem follows immediately from the Division Algebra–Parallelizable $\mathbb{R}\mathrm{P}^n$-Proposition 275.2 together with the Parallelizability-of-$\mathbb{R}\mathrm{P}^n$-Proposition 275.4. ∎

275.3. **Immersions of smooth manifolds into Euclidean space.** We start out by recalling the following theorem:

Theorem 262.9. (Weak Whitney Immersion Theorem) Every closed m-dimensional smooth manifold M admits an immersion $f\colon M \to \mathbb{R}^{2m}$.

[2946] We proved the claim already on page 4110, for the reader's convenience we recall the argument.

Sketch of proof. Let M be a closed m-dimensional smooth manifold. The proof that M admits an immersion $f\colon M \to \mathbb{R}^{2m}$ went through two relatively straightforward steps:

(1) In the Euclidean Embedding Theorem 33.2 we showed that M admits a smooth embedding into some $\mathbb{R}^N$.
(2) Using the Generic Vector Lemma 262.10 we showed that if we are given an immersion $f\colon M \to \mathbb{R}^N$ for some $N > 2m$, then we can find a non-zero vector $v \in \mathbb{R}^n$ such that the composition of $f\colon M \to \mathbb{R}^N$ with the orthogonal projection $\mathbb{R}^N \to (\mathbb{R}\cdot v)^\perp$ from $\mathbb{R}^N$ to the $(N-1)$-dimensional vector subspace $(\mathbb{R}\cdot v)^\perp$ is still an immersion. Iterating this procedure gave us the desired conclusion. ■

Later we quoted the following strengthening of the Weak Whitney Immersion Theorem 262.9:

Theorem 262.13. (Whitney Immersion Theorem) Every smooth manifold of dimension $m \geq 2$ admits an immersion into $\mathbb{R}^{2m-1}$.

Proof. Unfortunately the proof of this theorem is beyond the abilities of this author. Thus we refer to [**Why1944b**, p. 247] and [**Wall2016**, Theorem 6.3.6] for a proof. ■

It is natural to ask whether one do even better than the Whitney Immersion Theorem 262.13. In this section we will see whether we can go any lower than the general bound of $2m-1$.

We start out with the following lemma, which essentially is the content of Exercise 203.9:

Lemma 275.5. (Inverse-in-Cohomology Ring) Let X be a topological space and let R be a commutative ring. For the purpose of this lemma we consider the "extended" cohomology ring[2947]

$$\mathrm{H}^{**}(X;R) \;:=\; \prod_{k\in\mathbb{N}_0} \mathrm{H}^k(X;R),$$

which is a ring by the cup product. Let $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathrm{H}^{**}(X;R)$. If $\alpha_0 = 1$, then there exists a *unique* $\beta = (\beta_0, \beta_1, \dots) \in \mathrm{H}^{**}(X;R)$ with $\alpha \cup \beta = 1$. Furthermore $\beta_0 = 1$.

Proof. Let X be a topological space and let R be a commutative ring. Furthermore let $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathrm{H}^{**}(X;R)$.

First note that the proof of the Cup Product-Ring Proposition 203.25 shows that $\mathrm{H}^{**}(X;R)$ is a ring. It now follows easily from the Cup Product-Commutativity Proposition 203.20 (see also Exercise 275.7) that in the extended cohomology ring, a left inverse is also a right inverse (and vice versa) and that left and right inverses are (if they exist) unique. Thus it remains to prove the following claim:

Claim. If $\alpha_0 = 1$, then there exists a $\beta = (\beta_0, \beta_1, \dots) \in \mathrm{H}^{**}(X;R)$ with $\beta_0 = 1$ such that $\alpha \cup \beta = 1$.

Proof. Clearly it suffices to define a sequence $\beta = (\beta_0, \beta_1, \dots) \in \mathrm{H}^{**}(X;R)$ with $\beta_0 = 1$ such that for each $k \in \mathbb{N}_0$ we have

$$\left(\sum_{i=0}^{k} \alpha_i\right) \cup \left(\sum_{j=0}^{k} \beta_j\right) \;=\; 1 + \text{terms of degree} \geq k+1.$$

We do so iteratively as follows:

- We set $\beta_0 = 1$. Since $\alpha_0 = 1$ this has the right properties.

[2947] It is not clear to the author how standard this notation is. It does get used for example in [**OsbH1982**, p. 2].

- Suppose we already defined $\beta_0, \ldots, \beta_k$. Note that

$$(*) \qquad \left(\sum_{i=0}^{k} \alpha_i\right) \cup \left(\sum_{j=0}^{k} \beta_j\right) \;=\; 1 + \underbrace{\sum_{h=1}^{k} \alpha_h \cup \beta_{k+1-h}}_{\text{term in degree } k+1} + \text{terms of degree} \geq k+2.$$

We set

$$\beta_{k+1} \;:=\; -\alpha_{k+1} - \sum_{h=1}^{k} \alpha_h \cup \beta_{k+1-h}.$$

We now see that

$$\left(\sum_{i=0}^{k+1} \alpha_i\right) \cup \left(\sum_{j=0}^{k+1} \beta_j\right) \overset{\text{follows from elementary algebra and the fact that } \alpha_0=\beta_0=1}{=} \left(\sum_{i=0}^{k} \alpha_i\right) \cup \left(\sum_{j=0}^{k} \beta_j\right) + \alpha_{k+1} + \beta_{k+1} + \text{terms of degree} \geq k+2$$
$$\overset{\text{follows immediately from } (*) \text{ and the definition of } \beta_{k+1}}{=} 1 + \text{terms of degree} \geq k+2.$$

■

Example.

(1) Let $n \in \mathbb{N}$. In the cohomology ring $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$ we see that

$$w(\mathrm{T}\mathbb{R}\mathrm{P}^n)^{-1} \overset{\text{by the } \mathbb{R}\mathrm{P}^n\text{-SW Proposition 274.21}}{=} ((1+x)^{n+1})^{-1} \overset{\text{holds in any ring}}{=} ((1+x)^{-1})^{n+1} \overset{\text{since in } \mathbb{F}_2[x]/(x^{n+1}) \text{ have the equality } (1+x)\cdot(1+x+\cdots+x^n) = 1+x^{n+1} = 1}{=} (1+x+\cdots+x^n)^{n+1}.$$

(2) Let $p\colon E \to B$ be a real vector bundle over a topological space that is Hausdorff and paracompact. By Definition 274.14 we know that $w_0(E) = 1$. It thus follows from the Inverse-in-Cohomology Ring 275.5 that we have an inverse $w(E)^{-1} \in \mathrm{H}^{**}(B; \mathbb{F}_2)$.

(3) Let M be a smooth manifold. It follows from the proof of the Euclidean Immersion–Tangent Bundle Corollary 269.19 and the Stiefel-Whitney-Multiplicativity Theorem 274.19 that $w(M)^{-1}$ is the total Stiefel-Whitney class of the stable normal vector bundle which we introduced in Definition 269.23. In the literature, e.g. in [**MiS1974**, p. 49], the inverse $w(M)^{-1}$) is sometimes called the total dual Stiefel-Whitney class of M. □

We can now relate immersions to Stiefel-Whitney classes:

Lemma 275.6. (Euclidean Immersion Lemma) Let M be an m-dimensional smooth manifold. If there exists an immersion $f\colon M \to \mathbb{R}^n$ then $w(M)^{-1} \in \bigoplus_{i=0}^{n-m} \mathrm{H}^i(M; \mathbb{F}_2)$.[2948]

Proof. Let $f\colon M \to \mathbb{R}^n$ be an immersion. The Euclidean Immersion–Tangent Bundle Corollary 269.19 tells us that there exists an $(n-m)$-dimensional real vector bundle $E \to M$ such that $\mathrm{T}M \oplus E$ is trivializable. We now see

$$\overbrace{w(M)}^{=w(\mathrm{T}M)} \cup w(E) \overset{\text{by the Stiefel-Whitney Multiplicativity Theorem 274.19}}{=} w(\mathrm{T}M \oplus E) \;=\; w(\text{trivializable bundle}) \overset{\text{by the Trivializable SW-Lemma 274.16}}{=} 1.$$

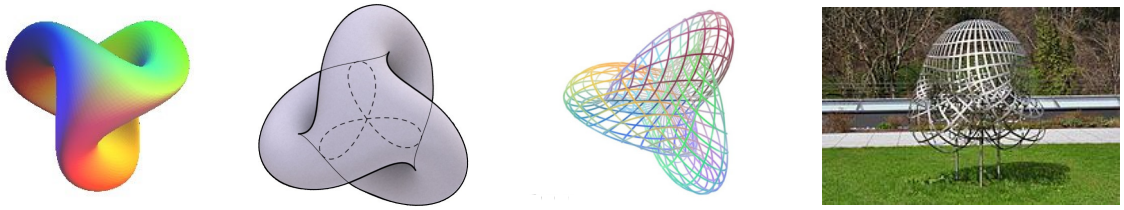
[2948] Note that this lemma, together with the Whitney Immersion Theorem 262.13 implies that we always have $w(M)^{-1} \in \bigoplus_{i=0}^{n-1} \mathrm{H}^i(M; \mathbb{F}_2)$. We will give an alternative proof of this statement in Lemma **??**.

It follows that $w(E) = w(M)^{-1}$. Since E is an $(n-m)$-dimensional real vector bundle we obtain immediately from Definition 274.14 (2b) that $w(M)^{-1} \in \bigoplus_{i=0}^{n-m} \mathrm{H}^i(M;\mathbb{F}_2)$. ■

Example. Not surprisingly we consider the real projective space $\mathbb{R}\mathrm{P}^n$. Recall that by the $\mathbb{R}\mathrm{P}^n$-SW-Proposition 274.21 we know that

$$w(\mathrm{T}\mathbb{R}\mathrm{P}^n) \;=\; (1+x)^{n+1} \;\in\; \mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \;=\; \mathbb{F}_2[x]/(x^{n+1}).$$

(1) We start out by considering the real projective plane $\mathbb{R}\mathrm{P}^2$. By the above formula we have $w(\mathbb{R}\mathrm{P}^2) = (1+x)^3 = 1+3x+3x^2+x^3 = 1+x+x^2 \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^3)$. This implies that $w(\mathbb{R}\mathrm{P}^2)^{-1} = 1+x$. Note that this calculation is consistent with the Euclidean Immersion Lemma 275.6 and the fact, first observed on page 788, that we can immerse $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$.



Boy's surface at Oberwolfach

In the figure above we show the image of the immersion of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$ that was first found by Werner Boy [**Boy1903**].

(2) Next we consider the 9-dimensional real projective space $\mathbb{R}\mathrm{P}^9$. Note that by the above formula we have $w(\mathbb{R}\mathrm{P}^9) = (1+x)^{10} = 1+x^2+x^8 \in \mathrm{H}^*(\mathbb{R}\mathrm{P}^9;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^{10})$. An elementary calculation shows that $w(\mathbb{R}\mathrm{P}^9)^{-1} = 1+x^2+x^4+x^6$. It follows from the Euclidean Immersion Lemma 275.6 that if $\mathbb{R}\mathrm{P}^9$ can be immersed in some $\mathbb{R}\mathrm{P}^{9+k}$, then $k \geq 6$.

(3) Finally we consider the case $n = 2^r$. We see that we have the following equality in $\mathrm{H}^*(\mathbb{R}\mathrm{P}^{2^r};\mathbb{F}_2) = \mathbb{F}_2[x]/(x^{2^r+1})$:

$$w(\mathbb{R}\mathrm{P}^{2^r}) \;=\; (1+x)^{2^r+1} \;=\; (1+x)^{2^r}\cdot(1+x) \;=\; (1+x^{2^r})\cdot(1+x) \;=\; 1+x+x^{2^r}.$$

An elementary calculation shows that

$$w(\mathbb{R}\mathrm{P}^{2^r})^{-1} \;=\; 1+x+x^2+\cdots+x^{2^r-1}.$$

Note that the combination of this calculation and the Euclidean Immersion Lemma 275.6 shows that in this case $m = 2n-1$ is the lowest possible dimension for which we can find an immersion $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^m$. In other words, for $n = 2^r$ the Whitney Immersion Theorem 262.13 is optimal. □

275.4. **Appendix: Massey's Vanishing Theorem and Cohen's Immersion Theorem.** In this appendix we will discuss a refinement of the Whitney Immersion Theorem 262.13. To formulate this statement we will need the following notation:

Notation 275.7. Given $n \in \mathbb{N}_0$ we denote by $\alpha(n)$ the number of ones in the dyadic expansion of n.

Example.

(1) For $n = 2^m$ we have $\alpha(2^m) = 1$.
(2) For $n = 2^m - 1$ we have $\alpha(2^m) = m$.
(3) For any non-zero $n \in \mathbb{N}$ we have $\alpha(n) \geq 1$. □

The following theorem was proved by William Massey in 1960:

Theorem 275.8. (Massey Vanishing Theorem) Let $n \in \mathbb{N}$. For every compact n-dimensional smooth manifold we have $w(M)^{-1} \in \bigoplus_{i=0}^{n-\alpha(n)} \mathrm{H}^i(M;\mathbb{F}_2)$.

Proof. The theorem was proved by William Massey [**Mass1960**, Theorem 1] (see also [**CoheR1982**, p. 3390] and [**CoheR1985**, p. 238]). ■

Example.

(1) As we mentioned above, for any $n \in \mathbb{N}$ we have $\alpha(n) \geq 1$. The Massey Vanishing Theorem 275.8 thus saws that for any compact smooth manifold of dimension ≥ 1 we have $w(M)^{-1} \in \bigoplus_{i=0}^{n-1} \mathrm{H}^i(M;\mathbb{F}_2)$. This is consistent with the combination of the Whitney Immersion Theorem 262.13 and the Euclidean Immersion Lemma 275.6.
(2) Let $n \in \mathbb{N}$. We write $n = 2^{k_1} + \cdots + 2^{k_r}$ where $k_1 < k_2 < \cdots < k_r$. Note that $\alpha(n) = r$. We consider $M = \mathbb{R}\mathrm{P}^{2^{k_1}} \times \cdots \times \mathbb{R}\mathrm{P}^{2^{k_r}}$. An elementary calculation, using the calculation on page 5308 and using the Manifold Product-SW-Lemma 274.20, shows that the $(n-\alpha(n))$-degree term of $w(M)^{-1}$ is non-zero.[2949] This shows that the degree bound in the Massey Vanishing Theorem 275.8 is optimal. □

The combination of the Massey Vanishing Theorem 275.8 and the Euclidean Immersion Lemma 275.6 raised the question, whether any compact n-dimensional smooth manifolds admits an immersion into $\mathbb{R}^{2n-\alpha(n)}$. In 1982 this was shown to indeed be the case, i.e. we have the following strengthening of the Massey Vanishing Theorem 275.8:

Theorem 275.9. (Cohen Immersion Theorem) Let $n \in \mathbb{N}$. Every compact n-dimensional smooth manifold admits an immersion into $\mathbb{R}^{2n-\alpha(n)}$.

Proof. This theorem was proved by Ralph Cohen, see [**CoheR1982**], [**CoheR1985**, p. 238] and [**CoT1989**, p. 78]. ■

Remark. The above discussion of $M = \mathbb{R}\mathrm{P}^{2^{k_1}} \times \cdots \times \mathbb{R}\mathrm{P}^{2^{k_r}}$ together with the Euclidean Immersion Lemma 275.6 shows that the Cohen Immersion Theorem 275.9 is optimal. □

275.5. Exercises for Chapter 275.

Exercise 275.1. Let $n \in \mathbb{N}$. We write $n + 1 = 2^r \cdot m$ where m is odd. Show that there do not exist 2^r vector fields on $\mathbb{R}\mathrm{P}^n$ which are linearly independent at every point.

Exercise 275.2. Show that every compact 2-dimensional smooth manifold admits an immersion into $\mathbb{R}^3$.

Exercise 275.3. Which compact 2-dimensional smooth manifolds admit an immersion into $\mathbb{R}^2$?

[2949] The details will surely be filled in in Exercise 275.6.

Exercise 275.4. Provide an example of some $n \in \mathbb{N}$, a closed n-dimensional smooth manifold, a commutative ring R and of $a, b \in \mathrm{H}^*(M; R)$ such that $a \cup b = 1$ and such that neither a nor b is contained in $\bigoplus_{i=0}^{n-1} \mathrm{H}_i(M; R)$.

Exercise 275.5.

(a) Let M be an n-dimensional smooth manifold. Show that if there exists an immersion of M into $\mathbb{R}^{n+1}$, then for each $i \in \mathbb{N}_0$ we have $w_i(M) = w_1(M)^i \in \mathrm{H}^i(M; \mathbb{F}_2)$.

(b) Show that if $\mathbb{R}\mathrm{P}^n$ admits an immersion into $\mathbb{R}^{n+1}$, then n is of the form $2^r - 1$ or $2^r - 2$.

Exercise 275.6. Let $n \in \mathbb{N}$. We write $n = 2^{k_1} + \cdots + 2^{k_r}$ where $k_1 < k_2 < \cdots < k_r$. We consider $M = \mathbb{R}\mathrm{P}^{2^{k_1}} \times \cdots \times \mathbb{R}\mathrm{P}^{2^{k_r}}$. Show that the $(n-r)$-degree term of $w(M)^{-1}$ is non-zero.

Hint. Make use of the calculation of $w(\mathbb{R}\mathrm{P}^{2^k})^{-1}$ on page 5308 and make use of the Manifold Product-SW-Lemma 274.20.

Remark. The $\mathbb{F}_2$-Cohomology Ring-of-Product Proposition 208.16 together with the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 allows us to make the identifications

$$\mathrm{H}^*(\mathbb{R}\mathrm{P}^{m_1} \times \cdots \times \mathbb{R}\mathrm{P}^{m_r}; \mathbb{F}_2) \;=\; \mathbb{F}_2[x_1, \ldots, x_r]/(x_1^{m_1+1}, \ldots, x_r^{m_r+1}).$$

Exercise 275.7.

(a) Let S be a ring and let $a, b \in S$. If $a \cdot b = 1$, does it follow that $b \cdot a = 1$?

(b) Let X be a topological space and let R be a commutative ring. As in the Inverse-in-Cohomology Ring 275.5 we consider the "extended" cohomology ring

$$\mathrm{H}^{**}(X; R) \;:=\; \prod_{k \in \mathbb{N}_0} \mathrm{H}^k(X; R),$$

where multiplication is given by the cup product. Let $\alpha, \beta \in \mathrm{H}^{**}(X; R)$ such that $\alpha \cup \beta = 1$. Show that $\beta \cup \alpha = 1$.

Hint. Make use of the Cup Product-Commutativity Proposition 203.20.

Remark. This exercise is very similar to Exercise 203.5.

CHAPTER 276

Pontryagin classes

In the previous chapter we introduced the Chern classes $c_k \in \mathrm{H}^{2k}(B;\mathbb{Z})$ of a complex vector bundle $p\colon E \to B$ and we introduced the Stiefel-Whitney $w_k \in \mathrm{H}^k(B;\mathbb{F}_2)$ of a real vector bundle $p\colon E \to B$. It feels a little unfair that to a real vector bundle we can only associate $\mathbb{F}_2$-valued cohomology classes. In this chapter we will use an embarrassingly cheap trick to associate integer cohomology classes to a real vector bundle $p\colon E \to B$: We will just consider the Chern classes of the complexification $E \otimes_{\mathbb{R}} \mathbb{C} \to B$. These will be called the Pontryagin classes of $p\colon E \to B$. The Pontryagin classes are of course interesting in their own right. But they will develop their full usefulness only a little later, namely in Chapter 281, when we will prove the Hirzebruch Signature Theorem 281.14.

This chapter is organized as follows:

- In Section 276.1 we will see (again) that by "fiberwise tensoring with $\mathbb{C}$" we can associate to any real vector bundle (miraculously!) a complex vector bundle.
- In Section 276.2 we will introduce Pontryagin classes of real vector bundles.
- In Section 276.3 we will determine the Pontryagin classes of the tangent bundle of the complex projective spaces.
- In Section 276.4 we will provide exercises for Chapter 276.

In the next Chapter 277 we will formulate and prove the Thom Isomorphism Theorem for vector bundles.

276.1. **Tensoring real vector bundles with $\mathbb{C}$.** The idea behind the Pontryagin classes is that given a real vector bundle $p\colon E \to B$ we consider the complexification $p\colon E \otimes_{\mathbb{R}} \mathbb{C} \to B$ and to consider the corresponding Chern classes. For the reader's convenience we now recall some of the definitions and facts about complexifications:

Lemma 268.13 (Tensor-with-$\mathbb{C}$-Lemma)

(1) Let V be a real vector space. We consider the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$. The map

$$\begin{array}{rcl} V \otimes_{\mathbb{R}} \mathbb{C} \times \mathbb{C} & \to & V \otimes_{\mathbb{R}} \mathbb{C} \\ \left(\sum_{k=1}^{m} v_i \otimes z_i, w \right) & \mapsto & \sum_{k=1}^{m} v_i \otimes (z_i \cdot_{\mathbb{C}} w) \end{array}$$

defines a $\mathbb{C}$-vector space structure on $V \otimes_{\mathbb{R}} \mathbb{C}$. We refer to $V \otimes_{\mathbb{R}} \mathbb{C}$ as the complexification of V.

(2) The maps

$$\begin{array}{rcl} V & \mapsto & V \otimes_{\mathbb{R}} \mathbb{C} \\ (f\colon V \to W) & \mapsto & (f \otimes \mathrm{id}_{\mathbb{C}}\colon V \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}) \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces to the category $\mathcal{V}ec_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces. This functor restricts to the subcategories given by finite-dimensional vector spaces and where we only consider isomorphisms.

The tensor product gives us a functorial way to turn real vector spaces into complex vector spaces and bundles. We already saw in Definition 268.29 that this procedure also works for vector bundles:

Definition 268.29. Let $p\colon E \to B$ be an n-dimensional (smooth) $\mathbb{R}$-vector bundle. We refer to the vector bundle

$$E \otimes_{\mathbb{R}} \mathbb{C} \coloneqq \bigsqcup_{b\in B} \underbrace{E_b \otimes_{\mathbb{R}} \mathbb{C}}_{\substack{n\text{-dimensional}\\ \mathbb{C}\text{-vector space}}} \qquad \text{with the obvious map } r\colon E \otimes_{\mathbb{R}} \mathbb{C} \to B$$

as the complexification of the $\mathbb{R}$-vector bundle E.

We continue by recalling one more lemma from earlier on:

Lemma 268.30. (Bundle–Tensor-with-$\mathbb{C}$-Lemma)

(1) Let $p\colon E \to B$ be a real vector bundle. We consider the $\mathbb{R}$-vector bundle $q\colon E\otimes_{\mathbb{R}}\mathbb{C} \to B$ from Definition 268.29. The fiberwise multiplication given by the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.13 turns $E \otimes_{\mathbb{R}} \mathbb{C}$ into a complex vector bundle.

(2) (a) Given a morphism

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C \end{array}$$

of real vector bundles we obtain an obvious induced morphisms of complex vector bundles.

(b) The map $E \mapsto E \otimes_{\mathbb{R}} \mathbb{C}$ defines a covariant functor from $\mathcal{VectBund}_{\mathbb{R}}$ to $\mathcal{VectBund}_{\mathbb{C}}$.

The following lemma shows that the complexification behaves exactly as suspected:

Lemma 276.1. (Bundle–Tensor-with-$\mathbb{C}$–Properties Lemma)

(1) The complexification of a trivializable real vector bundle is a trivializable complex vector bundle.

(2) Let $p\colon E \to B$ and $q\colon F \to B$ be two real vector bundles. There exists a natural isomorphism

$$E \otimes_{\mathbb{R}} \mathbb{C} \oplus F \otimes_{\mathbb{R}} \mathbb{C} \;\to\; (E \oplus F) \otimes_{\mathbb{R}} \mathbb{C}$$

of complex vector bundles.

(3) Let

$$\begin{array}{ccc} E & \xrightarrow{H} & F \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C \end{array}$$

be a morphism of real vector bundles. There exists a natural isomorphism

$$E \otimes_{\mathbb{R}} \mathbb{C} \;\to\; h^*(F \otimes \mathbb{C})$$

of complex vector bundles over B.

Sketch of proof.

(1) First of all note that the map

$$\begin{array}{rcl} (B \times \mathbb{R}^m) \otimes_{\mathbb{R}} \mathbb{C} & \to & B \times \mathbb{C}^m \\ \sum_{i=1}^{k} (b, v_i) \otimes z_i & \mapsto & \left(b, \sum_{i=1}^{k} v_i \cdot z_i\right) \end{array}$$

is an isomorphism of $\mathbb{C}$-vector bundles. The statement for trivializable bundles now follows from the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.30.

(2) The natural injections $E \to E \oplus F$ and $F \to E \oplus F$ induce maps $E \otimes_{\mathbb{R}} \mathbb{C} \to (E \oplus F) \otimes_{\mathbb{R}} \mathbb{C}$ and $F \otimes_{\mathbb{R}} \mathbb{C} \to (E \oplus F) \otimes_{\mathbb{R}} \mathbb{C}$. From these maps we get a morphism

$$E \otimes_{\mathbb{R}} \mathbb{C} \oplus F \otimes_{\mathbb{R}} \mathbb{C} \quad \to \quad (E \oplus F) \otimes_{\mathbb{R}} \mathbb{C}$$

of $\mathbb{C}$-vector bundles. Note that it follows from the Vector Bundle Isomorphism Criterion 265.40 that this map is an isomorphism of vector bundles.

(3) This lemma follows almost immediately from the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.30 (2) together with the Vector Bundle–Pullback Proposition 265.36. ■

We conclude this section with a much more interesting lemma:

Lemma 276.2. (Complex Bundle–Tensor-with-$\mathbb{C}$-Lemma) Let $p\colon E \to B$ be an n-dimensional complex vector bundle. We can also view it as a $2n$-dimensional real vector bundle. We have a natural isomorphism

$$E \otimes_{\mathbb{R}} \mathbb{C} \quad \xrightarrow{\cong} \quad E \underset{\uparrow}{\oplus} \overline{E}$$

in Definition 268.33 we introduced the complex conjugate vector bundle $\overline{E} \to B$

of complex vector bundles.

Proof. For each finite-dimensional complex vector space V we consider the map

$$\begin{array}{rcl} V \otimes_{\mathbb{R}} \mathbb{C} & \to & V \oplus \overline{V} \\ v \otimes z & \mapsto & (\underset{\uparrow}{v \cdot z}, \underset{\uparrow}{v \cdot \overline{z}}). \end{array}$$

multiplication by complex scalars in the complex vector space V

It is elementary to verify that this map is an isomorphism of $\mathbb{C}$-vector spaces.[2950] These maps clearly define a natural isomorphism between the smooth functor $V \mapsto V \otimes_{\mathbb{R}} \mathbb{C}$ and the smooth functor $V \mapsto V \oplus \overline{V}$ from the category $\mathcal{V}ec_{\mathbb{C}}$ of complex vector spaces to itself.

It follows from the Smooth Natural Isomorphism–Vector Bundle Lemma 268.25 that this natural isomorphism gives rise to a natural isomorphism

$$E \otimes_{\mathbb{R}} \mathbb{C} \quad \xrightarrow{\cong} \quad E \oplus \overline{E}$$

of complex vector bundles. ■

276.2. **Pontryagin classes.** The discussion of the previous section leads us to the following idea: Given a real vector bundle $p\colon E \to B$ we can consider the corresponding complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C} \to B$ which allows us to consider the corresponding Chern classes $c_k(E \otimes_{\mathbb{R}} \mathbb{C}) \in \mathrm{H}^{2k}(B; \mathbb{Z})$. This seems a little bit like creating mathematical objects ex nihilo. In fact the following lemma says that for odd $k \in \mathbb{N}$ we do indeed fail to get anything really intersecting:

[2950] Clearly this map is a homomorphism of $\mathbb{C}$-vector spaces. Since both sides are complex vector spaces of the same dimension it remains to show that the map is an epimorphism. This in turn can be seen as follows: With $z = 1$ one can easily see that all elements of the form (v, v) lie in the image of the map and with $z = \mathrm{i}$ one can easily see that all elements of the form $(v, -v)$ lie in the image of the map.

Lemma 276.3. (Odd Chern-2-Torsion Lemma) Let $p\colon E \to B$ be a real vector bundle over a topological space that is Hausdorff and paracompact. For every *odd* $k \in \mathbb{N}$ we have $2 \cdot c_k(E \otimes_{\mathbb{R}} \mathbb{C}) = 0 \in \mathrm{H}^{2k}(B;\mathbb{Z})$.

Remark. We just showed that the Chern classes $c_{2k+1}(E \otimes_{\mathbb{R}} \mathbb{C})$ are 2-torsion. This does not yet completely disqualify them. But according to [**Hat2**, p. 97] these Chern classes can also be expressed in terms of the Stiefel-Whitney classes of E. More precisely: We consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{F}_2 \to 0$. As in **??** (or as in Exercise 197.8) this gives rise to Bockstein homomorphisms $\delta_m\colon \mathrm{H}^m(B;\mathbb{F}_2) \to \mathrm{H}^{m+1}(B;\mathbb{Z})$. With this notation [**Hat2**, p. 97] says that

$$c_{2k+1}(E \otimes_{\mathbb{R}} \mathbb{C}) \;=\; \beta_{4k+1}(\underbrace{w_{2k}(E) \cup w_{2k+1}(E)}_{\in \mathrm{H}^{4k+1}(B;\mathbb{F}_2)}) \;\in\; \mathrm{H}^{4k+2}(B;\mathbb{Z}).$$

So we do not gain any new information from these Chern classes and we will henceforth ignore them. □

Proof. [2951] Recall that in Definition 268.33 we introduced, given any complex vector bundle $F \to B$, the complex conjugate complex vector bundle $\overline{F} \to B$. We now see that for any $k \in \mathbb{N}_0$ we have

$$c_k(E \otimes_{\mathbb{R}} \mathbb{C}) \underset{\substack{\uparrow \\ \text{one can easily verify that the map } v \otimes z \mapsto v \otimes \overline{z} \\ \text{is an isomorphism of complex vector bundles}}}{=} c_k(\overline{E \otimes_{\mathbb{R}} \mathbb{C}}) \underset{\substack{\uparrow \\ \text{by the Complex Conjugate-Chern-Lemma 274.8}}}{=} (-1)^k \cdot c_k(E \otimes_{\mathbb{R}} \mathbb{C}).$$

For odd k we obtain that $c_k(E \otimes_{\mathbb{R}} \mathbb{C}) = -c_k(E \otimes_{\mathbb{R}} \mathbb{C})$, which gives the promised result. ■

The Odd Chern-2-Torsion Lemma 276.3 says that we don't really need to bother about odd Chern classes $c_{2k+1}(E \otimes_{\mathbb{R}} \mathbb{C})$. Thus we will now concentrate on the even Chern classes $c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C})$. This leads us to the main definition of this chapter:

Definition 276.4. Let $E \to B$ be an n-dimensional real vector bundle over a topological space that is Hausdorff and paracompact. By the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.30 we can consider the corresponding n-dimensional complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C}$. Given $k \in \mathbb{N}_0$ we define the k-th Pontryagin[2958] class of E as[2959]

$$p_k(E) \;:=\; (-1)^k \cdot c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \;\in\; \mathrm{H}^{4k}(B;\mathbb{Z}).$$

Example. If $E \to B$ is a 1-dimensional real vector bundle, then it follows immediately from Definitions 274.1 and 276.4 that $p_k(E) = 0$ for $k \in \mathbb{N}$. □

Not surprisingly our results on Chern classes from Section 274.1 lead us to analogous statements on Pontryagin classes. For example we have the following analog of the Naturality-of-Chern-Proposition 274.2 and of the Naturality-of-SW-Proposition 274.15.

[2951] In Exercise 276.4 we will see that one can also use the Complex Bundle–Tensor-with-$\mathbb{C}$-Lemma 276.2 to give a proof of the Odd Chern-2-Torsion Lemma 276.3.

[2958] Pontryagin classes were introduced in 1947 by the Soviet mathematician Lev Pontryagin [**Pon1947**]. Pontryagin (1908-1988) was completely blind after a failed eye surgery at the age of 14.

[2959] Following [**MiS1974**, p. 174] and [**Hat2**, p. 94] we put in the factor $(-1)^k$. This will make some formulas involving Pontryagin classes prettier.

Proposition 276.5. (Naturality-of-Pontryagin-Proposition) Let $k \in \mathbb{N}_0$. Furthermore let B and C be topological spaces that are Hausdorff and paracompact.

(1) For any morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of real vector bundles we have $h^*(p_k(E)) = p_k(F) \in \mathrm{H}^{4k}(C;\mathbb{Z})$.

(2) (a) If $p\colon E \to B$ and $p'\colon E' \to B$ are real vector bundles that are isomorphic over B, then $p_k(E) = p_k(E') \in \mathrm{H}^{4k}(B;\mathbb{Z})$.

(b) Let $p\colon E \to B$ be a real vector bundle. For any continuous map $h\colon C \to B$ between the topological spaces C and B we have the following equality:[2960]

$$p_k(h^*E) \;=\; h^*(p_k(E)) \;\in\; \mathrm{H}^{4k}(C;\mathbb{Z}).$$

Proof. As we will see, the statements follow easily from the Naturality-of-Chern-Proposition 274.2.

(2) (a) This statement follows almost immediately from the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.30 together with the Naturality-of-Chern-Proposition 274.2 (2a).

(b) We have

$$\begin{aligned} p_k(h^*(E)) &\underset{\text{Definition 276.4}}{=} (-1)^k \cdot c_k(h^*(E) \otimes_{\mathbb{R}} \mathbb{C}) \\ &\underset{\substack{\text{by the Bundle–Tensor-with-}\mathbb{C}\text{-Properties} \\ \text{Lemma 276.1 (3)}}}{=} (-1)^k \cdot c_k(h^*(F \otimes_{\mathbb{R}} \mathbb{C})) \\ &\underset{\text{by the Naturality-of-Chern Proposition 274.2}}{=} (-1)^k \cdot h^*(c_k(E \otimes_{\mathbb{R}} \mathbb{C})) \\ &\underset{\text{by Definition 276.4}}{=} h^*(p_k(E)). \end{aligned}$$

(1) This statement follows immediately from (2) and the Vector Bundle–Pullback Proposition 265.36 (2). ∎

We continue with the following analog of the Trivializable-Chern-Lemma 274.3 and of the Trivializable-SW-Lemma 274.16:

Lemma 276.6. (Trivializable-Pontryagin-Lemma) Let $E \to B$ be a trivializable real vector bundle over a topological space that is Hausdorff and paracompact. For every $k \in \mathbb{N}$ we have $p_k(E) = 0$.

Proof. This lemma can be proved by the same argument as in the Trivializable-Chern-Lemma 274.3. Alternatively one deduce the lemma from the Trivializable-Chern-Lemma 274.3 and the Bundle–Tensor-with-$\mathbb{C}$-Properties Lemma 276.1 (1). ∎

Before we can formulate the multiplicativity theorem for Pontryagin classes we need to introduce the following analog of the Definition 274.5 of the total Chern class and of the Definition 274.18 of the total Stiefel-Whitney class:

Definition 276.7. Given a real vector bundle $p\colon E \to B$ over a topological space that is Hausdorff and paracompact we refer to

$$p(E) \;:=\; (p_0(E), 0, 0, 0, p_1(E), 0, 0, 0, p_2(E), \dots) \;\in\; \mathrm{H}^*(B;\mathbb{Z}) \;:=\; \bigoplus_{i=0}^{\infty} \mathrm{H}^i(B;\mathbb{Z})$$

as the total Pontryagin class.

The following result is an analog of the Chern-Multiplicativity Theorem 274.6 and of the SW-Multiplicativity Theorem 274.19:

Theorem 276.8. (Pontryagin-Multiplicativity Theorem)

(1) The total Pontryagin class is multiplicative with respect to the direct sum of real vector bundles *up to 2-torsion in* $\mathrm{H}^*(B;\mathbb{Z})$. More precisely, if $e\colon E \to B$ and $f\colon F \to B$ are two real vector bundles over a topological space that is Hausdorff and paracompact, then[2961]
$$p(E \oplus F) \;=\; p(E) \cup p(F) \;\in\; \mathrm{H}^*(B;\mathbb{Z})/2\text{-torsion}.$$

(2) Let B be a topological space that is Hausdorff and paracompact. Furthermore let $e\colon E \to B$ be a real vector bundle and let $f\colon F \to B$ be a *trivializable* real vector bundle. Then
$$p(E \oplus F) \;=\; p(E) \;\in\; \mathrm{H}^*(B;\mathbb{Z}).$$

Proof.

(1) We perform the following calculation in $\mathrm{H}^*(B;\mathbb{Z})/2$-torsion:

$$\begin{array}{l} p(E\oplus F) \;\overset{\text{by Definition 276.7}}{=}\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot c_{2n}((E\oplus F)\otimes_{\mathbb{R}}\mathbb{C}) \;\overset{\text{by the Bundle–Tensor-with-}\mathbb{C}\text{–Properties Lemma 276.1 (2)}}{=}\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot c_{2n}(E\otimes_{\mathbb{R}}\mathbb{C}\oplus F\otimes_{\mathbb{R}}\mathbb{C}) \\ \overset{\uparrow}{=}\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \sum_{k\in\mathbb{N}_0} c_{2k}(E\otimes_{\mathbb{R}}\mathbb{C}) \cup c_{2n-2k}(F\otimes_{\mathbb{R}}\mathbb{C}) \\ \quad \text{follows from the Chern-Multiplicativity Theorem 274.6 and since by the Odd Chern-2-Torsion Lemma 276.3 we can ignore the contributions of the Chern classes of odd index, as we work modulo 2-torsion} \\ \overset{\uparrow}{=}\; \Big(\sum_{i\in\mathbb{N}_0} (-1)^i \cdot c_{2i}(E\otimes_{\mathbb{R}}\mathbb{C})\Big) \cup \Big(\sum_{j\in\mathbb{N}_0} (-1)^j \cdot c_{2j}(F\otimes_{\mathbb{R}}\mathbb{C})\Big) \;\overset{\uparrow}{=}\; p(E)\cup p(F). \\ \quad \text{elementary algebra} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Definition 276.7} \end{array}$$

(2) Let $f\colon F \to B$ is a trivializable real vector bundle over a topological space that is Hausdorff and paracompact. We know by the Trivializable-Chern-Lemma 274.3 that the Chern classes of $F\otimes_{\mathbb{R}}\mathbb{C}$ vanish in degree ≥ 1. One can now easily verify that the calculation in (1) gives the desired result, without the need of modding out 2-torsion. ∎

Example. Let $n \in \mathbb{N}$. Let M be a closed orientable submanifold of $\mathbb{R}^n$ of codimension one. For example we could take $M = S^{n-1} \subseteq \mathbb{R}^n$. We want to determine the total Pontryagin class of the tangent bundle $\mathrm{T}M$. As before we denote by $\mathrm{N}M$ the normal vector bundle of $M \subseteq \mathbb{R}^n$. Then

$$\begin{array}{l} p(\mathrm{T}M) \;\overset{\uparrow}{=}\; p(\mathrm{T}M)\cup p(\mathrm{N}M) \;\overset{\uparrow}{=}\; p(\mathrm{T}M \oplus \mathrm{N}M) \\ \text{by the Trivializable-Pontryagin Lemma 276.6 since the real vector bundle } \mathrm{N}M \text{ is trivializable}[2962] \qquad \text{by the Pontryagin Multiplicativity Theorem 276.8 (2)} \\ \quad \overset{\uparrow}{=}\; p(\text{trivial vector bundle } M\times\mathbb{R}^n) \;\overset{\uparrow}{=}\; 1. \\ \quad \text{see page 5170} \qquad\qquad\qquad\qquad\qquad\qquad \text{by Trivializable Pontryagin-Lemma 276.6} \end{array}$$

[2961] For an abelian group A and $n \in \mathbb{N}$ we mean by "n-torsion" the subgroup $\{a \in A \,|\, n\cdot a = 0\}$.

In particular we see that the Pontryagin classes of the tangent bundles of spheres S^m are trivial. □

We conclude this discussion of the properties of Pontryagin classes with the following lemma which is germane to the theory of Pontryagin classes, and which shows that the Pontryagin classes of complex vector bundles are completely determined by Chern classes:

Lemma 276.9. (Complex Vector Bundle–Pontryagin-Lemma) Given any *complex* vector bundle $E \to B$ over a topological space that is Hausdorff and paracompact we have for any $k \in \mathbb{N}_0$ the equality

$$p_k(E) \;=\; (-1)^k \cdot \sum_{i=0}^{2k} (-1)^i \cdot c_i(E) \cup c_{2k-i}(E).$$

In other words, we have

$$\sum_{k\in\mathbb{N}_0} (-1)^k \cdot p_k(E) \;=\; \Big(\sum_{i\in\mathbb{N}_0} (-1)^i \cdot c_i(E)\Big) \cup \Big(\sum_{j\in\mathbb{N}_0} c_j(E)\Big).$$

Proof. Note that we have the following equalities in $\mathrm{H}^{4k}(B;\mathbb{Z})$:

$$(-1)^k \cdot p_k(E \otimes_{\mathbb{R}} \mathbb{C}) \underset{\uparrow}{=} c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \underset{\uparrow}{=} c_{2k}(E \oplus \overline{E}) = \sum_{i=0}^{2k} c_i(E) \cup c_{2k-i}(\overline{E})$$

by Definition 276.7; by the Complex Bundle Tensor-with-$\mathbb{C}$-Lemma 276.2; by the Chern-Multiplicativity Theorem 274.6

$$= \underset{\uparrow}{\sum_{i=0}^{2k}} (-1)^i \cdot c_i(E) \cup c_{2k-i}(E).$$

by the Complex Conjugate-Chern Lemma 274.8

Finally the second equality of the lemma is an almost immediate consequence of the first equality. ■

276.3. **Pontryagin classes of complex projective spaces.** In the $\mathbb{C}\mathrm{P}^n$-Chern-Proposition 274.9 we calculated the Chern classes of the tangent bundles of the complex projective spaces. We will now use this calculation to calculate the Pontryagin classes of the tangent bundles of complex projective spaces:

Proposition 276.10. ($\mathbb{C}\mathbf{P}^n$-Pontryagin-Proposition) Let $n \in \mathbb{N}_0$. As usual we denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ the standard generator as introduced in Notation 213.6 and in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20. Then

$$p(\mathrm{T}\mathbb{C}\mathrm{P}^n) \;=\; (1+x^2)^{n+1} \;\in\; \mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}).$$

In other words, for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$ we have[2963]

$$p_k(\mathrm{T}\mathbb{C}\mathrm{P}^n) \;=\; \binom{n+1}{k} \cdot x^{2k} \;\in\; \mathrm{H}^{4k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;=\; \mathbb{Z} \cdot x^{2k}.$$

and all other Pontryagin classes are zero.

[2962] One could also use the fact that the Pontryagin class of any 1-dimensional real vector bundle is necessarily zero.

Example. The $\mathbb{C}\mathrm{P}^n$-Pontryagin-Proposition 276.10 implies in particular that

$$p(\mathbb{C}\mathrm{P}^1) \;=\; 1, \quad p(\mathbb{C}\mathrm{P}^2) \;=\; 1 + 3x^2, \quad p(\mathbb{C}\mathrm{P}^3) \;=\; 1 + 4x^2, \quad p(\mathbb{C}\mathrm{P}^4) \;=\; 1 + 5x^2 + 10x^4. \quad \square$$

Proof. The proposition follows from the following calculation in $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$:

$$\sum_{k\in\mathbb{N}_0} (-1)^k \cdot p_k(\mathrm{T}\mathbb{C}\mathrm{P}^n) \;\overset{\text{by the Tangent Bundle Proposition 265.10 we know that } \mathrm{T}\mathbb{C}\mathrm{P}^n \text{ is a complex vector bundle thus the equality follows from the Complex Vector Bundle–Pontryagin-Lemma 276.9}}{=}\; \Big(\sum_{i\in\mathbb{N}_0} (-1)^i \cdot c_i(\mathrm{T}\mathbb{C}\mathrm{P}^n)\Big) \cup \Big(\sum_{j\in\mathbb{N}_0} c_j(\mathrm{T}\mathbb{C}\mathrm{P}^n)\Big)$$

$$= \;(1-x)^{n+1} \cup (1+x)^{n+1} \;=\; (1-x^2)^{n+1} \;=\; \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \binom{n+1}{k} \cdot x^{2k}.$$

this follows easily from the $\mathbb{C}\mathrm{P}^n$-Chern-Proposition 274.9

■

276.4. Exercises for Chapter 276.

Exercise 276.1. In the proof of the Complex Bundle–Tensor-with-$\mathbb{C}$-Lemma 276.2 we considered the map

$$\begin{array}{rcl} V \otimes_{\mathbb{R}} \mathbb{C} & \to & V \oplus \overline{V} \\ v \otimes z & \mapsto & (v \cdot z, v \cdot \overline{z}). \end{array}$$

multiplication by complex scalars in the complex vector space V

Provide an explicit inverse of this map.

Exercise 276.2. Show that there exists a 2-dimensional real vector bundle $E \to B$ such that $p_1(E) \in \mathrm{H}^4(B;\mathbb{Z})$ is non-zero.

Exercise 276.3. Which formulas involving Pontryagin classes do become prettier by introducing the $(-1)^k$-factor in Definition 276.4?

Exercise 276.4. Use the Complex Bundle–Tensor-with-$\mathbb{C}$-Lemma 276.2 to give an alternative proof of the Odd Chern-2-Torsion Lemma 276.3.

Exercise 276.5. Is every complex vector bundle the complexification of a real vector bundle?

Exercise 276.6. If all Pontryagin classes vanish and if the cohomology groups have no 2-torsion, is the vector bundle trivial?

Exercise 276.7. Let M and N be two smooth manifolds such that in both cases all Pontryagin of degree ≥ 1 are zero. Show that the Pontryagin classes of $M \times N$ of degree ≥ 1 are zero in $\mathrm{H}^*(M \times N;\mathbb{Z})/2$-torsion.

Exercise 276.8.

(a) Let M and N be two closed oriented smooth manifolds. Let $\mathrm{pr}_M \colon M \times N \to M$ and $\mathrm{pr}_N \colon M \times N \to N$ the two natural projections. Show that we have the following equality in $\mathrm{H}^*(M \times N;\mathbb{Z})$:

$$p(M \times N) \;=\; \mathrm{pr}_M^*(p(M)) \cup \mathrm{pr}_N^*(p(N)).$$

(b) Let $m, n \in \mathbb{N}_0$. Which Pontryagin numbers of $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$ are non-zero?

Exercise 276.9. Let $\gamma\colon \mathbb{Z} \to \mathbb{F}_2$ be the unique epimorphism. It follows from the Cohomology-Coefficients-Functor Lemma 197.21 that for each real vector bundle $p\colon E \to B$ over a topological space B that is Hausdorff and paracompact and each $k \in \mathbb{N}_0$ we can now consider the cohomology classes $\gamma_*(p_k(B)) \in \mathrm{H}^{4k}(B;\mathbb{F}_2)$.

(a) Show that there exists a polynomial $f_k(x_1,\dots,x_{4k}) \in \mathbb{F}_2[x_1,\dots,x_{4k}]$ such that for any such real vector bundle we have $\gamma_*(p_k(E)) = f_k(w_1(E),\dots,w_{4k}(E))$.

(b) Determine these polynomials explicitly for as many k as possible.

CHAPTER 277

The Thom Isomorphism Theorem for vector bundles

In Chapter 244, given an $(\mathrm{B}^n, \mathrm{O}(n))$-bundle $p\colon E \to B$ we introduced the corresponding Thom class $\Theta(E) \in \mathrm{H}^n(B; \mathbb{F}_2)$. In this chapter we will introduce the analogous concept for n-dimensional vector bundles.

This chapter is organized as follows:

- In Section 277.1 we will discuss some elementary concepts for real vector bundles and we will discuss the relationship between real vector bundles and linear disk bundles.
- In Section 277.2 we will formulate the Thom Isomorphism Theorem 277.9.
- In Section 277.3 we will provide the proof of the Thom Isomorphism Theorem 277.9.
- In Section 277.4 we will calculate the Thom classes of Whitney sums.
- In Section 277.5 we will provide exercises for Chapter 277.

In the next Chapter 278 we will use Thom classes to introduce the Euler class of a real vector bundle. Afterwards, in Chapter 279 we will use the Euler class and Steenrod operations to give a new, alternative definition of the Stiefel-Whitney classes of real vector bundles.

277.1. **Vector bundles and disk bundles.** We start out by introducing some definitions for oriented real vector spaces:

Definition 277.1. Let $n \in \mathbb{N}_0$ and let V be an oriented n-dimensional real vector space.[2964]

(1) We write $V_{\neq 0} := V \setminus \{0\}$.

(2) We pick an orientation-preserving isomorphism $\Phi\colon \mathbb{R}^n \to V$. Let $\theta \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0}) \cong \mathbb{Z}$ be the standard generator from page 3683. We refer to $[V] := \Phi_*([\theta]) \in \mathrm{H}_n(V, V_{\neq 0})$ as the standard generator of $\mathrm{H}_n(V, V_{\neq 0})$.[2965] [2966]

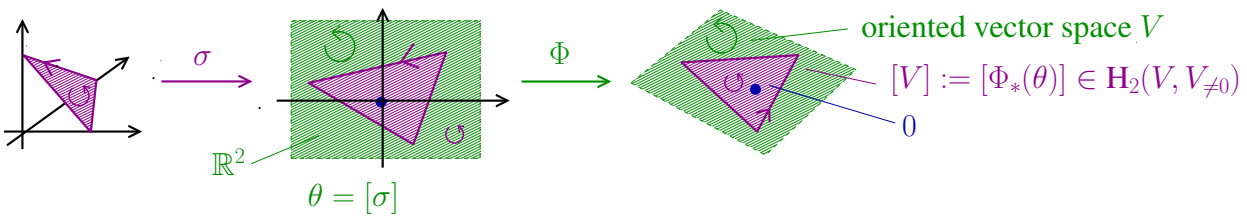
(3) It follows easily from the Universal Coefficient-H^*-Theorem 201.23 that $\mathrm{H}^n(V, V_{\neq 0}) \cong \mathbb{Z}$ and that there exists a unique generator, denoted by $[V]^*$, such $\langle [V]^*, V\rangle = 1$. We refer to $[V]^*$ as the standard generator of $\mathrm{H}^n(V, V_{\neq 0}; \mathbb{Z})$.

We continue with some notation regarding real vector bundles:

[2964] We refer to Definition 31.3 and Definition 31.4 for the definition of the orientation of a real vector space. As always the case of a 0-dimensional vector space needs to be treated slightly differently. At times we will leave it to the reader to make the elementary adjustments for the 0-dimensional case.

[2965] We need to show that this definition of $[V]$ does not depend on the choice of Φ: First note that given any two choices $\Phi_1, \Phi_2\colon \mathbb{R}^n \to V$ of orientation-preserving isomorphisms there exists an $A \in \mathrm{GL}_+(n, \mathbb{R})$ such that $\Phi_1 = \Phi_2 \circ \rho_A$ where $\rho_A\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by multiplication by A. By the Matrix Group-π_0-Proposition 4.54 (1) we know that $\mathrm{GL}_+(n, \mathbb{R})$ is path-connected. This implies that ρ_A is isotopic to $\rho_{\mathrm{id}} = \mathrm{id}$ rel $\{0\}$. It now follows easily from the Homotopy-H_*-Proposition 146.11 that we have an equality $\Phi_{1*}([\theta]) = \Phi_{2*}([\theta]) \in \mathrm{H}_n(V, V_{\neq 0})$.

[2966] It follows from the Manifold-Local R-Homology Lemma 191.2, or many alternative arguments, that $\mathrm{H}_k(V, V_{\neq 0}) = 0$ for $k \neq n$ and that $\mathrm{H}_n(V, V_{\neq 0}) = \mathbb{Z} \cdot \Phi_*([\theta])$.



Notation 277.2. Let $p\colon E \to B$ be a real vector bundle.

(1) Let $b \in B$. As in Definition 265.1 we write $E_b := p^{-1}(b)$. By definition this is an n-dimensional real vector space. Following Definition 277.1 we denote by $E_{b,\neq 0} := E_b \setminus \{0\}$ the set of *non-zero* vectors in the vector space E_b.

(2) We denote by $E_{\neq 0}$ the union of all $E_{b,\neq 0}$, i.e. $E_{\neq 0}$ is the set of non-zero vectors in each fiber.[2967]

Our next goal is to elucidate the relationship between vector bundles and linear disk bundles. To do so it is convenient to work with an alternative point of view regarding vector bundles. More precisely we will make use of the following variation on the Vector Bundle Definition Lemma 266.2:[2968]

Lemma 277.3. (Vector Bundle Definition Lemma)

(i) (1) Let $(p\colon Y \to B, \{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{R}^n\}_{i\in I})$ be an $(\mathbb{R}^n, \mathrm{GL}(n,\mathbb{R}))$-bundle in the sense of Definition 240.8. For every $b \in B$ we can define a unique $\mathbb{R}$-vector space structure on the fiber $p^{-1}(b)$ such that for each $i \in I$ and each $b \in U_i$ the following map is an isomorphism of $\mathbb{R}$-vector spaces:

$$p^{-1}(b) \xrightarrow{\Phi_i|_{p^{-1}(b)}} \{b\} \times \mathbb{R}^n \xrightarrow{(b,f)\mapsto f} \mathbb{R}^n.$$

These vector space structures turn $p\colon Y \to B$ into an n-dimensional $\mathbb{R}$-vector bundle in the sense of Definition 265.1.

(2) Let $p\colon Y \to B$ be an n-dimensional $\mathbb{R}$-vector bundle in the sense of Definition 265.1. The family of all $\mathbb{R}$-vector bundle trivializations of $p\colon Y \to B$ is an $(\mathbb{R}^n, \mathrm{GL}(n,\mathbb{R}))$-bundle.

These two maps define a weak equivalence of categories (in the sense of Definition 5.29) between the category $\mathcal{B}\mathit{und}_{(\mathbb{R}^n,\mathrm{GL}(n,\mathbb{R}))}$ of $(\mathbb{R}^n, \mathrm{GL}(n,\mathbb{R}))$-bundles and the category $\mathcal{V}\mathit{ectBund}_{\mathbb{R}^n}$ of n-dimensional $\mathbb{R}$-vector bundles.

(ii) As in (i) we have the following correspondences:[2969]

(a) $(\mathbb{R}^n, \mathrm{GL}_+(n,\mathbb{R}))$-bundles $\iff$ oriented $\mathbb{R}^n$-vector bundles.

(b) $(\mathbb{R}^n, \mathrm{O}(n))$-bundles $\iff$ $\mathbb{R}^n$-vector bundles equipped with an inner product.

(c) $(\mathbb{R}^n, \mathrm{SO}(n))$-bundles $\iff$ oriented $\mathbb{R}^n$-vector bundles with an inner product.

The above correspondences also hold if we restrict ourselves to smooth bundles.

[2967] In the Vector Bundle-Section Lemma 265.19 we introduced the 0-section $\sigma\colon B \to E$. By definition we have $E_{\neq 0} = E \setminus \sigma(B)$.

[2968] The following lemma gives in particular a solution to Exercises 266.1 and 266.3.

[2969] Recall that in Definition 269.8 we introduced the notion of an inner product on a vector bundle. Also recall that in the Inner Product-on-Vector Bundle Proposition 269.10 we showed that such an inner product exists if the base space is paracompact and Hausdorff.

Sketch of proof. Statement (i) is the content of the original Vector Bundle Definition Lemma 277.3. The other statements are proved in a similar fashion:

(a) For an $(\mathbb{R}^n, \mathrm{GL}_+(n,\mathbb{R}))$-bundle we use this atlas and the standard orientation on $\mathbb{R}^n$ to equip the fibers with an orientation. Conversely, given an oriented $\mathbb{R}^n$-vector bundle we only consider orientation-preserving trivializations and we obtain an $(\mathbb{R}^n, \mathrm{GL}_+(n,\mathbb{R}))$-bundle.

(b) For an $(\mathbb{R}^n, \mathrm{O}(n))$-bundle we use this atlas and the standard Euclidean inner product on $\mathbb{R}^n$ to equip the fibers with an inner product. Conversely, given an $\mathbb{R}^n$-vector bundle that is equipped with an inner product we only consider trivializations that are fiberwise isometries. This way we obtain an $(\mathbb{R}^n, \mathrm{O}(n))$-bundle.

(c) This statement follows by combining the ideas from (a) and (b).

We leave it to the reader to fill in the details. ■

Next recall that by Definition 57.4 and Definition 244.11 a linear $\overline{B}^n$-bundle is exactly the same as a $(\overline{B}^n, \mathrm{O}(n))$-bundle. In Definition 244.2 we introduced in particular the notion of an orientation of an $\overline{B}^n$-bundle. To deal with this setup it is convenient to recall the following definition & lemma:

Lemma 244.4. (Disk Bundle Definition Lemma)

(i) By Definition 57.4 and Definition 244.11 we have the following equality:

$$(\overline{B}^n, \mathrm{O}(n,\mathbb{R}))\text{-bundles} \quad = \quad \text{linear } \overline{B}^n\text{-vector bundles.}$$

(ii) As in the Vector Bundle Definition Lemma 277.3 (ii-a) we have the following correspondence:

$$(\overline{B}^n, \mathrm{SO}(n,\mathbb{R}))\text{-bundles} \quad \Longleftrightarrow \quad \text{oriented linear } \overline{B}^n\text{-vector bundles.}$$

The following proposition shows that an (oriented) vector bundle that is equipped with an inner product gives rise to an (oriented) linear disk bundle:

Proposition 277.4. (Vector Bundle-to-Disk Bundle Proposition) Let $p\colon E \to X$ be an n-dimensional real vector bundle that is equipped with an inner product φ or equivalently, let $p\colon E \to X$ be an $(\mathbb{R}^n, \mathrm{O}(n))$-bundle. We consider the following subspaces of E:

$$\begin{aligned} \mathrm{B}E &:= \{e \in E \,|\, \varphi(e,e) \leq 1\}, \\ \mathrm{S}E &:= \{e \in E \,|\, \varphi(e,e) = 1\}. \end{aligned}$$

The following statements hold:

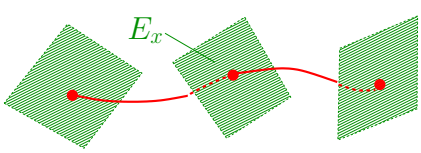
(1) The atlas for the $(\mathbb{R}^n, \mathrm{O}(n))$-bundle $p\colon E \to X$ restricts to an atlas for $p\colon \mathrm{B}E \to X$ as a $(\overline{B}^n, \mathrm{O}(n))$-bundle. In particular $p\colon \mathrm{B}E \to X$ is a linear $\overline{B}^n$-bundle with associated sphere bundle $\mathrm{S}E$.

(2) (a) If $p\colon E \to X$ is a smooth real vector bundle and if the inner product φ is smooth, then $p\colon \mathrm{B}E \to X$ is naturally a smooth linear $\overline{B}^n$-bundle.

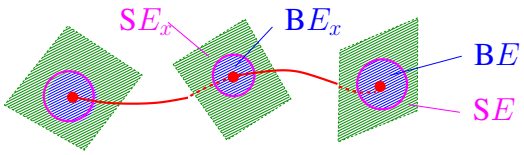
(b) If $p\colon E \to X$ is oriented, then the induced orientations on the fibers of $\mathrm{B}E$ turn $p\colon \mathrm{B}E \to X$ into an oriented $\overline{B}^n$-bundle.

(3) (a) The inclusion $(\mathrm{B}E, \mathrm{S}E) \to (E, E_{\neq 0})$ induces isomorphisms on homology and cohomology groups.

(b) For each $x \in X$ the inclusion $(\mathrm{B}E_x, \mathrm{S}E_x) \to (E_x, E_{x0})$ induces isomorphisms on homology and cohomology groups.



vector bundle $p\colon E \to X$



linear $\overline{B}^2$-bundle $p\colon \mathrm{B}E \to X$

Sketch of proof.

(1) This statement follows easily from the definitions.

(2) (a) This statement follows easily from the definitions.

(b) This statement follows easily from the definitions together with the Vector Bundle Definition Lemma 277.3 (ii-a) and the Disk Bundle Definition Lemma 244.4.

(3) As in the Radial Shrinking Lemma 57.8 (2) we see that the 0-section is a deformation retract of E and also of $\mathrm{B}E$. Similarly it is clear that the map

$$\begin{array}{rcl} E_{\neq 0} \times [0,1] & \to & E_{\neq 0} \\ (x,t) & \mapsto & x \cdot ((1-t) + \frac{1}{|x|} \cdot t) \end{array}$$

is a deformation retraction from $E_{\neq 0}$ to $\mathrm{S}E$. It thus follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10, the Homotopy Equivalence-Relative-H_*-Corollary 146.8 and the Homotopy Equivalence-Relative-H^*-Corollary 198.14 that the inclusion $(\mathrm{B}E, \mathrm{S}E) \to (E, E_{\neq 0})$ induces isomorphisms on homology and cohomology groups. Since the above logic also works "fiberwise" we see that for each $x \in X$ the inclusion $(\mathrm{B}E_x, \mathrm{S}E_x) \to (E_x, E_{x0})$ also induces isomorphisms on homology and cohomology groups. ■

Our next goal is to formulate a suitable converse to the Vector Bundle-to-Disk Bundle Proposition 277.4. This will require the following notation:

Notation 277.5. As on page 652 we consider the maps

$$\begin{array}{rcl} \alpha\colon B^n & \to & \mathbb{R}^n \\ v & \mapsto & \dfrac{v}{\sqrt{1-\|v\|^2}} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\colon \mathbb{R}^n & \to & B^n \\ v & \mapsto & \dfrac{v}{\sqrt{1+\|v\|^2}}. \end{array}$$

One can easily verify that they are smooth and that they are inverses of one another. This shows that both maps are diffeomorphisms.

Now we can explain how to turn linear disk bundles into vector bundles:

Proposition 277.6. (Disk Bundle-to-Vector Bundle Proposition) Let $p\colon T \to X$ be a linear $\overline{B}^n$-bundle over a topological space X, i.e. T is a $(\overline{B}^n, \mathrm{O}(n))$-bundle. As in the Boundary Fiber Bundle Lemma 244.1 we consider $\mathrm{S}T$, which is given by taking fiberwise the boundary. We write

$$\mathrm{V}T \;:=\; T \setminus \mathrm{S}T.$$

(1) Let $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \overline{B}^n\}_{i\in I}$ be the atlas from the $(\overline{B}^n, \mathrm{O}(n))$-bundle. Then the maps $\{\Phi_i\colon p^{-1}(U_i) \cap (T \setminus \mathrm{S}T) \to U_i \times B^n \xrightarrow{(x,v)\mapsto(x,\alpha(v))} U_i \times \mathbb{R}^n\}_{i\in I}$ form an atlas for $\mathrm{V}T$ as an $(\mathbb{R}^n, \mathrm{O}(n))$-bundle, i.e. as an n-dimensional real vector bundle that is equipped with an inner product.

(2) If $p\colon T \to X$ is oriented, then the induced orientations on the fibers of $\mathrm{B}E$ turn $p\colon \mathrm{V}T \to X$ into an oriented real vector bundle.

Proof. The first part of the proposition follows easily from the elementary observation that the diffeomorphism $\alpha\colon B^n \to \mathbb{R}^n$ has the convenient feature that it commutes with multiplication by any $A \in \mathrm{O}(n)$. The second statement follows easily from the Vector Bundle Definition Lemma 277.3 (ii-a), the Disk Bundle Definition Lemma 244.4 and the observation that the diffeomorphism α is orientation-preserving. ■

The following proposition shows that the constructions from the Vector Bundle-to-Disk Bundle Proposition 277.4 and the Disk Bundle-to-Vector Bundle Proposition 277.6 are essentially inverses of one another:

Proposition 277.7. (Disk Bundle–Vector Bundle Proposition)[2970]

(1) Let $n \in \mathbb{N}_0$ and let $p\colon E \to X$ be an $(\mathbb{R}^n, \mathrm{O}(n))$-bundle. The corresponding $(\mathbb{R}^n, \mathrm{O}(n))$-bundle $\mathrm{VB}E \to X$ is naturally isomorphic to the original $(\mathbb{R}^n, \mathrm{O}(n))$-bundle.

(2) Let $n \in \mathbb{N}_0$ and let $p\colon T \to X$ be a $(\overline{B}^n, \mathrm{O}(n))$-bundle. The corresponding $(\overline{B}^n, \mathrm{O}(n))$-bundle $\mathrm{BV}T \to X$ is naturally isomorphic to the original $(\overline{B}^n, \mathrm{O}(n))$-bundle.

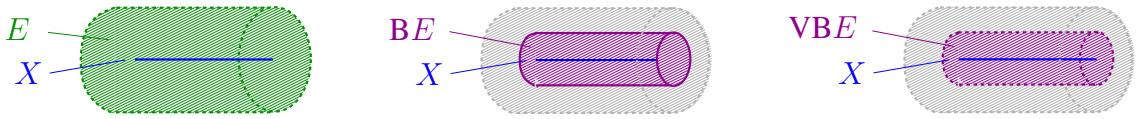
The analogous statements also hold in the oriented setting.

Proof.

(1) Let $p\colon E \to X$ be an $(\mathbb{R}^n, \mathrm{O}(n))$-bundle where the corresponding atlas is given by $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \mathbb{R}^n\}_{i\in I}$. One can easily verify that the maps

$$p^{-1}(U_i) \xrightarrow{\Phi_i} U_i \times \mathbb{R}^n \xrightarrow[\mathrm{diag}(\mathrm{id},\beta)]{(x,v)\mapsto \left(x, \frac{v}{\sqrt{1+\|v\|^2}}\right)} U_i \times B^n \xrightarrow{\Phi_i^{-1}} \underbrace{(p|_{\mathrm{VB}E})^{-1}(U_i)}_{=p^{-1}(U_i)\cap \mathrm{VB}E}$$

form an isomorphism of $(\mathbb{R}^n, \mathrm{O}(n))$-bundles.



(2) Let $p\colon T \to X$ be a $(\overline{B}^n, \mathrm{O}(n))$-bundle where the corresponding atlas is given by $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \overline{B}^n\}_{i\in I}$. One can easily verify that the maps

$$p^{-1}(U_i) \xrightarrow{\Phi_i} U_i \times \overline{B}^n \xrightarrow{(x,v)\mapsto \left(x, \frac{v}{\sqrt{2}}\right)} U_i \times B^n_{\frac{1}{\sqrt{2}}} \xrightarrow{\Phi_i^{-1}} \underbrace{(p|_{\mathrm{BV}T})^{-1}(U_i)}_{=p^{-1}(U_i)\cap \mathrm{BV}T}$$

form an isomorphism of $(\overline{B}^n, \mathrm{O}(n))$-bundles.

Of course the same argument also works if we replace $\mathrm{O}(n)$ by $\mathrm{SO}(n)$. ■

A smooth disk bundle gives naturally rise to two vector bundles. The following shows that these two are fortunately essentially the same:

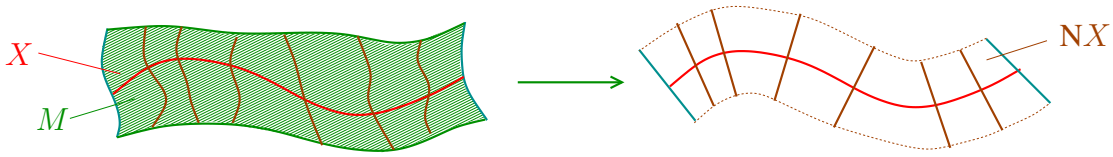
Proposition 277.8. (Normal Vector Bundle–Disk Bundle Proposition) Let $p\colon M \to X$ be a linear smooth $\overline{B}^n$-bundle over a smooth manifold X. As in Definition 244.11 we

[2970] In fact the proposition shows that the assignments $E \mapsto \mathrm{B}E$ and $T \mapsto \mathrm{V}T$ are equivalences of categories in the sense of Definition 5.29.

view X as a proper smooth submanifold of M by identifying each $x \in X$ with the origin in the fiber of x. We consider the following two real vector bundles over X:

(1) The vector bundle $\mathrm{V}M \to X$ from the Disk Bundle-to-Vector Bundle Proposition 277.6.
(2) The normal vector bundle $\mathrm{N}X$ of $X \subseteq M$ that is provided by Definition 269.15.

These two vector bundles are isomorphic.



Proof. Let $p\colon M \to X$ be a linear smooth $\overline{B}^n$-bundle over a smooth manifold X. We introduce the following notation:

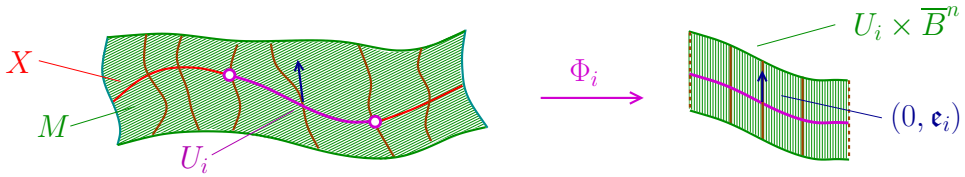
- Let $\{\Phi_i\colon p^{-1}(U_i) \to U_i \times \overline{B}^n\}_{i\in I}$ be the atlas of the $(\overline{B}^n, \mathrm{O}(n))$-bundle.
- We write $q := p|_{\mathrm{V}M}\colon \mathrm{V}M \to X$.
- For $i \in I$ we denote by $\widetilde{\Phi}_i\colon q^{-1}(U_i) \to U_i \times \mathbb{R}^n$ the trivialization from the Disk Bundle-to-Vector Bundle Proposition 277.6, i.e. the map that is given by the composition of the map $\Phi_i|_{q^{-1}(U_i)} \to U_i \times B^n$ with the map

$$\begin{aligned} \mathrm{diag}(\mathrm{id}, \alpha)\colon U_i \times B^n &\to U_i \times \mathbb{R}^n \\ (x, v) &\mapsto \left(x, \tfrac{v}{\sqrt{1-\|v\|^2}}\right). \end{aligned}$$

- For $i \in I$ we consider the map:

$$\begin{aligned} \Theta_i\colon U_i \times \mathbb{R}^n &\to \mathrm{T}M|_{U_i} \\ (x, (\lambda_1, \ldots, \lambda_n)) &\mapsto \Big(x, \sum_{i=1}^n \lambda_i \cdot \underbrace{\mathrm{D}(\Phi_i^{-1})_{(x,0)}}_{\substack{\text{homomorphism} \\ \mathrm{T}_x U_i \times \mathbb{R}^n \to \mathrm{T}_x M}} (0, \mathfrak{e}_i)\Big). \end{aligned}$$

- For $i \in I$ we consider the map $\Xi := \Theta_i \circ \widetilde{\Phi}_i\colon q^{-1}(U_i) \to \mathrm{T}M|_{U_i}$. Note that it follows easily from the definitions that this map is linear on the fibers.



Claim. For $i, j \in I$ the maps Ξ_i and Ξ_j agree on $q^{-1}(U_i) \cap q^{-1}(U_j)$.

Proof. Let $i, j \in I$. By Definition 240.8 of a $(\overline{B}^n, \mathrm{O}(n))$-bundle we know that there exists a continuous map $\gamma\colon U_i \cap U_j \to \mathrm{O}(n)$ such that the following diagram commutes:

$$(U_i \cap U_j) \times \overline{B}^n \xleftarrow[\cong]{\Phi_i} p^{-1}(U_i \cap U_j) \xrightarrow[\cong]{\Phi_j} (U_i \cap U_j) \times \overline{B}^n.$$
$$(x,v) \mapsto (x, \gamma(x)(v))$$

We now consider the following diagram:

$$\begin{array}{ccccccc}
 & \xrightarrow{\Phi_i|_{q^{-1}(U_i\cap U_j)}} & (U_i\cap U_j)\times B^n & \xrightarrow{\mathrm{diag}(\mathrm{id},\alpha)} & (U_i\cap U_j)\times\mathbb{R}^n & \xrightarrow{\Theta_i} & \mathrm{T}M|_{U_i\cap U_j} \\
q^{-1}(U_i\cap U_j) & & \downarrow (x,v)\mapsto(x,\gamma(x)(v)) & & \downarrow (x,v)\mapsto(x,\gamma(x)(v)) & & \downarrow \mathrm{id} \\
 & \xrightarrow{\Phi_j|_{q^{-1}(U_i\cap U_j)}} & (U_i\cap U_j)\times B^n & \xrightarrow{\mathrm{diag}(\mathrm{id},\alpha)} & (U_i\cap U_j)\times\mathbb{R}^n & \xrightarrow{\Theta_j} & \mathrm{T}M|_{U_i\cap U_j}.
\end{array}$$

The claim now follows from the following observations:

- The triangle commutes by definition.
- The first square commutes since α commutes with multiplication by orthogonal matrices.
- It follows easily from the chain rule and the discussion on page 651 that the second square commutes. ⊞

It follows from the claim and the Pasting Proposition 4.7 that the maps Ξ_i define a continuous map $\Xi\colon \mathrm{V}M \to \mathrm{T}M$. Finally we consider the map

$$\begin{array}{rcl}
\mathrm{T}X \oplus \mathrm{V}M & \to & \mathrm{T}M \\
((x,v),(x,w)) & \mapsto & \underbrace{(x,v)+(\Xi(x,w))}_{\text{addition in the fiber } \mathrm{T}_xM}.
\end{array}$$

Note that this map is fiberwise a homomorphism and that for each $x \in X$ we have $\mathrm{T}_xX \oplus \Xi(\mathrm{V}M_x) = \mathrm{T}_xM$. It is now clear that this map is fiberwise an isomorphism. In other words, the above map is an isomorphism of vector bundles over M. It now follows from the Normal Vector Bundle Proposition 269.13 (3) that the vector bundle $\mathrm{V}M$ is isomorphic to the normal vector bundle $\mathrm{N}X$. ■

277.2. **The Thom Isomorphism Theorem.** The following theorem is an analog of the Thom–Disk Bundle Isomorphism Theorem 244.5:

Theorem 277.9. (Thom Isomorphism Theorem) Let $p\colon E \to B$ be an n-dimensional real vector bundle.

(1) (a) There exists a unique $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2)$ with the property that for each $b \in B$ the restriction to $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2)$ is non-zero.[2971]

(b) For each $k \in \mathbb{Z}$ the map[2972] [2973]

$$\begin{array}{rcl}
\mathrm{H}^k(B;\mathbb{F}_2) & \to & \mathrm{H}^{n+k}(E, E_{\neq 0}; \mathbb{F}_2) \\
\varphi & \mapsto & \Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi)
\end{array}$$

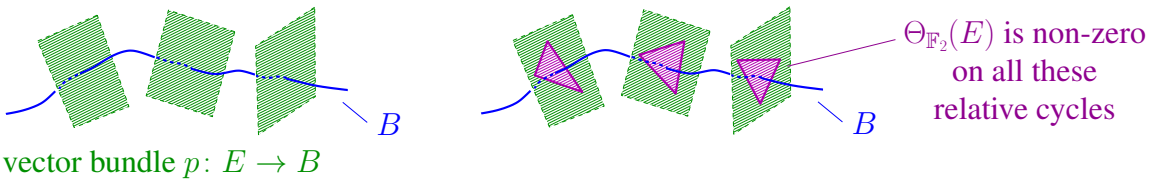
is an isomorphism.

(2) (a) If the real vector bundle E is oriented in the sense of Definition 265.23, then there exists a unique $\Theta_{\mathbb{Z}}(E) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z})$ with the property that for each $b \in B$ the restriction to $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{Z})$ is the standard generator given by the orientation and Definition 277.1.

(b) For each $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl}
\mathrm{H}^k(B;\mathbb{Z}) & \to & \mathrm{H}^{n+k}(E, E_{\neq 0}; \mathbb{Z}) \\
\varphi & \mapsto & \Theta_{\mathbb{Z}}(E) \cup p^*(\varphi)
\end{array}$$

is an isomorphism.



Definition 277.10. Let $p\colon E \to B$ be an n-dimensional real vector bundle. The cohomology classes $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z})$ and $\Theta_{\mathbb{Z}}(E) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z})$ from the Thom Isomorphism Theorem 277.9 are both called Thom class and both isomorphisms from the Thom Isomorphism Theorem 277.9 are called Thom isomorphism.[2974]

We postpone the proof of the Thom Isomorphism Theorem 277.9 to Section 277.3. Before we deal with the proof of the Thom Isomorphism Theorem 277.9 we first want to discuss a few properties of Thom classes. First of all, let us state and prove the following proposition, which is the analog of the Thom Class–Natural Proposition 244.7:

Proposition 277.11. (Thom Class–Natural Proposition)

(1) Let

$$\begin{array}{ccc} E & \xrightarrow{G} & F \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & C \end{array}$$

be a morphism of n-dimensional real vector bundles. Then

$$\Theta_{\mathbb{F}_2}(E) \;=\; G^*(\Theta_{\mathbb{F}_2}(F)) \;\in\; \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2).$$

(2) For a morphism of oriented n-dimensional real vector bundles (in the sense of Definition 265.23) we have the analogous equality

$$\Theta_{\mathbb{Z}}(E) \;=\; G^*(\Theta_{\mathbb{Z}}(F)) \;\in\; \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z}).$$

Proof.

(1) Let $\Theta_{\mathbb{F}_2}(F) \in \mathrm{H}^n(F, F_0; \mathbb{F}_2)$ be the Thom class. We claim that we have the equality $\Theta_{\mathbb{F}_2}(E) = G^*(\Theta_{\mathbb{F}_2}(F)) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2)$. By the uniqueness statement of the Thom Isomorphism Theorem 277.9 (1) it suffices to prove the following claim:

Claim. For each $b \in B$ the restriction of $G^*(\Theta_{\mathbb{F}_2}(F))$ to $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2)$ is non-zero.

[2971] As the reader is surely aware, $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2) \cong \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0}; \mathbb{F}_2) \cong \mathbb{F}_2$. Thus there is a unique non-zero element in this cohomology group.

[2972] Note that $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2)$ and that $p^*(\varphi) \in \mathrm{H}^k(E; \mathbb{F}_2)$. Therefore the relative cup product $\Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi)$ from Definition 204.7 lies in $\mathrm{H}^{n+k}(E, E_{\neq 0}; \mathbb{F}_2)$.

[2973] For $k \in \{-n, \ldots, -1\}$ the statement implies that $\mathrm{H}^{n+k}(E, E_{\neq 0}; \mathbb{F}_2) \cong \mathrm{H}^k(B; \mathbb{F}_2) = 0$.

[2974] Note that in [**MiS1974**, p. 90] this is called the *fundamental class*.

Proof. Let $b \in B$. We set $c := g(b)$. Note that we have the following commutative diagram

$$\begin{array}{ccc} E_b & \xrightarrow[\cong]{G|_{E_b}} & F_c \\ \downarrow & & \downarrow \\ E & \xrightarrow{G} & F \end{array} \qquad \text{which induces the diagram} \qquad \begin{array}{ccc} \mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2) & \xleftarrow{\cong} & \mathrm{H}^n(F_c, F_{c0}; \mathbb{F}_2) \\ \uparrow & & \uparrow \\ \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2) & \xleftarrow{G^*} & \mathrm{H}^n(F, F_0; \mathbb{F}_2). \\ & & \Theta_{\mathbb{F}_2}(F) \end{array}$$

By functoriality of cohomology the diagram to the right also commutes. Since the top horizontal map is an isomorphism it is now clear that the restriction of $G^*(\Theta_{\mathbb{F}_2}(F))$ to $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2)$ is non-zero. ⊞

(2) The proof of the statement for a morphism of oriented n-dimensional real vector bundles is almost the same as the above proof. We leave it to the reader to fill in the details. ■

Note that for an oriented real vector bundle $p\colon E \to B$ we now have two Thom classes, namely $\Theta_{\mathbb{F}_2}(E)$ and $\Theta_{\mathbb{Z}}(E)$. The following proposition, which is analogous to the Thom Class Relationship Proposition 244.8, gives us the utterly predictable connection between these two cohomology class:

Proposition 277.12. (Thom Class Relationship Proposition) Let $p\colon E \to B$ be an n-dimensional real vector bundle. We denote by $\gamma_*\colon \mathbb{Z} \to \mathbb{F}_2$ the obvious ring homomorphism. From the Cohomology-Coefficients-Functor Lemma 197.21 we obtain an induced homomorphism $\gamma_*\colon \mathrm{H}^k(E, E_{\neq 0}; \mathbb{Z}) \to \mathrm{H}^k(E, E_{\neq 0}; \mathbb{F}_2)$. If E is oriented, then we have the equality

$$\Theta_{\mathbb{F}_2}(E) \;=\; \gamma_*(\Theta_{\mathbb{Z}}(E)) \;\in\; \mathrm{H}^n(B; \mathbb{F}_2).$$

Proof. Let $\Theta_{\mathbb{Z}}(F) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z})$ be the Thom class. We claim that we have the equality $\Theta_{\mathbb{F}_2}(E) = \gamma_*(\Theta_{\mathbb{Z}}(E)) \in \mathrm{H}^n(B; \mathbb{F}_2)$. By the uniqueness statement of the Thom Isomorphism Theorem 277.9 (1) it suffices to prove the following claim:

Claim. For each $b \in B$ the restriction of $\gamma_*(\Theta_{\mathbb{Z}}(E)) \in \mathrm{H}^n(B; \mathbb{F}_2)$ to $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2)$ is non-zero.

Proof. Let $b \in B$. We denote by $\iota_b\colon E_b \to E$ the inclusion. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z}) & \xrightarrow{\iota_b^*} & \mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{Z}) \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2) & \xrightarrow{\iota_b^*} & \mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{F}_2). \end{array} \qquad \begin{array}{c} \Theta_{\mathbb{Z}}(E) \\ \\ \\ \\ [E_b]^* \end{array}$$

We make the following observations:

- It follows from the Cohomology-Coefficients-Functor Lemma 197.21 (2) that the diagram commutes.
- By definition of $\Theta_{\mathbb{Z}}(E)$ we know that $\iota_b^*(\Theta_{\mathbb{Z}}(E))$ is a generator of $\mathrm{H}^n(E_b, E_{b,\neq 0}; \mathbb{Z})$.
- It follows easily from the H^*–Change-of-Coefficients Theorem 202.10 together with the H^*–Coefficient Change Lemma 202.11 that the right vertical map is an epimorphism.

It is now clear that $\iota_b^*(\gamma_*(\Theta_{\mathbb{Z}}(E)))$ is the unique non-zero element. ■

We continue this discussion of basic properties of the Thom class with the following elementary lemma:

Lemma 277.13. (Orientation–Thom Lemma Let $p\colon E \to B$ be an oriented n-dimensional real vector bundle. For the purpose of this lemma we denote by $-E$ the same real vector bundle but with the opposite orientation on each fiber. Then

$$\Theta_{\mathbb{Z}}(-E) \;=\; -\Theta_{\mathbb{Z}}(E) \;\in\; \mathrm{H}^n(w, E_{\neq 0}; \mathbb{Z}).$$

Proof. This lemma follows easily from the definition of the Thom class in the Thom Isomorphism Theorem 277.9 (2) and the Fundamental Class–Opposite Orientation Lemma 194.4. ■

Finally, as the reader will surely have noticed, by now we have defined Thom classes for (oriented) $\overline{B}^n$-bundles and for (oriented) real vector bundles. The following proposition tells us that these two concepts are essentially the same:

Proposition 277.14. (Thom-equals-Thom Proposition) Let $p\colon E \to B$ be an n-dimensional real vector bundle over a topological space B. We assume that the vector bundle has a finite atlas. Let φ be an inner product on E.[2975] Next let $p\colon \mathrm{B}E \to B$ be the corresponding $\overline{B}^n$-bundle from the Vector Bundle-to-Disk Bundle Proposition 277.4.

(1) We apply the Thom–Disk Bundle Isomorphism Theorem 244.5 and we obtain a Thom class $\Theta_{\mathbb{F}_2}(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2) \in \mathrm{H}^n(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2)$. The inclusion induced isomorphism

$$\iota^*\colon \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2) \;\to\; \mathrm{H}^n(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2)$$

from the Vector Bundle-to-Disk Bundle Proposition 277.4 has the property that it sends

$$\Theta_{\mathbb{F}_2}(E \to B) \;\mapsto\; \Theta_{\mathbb{F}_2}(\mathrm{B}E \to B).$$

(2) If we start out with an oriented real vector bundle, then an analogous statement holds for the integer Thom classes.

Proof. We provide the proof for Statement (2). The proof for Statement (1) is essentially a slight simplification of the argument.

By the Thom–Disk Bundle Isomorphism Theorem 244.5 it suffices to prove the following claim:

Claim. For each $b \in B$ the restriction of $\iota^*(\Theta_{\mathbb{Z}}(E \to B))$ to $\mathrm{H}^n(\mathrm{B}E_b, \mathrm{SB}E_b; \mathbb{Z})$ is the fundamental class.

Proof. Note that we have the following commutative diagram of inclusion maps:

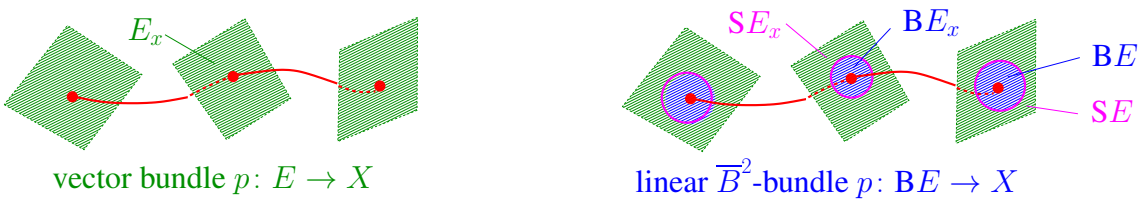
$$\begin{array}{ccc} (\mathrm{B}E_b, \mathrm{S}E_b) & \longrightarrow & (\mathrm{B}E, \mathrm{S}E) \\ \downarrow & & \downarrow \\ (E_b, E_{0b}) & \xrightarrow{\iota} & (E, E_{\neq 0}) \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^n(\mathrm{B}E_b, \mathrm{S}E_b; \mathbb{Z}) & \xleftarrow[\iota^*]{} & \mathrm{H}^n(\mathrm{B}E, \mathrm{S}E; \mathbb{Z}) \\ \cong\uparrow & & \uparrow \\ \mathrm{H}^n(E_b, E_{0b}; \mathbb{Z}) & \longleftarrow & \mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z}). \\ & & \Theta_{\mathbb{Z}}(E \to B) \end{array}$$

By the Vector Bundle-to-Disk Bundle Proposition 277.4 we know that in the diagram to the right the left vertical map is an isomorphism and that it sends the fundamental of $\mathrm{H}^n(E_b, E_{0b}; \mathbb{Z})$ to the standard generator of $\mathrm{H}^n(\mathrm{B}E_b, \mathrm{S}E_b; \mathbb{Z})$.
It is now clear that the restriction of $\iota^*(\Theta_{\mathbb{Z}}(E \to B))$ to $\mathrm{H}^n(\mathrm{B}E_b, \mathrm{SB}E_b; \mathbb{Z})$ is the fundamental class. ■

[2975] By the Inner Product-on-Vector Bundle Proposition 269.10 we know that such an inner product exists if the base space is Hausdorff and paracompact.



277.3. **Proof of the Thom Isomorphism Theorem 277.9.** There are two approaches to proving the Thom Isomorphism Theorem 277.9:

(i) One can either reduce it to the Thom–Disk Bundle Isomorphism Theorem 244.5, or
(ii) one can adapt the proof of the Thom–Disk Bundle Isomorphism Theorem 244.5.

First we sketch a proof using approach (i). Thus let $p\colon E \to B$ be an n-dimensional real vector bundle over a topological space B. In this approach we now assume that B is Hausdorff and paracompact and that the bundle has a finite atlas. Note that it follows from the Inner Product-on-Vector Bundle Proposition 269.10 that there exists an inner product on the real vector bundle. Next let $p\colon \mathrm{B}E \to B$ be a corresponding $\overline{B}^n$-bundle as in the Vector Bundle-to-Disk Bundle Proposition 277.4. We apply the Thom–Disk Bundle Isomorphism Theorem 244.5 and we obtain a Thom class $\Theta_{\mathbb{F}_2}(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2) \in \mathrm{H}^n(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2)$. By the Vector Bundle-to-Disk Bundle Proposition 277.4 we know that the inclusion induced map $\iota^*\colon \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2) \to \mathrm{H}^n(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2)$ is an isomorphism. We already showed in the Thom-equals-Thom Proposition 277.14 that $(\iota^*)^{-1}(\Theta_{\mathbb{F}_2}(\mathrm{B}E, \mathrm{S}E; \mathbb{F}_2))$ has the desired properties. Finally essentially the same argument gives us also the uniqueness statement.

We now turn to approach (i), i.e. we turn to the modification of the proof of the Thom–Disk Bundle Isomorphism Theorem 244.5.

In the following lemma we first deal with the case that $E = B \times \mathbb{R}^n$ is the trivial n-dimensional real vector bundle over some topological space B. More precisely, we first prove the following analog of the Trivializable–Thom Lemma 244.9:

Lemma 277.15. (Trivializable–Thom Lemma) Let $n \in \mathbb{N}_0$ and let B be a topological space. The statement of the Thom–Disk Bundle Isomorphism Theorem 244.5 holds for the trivial n-dimensional real vector bundle $B \times \mathbb{R}^n \to B$ where we equip each $\{b\} \times \mathbb{R}^n$ with the obvious orientation. In fact we can describe the Thom classes precisely: We denote by $q\colon X \times \mathbb{R}^n \to \mathbb{R}^n$ the natural projection.

(1) If we denote by $[\mathbb{R}^n]^*_{\mathbb{F}_2} \in \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0}; \mathbb{F}_2)$ the unique non-zero element, then the cohomology class $\Theta := q^*([\mathbb{R}^n]^*_{\mathbb{F}_2}) \in \mathrm{H}^n(X \times \mathbb{R}^n, X \times \mathbb{R}^n_{\neq 0}; \mathbb{F}_2)$ has the desired properties and it is the unique such element.
(2) If we denote by $[\mathbb{R}^n]^* \in \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0}; \mathbb{Z})$ the standard generator from Definition 277.1, then the cohomology class $\Theta := q^*([\mathbb{R}^n]^*) \in \mathrm{H}^n(X \times \mathbb{R}^n, X \times \mathbb{R}^n_{\neq 0}; \mathbb{Z})$ has the desired properties and it is the unique such element.

Proof. [2976] In the following we deal with $\mathbb{Z}$-coefficients. The proof for $\mathbb{F}_2$-coefficients is essentially the same. We start out with the following claim:

[2976] The following proof is a carbon-copy of the proof of the Trivializable–Thom Lemma 244.9.

Claim 1. Let W be a topological space. In the following we denote by $p\colon W \times \mathbb{R}^n \to W$ and $q\colon W \times \mathbb{R}^n \to \mathbb{R}^n$ the natural projections. For each $k \in \mathbb{Z}$ the following map is an isomorphism[2977]

$$\begin{array}{rcl} \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0};\mathbb{Z}) & \to & \mathrm{H}^{k+n}(W \times \mathbb{R}^n, W \times \mathbb{R}^n_{\neq 0};\mathbb{Z}) \\ \varphi \otimes \psi & \mapsto & p^*(\varphi) \cup q^*(\psi). \end{array}$$

Proof. We consider the following diagram:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^{n-1}(\mathbb{R}^n_{\neq 0};\mathbb{Z}) & \xrightarrow{\varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} & \mathrm{H}^{k+n-1}(W \times \mathbb{R}^n_{\neq 0};\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\delta_{n-1}} & & \downarrow{\scriptstyle \delta_{k+n-1}} \\ \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0};\mathbb{Z}) & \xrightarrow{\varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} & \mathrm{H}^{k+n}(W \times \mathbb{R}^n, W \times \mathbb{R}^n_{\neq 0};\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\text{я}} & & \downarrow{\scriptstyle \text{я}} \\ \mathrm{H}^k(W;\mathbb{Z}) \otimes \mathrm{H}^n(\mathbb{R}^n;\mathbb{Z}) & \xrightarrow{\varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} & \mathrm{H}^{k+n}(W \times \mathbb{R}^n;\mathbb{Z}) \\ \downarrow & & \downarrow \end{array}$$

We make the following clarifications and observations:

- The right vertical sequence is just given by the LES-of-Pair Proposition 146.7 corresponding to the pair $(W \times \mathbb{R}^n, W \times \mathbb{R}^n_{\neq 0})$. In particular it is exact.
- The left vertical sequence is given by tensoring the exact sequence of the LES-of-Pair Proposition 146.7 corresponding to the pair $(\mathbb{R}^n, \mathbb{R}^n_{\neq 0})$ with the group $\mathrm{H}^k(W;\mathbb{Z})$. It follows from the Algebraic Universal Coefficient Theorem 162.15 that the left vertical sequence is also exact.
- It follows easily from the naturality of the connecting homomorphism in the LES-of-Triple Proposition 198.13, the Cup Product-Connecting Homomorphism Lemma 204.18 and the Cup Product-Naturality Lemma 204.8 that the diagram commutes up to a sign.
- It follows from the Künneth-H*-Theorem 208.1 together with the Tor-Properties Lemma 162.13 (3) that the top and bottom horizontal maps are isomorphisms (here we use that $\mathbb{R}^n$ and $\mathbb{R}^n_{\neq 0}$ both satisfy Condition $(*)$ from the Künneth-H*-Theorem 208.1).

It follows from the above together with the Five Lemma 145.11 that the middle horizontal map is (as promised) also an isomorphism. ⊞

Now we turn to the actual proof of the lemma. Thus let B be a topological space.

(a) As usual we denote by $[\mathbb{R}^n]^* \in \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0};\mathbb{Z})$ the standard generator from Definition 277.1.

Claim 2. The cohomology class $\Theta := q^*([\mathbb{R}^n]^*) \in \mathrm{H}^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_{\neq 0};\mathbb{Z})$ has the property that for each $b \in B$ the restriction of Θ to $\mathrm{H}^n(\{b\} \times \mathbb{R}^n, \{b\} \times \mathbb{R}^n_{\neq 0};\mathbb{Z})$ equals the dual fundamental class $[\{b\} \times \mathbb{R}^n]^*$.

[2977]Here we use the relative cup product from Definition 204.7.

Proof. Let $b \in B$. We denote by $\iota_b \colon \{b\} \times \mathbb{R}^n \to B \times \mathbb{R}^n$ the inclusion and we denote by $f_b \colon \mathbb{R}^n \xrightarrow{v \mapsto (x,v)} \{b\} \times \mathbb{R}^n$ the obvious homeomorphism. We now see that

$$\begin{array}{rcl} \langle \iota_b^*(\Theta), f_{b*}([\mathbb{R}^n]) \rangle_{\{b\}\times\mathbb{R}^n} & \overset{\text{since } \Theta := q^*([\mathbb{R}^n]^*)}{=} & \langle (q \circ \iota_b)^*([\mathbb{R}^n]^*), f_{b*}([\mathbb{R}^n]) \rangle_{\{b\}\times\mathbb{R}^n} \\ & = & \langle [\mathbb{R}^n]^*, \underbrace{(q \circ \iota_b \circ f_b)}_{=\mathrm{id}_{\mathbb{R}^n}}{}_*([\mathbb{R}^n]) \rangle_{\mathbb{R}^n} \\ & \overset{\text{by the Kronecker Pairing–Natural Lemma 199.9 (1)}}{=} & \langle [\mathbb{R}^n]^*, [\mathbb{R}^n] \rangle_{\mathbb{R}^n} \ = \ 1. \end{array}$$

This shows that $\iota_b^*(\Theta)$ equals the standard generator of $\mathrm{H}^n(\{b\} \times \mathbb{R}^n, \{b\} \times \mathbb{R}^n_{\neq 0}; \mathbb{Z})$. ⊞

Claim 3. The cohomology class $\Theta := q^*([\mathbb{R}^n]^*) \in \mathrm{H}^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_{\neq 0}; \mathbb{Z})$ is the *unique* cohomology class with the property from Claim 2.

Proof. First note that it follows easily from the Path-Component-H^*-Lemma 198.8 that it suffices to deal with the case that B is path-connected and non-empty. Now let $b \in B$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_{\neq 0}; \mathbb{Z}) & \xrightarrow{\mathrm{diag}(\iota,\mathrm{id})^*} & \mathrm{H}^n(\{b\} \times \mathbb{R}^n, \{b\} \times \mathbb{R}^n_{\neq 0}; \mathbb{Z}) \\ {\scriptstyle \varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} \uparrow & & \uparrow {\scriptstyle \varphi\otimes\psi \mapsto p^*(\varphi)\cup q^*(\psi)} \\ \mathrm{H}^0(B; \mathbb{Z}) \otimes \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0}; \mathbb{Z}) & \xrightarrow{\iota^*\otimes\mathrm{id}} & \mathrm{H}^0(\{b\}; \mathbb{Z}) \otimes \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0}; \mathbb{Z}) \end{array}$$

where the horizontal maps are induced by the inclusion $\iota \colon \{b\} \to B$. Clearly it suffices to show that the top horizontal map is an isomorphism. This in turn follows from the following three observations:

- It follows easily from the Cup Product-Naturality Lemma 204.8 that the diagram commutes.
- By Claim 1 we know that the vertical maps are isomorphisms.
- Finally note that it follows easily from the fact that X is path-connected and non-empty, together with the H^0-Proposition 198.9 that the bottom horizontal map is an isomorphism. ⊞

(b) It follows from Claim 1 that $\Theta := q^*([\mathbb{R}^n]^*) \in \mathrm{H}^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_{\neq 0}; \mathbb{Z})$ has the desired property (b). ■

We now turn to the proof of the Thom Isomorphism Theorem 277.9. We will first discuss the case that the bundle has a finite atlas. Afterwards we will discuss the general case.

Proof of the Thom Isomorphism Theorem 277.9 for a finite atlas. [2978] In the following we will prove Statement (1) of the theorem which deals with $\mathbb{F}_2$-coefficients. The proof of Statement (2), which deals with $\mathbb{Z}$-coefficients, is essentially the same. In fact, in the remainder of the proof we will, for space reasons, ignore coefficients, which makes the two statements almost indistinguishable.

For the purpose of this proof we introduce the following language: We say that an n-dimensional real vector bundle $p \colon E \to B$ satisfies $(*)$ if $p \colon E \to B$ satisfies the conclusion

[2978] The following proof is a copy-replace version of the proof of the Thom Isomorphism Theorem 277.9.

of the Thom–Disk Bundle Isomorphism Theorem 244.5, i.e. if the following two conditions hold:

(a) There exists a unique $\Theta(E) \in \mathrm{H}^n(E, E_{\neq 0})$ with the property that for each $b \in B$ the restriction to $\mathrm{H}^n(E_b, E_{b,\neq 0})$ is non-zero.
(b) For each $k \in \mathbb{Z}$ the map[2979]

$$\begin{array}{rcl} \mathrm{H}^k(B) & \to & \mathrm{H}^{n+k}(E, E_{\neq 0}) \\ \varphi & \mapsto & \Theta(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

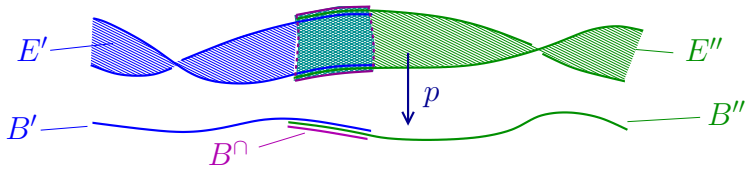
Claim 1. The statement (∗) holds if $p\colon E \to B$ is a trivializable n-dimensional real vector bundle.

Proof. This statement follows easily from the Trivializable–Thom Lemma 277.15 together with the Cup Product-Naturality Lemma 204.8. ⊞

Claim 2. Let $p\colon E \to B$ be an n-dimensional real vector bundle. We suppose that there exist two open subsets $B', B'' \subseteq B$ with $B' \cup B'' = B$ and such that the restrictions of p to B' and B'' satisfy (∗) and such that the restriction of p to $B' \cap B''$ also satisfies (∗). Then $p\colon E \to B$ also satisfies (∗).

Proof. We introduce the following notation:

- We write $B^\cap := B' \cap B''$.
- We write $E' := p^{-1}(B')$, $E'' := p^{-1}(B'')$ and $E^\cap := p^{-1}(B^\cap)$.
- We denote by $\iota'\colon B^\cap \to B'$ and $\iota''\colon B^\cap \to B''$ the inclusions. By a slight abuse of notation we denote by $\iota'\colon E^\cap \to E'$ and $\iota''\colon E^\cap \to E''$ also the inclusions.
- We denote by $\widetilde{\iota}'\colon B' \to B$ and $\widetilde{\iota}''\colon B'' \to B$ the inclusions. Similarly we denote by $\widetilde{\iota}'\colon E' \to E$ and $\widetilde{\iota}''\colon E'' \to E$ also the inclusions.



Next note that by a variation on the Mayer–Vietoris-H*-Theorem 200.11 we have the following Mayer–Vietoris sequence

$$\to \underbrace{\mathrm{H}^{n-1}(E^\cap, E^\cap_{\neq 0})}_{\cong \mathrm{H}^{-1}(B^\cap)=0} \xrightarrow{\delta} \mathrm{H}^n(E, E_{\neq 0}) \xrightarrow{(\widetilde{\iota}'^*, -\widetilde{\iota}''^*)} \mathrm{H}^n(E', E'_{\neq 0}) \oplus \mathrm{H}^n(E'', E''_{\neq 0}) \xrightarrow{\iota'^* \oplus \iota''^*} \mathrm{H}^n(E^\cap, E^\cap_{\neq 0}) \to$$

(Θ' under $\mathrm{H}^n(E', E'_{\neq 0})$, Θ'' under $\mathrm{H}^n(E'', E''_{\neq 0})$)

We make the following observations:

(i) By hypothesis we have unique elements $\Theta' \in \mathrm{H}^n(E', E'_{\neq 0})$ and $\Theta'' \in \mathrm{H}^n(E'', E''_{\neq 0})$ which satisfy (a) and (b) for the two vector bundles $E' \to X'$ and $E'' \to X''$. As in the Thom Class–Natural Proposition 277.11 we see that $\iota'^*(\Theta') \in \mathrm{H}^n(E^\cap, E^\cap_{\neq 0})$ and also $\iota''^*(\Theta') \in \mathrm{H}^n(E^\cap, E^\cap_{\neq 0})$ both satisfy (a) and (b) for $p\colon E^\cap \to B^\cap$. Since we assume the uniqueness statement for $B^\cap$ we obtain that $\iota'^*(\Theta') = \iota''^*(\Theta'') \in \mathrm{H}^n(E^\cap, E^\cap_{\neq 0})$. We denote this element by $\Theta^\cap \in \mathrm{H}^n(E^\cap, E^\cap_{\neq 0})$.
(ii) Since $p\colon E^\cap \to B^\cap$ satisfies (∗) we see that $\mathrm{H}^{n-1}(E^\cap, E^\cap_{\neq 0}) \cong \mathrm{H}^{-1}(B^\cap) = 0$.

(iii) It follows from (i) and (ii) that there exists a unique cohomology class $\Theta \in \mathrm{H}^n(E, E_{\neq 0})$ with $\widetilde{\iota}^*(\Theta) = \Theta' \in \mathrm{H}^n(E', E'_{\neq 0})$ and with $\widetilde{\iota}''^*(\Theta) = \Theta'' \in \mathrm{H}^n(E'', E''_{\neq 0})$.

It follows easily from the above discussion and the functoriality of cohomology that Θ is the unique class that satisfies (a) for the n-dimensional real vector bundle $p\colon E \to B$.

It remains to show that Θ satisfies (b). Thus let $k \in \mathbb{Z}$. We consider the following diagram:

$$\begin{array}{ccccccccc}
\longrightarrow \mathrm{H}^{k-n-1}(B^\cap) & \xrightarrow{\delta} & \mathrm{H}^{k-n}(B) & \xrightarrow{(\widetilde{\iota}'^*, -\widetilde{\iota}''^*)} & \mathrm{H}^{k-n}(B') \oplus \mathrm{H}^{k-n}(B'') & \xrightarrow{\iota'^* \oplus \iota''^*} & \mathrm{H}^{k-n}(B^\cap) & \longrightarrow \\
\downarrow{\scriptstyle \varphi \mapsto \Theta^\cap \cup p^*(\varphi)} & & \downarrow{\scriptstyle \varphi \mapsto \Theta \cup p^*(\varphi)} & & \downarrow{\scriptstyle \mathrm{diag}(\ldots,\ldots)} & & \downarrow{\scriptstyle \varphi \mapsto \Theta^\cap \cup p^*(\varphi)} & \\
\to \mathrm{H}^{k-1}(E^\cap, E^\cap_{\neq 0}) & \xrightarrow{\delta} & \mathrm{H}^k(E, E_{\neq 0}) & \xrightarrow{(\widetilde{\iota}'^*, -\widetilde{\iota}''^*)} & \mathrm{H}^k(E', E'_{\neq 0}) \oplus \mathrm{H}^k(E'', E''_{\neq 0}) & \xrightarrow{\iota'^* \oplus \iota''^*} & \mathrm{H}^k(E^\cap, E^\cap_{\neq 0}) & \to
\end{array}$$

We make the following observations:

(a) By hypothesis we know that the vertical maps corresponding to $B^\cap$ and B', B'' are isomorphisms.

(b) We claim that the diagram commutes up to a sign:

- It follows easily from the Cup Product-Naturality Lemma 204.8 that the middle square and the square to the right both commute.
- It follows fairly easily from the fact that the connecting homomorphism in the Mayer–Vietoris sequence is defined via the connecting homomorphism in a long exact sequence, the naturality of the connecting homomorphism in the LES-of-Triple Proposition 198.13, the Cup Product-Connecting Homomorphism Lemma 204.18 and the fact that the restriction of Θ to $E^\cap$ equals $\Theta^\cap$, that the left square commutes up to a sign. We leave it to the reader to fill in the details.

It now follows from the Five Lemma 145.11 that the vertical map corresponding to B is also an isomorphism. This concludes the proof that Θ satisfies (b). ⊞

Claim 3. Let $p\colon E \to B$ be an n-dimensional real vector bundle. If B is covered by open subsets $B_1, \ldots, B_k$ such that each the restriction to each X_i is trivializable, then (∗) holds for $p\colon E \to B$.

Proof. We prove the claim by induction on $k \in \mathbb{N}$. The case $k = 0$ is trivial and the case $k = 1$ is just the content of Claim 1. Now we assume that the statement holds for some $k \in \mathbb{N}$. Next let $p\colon E \to B$ be an n-dimensional real vector bundle such that B is covered by open subsets $B_1, \ldots, B_{k+1}$ such that the restriction to each B_i is trivializable. We need to show that (∗) holds for $p\colon E \to B$.

We set $B' := B_1 \cup \cdots \cup B_k$ and $B'' = B_{k+1}$. Note that $B^\cap := B' \cap B''$ can be written as $B^\cap = (B_1 \cap B_{k+1}) \cup \cdots \cup (B_k \cap B_{k+1})$. Thus $B^\cap$ is covered by the k open subsets $B_i \cap B_{k+1}$. Note that the restriction of $p\colon E \to B$ to each open subset $B_i \cap B_{k+1}$ is also trivializable. It thus follows from our induction hypothesis that (∗) holds for B', B'' and also for $B' \cap B''$. Thus it follows from Claim 2 that (∗) holds for B. ■

Proof of the Thom Isomorphism Theorem 277.9 for the general case. The proof of the general statement is worked out in great detail in [**MiS1974**, Chapter 10]. For brevity's sake we will only provide a sketch of an argument for the $\mathbb{F}_2$-case.

Thus let $p\colon E \to B$ be an n-dimensional real vector bundle. As in Definition 209.14 we denote by $\mathcal{K}(B)$ the set of all compact subsets of B. As we discussed on page 4011, it follows

easily from the Compact–Union-Intersection Lemma 3.41 that the set $\mathcal{K}(B)$, together with the relation "$\subseteq$" defined by inclusion, is a directed set in the sense of Definition 104.1 and Definition 209.8.

We continue with the following claim, which might be of independent interest:

Claim. For every $j \in \mathbb{N}_0$ and every field $\mathbb{F}$ the natural map

$$\mathrm{H}^j(B;\mathbb{F}) \;\to\; \varprojlim_{K\in\mathcal{K}(B)} \mathrm{H}^j(K;\mathbb{F})$$

is an isomorphism.[2980]

Proof. We start out with an observation: By the Compact Image Lemma 4.15 we know that the image of any singular simplex is compact. It thus follows easily from the argument of the Exhaustion-H_*-Proposition 154.2 that the natural map

$$\varinjlim_{K\in\mathcal{K}(B)} \mathrm{H}_j(K;\mathbb{F}) \;\to\; \mathrm{H}_j(B;\mathbb{F})$$

is an isomorphism. We now consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^j(B;\mathbb{F}) & & \longrightarrow & & \varprojlim \mathrm{H}^j(K;\mathbb{F}) \\
{\scriptstyle \mathrm{ev}}\downarrow{\scriptstyle \cong} & & & & {\scriptstyle \cong}\uparrow{\scriptstyle \mathrm{ev}} \\
\mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_j(B;\mathbb{F})) & \xleftarrow{\cong} & \mathrm{Hom}_{\mathbb{F}}\left(\varinjlim \mathrm{H}_j(K;\mathbb{F});\mathbb{F}\right) & \xleftarrow{\cong} & \varprojlim \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_j(K;\mathbb{F});\mathbb{F}).
\end{array}$$

The claim follows from the following observations:

- It follows from the Universal Coefficient-H^*–Field Theorem 202.2 that the evaluation maps are isomorphisms. In other words, the left vertical maps are both isomorphism.
- It follows from the observation at the beginning of the proof of the claim that the bottom left horizontal map is an isomorphism.
- It follows from the Hom-of-Direct Colimit Lemma 229.7 that the bottom right horizontal map is an isomorphism.
- It follows (hopefully) easily from the various definitions that the diagram commutes. ⊞

For each compact subspace $K \subseteq B$ we have the following diagram

$$\begin{array}{ccc}
\mathrm{H}^j(E;\mathbb{F}_2) & \xrightarrow{\cup\Theta_{\mathbb{F}_2}(E\to B)} & \mathrm{H}^{j+n}(E, E_{\neq 0};\mathbb{F}_2) \\
\downarrow & & \downarrow \\
\mathrm{H}^j(p^{-1}(K)) & \xrightarrow{\cup\Theta_{\mathbb{F}_2}(E\to B)} & \mathrm{H}^{j+n}(p^{-1}(K), p^{-1}(K)_{\neq 0};\mathbb{F}_2).
\end{array}$$

It follows from the Cup Product-Naturality Lemma 204.8 that the diagram commutes. ■

277.4. **The Thom class of Whitney sums.** In this section we will see that the Thom classes behave well with respect to Whitney sums:

[2979] This claim is essentially the content of Exercise 230.6.

[2980] The Exhaustions-H^*-Proposition 230.11 indicates that the analogous statement for $\mathbb{Z}$-coefficients does *not* hold.

Proposition 277.16. (Whitney Sum–Thom Class Proposition) Let B be a topological space. Furthermore let $p\colon E \to B$ be an m-dimensional real vector bundle and let $q\colon F \to B$ be an n-dimensional real vector bundle. We consider the Whitney sum $E \oplus F$ from Definition 268.3 and we consider the natural projections

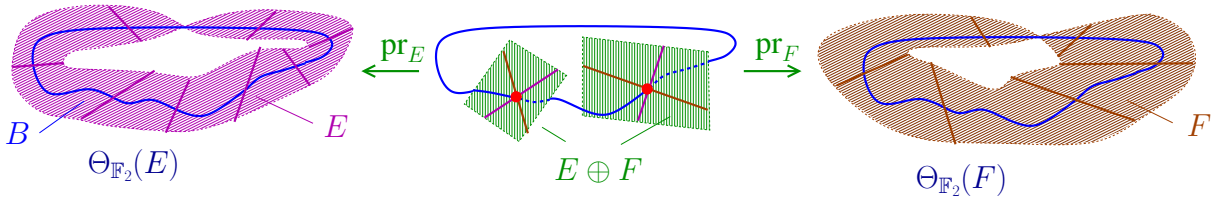
$$\begin{array}{rcl} \mathrm{pr}_E\colon E\oplus F & \to & E \\ (b,(e,f)) & \mapsto & e \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{pr}_F\colon E\oplus F & \to & F \\ (b,(e,f)) & \mapsto & f. \end{array}$$

(1) We have the equality

$$\Theta_{\mathbb{F}_2}(E\oplus F) \;=\; \mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E)) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F)) \;\in\; \mathrm{H}^{m+n}(E\oplus F, (E\oplus F)_{\neq 0}; \mathbb{F}_2).$$

(2) If E and F are oriented, then we equip $E \oplus F$ with the obvious orientation and we obtain the equality[2981]

$$\Theta_{\mathbb{Z}}(E\oplus F) \;=\; \mathrm{pr}_E^*(\Theta_{\mathbb{Z}}(E)) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{Z}}(F)) \;\in\; \mathrm{H}^{m+n}(E\oplus F, (E\oplus F)_{\neq 0}; \mathbb{Z}).$$



Proof. [2982] In the following we provide the proof of Statement (2). The proof of Statement (1) is obviously almost the same. In fact, to simplify the notation we will just write $\Theta = \Theta_{\mathbb{Z}}$, so the two statements are almost indistinguishable anyway.

So let $p\colon E \to B$ be an oriented m-dimensional real vector bundle and let $q\colon F \to B$ be an oriented n-dimensional real vector bundle.

By the Thom Isomorphism Theorem 277.9 (2) it remains to prove the following claim:

Claim. For each $b \in B$ we have

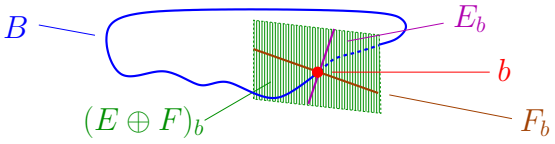
$$\langle \iota_b^*(\mathrm{pr}_E^*(\Theta(E)) \cup \mathrm{pr}_F^*(\Theta(F))), [(E\oplus F)_b]\rangle \;=\; 1.$$

(Here ι_b is of course the restriction to the fiber over b.)

Proof. Let $b \in B$. We start out with the following preparations.

(a) By Definition 277.10 we have $\langle \iota_b^*(\Theta(E)), [E_b]\rangle = 1$ and $\langle \iota_b^*(\Theta(F)), [F_b]\rangle = 1$.

(b)
- We denote by $\mathrm{pr}_{E_b}\colon (E\oplus F)_b \to E_b$ the projection.
- We denote by $\mathrm{pr}_{F_b}\colon (E\oplus F)_b \to F_b$ the projection.



We perform the following calculation:

[2981] We set $U := \{(b,(e,f)) \in E\oplus F \,|\, e \neq 0\}$ and $V := \{(b,(e,f)) \in E\oplus F \,|\, f \neq 0\}$. These are open subsets of $E\oplus F$. Note that $\mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E)) \in \mathrm{H}^m(E\oplus F, U; \mathbb{F}_2)$ and $\mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F)) \in \mathrm{H}^m(E\oplus F, V; \mathbb{F}_2)$. Since $U \cup V = (E\oplus F)_{\neq 0}$ we see that the cup product does take values in $\mathrm{H}^{m+n}(E\oplus F, (E\oplus F)_{\neq 0}; \mathbb{F}_2)$.

[2982] The proof shares some features with the proof of the Cup Product–Intersection Manifold Theorem 244.16. But fortunately the current proof is much easier.

$$\begin{aligned}
\langle \iota_b^*(\mathrm{pr}_E^*(\Theta(E)) \cup \mathrm{pr}_F^*(\Theta(F))), [(E \oplus F)_b] \rangle &\overset{\substack{\text{by the Cup Product}\\ \text{Naturality Lemma 204.8}}}{\overset{\downarrow}{=}} \langle \iota_b^*(\mathrm{pr}_E^*(\Theta(E))) \cup \iota_b^*(\mathrm{pr}_F^*(\Theta(F))), [(E \oplus F)_b] \rangle \\
&\underset{\substack{\text{since } \mathrm{pr}_E \circ \iota_b = \iota_b \circ \mathrm{pr}_{E_b} \colon (E \oplus F)_b \to E \text{ and}\\ \text{since } \mathrm{pr}_F \circ \iota_b = \iota_b \circ \mathrm{pr}_{F_b} \colon (E \oplus F)_b \to F}}{=} \langle \mathrm{pr}_{E_b}^*(\iota_b^*(\Theta(E))) \cup \mathrm{pr}_{F_b}^*(\iota_b^*(\Theta(F))), [(E \oplus F)_b] \rangle \\
&\underset{\text{follows from (a)}}{=} \langle \mathrm{pr}_{E_b}^*([E_b]^*) \cup \mathrm{pr}_{F_b}^*([F_b]^*), [(E \oplus F)_b] \rangle \underset{\substack{\text{follows easily from the Fundamental}\\ \text{Class Product Proposition 207.2 (1)}}}{=} 1.
\end{aligned}$$

■

277.5. Exercises for Chapter 277.

Exercise 277.1. Let

$$\begin{aligned} p\colon M := ([0,1] \times \mathbb{R})/(0,v) \sim (1,-v) &\to S^1 \\ [(t,v)] &\mapsto \exp(2\pi \mathrm{i} t) \end{aligned}$$

be the Möbius vector bundle which we first introduced on page 5119. Is the corresponding Thom class $\Theta_{\mathbb{F}_2}(p\colon M \to S^1) \in \mathrm{H}^1(M, M_0; \mathbb{F}_0)$ zero?

Exercise 277.2. In this exercise we will prove a homological analog of the Thom isomorphism. More precisely, let $p\colon E \to B$ be an n-dimensional real vector bundle. Use the Thom Isomorphism Theorem 277.9 (1) and the Universal Coefficient-H*–Field Theorem 202.2 to show that for each $k \in \mathbb{Z}$ the map

$$\mathrm{H}_{n+k}(E, E_{\neq 0}; \mathbb{F}_2) \xrightarrow{\sigma \mapsto \sigma \cap \Theta_{\mathbb{F}_2}(E)} \mathrm{H}_k(E; \mathbb{F}_2)$$

is an isomorphism.

Remark. If the vector bundle is oriented, then [**MiS1974**, Corollary 10.7] says the map

$$\mathrm{H}_{n+k}(E, E_{\neq 0}; \mathbb{Z}) \xrightarrow{\sigma \mapsto \sigma \cap \Theta_{\mathbb{Z}}(E)} \mathrm{H}_k(E; \mathbb{Z})$$

is an isomorphism. A proof of this statement is also given in [**Shas2014**, Chapter 11.2].

Exercise 277.3. Let $n \in \mathbb{N}$. We consider the following subspaces of $S^n \times S^n$:

$$\begin{aligned} \text{“diagonal”} \quad D &:= \{(x,x) \in S^n \times S^n \mid x \in S^n\}, \\ \text{“anti-diagonal”} \quad A &:= \{(x,-x) \in S^n \times S^n \mid x \in S^n\}. \end{aligned}$$

Furthermore we consider the map

$$\begin{aligned} f\colon (S^n \times S^n) \setminus A &\to \mathbb{R}^{n+1} \\ (x,y) &\mapsto \begin{array}{l}\text{the unique intersection point of } \mathrm{T}_x S^n \text{ with}\\ \text{the line through the two points } -x \text{ and } y\end{array} \end{aligned}$$

and we consider the following map:

$$\begin{aligned} \varphi\colon (S^n \times S^n) \setminus A &\to \mathrm{T}S^n \\ (x,y) &\mapsto (x, f(x,y)). \end{aligned}$$

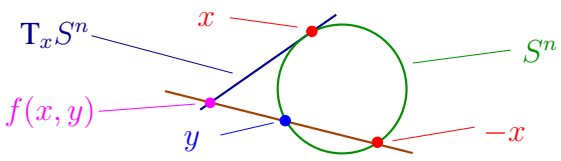
(a) Show that the map $\varphi\colon (S^n \times S^n) \setminus A \to \mathrm{T}S^n$ is a homeomorphism that restricts to a homeomorphism $(S^n \times S^n) \setminus (A \cup D) \to (\mathrm{T}S^n)_{\neq 0}$.

(b) Let $\rho\colon S^n \times S^n \to S^n \times S^n$ be the map that is given by $\rho(x,y) = \rho(x,-y)$. Note that $\rho(D) = A$. Furthermore for $i = 1,2$ let $p_i\colon S^n \times S^n \to S^n$ be the projection onto the

i-th factor. Show that we have the following equality in $\mathrm{H}^n(\mathrm{T}S^n)$:

$$((f \circ \rho)^{-1})^*(p_1^*([S^n]^*) - p_2^*([S^n]^*)) \;=\; я(\Theta_{\mathbb{Z}}(\mathrm{T}S^n)).$$

Remark. A few more details are provided in [**Hat2**, Proposition 4.2].



CHAPTER 278

The Euler class of vector bundles

Let $p\colon W \to B$ be an n-dimensional real vector bundle. In the previous Chapter 277 we proved the Thom Isomorphism Theorem 277.9. In this chapter we will see that this theorem allows us to define the $\mathbb{F}_2$-Euler class $e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(B;\mathbb{F}_2)$. Furthermore, if $p\colon W \to B$ is in fact oriented, then we can define the integral Euler class $e_{\mathbb{Z}}(W) \in \mathrm{H}^n(B;\mathbb{Z})$. The name *Euler* class stems from the fact that for the tangent bundles of closed oriented smooth manifolds these Euler classes are essentially the same as the Euler characteristic.

This chapter is organized as follows:

- In Section 278.1 we will introduce the Euler class of a real vector bundle.
- In Section 278.2 we will see that the Euler classes for vector bundles are essentially the same concept as the Euler classes of disk bundles which we introduced in Definition 245.1.
- In Section 278.3 we will discuss basic properties of the Euler class and we will see that for the most part these match similar properties of Chern classes and Stiefel-Whitney classes.
- In Section 278.4 we will relate the Euler class to the Euler number and to Euler characteristics.
- In Section 278.5 we will relate the Euler classes to Stiefel-Whitney classes and Chern classes.
- In Section 278.6 we will relate Euler classes to Pontryagin classes.
- In Section 278.7 we will provide exercises for Chapter 278.

In the next Chapter 279 we will use the $\mathbb{F}_2$-Euler class $e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(B;\mathbb{F}_2)$ and Steenrod operations to (re-) introduce the Stiefel-Whitney classes of a real vector bundle.

278.1. **Definition of the Euler class.** Similar to Definition 245.1 we will now use the Thom classes from Theorem 277.9 to introduce the Euler classes of real vector bundles:

Definition 278.1.

(1) Let $p\colon W \to B$ be an n-dimensional real vector bundle. We start out with some observations and some notation:
   - It follows from the argument of the Radial Shrinking Lemma 57.8 (2a) that the projection $p\colon W \to B$ is a homotopy equivalence. Thus p induces isomorphisms on cohomology groups.
   - We denote by $\text{я}\colon \mathrm{H}^n(W, W_{\neq 0};\mathbb{F}_2) \to \mathrm{H}^n(W;\mathbb{F}_2)$ the natural map.
   - We denote by $\Theta_{\mathbb{F}_2}(p\colon W \to B) \in \mathrm{H}^n(W, W_{\neq 0};\mathbb{F}_2)$ the Thom class from the Thom Isomorphism Theorem 277.9.

   We refer to

$$e_{\mathbb{F}_2}(W) \;:=\; ((p^*)^{-1} \circ \text{я})(\underbrace{\Theta_{\mathbb{F}_2}(W)}_{\in \mathrm{H}^n(W, W_{\neq 0}; \mathbb{F}_2)}) \;\in\; \mathrm{H}^n(B; \mathbb{F}_2)$$

as the $\mathbb{F}_2$-Euler class of $p\colon W \to B$.

(2) Let $p\colon W \to B$ be an oriented n-dimensional real vector bundle. We refer to

$$e_{\mathbb{Z}}(W) \;:=\; ((p^*)^{-1} \circ \text{я})(\underbrace{\Theta_{\mathbb{Z}}(W)}_{\in \mathrm{H}^n(W, W_{\neq 0}; \mathbb{Z})}) \;\in\; \mathrm{H}^n(B; \mathbb{Z})$$

as the $\mathbb{Z}$-Euler class of $p\colon W \to B$.

Note that for an oriented real vector bundle $p\colon W \to B$ we now have two Euler classes, namely $e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(B; \mathbb{F}_2)$ and $e_{\mathbb{Z}}(W) \in \mathrm{H}^n(B; \mathbb{Z})$. The following lemma, which is the analog of the Euler Class Relationship Lemma 245.2, gives us the non-surprising connection between these two cohomology classes:

Lemma 278.2. (Euler Class Relationship Lemma) Let $p\colon W \to B$ be an oriented n-dimensional real vector bundle. We denote by $\gamma_*\colon \mathbb{Z} \to \mathbb{F}_2$ the obvious ring homomorphism. From the Cohomology-Coefficients-Functor Lemma 197.21 we obtain an induced homomorphism $\gamma_*\colon \mathrm{H}^n(B; \mathbb{Z}) \to \mathrm{H}^n(B; \mathbb{F}_2)$. With this notation we have the equality

$$e_{\mathbb{F}_2}(W) \;=\; \gamma_*(e_{\mathbb{Z}}(W)) \;\in\; \mathrm{H}^n(B; \mathbb{F}_2).$$

Proof. Let $p\colon W \to B$ be an oriented n-dimensional real vector bundle. We consider the following diagram:

$$\begin{array}{ccccc}
 & \overset{\Theta_{\mathbb{Z}}(p\colon W \to B)}{} & & & \overset{e_{\mathbb{Z}}(p\colon W \to B)}{} \\
\mathrm{H}^n(W, W_{\neq 0}; \mathbb{Z}) & \xrightarrow{\text{я}} & \mathrm{H}^n(W; \mathbb{Z}) & \xleftarrow{p^*} & \mathrm{H}^n(B; \mathbb{Z}) \\
\downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\
\mathrm{H}^n(W, W_{\neq 0}; \mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^n(W; \mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}^n(B; \mathbb{F}_2)\,. \\
\underset{\Theta_{\mathbb{F}_2}(p\colon W \to B)}{} & & & & \underset{e_{\mathbb{F}_2}(p\colon W \to B)}{}
\end{array}$$

The lemma follows from Definition 278.1 and the following two observations:

(1) It follows from the Cohomology-Coefficients Functor Lemma 197.21 (2) that the diagram commutes.

(2) It follows from the Thom Class Relationship Proposition 277.12 that we have the equality $\gamma_*(\Theta_{\mathbb{Z}}(p\colon W \to B)) = \Theta_{\mathbb{F}_2}(p\colon W \to B)$. ∎

We continue with a simple criterion for the Euler class to vanish:

Proposition 278.3. (Non-Vanishing Section–Euler Proposition) Let $p\colon W \to B$ be a real vector bundle such that there exists a nowhere-vanishing section $\sigma\colon B \to W$. The following statements hold:

(1) $e_{\mathbb{F}_2}(W) = 0$.

(2) If the real vector bundle is oriented, then $e_{\mathbb{Z}}(W) = 0$.

Example.

(1) It follows from the Non-Vanishing Section–Euler Proposition 278.3 that for any real vector bundle E we have $w(E \oplus \mathrm{Tr}(E)) = 0$. This is in contract to the setting of Stiefel-Whitney which we discussed on page 5289.

(2) In Exercise 278.6 we will see that the converse to the Non-Vanishing Section–Euler Proposition 278.3 does not hold. □

Proof. We denote by $\sigma_0 \colon B \to W$ the 0-section and we denote by $\iota \colon W_{\neq 0} \to W$ the inclusion. Note that the nowhere vanishing section $\sigma \colon B \to W$ really defines a continuous map $\sigma \colon B \to W_{\neq 0}$. Next we consider the following diagram:

$$\begin{array}{ccccc} \overset{\Theta_{\mathbb{F}_2}(W)}{\mathrm{H}^n(W, W_{\neq 0}; \mathbb{F}_2)} & \xrightarrow{\text{я}} & \mathrm{H}^n(W; \mathbb{F}_2) & \xrightarrow{\iota^*} & \overset{0}{\mathrm{H}^n(W_{\neq 0}; \mathbb{F}_2)} \\ & & \uparrow{\scriptstyle p^*} \cong \downarrow{\scriptstyle \sigma_0^*} & \swarrow{\scriptstyle \sigma^*} & \\ & & \mathrm{H}^n(B; \mathbb{F}_2). \; \underset{e_{\mathbb{F}_2}(W)}{} & & \end{array}$$

We make the following clarifications and observations:

(a) The top sequence is just the long exact sequence in cohomology of the pair $(W, W_{\neq 0})$. In particular the composition of the two top horizontal maps is the zero map.

(b) Note that $\sigma_0 \colon B \to W$ and $\iota \circ \sigma \colon B \to W$ are clearly homotopic. It thus follows from the Homotopy-H*-Proposition 198.11 that $\sigma_0^* = \sigma^* \circ \iota^*$. This implies that the triangle commutes.

(c) Note that $p \circ \sigma_0 = \mathrm{id}$ and that p is a homotopy equivalence. Thus we have $p^* = (\sigma_0^*)^{-1}$. In particular the vertical map is an isomorphism.

We now see that

$$e_{\mathbb{F}_2}(W) \underset{\text{by Definition 278.1}}{=} ((p^*)^{-1} \circ \text{я})(\Theta_{\mathbb{F}_2}(W)) \underset{\text{by (c)}}{=} (\sigma_0^* \circ \text{я})(\Theta_{\mathbb{F}_2}(W)) \underset{\text{by (b)}}{=} (\sigma^* \circ \iota^* \circ \text{я})(\Theta_{\mathbb{F}_2}(W)) \underset{\text{by (a)}}{=} 0. \quad \blacksquare$$

We conclude this initial discussion of the Euler class with a simple-minded lemma:

Lemma 278.4. (Orientation–Euler Lemma Let $p \colon W \to B$ be an oriented n-dimensional real vector bundle. For the purpose of this lemma we denote by $-W$ the same real vector bundle but with the opposite orientation on each fiber. Then

$$e_{\mathbb{Z}}(-W) \;=\; -e_{\mathbb{Z}}(W) \;\in\; \mathrm{H}^n(B; \mathbb{Z}).$$

Proof. We have the following equalities in $\mathrm{H}^n(B; \mathbb{Z})$:

$$e_{\mathbb{Z}}(-W) \underset{\text{see Definition 278.1}}{=} (\text{я} \circ (p^*)^{-1})(\Theta_{\mathbb{Z}}(-W)) \underset{\substack{\text{by the Orientation–Thom} \\ \text{Lemma 277.13}}}{=} -(\text{я} \circ (p^*)^{-1})(\Theta_{\mathbb{Z}}(W)) \underset{\text{see Definition 278.1}}{=} -e_{\mathbb{Z}}(W). \quad \blacksquare$$

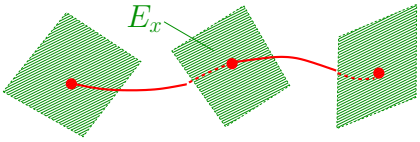
278.2. **The Euler classes for vector bundles and disk bundles.** As the reader will surely have noticed, by now we have defined Euler classes for (oriented) $\overline{B}^n$-bundles and for (oriented) real vector bundles. The following proposition tells us that these two concepts are essentially the same:

Proposition 278.5. (Euler-equals-Euler Proposition) Let $p \colon E \to B$ be an n-dimensional real vector bundle. We assume that the bundle has a finite atlas.[2983] Let φ be an

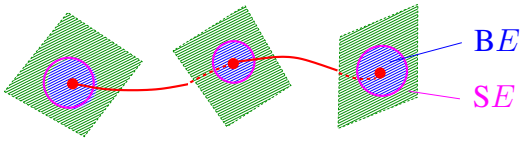
inner product on E.[2984] Next let $p\colon \mathrm{B}E \to B$ be the corresponding $\overline{B}^n$-bundle from the Vector Bundle-to-Disk Bundle Proposition 277.4.

(1) We have
$$\underbrace{e_{\mathbb{F}_2}(E \to B)}_{\text{see Definition 278.1}} = \underbrace{e_{\mathbb{F}_2}(\mathrm{B}E \to B)}_{\text{see Definition 245.1}}.$$

(2) If we start out with an oriented real vector bundle, then
$$\underbrace{e_{\mathbb{Z}}(E \to B)}_{\text{see Definition 278.1}} = \underbrace{e_{\mathbb{Z}}(\mathrm{B}E \to B)}_{\text{see Definition 245.1}}.$$



vector bundle $p\colon E \to B$



linear $\overline{B}^2$-bundle $p\colon \mathrm{B}E \to B$

Proof. We provide the proof for Statement (2). The proof for Statement (1) is essentially a slight simplification of the argument. We consider the following diagram:

$$\begin{array}{ccccc}
 & {\scriptstyle \Theta_{\mathbb{Z}}(E\to B)} & & & \\
\mathrm{H}^n(E, E_{\neq 0}; \mathbb{Z}) & \xrightarrow{\text{я}} & \mathrm{H}^n(E;\mathbb{Z}) & \xleftarrow{p_*} & \mathrm{H}^n(B;\mathbb{Z}) \\
\downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^n(\mathrm{B}E, \mathrm{S}E; \mathbb{Z}) & \xrightarrow{\text{я}} & \mathrm{H}^n(\mathrm{B}E;\mathbb{Z}) & \xleftarrow{p_*} & \mathrm{H}^n(B;\mathbb{Z}). \\
{\scriptstyle \Theta_{\mathbb{Z}}(\mathrm{B}E\to B)} & & & & {\scriptstyle e_{\mathbb{Z}}(\mathrm{B}E\to B)}
\end{array}$$

We make the following clarifications and observations:

- We denote by $\iota\colon \mathrm{B}E \to E$ the inclusion.
- By the Thom-equals-Thom Proposition 277.14 from the previous chapter we know that we have the equality $\iota^*(\Theta_{\mathbb{Z}}(E \to B)) = \Theta_{\mathbb{Z}}(\mathrm{B}E \to B)$.
- It follows from the functoriality of cohomology that the diagram commutes.

It now follows easily from Definition 245.1 and Definition 278.1 that the two Euler classes agree. ∎

278.3. **Properties of the Euler class.** In Chapter 274 we introduced the Chern classes of complex vector bundles c_i and we introduced the Stiefel-Whitney classes w_i of real vector bundles. We showed that they are natural, that they vanish for trivial vector bundles and that they satisfy a multiplicativity rule. We will now see that the Euler classes have analogous properties.[2985]

We start out with the following proposition, which mirrors the Naturality-of-Euler-Proposition 245.3 and which is an analog of the Naturality-of-Chern-Proposition 274.2, the Naturality-of-SW-Proposition 274.15 and the Naturality-of-Pontryagin-Proposition 276.5:

[2983]This condition is only needed since in Definition 245.1 we restricted ourselves, out of laziness, to this setup.

[2984]By the Inner Product-on-Vector Bundle Proposition 269.10 we know that such an inner product exists if the base space is Hausdorff and paracompact.

[2985]In fact in the Chern Class Characterization Proposition 274.7 and the Stiefel-Whitney Class Characterization Proposition 274.22 we showed that these three properties, together with the calculation for tautological vector bundles, completely characterize the Chern classes and the Stiefel-Whitney classes.

Proposition 278.6. (Naturality-of-Euler-Proposition)

(1) For any morphism

$$\begin{array}{ccc} W & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of n-dimensional real vector bundles we have $h^*(e_{\mathbb{F}_2}(V)) = e_{\mathbb{F}_2}(W) \in \mathrm{H}^n(C;\mathbb{F}_2)$.

(2) For any morphism

$$\begin{array}{ccc} W & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of oriented n-dimensional real vector bundles we have $h^*(e_{\mathbb{Z}}(V)) = e_{\mathbb{Z}}(W) \in \mathrm{H}^n(C;\mathbb{Z})$.

Proof. In the following we prove Statement (1). The proof of Statement (2) is essentially the same. Thus let

$$\begin{array}{ccc} W & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

be a morphism of n-dimensional real vector bundles. We consider the following diagram:

$$\begin{array}{ccccc} & \overset{\Theta_{\mathbb{F}_2}(p\colon W\to C)}{} & & & \overset{e_{\mathbb{F}_2}(p\colon W\to C)}{} \\ \mathrm{H}^n(W, W_{\neq 0};\mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^n(W;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}^n(C;\mathbb{F}_2) \\ \downarrow{\scriptstyle H^*} & & \downarrow{\scriptstyle H^*} & & \downarrow{\scriptstyle h^*} \\ \mathrm{H}^n(V, V_{\neq 0};\mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^n(V;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}^n(B;\mathbb{F}_2)\,. \\ & \underset{\Theta_{\mathbb{F}_2}(p\colon V\to B)}{} & & & \underset{\Theta_{\mathbb{F}_2}(p\colon V\to B)}{} \end{array}$$

Note that the diagram commutes by the functoriality of cohomology. This proposition now easily from the Thom Class–Natural Proposition 277.11. ∎

Using the Naturality-of-Euler-Proposition 278.6 we can easily prove the following, perhaps slightly disappointing, proposition:[2986]

Proposition 278.7. (Odd Dimension–Euler Class Proposition) Let $p\colon W \to B$ be an oriented n-dimensional real vector bundle. If n is odd, then $2 \cdot e_{\mathbb{Z}}(W) = 0 \in \mathrm{H}^n(B;\mathbb{Z})$.

Proof. Let $p\colon W \to B$ be an oriented n-dimensional real vector bundle. We consider the map

$$\begin{array}{ccc} W & \to & -W \\ w & \mapsto & -w. \\ & & \uparrow \\ & & \text{minus in the fiber} \end{array}$$

It follows from $\det(-\mathrm{id}_n) = (-1)^n$ and our hypothesis that n is odd that the above map is an morphism of oriented real vector bundles over B. Thus we see that

$$\begin{array}{ccccccccc} e_{\mathbb{Z}}(W) + e_{\mathbb{Z}}(W) & \underset{\uparrow}{=} & e_{\mathbb{Z}}(W) + e_{\mathbb{Z}}(-W) & \underset{\uparrow}{=} & e_{\mathbb{Z}}(W) - e_{\mathbb{Z}}(W) & = & 0 & \in & \mathrm{H}^n(B;\mathbb{Z}). \\ \text{follows from the above isomorphism together with the Naturality-of-Euler–Proposition 278.6} & & & & \text{by the Orientation–Euler Lemma 278.4} & & & & \end{array}$$

∎

We continue with the following lemma which mirrors the Trivializable-Euler Lemma 245.4 and which is the obvious analog of the Trivializable-Chern-Lemma 274.3, the Trivializable-SW-Lemma 274.16 and the Trivializable-Pontryagin-Lemma 276.6:

[2986] Note that the proposition is the analog of Exercise 245.2.

Lemma 278.8. (Trivializable-Euler Lemma) Let $p\colon W \to B$ be a real vector bundle.
(1) If p is trivializable, then $e_{\mathbb{F}_2}(W) = 0$.
(2) If W is trivializable, then for any orientation we have $e_{\mathbb{Z}}(W) = 0$.

Proof. In the following we provide a proof of Statement (1). The proof of Statement (2) is evidently almost the same. Thus let $p\colon W \to B$ be a trivializable n-dimensional real vector bundle. For fun we provide two slightly different arguments:

(a) We consider the trivial vector bundle $\{x\} \times \mathbb{R}^n \to \{x\}$ over the singleton $\{x\}$. Let $h\colon B \to \{x\}$ be the unique map. We now see that we have the following equalities in $\mathrm{H}^n(B;\mathbb{F}_2)$:

$$e_{\mathbb{F}_2}(p\colon W \to B) \underset{\substack{\uparrow\\ \text{since } W \text{ is trivializable we obtain from the discussion on page 5145 that the two vector bundles are isomorphic}}}{=} e_{\mathbb{F}_2}(h^*(\{x\} \times \mathbb{R}^n)) \underset{\substack{\uparrow\\ \text{by the Naturality-of-Euler Proposition 278.6}}}{=} h^*(e_{\mathbb{F}_2}(\{x\} \times \mathbb{R}^n)) \underset{\substack{\uparrow\\ \text{since } \mathrm{H}^n(\{x\} \times \mathbb{R}^n;\mathbb{F}_2) = 0}}{=} h^*(0) = 0.$$

(b) It follows easily from the Naturality-of-Euler-Proposition 245.3 that we only have to consider the case that we are dealing with the trivial n-dimensional real vector bundle $p\colon B \times \mathbb{R}^n \to B$. We denote by $q\colon B \times \mathbb{R}^n \to \mathbb{R}^n$ the projection. Now we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_{\neq 0};\mathbb{F}_2) & \xrightarrow{\text{я}} & \mathrm{H}^n(B \times \mathbb{R}^n;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}^n(B;\mathbb{F}_2) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \\ \mathrm{H}^n(\mathbb{R}^n, \mathbb{R}^n_{\neq 0};\mathbb{F}_2) & \xrightarrow{\text{я}} & \underbrace{\mathrm{H}^n(\mathbb{R}^n;\mathbb{F}_2)}_{=0}. & & \end{array}$$

$\Theta_{\mathbb{F}_2}(p\colon B\times\mathbb{R}^n\to B)$ over the top-left; $e_{\mathbb{F}_2}(p\colon B\times\mathbb{R}^n\to B)$ over the top-right; $[\mathbb{R}^n]^*$ under the bottom-left.

The desired conclusion follows from the following observations:
- The diagram commutes by functoriality of cohomology.
- By the Trivializable–Thom Lemma 277.15 we have $\Theta_{\mathbb{F}_2}(p\colon B\times\mathbb{R}^n \to B) = q^*([\mathbb{R}^n]^*)$.
- Since $n \in \mathbb{N}$ we have $\mathrm{H}^n(\mathbb{R}^n;\mathbb{F}_2) = 0$. ∎

Next we have the following analog of the Tautological Bundle-Chern-Lemma 274.4 and of the Tautological Bundle-SW-Lemma 274.17:

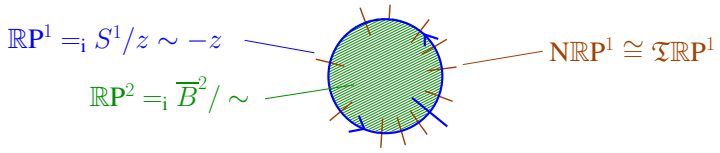
Lemma 278.9. (Tautological Bundle-Euler Class-Lemma) Let $n \in \mathbb{N} \cup \{\infty\}$.
(1) Let $p\colon \mathfrak{T}\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ be the tautological real line bundle that we introduced in the Tautological Line Bundle Lemma 265.6. In this case we have
$$e_{\mathbb{F}_2}(\mathfrak{T}\mathbb{R}\mathrm{P}^n) = [\mathbb{R}\mathrm{P}^1]^* \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2).$$
(2) Let $p\colon \mathfrak{T}\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be the tautological complex line bundle. In this case we have
$$e_{\mathbb{Z}}(\mathfrak{T}\mathbb{C}\mathrm{P}^n) = [\mathbb{C}\mathrm{P}^1]^* \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Proof. Let $n \in \mathbb{N} \cup \{\infty\}$. In the following we will prove Statement (1), since this is the statement we will need later on. The proof of Statement (2) is very similar, for the most

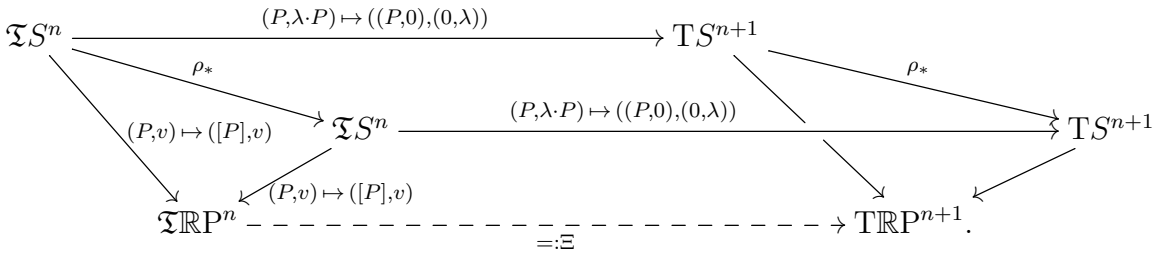
part we just need to replace the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19 by the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 (if we ignore signs) or the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20 (if we care about signs.)

In the following we first deal with the case $n \in \mathbb{N}$. We consider $\mathbb{R}\mathrm{P}^n$ as a submanifold of $\mathbb{R}\mathrm{P}^{n+1}$ via the smooth embedding $[v_0 : v_1 : \cdots : v_n] \mapsto [v_0 : v_1 : v_2 : \cdots : v_n : 0]$. We start out with the following claim which is perhaps of independent interest:

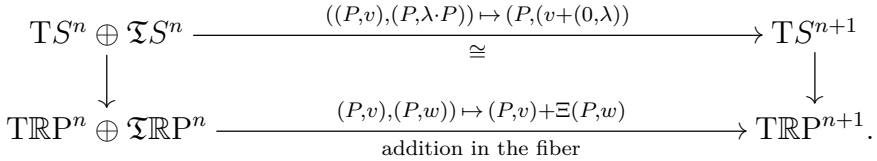
Claim 1. The tautological bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^n$ is isomorphic to the normal vector bundle $\mathrm{N}\mathbb{R}\mathrm{P}^n$ of the submanifold $\mathbb{R}\mathrm{P}^n \subseteq \mathbb{R}\mathrm{P}^{n+1}$.



Proof. We consider the tautological bundle $\mathfrak{T}S^n := \{(P, v) \,|\, P \in S^n, v \in \mathbb{R} \cdot P\}$. Furthermore we consider the map $\rho\colon S^n \to S^n$ that is given by $\rho(P) = -P$. We consider the following diagram:

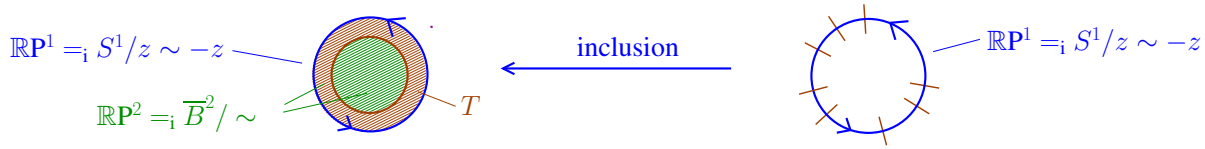


One can easily verify that the diagram commutes. It follows that the top horizontal maps induce a well-defined morphism $\Xi\colon \mathfrak{T}\mathbb{R}\mathrm{P}^n \to \mathrm{T}\mathbb{R}\mathrm{P}^{n+1}$ of vector bundles. It follows easily from the discussion that the following diagram commutes:



One can easily verify that the top horizontal and the vertical maps are fiberwise isomorphisms. Thus we see that the bottom horizontal map is also fiberwise an isomorphism. It now follows from the Normal Vector Bundle Proposition 269.13 that the tautological bundle $\mathfrak{T}\mathbb{R}\mathrm{P}^n$ is isomorphic to the normal vector bundle $\mathrm{N}\mathbb{R}\mathrm{P}^n$ of $\mathbb{R}\mathrm{P}^n \subseteq \mathbb{R}\mathrm{P}^{n+1}$. ⊞

Next we apply the Tubular Neighborhood–Existence Theorem 57.11 to obtain a tubular neighborhood $T \to \mathbb{R}\mathrm{P}^n$ of $\mathbb{R}\mathrm{P}^n \subseteq \mathbb{R}\mathrm{P}^{n+1}$.



We now see that

$$e_{\mathbb{F}_2}(\mathfrak{T}\mathbb{R}\mathrm{P}^n) \underset{\text{by Claim 1}}{=} e_{\mathbb{F}_2}(\mathrm{N}\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n) \underset{\text{by the Euler-equals-Euler Proposition 278.5}}{=} e_{\mathbb{F}_2}(T \to \mathbb{R}\mathrm{P}^n) \underset{\text{by the Projective Tubular Neighborhood–Euler Proposition 245.5 alternatively one can use Exercise 245.6}}{=} [\mathbb{R}\mathrm{P}^1]^* \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2).$$

Finally it remains to deal with the case $n = \infty$. Let $\iota\colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^\infty$ be the usual embedding. We see that

$$[\mathbb{R}\mathrm{P}^1]^* \underset{\text{by the above}}{=} e_{\mathbb{F}_2}(\mathfrak{T}\mathbb{R}\mathrm{P}^1) \underset{\text{by the Tautological Bundle Lemma 270.8}}{=} e_{\mathbb{F}_2}(\iota^*(\mathfrak{T}\mathbb{R}\mathrm{P}^\infty)) \underset{\text{by the Naturality-of-Euler Proposition 245.3}}{=} \iota^*(e_{\mathbb{F}_2}(\mathfrak{T}\mathbb{R}\mathrm{P}^\infty)).$$

By the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 213.17 we know that $\iota^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^1;\mathbb{F}_2)$ is an isomorphism. Thus we see that $e_{\mathbb{F}_2}(\mathfrak{T}\mathbb{R}\mathrm{P}^\infty) = [\mathbb{R}\mathrm{P}^1]^* \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$. ■

Finally we have the following analog of the Chern-Multiplicativity Theorem 274.6 and the SW-Multiplicativity Theorem 274.19:

Theorem 278.10. (Euler–Multiplicativity Theorem) Let B be a topological space. Furthermore let $p_V\colon V \to B$ be an m-dimensional real vector bundle and let $p_W\colon W \to B$ be an n-dimensional real vector bundle. We consider the Whitney sum $V \oplus W$ from Definition 268.3.

(1) We have the equality

$$e_{\mathbb{F}_2}(V \oplus W) = e_{\mathbb{F}_2}(V) \cup e_{\mathbb{F}_2}(W) \in \mathrm{H}^{m+n}(B;\mathbb{F}_2).$$

(2) If V and W are actually oriented, then we equip $V \oplus W$ with the obvious direct sum orientation and we obtain the equality

$$e_{\mathbb{Z}}(V \oplus W) = e_{\mathbb{Z}}(V) \cup e_{\mathbb{Z}}(W) \in \mathrm{H}^{m+n}(B;\mathbb{Z}).$$

Proof. In the following we provide the proof of Statement (2). The proof of Statement (1) is exactly the same, except that we need to replace "$\mathbb{Z}$" by "$\mathbb{F}_2$".

Let B be a topological space. Furthermore let $p_V\colon V \to B$ be an m-dimensional real vector bundle and let $p_W\colon W \to B$ be an n-dimensional real vector bundle. We consider the Whitney sum $p_{V\oplus W}\colon V \oplus W \to B$.

By Definition 278.1 of the Euler class we have to prove the following claim:

Claim. We have the following equality in $\mathrm{H}^{m+n}(V \oplus W;\mathbb{Z})$:

$$p_{V\oplus W}^*(e_{\mathbb{Z}}(V) \cup e_{\mathbb{Z}}(W)) = \text{я}(\Theta_{\mathbb{Z}}(V \oplus W \to B)).$$

Proof. We start out with some preparations:

(a) We denote by $\mathrm{pr}_V\colon V \oplus W \to V$ and $\mathrm{pr}_W\colon V \oplus W \to W$ the natural projections.

(b)
- We have $p_V \circ \mathrm{pr}_V = p_{V\oplus W}\colon V \oplus W \to B$, thus $\mathrm{pr}_V^* \circ p_V^* = p_{V\oplus W}^*$.
- We have $p_W \circ \mathrm{pr}_W = p_{V\oplus W}\colon V \oplus W \to B$, thus $\mathrm{pr}_W^* \circ p_W^* = p_{V\oplus W}^*$.

(c) From the Whitney Sum–Thom Class Proposition 277.16 we obtain the equality

$$\Theta_{\mathbb{Z}}(V \oplus W) = \mathrm{pr}_V^*(\Theta_{\mathbb{Z}}(V)) \cup \mathrm{pr}_W^*(\Theta_{\mathbb{Z}}(W)) \in \mathrm{H}^{m+n}(V \oplus W, (V \oplus W)_{\neq 0};\mathbb{Z}).$$

We now see that

$$
\begin{array}{l}
\text{by the Cup Product-Naturality}\\
\text{Lemma 204.8}\\
\downarrow\\
p^*_{V\oplus W}(e_{\mathbb{Z}}(V)\cup e_{\mathbb{Z}}(W)) \overset{=}{} p^*_{V\oplus W}(e_{\mathbb{Z}}(V))\cup p^*_{V\oplus W}(e_{\mathbb{Z}}(W))\\
\quad = \ \mathrm{pr}_V^*(p_V^*(e_{\mathbb{Z}}(V)))\cup \mathrm{pr}_W^*(p_W^*(e_{\mathbb{Z}}(W))) \quad = \ \mathrm{pr}_V^*(\text{я}(\Theta_{\mathbb{Z}}(V)))\cup \mathrm{pr}_W^*(\text{я}(\Theta_{\mathbb{Z}}(W)))\\
\quad\uparrow \qquad\qquad\qquad\

Proposition 278.12. (Euler Class–Euler Number Proposition) Let M be a closed connected non-empty m-dimensional smooth manifold and let $p\colon W \to M$ be a smooth m-dimensional real vector bundle over M.

(1) Then $$e_{\mathbb{F}_2}(W) \;=\; \chi_{\mathbb{F}_2}(W) \cdot [M]^*_{\mathbb{F}_2} \;\in\; \mathrm{H}^m(M;\mathbb{F}_2).$$

(2) We now assume that M and W are oriented. Then $$e_{\mathbb{Z}}(W) \;=\; \chi(W) \cdot [M]^* \;\in\; \mathrm{H}^m(M;\mathbb{Z}).$$

Proof. In the following we prove the more interesting Statement (2). The proof of Statement (1) is of course essentially the same. Let M be a closed oriented connected non-empty m-dimensional smooth manifold and let $p\colon W \to M$ be an oriented smooth m-dimensional real vector bundle over M. Since M is oriented, connected and non-empty it follows from Definition 201.27 that it suffices to prove the following claim:

Claim. $$M \bullet_W M \;=\; \langle e_{\mathbb{Z}}(W), [M]\rangle_M.$$

Proof. We introduce the following notation:

- Let $p\colon \mathrm{B}W \to M$ be the corresponding oriented linear $\overline{B}^m$-bundle from the Vector Bundle-to-Disk Bundle Proposition 277.4.
- We denote by $i\colon M \to \mathrm{B}W$ and $i\colon M \to W$ the inclusions, which is the same as the 0-section.

The proposition follows from the following equalities

$$\begin{aligned}
M \bullet_W M &\overset{\text{follows easily from the definitions}^{2988}}{=} M \bullet_{\mathrm{B}W} M \\
&= \epsilon((\mathrm{PD}_{\mathrm{B}W}([M]) \cup \mathrm{PD}_{\mathrm{B}W}([M])) \cap [\mathrm{B}W]) \\
&\overset{\text{by the Cap Product-Kronecker Lemma 206.12}}{=} \epsilon(\mathrm{PD}_{\mathrm{B}W}([M]) \cap [M]) \\
&\overset{\text{by Definition 210.2}}{=} \langle \Theta_{\mathbb{Z}}(p\colon \mathrm{B}W \to M), i_*([M])\rangle_{\mathrm{B}W} \\
&\overset{\text{by the Cap Product Kronecker Lemma 206.12}}{=} \langle i^*(\Theta_{\mathbb{Z}}(p\colon W \to M)), [M]\rangle_M \\
&\overset{\text{follows easily from the Thom-equals-Thom Proposition 277.14 and the functoriality of cohomology}}{=} \langle e_{\mathbb{Z}}(W), [M]\rangle_M.
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{by the Cup Product–Intersection Number Theorem 220.10}}{=} \langle \mathrm{PD}_{\mathrm{B}W}([M]) \cup \mathrm{PD}_{\mathrm{B}W}([M]), [\mathrm{B}W]\rangle_{\mathrm{B}W} \\
&= \epsilon(\mathrm{PD}_{\mathrm{B}W}([M]) \cap (\mathrm{PD}_{\mathrm{B}W}([M]) \cap [\mathrm{B}W])) \\
&\overset{\text{by the Cup-Cap Lemma 206.15}}{=} \epsilon(\Theta_{\mathbb{Z}}(p\colon \mathrm{B}W \to M) \cap [M]) \\
&\overset{\text{by the Smooth Disk Bundle–Thom Class Proposition 244.13}}{=} \langle i^*(\Theta_{\mathbb{Z}}(p\colon \mathrm{B}W \to M)), [M]\rangle_M \\
&\overset{\text{by the Kronecker Pairing Natural Lemma 199.9 (1)}}{=} \langle e_{\mathbb{Z}}(W), [M]\rangle_M. \\
&\overset{\text{by Definition 245.1, here we use that } i^* = (p^*)^{-1}}{}
\end{aligned}$$

■

The following proposition finally explains where the name *Euler* class comes from:

[2988] Alternatively this equality follows from the more fancy Intersection Form-Submanifold Proposition 225.10.

Proposition 278.13. (Euler Class–Euler Characteristic Proposition) Let M be a closed n-dimensional smooth manifold.

(1) We have the equality

$$e_{\mathbb{F}_2}(\mathrm{T}M) \;=\; \chi(M)\cdot[M]^*_{\mathbb{F}_2} \;\in\; \mathrm{H}^n(M;\mathbb{F}_2).$$

(2) If M is oriented, then we have the equality

$$e_{\mathbb{Z}}(\mathrm{T}M) \;=\; \chi(M)\cdot[M]^* \;\in\; \mathrm{H}^n(M;\mathbb{Z}).$$

Proof. Here is the proof of Statement (2): We have the following equalities:

$$e_{\mathbb{Z}}(\mathrm{T}M) \underset{\text{by the Euler Class–Euler Number Proposition 278.12}}{=} \chi(\mathrm{T}M)\cdot[M] \underset{\text{by the Euler Number -of-Tangent Bundle Proposition 265.34}}{=} \chi(M)\cdot[M]^* \;\in\; \mathrm{H}^n(M;\mathbb{Z}).$$

The proof of Statement (1) is obviously almost the same. ■

Example. First recall that it follows from the discussion on page 5290 and on page 5317 that the Stiefel-Whitney classes of S^2 and the Pontryagin classes of $\mathrm{T}S^2$ are trivial.

(1) In contrast note that it follows from the Euler Class–Euler Characteristic Proposition 278.13 that for *even* $n \in \mathbb{N}$ we have $e_{\mathbb{Z}}(\mathrm{T}S^n) = 2\cdot[S^n]^*$.[2989] This shows that the Euler class is indeed a new invariant of vector bundles, in the sense that it is not determined by the Stiefel-Whitney classes and the Pontryagin classes.

(2) The discussion in (1) might suggest that perhaps now we have found complete invariants of real vector bundles. But this is not the case. For example it follows from the above and the Euler Class–Euler Characteristic Proposition 278.13 that all known characteristic classes of S^5 are trivial. But by the Sphere–Parallelizability Theorem 265.35 we know that the tangent bundle of S^5 is nonetheless a non-trivial vector bundle. An alternative example will be given in Exercise 278.7. □

278.5. **Euler classes, Stiefel-Whitney classes and Chern classes.** Let B be a topological space that is Hausdorff and paracompact. Recall that in Definition 274.1 we associated to any complex vector bundle $p\colon E \to B$ Chern classes $c_n(E) \in \mathrm{H}^{2n}(B;\mathbb{Z})$. Furthermore, in Definition 274.14 we associated to any real vector bundle $p\colon E \to B$ the Stiefel-Whitney classes $w_n(E) \in \mathrm{H}^n(B;\mathbb{F}_2)$. The main goal of this section is to relate our new best friends, the Euler classes, to Chern classes and Stiefel-Whitney classes.

This will be achieved by proving the following two propositions:

Proposition 278.14. (Euler–Chern Class Proposition) Let $p\colon V \to B$ be an n-dimensional complex vector bundle over a topological space B that is Hausdorff and paracompact. Recall that by Complex Vector Bundle–Orientable Lemma 265.25 we can view it as an oriented $2n$-dimensional real vector bundle. We have the equality

$$c_n(V) \;=\; e_{\mathbb{Z}}(V) \;\in\; \mathrm{H}^{2n}(B;\mathbb{Z}).$$

Proposition 278.15. (Euler–Stiefel-Whitney Class Proposition) Let $p\colon V \to B$ be an n-dimensional real vector bundle over a topological space B that is Hausdorff and

[2989] An alternative argument for this equality is given in [**Hat2**, Proposition 3.14].

paracompact. We have the equality

$$w_n(V) \;=\; e_{\mathbb{F}_2}(V) \;\in\; \mathrm{H}^n(B;\mathbb{F}_2).$$

Example. Let Σ be a closed oriented 2-dimensional smooth manifold of some genus $g \in \mathbb{N}_0$. It follows from the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Proposition 40.4, the Complex Torus Lemma 40.9 and the Surface–Möbius Structure Proposition 132.4 that Σ_g admits the structure of a 1-dimensional complex manifold.[2990] It follows from the Tangent Bundle Proposition 265.10 that the tangent bundle $\mathrm{T}\Sigma$ is a 1-dimensional complex vector bundle. We now see that

$$c_1(\mathrm{T}\Sigma) \underset{\text{by the Euler–Chern Class Proposition 278.14}}{\overset{\uparrow}{=}} e_{\mathbb{Z}}(\mathrm{T}\Sigma) \underset{\text{follows from the Euler Class Euler Characteristic Proposition 278.13}}{\overset{\uparrow}{=}} \chi(\Sigma)\cdot[\Sigma]^*.$$

This matches the calculation from the Chern-of-Surface Proposition 274.12. □

Example. We now have the tools to calculate the Stiefel-Whitney classes of the tangent bundles of the closed connected non-empty 2-dimensional smooth manifolds:

- Let Σ_g be the surface of genus g from page 342.[2991]
  (1) We have $w_1(\mathrm{T}\Sigma_g) = 0$: This follows from the Bundle Orientation-w_1-Proposition 274.23.
  (2) We have $w_2(\mathrm{T}\Sigma_g) = 0$: This follows from the Euler–Stiefel-Whitney Class Proposition 278.15 together with the Euler Class–Euler Characteristic Proposition 278.13 since by the discussion on page 2987 we know that $\chi(\Sigma_g) = 2 - 2g \equiv 0 \bmod 2$.
- Let $k \in \mathbb{N}$ and let N_k be the non-orientable surface of genus k from Definition 9.19.
  (1) We have $w_1(\mathrm{T}\Sigma_g) \neq 0$: This follows from the Bundle Orientation-w_1-Proposition 274.23. A complete calculation of $w_1(\mathrm{T}\Sigma_g) \in \mathrm{H}^1(N_k;\mathbb{F}_2)$ will be provided in Exercise 278.8.
  (2) We have $w_2(\mathrm{T}N_k) = k\cdot[N_k]^*$: This follows from the same arguments as above together with the observation from page 2987 that $\chi(N_k) = 2 - k \equiv k \bmod 2$. We also get that $w_2(\Sigma_g) = 0$. □

In the following we will put, for clarity of exposition, the focus on the proof of the Euler–Chern Class Proposition 278.14, since the proof of the Euler–Stiefel-Whitney Class Proposition 278.15 is essentially the same, and just mildly easier.

The proof of the Euler–Chern Class Proposition 278.14 rests on the following variation on the Chern Class Characterization Proposition 274.7:

Proposition 278.16. (Top Chern Class Characterization Proposition) Suppose we have a sequence of maps $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ such that ϵ_k associates to any k-dimensional complex

[2990] As we saw in the, Complex Tori–Classification Proposition 129.2, this structure is not in anyway unique.
[2991] On page 5290 we gave an alternative calculation of $w(\Sigma_g)$.

vector bundle $p\colon V \to B$ over a topological space that is Hausdorff and paracompact a class $\epsilon_k(V) \in \mathrm{H}^{2k}(B;\mathbb{Z})$ such that the following three statements hold:[2992]

(2) Each ϵ_k is natural in the sense that for every morphism

$$\begin{array}{ccc} F & \xrightarrow{H} & V \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{h} & B \end{array}$$

of complex vector bundles we have $h^*(\epsilon_k(V)) = \epsilon_k(F) \in \mathrm{H}^{2k}(C;\mathbb{Z})$.

(3) Let B be a topological space that is Hausdorff and paracompact. Furthermore let $p\colon V \to B$ be an m-dimensional complex vector bundle and let $q\colon W \to B$ be an n-dimensional complex vector bundle. Then

$$\epsilon_{m+n}(V \oplus W) \;=\; \epsilon_m(V) \cup \epsilon_n(W).$$

(4) For the tautological complex vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ we have the equality

$$\epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) \;=\; [\mathbb{C}\mathrm{P}^1]^* \;\in\; \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}).$$

If we have another such sequence $\widetilde{\epsilon}_k$ with the same properties, then $\widetilde{\epsilon}_k = \epsilon_k$ for all $k \in \mathbb{N}_0$.

Proof. Suppose we have two sequences of maps $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ and $\{\widetilde{\epsilon}_k\}_{k\in\mathbb{N}_0}$ that satisfy the above Conditions (2)–(4). We need to show that for all $k \in \mathbb{N}_0$ and for every complex vector bundle $p\colon V \to B$ over a topological space that is Hausdorff and paracompact we have $\widetilde{\epsilon}_k(V) = \epsilon_k(V)$. By (4) we know that the equality holds for the tautological vector bundle $\mathfrak{T}\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. In the following claims we will expand the class of complex vector bundles where equality holds:

Claim 1. We have $\widetilde{\epsilon}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) = \epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

Proof. First note that it follows from the Tautological Bundle Lemma 270.8 and the Vector Bundle–Pullback Proposition 265.36 (2) that the pullback of $\mathfrak{T}\mathbb{C}\mathrm{P}^\infty$ under the usual embedding $\iota\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ is isomorphic to $\mathfrak{T}\mathbb{C}\mathrm{P}^1$. Thus we see that

$$\begin{array}{ccccccc} \iota^*(\widetilde{\epsilon}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) & = & \widetilde{\epsilon}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) & = & \epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^1) & = & \iota^*(\epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the above discussion} & & \text{since } \widetilde{\epsilon}_1 \text{ and } \epsilon_1 & & \text{by the above discussion} & \\ & \text{and since } \widetilde{\epsilon}_1 \text{ satisfies (2)} & & \text{both satisfy (4)} & & \text{and since } \epsilon_1 \text{ satisfies (2)} & \end{array}$$

By the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 213.5 we know that the induced map $\iota^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ is an isomorphism. This fact together with the above calculation gives us the desired equality $\widetilde{\epsilon}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) = \epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)$. ⊞

Next we consider the tautological line bundle $p\colon \mathfrak{T}G_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ from the Tautological Bundle Lemma 270.8:

Claim 2. For every $n \in \mathbb{N}$ we have $\widetilde{\epsilon}_n(\mathfrak{T}G_n(\mathbb{C}^\infty)) = \epsilon_n(\mathfrak{T}G_n(\mathbb{C}^\infty)) \in \mathrm{H}^n(G_n(\mathbb{C}^\infty);\mathbb{Z})$.

[2992] We try to maximize the commonality between this proposition and the Chern Class Characterization Proposition 274.7. For this reason we use this somewhat weird numbering system, which starts at (2).

Proof. Let $n \in \mathbb{N}$. We start out with a few observations:

(a) It follows from the Grassmann Product Pullback Proposition 273.3 together with the Splitting Theorem 273.6 that there exists a continuous map $\mu\colon (\mathbb{C}\mathrm{P}^\infty)^n \to G_n(\mathbb{C}^\infty)$ with the following two properties:
  (i) We have an isomorphism $\mathfrak{T}\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathfrak{T}\mathbb{C}\mathrm{P}^\infty \xrightarrow{\cong} \mu^*(\mathfrak{T}G_n(\mathbb{C}^\infty))$ of complex vector bundles.
  (ii) The induced map $\mu^*\colon \mathrm{H}^*(G_n(\mathbb{C}\mathrm{P}^\infty);\mathbb{Z}) \to \mathrm{H}^*((\mathbb{C}\mathrm{P}^\infty)^n;\mathbb{Z})$ is a monomorphism.

(b) Let $\pi_1, \dots, \pi_n\colon (\mathbb{C}\mathrm{P}^\infty)^n \to \mathbb{C}\mathrm{P}^\infty$ be the n natural projections. Recall that by the Bundle Product-as-Whitney Sum Lemma 268.5 we know that $\mathfrak{T}(\mathbb{C}\mathrm{P}^\infty) \times \cdots \times \mathfrak{T}\mathbb{C}\mathrm{P}^\infty$ is isomorphic to the Whitney sum $\pi_1^*(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \oplus \cdots \oplus \pi_n^*(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)$.

We now see that we have the following equality in $\mathrm{H}^*(G_n(\mathbb{C}\mathrm{P}^\infty);\mathbb{Z})$:

$$\begin{aligned}
\mu^*(\widetilde{\epsilon}_n(\mathfrak{T}G_n(\mathbb{C}^\infty))) &\overset{\text{since } \widetilde{\epsilon}_n \text{ satisfies (2)}}{=} \widetilde{\epsilon}_n(\mu^*(\mathfrak{T}G_n(\mathbb{C}^\infty))) \\
&\overset{\text{by (b)}}{=} \widetilde{\epsilon}_n(\pi_{1*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \oplus \cdots \oplus \pi_{n*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \\
&\overset{\text{since } \widetilde{\epsilon}_1 \text{ satisfies (2)}}{=} \pi_{1*}(\widetilde{\epsilon}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \cup \cdots \cup \pi_{n*}(\widetilde{\epsilon}_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \\
&\overset{\text{same argument backwards}}{=} \dots = \mu^*(\epsilon_n(\mathfrak{T}G_n(\mathbb{C}^\infty))).
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{by (a-i)}}{=} \widetilde{\epsilon}_n(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathfrak{T}\mathbb{C}\mathrm{P}^\infty) \\
&\overset{\text{since the sequence } \widetilde{\epsilon}_k \text{ satisfies (3)}}{=} \widetilde{\epsilon}_1(\pi_{1*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \cup \cdots \cup \widetilde{\epsilon}_1(\pi_{n*}(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \\
&\overset{\text{by Claim 1}}{=} \pi_{1*}(\epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty)) \cup \cdots \cup \pi_{n*}(\epsilon_1(\mathfrak{T}\mathbb{C}\mathrm{P}^\infty))
\end{aligned}$$

Finally note that it follows from this calculation together with the injectivity of μ^* from (a-ii) that we obtain the desired equality $\widetilde{\epsilon}_n(\mathfrak{T}G_n(\mathbb{C}^\infty)) = \epsilon_n(\mathfrak{T}G_n(\mathbb{C}^\infty)) \in \mathrm{H}^n(G_n(\mathbb{C}^\infty);\mathbb{Z})$. We have thus achieved our goal. ⊞

The following claim gives us finally the desired statement:

Claim 3. For every complex vector bundle $p\colon V \to B$ over a topological space B that is Hausdorff and paracompact we have $\widetilde{\epsilon}_n(V) = \epsilon_n(V)$ for all $n \in \mathbb{N}_0$.

Proof. Let $p\colon V \to B$ be an n-dimensional complex vector bundle over a topological space B that is Hausdorff and paracompact. By the Grassmannian–Universal Bundle Theorem 270.10 there exists a continuous map $f\colon B \to G_n(\mathbb{C}^\infty)$ such that V is isomorphic to $f^*\mathfrak{T}G_n(\mathbb{C}^\infty)$. We now see that

$$\widetilde{\epsilon}_n(V) \overset{\text{since } \widetilde{\epsilon}_n \text{ satisfies hypothesis (2)}}{=} f^*(\widetilde{\epsilon}_n(\mathfrak{T}G_n(\mathbb{C}^\infty))) \overset{\text{by Claim 2}}{=} f^*(\epsilon_n(\mathfrak{T}G_n(\mathbb{C}^\infty))) \overset{\text{since } \epsilon_n \text{ satisfies hypothesis (2)}}{=} \epsilon_n(V).$$

■

Proof of the Euler–Chern Class Proposition 278.14. By the Top Chern Class Characterization 278.16 it suffices to verify that the top Chern class and the Euler class have properties (2), (3) and (4). This is done in the following two claims:

Claim 1. The Chern class c_n of n-dimensional complex vector bundles has properties (2), (3) and (4).

Proof.

(2) This follows immediately from the Naturality-of-Chern-Proposition 274.2.

(3) This follows easily from the Chern-Multiplicativity Theorem 274.6 together with the dimension conditions from Definition 274.1.
(4) This follows immediately from the Tautological Bundle-Chern-Lemma 274.4. ⊞

Claim 2. The Euler class, applied to complex vector bundles, has properties (2), (3) and (4).

Proof.

(2) This follows immediately from the Naturality-of-Euler-Proposition 278.6.
(3) This follows immediately from the Euler–Multiplicativity Theorem 278.10.
(4) This follows immediately from the Tautological Bundle-Euler Class-Lemma 278.9. ■

Proof of the Euler–Stiefel-Whitney Class Proposition 278.15. The proof of this proposition is very similar to the proof of the Euler–Chern Class Proposition 278.14. We leave it to the reader to look up all the corresponding statements for Stiefel-Whitney classes and to make the necessary, mostly minute, changes. ■

278.6. **Euler classes and Pontryagin classes.** In this bonus section we will relate the Euler class to the Pontryagin classes:

Proposition 278.17. (Euler–Pontryagin Class Proposition) Let $p\colon V \to B$ be an oriented $2n$-dimensional real vector bundle over a topological space B that is Hausdorff and paracompact. We have the equality

$$p_n(V) \;=\; e_{\mathbb{Z}}(V) \cup e_{\mathbb{Z}}(V) \;\in\; \mathrm{H}^{4n}(B;\mathbb{Z}).$$

Proof. Let $p\colon V \to B$ be an oriented $2n$-dimensional real vector bundle over a topological space B that is Hausdorff and paracompact. We consider the oriented real vector bundle $V \oplus V$ from Definition 268.29 and we consider the complex vector bundle $V \otimes_{\mathbb{R}} \mathbb{C}$ which we equip with the natural orientation from the Complex Vector Bundle–Orientable Lemma 265.25.

We start out with an elementary claim from linear algebra:

Claim. For any oriented real vector space U we consider the isomorphism

$$\begin{array}{rccc} \Phi\colon & \underbrace{U \oplus U}_{\substack{\text{equipped with the}\\ \text{orientation from}\\ \text{Definition 31.13}}} & \to & \underbrace{U \otimes_{\mathbb{R}} \mathbb{C}}_{\substack{\text{equipped with the}\\ \text{orientation from}\\ \text{Definition 40.17}}} \\ & (u, u') & \mapsto & u + \mathrm{i}u'. \end{array}$$

This isomorphism is orientation-preserving if and only if $\dim(U)$ is even.

Proof. [2993] Let $(u_1, \dots, u_n)$ be a positive basis for U. First note that it follows easily from Definition 31.13 that $((u_1, 0), \dots, (u_n, 0), (0, u_1), \dots, (0, u_n))$ is a positive basis for $U \oplus U$. Next note that it follows immediately from Definition 40.17 that $(u_1, \mathrm{i}u_1, \dots, u_n, \mathrm{i}u_n)$ is a positive basis for $U \otimes_{\mathbb{R}} \mathbb{C}$. The claim now follows from the observation that the base change matrix from $(\Phi(u_1, 0), \dots, \Phi(u_n, 0), \Phi(0, u_1), \dots, \Phi(0, u_n))$ to $(u_1, \mathrm{i}u_1, \dots, u_n, \mathrm{i}u_n)$ has positive determinant if and only if n is even. ⊞

[2993] This gives in particular a solution to Exercise 40.4.

We now see that

$$p_n(V) \underset{\uparrow}{=} (-1)^n \cdot c_{2n}(V \otimes_{\mathbb{R}} \mathbb{C}) \underset{\uparrow}{=} (-1)^n \cdot e_{\mathbb{Z}}(V \otimes_{\mathbb{R}} \mathbb{C}) \underset{\uparrow}{=} e_{\mathbb{Z}}(V \oplus V)$$

by Definition 276.4; by the Euler–Chern Class Proposition 278.14; follows easily from the claim together with the Orientation–Euler Lemma 278.4

$$\underset{\uparrow}{=} e_{\mathbb{Z}}(V) \cup e_{\mathbb{Z}}(V).$$

by the Euler–Multiplicativity Theorem 278.10

■

278.7. Exercises for Chapter 278.

Exercise 278.1. Show that given any *even* $k \in \mathbb{Z}$ there exists an oriented 4-dimensional real vector bundle $p\colon W \to S^4$ with $e_{\mathbb{Z}}(W) = k \cdot [S^4]^*$.

Exercise 278.2. Let $g \in \mathbb{N}_0$. We denote by Σ_g the surface of genus g. Show that given any $k \in \mathbb{Z}$ there exists an oriented 2-dimensional real vector bundle $p\colon W \to \Sigma_g$ with $e_{\mathbb{Z}}(W) = k \cdot [\Sigma_g]^*$.

Exercise 278.3. Let $g \in \mathbb{N}_0$. Show that there exists a 3-dimensional real vector bundle $p\colon W \to \Sigma_g \times S^1$ such that $e_{\mathbb{F}_2}(W) \neq 0$.

Exercise 278.4. Let M be a closed orientable $2n$-dimensional smooth submanifold of non-zero Euler characteristic. Show that $\mathrm{T}M$ does not admit an odd-dimensional vector subbundle in the sense of Definition 269.12.

Exercise 278.5. Let M be a complex manifold of real dimension $4n+2$. Show that if the complex tangent bundle $\mathrm{T}M$ is isomorphic to the complex conjugate vector bundle $\overline{\mathrm{T}M}$ (see Definition 268.33), then $\chi(M) = 0$.

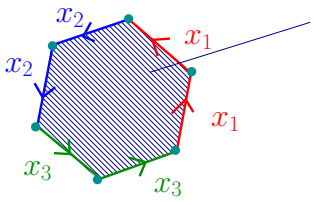
Exercise 278.6. Use the Real Vector Bundle-over-Sphere Proposition 267.9 to construct an orientable 3-dimensional real vector bundle $p\colon W \to S^4$ that does not split as the Whitney sum of lower-dimensional real vector bundles.
Hint. Recall that $\pi_3(\mathrm{GL}_+(3,\mathbb{R})) \cong \pi_3(\mathrm{SO}(3)) \cong \pi_3(\mathbb{R}\mathrm{P}^3) \cong \pi_3(S^3) \cong \mathbb{Z}$.
Remark. Note that it follows from $\mathrm{H}^3(S^4;\mathbb{Z}) = 0$ that $e_{\mathbb{Z}}(W) = 0$. Thus the exercise shows that the converse to the Non-Vanishing Section–Euler Proposition 278.3 does not hold.

Exercise 278.7. Use the Real Vector Bundle-over-Sphere Proposition 267.9 to construct an orientable 3-dimensional real vector bundle $p\colon W \to S^5$ such that all our characteristic classes (Stiefel-Whitney, Euler, Pontryagin) are zero, but which is non-trivializable.
Hint. Note that it follows from the Steenrod Operations-π_n-Theorem 236.19 (and various other results) that $\pi_4(\mathrm{GL}_+(3,\mathbb{R})) \cong \pi_4(\mathrm{SO}(3)) \cong \pi_4(\mathbb{R}\mathrm{P}^3) \cong \pi_4(S^3) \neq 0$.

Exercise 278.8. Let $k \in \mathbb{N}_{\geq 2}$. We consider the non-orientable surface N_k of genus k from Definition 9.19.

(a) Show that the closed curves $x_1, \ldots, x_k$ shown below define a basis for $\mathrm{H}_1(N_k;\mathbb{F}_2)$.
(b) For $i = 1, \ldots, k$ determine the Kronecker pairing $\langle w_1(\mathrm{T}N_k), [x_i] \rangle_{N_k}$.

Remark. This exercise gives a complete calculation of the first Stiefel-Whitney class $w_1(\mathrm{T}N_k) \in \mathrm{H}^1(N_k;\mathbb{F}_2) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{H}_1(N_k;\mathbb{F}_2),\mathbb{F}_2)$. Note that this exercise is closely related to Exercise 102.12.



non-orientable surface N_3 of genus 3

$$\mathrm{H}_1(N_k;\mathbb{F}_2) = \mathbb{Z}_2 \cdot [x_1] \oplus \cdots \oplus \mathbb{Z}_2 \cdot [x_k]$$

CHAPTER 279

Stiefel-Whitney classes via Steenrod operations

Let $p\colon E \to B$ be an n-dimensional real vector bundle. In the previous Chapter 278 we used the Thom Isomorphism Theorem 277.9 to define the $\mathbb{F}_2$-Euler class $e_{\mathbb{F}_2}(E) \in \mathrm{H}^n(B;\mathbb{F}_2)$. In this chapter we will combine the Thom Isomorphism Theorem 277.9 with Steenrod operations to give a new, alternative definition of Stiefel-Whitney classes.

This chapter is organized as follows:

- In Section 279.1 we will give an alternative characterization of Euler classes. This point of view will motivate our new definition of Stiefel-Whitney classes.
- In Section 279.2 we will recall the key properties of Steenrod operations from Chapter 236.
- In Section 279.3 we will give an alternative definition of Stiefel-Whitney classes and we will use the properties of Steenrod operations and the properties of Thom classes to show that the "new" Stiefel-Whitney classes satisfy the same properties as the "old" Stiefel-Whitney classes.
- In Section 279.4 we will see that the two definitions of Stiefel-Whitney classes are consistent.
- In Section 279.5 we will provide exercises for Chapter 279.

In the next **??** we will study Stiefel-Whitney classes of smooth manifolds.

279.1. **An alternative definition of Euler classes.** For clarity's sake let us recall the statement of the Thom Isomorphism Theorem 277.9:

Theorem 277.9. (Thom Isomorphism Theorem) Let $p\colon E \to B$ be an n-dimensional real vector bundle. We denote by $E_{\neq 0} \subseteq E$ the set of all non-zero vectors and for each $b \in B$ we write $E_b := p^{-1}(E_b)$ and $E_{b,\neq 0} := E_b \setminus \{0\}$.

(1) (a) There exists a unique $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E, E_{\neq 0};\mathbb{F}_2)$ with the property that for each $b \in B$ the restriction to $\mathrm{H}^n(E_b, E_{b,\neq 0};\mathbb{F}_2)$ is non-zero.

(b) For each $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl} \mathrm{H}^k(B;\mathbb{F}_2) & \to & \mathrm{H}^{n+k}(E, E_{\neq 0};\mathbb{F}_2) \\ \varphi & \mapsto & \Theta_{\mathbb{F}_2}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

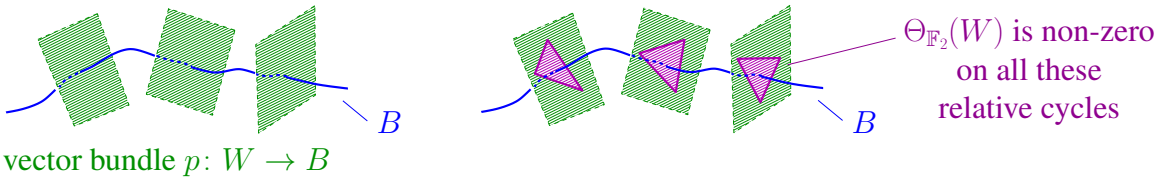
(2) (a) If the real vector bundle E is oriented in the sense of Definition 265.23, then there exists a unique $\Theta_{\mathbb{Z}}(E) \in \mathrm{H}^n(E, E_{\neq 0};\mathbb{Z})$ with the property that for each $b \in B$ the restriction to $\mathrm{H}^n(E_b, E_{b,\neq 0};\mathbb{Z})$ is the standard generator given by the orientation of the vector space E_b and Definition 277.1.

(b) For each $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl} \mathrm{H}^k(B;\mathbb{Z}) & \to & \mathrm{H}^{n+k}(E, E_{\neq 0};\mathbb{Z}) \\ \varphi & \mapsto & \Theta_{\mathbb{Z}}(E) \cup p^*(\varphi) \end{array}$$

is an isomorphism.

Both classes $\Theta_{\mathbb{F}_2}(E)$ and $\Theta_{\mathbb{Z}}(E)$ are called Thom class and both isomorphisms are called Thom isomorphism.



The following lemma can be used to give an alternative definition of the Euler class of a real vector bundle:

Lemma 279.1. (Euler Class–Calculation Lemma) Let $p\colon E \to B$ be an n-dimensional real vector bundle.

(1) By the Thom Isomorphism Theorem 277.9 (1) there exists a unique $\alpha_{\mathbb{F}_2} \in \mathrm{H}^n(B;\mathbb{F}_2)$ such that

$$\Theta_{\mathbb{F}_2}(E) \cup p^*(\alpha_{\mathbb{F}_2}(E)) \;=\; \Theta_{\mathbb{F}_2}(E) \cup \Theta_{\mathbb{F}_2}(E) \;\in\; \mathrm{H}^{2n}(E, E_{\neq 0};\mathbb{F}_2).$$

In fact $\alpha_{\mathbb{F}_2}$ equals the Euler class $e_{\mathbb{F}_2}(E) \in \mathrm{H}^n(B;\mathbb{F}_2)$ from Definition 278.1.

(2) If the vector bundle E is oriented, then it follows from the Thom Isomorphism Theorem 277.9 (2) that there exists a unique $\alpha_{\mathbb{Z}} \in \mathrm{H}^n(B;\mathbb{Z})$ with

$$\Theta_{\mathbb{Z}}(E) \cup p^*(\alpha_{\mathbb{Z}}(E)) \;=\; \Theta_{\mathbb{Z}}(E) \cup \Theta_{\mathbb{Z}}(E) \;\in\; \mathrm{H}^{2n}(E, E_{\neq 0};\mathbb{Z}).$$

In fact α equals the Euler class $e(E) \in \mathrm{H}^n(B;\mathbb{Z})$ from Definition 278.1.

Proof. Let $R = \mathbb{F}_2$ or $R = \mathbb{Z}$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^n(E, E_{\neq 0}; R) \times \mathrm{H}^n(E, E_{\neq 0}; R) & \xrightarrow{\cup} & \mathrm{H}^{2n}(E, E_{\neq 0}; R) \\ \downarrow{\scriptstyle \mathrm{id}} \quad \downarrow{\scriptstyle я} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{H}^n(E, E_{\neq 0}; R) \times \mathrm{H}^n(E; R) & \xrightarrow{\cup} & \mathrm{H}^{2n}(E, E_{\neq 0}; R). \end{array}$$

Note that it follows from the Cup Product-Naturality Lemma 204.8 that this diagram commutes. We now see that

$$\Theta_R(E) \cup p^*(((p^*)^{-1} \circ я)(\Theta_R(E))) \;=\; \Theta_R(E) \cup я(\Theta_R(E)) \underset{\substack{\uparrow \\ \text{by the above commutative diagram}}}{=} \Theta_R(E) \cup \Theta_R(E).$$

We have thus shown that $((p^*)^{-1} \circ я)(\Theta_R(E))$ agrees with the cohomology class α that we introduced in the lemma. Finally note that $((p^*)^{-1} \circ я)(\Theta_R(E))$ is exactly the Euler class $e_{\mathbb{F}_2}(E) \in \mathrm{H}^n(B;\mathbb{F}_2)$ from Definition 278.1. ■

The Euler Class–Calculation Lemma 279.1 and the Thom Isomorphism Theorem 277.9 gives us an interesting idea for defining more characteristic classes: All we need to do is to "produce" more cohomology classes out of $\Theta_{\mathbb{F}_2}(E)$ and $\Theta_{\mathbb{Z}}(E)$ and put them on the right hand side of the Thom Isomorphism Theorem 277.9 (2). In the above Euler Class–Calculation Lemma 279.1 we used the cup product to produce such new cohomology classes.

Next we will do something more clever, namely we will use Steenrod operations. In the following section we summarize some key facts about Steenrod operations from Chapter 236.

279.2. **Interlude: Reminder of Steenrod operations.** Since Steenrod operations are at best a faint memory, we will now summarize the key properties thereof. To do so, we first need to recall the following definition:

Definition 236.1.

(1) Let G and H be abelian groups and let $m, n \in \mathbb{N}_0$. Any natural transformation from the contravariant functor

$$\mathrm{H}^m(-;G)\colon \text{category of topological spaces} \quad \to \quad \text{category of sets}$$

to the contravariant functor

$$\mathrm{H}^n(-;H)\colon \text{category of topological spaces} \quad \to \quad \text{category of sets}$$

is called a cohomology operation of degree $n-m$. In other words, a cohomology operation consists for any topological space X of a map $\Phi_X\colon \mathrm{H}^m(X;G) \to \mathrm{H}^n(X;H)$ of sets such that for every continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^m(Y;G) & \xrightarrow{f^*} & \mathrm{H}^m(X;G) \\ \downarrow{\scriptstyle \Phi_Y} & & \downarrow{\scriptstyle \Phi_X} \\ \mathrm{H}^n(Y;H) & \xrightarrow{f^*} & \mathrm{H}^n(X;H). \end{array}$$

(2) A cohomology operation is called additive if each map $\Phi_X\colon \mathrm{H}^m(X;G) \to \mathrm{H}^n(X;H)$ is a homomorphism.

With the obvious modifications the same definitions also apply to relative cohomology viewed as functors from the category of pairs of topological spaces to the category of sets.

Example. It follows from the Cup Product-Naturality Lemma 204.8 that for any commutative ring R and any $m \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} \mathrm{H}^m(X;R) & \to & \mathrm{H}^{2m}(X;R) \\ \varphi & \mapsto & \varphi \cup \varphi \end{array}$$

define a cohomology operation. Note that if $1 = 1 \in R$, then it follows from Freshman's dream that this cohomology operation is additive. But also note that in general this cohomology operation is not additive. □

Theorem 236.3. (Steenrod Theorem) There exists a family

$$\mathrm{Sq}^k\colon \mathrm{H}^n(X,A;\mathbb{F}_2) \ \to \ \mathrm{H}^{n+k}(X,A;\mathbb{F}_2) \quad \text{with } k \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}_0$$

of additive cohomology operations (called Steenrod operations) which satisfies the following axioms:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.
(2) For any $\varphi \in \mathrm{H}^k(X,A;\mathbb{F}_2)$ we have $\mathrm{Sq}^i(\varphi) = \varphi^2 = \varphi \cup \varphi \in \mathrm{H}^{2k}(X,A;\mathbb{F}_2)$.
(3) For any $\varphi \in \mathrm{H}^j(X,A;\mathbb{F}_2)$ with $k > j$ we have $\mathrm{Sq}^k(\varphi) = 0$.

(4) Let (X, A, B) be an excisive triad of topological spaces in the sense of Definition 158.18. For any $\varphi \in \mathrm{H}^r(X, A; \mathbb{F}_2)$ any $\psi \in \mathrm{H}^s(X, B; \mathbb{F}_2)$ and any $k \in \mathbb{N}_0$ we have the equality

$$\mathrm{Sq}^k(\varphi \cup \psi) \;=\; \sum_{i=0}^{k} \underbrace{\mathrm{Sq}^i(\varphi)}_{\in \mathrm{H}^{r+i}(X,A;\mathbb{F}_2)} \cup \underbrace{\mathrm{Sq}^{k-i}(\psi)}_{\in \mathrm{H}^{s+k-i}(X,B;\mathbb{F}_2)} \;\in\; \mathrm{H}^{r+s+k}(X, A \cup B; \mathbb{F}_2).$$

This equality is called the Cartan formula. Note that if for $\alpha \in \mathrm{H}^n(X, A; \mathbb{F}_2)$ we set $\mathrm{Sq}(\alpha) := \sum_{i=0}^{n} \mathrm{Sq}^i(\alpha)$, then the Cartan formula says that

$$\mathrm{Sq}(\varphi \cup \psi) \;=\; \mathrm{Sq}(\varphi) \cup \mathrm{Sq}(\psi).$$

For completeness' sake we also recall the following theorem, even though we will not make use of it:

Theorem 236.4. (Steenrod Operations–Uniqueness Theorem) Any two families of additive cohomology operations that satisfy Axioms (1) to (4) are identical.

279.3. **Alternative definition of Stiefel-Whitney class.** Using the Steenrod operations from the last section we can now give a new (or alternative) definition of the Stiefel-Whitney classes of a real vector bundle:

Definition 279.2. Let $n \in \mathbb{N}_0$, let $p\colon E \to B$ be an n-dimensional real vector bundle and let $k \in \mathbb{N}_0$. We denote by $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(E, E_{\neq 0}; \mathbb{F}_2)$ the corresponding Thom class. By the Thom Isomorphism Theorem 277.9 (1) there exists a unique $\widetilde{w}_k(E) \in \mathrm{H}^k(B; \mathbb{F}_2)$ with

$$\Theta_{\mathbb{F}_2}(E) \cup p^*(\widetilde{w}_k(E)) \;=\; \underbrace{\mathrm{Sq}^k(\Theta_{\mathbb{F}_2}(E))}_{\in \mathrm{H}^{n+k}(E, E_{\neq 0}; \mathbb{F}_2)}.$$

We refer to $\widetilde{w}_k(E) \in \mathrm{H}^k(B; \mathbb{F}_2)$ as the k-th Stiefel-Whitney class of the real vector bundle $p\colon E \to B$.

Remark. This alternative definition of Stiefel-Whitney classes was first introduced by Wenjun Wu [**Wu1950**]. Of course we will see shortly that these Stiefel-Whitney classes agree with the ones we introduced in Definition 274.14. □

The following proposition shows that at least one of the Stiefel-Whitney classes is a by now familiar concept:

Proposition 279.3. (Euler–Stiefel-Whitney Class Proposition) Let $p\colon E \to B$ be an n-dimensional real vector bundle. We have the following equality

$$\widetilde{w}_n(E) \;=\; e_{\mathbb{F}_2}(E) \;\in\; \mathrm{H}^n(B; \mathbb{F}_2).$$

Proof. Let $p\colon E \to B$ be an n-dimensional real vector bundle. Note that

$$\Theta_{\mathbb{F}_2}(E) \cup p^*(e_{\mathbb{F}_2}(E)) \underset{\text{by the Euler Class Calculation Lemma 279.1}}{=} \Theta_{\mathbb{F}_2}(E) \cup \Theta_{\mathbb{F}_2}(E) \underset{\text{by the Steenrod Theorem 236.3 (2)}}{=} \mathrm{Sq}^n(\Theta_{\mathbb{F}_2}(E)) \underset{\text{by Definition 279.2}}{=} \Theta_{\mathbb{F}_2}(E) \cup p^*(\widetilde{w}_n(E)).$$

It now follows from the Thom Isomorphism Theorem 277.9 (1) that we have the desired equality $\widetilde{w}_n(E) = e_{\mathbb{F}_2}(E) \in \mathrm{H}^n(B; \mathbb{F}_2)$. ■

As the name suggests, surely these Stiefel-Whitney classes $\widetilde{w}_k$ should equal the Stiefel-Whitney classes that we introduced in Definition 274.14. The idea for showing equality is to use the Stiefel-Whitney Class Characterization Proposition 274.22. Thus we need to

verify that the Stiefel-Whitney classes $\widetilde{w}_k$ have the same four key properties of the original Stiefel-Whitney classes w_k. More precisely, in the following four subsections we will see that the classes $\widetilde{w}_k$ have the following properties:

(1) the dimension conditions as for the Stiefel-Whitney classes from Definition 274.14 (2a) and (2b),
(2) the analog of the Naturality-of-SW-Proposition 274.15,
(3) the analog of the Stiefel-Whitney-Multiplicativity Theorem 274.19 and
(4) the analog of the Tautological Bundle-SW-Lemma 274.17.

As we will see, these statements follow fairly easily from the properties of the Thom classes that we established in Chapter 277 together with the really-quite-powerful Steenrod Theorem 236.3.

279.3.1. **Dimension conditions on Stiefel-Whitney classes.** We start out with the following lemma, which gives us the desired analog of Definition 274.14 (2a) and (2b):

Lemma 279.4. (Dimensions-of-SW-Lemma) Let $p\colon E \to B$ be an n-dimensional real vector bundle. We have the following equalities:
(a) We have $\widetilde{w}_0(E) = 1 \in \mathrm{H}^0(B;\mathbb{F}_2)$.
(b) We have $\widetilde{w}_k(E) = 0$ for $k > n$.

Proof.

(a) By the Steenrod Theorem 236.3 (1) we know that $\mathrm{Sq}^0 = \mathrm{id}$. It follows immediately from Definition 279.2 that $\widetilde{w}_0(E) = 1$.
(b) Let $k > n$. Since $\Theta_{\mathbb{F}_2}(E) \in \mathrm{H}^n(B;\mathbb{F}_2)$ it follows from the Steenrod Theorem 236.3 (3) that $\mathrm{Sq}^n(\Theta_{\mathbb{F}_2}(E)) = 0$. Thus it follows immediately from Definition 279.2 that $\widetilde{w}_k(E) = 0$. ∎

279.3.2. **Naturality of Stiefel-Whitney classes.** Next we have the following analog of the Naturality-of-SW-Proposition 274.15:

Proposition 279.5. (Naturality-of-SW-Proposition) Let $k \in \mathbb{N}_0$.
(1) For any morphism

$$\begin{array}{ccc} F & \xrightarrow{G} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{g} & B \end{array}$$

of real vector bundles we have $g^*(\widetilde{w}_k(E)) = \widetilde{w}_k(F) \in \mathrm{H}^k(C;\mathbb{F}_2)$.
(2) (a) If $p\colon E \to B$ and $p'\colon E' \to B$ are real vector bundles that are isomorphic over B, then $\widetilde{w}_k(E) = \widetilde{w}_k(E') \in \mathrm{H}^k(B;\mathbb{F}_2)$.
(b) Let $p\colon E \to B$ be a real vector bundle. For any continuous map $\varphi\colon C \to B$ between topological spaces we have the following equality:

$$\widetilde{w}_k(\varphi^*E) \;=\; \varphi^*(\widetilde{w}_k(E)) \;\in\; \mathrm{H}^k(C;\mathbb{F}_2).$$

Proof. One can easily see that Statement (2) is a straightforward consequence of Statement (1). In the following we now prove Statement (1). Let

$$\begin{array}{ccc} F & \xrightarrow{G} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{g} & B \end{array}$$

be a morphism of real vector bundles. Recall that we need to show that we have the equality $g^*(\widetilde{w}_k(E)) = \widetilde{w}_k(F) \in \mathrm{H}^k(C;\mathbb{F}_2)$. By Definition 279.2 of $\widetilde{w}_k(F)$ it suffices to prove the following claim:

Claim. $\Theta_{\mathbb{F}_2}(F) \cup q^*(g^*(\widetilde{w}_k(E))) \;=\; \mathrm{Sq}^k(\Theta_{\mathbb{F}_2}(F))$.

Proof. We perform the following calculation:

$$\begin{array}{lll} & \overset{\text{follows from } G^*G \cong E, \text{ the Thom Class–Natural Proposition 277.11 and the functoriality of cohomology}}{\downarrow} & \overset{\text{by the Cup Product-Naturality Lemma 204.8}}{\downarrow} \\ \Theta_{\mathbb{F}_2}(F) \cup q^*(g^*(\widetilde{w}_k(E))) & = G^*(\Theta_{\mathbb{F}_2}(E)) \cup G^*(p^*(\widetilde{w}_k(E))) & = G^*(\Theta_{\mathbb{F}_2}(E) \cup p^*(\widetilde{w}_k(E))) \\ & = G^*(\mathrm{Sq}^k(\Theta_{\mathbb{F}_2}(E))) & = \mathrm{Sq}^k(G^*(\Theta_{\mathbb{F}_2}(E))) \\ & \underset{\text{by Definition 279.2 of } \widetilde{w}_k(E)}{\uparrow} & \underset{\text{by Definition 236.1 we know that cohomology operations are natural}}{\uparrow} \\ & = \mathrm{Sq}^k(\Theta_{\mathbb{F}_2}(F)). & \\ & \underset{\text{by the Thom Class–Natural Proposition 277.11}}{\uparrow} & \end{array}$$

■

The following lemma is almost indistinguishable from the Trivializable-SW-Lemma 274.16:

Lemma 279.6. (Trivializable-SW-Lemma) Let $p\colon F \to B$ be a trivializable real vector bundle over a topological space. For every $k \in \mathbb{N}$ we have $\widetilde{w}_k(F) = 0$.

Proof. [2994] Let $p\colon W \to B$ be a trivializable n-dimensional real vector bundle. We consider the trivial vector bundle $\{x\} \times \mathbb{R}^n \to \{x\}$ over the singleton $\{x\}$. Let $g\colon B \to \{x\}$ be the unique map. We now see that

$$\begin{array}{lll} w_k(p\colon W \to B) & = w_k(g^*(\{x\} \times \mathbb{R}^n)) & = g^*(w_k(\{x\} \times \mathbb{R}^n)) \;=\; g^*(0) \;=\; 0. \\ \underset{\text{since } W \text{ is trivializable we see that the two vector bundles are isomorphic}}{\uparrow} & \underset{\text{by the Naturality-of-SW Proposition 279.5}}{\uparrow} & \underset{\text{since } \mathrm{H}^k(\{x\} \times \mathbb{R}^n;\mathbb{F}_2) = 0 \text{ (here we use that } k \in \mathbb{N})}{\uparrow} \end{array}$$

■

279.3.3. **Multiplicativity of Stiefel-Whitney classes.** Our next goal is to state a multiplicativity result. At this stage it is convenient to introduce the following analog of Definition 274.18:

Definition 279.7. Given a real vector bundle $p\colon E \to B$ we refer to

$$\widetilde{w}(E) \;:=\; (\widetilde{w}_0(E), \widetilde{w}_1(E), \widetilde{w}_2(E), \dots) \;\in\; \mathrm{H}^*(B;\mathbb{F}_2) \;:=\; \bigoplus_{i=0}^{\infty} \mathrm{H}^i(B;\mathbb{F}_2)$$

as the total Stiefel-Whitney class.

The following theorem is the analog of the SW-Multiplicativity Theorem 274.19:

Theorem 279.8. (SW-Multiplicativity Theorem) Let B be a topological space. Furthermore let $p_E\colon E \to B$ be an m-dimensional real vector bundle and let $p_F\colon F \to B$ be an n-dimensional real vector bundle. We consider the Whitney sum $E \oplus F$ from Definition 268.3. For each $k \in \mathbb{N}_0$ we have[2995]

$$\widetilde{w}_k(E \oplus F) \;=\; \sum_{i=0}^{k} \widetilde{w}_i(E) \cup \widetilde{w}_{k-i}(F).$$

[2994] We follow the logic from the Trivializable-Euler Lemma 278.8.

In other words we have

$$\widetilde{w}(E \oplus F) \;=\; \widetilde{w}(E) \cup \widetilde{w}(F).$$

Proof. Let $p_{E\oplus F}\colon E \oplus F \to B$ be the Whitney sum of $p_E\colon E \to B$ and $p_F\colon F \to B$. It follows easily from Definition 279.2 that it suffices to prove the following claim:

Claim. $\Theta_{\mathbb{F}_2}(E \oplus F) \cup p_{E\oplus E}^*(\widetilde{w}(E) \cup \widetilde{w}(F)) \;=\; \mathrm{Sq}(\Theta_{\mathbb{F}_2}(E \oplus F))$.

Proof. We start out with some preparations:

(a) We denote by $\mathrm{pr}_E\colon E \oplus F \to E$ and $\mathrm{pr}_F\colon E \oplus F \to F$ the natural projections.

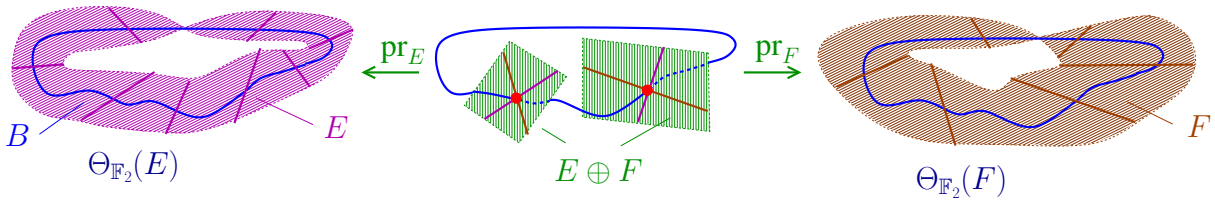
(b)
- We have $p_E \circ \mathrm{pr}_E = p_{E\oplus F}\colon E \oplus F \to B$, thus $\mathrm{pr}_E^* \circ p_E^* = p_{E\oplus F}^*$.
- We have $p_F \circ \mathrm{pr}_F = p_{E\oplus F}\colon E \oplus F \to B$, thus $\mathrm{pr}_F^* \circ p_F^* = p_{E\oplus F}^*$.

(c) From the Whitney Sum–Thom Class Proposition 277.16 (1) we obtain the equality

$$\Theta_{\mathbb{F}_2}(E \oplus F) \;=\; \mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E)) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F)) \quad \in\; \mathrm{H}^{m+n}(E \oplus F, (E \oplus F)_{\neq 0}; \mathbb{F}_2).$$

(d) By Definition 279.2 we have

$$\begin{aligned} \Theta_{\mathbb{F}_2}(E) \cup p_E^*(\widetilde{w}(E)) &\;=\; \mathrm{Sq}(\Theta_{\mathbb{F}_2}(E)) \quad \in\; \mathrm{H}^*(E, E_{\neq 0}; \mathbb{F}_2), \\ \Theta_{\mathbb{F}_2}(F) \cup p_F^*(\widetilde{w}(F)) &\;=\; \mathrm{Sq}(\Theta_{\mathbb{F}_2}(F)) \quad \in\; \mathrm{H}^*(F, F_{\neq 0}; \mathbb{F}_2). \end{aligned}$$



We now see that

$$\begin{aligned} &\Theta_{\mathbb{F}_2}(E \oplus F) \cup p_{E\oplus F}^*(\widetilde{w}(E) \cup \widetilde{w}(F)) \;= \\ &\quad= \mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E)) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F)) \cup p_{E\oplus F}^*(\widetilde{w}(E) \cup \widetilde{w}(F)) \\ &\qquad\uparrow \text{ here we use (c)} \\ &\quad= \mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E)) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F)) \cup \mathrm{pr}_E^*(p_E^*(\widetilde{w}(E))) \cup \mathrm{pr}_F^*(p_F^*(\widetilde{w}(F))) \\ &\qquad\uparrow \text{ follows from the Cup Product-Naturality Lemma 204.8 together with (b)} \\ &\quad= \mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E) \cup p_E^*(\widetilde{w}(E))) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F) \cup p_F^*(\widetilde{w}(F))) \\ &\qquad\uparrow \text{ by the Cup Product-Naturality Lemma 204.8 and the Cup Product-Commutativity Proposition 204.10} \\ &\quad= \mathrm{pr}_E^*(\mathrm{Sq}(\Theta_{\mathbb{F}_2}(E))) \cup \mathrm{pr}_F^*(\mathrm{Sq}(\Theta_{\mathbb{F}_2}(F))) \\ &\qquad\uparrow \text{ follows from (d)} \\ &\quad= \mathrm{Sq}(\mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E))) \cup \mathrm{Sq}(\mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F))) \\ &\qquad\uparrow \text{ follows from the naturality of Steenrod operations} \\ &\quad= \mathrm{Sq}(\mathrm{pr}_E^*(\Theta_{\mathbb{F}_2}(E)) \cup \mathrm{pr}_F^*(\Theta_{\mathbb{F}_2}(F))) \\ &\qquad\uparrow \text{ by the Cartan Formula from the Steenrod Theorem 236.3 (4)} \\ &\quad= \mathrm{Sq}(\Theta_{\mathbb{F}_2}(E \oplus F)). \\ &\qquad\uparrow \text{ follows from (c)} \end{aligned}$$

[2995] Note that it follows from the Euler–Stiefel-Whitney Class Proposition 279.3 that the case $k = m + n$ is exactly the statement of the Euler–Multiplicativity Theorem 278.10.

■

279.3.4. **Stiefel-Whitney classes of the tautological bundle.** Finally we have the following analog of the Tautological Bundle-SW-Lemma 274.17:

Lemma 279.9. (Tautological Bundle-SW-Lemma) Let $n \in \mathbb{N} \cup \{\infty\}$. Furthermore let $p\colon \mathfrak{T}\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ be the tautological real line bundle that we introduced in the Tautological Line Bundle Lemma 265.6. In this case we have

$$\widetilde{w}_1(\mathfrak{T}\mathbb{R}\mathrm{P}^n) \;=\; x \;\in\; \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$$

where $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ is the unique non-trivial element.

Proof. A somewhat different proof is given in [**MiS1974**, p. 93]. We have the following equalities in $\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$:

$$\widetilde{w}_1(\mathfrak{T}\mathbb{R}\mathrm{P}^n) \underset{\substack{\uparrow\\ \text{by the Euler–Stiefel-Whitney}\\ \text{Class Proposition 279.3}}}{=} e_{\mathbb{F}_2}(\mathfrak{T}\mathbb{R}\mathrm{P}^n) \underset{\substack{\uparrow\\ \text{by the Tautological}\\ \text{Bundle-Euler}\\ \text{Class-Lemma 278.9}}}{=} x.$$

■

279.4. **Comparing the two definitions of Stiefel-Whitney classes.** The following proposition now shows that Definition 274.14 and Definition 279.2 of Stiefel-Whitney classes agree for any real vector bundle where both apply, namely for real vector bundles over topological spaces that are Hausdorff and paracompact:

Proposition 279.10. (Stiefel-Whitney Classes Agree Proposition) Let $p\colon E \to B$ be a real vector bundle over a topological space that is Hausdorff and paracompact. For any $k \in \mathbb{N}_0$ we have $w_k(E) = \widetilde{w}_k(E)$. In other words, we have

$$\Theta_{\mathbb{F}_2}(E) \cup p^*(w_k(E)) \;=\; \underbrace{\mathrm{Sq}^k(\Theta_{\mathbb{F}_2}(E))}_{\in \mathrm{H}^{n+k}(E,E_{\neq 0};\mathbb{F}_2)}.$$

In other words, one can view Definition 274.14 as an extension of Definition 279.2 from real vector bundles over topological spaces that are Hausdorff and paracompact to real vector bundles over arbitrary base spaces. For whatever that might be tworth.

Proof. This follows from the Stiefel-Whitney Class Characterization Proposition 274.22 and the fact that $\widetilde{w}_k(E)$ satisfies the four Properties (1)–(4):

(1) This holds by the Dimensions-of-SW-Lemma 279.4.
(2) This holds by the Naturality-of-SW-Proposition 279.5.
(3) This holds by the SW-Multiplicativity Theorem 279.8.
(4) This hold by the Tautological Bundle-SW-Lemma 279.9. ■

For clarity let us state the following proposition, which is a slight generalization of the Euler–Stiefel-Whitney Class Proposition 278.15:

Proposition 279.11. (Euler–Stiefel-Whitney Class Proposition) Let $p\colon E \to B$ be an n-dimensional real vector bundle over a topological space that is Hausdorff and paracompact. Then we have the equality

$$w_n(E) \;=\; e_{\mathbb{F}_2}(E) \;\in\; \mathrm{H}^n(B;\mathbb{F}_2).$$

Proof. This proposition follows immediately from combining the earlier Euler–Stiefel-Whitney Class Proposition 278.15 with the Stiefel-Whitney Classes Agree Proposition 279.10. ■

279.5. **Exercises for Chapter 279.**

Exercise 279.1. Note that it follows from the Cup Product-Naturality Lemma 204.8 that for any commutative ring R and any $m \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} \mathrm{H}^m(X;R) & \to & \mathrm{H}^{2m}(X;R) \\ \varphi & \mapsto & \varphi \cup \varphi \end{array}$$

define a cohomology operation.

(a) Show that if $1+1=0 \in R$, then this cohomology operation is additive.
(b) Show that if m is odd, then this cohomology operation is additive.
(c) Show for any ring R with $1+1 \neq 0 \in R$ and any even $m \in \mathbb{N}$ the above cohomology operation is not additive.

Exercise 279.2. Let M be a closed n-dimensional smooth manifold and let $i_1, \dots, i_r \in \mathbb{N}_0$ such that $i_1 + \cdots + i_r = n$. We consider

$$\langle \underbrace{w_{i_1}(M) \cup \cdots \cup w_{i_r}(M)}_{\in \mathrm{H}^n(M;\mathbb{F}_2)}, [M]_{\mathbb{F}_2} \rangle \ \in \ \mathbb{F}_2.$$

Show that if M is the boundary of a compact $(n+1)$-dimensional smooth manifold F, then $\langle w_{i_1}(M) \cup \cdots \cup w_{i_r}(M), [M]_{\mathbb{F}_2} \rangle = 0$.
Hint. What is the restriction of the $(n+1)$-dimensional real vector bundle $\mathrm{T}F$ to M?
Remark. This exercise is identical to Exercise 274.5. Recall that in the Boundary-χ-Corollary 210.10 we showed that if F is a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold, then $\chi(F)$ is even. Thus, in light of the Euler Class–Euler Number Proposition 278.12, in the context of smooth manifolds the above exercise can be viewed as a generalization of the Boundary-χ-Corollary 210.10.

Part XXXI

The Hirzebruch signature theorem and exotic spheres

In this part of the lecture notes we will use Pontryagin classes to formulate (and partly prove) the Hirzebruch Signature Theorem 281.14. Afterwards we will use the Hirzebruch Signature Theorem to show that there exist 7-dimensional smooth manifolds that are homeomorphic to S^7 but that are not diffeomorphic. These smooth manifolds are the famous exotic 7-spheres that gained John Milnor a Field medal in 1958.

This part of the lecture notes contains the following chapters:

- In Chapter 280 we will introduce and study the cobordism group.
- In Chapter 281, we will state and partially prove the Hirzebruch Signature Theorem 281.14.
- In Chapter 282 we will use the Hirzebruch Signature Theorem 281.14 together with the High-Dimensional Poincaré Conjecture 264.7 to show that there exists a 7-dimensional smooth manifold that is homeomorphic to S^7 but that is not diffeomorphic to S^7.

CHAPTER 280

Cobordisms

In the Manifold–Non-Classification Theorem 103.4 we saw that it is a hopeless task to try to classify closed smooth manifolds of a fixed dimension $n \geq 4$ up to *diffeomorphism*. In this chapter we study a weaker notion of equivalence, namely *cobordism* instead of *diffeomorphism*. The rough idea is that two closed oriented n-dimensional smooth manifolds S and T are called *cobordant* if there exists a compact oriented smooth manifold W with $\partial W = S \sqcup (-T)$. We will see that closed oriented n-dimensional smooth manifolds up to *cobordism* lead to a group under the disjoint union operation. This group is called the *oriented cobordism group* Ω_n.

In this chapter we will introduce this oriented cobordism group Ω_n and its "unoriented" cousin $\mathcal{N}_n$. Furthermore we will state and discuss several calculations of these cobordism groups. The high point of this chapter is the Thom Cobordism Theorem 280.27. Unfortunately we will lack the expertise to prove said theorem. But we will make very good use of it in the next chapter, when we will use it to prove the Hirzebruch Signature Theorem 281.14.

This chapter is organized as follows:

- In Section 280.1 we will introduce the (un-) oriented cobordism groups Ω_n and $\mathcal{N}_n$.
- In Section 280.2 we will use Euler characteristics to show that some smooth manifolds are non-trivial in the cobordisms groups.
- In Section 280.3 we will collect some basic facts about the cobordism groups in dimensions $1, 2$ and 3.
- In Section 280.4 we will use the signature (which is defined via the intersection form) to study the cobordism groups Ω_{4k} in dimensions $4k$.
- In Section 280.5 we will see that the individual cobordism groups Ω_n and $\mathcal{N}_n$ give rise to cobordism ring Ω_* and $\mathcal{N}_*$.
- In Section 280.6 we will formulate the Thom Cobordism Theorem 280.27.
- In Section 280.7 we will provide exercises for Chapter 280.

In the next Chapter 281 we will state and partially prove the Hirzebruch Signature Theorem 281.14.

280.1. **The cobordism groups.** In the following subsections we will first introduce the oriented cobordism groups Ω_n, $n \in \mathbb{N}_0$ and afterwards we will introduce the unoriented cobordism groups $\mathcal{N}_n$, $n \in \mathbb{N}_0$.

280.1.1. **The disjoint union monoid.** Here is a naive idea: Given two closed n-dimensional smooth manifolds M and N we obtain a new closed n-dimensional smooth manifold by taking the disjoint union $M \sqcup N$. This operation seems to have some structure:

- Taking the disjoint union with the empty smooth manifold has no real effect.
- At first glance the operation looks abelian and associative.

Thus it looks that all closed n-dimensional smooth manifolds together with the operation "$\sqcup$" form a monoid.

For the reader's convenience we recall the Definition 2.1 of a monoid.

Definition 280.1. A monoid is a *set* S together with a map

$$\begin{array}{ccc} S \times S & \to & S \\ (x, y) & \mapsto & x \cdot y \end{array}$$

such that the following two conditions are satisfied:[2996]

(1) there exists a neutral element $e \in S$, i.e. an element which satisfies $e \cdot x = x \cdot e = x$ for all $x \in S$,
(2) the operation "$\cdot$" is associative, i.e. for all $x, y, z \in S$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Furthermore:

(3) The monoid is called abelian if for all $x, y \in S$ we have $x \cdot y = y \cdot x$.

So, if we want to turn "all n-dimensional smooth manifolds" into a monoid, we first have to ensure that we are actually dealing with a set. This can be arranged by the following lemma:

Lemma 280.2. (Isomorphism Classes-of-Manifold Lemma) Let $n \in \mathbb{N}_0$.

(1) The homeomorphism classes of n-dimensional topological manifolds form a set.
(2) The diffeomorphism classes of n-dimensional smooth manifolds form a set.

The analogous statement also holds if we look at oriented topological or smooth manifolds.

Remark. Note that here it is essential that topological manifolds are assumed to be second-countable. If we did not make this assumption, then any set with the discrete topology would be a 0-dimensional topological manifold. It follows from the discussion on page 624 that for 0-dimensional topological manifolds a diffeomorphism is the same as a bijection. But the bijection classes of sets seemingly do not form a set. □

Sketchy proof. [2997]

(1) It follows from the Euclidean Embedding Theorem 33.2 (1) that every n-dimensional topological manifold is homeomorphic to a subspace of some $\mathbb{R}^k$.



Thus we can identify "homeomorphism classes of n-dimensional topological manifolds" with "homeomorphism classes of subspaces of some $\mathbb{R}^k$ that are n-dimensional topological manifolds". Since the power set of any $\mathbb{R}^k$ is a set, since the union of a family of sets is again a set and since given an equivalence relation, the equivalence classes form a set, we see that the homeomorphism classes of n-dimensional topological manifolds form a set.

(2) As in (1) we see, using the Euclidean Embedding Theorem 33.2 (2) and using the Smooth Embedding Theorem 30.15, that it suffices to show that the diffeomorphism classes of n-dimensional smooth manifolds, where the underlying topological space is a

[2996] A *semi-group* is defined similarly, except that we do not demand the existence of a neutral element.
[2997] An alternative approach to proving the lemma is sketched in Exercise 280.1.

subset of some $\mathbb{R}^k$, form a set. But this follows from the fact that given a submanifold of some $\mathbb{R}^k$ the possible maximal atlases form a set.

We leave it to our friends the logicians to make the term "form a set precise" and to turn the above sloppy argument into a proper proof. ■

The Isomorphism Classes-of-Manifold Lemma 280.2 motivates the following notation:

Definition 280.3. Let $n \in \mathbb{N}_0$.

(1) We denote by $\mathcal{M}_n$ the set of all diffeomorphism classes of closed n-dimensional smooth manifolds.

(2) We denote by $\mathcal{M}_n^{\text{or}}$ the set of all diffeomorphism classes of closed oriented n-dimensional smooth manifolds.

Now that we have a set to play with, let us recall Notation 1.12 of the disjoint union of two sets:

Notation 280.4. Given two sets A and B we refer to[2998]

$$A \sqcup B \;:=\; (A \times \{1\}) \cup (B \times \{2\})$$

as the disjoint union of A and B

We are now ready to formulate and prove the following not overly exciting lemma:

Lemma 280.5. (Manifold–Disjoint Union–Monoid Lemma) Let $n \in \mathbb{N}_0$. The sets $\mathcal{M}_n$ and $\mathcal{M}_n^{\text{or}}$ with the disjoint union operation form an abelian monoid where the neutral element is represented by the empty smooth manifold $\varnothing$.

Proof. In the following we deal with $\mathcal{M}_n$. The proof for $\mathcal{M}_n^{\text{or}}$ is almost the same.

(1) Given any n-dimensional smooth manifold M we have the diffeomorphisms

$$\begin{array}{ccc} M \sqcup \varnothing & \to & M \\ (P,1) & \mapsto & P \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \varnothing \sqcup M & \to & M \\ (P,2) & \mapsto & P. \end{array}$$

This shows that $[\varnothing]$ is indeed a neutral element.

(2) One can easily show that given any three n-dimensional smooth manifolds X, Y, Z we have a diffeomorphism $(X \sqcup Y) \sqcup Z \to X \sqcup (Y \sqcup Z)$. This shows that $(\mathcal{M}_n, \sqcup)$ is associative.

(3) One can easily show that given any two n-dimensional smooth manifolds X, Y we have a diffeomorphism $X \sqcup Y \to Y \sqcup X$. This shows that $(\mathcal{M}_n, \sqcup)$ is abelian. ■

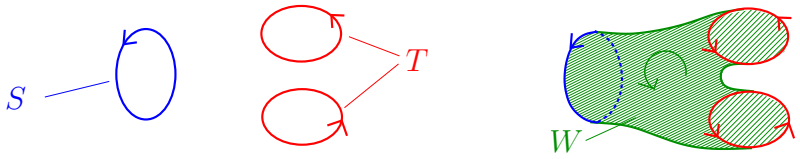
280.1.2. **The oriented cobordism group.** Surely there are people who a very excited about abelian monoids. But it does not feel like the monoids $(\mathcal{M}_n, \sqcup)$ and $(\mathcal{M}_n^{\text{or}}, \sqcup)$ tell us much about (oriented) n-dimensional smooth manifolds. To make things a little more interesting let us recall Definitions 263.1 and **??**:

[2998] Note that $A \times \{1\}$ and $B \times \{2\}$ are subsets of $A \times \{1,2\}$. Thus it make sense to take their disjoint union as in Notation 1.6.

Definition 280.6. Let $n \in \mathbb{N}_0$ and let S and T be two closed oriented n-dimensional smooth manifolds.

(1)
- A cobordism between S and T is a pair $(W, \Theta\colon \partial W \to S \sqcup -T)$ consisting of a compact oriented $(n+1)$-dimensional smooth manifold W together with an orientation-preserving diffeomorphism[2999] $\Theta\colon \partial W \xrightarrow{\cong} S \sqcup -T$.[3000]
- We refer to the maps $S \to S \sqcup -T \xrightarrow{\Theta^{-1}} \partial W$ and $T \to S \sqcup -T \xrightarrow{\Theta^{-1}} \partial W$ as the natural smooth embeddings.

(2) We say that S and T are cobordant if there exists a cobordism between S and T.



Example. Let S and T be two closed oriented n-dimensional smooth manifolds. If there exists an orientation-preserving diffeomorphism $f\colon S \to T$, then S and T are cobordant. Indeed, a cobordism is given by $W = [0,1] \times S$ together with the orientation-preserving diffeomorphism from $\partial W = \partial([0,1] \times S) = (\{1\} \times S) \cup (-\{0\} \times S)$ to $S \cup -T$ that is given by the maps $\{1\} \times S \xrightarrow{(1,s) \mapsto s} S$ and $\{0\} \times S \xrightarrow{(0,s) \mapsto f(s)} T$. □

Lemma 280.7. (Cobordism Group Lemma) Let $n \in \mathbb{N}_0$.

(1) "Being cobordant" defines an equivalence relation "$\sim$" on the set $\mathcal{M}_n^{\mathrm{or}}$.

(2) The map
$$\begin{array}{rcl} \mathcal{M}_n^{\mathrm{or}} \times \mathcal{M}_n^{\mathrm{or}} & \to & \mathcal{M}_n^{\mathrm{or}} \\ ([S],[T]) & \mapsto & [S \sqcup T] \end{array}$$
is well-defined and it descends to an abelian group structure on the set $\mathcal{M}_n^{\mathrm{or}} / \sim$[3001] where the neutral element is given by the empty smooth manifold and where given $M \in \mathcal{M}_n^{\mathrm{or}}$ the inverse of $[M]$ is given by $[-M]$.

Proof.

(1) First we need to get the following claim out of the way:

Claim. "Being cobordant" defines a relation "$\sim$" on the set $\mathcal{M}_n^{\mathrm{or}}$.

Proof. We need to show that if we have two pairs of diffeomorphic oriented smooth manifolds M_1, M_2 and N_1, N_2 and if M_1 is cobordant to N_1, then M_2 is also cobordant to N_2. Thus let $f\colon M_1 \to M_2$ be a diffeomorphism and let $g\colon N_1 \to N_2$ be a diffeomorphism. If $(W, \Theta\colon \partial W \xrightarrow{\cong} M_1 \sqcup -N_1)$ is a cobordism between the oriented smooth manifolds M_1 and N_1, then clearly $(W, (f \sqcup g) \circ \Theta\colon \partial W \to M_2 \sqcup -N_2)$ is a cobordism between the oriented smooth manifolds M_2 and N_2. ⊞

[3000] Recall that it follows from the Boundary Orientation Lemma 31.38 that the boundary ∂W inherits a natural orientation from the orientation of W.

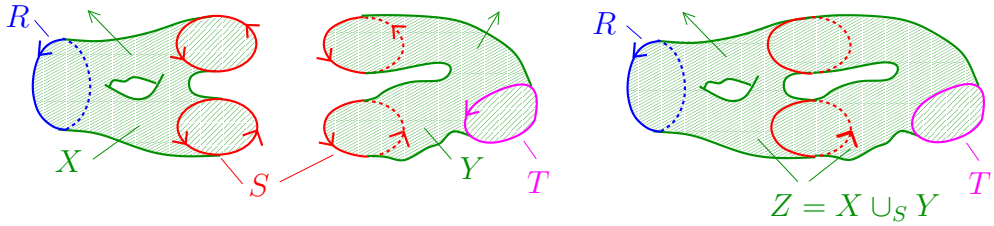
[3001] By the Isomorphism Classes-of-Manifold Lemma 280.2 we know that $\mathcal{M}_n^{\mathrm{or}}$ and thus also $\mathcal{M}_n^{\mathrm{or}}/\sim$ are a set.

Next we will show that the relation $\sim$ on $\mathcal{M}_n^{\mathrm{or}}$ is an equivalence relation in the sense of Definition 1.16. In other words, we will show that the relation is reflexive, symmetric and transitive:

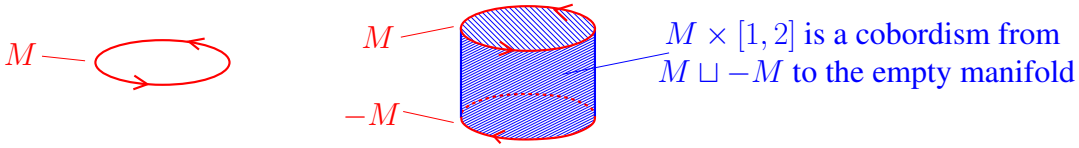
(a) Given $[M] \in \mathcal{M}_n^{\mathrm{or}}$ we have $[M] \sim [M]$ since, by the discussion on page 832, we have an equality $\partial([1,2] \times M) = \{1\} \times (-M) \cup \{2\} \times M$ of oriented smooth manifolds.

(b) Suppose that we have $[S], [T] \in \mathcal{M}_n^{\mathrm{or}}$ with $[S] \sim [T]$. We need to show that $[T] \sim [S]$. Our hypothesis implies that there exists a compact oriented $(n+1)$-dimensional smooth manifold W together with an orientation-preserving diffeomorphism $\Theta\colon \partial W \xrightarrow{\cong} S \sqcup (-T)$. We now consider $-W$. Clearly we have an orientation diffeomorphism $\partial(-W) \xrightarrow{\Theta} -S \sqcup T \xrightarrow{\cong} T \sqcup (-S)$ which shows that $[T] \sim [S]$.

(c) Finally suppose that we have $[R], [S], [T] \in \mathcal{M}_n^{\mathrm{or}}$ with $[R] \sim [S]$ and $[S] \sim [T]$. We claim that $[R] \sim [T]$, i.e. we claim that R is cobordant to T. Our hypothesis means that there exist compact oriented $(n+1)$-dimensional smooth manifolds X and Y with an orientation-preserving diffeomorphism $f\colon \partial X \xrightarrow{\cong} R \sqcup -S$ and an orientation-preserving diffeomorphism $g\colon \partial Y \xrightarrow{\cong} S \sqcup -T$. As in the proof of the Cobordant–Transitive Lemma **??** we set $Z \coloneqq X \cup_S Y$ where we glue S to X and Y via the maps $f^{-1}|_S$ and $g^{-1}|_S$. It follows from the Gluing-Smooth Manifolds-Proposition 54.5 that Z is a compact orientable $(n+1)$-dimensional smooth manifold which admits an orientation-preserving diffeomorphism $\partial Z \xrightarrow{\cong} R \sqcup -T$.



(2) We first observe that the disjoint union operation clearly descends to a well-defined operation on $\mathcal{M}_n^{\mathrm{or}}$. Next we need to show that $(\mathcal{M}_n^{\mathrm{or}}, \sqcup)$ is an abelian group. It follows easily from the Manifold–Disjoint Union–Monoid Lemma 280.5 that $(\mathcal{M}_n^{\mathrm{or}}, \sqcup)$ is an abelian monoid. It remains to show that $(\mathcal{M}_n^{\mathrm{or}}, \sqcup)$ is actually a group. In other words, we need to show that any $[M] \in \mathcal{M}_n^{\mathrm{or}}$ has an inverse element. In fact the inverse is represented by the oriented smooth manifold $-M$, since by the discussion on page 832, we have $\partial([1,2] \times M) = \{1\} \times (-M) \cup \{2\} \times M$ which admits an obvious orientation-preserving diffeomorphism to $(M \sqcup -M) \sqcup \varnothing$.



This shows that the smooth manifold $[1,2] \times M$ defines a cobordism from $M \sqcup -M$ to the empty smooth manifold, i.e. $[M \sqcup -M] \sim [\varnothing]$. ■

The Cobordism Group Lemma 280.7 leads us to the following definition.

Definition 280.8. Let $n \in \mathbb{N}_0$. We refer to

$$\Omega_n \;\coloneqq\; (\mathcal{M}_n^{\mathrm{or}}/\!\sim, \sqcup)$$

as the n-th oriented cobordism group. Very often we will also just talk of the n-th cobordism group.

Convention 280.9. In the following we will often be slightly sloppy in our notation, meaning that on many occasions we will not distinguish between a (closed oriented) smooth manifold M, its diffeomorphism class $[M] \in \mathcal{M}_n^{\mathrm{or}}$ and its cobordism class $[M] \in \Omega_n$. For example, given oriented smooth manifolds W, S and T we often write $\partial W = S \sqcup -T$, even though strictly speaking we only have an orientation-preserving diffeomorphism $\partial W \to S \sqcup -T$. In other words, we will use the natural smooth embeddings from Definition 280.6 to identify S and T with submanifolds of ∂W.

We conclude this introduction to the oriented cobordism group with an extended observation: We have another natural operation on oriented connected smooth manifolds, namely the connected sum operation. The following lemma says that up to cobordisms the connected sum operation is the same as the disjoint union:

Lemma 280.10. (Cobordism Sum Lemma) Let $n \in \mathbb{N}_0$ and let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds, then

$$M \# N \;=\; M \sqcup N \;\in\; \Omega_n.$$

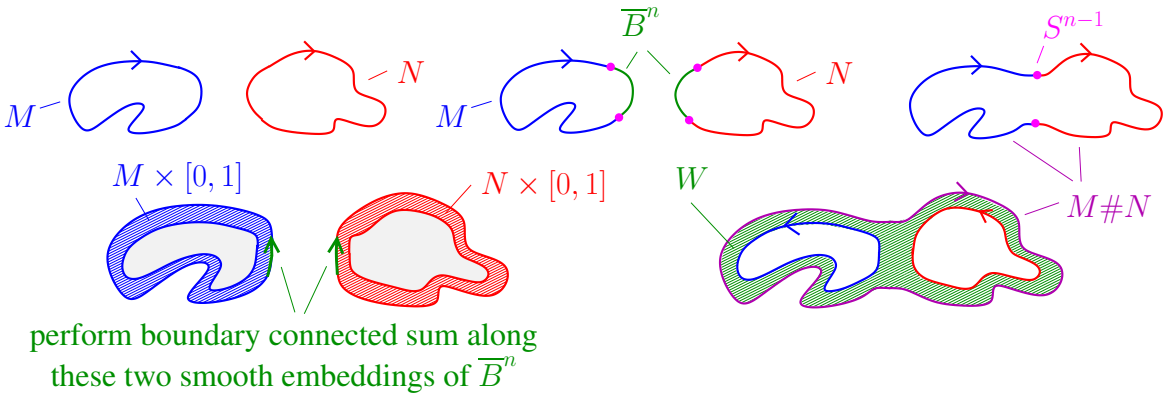
Sketch of proof. A proof of this lemma is provided by the combination of the Connected Sum–Surgery Lemma **??** and of the Trace-of-Surgery Proposition **??**. For the reader's convenience we also provide a direct proof. Thus let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to N$ be smooth embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing.

We consider $[0,1] \times M$ and $[0,1] \times N$. Note that by the discussion on page 832, we have $\partial([0,1] \times M) = (\{0\} \times (-M)) \cup (\{1\} \times M)$ and $\partial([0,1] \times N) = (\{0\} \times (-N)) \cup (\{1\} \times N)$.

We use the smooth embeddings $\overline{B}^n \xrightarrow{x \mapsto (\varphi(x),1)} M \times \{1\}$ and $\overline{B}^n \xrightarrow{x \mapsto (\psi(x),1)} N \times \{1\}$ to form the boundary connected sum

$$\begin{aligned} W &:= ([0,1] \times M) \#_{\mathrm{b}} ([0,1] \times N) \\ &:= ([0,1] \times M) \sqcup ([0,1] \times N) / (\varphi(x),1) = (\psi(x),1) \text{ for } x \in \overline{B}^n \end{aligned}$$

that we introduced in Definition 61.10.



By the Boundary Connected Sum Proposition 61.11 (1) and (2) we know that W admits the structure of an orientable $(n+1)$-dimensional smooth manifold such that

$$\partial W \;=\; \underbrace{(\{1\}\times M)\#(\{1\}\times N)}_{=M\#N} \sqcup \underbrace{(-\{0\}\times M)}_{=-M} \sqcup \underbrace{(-\{0\}\times N)}_{=-N}.$$

We have thus found the desired cobordism. ∎

280.1.3. **The unoriented cobordism group.** In the previous subsection we saw that closed oriented n-dimensional smooth manifolds give rise to the oriented cobordism group Ω_n. In this subsection we will drop the adjective "oriented" and we are lead to the notion of the unoriented cobordism group:

Definition 280.11. If we use exactly the same definitions as before except that we drop the condition that the smooth manifolds are orientable and we drop the condition that the cobordisms are orientable, then given $n \in \mathbb{N}_0$ we obtain the n-th unoriented cobordism group

$$\mathcal{N}_n \;:=\; (\mathcal{M}_n/\,\text{(unoriented) cobordism}, \sqcup).$$

The following lemma shows that the unoriented cobordism groups are fairly dull abelian groups:

Lemma 280.12. (Unoriented Cobordism–Order Two-Lemma) Let $n \in \mathbb{N}_0$. Every non-trivial element in $\mathcal{N}_n$ is of order two, i.e. given any $M \in \mathcal{N}_n$ we have $M \sqcup M = 0 \in \mathcal{N}_n$.

Proof. Let $n \in \mathbb{N}_0$ and let $M \in \mathcal{N}_n$. The lemma follows from the following equalities in the group $\mathcal{N}_n$:

$$2\cdot M \;=\; M \sqcup M \;\underset{\text{by Notation 280.4}}{\uparrow}{=}\; (\{1\}\times M)\sqcup(\{2\}\times M) \;\underset{\text{by the Smooth Manifold Product Proposition 25.34 we have this equality of unoriented smooth manifolds}}{\uparrow}{=}\; \partial([1,2]\times M) \;=\; 0 \;\in\; \mathcal{N}_n.$$

∎

We conclude this subsection with the following lemma which is the obvious analog of the Cobordism Sum Lemma 280.10:

Lemma 280.13. (Unoriented Cobordism Sum Lemma) Let $n \in \mathbb{N}_0$. Let M and N be two closed connected n-dimensional smooth manifolds.[3002] Then

$$M\#N \;=\; M \sqcup N \;\in\; \mathcal{N}_n.$$

Proof. The proof of this lemma is verbatim the same as in the Cobordism Sum Lemma 280.10. ∎

280.2. **The cobordism groups and Euler characteristics.** In this section we will see that the Euler characteristic can be used to show that at least some cobordism groups are non-trivial:

Proposition 280.14. (Cobordism Group-χ-Proposition)
(1) Let $m \in \mathbb{N}_0$. The map

$$\begin{array}{rccc} \chi\colon & \mathcal{N}_{2m} & \to & \mathbb{Z}/2\mathbb{Z} \\ & [M] & \mapsto & \chi(M) \end{array}$$

[3002] If M and N are both orientable, then the connected sum $M\#N$ is only well-defined once we pick an orientation for M and N. If M and N are orientable, then the equality is understood as holding for the Definition 54.12 of the connected sum for any choice of orientation for M and N.

is well-defined and it is surjective, in fact $\chi(\mathbb{R}\mathrm{P}^{2m}) = 1$. In other words, the map is an epimorphism.

(2) Let $m \in \mathbb{N}_0$. The map $\Omega_{4m} \to \mathcal{N}_{4m} \xrightarrow{\chi} \mathbb{Z}/2\mathbb{Z}$ is an epimorphism.

Proof.

(1) First note that for any two closed smooth manifolds M and N we clearly have the equality $\chi(M \sqcup N) = \chi(M) + \chi(N)$. It follows easily from this observation together with the following claim that the map $\chi\colon \mathcal{N}_{2m} \to \mathbb{Z}/2\mathbb{Z}$ is well-defined.

Claim. If X is a compact smooth manifold of dimension $2m+1$, then $\chi(\partial X)$ is even.

Proof. This claim is just a reformulation of the Boundary-χ-Corollary 210.10. For the reader's convenience we sketch the argument again. We perform the following calculation:

$$\chi(X) \underset{\uparrow}{=} \sum_{i=0}^{2n+1} (-1)^i \cdot \dim_{\mathbb{F}_2}(\mathrm{H}_i(X;\mathbb{F}_2)) \underset{\uparrow}{=} \sum_{i=0}^{2n+1} (-1)^i \cdot \dim_{\mathbb{F}_2}(\mathrm{H}_{2n+1-i}(X,\partial X;\mathbb{F}_2))$$

by the Manifold-χ Proposition 190.24 (4)

by the Homology Symmetry Theorem 210.6, which is a consequence of the Poincaré Duality Theorem 210.1 and the Universal Coefficient-H*–Field Theorem 202.2

$$\underset{\uparrow}{=} -\sum_{j=0}^{2n+1} (-1)^j \cdot \dim_{\mathbb{F}_2}(\mathrm{H}_j(X,\partial X;\mathbb{F}_2)) \underset{\uparrow}{=} -\chi(X,\partial X) \underset{\uparrow}{=} -\chi(X) + \chi(\partial X).$$

follows from the substitution $j = 2n+1-i$ together with the fact that $(-1)^{2n+1} = -1$

by the Manifold-χ Proposition 190.24 (3)

follows from the Relative-χ-Lemma 156.8

Solving for $\chi(\partial X)$ we see that $\chi(\partial X) = 2 \cdot \chi(X)$ is indeed even. ⊞

We already mentioned that $\chi(M \sqcup N) = \chi(M) + \chi(N)$. This of course implies that $\chi\colon \mathcal{N}_{2m} \to \mathbb{Z}/2\mathbb{Z}$ is actually a homomorphism. Finally recall that on page 2988 we showed that we have $\chi(\mathbb{R}\mathrm{P}^{2n}) = \frac{1}{2}(1 + (-1)^{2n}) = 1$. This shows that $\chi\colon \mathcal{N}_{2m} \to \mathbb{Z}/2\mathbb{Z}$ is indeed an epimorphism.

(2) On page 2988 we showed that $\chi(\mathbb{C}\mathrm{P}^{2m}) = 2m+1$. Since $\mathbb{C}\mathrm{P}^{2m}$ is a closed *orientable* $4m$-dimensional smooth manifold we see that the map $\Omega_{4m} \to \mathcal{N}_{4m} \xrightarrow{\chi} \mathbb{Z}/2\mathbb{Z}$ is an epimorphism. ■

280.3. **Cobordism group in low dimensions.** In the following proposition we summarize the computations of the low-dimensional (unoriented) cobordism groups.

Proposition 280.15. (Low Cobordism Group Proposition) We have the following table of isomorphism types of the (unoriented) cobordism groups for $n = 0, 1, 2, 3$:

	$n=0$	$n=1$	$n=2$	$n=3$
Ω_n	$\mathbb{Z}$	0	0	0
$\mathcal{N}_n$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z} \cdot [\mathbb{R}\mathrm{P}^2]$	0

Proof.

($n=0$) It follows easily from the Smooth 1-Manifold Classification Theorem 28.8 together with the discussion on page 825 that the maps[3003]

$$\begin{array}{rcl} \Omega_0 & \to & \mathbb{Z} \\ \coprod_{i=1}^{k} \{(P_i, \epsilon_i)\} & \mapsto & \sum_{i=1}^{k} \epsilon_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{N}_0 & \to & \mathbb{Z}/2\mathbb{Z} \\ \coprod_{i=1}^{k} \{P_i\} & \mapsto & k \bmod 2 \end{array}$$

are isomorphisms.

($n=1$) In the Smooth 1-Manifold Classification Theorem 28.8 we saw that every closed connected non-empty 1-dimensional smooth manifold is diffeomorphic to S^1. Evidently $S^1 = \partial \overline{B}^2$[3004]. It follows that every (oriented) compact 1-dimensional smooth manifold represents the trivial element in Ω_1 and in $\mathcal{N}_1$. Therefore we see that $\Omega_1 = 0$ and $\mathcal{N}_1 = 0$.

($n=2$) In the Surface Classification Theorem 101.16 we saw that every closed orientable 2-dimensional smooth manifold is diffeomorphic to the surface of some genus $g \in \mathbb{N}_0$. By the Boundary Connected Sum Proposition 61.11 (3) the surface of genus g is the boundary of an orientable 3-dimensional genus g handlebody in the sense of Definition 247.15. Thus we see that $\Omega_2 = 0$.

We turn to the calculation of $\mathcal{N}_2$. We want to prove the following claim:

Claim. The map

$$\begin{array}{rcl} \varphi\colon \mathbb{Z}/2\mathbb{Z} & \to & \mathcal{N}_2 \\ [n] & \mapsto & n \cdot \mathbb{R}\mathrm{P}^2 \end{array}$$

is an isomorphism.

Proof. We prove the claim in several steps:

(a) First we show that φ is a homomorphism: We denote by $\psi\colon \mathbb{Z} \to \mathcal{N}_2$ the map that is given by $n \mapsto n \cdot \mathbb{R}\mathrm{P}^2$. Since $\mathbb{Z}$ is a free group this is a well-defined homomorphism. It remains to show that ψ in fact factors through the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$. But that is an immediate consequence of the Unoriented Cobordism–Order Two-Lemma 280.12.

(b) Next we show that $\varphi\colon \mathbb{Z}/2\mathbb{Z} \to \mathcal{N}_2$ is a monomorphism. We only need to show that $\varphi(1) = \mathbb{R}\mathrm{P}^2$ represents a non-trivial element in $\mathcal{N}_2$. But this follows immediately from the Cobordism Group-χ-Proposition 280.14 and the observation that $\chi(\mathbb{R}\mathrm{P}^2) = 1 \neq 0 \in \mathbb{Z}/2\mathbb{Z}$.

(c) It remains to show that φ is an epimorphism. So let M be a compact 2-dimensional smooth manifold representing an element in $\mathcal{N}_2$. It follows from the Unoriented Cobordism Sum Lemma 280.13 that without loss of generality we can assume that M is connected. If M is orientable, then we already saw above that M is the boundary of a compact 3-dimensional smooth manifold, hence $[M] = 0 \in \mathcal{N}_2$.

Finally we consider the case that M is non-orientable. We have the following equalities in $\mathcal{N}_2$:

[3003]Recall that an orientation for a 0-dimensional smooth manifold M consists of a choice of a sign in $\{\pm 1\}$ for each point in M, see the remark on page 810.

[3004]This is an equality of oriented smooth manifolds if we equip S^1 and $\overline{B}^2$ with the usual orientation and also if we equip both smooth manifolds with the opposite orientation.

$$[M] \underset{\substack{\uparrow\\ \text{by the Surface Classification Theorem 101.16 we know that there exists an } n\in\mathbb{N}_0 \text{ such that } M \text{ is diffeomorphic to the connected sum of } n \text{ copies of } \mathbb{R}\mathrm{P}^2}}{=} [\text{connected sum of } n \text{ copies of } \mathbb{R}\mathrm{P}^2] \overset{\substack{\downarrow\\ \text{by the Unoriented Cobordism Sum Lemma 280.13}}}{=} n\cdot[\mathbb{R}\mathrm{P}^2] \underset{\substack{\uparrow\\ \text{by the Unoriented Cobordism Order Two-Lemma 280.12}}}{=} (n \bmod 2)\cdot[\mathbb{R}\mathrm{P}^2].$$

This completes the proof that φ is an epimorphism.

($n=3$) Later, in the 3-Manifolds Bound Theorem **??** we will show that $\Omega_3 = 0$. The fact that $\mathcal{N}_3 = 0$ was shown by Albrecht Dold [**Dol1956**] in 1956. An alternative proof is given in [**Lic1963**, Theorem 4.]. ■

280.4. **The signature and the cobordism group in dimension $4k$.** In the last section we have just seen that the (unoriented) cobordism groups for $n = 0, 1, 2, 3$ are not particularly exciting. Our next goal is to study Ω_4. In the Cobordism Group-χ-Proposition 280.14 we showed that the Euler characteristic defines an epimorphism $\chi\colon \Omega_4 \to \mathbb{Z}/2\mathbb{Z}$ and on page 2988 we saw that $\chi(\mathbb{C}\mathrm{P}^2) = 3 \equiv 1 \neq 0 \in \mathbb{Z}/2\mathbb{Z}$. But from the Cobordism Group-χ-Proposition 280.14 we cannot conclude whether or not $\mathbb{C}\mathrm{P}^2$ is an element of infinite order in Ω_4.

To settle this question it is necessary to recall the Definition 224.12 of the intersection form and Definition 226.15 of the signature of a $4k$-dimensional topological manifold:

Definition 280.16. Let $k \in \mathbb{N}_0$.

(1) Let M be a compact oriented $2k$-dimensional topological manifold. We refer to the map

$$\begin{aligned} Q_M\colon \mathrm{FH}_k(M)\times\mathrm{FH}_k(M) &\to \mathbb{Z}\\ (a,b) &\mapsto \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2k}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{aligned}$$

as the intersection form of M.

(2) Let M be a compact oriented $4k$-dimensional topological manifold. Recall that this form is $(-1)^k$-symmetric. We define the signature of M to be

$$\mathrm{sign}(M) \coloneqq \mathrm{sign}(Q_M) \coloneqq \mathrm{sign}(Q_M \otimes \mathbb{R}).$$

Here we refer to Definition 226.14 for the definition of the real form $Q_M \otimes \mathbb{R}$ and we refer to Definition 226.8 for the definition of the signature of a form over $\mathbb{R}$.

The following proposition is the key to showing that $\mathbb{C}\mathrm{P}^2$ represents in fact an element of infinite order in Ω_4:

Proposition 280.17. (Boundary-Signature Proposition) Let $k \in \mathbb{N}_0$ and let M be a closed oriented $4k$-dimensional topological manifold.[3005] If there exists a compact orientable topological manifold X with $\partial X = M$, then $\mathrm{sign}(M) = 0$.[3006]

It will take us a while to prove the Boundary-Signature Proposition 280.17, therefore we first provide the following corollary:

Corollary 280.18. (Signature–Cobordism Corollary) For any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} \text{sign}\colon & \Omega_{4k} & \to & \mathbb{Z} \\ & [M] & \mapsto & \text{sign}(M) \end{array}$$

is well-defined, it is a homomorphism and it is surjective, in fact we have $\text{sign}(\mathbb{C}\text{P}^{2k}) = 1$. In other words, the map is an epimorphism.[3007]

Proof of Corollary 280.18. First we need to show that the map $\text{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is well-defined. Thus suppose that we are given closed oriented $4k$-dimensional smooth manifolds M and N with $[M] = [N] \in \Omega_{4k}$. By Definition 280.8 of Ω_{4k} there exists a compact oriented smooth manifold X with $\partial X = M \sqcup -N$. We have the following equalities:

$$\begin{array}{ccccccccc} 0 & \underset{\uparrow}{=} & \text{sign}(\partial X) & = & \text{sign}(M \sqcup (-N)) & \underset{\uparrow}{=} & \text{sign}(M) + \text{sign}(-N) & \underset{\uparrow}{=} & \text{sign}(M) - \text{sign}(N). \\ & & \text{by the Boundary-Signature Proposition 280.17} & & & & \text{by the Manifold Signature Additivity Lemma 226.17 together with the Manifold–Orientation–Signature Lemma 226.16 (1)} & & \end{array}$$

This shows that the map $\text{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is well-defined. It follows from the Manifold Signature Additivity Lemma 226.17 that the map $\text{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is a homomorphism. Finally it follows immediately from the Cup Product-$\mathbb{C}\text{P}^n$-Proposition 213.7 that $\text{sign}(\mathbb{C}\text{P}^{2k}) = \pm 1$,[3008] which shows that the map $\text{sign}\colon \Omega_{4k} \to \mathbb{Z}$ is an epimorphism. ■

The proof of the Boundary-Signature Proposition 280.17 will require the remainder of this section. In the next Section 280.5 we will introduce the cobordism rings Ω_* and $\mathcal{N}_*$.

The remainder of this section is organized as follows:

- In Subsection 280.4.1 we will (re-) introduce the purely algebraic notion of a metabolizer.
- In Subsection 280.4.2 we will formulate and prove the Half-Live Half-Die Theorem 280.22.
- In Subsection 280.4.3 we will conclude the proof of the Boundary-Signature Proposition 280.17.

280.4.1. **Metabolizers.** We start out with the following algebraic concept:

Definition 280.19. Let $(V, g\colon V \times V \to \mathbb{R})$ be a symmetric form over $\mathbb{R}$.

(1) A metabolizer for (V, g) is a vector subspace W with $g|_W = 0$ and which satisfies $\dim_{\mathbb{R}}(W) = \frac{1}{2}\dim_{\mathbb{R}}(V)$.
(2) We say that the form is metabolic if it admits a metabolizer.[3009]

Next we can recall a lemma from earlier on:

Lemma 227.6. (Vanishing Signature Lemma) Let (V, g) be a non-singular symmetric form over $\mathbb{R}$. If the form is metabolic, then $\text{sign}(g) = 0$.

Proof of Lemma 227.6. For the reader's convenience we provide the proof of the lemma again. Note that it follows from our hypothesis that $\dim_{\mathbb{R}}(W) = \frac{1}{2}\dim_{\mathbb{R}}(V)$ that we can find a basis $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ for V such that $\{a_1, \ldots, a_k\}$ is a basis for W. Since

[3007] In fact the proof of the Low Cobordism Group Proposition 280.15 shows that for $k = 0$ this homomorphism is in fact an isomorphism.
[3008] In fact the more fancy Cup Product-of-Duals-in-$\mathbb{C}\text{P}^n$-Lemma 220.21 says that $\text{sign}(\mathbb{C}\text{P}^{2k}) = +1$, but this is irrelevant for the proof.
[3009] One can easily verify that this definition is consistent with Definition 227.5.

$g|_W = 0$ and since g is symmetric we see that with respect to this basis the form g is given by a matrix of the form $\begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & Q \end{pmatrix}$ where Q is a symmetric matrix. It follows from the hypothesis that g is non-singular and the Non-Singularity Lemma 213.2 (1) that $\det(P) \neq 0$Now we consider the following three transformations

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & P \\ P^{\mathrm{T}} & Q \end{pmatrix} \begin{pmatrix} (P^{\mathrm{T}})^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix},$$
$$\begin{pmatrix} \mathrm{id} & 0 \\ -\frac{1}{2}Q & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\frac{1}{2}Q \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix}.$$

It follows from the Form–Matrix-Lemma 224.5 (1b) that the three matrices to the right all define isometric forms. Considering the last matrix we obtain from the Signature-Eigenvalues Proposition 226.11 that the signature is zero. ∎

280.4.2. **The Half-Live Half-Die Theorem.** Before we get into the heart of this subsection, let us introduce the following variation on the usual intersection form Q_M:

Definition 280.20. Let $k \in \mathbb{N}_0$ and let M be a compact oriented $2k$-dimensional topological manifold. For any field $\mathbb{F}$ we refer to the map

$$\begin{aligned} Q_M^{\mathbb{F}} \colon \mathrm{H}_k(M;\mathbb{F}) \times \mathrm{FH}_k(M;\mathbb{F}) &\to \mathbb{F} \\ (a,b) &\mapsto \big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{F})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{F})}}_{\in \mathrm{FH}^{2k}(M,\partial M;\mathbb{F})}, [M] \big\rangle \end{aligned}$$

as the $\mathbb{F}$-valued intersection form of M.

The Vanishing Signature Lemma 227.6 teaches us that if we want to show that the signature of a form is zero, then we should better find a metabolizer. The next proposition gives us subspaces on which the various flavors of intersection forms do indeed vanish:

Proposition 280.21. (Intersection Form–Kernel-Proposition) Let $k \in \mathbb{N}$ and let X be a compact orientable topological manifold of dimension $2k+1$. As usual we denote by $i \colon \partial X \to X$ the inclusion map.

(1) For any $a, b \in \ker(i_* \colon \mathrm{H}_k(\partial X) \to \mathrm{H}_k(X))$ we have $Q_{\partial X}(a,b) = 0$.[3010]
(2) For any field $\mathbb{F}$ and any two elements $a, b \in \ker(i_* \colon \mathrm{H}_k(\partial X;\mathbb{F}) \to \mathrm{H}_k(X);\mathbb{F})$ we have $Q_{\partial X}^{\mathbb{F}}(a,b) = 0$.[3011]

Proof. Let $k \in \mathbb{N}$ and let X be a compact orientable topological manifold of dimension $2k+1$. As usual we denote by $i \colon \partial X \to X$ the inclusion map. In the following we will only deal with Statement (1) regarding $\mathbb{Z}$-coefficients. The reader will have no troubles with making the straightforward adjustments needed to obtain a proof for Statement (2) that deals with $\mathbb{F}$-coefficients.

[3011] Here $Q_{\partial X}$ denotes the $\mathbb{Z}$-valued intersection form that we introduced on Definition 222.3. Also recall that in Definition 226.33 we introduced the $\mathbb{F}_2$-valued intersection form.

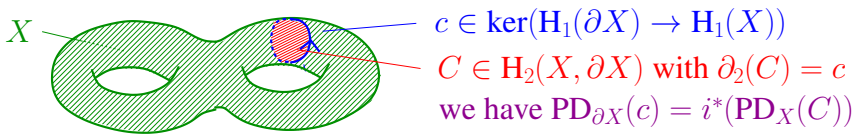
We start our proof of Statement (1) with the following claim:

Claim. Given any $c \in \ker(\mathrm{H}_k(\partial X) \to \mathrm{H}_k(X))$ there exists a cohomology class $\theta \in \mathrm{H}^k(X;\mathbb{Z})$ such that $\mathrm{PD}_{\partial X}(c) = i^*(\theta)$.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}^k(X,\partial X;\mathbb{Z}) & \longrightarrow & \mathrm{H}^k(X;\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{H}^k(\partial X;\mathbb{Z}) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(X,\partial X;\mathbb{Z}) & \longrightarrow & \ldots \\
 & & \cong\downarrow{\scriptstyle -\cap[X]} & & \cong\downarrow{\scriptstyle -\cap[X]} & & \cong\downarrow{\scriptstyle -\cap[\partial X]} & & \cong\downarrow{\scriptstyle -\cap[X]} & & \\
\ldots & \longrightarrow & \mathrm{H}_{k+1}(X) & \longrightarrow & \mathrm{H}_{k+1}(X,\partial X) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(\partial X) & \xrightarrow[c\mapsto 0]{i_*} & \mathrm{H}_k(X) & \longrightarrow & \ldots
\end{array}$$

where the horizontal sequences are the long exact sequences of the pair $(X,\partial X)$ in homology and cohomology.



By the Poincaré Duality Theorem 210.1 and by the Poincaré Duality–Boundary Proposition 211.14 we know that the vertical maps are isomorphisms and we know that the diagram commutes up to signs. The claim follows almost immediately from the above discussion. ⊞

Now let $a, b \in \ker(\mathrm{H}_k(\partial X) \to \mathrm{H}_k(X))$. By the claim there exist cohomology classes $\varphi, \psi \in \mathrm{H}^k(X;\mathbb{Z})$ such that $\mathrm{PD}_{\partial X}(a) = i^*(\varphi)$ and $\mathrm{PD}_{\partial X}(b) = i^*(\psi)$. Now we calculate that

$$\begin{array}{rcl}
Q_{\partial X}(a,b) & = & \langle \mathrm{PD}_{\partial X}(a) \cup \mathrm{PD}_{\partial X}(b), [\partial X]\rangle_{\partial X} \\
 & = & \langle i^*(\varphi) \cup i^*(\psi), [\partial X]\rangle_{\partial X} \\
 & = & \langle \varphi \cup \psi, i_*([\partial X])\rangle_X \\
 & = & \langle \varphi \cup \psi, 0\rangle_X \\
 & = & 0.
\end{array}$$

by the Cup Product Naturality Lemma 203.9

by the Kronecker Pairing Natural Lemma 199.9 (1)

by the Topological Kernel-of-Boundary Corollary 194.14

■

The Intersection Form–Kernel-Proposition 280.21 raises the question: How big are these kernels? The following theorem gives us the satisfying answer, that they have just the right size:

Theorem 280.22. (Half-Live Half-Die Theorem) Let X be a compact orientable topological manifold of dimension $2k+1$. We denote by $i\colon \partial X \to X$ the inclusion map.

(1) We have the following equality:[3012]

$$\mathrm{rank}(\ker(i_*\colon \mathrm{H}_k(\partial X) \to \mathrm{H}_k(X))) = \tfrac{1}{2}\,\mathrm{rank}(\mathrm{H}_k(\partial X)).$$

(2) For any field $\mathbb{F}$ we have the following equalities:

$$\dim_{\mathbb{F}}(\ker(i_*\colon \mathrm{H}_k(\partial X;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F}))) = \tfrac{1}{2}\dim_{\mathbb{F}}(\mathrm{H}_k(\partial X;\mathbb{F})).$$

Remark. Given any homomorphism $\varphi\colon V \to W$ between finite-dimensional vector spaces the equality $\dim_{\mathbb{F}}(\ker(\varphi)) + \dim_{\mathbb{F}}(\mathrm{im}(\varphi)) = \dim_{\mathbb{F}}(V)$ holds. Thus statement (2) of the

[3012] We will see in Exercise 280.10 that in general it is not true that $\ker(i_*\colon \mathrm{H}_k(\partial X) \to \mathrm{H}_k(X))$ is a subsummand of $\mathrm{H}_k(\partial X)$.

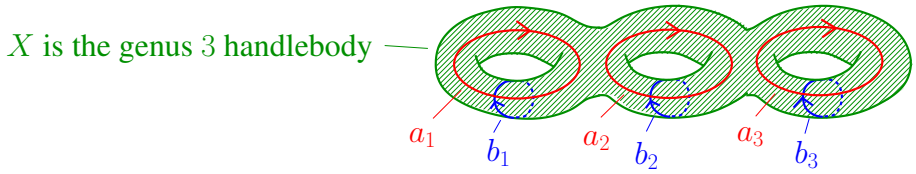
theorem can be reformulated as saying that

$$\dim_{\mathbb{F}}\big(\mathrm{im}(i_*\colon \mathrm{H}_k(\partial X;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F}))\big) \;=\; \dim_{\mathbb{F}}\big(\ker(i_*\colon \mathrm{H}_k(\partial X;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F}))\big).$$

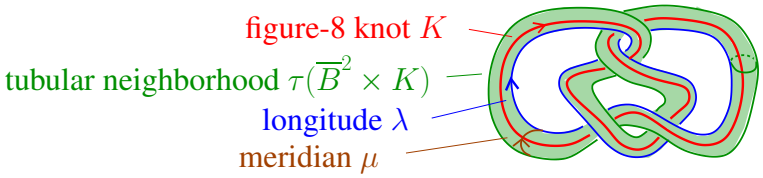
This explains why the theorem is called the "Half-Live Half-Die Theorem". □

Examples.

(1) We consider the 3-dimensional genus 3 handlebody X that is shown in the figure below. In the figure below we also show the usual basis $a_1, a_2, a_3, b_1, b_2, b_3$ for $\mathrm{H}_1(\partial X;\mathbb{R})$. In this case the kernel of the inclusion-induced map $\mathrm{H}_1(\partial X;\mathbb{R}) \to \mathrm{H}_1(X;\mathbb{R})$ is the three dimensional vector subspace that is generated by b_1, b_2 and b_3.



(2) Let $K \subseteq S^3$ be a knot. We pick a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ and as in Definition 108.30 we consider the corresponding knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$.



The boundary of X_K is a torus. In Exercise 280.11 we will see that there exist two closed oriented smoothly embedded curves μ and λ on ∂X_K such that $[\mu], [\lambda]$ form a basis of $\mathrm{H}_1(\partial X_K)$ and such that $\ker(\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K)) = \mathbb{Z} \cdot \lambda$. □

Proof. Let X be a compact orientable $(2k+1)$-dimensional topological manifold. We denote by $i\colon \partial X \to X$ the inclusion map and we write $M = \partial X$. In the following we first prove Statement (2) of the theorem.

Thus let $\mathbb{F}$ be any field. Until further notice, we henceforth work with $\mathbb{F}$-coefficients for all (co-) homology groups. For space reasons we do not mention the $\mathbb{F}$-coefficients explicitly. We consider the long exact sequence in homology of the pair (X, M):

$$\to \mathrm{H}_{k+1}(M) \to \mathrm{H}_{k+1}(X) \to \mathrm{H}_{k+1}(X,M) \xrightarrow{\partial_{k+1}} \mathrm{H}_k(M) \xrightarrow{i_*} \mathrm{H}_k(X) \to \mathrm{H}_k(X,M) \to \mathrm{H}_{k-1}(M) \to$$

The key observation is that this long exact sequence is "symmetric" around $\mathrm{H}_k(M;\mathbb{F})$. More precisely, by the Homology Symmetry Theorem 210.6 (2)[3013] we have the following isomorphisms:

$$\to \overbrace{\mathrm{H}_{k+1}(M)}^{\cong \mathrm{H}_{k-1}(M)} \to \overbrace{\mathrm{H}_{k+1}(X)}^{\cong \mathrm{H}_k(X,M)} \to \overbrace{\mathrm{H}_{k+1}(X,M)}^{\cong \mathrm{H}_k(X)} \xrightarrow{\partial_{k+1}} \mathrm{H}_k(M) \xrightarrow{i_*} \mathrm{H}_k(X) \to \mathrm{H}_k(X,M) \to \mathrm{H}_{k-1}(M) \to$$

Therefore it suffices to prove the following purely algebraic claim.

[3013] Even though it is irrelevant we would like to stress that the isomorphisms $\mathrm{H}_k(M;\mathbb{F}) \cong \mathrm{H}_{2k-k}(M;\mathbb{F})$, $\mathrm{H}_k(X;\mathbb{F}) \cong \mathrm{H}_{2k+1-k}(X,M;\mathbb{F})$ and $\mathrm{H}_k(X,M;\mathbb{F}) \cong \mathrm{H}_{2k+1-k}(X;\mathbb{F})$ are not natural.

Claim. Let $m \in \mathbb{N}_0$. If

$$0 \to A_m \xrightarrow{\beta_m} A_{m-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{m-1} \xrightarrow{\alpha_m} A_m \to 0$$

is an exact sequence of finite-dimensional $\mathbb{F}$-vector spaces,[3014] then

$$\dim_{\mathbb{F}}(\ker(\gamma\colon B \to A_1)) \;=\; \tfrac{1}{2} \cdot \dim_{\mathbb{F}}(B).$$

Proof. We prove the claim by induction on $m \in \mathbb{N}_0$. The case $m = 0$ is clear, since in this case $B = 0$. The case $m = 1$ also follows easily from elementary linear algebra. Now assume that the statement holds for some $m - 1 \in \mathbb{N}$. Next we consider a symmetric sequence

$$0 \to A_m \xrightarrow{\beta_m} A_{m-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{m-1} \xrightarrow{\alpha_m} A_m \to 0.$$

as above. It follows from the Break-into-SES Lemma 145.6 that the following sequence is also exact

$$0 \to A_{m-1}/\beta_m(A_m) \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to \ker(A_{m-1} \xrightarrow{\alpha_m} A_m) \to 0.$$

But we have

$$\dim_{\mathbb{F}}(A_{m-1}/\beta_m(A_m)) \underset{\substack{\uparrow \\ \text{since } \beta_m \text{ is injective}}}{=} \dim_{\mathbb{F}}(A_{m-1}) - \dim_{\mathbb{F}}(A_m) \underset{\substack{\uparrow \\ \text{since } \alpha_m \text{ is surjective}}}{=} \dim_{\mathbb{F}}(\ker(A_{m-1} \xrightarrow{\alpha_m} A_m)).$$

So the new shorter sequence is also symmetric. So we can apply the induction hypothesis to get the desired equality. This concludes the proof of the claim and thus the proof of Statement (2) of the theorem. ⊞

The proof of Statement (1) is almost the same, except that we need to replace statements about dimensions with statements about ranks. We leave it to the reader to make the minor adjustments that are needed. ■

280.4.3. **Proof of the Boundary-Signature Proposition 280.17.** Finally we can turn to the actual proof of the Boundary-Signature Proposition 280.17:

Proof of the Boundary-Signature Proposition 280.17. Let X be a compact oriented $(4k+1)$-dimensional topological manifold with $k \geq 1$. We write $M = \partial X$. We want to show that $\operatorname{sign}(M) = 0$. We start out with the following very general claim:

Claim. The forms $Q_M \otimes \mathbb{R}$ and $Q_M^{\mathbb{R}}$ are isometric.

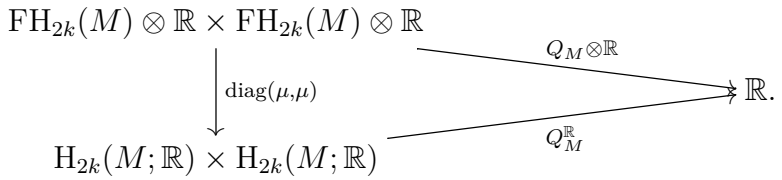
Proof. We consider the map

$$\begin{array}{rccc} \mu\colon & \mathrm{FH}_{2k}(M) \otimes \mathbb{R} & \to & \mathrm{H}_{2k}(M;\mathbb{R}) \\ & [\sigma] \otimes r & \mapsto & [\sigma \otimes r] \end{array}$$

from the Homology-Multiplication Lemma 161.7. It follows easily from the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$ Lemma 159.10 together with the Torsion–Free Coefficients Proposition 161.8 that this map is an isomorphism.

[3014] Recall that by the Topological Manifolds-Invariants Proposition 189.14 (2) together with the Relative-H_*-Finite Generation Lemma 179.11 (1) we know that the real homology groups of X, M and (X, M) are finite-dimensional vector spaces.

Next note that it follows immediately from the Cup Product-Coefficients Lemma 204.9 and the Kronecker Pairing–Natural Lemma 199.9 (2) that the following diagram commutes:



The claim follows immediately from the above considerations. ⊞

It follows from the claim that it suffices to show that $\mathrm{sign}(Q_M^{\mathbb{R}}) = 0$. To show this we now consider $P := \ker(\mathrm{H}_{2k}(M;\mathbb{R}) \to \mathrm{H}_{2k}(X;\mathbb{R}))$. It follows from the Intersection Form–Kernel-Proposition 280.21 together with the Half-Live Half-Die Theorem 280.22 that P is a metabolizer for $Q_M^{\mathbb{R}}$. By the Vanishing Signature Lemma 227.6 this implies that we have indeed $\mathrm{sign}(Q_M^{\mathbb{R}}) = 0$. ■

280.5. **The cobordism ring.** In this section we will see that the cobordism groups $\{\Omega_n\}_{n\in\mathbb{N}_0}$ and $\{\mathcal{N}_n\}_{n\in\mathbb{N}_0}$ give rise to graded rings. To formulate this precisely it is convenient to recall Definition 203.23:

Definition 280.23. A superalgebra is a graded ring[3015] $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for every $a \in R_k$ and $b \in R_l$ we have $b \cdot a = (-1)^{k\cdot l} \cdot a \cdot b$.

Now we can state the promised lemma:

Lemma 280.24. (Cobordism Ring Lemma)

(1) For any $m, n \in \mathbb{N}_0$ the maps

$$\begin{array}{ccc} \Omega_m \times \Omega_n & \to & \Omega_{m+n} \\ ([M],[N]) & \mapsto & [M \times N] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{N}_m \times \mathcal{N}_n & \to & \mathcal{N}_{m+n} \\ ([M],[N]) & \mapsto & [M \times N] \end{array}$$

are well-defined.

(2) (a) $\Omega_* := \bigoplus_{n\in\mathbb{N}_0} \Omega_n$ with the multiplication from (1) is a superalgebra.

(b) $\mathcal{N}_* := \bigoplus_{n\in\mathbb{N}_0} \mathcal{N}_n$ with the multiplication from (1) is a commutative ring.

Proof.

(1) We prove the statement regarding the Ω_k's, the proof for the $\mathcal{N}_k$'s is verbatim the same. So suppose that we have $[M] = [M'] \in \Omega_m$ and $[N] \in \Omega_n$. This means that there exists a compact oriented $(m+1)$-dimensional smooth manifold W with $\partial W = M \sqcup (-M')$. But then we also have the following natural diffeomorphisms of *oriented* smooth manifolds:

$$\partial(W \times N) \underset{\uparrow}{=} \partial W \times N = (M \sqcup (-M')) \times N \underset{\uparrow}{=} (M \times N) \sqcup -(M' \times N).$$

by the Smooth Manifold Product (2) Proposition 25.34 together with the Product Orientation Proposition 31.48

since $(-M') \times N = -(M' \times N)$

This shows that $[M \times N] = [M' \times N]$. The same argument shows that the map does not depend on the representative of $[N] \in \Omega_n$.

[3015] We refer to Definition 2.6 for our definition of a ring.

(2) (a) The verification of this statement is for the most part totally elementary. The only aspect which deserves discussing is the question to what degree the product is commutative. Thus let M and N be closed oriented smooth manifolds. It follows from elementary linear algebra (which we worked out in Exercise 31.13) that the diffeomorphism

$$\begin{array}{rcl} M \times N & \to & N \times M \\ (x, y) & \mapsto & (y, x) \end{array}$$

is orientation-preserving if and only if $\dim(M) \cdot \dim(N)$ is even. It follows easily from this observation that we have a superalgebra in our hands.

(b) Since we now no longer worry about orientations we see that orientation-issue from (a) plays no role and multiplication is $\mathcal{N}_*$ is in fact commutative. It is now clear that $\mathcal{N}_*$ is a commutative ring. ■

The Cobordism Ring Lemma 280.24 leads us to the following definition:

Definition 280.25. We define

$$\Omega_* := \bigoplus_{n \in \mathbb{N}_0} \Omega_n \quad \text{and} \quad \mathcal{N}_* := \bigoplus_{n \in \mathbb{N}_0} \mathcal{N}_n.$$

Together with the maps defined in the Cobordism Ring Lemma 280.24 we can view these groups as commutative rings. We refer to Ω_* as the oriented cobordism ring and we refer to $\mathcal{N}_*$ as the unoriented cobordism ring.

Just for fun we state the following lemma, even though we will not make use of it:

Lemma 280.26. (Cobordism Ring Homomorphism Lemma) The maps

$$\begin{array}{rcl} \Omega_* & \to & \mathbb{Z} \\ M & \mapsto & \begin{cases} 0, & \text{if } \dim(M) \not\equiv 0 \bmod 4, \\ \operatorname{sign}(M), & \text{if } \dim(M) \equiv 0 \bmod 4 \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{N}_* & \to & \mathbb{Z}/2\mathbb{Z} \\ M & \mapsto & \chi(M) \end{array}$$

are ring homomorphisms.

Proof. The statement about the signature follows immediately from the Signature–Product Theorem 227.4 and the Manifold Signature Additivity Lemma 226.17. Furthermore, the statement about the Euler characteristic follows immediately from the Manifold-χ-Proposition 190.24 (1) and (2). ■

280.6. **The Thom Cobordism Theorem.** We are now ready to formulate the following theorem which summarizes some of the most important results on the oriented cobordism ring Ω_*. Most of the statements in the theorem are due to René Thom [**Tho1954b, Tho2007**].

Theorem 280.27. (Thom Cobordism Theorem)

(1) Each cobordism group Ω_n is finitely generated.

(2) The ring homomorphism[3016 30173018]

$$\begin{array}{rcl} \mathbb{Q}[\{x_m\}_{m \in \mathbb{N}}] & \to & \Omega_* \otimes \mathbb{Q} \\ x_m & \mapsto & \mathbb{C}\mathrm{P}^{2m} \end{array}$$

that is determined by

is an isomorphism.

(3) For small n the isomorphism types of Ω_n are given by the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11
Ω_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

(a) We have $\Omega_4 = \mathbb{Z}\cdot[\mathbb{C}\mathrm{P}^2]$. In particular the homomorphism

$$\begin{aligned}\mathrm{sign}\colon \Omega_4 &\to \mathbb{Z}\\ [M] &\mapsto \mathrm{sign}(M)\end{aligned}$$

from the Signature–Cobordism Corollary 280.18 is actually an isomorphism.

(b) We have $\Omega_8 \cong \mathbb{Z}^2$ and a basis for Ω_8 is given by $\{\mathbb{C}\mathrm{P}^4, \mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2\}$.[3019]

(4) The group Ω_n is non-trivial for every $n \geq 8$.

(5) Every torsion element in Ω_n has order 2.



Proof.

(1) This statement was proved by René Thom [**Tho1954b**, Thèoréme IV.15][**Tho2007**, Theorem IV.15] building on work of Jean-Pierre Serre [**Ser1953**], see also [**Stong1968**, p. 177].

(2) This statement was proved by René Thom [**Tho1954b**, Chapitre IV][**Tho2007**, Chapter IV], see also [**Stong1968**, p. 177]. Alternatively an outline of the proof is presented in [**Kupe2017c**].

(3) The calculation of the cobordism groups $\Omega_0,\dots,\Omega_3$ was given in the Low Cobordism Group Proposition 280.15, the values $\Omega_4,\dots,\Omega_7$ are given by René Thom in [**Tho1954b**, Thèoréme IV.13][**Tho2007**, Theorem IV.13]. For $\Omega_8,\dots,\Omega_{11}$ we refer to [**MiS1974**, p. 203].

(4) This statement is proved in [**Dol1956**, p. 34].

(5) The last statement was proved by Terrence Wall [**Wall1960**], see also [**Stong1968**, p. 182]. ■

We conclude this chapter with the unoriented analog of the Thom Cobordism Theorem 280.27 which gives us the complete calculation of the unoriented cobordism ring $\mathcal{N}_*$.

[3015] On the left-hand side $\mathbb{Q}[\{x_m\}_{m\in\mathbb{N}}]$ denotes the polynomial ring over $\mathbb{Q}$ with variables $\{x_m\}_{m\in\mathbb{N}}$.

[3016] In Chapter 159 we introduced the tensor product of two abelian groups. Note that the tensor product of two commutative rings is again a commutative ring with the "naive multiplication". In particular the right-hand side $\Omega_*\otimes\mathbb{Q}$ is a commutative ring.

On page 3996 we introduced the notion of the tensor product of two superalgebras. Since commutative rings (with the trivial grading) are superalgebras this definition also gives us the notion of the tensor product of the superalgebra Ω_* with the commutative ring $\mathbb{Q}$.

[3017] If we give each variable x_m the grading $4m$, and if we equip $\Omega_*\otimes\mathbb{Q}$ with the obvious grading, then this is an isomorphism of graded rings.

[3018] Note that in the ring homomorphism from (1) we have:

- in degree 4: $x_1 \mapsto [\mathbb{C}\mathrm{P}^2]$,
- in degree 8: $x_2 \mapsto [\mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2]$ and also $x_2 \mapsto [\mathbb{C}\mathrm{P}^4]$.

[3019] In the proof of the Pontryagin Number Isomorphism Theorem 281.12 we will in particular explain why $\mathbb{C}\mathrm{P}^4$ and $\mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2$ are linearly independent in the abelian group Ω_8.

Theorem 280.28. (Thom-Dold Cobordism Theorem) In the following we consider the set $I := \{n \in \mathbb{N} \,|\, n \neq 2^i - 1\} = \{2, 4, 5, 6, 8, 9, \dots\}$. There exists a ring isomorphism

$$\mathbb{Z}/2\mathbb{Z}[\{x_i\}_{i\in I}] \;\to\; \mathcal{N}_*$$

which is given by sending x_{2k} to $\mathbb{R}\mathrm{P}^k$ and given by sending x_i for odd i to an appropriate i-dimensional Dold manifold. Here a Dold manifold is a smooth manifold of the form[3020]

$$P(m,n) \;:=\; (S^m \times \mathbb{C}\mathrm{P}^n)/(x,z) \sim (-x,\overline{z}) \quad \text{for } x \in S^m \text{ and } z \in \mathbb{C}\mathrm{P}^n$$

for some $n, m \in \mathbb{N}_0$.

Proof. This theorem was proved by René Thom [**Tho1954b**, Thèoréme IV.12] [**Tho2007**, Theorem IV.12] and independently by Albrecht Dold [**Dol1956**, p. 32]. ∎

280.7. Exercises for Chapter 280.

Exercise 280.1. Let $n \in \mathbb{N}_0$. Let I be a countable set, let $\{B_i\}_{i\in I}$ be a family of open subsets of $\mathbb{R}^n$ and let $\{\Theta_{ij} \colon C_i^j \to D_i^j\}_{i,j\in I}$ be a family of homeomorphisms between open subsets $C_i^j \subseteq B_i$ and $D_i^j \subseteq B_j$ such that for every $i, j, k \in I$ and every $b \in C_i^j \cap \Theta_{ij}^{-1}(C_j^k)$ we have $b \in C_k^k$ and we have $\Theta_{jk}(b) \circ \Theta_{ij}(b) = \Theta_{ik}(b)$. We set

$$M(\Theta_{ij}) \;:=\; \Big(\coprod_{i\in I} B_i\Big)\Big/\overset{\text{where } b \in C_i^j}{\overset{\downarrow}{(b,i)} \sim (\Theta_{ij}(b), j)}.$$

(a) Show that if $M(\Theta_{ij})$ is Hausdorff, then it is an n-dimensional topological manifold with empty boundary.
(b) Show that any n-dimensional topological manifold with empty boundary is homeomorphic to a topological manifold of the above form.
(c) Show that the class of all homeomorphism types of n-dimensional topological manifolds with empty boundary forms a set.

Remark 1. The idea here is that one can describe topological manifolds (up to homeomorphism) by the transition maps between their charts.
Remark 2. This exercise is closely related to the Bundle–Transition Map Lemma 240.18 and the Bundle-Complete Construction Lemma 240.19.

Exercise 280.2. We consider the Klein bottle

$$K \;=\; (S^1 \times [0,1])/\!\sim \quad \text{where } (z,0) \;\sim\; (\overline{z},1) \text{ for all } z \in S^1.$$

Recall that in the Surface–Connected Sum Lemma 54.15 (1) we showed that K is diffeomorphic to $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$. Thus it follows from the Unoriented Cobordism Sum Lemma 280.13 that $[K] = [\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2] = [\mathbb{R}\mathrm{P}^2 \sqcup \mathbb{R}\mathrm{P}^2] = 2 \cdot [\mathbb{R}\mathrm{P}^2] = 0 \in \mathcal{N}_2$. Write down an explicit compact 3-dimensional smooth manifold M with $\partial M = K$.

Exercise 280.3. Provide a proof for the Vanishing Signature Lemma 227.6 that does not use bases.[3021] Let us elaborate. Let (V, g) be a non-singular symmetric form over $\mathbb{R}$ and

[3020] It follows from the Manifold–Quotient Proposition 25.39 that each $P(m,n)$ is in fact a closed smooth manifold.
[3021] As the reader surely knows, introducing bases is morally wrong and should be avoided at all costs.

suppose that there exists a vector subspace W of V with $g|_W = 0$ and with $\dim_{\mathbb{R}}(W) = \frac{1}{2}\dim_{\mathbb{R}}(V)$. Show that given any subspace A of V such that $g|_A$ is positive (respectively negative) definite there exists a subspace B of V such that $g|_B$ is negative (respectively positive) definite and which satisfies $\dim_{\mathbb{R}}(A) = \dim_{\mathbb{R}}(B)$. It follows immediately from Definition 226.8 that a proof of this statement implies that $\operatorname{sign}(g) = 0$.
Remark. This exercise is identical to Exercise 227.5.

Exercise 280.4. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link in S^3. We denote by X_L its exterior which we introduced in Definition 108.30. The boundary of X_L consists of m tori $T_1, \dots, T_m$. Let $k \in \{1, \dots, m\}$. We denote by $i\colon T_k \to X_L$ the inclusion map.

(a) Show that $\dim_{\mathbb{R}}(\operatorname{im}(i_*\colon \mathrm{H}_1(T_k;\mathbb{R}) \to \mathrm{H}_1(X_L;\mathbb{R}))) \geq 1$.
(b) Do we necessarily have $\dim_{\mathbb{R}}(\operatorname{im}(i_*\colon \mathrm{H}_1(T_k;\mathbb{R}) \to \mathrm{H}_1(X_L;\mathbb{R}))) = 1$?

Exercise 280.5. Let (V, g) be a symmetric form over $\mathbb{R}$. Show that if $\operatorname{sign}(g) = 0$, then the form is metabolic.
Remark. This exercise gives us a converse to the Vanishing Signature Lemma 227.6. Also note that this exercise is almost the same as Exercise 227.2.

Exercise 280.6. Does there exist a closed oriented 4-dimensional smooth manifold M that has infinite order in Ω_4 and that has zero Euler characteristic?

Exercise 280.7.

(a) Let M be an orientable n-dimensional smooth manifold with $n \geq 4$ and let C be a closed connected one-dimensional submanifold that admits a diffeomorphism $f\colon S^1 \to C$. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) there exists a tubular map $\tau\colon \overline{B}^{n-1} \times C \to M$. It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 and the Gluing-Smooth Manifolds-Proposition 54.5 that
$$W \coloneqq (M \setminus \tau(C \times B^{n-1})) \cup_{(\tau \circ \operatorname{diag}(f,\mathrm{id}))(S^1 \times S^{n-2}) = S^1 \times S^{n-2}} (\overline{B}^2 \times S^{n-2})$$
is an n-dimensional smooth manifold. Compute $\pi_1(W)$ in terms of $\pi_1(M)$ and C. (This approach to modifying the smooth manifold M is sometimes called a *surgery*.)
(b) Let $n \geq 3$ and let X be a closed n-dimensional smooth manifold. Suppose that there exists a compact orientable $(n+1)$-dimensional M with $\partial M = X$. Show that there exists also a compact orientable *simply connected* $(n+1)$-dimensional smooth manifold N with $\partial N = X$.
(c) Does (b) also hold for $n = 2$?

Exercise 280.8. Let V be an $\mathbb{F}_2$-vector space of dimension $2k$ and let $\langle\,,\,\rangle\colon V \times V \to \mathbb{F}_2$ be a symmetric non-singular form. Suppose we are given vectors $a_1, \dots, a_k \in V$ such that $\langle a_i, a_j\rangle = 0$ for all $i, j \in \{1, \dots, k\}$. Show that there exist vectors $b_1, \dots, b_k \in V$ such that the following conditions are satisfied:

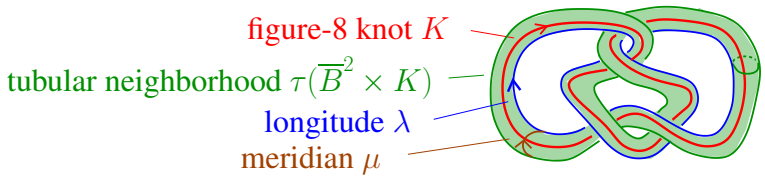
(1) The vectors $a_1, \dots, a_k, b_1, \dots, b_k$ are a basis for V.
(2) For each $i, j \in \{1, \dots, k\}$ we have $\langle a_i, b_j\rangle = \delta_{ij}$.
(3) For each $i, j \in \{1, \dots, k\}$ we have $\langle b_i, b_j\rangle = 0$.

Remark. Such a basis for V is often called a *symplectic basis*.

Exercise 280.9. Let M be a closed smooth manifold that admits a smooth fixed-point free involution $\tau\colon M \to M$. Show that there exists a compact smooth manifold W such that M is diffeomorphic to ∂W.

Exercise 280.10. We consider $\mathbb{R}\mathrm{P}^3$ and we consider the submanifold K that is given by the standard embedding of $\mathbb{R}\mathrm{P}^1$ into $\mathbb{R}\mathrm{P}^3$. We pick a tubular neighborhood N for K and we set $X := \mathbb{R}\mathrm{P}^3 \setminus \mathring{N}$. Show that $\ker(i_*\colon \mathrm{H}_1(\partial X) \to \mathrm{H}_1(X))$ is not a subsummand of $\mathrm{H}_1(\partial X)$.
Remark. This exercise shows that read the statement of the Half-Live Half-Die Theorem 280.22 (1) carefully.

Exercise 280.11. Let $K \subseteq S^3$ be a knot. We pick a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ and as in Definition 108.30 we consider the corresponding knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$. Show that there exist two closed oriented smoothly embedded curves μ and λ on ∂X_K such that $[\mu], [\lambda]$ form a basis of $\mathrm{H}_1(\partial X_K)$ and such that $\ker(\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K)) = \mathbb{Z} \cdot \lambda$.



Exercise 280.12. Let $m \in \mathbb{N}_0$.

(a) Show that there exists a smooth map $p\colon \mathbb{R}\mathrm{P}^{2m+1} \to \mathbb{C}\mathrm{P}^m$ which is an $(S^1, \mathrm{SO}(2))$-fiber bundle.

(b) Show that there exists a compact orientable smooth manifold W with $\partial W = \mathbb{R}\mathrm{P}^{2m+1}$.

Remark. This shows that $[\mathbb{R}\mathrm{P}^{2m+1}] = 0 \in \Omega_{2m+1}$ and in particular $[\mathbb{R}\mathrm{P}^{2m+1}] = 0 \in \mathcal{N}_{2m+1}$. This in contrast to the fact that the Cobordism Group-χ-Proposition 280.14 implies that $[\mathbb{R}\mathrm{P}^{2m}] \neq 0 \in \mathcal{N}_{2m}$.

Exercise 280.13. Let $m \in \mathbb{N}_0$.

(a) Show that there exists a smooth map $p\colon \mathbb{C}\mathrm{P}^{2m+1} \to \mathbb{H}\mathrm{P}^m$ which is an $(S^2, \mathrm{SO}(3))$-fiber bundle.

(b) Show that there exists a compact orientable smooth manifold W with $\partial W = \mathbb{C}\mathrm{P}^{2m+1}$.

Remark. This shows that $[\mathbb{C}\mathrm{P}^{2m+1}] = 0 \in \Omega_{4m+2}$. This in contrast to the fact that the Signature–Cobordism Corollary 280.18 implies that $[\mathbb{C}\mathrm{P}^{2m}]$ generates a $\mathbb{Z}$-summand in Ω_{4m}.

Exercise 280.14. Let $n \in \mathbb{N}_0$. A *partition of* n is a set $I = \{i_1, \dots, i_r\}$ of numbers in $\mathbb{N}$ such that $n = i_1 + \cdots + i_r$. Often we just write, $I = i_1 + \cdots + i_r$ *is a partition of* n. Use the Thom Cobordism Theorem 280.27 to show that for $n \in \mathbb{N}_0$ the dimension of $\Omega_{4n} \otimes \mathbb{Q}$ equals the number of partitions of n.

CHAPTER 281

The Hirzebruch Signature Theorem

In this chapter we will first use the Pontryagin classes from Definition 276.4 to introduce the Pontryagin numbers of a closed oriented $4n$-dimensional smooth manifold and we will show that these give rise to homomorphisms from the $4n$-dimensional cobordism groups Ω_{4n} to $\mathbb{Z}$. Afterwards we will use the Thom Cobordism Theorem 280.27 to (partially) prove the Hirzebruch Signature Theorem 281.14, which amazingly, shows that the Pontryagin numbers determine the signature of a given closed oriented $4n$-dimensional smooth manifold. Finally we will use the Hirzebruch Signature Theorem 281.14 to show that the signature of a closed oriented $4n$-dimensional smooth manifold is multiplicative under finite coverings.

This chapter is organized as follows:

- In Section 281.1 we will collect a few basic facts about Pontryagin classes of smooth manifolds and we will introduce the Pontryagin numbers of closed oriented $4n$-dimensional smooth manifolds.
- In Section 281.2 we will cite some results which show that in general the Pontryagin classes depend on the smooth structure of a smooth manifold, i.e. we will see that they are in general not homeomorphism invariants.
- In Section 281.3 we will see that Pontryagin numbers give rise to homomorphisms from cobordism groups to $\mathbb{Z}$.
- In Section 281.4 we will state and partially prove the Hirzebruch Signature Theorem 281.14.
- In Section 281.5 we will use the Hirzebruch Signature Theorem 281.14 to show that the signature is multiplicative under finite coverings.
- In Appendix 281.6 we will discuss the signature of fiber bundles.
- In Section 281.7 we will provide exercises for Chapter 281.

In the next Chapter 282 we will construct exotic 7-dimensional spheres.

281.1. **Pontryagin numbers of smooth manifolds.** We start out by recalling the Definition 276.4 of Pontryagin classes: Thus let $E \to B$ be an n-dimensional real vector bundle over a topological space that is Hausdorff and paracompact. By the Bundle–Tensor-with-$\mathbb{C}$-Lemma 268.30 we can consider the corresponding n-dimensional complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C}$. Given $k \in \mathbb{N}_0$ we define the k-th Pontryagin[3028] class of E as

$$p_k(E) \;:=\; (-1)^k \cdot c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \;\in\; \mathrm{H}^{4k}(B;\mathbb{Z}).$$

(c_{2k}: Chern class from Definition 274.1)

In this chapter we are mostly interested in tangent bundles of smooth manifolds. In this section we will lay the foundations for this study. This section consists of two subsections:

- In Subsection 281.1.1 we will discuss a few basic properties of Pontryagin classes of tangent bundles of smooth manifolds.
- In Subsection 281.1.2 we will introduce the Pontryagin numbers of closed oriented $4n$-dimensional smooth manifolds.

281.1.1. **Pontryagin classes of smooth manifolds.** Since any smooth manifold comes automatically with a real vector bundle, namely the tangent bundle, it makes sense to introduce the following definition:

Definition 281.1. Given a smooth manifold M we define its Pontryagin classes as the Pontryagin classes of its tangent bundle $\mathrm{T}M$, i.e. for $k \in \mathbb{N}_0$ we write[3029]

$$p_k(M) \coloneqq p_k(\mathrm{T}M) \in \mathrm{H}^{4k}(M;\mathbb{Z}).$$

Furthermore, as in Definition 276.7 we refer to

$$p(M) \coloneqq (p_0(E),0,0,0,p_1(M),0,0,0,p_2(M),\dots) \in \mathrm{H}^*(M;\mathbb{Z}) = \bigoplus_{i=0}^{\infty} \mathrm{H}^i(B;\mathbb{Z})$$

as the total Pontryagin class of M.

Next we recall the following calculation which will play an important role in this chapter:

Proposition 276.10. ($\mathbb{C}\mathrm{P}^n$-Pontryagin-Proposition) Let $n \in \mathbb{N}_0$. As usual we denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ the standard generator as introduced in Notation 213.6 and in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 220.20. Then, using Definition 281.1, we have

$$p(\mathbb{C}\mathrm{P}^n) = (1+x^2)^{n+1} \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}).$$

In other words, for $k = 0,\dots,\lfloor \frac{n}{2} \rfloor$ we have

$$p_k(\mathbb{C}\mathrm{P}^n) = \binom{n+1}{k} \cdot x^{2k} \in \mathrm{H}^{4k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) = \mathbb{Z} \cdot x^{2k}.$$

and all other Pontryagin classes are zero.

Example. The $\mathbb{C}\mathrm{P}^n$-Pontryagin-Proposition 276.10 implies in particular that

$$p(\mathbb{C}\mathrm{P}^1) = 1 \quad \text{and} \quad p(\mathbb{C}\mathrm{P}^2) = 1+3x^2. \qquad \square$$

The following proposition (which incidentally gives a solution to Exercise 276.8) allows us to calculate the Pontryagin classes of products of smooth manifolds:

Proposition 281.2. (Pontryagin-of-Product Proposition) Let M and N be two closed oriented smooth manifolds. Let $\mathrm{pr}_M \colon M \times N \to M$ and $\mathrm{pr}_N \colon M \times N \to N$ be the two natural projections. Then we have the following equality in $\mathrm{H}^*(M \times N;\mathbb{Z})$:

$$p(M \times N) = \mathrm{pr}_M^*(p(M)) \cup \mathrm{pr}_N^*(p(N)).$$

[3029]Recall that we showed in the Manifold+CW-Complex–Paracompact Proposition 241.4 that all smooth manifolds are Hausdorff and paracompact. Thus the Pontryagin classes of the tangent bundle are indeed defined.

Example. Given $m, n \in \mathbb{N}_0$ we consider the product $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$. In the following let $\mathrm{pr}_x \colon \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$ and $\mathrm{pr}_y \colon \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be the two natural projections. It follows from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 213.7 together with the Cohomology Ring-of-Product Proposition 208.12 (which we can apply by the CW–Chain Complex Condition Lemma 208.2) that we can make the identification $\mathrm{H}^*(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[x,y]/(x^{m+1}, y^{n+1})$ where x and y are the pullbacks of the standard generators of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$ and $\mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ under the two natural projections pr_x and pr_y. We now see that

$$p(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) \underset{\text{by the Pontryagin-of-Product Proposition 281.2}}{\overset{\uparrow}{=}} \mathrm{pr}_x^*(\mathbb{C}\mathrm{P}^m) \cup \mathrm{pr}_y^*(\mathbb{C}\mathrm{P}^n) \underset{\text{by the } \mathbb{C}\mathrm{P}^n\text{-Pontryagin Proposition 276.10}}{\overset{\uparrow}{=}} (1+x^2)^{m+1} \cdot (1+y^2)^{n+1} \in \mathbb{Z}[x,y]/(x^{m+1}, y^{n+1}).$$

To make things more concrete, later in this chapter we will make use of the special case $m = n = 2$. In this case we see that

$$p(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) \underset{\text{by the above}}{\overset{\uparrow}{=}} (1+x^2)^3 \cdot (1+y^2)^3 \underset{\text{equality in the ring } \mathrm{H}^*(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2; \mathbb{Z}) = \mathbb{Z}[x,y]/(x^3, y^3)}{\overset{\uparrow}{=}} (1+3x^2) \cdot (1+3y^2) = 1 + \underbrace{3(x^2+y^2)}_{=p_1} + \underbrace{9x^2y^2}_{=p_2}.$$

□

Proof. Let M and N be two closed oriented smooth manifolds. Let $\mathrm{pr}_M \colon M \times N \to M$ and $\mathrm{pr}_N \colon M \times N \to N$ be the two natural projections. We have the following equalities in $\mathrm{H}^*(M \times N; \mathbb{Z})$:

$$\begin{array}{rcl} & \text{by Definition 281.1} & \text{by the Tangent Bundle-of-Product Proposition 268.6} \\ p(M \times N) & \overset{\downarrow}{=} p(\mathrm{T}(M \times N)) & \overset{\downarrow}{=} p(\mathrm{pr}_M^* \mathrm{T}M \oplus \mathrm{pr}_N^* \mathrm{T}N) \\ & = p(\mathrm{pr}_M^* \mathrm{T}M) \cup p(\mathrm{pr}_N^* \mathrm{T}N) & = \mathrm{pr}_M^* p(\mathrm{T}M) \cup \mathrm{pr}_N^* p(\mathrm{T}N). \\ & \overset{\uparrow}{=} & \overset{\uparrow}{=} \\ & \text{by the Pontryagin Multiplicativity Theorem 276.8} & \text{by the Naturality-of Pontryagin-Proposition 276.5} \end{array}$$

■

We conclude this subsection with the following lemma, which shows how (local) diffeomorphisms interact with Pontryagin classes:

Lemma 281.3. (Pontryagin Class–Diffeomorphism Lemma) If $f \colon M \to N$ is a local diffeomorphism between two smooth manifolds, then for any $k \in \mathbb{N}_0$ we have

$$\underbrace{p_k(\mathrm{T}M)}_{=p_k(M)} = f^*(\underbrace{p_k(\mathrm{T}N)}_{=p_k(N)}) \in \mathrm{H}^{4k}(M; \mathbb{Z}).$$

Proof. For any $k \in \mathbb{N}_0$ we have the following equalities in $\mathrm{H}^{4k}(M; \mathbb{Z})$:

$$p_k(\mathrm{T}M) \underset{\text{by the Tangent Bundle Pullback Lemma 265.41 (1)}}{\overset{\uparrow}{=}} p_k(f^*\mathrm{T}N) \underset{\text{by Naturality-of-Pontryagin Proposition 276.5}}{\overset{\uparrow}{=}} f^*(p_k(\mathrm{T}N)).$$

■

281.1.2. **Pontryagin numbers.** In this subsection we will see that we can use Pontryagin classes to define integer-valued invariants of compact oriented $4k$-dimensional smooth manifolds.

In preparation we provide the following definition from number theory:

Definition 281.4. Let $n \in \mathbb{N}_0$. A partition of n is a set $I = \{i_1, \ldots, i_r\}$ of numbers in $\mathbb{N}$ such that $n = i_1 + \cdots + i_r$. Often we just write, $I = i_1 + \cdots + i_r$ is a partition of n.

Example.

(1) The natural number 4 has 5 partitions, namely 1+1+1+1, 2+1+1, 2+2, 3+1 and 4.
(2) The natural number 0 has precisely one partition, namely the empty partition.
(3) Just for fun we mention that Hardy and Ramanujan showed that for large $r \in \mathbb{N}$ the number of partitions of r is approximately $\frac{1}{4r\sqrt{3}} \exp(\pi\sqrt{2r/3})$. We refer to [**Ostm1956**, p. 64] for a precise formulation of this result and for details. □

We continue with the definition of the Pontryagin numbers of a smooth manifold:

Definition 281.5. Let $n \in \mathbb{N}_0$ and let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n. Given a compact oriented $4n$-dimensional smooth manifold we define its I-Pontryagin class as

$$p_I(M) \coloneqq p_{i_1}(M) \cup \cdots \cup p_{i_r}(M) \in \mathrm{H}^{4n}(M;\mathbb{Z})$$

and we define the corresponding I-Pontryagin number as

$$\widetilde{p}_I(M) \coloneqq \langle p_I(M), [M] \rangle \in \mathbb{Z}.$$

We conclude this subsection with the following subtle lemma, which says that Pontryagin numbers are diffeomorphism invariants:

Lemma 281.6. (Pontryagin Number–Diffeomorphism Lemma) Let $n \in \mathbb{N}_0$ and let M and N be two compact oriented $4n$-dimensional smooth manifolds. If there exists an orientation-preserving diffeomorphism between M and N, then for any partition I of n we have

$$\widetilde{p}_I(M) = \widetilde{p}_I(N).$$

Proof. Let $f\colon M \to N$ be an orientation-preserving diffeomorphism between two compact oriented $4n$-dimensional smooth manifolds M and N. Furthermore let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n. We see that

$$\begin{array}{lll}
 & \text{by Definition 281.5} & \text{by Definition 281.5} \\
\widetilde{p}_I(M) \overset{\downarrow}{=} \langle p_I(M), [M] \rangle_M & & \overset{\downarrow}{=} \langle p_{i_1}(M) \cup \cdots \cup p_{i_r}(M), [M] \rangle_M \\
\overset{\uparrow}{=} \langle f^* p_{i_1}(N) \cup \cdots \cup f^* p_{i_r}(N), [M] \rangle_M & & \overset{\uparrow}{=} \langle f^*(p_{i_1}(N) \cup \cdots \cup p_{i_r}(N)), [M] \rangle_M \\
 & \text{by the Pontryagin Class Diffeomorphism Lemma 281.3} & \text{by the Cup Product Naturality Lemma 203.9} \\
\overset{\uparrow}{=} \langle p_{i_1}(N) \cup \cdots \cup p_{i_r}(N), f_*([M]) \rangle_N & & \overset{\uparrow}{=} \langle p_{i_1}(N) \cup \cdots \cup p_{i_r}(N), [N] \rangle_N \\
 & \text{by the Kronecker Pairing Natural Lemma 199.9 (1b)} & \text{by the discussion on page 3773, note that here we use that } f \text{ is orientation-preserving} \\
\overset{\uparrow}{=} \langle p_I(N), [N] \rangle_N & & \overset{\uparrow}{=} \widetilde{p}_I(N). \\
 & \text{by Definition 281.5} & \text{by Definition 281.5}
\end{array}$$

■

281.2. **Interlude: Pontryagin numbers as invariants of smooth manifolds.** So far, throughout these lecture notes we have basically only studied invariants of topological spaces. Even the de Rham cohomology groups that we introduced in Section 78.1 were shown in the Smooth Singular-H*-Isomorphism Proposition 231.12 to be topological invariants, i.e. their isomorphism types depend only on the homeomorphism type of the underlying topological space.

The Pontryagin classes of a smooth manifold M turn out to be more subtle invariants: they are defined using the tangent bundle of M. The definition of the tangent bundle in turn relies on the choice of the smooth structure. Put differently, a priori the Pontryagin

classes $p_k(M) \in \mathrm{H}^{4k}(M;\mathbb{Z})$, and thus also the Pontryagin numbers $p_I(M) \in \mathbb{Z}$, depend on the smooth structure on a given topological manifold.

In a similar vein, in the Pontryagin Number–Diffeomorphism Lemma 281.6 we showed that, given two closed oriented $4k$-dimensional smooth manifolds M and N, the Pontryagin numbers are the same, if there exists an orientation-preserving diffeomorphism.

So the question arises, can Pontryagin classes and Pontryagin numbers detect different smooth structures on a topological manifold? In other words, are they homeomorphism invariants? In this section we will gather a few facts regarding this question. Unfortunately we will not be able to prove a single of these statements. But we will also not make use of these results, so this section can be seen as a guide to further reading in the literature.

First of all the following theorem demonstrates that Pontryagin classes contain information on smooth structures on topological manifolds:

Theorem 281.7. (Pontryagin Number–Smooth Structure Dependence Theorem) For any $n \geq 2$ there exist two closed smooth manifolds M and N with the following properties:

(1) the smooth manifolds M and N are homeomorphic,
(2) the n-th Pontryagin class of M is trivial, whereas
(3) the n-th Pontryagin class of N is non-zero.

In particular it follows from the Pontryagin Class–Diffeomorphism Lemma 281.3 that the smooth manifolds M and N are not diffeomorphic.

Proof. This theorem is proved in [**KrL2005**, Theorem 4.8], see also [**KrL2005**, Proof of Proposition 4.7]. Examples of non-compact smooth manifolds M and N with the same properties (1), (2) and (3) were given before by John Milnor [**Miln1964a**, p. 78] in 1964, see also [**Rudy2016**, p. 87]. Furthermore, similar ideas had already been used before by John Milnor [**Miln1956a**] in the proof of the Milnor Exotic Sphere Theorem 282.6 where he showed that there exists a smooth manifold that is homeomorphic but not diffeomorphic to S^7. ■

In fact the non-zero Pontryagin classes of the smooth manifolds M and N in [**Miln1964a**, p. 78] and [**KrL2005**, Theorem 4.8] happen to be torsion elements, so the Pontryagin classes of M and N differ by some torsion element in cohomology. Put differently, the images of the Pontryagin classes in *rational* cohomology groups agree. This observation leads us to the definition of the *rational Pontryagin* classes of a smooth manifold.

Definition 281.8. Given a smooth manifold M and $i \in \mathbb{N}_0$ we define the i-th rational Pontryagin class $p_i(M;\mathbb{Q}) \in \mathrm{H}^{4i}(M;\mathbb{Q})$ of M to be the image of $p_i(M) \in \mathrm{H}^{4i}(M;\mathbb{Z})$ under the homomorphism $\mathrm{H}^{4i}(M;\mathbb{Z}) \to \mathrm{H}^{4i}(M;\mathbb{Q})$ from the Cohomology-Coefficients-Functor Lemma 197.21.

The following theorem says that rational classes can *not* be used to find examples of smooth manifolds that are homeomorphic but non-diffeomorphic.

Theorem 281.9. (Rational Pontryagin-Homeomorphism Invariance Theorem) Let M and N be two closed smooth manifolds. Let $f\colon M \to N$ be a homeomorphism between closed smooth manifolds. We denote by $f^*\colon \mathrm{H}^*(N;\mathbb{Q}) \to \mathrm{H}^*(M;\mathbb{Q})$ the induced

map on rational cohomology. Then for any $i \in \mathbb{N}_0$ we have

$$p_i(M;\mathbb{Q}) \;=\; f^*(p_i(N;\mathbb{Q})).$$

Proof. The theorem was first proved by Sergei Novikov [**Nov1965**] in 1965.[3030] See also [**Gram1966**] for an alternative proof. ■

In fact, using the Rational Pontryagin-Homeomorphism Invariance Theorem 281.9 one can show that for each $k \in \mathbb{N}$ there exists an $e_k \in \mathbb{N}$ such that $e_k \cdot p_k$ is a topological invariant, i.e. preserved under homeomorphism. These numbers e_k were determined in [**Sharm1985**]. For example it is shown in [**Sharm1985**, Theorem 1.6] that $e_1 = 1$, which explains why in the Pontryagin Number–Smooth Structure Dependence Theorem 281.7 we need to assume that $n \geq 2$.

281.3. **Pontryagin numbers and cobordisms.** The following proposition relates Pontryagin classes to the previous Chapter 280, which dealt with cobordisms. The proposition was first proved by Pontryagin.

Proposition 281.10. (Pontryagin Number–Bordism Invariant Proposition) In the following let $n \in \mathbb{N}_0$ and let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n.

(1) Let M be a closed oriented $4n$-dimensional smooth manifold. If M is the boundary of a compact orientable smooth manifold W, then $\widetilde{p}_I(M) = 0$.

(2) The map

$$\begin{array}{rccc} \widetilde{p}_I\colon & \Omega_{4n} & \to & \mathbb{Z} \\ & M & \mapsto & \widetilde{p}_I(M) \end{array}$$

is a well-defined homomorphism.

The proof of the Pontryagin Number–Bordism Invariant Proposition 281.10 makes use of the following lemma, which is also of independent interest:[3031]

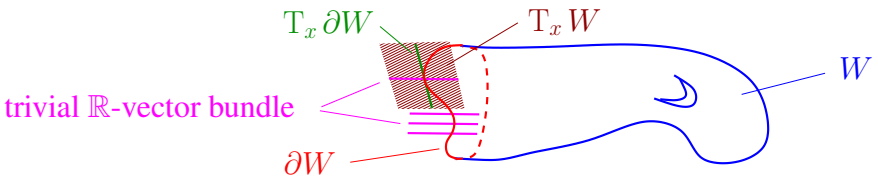
Lemma 281.11. (Tangent Bundle-of-Boundary Lemma) Let W be a smooth manifold. There exists an isomorphism

$$i^*(\mathrm{T}W) \;\cong\; \mathrm{T}\partial W \oplus \underbrace{\mathrm{Tr}_{\partial W}\mathbb{R}}_{\substack{\uparrow \\ \text{the trivial one-dimensional} \\ \text{vector bundle over } \partial W}}$$

of vector bundles over ∂W.

Proof of the Tangent Bundle-of-Boundary Lemma 281.11. We give two slightly different, but related proofs for the claim:

(i) By the Smooth Collar Neighborhood Theorem 35.6 we can find a collar neighborhood $\partial W \times [0,1] \subseteq W$ for ∂W.



[3030] Sergei Novikov (1938-2024) is a Russian mathematician who obtained a Fields medal in 1970.

[3031] Note that the Tangent Bundle-of-Boundary Lemma 281.11 gives a solution to Exercise 269.4.

We have the following isomorphisms of vector bundles over ∂W:

$$i^*(\mathrm{T}W) \underset{\uparrow}{\cong} i^*(\mathrm{T}(\partial W \times [0,1])) \underset{\uparrow}{\cong} \mathrm{T}\partial W \oplus i^*(\overbrace{\mathrm{T}[0,1]}^{\substack{\text{trivializable,}\\ \text{see page 5128}}}) \underset{\uparrow}{\cong} \mathrm{T}\partial W \oplus \mathrm{Tr}_{\partial W}\mathbb{R}.$$

by the Tangent Bundle Pullback Lemma 265.41 (1) — by the Tangent Bundle -of-Product Proposition 268.6 — by the Vector Bundle Pullback Functor Lemma 265.38

(ii) By the Inner Product-on-Vector Bundle Proposition 269.10 we can equip $i^*\mathrm{T}W$ with an inner product. One can easily verify that $\mathrm{T}\partial W$ defines a subbundle of $i^*\mathrm{T}W$. By the Normal Vector Bundle Proposition 269.13 we consider the corresponding normal subbundle $(\mathrm{T}\partial W)^\perp$ which has in particular the property that $\mathrm{T}\partial W \oplus (\mathrm{T}\partial W)^\perp \cong i^*\mathrm{T}W$. Thus it remains to show that $(\mathrm{T}\partial W)^\perp$ is trivializable. For each $x \in \partial W$ we pick the unique normal vector of length one that points inward in the sense of Definition 31.37. One can easily show these vectors define an orientation on $(\mathrm{T}\partial W)^\perp$. Thus we see that $(\mathrm{T}\partial W)^\perp$ is a 1-dimensional orientable vector bundle. It follows from Exercise 265.17 that it is trivializable. ■

Proof of the Pontryagin Number–Bordism Invariant Proposition 281.10. Let W be a compact oriented $(4n+1)$-dimensional smooth manifold. We set $M = \partial W$ and we denote by $i\colon M \to W$ the inclusion map. Let $I = i_1 + i_2 + \cdots + i_r$ be a partition of n.

(1) We calculate that

$$\widetilde{p}_I(M) = \langle p_I(\mathrm{T}M), [M]\rangle \underset{\uparrow}{=} \langle p_I(\mathrm{T}M \oplus \mathrm{Tr}_M\mathbb{R}), [M]\rangle \underset{\uparrow}{=} \langle p_I(i^*\mathrm{T}M), [M]\rangle$$

by the Trivializable-Pontryagin-Lemma 276.6 and by the Pontryagin-Multiplicativity Theorem 276.8 — by the Tangent Bundle of-Boundary Lemma 281.11

$$\underset{\uparrow}{=} \langle i^*p_I(\mathrm{T}M), [M]\rangle \underset{\uparrow}{=} \langle p_I(\mathrm{T}M), i_*([M])\rangle \underset{\uparrow}{=} \langle p_I(\mathrm{T}M), 0\rangle = 0.$$

follows from the Naturality-of Pontryagin-Proposition 276.5 together with the Cup Product Naturality Lemma 203.9 — by the Kronecker Pairing–Natural Lemma 199.9 (1) — by the Topological Kernel-of-Boundary Inclusion Corollary 194.14

(2) Statement (2) is an immediate consequence of Statement (1) and the trivial[3032] observation that for two closed oriented $4n$-dimensional smooth manifolds M and N we have the equality

$$\widetilde{p}_I(M \sqcup N) = \widetilde{p}_I(M) + \widetilde{p}_I(N).$$ ■

The following theorem is much more exciting: It gives us a converse to the Pontryagin Number–Bordism Invariant Proposition 281.10:

Theorem 281.12. (Pontryagin Number–Isomorphism Theorem) Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}_n$ the set of all partitions of n.

(1) The map[3033]

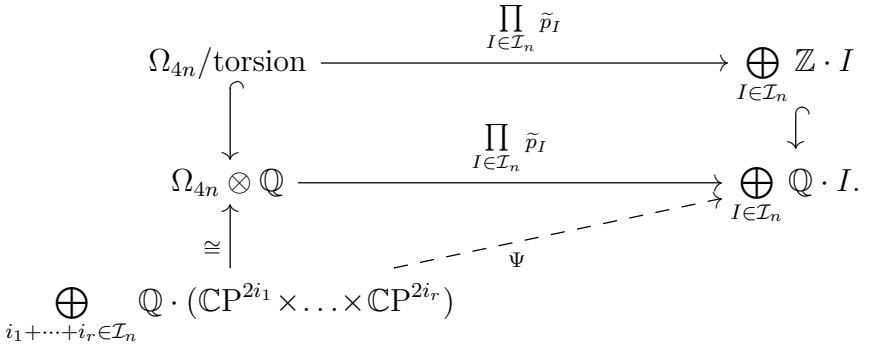
$$\begin{aligned} \prod_{I\in\mathcal{I}_n} \widetilde{p}_I\colon \Omega_{4n}\otimes\mathbb{Q} &\to \bigoplus_{I\in\mathcal{I}_n} \mathbb{Q}\cdot I \\ [M]\otimes z &\mapsto \{\widetilde{p}_I(M)\cdot z\}_{I\in\mathcal{I}_n} \end{aligned}$$

is an isomorphism.

[3032] Why is this "trivial"? How can one prove this "obvious" equality?

(2) Let M be a closed oriented $4n$-dimensional smooth manifold. If all Pontryagin numbers of M vanish, then $[M] = 0 \in \Omega_{4n}/\text{torsion}$.

Sketch of proof. From the Pontryagin Number–Bordism Invariant Proposition 281.10 (2) we obtain the following diagram of homomorphisms:



First note that it follows from the fact that the cobordism groups are finitely generated (see Thom Cobordism Theorem 280.27 (1)) and Exercise 159.3 that the left upper vertical map is a monomorphism. Next note that the right upper vertical map is evidently a monomorphism. It is clear that the diagram commutes. Thus we see that Statement (2) of the theorem is an immediate consequence of Statement (1).

Put differently, it remains to show that the bottom horizontal map is an isomorphism. Note that the lower vertical map is the isomorphism that is given by the Thom Cobordism Theorem 280.27. Thus it suffices to show that the diagonal map is an isomorphism. As a reality check, note that the dimensions agree, in both cases the dimension is just given by the cardinality of $\mathcal{I}_n$.

In the following claim we carry out the arguments for $n = 1$ and $n = 2$ "by hand". The general case needs more clever arguments, for which we refer to [**Tho2007**, Corollary IV.16].

Claim.

(1) Let $n = 1$. With respect to the bases $\{\mathbb{C}\mathrm{P}^2\}$ and $\mathcal{I}_1 = \{I = 1\}$ the diagonal map Ψ is represented by the (1×1)-matrix (3).

(2) Let $n = 2$. With respect to the bases $\{\mathbb{C}\mathrm{P}^4, \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2\}$ and $\mathcal{I}_2 = \{I = 2, I = 1 + 1\}$ the diagonal map Ψ is represented by the (2×2)-matrix $\begin{pmatrix} 10 & 9 \\ 25 & 18 \end{pmatrix}$.

In particular in both cases the matrix is invertible.

Proof. We now deal with the two cases:

(1) The case $n = 1$ is easy: We make the usual identification $\mathrm{H}^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[x]/(x^3)$, and we calculate that

$$\widetilde{p}_1(\mathbb{C}\mathrm{P}^2) \underset{\uparrow}{=} \langle p_1(\mathbb{C}\mathrm{P}^2), [\mathbb{C}\mathrm{P}^2]\rangle \underset{\uparrow}{=} \langle 3x^2, [\mathbb{C}\mathrm{P}^2]\rangle \underset{\uparrow}{=} 3.$$

see Definition 281.5 — by the $\mathbb{C}\mathrm{P}^n$-Pontryagin Proposition 276.10 — by the Cup Product-$\mathbb{C}\mathrm{P}^n$ Proposition 220.20

[3033] We discussed in Exercise 280.14 that the $\mathbb{Q}$-vector spaces to the left and right have the same dimension.

(2) The case $n = 2$ is slightly more involved. For clarity let us point out that we need to show that we have the following table:

	$\mathbb{C}\mathrm{P}^4$	$\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$
p_2	10	9
$p_1 \cup p_1$	25	18.

We first deal with the left column: Note that by the $\mathbb{C}\mathrm{P}^n$-Pontryagin-Proposition 276.10 we have $p(\mathbb{C}\mathrm{P}^4) = 1 + 5x^2 + 10x^4 \in \mathrm{H}^*(\mathbb{C}\mathrm{P}^4;\mathbb{Z}) = \mathbb{Z}[x]/(x^5)$. Thus as above we see that $\widetilde{p}_2(\mathbb{C}\mathrm{P}^4) = 10$. Furthermore, with similar arguments as above we see that

$$\widetilde{p}_{1+1}(\mathbb{C}\mathrm{P}^4) \underset{\substack{\uparrow \\ \text{see Definition 281.5}}}{=} \langle p_1(\mathbb{C}\mathrm{P}^4) \cup p_1(\mathbb{C}\mathrm{P}^4), [\mathbb{C}\mathrm{P}^4]\rangle \underset{\substack{\uparrow \\ \text{we just saw that } p_1(\mathbb{C}\mathrm{P}^4) = 5x^2}}{=} \langle 5x^2 \cdot 5x^2, [\mathbb{C}\mathrm{P}^4]\rangle = 25.$$

This concludes the calculation of the first column of the matrix.

It remains to deal with the second column. In other words, we now consider $\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$. As we discussed on page 5390 we have the following equality in the cohomology ring $\mathrm{H}^*(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \mathbb{Z}[x,y]$:

$$p(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = (1+3x^2)\cdot(1+3y^2) = 1 + \underbrace{3(x^2+y^2)}_{=p_1} + \underbrace{9x^2y^2}_{=p_2}.$$

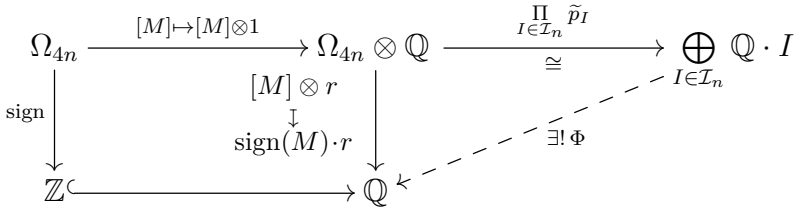
In particular $p_1(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = 3x^2 + 3y^2$ and $p_2(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = 9x^2y^2$. It follows that $p_{1+1}(\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2) = (3x^2+3y^2)^2 = 18x^2y^2$. The claim for $n = 2$ now follows from the observation that the Fundamental Class-Product Proposition 207.2 tells us that $\langle x^2 \cdot y^2, [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]\rangle = 1$. ■

281.4. **The Hirzebruch Signature Theorem.** Our next goal is to use the Pontryagin Number–Isomorphism Theorem 281.12 to prove the following proposition which says that the signature of a closed oriented smooth manifold can be determined from its Pontryagin numbers.

Proposition 281.13. (Signature-via-Pontryagin Numbers Proposition) Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}_n$ the set of all partitions of n. There exists a family $\{a_I\}_{I\in\mathcal{I}_n}$ of rational numbers such that for every closed oriented $4n$-dimensional smooth manifold M we have

$$\mathrm{sign}(M) = \sum_{I\in\mathcal{I}_n} a_I \cdot \widetilde{p}_I(M).$$

Proof. Let $n \in \mathbb{N}_0$. We denote by $\mathcal{I}_n$ the set of all partitions of n. We consider the following diagram:



First note that it follows from the Signature–Cobordism Corollary 280.18 that the left-hand vertical map is a homomorphism. Next note that it follows from the Pontryagin Number–Bordism Invariant Proposition 281.10 that the top right map is an isomorphism

of $\mathbb{Q}$-vector spaces. This implies that there exists a diagonal homomorphism Φ that makes the diagram commute. Given a partition $I \in \mathcal{I}_n$ we define $a_I \in \mathbb{Q}$ to be the value of Φ at $1 = 1 \cdot I \in \mathbb{Q} \cdot I \subseteq \bigoplus_{I \in \mathcal{I}_n} \mathbb{Q} \cdot I$. In other words, the map

$$\bigoplus_{I \in \mathcal{I}_n} \mathbb{Q} \cdot I \quad \to \quad \mathbb{Q}$$

is given by

$$\bigoplus_{I \in \mathcal{I}_n} x_I \cdot I \quad \mapsto \quad \sum_{I \in \mathcal{I}_n} x_I \cdot a_I.$$

The desired equality $\operatorname{sign}(M) = \sum_{I \in \mathcal{I}_n} a_I \cdot \widetilde{p}_I(M)$ now follows immediately from the commutativity of the previous diagram. ■

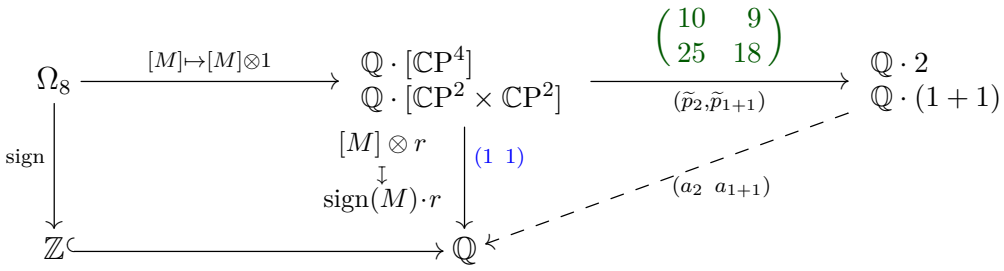
The coefficients in the Signature-via-Pontryagin Numbers Proposition 281.13 were completely determined by Friedrich Hirzebruch [**Hirz1953**]. The low-dimensional terms are given in the following theorem.

Theorem 281.14. (Hirzebruch Signature Theorem) Let $n \in \{1, 2, 3, 4\}$ and let M be a closed oriented $4n$-dimensional smooth manifold. Given $k \in \mathbb{N}_0$ we write in the following $p_k \coloneqq p_k(M) \in \mathrm{H}^{4k}(M; \mathbb{Z})$. The signature of M can be determined as follows:

n	$\operatorname{sign}(M)$ equals the Kronecker pairing of $[M]$ with the cohomology class
1	$\frac{1}{3}p_1$
2	$\frac{1}{45}(7p_2 - p_1^2)$
3	$\frac{1}{945}(62p_3 - 13p_2 \cup p_1 + 2p_1^3)$
4	$\frac{1}{14175}(381p_4 - 71p_3 \cup p_1 - 19p_2^2 + 22p_2 \cup p_1^2 - 3p_1^4)$.

Remark. The Hirzebruch Signature Theorem 281.14 can also be proved, in a very different fashion, via the Atiyah-Singer Index Theorem, see [**AtS1963**, p. 427] and [**AtS1968**, Chapter 6]. □

Proof. We carry out the case $n = 2$. We adopt the notation from the proof of the Signature-via-Pontryagin Numbers Proposition 281.13. By the Thom Cobordism Theorem 280.27 we have $\Omega_8 \otimes \mathbb{Q} = \mathbb{Q} \cdot [\mathbb{C}\mathrm{P}^4] \oplus \mathbb{Q} \cdot [\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2]$. The diagram from the proof of the Signature-via-Pontryagin Numbers Proposition 281.13 now looks as follows:



Now let M be a closed oriented 8-dimensional smooth manifold. It follows from the above diagram (or the proof of the Signature-via-Pontryagin Numbers Proposition 281.13) that

we have $\operatorname{sign}(M) = a_2 \cdot \widetilde{p}_2(M) + a_{1+1} \cdot \widetilde{p}_{1+1}(M)$ where

$$\begin{pmatrix} a_2 \\ a_{1+1} \end{pmatrix} = \begin{array}{c}\text{the } (1\times 2)\text{-matrix that is given by} \\ \text{the value of the signature applied} \\ \text{to the basis } \mathbb{C}\mathrm{P}^4 \text{ and } \mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2\end{array} \cdot \begin{array}{c}\text{inverse of the } (2\times 2)\text{-matrix that} \\ \text{expresses } p_2, p_{1+1} \text{ for the smooth} \\ \text{manifolds } \mathbb{C}\mathrm{P}^4 \text{ and } \mathbb{C}\mathrm{P}^2\times\mathbb{C}\mathrm{P}^2\end{array}$$

$$= \underset{\uparrow}{(1 \quad 1)\begin{pmatrix} 10 & 9 \\ 25 & 18 \end{pmatrix}^{-1}} = \tfrac{-1}{45}(1 \quad 1)\begin{pmatrix} 18 & -9 \\ -25 & 10 \end{pmatrix} = \tfrac{1}{45}\begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

by the Cup Product-of-Duals-in-$\mathbb{C}\mathrm{P}^n$-Lemma 220.21 and the Signature–Product Theorem 227.4 we know that the signatures of these two smooth manifolds are $+1$
for the second matrix we use the calculation from the proof of the Pontryagin Number–Isomorphism Theorem 281.12

This gives precisely the desired result for $n = 2$. The case $n = 1$ is evidently even easier.

The higher-dimensional cases are dealt with, more cleverly, in [**Hirz1953**] or alternatively in [**MiS1974**, p. 224]. These references also contain a precise formula for all $n \in \mathbb{N}_0$. ■

Remark.

(1) The expressions in the Hirzebruch Signature Theorem 281.14 might seem random, but they do arise in a cogent way from the coefficients of the power series $\frac{\sqrt{t}}{\tanh(t)}$. This connection is explained in [**MiS1974**, Chapter 19].

(2) The signature of a smooth manifold is of course an integer. Thus we obtain from the Hirzebruch Signature Theorem 281.14, applied to $n = 1$, the slightly mysterious statement that for any closed oriented 4-dimensional smooth manifold the integer $\langle p_1(M), [M]\rangle$ is divisible by 3. □

The Hirzebruch Signature Theorem 281.14 has many fascinating and often unintuitive consequences. Let us start out with the following result:

Proposition 281.15. (b_6-Even Proposition) Let M be a closed orientable 12-dimensional smooth manifold. If $b_4(M) = 0$, then $b_6(M)$ is even.

Proof. Let M be a closed orientable 12-dimensional smooth manifold with $b_4(M) = 0$. We see that modulo 2 we have the following equalities:

$$b_6(M) \underset{\downarrow}{\equiv} \operatorname{sign}(M) \underset{\uparrow}{=} \frac{1}{945}(62p_3(M) - 13p_2(M)\cup p_1(M) + 2p_1(M)^3) \underset{\uparrow}{=} 2\cdot\frac{31}{945}\cdot p_3(M).$$

by the Intersection Form Proposition 224.13 and the Signature-Eigenvalues Proposition 226.11

by the Hirzebruch Signature Theorem 281.14 applied for $n = 3$

since $b_4(M) = 0$ we know by the Universal Coefficient-H*-Theorem 201.23 that $\mathrm{H}^4(M;\mathbb{Z})$ is torsion, hence we have $p_2(M)\cup p_1(M) = p_1^3(M) = 0$

Since $b_6(M)$ and $p_3(M)$ are integers we obtain from the equality $b_6(M) = 2\cdot\frac{31}{945}\cdot p_3(M)$ that $b_6(M)$ is necessarily even. ■

281.5. **Signatures of finite coverings.** In the Covering-χ-Proposition 156.13 we saw that the Euler characteristic of a finite CW-complex is multiplicative under finite covers. In fact

we saw that this statement follows almost immediately from the definition of a CW-complex. The following theorem gives the analogous statement that the signature of a *closed* oriented $4n$-dimensional smooth manifold is multiplicative under finite covers:

Theorem 281.16. (Signature–Covering Theorem) Let $n \in \mathbb{N}_0$. Let M be a *closed* oriented $4n$-dimensional smooth manifold and let $\widetilde{M} \to M$ be a finite covering. Then

$$\operatorname{sign}(\widetilde{M}) \;=\; [\widetilde{M} : M] \cdot \operatorname{sign}(M).$$

Remark.

(1) Perhaps it is initially difficult to appreciate the significance of the Signature-Multiplicativity Theorem 281.16. If we only consider (co-) homological invariants, then there are basically only two statements that relate the invariants of a (reasonable) topological space X and an n-fold covering $p\colon \widetilde{X} \to X$ thereof:
   (a) as pointed out above, if X is a finite CW-complex, then $\chi(\widetilde{X}) = n \cdot \chi(X)$,
   (b) the Transfer–Projection Proposition 165.4 shows that $p_*\colon \mathrm{H}_*(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_*(X;\mathbb{Q})$ is an epimorphism.

   Besides these observations, (co-) homological invariants behave fairly erratically under finite covers. One example of such "random" behavior is given in Exercise 281.2.

(2) One might think that one should be able to prove the Signature–Covering Theorem 281.16 in a naive way. But this is not the case. This can be seen as follows:
   - It is clear that our sketch of the proof of the Signature–Covering Theorem 281.16 works only for smooth manifolds. Our proof does not apply to topological manifolds.
   - The statement of the Signature–Covering Theorem 281.16 does in fact hold for topological manifolds, but the proof is even harder, see e.g. [**SchafJ1970**, Theorem 8].
   - Nonetheless one cannot prove the Signature–Covering Theorem 281.16 just using tools from algebraic topology, in particular one cannot deduce the result from Poincaré duality. For example in [**Wall1967**, Corollary 5.4.1] it is shown that the statement does *not* hold for "Poincaré complexes", which implies that any naive approach to proving the multiplicativity result is doomed to fail.

(3) Note that for compact $4n$-dimensional smooth manifolds with *non-empty boundary* the signature is in general *not* multiplicative under finite covers. We refer to [**APS1975**] for details. □

Proof. Let $n \in \mathbb{N}_0$. As usual we denote by $\mathcal{I}_n$ the set of all partitions of n. By the Signature-via-Pontryagin Numbers Proposition 281.13 we know that there exist rational numbers $\{a_I\}_{I\in\mathcal{I}_n}$ such that for every closed oriented $4n$-dimensional smooth manifold X the following equality holds:

$$(*) \qquad \operatorname{sign}(X) \;=\; \sum_{I\in\mathcal{I}_n} a_I \cdot \langle p_I(X), [X]\rangle_X.$$

Now let M be a closed oriented $4n$-dimensional smooth manifold and let $q\colon \widetilde{M} \to M$ be a finite covering. We have the following equalities:

$$\begin{aligned}
\operatorname{sign}(\widetilde{M}) &= \sum_{I\in\mathcal{I}_n} a_I \cdot \langle p_I(\mathrm{T}\widetilde{M}), [\widetilde{M}]\rangle_{\widetilde{M}} && = \sum_{I\in\mathcal{I}_n} a_I \cdot \langle (p_I(q^*\mathrm{T}M)), [\widetilde{M}]\rangle_{\widetilde{M}} \\
&\uparrow && \uparrow \\
&\text{by (}*\text{)} && \text{by the Tangent Bundle–Pullback Lemma 265.41 (2)} \\
& && \text{we know that } q^*\mathrm{T}M = \mathrm{T}\widetilde{M} \\
&= \sum_{I\in\mathcal{I}_n} a_I \cdot \langle q^*(p_I(\mathrm{T}M)), [\widetilde{M}]\rangle_{\widetilde{M}} && = \sum_{I\in\mathcal{I}_n} a_I \cdot \langle p_I(\mathrm{T}M), q_*([\widetilde{M}])\rangle_M \\
&\uparrow && \uparrow \\
&\text{by the Naturality-of-Pontryagin-Proposition 276.5,} && \text{by the Kronecker Pairing} \\
&\text{and the Cup Product-Naturality Lemma 203.9} && \text{Natural Lemma 199.9 (1)} \\
&= [\widetilde{M} : M] \cdot \sum_{I\in\mathcal{I}_n} a_I \cdot \langle p_I(\mathrm{T}M), [M]\rangle_M && = [\widetilde{M} : M] \cdot \operatorname{sign}(M). \\
&\uparrow && \uparrow \\
&\text{by the Fundamental Class-Covering Proposition 194.11} && \text{by (}*\text{)} \\
&\text{and the Transfer–Projection Proposition 165.4} &&
\end{aligned}$$

■

281.6. **Appendix: Signature of fiber bundles.** We mentioned already on several occasions, in particular in Chapter 227 and just recently in Section 281.5, that the signature shares many features with the Euler characteristic. For example both are additive and multiplicative under suitable hypothesis.[3034]

In this appendix we want to consider the behavior of these invariants for fiber bundles. First note that it follows easily from the Simplicial Complex–Fiber Bundle-χ-Proposition 242.8 or alternatively from the Manifold–Fiber Bundle-χ-Theorem 250.15, that if $p\colon M \to B$ is a fiber bundle where the fiber F and the base manifold B are both compact smooth manifolds, then

$$\chi(M) \;=\; \chi(F)\cdot\chi(B).$$

It is natural to ask whether an analogous statement holds for signatures.

To make sense of this question, let us recall Definition 227.3, which extends in a very unimaginative way the notion of the signature for $4k$-dimensional smooth manifolds to smooth manifolds of arbitrary dimensions:

Definition 227.3. Let M be a compact oriented smooth manifold. If $\dim(M)$ is not divisible by 4, then we set $\operatorname{sign}(M) := 0$.

The following question now makes sense:

Question 281.17. Let $p\colon M \to B$ be a smooth bundle map with fiber F. If B, M, F are closed oriented smooth manifolds such that the orientations are compatible[3035], do we have an equality

$$\operatorname{sign}(M) \;=\; \operatorname{sign}(F)\cdot\operatorname{sign}(B)\,?$$

Next we recall some examples of smooth fiber bundles where the base manifold, the total manifold and the fiber are closed oriented smooth manifolds:

(1) Given closed oriented smooth manifolds F and B the projection $B\times F \to B$ is a smooth F-bundle.

[3034]These statements we proved in the Union-χ-Proposition 156.10, the Product-χ-Proposition 156.12, the Novikov Additivity Theorem 227.11 and the Signature–Product Theorem 227.4.

[3035]We pick $b \in B$ and we consider the fiber $F := p^{-1}(b)$. For each $x \in F$ we get a short exact sequence $0 \to \mathrm{T}_xF \to \mathrm{T}_xM \to \mathrm{T}_bB \to 0$. We demand that the orientations of these real vector spaces are compatible in the sense of Definition 31.11.

(2) By the Covering-as-Bundle Lemma 238.2 we know that a smooth bundle map $p\colon M \to B$ such that the fiber $F := p^{-1}(b)$ is a closed 0-dimensional smooth manifold is exactly the same as a finite covering $p\colon M \to B$.

(3) Recall that given any closed oriented smooth manifold F and given any orientation-preserving diffeomorphism $\varphi\colon F \to F$ we can consider the corresponding mapping torus $\mathrm{Tor}(F,\varphi) := ([0,1] \times F)/(0,x) \sim (1,\varphi(x))$ from Definition 21.19 together with the natural projection $\mathrm{Tor}(F,\varphi) \to S^1$ that is given by $[(x,t)] \mapsto \exp(2\pi\,\mathrm{i}t)$. By the Mapping Torus–Bundle Lemma 238.14 this is a smooth F-bundle over S^1 and any smooth bundle over S^1 is of that form.

It turns out that in all of the above examples we do have a multiplicativity statement for signatures:

(1) In the Signature–Product Theorem 227.4 we showed that the signature is multiplicative under products of closed oriented smooth manifolds.

(2) It follows from the Signature–Covering Theorem 281.16 that the signature is multiplicative for fiber bundles of closed oriented smooth manifolds where the fiber is 0-dimensional.

(3) It follows easily from the Novikov Additivity Theorem 227.11, see also Exercise 227.8, that the signature of any mapping torus as above is zero.[3036]

But what about the general case? Here things become more difficult. In the following we will state three theorems, which unfortunately we will not be able to prove in these lecture notes. The first theorem says that the answer to Question 281.17 is yes, if the base manifold is simply connected:

Theorem 281.18. (Simply Connected Base–Signature Multiplicativity Theorem) Let $p\colon M \to B$ be a smooth bundle map with fiber F. If B, M, F are closed oriented smooth manifolds such that the orientations are compatible and if B is simply connected, then

$$\mathrm{sign}(M) \;=\; \mathrm{sign}(F) \cdot \mathrm{sign}(B)\,?$$

Proof. This theorem was proved by Shiing-Shen Chern, Friedrich Hirzebruch and Jean-Pierre Serre [**CHS1957**, p. 587]. ■

The next theorem says that in general the answer to Question 281.17 is no:

Theorem 281.19. (Signature–Non Multiplicativity Theorem) There exists a smooth bundle map $p\colon M \to B$ where the fiber and the base are closed oriented 2-dimensional smooth manifolds and such that the total space M is a closed oriented 4-dimensional smooth manifold, such that $\mathrm{sign}(M) \neq 0$. In particular we have

$$\mathrm{sign}(M) \;\neq\; \mathrm{sign}(F) \cdot \mathrm{sign}(B).$$

Proof. This theorem was proved by Michael Atiyah [**At1969**, p. 78] and independently by Friedrich Hirzebruch [**Hirz1969**, p. 262]. ■

Finally the last theorem says that the equality of Question 281.17 holds modulo 4:

Theorem 281.20. (Signature-mod-4–Multiplicativity Theorem) Let $p\colon M \to B$ be a smooth bundle map with fiber F. If B, M, F are closed oriented smooth manifolds

[3036] We hope that the reader worked out the details in Exercise 227.8.

such that the orientations are compatible, then we have the equality

$$\operatorname{sign}(M) \equiv \operatorname{sign}(F) \cdot \operatorname{sign}(B) \mod 4.$$

Proof. This theorem was proved by Ian Hambleton, Andrew Korzeniewski and Andrew Ranicki [**HKR2007**, Theorem A]. ■

281.7. Exercises for Chapter 281.

Exercise 281.1. In the proof of the Pontryagin Number–Isomorphism Theorem 281.12 determine the homomorphism Ψ for $n = 3$.

Exercise 281.2. Show that given any $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 2}$ there exists a closed smooth manifold M with $\mathrm{H}_k(M;\mathbb{Q}) = 0$ and that admits a 2-fold covering $\widetilde{M}$ with $\mathrm{H}_k(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}^n$.
Hint. You can take M to be a continuous mapping torus of some high-dimensional torus. Use the Mapping Torus-H_*-Proposition 153.6 to compute homology groups.
Remark. This shows that in contrast to the Euler characteristic and the signature, the individual homology groups can behave quite erratically under finite coverings.

Exercise 281.3. Let $p\colon M \to B$ be a smooth bundle map with fiber F. We suppose that B, M, F are closed oriented smooth manifolds such that the orientations are compatible. Show that we have the equality

$$\operatorname{sign}(M) \equiv \operatorname{sign}(F) \cdot \operatorname{sign}(B) \mod 2.$$

Hint. Establish a connection between the Euler characteristic and the signature modulo 2.

CHAPTER 282

Exotic spheres

In this chapter we will use the High-Dimensional Poincaré Conjecture 264.7 together with the Hirzebruch Signature Theorem 281.14 to prove what is arguably the most amazing result of these lecture notes, namely the Exotic 7-Sphere Theorem 282.6 which says that the 7-dimensional sphere S^7 admits two inequivalent smooth structures. In mathematical colloquial language one says that there exists an exotic smooth structure on S^7.

More precisely, we will construct an 8-dimensional smooth manifold $M_8(E_8)$ that is given by carefully attaching eight 4-handles to $\overline{B}^8$ along the "E_8-pattern". Using the High-Dimensional Poincaré Conjecture 264.7 we will see that the boundary $\partial M_8(E_8)$ is homeomorphic to S^7 and by a clever application of the Hirzebruch Signature Theorem 281.14 we will see that the boundary $\partial M_8(E_8)$ is *not* diffeomorphic to S^7. Thus we have found a smooth manifold that is homeomorphic to but not diffeomorphic to S^7. This statement is logically equivalent to the Exotic 7-Sphere Theorem 282.6

This chapter is organized as follows:

- In Section 282.1 we will perform a somewhat technical, but crucial construction, namely we will show how one can attach a 4-handle to $\overline{B}^8$ to obtain a smooth manifold with several desirable properties, in particular with vanishing first Pontryagin class.
- In Section 282.2 we will use the construction from Section 282.1 to construct the smooth manifold $M_8(E_8)$. We will show that $M_8(E_8)$ is homeomorphic to, but not diffeomorphic to S^7. In other words, we will prove the Exotic 7-Sphere Theorem 282.6.
- In Section 282.3 we will discuss several consequences of the Exotic 7-Sphere Theorem 282.6.
- In Section 282.4 we will provide exercises for Chapter 282.

In the next **??** we will discuss the generalized Poincaré Conjecture.

282.1. **Preparation: attaching 4-handles to $\overline{B}^8$.** The only goal of this section is to prove the following slightly technical proposition, which will be an essential ingredient in the next section:

Proposition 282.1. (4-Handle Attachment Proposition) For each trivial[3037] 3-dimensional knot $K \subseteq S^7$ there exists a thickened 3-sphere $\varphi\colon \overline{B}^4 \times S^3 \to S^7 = \partial\overline{B}^8$ with the following three properties:[3038]

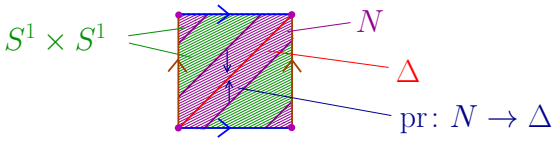
(1) $\varphi(\{0\} \times S^3) = K$.

(2) We have $\mathrm{slk}(\varphi) = 2$.

(3) We have $p_1(\overline{B}^8 \cup_\varphi h^8) = 0$.

The key to proving the 4-Handle Attachment Proposition 282.1 is the following lemma:

Lemma 282.2. (Diagonal–Tubular Neighborhood Lemma) Let $n \in \mathbb{N}_0$. We consider the smooth manifold $S^n \times S^n$ and the diagonal $\Delta := \{(x,x) \in S^n \times S^n \,|\, x \in S^n\}$. We consider
$$N := \{(x,y) \in S^n \times S^n \,|\, \|x-y\| \leq \tfrac{1}{2}\}.$$
(0) (a) The diagonal Δ is an n-dimensional smooth submanifold of $S^n \times S^n$. We give Δ^n the orientation which turns the map $S^n \to S^n \times S^n$ given by $x \mapsto (x,x)$ into an orientation-preserving diffeomorphism.
(b) N is a codimension-zero submanifold of $S^n \times S^n$.
(1) The map $\mathrm{pr}\colon N \to \Delta$ that is given by $(x,y) \mapsto (x,x)$ defines a smooth $(\overline{B}^n, \mathrm{SO}(n))$-bundle.[3037]
(2) We have $Q_N([\Delta],[\Delta]) = 1 + (-1)^n$.[3038]
(3) For all $k \in \mathbb{N}$ we have $p_k(N) = 0$.[3039]



Proof.

(0) (a) The experienced reader will by now have no troubles with showing that the diagonal is a smooth submanifold. Otherwise, see Exercise 25.30.
(b) We consider the map
$$\begin{array}{rcl} f\colon S^n \times S^n & \to & \mathbb{R} \\ (x,y) & \mapsto & \|x-y\|. \end{array}$$
We leave it to the reader to verify that any $s \in (-1,1)$ is a regular value of f. It thus follows from the Transversal Preimage Theorem 50.8 (or alternatively from Exercise 32.8) that $N = f^{-1}([-\frac{1}{2}, \frac{1}{2}])$ is a codimension-zero submanifold of $S^n \times S^n$.

[3036]Recall that in Definition 258.2 (1) we introduced the *standard 3-dimensional trivial knot* in S^7. Furthermore, by Definition 258.2 (2) a 3-dimensional knot in S^7 is called *trivial*, if it is smoothly isotopic (in the sense of Definition 41.10) to the above standard trivial link.
[3036]Recall that in Definition 259.2 we introduced the self-linking number $\mathrm{slk}(\varphi)$ of a thickened sphere φ. Furthermore recall that $p_1(M) \in \mathrm{H}^4(M;\mathbb{Z})$ stands for the first Pontryagin class of the tangent bundle of a smooth manifold M. Finally recall that according to Definition 246.4 the notation $\overline{B}^8 \cup_\varphi h^8$ stands for attaching a 4-handle to $\overline{B}^*$ along φ.
[3037]In other words, N is a tubular neighborhood of the submanifold $\Delta \subseteq S^n \times S^n$ in the sense of Definition 57.9. In fact one could argue that this statement follows from the proof of the Tubular Neighborhood–Existence Theorem 57.11.
[3038]In other words, the Euler number of $\mathrm{pr}\colon N \to \Delta$ (as introduced in Definition 265.32) equals 2.
[3039]In [**Miln1959b**, p. 3 p. 14] it is explained that N is in fact parallelizable. For our purposes it is enough to know that $p_1(N) = 0$.

(1) We introduce the following concepts:

- In Exercise 282.2 we will show that for each $Q \in S^n$ there exists an open neighborhood U_Q of $Q \in S^n$ and a smooth map $\Theta_Q \colon B^n \to \mathrm{SO}(n+1)$ such that[3040]

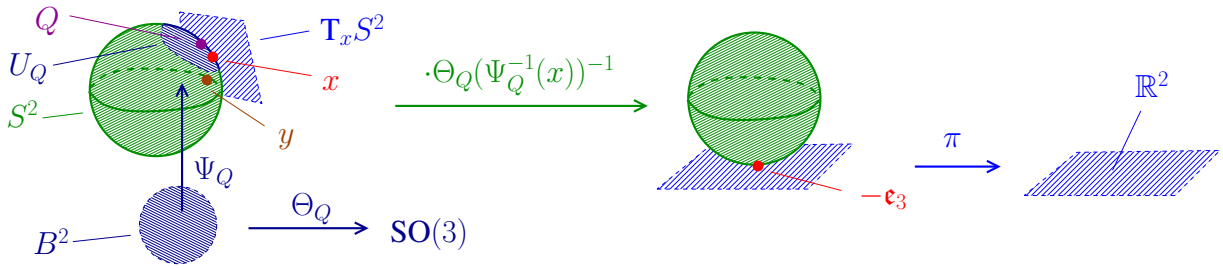
$$\begin{array}{rcl} \Psi_Q \colon B^n & \to & U_Q \\ x & \mapsto & \Theta_Q(x) \cdot (-\mathfrak{e}_{n+1}) \end{array}$$

  is a diffeomorphism.
- We consider the isomorphism $\pi \colon \mathrm{T}S^n_{-\mathfrak{e}_{n+1}} = \mathbb{R}^n \times \{0\} \xrightarrow{(x,0)\mapsto x} \mathbb{R}^n$.

For each $Q \in S^n$ we consider the map

$$\begin{array}{rcl} \Phi_Q \colon \mathrm{pr}^{-1}(U_Q) & \to & U_Q \times \overline{B}^n \\ (x,y) & \mapsto & \Big(x, \sqrt{\frac{7}{16}} \cdot \pi(\underbrace{\Theta_Q(\Psi_Q^{-1}(x))^{-1}}_{\in \mathrm{SO}(n+1)}(\text{orthogonal projection of } y-x \text{ onto } \mathrm{T}_x S^n))\Big). \end{array}$$

We leave it to the reader to show that this map $\Phi_Q \colon \mathrm{pr}^{-1}(U_Q) \to U_Q \times \overline{B}^n$ is a local smooth trivialization of $pr \colon N \to S^n$ as a $\overline{B}^n$-bundle and that the transition maps for different $Q \in S^n$ are $\mathrm{SO}(n)$-compatible.



(2) We denote by $[\Delta] \in \mathrm{H}_n(N)$ the fundamental class of the diagonal $\Delta \subseteq N$. Then

$$\begin{array}{ccccc} Q_N([\Delta],[\Delta]) & = & Q_{S^n \times S^n}([\Delta],[\Delta]) & = & Q_{S^n \times S^n}([0 \times S^n] + [S^n \times 0], [0 \times S^n] + [S^n \times 0]) \\ & & \uparrow & & \uparrow \end{array}$$

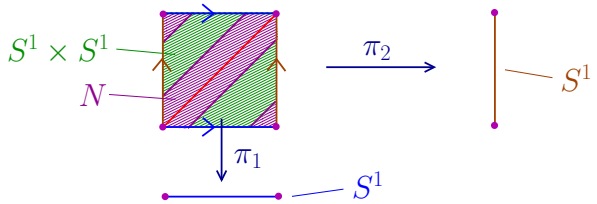
follows from the Intersection Form Submanifold Proposition 225.9

by the $(S^k \times S^l)$-H_*-Lemma 164.7, see page 3142 for details

$$= \underset{\uparrow}{} (1 \quad 1) \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + (-1)^n.$$

follows from the Product-of-Spheres–Cup Product Lemma 207.3, see also the discussion on page 4331

(3) Let $\iota \colon N \to S^n \times S^n$ be the inclusion. We denote by $\pi_i \colon S^n \times S^n \to S^n$, $i = 1, 2$ the two natural projections.



[3040] We picked $-\mathfrak{e}_{n+1}$ instead of $\mathfrak{e}_{n+1}$, since this leads to a clear illustration in the figure below.

We can now perform the following calculation:[3041][3042]

$$
\begin{array}{lllll}
 & \text{by Definition 281.1} & \text{by the Tangent Bundle Pullback Lemma 265.41} & & \text{by the Naturality-of Pontryagin-Proposition 276.5} \\
p(N) & \overset{\downarrow}{=} p(\mathrm{T}N) & \overset{\downarrow}{=} p(\iota^*\mathrm{T}(S^n \times S^n)) & & \overset{\downarrow}{=} \iota^*(p(\mathrm{T}(S^n \times S^n))) \\
 & & \underset{\uparrow}{=} \iota^*(p(\pi_1^*\mathrm{T}S^n \oplus \pi_2^*\mathrm{T}S^n)) & & \underset{\uparrow}{=} \iota^*(p(\pi_1^*\mathrm{T}S^n) \cup p(\pi_2^*\mathrm{T}S^n)) \\
 & & \text{by the Tangent Bundle-of Product Proposition 268.6} & & \text{by the Pontryagin Multiplicativity Theorem 276.8 (2)} \\
 & & \underset{\uparrow}{=} \iota^*(\pi_1^*(p(\mathrm{T}S^n)) \cup \pi_2^*(p(\mathrm{T}S^n))) & & \underset{\uparrow}{=} \iota^*(\pi_1^*(1) \cup \pi_2^*(1)) \quad = \quad 1. \\
 & & \text{by the Naturality-of Pontryagin-Proposition 276.5} & & \text{see page 5317}
\end{array}
$$

This is equivalent to saying that $p_k(N) = 0$ for all $k \in \mathbb{N}$. ■

Proof of the 4-Handle Attachment Proposition 282.1. We continue with the notation from the Diagonal–Tubular Neighborhood Lemma 282.2. First note that the projection $\pi_1 \colon \Delta = \{(x,x) \in S^4 \times S^4 \,|\, x \in S^4\} \to S^4$ onto the first coordinate is clearly a diffeomorphism. This shows that the map $\pi_1 \circ \mathrm{pr} \colon N \to S^4$ has the following properties:

(1) It defines a smooth $(\overline{B}^4, \mathrm{SO}(4))$-bundle $\pi_1 \circ \mathrm{pr} \colon N \to S^4$.
(2) If $\sigma \colon S^4 \to N$ is the 0-section, then $Q_N([\sigma(S^4)], [\sigma(S^4)]) = 1 + (-1)^4 = 2$.
(3) For all $k \in \mathbb{N}$ we have $p_k(N) = 0$.

It follows from this observation together with the Sphere Bundle-over-Sphere Proposition 241.20 (1), the Whitney Approximation Theorem 36.1 and the Ball–Handle Attachment Lemma 246.9 (2) that there exists a thickened 3-sphere $\varphi \colon \overline{B}^4 \times S^3 \to S^7 = \partial \overline{B}^8$ such that $\overline{B}^8 \cup_\varphi h^4$ is diffeomorphic to N and such that $\varphi(\{0\} \times S^3)$ is the standard trivial knot U from Definition 258.2.



It follows from the above together with the Middle Dimension Handles Proposition 259.4 (3) that φ has the following properties:

(1) $\varphi(\{0\} \times S^3) = U$.
(2) We have $\mathrm{slk}(\varphi) = 2$.
(3) For all $k \in \mathbb{N}$ we have $p_1(\overline{B}^8 \cup_\varphi h^4) = 0$.

Finally let K be any trivial 3-dimensional knot $K \subseteq S^7$. By the Isotopy Extension Theorem 44.2 there exists a diffeotopy $F \colon S^7 \times [0,1] \to S^7$ with $F_0 = \mathrm{id}$ and such that F_1 restricts to a diffeomorphism $U \to K$. It is now clear that $F_1 \circ \varphi \colon \overline{B}^4 \times S^3 \to S^7$ has all the desired properties. ■

282.2. **Existence of exotic spheres.** In this section we will reach the high point of the lecture notes, namely the construction of exotic 7-spheres. For the construction we once again to need to roll out the E_8-matrix:

[3041] In the argument below we show in particular that the Pontryagin classes of $S^n \times S^n$ are trivial. This can also be shown using Exercise 268.8, where we showed that the product of two stably parallelizable smooth manifolds is again stable parallelizable.
[3042] Using the Pontryagin-of-Product Proposition 281.2 one could also shorten the argument below.

Notation 282.3. We set

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

The E_8-matrix allows us to introduce a remarkable smooth manifold:

Definition 282.4. It follows from the Linking Number–Realization Proposition 258.11 and the 4-Handle Attachment Proposition 282.1 that there exist orientation-preserving thickened 3-spheres $\varphi_1, \dots, \varphi_8 \colon \overline{B}^4 \times S^3 \to S^7$ with disjoint images with the following properties:

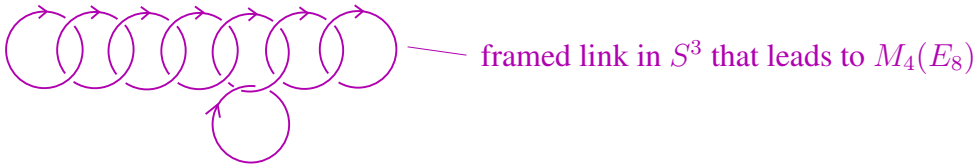
(1) The corresponding linking matrix[3043] $\mathrm{lk}(\varphi_1, \dots, \varphi_8)$ equals E_8,
(2) Each $\varphi_i|_{\{0\}\times S^3}$ is smoothly isotopic to the standard embedding of S^3 into S^7.
(3) For each $i \in \{1, \dots, 8\}$ we have $p_1(\overline{B}^4 \cup_{\varphi_i} h^4) = 0$.

We refer to

$$M_8(E_8) := \overline{B}^8 \cup_{\varphi_1} h^4 \cup \dots \cup_{\varphi_8} h^4.$$

as an 8-dimensional E_8-manifold.[3044]

Remark. In Definition 259.16 we studied the 4-dimensional analog of Definition 282.4, namely the 4-dimensional smooth manifold $M_4(E_8)$.



The really interesting aspect of the smooth manifold $M_4(E_8)$ was its boundary, which by the Poincaré-$\mathbb{Z}$HS^3–E_8-Manifold Proposition 259.17 turned out to be the Poincaré Homology Sphere which we originally introduced in Definition 183.27. We will soon see that the boundary of any $M_8(E_8)$-manifold is also exceedingly interesting. □

First we collect what we know about various invariants of the 8-dimensional E_8-manifold $M_8(E_8)$ and of its boundary:

Proposition 282.5. ($M_8(E_8)$-Invariants Proposition) Let $M_8(E_8)$ be an 8-dimensional E_8-manifold as in Definition 282.4.

(1) $M_8(E_8)$ is a compact 8-dimensional smooth manifold that admits a unique orientation such that the inclusion $\overline{B}^8 \to M_8(E_8)$ is orientation-preserving.
(2) (a) For every $i \neq 0$ the map

$$\bigoplus_{k=1}^{8} \iota_{k*} \colon \bigoplus_{k=1}^{8} \mathrm{H}_i(\overline{B}^8 \cup_{\varphi_k} h^4) \to \mathrm{H}_i(M_8(E_8)),$$

[3043] Recall that we introduced the self-linking matrix in Definition 259.5.
[3044] Note that we do not claim that $M_8(E_8)$ is uniquely defined by the conditions.

which is induced by the inclusions $\iota_k \colon \overline{B}^8 \cup_{\varphi_k} h^4 \to M_8(E_8)$, is an isomorphism.
(b) We have $\mathrm{H}_i(M_8(E_8)) = 0$ for $i \neq 0, 4$.
(3) The intersection form $Q_{M_8(E_8)}$ is represented by the E_8-matrix.
(4) We have $p_1(M_8(E_8)) = 0 \in \mathrm{H}^4(M_8(E_8); \mathbb{Z})$.

Proof. The first three statements follow immediately from the Middle Dimension Handles Proposition 259.4. Thus it remains to prove the last statement, i.e. we need to show that $p_1(M_8(E_8)) = 0 \in \mathrm{H}^4(M_8(E_8); \mathbb{Z})$. For $k = 1, \dots, 8$ we denote by $\iota_k \colon \overline{B}^8 \cup_{\varphi_k} h^4 \to M_8(E_8)$ the inclusion map.

We start out with the following cohomological analog of Statement (2a):

Claim 1. For each $i \in \mathbb{N}$ the map

$$\prod_{k=1}^{8} \iota_k^* \colon \mathrm{H}^i(M_8(E_8); \mathbb{Z}) \to \bigoplus_{k=1}^{8} \mathrm{H}^i(\overline{B}^8 \cup_{\varphi_k} h^4; \mathbb{Z})$$

is an isomorphism.

Proof. We consider the following diagram:

$$\begin{array}{ccccc}
\bigoplus_{k=1}^{8} \mathrm{H}^i(\overline{B}^8 \cup_{\varphi_k} h^4; \mathbb{Z}) & & \xleftarrow{\prod_{k=1}^{8} \iota_k^*} & & \mathrm{H}^i(M_8(E_8); \mathbb{Z}) \\
\downarrow{\scriptstyle \oplus \mathrm{ev}}_{\cong} & & & & \downarrow{\scriptstyle \mathrm{ev}}_{\cong} \\
\bigoplus_{k=1}^{8} \mathrm{Hom}(\mathrm{H}_i(\overline{B}^8 \cup_{\varphi_k} h^4), \mathbb{Z}) & \longrightarrow & \mathrm{Hom}\left(\bigoplus_{k=1}^{8} \mathrm{H}_i(\overline{B}^8 \cup_{\varphi_k} h^4), \mathbb{Z}\right) & \xleftarrow{(\oplus \iota_k)^*} & \mathrm{Hom}(\mathrm{H}_i(M_8(E_8)), \mathbb{Z}).
\end{array}$$

$$\mathrm{Hom}\left(\bigoplus_{k=1}^{8} \mathrm{H}_i(\overline{B}^8 \cup_{\varphi_k} h^4), \mathbb{Z}\right) \xrightarrow{\oplus(\iota_k^*)} \bigoplus_{k=1}^{8} \mathrm{Hom}(\mathrm{H}_i(\overline{B}^8 \cup_{\varphi_k} h^4), \mathbb{Z})$$

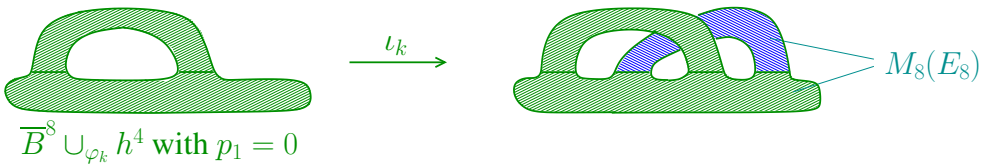
We make the following observations:

- It follows from (2a) that the bottom right horizontal map is an isomorphism.
- The bottom left horizontal isomorphism is just given by elementary algebra.[3045]
- Since all homology groups are torsion-free we obtain from the Universal Coefficient-H*-Theorem 201.23 that the vertical (which are just evaluations maps) maps are isomorphisms.
- The diagram commutes by inspection.

It follows that the composition of the two bottom horizontal maps is an isomorphism. But that was our goal all along. ⊞

By Claim 1 it remains to prove the following claim:

Claim 2. For each $k \in \{1, \dots, 8\}$ we have $\iota_k^*(p_1(M_8(E_8))) = 0 \in \mathrm{H}^4(\overline{B}^8 \cup_{\varphi_k} h^4; \mathbb{Z})$.



[3045]Note though that we do use that for abelian groups finite products are the same as finite direct sums.

Proof. Let $k \in \{1, \ldots, 8\}$. We have the following equalities in $\mathrm{H}^4(\overline{B}^8 \cup_{\varphi_k} h^4; \mathbb{Z})$:[3046]

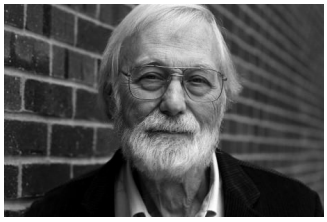
$$\iota_k^*(p_1(M_8(E_8))) \overset{\text{by Definition 281.1}}{=} \iota_k^*(p_1(\mathrm{T}M_8(E_8))) \overset{\text{by the Naturality-of Pontryagin-Proposition 276.5}}{=} p_1(\iota_k^*\mathrm{T}M_8(E_8)) \overset{\text{by the Tangent Bundle Pullback Lemma 265.41}}{=} p_1(\mathrm{T}(\overline{B}^8 \cup_{\varphi_k} h^4)) \overset{\text{by construction of } M_8(E_8)}{=} 0.$$

■

We are now finally ready to prove the following theorem:

Theorem 282.6. (Exotic 7-Sphere Theorem) There exists a closed 7-dimensional smooth manifold that is homeomorphic to S^7 but that is not diffeomorphic to S^7.

Remark. The Exotic 7-Sphere Theorem 282.6 is easily one of the most influential theorems in geometric topology. It was first proved by John Milnor [**Miln1956a**] in 1956 and it gained John Milnor a well-deserved Fields medal in 1962. (The story of finding an exotic 7-dimensional sphere is also told in [**Miln2000**].)



John Milnor (∗1931)



The initial construction by John Milnor was given by considering S^3-bundles over S^4. In our proof we follow the approach from [**Miln1964b**]. □

Proof. Let $M_8(E_8)$ be an 8-dimensional E_8-manifold as in Definition 282.4. The theorem follows from the following claim:

Claim. The boundary $\partial M_8(E_8)$ is a closed 7-dimensional smooth manifold that has the following two properties:
(1) $\partial M_8(E_8)$ is homeomorphic to S^7.
(2) $\partial M_8(E_8)$ is not diffeomorphic to S^7.

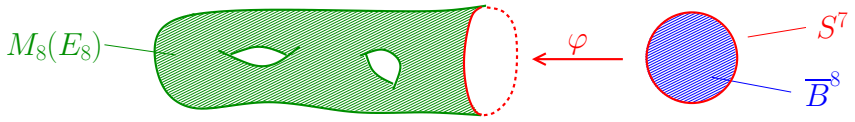
Proof. We now prove the two properties of $\partial M_8(E_8)$:

(1) First note that it follows from the $M_8(E_8)$-Invariants Proposition 282.5, the E_8-Lemma 259.15 (which says that the matrix E_8 is invertible) and the Boundary Invariants Proposition 259.11 (4) that $\partial M_8(E_8) = \partial(\overline{B}^8 \cup_{\varphi_1} h^4 \cup \cdots \cup_{\varphi_8} h^4)$ is a smooth homotopy 7-sphere.

Using the deep fact that $7 \geq 6$ we obtain from the High-Dimensional Poincaré Conjecture 264.7 that $\partial M_8(E_8)$ is homeomorphic to S^7.

[3046]There is a slight cheat in the argument: The subspace $H_k := \overline{B}^8 \cup_{\varphi_k} h^4$ is a smooth submanifold of $M_8(E_8)$ only after introducing corners. There are two approaches to get around this issue: The easier one is to consider the subspace $H_k \setminus \partial H_k$, which is a codimension-zero submanifold of both H_k and $M_8(E_8)$ and which is homotopy equivalent to H_k. Alternatively one could introduce Pontryagin classes of smooth manifolds with corner and study their properties.

(2) We need to show that $\partial M_8(E_8)$ is not diffeomorphic to S^7. We perform a proof by contradiction. So let us assume that there exists a diffeomorphism $\varphi\colon S^7 \to \partial M_8(E_8)$. Then we could use the Gluing-Smooth Manifolds-Proposition 54.5 to glue $\overline{B}^8$ along φ to obtain a closed 8-dimensional smooth manifold $W := M_8(E_8) \cup_\varphi \overline{B}^8$.



We make the following observations regarding W:

(a) Note that

$$\operatorname{sign}(W) \;=\; \operatorname{sign}(M_8(E_8) \cup_\varphi \overline{B}^8) \underset{\text{as in the Manifold Signature Additivity Lemma 226.17}}{\uparrow =} \operatorname{sign}(M_8(E_8)) \underset{\text{by the } M_8(E_8) \text{ Invariants Proposition 282.5}}{\uparrow =} \operatorname{sign}(E_8) \underset{\text{by the } E_8\text{-Lemma 259.15}}{\uparrow =} 8.$$

(b) We denote by $\iota\colon M_8(E_8) \to W$ the natural smooth embedding. We have the following equalities in $\mathrm{H}^4(M_8(E_8);\mathbb{Z})$:

$$\iota^*(p_1(W)) \underset{\text{by Definition 281.1}}{\downarrow =} \iota^*(p_1(\mathrm{T}W)) \underset{\text{by the Naturality-of Pontryagin-Proposition 276.5}}{\uparrow =} p_1(\iota^*\mathrm{T}W) \underset{\text{by the Tangent Bundle Pullback Lemma 265.41}}{\downarrow =} p_1(\mathrm{T}M_8(E_8)) \underset{\text{by the } M_8(E_8)\text{-Invariants Proposition 282.5}}{\uparrow =} 0.$$

An elementary argument, say using the Mayer–Vietoris-H*-Theorem 200.11, shows that the induced map $\iota^*\colon \mathrm{H}^4(W;\mathbb{Z}) \to \mathrm{H}^4(M_8(E_8);\mathbb{Z})$ is an isomorphism. It thus follows from the above calculation that $p_1(W) = 0$.

We now see that

$$8 \underset{\text{by (a)}}{\uparrow =} \operatorname{sign}(M_8(E_8)) \underset{\text{by the Hirzebruch Signature Theorem 281.14}}{\uparrow =} \langle \tfrac{1}{45}(7p_2(W) - p_1(W)^2), [W]\rangle \underset{\text{by (b)}}{\uparrow =} \tfrac{1}{45} \cdot 7 \cdot \langle p_2(W), [W]\rangle.$$

This implies that $\langle p_2(W), [W]\rangle = \frac{8\cdot 45}{7}$, which is of course absurd, since $p_2(W)$ needs to be an integer. ∎

282.3. **Consequences of the Exotic 7-Sphere Theorem 282.6.** The Exotic 7-Sphere Theorem 282.6 has many consequences. For example it gives an affirmative answer to our earlier Question 259.12 (2) and it gives a negative answer to our earlier Question 264.8. In this section we will summarize several other consequences of the Exotic 7-Sphere Theorem 282.6. To formulate the first result it is convenient to recall Definition 25.21:

Definition 282.7. Let M be a topological manifold. We say that two smooth structures $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exists a diffeomorphism between $(M, \mathcal{A})$ and $(M, \mathcal{B})$.

We can now formulate the following theorem, which is easily seen to be logically equivalent to the Exotic 7-Sphere Theorem 282.6:

Theorem 282.8. (7-Sphere–Smooth Structures Theorem) The topological manifold S^7 admits two different smooth structures.

Proof. Let $\mathcal{A}$ be the standard smooth structure on S^7. By the Exotic 7-Sphere Theorem 282.6 we know that there exists a closed 7-dimensional smooth manifold $(\Sigma, \mathcal{B})$ that admits a homeomorphism $f\colon \Sigma \to S^7$, but such that Σ is not diffeomorphic to S^7.



Using the homeomorphism $f^{-1}\colon S^7 \to \Sigma$ we can turn the smooth atlas $\mathcal{B}$ on Σ into a smooth atlas $(f^{-1})^*\mathcal{B}$ on S^7 by precomposing every chart in $\mathcal{B}$ by f^{-1}. Evidently the map $f\colon (S^7, f^{-1})^*\mathcal{B}) \to (\Sigma, \mathcal{B})$ is a diffeomorphism. Since being diffeomorphic is an equivalence relation we now see that $(S^7, f^{-1})^*\mathcal{B})$ is not diffeomorphic to $(S^7, \mathcal{A})$. ■

In the Glue-in-Disk Lemma 255.7 we saw that for a compact 2-dimensional smooth manifold there is (up to diffeomorphism) a unique way to capping of the boundary circles. Now we see that the higher-dimensional analog does not hold:

Proposition 282.9. (Ball Gluing–Non-Uniqueness Proposition) There exists an orientation-preserving diffeomorphism $\varphi\colon S^6 \to S^6$ such that the resulting smooth manifold $\overline{B}^7 \cup_\varphi \overline{B}^7 =: \overline{B}^7 \cup_\varphi h^7$ is not diffeomorphic to $\overline{B}^7 \cup_{\mathrm{id}} \overline{B}^7 =: \overline{B}^7 \cup_{\mathrm{id}} h^7$.

Proof. Let M be an exotic 7-dimensional sphere. By the High-Dimensional Poincaré Conjecture 264.7 (1) we know that M admits a handle decomposition consisting of a single 0-handle and a single 7-handle. ■

Proof. First recall that by the Exotic 7-Sphere Theorem 282.6 we know that there exists a homotopy 7-sphere Σ that is not diffeomorphic to S^7. Next note that it follows from the Homotopy Ball-Diffeomorphism Theorem 264.2 and the Homotopy Sphere–Handle Decomposition Theorem 264.4 that there exists an orientation-preserving diffeomorphism $\varphi\colon S^6 \to S^6$ such that Σ is diffeomorphic to $\overline{B}^7 \cup_\varphi \overline{B}^7$.

On the other hand we showed on page 1229 that the sphere S^7 is diffeomorphic to $\overline{B}^7 \cup_{\mathrm{id}} \overline{B}^7$. Since S^7 is not diffeomorphic to Σ we have achieved our goal. ■

Proposition 282.10. (Exotic Diffeomorphism Proposition) There exists an orientation-preserving diffeomorphism $\varphi\colon S^6 \to S^6$ that is not diffeotopic to the identity.

Proof. By the Ball Gluing–Non-Uniqueness Proposition 282.9 we know that there exists an orientation-preserving diffeomorphism $\varphi\colon S^6 \to S^6$ such that the resulting smooth manifold $\overline{B}^7 \cup_\varphi \overline{B}^7 =: \overline{B}^7 \cup_\varphi h^7$ is not diffeomorphic to $\overline{B}^7 \cup_{\mathrm{id}} \overline{B}^7 =: \overline{B}^7 \cup_{\mathrm{id}} h^7$. It follows from the Diffeotopic-Gluings Lemma 54.7 (or alternatively from the Handle Attachment Proposition 246.3 (5)) that φ is not diffeotopic to id. ■

Let $n \in \mathbb{N}_0$. In the Alexander Trick Proposition 264.6 we showed that every homeomorphism $\varphi\colon S^n \to S^n$ extends to a homeomorphism of $\overline{B}^{n+1}$. In Question 250.13 we asked whether the analogous statement also holds if we replace "homeomorphism" by "diffeomorphism". The following proposition gives a negative answer to this question in dimension 6:

Proposition 282.11. (Diffeomorphism–Non Extension Proposition) There exists a diffeomorphism $\varphi\colon S^6 \to S^6$ that does not extend to a diffeomorphism of $\overline{B}^7$.

Proof. This result follows from the Exotic 7-Sphere Theorem 282.6, the Homotopy Sphere–Handle Decomposition Theorem 264.4 and the Two Handles Manifold Proposition 264.5 (S). ■

The above results show that some statements which hold in low dimensions do not extend to higher dimensions.

282.4. Exercises for Chapter 282.

Exercise 282.1. Give an example of two closed oriented connected n-dimensional smooth manifolds M and M', submanifolds $X \subseteq M$ and $X' \subseteq M'$ that are diffeomorphic to $\overline{B}^n$ and an orientation-reserving diffeomorphism $\varphi\colon \partial X \to \partial X'$ such that the smooth manifold $(M \setminus \mathring{X}) \cup_{\varphi\colon \partial X \to \partial X'} (M \setminus \mathring{X}')$ is *not* diffeomorphic to the connected sum $M \# N$.
Remark 1. This exercise is the companion to Exercise 224.10 where we showed that in the setting of topological manifolds the analog of the above construction does in fact give a topological manifold that is homeomorphic to $M \# N$.
Remark 2. This exercise shows that in the Definition 54.12 of the connected sum it would be unsafe to glue the two boundary spheres via any old orientation-preserving diffeomorphism.

Exercise 282.2. Let $n \in \mathbb{N}_0$. Show that for each $Q \in S^n$ there exists an open neighborhood U of $Q \in S^n$ and a smooth map $\Theta\colon B^n \to \mathrm{SO}(n+1)$ such that the map

$$\begin{array}{rcl} B^n & \to & U \\ x & \mapsto & \Theta(x) \cdot \mathfrak{e}_{n+1} \end{array}$$

is a diffeomorphism.
Hint. First consider the case $Q = \mathfrak{e}_{n+1}$.

Exercise 282.3. Is it possible to attach eight 2-handles to $\overline{B}^4$ along the E_8-matrix such that the boundary of the resulting 4-dimensional smooth manifold is simply connected?
Hint. Look up facts on 3-dimensional and 4-dimensional smooth manifolds.

Exercise 282.4. Show that S^{11} admits two inequivalent smooth structures.

Exercise 282.5. Let $n \in \mathbb{N}_0$ and let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. We recall Definition 54.12 of the connected sum $M \# N$: We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N$. We define the connected sum of M and N as

$$M \# N \coloneqq (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n))/\sim \quad \text{where } P \sim (\psi \circ \varphi^{-1})(P) \text{ for all } P \in \varphi(S^{n-1}).$$

Here note that $\psi \circ \varphi^{-1}\colon \varphi(S^{n-1}) \to \psi(S^{n-1})$ is an orientation-reserving diffeomorphism. By the Smooth Connected Sum-Proposition 54.13 (5) we know that the diffeomorphism type of $M \# N$ does not depend on the choice of φ and ψ (with the correct orientation behavior). Give an example which shows, that a different choice of orientation-reversing diffeomorphism $\theta\colon \varphi(S^{n-1}) \to \psi(S^{n-1})$ can lead to a smooth manifold that is not diffeomorphic to $M \# N$.

Part XXXII

Knots and links: Homology and cohomology

In Part XII we introduced knots and links. We discussed many examples and we introduced several important notions. Furthermore we used the fundamental group of the complement of links to distinguish links. The most sophisticated concept we introduced was the Alexander polynomial of an oriented link, which we defined using fundamental groups and Fox calculus.

Now, many parts later, we are by now well acquainted with homology and cohomology groups. In this part we will see what these new invariants can tell us about knots and links. As it turns out, we were so successful in Part XII, that at this stage we gain relatively little from homology and cohomology. The most interesting aspect is that we introduce the notion of the Alexander module of an oriented link and we use it to give a new (and equivalent) definition of the Alexander polynomial. For knots this gives us a new and fun method for calculating the Alexander polynomial, which in many settings is much more efficient than our previous method.

This part is mostly independent of the Part XII, we will only make use of some very basic notions which we introduced in Part XII. The discussion of (co-) homology-based invariants of knots and links in this part is for the most part independent of the discussion of fundamental group-based invariants in Part XII.

This part of the lecture notes contains the following chapters:

- In Chapter 283 we will give another short introduction to knots and links and we will calculate the homology groups of complements of knots and links.
- In Chapter 284 we will use the calculation of homology groups to (re-) introduce linking numbers.
- In Chapter 285 we will (re-) introduce Seifert surfaces and we will give a new proof that any link in S^3 admits a Seifert surface.
- In Chapter 286 we will introduce the rather general concept of Alexander modules.
- In Chapter 287 we will use the Alexander modules to introduce Alexander polynomials.
- In Chapter 288 we will apply the general concept of Alexander modules and Alexander polynomials to oriented knots and links.
- In Chapter 289 we will introduce Seifert matrices and we will see that these can be used to calculate the single-variable Alexander polynomials of oriented links.
- In Chapter 290 we will discuss some of the topological content of the Alexander polynomial, for example, we will relate it to the knot genus and the fiberedness of knots.
- In Chapter 291 we will show that any two Seifert surfaces of an oriented knot are tube-equivalent. This will allow us to show that any two Seifert matrices are S-equivalent.
- In Chapter 292 we will use Seifert matrices to introduce the Levine-Tristram signatures of an oriented link.
- In Chapter 293 we will see that the Levine-Tristram signatures give rise to lower bounds on the Gordian distance between links and on the smooth 4-genus of a knot.
- In Chapter 294 we will introduce the knot concordance group and we will use Levine-Tristram signatures to show that the knot concordance group is non-trivial.
- In Chapter 295 we will use Seifert matrices to (re-) introduce the Conway polynomial of any oriented link.

CHAPTER 283

Introduction to knots and links II

In Chapter 108 we already gave an introduction to knots and links in S^3. To keep this part of the lecture notes largely self-contained we will recall some of the concepts and results from Chapter 108. Furthermore we will provide a calculation of the homology groups of the complements of knots and links.

This chapter is organized as follows:

- In Section 283.1 we will (re-) introduce knots and links.
- In Section 283.2 we will (re-) introduce tubular maps and exteriors of links.
- In Section 283.3 we will (re-) introduce the notion of a meridian of an oriented knot.
- In Section 283.4 we will determine the homology groups of link complements. We will also give explicit bases of these homology groups.
- In Section 283.5 we will provide exercises for Chapter 283.

In the next Chapter 284 we will introduce the linking number of two disjoint oriented knots in S^3.

283.1. **(Re-) introduction to knots and links.** We start out by recalling Definition 108.5 of knots and links:

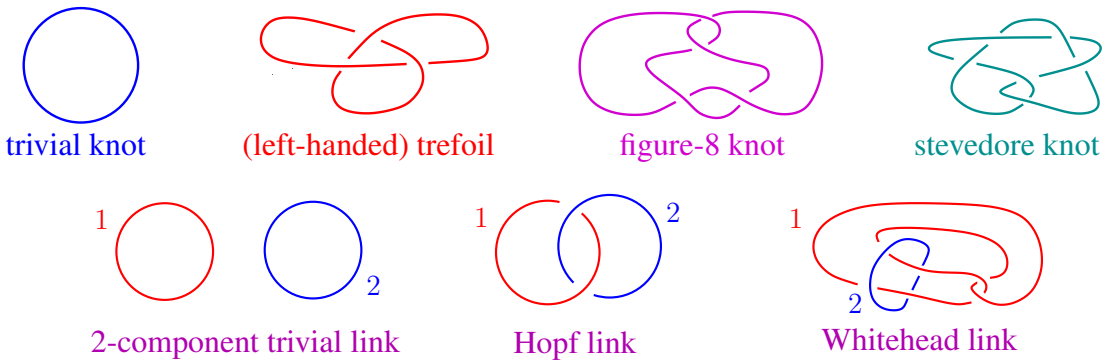
Definition 108.5. Let $m \in \mathbb{N}_0$.

(1) (a) An (oriented) m-component link is an (oriented) submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1 together with a bijection from $\{1, \ldots, m\}$ to the set of components of L.

(b) A 1-component link is called a knot.

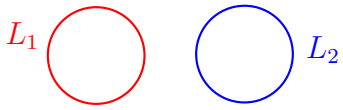
(2) Given an m-component link $L \subseteq S^3$ and given $i \in \{1, \ldots, m\}$ we denote by L_i the i-th component of L.

Example. In the following figures we provide some examples of knots and links:



We will use Identifiation 108.2 to make the identification $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ and we will view $\mathbb{R}^3$ as a subspace of S^3. □

Example. In most cases we will not bother with giving a completely rigorous definition of a knot or link under consideration.



trivial 2-component link

But we make one exception: Let $m \in \mathbb{N}_0$. As in Definition 108.6 we define the

$$\text{standard trivial } m\text{-component link} := \bigsqcup_{i=1}^{m} \{(x, y + 3(i-1), 0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$$

as a submanifold of $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$. The trivial knot is sometimes also referred to as the *unknot*. □

Next we recall Definition 108.7:

Definition 108.7. Let $m \in \mathbb{N}_0$. We say that two m-component links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_m$ are smoothly isotopic if there exists a smooth isotopy from L to $\widetilde{L}$, i.e. if there exists a smooth map

$$\begin{array}{rcl} F\colon L \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:

(1) For each $t \in [0,1]$ the map $F_t\colon L \to S^3$ is a smooth embedding.
(2) We have $F_0 = \mathrm{id}$ and for each $i \in \{1, \ldots, m\}$ we have $F_1(L_i) = \widetilde{L}_i$.

If L and $\widetilde{L}$ are oriented links, then we also demand that the smooth isotopy is orientation-preserving, i.e. the map $F_1\colon L \to \widetilde{L}$ is orientation-preserving.

Example. One can easily convince oneself that the three knots shown in the figure below are smoothly isotopic.



Usually we do not distinguish between two knots or links if they are smoothly isotopic. In particular we say that an m-component link is *trivial* if it is smoothly isotopic to the above standard trivial link. For instance, all three knots in the above figure are called trivial knot. □

The following proposition gives us a criterion for two links to be smoothly isotopic:

Proposition 283.1. (Smooth Isotopy Characterization Proposition) Let L and $\widetilde{L}$ be two m-component links in S^3. The following two statements are equivalent:

(1) The links L and $\widetilde{L}$ are smoothly isotopic.
(2) There exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \mathrm{id}$ and such that for each $i \in \{1, \ldots, m\}$ we have $F_1(L_i) = \widetilde{L}_i$.
(3) There exists an orientation-preserving self-diffeomorphism $f\colon S^3 \to S^3$ such that for each $i \in \{1, \ldots, m\}$ we have $f(L_i) = \widetilde{L}_i$.

Proof. The direction (2)⇒(1) is clear. The direction (1)⇒(2) follows from the Isotopy Extension Theorem 44.2. The direction (2)⇒(3) is again clear. The direction (3)⇒(2) is proved in the Smooth Isotopy–Diffeomorphism Proposition 108.11. ■

Next we recall two basic constructions of new knots and links:

Definition 108.21.

(1) Let $K \subseteq S^3$ be an oriented knot. We define the reverse K^{rev} of K to be the knot K with the opposite orientation.

(2) A knot K is called reversible if the two orientations on K give smoothly isotopic oriented knots, otherwise we call the knot irreversible.

Example. In the figure below we show the oriented trefoil and its reverse.



the oriented trefoil K and its reverse K^{rev}

It is a priori not clear whether the trefoil is reversible, i.e. it is not clear whether the above two oriented knots are smoothly isotopic. The goal is to detect methods for determining whether or not a knot is reversible. □

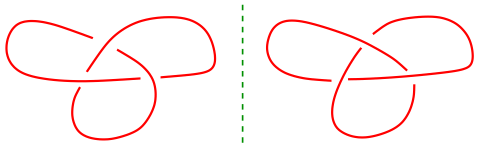
Next we summarize the content of Definition 108.23 and Definition 108.26:

Definition 283.2. Given a link $L \subseteq S^3$.

(1) The reflection of L in any hyperplane of $\mathbb{R}^4$ is called mirror of L and it is denoted by L^{mir}.[3047]

(2) We say that a link L in S^3 is amphichiral if it is smoothly isotopic to its mirror, otherwise we call the link L chiral.

Example. In the figure below we show the trefoil and its mirror.



the trefoil K and its mirror K^{mir}

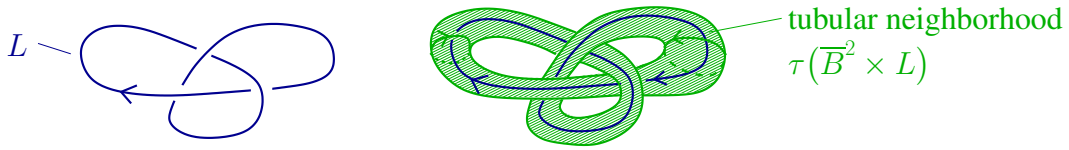
It is a priori not clear whether the trefoil is amphichiral, i.e. it is not clear whether the above two knots are smoothly isotopic. As above the goal is to detect methods for determining whether or not a knot is amphichiral. □

283.2. **Tubular maps and the link exterior.** While we are at it, let us also recall the content of Definition 108.28 and Definition 259.7:

[3047] In the S^3-Link Mirror Lemma 108.24 we showed that the reflections in any two hyperplanes of $\mathbb{R}^4$ give rise to smoothly isotopic links. We also showed if we start out with a link $L \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$, then we can also just take the reflection in any affine hyperplane of $\mathbb{R}^3$.

Definition 283.3. Let $L \subseteq S^3$ be a link.

(1) A tubular map is a smooth embedding $\tau\colon \overline{B}^2 \times L \to S^3$ such that for every $P \in L$ we have $\tau(0, P) = P$.
(2) The image of a tubular map is called a tubular neighborhood of L.
(3) If L is oriented, then an orientation-preserving tubular map is called a thickening of L.[3048]



Next we recall the following theorem, which says that in our setting tubular maps always exist and that they are unique in a suitable sense.[3049]

Theorem 108.29. (Link Tubular Map Theorem) Let $L \subseteq S^3$ be an oriented link.

(1) There exists a thickening $\tau\colon \overline{B}^2 \times L \to S^3$.
(2) Let $\sigma, \tau\colon \overline{B}^2 \times L \to S^3$ be two thickenings for L. In this setting there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ and a smooth map $g\colon L \to \mathrm{SO}(2)$ with the following properties:
   (a) $F_0 = \mathrm{id}$.
   (b) F is a homotopy rel L.
   (c) The diffeomorphism $F_1\colon S^3 \to S^3$ restricts to the map[3050]

$$\begin{array}{rcl} \sigma(\overline{B}^2 \times L) & \to & \tau(\overline{B}^2 \times L) \\ \sigma(v, x) & \mapsto & \tau(g(x) \cdot v, x). \end{array}$$

Next let us remind ourselves of Definition 108.30:

Definition 108.30. Let L be a link in S^3. By the Link Tubular Map Theorem 108.29 (1) there exists a tubular map $\tau\colon \overline{B}^2 \times L \to S^3$. We refer to $X_L \coloneqq S^3 \setminus \tau(B^2 \times L)$ as the exterior of L.

Let us recall one more lemma:

Lemma 108.31. (Link Exterior Lemma) Let L be a link and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map.

(1) The exterior $X_L \coloneqq S^3 \setminus \tau(B^2 \times L)$ of L is a compact smooth submanifold of S^3 with $\partial X_L = \tau(S^1 \times L)$. In particular X_L inherits an orientation from S^3.
(2) The exterior X_L is well-defined up to an orientation-preserving diffeomorphism.
(3) If L and L' are smoothly isotopic links, then there exists an orientation-preserving diffeomorphism between the link exteriors X_L and $X_{L'}$.
(4) The inclusion $X_L \to S^3 \setminus L$ is a homotopy equivalence.

[3048] We equip S^3 with the standard orientation from page 817 and we equip $\overline{B}^2 \times L$ with the product orientation that we introduced in Definition 192.8.

[3049] In fact the next theorem is just a convenient repacking of the Product Tubular Neighborhood–Existence Theorem 58.4 (2) together with the Tubular Map Lemma 58.3 and the Tubular Map–Uniqueness Theorem 58.5.

283.3. **The meridian of an oriented knot.** Next we recall the definition of a meridian:

Definition 109.1. Let K be an oriented knot in S^3. Let $\mu_K \subseteq S^3 \setminus K$ be an oriented smoothly embedded loop. We say that μ_K is a meridian of K if there exists a smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ with the following three properties:

(1) The map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \mu_K$.
(2) The image $\varphi(\overline{B}^2)$ intersects K transversally in a single point P.
(3) A positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ followed by a positive basis for $\mathrm{T}_P K$ gives a positive basis for $\mathrm{T}_P S^3$.

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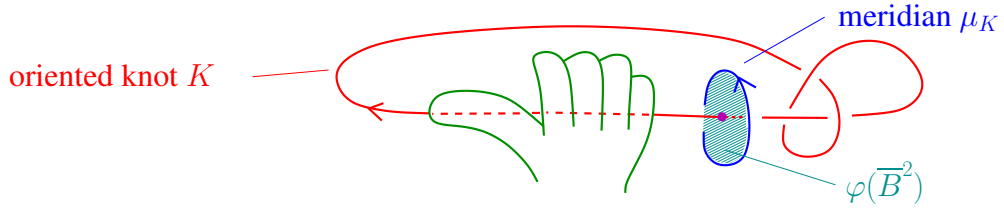
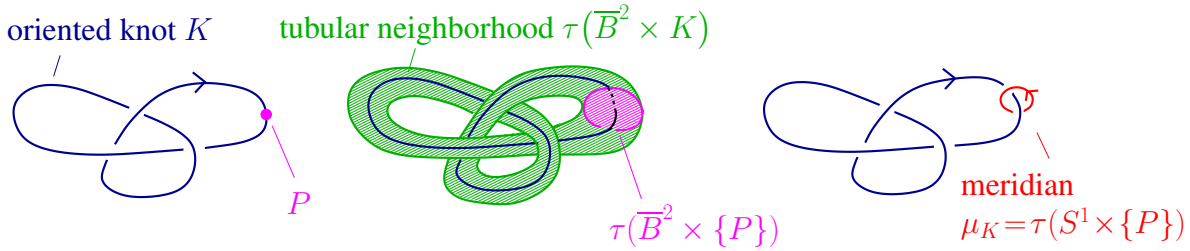


Evidently this is one more occasion where we want to recall a result:

Proposition 109.2. (Meridian Proposition) Let L be an oriented link in S^3 and let K be a component of L.

(1) K admits a meridian.
(2) Let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening. For any $P \in K$ the image $\tau(S^1 \times \{P\})$ is a meridian of K.
(3) Any two meridians for K in $S^3 \setminus L$ are smoothly isotopic in $S^3 \setminus L$.
(4) Given any meridian μ of the component K there exists a thickening $\tau\colon \overline{B}^2 \times L \to S^3$ and a $P \in K$ with $\mu = \tau(S^1 \times \{P\})$.

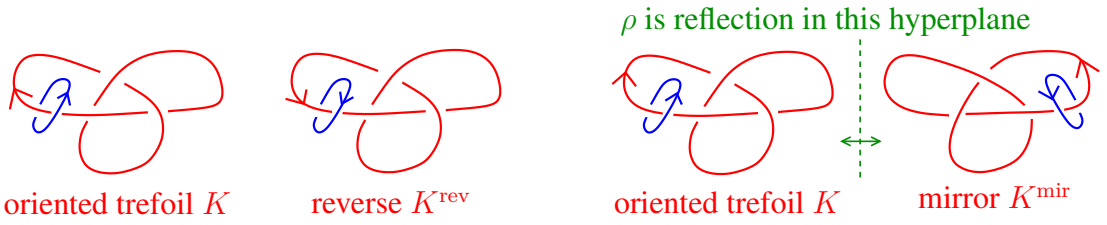


We conclude this short section with a discussion of meridians and symmetries. More precisely, we recall the following lemma:

Lemma 109.3. (Meridian–Symmetries Lemma) Let $K \subseteq S^3$ be an oriented knot with meridian μ_K.

(1) A meridian for the reverse K^{rev} is given by $(\mu_K)^{\mathrm{rev}}$.
(2) Let $\rho\colon S^3 \to S^3$ be a reflection in a hyperplane. Then a meridian for the mirror $K^{\mathrm{mir}} = \rho(K)$ is given by $\rho(\mu_K)^{\mathrm{rev}}$.

Example. In the following figure we consider the trefoil, its meridian and the reverse and the mirror of the trefoil.
This illustrates the Meridian–Symmetries Lemma 109.3. □



283.4. **Homology of link complements.** Next we want to calculate the homology groups of a link complement. Here by "calculate" we mean that we do not only want to determine the isomorphism type of the homology groups, but we also want to provide a basis for the homology groups:[3051] [3052] [3053]

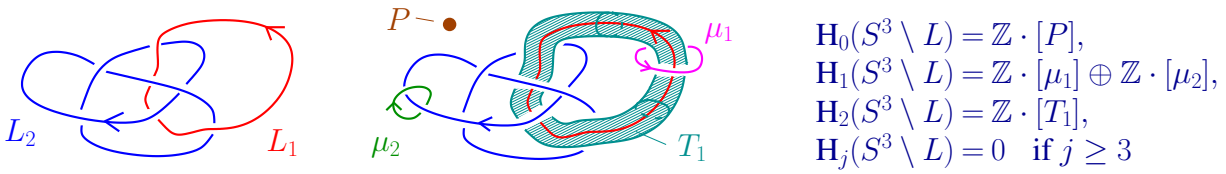
Proposition 283.4. (Link Complement-H_*–Basis-Proposition) Let $m \in \mathbb{N}$ and let $L = L_1 \sqcup \cdots \sqcup L_m$ be an oriented m-component link in S^3. Furthermore let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening for L. The isomorphism types of the homology groups of $S^3 \setminus L$ and of the link exterior $X_L = S^3 \setminus \tau(B^2 \times L)$ are given as follows:

$$\mathrm{H}_n(X_L) \xrightarrow[\text{induced by the inclusion } X_L \to S^3 \setminus L]{\cong} \mathrm{H}_n(S^3 \setminus L) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^m, & \text{if } n = 1, \\ \mathbb{Z}^{m-1}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

We can also provide bases for these homology groups:

(1) A basis for $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^m$ is given by the homology classes that are represented by the meridians $\mu_1, \ldots, \mu_m$ of $L_1, \ldots, L_m$.[3054]

(2) A basis for $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^{m-1}$ is given by considering the fundamental classes of all but one of the boundary tori $T_i := \tau(S^1 \times L_i)$, $i \in \{1, \ldots, m\}$. More precisely, for any $j \in \{1, \ldots, m\}$ a basis for $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^{m-1}$ is given by $[T_1], \ldots, \widehat{[T_j]}, \ldots, [T_m]$.[3055]



Before we turn to the proof of the Link Complement-H_*–Basis-Proposition 283.4 let us mention the following consequence:

[3051] In the Knot Complement-H_*-Proposition 185.29 we computed the isomorphism type of the homology groups of a knot complement. In the Link Complement-H_*-Proposition 217.11 we used the Sphere–Alexander Duality 217.8 to determine the isomorphism type of the homology groups of a link complement. Unfortunately the approach using the Sphere–Alexander Duality 217.8 did not give us a basis for the homology groups.

[3052] Note that the proposition gives in particular an answer to Question 217.12.

[3053] Also note that in the Link Complement-H_*-Lemma 258.6 we proved a more general statement, but for the convenience of the reader we now provide the calculation again.

[3054] It follows from the Meridian Proposition 109.2 (3) together with the Homotopy-H_*-Proposition 144.12 that any two meridians for a component define the same homology class.

[3055] We equip each T_i with the orientation coming from the Boundary Orientation Lemma 31.38.

Proposition 283.5. (Link Complement-Boundary H_*–Proposition) Let L be an oriented non-empty link in S^3. Furthermore let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening for L. We denote by $\iota\colon \partial X_L \to X_L$ the inclusion. There exists a short exact sequence of the following form:

$$0 \to \mathrm{H}_2(X_L, \partial X_L) \xrightarrow{\partial_2} \mathrm{H}_1(\partial X_L) \xrightarrow{\iota_*} \mathrm{H}_1(X_L) \to 0.$$

Furthermore this exact sequence splits in the sense of Definition 145.4.

Proof. We consider the long exact sequence of homology groups of the pair $(X_L, \partial X_L)$:

$$\ldots \to \mathrm{H}_2(\partial X_L) \xrightarrow{\iota_*} \mathrm{H}_2(X_L) \xrightarrow{\text{я}} \mathrm{H}_2(X_L, \partial X_L) \xrightarrow{\partial_2} \mathrm{H}_1(\partial X_L) \xrightarrow{\iota_*} \mathrm{H}_1(X_L) \to \ldots$$

We make the following observations:

(1) It follows from the Link Complement-H_*–Basis-Proposition 283.4 (1) that $\mathrm{H}_1(X_L)$ is generated by the meridians. This implies that the map $i_*\colon \mathrm{H}_1(\partial X_L) \to \mathrm{H}_1(X_L)$ is an epimorphism.

(2) It follows from the Link Complement-H_*–Basis-Proposition 283.4 (2) that $\mathrm{H}_1(X_L)$ is generated by $[T_1], \ldots, [T_m]$. This implies that the map $i_*\colon \mathrm{H}_2(\partial X_L) \to \mathrm{H}_2(X_L)$ is an epimorphism.

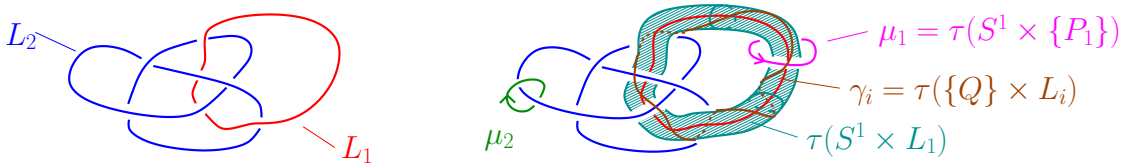
It follows from these observations that the sequence splits. By the Link Complement-H_*–Basis-Proposition 283.4 (1) we know that $\mathrm{H}_1(X_L)$ is a free abelian group. It thus follows from the Free Abelian Group–Uniqueness Lemma 97.6 that the short exact sequence splits. ■

The proof of the Link Complement-H_*–Basis-Proposition 283.4 will require the remainder of this section.

In the following we will first provide a proof of the Link Complement-H_*–Basis-Proposition 283.4. Afterwards we will give (just for fun) a separate argument that illustrates the somewhat counterintuitive fact that the fundamental classes of all but one of the boundary tori form a basis for $\mathrm{H}_2(X_L)$.

Proof of Proposition 283.4. [3056] Let $m \in \mathbb{N}$, let $L = L_1 \sqcup \cdots \sqcup L_m$ be an oriented m-component link in S^3 and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening for L. As usual we consider the link exterior $X_L := S^3 \setminus \tau(B^2 \times L)$. Next note that it follows from the Link Exterior Lemma 108.31 together with the Homotopy Equivalence-H_*-Proposition 144.14 (1b) that the inclusion $X_L \to S^3 \setminus L$ induces an isomorphism on all homology groups. Thus it remains to determine the homology groups of X_L.

For $i = 1, \ldots, m$ we pick a point $P_i \in L_i$. We also pick $Q \in S^1$. For $i = 1, \ldots, m$ we consider the meridian $\mu_i := \tau(S^1 \times \{P_i\})$ and we set $\gamma_i := \tau(\{Q\} \times L)$ and $T_i = \tau(S^1 \times L_i)$.



[3056] This proof is very similar to the proof of the Link Complement-H_*-Lemma 258.6.

We consider the decomposition

$$S^3 \;=\; \underbrace{(S^3 \setminus \tau(B^2 \times L))}_{=X_L} \cup_{\tau(S^1 \times L)} \tau(\overline{B}^2 \times L).$$

It follows from the Codimension-Zero Smooth Embedding Proposition 30.21 that we can apply the Mayer-Vietoris Theorem 153.2 (3) to the above decomposition of S^3. Together with the Path-Component–H_*–Lemma 143.21 we obtain the following long exact sequence:

$$\begin{aligned}
&\to \bigoplus_{i=1}^m \mathrm{H}_3(T_i) \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_3(X_L) \oplus \bigoplus_{i=1}^m \mathrm{H}_3(\tau(\overline{B}^2 \times L_i)) \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \mathrm{H}_3(S^3) \xrightarrow{\partial_3} \\
&\to \bigoplus_{i=1}^m \mathrm{H}_2(T_i) \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_2(X_L) \oplus \bigoplus_{i=1}^m \mathrm{H}_2(\tau(\overline{B}^2 \times L_i)) \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \mathrm{H}_2(S^3) \xrightarrow{\partial_2} \\
&\to \bigoplus_{i=1}^m \mathrm{H}_1(T_i) \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_1(X_L) \oplus \bigoplus_{i=1}^m \mathrm{H}_1(\tau(\overline{B}^2 \times L_i)) \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \mathrm{H}_1(S^3) \xrightarrow{\partial_1} \\
&\to \bigoplus_{i=1}^m \mathrm{H}_0(T_i) \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_0(X_L) \oplus \bigoplus_{i=1}^m \mathrm{H}_0(\tau(\overline{B}^2 \times L_i)) \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \mathrm{H}_0(S^3) \xrightarrow{\partial_0} 0.
\end{aligned}$$

Now we throw in everything we know about the above setting:

$$\begin{aligned}
&\to \bigoplus_{i=1}^m \underbrace{\mathrm{H}_3(T_i)}_{=0} \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_3(X_L) \oplus \bigoplus_{i=1}^m \underbrace{\mathrm{H}_3(\tau(\overline{B}^2 \times L_i))}_{=0} \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \underbrace{\mathrm{H}_3(S^3)}_{=\mathbb{Z}\cdot[S^3]} \xrightarrow{\partial_3} \\
&\to \bigoplus_{i=1}^m \underbrace{\mathrm{H}_2(T_i)}_{=\mathbb{Z}\cdot[T_i]} \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_2(X_L) \oplus \bigoplus_{i=1}^m \underbrace{\mathrm{H}_2(\tau(\overline{B}^2 \times L_i))}_{=0} \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \underbrace{\mathrm{H}_2(S^3)}_{=0} \xrightarrow{\partial_2} \\
&\to \bigoplus_{i=1}^m \underbrace{\mathrm{H}_1(T_i)}_{=\mathbb{Z}\cdot[\mu_i]\oplus\mathbb{Z}\cdot[\gamma_i]} \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_1(X_L) \oplus \bigoplus_{i=1}^m \underbrace{\mathrm{H}_1(\tau(\overline{B}^2 \times L_i))}_{=\mathbb{Z}\cdot[\gamma_i]} \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \underbrace{\mathrm{H}_1(S^3)}_{=0} \xrightarrow{\partial_1} \\
&\qquad\qquad \uparrow \\
&\qquad\qquad \text{for each } i \in \{1,\dots,m\} \\
&\qquad\qquad \text{we have } [\mu_i] \mapsto ([\mu_i], 0) \text{ and} \\
&\qquad\qquad \text{we have } [\gamma_i] \mapsto (?, [\gamma_i]) \\
&\to \bigoplus_{i=1}^m \underbrace{\mathrm{H}_0(T_i)}_{\mathbb{Z}\cdot[\tau(Q,P_i)]} \xrightarrow{(\oplus i_*, -\operatorname{diag}(i_*))} \mathrm{H}_0(X_L) \oplus \bigoplus_{i=1}^m \underbrace{\mathrm{H}_0(\tau(\overline{B}^2 \times L_i))}_{=\mathbb{Z}\cdot[\tau(Q,P_i)]} \xrightarrow{i_* \oplus \bigoplus \oplus i_*} \underbrace{\mathrm{H}_0(S^3)}_{=\mathbb{Z}\cdot[\tau(P_1,Q_1)]} \xrightarrow{\partial_0} 0.
\end{aligned}$$

To better understand this exact sequence we make the following clarifications and observations:

- By the Torus-H_*-Proposition 153.9 we know that $\mathrm{H}_1(T_i) = \mathbb{Z} \cdot [\mu_i] \oplus \mathbb{Z} \cdot [\gamma_i]$.
- By the discussion on page 3751 we know that $[\partial X_L] = [T_1] + \cdots + [T_m]$.
- It follows from the Manifold-Top-H_*-Theorem 193.1 and the Path-Component-H_*-Lemma 143.21 that $\mathrm{H}_2(\partial X_L) = \mathbb{Z} \cdot [T_1] \oplus \cdots \oplus \mathbb{Z} \cdot [T_m]$.
- It follows from the Manifold–MV-Connecting Homomorphism Corollary 194.16 that the connecting homomorphism ∂_3 sends $[S^3]$ to $[\partial X_L] \in \mathrm{H}_2(\partial X_L)$.

We now use the above data to perform the calculations of the homology groups of X_L:

(3) It follows from the above discussion that ∂_3 is a monomorphism which implies that $\mathrm{H}_3(X_L) = 0$.

(2) It follows easily from the above discussion of the connecting homomorphism ∂_3 that we have an isomorphism $(\mathbb{Z} \cdot [T_1] \oplus \cdots \oplus \mathbb{Z} \cdot [T_m])/\mathbb{Z} \cdot ([T_1] + \cdots + [T_m]) \xrightarrow{\cong} \mathrm{H}_2(X_L)$. This implies that we have an isomorphism $\mathrm{H}_2(X_L) \cong \mathbb{Z}^{m-1}$ and that for any $i \in \{1,\dots,m\}$ a basis for $\mathrm{H}_2(X_L)$ is given by the fundamental classes $[T_1], \dots, \widehat{[T_i]}, \dots, [T_m]$.

(1) The second to last line of the exact sequence gives us an isomorphism of the form

$$\bigoplus_{i=1}^{m} (\mathbb{Z} \cdot [\mu_i] \oplus \mathbb{Z} \cdot [\gamma_i]) \xrightarrow[\cong]{\substack{[\mu_i] \mapsto ([\mu_i], 0) \\ [\gamma_i] \mapsto (?, [\gamma_i])}} \mathrm{H}_1(X_L) \oplus \bigoplus_{i=1}^{m} \mathbb{Z} \cdot [\gamma_i]$$

It follows from the Cancellation Proposition 97.26 that $\mathrm{H}_1(X_L) \cong \mathbb{Z}^m$. Since a matrix of the form $\begin{pmatrix} A & B \\ 0 & \mathrm{id} \end{pmatrix}$ is invertible if and only if A is invertible we see that $[\mu_1], \dots, [\mu_m]$ is a basis of $\mathrm{H}_1(X_L) \cong \mathbb{Z}^m$.

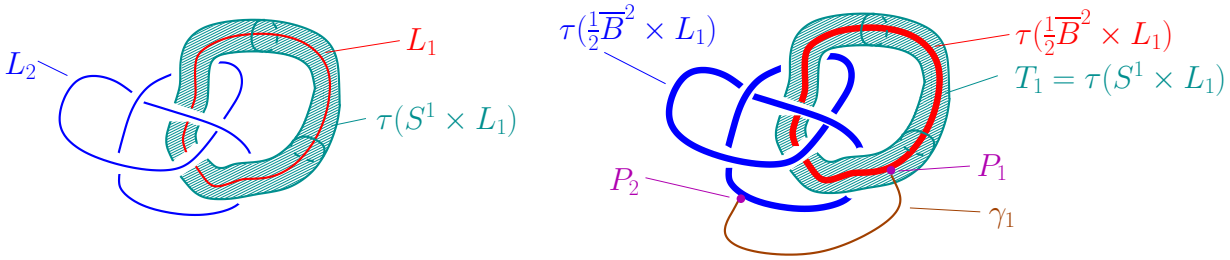
(0) Finally note that it follows easily from the bottom horizontal sequence and the H_0-Proposition 143.25 that $\mathrm{H}_0(X_L) \cong \mathbb{Z}$. ■

Alternative sketch of a proof of Proposition 283.4. Let $L = L_1 \sqcup \dots \sqcup L_m$ be an m-component link in S^3. We know from the Link Complement-H_*-Proposition 217.11 that we have isomorphisms $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^m$ and that $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^{m-1}$. It remains to identify bases of the homology groups.

We pick a thickening $\tau\colon \overline{B}^2 \times L \to S^3$ for L. We consider the corresponding exterior $X_L := S^3 \setminus \tau(B^2 \times L)$.

(1) There are many ways how one can show that the meridians form a basis for $\mathrm{H}_1(X_L)$. The easiest approach might just be the argument in the proof above, which does not build on any fancy results on fundamental classes. But there are other approaches, for example one can also use the Link Group-Abelianization Proposition 113.9 together with the Hurewicz Theorem 166.10.[3057]

(2) We now turn to the much more interesting question, why the fundamental classes of all but one of the boundary tori of X_L give a basis for $\mathrm{H}_2(X_L)$. To simplify the notation we consider the case $j = m$. We write $X' := S^3 \setminus \tau(B^2_{\frac{1}{2}} \times L)$. Furthermore, for $i = 1, \dots, m$ we write $T_i := \tau(S^1 \times L_i)$. For $i = 1, \dots, m$ we pick a point P_i on $\tau(S^1_{\frac{1}{2}} \times L_i) \subseteq \partial X'$. Since $S^3 \setminus \tau(B^2 \times L)$ is path-connected we can pick for each $i \in \{1, \dots, m-1\}$ a path γ_i from P_i to P_m that intersects T_i transversally in a single point and that does not intersect any of the T_j with $j \neq i, m$.



Now we consider the asymmetric intersection pairing

$$\begin{aligned} Q^{\mathrm{as}}_{X',\partial X',\varnothing}\colon \mathrm{H}_1(X', \partial X') \times \mathrm{H}_2(X') &\to \mathbb{Z} \\ (\sigma, \tau) &\mapsto \langle \mathrm{PD}_{X'}(\sigma) \cup \mathrm{PD}_{X'}(\tau), [X'] \rangle \end{aligned}$$

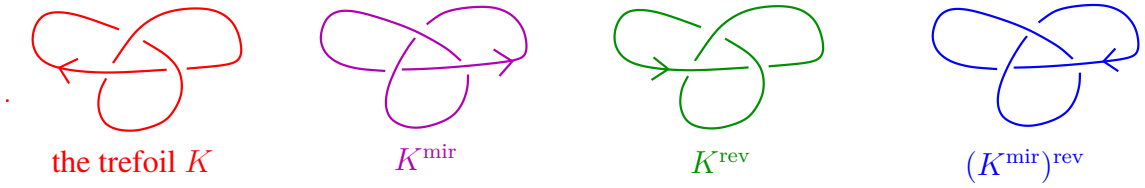
from Definition 225.3. It follows easily from the Maps-Cup Product–Intersection Number Theorem 221.16 that for any choice of $i, j \in \{1, \dots, m-1\}$ we have the equality $Q^{\mathrm{as}}_{X',\partial X',\varnothing}([\gamma_i], [T_j]) = \delta_{ij}$. Since we already know that $\mathrm{H}_2(X_L) \cong \mathbb{Z}^{m-1}$ we now obtain

[3057] One more approach would be to apply the Sphere–Alexander Duality-via-Linking Numbers Lemma 285.10 to $C_i = \{0\} \times L_i$, $i = 1, \dots, m$ and $D_i := \mu_i$, $i = 1, \dots, m$.

from the Kronecker-Basis Lemma 199.10 (1) that $[T_1], \dots, [T_{m-1}]$ do indeed form a basis of $\mathrm{H}_2(X')$. ■

283.5. Exercises for Chapter 283.

Exercise 283.1. In the figure below we show four oriented knots. Find a pair of oriented knots that are smoothly isotopic.



Exercise 283.2. Let $L = L_1 \sqcup \dots \sqcup L_m \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be an m-component link. Determine the homology groups $\mathrm{H}_k(\mathbb{R}^3 \setminus L)$ and describe a basis of these homology groups.

Exercise 283.3. Let K be a knot in $\mathbb{RP}^3$. Determine the homology groups $\mathrm{H}_k(\mathbb{RP}^3 \setminus K)$.
Hint. Your result will depend on the homology class $[K] \in \mathrm{H}_1(\mathbb{RP}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 283.4. We consider the two disjoint knots J and K shown below. Is the link $L = J \cup K$ smoothly isotopic to the link $L = K \cup J$ (with the opposite ordering of the two components)?
Remark. This exercise is identical to Exercise 62.9.

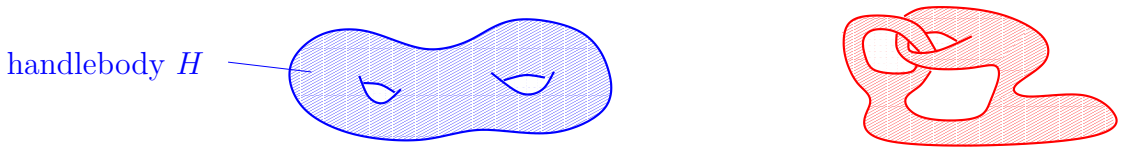


Exercise 283.5. For the purpose of this exercise we say that two knots K and $\widetilde{K}$ are *weakly equivalent* if there exists a continuous map

$$\begin{array}{rcl} F\colon K \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following two conditions hold:
(1) For each $t \in [0,1]$ the map $F_t\colon K \to S^3$ is a smooth embedding.
(2) We have $F_0 = \mathrm{id}$ and we have $F_1(K) = \widetilde{K}$.
If two knots are weakly equivalent, are they also smoothly isotopic in the sense of Definition 108.7?
Remark. The two definitions of "weakly equivalent" and "smoothly isotopic" are almost the same, except that in Definition 108.7 we demanded that F is smooth.
Remark. This exercise is identical to Exercise 108.3.

Exercise 283.6. In the figure below we show two smooth embeddings of the genus two handlebodies into $\mathbb{R}^3$. Are these two embeddings smoothly isotopic?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CHAPTER 284

The linking number

In the previous Chapter 283 we just (re-) introduced knots and links. In this chapter we will introduce the linking number of two oriented disjoint knots in S^3. This is one of the oldest and most central invariants of topology. We will see that one can calculate the linking number from a diagram and we will use the linking number to introduce the longitude of a knot.

This chapter is organized as follows:

- In Section 284.1 we will (re-) introduce the linking number $\mathrm{lk}(K, J)$ of two disjoint oriented knots K and J in S^3.
- In Section 284.2 we will see that one can calculate linking numbers from diagrams. We will use this calculation to show that the linking number is symmetric.
- In Section 284.3 we will use the linking number to define the self-linking number of the thickening of a knot. This notion will allow us to formulate a uniqueness statement for thickenings.
- In Section 284.4 we will use the linking number to (re-) introduce the longitude of an oriented knot.
- In Section 284.5 we will prove some results regarding the interplay between longitudes and meridians.
- In Appendix 284.6 we will use the interplay between the meridian and the longitude to give an alternative proof that linking numbers are symmetric.
- In Section 284.7 we will provide exercises for Chapter 284.

In the next Chapter 285 we will introduce Seifert surfaces and we will study their relationship to linking numbers.

284.1. **Linking numbers.** We already defined the notion of the linking number between suitable oriented submanifolds on various occasions, e.g. in Definitions 52.2, 86.10, 114.1 and 258.7. In this section we give the definition that we will work with in this chapter:

Definition 284.1. Let $K \subseteq S^3$ be an oriented knot.

(1) We denote by μ_K the meridian from Definition 109.1.
(2) By the Link Complement-H_*–Basis-Proposition 283.4 we know that there exists a unique isomorphism $\Phi_K \colon \mathrm{H}_1(S^3 \setminus K) \to \mathbb{Z}$ with $\Phi_K([\mu_K]) = 1$.
(3) It follows from (2) and the H^1-Hom-Proposition 201.24 that there exists a unique generator $[\mu_K]^* \in \mathrm{H}^1(S^3 \setminus K; \mathbb{Z})$ with[3058] $\langle [\mu_K]^*, [\mu_K] \rangle = 1$.
(4) Let $J \subseteq S^3$ be an oriented knot that is disjoint from K. We denote by $[J] \in \mathrm{H}_1(S^3 \setminus K)$ the image of the fundamental class of the oriented smooth manifold J under the

inclusion $J \to S^3 \setminus K$. We define the linking number of K and J to be

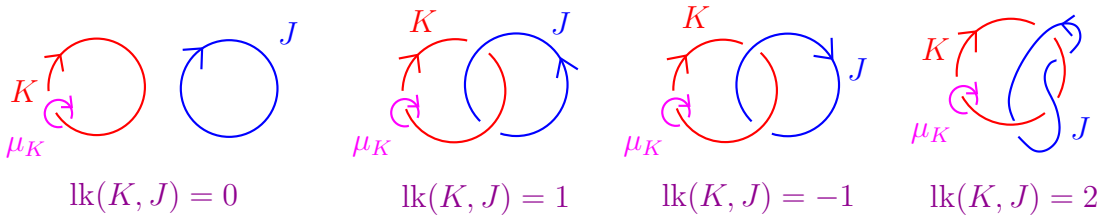
$$\mathrm{lk}(K,J) \;:=\; \Phi_K([J]) \;\underset{\uparrow}{=}\; \langle [\mu_K]^*, [J] \rangle$$

equality holds by the above definitions

Example.

(1) Let $K \subseteq S^3$ be an oriented knot. It follows immediately from Definition 284.1 that for any meridian μ_K of K we have $\mathrm{lk}(K, \mu_K) = +1$.

(2) If K is the trivial knot, then we saw in the Trivial Knot–Complement Proposition 111.11 that $S^3 \setminus K$ is diffeomorphic to the open solid torus $B^2 \times S^1$.

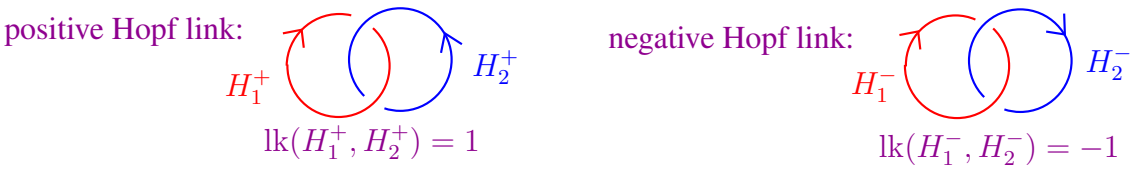


Using this diffeomorphism it is fairly straightforward to determine $\Phi_K \colon \mathrm{H}_1(S^3 \setminus K) \to \mathbb{Z}$ applied to a given oriented knot $J \subseteq S^3 \setminus K$. In particular we can easily obtain the linking numbers of the oriented links shown in the figure above. □

In the following we will state some basic properties of the linking number. Some of these might already be familiar to the reader. For example the following lemma is essentially identical to the Linking Number–Isotopy-Lemma 114.2:

Lemma 284.2. (Linking Number–Isotopy-Lemma) Let K and J be a pair of two oriented disjoint knots and let $\widetilde{K}$ and $\widetilde{J}$ be another pair of two oriented disjoint knots. If the oriented link $K \sqcup J$ is smoothly isotopic to the oriented link $\widetilde{K} \sqcup \widetilde{J}$, then $\mathrm{lk}(K,J) = \mathrm{lk}(\widetilde{K}, \widetilde{J})$.

Example. In the figure below we consider the two Hopf links H^- and H^+:



It follows from the Linking Number–Isotopy-Lemma 284.2 together with the calculation of the linking numbers in the figure preceding the lemma that the two Hopf links H^- and H^+ are not smoothly isotopic. □

Proof. This statement follows easily from Smooth Isotopy Characterization Proposition 283.1 (1)⇒(3) and the definitions. ■

The Linking Number–Isotopy-Lemma 284.2 gave us the unsurprising statement that if we change two disjoint oriented knots by a *smooth isotopy*, then the linking number is

[3058] Let G be an abelian group. If (X, A) is a pair of topological spaces, then the map

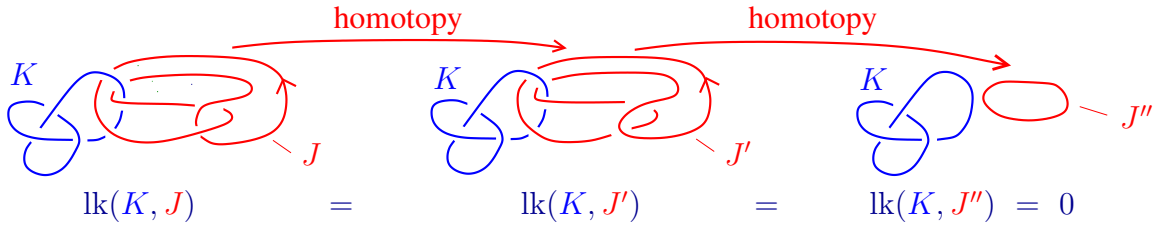
$$\begin{aligned} \langle\ ,\ \rangle_X := \langle\ ,\ \rangle_{(X,A)} \colon \mathrm{H}^n(X, A; G) \times \mathrm{H}_n(X, A) &\to G \\ ([\varphi], [c]) &\mapsto \langle [\varphi], [c] \rangle_X := \varphi(c) \end{aligned}$$

is the Kronecker pairing from the Kronecker Pairing Lemma 199.7.

unchanged. The following lemma says that if we keep the knot K fixed, then we can change the knot J much more drastically, namely by a *homotopy*, without affecting the linking number:[3059]

Lemma 284.3. (Linking Number–Homotopy-Lemma) Let $K \subseteq S^3$ be an oriented knot and let J and J' be two oriented knots in the complement of K. If J and J' are homotopic in the sense of Definition 220.11 in the complement of K, i.e. if there exists a continuous map $F\colon J \times [0,1] \to S^3 \setminus K$ such that $F_0 = \mathrm{id}$ and such that $F_1\colon J \to J'$ is an orientation-preserving diffeomorphism, then

$$\mathrm{lk}(K,J) \;=\; \mathrm{lk}(K,J').$$



Proof. We have the following equalities:

$$\mathrm{lk}(K,J) \underset{\text{Definition 284.1}}{=} \Phi_K([J]) \underset{\substack{\text{by the Homotopy-H}_*\\ \text{Proposition 144.12 we have}\\ [J]=[J'] \in \mathrm{H}_1(S^3\setminus K)}}{=} \Phi_K([J']) \underset{\text{Definition 284.1}}{=} \mathrm{lk}(K,J').$$

■

The next lemma is a variation on the Linking Number–Orientation Lemma 114.4:

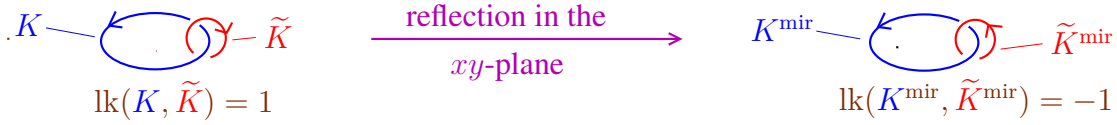
Lemma 284.4. (Linking Number–Orientation Lemma) Let K, J be two disjoint oriented knots in S^3. If we change the orientation of precisely one of the two knots, then the sign of the linking number flips.

Proof. This statement follows easily from the definitions and the Fundamental Class-Opposite Orientation Lemma 194.4 which says fact that for a compact oriented n-dimensional smooth manifold we have $[-M] = -[M] \in \mathrm{H}_n(M, \partial M)$. ■

Our next lemma is an analog of the Linking Number–Mirror Image Lemma 114.5:

Lemma 284.5. (Linking Number–Mirror Image Lemma) Let K, J be two disjoint oriented knots in S^3. Furthermore let $\rho\colon S^3 \to S^3$ be a reflection in a hyperplane. Then

$$\mathrm{lk}(\rho(K),\rho(J)) \;=\; -\mathrm{lk}(K,J).$$



[3059]Note that the following lemma is essentially identical to the Linking Number–Homotopy-Lemma 114.3.

Proof. We write $m = \mathrm{lk}(K, J)$. Note that by Definition 284.1 this means that we have the equality $[J] = m \cdot [\mu_K] \in \mathrm{H}_1(S^3 \setminus K)$. We now see that

$$\begin{array}{rcccl}
\mathrm{lk}(\rho(K), \rho(J)) & \overset{\text{by Definition 284.1}}{\downarrow} & & \overset{\text{essentially by definition}}{\downarrow} & \\
& = \langle [\mu_{\rho(K)}]^*, [\rho(J)] \rangle & & = \langle [\mu_{\rho(K)}]^*, \rho_*([J]) \rangle & \\
& = \langle [\mu_{\rho(K)}]^*, \rho_*(m \cdot [\mu_K]) \rangle & & = m \cdot \langle [\mu_{\rho(K)}]^*, [\rho(\mu_K)] \rangle & \\
& \overset{\text{definition of } m}{\uparrow} & & \overset{\text{(bi-) linearity of } \Phi_K \text{ and } \langle -,- \rangle}{\uparrow} & \\
& = m \cdot \langle [\mu_{\rho(K)}]^*, [-\mu_{\rho(K)}] \rangle & = -m \cdot \underbrace{\langle [\mu_{\rho(K)}]^*, [\mu_{\rho(K)}] \rangle}_{=1} & = -m = -\mathrm{lk}(K, J). & \\
& \overset{\text{by the Meridian–Symmetries Lemma 109.3}}{\uparrow} & & & \blacksquare
\end{array}$$

284.2. **Linking numbers from diagrams.** In this section we will see that if two disjoint oriented knots are given by a diagram, then there is a straightforward way to read off the linking number from the given diagram. Recall that we gave a formal definition of a diagram in Definition 112.4. For the purpose of this section we now give an informal definition:

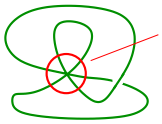
Definition 284.6. (Informal) A link diagram is a collection of m closed immersed curves in $\mathbb{R}^2$ with the following properties:

- At any crossings only two strands meet.
- All crossings are transverse.

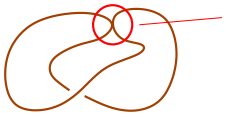
For each crossing we indicate pictorially which strand goes under or over. For $m = 1$ we speak of course of a knot diagram. To any such diagram we associated in the Diagram-to-Link Lemma 112.5 an m-component link in $\mathbb{R}^2 \times [-1, 0]$ in the "obvious way".



knot diagram

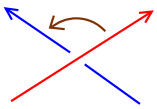


triple point is not allowed

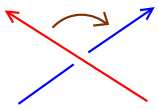


tangency is not allowed

Definition 284.7. Given an oriented diagram we say a crossing is positive (respectively negative) if the direction of the upper strand together with the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$.



positive crossing



negative crossing

To enhance readability we will formulate the next proposition, which is identical to the Linking Number-via-Crossings Proposition 114.9, in a slightly informal way:

Proposition 284.8. (Linking Number-via-Crossings Proposition) Let K, J be two disjoint oriented knots in S^3. We suppose that the link $K \sqcup J$ is given by a diagram in the xy-plane.

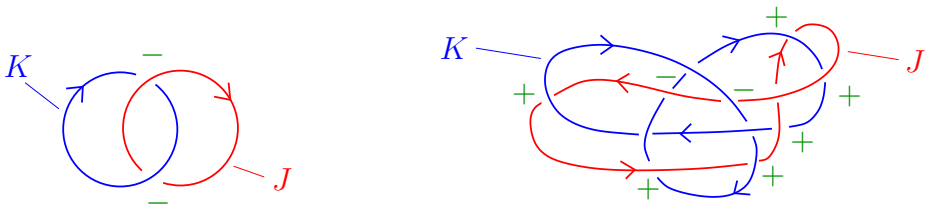
(1) We have the following equality:

$$\begin{aligned} \mathrm{lk}(K,J) \;=\; & \text{number of positive crossings where } K \text{ crosses over } J \\ - \;& \text{number of negative crossings where } K \text{ crosses over } J. \end{aligned}$$

(2) More symmetrically we also have the following equality:

$$\begin{aligned} \mathrm{lk}(K,J) \;=\; & \tfrac{1}{2} \cdot \text{number of positive crossings between } K \text{ and } J \\ - \;& \tfrac{1}{2} \cdot \text{number of negative crossings between } K \text{ and } J. \end{aligned}$$

Example. In the figure below we show two pairs of disjoint oriented knots K, J and we compute the linking number using the Linking Number-via-Crossings Proposition 284.8 by considering the signs of the crossings between K and J.



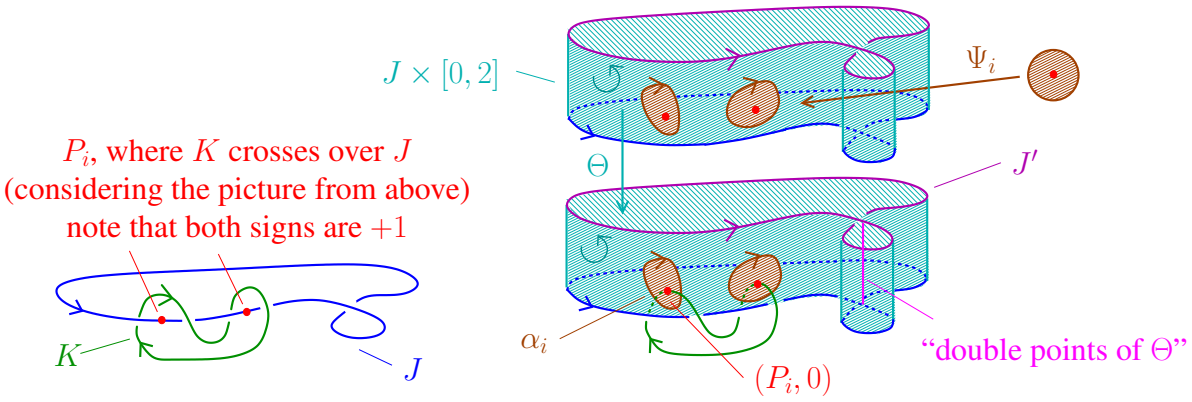
linking number $= -1$ linking number $= 2$

As a reality check, note that on the left we consider the Hopf link that we had already considered in the figure on page 5427. Fortunately in both calculations we obtain that the linking number equals -1. □

In the following we prove the two statements of the Linking Number-via-Crossings Proposition 284.8 separately.

Proof of Proposition 284.8 (1). Let K, J be two disjoint oriented knots in S^3. We suppose that the link $K \sqcup J$ is given by a diagram in the xy-plane. Recall that this implies that $K \sqcup J \subseteq \mathbb{R}^2 \times [-1, 0]$. We introduce some notation:

- We denote by $P_1, \ldots, P_m \subseteq \mathbb{R}^2$ the crossings where K crosses over J. We denote the corresponding signs by $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$.
- We consider the map
$$\begin{aligned} \Theta \colon J \times [0,2] &\to \mathbb{R}^3 \\ (j,t) &\mapsto j + (0,0,t). \end{aligned}$$
We equip the smooth manifold $J \times [0,2]$ with the orientation such that as oriented smooth manifolds we have $\partial(J \times [0,2]) = (J \times \{0\}) \sqcup (-J \times \{2\})$.
- Note that $\Theta(J \times [0,2]) \cap K = \{(P_1, 0), \ldots, (P_m, 0)\}$. One can show fairly easily that we can pick *orientation-reversing* smooth embeddings $\Psi_i \colon \overline{B}^2 \to J \times (0,2)$, $i = 1, \ldots, m$ with disjoint images such that for $i = 1, \ldots, m$ the map $\Theta \circ \Psi_i \colon \overline{B}^2 \to \mathbb{R}^3$ is a smooth embedding with $(\Theta \circ \Psi_i)(0)) = (P_i, 0)$.
- For $i = 1, \ldots, m$ we consider the oriented smooth manifold $\alpha_i := (\Theta \circ \Psi_i)(S^1)$.
- We set $J' := \Theta(J \times \{2\})$.



Claim.

(1) We have the following equality in $\mathrm{H}_1(S^3 \setminus K)$:

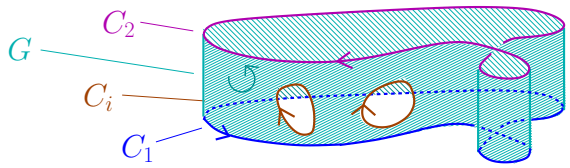
$$[J] \;=\; [J'] + [\alpha_1] + \cdots + [\alpha_m].$$

(2) Let $i \in \{1, \ldots, m\}$. The following statement holds:
- If $\epsilon_i = 1$, then α_i is a meridian of K.
- If $\epsilon_i = -1$, then α_i^{rev} is a meridian of K.

(3) $J' \subseteq \mathbb{R}^2 \times [1, \infty) \subseteq S^3 \setminus K$.

Proof.

(1) First we recall the following more general statement: Let G be a compact oriented n-dimensional smooth manifold with boundary components $C_1, \ldots, C_k$. We equip $C_1, \ldots, C_k$ with the corresponding boundary orientation from the Boundary Orientation Lemma 31.38. It follows from the Topological Kernel-of-Boundary Inclusion Corollary 194.14 (1) that $[C_1] + \cdots + [C_k] = 0 \in \mathrm{H}_{n-1}(G)$. We now apply this logic to $G := (J \times [0,2]) \setminus (\Psi_1(B^2) \cdots \cup \ldots \Phi_m(B^2))$ and its $m+2$ boundary components.



$$[C_1] + \cdots + [C_{m+2}] = 0 \in \mathbf{H}_1(G)$$

The desired statement now follows from the Opposite Side–Opposite Orientation Lemma 31.40 which tells us that the boundary orientation of the boundary component $\Psi_i(S^1) \subseteq \partial G$ is the opposite of the orientation of α_i.

(2) First note that it follows almost immediately from Definition 109.1 that, up to orientation, α_i is indeed a meridian of K. We leave it to the reader to figure out, using the various orientations, that we obtain the promised result.

(3) This statement follows from the observation that we have $K \sqcup J \subseteq \mathbb{R}^2 \times [-1, 0]$, thus, if we lift J by $(0, 0, 2)$ into the z-direction we are safely in the subspace $\mathbb{R}^2 \times [1, \infty)$ which is disjoint from K. ⊞

We now see that

$$\mathrm{lk}(K,J) \overset{\text{Definition 284.1}}{\downarrow =} \Phi_K([J]) \overset{\text{by Claim (1)}}{\downarrow =} \Phi_K([J'] + [\alpha_1] + \cdots + [\alpha_m]) = \Phi_K([J]) + \sum_{i=1}^{m} \Phi_K([\alpha_i]) = \sum_{i=1}^{m} \epsilon_i.$$

$\Phi_K([J]) = 0$ since Φ_K is linear; $\Phi_K([\alpha_i]) = \epsilon_i$ by Claim (2) and (3), here we use the discussion on page 5427 and we use that $\mathrm{H}_1(\mathbb{R}^2 \times [1,\infty)) = 0$ ■

Proof of Proposition 284.8 (2). Let K, J be two disjoint oriented knots in S^3. We suppose that the link $K \sqcup J$ is given by a diagram in the xy-plane. We just showed that

$$\mathrm{lk}(K,J) = \begin{array}{l}\text{number of positive crossings where } K \text{ crosses over } J \\ - \text{ number of negative crossings where } K \text{ crosses over } J.\end{array}$$

The next claim gives us a subtle modification of this equality:

Claim. We also have the following equality:

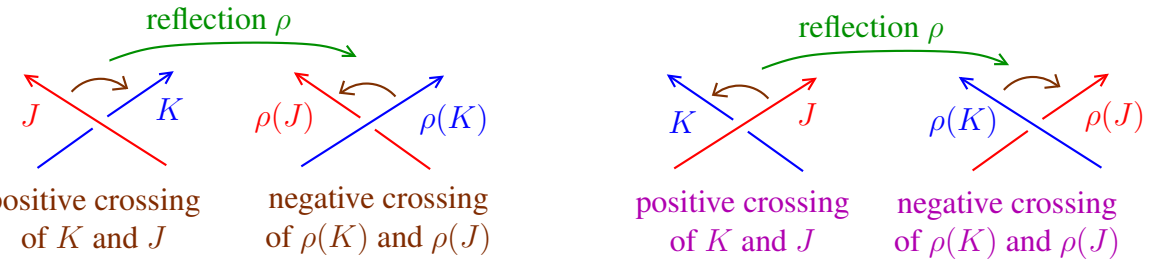
$$\mathrm{lk}(K,J) = \begin{array}{l}\text{number of positive crossings where } K \text{ crosses under } J \\ - \text{ number of negative crossings where } K \text{ crosses under } J.\end{array}$$

Proof. There are several ways how one can prove the claim:

- We can apply a little trick: Namely, let $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the xy-plane. We then see that

$$\mathrm{lk}(K,J) \overset{\text{by the Linking Number–Mirror Image Lemma 284.5}}{\downarrow =} -\mathrm{lk}(\rho(K),\rho(J)) = \begin{array}{l} - \text{ number of positive crossings where } \rho(K) \text{ crosses over } \rho(J) \\ \overset{\text{by Statement (1)}}{\uparrow =} \text{number of negative crossings where } \rho(K) \text{ crosses over } \rho(J). \end{array}$$

$$= \begin{array}{l} \text{number of positive crossings where } K \text{ crosses under } J \\ \overset{\text{see the figure below}}{\uparrow =} - \text{ number of negative crossings where } K \text{ crosses under } J. \end{array}$$



- Alternatively one can modify the above proof of Statement (1), where instead of "dragging J upwards", i.e. instead of using the map

$$\Theta_{\text{up}}\colon J \times [0,2] \to \mathbb{R}^3,\ (j,t) \mapsto j + (0,0,t) \qquad \text{we use the map} \qquad \Theta_{\text{down}}\colon J \times [0,2] \to \mathbb{R}^3,\ (j,t) \mapsto j - (0,0,t).$$

The reader can easily verify that this way one obtains the desired result. ⊞

By adding the above two calculations of $\mathrm{lk}(K,J)$ we obtain the equality:

$$2\cdot \mathrm{lk}(K,J) \;=\; \begin{array}{l} \text{number of positive crossings between } K \text{ and } J \\ - \;\; \text{number of negative crossings between } K \text{ and } J, \end{array}$$

which gives us the desired equality. ■

The above Definition 284.1 of the linking number $\mathrm{lk}(K,J) = \Phi_K([J])$ is a priori awfully asymmetric in K and J. Rather surprisingly the following proposition, which is identical to the Linking Number-Symmetry Proposition 114.10, says that the linking number $\mathrm{lk}(K,J)$ is symmetric in the oriented knots K and J:

Proposition 284.9. (Linking Number-Symmetry Proposition) For any two disjoint oriented knots K and J in S^3 we have

$$\mathrm{lk}(J,K) \;=\; \mathrm{lk}(K,J).$$

Proof. Let K and J be two disjoint oriented knots. It follows from the Link Diagram Existence Proposition 112.7 and the Linking Number–Isotopy-Lemma 284.2 that we can assume that the link $K \sqcup J$ is given by a diagram.

The desired statement now follows immediately from the fact that the formula in Statement (2) of the Linking Number-via-Crossings Proposition 284.8 is symmetric in the knots K and J. ■

Remark.

(1) The proof of the Linking Number-Symmetry Proposition 284.9 via the Linking Number-via-Crossings Proposition 284.8 is very intuitive. It does have the disadvantage though that it requires a longish detour through diagrams. In Appendix 284.6 we will give an alternative proof of the Linking Number-Symmetry Proposition 284.9 which is very formal and dry, but which has for example the advantage that it easily generalizes to more general contexts, like knots in integral homology spheres.

(2) In the next chapter we will prove the Linking Number-via–Surfaces Proposition 285.7 which gives us another proof of the Linking Number-Symmetry Proposition 284.9. □

284.3. **Self-linking number of thickenings of knots.** In this section we will use the linking number to have a closer look at thickenings of knots. We start out with the following definition which is essentially just a slight variation on Definition 259.2.

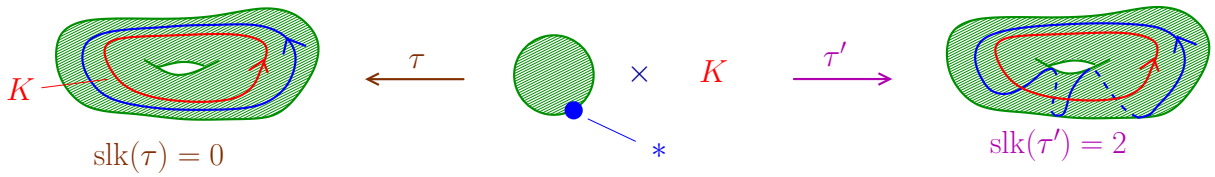
Definition 284.10. Let $K \subseteq S^3$ be an oriented knot. Given a thickening $\tau\colon \overline{B}^2 \times K \to S^3$ we define

$$\text{self-linking number of } \tau \;:=\; \mathrm{slk}(\tau) \;:=\; \mathrm{lk}(K,\, \underbrace{\tau(\{*\}\times K)}_{\substack{\text{oriented knot}\\ \text{that is}\\ \text{disjoint from } K}}),$$

where $* \in \overline{B}^2 \setminus \{0\}$ is a random point.[3060]

Example. In the figure below we show two thickenings of a trivial knot K.

[3060] It follows easily from the Linking Number–Homotopy-Lemma 284.3 that this definition does not depend on the choice of the point $* \in \overline{B}^2 \setminus \{0\}$.



We determined the self-linking number via the Linking Number-via-Crossings Proposition 284.8. □

The following lemma can be viewed a refinement of the Link Tubular Map Theorem 108.29:

Lemma 284.11. (Knot Thickening–Self Linking Lemma) Let K be an oriented knot in S^3.

(1) Let $\tau\colon \overline{B}^2 \times K \to S^3$ be a thickening of K and let $d \in \mathbb{Z}$. Furthermore let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism. Similar to the Linear $\overline{B}^k$-Bundle-Isomorphism Lemma 58.6 we consider the diffeomorphism

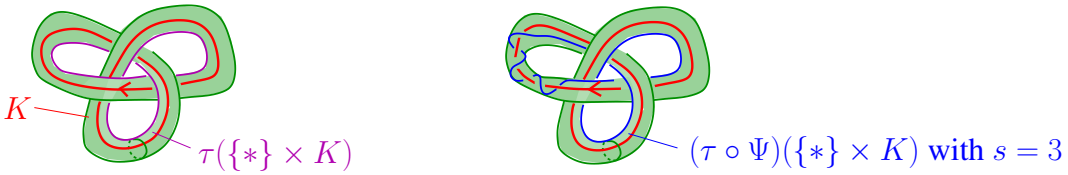
$$\begin{array}{rcl} \Psi_d\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ (w, \gamma(z)) & \mapsto & (\underbrace{z^d \cdot_{\mathbb{C}} w}_{\substack{\text{multiplication in } \mathbb{C}, \\ \text{where } w \in \overline{B}^2 \subseteq \mathbb{C} \\ \text{and where } z^d \in \mathbb{C}}}, \gamma(z)). \\ \uparrow & & \\ \text{where } z \in S^1 & & \end{array}$$

The map $\tau \circ \Psi_d$ is a thickening with $\operatorname{slk}(\tau \circ \Psi_d) = \operatorname{slk}(\tau) + d$.

(2) Given any $n \in \mathbb{Z}$ there exists a thickening of K with self-linking number n.

(3) If $\tau, \tau'\colon \overline{B}^2 \times K \to S^3$ are two thickenings of K with the same self-linking number, then there exists a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ rel K with $H_0 = \mathrm{id}$ and $H_1 \circ \tau' = \tau$.[3061]

Given $K \subseteq \mathbb{R}^3 \subseteq S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ the analogous statements hold if we consider thickenings that are contained in $\mathbb{R}^3$. Furthermore, in this setting the diffeotopy in (3) of $\mathbb{R}^3$ can be assumed to have compact support.



Sketch of proof. Let K be an oriented knot and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism.

(1) Let $\tau\colon \overline{B}^2 \times K \to S^3$ be a thickening of K and let $d \in \mathbb{Z}$. We denote by m the self-linking number of τ. We write $\lambda := \tau(\{1\} \times K)$, we write $\mu := \tau(S^1 \times \{*\})$ and we view λ and μ as oriented 1-dimensional submanifolds of S^3 with the orientations given by K and S^1. Note that it follows from the Meridian Proposition 109.2 that μ is a meridian of K.

[3061] Note that this uniqueness statement is much prettier than the corresponding statement in the Link Tubular Map Theorem refthm:knot-tubular-map.

As in Definition 284.1 we denote by $\Phi_K\colon \mathrm{H}_1(S^3 \setminus K) \to \mathbb{Z}$ the unique isomorphism with $\Phi_K([\mu_K]) = 1$. We now perform the following calculation:

$$\begin{array}{lll} & \text{by Definition 284.10 and by Definition 284.1} & \text{follows from the Action-on-Torus Homology Lemma 166.12 (2)} \\ \mathrm{slk}(\tau \circ \Psi_d) & \overset{\downarrow}{=} \Phi_K((\tau \circ \Psi_d)([\{1\} \times K])) & \overset{\downarrow}{=} \Phi_K(\tau_*(d \cdot [S^1 \times \{*\}] + [\{1\} \times K])) \\ & = \Phi_K(d \cdot [\mu] + [\lambda]) & = d + \Phi_K([\tau(\{1\} \times K]) \quad \underset{\uparrow}{=} \; d + \mathrm{slk}(\tau). \\ & \underset{\uparrow}{} & \underset{\uparrow}{} \\ & \text{by definition of } \lambda \text{ and } \mu & \text{since } \Phi_K(\mu) = 1 \text{ and by definition of } m \quad \text{by Definition 284.10 and by Definition 284.1} \end{array}$$

(2) Recall that it follows from the Link Tubular Map Theorem 108.29 that there exists a thickening $\tau\colon \overline{B}^2 \times K \to S^3$ of K. We denote by m the self-linking number of τ. The desired statement now follows from (1) applied to $d := n - m$.

(3) Let $\tau, \tau'\colon \overline{B}^2 \times K \to S^3$ be two thickenings of K with the same self-linking number.[3062] First note that it follows from the Link Tubular Map Theorem 108.29 that there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ and a smooth map $g\colon K \to \mathrm{SO}(2)$ with the following properties:

(i) $F_0 = \mathrm{id}$.

(ii) F is a homotopy rel K.

(iii) The diffeomorphism $F_1\colon S^3 \to S^3$ restricts to the map

$$\begin{array}{rcl} \tau(\overline{B}^2 \times K) & \to & \tau'(\overline{B}^2 \times K) \\ \tau(v,k) & \mapsto & \tau'(g(k) \cdot v, k). \end{array}$$

Claim. The map $g\colon K \to \mathrm{SO}(2)$ is homotopic to a constant map.

Proof. We make a few preparations:

(a) Since $\gamma\colon S^1 \to K$ is a diffeomorphism it suffices to show that $g \circ \gamma\colon S^1 \to \mathrm{SO}(2)$ is homotopic to a constant map.

(b) We consider the map

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ (v,k) & \mapsto & (g(k) \cdot v, k). \end{array}$$

Note that $F_1 \circ \tau = \tau' \circ \Theta$.

(c) We make the usual identifications $S^1 =_{\mathrm{i}} \mathrm{U}(1) =_{\mathrm{i}} \mathrm{SO}(2)$.

(d) Note that it follows from the Degree Isomorphism Proposition 169.2 that the map $g \circ \gamma\colon S^1 \to \mathrm{SO}(2) =_{\mathrm{i}} S^1$ is homotopic to a map of the form $z \mapsto z^d$. By (a) it thus remains to show that $d = 0$.

(e) As in (1) we consider the diffeomorphism

$$\begin{array}{rcl} \Psi_d\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ (v,k) & \mapsto & ((\gamma^{-1}(k))^d \cdot_{\mathbb{C}} v, k). \end{array}$$

We now perform the following calculation:

[3062] The following argument has some similarities with the proof of the Band-in-Solid Torus Lemma 115.5.

$$\begin{array}{rcll} \text{self-linking number of } \tau & \overset{\text{by Definition 284.10}}{=} & \mathrm{lk}(K, \tau(\{1\} \times K)) & \overset{\text{follows from the Linking Number Homotopy-Lemma 284.3 applied to the homotopy } F}{=} \mathrm{lk}(K, (\tau' \circ \Theta)(\{1\} \times K)) \\ & \overset{\text{it follows easily from (d) that } (\tau \circ \Theta)(\{1\} \times K) \text{ is homotopic in } S^3 \setminus K \text{ to } (\tau \circ \Psi_d)(\{1\} \times K), \text{ thus the equality follows again from the Linking Number–Homotopy-Lemma 284.3}}{=} & \mathrm{lk}(K, (\tau' \circ \Psi_d)([\{1\} \times K])) & \overset{\text{by the calculation in (1)}}{=} d + \mathrm{lk}(K, \tau'(\{1\} \times K)) \\ & \overset{\text{by Definition 284.10}}{=} & d + \text{self-linking number of } \tau'. & \end{array}$$

Since τ and τ' have by hypothesis the same self-linking number we see that $d = 0$. ⊞
We just learned in the claim that the map $g\colon K \to \mathrm{SO}(2)$ is homotopic to a constant map. Since $\mathrm{SO}(2)$ is path-connected we can find a homotopy from the constant map $k \to \mathrm{id}_2$ to g. In fact, by the Replace-by-Smooth Homotopy Proposition 36.6 we can even find a *smooth* homotopy $G\colon K \times [0,1] \to \mathrm{SO}(2)$ from $k \to \mathrm{id}_2$ to g. This smooth homotopy gives rise to the smooth isotopy

$$\begin{array}{rcl} H\colon \tau(\overline{B}^2 \times K) \times [0,1] & \to & S^3 \\ (\tau(v,k),t) & \mapsto & F_t(\tau(G_t(k)^{-1} \cdot v)) \end{array}$$

rel K from $H_0 = \mathrm{id}$ to $H_1 = \tau' \circ \tau^{-1}$.[3063] The desired statement now follows from the Isotopy Extension Theorem 44.2.

Evidently the same arguments as above also apply if we replace S^3 by $\mathbb{R}^3$. ■

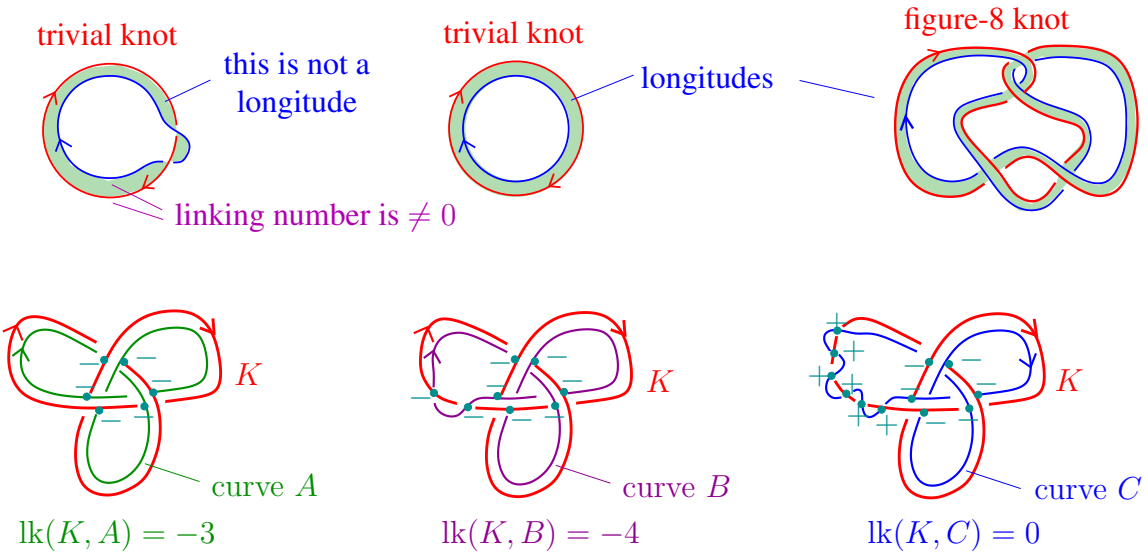
284.4. **The longitude.** We continue by recalling Definition 115.1:

Definition 115.1. Let L be an oriented link in S^3 and let K be a component of L. Let $\lambda_K \subseteq S^3 \setminus K$ be an oriented smoothly embedded loop. We say that λ_K is a longitude of K if there exists a smooth embedding $\varphi\colon S^1 \times [0,1] \to S^3$ with the following four properties:
(1) The map $\varphi_0\colon S^1 \to S^3$ defines an orientation-preserving diffeomorphism $S^1 \to K$.
(2) The map $\varphi_1\colon S^1 \to S^3$ defines an orientation-preserving diffeomorphism $S^1 \to \lambda_K$.
(3) The image of φ does not intersect any other component of L.
(4) We have $\mathrm{lk}(K, \lambda_K) = 0$.
The image $\varphi(S^1 \times [0,1])$ is often called a smoothly embedded band.

Examples.

(1) In the figure below we show a longitude for the trivial knot and the figure-8 knot. In each case we can use the Linking Number-via-Crossings Proposition 284.8 to verify that the linking number is indeed zero.
(2) In the figure below we show the trefoil with three wannabe longitudes. All of them cobound a smoothly embedded annulus with K.
Using the Linking Number-via-Crossings Proposition 284.8 we calculate the linking numbers of the oriented knot K with the oriented curves A, B and C. We obtain

[3063] Since the author got rather confused at this point, let us verify that the claim is correct: We have $H_1(\tau(v,k)) = F_1(\tau(g(k)^{-1} \cdot v)) = \tau'(g(k) \cdot g(k)^{-1} \cdot v, k) = \tau'(v,k)$.



that C is the only curve with linking number zero. Thus among A, B, C only C is a longitude of K. □

Next we recall the following analog of the Meridian Proposition 109.2:

Proposition 115.2. (Longitude Proposition) Let L be an oriented link in S^3 and let K be a component of L.

(1) K admits a longitude.

(2) Let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening. There exists a unique $m \in \mathbb{Z}$ such that the image of the map

$$\begin{array}{rcl} S^1 & \to & S^3 \\ z & \mapsto & \tau(z^m, \gamma(z)) \end{array}$$

is a longitude of K.

(3) Any two longitudes for K are smoothly isotopic in $S^3 \setminus L$.

(4) Given any longitude λ of the component K there exists a thickening $\tau\colon \overline{B}^2 \times L \to S^3$ such that λ is of the form described in (2).

We conclude this section with the following lemma which is an extension of the Meridian–Symmetries Lemma 109.3:

Lemma 115.7. (Meridian-Longitude–Symmetries Lemma) Let $K \subseteq S^3$ be an oriented knot with meridian μ_K and longitude λ_K. Furthermore let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane. We obtain the following table:

	meridian	longitude
original knot K	μ_K	λ_K
reverse K^{rev}	μ_K^{rev}	λ_K^{rev}
mirror $K^{\text{mir}} = \rho(K)$	$\rho(\mu_K)^{\text{rev}}$	$\rho(\lambda_K)$
inverse $K^{\text{inv}} = \rho(K)^{\text{rev}}$	$\rho(\mu_K)$	$\rho(\lambda_K)^{\text{rev}}$.

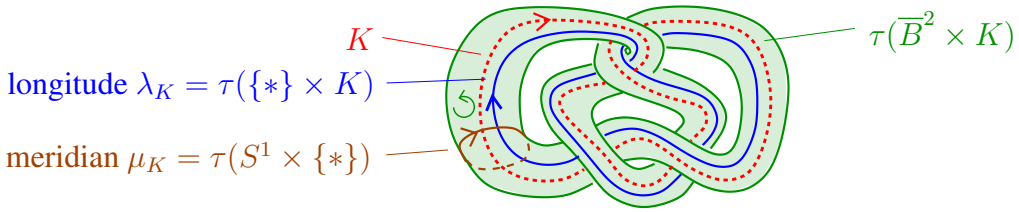
284.5. **The interplay between longitudes and meridians.** On several occasions the following lemma (which gives in particular a solution to Exercise 280.11) will come in handy:

Lemma 284.12. (Longitude-Meridian Lemma) Let $K \subseteq S^3$ be an oriented knot. Suppose we are given a thickening $\tau\colon \overline{B}^2 \times K \to S^3$ with self-linking number zero. As usual we denote by $X_K = S^3 \setminus \tau(B^2 \times K)$ the corresponding knot exterior.

(1) $\mu_K = \tau(S^1 \times \{*\})$ is a meridian of K and $\lambda_K = \tau(\{*\} \times K)$ is a longitude of K.
(2) The classes $[\mu_K]$ and $[\lambda_K]$ form a basis for $\mathrm{H}_1(\partial X_K)$.
(3) We denote by $\iota\colon \partial X_K \to X_K$ the inclusion. There exists a short exact sequence of the following form

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}_2(X_K, \partial X_K) & \xrightarrow{\partial_2} & \overbrace{\mathrm{H}_1(\partial X_K)}^{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\lambda_K]} & \xrightarrow{\iota_*} & \overbrace{\mathrm{H}_1(X_K)}^{=\mathbb{Z}\cdot[\mu_K]} & \to & 0 \\ & & & & [\mu_K] & \mapsto & [\mu_K] & & \\ & & & & [\lambda_K] & \mapsto & 0. & & \end{array}$$

(4) (a) The longitude λ_K is (up to smooth isotopy) the unique closed oriented smoothly embedded curve on the torus $\tau(S^1 \times K)$ that is homologous to $K = \tau(\{0\} \times K)$ in $\tau(\overline{B}^2 \times K)$ and that is null-homologous in X_K.
(b) The kernel of the inclusion induced map $\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(\tau(\overline{B}^2 \times K)$ is given by $[\mu_K]$.
(5) If we view μ_K and λ_K as closed oriented smoothly embedded curves on the oriented torus $\tau(S^1 \times K)$, then the algebraic intersection number equals $\mu_K \cdot \lambda_K = +1$.[3064]



Proof. Let $K \subseteq S^3$ be an oriented knot and let $\tau\colon \overline{B}^2 \times K \to S^3$ be a thickening with self-linking number zero. We set $X_K = S^3 \setminus \tau(B^2 \times K)$, $\mu_K = \tau(S^1 \times \{k_0\})$ and $\lambda_K = \tau(\{x_0\} \times K)$ for some $k_0 \in K$ and some $x_0 \in S^1$.

(1) It follows immediately from the Meridian Proposition 109.2 (2) that $\mu_K = \tau(S^1 \times \{k_0\})$ is a meridian of K. It remains to show that $\lambda_K = \tau(\{x_0\} \times K)$ is a longitude of K in the sense of Definition 115.1. We pick an orientation-preserving diffeomorphism $\gamma\colon S^1 \to K$. Next we consider the smooth embedding

$$\begin{array}{rcl} \varphi\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & \tau(t \cdot x_0, \gamma(z)). \end{array}$$

[3064]It follows from the Opposite Side–Opposite Orientation Lemma 31.40 that if we consider μ_K and λ_K as submanifolds of ∂X_K, equipped with the boundary orientation, then $\mu_K \cdot \lambda_K = -1$. This sign confusion never ceases to be a course of hilarious entertainment.

Note that $\varphi_0\colon S^1 \to K$ and $\varphi_1\colon S^1 \to \tau(\{x_0\} \times K)$ are orientation-preserving diffeomorphisms. Finally note that

$$\begin{array}{ccccc} \mathrm{lk}(K, \lambda_K) & = & \mathrm{lk}(K, \tau(\{x_0\} \times S^1)) & \underset{\text{see Definition 284.10}}{=} & \mathrm{slk}(\tau) & \underset{\text{by hypothesis on } \tau}{=} & 0. \end{array}$$

Thus we have verified that $\lambda_K = \tau(\{x_0\} \times K)$ is a longitude of K in the sense of Definition 115.1.

(2) This statement follows almost immediately from the Torus-H_*-Proposition 153.9 (2) which says that $\mathrm{H}_1(S^1 \times S^1) = \mathbb{Z} \cdot [\{1\} \times S^1] \oplus \mathbb{Z} \cdot [S^1 \times \{1\}]$.

(3) As we pointed out in the Link Complement-Boundary H_*–Proposition 283.5 the long exact sequence of homology groups of the pair $(X_K, \partial X_K)$ looks as follows:

$$\ldots \to \underbrace{\mathrm{H}_2(X_K)}_{\substack{= 0 \text{ by the Link} \\ \text{Complement-H}_* \\ \text{Basis-Proposition 283.4}}} \xrightarrow{\jmath} \mathrm{H}_2(X_K, \partial X_K) \xrightarrow{\partial_2} \mathrm{H}_1(\partial X_K) \xrightarrow{\iota_*} \underbrace{\mathrm{H}_1(X_K)}_{\substack{= \mathbb{Z} \cdot [\mu_K] \text{ by the Link} \\ \text{Complement-H}_* \\ \text{Basis-Proposition 283.4}}} \to \ldots$$

Note that it follows from (1) that $\mathrm{H}_1(X_K) = \mathbb{Z} \cdot [\mu_K]$ and that it follows from Definition 115.1 that $\iota_*([\lambda_K]) = 0$. Furthermore we know from the Link Complement-H_*–Basis-Proposition 283.4 that the right-hand map is an epimorphism. The statement follows immediately from these observations.

(4) (a) It follows from (3) that the kernel of $\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K)$ equals $\mathbb{Z} \cdot [\lambda_K]$. Since $\mathrm{H}_1(\tau(\overline{B}^2 \times K)) = \mathbb{Z} \cdot [K] = \mathbb{Z} \cdot [\lambda_K]$ we see that there exists a unique homology class that has the properties we are interested in. The desired statement now follows from the fact, shown in the Curves-on-Torus Lemma 234.25 (1), that two closed oriented connected curves on a torus are smoothly isotopic if and only if they are homologous.

(b) This statement follows from (2), (a) and the fact that μ_K bounds a disk in $\tau(\overline{B}^2 \times K)$.

(5) This statement follows from (1) and the easy observation that the oriented submanifolds $S^1 \times \{*\}$ and $\{*\} \times S^1$ of the oriented smooth manifold $S^1 \times S^1$ have algebraic intersection number one. ∎

284.6. **Appendix: The symmetry of the linking number.** The goal of this appendix is to give an alternative, arguably more formal, proof that linking numbers are symmetric, i.e. we give a new proof of the following proposition:

Proposition 284.9. (Linking Number-Symmetry Proposition) For any two disjoint oriented knots K and J in S^3 we have

$$\mathrm{lk}(J, K) = \mathrm{lk}(K, J).$$

In the proof of the Linking Number-Symmetry Proposition 284.9 we will need the following addendum to the Longitude-Meridian Lemma 284.12:

Lemma 284.12 (Longitude-Meridian Lemma) Let K be an oriented knot in S^3 and let $\tau\colon \overline{B}^2 \times K \to S^3$ be a thickening with self-linking number zero. As usual we denote

by $X_K = S^3 \setminus \tau(B^2 \times K)$ the corresponding knot exterior. Furthermore we consider the meridian $\mu_K = \tau(S^1 \times \{*\})$ and the longitude $\lambda_K = \tau(\{*\} \times K)$.

(6) If denote by $i\colon \partial X_K \to X_K$ the inclusion map, then

$$i^*([\mu_K]^*) \cap [\partial X_K] \;=\; -[\lambda_K] \;\in\; \mathrm{H}_1(\partial X_K).$$

Here we consider $[\mu_K]^* \in \mathrm{H}^1(X_K;\mathbb{Z})$ from Definition 284.1.

Proof of the Longitude-Meridian Lemma 284.12 (6).

(6) We denote by $p\colon S^1 \times K \to S^1$ the projection. Now we calculate that

$$i^*([\mu_K]^*) \cap [\partial X_K] \underset{\uparrow}{=} i^*([\mu_K]^*) \cap \tau_*(-[S^1 \times K]) \underset{\uparrow}{=} -\tau_*\big(\tau^*(i^*([\mu_K]^*)) \cap [S^1 \times K]\big)$$

it follows from the fact that τ is orientation-preserving and the Opposite Side–Opposite Orientation Lemma 31.40 that we have the equality $\tau_*([S^1 \times S^1]) = [\partial\tau(\overline{B}^2 \times K)] = -[\partial X_K]$

by the Cap Product-Naturality Lemma 206.13

$$= -\tau_*\big(p^*([S^1]^*) \cap [S^1 \times K]\big) \underset{\uparrow}{=} -\tau_*([\{*\} \times K])$$

it follows easily from the hypothesis that τ has self-linking number zero that $\tau^*(i^*([\mu_K]^*)) = p^*([S^1]^*)$

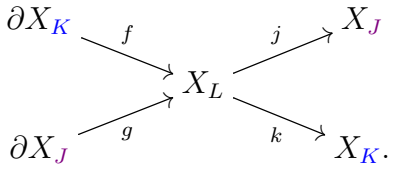
see page 3955, or alternatively use the Fundamental Class-Product Proposition 207.2

$$= -[\lambda_K].$$

by the Longitude Proposition 115.2 (2) and (3)

■

Proof of the Linking Number-Symmetry Proposition 284.9. [3065] We pick a thickening for the link $L = J \sqcup K$ and we use it to define the exteriors X_L, X_K and X_J. Throughout the proof we consider the following inclusion maps



Given a topological space Y we denote by $\epsilon_Y\colon \mathrm{H}_0(Y) \to \mathbb{Z}$ the natural augmentation homomorphism from the Augmentation Lemma 143.24 and the H_0-Proposition 143.25. We start out with the following claim:

Claim. We have $\mathrm{lk}(K, J) \;=\; \epsilon_{X_L}\big(\big(\underbrace{k^*([\mu_K]^*)}_{\in \mathrm{H}^1(X_L;\mathbb{Z})} \cup \underbrace{j^*([\mu_J]^*)}_{\in \mathrm{H}^1(X_L;\mathbb{Z})}\big) \cap \underbrace{g_*([\partial X_J])}_{\in \mathrm{H}_2(X_L)}\big).$

[3065] This argument was developed with Gerrit Herrmann.

Proof. We denote by $[J] \in \mathrm{H}_1(J)$ the fundamental class. We have the following equalities:

$$\begin{array}{rl}
& \epsilon_{X_L}((k^*([\mu_K]^*) \cup j^*([\mu_J]^*)) \cap g_*([\partial X_J])) \\
\overset{\uparrow}{=} & -\epsilon_{X_L}(k^*([\mu_K]^*) \cap (j^*([\mu_J]^*) \cap g_*([\partial X_J]))) \\
\text{by the Cup-Cap Lemma 206.15} & \\
= & \epsilon_{X_L}(k^*([\mu_K]^*) \cap g_*([\lambda_J])) \\
\overset{\uparrow}{=} & \\
\text{by the Longitude-Meridian Lemma 284.12 (6) from page 5439} & \\
= & \langle [\mu_K]^*, [J] \rangle_{X_K} \\
\overset{\uparrow}{=} & \\
\text{since } [\lambda_J] = [J] \text{ in } \mathrm{H}_1(X_K) & \\
\end{array}$$

$$\begin{array}{rl}
\text{by the Cup Product-Commutativity Proposition 204.10} & \\
\overset{\downarrow}{=} & -\epsilon_{X_L}((j^*([\mu_J]^*) \cup k^*([\mu_K]^*)) \cap g_*([\partial X_J])) \\
= & -\epsilon_{X_L}(k^*([\mu_K]^*) \cap g_*(g^* j^*[\mu_J]^* \cap [\partial X_J])) \\
\overset{\uparrow}{=} & \\
\text{by the Cup Product-Naturality Lemma 203.9 applied to } g \text{ and the naturality of the augmentation map} & \\
= & \langle [\mu_K]^*, k_* g_*([\lambda_J]) \rangle_{X_K} \\
\overset{\uparrow}{=} & \\
\text{by the Kronecker Pairing–Natural Lemma 199.9 (1) and the Cap Product-Kronecker Lemma 206.12} & \\
= & \mathrm{lk}(K, J). \\
\overset{\uparrow}{=} & \\
\text{see Definition 284.1} & \boxplus
\end{array}$$

Now we conclude that

$$\begin{array}{rl}
\mathrm{lk}(K, J) & = \epsilon_{X_L}((j^*([\mu_J]^*) \cup k^*([\mu_K]^*)) \cap g_*([\partial X_J])) \\
& \overset{\uparrow}{=} \\
& \text{by the claim} \\
& = \epsilon_{X_L}((-k^*([\mu_K]^*) \cup j^*([\mu_J]^*)) \cap (-f_*([\partial X_K]))) \quad = \mathrm{lk}(J, K). \\
& \overset{\uparrow}{=} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\uparrow}{=} \\
& \text{by the Cup Product-Commutativity Proposition 204.10 and since by the Topological Kernel-of-Boundary Inclusion Corollary 194.14 we have } g_*([\partial X_J]) + f_*([\partial X_K]) = 0 \in \mathrm{H}_2(X_L) \qquad \text{by the claim}
\end{array}$$

■

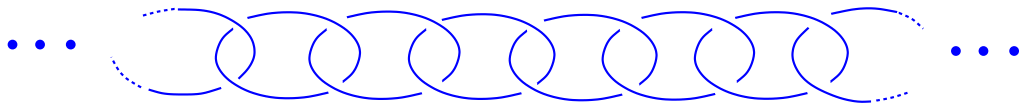
284.7. Exercises for Chapter 284.

Exercise 284.1. Provide an example of an oriented 3-component link $L = L_1 \sqcup L_2 \sqcup L_3$ such that for every $i \neq j$ the following two statements hold:
(1) The linking number $\mathrm{lk}(L_i, L_j)$ is zero.
(2) The link $L_i \sqcup L_j$ is non-trivial.

Exercise 284.2. For which matrices $A = (a_{ij}) \in \mathrm{M}(n \times n, \mathbb{Z})$ does there exist an oriented n-component link $L = L_1 \sqcup \cdots \sqcup L_n$ such that for every $i \neq j$ we have $\mathrm{lk}(L_i, L_j) = a_{ij}$?

Exercise 284.3. Let $L = L_1 \sqcup L_2 \subseteq S^3$ be a 2-component link. Recall that in Definition 109.5 we said that the link L is *splittable*, if there exist disjoint smoothly embedded closed balls $B_1, B_2 \subseteq S^3$ such that each L_i is contained in B_i. Show that if $L = L_1 \sqcup L_2$ is an oriented 2-component link that is splittable, then $\mathrm{lk}(L_1, L_2) = 0$.

Exercise 284.4. We consider the infinite Hopf link H_∞ that is shown in the figure below. Does there exist a smooth embedding $\varphi\colon S^2 \to S^3 \setminus H_\infty$ such that the two components of $S^3 \setminus \varphi(S^2)$ each contains at least one component of H_∞?

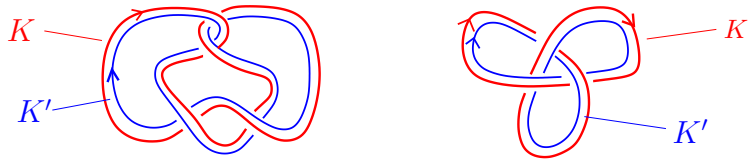


infinite Hopf link H_∞

Exercise 284.5. Let K be an oriented knot that is given by a diagram D. Let K' be the knot that is given by taking a parallel copy as in the figure below. Express $\mathrm{lk}(K, K')$ in

terms of the crossings of the diagram D.

Remark. Another way of putting the exercises is as follows: The knot K' is the "obvious" candidate for a longitude of K. For the usual diagram of the figure-8 knot we saw that we do indeed obtain the longitude, but we also saw that for the usual diagram of the trefoil we do not get a longitude.



Exercise 284.6. Let M be a closed oriented 3-dimensional smooth manifold. Furthermore let $K, J \subseteq M^3$ be two disjoint oriented 1-dimensional submanifolds of M that are diffeomorphic to S^1. Use the Manifold–Knot Complement-H_*-Proposition 217.18 to define a linking number $\mathrm{lk}_M(K, J) \in \mathbb{Z}$ which has the following naturality property: If we start out with $K, J \subseteq B^3$ and if $\Phi\colon \overline{B}^3 \to M$ and $\Psi\colon \overline{B}^3 \to S^3$ are two orientation-preserving smooth embeddings, then

$$\mathrm{lk}_M(\Phi(K), \Phi(J)) \;=\; \underbrace{\mathrm{lk}(\Psi(K), \Psi(J))}_{\text{see Definition 284.1}}.$$

Exercise 284.7. We consider the two links shown in the figure below. Show that the link complements are diffeomorphic.



Exercise 284.8. Let $L = K \sqcup J$ and $L' = K' \sqcup J'$ be two oriented links such that the complements are diffeomorphic. Does it follow that $\mathrm{lk}(K, J) = \pm\,\mathrm{lk}(K', J')$?

Exercise 284.9. Let K be a knot together with a diagram. In Section 113.2 we gave an explicit algorithm that determines the Wirtinger presentation for $\pi_1(S^3 \setminus K)$. Explain what element in this presentation corresponds to the longitude. Alternatively, for readers who like more explicit questions, determine the group system $(\pi_1(S^3 \setminus K), \mu_K, \lambda_K)$ for the trefoil and the figure-8 knot.

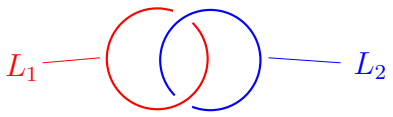
Remark. You might want to use Exercise 284.3.

Exercise 284.10. We consider the unoriented Hopf link that is shown in the figure below. There are a priori eight different oriented links that we can produce out of the Hopf link:

- We have two choices of orientations for L_1.
- We have two choices of orientations for L_2.
- For any of the four choices we can take the mirror image of the oriented link.

How many different smooth isotopy types of oriented links do we get that way?

Remark. This exercise is identical to Exercise 114.6.



Exercise 284.11. We say that two oriented links $L = L_1 \sqcup L_2 \subseteq S^3$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \widetilde{L}_2 \subseteq S^3$ are *link homotopic* if there exists a link homotopy between L and $\widetilde{L}$, i.e. a smooth map $F\colon L \times [0,1] \to S^3$ with the following properties:

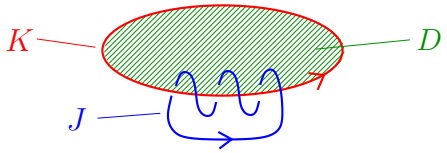
- We have $F_0 = \mathrm{id}$ and we have $F_1(L) = \widetilde{L}$ as oriented links.
- For each $t \in [0,1]$ we have $F_t(L_1) \cap F_t(L_2) = \varnothing$.
- There exist only finitely many $t \in [0,1]$ for which $F_t\colon L \to S^3$ is not a smooth embedding.

Show that $\mathrm{lk}(L_1, L_2) = \mathrm{lk}(\widetilde{L}_1, \widetilde{L}_2)$.

Hint. Show that if there exists such a link homotopy F, then we can find a link homotopy G such that there exist $0 = t_0 < t_1 < \cdots < t_m = 1$ such that on each interval $[t_i, t_{i+1}]$ either $G|_{L_1 \times [t_i, t_{i+1}]}$ is stationary or $G|_{L_2 \times [t_i, t_{i+1}]}$ is stationary.

Exercise 284.12. We will use the notion of link homotopy from Exercise 284.11. Let (K, J) and $(\widetilde{K}, \widetilde{J})$ be two pairs of oriented disjoint knots in S^3. In Exercise 284.11 we showed that if $K \sqcup J$ is link-homotopic to $\widetilde{K} \sqcup \widetilde{J}$, then $\mathrm{lk}(K, J) = \mathrm{lk}(\widetilde{K}, \widetilde{J})$. Our goal is to show that if $\mathrm{lk}(K, J) = \mathrm{lk}(\widetilde{K}, \widetilde{J})$, then $K \sqcup J$ is link-homotopic to $\widetilde{K} \sqcup \widetilde{J}$. Equivalently, we want to show that if we are a given a link $K \sqcup J$ with $\mathrm{lk}(K, J) = m$, then it is link-homotopic to the standard link shown below.

- Show that after a link homotopy we can assume that K is the standard trivial knot.
- Let $D \subseteq S^3$ be the standard disk that bounds the standard trivial knot K. Show that after a link homotopy we can assume that J intersects D transversally in precisely m points.
- Show that $L = K \sqcup J$ is link-homotopic to the standard link shown below.



CHAPTER 285

Seifert surfaces and linking numbers

In this chapter we will (re-) introduce the notion of a Seifert surface of a knot and we will use our results from cohomology theory to give a very clean proof that every knot in S^3 admits a Seifert surface. We will also see that often Seifert surfaces give a convenient way for calculating linking numbers. In fact towards the end of the chapter we will show that the linking number admits many equivalent definitions.

This chapter is organized as follows:

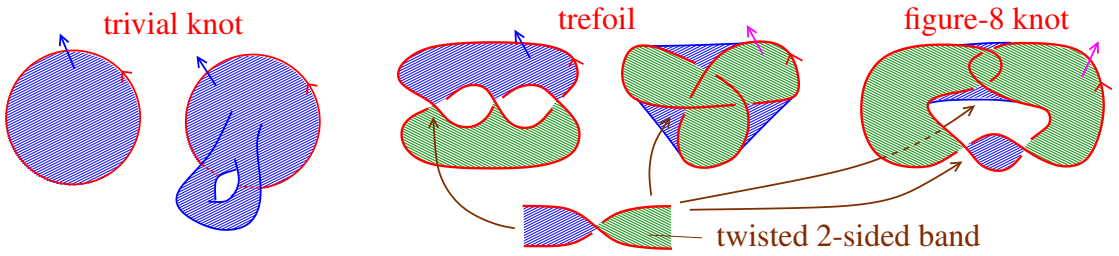
- In Section 285.1 we will (re-) introduce the notion of a Seifert surface. We will use cohomology to give a new proof that Seifert surfaces always exist.
- In Section 285.2 we will see that one can calculate the linking number using Seifert surfaces and also using surfaces in the 4-ball.
- In Section 285.3 we will summarize all the many different ways to define and calculate the linking number.
- In Appendix 285.4 we will relate Alexander duality to linking numbers.
- In Section 285.5 we will provide exercises for Chapter 285.

In the next Chapter 286 we will introduce the very general concept of an Alexander module.

285.1. **Seifert surfaces.** In Chapter 121 we introduced the notion of a Seifert surface, we showed that any knot admits a Seifert surface and we discussed many examples and properties of Seifert surfaces. In this section we will give a new proof that Seifert surfaces exact. We start out by recalling Definition 121.1, so that we know what we are talking about:

Definition 285.1. Let $K \subseteq S^3$ be a knot. A Seifert surface for K is a compact orientable connected 2-dimensional smooth submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented smooth manifolds, in the sense of Convention 31.39.

Example. In the figure below we show Seifert surfaces for the trivial knot, the trefoil and the figure-8 knot.



The colors indicate that the surfaces are two-sided, i.e. they are orientable. □

The following theorem is arguably one of the most frequently used results in knot theory.

Theorem 285.2. (Seifert Surface Existence Theorem)
(a) Every knot $K \subseteq S^3$ admits a Seifert surface.
(b) Every knot $K \subseteq \mathbb{R}^3$ admits a Seifert surface in $\mathbb{R}^3$.

Remark.
(1) The above theorem was first proved in 1930 by F. Frankl and Lev Pontryagin [**FraP1930**]. Shortly afterwards Herbert Seifert [**Sei1934**] gave a different proof, namely essentially the one we presented on page 2336. Our earlier proof on page 2336 was rather hands on and constructive, but depending on one's taste, the proof was perhaps a little ad hoc and not completely rigorous. The proof we will now give is completely rigorous and it can easily be generalized to many other settings, but the proof below will be much less explicit than the previous proof on page 2336.
(2) In the Seifert Surface Existence Theorem 288.12 we will formulate and prove an extension of the above Seifert Surface Existence Theorem 285.2 to links.
(3) A slight generalization of the argument below proves the following stronger statement: Given any $n \in \mathbb{N}$ and given any closed orientable n-dimensional smooth manifold $V \subseteq S^{n+2}$ there exists a compact orientable $(n+1)$-dimensional submanifold $W \subseteq S^{n+2}$ such that $\partial W = V$. We refer either to Exercise 285.8 or to [**Bre1993**, Theorem VII.14.7], [**Er1969**, Lemma 2.2] and [**BDR2017**] for details. □

Our new proof of the Seifert Surface Existence Theorem 285.2 will require the remainder of this section. In the next Section 285.2 we will learn how to calculate linking numbers using Seifert surfaces.

Our proof of the Seifert Surface Existence Theorem 285.2 builds on the following proposition:

Proposition 285.3. (Smooth Map-to-S^1-Proposition) Let M be a smooth manifold.
(1) Given any $\Phi \in \mathrm{H}^1(M;\mathbb{Z})$ there exists a smooth map $f\colon M \to S^1$ with $f^*([S^1]^*) = \Phi$.
(2) If $f, g\colon M \to S^1$ are two smooth maps with $f^*([S^1]^*) = g^*([S^1]^*) \in \mathrm{H}^1(M;\mathbb{Z})$, then f and g are smoothly homotopic.

Proof of the Smooth Map-to-S^1-Proposition 285.3. Both statements follow from the Maps-to-S^1-Proposition 233.4. Alternatively both statements also follow from the future Maps-to-K(π, n)-Proposition 307.18 together with the Abelian-π_1–Drop Base Point Proposition 308.42, the observation on page 5937 that S^1 is a model for $\mathrm{K}(\mathbb{Z}, 1)$, the Smooth Manifold–CW-Theorem 179.9 and the Whitney Approximation Theorem 36.1. ■

Using the Smooth Map-to-S^1-Proposition 285.3 we can prove the following proposition, which is perhaps of independent interest:

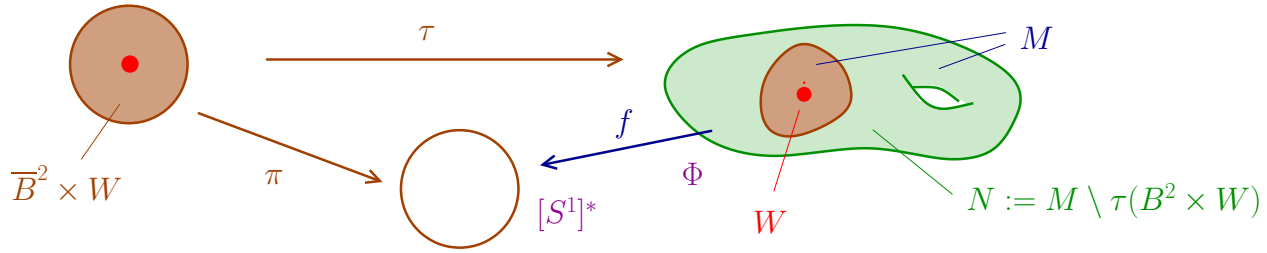
Proposition 285.4. (Extend-Tubular Projection Proposition) Let M be an n-dimensional smooth manifold, let $W \subseteq M$ be a connected proper $(n-2)$-dimensional smooth manifold and let $\tau\colon \overline{B}^2 \times W \to M$ be tubular map. We consider the map

$$\begin{array}{rcl} \pi\colon (\overline{B}^2 \setminus \{0\}) \times W & \to & S^1 \\ (z, w) & \mapsto & \frac{z}{|z|}. \end{array}$$

Furthermore let $\Phi \in \mathrm{H}^1(M \setminus W; \mathbb{Z})$ be a cohomology class such that we have the equality $(\tau|_{S^1 \times W})^*(\Phi) = \pi^*([S^1]^*) \in \mathrm{H}^1(S^1 \times W; \mathbb{Z})$. In this setting there exists a smooth map $f\colon M \setminus W \to S^1$ with the following two properties:

- We have $f^*([S^1]^*) = \Phi \in \mathrm{H}^1(M \setminus W; \mathbb{Z})$.
- The following diagram commutes:

$$\begin{array}{ccc} (\overline{B}^2_{\frac{1}{2}} \setminus \{0\}) \times W & \xrightarrow{\quad\tau\quad} & M \setminus W \\ & {\scriptstyle \pi,\ (z,w) \mapsto \frac{z}{|z|}} \searrow \quad \swarrow {\scriptstyle f} & \\ & S^1. & \end{array}$$



Proof of the Extend-Tubular Projection Proposition 285.4. We start out with several preparations:

(a) We consider $N := M \setminus \tau(B^2 \times W)$. It follows from the Codimension-Zero Submanifold Proposition 25.49 that N is an n-dimensional smooth manifold.

(b) It follows from the Smooth Map-to-S^1-Proposition 285.3 (1) that there exists a smooth map $g\colon N \to S^1$ such that $g^*([S^1]^*) = \Phi|_N$.

The task now is to "connect" the maps $g\colon N \to S^1$ and $\pi \circ \tau^{-1}\colon \tau((\overline{B}^2_{\frac{1}{2}} \setminus \{0\}) \times W) \to S^1$. The key to doing so is the following claim:

Claim. There exists a smooth map $h\colon (\overline{B}^2 \setminus B^2_{\frac{1}{2}}) \times W \to S^1$ which is given by $g \circ \tau$ on $S^1 \times W$ and which is given by $z \mapsto \frac{z}{|z|}$ on $S^1_{\frac{1}{2}} \times W$.

Proof. We consider the smooth maps

$$\begin{array}{rcl} g \circ \tau\colon S^1 \times W & \to & S^1 \\ (z,w) & \mapsto & (g \circ \tau)(z,w) \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi\colon S^1 \times W & \to & S^1 \\ (z,w) & \mapsto & z. \end{array}$$

Note that we have the following equalities in $\mathrm{H}^1(S^1 \times W; \mathbb{Z})$:

$$(g \circ \tau)^*([S^1]^*) = \tau^*(g^*([S^1]^*)) \underset{\text{by (b)}}{\overset{\uparrow}{=}} \tau^*([\Phi]_N) \underset{\text{by hypothesis}}{\overset{\uparrow}{=}} \pi^*([S^1]^*).$$

It thus follows from the Smooth Map-to-S^1-Proposition 285.3 (2) that there exists a smooth homotopy $F\colon (S^1 \times W) \times [0,1] \to S^1$ with $F_0 = g \circ \tau$ and $F_1 = \pi$. The map

$$\begin{array}{rcccl} h\colon (\overline{B}^2 \setminus B^2_{\frac{1}{2}}) \times W & \xrightarrow{\cong} & (S^1 \times W) \times [0,1] & \xrightarrow{H} & S^1 \\ (z,w) & \mapsto & (\left(\frac{z}{|z|}, w\right), 1 - 2 \cdot |z|) & & \end{array}$$

has the desired properties. ⊞

Next we consider the map

$$\begin{array}{rcl} \widetilde{f}\colon M \setminus W & \to & S^1 \\ P & \mapsto & \begin{cases} g(P), & \text{if } P \in N, \\ h(\tau^{-1}(P)), & \text{if } P \in \tau((\overline{B}^2 \setminus B^2_{\frac{1}{2}}) \times W), \\ \frac{z}{|z|}, & \text{if } P = \tau(z,w) \text{ for some } (z,w) \in (\overline{B}^2_{\frac{1}{2}} \setminus \{0\}) \times W. \end{cases} \end{array}$$

Note that it follows from the Pasting Proposition 4.7 (2) that this map $\widetilde{f}$ is continuous. Also note that this map $\widetilde{f}$ is smooth on the closed subset $\tau((\overline{B}^2_{\frac{1}{2}} \setminus \{0\}) \times W)$. It now follows from the Whitney Approximation Theorem 36.1 that $\widetilde{f}$ is homotopic to a smooth map f which agrees with $\widetilde{f}$ on the closed subset $\tau((\overline{B}^2_{\frac{1}{2}} \setminus \{0\}) \times W)$. This map has all the desired properties. ■

Finally we are in a position to prove the Seifert Surface Existence Theorem 285.2:

Proof of the Seifert Surface Existence Theorem 285.2 (1). Let $K \subseteq S^3$ be a knot. We need to show that there exists a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$.

First we introduce a few objects:

- We pick an orientation for K.
- It follows from the Knot Thickening–Self Linking Lemma 284.11 (2) that we can pick a thickening $\tau\colon \overline{B}^2 \times K \to S^3$ for the knot K with self-linking number zero.
- We consider the meridian $\mu_K = \tau(S^1 \times \{k\})$ and the longitude $\lambda_K = \tau(\{*\} \times K)$.
- We denote by $\Phi_K \in \mathrm{H}^1(S^3 \setminus K; \mathbb{Z})$ the unique element with $\langle \Phi_K, [\mu_K] \rangle = 1$.
- We denote by $\pi\colon S^1 \times K \to S^1$ the natural projection.

Claim 1. We have the following equality in $\mathrm{H}^1(S^1 \times K; \mathbb{Z})$:

$$(\tau|_{S^1 \times K})^*(\Phi_K) \;=\; \pi^*([S^1]^*).$$

Proof. First note that it follows from the Universal Coefficient-H*-Theorem 201.23 that it suffices to show that for all $\sigma \in \mathrm{H}_1(\tau(S^1 \times K))$ we have

$$\langle (\tau|_{S^1 \times K})^*(\Phi_K), \sigma \rangle \;=\; \langle \pi^*([S^1]^*), \sigma \rangle.$$

In fact it suffices to prove the equality for all elements of a generating set of $\mathrm{H}_1(\tau(S^1 \times K))$. In fact a basis for $\mathrm{H}_1(\tau(S^1 \times K))$ is given by $[\tau(S^1 \times \{k\})]$ and $[\tau(\{*\} \times K)]$. The claim thus follows from the following calculations:

$$\begin{array}{rcccccl} \langle (\tau|_{S^1 \times K})^*(\Phi_K), [\tau(S^1 \times \{k\})] \rangle & = & \langle \Phi_K, [\tau(S^1 \times \{k\})] \rangle & = & \langle \Phi_K, [\mu_K] \rangle & = & 1, \\ \langle \pi^*([S^1]^*), [\tau(S^1 \times \{k\})] \rangle & = & \langle [S^1]^*, [\pi(S^1 \times \{k\})] \rangle & = & \langle [S^1]^*, [S^1] \rangle & = & 1, \\ \langle (\tau|_{S^1 \times K})^*(\Phi_K), [\tau(\{*\} \times K)] \rangle & = & \langle \Phi_K, [\tau(\{*\} \times K)] \rangle & = & \langle \Phi_K, [\lambda_K] \rangle & = & 0, \\ \langle \pi^*([S^1]^*), [\tau(\{*\} \times K)] \rangle & = & \langle [S^1]^*, \pi_*([\{*\} \times K]) \rangle & = & \langle [S^1]^*, 0 \rangle & = & 0. \end{array}$$

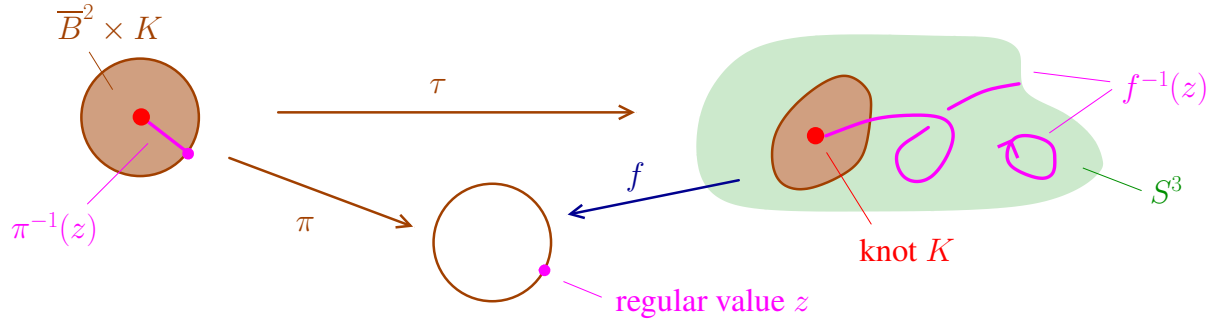
Here the first equality is always given by the Kronecker Pairing–Natural Lemma 199.9 (1). ⊞

Now note that it follows from Claim 1 and the Extend-Tubular Projection Proposition 285.4 that there exists a smooth map $f\colon S^3 \setminus K \to S^1$ such that following diagram

commutes:

$$\begin{array}{ccc} (\overline{B}^2 \setminus \{0\}) \times K & \xrightarrow{\tau} & S^3 \setminus K \\ & \underset{(z,w)\mapsto \frac{z}{|z|}}{\pi} \searrow \quad \swarrow \underset{f}{} & \\ & S^1. & \end{array}$$

Next note that it follows from Sard's Theorem 38.1 together with the Full Measure-Density Proposition 37.4 that f admits a regular value $z \in S^1$. Finally note that it follows from the Regular Value Theorem 32.5 (1) and (3) that $f^{-1}(z)$ is an orientable proper smooth submanifold of $S^3 \setminus K$.



Claim 2. The union $F := f^{-1}(z) \cup K$ is a compact orientable 2-dimensional smooth submanifold of S^3 with $\partial F = K$.

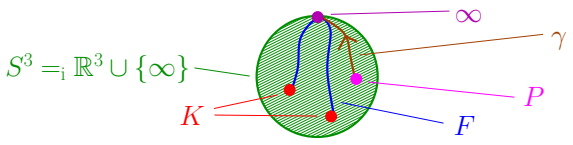
Proof. Note that $F \cap (S^3 \setminus K)$ is by construction a proper smooth submanifold of $S^3 \setminus K$. Furthermore note that $F \cap \tau(B^2 \times K) = \tau(\{r \cdot z \,|\, r \in [0,1)\} \times K)$. Clearly this is a proper smooth submanifold of $\tau(B^2 \times K)$. Since $S^3 \setminus K$ and $\tau(B^2 \times K)$ form an open cover of S^3 we see, using the Pasting Proposition 4.7 (2), that $F := f^{-1}(z) \cup K$ is a compact orientable 2-dimensional smooth submanifold of S^3 with $\partial F = K$. ⊞

It is now clear that the component of F that contains K is indeed a Seifert surface for the knot K. ■

Proof of the Seifert Surface Existence Theorem 285.2 (2). Now let $K \subseteq \mathbb{R}^3$ be a knot. By (1) we know of course that K admits a Seifert surface $F \subseteq S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. If $F \subseteq \mathbb{R}^3$, then we are done. Now assume that $\infty \in F$. Since F is a 2-dimensional submanifold we can pick $P \in \mathbb{R}^3 \setminus F$.

Claim. There exists a smooth diffeotopy $H\colon S^3 \times [0,1] \to S^3$ rel K with $H_0 = \mathrm{id}$ and $H_1(P) = \infty$.

Proof. [3066] Since $S^3 \setminus K$ is path-connected we obtain from the Smooth Path-Connectivity Proposition 25.32 a smooth path $\gamma\colon [0,1] \to S^3 \setminus K$ with $\gamma(0) = P$ and $\gamma(1) = \infty$. We view γ as a smooth isotopy.



[3066] The claim and its proof are close in spirit to the Link-in-$\mathbb{R}^3$-S^3-Lemma 108.17.

It follows from the Isotopy Extension Theorem 44.2 that there exists a smooth diffeotopy $H\colon (S^3 \setminus K) \times [0,1] \to S^3 \setminus K$ with $H_0 = \mathrm{id}$ and with $H_1(P) = \infty$ and such that H has compact support. This implies that H can be extended to a smooth diffeotopy on S^3 that is stationary on K. ⊞

Clearly $H_1(F)$ is now a Seifert surface with $H_1(F) \subseteq \mathbb{R}^3$. ■

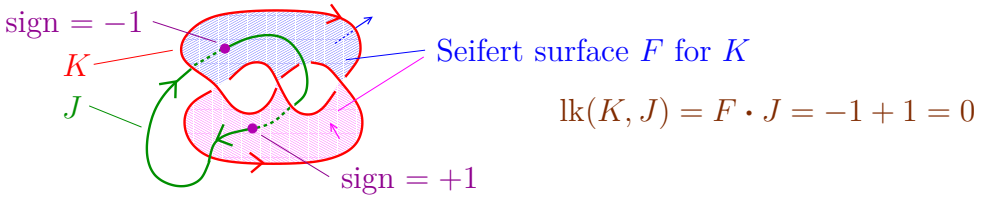
285.2. **Linking numbers and intersection numbers of surfaces.** In this section we will get to know two ways how one can calculate linking numbers as intersection numbers of suitable submanifolds.

Proposition 285.5. (Linking #-via-Intersection #-Proposition) Let K, J be two disjoint oriented knots in S^3. Furthermore let F be a Seifert surface for K that is transverse to J. Then

$$\mathrm{lk}(K,J) \;=\; F \bullet_{S^3} J$$

where $F \bullet J$ denotes the algebraic intersection number of F and J as introduced in Definitions 51.4 and 220.5.[3067]

Example. In the figure below we show two disjoint oriented knots K, J and a Seifert surface F for K.



Note that $F \bullet J = -1 + 1$, thus it follows from the Linking #-via-Intersection #-Proposition 285.5 that $\mathrm{lk}(K,J) = 0$. □

In the proof of the Linking #-via-Intersection #-Proposition 285.5 we will need the following somewhat technical lemma:

Lemma 285.6. (Seifert Surface–Tubular Map Lemma) Let $K \subseteq S^3$ be a knot. Given any Seifert surface F for the knot K there exists a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ such that $\tau(\overline{B}^2 \times K) \cap F = \tau([0,1] \times K)$, where $[0,1] \subseteq \overline{B}^2 \subseteq \mathbb{C}$ is the unit interval.



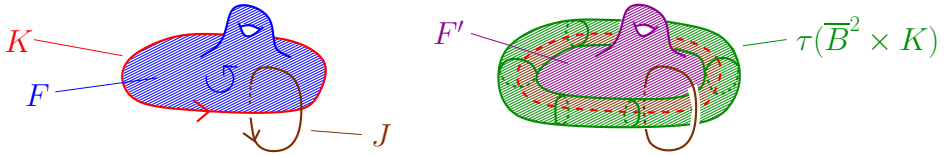
Proof of the Seifert Surface–Tubular Map Lemma 285.6. This statement follows immediately from the Product Tubular Neighborhood–Existence Theorem 58.4 (2) and the Tubular Map Lemma 58.3 together with the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33. ■

Proof of the Linking #-via-Intersection #-Proposition 285.5. Let K, J be two disjoint oriented knots in S^3 and let F be a Seifert surface for K that is transverse to J. It

[3067]Recall that in the Cup Product–Intersection Number Theorem 220.10 we showed that intersection pairings can be calculated in terms of algebraic intersection numbers of embedded transverse submanifolds. Using this fact it is now straightforward to see that the Linking #-via-Intersection #-Proposition 285.5 is a special case of our earlier Linking #–Sphere Intersection #- Proposition 258.13.

follows easily from the above Seifert Surface–Tubular Map Lemma 285.6 that we can pick a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ for K such that $\tau(S^1 \times K)$ intersects F transversally in a single curve and such that $\tau(\overline{B}^2 \times K)$ does not intersect J.

We write $X_K \coloneqq S^3 \setminus \tau(B^2 \times K)$, we set $\mu_K = \tau(S^1 \times \{*\}$ and we write $F' \coloneqq F \cap X_K$. It follows from our choice of tubular neighborhood and from the Submanifold Transversal Intersection Theorem 50.4 that F' is a compact oriented proper submanifold of X_K. In particular we obtain a homology class $[F'] \in \mathrm{H}_2(X_K, \partial X_K)$.



We start out with the following claim:

Claim. We have $\mathrm{PD}_{X_K}([F']) = [\mu_K]^* \in \mathrm{H}^1(X_K; \mathbb{Z})$.

Proof. First we point out that it follows from Definition 210.2 of the Poincaré Duality isomorphism PD_{X_K} the claim is equivalent to showing that $[\mu_K]^* \cap [X_K] = [F'] \in \mathrm{H}_2(X_K, \partial X_K)$. By the Link Complement-Boundary H_*–Proposition 283.5 we know that the connecting homomorphism $\partial_2\colon \mathrm{H}_2(X_K, \partial X_K) \to \mathrm{H}_1(\partial X_K)$ is a monomorphism. Therefore it suffices to show that $\partial_2([\mu_K]^* \cap [X_K]) = \partial_2([F'])$. We denote by $i\colon \partial X_K \to X_K$ the inclusion map. Note that we have the following equalities in $\mathrm{H}_1(\partial X_K)$:

$$\partial_2([\mu_K]^* \cap [X_K]) \underset{\text{by the Poincaré Duality Boundary Proposition 211.14}}{\overset{\uparrow}{=}} -i^*([\mu_K]^*) \cap [\partial X_K] \overset{\downarrow}{=} [\lambda_K] \overset{\downarrow}{=} [\partial F'] \underset{\text{by the Fundamental Class-of-}\partial\text{ Proposition 194.13}}{\overset{\uparrow}{=}} \partial_2([F']).$$

by the Longitude-Meridian Lemma 284.12 (5) and by the Longitude-Meridian Lemma 284.12 (6) from page 5439

⊞

Now we calculate that

$$\begin{aligned} F \cdot J \overset{\downarrow}{=} J \cdot F \overset{\downarrow}{=} J \cdot F' &\overset{\downarrow}{=} \epsilon_{X_K}((\mathrm{PD}_{X_K}([J]) \cup \mathrm{PD}_{X_K}([F'])) \cap [X_K]) \\ &\underset{\uparrow}{=} \epsilon_{X_K}(\mathrm{PD}_{X_K}([F']) \cap (\mathrm{PD}_{X_K}([J]) \cap [X_K])) \\ &\underset{\uparrow}{=} \epsilon_{X_K}([\mu_K]^* \cap [J]) \\ &\underset{\uparrow}{=} \langle [\mu_K]^*, [J] \rangle \underset{\uparrow}{=} \mathrm{lk}(K, J). \end{aligned}$$

by the Intersection Number Symmetry Lemma 220.8

since $J \subseteq X_K$

follows from the Intersection Number via-Cup Product Theorem 220.10 and the Cap Product-Kronecker Lemma 206.12

by the Cup-Cap Lemma 206.15

by the claim we have $\mathrm{PD}_{X_K}([F']) = [\mu_K]^*$ and by Definition 210.2 we have $\mathrm{PD}_{X_K}([J]) \cap [X_K] = [J]$

by the Cap Product Kronecker Lemma 206.12

see Definition 284.1

■

In the following proposition we give a 4-dimensional interpretation of the linking number:

Proposition 285.7. (Linking Number-via–Surfaces Proposition) Let K, K' be two disjoint oriented knots in S^3.

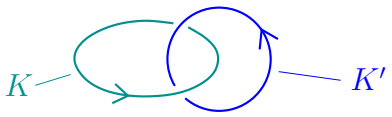
(1) There exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse.

(2) Let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. If S and S' are transverse, then[3068] [3069]

$$\mathrm{lk}(K, K') \;=\; S \bullet_{\overline{B}^4} S'.$$

Example. We consider the Hopf link $H := \{(z,w) \in \mathbb{C}^2 \,|\, |z| = 1 \text{ or } |w| = 1\}$ in the 3-sphere $S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\}$ with the two components that are given by $K := \{(z,0) \in \mathbb{C}^2 \,|\, |z| = 1\}$ and $K' := \{(0,w) \in \mathbb{C}^2 \,|\, |w| = 1\}$, which we give the obvious orientation.

Hopf link $H = K \sqcup K'$



In the following we consider the closed disk $S := \{(z,0) \in \mathbb{C}^2 \,|\, |z| \leq 1\}$ and we consider the closed disk $S' := \{(0,w) \in \mathbb{C}^2 \,|\, |w| \leq 1\}$. Clearly $\partial S = K$ and $\partial S' = K'$. We now see that

$$\mathrm{lk}(K, K') \;\underset{\text{by the Linking Number via–Surfaces Proposition 285.7}}{=}\; S \bullet_{\overline{B}^4} S' \;\underset{\text{since we have } S \cap S' = \{(0,0)\} \text{ and since the orientations match}}{=}\; 1.$$

□

Proof. Let K, K' be two disjoint oriented knots.

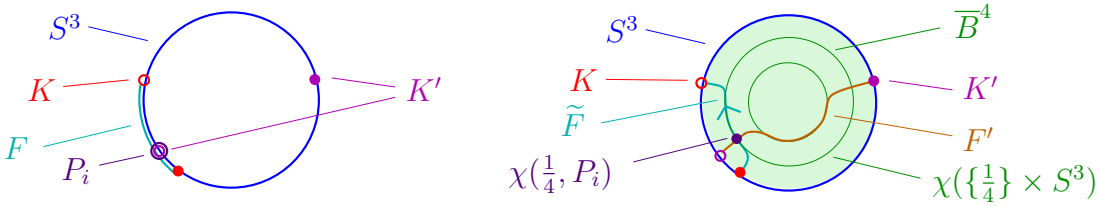
(1) We start out with a few preparations:

- It follows from the Seifert Surface Existence Theorem 285.2 and the Transversality Theorem 50.9 (2) that there exists a Seifert surface F for K that is transverse to K'.
- We denote by $P_1, \ldots, P_m$ the intersection points of F and K' and we denote by $\epsilon_1, \ldots, \epsilon_m$ the corresponding signs.
- We apply the Seifert Surface Existence Theorem 285.2 to obtain a Seifert surface F' for K'.
- Let
$$\begin{array}{rcl} \chi\colon [0,1] \times S^3 & \to & S^3 \\ (t,x) & \mapsto & (1 - \frac{1}{2}t) \cdot x \end{array}$$
be the "obvious" collar for $S^3 = \partial \overline{B}^4$.
- We use the collar $\chi\colon [0, \frac{1}{2}] \times S^3 \to S^3$ to construct the push-in $\widetilde{F}$ of F as in Definition 35.11. Similarly we use the collar $\chi\colon [0,1] \times S^3 \to S^3$ to define the push-in $\widetilde{F}'$ of F'.

It follows from the Push-In Lemma 35.12 that $\widetilde{F}$ and $\widetilde{F}'$ are oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial \widetilde{F} = K$ and $\partial \widetilde{F}' = \widetilde{K}$. Furthermore, one can easily

[3068] Note that this statement together with the Intersection Number–Symmetry Lemmas 51.6 and 220.8 gives in particular a new proof of the Linking Number-Symmetry Proposition 284.9.

[3069] In the earlier Linking #–Ball Intersection #-Proposition 258.14 we proved a very similar statement in all dimensions. In fact the Cup Product–Intersection Number Theorem 220.10 shows that the earlier proposition implies the above Linking Number-via–Surfaces Proposition 285.7. The proof of the earlier proposition is arguably mathematically purer than the argument that we will give below. The argument below in turn has the advantage that it is perhaps more intuitive and more geometric.



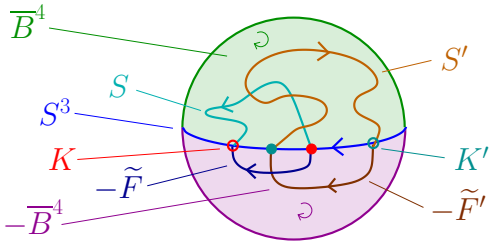
verify, using the explicit Definition 35.11 that $\widetilde{F}$ and $\widetilde{F}'$ intersect precisely in the points $\chi(\frac{1}{4}, P_1), \dots, \chi(\frac{1}{4}, P_m)$ where the signs are again given by $\epsilon_1, \dots, \epsilon_m$.

(2) We continue with the notation from (1). Note that

$$\operatorname{lk}(K, K') \underset{\uparrow}{=} F \bullet_{S^3} K' \underset{\uparrow}{=} \sum_{i=1}^{m} \epsilon_i \underset{\uparrow}{=} \widetilde{F} \bullet_{\overline{B}^4} \widetilde{F}' \underset{\uparrow}{=} -(-\widetilde{F}) \bullet_{-\overline{B}^4} (-\widetilde{F}').$$

by the Linking Numbers-via Intersection Proposition 285.5 — follows from the discussion of the intersection points in (1) — follows easily from the definitions

Now let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. We think of $S^4 =_{\mathrm{i}} \overline{B}^4 \cup (-\overline{B}^4)$ as the union of two copies of $\overline{B}^4$ with opposite orientations, the first copy contains S, S' and the second contains $-\widetilde{F}, -\widetilde{F}'$.



It follows from the Gluing Submanifolds Proposition 56.12 that we can arrange, without changing the intersection points, that $S \cup -\widetilde{F}$ and $S' \cup -\widetilde{F}'$ are closed oriented smooth submanifolds of S^4. We calculate that

$$S \bullet_{\overline{B}^4} S' - \operatorname{lk}(K, K') \underset{\downarrow}{\text{by the above}} = S \bullet_{\overline{B}^4} S' + (-\widetilde{F}) \bullet_{-\overline{B}^4} (-\widetilde{F}') = (S \cup -\widetilde{F}) \bullet_{S^4} (S' \cup -\widetilde{F}') \underset{\uparrow}{=} 0.$$

since we have the equalities $S \cap \widetilde{F}' = S' \cap \widetilde{F} = K \cap K' = \varnothing$ — follows from the Cup Product Intersection Number Theorem 220.10 since $\mathrm{H}_2(S^4) = 0$

We have thus shown the desired equality $S \bullet_{\overline{B}^4} S' = \operatorname{lk}(K, K')$. ∎

285.3. **Comparisons of the definition of the linking number.** Given two disjoint oriented knots K and J in $\mathbb{R}^3$ we have now seen many different ways to define and think about the linking number. In this section we want to summarize all definitions from these lecture notes and we want to show that they agree.

As the reader might remember, one of the many definitions builds on the notion of the degree of a map between two closed oriented connected non-empty smooth manifolds of the same dimension.

For the purpose of this section it is arguably best to work with the following definition of the degree:[3070]

Definition 46.3. Let $f\colon M \to N$ be a smooth map between two closed oriented connected non-empty smooth manifolds of the same dimension.

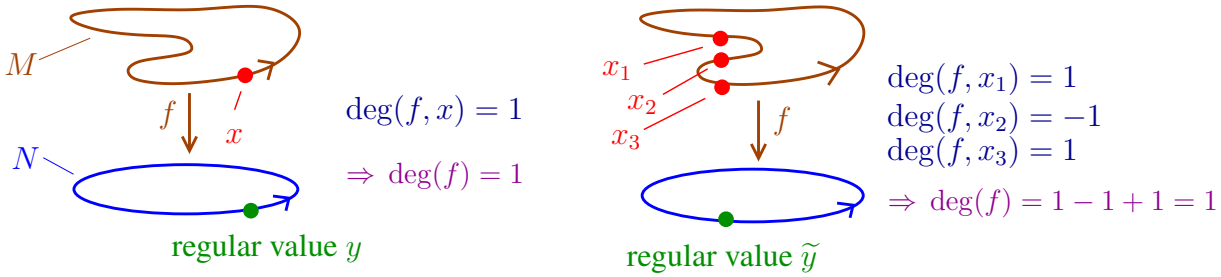
(1) If f is a local diffeomorphism at a point $x \in M$, then we define the local degree of f at the point x as

$$\deg(f,x) \coloneqq \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

(2) It follows from Sard's Theorem 38.1 that there exists a regular value $y \in N$ of the smooth map $f\colon M \to N$. We define the degree of the smooth map f as follows:[3071]

$$\deg(f) \coloneqq \sum_{x \in f^{-1}(y)} \deg(f,x) \in \mathbb{Z}.$$

It follows from the Degree–Well-Defined Theorem 46.2 that this definition does not depend on the choice of the regular value y.



Now we can formulate the following proposition which collects the various definitions of the linking number and shows that they are all the same.[3072]

Proposition 285.8. (Linking Number Reformulation Proposition) Let K, J be two disjoint oriented knots in $\mathbb{R}^3 \subseteq S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. First we introduce some extra objects:

(a) Let μ_K be a meridian of K.
- We denote by $\Phi_K^{\pi_1} \in \pi_1(S^3 \backslash K) \to \mathbb{Z}$ the unique homomorphism with $\Phi_K([\mu_K]) = 1$.
- We denote by $\Phi_K^{\mathrm{H}_1} \in \mathrm{H}_1(S^3 \backslash K) \to \mathbb{Z}$ the unique homomorphism with $\Phi_K([\mu_K]) = 1$.

(b) Let F be a Seifert surface for J that intersects K transversally.

(c) Let $\widetilde{F}, \widetilde{G}$ be proper oriented submanifolds of $\overline{B}^4$ with $\partial\widetilde{F} = J$ and $\partial\widetilde{G} = K$.

(d) Given a continuous map $f\colon M \to N$ between closed oriented connected non-empty smooth manifolds of the same dimension we denote by $\deg(f\colon M \to N) \in \mathbb{Z}$ the degree from Definition 46.3.

(e) Let $f\colon [0,2\pi] \to J$ and $g\colon [0,2\pi] \to K$ be orientation-preserving parametrizations.

[3070] We also introduced the notion of the degree in Definitions 86.2, 186.1 and 195.1. In the Degrees-Agree Proposition 231.19 we showed that all these definitions of the degree agree.

[3071] Since $\dim(M) = \dim(N)$ we see that for a regular value x the differential is an isomorphism. By the Inverse Mapping Theorem 30.1 this implies that f is a local diffeomorphism at x. Thus the local degree is defined.

[3072] The proposition can be viewed as a variation on the Linking Number Reformulation Proposition 258.19.

(f) Let $$\mathrm{AD}_K \colon \mathrm{H}_1(S^3 \setminus K) \;\to\; \mathrm{H}^1(K;\mathbb{Z})$$ be the isomorphism from the Sphere–Alexander Duality Theorem 217.8 or alternatively from the Sphere–Alexander Duality Theorem 218.2.[3073]

The following numbers are all equal:[3074]

(1) $\Phi_K^{\pi_1}([J])$ (Definition 114.1 of $\mathrm{lk}(K,J)$).

(2) $\Phi_K^{\mathrm{H}_1}([J])$ (Definitions 258.7 and 284.1 of $\mathrm{lk}(K,J)$)

(3) The signed count of crossings as in the Linking Number-via-Crossings Propositions 114.9 and 284.8.

(4) The intersection number $K \bullet_{S^3} F$.

(5) The intersection number $\widetilde{G} \bullet_{\overline{B}^4} \widetilde{F}$.

(6) $$-\deg \begin{pmatrix} K \times J & \to & S^2 \\ (x,y) & \mapsto & \frac{x-y}{\|x-y\|} \end{pmatrix}$$ (Definitions 52.2 and 86.10 of $\mathrm{lk}(K,J)$).

(7) $$\frac{1}{4\pi} \int_{s=0}^{s=2\pi} \int_{t=0}^{t=2\pi} \frac{\det \overbrace{\begin{pmatrix} (f(s)-g(t)) & f'(s) & g'(t) \end{pmatrix}}^{3\times 3\text{–matrix}}}{\|f(s)-g(t)\|^3}\, dt\, ds.$$

(8) $$-\langle \underbrace{\mathrm{AD}_K([J])}_{\in \mathrm{H}^1(K;\mathbb{Z})}, [K] \rangle_K.$$

Remark. This proposition is loosely modelled on the well-known [**Rolf1990**, Theorem 5.D.5 on p. 133]. □

Proof. We consider in the following various equalities:

(1)=(2) This equality follows from the Hurewicz Theorem 166.10.

(2)=(3) This equality is the content of the above Linking Number-via-Crossings Proposition 284.8.[3075]

(2)=(4) This equality follows from the Linking #-via-Intersection #-Proposition 285.5.[3076]

(2)=(5) This equality follows from the Linking Number-via–Surfaces Proposition 285.7.[3077]

(3)=(6) This equality was shown, with slightly different language, in the Linking Number–Crossings Proposition 52.5. The key idea was to work with Definition 46.3: One picks the regular value $z = (0,0,1)$, and one shows that the preimages under the map $K \times J \to S^2$ correspond precisely to the crossings. Furthermore the local degrees correspond to the signs of the crossings.

(4)=(6) We proved this equality in the Linking #-via-Intersection #-Proposition 52.8.[3078]

(6)=(7) This equality was proved in the Gauß Linking Number Theorem 86.13.

[3074] We showed in the Sphere–Alexander Duality Isomorphisms Proposition 218.23 that the isomorphisms from these two theorems are the same.

[3075] Alternatively note that the equality (1)=(3) was proved in the Linking Number-via-Crossings Proposition 284.8.

[3076] Alternatively this equality follows easily from the Linking #–Sphere Intersection #-Proposition 258.13 together with the Cup Product–Intersection Number Theorem 220.10.

[3077] Alternatively this equality follows from the Linking #–Ball Intersection #-Proposition 258.14 together with the Cup Product–Intersection Number Theorem 220.10.

[3078] Depending on your favorite definition of the degree you might also need the Degrees-Agree Proposition 231.19 to obtain this equality.

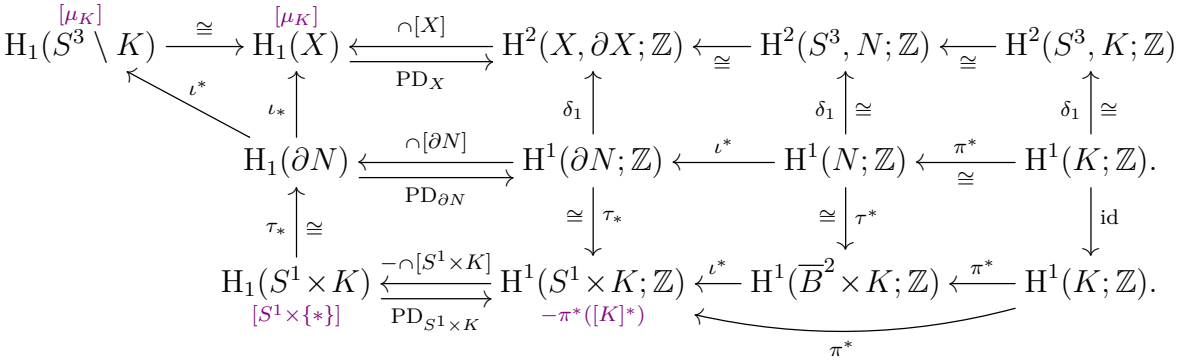
(2)=(8) We need to show that the two *homomorphisms*

$$\begin{array}{rcl} \mathrm{H}_1(S^3 \setminus K) & \to & \mathbb{Z} \\ \varphi & \mapsto & \langle \mathrm{AD}_K(\varphi), [K] \rangle \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Phi_K \colon \mathrm{H}_1(S^3 \setminus K) & \to & \mathbb{Z} \\ m \cdot [\mu_K] & \mapsto & m \end{array}$$

agree up to a minus sign. We pick a thickening $\tau \colon \overline{B}^2 \times K \to S^3$ of K. We set $\mu_K := \tau(S^1 \times \{*\})$. Since $\mathrm{H}_1(S^3 \setminus K) = \mathbb{Z} \cdot [\mu_K]$ it suffices to prove the following claim:

Claim. We have $\mathrm{AD}_K([\mu_K]) = -[K]^* \in \mathrm{H}^1(K; \mathbb{Z})$.

Proof. We write $N := \tau(\overline{B}^2 \times K)$ and $X := S^3 \setminus \tau(B^2 \times K)$. We consider the following diagram:



We make the following observations and clarifications:

(a) The symbol ι always stands for an inclusion.

(b) We denote by $\pi \colon S^1 \times K \to K$ the projection.

(c) It follows from the functoriality of cohomology that the triangle and the semilune commute.

(d) The maps with no names are induced by the obvious maps between (pairs of) topological spaces.

(e) By Definition 210.2 the Poincaré duality maps are the inverses of the maps given by capping with the fundamental class.

(f) On page 4150 we showed that all maps that are decorated with "≅" are indeed isomorphisms.

(g) The maps δ_1 are the connecting homomorphisms of long exact sequences of pairs in cohomology. As we pointed out in the LES-of-Triple Proposition 198.13 these are natural. Thus we see that the top middle and the top right square commute.

(h) It follows from the Poincaré Duality–Connecting Homomorphism Proposition 211.15 that the top left square commutes.

(i) It follows from the Degrees-Agree Proposition 231.19 and the discussion on page 4150 that the "right-down" route from $\mathrm{H}_1(S^3 \setminus K) \to \mathrm{H}^1(K; \mathbb{Z})$ via $\mathrm{H}^2(S^3, K; \mathbb{Z})$ is precisely the Alexander duality isomorphism AD_K.

(j) By Definition 283.3 we know that the thickening $\tau \colon \overline{B}^2 \times K \to S^3$ is orientation-preserving. Note that it follows from the Opposite Side–Opposite Orientation Lemma 31.40 that the orientation of $\tau(S^1 \times K)$ coming from $\tau(\overline{B}^2 \times K)$ is the

opposite of the orientation coming from $X = S^3 \setminus \tau(B^2 \times K)$. It now follows that the lower left square commutes.

(k) It follows from the functoriality of cohomology that the lower middle and lower right square commute.

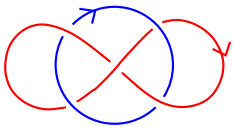
(l) It follows immediately from the Fundamental Class-Product Proposition 207.2 that $\pi^*([K]^*) \cap [S^1 \times K] = [S^1 \times \{*\}] \in \mathrm{H}_1(S^1 \times K)$.

We now see that

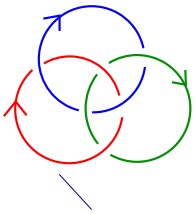
$$\mathrm{AD}_K([\mu_K]) \underset{\text{by definition of } \mu_K}{=} \mathrm{AD}_K((\iota_* \circ \tau_*)([S^1 \times \{*\}])) \underset{\text{follows easily from the above discussion}}{=} -[K^*].$$

■

Example. Given any topological invariant the question that arises is, how good is it at distinguishing objects? It is quite easy to find examples of links which show that the linking number is far from being a complete invariant. For example, in the figure below we show the Whitehead link[3079] and the Borromean rings.[3080]



Whitehead link



Borromean rings



coat of arms of the Borromeo family

Both are links with zero linking numbers and unknotted components but which are appear to be non-trivial. □

We record the following question for later on.

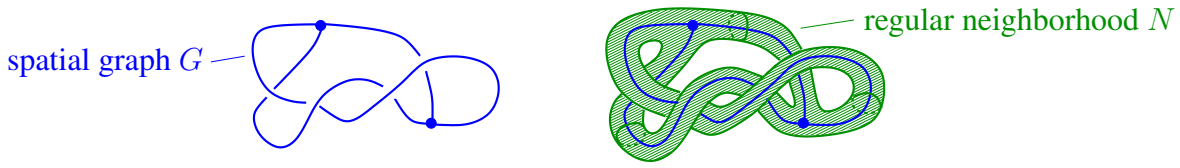
Question 285.9. How can we show that neither the Whitehead link nor the Borromean rings are smoothly isotopic to the trivial link?

Remark. We can of course try to determine the fundamental groups of the complements of the trivial links and the Whitehead link and the Borromean rings. But then the question arises, how can we distinguish these groups? □

285.4. **Appendix: Alexander duality and linking numbers.** Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood in the sense of Definition 217.2. (For example, by the Spatial Graph–Regular Neighborhood Proposition 217.7, the Product Tubular Neighborhood–Existence Theorem 58.4 (2) and the Basics-of-Tubular Neighborhood Lemma 57.10 the subset Y could be a link or a spatial graph.)

[3079] The Whitehead link is named after John Henry Constantine Whitehead who also introduced the concept of CW-complexes.

[3080] The Borromean rings are named after the Borromeo family that carries the Borromean rings in its coat of arms since medieval days.



We can compute the isomorphism types of the first homology group of the complement $S^3 \setminus Y$ as follows:

$$\mathrm{H}_1(S^3 \setminus Y) \overset{\cong}{\underset{\uparrow}{\rightarrow}} \mathrm{H}^1(Y;\mathbb{Z}) \overset{\cong}{\underset{\uparrow}{\rightarrow}} \mathrm{Hom}(\mathrm{H}_1(Y),\mathbb{Z}) \overset{\cong}{\underset{\uparrow}{}} \mathrm{FH}_1(Y) := \mathrm{H}_1(Y)/\text{torsion}.$$

by the Sphere–Alexander Duality Theorem 217.8

by the Universal Coefficient Theorem 201.23

non-natural isomorphism (here we assume that $\mathrm{H}_1(Y)$ is finitely generated)

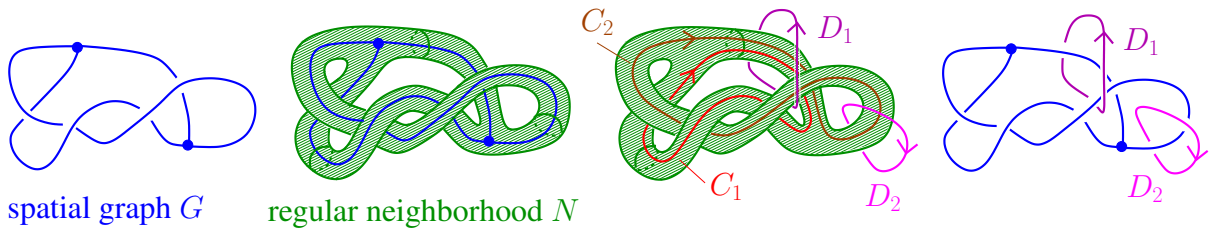
The catch with the Sphere–Alexander Duality Theorem 217.8 is that it is quite hard to "see" the isomorphism. For example, suppose that we are given a basis for $\mathrm{H}_1(Y)/\text{torsion}$, how do we get a basis for $\mathrm{H}_1(S^3 \setminus Y)$? In many situations the following lemma resolves that issue.

Lemma 285.10. (Sphere–Alexander Duality-via-Linking Numbers Lemma) Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N(Y)$. We suppose that $\mathrm{H}_1(Y)$ is a free abelian group[3081] of rank k. Let $C_1, \ldots, C_n$ be oriented curves in $N(Y)$ and let $D_1, \ldots, D_k$ be oriented curves in $S^3 \setminus N(Y)$. If

$$\det\Big(\underbrace{(\mathrm{lk}(C_i, D_j)_{i,j=1,\ldots,k})}_{(k\times k)\text{–matrix}}\Big) \;=\; \pm 1,$$

then $[D_1], \ldots, [D_k]$ form a basis of $\mathrm{H}_1(S^3 \setminus Y)$.

Example. In the figure below we consider a spatial graph G together with a regular neighborhood N.



By the Graph-H_*-Lemma 156.9 we know that $\mathrm{H}_1(G) \cong \mathbb{Z}^2$. In the figure above we also see two oriented curves C_1 and C_2 in N and two oriented curves D_1 and D_2 in $S^3 \setminus \mathring{N}$. Using the Linking Number-via-Crossings Proposition 284.8 one can easily calculate that

$$\begin{pmatrix} \mathrm{lk}(C_1, D_1) & \mathrm{lk}(C_1, D_2) \\ \mathrm{lk}(C_2, D_1) & \mathrm{lk}(C_2, D_2) \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It follows from the Sphere–Alexander Duality-via-Linking Numbers Lemma 285.10 that $[D_1], [D_2]$ form a basis for $\mathrm{H}_1(S^3 \setminus G)$. □

[3081] Is it possible that $\mathrm{H}_1(Y)$ is not a free abelian group?

Proof. Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N = N(Y)$. We denote by $\mathring{N}$ the interior of N.

Claim. The map

$$\begin{aligned} \Omega\colon \mathbb{Z}^k &\to \mathrm{H}_1(S^3 \setminus \mathring{N}) \\ (a_1, \dots, a_k) &\mapsto \sum_{i=1}^{k} a_i \cdot [D_i] \end{aligned}$$

is an isomorphism.

Proof. We also consider the homomorphism

$$\begin{aligned} \Theta\colon \mathrm{H}_1(S^3 \setminus \mathring{N}) &\to \mathbb{Z}^k \\ \sigma &\mapsto (\Phi_{C_1}(\sigma), \dots, \Phi_{C_k}(\sigma)). \end{aligned}$$

We make the following clarifications and observations:

(a) The homomorphisms Φ_{C_i} are given by Definition 284.1.

(b) It follows from our hypothesis and the Definition 284.1 of linking numbers that the map $\Theta \circ \Omega\colon \mathbb{Z}^k \to \mathbb{Z}^k$ is an epimorphism.

(c) By the Sphere–Alexander Duality Theorem 217.8 and the Universal Coefficient-H*-Theorem 201.23 we know that $\mathrm{H}_1(S^3 \setminus \mathring{N})$ is also isomorphic to $\mathbb{Z}^k$.

(d) By the Epi-Automorphism Lemma 97.27 an epimorphism between two free abelian groups of the same rank is an isomorphism.

(e) It follows from (b), (c) and (d) that the epimorphism $\Theta\colon \mathrm{H}_1(S^3 \setminus \mathring{N}) \to \mathbb{Z}^k$ is an isomorphism.

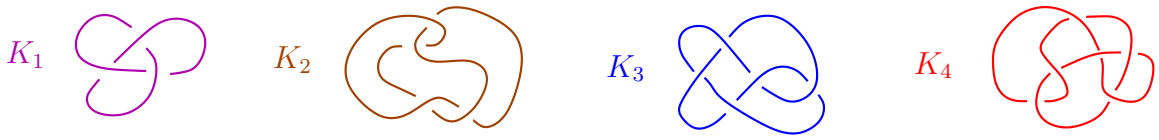
(f) It follows from (b) and (e) that $\Omega\colon \mathbb{Z}^k \to \mathrm{H}_1(S^3 \setminus \mathring{N})$ is an isomorphism. ⊞

Note that $\{[D_1], \dots, [D_k]\}$ are the image under Ω of the standard basis of $\mathbb{Z}^k$. It thus follows from the claim that $[D_1], \dots, [D_k]$ form a basis for $\mathrm{H}_1(S^3 \setminus \mathring{N})$. But it follows from the Basics-of-Regular Neighborhood Lemma 217.4 together with the Homotopy Equivalence-H$_*$-Proposition 144.14 that the inclusion-induced map $\mathrm{H}_1(S^3 \setminus Y) \to \mathrm{H}_1(S^3 \setminus \mathring{N})$ is an isomorphism. Thus we see that $[D_1], \dots, [D_k]$ form a basis of $\mathrm{H}_1(S^3 \setminus Y)$.[3082] ■

285.5. Exercises for Chapter 285.

Exercise 285.1. Find explicit Seifert surfaces for the knots knots K_1, K_2, K_3 and K_4 that are shown in the figure below.

Remark. The answer to a question can only be as rigorous as the input. Since the input is a picture it suffices here of course to draw a convincing picture of a Seifert surface.



Exercise 285.2. Let $K \subseteq S^3$ be a knot. Does there always exist an embedding $\varphi\colon \overline{B}^2 \to \overline{B}^4$ such that $\varphi(S^1) = K$?

Remark. Recall that an embedding is a homeomorphism onto its image. Not more and not less.

[3082] The alert reader will have noticed that the proof resembles the argument given in the Kronecker-Basis Lemma 199.10 (2).

Exercise 285.3. An m-component link $L = L_1 \sqcup \cdots \sqcup L_m \subseteq S^3$ is called *split link* if there exist disjoint closed smoothly embedded balls $B_1, \dots, B_m \subseteq S^3$ such that each L_i is contained in B_i. Furthermore L is called a *boundary link* if there exist m disjoint Seifert surfaces $F_1, \dots, F_m$.

(a) A split link is easily seen to be a boundary link. Do there exist boundary links that are not split?

(b) Show that if $L = L_1 \sqcup \cdots \sqcup L_m$ is a boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group of rank m.
Hint. You could try to find an interesting map from $S^3 \setminus L$ onto the wedge of m circles.

Exercise 285.4. Let $K, J \subseteq S^3$ be two oriented disjoint knots in S^3 with $\mathrm{lk}(K, J) = 0$. Show that there exists a Seifert surface for K that is disjoint from J.
Hint. By the Seifert Surface Existence Theorem 285.2 we know that K admits a Seifert surface. Using the Transversality Theorem 50.10 one can show that K admits a Seifert surface that is transverse to J. Now we still need to make it disjoint from J.

Exercise 285.5. Is every knot $K \subseteq S^3$ the boundary of a *non-orientable* compact surface that is smoothly embedded in S^3?

Exercise 285.6. One can also extend the notion of a Seifert surface to oriented links: A Seifert surface for an oriented link L is a compact oriented 2-dimensional smooth submanifold $F \subseteq S^3$ with $\partial F = L$ and with no closed components. Find explicit connected Seifert surfaces for the oriented links that are shown in the figure below.



Exercise 285.7. We consider the Hopf link $H = H_1 \cup H_2 \subseteq \mathbb{R}^3 \subseteq S^3$ as shown in the figure below. We can draw pictures in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ by using "redness" to indicate the fourth-dimension. For example in the figure we show to the right a disk F in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ which bounds H_2 but which does not intersect H_1 since the "redness coordinate" of the intersection point in $\mathbb{R}^3$ is different. Draw pictures of two properly smoothly embedded disks F_1, F_2 in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ such that $\partial F_i = H_i$ and such that F_1 and F_2 intersect in a single point.



Exercise 285.8. Let $n \geq 4$ and let $K \subseteq S^n$ be a closed orientable smooth submanifold of codimension 2. Show that there exists a compact orientable submanifold W of S^n with $\partial W = K$.
Remark. Note that in general the analogous statement is not true if we do not impose any conditions on the codimension. For example by the Euclidean Embedding Theorem 33.2 and the Smooth Embedding Theorem 30.15 we can view $\mathbb{C}\mathrm{P}^2$ as a smooth submanifold of some $S^n = \mathbb{R}^n \cup \{\infty\}$. But by the proof of the Signature–Cobordism Corollary 280.18 we know that $\mathbb{C}\mathrm{P}^2$ is not the boundary of any compact orientable smooth manifold, let alone one which is embedded in S^n.

Exercise 285.9. The Borromean rings, shown in the figure on page 5456, have the property that if you remove one component, then the remaining two knots form a trivial link. Does there exist a four component link with the property that if you remove any of the four components, then you get a trivial three-component link?

Exercise 285.10. Let $K \subseteq S^3$ be a knot and let F be a Seifert surface for K. Show that for any smoothly embedded closed curve $J \subseteq F \setminus K$ we have $\operatorname{lk}(K, J) = 0$.
Hint. Consider a push-off of J and use the Linking #-via-Intersection #-Proposition 285.5.

Exercise 285.11. Let π be a group. We define the *derived series* of π as follows: We set $\pi^{(0)} := \pi$ and iteratively we set $\pi^{(i+1)} := [\pi^{(i)}, \pi^{(i)}]$ to be the commutator subgroup of $\pi^{(i)}$. Our goal is to show that given any knot K the longitude lies in the second derived subgroup of $\pi := \pi_1(S^3 \setminus K)$. We do so as follows:

(a) Show that for any compact orientable connected 2-dimensional smooth manifold F with non-empty connected boundary the element $[\partial F]$ defines an element of the commutator subgroup of $\pi_1(F)$.
(b) Show that $\pi^{(1)} = [\pi, \pi] = \ker(\Phi_K \colon \pi \to \mathbb{Z}) = \ker(\pi \xrightarrow{[J] \mapsto \operatorname{lk}(K,J)} \mathbb{Z})$.
(c) Show that $\pi_1(F \setminus K)$ lies in the commutator subgroup of $\pi = \pi_1(S^3 \setminus K)$.
Hint. Use Exercise 285.10.
(d) Show that the longitude λ_K lies in the second derived subgroup of π.
Remark. Note that it follows from the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33 that we can assume that the longitude lies on the Seifert surface of K.

CHAPTER 286

Alexander modules

In the last chapters we (re-) introduced knots and links and we studied in great detail the linking number of two disjoint oriented knots. Our next major goal is to define more powerful invariants of knots and links.

To achieve this goal we will develop in this chapter the general theory of Alexander modules. More precisely, in this chapter we will do the following: Let X be a connected topological space (that satisfies a few mild technical hypothesis), let $x_0 \in X$ and let $\Phi\colon \pi_1(X, x_0) \to H$ be an epimorphism onto an abelian group. We consider the covering $\widetilde{X}_\Phi$ corresponding to $\ker(\Phi)$ and we will see that its homology groups are modules over the group ring $\mathbb{Z}[H]$. These are called the Alexander modules of (X, Φ). We will calculate these Alexander modules in several instances.

This chapter is organized as follows:

- In Section 286.1 we will remind the reader of various results on the existence and uniqueness of coverings.
- In Section 286.2 we will reintroduce the deck transformation group of a covering.
- In Section 286.3 we will consider a natural covering of any mapping torus $\mathrm{Tor}(X, f)$.
- In Section 286.4 we will recall the concept of a group ring.
- In Section 286.5 we will see that the homology groups of coverings can be viewed as modules over group rings. We will refer to those as Alexander modules.
- In Section 286.6 we will prove two more theoretical statements about Alexander modules.
- In Section 286.7 we will provide exercises for Chapter 286.

In the next Chapter 287 we will use the Alexander modules to define Alexander polynomials.

286.1. **Reminder of covering theory.** In this section we recall same aspects of covering theory. For the reader's convenience we start out by recalling the definition of a covering:

Definition 91.8. Let $p\colon \widetilde{X} \to X$ be a continuous map between topological spaces.

(1) An open subset $U \subseteq X$ is called uniformly covered, if $p^{-1}(U) \subseteq \widetilde{X}$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of $\widetilde{X}$ with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.

(2) We say that the map $p\colon \widetilde{X} \to X$ is a covering, if p is surjective and if for every $x \in X$ there exists an open neighborhood U of $x \in X$ which is uniformly covered.

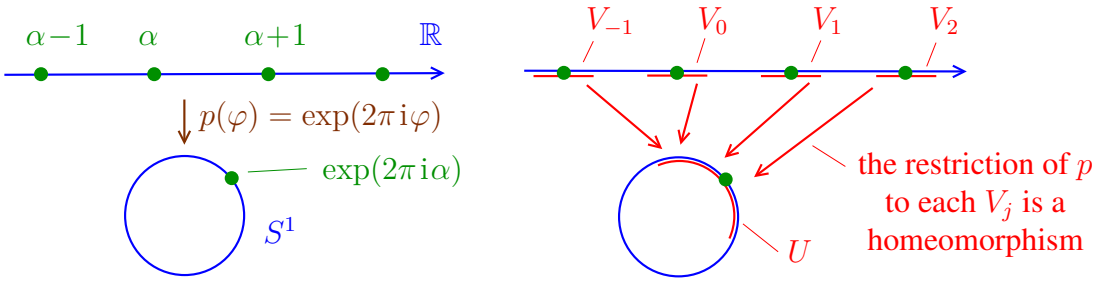
(3) (a) A pointed topological space is a pair (Y, y_0) consisting of a topological space Y and some $y_0 \in Y$.

(b) We say $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering of pointed topological spaces if the map $p\colon \widetilde{X} \to X$ is a covering with $p(\widetilde{x}_0) = x_0$.

Example. Let $n \in \mathbb{N}_0$. It follows easily from the Action–Covering Proposition 91.11 and the usual diffeomorphism $\mathbb{R}^n/\mathbb{Z}^n \to (S^1)^n$ that the maps

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ (t_1,\dots,t_n) & \mapsto & [(t_1,\dots,t_n)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{R}^n & \to & (S^1)^n \\ (t_1,\dots,t_n) & \mapsto & (\exp(2\pi\,\mathrm{i}\,t_1),\dots,\exp(2\pi\,\mathrm{i}\,t_n)) \end{array}$$

are coverings.



For $n = 1$ the proof that $\mathbb{R} \to S^1$ is a covering is illustrated in the figure above. □

The following proposition is one of the first non-trivial results of covering theory:

Proposition 92.7. (Covering-π_1-Monomorphism Proposition) Let $p\colon \widetilde{X} \to X$ be a covering. Given any $\widetilde{x}_0 \in \widetilde{X}$ the induced map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, p(\widetilde{x}_0))$ is a monomorphism.

The Covering-π_1-Monomorphism Proposition 92.7 shows that coverings give rise to subgroups of the fundamental group. The following theorem makes the more interesting statement that, under mild technical hypothesis on the topological space, any subgroup of a fundamental group comes from a covering:

Theorem 127.10. (Covering Existence Theorem) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected[3083]. Furthermore let $x_0 \in X$ and let $\Gamma \subseteq \pi = \pi_1(X, x_0)$ be a subgroup. There exists a path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of pointed topological spaces such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$.

In the following we will provide a sketch of the proof of the Covering Existence Theorem 127.10. We start out by recalling the following definition:

Definition 127.11. Let X be a topological space, let $x_0 \in X$ and let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup.

(1) As in Definition 16.2 we denote by $X^{[0,1]}$ the set of all continuous maps $[0,1] \to X$.

(2) We consider the path space

$$P_{x_0} \;:=\; \{\gamma \in X^{[0,1]} \,|\, \gamma(0) = x_0\} \;=\; \{\text{all paths in } X \text{ with starting point } x_0\}.$$

[3083] We introduced the mystifying concept of "semi-locally simply connected" in Definition 127.9. The precise definition is right now irrelevant for us, since in all examples we are only interested in CW-complexes and topological spaces, and both are locally contractible, in particular they are locally simply connected and semi-locally simply connected. (This was shown in the Manifold–Local Properties Lemma 23.8 and in the CW-Complex–Local Properties Proposition 137.8).

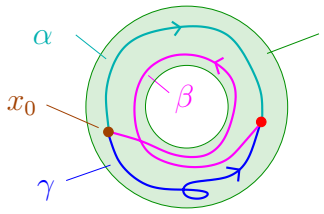
(3) For $u, v \in P_{x_0}$ we write

$$\alpha \sim_\Gamma \beta \quad :\iff \quad \alpha \text{ and } \beta \text{ have the same endpoint and } [\ \underbrace{\alpha * \overline{\beta}}_{\text{loop in } (X,x_0)}\ ] \in \Gamma.$$

(4) We write $\widetilde{X}_{x_0,\Gamma} := P_{x_0}/\sim_\Gamma$ and $\widetilde{x}_0 := [\text{const}_{x_0}]$, i.e. $\widetilde{x}_0$ is the equivalence class of the constant path.

(5) We consider the evaluation map

$$\begin{aligned} p\colon \widetilde{X}_{x_0,\Gamma} = P_{x_0}/\sim_\Gamma &\to X \\ [\alpha] &\mapsto \alpha(1). \end{aligned}$$



X is an annulus

we consider $\Gamma = 2\mathbb{Z} \subseteq \mathbb{Z} =_{\text{i}} \pi_1(X, x_0)$,
then α and β are equivalent,
but α and γ are inequivalent

The Covering Existence Theorem 127.10 is now a consequence of the following lemma, which gives an explicit construction of the desired covering:

Lemma 127.14 (CO-Topology Lemma) Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup.

(i) We equip $X^{[0,1]} :=$ set of all continuous maps $[0,1] \to X$ with the compact-open topology from Definition 16.2.

(ii) We equip $P_{x_0} := \{\gamma \in X^{[0,1]} \mid \gamma(0) = x_0\}$ with the subspace topology.

(iii) We equip $\widetilde{X}_{x_0,\Gamma} := P_{x_0}/\sim_\Gamma$ with the quotient topology.[3084]

We write $\widetilde{X} := \widetilde{X}_{x_0,\Gamma}$. If X is locally path-connected and semi-locally simply connected, then the following statements hold:

(1) The above evaluation map $p\colon \widetilde{X} \to X$ is continuous and open.

(2) The map $p\colon \widetilde{X} \to X$ is a covering.

(3) The topological space $\widetilde{X}$ is path-connected.

(4) We have $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$.

The following proposition gives us one of the key features of coverings:

Proposition 127.5. (Map Lifting Criterion) Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z, z_0) \to (X, x_0)$ be a continuous map. If Z is locally path-connected, then we have the following equivalence of statements:

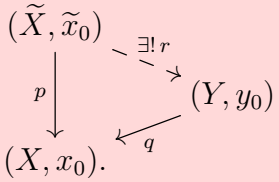
$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{x}_0) \\ & \overset{\text{there exists a lift } \widetilde{f}}{\nearrow} & \downarrow p \\ (Z, z_0) & \xrightarrow[f]{} & (X, x_0) \end{array} \qquad \iff \qquad \operatorname{im}(f_*) \subseteq \operatorname{im}(p_*).$$

[3084] As we explain in the Whisker Topology Lemma 127.12, we could also equip $\widetilde{X}_{x_0,\Gamma} := P_{x_0}/\sim_\Gamma$ with the "whisker topology".

Sketch of proof. The "⇒"-direction follows easily from the functoriality of fundamental groups. We turn to the proof of the "⇐"-direction: We assume that $\operatorname{im}(f_*) \subseteq \operatorname{im}(p_*)$. Let $z \in Z$. We pick a path $\gamma\colon [0,1] \to Z$ with $\gamma(0) = z_0$ and $\gamma(1) = z$. We lift the path $\delta := f \circ \gamma\colon [0,1] \to X$ to a path $\widetilde{\delta}\colon [0,1] \to \widetilde{X}$ with $\widetilde{\delta}(0) = \widetilde{x}_0$. We set $\widetilde{f}(z) := \widetilde{\delta}(1)$. It follows from the hypothesis $\operatorname{im}(f_*) \subseteq \operatorname{im}(p_*)$ that the map is well-defined. ■

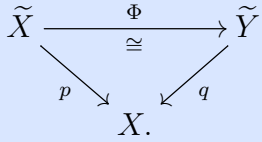
The following theorem is a relatively straightforward consequence of the Map Lifting Criterion 127.5:

Theorem 127.15. (Covering–Covering Theorem) Let X be a locally path-connected topological space, let $x_0 \in X$ and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path-connected covering of pointed topological spaces. For every other path-connected covering $q\colon (Y, y_0) \to (X, x_0)$ which satisfies the inclusion $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq q_*(\pi_1(Y, y_0))$ there exists a *unique* covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ of pointed topological spaces such that the following diagram commutes:



Our next goal is to discuss to what degree coverings are unique. To make sense of this let us recall the following definition:

Definition 127.16. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to X$ be two coverings of a topological space X. We say that p and q are isomorphic, if there exists a homeomorphism $\Phi\colon \widetilde{X} \to \widetilde{Y}$ such that the following diagram commutes:



Similarly we define an isomorphism of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

are isomorphic. In fact the desired map $\Phi\colon \mathbb{R}/n\mathbb{Z} \to S^1$ is given by $[t] \mapsto \exp(2\pi\,\mathrm{i}\,\frac{t}{n})$. □

Now we can formulate the promised uniqueness theorem:

Theorem 127.17. (Covering Unique Existence Theorem) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$ and let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup. There exists, up to isomorphism, a *unique* path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of pointed topological spaces such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$.

Proof. The existence statement of the theorem is the content of the Covering Existence Theorem 127.10. The uniqueness statement of the theorem follows almost immediately from the Covering–Covering Theorem 127.15. ■

This theorem leads us to the following definition:

Definition 127.18. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $x_0 \in X$ and let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup. We refer to any path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of pointed topological spaces with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$ as a covering of X corresponding to Γ. By the Covering Unique Existence Theorem 127.17 any two such coverings are isomorphic, in particular any such covering is equivalent to the natural covering from the CO-Topology Lemma 127.14. In practice usually we will not distinguish between coverings corresponding to the same subgroup.

At times we will be interested in the following special case of Definition 127.18:

Definition 127.21 Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $x_0 \in X$. We refer to any covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering of (X, x_0).

Example.

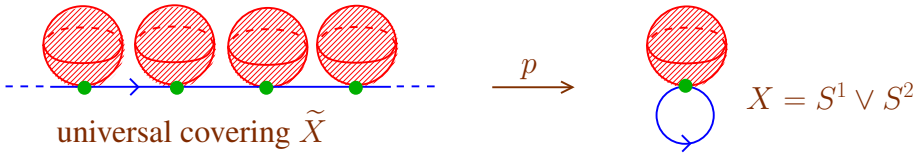
(1) Let $n \in \mathbb{N}_0$. Since $\mathbb{R}^n$ is simply connected we see that the covering

$$\begin{array}{rcl} \mathbb{R}^n & \to & (S^1)^n \\ (t_1, \ldots, t_n) & \mapsto & (\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)) \end{array}$$

from page 5462 is in fact a universal covering of the n-dimensional torus.

(2) We consider $X = S^1 \vee S^2$ where we identify $\mathrm{i} \in S^1$ with $(0, 0, -1) \in S^2$. We also consider the topological space

$$\widetilde{X} = \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) / \sim \qquad \text{where } n \sim (n, (0, 0, -1)) \text{ for all } n \in \mathbb{Z}.$$



It is pretty straightforward to see that the map

$$\begin{array}{rcl} p\colon \widetilde{X} & \to & S^1 \vee S^2 \\ [P] & \mapsto & \begin{cases} [\exp(\mathrm{i}(2\pi P + \frac{\pi}{2}))], & \text{if } P \in \mathbb{R} \\ [Q], & \text{if } P = (n, Q) \in \mathbb{Z} \times S^2 \end{cases} \end{array}$$

is well-defined and in fact a covering. On page 1945 we showed that $\widetilde{X}$ is simply connected. Therefore $p\colon \widetilde{X} \to X$ is a universal covering of $X = S^1 \vee S^2$.

(3) We consider the wedge $X = S^1 \vee S^1$. Let $\widetilde{X}$ be the infinite graph that is shown below. In the Wedge-of-Circles–Universal Covering Proposition 127.24 we showed that the above map $\widetilde{X} \to X$ is a universal covering of $X = S^1 \vee S^1$. □

286.2. **The deck transformation group.** We start out by recalling some aspects from Definition 128.1:

Definition 286.1. Let $p\colon \widetilde{X} \to X$ be a covering. A deck transformation is a homeomorphism $d\colon \widetilde{X} \to \widetilde{X}$ such that $p(d(x)) = p(x)$ for all $x \in \widetilde{X}$, i.e. such that the following

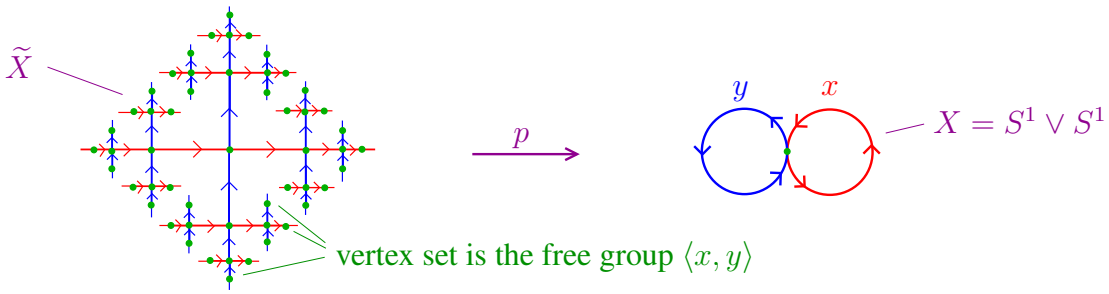
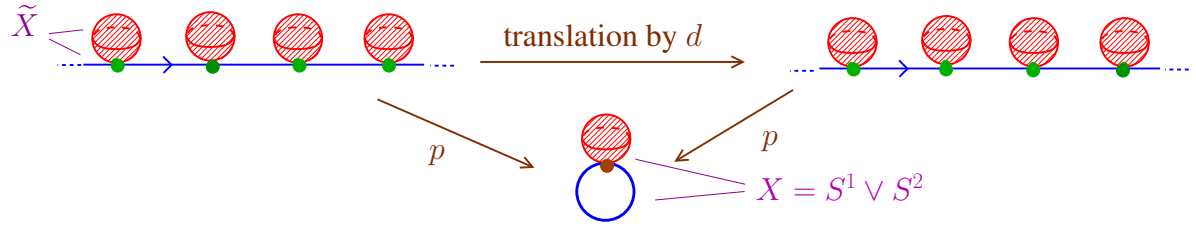


diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow[\cong]{d} & \widetilde{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X. & \end{array}$$

The group of deck transformations is called the deck transformation group of the covering $p\colon \widetilde{X} \to X$ and we denote it by $\mathrm{Deck}(p\colon \widetilde{X} \to X)$.

Example. In the figure below we show again this chapter's most popular covering:



One can easily verify that for any $d \in \mathbb{Z}$ "translation by d" is a deck transformation and that these are all deck transformations of the covering $p\colon \widetilde{X} \to S^1 \vee S^2$. □

The next proposition gives us a convenient tool for determining the deck transformation group:

Proposition 286.2. (Deck Transformation–Normal Proposition) Let $\widetilde{X}$ and X be two topological spaces that are both path-connected and locally path-connected. Furthermore let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering of pointed topological spaces. In the following we write $\pi := \pi_1(X, x_0)$ and $\Gamma := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. We assume that Γ is a normal subgroup of π. The map

$$\begin{array}{rccl} \Xi_{(\widetilde{X},\widetilde{x}_0)}\colon \mathrm{Deck}(p\colon \widetilde{X} \to X) & \to & \pi/\Gamma \\ d & \mapsto & \big[\underbrace{p \circ (\text{path in } \widetilde{X} \text{ from } \widetilde{x}_0 \text{ to } d(\widetilde{x}_0))}_{\substack{\text{this is a loop in } (X, x_0), \text{ thus this} \\ \text{defines an element of } \pi = \pi_1(X, x_0)}}\big] \end{array}$$

is a natural isomorphism of groups where the inverse is given by

$$\begin{array}{rcl} \pi/\Gamma & \to & \mathrm{Deck}(p\colon \widetilde{X} \to X) \\ [f\colon [0,1] \to X] & \mapsto & \text{we construct a deck transformation as follows:} \\ & & \text{let } \widetilde{f}\colon [0,1] \to \widetilde{X} \text{ be the unique lift of } f \text{ with } \widetilde{f}(0) = \widetilde{x}_0, \\ & & \text{by the Covering–Covering Theorem 127.15 there exists} \\ & & \text{a unique lift of } p\colon \widetilde{X} \to X \text{ to } \widetilde{p}\colon \widetilde{X} \to \widetilde{X} \text{ with } \widetilde{p}(\widetilde{x}_0) = \widetilde{f}(1), \\ & & \text{this map } \widetilde{p} \text{ is the desired deck transformation.} \end{array}$$

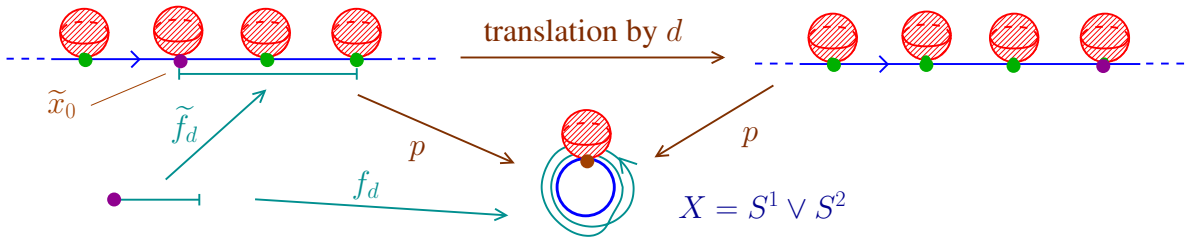
Proof. This proposition is just a special case of the Deck Transformation–Normalizer Proposition 128.9. ■

Before we consider examples for the Deck Transformation–Normal Proposition 286.2 we introduce the following convention:

Convention 286.3. Let (X, x_0) be a pointed topological space that is path-connected and locally path-connected. Furthermore let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a group G. Finally let $p\colon (\widetilde{X}, x_0) \to (X, x_0)$ be a covering of pointed topological spaces such that $p_*(\pi_1(\widetilde{X}_0, x_0)) = \ker(\varphi)$.[3085] Then $\varphi\colon \pi_1(X, x_0) \to G$ descends to an isomorphism $\pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}_0, x_0)) \to G$ which, together with the Deck Transformation–Normal Proposition 286.2 defines an action of G on $\widetilde{X}$. We will always equip $\widetilde{X}$ with this G-action.

Example.

(1) We consider again $X = S^1 \vee S^2$ with the base point given by the wedge point x_0. Let $\varphi\colon \pi_1(S^1 \vee S^1, x_0) \to \mathbb{Z}$ be the "obvious" isomorphism. Note that the covering corresponding to $\ker(\varphi)$ is given by the ever popular covering $p\colon \widetilde{X} \to S^1 \vee S^2$ that is shown below. We equip $\widetilde{X}$ with the base point $\widetilde{x}_0 \in \widetilde{X}$ that is given by $[0]$.



Next we want to understand the action of $\mathbb{Z}$ on $\widetilde{X}$ that we introduced in the Deck Transformation–Normal Proposition 286.2. Thus let $d \in \mathbb{Z}$. We consider the loop

$$\begin{array}{rcl} f_d\colon [0,1] & \to & S^1 \vee S^2 \\ t & \mapsto & [\exp(2\pi\,\mathrm{i}d \cdot t + \frac{\pi}{2})] \end{array}$$

which corresponds to d under $\varphi\colon \pi_1(S^1 \vee S^2, x_0) \to \mathbb{Z}$. The lift of f_d to the starting point $[0] \in \widetilde{X}$ is given by the map

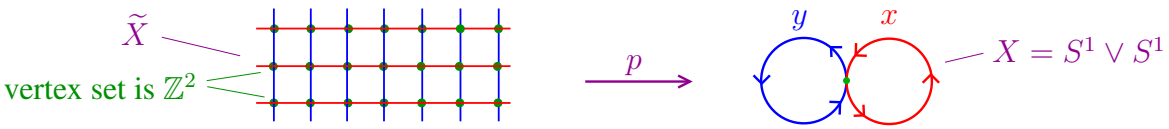
$$\begin{array}{rcl} \widetilde{f}\colon [0,1] & \to & \widetilde{X} \\ t & \mapsto & [d \cdot t]. \end{array}$$

Its endpoint is $\widetilde{f}(1) = [d]$. Thus, under the isomorphism from the Deck Transformation–Normal Proposition 286.2, we associate to d the deck transformation that turns $[0]$ into

[3085] Recall that in the Covering Existence Theorem 127.10 we showed that such a covering exists under rather mild technical hypothesis on $\widetilde{X}$.

$[d]$, i.e. we consider the deck transformation that is given by "translation by d to the right".

(2) We consider again the wedge $S^1 \vee S^1$, but this time we consider the "obvious" epimorphism $\varphi\colon \pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle \to \mathbb{Z}^2$ which corresponds to the abelianization.[3086]

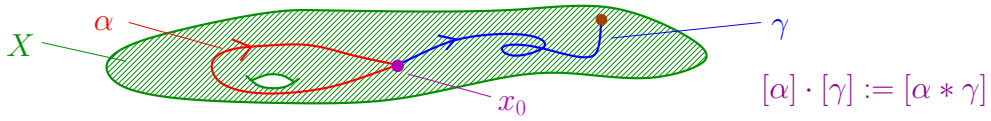


In the figure above we consider the "grid" $\widetilde{X} := \{(s,t) \in \mathbb{R}^2 \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ together with the covering map

$$\begin{array}{rcl} \widetilde{X} & \to & S^1 \vee S^1 \overset{\text{see Definition 96.1 and Notation 1.12}}{=} ((S^1 \times \{a\}) \cup (S^1 \times \{b\}))/(1,a) \sim (1,b) \\ (s,t) & \mapsto & \begin{cases} [(\exp(2\pi\,\mathrm{i}\,s), a)], & \text{if } t \in \mathbb{Z}, \\ [(\exp(2\pi\,\mathrm{i}\,t), b)], & \text{if } s \in \mathbb{Z}. \end{cases} \end{array}$$

In Exercise 286.1 we will show that $p_*(\widetilde{X}, \widetilde{x}_0) = \ker(\varphi)$. As in (1) one can easily verify that the action by $\mathbb{Z}^2$ on $\widetilde{X}$ is given by translation.

(3) Let (X, x_0) be a pointed path-connected topological space and let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism. As before we assume that X is locally path-connected and semi-locally simply connected. We denote by $p\colon \widetilde{X}_{x_0,\ker(\varphi)} = P_{x_0}/\sim_{\ker(\varphi)} \to X$ the covering of X corresponding to $\ker(\varphi)$ from Definition 127.11.



It follows easily from the definitions that the map from Convention 286.3 is given by

$$\begin{array}{rcl} G \times \widetilde{X}_{x_0,\ker(\varphi)} & \to & \widetilde{X}_{x_0,\ker(\varphi)} \\ (g, [\gamma]) & \mapsto & [\alpha * \gamma]. \end{array}$$

where $\varphi([\alpha]) = g$

□

286.3. **The natural covering of the mapping torus.** In this section we consider mapping tori, their coverings and the corresponding deck transformations. We start out by recalling the following concepts from Definition 21.19.

Definition 286.4. Let X be a path-connected topological space and let $f\colon X \to X$ be a continuous map.

(1) We call

$$\mathrm{Tor}(X, f) := (X \times [0,1]) / \sim \quad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

the mapping torus of (X, f).

(2) We refer to the map

$$\begin{array}{rcl} p\colon \mathrm{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}\,t) \end{array}$$

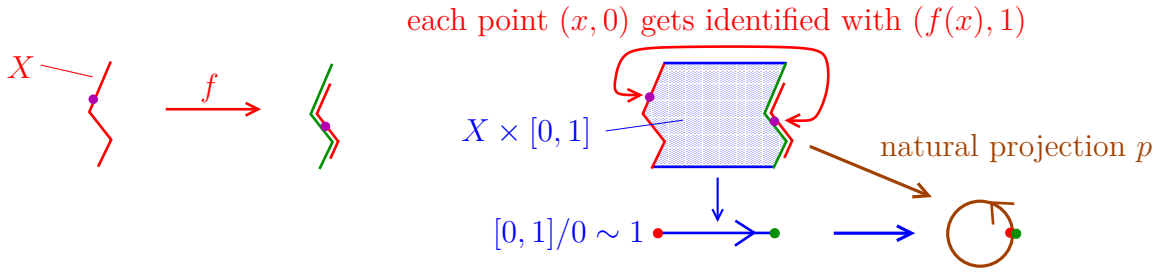
as the natural projection onto S^1.

[3086] Note that this discussion gives a solution to Exercise 127.14.

(3) For any $x_0 \in X$ we refer to the map

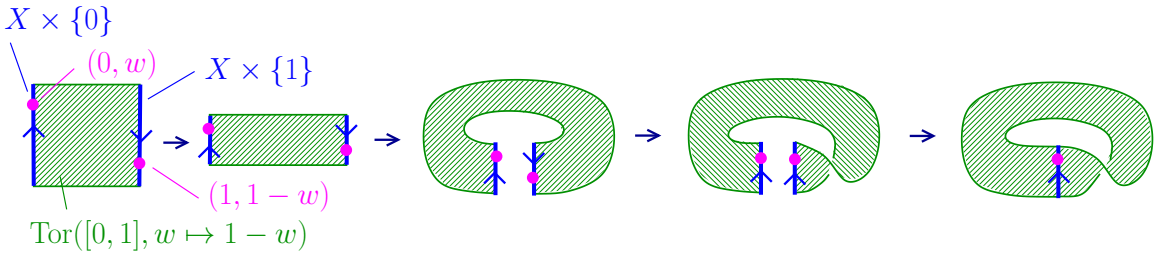
$$\begin{array}{ccccc} \pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) & \xrightarrow{p_*} & \pi_1(S^1,1) & \xrightarrow[\cong]{\text{isomorphism by the Circle-}\pi_1\text{-Theorem 88.22}} & \mathbb{Z} \\ & & [s \mapsto e^{2\pi i s}] & \mapsto & 1 \end{array}$$

as the natural epimorphism to $\mathbb{Z}$.[3087]



Example.

(1) We consider the topological space $[0,1]$ and the self-homeomorphism $f\colon [0,1] \to [0,1]$ that is given by $f(w) = 1 - w$. It follows immediately from the definitions that the corresponding mapping torus $\mathrm{Tor}([0,1],f)$ is precisely the Möbius band as defined on page 310.



(2) Let $X = S^1$ and let $f\colon S^1 \to S^1$ be the homeomorphism that is given by $f(z) = \overline{z}$. On page 575 we pointed out that the mapping torus $\mathrm{Tor}(S^1,f)$ is homeomorphic to the Klein bottle from page 312.

(3) Let X be a topological space. In the Mapping Torus-of-id Lemma 21.22 we showed that the map

$$\begin{array}{rccc} \Phi\colon \mathrm{Tor}(X,\mathrm{id}_X) & \to & X \times S^1 \\ [(x,t)] & \mapsto & (x, \exp(2\pi i t)) \end{array}$$

is a natural homeomorphism. □

Now we can provide our next example of coverings and deck transformations:

Proposition 286.5. (Mapping Torus-Covering Proposition) Let X be a path-connected topological space, let $f\colon X \to X$ be a *homeomorphism* and let $x_0 \in X$. We assume that X is locally path-connected and semi-locally simply connected. We consider the natural epimorphism $\Phi\colon \pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) \to \mathbb{Z}$ from Definition 286.4. There exists a covering

$$p\colon X \times \mathbb{R} \to \mathrm{Tor}(X,f)$$

[3087] Since X is path-connected we obtain from an elementary argument, see Exercise 88.15, that this map is indeed an epimorphism.

corresponding to $\ker(\Phi)$ such that the $\mathbb{Z}$-action from Convention 286.3 is given by the action

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x, s)) & \mapsto & (f^n(x), s + n). \end{array}$$

The proof of the Mapping Torus-Covering Proposition 286.5 relies on the following lemma:

Lemma 93.8. (Mapping Torus-of-Homeomorphism Lemma) Let X be a topological space and let $f\colon X \to X$ be a *homeomorphism*. Then the following hold:

(1) The map

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x, t)) & \mapsto & (f^n(x), t + n) \end{array}$$

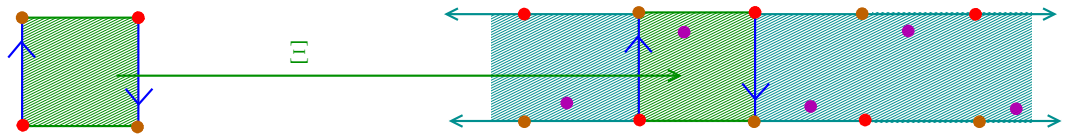
defines a continuous, proper and discrete action of the group $\mathbb{Z}$ on $X \times \mathbb{R}$.

(2) The map

$$\begin{array}{rcl} \mathrm{Tor}(X, f) = (X \times [0, 1])/\sim & \to & (X \times \mathbb{R})/\mathbb{Z} \\ [(x, t)] & \mapsto & [(x, t)] \end{array}$$

is a homeomorphism.

Example. We consider the case that $X = [-1, 1]$ and $f(x) = -x$. The corresponding $\mathbb{Z}$-action on $[-1, 1] \times \mathbb{R}$ is given by $n \cdot (x, t) := ((-1)^n \cdot x, t + n)$.



$\mathrm{Tor}([-1, 1], x \mapsto 1 - x)$ $\qquad$ $n \cdot (x, t) = ((-1)^n \cdot x, t + n)$

In the figure we attempt to illustrate the mapping torus $\mathrm{Tor}([-1, 1], x \mapsto -x)$ and we illustrate $[-1, 1] \times \mathbb{R}$ with the $\mathbb{Z}$-action. □

Proof of the Mapping Torus-Covering Proposition 286.5. Let X be a path-connected topological space, let $f\colon X \to X$ be a homeomorphism and let $x_0 \in X$. We assume that X is locally path-connected and semi-locally simply connected. We consider the natural epimorphism $\Phi\colon \pi_1(\mathrm{Tor}(X, f), [(x_0, 0)]) \to \mathbb{Z}$ from Definition 286.4.

Next we consider the following diagram:

$$\begin{array}{ccc} & & X \times \mathbb{R} \\ & & {\scriptstyle =:q}\big\downarrow{\scriptstyle (x,s)\mapsto[(x,s)]} \\ \mathrm{Tor}(X, f) = (X \times [0, 1])/\sim & \xrightarrow[\cong]{\Xi} & (X \times \mathbb{R})/\mathbb{Z}. \end{array}$$

We make the following clarifications and observations:

(a) The horizontal map is the homeomorphism from the Mapping Torus-of-Homeomorphism Lemma 93.8 (2).

(b) It follows from the Mapping Torus-of-Homeomorphism Lemma 93.8 (1) and the Action–Covering Proposition 91.11 that the vertical map is a covering map.

(c) It follows from (a) and (b) that $p := \Xi^{-1} \circ q\colon X \times \mathbb{R} \to \mathrm{Tor}(X, f)$ is a covering map.

(d) Using the Mapping Torus-π_1-Proposition 93.13 one can easily verify that we have an equality $p_*(\pi_1(X \times \mathbb{R}, (x_0, 0))) = \ker(\Phi)$.

It remains to prove the following claim:

Claim. The $\mathbb{Z}$-action from Convention 286.3 is given by the action

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x, s)) & \mapsto & (f^n(x), s + n). \end{array}$$

Proof. Let $n \in \mathbb{Z}$. We first introduce a few objects:

- Since X is path-connected we can pick a path $\alpha\colon [0,1] \to X$ from the starting point $\alpha(0) = x_0$ to the endpoint $\alpha(1) = f(x_0)$.
- We consider the path $\beta\colon [0,1] \to X \times [0,1]$ that is given by $\beta(t) = (\alpha(t), t)$.
- We consider the path $\gamma := \beta * \cdots * \beta$ that is given by the concatenation of n copies of β.
- It follows easily from Definition 286.4 that $\Phi([\gamma]) = n$.
- For $k \in \mathbb{Z}$ let $\mu_k\colon X \times [0,1] \to X \times \mathbb{R}$ be the map that is given by $\mu_k(x,t) = (x, k + t)$.
- Note that $\widetilde{\gamma} := \beta * (\mu_1 \circ \operatorname{diag}(\mathrm{id}, f)) \circ \beta * \cdots * (\mu_{n-1} \circ \operatorname{diag}(\mathrm{id}, f^{n-1})) \circ \beta$ is a lift of γ with $\widetilde{\gamma}(0) = (x_0, 0)$
- Finally note that $\widetilde{\gamma}(1) = (f^n(x_0), n)$.

Since the map from the claim is a deck transformation which sends $(x_0, 0)$ to $(f^n(x_0), n)$ we have proved the desired statement. The case $n \in \mathbb{Z}_{<0}$ is proved in essentially the same way. ∎

286.4. **Interlude: Group rings.** In this section we will recall the definition of the group ring and we will recall some basic statements about group rings. We start out by recalling Definition 97.30:

Definition 286.6. Given a group G we set

$$\mathbb{Z}[G] \;:=\; \left\{ \sum_{i=1}^{m} r_i \cdot g_i \;\middle|\; r_1, \ldots, r_m \in \mathbb{Z} \text{ and } g_1, \ldots, g_m \in G \right\}.$$

We equip $\mathbb{Z}[G]$ with the multiplication that is given by

$$\left(\sum_{i=1}^{m} r_i \cdot g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \;:=\; \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_{\mathbb{Z}} s_j}_{\in \mathbb{Z}} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

By the Group Ring Lemma 97.31 we know that this multiplication turns $\mathbb{Z}[G]$ into an associative ring. We refer to $\mathbb{Z}[G]$ as the group ring of G.

Example.

(1) We consider the infinite cyclic group $\langle t \rangle := \{\ldots, t^{-2}, t^{-1}, 1, t, t^2, \ldots\}$. In this case the group ring $\mathbb{Z}[\langle t \rangle]$ is isomorphic to the ring $\mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm 1}] & \to & \mathbb{Z}[\langle t \rangle] \\ \sum_{i=r}^{i=s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{i=s} a_i \cdot t^i \end{array}$$

is clearly a ring isomorphism. We will use this isomorphism to identify these two rings.

(2) Let $m \in \mathbb{N}$. We consider the free group $\langle t_1, \ldots, t_m \rangle$ and the abelianization $\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$. We can think of $\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$ as the group of all monomials $t_1^{r_1} \cdot \ldots \cdot t_m^{r_m}$ with $r_1, \ldots, r_m$ with the obvious multiplication. In other words, $\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$ is a multiplicative version of the free abelian group $\mathbb{Z}^m$. In this case we see, as in (1), that the map

$$\begin{array}{rcl} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] & \to & \mathbb{Z}[\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}] \\ \sum_{i=r}^{i=s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{i=s} a_i \cdot t^i \end{array}$$

is a ring isomorphism. We will use this isomorphism to identify these two rings. □

We recall the following, essentially trivial lemma:

Lemma 97.33. (Group Ring Functor Lemma) The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{Group}) & \to & \mathrm{Ob}(\mathcal{Ring}) \\ G & \mapsto & \mathbb{Z}[G] \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Group}}(G,H) & \to & \mathrm{Mor}_{\mathcal{Ring}}(\mathbb{Z}[G],\mathbb{Z}[H]) \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \varphi_*\colon \mathbb{Z}[G] & \to & \mathbb{Z}[H] \\ \sum_{i=1}^m r_i \cdot g_i & \mapsto & \sum_{i=1}^m r_i \cdot \varphi(g_i) \end{array} \right) \end{array}$$

define a covariant functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{Ring}$ of rings.

We conclude this short section with the following elementary lemma:

Lemma 97.34. (Group Ring–Module Lemma) Let G be a group and let A be an abelian group (with additive notation for the group structure). Suppose we are given a G-action on the group A.[3088] Then the map

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^m r_i \cdot g_i, a \right) & \mapsto & \sum_{i=1}^m r_i \cdot (\underbrace{g_i \cdot a}_{\in A}) \end{array}$$

defines a natural $\mathbb{Z}[G]$-left module[3089] structure on A.

286.5. **Group rings and modules arising from coverings.** In this section we will see that homology groups of coverings should really be viewed as modules over group rings:

Lemma 286.7. (Homology-of-Covering Lemma) Let (X, x_0) be a pointed connected topological space and let $\varphi\colon \pi_1(X,x_0) \to G$ be an epimorphism onto a group G. As before we assume that X is path-connected, locally path-connected and semi-locally simply connected.

(1) Let $p\colon \widetilde{X} \to X$ be any covering corresponding to $\ker(\varphi)$. Let $k \in \mathbb{N}_0$. The action from Convention 286.3 defines a map

$$\begin{array}{rcl} \mathbb{Z}[G] \times \mathrm{H}_k(\widetilde{X}) & \to & \mathrm{H}_k(\widetilde{X}) \\ (g, \sigma) & \mapsto & \underbrace{(g \cdot -)_*(\sigma)}_{\uparrow} \end{array}$$

here $(g \cdot -)$ is the map $\widetilde{X} \to \widetilde{X}$ given by the g-action, and $(g \cdot -)_*\colon \mathrm{H}_k(\widetilde{X}) \to \mathrm{H}_k(\widetilde{X})$ is the induced homomorphism

which turns $\mathrm{H}_k(\widetilde{X})$ into a $\mathbb{Z}[G]$-left module.

(2) The isomorphism type of the $\mathbb{Z}[G]$-left module from (1) does not depend on the choice of the covering corresponding to $\ker(\varphi)$.

[3088] A G-action on an abelian group A is by Convention 2.40 a homomorphism $G \to \mathrm{Aut}(A)$.

[3089] Since $\mathbb{Z}[G]$ is in general not a commutative ring we need to distinguish between $\mathbb{Z}[G]$-left modules and $\mathbb{Z}[G]$-right modules. In this chapter we will mostly deal with the case that G is abelian, where this distinction becomes immaterial.

Proof.

(1) By Convention 286.3 we have an action of G on $\widetilde{X}$. By functoriality of homology groups this gives us an action of G on the abelian group $\mathrm{H}_k(\widetilde{X})$. It now follows from the Group Ring–Module Lemma 97.34 that the given map defines a $\mathbb{Z}[G]$-left module structure on the homology group $\mathrm{H}_k(\widetilde{X})$.

(2) In the following let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ be two coverings such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \ker(\varphi) = q_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$. It follows from the Covering Unique Existence Theorem 127.17 that there exists a homeomorphism $\Theta\colon (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{Y}, \widetilde{y}_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow[\cong]{\Theta} & \widetilde{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X. & \end{array}$$

Next note that it follows easily from the explicit description of the isomorphism of the Deck Transformation–Normal Proposition 286.2 that for any $g \in \pi_1(X, x_0)/\ker(\varphi)$ and any $\widetilde{x} \in \widetilde{X}$ we have $\Theta(g \cdot x) = g \cdot \Theta(x)$. It thus follows that the induced map

$$\Theta_*\colon \mathrm{H}_k(\widetilde{X}) \ \to \ \mathrm{H}_k(\widetilde{Y})$$

is an isomorphism of $\mathbb{Z}[G]$-modules. ■

The construction of the module in the Homology-of-Covering Lemma 286.7 depends a priori on the choice of a base point. The dependence on a choice of a base point is a constant nuisance. But as we will see now, if G is abelian, then we can get around this issue:

Lemma 286.8. (Abelian Covering Lemma) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto an abelian group H. Given $x_0 \in X$ we denote by

$$\begin{array}{rcl} \Xi_{x_0}\colon \pi_1(X, x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon (S^1, 1) \to (X, x_0)] & \mapsto & \gamma_*([S^1]) \end{array}$$

the homomorphism from the Hurewicz Theorem 166.10. Given x_0 let $\widetilde{X}_{\Phi\circ\Xi_{x_0}}$ be any covering of X corresponding to the kernel of the homomorphism $\Phi \circ \Xi_{x_0}\colon \pi_1(X, x_0) \to H$. The isomorphism type of the $\mathbb{Z}[H]$-module $\mathrm{H}_k(\widetilde{X}_{\Phi\circ\Xi_{x_0}})$ only depends on the choice of Φ.

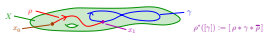
Proof. First note that it follows from the Homology-of-Covering Lemma 286.7 that it remains to show that the isomorphism type of the $\mathbb{Z}[H]$-module $\mathrm{H}_k(\widetilde{X}_{\Phi\circ\Xi_{x_0}})$ does not depend on the choice of x_0. Thus let $x_0, x_1 \in X$ be two points. We make a few preparations:

- Let $p\colon (\widetilde{X}, \widetilde{x}_1) \to (X, x_1)$ be a covering of X corresponding to the kernel of the homomorphism $\Phi\circ\Xi_{x_1}\colon \pi_1(X, x_1) \to H$, i.e. we have $p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)) = \ker(\Phi\circ\Xi_{x_1}\colon \pi_1(X, x_1) \to H)$.
- Since X is path-connected we can find a path $\rho\colon [0, 1] \to X$ from the point $x_0 = \rho(0)$ to the point $x_1 = \rho(1)$.
- Let $\widetilde{\rho}\colon [0, 1] \to \widetilde{X}$ be a lift of ρ with $\rho(x_1) = \widetilde{x}_1$. We set $\widetilde{x}_0 := \widetilde{\rho}(1)$.
- As in the Change-of-Base Point Proposition 88.14 we consider the isomorphism

$$\begin{array}{rcl} \rho^*\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\gamma] & \mapsto & [\,\rho * \gamma * \overline{\rho}\,]. \end{array}$$

Similarly we get an induced homomorphism

$$\widetilde{\rho}^*\colon \pi_1(\widetilde{X},\widetilde{x}_1) \;\to\; \pi_1(\widetilde{X},\widetilde{x}_0).$$



- It follows easily from the definitions and the Hurewicz Homomorphism Proposition 166.3 (3) that the following diagram commutes:

$$\begin{array}{ccccc}
\pi_1(\widetilde{X},\widetilde{x}_1) & & \xrightarrow{\quad\widetilde{\rho}^*\quad} & & \pi_1(\widetilde{X},\widetilde{x}_0) \\
\downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle p_*} \\
\pi_1(X,x_1) & & \xrightarrow{\quad\rho^*\quad} & & \pi_1(X,x_0) \\
& {\scriptstyle \Xi_{x_1}}\searrow & & \swarrow{\scriptstyle \Xi_{x_0}} & \\
& & \mathrm{H}_1(X) & & \\
& & \downarrow{\scriptstyle \Phi} & & \\
& & H. & &
\end{array}$$

It is now clear that $p\colon \widetilde{X} \to X$ is also a covering corresponding to the kernel of the homomorphism $\Phi \circ \Xi_{x_0}\colon \pi_1(X,x_0) \to H$. Finally one can easily show that the H-action on $\widetilde{X}$ from Convention 286.3, defined via x_0 agrees with the H-action defined via x_1. We leave it to the reader to fill in the details. ■

The Homology-of-Covering Lemma 286.7 together with the Abelian Covering Lemma 286.8 allow us to introduce the following definition:

Definition 286.9. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto an abelian group H. Let $x_0 \in X$ and let $\widetilde{X}_\Phi$ be any covering of X corresponding to the kernel of the homomorphism $\pi_1(X,x_0) \to \mathrm{H}_1(X) \xrightarrow{\Phi} H$. Given any $k \in \mathbb{N}_0$ we set[3090]

$$\mathrm{H}_k^\Phi(X;\mathbb{Z}[H]) \;:=\; \underbrace{\mathrm{H}_k(\widetilde{X}_\Phi)}_{\substack{\text{viewed as a}\\ \mathbb{Z}[H]\text{-module}}}.$$

It follows from the Homology-of-Covering Lemma 286.7 together with the Abelian Covering Lemma 286.8 that the isomorphism type of the $\mathbb{Z}[H]$-module $\mathrm{H}_k^\Phi(X;\mathbb{Z}[H])$ depends neither on the choice of x_0 nor on the choice of the covering. For want of a better name we refer to the $\mathbb{Z}[H]$-module $\mathrm{H}_k^\Phi(X;\mathbb{Z}[H])$ as the k-th Alexander module of (X,Φ).

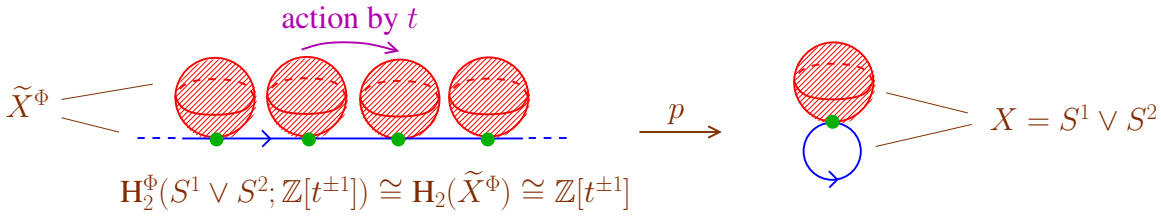
Example.

(1) We consider $X := S^1 \vee S^2$ and the isomorphism $\Phi\colon \mathrm{H}_1(S^1 \vee S^2) \to \langle t\rangle$ that is given by sending the standard generator of $\mathrm{H}_1(S^1)$ to t.[3091] We consider the usual covering

[3090] This notation indicates, that later in the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 we will see that $\mathrm{H}_k^\Phi(X;\mathbb{Z}[H])$ is naturally isomorphic to the "twisted homology module" $\mathrm{H}_k^{\mathrm{tw}}(X;\mathbb{Z}[H])$, where we view $\mathbb{Z}[H]$ as a $\mathbb{Z}[\pi_1(X)]$-module via Φ.

[3091] Since we now work with group rings $\mathbb{Z}[H]$ we now mostly think of H as a multiplicative group instead of an additive group.

$\widetilde{X} \to X = S^1 \vee S^2$ that is shown below. Note that this is the covering corresponding to $\ker(\Phi) = \{e\}$.



We now see that

$$\mathrm{H}_1^\Phi(S^1 \vee S^2; \mathbb{Z}[t^{\pm 1}]) \underset{\uparrow}{=} \mathrm{H}_1(\widetilde{X}) \underset{\uparrow}{\overset{\cong}{\leftarrow}} \bigoplus_{k \in \mathbb{Z}} \mathrm{H}_2(\{k\} \times S^2) = \bigoplus_{k \in \mathbb{Z}} \overbrace{\mathrm{H}_2(\{k\} \times S^2)}^{\cong \mathbb{Z}} \cong \mathbb{Z}[t^{\pm 1}].$$

see Definition 286.9 — follows from the Mayer–Vietoris Theorem 153.2 and the Path-Component–H_*–Lemma 143.21, here the $\langle t \rangle$-action is the obvious one

(2) Let X be a topological space that is non-empty, path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto an abelian group. We consider a corresponding covering $p\colon \widetilde{X} \to X$. We now see that

$$\mathrm{H}_0^\Phi(X; \mathbb{Z}[H]) \underset{\uparrow}{=} \mathrm{H}_0(\widetilde{X}) \underset{\uparrow}{\cong} \mathbb{Z}, \text{ where each } h \in H \text{ acts trivially.}$$

see Definition 286.9 — by the H_0-Proposition 143.25 and since X is path-connected and non-empty

We formulate our next example of an Alexander module as a lemma:

Lemma 286.10. (Mapping Torus-Alexander Module Lemma) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $f\colon X \to X$ be a self-homeomorphism. We consider the natural epimorphism $\Phi\colon \mathrm{H}_1(\mathrm{Tor}(X, f)) \to \langle t \rangle$ from Definition 286.4. For any $k \in \mathbb{N}_0$ we have the following isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules[3092]

$$\mathrm{H}_k^\Phi(\mathrm{Tor}(X, f), \mathbb{Z}[t^{\pm 1}]) \cong \mathrm{H}_k(X)$$

where the t-action on $\mathrm{H}_k(X)$ is given by $f_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(X)$.

Proof. We have the following two isomorphisms of $\mathbb{Z}[t^{\pm 1}]$-modules:

$$\mathrm{H}_k^\Phi(\mathrm{Tor}(X, f), \mathbb{Z}[t^{\pm 1}]) \overset{\text{(a)}}{:=} \underbrace{\mathrm{H}_k(X \times \mathbb{R})}_{\substack{\text{the } t\text{-action is} \\ \text{induced by the map} \\ (x, s) \mapsto (f(x), s+1)}} \overset{\text{(b)}}{\longrightarrow} \underbrace{\mathrm{H}_k(X)}_{\substack{\text{the } t\text{-action is} \\ \text{induced by the} \\ \text{map } f\colon X \to X}}$$

We explain the two isomorphisms in greater detail:

(a) This map comes directly from Definition 286.9 together with the Mapping Torus-Covering Proposition 286.5.

(b) We denote by $\pi\colon X \times \mathbb{R} \to X$ the natural projection. Clearly this is a homotopy equivalence. Thus it follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8

[3092] One can easily verify that $\mathrm{Tor}(X, f)$ is also path-connected, locally path-connected and semi-locally simply connected. Thus it makes sense to talk of the Alexander module $\mathrm{H}_k^\Phi(\mathrm{Tor}(X, f), \mathbb{Z}[t^{\pm 1}])$.

that π induces an isomorphism $\pi_*\colon \mathrm{H}_k(X \times \mathbb{R}) \to \mathrm{H}_k(X)$ on homology groups. It remains to show that it induces an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules, in other words, it remains to prove the following claim:

Claim. The isomorphism $\pi_*\colon \mathrm{H}_k(X \times \mathbb{R}) \to \mathrm{H}_k(X)$ commutes with the t-action on both sides.

Proof. The claim follows immediately from the functoriality of homology and the observation that the following commutative diagram commutes:

$$\begin{array}{ccc} X \times \mathbb{R} & \xrightarrow{(x,s) \mapsto (f(x),s+1)} & X \times \mathbb{R} \\ {\scriptstyle =:\pi}\downarrow{\scriptstyle (x,s) \mapsto x} & & {\scriptstyle =:\pi}\downarrow{\scriptstyle (x,s) \mapsto x} \\ X & \xrightarrow{x \mapsto f(x)} & X. \end{array}$$

■

286.6. **Properties of Alexander modules.** We start out with the following unsurprising result:

Lemma 286.11. (Alexander Module–Homotopy Lemma)[3093] Let X and Y be path-connected, locally path-connected and semi-locally simply connected topological spaces and let $\Psi\colon \mathrm{H}_1(Y) \to H$ be an epimorphism onto an abelian group H. Furthermore let $f\colon X \to Y$ be a continuous map. If f is a homotopy equivalence, then for each $k \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_k^{\Psi \circ f_*}(X; \mathbb{Z}[H]) \;\cong\; \mathrm{H}_k^{\Psi}(Y; \mathbb{Z}[H])$$

of $\mathbb{Z}[H]$-modules.

Sketch of proof. [3094] We make a few preparations:

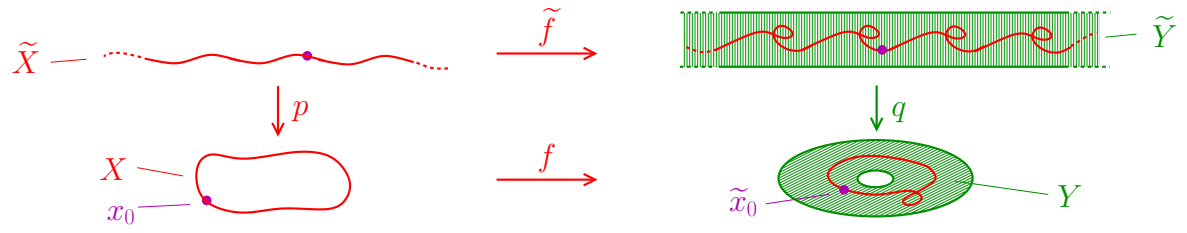
(a) We pick $x_0 \in X$ and we set $y_0 := f(x_0)$.
(b) Since f is a homotopy equivalence it follows from the Homotopy-π_1-Proposition 90.8 that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.
(c) We pick a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ corresponding to $\ker(\Psi \circ f_*)$.
(d) We pick a covering $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ corresponding to $\ker(\Psi)$.
(e) We consider the diagram

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{f}}{\dashrightarrow} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

It follows from (b), (c) and (d) that $\mathrm{im}(f \circ p)_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq q_*(\pi_1(\widetilde{Y}, \widetilde{y}_0))$.[3095] It thus follows the Map Lifting Criterion 127.5 that the map $f \circ p\colon \widetilde{X} \to Y$ lifts to a continuous map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ with $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$.

[3093] This lemma foreshadows the much more general Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.6 and the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1.
[3094] This proof also gives a solution for Exercise 127.30.
[3095] Indeed: We have

$$\mathrm{im}(f \circ p)_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \underset{\text{by (c)}}{=} \mathrm{im}(f_*(\ker(\Psi \circ f_*))) \underset{\text{by (b)}}{=} \ker(\Psi) \underset{\text{by (d)}}{=} q_*(\pi_1(\widetilde{Y}, \widetilde{y}_0)).$$



Claim. The map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is a homotopy equivalence.

Proof. Since $f\colon X \to Y$ is a homotopy equivalence we know by Definition 19.9 of a homotopy equivalence that there exists a homotopy inverse $g\colon Y \to X$ of f. Recall that this means that there exists a homotopy $F\colon X \times [0,1] \to X$ from $F_0 = g \circ f$ to $F_1 = \mathrm{id}_X$ and a homotopy $G\colon Y \times [0,1] \to Y$ from $G_0 = f \circ g$ to $G_1 = \mathrm{id}_Y$. It follows from the Map Lifting Criterion 127.5 that these homotopy lift to homotopies $F\colon X \times [0,1] \to X$ from $F_0 = g \circ f$ to $F_1 = \mathrm{id}_X$ and a homotopy $G\colon Y \times [0,1] \to Y$ from $G_0 = f \circ g$ to $G_1 = \mathrm{id}_Y$. ⊞

It follows from the Homotopy Equivalence-H_*-Proposition 144.14 that the induced map

$$\widetilde{f}_*\colon \mathrm{H}_k(\widetilde{X}) \;\to\; \mathrm{H}_k(\widetilde{Y})$$

is an isomorphism of abelian groups. Finally it is elementary to show that this isomorphism is equivariant with respect to the H-action from Convention 286.3. It thus follows that $\widetilde{f}_*\colon \mathrm{H}_k(\widetilde{X}) \to \mathrm{H}_k(\widetilde{Y})$ is an isomorphism of $\mathbb{Z}[H]$-modules. But that is exactly what we had to show. ■

In the Singular-Cellular-H_*-Proposition 155.8 we learned that one can calculate singular homology using cellular chain complexes. The following proposition gives us a corresponding statement for Alexander modules:

Proposition 286.12. (Alexander Module–CW Proposition) Let X be a connected CW-complex and let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto an abelian group H. Furthermore let $p\colon \widetilde{X}_\Phi \to X$ be a covering corresponding to $\ker(\Phi)$. We make the following observations:

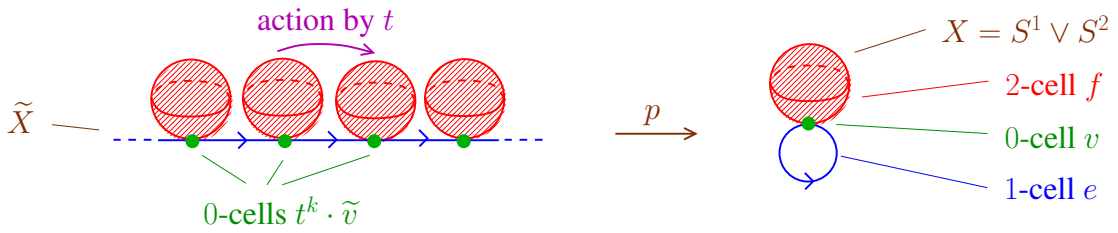
- We use the CW-Complex Covering Proposition 138.1 (1) to equip $\widetilde{X}_\Phi$ with the corresponding natural CW-structure.
- It is clear from the definition of the CW-structure that the H-action on $\widetilde{X}_\Phi$ preserves this CW-structure.
- It follows from the naturality of the boundary homomorphism and the Group Ring–Module Lemma 97.34 that we can view the cellular chain complex $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi)$ as a chain complex of $\mathbb{Z}[H]$-modules.

We have an isomorphism

$$\mathrm{H}_k^\Phi(X;\mathbb{Z}[H]) \;\cong\; \frac{\ker\left(\partial_k\colon \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X}_\Phi)\right)}{\mathrm{im}\left(\partial_{k+1}\colon \mathrm{C}_{k+1}^{\mathrm{CW}}(\widetilde{X}_\Phi) \to \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi)\right)}$$

of $\mathbb{Z}[H]$-modules.

Example. We consider once again $X = S^1 \vee S^2$, which we equip with the obvious CW-structure consisting of one 0-cell v, one 1-cell e and one 2-cell f. As always we consider the covering $p\colon \widetilde{X} \to S^1 \vee S^2$ corresponding to the usual isomorphism $\Phi\colon \pi_1(S^1 \vee S^2, v) \to \langle t \rangle$.



It follows from the CW-Complex Covering Proposition 138.1, that we can equip $\widetilde{X}$ with a CW-structure with 0-cells $t^k \cdot \widetilde{v}$, 1-cells $t^k \cdot \widetilde{e}$ and 2-cells $t^k \cdot \widetilde{f}$ where $k \in \mathbb{Z}$ and such that the cellular chain complex is given by

$$0 \to \underbrace{\mathrm{C}_2^{\mathrm{CW}}(\widetilde{X})}_{=\mathbb{Z}[t^{\pm 1}]\cdot\widetilde{f}} \xrightarrow[\partial_2]{\widetilde{f}\mapsto 0} \underbrace{\mathrm{C}_1^{\mathrm{CW}}(\widetilde{X})}_{=\mathbb{Z}[t^{\pm 1}]\cdot\widetilde{e}} \xrightarrow[\partial_1]{\widetilde{e}\mapsto (t-1)\cdot\widetilde{v}} \underbrace{\mathrm{C}_0^{\mathrm{CW}}(\widetilde{X})}_{=\mathbb{Z}[t^{\pm 1}]\cdot\widetilde{v}} \to 0.$$

Using the Alexander Module–CW Proposition 286.12 we now see that

$$\begin{aligned}
\mathrm{H}_2^{\Phi}(S^1 \vee S^2; \mathbb{Z}[t^{\pm 1}]) &= \frac{\ker(\partial_2)}{\mathrm{im}(\partial_3)} \cong \mathbb{Z}[t^{\pm 1}] \cdot \widetilde{f}, \\
\mathrm{H}_1^{\Phi}(S^1 \vee S^2; \mathbb{Z}[t^{\pm 1}]) &= \frac{\ker(\partial_1)}{\mathrm{im}(\partial_2)} \cong \frac{\ker(\cdot(t-1)\cdot\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}])}{0} = 0. \\
\mathrm{H}_0^{\Phi}(S^1 \vee S^2; \mathbb{Z}[t^{\pm 1}]) &= \frac{\ker(\partial_0)}{\mathrm{im}(\partial_1)} \cong \frac{\mathbb{Z}[t^{\pm 1}]\cdot\widetilde{v}}{(\mathbb{Z}[t^{\pm 1}]\cdot(t-1)\cdot\widetilde{v})} \cong \frac{\mathbb{Z}[t^{\pm 1}]}{(t-1)\mathbb{Z}[t^{\pm 1}]}.
\end{aligned}$$

□

Proof. Let X be a connected CW-complex, let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto an abelian group H and let $p\colon \widetilde{X}_\Phi \to X$ be a covering corresponding to $\ker(\Phi)$. Let $k \in \mathbb{N}_0$. By the Singular-Cellular-H_*-Proposition 155.8 we have a *natural* isomorphism

$$\Theta\colon \mathrm{H}_k(\widetilde{X}_\Phi) \to \mathrm{H}_k^{\mathrm{CW}}(\widetilde{X}_\Phi).$$

The "natural" part implies that this isomorphism commutes with the action of H on the singular and cellular homology groups. [3096] In other words, the above isomorphism is an isomorphism of $\mathbb{Z}[H]$-modules.

We have thus shown that we have isomorphisms

$$\mathrm{H}_k^{\Phi}(X; \mathbb{Z}[H]) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{CW}}(\widetilde{X}_\Phi) = \frac{\ker\left(\partial_k\colon \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X}_\Phi)\right)}{\mathrm{im}\left(\partial_{k+1}\colon \mathrm{C}_{k+1}^{\mathrm{CW}}(\widetilde{X}_\Phi) \to \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi)\right)}$$

of $\mathbb{Z}[H]$-modules. ■

286.7. Exercises for Chapter 286.

[3096] More precisely, the Singular-Cellular-H_*-Proposition 155.8 says that for any CW-complex Y we have an isomorphism

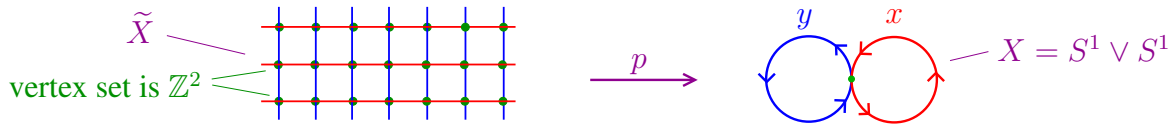
$$\Theta_Z\colon \mathrm{H}_k(Z) \to \mathrm{H}_k^{\mathrm{CW}}(Z)$$

such that for any cellular continuous map $f\colon Y \to Z$ between CW-complexes the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_k(Y) & \xrightarrow[\cong]{\Theta_Y} & \mathrm{H}_k^{\mathrm{CW}}(Y) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{H}_k(Z) & \xrightarrow[\cong]{\Theta_Z} & \mathrm{H}_k^{\mathrm{CW}}(Z).
\end{array}$$

We apply this fact to $Y = Z = \widetilde{X}_\Phi$ and to the maps that are given by the G-action.

Exercise 286.1. We consider again the wedge $S^1 \vee S^1$ and we consider the "obvious" epimorphism $\varphi\colon \pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle \to \mathbb{Z}^2$ which corresponds to the abelianization. Show that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \ker(\varphi)$.



Exercise 286.2. Let H be an abelian group (possibly not finitely generated). Show that $\mathbb{Z}[H]$ is a domain if and only if H is torsion-free.

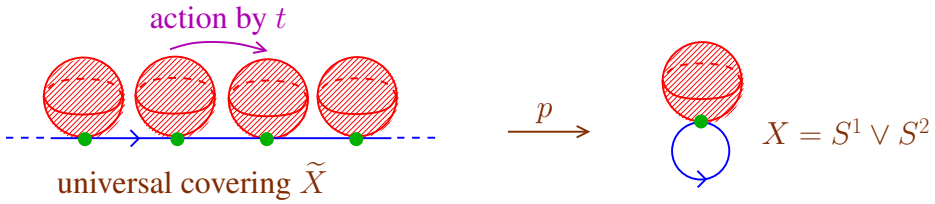
Exercise 286.3. Let $n \in \mathbb{N}_{\geq 2}$. We consider $X := \mathbb{R}\mathrm{P}^n \vee S^2$. Let $\Phi\colon \mathrm{H}_1(X) \to \mathbb{Z}/2\mathbb{Z}$ be the unique isomorphism. Describe the Alexander modules $\mathrm{H}_k^{\Phi}(\mathbb{R}\mathrm{P}^n \vee S^2; \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}])$ for $k \in \mathbb{N}_0$.

Exercise 286.4. Let K be the Klein bottle.

(a) Let $\Phi, \Psi\colon \mathrm{H}_1(K) \to \langle t\rangle \cong \mathbb{Z}$ be two epimorphisms. Are the Alexander modules $\mathrm{H}_1^{\Phi}(K; \mathbb{Z}[t^{\pm 1}])$ and $\mathrm{H}_1^{\Psi}(K; \mathbb{Z}[t^{\pm 1}])$ necessarily isomorphic?

(b) Let $\Phi, \Psi\colon \mathrm{H}_1(K) \to \langle t \,|\, t^2\rangle \cong \mathbb{Z}/2\mathbb{Z}$ be two epimorphisms. Are the Alexander modules $\mathrm{H}_1^{\Phi}(K; \mathbb{Z}[t]/(t^2))$ and $\mathrm{H}_1^{\Psi}(K; \mathbb{Z}[t]/(t^2))$ necessarily isomorphic?

Exercise 286.5. Once again we consider the covering $\widetilde{X} \to X = S^1 \vee S^2$ that is shown below. Instead of considering the homology group $\mathrm{H}_1(\widetilde{X})$ as a $\mathbb{Z}[t^{\pm 1}]$-module we could also consider the cohomology $\mathrm{H}^1(\widetilde{X}; \mathbb{Z})$ as a $\mathbb{Z}[t^{\pm 1}]$-module. What can you say about it? Is it a free $\mathbb{Z}[t^{\pm 1}]$-module? If yes, what is the rank?



Exercise 286.6. Give an example of a covering $p\colon \widetilde{X} \to X$ between path-connected topological spaces together with an isomorphism $\Phi\colon \mathrm{Deck}(p\colon \widetilde{X} \to X) \cong \langle s, t\rangle_{\mathrm{ab}}$ such that the following two conditions are satisfied:

- $\mathrm{H}_1(\widetilde{X})$ is a finitely generated abelian group.
- The actions of s and t on $\mathrm{H}_1(\widetilde{X})$ are both non-trivial.

Hint. You could consider "double mapping tori" corresponding to two suitable self-homeomorphisms of a topological space Y.

Exercise 286.7. We consider again the wedge $S^1 \vee S^1$ and we consider the "obvious" epimorphism $\varphi\colon \pi_1(S^1 \vee S^1, *) =_{\mathrm{i}} \langle x, y\rangle \to \langle x, y\rangle_{\mathrm{ab}}$ which corresponds to the abelianization.

(a) Use the Alexander Module–CW Proposition 286.12 to calculate the Alexander modules $\mathrm{H}_k(S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}])$.

(b) Given any module M over $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ we can define its rank as the dimension of the $\mathbb{Q}(t_1, \ldots, t_m)$-vector space[3097] $M \otimes_{\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]} \mathbb{Q}(t_1, \ldots, t_m)$- What is the rank of the modules in (a)?

[3097] We will introduce this more general tensor product in Section 312.2.

Exercise 286.8. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. We write $\pi := \pi_1(X, x_0)$. Furthermore let $\Phi\colon \pi \to H$ be an epimorphism onto an abelian group H. We write $\Gamma := \ker(\Phi)$.

(a) Show that $\Gamma/[\Gamma, \Gamma]$ is isomorphic to $\mathrm{H}_1^\Phi(X; \mathbb{Z}[H])$ as an abelian group.
(b) First show that H acts on $\Gamma/[\Gamma, \Gamma]$ by conjugation. Then show that $\Gamma/[\Gamma, \Gamma]$ is isomorphic to $\mathrm{H}_1^\Phi(X; \mathbb{Z}[H])$ as a $\mathbb{Z}[H]$-left module.

Remark. This shows that the first Alexander module $\mathrm{H}_1^\Phi(X; \mathbb{Z}[H])$ is completely determined by the fundamental group $\pi = \pi_1(X, x_0)$.

Exercise 286.9. We consider the torus $X = S^1 \times S^1$ together with the homomorphism $\Phi\colon \mathrm{H}_1(S^1 \times S^1) \to \langle t \rangle$ that is given by $\Phi([\{1\} \times S^1]) = \Phi([S^1 \times \{1\}]) = t$. Determine the isomorphism types of the Alexander modules $\mathrm{H}_k^\Phi(S^1 \times S^1; \mathbb{Z}[t^{\pm 1}])$.

CHAPTER 287

Alexander polynomials

Let X be a topological space that satisfies a few reasonable technical hypothesis and let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto an abelian group. In Chapter 286 we introduced the corresponding Alexander modules $\mathrm{H}_k^\Phi(X;\mathbb{Z}[H])$ which are modules over the group ring $\mathbb{Z}[H]$. In this chapter we will see that these $\mathbb{Z}[H]$-modules are often finitely generated. Furthermore, given a free abelian group H and a finitely generated $\mathbb{Z}[H]$-module M we will introduce the order $\mathrm{ord}(M) \in \mathbb{Z}[H]$, which is well-defined up to multiplication by a unit in $\mathbb{Z}[H]$. This will allow us to introduce the Alexander polynomials of a pair (X, Φ) as above.

This chapter is organized as follows:

- In Section 287.1 we will introduce the concept of a Noetherian ring.
- In Section 287.2 we will study presentation matrices of finitely generated modules.
- In Section 287.3 we will see that in many reasonable settings Alexander modules are finitely generated.
- In Section 287.4 we will introduce the concept of a unique factorization domain (better known under the name UFD).
- In Section 287.5 we will introduce elementary ideals of modules and of orders of modules over UFDs.
- In Section 287.6 we will study orders of Alexander modules, we will refer to those as Alexander polynomials.
- In Section 287.7 we will calculate Alexander polynomials of mapping tori.
- In Appendix 287.8 we will show that the order is multiplicative under short exact sequences.
- In Section 287.9 we will provide exercises for Chapter 287.

In the next Chapter 288 we will use the concepts from this chapter to introduce the Alexander modules and Alexander polynomials of oriented knots and links.

287.1. **Interlude: Noetherian rings.** In Chapter 286 we introduced the Alexander modules $\mathrm{H}_k^\Phi(X;\mathbb{Z}[H])$ which are modules over the group ring of a finitely generated abelian group H. To get a better grip on these modules we need to look up various aspects from commutative algebra.

We start out with the following definition, which is consistent with Definition 97.19:

Definition 287.1. Let R be a commutative ring and let M be an R-module. We say M is finitely generated if there exists a subset $X \subseteq M$ such that the natural homomorphism $R^{(X)} \to M$ of R-modules is an epimorphism.

The following definition might be familiar from a course in commutative algebra:

Definition 287.2. Let R be a commutative ring. If every R-submodule of R is finitely generated, then we call R Noetherian.

Remark. The polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ on infinitely many variables is not Noetherian. For example the ideal generated by $\{x_1, x_2^2, x_3^3, \dots\}$ is not finitely generated as a $\mathbb{Z}[x_1, x_2, \dots]$-module. □

The following proposition gives us many examples of Noetherian rings:

Proposition 287.3. (Noetherian Rings Proposition)

(1) (a) Every field is Noetherian.
(b) $\mathbb{Z}$ is Noetherian.
(2) If A is a Noetherian ring, then the polynomial ring $A[t]$ is Noetherian.
(3) Let A be a Noetherian ring. If $S \subseteq A$ is a multiplicatively closed subset, then the ring $S^{-1}A$, which is given by the localization of A at S, is Noetherian.
(4) If A is a Noetherian ring and $I \subseteq A$ is an ideal, then the ring A/I is also Noetherian.

Proof.

(1) (a) Clearly every submodule of a field is either zero or the whole field.
(b) This statement can be proved using the Euclidean algorithm.
(2) If A is Noetherian, then it follows from [**Lang2002b**, Theorem IV.4.1] that $A[t]$ is Noetherian.
(3) This statement is [**Lang2002b**, Proposition X.1.6].
(4) This statement is an elementary exercise in algebra. ■

The following proposition now shows that our favorite rings are indeed Noetherian:

Proposition 287.4. (Laurent Polynomial Ring–Noetherian Proposition) Let A be a Noetherian ring (e.g. $A = \mathbb{Z}$ or A is field).

(1) Every multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ is Noetherian.
(2) For any finitely generated abelian group H the group ring $A[H]$ is Noetherian.

Proof. Let A be a Noetherian ring.

(1) It follows from the Noetherian Rings Proposition 287.3 (2) applied altogether m times, that $A[t_1, \dots, t_m]$ is Noetherian. Note that we obtain the multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ by inverting the multiplicatively closed subset that is given by all monomials in $t_1, \dots, t_k$. It thus follows from the Noetherian Rings Proposition 287.3 (3) that $A[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ is Noetherian.[3098]
(2) Let H be a finitely generated abelian group. By the Abelian Group-Classification Theorem 97.24 we know that H is isomorphic to $\mathbb{Z}^m \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/k_n\mathbb{Z}$ for some $m, n \in \mathbb{N}_0$ and some $k_1, \dots, k_n \in \mathbb{N}$. One can easily see that the group ring $A[H]$ is isomorphic to $A[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1, \dots, y_n]/(y_1^{k_1}, \dots, y_n^{k_n})$. Similar to (1) it thus follows from the Noetherian Rings Proposition 287.3 (2), (3) and (4) that the group ring $A[H]$ is Noetherian. ■

[3098] Alternatively one could argue that the Laurent polynomial $A[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ is isomorphic to the ring $A[t_1, s_1, \dots, t_k, s_k]/(s_1 \cdot t_1 - 1, \dots, s_k \cdot t_k - 1)$. It now follows from the Noetherian Rings Proposition 287.3 (2) and (4) that $A[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ is Noetherian.

The following proposition states the key property of Noetherian rings that we are interested in.

Proposition 287.5. (Noetherian Ring–Submodule Proposition) Let R be a Noetherian ring. If P is a finitely generated R-module, then every R-submodule of P is also finitely generated.

Proof. This statement is proved in [**Lang2002b**, Proposition X.1.1]. ■

287.2. **Presentation matrices.** We start out with an analog of a finite presentation for a group:

Definition 287.6. Let R be a commutative ring and let M be an R-module. A finite presentation for M is a finite exact sequence of R-homomorphisms of the form

$$R^k \xrightarrow{v \mapsto A \cdot v} R^l \xrightarrow{\psi} M \to 0$$

where $A \in \mathrm{M}(l \times k, R)$. Any such matrix A is called a presentation matrix for M.

Remark. Let R be a commutative ring and let M be an R-module. Note that a matrix $A \in \mathrm{M}(l \times k, R)$ is a presentation matrix for M if and only if there exists an isomorphism $R^l / A \cdot R^k \to M$ of R-modules. □

Example. We consider the ring $\mathbb{Z}[t]$ and the $\mathbb{Z}[t]$-module $\mathbb{Z}/2\mathbb{Z}$ where t acts trivially. One can easily show that the (1×2)-matrix $(2 \quad t-1)$ is a presentation matrix for M. □

The following proposition shows that over a Noetherian ring every finitely generated module admits a finite presentation:

Proposition 287.7. (Presentation Matrix–Existence Proposition) Let R be a Noetherian ring. Every finitely finitely generated R-module admits a presentation matrix.

Proof. Let M be a finitely generated R-module. Since M is a finitely generated R-module there exists an epimorphism $\psi\colon R^l \to M$ of R-modules. Since R is Noetherian it follows from the Noetherian Ring–Submodule Proposition 287.5 that $\ker(\psi)$ is also a finitely generated R-module. This means that there exists an epimorphism $\varphi\colon R^k \to \ker(\psi)$. In summary we have obtained the following little diagram:

$$\begin{array}{ccccccc} R^k & \xrightarrow{\iota \circ \varphi} & R^l & \xrightarrow{\psi} & M & \longrightarrow & 0. \\ & \searrow^{\varphi} & \uparrow{\scriptstyle =:\iota} & & & & \\ & & \ker(\psi) & & & & \end{array}$$

Note that the horizontal sequence is exact. Any matrix representing φ is a presentation matrix for M. ■

It is easy to see that presentation matrices are not unique. But the following proposition, which can be viewed as an analog of the Tietze Theorem 100.25, shows that any two presentation matrices are related by a few basic moves:

Proposition 287.8. (Presentation Matrix–Uniqueness Proposition) Let R be a commutative ring and let A and $\widetilde{A}$ be two matrices over R. The following two statements are equivalent:

(1) A and $\widetilde{A}$ are presentation matrices for isomorphic R-modules.

(2) The matrices A and $\widetilde{A}$ are related by a finite sequence of the following four types of moves or their inverses:

(i) (a) Permutation of the rows.
(b) Permutation of the columns.

(ii) Replacement of a matrix C by $\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$.

(iii) Adding an extra column of zeros to a matrix.[3099]

(iv) (a) Adding an R-multiple of a row to another row.
(b) Adding an R-multiple of a column to another column.

Proof. The proof of the (2)⇒(1)-direction is an elementary linear algebra exercise that we leave to the reader. It is better to spend our energy on the proof of (1)⇒(2)-direction. We denote by $\sim$ the equivalence relation on matrices over R that is generated by (i)–(iv). Let A and $\widetilde{A}$ be two presentation matrices for isomorphic R-modules M and $\widetilde{M}$. We need to show that $A \sim \widetilde{A}$. By hypothesis we have a diagram of the following form

$$\begin{array}{ccccccccc} R^k & \xrightarrow{A} & R^l & \xrightarrow{\psi} & M & \longrightarrow & 0 \\ & & & & \cong \downarrow f & & \\ R^{\tilde{k}} & \xrightarrow{\widetilde{A}} & R^{\tilde{l}} & \xrightarrow{\tilde{\psi}} & \widetilde{M} & \longrightarrow & 0 \end{array}$$

where $f\colon M \to \widetilde{M}$ is an isomorphism and where the two horizontal sequences are exact. Since the modules to the left of M and $\widetilde{M}$ are free R-modules we obtain from the argument of the Homological Algebra–Fundamental Theorem 158.4[3100] that there exist matrices $P, Q, \widetilde{P}, \widetilde{Q}$ over R which make the following diagrams commute:

$$\begin{array}{ccccccccc} R^k & \xrightarrow{A} & R^l & \xrightarrow{\psi} & M & \longrightarrow & 0 \\ \downarrow P & & \downarrow Q & & \downarrow f & & \\ R^{\tilde{k}} & \xrightarrow{\widetilde{A}} & R^{\tilde{l}} & \xrightarrow{\tilde{\psi}} & \widetilde{M} & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} R^k & \xrightarrow{A} & R^l & \xrightarrow{\psi} & M & \longrightarrow & 0 \\ \uparrow \widetilde{P} & & \uparrow \widetilde{Q} & & \uparrow f^{-1} & & \\ R^{\tilde{k}} & \xrightarrow{\widetilde{A}} & R^{\tilde{l}} & \xrightarrow{\tilde{\psi}} & \widetilde{M} & \longrightarrow & 0. \end{array}$$

[3099] Note that in (iii) we break the symmetry between rows and columns.

[3100] We formulated and proved the Homological Algebra–Fundamental Theorem 158.4 for free abelian groups, i.e. for free $\mathbb{Z}$-modules. The statements and proofs generalize to the setting of free R-modules in an obvious way.

Next note that

$$A \overset{\text{by (ii)}}{\sim} \begin{pmatrix} A & 0 \\ 0 & \mathrm{id}_{\widetilde{l}} \end{pmatrix} \overset{\text{by (iv-a)}}{\sim} \begin{pmatrix} A & \widetilde{Q} \\ 0 & \mathrm{id}_{\widetilde{l}} \end{pmatrix} \sim \begin{pmatrix} A & \widetilde{Q} & A \cdot P \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} \end{pmatrix} \overset{\text{since the above diagram commutes}}{\uparrow}$$

$$\overset{\text{by (iii)}}{\sim} \begin{pmatrix} A & \widetilde{Q} & 0 \\ 0 & \mathrm{id}_{\widetilde{l}} & 0 \end{pmatrix} \sim \begin{pmatrix} A & \widetilde{Q} & 0 \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} \end{pmatrix} \overset{\text{by (iv-b)}}{\uparrow}$$

$$\overset{\text{by (iv-b)}}{\sim} \begin{pmatrix} A & \widetilde{Q} & \widetilde{Q} \cdot \widetilde{A} \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} \end{pmatrix} \sim \begin{pmatrix} A & \widetilde{Q} & 0 & \widetilde{Q} \cdot Q \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} & Q \end{pmatrix}. \overset{\text{by (iii) and (iv-b)}}{\uparrow}$$

Claim. There exists a matrix $D \in \mathrm{M}(k \times l, R)$ with $A \cdot D = \widetilde{Q} \cdot Q - \mathrm{id}_l$.

Proof. First note that we have the following commutative diagram:

$$\begin{array}{ccccccc} R^k & \xrightarrow{A} & R^l & \xrightarrow{\psi} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle f} & & \\ R^{\widetilde{k}} & \xrightarrow{\widetilde{A}} & R^{\widetilde{l}} & \xrightarrow{\widetilde{\psi}} & \widetilde{M} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \widetilde{P}} & & \downarrow{\scriptstyle \widetilde{Q}} & & \downarrow{\scriptstyle f^{-1}} & & \\ R^k & \xrightarrow{A} & R^l & \xrightarrow{\psi} & M & \longrightarrow & 0. \end{array}$$

It follows from the above diagram that for any $v \in R^l$ we have $\psi(v) = \widetilde{\psi}(Q \cdot v) = \psi(\widetilde{Q} \cdot Q \cdot v)$. In other words we have the inclusion $\mathrm{im}(\widetilde{Q} \cdot Q - \mathrm{id}) \subseteq \ker(\psi)$. Since the top horizontal sequence is exact we see that $\mathrm{im}(\widetilde{Q} \cdot Q - \mathrm{id}) \subseteq \mathrm{im}(A)$. This means that for each $i \in \{1, \dots, k\}$ there exists a $d_i \in R^l$ with $A \cdot d_i = ((\widetilde{Q} \cdot Q) - \mathrm{id}) \cdot \mathfrak{e}_i$. We set $D \coloneqq (d_1 \ \dots \ d_k)$. By construction this matrix has the desired properties. ⊞

We now see that

$$A \overset{\text{by the above}}{\sim} \begin{pmatrix} A & \widetilde{Q} & 0 & \widetilde{Q} \cdot Q \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} & Q \end{pmatrix} \overset{\text{by the claim}}{\sim} \begin{pmatrix} A & \widetilde{Q} & 0 & A \cdot D + \mathrm{id}_l \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} & Q \end{pmatrix} \sim \begin{pmatrix} \widetilde{A} & Q & 0 & \mathrm{id}_{\widetilde{l}} \\ 0 & \mathrm{id}_l & A & \widetilde{Q} \end{pmatrix} \overset{\text{by (i)}}{\uparrow}$$

$$\overset{\text{by (iv-b)}}{\sim} \begin{pmatrix} A & \widetilde{Q} & 0 & \mathrm{id}_l \\ 0 & \mathrm{id}_{\widetilde{l}} & \widetilde{A} & Q \end{pmatrix} \sim \widetilde{A}. \overset{\text{by the same argument backwards}}{\uparrow}$$

■

287.3. **Finite presentations for Alexander modules.** In this section we will use the results from the previous section to show that Alexander modules are often finitely generated:

Proposition 287.9. (Alexander Module–Finite Presentation Proposition) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $\Phi \colon \mathrm{H}_1(X) \to H$ be an epimorphism onto a finitely generated abelian group. If X is homotopy equivalent to a finite CW-complex, then each Alexander module $\mathrm{H}_k(\widetilde{X}_\Phi)$ is a finitely generated $\mathbb{Z}[H]$-module.

Example. Let X be a compact smooth manifold. It follows from the Smooth Manifold-CW-Theorem 179.9 that X admits a finite CW-structure. □

Proof. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto a finitely generated abelian group. We assume that X is homotopy equivalent to a finite CW-complex. In fact, it follows from the Alexander Module–Homotopy Lemma 286.11 that we might as well assume that X is a finite CW-complex.

Similar to the Alexander Module–CW Proposition 286.12 we make the following observation:

(a) We use the CW-Complex Covering Proposition 138.1 (1) to equip $\widetilde{X}_\Phi$ with the corresponding natural CW-structure.
(b) It is clear from the definition of the CW-structure that the H-action on $\widetilde{X}_\Phi$ preserves this CW-structure.
(c) It follows from the naturality of the boundary homomorphism and the Group Ring–Module Lemma 97.34 that we can view the cellular chain complex $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi)$ as a chain complex of $\mathbb{Z}[H]$-modules.
(d) In fact one can show easily that $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi)$ is a free $\mathbb{Z}[H]$-module where the rank is given by the number of k-cells of X.

Next recall that in the Alexander Module–CW Proposition 286.12 we showed that we have an isomorphism

$$\mathrm{H}_k^\Phi(X;\mathbb{Z}[H]) \cong \frac{\ker\left(\partial_k\colon \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X}_\Phi)\right)}{\mathrm{im}\left(\partial_{k+1}\colon \mathrm{C}_{k+1}^{\mathrm{CW}}(\widetilde{X}_\Phi) \to \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_\Phi)\right)}$$

of $\mathbb{Z}[H]$-modules. Finally note that it follows from (d), the Laurent Polynomial Ring–Noetherian Proposition 287.4 and the Noetherian Ring–Submodule Proposition 287.5 that the right hand side is a finitely generated $\mathbb{Z}[H]$-module. ■

287.4. **Interlude: Unique Factorization Domains.** Our next goal is to associate to finitely generated $\mathbb{Z}[H]$-modules an invariant. To do so we will need the notion of the greatest common divisor in a ring. To make sense of this notion we need to introduce a second type of ring:

Definition 287.10. Let R be a commutative ring.

(1) We say that $g \in R$ is irreducible if g is not a unit and if whenever we can write $g = a \cdot b$ for some $a, b \in R$, then a or b is a unit in R.
(2) We say that R is a unique factorization domain (UFD) if R is a domain and if every non-zero element of R has a unique (up to units and permutations) factorization into irreducible elements.

The following propositions gives us many examples of UFDs:

Proposition 287.11. (UFD Proposition)

(1) (a) Every field is a UFD.
    (b) $\mathbb{Z}$ is a UFD.
(2) If A is a UFD, then the polynomial ring $A[t]$ is a UFD.
(3) Let A be a UFD. If $S \subseteq A$ is a multiplicatively closed subset, then the ring $S^{-1}A$, which is given by the localization A at S, is a UFD.

Proof.

(1) (a) This statement is clear since every non-zero element is now a unit.
(b) This statement is usually referred to as the Fundamental Theorem of Arithmetic. It was first proved by Gauss in [**Gaus1801**]. A modern proof is given in [**Lang2002b**, Theorem II.5.2].
(2) If A is UFD, then it follows from [**Lang2002b**, Theorem IV.2.3] that $A[t]$ is a UFD.
(3) This statement is [**Lang2002b**, Exercise 5 on p. 115]. ■

The following proposition now shows that many of our favorite rings are also unique factorization domains:

Proposition 287.12. (Laurent Polynomial Ring–UFD Proposition) Let A be a UFD (e.g. $A = \mathbb{Z}$ or A is field). Then every multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ is a UFD.

Proof. Let A be a UFD. It follows from the UFD Proposition 287.11 (2) applied altogether m times, that $A[t_1, \ldots, t_m]$ is a UFD. Again we point out that we obtain the multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ by inverting the multiplicatively closed subset that is given by all monomials in $t_1, \ldots, t_k$. It thus follows from the UFD Proposition 287.11 (3) that $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ is a UFD. ■

We move on to the following definition which we first encountered as Definition 116.20:

Definition 287.13. Let R be a commutative ring.

(1) Let $m \in R$. We say that $d \in R$ is a divisor of m if there exists an $r \in R$ with $d \cdot r = m$.
(2) Let $M \subseteq R$ be a subset.
(a) We say that $d \in R$ is a divisor of M if d is a divisor of every $m \in M$.
(b) We say that $g \in R$ is a greatest common divisor of M if the following two conditions are satisfied:
- g is a divisor of M.
- If h is a divisor of M, then h is also a divisor of g.

Lemma 287.14. (UFD-GCD Lemma) Let R be a commutative ring and let $M \subseteq R$. If R is a UFD, then there exists a greatest common divisor $\gcd(M) \in R$ of M and it is unique up to multiplication by a unit in R.

Proof. We leave the elementary proof of this lemma to the reader. ■

Example.

(1) Let R be a UFD. It is a good mental exercise to show that $\gcd(\{0\}) = 0$.
(2) Let R be a UFD and let $M \subseteq R$ be a subset. We consider the ideal $I(M)$ that is generated by M. It follows easily from the definitions that $\gcd(M) = \gcd(I(M))$. □

In the UFD-GCD Lemma 287.14 we just saw that the greatest common divisor is only well-defined up to multiplication by a unit. This motivates the following notation, which is closely related to the notation we introduced in Notation 116.13:

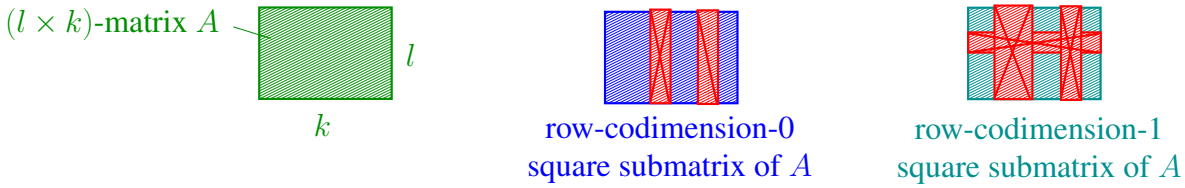
Notation 287.15. Let R be a ring. Given $p, q \in R$ we write $p \doteq q$ if there exists a unit $\epsilon \in R$ such that $p = \epsilon \cdot q$.

287.5. **Elementary ideals and orders of modules.** In this section we will introduce elementary ideals and orders of finitely generated modules over Noetherian UFDs. We start out with the following elementary, but perhaps slightly confusing definition:

Definition 287.16. Let R be a ring and let $A \in \mathrm{M}(l \times k, R)$ be a matrix.

(1) Let $i \in \{0, \ldots, l\}$ with $i + k - l \geq 0$. We refer to any $(l - i) \times (l - i)$-matrix that is obtained from A by crossing out i rows and $(i + k - l)$ columns as a row-codimension-i submatrix of A.[3101]

(2) We now suppose that R is a commutative ring. We refer to the ideal generated by the determinants of all square submatrices of row-codimension $\leq i$ of A as the i-th elementary ideal $E_i(A) \subseteq R$ of A.



The above Definition 287.18 on its own is perhaps not very interesting. But the following proposition turns elementary ideals into a useful concept:

Proposition 287.17. (Module–Elementary Ideal Proposition)[3102] Let R be a Noetherian ring and let M be a finitely generated R-module. Furthermore let $k \in \mathbb{N}_0$. By the Presentation Matrix–Existence Proposition 287.7 we know that M admits a presentation matrix A. The k-th elementary ideal $E_k(A) \subseteq R$ does not depend on the choice of the presentation matrix.

Proof. Let A and B be two presentation matrices for M. It follows from the Presentation Matrix–Uniqueness Proposition 287.8 that A and B are related by certain moves (i)–(iv) and their inverses. Using the multilinearity of determinants (see the Determinant–Characterization Proposition 2.18) one can easily verify that the k-th elementary ideal is unaffected by any of the moves. We only mention two moves explicitly:

(ii) Replacement of a matrix C by $\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$: In this case the new row-codimension-k square matrices are of one of the following three types:

- $\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$, where D is a row-codimension-k square submatrix of C.
- a row-codimension-$(k-1)$ square submatrix of $(C \;\; 0)$.
- a row-codimension-$(k-1)$ square submatrix of C.

Note that it follows from the Leibniz Rule 2.18 that in the third case the determinant is a linear combination of determinants of row-codimension-k square submatrices of C. Using these observations it is now elementary to see that the elementary ideals are unchanged.

[3101] Note that this notion is a bit risque, since it is not symmetric in the columns and rows.

[3102] This statement has more than a passing resemblance with the Independence-of-Generator Lemma 116.21. The fact that the roles of rows and columns are swapped is due to the fact that currently we think of elements of R^k as column vectors, whereas in the earlier setting we implicitly worked with row vectors.

(iii) Adding an extra column of zeros to a matrix: This is the one move which is not symmetric in rows and columns. The above Definition 287.18 is not symmetric either in rows and columns, and one can now easily see that our definition is set up in such a way as to compensate for (iii). We leave it to the reader to fill in all the remaining details. ■

The Module–Elementary Ideal Proposition 287.17 allows us to introduce the following definition:

Definition 287.18. Let R be a Noetherian UFD and let M be a finitely generated R-module. By the Presentation Matrix–Existence Proposition 287.7 we know that M admits a presentation matrix A

(1) Let $k \in \mathbb{N}_0$. We refer to
$$E_k(M) \coloneqq E_k(A) \subseteq R$$
as the k-th elementary ideal of the R-module M. It follows from the Module–Elementary Ideal Proposition 287.17 that $E_k(M)$ is well-defined.

(2) We refer to[3103]
$$\operatorname{ord}_R(M) \coloneqq \gcd_R(E_0(A)) \in R$$
as the order of the R-module M. It follows from the Module–Elementary Ideal Proposition 287.17 together with the UFD-GCD Lemma 287.14 that the order is well-defined up to "$\doteq$", i.e. up to multiplication by a unit in R.

Also note that it follows immediately from the definitions that for isomorphic R-modules the elementary ideals agree and that the orders agree up to "$\doteq$".

Remark. Elementary ideals of modules were first introduced by the German mathematician Hans Fitting [**Fit1936**]. □

Example.

(1) Let R be a Noetherian UFD. If an R-module M admits a square presentation matrix A, then $E_0(M) = (\det(A))$ and thus $\operatorname{ord}_R(M) = \det(A)$. Thus in this lucky setting we do not need to deal with greatest common divisors.

(2) Let R be a Noetherian UFD.

(a) Let M be the zero R-module. The (0×0)-matrix is a presentation matrix for the zero R-module.

- Note that the (0×0)-matrix has one row-codimension-0 submatrix, namely the (0×0)-matrix itself. As we pointed out in Footnote 29, the determinant of the 0×0-matrix is 1. Thus we see that $\operatorname{ord}_R(\{0\}) = 1$.[3104]
- Now let $k \in \mathbb{N}$. There is no row-codimension-k submatrix of the (0×0)-matrix. But by our (slightly subtle) definition of $E_k(\{0\})$ we nonetheless have the equality $E_k(\{0\}) = E_0(\{0\}) = \{0\}$.

(b) Let M be a free R-module of rank $d \in \mathbb{N}$. A presentation matrix is given by the $(d \times 0)$-matrix A. Since this matrix has no row-codimension-0 square submatrices

[3103] Of course one could also consider the orders of the higher elementary ideals. We will do so in some of the exercises, but for the most part the 0-th elementary ideal is all that we will care about.

[3104] Of course any identity matrix is a presentation matrix. This gives a clearer, but less minimalist proof that $\operatorname{ord}_R(\{0\}) = \det(\mathrm{id}) = 1$.

we see that the 0-th elementary ideal is given by $E_0(M) = E_0(A) = I(\varnothing) = \{0\}$. Thus we see that $\operatorname{ord}_R(M) = \gcd_R(\{0\}) = 0$.[3105]

(3) We consider the ring $R = \mathbb{Z}$.

(a) Let M be a finitely generated abelian group. It follows from the Abelian Group-Classification Theorem 97.24 that $M \cong \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_m\mathbb{Z}$ for some $k_1, \ldots, k_m \in \mathbb{N}_0$. Thus $\operatorname{diag}(k_1, \ldots, k_m)$ is a presentation matrix for M and we see that

$$\operatorname{ord}_{\mathbb{Z}}(M) \;=\; \det(\operatorname{diag}(k_1, \ldots, k_m)) \;=\; k_1 \cdot \ldots \cdot k_m \;=\; \begin{cases} |M|, & \text{if } M \text{ is finite,} \\ 0, & \text{otherwise.} \end{cases}$$

(b) Let M be a $\mathbb{Z}$-module with presentation matrix $A = \begin{pmatrix} 15 & 0 \\ 9 & 18 \\ 3 & 12 \end{pmatrix}$. Then

$$\operatorname{ord}_{\mathbb{Z}}(M) \;=\; \gcd\Big\{ \underbrace{\det\begin{pmatrix} \cancel{15} & \cancel{0} \\ 9 & 18 \\ 3 & 12 \end{pmatrix}}_{=54}, \underbrace{\det\begin{pmatrix} 15 & 0 \\ \cancel{9} & \cancel{18} \\ 3 & 12 \end{pmatrix}}_{=180}, \underbrace{\det\begin{pmatrix} 15 & 0 \\ 9 & 18 \\ \cancel{3} & \cancel{12} \end{pmatrix}}_{=240} \Big\} \;=\; 6.$$

By (a) we now know that M has 6 elements. □

For the record we state the following elementary lemma:

Lemma 287.19. (Order–Functorial Lemma) Let R be a Noetherian UFD and let M be a finitely generated R-module.

(1) If N is an R-module that is isomorphic to M, then $\operatorname{ord}_R(M) \doteq \operatorname{ord}_R(N)$.

(2) Let $\varphi\colon S \to R$ be an isomorphism of rings. If we use φ to view M as an S-module, then

$$\operatorname{ord}_R(R\text{-module } M) \;\doteq\; \varphi(\operatorname{ord}_S(S\text{-module } M)).$$

Proof.

(1) This statement follows from the observation that any presentation matrix for M is also a presentation matrix for N and vice versa.

(2) This statement follows easily from the definitions and the fact that the determinant commutes with ring homomorphisms. ■

We continue with a slightly non-trivial calculation of an order:

Proposition 287.20. (Finite–Order One Proposition) Let H be a finitely generated free abelian group. If M is a finitely generated $\mathbb{Z}[H]$-module, if $\operatorname{rank}(H) \geq 1$ and if M is finite (as a set), then $\operatorname{ord}_{\mathbb{Z}[H]}(M) = 1$.

Proof. Let M be a finitely generated $\mathbb{Z}[H]$-module which is a finite set. We pick a finite presentation

$$\mathbb{Z}[H]^k \;\xrightarrow{A}\; \mathbb{Z}[H]^l \;\xrightarrow{\psi}\; M \;\to\; 0.$$

We make two basic observations:

- Since M is finite we have $n := |M| \in \mathbb{N}$.
- Since $\operatorname{rank}(H) \geq 1$ we can pick a non-trivial $h \in H$. Since M is finite the automorphism of M that is given by multiplication by h has finite order. This means that there exists an $m \in \mathbb{N}$ such that multiplication by h^m on M equals id_M.

[3105] The $(d \times d)$-matrix consisting of zeros is also a presentation matrix for R^d. This gives again a clearer, but less minimalist proof that $\operatorname{ord}_R(R^d) = \det(0) = 0$.

Claim. The sequence
$$\mathbb{Z}[H]^{2l+k} \xrightarrow[=:\varphi]{((h^m-1)\cdot \mathrm{id}_l \quad n\cdot \mathrm{id}_l \quad A)} \mathbb{Z}[H]^l \xrightarrow{\psi} M \to 0$$
is also a presentation for M.

Proof. This statement follows immediately from the following two observations:

- For $i = 1, \dots, k$ we have $\psi(\varphi(\mathfrak{e}_i)) = \psi((h^m - 1) \cdot \mathfrak{e}_i) = h^m \cdot \psi(\mathfrak{e}_i) - \psi(\mathfrak{e}_i) = 0 \in M$.
- For $i = k+1, \dots, 2k$ we have $\psi(\varphi(\mathfrak{e}_i)) = \psi(n \cdot \mathfrak{e}_{i-k}) = n \cdot \psi(\mathfrak{e}_{i-k}) = 0 \in M$. ⊞

Note that $(h^m - 1) \cdot \mathrm{id}_l$ and $n \cdot \mathrm{id}_l$ are both row-codimension-0 square submatrices of the new presentation matrix. It follows from Definition 287.18 that $(h^m-1)^l = \det((h^m - 1) \cdot \mathrm{id}_l)$ divides $\mathrm{ord}_{\mathbb{Z}[H]}(M)$ and that $n^l = \det(n \cdot \mathrm{id}_l)$ also divides $\mathrm{ord}_{\mathbb{Z}[H]}(M)$. Since $h \in H$ is non-trivial it follows from an elementary algebraic argument, see e.g. Exercise 287.6, that $\gcd_{\mathbb{Z}[H]}(n^l, (h^m - 1)^l) = 1$. It thus follows that $\mathrm{ord}_{\mathbb{Z}[H]}(M) \doteq 1$. ■

The following proposition shows that the order is multiplicative for short exact sequences:

Proposition 287.21. (Order–Exactness Proposition) Let R be a Noetherian UFD.

(1) For any short exact sequence $0 \to K \to L \to M \to 0$ of finitely generated R-modules we have
$$\mathrm{ord}_R(L) \doteq \mathrm{ord}_R(K) \cdot \mathrm{ord}_R(M).$$

(2) For any two finitely generated R-modules M and N we have
$$\mathrm{ord}_R(M \oplus N) \doteq \mathrm{ord}_R(M) \cdot \mathrm{ord}_R(N).$$

Example. Let $0 \to K \to L \to M \to 0$ be a short exact sequences of finite abelian groups. It follows from the discussion on page 5490 that in this setup the Order–Exactness Proposition 287.21 (1) just gives us the familiar statement that $|L| = |K| \cdot |M|$. □

Proof.

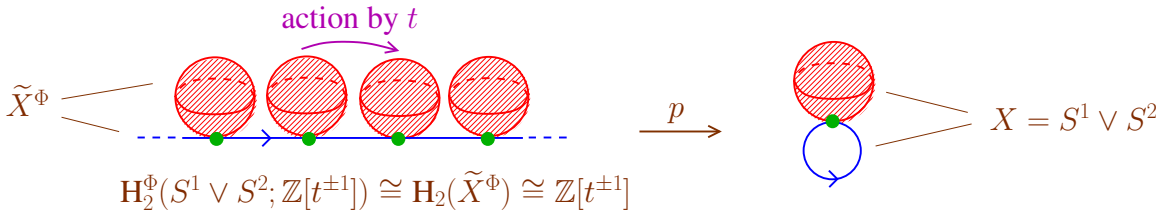
(1) We outsource the somewhat lengthy proof of this statement to Appendix 287.8.

(2) This statement follows immediately from Statement (1). We will provide a more direct proof in Exercise 287.16. ■

287.6. **Alexander modules and Alexander polynomials.** The ring and module theory which we just developed allows us to introduce the following rather general definition:

Definition 287.22. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. We assume that X is homotopy equivalent to a finite CW-complex. Let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto a finitely generated free abelian group. Given any $k \in \mathbb{N}_0$ we consider the Alexander module $\mathrm{H}_k^\Phi(X; \mathbb{Z}[H])$ from Definition 286.9. By the Alexander Module–Finite Presentation Proposition 287.9 we know that $\mathrm{H}_k^\Phi(X; \mathbb{Z}[H])$ is a finitely generated $\mathbb{Z}[H]$-module. Since H is a finitely generated *free* abelian group we know by the Laurent Polynomial Ring–UFD Proposition 287.12 that $\mathbb{Z}[H]$ is a UFD. Thus we can use the order from Definition 287.18 to define the k-th Alexander polynomial of (X, Φ) as
$$\Delta_k^\Phi(X) := \mathrm{ord}_{\mathbb{Z}[H]}(\mathrm{H}_k^\Phi(X; \mathbb{Z}[H])) \in \mathbb{Z}[H].$$
Note that it follows from Definition 287.18 that $\Delta_k^\Phi(X)$ is well-defined up to "$\doteq$".

Example. We consider again the wedge $X = S^1 \vee S^2$ with the obvious isomorphism $\Phi\colon \mathrm{H}_1(S^1 \vee S^2) \to \mathbb{Z}$. The covering $\widetilde{X}^\Phi \to X = S^1 \vee S^2$ is shown below.



We now see that

$$\Delta_2^\Phi(S^1 \vee S^2) \underset{\text{by Definition 287.22}}{=} \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_2^\Phi(S^1 \vee S^2; \mathbb{Z}[t^{\pm 1}])) \underset{\text{see page 5475}}{=} \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}]) \underset{\text{see page 5490}}{=} 0. \qquad \square$$

We formulate our next example as a lemma:

Lemma 287.23. (Zeroth-Alexander Polynomial Lemma) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. We assume that X is homotopy equivalent to a finite CW-complex. Let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto a free abelian group of rank ≥ 1. Then

$$\Delta_0^\Phi(X) = \begin{cases} h-1, & \text{if } H = \langle h \rangle \text{ has rank 1,} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. As on page 5475 we make the following observation:

$$\mathrm{H}_0^\Phi(X; \mathbb{Z}[H]) \underset{\text{see Definition 286.9}}{=} \mathrm{H}_0(\widetilde{X}) \underset{\substack{\text{by the } \mathrm{H}_0\text{-Proposition 143.25 and since } X \text{ is} \\ \text{path-connected and non-empty}}}{\cong} \mathbb{Z}, \text{ where each } h \in H \text{ acts trivially.}$$

It follows that it suffices to find a presentation matrix for the $\mathbb{Z}[H]$-module $\mathbb{Z}$.

Claim. We pick a basis $h_1, \ldots, h_m$ for H. The maps

$$\mathbb{Z}[H]^m \xrightarrow[=:\varphi]{(h_1-1 \ \ldots \ h_m-1)} \mathbb{Z}[H] \xrightarrow[=:\psi]{h \mapsto 1} \mathbb{Z} \to 0$$

form a presentation for the above $\mathbb{Z}[H]$-module $\mathbb{Z}$.

Proof. It is clear that ψ is an epimorphism and that $\psi \circ \varphi = 0$. Thus it remains to show that the map

$$\begin{aligned} \alpha\colon \mathbb{Z}[H]/(h_1 - 1, \ldots, h_m - 1) &\to \mathbb{Z} \\ [p] &\mapsto \psi(p) \end{aligned}$$

is an isomorphism. We consider the map $\beta\colon \mathbb{Z} \to \mathbb{Z}[H]/(h_1 - 1, \ldots, h_m - 1)$ that is given by $\beta(1) = [1]$. It is clear that $\alpha \circ \beta = \mathrm{id}_\mathbb{Z}$. Furthermore one can easily verify that for all $i = 1, \ldots, m$ we have $(\beta \circ \alpha)([h_i]) = [1] = [h_i]$. Since $[h_1], \ldots, [h_m]$ is a generating set for the ring $\mathbb{Z}[H]/(h_1 - 1, \ldots, h_m - 1)$ we see that $\beta \circ \alpha = \mathrm{id}_{\mathbb{Z}[H]/(h_1-1,\ldots,h_m-1)}$. $\boxplus$

We now see that

$$\Delta_0^\Phi(X) \underset{\text{by the above}}{=} \mathrm{ord}_{\mathbb{Z}[H]}(\mathbb{Z}) \underset{\substack{\text{follows from Definition 287.18} \\ \text{and the above claim}}}{=} \mathrm{gcd}_{\mathbb{Z}[H]}\{h_1 - 1, \ldots, h_m - 1\} \underset{\substack{\text{follows from elementary} \\ \text{algebra}}}{=} \begin{cases} h-1, & \text{if } H = \langle h \rangle \text{ has rank 1,} \\ 1, & \text{otherwise.} \end{cases}$$

$\blacksquare$

On many occasions we will explicitly or implicitly use the following lemma:

Lemma 287.24. (Alexander Module–Homotopy Lemma) Let X and Y be two path-connected, locally path-connected and semi-locally simply connected topological spaces that are both homotopy equivalent to a finite CW-complex. Also let $\Psi\colon \mathrm{H}_1(Y) \to H$ be an epimorphism onto a free abelian group. Finally let $f\colon X \to Y$ be a continuous map. If f is a homotopy equivalence (e.g. f could be a homeomorphism), then for each $k \in \mathbb{N}_0$ we have the following equality in $\mathbb{Z}[H]$:

$$\Delta_k^{\Phi\circ f_*}(X) \doteq \Delta_k^{\Phi}(Y).$$

Proof. We have the following equalities in $\mathbb{Z}[H]$:

$$\Delta_k^{\Phi\circ f_*}(X) \underset{\text{by Definition 287.22}}{\overset{\uparrow}{=}} \mathrm{ord}_{\mathbb{Z}[H]}(\mathrm{H}_k^{\Phi\circ f_*}(X;\mathbb{Z}[H])) \underset{\text{follows from the Alexander Module Homotopy Lemma 286.11}}{\overset{\uparrow}{\doteq}} \mathrm{ord}_{\mathbb{Z}[H]}(\mathrm{H}_k^{\Phi}(Y;\mathbb{Z}[H])) \underset{\text{by Definition 287.22}}{\overset{\uparrow}{=}} \Delta_k^{\Phi}(Y).$$

■

287.7. **Alexander polynomials of mapping tori.** In this section we will consider Alexander polynomials of mapping tori. Before we state the main result let us recall the following notions on Laurent polynomials from Definition 123.1.

Definition 287.25.

(1) Given a non-zero Laurent polynomial $p(t) = \sum_{i=r}^{s} a_i \cdot t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its degree as

$$\deg(p(t)) = \deg\left(\sum_{i=r}^{s} a_i \cdot t^i\right) = s - r.$$

We extend this definition to $\deg(0) = -\infty$. Note that if we have $p(t), q(t) \in \mathbb{Z}[t^{\pm 1}]$ with $p \doteq q$, then $\deg(p(t)) = \deg(q(t))$.

(2) We say that a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic if the highest coefficient is ± 1. We say a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is bimonic if the lowest and the highest coefficient is ± 1.

Example. The Laurent polynomial $p(t) = -5t^{-2} + 3t^{-1} + 7t^4 - t^5$ has degree 7, it is monic but it is not bimonic. □

Using the Mapping Torus-Covering Proposition 286.5 we can determine the Alexander polynomials of mapping tori:

Proposition 287.26. (Mapping Torus-Alexander Polynomial Proposition) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and that is homotopy equivalent to a finite CW-complex. Furthermore let $f\colon X \to X$ be a self-homeomorphism. We consider the natural epimorphism $\Phi\colon \mathrm{H}_1(\mathrm{Tor}(X,f)) \to \langle t\rangle$ from Definition 286.4.

(1) We write $\mathrm{FH}_k(X) := \mathrm{H}_k(X)/\text{torsion}$. We pick a basis $v_1, \ldots, v_m \in \mathrm{FH}_k(X)$. Let $A \in \mathrm{GL}(m,\mathbb{Z})$ be the matrix that represents the induced map $f_*\colon \mathrm{FH}_k(X) \to \mathrm{FH}_k(X)$ with respect to the basis $v_1, \ldots, v_m$. With this notation we have[3106]

$$\Delta_k^{\Phi}(\mathrm{Tor}(X,f)) = \text{characteristic polynomial of } f_* = \det(t \cdot \mathrm{id}_m - A).$$

(2) The Alexander polynomial $\Delta_k^{\Phi}(\mathrm{Tor}(X,f))$ is bimonic and its degree equals the k-th Betti number $b_k(X) := \mathrm{rank}(\mathrm{H}_k(X)) = m$.

Example.

(1) Let X be a finite CW-complex. We consider the product $X \times S^1$ with the epimorphism $\mathrm{H}_1(X \times S^1) \to \langle t \rangle$ that is induced by the natural projection $X \times S^1 \to S^1$. Let $k \in \mathbb{N}_0$. We set $b := b_k(X) = \mathrm{rank}(\mathrm{H}_k(X))$. Then we see that

$$\Delta_k^\Phi(X \times S^1) \underset{\substack{\uparrow \\ \text{see page 5469}}}{=} \Delta_k^\Phi(\mathrm{Tor}(X, \mathrm{id})) \underset{\substack{\uparrow \\ \text{by the Mapping Torus-Alexander} \\ \text{Polynomial Proposition 287.26}}}{=} \det(t \cdot \mathrm{id}_b - \mathrm{id}_b) = (t-1)^b.$$

(2) Let K be the Klein bottle. For the (up to a sign) unique epimorphism $\Phi\colon \mathrm{H}_1(K) \to \langle t \rangle$ we see that

$$\Delta_1^\Phi(K) \underset{\substack{\uparrow \\ \text{see page 5469}}}{=} \Delta_1^\Phi(\mathrm{Tor}(S^1, f(z) = \overline{z})) \underset{\substack{\uparrow \\ \text{by the Mapping Torus-Alexander} \\ \text{Polynomial Proposition 287.26 and since} \\ \text{the map } f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1) \text{ is} \\ \text{given by multiplication by } -1}}{=} \det(t \cdot 1 - (-1)) = 1 + t.$$

□

The proof of the Mapping Torus-Alexander Polynomial Proposition 287.26 rests on the following purely algebraic lemma:

Lemma 287.27. (Order–Characteristic Polynomial Lemma) Let V be a finitely generated free abelian group and let $\varphi\colon V \to V$ be an isomorphism. We equip V with the $\langle t \rangle$-action given by $t^i \cdot h = \varphi^i(h)$ and we use this action to view V as a $\mathbb{Z}[t^{\pm 1}]$-module.

(1) Let $v_1, \dots, v_m$ be a basis for V. We denote by $A \in \mathrm{GL}(m, \mathbb{Z})$ the matrix that represents the automorphism φ with respect to the basis $v_1, \dots, v_m$. Then

$$\mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(V) \doteq \text{characteristic polynomial of } \varphi \doteq \det(t \cdot \mathrm{id}_m - A).$$

(2) The Laurent polynomial $\mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(V)$ is bimonic and its degree equals m.

Sketch of proof of Lemma 287.27.

(1) We consider the following sequence of $\mathbb{Z}[t^{\pm 1}]$-module homomorphisms:

$$\mathbb{Z}[t^{\pm 1}]^m \xrightarrow{t \cdot \mathrm{id}_m - A} \mathbb{Z}[t^{\pm 1}]^m \xrightarrow{\mathfrak{e}_i \mapsto v_i} V \to 0.$$

In Exercise 287.15 we will see that this sequence is exact. Thus we see that $t \cdot \mathrm{id}_m - A$ is a presentation matrix for the $\mathbb{Z}[t^{\pm 1}]$-module V. Since this is a square matrix we obtain from the elementary discussion on page 5489 that $\mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(V) \doteq \det(t \cdot \mathrm{id}_m - A)$.

(2) It follows easily from the Leibniz Formula 2.18 for determinants (see also the Polynomial Matrix–Determinant Lemma 123.11) that for $A \in \mathrm{GL}(m, \mathbb{Z})$ we have

$$\begin{aligned} t^m\text{-coefficient of } \det(t \cdot \mathrm{id}_m - A) &= \det(\mathrm{id}_m) &&= 1, \\ t^0\text{-coefficient of } \det(t \cdot \mathrm{id}_m - A) &= (-1)^m \cdot \det(A), \end{aligned}$$

and that all coefficients of degree $> m$ are zero and that all coefficients of negative degree are zero. Together with Statement (1) these observations imply the promised statements. ■

Proof of Proposition 287.26. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $f\colon X \to X$ be a

[3106] For peace of mind we point out that it follows from the Cellular Approximation Theorem 141.2, Exercise 21.17 and Exercise 139.11 that $\mathrm{Tor}(X, f)$ is also homotopy equivalent to a finite CW-complex.

self-homeomorphism. We have the following equalities:

$$\begin{array}{rl}
& \text{by Definition 287.22} \\
\Delta_k^\Phi(\mathrm{Tor}(X,f)) & \overset{\downarrow}{=} \mathrm{ord}_{\mathbb{Z}[t^{\pm1}]}(\mathrm{H}_k^\Phi(\mathrm{Tor}(X,f),\mathbb{Z}[t^{\pm1}])) \\
& = \mathrm{ord}_{\mathbb{Z}[t^{\pm1}]}(\mathrm{H}_k(X) \text{ where } t \text{ acts via } f_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(X)) \\
& \text{by the Mapping Torus Alexander} \\
& \text{Module Lemma 286.10} \\
& \overset{\uparrow}{=} \mathrm{ord}_{\mathbb{Z}[t^{\pm1}]}(\mathrm{FH}_k(X) \text{ where } t \text{ acts via } f_*\colon \mathrm{FH}_k(X) \to \mathrm{FH}_k(X)) \\
& \text{follows from the Order–Exactness Proposition 287.21 (1), applied to} \\
& \text{the short exact sequence } 0 \to \mathrm{Tor}\,\mathrm{H}_k(X) \to \mathrm{H}_k(X) \to \mathrm{FH}_k(X) \to 0 \\
& \text{of } \mathbb{Z}[t^{\pm1}]\text{-modules, together with the Finite–Order One Proposition 287.20} \\
& \overset{\uparrow}{=} \text{characteristic polynomial of } f_* \qquad \overset{\uparrow}{=} \det(V - t\cdot \mathrm{id}_m). \\
& \text{this follows from the Order–Characteristic Polynomial Lemma 287.27}
\end{array}$$

Finally note that it follows from the Order–Characteristic Polynomial Lemma 287.27 that $\Delta_k^\Phi(\mathrm{Tor}(X,f))$ is binomic of degree $b_k(X) = m$. ∎

287.8. **Appendix: Multiplicativity of orders.** The only goal of this appendix is to prove the following multiplicativity property of orders:

Proposition 287.28. (Order–Exactness Proposition) Let R be a Noetherian UFD. For any short exact sequence $0 \to A \to B \to C \to 0$ of finitely generated R-modules we have

$$\mathrm{ord}_R(B) \doteq \mathrm{ord}_R(A)\cdot\mathrm{ord}_R(C).$$

In the proof of the Order–Exactness Proposition 287.28 we will need the following lemma:

Lemma 287.29. (Middle Presentation Matrix Lemma) Let R be a Noetherian ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules.

(1) If X is a $(k \times \widetilde{k})$-presentation matrix for A and if Z is an $(l \times \widetilde{l})$-presentation matrix for C, then there exists a presentation matrix for B of the form

$$Y = \begin{pmatrix} X & S \\ 0 & Z \end{pmatrix}.$$

(2) The order $\mathrm{ord}_R(B)$ is a divisor of $\mathrm{ord}_R(A)\cdot\mathrm{ord}_R(C)$.

(3) Let $s_1,\dots,s_{\widetilde{l}} \in R^k$ be the columns of the matrix S and let $z_1,\dots,z_{\widetilde{l}} \in R^l$ be the columns of the matrix Z. If we have $r_1,\dots,r_{\widetilde{l}} \in R$ with $r_1\cdot z_1 + \cdots + r_{\widetilde{l}}\cdot z_{\widetilde{l}} = 0 \in R^{\widetilde{l}}$, then $[r_1\cdot s_1 + \cdots + r_{\widetilde{l}}\cdot s_{\widetilde{l}}] = 0 \in R^k/X\cdot R^{\widetilde{k}}$.

Sketch of proof of the Middle Presentation Matrix Lemma 287.29.

(1) We outsource the elementary proof to Exercise 287.17.

(2) We use the notation from (1) and we call Y the resulting presentation matrix for B. Let V be a codimension-0 square submatrix of Y. If $\det(V) \neq 0$, then it follows easily that $V = \left(\begin{smallmatrix} U & * \\ 0 & W \end{smallmatrix}\right)$ where U is a codimension-0 square submatrix of X and W is a codimension-0 square submatrix of Z. Since $\det(V) = \det(U)\cdot\det(W)$ we now see that $E_0(Y) \subseteq E_0(X)\cdot E_0(Z)$. It now follows easily that $\mathrm{ord}_R(B)$ is a divisor of $\mathrm{ord}_R(A)\cdot\mathrm{ord}_R(C)$.

(3) We consider the following diagram:

$$\begin{array}{ccc} R^{\tilde{k}} & \xrightarrow{v \mapsto \binom{v}{0}} & R^{\tilde{k}+\tilde{l}} \\ \downarrow{\scriptstyle X\cdot} & & \downarrow{\scriptstyle Y\cdot} \\ R^{k} & \xrightarrow{v \mapsto \binom{v}{0}} & R^{k+l} \\ \downarrow & & \downarrow \\ A = \operatorname{coker}(X) & \xrightarrow{[v] \mapsto \left[\binom{v}{0}\right]} & B = \operatorname{coker}(Y). \end{array}$$

Note that the diagram commutes. The statement now follows easily from the observation that the bottom horizontal map is a monomorphism. ■

Proof of the Order–Exactness Proposition 287.28. [3107] Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence of finitely generated R-modules. We need to show that

$$\operatorname{ord}_R(B) \doteq \operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C).$$

We prove this equality by induction on the number of generators of the R-module A.

(1) First we deal with the case that A is generated by a single element $a \in A$. Note that this means that A admits a $(1 \times s)$-presentation matrix of the form $(\alpha_1 \ \dots \ \alpha_s)$. We pick an $(m \times t)$-presentation matrix $(\gamma_1 \ \dots \ \gamma_t)$ for C. Note that it follows from the Middle Presentation Matrix Lemma 287.29 (1) that B has an $(m+1) \times (s+t)$-presentation matrix of the form

$$M = \begin{pmatrix} \alpha_1 & \dots & \alpha_s & \beta_1 & \dots & \beta_t \\ 0 & \dots & 0 & \gamma_1 & \dots & \gamma_t \end{pmatrix}$$

Next note that it follows from the Middle Presentation Matrix Lemma 287.29 (2) that $\operatorname{ord}_R(B)$ is a divisor of $\operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C)$. It follows easily from Definition 287.16 and Definition 287.16 that it remains to prove the following claim:

Claim. For every row-codimension-0 square submatrix P of M the product $\operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C)$ is a divisor of $\det(P)$.

Proof. First note that if P involves any of the first s columns, then $\det(P) = 0$, and thus the statement holds trivially. It follows from this observation that we only have to deal with the case that P is built out of the last t columns. To simplify the notation we suppose that P contains precisely the columns $\binom{\beta_1}{\gamma_1}, \dots, \binom{\beta_{m+1}}{\gamma_{m+1}}$. Since $\gamma_1, \dots, \gamma_{m+1}$ lie in R^m and since R is a domain we see that there there exist non-zero $r_1, \dots, r_{m+1} \in R$ such that

$$(*) \qquad r_1 \cdot \gamma_1 + \dots + r_{m+1} \cdot \gamma_{m+1} = 0.$$

In fact, after dividing by the greatest common divisor we can assume that $r_1, \dots, r_{m+1}$ are coprime, i.e. the greatest divisor of $\{r_1, \dots, r_{m+1}\}$ equals 1. Next note that it follows from $(*)$ and the Middle Presentation Matrix Lemma 287.29 (3) that we have following equality in A:

$$(r_1 \cdot \beta_1 + \dots + r_{m+1} \cdot \beta_{m+1}) \cdot a = 0.$$

[3107] In the following we follows closely the argument provided in [**Levine1967**, Lemma 5].

This implies that $\beta := r_1 \cdot \beta_1 + \cdots + r_{m+1} \cdot \beta_{m+1}$ is divisible by $\operatorname{ord}_R(A)$.[3108]

Next, for $k \in \{1, \ldots, m\}$ we consider the matrix

$$Q_k = \underbrace{\begin{pmatrix} \beta_1 & \ldots & \beta_t \\ \gamma_1 & \ldots & \gamma_t \end{pmatrix}}_{=P} \cdot \begin{pmatrix} 1 & 0 & r_1 & 0 & 0 \\ 0 & \ddots & \vdots & & \vdots \\ \vdots & & r_k & 0 & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 0 & 0 & r_{m+1} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \beta & 0 & 0 \\ \gamma_1 & \ldots & * & \ldots & \gamma_{m+1} \end{pmatrix}$$

for some $* \in R^m$. Note that it follows from the above that

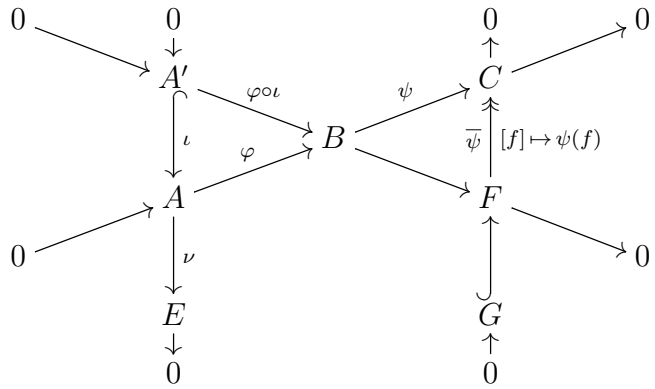
$$r_k \cdot \det(P) = \det(Q_k) = \beta \cdot \underbrace{\det(\gamma_1 \ldots \widehat{*} \ldots \gamma_{m+1})}_{\in R} = \beta \cdot \det(\gamma_1 \ \ldots \ \widehat{\gamma_k} \ \ldots \gamma_{m_1}).$$

By the above we know that $\operatorname{ord}_R(A)$ is a divisor of β. Furthermore it follows from Definition 287.16 that $\operatorname{ord}_R(C)$ is a divisor of $\det(\gamma_1 \ldots \widehat{\gamma_k} \ldots \gamma_{m_1})$. This shows that $\operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C)$ is a divisor of $r_k \cdot \det(P)$. Since the $r_1, \ldots, r_k$ are coprime we now see that $\operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C)$ is indeed a divisor of $\det(P)$. ⊞

(2) Now we turn to the induction step. We assume that the statement holds if A is generated by $n \in \mathbb{N}$ elements. Now we assume that A is generated by $n+1$ elements. Note that we can find an exact sequence

$$0 \to A' \xrightarrow{\iota} A \xrightarrow{\nu} E \to 0$$

of R-modules where A' is generated by n elements and where E is generated by one element. We set $F := \operatorname{coker}(\varphi \circ \iota \colon A' \to B)$ and we set $G := \ker(\overline{\psi} \colon F \to C)$. Next we consider the following diagram



We make the following observations:

(a) It follows easily from the definitions that the four sequences are exact.

(b) A standard diagram chase argument shows that E is isomorphic to G, in particular $\operatorname{ord}_R(E) = \operatorname{ord}_R(G)$.

(c) By induction, and since we dealt with the case of one generator, we obtain the equalities

[3108] This can be seen as follows: Let $d \in R$ with $d \cdot a = 0$. It follows that $(\alpha_1 \ \ldots \ \alpha_s \ d)$ is also a presentation matrix for A. But this implies that $\operatorname{ord}_R(A)$ is a divisor of d.

$$\begin{array}{lrcl} \text{(i)} & \operatorname{ord}_R(A) & = & \operatorname{ord}_R(A') \cdot \operatorname{ord}_R(E) \\ \text{(ii)} & \operatorname{ord}_R(B) & = & \operatorname{ord}_R(A') \cdot \operatorname{ord}_R(F) \\ \text{(iii)} & \operatorname{ord}_R(F) & = & \operatorname{ord}_R(G) \cdot \operatorname{ord}_R(C). \end{array}$$

(d) Note that (c) implies that

$$\begin{aligned} \operatorname{ord}_R(B) \cdot \operatorname{ord}_R(E) &\overset{\text{(ii)}}{=} \operatorname{ord}_R(A') \cdot \operatorname{ord}_R(E) \cdot \operatorname{ord}_R(F) \overset{\text{(i)}}{=} \operatorname{ord}_R(A) \cdot \operatorname{ord}_R(F) \\ &\overset{\text{(iii)}}{=} \operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C) \cdot \operatorname{ord}_R(G). \end{aligned}$$

Now we can conclude the argument:

- If $\operatorname{ord}_R(E) = \operatorname{ord}_R(G = 0$, then it follows from (i) that $\operatorname{ord}_R(A) = 0$, it follows from (iii) that $\operatorname{ord}_R(F) = 0$ and thus it follows from (ii) that $\operatorname{ord}_R(B) = 0$. Thus $\operatorname{ord}_R(B) = 0 = \operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C)$.
- If $\operatorname{ord}_R(E) = \operatorname{ord}_R(G \neq 0$, then we obtain from the equality in (d) together with (b) that $\operatorname{ord}_R(B) = \operatorname{ord}_R(A) \cdot \operatorname{ord}_R(C)$. ■

287.9. Exercises for Chapter 287.

Exercise 287.1. We consider the ring $R = \mathbb{Z}$. Let M be a $\mathbb{Z}$-module with presentation matrix

$$A = \begin{pmatrix} 15 & 0 \\ 9 & 18 \\ 3 & 12 \end{pmatrix}.$$

Use the moves from the Presentation Matrix–Uniqueness Proposition 287.8 to turn this matrix into a diagonal matrix.

Exercise 287.2. Let R be a Noetherian UFD. Does every finitely generated R-module necessarily admit a square presentation matrix?

Exercise 287.3. We consider the homomorphism

$$\begin{array}{rcl} f\colon \mathbb{Z}[x,y]^2 & \to & \mathbb{Z}[x,y] \\ p & \mapsto & p \cdot (x-1) + p \cdot (y-1). \end{array}$$

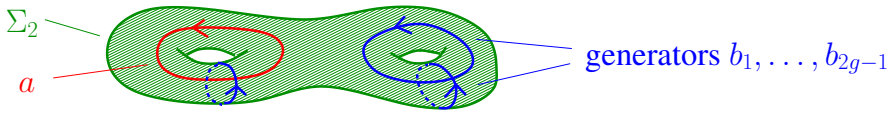
Find a presentation matrix for the $\mathbb{Z}[x,y]$-module $\ker(f)$.

Exercise 287.4. We consider the homomorphism

$$\begin{array}{rcl} f\colon \mathbb{Z}[x] & \to & \mathbb{Z}/2\mathbb{Z} \\ p(x) & \mapsto & [p(1)]. \end{array}$$

Find a presentation matrix for the $\mathbb{Z}[x]$-module $\ker(f)$.

Exercise 287.5. Let $g \in \mathbb{N}_{\geq 2}$. Let $\Phi\colon \mathrm{H}_1(\Sigma_g) \to \langle t \rangle$ be the epimorphism that is given by $a \mapsto t$ and by sending all other generators $b_1, \ldots, b_{2g-1}$ to the trivial element. We consider the Alexander module $\mathrm{H}_1(\Phi; \mathbb{Z}[t^{\pm 1}])$. What can you say about it? Is it a free $\mathbb{Z}[t^{\pm 1}]$-module? If yes, what is the rank?



Exercise 287.6. We introduce the following concept: Let V be a real vector space V and let $X = \{P_1, \dots, P_n\} \subseteq V$ be a finite subset. As on page 188 we consider the convex hull

$$\mathrm{Hull}(X) \;\coloneqq\; \Big\{ \sum_{i=1}^{n} t_i \cdot P_i \,\Big|\, t_1, \dots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \Big\}.$$

Furthermore, given two subset $A, B \subseteq V$ we define its *Minkowski sum*

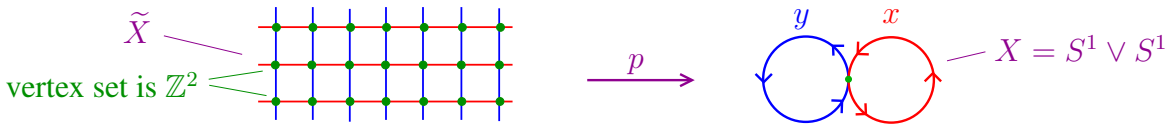
$$A + B \;\coloneqq\; \{a + b \,|\, a \in A \text{ and } b \in B\}.$$

Now let H be a free abelian group and let $p = \sum_{i=1}^{m} a_i \cdot h_i \in \mathbb{Z}[H]$ where $a_1, \dots, a_m \in \mathbb{Z} \setminus \{0\}$. We define the *Newton polytope*

$$N(p) \;\coloneqq\; \mathrm{Hull}(\{h_1 \otimes 1, \dots, h_m \otimes 1\}) \;\subseteq\; H \otimes \mathbb{R}.$$

(a) Show that for $p, q \in \mathbb{Z}[H]$ we have $N(p \cdot q) = N(p) + N(q)$.
(b) Let $G \subseteq H$ be a subsummand, let $p \in \mathbb{Z}[G]$ and let $q \in \mathbb{Z}[H]$. Show that if $q|p$, then there exists an $h \in H$ such that $h \cdot q \in \mathbb{Z}[G]$.
(c) Let $h \in H$ be non-trivial, let $p(h)$ be a polynomial in h such that the highest h^k-coefficient is ± 1 and let $m \in \mathbb{N}$. Show that $\gcd(m, p(h)) \doteq 1$.

Exercise 287.7. As on page 5468 we consider the wedge $S^1 \vee S^1$ with the "obvious" epimorphism $\Phi\colon \mathrm{H}_1(S^1 \vee S^1) \to \langle x, y \rangle_{\mathrm{ab}}$.
(a) Find a resolution for $\mathrm{H}_1^{\Phi}(S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}])$.
(b) Determine $\mathrm{ord}_{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]}(\mathrm{H}_1^{\Phi}(S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]))$.



Exercise 287.8. We consider the wedge $S^1 \vee S^1 \vee S^1$ of *three* circles with the "obvious" epimorphism $\Phi\colon \mathrm{H}_1(S^1 \vee S^1 \vee S^1) \to \langle x, y, z \rangle_{\mathrm{ab}}$.
(a) Find a resolution for $\mathrm{H}_1^{\Phi}(S^1 \vee S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}])$.
(b) Determine $\mathrm{ord}_{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}(\mathrm{H}_1^{\Phi}(S^1 \vee S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]))$.

Remark. This exercise is evidently just a variation on Exercise 287.7. But it turns out that the case of the wedge of three circles is somewhat more difficult than the case of the wedge of two circles. Thus this deserves to be a separate exercise.

Exercise 287.9. Let X be a path-connected topological space and let $\Phi\colon \mathrm{H}_1(X) \to H$ be a homomorphism to an abelian group. Can you always find a homeomorphism $f\colon X \to X$ with $\Phi \circ f_* = -\Phi\colon \mathrm{H}_1(X) \to H$?

Exercise 287.10. Let R be a Noetherian UFD and let M be a finitely generated R-module with $\mathrm{ord}_R(M) \doteq 1$. Does it follow that $M = R$?

Exercise 287.11. Are there examples of a Noetherian UFD R and of two finitely generated R-modules M and N with the following two properties:

- We have $\gcd_R(E_0(M)) \doteq \gcd_R(E_0(N))$.
- We have $\gcd_R(E_1(M)) \not\doteq \gcd_R(E_1(N))$?

Exercise 287.12. Let R be a Noetherian UFD and let $d \in \mathbb{N}_0$. Determine all the elementary ideals $E_k(R^d)$.

Exercise 287.13. Let X be a path-connected topological space and let H be a free abelian group. Is it possible that there exist two epimorphisms $\Phi, \Psi \colon \mathrm{H}_1(X) \to H$ such that $\Delta_1^\Phi(X) = 0$ and $\Delta_1^\Psi(X) \neq 0$?

Exercise 287.14. We consider the 2-dimensional torus $T := S^1 \times S^1$. Given a homeomorphism $f \colon T \to T$ and given $k \in \mathbb{N}_0$ we can consider the Alexander polynomial $\Delta_k^\Phi(\mathrm{Tor}(T,f)) = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_k^\Phi(\mathrm{Tor}(T,f)); \mathbb{Z}[t^{\pm 1}])$.

(a) Find a homeomorphism $f \colon T \to T$ such that $\Delta_1^\Phi(\mathrm{Tor}(T,f)) \doteq 1 - t + t^2$.
(b) Find a homeomorphism $f \colon T \to T$ such that $\Delta_1^\Phi(\mathrm{Tor}(T,f)) \doteq 1 - 3t + t^2$.
(c) Show that if $f \colon T \to T$ is a homeomorphism such that $\mathrm{H}_1(\mathrm{Tor}(T,f)) \cong \mathbb{Z}$, then the Alexander polynomial $\Delta_1^\Phi(\mathrm{Tor}(T,f))$ is one of the two cases considered above.
(d) What are the possible values for $\Delta_2^\Phi(\mathrm{Tor}(T,f))$?

Exercise 287.15. Let $A \in \mathrm{GL}(m,\mathbb{Z})$. We consider $\mathbb{Z}^m$ as a $\mathbb{Z}[t^{\pm 1}]$-module where the t-action is given by multiplication by A. Show that the following sequence of $\mathbb{Z}[t^{\pm 1}]$-module homomorphisms

$$\mathbb{Z}[t^{\pm 1}]^m \xrightarrow{t\cdot \mathrm{id}_m - A} \mathbb{Z}[t^{\pm 1}]^m \xrightarrow{\mathfrak{e}_i \mapsto \mathfrak{e}_i} \mathbb{Z}^m \to 0$$

is exact.

Exercise 287.16. Let R be a commutative ring which is a Noetherian UFD and let M and N be finitely generated R-modules. Show, just using the definitions, that

$$\mathrm{ord}(M \oplus N) \doteq \mathrm{ord}(M) \cdot \mathrm{ord}(N).$$

Exercise 287.17. Let R be a Noetherian ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules.

(a) Show that if X is a presentation matrix for A and if Z is a presentation matrix for C, then there exists a presentation matrix for B of the form

$$\begin{pmatrix} X & * \\ 0 & Z \end{pmatrix}.$$

(b) Provide a short self-contained proof of the Order–Exactness Proposition 287.28 if A and C both admit square presentation matrices.

Exercise 287.18. Let R be a principal ideal domain (PID).

(a) Show that R is Noetherian.
(b) Show that R is a UFD.
(c) Show that for any short exact sequence $0 \to A \to B \to C \to 0$ of finitely generated R-modules we have

$$\mathrm{ord}_R(B) \doteq \mathrm{ord}_R(A) \cdot \mathrm{ord}_R(C).$$

Remark. Evidently you are supposed to give a self-contained proof that does not make use of the Order–Exactness Proposition 287.28.

Remark. You might want to make use of the classification of finitely generated modules over a PID.

CHAPTER 288

Alexander polynomials of knots and links

In this chapter we will apply the general concepts from the previous two chapters to oriented knots and links in S^3. More precisely: Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link with meridians $\mu_{L_1}, \dots, \mu_{L_m}$. We can consider the link complement $S^3 \setminus L$ together with either the epimorphism

$$\begin{array}{rcl} \Phi_L \colon \pi_1(S^3 \setminus L) & \to & \langle t_1, \dots, t_m \rangle_{\text{ab}} \\ \mu_{L_i} & \mapsto & t_i \end{array} \qquad \text{or with the epimorphism} \qquad \begin{array}{rcl} \widehat{\Phi}_L \colon \pi_1(S^3 \setminus L) & \to & \langle t \rangle \\ \mu_{L_i} & \mapsto & t. \end{array}$$

The machinery from the last chapters gives us corresponding Alexander modules and corresponding Alexander polynomials $\Delta_L(t_1, \dots, t_m) \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1}]$. We will calculate these Alexander polynomials for some basic examples. For more complicated situations we will need the notion of a Seifert matrix, which we will develop in the next Chapter 289.

This chapter is organized as follows:

- In Section 288.1 we will introduce the multivariable Alexander modules and polynomials of knots and links.
- In Section 288.2 we will introduce the total linking number of an oriented link $L \subseteq S^3$ with an oriented knot $K \subseteq S^3 \setminus L$.
- In Section 288.3 we will use the total linking number to introduce the single-variable Alexander modules and polynomials of knots and links.
- In Section 288.4 we will introduce the notion of a Seifert surface for an oriented link.
- In Section 288.5 we will use Seifert surfaces to construct explicit infinite-cyclic coverings of links. This construction will be an essential step in the calculation of the single-variable Alexander polynomial of knots and links.
- In Section 288.6 we will provide exercises for Chapter 288.

In the next Chapter 289 we will introduce Seifert matrices and we will see that these can be used to calculate the single-variable Alexander polynomials.

288.1. **Alexander modules of knots and links.** As promised we will now apply the concepts from the last two chapters to oriented knots and links:

Definition 288.1. Let $L = L_1 \sqcup \cdots \sqcup L_m \subseteq S^3$ be an oriented m-component link in S^3. As usual we denote the meridians by $\mu_{L_1}, \dots, \mu_{L_m}$.

(1) It follows from the Link Complement-H_*-Proposition 283.4 that there exists a unique epimorphism

$$\Phi_L \colon \mathrm{H}_1(S^3 \setminus L) \to \langle t_1, \dots, t_m \rangle_{\text{ab}}$$

such that for $i = 1, \dots, m$ we have $\Phi_L([\mu_{L_i}]) = t_i$.

(2) Let $k \in \mathbb{N}_0$. By Definition 286.9 we can consider the corresponding multivariable Alexander module[3109 3110]

$$\mathrm{H}_k(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]) \ := \ \mathrm{H}_k^{\Phi_L}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]).$$

(3) We define the multivariable Alexander polynomial of L as[3111 3112 3113]

$$\Delta_L(t_1, \dots, t_m) \ := \ \Delta_1(S^3 \setminus L, \Phi_L) \ := \ \underset{\substack{\uparrow \\ \text{see Definition 287.22 and} \\ \text{see Definition 287.18}}}{} \mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])).$$

As in Definition 287.18 we see that the Alexander polynomial $\Delta_L(t_1, \dots, t_m)$ is well-defined up to "$\doteq$", i.e. it is well-defined up to a sign and up to multiplication by a monomial in $t_1, \dots, t_m$.

Remark.

(1) We already introduced the (multi-variable) Alexander polynomial of an oriented link in Definition 117.2 and Definition 117.4. At first glance the definitions look quite different. But we will see in the Alexander Polynomials Agree Proposition **??** that the two definitions agree. In this and the following chapters we will only work with Definition 288.1.

(2) The above definition of the Alexander polynomial is the one which is most commonly used in textbooks, see e.g. [**BZH2014**, Definition 8.18], [**Rolf1990**, p. 206] and [**Lic1997a**, p. 55]. □

In the following we will consider the simplest knots and links, namely the unknot, the Hopf link and the trivial link:

Lemma 288.2. (Link Alexander Polynomial–Example Lemma)

(1) For the unknot U we have $\Delta_U(t) = 1$.
(2) For the Hopf link H we have $\Delta_H(t_1, t_2) = 1$.
(3) If $m \geq 2$, then for the trivial m-component link L we have $\Delta_L(t_1, \dots, t_m) = 0$.

Proof.

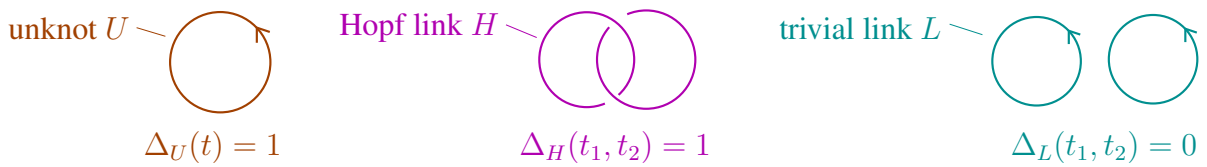
[3109]Note that we have an obvious ring isomorphism from the group ring $\mathbb{Z}[\langle t_1, \dots, t_m \rangle_{\mathrm{ab}}]$ to the Laurent-polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we will identify these two rings. We will discuss this observation in slightly greater detail later on page 5981.

[3110]It is the author's understanding that algebraists prefer the term "multivariate polynomial", but knot theorists (for whatever reason) settled for the term "multivariable Alexander polynomial".

[3111]The only content of this notation is that we drop the Φ_L from the notation, since for an oriented link it is clear which epimorphism to $\langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$ we mean.

[3112]Note that $S^3 \setminus L$ is an open subset of S^3, thus it is path-connected, locally path-connected and semi-locally simply connected. Furthermore, by the Link Exterior Lemma 108.31 (4) we know that $S^3 \setminus L$ is homotopy equivalent to the link exterior, which in turn is a compact smooth manifold. By the Smooth Manifold-CW-Theorem 179.9 this discussion implies that $S^3 \setminus L$ is homotopy equivalent to a finite CW-complex. Thus it makes sense to study the order of the Alexander module.

[3113]In principle one could also consider $\mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]))$, but we will see later on, that this order is not very interesting.



(1) Let $U \subseteq S^3$ be the unknot. By the Trivial Knot–Complement Proposition 111.11 we know that $S^3 \setminus U$ is diffeomorphic to the open solid torus $B^2 \times S^1$. It thus follows that

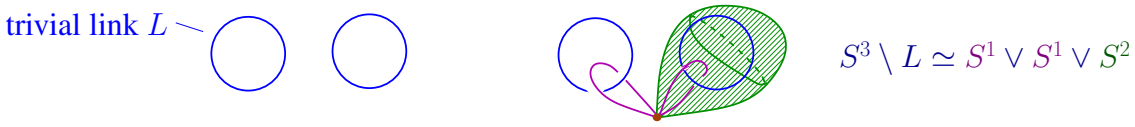
$$\mathrm{H}_1^{\Phi_K}(S^3 \setminus U; \mathbb{Z}[t^{\pm 1}]) = \mathrm{H}_1\left(\begin{array}{c}\text{covering of } B^2 \times S^1 \\ \text{corresponding to } \ker(\Phi_K)\end{array}\right) \cong \mathrm{H}_1(B^2 \times \mathbb{R}) = 0.$$

This of course implies that $\Delta_U(t) = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_1^{\Phi_K}(S^3 \setminus U; \mathbb{Z}[t^{\pm 1}])) \doteq 1$.

(2) Let $H = H_1 \cup H_2 \subseteq S^3$ be the Hopf link. By the Hopf Link-Complement-Proposition 111.23 we know that $S^3 \setminus H$ is diffeomorphic to $S^1 \times S^1 \times (-1, 1)$. The corresponding covering is given by $\mathbb{R}^2 \times (-1, 1)$. Thus, as in (1) it now follows easily from the definitions that $\Delta_H(t_1, t_2) \doteq 1$.

(3) Let $m \in \mathbb{N}$ and let $L = L_1 \sqcup \cdots \sqcup L_m$ be the trivial m-component link. We start out with the following claim:

Claim. The complement $S^3 \setminus L$ is homotopy equivalent to the wedge of m copies of S^1 and $m - 1$ copies of S^2.



Proof. We outsource the proof of this statement to Exercise 288.2. ⊞

In the following we deal with the case $m = 2$. The general case is proved in a very similar fashion. In the case $m = 2$ we have

$$\begin{array}{ccc} & \overset{\text{by Definition 288.1}}{\downarrow} & \overset{\text{by the Alexander Module Homotopy Lemma 286.11}}{\downarrow} \\ \Delta_L(t_1, t_2) & \doteq \mathrm{ord}(\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}])) & \doteq \mathrm{ord}(\mathrm{H}_1(S^1 \vee S^1 \vee S^2; \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}])) \\ & = \mathrm{ord}(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]) & = 0. \\ & \underset{\text{follows easily from the Alexander Module–CW Proposition 286.12 and Exercise 287.3}}{\uparrow} & \underset{\text{see page 5490}}{\uparrow} \end{array}$$

■

The following proposition is an analog of the Knot–Alexander Polynomial Proposition 117.3 and of the Link–Alexander Polynomial Proposition 117.5:

Proposition 288.3. (Link Isotopy–Alexander Polynomial Proposition) Let $m \in \mathbb{N}$. If L and $\widetilde{L}$ are two oriented m-component links in S^3 that are smoothly isotopic, then the multivariable Alexander polynomials of L and $\widetilde{L}$ agree up "$\doteq$".

Proof. Let $m \in \mathbb{N}$. Let L and $\widetilde{L}$ be two oriented m-component links that are smoothly isotopic. It follows from the Smooth Isotopy Characterization Proposition 283.1 that there exists an orientation preserving diffeomorphism $f\colon S^3 \to S^3$ that restricts to orientation-preserving diffeomorphisms $f\colon L_i \to \widetilde{L}_i$. It follows easily from Definition 109.1 that this

map f carries oriented meridians to meridians. Note that this shows that we get a diffeomorphism $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(S^3 \setminus L) & \xrightarrow[\cong]{f_*} & \mathrm{H}_1(S^3 \setminus \widetilde{L}) \\ & {\scriptstyle \Phi_L} \searrow \quad \swarrow {\scriptstyle \Phi_{\widetilde{L}}} & \\ & \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}. & \end{array}$$

We now see that

$$\begin{aligned} \Delta_{\widetilde{L}}(t_1, \dots, t_m) &\overset{\text{by Definition 288.8}}{=} \mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1^{\Phi_{\widetilde{L}}}(S^3 \setminus \widetilde{L}; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) \\ &= \mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1^{\Phi_{\widetilde{L}} \circ f_*}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) \\ &\overset{\text{by the Alexander Module Homeomorphism Lemma 286.11}}{=} \mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1^{\Phi_L}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) \\ &\overset{\text{by the above commutative diagram}}{=} \Delta_L(t_1, \dots, t_m). \end{aligned} \qquad \overset{\text{by Definition 288.8}}{=}$$

■

The following lemma, which is an analog of the Alexander Polynomial–Orientation Lemma 117.6, shows that orientations only have a mild impact on the Alexander polynomial:

Lemma 288.4. (Alexander Polynomial–Orientation Lemma) Let L be an oriented m-component link in S^3. Let $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$. We denote by $L^{(\epsilon_1, \dots, \epsilon_m)}$ the link L where we flipped the orientation of the i-th component precisely if $\epsilon_i = -1$. Then we have the following equality of multivariable Alexander polynomials:

$$\Delta_{L^{(\epsilon_1, \dots, \epsilon_m)}}(t_1, \dots, t_m) \doteq \Delta_L(t_1^{\epsilon_1}, \dots, t_m^{\epsilon_m}).$$

Proof. Let L be an oriented m-component link. Let $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$. We make a few preparations:

(a) We write $L' \coloneqq L^{(\epsilon_1, \dots, \epsilon_m)}$.
(b) We write $H \coloneqq \mathrm{H}_1(S^3 \setminus L) = \mathrm{H}_1(S^3 \setminus L')$.
(c)
- Let $\mu_1, \dots, \mu_m$ be meridians for the oriented link L.
- Let $\mu'_1, \dots, \mu'_m$ be the corresponding meridians for the oriented link L'.

(d) It follows from the Meridian–Symmetries Lemma 109.3 that $\mu'_i = \mu_i$ if $\epsilon_i = 1$ and that $\mu'_i = \mu_i^{\mathrm{rev}}$ if $\epsilon_i = -1$.
(e) As in Definition 288.1 we introduce the following notation:
- We denote by $\Phi_L\colon H \to \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$ the epimorphism given by $\Phi_L([\mu_i]) = t_i$.
- We denote by $\Phi_{L'}\colon H \to \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$ the epimorphism given by $\Phi_{L'}([\mu'_i]) = t_i$.

(f) We denote by $\Theta\colon \langle t_1, \dots, t_m \rangle_{\mathrm{ab}} \to \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$ the isomorphism which is given by $t_i \mapsto t_i^{\epsilon_i}$. Note that it follows from (d) that the following diagram commutes:

$$\begin{array}{ccc} & H & \\ & {\scriptstyle \Phi_L} \swarrow \quad \searrow {\scriptstyle \Phi_{L'}} & \\ \langle t_1, \dots, t_m \rangle_{\mathrm{ab}} & \xrightarrow[t_i \mapsto t_i^{\epsilon_i}]{\Theta} & \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}. \end{array}$$

(g) We denote by Θ_* the ring automorphism of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ that is induced by Θ.

We now see that

$$\begin{array}{rcl}
\Delta_{L'}(t_1,\dots,t_m) & \overset{\text{by Definition 288.8}}{=} & \operatorname{ord}(\mathrm{H}_1^{\Phi_{L'}}(S^3\setminus L;\mathbb{Z}[t_1^{\pm1},\dots,t_m^{\pm1}])) \\
& = & \operatorname{ord}(\mathrm{H}_1^{\Theta\circ\Phi_L}(S^3\setminus L;\mathbb{Z}[t_1^{\pm1},\dots,t_m^{\pm1}])) \\
& \overset{\text{by (f)}}{=} & \Theta_*(\operatorname{ord}(\mathrm{H}_1^{\Phi_L}(S^3\setminus L;\mathbb{Z}[t_1^{\pm1},\dots,t_m^{\pm1}]))) \\
& \overset{\substack{\text{follows from the Group Ring Functor Lemma 97.33}\\ \text{and the Order–Functorial Lemma 287.19}}}{=} & \Theta_*(\Delta_L(t_1,\dots,t_m)) \\
& \overset{\text{by Definition 288.8}}{=} & \Delta_L(t_1^{\epsilon_1},\dots,t_m^{\epsilon_m}).
\end{array}$$

since $\Theta_*(t_i)=t_i^{\epsilon_i}$ ■

In the Link Alexander Polynomial–Example Lemma 288.2 we managed to calculate the multivariable Alexander polynomials of the unknot, of the Hopf link and of trivial links. In each case we used that we have a pretty good understanding of the homotopy type of the corresponding complement. But otherwise there are not so many knots and links for which we have a really good understanding of the complements. This makes calculating multivariable Alexander polynomials a difficult task. Thus in the following we will introduce a single-variable Alexander polynomial that is easier to calculate.

288.2. **The total linking number.** Our next goal is to introduce a single-variable Alexander polynomial for an oriented link. Before we introduce this concept it is convenient to introduce the notion of the total linking number of an oriented link $L\subseteq S^3$ with a knot $J\subseteq S^3\setminus L$:

Definition 288.5. Let $L=L_1\sqcup\cdots\sqcup L_m\subseteq S^3$ be an oriented m-component link in S^3. As usual we denote the meridians by $\mu_{L_1},\dots,\mu_{L_m}$.

(1) It follows from the Link Complement-H_*-Proposition 283.4 that there exists a unique homomorphism
$$\widehat{\Phi}_L\colon \mathrm{H}_1(S^3\setminus L)\ \to\ \mathbb{Z}$$
such that for $i=1,\dots,m$ we have $\widehat{\Phi}_L([\mu_{L_i}])=1$.[3114]

(2) Let $J\subseteq S^3\setminus L$ be an oriented knot. We refer to
$$\operatorname{lk}(L,J)\ =\ \widehat{\Phi}_L([J])$$
as the total linking number of L and J. If L is knot, then this definition equals of course Definition 284.1.

The following lemma justifies the name "total linking number":

Lemma 288.6. (Total Linking Number Lemma) Let $L=L_1\sqcup\cdots\sqcup L_m\subseteq S^3$ be an oriented m-component link in S^3. For any oriented knot $J\subseteq S^3\setminus L$ we have
$$\operatorname{lk}(L,J)\ =\ \operatorname{lk}(L_1,J)+\cdots+\operatorname{lk}(L_m,J).$$

Proof. Let $\mu_{L_1},\dots,\mu_{L_m}$ be the meridians of L. For $j=1,\dots,m$ let $\iota_j\colon S^3\setminus L\to S^3\setminus L_j$ be the inclusion. It follows immediately from Definition 284.1 and Definition 288.5 that it suffices to prove the following claim:

Claim. The following diagram commutes:

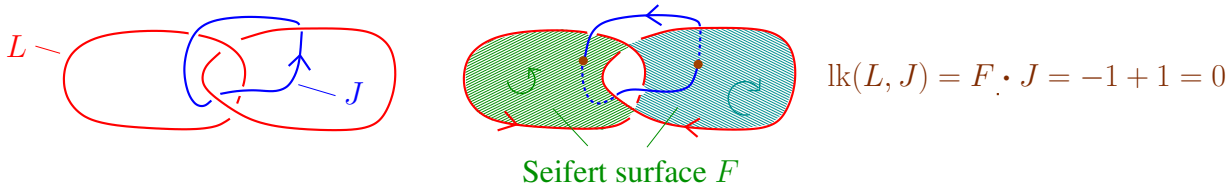
$$\begin{array}{ccc} \mathrm{H}_1(S^3 \setminus L) & \xrightarrow{(\iota_{1*},\dots \iota_{m*})} & \bigoplus_{k=1}^{m} \mathrm{H}_1(S^3 \setminus L_j) \\ & \searrow^{\widehat{\Phi}_L} \quad \swarrow_{\Phi_{L_1}+\cdots+\Phi_{L_m}} & \\ & \mathbb{Z}. & \end{array}$$

Proof. By the Link Complement-H_*–Basis-Proposition 283.4 we know that the meridians generate a basis for $\mathrm{H}_1(S^3 \setminus L)$. Thus it suffices to show that the diagram commutes for each $[\mu_{L_i}]$. This in turn follows immediately from the definitions and the observation that for $i \neq j$ the meridian μ_{L_i} bounds a disk in $S^3 \setminus L_j$, thus $\iota_{j*}([\mu_i]) = 0 \in \mathrm{H}_1(S^3 \setminus L_j)$. ■

The following proposition is a generalization of the Linking #-via-Intersection #-Proposition 285.5. It gives us a geometric approach to calculating the total linking number:

Proposition 288.7. (Total Linking #-via-Intersection #-Proposition) Let $L \subseteq S^3$ be an oriented link and let F be a Seifert surface for L. For any oriented knot $J \subseteq S^3 \setminus L$ that is transverse to F we have the equality

$$\mathrm{lk}(L, J) \;=\; \underbrace{F \cdot_{S^3} J}_{\substack{\text{algebraic} \\ \text{intersection} \\ \text{number}}} .$$



$\mathrm{lk}(L, J) = F \cdot J = -1 + 1 = 0$

Proof. First note that it follows from the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33 that we can pick a thickening $\tau\colon \overline{B}^2 \times L \to S^3 \setminus J$ for L such that $\tau(\overline{B}^2 \times L) \cap F$ is an annulus and such that $\tau(S^1 \times L) \cap F = \tau(\{1\} \times L)$. We introduce the following notation:

(a) We consider the link exterior $X_L := S^3 \setminus \tau(B^2 \times L)$.
(b) We set $F' := F \cap X_L$.
(c) For $i = 1, \dots, m$ we consider the boundary torus $T_i := \tau(S^1 \times L_i)$.
(d) For $i = 1, \dots, m$ we consider the meridian $\mu_{L_i} = \tau(S^1 \times \{x_i\})$ where $x_i \in L_i$.
(e) By the Link Complement-H_*–Basis-Proposition 283.4 we know that $[\mu_{L_1}], \dots, [\mu_{L_m}]$ is a basis for $\mathrm{H}_1(X_L)$.
(f) By (e) there exist $\lambda_1, \dots, \lambda_m \in \mathbb{Z}$ such that $[J] = \lambda_1 \cdot [\mu_{L_1}] + \cdots + \lambda_m \cdot [\mu_{L_m}]$.

Now we calculate that

$$\begin{array}{lll}
\text{by the Intersection Number Symmetry Lemma 220.8} & \text{since } J\subseteq X_L & \text{follows from the Intersection Number via-Cup Product Theorem 220.10, and the discussion on page 4235}\\
F \bullet_{S^3} J \overset{\downarrow}{=} J \bullet_{S^3} F \overset{\downarrow}{=} J \bullet_{X_L} F' & & \overset{\downarrow}{=} \langle \mathrm{PD}_{X_L}([F']),[J]\rangle_{X_L}\\
= \left\langle \mathrm{PD}_{X_L}([F']), \sum_{i=1}^m \lambda_i\cdot[\mu_{L_i}]\right\rangle_{X_L} & & = \sum_{i=1}^m \lambda_i\cdot\langle \mathrm{PD}_{X_L}([F']),[\mu_{L_i}]\rangle_{X_L}\\
\uparrow & & \uparrow\\
\text{follows from (f)} & & \text{since } \langle -,\rangle \text{ is bilinear}\\
= \sum_{i=1}^m \lambda_i\cdot \underbrace{F'\bullet_{X_L}\mu_{L_i}}_{=1} & & = \sum_{i=1}^m \lambda_i = \widehat{\Phi}_{L_i}([J]) = \mathrm{lk}(L,J).\\
\uparrow & & \uparrow\\
\text{same argument as above} & & \text{since } F' \text{ and } \mu_{L_i}=\tau(\{1\}\times L) \text{ intersect positively in the point } \tau(1,x_i)
\end{array}$$

■

288.3. **The single-variable Alexander polynomial.** We will now use the notion of the total linking number to introduce the single-variable Alexander polynomial of an oriented link:

Definition 288.8. Let $m\in\mathbb{N}$ and let $L=L_1\sqcup\cdots\sqcup L_m\subseteq S^3$ be an oriented m-component link in S^3. As usual we denote the meridians by $\mu_{L_1},\dots,\mu_{L_m}$.

(1) As in Definition 288.5 we see that it follows from the Link Complement-H_*-Proposition 283.4 that there exists a unique epimorphism
$$\widehat{\Phi}_L\colon \mathrm{H}_1(S^3\setminus L) \to \langle t\rangle$$
such that for $i=1,\dots,m$ we have $\widehat{\Phi}_L([\mu_{L_i}])=t$.[3115]

(2) Any covering $\widetilde{X}\to S^3\setminus L$ corresponding to $\ker(\widehat{\Phi}_L\colon \mathrm{H}_1(S^3\setminus L)\to\langle t\rangle)$ is called the total linking covering of $S^3\setminus L$.

(3) Let $k\in\mathbb{N}_0$. By Definition 286.9 we can consider the corresponding single-variable Alexander module
$$\mathrm{H}_k(S^3\setminus L;\mathbb{Z}[t^{\pm1}]) := \mathrm{H}_k^{\widehat{\Phi}_L}(S^3\setminus L;\mathbb{Z}[t^{\pm1}]).$$

(4) Using the order from Definition 287.18 we can define the single-variable Alexander polynomial of L as
$$\Delta_L(t) := \mathrm{ord}_{\mathbb{Z}[t^{\pm1}]}(\mathrm{H}_1(S^3\setminus L;\mathbb{Z}[t^{\pm1}])).$$
As in Definition 287.18 we see that the Alexander polynomial $\Delta_L(t)$ is well-defined up to "$\doteq$", i.e. it is well-defined up to multiplication by some $\pm t^k$.

Remark. It is natural to ask, given an oriented m-component link $L\subseteq S^3$, what is the relationship between the multivariable Alexander polynomial $\Delta_L(t_1,\dots,t_m)\in\mathbb{Z}[t_1^{\pm1},\dots,t_m^{\pm1}]$ and the single-variable Alexander polynomial $\Delta_L(t)\in\mathbb{Z}[t^{\pm1}]$. We will return to this question much later in the Multi-Single Variable-Milnor Torsion Proposition **??**. □

The following proposition is the obvious analog of the above Link Isotopy–Alexander Polynomial Proposition 288.3:

[3115]For an oriented link $J\subseteq S^3\setminus L$ th number $\widehat{\Phi}_L([J])\in\mathbb{Z}$ is sometimes called the *total linking number* of L and J.

Proposition 288.9. (Link Isotopy–Alexander Polynomial Proposition) If L and $\widetilde{L}$ are two oriented non-empty links in S^3 that are smoothly isotopic, then the single-variable Alexander polynomials of L and $\widetilde{L}$ agree up "$\doteq$" in $\mathbb{Z}[t^{\pm 1}]$.

Proof. The proof is a slight variation on the proof of the Link Isotopy–Alexander Polynomial Proposition 288.3. ■

Finally the following lemma is an analog of the Alexander Polynomial–Orientation Lemma 288.4:

Lemma 288.10. (Alexander Polynomial–Orientation Lemma) Let L be an oriented non-empty link in S^3. Similar to Definition 108.21 we denote by L^{rev} the link where we flipped the orientations of *all* components. We have the following equality of single-variable Alexander polynomials:

$$\Delta_{L^{\text{rev}}}(t) \doteq \Delta_L(t^{-1}).$$

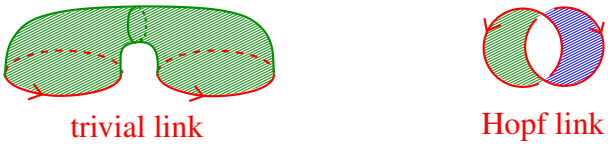
Proof. The proof is a slight variation on the proof of the Alexander Polynomial–Orientation Lemma 288.4. ■

288.4. **The Seifert surface for links.** Our next goal is to find a practical way to calculate the single-variable Alexander polynomials $\Delta_L(t)$ for any oriented link L in S^3. In the next section we will perform a key step towards this goal, namely we will explicitly construct the total linking covering in terms of Seifert surfaces.

For knots we already know what we mean by a connected Seifert surface. The following definition is the extension of Definition 285.1 to links:

Definition 288.11. Let $L \subseteq S^3$ be an oriented link. A Seifert surface for L is a compact oriented 2-dimensional submanifold F of S^3 with no closed components such that $\partial F = L$ as oriented smooth manifolds, in the sense of Convention 31.39.[3116]

Example. In the figure below we show connected Seifert surfaces for the trivial 2-component link and for the Hopf link.



The colors indicate that the surfaces are two-sided, i.e. they are orientable. □

The following theorem is a slight extension of the Seifert Surface Existence Theorem 285.2:

Theorem 288.12. (Seifert Surface Existence Theorem) Every oriented link $L \subseteq S^3$ admits a connected Seifert surface.

As we will see, the proof of the Seifert Surface Existence Theorem 288.12 is mostly the same as the proof of the earlier Seifert Surface Existence Theorem 285.2. But we will need the following extra ingredient, which is perhaps of independent interest:

Lemma 288.13. (Make-Surface-Connected Lemma) Let $G \subseteq S^3$ be a compact orientable 2-dimensional smooth submanifold with no closed components. There exists a

[3116] Note that for a knot the condition "no closed components" is equivalent to "connected". Thus we see that this definition is consistent with Definition 285.1.

compact orientable *connected* 2-dimensional smooth submanifold of S^3 with no closed components with the same boundary as G.

Proof of the Make-Surface-Connected Lemma 288.13. We start out with the following claim:

Claim 1. For any compact orientable 2-dimensional smooth submanifold G of S^3 with no closed components the complement $S^3 \setminus G$ is path-connected.

Proof. In Exercise 288.4 we will sketch an argument that is perhaps less rigorous but arguably more intuitive. We perform the following calculation:

$$\mathrm{H}_0(S^3 \setminus G) \underset{\text{by the Reduced Homology Lemma 147.3 (0)}}{\cong} \mathbb{Z} \oplus \widetilde{\mathrm{H}}_0(S^3 \setminus G) \underset{\text{by the General Alexander Duality Theorem 218.2}}{\cong} \mathbb{Z} \oplus \mathrm{H}^2(G;\mathbb{Z}) \underset{\text{by the Top Cohomology Theorem 201.26, since } G \text{ has no closed components}}{\cong} \mathbb{Z} \oplus 0.$$

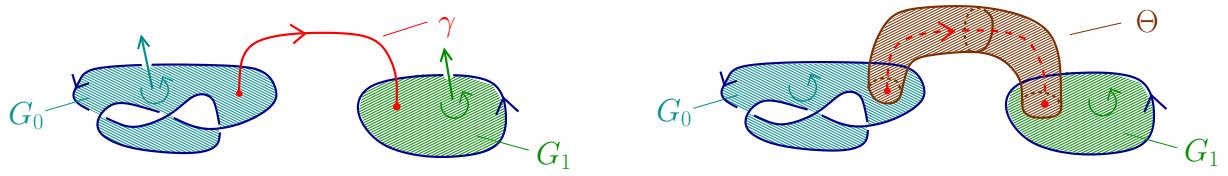
It follows from the H_0-Proposition 143.25 that $S^3 \setminus G$ is indeed connected. ⊞

The lemma now follows from Claim (1) and iteratively applying the following claim:

Claim 2. Let G be a compact orientable 2-dimensional smooth submanifold G of S^3 with no closed components. If G is disconnected, then there exists a compact orientable 2-dimensional smooth submanifold G' of S^3 with no closed components, with $\partial G' = \partial G$ and with fewer components that G.

Proof. [3117] Let G_0, G_1 be two components of G. By Claim 1 we know that $S^3 \setminus G$ is path-connected. Thus we can pick a smooth embedding $\gamma\colon [0,1] \to S^3$ with the following properties:

- The image $\gamma(0,1)$ is disjoint from G.
- We have $\gamma(0) \in G_0$ and $\gamma'(0)$ is a positive normal vector to G_0.
- We have $\gamma(1) \in G_1$ and $\gamma'(1)$ is a negative normal vector to G_1.



It follows from the Product Tubular Neighborhood–Existence Theorem 58.4 that we can thicken γ to a smooth embedding $\Theta\colon \overline{B}^2 \times [0,1] \to S^3$ with the following properties:

- We have $\Theta(0,t) = \gamma(t)$ for all $t \in [0,1]$ and the image $\Theta(\overline{B}^2 \times (0,1))$ is disjoint from G.
- The map Θ restricts to an orientation-preserving smooth embedding $\Theta_0\colon \overline{B}^2 \to G_0$.
- The map Θ restricts to an orientation-preserving smooth embedding $\Theta_1\colon \overline{B}^2 \to G_1$.

We now set $$G' := (G \setminus (\Theta_0(B^2) \cup \Theta_1(B^2)) \cup \Theta(S^1 \times [0,1]).$$

After "rounding corners" one sees that G' has all the desired properties. We leave it to the reader to turn this discussion into lean code. ■

[3117] The following argument is very similar to the Internal Connected Sum Construction Construction 187.28.

Proof of the Seifert Surface Existence Theorem 288.12. Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented link. By the Make-Surface-Connected Lemma 288.13 it remains to prove the following claim:

Claim. There exists a compact orientable 2-dimensional smooth submanifold F of S^3 with $\partial F = L$ and such that F has no closed components.

Proof. For each component L_i we apply the Knot Thickening–Self Linking Lemma 284.11 (2) to find a thickening $\tau_i \colon \overline{B}^2 \times L_i \to S^3$ with disjoint images.

The argument in the proof of the Seifert Surface Existence Theorem 285.2 then gives us a smooth map $f \colon S^3 \setminus L \to S^1$ such that for a regular value $z \in S^1$ the preimage $f^{-1}(z)$ together with L is a compact orientable 2-dimensional smooth submanifold F of S^3 with $\partial F = L$. After throwing out all closed components we have obtained the desired submanifold. ■

288.5. **The infinite-cyclic covering of a link complement.** Our goal is to find a practical way to calculate the single-variable Alexander polynomials $\Delta_L(t)$ for any oriented link L in S^3. By definition $\Delta_L(t)$ is defined via the first homology of the infinite-cyclic covering of $S^3 \setminus L$ corresponding to $\ker(\widehat{\Phi}_L \colon \mathrm{H}_1(S^3 \setminus L) \to \langle t \rangle)$. In this section we will give an explicit construction of these coverings. In the next Chapter 289 we will see how one can calculate the corresponding $\mathbb{Z}[t^{\pm 1}]$-homology modules via "Seifert matrices".

The construction of the total linking covering will require some preparations. First of all in this section we will use the following notation, which we already used in the proof of the Genus–Alexander Polynomial Proposition 123.2:

Notation 288.14. Given a Seifert surface F for a link L we refer to $F' := F \setminus L$ as an open Seifert surface for L.

Lemma 288.15. (Open Seifert Surface Lemma) Let L be an oriented non-empty link and let F be a Seifert surface for L.

(1) The corresponding open Seifert surface F' is a proper submanifold of $S^3 \setminus L$.
(2) The inclusion map $F' \to F$ is a homotopy equivalence.

Proof.

(1) Since $F \subseteq S^3$ is compact, it is a closed subset, and thus $F' \subseteq S^3 \setminus L$ is closed by definition of the subspace topology. Since neither F' nor $S^3 \setminus L$ has non-empty boundary we see that F' is indeed a proper submanifold of $S^3 \setminus L$ in the sense of Definition 23.18.
(2) Using a collar $\chi \colon \partial F \times [0,1] \to F$ of ∂F one can show that F and F' have a common deformation retract, namely $F \setminus \chi(F \times [0,1))$. The details of this argument are worked out in the proof of the Push-Away-from-Boundary Corollary 35.8. ■

Next we recall Definition 55.1:

Definition 288.16. Let M be a smooth manifold and let $\Sigma \subseteq M$ be a proper codimension-one submanifold of M with empty boundary. A bicollar for Σ is a smooth embedding[3118] $\beta \colon F \times [-1,1] \to M \setminus \partial M$ that has the following two properties:

(a) The restriction of β to $F \times \{0\}$ is the obvious map $(P,0) \mapsto P$.
(b) The image $\beta(F \times [-1,1])$ is a closed subset of M.



Next we recall the following theorem:

Theorem 55.2. (Bicollar Neighborhood Theorem) Let M be an oriented smooth manifold and let $\Sigma \subseteq M$ be a proper oriented codimension-one submanifold of M with empty boundary. There exists an orientation-preserving bicollar $[-1,1] \times \Sigma \to M$.

The following lemma will give us the key idea for constructing the total linking covering of a link complement $S^3 \setminus L$:

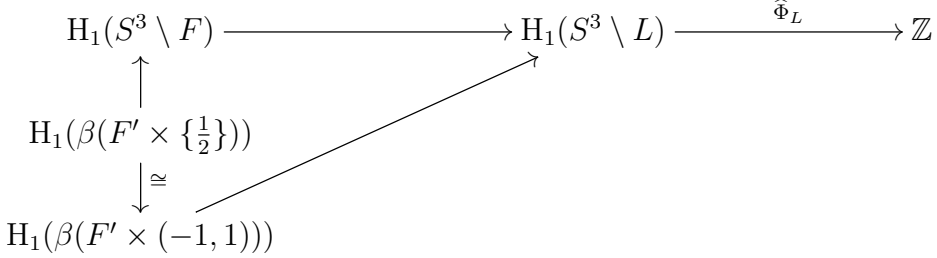
Lemma 288.17. (Total Linking Number-Vanishing Lemma) Let $L \subseteq S^3$ be an oriented non-empty link and let F be a Seifert surface for L. We write $F' := F \setminus L$. Furthermore let $\beta\colon F' \times [-1,1] \to S^3 \setminus L$ be an orientation-preserving bicollar for $F' \subseteq S^3 \setminus L$. The following two maps are both the zero map:

$$\text{(a)} \qquad \mathrm{H}_1(S^3 \setminus F) \to \mathrm{H}_1(S^3 \setminus L) \xrightarrow{\widehat{\Phi}_L} \mathbb{Z}$$

$$\text{(b)} \qquad \mathrm{H}_1(\beta(F' \times (-1,1))) \to \mathrm{H}_1(S^3 \setminus L) \xrightarrow{\widehat{\Phi}_L} \mathbb{Z}.$$

Here in both cases the horizontal map to the left is induced by the obvious two inclusions.

Proof. We consider the following diagram:



The lemma follows from the following clarifications and observations:

- All maps with no names are induced by inclusions.
- The diagram commutes by the functoriality of homology.
- It follows from the Represent-by-1-Manifold Corollary 187.19 that any class in $\mathrm{H}_1(S^3 \setminus F)$ can be represented by an oriented knot $J \subseteq S^3 \setminus F$. It now follows from the Total Linking #-via-Intersection #-Proposition 288.7 that the composition of the two horizontal maps on top is zero.
- Since the inclusion $F' \times \{\frac{1}{2}\} \to F' \times (-1,1)$ is a homotopy equivalence we obtain from the Homotopy Equivalence-H_*-Proposition 144.14 that the bottom vertical map is an isomorphism. ■

[3118] In the original definition we wrote $[-1,1] \times F$ instead of $F \times [-1,1]$. Of course this difference is basically immaterial. In this chapter, for psychological reasons, it is better to adopt the latter point of view.

Note that the Total Linking Number-Vanishing Lemma 288.17 gives us a neat idea for constructing explicitly the total linking covering of $S^3 \setminus L$ corresponding to the kernel of $\widehat{\Phi}_L \colon \mathrm{H}_1(S^3 \setminus L) \to \mathbb{Z}$:

(a) We pick a Seifert surface F.
(b) The link complement $S^3 \setminus L$ admits an open covering by $S^3 \setminus F$ and $\beta(F' \times (-1,1))$.
(c) If we have a covering $p \colon \widetilde{X} \to S^3 \setminus L$ corresponding to the kernel of $\widehat{\Phi}_L \colon \mathrm{H}_1(S^3 \setminus L) \to \mathbb{Z}$, then it follows from the Linking Number-Vanishing Lemma 288.17 and the Map Lifting Criterion 127.5 that we get "$\mathbb{Z}$-many" lifts

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\mathbb{Z}\text{-lifts}}{\nearrow} & \big\downarrow p \\ S^3 \setminus F & \hookrightarrow & S^3 \setminus L \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & \widetilde{X} \\ & \overset{\mathbb{Z}\text{-lifts}}{\nearrow} & \big\downarrow p \\ F' \times (-1,1) & \underset{\beta}{\hookrightarrow} & S^3 \setminus L. \end{array}$$

(d) Conversely this suggests that we can reconstruct $\widetilde{X}$ by considering $\mathbb{Z}$-copies of $S^3 \setminus F$ and $\mathbb{Z}$-copies of $F' \times (-1,1)$ and by gluing appropriately.

The following proposition turns the above idea into mathematical reality. This gives us an explicit construction of the total linking covering of the complement of an oriented link in S^3.[3119]

Proposition 288.18. (Link–Infinite Cyclic Covering Proposition) Let $L \subseteq S^3$ be an oriented non-empty link and let F be a Seifert surface for L. We write $F' := F \setminus L$. Furthermore let $\beta \colon F' \times [-1,1] \to S^3 \setminus L$ be an orientation-preserving bicollar for the proper submanifold $F' \subseteq S^3 \setminus L$.[3120] We consider

$$\begin{array}{rrcll} & \widetilde{X} & := & ((F' \times (-1,1)) \times \mathbb{Z}) \sqcup ((S^3 \setminus F) \times \mathbb{Z}) / \sim & \\ \text{where} & ((x,t),k) & \sim & (\beta(x,t),k) & \text{for all } ((x,t),k) \in (F' \times (-1,0)) \times \mathbb{Z}, \\ & ((x,t),k) & \sim & (\beta(x,t),k+1) & \text{for all } ((x,t),k) \in (F' \times (0,1)) \times \mathbb{Z}. \end{array}$$

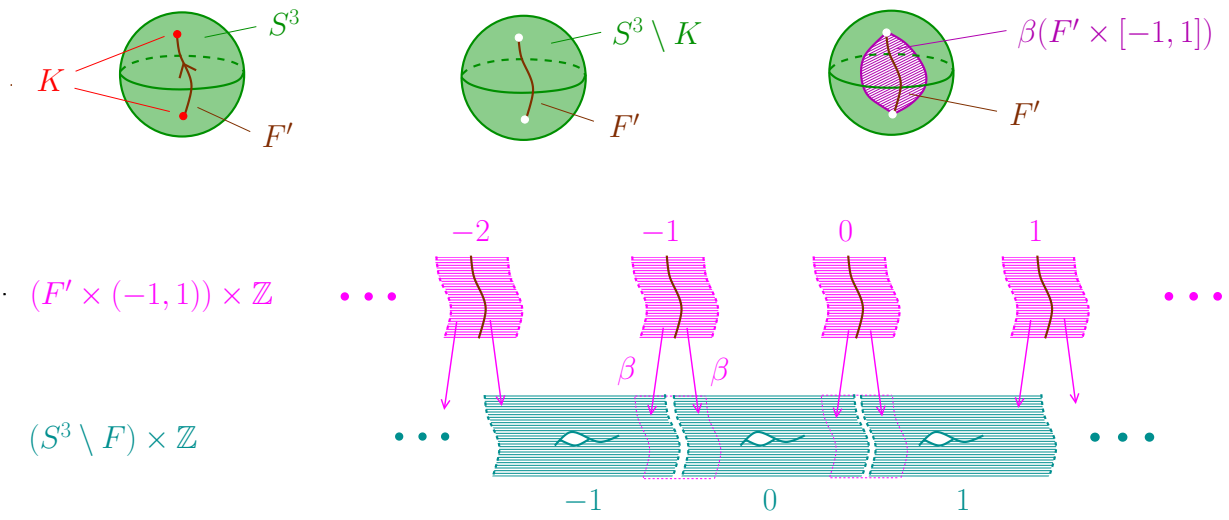
The maps

$$\begin{array}{ccc} (S^3 \setminus F) \times \mathbb{Z} & \to & S^3 \setminus L \\ (P,k) & \mapsto & P \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (F' \times (-1,1)) \times \mathbb{Z} & \to & S^3 \setminus L \\ ((P,t),k) & \mapsto & \beta(P,t) \end{array}$$

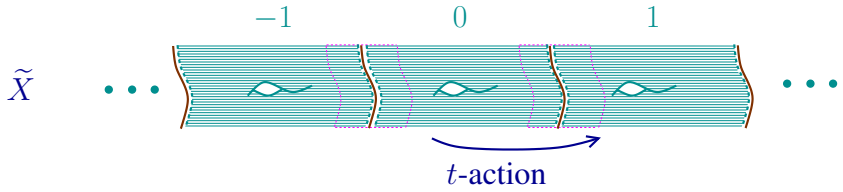
define a continuous map $p \colon \widetilde{X} \to S^3 \setminus L$ which is the total linking covering corresponding to $\ker(\widehat{\Phi}_L \colon \mathrm{H}_1(S^3 \setminus L) \to \langle t \rangle)$ such that the action by t^k from Convention 286.3 is given by the k-translation on the $\mathbb{Z}$-factor.

Illustration. In the figure below we show a dimensionally reduced picture of an oriented knot $K \subseteq S^3$ together with a Seifert surface F for K and an orientation-preserving bicollar $\beta \colon F \times [-1,1] \to S^3$.
In the following figure we show $\mathbb{Z}$-many copies of $F' \times (-1,1)$ and $\mathbb{Z}$-many copies of $S^3 \setminus F$, where the k-th copy of $F' \times (-1,0)$ gets glued via β to the k-th copy of $S^3 \setminus F$ "to the left" and the k-th copy of $F' \times (0,1)$ gets glued via β to the $(k+1)$-st copy of $S^3 \setminus F$ "to the right".

[3119] Note that the proposition is really just a variation on the Cyclic-Covering Proposition 234.21, which dealt with a very similar problem.
[3120] Note that it follows from the Open Seifert Surface Lemma 288.15 (1) and the Bicollar Neighborhood Theorem 55.2 that $F' \subseteq S^3 \setminus L$ admits such a bicollar.



The resulting topological space $\widetilde{X}$ is shown in the figure below:



In the figure above we also illustrate the fact that the t-action is given by shifting "everything to the right by one". □

Proof. Let $L \subseteq S^3$ be an oriented non-empty link and let F be a Seifert surface for L. We write $F' \coloneqq F \setminus L$. Let $\beta\colon F' \times [-1,1] \to S^3 \setminus L$ be an orientation-preserving bicollar. We consider

$$\widetilde{X} \coloneqq ((F' \times (-1,1)) \times \mathbb{Z}) \sqcup ((S^3 \setminus F) \times \mathbb{Z})/\!\sim$$

where

$$\begin{aligned} ((x,t),k) &\sim (\beta(x,t),k) && \text{for all } ((x,t),k) \in (F' \times (-1,0)) \times \mathbb{Z}, \\ ((x,t),k) &\sim (\beta(x,t),k+1) && \text{for all } ((x,t),k) \in (F' \times (0,1)) \times \mathbb{Z}. \end{aligned}$$

We start out with the following claim:

Claim 1. The images of $(F' \times (-1,1)) \times \{k\}$ and $(S^3 \setminus F) \times \{k\}$ in $\widetilde{X}$ are open subsets.

Proof. This statement follows easily from the Codimension-Zero Smooth Embedding Proposition 30.21 (1), the Topological-Quotient Proposition 8.2 (0a) and the Topological-Coproduct Proposition 7.26 (0a). ⊞

Claim 2. The map $p\colon \widetilde{X} \to S^3 \setminus L$ is a covering.

Proof. First note that it follows easily from the Pasting Proposition 4.7 that the map $p\colon \widetilde{X} \to S^3 \setminus L$ is continuous. Next note that $S^3 \setminus L = \beta(F' \times (-1,1)) \cup (S^3 \setminus F)$. Thus we see that the map p is surjective. Finally note that it is elementary to verify that the two open subsets $\beta(F' \times (-1,1))$ and $P \in S^3 \setminus F$ of $S^3 \setminus L$ are uniformly covered in the sense of Definition 91.8. Since these two open sets form an open cover of $S^3 \setminus L$ we see that p is a covering. ⊞

Claim 3. We have $p_*(\mathrm{H}_1(\widetilde{X}_L)) \subseteq \ker(\widehat{\Phi}_L)$.

Proof. Note that it follows easily from the Mayer–Vietoris Theorem 153.2 and the Path-Component–H_*–Lemma 143.21 (which we can apply by Claim 1) that we have an exact sequence of the following form:

$$\begin{aligned} &\to \bigoplus_{k\in\mathbb{Z}} \mathrm{H}_1(F'\times(-1,1)) \oplus \bigoplus_{k\in\mathbb{Z}} \mathrm{H}_1(S^3\setminus F) \to \mathrm{H}_1(\widetilde{X}) \\ \xrightarrow{\partial_1} &\bigoplus_{k\in\mathbb{Z}} \mathrm{H}_0(F'\times((-1,0)\cup(0,1))) \to \bigoplus_{k\in\mathbb{Z}} \mathrm{H}_0(F'\times(-1,1)) \oplus \bigoplus_{k\in\mathbb{Z}} \mathrm{H}_0(S^3\setminus F) \to \dots \end{aligned}$$

Using the fact that F' is connected and using the H_0-Proposition 143.25 one can easily verify that the bottom left map at the H_0-level is a monomorphism. This implies that the connecting homomorphism ∂_1 is the zero map. Thus we see that $\mathrm{H}_1(\widetilde{X})$ is generated by the homology groups of the copies of $S^3\setminus F$ and $\beta(F'\times(-1,1))$. But it follows from the Total Linking Number-Vanishing Lemma 288.17 that the maps $\Phi_L\circ p_*\colon \mathrm{H}_1((F'\times(-1,1))\times\{k\}) \to \mathbb{Z}$ and $\Phi_L\circ p_*\colon \mathrm{H}_1((S^3\setminus F)\times\{k\}) \to \mathbb{Z}$ are both zero. ⊞

Claim 4. We have $p_*(\mathrm{H}_1(\widetilde{X}_L)) = \ker(\widehat{\Phi}_L)$.

Proof. First note that there is an obvious $\mathbb{Z}$-action on $\widetilde{X}$ such that p descends to a homeomorphism $\widetilde{X}/\mathbb{Z} \to S^3\setminus L$. We now consider the following homomorphisms:

$$\mathbb{Z} \xrightarrow{\cong} \pi_1(X/\mathbb{Z})/p_*(\pi_1(\widetilde{X})) \xrightarrow{\cong} \pi_1(S^3\setminus L)/p_*(\pi_1(\widetilde{X})) \xrightarrow{\widehat{\Phi}_L} \mathbb{Z}.$$

We make the following clarifications and observations:

(1) The isomorphism to the left is provided by the Action–π_1-Theorem 93.1.
(2) The second isomorphism is provided by the above homeomorphism $\widetilde{X}/\mathbb{Z} \to S^3\setminus L$, which is induced by the projection $p\colon \widetilde{X} \to S^3\setminus L$.
(3) It follows from the above Claim 3 and the Hurewicz Theorem 166.10 that the epimorphism $\widehat{\Phi}_L\colon \pi_1(S^3\setminus L) \to \mathbb{Z}$ descends to an epimorphism $\widehat{\Phi}_L\colon \pi_1(S^3\setminus L)/p_*(\pi_1(\widetilde{X})) \to \mathbb{Z}$.
(4) It follows from (1), (2) and the Epi-Automorphism Lemma 97.27 that the induced epimorphism $\widehat{\Phi}_L\colon \pi_1(X)/p_*(\pi_1(\widetilde{X})) \to \mathbb{Z}$ is in fact an isomorphism. But this implies directly that $\pi_1(\widetilde{X}) = \ker(\widehat{\Phi}_L)$. ⊞

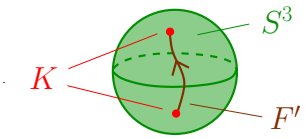
It remains to prove the following claim:

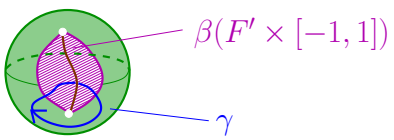
Claim 5. The action by t^k on $\widetilde{X}$ from Convention 286.3 is given by the k-translation on the $\mathbb{Z}$-factor.

Proof. Let $P \in F'$. We consider any loop $\gamma\colon [0,1] \to S^3\setminus L$ that is given by the concatenation of the following three paths:

- the path $t \mapsto \beta(P, 2t)$,
- any path in $S^3\setminus F$ from $\beta(P,1)$ to $\beta(P,-1)$ and
- the path $t \mapsto \beta(P,(2t-1))$.

Note that $\widehat{\Phi}_L([\gamma]) = F \cdot_{S^3\setminus L} \gamma = 1$.





Furthermore note that $\gamma\colon [0,1] \to S^3 \setminus L$ admits an obvious lift to a path $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ with $\widetilde{\gamma}(0) = [(P,0)]$ and $\widetilde{\gamma}(1) = [(P,1)]$. This shows that the t-action from Convention 286.3 is given translation by $+1$ in the $\mathbb{Z}$-factor. The case $k \in \mathbb{Z}$ follows immediately from this observation. ■

Let $L \subseteq S^3$ be an oriented non-empty link and let F be a Seifert surface for L. In the Link–Infinite Cyclic Covering Proposition 288.18 we gave an explicit description of the total linking number covering $\widetilde{X} \to S^3 \setminus L$ in terms of $\mathbb{Z}$-copies of $S^3 \setminus F$ and $\mathbb{Z}$-copies of $F' \times (-1,1)$. Furthermore, in the proof of the Link–Infinite Cyclic Covering Proposition 288.18 we saw that we can calculate the homology groups of $\widetilde{X}$ in terms of the following three pieces of data:

(a) The homology groups of F.
(b) The homology groups of $S^3 \setminus F$ (which we can obtain from the General Alexander Duality Theorem 218.2).
(c) The inclusion induced maps $\mathrm{H}_1(\beta(F' \times (-1,0))) \to \mathrm{H}_1(S^3 \setminus F)$ and also the inclusion induced map $\mathrm{H}_1(\beta(F' \times (0,1))) \to \mathrm{H}_1(S^3 \setminus F)$.

Determining the last piece of data p poses a serious problem that we cannot solve in this chapter. We will deal with this issue in the next Chapter 289, once we have introduced Seifert forms and Seifert matrices.

288.6. Exercises for Chapter 288.

Exercise 288.1. Let L be an oriented m-component link in S^3. We denote by L^{mir} a mirror of L. What is the relationship between the multivariable Alexander polynomials of L and L^{mir}?
Remark. This exercise is an analog of the Alexander Polynomial–Mirror Lemma 117.15.

Exercise 288.2. Let $m \in \mathbb{N}$ and let L be a trivial m-component link in S^3. Show that $S^3 \setminus L$ is homotopy equivalent to the wedge of m copies of S^1 and $m-1$ copies of S^2.

Exercise 288.3. Let $m \in \mathbb{N}$ and let L be a trivial m-component link in S^3. What is the lowest $k \in \mathbb{N}_0$ such that the k-th elementary ideal of the multi-variable Alexander module $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$ is non-zero?
Remark. You might want to build on Exercise 288.2.

Exercise 288.4. Let $G \subseteq S^n$ be a compact orientable $(n-1)$-dimensional smooth submanifold with no closed components. We want to show that the complement $S^n \setminus G$ is connected without making use of Alexander duality. You can proceed as follows:

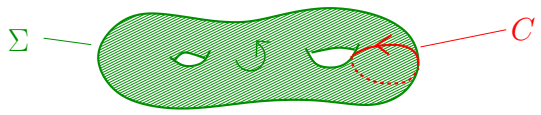
- Let $P, Q \in S^n \setminus G$.
- Pick a smoothly embedded path γ from P to Q that is transverse to G.
- Modify γ to make it disjoint from G.

Exercise 288.5. Calculate the single-variable Alexander polynomial for the Hopf link and the trivial 2-component link.

Exercise 288.6. We consider the oriented surface Σ of genus two together with the closed oriented submanifold C that are shown below. We consider the covering $p\colon \widetilde{\Sigma} \to \Sigma$ corresponding to $\mathrm{PD}_\Sigma([C]) \in \mathrm{H}^1(\Sigma;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\mathrm{H}_1(\Sigma),\mathbb{Z})$. Construct an explicit model of such

a covering.

Hint. Imitate the Link–Infinite Cyclic Covering Proposition 288.18.



CHAPTER 289

Seifert matrices and Alexander modules

Given an oriented link $L \subseteq S^3$ we introduced in the previous Chapter 288 the total linking covering $\widetilde{X} \to S^3 \setminus L$, which in turn we used to define the single-variable Alexander module $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$ and the single-variable Alexander polynomial $\Delta_L(t)$. Furthermore, in Chapter 288 we used connected Seifert surfaces to give an explicit construction of the total linking covering $\widetilde{X} \to S^3 \setminus L$. In this chapter we will calculate the homology groups of $\widetilde{X}$ in terms of "Seifert matrices". In other words, we will calculate the single-variable Alexander module $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$ and thus also the single-variable Alexander polynomial $\Delta_L(t)$ in terms of "Seifert matrices".

This chapter is organized as follows:

- In Section 289.1 we will introduce the concept of a push-off.
- In Section 289.2 we will fit the single-variable Alexander module of an oriented link into a long exact sequence of homology groups.
- In Section 289.3 we will introduce a generalized linking form.
- In Section 289.4 we will introduce the Seifert form and Seifert matrices.
- In Section 289.5 we will express the Alexander modules in terms of Seifert matrices.
- In Section 289.6 we will use Seifert surfaces and Seifert matrices to prove two theoretical statements about single-variable Alexander polynomials.
- In Section 289.7 we will express the homology groups of finite cyclic coverings of knot complements in terms of Seifert matrices.
- In Section 289.8 we will provide exercises for Chapter 289.

In the next Chapter 290 we will use the results from this chapter to calculate the Alexander polynomial of connected sums and to relate the Alexander polynomial to the knot genus and to fiberedness.

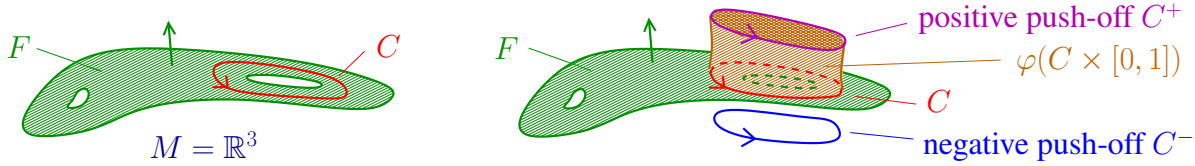
289.1. **Push-offs.** We start out with the following very geometric definition:

Definition 289.1. Let M be an oriented 3-dimensional smooth manifold with $\partial M = \varnothing$ and let $F \subseteq M$ be a compact oriented submanifold. Furthermore let $C \subseteq F \setminus \partial F$ be a closed oriented 1-dimensional smooth submanifold.

(1) We say that a closed oriented 1-dimensional smooth submanifold $C^+ \subseteq M \setminus F$ is a positive push-off of C if there exists a smooth embedding $\varphi\colon C \times [0,1] \to M$ with the following four properties:
   (a) We have $\varphi_0 = \mathrm{id}_C$.
   (b) The map $\varphi_1\colon C \to C^+$ is an orientation-preserving diffeomorphism.
   (c) $\varphi(C \times (0,1]) \cap F = \varnothing$.

(d) For each point $P \in C$ a positive basis of $\mathrm{T}_P F$ followed by the vector $\mathrm{D}\varphi_{(P,0)}(0,\mathfrak{e})$ is a positive basis of $\mathrm{T}_P M$.

(2) A negative push-off is defined the same way, except that we need to replace "positive basis" by "negative basis".



The following lemma shows that push-offs exist and that they are essentially unique:

Lemma 289.2. (Loop Push-Off Lemma) Let M be an oriented 3-dimensional smooth manifold with $\partial M = \varnothing$ and let $F \subseteq M$ be a compact oriented 2-dimensional submanifold. Furthermore let $C \subseteq F' := F \setminus \partial F$ be a closed oriented 1-dimensional smooth submanifold.

(1) (a) Let $\beta\colon F' \times [-1,1] \to S^3 \setminus \partial F$ be an orientation-preserving bicollar of the submanifold $F' = F \setminus \partial F$ of $S^3 \setminus \partial F$.[3121] Then $\beta(C \times \{1\})$ is a positive push-off of C.

(b) C admits a positive push-off.

(2) Any two positive push-offs of C are smoothly isotopic in $M \setminus F$.

The analogous statements of course also hold for "negative" instead of "positive".

Sketch of proof.

(1) (a) Let $\beta\colon F' \times [-1,1] \to S^3 \setminus \partial F$ be an orientation-preserving bicollar. It is essentially clear that the map $\varphi := \beta|_{C\times[0,1]}$ has the properties to witness that the closed oriented 1-dimensional smooth submanifold $\beta(C,1)$ is a positive push-off of C.

(b) This statement follows from (a) together with the Bicollar Neighborhood Theorem 55.2 applied to $F' \subseteq S^3 \setminus \partial F$.

(2) We start out with a few preparations:[3122]

- Let $\beta\colon F' \times [-1,1] \to S^3 \setminus \partial F$ as in (1) be an orientation-preserving bicollar. Furthermore let C^+ be a positive push-off of C. It follows from (1a) and the Smooth Isotopy-Equivalence Relation Proposition 41.5 that it suffices to show that $\beta(C \times \{1\})$ and C^+ are smoothly isotopic in $M \setminus F$.
- We pick a smooth embedding $\varphi\colon C \times [0,1] \to M$ as in Definition 289.1 such that $\varphi(\{1\} \times C) = C^+$.
- Since F is orientable we obtain from the Product Tubular Neighborhood–Existence Theorem 58.4 (2) that we can pick a tubular map $\tau\colon C \times [-1,1] \to F$.
- We consider the projection

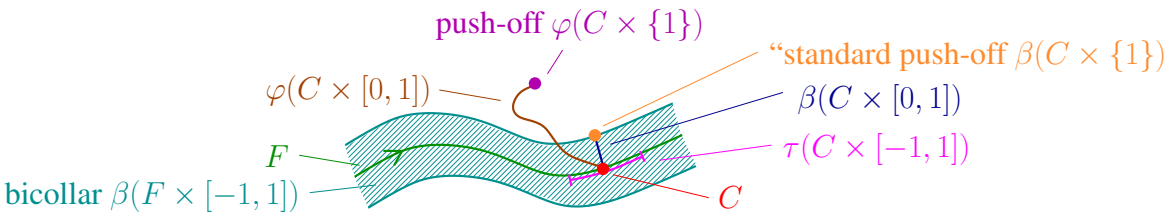
$$\begin{array}{rcl} \pi\colon \beta(\tau(C \times [-1,1]) \times [0,1]) & \to & \beta(C \times [0,1]) \\ \beta(\tau(c,s),t) & \mapsto & \beta(c,t). \end{array}$$

We now prove the desired statement in several steps:

(i) • As we pointed out in Footnote 606, it follows easily from the Codimension-Zero Smooth Embedding Proposition 30.21 (1) that $\beta(\tau(C \times (-1,1)) \times (-1,1))$ is an

[3121] We equip $F' \times [-1,1]$ with the product orientation from the Product Orientation Proposition 31.48.

[3122] The logic of the uniqueness statement is similar to the logic of the proof of the Smooth Collar-Uniqueness Proposition 54.2.



open subset of M that contains $C = \varphi_0(C)$. Since $C \subseteq F'$ is compact it follows from the Tube Lemma 7.12 that there exists an $\epsilon > 0$ such that we have the inclusion $\varphi(C \times [0,\epsilon]) \subseteq \beta(\tau(C \times (-1,1)) \times (-1,1))$. After a smooth isotopy we might as well assume that $\epsilon = 1$.

- It follows fairly easily from the hypothesis that $\varphi(C \times \{1\})$ is a positive-push off that $\varphi(C \times [0,1])$ is actually contained in $\beta(\tau(C \times (-1,1)) \times [0,1))$.

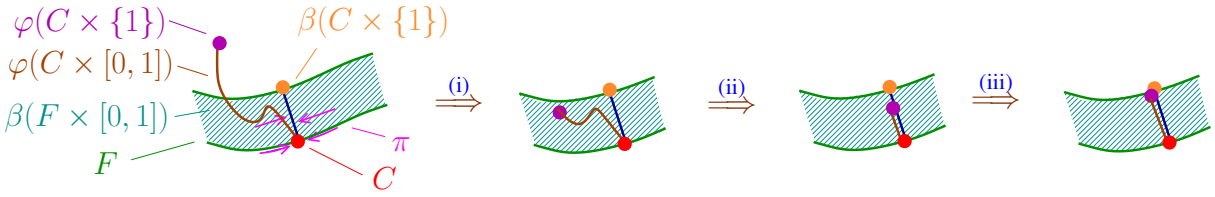
(ii)
- By (i) we can consider the map

$$\begin{array}{rcl} \pi\colon \varphi(C \times [0,1]) & \to & \beta(C \times [0,1]) \\ \varphi(c,t) & \mapsto & \pi(\varphi(c,t)). \end{array}$$

It follows easily from the choice of the map φ that for each $c \in C$ we have $\pi(c) = \pi(\varphi(c,0)) = c$ and that the differential of π at $(c,0) \in C \times [0,1]$ is an isomorphism. It now follows from the Inverse Mapping Theorem 30.1 and the Neighborhood Smooth Embedding Theorem 27.16 that a restriction of π to some $\varphi(C \times [0,\epsilon])$ is a smooth embedding. After rescaling we can assume that $\pi\colon \varphi(C \times [0,1]) \to \beta(C \times [0,1])$ is a smooth embedding.

- After a smooth isotopy (along the fibers of π) we can in fact assume that $\varphi(C \times [0,1]) \subseteq \beta(C \times [0,1])$.

(iii) After rescaling we can in fact assume that $\underbrace{\varphi(C \times \{1\})}_{=C^+} = \beta(C \times \{1\})$.



All these modifications correspond to smooth isotopies. Thus we have turned the push-off $\varphi(C \times \{1\})$ into the "standard push-off" $\beta(C \times \{1\})$.

The proof for "negative push-off" instead of "positive push-off" is evidently basically identical. ■

The following lemma gives us a "homological push-off", which does not depend on any choices:

Lemma 289.3. (Homology Push-Off Lemma) Let M be an oriented 3-dimensional smooth manifold with $\partial M = \varnothing$ and let $F \subseteq M$ be a compact oriented 2-dimensional

submanifold. The positive pushing-positive homomorphism

$$\begin{array}{rcl} \iota_+\colon \mathrm{H}_1(F) & \to & \mathrm{H}_1(S^3 \setminus F) \\ \left[\begin{array}{c}\text{closed oriented 1-dimensional}\\ \text{smooth submanifold } C \text{ of } F \setminus \partial F\end{array}\right] & \mapsto & [\underbrace{\text{positive push-off } C^+ \text{ of } C}_{\text{see Definition 289.1}}] \end{array}$$

is well-defined. Using "negative push-offs" we also get a well-defined negative pushing-positive homomorphism $\iota_-\colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$.

Proof. As before we set $F' := F \setminus \partial F$. Recall that it follows from the Open Seifert Surface Lemma 288.15 (2) and the Homotopy Equivalence-H_*-Proposition 144.14 that the inclusion induced map $\mathrm{H}_1(F') \to \mathrm{H}_1(F)$ is an isomorphism. Thus it suffices to show that the map

$$\begin{array}{rcl} \iota_+\colon \mathrm{H}_1(F') & \to & \mathrm{H}_1(S^3 \setminus F) \\ \left[\begin{array}{c}\text{closed oriented 1-dimensional}\\ \text{smooth submanifold } C \text{ of } F'\end{array}\right] & \mapsto & [\,\text{positive-push off } C^+ \text{ of } C\,] \end{array}$$

is well-defined. First note that it follows from the Represent-by-1-Manifold Corollary 187.19 that any homology class in $\mathrm{H}_1(F')$ can indeed be represented by a closed oriented 1-dimensional smooth submanifold in F'. Next note that it follows from the Bicollar Neighborhood Theorem 55.2 that we can pick an orientation-preserving bicollar $\beta\colon F' \times [-1,1] \to S^3 \setminus L$. If we use this bicollar we see that the map

$$\begin{array}{rcl} \iota_+^\beta\colon \mathrm{H}_1(F') & \to & \mathrm{H}_1(S^3 \setminus F) \\ \left[\begin{array}{c}\text{closed oriented 1-dimensional}\\ \text{smooth submanifold } C \text{ of } F'\end{array}\right] & \mapsto & \beta_{1*}([C]) = [\beta(C \times \{1\})] \end{array}$$

is just the map that is induced by the map $\beta_1\colon F' \to S^3 \setminus L$, in particular it is well-defined (i.e. independent of the choice of C) and a homomorphism.

Finally, given an oriented oriented closed oriented 1-dimensional smooth submanifold C in F' it follows from the Loop Push-Off Lemma 289.2 that $\iota_+ = \iota_+^\beta$, i.e. the definition does not depend on the choice of the push-off. ∎

289.2. **The Alexander module in terms of connected Seifert surfaces.** The following theorem shows that the single-variable Alexander module of an oriented link fits into a long exact sequence:

Theorem 289.4. (Alexander Module–Seifert Surface Theorem) Let $L \subseteq S^3$ be an oriented non-empty link and let F be a connected Seifert surface for L. Let $\iota_\pm\colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ be the two homomorphisms from the Homology Push-Off Lemma 289.3. Then there exists an exact sequence of $\mathbb{Z}[t^{\pm 1}]$-modules of the form

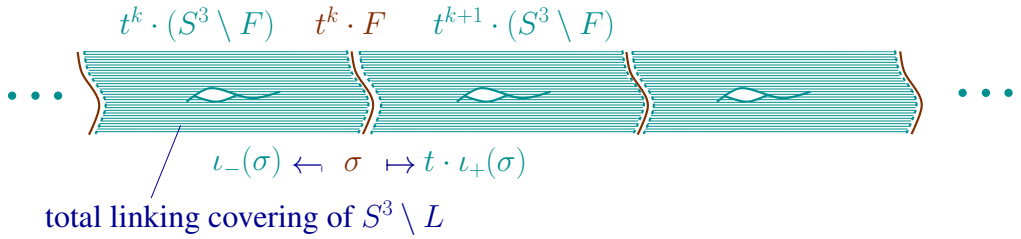
$$\begin{array}{rcl} & \dots & \to \mathrm{H}_2(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\partial_2} \\ \to \mathrm{H}_1(F) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\cdot\iota_+ - \iota_-} \mathrm{H}_1(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\partial_1} \\ \to \mathrm{H}_0(F) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\cdot\iota_+ - \iota_-} \mathrm{H}_0(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_0(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) \to 0. \end{array}$$

We make two clarifications:

- To the left we consider the maps

$$\begin{array}{ccc} \mathrm{H}_j(F) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{t\cdot\iota_+ - \iota_-} & \mathrm{H}_j(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] \\ \sigma \otimes 1 & \mapsto & \iota_+(\sigma) \otimes t - \iota_-(\sigma) \otimes 1. \end{array}$$

- The map $\mathrm{H}_1(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] \to \mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$ is induced by the inclusions of the various copies of $S^3 \setminus F$ into the total linking covering of $S^3 \setminus L$ as constructed in the Link–Infinite Cyclic Covering Proposition 288.18.



The proof of the Alexander Module–Seifert Surface Theorem 289.4 will require the remainder of this section. In the next Section 289.3 we will start with the task of calculating the homology groups and homomorphisms that appear in the above long exact sequence.

In the proof of the Alexander Module–Seifert Surface Theorem 289.4 we will need the following elementary algebraic lemma, which is essentially the same as the Canceling Exact Sequence Lemma 153.7:

Lemma 289.5. (Canceling Exact Sequence Lemma) Let R be a commutative ring. Suppose we are given the following commutative diagram of homomorphisms of R-modules:

$$\begin{array}{ccccccccccc} \longrightarrow & \begin{matrix} A_n \\ A_n \end{matrix} & \xrightarrow{\begin{pmatrix} \varphi_n & \psi_n \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{matrix} B_n \\ A_n \end{matrix} & \xrightarrow{(\beta_n\ \alpha_n)} & C_n & \xrightarrow{\partial_n} & \begin{matrix} A_{n-1} \\ A_{n-1} \end{matrix} & \xrightarrow{\begin{pmatrix} \varphi_{n-1} & \psi_{n-1} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{matrix} B_{n-1} \\ A_{n-1} \end{matrix} & \longrightarrow \\ & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & \\ \longrightarrow & A_n & \xrightarrow{\psi_n - \varphi_n} & B_n & \xrightarrow{\beta_n} & C_n & \xrightarrow{\widetilde{\partial}_n} & A_{n-1} & \xrightarrow{\psi_{n-1} - \varphi_{n-1}} & B_{n-1} & \longrightarrow \end{array}$$

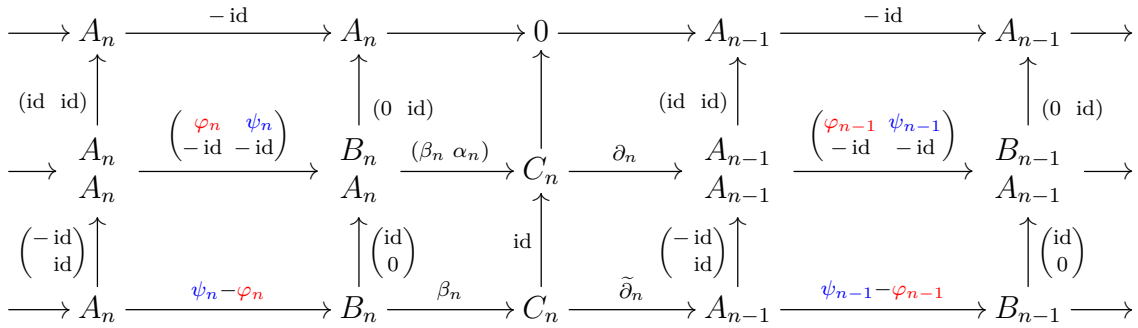
If the top sequence is exact, then there exist unique homomorphisms $\widetilde{\partial}_n \colon C_n \to A_{n-1}$ of R-modules that make the diagram commute. With these homomorphisms the bottom sequence is also exact.

Proof of the Canceling Exact Sequence Lemma 289.5. [3123] First note that we know that the composition of the two top right homomorphisms is zero. It follows that for any $c \in C_n$ there exists a unique $a \in A_{n-1}$ such that $\partial_n(c) = \begin{pmatrix} -a \\ a \end{pmatrix}$. We now set $\widetilde{\partial}_n(c) := a$. One can easily see that this is the unique homomorphism that makes the diagram commute.

It remains to show that the bottom sequence is exact. One can do so directly, using a diagram chase. In the following we perform a little trick to outsource the diagram chase to

[3123] The proof is literally the same as the proof of the Canceling Exact Sequence Lemma 153.7. We provide the proof again to keep this chapter self-contained.

a case we already dealt with: We consider the following vertically expanded diagram:



By construction the bottom part of the diagram commutes. It follows easily from the definitions that the top part also commutes. Furthermore note that the vertical sequences, with 0 added on top and bottom, are exact.

Next note that evidently the top sequence is exact. Furthermore by hypothesis the middle sequence is exact. Since the bottom vertical maps are monomorphisms we see that the bottom sequence is a chain complex.

It follows from the above discussion that all these maps form a "vertical" short exact sequence of "horizontal" chain complexes. Clearly the homology groups of the top chain complex are zero. By hypothesis the homology groups of the middle chain complex are also zero. It follows from the LES-Proposition 145.10 that the homology groups of the bottom chain complex are also zero. But this means that the bottom sequence is exact. ∎

Proof of the Alexander Module–Seifert Surface Theorem 289.4. Let $L \subseteq S^3$ be an oriented non-empty link and let F be a connected Seifert surface for L. We will construct the desired long exact sequence involving $\mathrm{H}_*(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$ by considering the explicit total linking covering from the Link–Infinite Cyclic Covering Proposition 288.18. We introduce the following notation and objects:

(a) We write $F' := F \setminus L$.
(b) By the Bicollar Neighborhood Theorem 55.2 we can pick an orientation-preserving bicollar $\beta\colon F' \times [-1,1] \to S^3 \setminus L$.
(c) We write $I := (-1,1)$, $I_- := (-1,0)$ and $I_+ := (0,1)$.
(d) As in the Link–Infinite Cyclic Covering Proposition 288.18 we consider

$$\widetilde{X} \ := \ ((F' \times I) \times \mathbb{Z}) \sqcup ((S^3 \setminus F) \times \mathbb{Z})/\!\sim$$

where
$$\begin{array}{lcll} ((x,t),k) & \sim & (\beta(x,t),k) & \text{for all } ((x,t),k) \in (F' \times I_-) \times \mathbb{Z}, \\ ((x,t),k) & \sim & (\beta(x,t),k+1) & \text{for all } ((x,t),k) \in (F' \times I_+) \times \mathbb{Z}. \end{array}$$

By the Link–Infinite Cyclic Covering Proposition 288.18 we know that the maps

$$\begin{array}{rcl} (S^3 \setminus F) \times \mathbb{Z} & \to & S^3 \setminus L \\ (P,k) & \mapsto & P \end{array} \qquad \text{and} \qquad \begin{array}{rcl} (F \times I) \times \mathbb{Z} & \to & S^3 \setminus L \\ ((P,t),k) & \mapsto & \beta(P,t) \end{array}$$

define a continuous map $p\colon \widetilde{X} \to S^3 \setminus L$ which is the total linking covering corresponding to $\ker(\widehat{\Phi}_L\colon \mathrm{H}_1(S^3 \setminus L) \to \langle t \rangle)$. Furthermore we know that the action by t^k on $\widetilde{X}$ is given by the k-translation on the $\mathbb{Z}$-factor.

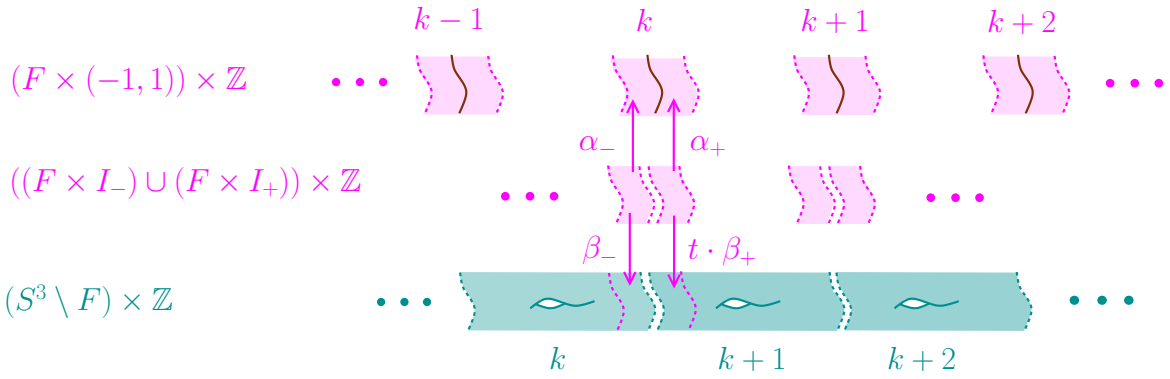
(e) Given any topological space W we obtain from the Path-Component–H_*–Lemma 143.21 a natural isomorphism

$$\begin{array}{rcl} \mathrm{H}_k(W) \otimes \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_k(W \times \mathbb{Z}) \\ \sigma \otimes t^k & \mapsto & f_{k*}(\sigma) \end{array}$$

here $f_k \colon W \to W \times \{k\}$ is the obvious map

(f) We denote by $\alpha_\pm \colon F' \times I_\pm \to F' \times I$ the inclusions.

(g) We denote by $\beta_\pm \colon F' \times I_\pm \to S^3 \backslash F$ the restrictions of the bicollar $\beta \colon F' \times [-1,1] \to S^3 \backslash L$ to the subspaces $F' \times I_\pm$.



It follows easily from (d), (e), the Mayer–Vietoris Theorem 153.2 and the Path-Component–H_*–Lemma 143.21 that we have an exact sequence of $\mathbb{Z}[t^{\pm 1}]$-modules of the following form

$$\to \begin{array}{l} \mathrm{H}_1(F' \times I_-) \otimes \mathbb{Z}[t^{\pm 1}] \\ \mathrm{H}_1(F' \times I_+) \otimes \mathbb{Z}[t^{\pm 1}] \end{array} \xrightarrow{\begin{pmatrix} \alpha_- & \alpha_+ \\ -\beta_- & -t \cdot \beta_+ \end{pmatrix}} \begin{array}{l} \mathrm{H}_1(F' \times I) \otimes \mathbb{Z}[t^{\pm 1}] \\ \mathrm{H}_1(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] \end{array} \xrightarrow{(\xi,\zeta)} \underbrace{\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])}_{:= \mathrm{H}_1(\widetilde{X}) \text{ as a } \mathbb{Z}[t^{\pm 1}]\text{-module}} \to$$

where the maps ξ and ζ are induced in a suitable sense by inclusions. The idea is to rewrite the above exact sequence to turn it into the desired exact sequence. This requires some extra notation:

- We consider the maps $\mu_\pm \colon F' \to F' \times I_\pm$ that are given by $x \mapsto (x, \pm\frac{1}{2})$.
- We consider the map $\mu \colon F' \to F' \times I$ that is given by $x \mapsto (x, 0)$.
- We denote by $i \colon F' \to F$ the inclusion.
- Let $\iota_\pm \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ be the two homomorphisms from the Homology Push-Off Lemma 289.3.

Now we consider the following diagram:

$$\begin{array}{ccccccc}
\longrightarrow & \begin{matrix}\mathrm{H}_1(F'\times I_-)\otimes\mathbb{Z}[t^{\pm1}]\\ \mathrm{H}_1(F'\times I_+)\otimes\mathbb{Z}[t^{\pm1}]\end{matrix} & \xrightarrow{\begin{pmatrix}\beta_- & t\cdot\beta_+\\ -\alpha_- & -\alpha_+\end{pmatrix}} & \begin{matrix}\mathrm{H}_1(S^3\setminus F)\otimes\mathbb{Z}[t^{\pm1}]\\ \mathrm{H}_1(F'\times I)\otimes\mathbb{Z}[t^{\pm1}]\end{matrix} & \xrightarrow{(\xi,\zeta)} & \mathrm{H}_1(S^3\setminus L;\mathbb{Z}[t^{\pm1}]) & \xrightarrow{\partial_1}\\
 & \begin{pmatrix}\mu_{-*} & 0\\ 0 & \mu_{+*}\end{pmatrix}\uparrow\cong & & \begin{pmatrix}\mathrm{id} & 0\\ 0 & \mu_*\end{pmatrix}\uparrow\cong & & \uparrow\mathrm{id} & \\
\longrightarrow & \begin{matrix}\mathrm{H}_1(F')\otimes\mathbb{Z}[t^{\pm1}]\\ \mathrm{H}_1(F')\otimes\mathbb{Z}[t^{\pm1}]\end{matrix} & \xrightarrow{\begin{pmatrix}\beta_-\circ\mu_- & t\cdot\beta_+\circ\mu_+\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}\mathrm{H}_1(S^3\setminus F)\otimes\mathbb{Z}[t^{\pm1}]\\ \mathrm{H}_1(F')\otimes\mathbb{Z}[t^{\pm1}]\end{matrix} & \xrightarrow{(\xi,\zeta)} & \mathrm{H}_1(S^3\setminus L;\mathbb{Z}[t^{\pm1}]) & \xrightarrow{\partial_1}\\
 & \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}\uparrow & & \begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}\uparrow & & \uparrow\mathrm{id} & \\
\longrightarrow & \mathrm{H}_1(F')\otimes\mathbb{Z}[t^{\pm1}] & \xrightarrow{t\cdot\beta_+\circ\mu_+-\beta_-\circ\mu_-} & \mathrm{H}_1(S^3\setminus F)\otimes\mathbb{Z}[t^{\pm1}] & \xrightarrow{\xi} & \mathrm{H}_1(S^3\setminus L;\mathbb{Z}[t^{\pm1}]) & \xrightarrow{\widetilde{\partial}_1}\\
 & \cong\downarrow i_* & & \mathrm{id}\downarrow & & \downarrow\mathrm{id} & \\
\longrightarrow & \mathrm{H}_1(F)\otimes\mathbb{Z}[t^{\pm1}] & \xrightarrow{t\cdot\iota_+-\iota_-} & \mathrm{H}_1(S^3\setminus F)\otimes\mathbb{Z}[t^{\pm1}] & \xrightarrow{\xi} & \mathrm{H}_1(S^3\setminus L;\mathbb{Z}[t^{\pm1}]) & \xrightarrow{\widetilde{\partial}_1}
\end{array}$$

We make the following clarifications and observations:

(a) The maps $\mu_\pm\colon F'\to F'\times I_\pm$ and $\mu\colon F'\to F'\times I$ are all homotopy equivalences. It follows from the Homotopy Equivalence-H_*-Proposition 144.14 that the top vertical maps are isomorphisms.

(b) The maps $\alpha_\pm\circ\mu_\pm\colon F'\to F'\times I$ and $\mu\colon F'\to F'\times I$ are clearly homotopic. It follows from the Homotopy-H_*-Proposition 144.12 that the top left square of the diagram commutes.

(c) Since the top horizontal line is exact it now follows from (a) and (b) that the second horizontal line is also exact.

(d) It follows from the Canceling Exact Sequence Lemma 289.5 that we have connecting homomorphisms $\widetilde{\partial}_*$ such that the third horizontal line is exact.

(e) It follows immediately from the definition of the maps $\iota_\pm$ and the Loop Push-Off Lemma 289.2 that we have the equality $\iota_\pm\circ i_* = \beta_{\pm*}\circ\mu_\pm\colon \mathrm{H}_i(F')\to\mathrm{H}_i(S^3\setminus F)$. This shows that the bottom left square commutes.

(f) It follows from the Open Seifert Surface Lemma 288.15 (2) together with the Homotopy Equivalence-H_*-Proposition 144.14 that the maps $i_*\colon \mathrm{H}_k(F')\to\mathrm{H}_k(F)$ are isomorphisms. Thus we obtain from (d) and (e) that the bottom horizontal line is also exact. ■

289.3. **Generalized linking.** Our next goal is to determine the homology groups and homomorphisms that appear in the Alexander Module–Seifert Surface Theorem 289.4. The following theorem allows us, in particular, to determine the isomorphism types of the homology groups $\mathrm{H}_k(S^3\setminus F)$:

Theorem 289.6. (Sphere–Alexander Duality Theorem) Let $n\in\mathbb{N}$ and let $X\subsetneq S^n$ be a compact non-empty submanifold. There exists a natural isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_j(S^n\setminus X)\xrightarrow{\cong}\widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

Here "natural" means the following two statements:

(i) If $X \subseteq Y \subsetneq S^n$ are two compact non-empty submanifolds of S^n (not necessarily of the same dimension), then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(S^n \setminus Y) & \xrightarrow[\cong]{\mathrm{AD}_Y} & \widetilde{\mathrm{H}}^{n-j-1}(Y;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}_j(S^n \setminus X) & \xrightarrow[\cong]{\mathrm{AD}_X} & \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}), \end{array}$$

here the left vertical map is induced by the inclusion $S^n \setminus Y \to S^n \setminus X$ and the right vertical map is induced by the inclusion $X \to Y$.

(ii) Let $f\colon S^n \to S^n$ be a diffeomorphism. We set $\epsilon := 1$ if f is orientation-preserving and we set $\epsilon := -1$ if f is orientation-reversing. Then the following diagram commutes up to ϵ:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_j(S^n \setminus X) & \xrightarrow[\cong]{\mathrm{AD}_X} & \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle f^*} \\ \widetilde{\mathrm{H}}_j(S^n \setminus f(X)) & \xrightarrow[\cong]{\mathrm{AD}_{f(X)}} & \widetilde{\mathrm{H}}^{n-j-1}(f(X);\mathbb{Z}). \end{array}$$

Proof. This is just a special case of the General Alexander Duality Theorem 218.2 or alternatively of the Sphere–Alexander Duality Theorem 218.7. ■

The Sphere–Alexander Duality Theorem 289.6 allows us to introduce the following concept:

Definition 289.7. Let $X \subsetneq S^3$ be a compact non-empty submanifold. We define the linking form

$$\begin{array}{rcl} \mathrm{lk}_X\colon \mathrm{H}_1(X) \times \mathrm{H}_1(S^3 \setminus X) & \to & \mathbb{Z} \\ (\alpha, \beta) & \mapsto & \langle \underbrace{\mathrm{AD}_X(\beta)}_{\in \mathrm{H}^1(X;\mathbb{Z})}, \alpha \rangle_X. \end{array}$$

Example. Let K and L be two disjoint oriented knots in S^3. Then we have the following equalities:

$$\underbrace{\mathrm{lk}(K,J)}_{\text{see Definition 284.1}} \quad \underset{\text{by the Linking Number Reformulation Proposition 285.8}}{=} \quad \langle \mathrm{AD}_K(\overbrace{[J]}^{\in \mathrm{H}_1(S^3 \setminus K)}), [K] \rangle \quad = \quad \underbrace{\mathrm{lk}_K([K],[J])}_{\text{see the above Definition 289.7}}.$$

□

The naturality of the Alexander duality isomorphism translates into the following lemma:

Lemma 289.8. (Naturality-of-Linking Lemma)

(i) Let $X \subseteq Y \subsetneq S^3$ be two compact non-empty submanifolds. We denote by $i\colon X \to Y$ and $j\colon S^3 \setminus Y \to S^3 \setminus X$ the inclusions. For any $\sigma \in \mathrm{H}_1(X)$ and any $\tau \in \mathrm{H}_1(S^3 \setminus Y)$ we have the equality

$$\mathrm{lk}_Y(i_*(\sigma), \tau) \;=\; \mathrm{lk}_X(\sigma, j_*(\tau)).$$

In other words, the following diagram commutes in the sense of Convention 204.17:

$$\begin{array}{ccc} \mathrm{H}_1(X) \times \mathrm{H}_1(S^3 \setminus X) & \xrightarrow{\mathrm{lk}_X} & \mathbb{Z} \\ {\scriptstyle i_*}\downarrow \quad \uparrow{\scriptstyle j_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{H}_1(Y) \times \mathrm{H}_1(S^3 \setminus Y) & \xrightarrow{\mathrm{lk}_Y} & \mathbb{Z}. \end{array}$$

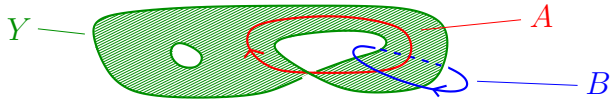
(ii) Let $X \subsetneq S^3$ be a compact non-empty submanifold and let $f\colon S^3 \to S^3$ be a diffeomorphism. We set $\epsilon := +1$ if f is orientation-preserving, otherwise we set $\epsilon := -1$. For any $\sigma \in \mathrm{H}_1(X)$ and any $\tau \in \mathrm{H}_1(S^3 \setminus X)$ we have the equality

$$\mathrm{lk}_X(\sigma, \tau) \;=\; \epsilon \cdot \mathrm{lk}_{f(X)}(f_*(\sigma), f_*(\tau)).$$

Proof. Both statements follow immediately from the naturality of AD_X (see the Sphere Alexander Duality Theorem 289.6) and the Kronecker Pairing–Natural Lemma 199.9 (1). ■

The following example shows us how one can use the Naturality-of-Linking Lemma 289.8 (i) to reduce the calculation of linking forms to the classical case of the linking number between two disjoint oriented knots:

Example. Let $Y \subseteq S^3$ be a compact submanifold and let $a \in \mathrm{H}_1(Y)$ and let $b \in \mathrm{H}_1(S^3 \setminus Y)$. Furthermore let $A \subseteq Y$ and $B \subseteq S^3 \setminus Y$ be smoothly embedded oriented loops such that $[A] = a \in \mathrm{H}_1(Y)$ and $[B] = b \in \mathrm{H}_1(S^3 \setminus Y)$.



Then $\mathrm{lk}_Y(a,b) \;=\; \mathrm{lk}_Y([A],[B]) \underset{\uparrow}{=} \mathrm{lk}_A([A], j_*([B])) \underset{\uparrow}{=} \mathrm{lk}(A,B).$

by the Naturality-of-Linking Lemma 289.8 (i) applied to $A \subseteq Y$, here $j\colon S^3 \setminus A \to S^3 \setminus Y$ is the inclusion

see page 5525

□

289.4. **The Seifert form and Seifert matrices.** Using the linking form from the previous section we can now introduce the notion of a Seifert form and of a Seifert matrix:

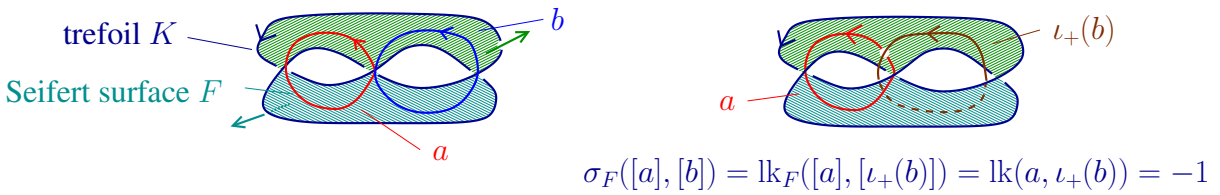
Definition 289.9. Let $L \subseteq S^3$ be an oriented non-empty link and let $F \subseteq S^3$ be a connected Seifert surface for L which we equip with the orientation such that $\partial F = L$ as oriented smooth manifolds.[3124] We consider the positive pushing-off homomorphism $\iota_+\colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ from the Homology Push-Off Lemma 289.3.

(1) The Seifert form of F is defined as the form[3125]

$$\begin{array}{rcl} \sigma_F\colon \mathrm{H}_1(F) \times \mathrm{H}_1(F) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \mathrm{lk}_F(a, \underbrace{\iota_+(b)}_{\in \mathrm{H}_1(S^3 \setminus F)}). \end{array}$$

[3124] Recall that in the Boundary Orientation Lemma 31.38 we showed that every orientation on a smooth manifold endows the boundary with a natural orientation.

[3125] Here we follow the convention of [**Rolf1990**, p. 201], [**Lic1997a**, p. 53] and [**BZH2014**, p. 117], which is the transpose of the convention used in other sources, e.g. [**Levine1977**].



$$\sigma_F([a],[b]) = \mathrm{lk}_F([a],[\iota_+(b)]) = \mathrm{lk}(a,\iota_+(b)) = -1$$

Definition 289.9.

(2) It follows from the Surface-H_*-Proposition 155.17 and the Surface Classification Theorem 101.16 that $\mathrm{H}_1(F)$ is a free abelian group of some rank k. Any $(k \times k)$-matrix representing the above Seifert form σ_F[3126] is called a Seifert matrix for the link L. More explicitly: If $v_1, \dots, v_k \in \mathrm{H}_1(F)$ is a basis, then we refer to

$$A \;:=\; (\sigma_F(v_r, v_s))_{r,s=1,\dots,k} \;=\; (\mathrm{lk}_F(v_r, \iota_+(v_s)))_{r,s=1,\dots,k}$$

as a Seifert matrix for the link L.

In the following we will see that Seifert forms can be calculated using linking numbers of disjoint oriented knots:

Lemma 289.10. (Seifert Form–Calculation Lemma) Let $L \subseteq S^3$ be an oriented non-empty link and let F be a connected Seifert surface for L.

(1) Let $C, D \subseteq F \setminus \partial F$ be smoothly embedded oriented loops.[3127] Then for any positive push-off D^+ of D we have

$$\sigma_F([C],[D]) \;=\; \mathrm{lk}(C, D^+).$$

(2) Let $C \subseteq F \setminus \partial F$ be a smoothly embedded oriented loop. We say $C^* \subseteq F$ is a parallel copy of C if there exists a smooth embedding $\varphi\colon C \times [0,1] \to F$ with $\varphi_0(C) = C$ and $\varphi_1(C) = C^*$ are oriented smooth manifolds. Now let $C^* \subseteq F$ be a parallel copy of C. Then

$$\sigma_F([C],[C]) \;=\; \mathrm{lk}(C, C^+) \;=\; \mathrm{lk}(C, C^*).$$

Proof. Let $L \subseteq S^3$ be an oriented non-empty link and let F be a connected Seifert surface for L.

(1) Let $C, D \subseteq F \setminus \partial F$ be smoothly embedded oriented loops. We have the following equalities:

$$\sigma_F([C],[D]) \underset{\text{by Definition 289.9}}{=} \mathrm{lk}_F([C],\iota_+([D])) \underset{\substack{\text{by the definition of } \iota_+ \\ \text{in the Homology} \\ \text{Push-Off Lemma 289.3}}}{=} \mathrm{lk}_F([C],[D^+]) \underset{\text{see page 5526}}{=} \mathrm{lk}(C, D^+).$$

(2) Let $C \subseteq F \setminus \partial F$ be a smoothly embedded oriented loop. Let C^* be a parallel copy of C *on* F. Using the smooth embedding $\varphi\colon C \times [0,1] \to F$ from the definition of C^* and using a tubular neighborhood of $F \subseteq S^3$ one can easily verify that C^+ and C^* are smoothly isotopic in the complement of C. Thus it follows from the Linking Number–Isotopy-Lemma 284.2 that we have the promised equality $\mathrm{lk}(C, C^+) = \mathrm{lk}(C, C^*)$. ■

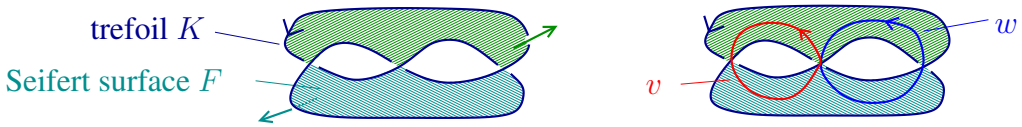
[3126] We refer to Definition 2.26 and Definition 224.4 for the definition of a matrix representing a form.

[3127] It follows from the Represent-by-1-Manifold Corollary 187.19 that any *primitive* element of $\mathrm{H}_1(F)$ can be represented by a smoothly embedded oriented loop.

Example. Let $L \subseteq S^3$ be an oriented non-empty link and let $F \subseteq S^3$ be a connected Seifert surface. The Seifert Form–Calculation Lemma 289.10 now gives us a pleasant way to calculate a corresponding Seifert matrix: Let $C_1, \dots, C_k \subseteq F \setminus \partial F$ be smoothly embedded oriented loops such that $[C_1], \dots, [C_k]$ form a basis for $\mathrm{H}_1(F)$. It follows from the Seifert Form–Calculation Lemma 289.10 that $(\mathrm{lk}(C_i, C_j^+))_{i,j=1,\dots,k}$ is a Seifert matrix for L. We now apply this approach to some of our favorite examples:

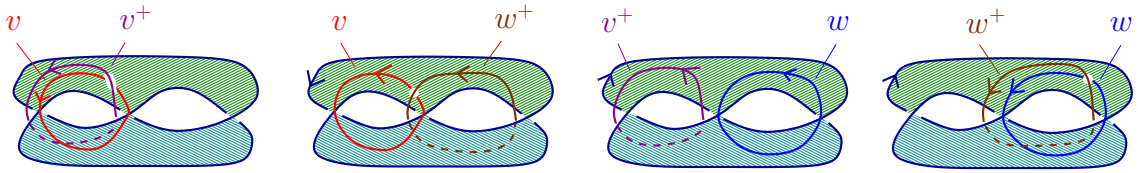
(1) Let K be the unknot. We pick the disk D as its connected Seifert surface. Since $\mathrm{H}_1(D) = 0$ we obtain the (0×0)-matrix as a Seifert matrix for the unknot.

(2) We consider the oriented trefoil K and the connected Seifert surface F for K that are shown in the figure below. Furthermore we consider the smoothly embedded oriented loops $v, w \subseteq F \setminus \partial F$ that are shown below. Evidently these give rise to a basis of $\mathrm{H}_1(F)$.



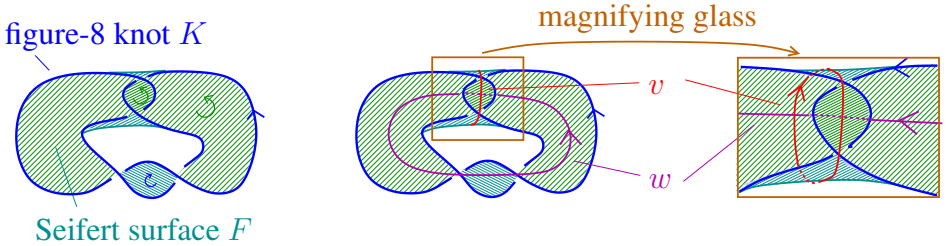
Using the Seifert Form–Calculation Lemma 289.10 and using the Linking Number-via-Crossings Proposition 284.8 it is elementary to verify that the corresponding Seifert matrix is given by

$$A = \begin{pmatrix} \mathrm{lk}(v, \iota_+(v)) & \mathrm{lk}(v, \iota_+(w)) \\ \mathrm{lk}(w, \iota_+(v)) & \mathrm{lk}(w, \iota_+(w)) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$



$\mathrm{lk}(v, v^+) = +1$ $\mathrm{lk}(v, w^+) = -1$ $\mathrm{lk}(w, v^+) = 0$ $\mathrm{lk}(w, w^+) = +1$

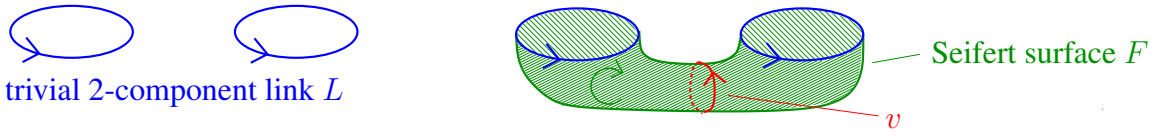
(3) We consider the oriented figure-8 knot K and the connected Seifert surface F for K that are shown in the figure below. Furthermore we show smoothly embedded loops $v, w \subseteq F \setminus \partial F$ that define a basis for $\mathrm{H}_1(F)$.



Similar to (2) we see that a Seifert matrix for the figure-8 knot is given by

$$A = \begin{pmatrix} \mathrm{lk}(v, \iota_+(v)) & \mathrm{lk}(v, \iota_+(w)) \\ \mathrm{lk}(w, \iota_+(v)) & \mathrm{lk}(w, \iota_+(w)) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

(4) (a) We consider the trivial 2-component link L and the connected Seifert surface F for L that are shown in the figure below. Furthermore we show a smoothly embedded loop $v \subseteq F \setminus \partial F$ that defines a basis for $\mathrm{H}_1(F)$.

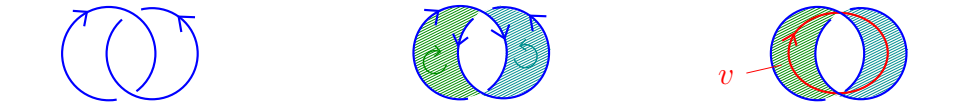


Similar to (2) and (3) we see that a Seifert matrix for L is given by

$$A \;=\; (\mathrm{lk}(v, \iota_+(v))) \;=\; (0).$$

(b) Let $m \in \mathbb{N}_{\geq 2}$. Similar to (a) one sees that for the trivial m-component link that a Seifert matrix is given by the $(m-1) \times (m-1)$-matrix consisting entirely of zeros.

(5) Finally we consider the negative Hopf link that is shown below.



In the figure we show a Seifert surface F and a generator $v \in \mathrm{H}_1(F)$. Using the Seifert Form–Calculation Lemma 289.10 (2) one can easily verify that $\mathrm{lk}(v, \iota_+(v)) = +1$. Thus $(+1)$ is a Seifert matrix for the negative Hopf link. Similarly (-1) is a Seifert matrix for the positive Hopf link. It's all very confusing. □

In the definition of the Seifert form we used the positive pushing-off homomorphism. This is a little arbitrary and one might ask, what happens if we use instead the negative pushing-off homomorphism. The following lemma gives a complete answer to this question:

Lemma 289.11. (Seifert Form–Negative Push-Off Lemma) Let $L \subseteq S^3$ be an oriented non-empty link and let $F \subseteq S^3$ be a connected Seifert surface for L which we equip with the orientation such that $\partial F = L$ as oriented smooth manifolds. We consider the pushing-off homomorphisms $\iota_\pm \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ from the Homology Push-Off Lemma 289.3. For any $a, b \in \mathrm{H}_1(F)$ we have the equality

$$\underbrace{\mathrm{lk}_F(a, \iota_+(b))}_{=\sigma_F(a,b)} \;=\; \mathrm{lk}_F(b, \iota_-(a)).$$

Proof. Let $a, b \in \mathrm{H}_1(F)$. We start out with a few preparations:

(a) We write $F' := F \setminus \partial F$.

(b) We denote by $i \colon F' \to F$ the inclusion.

(c) It follows from the Open Seifert Surface Lemma 288.15 (2) and the Homotopy Equivalence-H_*-Proposition 144.14 that the inclusion induced map $\mathrm{H}_1(F') \to \mathrm{H}_1(F)$ is an isomorphism.

(d) It follows from (c) and the Represent-by-1-Manifold Corollary 187.19 that we can find closed oriented 1-dimensional smooth submanifolds $A, B \subseteq F'$ such that $[A] = a$ and $[B] = b$. Using the fact that lk_F is bilinear we see that we can restrict ourselves to the case that A and B are connected.

(e) It follows from the Bicollar Neighborhood Theorem 55.2 that we can pick a bicollar $F' \times [-1, 1] \to S^3 \setminus L$.

(f) For $t \in [-1,1]$ consider the map

$$\begin{array}{rcl} \beta_t \colon F' & \to & S^3 \setminus L \\ x & \mapsto & \beta_t(x) := \beta(x,t). \end{array}$$

We now see that

$$\begin{array}{ccccccc} \mathrm{lk}_F(a, \iota_+(b)) & \overset{\text{by the Homology Push-Off Lemma 289.3 and the Loop Push-Off Lemma 289.2}}{\downarrow =} & \mathrm{lk}_F([A],[\beta_1(B)]) & \overset{\text{see page 5526}}{\downarrow =} & \mathrm{lk}(A, \beta_1(B)) & & \\ & \overset{\text{by the Linking Number–Isotopy Lemma 284.2, since } \beta \text{ defines a smooth isotopy from the link } \beta_0(A) \sqcup \beta_1(B) = A \sqcup \beta_1(B) \text{ to the link } \beta_{-1}(A) \sqcup \beta_0(B) = \beta_{-1}(A) \sqcup B}{\uparrow =} & \mathrm{lk}(\beta_{-1}(A), B) & \overset{\text{by the Linking Number Symmetry Proposition 284.9}}{\uparrow =} & \mathrm{lk}(B, \beta_{-1}(A)) & \overset{\text{by the same argument backwards}}{\uparrow =} & \mathrm{lk}_F(b, \iota_-(a)). \end{array}$$

■

289.5. **Alexander modules and Seifert matrices.** The following theorem gives us the raison d'être of Seifert matrices:

Theorem 289.12. (Alexander Module–Seifert Matrix Theorem) Let $L \subseteq S^3$ be an oriented non-empty link and let A be a Seifert matrix for L of size k.

(1) The matrix $t \cdot A - A^{\mathrm{T}} \in \mathrm{M}(k \times k, \mathbb{Z}[t^{\pm 1}])$ is a presentation matrix for the Alexander module $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$.

(2) We have $\Delta_L(t) \doteq \det(t \cdot A - A^{\mathrm{T}})$.

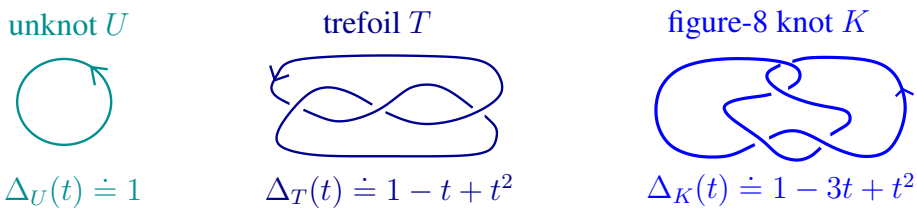
Remark. Note that we are now in a very pleasant setting, namely we now have all the tools to calculate the single-variable Alexander polynomial of an oriented link algorithmically: Let L be an oriented non-empty link that is given by a diagram. In Chapter 121 we learned about Seifert's algorithm which gives us an explicit connected Seifert surface F for L. We then determine a set of smoothly embedded loops $C_1, \dots, C_k$ on F that represent a basis for $\mathrm{H}_1(F)$. Using the Linking Number-via-Crossings Proposition 284.8 we can then calculate the corresponding Seifert matrix $A := (\sigma_F(C_i, C_j^+)_{i,j=1,\dots,k}$. By the Alexander Module–Seifert Matrix Theorem 289.12 the single-variable Alexander polynomial can then be calculated as $\Delta_L(t) \doteq \det(t \cdot A - A^{\mathrm{T}})$. □

Examples.

(1) We consider our favorite three knots: the unknot, the trefoil and figure-8 knot. From the discussion on page 5528 and the Alexander Module–Seifert Matrix Theorem 289.12 we obtain the following table:

knot	Seifert matrix	$t \cdot A - A^{\mathrm{T}}$	Alexander polynomial
unknot U	(0×0)-matrix	(0×0)-matrix	1
trefoil T	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} t-1 & -t \\ 1 & t-1 \end{pmatrix}$	$t^2 - t + 1$
figure-8 knot K	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -t+1 & -t \\ 1 & t-1 \end{pmatrix}$	$-t^2 + 3t - 1.$

It follows from these calculations, together with the Link Isotopy–Alexander Polynomial Proposition 288.3, that the unknot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.



(2) Similar to (1) we see that for the trivial 2-component link L we have the equalities $\Delta_L(t) = \det(t \cdot 0 - 0) = 0$. We will consider more links in the exercises. □

The proof of the Alexander Module–Seifert Matrix Theorem 289.12 will require the remainder of this section. In the next Section 289.6 we will apply the Alexander Module–Seifert Matrix Theorem 289.12 to prove some general structure results about Alexander polynomials.

The proof of the Alexander Module–Seifert Matrix Theorem 289.12 rests on two lemmas:

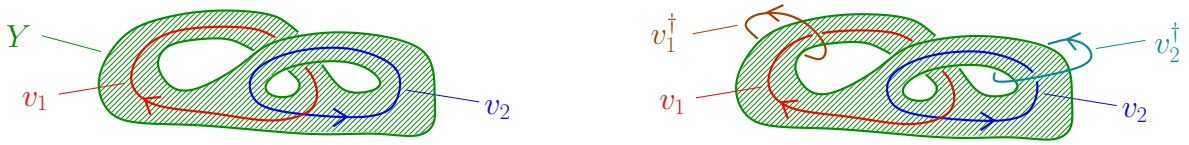
Lemma 289.13. (Linking–Dual Basis Lemma) Let $Y \subsetneq S^3$ be a compact non-empty smooth submanifold and let $v_1, \dots, v_k$ be a basis for $\mathrm{H}_1(Y)$.[3128]

(a) There exists a unique basis $v_1^\dagger, \dots, v_k^\dagger$ of the homology group $\mathrm{H}_1(S^3 \setminus Y)$ such that for any choice of $i, j \in \{1, \dots, k\}$ we have

$$\mathrm{lk}_Y(v_i, v_j^\dagger) \;=\; \delta_{ij}.$$

(b) For any $\sigma \in \mathrm{H}_1(S^3 \setminus Y)$ we have

$$\sigma \;=\; \sum_{i=1}^{k} \mathrm{lk}_Y(v_i, \sigma) \cdot v_i^\dagger.$$



Proof of the Linking–Dual Basis Lemma 289.13. Let $v_1, \dots, v_k$ be a basis for $\mathrm{H}_1(Y)$.

(1) We proceed in two steps:

- It follows from the H^1-Hom-Proposition 201.24 that the evaluation map

$$\begin{array}{rcl} \mathrm{H}^1(Y;\mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{H}_1(Y),\mathbb{Z}) \\ \varphi & \mapsto & (\sigma \mapsto \langle \varphi, \sigma \rangle_Y \end{array}$$

is an isomorphism. This implies in particular that there exists a unique basis $v_1^*, \dots, v_k^* \in \mathrm{H}^1(Y;\mathbb{Z})$ such that for all $i, j \in \{1, \dots, k\}$ we have $\langle v_i^*, v_j \rangle_Y = \delta_{ij}$.

[3128] We will only apply the lemma in the case that Y is a compact connected 2-dimensional smooth manifold with non-empty boundary where it is clear that such a basis exists. For completeness we now explain why such a basis always exists: Since $Y \subseteq S^3$ is a compact, weakly locally contractible subspace we obtain from the Finite-H_*-H^*-Lemma 218.19 that all homology and all cohomology groups are finitely generated. Next note that it follows from the General Alexander Duality Theorem 218.2, Exercise 217.6 and the H^1-Hom-Proposition 201.24 that $\mathrm{H}_1(Y) \cong \mathrm{H}^1(S^3 \setminus Y;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(S^3 \setminus Y),\mathbb{Z})$, which is a free abelian group.

- We consider the isomorphism $\mathrm{AD}_Y\colon \mathrm{H}_1(S^3\setminus Y)\to \mathrm{H}^1(Y;\mathbb{Z})$ from the Sphere–Alexander Duality Theorem 289.6. Since this is an isomorphism there exists a unique basis $v_1^\dagger,\dots,v_k^\dagger$ of $\mathrm{H}_1(S^3\setminus Y)$ such that for $i=1,\dots,k$ we have $\mathrm{AD}_Y(v_i^\dagger)=v_i^*$.

Finally note that for any $i,j\in\{1,\dots,k\}$ we have the following equalities:

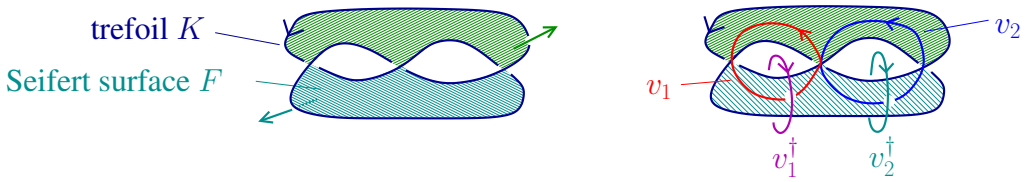
$$\mathrm{lk}_Y(v_i,v_j^\dagger) \underset{\substack{\uparrow\\ \text{see Definition 289.7}\\ \text{of the linking form } \mathrm{lk}_Y}}{=} \langle \mathrm{AD}_Y(v_j^\dagger), v_i\rangle_Y \underset{\substack{\uparrow\\ \text{by definition}\\ \text{of } v_j^\dagger\in \mathrm{H}_1(S^3\setminus Y)}}{=} \langle v_j^*, v_i\rangle_Y \underset{\substack{\uparrow\\ \text{by the above}\\ \text{definition of } v_j^*}}{=} \delta_{ij}.$$

(2) Since $\mathrm{lk}_Y\colon \mathrm{H}^1(Y;\mathbb{Z})\times \mathrm{H}_1(S^3\setminus Y)\to\mathbb{Z}$ is bilinear and since $v_1^\dagger,\dots,v_k^\dagger$ is a basis for $\mathrm{H}_1(S^3\setminus Y)$ it suffices to prove the desired equality for each $\sigma=v_j^\dagger$. But in this case the equality holds by the defining properties of the $v_j^\dagger$. ■

Using the Linking–Dual Basis Lemma 289.13 we can now formulate and prove the following essential lemma:

Lemma 289.14. (Seifert Surface-Complement Lemma) Let $L\subseteq S^3$ be an oriented non-empty link, let F be a Seifert surface (not necessarily connected) for L and let $v_1,\dots,v_k$ be a basis of $\mathrm{H}_1(F)$. We denote by $A=(\sigma_F(v_r,v_s))_{r,s=1,\dots,k}$ the corresponding Seifert matrix. Let $v_1^\dagger,\dots,v_k^\dagger$ be the basis of $\mathrm{H}_1(S^3\setminus F)$ from the Linking–Dual Basis Lemma 289.13. With respect to the bases $v_1,\dots,v_k$ of $\mathrm{H}_1(F)$ and $v_1^\dagger,\dots,v_k^\dagger$ of $\mathrm{H}_1(S^3\setminus F)$ the following two statements hold:

(a) The map $\iota_+\colon \mathrm{H}_1(F)\to \mathrm{H}_1(S^3\setminus F)$ is represented by the matrix A.
(b) The map $\iota_-\colon \mathrm{H}_1(F)\to \mathrm{H}_1(S^3\setminus F)$ is represented by the matrix A^{T}.



Proof. Let $L\subseteq S^3$ be an oriented non-empty link, let F be a connected Seifert surface for L, let $v_1,\dots,v_k$ be a basis of $\mathrm{H}_1(F)$ and let $A=(\sigma_F(v_r,v_s))_{r,s=1,\dots,k}$ be the corresponding Seifert matrix.

(a) Let $r\in\{1,\dots,k\}$. We perform the following basic calculation:

$$\iota_+(v_r) \underset{\substack{\uparrow\\ \text{by the Linking–Dual}\\ \text{Basis Lemma 289.13 (2)}}}{=} \sum_{s=1}^k \mathrm{lk}_F(v_s,\iota_+(v_r))\cdot v_s^\dagger \underset{\substack{\uparrow\\ \text{by Definition 289.9}\\ \text{of the Seifert form } \sigma_F}}{=} \sum_{s=1}^k \sigma_F(v_s,v_r)\cdot v_s^\dagger$$

It follows from Definition 2.11 that the matrix $(\sigma_F(v_s,v_r))_{s,r=1,\dots,k}=A$ is a presentation matrix for the homomorphism ι_+.

(b) Let $r\in\{1,\dots,k\}$. Similar to (1) we perform the following calculation:

$$\iota_-(v_r) \underset{\substack{\uparrow\\ \text{by the Linking–Dual}\\ \text{Basis Lemma 289.13 (2)}}}{=} \sum_{s=1}^k \mathrm{lk}_F(v_s,\iota_-(v_r))\cdot v_s^\dagger \underset{\substack{\uparrow\\ \text{by the Seifert Form–Negative}\\ \text{Push-Off Lemma 289.11}}}{=} \sum_{s=1}^k \mathrm{lk}_F(v_r,\iota_+(v_s))\cdot v_s^\dagger \underset{\substack{\uparrow\\ \text{by Definition 289.9}\\ \text{of the Seifert form } \sigma_F}}{=} \sum_{s=1}^k \sigma_F(v_r,v_s)\cdot v_s^\dagger$$

It follows again from Definition 2.11 that the matrix $(\sigma_F(v_r, v_s))_{s,r=1,\dots,k} = A^{\mathrm{T}}$ is a presentation matrix for the homomorphism ι_-. ■

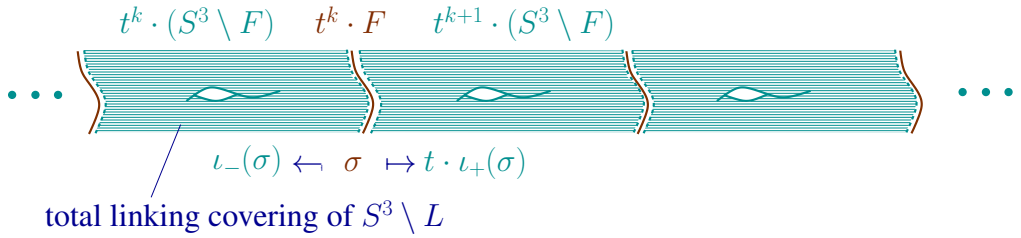
Proof of the Alexander Module–Seifert Matrix Theorem 289.12. Let $L \subseteq S^3$ be an oriented non-empty link and let A be a Seifert matrix for L of size k.

(1) We pick a connected Seifert surface F together with a basis $v_1, \dots, v_k$ of $\mathrm{H}_1(F)$ such that A is the corresponding Seifert matrix. Our goal is to show that $t \cdot A - A^{\mathrm{T}}$ is a presentation matrix for the Alexander module $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$.

As usual let $\iota_\pm \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ be the two homomorphisms from the Homology Push-Off Lemma 289.3. We start out with the following claim:

Claim. There exists an exact sequence of the form

$$\dots \to \mathrm{H}_1(F) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t \cdot \iota_+ - \iota_-} \mathrm{H}_1(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] \to \mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) \to 0.$$



Proof. It follows from the Alexander Module–Seifert Surface Theorem 289.4 that there exists an exact sequence of the form

$$\begin{array}{ccccccc} & & & \dots & \to & \mathrm{H}_2(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) & \xrightarrow{\partial_2} \\ \to & \mathrm{H}_1(F) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{t \cdot \iota_+ - \iota_-} & \mathrm{H}_1(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) & \xrightarrow{\partial_1} \\ \to & \mathrm{H}_0(F) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{t \cdot \iota_+ - \iota_-} & \mathrm{H}_0(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_0(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) & \to 0. \end{array}$$

Comparing this sequence with the statement of the claim we see that we need to show that the connecting homomorphism ∂_1 is the zero map. To do so we need to consider the bottom part of the above sequence. We start out with two observations:

- By definition of a connected Seifert surface we know that F is connected. Since L is non-empty we see that F is non-empty.
- As on page 5509 we calculate that

$$\begin{array}{ccccc} \mathrm{H}_0(S^3 \setminus F) & \cong & \mathbb{Z} \oplus \widetilde{\mathrm{H}}_0(S^3 \setminus F) & \cong & \mathbb{Z} \oplus \mathrm{H}^2(F; \mathbb{Z}) & \cong & \mathbb{Z} \oplus 0. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the Reduced Homology Lemma 147.3 (0)} & & \text{by the Sphere–Alexander Duality Theorem 289.6} & & \text{by the Top Cohomology Theorem 201.26, since } F \text{ is connected and since } \partial F \neq \varnothing & \end{array}$$

Now we consider the following diagram:

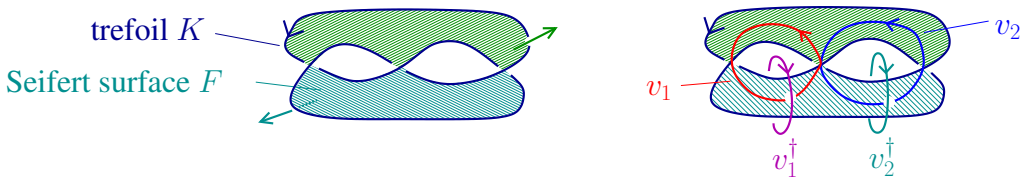
$$\begin{array}{ccc} \mathrm{H}_0(F) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{t \cdot \iota_+ - \iota_-} & \mathrm{H}_0(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] \\ {\scriptstyle \epsilon_F \otimes \mathrm{id}} \downarrow \cong & & {\scriptstyle \epsilon_{S^3 \setminus F} \otimes \mathrm{id}} \downarrow \cong \\ \mathbb{Z} \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{\cdot (t-1)} & \mathbb{Z} \otimes \mathbb{Z}[t^{\pm 1}]. \end{array}$$

We make the following observations:

(a) The vertical maps ϵ_Z are the natural augmentation maps $\mathrm{H}_0(Z) \to \mathbb{Z}$ from the Augmentation Lemma 143.24 that are given by $[P] \mapsto 1$.

(b) Since F and $S^3 \setminus F$ are non-empty and path-connected we obtain from the H_0-Proposition 143.25 that the vertical maps are isomorphisms.
(c) It follows from the naturality of the augmentation maps that the diagram commutes.
(d) It is clear that the bottom horizontal map is a monomorphism.
(e) Note that it follows from (b), (c) and (d) that the top horizontal homomorphism $t \cdot \iota_+ - \iota_- \colon \mathrm{H}_0(F) \otimes \mathbb{Z}[t^{\pm 1}] \to \mathrm{H}_0(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}]$ is a monomorphism.
(f) It follows from (e) that the connecting homomorphism ∂_1 is indeed as desired the zero map. ⊞

In the following let $v_1^\dagger, \dots, v_k^\dagger$ be the basis of $\mathrm{H}_1(S^3 \setminus F)$ from the Linking–Dual Basis Lemma 289.13 that is dual to the basis $v_1, \dots, v_k$ of $\mathrm{H}_1(F)$.



By the Seifert Surface-Complement Lemma 289.14 we know that with respect to the bases $v_1, \dots, v_k$ of $\mathrm{H}_1(F)$ and $v_1^\dagger, \dots, v_k^\dagger$ of $\mathrm{H}_1(S^3 \setminus F)$ the following two statements hold:
(a) The map ι_+ is represented by the matrix A.
(b) The map ι_- is represented by the matrix A^{T}.

It follows from this discussion that we have the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccc} & & \mathbb{Z}[t^{\pm 1}]^k & \xrightarrow{t \cdot A - A^{\mathrm{T}}} & \mathbb{Z}[t^{\pm 1}]^k & & \\ & & \cong \downarrow \mathfrak{e}_i \mapsto v_i \otimes 1 & & \cong \downarrow \mathfrak{e}_i \mapsto v_i^\dagger \otimes 1 & & \\ \dots \longrightarrow & \mathrm{H}_1(F) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{t \cdot \iota_+ - \iota_-} & \mathrm{H}_1(S^3 \setminus F) \otimes \mathbb{Z}[t^{\pm 1}] & \longrightarrow & \mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}]) & \longrightarrow 0. \end{array}$$

By the claim we know that the bottom sequence is exact. It follows that $t \cdot A - A^{\mathrm{T}}$ is a presentation matrix for the single-variable Alexander module $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$.

(2) We now see that

$$\Delta_L(t) \underset{\text{by Definition 288.8}}{\overset{\uparrow}{=}} \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])) \underset{\substack{\text{follows from Statement (1)} \\ \text{and the discussion on page 5489}}}{\overset{\uparrow}{=}} \det(t \cdot A - A^{\mathrm{T}}).$$

■

289.6. **Applications of connected Seifert surfaces and Seifert matrices.** As we will see in this section, the Alexander Module–Seifert Matrix Theorem 289.12 also allows us to prove some general structure results about Alexander polynomials. First of all we have the following analog of the Alexander Polynomial–Symmetry Theorem 117.7:

Proposition 289.15. (Alexander Polynomial–Symmetry Proposition) Given any oriented link L we have $\Delta_L(t) \doteq \Delta_L(t^{-1})$.

Remark.

(1) Note that the Alexander Polynomial–Symmetry Proposition 289.15 together with the Alexander Polynomial–Orientation Lemma 288.10 shows that the Alexander polynomial is an invariant of *unoriented knots*.
(2) Note that for the multivariable Alexander polynomial of links our current approach for showing the symmetry of Alexander polynomials does not work. There are several approaches to fixing this issues: In the Alexander Polynomial–Symmetry Theorem 117.7 we already gave a very different argument. In the Alexander Polynomial–Symmetry Theorem **??** we will see that one can prove symmetry using "C-complexes", which are a generalization of connected Seifert surfaces. Furthermore much later we will use Reidemeister torsion to prove symmetry, we refer to the Alexander Polynomials Agree Proposition **??** and the Alexander Torsion–Symmetry Theorem **??** for details. □

Proof. Let $L \subseteq S^3$ be an oriented link. Note that it follows from the Seifert Surface Existence Theorem 288.12 that we can pick a $(k \times k)$-Seifert matrix A for L. We see that we have the following equalities:

$$\Delta_L(t) \doteq \det(t \cdot A - A^{\mathrm{T}}) \doteq \det(-A + t^{-1} \cdot A^{\mathrm{T}}) \doteq \det(t^{-1} \cdot A^{\mathrm{T}} - A)$$

$\doteq$ by the Alexander Module Seifert Matrix Theorem 289.12

$\doteq$ since for any $(k \times k)$-matrix W we have $\det((-t^{-1}) \cdot W) = (-t^{-1})^k \cdot \det(W)$

$$\doteq \det(t^{-1} \cdot A^{\mathrm{T}} - A) \doteq \Delta_L(t^{-1}).$$

$\doteq$ since $\det(W^{\mathrm{T}}) = \det(W)$

$\doteq$ follow from the Alexander Module Seifert Matrix Theorem 289.12 ■

The following proposition relates any Seifert matrix to the intersection form on a connected Seifert surface :

Proposition 289.16. (Seifert Matrix–Intersection Form Proposition) Let $m \in \mathbb{N}$ and let L be an oriented m-component link with connected Seifert surface F. Furthermore let $v_1, \dots, v_k \in \mathrm{H}_k(F)$ be a basis. We denote by A the corresponding Seifert matrix.

(1) The matrix $A - A^{\mathrm{T}}$ represents the intersection form $Q_F \colon \mathrm{H}_1(F) \times \mathrm{H}_1(F) \to \mathbb{Z}$ on F with respect to the basis $v_1, \dots, v_k$.
(2) Let g be the genus of F. There exists a $P \in \mathrm{GL}(k, \mathbb{Z})$ such that $P^{\mathrm{T}}(A - A^{\mathrm{T}})P$ is the block diagonal sum of g copies of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $m - 1$ zeros on the diagonal.
(3) (a) If L is knot, then we have $\det(A - A^{\mathrm{T}}) = 1$.
(b) If L has at least two components, then we have $\det(A - A^{\mathrm{T}}) = 0$.

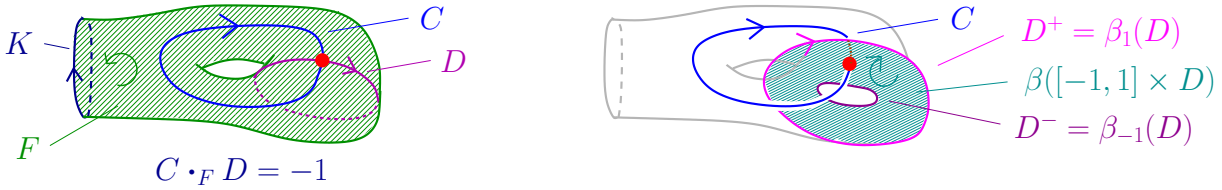
Proof. Let $L \subseteq S^3$ be an oriented link with $m \in \mathbb{N}$ components and let F be a connected Seifert surface for K.

(1) As usual we set $F' := F \setminus L$. We start out with the following claim:

Claim 1. Let $C, D \subseteq F'$ be two smoothly embedded oriented transverse loops. Then we have the following equality:

$$\underbrace{C \cdot_F D}_{\substack{\text{algebraic} \\ \text{intersection} \\ \text{number}}} = \underbrace{\sigma_F([C],[D])}_{\substack{\text{Seifert} \\ \text{form}}} - \underbrace{\sigma_F([D],[C])}_{\substack{\text{Seifert} \\ \text{form}}}.$$

Proof. Let $C, D \subseteq F'$ be two smoothly embedded oriented transverse loops. To prove the desired above equality it is convenient to pick an orientation-preserving bicollar $\beta\colon [-1,1] \times F' \to S^3 \setminus L$ for $F' \subseteq S^3 \setminus L$.



We have the following equalities:

$$\sigma_F([C],[D]) - \sigma_F([D],[C]) = \underset{\uparrow}{=} \operatorname{lk}(C,\beta_1(D)) - \operatorname{lk}(C,\beta_{-1}(D)) \underset{\uparrow}{=} \operatorname{lk}(\beta_1(D),C) + \operatorname{lk}(\beta_{-1}(D)^{\mathrm{rev}},C)$$

by the Loop Push-Off Lemma 289.2, the Seifert Form–Calculation Lemma 289.10, and the Seifert Form–Negative Push-Off Lemma 289.11

this follows from the Linking Number-Symmetry Proposition 284.9 and the Linking Number–Orientation Lemma 284.4

$$\underset{\uparrow}{=} \operatorname{lk}(\overbrace{\beta_1(D) \cup \beta_{-1}(D)^{\mathrm{rev}}}^{\text{oriented link}}, C) \qquad \underset{\uparrow}{=} \operatorname{lk}(\overbrace{\beta(\partial([-1,1] \times D))}^{\text{oriented link}}, C)$$

by the Total Linking Number Lemma 288.6

by the Product Orientation Proposition 31.48 (3d) we have $\partial([-1,1] \times D) = (D \times \{1\}) \sqcup -(D \times \{-1\})$ as oriented smooth manifolds

$$\underset{\uparrow}{=} \beta([-1,1] \times D) \cdot_{S^3} C \qquad \underset{\uparrow}{=} C \cdot_F D.$$

by the Total Linking #-via-Intersection # Proposition 288.7, here we use that $\beta([-1,1] \times D)$ intersects C transversally

the intersection points are the same and the signs also agree

⊞

The following claim gives us the "homological version" of Claim 1:

Claim 2. Given any $c, d \in \mathrm{H}_1(F)$ we have

$$Q_F(c,d) = \sigma_F(c,d) - \sigma_F(d,c).$$

Proof. Let $c, d \in \mathrm{H}_k(F)$. Since both sides are bilinear it suffices to deal with the case that $c, d \in \mathrm{H}_1(F)$ are primitive, i.e. they are not non-trivial multiples of other homology classes. Since c, d are primitive we obtain from the Represent-by-1-Manifold Corollary 187.19 that we can represent c, d by smoothly embedded oriented loops C and D. Furthermore note that it follows from the Transversality Theorem 50.9 and the Homotopy-H_*-Proposition 144.12 that we can assume that C and D are transverse. Now we see that

$$Q_F(c,d) \underset{\downarrow}{=} Q_F([C],[D]) \underset{\downarrow}{=} C \cdot_F D = \underset{\uparrow}{=} \sigma_F([C],[D]) - \sigma_F([D],[C]) \underset{\uparrow}{=} \sigma_F(c,d) - \sigma_F(d,c).$$

by the choice of C and D

by the Cup Product–Intersection Number Theorem 220.10

by the claim

by the choice of C and D

⊞

Now let $v_1, \dots, v_k \in \mathrm{H}_k(F)$ be a basis. We denote by A the corresponding Seifert matrix. The following claim gives us the desired result:

Claim 3. The matrix $A - A^{\mathrm{T}}$ represents the intersection form $Q_F\colon \mathrm{H}_1(F) \times \mathrm{H}_1(F) \to \mathbb{Z}$ on F with respect to the basis $v_1, \dots, v_k$.

Proof. Let $i, j \in \{1, \dots, k\}$. We see that

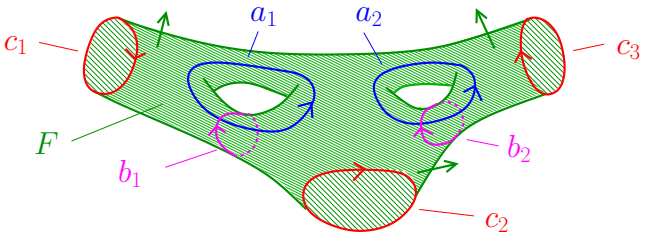
$$Q_F(v_i, v_j) \underset{\text{by Claim 2}}{=} \sigma_F(v_i, v_j) - \sigma_F(v_j, v_i) \underset{\text{by Definition 289.9}}{=} ij\text{-entry of } A - ij\text{-entry of } A^{\mathrm{T}}.$$

⊞

(2) It follows from the Surface Classification Theorem 101.16 that the connected Seifert surface F is diffeomorphic to some surface of genus g minus m open disks. The promised statement thus follows from Statement (1), the Form–Matrix-Lemma 224.5 (1b) and the following claim:

Claim. Let F be the surface of genus g minus m open disks. The intersection form of F is represented by the matrix which is the block diagonal sum of g copies of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $m - 1$ zeros on the diagonal.

Proof. We calculated the intersection form on page 4330 and, fortunately, we got the result we just claimed.



surface of genus two minus three open disks

$$Q_F = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & 0 & 0 \\ 0 & 0 & & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \\ 0 & 0 & & & \end{pmatrix}$$

(g blocks; block of $m - 1$ zeros)

Alternatively we refer to the figure above to remind the reader of the calculation of the intersection form of F. ⊞

(3) Let g be the genus of F. By (2) there exists a $P \in \mathrm{GL}(k, \mathbb{Z})$ such that $P^{\mathrm{T}}(A - A^{\mathrm{T}})P$ is the block diagonal sum of g copies of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $m - 1$ zeros on the diagonal. We now see that

$$\det(A - A^{\mathrm{T}}) \underset{\text{since } \det(P) = \pm 1}{=} \det(P^{\mathrm{T}}(A - A^{\mathrm{T}})P) \underset{\text{by a direct calculation}}{=} \begin{cases} 1, & \text{if } m = 1, \\ 0, & \text{if } m \geq 2. \end{cases}$$

■

We can now prove the following analog and generalization of the Alexander Polynomial-at-1 Proposition 118.13:

Proposition 289.17. (Alexander Polynomial-at-1 Proposition) Let $m \in \mathbb{N}$ and let L be an oriented m-component link in S^3. Then

$$\Delta_L(1) = \begin{cases} \pm 1, & \text{if } L \text{ is a knot,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $L \subseteq S^3$ be a link. It follows from the Seifert Surface Existence Theorem 288.12 that we can pick a Seifert matrix A for L. We now see that

$$\Delta_K(1) \underset{\substack{\uparrow \\ \text{by the Alexander Module} \\ \text{Seifert Matrix Theorem 289.12}}}{=} \pm\det(t\cdot A - A^{\mathrm{T}})(t=1) \overset{\substack{\text{follows from the Determinant} \\ \text{Ring Homomorphism Lemma 2.22} \\ \downarrow}}{=} \det(A - A^{\mathrm{T}}) \underset{\substack{\uparrow \\ \text{by the Seifert Matrix–Intersection} \\ \text{Form Proposition 289.16 (3)}}}{=} \begin{cases} 1, & \text{if } L \text{ is a knot,} \\ 0, & \text{otherwise.} \end{cases}$$

■

The Alexander Polynomial-at-1 Proposition 289.17 is a key ingredient in the proof of the following mathematical curiosity that was first proved by Richard Crowell [**Crow1963**, p. 353]:

Proposition 289.18. (Alexander Module-$\mathbb{Z}$-Torsion Free Proposition) For any oriented knot K the Alexander module $\mathrm{H}_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$ is a torsion-free abelian group.

Proof. The proposition follows from the Alexander Module–Seifert Matrix Theorem 289.12, the Alexander Polynomial-at-1 Proposition 289.17 and the following purely algebraic claim:

Claim. Let M be a $\mathbb{Z}[t^{\pm 1}]$-module that admits a square presentation matrix $P(t)$ which has the property that $\det(P(1)) = \pm 1$. Then M is a torsion-free abelian group.

Proof. Let $P(t) \in \mathrm{M}(k \times k, \mathbb{Z}[t^{\pm 1}])$ be a square matrix such that $\det(P(1)) = \pm 1$. We need to show that $\mathbb{Z}[t^{\pm 1}]^k / P(t) \cdot \mathbb{Z}[t^{\pm 1}]^k$ is a torsion-free abelian group. Clearly this is equivalent to showing that if for $n \in \mathbb{Z}$ and $v, w \in \mathbb{Z}[t^{\pm 1}]^k$ we have $n \cdot v = P(t) \cdot w$, then w is also a multiple of n.

So assume that we have $n \in \mathbb{Z}$ and $v = (v_1, \dots, v_k), w = (w_1, \dots, w_k) \in \mathbb{Z}[t^{\pm 1}]^k$ with $n \cdot v = P(t) \cdot w$. We introduce two algebraic objects:

(i) We denote by $\mathrm{Adj}(P(t)) \in \mathrm{M}(k \times k, \mathbb{Z}[t^{\pm 1}])$ the adjugate of $P(t)$ that we introduced in Definition 2.20. Recall that this means that $\mathrm{Adj}(P(t))$ is the unique matrix in $\mathrm{M}(k \times k, \mathbb{Z}[t^{\pm 1}])$ such that we have the equality $\mathrm{Adj}(P(t)) \cdot P(t) = \det(P(t)) \cdot \mathrm{id}_k$.

(ii) Given a non-zero polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ we define its *content* $\mathrm{cont}(f(t))$ as the greatest common divisor of its coefficients. Note that the Gauss lemma (see e.g. [**Lang2002b**, Theorem IV.2.1] and [**Gaus1801**, Article 42]) says that the content is multiplicative.

We now see that for each $i \in \{1, \dots, k\}$ we have the following equalities in $\mathbb{Z}$:

$$\begin{aligned} \mathrm{cont}(w_i(t)) &\overset{\substack{\text{by the Determinant–Ring} \\ \text{Homomorphism Lemma 2.22 we have} \\ (\det(P(t)))(t=1) = \det(P(1)) = \pm 1 \\ \downarrow}}{=} \mathrm{cont}(\det(P(t))) \cdot \mathrm{cont}(w_i(t)) \overset{\substack{\text{by the Gauss Lemma} \\ \downarrow}}{=} \mathrm{cont}(\det(P(t)) \cdot w_i(t)) \\ &\underset{\substack{\uparrow \\ \text{by (i)}}}{=} \mathrm{cont}(\mathrm{Adj}(P(t)) \cdot P(t) \cdot w_i(t)) \underset{\substack{\uparrow \\ \text{by choice of } v \text{ and } w}}{=} \mathrm{cont}(\mathrm{Adj}(P(t)) \cdot n \cdot v_i(t)) \\ &\underset{\substack{\uparrow \\ \text{by the Gauss Lemma or} \\ \text{elementary algebra}}}{=} n \cdot \mathrm{cont}(\mathrm{Adj}(P(t)) \cdot v_i(t)). \end{aligned}$$

We have thus shown that n divides the content of $w_i(t)$. But this implies that n divides the polynomial $w_i(t)$. ■

We conclude this section with a discussion of what polynomials can occur as Alexander polynomials of oriented knots and links. First let $K \subseteq S^3$ be a knot. By the Alexander Polynomial–Symmetry Proposition 289.15 and the Alexander Polynomial-at-1 Proposition 289.17 we now know that $\Delta_K(t) \doteq \Delta_K(t^{-1})$ and that $\Delta_K(1) = \pm 1$. The following proposition shows that these are the only two general properties of Alexander polynomials of knots:

Proposition 289.19. (Alexander Polynomial–Realization Proposition) Given any Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ with $p(t) \doteq p(t^{-1})$ and with $p(1) = \pm 1$ there exists a knot $K \subseteq S^3$ with $\Delta_K(t) \doteq p(t)$.

Proof. The proposition was first proved by Herbert Seifert [**Sei1934**, Satz 6]. An alternative proof was given by Jerome Levine [**Levine1965c**, p. 136]. A textbook proof is given in [**BZH2014**, Theorem 8.26]. ■

Interestingly we (i.e. humanity) are still at a loss at what multivariable polynomials can occur as Alexander polynomials of oriented links. In other words, the following question is still open:

Question 289.20. Which polynomials $p(t_1, \dots, t_m) \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ occur as the multivariable Alexander polynomial of an oriented m-component link?

289.7. **Homology groups of finite cyclic coverings of knot complements.** In this final section, given a knot K, we will relate the Alexander module to the homology groups of finite cyclic coverings of $S^3 \setminus K$. Furthermore we will use Seifert matrices to calculate the homology groups of these finite cyclic coverings. We start out with the following definition:

Notation 289.21. Let $K \subseteq S^3$ be an oriented knot and let $m \in \mathbb{N}$.

(1) Note that there exists (up to a sign) a unique epimorphism $\pi_1(S^3 \setminus K) \to \mathbb{Z}/m\mathbb{Z}$. As we explained in the Cocyclic Subgroup Lemma 113.16, the kernel of $\pi_1(S^3 \setminus K) \to \mathbb{Z}/m\mathbb{Z}$ is well-defined. The corresponding m-fold cyclic covering of the knot complement $E(K) := S^3 \setminus K$ is called the m-fold cyclic covering $p\colon E_m(K) \to S^3$ of $E(K)$.

(2) If K is oriented and if $\mu_K \subseteq S^3 \setminus K$ is a meridian of K, then we consider the submanifold $\widetilde{\mu}_K^m := p^{-1}(\mu_K)$ of $E_m(K)$.

Our goal will be to calculate the homology groups of the m-fold cyclic coverings $E_m(K)$ of $S^3 \setminus K$. We start out with the following slightly technical lemma:

Lemma 289.22. (Preimage-of-Meridian Lemma) Let $K \subseteq S^3$ be an oriented knot with meridian μ_K and let $m \in \mathbb{N}$. We consider the m-fold cyclic covering $p\colon E_m(K) \to S^3$ of the knot complement $S^3 \setminus K$.

(1) The submanifold $\widetilde{\mu}_K^m$ of $E_m(K)$ is connected and it is diffeomorphic to S^1.

(2) We consider the induced map $p_*\colon \mathrm{H}_1(E_m(K)) \to \mathrm{H}_1(S^3 \setminus K)$.
   (a) We have $p_*([\widetilde{\mu}_K^m]) = m \cdot [\mu_K]$.
   (b) We have $p_*(\mathrm{H}_1(E_m(K)) = \mathbb{Z} \cdot m \cdot [\mu_K]$.

(3) We denote by $i\colon \widetilde{\mu}_K^m \to E_m(K)$ the inclusion. The following sequence is exact and it splits:

$$0 \to \underbrace{\mathrm{H}_1(\widetilde{\mu}_K^m)}_{\cong \mathbb{Z}} \xrightarrow{i_*} \mathrm{H}_1(E_m(K)) \xrightarrow{\text{я}} \mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m) \to 0.$$

In particular we have an isomorphism $\mathrm{H}_1(E_m(K)) \cong \mathbb{Z} \oplus \mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m)$.

The Preimage-of-Meridian Lemma 289.22 says that the groups $\mathrm{H}_k(E_m(K))$ consist of a $\mathbb{Z}$-summand and another summand $\mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m)$ which presumably actually depends on K and m. Thus our goal, after the proof of Preimage-of-Meridian Lemma 289.22, will be to determine the second summand.[3129]

Proof. Let K be an oriented knot in S^3. As usual we denote by $\Phi_K \colon \mathrm{H}_1(S^3 \setminus K) \to \mathbb{Z}$ the map that is given by $\Phi_K([\mu_K]) = 1$.

(1) Recall that we know by the Link Complement-H_*–Basis-Proposition 283.4 that we have the equality $\mathrm{H}_1(S^3 \setminus K) = \mathbb{Z} \cdot [\mu_K]$. It follows from the Preimage-of-Covering Proposition 129.7 that $p^{-1}(\mu_K) \to \mu_K$ is just the covering corresponding to the kernel of the *epimorphism* $\pi_1(\mu_K) \to \pi_1(S^3 \setminus K) \to \mathrm{H}_1(S^3 \setminus K) \xrightarrow{\Phi_K} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. Thus the submanifold $\widetilde{\mu}_K^m$ of $E_m(K)$ is connected and it is diffeomorphic to S^1.

(2) We consider the following diagram:

$$\begin{array}{ccccc}
\pi_1(E_m(K)) & \xrightarrow[[\widetilde{\mu}_K^m] \mapsto [\mu_K]^m]{p_*} & \pi_1(S^3 \setminus K) & \xrightarrow{\Phi_K} & \mathbb{Z} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \mathrm{id} \\
\pi_1(E_m(K))_{\mathrm{ab}} & \xrightarrow{p_*} & \pi_1(S^3 \setminus K)_{\mathrm{ab}} & \xrightarrow{\Phi_K} & \mathbb{Z} \\
\big\downarrow \cong & & \big\downarrow \cong & & \big\downarrow \mathrm{id} \\
\mathrm{H}_1(E_m(K)) & \xrightarrow{p_*} & \mathrm{H}_1(S^3 \setminus K) & \xrightarrow{\Phi_K} & \mathbb{Z}.
\end{array}$$

We make the following clarifications and observations:

- Since $p \colon \widetilde{\mu}_K^m \to \mu_K$ is a connected m-fold covering we see that the generator of $\pi_1(\widetilde{\mu}_K^m)$ gets sent to the m-th power of the generator of $\pi_1(\mu_K)$. It follows that $(\Phi_K \circ p_*)([\widetilde{\mu}_K^m]) = m$.
- By definition of $E_m(K)$ we have $(\Phi_K \circ p_*)(\pi_1(E_m(K))) = m \cdot \mathbb{Z}$.
- The top left vertical maps are the natural epimorphisms from a group G to its abelianization G_{ab}.
- The vertical maps at the bottom are the isomorphisms from the Hurewicz Theorem 166.10.

The desired conclusions follow easily from the above discussion.

(3) We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_1(\widetilde{\mu}_K^m) & \xrightarrow{i_*} & \mathrm{H}_1(E_m(K)) & \xrightarrow{\text{я}} & \mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m) & \longrightarrow & 0 \\
\big\downarrow \cong & \searrow^{\Phi_K \circ p_*} & \big\downarrow p_* & & & & \\
 & & \mathrm{H}_1(S^3 \setminus K) & & & & \\
 & & \big\downarrow \Phi_K & & & & \\
m \cdot \mathbb{Z} & \hookrightarrow & \mathbb{Z}. & & & &
\end{array}$$

[3129] Later, in the Knot-Branched Covering-H_*-Proposition **??**, we will see that the group $\mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m)$ is isomorphic to the first homology group of the "m-fold branched covering along K" which we will introduce in **??**.

We make the following clarifications and observations:

(i) On top we see the long exact sequence in reduced homology of the pair $(E_m(K), \widetilde{\mu}_K^m)$ from the Reduced LES-of-Pair Proposition 147.8. By (1) we know that $\widetilde{\mu}_K^m$ is path-connected. Thus we obtain from the Reduced Homology Lemma 147.3 that $\widetilde{\mathrm{H}}_0(\widetilde{\mu}_K^m) = 0$. This explains why the exact sequence ends abruptly with a "0".

(ii) By (2b) we know that the image of the map $\Phi_K \circ p_* \colon \mathrm{H}_1(E_m(K)) \to \mathrm{H}_1(S^3 \setminus K)$ is given by $m \cdot \mathbb{Z}$.

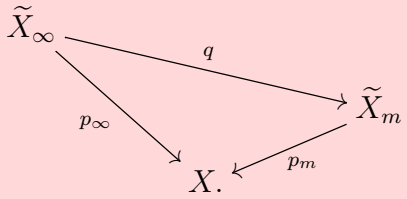
(iii) It follows from (ii) and (2a) that the left vertical map is an isomorphism.

It follows from the above that the map i_* is a monomorphism and that the top sequence is exact. Furthermore it follows from (iii) together with the Splitting Lemma 145.5 that the sequence splits and that the middle group is isomorphic to the direct sum of the two "outer" groups. ■

Our real goal is to calculate the homology groups $\mathrm{H}_1(E_m(K))$. The key to understanding these homology groups is the following proposition, which is perhaps of independent interest:

Proposition 289.23. (Infinite–Finite Cyclic Covering Proposition) Let X be a topological space that is Hausdorff, regionally compact, path-connected, locally path-connected and semi-locally simply connected. Let $\Phi \colon \mathrm{H}_1(X) \to \langle t \rangle$ be an epimorphism and let $m \in \mathbb{N}$. We denote by $\Phi_m \colon \mathrm{H}_1(X) \to \langle t \,|\, t^m \rangle$ the epimorphism that is given by the composition of Φ with the natural projection $\langle t \rangle \to \langle t \,|\, t^m \rangle$. We introduce the following notation:

- We denote by $p_\infty \colon \widetilde{X}_\infty \to X$ the covering corresponding to $\ker(\Phi \colon \mathrm{H}_1(X) \to \langle t \rangle)$.
- We denote by $p_m \colon \widetilde{X}_m \to X$ the covering corresponding to $\ker(\Phi_m \colon \mathrm{H}_1(X) \to \langle t \,|\, t^m \rangle)$.
- We denote by $q \colon \widetilde{X}_\infty \to \widetilde{X}_m$ the natural covering with $p_m \circ q = p_\infty$ that is given by the Covering–Covering Theorem 127.15. In summary, we have the following commutative diagram of coverings:



- We equip $\widetilde{X}_\infty$, and thus also $\mathrm{H}_k(\widetilde{X}_\infty)$, with the $\langle t \rangle$-action from Convention 286.3.

There exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_k(\widetilde{X}_\infty) \xrightarrow{\cdot(t^m-1)} \mathrm{H}_k(\widetilde{X}_\infty) \xrightarrow{q_*} \mathrm{H}_k(\widetilde{X}_m) \to \mathrm{H}_{k-1}(\widetilde{X}_\infty) \to \ldots$$

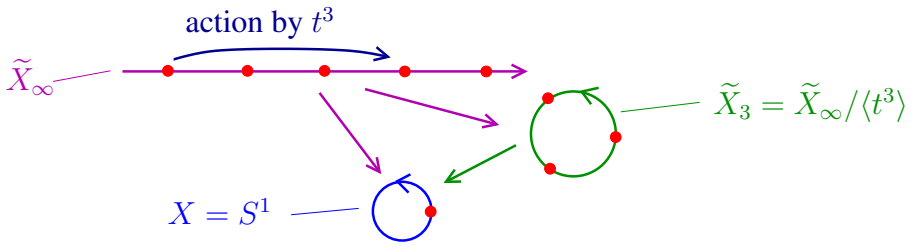
More generally, for any subspace $A \subseteq X$ there exists a long exact sequence of the form

$$\to \mathrm{H}_k(\widetilde{X}_\infty, p_\infty^{-1}(A)) \xrightarrow{\cdot(t^m-1)} \mathrm{H}_k(\widetilde{X}_\infty, p_\infty^{-1}(A)) \xrightarrow{q_*} \mathrm{H}_k(\widetilde{X}_m, p_m^{-1}(A)) \to \mathrm{H}_{k-1}(\widetilde{X}_\infty, p_\infty^{-1}(A)) \to$$

Sketch of proof. [3130] In the following we provide the proof for the case $A = \varnothing$. The proof for the general case is almost the same.

Most likely it will not come as a surprise to the reader that we will prove the existence of a long exact sequence of homology groups by constructing a short exact sequence of

[3130] We will give an alternative proof later in Section 323.6.



chain complexes. More precisely it follows from the LES-Proposition 145.10 that it suffices to prove the following statement:

Statement. For each $k \in \mathbb{N}_0$ the following sequence of chain maps is exact:

$$0 \;\to\; \mathrm{C}_k(\widetilde{X}_\infty) \xrightarrow{\cdot(t^m-1)} \mathrm{C}_k(\widetilde{X}_\infty) \xrightarrow{q_*} \mathrm{C}_k(\widetilde{X}_m) \;\to\; 0.$$

The key to proving this statement is to give an explicit description of $\widetilde{X}_m$:

Claim 1. Note that the subgroup $\langle t^m\rangle$ acts on $\widetilde{X}_\infty$. The topological space $\widetilde{X}_m$ is homeomorphic to $\widetilde{X}_\infty/\langle t^m\rangle$.

Proof. Note that it follows from the Deck Group–Discrete Action Lemma 128.2 that this action is discrete. Thus it follows from the Action–Covering Proposition 91.11 (2) that the map $q\colon \widetilde{X}_\infty/\langle t^m\rangle \to X$ is a covering. It now follows from the Action–π_1-Theorem 93.1 together with Convention 286.3 that $p_*(\pi_1(\widetilde{X}_\infty/\langle t^m\rangle)) = \ker(\Phi_m)$. In other words it now follows from the Covering Unique Existence Theorem 127.17 that we can assume that $\widetilde{X}_m = \widetilde{X}_\infty/\langle t^m\rangle$. ⊞

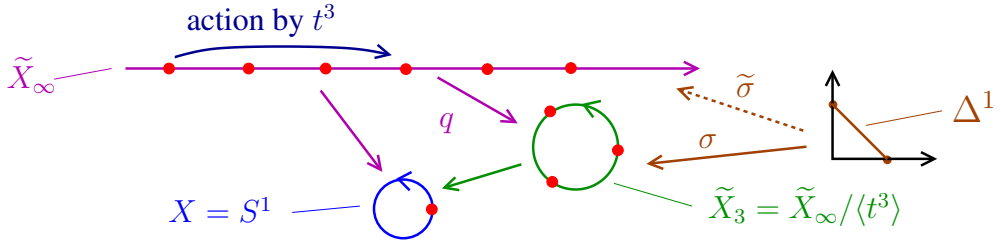
It follows from Claim 1 that it suffices to prove the following claim:

Claim 2. The following sequence is exact:

$$0 \;\to\; \mathrm{C}_k(\widetilde{X}_\infty) \xrightarrow{\cdot(t^m-1)} \mathrm{C}_k(\widetilde{X}_\infty) \xrightarrow{q_*} \mathrm{C}_k(\widetilde{X}_\infty/\langle t^m\rangle) \;\to\; 0.$$

Proof. This claim follows from the following observations:

- Since we assume that X is Hausdorff and regionally compact we obtain from Exercise 9.14 that the action of $\langle t^m\rangle$ on $\widetilde{X}_\infty$ is proper in the sense of Definition 9.4. Since the group $\langle t^m\rangle$ is infinite it follows from Exercise 9.15 that for any compact subset K we have $t^m \cdot K \not\subseteq K$. By applying this to the support of any chain in $\mathrm{C}_k(\widetilde{X}_\infty)$ we see that the map to the left is a monomorphism.
- Note that the t^m-action commutes with the projection $\widetilde{X}_\infty \to \widetilde{X}_\infty/\langle t^m\rangle$. It is thus clear that the composition of the two maps is the zero map.
- Let $\sigma\colon \Delta^k \to \widetilde{X}_\infty/\langle t^m\rangle$ be a singular k-simplex. It follows from the Map Lifting Criterion 127.5 that there exists a lift of σ to a continuous map $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}_\infty$. This shows that the map to the right is an epimorphism.



- Finally note that it follows immediately from the definitions that the homomorphism $q_*\colon \mathrm{C}_k(\widetilde{X}_\infty) \to \mathrm{C}_k(\widetilde{X}_\infty/\langle t^m\rangle)$ is given by identifying any two singular k-simplices in $\widetilde{X}_\infty$ that differ by the action of some t^{rm}, with $r \in \mathbb{Z}$. Using the observation that $t^{rm} - 1 = (t^m - 1) \cdot (1 + t^m + t^{2m} + t^{r(m-1)}) \in \mathbb{Z}[t^{\pm 1}]$ one can now easily verify that the kernel of p_* is contained in the map $\cdot(t^m - 1)$. ■

The following proposition, together with the Preimage-of-Meridian Lemma 289.22, gives us the calculation of the homology groups of the finite cyclic coverings of a knot complement:

Proposition 289.24. Knot–Finite Cyclic Covering-H_1-Proposition) Let $K \subseteq S^3$ be an oriented knot and let A be a Seifert matrix for K of size $2g$. Let $m \in \mathbb{N}$. We have an exact sequence of the form

$$(\mathbb{Z}^{2g})^m \xrightarrow{D_m} (\mathbb{Z}^{2g})^m \to \mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m) \to 0$$

where for $m \in \mathbb{N}_{\geq 2}$ we have

$$D_m = \begin{pmatrix} -A^{\mathrm{T}} & 0 & 0 & \dots & A \\ A & -A^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ \vdots & & & -A^{\mathrm{T}} & 0 \\ 0 & \dots & 0 & A & -A^{\mathrm{T}} \end{pmatrix}$$

and where for $m = 1$ we have $D_1 = A - A^{\mathrm{T}}$. Furthermore the t-action on $\mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m)$ corresponds to the obvious "block"-shift on $(\mathbb{Z}^{2g})^m$ "to the right".

Example. A heroic calculation, using the above Knot–Finite Cyclic Covering-H_1-Proposition 289.24 and using the Seifert matrices from page 5530, leads us to the following table:[3131]

$\mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m)$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
trefoil	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$
figure-8 knot	0	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$	...	

These homology groups appear rather random. But some times there is some structure. For example in Exercise 290.9 we will see that the homology groups for the trefoil are 6-periodic in m. □

Proof. We start out with the following variation on the Alexander Module–Seifert Matrix Theorem 289.12:

Claim. There exists an exact sequence of the form

$$(\mathbb{Z}[t^{\pm 1}]/(t^m - 1)) \xrightarrow{t \cdot A - A^{\mathrm{T}}} (\mathbb{Z}[t^{\pm 1}]/(t^m - 1)) \to \mathrm{H}_k(\widetilde{X}_m, \widetilde{\mu}_K^m) \to 0.$$

Proof. We consider the coverings $p_\infty\colon \widetilde{X}_\infty \to S^3 \setminus K$ and $p_m\colon E_m(K) \to S^3 \setminus K$. Let μ_K be a meridian for K. We write $\widetilde{\mu}_K^\infty = p_\infty^{-1}(\mu_K)$ and $\widetilde{\mu}_K^m = p_m^{-1}(\mu_K)$. Furthermore we denote by $q\colon \widetilde{X}_\infty \to \widetilde{X}_m$ the natural covering with $p_m \circ q = p_\infty$ that is given by the Covering–Covering

[3131] Note that it follows from the Hurewicz Theorem 166.10 that these calculations are consistent with the calculations on page 2209.

Theorem 127.15. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathrm{H}_1(\widetilde{X}^\infty, \widetilde{\mu}_K^\infty) & \xrightarrow{t^m-1} & \mathrm{H}_1(\widetilde{X}^\infty, \widetilde{\mu}_K^\infty) & \xrightarrow{q_*} & \mathrm{H}_1(\widetilde{X}_m, \widetilde{\mu}_K^m) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z}[t^{\pm 1}]^{2g} & \xrightarrow{t^m-1} & \mathbb{Z}[t^{\pm 1}]^{2g} & \longrightarrow & (\mathbb{Z}[t^{\pm 1}]/(t^m-1))^{2g} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle t\cdot A - A^{\mathrm{T}}} & & \uparrow{\scriptstyle t\cdot A - A^{\mathrm{T}}} & & \uparrow{\scriptstyle t\cdot A - A^{\mathrm{T}}} & & \\
0 & \longrightarrow & \mathbb{Z}[t^{\pm 1}]^{2g} & \xrightarrow{t^m-1} & \mathbb{Z}[t^{\pm 1}]^{2g} & \longrightarrow & (\mathbb{Z}[t^{\pm 1}]/(t^m-1))^{2g} & \longrightarrow & 0.
\end{array}$$

We make the following clarifications and observations:

(a) The top horizontal sequence is the long exact sequence from the Infinite–Finite Cyclic Covering Proposition 289.23. Here we use that it follows from the elementary Exercise 146.5 that $\mathrm{H}_0(\widetilde{X}^\infty, \widetilde{\mu}_K^\infty) = 0$.

(b) The bottom two horizontal sequences are clearly exact.

(c) The argument of the proof of the Preimage-of-Meridian Lemma 289.22 shows that $\widetilde{\mu}_K^\infty$ is the universal covering of μ_K, i.e. it is diffeomorphic to $\mathbb{R}$. It now follows from the Reduced LES-of-Pair Proposition 147.8 that the natural map $\text{я}\colon \mathrm{H}_1(\widetilde{X}^\infty) \to \mathrm{H}_1(\widetilde{X}^\infty, \widetilde{\mu}_K^\infty)$ is an isomorphism. The two sequences to the left are thus essentially the same as the exact sequence provided by the Alexander Module–Seifert Matrix Theorem 289.12.

(d) It now follows from an amusing diagram chase, see Exercise 289.16, that the right sequence is also exact. ⊞

Next we consider the following diagram:

$$\begin{array}{ccccccc}
(\mathbb{Z}^{2g})^m & & \longrightarrow & & (\mathbb{Z}^{2g})^m & & \\
\cong\downarrow & (v_1, v_2, \dots, v_m) & & & \cong\downarrow & (v_1, v_2, \dots, v_m) & \\
 & \downarrow & & & & \downarrow & \\
 & v_1 + \dots + t^{m-1}\cdot v_m & & & & v_1 + \dots + t^{m-1}\cdot v_m & \\
(\mathbb{Z}[t^{\pm 1}]/(t^m-1))^{2g} & & \xrightarrow{t\cdot A - A^{\mathrm{T}}} & & (\mathbb{Z}[t^{\pm 1}]/(t^m-1))^{2g} & \longrightarrow & \mathrm{H}_k(\widetilde{X}_m, \widetilde{\mu}_K^m) \longrightarrow 0.
\end{array}$$

We make the following observations:

- The bottom sequence is the exact sequence from the claim.
- Clearly the vertical maps are isomorphisms.

One can now easily verify that the corresponding matrix on top is given by the matrix in the statement of the proposition. Finally the statement regarding the t-action follows easily from the explicit isomorphisms. ■

We record the following special case of the Knot–Finite Cyclic Covering-H_1-Proposition 289.24:

Proposition 289.25. Knot–Double Covering-H_1-Proposition) Let $K \subseteq S^3$ be an oriented knot and let A be a Seifert matrix for K. The following two statements hold:

(1) (a) A presentation matrix for $\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$ is given by $A + A^{\mathrm{T}}$.
(b) The homology group $\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$ is a finite group of size $|\Delta_K(-1)|$, which is an odd number.

(2) The t-action on $\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$ is given by multiplication by -1.

Proof.

(1) (a) First note that it follows from the Finite Cyclic Covering Proposition 289.24 that $\begin{pmatrix} -A^{\mathrm{T}} & A \\ A & -A^{\mathrm{T}} \end{pmatrix}$ is a presentation matrix for $\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$. Next we say that two integer matrices are equivalent if they represent isomorphic abelian groups. Using the moves of the Presentation Matrix–Uniqueness Proposition 287.8 we now see that

$$\begin{pmatrix} -A^{\mathrm{T}} & A \\ A & -A^{\mathrm{T}} \end{pmatrix} \underset{\uparrow}{\sim} \begin{pmatrix} -A^{\mathrm{T}} & A - A^{\mathrm{T}} \\ A & A - A^{\mathrm{T}} \end{pmatrix} \underset{\uparrow}{\sim} \begin{pmatrix} -A - A^{\mathrm{T}} & 0 \\ A & A - A^{\mathrm{T}} \end{pmatrix} \underset{\uparrow}{\sim} (A + A^{\mathrm{T}}).$$

add first column to the second column

subtract second row from the first row

by the Seifert Matrix–Intersection Form Proposition 289.16 (3) we know that the matrix $A - A^{\mathrm{T}}$ is invertible

(b) We perform the following calculation modulo two:

$$\begin{array}{ccccccc} & \text{by (a)} & & & \text{by the Finite Matrix Quotient Lemma 97.23} & & \\ & \downarrow & & & \downarrow & & \\ |\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)| & = & |\mathbb{Z}^{2g}/(A + A^{\mathrm{T}}) \cdot \mathbb{Z}^{2g}| & = & |\det(A + A^{\mathrm{T}})| & & \\ & \equiv & \det(-A - A^{\mathrm{T}}) & = & \Delta_K(-1) & = & 1. \\ & \uparrow & & & \uparrow & & \uparrow \\ & \text{since the determinant commutes with ring homomorphisms} & & \text{by the Alexander Module Seifert Matrix Theorem 289.12} & & \text{by the Alexander Polynomial-at-1 Proposition 289.17} & \end{array}$$

(2) We will prove this statement in Exercise 289.19. ■

Remark. Let $K \subseteq S^3$ be a knot. In the Knot–Double Covering-H_1-Proposition 289.25 we just saw that $|\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)|$ is determined by the Alexander polynomial. In the Knot-Cyclic Covering-H_*-Proposition **??** we will express for any $m \in \mathbb{N}$ the "size" of the group $|\mathrm{H}_1(E_m(K), \widetilde{\mu}_K^2)|$ in terms of the Alexander polynomial. □

We conclude this chapter with the definition of what is arguably the most elementary non-trivial invariant of knots and links:

Definition 289.26. [3132] Let $L \subseteq S^3$ be an oriented non-empty link. We refer to

$$\det(L) \ := \ |\det(\Delta_L(-1))|$$

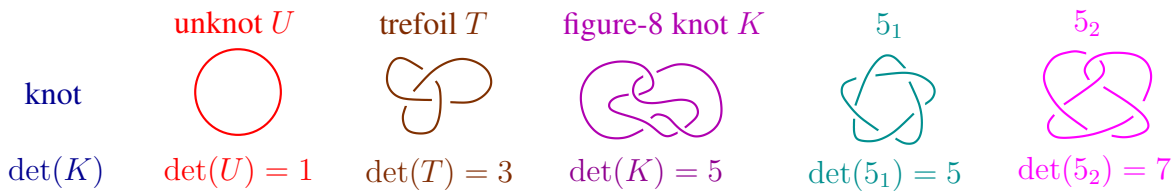
as the determinant of L. If L is in fact an oriented knot, then it follows from the Knot–Double Covering-H_1-Proposition 289.25 that $\det(L) \in \mathbb{N}$ is odd, in particular in this setting the determinant is non-zero.[3133]

Remark. This invariant does appear on various occasions. In fact in the Knot–Double Covering-H_1-Proposition 289.25 we already saw that the determinant is precisely the number of elements in $\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$. Furthermore, in Exercise 289.20 we will see that the determinant is related to the "n-colorings of knots". □

Example. In the following table we show the determinants of some "small knots":

[3132] We encountered this definition already in Exercise 117.20.

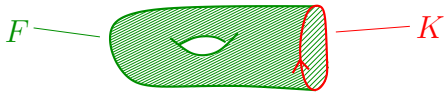
[3133] In Definition 289.26 we will introduce a refined version of the determinant of an oriented non-empty link L.



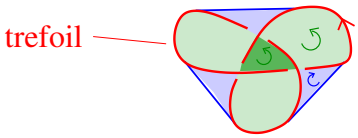
Here the determinants for the first three knots follow from the calculation of the corresponding Alexander polynomials on page 5530. The remaining values will be determined in Exercise 289.3. □

289.8. Exercises for Chapter 289.

Exercise 289.1. Let K be the trivial knot with the connected Seifert surface F that is shown below. Calculate a Seifert matrix corresponding to this connected Seifert surface.



Exercise 289.2. In the figure below we see the trefoil together with some connected Seifert surface. Determine a Seifert matrix that corresponds to this connected Seifert surface.

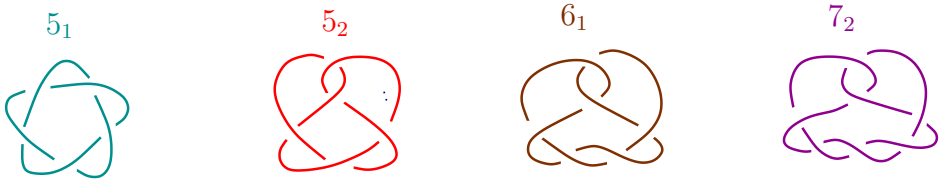


Exercise 289.3. We consider the knots 5_1, 5_2, 6_1 and 7_2 which are shown below.

(a) In each case find a connected Seifert surface.
Hint. In each case you can find a connected Seifert surface of genus one. These are somewhat similar to the connected Seifert surfaces for the trefoil (see page 5528) and the figure-8 knot (see page 5528).

(b) In each case calculate the Alexander polynomial.

(c) In each case calculate the determinant.



Exercise 289.4.

(a) Let $a_1, a_2 \in \mathbb{Z}$. We consider the knot K with the connected Seifert surface F that is shown below. Here the numbers a_1 and a_2 indicate how many full positive (respectively negative) twists we put into the bands. Calculate a Seifert matrix and calculate the corresponding Alexander polynomial.

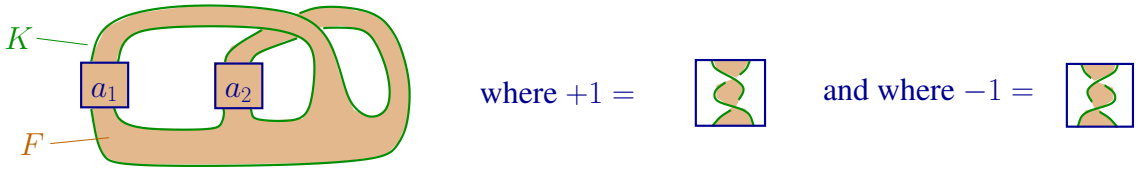
(b) Let $B \in \mathrm{M}(2g \times 2g, \mathbb{Z})$ be a matrix such that

$$B - B^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Show that there exists an oriented knot K such that B is a Seifert matrix for K.
Hint. Modify the construction of the knot in (a).

(c) Let $B \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix such that $\det(B - B^{\mathrm{T}}) = 1$. Show that there exists a $P \in \mathrm{GL}(n, \mathbb{Z})$ such that $P \cdot (B - B^{\mathrm{T}}) \cdot P^{\mathrm{T}}$ is the block diagonal sum of matrices of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(d) Let $B \in \mathrm{M}(2g \times 2g, \mathbb{Z})$ such that $\det(B - B^{\mathrm{T}}) = 1$. Show that there exists an oriented knot K such that B is a Seifert matrix for K.



Exercise 289.5. We consider the knots 6_1 and 9_{46} that are shown below.[3134]

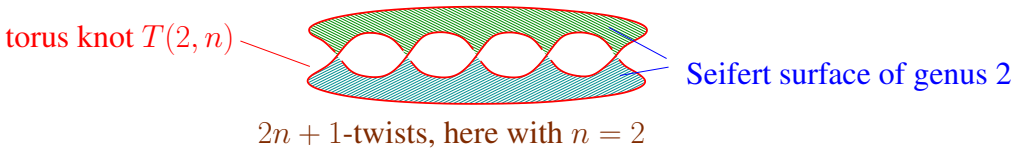
(a) Show that they have the same Alexander polynomial.
(b) Show that they can be distinguished using the first elementary ideal of the Alexander module.
(c) Show that they can be distinguished by considering the homology groups of the 2-fold cyclic coverings.



Exercise 289.6. Let $n \in \mathbb{N}$. We consider the $(2, 2n+1)$-torus knot $T(2, 2n_1)$ that is shown below in the figure.

(a) Show that $T(2, 2n_1)$ sits on the boundary of a smoothly embedded solid torus in S^3.[3135]
(b) Calculate the Alexander polynomial of the $(2, 2n + 1)$-torus knot.[3136]

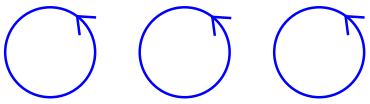
[3134] The knot 9_{46} is not shown in the standard way with a minimal number of crossings, but in a way that makes it easy to find a connected Seifert surface of genus one.
[3135] Whence the name "torus knot".
[3136] Helpfully we already drew a connected Seifert surface for $T(2, 2n + 1)$.



Exercise 289.7.

(a) Determine a Seifert matrix for the trivial 3-component link L and compute the single-variable Alexander polynomial $\Delta_L(t)$.

(b) Determine a Seifert matrix for the Hopf link H and compute the single-variable Alexander polynomial $\Delta_H(t)$.





Exercise 289.8. Let $m \in \mathbb{Z}$.

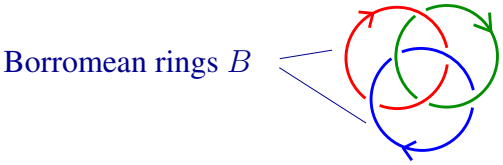
(a) Show that there exists an oriented 2-component link L such that $\Delta_L(t) = m \cdot (t-1)$.
Hint. Modify the Hopf link.

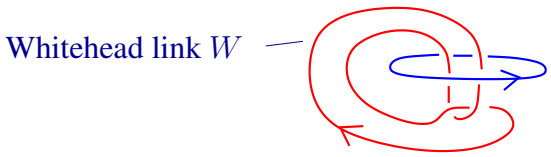
(b) We assume that m is odd. Show that there exists an oriented 2-component link L such that $\Delta_L(t) = t^m - 1$.
Hint. Let $n \in \mathbb{N}$.For the $(2, 2n+1)$-torus knot $T(2, 2n+1)$ we proved in Exercise 289.6 that $\Delta_{T(2,2n+1)}(t) = \frac{t^{2n+1}-1}{t-1}$. Turn the torus knot into a link with the right Alexander polynomial.

Exercise 289.9. Compute the 1-variable Alexander polynomial of the following links:

(a) The Borromean rings.

(b) The Whitehead link.





Exercise 289.10. Let R be a Noetherian UFD and let M and N be two finitely generated R-modules. Show that if there exists an epimorphism $f\colon M \to N$, then $\mathrm{ord}_R(M)$ is divisible by $\mathrm{ord}_R(N)$.

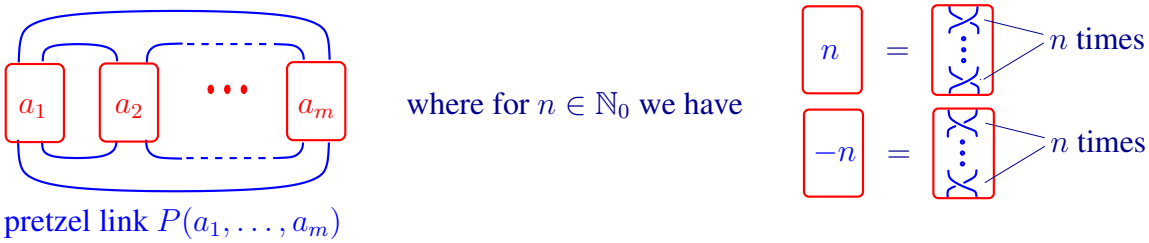
Exercise 289.11. Let K and J be two knots. We assume that there exists an epimorphism $\pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus J)$. Show that $\Delta_J(t)$ is a divisor of $\Delta_K(t)$.
Remark. Make use of Exercise 289.10.

Exercise 289.12. Let $m \in \mathbb{N}$ and let L be an oriented m-component link in S^3. Show that there exists a $p(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $\Delta_L(t) \doteq (t-1)^{m-1} \cdot p(t)$.
Hint. Make use of the Seifert Matrix–Intersection Form Proposition 289.16.

Exercise 289.13. Let L be an oriented link. We assume that there exists a *disconnected* compact oriented 2-dimensional smooth submanifold F with no closed components such that $\partial F = L$. Show that $\Delta_L = 0$.
Hint. In the proof of the Make-Surface-Connected Lemma 288.13 we saw that we can turn F into a *connected* submanifold by "adding tubes". Consider a Seifert corresponding to such a connected Seifert surface.

Exercise 289.14. Let $L \subseteq S^3$ be an oriented link. Show that $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$-module and show that it is completely determined by $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$.

Exercise 289.15. Let $a_1, \dots, a_m \in \mathbb{Z}$. We refer to the figure below for the definition of the *pretzel link* $P(a_1, \dots, a_m)$:
(a) Show that $P(a_1, \dots, a_m)$ is smoothly isotopic to $P(a_2, \dots, a_m, a_1)$.
(b) Show that the mirror of a pretzel link is again a pretzel link.
(c) For which $a_1, \dots, a_m$ is $P(a_1, \dots, a_m)$ a knot?
Hint. First show that the answer only depends on the parity of $a_1, \dots, a_m$.
(d) Show that the Alexander polynomial of $P(-3, 5, 7)$ equals $\doteq 1$.
Remark. This exercise is essentially the same as Exercise 117.12.



Exercise 289.16. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \uparrow & & \uparrow & & \uparrow & & \\ A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0. \end{array}$$

We assume that the three horizontal sequences are exact and that the two left vertical sequences are exact. Show that the right vertical sequence is also exact.
Remark. This exercise is identical to Exercise 145.15.

Exercise 289.17. Let K be an oriented knot in S^3. We consider the Alexander module $\mathrm{H}_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$. Show that multiplication by $t - 1$ is an isomorphism.
Hint. Use the Infinite–Finite Cyclic Covering Proposition 289.23.

Exercise 289.18. Explore the statement of the Infinite–Finite Cyclic Covering Proposition 289.23 for $X = S^1$, some $m \in \mathbb{N}$ and the usual isomorphism $\Phi\colon \mathrm{H}_1(S^1) \to \langle t \rangle$.

Exercise 289.19.

(a) Let $A \in \mathrm{M}(2g \times 2g, \mathbb{Z})$ be Seifert matrix for some knot K. We consider the module

$$\begin{matrix}\mathbb{Z}^{2g} \\ \mathbb{Z}^{2g}\end{matrix} \Big/ \begin{pmatrix} A & -A^{\mathrm{T}} \\ -A^{\mathrm{T}} & A \end{pmatrix} \begin{matrix}\mathbb{Z}^{2g} \\ \mathbb{Z}^{2g}\end{matrix}.$$

Show that the action given by permuting the two "blocks" is given by -1.

(b) Let $K \subseteq S^3$ be an oriented knot. Show that the t-action on $\mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$ is given by multiplication by -1.

Remark. This is just the content of the Knot–Double Covering-H_1-Proposition 289.25.

Exercise 289.20. Let $K \subseteq S^3$ be an oriented knot. We write $H := \mathrm{H}_1(E_2(K), \widetilde{\mu}_K^2)$. In the following let $\varphi\colon H \to H$ be the automorphism that is given by $\varphi(h) = -h$. As in Definition 93.11 we consider the semidirect product $H \rtimes \mathbb{Z}/2\mathbb{Z}$.

(a) Show that $\pi_1(S^3 \setminus K)$ admits an epimorphism onto $H \rtimes \mathbb{Z}/2\mathbb{Z}$.

(b) Let $n \in \mathbb{N}$. We consider the dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ from page 1815. Show that $\pi_1(S^3 \setminus K)$ admits an epimorphism onto the dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ if and only if $n | \det(K)$.

Remark. Together with Exercise 113.11 this shows that a knot K admits a non-trivial n-coloring if and only if $n | \det(K)$.

Exercise 289.21. Let $K \subseteq S^3$ be an oriented knot and let A be a $(2g \times 2g)$-Seifert matrix which is invertible over $\mathbb{Z}$, i.e. we have $\det(A) = \pm 1$. Show that for every $m \in \mathbb{N}$ the matrix $((A^{\mathrm{T}})^{-1} \cdot A)^m - \mathrm{id}_{2g}$ is a presentation matrix for $\mathrm{H}_1(E_m(K), \widetilde{\mu}_m^K)$.

Exercise 289.22. Let $K \subseteq S^3$ be an oriented knot. We want to determine the "size" of the homology group of the 3-fold covering of $S^3 \setminus K$. In the following we write $\xi := \exp(2\pi \mathrm{i}/3)$. Use the Knot–Finite Cyclic Covering-H_1-Proposition 289.24 to show that

$$|\mathrm{H}_1(E_3(K), \widetilde{\mu}_K^3)| \;=\; \Delta_K(\xi) \cdot \Delta_K(\xi^{-1}).$$

Exercise 289.23. Let K be an oriented knot, let $m \in \mathbb{N}$ and let F be a connected Seifert surface for the knot K.

(a) Imitate the proof of the Link–Infinite Cyclic Covering Proposition 288.18 to give an explicit description of the m-cyclic covering $E_m(K)$ of $S^3 \setminus K$.

(b) Using (a) imitate the proof of the Alexander Module–Seifert Matrix Theorem 289.12 to find a presentation matrix for $\mathrm{H}_1(E_m(K))$ in terms of a Seifert matrix corresponding to F.

Remark. This exercise gives a new proof of the Knot–Finite Cyclic Covering-H_1-Proposition 289.24

CHAPTER 290

Topological properties of knots and the Alexander polynomial

In the previous Chapter 289 we introduced the notion of a Seifert matrix and we showed that these can be used to calculate Alexander polynomials. In this chapter we will first study the effect of several operations on Seifert matrices. Afterwards we will relate the Alexander polynomial to the knot genus and to fiberedness.

This chapter is organized as follows:

- In Section 290.1 we will study the effect on Seifert matrices of reversing the orientation and taking the mirror image of an oriented link.
- In Section 290.2 we will (re-) introduce the connected sum of two oriented knots.
- In Section 290.3 we will see that the Seifert matrix is "additive" under the connected sum operation.
- In Section 290.4 we will (re-) introduce the genus of a knot and we will see that the Alexander polynomial polynomial gives rise to lower bounds on the knot genus.
- In Section 290.5 we will (re-) introduce the notion of fibered knot and we will discuss Alexander polynomials of fibered knots.
- In Section 290.6 we will provide exercises for Chapter 290.

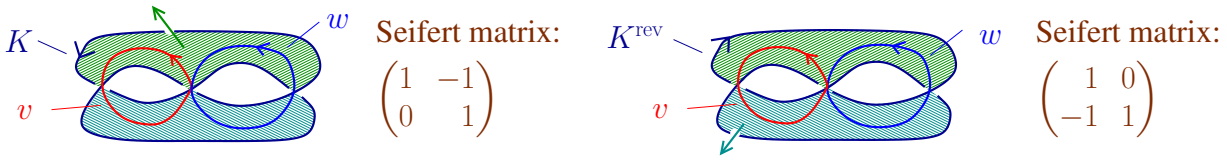
In the next Chapter 291 we will see that Seifert matrices are unique up "S-equivalence".

290.1. **Seifert matrices: Orientations and mirror images.** We start out with the following slight generalization of Definition 108.21:

Definition 290.1. Let K be an oriented link. We define the reverse L^{rev} of L to be the link L where we reversed the orientation of *each* component.

The following proposition tells us how switching to the reverse affects Seifert matrices:

Proposition 290.2. (Seifert Matrix–Reverse Proposition) Let $L \subseteq S^3$ be an oriented link. If A is a Seifert matrix for L, then the transpose A^{T} is a Seifert matrix for the reverse L^{rev}.



Proof. Let $L \subseteq S^3$ be an oriented link and let F be a connected Seifert surface for L. Recall that by Definition 288.11 this means that F is a compact oriented connected 2-dimensional submanifold F of S^3 such that $\partial F = L$ as oriented smooth manifolds, in the sense of Convention 31.39. Furthermore let $v_1, \ldots, v_k$ be a basis of $\mathrm{H}_1(F)$. We consider the corresponding Seifert matrix $A = \sigma_F(v_i, v_j)_{i,j=1,\ldots,k}$.

Next let L^{rev} be the reverse of L. We denote by F^{rev} the result of flipping the orientation of F. By the Boundary Orientation Lemma 31.38 (3) we have $\partial F^{\mathrm{rev}} = L^{\mathrm{rev}}$ as oriented smooth manifolds, i.e. F^{rev} is a connected Seifert surface for L^{rev}. By Definition 289.9 of a Seifert matrix it remains to prove the following claim:

Claim. For every choice of $i, j \in \{1, \dots, k\}$ we have

$$\sigma_{F^{\mathrm{rev}}}(v_i, v_j) \;=\; \sigma_F(v_j, v_i).$$

Proof. We start out with the following preparations and observations:

(a) We consider the oriented submanifolds $\widehat{F} := F \setminus L$ and $\widehat{F}^{\mathrm{rev}} := F^{\mathrm{rev}} \setminus L^{\mathrm{rev}}$.

(b) We pick an orientation-preserving bicollar $\beta\colon [-1,1] \times \widehat{F} \to S^3 \setminus L$.

(c) We denote by $\iota_\pm\colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ the homomorphisms from the Homology Push-Off Lemma 289.3.

(d) It follows from the Product Orientation Proposition 31.48 (3d) that

$$\begin{array}{rcl} \beta^{\mathrm{rev}}\colon [-1,1] \times \widehat{F}^{\mathrm{rev}} & \to & S^3 \setminus L \\ (t, x) & \mapsto & \beta(-t, x) \end{array}$$

is an orientation-preserving bicollar for $\widehat{F}$.

(e) We denote by $\iota^{\mathrm{rev}}_\pm\colon \mathrm{H}_1(F^{\mathrm{rev}}) \to \mathrm{H}_1(S^3 \setminus F)$ the homomorphisms from the Homology Push-Off Lemma 289.3.

(f) It follows easily from (d) and the Loop Push-Off Lemma 289.2 that $\iota^{\mathrm{rev}}_\pm = \iota_\mp$.

We now see that for any $i, j \in \{1, \dots, k\}$ we have the following equalities:

$$\sigma_{F^{\mathrm{rev}}}(v_i, v_j) \underset{\uparrow}{=} \mathrm{lk}_{F^{\mathrm{rev}}}(v_i, \iota^{\mathrm{rev}}_+(v_j)) \underset{\uparrow}{=} \mathrm{lk}_F(v_i, \iota_-(v_j)) \underset{\uparrow}{=} \mathrm{lk}_F(v_j, \iota_+(v_i)) \underset{\uparrow}{=} \sigma_F(v_j, v_i).$$

see Definition 289.9 — by (e) and since the linking form does *not* depend on the orientation of F — by the Seifert Form–Negative Push-Off Lemma 289.11 — see Definition 289.9 ∎

The Seifert Matrix–Reverse Proposition 290.2 leads us straight to the following analog of the Alexander Polynomial–Orientation Lemma 288.10 and of the Alexander Polynomial–Reverse Lemma 117.9:

Lemma 290.3. (Alexander Polynomial–Reverse Lemma) For any oriented non-empty link $L \subseteq S^3$ we have $\Delta_{L^{\mathrm{rev}}}(t) \doteq \Delta_L(t)$.

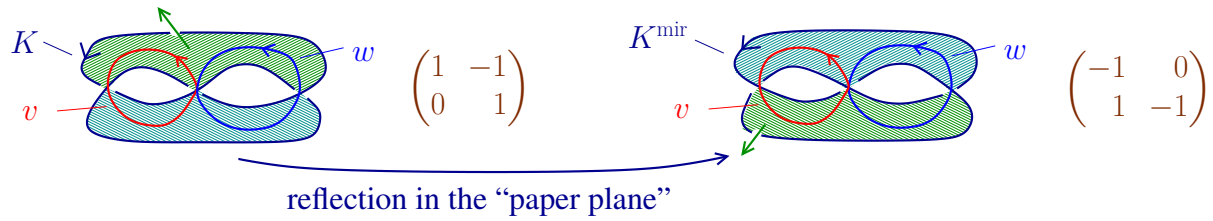
Proof. Let $L \subseteq S^3$ be an oriented non-empty link. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix A for L. We now see that

$$\Delta_{L^{\mathrm{rev}}}(t) \underset{\uparrow}{=} \det(t \cdot A^{\mathrm{T}} - (A^{\mathrm{T}})^{\mathrm{T}}) \;=\; \det(t \cdot A^{\mathrm{T}} - A) \underset{\uparrow}{=} \det(t \cdot A - A^{\mathrm{T}}) \underset{\uparrow}{=} \Delta_L(t).$$

by the Alexander Module Seifert Matrix Theorem 289.12 together with the Seifert Matrix Reverse Proposition 290.2 — this follows from $\det(W^{\mathrm{T}}) = \det(W)$ — by the Alexander Module–Seifert Matrix Theorem 289.12 ∎

The following proposition tells us how taking the mirror image affects Seifert matrices:

Proposition 290.4. (Seifert Matrix–Mirror Proposition) Let $L \subseteq S^3$ be an oriented link and let A be a Seifert matrix for L. The matrix $-A^{\mathrm{T}}$ is a Seifert matrix for the mirror L^{mir}.



Proof. Let $L \subseteq S^3$ be an oriented link and let F be a connected Seifert surface for the link L. Furthermore let $v_1, \ldots, v_k$ be a basis of $\mathrm{H}_1(F)$. We consider the corresponding Seifert matrix $A = \mathrm{lk}(v_i, v_j)_{i,j=1,\ldots,k}$.

Next let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane. We set $L^{\mathrm{mir}} := \rho(L)$. By the Boundary Orientation Lemma 31.38 (4) we have $\partial\rho(F) = \rho(\partial F) = \rho(L) = L^{\mathrm{mir}}$ as oriented smooth manifolds, i.e. $\rho(F)$ is a connected Seifert surface for $\rho(L)$. By Definition 289.9 of a Seifert matrix it remains to prove the following claim:

Claim. For every choice of $i, j \in \{1, \ldots, k\}$ we have

$$\sigma_{\rho(F)}(\rho_*(v_i), \rho_*(v_j)) \;=\; -\sigma_F(v_j, v_i).$$

Proof. We start out with the following preparations and observations:

(a) We set $\widehat{F} := F \setminus L$.

(b) We pick an orientation-preserving bicollar $\beta\colon [-1,1] \times \widehat{F} \to S^3 \setminus L$.

(c) We denote by $\iota_\pm\colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ the homomorphisms from the Homology Push-Off Lemma 289.3.

(d) It follows from the Product Orientation Preserving Lemma 31.50 that the map

$$\begin{array}{rcl} \beta^{\mathrm{mir}}\colon [-1,1] \times \rho(\widehat{F}) & \to & S^3 \setminus \rho(L) \\ (t, x) & \mapsto & \rho(\beta(-t, \rho^{-1}(x))) \end{array}$$

is an *orientation-preserving bicollar* for $\rho(\widehat{F})$.[3137]

(e) We denote by $\iota_\pm^{\mathrm{mir}}\colon \mathrm{H}_1(\rho(F)) \to \mathrm{H}_1(S^3 \setminus \rho(F))$ the homomorphisms from the Homology Push-Off Lemma 289.3.

(f) It follows easily from (d) and the Loop Push-Off Lemma 289.2 that $\rho_* \circ \iota_\pm^{\mathrm{mir}} = \iota_\mp \circ \rho_*$.

We now see that for any $i, j \in \{1, \ldots, k\}$ we have the following equalities:

$$\begin{array}{lll} \sigma_{\rho(F)}(\rho_*(v_i), \rho_*(v_j)) & = & \mathrm{lk}_{\rho(F)}(\rho_*(v_i), \iota_+(\rho_*(v_j))) \\ & \uparrow & \text{see Definition 289.9} \\ & = & -\mathrm{lk}_F(v_i, \iota_-(v_j)) \\ & \uparrow & \text{by the Naturality-of-Linking Lemma 289.8 (2), since the reflection } \rho \text{ is orientation-reversing} \\ & = & -\sigma_F(v_j, v_i). \\ & & \text{see Definition 289.9} \end{array} \qquad \begin{array}{lll} & = & \mathrm{lk}_{\rho(F)}(\rho_*(v_i), \rho_*(\iota_-(v_j))) \\ & \uparrow & \text{follows from (f)} \\ & = & -\mathrm{lk}_F(v_j, \iota_+(v_i)) \\ & \uparrow & \text{by Seifert Form Negative Push-Off Lemma 289.11} \end{array}$$

[3137] It takes some time to verify this statement: First note that the map $\rho^{-1}\colon \rho(\widehat{F}) \to F$ is orientation-preserving. It follows that the map $(t, x) \mapsto \beta(-t, \rho^{-1}(x))$ is orientation-reversing. But since ρ is also orientation-reversing we see that the given map is indeed orientation-preserving.

■

Similar to the above Alexander Polynomial–Reverse Lemma 290.3 we have the following lemma, which is an analog of the Alexander Polynomial–Mirror Lemma 117.15:

Lemma 290.5. (Alexander Polynomial–Mirror Lemma) For any oriented non-empty link $L \subseteq S^3$ we have $\Delta_{L^{\mathrm{mir}}}(t) \doteq \Delta_L(t)$.

Proof. Let $L \subseteq S^3$ be an oriented non-empty link. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix A for L. We now see that

$$\Delta_{L^{\mathrm{mir}}}(t) \underset{\uparrow}{=} \det(t \cdot (-A^{\mathrm{T}}) - (-A^{\mathrm{T}})^{\mathrm{T}}) \underset{\uparrow}{\doteq} \det(t \cdot A - A^{\mathrm{T}}) \underset{\uparrow}{=} \Delta_L(t).$$

by the Alexander Module Seifert Matrix Theorem 289.12 together with the Seifert Matrix Mirror Proposition 290.4

since $\det(W^{\mathrm{T}}) = \det(W)$

by the Alexander Module–Seifert Matrix Theorem 289.12

■

290.2. **The connected sum and Seifert matrices.** Let K and $\widetilde{K}$ be two oriented knots in S^3. In Definition 109.12 we introduced the corresponding connected sum $K \# \widetilde{K}$. We will not recall the precise definition, but we will present a way how one can build the connected sum in practice. This requires the following definition which combines aspects of Definition 63.3, Definition 63.4, Definition 109.14 and Definition 109.15:

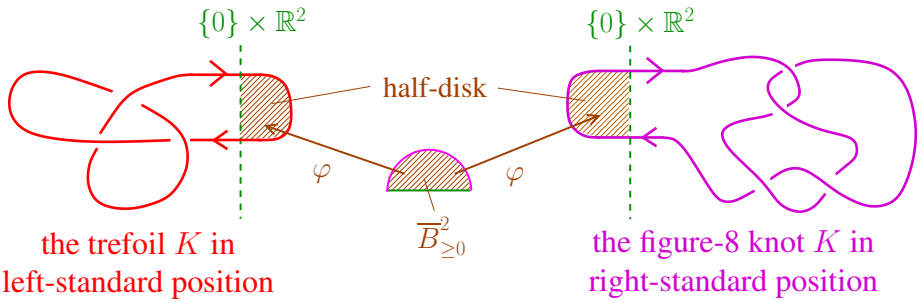
Definition 290.6. Let $K \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be an oriented knot. We say that K is in left-standard position if there exists an $\epsilon > 0$ such that the following conditions hold:

(1) $K \cap ([-\epsilon, \epsilon] \times \mathbb{R}^2) = [-\epsilon, \epsilon] \times \{(\pm 1, 0)\}$.

(2) The orientation of K is given as follows:
- The orientation of K points from $(-\epsilon, 1, 0)$ to $(\epsilon, 1, 0)$.
- The orientation of K points from $(\epsilon, -1, 0)$ to $(-\epsilon, -1, 0)$.

(3) There exists a smooth embedding $\varphi\colon \overline{B}^2_{\geq 0} \to \mathbb{R}_{\geq 0} \times \mathbb{R} \times \{0\}$ such that $\varphi(\overline{B}^2_{=0}) \subseteq \{0\} \times \mathbb{R}^2$ and such that $\varphi(S^1_{\geq 0}) = K \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$.

Similarly we define what it means for K to be in right-standard position.



We recall the following lemma which says that we can always put an oriented knot into a standard position:

Lemma 109.16. (Knot-Standard Position Lemma) Every oriented knot K in the 3-sphere $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ is smoothly isotopic to a knot in left-standard position and it is smoothly isotopic to a knot in right-standard position.

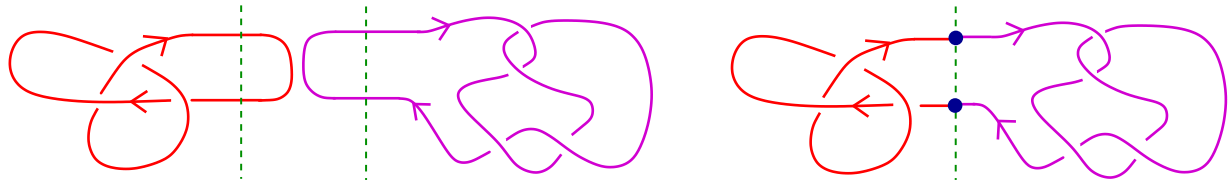
Using the Knot-Standard Position Lemma 109.16 we can now describe the connected sum of two oriented knots:

Definition 290.7. Let K and $\widetilde{K}$ be two oriented knots in S^3. By the Knot-Standard Position Lemma 109.16 we know that K is smoothly isotopic to a knot K^{lst} in left-standard position and we know that $\widetilde{K}$ is smoothly isotopic to a knot $\widetilde{K}^{\text{rst}}$ in right-standard position. We refer to

$$K\#\widetilde{K} := \underbrace{(K^{\text{lst}} \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2))}_{\subseteq \mathbb{S}^3_{\leq 0}} \cup \underbrace{(\widetilde{K}^{\text{rst}} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2))}_{\subseteq \mathbb{S}^3_{\geq 0}}$$

as the connected sum of K and $\widetilde{K}$.[3138] It follows from the Standard Connected Sum Lemma 109.17 that this definition is equivalent to the original Definition 109.12.

Example. In the figure below we show the trefoil K in left-standard position and we show the figure-8 knot $\widetilde{K}$ in right-standard position:



the trefoil K in left-standard position

the figure-8 knot $\widetilde{K}$ in right-standard position

connected sum $K\#\widetilde{K}$

In the figure above we show to the right the connected sum of the trefoil K and the figure-8 knot $\widetilde{K}$. □

Next we recall the following slightly surprising proposition:

Proposition 109.13. (Oriented Knot–Monoid Proposition) As in Notation 108.8 we denote by $\texttt{Knot}^{\text{or}}(S^3)$ the set of smooth isotopy classes of oriented knots. With this notation the map

$$\begin{array}{rcl} \texttt{Knot}^{\text{or}}(S^3) \times \texttt{Knot}^{\text{or}}(S^3) & \to & \texttt{Knot}^{\text{or}}(S^3) \\ [(K_1, K_2)] & \mapsto & [K_1\#K_2] \end{array}$$

is well-defined and it has the following properties:

(1) The sum operation has a neutral element, given by the trivial knot.
(2) The sum operation is associative.
(3) The sum operation is abelian.

In other words, $(\texttt{Knot}^{\text{or}}(S^3), \#)$ is an abelian monoid.

290.3. **The connected sum of knots and Seifert matrices.** In this section we want to study Seifert matrices and Alexander polynomials of connected sums:

Proposition 290.8. (Connected Sum–Seifert Matrix Proposition) Let K and $\widetilde{K}$ be two oriented knots with Seifert matrices A and $\widetilde{A}$. The block diagonal sum $\begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix}$ is a Seifert matrix for the connected sum $K\#\widetilde{K}$.

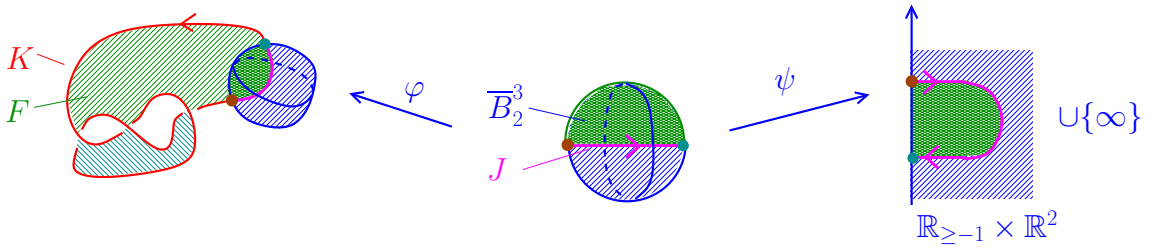
[3138] Note that by design $K\#\widetilde{K}$ is indeed a knot in S^3 and that the orientations of K and $\widetilde{K}$ endow $K\#\widetilde{K}$ with an obvious orientation.

Sketch of proof. We start out with a slightly more general discussion: We say that a connected Seifert surface G is in *left-standard position* if ∂G is in left-standard position in the sense of Definition 290.6, if there exists an $\epsilon > 0$ with $G \cap ([-\epsilon, 0] \times \mathbb{R}^2) = [-\epsilon, 0] \times [-1, 1] \times \{0\}$ and if $G \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$ is the image of a smooth embedding $\overline{B}^2_{\geq 0}$ as described in Definition 290.6.

Claim. Any connected Seifert surface is smoothly isotopic to a connected Seifert surface in left-standard position.

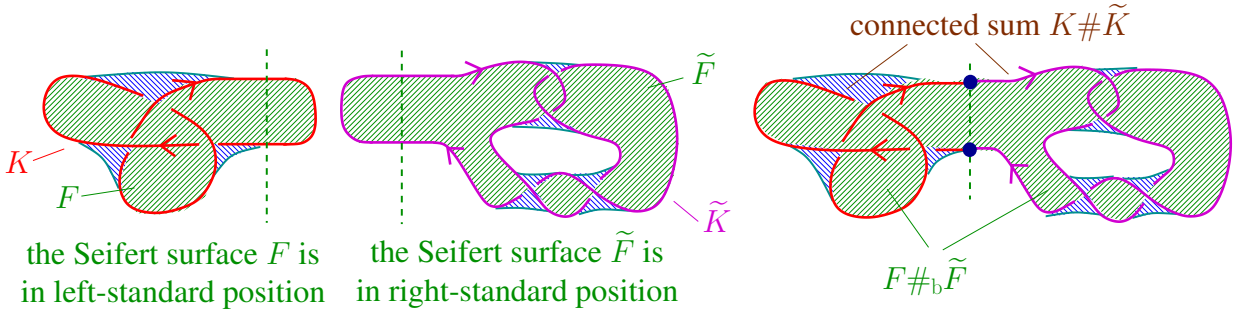
Proof. The proof of this claim is very similar to the proof of the Knot-Standard Position Lemma 109.16:

- Since F is a 2-dimensional smooth submanifold with boundary K we can pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ such that $\varphi^{-1}(F) = \overline{B}^2_{2,\geq 0} \times \{0\}$.
- One picks an orientation-preserving diffeomorphism $\psi\colon \overline{B}^3_2 \to (\mathbb{R}_{\geq -1} \times \mathbb{R}^2) \cup \{\infty\}$ such that $\psi(\overline{B}^2_{2,\geq 0} \times \{0\})$ is "standard".



The desired conclusion then follows from first applying the Smooth Ball Embedding Theorem 45.10 to the orientation-preserving smooth embeddings $\varphi, \psi\colon \overline{B}^3_2 \to S^3$ and from applying the Isotopy Extension Theorem 44.2 afterwards. We leave it to the reader to fill in the details. ⊞

Now we can start with the actual proof of the proposition. Let K and $\widetilde{K}$ be two oriented knots in S^3 with Seifert matrices A and $\widetilde{A}$. Let F and $\widetilde{F}$ be corresponding connected Seifert surfaces. It follows from the claim that we can assume that F is in left-standard position and that $\widetilde{F}$ is in right-standard position.[3139]



We obtain a connected Seifert surface for the oriented knot $K\#\widetilde{K}$ by taking the union of $F \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$ and $\widetilde{F} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$. We denote this connected Seifert surface by $F\#_{\mathrm{b}}\widetilde{F}$ since it is a boundary connected sum in the sense of Definition 61.10.[3140]

[3139] Surely nobody insists on providing the definition of a "right-standard position" in this context.

[3140] Note that $F\#_{\mathrm{b}}\widetilde{F}$ is essentially the same as the connected Seifert surface that we constructed in the Seifert Surface–Connected Sum Lemma 121.13.

Next let $C_1, \dots, C_{2g} \subseteq F \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$ be smoothly embedded oriented loops such that $[C_1], \dots, [C_{2g}]$ is a basis for $\mathrm{H}_1(F)$ which gives rise to the Seifert matrix A. Similarly we pick $\widetilde{C}_1, \dots, \widetilde{C}_{2\widetilde{g}} \subseteq \widetilde{F} \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$. A simple minded Mayer–Vietoris argument (see Exercise 160.22) shows that $C_1, \dots, C_{2g}, \widetilde{C}_1, \dots, \widetilde{C}_{2\widetilde{g}}$ represent a basis for $\mathrm{H}_1(F \#_{\mathrm{b}} \widetilde{F})$.

We can now easily calculate the Seifert form with respect to this basis and it is pretty clear that the resulting Seifert matrix is the block diagonal sum $\begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix}$. ■

Now we can easily prove the following analog of the Connected Sum–Alexander Polynomial Proposition 117.13:

Proposition 290.9. (Connected Sum–Alexander Polynomial Proposition) Let K and $\widetilde{K}$ be two oriented knots in S^3. The following equality holds:

$$\Delta_{K \# \widetilde{K}}(t) \doteq \Delta_K(t) \cdot \Delta_{\widetilde{K}}(t) \in \mathbb{Z}[t^{\pm 1}].$$

Proof. Let K and $\widetilde{K}$ be two oriented knots. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix A for K and we can pick a Seifert matrix $\widetilde{A}$ for $\widetilde{K}$. We now see that

$$\begin{array}{rcl} \Delta_{K \# \widetilde{K}}(t) & \overset{\text{follows from the Connected Sum–Seifert Matrix Proposition 290.8 together with the Alexander Module–Seifert Matrix Theorem 289.12}}{\doteq} & \det\left(t \cdot \begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix} - \begin{pmatrix} A^{\mathrm{T}} & 0 \\ 0 & \widetilde{A}^{\mathrm{T}} \end{pmatrix}\right) = \det\left(\begin{pmatrix} t \cdot A - A^{\mathrm{T}} & 0 \\ 0 & t \cdot \widetilde{A} - \widetilde{A}^{\mathrm{T}} \end{pmatrix}\right) \\ & \overset{\text{elementary algebra}}{=} & \det(t \cdot A - A^{\mathrm{T}}) \cdot \det(t \cdot \widetilde{A} - \widetilde{A}^{\mathrm{T}}) \overset{\text{by the Alexander Module Seifert Matrix Theorem 289.12}}{=} \Delta_K(t) \cdot \Delta_{\widetilde{K}}(t). \end{array}$$

■

290.4. **The Alexander polynomial and the knot genus.** In the following three subsections we will first (re-) introduce the notion of the genus of a compact orientable surface. Then we will (re-) introduce the genus of a knot $K \subseteq S^3$ and finally we will see that the Alexander polynomial of knot $K \subseteq S^3$ gives rise to a lower bound on the genus of the knot K.

290.4.1. **The genus of a surface.** We start out by recalling Notation 101.12 and 101.22:

Definition 290.10.

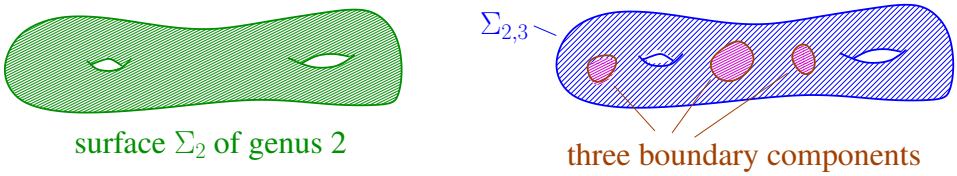
(1) Given $g, n \in \mathbb{N}_0$ we write[3141]

$$\Sigma_{g,n} \coloneqq \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Note that n is precisely the number of boundary components of $\Sigma_{g,n}$.

(2) Let M be a compact orientable connected non-empty 2-dimensional smooth manifold with n boundary components. By the Surface Classification Theorem 101.16 there exists a unique $g \in \mathbb{N}_0$ such that M is diffeomorphic to some $\Sigma_{g,n}$. We refer to g as the genus(M) of M.

[3141]Recall that according to Definition 101.10 by "minus n open disks" we mean the following: We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma_g$ with disjoint images. We refer to $\Sigma_g \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks. As we discussed in the Surface-Minus-Open Disks Lemma 101.11, the diffeomorphism type is independent of the choice of the smooth embeddings.



The following elementary lemma allows us to calculate the genus in terms of Euler characteristics:

Lemma 290.11. (Surface-χ-Genus Lemma) Let F be a compact orientable connected non-empty 2-dimensional smooth manifold with $n \in \mathbb{N}_0$ boundary components. Then

$$\text{genus of } F \;=\; \tfrac{1}{2} \cdot (2 - n - \chi(\Gamma)).$$

Proof. This lemma follows immediately from the calculation on page 2993 where we showed that $\chi(\Sigma_{g,n}) = 2 - 2g - n$. ■

In practical situations the following lemma is particularly helpful for determining the genus of a surface:

Lemma 290.12. (Genus-via-Graph Lemma) Let F be a compact orientable connected non-empty 2-dimensional submanifold with $n \in \mathbb{N}_0$ boundary components. If $\Gamma \subseteq F$ is a topological graph that is a deformation retract of F, then

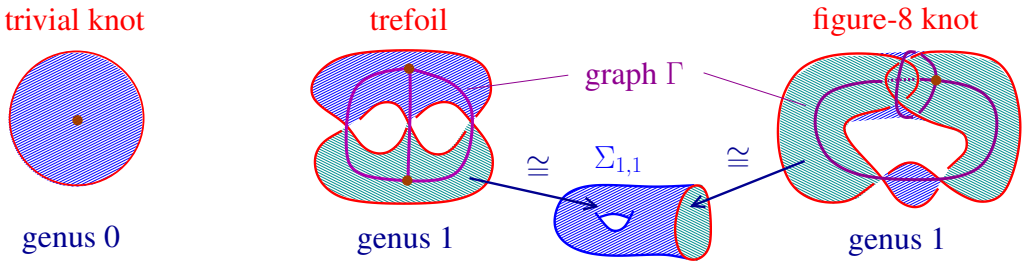
$$\text{genus of } F \;=\; \tfrac{1}{2} \cdot (2 - n - \chi(\Gamma)) \;=\; \tfrac{1}{2} \cdot (2 - n - \#\text{vertices of } \Gamma + \#\text{edges of } \Gamma).$$

Proof. We have the following equalities:

$$\begin{aligned} \text{genus of } F \;&\overset{\text{by the Surface-}\chi\text{-Genus Lemma 290.11}}{\downarrow}\; = \tfrac{1}{2} \cdot (2 - n - \chi(F)) \\ &\overset{\text{by the Homotopy Type-}\chi\text{ Lemma 156.3}}{\downarrow}\; = \tfrac{1}{2} \cdot (2 - n - \chi(\Gamma)) \\ &= \tfrac{1}{2} \cdot (2 - n - \#\text{vertices of } \Gamma + \#\text{edges of } \Gamma). \\ &\overset{\text{by Definition 156.1}}{\uparrow} \end{aligned}$$

■

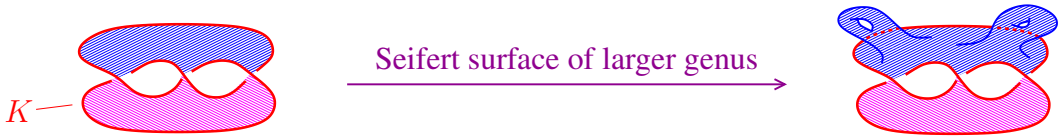
Example. In the figure below we show some of the Seifert surfaces from page 5444 and 5528.



With the Genus-via-Graph Lemma 290.12 it is straightforward to determine the genus in each case. □

290.4.2. **The genus of a knot.** Let K be a knot. By the Seifert Surface Existence Theorem 285.2 we know that K bounds a Seifert surface. The question arises: What genera can occur? Here is a simple minded observation: If the knot K bounds a Seifert surface of

genus g, then the construction sketched in the figure below shows pretty convincingly that K also bounds Seifert surfaces of any genus $h \geq g$.



So the interesting question is, what is the minimal genus of a Seifert surface for a given knot? This leads us to the following definition which is identical to Definition 121.8:

Definition 290.13. Given a knot $K \subseteq S^3$ we define[3142]

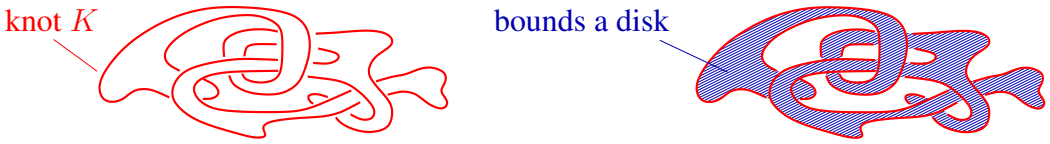
$$\text{genus}(K) := \text{minimal genus of a Seifert surface of } K.$$

As we pointed out in the Genus–Smooth Isotopy Invariant Lemma 121.9, it follows from the Isotopy Extension Theorem 44.2 that the genus is a smooth isotopy invariant of knots.

Next note that, almost by definition, a (Seifert) surface has genus zero if and only if it is diffeomorphic to a disk. For the reader's convenience we recall the following proposition which says that trivial knots are precisely the knots that admit a Seifert surface of genus zero:

Proposition 121.10. (Genus-Zero Knots Proposition) Let K be a knot in S^3. The following two statements are equivalent:

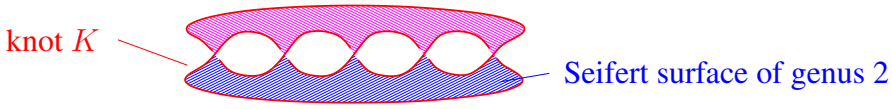
(i) The knot K has genus zero, i.e. K bounds a smoothly embedded disk in S^3.
(ii) The knot K is smoothly isotopic to the trivial knot.



Example.

(1) We consider again the trefoil. On page 5444 we showed that the trefoil admits a Seifert surface of genus 1. Furthermore on page 5530 we saw that the trefoil is not smoothly isotopic to the trivial knot. It follows from the Genus-Zero Knots Proposition 121.10 and Footnote 3142 that genus(trefoil) = 1.

(2) We consider the knot K and the Seifert surface F in the figure below.



Using the Genus-via-Graph Lemma 290.12 we can easily verify that F has genus two. The following question arises: Does K also admit a Seifert surface of genus one? In a few minutes we will have the tools to answer this question. □

[3142]Note that in the Genus–Smooth Isotopy Invariant Lemma 121.9 we pointed out that if K and $\widetilde{K}$ are smoothly isotopic knots, then it follows almost immediately from the Isotopy Extension Theorem 44.2 that $\text{genus}(K) = \text{genus}(\widetilde{K})$.

290.4.3. **The knot genus and the Alexander polynomial.** Our next goal is to show that the Alexander polynomial gives rise to a lower bound on the knot genus. To formulate the corresponding result we need to recall Definition 123.1:

Definition 290.14. Given a Laurent polynomial $p(t) = \sum_{i=r}^{s} a_i \cdot t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its degree as

$$\deg(p(t)) \;=\; \deg\Big(\sum_{i=r}^{s} a_i \cdot t^i\Big) \;=\; s - r.$$

We extend this definition to $\deg(0) = -\infty$. Note that if we have $p(t), q(t) \in \mathbb{Z}[t^{\pm 1}]$ with $p(t) \doteq q(t)$, then $\deg(p(t)) = \deg(q(t))$.[3143]

The following proposition, which in a way is identical to the Genus–Alexander Polynomial Proposition 123.2, allows us to address the above questions:

Proposition 290.15. (Genus–Alexander Polynomial Proposition) For every oriented knot $K \subseteq S^3$ we have

$$\deg(\Delta_K(t)) \;\leq\; 2 \cdot \mathrm{genus}(K).$$

Remark. This proposition was first proved by Herbert Seifert [**Sei1934**, Satz 3]. □

Proof. Let $K \subseteq S^3$ be a knot. By Definition 290.13 we need to prove the following claim:

Claim. If F is a Seifert surface of genus g for K, then $\deg(\Delta_K(t)) \leq 2g$.

Proof. Let F be a Seifert surface of genus g for K. We denote by A a corresponding Seifert matrix. Recall that by Surface-H_*-Proposition 155.17 we have $\mathrm{H}_1(F) \cong \mathbb{Z}^{2g}$. Thus $A \in \mathrm{M}(2g \times 2g, \mathbb{Z})$. Then

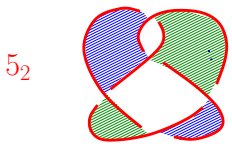
$$\deg(\Delta_K(t)) \;\underset{\uparrow}{=}\; \deg(\det(t \cdot A - A^{\mathrm{T}})) \;\underset{\uparrow}{\leq}\; 2g.$$

by the Alexander Module Seifert Matrix Theorem 289.12 — by the Polynomial Matrix–Determinant Lemma 123.11 and the fact that A is of size $2g \times 2g$

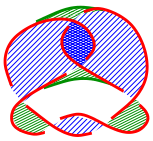
■

Example.

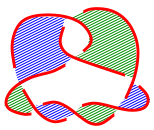
(1) We consider the knots 5_2, 6_1 and 7_2 which are shown below. We also show Seifert surfaces and we show their Alexander polynomials which were calculated in Exercise 289.3.



5_2 $\quad 2 - 3t + 2t^2$



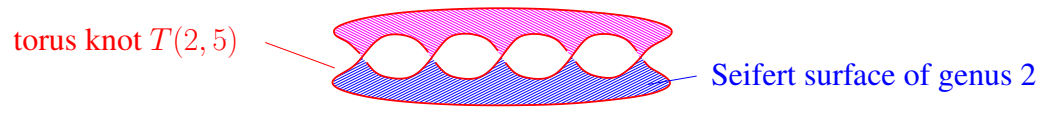
6_1 $\quad 2 - 5t + 2t^2$



7_2 $\quad 3 - 5t + 3t^2$

In all three examples the Seifert surface has genus one and we see that the inequality of the Genus–Alexander Polynomial Proposition 290.15 is in fact an equality.

(2) Let $n \in \mathbb{N}$ be odd. We consider the $(2, n)$-torus knot $T(2, n)$ that is shown below. In Exercise 289.6 we calculated that $\Delta_{T(2,n)}(t) = \frac{t^n+1}{t+1} = 1 - t + t^2 - \cdots + t^{n-1}$.



[3143] Evidently this definition is set up in such a way that it compensates for the indeterminacy of the single-variable Alexander polynomials of oriented links.

On the other hand one can easily see, using the Genus-via-Graph Lemma 290.12, that $T(2,n)$ admits a Seifert surface of genus $\frac{1}{2}(n-1)$. Thus we obtain from the Genus–Alexander Polynomial Proposition 290.15 that $\text{genus}(T(2,n)) = \frac{1}{2}(n-1)$. □

290.5. **The Alexander polynomial of fibered knots.** In the following three subsections we will first (re-) introduce the notion of a smooth bundle, then we will (re-) introduce fibered knots and finally we will study the Alexander polynomial of fibered knots.

290.5.1. **Fibered smooth manifolds.** We start out by recalling Definition 122.1 with the following general definition:

Definition 290.16.

(1) Let M and B be smooth manifolds. We say that a smooth map $p\colon M \to B$ is a smooth bundle map if there exists a smooth manifold F such that for every $b \in B$ there exists a smooth trivialization of p around b, i.e. there exists an open neighborhood U of $b \in B$ and a diffeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\Phi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,v)\mapsto x} & \\ & U. & \end{array}$$

For each $b \in B$ we refer to $p^{-1}(b)$ as a fiber of p. Note that each fiber is diffeomorphic to F.

(2) Let M and B be smooth manifolds. We say that M fibers smoothly over B if there exists a smooth bundle map $p\colon M \to B$.

Next we recall the following characterization of smooth bundles over S^1:

Lemma 122.3. (Smooth Mapping Torus–Bundle Lemma)

(1) Let F be a smooth manifold F and let $\mu\colon F \to F$ be a self-diffeomorphism. The natural projection

$$\begin{array}{rcl} q\colon \text{Tor}(F,\mu) = (F \times [0,1])/(f,0) \sim (\mu(f),1) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

is a smooth bundle map with fiber F.

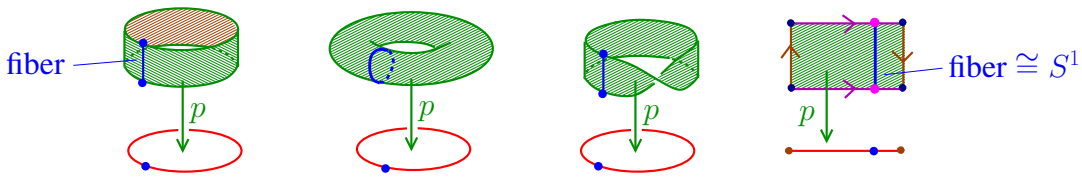
(2) Let M be a smooth manifold. A smooth map $p\colon M \to S^1$ is a smooth bundle map if and only if there exists a smooth manifold F, a diffeomorphism $\mu\colon F \to F$ and a diffeomorphism $\Theta\colon \text{Tor}(F,\mu) \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Tor}(F,\mu) & \xrightarrow[\cong]{\Theta} & M \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & {\scriptstyle [f,t]\mapsto \exp(2\pi\,\mathrm{i}t)} & \\ & S^1. & \end{array}$$

Example. In the figure below we illustrate that the annulus, the Möbius band, the torus and the Klein bottle fiber smoothly over S^1.[3144]

It is fairly elementary to show, using the Smooth Mapping Torus–Bundle Lemma 122.3, that all the remaining compact connected 2-dimensional smooth manifolds do *not* fiber smoothly over S^1. The details were worked out in Exercise 122.1. □

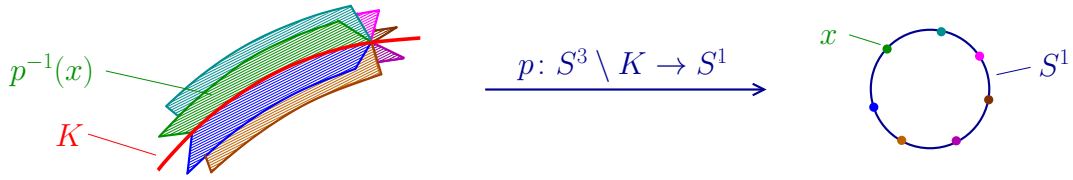
[3144] In fact we proved this statement already on page 575 and 575 and in the Mapping Torus-of-id-Lemma 21.22.



290.5.2. **Fibered knots.** In this subsection we return to the study of knots. We start out by recalling Definition 122.7:

Definition 122.7. Let $K \subseteq S^3$ be a knot.

(1) We say that K is fibered if there exists a smooth bundle map $p\colon S^3 \setminus K \to S^1$ such that each fiber $p^{-1}(x)$ together with K forms a Seifert surface for K.
(2) We say that a Seifert surface F is a fiber Seifert surface if $F \setminus K = F \setminus \partial F$ is the fiber of a smooth bundle map $S^3 \setminus K \to S^1$.



Example. Let $K \subseteq S^3$ be the standard trivial knot. By the Trivial Knot–Complement Proposition 111.11 we know that the complement $S^3 \setminus K$ admits an explicit diffeomorphism $S^3 \setminus K \to S^1 \times B^2$. Using this explicit diffeomorphism one can show easily that the trivial knot is fibered. □

To give more examples of fibered knots it is helpful to recall the following definition:

Definition 111.13.

(1) Let $p, q \in \mathbb{Z}$ be coprime. We refer to the image of the map

$$\begin{array}{rcl} \varphi_{p,q}\colon \mathbb{R} & \to & S^3 \\ t & \mapsto & \left(\frac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot p\cdot t), \frac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot q\cdot t)\right) \end{array}$$

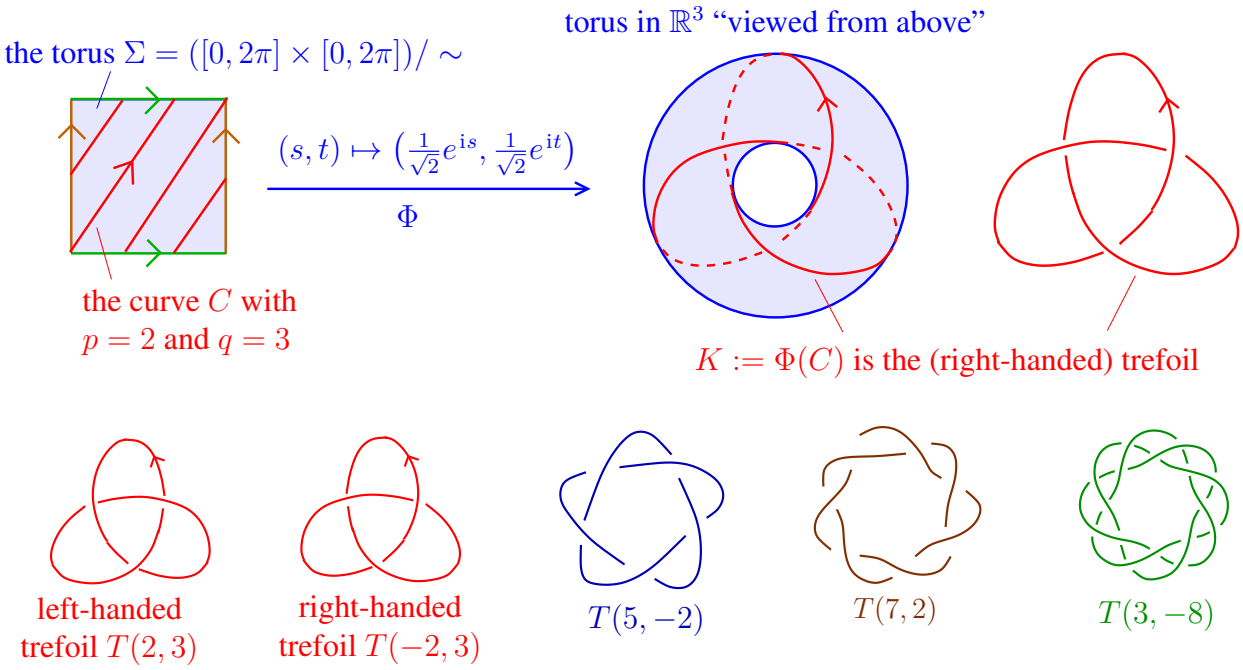
as the (p,q)-torus knot $T(p,q)$. We equip the torus knot with the orientation which turns the map $\varphi_{p,q}\colon \mathbb{R} \to T(p,q)$ into an orientation-preserving local diffeomorphism.
(2) We define the right-handed trefoil to be the $(2,3)$-torus knot and we define the left-handed trefoil to be the $(-2,3)$-torus knot.

Visualization.

(1) We give an alternative description of torus knots: Let $p, q \in \mathbb{Z}$ be coprime. We consider the oriented smoothly embedded closed curve $C = \{(p\cdot t, q\cdot t) \mid t \in \mathbb{R}\}$ on the torus $\Sigma = ([0,2\pi]\times[0,2\pi])/\sim$. By Definition 111.13 the (p,q)-torus knot $T(p,q)$ is the image of C under the map

$$\begin{array}{rcl} \Phi\colon \Sigma = ([0,2\pi]\times[0,2\pi])/\sim & \to & S^3 \\ (s,t) & \mapsto & \left(\frac{1}{\sqrt{2}}\exp(\mathrm{i}s), \frac{1}{\sqrt{2}}\exp(\mathrm{i}t)\right). \end{array}$$

(2) In the following figure we illustrate several examples of torus knots:



(3) A visualization of the (p, q)-torus knots is also given by Johannes Zander in

`https://www.desmos.com/3d/4ba6cb90d2`. □

The following proposition explains why we just introduced torus knots:

Proposition 122.9. (Torus Knot–Fibered Proposition) For any coprime $p, q \in \mathbb{Z}$ the corresponding torus knot $T(p, q)$ is fibered.

Sketch of proof. In the following we summarize the key steps in the proof of the Torus Knot–Fibered Proposition 122.9: Let $p, q \in \mathbb{Z}$ be coprime. We consider the map

$$\begin{array}{rcl} f\colon S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\} & \to & \mathbb{C} \\ (z,w) & \mapsto & z^q - w^p. \end{array}$$

One can easily see that $f^{-1}(0) = T(p, q)$ and that 0 is a regular value of f. This implies that $T(p, q) = f^{-1}(0)$ is indeed a 1-dimensional smooth submanifold of S^3. Next we consider the map

$$\begin{array}{rcl} \pi\colon S^3 \setminus T(p,q) = S^3 \setminus f^{-1}(0) & \to & S^1 \\ (z,w) & \mapsto & \frac{z^q - w^p}{|z^q - w^p|} = \frac{f(z,w)}{|f(z,w)|}. \end{array}$$

One can easily verify that for each $x \in S^3 \setminus T(p, q)$ the differential $D\pi_x$ is an epimorphism. Using the Ehresmann Fibration Theorem 122.6 one can now show that the above map $\pi\colon S^3 \setminus T(p, q) \to S^1$ is a smooth bundle map such that for each $z \in S^1$ the fiber $\pi^{-1}(z)$ together with $T(p, q)$ forms a Seifert surface for $T(p, q)$. ■

290.5.3. **The Alexander polynomial of fibered knots.** The main goal of this subsection is to prove the following proposition which is an analog of the earlier Fibered Knot–Alexander Polynomial Proposition 123.12:

Proposition 290.17. (Fibered Knot–Alexander Polynomial Proposition) In the following let $K \subseteq S^3$ be a fibered knot with fiber Seifert surface F as in Definition 122.7. Then the following two statements hold:

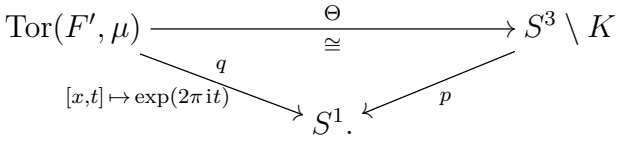
(1) The Alexander polynomial $\Delta_K(t)$ is bimonic in the sense of Definition 287.25, i.e. its highest and lowest non-zero coefficient are ± 1.
(2) We have $\deg(\Delta_K(t)) = 2 \cdot$ genus of a fiber Seifert surface F.
(3) Every Seifert matrix corresponding to the Seifert surface F is invertible over $\mathbb{Z}$.

Proof. Let $K \subseteq S^3$ be a fibered knot. By Definition 122.7 there exists a smooth bundle map $p\colon S^3 \setminus K \to S^1$ such that each fiber $p^{-1}(x)$ together with K forms a Seifert surface for K. We start out with some preparations:

(α) We set $F' := p^{-1}(1)$ and we consider the fiber Seifert surface $F := F' \cup K$.
(β) Recall that it follows from the Open Seifert Surface Lemma 288.15 (2) that the inclusion $i\colon F' \to F$ is a homotopy equivalence. It follows from the Homotopy Equivalence-H_*-Proposition 144.14 that $i_*\colon \mathrm{H}_1(F') \to \mathrm{H}_1(F)$ is an isomorphism.
(γ) We denote by g the genus of the fiber Seifert surface F.
(δ) It follows from the Surface-H_*-Proposition 155.17 that $\mathrm{rank}(\mathrm{H}_1(F)) = 2g$.

In the following we give two proofs of the three statements of the proposition:

(i) We start out our first proof with a few observations:
  (a) It follows from (β) that F' is homotopy equivalent to a finite CW-complex.
  (b) It follows from (β) and (δ) that $\mathrm{rank}(\mathrm{H}_1(F')) = 2g$.
  (c) It follows from the Smooth Mapping Torus–Bundle Lemma 122.3 that there exists a diffeomorphism $\mu\colon F' \to F'$ and a diffeomorphism $\Theta\colon \mathrm{Tor}(F,\mu) \to S^3 \setminus K$ such that the following diagram commutes:



We now see that

$$\Delta_K(t) \overset{\text{by Definition 288.8}}{\downarrow} = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])) \overset{\text{this follows from (c) and the Alexander Module–Homotopy Lemma 287.24}}{\downarrow} = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_1(\mathrm{Tor}(F',\mu); \mathbb{Z}[t^{\pm 1}]))$$

$$= \text{characteristic polynomial of the map } \mu\colon \mathrm{H}_1(F') \to \mathrm{H}_1(F') \quad = \det(t \cdot \mathrm{id}_{2g} - \mu_*).$$

$$\uparrow \text{follows from the Mapping Torus-Alexander Polynomial Proposition 287.26} \qquad \uparrow \text{follows from (b)}$$

$$= \text{bimonic polynomial of degree } 2g.$$

$$\uparrow \text{follows from the Polynomial Matrix–Determinant Lemma 123.11}$$

Finally let $A \in \mathrm{M}(2g \times 2g, \mathbb{Z})$ be a Seifert matrix corresponding to the Seifert surface F. We now see that

$$\det(t \cdot A - A^{\mathrm{T}}) \overset{\text{by the Alexander Module Seifert Matrix Theorem 289.12}}{=} \Delta_K(t) \overset{\text{by the above}}{=} \text{bimonic polynomial of degree } 2g.$$

It follows from this calculation and the Polynomial Matrix–Determinant Lemma 123.11 that $\det(A) = \pm 1$, i.e. A is invertible over $\mathbb{Z}$.

(ii) We now provide our alternative proof of the three statements of the proposition. We start out with the following claim:

Claim. The maps $\iota_\pm \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ from the Homology Push-Off Lemma 289.3 are isomorphisms.

Proof. We start out with the following observation:

(∗) It follows from the Smooth Bundles-Homotopies Proposition 243.10 (or in this case from an elementary argument) that for any smoothly embedded interval $I \subseteq S^1$ we have an orientation-preserving diffeomorphism $\Phi \colon p^{-1}(I) \to I \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(I) & \xrightarrow[\cong]{\Phi} & I \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (t,f)\mapsto t} & \\ & I. & \end{array}$$

We apply this fact to the smoothly embedded interval $I = \{\exp(\frac{\pi}{2}\mathrm{i}t) \mid t \in [-1,1]\} \subseteq S^1$ and we obtain from (∗) a bicollar $\beta \colon [-1,1] \times F' \to S^3 \setminus K$. Next we consider the following maps

$$\mathrm{H}_1(F) \underset{i_*}{\xleftarrow{\cong}} \mathrm{H}_1(F') \xrightarrow{\mathrm{id}} \mathrm{H}_1(\overbrace{p^{-1}(1)}^{=F'}) \xrightarrow[\cong]{\beta_{\pm 1 *}} \mathrm{H}_1(\overbrace{p^{-1}(S^1 \setminus \{1\})}^{=(S^3\setminus L)\setminus F' = S^3 \setminus F}) \xrightarrow{\mathrm{id}} \mathrm{H}_1(S^3 \setminus F).$$

$$\iota_\pm$$

The claim follows from the following two observations:

- Since β is an orientation-preserving bicollar we obtain from the Loop Push-Off Lemma 289.2 that the composition of the maps from left to right is indeed our homomorphism $\iota_\pm$.
- It follows from (∗) and the observation that $\{\frac{1}{2}\} \times F \to (S^1 \setminus \{1\}) \times F$ is a homotopy equivalence that the map $\beta_{\pm 1} \colon p^{-1}(1) \to S^3 \setminus p^{-1}(1))$ is a homotopy equivalence. By the Homotopy Equivalence-H_*-Proposition 144.14 this implies that $\beta_\pm$ induces an isomorphism on homology. ⊞

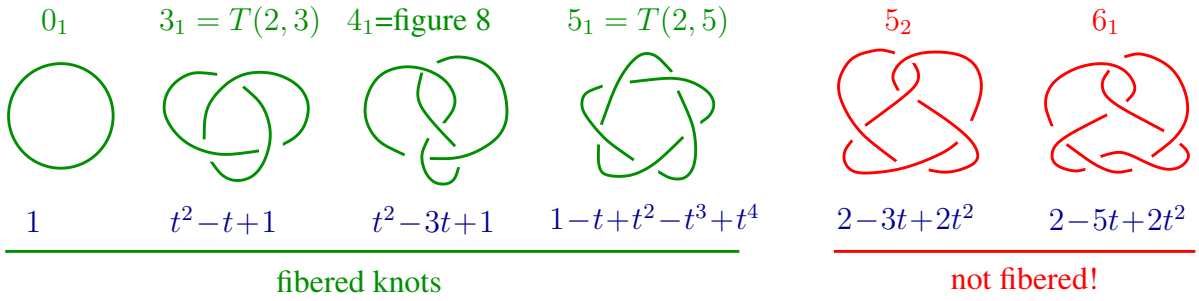
Next we pick a basis $v_1, \ldots, v_{2g}$ for $\mathrm{H}_1(F)$. Let A be the corresponding Seifert matrix. We equip $\mathrm{H}_1(S^3 \setminus F)$ with the dual basis $v_1^\dagger, \ldots, v_{2g}^\dagger$ given by the Linking–Dual Basis Lemma 289.13. By the Seifert Surface-Complement Lemma 289.14 we know that the homomorphism $\iota_+ \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus F)$ is represented by the Seifert matrix A. Note that it follows from this discussion and the claim that the Seifert matrix A is invertible.

Finally we see that

$$\begin{array}{ccccc} \Delta_K(t) & = & \det(t \cdot A - A^{\mathrm{T}}) & = & \text{bimonic polynomial of degree } 2g. \\ & \uparrow & & \uparrow & \\ & \text{by the Alexander Module Seifert Matrix Theorem 289.12} & & \text{by the Polynomial Matrix–Determinant Lemma 123.11 and the fact that } A \text{ and } A^{\mathrm{T}} \text{ have determinant } \pm 1 & \end{array}$$

■

Example. In the following figure we show some knots with low crossing numbers and we show their Alexander polynomials:



By the Torus Knot–Fibered Proposition 122.9 we know that the torus knots $0_1 = T(1,0)$, $T(2,3)$ and $T(5,2)$ are fibered. Furthermore we pointed out in the Fibered Genus-One Knots Theorem 122.15 that the figure-8 knot is fibered. The Alexander polynomials of the two knots to the right were (hopefully) calculated in Exercise 289.3. We see that the Alexander polynomials of 5_2 and 6_1 are not binomic. It thus follows from the Fibered Knot–Alexander Polynomial Proposition 290.17 that these two knots are not fibered. □

We conclude this section with a new proof of the following statement:

Corollary 123.13. (Fiber–Minimal Corollary) Let $K \subseteq S^3$ be a knot. If K is fibered, then for any fiber Seifert surface F we have the following equalities:

$$\text{genus}(K) \;=\; \text{genus}(F) \;=\; \tfrac{1}{2}\cdot \deg(\Delta_K(t)),$$

in particular F is a Seifert surface of minimal genus for K.

Proof. Let K be a fibered knot with corresponding fiber Seifert surface F. We see that

$$\begin{array}{ccccccc} \text{genus}(K) & \leq & \text{genus}(F) & = & \tfrac{1}{2}\cdot\deg(\Delta_K(t)) & \leq & \text{genus}(K). \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{by Definition 290.13} & & \text{by the Fibered Knot–Alexander Polynomial Proposition 290.17} & & \text{by the Genus–Alexander Polynomial Proposition 290.15} & & \end{array}$$

We see that all inequalities have to be equalities and we get the promised result. ■

290.6. Exercises for Chapter 290.

Exercise 290.1. Let K and $\widetilde{K}$ be two oriented knots in S^3.

(a) Show directly, just using Definition 288.1 and the Mayer-Vietoris Theorem 153.2, that there exists an isomorphism

$$\mathrm{H}_1(S^3 \setminus (K\#\widetilde{K}); \mathbb{Z}[t^{\pm 1}]) \;\cong\; \mathrm{H}_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) \oplus \mathrm{H}_1(S^3 \setminus \widetilde{K}; \mathbb{Z}[t^{\pm 1}]).$$

(b) Use (a) to show that following equality holds:

$$\Delta_{K\#\widetilde{K}}(t) \;\doteq\; \Delta_K(t)\cdot\Delta_{\widetilde{K}}(t) \;\in\; \mathbb{Z}[t^{\pm 1}].$$

Remark. This gives an alternative proof for the Connected Sum–Alexander Polynomial Proposition 290.9.

Exercise 290.2.

(a) How can you turn a knot that is in left-standard position into a knot that is in right-standard position?
Remark. Beware of orientations.

(b) Let K and $\widetilde{K}$ be two oriented knots that are in left-standard position. Use (a) to show that $K\#\widetilde{K}$ and $\widetilde{K}\#K$ are smoothly isotopic.

Remark. This gives an alternative proof for the Oriented Knot–Monoid Proposition 109.13 (3). Also note that this exercise is identical to Exercise 109.5.

Exercise 290.3. What are the Laurent polynomials $p(t) \in \mathbb{Z}[t^{\pm 1}]$ that occur as Alexander polynomials of genus one knots?

Exercise 290.4. Let $g \in \mathbb{N}$. Show that there exist infinitely many pairwise non-smoothly isotopic knots of genus g.

Exercise 290.5. The Genus–Alexander Polynomial Proposition 290.15 says that for every oriented knot $K \subseteq S^3$ we have the inequality

$$\deg(\Delta_K(t)) \;\leq\; 2 \cdot \mathrm{genus}(K).$$

Does there exist a generalization of this statement to oriented links? If yes, what is the statement?

Exercise 290.6. Let $L \subseteq S^3$ be an oriented m-component link.

(a) Let $p(t) \in \mathbb{Z}[t^{\pm 1}]$ be an Alexander polynomial of L. Show that there exists a $g \in \mathbb{N}_0$ such that $p(t) = t^{2g+m-1} \cdot p(t^{-1})$.
Hint. What can you say about the size of an Alexander matrix for L?

(b) Show that

$$\deg(\Delta_L(t)) \;=\; m - 1 \mod 2.$$

Exercise 290.7. Show that given any knot K and given any integer $m \in \mathbb{Z}$ there exists a Seifert surface for K such that for any corresponding Seifert matrix A the t^m-coefficient of $\det(t \cdot A - A^{\mathrm{T}})$ is non-zero.
Remark. The point of this exercise is that it shows that the degree of the highest power of t appearing in $\det(t \cdot A - A^{\mathrm{T}})$ contains no information about K.

Exercise 290.8. Let X be a topological space such that $\mathrm{H}_1(X)$ is a free abelian group of some rank k. Furthermore let $f\colon X \to X$ be a homeomorphism. Let P be a matrix that represents $f_*\colon \mathrm{H}_1(X) \to \mathrm{H}_1(X)$. We consider the mapping torus $\mathrm{Tor}(X, f)$ and the natural epimorphism $\Phi\colon \pi_1(\mathrm{Tor}(X, f)) \to \mathbb{Z}$. Given $n \in \mathbb{N}$ we denote by T_n the covering of $\mathrm{Tor}(X, f)$ corresponding to the kernel of the epimorphism $\pi_1(\mathrm{Tor}(X, f)) \xrightarrow{\Phi} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Show that

$$\mathrm{H}_1(T_n) \;\cong\; \mathbb{Z} \oplus \mathbb{Z}^k/(P^n - \mathrm{id}) \cdot \mathbb{Z}^k.$$

Exercise 290.9. Let K be the trefoil. Given any $n \in \mathbb{N}$ we denote by $E_n(K)$ the n-fold cyclic covering of $S^3 \setminus K$. Show that for any $m \in \mathbb{N}$ we have an isomorphism

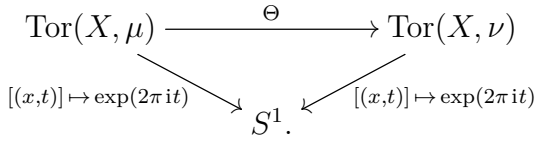
$$\mathrm{H}_1(E_m(K)) \;\cong\; \mathrm{H}_1(E_{m+6}(K)).$$

Remark. Make use of Exercise 290.8.

Exercise 290.10.

(a) Let X be a connected topological space and let $\mu, \nu\colon X \to X$ be two homeomorphisms such that there exists a homeomorphism $\Theta\colon \mathrm{Tor}(X, \mu) \to \mathrm{Tor}(X, \nu)$ which has the

property that the following diagram commutes:



Show that μ and ν are homeotopic.

(b) Let $p\colon M \to S^1$ be a smooth bundle map. We write $F := p^{-1}(1)$. By the Smooth Mapping Torus–Bundle Lemma 122.3 we can write $p\colon M \to S^1$ as a mapping torus $\mathrm{Tor}(F, \mu)$. It follows from (a) that the induced map $\mu_*\colon \mathrm{H}_k(F) \to \mathrm{H}_k(F)$ is independent of the choice of μ. We refer to this automorphism as the homological monodromy of $p\colon M \to S^1$ on k-th homology. Given a fibered knot determine the homological monodromy on first homology in terms of a Seifert matrix for K.

Exercise 290.11. Let $K \subseteq S^3$ be an oriented knot. Note that we can use the rational Alexander module $\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}])$ to define the rational Alexander polynomial

$$\Delta_K^{\mathbb{Q}}(t) \in \mathbb{Q}[t^{\pm 1}] \;:=\; \mathrm{ord}_{\mathbb{Q}[t^{\pm 1}]}(\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}]))$$

which is well-defined up to multiplication by $r \cdot t^n$ where $r \in \mathbb{Q} \setminus \{0\}$ and $n \in \mathbb{Z}$.

(a) Show that (up to the larger indeterminacy) we have $\Delta_K(t) \doteq \Delta_K^{\mathbb{Q}}(t)$.

(b) Show that $\deg(\Delta_K^{\mathbb{Q}}(t)) = \dim(\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}]))$.
Hint. Use that $\mathbb{Q}[t^{\pm 1}]$ is a PID.

(c) Let A be a finitely generated torsion-free abelian group. Show that the map $A \mapsto A \otimes \mathbb{Q}$ given by $a \mapsto a \otimes 1$ is injective.
Hint. Try to reduce the problem to the case that A is finitely generated.
Remark. This exercise already appeared as Exercise 159.15.

(d) Show that $\mathrm{H}_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) = 0$ if and only if $\Delta_K(t) \doteq 1$.
Hint. Recall that by the Alexander Module-$\mathbb{Z}$-Torsion Free Proposition 289.18 we know that $\mathrm{H}_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) = 0$ is torsion-free as an abelian group.

Exercise 290.12. Let $K \subseteq S^3$ be an oriented knot. As we pointed out in Exercise 290.11 we can use the rational Alexander module $\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}])$ to define the rational Alexander polynomial

$$\Delta_K^{\mathbb{Q}}(t) \in \mathbb{Q}[t^{\pm 1}] \;:=\; \mathrm{ord}_{\mathbb{Q}[t^{\pm 1}]}(\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}])).$$

Let $n \in \mathbb{N}$. As in Notation 289.21 we denote by $p\colon E_n(K) \to S^3$ the n-fold cyclic covering of $S^3 \setminus K$ and we write $\widetilde{\mu}_K^n := p^{-1}(\mu_K)$.

(a) Express the homology group $\mathrm{H}_1(E_n(K), \widetilde{\mu}_K^n; \mathbb{Q})$ in terms of $\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}])$.

(b) Show that the following statements are equivalent:
  (i) $\dim_{\mathbb{Q}}(\mathrm{H}_1(E_n(K), \widetilde{\mu}_K^n; \mathbb{Q})) > 0$.
  (ii) The polynomials $t^n - 1$ and $\Delta_K^{\mathbb{Q}}(t)$ are not coprime in $\mathbb{Q}[t^{\pm 1}]$.
  (iii) There exists an n-th root of unity that is a zero of $\Delta_K^{\mathbb{Q}}(t)$.
  Hint. Since $\mathbb{Q}[t^{\pm 1}]$ is a PID we can write $\mathrm{H}_1(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}])$ as a direct sum of cyclic $\mathbb{Q}[t^{\pm 1}]$-modules. Now make use of the rational analog of the Infinite–Finite Cyclic Covering Proposition 289.23.

(c) Show that the following statements are equivalent:
  (i) $\mathrm{H}_1(E_n(K), \widetilde{\mu}_K^n; \mathbb{Z})$ is infinite.
  (ii) There exists an n-th root of unity that is a zero of $\Delta_K(t)$.

(d) Show that if p is a prime, then for any $r \in \mathbb{N}$ the group $\mathrm{H}_1(E_{p^r}(K), \widetilde{\mu}_K^{p^r})$ is finite.
Hint. Use (c-ii), use the fact that $\Delta_K(1) = \pm 1$ and use facts about minimal polynomials of p^r-th roots of unity.

Exercise 290.13. Recall that in Definition 289.26 we introduced for an oriented knot $K \subseteq S^3$ the determinant $\det(K) := |\Delta_K(-1)|$. Show that the determinant of a knot is multiplicative under the connected sum operation.

Exercise 290.14. Let K and J be two oriented knots in S^3. Show that for any $m \in \mathbb{N}$ we have an isomorphism

$$\mathrm{H}_1(E_m(K\#J), \widetilde{\mu}_{K\#J}^m) \;\cong\; \mathrm{H}_1(E_m(K), \widetilde{\mu}_K^m) \oplus \mathrm{H}_1(E_m(J), \widetilde{\mu}_J^m).$$

You can prove this equality using Seifert matrices. Alternatively, and more elegantly, you could try to prove the equality using a Mayer–Vietoris type argument.

Exercise 290.15. Let K be a knot such that $\deg(\Delta_K(t)) = 2\mathrm{genus}(K)$. Show that if $\Delta_K(t)$ is irreducible, then K is prime.

Exercise 290.16. A *k-dimensional foliation* of an n-dimensional smooth manifold is a decomposition of M into a family of subspaces $\{F_i\}_{i\in I}$ such that for each $P \in M$ there exists an open neighborhood U of $P \in M$ and a diffeomorphism $\Phi\colon U \to V \times W$ where $V \subseteq \mathbb{R}^k$ is open and where $W \subseteq \mathbb{H}^{n-k}$ is open such that for each $w \in W$ the preimage $\Phi^{-1}(V \times \{w\})$ is contained in a single F_i. The subspaces F_i are called the *leaves* of the foliation.

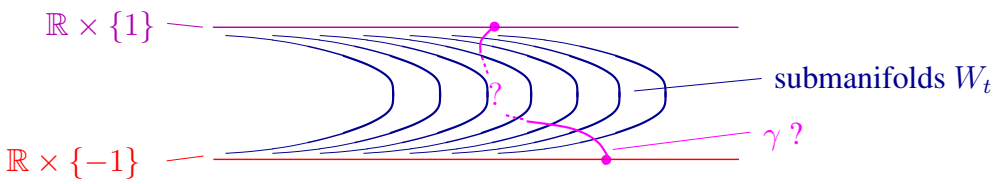
(a) Show that the complement of a fibered knot admits a foliation.
(b) Let $r, s \in \mathbb{R}$. Show that there exists a foliation on $S^1 \times S^1$ such that one of the leaves F_i is given by $\{(\exp(2\pi r\,\mathrm{i}\lambda), \exp(2\pi s\,\mathrm{i}\lambda)) \,|\, \lambda \in \mathbb{R}\}$.
(c) Under what condition on $r, s \in \mathbb{R}$ are the leaves in the foliation of (b) non-compact? Any such foliation of the torus with non-compact leaves is referred to as a *Kronecker foliation*.
(d) Does there exist a 1-dimensional foliation of the solid torus $\overline{B}^2 \times S^1$ with compact leaves that is not, up to diffeomorphism, given by the foliation $\{\{x\} \times S^1\}_{x\in\overline{B}^2}$?

Exercise 290.17.

(a) As in Exercise 50.7 we consider the smooth manifold $M = \mathbb{R} \times [-1, 1]$ together with the submanifolds

$$W_t \;:=\; \{(t + \tfrac{1}{y^2-1}, y) \,|\, y \in (-1, 1)\} \qquad \text{where } t \in \mathbb{R}.$$

Show that these submanifolds define a foliation in the sense of Exercise 290.16.
(b) Show that there is no smooth path $\gamma\colon [-1, 1] \to M$ that connects the bottom $\mathbb{R}\times\{-1\}$ with the top $\mathbb{R} \times \{1\}$ such that γ is transverse to every W_t, i.e. such that for every $s \in (-1, 1)$ with $\gamma(s) \in W_t$ we have $\gamma'(s) \notin \mathrm{T}_{\gamma(s)}W_t$.
Remark. This decomposition of the interior of $M = \mathbb{R} \times [-1, 1]$ into submanifolds is sometimes called a *Reeb foliation*.
(c) Show that there exists a foliation on the solid torus $\overline{B}^2 \times S^1$ such that the boundary $S^1 \times S^1$ is a leaf.
Hint. Modify the Reeb foliation.
(d) Let $K \subseteq S^3$ be a fibered knot with tubular map $\tau\colon \overline{B}^2 \times K \to S^3$. Show that there exists a foliation on S^3 such that $\tau(S^1 \times K)$ is a leaf.



Exercise 290.18. Let $W \subseteq \overline{B}^2 \times S^1$ be the oriented knot that is shown in the figure below. Given an oriented knot $K \subseteq S^3$ we pick an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ of self-linking number 0. Furthermore we pick an orientation-preserving diffeomorphism $\gamma\colon S^1 \to K$. We refer to

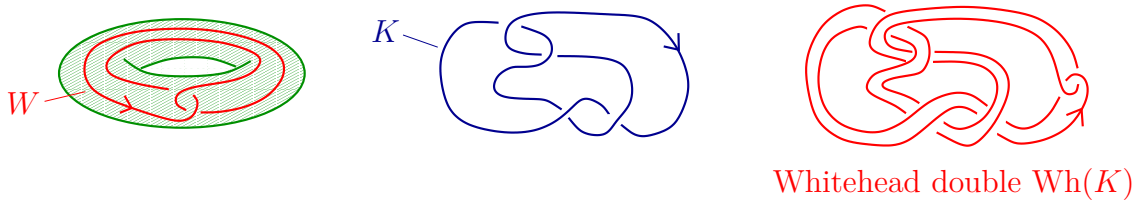
$$\mathrm{Wh}(K) \;:=\; \underbrace{(\tau \circ (\mathrm{id}_{\overline{B}^2} \times \gamma))}_{\text{map } \overline{B}^2 \times S^1 \to S^3}(W) \;\subseteq\; S^3$$

as the *Whitehead double of* K.

(a) Show that the Whitehead double $\mathrm{Wh}(K)$ admits a Seifert surface of genus 1.

(b) Show that the Alexander polynomial of the Whitehead double $\mathrm{Wh}(K)$ equals 1.

Remark. Later on page **??** we will show that if K is non-trivial, then so is the Whitehead double $\mathrm{Wh}(K)$.



CHAPTER 291

S-equivalence of Seifert matrices

First recall that by the Seifert Surface Existence Theorem 288.12 we know that every oriented link $L \subseteq S^3$ admits a Seifert surface. This in turn allows us to obtain a Seifert matrix A for L. Furthermore, in the Alexander Module–Seifert Matrix Theorem 289.12 we showed that the single-variable Alexander polynomial of L is given by $\Delta_L(t) \doteq \det(t \cdot A - A^{\mathrm{T}})$. This raises the question, is it possible to use the right hand side as the definition of the single-variable Alexander polynomial? This in turn raises the question, to what degree are Seifert surfaces and Seifert matrices actually unique?

In this chapter we will answer these questions: More precisely, we will first show that any two Seifert surfaces for a given link are "tube-equivalent", i.e. one can be turned into another by a sequence of attaching and removing tubes. We will then use this result to show that any two Seifert matrices of an oriented link are "S-equivalent".

This chapter is organized as follows:

- In Section 291.1 we will formulate the Tube Equivalence Theorem 291.4 which says that any two Seifert surfaces for a given oriented link are "tube-equivalent".
- In Section 291.2 we will discuss two general results about the interaction between codimension-one submanifolds and homology.
- In Section 291.3 we will provide the proof of the Tube Equivalence Theorem 291.4.
- In Section 291.5 we will study the impact of a tube attachment on Seifert matrices.
- In Section 291.6 we will see that any two Seifert matrices of an oriented link are S-equivalent.
- In Section 291.7 we will provide exercises for Chapter 291.

In the next Chapter 292 we will use the results of this chapter to introduce the Levine-Tristram signatures of an oriented link.

291.1. **Tube-equivalence of Seifert surfaces.** Let L be an oriented link. By the Seifert Surface Existence Theorem 288.12 we know that L admits a Seifert surface. Such a Seifert surface is certainly not unique. For example one could change it by a smooth isotopy. On page 5559 we saw also more drastic changes where we increased the genus of the Seifert surface.

The following definition describe one such construction of a new surface out of a given one:[3145]

[3145]Note that this definition is related to Construction 187.28 where we construct the internal connected sum of submanifolds.

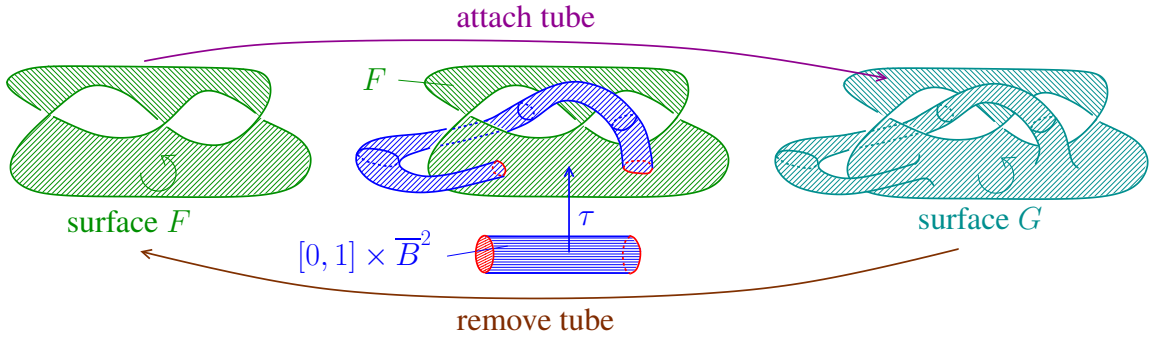
Definition 291.1. Let M be a 3-dimensional smooth manifold and let $F \subseteq M$ be an oriented 2-dimensional smooth submanifold. Furthermore let $\tau\colon [0,1] \times \overline{B}^2 \to M$ be a smooth embedding with the following properties:

(a) $\tau([0,1] \times \overline{B}^2) \cap F = \tau(\{0,1\} \times \overline{B}^2)$.
- $\tau\colon \{0\} \times \overline{B}^2 \to F$ is an orientation-reversing smooth embedding.
- $\tau\colon \{1\} \times \overline{B}^2 \to F$ is an orientation-preserving smooth embedding.[3146]

(b) $\tau([0,1] \times \overline{B}^2) \cap F \subseteq F \setminus \partial F$.

(c) $\tau([0,1] \times S^1)$ intersects F transversally.

We set
$$G \;=\; F \setminus \tau(\{0,1\} \times B^2) \cup \tau([0,1] \times S^1).$$
We say G is obtained from F by attaching a tube along τ. In the above setting we also say that F is obtained from G by removing a tube.



The following lemma implies in particular that attaching a tube to a Seifert surface for a link L is again a Seifert surface for L:

Lemma 291.2. (Tube Attachment Lemma) Let M be a 3-dimensional smooth manifold, let F be an oriented 2-dimensional submanifold of M and let $\tau\colon [0,1] \times \overline{B}^2 \to M$ be a smooth embedding as in Definition 291.1. Let
$$G \;=\; F \setminus \tau(\{0,1\} \times B^2) \cup \tau([0,1] \times S^1)$$
be the result of attaching a tube to F along τ.

(1) After rounding corners we can arrange that G is a smooth submanifold of M with $\partial G = \partial F$.

(2) G admits a unique orientation that coincides with the orientation on $F \setminus \tau(\{0,1\} \times B^2)$.

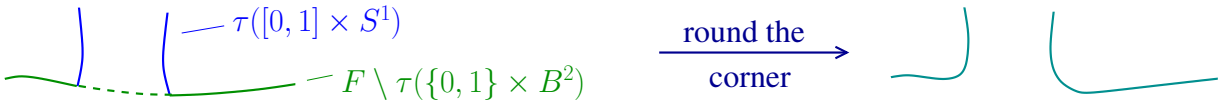
(3) If F is connected, then G is also connected.

Sketch of proof.

(1) Similar to the Collar-for-Submanifold Proposition 56.3 we can find a tubular neighborhood for F such that $\tau([0,1] \times S^1)$ is a product in an open neighborhood of F. Using a collar for $\tau(\{0,1\} \times S^1) \subseteq \partial(F \setminus \tau(\{0,1\} \times B^2))$ we can now round the corner of G along $\tau(\{0,1\} \times S^1)$. We leave it to the reader to fill in the details.

(2) This statement follows easily from the orientation condition in Definition 291.1 and the Gluing-Smooth Manifolds-Proposition 54.5 (4) together with the Product Orientation Proposition 31.48 (4).

[3146]These two statements can be summarized as saying that $\tau\colon \partial([0,1]) \times \overline{B}^2 \to F$ is an orientation-preserving smooth embedding.



(3) This statement follows from an elementary argument, see e.g. the Complement Connected Lemma 4.40 together with the Union Connected Lemma 4.39 (or alternatively the Surgery Proposition **??** (4)). ■

The following is one of the most important definitions of this chapter:

Definition 291.3. Let M be a 3-dimensional smooth manifold and let $F, G \subseteq M$ be two oriented 2-dimensional smooth submanifolds. We say F and G are tube-equivalent if we can turn F into G via a sequence of tube attachments, tube removals and smooth isotopies.

The following theorem is the key technical statement of this chapter:

Theorem 291.4. (Tube Equivalence Theorem) Let $L \subseteq S^3$ be an oriented link. Any two Seifert surfaces of L are tube-equivalent.

Remark. There are many different types of proofs in the literature, see e.g. [**Levine1970**, Theorem 1], [**GLi1978**, Theorem 11], [**Crom2004**, Theorem 5.5.1], [**Kau1987a**, p. 197], [**Lic1997a**, Theorem 8.2], [**Kaw1996**, Theorem 5.2.3], [**Kai1997**, Proposition 4.1.5 and Theorem 5.1.11] [**Muras2008**, Theorem 5.4.1] and [**Juh2023**, Proposition 4.11]. It is perhaps fair to say that few proofs are "lean ready". Our proof is a slight variation on the proof given by Gerrit Herrmann and José Pedro Quintanilha [**HQ2025**, Theorem 6.3]. □

The remainder of this chapter is organized as follows:

- In Section 291.2 we will prove two general results regarding codimension-one submanifolds and homology.
- In Section 291.3 use the results from Section 291.2 to provide the proof of the Tube Equivalence Theorem 291.4.
- In Section 291.5 we will study the impact of a tube attachment on Seifert matrices.
- In Section 291.6 we will use the Tube Equivalence Theorem 291.4 to show that any two Seifert matrices of an oriented link are S-equivalent.

291.2. **Codimension-one submanifolds and homology.** The goal of this section is to prove the following proposition which we will need in the proof of the Tube Equivalence Theorem 291.4, but which might be of independent interest:[3147]

Proposition 291.5. (Codimension-One Splitting Proposition) Let M be an oriented n-dimensional smooth manifold and let $F \subseteq M$ be a proper oriented smooth submanifold of codimension one. We suppose that there exists a $d \in \mathbb{N}$ and an $\alpha \in \mathrm{H}_{n-1}(M, \partial M)$ such that $[F] = d \cdot \alpha$. Then there exists a decomposition $F = F_1 \sqcup \cdots \sqcup F_d$ such that each F_i is a union of components of F and such that for each $i \in \{1, \ldots, d\}$ we have $[F_i] = \alpha \in \mathrm{H}_{n-1}(M, \partial M)$.

The proof of the Codimension-One Splitting Proposition 291.5 will require the remainder of this section. In the Section 291.3 we will use this proposition to provide the proof of the Tube Equivalence Theorem 291.4.

[3147]The following proposition gives in particular a solution to Exercise 234.10.

The proof of the above proposition will require a few preparations. We start out by recalling Definition 234.9:

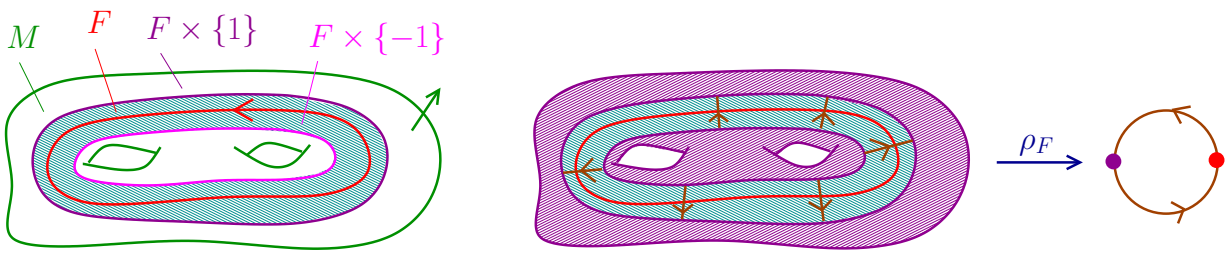
Definition 291.6. Let M be a compact oriented smooth manifold and let F be a proper oriented codimension-one submanifold of M. We introduce two objects:

- By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) together with the Tubular Map Lemma 58.3 we know that there exists an orientation-preserving tubular map $\tau\colon [-1,1]\times F\to M$.
- Using the Smooth Transition Function Lemma 24.9 one can easily find a smooth monotonously increasing map $\eta\colon [-1,1]\to[-1,1]$ with $\eta(t)=-1$ for $t\in[-1,-\frac{3}{4}]$, $\eta(t)=2t$ for $t\in[-\frac{1}{4},\frac{1}{4}]$ and $\eta(t)=1$ for $t\in[\frac{3}{4},1]$.

We refer to the map

$$\begin{array}{rccl}\rho_F\colon & M & \to & S^1\\ & P & \mapsto & \begin{cases}\exp(\pi\,\mathrm{i}\cdot\eta(t)), & \text{if } P=\tau(t,Q)\in[-1,1]\times F,\\ -1, & \text{otherwise}\end{cases}\end{array}$$

as the Thom-Pontryagin map. Using the Smooth Pasting Proposition 25.19 one can verify that ρ_F is smooth. One can easily verify that $1\in S^1$ is a regular value of ρ_F and that $\rho_F^{-1}(1)=F$.



Next we recall the following very useful lemma:

Lemma 234.7. (Preimage-of-Point-in-S^1-Lemma) Let M be a compact orientable n-dimensional smooth manifold, let $f\colon M\to S^1$ be a smooth map and let $z\in S^1$ be a regular value. We write $N:=f^{-1}(z)$. By the Regular Value Theorem 32.5 (1), (2), (3) we know that N is a compact orientable proper $(n-1)$-dimensional smooth manifold of M. In this setting we have the following equality in $\mathrm{H}^1(M;\mathbb{Z})$:

$$\mathrm{PD}_M(\underbrace{[N]}_{\in \mathrm{H}_{n-1}(M,\partial M)}) = f^*(\underbrace{[S^1]^*}_{\in \mathrm{H}^1(S^1;\mathbb{Z})}).$$

Now we can finally provide the proof of the above proposition:

Proof of the Codimension-One Splitting Proposition 291.5. [3148] Let M be an oriented n-dimensional smooth manifold and let $F\subseteq M$ be a proper oriented smooth submanifold of codimension one. We suppose that there exists a $d\in\mathbb{N}$ and an $\alpha\in\mathrm{H}_{n-1}(M,\partial M)$ such that $[F]=d\cdot\alpha$.

Let $\rho_F\colon M\to S^1$ be the Thom-Pontryagin map from Definition 291.6 that corresponds to F.

[3148] This proof is based on [**Thu1986a**, p. 103]. An alternative approach is essentially given in [**HQ2025**, Theorem 6.3].

Claim 1. We have $\rho_{F*}(\mathrm{H}_1(M)) \subseteq d \cdot \mathrm{H}_1(S^1) = \{d \cdot \sigma \,|\, \sigma \in \mathrm{H}_1(S^1)\}$.

Proof. We consider the following diagram:

$$\begin{array}{ccc} \text{set of continuous maps } M \to S^1 & \xrightarrow[=:\Xi]{(f\colon M\to S^1)\mapsto f^*([S^1]^*)} & \mathrm{H}^1(M;\mathbb{Z}) \\ \Big\downarrow{\scriptstyle (f\colon M\to S^1)\mapsto f_*} & & \Big\downarrow{\scriptstyle \mathrm{ev}} \\ \mathrm{Hom}(\mathrm{H}_1(M),\mathrm{H}_1(S^1)) & \xrightarrow[\text{induced by } [S^1]\mapsto 1\in\mathbb{Z}]{\cong} & \mathrm{Hom}(\mathrm{H}_1(M),\mathbb{Z}). \end{array}$$

We make the following observations:

(a) It follows easily from the definitions together with the Kronecker Pairing–Natural Lemma 199.9 (1) that the diagram commutes.
(b) As we discussed above, the point $1 \in S^1$ is a regular value of the Thom-Pontryagin map $\rho_F\colon M \to S^1$ and we have $\rho_F^{-1}(1) = F$. Thus it follows from the Preimage-of-Point-in-S^1-Lemma 234.7 that $\Xi([\rho_F]) = \mathrm{PD}_M([F])$.
(c) We set $\Phi := \mathrm{PD}_M([\alpha]) \in \mathrm{H}^1(M;\mathbb{Z})$. Note that it follows from our hypothesis and from (b) that $\Xi([\rho_F]) = \mathrm{PD}_M([F]) = \mathrm{PD}_M(d \cdot [\alpha]) = d \cdot \mathrm{PD}_M([\alpha])$.
(d) It follows from (a) and (c) that there exists a homomorphism $\psi\colon \mathrm{H}_1(M) \to \mathrm{H}_1(S^1)$ such that $\rho_{F*} = d \cdot \psi$.
(e) It follows from (d) that $\rho_{F*}(\mathrm{H}_1(M)) \subseteq d \cdot \mathrm{H}_1(S^1))$. ⊞

Claim 2. Let $\pi\colon S^1 \to S^1$ be the covering that is given by $z \mapsto z^d$ and let $x_0 \in F$. We have $\rho_{F*}(\pi_1(M,x_0)) \subseteq \pi_*(\pi_1(S^1,1))$.

Proof. First note that it follows from the Hurewicz Theorem 166.10 that we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(M,x_0) & \twoheadrightarrow & \mathrm{H}_1(M) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ \pi_1(S^1,1) & \xrightarrow{\cong} & \mathrm{H}_1(S^1). \end{array}$$

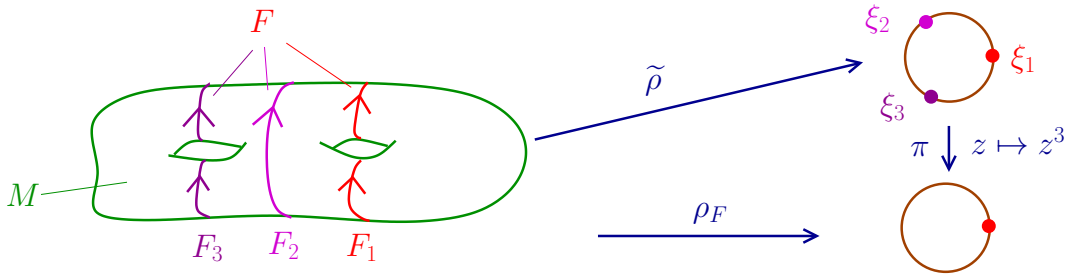
where the top horizontal map is an epimorphism and the bottom horizontal map is an isomorphism. This claim follows easily from this diagram together with Claim 1 and the discussion on page 1760 where we showed that $\pi_*(\pi_1(S^1,1)) = d \cdot \pi_1(S^1,1)$. ⊞

It follows from Claim 2 and the Map Lifting Criterion 127.5 that there exists a smooth map $\widetilde{\rho}\colon M \to S^1$ that makes the following diagram commute:

$$\begin{array}{ccc} & & S^1 \\ & \overset{\widetilde{\rho}}{\nearrow} & \Big\downarrow{\scriptstyle \pi\ z\mapsto z^d} \\ M & \xrightarrow[\rho_F]{} & S^1. \end{array}$$

For $j = 1,\dots,d$ we set $\xi_j := \exp(2\pi \mathrm{i} j/d)$ and we set $F_j := \widetilde{\rho}^{-1}(\xi_j)$.
The following claim shows that $F_1,\dots,F_d$ have the desired properties:

Claim 3. We have $F = F_1 \sqcup \cdots \sqcup F_d$, where each F_i is a union of components of F and for each $i \in \{1,\dots,d\}$ we have $[F_i] = \alpha \in \mathrm{H}_{n-1}(M,\partial M)$.



Proof. Since all F_i are closed subsets and since we only have finitely many F_j we see that each $F_i \subseteq F$ is closed and open in F. It follows from this discussion together with the Component Lemma 4.48 (4b) that each F_i is a union of components of F.

It remains for show that for each $i \in \{1, \dots, d\}$ we have $[F_i] = \alpha \in \mathrm{H}_{n-1}(M, \partial M)$. We do so in the following steps:

(a) Recall that 1 is a regular value of $\rho_F \colon M \to S^1$. It follows easily from the fact that π is a covering that each ξ_j is a regular value of $\widetilde{\rho} \colon M \to S^1$.

(b) It follows from (a) and the Preimage-of-Point-in-S^1-Lemma 234.7 that for all $i = 1, \dots, d$ we have $\mathrm{PD}_M([F_i]) = \mathrm{PD}_M([\widetilde{\rho}^{-1}(z_i)]) = \widetilde{\rho}^*([S^1]^*)$.

(c) It follows from (b) and Definition 210.2 that $[F_i] = [M] \cap \widetilde{\rho}^*([S^1]^*)$. This shows that $[F_i]$ does not depend on i.

(d) We now see that

$$d \cdot [F_i]) \underset{\text{by (c)}}{\uparrow} = [F_1] + \cdots + [F_d] \underset{\text{since } F = F_1 \sqcup \cdots \sqcup F_d}{\uparrow} = [F] \underset{\text{by hypothesis}}{\uparrow} = d \cdot \alpha.$$

(e) By the Manifold-H_{n-1}-Theorem 193.4 we know that $\mathrm{H}_{n-1}(M, \partial M)$ is torsion-free. It thus follows from (d) that $[F_i] = [\alpha]$. ∎

291.3. **Proof of the Tube Equivalence Theorem 291.4.** In this section we will finally provide the proof of the Tube Equivalence Theorem 291.4. But we need to start out with a few more preparations. This section is organized as follows.

- In Section 291.3.1 we will introduce standard Seifert surfaces.
- In Section 291.3.2 we will introduce truncated Seifert surfaces.
- In Section 291.3.3 we will introduce the Seifert surface graph for a given oriented link L and we will show that it is connected.
- In Section 291.3.4 we will conclude the proof of the Tube Equivalence Theorem 291.4.

291.3.1. **Standard Seifert surfaces.** We start out by recalling an aspect of Definition 288.5.

Definition 291.7. Let $L = L_1 \sqcup \cdots \sqcup L_m \subseteq S^3$ be an oriented m-component link in S^3. As usual we denote the meridians by $\mu_{L_1}, \dots, \mu_{L_m}$. It follows from the Link Complement-H_*-Proposition 283.4 that there exists a unique homomorphism

$$\widehat{\Phi}_L \colon \mathrm{H}_1(S^3 \setminus L) \to \mathbb{Z}$$

such that for $i = 1, \dots, m$ we have $\widehat{\Phi}_L([\mu_{L_i}]) = 1$.

Next we introduce our new favorite thickenings of oriented links:

Lemma 291.8. (Standard Thickening Lemma) Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link. There exists a thickening $\tau\colon \overline{B}^2 \times L \to S^3$ with the following property: For each $i \in \{1, \ldots, m\}$ the map

$$\mathrm{H}_1(\{1\} \times L) \xrightarrow{\tau_*} \mathrm{H}_1(S^3 \setminus L) \xrightarrow{\widehat{\Phi}_L} \mathbb{Z}$$

is the zero map. We refer to such a thickening as a standard thickening of L.

Proof. By the Link Tubular Map Theorem 108.29 we can pick a thickening $\sigma\colon \overline{B}^2 \times L \to S^3$. For $i = 1, \ldots, m$ we introduce the following objects:

- We set $T_i := \sigma(L_i \times S^1)$.
- We write $\lambda_i := \sigma(\{1\} \times L)$ and $\mu_i := \sigma(S^1 \times \{x_i\})$ where $x_i \in L_i$.
- We pick an orientation-preserving diffeomorphism $\gamma_i\colon S^1 \to L_i$.

By the Meridian Proposition 109.2 (2) we know that μ_i is a meridian of L_i. By Definition 291.7 this implies that $\widehat{\Phi}_L(\mu_i) = 1$. We set $d_i := \widehat{\Phi}_L(\lambda_i)$. As in the Knot Thickening–Self Linking Lemma 284.11 we see that if we precompose each $\sigma|_{\overline{B}^2 \times L_i}$ by the diffeomorphism

$$\begin{array}{rcl} \overline{B}^2 \times L_i & \to & \overline{B}^2 \times L_i \\ (z, \gamma_i(w)) & \mapsto & (w^{-d_i} \cdot z, \gamma_i(w)) \end{array}$$

then we obtain the desired thickening. ■

The Standard Thickening Lemma 291.8 leads us to the following definition:

Definition 291.9. Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a standard thickening. Let F be a Seifert surface for L. We say that the Seifert surface F is standard with respect to τ if for each $i \in \{1, \ldots, m\}$ there exists a $z_i \in S^1$ such that $F \cap \tau(\overline{B}^2 \times L_i) = \tau(\{r \cdot z_i \,|\, r \in [0,1]\} \times L)$. Often, when τ is understood we just drop the "with respect to τ".

The following lemma shows that Seifert surfaces can be turned into standard Seifert surfaces:

Lemma 291.10. (Standard Seifert Surface Lemma) Let $L \subseteq S^3$ be an oriented link and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a standard thickening of L. Every Seifert surface of L is smoothly isotopic rel L to a Seifert surface that is standard with respect to τ.

Proof. Let F be a Seifert surface for L. First note that it follows easily from the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33 together with the Product Tubular Neighborhood–Existence Theorem 58.4 and the Tubular Neighborhood–Uniqueness Theorem 57.12 that there exists a tubular map $\sigma\colon \overline{B}^2 \times L \to S^3$ such that $\sigma([0,1] \times L) = F$. After radial shrinking of σ we can assume that $\sigma(\overline{B}^2 \times L) \subseteq \tau(\overline{B}^2 \times L)$.

For $i = 1, \ldots, m$ we write $C_i := F \cap \sigma(S^1 \times L_i) = \sigma(\{1\} \times L_i)$.

Claim 1. We have $\widehat{\Phi}_L([C_i]) = 0$.

Proof. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(C_i) & \longrightarrow & \mathrm{H}_1(S^3 \setminus L) & \xrightarrow{\widehat{\Phi}_L} & \mathbb{Z} \\ & \searrow \quad \nearrow & & & \\ & \mathrm{H}_1(F) & & & \end{array}$$

where (as usual) all unnamed maps are induced by the inclusions. Note that the diagram commutes by the functoriality of homology. The claim now follows from the Total Linking Number-Vanishing Lemma 288.17 which says that the map $\mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus L) \to \mathbb{Z}$ is the zero map. ⊞

Next let $\Phi_{L_i} \colon \mathrm{H}_1(S^3 \setminus L_i) \to \mathbb{Z}$ be the usual epimorphism from Definition 284.1 that is given by sending any meridian of L_i to 1.

Claim 2. We have $\Phi_{L_i}([\sigma(\{1\} \times L_i)]) = \Phi_{L_i}([\tau(\{1\} \times L_i)])$.

Proof. Recall that we have the inclusion $\sigma(\overline{B}^2 \times L) \subseteq \tau(\overline{B}^2 \times L)$. Note that this implies that $[\sigma(\{1\} \times L_i)] \in \mathrm{H}_1(\tau((\overline{B}^2 \setminus \{0\}) \times L))$ and $[\tau(\{1\} \times L_i)] \in \mathrm{H}_1(\tau((\overline{B}^2 \setminus \{0\}) \times L))$ both lie in the kernel of the inclusion induced map $\mathrm{H}_1(\tau((\overline{B}^2 \setminus \{0\}) \times L)) \to \mathrm{H}_1(\tau(\overline{B}^2 \times L))$. Note that it follows easily from the Longitude-Meridian Lemma 284.12 (4b) that this kernel is generated by any meridian μ_i of L_i. It follows that in $\mathrm{H}_1(S^3 \setminus L_i) = \mathbb{Z} \cdot [\mu_i]$ we have $[\sigma(\{1\} \times L_i)] - [\tau(\{1\} \times L_i)] = k \cdot [\mu_i]$ for some $k \in \mathbb{Z}$. We now see that

$$\begin{aligned} \Phi_{L_i}([\sigma(\{1\} \times L_i)]) - \Phi_{L_i}([\tau(\{1\} \times L_i)]) &= \Phi_{L_i}([\sigma(\{1\} \times L_i)] - [\tau(\{1\} \times L_i)]) \\ &= \Phi_{L_i}([k \cdot \mu_i]) \quad = \quad \widehat{\Phi}_{L_i}(k \cdot [\mu_i]) \\ &\quad \uparrow \text{by the above} \qquad \uparrow \text{by the definition of } \widehat{\Phi}_L \text{ and } \Phi_{L_i} \\ &= \widehat{\Phi}_{L_i}([\sigma(\{1\} \times L_i)] - [\tau(\{1\} \times L_i)]) \\ &= \underbrace{\widehat{\Phi}_{L_i}([\sigma(\{1\} \times L_i)])}_{=0 \text{ by Claim 2}} - \underbrace{\widehat{\Phi}_{L_i}([\tau(\{1\} \times L_i)])}_{=0 \text{ by the Standard Thickening Lemma 291.8}} = 0. \end{aligned}$$

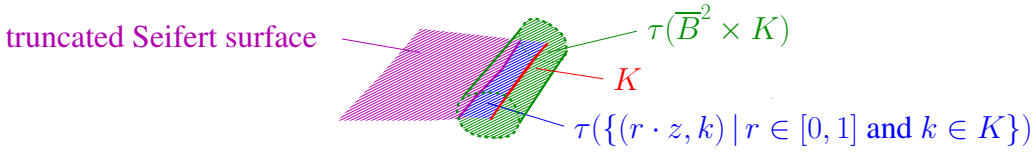
⊞

It follows from Claim 2 and the Knot Thickening–Self Linking Lemma 284.11 that there exists a diffeotopy of S^3 that turns the thickening σ into the thickening τ. ■

291.3.2. **Truncated Seifert surfaces.** For technical purposes it is in the following easier to work with the concept of a "truncated" Seifert surface :

Definition 291.11. Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link and let $\tau \colon \overline{B}^2 \times L \to S^3$ be a thickening.

(1) As in Definition 108.30 we refer to $X_L := S^3 \setminus \tau(B^2 \times L)$ as the exterior of L.
(2) A truncated Seifert surface is a proper submanifold F of the exterior X_L such that for each $i \in \{1, \ldots, m\}$ there exists a $z \in S^1$ such that $\partial F \cap T_i = \tau(\{z\} \times L_i)$.



Example. Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link. Furthermore let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening of L. If F is a Seifert surface that is standard with respect to τ, then $F \cap X_L$ is clearly a truncated Seifert surface. □

We will soon need the following lemma:

Lemma 291.12. (Truncated Seifert Surface–Arrange Boundary Lemma) In the following let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening. Let F be a truncated Seifert surface for L. Given any $z_1, \ldots, z_m \in S^1$ there exists a proper smooth isotopy from F to a truncated Seifert surface for L such that for $i = 1, \ldots, m$ we have $\partial F \cap T_i = \tau(\{z_i\} \times L_i)$.

Proof. Let $L = L_1 \cup \cdots \cup L_m \subseteq S^3$ be an oriented m-component link, let $\tau\colon \overline{B}^2 \times L \to S^3$ be a thickening and let F be a truncated Seifert surface for L. By Definition 291.11 there exist $w_1, \ldots, w_m \in S^1$ such that for $i = 1, \ldots, m$ we have $\partial F \cap T_i = \tau(\{w_i\} \times L_i)$.

Now let $z_1, \ldots, z_m \in S^1$. We pick $r_1, \ldots, r_m, s_1, \ldots, s_m \in \mathbb{R}$ such that for $i = 1, \ldots, m$ we have $w_i = \exp(2\pi \mathrm{i} r_i)$ and $z_i = \exp(2\pi \mathrm{i} s_i)$. Next we consider the smooth isotopy

$$\begin{array}{rcl} G\colon (\overline{B}^2 \times L) \times [0,1] & \to & S^3 \\ ((\xi, \zeta), t) & \mapsto & \tau(\xi \cdot \exp(2\pi \mathrm{i} \cdot (s_i - r_i) \cdot t), \zeta). \end{array}$$

Note that $G_0 = \tau$. Next note that it follows from the Isotopy Extension Theorem 44.2 that there exists a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ with $H_0 = \mathrm{id}$ and $H_1 \circ G_0 = G_1$. This diffeotopy restricts to a proper smooth isotopy from F to a truncated Seifert surface with the right boundary.[3149] ■

291.3.3. **The Seifert surface graph.** Given an oriented link L we can "organize" the truncated Seifert surfaces into a graph:

Definition 291.13. Let $L \subseteq S^3$ be an oriented link with thickening $\tau\colon \overline{B}^2 \times L \to S^3$ and exterior $X_L = S^3 \setminus \tau(B^2 \times L)$. We define a graph[3150] (V, E) as follows:

- The set of vertices V is given by the proper smooth isotopy[3151] classes of truncated Seifert surfaces of L.
- We say that two vertices v, v' are connected by an edge if there exist representatives F and F' of v and v' which are disjoint and if there exists a compact codimension-zero submanifold W of X_L such that $\partial_0 W = F \cup (-F')$.

This graph is called the Seifert surface graph $S(L)$ of L.

The following proposition is one of the key technical ingredients in the proof of the Tube Equivalence Theorem 291.4:

Proposition 291.14. (Seifert Surface Graph-Connected Proposition) Let $L \subseteq S^3$ be an oriented link with thickening $\tau\colon \overline{B}^2 \times L \to S^3$. The Seifert surface graph $S(L)$ is connected.

[3149] There are of course many other proofs for the lemma, for example one could "bend" F in a neighborhood of the boundary. But all other attempts will also need some extra input, like the Collar-for-Submanifold Proposition 56.3.

[3150] We refer to Definition 41.3 for the notion of a proper smooth isotopy.

[3151] Strictly speaking this is neither a graph in the sense of Definition 11.1 nor in the sense of Definition 11.12. But to the mature reader it should be clear what we mean.

Proof. [3152] Let $L \subseteq S^3$ be an oriented link with thickening $\tau\colon \overline{B}^2 \times L \to S^3$. We start out with the following claim:

Claim 1. Let F, F' be two truncated Seifert surfaces of L. There exists a truncated Seifert surface of L that is smoothly isotopic to F' and which is transverse to F.

Proof. Let F, F' be two truncated Seifert surfaces of L. First note that it follows immediately from the Truncated Seifert Surface–Arrange Boundary Lemma 291.12 that F' is properly smoothly isotopic to a truncated Seifert surface F'' such that $\partial F \cap \partial F'' = \varnothing$. The desired statement now follows from the Transversality Theorem 50.9 applied to the compact oriented proper submanifolds F and F'' of X_L. ⊞

Claim 2. Let F, F' be two truncated Seifert surfaces of L. If F is transverse to F', then $F \cap F'$ is a proper submanifold of X_L with finitely many components.

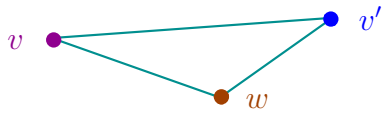
Proof. It follows from the Submanifold Transversal Intersection Theorem 50.4 that $F \cap F'$ is a proper submanifold of X_L. Since X_L is compact we obtain from the Compact-Closed Lemma 3.39 (1) that $F \cap F'$ itself is a compact smooth manifold. Thus it has finitely many components. ⊞

Given $v, v' \in S(L)$ we set

$$d(v, v') \;:=\; \min\{\#\pi_0(F \cap F') \in \mathbb{N}_0 \,|\, v = [F], v' = [F'] \text{ and } F \text{ and } F' \text{ are transverse}\}.$$

Note that it follows from Claim 1 and Claim 2 that $d(v, v') \in \mathbb{N}_0$ is indeed defined. Also note that $v, v' \in S(L)$ are connected by an edge if and only if $d(v, v') = 0$. It thus follows easily from elementary logic that it remains to prove the following claim:

Claim 3. Given $v, v' \in S(L)$ we can find $w \in S(L)$ such that $d(v, w) < d(v, v')$ and such that $d(w, v') < d(v, v')$.



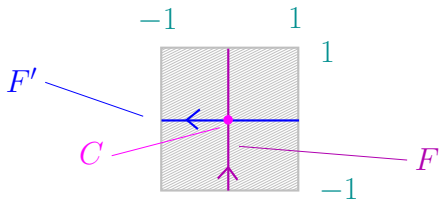
Proof. Let $v, v' \in S(L)$. We pick truncated Seifert surfaces F and F' with $v = [F]$, $v' = [F']$, which are transverse and such that $d(v, v') = \#\pi_0(F \cap F')$. Note that it follows from the Transversal Preimage Theorem 50.8 that $F \cap F'$ is a closed oriented 1-dimensional smooth submanifold of S^3. It follows easily from the Transverse Tubular Neighborhood Theorem 57.32, the Product Tubular Neighborhood–Existence Theorem 58.4 and the Tubular Neighborhood–Boundary Extension Theorem 57.29 that we can pick

(i) an *orientation-reversing tubular neighborhood* $F' \times [-1, 1]$ and
(ii) an *orientation-preserving tubular neighborhood* $[-1, 1] \times F$

such that on $F \cap F'$ we obtain a tubular neighborhood $[-1, 1] \times (F \cap F') \times [-1, 1]$ and such that $\partial F' \times [-1, 1] \subseteq \partial X_L$ and $[-1, 1] \times \partial F \subseteq \partial X_L$ preserve the product structure corresponding coming from the description $\partial X_L = \tau(S^1 \times L)$.
Next we construct a submanifold G in three steps:

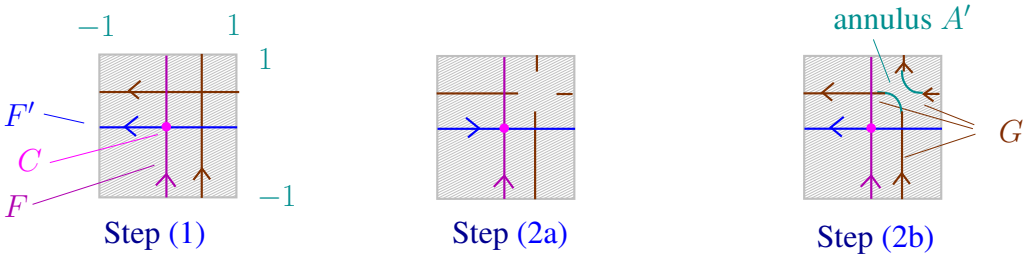
(1) We consider $(\{\frac{1}{2}\} \times F) \cup (F' \times \{\frac{1}{2}\})$, i.e. we consider the union of two positive push-offs of F and F'.

[3152] The argument we provide is based on the argument in [**HQ2025**, Chapter 3].



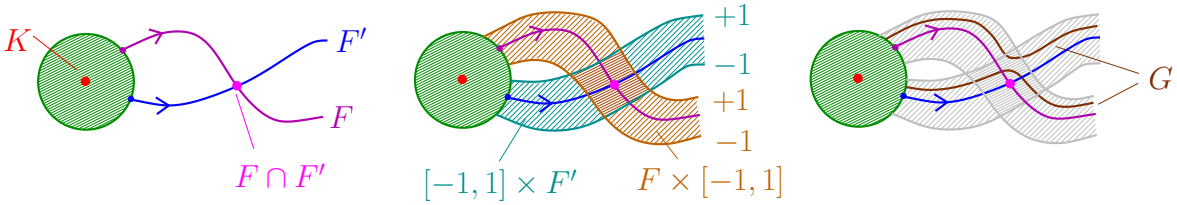
(2) For each component C of $F \cap F'$ we do the following:
   (a) We remove the two open annuli $\{\frac{1}{2}\} \times (\frac{1}{4}, \frac{3}{4}) \times C$ and $(\frac{1}{4}, \frac{3}{4}) \times \{\frac{1}{2}\} \times C$.
   (b) We do the following:
   - We pick a smoothly embedded annulus A' connecting $C \times \{\frac{1}{4}, \frac{1}{2}\}$ and $C \times \{\frac{1}{2}, \frac{3}{4}\}$.
   - We pick a smoothly embedded annulus A'' connecting $C \times \{\frac{1}{2}, \frac{1}{4}\}$ and $C \times \{\frac{3}{4}, \frac{1}{2}\}$.

   We can arrange that the annuli are disjoint, that they lie in $C \times [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$ and we can arrange them such that $(\{\frac{1}{2}\} \times F) \cup (F' \times \{\frac{1}{2}\}) \cup A' \cup A''$ is again a smooth submanifold. The local model is shown in the figure below.[3153] Note that we attached the annuli such that the orientations match, in other words, the resulting smooth manifold admits an orientation that restricts to the original orientations on $\{\frac{1}{2}\} \times F$ and on $F' \times \{\frac{1}{2}\}$.



(3) We throw out all closed components of the smooth manifold that was handed to us by Step (2).

We denote by G the resulting compact oriented proper submanifold of X_L.



Subclaim 1. We can write $G = G' \sqcup G''$ where G' and G'' are both standard Seifert surfaces for L. Both these standard Seifert surfaces are transverse to both F and F'.

Proof of subclaim. We start out with a few observations:

(a) First note that it follows from $F \cap F' \cap \partial X_L = \varnothing$ that $\partial G = (\{\frac{1}{2}\} \times \partial F) \cup (\partial F' \times \{\frac{1}{2}\})$. Next note that it follows from our boundary condition on the tubular neighborhoods

[3153] Using the local coordinates and using say the Smooth Transition Function Lemma 24.9 one can easily fill-in all the details.

that for each $i \in \{1, \dots, m\}$ the intersection $\partial G \cap \partial(S^1 \times L)$ consists of precisely two curves of the form $\tau(\{*\} \times L_i)$.

(b) It follows from the Link Complement-Boundary H_*–Proposition 283.5 and the Splitting Lemma 145.5 that the connecting homomorphism $\partial_2 \colon \mathrm{H}_2(X_L, \partial X_L) \to \mathrm{H}_1(\partial X_L)$ is a monomorphism that even admits a left-inverse $\varphi \colon \mathrm{H}_1(\partial X_L) \to \mathrm{H}_2(X_L, \partial X_L)$.

(c) We denote by $\alpha \in \mathrm{H}_1(\partial X_L)$ the homology class that is given by taking in each boundary torus the homology class represented by some $\tau(\{*\} \times L_i)$.

(d) It follows from (a), (b) and the Fundamental Class-of-∂-Proposition 194.13 that we have the equality $\partial_2([G]) = 2 \cdot \alpha$.

(e) Note that (b) and (d) imply that $[G] = \varphi(\partial_2([G])) = 2 \cdot \varphi(\alpha)$.

(f) It follows from (e) and the Codimension-One Splitting Proposition 291.5 that we can write $G = G' \sqcup G''$ where G' and G'' are unions of components of G such that $[G'] = [G'']$.

(g) It follows from $[G'] = [G'']$ and the Fundamental Class-of-∂-Proposition 194.13 that G' and G'' share the boundary curves in a most brotherly fashion, i.e. for every $i \in \{1, \dots, m\}$ the submanifolds G' and G'' each contain precisely one curve $\tau(\{*\} \times L_i)$ in the boundary torus $\tau(S^1 \times L_i)$.

The above discussion shows that G' and G'' are both standard Seifert surfaces for L.

Finally note that by construction G is transverse to both F and F'. This implies of course that G and G' are also both transverse to both F and F'. □

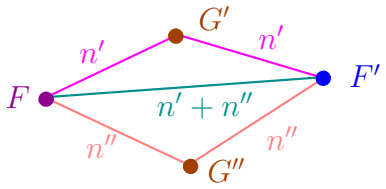
We move on to the second and last subclaim:

Subclaim 2. There exist $n', n'' \in \mathbb{N}_0$ with the following three properties:

(0) $\#\pi_0(F \cap F') = n' + n''$.

(1) $\#\pi_0(G' \cap F) = \#\pi_0(G' \cap F') = n'$.

(2) $\#\pi_0(G'' \cap F) = \#\pi_0(G'' \cap F') = n''$.



Note that the Subclaim 2 tells us that at least one of n', n'' is less than $\#\pi_0(F \cap F')$. The corresponding tilde-surface is thus the desired truncated Seifert surface.

Proof of subclaim. Let C be a component of $F \cap F'$. In the figure below we summarize again the construction of $G = G' \sqcup G''$.



As should be clear from the above picture one of the following two cases occurs:[3154]

(1) C either contributes a component to $G' \cap F$ *and to* $G' \cap F'$, or

(2) C contributes a component to $G'' \cap F$ *and to* $G'' \cap F'$.

[3154]Note that here we finally use the orientation conventions for the tubular neighborhoods from page 5580.

Since there are no other intersection points of G with F and F' we are done. ■

291.3.4. **Conclusion of the proof.** Now we are finally able to prove the Tube Equivalence Theorem 291.4:

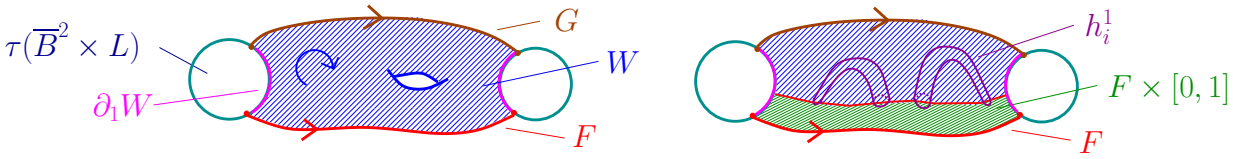
Theorem 291.4. (Tube Equivalence Theorem) Let $L \subseteq S^3$ be an oriented link. Then the following holds: Either L admits a disconnected Seifert surface or any two Seifert surfaces of L are tube-equivalent.

Sketch of proof. Let $L \subseteq S^3$ be an oriented link. By the Standard Thickening Lemma 291.8 we can pick a standard thickening $\tau\colon \overline{B}^2 \times L \to S^3$ for L. As in Definition 108.30 we consider the exterior $X_L := S^3 \setminus \tau(B^2 \times L)$.

It follows from the Standard Seifert Surface Lemma 291.10, the Truncated Seifert Surface Lemma 291.12 and the Seifert Surface Graph-Connected Proposition 291.14 that we only need to prove the following claim:

Claim. Let F and G be two truncated Seifert surfaces for L. If F and G are disjoint, then F and G are tube-equivalent in X_L.

Proof. By our hypothesis we can assume that F and G are connected. We pick a tubular map $\tau\colon (F \cup G) \times [0,1] \to X_L$. Let W be the component $X_L \setminus \tau((F \cup G) \times (-1,1))$ that contains $\tau(F \times \{1\})$. Since F is connected we see that W also contains an annulus on each boundary torus from a boundary curve of $F \times \{1\}$ to the boundary curve of $G \times \{-1\}$. Since G is connected this in turn implies that G is also contained in the boundary of W. In summary we see that $\partial_0 W = F \cup G$ and $\partial_1 W$ is a union of annuli, one for each boundary torus.



The above discussion shows that W, with the decomposition $\partial_- W = F$, $\partial_+ W = G$ and $\partial_v = \partial_1 W$, is a 3-dimensional cobordism with vertical boundary in the sense of Definition 253.1.[3155]

Since W is a cobordism with vertical boundary we obtain from the Handle Decomposition Theorem 253.8 a standard handle decomposition of W rel F. Since each component of W has non-empty intersection with F we obtain from a very mild generalization of the Minimal 0-Handle Proposition 247.13 that we can arrange that the handle decomposition has no 0-handles. Similarly it follows from a similarly mild generalization of the Minimal n-Handle Proposition 248.4 that we can furthermore arrange that the handle decomposition has no 3-handles. In summary we see that it is of the form

$$W = (F \times [0,1]) \cup h_1^1 \cdots \cup h_m^1 \cup h_1^2 \cdots \cup \ldots h_n^2.$$

We set $H := \partial_+((F \times [0,1]) \cup h_1^1 \cdots \cup h_m^1)$. Note that H is obtained from F via a smooth isotopy and m tube attachments.[3156] Furthermore note that it follows from a (evidently

[3155] This observation is similar in spirit to the statement of the Cut Along Surface–Vertical Boundary Lemma 253.2.

[3156] Here we implicitly use the remark on page 5572 which implies that attaching tubes does not destroy the connectedness.

rather mild) generalization of the Dual Handle Proposition 248.1 that H is obtained from G via a smooth isotopy and n tube attachments. This implies that F and G are tube-equivalent. ■

In Definition 11.12 we introduced the notion of an abstract graph

Definition 291.15. An abstract graph G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

Definition 291.16. An abstract graph $G = (V, E, \varphi)$ is called bipartite if we can find a decomposition $V = V' \sqcup V''$ such that each edge connects a point in V' to a point in V''.

Lemma 291.17. (Bipartite Criterion) Let $G = (V, E, \varphi)$ be a finite graph. The following two statements are equivalent:

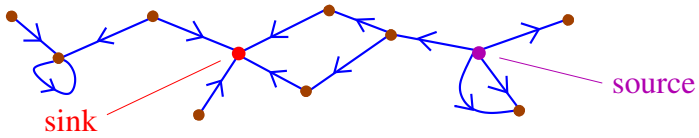
(i) The finite graph $G = (V, E, \varphi)$ is bipartite.
(ii) Each cycle in $G = (V, E, \varphi)$ has even length.

Next let us recall Definition 11.13:

Definition 291.18. Let $G = (V, E, \varphi)$ be an abstract graph. An orientation for G is a pair of maps $i, t\colon E \to V$ such that for each $e \in E$ we have $\varphi(e) = \{i(e), t(e)\}$.[3157]

Definition 291.19. Let $G = (V, E, \varphi, \mathcal{O} = (i, t))$ be an abstract graph in the sense of Definition 11.1.

- We say $v \in V$ is a sink if no edge exits v, i.e. if there is no $e \in E$ with $i(e) = v$.
- We say $v \in V$ is a source if no edge point to v, i.e. if there is no $e \in E$ with $t(e) = v$.



Definition 291.20. Let $G = (V, E, i, t)$ be an abstract graph.

- Let $v \in V$ be a sink. We obtain a new abstract graph by reversing the orientations of all edges with $t(e) = e$. We say that we pushed up a sink. Note that this construction turns a sink into a source.

We will need the following variation on Definition 11.2:

Definition 291.21. Let $G = (V, E, \varphi)$ be an oriented abstract graph.

(1) A path of length k is a sequence of edges $e_1, \ldots, e_k \in E$ such that there exist vertices $v = v_0, v_1, \ldots, v_k = w$ such that for each $i \in \{1, \ldots, k\}$ the vertices of e_i are v_{i-1}, v_i. If $v_0 = v_k$, then we say that the path is a loop.
(2) Now we assume that $G = (V, E, \varphi)$ is equipped with an orientation $\mathcal{O}$. We say that j-th edge of a path is *positive* if $v_j = i(e_j)$ and $v_{j+1} = t(e_j)$. Otherwise we say that the j-th edge is negative.

We will need the following purely combinatorial proposition which was first proved by Oliver Pretzel [**Pre1986**]:

[3157] Note that (V, E, i, t) is an oriented abstract graph in the sense of Definition 11.1.

Proposition 291.22. (Orientation-Reversal–Pushing-down-Sinks Proposition) Let $G = (V, E, i, t)$ be a finite abstract graph. The following two statements are equivalent:

(i) For every cycle in G the number of edges that it transverses in the positive direction equals the number of edges that it transverses in the negative direction.

(ii) There exists a sequence of orientations $\mathcal{O} = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_n = \mathcal{O}^{\mathrm{rev}}$ such that each $\mathcal{O}_i$ arises from $\mathcal{O}_{i-1}$ by pushing up a sink.

Proof. Given a path γ in (V, E, φ) and given an orientation $\mathcal{O}$ we denote by $\delta(\gamma, \mathcal{O})$ the number of positive edges minus the number of negative edges.

We start out with the easier (ii)⇒(i)-direction. This direction follows immediately from the following two observations:

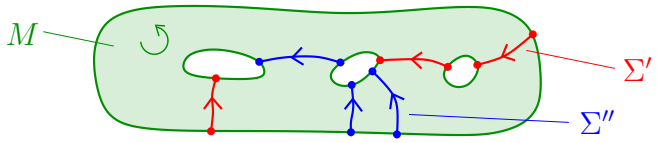
- Let γ be a loop and let $\mathcal{O}$ be an orientation. If we push up a sink to obtain an orientation $\mathcal{O}'$, then $\delta(\gamma, \mathcal{O}) = \delta(\gamma, \mathcal{O}')$. This can be seen as follows: A loop passes through v, then it will have a positive path follows by a negative path. After raising the sink the two signs get flipped.
- For any path γ we clearly have $\delta(\gamma, \mathcal{O}^{\mathrm{rev}}) = -\delta(\gamma, \mathcal{O})$.

Next we deal with the (i)⇒(ii)-direction. ■

Proposition 291.23. (Surface Cobordism Proposition) Let M be an n-dimensional compact oriented smooth manifold and let Σ', Σ'' be two disjoint proper oriented codimension-one submanifolds. Then the following two statements are equivalent:

(i) $[\Sigma'] = [\Sigma''] \in \mathrm{H}_{n-1}(M, \partial M)$.

(ii) Σ' and Σ'' are cobordism equivalent.



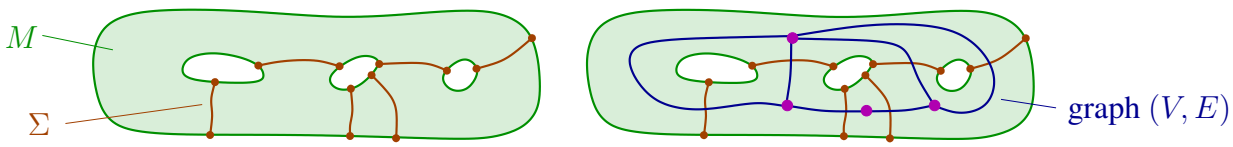
Proof. [3158] The (ii)⇒(i)-direction is the content of the Cobordant–Same Homology Class Proposition 212.12.

The remains of the proof is occupied with the (ii)⇒(i)-direction. We start out with a few preparations.

(a) We write $\Sigma := \Sigma' \sqcup \Sigma''$. We view Σ as an oriented submanifold.

(b) We denote by $\{\Sigma_e\}_{e \in E}$ the set of components of Σ. For each $F \subseteq E$ we set $\Sigma(F) := \bigcup_{f \in F} \Sigma_f$. Note that there exists $E' \subseteq E$ with $\Sigma' = \Sigma(E')$ and that there exists $E'' \subseteq E$ with $\Sigma''' = \Sigma(E')$.

(c) Let (V, E, φ) be the graph where the vertices are the components of $M \setminus\setminus \Sigma$ and where, as in (b), the edges correspond to the components of Σ.



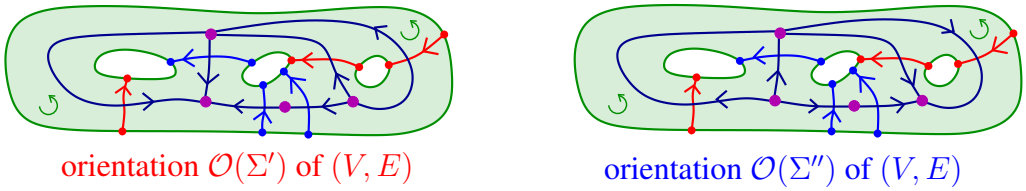
[3158] In this proof we follow the argument of [**FHMQZ2026**].

Now comes the subtle part of the preparations, where the fixed orientations of Σ and M play an essential role:

(a) Every $F \subseteq E$ defines an orientation $\mathcal{O}(F)$ of (V, E, φ) as follows:
- If $e \in E$ we define $i(e) \in V$ to be the component that contains Σ_{e+} and we define $t(e) \in V$ to be the component that contains Σ_{e-}.
- If $e \in E$ we define $i(e) \in V$ to be the component that contains Σ_{e-} and we define $t(e) \in V$ to be the component that contains Σ_{e+}.

Note that conversely any orientation $\mathcal{O}$ of (V, E, φ) defines a subset $F(\mathcal{O}) \subseteq E$.

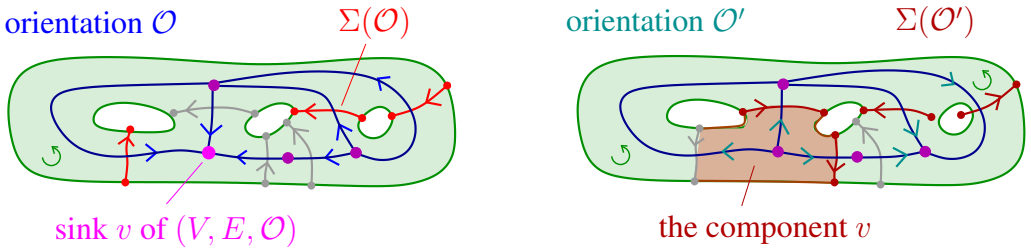
(b) Note that for $F \subseteq E$ we clearly have $\mathcal{O}(E \setminus F) = \mathcal{O}(F)^{\mathrm{rev}}$.



The following claim contains one of the key observation of this proof:

Claim 1. Let $\mathcal{O}$ be an orientation for (V, E) and let $v \in V$ be a sink of $(V, E, \mathcal{O})$. Let $\mathcal{O}'$ be the orientation of (V, E, φ) that we obtain by pushing up the sink of v. Then v defines an internal cobordism between $\Sigma(\mathcal{O})$ and $\Sigma(\mathcal{O}')$.

Proof.



⊞

Now we turn to the actual statement of the proposition:

Claim 2. We consider the oriented graph $(V, E, \mathcal{O}(J'))$. For every cycle in G the number of edges that it transverses in the positive direction equals the number of edges that it transverses in the negative direction.

Proof. This follows from $[J'] = [J'']$. ⊞

It follows from Claim 2 and the Orientation-Reversal–Pushing-down-Sinks Proposition 291.22 that there exists a sequence of orientations $\mathcal{O}(\Sigma') = \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_n = \mathcal{O}^{\mathrm{rev}} = \mathcal{O}(\Sigma'')$ such that each $\mathcal{O}_i$ arises from $\mathcal{O}_{i-1}$ by pushing up a sink. But it now follows from Claim 1 that for each $i \in \{1, \ldots, m\}$ the surface $\Sigma(\mathcal{O}_i)$ is cobordism equivalent to $\Sigma(\mathcal{O}_{i-1}$. This shows that $\Sigma' = \Sigma(\mathcal{O}(\Sigma'))$ is cobordism equivalent $\Sigma(\mathcal{O}(\Sigma'')) = \Sigma''$. ■

291.4. **Dealing with the disconnected case.** Let M be a compact oriented n-dimensional smooth manifold and let $Z \subseteq M$ be a disjoint oriented proper codimension-one submanifold. We pick an orientation-preserving tubular neighborhood $[-1, 1] \times Z$. Let

$\{Z_i\}_{i\in I}$ be the components of Z. We denote by $S(Z)$ the set of all oriented codimension-one submanifolds of M that are (as unoriented submanifolds) given by unions of parallel copies of the Z_i. Note that we allow the parallel copies to have different orientations.

Given $\Sigma \in S(Z)$ we denote by $z_i(\Sigma) \in \mathbb{Z}$ the number of copies of Z_i, counted with signs.

Definition 291.24. A cobordism in $S(Z)$ is an oriented codimension-zero submanifold C of M such that $\partial_0 C \in S(Z)$.

Definition 291.25. Let $\Sigma, \Sigma' \in S(Z)$. We say that they are related by a cobordism if there exists a cobordism C as in the previous definition with the following three properties:
(1) We have $\Sigma \setminus C = \Sigma' \setminus C$.
(2) $\Sigma \cap C = \Sigma \cap \partial C$ and $\Sigma' \cap C = \Sigma' \cap \partial C$
(3) $\partial_0 C = (\Sigma \cap \partial C) \sqcup ((-\Sigma') \cap \partial C)$ as oriented manifolds.
This leads to the notion of cobordism equivalence.

We want to prove the following proposition:

Proposition 291.26. Let X and Y be disjoint unions of components of Z. If $[X] = [Y]$, then X and Y are cobordism equivalent in the above sense.

Lemma 291.27. Let $\Sigma, \Sigma' \in S(Z)$. Then Σ and Σ' are cobordism equivalent if and only if $\Sigma \cup (-\Sigma')$ is cobordism equivalent to $\varnothing$.

Lemma 291.28. Let $\Sigma, \Sigma' \in S(Z)$. If for all $i \in I$ we have $z_i(\Sigma) = z_i(\Sigma')$, then Σ and Σ' are cobordism equivalent.

Proof. For innermost pair of opposite signs introduce the cobordism $[a, b] \times Z_i$. ■

Definition 291.29. A generalized cobordism in $S(Z)$ is a finite family $C = \{C_k\}_{k=1,\dots,m}$ of cobordisms in $S(Z)$ such that for each $k \leq l$ we have either $C_k \cap C_l = \varnothing$ or $C_k \subseteq C_l$.

Lemma 291.30. There exists a generalized cobordism $C = \{C_k\}_{k=1,\dots,m}$ in $S(Z)$ such that for each $i \in I$ we have $z_i(\partial C) = z_i(X \cup -Y)$.

Proof. That is the homological argument. Here we use that $[Z] = [Z]$. ■

Proof of Proposition 291.26. Let X and Y be disjoint unions of components of Z such that $[X] = [Y]$. We want to show that X and Y are cobordism equivalent. By Lemma 291.27 it suffices to show that $X \cup -Y$ is cobordism equivalent to $\varnothing$.

By Lemma 291.30 we can find a generalized cobordism $C = \{C_k\}_{k=1,\dots,m}$ in $S(Z)$ such that for each $i \in I$ we have $z_i(\partial C) = z_i(X \cup -Y)$. It follows from Lemma 291.28 that ∂C is cobordism equivalent to $X \cup -Y$.

So it suffices to show that ∂C is cobordism equivalent to $\varnothing$. But that is clear since $C_1, \dots, C_m$ define a sequence cobordisms from $\varnothing, \partial C_1, \partial C_1 \cup \partial_2, \dots, \partial C$. ■

Proposition 291.31. Let M be a compact oriented n-dimensional smooth manifold and let $X, Y \subseteq M$ be two disjoint oriented proper codimension-one submanifolds such that $[X] = [Y] \in \mathrm{H}_{n-1}(M)$.

Proof. Apply Proposition 291.26 to $Z = X \sqcup Y$. ■

291.5. **Seifert matrices and tube attachments.** In Definition 289.9 we introduced the Seifert matrix corresponding to a *connected* Seifert surface. We now give a generalization to the possibly disconnected Seifert surfaces.

Definition 291.32. Let L be an oriented non-empty link in S^3 and let F be a Seifert surface with d components. It follows from the Path-Component–H_*–Lemma 143.21, Surface-H_*-Proposition 155.17 and the Surface Classification Theorem 101.16 that $\mathrm{H}_1(F)$ is a free abelian group of some rank k. Let $v_1, \dots, v_k \in \mathrm{H}_1(F)$ be a basis. We set

$$\widetilde{A} \;:=\; (\sigma_F(v_r, v_s))_{r,s=1,\dots,k} \;=\; (\mathrm{lk}_F(v_r, \iota_+(v_s)))_{r,s=1,\dots,k}.$$

We refer to

$$\begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0_{(d-1)\times(d-1)} \end{pmatrix}$$

as the Seifert matrix corresponding to the basis $v_1, \dots, v_k$ for the link L. Note that if F consists of a single component, then this matches the original Definition 289.9.

The Tube Equivalence Theorem 291.4 raises the following question: How does a tube attachment affect generalized Seifert matrices? The following lemma give us a complete answer to this question:

Lemma 291.33. (Tube Attachment–Seifert Matrix Lemma) Let L be an oriented link in S^3 and let F be a generalized Seifert surface. Let G be a generalized Seifert surface for L that is obtained from F by a tube attachment.

(1) First we assume that the tube gets attached to a single component of F. Let $v_1, \dots, v_k$ be a basis for $\mathrm{H}_1(F)$. We denote by A the corresponding Seifert matrix. Then there exists a basis $w_1, \dots, w_k, w_{k+1}, w_{k+2}$ for $\mathrm{H}_1(G)$ and a $\xi \in \mathbb{Z}^k$, such that the corresponding Seifert matrix is of the form

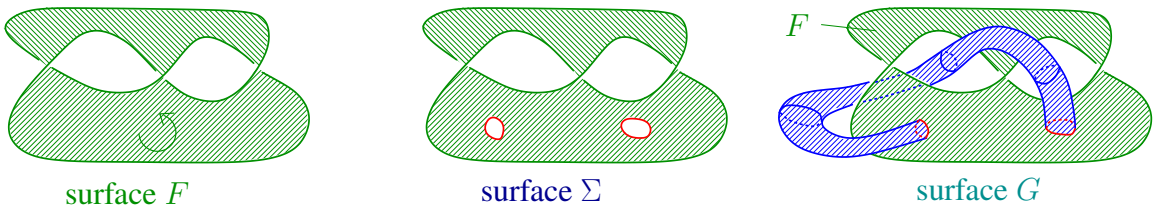
$$\begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or of the form} \quad \begin{pmatrix} A & 0 & 0 \\ \xi^{\mathrm{T}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(2) We now assume that the tube gets attached to two different components. Then there exists a basis $w_1, \dots, w_k, w_{k+1}$ for $\mathrm{H}_1(G)$ such that the corresponding Seifert matrix equals the original Seifert matrix.

The proof of the Tube Attachment–Seifert Matrix Lemma 291.33 will require the remainder of this section. In Section 291.6 we will use this result to show that any two Seifert matrices of an oriented link are "S-equivalent".

For clarity we prove the two statements of the Tube Attachment–Seifert Matrix Lemma 291.33 separately.

Proof of Lemma 291.33 (1). Let L be an oriented link in S^3 and let F be a generalized Seifert surface. Let G be a generalized Seifert surface for L that is obtained from F by a tube attachment where the attaching map $\tau\colon \{0,1\} \times \overline{B}^2 \to F$ takes values in a single component of F. To prove the desired statement we first need to understand the relationship between the homology groups of F and G. We denote by $\tau\colon [-1,1] \times \overline{B}^2 \to S^3$ the smooth embedding corresponding to the tube attachment. We set $\Sigma := F \setminus \tau(\{0,1\} \times B^2)$.

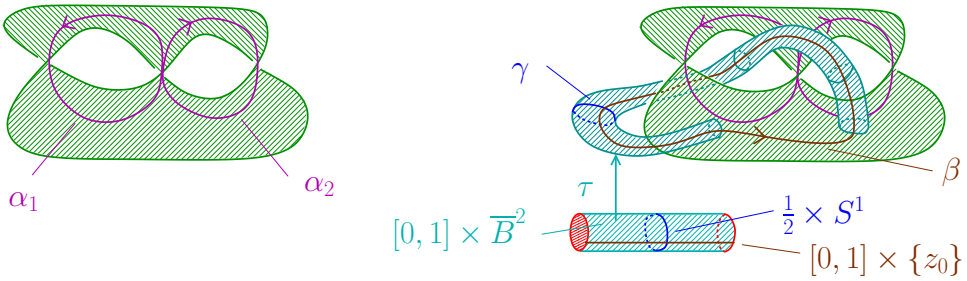


We introduce the following smoothly embedded oriented loops:

(a) By the Represent-by-1-Manifold Corollary 187.19 we can pick smoothly embedded oriented loops $\alpha_1, \dots, \alpha_k$ in F such that for $i = 1, \dots, k$ we have $[\alpha_i] = v_i \in \mathrm{H}_1(F)$. By an elementary argument using the Smooth Ball Embedding Theorem 45.10 (see also Exercise 45.7) we can arrange that the α_i miss $\tau(\{0,1\} \times \overline{B}^2)$. In other words, we can arrange that they lie in the interior of Σ.

(b) We pick $z_0 \in S^1$. Using the fact that $\tau\colon \{0,1\} \times \overline{B}^2 \to F$ takes values in a single component of F one can easily see, say using the Complement Connected Lemma 4.40, that $\alpha(1, z_0)$ to $\alpha(0, z_0)$ lie in the same component of Σ. We denote by β the union of the path $\tau([0,1] \times \{z_0\})$ with any smoothly embedded path in Σ from $\alpha(1, z_0)$ to $\alpha(0, z_0)$.

(c) We set $\gamma := \tau(\{\frac{1}{2}\} \times S^1)$.



Claim 1. The homology classes $[\alpha_1], \dots, [\alpha_k], [\beta], [\gamma]$ form a basis for $\mathrm{H}_1(G)$.

Proof. Our first goal is to find a suitable basis for the homology group $\mathrm{H}_1(\Sigma)$. In the following we write $S^1_\pm := \tau(\{\pm 1\} \times S^1)$. It follows from the Mayer–Vietoris Theorem 153.2 (3), applied to $\Sigma = F \cup (D_+ \cup D_-)$, that we have an exact sequence of the form:

$$\begin{aligned} 0 \;\to\; & \mathrm{H}_1(S^1_-) \oplus \mathrm{H}_1(S^1_+) \;\xrightarrow{[S^1_-]}\; \mathrm{H}_1(\Sigma) \oplus \overbrace{\mathrm{H}_1(D_-) \oplus \mathrm{H}_1(D_+)}^{=0} \;\xrightarrow{[\alpha_1],\dots,[\alpha_k]}\; \mathrm{H}_1(F) \;\xrightarrow{\partial_1}\; \\ \to\; & \mathrm{H}_0(S^1_-) \oplus \mathrm{H}_0(S^1_+) \;\to\; \mathrm{H}_0(\Sigma) \oplus \mathrm{H}_0(D_-) \oplus \mathrm{H}_0(D_+) \;\to\; \mathrm{H}_0(F) \;\to\; 0. \end{aligned}$$

We make the following observations:

- Since F has no closed components we obtain from the Surface-H_*-Proposition 155.17 that $\mathrm{H}_2(F) = 0$. This explains why to the top-left we start out with the group 0.
- Note that it follows from the H_0-Proposition 143.25 that the two inclusion induced maps $\mathrm{H}_0(S^1_\pm) \to \mathrm{H}_0(D_\pm)$ are isomorphisms. It follows easily that the sequence regarding the H_0-terms is exact. This implies that the connecting homomorphism ∂_1 is zero.

It now follows almost immediately that the homology classes $[\alpha_1], \dots, [\alpha_k], [S^1_-]$ and $[S^1_+]$ form a basis for $\mathrm{H}_1(\Sigma)$.

Next note that the Mayer–Vietoris Theorem 153.2 (3), applied to the decomposition $G = \Sigma \cup \tau([-1,1] \times S^1)$ gives us an exact sequence of the form:

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{H}_1(S^1_-) \oplus \mathrm{H}_1(S^1_+) & \to & \overset{[\alpha_1],\dots,[\alpha_k],[S^1_-],[S^1_+]}{\mathrm{H}_1(\Sigma)} & \oplus \mathrm{H}_1(\tau([-1,1] \times S^1)) & \to & \overset{[\beta]}{\mathrm{H}_1(G)} & \xrightarrow{\partial_1} & \\ & \to & \mathrm{H}_0(S^1_-) \oplus \mathrm{H}_0(S^1_+) & \to & \mathrm{H}_0(\Sigma) & \oplus \mathrm{H}_0(\tau([-1,1] \times S^1)) & \to & \mathrm{H}_0(G) & \to & 0. \end{array}$$

We make the following observations:

- Since G has no closed components we obtain from the Surface-H_*-Proposition 155.17 that $\mathrm{H}_2(G) = 0$. This explains why to the top-left we start out with the group 0.
- Note that $[S^1_-] = [S^1_+] \in \mathrm{H}_1(\tau([-1,1] \times S^1))$.
- It follows from the explicit description of the homomorphisms in the Mayer–Vietoris Theorem 153.2 (3), H_0-Proposition 143.25 and the fact that $\tau([-1,1] \times S^1)$ is path-connected and the fact that $S^1_\pm$ lie both in the same path-component of Σ that the kernel of the map $\mathrm{H}_0(S^1_-) \oplus \mathrm{H}_0(S^1_+) \to \mathrm{H}_0(\Sigma) \oplus \mathrm{H}_0(\tau([-1,1] \times S^1))$ is given by $\mathbb{Z} \cdot ([P_+] - [P_-])$ where $P_\pm \in S^1_\pm$ is a point.
- It follows from the calculation of the connecting homomorphism from Exercise 153.3 that $\partial_1([\beta]) = [P_+] - [P_-]$.

The desired statement follows easily from the above together with the earlier calculation of $\mathrm{H}_1(\Sigma)$. ⊞

Claim 2. With respect to the basis $[\alpha_1], \dots, [\alpha_k], [\beta], [\gamma]$ of $\mathrm{H}_1(G)$ the corresponding Seifert matrix is given by

$$\begin{pmatrix} A & \xi & 0 \\ \zeta^{\mathrm{T}} & * & \pm 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or it is given by} \quad \begin{pmatrix} A & \xi & 0 \\ \zeta^{\mathrm{T}} & * & 0 \\ 0 & \pm 1 & 0 \end{pmatrix}$$

where $\xi, \zeta \in \mathbb{Z}^k$ and $* \in \mathbb{Z}$.

Proof. The claim follows from the following observations:

- It follows from the Seifert Form–Calculation Lemma 289.10 that the first $(k \times k)$-quadrant is unchanged.
- Note that γ is the boundary of the smoothly embedded disk $\tau(\{\frac{1}{2}\} \times \overline{B}^2)$ which is disjoint from the α_i. It follows from the Linking #-via-Intersection #-Proposition 285.5 that all linking numbers with γ and the positive push-off γ^+ are zero, except for β. The two different options correspond to the possibility that β gets pushed into the solid cylinder $\tau([0,1] \times \overline{B}^2)$ or out of the solid cylinder $\tau([0,1] \times \overline{B}^2)$. ⊞

Note that it remains to prove the following claim:

Claim 3. The matrix

$$\begin{pmatrix} A & \xi & 0 \\ \zeta^{\mathrm{T}} & 0 & \pm 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{is congruent to} \quad \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and an analogous statement holds for the other type of Seifert matrix.

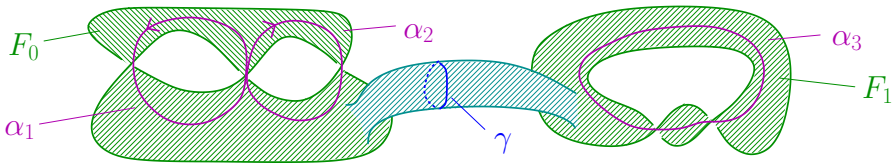
Proof. First note that after possibly multiplying both the last column and the last row by -1 we can arrange that the unique entry in the last column is $+1$. Next note that by adding

multiples of the last column to columns $1, \dots, k$ (and simultaneously adding multiples of the last row to rows $1, \dots, k$) we can dispose of ζ^{T}. ■

Proof of Lemma 291.33 (2). Let L be an oriented link in S^3 and let F be a generalized Seifert surface. Let G be a generalized Seifert surface for L that is obtained from F by a tube attachment where the attaching map $\tau\colon \{0,1\} \times \overline{B}^2 \to F$ takes values in two different components F_0 and F_1 of F. An elementary argument, say using the Complement Connected Lemma 4.40 and the Union Connected Lemma 4.39, shows that G has precisely one component less than F.

We set $\gamma := \tau(\{\frac{1}{2}\}) \times S^1$. An argument as in the proof of Case (1) shows easily that one can find smoothly embedded oriented loops $\alpha_1, \dots, \alpha_k \subseteq \Sigma := F \setminus \tau(\{0,1\} \times B^2)$ with the following properties:

- $[\alpha_1], \dots, [\alpha_k]$ form a basis for $\mathrm{H}_1(F)$.
- $[\alpha_1], \dots, [\alpha_k], [\gamma]$ form a basis for $\mathrm{H}_1(G)$.



The linking numbers of γ and $\iota_+(\gamma)$ with the α_i are all zero. Thus the Seifert matrix for G gained an extra row and column of zeros. This compensates for the fact that G has one component less. We now see that the Seifert matrices for F and for G, with respect to the above bases, are exactly the same. ■

291.6. **S-equivalence of Seifert matrices.** The Tube Attachment–Seifert Matrix Lemma 291.33 together with the Tube Equivalence Theorem 291.4 motivates the following algebraic definition that was first introduced by Kunio Murasugi [**Muras1965**, Definition 2.1]:

Definition 291.34.

(1) (a) Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and let $\xi \in \mathbb{Z}^k$. We refer to the $((k+2) \times (k+2))$-matrix

$$\begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

as a column enlargement of A.

(b) If a matrix B is obtained from A by a column enlargement, then we say that B is a column reduction of A.

(2) (a) Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and let $\xi \in \mathbb{Z}^k$. We refer to the $((k+2) \times (k+2))$-matrix

$$\begin{pmatrix} A & 0 & 0 \\ \xi^{\mathrm{T}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

as a row enlargement of A.

(b) If a matrix B is obtained from A by a row enlargement, then we say that B is a row reduction of A.

(3) We say that two integral square matrices are S-equivalent if we can obtain one from the other via a sequence of the following three types of moves:
  (i) column (or row) enlargements,
  (ii) column (or row) reductions,
  (iii) replacing A by $P^{\mathrm{T}}AP$ where P is an invertible matrix over $\mathbb{Z}$.

Now we can easily prove the main theorem of this chapter:

Theorem 291.35. (S-Equivalence Theorem) Let $L \subseteq S^3$ be an oriented link. Any two Seifert matrices for L are S-equivalent.

Remark.

(1) A somewhat different proof of the S-Equivalence Theorem 291.35 is given in [**Muras2008**, Chapter 5].

(2) Like essentially all other authors we will use the Tube Equivalence Theorem 291.4 to prove the S-Equivalence Theorem 291.35. At least for any oriented knot $K \subseteq S^3$ there is a completely different proof: In Definition 340.26 we will introduce the Blanchfield pairing

$$\mathrm{Bl}_K \colon \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

In the Blanchfield Pairing–Seifert Matrix Theorem 340.29 we will see that one can express the Blanchfield pairing in terms of any Seifert matrix. It then follows from a purely algebraic theorem of Hale Trotter [**Tro1973**, Main Theorem] (see also the discussion on page 6454) that all Seifert matrices of a knot are S-equivalent. □

Proof. Let $L \subseteq S^3$ be an oriented link. Note that by Definition 289.9 the Seifert matrix depends on two choices:

(a) Given a Seifert surface F we choose a basis for $\mathrm{H}_1(F)$.
(b) The choice of a Seifert surface in the first place.

We will now see that the definition of S-equivalence compensates for both:

(a) If we are given a Seifert surface F, then it follows immediately from Definition 289.9 and the elementary Form–Matrix-Lemma 224.5 that the corresponding Seifert matrix, up to congruence, does not depend on the choice of the basis of $\mathrm{H}_1(F)$.
(b) It follows from Definition 291.3, the Tube Equivalence Theorem 291.4 and Tube Attachment–Seifert Matrix Lemma 291.33 that different Seifert surfaces lead to S-equivalent Seifert matrices. ■

The following proposition gets used subconsciously all the time:

Proposition 291.36. (S-Equivalence–Smooth Isotopy Proposition) The S-equivalence class of a Seifert matrix is an invariant of the smooth isotopy type of an oriented link.

Proof. Let $L, \widetilde{L}$ be two oriented links in S^3 that are smoothly isotopic. We need to show that if A is a Seifert matrix for L, then it is also a Seifert matrix for $\widetilde{L}$. So let F be a Seifert surface for L and let $v_1, \dots, v_k \in \mathrm{H}_1(F)$ be a basis such that $A = (\sigma_F(v_i, \iota_+(v_j)))_{i,j=1,\dots,k}$. It follows from our hypothesis, together with the Isotopy Extension Theorem 44.2, that there exists an orientation-preserving diffeomorphism $\Phi \colon S^3 \to S^3$ that restricts to an orientation-preserving diffeomorphism $\Phi \colon L \to \widetilde{L}$. We consider the Seifert surface $\widetilde{F} := \Phi(F)$. It follows

easily from the Naturality-of-Linking Lemma 289.8 that for each $i, j \in \{1, \dots, k\}$ we have $\sigma_{\widetilde{F}}(\widetilde{\Theta}_*(v_i), \iota_+(\Theta_*(v_j))) = \sigma_F(v_i, \iota_+(v_j))$. Thus we have verified that A is a Seifert matrix for $\widetilde{L}$. ■

The name of the game is now to find invariants of matrices under S-equivalence. The following lemma does not really give us anything new, but it shows at least that we are on the right tracks:

Lemma 291.37. (Seifert Matrix–S-Equivalence–Polynomial Lemma) In the following let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and $B \in \mathrm{M}(l \times l, \mathbb{Z})$ be two matrices. If A and B are S-equivalent, then $\det(t \cdot A - A^{\mathrm{T}}) \doteq \det(t \cdot B - B^{\mathrm{T}})$.

Proof. It follows from Definition 291.34 that it suffices to prove the following claim:

Claim. Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and $B \in \mathrm{M}(l \times l, \mathbb{Z})$ be two matrices.

(1) If B is a column enlargement of A, then $\det(t \cdot B - B^{\mathrm{T}}) \doteq \det(t \cdot A - A^{\mathrm{T}})$.
(2) If B is a row enlargement of A, then $\det(t \cdot B - B^{\mathrm{T}}) \doteq \det(t \cdot A - A^{\mathrm{T}})$.
(3) If $B = P^{\mathrm{T}} A P$ for some $P \in \mathrm{GL}(k, \mathbb{Z})$, then $\det(t \cdot B - B^{\mathrm{T}}) \doteq \det(t \cdot A - A^{\mathrm{T}})$.

Proof. Evidently all statements follow from a direct calculation:

(1) Let B be a column enlargement of A as in Definition 291.34. We then see that

$$\begin{aligned} \det(t \cdot B - B^{\mathrm{T}}) &= \det\left(t \cdot \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}} \right) = \det \begin{pmatrix} t \cdot A - A^{\mathrm{T}} & t \cdot \xi & 0 \\ -\xi^{\mathrm{T}} & 0 & t \\ 0 & -1 & 0 \end{pmatrix} \\ &= t \cdot \det(t \cdot A - A^{\mathrm{T}}) \qquad \doteq \det(t \cdot A - A^{\mathrm{T}}). \\ &\uparrow \\ &\text{elementary linear algebra} \end{aligned}$$

(2) The proof of this statement is almost the same as the proof of (1).
(3) Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ and let $P \in \mathrm{GL}(n, \mathbb{Z})$. For $B = P^{\mathrm{T}} \cdot A \cdot P$ we see that

$$\begin{aligned} \det(t \cdot B - B^{\mathrm{T}}) &= \det(t \cdot P^{\mathrm{T}} \cdot A \cdot P - (P^{\mathrm{T}} \cdot A \cdot P)^{\mathrm{T}}) &&= \det(t \cdot P^{\mathrm{T}} \cdot A \cdot P - P^{\mathrm{T}} \cdot A^{\mathrm{T}} \cdot P) \\ &= \det(P^{\mathrm{T}} \cdot (t \cdot A - A^{\mathrm{T}}) \cdot P) &&= \det(t \cdot A - A^{\mathrm{T}}) \cdot \det(P)^2 \\ &= \det(t \cdot A - A^{\mathrm{T}}). \\ &\uparrow \\ &\text{since } P \in \mathrm{GL}(n, \mathbb{Z}) \end{aligned}$$

■

The reader is invited to come up with their own invariants of "matrices up to S-equivalence". We will present some invariants in the exercises and later in the next Chapter 292.

291.7. Exercises for Chapter 291.

Exercise 291.1. Let L be an oriented link with Seifert matrix A. We define the nullity of the link L as $\dim_{\mathbb{R}}(\ker(A + A^{\mathrm{T}}))$. Show that the nullity of L is well-defined, i.e. show that it does not depend on the choice of the Seifert matrix A.

Exercise 291.2. Let L be an oriented link in S^3 and let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a Seifert matrix for L. We define $\det(L) := \mathrm{i}^n \cdot \det(A + A^{\mathrm{T}})$. Show that this complex number is well-defined.

Exercise 291.3. Let $K \subseteq S^3$ be an oriented knot. We pick a $(2g \times 2g)$-Seifert matrix A. Note that it follows from the Knot–Double Covering-H_1-Proposition 289.25 that $\det(A + A^{\mathrm{T}}) \neq 0$, in particular, $A + A^{\mathrm{T}}$ is invertible over $\mathbb{Q}$. We consider the pairing

$$\begin{array}{rcl} \mathbb{Z}^{2g}/(A+A^{\mathrm{T}})\cdot\mathbb{Z}^{2g} \times \mathbb{Z}^{2g}/(A+A^{\mathrm{T}})\cdot\mathbb{Z}^{2g} & \to & \mathbb{Q}/\mathbb{Z} \\ ([v],[w]) & \mapsto & v^{\mathrm{T}}\cdot(A+A^{\mathrm{T}})^{-1}\cdot w. \end{array}$$

(a) Show that, given a Seifert matrix A, this pairing is well-defined and symmetric.
(b) Show that the isometry type of this pairing does not depend on the choice of the Seifert matrix A for K.
(c) How does the above pairing change if we replace K by its mirror?
(d) Let K be the trefoil (with some orientation). Compute this pairing for the trefoil K. *Hint.* In this case $\mathbb{Z}^{2g}/(A+A^{\mathrm{T}})\mathbb{Z}^{2g} \cong \mathbb{Z}/3\mathbb{Z}$.
(e) Let K still be the trefoil. Can the above bilinear pairing distinguish between K and K^{mir}?
(f) Show that the above form is additive under the connected sum operation.

Exercise 291.4. Let $K \subseteq S^3$ be an oriented knot. We pick a $(k\times k)$-Seifert matrix A. We consider the pairing

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm1}]^k/(tA-A^{\mathrm{T}})\cdot\mathbb{Z}[t^{\pm1}]^k \times \mathbb{Z}[t^{\pm1}]^k/(tA-A^{\mathrm{T}})\cdot\mathbb{Z}[t^{\pm1}]^k & \xrightarrow{\lambda(A)} & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}] \\ ([v],[w]) & \mapsto & v^{\mathrm{T}}\cdot(t-1)\cdot(A-tA^{\mathrm{T}})^{-1}\cdot\overline{w}. \end{array}$$

(a) Show that, given a Seifert matrix A, this pairing $\lambda(A)$ is well-defined, sesquilinear (in the sense of Definition 125.16) and Hermitian.
(b) For more courageous readers: Show that $\lambda(A)$ is non-singular.
(c) Show that the isometry type of this pairing does not depend on the choice of the Seifert matrix A for K.

Remark. The above form is called the *Blanchfield form of* K. Hale Trotter [**Tro1973**, p. 179] showed that for two oriented knots the corresponding Blanchfield forms are isometric if and only if the Seifert matrices are S-equivalent.

Exercise 291.5. Let L be an oriented link in S^3 and let F be a Seifert surface for L with Seifert matrix A. Furthermore let G be a Seifert surface that is obtained from F by a tube attachment along a smooth embedding $\tau\colon [0,1]\times\overline{B}^2 \to S^3$ to a single component of F. By the Tube Attachment–Seifert Matrix Lemma 291.33 we know that a Seifert matrix for G is of the form $\begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ or of the form $\begin{pmatrix} A & 0 & 0 \\ \xi^{\mathrm{T}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.
How can we determine from the gluing map τ whether we end up with the first or with the second Seifert matrix?

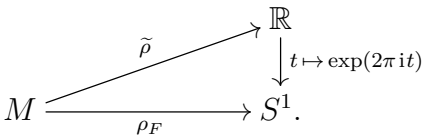
Exercise 291.6. Let $k \in \mathbb{N}_0$. We say a matrix $A \in \mathrm{M}(k\times k,\mathbb{Z})$ is a *Seifert matrix* if $\det(A-A^{\mathrm{T}}) = \pm 1$.

(a) Show that if $A \in \mathrm{M}(k\times k,\mathbb{Z})$ is a Seifert matrix, then k is even.
(b) Show that if $B \in \mathrm{M}(l\times l,\mathbb{Z})$ is S-equivalent to a Seifert matrix, then B is also a Seifert matrix.

Exercise 291.7. Let M be an oriented n-dimensional smooth manifold and let $F \subseteq M$ be a proper oriented *connected* smooth submanifold of codimension one such that we have $[F] = 0 \in \mathrm{H}_{n-1}(M, \partial M)$.

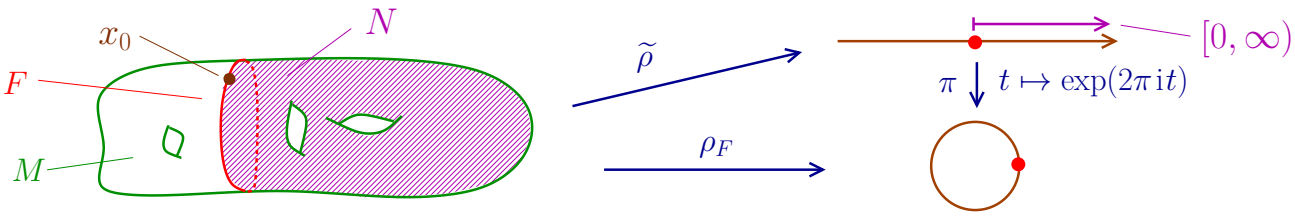
(a) Let $\rho_F \colon M \to S^1$ be the corresponding Thom-Pontryagin map from Definition 234.9. Show that $\rho_* \colon \pi_1(M) \to \pi_1(S^1)$ is the trivial map.

(b) It follows from (a) and the Map Lifting Criterion 127.5 that there exists a smooth map $\widetilde{\rho} \colon M \to \mathbb{R}$ with $\widetilde{\rho}(x_0) = 0$ that makes the following diagram commute:



By the Preimage-of-Interval Lemma 38.3 we know that $N := \widetilde{\rho}^{-1}([0, \infty))$ is a codimension-zero submanifold of M with $\partial_0 N = \widetilde{\rho}^{-1}(0)$. Show that $\widetilde{\rho}^{-1}(0) = F$.

Remark. This exercise gives a new proof for the Null Homologous Boundary Proposition 185.22.



Exercise 291.8. Given a compact orientable 2-dimensional smooth manifold S with components $S_1, \ldots, S_m$ we set

$$\chi_-(S) \;:=\; \sum_{i=1}^{m} \max\{-\chi(S_i), 0\}.$$

Let M be a compact oriented 3-dimensional smooth manifold. We consider the map

$$\begin{aligned} x_M \colon \mathrm{H}_2(M, \partial M) &\to \mathbb{N}_0 \\ \sigma &\mapsto \min\left\{ \chi_-(S) \;\middle|\; \begin{array}{c} S \text{ is a compact oriented proper 2-dimensional} \\ \text{submanifold of } M \text{ with } [S] = \sigma \end{array} \right\}. \end{aligned}$$

Now we can formulate the exercises:

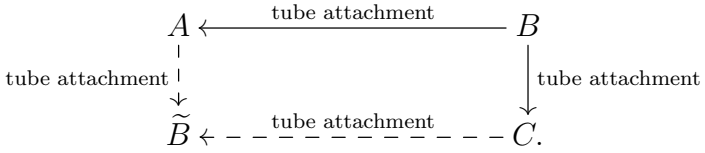
(a) Show that the map x_M is a seminorm, i.e. show that it has the following two properties:
- Given any $\sigma \in \mathrm{H}_2(M, \partial M)$ and $k \in \mathbb{Z}$ we have $x_M(k \cdot \sigma) = |k| \cdot x_M(\sigma)$.
  Hint. Use the Codimension-One Splitting Proposition 291.5.
- Given $\mu, \nu \in \mathrm{H}_2(M, \partial M)$ we have $x_M(\mu + \nu) \leq x_M(\mu) + x_N(\nu)$.
  Hint. Modify the construction in the proof of the Seifert Surface Graph-Connected Proposition 291.14 and make use of the Recombine Fundamental Class Proposition 187.27.

(b) Show that there exists a unique seminorm $x_M \colon \mathrm{H}_2(M, \partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ such that the map $\mathrm{H}_2(M, \partial M) \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1_{\mathbb{R}}]} \mathrm{H}_2(M, \partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ agrees with the above map x_M.

Remark. This seminorm $x_M \colon \mathrm{H}_2(M) \to \mathbb{R}_{\geq 0}$ is usually called the *Thurston norm of* M. It was introduced by William Thurston [**Thu1986a**] in 1976 and, as is shown in [**Thu1986a**] and [**Gab1983b**], it has many remarkable properties.

Remark. This exercise is closely related to Exercise 234.11.

Exercise 291.9. Let M be a 3-dimensional smooth manifold and let A, B, C be compact 2-dimensional submanifolds of M. We suppose that we obtain A via a tube attachment from B and we suppose that we obtain C also via a tube attachment from B. Show that there exist a proper submanifold $\widetilde{B}$ that is obtained via a tube attachment from A and that is obtained via a tube attachment from C. This can be summarized as follows



Hint. Try to show that the tube attachments to B can be done "disjointly".

Exercise 291.10. Let $L \subseteq S^3$ be an oriented link. Show that the graph of *connected* Seifert surfaces is connected.
Hint. Use the Tube Attachment Lemma 291.2 and use Exercise 291.9 generously.

CHAPTER 292

The Levine-Tristram signatures

Let L be an oriented link. In the previous Chapter 291 we showed in the S-Equivalence Theorem 291.35 that the Seifert matrices of L are unique up to S-equivalence. In other words, we show that any invariant of Seifert matrices up to S-equivalence defines an invariant of oriented links. For example, let A be a Seifert matrix for L. We could now define the Alexander polynomial as

$$\Delta_L(t) \ := \ \det(t \cdot A - A^{\mathrm{T}}).$$

It follows from the S-Equivalence Theorem 291.35 together with the Seifert Matrix–S-Equivalence Lemma 291.37 that this is a sensible definition, in the sense that this definition of the Alexander polynomial does not depend on the choice of the Seifert matrix A for L.

This begs the question, can we define other invariants of an oriented link using the S-equivalence class of a Seifert surface? In this section we will see that we can indeed do so, namely we will see that we can use this approach to define the Murasugi signature and the Levine-Tristram signatures of an oriented link L. We will see that these signatures can be used to distinguish many knots from their mirror images.

This chapter is organized as follows:

- In Section 292.1 we will (re-) introduce the signature of a Hermitian form.
- In Section 292.2 we will introduce the Murasugi signature of an oriented link.
- In Section 292.3 we will introduce the Levine-Tristram signatures of an oriented link.
- In Section 292.4 we will provide exercises for Chapter 292.

In the next Chapter 293 we will study many more applications of the Levine-Tristram signatures.

292.1. **The signature of a Hermitian form.** In the following three purely algebraic subsections we will recall some concepts from linear algebra.

292.1.1. **Hermitian matrices.** We start out by recalling the notion of a hermitian matrix.

Definition 292.1. Let $n \in \mathbb{N}_0$.
(1) A matrix $M \in \mathrm{M}(n \times n, \mathbb{C})$ is called Hermitian if $\overline{M} = M^{\mathrm{T}}$.
(2) We denote by $\mathrm{Herm}(n \times n, \mathbb{C})$ the set of Hermitian $(n \times n)$-matrices.
(3) We say that two Hermitian matrix M and N in $\mathrm{M}(n \times n, \mathbb{C})$ are complex congruent if there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P^{\mathrm{T}} \cdot M \cdot \overline{P} = N$.

We recall the following theorem:

Theorem 292.2. 2.35. (Spectral Theorem) If $B \in \mathrm{M}(n \times n, \mathbb{C})$ is a Hermitian matrix, then there exists a unitary matrix $P \in \mathrm{U}(n)$ such that $P^{\mathrm{T}} \cdot B \cdot \overline{P}$ is a diagonal matrix where all diagonal entries are real numbers. In particular all eigenvalues of B are real numbers.

The following proposition is similar to the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4:

Proposition 292.3. (Hermitian Matrices–Congruence Proposition) Let $n \in \mathbb{N}_0$ and let $M \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix. There exist $n, b^+, b^- \in \mathbb{N}_0$ and $P \in \mathrm{GL}(n, \mathbb{C})$ such that

$$P^{\mathrm{T}} \cdot M \cdot \overline{P} \;=\; \mathrm{diag}(\underbrace{0, \dots, 0}_{n \text{ times}}, \underbrace{1, \dots, 1}_{b^+ \text{ times}}, \underbrace{-1, \dots, -1}_{b^- \text{ times}}).$$

Proof. The proof of this proposition is very similar to the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 226.4. We leave it to you, the reader, to make the necessary adjustments. ■

292.1.2. **Hermitian forms.** We continue by recalling some concepts from Definition 2.29 and Definition 224.2:

Definition 292.4. Let V be a complex vector space.

(1) A pairing $\varphi\colon V \times V \to \mathbb{C}$ is called $\mathbb{C}$-sesquilinear if for all $\lambda \in \mathbb{C}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \overline{\lambda}$.

(2) A pairing $\varphi\colon V \times V \to \mathbb{C}$ is called Hermitian if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

A $\mathbb{C}$-sesquilinear Hermitian pairing $\varphi\colon V \times V \to \mathbb{C}$ over a finite-dimensional complex vector space is called a Hermitian form.

The following definition, which is modelled on Definition 224.4 and Definition 226.1, gives us many examples of Hermitian forms:

Definition 292.5. Let $M \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix. We introduce the following Hermitian form:

$$\begin{array}{rccc} \theta_{\mathbb{C}}(M)\colon & \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C} \\ & (v, w) & \mapsto & v^{\mathrm{T}} \cdot M \cdot \overline{w}. \end{array}$$

The following lemma, which is a variation on the Form–Matrix-Lemma 224.5, allows us to go back and forth between isometry classes of Hermitian forms and complex congruence classes of Hermitian matrices over $\mathbb{C}$.

Lemma 292.6. (Form–Matrix-Lemma)

(a) Let (V, ψ) be hermitian form. There exists a Hermitian matrix M such that (V, ψ) is isometric to $\theta_{\mathbb{C}}(M)$.

(b) Let M and N be two hermitian matrices. The forms $\theta_{\mathbb{C}}(M)$ and $\theta_{\mathbb{C}}(N)$ are isometric if and only if there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ with $N = P^{\mathrm{T}} \cdot M \cdot \overline{P}$.

Proof. The proof of this lemma is almost identical to the proof of the original Form–Matrix-Lemma 224.5. For illustration purposes we provide the proof of the "if"-direction of (b). Thus M and A be two hermitian matrices. We assume that there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ with $N = P^{\mathrm{T}} \cdot M \cdot \overline{P}$. We claim that the isomorphism $v \mapsto P \cdot v$ defines an isometry from the form $(\mathbb{C}^n, \theta_{\mathbb{C}}(P^{\mathrm{T}} N \overline{P}))$ to the form $(\mathbb{C}^n, \theta_{\mathbb{C}}(M))$. Indeed, for any $v, w \in \mathbb{C}^n$ we have

$$\theta_{\mathbb{C}}(M)(Pv, Pw) \;=\; (Pv)^{\mathrm{T}} M \overline{Pw} \;=\; v^{\mathrm{T}} (P^{\mathrm{T}} M \overline{P}) \overline{w} \;=\; \theta_{\mathbb{C}}(P^{\mathrm{T}} M \overline{P})(v, w).$$ ■

The following simply minded definition is a variation on Definition 226.5:

Definition 292.7. Let (V, φ) be a hermitian form.

(1) The vector subspace $\operatorname{rad}(\varphi) := \{v \in V \mid \theta(v, w) = 0 \text{ for all } w \in V\}$ is called the radical of the form φ.

(2) We refer to
$$\nu(\varphi) \;:=\; \dim_{\mathbb{F}}(\operatorname{rad}(\varphi)).$$
as the nullity $\nu(\varphi)$.

292.1.3. **The signature of a Hermitian form.** The following definition is a variation on Definition 226.6:

Definition 292.8. Let (V, φ) be a Hermitian form.

(1) We say that φ is positive-definite if $\varphi(v, v) \in \mathbb{R}_{>0}$ for all $v \neq 0$.[3159]

(2) We say that φ is negative-definite if $\varphi(v, v) \in \mathbb{R}_{<0}$ for all $v \neq 0$.

Next we recall Notation 226.7:

Notation 292.9. Given a Hermitian form $(V, \varphi\colon V \times V \to R)$ and given a vector subspace U of the vector space V we denote by $\varphi|_U$ the Hermitian form that is given by restricting the map $\varphi\colon V \times V \to \mathbb{C}$ to $U \times U \to \mathbb{C}$.

The following definition is modelled on Definition 226.8:

Definition 292.10. Let (V, φ) be a Hermitian form. We define

$b^+(\varphi) :=$ maximal dimension of a vector subspace U of V s.t. $\varphi|_U$ is positive-definite,
$b^-(\varphi) :=$ maximal dimension of a vector subspace U of V s.t. $\varphi|_U$ is negative-definite.

Furthermore we define the signature of φ as
$$\operatorname{sign}(\varphi) \;:=\; b^+(\varphi) - b^-(\varphi).$$

The following lemma, which often gets used subconsciously, is essentially the same as the Isometry Invariants Lemma 226.9:

Lemma 292.11. (Isometry Invariants Lemma) Let (V, φ) and (W, ψ) be two Hermitian forms. If (V, φ) and (W, ψ) are isometric, then $b_\pm(\varphi) = b_\pm(\psi)$ and $\operatorname{sign}(\varphi) = \operatorname{sign}(\psi)$.

Proof. The lemma is an immediate consequence of the definitions. ∎

In the following we will mostly work with Hermitian matrices. This leads us to the following convenient notation:

Notation 292.12. Let $M \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix.[3160] We define $\nu(M)$, $b_\pm(M)$ and $\operatorname{sign}(M)$ as the invariants of the corresponding Hermitian form, i.e. we set
$$\begin{aligned} \nu(M) &:= \nu(\theta_{\mathbb{C}}(M)), \\ b_\pm(M) &:= b_\pm(\theta_{\mathbb{C}}(M)), \\ \operatorname{sign}(M) &:= \operatorname{sign}(\theta_{\mathbb{C}}(M)). \end{aligned}$$

In the following proposition, which is almost identical to the Signature-Eigenvalues Proposition 226.11, we will determine the invariants $b^\pm$ and the signature of a Hermitian matrix in terms of the eigenvalues:

[3159] Note that it follows from the hypothesis that φ is Hermitian that for any $v \in V$ we have $\varphi(v, v) = \overline{\varphi(v, v)}$, i.e. we have $\varphi(v, v) \in \mathbb{R}$.

[3160] In many examples M will be a symmetric real matrix.

Proposition 292.13. (Signature-Eigenvalues Proposition) Let $M \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix over $\mathbb{C}$. Recall that we know from the Spectral Theorem 2.35 that all eigenvalues of M are real numbers and that M is diagonalizable. We have the following equalities:

$$\begin{array}{rcl} \nu(M) & = & \text{number of zero eigenvalues of } M\text{, counted with multiplicities,} \\ b^+(M) & = & \text{number of positive eigenvalues of } M\text{, counted with multiplicities,} \\ b^-(M) & = & \text{number of negative eigenvalues of } M\text{, counted with multiplicities.} \end{array}$$

In particular

$$\mathrm{sign}(M) \;=\; \begin{array}{l} \text{number of positive eigenvalues of } M \\ - \;\text{number of negative eigenvalues of } M. \end{array}$$

Furthermore it follows that $\nu(M) + b^+(M) + b^-(M) = n$.

Proof. The proof of this proposition is almost the same as the proof of the original Signature-Eigenvalues Proposition 226.11. We leave it to the reader to make the necessary adjustments. ■

We continue with a few basic results that can be helpful for calculations:

Lemma 292.14. (Signature Congruence Lemma) Let $M \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix. For any $P \in \mathrm{GL}(n, \mathbb{C})$ we have

$$\mathrm{sign}(P^{\mathrm{T}} \cdot M \cdot \overline{P}) \;=\; \mathrm{sign}(M).$$

Proof. This statement follows immediately from the Isometry Invariants Lemma 292.11 and the observation that the Form–Matrix-Lemma 292.6 tells us that the Hermitian forms $\theta_{\mathbb{C}}(P^{\mathrm{T}} \cdot M \cdot \overline{P})$ and $\theta_{\mathbb{C}}(M)$ are isometric. ■

Our next proposition, which is almost identical to the Signature Additivity Proposition 226.13, says that the signature of Hermitian matrices is additive under the block sum operation:

Proposition 292.15. (Signature Additivity Proposition) For any two Hermitian matrices A and B we have

$$\mathrm{sign} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;=\; \mathrm{sign}(A) + \mathrm{sign}(B).$$

Proof. This statement follows immediately from the Hermitian Matrices–Congruence Proposition 292.3 applied to A and B and the Signature-Eigenvalues Proposition 292.13. ■

We conclude this section with the following lemma, which is almost indistinguishable from the Vanishing Signature Lemma 227.6:

Lemma 292.16. (Vanishing Signature Lemma) Let $n \in \mathbb{N}_0$, let $P \in \mathrm{GL}(n, \mathbb{C})$ and let $Q \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix. We have

$$\mathrm{sign} \begin{pmatrix} 0 & P \\ \overline{P}^{\mathrm{T}} & Q \end{pmatrix} \;=\; 0.$$

Proof. We start out with a simple observation: It follows easily from the hypothesis that $Q \in \mathrm{M}(n \times n, \mathbb{C})$ is Hermitian that there exists a $Z \in \mathrm{M}(n \times n, \mathbb{C})$ such that $Q = Z + Z^{\mathrm{T}}$.

We consider the following three transformations

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & P \\ \overline{P}^{\mathrm{T}} & Q \end{pmatrix} \begin{pmatrix} (\overline{P}^{\mathrm{T}})^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix},$$
$$\begin{pmatrix} \mathrm{id} & 0 \\ -Z & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\overline{Z} \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix}.$$

We now see that

$$\operatorname{sign} \begin{pmatrix} 0 & P \\ \overline{P}^{\mathrm{T}} & Q \end{pmatrix} \underset{\substack{\uparrow \\ \text{by the above and the Signature} \\ \text{Congruence Lemma 292.14}}}{=} \operatorname{sign} \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix} \underset{\substack{\uparrow \\ \text{Signature-Eigenvalues} \\ \text{Proposition 292.13}}}{=} 0.$$

■

292.2. **The Murasugi signature of a link.** After this heavy dose of algebra we finally return to knot theory:

Proposition 292.17. (Link Signature Definition Proposition) Let $L \subseteq S^3$ be an oriented link and let A be a Seifert matrix A of L. The signature of the symmetric matrix $A + A^{\mathrm{T}}$ does not depend on the choice of the Seifert matrix of L and it is a smooth isotopy invariant of L.

The Link Signature Definition Proposition 292.17 leads us to the following invariant which was first introduced by Kunio Murasugi [**Muras1965**, Definition 2.2]:

Definition 292.18. Let $L \subseteq S^3$ be an oriented link. By the Seifert Surface Existence Theorem 288.12 we can find a Seifert matrix A for L. We refer to

$$\operatorname{sign}(L) := \operatorname{sign}(\underbrace{A + A^{\mathrm{T}}}_{\substack{\text{symmetric} \\ \text{real matrix}}})$$

as the (Murasugi) signature of L. It follows from the Link Signature Definition Proposition 292.17 that $\operatorname{sign}(L)$ is well-defined.

Proof. Let $L \subseteq S^3$ and $\widetilde{L} \subseteq S^3$ be an oriented link. By the S-Equivalence Theorem 291.35 and the S-Equivalence–Smooth Isotopy Proposition 291.36 we know that any two Seifert matrices for L and $\widetilde{L}$ are S-equivalent in the sense of Definition 291.34. Thus it suffices to prove the following claim:

Claim. Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and $B \in \mathrm{M}(l \times l, \mathbb{Z})$ be two matrices.

(1) If B is a column enlargement of A, then $\operatorname{sign}(B + B^{\mathrm{T}}) = \operatorname{sign}(A + A^{\mathrm{T}})$.
(2) If B is a row enlargement of A, then $\operatorname{sign}(B + B^{\mathrm{T}}) = \operatorname{sign}(A + A^{\mathrm{T}})$.
(3) If $B = P^{\mathrm{T}} A P$ for some $P \in \mathrm{GL}(k, \mathbb{Z})$, then $\operatorname{sign}(B + B^{\mathrm{T}}) = \operatorname{sign}(A + A^{\mathrm{T}})$.

Proof. Evidently all statements follow from a direct calculation:

(1) Let B be a column enlargement of A as in Definition 291.34. We then see that

$$\begin{aligned}\operatorname{sign}(B+B^{\mathrm{T}}) &= \operatorname{sign}\left(\begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}\right) = \operatorname{sign}\begin{pmatrix} A+A^{\mathrm{T}} & \xi & 0 \\ \xi^{\mathrm{T}} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \operatorname{sign}\begin{pmatrix} A+A^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad = \operatorname{sign}(A^{\mathrm{T}}+A) + \operatorname{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ &\text{follows from the Signature Congruence Lemma 292.14} \qquad \text{follows from the Signature Additivity Proposition 292.15} \\ &= \operatorname{sign}(A^{\mathrm{T}}+A). \\ &\uparrow \\ &\text{follows from the Vanishing Signature Lemma 292.16}\end{aligned}$$

(2) The proof of this statement is almost the same as the proof of (1).

(3) Let $A \in \mathrm{M}(n\times n,\mathbb{Z})$ and let $P \in \mathrm{GL}(n,\mathbb{Z})$. For $B = P^{\mathrm{T}}\cdot A\cdot P$ we see that

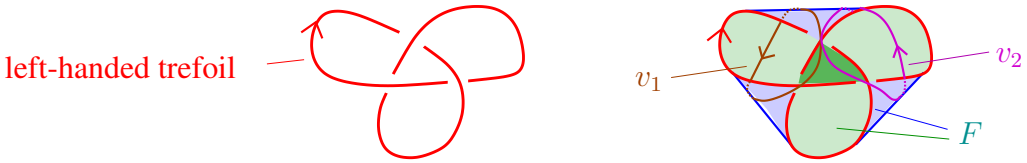
$$\begin{aligned}\operatorname{sign}(B+B^{\mathrm{T}}) &= \operatorname{sign}(P^{\mathrm{T}}\cdot A\cdot P + (P^{\mathrm{T}}\cdot A\cdot P)^{\mathrm{T}}) \qquad = \operatorname{sign}(P^{\mathrm{T}}\cdot A\cdot P + P^{\mathrm{T}}\cdot A^{\mathrm{T}}\cdot P) \\ &= \operatorname{sign}(P^{\mathrm{T}}\cdot (A+A^{\mathrm{T}})\cdot P) \qquad = \operatorname{sign}(A+A^{\mathrm{T}}). \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by the Signature Congruence Lemma 292.14}\end{aligned}$$

■

Example.

(1) Let U be the unknot. As we discussed on page 5528, the (0×0)-matrix is a Seifert matrix for U. Thus we see, not very surprisingly, that $\operatorname{sign}(U) = 0$.

(2) In the figure below we see the left-handed trefoil $T(2,3)$ together with some Seifert surface F and a basis $v_1, v_2 \in \mathrm{H}_1(F)$.[3161]



It is elementary[3162] to see that the corresponding Seifert matrix for the left-handed trefoil is given by $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. This implies that

$$\begin{aligned}\operatorname{sign}(\text{left-handed trefoil}) &= \operatorname{sign}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad = \operatorname{sign}\begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \qquad = 2. \\ &\uparrow \qquad\qquad\qquad\qquad \uparrow \qquad\qquad\qquad\qquad \uparrow \\ &\text{by Definition 292.18} \qquad \text{by the Signature Congruence Lemma 292.14} \qquad \text{by the Signature Eigenvalues Proposition 292.13}\end{aligned}$$

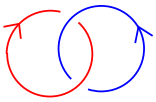
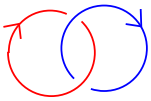
[3161] Recall that in Definition 111.13 we defined the left-handed trefoil as the torus knot $T(2,3)$ and we defined the right-handed trefoil as the torus knot $T(-2,3)$.

[3162] The author decided at some point to replace the formulation "one can easily see that a Seifert matrix is given by" by the above formulation.

(3) Next we consider the figure-8 knot. On page 5528 we showed that a Seifert matrix for the figure-8 knot is given by $A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$. This implies that

$$\text{sign(figure-8 knot)} \underset{\text{by Definition 292.18}}{=} \text{sign}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \underset{\substack{\text{by the Signature}\\ \text{Congruence}\\ \text{Lemma 292.14}}}{=} \text{sign}\begin{pmatrix} -2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \underset{\substack{\text{by the Signature}\\ \text{Eigenvalues}\\ \text{Proposition 292.13}}}{=} 0.$$

(4) We consider the two Hopf links H^- and H^+ that are shown in the figure below.

positive Hopf link H^+  negative Hopf link H^- 

On page 5529 we showed that a Seifert matrix for $H^\pm$ is given by (∓ 1). It follows from Definition 292.18 that, slightly annoyingly, $\text{sign}(H^\pm) = (\mp 1)$. □

We continue with some general properties of signatures:

Proposition 292.19. (Signature-Mirror-Reverse Proposition)
(1) For any oriented link L in S^3 we have $\text{sign}(L^{\text{mir}}) = -\text{sign}(L)$.
(2) For any oriented link L in S^3 we have $\text{sign}(L^{\text{rev}}) = \text{sign}(L)$.
Note that (2) implies that the signature for a knot does not depend on the choice of an orientation.

Proof. Let L be an oriented link in S^3 with Seifert matrix A.

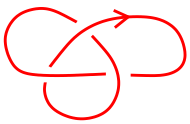
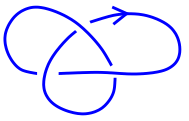
(1) By the Seifert Matrix–Mirror Proposition 290.4 we know that $-A^{\text{T}}$ is a Seifert matrix for L^{mir}. Thus we have the following equalities:

$$\text{sign}(L^{\text{mir}}) = \text{sign}(-A^{\text{T}} + (-A^{\text{T}})^{\text{T}}) = \text{sign}(-A - A^{\text{T}}) \underset{\substack{\text{it follows easily from the definitions that for a Hermitian matrix } H\\ \text{we have } b_\pm(-H) = b_\mp(H), \text{ thus } \text{sign}(-H) = -\text{sign}(H)}}{=} -\text{sign}(A + A^{\text{T}}) = -\text{sign}(L).$$

(2) By the Seifert Matrix–Reverse Proposition 290.2 we know that A^{T} is a Seifert matrix for L^{rev}. Thus we now see that

$$\text{sign}(L^{\text{rev}}) = \text{sign}(A^{\text{T}} + (A^{\text{T}})^{\text{T}}) = \text{sign}(A + A^{\text{T}}) = \text{sign}(L).$$ ■

Example. In the figure below we consider the left-handed trefoil $T = T(2,3)$ and the right-handed trefoil $T(-2,3)$. By the Torus Knot Lemma 111.14 we know that they are mirrors of one another. It thus follows from the calculation on page 5602 and the Signature-Mirror-Reverse Proposition 292.19 that the signature of the right-handed trefoil $T(-2,3) = T^{\text{mir}}$ equals -2.

left-handed trefoil $T = T(2,3)$  right-handed trefoil $T^{\text{mir}} = T(-2,3)$ 

$\text{sign}(T) = 2$ $\text{sign}(T^{\text{mir}}) = -2$

Thus we now finally managed to distinguish the left-handed trefoil from the right-handed trefoil. This discussion shows in particular that the trefoil is not smoothly isotopic to its

mirror, in other words, we showed that the trefoil is chiral in the sense Definition 108.23. This answers Question 108.27. □

In Definition 109.12 we introduced the connected sum $K\#\widetilde{K}$ of any two oriented knots $K, \widetilde{K} \subseteq S^3$. The following proposition shows that the signature is additive under this operation:

Proposition 292.20. (Signature-Additivity Proposition)

(1) For any oriented two knots K and $\widetilde{K}$ in S^3 we have the equality

$$\operatorname{sign}(K\#\widetilde{K}) \;=\; \operatorname{sign}(K) + \operatorname{sign}(\widetilde{K}).$$

(2) Let $(\mathtt{Knot}^{\mathrm{or}}(S^3), \#)$ be the monoid from the Oriented Knot–Monoid Proposition 109.13 that is given by the set $\mathtt{Knot}^{\mathrm{or}}(S^3)$ of smooth isotopy classes of oriented knots in S^3 under the connected sum operation. The map

$$\begin{array}{rcl} \mathtt{Knot}^{\mathrm{or}}(S^3) & \to & \mathbb{Z} \\ [K] & \mapsto & \operatorname{sign}(K) \end{array}$$

is a monoid homomorphism in the sense of Definition 2.2.

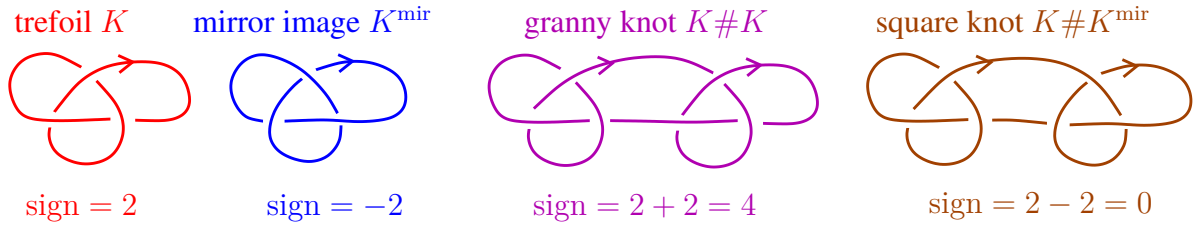
Proof.

(1) Let K and $\widetilde{K}$ be two oriented knots with Seifert matrices A and $\widetilde{A}$. By the Connected Sum–Seifert Matrix Proposition 290.8 we know that $\begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix}$ is a Seifert matrix for $K\#\widetilde{K}$. Thus

$$\operatorname{sign}(K\#\widetilde{K}) \;=\; \operatorname{sign}\begin{pmatrix} A+A^{\mathrm{T}} & 0 \\ 0 & \widetilde{A}+\widetilde{A}^{\mathrm{T}} \end{pmatrix} \underset{\substack{\uparrow \\ \text{by the Signature Additivity} \\ \text{Proposition 292.15}}}{=} \operatorname{sign}(A+A^{\mathrm{T}}) + \operatorname{sign}(\widetilde{A}+\widetilde{A}^{\mathrm{T}})$$
$$= \operatorname{sign}(K) + \operatorname{sign}(\widetilde{K}).$$

(2) This statement follows immediately from Statement (1) and the calculation on page 5602 where we showed that the signature of the unknot is zero. ■

Example. In the figure below we show again the trefoil and its "mirror image" K^{mir}:



The connected sum $K\#K$ is called the *granny knot* and the connected sum $K\#K^{\mathrm{mir}}$ is called the *square knot*.[3163] Note that

$$\begin{array}{rcccccl} \text{sign(granny knot)} & = & \text{sign}(K\#K) & \underset{\uparrow}{=} & \text{sign}(K)+\text{sign}(K) & \underset{\uparrow}{=} & 2+2 \quad = \quad 4. \\ & & & \text{by the Signature-Additivity Proposition 292.20} & & \text{by the above calculations} & \\ \text{sign(square knot)} & = & \text{sign}(K\#K^{\mathrm{mir}}) & \overset{\downarrow}{=} & \text{sign}(K)+\text{sign}(K^{\mathrm{mir}}) & \overset{\downarrow}{=} & 2+(-2) \quad = \quad 0. \end{array}$$

This calculation, together with the Signature-Mirror-Reverse Proposition 292.19 shows that the granny knot and the square knot are not smoothly isotopic (in fact not even up to mirror image). □

292.3. **Levine-Tristram signatures.** In this section we will see that we can generalize the (Murasugi) signatures to a whole family of signature invariants:

Definition 292.21. Let $L \subseteq S^3$ be an oriented link and let $z \in S^1$. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix $A \in \mathrm{M}(n \times n, \mathbb{Z})$ for L. We refer to

$$\sigma_L(z) \;:=\; \text{sign}(\underbrace{A \cdot (1-z) + A^{\mathrm{T}} \cdot (1-z^{-1})}_{(n\times n)\text{-Hermitian matrix}})$$

as the Levine-Tristram signature of L at z. Almost the same calculation as in the proof of the Link Signature Definition Proposition 292.17 shows that this definition does not depend on the choice of the Seifert matrix A and that this is a smooth isotopy invariant of L. The resulting map

$$\begin{array}{rccc} \sigma_L\colon & S^1 & \to & \mathbb{Z} \\ & z & \mapsto & \sigma_L(z) \end{array}$$

is called the Levine-Tristram signature function of L.

Remark. The Levine-Tristram signatures were introduced by Jerry Levine [**Levine1969a**, p. 242].



Jerry Levine (right) with somebody wearing a tie

A very similar concept was introduced around the same time by Andrew Tristram [**Tri1969**, p. 258]. □

[3163] It might seem more reasonable to refer to $K\#K$ as the square knot, but for whatever reason the naming convention is the one we just gave.

Example. Let $L \subseteq S^3$ be an oriented link.

(1) The following calculation shows that the Murasugi signature $\operatorname{sign}(L)$ equals the Levine-Tristram signature at $z = -1$:

$$\sigma_L(-1) \overset{\text{see Definition 292.21}}{=} \operatorname{sign}(A \cdot (1-(-1)) + A^{\mathrm{T}} \cdot (1-(-1))) = \operatorname{sign}(2 \cdot (A + A^{\mathrm{T}})) = \operatorname{sign}(A + A^{\mathrm{T}}) \overset{\text{see Definition 292.18}}{=} \operatorname{sign}(L).$$

(2) For $z = 1$ we always have $\sigma_L(1) = \operatorname{sign}(\text{zero-matrix}) = 0$. □

The Signature-Mirror-Reverse Proposition 292.19 and the Signature-Additivity Proposition 292.20 generalize in an obvious way from the Murasugi signature to the Levine-Tristram signatures. Surely there is no need to spell this out. Our time is better spent on the following proposition:

Proposition 292.22. (Levine-Tristram Signature Continuity Proposition) Let L be an oriented link in S^3. We denote by $Z := \{z \in S^1 \,|\, \Delta_L(t) = 0\}$ the set of zeros of the Alexander polynomial of L.

(1) The function
$$\begin{array}{rcl} S^1 \setminus (Z \cup \{1\}) & \to & \mathbb{Z} \\ z & \mapsto & \sigma_L(z) \end{array}$$
is continuous, in other words, it is locally constant.

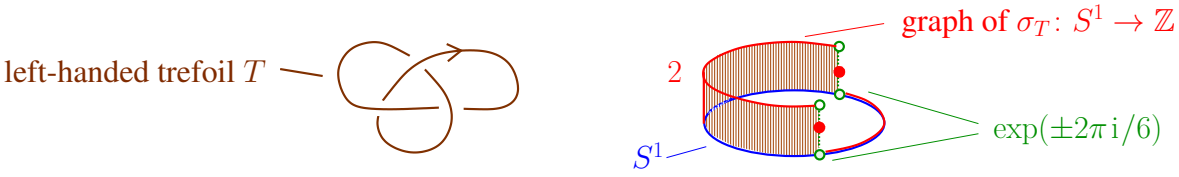
(2) For an oriented knot K the Levine-Tristram signature $\sigma_L \colon S^1 \to \mathbb{Z}$ function is also continuous at 1, in other words, it is continuous outside of the set of zeros of $\Delta_K(t)$.

Example.

(1) Evidently the Levine-Tristram signatures of the unknot are all zero.

(2) We consider again the left-handed trefoil T. We make the following observations:
- By the discussion on page 5602 and 5606 we know that $\sigma_T(-1) = \operatorname{sign}(T) = 2$ and $\sigma_T(1) = 0$.
- By the calculation on page 5530 we know that $\Delta_T(t) = t^2 - t + 1$. The only two zeros are thus $\exp(\pm 2\pi \mathrm{i}/6)$. It follows from the Levine-Tristram Signature Continuity Proposition 292.22 (2) that the signature function $\sigma_T \colon S^1 \to \mathbb{Z}$ is continuous outside of $\exp(\pm 2\pi \mathrm{i}/6)$.

It follows from these two observations that the Levine-Tristram signature function of T is $\equiv 2$ on the "interval" $\{\exp(2\pi \mathrm{i} t) \,|\, t \in (\frac{1}{6}, \frac{5}{6})\}$ and that it is $\equiv 0$ on the "interval" $\{\exp(2\pi \mathrm{i} t) \,|\, t \in (-\frac{1}{6}, \frac{1}{6})\}$. Thus we have determined all values of the Levine-Tristram signature function except for $\exp(\pm 2\pi \mathrm{i}/6)$. An elementary calculation, see Exercise 292.9, shows that these two values are $+1$.



(3) Let K be the figure-8 knot. Recall that on page 2107 we showed that K is amphichiral, i.e. K is smoothly isotopic to its mirror image. By the "Levine-Tristram analog" of

the Signature-Mirror-Reverse Proposition 292.19 this implies that the Levine-Tristram signatures of the figure-8 knot are all zero. □

The proof of the Levine-Tristram Signature Continuity Proposition 292.22 will require the remainder of this section. In the next Chapter 293 we will consider various applications of the Levine-Tristram signatures.

The proof of the Levine-Tristram Signature Continuity Proposition 292.22 makes use of the following lemma, which is also interesting in its own right:

Lemma 292.23. (Signature Continuity Lemma) Let $n \in \mathbb{N}_0$. As usual we denote by $\mathrm{Herm}(n \times n, \mathbb{C})$ the set of Hermitian $(n \times n)$-matrices. The map

$$\begin{array}{rcl} \mathrm{sign}\colon \mathrm{Herm}(n \times n, \mathbb{C}) \cap \mathrm{GL}(n, \mathbb{C}) & \to & \mathbb{Z} \\ A & \mapsto & \mathrm{sign}(A) \end{array}$$

is locally constant.

Sketch of proof. In the following we present two approaches to proving this lemma:

(1) The desired statement follows from the following three observations:
   (a) By the Signature-Eigenvalues Proposition 292.13 we can calculate the signature of a Hermitian matrix by counting positive and negative eigenvalues (with multiplicities).
   (b) The eigenvalues of a complex matrix A are given by the zeros (with multiplicities) of the characteristic polynomial $\chi_A(t) := \det(A - t \cdot \mathrm{id}) \in \mathbb{C}[t]$.
   (c) It follows from the Zeros Continuity Proposition 9.22 that the set of zeros of a polynomial (with multiplicities) is continuous (in a suitable sense) in the coefficients of the polynomial.

(2) Let $A \in \mathrm{Herm}(n \times n, \mathbb{C}) \cap \mathrm{GL}(n, \mathbb{C})$. We need to show that there exists an open neighborhood of A such that the signature is unchanged. We proceed as follows:
   (a) We write $b^{\pm} := b^{\pm}(A)$. Note that it follows from Definition 292.10 there exist vector subspaces $V^+, V^- \subseteq \mathbb{C}^n$ of dimension $b^{\pm}$ such that the restriction $\theta_{\mathbb{C}}(A)|_{V^+}$ is positive-definite and such that the restriction $\theta_{\mathbb{C}}(A)|_{V^-}$ is negative-definite.
   (b) We set $S_{\pm} := \{z \in \mathbb{C}^n \,|\, |z| = 1 \text{ and } z \in V_{\pm}\}$. Note that $S_{\pm}$ are closed subspaces of S^{2n-1}, thus they are compact.
   (c) Since $S_{\pm}$ are compact we see that there exists a $c > 0$ such that the function $v \mapsto \theta_{\mathbb{C}}(A)(v, v)$ is $\geq c$ on S_+ and $\leq -c$ on S_-.
   (d) It follows easily from the compactness of $S_{\pm}$, the Tube Lemma 7.12 and (c) that there exists an open neighborhood U of $A \in \mathrm{Herm}(n \times n, \mathbb{C}) \cap \mathrm{GL}(n, \mathbb{C})$ such that for every $B \in U$ the function $v \mapsto \theta_{\mathbb{C}}(B)(v, v)$ is $\geq \frac{c}{2}$ on S_+ and $\leq -\frac{c}{2}$ on S_-.
   (e) It follows from (d) and the sesquilinearity of hermitian forms that for every $B \in U$ the restriction $\theta_{\mathbb{C}}(B)|_{V^+}$ is positive-definite and the restriction $\theta_{\mathbb{C}}(B)|_{V^-}$ is negative-definite.
   (f) It follows from (e) that for every $B \in U$ we have $b^+(B) \geq b^+(A)$ and that we have $b^-(B) \geq b^-(A)$. Since the matrices A and B have non-zero determinant we know by the Signature-Eigenvalues Proposition 292.13 that $b^+(A) + b^-(A) = n$ and $b^+(B) + b^-(B) = n$. It now follows that $b^{\pm}(B) = b^{\pm}(A)$ and thus $\mathrm{sign}(B) = \mathrm{sign}(A)$.

For both approaches we leave it to the reader to fill in the details. ■

Proof of the Levine-Tristram Signature Continuity Proposition 292.22. Let L be an oriented link with Seifert matrix $A \in \mathrm{M}(n \times n, \mathbb{Z})$. We consider the Levine-Tristram signature function

$$\begin{array}{rcl} \sigma_L \colon S^1 & \to & \mathbb{Z} \\ z & \mapsto & \operatorname{sign}(A \cdot (1-z) + A^{\mathrm{T}} \cdot (1-z^{-1})) \end{array}$$

(1) We start out by observing that we have the following equality:

$$\det(A \cdot (1-t) + A^{\mathrm{T}} \cdot (1-t^{-1})) = \det((t^{-1}-1) \cdot (t \cdot A - A^{\mathrm{T}})) \underset{\uparrow}{=} (t^{-1}-1)^n \cdot \Delta_L(t).$$

by the Alexander Module Seifert Matrix Theorem 289.12

It thus follows from the Signature Continuity Lemma 292.23 that the Levine-Tristram signature function is indeed continuous outside of the set of zeros of Δ_L and away from 1.

(2) We still need to show that for an oriented knot K the Levine-Tristram signature $\sigma_K \colon S^1 \to \mathbb{Z}$ is continuous at $z = 1$. We can break this into two statements:
(a) We need to show that $\sigma_K \colon S^1_{\geq 0} \to \mathbb{Z}$ is continuous at $z = 1$.
(b) We need to show that $\sigma_K \colon S^1_{\leq 0} \to \mathbb{Z}$ is continuous at $z = 1$.
Statement (a) follows from the Signature Continuity Lemma 292.23 and the following three claims. (Statement (b) is evidently proved in almost the same way.)

Claim 1. The map

$$\begin{array}{rcl} \Theta \colon S^1_{\geq 0} & \to & \mathrm{Herm}(n \times n, \mathbb{C}) \\ z & \mapsto & \begin{cases} \frac{1}{|1-z|} \cdot (A \cdot (1-z) + A^{\mathrm{T}} \cdot (1-z^{-1})), & \text{if } z \neq 1, \\ \mathrm{i}(A - A^{\mathrm{T}}) \end{cases} \end{array}$$

is continuous at $z = 1$.

Proof. We start out with the following observation:

$$\begin{aligned} \lim_{t \searrow 0} \tfrac{1-e^{\mathrm{i}t}}{|1-e^{\mathrm{i}t}|} &= \lim_{t \searrow 0} \tfrac{1-\cos(t)-\mathrm{i}\sin(t)}{|1-\cos(t)-\mathrm{i}\sin(t)|} &&= -\mathrm{i}. \\ \lim_{t \searrow 0} \tfrac{1-e^{-\mathrm{i}t}}{|1-e^{\mathrm{i}t}|} &= \lim_{t \searrow 0} \tfrac{1-\cos(-t)-\mathrm{i}\sin(-t)}{|1-\cos(-t)-\mathrm{i}\sin(-t)|} &&= \mathrm{i}. \end{aligned}$$

We now see that

$$\begin{aligned} \lim_{t \searrow 0} \Theta(e^{\mathrm{i}t}) &= \lim_{t \searrow 0} \tfrac{1}{|1-e^{\mathrm{i}t}|}(A \cdot (1-e^{\mathrm{i}t}) + A^{\mathrm{T}} \cdot (1-e^{-\mathrm{i}t})) \\ &= A \cdot \underbrace{\lim_{t \searrow 0} \tfrac{1-e^{\mathrm{i}t}}{|1-e^{\mathrm{i}t}|}}_{=-\mathrm{i}} + A^{\mathrm{T}} \cdot \underbrace{\lim_{t \searrow 0} \tfrac{1-e^{-\mathrm{i}t}}{|1-e^{\mathrm{i}t}|}}_{=\mathrm{i}} = \mathrm{i}(-A + A^{\mathrm{T}}) = \Theta(1). \end{aligned}$$

⊞

Claim 2. For every $z \in S^1_{\geq 0}$ we have $\operatorname{sign}(\Theta(z)) = \sigma_L(z)$.

Proof. Let $z \in S^1_{\geq 0}$.

- For $z \neq 1$ the statement follows from the fact that multiplication by a positive real number does not affect the signature of a matrix.
- Now we consider the case $z = 1$. On page 5606 we pointed out that $\sigma_L(1) = 0$. Thus it remains to show that $\operatorname{sign}(\mathrm{i}(A - A^{\mathrm{T}})) = 0$. In Exercise 292.12 we will show that for any real matrix W with $W^{\mathrm{T}} = -W$ we have $\operatorname{sign}(\mathrm{i}W) = 0$. The desired statement now follows from this fact applied to $W = A - A^{\mathrm{T}}$. ⊞

Claim 3. If K is a knot, then $\Theta(1) \in \mathrm{GL}(n, \mathbb{C})$.

Proof. We need to show that $\det(\mathrm{i}(A - A^{\mathrm{T}})) \neq 0$. This follows from the following little calculation:

$$\det(\mathrm{i}(A - A^{\mathrm{T}})) \quad = \quad \mathrm{i}^n \cdot \det(A - A^{\mathrm{T}}) \quad \underset{\uparrow}{=} \quad \mathrm{i}^n \cdot \Delta_K(1) \quad \underset{\uparrow}{=} \quad \mathrm{i}^n \cdot (\pm 1).$$

by the Alexander Module Seifert Matrix Theorem 289.12

by the Alexander Polynomial-at-1 Proposition 289.17 ■

292.4. Exercises for Chapter 292.

Exercise 292.1. Show that the signature of a knot is always even.

Exercise 292.2. Let L be an oriented link with Seifert matrix A and let $z \in \mathbb{C}$. We define

$$\widetilde{\sigma}_L(z) \ := \ \mathrm{sign}(A \cdot z + A^{\mathrm{T}} \cdot \overline{z}).$$

(a) Show that $\widetilde{\sigma}_L(z)$ is well-defined.
(b) Do we get anything new? More precisely, are these signature invariants determined by the Levine-Tristram signatures?

Exercise 292.3.

(a) Show that given any $n \in \mathbb{Z}$ there exists an oriented link with signature n.
(b) Let $n \in \mathbb{Z}$. Does there exist an oriented 2-component link with signature n?

Exercise 292.4. Determine the Levine-Tristram signatures for the positive and the negative Hopf link that are shown on page 5603.

Exercise 292.5. Show that the Levine-Tristram signatures are additive under the split union of links.

Exercise 292.6. Let K be an oriented knot in S^3 and let $A \in \mathrm{M}(2g \times 2g, \mathbb{Z})$ be a Seifert matrix for L.

(a) We define $\det(K) := (-1)^g \cdot \det(A + A^{\mathrm{T}})$. Use the S-Equivalence Theorem 291.35 to show that this integer is well-defined.
Remark. This is essentially the content of Exercise 291.2.
(b) Show that the sign of the determinant is given by $(-1)^{\mathrm{sign}(K)/2}$.

Exercise 292.7. Let $n \in \mathbb{N}_0$, let $P \in \mathrm{GL}(n, \mathbb{C})$ and let $Q \in \mathrm{M}(n \times n, \mathbb{C})$ be a Hermitian matrix. We consider the form θ that is defined by the matrix $\begin{pmatrix} 0 & P \\ \overline{P}^{\mathrm{T}} & Q \end{pmatrix}$.

(a) Show that for each vector subspace $V \subseteq \mathbb{C}^{2n}$ such that $\theta|_V$ is positive-definite there exists a vector subspace $W \subseteq \mathbb{C}^{2n}$ of the same dimension such that $\theta|_W$ is negative.
(b) Show that $\mathrm{sign}(\theta) = 0$.

Remark. The point of this exercise is to give a new proof of the Vanishing Signature Lemma 292.16.

Exercise 292.8. For $m \in \mathbb{N}$ we consider the matrix

$$A_m \ = \ \begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}.$$

(a) Show that there exists an oriented knot K_m such that A_m is a Seifert matrix for K_m.
(b) Show that $\operatorname{sign}(K_m) = 2$.
(c) Provide an example of an oriented knot J such that Murasugi signature is zero but such that a Levine-Tristram signature is non-zero.

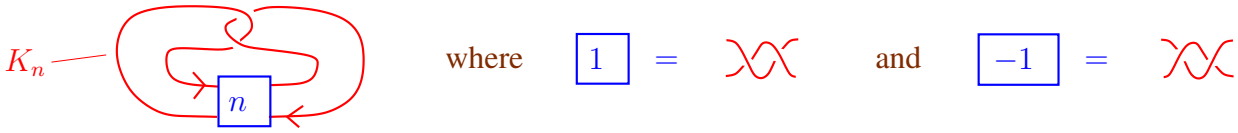
Exercise 292.9. Determine the Levine-Tristram signatures for the left-handed trefoil at $z_\pm = \exp(\pm 2\pi \mathrm{i}/6)$.

Exercise 292.10. Given $n \in \mathbb{Z}$ we consider the n-twist knot K_n that is shown in the figure below.

(a) For which values of $n \in \mathbb{Z}$ do you know the smooth isotopy of K_n?
(b) Determine the Alexander polynomial $\Delta_{K_n}(t)$.
(c) Determine the Levine-Tristram signature functions of K_n.
(d) Show that the image of the monoid homomorphism

$$\begin{array}{ccc} (\texttt{Knots}(S^3), \#) & \to & (\text{maps } S^1 \to \mathbb{Z}, +) \\ K & \mapsto & \sigma_K \end{array}$$

is an infinitely generated submonoid.



Exercise 292.11. Show that the signature for the $(2, p)$-torus knot equals $p - 1$.

Exercise 292.12. Let $W \in \mathrm{M}(n \times n, \mathbb{R})$ be a matrix with $W^{\mathrm{T}} = -W$.

(a) Show that there exists a $P \in \mathrm{GL}(n, \mathbb{R})$ such that $P^{\mathrm{T}} \cdot W \cdot P$ is a diagonal block matrix built out of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the zero matrix.
(b) Show that $\operatorname{sign}(\mathrm{i}W) = 0$.

Exercise 292.13.

(a) Let $K \subseteq S^3$ be an oriented knot and let A be a Seifert matrix for K. Show that if $\det(A) = 0$, then A is S-equivalent to a matrix B of smaller size.
(b) Show that the absolute values of the Levine-Tristram signatures are bounded by the degree of the Alexander polynomial.

CHAPTER 293

Applications of the Levine-Tristram signatures

Let L be an oriented link. In the previous Chapter 292 we introduced the Murasugi signature $\text{sign}(L) \in \mathbb{Z}$ and more generally the Levine-Tristram signature function $\sigma_L \colon S^1 \to \mathbb{Z}$. In this chapter we will study several applications of these signature functions: We will use them to give lower bounds on the "Gordian distance" between two oriented links and we will use them to provide lower bounds on the "smooth 4-genus" of a knot.

This chapter is organized as follows:

- In Section 293.1 we will introduce the Gordian distance between two oriented links and we will introduce the unknotting number of a knot K.
- In Section 293.2 we will see that the Levine-Tristram signatures can be used to give lower bounds on the Gordian distance between two oriented links and thus also on the unknotting number of a knot.
- In Section 293.3 we will introduce the smooth 4-genus of a knot and we will introduce the notion of a smoothly slice knot.
- In Section 293.4 we will formulate the 4-Genus–Signature Theorem 293.18 which says that Levine-Tristram signatures give lower bounds on the smooth 4-genus.
- In Section 293.5 we will provide a proof of the 4-Genus–Signature Theorem 293.18.
- In Section 293.6 we will provide exercises for Chapter 293.

In the next Chapter 294 we will introduce the knot concordance group and we will use the 4-Genus–Signature Theorem 293.18 to show that the knot concordance group is non-trivial.

293.1. **Gordian distance and the unknotting number.** In this section we will introduce the Gordian distance between two oriented links in S^3 and we will introduce the unknotting number of a knot K. To keep the section readable (and to make the section easier to write) we will at times be slightly informal in our exposition.

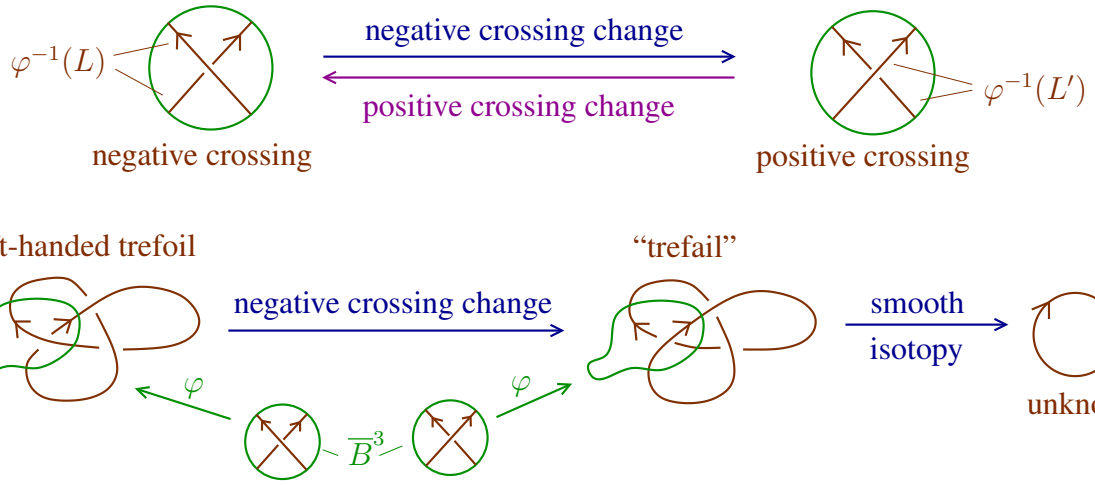
Definition 293.1. Let L and L' be two oriented links in S^3.

(a) We say that L' is obtained from L by a negative crossing change if (after possibly applying a smooth isotopy to L and L') there exists an orientation-preserving smooth embedding $\varphi \colon \overline{B}^3 \to S^3$ such that L and L' agree outside of $\varphi(B^3)$ and such that within $\varphi(\overline{B}^3)$ they differ by the images of the subspaces of $\overline{B}^3$ that are shown in the figure below.

(b) In the above setting we say that L is obtained from L' by a positive crossing change.

Remark. In Definition 113.1 we introduced the notion of a positive and a negative crossing of a diagram. If we consider a link that is associated to a diagram, then a *negative crossing change* corresponds to replacing a negative crossing with a positive crossing. □

Example. In the figure below we start out with a left-handed trefoil.



In the figure we also show a negative crossing change that turns the left-handed trefoil into the unknot. □

The notion of a crossing change leads us to the key definition of this section:

Definition 293.2. Let L and L' be two oriented links in S^3 with the same number of components. We define the Gordian distance $\mathrm{G}(L, L') \in \mathbb{N}_0 \cup \{\infty\}$ as the minimal length of a sequence $L = L_0, \ldots, L_g = L'$ of oriented links such that each L_i is obtained from L_{i-1} by a crossing change.[3164] [3165]



A special case of the Gordian distance deserves its own name:

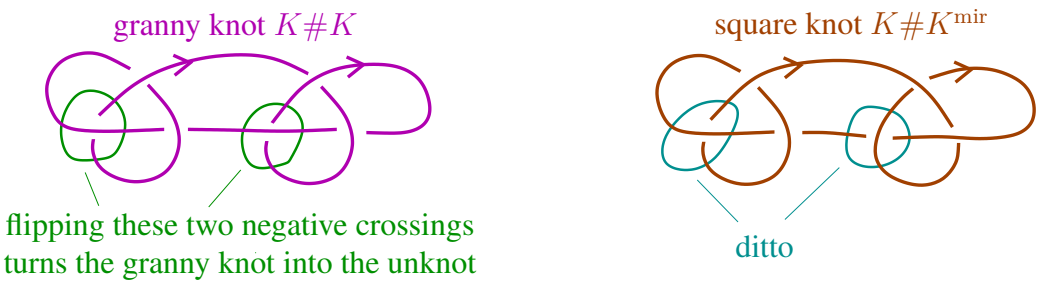
Definition 293.3. Let K be an oriented knot in S^3. We define the unknotting number

$$u(K) \;:=\; \mathrm{G}(K, \mathrm{unknot}).$$

Using the observation that the unknot is reversible one can easily see that the unknotting number of a knot K does not depend on the orientation of K.

Example. In the figure below we show the granny knot $K\#K$ and the square knot $K\#K^{\mathrm{mir}}$.
In each case we show two crossings which, if flipped to the other type of crossing, turn the knot into the unknot. This shows that $u(K\#K) \leq 2$ and $u(K\#K^{\mathrm{mir}}) \leq 2$. But in principle

[3164] By Exercise 113.12, or alternatively by the Unlinking Proposition 69.2, we know that the following statement holds: If L and L' are two oriented links in S^3 with the same number of components, then there exists a sequence of crossing changes that turns L into L'. This implies that the Gordian distance is actually a natural number. For the discussion in this chapter this fact is nice-to-have, but immaterial.
[3165] In Exercise 113.12 we gave a slightly different definition of a crossing change and Gordian distance. We leave it to the reader to show that these definitions are essentially equivalent.



it could be that there is a more clever single crossing change (e.g. after a complicated smooth isotopy), that turns one of the knots into the unknot. So right now we cannot conclude that the unknotting number equals 2. To show that we cannot do better than two crossing changes we need to provide lower bounds on the unknotting number. Some lower bounds will be provided in the next section. □

293.2. **The Gordian distance and the Levine-Tristram signatures.** The following proposition shows that the Levine-Tristram signatures $\sigma_L(z)$ from Definition 292.21 give us lower bounds on the Gordian distance and thus also on the unknotting number:

Proposition 293.4. (Gordian Distance–Signature Proposition) Let L and L' be two oriented links in S^3.

(1) If we can turn L into L' by c_+ positive crossing changes and c_- negative crossing changes, then for every $z \in S^1$ we have the following inequality:

$$\sigma_L(z) - \sigma_{L'}(z) \ \leq \ [-2c_+, 2c_-].$$

(2) For every $z \in S^1$ we have the following inequality:

$$|\sigma_L(z) - \sigma_{L'}(z)| \ \leq \ 2 \cdot \mathrm{G}(L, L').$$

We make two remarks:

- On page 5606 we discussed that the Levine-Tristram signatures of the unknot are zero. Thus we obtain from the above that for a knot K the Levine-Tristram signatures of K provide a lower bound on the unknotting number of K.
- By the discussion on page 5606 we know that $\sigma_L(-1) = \mathrm{sign}(L)$. Thus the above inequalities also hold for the Murasugi signatures of links.

Example.

(1) We return to the granny knot $K\#K$ and the square knot $K\#K^{\mathrm{mir}}$. It turns out that there is good news and bad news:

(☺) We start out with the good news: First recall that on page 5604 we showed that $\mathrm{sign}(K\#K) = 4$. It thus follows from the Gordian Distance–Signature Proposition 293.4 (2) that $u(K\#K) \geq 2$.[3166] Together with the above discussion on page 5612 we have now shown that $u(K\#K) = 2$.

[3166] In fact the Gordian Distance–Signature Proposition 293.4 (1) tells us that we need to perform at least two negative crossing changes to turn $K\#K$ into the unknot.

(☹) We move on to the bad news: On page 5604 we showed that $\operatorname{sign}(K\#K^{\mathrm{mir}}) = 0$. Thus we made no progress on the unknotting number of the square knot. We will deal with this issue in Exercise 293.4.

(2) In general it is quite difficult to determine the unknotting number of a given knot.

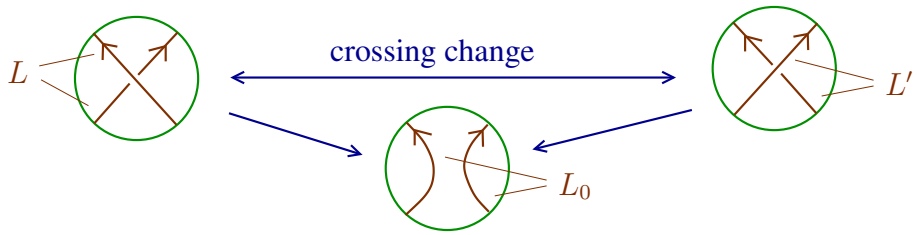


For example in the figure above we see the knot 10_{11} and it seems like it is unknown whether the unknotting number equals 2 or 3. □

The proof of the Gordian Distance–Signature Proposition 293.4 will require the remainder of this section. In the next Section 293.3 we will introduce the smooth 4-genus of a knot.

Before we can provide a proof of the Gordian Distance–Signature Proposition 293.4 we need to make a few preparations:

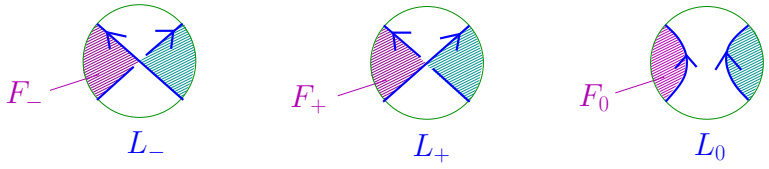
Definition 293.5. Let L and L' be two oriented links in S^3. Suppose we have an orientation-preserving smooth embedding $\varphi\colon \overline{B}^3 \to S^3$ such that L and L' differ by a crossing change along $\varphi(\overline{B}^3)$. We define a new link L_0 as shown in the figure below. We refer to it as the link obtained by resolving the crossing change.



link obtained by resolving the crossing change

The following lemma gives us coherent Seifert surfaces for the links arising at a crossing change:

Lemma 293.6. (Seifert Surface Triple Lemma) Let L_+ be an oriented link in S^3 that is obtained from an oriented link L_- by a negative crossing change. Let L_0 the link that we obtain by resolving the crossing change. We can find connected Seifert surfaces F_+, F_- and F_0 for L_+, L_- and L_0 that are identical outside of the "green ball" where they are of the form shown below.



Sketchy sketch of proof. [3167] Let L_+ be an oriented link that is obtained from an oriented link L_- by a negative crossing change via a smooth embedding $\varphi\colon \overline{B}^3 \to S^3$. We start out with a few preparations:

- Using the Smooth Ball Embedding Theorem 45.10 we can arrange, after a smooth isotopy of $L_\pm, L_0$, that the smooth embedding $\varphi\colon \overline{B}^3 \to S^3$ is the identity.
- The proof of the Link Diagram Existence Proposition 112.7 shows that almost all projections of L_+ and L_- will lead to a diagram. Thus, again after a smooth isotopy of $L_\pm, L_0$ we can assume that $L_\pm$ and L_0 are given by diagrams (in the sense of Definition 112.6) that differ as shown above.

Next we apply Seifert's algorithm, which we explained in the Seifert Surface Existence Theorem 121.2, to the diagram of L_0. Even though back then we formulated Seifert's algorithm only for diagrams of oriented knots it applies without changes to diagrams of oriented links. Up to symmetries the two possible outcomes of a Seifert surface for L_0 are shown in the figure below. (Using the approach of the Make-Surface-Connected Lemma 288.13 we can also arrange that F_0 is connected without affecting the surface in the ball.)



We now consider the two cases separately:

(a) In this case we just need to attach two bands to F_0 (with opposite twists) to obtain Seifert surfaces for F_- and F_+. Note that it follows from the Handle Attachment-π_0-Lemma 246.5 that F_- and F_+ are also connected.

(b) In this case we apply a smooth isotopy to the Seifert surface of L_0 as shown in the figure below:



This turns the Seifert surface into a Seifert surface as in (a), at the very low cost of having to work in a smaller ball. Now we can again attach bands to get Seifert surfaces for L_- and L_+. ■

The Seifert surfaces from the Seifert Surface Triple Lemma 293.6 will provide us with interesting Seifert matrices:

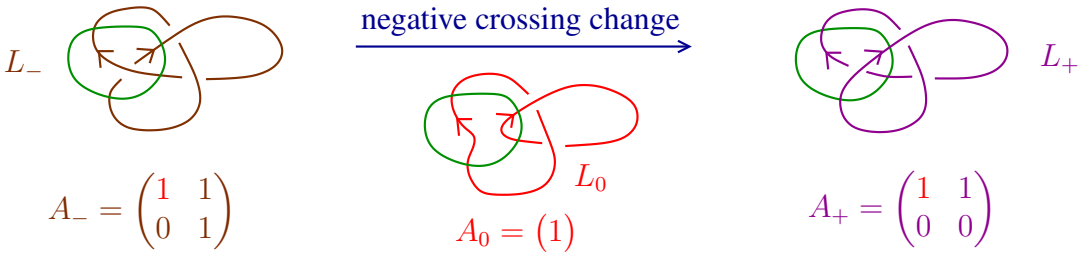
Lemma 293.7. (Seifert Matrix Triple Lemma) Let L_+ be an oriented link that is obtained from an oriented link L_- by a negative crossing change. As in the Seifert Surface Triple Lemma 293.6 we consider again the link L_0 that we obtain from Definition 293.5 by resolving the crossing change. There exists a Seifert matrix A_0 for L_0 and there exist

[3167] The lemma seems to be well-known and gets used implicitly and explicitly on many occasions, see [**BZH2014**, p. 154], [**Crom2004**, p. 152] and [**Lic1997a**, p. 82], [**Murak1990**, p. 285]. But there seem to be only few published reliable proofs in the literature.

$v, w \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$ such that

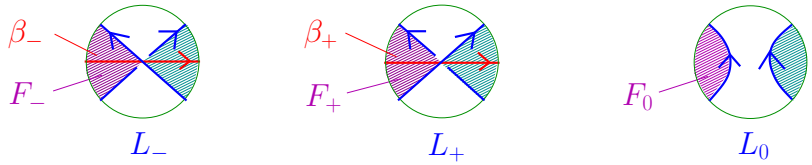
$$A_- := \begin{pmatrix} A_0 & v \\ w^{\mathrm{T}} & n+1 \end{pmatrix} \quad \text{is a Seifert matrix for } L_-,$$

$$A_+ := \begin{pmatrix} A_0 & v \\ w^{\mathrm{T}} & n \end{pmatrix} \quad \text{is a Seifert matrix for } L_+.$$



Sketch of proof. We pick connected Seifert surfaces $F_0, F_\pm$ as in the Seifert Surface Triple Lemma 293.6. Furthermore we pick a basis $a_1, \dots, a_k$ for $\mathrm{H}_1(F_0)$.

Since F_0 is connected we can find closed oriented smoothly embedded curves $\beta_+ \subseteq F_+$ and $\beta_- \subseteq F_-$ which are identical except that in the "green ball" they are as shown in the figure below.



We set $b_- := [\beta_-] \in \mathrm{H}_1(F_-)$ and $b_+ := [\beta_+] \in \mathrm{H}_1(F_+)$. An elementary Mayer-Vietoris argument shows that $a_1, \dots, a_k, b_-$ form a basis for $\mathrm{H}_1(F_-)$ and that $a_1, \dots, a_k, b_+$ form a basis for $\mathrm{H}_1(F_+)$. Next let A_0 be the Seifert matrix of F_0 corresponding to the basis $a_1, \dots, a_k$. It follows fairly easily from the definitions and the Seifert Form–Calculation Lemma 289.10 that there exist $v, w \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$ such that

$$A_- := \begin{pmatrix} A_0 & v \\ w^{\mathrm{T}} & n+1 \end{pmatrix} \quad \text{is the Seifert matrix for } F_- \text{ corresponding to } a_1, \dots, a_k, b_-,$$

$$A_+ := \begin{pmatrix} A_0 & v \\ w^{\mathrm{T}} & n \end{pmatrix} \quad \text{is the Seifert matrix for } F_+ \text{ corresponding to } a_1, \dots, a_k, b_+.$$ ■

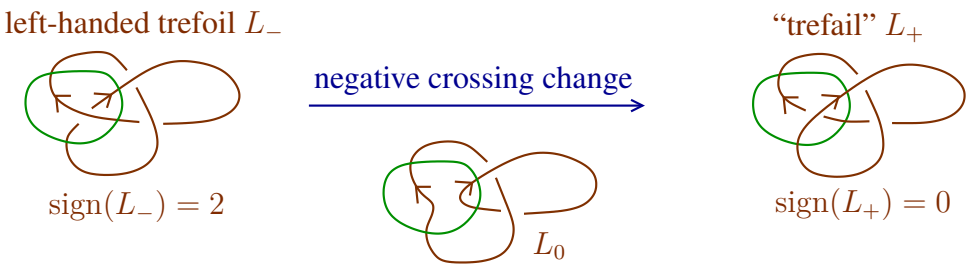
Now we are finally in a position to prove the Gordian Distance–Signature Proposition 293.4:

Proof of the Gordian Distance–Signature Proposition 293.4. First note that Statement (2) is a straightforward consequence of Statement (1). So we just need to prove Statement (1). In fact, by the triangle inequality and by symmetry we only have to prove the following:

Statement. Let L_+ and L_- be two oriented links in S^3. If L_+ is obtained from L_- by a negative crossing change, then for every $z \in S^1$ we have

$$\sigma_{L_-}(z) - \sigma_{L_+}(z) \ \in \ \{0, 1, 2\}.$$

To prove this statement we consider the link L_0 that we obtain from Definition 293.5 by resolving the crossing change.



By the Seifert Matrix Triple Lemma 293.7 there exists a $(k \times k)$-Seifert matrix A_0 for L_0 and $v, w \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$ such that

$$A_- := \begin{pmatrix} A_0 & v \\ w^{\mathrm{T}} & n+1 \end{pmatrix} \quad \text{is a Seifert matrix corresponding to } F_-,$$

$$A_+ := \begin{pmatrix} A_0 & v \\ w^{\mathrm{T}} & n \end{pmatrix} \quad \text{is a Seifert matrix corresponding to } F_+.$$

Given $z \in S^1$ and given $* \in \{+, -, 0\}$ we now write

$$P_*(z) \;=\; A_* \cdot (1 - z) + A_*^{\mathrm{T}} \cdot (1 - z^{-1}).$$

The following two claims show that $b^{\pm}(P_*(z))$ does not venture too far away from $b^{\pm}(P_0(z))$:

Claim 1. For $* \in \{-, +\}$ we have $b^{\pm}(P_*(z)) \in \{b^{\pm}(P_0(z)), b^{\pm}(P_0(z)) + 1\}$ and similarly for the nullity from Definition 292.7 we also have $\nu(P_*(z)) \in \{\nu(P_0(z)), \nu(P_0(z)) + 1\}$.

Proof. Let $* \in \{-, +\}$. We start out with one little observation: For any $v \in \mathbb{C}^k$ we have the equality

$$(\star) \qquad (\overline{v}^{\mathrm{T}} \;\; 0)\, P_*(z) \begin{pmatrix} v \\ 0 \end{pmatrix} \;=\; \overline{v}^{\mathrm{T}} P_0(z) v.$$

The claim for b^+ follows from the two inequalities that we will now prove:

- Let $V \subseteq \mathbb{C}^n$ be a vector subspace such that $\theta_{\mathbb{C}}(P_0(z))|_V$ is positive-definite. It follows almost immediately from $(\star)$ that $\theta_{\mathbb{C}}(P_*(z))|_{V \times \{0\}}$ is also positive-definite. This implies that $b^+(P_*(z)) \geq b^+(P_0(z))$.
- Let $V \subseteq \mathbb{C}^{n+1}$ be a vector subspaces such that $\theta_{\mathbb{C}}(P_*(z))|_V$ is positive-definite. It follows almost immediately from $(\star)$ that $\theta_{\mathbb{C}}(P_0(z))|_{V \cap (\mathbb{C}^n \times \{0\})}$ is also positive-definite. This implies that $b^+(P_0(z)) \geq b^+(P_*(z)) - 1$. In other words, we have the inequality $b^+(P_*(z)) \leq b^+(P_0(z)) + 1$.

Of course the same logic also applies to b^- and to the nullity ν. ⊞

Claim 2. For $* \in \{-, +\}$ we have $b^-(P_*(z)) = b^-(P_0(z))$ or $b^+(P_*(z)) = b^+(P_0(z))$.

Proof. This statement follows from Claim 1 and the fact that by the Signature-Eigenvalues Proposition 292.13 we have the equalities $b^-(P_*(z)) + b^+(P_*(z)) + \nu(P_*(z)) = k + 1$ and $b^-(P_0(z)) + b^+(P_0(z)) + \nu(P_0(z)) = k$. ⊞

Now we can finally relate the Levine-Tristram signatures of L_- and L_+:

Claim 3. For every $z \in S^1$ we have the inequality

$$|\sigma_{L_-}(z) - \sigma_{L_+}(z)| \;\leq\; 2.$$

Proof. The claim follows from the following (in-) equalities:

$$\begin{aligned} |\sigma_{L_-}(z) - \sigma_{L_+}(z)| &\overset{\text{by Definition 292.21}}{=} |\operatorname{sign}(P_-(z)) - \operatorname{sign}(P_+(z))| \\ &\leq \underbrace{|\operatorname{sign}(P_-(z)) - \operatorname{sign}(P_0(z))|}_{\in \{-1,0,1\}\text{, by Claim 1 and 2}} + \underbrace{|\operatorname{sign}(P_+(z)) - \operatorname{sign}(P_0(z))|}_{\in \{-1,0,1\}\text{, by Claim 1 and 2}} \leq 2. \end{aligned}$$

⊞

At this point it remains to prove the following claim:

Claim 4. For every $z \in S^1$ we have the inequality

$$\sigma_{L_-}(z) \geq \sigma_{L_+}(z).$$

Proof. First note that for any $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{C}^{n+1}$ we have

$$\begin{aligned} \theta_{\mathbb{C}}(P_-(z))(v,v) - \theta_{\mathbb{C}}(P_+(z))(v,v) &= \overline{v}^{\mathrm{T}} \cdot (P_-(z) - P_+(z)) \cdot v \\ &= \overline{v}^{\mathrm{T}} \cdot \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & (1-z) + (1-\overline{z}) \end{pmatrix} \cdot v \\ &= \underbrace{(2 - (z + \overline{z}))}_{\geq 0} \cdot \underbrace{\overline{v}_{n+1} \cdot v_{n+1}}_{\geq 0} \qquad \geq 0. \end{aligned}$$

Note that it follows immediately from this observation that we have the two inequalities $b^+(P_-(z)) \geq b^+(P_+(z))$ and $b^-(P_-(z)) \leq b^-(P_+(z))$. This of course gives us the desired inequality $\sigma_{L_-}(z) \geq \sigma_{L_+}(z)$. ■

293.3. **The smooth 4-genus of a knot.** In this section we will introduce the smooth 4-genus of a knot. We start out by recalling Definition 121.1 and Definition 285.1 for the 3-dimensional setting:

Definition 293.8. Let $K \subseteq S^3$ be a knot. A Seifert surface for K is a compact orientable connected 2-dimensional smooth submanifold F of S^3 such that $\partial F = K$.

By the Seifert Surface Existence Theorem 288.12 we know that any knot admits a Seifert surface. This fact lead us to Definition 290.13, which we repeat with slightly different notation:

Definition 293.9. Given a knot $K \subseteq S^3$ we define the 3-genus of K as

$$g_3(K) \coloneqq \operatorname{genus}(K) \coloneqq \text{ minimal genus of a Seifert surface } F \subseteq S^3 \text{ of } K.$$

It follows easily from the Isotopy Extension Theorem 44.2 that the 3-genus is a smooth isotopy invariant of knots.

In the following we will be interested in surfaces in the 4-ball. This leads us to the following two definitions:

Definition 293.10. Let $K \subseteq S^3$ be a knot. A smooth 4-ball surface for K is a compact orientable connected proper 2-dimensional smooth submanifold $F \subseteq \overline{B}^4$ with $\partial F = K$.

Definition 293.11. Let $K \subseteq S^3$ be a knot. We define the smooth 4-genus of K as[3168]

$$g_4^{\mathrm{sm}}(K) \coloneqq \text{ minimal genus of a smooth 4-ball surface } F \subseteq \overline{B}^4 \text{ for } K.$$

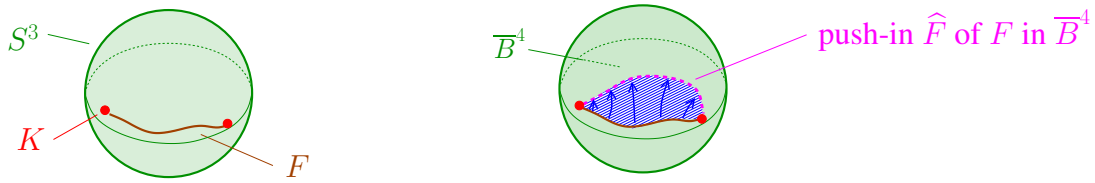
It follows again easily from the Isotopy Extension Theorem 44.2 that the smooth 4-genus is a smooth isotopy invariant of knots.

The following elementary lemma, in combination with the Seifert Surface Existence Theorem 288.12, shows in particular that the smooth 4-genus of a knot is indeed a natural number:

Lemma 293.12. (3D-4D-Genus Lemma) For any knot $K \subseteq S^3$ we have the inequality

$$g_4^{\mathrm{sm}}(K) \;\leq\; g_3(K).$$

Proof. Let $F \subseteq S^3$ be a Seifert surface for K. From Definition 35.11 we obtain a push-in $\widehat{F}$ of F into $\overline{B}^4$.



By the Push–In Lemma 35.12 we know that the push-in $\widehat{F}$ is a 4-ball surface for K that is diffeomorphic to F. It follows that $g_4^{\mathrm{sm}}(K) \leq g_3(K)$. ■

For completeness' sake we record the following lemma:

Lemma 293.13. (4-Genus-Mirror Lemma) Let $K \subseteq S^3$ be a knot. The smooth 4-genus is unchanged under taking mirrors.

Proof. Let $K \subseteq S^3$ be a knot. By Definition 108.23 the mirror K^{mir} is given by $\rho(K)$ where $\rho\colon S^3 \to S^3$ is the reflection in a hyperplane of $\mathbb{R}^4$. Evidently ρ extends to a self-diffeomorphism of $\overline{B}^4$. It is now clear that $g_4^{\mathrm{sm}}(\rho(K)) = g_4^{\mathrm{sm}}(K)$. ■

In many cases the knots we are dealing actually "sit" in $\mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. In this context the following lemma gives us a convenient reinterpretation of the smooth 4-genus:

Lemma 293.14. (Half-Euclidean 4-Genus Lemma) Let $N := (0,0,0,1) \in S^3$ be the North Pole and let $\Phi\colon S^3 \setminus \{N\} \to \mathbb{R}^3$ be the diffeomorphism from the Stereographic Projection Lemma 4.24 and which in Identifiation 4.25 we used to make the identification $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. Let $K \subseteq \mathbb{R}^3$ be a knot. Then

$$g_4^{\mathrm{sm}}(\underbrace{\Phi^{-1}(K)}_{\subseteq S^3}) \;=\; \begin{array}{c}\text{minimal genus of a compact orientable connected proper}\\ \text{2-dimensional submanifold } F \text{ of } \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \text{ with } \partial F = K \times \{0\}.\end{array}$$

Proof. Let $K \subseteq \mathbb{R}^3$ be a knot. For the purpose of this proof we write

$$g^{\mathrm{sm}}_{\mathbb{R}^3\times\mathbb{R}_{\geq 0}}(K) \;=\; \begin{array}{c}\text{minimal genus of a compact orientable connected proper}\\ \text{2-dimensional submanifold } F \text{ of } \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \text{ with } \partial F = K \times \{0\}.\end{array}$$

Similarly given a knot $J \subseteq S^3$ we write $g^{\mathrm{sm}}_{\overline{B}^4}(J)$. With this notation we want to show that for every knot $J \subseteq S^3 \setminus \{N\}$ we have $g^{\mathrm{sm}}_{\overline{B}^4}(J) = g^{\mathrm{sm}}_{\mathbb{R}^3\times\mathbb{R}_{\geq 0}}(\Phi(J))$.

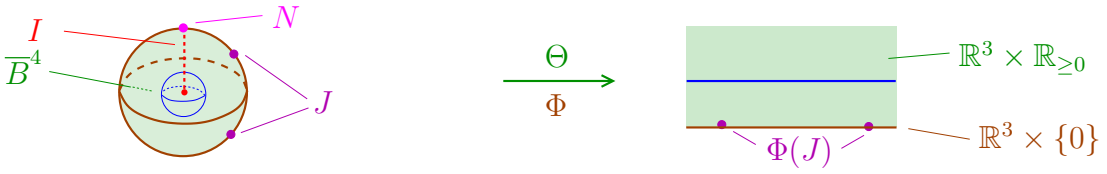
We start out with the following elementary claim:

[3168] One could also discuss *topological* submanifolds of $\overline{B}^4$ which then leads us to the topological 4-genus $g_4^{\mathrm{top}}(K)$ of K. Evidently $g_4^{\mathrm{top}}(K) \leq g_4^{\mathrm{sm}}(K)$. It turns out that there are cases where $g_4^{\mathrm{top}}(K) \leq g_4^{\mathrm{sm}}(K)$. But in these notes we just restrict ourselves to the smooth setting which is technically easier to deal with. This makes one wonder whether perhaps there is also a distinction between smooth and topological genera in dimension 3. But in **??** we will see that this is not the case.

Claim 1. We work with the following notation:

- As before let $\Phi\colon S^3 \setminus \{N\} \to \mathbb{R}^3 \times \{0\}$ be the stereographic projection from the Stereographic Projection Lemma 4.24.
- We consider the compact smooth submanifold $I := \{(0,0,0,t) \,|\, t \in [0,1]\} \subseteq \overline{B}^4$.

There exists a diffeomorphism $\Theta\colon \overline{B}^4 \setminus I \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ which restricts to the diffeomorphism $S^3 \setminus \{N\} \to \mathbb{R}^3 \times \{0\}$ that is given by $P \mapsto (\Phi(P), 0)$.



Proof. We consider the map

$$\begin{array}{rcl} \Theta\colon S^3 \setminus I & \to & \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \\ x = (x_1, x_2, x_3, x_4) & \mapsto & \left(\frac{x_1}{\|x\| - x_4}, \frac{x_2}{\|x\| - x_4}, \frac{x_3}{\|x\| - x_4}, \frac{1}{\|x\|} - 1 \right). \end{array}$$

In words this map is given by considering for each $t \in (0,1]$ the stereographic projection from $(0,0,0,t)$ that sends $S^3_t \setminus \{(0,0,0,t)\}$ to $\mathbb{R}^3$ followed by translation by $\frac{1}{t} - 1$ into the $(0,0,0,1)$-direction. We leave it to the reader to show that Θ is a diffeomorphism. It is now clear that $\Theta\colon S^3 \setminus I \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ has all the desired properties. ⊞

Note that it follows from Claim 1 that for every knot $J \subseteq S^3 \setminus \{N\}$ we have the equality $g^{\mathrm{sm}}_{\overline{B}^4 \setminus I}(J) = g^{\mathrm{sm}}_{\mathbb{R}^3 \times \mathbb{R}_{\geq 0}}(\Phi(J))$. Thus it remains to prove the following claim:

Claim 2. For every knot $J \subseteq S^3 \setminus \{N\}$ we have $g^{\mathrm{sm}}_{\overline{B}^4}(J) = g^{\mathrm{sm}}_{\overline{B}^4 \setminus I}(J)$.

Proof. Let $J \subseteq S^3 \setminus \{N\}$ be a knot. Evidently we have $g^{\mathrm{sm}}_{\overline{B}^4}(J) \leq g^{\mathrm{sm}}_{\overline{B}^4 \setminus I}(J)$. Thus it remains to prove the reverse inequality. To do so let F be a compact orientable 2-dimensional smooth submanifold F of $\overline{B}^4$ such that $\partial F = J$. As in Exercise 50.11 we first apply the Transversality Theorem 50.10 to F and the proper submanifold $\{0\} \subseteq \overline{B}^4$ to obtain a submanifold F' with $\partial F = \partial F'$ that is disjoint from $\{0\}$ and then we apply the Transversality Theorem 50.10 to $F' \subseteq \overline{B}^4 \setminus \{0\}$ and the proper submanifold $I \setminus \{0\} \subseteq \overline{B}^4 \setminus \{0\}$ to obtain a submanifold F'' with $\partial F'' = \partial F' = \partial F = J$ which is disjoint from I. Thus we found a surface in $\overline{B}^4 \setminus I$ of the same genus as before. It follows that $g^{\mathrm{sm}}_{\overline{B}^4 \setminus I}(J) \leq g^{\mathrm{sm}}_{\overline{B}^4}(J)$. ■

The case $g^{\mathrm{sm}}_4(K) = 0$ deserves a special name:

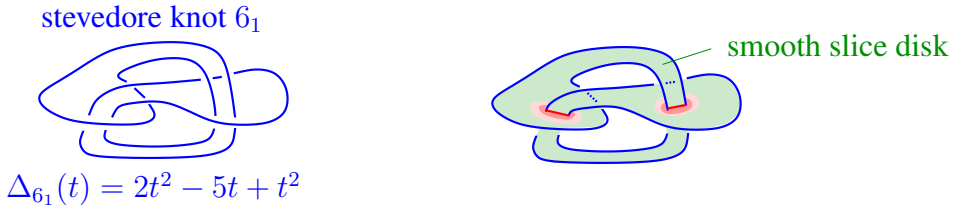
Definition 293.15. Let $K \subseteq S^3$ be a knot.

(1) A smooth slice disk for K is a proper smooth submanifold $D \subseteq \overline{B}^4$ that is diffeomorphic to $\overline{B}^2$ and which satisfies $\partial D = K$.
(2) The knot K is called smoothly slice if it bounds a smooth slice disk, in other words, K is called smoothly slice[3169] if $g^{\mathrm{sm}}_4(K) = 0$.

Example. We will now provide an example of a non-trivial knot that is smoothly slice. As in Exercises 121.13, 217.16 and 285.7, we will use "redness" to indicate the fourth

[3169] If we replace "smooth submanifold" by "topological submanifold" then we are lead to the notion of a *topological slice disk* and a *topologically slice knot*. We will not pursue the study of these notions.

dimension of a point in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$, i.e. the "redder" a point in a figure, the larger is the fourth coordinate.



In the figure above we show the stevedore knot $6_1 \subseteq \mathbb{R}^3$ which we can also view as a subspace of $\mathbb{R}^3 \times \{0\}$. We make two observations:

- A straightforward calculation shows that $\Delta_{6_1}(t) = 2t^2 - 5t + 2$. In particular 6_1 is not the trivial knot, which implies that $g_3(6_1) \geq 1$.
- In the figure we show an immersed disk $D \subseteq \mathbb{R}^3$ with various levels of "redness". If we apply the above convention we obtain a proper submanifold of $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ that is diffeomorphic to $\overline{B}^2$ and that cobounds K. It follows from the Half-Euclidean 4-Genus Lemma 293.14 that 6_1 is smoothly slice.[3170]

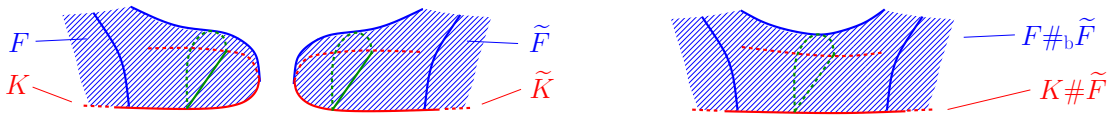
This discussion shows in particular that for $K = 6_1$ we have $g_4^{\mathrm{sm}}(K) < g_3(K)$. □

By the Genus–Connected Sum Proposition 121.12 we know that the 3-genus of oriented knots is additive under the connected sum operation. The following proposition shows that the smooth 4-genus is at least subadditive:

Proposition 293.16. (4-Genus–Subadditive Proposition) Let K and $\widetilde{K}$ be two oriented knots in S^3. We have the following inequality:

$$g_4^{\mathrm{sm}}(K\#\widetilde{K}) \;\leq\; g_4^{\mathrm{sm}}(K) + g_4^{\mathrm{sm}}(\widetilde{K}).$$

Sketch of proof. Let $F \subseteq \overline{B}^4$ be a smooth 4-ball surface for $K \subseteq S^3$ of minimal genus and let $\widetilde{F} \subseteq \overline{B}^4$ be a smooth 4-ball surface for $\widetilde{K} \subseteq S^3$ of minimal genus. Similar to the Seifert Surface–Connected Sum Lemma 121.13 we can form an internal boundary connected sum $F\#_{\mathrm{b}}\widetilde{F} \subseteq \overline{B}^4$ which cobounds $K\#\widetilde{K} \subseteq S^3$.



As we pointed out on page 2345 the genus is additive under the boundary connected sum operation. Thus we have found a smooth 4-ball surface $F\#_{\mathrm{b}}\widetilde{F} \subseteq \overline{B}^4$ for $K\#\widetilde{K} \subseteq S^3$ with genus $g_4^{\mathrm{sm}}(K) + g_4^{\mathrm{sm}}(\widetilde{K})$. This gives us the desired inequality. We leave it to the reader to fill in the details. ■

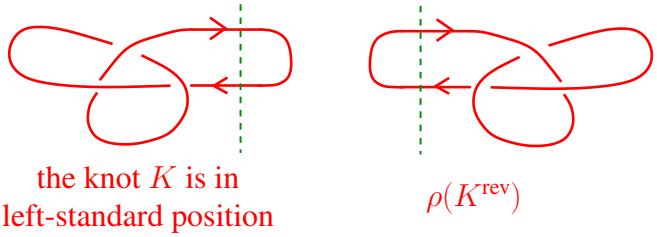
The following proposition gives in particular rise to many more examples of knots where the smooth 4-genus is strictly less than the 3-genus:

Proposition 293.17. ($(K\# - K)$-Slice Proposition) Let $K \subseteq S^3$ be an oriented knot. We set $-K := (K^{\mathrm{rev}})^{\mathrm{mir}}$.[3171] The knot $K\# - K$ is smoothly slice.

[3170] Note that this discussion solves Exercise 121.13.

Proof. Let $K \subseteq S^3$ be an oriented knot. It follows from the Knot-Standard Position Lemma 109.16 that, after a smooth isotopy, we can assume that K is left-standard in the sense of Definition 290.6.

Let $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection $(x,y,z) \mapsto (-x,y,z)$. Note that $\rho(K) = K^{\text{mir}}$. Also note that $\rho(K^{\text{rev}}) = (K^{\text{mir}})^{\text{rev}}$ is right-standard in the sense of Definition 290.6.



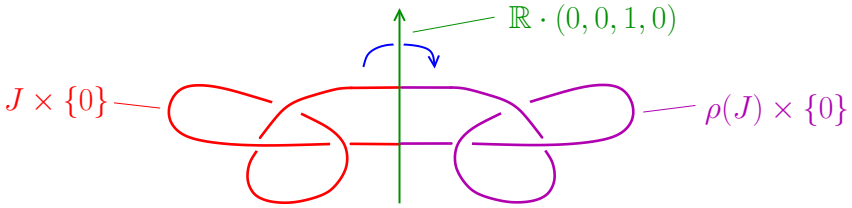
By Definition 109.12 we have the following equality of unoriented knots:

$$K\#\rho(K^{\text{rev}}) \;=\; \underbrace{(K \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2))}_{\subseteq \mathbb{S}^3_{\leq 0}} \cup \underbrace{(\rho(K) \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2))}_{\subseteq \mathbb{S}^3_{\geq 0}}.$$

It follows from this observation and the Half-Euclidean 4-Genus Lemma 293.14 that it suffices to prove the following claim:

Claim. We set $J := K \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$. There exists a proper smooth submanifold D in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ that is diffeomorphic to $\overline{B}^2$ and such that

$$\partial D \;=\; (J \cup \rho(J)) \times \{0\}.$$



Proof. The executive summary is given as follow: The desired disk is given by rotating J around the axis $\{(0,0)\} \times \mathbb{R} \times \{0\}$ by all angles in $[0,\pi]$. More precisely we consider the map

$$\begin{aligned} \theta\colon \mathbb{R}^3 \times [0,\pi] &\to \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \\ (v,t) &\mapsto (-v \cdot \cos(t), \sin(t)). \end{aligned}$$

We set $D := \theta(J \times [0,\pi])$. We claim that D has the desired properties:

- First note that using the Smooth Embedding Theorem 30.15 one can easily verify that $\theta(J \setminus (\{(0,0)\} \times \mathbb{R})) \times [0,\pi])$ is a proper smooth submanifold of the smooth manifold $(\mathbb{R}^3 \times \mathbb{R}_{\geq 0}) \setminus \mathbb{R} \cdot (0,0,1,0)$. Furthermore, since J is left-standard one can easily verify that in a neighborhood U of $\mathbb{R} \cdot (0,0,1,0) \subseteq \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ the intersection $\theta(J \times [0,\pi]) \cap U$ is a proper smooth manifold of U. It follows that $\theta(J \times [0,\pi])$ is indeed a proper smooth submanifold of $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$.

[3171]The notation $-K := (K^{\text{rev}})^{\text{mir}}$ might initially look rather weird. The rationale will be come clear once in Definition 294.3 we introduce the knot concordance group. Right now we can point out that it matches some other notation. For example let A be a Seifert matrix for K. It follows from the Seifert Matrix–Reverse Proposition 290.2 and the Seifert Matrix–Mirror Proposition 290.4 that $-A$ is a Seifert matrix for $-K$.

- It is not hard to see that $\theta(J \times [0,\pi])$ is simply connected. It thus follows from the Surface Classification Theorem 101.16 that $\theta(J \times [0,\pi])$ is diffeomorphic to the disk $\overline{B}^2$.
- Note that $\theta(J \times \{0\}) = J \times \{0\}$ and that $\theta(J \times \{\pi\}) = \rho(J) \times \{0\}$.

We have thus verified that $D := \theta(J \times [0,\pi])$ is a proper smooth submanifold of $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ that is diffeomorphic to $\overline{B}^2$ and such that $\partial D = (J \cup \rho(J)) \times \{0\}$. ■

293.4. **The 4-genus and Levine-Tristram signatures.** The only goal of this section is to formulate and discuss the following theorem which gives us lower bounds on the smooth 4-genus of a knot:

Theorem 293.18. (4-Genus–Signature Theorem) Let $K \subseteq S^3$ be an oriented knot. For any $z \in S^1$ with $\Delta_K(z) \neq 0$ we have[3172]

$$|\sigma_K(z)| \;\leq\; 2 \cdot g_4^{\mathrm{sm}}(K).$$

Recall that on page 5606 we showed that $\sigma_K(-1) = \mathrm{sign}(K)$ and recall that by the Alexander Polynomial-at-1 Proposition 289.17 we know that $\Delta_K(-1) \neq 0$. Thus the above inequality also holds for the Murasugi signature.

We will provide the proof of the 4-Genus–Signature Theorem 293.18 in the next Section 293.5. Afterwards, in Section 294.1 we will discuss Alexander polynomials of smoothly slice knots.

Example.

(1) For the trefoil we see that

$$\underset{\substack{\uparrow\\ \text{see page 5602}}}{2 \;=\; |\,\mathrm{sign}(\mathrm{trefoil})|} \;\underset{\substack{\uparrow\\ \text{by the 4-Genus Signature Theorem 293.18}}}{\leq}\; 2 \cdot g_4^{\mathrm{sm}}(\mathrm{trefoil}) \;\underset{\substack{\uparrow\\ \text{by the 3D-4D-Genus Lemma 293.12}}}{\leq}\; 2 \cdot g_3(\mathrm{trefoil}) \;\underset{\substack{\uparrow\\ \text{by elementary picture drawing}}}{\leq}\; 2 \cdot 1.$$

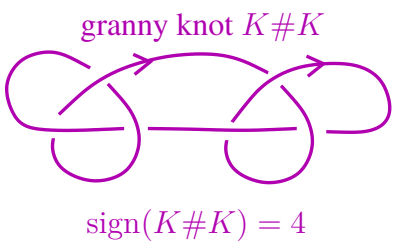
Thus we have shown that $g_4^{\mathrm{sm}}(\mathrm{trefoil}) = 1$. In particular we see that the trefoil is not smoothly slice.

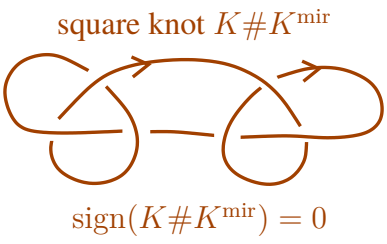
(2) On page 5603 and 5607 we showed that the Levine-Tristram signatures of the figure-8 knot are all disappointingly zero. So we do not obtain any meaningful information for the figure-8 knot. On page 5636 we will return to the question whether or not the figure-8 knot is smoothly slice. □

Next we turn to the granny knot and the square knot:

Example. Let K be the trefoil. In the figure below we show once again the granny knot $K \# K$ and the square knot $K \# K^{\mathrm{mir}}$.

[3172] Using the methods that are explained in [**FNOP2025**] one can upgrade the proof we provide from the smooth setting to the topological setting. In other words, one can show that for any $z \in S^1$ with $\Delta_K(z) \neq 0$ we have $|\sigma_K(z)| \leq 2 \cdot g_4^{\mathrm{top}}(K)$.





It turns out that in this case, in contrast to the discussion of the unknotting number on page 5613, there is only good news:

(☺) We start out with the granny knot $K\#K$. We have the following inequalities:

$$2 \underset{\uparrow}{=} \tfrac{1}{2}\cdot|\operatorname{sign}(K\#K)| \underset{\uparrow}{\leq} g_4^{\mathrm{sm}}(K\#K) \underset{\uparrow}{\leq} g_3(K\#K) \underset{\uparrow}{\leq} 2.$$

see page 5604 — by the 4-Genus–Signature Theorem 293.18 — by the 3D-4D-Genus Lemma 293.12 — by elementary picture drawing

Thus we see that all inequalities are in fact equalities. Fantastic.

(☺) We move on to the square knot $K\#K^{\mathrm{mir}}$. Recall that on page 5604 we showed that $\operatorname{sign}(K\#K^{\mathrm{mir}}) = 0$. Thus we do not get a meaningful lower bound on $g_4^{\mathrm{sm}}(K\#K^{\mathrm{mir}})$. But that is fine, since we have the following equalities:

$$g_4^{\mathrm{sm}}(K\#K^{\mathrm{mir}}) \underset{\uparrow}{=} g_4^{\mathrm{sm}}(K\#(K^{\mathrm{rev}})^{\mathrm{mir}}) = g_4^{\mathrm{sm}}(K\#-K) \underset{\uparrow}{=} 0.$$

Exercise 108.2 tells us that the trefoil K is reversible; by Definition 108.21 this means that $K^{\mathrm{rev}} = K$ — by the $(K\#-K)$-Slice Proposition 293.17 □

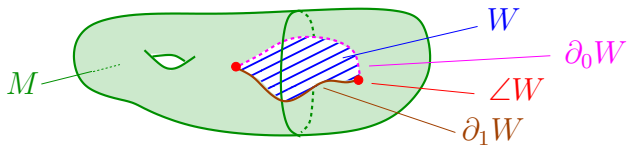
293.5. **Proof of the 4-Genus–Signature Theorem 293.18.** The proof of the 4-Genus–Signature Theorem 293.18 will require many preparations which we will spread over several subsections:

- In Section 293.5.1 we will state and prove the 4-Ball Surface Bounds Proposition 293.20.
- In Section 293.5.2 we will state and prove the Half-Live Half-Die Theorem 293.21.
- In Section 293.5.3 we will state and prove the 4D-Genus Seifert Matrix Proposition 293.22.
- In Section 293.5.4 we will state and prove the Signature Upper Bound Lemma 293.23.
- In Section 293.5.5 we will conclude the proof of the 4-Genus–Signature Theorem 293.18.

293.5.1. **3-manifolds cobounding a 4-ball surface.** First of all recall that in Definition 26.14 we introduced the notion of a submanifold with corner. Next we recall the following notation from Definition 26.15 and the Smooth Submanifold-with-Corner Lemma 26.16 (0):

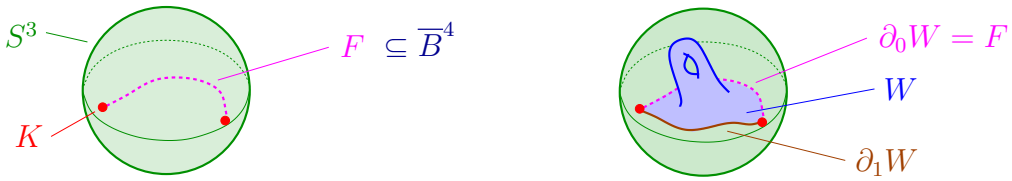
Definition 293.19. Let M be a smooth manifold and let W be a submanifold with corner of M. We write

$$\partial_0 W := \overline{\partial W \setminus \partial M} \quad \text{and} \quad \partial_1 W := \partial W \cap \partial M \quad \text{and} \quad \angle W = \partial_0 W \cap \partial_1 W.$$



Now we can formulate the result we are really interested in:

Proposition 293.20. (4-Ball Surface Bounds Proposition) Let $K \subseteq S^3$ be an oriented knot and let $F \subseteq \overline{B}^4$ be a smooth 4-ball surface for K. Then there exists a compact orientable 3-dimensional smooth submanifold with corner $W \subseteq \overline{B}^4$ such that $\partial_0 W = F$ and such that $\partial_1 W = W \cap S^3$ is a Seifert surface for K.



Proof. [3173] Let $F \subseteq \overline{B}^4$ be a smooth 4-ball surface for an oriented knot K. We start out with the following claim:

Claim 1. The inclusion induced map $\mathrm{H}_1(S^3 \setminus K) \to \mathrm{H}_1(\overline{B}^4 \setminus F)$ is an isomorphism.

Proof. By the Ball–Sphere–Alexander Duality Theorem **??** the Alexander duality isomorphisms produce a diagram of the form

$$\begin{array}{ccc} \mathrm{H}_1(S^3 \setminus K) & \xrightarrow{\quad i_* \quad} & \mathrm{H}_1(\overline{B}^4 \setminus F) \\ {\scriptstyle \mathrm{AD}_K}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{AD}_F}\downarrow{\scriptstyle \cong} \\ \mathrm{H}^1(K;\mathbb{Z}) & \xrightarrow{\quad \delta_1 \quad} & \mathrm{H}^2(F,K;\mathbb{Z}). \end{array}$$

that commutes up to a sign. Since F is connected we obtain from the discussion on page 3869 that the bottom horizontal map, which is just a connecting homomorphism, is an isomorphism. Thus the top horizontal map is also an isomorphism. ⊞

We continue with introducing a few objects:

- We consider the map
$$\begin{array}{rcl} \pi\colon (\overline{B}^2 \setminus \{0\}) \times F & \to & S^1 \\ (z,w) & \mapsto & \frac{z}{|z|}. \end{array}$$
- We denote by μ_K a meridian of K from Definition 109.1.
- It follows from Claim 1, the H^1-Hom-Proposition 201.24 and the Link Complement-H_*–Basis-Proposition 283.4 that there exists a unique $\Phi \in \mathrm{H}^1(\overline{B}^4 \setminus F;\mathbb{Z})$ with $\Phi([\mu_K]) = 1$.

Claim 2. There exists a tubular map $\tau\colon \overline{B}^2 \times F \to \overline{B}^4$ for the proper submanifold $F \subseteq \overline{B}^4$ such that the restriction $\tau|_{S^1\times F}\colon S^1 \times F \to \overline{B}^4 \setminus F$ satisfies the equality

$$(\tau|_{S^1\times F})^*(\Phi) \;=\; \pi^*([S^1]^*) \;\in\; \mathrm{H}^1(S^1 \times F;\mathbb{Z}).$$

Proof. If follows easily from the definition of π and the Künneth Theorem 164.6 that our task is equivalent to showing that there exists a tubular map $\tau\colon \overline{B}^2 \times F \to \overline{B}^4$ with the following two properties:

(a) We have $\langle (\tau|_{S^1\times F})^*(\Phi), [S^1 \times \{*\}]\rangle = 1$.

(b) The map
$$\mathrm{H}_1(F) \xrightarrow[f\mapsto(1,f)]{\text{induced by}} \mathrm{H}_1(\{1\}\times F) \xrightarrow{\tau_*} \mathrm{H}_1(\overline{B}^4 \setminus F) \xrightarrow{\sigma \mapsto \langle \Phi,\sigma\rangle} \mathbb{Z}$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{=:\varphi_\tau}$$

[3173] The proof is somewhat similar to the proof of the Seifert Surface Existence Theorem 285.2.

is the zero map.

By the Meridian Proposition 109.2 (2) we know that $\tau(S^1 \times \{*\})$ is a meridian. Thus it follows from the Kronecker Pairing–Natural Lemma 199.9 (1) that (a) is always satisfied. But it is not so clear that we can arrange (b).

To prove the existence of such a tubular map first recall that by Definition 293.10 we know that F is connected, orientable and that it has non-empty boundary. It thus follows from the Product Tubular Neighborhood–Existence Theorem 243.12 (2c) that we can find a tubular map $\sigma\colon \overline{B}^2 \times F \to \overline{B}^4$ for F.

There is no reason why we should expect that σ satisfies (b). The idea, as in the proof of the Knot Thickening–Self Linking Lemma 284.11, is to precompose σ by a suitable self-map of $\overline{B}^2 \times F$. More precisely, let $\varphi_\sigma\colon \mathrm{H}_1(F) \to \mathbb{Z}$ be the map as in (b). It follows from the Smooth Map-to-S^1-Proposition 285.3 that there exists a smooth map $\theta\colon F \to \mathrm{SO}(2)$ such that under the usual isomorphism $\mathrm{H}_1(\mathrm{SO}(2)) \to \mathbb{Z}$ we have $\theta_* = \varphi$. We now precompose σ by the diffeomorphism

$$\begin{array}{ccc} \overline{B}^2 \times F & \to & \overline{B}^2 \times F \\ (z, f) & \mapsto & (\theta(f)^{-1} \cdot z, f) \end{array}$$

to obtain a new tubular map $\tau\colon \overline{B}^2 \times F \to \overline{B}^4$. A calculation similar to the one in the proof of the Knot Thickening–Self Linking Lemma 284.11 shows that τ satisfies (b). ⊞

Claim 3. There exists a compact orientable 3-dimensional smooth submanifold with corner $W \subseteq \overline{B}^4$ such that $\partial_0 W = F$ and such that $\partial_1 W = W \cap S^3$ is a submanifold with $\partial(\partial_1 W) = K$.

Proof. First note that it follows from Claim 2 and the Extend-Tubular Projection Proposition 285.4 that there exists a smooth map $\gamma\colon \overline{B}^4 \setminus F \to S^1$ such that the following diagram commutes:

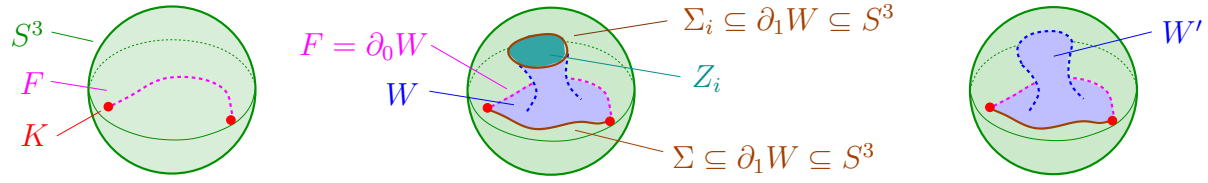
$$\begin{array}{ccc} (\overline{B}^2_{\frac{1}{2}} \setminus \{0\}) \times F & \xrightarrow{\ \tau\ } & \overline{B}^4 \setminus F \\ & {\scriptstyle (z,w) \mapsto \frac{z}{|z|}} \searrow \quad \swarrow {\scriptstyle \gamma} & \\ & S^1. & \end{array}$$

We now pick a regular value $z \in S^1$ and we consider $Z := \gamma^{-1}(z)$. It follows from the Regular Value Theorem 32.5 that Z is an orientable proper submanifold of $\overline{B}^4 \setminus F$ such that $\partial Z = \gamma^{-1}(z) \cap S^3$. Similar to the proof of the Seifert Surface Existence Theorem 285.2 we see that $W := Z \cup F$ is a compact orientable smooth submanifold with corner such that $\partial_0 W = F$ and such that $\partial(\partial_1 W) = K$. ⊞

It remains to prove the following claim:

Claim 4. There exists a compact orientable 3-dimensional smooth submanifold with corner $W \subseteq \overline{B}^4$ such that $\partial_0 W = F$ and such that $\partial_1 W = W \cap S^3$ is a Seifert surface for K.

Proof. What is the difference between Claim 3 and Claim 4 anyway? The issue is that $\partial_1 W = \partial W \cap S^3$ might have closed components. So let Σ be the component of $\partial_1 W$ with $\partial\Sigma = K$ and let $\Sigma_1, \dots, \Sigma_m$ be the other components of $\partial_1 W$.



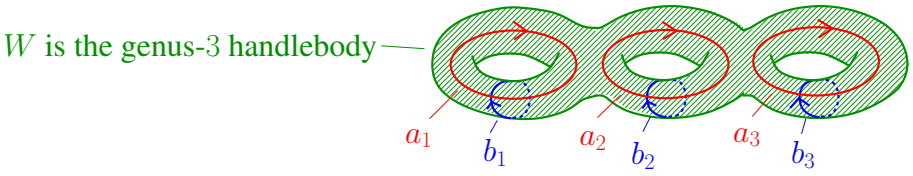
Since $\mathrm{H}_2(S^3) = 0$ and since Σ is connected we obtain from the Null Homologous Boundary Proposition 185.22 that each Σ_i bounds a unique compact codimension-zero submanifold Z_i of S^3 that is disjoint from Σ. Note that the Z_i are either disjoint or one is contained in another. Thus we can pick an innermost Z_i. We glue this to W and push it into the interior of the 4-ball. This way we can eliminate the $\Sigma_1, \ldots, \Sigma_m$ one after another to obtain a compact 3-dimensional smooth submanifold with corner $W' \subseteq \overline{B}^4$ such that $\partial_0 W' = F$ and such that $\partial_1 W' = \Sigma$. ■

293.5.2. **The Half-Live Half-Die Theorem.** The goal of this subsection is to prove the following theorem which is just a special case of the more general Half-Live Half-Die Theorem 280.22:

Theorem 293.21. (Half-Live Half-Die Theorem) Let W be a compact orientable topological manifold of dimension 3.[3174] We denote by $i\colon \partial W \to W$ the inclusion map. For any field $\mathbb{F}$ we have the following equality:

$$\dim_{\mathbb{F}}(\ker(i_*\colon \mathrm{H}_1(\partial W;\mathbb{F}) \to \mathrm{H}_1(W;\mathbb{F}))) \;=\; \tfrac{1}{2}\dim_{\mathbb{F}}(\mathrm{H}_1(\partial W;\mathbb{F})).$$

Example. We consider the 3-dimensional genus-3 handlebody W that is shown in the figure below. In the figure below we also show the usual basis $a_1, a_2, a_3, b_1, b_2, b_3$ for $\mathrm{H}_1(\partial W;\mathbb{Q})$.



In this case the kernel of the inclusion-induced map $\mathrm{H}_1(\partial W;\mathbb{Q}) \to \mathrm{H}_1(W;\mathbb{Q})$ is the three dimensional vector subspace that is generated by b_1, b_2 and b_3. □

Proof. [3175] Let W be a compact orientable 3-dimensional topological manifold. We denote by $i\colon \partial W \to W$ the inclusion map. Furthermore let $\mathbb{F}$ be any field. We consider the long exact sequence in homology of the pair $(W, \partial W)$:

$$\to \mathrm{H}_2(W;\mathbb{F}) \xrightarrow{\text{я}} \mathrm{H}_2(W,\partial W;\mathbb{F}) \xrightarrow{\partial_2} \mathrm{H}_1(\partial W;\mathbb{F}) \xrightarrow{i_*} \mathrm{H}_1(W;\mathbb{F}) \xrightarrow{\text{я}} \mathrm{H}_1(W,\partial W;\mathbb{F}) \to$$

The key observation is that this long exact sequence is "symmetric" around $\mathrm{H}_1(\partial W;\mathbb{F})$. More precisely, by the Homology Symmetry Theorem 210.6 (2) we have the following type of isomorphisms:

$$\to \overbrace{\mathrm{H}_2(W;\mathbb{F})}^{\cong \mathrm{H}_1(W,\partial W;\mathbb{F})} \to \overbrace{\mathrm{H}_2(W,\partial W;\mathbb{F})}^{\cong \mathrm{H}_1(W;\mathbb{F})} \xrightarrow{\partial_2} \mathrm{H}_1(\partial W;\mathbb{F}) \xrightarrow{i_*} \mathrm{H}_1(W;\mathbb{F}) \to \mathrm{H}_1(W,\partial W;\mathbb{F}) \to$$

Therefore it suffices to prove the following purely algebraic claim.

Claim. Let $m \in \mathbb{N}_0$. If

$$0 \to A_m \xrightarrow{\beta_m} A_0 \to \ldots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \ldots \to A_0 \xrightarrow{\alpha_m} A_m \to 0$$

[3174] By the Corner–Topological Manifold Lemma 26.5 we could take W to be a compact orientable smooth manifold with corner.

[3175] The following proof is essentially the same as the proof of the earlier Half-Live Half-Die Theorem 280.22. We just provide the proof again to keep this chapter somewhat self-contained.

is an exact sequence of finite-dimensional $\mathbb{F}$-vector spaces, then

$$\dim_{\mathbb{F}}(\ker(\gamma\colon B \to A_1)) \;=\; \tfrac{1}{2}\cdot\dim_{\mathbb{F}}(B).$$

Proof. We prove the claim by induction on $m \in \mathbb{N}_0$. The case $m = 0$ is clear, since in this case $B = 0$. The case $m = 1$ also follows easily from elementary linear algebra. Now we assume that the statement holds for some $m - 1 \in \mathbb{N}$. Next we consider a symmetric sequence

$$0 \to A_m \xrightarrow{\beta_m} A_{m-1} \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to A_{m-1} \xrightarrow{\alpha_m} A_m \to 0.$$

as above. It follows from the Break-into-SES Lemma 145.6 that the following sequence is also exact

$$0 \to A_{m-1}/\beta_m(A_m) \to \cdots \to A_2 \to A_1 \to B \xrightarrow{\gamma} A_1 \to A_2 \to \cdots \to \ker(A_{m-1} \xrightarrow{\alpha_m} A_m) \to 0.$$

But we have

$$\dim_{\mathbb{F}}(A_{m-1}/\beta_m(A_m)) \underset{\substack{\uparrow\\ \text{since } \beta_m \text{ is injective}}}{=} \dim_{\mathbb{F}}(A_{m-1}) - \dim_{\mathbb{F}}(A_m) \underset{\substack{\uparrow\\ \text{since } \alpha_m \text{ is surjective}}}{=} \dim_{\mathbb{F}}(\ker(A_{m-1} \xrightarrow{\alpha_m} A_m)).$$

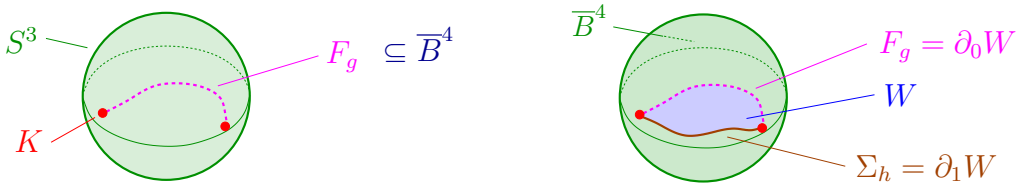
So the new shorter sequence is also symmetric. So we can apply the induction hypothesis to get the desired equality. ∎

293.5.3. **4-ball surfaces and the Seifert matrix.** Our next proposition finally relates the 4-genus to the Seifert matrix:

Proposition 293.22. (4D-Genus Seifert Matrix Proposition) Let K be an oriented knot that admits a smooth 4-ball surface of genus g. There exists a $(2h \times 2h)$-Seifert matrix for K of the form $\begin{pmatrix} 0_{(h-g)\times(h-g)} & X' \\ X'' & Y_{(h+g)\times(h+g)} \end{pmatrix}$.

Proof. Let K be an oriented knot that admits a smooth 4-ball surface F of genus g. We start by introducing a few objects:

- By the 4-Ball Surface Bounds Proposition 293.20 there exists a compact orientable 3-dimensional smooth submanifold with corner $W \subseteq \overline{B}^4$ such that $\partial_0 W = F$ and such that $\Sigma := \partial_1 W$ is a Seifert surface for K. We denote by h the genus of Σ.



- We set $V_{\mathbb{Q}} := \ker(\mathrm{H}_1(\Sigma;\mathbb{Q}) \to \mathrm{H}_1(W;\mathbb{Q}))$.
- We set $V_{\mathbb{Z}} := \ker(\mathrm{H}_1(\Sigma) \to \mathrm{H}_1(W))$.
- We set $V := \{v \in \mathrm{H}_1(\Sigma) \mid \text{ there exists an } n \in \mathbb{N} \text{ such that } n \cdot v \in V_{\mathbb{Z}}\}$.[3176]

Claim 1. The Seifert form σ_F vanishes on $V \times V$.

[3176] Recall that Exercise 280.10 indicates that there is no reason why $V_{\mathbb{Z}}$ should be a subsummand of $\mathrm{H}_1(\Sigma)$. In other words, in general we will have $V_{\mathbb{Z}} \subsetneq V$.

Proof. Since σ_F takes values in $\mathbb{Z}$ and since σ_F is a pairing we see that it suffices to show that the Seifert form σ_F vanishes on $V_{\mathbb{Z}} \times V_{\mathbb{Z}}$. Now let $v, w \in V_{\mathbb{Z}}$.

(a) It follows from the Represent-by-1-Manifold Corollary 187.19 that there exist closed oriented 1-dimensional smooth submanifolds $C \subseteq \Sigma \setminus K$ and $D \subseteq \Sigma \setminus K$ (possibly disconnected) with $[C] = v$ and $[D] = w$.

(b) It follows from the hypothesis that $[v], [w] \in \ker(\mathrm{H}_1(\partial\Sigma) \to \mathrm{H}_1(W))$ together with the Prescribed Boundary–Proposition 234.11 that there exist oriented proper 2-dimensional smooth submanifolds $F \subseteq W$ and $G \subseteq W$ with $\partial F = C$ and $\partial G = D$.[3177]

(c) Since W is orientable it follows from a mild generalization of the Product Tubular Neighborhood–Existence Theorem 58.4 (3) that there exists a smooth embedding $\tau\colon [-1,1] \times W$ with the following properties:
- For each $P \in W$ we have $\tau(0, P) = P$.
- We have $\tau^{-1}(S^3) = [-1,1] \times \partial_1 W$.

Note that the restriction of τ to $[-1,1] \times (\Sigma \setminus K) \to S^3$ is in particular a bicollar for $\Sigma \setminus K \subseteq S^3 \setminus K$.



We now see that

$$\sigma_F(v,w) \underset{\text{by a slight generalization of the Seifert Form Calculation Lemma 289.10 to links}}{=} \mathrm{lk}(C, \tau_1(D)) \underset{\text{by a slight generalization of the Linking Number-via–Surfaces Proposition 285.7 to links}}{=} F \bullet_{\overline{B}^4} \tau_1(G) \underset{\text{since } F \text{ and } \tau_1(G) \text{ are disjoint}}{=} 0.$$

⊞

Claim 2. The inclusions $\Sigma \to \partial W$ and $F \to \partial W$ induce an isomorphism

$$\underbrace{\mathrm{H}_1(\Sigma;\mathbb{Q})}_{\text{dimension } = 2h} \oplus \underbrace{\mathrm{H}_1(F;\mathbb{Q})}_{\text{dimension } = 2g} \to \mathrm{H}_1(\partial W;\mathbb{Q}).$$

Proof. Note that $\partial W = \Sigma \cup F$ with $\partial W \cap \Sigma = K = \partial W = \partial F$. The claim now follows from an elementary Mayer–Vietoris argument and the fact that ∂W is (as boundary of an orientable smooth manifold) orientable. We leave the details to the reader. ⊞

Claim 3. We have $\mathrm{rank}(V) \geq h - g$.

[3177] One could also bypass using such surfaces by replacing, in the argument below, the intuitively appealing Linking Number-via–Surfaces Proposition 285.7 by the more formal Linking #–Sphere Intersection #- Proposition 258.13.

Proof. We have the following equalities:

$$\begin{array}{ccccccc}
\operatorname{rank}(V) & \underset{\uparrow}{=} & \operatorname{rank}(V_{\mathbb{Z}}) & \overset{\downarrow}{=} & \dim_{\mathbb{Q}}(V_{\mathbb{Q}}) & \underset{\uparrow}{=} & \dim_{\mathbb{Q}}(\ker(\mathrm{H}_1(\Sigma;\mathbb{Q}) \to \mathrm{H}_1(W;\mathbb{Q})))\\
& & \text{by elementary algebra} & & & & \text{by definition of } V_{\mathbb{Q}}\\
& & & & & \underset{\uparrow}{=} & \dim_{\mathbb{Q}}(\ker(\mathrm{H}_1(\partial W;\mathbb{Q}) \to \mathrm{H}_1(W;\mathbb{Q}))) - 2g\\
& & & & & & \text{follows easily from Claim 2 and some elementary linear algebra}\\
& & & & & \underset{\uparrow}{\geq} & \tfrac{1}{2}\cdot \dim_{\mathbb{Q}}(\mathrm{H}_1(\partial W;\mathbb{Q})) - 2g\\
& & & & & & \text{follows immediately from the Half-Live Half-Die Theorem 293.21}\\
& & & & & \underset{\uparrow}{=} & \tfrac{1}{2}\cdot (2g+2h) - 2g \quad = \quad h-g.\\
& & & & & & \text{follows from Claim 2}
\end{array}$$

⊞

It remains to prove the following claim:

Claim 4. The subgroup V is a subsummand of $\mathrm{H}_1(\Sigma)$.

Proof. By definition of V we know that $\mathrm{H}_1(F)/V$ is torsion-free. It follows from the Abelian Group-Classification Theorem 97.24 that $\mathrm{H}_1(F)/V$ is in fact a free abelian group. It follows from this discussion together with the Free Abelian Group–Splitting Lemma 97.11 and the Splitting Lemma 145.5 that V is a subsummand of $\mathrm{H}_1(\Sigma)$. ■

293.5.4. **Upper bounds on signatures.** The next lemma allows us to turn the conclusion of the 4D-Genus Seifert Matrix Proposition 293.22 into a statement about signatures:

Lemma 293.23. (Signature Upper Bound Lemma) Suppose we are given a hermitian matrix $H \in \mathrm{Herm}(n\times n, \mathbb{C})$ of the form

$$H \;=\; \begin{pmatrix} 0_{k\times k} & X \\ \overline{X}^{\mathrm{T}} & Y_{(n-k)\times(n-k)} \end{pmatrix}$$

where the upper left is a 0-matrix of size $k\times k$. If $\det(H)\neq 0$, then

$$|\operatorname{sign}(H)| \;\leq\; n-2k.$$

Proof. First note that it follows from $\det(H)\neq 0$ that the matrix X has rank k. It now follows from elementary linear algebra that the matrix

$$H \;=\; \begin{pmatrix} 0_{k\times k} & X \\ \overline{X}^{\mathrm{T}} & Y_{(n-k)\times(n-k)} \end{pmatrix} \qquad \begin{array}{c}\text{is congruent to}\\ \text{a matrix of}\\ \text{the form}\end{array} \qquad \begin{pmatrix} 0_{k\times k} & X'_{k\times k} & 0 \\ \overline{X'}^{\mathrm{T}}_{k\times k} & Y'_{k\times k} & 0 \\ 0 & 0 & Y''_{(n-2k)\times(n-2k)} \end{pmatrix}$$

where $X'\in \mathrm{GL}(k,\mathbb{C})$. We then see that

$$\begin{array}{ccccccc}
\operatorname{sign}(H) & \underset{\uparrow}{=} & \operatorname{sign}\begin{pmatrix} 0_{k\times k} & X' & 0 \\ \overline{X'}^{\mathrm{T}} & Y' & 0 \\ 0 & 0 & Y'' \end{pmatrix} & \underset{\uparrow}{=} & \operatorname{sign}\begin{pmatrix} 0_{k\times k} & X' \\ \overline{X'}^{\mathrm{T}} & Y' \end{pmatrix} + \operatorname{sign}(Y'') & \underset{\uparrow}{=} & \operatorname{sign}(Y'').\\
& & \text{by the Signature Congruence Lemma 292.14} & & \text{by the Signature Additivity Proposition 292.15} & & \text{by the Vanishing Signature Lemma 292.16, here we use that } \det(X')\neq 0
\end{array}$$

Finally note that Y'' is a $(n-2k)\times(n-2k)$-matrix. It thus follows from the Signature-Eigenvalues Proposition 292.13 that $\operatorname{sign}(Y'') \leq n-2k$. ■

293.5.5. **Proof of the 4-Genus–Signature Theorem 293.18.** Finally we can provide the proof of the 4-Genus–Signature Theorem 293.18:

Proof of the 4-Genus–Signature Theorem 293.18. Let K be an oriented knot. We need to prove the following claim:

Claim. For every smooth 4-ball surface $F \subseteq \overline{B}^4$ for K of genus g and every $z \in S^1$ with $\Delta_K(z) \neq 0$ we have the inequality

$$|\sigma_K(z)| \;\leq\; 2\cdot g.$$

Proof. For $z=1$ the left hand side is zero by Definition 292.21, so in the following we can assume that $z \neq 1$. By the 4D-Genus Seifert Matrix Proposition 293.22 we know that there exists a $(2h\times 2h)$-Seifert matrix for K of the form

$$A \;=\; \begin{pmatrix} 0_{(h-g)\times(h-g)} & X' \\ X'' & Y_{(h+g)\times(h+g)} \end{pmatrix}.$$

Now we see that

$$|\sigma_K(z)| \underset{\text{by Definition 292.21}}{\overset{\uparrow}{=}} \operatorname{sign}\overbrace{(A\cdot(1-z)+A^{\mathrm{T}}\cdot(1-z^{-1}))}^{\text{by the discussion on page 5608 we know that the determinant } = (1-z)^{2h}\cdot\Delta_K(z)} \underset{\text{by the above discussion of } A}{\overset{\uparrow}{=}} \operatorname{sign}\begin{pmatrix} 0_{(h-g)\times(h-g)} & P \\ \overline{P}^{\mathrm{T}} & Q_{(h+g)\times(h+g)} \end{pmatrix}$$

$$\underset{\text{by the Signature Upper Bound Lemma 293.23 where we use that } \Delta_K(z)\neq 0 \text{ and } z\neq 1}{\overset{\uparrow}{\leq}} 2h-2(h-g) \qquad = \; 2g.$$

■

293.6. Exercises for Chapter 293.

Exercise 293.1. For $m \in \mathbb{N}$ we consider the matrix

$$A_m \;=\; \begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}.$$

(a) Show that there exists an oriented knot K_m such that A_m is a Seifert matrix for K_m.
(b) Show that $\operatorname{sign}(K_m) = 2$.
(c) Provide an example of an oriented knot J such that Murasugi signature is zero but such that a Levine-Tristram signature is non-zero.

Exercise 293.2.

(a) Compute the Levine-Tristram signatures for the knot J that is shown below.
(b) Show that for each $m \in \mathbb{N}$ there exists a knot for which one needs to perform at least m positive crossing changes *and* at least m negative crossing changes to turn it into the unknot.
Hint. Look at the Levine-Tristram signatures for different z and make use of say the trefoil and the above knot J.



Exercise 293.3. We consider the torus knot $T(5,2)$ that is shown in the figure below. What is its unknotting number?



Exercise 293.4. Given an oriented knot K with $(k \times k)$-Seifert matrix A we consider the abelian group $G(K) := \mathbb{Z}^k/(A + A^{\mathrm{T}}) \cdot \mathbb{Z}^k$. Furthermore given a finitely generated abelian group G we denote by $d(G)$ the minimal cardinality of a generating set for G.

(a) Show that the isomorphism type of $G(K)$ is an invariant of K.
(b) Show that $G(\text{trefoil}) \cong \mathbb{Z}/3\mathbb{Z}$.
(c) Show that $G(K)$ is additive under the connected sum operation and show that its isomorphism type is not changed by taking the mirror image.
(d) Let K and K' be two oriented knots that differ by a single crossing change. Show that $|d(G(K)) - d(G(K'))| \leq 1$.
Hint. This statement follows from the Seifert Matrix Triple Lemma 293.7 together with some elementary algebra.
(e) Show that for any oriented knot K we have $d(G(K)) \leq u(K)$.
(f) We consider again the square knot $K = T\#T^{\mathrm{mir}}$. Show that $u(\text{square knot}) = 2$.

Exercise 293.5. Let $F \subseteq S^3$ be a compact 2-dimensional smooth submanifold and let $\widehat{F}$ be the push-in of F into $\overline{B}^4$ that we introduced in Definition 35.11. Show that we have an isomorphism $\pi_1(\overline{B}^4 \setminus \widehat{F}) \cong \mathbb{Z}$.

Exercise 293.6. Show that the determinant of a smoothly slice knot is the square of a natural number.

Exercise 293.7. Let K be an oriented knot. We define the *stable smooth 4-genus* as

$$g_4^{\mathrm{sm,st}}(K) \;:=\; \limsup_{n\to\infty} \tfrac{1}{n} \cdot g_4^{\mathrm{sm}}(n \cdot K),$$

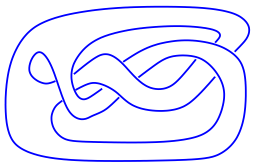
where $n \cdot K$ stands for the connected sum of n copies of K.

(a) Show that the above is actually a limit.
Remark. This will require some mild input from analysis.
(b) Give an example of a non-trivial knot with non-zero stable smooth 4-genus.
(c) Let K be a knot that is not smoothly slice. Is the stable smooth 4-genus necessarily non-zero?

Exercise 293.8. Show that every knot $K \subseteq \mathbb{R}^3 \times \{0\}$ bounds a smoothly embedded disk in $\mathbb{R}^4$.
Hint. By the Link Diagram Existence Proposition 112.7 we can assume that K is given by a diagram. Furthermore, by Exercise 113.12 we know that we can turn K into the unknot using crossing changes.

Exercise 293.9. We consider the two knots 6_1 and 9_{46} that are shown below.

(a) Show that in both cases the Alexander polynomial equals $2t^2 - 5t + 2$.
(b) Show that the Levine-Tristram signature function of 6_1 is zero except for two points.
(c) Show that the Levine-Tristram signature function of 9_{46} is identically zero.

6_1  9_{46} 

Exercise 293.10. In the following we identify S^3 with the equatorial sphere $S^4_{=0} \subseteq S^4$. A knot $K \subseteq S^3$ is called *doubly smoothly slice* if there exists a smooth embedding $\varphi\colon \overline{B}^3 \to S^4$ with the following properties:

- The map φ is transverse to $S^4_{=0}$.
- We have $\varphi(S^2) \cap S^4_{=0} = \varphi(S^2_{=0}) = K$.
- We have $\varphi(S^2_{\geq 0}) \subseteq S^4_{\geq 0}$.
- We have $\varphi(S^2_{\leq 0}) \subseteq S^4_{\leq 0}$.

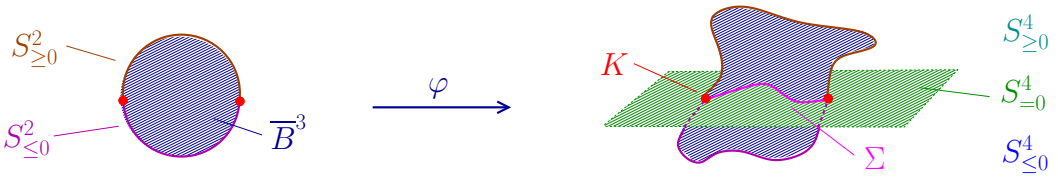
Note that K bounds the disks $\varphi(S^2_{\geq 0}) \subseteq S^4_{\geq 0}$ and $\varphi(S^2_{\leq 0}) \subseteq S^4_{\leq 0}$. We set $\Sigma := \varphi(\overline{B}^3) \cap S^3_{=0}$. Furthermore we set $W_{\geq 0} := \varphi(\overline{B}^3) \cap S^4_{\geq 0}$ and we set $W_{\leq 0} := \varphi(\overline{B}^3) \cap S^4_{\leq 0}$.

(a) We set $H_{\geq 0} := \ker(\mathrm{H}_1(\Sigma) \to \mathrm{H}_1(W_{\geq 0}))$ and we set $H_{\leq 0} := \ker(\mathrm{H}_1(\Sigma) \to \mathrm{H}_1(W_{\leq 0}))$. Show that $H_{\leq 0} \cap H_{\geq 0} = 0$ and that $H_{\leq 0} \oplus H_{\geq 0} = \mathrm{H}_1(\Sigma)$.
Remark. This is easier than it looks, you just need that $W_{\leq 0} \cap W_{\geq 0} = \Sigma$ and that $W_{\leq 0} \cup W_{\geq 0} = \overline{B}^3$.
(b) Show that K admits a $(2g \times 2g)$-Seifert matrix that is of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

where each block is a $(g \times g)$-matrix.
Hint. Use (a) and results from this chapter.
(c) Show that for all $z \in S^1$ we have $\sigma_L(z) = 0$.

Remark. This exercise together with Exercise 293.9 shows that the knot 6_1 is not smoothly slice. Well-trained eyes will spot that on the other hand the knot 9_{46} is in fact smoothly doubly slice.



Exercise 293.11. Let $K \subseteq S^3$ and let $\tau\colon \overline{B}^2 \times K \to S^3$ be a tubular map of self-linking number zero. We consider $X_0(K) := \overline{B}^4 \cup_\tau h^2$. Show that the following two statements are equivalent:

(i) The knot K is smoothly slice.
(ii) The smooth manifold $X_0(K)$ admits a smooth embedding into S^4.

Hint. Make use of the Smooth Ball Embedding Theorem 45.10, the Knot Thickening–Self Linking Lemma 284.11 and the Product Tubular Neighborhood–Existence Theorem 243.12.

CHAPTER 294

The smooth knot concordance group

First recall that in the Oriented Knot–Monoid Proposition 109.13 we introduced the abelian monoid $(\texttt{Knot}^{\text{or}}(S^3), \#)$ that is given by the set of smooth isotopy classes of oriented knots in S^3 together with the connected sum operation. In this chapter we will use the submonoid of smoothly slice knots to introduce a quotient of the abelian monoid $(\texttt{Knot}^{\text{or}}(S^3), \#)$, namely the smooth knot concordance group. We will use Levine-Tristram signatures and Alexander polynomials to show that this group is highly non-trivial.

This chapter is organized as follows:

- In Section 294.1 we will see that the Alexander polynomial can be used to show that certain knots are not smoothly slice.
- In Section 294.2 we will introduce the knot concordance group.
- In Section 294.3 we will use the Levine-Tristram signatures to show that the knot concordance group has in fact infinite rank.
- In Section 294.4 we will use the Alexander polynomial to obtain more results about the knot concordance group.
- In Section 294.5 we will provide exercises for Chapter 294.

In the next Chapter 69 we will use Seifert matrices to introduce the Conway polynomial of any oriented link.

294.1. **Alexander polynomials of smoothly slice knots.** For the reader's convenience we recall the following definition:

Definition 293.15. Let $K \subseteq S^3$ be a knot.

(1) A smooth slice disk for K is a proper smooth submanifold $D \subseteq \overline{B}^4$ that is diffeomorphic to $\overline{B}^2$ and which satisfies $\partial D = K$.

(2) The knot K is called smoothly slice if it bounds a smooth slice disk, in other words, K is called smoothly slice if $g_4^{\text{sm}}(K) = 0$.

By the 4-Genus–Signature Theorem 293.18 we know that Levine-Tristram signatures can be used to show that certain knots (e.g. the trefoils) are not smoothly slice. A key ingredient in the proof of the 4-Genus–Signature Theorem 293.18 was the 4D-Genus Seifert Matrix Proposition 293.22 which gives us some information about the Seifert matrix of a knot in terms of its smooth 4-genus.

This fact can also be used to prove the following theorem, which was first formulated and proved by Ralph Fox and John Milnor [**FMi1966**, p. 262]:

Theorem 294.1. (Fox-Milnor Theorem) Let $K \subseteq S^3$ be a knot. If K is smoothly slice, then there exists an $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that

$$\Delta_K(t) \doteq f(t) \cdot f(t^{-1}).$$

Example. We consider again the trefoil and the figure-8 knot, but this time in reverse order:

(2) Let K be the figure-8 knot. On page 5530 we showed that $\Delta_K(t) \doteq t^2 - 3t + 1$. It is elementary and straightforward to verify that this polynomial does *not* admit a factorization $t^2 - 3t + 1 = \pm t^k \cdot f(t) \cdot f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$. It thus follows from the Fox-Milnor Theorem 294.1 that the figure-8 knot is not smoothly slice. Using the 3D-4D-Genus Lemma 293.12 and remembering that on page 5528 we saw a genus-1 Seifert surface for the figure-8 knot we can now conclude that $g_4^{\mathrm{sm}}(\text{figure-8 knot}) = 1$.

(1) Let T be the trefoil. On page 5530 we showed that $\Delta_T(t) \doteq t^2 - t + 1$. Again this polynomial does not factor in the above way and we conclude that the trefoil is not smoothly slice and as in (1) we see that $g_4^{\mathrm{sm}}(\text{trefoil}) = 1$. This recovers the result from page 5623. □

Proof. Let $K \subseteq S^3$ be a knot that is smoothly slice. It follows from the 4D-Genus Seifert Matrix Proposition 293.22 that there exists a $(2h \times 2h)$-Seifert matrix for K of the form

$$A = \begin{pmatrix} 0_{h\times h} & X_{h\times h} \\ Y_{h\times h} & Z_{h\times h} \end{pmatrix}.$$

It follows that

$$\Delta_K(t) \underset{\uparrow}{\doteq} \det(t\cdot A - A^{\mathrm{T}}) \underset{\uparrow}{=} \begin{pmatrix} 0 & t\cdot X - Y^{\mathrm{T}} \\ t\cdot Y - X^{\mathrm{T}} & t\cdot Z - Z^{\mathrm{T}} \end{pmatrix}$$

by the Alexander Module Seifert Matrix Theorem 289.12; by the above

$$\underset{\uparrow}{\doteq} \det(t\cdot X - Y^{\mathrm{T}}) \cdot \det(t\cdot Y - X^{\mathrm{T}})$$

since X, Y, Z are square matrices of the same size

$$\underset{\uparrow}{\doteq} \det(t\cdot X - Y^{\mathrm{T}}) \cdot \det(t\cdot Y^{\mathrm{T}} - X) \underset{\uparrow}{\doteq} (-t)^h \cdot f(t) \cdot f(t^{-1}).$$

follows from $\det(W^{\mathrm{T}}) = \det(W)$; we set $f(t) := \det(t\cdot X - Y^{\mathrm{T}})$ ■

Remark. It seems like the Alexander polynomial does not give rise to any meaningful lower bounds on the smooth 4-genus that go beyond the Fox-Milnor Theorem 294.1. For example the knot 6_2 is known to have $g_4^{\mathrm{sm}}(K) = 1$ (see [**LiM**]) but the Alexander polynomial $\Delta_{6_2}(t) = 1 - 3t + 3t^2 - 3t^3 + t^4$ is irreducible. So a small smooth 4-genus does not lead to a factorization of the Alexander polynomial. In fact, much more dramatically, in [**Kon1979**, p. 558] it is shown that if K is a knot, then there exists another knot with the same Alexander polynomial, but with unknotting number and 4-genus equal to ≤ 1. □

294.2. **The definition of the knot concordance group.** As we mentioned in the beginning of the chapter, we will define the smooth knot concordance group as a suitable quotient of the abelian monoid $(\mathtt{Knot}^{\mathrm{or}}(S^3), \#)$. To define such a quotient monoid we will make use of the following completely formal concept:

Lemma 294.2. (Quotient Monoid Lemma) Let $(M,+)$ be an abelian monoid and let $S \subseteq M$ be a submonoid. Given $m, m' \in M$ we write

$$m \sim_S m' \quad :\Longleftrightarrow \quad \text{there exist } s, s' \in S \text{ with } m + s = m' + s'.$$

The following statements hold:

(1) The above relation $\sim_S$ is an equivalence relation on M.

(2) The set $M/S := M/\sim_S$ of equivalence classes inherits the structure of an abelian monoid. It is called the quotient monoid.

(3) If for every $m \in M$ there exists an $m' \in M$ such that $m + m' \in S$, then M/S is an abelian group.

(4) Given any $m \in M$ we see that

$$[m] = 0 \in M/S \qquad \Longleftrightarrow \qquad \text{there exists an } s \in S \text{ with } m + s \in S.$$

(5) Let G be an abelian monoid and let $\varphi\colon M \to G$ be a monoid homomorphism. If the restriction of φ to S is the trivial homomorphism, then φ descends to a well-defined monoid homomorphism

$$\begin{array}{rcl} M/\sim_S & \to & G \\ [m] & \mapsto & [\varphi(m)]. \end{array}$$

Proof. All statements follow easily from the definitions, the fact that M is abelian and the fact that S is a submonoid. We will not bore the reader by providing more details. In fact the reader will surely already have provided the details in Exercise 2.2. ■

We will now apply this very general scheme to knot theory:

Definition 294.3. We consider the following objects:

(1) As in the Oriented Knot–Monoid Proposition 109.13 we consider again the abelian monoid $(\mathtt{Knot}^{\mathrm{or}}(S^3), \#)$ of smooth isotopy classes of oriented knots in S^3 under the connected sum operation.

(2) Let $\mathcal{S}^{\mathrm{sm}}$ be the set of smooth isotopy classes of smoothly slice knots. It follows almost immediately from the 4-Genus–Subadditive Proposition 293.16 that $\mathcal{S}^{\mathrm{sm}}$ is a submonoid of $(\mathtt{Knot}^{\mathrm{or}}(S^3), \#)$.

(3) Given an oriented knot $K \subseteq S^3$ we set[3178]

$$-K := (K^{\mathrm{rev}})^{\mathrm{mir}}.$$

(4) It follows easily from the $(K\#-K)$-Slice Proposition 293.17 and the Quotient Monoid Lemma 294.2 that

$$\mathcal{C}^{\mathrm{sm}} := \mathtt{Knot}^{\mathrm{or}}(S^3)/\sim_{\mathcal{S}^{\mathrm{sm}}}$$

is an abelian group where for any given oriented knot $K \subseteq S^3$ we have $-[K] = [-K]$. The group $\mathcal{C}^{\mathrm{sm}} = \mathtt{Knot}^{\mathrm{or}}(S^3)/\sim_{\mathcal{S}^{\mathrm{sm}}}$ is called the smooth knot concordance group.[3179]

[3178] Given an oriented smooth manifold M we introduced in Definition 31.22 the smooth manifold $-M$ that is given by reversing the orientation of M. This notation seems inconsistent with the above definition of $-K$. The reason why they are consistent after all lies in the observation that we really should consider the pair (S^3, K). If we reverse the orientation of S^3 and K, then we end up with $(-S^3, -K)$ which is orientation-preservingly diffeomorphic as a pair to $(S^3, (K^{\mathrm{rev}})^{\mathrm{mir}})$.

[3179] If instead of smoothly slice knots one considers the monoid of topologically slice knots, then one ends up with the topological knot concordance group $\mathcal{C}^{\mathrm{sm}}$. We will not study this object.

The question that naturally arises is the following: Which oriented knots represent the trivial element in the smooth knot concordance group? The following, slightly non-trivial proposition, gives us the complete answer to this question:

Proposition 294.4. (Trivial-in-Smooth Concordance Group Proposition) In the following let $K \subseteq S^3$ be an oriented knot. Then

$$[K] = 0 \quad \in \mathcal{C}^{\mathrm{sm}} \qquad \Longleftrightarrow \qquad K \text{ is smoothly slice.}$$

The proof of the Trivial-in-Smooth Concordance Group Proposition 294.4 will require the remainder of this section. In the next Section 294.3 we will use Levine-Tristram signatures to show that the knot concordance group is rather non-trivial.

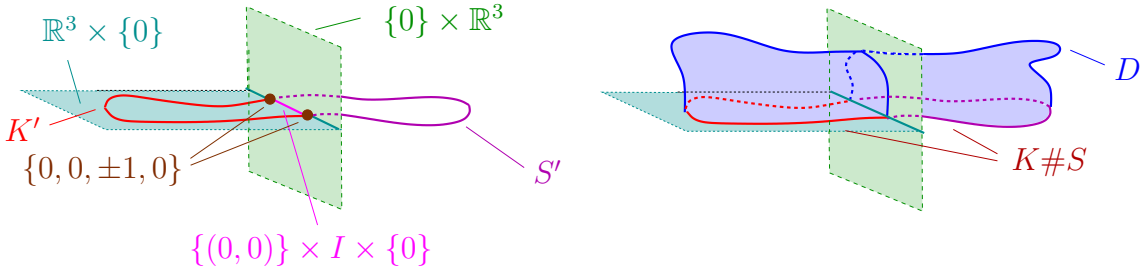
Sketch of proof. The "$\Leftarrow$"-direction follows immediately from the Quotient Monoid Lemma 294.2 (4). The "$\Rightarrow$"-direction follows again from the Quotient Monoid Lemma 294.2 (4), but this time we will also need the following claim:

Claim. Let $K \subseteq S^3$ be an oriented knot. If there exists an oriented smoothly slice knot S such that $K\#S$ is smoothly slice, then K itself is already smoothly slice.

Proof. We will use the Link-in-$\overline{B}^3$-$\mathbb{R}^3$-S^3-Proposition 108.15 together with the Half-Euclidean 4-Genus Lemma 293.14 to work with knots and disks in $\mathbb{R}^3$ and $\mathbb{R}^3_{\geq 0}$. This makes some of the notation a little bit easier.

So let $K \subseteq \mathbb{R}^3$ be an oriented knot and let $S \subseteq \mathbb{R}^3$ be an oriented smoothly slice knot such that $K\#S \subseteq \mathbb{R}^3$ is smoothly slice. We need to show that $K \times \{0\}$ bounds a properly smoothly embedded disk in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$. We start out with a few preparations.

- By the Knot-Standard Position Lemma 109.16 we can arrange that K is in left-standard position in the sense of Definition 290.6 and that S is in right-standard position.
- We set $K' := K \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$ and $S' := S \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$.
- We form the connected sum $K\#S = K' \cup S'$ as in Definition 109.12.
- We set $I := [-1, 1]$.
- By hypothesis we know that $K\#S$ is smoothly slice. It follows from the Half-Euclidean 4-Genus Lemma 293.14 and the Collar-for-Submanifold Proposition 56.3 that there exists a proper smooth submanifold $D \subseteq \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ that is diffeomorphic to the disk $\overline{B}^2$ and such that $D \cap (\mathbb{R}^3 \times [0, 1]) = (K\#S) \times [0, 1]$.



Subclaim. There exist a 2-dimensional smooth submanifold $E' \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \times \mathbb{R}_{\leq 0}$ and a 2-dimensional smooth submanifold $E'' \subseteq \{0\} \times \mathbb{R}^2 \times \mathbb{R}_{\leq 0}$ with the following properties:

- Both are homeomorphic to a disk.
- $\partial E' \cap \partial E''$ is diffeomorphic to an interval.

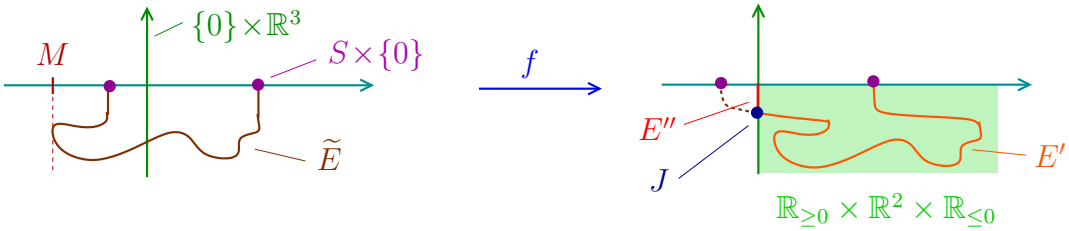
- $\partial E' = (E' \cap E'') \cup S'$.
- $\partial E'' = (E' \cap E'') \cup (\{(0,0)\} \times I \times \{0\})$.

Proof. Since S is smoothly slice we obtain, similar to the above discussion, a proper smooth submanifold $\widetilde{E} \subseteq \mathbb{R}^3 \times \mathbb{R}_{\leq 0}$ that is diffeomorphic to the disk and which has the property that $\widetilde{E} \cap (\mathbb{R}^3 \times [-2,0]) = S \times [-2,0]$. Using the Smooth Transition Function Lemma 24.9 we can find a smooth function $\eta\colon \mathbb{R}_{\leq 0} \to \mathbb{R}$ with the following properties:

- The function $\eta\colon \mathbb{R}_{\leq 0} \to \mathbb{R}$ is strictly monotonously increasing.
- We have $\eta(t) = 0$ for $t \in [-1,0]$.
- For $t \geq 2$ we have the inequality $\eta(t) > -M$, where M is the minimal x_1-coordinate of any point in $\widetilde{E}$.

We consider the diffeomorphism

$$\begin{array}{rcl} f\colon \mathbb{R}^3 \times \mathbb{R}_{\leq 0} & \to & \mathbb{R}^3 \times \mathbb{R}_{\leq 0} \\ (P,t) & \mapsto & (P, t+\eta(t)). \end{array}$$



We now set $E' := \Theta(\widetilde{E}) \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2 \times \mathbb{R}_{\leq 0})$. Using the fact that S is right-standard one can verify that E' is homeomorphic to a disk, that $\partial E' \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2 \times \{0\}) = S \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^2) =: S'$ and that $J := E' \cap (\{0\} \times \mathbb{R}^2 \times \mathbb{R}_{\leq 0})$ is a smoothly embedded interval. Furthermore, using the fact that S is right-standard one sees that there exists a smoothly embedded half-disk $E'' \subseteq \{0\} \times \mathbb{R}^2 \times \mathbb{R}$ such that $\partial E'' = J \cup (\{(0,0)\} \times [-1,1] \times \{0\})$. One can now easily verify that E' and E'' have the desired properties. ⊞

To construct our smooth slice disk for K we still have to make a few more preparations:

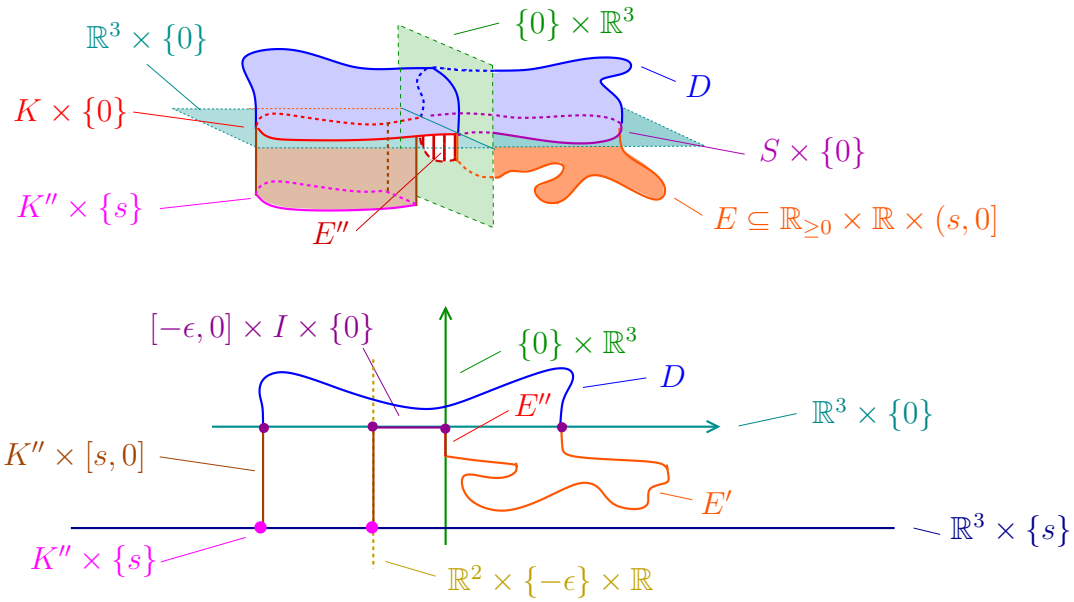
- Note that it follows in particular from Definition 290.6 that there exists an $\epsilon > 0$ such that $K \cap ((-\infty, \epsilon] \times \mathbb{R}^2) = [-\epsilon, 0] \times \{0\} \times \{\pm 1\}$.
- Since E is compact we can find an $s \in \mathbb{R}$ such that $E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \times (s, 0]$.
- We consider $K'' := (K \cap (-\infty, -\epsilon] \times \mathbb{R}^2) \cup \{(-\epsilon, 0)\} \times I \subseteq \mathbb{R}^3$. After rounding corners we can think of K'' as a smooth knot in $\mathbb{R}^3$. Using the fact that K is left-standard one can easily verify that K and K'' are smoothly isotopic.

Finally we consider

$$\Sigma \;:=\; (K'' \times [s,0]) \cup ([-\epsilon, 0] \times \{0\} \times I) \cup D \cup E' \cup E''$$

Unless the author managed to confuse himself this should now be a disk in $\mathbb{R}^3 \times [s, \infty)$ with $\Sigma \cap (\mathbb{R}^3 \times \{s\}) = K'' \times \{s\}$.
After smoothing the various corners of Σ, and pushing it upward by s, we obtain a smooth slice disk for K''. This shows that K'', and thus also K, is smoothly slice. ■

294.3. **The knot concordance group and Levine-Tristram signatures.** We just managed to turn oriented knots into an abelian group. Evidently we now want to get to know



this group better. The question on everybody's tip of the tongue is the following: Is this group non-trivial? If yes, how large is it? In this section we will give some answers to these questions.

294.3.1. **Levine-Tristram homomorphisms.** The following theorem is the key to showing that the knot concordance group is in fact infinitely generated:

Theorem 294.5. (Signature–Knot Concordance–Homomorphism Theorem) Let $\xi \in S^1$. We suppose that one of the following two conditions is satisfied:

(a) The complex number ξ is transcendental, or

(b) the complex number ξ is a prime power root of unity.[3180]

Then the map

$$\begin{array}{rccc} \widehat{\sigma}_\xi \colon & \mathcal{C}^{\mathrm{sm}} & \to & \mathbb{Z} \\ & [K] & \mapsto & \frac{1}{2} \cdot \sigma_K(\xi) \end{array}$$

is a well-defined homomorphism.

Example. Note that $\xi = -1$ is a prime power root of unity. It thus follows from the Signature–Knot Concordance–Homomorphism Theorem 294.5 that the map

$$\begin{array}{rccc} \widehat{\sigma}_\xi \colon & \mathcal{C}^{\mathrm{sm}} & \to & \mathbb{Z} \\ & [K] & \mapsto & \frac{1}{2} \cdot \mathrm{sign}(K) \end{array}$$

is a well-defined homomorphism. On page 5602 we showed that the signature of the left-handed trefoil T equals 2. It follows from the above that the left-handed trefoil spans a $\mathbb{Z}$-summand in the knot concordance group. □

Proof. Let $\xi \in S^1$. First note that it follows from the Levine-Tristram analog of the Signature-Additivity Proposition 292.20 that the map

$$\begin{array}{rccc} \widehat{\sigma}_\xi \colon & \mathtt{Knot}^{\mathrm{or}}(S^3) & \to & \mathbb{Z} \\ & [K] & \mapsto & \frac{1}{2} \cdot \sigma_K(\xi) \end{array}$$

[3180] We say $\xi \in S^1$ is a *prime power root of unity* if there exists a prime p and $m \in \mathbb{N}$ such that $\xi^{p^m} = 1$.

is a well-defined homomorphism. We need to verify the following two statements:

- The map σ_ξ takes values in $\mathbb{Z}$.
- The map σ_ξ descends to a map $\mathcal{C}^{\text{sm}} \to \mathbb{Z}$.

It follows from the Signature-Eigenvalues Proposition 292.13, Definition 294.3 and the Quotient Monoid Lemma 294.2 (5) that both issues are taken care of by the following claim:

Claim. Let $\xi \in S^1$. We suppose that one of the following two conditions is satisfied:
(a) The complex number ξ is transcendental, or
(b) the complex number ξ is a prime power root of unity.
Then for every smoothly slice knot K we have $\widehat{\sigma}_\xi(K) = 0$.

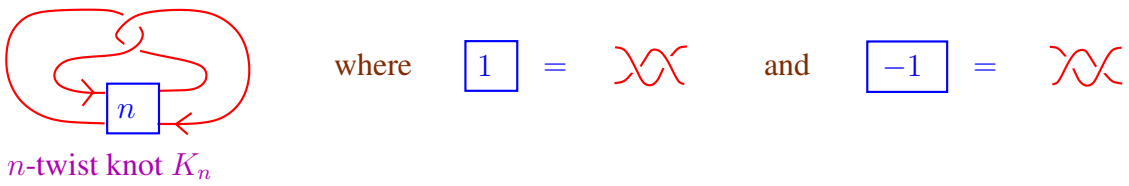
Proof. Let $\xi \in S^1$ and let $K \subseteq S^3$ be a smoothly slice knot. If $\xi = 1$, then it follows from the discussion on page 5606 that $\widehat{\sigma}_\xi(K) = \frac{1}{2}\sigma_K(\xi) = 0$. So we can now suppose that $\xi \neq 1$. By the 4-Genus–Signature Theorem 293.18 we know that $\widehat{\sigma}_\xi(K) = \frac{1}{2}\sigma_K(\xi) = 0$ *as long as* $\Delta_K(\xi) \neq 0$. Thus we need to verify that in our two setups we have $\Delta_K(\xi) \neq 0$.

(a) If ξ is transcendental, then it follows from the fact that $\Delta_K(\xi) \in \mathbb{Z}[t^{\pm 1}]$ has integral coefficients that $\Delta_K(\xi) \neq 0$.
(b) Now we suppose that $\xi \neq 1$ is a prime power root of unity. Recall that this means that there exists a prime p and a $d \in \mathbb{N}$ such that $\xi^{p^d} = 1$. Let $m_\xi(t) \in \mathbb{Z}[t^{\pm 1}]$ be the minimal polynomial of ξ. If we had $\Delta_K(\xi) = 0$, then, by definition of a minimal polynomial we would get that $m_\xi(t) \,|\, \Delta_K(t)$ in $\mathbb{Z}[t^{\pm 1}]$. In particular we would see that $m_\xi(1) \,|\, \Delta_K(1)$ in $\mathbb{Z}$. But the following two observations show that this is not the case:
  - Elementary algebra, see e.g. [**Lang2002b**, Chapter VI.3], tells us that $m_\xi(1) = p$.
  - By the Alexander Polynomial-at-1 Proposition 289.17 we know that $\Delta_K(1) = \pm 1$.

■

294.3.2. **Examples.** In this subsection we will see how much mileage we can squeeze out of the Signature–Knot Concordance–Homomorphism Theorem 294.5. To do so it it convenient to introduce the following family of knots:

Definition 294.6. Given $n \in \mathbb{Z}$ we consider the negative n-twist knot K_n that is shown in the figure below.



n-twist knot K_n

The following proposition shows in particular that the smooth knot concordance group has infinite rank.

Proposition 294.7. (Twist Knot–Concordance Group Proposition) Given $n \in \mathbb{N}$ we consider the negative n-twist knot K_n from Definition 294.6. These negative twist knots are linearly independent in the knot concordance group and they span a $\mathbb{Z}^\infty$-summand.

[3180]Note that for small values the negative n-twist knots are familiar knots. More precisely, for $n = 0$ we get the unknot, for $n = 1$ we get the left-handed trefoil and for $n = -1$ we get the figure-8 knot.

Proof. Given $n \in \mathbb{Z}$ we consider the negative n-twist knot K_n that is shown in the figure above. Similar to the discussion of the figure-8 knot on page 5528 we see that a Seifert matrix for K_n is given by

$$A_n \;=\; \begin{pmatrix} 1 & -1 \\ 0 & n \end{pmatrix}.$$

We use this Seifert matrix to calculate the Murasugi signature and the Alexander polynomial of each K_n:

$$\operatorname{sign}(K_n) \;\overset{\text{by Definition 292.18}}{=}\; \operatorname{sign}(A_n + A_n^{\mathrm{T}}) \;=\; \operatorname{sign}\begin{pmatrix} 2 & -1 \\ -1 & 2n \end{pmatrix} \;=\; \begin{cases} 2, & \text{if } n > 0, \\ 0, & \text{if } n \leq 0, \end{cases}$$

$$\Delta_{K_n}(t) \;=\; \det(t \cdot A_n - A_n^{\mathrm{T}}) \;=\; \det\begin{pmatrix} t-1 & -t \\ 1 & n \cdot (t-1) \end{pmatrix} \;=\; n \cdot (t-1)^2 + t$$

$$\text{by the Alexander Module–Seifert Matrix Theorem 289.12} \uparrow$$

Note that for $n > 0$ the signature is negative and that there exists a strictly monotonously increasing sequence $(r_n)_{n \in \mathbb{N}}$ of real numbers in $(0, \frac{1}{2})$ such that $\exp(\pm 2\pi \mathrm{i} r_n) \in S^1$ are precisely the two zeros of $\Delta_{K_n}(t)$.



For each $n \in \mathbb{N}$ we pick a transcendental $\xi_i \in S^1$ "between" $\exp(2\pi \mathrm{i} r_n)$ and $\exp(2\pi \mathrm{i} r_{n+1})$.

Claim. The map

$$\begin{aligned} \Xi\colon \mathcal{C}^{\mathrm{sm}} &\to \mathbb{Z}^\infty \\ [K] &\mapsto (\tfrac{1}{2} \cdot \sigma_K(\xi_1), \tfrac{1}{2} \cdot (\sigma_K(\xi_2) - \sigma_K(\xi_1)), \tfrac{1}{2} \cdot (\sigma_K(\xi_3) - \sigma_K(\xi_2)), \dots) \end{aligned}$$

is a well-defined group homomorphism.

Proof. First note that it follows from the Signature–Knot Concordance–Homomorphism Theorem 294.5 that the map is well-defined as a homomorphism with values in $\mathbb{Z}^{\mathbb{N}}$. But using the fact that the $r_i \in (0,1)$ are strictly monotonously increasing, using the Levine-Tristram Signature Continuity Proposition 292.22 and using the fact that each polynomial has only finitely many zeros one sees quite easily that the map takes values in $\mathbb{Z}^\infty$. ⊞

Next note that it follows quite easily from the Levine-Tristram Signature Continuity Proposition 292.22 and the above discussion of the Alexander polynomial and the Murasugi signature of K_n that for each $n \in \mathbb{N}$ we have

$$\Xi([K_n]) \;=\; (0, \dots, 0, \underset{\text{in the } n\text{-th entry}}{\overset{\uparrow}{1}}, 0, \dots) \;\in\; \mathbb{Z}^\infty.$$

Since $\mathbb{Z}^\infty$ is a free abelian group with basis $\mathfrak{e}_1, \mathfrak{e}_2, \dots$ we obtain from this calculation, together with the Splitting Lemma 145.5 and the Free Abelian Group–Splitting Lemma 97.11, that the K_n are linearly independent and that they span a $\mathbb{Z}^\infty$-summand in the knot concordance group $\mathcal{C}^{\mathrm{sm}}$. ■

Remark. Let $P \subseteq S^1$ be the set of prime power roots of unity. From the Signature–Knot Concordance–Homomorphism Theorem 294.5 we obtain the homomorphism

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{sm}} & \to & \mathbb{Z}^P \\ [K] & \mapsto & (\sigma_K(\xi))_{\xi \in P}. \end{array}$$

So how close is this homomorphism to being injective? It turns that there is good news and bad news:

- Let's start with the good news: The same definitions and concepts also make sense for n-dimensional knots $K \subseteq S^{n+2}$ where K is diffeomorphic to S^n. This allows us to define smooth high-dimensional knot concordance groups. For odd $n \geq 5$ it follows from the work of Jerry Levine [**Levine1969b**, Theorem 24] and Takao Matumoto [**Matu1977**, Theorem 1] that the above map in the high-dimensional setting is in fact a monomorphism modulo 2-torsion and 4-torsion.
- Now we move on to the bad news: In the "classical" setting that we are actually interested the above map is very far from being a monomorphism. [**CaG1986**]

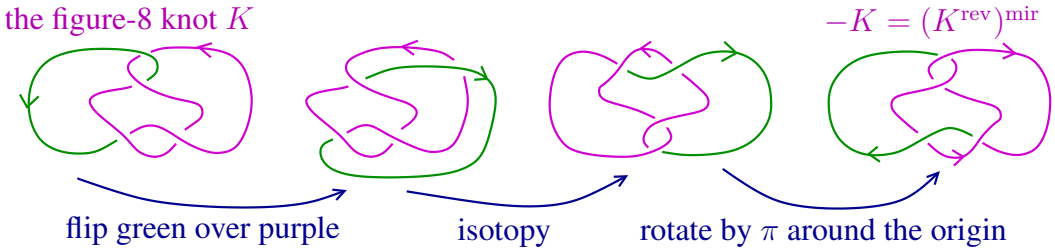
We refer to [**CaG1986, COT2003, Liv2005, Ras2010**] for more details on this story.□

294.4. **The knot concordance group and the Alexander polynomial.** In the last section we saw that, by now not very surprisingly, Levine-Tristram signatures give rise to "many" homomorphism from the smooth knot concordance group $\mathcal{C}^{\mathrm{sm}}$ to $\mathbb{Z}$. In this section we will use Alexander polynomials to define "many" homomorphisms from the smooth knot concordance group $\mathcal{C}^{\mathrm{sm}}$ to $\mathbb{Z}/2\mathbb{Z}$.

294.4.1. **Alexander polynomials as knot concordance homomorphisms.** One might wonder whether the smooth knot concordance group contains non-trivial elements of finite order. The following example shows that this is indeed the case:

Example. Let K be the figure-8 knot.

- On page 5636 we used the Alexander polynomial to show that the figure-8 knot is not smoothly slice. By the Trivial-in-Smooth Concordance Group Proposition 294.4 this implies that $[K] \neq 0 \in \mathcal{C}^{\mathrm{sm}}$.
- In the figure below we show that K is smoothly isotopic to $-K = (K^{\mathrm{rev}})^{\mathrm{mir}}$.



It follows that we have the following equalities in the knot concordance group $\mathcal{C}^{\mathrm{sm}}$:

$$[K] + [K] = [K\#K] \underset{\text{by the above}}{=} [K\#(K^{\mathrm{rev}})^{\mathrm{mir}}] \underset{\text{by Definition 294.3}}{=} [K\#(-K)] \underset{\text{by the } (K\#-K)\text{-Slice Proposition 293.17}}{=} 0 \in \mathcal{C}^{\mathrm{sm}}.$$

In summary we have shown that $[K]$ is an element of order two in the knot concordance group $\mathcal{C}^{\mathrm{sm}}$. □

In the next theorem we will see that Alexander polynomials can be used to define many homomorphisms from the knot concordance group $\mathcal{C}^{\mathrm{sm}}$ to $\mathbb{Z}/2\mathbb{Z}$. To formulate that statement it is convenient to introduce the following notation:

Notation 294.8. It follows from the UFD Proposition 287.11 together with the Laurent Polynomial Ring–UFD Proposition 287.12 that $\mathbb{Z}[t^{\pm 1}]$ is a UFD. One can now easily verify that for any irreducible $p(t) \in \mathbb{Z}[t^{\pm 1}]$ of degree ≥ 1 the map

$$\begin{array}{rcl} \Phi_{p(t)} \colon \mathbb{Z}[t^{\pm 1}] \setminus \{0\} & \to & \mathbb{Z} \\ f(t) & \mapsto & \underbrace{\max\{m \in \mathbb{N}_0 \,|\, p(t)^m \text{ divides } f(t)\}}_{=:\Phi_{p(t)}(f(t))} \end{array}$$

is a well-defined monoid homomorphism in the sense of Definition 2.2.

The above Notation 294.8 allows us to formulate the following theorem:

Theorem 294.9. (Alexander Polynomial–Knot Concordance–Homomorphism Theorem) Let $p(t) \in \mathbb{Z}[t^{\pm 1}]$ be an irreducible polynomial of degree ≥ 1 with $p(t) \doteq p(t^{-1})$. The map

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{sm}} & \to & \mathbb{Z}/2\mathbb{Z} \\ [K] & \mapsto & [\Phi_{p(t)}(\Delta_K(t))] \end{array}$$

is a well-defined homomorphism.

Proof. Let $p(t) \in \mathbb{Z}[t^{\pm 1}]$ be an irreducible polynomial. First note that it follows from the Link Isotopy–Alexander Polynomial Proposition 288.3, the Link Alexander Polynomial–Example Lemma 288.2, the Alexander Polynomial-at-1 Proposition 289.17 and the Connected Sum–Alexander Polynomial Proposition 290.9 that the map

$$\begin{array}{rcl} (\mathtt{Knot}^{\mathrm{or}}(S^3), \#) & \to & \mathbb{Z}[t^{\pm 1}] \setminus \{0\} \\ [K] & \mapsto & \Delta_K(t) \end{array}$$

is a well-defined monoid homomorphism. We now assume that $p(t) \doteq p(t^{-1})$ and that $\deg(p(t)) \geq 1$. We need to show that the monoid homomorphism

$$\begin{array}{rcl} (\mathtt{Knot}^{\mathrm{or}}(S^3), \#) & \to & \mathbb{Z}/2\mathbb{Z} \\ [K] & \mapsto & [\Phi_{p(t)}(\Delta_K(t))] \end{array}$$

descends to a monoid homomorphism on $\mathcal{C}^{\mathrm{sm}}$. By the Quotient Monoid Lemma 294.2 (5) this will follow from the following claim:

Claim. For every smoothly slice knot K we have $\Phi_{p(t)}(\Delta_K(t)) = 0 \in \mathbb{Z}/2\mathbb{Z}$.

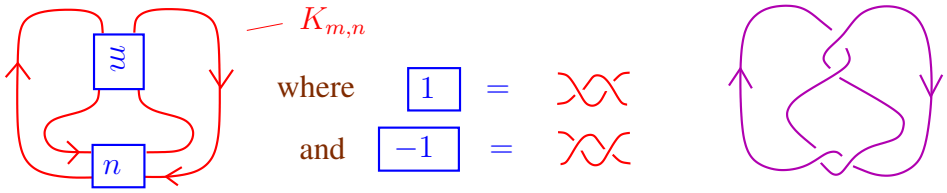
Proof. Let K be a smoothly slice knot. By the Fox-Milnor Theorem 294.1 there exists an $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $\Delta_K(t) \doteq f(t) \cdot f(t^{-1})$. We now see that

$$\begin{array}{ccccccc} & \text{by the above} & & \text{since } \Phi_{p(t)} \text{ is a monoid homomorphism} & & & \\ \Phi_{p(t)}(\Delta_K(t)) & \overset{\downarrow}{\doteq} & \Phi_{p(t)}(f(t) \cdot f(t^{-1})) & \overset{\downarrow}{=} & \Phi_{p(t)}(f(t)) \cdot \Phi_{p(t)}(f(t^{-1})) & & \\ & & & = & \Phi_{p(t)}(f(t)) \cdot \Phi_{p(t^{-1})}(f(t^{-1})) & = & 2 \cdot \Phi_{p(t)}(f(t)) = 0. \\ & & & \uparrow & & \uparrow & \\ & & & \text{since } p(t) \doteq p(t^{-1}) & & \text{since } t \mapsto t^{-1} \text{ induces a ring automorphism of } \mathbb{Z}[t^{\pm 1}] & \end{array}$$

■

294.4.2. **Examples.** To provide examples for the Alexander Polynomial–Knot Concordance–Homomorphism Theorem 294.9 we turn to the following generalization of Definition 294.6:

Definition 294.10. Given $m, n \in \mathbb{Z}$ we consider the (m,n)-double twist knot $K_{m,n}$ that is shown in the figure below. Note that for $m = -1$ we get the n-twist knot from Definition 294.6.



We have the following 2-torsion analog of the Twist Knot–Concordance Group Proposition 294.7.

Proposition 294.11. (Double Twist Knot–Concordance Group Proposition) Given $m \in \mathbb{N}$ we consider the $(m,-m)$-double twist knot $J_m := K_{m,-m}$ that we introduced in Definition 294.10. The homomorphism

$$\begin{array}{rcl} (\mathbb{Z}/2\mathbb{Z})^\infty & \to & \mathcal{C}^{\mathrm{sm}} \\ (r_1, r_2, \dots, r_m, 0, 0, \dots, 0) & \mapsto & \underbrace{r \cdot [J_1] + \dots + r_m \cdot [J_m]}_{=[r_1 \cdot J_1 \# \dots \# r_m \cdot J_m]} \end{array}$$

is well-defined and admits a left inverse. In particular the knots $\{J_m\}_{m\in\mathbb{N}}$ span a $(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$-summand of the smooth knot concordance group $\mathcal{C}^{\mathrm{sm}}$.

Proof. Let $m \in \mathbb{N}$. In Exercise 294.2 we will prove the following two statements:

(i) The $(m,-m)$-double twist knot $J_m = K_{m,-m}$ has the property that $-J_m$ is smoothly isotopic to J_m.
(ii) We have $\Delta_{J_m}(t) = p_m(t) := m \cdot t^2 - (2m+1) \cdot t + m$.
(iii) The polynomials $\Delta_{J_m}(t) = p_m(t)$ are irreducible in $\mathbb{Z}[t^{\pm 1}]$.

First note that it follows from (i) together with the $(K\# -K)$-Slice Proposition 293.17 and the discussion on page 5643 that the given map $\alpha\colon (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})} \to \mathcal{C}^{\mathrm{sm}}$ is actually a well-defined homomorphism.

Next note that it follows from the (ii) and (iii) and the Alexander Polynomial–Knot Concordance–Homomorphism Theorem 294.9 that the map

$$\begin{array}{rcl} \beta\colon \mathcal{C}^{\mathrm{sm}} & \to & \bigoplus_{m\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \\ [K] & \mapsto & ([\Phi_{p_m(t)}(\Delta_K(t))])_{m\in\mathbb{N}} \end{array}$$

is a well-defined homomorphism. Clearly this is a left-inverse to the homomorphism $\alpha\colon (\mathbb{Z}/2\mathbb{Z})^\infty \to \mathcal{C}^{\mathrm{sm}}$ we provided. It now follows from this discussion together with the Splitting Lemma 145.5 that $\alpha\colon (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})} \to \mathcal{C}^{\mathrm{sm}}$ is a monomorphism such that the image forms a summand. ■

We have now see that the knot concordance group $\mathcal{C}^{\mathrm{sm}}$ admits elements of infinite order and of order 2. This leads us to the following *open* question:

Question 294.12. Does the knot concordance group $\mathcal{C}^{\mathrm{sm}}$ admit elements of order $\neq 1, 2, \infty$?

294.5. Exercises for Chapter 294.

Exercise 294.1. Let $K \subseteq S^3$ be an oriented knot. We consider the following integral of the Levine-Tristram signatures

$$I(K) \;:=\; \int\limits_{\xi \in S^1} \sigma_K(\xi) \;:=\; \int\limits_{t=0}^{t=2\pi} \sigma_K(\exp(2\pi\,\mathrm{i}t))\,dt.$$

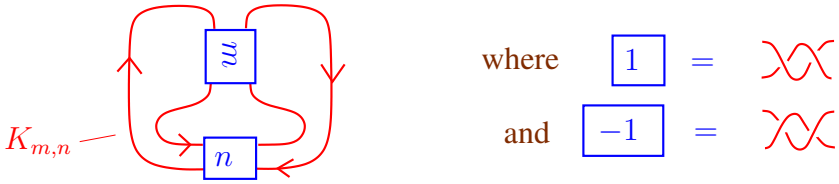
(a) Show that $I(K)$ is actually defined.
(b) Show that the map

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{sm}} & \to & \mathbb{R} \\ [K] & \mapsto & I(K) \end{array}$$

is a well-defined homomorphism and show that it is non-zero.

Exercise 294.2. Given $m, n \in \mathbb{Z}$ we consider the (m,n)-double twist knot $K_{m,n}$ that is shown in the figure below.

(a) Show that the $(m,-m)$-double twist knot $J_m = K_{m,-m}$ has the property that $-J_m$ is smoothly isotopic to J_m.
(b) Determine the Alexander polynomials of $K_{m,n}(t)$.
(c) In (b) you showed that $\Delta_{J_m}(t) = p_m(t) := m \cdot t^2 - (2m+1) \cdot t + m$. Show that these polynomials $\Delta_{J_m}(t) = p_m(t)$ are irreducible.



Exercise 294.3. We start out with a few definitions:

- A *Seifert matrix* is any matrix $A \in \mathrm{M}(k \times k, \mathbb{Z})$ such that $\det(A - A^{\mathrm{T}}) = \pm 1$. We consider the monoid M that is given by the congruence classes of Seifert matrices together with the block sum operation.
- A Seifert matrix $A \in \mathrm{M}(2g \times 2g, \mathbb{Z})$ is called *metabolic* if there exists a $P \in \mathrm{GL}(2g, \mathbb{Z})$ such that
$$P^{\mathrm{T}} A P \;=\; \begin{pmatrix} 0 & X_{g\times g} \\ Y_{g\times g} & Z_{g\times g} \end{pmatrix}.$$
One can easily verify that the congruence classes of metabolic Seifert matrices define a submonoid S of M.
- We refer to the monoid M/S as the *algebraic knot concordance group*.

Now we can formulate the following exercises:

(a) Show that M/S is an abelian group.
(b) Show that the map

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{sm}} & \to & M/S \\ [K] & \mapsto & [\text{Seifert matrix of } K] \end{array}$$

is a well-defined group homomorphism and show that it is actually an epimorphism.
(c) Show that M/S admits a summand that is isomorphic to $\mathbb{Z}^\infty$ and a summand that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$.

Remark. Jerry Levine [**Levine1969b**] showed that the algebraic knot concordance group M/S is isomorphic to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty$.

Exercise 294.4. Two oriented m-component links L and $\widetilde{L}$ in S^3 are called *smoothly concordant* if there exist disjoint properly smoothly embedded annuli $A_1, \dots, A_m \subseteq [0,1] \times S^3$ such that for each $i \in \{1, \dots, m\}$ we have $\partial A_i = (\{0\} \times (-L_i)) \cup (\{1\} \times \widetilde{L}_i)$.

(a) Show that a knot $K \subseteq S^3$ is smoothly concordant to the trivial knot if and only if K is smoothly slice.

(b) Show that if L and $\widetilde{L}$ are smoothly concordant, then for all $i \neq j \in \{1, \dots, m\}$ we have $\mathrm{lk}(L_i, L_j) = \mathrm{lk}(\widetilde{L}_i, \widetilde{L}_j)$.
Remark. This exercise is identical to Exercise 258.13.

Exercise 294.5. Let $K \subseteq S^3$ be a knot. A *ribbon disk for* K is defined as an immersion $\varphi\colon \overline{B}^2 \to S^3$ with the following properties:

(i) The restriction $\varphi|_{S^1}\colon S^1 \to K$ is a diffeomorphism.

(ii) There exist properly smoothly embedded intervals $I_1, \dots, I_m \subseteq \overline{B}^2$ and finitely many smoothly embedded intervals $J_1, \dots, J_m \subseteq B^2$ with the following properties:
- The two restrictions $\varphi\colon \overline{B}^2 \setminus (I_1 \cup \dots \cup I_m) \to S^3$ and $\varphi\colon \overline{B}^2 \setminus (J_1 \cup \dots \cup J_m) \to S^3$ are both smooth embeddings.
- For $i \neq j \in \{1, \dots, m\}$ we have $\varphi(I_i) \cap \varphi(J_j) = \varnothing$.
- For each $i \in \{1, \dots, m\}$ we have $\varphi(I_i) = \varphi(J_j)$.

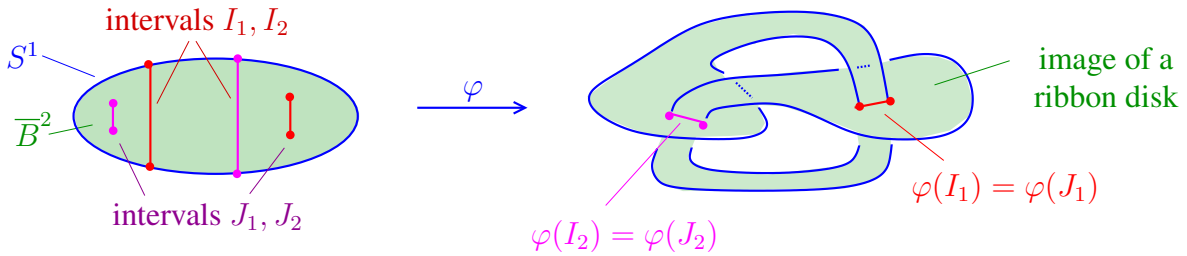
The images of $I_1, \dots, I_m$ are called *ribbon singularities*. A knot is called *ribbon* if it admits a ribbon disk.

(a) Show that every ribbon knot is also smoothly slice.
Remark. It is an open problem whether every smoothly slice knot is also a ribbon knot. This is known as the *Slice-Ribbon Conjecture*.

(b) Show that for every oriented knot $K \subseteq S^3$ the connected sum $K\# -K$ is ribbon.
Hint. Put K into left-standard position. Similar to the proof of the $(K\# -K)$-Slice Proposition 293.17 rotate $J := K \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^2)$ around the z-axis. The differ is, that this time we perform the rotation in $\mathbb{R}^3$ instead of $\mathbb{R}^4$. What object do you get by tracing out the rotation?

(c) Let φ be a ribbon disk as above with m ribbon singularities. Show that K admits a Seifert surface of genus m.

(d) Show that in (a) we can find a smooth disk D and that in (c) we can find a Seifert surface F such that $D \cup F$ is the boundary of a handlebody W of genus m that is embedded in $\overline{B}^4$.
Remark. This handlebody plays the role of the 4-Ball Surface Bounds Proposition 293.20 which, for a smoothly slice knot, we found with a big effort.



CHAPTER 295

The Conway polynomial via Seifert matrices

Let $L \subseteq S^3$ be an oriented link. By the Seifert Surface Existence Theorem 288.12 we know that L admits a Seifert matrix and by the S-Equivalence Theorem 291.35 we know that any two Seifert matrices of L are S-equivalent in the sense of Definition 291.34. In Definition 292.18 and Definition 292.21 we used this fact to define several new invariants of oriented links, namely the Murasugi signature $\mathrm{sign}(L)$ and the more general Levine-Tristram signature function $\sigma_L \colon S^1 \to \mathbb{Z}$. In this chapter we will use the S-Equivalence Theorem 291.35 to introduce a refinement of the single-variable Alexander polynomial $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1}]$, namely the Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$. This new invariant determines the Alexander polynomial but it has the added feature that it is well-defined "on the nose", i.e. without any indeterminacies. The key observation will be that the Conway polynomial satisfies a "skein relation", and that the Conway polynomial is in fact completely determined by the value of the unknot and the "skein relation". This will allow us to show that the Conway polynomial from this chapter is just the same as the Conway polynomial from Definition 68.5.

This chapter is organized as follows:

- In Section 295.1 we will introduce the Conway polynomial of an oriented link.
- In Section 295.2 we will discuss basic formal properties of the Conway polynomial.
- In Section 295.3 we will discuss basic topological properties of the Conway polynomial.
- In Section 295.4 we will formulate and prove the Alexander-Conway skein relation.
- In Section 295.5 we will discuss the axiomatic characterization of the Conway polynomial.
- In Section 295.6 we will provide exercises for Chapter 295.

295.1. **The Conway polynomial.** For the reader's convenience we recall the following definition:

Definition 291.34.

(1) (a) Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and let $\xi \in \mathbb{Z}^k$. We refer to the $((k+2) \times (k+2))$-matrix

$$\begin{pmatrix} A & \xi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

as a column enlargement of A.

(b) If a matrix B is obtained from A by a column enlargement, then we say that B is a column reduction of A.

(2) (a) Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and let $\xi \in \mathbb{Z}^k$. We refer to the $((k+2) \times (k+2))$-matrix

$$\begin{pmatrix} A & 0 & 0 \\ \xi^{\mathrm{T}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

as a row enlargement of A.

(b) If a matrix B is obtained from A by a row enlargement, then we say that B is a row reduction of A.

(3) We say that two integral square matrices are S-equivalent if we can obtain one from the other via a sequence of the following three types of moves:

(i) column (or row) enlargements,

(ii) column (or row) reductions,

(iii) replacing A by $P^{\mathrm{T}}AP$ where P is an invertible matrix over $\mathbb{Z}$.

In the S-Equivalence Theorem 291.35 we showed that for a given oriented link $L \subseteq S^3$ any two Seifert matrices for L are S-equivalent. This motivates the search of invariants of S-equivalence classes of matrices.

The following lemma is a slight but useful variation on the Seifert Matrix–S-Equivalence–Polynomial Lemma 291.37:

Lemma 295.1. (Seifert Matrix–S-Equivalence–Polynomial Lemma) In the following let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and $B \in \mathrm{M}(l \times l, \mathbb{Z})$ be two matrices. If A and B are S-equivalent, then we have the following equality in the Laurent polynomial ring $\mathbb{Z}[w]$:

$$\det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}) \;=\; \det(w^{-1} \cdot B - w \cdot B^{\mathrm{T}}).$$

Note that this equality holds "on the nose", no need to bother the hard-working "$\doteq$"-symbol.

Proof. This lemma follows from essentially the same calculation as in the Seifert Matrix–S-Equivalence–Polynomial Lemma 291.37. But, for the reader's convenience and peace of mind, we will now carry out the details. First note that it follows from Definition 291.34 that it suffices to prove the following claim:

Claim. Let $A \in \mathrm{M}(k \times k, \mathbb{Z})$ and $B \in \mathrm{M}(l \times l, \mathbb{Z})$ be two matrices.

(1) If B is a column enlargement of A, then $\det(w^{-1} \cdot B - w \cdot B^{\mathrm{T}}) \doteq \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}})$.

(2) If B is a row enlargement of A, then $\det(w^{-1} \cdot B - w \cdot B^{\mathrm{T}}) \doteq \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}})$.

(3) If $B = P^{\mathrm{T}}AP$ with $P \in \mathrm{GL}(k, \mathbb{Z})$, then $\det(w^{-1} \cdot B - w \cdot B^{\mathrm{T}}) \doteq \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}})$.

Proof. Evidently all statements follow from a direct calculation:

(1) Let B be a column enlargement of A as in Definition 291.34. We then see that

$$\begin{aligned}
\det(w^{-1}\cdot B - w\cdot B^{\mathrm{T}}) &= \det\left(w^{-1}\cdot\begin{pmatrix} A & \xi & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} - w\cdot\begin{pmatrix} A & \xi & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}^{\mathrm{T}}\right)\\
&= \det\begin{pmatrix} w^{-1}\cdot A - w\cdot A^{\mathrm{T}} & w^{-1}\cdot\xi & 0\\ -w\cdot\xi^{\mathrm{T}} & 0 & w^{-1}\\ 0 & -w & 0\end{pmatrix}\\
&= \det\begin{pmatrix} w^{-1}\cdot A - w\cdot A^{\mathrm{T}} & 0 & 0\\ 0 & 0 & w^{-1}\\ 0 & -w & 0\end{pmatrix} = \det(w^{-1}\cdot A - w\cdot A^{\mathrm{T}}).\\
&\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow\\
&\text{these two equalities follow easily from elementary linear algebra}
\end{aligned}$$

(2) The proof of this statement is almost the same as the proof of (1).

(3) Let $A\in \mathrm{M}(n\times n,\mathbb{Z})$ and let $P\in \mathrm{GL}(n,\mathbb{Z})$. For $B = P^{\mathrm{T}}\cdot A\cdot P$ we see that

$$\begin{aligned}
\det(w^{-1}\cdot B - w\cdot B^{\mathrm{T}}) &= \det(w^{-1}\cdot P^{\mathrm{T}}AP - w\cdot(P^{\mathrm{T}}AP)^{\mathrm{T}}) &&= \det(w^{-1}\cdot P^{\mathrm{T}}AP - w\cdot P^{\mathrm{T}}A^{\mathrm{T}}P)\\
&= \det(P^{\mathrm{T}}\cdot(w^{-1}\cdot A - w\cdot A^{\mathrm{T}})\cdot P) &&= \det(w^{-1}\cdot A - w\cdot A^{\mathrm{T}})\cdot\det(P)^2\\
&= \det(w^{-1}\cdot A - w\cdot A^{\mathrm{T}}).\\
&\uparrow\\
&\text{since } P\in \mathrm{GL}(n,\mathbb{Z})
\end{aligned}$$

■

The Seifert Matrix–S-Equivalence–Polynomial Lemma 295.1 together with the S-Equivalence Theorem 291.35 would already lead to a new invariant of oriented links. But it is better to wait for a second and to first prove two elementary algebraic lemmas, which will allow us to give a particularly elegant formulation of the new invariant.

Lemma 295.2. (Polynomial Anti-Symmetry Lemma) Let $A\in \mathrm{M}(m\times m,\mathbb{Z})$. The Laurent polynomial

$$f(w) \;=\; \det(w^{-1}\cdot A - w\cdot A^{\mathrm{T}}) \in \mathbb{Z}[w^{\pm 1}]$$

has the property that $f(-w^{-1}) = f(w)$.

Proof. [3181] The lemma follows from the following calculation:

$$\begin{aligned}
f(w) \;&= \det(w^{-1}\cdot A - w\cdot A^{\mathrm{T}}) \overset{\text{since } \det(W^{\mathrm{T}})=\det(W)}{\overset{\downarrow}{=}} \det(w^{-1}\cdot A^{\mathrm{T}} - w\cdot(A^{\mathrm{T}})^{\mathrm{T}})\\
&= \det((-w)\cdot A - (-w^{-1})\cdot A^{\mathrm{T}}) \;=\; f(-w^{-1}).
\end{aligned}$$

■

This is as good a moment as any to remind the reader of the follow lemma:

Lemma 68.4. (Rewriting Polynomial Lemma) We consider the map

$$\begin{aligned}
\Theta\colon \mathbb{Z}[z] &\to \mathbb{Z}[w^{\pm 1}]\\
p(z) &\mapsto p(w - w^{-1}).
\end{aligned}$$

This map has the following properties:

(1) Θ is a ring homomorphism.

[3181] The proof of the lemma is almost identical to the elementary proof of the Alexander Polynomial–Symmetry Proposition 289.15.

(2) The map Θ is injective.
(3) The image of Θ is given by all $f(w) \in \mathbb{Z}[w^{\pm 1}]$ with $f(-w^{-1}) = f(w)$.

We are now in a position to introduce the Conway polynomial of oriented links:

Definition 295.3. Let $L \subseteq S^3$ be an oriented non-empty link. By the Seifert Surface Existence Theorem 288.12 we know that L admits a Seifert matrix A. It follows from the Polynomial Anti-Symmetry Lemma 295.2 together with the Rewriting Polynomial Lemma 68.4 that there exists a unique polynomial $\nabla_L(z) \in \mathbb{Z}[z]$ with[3182]

$$\nabla_L(w - w^{-1}) \;=\; \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}).$$

It follows from the S-Equivalence Theorem 291.35, the S-Equivalence–Smooth Isotopy Proposition 291.36 and the Seifert Matrix–S-Equivalence–Polynomial Lemma 295.1 that $\nabla_L(z)$ is well-defined, i.e. it is independent of the choice of the Seifert matrix A, and that it is a smooth isotopy invariant. We refer to $\nabla_L(z)$ as the Conway polynomial of L.[3183]

Remark. The Conway polynomial was introduced, in a somewhat different language, by John Conway [**Conw1970**, Section 6]. The above definition of the Conway polynomial was first given by Louis Kauffman [**Kau1981**, p. 101]. □

Example. As always we first consider the trinity of the unknot U, the trefoil T and the figure-8 knot K:

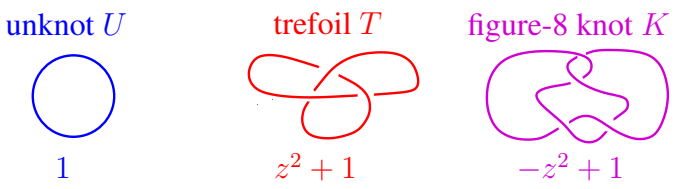
(1) We start out with the unknot U. By the discussion on page 5528 we know that the (0×0)-matrix is a Seifert matrix for U. It follows, not surprisingly, that $\nabla_U(z) = 1$.
(2) We consider the trefoil T. On page 5528 we showed that a Seifert matrix for T is given by $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Next note that

$$\det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}) \;=\; \det \begin{pmatrix} w^{-1} - w & -w^{-1} \\ w & w^{-1} - w \end{pmatrix} \;=\; (w - w^{-1})^2 + 1.$$

It follows from Definition 295.3 that $\nabla_T(z) = z^2 + 1$.
(3) Next we turn to the figure-8 knot K. On page 5528 we showed that a Seifert matrix for K is given by $A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$. This time note that

$$\det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}) \;=\; \det \begin{pmatrix} -w^{-1} + w & -w^{-1} \\ w & w^{-1} - w \end{pmatrix} \;=\; -(w - w^{-1})^2 + 1.$$

It follows from Definition 295.3 that $\nabla_T(z) = -z^2 + 1$.

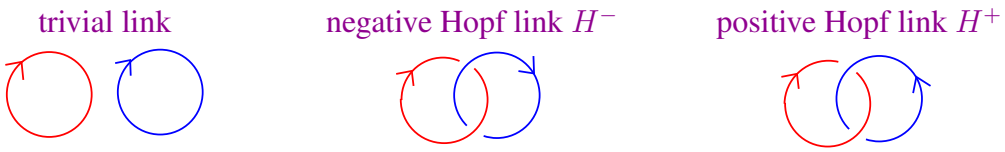


[3182] A careless author might be tempted to define the Conway polynomial via $\det(w \cdot A - w^{-1} \cdot A^{\mathrm{T}})$. The wisdom of the above choice will become apparent in the proof of the Alexander-Conway Skein Theorem 295.15.
[3183] Sometimes in the literature the Conway polynomial is referred to as the *Conway potential function*.

Now we turn to links:

(4) Let $m \in \mathbb{N}_{\geq 2}$ and let L be the trivial m-component link. On page 5529 we showed that a Seifert matrix for L is given by the $((m-1)\times(m-1))$-matrix consisting entirely of zeros. It follows easily from Definition 295.3 that $\nabla_{\text{trivial link } L}(z) = 0$.

(5) In the figure below we consider the positive Hopf link H^+ and the negative Hopf link H^-. On page 5529 we showed that a Seifert matrix for $H^\pm$ is given by $A_\pm = (\mp 1)$. Note that $\det(w^{-1}\cdot A_\pm - w\cdot A_\pm^{\mathrm{T}}) = \det(w^{-1}\cdot(\mp 1) - w\cdot(\mp 1)) = \pm\cdot(w - w^{-1})$. It follows from Definition 295.3 that $\nabla_{H^\pm}(z) = \pm z$.



Note that these calculations match the calculations from page 1440. □

295.2. **Basic formal properties of the Conway polynomial.** The following lemma shows that the single-variable Alexander polynomial is determined by the Conway polynomial:

Lemma 295.4. (Alexander–Conway Polynomial Lemma) Given any oriented non-empty link $L \subseteq S^3$ we have the equality

$$\Delta_L(t^2) \doteq \nabla_L(t^{-1} - t).$$

Proof. Let $L \subseteq S^3$ be an oriented non-empty link. We pick a Seifert matrix A for L. Then

$$\Delta_L(t^2) \underset{\uparrow}{\doteq} \det(t^2\cdot A - A^{\mathrm{T}}) \underset{\uparrow}{\doteq} \det(t\cdot A - t^{-1}\cdot A^{\mathrm{T}}) \underset{\uparrow}{=} \nabla_L(t^{-1} - t).$$

by the Alexander Module Seifert Matrix Theorem 289.12 — by elementary algebra — by Definition 295.3 and since $t \mapsto t^{-1}$ defines a ring automorphism of $\mathbb{Z}[t^{\pm 1}]$ ■

In Definition 289.26 we defined the determinant of an oriented link L as $\det(L) := |\Delta_L(-1)|$. It is now essentially clear that the Conway polynomial also determines the determinant of an oriented link:

Lemma 295.5. (Conway Polynomial–Determinant Lemma) Given any oriented non-empty link $L \subseteq S^3$ we have the equality

$$\det(L) = |\nabla_L(-2\mathrm{i})|.$$

Proof. Let $L \subseteq S^3$ be an oriented non-empty link. We now see that

$$\det(L) \underset{\uparrow}{=} |\Delta_L(-1)| = |\Delta_L(\mathrm{i}^2)| \underset{\uparrow}{=} |\nabla_L(\mathrm{i}^{-1} - \mathrm{i})| = \nabla_L(-2\mathrm{i}).$$

by Definition 289.26 — by the Alexander–Conway Polynomial Lemma 295.4 ■

This lemma motivates the following refinement of Definition 289.26:

Definition 295.6. Let $L \subseteq S^3$ be an oriented non-empty link with an $(n \times n)$-Seifert matrix A. We refer to

$$\det(L) \;:=\; \nabla_L(-2\mathrm{i}) \underset{\substack{\uparrow \\ \text{by Definition 295.3,} \\ \text{since } -2\mathrm{i} = \mathrm{i}^{-1} - \mathrm{i}}}{=} \det(\mathrm{i} \cdot A - \mathrm{i}^{-1} \cdot A^T) \underset{\substack{\uparrow \\ \text{elementary algebra}}}{=} \mathrm{i}^n \cdot \det(A + A^{\mathrm{T}})$$

as the determinant of the oriented link L.[3184] [3185]

Next we turn to the symmetries of the Conway polynomial. The following proposition is an analog of the Alexander Polynomial–Symmetry Proposition 289.15:

Proposition 295.7. (Conway Polynomial–Symmetry Proposition) Let $L \subseteq S^3$ be an oriented non-empty link with m components. Then

$$\nabla_L(-z) \;=\; (-1)^{m-1} \cdot \nabla_L(z).$$

Proof. Let L be an oriented non-empty link with m components. By the Seifert Surface Existence Theorem 288.12 we can pick a connected Seifert surface F for L. We denote by g the genus of F. Recall that it follows from the Surface-H_*-Proposition 155.17 that $\mathrm{H}_1(F)$ has rank $2g+m-1$. Let $A \in \mathrm{M}((2g+m-1) \times (2g+m-1), \mathbb{Z})$ be a corresponding Seifert matrix for F. We see that we have the following equalities:

$$\begin{aligned}
\nabla_L(-(w - w^{-1})) \;&=\; \nabla_L(w^{-1} - w) \underset{\substack{\uparrow \\ \text{follows easily from Definition 295.3} \\ \text{and since } w \mapsto w^{-1} \text{ is a ring} \\ \text{automorphism of } \mathbb{Z}[w^{\pm 1}]}}{=} \det(w \cdot A - w^{-1} \cdot A^{\mathrm{T}}) \underset{\substack{\uparrow \\ \text{since } \det(W^{\mathrm{T}}) = \det(W)}}{=} \det(w \cdot A^{\mathrm{T}} - w^{-1} \cdot A) \\
&\underset{\substack{\uparrow \\ \text{since } A \text{ is a square matrix} \\ \text{of size } 2g+m-1}}{=} (-1)^{2g+m-1} \cdot \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}) \underset{\substack{\uparrow \\ \text{by Definition 295.3}}}{=} (-1)^{m-1} \cdot \nabla_L(w - w^{-1}).
\end{aligned}$$

This calculation together with the Rewriting Polynomial Lemma 68.4 (2) gives us the desired equality $\nabla_L(-z) = (-1)^{m-1} \cdot \nabla_L(z)$. ∎

Next we will use the Conway Polynomial–Symmetry Proposition 295.7 to prove a result about the coefficients of the Conway polynomial. To state the result it is convenient to (re-) introduce Definition 69.7:

Definition 295.8. Let $L \subseteq S^3$ be an oriented non-empty link. Given $i \in \mathbb{N}_0$ we write

$$a_i(L) \;:=\; \text{the } z^i\text{-coefficient of } \nabla_L(z).$$

For peace of mind we extend this definition to $a_i(L) = 0$ for all $i < 0$.

We can now formulate the following analog of the Conway Polynomial–Low Coefficients Proposition 69.8:

[3184] It follows from the Conway Polynomial–Determinant Lemma 295.5 that the norm of the above "new determinant of L" equals the "old determinant of L" from Definition 289.26. In practice this should not cause any real confusion.

[3185] Note that this definition is consistent with Exercise 291.2.

Proposition 295.9. (Conway Polynomial–Zero Coefficients Proposition) Let $L \subseteq S^3$ be an oriented non-empty link with m components.

(1) If m is odd (e.g. if L is a knot), then all odd coefficients $a_{2k+1}(L)$ are zero.
(2) If m is even, then all even coefficients $a_{2k}(L)$ are zero.

Proof. Let L be an oriented non-empty link with m components. It follows immediately from the Conway Polynomial–Symmetry Proposition 295.7 that

$$(-1)^i \cdot a_i(L) \;=\; (-1)^{m-1} \cdot a_i(L).$$

Evidently this implies the promised statements. ■

We conclude this section with the following proposition that plays the role of the Alexander Polynomial-at-1 Proposition 289.17 and Exercise 295.10:

Proposition 295.10. (Conway Polynomial–Low Coefficients Proposition)

(1) For any oriented knot $K \subseteq S^3$ we have $a_0(K) = 1$.
(2) Let $m \in \mathbb{N}$ and let L be an oriented m-component link in S^3. We have $a_i(L) = 0$ for $i = 0, \ldots, m-2$.

Proof.

(1) Let $K \subseteq S^3$ be an oriented knot. We pick a Seifert matrix A for K. By the Seifert Matrix–Intersection Form Proposition 289.16 (3) we know that $\det(A - A^{\mathrm{T}}) = 1$. We now see that

$$a_0(K) \underset{\text{see Definition 295.8}}{\uparrow} = \nabla_K(0) \;=\; \nabla_K(1^{-1} - 1) \underset{\text{see Definition 295.3}}{\uparrow} = \det(\overbrace{1 \cdot A - 1^{-1} \cdot A^{\mathrm{T}}}^{=A-A^{\mathrm{T}}}) \underset{\text{by the above}}{\uparrow} = 1.$$

(2) Let $m \in \mathbb{N}$ and let L be an oriented m-component link in S^3. We pick a connected Seifert surface F of genus g for L. By the Seifert Matrix–Intersection Form Proposition 289.16 (3) we can find a Seifert matrix A for L that satisfies

$$A - A^{\mathrm{T}} \;=\; \begin{pmatrix} Q_{2g\times Q_{2g}} & 0 \\ 0 & 0_{(m-1)\times(m-1)} \end{pmatrix}$$

where $\det(Q_{2g\times 2g}) = 1$. It follows that we can write

$$A \;=\; \begin{pmatrix} B_{2g\times 2g} & C_{(m-1)\times 2g} \\ C^{\mathrm{T}}_{2g\times(m-1)} & D_{(m-1)\times(m-1)} \end{pmatrix}$$

where $D = \overline{D}^{\mathrm{T}}$, i.e. D is symmetric. It follows that

$$\begin{aligned} \nabla_L(w^{-1} - w) &\underset{\text{see Definition 295.3}}{\downarrow} = \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}) \underset{\text{since } D \text{ is symmetric}}{\downarrow} = \det \begin{pmatrix} w^{-1} \cdot B - w \cdot B^{\mathrm{T}} & (w^{-1} - w) \cdot C \\ (w^{-1} - w) \cdot C^{\mathrm{T}} & (w^{-1} - w) \cdot D \end{pmatrix} \\ &= (-(w - w^{-1}))^{m-1} \cdot \underbrace{\det \begin{pmatrix} w^{-1} \cdot B - w \cdot B^{\mathrm{T}} & C \\ (w^{-1} - w) \cdot C^{\mathrm{T}} & D \end{pmatrix}}_{=:\varphi(w)}. \\ &\quad \underset{\substack{\text{multilinearity of the} \\ \text{determinant applied to} \\ \text{the last } m-1 \text{ columns}}}{\uparrow} \end{aligned}$$

Claim. The Laurent polynomial $\varphi(w)$ is anti-symmetric, i.e. $\varphi(w) = \varphi(-w^{-1})$.

Proof. Let $\xi\colon \mathbb{Z}[w^{\pm 1}] \to \mathbb{Z}[w^{\pm 1}]$ be the automorphism that is given by $w \mapsto -w^{-1}$. We need to show that $\xi(\varphi) = \varphi$. Note that

$$\begin{array}{ccccc} (-(w-w^{-1}))^{m-1} \cdot \xi(\varphi) & \overset{\text{since } \xi \text{ is a ring homomorphism}}{\underset{\uparrow}{=}} & \xi((-(w-w^{-1}))^{m-1} \cdot \varphi) & \overset{\text{by definition of } \varphi}{\underset{\uparrow}{=}} & \xi(\nabla_L(w-w^{-1})) \\ & \underset{\text{since } \xi(w-w^{-1}) = w - w^{-1}}{=} & \nabla_L(w-w^{-1}) & \underset{\text{by definition of } \varphi}{=} & (-(w-w^{-1}))^{m-1} \cdot \varphi. \end{array}$$

Since $\mathbb{Z}[w^{\pm 1}]$ is a domain we obtain from this calculation that $\xi(\varphi) = \varphi$. ⊞

It follows from the claim and the Rewriting Polynomial Lemma 68.4 (3) that there exists a $p(z) \in \mathbb{Z}[z]$ such that $p(w-w^{-1}) = \varphi(w)$. Thus we have shown that

$$\nabla_L(w-w^{-1}) \;=\; (-(w-w^{-1}))^{m-1} \cdot p(w-w^{-1}).$$

It now follows from the Rewriting Polynomial Lemma 68.4 (2) that $\nabla_L(z) = z^{m-1} \cdot p(z)$. Clearly this implies that the first $m-2$ coefficients of $\nabla_L(z)$ are zero. ■

295.3. **Basic topological properties of the Conway polynomial.** In this section we will collect several straightforward and mostly uninspiring topological properties of the Conway polynomial, which match the properties of the single-variable Alexander polynomial.

First of all we have following lemma, which is the analog of the Alexander Polynomial–Reverse Lemma 290.3:

Lemma 295.11. (Conway Polynomial–Reverse Lemma) For any oriented non-empty link $L \subseteq S^3$ we have

$$\nabla_{L^{\mathrm{rev}}}(z) \;=\; \nabla_L(z).$$

This shows in particular that the Conway polynomial is an invariant of *unoriented* knots.

Proof. Let $L \subseteq S^3$ be an oriented non-empty link. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix A for L. We make the following observation:

(⋆) The Seifert Matrix Reverse Proposition 290.2 says that A^{T} is a Seifert matrix for L^{rev}.

We now see that

$$\begin{array}{ccccc} \nabla_{L^{\mathrm{rev}}}(w-w^{-1}) & \underset{\text{by Definition 295.3 and } (\star)}{\underset{\uparrow}{=}} & \det(w^{-1} \cdot A^{\mathrm{T}} - w \cdot (A^{\mathrm{T}})^{\mathrm{T}}) & \underset{\text{elementary algebra}}{\underset{\uparrow}{=}} & \det(w^{-1} \cdot A^{\mathrm{T}} - w \cdot A) \\ & \underset{\text{since } \det(W^{\mathrm{T}}) = \det(W)}{\underset{\uparrow}{=}} & \det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}}) & \underset{\text{by Definition 295.3}}{\underset{\uparrow}{=}} & \nabla_L(w-w^{-1}). \end{array}$$

It follows from the Rewriting Polynomial Lemma 68.4 (1) that $\nabla_{L^{\mathrm{rev}}}(z) = \nabla_L(z)$. ■

Not surprisingly we also have an analog of the Alexander Polynomial–Mirror Lemma 290.5:

Lemma 295.12. (Conway Polynomial–Mirror Lemma) Let $m \in \mathbb{N}$. For any oriented m-component link $L \subseteq S^3$ we have the following equality in $\mathbb{Z}[z]$:

$$\nabla_{L^{\mathrm{mir}}}(z) \;=\; \nabla_L(-z) \;\underset{\text{by the Conway Polynomial Symmetry Proposition 295.7}}{\underset{\uparrow}{=}}\; (-1)^{m-1} \cdot \nabla_L(z).$$

Proof. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix A for L. We make the following observation:

(⋆) The Seifert Matrix Mirror Proposition 290.4 says that $-A^{\mathrm{T}}$ is a Seifert matrix for L^{mir}.

We now see that

$$\nabla_{L^{\mathrm{mir}}}(w - w^{-1}) \underset{\substack{\uparrow \\ \text{by Definition 295.3 and } (*)}}{=} \det(w^{-1} \cdot (-A^{\mathrm{T}}) - w \cdot (-A^{\mathrm{T}})^{\mathrm{T}}) \underset{\substack{\uparrow \\ \text{since } \det(W^{\mathrm{T}}) = \det(W)}}{=} \det((-w^{-1}) \cdot A - (-w) \cdot A^{\mathrm{T}})$$

$$\underset{\substack{\uparrow \\ \text{follows from Definition 295.3 and} \\ \text{the observation that } w \mapsto -w \text{ defines a} \\ \text{ring automorphism of } \mathbb{Z}[w^{\pm 1}]}}{=} \nabla_L(-(w - w^{-1})).$$

This calculation together with the Rewriting Polynomial Lemma 68.4 (1) implies that $\nabla_{L^{\mathrm{mir}}}(z) = \nabla_L(-z)$. ■

Next we turn to the two main constructions of new links, namely split unions and connected sums. First of all we have the following analog of the Conway Polynomial–Splittable Proposition 295.13:

Proposition 295.13. (Conway Polynomial–Splittable Proposition) Let $L \subseteq S^3$ be an oriented link. If L is splittable in the sense of Definition 109.5, i.e. if L is the split union of two non-empty links, then $\nabla_L(z) = 0$.

Proof. The proposition follows easily from the definitions and constructing a suitable *connected* Seifert surface for L. We outsource the details to Exercise 295.1. ■

We conclude this section with the following analog of the Connected Sum–Alexander Polynomial Proposition 290.9 and of the Connected Sum–Conway Polynomial Proposition 295.14:

Proposition 295.14. (Connected Sum–Conway Polynomial Proposition) Let K and $\widetilde{K}$ be two oriented knots in S^3. The following equality holds:

$$\nabla_{K \# \widetilde{K}}(z) \;=\; \nabla_K(z) \cdot \nabla_{\widetilde{K}}(z) \;\in\; \mathbb{Z}[z].$$

Proof. Let K and $\widetilde{K}$ be two oriented knots in S^3. By the Seifert Surface Existence Theorem 288.12 we can pick a Seifert matrix A for K and we can pick a Seifert matrix $\widetilde{A}$ for $\widetilde{K}$.

(⋆) The Connected Sum–Seifert Matrix Proposition 290.8 says that $\begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix}$ is a Seifert matrix for the connected sum $K \# \widetilde{K}$.

We now see that

$$\begin{aligned}\nabla_{K\#\widetilde{K}}(w-w^{-1}) &\overset{\text{by Definition 295.3 and } (*)}{=} \det\left(w^{-1}\cdot\begin{pmatrix}A & 0\\ 0 & \widetilde{A}\end{pmatrix} - w\cdot\begin{pmatrix}A^{\mathrm{T}} & 0\\ 0 & \widetilde{A}^{\mathrm{T}}\end{pmatrix}\right)\\ &= \det\left(\begin{pmatrix}w^{-1}\cdot A - w\cdot A^{\mathrm{T}} & 0\\ 0 & w^{-1}\cdot\widetilde{A} - w^{-1}\cdot\widetilde{A}^{\mathrm{T}}\end{pmatrix}\right)\\ &= \det(w^{-1}\cdot A - w\cdot A^{\mathrm{T}})\cdot\det(w^{-1}\cdot\widetilde{A} - w\cdot\widetilde{A}^{\mathrm{T}}) \overset{\text{by Definition 295.3}}{=} \nabla_K(w-w^{-1})\cdot\nabla_{\widetilde{K}}(w-w^{-1}).\end{aligned}$$

$\uparrow$ elementary algebra

This calculation together with the Rewriting Polynomial Lemma 68.4 (1) and (2) gives us the desired equality $\nabla_{K\#\widetilde{K}}(z) = \nabla_K(z)\cdot\nabla_{\widetilde{K}}(z)$. ■

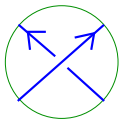
295.4. **The Alexander-Conway skein relation.** Note that, in contrast to the Alexander polynomial, the Conway polynomial of an oriented link has no indeterminacy. This implies in particular that it makes sense to talk of the sum of two Conway polynomials. The following theorem makes full use of this feature:

Theorem 295.15. (Alexander-Conway Skein Theorem) The Conway polynomial has the following property: Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same *diagram*, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown below. Then we have the Alexander-Conway skein relation

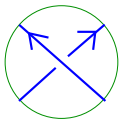
$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\cdot\nabla_{L_0}(z).$$

In other words, using Definition 295.8 we see that for each $i\in\mathbb{N}_0$ we have[3186]

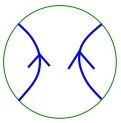
$$a_i(L_+) - a_i(L_-) = a_{i-1}(L_0).$$



L_+



L_-



L_0

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\cdot\nabla_{L_0}(z)$$
$$a_i(L_+) - a_i(L_-) = a_{i-1}(L_0)$$

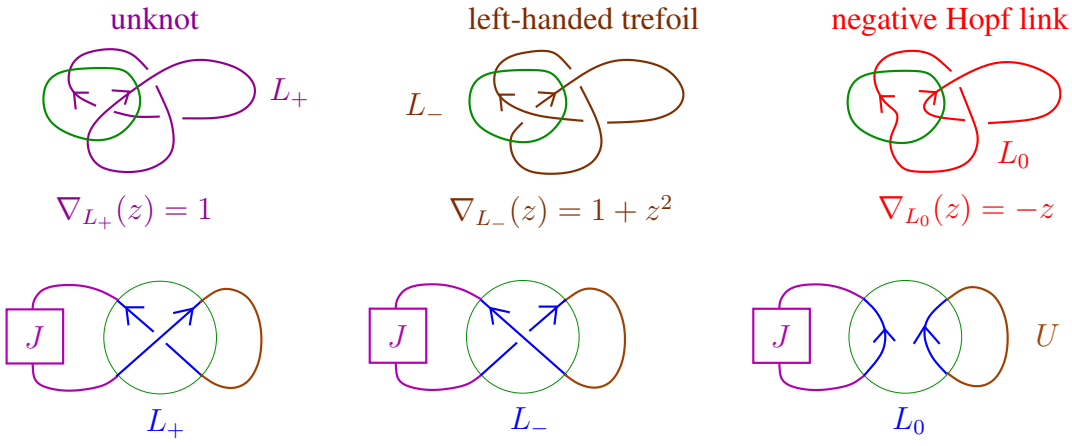
Remark. Interestingly enough James Alexander himself was essentially aware of this theorem, see [**Ale1928**, p. 301]. But it was John Conway [**Conw1970**] who first realized the relevance of this result, namely he realized that this skein relation essentially characterizes the Conway polynomial. We will discuss this feature in the Conway Polynomial-Characterization Theorem 69.5. □

Example.

(1) We consider the oriented links L_+, L_- and L_0 that are shown below. We also show the Conway polynomials of these knots and links which conveniently enough we already calculated on page 5651.
We see that we indeed have $\underbrace{\nabla_{L_+}(z)}_{=1} - \underbrace{\nabla_{L_-}(z)}_{=1+z^2} = z\cdot\underbrace{\nabla_{L_0}(z)}_{-1}$. Magic!

(2) As a "proof of concept" let us consider the split union L_0 of a non-empty link J together with an unknot U as shown below to the right.

[3186] Note that by definition a_{-1} is always zero. Thus for $i=0$ we see that $a_0(L_+) - a_0(L_-) = 0$.



We now see that

$$z \cdot \nabla_{J \sqcup U}(z) \quad = \quad z \cdot \nabla_{L_0}(z) \quad \underset{\uparrow}{=} \quad \nabla_{L_+}(z) - \nabla_{L_-}(z) \quad \underset{\uparrow}{=} \quad \nabla_J(z) - \nabla_J(z) \quad = \quad 0.$$

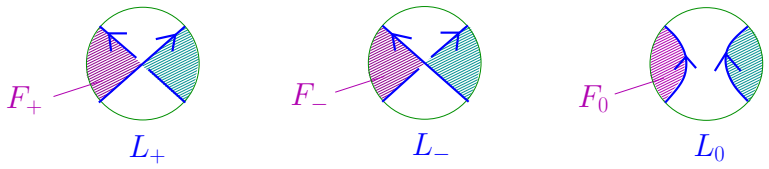
by the Alexander-Conway Skein Theorem 295.15 — since $L_\pm$ are smoothly isotopic to J

This implies that $\nabla_{J \sqcup U}(z) = 0$. This shows in particular that for any trivial link with ≥ 2 components the Conway polynomial vanishes. □

The proof of the Alexander-Conway Skein Theorem 295.15 will occupy the remainder of this section. In the next Section 69.1 we will study unlinking sequences, which will be essential in the proof of the characterization of the Conway polynomial.

As we will see, the proof of the Alexander-Conway Skein Theorem 295.15 shares some features with the proof of the Gordian Distance–Signature Proposition 293.4. In particular we will need the following slight variation on the Seifert Surface Triple Lemma 293.6:

Lemma 295.16. (Seifert Surface Triple Lemma) Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same diagram, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown below. We can find connected Seifert surfaces F_+, F_- and F_0 for L_+, L_- and L_0 that are identical outside of the smoothly embedded ball where they are of the form shown below.



Proof. [3187] Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to diagrams, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown above. We apply Seifert's algorithm, which we explained in the Seifert Surface Existence Theorem 121.2, to the diagram of L_0. Up to symmetries the two possible outcomes of a Seifert surface for L_0 are shown in the figure below. (Using the

[3187] The above lemma is a little more simple-minded than the original Seifert Surface Triple Lemma 295.16 since this time we only consider links that correspond to diagrams.

approach of the Make-Surface-Connected Lemma 288.13 we can also arrange that F_0 is connected.)



We now consider the two cases separately:

(a) We just need to attach two bands to F_0 to obtain Seifert surfaces for F_+ and F_-.
(b) We apply a smooth isotopy to the Seifert surface of L_0 as shown in the figure below:



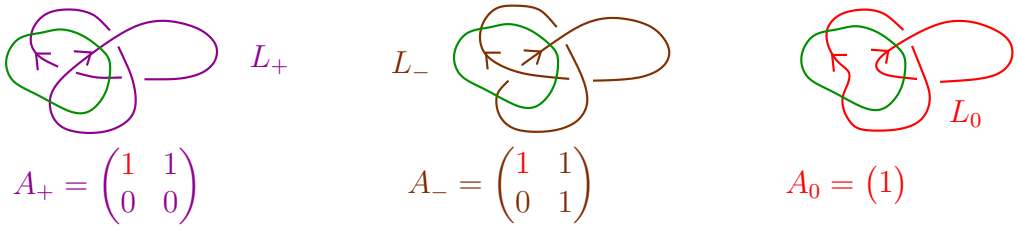
This turns the Seifert surface into a Seifert surface as in (a). Now we can again attach bands to get Seifert surfaces for L_+ and L_-. ■

The following lemma is almost identical to the Seifert Matrix Triple Lemma 293.7:

Lemma 295.17. (Seifert Matrix Triple Lemma) Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same diagram, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown above. There exists a Seifert matrix A_0 for L_0 and there exist $\zeta, \xi \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$ such that

$$A_+ := \begin{pmatrix} A_0 & \zeta \\ \xi^{\mathrm{T}} & n \end{pmatrix} \qquad \text{is a Seifert matrix for } L_+,$$

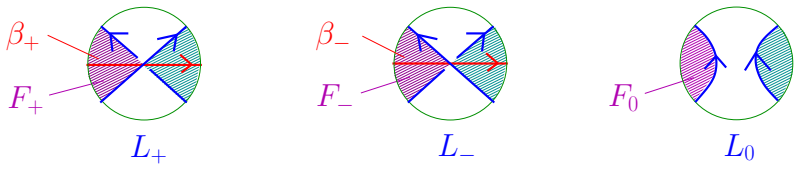
$$A_- := \begin{pmatrix} A_0 & \zeta \\ \xi^{\mathrm{T}} & n+1 \end{pmatrix} \qquad \text{is a Seifert matrix for } L_-.$$



Proof. [3188] First we pick connected Seifert surfaces $F_0, F_\pm$ as in the Seifert Surface Triple Lemma 293.6. Furthermore we pick a basis $a_1, \ldots, a_k$ for $\mathrm{H}_1(F_0)$.

Since F_0 is connected we can find closed oriented smoothly embedded curves $\beta_- \subseteq F_-$ and $\beta_+ \subseteq F_+$ which are identical except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown below.

[3188] The following proof is of course almost the same as the proof of the Seifert Matrix Triple Lemma 293.7. We provide the argument again to make this chapter more self-contained.



We set $b_+ := [\beta_+] \in \mathrm{H}_1(F_+)$ and $b_- := [\beta_-] \in \mathrm{H}_1(F_-)$. A Mayer-Vietoris argument shows that $a_1, \dots, a_k, b_+$ form a basis for $\mathrm{H}_1(F_+)$ and that $a_1, \dots, a_k, b_-$ form a basis for $\mathrm{H}_1(F_-)$. Next let A_0 be the Seifert matrix of F_0 corresponding to the basis $a_1, \dots, a_k$. It follows from the definitions and the Seifert Form–Calculation Lemma 289.10 that there exist $\zeta, \xi \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$ such that

$$A_+ := \begin{pmatrix} A_0 & \zeta \\ \xi^{\mathrm{T}} & n \end{pmatrix} \qquad \text{is the Seifert matrix for } F_+ \text{ corresponding to } a_1, \dots, a_k, b_+,$$

$$A_- := \begin{pmatrix} A_0 & \zeta \\ \xi^{\mathrm{T}} & n+1 \end{pmatrix} \qquad \text{is the Seifert matrix for } F_- \text{ corresponding to } a_1, \dots, a_k, b_-. \qquad \blacksquare$$

Proof of the Alexander-Conway Skein Theorem 295.15. Suppose that we are given three oriented links L_+, L_-, L_0 that are associated to the same *diagram*, except that in a smoothly embedded closed 3-ball they are–up to smooth isotopy–of the form shown above on numerous occasions. Recall that we need to show that

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) \;=\; z \cdot \nabla_{L_0}(z).$$

It follows from the Seifert Matrix Triple Lemma 295.17 that there exists a Seifert matrix A_0 for L_0 and that there exist $\zeta, \xi \in \mathbb{Z}^k$ and $n \in \mathbb{Z}$ such that

$$A_+ := \begin{pmatrix} A_0 & \zeta \\ \xi^{\mathrm{T}} & n \end{pmatrix} \qquad \text{is a Seifert matrix for } L_-,$$

$$A_- := \begin{pmatrix} A_0 & \zeta \\ \xi^{\mathrm{T}} & n+1 \end{pmatrix} \qquad \text{is a Seifert matrix for } L_+.$$

We now see that

$$\begin{aligned}
&\Delta_{L_+}(w - w^{-1}) - \Delta_{L_-}(w - w^{-1}) \quad \overset{\text{see Definition 295.3}}{=} \\
&= \det(w^{-1} \cdot A_+ - w \cdot A_+^{\mathrm{T}}) - \det(w^{-1} \cdot A_- - w \cdot A_-^{\mathrm{T}}) \\
&= \det \begin{pmatrix} w^{-1} \cdot A_0 - w \cdot A_0^{\mathrm{T}} & w^{-1} \cdot \zeta - w \cdot \xi \\ w^{-1} \cdot \xi^{\mathrm{T}} - w \cdot \zeta^{\mathrm{T}} & n \cdot (w^{-1} - w) \end{pmatrix} - \det \begin{pmatrix} w^{-1} \cdot A_0 - w \cdot A_0^{\mathrm{T}} & w^{-1} \cdot \zeta - w \cdot \xi \\ w^{-1} \cdot \xi^{\mathrm{T}} - w \cdot \zeta^{\mathrm{T}} & (n+1) \cdot (w^{-1} - w) \end{pmatrix} \\
&= \det \begin{pmatrix} w^{-1} \cdot A_0 - w \cdot A_0^{\mathrm{T}} & 0 \\ w^{-1} \cdot \xi^{\mathrm{T}} - w \cdot \zeta^{\mathrm{T}} & -(w^{-1} - w) \end{pmatrix} \quad = \quad (w - w^{-1}) \cdot \det(w^{-1} \cdot A_0 - w \cdot A_0^{\mathrm{T}}) \\
&\quad \uparrow \text{follows from the multilinearity of the applied to the last column} \qquad \uparrow \text{by elementary linear algebra and since } -(w^{-1} - w) = w - w^{-1} \text{3189} \\
&= (w - w^{-1}) \cdot \Delta_{L_0}(w - w^{-1}). \\
&\quad \uparrow \text{see Definition 295.3}
\end{aligned}$$

It follows from this calculation together with the uniqueness statement of the Rewriting Polynomial Lemma 68.4 (2) that

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) \;=\; z \cdot \nabla_{L_0}(z).$$ ■

295.5. **The axiomatic characterization of the Conway polynomial.** We start out with the following variation on the Conway Polynomial-Characterization Theorem 69.5:

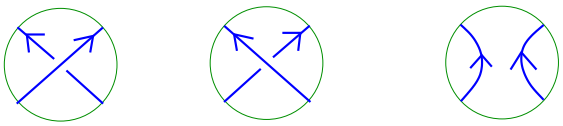
Theorem 295.18. (Conway Polynomial-Characterization Theorem) The Conway polynomial from Definition 295.3 is the only map

$$\texttt{Link}^{\text{or}}_{\neq\varnothing}(S^3) \;\to\; \mathbb{Q}(z)$$

which satisfies the following two Conway axioms:

(C1) The invariant of the unknot equals 1.

(C2) The invariant satisfies the skein relation from the Alexander-Conway Skein Theorem 295.15, which is also summarized in the figure below.



(C1) $\nabla_{\text{unknot}}(z) = 1$

(C2) $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \cdot \nabla_{L_0}(z)$

Proof. First note that it follows from the calculation on page 5651 and the Alexander-Conway Skein Theorem 68.7 that the Conway polynomial from Definition 295.3 satisfies Properties (C1) and (C2).

The uniqueness statement follows of course from the Link Invariant–Uniqueness Proposition 69.6 applied to $\Theta\colon \texttt{Link}^{\text{or}}_{\neq\varnothing}(S^3) \to \mathbb{Q}(z)$ with $a_+ = 1$, $a_- = -1$, $a_0 = z$ and with $u = 1$. ■

This theorem has two interesting consequences:

- It shows that the two definitions of the Conway polynomial in Definition 68.5 and Definition 295.3 lead to the same invariant.
- As we saw in Chapter 69 many interesting properties of the Conway polynomial can now be deduced just using axioms C1 and C2.

295.6. **Exercises for Chapter 295.**

Exercise 295.1. Let $L \subseteq S^3$ be an orientable link that is splittable in the sense of Definition 109.5. Show, using Definition 295.3, that $\nabla_L(z) = 0$.

Exercise 295.2. Reformulate the Genus–Alexander Polynomial Proposition 290.15 in terms of the Conway polynomial.

Exercise 295.3. Reformulate the Fibered Knot–Alexander Polynomial Proposition 290.17 in terms of the Conway polynomial.

[3189] That is where it helps that we work with $\det(w^{-1} \cdot A - w \cdot A^{\mathrm{T}})$ instead of the initially more tempting $\det(w \cdot A - w^{-1} \cdot A^{\mathrm{T}})$

Exercise 295.4. Reformulate the Fox-Milnor Theorem 294.1 in terms of the Conway polynomial.

Exercise 295.5. Let $L = L_1 \cup L_2$ be an oriented 2-component link in S^3. We define the *splitting number* $\mathrm{sp}(L)$ of L as the minimal number of crossing changes (in the sense of Definition 293.1) that are needed to turn L into a link that is totally split in the sense of Definition 69.1.
(a) Show that $|\mathrm{lk}(L_1, L_2)| \leq \mathrm{sp}(L)$.
(b) Provide an example of an oriented 2-component link with $|\mathrm{lk}(L_1, L_2)| < \mathrm{sp}(L)$.

Exercise 295.6. In Definition 295.6 we defined for any oriented non-empty link L its determinant as $\det(L) = \nabla_L(2\mathrm{i})$. Give an axiomatic characterization of the determinant of oriented non-empty links.

Exercise 295.7. Let $m \in \mathbb{N}$ and let L be an oriented m-component link in S^3. Use the Conway axioms (C1) and (C2) to show that $a_i(L) = 0$ for $i = 0, \ldots, m-2$.
Hint. Prove the statement iteratively for $m = 1, 2, 3, \ldots$. For a fixed m, use the Unlinking Proposition 69.2 (2), the Conway axiom (C2) and the fact that the Conway polynomial of a link with ≥ 2 components is zero.
Remark. This gives a new proof of the Conway Polynomial–Low Coefficients Proposition 295.10 (2).

Part XXXIII

Homotopy Theory

In this part of the lecture notes we develop more tools from homotopy theory. A particular focus will be on calculating more homotopy groups of spheres, using various tools.

This part of the lecture notes is organized as follows:

- In Chapter 296 we will introduce the relative homotopy groups $\pi_n(X, A, x_0)$ of a pointed pair of topological spaces.
- In Chapter 297 we will introduce the notion of a fibration. Using relative homotopy groups we will show that any Serre fibration $p\colon Y \to B$ with fiber F gives rise to a long exact sequence of homotopy groups relating the homotopy groups of F, Y and B.
- In Chapter 298 we return to the study of fiber bundles, since these form a particularly important class of fibrations. We will consider several old and new examples of bundles.
- In Chapter 299 we will introduce Lie groups and we will consider the homotopy groups of Lie groups. We will in particular calculate $\pi_1(\mathrm{SO}(n))$ for any $n \geq 3$. This calculation allows us to introduce the Spin groups $\mathrm{Spin}(n)$.
  In this chapter we will also state several results, e.g. the Bott Periodicity Theorem 299.21, which unfortunately we will not be able to prove for space reasons.
- In Chapter 300 we give an introduction to Thom-Pontryagin theory which makes it possible to calculate homotopy groups of spheres using "framed cobordism groups". We will use this approach to give a new proof that $\pi_n(S^n) \cong \mathbb{Z}$ and that $\pi_3(S^2) \cong \mathbb{Z}$. Furthermore we will use this approach to show that the homotopy groups $\pi_{4n-1}(S^n)$ are infinite.
- In Chapter 301 we will further develop Thom-Pontryagin theory. We will use it to prove the Freudenthal Suspension Theorem 301.12 which says that for $n + k \geq 2$ the suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism.
- In Chapter 302 we will use Thom-Pontryagin theory to show that for every $n \geq 3$ we have an isomorphism $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$.
- In Chapter 303 we will summarize all results on homotopy groups of spheres that we proved throughout these lecture notes.

CHAPTER 296

Relative homotopy groups

While studying homology groups we saw at some point that it was convenient to introduce the relative homology groups $\mathrm{H}_k(X, A)$ corresponding to a pair of topological spaces (X, A). In our latest attempt to understand homotopy groups we will now, perhaps not surprisingly, introduce relative homotopy groups.

These have formal properties that are similar to relative homology groups, for example we will see that there exists a long exact sequence of (relative) homotopy groups of a pointed pair (X, A, x_0) of topological spaces.

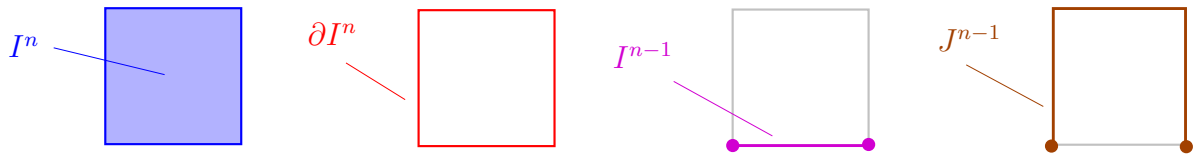
This chapter is organized as follows:

- In Section 296.1 we will introduce the relative homotopy groups $\pi_n(X, A, x_0)$.
- In Section 296.2 we will formulate and prove the long exact homotopy sequence of a pair.
- In Section 296.3 we will study how paths act on relative homotopy groups.
- In Section 296.4 we will give an alternative description of relative homotopy groups.
- In Section 296.5 we will study connected (pairs of) topological spaces.
- In Section 296.6 we will formulate the relative Hurewicz Theorem 296.25.
- In Section 296.7 we will discuss the question whether there is an excision type theorem for relative homotopy groups.
- In Section 296.8 we will provide exercises for Chapter 296.

In the next Chapter 297 we will introduce fibrations.

296.1. **The definition of relative homotopy groups.** We start out with introducing the following notation:

Notation 296.1. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Furthermore, given $n \in \mathbb{N}_0$ we use the following notation:
(1) We denote by $I^n \subseteq \mathbb{R}^n$ the cube in $\mathbb{R}^n$.
(2) We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
(3) By a slight abuse of notation we make the identification $I^{n-1} =_{\mathrm{i}} I^{n-1} \times \{0\}$.
(4) We write $J^{n-1} := \overline{\partial I^n \setminus I^{n-1}}$.

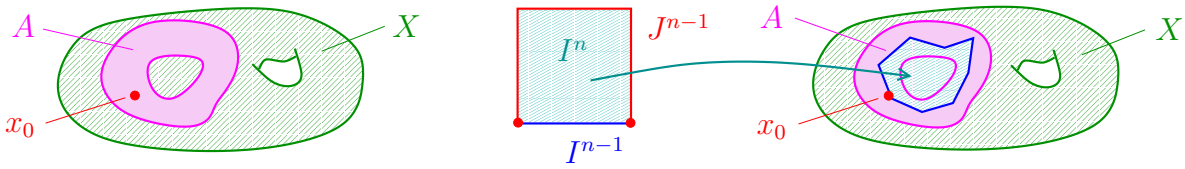


The following definition is a variation on the notion of a pointed topological space that we introduced in Exercise 7.27.

Definition 296.2. A pointed pair of topological spaces is a triple (X, A, x_0), where X is a topological space, A is a subset of X and x_0 is a point in A. A morphism between pointed pairs of topological spaces (X, A, x_0) and (Y, B, y_0) is a continuous map $f\colon X \to Y$ with $f(A) \subseteq B$ and $f(x_0) = y_0$. We refer to the corresponding category as the category of pointed pairs of topological spaces.

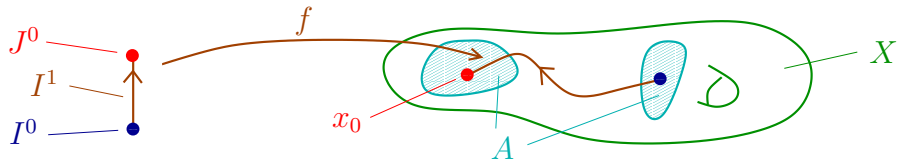
Definition 296.3. Let (X, A, x_0) be a pointed pair of topological spaces. Given $n \geq 1$ we define the n-th relative homotopy group of (X, A, x_0) to be[3190]

$$\pi_n(X, A, x_0) \ := \ \text{homotopy classes of continuous maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\}).$$



Examples.

(1) Let (X, A, x_0) be a pointed pair of topological spaces. We want to get a better understanding of elements in $\pi_1(X, A, x_0)$. For $n = 1$ we have $I^1 = [0, 1]$, $\partial I^1 = \{0, 1\}$ and $J^0 = \{1\}$. A continuous map $(I^1, \partial I^1, J^0) \to (X, A, \{x_0\})$ is thus the same as a path in X that starts at a point in A and that ends in x_0. We refer to the figures below for an illustration.





the maps define the same element in $\pi_1(X, A, x_0)$

the maps define (presumably) three different elements in $\pi_1(X, A, x_0)$

(2) Let (X, x_0) be a pointed topological space. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, \{x_0\}, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, \{x_0\}, \{x_0\})] \end{array}$$

defines, basically by definition, a natural bijection. In the following we will use this natural bijection to make the identification $\pi_n(X, x_0) =_{\mathrm{i}} \pi_n(X, \{x_0\}, x_0)$.

[3190] In Definition 19.24 we defined the notion of continuous maps between pairs of topological spaces and we defined the notion of a homotopy between such maps. These notions extend in an obvious way to triples of topological spaces.

(3) Let X be a topological space, let A be a subset of X and let $x_0 \in A$ be a point. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, A, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] \end{array}$$

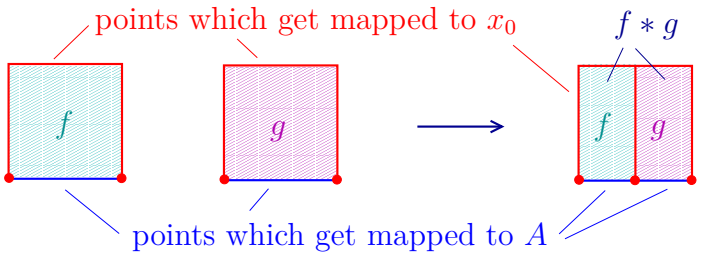
is a natural map from absolute homotopy groups to relative homotopy groups. □

In the previous discussion we already referred to $\pi_n(X, A, x_0)$ as a "relative homotopy group". This was slightly premature, but now we will actually deliver the goods and provide a group structure.

Definition 296.4. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 2$.[3191] Given two continuous maps $f, g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ we define[3192]

$$\begin{array}{rcl} f * g\colon I^n & \to & X \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f(2t_1, t_2, \ldots, t_n), & \text{if } t_1 \in [0, \frac{1}{2}], \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{if } t_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Note that $f * g$ is again a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ of triples. The definition of $f * g$ is illustrated in the figure below.



The following proposition finally gives the justification for the name *relative homotopy group*.

Proposition 296.5. (Relative Homotopy Group Proposition) Let (X, A, x_0) be a pointed pair of topological spaces.

(1) Let $n \geq 2$. The map

$$\begin{array}{rcl} \pi_n(X, A, x_0) \times \pi_n(X, A, x_0) & \to & \pi_n(X, A, x_0) \\ ([f], [g]) & \mapsto & [f * g] \end{array}$$

is well-defined and it defines a group structure on $\pi_n(X, A, x_0)$. Furthermore the following two statements hold:

(a) The trivial element in $\pi_n(X, A, x_0)$ is represented by the constant map $I^n \to X$ that sends every point in I^n to x_0.

(b) The inverse of the relative homotopy class $[f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})]$ is represented by the map

$$\begin{array}{rcl} (I^n, \partial I^n, J^{n-1}) & \to & (X, A, \{x_0\}) \\ (t_1, t_2, \ldots, t_n) & \mapsto & (1 - t_1, t_2, \ldots, t_n). \end{array}$$

[3191] What, if anything, goes wrong for $n = 1$?

[3192] This map is continuous by the Pasting Proposition 4.7 (2).

(2) For $n \geq 2$ the above natural bijection $\pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, \{x_0\}, x_0)$ is in fact a group isomorphism.
(3) For $n \geq 2$ the above natural map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is a group homomorphism.
(4) For $n \geq 3$ the group $\pi_n(X, A, x_0)$ is abelian.
(5) Let $n \in \mathbb{N}$. Given a morphism $f\colon (X, A, x_0) \to (Y, B, y_0)$ of pointed pairs of topological spaces the map

$$\begin{array}{rcl} f_*\colon \pi_n(X, A, x_0) & \to & \pi_n(Y, B, y_0) \\ [\varphi] & \mapsto & [f \circ \varphi] \end{array}$$

is well-defined. Furthermore the following four statements hold:
(a) For $n \geq 2$ the induced map f_* is a group homomorphism.
(b) The maps f_* define a covariant functor from the category of pointed pairs of topological spaces to the category of sets (respectively groups for $n \geq 2$).
(c) Let $n \geq 1$ and let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces. If $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$ is a bijection for all $x_0 \in A$ and if g is homotopic[3193] to f, then $g_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, g(x_0))$ is also a bijection for all $x_0 \in A$.
(d) Let $f\colon (X, A) \to (Y, B)$ be a continuous map between pairs of topological spaces and let $x_0 \in A$. If f is a homotopy equivalence, then the map

$$f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$$

is a bijection (in particular it is an isomorphism if $n \geq 2$).

Sketch of proof.

(1) The proof of Statement (1) is quite similar to the proof of the π_n-Group Proposition 134.6. We leave it to the reader to fill in the details.
(3) This statement requires just slightly more thought than one might think. More precisely, given two continuous maps f and g from $(I^n, \partial I^n, J^{n-1})$ to $(X, A, \{x_0\})$, the "product maps" $f * g$ considered in Definition 134.4 for $\pi_n(X, \{x_0\})$ (and illustrated in the figure on page 2575) and in the above proposition for $\pi_n(X, A, x_0)$ are not identical. Nonetheless, using the Shrinking Lemma 134.8 or the Addition-in-π_n-Proposition 167.25 it is straightforward to show that they are homotopic. This observation shows that the above identification of sets is in fact an identification of groups.
(2) We already know that the map is a bijection. Thus it suffices to show that it is a group homomorphism. But this follows from (3) applied to $A = \{x_0\}$.
(4) The proof of this statement is an entertaining variation on the proof of the π_n-Abelian Proposition 134.11. The proof is the content of Exercise 296.2.
(5) Statements (a) and (b) follow basically immediately from the definitions. Statement (c) and (d) are proved in a similar fashion to the Homotopy Equivalence-π_n-Proposition 134.17. ■

Remark.

(1) We will see shortly that the relative homotopy group $\pi_2(X, A, x_0)$ is in general not abelian.

[3193]Here we mean of course, "homotopic as a map between pairs of topological spaces".

(2) For want of a better idea, given a pointed pair of topological spaces (X, A, x_0) we refer to $\pi_1(X, A, x_0)$ also as a *relative homotopy group* even though strictly speaking we did not define a group structure on $\pi_1(X, A, x_0)$. □

In light of the Relative Homotopy Group Proposition 296.5 we adopt the following convention:

Convention 296.6. For $n \geq 3$ we usually use additive notation for the group structure on the relative homotopy groups $\pi_n(X, A, x_0)$.

Let X be a topological space and let $A \subseteq B \subseteq X$ be two subsets. In the LES-of-Triple Proposition 146.5 we saw that this data gives rise to a long exact sequence of homology groups. Shortly we will see that a similar statement holds for relative homotopy groups. We did not define a group structure on $\pi_1(X, A, x_0)$ and we did not define a group structure on $\pi_0(X, x_0)$. (In fact we will see in Exercise 296.3 that there is no natural way to turn these sets into groups.) Nonetheless we would like to include these objects at the lower end of the long exact sequence of (relative) homotopy groups. In fact, as it turns out, $\pi_1(X, A, x_0)$ and $\pi_0(X, x_0)$ have ever so slightly more structure than just the structure of a set:

Definition 296.7.

(1) A pointed set is a pair (S, s_0) where S is a set and $s_0 \in S$ is a point. We call s_0 the trivial element in S.
(2) A morphism from a pointed set (S, s_0) to a pointed set (T, t_0) is a map $f\colon S \to T$ with $f(s_0) = t_0$. Furthermore we adopt the following language:
   (a) Given a morphism $f\colon (S, s_0) \to (T, t_0)$ we refer to $\ker(f) \coloneqq f^{-1}(t_0)$ as the kernel of f.
   (b) We say that a morphism $f\colon (S, s_0) \to (T, t_0)$ is trivial if $\ker(f) = S$, put differently, if all elements in S get sent to t_0.
   (c) We say that a morphism $f\colon (S, s_0) \to (T, t_0)$ is a monomorphism if $\ker(f) = \{s_0\}$.
   (d) We say that a morphism $f\colon (S, s_0) \to (T, t_0)$ is an epimorphism if f is surjective.
(3) We say that a sequence
$$\ldots \xrightarrow{f_{n+2}} (S_{n+1}, s_{n+1}) \xrightarrow{f_{n+1}} (S_n, s_n) \xrightarrow{f_n} (S_{n-1}, s_{n-1}) \xrightarrow{f_{n-1}} \ldots$$
of morphisms of pointed sets is exact if for every $n \in \mathbb{N}_0$ we have the usual equality $\ker(f_n) = \operatorname{im}(f_{n+1})$.

Examples.

(1) Any group G gives rise to the pointed set $(G, \{e\})$ where $e \in G$ evidently denotes the trivial element. We will thus always view a group as a pointed set. Any group homomorphism is also a morphism of the corresponding pointed sets. Furthermore an exact sequence of homomorphisms of groups is also an exact sequence of pointed sets.
(2) If H is a group and G is a subgroup, then the set of left cosets H/G is a pointed set with trivial element given by $G \in H/G$. The natural projection $H \to H/G$ is a morphism of pointed sets. □

Lemma 296.8. (Pointed Set–SES Lemma) Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that

we have an exact sequence

$$\{e\} \to G \xrightarrow{\varphi} H \xrightarrow{\psi} (S, s_0) \to \{e\}$$

where ψ is a morphism of pointed sets that is H-equivariant, i.e. for any $h, h' \in H$ we have $\psi(h \cdot h') = h \cdot \psi(h')$. Then $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Proof. We will prove the lemma in Exercise 296.4. ■

Example. The previous lemma might sound rather artificial. But we have seen it before in group theory: Suppose that H is a group and G is a subgroup. As above we view the set of left cosets H/G as a pointed set. Left-multiplication by H defines an action on H/G. Evidently the projection $H \to H/G$ is H-equivariant and we get a short exact sequence $\{e\} \to G \to H \to H/G \to \{e\}$ as in the Pointed Set–SES Lemma 296.8. □

We continue with pointed sets that arise in topology.

Examples.

(1) Let (X, x_0) be a pointed topological space. Recall that on page 2574 we defined $\pi_0(X, x_0)$ to be the set of homotopy classes of continuous maps from the point I^0 to X. In other words, $\pi_0(X, x_0)$ is the set of path-components of X. We view $\pi_0(X, x_0)$ as a pointed set where the trivial element is given by the map that sends the unique point in I^0 to x_0. A continuous map $f\colon X \to Y$ between topological spaces defines a morphism $f_*\colon \pi_0(X, x_0) \to \pi_0(Y, f(x_0))$ of pointed sets in the obvious way. Sometimes, by an unfortunate abuse of language, we refer to $\pi_0(X, x_0)$ as a homotopy group of X.

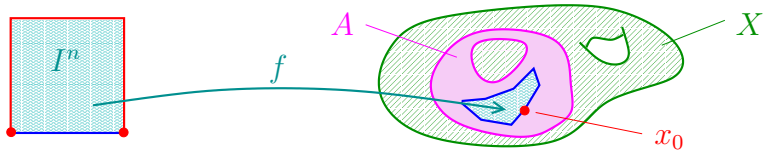
(2) Similarly to the above, given a pointed pair of topological spaces (X, A, x_0) we view $\pi_1(X, A, x_0)$ as pointed set where the trivial element is once again represented by the constant path $I \to X$ given by sending all points to x_0. □

We conclude this section with the following elementary lemma:

Lemma 296.9. (Zero-in-Relative Homotopy Group Lemma) Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 1$. Let $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ be a continuous map. If $f(I^n) \subseteq A$, then f represents the trivial element in $\pi_n(X, A, x_0)$.[3194] [3195]

Proof. The lemma will be proved in Exercise 296.6. ■

Since it is much easier to convince oneself of a statement if one sees an explicit example, we show such a map with $f(I^n) \subseteq A$ in the figure below.



[3194] Put differently, as for relative homology groups, all objects that "land in A" are in fact trivial in the relative homotopy group.

[3195] Note that by the above discussion of pointed sets the statement of the Zero-in-Relative Homotopy Group Lemma 296.9 is also meaningful for $n = 1$.

296.2. **The long exact homotopy sequence of a pair.** In the LES-of-Triple Proposition 146.5 we saw that we can associate to a pair (X, A) of topological spaces a long exact sequence of (relative) homology groups. The goal of this section is to do the same for homotopy groups. More precisely we will associate to a pointed pair (X, A, x_0) of topological spaces a long exact sequence of (relative) homotopy groups.

Proposition 296.10. (Homotopy Groups-LES-Proposition) Let (X, A, x_0) be a pointed pair of topological spaces.

(1) For any $n \geq 1$ the map

$$\begin{array}{rcl} \partial_n : \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ [f : (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}) & \to & (A, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array} \right] \end{array}$$

is well-defined. (See the figure below.) We refer to it as the connecting morphism. It has the following two properties:

(a) The connecting morphism is a morphism of pointed sets and it is a group homomorphism for $n \geq 2$.

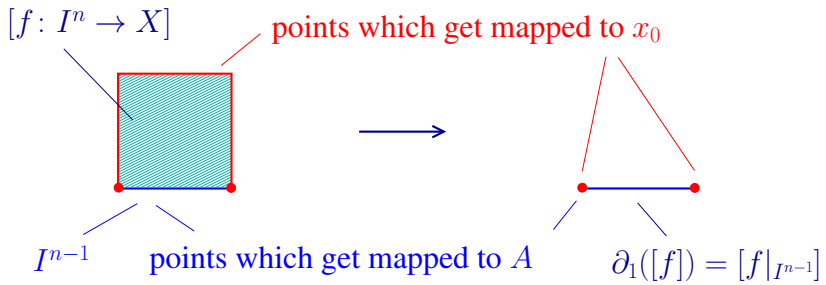
(b) The connecting morphism is a natural transformation from the covariant functor $\pi_n(X, A, x_0)$ to the covariant functor $\pi_{n-1}(A, x_0)$. In everyday language this just simply means that for a continuous map $f : (X, A, x_0) \to (Y, B, y_0)$ between pointed pairs of topological spaces the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{\partial_n} & \pi_{n-1}(A, x_0) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_n(Y, B, y_0) & \xrightarrow{\partial_n} & \pi_{n-1}(B, y_0). \end{array}$$

(2) The following sequence is exact

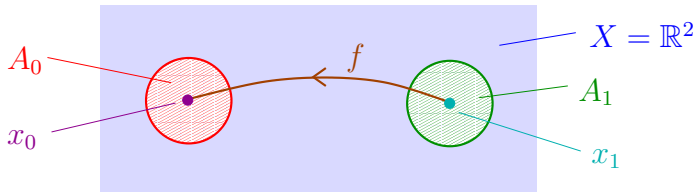
$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0).$$

Here $i : A \to X$ is the inclusion map and the map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is the natural map introduced on page 5667.



Examples.

(1) We consider $X = \mathbb{R}^2$ and we consider $A = A_0 \sqcup A_1$ which is the union of disjoint closed disks A_0 and A_1 in $X = \mathbb{R}^2$. We denote by $x_0 \in A_0$ and $x_1 \in A_1$ the centers of the two disks. Furthermore we denote by $f : I = [0, 1] \to X$ a map with $f(0) = x_1$ and $f(1) = x_0$.

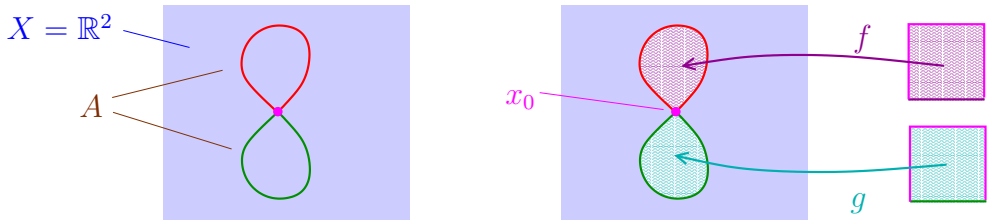


Finally, for any $n \in \mathbb{N}_0$ and any $i \in \{0, 1\}$ we denote by $c_{x_i} \colon I^n \to A_i$ the constant map that sends all points to x_i. It follows from the Deformation Retract–Homotopy Equivalence Lemma 19.10 together with the Homotopy-Equivalence π_0-Lemma 19.17 and the Homotopy Equivalence-π_1-Proposition 90.9 (1) and the Homotopy Equivalence-π_n-Proposition 134.17 (1) that all the homotopy groups of A and X are generated by $[c_{x_0}]$ except for $\pi_0(A, x_0)$ that equals the set $\{[c_{x_0}], [c_{x_1}]\}$.

As discussed on page 5666, f defines an element in $\pi_1(X, A, x_0)$. Using the definitions one sees almost immediately that $\partial[f] = [c_{x_1}] \in \pi_0(A, x_0)$. It follows easily from this data and the Homotopy Groups-LES-Proposition 296.10 (2) that $\pi_1(X, A, x_0)$ consists of precisely two elements, namely $[c_{x_0}]$ and $[f]$.

(2) Let $X = \mathbb{R}^2$. We denote by A the wedge of two circles, viewed as a subset of $X = \mathbb{R}^2$. Furthermore we denote by x_0 the wedge point. It follows from the Homotopy Groups-LES-Proposition 296.10 (2) that $\pi_2(X, A, x_0) \xrightarrow{\cong} \pi_1(A, x_0)$. In the Surface Group–Nonabelian Lemma 101.2 we saw that $\pi_1(A, x_0)$ is non-abelian, thus $\pi_2(X, A, x_0)$ is also non-abelian.

In fact we can identify two elements in $\pi_2(X, A, x_0)$ that do not commute. We consider the two maps $f, g \colon (I^2, \partial I^2, J^1) \to (X, A, \{x_0\})$ that are illustrated in the figure below.



It follows easily from the definitions that $\partial[f], \partial[g]$ are the standard generators of the free group[3196] $\pi_1(S^1 \vee S^1, x_0)$. Since these do not commute we see that $[f]$ and $[g]$ do not commute either. □

Proof. We start out with the proof of Statement (1). In fact this statement follows mostly easily from the definitions. One only needs to think for a minute about the question why the connecting homomorphism is in fact a homomorphism.

We turn to the proof of Statement (2). In fact we prove a slight variation on Statement (2). More precisely, let X be a topological space, let A be a subset of X and let B be a

[3196]We refer to the Wedge-of-Circles-π_1-Lemma 99.11 for the calculation of the fundamental group of $S^1 \vee S^1$.

subset of A. Finally let $x_0 \in B$. For any $n \geq 2$ the map

$$\begin{array}{rcl} \partial_n \colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, B, x_0) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \to & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}, J^{n-2}) & \to & (A, B, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array} \right] \end{array}$$

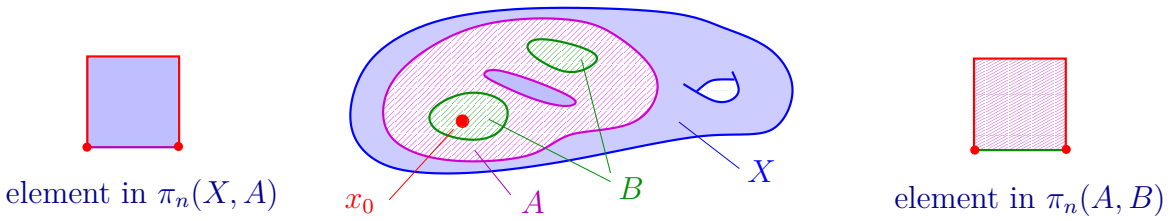
is easily seen to be a well-defined morphism of pointed sets. Our goal is to prove the following claim:

Claim. The following sequence is exact[3197]

$$\ldots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, B, x_0) \to \ldots \to \pi_1(X, A, x_0).$$

where $i \colon (A, B) \to (X, B)$ and $j \colon (X, B) \to (X, A)$ are the obvious morphisms of pointed pairs of topological spaces.

Note that this claim, applied to $B = \{x_0\}$ together with the Relative Homotopy Group Proposition 296.5 (2) almost implies that the sequence of the proposition is exact. The only problem is that the sequence of the claim stops slightly earlier, more precisely it stops at the subindex one instead of subindex zero. Thus the only piece missing is the exactness of the sequence of the proposition at the last three pointed sets. We leave the verification of exactness at these three spots as an elementary exercise to the conscientious reader.



Proof. Not surprisingly the proof of the claim breaks up into three separate proofs, namely we need to verify exactness at the three different groups.

Exactness at $\pi_n(X, B, x_0)$. It follows easily from the Zero-in-Relative Homotopy Group Lemma 296.9 that $j_* \circ i_* = (j \circ i)_*$ is the trivial morphism. We leave it to the reader to verify that $\ker(j_*) \subseteq \operatorname{im}(i_*)$.

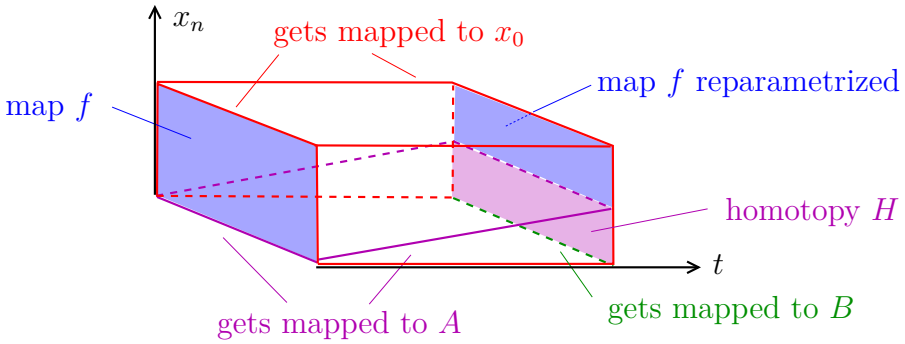
Exactness at $\pi_n(X, A, x_0)$. It follows again easily from the Zero-in-Relative Homotopy Group Lemma 296.9 and the definition of the connecting homomorphism ∂ that $\partial \circ j_*$ is the trivial morphism. Now suppose that we are given $f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ such that the restriction of f to $I^{n-1} \times \{0\}$ represents the trivial element in $\pi_{n-1}(A, B, x_0)$. This means that there exists a homotopy $H \colon I^{n-1} \times I \to A$ (with all the required restrictions) from $f|_{I^n \times \{0\}}$ to the constant map. The map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f\big(x_1, \ldots, x_{n-1}, \frac{t}{2} \cdot x_n + (1 - x_n)\big), & \text{if } x_n \in \left[\frac{t}{2}, 1\right] \\ H\big(x_1, \ldots, x_{n-1}, 2x_n - t\big), & \text{if } x_n \in \left[0, \frac{t}{2}\right] \end{cases} \end{array}$$

is continuous by the Homotopy Combination Lemma 18.8 (1) and it defines a homotopy (again with all the desired extra properties) from f to a map that represents an element

[3197] This sequence plays of course the role of the long exact sequence in homology groups of a triple of topological spaces, see the LES-of-Triple Proposition 146.5.

in $\pi_n(X, B, x_0)$. The definition of the homotopy is sketched, to the best of my abilities, in the figure below.

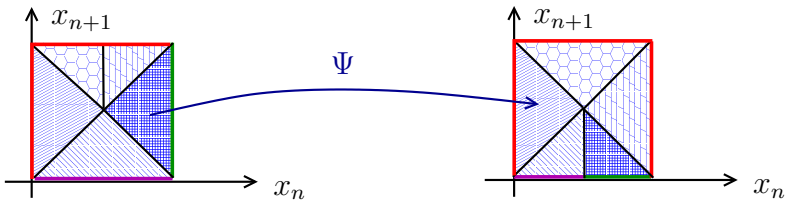


Exactness at $\pi_n(A, B, x_0)$. First note that $i_* \circ \partial$ is the trivial morphism. This can be seen as follows: If $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, \{x_0\})$ is any continuous map, then the map $f\colon (I^n \times [0,1], \partial I^n \times [0,1]) \to (X, x_0)$ defines a homotopy from $\partial f = f|_{I^n \times \{0\}}$ to the constant map c_{x_0}. Thus $(i_* \circ \partial)[f] = [i_* \circ \partial f]$ is the trivial element in $\pi_n(X, B, x_0)$.

Now suppose that we are given a continuous map $f\colon (I^n, \partial I^n, J^{n-1}) \to (A, B)$ such that $j \circ f$ represents the trivial element in $\pi_n(X, B, x_0)$. This means that there exists a homotopy H (with all the obvious conditions) from $j \circ f$ to the constant map c_{x_0}. We define

$$\begin{array}{rcl} g\colon I^n & \to & B \\ (x_1, \ldots, x_n) & \mapsto & H(x_1, \ldots, x_{n-1}, 0, x_n). \end{array}$$

We pick a homeomorphism $\Psi\colon I^2 \to I^2$ with the properties as illustrated in the figure below.[3198] (Here it is psychologically convenient to think of I^2 as the square in the plane with x_n and x_{n+1}-coordinates.)



It follows immediately from the definitions that the map

$$\begin{array}{rcl} I^{n+1} & \to & X \\ (x_1, \ldots, x_n, x_{n+1}) & \to & H(x_1, \ldots, x_{n-1}, \Psi(x_n, x_{n+1})) \end{array}$$

defines an element in $\pi_{n+1}(X, A, x_0)$ such that the connecting morphism sends it to the element in $\pi_n(A, B, x_0)$ represented by the map[3199]

$$\begin{array}{rcl} f * g\colon I^n & \to & X \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f(t_1, \ldots, t_{n-1}, 2t_n), & \text{if } t_n \in [0, \frac{1}{2}], \\ g(t_1, \ldots, t_{n-1}, 2t_n - 1), & \text{if } t_n \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

[3198] The triangles in the figure below indicate how one could write down an explicit homeomorphism.

[3199] Strictly speaking this definition of $f * g$ differs from Definition 296.4 since we use a different coordinate to perform the product. But it is convenient to use the same notation.

It remains to show that $f * g$ and f represent the same element in $\pi_n(A, B, x_0)$. This can be done in a fashion similar to the above proof of exactness. More precisely, a homotopy between f and $f * g$, again with all the desired properties, is given by

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f\big(x_1, \ldots, x_{n-1}, \frac{t}{2} \cdot x_n + (1 - x_n)\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ H\big(x_1, \ldots, x_{n-1}, 2x_n - t\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big] \end{cases} \end{array}$$ ■

296.3. **Paths acting on relative homotopy groups.** Let (X, x_0) be a pointed topological space. In Section 169.2 we saw that the fundamental group $\pi_1(X, x_0)$ of X acts on any homotopy group $\pi_n(X, x_0)$. In this section we will discuss an analogous construction for relative homotopy groups. In particular, given a pointed pair (X, A, x_0) of topological spaces we will see that the fundamental group $\pi_1(A, x_0)$ of A (let us stress that here we write A instead of X) acts on any relative homotopy group $\pi_n(X, A, x_0)$.

The following definition is quite similar to Definition 169.3.

Definition 296.11. Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.[3200] Furthermore let $\gamma\colon [0,1] \to A$ be a path in the subset A[3201] from a point x_0 to a point x_1. Finally let

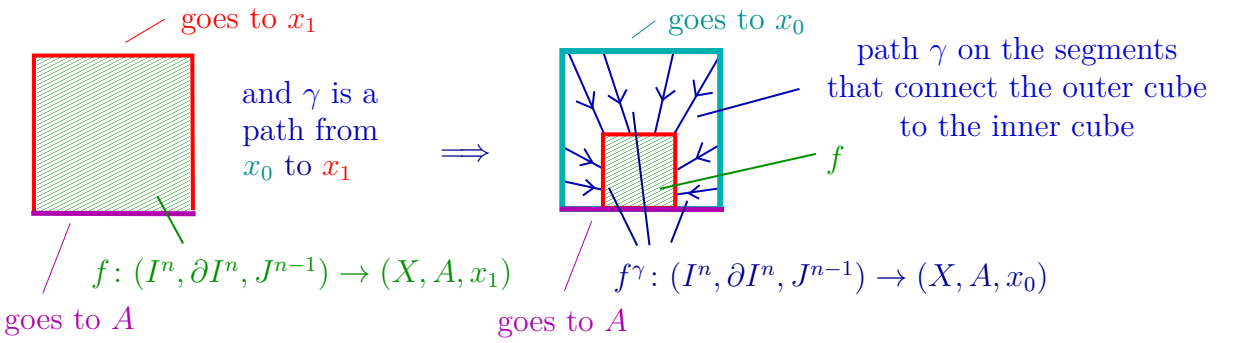
$$f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_1\})$$

be a continuous map. We define

$$f^\gamma\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$$

to be the map that is illustrated in the figure below. For our particularly pedantic readers we now also give a super-rigorous definition of f^γ: First of all, given $t \in [0,1]$ we define $\mathrm{C}_t = \big[\frac{t}{4}, \frac{4-t}{4}\big]^{n-1} \times [0, 1 - \frac{t}{2}]$. Now we define

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n, J^{n-1}) & \to & (X, A, x_0) \\ (x_1, \ldots, x_n) & \to & \begin{cases} f\big(2x_1 - \frac{1}{2}, \ldots, 2x_n\big), & \text{if } (x_1, \ldots, x_n) \in \mathrm{C}_1 = \big[\frac{1}{4}, \frac{3}{4}\big]^{n-1} \times [0, \frac{1}{2}], \\ \gamma(t), & \text{if } (x_1, \ldots, x_n) \in \partial \mathrm{C}_t \text{ for } t \in [0,1]. \end{cases} \end{array}$$



The following proposition can be viewed as the "relative analog" of the Change-of-Base Point Proposition 134.12.

[3200] What is wrong with $n = 1$?

[3201] Note that in contrast to the definition in Definition 169.3 we only consider paths in A. At this point we do not allow paths in X.

Proposition 296.12. (Change-of-Base Point Proposition) Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.

(1) If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X,A,x_1) & \to & \pi_n(X,A,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is a well-defined map that is a group homomorphism.

(2) If x_0 and x_1 are two points in A which lie in the same path-component of A, we can pick a path γ from x_0 to x_1 and the map $\gamma_*\colon \pi_n(X,A,x_1) \to \pi_n(X,A,x_1)$ is an isomorphism.

(3) Let $\varphi\colon (X,A) \to (Y,B)$ be a continuous map between pairs of topological spaces.

(a) If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X,A,x_1) & \xrightarrow{\gamma_*} & \pi_n(X,A,x_0) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \pi_n(Y,B,\varphi(x_1)) & \xrightarrow{(\varphi\circ\gamma)_*} & \pi_n(Y,B,\varphi(x_0)). \end{array}$$

(b) If $\varphi_*\colon \pi_n(X,A,x_0) \to \pi_n(Y,B,\varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in A$, then for any other point x_1 in the same path component of A the map $\varphi_*\colon \pi_n(X,A,x_1) \to \pi_n(Y,B,\varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).

(4) Let γ and δ be two paths in A from x_0 to x_1. If γ and δ are path-homotopic in A, then the corresponding maps γ_* and δ_* from $\pi_n(X,A,x_1)$ to $\pi_n(X,A,x_0)$ agree.

(5) If γ is a path in A from x_0 to x_1 and if δ is a path in A from x_1 to x_2, then[3202]

$$(\underbrace{\gamma * \delta}_{\text{path in } A \text{ from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X,A,x_2) \to \pi_n(X,A,x_0).$$

(6) The map

$$\begin{array}{rcl} \pi_1(A,x_0) \times \pi_n(X,A,x_0) & \to & \pi_n(X,A,x_0) \\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f] := [f^\gamma] \end{array}$$

defines an action (in the sense of Definition 2.36) of the group $\pi_1(A,x_0)$ on the group $\pi_n(X,A,x_0)$.

(7) We denote by $i\colon A \to X$ the inclusion map and we denote by $\varphi\colon \pi_n(X,x_0) \to \pi_n(X,A,x_0)$ the natural map. Then, with Convention 204.17 in mind, the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_0) \times \pi_n(X,x_0) & \longrightarrow & \pi_n(X,x_0) \\ {\scriptstyle i_*}\uparrow \quad \downarrow{\scriptstyle \varphi} & & {\scriptstyle \varphi}\downarrow \\ \pi_1(A,x_0) \times \pi_n(X,A,x_0) & \longrightarrow & \pi_n(X,A,x_0). \end{array}$$

(8) If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X,A,x_1) & \xrightarrow{[f]\mapsto[f^\gamma]} & \pi_n(X,A,x_0) \\ \downarrow{\scriptstyle \partial_n} & & {\scriptstyle \partial_n}\downarrow \\ \pi_{n-1}(A,x_1) & \xrightarrow{[f]\mapsto[f^\gamma]} & \pi_{n-1}(A,x_0). \end{array}$$

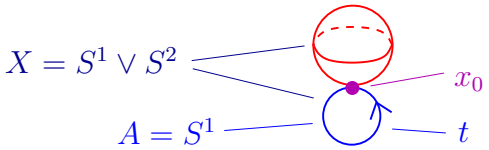
Proof. The proof of the first six statements of the proposition is basically identical to the proof of the Change-of-Base Point Proposition 169.5. The last two statements follow easily from the definitions. To protect the trees we will not carry out the details. ■

Remark. Let X be a topological space, let A be a non-empty subset and let $n \in \mathbb{N}_{\geq 2}$. If A is path-connected, then it follows from the Change-of-Base Point Proposition 296.12 (2) that the isomorphism type of the relative higher homotopy groups $\pi_n(X, A, x_0)$ does not depend on the choice of a base point in A. Thus if we are only interested in the isomorphism type of the n-th relative homotopy group of (X, A), then sometimes we just write $\pi_n(X, A)$, i.e. we suppress the base point from the notation. □

Recall that for the relative homotopy groups $\pi_n(X, A, x_0)$ with $n \geq 3$ we use the additive notation for the group structure. Thus the Group Ring–Module Lemma 97.34, the π_n-Abelian Proposition 134.11, the Relative Homotopy Group Proposition 296.5 (4) and the Change-of-Base Point Proposition 296.12 lead us to the following observation:

Observation. Let (X, A, x_0) be a pointed pair of topological spaces. If $n \geq 3$, then the abelian group $\pi_n(X, A, x_0)$ admits a natural $\mathbb{Z}[\pi_1(A, x_0)]$-left module structure.

Example. We consider the wedge $X = S^1 \vee S^2$ where we identify the point $\mathrm{i} \in S^1$ with the point $(0, 0, -1) \in S^2$. We denote by x_0 the wedge point. Furthermore let $A = S^1 \subseteq X$. We denote by $t \in \pi_1(S^1, i)$ the standard generator.



By the Wedge-π_1-Proposition 99.10 we know that the inclusion $A \to X$ induces an isomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$. We use this isomorphism to make the identification $\pi_1(A, x_0) = \pi_1(X, x_0) = \langle t \rangle$. Note that it follows from the Homotopy Groups-LES-Proposition 296.10 and the Change-of-Base Point Proposition 296.12 (7) that the natural map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules. In particular, by the $S^1 \vee S^2$–π_2–Proposition 169.9, we have

$$\pi_2(X, A, x_0) \xleftarrow{\cong} \pi_2(S^1 \vee S^2) \xrightarrow{\cong} \mathbb{Z}[t^{\pm 1}].$$ □

296.4. **Alternative description of relative homotopy groups.** Let (X, x_0) be a pointed topological space. Recall that on page 2578 we had in particular introduced the following identifications:

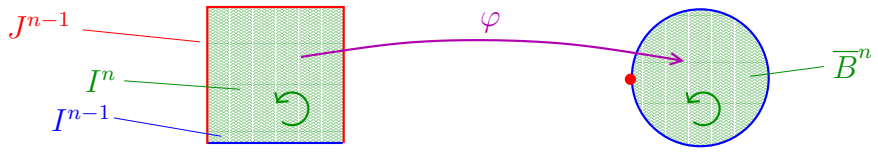
$$\pi_n(X, x_0) = \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of continuous maps}\\ (I^n, \partial I^n) \to (X, x_0)\end{array} = \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of continuous maps}\\ (S^n, *) \to (X, x_0).\end{array}$$

These identifications allowed us to go back and forth between different view points regarding homotopy groups.

[3202]Here it is worth remembering that by Definition 88.7 the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.

Before we continue with our discussion of relative homotopy groups we also want to introduce a second view point regarding relative homotopy groups. The key to doing so is the following elementary lemma:

Lemma 296.13. (Square-Ball-Homeomorphism Lemma) Let $n \in \mathbb{N}$. There exists an explicit[3203] map $\varphi\colon I^n \to \overline{B}^n$ with the following three properties:
(1) We have $\varphi(J^{n-1}) = * := (0,\dots,0,1)$ and $\varphi(I^{n-1}) = S^{n-1}$.
(2) The map φ descends to a homeomorphism $(I^n/J^{n-1}, I^{n-1}/\partial I^{n-1}) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$.
(3) The restriction of φ to the map $I^n \setminus \partial I^n \to B^n$ is an orientation-preserving diffeomorphism.



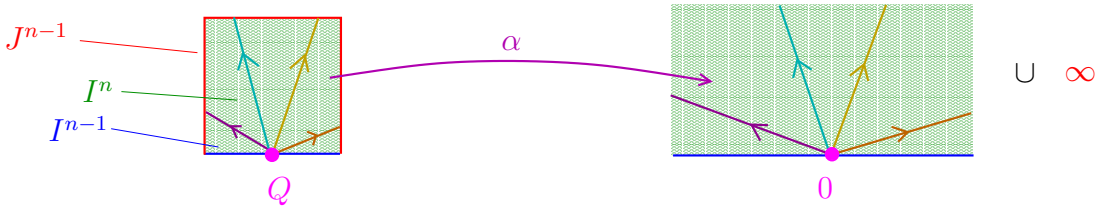
Sketch of a proof for Lemma 296.13. We write $Q = (\frac{1}{2},\dots,\frac{1}{2},0) \in I^n$. Given any point $x \in I^n \setminus \{Q\}$ we define

$$\rho(x) \;:=\; \sup\big\{\|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in I^n\big\}.$$

We consider the map

$$\begin{array}{rcl} \alpha\colon I^n/J^{n-1} & \to & \overbrace{\{(x_1,\dots,x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}}^{=H_n} \cup \{\infty\} \\ [x] & \mapsto & \begin{cases} \frac{x-Q}{\|x-Q\|} \cdot \tan\Big(\frac{\pi}{2} \cdot \frac{\|x-Q\|}{\rho(x)}\Big), & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

This map stretches each of the Euclidean segments from Q to a point on J^{n-1} to an infinite ray emanating from the origin. In particular all points in J^{n-1} get sent to infinity, i.e. the map is well-defined. We refer to the figure below for an illustration. Similar to the proof



of the Convex-to-Ball Proposition 4.28 we see that the map α is a homeomorphism that restricts to a homeomorphism $I^{n-1}/\partial I^{n-1} \to (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$. Let $\beta\colon \mathbb{R}^n \cup \{\infty\} \to S^n$ be the inverse of the stereographic projection that we introduced in the Stereographic Projection Lemma 4.24. Note that $\beta(\infty) = (0,\dots,0,1)$. Furthermore note that the homeomorphism β restricts to a homeomorphism[3204]

$$\beta\colon H^n \cup \{\infty\} \;\to\; S^n_{\geq 0} := \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_n \geq 0\}.$$

[3203]"Explicit" means that in the proof we explicitly write down such a map that does not depend on any choices. No reader will ever want to see the precise definition, but the fact that the map is explicitly given implies that there is no ambiguity involved when we write "let φ be the map from the Square-Ball-Homeomorphism Lemma 296.13."
[3204]Note that the condition $x_n \geq 0$ is not a typo.

Finally we consider the following map:

$$\begin{array}{ccccccc} I^n & \to & I^n/J^{n-1} & \xrightarrow{\alpha} & H_n \cup \{\infty\} & \xrightarrow{\beta} & S^n_{\geq 0} & \to & \overline{B}^n \\ & & & & (x_1, \ldots, x_{n-1}, x_n, x_{n+1}) & \mapsto & (x_1, \ldots, x_{n-1}, x_{n+1}). & & \end{array}$$

Using Exercise 31.8 one can easily verify that this map has all the desired properties.[3205] ■

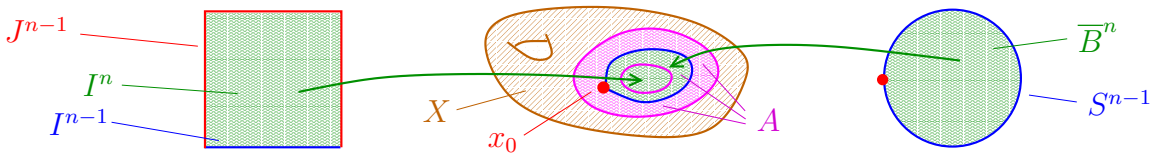
Convention 296.14. Using the map

$$\varphi\colon (I^n, \partial I^n, J^{n-1}) \to (\overline{B}^n, S^{n-1}, *)$$

from the Square-Ball-Homeomorphism Lemma 296.13 we make the following identifications

$$\pi_n(X, A, x_0) = \begin{array}{c}\text{set (of homotopy classes) of} \\ \text{maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\end{array} =_{\text{i}} \begin{array}{c}\text{set (of homotopy classes) of} \\ \text{maps } (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0).\end{array}$$

We refer to the figure below for an illustration.



Remark. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. As above we make the identifications

$$\pi_n(X, A, x_0) = \left\{ \begin{array}{c}\text{homotopy classes of} \\ \text{continuous maps} \\ (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)\end{array} \right\} \text{ and } \pi_{n-1}(A, x_0) = \left\{ \begin{array}{c}\text{homotopy classes of} \\ \text{continuous maps} \\ (S^{n-1}, *) \to (A, x_0)\end{array} \right\}.$$

We leave it to the reader to verify that in this setting the connecting homomorphism $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ from the Homotopy Groups-LES-Proposition 296.10 is given by the map

$$\begin{array}{ccc} \partial_n\colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ [f\colon (\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)] & \mapsto & [f|_{S^{n-1}}\colon (S^{n-1}, *) \to (A, x_0)]. \end{array}$$

□

296.5. **Connected (pairs of) topological spaces.** For convenience we recall Definition 167.8:

Definition 296.15. Let $k \in \mathbb{N}_0$. We say a topological space X is k-connected, if $\pi_i(X) = 0$ for $i = 0, \ldots, k$.

This definition has the following obvious "relative" analog.

Definition 296.16. Let X be a topological space, let $A \subseteq X$ and let $k \in \mathbb{N}_0$. We say that (X, A) is k-connected if every path-component of X contains a point of A and if for every $i \in \{1, \ldots, k\}$ and every $x_0 \in A$ we have $\pi_i(X, A, x_0) = 0$.

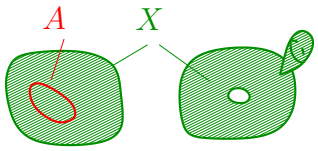
The following lemma provides a convenient reformulation of the previous definition.

[3205] More precisely, the author is somewhat optimistic that the signs are right and that the map is indeed orientation-preserving.

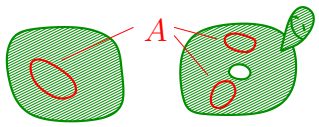
Lemma 296.17. (Pair k-Connected Characterisation Lemma) Let X be a topological space, let $A \subseteq X$ and let $k \in \mathbb{N}$. The following two statements are equivalent:

(1) The pair (X, A) is k-connected.

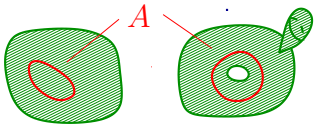
(2) The inclusion-induced map $\pi_0(A) \to \pi_0(X)$ is a bijection, furthermore for every $a \in A$ and every $j \in \{1, \ldots, k-1\}$ the inclusion-induced homomorphism $\pi_j(A, a) \to \pi_j(X, a)$ is an isomorphism and the inclusion-induced homomorphism $\pi_k(A, a) \to \pi_k(X, a)$ is an epimorphism.



(X, A) is not 0-connected



(X, A) is 0-connected but not 1-connected (for two reasons)



(X, A) is 1-connected but not 2-connected (for two reasons)

Example. Let X be a topological space and let $A \subseteq X$ be a deformation retract. First note that it follows from the elementary Homotopy-Equivalence π_0-Lemma 19.17 that the inclusion-induced map $\pi_0(A) \to \pi_0(X)$ is a bijection. Next note that it follows from the Homotopy Equivalence-π_n-Proposition 134.17 (1) that for any $x_0 \in A$ and any $i \in \mathbb{N}_0$ the inclusion-induced map $\pi_i(A, x_0) \to \pi_i(X, x_0)$ is an isomorphism. It follows from the Pair k-Connected Characterisation Lemma 296.17 that (X, A) is k-connected for every $k \in \mathbb{N}$. □

Proof. We denote by $i\colon A \to X$ the inclusion map. Let $a \in A$. By the Homotopy Groups-LES-Proposition 296.10 we have the following long exact sequence of homotopy groups (and pointed sets):

$$\ldots \to \pi_k(A, a) \xrightarrow{i_*} \pi_k(X, a) \to \pi_k(X, A, a) \xrightarrow{\partial_k} \pi_{k-1}(A, a) \xrightarrow{i_*} \pi_{k-1}(X, a) \to \ldots$$

The lemma is an immediate consequence of the existence of this long exact sequence. ■

The following proposition gives a convenient characterization of the vanishing relative homotopy groups.

Proposition 296.18. (Relative Homotopy Group–Vanishing Proposition) Let X be a topological space and let $A \subseteq X$. For any $n \in \mathbb{N}$ the following two statements are equivalent:

(1) Every continuous map $(\overline{B}^n, S^{n-1}) \to (X, A)$ is homotopic rel S^{n-1} to a map $\overline{B}^n \to A$,

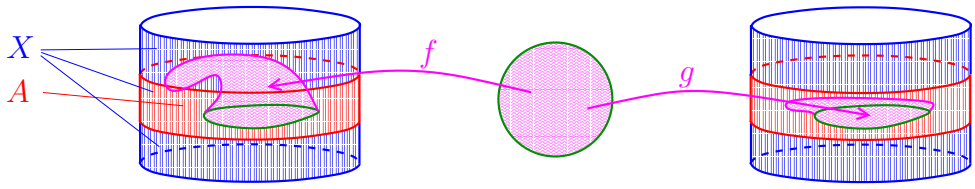
(2) for every $x_0 \in A$ we have $\pi_n(X, A, x_0) = 0$.

Furthermore, for $n = 0$ the above statement (1) is equivalent to

(2′) every path-component of X contains a point of A.

Example. We consider the infinite cylinder $X = S^1 \times \mathbb{R}$ and the annulus $A = S^1 \times [-1, 1]$. Evidently A is a deformation retract of X.
Thus, by the previous example the pair (X, A) is k-connected for every $k \in \mathbb{N}_0$. In the figure below we show a map $f\colon (\overline{B}^2, S^1) \to (X, A)$. Furthermore we show a map $g\colon \overline{B}^2 \to A$ that is homotopic, rel S^1, to f. □

Proof of Proposition 296.18. First note that it is elementary to see that for $n = 0$ the statements (1) and (2′) are equivalent. Now let $n \geq 1$. The implication (1)⇒(2)



is an immediate consequence of the Zero-in-Relative Homotopy Group Lemma 296.9 and the fact, proved in the Square-Ball-Homeomorphism Lemma 296.13, that there exists a homeomorphism $(I^n, \partial I^n) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$.

The most interesting statement of the proposition is the implication (2)⇒(1). Thus we now assume that $\pi_n(X, A, x_0) = 0$ for every $x_0 \in A$. Let $f\colon (\overline{B}^n, S^{n-1}) \to (X, A)$ be a continuous map. We need to show that f is homotopic rel S^{n-1} to a map that takes values only in A. Let $\varphi\colon I^n \to \overline{B}^n$ be the map from the Square-Ball-Homeomorphism Lemma 296.13 with $\varphi(J^{n-1}) = (0, \ldots, 0, 1)$. We set $x_0 \coloneqq f(0, \ldots, 0, 1)$. Note that the map $f \circ \varphi\colon I^n \to X$ is actually a map

$$f \circ \varphi\colon (I^n, \partial I^n, J^{n-1}) \;\to\; (X, A, \{x_0\})$$

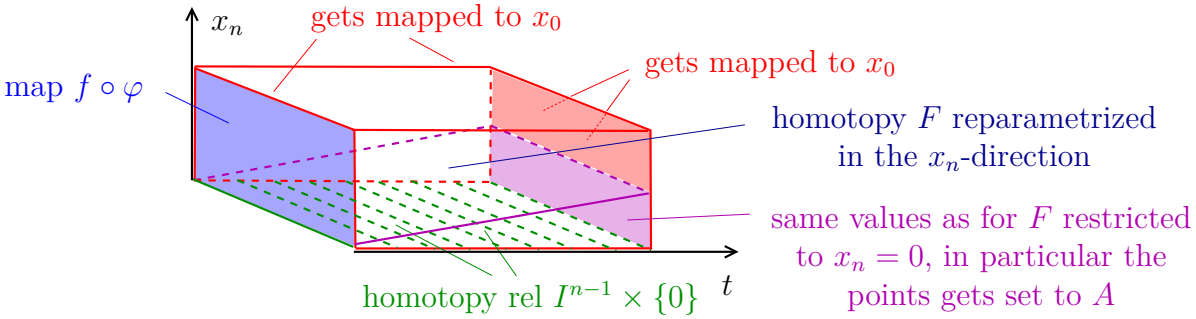
of triples. Since $\pi_n(X, A, x_0) = 0$ we see that there exists a homotopy

$$F\colon (I^n, \partial I^n, J^{n-1}) \times [0,1] \;\to\; (X, A, \{x_0\})$$

which satisfies $F_0 = f \circ \varphi$ and $F_1 = c_{x_0}$. Now we consider the map

$$\begin{aligned} I^n \times [0,1] &\to X \\ (x_1, \ldots, x_n, t) &\mapsto \begin{cases} F\big(x_1, \ldots, x_{n-1}, \frac{1}{1-\frac{t}{2}} \cdot (x_n - \frac{t}{2}), t\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ F\big(x_1, \ldots, x_{n-1}, 0, 2 \cdot x_n\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big]. \end{cases} \end{aligned}$$

The definition of the homotopy is sketched, with moderate success, in the figure below.



This map has the following properties:

(0) the map is continuous by the Homotopy Combination Lemma 18.8 (1),
(1) the map is a homotopy rel $I^{n-1} \times \{0\}$,
(2) the map is a homotopy from $f \circ \varphi$ to a map that takes values in A,
(3) the map sends $J^{n-1} \times [0,1]$ to x_0.

It follows from Property (3), and using the above map $\varphi\colon (I^n, J^{n-1}) \to (\overline{B}^n, (0, \ldots, 0, 1))$, that F defines a homotopy $G\colon \overline{B}^n \times [0,1] \to X$. Property (1) implies that G is a homotopy rel S^{n-1}. Furthermore Property (2) says that G is a homotopy from f to a map that takes values in A. Thus G is the desired a homotopy. ∎

The following proposition gives us an important example of a k-connected pair of topological spaces.

Proposition 296.19. (CW-Pair k-Connected Proposition) Let X be a CW-complex and let $k \in \mathbb{N}_0$.

(1) If A is a subcomplex such that all open cells in $X \setminus A$ have dimension greater than k, then the pair (X, A) is k-connected.
(2) The pair (X, X^k) is k-connected.[3206]

Proof.

(1) Let X be a CW-complex, let A be a subcomplex and let $k \in \mathbb{N}_0$. Suppose that all cells in $X \setminus A$ have dimension greater than k. This implies in particular that every path-component of X contains at least one point in A. The remainder of the proof is almost the same as the proof of the CW-Skeleton-π_n-Proposition 141.3. The key difference is that we need to replace the Cellular Approximation Theorem 141.2 by the Cellular Approximation Theorem 141.5 for Pairs of CW-complexes and that now we use the (1)⇒(2) implication of the Relative Homotopy Group–Vanishing Proposition 296.18. We leave it to the reader to carry out the modest modifications of the proof of the CW-Skeleton-π_n-Proposition 141.3.
(2) This statement is of course an immediate consequence of Statement (1). ■

The following corollary is an extremely useful consequence of the Pair k-Connected Characterisation Lemma 296.17 and the CW-Pair k-Connected Proposition 296.19 (2). We had proved it earlier in the guise of the CW-Skeleton-π_1-Proposition 136.22 and the CW-Skeleton-π_n-Proposition 141.3.

Corollary 296.20. (CW-Skeleton-π_n-Corollary) Let X be a CW-complex, let $k \in \mathbb{N}_0$ and let $x \in X^k$. The following two statements hold:

(1) For $j = 1, \ldots, k-1$ the inclusion-induced map $\pi_j(X^k, x) \to \pi_j(X, x)$ is an isomorphism.
(2) The inclusion-induced map $\pi_k(X^k, x) \to \pi_k(X, x)$ is an epimorphism.

296.6. **The relative Hurewicz Theorem.** In this section we state the "Relative Hurewicz Theorem" that generalizes the original Hurewicz Theorem 167.9 to the relative setting.

Definition 296.21. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. We denote by $[I^n] \in \mathrm{H}_n(I^n, \partial I^n)$ the fundamental class of I^n.[3207] We refer to the map

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & \mathrm{H}_n(X, A) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

as the relative Hurewicz homomorphism.

The following lemma summarizes some elementary properties of the relative Hurewicz homomorphism.

Lemma 296.22. (Relative Hurewicz Homomorphism Lemma)

[3206] Recall that X^k denotes the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.
[3207] Here we endow $I^n \subseteq \mathbb{R}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(1) Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. The map

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & \mathrm{H}_n(X, A) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

is well-defined and it is a homomorphism.

(2) Let (X, x_0) be a pointed topological space and let $n \geq 2$. The following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X) \\ \cong\downarrow & & \downarrow\cong \\ \pi_n(X, \{x_0\}, x_0) & \xrightarrow{\Phi_{(X,\{x_0\},x_0)}} & \mathrm{H}_n(X, \{x_0\}). \end{array}$$

Here the left-hand vertical map is given by the identification from page 5666 and the right-hand vertical map is the obvious map from absolute to relative homology groups.

(3) Let (X, A, x_0) be a pointed pair of topological spaces. The following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) & \longrightarrow \ldots \to & \pi_1(X, x_0) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) & \longrightarrow \ldots \to & \mathrm{H}_1(X). \end{array}$$

Here the horizontal sequences are the long exact sequences given by the LES-of-Pair Proposition 146.7 and by the Homotopy Groups-LES-Proposition 296.10 and the vertical maps are the Hurewicz homomorphisms.

Sketch of proof.

(1) The proof of Statement (1) is evidently just a straightforward variation on the proof of the Hurewicz Homomorphism Proposition 167.3. We refer to [**Bre1993**, Lemma VII.10.2] or [**Hat2002**, Proposition 4.36] for details.

(3) We have to show that each of the three squares commutes.
  (a) The first square commutes basically by definition, see also the Hurewicz Homomorphism Proposition 167.3 (3).
  (b) The fact that the second square commutes is an immediate consequence of the definitions of the "absolute" and the "relative" Hurewicz homomorphisms, see above and Definition 167.2.
  (c) Finally it follows fairly easily from the Codimension Zero-Fundamental Class Lemma 194.10 and the Fundamental Class-of-∂-Proposition 194.13 that the third square commutes.

(2) This is just a special case of the second square considered in (3). ■

Again the following question arises: to what degree is the relative Hurewicz homomorphism in fact an isomorphism? One might hope that if a pair (X, A) is $(n-1)$-connected, then the Hurewicz homomorphism $\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \to \mathrm{H}_n(X, A)$ is an isomorphism. But the following example shows that this is *not* the case.

Example. We consider again $X = S^1 \vee S^2$ and $A = S^1 \subseteq X$. We denote by x_0 the wedge point. It follows easily from the long exact sequence of homotopy groups provided by the

Homotopy Groups-LES-Proposition 296.10 that (X, A) is 1-connected.[3208] We see that in this case the map

$$\Phi_{(X,A,x_0)}\colon \underbrace{\pi_2(X,A,x_0)}_{\cong \mathbb{Z}^{(\mathbb{N})}, \text{ see page 5677}} \to \underbrace{\mathrm{H}_2(X,A)}_{\cong \mathbb{Z}}$$

is *not* an isomorphism. □

The following lemma is the key to understanding why the relative Hurewicz homomorphism fails to be a monomorphism.

Lemma 296.23. (Relative Hurewicz–π_1-Action Lemma) Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\sigma \in \pi_n(X, A, x_0)$. For any $g \in \pi_1(A, x_0)$ we have

$$\Phi_{(X,A,x_0)}(g \cdot \sigma) \;=\; \Phi_{(X,A,x_0)}(\sigma).$$

Proof. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $[f] \in \pi_n(X, A, x_0)$ and let $[\gamma] \in \pi_1(A, x_0)$. As in the Change-of-Base Point Proposition 169.5 (7) we see that the maps $f\colon (I^n, \partial I^{n-1}) \to (X, A)$ and $f^\gamma\colon (I^n, \partial I^{n-1}) \to (X, A)$ are homotopic. We obtain that

$$\Phi_{(X,A,x_0)}([\gamma]\cdot[f]) \;=\; \Phi_{(X,A,x_0)}([f^\gamma]) \;=\; (f^\gamma)_*([I^n]) \underset{\uparrow}{=} f_*([I^n]) \;=\; \Phi_{(X,A,x_0)}([f]).$$

by the Homotopy-H_*-Proposition 146.11, since f and f^γ are homotopic as maps $(I^n, \partial I^n) \to (X, A)$ ■

The above lemma leads us to the following definition:

Definition 296.24. Let (X, A, x_0) be a pointed pair of topological spaces.[3209]

(1) For $n \in \mathbb{N}_{\geq 3}$ we define $\pi'_n(X, A, x_0)$ to be the quotient group[3210]

$$\pi'_n(X,A,x_0) \;\coloneqq\; \pi_n(X,A,x_0)\big/\langle\{g\cdot\sigma-\sigma \,|\, g\in\pi_1(A,x_0) \text{ and } \sigma\in\pi_n(X,A,x_0)\}\rangle.$$

(2) We define[3211]

$$\pi'_2(X,A,x_0) \;\coloneqq\; \pi_2(X,A,x_0)\big/\langle\langle\{(g\cdot\sigma)\cdot\sigma^{-1} \,|\, g\in\pi_1(A,x_0), \sigma\in\pi_2(X,A,x_0)\}\rangle\rangle.$$

(3) By the Relative Hurewicz–π_1-Action Lemma 296.23 the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)}\colon \pi_n(X,A,x_0) \;\to\; \mathrm{H}_n(X,A)$$

descends to a homomorphism

$$\Phi_{(X,A,x_0)}\colon \pi'_n(X,A,x_0) \;\to\; \mathrm{H}_n(X,A).$$

We refer to this homomorphism again as the Hurewicz homomorphism.

[3208] Alternatively we could view X as a CW-complex in the usual way. Then we obtain from the CW-Pair k-Connected Proposition 296.19 that the pair (X, A) is 1-connected.

[3210] Here we use the common, but sometimes confusing, convention that we write the group structure on the abelian group $\pi_n(X, A, x_0)$, $n \geq 3$, additively whereas we write the group structure on $\pi_2(X, A, x_0)$ multiplicatively. Evidently there is no rule without an exception, hence, if $\pi_2(X, A, x_0)$ happens to be abelian after all, then we also like to use additive notation for $\pi_2(X, A, x_0)$.

[3211] Recall that given a group π and a subset $X \subseteq \pi$ we denote by $\langle\langle X\rangle\rangle$ the subgroup of π normally generated by X. We refer to Definition 98.20 for details.

Remark. Let (X, A, x_0) be a pointed pair of topological spaces. It is a fun exercise to show "by hand" that $\pi_2'(X, A, x_0)$ is an abelian group. We refer to [**Hat2002**, Lemma 4.39] for details. □

Example. Once again we consider $X = S^1 \vee S^2$ and $A = S^1 \subseteq X$. We denote by x_0 the wedge point. As usual we write $\pi_1(A, x_0) = \langle t \rangle$. On page 5677 we already saw that $\pi_2(X, A, x_0) \cong \mathbb{Z}[t^{\pm 1}]$ as $\mathbb{Z}[\pi_1(A, x_0)] = \mathbb{Z}[t^{\pm 1}]$-modules. Thus we have an isomorphism

$$\pi_2'(X, A, x_0) \;\cong\; \mathbb{Z}[t^{\pm 1}]/\{(t^k - 1) \cdot f \mid f \in \mathbb{Z}[t^{\pm 1}] \text{ and } k \in \mathbb{Z}\}.$$

It is an amusing algebra exercise to show that the group on the right-hand side is isomorphic to $\mathbb{Z}$. □

The following theorem is a generalization of the Hurewicz Theorem 167.9 to the relative case.

Theorem 296.25. (Relative Hurewicz Theorem) Let (X, A, x_0) be a pointed pair of topological spaces such that X and A are path-connected. Let $n \geq 2$. If the pair (X, A) is $(n-1)$-connected, then the following two statements hold:

(1) We have $\mathrm{H}_k(X, A) = 0$ for $k = 1, \ldots, n-1$.

(2) The Hurewicz homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi_n'(X, A, x_0) \;\to\; \mathrm{H}_n(X, A)$$

is an isomorphism. In particular, if $\pi_1(A, x_0) = 0$, then the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \;\to\; \mathrm{H}_n(X, A)$$

is an isomorphism.

Proof. We will not provide a proof of the Relative Hurewicz Theorem. A full proof is given in [**Bre1993**, Theorem VII.10.7] or alternatively in [**Hat2002**, Theorem 4.37] and [**Spa1995**, Proposition 7.5.1]. If A is simply connected, then it is a great exercise to figure out to what degree the proof of the Hurewicz Theorem 167.9 can be modified to deal with the relative case. ■

The following theorem, which can be viewed as a corollary to the Relative Hurewicz Theorem 296.25, is a nice illustration of the usefulness of relative homotopy groups.

Theorem 296.26. (Homology–Homotopy-Isomorphism Theorem) Let X and Y be two topological spaces and furthermore let $f \colon X \to Y$ be a continuous map. We pick a base point $x_0 \in X$ and we write $y_0 = f(x_0)$.

(1) If for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \pi_i(X, x_0) \;\xrightarrow{\cong}\; \pi_i(Y, y_0)$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \mathrm{H}_i(X) \;\xrightarrow{\cong}\; \mathrm{H}_i(Y)$$

is also an isomorphism.

(2) If X and Y are simply connected, then the converse to (1) holds. More precisely, in this case, if for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \mathrm{H}_i(X) \;\xrightarrow{\cong}\; \mathrm{H}_i(Y)$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

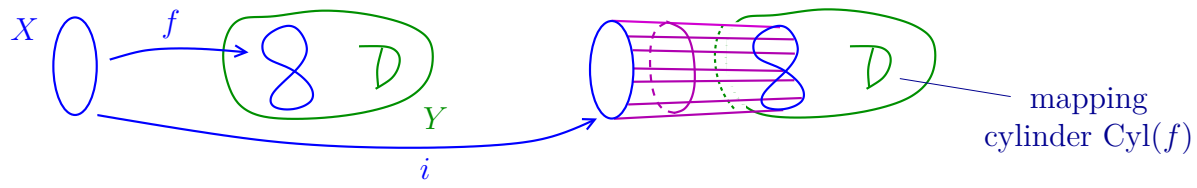
$$f_*\colon \pi_i(X, x_0) \xrightarrow{\cong} \pi_i(Y, y_0)$$

is also an isomorphism.

Remark. In some textbooks, see e.g. [**Bre1993**, p. 481], [**DaK2001**, p. 162] and [**Spa1995**, p. 399], this theorem is referred to as the "Whitehead Theorem". We will reserve the name "Whitehead Theorem" for a much more interesting later result, namely *the* Whitehead Theorem 305.1. □

Proof. The idea is to "replace" the map $f\colon X \to Y$ by an inclusion map so that we can apply the commutative diagram from the Relative Hurewicz Homomorphism Lemma 296.22. The key to doing so is the notion of the mapping cylinder that we introduced in Definition 21.1. Recall that the mapping cylinder corresponding to f is defined as the topological space

$$\mathrm{Cyl}(f) \coloneqq ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$



We consider the maps

$$i\colon X \to \mathrm{Cyl}(f),\ x \mapsto (x,0) \quad \text{respectively} \quad r\colon \mathrm{Cyl}(f) \to Y,\ P \mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases}$$

Note that we have $r \circ i = f\colon X \to Y$. Also note that in the Mapping Cylinder–Homotopy Lemma 21.5 (2a) we showed that the retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. We use the inclusion $i\colon X \to \mathrm{Cyl}(f)$ to identify X with $i(X)$, i.e. we view X as a subset of $\mathrm{Cyl}(f)$.

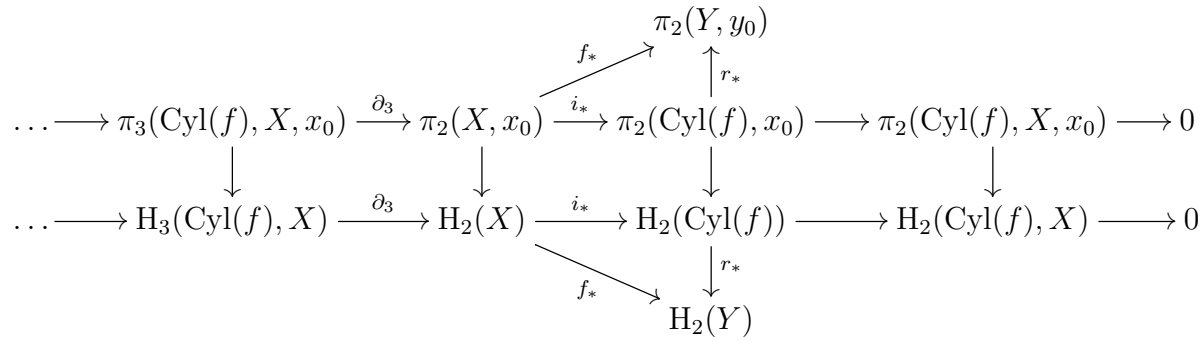
In the following we prove Statement (2) of the theorem. Thus we assume that X and Y are simply connected and we assume that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \mathrm{H}_i(X) \xrightarrow{\cong} \mathrm{H}_i(Y)$$

is an isomorphism. We need to show that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \pi_i(X, x_0) \xrightarrow{\cong} \pi_i(Y, y_0)$$

is also an isomorphism. We consider the following maps:



where the vertical maps between the second and the third row are the Hurewicz homomorphisms. The desired statement follows from the following observations:

(a) The triangles commute since $f = r \circ i$. Furthermore the squares commute by the Relative Hurewicz Homomorphism Lemma 296.22.

(b) By our hypothesis the maps $f_*\colon \mathrm{H}_*(X) \to \mathrm{H}_*(Y)$ are isomorphisms. Furthermore, since r is a deformation retraction we obtain from the Homotopy Equivalence-H_*-Proposition 144.14 (1b) that the maps $r_*\colon \mathrm{H}_*(\mathrm{Cyl}(f)) \to \mathrm{H}_*(Y)$ are isomorphisms.

(c) Since $f_* = (r \circ i)_* = r_* \circ i_*$ we obtain immediately from our result in (b) that the maps $i_*\colon \mathrm{H}_*(X) \to \mathrm{H}_*(\mathrm{Cyl}(f))$ are also isomorphisms.

(d) It follows from (c) and the bottom long exact sequence that all the relative homology groups $\mathrm{H}_*(\mathrm{Cyl}(f), X)$ vanishes.

(e) It follows easily from (d), the hypothesis that X is simply connected and the Relative Hurewicz Theorem 296.25 that all the relative homotopy groups $\pi_*(\mathrm{Cyl}(f), X, x_0)$ vanish.

(f) Using (e) we obtain immediately from the top long exact sequence that the maps $i_*\colon \pi_*(X, x_0) \to \pi_*(\mathrm{Cyl}(f), x_0)$ are isomorphisms.

(g) By the Homotopy Equivalence-π_n-Proposition 134.17 (1) we know that the induced maps $r_*\colon \pi_*(\mathrm{Cyl}(f), x_0) \to \pi_*(Y, y_0)$ are isomorphisms.

(h) Finally note that it follows from $f_* = r_* \circ i_*$ together with (f) and (g) that the maps $f_*\colon \pi_*(X, x_0) \to \pi_*(Y, y_0)$ are indeed isomorphisms.

The proof of Statement (1) is very similar to the above argument. We leave it to the reader to fill in the details. ■

We conclude this section with a discussion of homotopy spheres. Recall that by Definition 210.19 a homotopy n-sphere Σ is a closed n-dimensional topological manifold that is $(n-1)$-connected, i.e. the homotopy groups of Σ agree with the homotopy groups of S^n up to degree $n-1$.

The following pleasant corollary shows that in fact all homotopy groups of Σ are isomorphic to the corresponding homotopy groups of S^n.

Corollary 296.27. (Homotopy Spheres-π_*-Corollary) Let $n \in \mathbb{N}_{\geq 2}$. If Σ is a homotopy n-sphere, then for every $i \in \mathbb{N}_0$ we have $\pi_i(\Sigma) \cong \pi_i(S^n)$.

Proof. Let Σ be a homotopy n-sphere. Note that by the Homology–Homotopy-Isomorphism Theorem 296.26 (2) it suffices to find a continuous map $f\colon S^n \to \Sigma$ such that for every $i \in \mathbb{N}_{i \geq 2}$ the induced map $f_*\colon \mathrm{H}_i(S^n) \to \mathrm{H}_i(\Sigma)$ is an isomorphism.

So let us try to find such a map f. We pick a base point $x_0 \in \Sigma$. By the definition and the Homotopy Sphere-Characterization Proposition Proposition **??** we know that Σ is a closed connected non-empty n-dimensional topological manifold with $\mathrm{H}_i(\Sigma) \cong \pi_i(\Sigma, x_0) = 0$ for $i = 1, \dots, n-1$. It follows from the Hurewicz Theorem 167.9 that the Hurewicz homomorphism

$$\begin{array}{rcl} \Phi\colon \pi_n(\Sigma, x_0) & \to & \mathrm{H}_n(\Sigma) \\ [f\colon (S^n, *) \to (\Sigma, x_0)] & \mapsto & f_*([S^n]) \end{array}$$

is an isomorphism. Note that the fact that $\pi_1(\Sigma, x_0) = 0$ together with the Non-Orientable $\mathbb{Z}/2\mathbb{Z}$-Epimorphism Corollary 191.32 implies that Σ is orientable. Therefore by the Orientation-Top-H_*-Theorem 193.3 we know that $\mathrm{H}_n(\Sigma) \cong \mathbb{Z}$. Since the Hurewicz homomorphism is an isomorphism we can pick a continuous map $f\colon S^n \to \Sigma$ such that $f_*([S^n]) \in \mathrm{H}_n(\Sigma) \cong \mathbb{Z}$ is a generator.

We claim that f has the desired properties. By choice of f we know that the map $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(\Sigma)$ is an epimorphism. But since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism. Since S^n and Σ are path-connected we obtain from the H_0-Proposition 143.25 that $f_*\colon \mathrm{H}_0(S^n) \to \mathrm{H}_0(\Sigma)$ is also an isomorphism. Furthermore for all other dimensions f_* induces an isomorphism of homology groups for the rather banal reasons that all other homology groups of S^n and Σ are zero. (Indeed, we saw above that the relative homology groups in dimension less than n vanish and it follows from the Manifold-Top-H_*-Theorem 193.1 that the homology groups vanish in dimension greater than n.) ∎

296.7. **Excision for relative homotopy groups?** We have now seen that the (relative) homotopy groups and (relative) homology groups have some properties in common:

(1) both invariants are functorial,
(2) in both cases homotopic maps induce the same maps on the invariants,
(3) a (pointed) pair of topological spaces gives rise to a long exact sequence.

These formal similarities between (relative) homotopy and (relative) homology groups makes one wonder whether an analog of the Excision Theorem 148.1 holds for relative homotopy groups.

For better or worse this is not the case. More precisely, the next lemma shows that a naive analog of the Excision Theorem 148.1 fails even in very simple situations.

Lemma 296.28. (π_n-No Excision Lemma) In the following we consider the 2-dimensional sphere $X = S^2$ together with the subset $A := S^2_{\leq 0} = \{(x, y, z) \in S^2 \mid z \leq 0\}$ given by the lower hemisphere and together with the subset $Z = \{(0, 0, -1)\}$ that is given by the South Pole. Then[3212]

$$\pi_3(X, A) \ncong \pi_3(X \setminus Z, A \setminus Z).$$

Proof. We start out with the following observations:

(1) On page 183 we saw that the lower hemisphere $A = S^2_{\leq 0}$ is homeomorphic to $\overline{B}^2$. In particular A is contractible which implies by the Homotopy Equivalence-π_n-Proposition 134.17 (2) that $\pi_2(A) = \pi_3(A) = 0$.

[3212]Here evidently the closure of Z is contained in the interior of A, i.e. the hypotheses for the Excision Theorem 148.1 for homology groups are satisfied.



(2) The above homeomorphism also shows that $A \setminus Z$ is homeomorphic to $\overline{B}^2 \setminus \{(0)\}$ which is homotopy equivalent to S^1. This implies that $\pi_2(A \setminus Z) = \pi_3(A \setminus Z) = 0$.
(3) Finally in the Stereographic Projection Lemma 4.24 we saw that $X \setminus Z = S^2 \setminus \{(0,0,-1)\}$ is homeomorphic to $\mathbb{R}^2$ which implies that $\pi_2(X \setminus Z) = \pi_3(X \setminus Z) = 0$.

Note that it follows from these observations and the long exact sequence of homotopy groups, provided by the Homotopy Groups-LES-Proposition 296.10, that we have isomorphisms $\pi_3(X \setminus Z, A \setminus Z) \cong \pi_2(A \setminus Z) \cong \pi_2(S^1) = 0$ and furthermore that $\pi_3(X, A) \cong \pi_3(S^2)$. But in the Hopf Map–Homotopy Theorem 216.6 we saw that $\pi_3(S^2)$ is non-trivial. We have thus shown that $\pi_3(X, A)$ is not isomorphic to $\pi_3(X \setminus Z, A \setminus Z)$. ■

Remark. We just used the fact that $\pi_3(S^2) \neq 0$ to show that relative homotopy groups do not satisfy excision. It is a good moment to remind ourselves of the Eilenberg-Steenrod Uniqueness Theorem 160.19. The theorem suggests that if relative homotopy groups did satisfy excision, then it would follow from the Homotopy Groups-LES-Proposition 296.10 that, at least morally, (abelianized) homotopy groups would agree with homology groups.[3213] But the calculations $\pi_3(S^2) \neq 0$ and $\mathrm{H}_3(S^2) = 0$ show that this is not the case. □

In much more restricted circumstances there is in fact an excision theorem. For the record we state the following theorem. We will not make use of it and therefore we will not provide the proof (which by all accounts seems to be somewhat messy.)

Theorem 296.29. (Blakers-Massey Theorem)
(1) Let X be a topological space and let A and B be subsets such that $X = \mathring{A} \cup \mathring{B}$. If $(A, A \cap B)$ is n-connected and if $(B, A \cap B)$ is m-connected, then the inclusion-induced map
$$\pi_k(A, A \cap B) \;\to\; \pi_k(X, B)$$
is an isomorphism for $k < m + n$ and it is an epimorphism for $k = m + n$.
(2) The conclusion of (1) also holds if X is a CW-complex and A and B are two subcomplexes of X with $X = A \cup B$ such that $A \cap B$ is non-empty and connected.

Proof. Statement (1) was originally proved in 1952 by Albert Blakers and William Massey [**BlM1952**]. Proofs can also be found in [**GrayB1975**, Corollary 16.27], [**tD2008**, Theorem 6.4.1], [**tDKP1970**, p. 211] or [**WhdG1978**, Chapter VII.7]. Statement (2) can presumably be deduced from Statement (1). Alternatively see [**Hat2002**, Theorem 4.23]. ■

296.8. Exercises for Chapter 296.

Exercise 296.1. Show that the three paths to the right of the figure on page 5666 define three different elements in $\pi_1(X, A, x_0)$.

[3213] To turn this logic into a proper argument might be a little tricky because of base point issues.

Exercise 296.2. Let (X, A, x_0) be a pointed pair of topological spaces. Show that if $n \geq 3$, then the relative homotopy group $\pi_n(X, A, x_0)$ is abelian.

Exercise 296.3. Let X be a topological space and $x_0 \in X$. We consider the set $\pi_0(X, x_0)$. We all know that $\pi_0(X, x_0)$ is "not a group". But why not? After all, by [**HK1973**, p. 340] it follows from the Axiom of Choice 1.1 that every non-empty set can be equipped with a group structure So why is not possible to equip $\pi_0(X, x_0)$ with a group structure in a "meaningful way"? More precisely, if we could equip it with a group structure in a reasonable sense, what properties would we demand? Why can these not be satisfied?

Exercise 296.4. Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence

$$\{e\} \ \to \ G \ \xrightarrow{\varphi} \ H \ \xrightarrow{\psi} \ (S, s_0) \ \to \ \{e\}$$

where ψ is a morphism of pointed sets that is H-equivariant, i.e. for any $h, h' \in H$ we have $\psi(h \cdot h') = h \cdot \psi(h')$. Show that $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.
Remark. This exercise is identical to Exercise 239.5.

Exercise 296.5. Is a monomorphism of pointed sets necessarily injective?

Exercise 296.6. Provide a proof for the Zero-in-Relative Homotopy Group Lemma 296.9.

Exercise 296.7. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. By the convention of page 5679 we can view elements in $\pi_n(X, A, x_0)$ as homotopy classes of continuous maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, \{x_0\})$. Using this point of view, how can we visualize the action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$?

Exercise 296.8. Give an example of a connected CW-complex X, a connected subcomplex $A \subseteq X$ and an $n \in \mathbb{N}$ such that the homotopy group $\pi_n(X/A)$ is not isomorphic to the relative homotopy group $\pi_n(X, A)$.

CHAPTER 297

Fibrations

In this chapter we introduce fibrations and we show that fibrations give rise to a long exact sequence of homotopy groups. In the subsequent two chapters we will give many more examples of fibrations and we will see that fibrations allow us to compute various interesting higher homotopy groups.

This chapter is organized as follows:

- In Section 297.1 we will introduce fibrations and we will discuss some basic examples.
- In Section 297.2 we will introduce the homotopy fiber of a fibration.
- In Section 297.3 we will formulate the Fibration-LES Theorem 297.11 which says hat a Serre fibration gives rise to a long exact sequence of homotopy groups.
- In Section 297.4 we will prove the Fibration-LES Theorem 297.11.
- In Section 297.5 we will introduce the path space fibration.
- In Section 297.6 we will see that every continuous map can be replaced by a fibration.
- In Section 297.7 we will introduce adjoint functors.
- In Section 297.8 we will provide exercises for Chapter 297.

In the next Chapter 298 we will (re-) introduce fiber bundles and we will see that they are in particular fibrations.

297.1. **Definition and examples of fibrations.** The following is the key definition of this chapter.

Definition 297.1. Let $p\colon Y \to B$ be a continuous map between topological spaces.

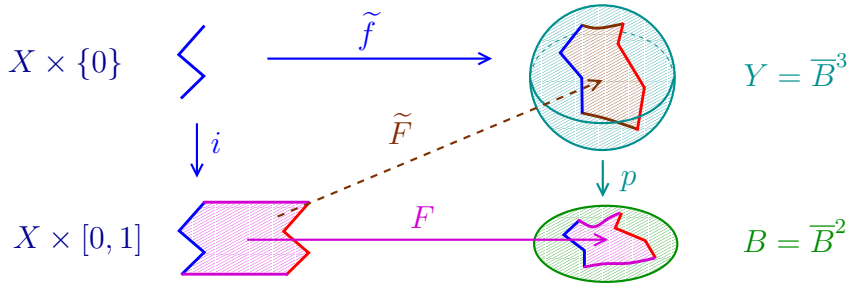
(1) Let X be a topological space. We say that the map $p\colon Y \to B$ has the homotopy lifting property with respect to X if given any continuous map $F\colon X \times [0,1] \to B$ and given any continuous map $\widetilde{f}\colon X \times \{0\} \to Y$ with $p \circ \widetilde{f} = F \circ i$ there exists a continuous map $\widetilde{F}\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\widetilde{f}} & Y \\ \downarrow{\scriptstyle i} & \overset{\widetilde{F}}{\nearrow ??} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[F]{} & B. \end{array}$$

Sometimes we refer to $\widetilde{F}\colon X \times [0,1] \to Y$ as a lift of $F\colon X \times [0,1] \to B$. (Note that we do *not* demand that the lift $\widetilde{F}$ is in any way unique.) We refer to the figure below for an illustration of the definition.

(2) We say that the map $p\colon Y \to B$ is a fibration if p has the homotopy lifting property with respect to all topological spaces.

(3) We say that the map $p\colon Y \to B$ is a Serre fibration if p has the homotopy lifting property with respect to all closed balls $\overline{B}^n$, $n \in \mathbb{N}_0$. So perhaps somewhat confusingly a Serre fibration is not necessarily a fibration.



Remark. Different textbooks mean different objects when they talk about "fibrations". Our definition coincides with the language used in [**Hat2002, Rot1988, DaK2001**]. In [**Bre1993**], what we call a "fibration" is called a "Hurewicz fibration".

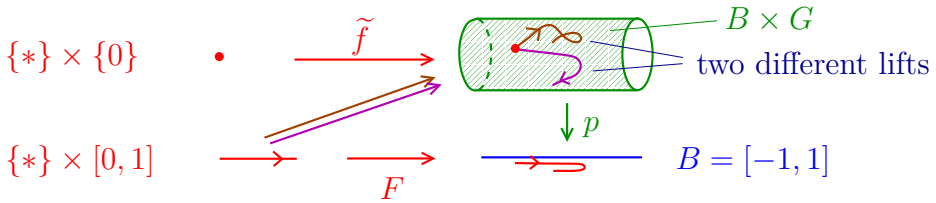
Examples.

(1) As we saw in Exercise 127.6, it follows easily from the Map Lifting Criterion 127.5 that every covering $p\colon Y \to B$ has the homotopy lifting property (in fact, by the Lift Uniqueness Proposition 92.2, with a unique lift) with respect to any closed ball $\overline{B}^n$. This implies in particular that every covering is a Serre fibration.

(2) Let B and G be topological spaces. We denote by $p\colon B\times G \to B$ and $q\colon B\times G \to G$ the two obvious projections. In the following we show that the projection $p\colon B \times G \to B$ is a fibration. Indeed, given any continuous map $F\colon X \times [0,1] \to B$ and given any continuous map $\widetilde{f}\colon X \times \{0\} \to B \times G$ the map

$$\begin{array}{rcl} \widetilde{F}\colon X \times [0,1] & \to & B \times G \\ (x,t) & \mapsto & (F(x,t), q(\underbrace{\widetilde{f}(x,0)}_{\in B\times G})) \end{array}$$

makes the diagram commute and it follows easily from the Topological-Product Proposition 7.2 that this map is continuous. As is shown in the figure below, in contrast to the first example, the lift $\widetilde{F}$ is in general not unique.



(3) For every $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & \overline{B}^n \\ (x_1, \ldots, x_n, x_{n+1}) & \mapsto & (x_1, \ldots, x_n) \end{array}$$

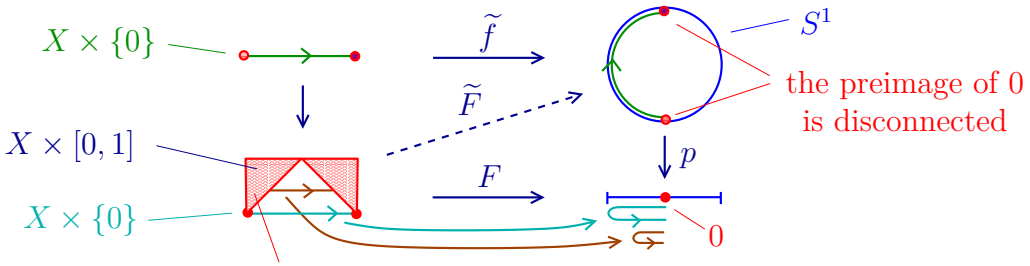
is a Serre fibration. With some effort this statement can be proved "by hand". The case $n = 1$ is precisely the content of Exercise 297.1. □

We conclude the list of examples with a "non-example". To make this example more readable we package it is as a lemma.

Lemma 297.2. (Non-Serre Fibration Lemma) The map $p\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate is not a Serre fibration (let alone a fibration).

Proof. We consider the topological space $X = \overline{B}^1 = [-1,1]$ together with the maps $\widetilde{f}\colon X \times \{0\} \to S^1$ and $F\colon [-1,1] \times [0,1] \to [-1,1]$ that are illustrated in the figure below. We claim that there is no lifting $\widetilde{F}\colon X \times [0,1] \to S^1$. More precisely, we prove the following claim:

Claim. If $\widetilde{F}\colon X \times [0,1] \to S^1$ is a continuous map with $p \circ \widetilde{F} = F$, then $\widetilde{F}|_{X\times\{0\}}$ does not agree with $\widetilde{f}$.



Proof. Thus let $\widetilde{F}\colon X \times [0,1] \to S^1$ be a continuous map with $p \circ \widetilde{F} = F$. We make the following observations:

(1) The preimage $F^{-1}(0)$ is connected by the definition of F.
(2) We obtain from (1) and the Image-Connected Lemma 4.34 that the image $\widetilde{F}(F^{-1}(0))$ is also connected.
(3) The fact that $p \circ \widetilde{F} = F$ implies that $\widetilde{F}(F^{-1}(0)) \subseteq p^{-1}(0)$.[3214]
(4) By definition of the map p we have $p^{-1}(0) = \{(0, \pm 1)\}$.
(5) It follows from (2), (3) and (4) that $\widetilde{F}$ restricted to $F^{-1}(0)$ is constant.
(6) The map $\widetilde{f}$ is not constant on $F^{-1}(0) \cap (X \times \{0\})$.

The combination of (5) and (6) proves our claim. ∎

Since we distinguish between a "fibration" and a "Serre fibration", the reader will have an inkling, that the two concepts are most likely not the same. In fact many such examples are given in [**BrownR1966**, p. 575] and [**StW1984**, p. 573] (see also Exercise 297.2). On the other hand, the following theorem shows that in many reasonable settings the two concepts turn out to be the same:

Theorem 297.3. (Serre Fibration–Fibration Theorem) Let $p\colon Y \to B$ be a Serre fibration such that Y and B are CW-complexes.

(1) The map $p\colon Y \to B$ has the homotopy lifting property with respect to all compactly generated[3215] topological spaces X.
(2) If the product $Y \times B$ is compactly generated, then $p\colon Y \to B$ is a fibration.

[3214]Since this particular authors always find these arguments confusing let us give the argument: Let $y \in \widetilde{F}(F^{-1}(0))$. This means that there exists a $P \in F^{-1}(0)$ with $y = \widetilde{F}(P)$. We have $p(y) = p(\widetilde{F}(P)) = F(P)$, but $P \in F^{-1}(0)$, thus $p(y) = 0$. In other words, we have $y \in p^{-1}(0)$.

[3215]Compactly generated topological spaces were introduced in Definition 139.30. As an example, in the CW-CGWH Proposition 139.31 we showed that CW-complexes are compactly generated.

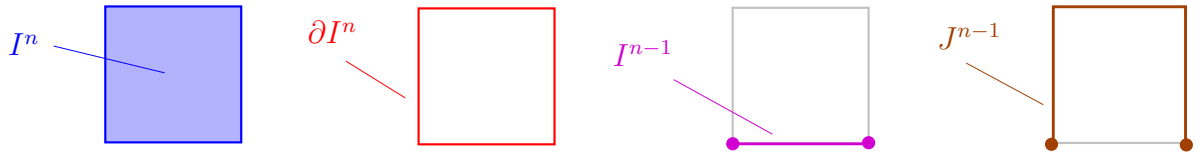
Proof. Since we will not make use of this theorem, we refer to [**StW1984**, Theorem 1] and [**Cau1992**, p. 5] for a proof. ■

We continue with introducing some language and notation.

Notation 297.4.

(1) Given a fibration or a Serre fibration $p\colon Y \to B$ we sometimes refer to B as the base space and to Y as the total space.[3216] Furthermore, given a point $* \in B$ we refer to $F := p^{-1}(*)$ as a fiber. The last definition is somewhat dangerous since, as we see in the last of the above examples, the homeomorphism type $p^{-1}(*)$ depends on the choice of $*$. We will shortly discuss "uniqueness" of the fiber. Finally throughout this chapter we denote by $i\colon F \to Y$ the inclusion map.

(2) Given $n \in \mathbb{N}_0$ we define I, I^n and J^n as in Notation 296.1 and as indicated in the figure below.



Before we can formulate the main theorem of this chapter we need to make some preparations.

Auxiliary Lemma 297.5.

(1) For each $n \in \mathbb{N}$ there exists an orientation-preserving homeomorphism

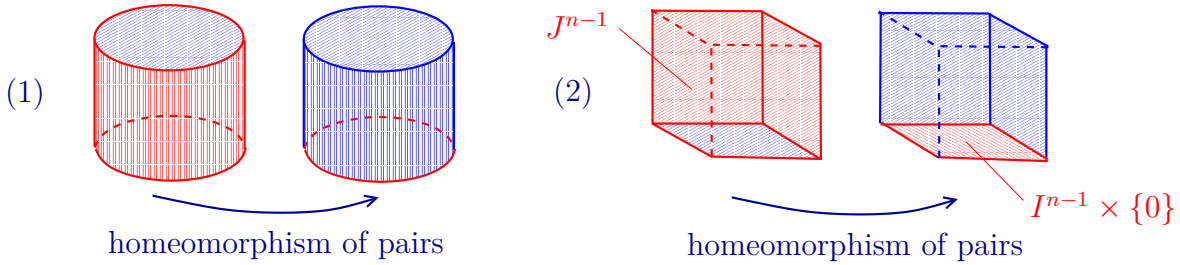
$$(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1])) \xrightarrow{\cong} (\overline{B}^n \times [0,1], \overline{B}^n \times \{0\})$$

of pairs of topological spaces.

(2) For each $n \in \mathbb{N}$ there exists a homeomorphism

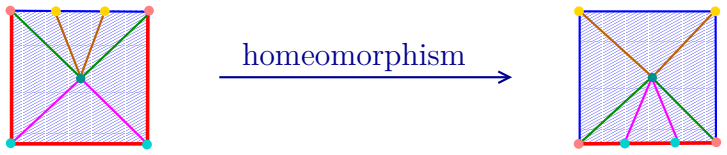
$$(I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\}).$$

of pairs of topological spaces.



Sketch of proof.

(1) For $n = 1$ such a homeomorphism is illustrated in the figure below.



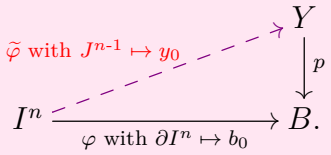
[3216] This mirrors the language that we introduced in Definition 265.1 for vector bundles.

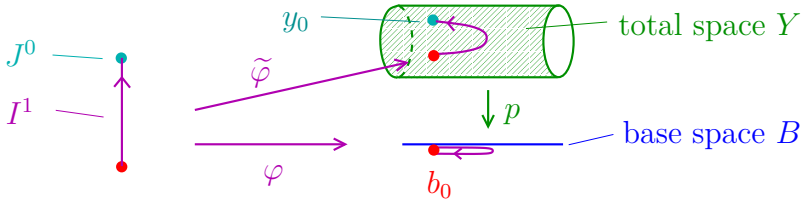
The general case is given by "rotating this homeomorphism around the $x_1 = 0$-axis in $\mathbb{R}^n$".

(2) Using the homeomorphism $\overline{B}^{n-1} \cong I^{n-1}$ from the Convex-to-Ball Proposition 4.28 (2) one can easily reduce the proof of (2) to the statement that we had just proved in Statement (1). ∎

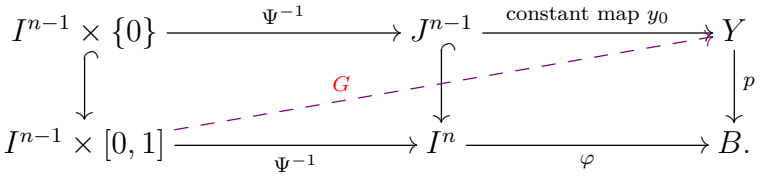
We will also soon need the following elementary lemma:

Auxiliary Lemma 297.6. Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Furthermore let $n \in \mathbb{N}$. Given any continuous map $\varphi\colon (I^n, \partial I^n) \to (B, \{b_0\})$ there exists a continuous map $\widetilde{\varphi}\colon (I^n, J^{n-1}) \to (Y, \{y_0\})$ such that the following diagram commutes:





Proof. Let $\Psi\colon (I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\})$ be a homeomorphism as in Lemma 297.5 (2). We consider the diagram



Note that the diagram without the red dashed arrow commutes. Since p is a Serre fibration and since I^{n-1} is homeomorphic to the $(n-1)$-ball, see the Convex-to-Ball Proposition 4.28 (2), it follows that a map $G\colon I^{n-1} \times I \to Y$ exists that makes the diagram commute. It is now straightforward to verify that $G \circ \Psi\colon I^n \to B$ has the desired properties. ∎

297.2. **The homotopy fiber of a fibration.** First we recall the following notation from Definition 19.11.

Notation 297.7. Given two topological spaces X and Y we define $[X, Y]$ to be the set of homotopy equivalence classes of continuous maps from X to Y.[3217]

Remark. Let X, Y and Z be topological spaces. Recall that it follows from the Homotopy Composition Lemma 18.10 (2) that the map

$$\begin{array}{rcl} [Y, Z] \times [X, Y] & \mapsto & [X, Z] \\ ([g], [f]) & \mapsto & [g \circ f] \end{array}$$

[3217]Recall that in the Homotopy-Equivalence Relation Lemma 18.9 we saw that "being homotopic" is an equivalence relation.

is well-defined. □

Now we can formulate the key technical proposition of this section.

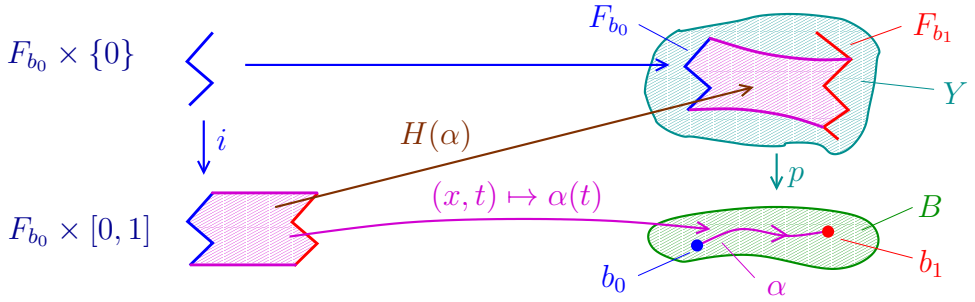
Proposition 297.8. (Fiber Homotopy Proposition) Let $p\colon Y \to B$ be a fibration. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram:

$$\begin{array}{ccc} F_{b_0} \times \{0\} & \xrightarrow{(x,0)\mapsto x} & Y \\ \downarrow{\scriptstyle i} & \overset{H(\alpha)}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{b_0} \times [0,1] & \xrightarrow[(x,t)\mapsto \alpha(t)]{} & B. \end{array}$$

Since $p\colon Y \to B$ is a fibration there exists a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$.[3218] (We refer to the figure below for an illustration.) We define $\Theta(\alpha) := H(\alpha)_1$. The following statements hold:

(1) The map $\Theta(\alpha)$ goes from F_{b_0} to F_{b_1}.
(2) The map $\Theta(\alpha)$, viewed as an element in $[F_{b_0}, F_{b_1}]$, is well-defined.
(3) The map $\Theta(\alpha)\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence.
(4) If two paths $\alpha, \beta\colon [0,1] \to B$ from b_0 to b_1 are path-homotopic, then $[\Theta(\alpha)] = [\Theta(\beta)]$.
(5) If c_b is the constant path at $b \in B$, then $[\Theta(c_b)] = [\mathrm{id}_{F_b}] \in [F_b, F_b]$.
(6) If α is a path in B from b_0 to b_1 and β is a bath in B from b_1 to b_2, then

$$\underbrace{[\Theta(\alpha * \beta)]}_{\in [F_{b_0}, F_{b_2}]} = \underbrace{[\Theta(\beta)]}_{\in [F_{b_1}, F_{b_2}]} \circ \underbrace{[\Theta(\alpha)]}_{\in [F_{b_0}, F_{b_1}]}.$$



Remark. The picture behind the statement of the Fiber Homotopy Proposition 297.8 is that while walking along B we can drag the fiber along. Thus any path defines a map between the corresponding fibers. □

Proof. Let $p\colon Y \to B$ be a fibration over a topological space B. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the

[3218] Recall that in general $H(\alpha)$ is not uniquely determined by the data, i.e. there might be many such maps.

following diagram:

$$\begin{array}{ccc} F_{b_0} \times \{0\} & \xrightarrow{(p,0)\mapsto p} & Y \\ \downarrow{\scriptstyle i} & \overset{H(\alpha)}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{b_0} \times [0,1] & \xrightarrow[(x,t)\mapsto\alpha(t)]{} & B. \end{array}$$

Since $p\colon Y \to B$ is by hypothesis a fibration there exists in fact such a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$. Note that the map $H(\alpha)_0\colon F_{b_0} \to B$ is just the obvious inclusion. We turn to the proofs of the statements, albeit perhaps in an unexpected order:

(1) Since the above diagram commutes we see immediately that the map $H(\alpha)_1$ is a continuous map from $F_{b_0} \to F_{b_1}$.

(4) Let $\alpha, \beta\colon [0,1] \to B$ be two paths from b_0 to b_1. We pick maps $H(\alpha)$ and $H(\beta)$ as above, keeping in mind that $H(\alpha)$ and $H(\beta)$ are not uniquely determined. Now we assume that α and β are path-homotopic. We need to show the following claim:

Claim. The maps $H(\alpha)_1, H(\beta)_1\colon F_{b_0} \to F_{b_1}$ are homotopic.

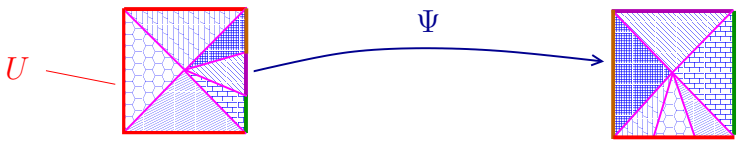
Proof. We start out with the following observation. It follows immediately from Lemma 297.5 (2), applied to $n = 2$, that there exists a self-homeomorphism

$$\Psi\colon [0,1] \times [0,1] \;\to\; [0,1] \times [0,1]$$

that restricts to a homeomorphism

$$\underbrace{([0,1] \times \{0,1\}) \cup (\{0\} \times [0,1])}_{=:U} \;\to\; [0,1] \times \{0\}.$$

For reassurance the construction of such a map is also sketched in the figure below.



Now we pick a homotopy $G\colon [0,1] \times [0,1] \to B$ rel $\{0,1\}$ from α to β. We consider the following diagram:

$$\begin{array}{ccccc} F_{b_0}\times[0,1]\times\{0\} & \xrightarrow{\mathrm{id}_{F_{b_0}}\times\Psi^{-1}} & \overbrace{(F_{b_0}\times[0,1]\times\{0,1\}) \cup (F_{b_0}\times\{0\})\times[0,1]}^{=F_{b_0}\times U} & \xrightarrow{\Omega} & Y \\ \downarrow & & \downarrow & \overset{\Xi}{\nearrow}\ \overset{\Xi\circ(\mathrm{id}_{F_{b_0}}\times\Psi)}{\nearrow} & \downarrow{\scriptstyle p} \\ F_{b_0}\times[0,1]\times[0,1] & \xrightarrow[\mathrm{id}_{F_{b_0}}\times\Psi^{-1}]{} & F_{b_0}\times[0,1]\times[0,1] & \xrightarrow[(P,s,t)\mapsto G(s,t)]{} & B. \end{array}$$

$$\mathrm{id}_{F_{b_0}}\times\Psi$$

We make the following clarifications and observations:

(a) The map Ω is defined as follows:[3219]
- on $F_{b_0} \times [0,1] \times \{0\} = F_{b_0} \times [0,1]$ the map Ω is given by $H(\alpha)$,
- on $F_{b_0} \times [0,1] \times \{1\} = F_{b_0} \times [0,1]$ the map Ω is given by $H(\beta)$,
- on $F_{b_0} \times \{0\} \times [0,1]$ the map Ω is given by the projection onto F_{b_0} followed by the inclusion into Y.

(b) It follows immediately from the definitions that the diagram commutes.

(c) Since $p\colon Y \to B$ is a fibration we see that there exists a continuous map Ξ that makes the outer diagram commute.

(d) The map $\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi)\colon F_{b_0} \times [0,1] \times [0,1] \to Y$ is a homotopy between $H(\alpha)$ and $H(\beta)$. In particular the map $(\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi))_1\colon F_{b_0} \times [0,1] \to Y$ is the desired homotopy between $H(\alpha)_1$ and $H(\beta)_1$. ⊞

(5) This statement follows from the observation that for the constant path c_b at a point $b \in B$ we can choose $H(c_b)\colon F_b \times [0,1] \to F_b$ to be the projection. Evidently we obtain that $\Theta(c_b) = H(c_b)_1 = \mathrm{id}_{F_b}$.

(6) Let α be a path in B from b_0 to b_1 and let β be a bath in B from b_1 to b_2. Almost the same way that we define the product $\alpha * \beta$ we combine $H(\alpha)$ and $H(\beta)$. More precisely, we consider the map[3220]

$$\begin{array}{rcl} H(\alpha * \beta)\colon F_{b_0} \times [0,1] & \to & X \\ (P,t) & \mapsto & \begin{cases} H(\alpha)(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ H(\beta)\big(\underbrace{H(\alpha(P),1)}_{\in F_{b_1}}, 2t-1\big), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

One can easily verify that this map has all the properties that we expect from a map that is called $H(\alpha * \beta)$. The desired statement is now an immediate consequence of the observation that $H(\alpha * \beta)_1 = H(\beta)_1 \circ H(\alpha)_1$.

(3) Let $\alpha\colon [0,1] \to B$ be a path from the point b_0 to the point b_1. We pick a continuous map $H(\alpha)\colon F_{b_0} \times [0,1] \to X$ as above. We want to show that the corresponding map $H(\alpha)_1\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence. We denote by $\overline{\alpha}\colon [0,1] \to B$ the inverse path of α, i.e. the path from b_1 to b_0 that is given by $\overline{\alpha}(t) = \alpha(1-t)$. We pick a continuous map $H(\overline{\alpha})\colon F_{b_1} \times [0,1] \to X$ as above. It remains to prove the following claim:

Claim.

(a) The map $H(\overline{\alpha})_1 \circ H(\alpha)_1\colon F_{b_0} \to F_{b_0}$ is homotopic to the identity of F_{b_0}.

(b) The map $H(\alpha)_1 \circ H(\overline{\alpha})_1\colon F_{b_1} \to F_{b_1}$ is homotopic to the identity of F_{b_1}.

Proof. We prove Statement (1). The proof of Statement (2) is verbatim the same. We have

$$\begin{array}{ccccc} [\, H(\overline{\alpha})_1 \circ H(\alpha)_1 \,] & = & [\, H(\alpha * \overline{\alpha}) \,] & = & [\, H(c_{b_0}) \,] & = & [\, \mathrm{id}_{F_{b_0}} \,]. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by (6)} & & \text{by (4) and the Path-Homotopy} & & \text{by (5)} & \\ & & & \text{Product Proposition 88.10 (3)} & & & \end{array}$$

⊞

[3219] As on so many other occasions we can use the Pasting Proposition 4.7 (2) to show that the map Ω is indeed continuous.

[3220] It follows again from the Homotopy Combination Lemma 18.8 (1) that this map is continuous.

(2) We apply the argument from (4) to $\beta = \alpha$ together with two maps $H(\alpha)$ and $H(\beta)$ that make the diagram commute. We obtain by (3) that $H(\alpha)$ and $H(\beta)$ are homotopic, i.e. we see that $[H(\alpha)] \in [F_{b_0}, F_{b_1}]$ is well-defined. ■

Using the Fiber Homotopy Proposition 297.8, and the proof thereof, we almost immediately obtain the following proposition:

Proposition 297.9. (Fiber-Homotopy Type Proposition) Let $p\colon Y \to B$ be a continuous map between topological spaces. We assume that B is path-connected.

(1) If $p\colon Y \to B$ is a fibration, then for any two points $b_0, b_1 \in B$ the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.
(2) If $p\colon Y \to B$ is a Serre fibration, then for any two points $b_0, b_1 \in B$ such that the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are discrete subsets, the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.

Proof.

(1) This statement follows immediately from the Fiber Homotopy Proposition 297.8 (3) and the hypothesis that B is path-connected.
(2) Let $p\colon Y \to B$ be a Serre fibration with B a path-connected topological space. Let $b_0, b_1 \in B$ such that the fibers $F_{b_0} \coloneqq p^{-1}(b_0)$ and $F_{b_1} \coloneqq p^{-1}(b_1)$ are discrete subsets. In the proof of the Fiber Homotopy Proposition 297.8 (3) we used the homotopy lifting property for the topological spaces $X = F_{b_i} \times [0,1]^n$ with $i \in \{0,1\}$ and $n \in \{1,2\}$. Since F_{b_0} and F_{b_1} are discrete we see that each component of $X = F_{b_i} \times [0,1]^n$ is homeomorphic to $[0,1]^n$, i.e. each component is homeomorphic to $\overline{B}^n$. Since we can do the lifting component-wise it suffices in our context that we are given a Serre fibration. We leave it to the reader to fill in the details. ■

The Fiber-Homotopy Type Proposition 297.9 (1) allows us to make the following definition:

Definition 297.10. Let $p\colon Y \to B$ be a fibration over a path-connected topological space B. We refer to the homotopy type of the fibers as the homotopy fiber of the fibration.

Remark. A more precise statement of the Fiber-Homotopy Type Proposition 297.9 (1) holds: any path $\gamma\colon [0,1] \to B$ from b_0 to b_1 defines naturally a homotopy equivalence between the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$. We refer to [**DaK2001**, Theorem 6.12] for the precise formulation and proof of this statement. □

Examples.

(1) Let $p\colon \widetilde{X} \to X$ be a covering map over a path-connected topological space. As discussed on page 5692 such a map is a Serre fibration. It follows immediately from the definition of a covering map that for each $x \in X$ the fiber $p^{-1}(x)$ is a discrete subset of $\widetilde{X}$. Therefore it follows from the Homotopy-Equivalence π_0-Lemma 19.17 and the Fiber-Homotopy Type Proposition 297.9 (2) that given any two points $x_0, x_1 \in X$ there exists a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$. In other words, we have found a new proof for the Covering–Preimage Lemma 91.9.

(2) Let us consider again the "non-example" from the Non-Serre Fibration Lemma 297.2, i.e. we consider the map

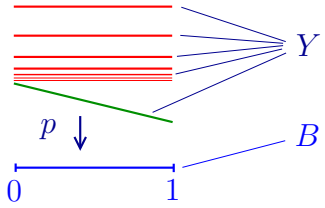
$$\begin{array}{rcl} p\colon Y = S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 = 1\} & \to & B = [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

In this case $p^{-1}(-1) = \{(-1,0)\}$ and $p^{-1}(0) = \{(0,\pm 1)\}$ are discrete subsets, but the number of components is different. From the Homotopy-Equivalence π_0-Lemma 19.17 we deduce that $p^{-1}(-1)$ is not homotopy equivalent to $p^{-1}(0)$. Thus the Fiber-Homotopy Type Proposition 297.9 (2) gives a new proof (after the more hands-on proof provided by the Non-Serre Fibration Lemma 297.2) that p is not a Serre fibration.

(3) We consider the map $p\colon Y \to B$ given by the total space

$$Y \;:=\; \{\underbrace{(x,y)}_{\in\mathbb{R}^2} \,\big|\, y = 1-x \text{ and } x \in [0,1]\} \cup \bigcup_{n\in\mathbb{N}} \{\underbrace{(x,y)}_{\in\mathbb{R}^2} \,|\, x \in [0,1] \text{ and } y = 1+\tfrac{1}{n}\}$$

the base space $B = [0,1]$ and the obvious projection $p\colon Y \to B$.



It is an amusing exercise to show that $p\colon Y \to B$ is a Serre fibration. We saw in Exercise 19.21 that the fibers $p^{-1}(0)$ and $p^{-1}(1)$ are not homotopy equivalent. It follows from the Fiber-Homotopy Type Proposition 297.9 (1) that $p\colon Y \to B$ is not a fibration.[3221]

We can also show more directly that p is not a fibration: We consider $X = p^{-1}(0)$, the usual projection $f\colon X \times [0,1] \to B = [0,1]$ together with the obvious inclusion $\widetilde{f}\colon X \times \{0\} \to Y$. It is once again an amusing elementary exercise to show that there is no continuous map $\widetilde{F}\colon X \times [0,1] \to Y$ with all the desired properties. This shows that $p\colon Y \to B$ is not a fibration. □

297.3. **The long exact homotopy sequence of a Serre fibration.** For us the most interesting aspect of Serre fibrations is that they give rise to long exact sequences of homotopy groups. More precisely, we have the following theorem.

[3221]The discussion of this example would be more intelligible if we followed the convention of [**Bre1993**, p. 450] and if we referred to "fibrations" as "Hurewicz fibrations" to better distinguish them from "Serre fibrations".

Theorem 297.11. (Fibration-LES Theorem) Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$.

(1) Given any $n \in \mathbb{N}$ the following map is well-defined:

$$\begin{aligned} \partial_n\colon \pi_n(B,b_0) &\to \pi_{n-1}(F,y_0) \\ [\varphi\colon (I^n,\partial I^n) \to (B,b_0)] &\mapsto \left[\begin{array}{c} \text{pick a continuous map} \\ \widetilde{\varphi}\colon (I^n,J^{n-1}) \to (Y,\{y_0\}) \text{ with } p\circ\widetilde{\varphi} = \varphi \\ \text{(by Lemma 297.6 we know that such a map exists)} \\ \text{and then restrict the map } \widetilde{\varphi} \text{ to } I^{n-1} = I^{n-1}\times\{0\} \end{array}\right]. \end{aligned}$$

(The definition of ∂_n is illustrated moderately successfully in the figure below.) Furthermore, these maps ∂_n have the following properties:

(a) The maps are morphisms of groups (if $n \geq 2$) and of pointed sets (if $n = 1$).

(b) The maps are natural in the following two senses: Suppose we are given another Serre fibration $p'\colon Y' \to B'$ and suppose that we are given a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & B \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y' & \xrightarrow{p'} & B'. \end{array}$$

If we write $y_0' = \Phi(y_0), b_0' = \phi(b_0)$ and $F' = (p')^{-1}(b_0')$, then for any $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{array}{ccc} \pi_n(B,b_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F,y_0) \\ {\scriptstyle\phi_*}\downarrow & & \downarrow{\scriptstyle(\Phi|_F)_*} \\ \pi_n(B',b_0') & \xrightarrow{\partial_n} & \pi_{n-1}(F',y_0'). \end{array}$$

(c) Let $\alpha\colon [0,1] \to B$ be a path from b_0 to some point b_1.[3222] We write $F_1 = p^{-1}(b_1)$. If $\Theta(\alpha)\colon F \to F_1$ is a continuous map as in the Fiber Homotopy Proposition 297.8, then the following diagram commutes:

$$\begin{array}{ccc} \pi_n(B,b_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F,y_0) \\ {\scriptstyle\alpha_*}\downarrow & & \downarrow{\scriptstyle\Theta(\alpha)_*} \\ \pi_n(B,b_1) & \xrightarrow{\partial_n} & \pi_{n-1}(F_1,y_1) \end{array}$$

where $y_1 = \Theta(\alpha)(y_0)$.

(2) The following sequence is exact:[3223]

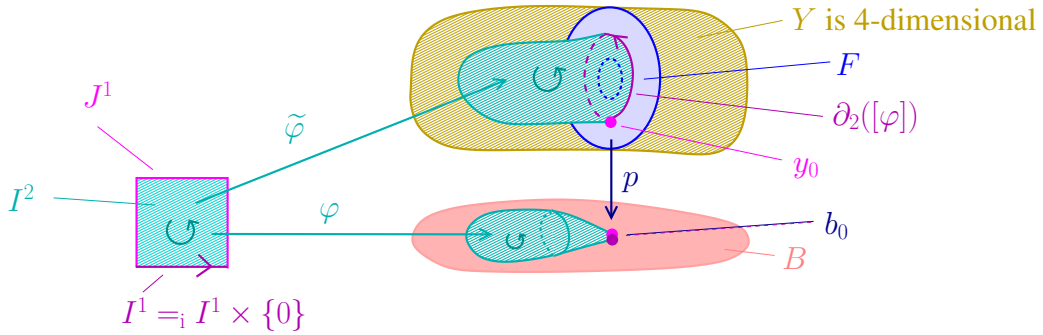
$$\begin{aligned} \ldots \xrightarrow{\partial_3} \pi_2(F,y_0) \xrightarrow{i_*} \pi_2(Y,y_0) \xrightarrow{p_*} \pi_2(B,b_0) \xrightarrow{\partial_2} \\ \xrightarrow{\partial_2} \pi_1(F,y_0) \xrightarrow{i_*} \pi_1(Y,y_0) \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial_1} \\ \xrightarrow{\partial_1} \pi_0(F,y_0) \xrightarrow{i_*} \pi_0(Y,y_0) \xrightarrow{p_*} \pi_0(B,b_0). \end{aligned}$$

(3) If the inclusion map $i\colon F \to Y$ is actually homotopic rel y_0 to the constant map $c_{y_0}\colon F \to \{y_0\}$, then for any $n \in \mathbb{N}_{\geq 2}$ the above long exact sequence breaks into a short exact sequence

$$0 \to \pi_n(Y,y_0) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial_n} \pi_{n-1}(F,y_0) \to 0.$$

Furthermore the short exact sequence splits, which implies by the Splitting Lemma 145.5 (1)⇒(3) that for any $n \in \mathbb{N}_{\geq 2}$ we have an isomorphism

$$\pi_n(B, b_0) \;\cong\; \pi_n(Y, y_0) \oplus \pi_{n-1}(F, y_0).$$



Remark. In the Homological Leray-Serre Spectral Sequence 341.3 we will see that for a fiber bundle $p\colon Y \to B$ with fiber F there exists also a rather subtle interplay between homology groups of Y, B and F. □

Before we head towards the proof of the Fibration-LES Theorem 297.11 we want to see what the theorem tells us in two types of fibrations that we considered above, namely for coverings and products.

Example. Let $p\colon Y \to B$ be a covering of path-connected topological spaces. As mentioned above, it follows from the Map Lifting Criterion 127.5 that p is a Serre fibration. We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. It follows immediately from the definition of a covering map that F, equipped with the subspace topology coming from Y, is a discrete space, in particular $\#\pi_0(F, y_0) = \#F$ and $\pi_i(F, y_0) = 0$ for $i \geq 1$. From the long exact sequence provided by the Fibration-LES Theorem 297.11 we obtain the following two pieces of information:

(1) At the lower end the long exact sequence looks as follows:

$$\underbrace{\pi_1(F, y_0)}_{=\{e\}} \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \underbrace{\pi_0(F, y_0)}_{\#\pi_0(F,y_0)=\#F} \to \underbrace{\pi_0(Y, y_0)}_{=\{e\}} \to \underbrace{\pi_0(B, b_0)}_{=\{e\}}.$$

In other words we obtain the short exact sequence

$$\{e\} \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \pi_0(F, y_0) \to \{e\}.$$

We draw the following two conclusions:

(a) The map $p_*\colon \pi_1(Y, y_0) \to \pi_1(B, b_0)$ is a monomorphism.

(b) With a little bit of extra thought[3224] we obtain from the short exact sequence, the Fiber Homotopy Proposition 297.8, the Fibration-LES Theorem 297.11 (1c) and the Pointed Set–SES Lemma 296.8 that $p_*(\pi_1(Y, y_0))$ is a subgroup of $\pi_1(B, b_0)$ of index $\#\pi_0(F, y_0) = \#F = [Y : B]$.

These two statements are precisely the content of Lemma 90.7 and the Covering-Degree Proposition 92.10.

[3222] Here we allow of course that $b_1 = b_0$.

[3223] Here, at the lower end, we mean "exact" in the sense of Definition 296.7.

[3224] We leave this extra thinking as a mental exercise to the reader.

(2) Since all other homotopy groups of F vanish we see that for any $n \geq 2$ the map $p_*\colon \pi_n(Y, x_0) \to \pi_n(B, b_0)$ is an isomorphism. This result recovers the Covering-π_n-Isomorphism Proposition 134.22. □

We move on to the next application. More precisely we want to apply the Fibration-LES Theorem 297.11 to the case of a Serre fibration of the form $X \times Y \to X$. This leads to a new proof of the Product-π_1-Proposition 90.13 and the Product-π_n-Proposition 134.18, at least for finite products.

Proposition 134.18. (Product-π_n-Proposition) Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. We consider the projections

$$\begin{array}{rcl} p\colon X \times Y & \to & X \\ (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon X \times Y & \to & Y \\ (x,y) & \mapsto & y. \end{array}$$

For any $n \in \mathbb{N}$ the map[3225]

$$\begin{array}{rcl} \pi_n(X \times Y, (x_0, y_0)) & \to & \pi_n(X, x_0) \times \pi_n(Y, y_0) \\ \psi & \to & (p_*(\psi), q_*(\psi)) \end{array}$$

is an isomorphism.

Proof. Throughout this proof we denote by $j\colon Y \to X \times Y$ the embedding that is given by $y \mapsto (x_0, y)$. Next note that by the discussion on page 5692 we know that the projection $p\colon X \times Y \to X$ is a fibration with fiber $p^{-1}(x_0) = \{x_0\} \times Y = j(Y)$. We obtain from the Fibration-LES Theorem 297.11 the long exact sequence

$$\ldots \longrightarrow \pi_n(Y, y_0) \xrightarrow[\overleftarrow{q_*}]{j_*} \pi_n(X \times Y, (x_0, y_0)) \xrightarrow{p_*} \pi_n(X, x_0) \xrightarrow{\partial_n} \pi_{n-1}(Y, y_0) \xrightarrow{j_*} \ldots.$$

Evidently we have $q \circ j = \mathrm{id}_Y$ which implies that $q_* \circ j_* = \mathrm{id}_{\pi_n(Y, y_0)}$. It follows that for every $n \in \mathbb{N}_0$ the map $j_*\colon \pi_n(Y, y_0) \to \pi_n(X \times Y, (x_0, y_0))$ is a monomorphism. Thus we obtain from the Break-into-SES Lemma 145.6 that for each $n \in \mathbb{N}$ the sequence

$$0 \;\to\; \pi_n(Y, y_0) \;\xrightarrow{j_*}\; \pi_n(X \times Y, (x_0, y_0)) \;\xrightarrow{p_*}\; \pi_n(X, x_0) \;\to\; 0$$

is exact. We again use that $q_* \circ j_* = \mathrm{id}$ and we obtain with the same argument[3226] as in the Splitting Lemma 145.5 (2)⇒(3) that the given map

$$\begin{array}{rcl} \Phi\colon \pi_n(X \times Y, (x_0, y_0)) & \to & \pi_n(X, x_0) \times \pi_n(Y, y_0) \\ \psi & \to & (p_*(\psi), q_*(\psi)) \end{array}$$

is indeed an isomorphism. ■

Even these modest applications should already convince the reader that it is worth proving the Fibration-LES Theorem 297.11. After the proof we will give many more examples of fibrations and we will find many more interesting applications.

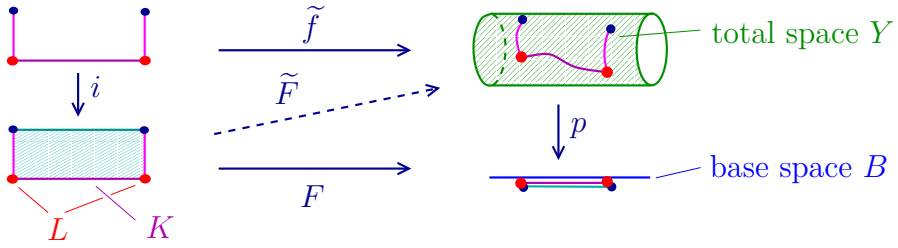
[3225] Note that for $n \geq 2$ the map is a group homomorphism since in this case the group on the right-hand side is abelian. It is a priori not completely clear that the map is also a homomorphism for $n = 1$, but we will see in the proof that the map is indeed a homomorphism.

[3226] Note that for simplicity we formulated the Splitting Lemma 145.5 for abelian groups. But if one looks at the proof one realizes that we used the hypothesis that the groups are abelian only in the proof of the (1)⇒(2) statement. The (2)⇒(3)-direction still holds with the same proof.

297.4. **Proof of the Fibration-LES Theorem 297.11.** The proof of the Fibration-LES Theorem 297.11 requires some preparations, i.e. we first need to prove some propositions.

Proposition 297.12. (CW–Homotopy–Lift Proposition) In the following let $p\colon Y \to B$ be a Serre fibration and let (K, L) be a pair of CW-complexes. We denote by $i\colon (K \times \{0\}) \cup (L \times [0,1]) \to K \times [0,1]$ the inclusion map. Given any continuous map $F\colon K \times [0,1] \to B$ and given any continuous map $\widetilde{f}\colon (K \times \{0\}) \cup (L \times [0,1]) \to Y$ with $p \circ \widetilde{f} = F \circ i$ there exists a continuous map $\widetilde{F}\colon K \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} (K \times \{0\}) \cup (L \times [0,1]) & \xrightarrow{\widetilde{f}} & Y \\ \downarrow{\scriptstyle i} & \overset{\widetilde{F}}{\nearrow ??} & \downarrow{\scriptstyle p} \\ K \times [0,1] & \xrightarrow[F]{} & B. \end{array}$$



Proof. We start out with a reformulation of our task: Given $n = 0, 1, 2, 3, \ldots$ we need to define maps

$$\widetilde{F}^n\colon (K \times \{0\}) \cup (K^n \cup L) \times [0,1] \;\to\; Y$$

that make the above diagram commute and which have the property that for $n < m$ the maps $\widetilde{F}^n$ and $\widetilde{F}^m$ agree on the common domain.

We set $\widetilde{F}^{-1} = \widetilde{f}$. Suppose we have already defined maps $\widetilde{F}^{-1}, \ldots, \widetilde{F}^{n-1}$ as above. Let $\{\Phi_i\colon (\overline{B}^n, S^{n-1}) \to (K^n, L^{n-1})\}_{i \in I}$ be a full set of characteristic maps of the n-cells of K that do not completely lie in L. Recall that by Lemma 297.5 (1) there exists a homeomorphism

$$\Psi\colon \big(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1])\big) \;\overset{\cong}{\to}\; \big(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\}\big).$$

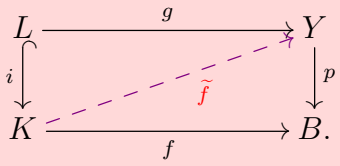
For each $i \in I$ we consider the following diagram:

$$\begin{array}{ccccc} \overline{B}^n \times \{0\} & \xrightarrow{\Psi^{-1}} & \overline{B}^n \times \{0\} \cup S^{n-1} \times \{1\} & \xrightarrow{\widetilde{F}^{n-1} \circ \Phi_i} & Y \\ \downarrow & & \downarrow & \overset{\Xi_i}{\nearrow} & \downarrow{\scriptstyle p} \\ \overline{B}^n \times [0,1] & \xrightarrow[\Psi^{-1}]{} & \overline{B}^n \times [0,1] & \xrightarrow[F \circ \Phi_i]{} & B. \end{array}$$

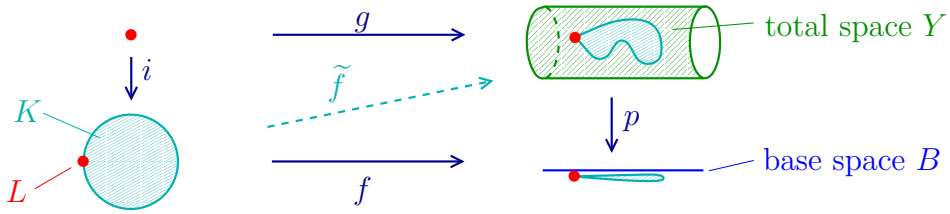
One can easily verify that the two rectangles commute. Since $p\colon Y \to B$ is a Serre fibration there exists a continuous map $\Xi_i\colon \overline{B}^n \times [0,1] \to Y$ that makes the diagram commute. These maps $\{\Xi_i\}_{i \in I}$ can now be used to extend $\widetilde{F}^{n-1}$ to the desired map $\widetilde{F}^n$. ∎

Proposition 297.13. (CW-Deformation Retract–Lift Proposition) Let $p\colon Y \to B$ be a Serre fibration and let (K, L) be a pair of CW-complexes such that L is a deformation retract of K. We denote by $i\colon L \to K$ the inclusion map. Given any continuous map

$f\colon K \to B$ and given any continuous map $g\colon L \to Y$ with $f \circ i = p \circ g\colon L \to B$ there exists a continuous map $\tilde{f}\colon K \to Y$ such that the following diagram commutes:



Example. Later on we will apply the CW-Deformation Retract–Lift Proposition 297.13 to the special case that $K = \overline{B}^n$ and that $L = \{(0,\ldots,0,1)\}$.



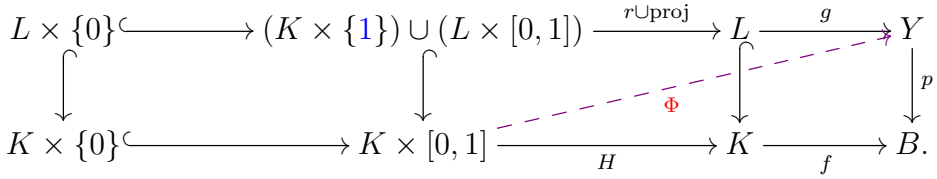
We refer to the figure above for an illustration. □

Proof. We make the following preparations:

(1) Since L is a deformation retract of K there exists by definition of a deformation retraction, see Definition 19.3, a homotopy $H\colon K \times [0,1] \to K$ rel L such that $H_0 = \mathrm{id}_K$ and $H_1(K) \subseteq L$. We denote by $r = H_1\colon K \to L$ the corresponding retraction.

(2) We denote by $\mathrm{proj}\colon L \times [0,1] \to L$ the obvious projection.

Now we consider the following commutative diagram:



To be somewhat more precise, the map $r \cup \mathrm{proj}\colon (K \times \{1\}) \cup (L \times [0,1])$ is defined via the retraction r on $K = K \times \{1\}$ and the projection $L \times [0,1] \to L$. The right-hand square commutes by our hypothesis. One easily verifies that the middle square commutes. It follows from the CW–Homotopy–Lift Proposition 297.12 that there exists a continuous map $\Phi\colon K \times [0,1] \to Y$ that makes the diagram commute. For any $k \in K$ we have

$$(p \circ \Phi)(k,0) \;=\; f(H(k,0)) \;=\; f(\mathrm{H}_0(k)) \;=\; f(k)$$

and for any $l \in L$ we have

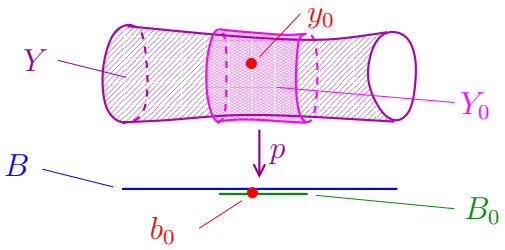
$$\Phi(l,0) \;=\; (g \circ \mathrm{proj})(l,0) \;=\; g(\mathrm{proj}(l,0)) \;=\; g(l).$$

This shows that the restriction of Φ to $K = K \times \{0\}$ has the desired properties. ■

Proposition 297.14. (Fibration–Homotopy Group Isomorphism Proposition) Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Let $B' \subseteq B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Then

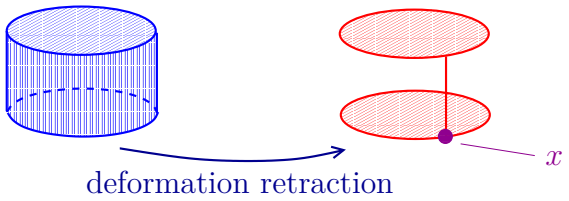
$$p_*\colon \pi_n(Y, Y', y_0) \;\to\; \pi_n(B, B', b_0)$$

is an isomorphism for all $n \in \mathbb{N}$.



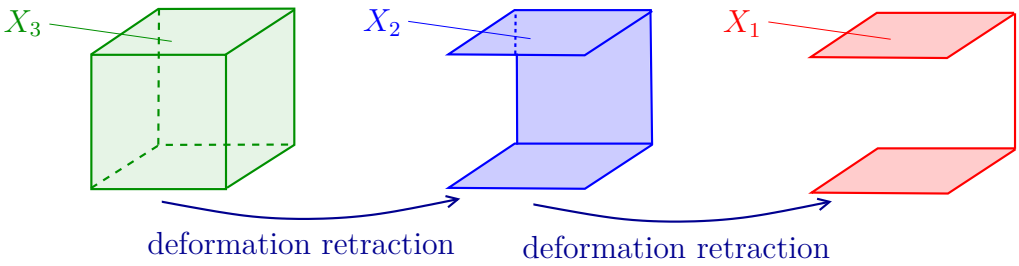
In the proof of the Fibration–Homotopy Group Isomorphism Proposition 297.14 we will need the following elementary lemma:

Lemma 297.15. (Cylinder-Retraction Lemma) Let $n \in \mathbb{N}$. For any point $x \in S^{n-1}$ the subset $(\overline{B}^n \times \partial([0,1])) \cup (\{x\} \times [0,1])$ is a deformation retract of $\overline{B}^n \times [0,1]$.



Proof of the Cylinder-Retraction Lemma 297.15. As usual we write $I = [0,1]$. Given $k \in \{1, \ldots, n\}$ we consider

$$X_k \;\coloneqq\; (I^{n-1} \times \partial I) \;\cup\; (\{\underbrace{(1,1,\ldots,1)}_{\in \mathbb{R}^{n-k}}\} \times I^k).$$



Using the Convex-to-Ball Proposition 4.28 (2) and the Orthogonal Action Lemma 2.37 (3) one can show easily that there exists a homeomorphism

$$(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0,1\}) \cup (\{x\} \times [0,1])) \;\overset{\cong}{\to}\; (X_n, X_1).$$

Thus it suffices to show that each X_i is a deformation retract of X_{i+1}. But such deformation retractions one can actually write down by hand. ∎

Proof of Proposition 297.14. Let $p\colon Y \to B$ be a Serre fibration and let $n \in \mathbb{N}$. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$. Finally let $B' \subseteq B$ be any subset with $b_0 \in B'$. We set $Y' \coloneqq p^{-1}(B')$. Throughout this proof we use the identification from page 5679, i.e. we view elements in a relative homotopy group $\pi_n(X, A, x_0)$ as homotopy classes of continuous maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)$.

First we show that $p_*\colon \pi_n(Y, Y', y_0) \to \pi_n(B, B', b_0)$ is an epimorphism. Thus suppose that we are given a continuous map $f\colon (\overline{B}^n, S^{n-1}, *) \to (B, B', b_0)$. It follows from the

CW-Deformation Retract–Lift Proposition 297.13, applied to $(K, L) = (\overline{B}^n, *)$, that there exists a continuous map $g\colon \overline{B}^n \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} \{*\} & \xrightarrow{*\mapsto y_0} & Y \\ \downarrow & \overset{g}{\nearrow} & \downarrow p \\ \overline{B}^n & \xrightarrow[f]{} & B. \end{array}$$

Basically by definition g is a continuous map $(\overline{B}^n, S^{n-1}, *) \to (Y, Y', y_0)$. Since $f = p \circ g$ we have $[f] = p_*([g])$.

Next we show that $p_*\colon \pi_n(Y, Y', y_0) \to \pi_n(B, B', b_0)$ is also a monomorphism. Therefore suppose that we are given two continuous maps $g_0, g_1\colon (\overline{B}^n, S^{n-1}, *) \to (Y, Y', y_0)$ such that the equality $p_*([g_0]) = p_*([g_1]) \in \pi_n(B, B', b_0)$ holds. This equality means that $p \circ g_0$ is homotopic to $p \circ g_1$ via a homotopy $F\colon (\overline{B}^n, S^{n-1}, *) \times [0,1] \to (B, B', b_0)$. We consider the following commutative diagram (first without the dashed arrow):

$$\begin{array}{ccc} (\overline{B}^n \times \{0,1\}) \cup (\{*\} \times [0,1]) & \xrightarrow{\text{via } g_0, g_1} & Y \\ \downarrow & \overset{\widetilde{F}}{\nearrow} & \downarrow p \\ \overline{B}^n \times [0,1] & \xrightarrow[F]{} & B. \end{array}$$

By the CW-Deformation Retract–Lift Proposition 297.13 and the Cylinder-Retraction Lemma 297.15 there exists a continuous map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes. This map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ is easily seen to be a homotopy between g_0 and g_1 which shows that $[g_0] = [g_1] \in \pi_n(Y, Y', y_0)$. ■

With these preparations it is surprisingly easy to provide a proof of the Fibration-LES Theorem 297.11 (1) and (2).

Proof of the Fibration-LES Theorem 297.11 (1) and (2). Let $p\colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i\colon F \to Y$ the inclusion map. We consider the following diagram of maps

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \pi_n(F, y_0) & \xrightarrow{i_*} & \pi_n(Y, y_0) & \longrightarrow & \pi_n(Y, F, y_0) & \xrightarrow{\partial_n} & \pi_{n-1}(F, y_0) & \xrightarrow{i_*} & \ldots \\ & & & & & & \cong \downarrow p_* & & & & \\ & & & & & & \pi_n(B, \{b_0\}, b_0) & & & & \\ & & & & & & \downarrow \cong & & & & \\ & & & & & & \pi_n(B, b_0). & & & & \end{array}$$

(with p_* from $\pi_n(Y, y_0)$ to $\pi_n(B, b_0)$, and $\partial_n \circ p_*^{-1}$ from $\pi_n(F, y_0)$ to $\pi_n(B, b_0)$)

We make the following comments:

(a) The top sequence of maps is the long exact sequence of homotopy groups of the pointed pair of topological spaces (Y, F, y_0), see the Homotopy Groups-LES-Proposition 296.10 for details.

(b) The top vertical map is an isomorphism by the Fibration–Homotopy Group Isomorphism Proposition 297.14.

(c) The bottom vertical map is the identification given on page 5666.

(d) It follows immediately from the definitions that the left-hand triangle commutes.

(e) The right-hand triangle commutes by definition of the diagonal map.

It follows immediately from the definition of p_* that given any $n \in \mathbb{N}$ the map

$$\partial_n \circ p_*^{-1} \colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$$

is given by

$$[\varphi \colon (I^n, \partial I^n) \to (B, b_0)] \mapsto \left[\begin{array}{c} \text{the restriction of } \widetilde{\varphi} \text{ to } I^{n-1} = I^{n-1} \times \{0\} \text{ where} \\ \widetilde{\varphi} \text{ is a continuous map } (I^n, J^{n-1}) \to (Y, \{y_0\}) \\ \text{with } p \circ \widetilde{\varphi} = \varphi \end{array} \right].$$

This observation shows that the map given in Statement (1) of the theorem is well-defined and that it is a homomorphism (respectively a morphism of pointed sets). We leave it to the few readers who made it that far to verify that the maps ∂_n have the two properties stated in (1b) and (1c).

Furthermore, since the top sequence is exact and since the vertical maps are isomorphisms we see that the sequence given in Statement (2) of the theorem is indeed exact. ∎

Proof of the Fibration-LES Theorem 297.11 (3). Let $p \colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i \colon F \to Y$ the inclusion map. Now we suppose that the inclusion map $i \colon F \to Y$ is homotopic rel y_0 to the constant map $c_{y_0} \colon F \to \{y_0\}$. Put differently, we suppose that there exists a homotopy $G \colon F \times [0,1] \to Y$ rel y_0 such that $G_0 = i$ and $G_1(f) = y_0$ for every $f \in F$. We need to prove the following claim:

Claim. For any $n \in \mathbb{N}_{\geq 2}$ the long exact sequence of homotopy groups breaks into a short exact sequence

$$0 \to \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_n} \pi_{n-1}(F, y_0) \to 0$$

and the short exact sequence splits.

Proof. First note that it follows almost immediately from the definition of a split exact sequence (see Definition 145.4), the Splitting Lemma 145.5 (1)⇒(2) and the Break-into-SES Lemma 145.6 that it suffices to show that for every $n \in \mathbb{N}_{\geq 2}$ there exists a homomorphism $\lambda \colon \pi_{n-1}(F, y_0) \to \pi_n(B, b_0)$ such that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F, y_0)}$. We consider the map[3227]

$$\begin{array}{rcl} \lambda \colon \pi_{n-1}(F, y_0) \to & \pi_n(Y, F, y_0) & \xrightarrow{p_*} \pi_n(B, \{b_0\}, b_0) = \pi_n(B, b_0) \\ [\varphi \colon (I^{n-1}, \partial I^{n-1}) \to (F, y_0)] \mapsto & \left[\begin{array}{c} (I^n, \partial I^n, J^{n-1}) \to (Y, F, \{y_0\}) \\ (x,t) \mapsto G(\varphi(x), t) \end{array} \right] & \end{array}$$

It is elementary to verify that λ is in fact a homomorphism. It follows almost immediately from the definitions that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F, y_0)}$. ∎

297.5. **The path space fibration.** Next we recall, the long dormant, Definition 16.2:

Definition 297.16. Let X and Y be topological spaces.

(1) We denote by $X^Y = C(Y, X)$ the set of all continuous maps from Y to X.
(2) The compact-open topology on X^Y is the topology generated, in the sense of Definition 3.56, by all the sets of the form

$$M(K, U) := \{f \in X^Y \mid f(K) \subseteq U\}$$

[3227] It takes a second to actually verify that the map $(x,t) \mapsto G(\varphi(x), t)$ defines in fact a map $(I^n, \partial I^n, J^{n-1}) \to (Y, F, \{y_0\})$.

where $K \subseteq Y$ is compact and where $U \subseteq X$ is open.

In the following we state slightly simplified versions of two technical results which we proved a long time ago.

Proposition 16.9. (Evaluation Map Proposition) Let X be a topological space.

(1) The evaluation map
$$\begin{array}{rcl} e\colon X^{[0,1]} \times [0,1] & \to & X \\ ((f\colon [0,1] \to X), y) & \mapsto & f(y) \end{array}$$
is continuous.

(2) For each $y \in [0,1]$ the evaluation map
$$\begin{array}{rcl} e\colon X^{[0,1]} & \to & X \\ (f\colon [0,1] \to X) & \mapsto & f(y) \end{array}$$
is continuous.

Proposition 16.10. (Compact-Open Exponential Proposition) Let X and T be two topological spaces. Let $H\colon T \times [0,1] \to X$ be a continuous map. Given $t \in T$ we denote by $H_t\colon [0,1] \to X$ the map defined by $H_t(y) = H(t,y)$. Then

$$H\colon T \times [0,1] \to X \text{ is continuous} \Leftrightarrow \begin{array}{l} \text{(1) each } H_t\colon [0,1] \to X \text{ is continuous, and} \\ \text{(2) the map } T \to X^{[0,1]} \text{ given by } t \mapsto H_t \text{ is continuous.} \end{array}$$

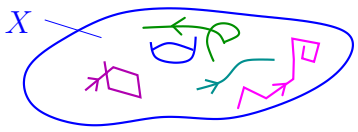
This reminder allows us to introduce our latest best friends:

Definition 297.17. Let X be a topological space.

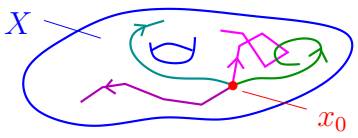
(1) We define
$$\text{free path space } X^{[0,1]} \quad := \quad \text{set of all continuous maps } [0,1] \to X$$
where $X^{[0,1]}$ is as usual equipped with the compact-open topology. Furthermore we introduce the
$$\text{free loop space } \Omega(X) \quad := \quad \{f \in X^{[0,1]} \mid f(0) = f(1)\}.$$
We equip the free loop space with the compact-open topology coming from $X^{[0,1]}$.

(2) Suppose we are also given a base point $x_0 \in X$. We introduce the
$$\text{path space } P(X,x_0) \quad := \quad \{f \in X^{[0,1]} \mid f(0) = x_0\}$$
and the
$$\text{loop space } \Omega(X,x_0) \quad := \quad \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}.$$
We equip both sets with the compact-open topology coming from $X^{[0,1]}$.

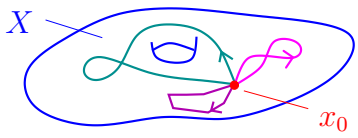
(3) We denote by $c_{x_0}\colon [0,1] \to X$ the constant path given by sending all t's to x_0.



elements in the free path space $X^{[0,1]}$



elements in the path space $P(X,x_0)$



elements in the loop space $\Omega(X,x_0)$

Examples.

(1) Let (X, x_0) be a pointed topological space and let $\alpha, \beta \in \Omega(X, x_0)$. It follows from the Compact-Open Exponential Proposition 16.10 that a path in $\Omega(X, x_0)$ from α to β is basically the same as a homotopy between the maps α and β viewed as loops in (X, x_0). Thus we see that we have a natural identification

$$\pi_0(\Omega(X, x_0)) \ =_{\mathrm{i}} \ \pi_1(X, x_0).$$

(2) Let (X, x_0) be a pointed topological space. It follows from the Evaluation Map Proposition 16.9 that the map

$$\begin{array}{rcl} e\colon P(X, x_0) & \to & X \\ f & \mapsto & f(1) \end{array}$$

is continuous. We can ask whether this map has a right inverse. More precisely, the question is whether there exists a continuous map $s\colon X \to P(X, x_0)$ such that $e(s(x)) = x$ for all $x \in X$. This sounds like a rather random question, so let us reformulate it. The question is, can we *continuously* assign to each $x \in X$ a path from x_0 to x? Clearly that is possible say for $(X, x_0) = (\mathbb{R}^2, 0)$, we just assign to each $x \in \mathbb{R}^2$ the direct path $e(x)$ that is given by $t \mapsto t \cdot x$. But can we do so say for $X = S^1$ or $X = S^2$?[3228] We will answer this question in Exercise 297.3. □

For the record we state the following lemma:

Lemma 297.18. (Loop Space Functor Lemma) The maps

$$\begin{array}{rcl} (X, x_0) & \mapsto & (\Omega(X, x_0), c_{x_0}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & \left(\begin{array}{rcl} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f \circ \gamma\colon [0,1] \to Y) \end{array} \right) \end{array}$$

define a covariant functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{PTop}$ of pointed topological spaces. We refer to this functor as the *loop space functor*.

Proof. Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map between pointed topological spaces. It follows from the Compact-Open Functoriality Lemma 16.5 (1) that the map

$$\begin{array}{rcl} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f \circ \gamma\colon [0,1] \to Y) \end{array}$$

is actually continuous. With this observation out of the way it is straightforward to prove the lemma. ■

The following lemma says that in some sense (free) path spaces, in isolation, are not particularly interesting examples of topological spaces.

Lemma 297.19. (Path Space–Contractible Lemma)

(1) Let X be a topological space. We view X as a subset of the free path space $X^{[0,1]}$ via the inclusion $x \mapsto c_x$.[3229] Then X is a deformation retract of the free path space $X^{[0,1]}$.

(2) If (X, x_0) is a pointed topological space, then $\{c_{x_0}\}$ is a deformation retract of the path space $P(X, x_0)$, in particular the path space $P(X, x_0)$ is contractible.

[3228] This question is somewhat similar to Question 94.6 and it also falls into the vaguely defined realm of "topological robotics".

[3229] Note that this inclusion $X \to X^{[0,1]}$ is by the Constant Map Lemma 16.4 an embedding.



Proof. We prove Statement (2). The proof of Statement (1) is basically identical. Thus let (X, x_0) be a pointed topological space. We consider the "obvious" deformation retraction given by

$$\begin{array}{rcl} F\colon P(X,x_0)\times[0,1] & \to & P(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \begin{pmatrix} [0,1] & \to & X \\ s & \mapsto & f(s\cdot(1-t)) \end{pmatrix} \end{array}$$

The only reason why we are not already done is that we still need to verify that F is in fact continuous. As we will see this is slightly delicate. We consider the map

$$G\colon \underbrace{X^{[0,1]}\times[0,1]}_{=:T} \to X^{[0,1]}$$

that is defined in exactly the same way as F. Since $F = G|_{P(X,x_0)\times[0,1]}$ it suffices to show that G is continuous. By the"$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 16.10 applied to $T = X^{[0,1]}\times[0,1]$ it suffices to show that the map

$$\begin{array}{rcl} H\colon [0,1]\times\overbrace{(X^{[0,1]}\times[0,1])}^{=T} & \to & X \\ (s,((g\colon[0,1]\to X),x)) & \mapsto & g(s\cdot x) \end{array}$$

is continuous. But the map H can be written as the composition

$$\begin{array}{rcccl} H\colon [0,1]\times X^{[0,1]}\times[0,1] & \to & X^{[0,1]}\times[0,1] & \to & X \\ (s,g,x) & \mapsto & (g,s\cdot(1-x)) & & \\ & & (g,t) & \mapsto & g(t). \end{array}$$

The first map is continuous by the Topological-Product Proposition 7.2 (1) and the second map is continuous by the Evaluation Map Proposition 16.9 (1). Thus we see that H itself is continuous. ∎

In contrast to the previous proposition the following theorem shows that (free) path spaces give rise to interesting *maps*.

Theorem 297.20. (Path Space-Fibration Theorem)

(1) Let X be a topological space. The evaluation map

$$\begin{array}{rcl} p\colon X^{[0,1]} & \to & X \\ (f\colon[0,1]\to X) & \mapsto & f(1) \end{array}$$

is a fibration. Furthermore, given any $x_0\in X$ the fiber $p^{-1}(x_0)$ is the topological space of all paths $[0,1]\to X$ that *end* in x_0.

(2) Let (X, x_0) be a pointed topological space. The evaluation map

$$\begin{array}{rcl} p\colon P(X,x_0) & \to & X \\ (f\colon[0,1]\to X) & \mapsto & f(1) \end{array}$$

is a fibration with fiber $p^{-1}(x_0) = \Omega(X, x_0)$.

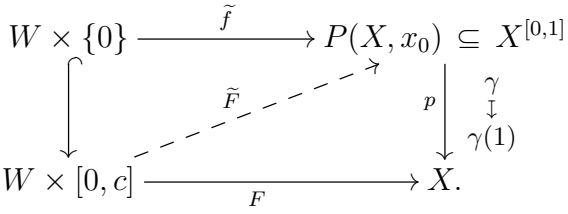
The Path Space-Fibration Theorem 297.20 leads us to the following definition:

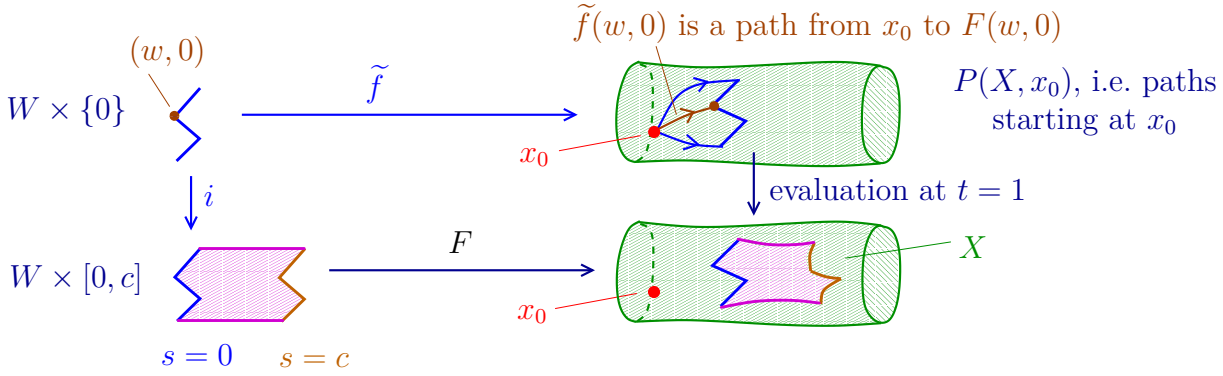
Definition 297.21.

(1) Let X be a topological space. We refer to the above fibration $p\colon X^{[0,1]} \to X$ as the free path space fibration of X.

(2) Let (X, x_0) be a pointed topological space. In the following we refer to the above fibration $p\colon P(X, x_0) \to X$ as the path space fibration of (X, x_0).

Proof. We prove Statement (2). The proof of Statement (1) is basically identical and is thus left to the reader. Thus let (X, x_0) be a pointed topological space. First note that the map $p\colon P(X, x_0) \to X$ is continuous by the Evaluation Map Proposition 16.9 (2). Next recall that we need to solve the following type of extension problem for some topological space W and $c > 0$: [3230]





The idea for the proof is quite simple: Given $(w, s) \in W \times [0, c]$ we have to "systematically" find an element $\gamma \in P(X, x_0)$ with $p(\gamma) = F(w, s)$, i.e. we have to find a path $\gamma\colon [0, 1] \to X$ from x_0 to $F(w, s)$. But there is an obvious path, namely we first travel along $\widetilde{f}(w, 0)$ from x_0 to $F(w, 0)$ and then we use the path $t \mapsto F(w, t)$ to get from $F(w, 0)$ to $F(w, s)$. That is exactly what we will do, except that we need to be a little careful with our parametrizations.

Now we have to turn this idea into a proper proof. We consider the map

$$\begin{array}{rcl} \widetilde{F}\colon W \times [0, c] & \to & P(X, x_0) \\ (w, s) & \mapsto & \left(\begin{array}{rcl} [0, 1] & \to & X \\ t & \mapsto & \begin{cases} \widetilde{f}(w, 0)\left(\frac{t}{1 - \frac{s}{2c}}\right), & \text{if } t \in [0, 1 - \frac{s}{2c}], \\ F(w, 2 \cdot (t - 1 + \frac{s}{2c})), & \text{if } t \in [1 - \frac{s}{2c}, 1] \end{cases} \end{array} \right). \end{array}$$

[3230] In the diagram we write $W \times [0, c]$ instead of the more customary $W \times [0, 1]$ for purely psychological reasons, it makes it easier to distinguish the two types of intervals which arise throughout the proof and which play very different roles.

We claim that this map has all the desired properties:

(1) Let $(w,s) \in W \times [0,c]$. We make the following observations:
   (1) The given map $[0,1] \to X$ starts at x_0.
   (2) It follows easily from the Pasting Proposition 4.7 (2) that the given map $[0,1] \to X$ is indeed continuous.

   In other words, the given map $[0,1] \to X$ does indeed define an element in $P(X,x_0)$.
(2) For each $(w,s) \in W \times [0,c]$ we have by construction that $\widetilde{F}(w,s)(t=1) = F(w,s)$. In other words, we have $p \circ \widetilde{F} = F$.

Thus it remains to prove the following claim:

Claim. The map $\widetilde{F}\colon W \times [0,c] \to P(X,x_0)$ is continuous.

Proof. We consider the map

$$\begin{array}{rcl} \widetilde{G}\colon W \times [0,c] \times [0,1] & \to & X \\ (w,s,t) & \mapsto & \widetilde{F}(w,s)(t) = \begin{cases} \widetilde{f}(w,0)\left(\frac{t}{1-\frac{s}{2c}}\right), & \text{if } t \in [0,1-\frac{s}{2c}], \\ F(w,2\cdot(t-1+\frac{s}{2c})), & \text{if } t \in [1-\frac{s}{2c},1]. \end{cases} \end{array}$$

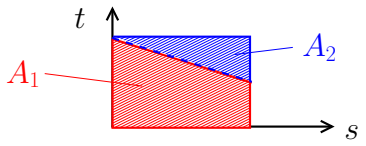
It follows easily from the fact that F and $\widetilde{f}$ are continuous that the restrictions of $\widetilde{G}$ to the closed subsets

$$A_1 := W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [0,1-\tfrac{s}{2c}]\},$$

and

$$A_2 := W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [1-\tfrac{s}{2c},1]\}$$

are continuous.



Therefore it follows from the Pasting Proposition 4.7 (2) that $\widetilde{G}$ is continuous. Finally we deduce from the"$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 16.10 applied to $T = W \times [0,c]$, $Y = [0,1]$ and $H = \widetilde{G}$ that $\widetilde{F}$ is also continuous. ∎

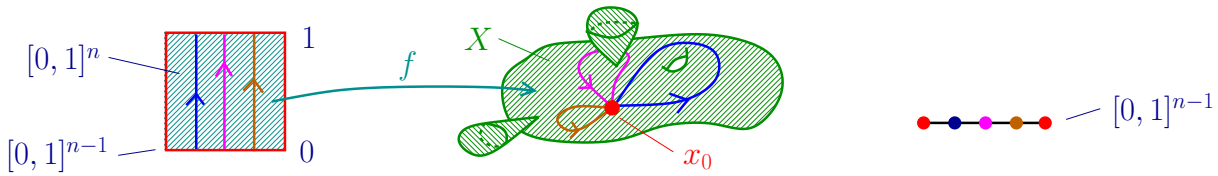
We obtain the following amusing corollary which later on will turn out to play an important role.

Corollary 297.22. (Loop Space-Homotopy Groups Corollary) Given any pointed topological space (X,x_0) and any $n \geq 1$ the map[3231]

$$\begin{array}{rcl} \partial_n\colon \pi_n(X,x_0) & \overset{\cong}{\to} & \pi_{n-1}(\Omega(X,x_0),c_{x_0}) \\ [f\colon ([0,1]^n,\partial([0,1]^n)) \to (X,x_0)] & \mapsto & \left[\begin{array}{r} ([0,1]^{n-1},\partial([0,1]^{n-1})) \to (\Omega(X,x_0),c_{x_0}) \\ x \mapsto \left(\begin{array}{r} [0,1] \to X \\ t \mapsto f(x,t) \end{array}\right) \end{array}\right] \end{array}$$

is a natural isomorphism.

[3231]Using the Compact-Open Exponential Proposition 16.10 one can show directly, with some slight effort, that the map ∂_n is well-defined.



Remark. For $n = 1$ the statement of the Loop Space-Homotopy Groups Corollary 297.22 recovers the result from page 5710 where we saw that there exists a natural identification

$$\pi_0(\Omega(X, x_0), c_{x_0}) \ =_i \ \pi_1(X, x_0).$$

□

Proof. There are two approaches to proving the corollary, namely one can prove it by "the machine" and one can prove it "by hand".

(1) Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map between pointed topological spaces. It is clear that we obtain a commutative diagram

$$\begin{array}{ccc} P(X, x_0) & \xrightarrow{\gamma \mapsto f \circ \gamma} & P(Y, y_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

It follows from the Path Space-Fibration Theorem 297.20 (1) and (2) and the Fibration-LES Theorem 297.11 that we have commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} \ldots \to \pi_n(P(X, x_0), c_{x_0}) & \to & \pi_n(X, x_0) & \xrightarrow{\partial_n} & \pi_{n-1}(\Omega(X, x_0), c_{x_0}) & \to & \pi_{n-1}(P(X, x_0), c_{x_0}) & \to & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots \to \pi_n(P(Y, y_0), c_{y_0}) & \to & \pi_n(Y, y_0) & \xrightarrow{\partial_n} & \pi_{n-1}(\Omega(Y, y_0), c_{y_0}) & \to & \pi_{n-1}(P(Y, y_0), c_{y_0}) & \to & \ldots \end{array}$$

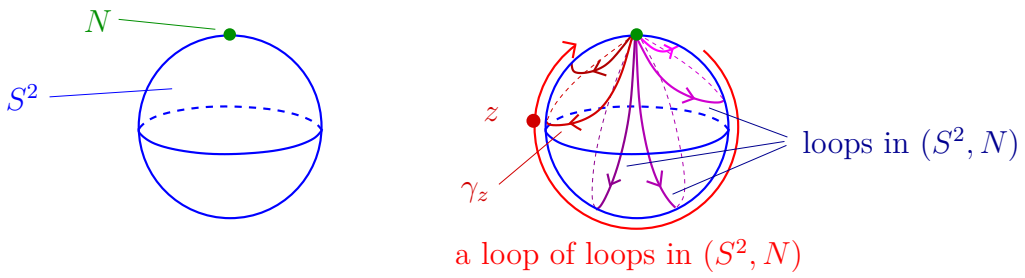
We obtain from the Path Space–Contractible Lemma 297.19 and the Homotopy Equivalence-π_n-Proposition 134.17 (2) that the homotopy groups of the path spaces $P(X, x_0)$ and $P(Y, y_0)$ are trivial. Thus the connecting homomorphisms are *natural* isomorphisms. We leave it to the reader to verify that in our situation the connecting homomorphisms from the Fibration-LES Theorem 297.11 are precisely the maps that we wrote down. Incidentally this then also shows that the maps, as written down, actually do make sense.

(2) In Exercise 297.6 we will prove the corollary "by hand". ■

Example. We consider the loop space of the 2-dimensional sphere with the base point given by the North Pole $N = (0, 0, 1)$. We calculate that

$$\pi_1(\Omega(S^2, N), c_N) \xleftarrow[\text{by the Loop Space-Homotopy Groups Corollary 297.22}]{\cong} \pi_2(S^2, N) \xrightarrow[\text{by the } \pi_n(S^n)\text{–Proposition 167.10}]{\cong} \mathbb{Z}.$$

In the figure below we see a loop $\{\gamma_z\}_{z \in S^1}$ of loops in the pointed topological space (S^2, N). Put differently, we see a loop in the pointed topological space $(\Omega(S^2, N), c_N)$. It is an amusing and instructive exercise to run through the above two isomorphisms to verify that our loop in the pointed topological space $(\Omega(S^2, N), c_N)$ is actually a generator of $\pi_1(\Omega(S^2, N), c_N)$.



So what does this example actually really mean? It means that even though each individual loop γ_z is null-homotopic in S^2, we cannot continuously deform all the loops γ_z altogether into the constant loop. It is a splendid idea to think about what it would mean for the loop $[z \mapsto \gamma_z] \in \pi_1(\Omega(S^2, N), c_N)$ to be trivial and why, pictorially, this is not the case in our situation. □

Note that the statement of the Loop Space-Homotopy Groups Corollary 297.22 can be summarized as follows: The loop space functor

$$\begin{array}{rccc} \Omega\colon & \text{category } \mathcal{PTop} \text{ of pointed topological spaces} & \to & \text{category } \mathcal{PTop} \text{ of pointed topological spaces} \\ & (X, x_0) & \mapsto & (\Omega(X, x_0), c_{x_0}) \end{array}$$

shifts homotopy groups down by one degree, i.e. for every pointed topological space (X, x_0) and every $n \in \mathbb{N}$ we have a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

This should be compared to the fact that in the Suspension-H_*-Proposition 149.3 we saw that the suspension functor

$$\begin{array}{rccc} \Sigma\colon & \text{category } \mathcal{Top} \text{ of topological spaces} & \to & \text{category } \mathcal{Top} \text{ of topological spaces} \\ & X & \mapsto & \Sigma(X) \end{array}$$

shifts reduced homology groups up by one degree, i.e. for every topological space X and any $n \in \mathbb{N}$ we have a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)).$$

In Exercise 297.10 we will discuss whether we can also shift homotopy groups *inward* and whether we can shift reduced homology groups *downward* in a covariant functorial way.

297.6. **Every continuous map can be replaced by a fibration.** In this section we will see that one can "replace" any continuous map $f\colon X \to Y$ between topological spaces by a fibration $P_f \to Y$ that has very similar properties to the map f.
We start out with the notion of the pullback of a fibration. Perhaps not surprisingly the definition is almost identical to the definition of the pullback of a vector bundle that we gave in Definition 265.37.

Lemma 297.23. (Fibration-Pullback Lemma) Let $f\colon A \to B$ be a continuous map between topological spaces and let $q\colon Z \to B$ be a fibration. As in the Topological-Pullback

Proposition 7.22 we consider the pullback

$$f^*Z \;\coloneqq\; \{(a,z)\,|\,a\in A \text{ and } \overbrace{z\in q^{-1}(f(a))}^{\text{i.e. } q(z)=f(a)}\} \;\subseteq\; A\times Z.$$

The following statements hold:

(1) The pullback map

$$\begin{aligned} p\colon f^*Z \;\coloneqq\; \{(a,z)\,|\,a\in A \text{ and } z\in q^{-1}(f(a))\} &\to A\\ (a,z) &\mapsto a \end{aligned}$$

is also a fibration. Furthermore for any $a\in A$ the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$.

(2) The following diagram commutes:

$$\begin{array}{ccc} f^*Z & \xrightarrow{(a,z)\mapsto z} & Z\\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q}\\ A & \xrightarrow{f} & B. \end{array}$$

Proof.

(2) It is straightforward to verify that the diagram commutes.

(1) Let X be any topological space. We consider the following diagram where we initially ignore the diagonal arrows:

$$\begin{array}{ccccc} X\times\{0\} & \xrightarrow{\widetilde{g}} & f^*Z & \xrightarrow{(a,z)\mapsto z} & Z\\ \downarrow{\scriptstyle i} & \overset{?\widetilde{G}?}{\nearrow} & \downarrow{\scriptstyle p} & \overset{\widetilde{f\circ G}}{\nearrow} & \downarrow{\scriptstyle q}\\ X\times[0,1] & \xrightarrow[G]{} & A & \xrightarrow[f]{} & B. \end{array}$$

Here the maps $G\colon X\times[0,1]\to A$ and $\widetilde{g}\colon X\times\{0\}\to f^*Z$ are "the challenge" we have to deal with in the definition of a fibration. Since $q\colon Z\to B$ is a fibration there exists a continuous map $\widetilde{f\circ G}\colon X\times[0,1]\to Z$ that makes the diagram commute. It is now straightforward to verify that the map

$$\begin{aligned} X\times[0,1] &\to f^*Z=\{(a,z)\,|\,a\in A \text{ and } z\in q^{-1}(f(a))\}\\ (x,t) &\mapsto \big(G(x,t),\big(\widetilde{f\circ G}\big)(x,t)\big) \end{aligned}$$

makes the above diagram commute. We have thus shown that $p\colon f^*Z\to A$ is also a fibration.

Finally we have to prove the statement regarding the fibers. Thus let $a\in A$. It follows immediately from the definitions that $p^{-1}(a)=\{a\}\times q^{-1}(f(a))\subseteq A\times Z$. It follows from the Product Topology-Basics Lemma 7.6 (1a) and (1b) that the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$. ∎

Definition 297.24. Using the notation from the Fibration-Pullback Lemma 297.23 we refer to $f^*Z\to A$ as the pullback of the fibration $q\colon Z\to B$ under the map $f\colon A\to B$.

Now we turn to the actual problem that we had stated in the beginning of this section.

Definition 297.25. Let $f\colon X \to Y$ be a continuous map between topological spaces. Similar to the free path space fibration we now consider the fibration[3232]

$$\begin{array}{rcl} q\colon Y^{[0,1]} & \to & Y \\ \gamma & \mapsto & \gamma(0). \end{array}$$

We refer to the map

$$\begin{array}{rcl} p\colon P_f := f^*(Y^{[0,1]}) = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1) \end{array}$$

as the mapping path fibration.[3233]

In the Replace-by-Cofibration Proposition 22.8 we saw that any continuous map $f\colon X \to Y$ between topological spaces can be "replaced" by a cofibration $X \to \mathrm{Cyl}(f)$. The following proposition is the analog of the Replace-by-Cofibration Proposition 22.8 for fibrations.

Proposition 297.26. (Replace-by-Fibration Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. We denote by $p\colon P_f \to Y$ the corresponding mapping path fibration from Definition 297.25.

(1) The map

$$\begin{array}{rcl} h\colon X & \to & P_f \\ x & \mapsto & (x, \text{constant path } c_{f(x)}) \end{array}$$

is a homotopy equivalence and it has the property that it makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{h} & P_f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y. & \end{array}$$

(2) The map $p\colon P_f \to Y$ is a fibration.[3234] Given any $y \in Y$ the fiber is given by

$$\{(x,\gamma) \,|\, x \in X \text{ and } \gamma\colon [0,1] \to Y \text{ is a path from } f(x) \text{ to } y\}.$$

Definition 297.27. Let $f\colon X \to Y$ be a continuous map between topological spaces. We suppose that Y is path-connected. We refer to the homotopy type of the fiber of the corresponding mapping path fibration $P_f \to Y$[3235] as the homotopy fiber of f.

Remark. If $f\colon X \to Y$ is a fibration, then it is shown in [**DaK2001**, Theorem 6.18] that the map $h\colon X \to P_f$ in the Replace-by-Fibration Proposition 297.26 (1) is a "fiber homotopy equivalence", thus, loosely speaking, the original fibration $f\colon X \to Y$ and the new fibration $p\colon P_f \to Y$ are essentially the same. We will not make use of this fact. □

Example. The Replace-by-Fibration Proposition 297.26 thus says that any continuous map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration. As a reality check let us consider again the "non-example" from the Non-Serre Fibration Lemma 297.2,

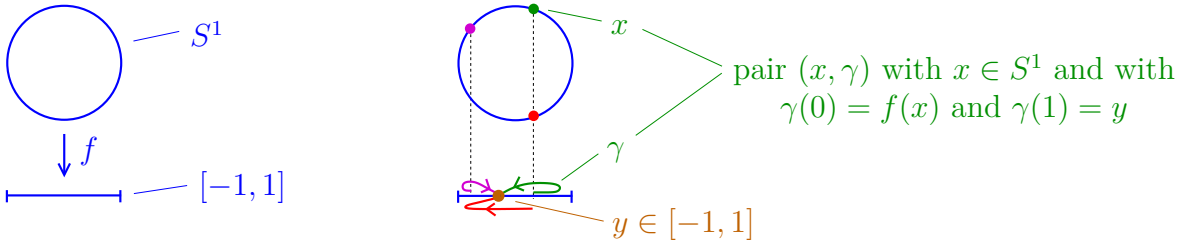
[3232] This map is basically the same as the free path space fibration except that now we consider the evaluation $\gamma(0)$ instead of the evaluation $\gamma(1)$.
[3233] Note that it follows from the Evaluation Map Proposition 16.9 (2) that this map is in fact continuous.
[3234] This statement justifies the name "mapping path *fibration*".
[3235] Recall that the Fiber-Homotopy Type Proposition 297.9 (1) says that the homotopy type of the fiber of a fibration over a path-connected topological space is well-defined.

i.e. we consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. The Non-Serre Fibration Lemma 297.2 says that f is not a fibration. Let us see what the above construction does in this case.

As pointed out in the Replace-by-Fibration Proposition 297.26 (2), given any $y \in [-1,1]$ the preimage of y under the map $p\colon P_f \to Y = [-1,1]$ is given by

$$p^{-1}(y) \;=\; \{(x,\gamma)\,|\,x \in S^1 \text{ and } \gamma\colon [0,1] \to [-1,1] \text{ is a path from } f(x) \text{ to } y\}.$$

In the figure below we strive to illustrate three points in the preimage of a point $y \in [-1,1]$.



In Exercise 297.5 we will see that for each $y \in [-1,1]$ the preimage $p^{-1}(y)$ is homotopy equivalent to S^1. In particular this shows that the homotopy fiber of $f\colon S^1 \to [-1,1]$ is S^1. This exercise does not quite show yet that the map $p\colon P_f \to [-1,1]$ is a fibration, but at the very least it is consistent with the Fiber-Homotopy Type Proposition 297.9 (1). □

Proof of the Replace-by-Fibration Proposition 297.26. Let $f\colon X \to Y$ be a continuous map between topological spaces. We consider the map

$$\begin{array}{rcl} p\colon P_f = f^*(Y^{[0,1]}) = \{(x,\gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]}\,|\,\gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1). \end{array}$$

We turn to the proof of the two statements:

(1) Given $y \in Y$ we denote, as usual, by $c_y\colon [0,1] \to Y$ the constant path given by $c_y(t) = y$ for all $t \in [0,1]$. We consider the map $h\colon X \to P_f$ that is given by $x \mapsto (x, c_{f(x)})$.[3236] We claim it has the desired properties. It is obvious that $p \circ h = f$, i.e. the diagram given in the proposition commutes. So it remains to show that h is a homotopy equivalence. We consider the map $g\colon P_f \to X$ given by $g(x,\gamma) = x$. Evidently we have $g \circ h = \mathrm{id}_X$. We are done once we have shown that $h \circ g$ is homotopic to id_{P_f}. In fact such a homotopy is given by[3237]

$$\begin{array}{rcl} H\colon P_f \times [0,1] & \to & P_f \\ ((x,\gamma),t) & \mapsto & \left(x, \left(\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \gamma(s \cdot t) \end{array}\right)\right). \\ \uparrow & & \\ \gamma\colon [0,1] \to Y \text{ is a} & & \\ \text{path with } \gamma(0) = f(x) & & \end{array}$$

One easily verifies that $H_0 = h \circ g$ and that $H_1 = \mathrm{id}_{P_f}$.

[3236] It follows from the Topological-Product Proposition 7.2 (1) and the Constant Map Lemma 16.4 that this map is continuous.

[3237] An argument as in the proof of the Path Space–Contractible Lemma 297.19 (2) shows that the map H is indeed continuous.

(2) We want to show that the map $p\colon P_f \to Y$ is a fibration. Thus let A be any topological space. Let $c > 0$. We consider the following "challenge diagram":[3238]

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\quad \widetilde{g} \quad} & P_f = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \mid \gamma(0) = f(x)\} \\ \Big\downarrow{\scriptstyle i} & \overset{\widetilde{G}}{\underset{??}{\nearrow}} & \Big\downarrow{\scriptstyle p} \quad (x,\gamma) \mapsto \gamma(1) \\ A \times [0,c] & \xrightarrow[\quad G \quad]{} & Y. \end{array}$$

To define $\widetilde{G}$ we first split up the map $\widetilde{g}$ into its components. More precisely, we define maps $\widetilde{g}_1\colon A \to X$ and $\widetilde{g}_2\colon A \to Y^{[0,1]}$ via the equality

$$\widetilde{g}(a,0) \;=\; \big(\widetilde{g}_1(a), \widetilde{g}_2(a)\big) \;\in\; P_f \subseteq X \times Y^{[0,1]} \qquad \text{for all } a \in A.$$

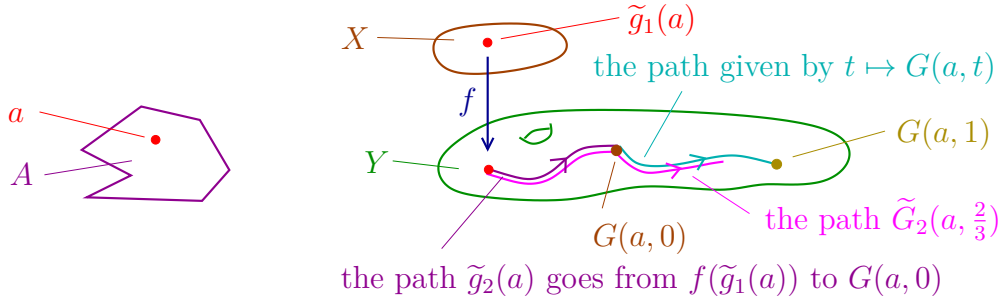
Note that by definition of P_f the path $\widetilde{g}_2(a) \in Y^{[0,1]}$ is a path from $f(\widetilde{g}_1(a))$ to $G(a,0)$. We need to define a suitable map $\widetilde{G}\colon A \times [0,c] \to P_f \subseteq X \times Y^{[0,1]}$. We consider the two factors separately:

(a) We already have the map $A \times [0,c] \to A \xrightarrow{\widetilde{g}_1} X$.

(b) To each $(a,s) \in A \times [0,c]$ we need to assign "systematically" a path $[0,1] \to Y$ from $f(x)$ to $G(a,s)$. The idea is quite simple: Given $(a,s) \in A \times [0,c]$ we consider the path $[0,1] \to Y$ that is given by first traveling along the path $\widetilde{g}_2(a)$ from $f(\widetilde{g}_1(a))$ to $G(a,0)$ and then we travel along the path $t \mapsto G(a,t)$ from $G(a,0)$ to $G(a,s)$. More precisely, similar to the proof of the Path Space-Fibration Theorem 297.20 we consider the map[3239]

$$\begin{array}{rcl} \widetilde{G}_2\colon A \times [0,c] & \to & Y^{[0,1]} \\ (a,s) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & \begin{cases} \widetilde{g}_2(a)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0, 1-\frac{s}{2c}], \\ G(a, 2\cdot(t-1+\frac{s}{2c})), & \text{if } t \in [1-\frac{s}{2c}, 1] \end{cases} \end{array} \right). \end{array}$$

We refer to the figure below for an illustration of the definition of $\widetilde{G}_2$.



Finally we consider the following map:

$$\begin{array}{rcl} \widetilde{G}\colon A \times [0,c] & \to & P_f \\ (a,s) & \mapsto & \big(\widetilde{g}_1(a), \widetilde{G}_2(a,s)\big). \end{array}$$

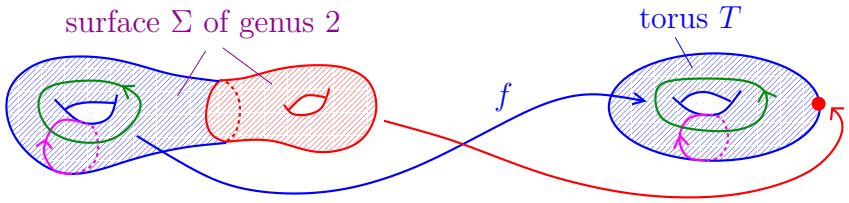
[3238] As in the proof of the Path Space-Fibration Theorem 297.20 we prefer to work, for purely psychological reasons, with $A \times [0,c]$ instead of $A \times [0,1]$.

[3239] The argument in the proof of the Path Space-Fibration Theorem 297.20 also shows that the map $\widetilde{G}_2$ does indeed take values in $Y^{[0,1]}$ and that the map $\widetilde{G}_2\colon A \times [0,c] \to Y^{[0,1]}$ is continuous.

We make the following observations:
(a) It follows easily from the definitions that $\widetilde{G}$ takes values in P_f.
(b) It follows from the above and the Topological-Product Proposition 7.2 (1) that $\widetilde{G}$ is continuous.
(c) It is straightforward to verify that $\widetilde{G}$ makes the diagram commute.
We have thus verified that $\widetilde{G}$ has all the desired properties. ■

The statement of the Replace-by-Fibration Proposition 297.26 that any continuous map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration is quite surprising and perhaps also somewhat puzzling. Therefore we want to consider one more example.

Example. We denote by Σ the surface of genus 2 and we denote by T the torus. We consider the map $f\colon \Sigma \to T$ that is illustrated in the figure below. We denote by $p\colon P_f \to T$ the mapping path fibration and we denote by F the corresponding homotopy fiber. So what can we say about the mysterious homotopy fiber F?



We have the following diagram:

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \pi_2(F) & \longrightarrow & \pi_2(P_f) & \xrightarrow{p_*} & \pi_2(T) & \xrightarrow{\partial} & \pi_1(F) & \longrightarrow & \pi_1(P_f) & \xrightarrow{p_*} & \pi_1(T) & \longrightarrow & \ldots \\ & & & & \uparrow{\scriptstyle h_*}\cong & \nearrow{\scriptstyle f_*} & & & & & \uparrow{\scriptstyle h_*}\cong & \nearrow{\scriptstyle f_*} & & & \\ & & & & \pi_2(\Sigma) & & & & & & \pi_1(\Sigma). & & & & \end{array}$$

We make the following explanations and observations:

(1) The top sequence is the long exact sequence coming from the Fibration-LES Theorem 297.11. The vertical maps are induced by the homotopy equivalence $h\colon \Sigma \to P_f$ from the Replace-by-Fibration Proposition 297.26 (1). In particular the vertical maps are isomorphisms by the Homotopy Equivalence-π_n-Proposition 134.17 (1).
(2) Since $p \circ h = f$ we see that the diagram commutes.
(3) By the discussion on page 2587 we know that $\pi_n(T) = \pi_n(\Sigma) = 0$ for $n \geq 2$.
(4) It is pretty clear that the map $f_*\colon \pi_1(\Sigma) \to \pi_1(T)$ is an epimorphism.
(5) We obtain from the above discussion and from the Surface Group–Free Subgroup Proposition 130.9 that F is path-connected, that $\pi_n(F) = 0$ for $n \geq 2$ and that $\pi_1(F)$ is a free group that is *not finitely generated*.

Thus we see that the homotopy groups of F are isomorphic to the homotopy groups of an infinite wedge of circles. □

297.7. **Adjoint functors.** In Chapter 22 we introduced the notion of a "cofibration" and we just got acquainted with the notion of a "fibration". In this section we will discuss the relationship between these two notions.

First we point out a formal similarity between fibrations and cofibrations, namely they both give rise to long exact sequences:

(1) Let $p\colon Y \to B$ be a fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and we denote by $F = p^{-1}(b_0)$ the homotopy fiber of p. Let $i\colon F \to Y$ be the inclusion map. By the Fibration-LES Theorem 297.11 we obtain the following long exact sequence of homotopy groups:

$$\ldots \xrightarrow{\partial_{n+1}} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_n} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \ldots.$$

(2) Let $i\colon A \to X$ be a cofibration. We refer to the mapping cone $C := \mathrm{Cone}(i)$ as the *cofiber*. If $i(A)$ is a closed non-empty subset of X, then it follows easily from the Reduced LES-of-Pair Proposition 147.8 and the Cofibration–Relative-to-Quotient-H_*-Corollary 153.16 that we obtain a long exact sequence of reduced homology groups of the following form:

$$\ldots \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(C) \xrightarrow{\partial_n} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \ldots.$$

This basically recovers the statement of the Mapping Cone-H_*-Lemma 153.17.

Here is another similarity between fibrations and cofibrations:

(1) Let $p\colon Y \to B$ be a fibration and let $f\colon A \to B$ be a continuous map. As in the Topological-Pullback Proposition 7.22 we consider the pullback

$$\begin{array}{ccc} f^*Y & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow[f]{} & B \end{array}$$

where $f^*Y = \{(a, y) \in A \times Y \mid f(a) = p(y)\}$. In the Fibration-Pullback Lemma 297.23 we saw that the pullback $f^*Y \to A$ is also a fibration.

(2) Now let $i\colon A \to X$ be a cofibration and let $f\colon A \to B$ be a continuous map to another topological space. As in Definition 8.13 we consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & f_*X \end{array}$$

where $f_*X = (X \sqcup B)/i(a) \sim f(a)$ for all $a \in A$. By the Opposite Map-Cofibration Proposition 22.23 (1) we will know that the pushout $B \to f_*X$ is again a cofibration.

So, to summarize: *the pullback of a fibration is again a fibration* and *the pushout of a cofibration is again a cofibration.*

The main goal of this section though is to formulate an arguably more direct relationship between the notions of a fibration and a cofibration. The key to doing so is Definition 6.33 from category theory.

Definition 297.28. We say that two covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ form an adjoint pair if given any $c \in \mathrm{Ob}(\mathcal{C})$ and $d \in \mathrm{Ob}(\mathcal{D})$ there exists a bijection

$$\mathrm{Mor}_{\mathcal{D}}(F(c), d) \to \mathrm{Mor}_{\mathcal{C}}(c, G(d))$$

that is natural in c and d.[3240] We also say that the functor F is the left adjoint of G and that G is the right adjoint of F.

Examples.

(1) Let $\mathcal{AbGroup}$ be the category of abelian groups and let A be an abelian group. The Hom–Tensor Adjunction Lemma 312.13 says that the two functors

$$\begin{array}{rcl} F\colon \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ B & \mapsto & A \otimes B \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{AbGroup} & \to & \mathcal{AbGroup} \\ B & \mapsto & \operatorname{Hom}(A, B) \end{array}$$

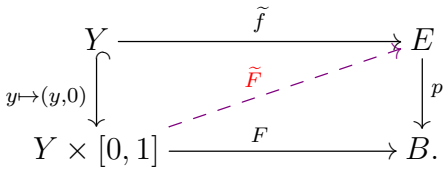
form an adjoint pair.

(2) Let $\mathcal{Top}$ be the category of topological spaces. Rather similarly to the previous example we consider the two functors[3241]

$$\begin{array}{rcl} F\colon \mathcal{Top} & \to & \mathcal{Top} \\ X & \mapsto & X \times [0,1] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{Top} & \to & \mathcal{Top} \\ X & \mapsto & X^{[0,1]}. \end{array}$$

In the Cylinder–Path Space Adjunction Lemma 16.11 we showed that the functors F and G are adjoint. □

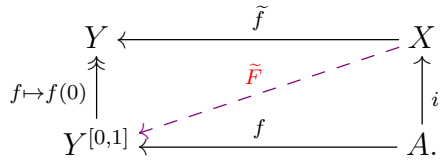
Now we can explain the formal relationship between fibrations and cofibrations. Let E and B be topological spaces and let $p\colon E \to B$ be a continuous map. Recall that by definition $p\colon E \to B$ is a fibration if we can always complete the following diagram:



Now we "dualize" this diagram as follows:

(1) we reverse the orientation of each arrow,
(2) we replace monomorphisms by epimorphisms,
(3) we replace $Y \times [0,1]$ by $Y^{[0,1]}$,
(4) for purely psychological reasons we replace $p\colon E \to B$ by $i\colon A \to X$.

We obtain the following diagram:

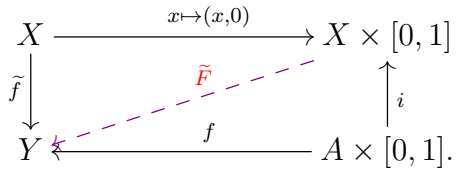


Finally recall that it follows from the Cylinder–Path Space Adjunction Lemma 16.11 that given two topological spaces A and B we have a natural identification

$$\operatorname{Mor}(A \times [0,1], B) \;=\; \operatorname{Mor}(A, B^{[0,1]}).$$

[3240] We leave it to our reader to figure out what "natural in c and d" means.
[3241] Here we equip $X \times [0,1]$ with the product topology and we equip $X^{[0,1]}$ with the compact-open topology.

Thus (with a few other more or less mysterious changes) the previous challenge diagram translates into the following challenge diagram:



But this challenge diagram looks familiar: it is precisely the diagram that appears in the definition of a cofibration.

We conclude this chapter with three quotes:

(1) "In the mathematical disciplines of algebraic topology and homotopy theory, Eckmann–Hilton duality in its most basic form, consists of taking a given diagram for a particular concept and reversing the direction of all arrows [...]."[3242] [3243]

(2) "Eckmann–Hilton duality can be extremely helpful as an organizational principle, reducing significantly what one has to remember, and providing valuable hints on how to proceed in various situations." [**Hat2002**, p. 464]

(3) "In mathematics [...] currying[3244] is the technique of translating the evaluation of a function that takes multiple arguments into evaluating a sequence of functions, each with a single argument. For example, a function that takes two arguments, one from X and one from Y, and produces outputs in Z, by currying is translated into a function that takes a single argument from X and produces as outputs functions from Y to Z."[3245]

So we see that in the above discussion we performed Eckmann-Hilton duality combined with some currying.

297.8. Exercises for Chapter 297.

Exercise 297.1. Show that the projection

$$\begin{array}{ccc} \overline{B}^2 & \to & \overline{B}^1 = [-1,1] \\ (x,y) & \mapsto & x \end{array}$$

is a Serre fibration.

Exercise 297.2. We consider

$$Y := \{(y-1,y) \,|\, y \in [0,1]\} \cup \{(\tfrac{1}{n},y) \,|\, y \in [0,1] \text{ and } n \in \mathbb{N}\}$$

together with the projection $p\colon Y \xrightarrow{(x,y)\mapsto y} [0,1]$.

(a) Show that $p\colon Y \to [0,1]$ is a Serre fibration.

[3242] https://en.wikipedia.org/wiki/Eckmann-Hilton_duality

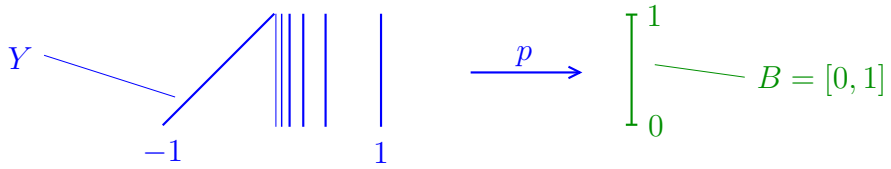
[3243] Note that the notion of pullback and pushout are related by Eckmann-Hilton duality.

[3244] This notion is named after the American mathematician Haskell Brooks Curry (1900-1982) who was an American mathematician and logician. There are three programming languages named after him, namely Haskell, Brook and Curry.

[3245] https://en.wikipedia.org/wiki/Currying#Algebraic_topology

(b) Show that $p\colon Y \to [0,1]$ is not a fibration.
Hint. We set $F := p^{-1}(1)$. We consider the inclusion $i\colon F \to Y$. Show that there exists a homotopy from $p \circ i = \mathrm{const}_1\colon F \to F$ to the constant map $\mathrm{const}_0\colon F \to F$ cannot be lift to a homotopy of i.

Remark. This is the example from [**BrownR1966**, p. 575]. You should not expect to find more details in that reference.



Exercise 297.3. . Let $n \in \mathbb{N}_0$ and let x_0 be a point on S^n. It follows from the Evaluation Map Proposition 16.9 that the map

$$\begin{array}{rcl} e\colon P(S^n, x_0) & \to & S^n \\ \{f\colon [0,1] \to S^n \,|\, f(0) = x_0\} & \mapsto & f(1) \end{array}$$

is continuous. For which $n \in \mathbb{N}_0$ does e admit a right inverse?

Exercise 297.4. In the proof of the Doppelgänger Corollary 213.11 we used cup products to show that the product $S^2 \times S^4$ and the complex projective space $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent. Reprove this result using homotopy groups.

Exercise 297.5. We consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. We define the mapping path fibration $q\colon P_f \to [-1,1]$ as in the Replace-by-Fibration Proposition 297.26. Show that for each $y \in [-1,1]$ the preimage $q^{-1}(y)$ admits a deformation retraction to a circle.
Remark. See the figure on page 5718 for an illustration of the mapping path fibration.

Exercise 297.6. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. Show "with your bare hands" that the map

$$\begin{array}{rcl} \partial_n\colon \pi_n(X,x_0) & \xrightarrow{\cong} & \pi_{n-1}(\Omega(X,x_0), c_{x_0}) \\ [f\colon ([0,1]^n, \partial([0,1]^n)) \to (X,x_0)] & \mapsto & \left[\begin{array}{l} ([0,1]^{n-1}, \partial([0,1]^{n-1})) \to (\Omega(X,x_0), c_{x_0}) \\ \qquad x \mapsto \left(\begin{array}{l} [0,1] \to X \\ \quad t \mapsto f(x,t) \end{array}\right) \end{array}\right] \end{array}$$

is a natural isomorphism.
Remark. This is of course the statement of the Loop Space-Homotopy Groups Corollary 297.22.

Exercise 297.7. Let (X, x_0) be a pointed topological space. In the Loop Space-Homotopy Groups Corollary 297.22 we give for any $n \geq 1$ a natural isomorphism

$$\pi_n(X,x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X,x_0), c_{x_0}).$$

What does this statement mean for $n = 1$? It seems to say that $\pi_0(\Omega(X,x_0), c_{x_0})$ admits a natural group structure. Can you define such a natural group structure without using the above isomorphism?

Exercise 297.8. Convince yourself, without using the Loop Space-Homotopy Groups Corollary 297.22, that the loop in the pointed topological space $(\Omega(S^2, N), c_N)$ that is shown in the figure on page 5715 is not null-homotopic.

Exercise 297.9. Let X be a topological space and let $x_0 \in X$ be a base point. Recall that the loop space $\Omega(X, x_0)$ is defined as the set $\Omega(X, x_0) = \{f \in X^{[0,1]} \,|\, f(0) = f(1) = x_0\}$ together with the compact-open topology. We denote by $\sim$ the equivalence relation on $\Omega(X, x_0)$ given by homotopy of paths. By definition we have $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$. We equip $\pi_1(X, x_0)$ with the quotient topology. We can thus view $\pi_1(X, x_0)$ as a topological space. Show that $\pi_1(S^1, 1)$ is a discrete space.
Hint. It is possibly easier to show that for every locally simply pointed connected topological space X and any $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$ is a discrete space. For this you can make use of Locally Path-Connected Lemma 4.57 and the Compact-Open Exponential Proposition 16.10.
Remark. There are examples of pointed topological spaces for which $\pi_1(X, x_0)$ is not a discrete space. For example the fundamental group of the Hawaiian Earrings, introduced in Exercise 105.1, is not a discrete space. We refer to [**Fab2005, Fab2011**] for details.

Exercise 297.10.

(a) Show that there is no functor

$$\Phi\colon \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed}\\ \text{topological spaces}\end{array} \to \begin{array}{c}\text{category of } \mathcal{PTop} \text{ pointed}\\ \text{topological spaces}\end{array}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n+1}(\Phi(X, x_0)).$$

(b) Show that there is no functor

$$\Psi\colon \text{category of topological spaces} \to \text{category of topological spaces}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n-1}(\Psi(X)).$$

Hint. In Exercise 147.3 (b) we proved the following statement: If X is a topological space with $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}$, then given any continuous map $f\colon X \to X$ the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

Exercise 297.11. Let X be a topological space. We consider the map

$$\begin{array}{rcl} p\colon X^{[0,1]} & \to & X \times X \\ (f\colon [0,1] \to X) & \mapsto & (f(0), f(1)) \end{array}$$

(a) Show that $p\colon X^{[0,1]} \to X \times X$ is a fibration.
(b) Show that there exists a right inverse to p, i.e. a continuous map $s\colon X \times X \to X^{[0,1]}$ with $p \circ s = \mathrm{id}$, if and only if X is contractible.

Exercise 297.12. Let X be a CW-complex and let A be a subcomplex. Furthermore let Y be a topological space. Show that the map

$$\begin{array}{rcl} Y^X & \to & Y^A \\ (f\colon X \to Y) & \mapsto & (f|_A\colon A \to Y) \end{array}$$

is a fibration where the fiber is homotopy equivalent to $Y^{X/A}$.
Hint. Use the Homotopy Extension Theorem 140.2.

Exercise 297.13. Let X be a topological space and let $x_0 \in X$. Determine the homotopy fiber of the inclusion $\{x_0\} \to X$.

Exercise 297.14. Make sense of the statement that the homotopy fiber is functorial.

CHAPTER 298

Fiber bundles and fibrations

In this chapter we will reintroduce fiber bundles. For us they will be the most important source of fibrations. In this and the following chapter one of our favorite games will be to find more and more fiber bundles to obtain lots of non-trivial information on homotopy groups of various topological spaces.

This chapter is organized as follows:

- In Section 298.1 we will recall the definition of fiber bundles.
- In Section 298.2 we will provide some examples of fiber bundles.
- In Section 298.3 we will discuss properties of fiber bundles.
- In Section 298.4 we will discuss the Hopf fiber bundle s arising from complex and quaternionic projective spaces.
- In Section 298.5 we will discuss the Hopf fiber bundle arising from the octonions.
- In Section 298.6 we will provide exercises for Chapter 298.

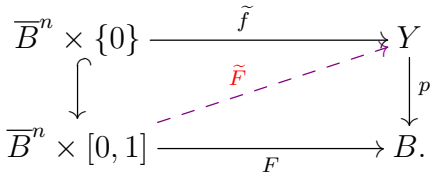
In the next Chapter 299 we will discuss homotopy groups of some Lie groups.

298.1. **The definition of fiber bundles.** The following proposition says that being a Serre fibration is a "local property".

Proposition 298.1. (Local Serre Fibration Proposition) Let $p\colon Y \to B$ be a continuous map between topological spaces. If for every $b \in B$ there exists an open neighborhood U of b such that $p\colon p^{-1}(U) \to U$ is a Serre fibration, then $p\colon Y \to B$ itself is a Serre fibration.

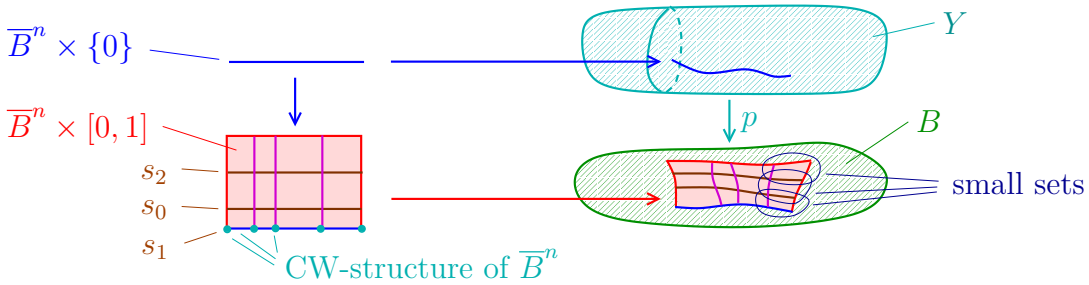
Proof. Let $p\colon Y \to B$ be a continuous map between topological spaces. In this proof we say that $U \subseteq B$ is *small* if $p\colon p^{-1}(U) \to U$ is a Serre fibration.

Now assume that $p\colon Y \to B$ has the property that B can be covered by small open subsets. We want to show that this hypothesis implies that the map $p\colon Y \to B$ is a Serre fibration. Therefore let $n \in \mathbb{N}_0$, let $F\colon \overline{B}^n \times [0,1] \to B$ and let $\widetilde{f}\colon \overline{B}^n \times \{0\} \to Y$ be continuous maps such that the rectangle in the following diagram commutes:



We need to show that there exists a diagonal map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes.

Claim. There exists a CW-structure X on $\overline{B}^n$ and real numbers $0 = s_0 < s_1 < \cdots < s_k = 1$ such that for each cell e of X and each $i \in \{0, \dots, k-1\}$ the image $F(e \times [s_i, s_{i+1}])$ is small.



Proof. By the Convex-to-Ball Proposition 4.28 (2) we can identify $\overline{B}^n$ with $[0,1]^n$. Since B is by hypothesis covered by small open subsets and since any subset of a small set is again small we obtain from the Lebesgue Lemma 10.9 that there exists a $k \in \mathbb{N}$ such that each

$$F\Big(\underbrace{\left[\tfrac{a_1}{k}, \tfrac{a_1+1}{k}\right] \times \ldots \left[\tfrac{a_n}{k}, \tfrac{a_n+1}{k}\right]}_{\subseteq [0,1]^n = \overline{B}^n} \times \underbrace{\left[\tfrac{a_{n+1}}{k}, \tfrac{a_{n+1}+1}{k}\right]}_{\subseteq [0,1]}\Big)$$

is small for all $a_1, \dots, a_{n+1} \in \{0, \dots, k-1\}$. Similar to the discussion on page 2690 we equip $[0,1]^n = \overline{B}^n$ with the CW-structure given by the "subcubes" of side length $\frac{1}{k}$. Furthermore we set $s_i = \frac{i}{k}$, $i = 0, \dots, k$. By design this CW-structure and this choice of s_i has the desired properties. ⊞

We continue with the following claim:

Claim. Let $i \in \{0, \dots, k-1\}$. Given a continuous map $\widetilde{F}_i \colon \overline{B}^n \times \{s_i\} \to Y$ with $p \circ \widetilde{F}_i = F|_{\overline{B}^n \times \{s_i\}}$ there exists a continuous map $\widetilde{F}_{i+1} \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{B}^n \times \{s_i\} & \xrightarrow{\widetilde{F}_i} & Y \\ \downarrow & \overset{\widetilde{F}_{i+1}}{\nearrow} & \downarrow p \\ \overline{B}^n \times [s_i, s_{i+1}] & \xrightarrow[F]{} & B. \end{array}$$

Proof. In fact the claim follows immediately from the argument of the proof of the CW–Homotopy–Lift Proposition 297.12 applied to the CW-complex $K := X = \overline{B}^n$ and the subcomplex $L = \varnothing$. Indeed, in the proof of the CW–Homotopy–Lift Proposition 297.12 we did not need to assume that $p \colon Y \to B$ is a Serre fibration, all we ever used is that for every cell e of B the image $F(e)$ is small in the above sense. But in our present setting this is guaranteed by our choice of the CW-structure on $\overline{B}^n$. ⊞

We set $\widetilde{F}_0 = \widetilde{f}_0 \colon \overline{B}^n \times \{0\} \to Y$. Iteratively applying the previous claim we obtain maps $\widetilde{F}_i \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ that agree on the overlaps of the domains. It follows from the Homotopy Combination Lemma 18.8 (1) that all these maps together define a continuous map $\widetilde{F} \colon \overline{B}^n \times [0,1] \to Y$ which by construction has the desired properties. ■

As we saw on page 5692, every projection $p \colon B \times F \to B$ is a (Serre) fibration. This observation, together with the Local Serre Fibration Proposition 298.1 leads us naturally to the following definition, which we already encountered in Definition 238.1:

Definition 298.2. Let F be a topological space.

(1) A continuous map $p\colon Y \to B$ between two topological spaces is called an F-bundle over B if given any $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

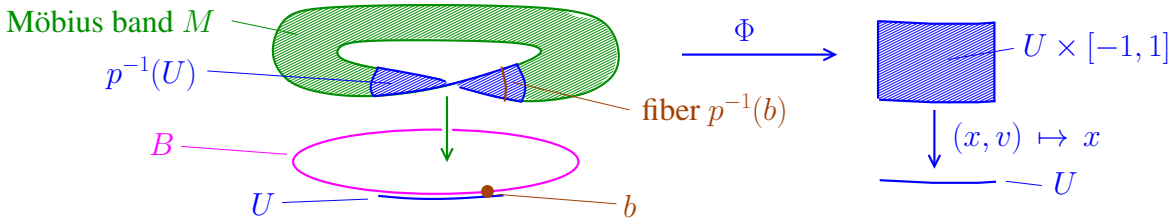
$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f)\mapsto x} & \\ & U. & \end{array}$$

Given an F-bundle $p\colon Y \to B$ we adopt the following language:

- We refer to B as the base space, to Y as the total space and to F as the fiber of the F-bundle.
- We refer to a homeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ as above as trivialization.
- At times we refer to the map p also a bundle map.

(2) A continuous map $p\colon Y \to B$ between two topological spaces is called a bundle map if there exists a topological space F such that p is an F-bundle.

Example. In the following figure we show a map $p\colon M \to B$ from the Möbius band to a circle B.



In the figure above we illustrate that this map is a $[-1,1]$-bundle. □

For clarity we state now the obvious as a proposition:

Proposition 298.3. (Fiber Bundle–Serre Fibration Proposition) Let F be a topological space and let $p\colon Y \to B$ be an F-bundle in the sense of Definition 298.2.

(1) All fibers $p^{-1}(b)$ with $b \in B$ are homeomorphic to F.

(2) The map $p\colon Y \to B$ is a Serre fibration $F \to Y \to B$ in the sense of Definition 297.1.

Proof.

(1) Let $b \in B$. By definition of a bundle there exists an open neighborhood U of b and a homeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (x,f)\mapsto x} & \\ & U. & \end{array}$$

We now see that $p^{-1}(b)$ is homeomorphic to $\{b\} \times F$, equipped with the subspace topology coming from the product topology on $U \times F$. By the Product Topology-Basics Lemma 7.6 (1) we know that the map $F \to U \times F$, given by $x \mapsto (b, f)$ is an embedding. This shows that F is homeomorphic to $\{b\} \times F$ and thus to $p^{-1}(b)$.

(2) By the discussion on page 5692 we know that a projection $U \times F \to U$ from a product is a fibration. It is now clear that a fiber bundle is a local Serre fibration. It now follows from the Local Serre Fibration Proposition 298.1 that any F-bundle is also Serre fibration. ■

The following proposition, which is almost identical to the Fiber-LES-Proposition 239.8, will be the incarnation of the Fibration-LES Theorem 297.11 that we will use most often.

Proposition 298.4. (Fiber-LES-Proposition) Let $p\colon Y \to B$ be a bundle map with fiber F.

(1) We assume that B is non-empty. We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$. By the Fiber Bundle–Serre Fibration Proposition 298.3 (1) we can pick an identification $F =_{\mathrm{i}} p^{-1}(b_0)$. We denote by $i\colon F =_{\mathrm{i}} p^{-1}(b_0) \to Y$ the inclusion. By Fiber Bundle–Serre Fibration Proposition 298.3 (2) we know that $p\colon Y \to B$ is a Serre fibration to define maps $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$, $n \in \mathbb{N}$ (which are homomorphisms except for $n = 0$) as in the Fibration-LES Theorem 297.11. The following sequence is exact:

$$\begin{aligned}
\ldots \xrightarrow{\partial_3} \pi_2(F, y_0) \xrightarrow{i_*} \pi_2(Y, y_0) \xrightarrow{p_*} \pi_2(B, b_0) \xrightarrow{\partial_2} \\
\xrightarrow{\partial_2} \pi_1(F, y_0) \xrightarrow{i_*} \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \\
\xrightarrow{\partial_1} \pi_0(F, y_0) \xrightarrow{i_*} \pi_0(Y, y_0) \xrightarrow{p_*} \pi_0(B, b_0).
\end{aligned}$$

(2) The maps $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$ are natural in the sense that if we are given a commutative diagram

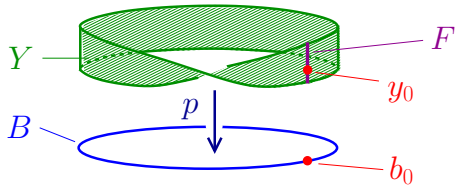
$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

where $p'\colon Y' \to B'$ is a bundle map, then given any base point $y'_0 \in Y'$ the following diagram commutes:

$$\begin{array}{ccc} \pi_n(B', p'(y'_0)) & \xrightarrow{\partial_n} & \pi_{n-1}(F', y'_0) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} \\ \pi_n(B, p(y_0)) & \xrightarrow{\partial_n} & \pi_{n-1}(F, y_0), \end{array}$$

where $y_0 := g(y'_0)$, $F := p^{-1}(y_0)$ and $F' := (p')^{-1}(y'_0)$.

(3) If the fiber F and the base space B are path-connected, then the total space Y is also path-connected.



Proof.

(1) This statement follows immediately from the Fiber Bundle–Serre Fibration Proposition 298.3 together with the Fibration-LES Theorem 297.11 (2).

(2) This statement follows easily from the Fibration-LES Theorem 297.11 (1b).

(3) This statement follows immediately from (1). ■

298.2. **Examples of fiber bundles.** The Fiber-LES-Proposition 298.4 sparks (or rekindles) our interest in fiber bundles. Before we discuss examples in detail let us recall the following definition:

Definition 298.5.

(1) For any two topological spaces B and F the projection $p\colon B \times F \to B$ onto the first factor is evidently an F-bundle. We refer to it as the trivial F-bundle over B.

(2) Let B be a topological space and let $p\colon Y \to B$ and $q\colon Z \to B$ be two F-bundles over B. An isomorphism of the two F-bundles over B is a homeomorphism $f\colon Y \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & Z \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B. & \end{array}$$

(3) Any F-bundle over some topological space B that is isomorphic to the above trivial bundle $B \times F \to F$ from (1) is called trivializable, otherwise it is called non-trivializable.

With this language it now makes sense to consider more examples:

Examples.

(1) The "obvious projection" from the Möbius band to the circle S^1, i.e. the map

$$\begin{array}{rcl} Y = ([0,1] \times [-1,1])/(0,y) \sim (1,-y) & \to & S^1 \\ [(x,y)] & \mapsto & \exp(2\pi \mathrm{i} x) \end{array}$$

is easily seen to be a fiber bundle with fiber $[-1,1]$. Since the Möbius band is not homeomorphic to the trivial bundle $S^1 \times [-1,1]$ we see that the bundle is non-trivializable.

(2) (a) The same argument as in (2) shows that the Klein bottle is a non-trivializable S^1-bundle over S^1. In the Surface–Connected Sum Lemma 54.15 (1) we saw that the connected sum $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle.

(b) In (a) we have thus shown that $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is a non-trivializable S^1-bundle over S^1.

(c) In the Projective Space–Connected Sum Proposition 238.10 we generalized this statement and we showed that the connected sum $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$ is a non-trivializable S^2-bundle over S^2 and that the connected sum $\mathbb{H}\mathrm{P}^2 \# \mathbb{H}\mathrm{P}^2$ is a non-trivializable S^4-bundle over S^4.

(3) An n-dimensional vector bundle $p\colon W \to B$, in the sense of Definition 265.1, is by definition in particular an $\mathbb{R}^n$-bundle over B. □

We continue with more examples of fiber bundles.

Examples.

(1) As we pointed out above, any n-dimensional vector bundle $p\colon W \to B$ is by definition an $\mathbb{R}^n$-bundle over B. In fact p gives rise to two other bundles which are much more interesting topological objects:

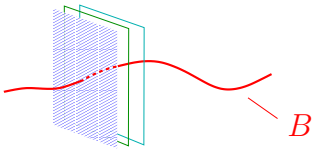
(a) We denote by $s\colon B \to W$ the zero-section, i.e. the map that is given by sending each $b \in B$ to the zero-element of the vector space $p^{-1}(y)$. We write $W^* \coloneqq W \setminus s(B)$.

It follows easily from the definitions that $p\colon W^* \to B$ is a fiber bundle with fiber $\mathbb{R}^n \setminus \{0\}$.

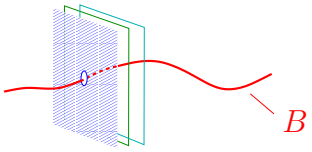
(b) Let $\lambda \in \mathbb{R}$. By definition of a vector bundle we have a well-defined multiplication by λ on each fiber. We define

$$SW \;:=\; W^*/\sim \qquad \begin{array}{l}\text{where } v \sim w \text{ if } p(v) = p(w) \text{ and} \\ \quad \text{if there exists a } \lambda > 0 \text{ with } \lambda v = w.\end{array}$$

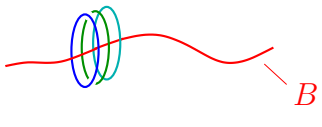
The map $p\colon W^* \to B$ descends to a map $p\colon SW \to B$. We leave it as an entertaining exercise to verify that $p\colon SW \to B$ is an S^{n-1}-bundle.



$\mathbb{R}^n$-bundle $p\colon W \to B$



$(\mathbb{R}^n\setminus\{0\})$-bundle $p\colon W^* \to B$



S^{n-1}-bundle $p\colon SW \to B$

(2) A smooth $\overline{B}^n$-bundle $p\colon W \to M$ over a smooth manifold M, in the sense of Definition 57.1, is by definition a $\overline{B}^n$-bundle in the above sense. Slightly more interesting: If the smooth manifold M has no boundary and if we restrict p to a map $\partial W \to M$, then it follows easily from Proposition 25.34 that the restriction of p to ∂W defines an S^{n-1}-bundle $p\colon \partial W \to M$. □

298.3. **Properties of fiber bundles.** The following lemma is a special case of the Bundle–Pullback Proposition 240.21:

Lemma 298.6. (Bundle Pullback Lemma) Let $g\colon C \to B$ be a continuous map between topological spaces and let $p\colon Y \to B$ be a fiber bundle with fiber F. As in the Topological-Pullback Proposition 7.22 we consider the pullback

$$g^*Y \;:=\; \{(c,f)\,|\,c \in C \text{ and } f \in p^{-1}(g(c))\} \;\subseteq\; C \times Y,$$

which is equipped with the subspace topology. Then the map

$$\begin{array}{rcl} q\colon g^*Y \;:=\; \{(c,f)\,|\,c \in C \text{ and } f \in p^{-1}(g(c))\} & \to & C \\ (c,f) & \mapsto & c \end{array}$$

is also an F-bundle.

Proof. The proof is, not surprisingly, also almost identical to the proof of the Vector Bundle–Pullback Proposition 265.36. ■

Remark. The pullback of bundles satisfies basically the same statements as the pullback of vector bundles, see the Vector Bundle–Pullback Proposition 265.36 and the Vector Bundle-Pullback–Functorial Lemma 265.39. There is surely no need to spell out these properties again. □

In the Mapping Torus–Bundle Lemma 238.14 and the Mapping Torus–$(F,\mathcal{G})$-Bundle Lemma 241.11 we already classified bundles over S^1. For the reader's convenience we now prove the next lemma, which gives a down-to-earth classification of bundles over compact intervals $[a,b]$ and also over the circle S^1.

Lemma 298.7. (Bundle-over-Interval Lemma)
(1) Every bundle over a compact interval $[a, b]$ is trivializable.
(2) If $p\colon Y \to S^1$ is a fiber bundle with fiber X, then there exists a homeomorphism $f\colon X \to X$ such that p is isomorphic to the above fiber bundle $\mathrm{Tor}(X, f) \to S^1$.

Sketch of proof.
(1) Since $[a, b]$ is contractible and compact we see that this statement is just a special case of the Bundles-Homotopies Proposition 241.10. In Exercise 298.3 we will see that in this particular case one can also give a down-to-earth proof.
(2) Suppose we are given a fiber bundle $p\colon Y \to S^1$ with fiber X. Let $\varphi\colon [0, 2\pi] \to S^1$ be the map that is given by $\varphi(t) = \exp(\mathrm{i}t)$. It follows from (1) that the pullback bundle

$$q\colon \varphi^*Y = \{(t, v) \in [0, 2\pi] \times Y \mid v \in p^{-1}(\varphi(t))\} \;\to\; [0, 2\pi]$$

is trivializable, i.e. there exists a homeomorphism $\Phi\colon [0, 2\pi] \times X \to \varphi^*Y$ such that the following diagram commutes:

$$\begin{array}{ccc} [0,2\pi]\times X & \xrightarrow{\quad\Phi\quad} & \varphi^*Y \\ & {\scriptstyle (t,x)\mapsto t}\searrow \quad \swarrow{\scriptstyle q} & \\ & [0,2\pi]. & \end{array}$$

We define $f\colon X \to X$ to be the homeomorphism that is given by

$$\begin{array}{ccccccccccc} & & & & & & \varphi(0)=1=\varphi(2\pi) & & & & \\ & & & & & & \downarrow & & & & \\ X & \xrightarrow{x\mapsto(0,x)} & \{0\}\times X & \xrightarrow{\Phi} & q^{-1}(0) & \to & p^{-1}(1) & \leftarrow & q^{-1}(2\pi) & \xleftarrow{\Phi} & \{2\pi\}\times X \xrightarrow{(2\pi,x)\mapsto x} X. \\ & & & & (0,v) & \mapsto & v & & & & \\ & & & & & & v & \mapsto & (2\pi, v). & & \end{array}$$

We leave it as a mildly interesting exercise to the reader to verify that $p\colon Y \to S^1$ is isomorphic to the fiber bundle $\mathrm{Tor}(X, f) \to S^1$. ■

We conclude this section with an example that shows how the above results can be used in practice. Thus let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the smooth manifold S^n. We calculate that for $i = 1, \ldots, n-k-2$ we have

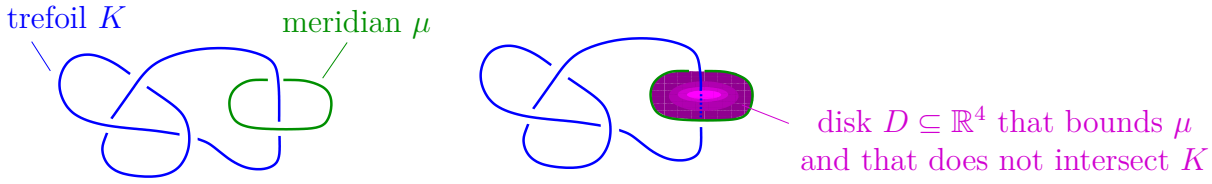
$$\begin{array}{ccccc} \mathrm{H}_i(S^n \setminus K; \mathbb{Z}) & \cong & \mathrm{H}^{n-i-1}(K; \mathbb{Z}) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{by the Sphere–Alexander Duality Theorem 217.8} & & \text{by the Manifold-Top-H}^*\text{-Theorem 201.26 (1) and the fact that } n-i-1 > k = \dim(K) & \end{array}$$

The following proposition shows that not only do the first $n-k-2$ homology vanish, but the first $n-k-2$ homotopy groups also vanish.

Proposition 298.8. (Sphere–Complement-π_*-Proposition) Let $n \in \mathbb{N}$ and let K be a closed k-dimensional submanifold of the smooth manifold S^n. For $i = 1, \ldots, n-k-2$ we have $\pi_i(S^n \setminus K) = 0$.

Example. Let K be the trefoil in $\mathbb{R}^3$ and let $\mu \subseteq \mathbb{R}^3$ be a meridian from Definition 109.1. (See also the figure below to the left.) In the Knot Complement-H_*-Proposition 185.29 we saw that μ represents a non-zero element in $\mathrm{H}_1(S^3 \setminus K)$. It follows from the naturality of the Hurewicz homomorphism, see the Hurewicz Homomorphism Proposition 166.3 (3),

that μ also represents a non-trivial element in $\pi_1(S^3 \setminus K)$. Via the usual smooth embedding $\mathbb{R}^3 \to \mathbb{R}^3 \times \{0\} \to \mathbb{R}^4$ we can view K and μ as submanifolds of $S^4 = \mathbb{R}^4 \cup \{\infty\}$. By the Sphere–Complement-π_*-Proposition 298.8 we have $\pi_1(S^4 \setminus K) = 0$. In particular we obtain that μ is trivial in $\pi_1(S^4 \setminus K)$. In fact in the figure below we show an embedded disk D in $\mathbb{R}^4 \subseteq S^4$ whose boundary is given by μ and that does not intersect K.



Here we use the page **??**, i.e. "pinkness" of a point measures how large is the fourth coordinate. □

Proof. Let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the smooth manifold S^n. If $n-k-2 \leq 0$, then there is nothing to prove. Thus we assume that $n-k-2 \geq 1$, i.e. we assume that $n \geq k+3$.

We want to show that $\pi_i(S^n \setminus K) = 0$ for $i = 1, \ldots, n-k-2$. We already know that the corresponding homology groups vanish. Thus by the Hurewicz Corollary 167.13 it suffices to show that $\pi_1(S^n \setminus K) = 0$. That is precisely what we will do in the remainder of this proof.

In the following we consider the case that K is connected. We leave it to the reader to modify the proof to deal with the case that K is disconnected. By the Tubular Neighborhood–Existence Theorem 57.11 the closed submanifold K admits a tubular neighborhood $p\colon W \to K$. Recall that W is a compact n-dimensional submanifold of S^n and that p is in particular a $\overline{B}^{n-k}$-bundle over K. As we pointed out on page 5732, the projection $p\colon \partial W \to K$ is an S^{n-k-1}-bundle.

We write $X = S^n \setminus \mathring{W}$. It follows from the Basics-of-Tubular Neighborhood Lemma 57.10 together with the Homotopy Equivalence-π_1-Proposition 90.9 (1a) that $\pi_1(S^n \setminus K)$ is isomorphic to $\pi_1(X)$. Thus it suffices to show that X is simply connected.

By the Codimension-Zero Submanifold Proposition 25.49 we know that X is a compact n-dimensional submanifold of S^n with $W \cap X = \partial W = \partial X$. From the Fiber-LES-Proposition 298.4 (3) we know that ∂W is connected. We pick a base point $x_0 \in \partial W$. By the Manifold–Seifert-van Kampen Theorem 102.1 we know that the inclusion-induced map

$$\pi_1(X, x_0) *_{\pi_1(\partial W, x_0)} \pi_1(W, x_0) \to \underbrace{\pi_1(S^n, x_0)}_{=0}$$

is an isomorphism. We recall that the Amalgamated Product Lemma 98.24 (2a) says that $A *_G G \cong A$. Thus, if we want to show that $\pi_1(X, x_0) = 0$ it follows from the Amalgamated Product Lemma 98.24 (2a) that it suffices to prove the following claim:

Claim. The inclusion-induced map $\pi_1(\partial W, x_0) \to \pi_1(W, x_0)$ is an isomorphism.

Proof. We denote both of the inclusions $\partial W \to W$ and $S^{n-k-1} \to \overline{B}^{n-k}$ by i. We have a commutative diagram

$$\begin{array}{ccc} \partial W & \xrightarrow{\quad i \quad} & W \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

of fiber bundles with fibers S^{n-k-1} and $\overline{B}^{n-k}$. From the Fiber-LES-Proposition 298.4 (2) we obtain the following commutative diagram of long exact sequences of morphisms between pointed sets:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \pi_2(K) & \xrightarrow{\partial_2} & \overbrace{\pi_1(S^{n-k-1})}^{=0} & \longrightarrow & \pi_1(\partial W) & \longrightarrow & \pi_1(K) & \xrightarrow{\partial_1} & \overbrace{\pi_0(S^{n-k-1})}^{=0} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i_*} & & \\ \dots & \longrightarrow & \pi_2(K) & \xrightarrow{\partial_2} & \underbrace{\pi_1(\overline{B}^{n-k})}_{=0} & \longrightarrow & \pi_1(W) & \longrightarrow & \pi_1(K) & \xrightarrow{\partial_1} & \underbrace{\pi_0(\overline{B}^{n-k})}_{=0} & \longrightarrow & \dots \end{array}$$

Since $n-k \geq 3$ we have $n-k-1 \geq 2$, thus $\pi_1(S^{n-k-1}) = 0$ by the Sphere-π_1-Proposition 88.24. Furthermore, since we assume that $n-k-3$ we see that S^{n-k-1} and $\overline{B}^{n-k}$ are path-connected. Thus the pointed sets to the right consist of a single element, thus the boundary maps ∂_1 are actually group homomorphism and the right vertical map is a group homomorphism. Since all maps are group homomorphisms we can thus apply the Five Lemma 145.11 and we see that the middle vertical map $i_*\colon \pi_1(\partial W) \to \pi_1(W)$ is an isomorphism. ∎

Remark. In this remark we want to sketch an alternative approach to proving the above Sphere–Complement-π_*-Proposition 298.8. Thus let $n \in \mathbb{N}$, let K be a closed k-dimensional submanifold of the smooth manifold S^n and finally let $i \in \{1, \dots, n-k-2\}$. We want to show that for any base point $x_0 \in S^n \setminus K$ w have $\pi_i(S^n \setminus K, x_0) = 0$. In the following let $[\varphi\colon (S^i, *) \to (S^n \setminus K, x_0)] \in \pi_i(S^n \setminus K, x_0)$ be a random element. We make the following observations:

(1) By the Whitney Approximation Theorem 36.1 we can assume that φ is smooth.
(2) Since $i < n$ we know that $\pi_i(S^n, x_0) = 0$. It follows from the Trivial-in-π_n-Lemma 134.10 that there exists a continuous map $\psi\colon \overline{B}^{i+1} \to S^n$ with $\psi|_{S^i} = \varphi$.
(3) Once again, by the Whitney Approximation Theorem 36.1 we can assume that ψ is also smooth.
(4) Since $k+i+1 < n$ one would like to argue, similar to the Transversality Theorem 50.9, that we can homotop ψ rel S^i to a map that does not intersect K, i.e. we would like to homotop ψ rel S^i to a map $\overline{B}^{i+1} \to S^n \setminus K$. By the Trivial-in-$\pi_n$-Lemma 134.10 this would then imply that $[\varphi] = 0 \in \pi_i(S^n \setminus K, x_0)$.

This approach can be turned into a proper proof, but we will not do so. □

298.4. **Hopf fiber bundle s arising from complex and quaternionic projective spaces.** In this section we will see that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the quaternionic projective spaces[3246] $\mathbb{H}\mathrm{P}^n$ give rise to interesting fiber bundles. In particular we

[3246]We refer to Definition 215.2 for the slightly subtle definition of the quaternionic projective spaces.

will see that the Hopf map $S^3 \to S^2$ that we introduced in Definition 134.24 is in fact an S^1-bundle.

In the following we introduce two straightforward higher-dimensional analogs of the original Hopf map.

Definition 298.9. Let $n \in \mathbb{N}$. We refer to the maps

$$\begin{array}{rcl} p\colon S^{2n+1} = \{(z_0,\dots,z_n) \in \mathbb{C}^{n+1} \,|\, |z_0|^2 + \cdots + |z_n|^2 = 1\} & \to & \mathbb{C}\mathrm{P}^n \\ (z_0,\dots,z_n) & \mapsto & [z_0 : \cdots : z_n] \end{array}$$

and[3247]

$$\begin{array}{rcl} q\colon S^{4n+3} = \{(h_0,\dots,h_n) \in \mathbb{H}^{n+1} \,|\, |h_0|^2 + \cdots + |h_n|^2 = 1\} & \to & \mathbb{H}\mathrm{P}^n \\ (h_0,\dots,h_n) & \mapsto & [h_0 : \cdots : h_n] \end{array}$$

as Hopf maps.

The following proposition is a straightforward generalization of the Hopf-Bundle Proposition 238.4. This proposition together with the Fiber-LES-Proposition 298.4, explains our renewed interest in the Hopf maps.[3248]

Proposition 298.10. (Hopf-Bundle Proposition) Let $n \in \mathbb{N}$.
(1) The Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle.
(2) The Hopf map $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle.

Proof. [3249] We only provide the proof of (2). The proof of (1) is a straightforward modification of the proof of (2). Given $j \in \{0,\dots,n\}$ we define

$$V_j \coloneqq \{[h_0 : \cdots : h_n] \in \mathbb{H}\mathrm{P}^n \,|\, h_j \neq 0\}.$$

The same argument as in the proof of the Projective Space–Manifold Proposition 25.13 shows that V_j is an open subset of $\mathbb{H}\mathrm{P}^n$.

Claim. The map

$$\begin{array}{rcl} \Phi_j\colon q^{-1}(V_j) = \left\{(h_0,\dots,h_n) \in \mathbb{H}^{n+1} \,\middle|\, \sum_{i=0}^n |h_i|^2 = 1 \text{ and } h_j \neq 0\right\} & \to & V_j \times S^3 \\ (h_0,\dots,h_n) & \mapsto & \left([h_0 : \cdots : h_n], \frac{1}{|h_j|}\cdot h_j\right) \end{array}$$

is a homeomorphism.

Proof. We consider the map

$$\begin{array}{rcl} \Psi_j\colon V_j \times S^3 & \to & q^{-1}(V_j) \\ ([h_0 : \cdots : \underbrace{h_j}_{\neq 0} : \cdots : h_n], u) & \mapsto & \text{the normalization to length one of the non-zero vector} \\ & & \left(u\cdot h_j^{-1}\cdot h_0, \dots, \underbrace{u\cdot h_j^{-1}\cdot h_j}_{=u}, \dots, u\cdot h_j^{-1}\cdot h_n\right) \in \mathbb{H}^{n+1}. \end{array}$$

This map Ψ_j is well-defined. Indeed, a short calculation shows that if we pick a different representative for $[h_0 : \cdots : h_n]$, i.e. if we plug in $[\lambda\cdot h_0 : \cdots : \lambda\cdot h_n]$ for some $\lambda \in \mathbb{H}\setminus\{0\}$

[3247] Recall that in Definition 171.2 we defined the norm of a quaternion $h = a\cdot 1 + b\cdot \mathrm{i} + c\cdot \mathrm{j} + d\cdot \mathrm{k}$ as $|h| = \sqrt{a^2+b^2+c^2+d^2}$.
[3248] For mysterious reasons the Hopf maps are almost invariably called "Hopf fiber bundle s", even though the name "Hopf fiber bundle" would be more precise.
[3249] This gives essentially a solution to Exercise 240.10.

then we obtain the same result.[3250] Using the Topological-Quotient Proposition 8.2 (1b) one can show that the map Ψ_j is continuous. One easily verifies that Ψ_j and Φ_j are inverses of one another. ⊞

It is clear that the open subsets V_j cover all of $\mathbb{H}\mathrm{P}^n$ and that the triangles

$$\begin{array}{ccc} q^{-1}(V_j) & \xrightarrow{\Phi_j} & V_j \times S^3 \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle (v,h)\mapsto h} & \\ & S^3 & \end{array}$$

commute. This shows, according to Definition 298.2, that $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle. ■

The Hopf-Bundle Proposition 298.10, combined with the Fiber-LES-Proposition 298.4 gives us two interesting theorems:

Theorem 239.12. **($\pi_n(S^3)=\pi_n(S^2)$)–Theorem**
(1) For all $n \geq 3$ we have $\pi_n(S^3) \cong \pi_n(S^2)$.
(2) We have $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_3(S^2)$ is generated by the Hopf map $S^3 \to S^2$.[3251]

Proof.

(1) On page 340 we gave an identification of $\mathbb{C}\mathrm{P}^1$ with S^2. Thus the Hopf-Bundle Proposition 298.10 (1), applied to $n = 1$, tells us that the Hopf map $p\colon S^3 \to S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle. The Fiber-LES-Proposition 298.4 gives us the long exact sequence[3252]

$$\ldots \to \pi_n(S^1) \to \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \xrightarrow{\partial_n} \pi_{n-1}(S^1) \to \pi_{n-1}(S^3) \to \ldots$$

The desired statement now follows from the example on page 2587 where we saw that $\pi_k(S^1) = 0$ for $k \geq 2$.

(2) In the $\pi_n(S^n)$–Proposition 167.10 we saw that $\pi_3(S^3) = \mathbb{Z} \cdot [\mathrm{id}_{S^3}]$. By (1) the Hopf map $p\colon S^3 \to S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$ induces an isomorphism $p_*\colon \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$. The image of $[\mathrm{id}_{S^3}]$ under p_* is by definition precisely the class in $\pi_3(S^2)$ represented by the Hopf map. Thus we see that $\pi_3(S^2) \cong \mathbb{Z}$ and that $\pi_3(S^2)$ is generated by the Hopf map. ■

We move on to our second theorem:

Theorem 298.11. (Hopf Fiber Bundle–Homotopy Group Theorem)
(1) For all $n \geq 2$ we have $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$.
(2) For $n \in \{1, \ldots, 6\}$ we have $\pi_n(S^4) \cong \pi_{n-1}(S^3)$.

Remark. From the $\pi_n(S^n)$–Proposition 167.10 we know that $\pi_7(S^7) \cong \mathbb{Z}$. Therefore it follows from the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 (1) that the group

[3250] Note though that the order of the factors in the definition of Ψ_j is important since $\mathbb{H}$ is not commutative.
[3251] This statement gives us an answer to Question 134.25 and to Question 216.24 (1).
[3252] It is actually worth having a glance at the lower end of this sequence. We have the exact sequence

$$\pi_2(S^3) \xrightarrow{p_*} \pi_2(S^2) \xrightarrow{\partial_2} \pi_1(S^1) \to \pi_1(S^3).$$

From the Sphere–Low-π_n-Proposition 134.19 we know that $\pi_1(S^3) = \pi_2(S^3) = 0$ and the Circle-π_1-Theorem 88.22 we know that $\pi_1(S^1) \cong \mathbb{Z}$. Thus we obtain that $\pi_2(S^2) \cong \mathbb{Z}$. In Exercise 298.11 we will see that using this exact sequence we can obtain a new proof that id_{S^2} is a generator of $\pi_2(S^2)$.

$\pi_7(S^4) \cong \pi_7(S^7) \oplus \pi_6(S^3)$ admits an epimorphism onto $\mathbb{Z}$. In fact in the Hopf Invariant One-Corollary 216.15 we already saw that the Hopf invariant gives us an explicit epimorphism $\pi_7(S^4) \to \mathbb{Z}$. □

Proof.

(1) By the Quaternion-Projective Space Lemma 215.3 (1) we can identify the quaternionic projective space $\mathbb{H}\mathrm{P}^1$ with S^4. Therefore if we apply the Hopf-Bundle Proposition 298.10 (2) with $n = 1$ we obtain an S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$. We pick $y_0 \in S^7$ and we write $b_0 = q(y_0)$. We pick an identification $S^3 = q^{-1}(b_0)$ and we denote by $i\colon S^3 \to S^7$ the map that is given by the inclusion of the fiber $F = S^3$ into S^7. From the Fiber-LES-Proposition 298.4 we obtain the following long exact sequence

$$\cdots \to \pi_n(S^3, y_0) \xrightarrow{i_*} \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial_n} \pi_{n-1}(S^3, y_0) \xrightarrow{i_*} \pi_{n-1}(S^7, y_0) \to \dots.$$

By the Sphere–Low-π_n-Proposition 134.19 we know that $\pi_3(S^7, y_0) = 0$. Recall that by definition this implies that the map $i\colon S^3 \to S^7$ is homotopic rel y_0 to the constant map $c_{y_0}\colon S^3 \to \{y_0\}$. By the Fibration-LES Theorem 297.11 (3) this implies that the above long exact sequence splits into short exact sequences

$$0 \to \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial_n} \pi_{n-1}(S^3, y_0) \to 0.$$

Furthermore the Fibration-LES Theorem 297.11 (3) says that these short exact sequences split, therefore we obtain the promised isomorphisms

$$\pi_n(S^4, b_0) \cong \pi_n(S^7, y_0) \oplus \pi_{n-1}(S^3, y_0).$$

(2) The statement holds trivially the Sphere–Low-π_n-Proposition 134.19 for $n = 1$. For $n \geq 2$ this statement follows from (1) and the Sphere–Low-π_n-Proposition 134.19 which says in this context that $\pi_n(S^7) = 0$ for $n = 1, \dots, 6$. ■

As we will see, the Fiber-LES-Proposition 298.4 is a goose that just cannot stop laying golden eggs. For example we have the following theorem:

Theorem 298.12. ($\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem) We use the homeomorphism from page 340 to make the identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ and as on page 338 we view $\mathbb{C}\mathrm{P}^1$ as a subspace of each $\mathbb{C}\mathrm{P}^n$ and of $\mathbb{C}\mathrm{P}^\infty$.

(1) Let $n \in \mathbb{N}$. If we denote by $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ the Hopf map from Definition 298.9, then

(2)
$$\pi_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, \dots, 2n, \\ \mathbb{Z} \cdot [p], & \text{if } k = 2n+1. \end{cases}$$

$$\pi_k(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, 6, \dots. \end{cases}$$

Remark. Let $k < l$. In Question 167.7 we asked whether the complex projective space $\mathbb{C}\mathrm{P}^k$ is a retract of $\mathbb{C}\mathrm{P}^l$. In the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 213.15 we used the calculation of the cup product to give a negative answer. Now we can also use homotopy groups for the same purpose. More precisely, from the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12, we obtain that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^k) \cong \mathbb{Z}$ but that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^l) = 0$. The usual argument, see the Circle–Non Retract Proposition 90.6 implies that once again we obtain a negative answer to Question 167.7, i.e. $\mathbb{C}\mathrm{P}^k$ is not a retract of $\mathbb{C}\mathrm{P}^l$. □

Proof.

(1) By the Hopf-Bundle Proposition 298.10 (1) we know that for every $n \in \mathbb{N}$ the Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle. Thus we obtain from the Fiber-LES-Proposition 298.4 the following long exact sequence of homotopy groups:

$$\ldots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \xrightarrow{p_*} \pi_k(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial_k} \pi_{k-1}(S^1) \to \ldots$$

We recall that $\pi_k(S^{2n+1}) = 0$ for $k = 1, \ldots, 2n$, that $\pi_{2n+1}(S^{2n+1}) \cong \mathbb{Z} \cdot [\mathrm{id}_{S^{2n+1}}]$ and that $\pi_k(S^1) = 0$ for $k \geq 2$. Putting everything together implies immediately that $\pi_k(\mathbb{C}\mathrm{P}^n) = 0$ for $k = 3, \ldots, 2n$ and that $\pi_{2n+1}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z} \cdot [p]$.[3253] Only the lower end of the long exact sequence requires slightly more thought. More precisely, we have the exact sequence

$$\underbrace{\pi_2(S^{2n+1})}_{=0} \to \pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial_2} \pi_1(S^1) \to \underbrace{\pi_1(S^{2n+1})}_{=0} \to \pi_1(\mathbb{C}\mathrm{P}^n) \to \underbrace{\pi_0(S^1)}_{=0} \to \pi_0(S^{2n+1}).$$

It follows that $\pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} \pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$.[3254] The only thing which is not entirely clearly from this argument is why $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$. One could either heroically fight one's way through the above isomorphism, in the same vein as in Exercise 298.11, and use that we know explicitly the generator of $\pi_1(S^1)$.

But fortunately we do not have to go down that route since in the $\pi_2(\mathbb{C}\mathrm{P}^n)$–Corollary 167.12 we already saw that $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$.

(2) We calculate that

$$\pi_k(\mathbb{C}\mathrm{P}^\infty) = \pi_k\Big(\bigcup_{n\in\mathbb{N}} \mathbb{C}\mathrm{P}^n\Big) \underset{\uparrow}{=} \operatorname*{colim}_{\longrightarrow} \pi_k(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{=} \begin{cases} \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, \ldots. \end{cases}$$

by the CW-Direct Colimit-π_n-Proposition 136.25 since by the discussion on page 2628 we can equip $\mathbb{C}\mathrm{P}^\infty$ with a CW-structure where each n-skeleton is given by $\mathbb{C}\mathrm{P}^n \subseteq \mathbb{C}\mathrm{P}^\infty$

by (1)

■

298.5. **The Hopf fiber bundle arising from the octonions.** In the Hopf-Bundle Proposition 298.10 we had in particular seen that the Hopf map $p\colon S^3 \to \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ is an S^1-bundle and that the Hopf map $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ is an S^3-bundle. We exploited these fiber bundles in the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12 and the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 to obtain non-trivial information on higher homotopy groups of spheres.

In our discussion of division algebras in Section 171.1 we gave the quaternions as a first example of a non-commutative division algebra. The algebra structure on the quaternions is still nonetheless quite reasonable, in particular the algebra structure is associative which was enough to define the quaternionic projective spaces in Definition 215.2.

Next on the list of ever weirder division algebras were the octonions $\mathbb{O}$ that we introduced in Definition 214.11. Recall that the underlying vector space of the octonions is $\mathbb{R}^8$ but that this time the algebra structure is neither commutative nor associative. The only redeeming feature is that $\mathbb{O}$ is still a division algebra.

[3253] Here we use the trivial, but useful observation that $p_*([\mathrm{id}_{S^n}]) = [p]$.

[3254] Of course we already saw in the Projective Space–π_1-Proposition 135.8 (2) that $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$.

Since the octonions are no longer associative one cannot just copy-paste the definition of the quaternionic projective spaces in Definition 215.2 to define octonionic projective spaces in any obvious way. But in Definition 215.6 we performed a little miracle, namely we constructed an 8-dimensional dimensional smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^1$. With some effort one can surely show, as in the Hopf-Bundle Proposition 298.10, that it gives rise to a Hopf fiber bundle $r\colon S^{15} \to \mathbb{O}\mathrm{P}^2 = S^8$ with fiber S^7. We will not go down that route, instead we will construct such a fiber bundle in a somewhat different way and we still claim the following proposition:

Proposition 298.13. (Fiber Bundle-S^{15}-S^8-Proposition) There exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7.

The proof of the Fiber Bundle-S^{15}-S^8-Proposition 298.13 is rather longish, thus we will first point out that it allows us to prove the following theorem.

Theorem 298.14. (Octonionic Hopf Fiber Bundle–Homotopy Group Theorem)

(1) For all $n \geq 2$ we have $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.
(2) For $n \in \{1, \ldots, 14\}$ we have $\pi_n(S^8) \cong \pi_{n-1}(S^7)$.

Proof. The proof is basically the same as the proof of the Hopf Fiber Bundle–Homotopy Group Theorem 298.11. We just need to replace the Hopf-Bundle Proposition 298.10 (2) by the Fiber Bundle-S^{15}-S^8-Proposition 298.13. ■

We can copy-paste the remark after the Hopf Fiber Bundle–Homotopy Group Theorem 298.11:

Remark. By the $\pi_n(S^n)$–Proposition 167.10 we know that $\pi_{15}(S^{15}) \cong \mathbb{Z}$. Thus it follows from the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 (1) that $\pi_{15}(S^8)$ admits an epimorphism onto $\mathbb{Z}$. In fact in the Hopf Invariant One-Corollary 216.15 we already saw that the Hopf invariant is an epimorphism $\pi_{15}(S^8) \to \mathbb{Z}$. □

The remainder of this chapter is occupied with providing a proof of the Fiber Bundle-S^{15}-S^8-Proposition 298.13. We will need the notion of the join of two topological spaces that we introduced in Definition 20.1.

Definition 298.15. Let X and Y be two non-empty topological spaces. We define the join $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define $X * Y$ to be the other topological space.

We remind the reader of the following lemma:

Lemma 20.4. (Join-of-Spheres Lemma) Given any $m, n \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.

The following proposition explains our sudden interest in the join of two topological spaces.

Proposition 298.16. (Join–Suspension Fiber Bundle Proposition) Let X be a topological space that is compact and Hausdorff. Furthermore let $\varphi\colon X \times X \to X$ be a continuous map. Given $x \in X$ we consider the maps[3255]

$$\begin{array}{rcl} L_x\colon X & \to & X \\ a & \mapsto & \varphi(x,a) \end{array} \quad \text{and} \quad \begin{array}{rcl} R_x\colon X & \to & X \\ a & \mapsto & \varphi(a,x). \end{array}$$

If each map $L_x\colon X \to X$ and each map $R_x\colon X \to X$ is a homeomorphism, then the map

$$\begin{array}{rcl} h_\varphi\colon X * X & \to & \Sigma(X) \\ [(x,t,y)] & \mapsto & [(\varphi(x,y), 2t-1)] \end{array}$$

is a fiber bundle with fiber X.[3256]

Example. We write $S^3 = \{z \in \mathbb{H} \,|\, |z| = 1\}$. By the Quaternion Lemma 171.3 (7) we know that for any $w, z \in \mathbb{H}$ we have $|w \cdot z| = |w| \cdot |z|$. Thus multiplication of quaternions defines a map $\varphi\colon S^3 \times S^3 \to S^3$. Given any $z \in S^3$ the map L_z is a homeomorphism. In fact an inverse is given by $L_{z^{-1}}$ since for any $x \in S^3$ we have

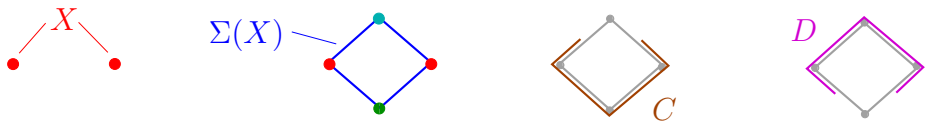
$$(L_{z^{-1}} \circ L_z)(x) \;=\; z^{-1} \cdot (z \cdot x) \;\underset{\text{since } \mathbb{H} \text{ is associative}}{=}\; (z^{-1} \cdot z) \cdot x \;=\; x,$$

and similarly we have $L_z \circ L_{z^{-1}} = \mathrm{id}_{S^3}$. The same argument shows that R_z is also a homeomorphism. Thus we obtain from the Join–Suspension Fiber Bundle Proposition 298.16 and the Join-of-Spheres Lemma 20.4 a fiber bundle $p\colon S^7 \to S^4$ with fiber S^3. We leave it to the heroically inclined reader to verify that this S^3-bundle is isomorphic to the S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ that we had constructed in the proof of the Hopf-Bundle Proposition 298.10. □

Proof. We consider the following two subsets of $\Sigma(X)$:

$$C \;:=\; ([-1, \tfrac{1}{2}] \times X) \,/\, \{-1\} \times X \quad \text{and} \quad D \;:=\; ([-\tfrac{1}{2}, 1] \times X) \,/\, \{1\} \times X.$$

We refer to the figure below for an illustration. Since the interiors of these two sets cover $\Sigma(X)$ it suffices to find trivializations of the map $h_\varphi\colon X * X \to \Sigma(X)$ with fiber X over C and over D.



We start out with finding a trivialization over C.

Claim 1. The map

$$\begin{array}{rcl} \Psi\colon X \times C & \to & X * X \\ (x, [(y,t)]) & \mapsto & [(x, \frac{2}{3}(t+1), L_x^{-1}(y))] \end{array}$$

is continuous and it restricts to a homeomorphism onto $h_\varphi^{-1}(C)$.

[3255]Here the mnemonic device is that "L" should suggest "left-multiplication" and "R" should suggest "right-multiplication".

[3256]It is a good exercise to first verify that the map is actually well-defined.

Proof. By now the reader is surely aware of the fact that whenever we mix quotients with products we need to tread carefully. Thus let us first consider the maps

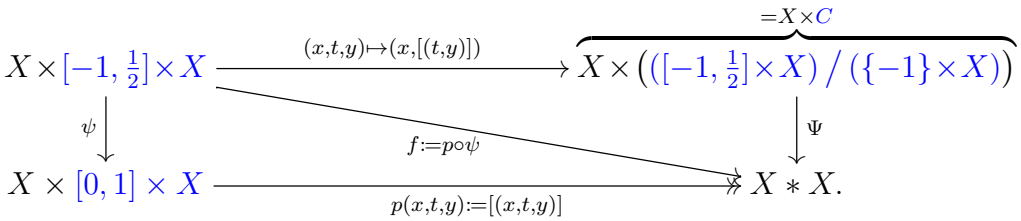
$$\begin{array}{rcl} \alpha\colon X\times X & \to & X\times X \\ (x,y) & \mapsto & (x,\varphi(x,y))=(x,L_x(y)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon X\times X & \to & X\times X \\ (x,y) & \mapsto & (x,L_x^{-1}(y)). \end{array}$$

Note that these two maps are inverses of one another. Also note that α is continuous by the Topological-Product Proposition 7.2 (1). Furthermore note that α is a bijection, in fact the inverse is given by β. Since X is compact and Hausdorff we obtain from the Product Topology-Properties Proposition 7.9 and the Compact-Hausdorff Proposition 4.23 (3) that α is a homeomorphism. In particular $\beta=\alpha^{-1}$ is also continuous.

Next we consider the map

$$\begin{array}{rcl} \psi\colon X\times([-1,\frac{1}{2}]\times X) & \to & X\times[0,1]\times X \\ (x,t,y) & \mapsto & (x,\frac{2}{3}(t+1),L_x^{-1}(y)). \end{array}$$

Since $\beta(x,y)=(x,L_x^{-1}(y))$ and since β is continuous we see that ψ is also continuous. Finally we consider the following commutative diagram:



Claim 2. The map Ψ is continuous.

Proof. We had just seen that $f=p\circ\psi$ is continuous. Note that by the Projection–Quotient Map Lemma 17.4 the projection $q\colon[-1,\frac{1}{2}]\times X\to([-1,\frac{1}{2}]\times X)\,/\,(\{-1\}\times X)$ is a quotient map. Since X is compact and Hausdorff we know by the Compact Hausdorff-Regionally Compact Lemma 4.58 that X is regionally compact.[3257] Thus it follows from the Whitehead Quotient Theorem 17.6 that the top horizontal map $\mathrm{id}_X\times q$ is a quotient map. Thus it follows from the Quotient Map-Composition Lemma 17.7 (2) that ψ is continuous. ⊞

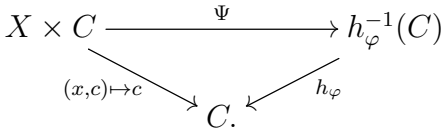
One easily verifies that Ψ defines a bijection

$$\Psi\colon \underbrace{X\times C}_{\subseteq X*X} \to \underbrace{h_\varphi^{-1}(C)}_{\subseteq\Sigma(X)}.$$

Since X is compact, we know by the Join Topology-Properties Proposition 20.2 (1) that the left-hand side is compact. By the Suspension Topology-Properties Proposition 20.26 (3) and our hypothesis that X is Hausdorff we know that the right-hand side is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 4.23 (3) that the continuous bijection $\Psi\colon X\times C\to h_\varphi^{-1}(C)$ is indeed a homeomorphism. ⊞

[3257] If the reader is unhappy about using the Compact Hausdorff-Regionally Compact Lemma 4.58 we point out that in our application of the proposition the topological space will be $X=S^n$ and thus obviously regionally compact.

We consider the following diagram:



It is straightforward to verify that this diagram commutes. Since the top map is homeomorphism by the previous claim we have found a trivialization of h_φ over C. We leave it to the reader to make the minor modifications needed to find a trivialization of h_φ over D.[3258] ■

Evidently the idea now is to replace the quaternions in the example on page 5741 by the octonions. In the following we recall the most salient bits from the discussion of octonions in Definition 214.11. The octonions $\mathbb{O}$ are $\mathbb{R}^8$ together with an algebra structure over $\mathbb{R}$ that has the following properties:

(1) The octonions $\mathbb{O}$ are a division algebra, i.e. for all $a \neq 0$ and all $b \in \mathbb{O}$ there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$.
(2) The algebra structure is neither commutative nor associative.
(3) Multiplication on $\mathbb{O}$ respects the Euclidean norm, more precisely, given $a, b \in \mathbb{O}$ we have $\|a \cdot b\| = \|a\| \cdot \|b\|$.

It follows from (3) that the algebra multiplication induces a map $\varphi\colon S^7 \times S^7 \to S^7$.

Auxiliary Lemma 298.17. Given any $z \in S^7$ the maps $L_z\colon S^7 \to S^7$ and $R_z\colon S^7 \to S^7$, as defined in the Join–Suspension Fiber Bundle Proposition 298.16, are homeomorphisms.

Proof. First note that we can *not* recycle the argument from page 5741 since there we used that the quaternions are associative.

By the Closed Manifold–Injection Corollary 170.10 (2) it suffices to show that the maps L_z and R_z are injective. First we show that given any $z \in S^7$ the map $L_z\colon S^7 \to S^7$ is injective. So suppose that we have $a, b \in S^7$ with $L_z(a) = L_z(b)$. We obtain that

$$\|a-b\| \underset{\text{since } z\in S^7}{=} \|z\| \cdot \|a-b\| \underset{\text{by (3)}}{=} \|z\cdot(a-b)\| \underset{\substack{\text{since } \mathbb{O} \text{ is}\\ \text{an algebra}}}{=} \|z\cdot a - z\cdot b\| \underset{\substack{\text{by definition}\\ \text{of the map } L_z}}{=} \|L_z(a) - L_z(b)\| \underset{\substack{\text{since we have}\\ L_z(a)=L_z(b)}}{=} 0.$$

Thus we see that $a = b$. Almost the same argument shows that R_z is injective.[3259] ■

Now we are basically done with the proof of the Fiber Bundle-S^{15}-S^8-Proposition 298.13.

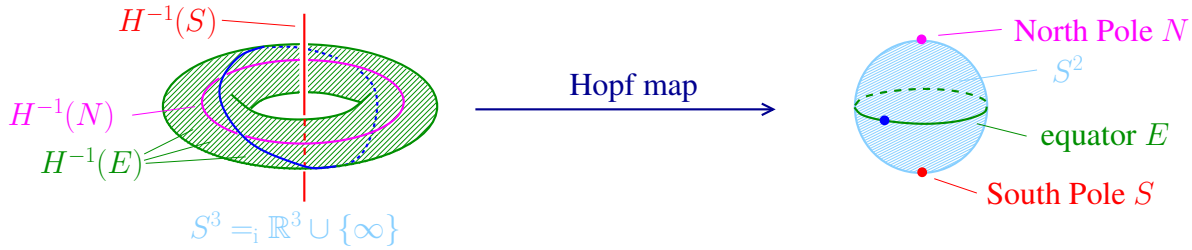
Proof of the Fiber Bundle-S^{15}-S^8-Proposition 298.13. It follows immediately from Lemma 298.17, the Join–Suspension Fiber Bundle Proposition 298.16 and the Join-of-Spheres Lemma 20.4 that there exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7. ■

[3258] Note that in this second part we will need to use that the maps $R_x\colon X \to X$ are homeomorphisms.
[3259] At first glance it might appear odd that we never used that the octonions are a division algebra, we only used the innocuous looking third property. But it is clear that (3) already implies that $\mathbb{O}$ has no zero-divisors. By Lemma 171.7 this forces $\mathbb{O}$ to be a division algebra.

298.6. Exercises for Chapter 298.

Exercise 298.1. Let F be a topological space and let $p\colon Y \to B$ be an F-bundle. Suppose that the fiber F and the base space B are path-connected. Show directly, without making use of the Fiber-LES-Proposition 298.4 (1), that the total space Y is also path-connected.
Remark. This exercise is identical to Exercise 238.6.

Exercise 298.2. We consider the Hopf map $H\colon S^3 \to S^2$ as illustrated below. We also show a continuous map $f\colon (I^2, \partial I^2) \to (S^2, N)$ which gives a generator of $\pi_2(S^2, N)$. Give an explicit lift $\widetilde{f}\colon I^2 \to S^3$ of f which is constant on $J^1 \subseteq I^2$.



Remark. This exercise is identical to Exercise 239.4.

Exercise 298.3. Let $p\colon Y \to [0,1]$ be a bundle with fiber F. Show with your bare hands (i.e. without using the Bundles-Homotopies Proposition 241.10) that p is trivial, i.e. show that p is isomorphic to the trivial bundle $F \times [0,1] \to [0,1]$.
Remark. This exercise is identical to Exercise 238.2.

Exercise 298.4. We consider the octonionic projective space $\mathbb{O}\mathrm{P}^1$ from Definition 215.6. Modify the proof of the Hopf-Bundle Proposition 298.10 to show that $\mathbb{O}\mathrm{P}^1$ gives rise to a Hopf fiber bundle $r\colon S^{15} \to \mathbb{O}\mathrm{P}^2 = S^8$ with fiber S^7.

Exercise 298.5. As in Definition 134.24 we consider the Hopf map:

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

(a) Let $P, Q \in S^2$ be two distinct points. Show that $X \times Y(P) \cup X \times Y(Q)$ is isotopic to a Hopf link, that we introduced in Definition 111.22.
(b) Let $P, Q, R \in S^2$ be two distinct points. Sketch the link $X \times Y(P) \cup X \times Y(Q) \cup X \times Y(R)$.

Exercise 298.6. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi\colon \mathrm{U}(n) = \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, A \cdot \overline{A}^{\mathrm{T}} = \mathrm{id}\} & \to & \mathbb{C} \setminus \{0\} \\ A & \mapsto & \det(A). \end{array}$$

(a) Show that φ takes values in S^1.
(b) Show that the map $\varphi\colon \mathrm{U}(n) \to S^1$ is a fiber bundle.
Hint. You can use the Ehresmann Fibration Theorem 238.6.

Exercise 298.7. Let $n \in \mathbb{N}$. Use the Ehresmann Fibration Theorem 238.6 to show that there exists a fiber bundle $p\colon \mathrm{O}(n) \to S^{n-1}$ with fiber $\mathrm{O}(n-1)$.

Exercise 298.8. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. By the Mapping Torus–Bundle Lemma 238.14 together with the Fiber-LES-Proposition 298.4

we know that the map

$$\begin{aligned} \operatorname{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) &\to S^1 \\ [(x,t)] &\mapsto \exp(2\pi \mathrm{i} t) \end{aligned}$$

is a fibration. We make the obvious identification $X = X \times \{0\} \subseteq \operatorname{Tor}(X,f)$. Let $k \in \mathbb{N}_0$ and let $\alpha\colon [0,1] \to S^1$ be the usual loop given by $\alpha(t) = \exp(2\pi \mathrm{i} t)$. By the Fiber Homotopy Proposition 297.8 and the Chain Homotopy–H_*-Lemma 144.5 we get a well-defined map

$$\Theta(\alpha)_*\colon \mathrm{H}_k(X;\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z}).$$

Show that $\Theta(\alpha)_* = (f_*)^{-1}$.

Exercise 298.9. Let W be an n-dimensional smooth manifold and let K be a compact connected proper k-dimensional submanifold with $k \leq n-3$. Show that the inclusion-induced map $\pi_1(W \setminus K) \to \pi_1(W)$ is an isomorphism.
Hint. Use the Amalgamated Product Lemma 98.24 (2a).

Exercise 298.10. The same way that we defined the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$, see page 2039, we define the infinite quaternionic projective space $\mathbb{H}\mathrm{P}^\infty$ as the direct colimit of the finite-dimensional quaternionic projective spaces $\mathbb{H}\mathrm{P}^k$. Show that for any $i \in \mathbb{N}_{\geq 2}$ we have an isomorphism $\pi_i(\mathbb{H}\mathrm{P}^\infty) \cong \pi_{i-1}(S^3)$.

Exercise 298.11. Let $p\colon S^3 \to S^2$ the Hopf fiber bundle . Recall that in the Hopf-Bundle Proposition 298.10 we saw that this map is an S^1-fiber bundle. We consider the connecting homomorphism $\partial_2\colon \pi_2(S^2) \to \pi_1(S^1)$ of the long exact sequence of homotopy groups from the Fiber-LES-Proposition 298.4 (1). Show, just using the definition of the connecting homomorphism, that $\partial([\mathrm{id}_{S^2}]) = [\mathrm{id}_{S^1}]$.
Remark. This exercise is identical to Exercise 239.8.

Exercise 298.12. Show that the homotopy groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are isomorphic.

Exercise 298.13. Let $m,k \in \mathbb{N}$ and let $p\colon M \to S^k$ be a fiber bundle with fiber S^m. We denote by $i\colon S^m \to M$ the homeomorphism to any of the fibers.

(a) Show that M is a closed $(m+k)$-dimensional topological manifold.

(b) We suppose that $m < k$ and we suppose that M is orientable. Show that the maps $p_*\colon \mathrm{H}_k(M) \to \mathrm{H}_k(S^k)$ and $i_*\colon \mathrm{H}_m(S^m) \to \mathrm{H}_m(M)$ are isomorphisms and show that $\mathrm{H}_i(M) = 0$ for $i \neq 0,k,m,m+n$.
Hint. Use the Fibration-LES Theorem 297.11 and the Homology–Homotopy-Isomorphism Theorem 296.26 and use Poincaré Duality.

(c) We suppose again that M is orientable. Show that the conclusion in (b) does not necessarily hold if we drop the hypothesis that $m < k$.

CHAPTER 299

Homotopy groups of Lie groups

In this chapter our focus will lie on the study of the homotopy groups of the matrix groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. We will see that it is convenient to define a third type of groups, namely the groups $\mathrm{Sp}(n)$, which can be viewed as the quaternionic analog of $\mathrm{O}(n)$ and $\mathrm{U}(n)$.

Since these groups are Lie groups and since this structure is helpful in our study of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ we first give a short (re-) introduction to Lie groups Afterwards we will use several bundle maps to study the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

This chapter is organized as follows:

- In Section 299.1 we will (re-) introduce topological groups and Lie groups.
- In Section 299.2 we will discuss the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.
- In Section 299.3 we will continue with the discussion of the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.
- In Section 299.4 we will introduce Spin groups.
- In Section 299.5 we will state the Bott Periodicity Theorem 299.21.
- In Section 299.6 we will introduce H-spaces.
- In Section 299.7 we will state and prove the Eckmann-Hilton Theorem 299.32.
- In Section 299.8 we will provide exercises for Chapter 299.

In the next Chapter 300 we will introduce the basic ideas of the Thom-Pontryagin Theory.

299.1. **Topological groups and Lie groups.** First we recall Definition 7.30:

Definition 299.1.

(1) A topological group is a topological space X together with a group structure such that the two maps

$$\begin{array}{ccc} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous.

(2) A morphism between topological groups X and Y is a group homomorphism $f\colon X\to Y$ that is furthermore continuous.

(3) We denote by $\mathcal{TopGr}$ the category of topological groups.

Examples.

(1) On page 299 we already saw that for any $n\in\mathbb{N}$ the matrix groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

(2) Let X be a topological space and let $x_0\in X$ be a base point. We consider the loop space $\Omega(X,x_0)=\{\gamma\colon[0,1]\to X\,|\,\gamma(0)=\gamma(1)=x_0\}$, which we equip with the compact-open topology. Next we consider the fundamental group

$$\pi(X,x_0) \;=\; \Omega(X,x_0)/\!\sim \qquad \text{where } f\sim g \text{ if } f,g \text{ are homotopic rel endpoints.}$$

As in Exercise 297.9 we can equip the set $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$ with the quotient topology, i.e. we view $\pi_1(X, x_0)$ as a topological space. As we know very well, the set $\pi_1(X, x_0)$ admits a group structure. But note that, rather surprisingly, the multiplication map $\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$ is in general *not* continuous. In fact such an example is given by the Hawaiian Earrings, introduced in Exercise 105.1. We refer to [**Fab2011**] for details.[3260] The upshot is that not every topological space with a reasonable looking group structure is necessarily a topological group. □

Next we recall Definition 34.2 of a Lie group, which is quite similar to the above definition of a topological group.

Definition 299.2.

(1) A Lie group is a smooth manifold X together with a group structure such that the two maps[3261]
$$\begin{array}{ccc} X \times X & \to & X \\ (x, y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$
are smooth.

(2) A morphism between Lie groups X and Y is a group homomorphism $f\colon X \to Y$ that is furthermore smooth.

(3) We denote by $\mathcal{LieGr}$ the category of Lie groups.

Remark.

(1) Recall that by Definition 25.15 a smooth map is in particular continuous. It follows that a Lie group is also a topological group.

(2) As we discussed on page 873 it is elementary to see that Lie groups have empty boundary. □

Example.

(1) It is straightforward to see that the smooth manifold $\mathbb{R}$ with the group structure given by addition and the smooth manifold S^1 with the group structure given by multiplication are Lie groups.

(2) In the Matrix–Lie Group Lemma 34.3 we saw that for any $n \in \mathbb{N}$ the matrix groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are not only topological groups, they are even Lie groups. □

As we mentioned in the beginning of this chapter, in this chapter we are mostly interested in the homotopy groups of the Lie groups
$$\begin{aligned} \mathrm{O}(n) &= \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, A^{\mathrm{T}} A = \mathrm{id}\} \\ \mathrm{U}(n) &= \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, \overline{A}^{\mathrm{T}} A = \mathrm{id}\}. \end{aligned}$$
As we will see later on, even if we are only interested in these groups it makes sense to extend our gaze and to consider the quaternionic analog of $\mathrm{O}(n)$ and $\mathrm{U}(n)$:

[3260] In fact, as is pointed out in [**Fab2011**], the failure of $\pi_1(X, x_0)$ to be a topological group is another instance of the phenomenon, discussed in Section 17.1, that in general product and quotient topologies do not mix well.

[3261] Here we equip $X \times X$ with the product smooth manifold structure introduced in the Smooth Manifold Product Proposition 25.34 and in the Smooth Manifold Product Proposition 26.12.

Definition 299.3. Let $n \in \mathbb{N}$. We set[3262]

$$\mathrm{Sp}(n) \;:=\; \{A \in \mathrm{M}(n \times n, \mathbb{H}) \,|\, \overline{A}^{\mathrm{T}} \cdot A = \mathrm{id}\}.$$

We call $\mathrm{Sp}(n)$ the n-th symplectic group.

We will first discuss the symplectic groups $\mathrm{Sp}(n)$ a little bit before we explain the slightly weird name:

Lemma 299.4. (Symplectic Group Lemma)

(1) We have $\mathrm{Sp}(1) = S^3$.
(2) Let $n \in \mathbb{N}$. Matrix multiplication turns $\mathrm{Sp}(n)$ into a group.
(3) Let $n \in \mathbb{N}$. We write $J = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$. The map

$$\begin{array}{rccl} \Phi\colon & \mathrm{Sp}(n) & \to & \{X \in \mathrm{U}(2n) \,|\, J \cdot X \cdot J^{-1} = \overline{X}\} \\ & P & \mapsto & \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \quad \text{where } A, B \in \mathrm{M}(n \times n, \mathbb{C}) \text{ with}^{3263}\ P = A + B\,\mathrm{j} \end{array}$$

is a group isomorphism.

Proof.

(1) In the Quaternion Lemma 171.3 (4) we saw that for any quaternion $h \in \mathbb{H}$ we have $h \cdot \overline{h} = \|h\|^2$. It follows immediately that $\mathrm{Sp}(1) = S^3$.
(2) We start out with the following observations:
  (i) First note that a straightforward calculation[3264] shows that for any two matrices $A, B \in \mathrm{M}(n \times n, \mathbb{H})$ we have $\overline{(A \cdot B)}^{\mathrm{T}} = \overline{B}^{\mathrm{T}} \cdot \overline{A}^{\mathrm{T}}$.
  (ii) If $\mathbb{F}$ is a skew field and if A and B are matrices in $\overline{M}(n \times n, \mathbb{F})$ with $A \cdot B = \mathrm{id}$, then $B \cdot A = \mathrm{id}$.

  Now we turn to the proof that $\mathrm{Sp}(n)$ is a group:
  (a) The fact that $\mathbb{H}$ is associative implies immediately that the multiplication of matrices in $\mathrm{M}(n \times n, \mathbb{H})$ is associative.
  (b) Evidently the identity matrix lies in $\mathrm{Sp}(n)$ and it is a neutral element.
  (c) Given $A \in \mathrm{Sp}(n)$ we have by definition that $\overline{A}^{\mathrm{T}} \cdot A = \mathrm{id}$. It follows from Observation (ii) that also $A \cdot \overline{A}^{\mathrm{T}} = \mathrm{id}$. Next note that $\overline{(\overline{A}^{\mathrm{T}})}^{\mathrm{T}} = A$. These two observations show that $\overline{A}^{\mathrm{T}} \in \mathrm{Sp}(n)$ and that $\overline{A}^{\mathrm{T}}$ is a left- and a right-inverse for A.
  (d) It follows from the above Observation (i) that $\mathrm{Sp}(n)$ is closed under matrix multiplication.

  We have thus shown that $\mathrm{Sp}(n)$ is indeed a group.
(3) We will only use this isomorphism to justify the name for $\mathrm{Sp}(n)$, otherwise it will not play a role. Therefore we will not provide the fairly elementary proof of this statement, instead we refer to [**Bre1993**, p. 103] for the proof. ∎

[3262] Recall that the conjugate of a quaternion $h = a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ is defined as $\overline{h} = a \cdot 1 - b \cdot \mathrm{i} - c \cdot \mathrm{j} - d \cdot \mathrm{k}$. Given a matrix $A = (a_{ij})$ over $\mathbb{H}$ we define $\overline{A} = (\overline{a}_{ij})$.

[3263] It follows from $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ that any quaternion $a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ can be uniquely written as $(a \cdot 1 + b \cdot \mathrm{i}) + (c \cdot 1 + d \cdot \mathrm{i}) \cdot \mathrm{j}$ where $a \cdot 1 + b \cdot \mathrm{i}$ and $c \cdot 1 + d \cdot \mathrm{i}$ are complex numbers. The same statement applies to matrices.

[3264] In the calculation we need to use the fact, mentioned in the Quaternion Lemma 171.3 (4), that for quaternions $w, z \in \mathbb{H}$ we have $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$.

Not surprisingly we will now see that $\mathrm{Sp}(n)$ is a Lie group. More precisely, in the following lemma we will discuss the three groups $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, so that we can compare their features:

Lemma 299.5. (Matrix–Lie Group Lemma)[3265]

(1) Let $n \in \mathbb{N}$.
   (a) The set $\mathrm{O}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{R}) =_{\mathrm{i}} \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n-1)$.
   (b) The set $\mathrm{U}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{C}) =_{\mathrm{i}} \mathbb{C}^{n^2} =_{\mathrm{i}} \mathbb{R}^{2n^2}$ of dimension n^2.
   (c) The set $\mathrm{Sp}(n)$ is a submanifold of $\mathrm{M}(n \times n, \mathbb{H}) =_{\mathrm{i}} \mathbb{H}^{n^2} =_{\mathrm{i}} \mathbb{R}^{4n^2}$ of dimension $2n^2+n$.
   Furthermore, if we view $\mathrm{O}(n)$ as a subspace of $\mathrm{O}(n+1)$ in the obvious way, then $\mathrm{O}(n)$ is a submanifold of $\mathrm{O}(n+1)$. The same statements apply to $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.
(2) $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, viewed as smooth manifolds in their own right, are closed, in particular they are compact.
(3) The usual matrix multiplication turns $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ into Lie groups.

Remark.

(1) As we mentioned in the Symplectic Group Lemma 299.4 (1) we have $\mathrm{Sp}(1) = S^3$. The proof of Matrix–Lie Group Lemma 299.5 says (as in Exercise 171.3) that quaternion multiplication turns $\mathrm{Sp}(1) = S^3$ into a Lie group.
(2) Recall that in **??** we defined the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ as follows:
$$\mathrm{Sp}(2g, \mathbb{Z}) \;=\; \{A \in \mathrm{GL}(2g, \mathbb{Z}) \,|\, A^{\mathrm{T}} \cdot J \cdot A = J\} \quad \text{where} \quad J = \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}.$$
This definition, together with the Symplectic Group Lemma 299.4 and the Matrix–Lie Group Lemma 299.5, leads us to calling $\mathrm{Sp}(n)$ the *n-th compact symplectic group*. □

Proof. In the Matrix Group-Smooth Manifold Proposition 32.10 we had basically proved statements (1) and (2) for the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. The only aspect that we did not mention explicitly so far is that $\mathrm{O}(n)$ is a submanifold of $\mathrm{O}(n+1)$. We leave it to the reader to verify that this statement holds. The proof of (1) and (2) for $\mathrm{Sp}(n)$ is basically the same.

In the Matrices–Topological Group Lemma 7.31 we showed that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are topological groups. Almost the same argument given in that proof shows that matrix multiplication and taking inverses are smooth maps. This implies that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are Lie groups. Furthermore it is clear that multiplication in $\mathrm{Sp}(n)$ is smooth. It remains to show that the map
$$\begin{array}{ccc} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$
is smooth. We will show this fact in Exercise 299.2. ■

The following lemma (which is identical to the Lie Group–Parallelizability Lemma 265.14), gives an easy way to rule out the existence of Lie group structures on some smooth manifolds.

Lemma 299.6. (Lie Group–Parallelizability Lemma) Every Lie group is parallelizable.[3266]

[3265] This lemma is a slight generalization of the earlier Matrix–Lie Group Lemma 34.3.

[3266] Recall that according to Definition 265.13 a smooth manifold is called *parallelizable* if its tangent bundle is trivializable.

Example.

(1) Recall that on page 5141 we showed that for $g \neq 1$ the surface of genus g is not parallelizable. Thus it follows from the Lie Group–Parallelizability Lemma 299.6 that these surfaces cannot be equipped with a Lie group structure.

(2) It is natural to ask what spheres admit Lie group structures. Here is what we know:
- Basically by definition we have $\mathrm{O}(1) = S^0$ and $\mathrm{U}(1) = S^1$, furthermore by the Symplectic Group Lemma 299.4 (1) we have $\mathrm{Sp}(1) = S^3$. Thus it follows from the Matrix–Lie Group Lemma 299.5 that these three spheres can be viewed as Lie groups.
- On page 5135 we saw that $2n$-dimensional spheres with $n \geq 1$ are not parallelizable. It follows from the Lie Group–Parallelizability Lemma 299.6 that these cannot admit Lie group structures.
- By the Sphere–Parallelizability Theorem 265.35 (which we did not prove) we know that S^n is parallelizable if and only if $n = 0, 1, 3, 7$.
- By the above the only question that remains is whether S^7 has a Lie group structure. It turns out that this is *not* the case, we refer to [**HoM2020**, Theorem 6.95] for details. □

Sketch of proof. Let G be an n-dimensional Lie group with trivial element e. We denote by $R_g \colon G \to G$ the map that is given by right-multiplication by g. Since G is a Lie group this map is in fact a diffeomorphism. Next we consider the map

$$\begin{array}{rcl} G \times \mathrm{T}_e G & \to & \mathrm{T}G = \{(g, w) \,|\, g \in G \text{ and } w \in \mathrm{T}_g G\} \\ (g, v) & \mapsto & (g, (R_g)_*(v)). \end{array}$$

Using the fact that the Lie group multiplication is smooth one can show, basically by the argument of the Lie Group–Orientable Lemma 34.4, that this map is smooth. In particular it is an isomorphism of smooth vector bundles. Since the left hand side is isomorphic to the trivial vector bundle $M \times \mathbb{R}^m$ we have shown that the tangent bundle $\mathrm{T}M$ is trivializable. ■

Next we turn to quotients of Lie groups. We recall the following convention:

Convention 34.7. Let G be a topological group and let H be a subgroup. We denote by G/H the topological space that is given by the equivalence relation

$$g_1 \sim g_2 \iff \text{there exists an } h \in H \text{ with } g_1 \cdot h = g_2.$$

We also recall the following lemma, which gives us more examples of Lie groups.

Lemma 34.8. (Lie Group–Quotient Lemma) Let G be a Lie group and let K be a finite normal subgroup. The quotient G/K admits a unique smooth manifold structure such that the projection $G \to G/K$ is a local diffeomorphism. Furthermore the group structure on the quotient group G/K defines a Lie group structure on G/K.

Example. It follows from the Lie Group–Quotient Lemma 34.8 that for any even $n \in \mathbb{N}$ the quotients $\mathrm{SU}(n)/\{\pm \mathrm{id}\}$ and $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\{\pm(\mathrm{id}, \mathrm{id})\}$ are Lie groups. □

Proof. For the reader's convenience we recall the proof. In the elementary Topological Group–Subgroup Action Lemma 34.6 we already saw that the action of K on G is free and continuous. Since G is a Lie group one also sees that the action is smooth. Finally,

since G is finite we get for free that the action is proper. Thus we obtain from the Manifold–Quotient Proposition 25.39 that G/K admits a unique smooth manifold structure (with empty boundary) such that the projection $G \to G/K$ is a local diffeomorphism. Next note that clearly $G/K \times G/K$ is diffeomorphic to $(G \times G)/(K \times K)$ and that the projection $G \times G \to (G \times G)/(K \times K)$ is a local diffeomorphism. It follows easily from these observations that the induced multiplication map $G/K \times G/K \to G/K$ is again smooth. ■

Above we have seen that subgroups of matrix groups often give rise to Lie groups. For the reader's convenience we recall the following theorem which says that, in a sense, all compact Lie groups arise that way:

Theorem 34.5. (Compact Lie Group Linearity Theorem) Every compact[3267] Lie group is isomorphic (in the obvious sense) to a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$.

On page 5758 we will point out an example of a path-connected *non-compact* Lie group that is *not* isomorphic to a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$.

299.2. **The homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.** In this and the following section we want to see what we can say about the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Convention 299.7.

(1) Given $n \in \mathbb{N}$ we have the obvious monomorphism

$$\begin{array}{rcl} \mathrm{O}(n) & \to & \mathrm{O}(n+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Often we will use it to identify $\mathrm{O}(n)$ with its image in $\mathrm{O}(n+1)$. Similarly we view $\mathrm{U}(n)$ as a subgroup of $\mathrm{U}(n+1)$ and we view $\mathrm{Sp}(n)$ as a subgroup of $\mathrm{Sp}(n+1)$.

(2) We will view any subgroup of $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{H})$ as a topological space that is equipped with the base point given by the identity matrix. Thus in the remainder of this chapter we will suppress the base point in the discussion of fundamental groups.

The following proposition, which generalizes the earlier Matrix Group-Fiber Proposition 238.12, gives us new interesting examples of fiber bundles:

Proposition 299.8. (Matrix Group-Fiber Proposition) Let $n \in \mathbb{N}$. The maps

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1) & \to & S^{4n+3} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array}$$

are bundle maps with fibers given by $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ respectively.

Remark. Without too much effort one can show that the bundle maps $\mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$ and $\mathrm{U}(n+1) \to S^{2n+1}$ with fiber $\mathrm{U}(n)$ from the proposition restrict to a bundle map $\mathrm{SO}(n+1) \to S^n$ with fiber $\mathrm{SO}(n)$ and to a bundle map $\mathrm{SU}(n+1) \to S^{2n+1}$ with fiber $\mathrm{SU}(n)$. □

We will outline three approaches to proving the Matrix Group-Fiber Proposition 299.8.

First proof of the Matrix Group-Fiber Proposition 299.8. In the following we deal with the map $q\colon \mathrm{O}(n+1) \to S^n$. The other two maps are dealt with in a similar fashion.

First note that

$$q^{-1}((0,\dots,0,1)) \;=\; \{(v_1 \;\dots\; v_{n+1}) \in \mathrm{O}(n+1) \,|\, v_{n+1} = \mathfrak{e}_{n+1}\} \;=\; \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, A \in \mathrm{O}(n) \right\}.$$

Thus we see that the "fiber" $q^{-1}((0,\dots,0,1))$ is diffeomorphic to $\mathrm{O}(n)$. Next note that by the Matrix–Lie Group Lemma 299.5 we know that $\mathrm{O}(n+1)$ is a smooth submanifold of $\mathrm{M}((n+1)\times(n+1),\mathbb{R})$ and that $\mathrm{O}(n+1)$ is a closed smooth manifold. Since S^n is connected it follows from the Ehresmann Fibration Theorem 238.6 that it remains to prove the following claim:

Claim. The given map $q\colon \mathrm{O}(n+1) \to S^n$ is a submersion.

Proof. Evidently the projection $\mathrm{M}((n+1)\times(n+1),\mathbb{R}) \to \mathbb{R}^{n+1}$ onto the first column is smooth. It follows from the Submanifold Smooth Structure Lemma 25.23 that the given map $q\colon \mathrm{O}(n+1) \to S^n$ is also smooth.

Let $A \in \mathrm{O}(n+1)$. We need to show that the differential $\mathrm{D}q_A\colon \mathrm{T}_A\,\mathrm{O}(n+1) \to \mathrm{T}_{q(A)}S^n$ is an epimorphism. First note that by the Tangent Space-Isomorphism Proposition 29.24 this is equivalent to showing that the differential $\mathrm{D}q_A\colon \widetilde{\mathrm{T}}_A\,\mathrm{O}(n+1) \to \widetilde{\mathrm{T}}_{q(A)}S^n$ is an epimorphism.

Next note that it follows easily from the Physical Tangent Spaces-Lemma 29.2 (3) that a basis for $\widetilde{\mathrm{T}}_{q(A)}S^n = A \cdot (\widetilde{\mathrm{T}}_{q(\mathrm{id})}S^n)$ is given by the vectors $A\cdot\mathfrak{e}_1,\dots,A\cdot\mathfrak{e}_n$. It remains to show that the basis vector $A\cdot\mathfrak{e}_1$, $i=1,\dots,n$ are contained in the image of $\mathrm{D}q_A$. For $i=1,\dots,n$ we consider the curve

$$\begin{array}{rcl} \gamma_i\colon \mathbb{R} & \to & \mathrm{O}(n+1) \\ t & \mapsto & A \cdot \begin{pmatrix} \mathrm{id}_{i-1} & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) \\ 0 & 0 & \mathrm{id}_{n-i} & 0 \\ 0 & -\sin(t) & 0 & \cos(t) \end{pmatrix}. \end{array}$$

Note that $\gamma_i(0) = A$ and note that

$$\mathrm{D}q_A(\gamma_i'(0)) \;\underset{\text{Chain Rule}}{=}\; (q\circ\gamma_i)'(0) \;\underset{\substack{\text{definition of} \\ q \text{ and } \gamma_i}}{=}\; A \cdot \begin{pmatrix} 0_{i-1} \\ \sin(t) \\ 0 \\ \cos(t) \end{pmatrix}' (0) \;=\; A \cdot \begin{pmatrix} 0_{i-1} \\ \cos(0) \\ 0 \\ \sin(0) \end{pmatrix} \;=\; A\cdot\mathfrak{e}_i.$$

In summary we have now shown that the differential $\mathrm{D}q_A\colon \mathrm{T}_A\,\mathrm{O}(n+1) \to \mathrm{T}_{q(A)}S^n$ is indeed an epimorphism. ∎

Second proof of the Matrix Group-Fiber Proposition 299.8. In the following we give an outline of a "bare hands only" proof that the first projection $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a bundle map with fiber $\mathrm{O}(n)$. We leave it to the reader to modify the proof to deal with the two other cases.

Thus let $n \in \mathbb{N}$. As usual we equip $\mathbb{R}^{n+1}$ with the Euclidean inner product $\langle\,,\,\rangle$. In this proof we identify $\mathrm{O}(n+1)$ with the group of isometries of $(\mathbb{R}^{n+1},\langle\,,\,\rangle)$. More precisely, we

do not distinguish in the notation between an isometry of $(\mathbb{R}^{n+1}, \langle\,,\,\rangle)$ and the matrix it represents with respect to the standard basis of $\mathbb{R}^{n+1}$.

We consider the two compact subsets

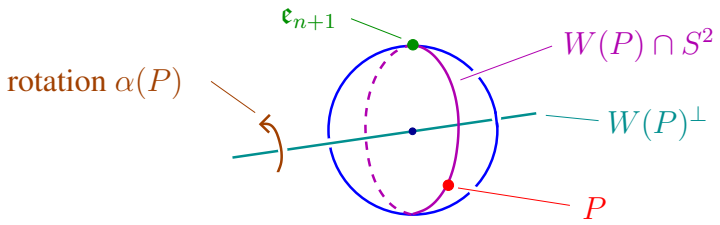
$$V \;=\; \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \le \tfrac{1}{2}\} \quad \text{and } W \;=\; \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \ge -\tfrac{1}{2}\}.$$

We continue with the following claim:

Claim. Let $P \in V$. We write $W(P) \coloneqq \operatorname{span}\{\mathfrak{e}_{n+1}, P\}$. There exists a unique orientation-preserving isometry $\alpha(P)$ of $\mathbb{R}^{n+1} = W(P) \oplus W(P)^\perp$ that satisfies the following two conditions:

(1) the restriction of $\alpha(P)$ to $W(P)$ is an orientation-preserving isometry of $W(P)$ that sends $\mathfrak{e}_{n+1}$ to P,
(2) it equals the identity on $W(P)^\perp$.

Proof. We illustrate the definition of $\alpha(P)$ in the figure below. In the 3-dimensional setting the isometry $\alpha(P)$ is the unique rotation around "the $W(P)^\perp$-axis" that sends $\mathfrak{e}_{n+1}$ to P.



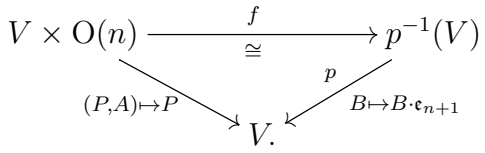
Note that $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are both positive-definite and symmetric forms. Thus it follows from Exercise 226.2 that both forms are also non-singular. Therefore we obtain from Sylvester's Theorem 226.12 that the two forms $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are isometric. In the latter case the orientation-preserving isometries are given by $\mathrm{SO}(2)$. Since $\mathrm{SO}(2) =_{\mathrm{i}} S^1$ acts freely and transitively on S^1 we see that given any two points $X, Y \in \mathbb{R}^2$ with $\|X\| = \|Y\|$ there exists a unique orientation-preserving isometry β of $(\mathbb{R}^2, \langle\,,\,\rangle)$ with $\beta(X) = Y$. Since $(W(P), \langle\,,\,\rangle)$ is isometric to $(\mathbb{R}^2, \langle\,,\,\rangle)$ the same uniqueness statement applies to $W(P)$ and the two points $\mathfrak{e}_{n+1}, P \in W(P)$. The claim immediately follows from this observation. ⊞

We leave it to the reader to verify that the map $\alpha\colon V \to \mathrm{O}(n+1)$ that we had just defined is indeed continuous. Next we consider the map

$$\begin{aligned} f\colon V \times \mathrm{O}(n) &\to \mathrm{O}(n+1) \\ (P, A) &\mapsto \alpha(P) \circ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

An entertaining exercise in linear algebra shows that the map f restricts to a bijection $V \times \mathrm{O}(n) \to p^{-1}(V)$. Since V and $\mathrm{O}(n)$ are compact we obtain from the Compact-Hausdorff Proposition 4.23 (3) that f defines a homeomorphism $V \times \mathrm{O}(n) \to p^{-1}(V)$.

Now recall that for any $P \in V$ we have, by design, that $\alpha(P)(\mathfrak{e}_{n+1}) = P$. Using this observation we see almost immediately that the following diagram commutes:



This shows that we have found a trivialization of p over V. With very minor modifications one can also find a trivialization of p over W. Since the interiors of V and W cover all of S^n we see that these two trivializations show that $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a fiber bundle with fiber $\mathrm{O}(n)$. ■

Our third approach to proving the Matrix Group-Fiber Proposition 299.8 is unfortunately much less self-contained, since it relies on the following proposition, that shows more generally that Lie groups (and sometimes topological groups) give rise to fiber bundles.

Proposition 299.9. (Lie Group-Fiber Proposition)

(1) Let G be a Lie group. If H is a closed subgroup of G, then the projection $G \to G/H$ is a bundle map with fiber H.
(2) Let G be a topological group and H be a closed subgroup of G. If H admits the structure of a Lie group, then the projection $G \to G/H$ is a bundle map with fiber H.

Remark. One might hope that the Lie Group-Fiber Proposition 299.9 (1) generalizes to all topological groups and subgroups. But as shown in [**Karu1958**, p. 347], there exists a topological group G and a closed subgroup H such that the projection $G \to G/H$ is not a bundle map. □

Proof of the Lie Group-Fiber Proposition 299.9.

(1) If G is compact and if H is a Lie subgroup (in particular a smooth submanifold), then this statement can be proved using the Ehresmann Fibration Theorem 238.6. The general statement is proved in [**Kiri2008**, Theorem 2.11] or [**War1983**, Theorem 3.58]. Alternatively, if G is a closed subgroup of some $\mathrm{GL}(n,\mathbb{C})$, then the proposition is proved in [**Bre1993**, p. 110].
(2) This statement is proved in [**Pal1961**, Corollary 4.1]. ■

Our third proof of the Matrix Group-Fiber Proposition 299.8 also needs the following lemma:

Lemma 299.10. (Matrix Group-Quotient Lemma) Let $n \in \mathbb{N}$. The maps

$$\begin{array}{ccc} \mathrm{O}(n+1)/\mathrm{O}(n) \xrightarrow{p} S^n & & \mathrm{U}(n+1)/\mathrm{U}(n) \xrightarrow{q} S^{2n+1} \\ [(v_1 \ldots v_{n+1})] \mapsto v_{n+1} & & [(v_1 \ldots v_{n+1})] \mapsto v_{n+1} \end{array} \quad \text{and} \quad \begin{array}{c} \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \xrightarrow{r} S^{4n+3} \\ [(v_1 \ldots v_{n+1})] \mapsto v_{n+1} \end{array}$$

are well-defined and they are homeomorphisms.

Proof.

(1) First we show that the given map $p\colon \mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ is well-defined. To do so we consider the map[3268]

[3268]It follows easily from $A^{\mathrm{T}}A = \mathrm{id}$ that every column v of A satisfies $\|v\| = 1$, i.e. every column lies in S^n.

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ V := (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} = V \cdot \mathfrak{e}_{n+1}. \end{array}$$

Note that if we multiply V on the right by a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then the multiplication with $\mathfrak{e}_{n+1}$ is not affected. Therefore the above map $\mathrm{O}(n+1) \to S^n$ descends to a map $\mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ which is exactly the map p we consider in the lemma. This shows that p is well-defined.

Also note that basic linear algebra tells us that given any point $v_{n+1} \in S^n$ there exist vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $v_1, \ldots, v_n, v_{n+1}$ form an orthonormal basis $\mathbb{R}^{n+1}$. This shows that the given map p is a surjection. Similarly, elementary linear algebra shows that p is injective. Finally note that S^n is Hausdorff and note that the Matrix–Lie Group Lemma 299.5 and the Compact Image Lemma 4.15 imply that $\mathrm{O}(n+1)/\mathrm{O}(n)$ is compact. Therefore it follows from the Compact-Hausdorff Proposition 4.23 (3) that the map p is indeed a homeomorphism.

(2),(3) The proof for (2) and (3) is basically identical to the proof in (1), except that the linear algebra aspects require slightly more thought. ■

Third proof of the Matrix Group-Fiber Proposition 299.8. These three statements are an immediate consequence of the Matrix Group-Quotient Lemma 299.10 and the Lie Group-Fiber Proposition 299.9 (1). ■

299.3. **The homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ II.** After the long discussion of the Matrix Group-Fiber Proposition 299.8 we continue with the actual study of the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$. The following proposition generalizes our earlier Matrix Group–Inclusion–π_n-Proposition 239.14:

Proposition 299.11. (Matrix Group–Inclusion–π_n-Proposition) Let $n \in \mathbb{N}$.

(1) The inclusion-induced map[3269]

$$\pi_k(\mathrm{O}(n)) \to \pi_k(\mathrm{O}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < n-1, \\ \text{epimorphism for } k = n-1. \end{cases}$$

(2) The inclusion-induced map

$$\pi_k(\mathrm{U}(n)) \to \pi_k(\mathrm{U}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < 2n, \\ \text{epimorphism for } k = 2n. \end{cases}$$

(3) The inclusion-induced map

$$\pi_k(\mathrm{Sp}(n)) \to \pi_k(\mathrm{Sp}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < 4n+2, \\ \text{epimorphism for } k = 4n+2. \end{cases}$$

Example.

[3269]Recall that in the Matrix Group-π_0-Proposition 4.54 (3) we saw that $\mathrm{O}(n)$ is disconnected. Thus in principle it is not completely clear what we mean by $\pi_k(\mathrm{O}(n))$. But, as we pointed out on page 5751, we always consider $\mathrm{O}(n)$ as equipped with the base point that is given by the identity. Also note that it follows easily from the Components-of-Topological Group Lemma 7.34 that for a topological group the isomorphism type of the homotopy groups does not depend on the choice of a base point.

(1) First note that

$$\pi_2(\mathrm{SO}(3)) \underset{\text{by the SU(2)-SO(3) Covering Theorem 94.3 (2)}}{\cong} \pi_2(\mathbb{R}\mathrm{P}^3) \underset{\text{by the Projective Space Avatar Lemma 9.16 (2)}}{\cong} \pi_2(S^3/x \sim -x) \underset{\text{by the Covering } \pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_2(S^3) \underset{\text{by the Sphere Low-}\pi_n \text{ Proposition 134.19}}{=} 0.$$

Next note that it follows from the Matrix Group–Inclusion–π_n-Proposition 239.14 we know that for every $n \geq 3$ the inclusion-induced map $\pi_2(\mathrm{O}(n)) \to \pi_2(\mathrm{O}(n+1))$ is an epimorphisms. It follows that for all $n \geq 3$ we have $\pi_2(\mathrm{O}(n)) = 0$.

(2) We know that $\mathrm{U}(1) = S^1$. Thus we obtain from the Matrix Group–Inclusion–π_n-Proposition 299.11 (2) that all unitary groups $\mathrm{U}(n)$ are path-connected.

(3) Recall that in the Symplectic Group Lemma 299.4 (1) we already pointed out that $\mathrm{Sp}(1) = S^3$. Thus we obtain from the Matrix Group–Inclusion–π_n-Proposition 299.11 (3) that all compact symplectic groups $\mathrm{Sp}(n)$ are simply connected. □

Proof. We prove Statement (1). The proofs of the remaining two statements are almost identical.[3270] First recall that it follows from the Matrix Group-Fiber Proposition 299.8 that we have a bundle map $p\colon \mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$. From the Fiber-LES-Proposition 298.4 we thus obtain a long exact sequence of the form

$$\ldots \to \pi_{k+1}(S^n) \xrightarrow{\partial_{k+1}} \pi_k(\mathrm{O}(n)) \to \pi_k(\mathrm{O}(n+1)) \to \underbrace{\pi_k(S^n)}_{=0 \text{ for } k<n} \xrightarrow{\partial_k} \ldots$$

The desired statement now follows from the familiar fact that by the Sphere–Low-π_n-Proposition 134.19 we have $\pi_i(S^k) = 0$ for $i < k$. ■

As a corollary to the Matrix Group–Inclusion–π_n-Proposition 299.11 (1) we can now finally answer Question 93.5. More precisely, we have the following proposition (which is a doppelgänger of the Matrix Group-π_1-Proposition 239.15.)

Proposition 299.12. (Matrix Group-π_1-Proposition)

(0) For any $n \in \mathbb{N}$ the inclusions $i\colon \mathrm{SO}(n) \to \mathrm{SL}(n,\mathbb{R})$ and $j\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}_+(n,\mathbb{R})$[3271] are homotopy equivalences.

(1) Let $n \in \mathbb{N}_{\geq 2}$. We have isomorphisms

$$\pi_1(\mathrm{SO}(n)) \underset{i_*}{\overset{\cong}{\to}} \pi_1(\mathrm{SL}(n,\mathbb{R})) \underset{j_*}{\overset{\cong}{\to}} \pi_1(\mathrm{GL}_+(n,\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \geq 3. \end{cases}$$

(2) In each of the three fundamental groups of (1) a generator of the group is represented by the loop

$$\begin{array}{ccc} S^1 & \to & \mathrm{SO}(n) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Proof.

(0) This statement follows from the Matrix Groups Deformation Retract Proposition 19.4 together with the Deformation Retract–Homotopy Equivalence Lemma 19.10.

[3270] The only catch is that one needs to be careful about the degrees involved.

[3271] Recall that we write $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$.

(1) First we consider the case $n = 2$. We have isomorphisms

$$\underset{\text{by the Circle-}\pi_1\text{-Theorem 88.22}}{\mathbb{Z} \overset{\cong}{\uparrow} \pi_1(S^1, \mathrm{id})} = \pi_1(\mathrm{SO}(2), \mathrm{id}) \underset{\text{by (0) and the Homotopy-}\pi_1\text{-Proposition 90.8}}{\overset{\cong}{\uparrow} \pi_1(\mathrm{SL}(2,\mathbb{R}), \mathrm{id}) \overset{\cong}{\uparrow} \pi_1(\mathrm{GL}_+(2,\mathbb{R}), \mathrm{id})}.$$

Now we consider the case $n \geq 3$. We have isomorphisms

$$\underset{\text{by the SO(3)–}\pi_1\text{-Proposition 93.4}}{\mathbb{Z}/2\mathbb{Z} \overset{\cong}{\downarrow} \pi_1(\mathrm{SO}(3), \mathrm{id})} \underset{\text{by the Matrix Group–Inclusion–}\pi_n\text{-Proposition 299.11 and since by the Matrix Group Path-Component Proposition 4.54 (2) we know that SO}(k)\text{ is the path-component of O}(k)\text{ that contains id}}{\cong \pi_1(\mathrm{SO}(n), \mathrm{id})} \underset{\text{follows from (0) together with the Homotopy-}\pi_1\text{-Proposition 90.8}}{\overset{\cong}{\downarrow} \pi_1(\mathrm{SL}(n,\mathbb{R}), \mathrm{id}) \overset{\cong}{\downarrow} \pi_1(\mathrm{GL}_+(n,\mathbb{R}), \mathrm{id})}.$$

(2) We consider the loop

$$\begin{array}{rcl} \gamma\colon S^1 & \to & \mathrm{SO}(2) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \end{array}$$

Since $\mathrm{SO}(2) =_{\mathrm{i}} S^1$ we know by the Circle-π_1-Theorem 88.22 that γ represents a generator of $\pi_1(\mathrm{SO}(2))$. By the Matrix Group–Inclusion–π_n-Proposition 299.11 we know that the inclusion $\mathrm{SO}(2) \to \mathrm{SO}(n)$ induces an epimorphisms of fundamental groups. It follows immediately that the loop given in (1) is indeed a generator of $\pi_1(\mathrm{SO}(n))$. Since the inclusions i and j induce isomorphisms of fundamental groups we see that the given loop also represents a generator of $\pi_1(\mathrm{SL}(n,\mathbb{R}))$ and $\pi_1(\mathrm{GL}(n,\mathbb{R}))$. ■

299.4. **Spin groups.** Let $n \geq 3$. Recall that in the Matrix Group-π_0-Proposition 4.54 (3) we showed that $\mathrm{SO}(n)$ is path-connected and in the Matrix Group-π_1-Proposition 299.12 (1) we had just seen that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$. By the Covering-Degree Proposition 92.10 this implies that the universal covering, as introduced in Definition 127.21, of $\mathrm{SO}(n)$ is a 2-fold covering of $\mathrm{SO}(n)$.[3272] For example in the SU(2)-SO(3)-Covering Theorem 94.3 we showed that the 2-fold covering of SO(3) is diffeomorphic to SU(2).

hus the following question arises:

Question 299.13. Let $n \geq 3$.

(1) Is the universal covering of $\mathrm{SO}(n)$ a group?
(2) If yes, is it a group "we already know"?

It turns out that the answer to Question 299.13 (1) is yes. In fact, the following, initially perhaps somewhat surprising proposition holds. The proof turns out to be almost disappointingly easy.

Proposition 299.14. (Covering-of-Lie Group Proposition)[3273] Let G be a connected Lie group and let $p\colon \widetilde{G} \to G$ be some covering (e.g. it could be the universal covering). We denote by 1_G the neutral element in G. We pick a point $1_{\widetilde{G}} \in p^{-1}(1_G)$. Then $\widetilde{G}$ admits a unique Lie group structure such that $p\colon \widetilde{G} \to G$ is a smooth map and a homomorphism and such that $1_{\widetilde{G}}$ is the neutral element in $\widetilde{G}$.

[3272] Note that here we use that $\mathrm{SO}(n)$ is a smooth manifold, which implies that the universal covering does indeed exist.

[3273] This proposition is identical to the earlier Covering-of-Lie Group Proposition **??**.

Remark.

(1) Let G be a topological group that is connected and locally path-connected. Note that the proof of the Covering-of-Lie Group Proposition 299.14 also shows that given any covering $p\colon \widetilde{G} \to G$ of G together with a choice of $\widetilde{e} \in \widetilde{G}$ with $p(\widetilde{e}) = e$ there exists a unique topological group structure on $\widetilde{G}$ with trivial element $\widetilde{e}$ which turns p into a group homomorphism. This statement was the content of Exercise 127.28.

(2) Let $k \in \mathbb{N}_{\geq 2}$. By the Covering-of-Lie Group Proposition 299.14 the universal covering $\mathrm{SL}(k,\mathbb{R})$ of the Lie group $\mathrm{SL}(k,\mathbb{R})$ is again Lie group. In [**HallB2015**, Theorem C.12] it is shown that $\mathrm{SL}(k,\mathbb{R})$ is *not* isomorphic to any subgroup of $\mathrm{GL}(n,\mathbb{R})$ for any $n \in \mathbb{N}$. This shows that the Compact Lie Group Linearity Theorem 34.5 does not extend to path-connected non-compact Lie groups. □

Sketch of the proof. First note that we saw in the Manifold Covering Proposition 95.4 that $\widetilde{G}$ is a topological manifold that admits a unique smooth structure such that $p\colon \widetilde{G} \to G$ is a smooth map. Now we turn towards defining the Lie group structure on the smooth manifold $\widetilde{G}$. In fact we will give two equivalent definitions of the group structure on $\widetilde{G}$:

(1) We consider the map

$$\begin{array}{rccc} \mu\colon & \widetilde{G} \times \widetilde{G} & \to & G \\ & (x,y) & \mapsto & p(x) \cdot p(y). \end{array}$$

Claim. We have $\mu_*(\pi_1(\widetilde{G} \times \widetilde{G}, (1_{\widetilde{G}}, 1_{\widetilde{G}}))) \subseteq p_*(\pi_1(\widetilde{G}, 1_{\widetilde{G}}))$.

Proof. We consider the two continuous maps $i, j\colon \widetilde{G} \to \widetilde{G} \times \widetilde{G}$ that are given by $i(g) = (g, 1_{\widetilde{G}})$ and $j(g) = (1_{\widetilde{g}}, g)$. First note that it follows easily from $p(1_{\widetilde{G}}) = 1_G$ that $\mu \circ i = p$ and $\mu \circ j = p$. Next note that this implies that $\mu_*(i_*(\pi_1(\widetilde{G}, 1_{\widetilde{G}}))) \subseteq p_*(\pi_1(\widetilde{G}, 1_{\widetilde{G}}))$ and that $\mu_*(j_*(\pi_1(\widetilde{G}, 1_{\widetilde{G}}))) \subseteq p_*(\pi_1(\widetilde{G}, 1_{\widetilde{G}}))$. The claim now follows from the observation that by the Product-π_1-Proposition 90.13 we know that the images of i_* and j_* generate $\pi_1(\widetilde{G} \times \widetilde{G}, (1_{\widetilde{G}}, 1_{\widetilde{G}}))$. ⊞

We consider the maps

$$\begin{array}{ccc} & & \widetilde{G} \\ & & \downarrow{\scriptstyle p} \\ \widetilde{G} \times \widetilde{G} & \xrightarrow[(x,y)\mapsto p(x)\cdot p(y)]{} & G. \end{array}$$

It follows from the claim and the Map Lifting Criterion 127.5 that there exists a unique lift

$$\begin{array}{ccc} & \overset{(1_{\widetilde{G}}, 1_{\widetilde{G}}) \mapsto 1_{\widetilde{G}}}{\nearrow} & \widetilde{G} \\ & & \downarrow{\scriptstyle p} \\ \widetilde{G} \times \widetilde{G} & \xrightarrow[(x,y)\mapsto p(x)\cdot p(y)]{} & G. \end{array}$$

(2) We set $\Gamma \subseteq p_*(\pi_1(\widetilde{G}, 1_{\widetilde{G}}))$. By the Whisker Topology Lemma 127.12 and the Universal Covering Proposition 127.20 we can assume that $\widetilde{G}$ equals

$$\widetilde{G} \;=\; \{\text{paths in } G \text{ starting at } 1_G\}/\sim$$

where we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [\underbrace{u * \overline{v}}_{\text{loop in } (G, 1_G)}] \in \Gamma.$$

With that explicit description we define

$$\begin{array}{rccc} \mu\colon & \widetilde{G} \times \widetilde{G} & \to & \widetilde{G} \\ & ([u],[v]) & \mapsto & \left[\begin{array}{ccc} [0,1] & \to & G \\ t & \mapsto & u(t) \cdot v(t) \end{array}\right]. \end{array}$$

Using the fact that the multiplication map $G \times G \to G$ is continuous one can show fairly easily that this map is well-defined.

We leave it as a charming exercise to show that the two given definitions of the group structure agree and that in both cases we define a Lie group structure on $\widetilde{G}$.[3274]

It remains to show that the group structure on $\widetilde{G}$ is the unique Lie group structure on the smooth manifold $\widetilde{G}$ for which $1_{\widetilde{G}}$ is the neutral element and that turns p into a homomorphism. Suppose we are given a smooth map $\varphi\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ that has the same properties. We obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{G} & \xrightarrow{\varphi} & \widetilde{G} \\ {\scriptstyle p\times p}\downarrow & \searrow^{(x,y)\mapsto p(x)\cdot p(y)} & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow[(x,y)\mapsto x\cdot y]{} & G. \end{array}$$

Since the diagram commutes we see that $\varphi\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is precisely a lift of the blue map $\widetilde{G} \times \widetilde{G} \to G$ that sends $(1_{\widetilde{G}}, 1_{\widetilde{G}})$ to $1_{\widetilde{G}}$. Thus by the uniqueness of the lift we see that the Lie group structure on $\widetilde{G}$ is uniquely determined by our wish list. ■

This proposition leads us to the following definition:

Definition 299.15. Let $n \geq 3$. We refer to the natural universal covering[3275] of the pointed topological space $(\mathrm{SO}(n), \mathrm{id})$ as the spin group $\mathrm{Spin}(n)$. By the Covering-of-Lie Group Proposition 299.14 there exists a unique Lie group structure on $\mathrm{Spin}(n)$ for which the canonical base point is the neutral element and which has the property that the projection $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a smooth map and a group homomorphism.

The combination of the Matrix Group-π_1-Proposition 299.12 (1), Proposition 299.14 and the Universal Covering Proposition 127.20 shows that $\mathrm{Spin}(n)$ is indeed a Lie group and that $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a 2-fold covering that is a group homomorphism.

Remark. By the Universal Covering Proposition 127.20 the universal covering of a smooth manifold, even if it is equipped with a base point, is only well-defined up to "equivalence". This slightly awkward issue can be circumvented in two different ways:

(1) either we use the explicit model of the universal covering of a based smooth manifold in terms of equivalence classes of paths, or
(2) in this particular situation there is a more elegant solution, one can give a purely algebraic explicit description of $\mathrm{Spin}(n)$. We refer to [**Mein2013**, Chapter 3.1] for details. □

It is harder to give an answer to Question 299.13 (2). For reasonably low dimensions the following proposition gives a positive answer.

Proposition 299.16. (Matrix Group–Isomorphisms Proposition)

[3274]The smoothness of the group multiplication is perhaps easier to prove with the first point of view, the fact that it is a group structure is possibly less confusing to verify with the second point of view.
[3275]We refer to Definition 127.21 for the definition of the canonical universal covering of a suitable pointed topological space and to the definition of the corresponding canonical base point.

(1) There exist Lie group isomorphisms[3276]

(a) $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm \mathrm{id}\}$
(b) $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(\mathrm{id}, \mathrm{id})\}$
(c) $\mathrm{SO}(5) \cong \mathrm{Sp}(2)/\{\pm \mathrm{id}\}$.

(2) There exist Lie group isomorphisms

(a) $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$
(b) $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$
(c) $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$
(d) $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.

The Matrix Group–Isomorphisms Proposition 299.16 (b) thus says that for $n = 3, 4, 5, 6$ the spin group $\mathrm{Spin}(n)$ is isomorphic to Lie groups that we are already familiar with. But according to [**Mein2013**, p. 53] "for $n \geq 7$, there are no further accidental isomorphisms of this type." In other words, in general the answer to Question 299.13 (2) is negative.

Proof.

(1) (a) This statement follows immediately from the SU(2)-Action Lemma 94.2 and the SU(2)-SO(3)-Covering Theorem 94.3.

(b) In the Quaternion Unit Proposition 171.5 (1) we saw that the quaternion multiplication defines a group structure on $S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$ where the trivial element is given by $1 \in \mathbb{H}$. Furthermore in the Quaternion Unit Proposition 171.5 (2) we had also seen that the Lie groups SU(2) and S^3 are isomorphic. Thus it suffices to show that the Lie groups $(S^3 \times S^3)/\{\pm(1,1)\}$ and SO(4) are diffeomorphic. We consider the following map:

$$\begin{array}{rcl} \Phi\colon S^3 \times S^3 & \to & \mathrm{SO}(4) \\ (q_1, q_2) & \mapsto & \left(\begin{array}{rcl} \mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1} \end{array} \right). \end{array}$$

We make the following observations:

(a) In Exercise 171.4 we showed that the map $z \mapsto q_1 z q_2^{-1}$ does indeed define an element in SO(4).

(b) One can easily verify that the map Φ is smooth and that it is a homomorphism.

(c) We leave it to the energetic reader to verify that the differential of Φ at each point is invertible.

(d) We note that the kernel of this homomorphism is given by $\{\pm(1,1)\}$. Indeed, the kernel of this homomorphism is given by the set of all $(q_1, q_2) \in S^3 \times S^3$ such that $q_1 z q_2^{-1} = z$ for all $z \in \mathbb{H}$. By setting $z = 1$ we obtain that $q_1 = q_2$. Furthermore, since $q_1 z q_1^{-1} = z$ for all $z \in S^3$ we see that $q_1 = q_2$ lies in the center of the group S^3. An elementary calculation shows that the center of S^3 is given by ± 1. Thus the kernel of Φ contains precisely the two elements $(1, 1)$ and $(-1, -1)$.

[3276]Note that it follows from the Lie Group–Quotient Lemma 34.8 that the groups on the right-hand side are indeed Lie groups.

It follows from the above, the Topological-Quotient Proposition 8.2 (1b) and the Lie Group–Quotient Lemma 34.8 and elementary group theory[3277] that the above map Φ descends to a smooth and injective map

$$\overline{\Phi}\colon (S^3 \times S^3)/\{\pm(1,1)\} \;\to\; \mathrm{SO}(4)$$

such that the differential at each point is invertible. By the Matrix–Lie Group Lemma 299.5 and the Lie Group–Quotient Lemma 34.8 both sides are closed connected 6-dimensional smooth manifolds. Thus it follows from the Closed Manifold–Injection Corollary 170.10 (2) that the map $\overline{\Phi}$ is also surjective. It follows from the above results and the Inverse Mapping Theorem 30.1 (2) that $\overline{\Phi}$ is in fact a diffeomorphism.

(c) Let V be the real vector space of quaternionic Hermitian 2×2-matrices with trace 0. In other words we consider

$$V \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2\times 2, \mathbb{H}) \;\middle|\; a+d=0, a, d \in \mathbb{R} \text{ and } b = \overline{c} \right\}.$$

Note that V is a 5-dimensional real vector space. The group $\mathrm{Sp}(2)$ acts by conjugation on V.[3278] One easily verifies that the kernel is given by $\{\pm \mathrm{id}\}$. Furthermore by [**Pro2007**, Chapter 4.5.3] there exists a positive-definite symmetric form on V that is preserved by this action. Thus, by Sylvester's Theorem 226.12 this action defines a homomorphism $\mathrm{Sp}(2) \to \mathrm{SO}(5)$. Similar to (b) we obtain a diffeomorphism $\mathrm{Sp}(2)/\{\pm \mathrm{id}\} \xrightarrow{\cong} \mathrm{SO}(5)$.

(2) (a) By (2a) we can make the identification $\mathrm{SO}(3) =_{\mathrm{i}} \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ of Lie groups. By the SU(2)-SO(3)-Covering Theorem 94.3 we know that $\mathrm{SU}(2)$ is homeomorphic to S^3, in particular we see that $\mathrm{SU}(2)$ is simply connected. By the Lie Group–Quotient Lemma 34.8 the quotient map $\mathrm{SU}(2) \to \mathrm{SU}(2)/\{\pm 1\}$ is a covering map, it is smooth and it is a homomorphism. Thus we see that $\mathrm{SU}(2)$ is in fact the universal covering of $\mathrm{SU}(2)/\{\pm \mathrm{id}\} =_{\mathrm{i}} \mathrm{SO}(3)$ and we deduce from the Covering-of-Lie Group Proposition 299.14 that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

(b) By the same argument as in (2a) this statement can be deduced from (1b).

(c) By the same argument as in (2a) this statement can be deduced from (1c).

(d) We will not make use of this result, hence we will not provide a proof, instead we refer to [**Mein2013**, p. 84] for details. ■

Remark. Initially the spin groups might appear as amusing curiosities. In fact they arise naturally in many parts of mathematics and physics. We refer to [**LaM1989, Morg1996, Mein2013**] for more details. □

299.5. **The Bott Periodicity Theorem.** We continue with the following definition:

[3277] More precisely, we use the elementary fact that if $\alpha\colon G \to H$ is a group homomorphism, then the induced map $G/\ker(\alpha) \to H$ is injective.

[3278] The group $\mathrm{Sp}(2)$ acts evidently by conjugation on all quaternionic 2×2-matrices, in principle one now needs to verify that this action restricts to an action on V.

Definition 299.17. We define[3279]

$$\mathrm{O} \coloneqq \varinjlim \mathrm{O}(n) = \left\{ A \in \underbrace{\mathrm{M}(\mathbb{N}\times\mathbb{N}, R)}_{\text{see Notation 2.14}} \;\middle|\; \begin{array}{l}\text{there exists an } n\in\mathbb{N}\\ \text{and a } B\in\mathrm{O}(n) \text{ with}\end{array} A = \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} \right\}$$

where O has the obvious group structure and where O has the topology that is defined by the condition that subset $U \subseteq \mathrm{O}$ is open if and only if for each n the intersection $U \cap \mathrm{O}(n)$ is open. Similarly we define

$$\mathrm{U} \coloneqq \varinjlim \mathrm{U}(n), \quad \mathrm{Sp} \coloneqq \varinjlim \mathrm{Sp}(n)$$

In the Algebraic–Direct Colimit Existence Proposition 104.7 we saw that every direct system $\{G_i\}_{i\in I}$ of groups admits a direct colimit $\varinjlim G_i$. In fact for the above direct colimit $\mathrm{O} = \varinjlim \mathrm{O}(n)$ the group structure is the "obvious" one given above. The following shows that these group structure are continuous, i.e. O, U and Sp are topological groups.

Lemma 299.18. (Group Structure-O-U-Sp-Lemma) The above group structures on O, U and Sp are continuous.

Remark. Initially one might be tempted to dismiss the (Group Structure-O-U-Sp-Lemma 299.18 as obvious. But in general there is a subtle issue. If $\{G_i\}_{i\in I}$ is a direct system of topological groups, then the above multiplication on the direct colimit $\varinjlim G_i$ might not be continuous. Such examples are for example provided in [**TSH1998**]. If one thinks about it for a while one realizes that this failure for the multiplication map to be continuous is another instance of the observation made in Section 17.1, that mixing product and quotient topologies is fraught with dangers. □

Proof. This lemma can surely be proved, with a little effort, "by hand". Alternatively one can use [**TSH1998**, Theorem 2.7] which shows that for a direct system $\{X_i\}_{i\in\mathbb{N}}$ of compact topological groups the above multiplication on $\varinjlim X_i$ is indeed continuous. ■

For later usages of O, U and Sp we have to get the following technical lemma out of the way.

Lemma 299.19. (CW Structure-O-U-Sp-Lemma) The topological spaces O, U and Sp admit CW-structures such that each $\mathrm{O}(n) \subseteq \mathrm{O}$, $\mathrm{U}(n) \subseteq \mathrm{U}$ and $\mathrm{Sp}(n) \subseteq \mathrm{Sp}$ is a subcomplex.

Proof. In [**WhdJ1944**] it is shown that we can equip the $\mathrm{O}(n)$ with CW-structures such that for each n the subset $\mathrm{O}(n) \subseteq \mathrm{O}(n+1)$ is actually a subcomplex. Analogous statements for $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are proved in [**Yok1955**, Theorem 5.3 and 5.5] or alternatively in [**Stro2011**, Chapter 9.3.3]. It follows from the Direct Colimit–CW Lemma 139.13 that O is a CW-complex. ■

Now we turn to the study of the homotopy groups of O, U and Sp.

[3279]When we talk about the direct colimit we have to say what category we consider. Here we want to consider the category *Group* of groups and of topological spaces at the same time. Since we have obvious inclusions maps $\mathrm{O}(n) \to \mathrm{O}(n+1)$ we obtain from the Topological-Direct Colimit Proposition 104.10 that in both settings we obtain the same underlying set, which we can then equip with a group structure and a topology. Together with the Algebraic–Direct Colimit Existence Proposition 104.7 we obtain that the object to the right is indeed a model for the direct colimit.

Lemma 299.20. (Homotopy Group-O-U-Sp-Lemma) Given any $k \in \mathbb{N}$ we have

(1) $\pi_k(\mathrm{O}) = \varinjlim \pi_k(\mathrm{O}(n)) \overset{\cong}{\leftarrow} \pi_k(\mathrm{O}(N))$ for any $N \in \mathbb{N}$ with $N > k+1$,

(2) $\pi_k(\mathrm{U}) = \varinjlim \pi_k(\mathrm{U}(n)) \overset{\cong}{\leftarrow} \pi_k(\mathrm{U}(N))$ for any $N \in \mathbb{N}$ with $2N > k$,

(3) $\pi_k(\mathrm{Sp}) = \varinjlim \pi_k(\mathrm{Sp}(n)) \overset{\cong}{\leftarrow} \pi_k(\mathrm{Sp}(N))$ for any $N \in \mathbb{N}$ with $4N > k-2$.

Proof. We prove (1). The proofs of (2) and (3) are almost identical. We start out with the following claim:

Claim. Given any compact subset K of O there exists an $n \in \mathbb{N}$ with $K \subseteq \mathrm{O}(n)$.

Proof. Let K be a compact subset of O. Given $n \in \mathbb{N}$ we write $X_n := \{1, \dots, n\} \times \{1, \dots, n\}$. First recall that $\mathrm{O}(n)$ is equipped with the subspace topology coming from the inclusion map $g_n \colon \mathrm{O}(n) \subseteq \mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{X_n} = \mathbb{R}^{n^2}$. Given $n \in \mathbb{N}$ we denote by $f_n \colon \mathbb{R}^{X_n} \to \mathbb{R}^{X_{n+1}}$ the obvious inclusion map. These have the property that for any $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{O}(n) & \xrightarrow{g_n} & \mathbb{R}^{X_n} \\ \downarrow & & \downarrow f_n \\ \mathrm{O}(n+1) & \xrightarrow{g_{n+1}} & \mathbb{R}^{X_{n+1}}. \end{array}$$

The above inclusion maps $g_n \colon \mathrm{O}(n) \to \mathbb{R}^{X_n}$ induce a map $g \colon \mathrm{O} = \varinjlim \mathrm{O}(n) \hookrightarrow \varinjlim \mathbb{R}^{X_n}$. By our hypothesis that K is compact and the Compact Image Lemma 4.15 we know that $g(K) \subseteq \varinjlim \mathbb{R}^{X_n}$ is also compact. Basically the same argument as in the CW-Complex-Finiteness Theorem 136.16 (see also Exercise 104.18) shows that there exists an $n \in \mathbb{N}$ with $g(K) \subseteq \mathbb{R}^{X_n}$. But this implies that $K \subseteq \mathrm{O}(n)$. ⊞

Now let $N \in \mathbb{N}$ with $N > k+1$. We have

$$\begin{array}{ccccc} \pi_k(\mathrm{O}) & = & \varinjlim \pi_k(\mathrm{O}(n)) & = & \pi_k(\mathrm{O}(N)). \\ \uparrow & & & & \uparrow \\ \text{by the CW-Direct Colimit-}\pi_n\text{-} & & & & \text{by the Matrix Group–Inclusion–}\pi_n\text{-Proposition 299.11 (1)} \\ \text{Proposition 136.25 and the claim} & & & & \text{and the Direct Colimit–Stabilization Lemma 104.5} \end{array}$$

■

We will now state two versions of the Bott Periodicity Theorem which implies in particular that the homotopy groups of O, U and Sp are periodic.
The first theorem requires just a little bit of preparation.

(1) As above we equip U, Sp and O with the base point given by the infinite identity matrix. We use these base points to form the loop spaces ΩU, $\Omega\,\mathrm{Sp}$ and ΩO.
(2) Given a point x in a topological space X we denote by c_x the corresponding constant path.
(3) We use notation (2) to equip each loop space $\Omega(X, x_0)$ with the base c_{x_0} and we can thus iterate the loop space construction.
(4) The obvious isomorphism $\mathbb{H} \overset{\cong}{\to} \mathbb{R}^4$ of real vector spaces defines the two inclusion maps $\mathrm{Sp}(n) \to \mathrm{O}(4n)$ and $\mathrm{Sp} \to \mathrm{O}$.
(5) The inclusion $\mathbb{R} \to \mathbb{H}$ defines inclusions $\mathrm{O}(n) \to \mathrm{Sp}(n)$ and thus $\mathrm{O} \to \mathrm{Sp}$.

Theorem 299.21. (Bott Periodicity Theorem) There exist homotopy equivalences

(1) $\mathrm{U} \to \Omega^2 \mathrm{U}$ (2) $\mathrm{Sp} \to \Omega^4 \mathrm{O}$ and (3) $\mathrm{O} \to \Omega^4 \mathrm{Sp}$.

Before we talk about the proof of the Bott Periodicity Theorem 299.21 let us move on to the following direct consequence.

Theorem 299.22. (Bott Periodicity Theorem) For any $i \in \mathbb{N}_0$ we have[3280]

$$\begin{array}{lrcl} (1) & \pi_{i+2}(\mathrm{U}) & \cong & \pi_i(\mathrm{U}), \\ (2) & \pi_{i+4}(\mathrm{Sp}) & \cong & \pi_i(\mathrm{O}), \\ (3) & \pi_{i+4}(\mathrm{O}) & \cong & \pi_i(\mathrm{Sp}). \end{array}$$

In particular, combining (2) and (3) we have for any $i \in \mathbb{N}_0$ that

$$\begin{array}{lrcl} (4) & \pi_{i+8}(\mathrm{O}) & \cong & \pi_i(\mathrm{O}), \\ (5) & \pi_{i+8}(\mathrm{Sp}) & \cong & \pi_i(\mathrm{Sp}). \end{array}$$

Proof of the Bott Periodicity Theorems 299.21 and 299.22. First note that the Bott Periodicity Theorem 299.22 follows immediately from Theorem 299.21 together with the Loop Space-Homotopy Groups Corollary 297.22. All the known proofs are non-trivial and go well-beyond what we can do in these modest notes.

Theorem 299.21 was first proved by Raoul Bott [**Bot1957, Bot1959, Bot1970**] in 1957. Another proof, using Morse theory, is given in [**Miln1963a**, p. 129, 142]. There are explicit continuous maps that give these homotopy equivalences, they are described well in [**DL61**]. Nonetheless the description is too long for us to write down at this point.

There are many proofs of the Bott Periodicity Theorem 299.22 which use a slightly different language, in particular it is at first glance not entirely clear whether they also imply Theorem 299.21. We refer to [**Stro2011**, Chapter 32.6], [**AGP2002**, Chapter 9.5], [**APr1999**] and [**Hat2**, Theorem 2.16] for a discussions and proofs of the Bott Periodicity Theorem 299.22. ■

Using the Bott Periodicity Theorem 299.22 we can completely determine the homotopy groups of O, U and Sp.

Proposition 299.23. (Homotopy Group-O-U-Sp-Proposition) The homotopy groups of O, U and Sp are as follows:[3281]

$i \bmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{O})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(\mathrm{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$.

Proof. The homotopy groups of U are straightforward to determine. More precisely, let $i \in \mathbb{N}_0$. We have

[3280] For $i = 0$ we actually have an isomorphism of pointed sets.

[3281] Recall that by the Homotopy Group-O-U-Sp-Lemma 299.20 this table gives us also the groups $\pi_k(\mathrm{O}(n))$ for $n > k + 1$, $\pi_k(\mathrm{U}(n))$ for $2n > k$ and $\pi_k(\mathrm{Sp}(n))$ for $4n > k - 2$.

$$\pi_i(\mathrm{U}) \underset{\text{by the Bott Periodicity Theorem 299.22}}{=} \pi_{i \bmod 2}(\mathrm{U}) \underset{\text{by the Homotopy Group O-U-Sp-Lemma 299.20 (2)}}{\cong} \pi_{i \bmod 2}(\mathrm{U}(1)) \underset{\text{since } \mathrm{U}(1) = S^1}{=} \pi_{i \bmod 2}(S^1) \underset{\text{see the Matrix Group Path-Component Proposition 4.54 and the Circle-}\pi_1\text{-Theorem 88.22}}{\cong} \begin{cases} 0, & \text{if } i = 0 \bmod 2, \\ \mathbb{Z}, & \text{if } i = 1 \bmod 2. \end{cases}$$

Now we consider the homotopy groups of O and Sp. By the Bott Periodicity Theorem 299.22 it suffices to compute the homotopy groups $\pi_i(\mathrm{O})$ and $\pi_i(\mathrm{Sp})$ for $i = 0, 1, 2, 3$.[3282] So let $i \in \{0, 1, 2, 3\}$. We have

$$\pi_i(\mathrm{Sp}) \underset{\text{by the Homotopy Group-O-U-Sp Lemma 299.20 (3)}}{\cong} \pi_i(\mathrm{Sp}(1)) \underset{\text{since } \mathrm{Sp}(1) = S^3}{=} \pi_i(S^3) \underset{\text{by the Sphere–Low-}\pi_n\text{-Proposition 134.19 and the } \pi_n(S^n)\text{–Proposition 167.10}}{\cong} \begin{cases} 0, & \text{if } i = 0, 1, 2, \\ \mathbb{Z}, & \text{if } i = 3. \end{cases}$$

We turn to the calculation of the homotopy groups of O. First we note that for any $i \geq 1$ we have

$$\pi_i(\mathrm{O}) \underset{\text{by the Homotopy Group-O-U-Sp Lemma 299.20 (3)}}{\cong} \pi_i(\mathrm{O}(i+2)) \underset{\text{by the Matrix Group-}\pi_0\text{-Proposition 4.54 and the Components-of-Topological Group Lemma 7.34}}{\cong} \pi_i(\mathrm{SO}(i+2)).$$

Thus it remains to prove the following claim:

Claim. (a) $\pi_0(\mathrm{O}(2)) \cong \mathbb{Z}/2\mathbb{Z}$ (b) $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$ (c) $\pi_2(\mathrm{SO}(4)) \cong 0$ (d) $\pi_3(\mathrm{SO}(5)) \cong \mathbb{Z}$.

Proof.

(a) By the Matrix Group-π_0-Proposition 4.54 (3) we know that $\mathrm{O}(2)$ consists of two path components, thus $\pi_0(\mathrm{O}(2))$ is by definition a pointed set with two elements.
(b) By the Matrix Group-π_1-Proposition 299.12 (1) we know that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
(c) We have the following isomorphisms:

$$\pi_2(\mathrm{SO}(4)) \underset{\text{by the Matrix Group–Isomorphisms Proposition 299.16 (1b) and the Quaternion Units Proposition 171.5 (2)}}{\cong} \pi_2((S^3 \times S^3)/\mathbb{Z}/2\mathbb{Z}) \underset{\text{by the Lie Group–Quotient Lemma 34.8 and the Covering-}\pi_n\text{-Isomorphism Proposition 134.22}}{\cong} \pi_2(S^3 \times S^3) \underset{\text{by the Product-}\pi_n \text{ Proposition 134.18}}{\cong} \pi_2(S^3) \oplus \pi_2(S^3) \underset{\text{the Sphere–Low-}\pi_n \text{ Proposition 134.19}}{=} 0.$$

[3282] We will see that $\pi_0(\mathrm{O})$ is the pointed set with two elements. It follows from the Bott Periodicity Theorem 299.22 that $\pi_{4+8k}(\mathrm{Sp})$ and $\pi_{8k}(\mathrm{O})$, $k \geq 1$, are also pointed sets with two elements. But these pointed sets are also groups. Since there is only one group structure on a pointed set with two elements we see that all these groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(d) We have the following isomorphisms:

$$\begin{array}{ccccccccc} & & & & \text{by the Lie Group–Quotient Lemma 34.8 and} & & & & \\ & & & & \text{the Covering-}\pi_n\text{-Isomorphism Proposition 134.22} & & & & \\ & & & & \downarrow\cong & & & & \\ \pi_3(\mathrm{SO}(5)) & \cong & \pi_3(\mathrm{Sp}(2)/\{\pm\mathrm{id}\}) & \cong & \pi_3(\mathrm{Sp}(2)) & \cong & \pi_3(\mathrm{Sp}) & \cong & \mathbb{Z}. \\ \uparrow & & & & & & \uparrow & & \uparrow \\ \text{by the Matrix Group–Isomorphisms} & & & & & & \text{by the Matrix Group–Inclusion} & & \text{see above} \\ \text{Proposition 299.16 (1c)} & & & & & & \pi_n\text{-Proposition 299.11} & & \end{array}$$

■

299.6. **H-spaces.** In this final section of this chapter we want to state two general statements about homotopy groups of Lie groups. In fact these statements hold for a much wider class of topological spaces that we introduce now:

Definition 299.24. We say that a topological space X is an H-space[3283] if there exists a continuous map

$$\mu\colon X\times X \to X$$

and an element $e\in X$ such that $\mu(e,e)=e$ and such that the maps

$$\begin{array}{ccc} (X,e) & \to & (X,e) \\ x & \mapsto & \mu(x,e) \end{array} \quad\text{and}\quad \begin{array}{ccc} (X,e) & \to & (X,e) \\ x & \mapsto & \mu(e,x) \end{array}$$

of pairs of topological spaces are homotopic to the identity.

Examples.

(1) Evidently Lie groups and more generally, topological groups are H-spaces, where μ is given by the group structure and e is the trivial element of the group. In particular we see that the Lie groups S^1 and S^3 are H-spaces.

(2) The topological space S^7 together with the multiplication given by the octonion structure on $\mathbb{O}=\mathbb{R}^8$ and with $e=1=(1,0,\dots,0)\in S^7$ is an H-space.[3284] Since octonion multiplication is not associative we see that, at least with this multiplication, S^7 is not a topological group. □

We have just seen that the spheres S^1, S^3 and S^7 are H-space. Before we continue with more examples we point out that we have all the tools to show that no other spheres have H-space structures.

Theorem 299.25. (Sphere-H-Space Theorem) Let $n\in\mathbb{N}$. If S^n admits the structure of an H-space, then $n\in\{1,3,7\}$.[3285]

Proof. Let $\mu\colon S^n\times S^n\to S^n$ be the multiplication map of an H-space structure and let $e\in X$ be the "neutral" element. It follows immediately from the definition of the bidegree, see Definition 216.18, together with the Homotopy-H_*-Proposition 144.12 and the existence of e, that the map μ has bidegree $(1,1)$. From the Bidegree–Hopf Invariant Proposition 216.21 we obtain a map $S^{2n+1}\to S^{n+1}$ with Hopf invariant ±1. By the Adams Theorem 237.12[3286] we know that $n+1\in\{2,4,8\}$. ■

[3283] According to [**Hub1999**, p. 747] the term H-space was coined by Jean-Pierre Serre to honor the work of Heinz Hopf.

[3284] Recall that by the discussion in Definition 214.11 we know that for octonions $w,z\in\mathbb{O}=\mathbb{R}^8$ we have $\|w\cdot z\|=\|w\|\cdot\|z\|$, in particular the multiplication restricts to a map on S^7. Also note that multiplication by $e=1$ is the identity.

[3285] In Exercise 208.6 we showed the much weaker statement that n has to be even.

[3286] We partially proved the somewhat weaker Hopf Invariant–Power-of-2 Theorem 237.11.

We continue with one more amusing example of an H-space.

Example. Given a non-zero polynomial $p(t) = \sum_{i=0}^{n} a_i \cdot t^i \in \mathbb{C}[t]$ we define

$$[p(t)] \;:=\; [a_0 : a_1 : \cdots : a_n : 0 : \dots] \;\in\; \mathbb{C}\mathrm{P}^\infty.$$

Evidently every element in $\mathbb{C}\mathrm{P}^\infty$ is of that form. We consider the "multiplication" map

$$\begin{array}{rcl} \mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty & \to & \mathbb{C}\mathrm{P}^\infty \\ ([p(t)], [q(t)]) & \mapsto & [\underbrace{p(t)\cdot q(t)}_{\in \mathbb{C}[t]}]. \end{array}$$

We leave it to the reader to verify that μ is continuous. It is clear that left and right "multiplication" by $e = [1] = [1 : 0 : \dots]$ is the identity. This shows that $\mathbb{C}\mathrm{P}^\infty$ together with μ and e is an H-space. Note that this multiplication μ is commutative and associative, but it does not have inverses[3287], thus μ does not define a group structure on $\mathbb{C}\mathrm{P}^\infty$. □

We formulate the last example as a lemma.

Lemma 299.26. (Loop-Space–H Space Lemma) Let (X, x_0) be a pointed topological space. We denote by $\Omega(X, x_0)$ the loop space from Definition 297.17.

(1) The map

$$\begin{array}{rcl} \mu\colon \Omega(X,x_0) \times \Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f\colon [0,1] \to X, g\colon [0,1] \to X) & \mapsto & (\quad \underbrace{f * g}_{\substack{\text{multiplication of loops} \\ \text{as introduced in Definition 88.7}}} \quad : [0,1] \to X) \end{array}$$

is continuous.

(2) The map $\mu\colon \Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$ from (1) together with the constant loop given by $e(t) \equiv x_0$, turns $\Omega(X, x_0)$ into an H-space.

Remark.

(1) Note that for the H-space $\Omega(X, x_0)$ "multiplication" by e is not the identity, it is only homotopy equivalent to the identity.

(2) We had hinted at the existence of a multiplication map on the loop space $\Omega(X, x_0)$ in Exercise 297.7. □

Proof. Let (X, x_0) be a pointed topological space. To simplify the notation we write ΩX instead of $\Omega(X, x_0)$.

(1) We need to show that the map $\mu\colon \Omega X \times \Omega X \to \Omega X$ is in fact continuous. By the"$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 16.10, applied to the topological space $T = \Omega X \times \Omega X$ and the regionally compact space $Y = [0, 1]$, it suffices to show that the following map is continuous:

$$\Phi\colon (\Omega X \times \Omega X) \times [0,1] \xrightarrow{\mu\times \mathrm{id}} \Omega X \times [0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

On the subset $(\Omega X \times \Omega X) \times [0, \frac{1}{2}]$ this map is equal to

$$(\Omega X \times \Omega X) \times [0, \tfrac{1}{2}] \xrightarrow{((f,g)\mapsto f)\times(t\mapsto 2t))} \Omega X \times [0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

[3287] Why are there elements in $\mathbb{C}\mathrm{P}^\infty$ which do not have multiplicative inverses?

The first map is evidently continuous. Furthermore, the second map is continuous by the Evaluation Map Proposition 16.9. Thus we have shown that the restriction of Φ to the subset $(\Omega X \times \Omega X) \times [0, \frac{1}{2}]$ is continuous. Basically the same argument shows that the restriction of Φ to $(\Omega X \times \Omega X) \times [\frac{1}{2}, 1]$ is continuous. it follows from the Homotopy Combination Lemma 18.8 (1) that Φ itself is continuous.

(2) Let $e \in \Omega(X, x_0)$ be the constant loop given by $e(t) \equiv x_0$. It is clear that $\mu(e, e) = e$. Thus it remains to show that the maps

$$\begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(f,e) \end{array} \quad \text{and} \quad \begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(e,f) \end{array}$$

of pairs of topological spaces are homotopic to the identity. We consider the map

$$\begin{array}{rcl} H\colon \Omega(X,x_0)\times[0,1] & \to & \Omega(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{2s}{t+1}), & \text{if } s \in [0, \frac{t+1}{2}], \\ x_0, & \text{otherwise.} \end{cases} \end{array} \right) \end{array}$$

We leave it to the underchallenged reader to show, using the"$\Rightarrow$"-direction of the Compact-Open Exponential Proposition 16.10, that this map H is in fact continuous. Note that we have $H(e, t) = e$ for all $t \in [0, 1]$. With these observations it is basically clear that it is the desired homotopy between the map $f \mapsto \mu(f, e)$ and the identity.

Almost exactly the same way we deal with the second map $f \mapsto \mu(e, f)$. ■

The following proposition, which is a generalization of the Topological Group-π_1-Proposition 90.16, is the only statement on H-spaces that we can prove with our limited space resources.

Proposition 299.27. (H-Space-π_1-Proposition) If X is a path-connected H-space (e.g. if X is a topological group), then $\pi_1(X)$ is abelian.

Remark. Let (Y, y_0) be a pointed topological space. We recall the following two facts:

(a) Recall that in the Loop Space-Homotopy Groups Corollary 297.22 we saw that given any $k \in \mathbb{N}$ there exists an isomorphism $\pi_{k+1}(Y, y_0) \xrightarrow{\cong} \pi_k(\Omega(Y, y_0))$.

(b) We pointed out above that the loop space $\Omega(Y, y_0)$ is an H-space.

The combination of (a), (b) and the H-Space-π_1-Proposition 299.27 implies that given any $k \in \mathbb{N}_{\geq 2}$ we have

$$\pi_k(Y,y_0) \underset{\text{by (a)}}{\overset{\uparrow}{\cong}} \pi_{k-1}(\Omega(Y,y_0)) \cong \ldots \underset{\text{by (a)}}{\overset{\uparrow}{\cong}} \pi_1(\overbrace{\Omega(\Omega\ldots\Omega}^{k \text{ times}}(Y,y_0))) \underset{\text{by (b) and since } k\geq 2}{\overset{\uparrow}{=}} \pi_1(\text{H-space}) \overset{\text{H-Space-}\pi_1\text{-Proposition 299.27}}{\overset{\downarrow}{=}} \text{abelian}.$$

We have thus shown that $\pi_k(Y, y_0)$ is abelian for every $k \geq 2$. In other words, we have obtained a new proof for the π_n-Abelian Proposition 134.11. □

Just for kicks we provide two different proofs for the H-Space-π_1-Proposition 299.27. The first proof, which we provide here is "down to earth". The second proof, which we will provide in the next section, is more structural.

First proof of the H-Space-π_1-Proposition 299.27. We denote by $\mu\colon X \times X \to X$ and $e \in X$ the objects in the definition of an H-space. Let $f, g\colon [0,1] \to X$ be loops in e. We need to show that the path $f * g$ is path-homotopic to the path $g * f$.

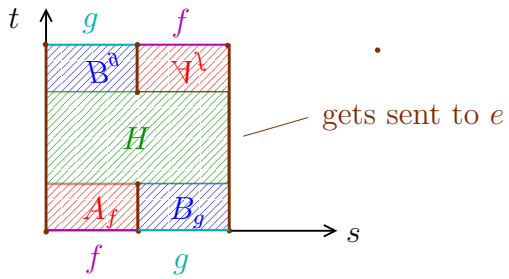
We denote by $\mu(f, e)$ the path $t \mapsto \mu(f(t), e)$ and similarly we define $\mu(e, g)$. We consider the map

$$\begin{array}{rccc} F\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & \mu(f(s), g(t)). \end{array}$$

Note that this map is continuous by the Topological-Product Proposition 7.2 (1) and the fact that μ is continuous. Furthermore we consider the following path-homotopy:[3288]

$$\begin{array}{rccc} H\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & \begin{cases} F(2s\cdot(1-t,t)), & \text{if } s \le \frac{1}{2} \\ F((2-2s)\cdot(1-t,t) + (2s-1)\cdot(1,1)), & \text{if } s > \frac{1}{2}. \end{cases} \end{array}$$

By carefully plugging in $t = 0$ and $t = 1$ we see that H is a path-homotopy from the loop $\mu(f, e) * \mu(e, g)$ to the loop $\mu(e, g) * \mu(f, e)$.



By definition of an H-space there exists a homotopy $A\colon X \times [0,1] \to X$ rel $\{e\}$ with $A_0 = \mathrm{id}$ and $A_1 = \mu(-, e)$ and there exists a homotopy $B\colon X \times [0,1] \to X$ rel $\{e\}$ with $B_0 = \mathrm{id}$ and $B_1 = \mu(e, -)$. We consider the maps

$$\begin{array}{rccc} A_f\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & A(\underbrace{f(s)}_{\in X}, t) \end{array} \quad \text{and} \quad \begin{array}{rccc} B_g\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & B(\underbrace{g(s)}_{\in X}, t). \end{array}$$

Note that A_f is a path-homotopy from the loop f to the loop $\mu(f, e)$ and B_g is a path-homotopy from the loop g to the loop $\mu(e, g)$. The path-homotopy between the loops $f * g$ and $g * f$ is given by combining H with A_f and B_g as indicated in the figure below. We leave it to the reader to conscientiously write down the precise homotopy. ∎

The following theorem was first proved by William Browder [**Browd1960**, Theorem 2] [**Browd1961**, Theorem 6.11] in 1961.

Theorem 299.28. (H-Space-π_n-Theorem) Let X be a path-connected H-space. If the group $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X;\mathbb{Z})$ is finitely generated, then

$$\min\{m \in \mathbb{N}_{\ge 2} \mid \pi_m(X) \neq 0\}$$

is odd. In particular we have $\pi_2(X) = 0$.

Examples.

[3288] It follows easily from the Pasting Proposition 4.7 (2′) that this map is continuous.

(1) On page 5767 we saw that $\mathbb{C}\mathrm{P}^\infty$ is an H-space. Furthermore in the $\pi_2(\mathbb{C}\mathrm{P}^n)$–Corollary 167.12 we saw that $\pi_2(\mathbb{C}\mathrm{P}^\infty) \neq 0$. Thus we see that in the statement of the H-Space-π_n-Theorem 299.28 we cannot drop the hypothesis that $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X;\mathbb{Z})$ is a finitely generated group.
(2) Let $k \in \mathbb{N}$. It follows from the H-Space-π_n-Theorem 299.28, together with the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 and the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 155.9 that the finite complex projective space $\mathbb{C}\mathrm{P}^k$ does *not* admit an H-space structure. □

We conclude this chapter with a corresponding theorem for Lie groups.

Theorem 299.29. (Lie Group-π_2-Theorem) If G is a path-connected Lie group, then

$$\min\{m \in \mathbb{N}_{\geq 2} \,|\, \pi_m(G) \neq 0\}$$

is odd. In particular $\pi_2(G) = 0$.

Remark. There are many direct proofs in the literature for the statement that given a path-connected Lie group G the second homotopy group $\pi_2(G)$ vanishes, see e.g. [**Carta1936**] or [**Miln1963a**, Theorem 21.7] or [**BtD1995**, Proposition V.7.5]. □

Note that the Lie Group-π_2-Theorem 299.29 has no conditions on the Lie group except that it needs to be path-connected. The reason why we can get away with so few conditions is given by the following proposition:

Proposition 299.30. (Maximal Compact Lie Subgroup Proposition) Let G be a Lie group. There exists a compact Lie subgroup K[3289] such that G is homeomorphic to $K \times \mathbb{R}^n$ for some $n \in \mathbb{N}_0$.

Proof of Proposition 299.30. Like many other foundational results on Lie groups this proposition has its origins in the work of Henri Cartan [**Carta1936**]. A full proof is given in [**Mal1945**, Theorem 11] or [**Mos1949**, Theorem 2]. A sketch of the proof is provided in [**Sam1952**, Chapter 7]. ■

Example. We consider the group $\mathrm{Isom}(\mathbb{R}^n)$ of isometries of the vector space $\mathbb{R}^n$ equipped with the usual inner product. It is an elementary, but not totally trivial fact, see [**Rees1983**, p. 11], that

$$\mathrm{Isom}(\mathbb{R}^n) \;=\; \{\text{all maps of the form } x \mapsto Ax + v \text{ where } A \in \mathrm{O}(n) \text{ and } v \in \mathbb{R}^n\}.$$

The maps

$$\begin{array}{rcl} \mathrm{O}(n) \times \mathbb{R}^n & \to & \mathrm{Isom}(\mathbb{R}^n) \\ (A, v) & \mapsto & (x \mapsto Ax + v) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{Isom}(\mathbb{R}^n) & \to & \mathrm{O}(n) \times \mathbb{R}^n \\ f & \mapsto & (\underbrace{(x \mapsto (f(x) - f(0))}_{\text{represented by matrix in } \mathrm{O}(n)}, f(0)) \end{array}$$

are inverses of one another. Thus we can use the smooth manifold structure on $\mathrm{O}(n) \times \mathbb{R}^n$ to turn $\mathrm{Isom}(\mathbb{R}^n)$ into a smooth manifold and it is quasi-obvious to see that this turns $\mathrm{Isom}(\mathbb{R}^n)$ into a Lie group. We can view $\mathbb{R}^n$ and $\mathrm{O}(n)$ as subgroups of $\mathrm{Isom}(\mathbb{R}^n)$ in an obvious way. But one easily verifies that the map $\mathrm{O}(n) \times \mathbb{R}^n \to \mathrm{Isom}(\mathbb{R}^n)$ is not an

[3289] In fact a Lie group contains a maximal compact Lie subgroup which is furthermore unique up to conjugation. We will not make use of this fact.

isomorphism. In other words, as predicted by the Maximal Compact Lie Subgroup Proposition 299.30 the Lie group $\mathrm{Isom}(\mathbb{R}^n)$ splits topologically into a product. But the Lie group does not split as a Lie group. □

Proof of the Lie Group-π_2-Theorem 299.29. Let G be a path-connected Lie group. By the Maximal Compact Lie Subgroup Proposition 299.30 we know that G is homotopy equivalent to a compact path-connected Lie group K. By the Homotopy Equivalence-π_n-Proposition 134.17 (1) it suffices to prove the desired statement for K. Since K is in particular a compact smooth manifold we obtain from the Smooth Manifolds-Invariants Proposition 179.10 that $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X;\mathbb{Z})$ is a finitely generated abelian group. Thus we can conclude the proof by appealing to Theorem 299.28. ■

299.7. **The Eckmann-Hilton Theorem.** In this section we will state and prove a theorem that was provided by Bruno Eckmann and Peter Hilton [**EcH1962**] in 1962. The theorem is totally elementary, but at the same time it is surprisingly useful. In particular it will be the key to giving the second proof of the H-Space-π_1-Proposition 299.27.

The formulation of the Eckmann-Hilton Theorem requires the following harmless definition.

Definition 299.31. Let X be a set.

(1) A binary operation is nothing but a map $X \times X \to X$.
(2) A binary operation $\mu\colon X \times X \to X$ is called unital if there exists an element $1 \in X$ such that $\mu(1,x) = \mu(x,1) = x$ for all $x \in X$.

Example. The multiplication map on a group is evidently a unital binary operation. □

The following theorem is in some sense quite elementary, but the cleverness lies in its formulation.

Theorem 299.32. (Eckmann-Hilton Theorem) Let X be a set equipped with two binary operations "$\circ$" and "$\otimes$". Suppose the following two conditions are satisfied:

(a) "$\circ$" and "$\otimes$" are both unital.
(b) For every $a,b,c,d \in X$ we have the equality

$$(a \otimes b) \circ (c \otimes d) \;=\; (a \circ c) \otimes (b \circ d).$$

Then "$\circ$" and "$\otimes$" are the same and they are commutative and associative.

Proof. [3290] First we show that the units of the two operations "$\circ$" and "$\otimes$" coincide:

$$1_\circ = 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) \underset{\text{by (b)}}{=} (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes.$$

Next we show that the two operations "$\circ$" and "$\otimes$" are the same and that they are commutative. Thus let $x,y \in X$. Then

$$x \circ y = (1 \otimes x) \circ (y \otimes 1) \underset{\text{by (b)}}{=} (1 \circ y) \otimes (x \circ 1) = y \otimes x = (y \circ 1) \otimes (1 \circ x) \underset{\text{again by (b)}}{=} (y \otimes 1) \circ (1 \otimes x) = y \circ x.$$

[3290] This is of course the solution to Exercise 134.10.

Finally we show that the operations are associative. Thus let $x, y, z \in X$. We have the following equality:

$$(x \otimes y) \otimes z \;=\; (x \otimes y) \otimes (1 \otimes z) \underset{\text{one last time by (b)}}{\overset{\uparrow}{=}} (x \otimes 1) \otimes (y \otimes z) \;=\; x \otimes (y \otimes z).$$

■

Now we can prove the following lemma, which in particular gives the promised second, more structural proof of the H-Space-π_1-Proposition 299.27.

Lemma 299.33. (Binary Operations-on-π_1-Lemma) Let X be a path-connected H-space and let $x_0 \in X$. In the following we denote by $\mu\colon X \times X \to X$ the multiplication map of the H-space. We consider the following two binary operations on $\pi_1(X, x_0)$:

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \overset{\circ}{\to} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & [f * g] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \overset{\otimes}{\to} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \mu(f(t),g(t)) \end{array}\right]. \end{array}$$

The following two statements hold:

(1) The two binary operations agree.

(2) The group $\pi_1(X, x_0)$ is abelian.

Proof. Let X be a path-connected H-space and let $x_0 \in X$. We denote by $\mu\colon X \times X \to X$ the multiplication map of the H-space. Since μ is an H-space structure one sees easily that "$\otimes$" is a unital binary operation. Furthermore we have known ever since the π_1–Group Proposition 88.12 that "$\circ$" is a unital binary operation. In Exercise 299.7 we will show that Condition (b) of the Eckmann-Hilton Theorem 299.32 is satisfied. Thus we obtain from the Eckmann-Hilton Theorem 299.32 the amusing fact that the two operations agree and we obtain that both operations are commutative. In particular this means that $\pi_1(X, x_0)$ is abelian. ■

299.8. Exercises for Chapter 299.

Exercise 299.1. Let $n \in \mathbb{N}$. We define

$$\mathrm{GL}(n,\mathbb{H}) \;=\; \left\{ A \in \mathrm{M}(n \times n, \mathbb{H}) \;\middle|\; \begin{array}{c} \text{there exist matrices } X, Y \in \mathrm{M}(n \times n, \mathbb{H}) \\ \text{such that } AX = YA = \mathrm{id} \end{array} \right\}.$$

Is $\mathrm{GL}(n, \mathbb{H})$ an open subset of $\mathrm{M}(n \times n, \mathbb{H})$?

Exercise 299.2. Let $n \in \mathbb{N}$. In the proof of the Matrix–Lie Group Lemma 299.5 we already saw that $\mathrm{Sp}(n)$ is a smooth manifold and that matrix multiplication turns $\mathrm{Sp}(n)$ into a group. Show that the map

$$\begin{array}{rcl} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$

is a smooth map.

Note: The argument in the proof of the Matrices–Topological Group Lemma 7.31 can be used to show that taking inverses in $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ is smooth. Does this help us in any way?

Remark. In principle one could use Exercise 34.1. But what do readers do who are still traumatized by the Implicit Function Theorem?

Exercise 299.3. Show that the Lie group $\mathrm{SU}(n) \times S^1$ is diffeomorphic to a finite covering of $\mathrm{U}(n)$.

Exercise 299.4. Determine all compact connected 2-dimensional smooth manifolds that admit a Lie group structure.

Exercise 299.5. We define $\mathrm{SU} = \operatorname{colim}\limits_{\longrightarrow} \mathrm{SU}(n)$ the same way as we defined U. Determine the homotopy groups of SU.
Remark. You might want to use Exercise 298.6.

Exercise 299.6.

(a) Let X be a CW-complex and let $e \in X^0$ be a point in the 0-skeleton. Suppose we are given a continuous map $\mu\colon X \times X \to X$ such that the two maps

$$\begin{array}{ccc} X & \to & X \\ x & \mapsto & \mu(x,e) \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \to & X \\ x & \mapsto & \mu(e,x) \end{array}$$

are homotopic to the identity. Show that μ is homotopic to a map $\nu\colon X \times X \to X$ such that $\nu(x,e) = \nu(e,x) = x$ for all $x \in X$.

(b) Show that ν in (a) defines an H-space structure on X.

(c) Let X be a CW-complex. Suppose that there exists a homotopy equivalence $f\colon X \to Y$ to an H-space Y. Show that X itself admits an H-space structure.

Exercise 299.7. Let (X, x_0) be a path-connected non-empty H-space. Let $\mu\colon X \times X \to X$ be the multiplication of the H-space structure. We consider the following two binary operations on $\pi_1(X, x_0)$:

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \xrightarrow{\circ} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & [f * g] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \xrightarrow{\otimes} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & \begin{pmatrix} [0,1] \to X \\ t \mapsto \mu(f(t),g(t)) \end{pmatrix}. \end{array}$$

Show that "$\circ$" and "$\otimes$" satisfy Condition (b) of the Eckmann-Hilton Theorem 299.32.

Exercise 299.8. Let (X, x_0) be a pointed topological space and let $k \in \mathbb{N}_{\geq 2}$. Define the multiplication $*\colon \pi_k(X,x_0) \times \pi_k(X,x_0) \to \pi_k(X,x_0)$ as in Definition 134.4. Provide a different multiplication $\otimes\colon \pi_k(X,x_0) \times \pi_k(X,x_0) \to \pi_k(X,x_0)$ such that "$*$" and "$\otimes$" satisfy the conditions of the Eckmann-Hilton Theorem 299.32. Conclude, once again, that $\pi_k(X,x_0)$ is abelian.

Exercise 299.9. Let $n \in \mathbb{N}_{\geq 3}$. In the Matrix Group-$\pi_1$-Proposition 299.12 (1) we saw that $\pi_1(\mathrm{SO}(n), \mathrm{id}) \cong \mathbb{Z}/2\mathbb{Z}$. Give an example of a loop that is a generator of $\pi_1(\mathrm{SO}(n), \mathrm{id})$ and show explicitly why it has order 2 in the fundamental group.

Exercise 299.10. We start out with the following three observations:

(1) It follows from the Matrix Group-Fiber Proposition 299.8 that we have a bundle map $p\colon \mathrm{SO}(3) \to S^2$ with fiber $\mathrm{SO}(2)$.

(2) On page 202 we saw that $\mathrm{SO}(2)$ is homeomorphic to S^1.

(3) It follows from (1) and the Fiber-LES-Proposition 298.4 that we have a long exact sequence

$$\underbrace{\pi_2(S^2)}_{\cong\mathbb{Z}} \xrightarrow{\partial} \underbrace{\pi_1(\mathrm{SO}(2))}_{\cong\mathbb{Z}} \to \pi_1(\mathrm{SO}(3)) \to \underbrace{\pi_1(S^2)}_{=0} \to \dots.$$

Analyze the connecting homomorphism $\pi_2(S^2) \to \pi_1(\mathrm{SO}(2))$ to give a new proof that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 299.11. Let $n \in \mathbb{N}$. Use the approach of the proof of the Matrix Group–Inclusion–π_n-Proposition 299.11 to show that $\mathrm{SO}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are path-connected.
Remark. This exercise provides a new and perhaps more satisfactory proof to some of the statements of the Matrix Group-π_0-Proposition 4.54.

Exercise 299.12.
(a) For every $n \in \mathbb{N}$ compute the fundamental group $\pi_1(\mathrm{U}(n))$.
(b) For every $n \in \mathbb{N}$ compute the second homotopy group $\pi_2(\mathrm{U}(n))$.

CHAPTER 300

Thom-Pontryagin Theory I: The basics

In this chapter we will introduce the Thom-Pontryagin approach to calculating homotopy groups of spheres. Furthermore we will see that it allows us to reprove several earlier results. In Chapters 301 and 302 we will use the Thom-Pontryagin approach to obtain new results.

This chapter is organized as follows:

- In Section 300.1 we will introduce the generalized Thom-Pontryagin construction.
- In Section 300.2 we will relate the Thom-Pontryagin construction to homotopy groups.
- In Section 300.3 we will introduce the thickened cobordism groups and will state and prove the Thom-Pontryagin Theorem 300.19 which says that the thickened cobordism groups are isomorphic to suitable homotopy groups of spheres.
- In Section 300.4 we will calculate the thickened cobordism group $\Omega_1^{\text{th}}(\mathbb{R}^3)$. show that the thickened cobordism group $\Omega_{2n-1}^{\text{th}}(\mathbb{R}^{4n-1})$ admits an epimorphism onto $\mathbb{Z}$.
- In Section 300.5 we will recall and generalize the definition of the linking number.
- In Section 300.6 we will use the linking number to define the Hopf invariant on thickened cobordism groups. We will use the Hopf invariant to show that for any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.
- In Section 300.7 we will provide exercises for Chapter 300.

In the next Chapter 301 we will state and prove the Freudenthal Suspension Theorem 301.12.

Before we start with discussing the Thom-Pontryagin theory we fix some notations and conventions.

Notation 300.1. Let $n \in \mathbb{N}_0$.

(1) We denote by $*$ the point in $\overline{B}^n/S^{n-1}$ that is given by $[S^{n-1}]$.
(2) We denote by $*$ the "North Pole" $(0,\ldots,0,1) \in S^n$.

We make the following identifications:

(a) We use the homeomorphism $S^n \to \mathbb{R}^n \cup \{\infty\}$ from the Stereographic Projection Lemma 4.24, which is given by stereographic projection, that sends the North Pole $*$ to ∞ to make the identification $S^n =_{\text{i}} \mathbb{R}^n \cup \{\infty\}$.
(b) In the Ball Quotient-Sphere Lemma 8.10 we wrote down an explicit homeomorphism $\overline{B}^n/S^{n-1} \to S^n$. Note that this homeomorphism sends $*$ to $(0,\ldots,0,1)$.
(c) Putting the two homeomorphisms from (a) and (b) together we obtain a homeomorphism

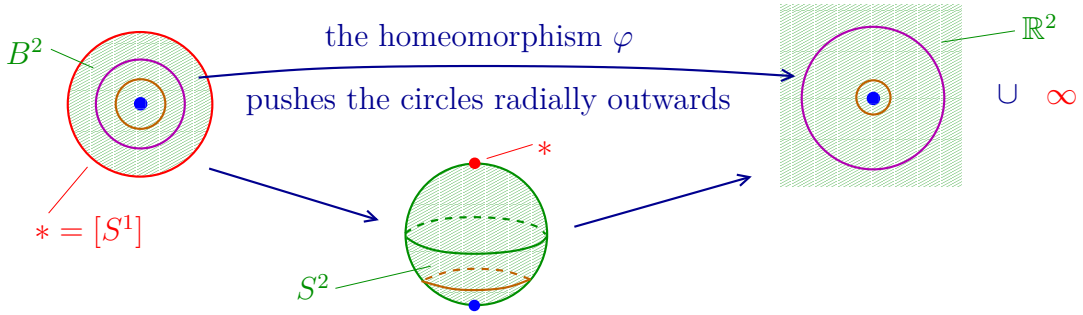
$$\varphi\colon \overline{B}^n/S^{n-1} \;\to\; \mathbb{R}^n \cup \{\infty\}.$$

It follows fairly easily from the definitions of the two maps in (a) and (b) that φ restricts to a diffeomorphism $B^n \to \mathbb{R}^n$ and that φ is of the form

$$x \;\mapsto\; \begin{cases} \eta(\|x\|)\cdot x, & \text{if } x \in B^n, \\ \infty, & \text{if } x = *, \end{cases}$$

where $\eta\colon [0,1) \to [0,\infty)$ is a strictly monotonously increasing function with $\eta(0) = 0$ and $\lim_{t\to 1} \eta(t) = \infty$.

Summarizing we have the identifications $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} \overline{B}^n/S^{n-1}$ and under these identifications the points $* \in S^n$, ∞ and $* \in \overline{B}^n/S^{n-1}$ get identified. We refer to the figure below for an illustration.



300.1. **The generalized Thom-Pontryagin construction.** In Definition 234.9 we saw that a proper oriented codimension-one submanifold of a compact oriented smooth manifold M gives rise, via the Thom-Pontryagin construction, to a map $M \to S^1$. In this section we will generalize this construction.

The key objects in the more general construction of Thom-Pontryagin maps are "thickened" submanifolds, which generalize the notion of a thickened knot from Definition 283.3:

Definition 300.2. Let M be an orientable n-dimensional smooth manifold.

(1) Let N be a compact orientable proper k-dimensional submanifold of M. A thickening of N is a smooth embedding $g\colon \overline{B}^{n-k} \times N \to M$ with the following properties:[3291 32923293]
   (1) For all $P \in N$ we have $g(0,P) = P$.
   (2) The image $g(\overline{B}^{n-k} \times N)$ is a submanifold with corner of M.
   (3) We have $g(\overline{B}^{n-k} \times N) \cap \partial M = g(\overline{B}^{n-k} \times \partial N)$.

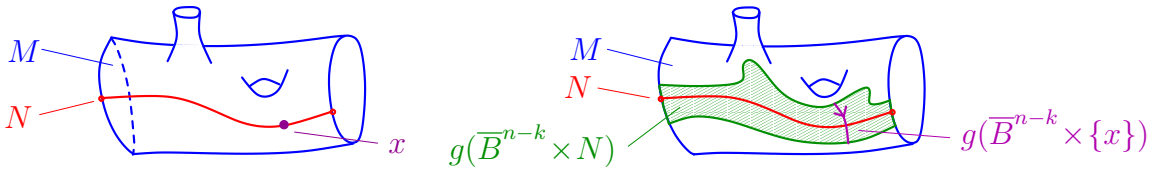
   We refer to the figure below for an illustration.

(2) A thickened submanifold of M is a pair (N,g) where N is a compact orientable proper[3294] submanifold of M and g is a thickening for N.

(3) Let $\varphi\colon M \to W$ be a smooth embedding of M into a smooth manifold W of the same dimension. Given a thickened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N)$ of $M \setminus \partial M$ we equip $\varphi(N)$ with the "obvious thickening", namely with the thickening that is given by

$$\begin{aligned} \varphi_*(g)\colon \overline{B}^{n-k} \times \varphi(N) &\;\to\; W \\ (v,P) &\;\mapsto\; g(v, \varphi^{-1}(P)). \end{aligned}$$

Evidently $(\varphi(N), \varphi_*(g))$ is a thickened submanifold of W.

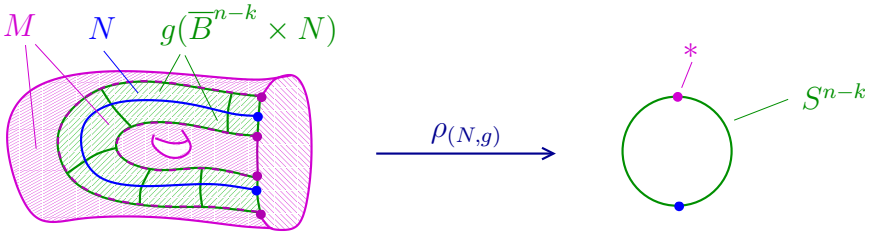


The following definition can be viewed as a generalization of the Thom-Pontryagin map introduced in Definition 234.9.

Definition 300.3. Let M be an orientable n-dimensional smooth manifold. Given a thickened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k}\times N\to M)$ we refer to the map[3295]

$$\begin{aligned}\rho_{(N,g)}\colon M &\to S^{n-k}=\overline{B}^{n-k}/S^{n-k-1}\\ P &\mapsto \begin{cases}[x], & \text{if } P=g(x,y) \text{ with } (x,y)\in\overline{B}^{n-k}\times N,\\ *, & \text{otherwise}\end{cases}\end{aligned}$$

as the Thom-Pontryagin map. (We refer to the figure below for an illustration.) Sometimes we refer to this approach of assigning to a thickened k-dimensional submanifold a map $M\to S^{n-k}$ as the Thom-Pontryagin construction.



Remark.

(1) Let N be a compact *oriented* proper codimension-one submanifold of an *oriented* smooth manifold M. It follows immediately from the Product Tubular Neighborhood–Existence Theorem 58.4 (3) together with the Tubular Map Lemma 58.3 that N admits a thickening $[-1,1]\times N$ such that the inclusion $(-1,1)\times N\to M$ is orientation-preserving.[3296] The map $\rho_N\colon M\to S^1$ that we defined in Definition 234.9 is basically the same as the above map $\rho_{(N,g)}\colon M\to S^1$. The only difference is that the two maps differ by multiplication by $\mathrm{i}\in S^1$.

[3291]Here one really would like to say that $g\colon \overline{B}^{n-k}\times N\to M$ is an "smooth embedding", but if N has non-empty boundary, then it is not entirely clear in what sense $\overline{B}^{n-k}\times N$ is a smooth manifold and what "smooth embedding" is supposed to mean.

[3292]The definition looks very similar to the definition of a tubular neighborhood. In fact a thickening is the same data as a product tubular neighborhood, it is just phrased in slightly different words.

[3293]In Definition 283.3 we introduced the thickening of an *oriented* knot. For knots the present definition is the same as before, except that we no longer demand that the knot is oriented.

[3294]Note that here, in the definition of a thickened submanifold, we sneaked in the adjectives "compact", "orientable" and "proper", so that we will not have to carry them around all the time.

[3295]It follows easily from the Smooth Submanifold-with-Corner Lemma 26.16 (3) and the Pasting Proposition 4.7 (2) that this map is indeed continuous.

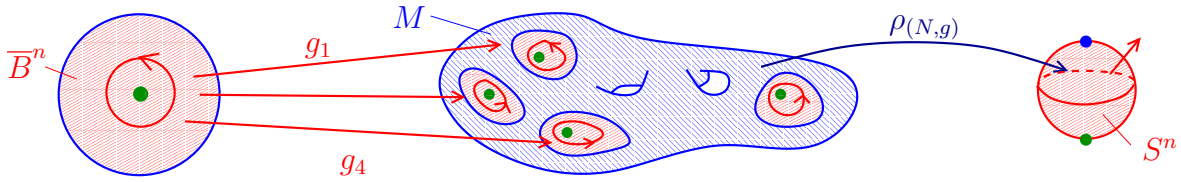
[3296]Furthermore we saw in the Tubular Neighborhood–Uniqueness Theorem 57.12 that this thickening is essentially unique.

(2) We saw in the Non-Product Tubular Neighborhood Proposition 220.22 that the submanifold $\mathbb{C}\mathrm{P}^1$ of $\mathbb{C}\mathrm{P}^2$ does *not* admit a product tubular neighborhood, in particular it does not admit a thickening. □

Example. Let $n \in \mathbb{N}$ and let M be a closed orientable n-dimensional smooth manifold together with a 0-dimensional submanifold $N = \{P_1, \ldots, P_m\}$. Suppose that N is equipped with a thickening

$$g = g_1 \sqcup \cdots \sqcup g_m \colon (\overline{B}^n \times \{P_1\}) \sqcup \cdots \sqcup (\overline{B}^n \times \{P_m\}) \;\to\; M.$$

From the Thom-Pontryagin construction we obtain a map $\rho_{(N,g)} \colon M \to S^n$.



It might not be clear what this is good for, but it is kind of fun. This example is illustrated in the figure above. □

The following lemma gives us a first interesting example of a Thom-Pontryagin map.

Lemma 300.4. (Point-in-Ball-Thickening Lemma) Let $n \in \mathbb{N}$ and let $P \in B^n$. We equip the 0-dimensional smooth manifold $\{P\}$ with any thickening $g \colon \overline{B}^n \times \{P\} \to B^n$ that is orientation-preserving[3297]. The corresponding map

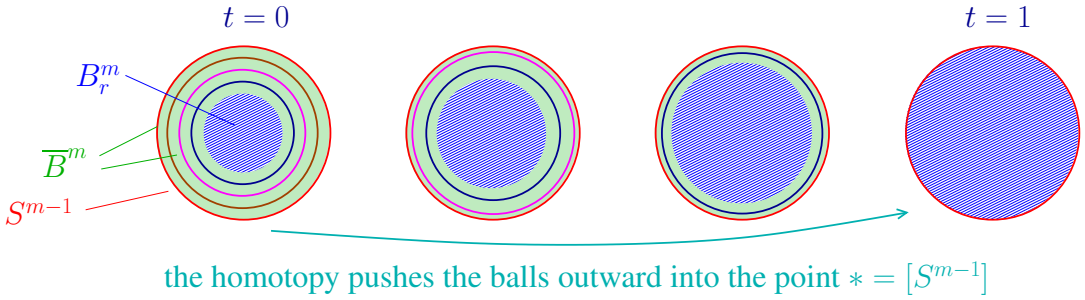
$$\rho_{(\{P\},g)} \colon \overline{B}^n / S^{n-1} \;\to\; \overline{B}^n / S^{n-1}$$

is homotopic rel $*$ to the identity map of $\overline{B}^n / S^{n-1}$.

The proof of the Point-in-Ball-Thickening Lemma 300.4 relies on the following lemma which we will use on several occasions.

Lemma 300.5. (Radial Pushing Lemma) Let $m \in \mathbb{N}$. Given any real number $r \in (0,1)$ there exists a homotopy $F \colon \overline{B}^m / S^{m-1} \times [0,1] \to \overline{B}^m / S^{m-1}$ rel $*$ with the following three properties:

(1) $F_0 = \mathrm{id}$.
(2) The restriction of F_1 to B^m_r is given by multiplication by $\frac{1}{r}$.
(3) For any point $x \notin B^m_r$ we have $F_1(x) = *$.



[3297] We equip $\overline{B}^n$ and B^n with the usual orientation.

Proof of the Radial Pushing Lemma 300.5. Let $r \in (0,1)$. We consider the map

$$\begin{array}{rcl} (\overline{B}^m/S^{m-1}) \times [0,1] & \to & \overline{B}^m/S^{m-1} \\ ([x],t) & \mapsto & \begin{cases} [x \cdot (1-t+t\cdot\frac{1}{r})], & \text{if } x \in B^m \text{ and } t < \frac{1-\|x\|}{(\frac{1}{r}-1)\cdot\|x\|} \\ *, & \text{otherwise.} \end{cases} \end{array}$$

We leave as to the engaged reader to verify that this map is indeed continuous. It is clear that F has the desired properties. ■

Now we can easily prove the Point-in-Ball-Thickening Lemma 300.4.

Sketch of proof of Lemma 300.4. It follows quite easily[3298] from the Smooth Ball Embedding Theorem 45.10 that we can assume without loss of generality that $P = 0$ and that we can assume that g is the thickening $g\colon \overline{B}^n \times \{0\} \to B^n$ given by $g(v,0) = \frac{1}{2}v$. By the Radial Pushing Lemma 300.5 there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following three properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to $B^n_{\frac{1}{2}}$ is given by the map $x \mapsto 2 \cdot x$,
(3) for any point $x \notin B^n_{\frac{1}{2}}$ we have $F_1(x) = *$.

If one thinks about it for a second, then one realizes that F_1 equals $\rho_{(0,g)}$ "on the nose". Thus we have found the desired homotopy. ■

300.2. **The Thom-Pontryagin construction and homotopy groups.** The following is the key observation which justifies our renewed interest in the Thom-Pontryagin construction.

Observation. Let $n,k \in \mathbb{N}_0$ and let $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ be a thickened k-dimensional submanifold of $\mathbb{R}^{n+k}$. We can view $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ also as a thickened submanifold of $\mathbb{R}^{n+k} \cup \{\infty\}$.[3299] Note that the corresponding Thom-Pontryagin map

$$\begin{array}{rcl} \rho_{(N,g)}\colon S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} & \to & \overline{B}^n/S^{n-1} = S^n \\ P & \mapsto & \begin{cases} [x], & \text{if } P = g(x,y) \text{ with } (x,y) \in \overline{B}^n \times N, \\ *, & \text{otherwise,} \end{cases} \end{array}$$

sends ∞ to $*$. In particular $\rho_{(N,g)}$ defines an element in $\pi_{n+k}(S^n,*)$. It is worth repeating this observation:

thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to elements in $\pi_{n+k}(S^n,*)$.

The following questions immediately arise:

Question 300.6. Let $n,k \in \mathbb{N}_0$.

(1) Which elements in $\pi_{n+k}(S^n,*)$ can be realized by Thom-Pontryagin maps?
(2) When do two thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to the same element in $\pi_{n+K}(S^n,*)$.

Before we address this question we intend to discuss two examples in greater detail:

(1) the case of thickened 0-dimensional submanifolds of some $\mathbb{R}^n$, and

[3298] The slightly hesitant "quite easily" refers to the fact that, if one thinks about it, we need that the diffeotopy of $\mathbb{R}^{m+k}$ given by the Smooth Ball Embedding Theorem 45.10 has compact support.
[3299] Why can we do so?

(2) the case of thickened 1-dimensional submanifolds of $\mathbb{R}^3$.

We start out with the first case. Dealing with this case requires one definition.

Definition 300.7. Let $P \in \mathbb{R}^n$ be a point and let $g\colon \{P\} \times \overline{B}^n \to \mathbb{R}^n$ be a thickening of P. We equip $\{P\} \times \overline{B}^n = \overline{B}^n$ and $\mathbb{R}^n$ with the standard orientations. We define

$$\text{sign}(g) \;:=\; \begin{cases} +1, & \text{if } g \text{ is orientation-preserving,} \\ -1, & \text{if } g \text{ is orientation-reversing.} \end{cases}$$

Now we can discuss the case of thickened 0-dimensional submanifolds:

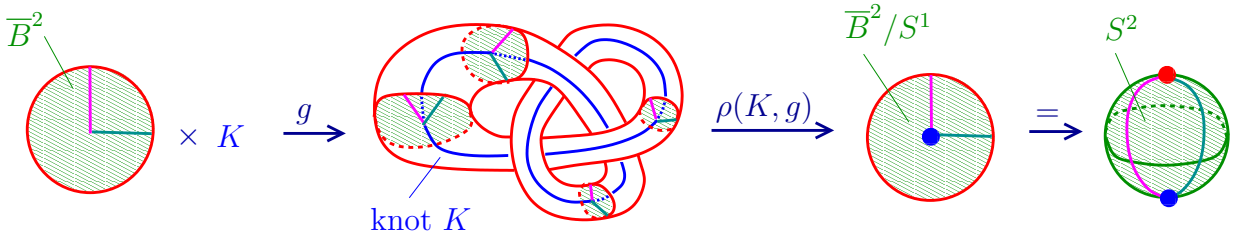
Example. Let $N = \{P_1, \dots, P_m\}$ be a 0-dimensional submanifold of some $\mathbb{R}^n$ and suppose N is equipped with a thickening

$$g = g_1 \sqcup \dots \sqcup g_m\colon (\overline{B}^n \times P_1) \sqcup \dots \sqcup (\overline{B}^n \times P_m) \;\to\; \mathbb{R}^n.$$

It follows almost immediately from the Addition-in-π_n-Proposition 167.25 that

$$[\rho_{(N,g)}] \;=\; \sum_{i=1}^{m} \text{sign}(g_i) \cdot [\text{id}_{S^n}] \;\in\; \pi_n(S^n, *). \qquad \square$$

We continue our discussion of examples of the Thom-Pontryagin construction with knots and links. By the above discussion every thickened knot or link in $\mathbb{R}^3$, i.e. every thickened 1-dimensional submanifold of $\mathbb{R}^3$, gives rise to an element in $\pi_3(S^2, *)$. For the trefoil together with some thickening we illustrate the corresponding map $S^3 \to S^2$ in the figure below.



Before we continue we need to introduce the following slight variation on Definition 284.10.

Definition 300.8. Let K be a knot and let $g\colon \overline{B}^2 \times K \to S^3$ be a thickening. We pick an orientation for K and we equip $g(\{1\} \times K)$ with the corresponding definition. We define[3300]

$$\text{self-linking number of } g \;:=\; \text{lk}(K, g(\{1\} \times K)).$$

It follows from the Linking Number–Orientation Lemma 284.4 that this definition does not depend on the choice of the orientation of K.

The following lemma deals with two basic, but important examples. Later on both examples will follow (at least up to sign) from a more general result, but for the geometrically inclined person it can be fun to consider the more direct arguments provided in the proof of the lemma.

[3300]We refer to Definition 114.1 or alternatively to Definition 284.1 for the definition of the linking number $\text{lk}(K, J)$ of two disjoint oriented knots K and J.

Lemma 300.9. (Trivial Knot–Thom Pontryagin Map Lemma) Let K be the trivial knot and let $g\colon \overline{B}^2 \times K \to \mathbb{R}^3$ be a thickening.

(1) If g is a thickening with self-linking number zero, then $[\rho_{(K,g)}] = 0 \in \pi_3(S^2, *)$.
(2) If g is a thickening with self-linking number one, then[3301] $[\rho_{(K,g)}] = [H] \in \pi_3(S^2, *)$ where $H\colon (S^3, *) \to (S^2, *)$ denotes the Hopf map, that we introduced in Definition 134.24.

Sketch of proof. It follows easily from the Knot Thickening–Self Linking Lemma 284.11[3302] that any two thickenings with the same self-linking number give rise to the same element in $\pi_3(S^2, *)$. Furthermore it follows almost immediately from the Sphere–Solid Tori-Decomposition Lemma 111.6 that we can make the identification

$$S^3 \;=\; (S^1 \times \overline{B}^2) \cup_{S^1 \times S^1} (\overline{B}^2 \times S^1)$$

such that $K = \{0\} \times S^1$. Furthermore it follows from the Torus Embeddings–Isotopic Lemma 111.8 that the obvious inclusion of $\overline{B}^2 \times S^1$ into the above decomposition of S^3 is a thickening of the trivial knot $K = S^1$ with self-linking number zero.

(1) Let g be the thickening of the trivial knot K with self-linking number zero. We want to show that $[\rho_{(K,g)}] = 0 \in \pi_3(S^2, *)$. By the Change-of-Base Point Proposition 169.5 it suffices to show that the map $\rho_{(K,g)}\colon S^3 \to S^2$ is homotopic to a constant map. (More precisely, the Change-of-Base Point Proposition 169.5 (7) says that in this context we do not need to worry about base points since S^2 is path-connected.) Using the above preparations we will prove in Exercise 300.1 that the map $\rho_{(K,g)}\colon S^3 \to S^2$ is indeed homotopic to a constant map.
(2) First we note that under the above identification of S^3 the Hopf map is given as follows:[3303]

$$\begin{array}{rcl} H\colon \overbrace{(S^1 \times \overline{B}^2) \cup_{S^1 \times S^1} (\overline{B}^2 \times S^1)}^{=S^3} & \to & \mathbb{C}\mathrm{P}^1 \\ [(z,w)] & \mapsto & \begin{cases} [\frac{w}{z} : 1], & \text{if } z \in S^1 \text{ and } w \in \overline{B}^2, \\ [1 : \frac{z}{w}], & \text{if } z \in \overline{B}^2 \text{ and } w \in S^1. \end{cases} \end{array}$$

Now note that it follows from the Knot Thickening–Self Linking Lemma 284.11 (1) that the map

$$\begin{array}{rcl} g\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \;\subseteq\; S^3 \;=\; (S^1 \times \overline{B}^2) \cup_{S^1 \times S^1} (\overline{B}^2 \times S^1) \\ (z,w) & \mapsto & (z \cdot w, w) \end{array}$$

[3301] The sign should be correct. Hopefully. Perhaps.
[3302] Note that here we need that in the statement of the Knot Thickening–Self Linking Lemma 284.11 for $\mathbb{R}^3$ we stated that there exists a diffeotopy of $\mathbb{R}^3$ with compact support.
[3303] As a reality check one can verify that the map is well-defined, i.e. for $[(z,w)]$ with $(z,w) \in S^1 \times S^1$ we get consistent answers.

is a thickening of the trivial knot of self-linking number one. Furthermore by definition the corresponding map $\rho_{(K,g)}$ is easily seen to be the map[3304]

$$\begin{array}{rcl} \rho_{(K,g)}\colon (S^1 \times \overline{B}^2) \cup_{S^1\times S^1} (\overline{B}^2 \times S^1) & \to & \overline{B}^2/S^1 \\ [(z,w)] & \mapsto & \begin{cases} \infty = [S^1], & \text{if } z \in S^1 \text{ and } w \in \overline{B}^2, \\ \left[\frac{z}{w}\right], & \text{if } z \in \overline{B}^2 \text{ and } w \in S^1.. \end{cases} \end{array}$$

Comparing the definitions of the Hopf map H and the map $\rho_{(K,g)}$ on $\overline{B}^2 \times S^1$ we see that we are perhaps not that far off.

Next we need to relate the targets $\mathbb{C}\mathrm{P}^1$ and $\overline{B}^2/S^1$. To do so we consider the map $\Theta\colon \mathbb{C}\mathrm{P}^1 \to \overline{B}^2/S^1$ that is given by $\Theta([1:z]) = \frac{z}{1+|z|}$ for $z \in \mathbb{C}$ and that is given by $\Theta([0:1]) = [S^1] = * \in \overline{B}^2/S^1$. It follows easily from the Compact-Hausdorff Proposition 4.23 (3) that Θ is a homeomorphism. The particularly audacious reader can verify that this map is homotopic to the "official" identification $\mathbb{C}\mathrm{P}^1 = \overline{B}^2/S^1$ which is given by the identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ from page 340 and the above identification $S^2 =_{\mathrm{i}} \overline{B}^2/S^1$. Note that the image of the first solid torus $\overline{B}^2 \times S^1$ under the map $\Theta \circ H$ equals the disk of radius $\frac{1}{2}$, i.e. it equals $\overline{B}^2_{\frac{1}{2}} \subseteq \overline{B}^2/S^1$.

We apply the Radial Pushing Lemma 300.5 to obtain a homotopy F of $\overline{B}^2/S^1$ with the following properties:

(a) $F_0 = \mathrm{id}$,

(b) the restriction of F_1 to $B^2_{\frac{1}{2}}$ is given by multiplication by 2,

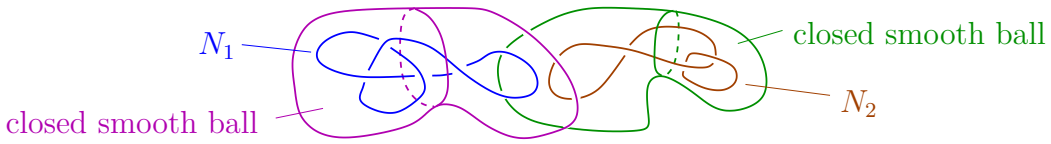
(c) for any point $x \notin B^2_{\frac{1}{2}}$ we have $F_1(x) = *$.

The map $F_0 \circ \Theta \circ H = \Theta \circ H$ is homotopic rel $*$ to $F_1 \circ \Theta \circ H$. It is now straightforward to show that $F_1 \circ \Theta \circ H$ is homotopic rel $*$ to $\rho_{(K,g)}$. We leave it to the reader to fill in the details. ■

We conclude this list of basic examples of the Thom-Pontryagin construction with the following lemma:

Lemma 300.10. (Thom-Pontryagin Additivity Lemma) In the following let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. Let (N_i, g_i), $i = 1, \ldots, m$ be thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$. If the images $g_i(\overline{B}^n \times N_i)$ are contained in m disjoint closed smooth $(n+k)$-dimensional balls[3305], then

$$\rho_{(N_1 \sqcup \cdots \sqcup N_m, g_1 \sqcup \cdots \sqcup g_m)} \;=\; \rho_{(N_1,g_1)} + \cdots + \rho_{(N_m,g_m)} \;\in\; \pi_{n+k}(S^n, *).$$



Proof. This lemma follows easily from the Addition-in-π_n-Proposition 167.25. We leave it to the reader to fill in the details. ■

The following proposition gives an affirmative answer to Question 300.6.

[3304] Why does it say $[\frac{z}{w}]$ instead of $[z \cdot w]$, which might look more natural?

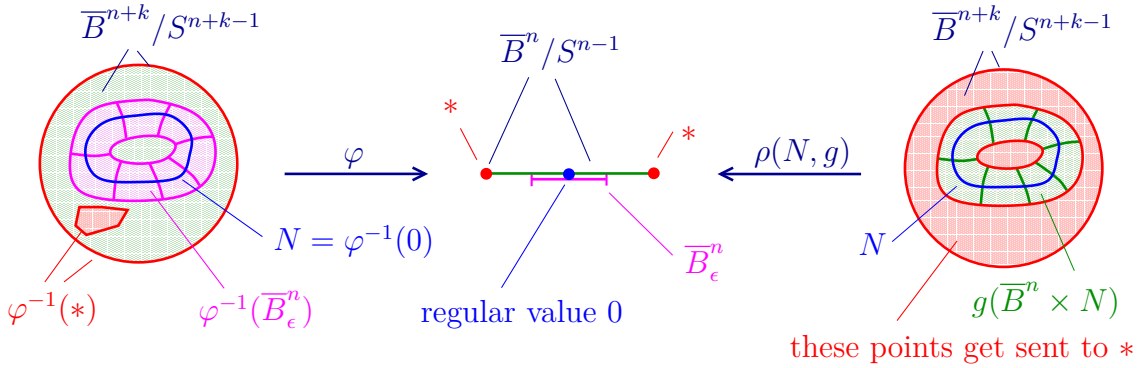
[3305] Here a "closed smooth $(n+k)$-dimensional ball" is shorthand for the image of a smooth embedding of $\overline{B}^{n+k}$ in $\mathbb{R}^{n+k}$.

Proposition 300.11. (Homotopy Class-Realization Proposition) Let $k, n \in \mathbb{N}_0$. Given any $\varphi \in \pi_{n+k}(S^n, *)$ there exists a thickened k-dimensional submanifold (N, g) of $\mathbb{R}^{n+k}$ with $[\rho_{(N,g)}] = \varphi$.

Proof. Throughout this proof we find it convenient to use Identifiation 8.11 to view each m-dimensional sphere as the quotient $\overline{B}^m/S^{m-1}$. So suppose that we are given a continuous map $\varphi\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$ with $\varphi(*) = *$. As we pointed out in the proof of the Sphere–Low-π_n-Proposition 134.19, it is a consequence of the Whitney Approximation Theorem 36.1 and the Replace-by-Smooth Homotopy Proposition 36.6 that we can arrange that φ is in fact a smooth map. It follows immediately from Sard's Theorem 38.1 and the Full Measure-Density Proposition 37.4 (2) that there exists a regular value $z \in \overline{B}^n/S^{n-1}$ such that $\varphi^{-1}(z)$ does not contain the point $*$. By the Homogeneous-via-Diffeotopies Proposition 44.6 (2)[3306] we can without loss of generality assume that the origin $z = 0$ is the regular value. By the Regular Value Theorem 32.5 we know that the preimage $N := \varphi^{-1}(0)$ is a closed orientable k-dimensional submanifold of B^{n+k}. Furthermore note that by Statement (4) of the Tubular-Regular Value Theorem 58.13 we know that there exists an $\epsilon > 0$ and a thickening $g\colon \overline{B}^n \times N \to B^{n+k}$ such that $\varphi^{-1}(\overline{B}^n_\epsilon) = g(\overline{B}^n \times N)$ and such that the following diagram commutes:[33073308]

$$\begin{array}{ccc} \overline{B}^n \times N & \xrightarrow{\quad g \quad} & B^{n+k} \\ \downarrow{\scriptstyle (p,x)\mapsto \epsilon\cdot p} & & \downarrow{\scriptstyle \varphi} \\ \overline{B}^n_\epsilon & \hookrightarrow & \overline{B}^n/S^{n-1}. \end{array}$$

This thickening g allows us to consider the map $\rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$.



Note that in general φ and $\rho_{(N,g)}$ are quite different maps. But that does not bother us too much, since it suffices to prove the following claim:

Claim. The maps φ and $\rho_{(N,g)}$ from $\overline{B}^{n+k}/S^{n+k-1}$ to $\overline{B}^n/S^{n-1}$ are homotopic rel $*$.

[3306] To be more precise, we apply the Homogeneous-via-Diffeotopies Proposition 44.6 to the smooth manifold $\overline{B}^n/S^{n-1} = \mathbb{R}^n \cup \{\infty\}$ with $P_1 = z$, $P_2 = \infty$ and $Q_1 = 0$ and $Q_2 = \infty$.

[3307] Note the discretely hidden multiplication by ϵ in the left-vertical map.

[3308] Note that the above information implies in particular that $\varphi^{-1}(B^n_\epsilon) = g(B^n \times N)$.

Proof. First we remind the reader that in the Radial Pushing Lemma 300.5 we showed that there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to B^n_ϵ is given by multiplication by $\frac{1}{\epsilon}$,
(3) for any point $x \notin B^n_\epsilon$ we have $F_1(x) = *$.

We define $\psi = F_1 \circ \varphi$. It follows from (1) that ψ is homotopic rel $*$ to φ. Therefore we are done once we have shown that ψ in fact equals $\rho_{(N,g)}$. As a reminder, see Definition 300.3, the map $\rho_{(N,g)}$ is given as follows:[3309]

$$\begin{array}{rcl} \rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} & \to & S^n = \overline{B}^n/S^{n-1} \\ P & \mapsto & \begin{cases} x, & \text{if } P = g(x,y) \text{ with } (x,y) \in B^n \times N, \\ *, & \text{otherwise.} \end{cases} \end{array}$$

Now we verify that ψ does indeed agree with $\rho_{(N,g)}$. Thus first suppose that we are given a point $P = g(x,y)$ with $(x,y) \in B^n \times N$. We have

$$\psi(P) \underset{\text{definition of } \psi}{=} F_1(\varphi(g(x,y))) \underset{\text{by the above commutative diagram}}{=} F_1(\epsilon \cdot x) \underset{\text{by Property (2) of } F_1 \text{ and since } \epsilon \cdot x \in B^n_\epsilon}{=} \tfrac{1}{\epsilon} \cdot \epsilon \cdot x = x = \rho_{(N,g)}(g(x,y)) = \rho_{(N,g)}(P).$$

Finally suppose that we are given a point $P \in \overline{B}^{n+k}/S^{n+k-1}$ that does not lie in the subset $g(B^n \times N)$. In this case we calculate that

$$\psi(P) \underset{\text{by definition of the map } \psi}{=} F_1(\varphi(P)) \underset{\text{since } \varphi^{-1}(B^n_\epsilon) = g(B^n \times N) \text{ and by choice of } P}{=} F_1(\text{point that does not lie in } B^n_\epsilon) \underset{\text{by Property (3) of } F_1}{=} * = \rho_{(N,g)}(P).$$

■

300.3. **The thickened cobordism groups and the Thom-Pontryagin Theorem.** Recall that the Thom-Pontryagin construction from Definition 300.3 makes the assignment

$$\text{thickened } k\text{-dimensional submanifold of } \mathbb{R}^{n+k} \rightsquigarrow \text{map } (S^{n+k}, *) \to (S^n, *).$$

Now we would like to answer Question 300.6 (2), i.e. we would like to determine when do two thickened k-dimensional submanifolds give rise to the same homotopy class of continuous maps $(S^{n+k}, *) \to (S^n, *)$. Loosely speaking we would like to fill in the left-hand side of

$$?? \rightsquigarrow \text{map } (S^{n+k} \times [0,1], * \times [0,1]) \to (S^n, *).$$

A quick look at the Thom-Pontryagin construction suggests that the left-hand side should be filled in with a suitable thickened $(k+1)$-dimensional submanifold of $\mathbb{R}^{n+k} \times [0,1]$.

This makes sense, except that in the Product Orientation Proposition 31.48 (5) we saw that for products of two smooth manifolds it is advisable to have the smooth manifold with boundary as a first factor. Thus we will consider submanifolds of $[0,1] \times \mathbb{R}^{n+k}$.[3310] Before we continue we introduce the following straightforward convention.

[3309] Eyes which profited from a carrot-rich diet might notice that the description is not verbatim the same as in Definition 300.3. But with either description of the map the points $P = g(x,y)$ with $x \in S^{n-1}$ and $y \in N$ get sent to $*$. So we really do consider the same map as before.

[3310] Perhaps in hindsight we should have defined homotopies as maps $[0,1] \times X \to Y$ instead of defining them as maps $X \times [0,1] \to Y$?

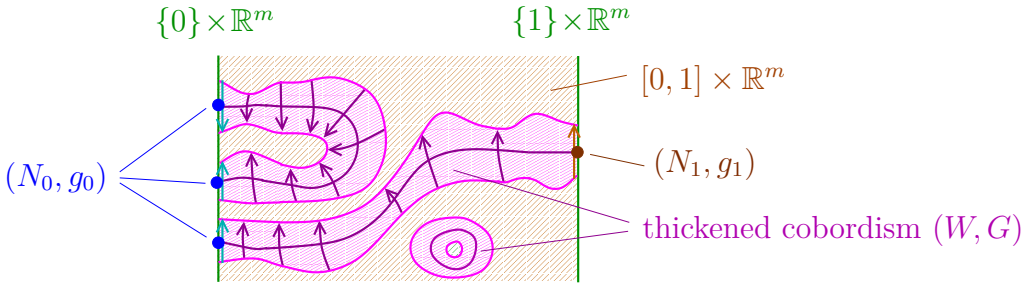
Convention 300.12. Given a topological space X and an interval I and some $t \in I$ we make at times the identification $X =_{\mathrm{i}} \{t\} \times X$ via the obvious bijection.

The above observation leads us to the following definition which looks awfully similar to the definition of the notion of a cobordism of smooth manifolds that we introduced in **??**.

Definition 300.13. Let $k, m \in \mathbb{N}_0$.

(1) Let N_0 and N_1 be two closed orientable k-dimensional submanifolds of $\mathbb{R}^m$. A cobordism between N_0 and N_1 is a compact orientable proper $(k+1)$-dimensional submanifold W of $[0,1] \times \mathbb{R}^m$ with $\partial W = (\{0\} \times N_0) \cup (\{1\} \times N_1)$.[3311]

(2) Let (N_0, g_0) and (N_1, g_1) be two thickened k-dimensional submanifolds of $\mathbb{R}^m$. A thickened cobordism between (N_0, g_0) and (N_1, g_1) is a thickened $(k+1)$-dimensional submanifold (W, G) of $[0,1] \times \mathbb{R}^m$ where W is a cobordism between N_0 and N_1 and G is a thickening of W with the following two properties:[3312]
   (a) $G|_{\{0\}\times\overline{B}^m} = g_0$ under the identification $\{0\} \times \overline{B}^m = \overline{B}^m$,
   (b) $G|_{\{1\}\times\overline{B}^m} = g_1$ under the identification $\{1\} \times \overline{B}^m = \overline{B}^m$.
   If such a thickened cobordism exists, then we say that (N_0, g_0) and (N_1, g_1) are cobordant.[3313] If (N, g) is cobordant to the empty submanifold, then we say that N is a thickened null-cobordism and we say that (N, g) is null-cobordant. The definition of a thickened cobordism is illustrated in the figure below.



The following lemma shows that diffeotopies give rise to thickened cobordisms.

Lemma 300.14. (Diffeotopy–Cobordism Lemma) Let (M, g) be a thickened submanifold of $\mathbb{R}^m$ and let φ be a diffeomorphism of $\mathbb{R}^m$. If φ is diffeotopic to the identity, then $(\varphi(M), \varphi_*(g))$ is cobordant to (M, g).

Example. Let $m \in \mathbb{N}$. Any map $\mathbb{R}^m \to \mathbb{R}^m$ of the form $x \mapsto r \cdot x + P$ for some $r > 0$ and $P \in \mathbb{R}^m$ is clearly diffeotopic to the identity. Thus the Diffeotopy–Cobordism Lemma 300.14 says that given a thickened submanifold (M, g) we can translate and shrink it without changing the thickened cobordism class. □

Proof. Let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a thickened k-dimensional submanifold of $\mathbb{R}^m$ and furthermore let $F\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ be a diffeotopy from the identity to a diffeomorphism

[3311] Note that we work with submanifolds that are orientable, but we do not assume that they are equipped with an orientation.

[3312] Note that it follows from property (d) of a thickened submanifold, see Definition 300.2, that we know that in fact $G|_{\{0\}\times\overline{B}^m} \subseteq \{0\} \times \mathbb{R}^m$ and $G|_{\{1\}\times\overline{B}^m} \subseteq \{1\} \times \mathbb{R}^m$.

[3313] We spare the reader the even more logical name "thickened cobordant".

φ. It is straightforward to verify that the map

$$\begin{array}{rcl} [0,1]\times M & \to & [0,1]\times\mathbb{R}^m \\ (t,P) & \mapsto & (t,F(P,t)) \end{array}$$

is a smooth embedding. We write $W := F([0,1]\times M)$. It is straightforward to verify that W together with the thickening given by

$$\begin{array}{rcl} \overline{B}^{m-k}\times W & \to & [0,1]\times\mathbb{R}^m \\ (v,(t,F(P,t))) & \mapsto & (t,F(g(v,P),t)) \end{array}$$

is a thickened cobordism between (M,g) and $(\varphi(M),\varphi_*(g))$. ■

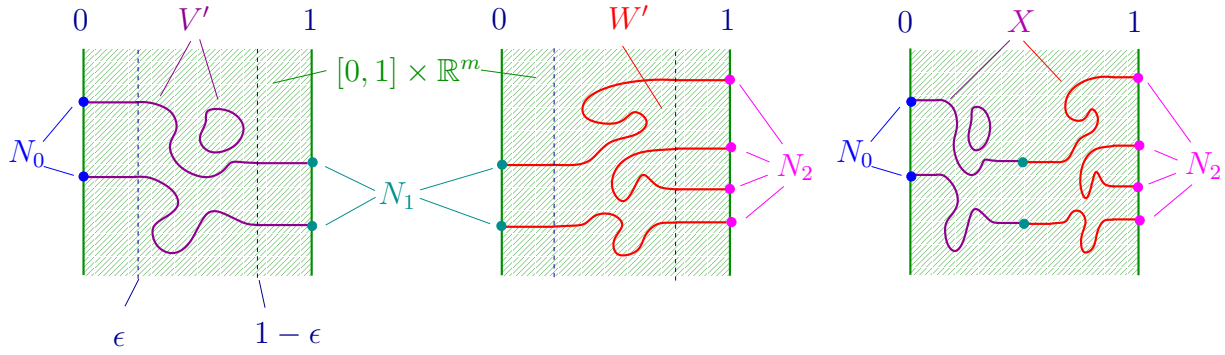
Lemma 300.15. (Thickened Submanifold–Cobordism–Equivalence Relation Lemma) "Being cobordant" is an equivalence relation on the set of thickened k-dimensional submanifolds of $\mathbb{R}^m$.

Sketch of proof. To simplify the discussion we ignore throughout the proof the thickenings. The reader should have no major troubles modifying the proof to also include the thickenings.

It is straightforward to see that "being cobordant" is reflexive and symmetric: Indeed, given a closed k-dimensional submanifold N a cobordism from N to itself is given by $[0,1]\times N\subseteq[0,1]\times\mathbb{R}^m$. Furthermore, suppose that there exists a cobordism W between N_0 and N_1. "Flipping" W along $\mathbb{R}^m\times\{\frac{1}{2}\}$ we obtain a cobordism between N_1 and N_0.

The proof of transitivity takes more efforts. Initially the idea is quite simple. Given a cobordism V from N_0 to N_1 and another cobordism W from N_1 to N_2 one would like to stick them together to obtain a cobordism from N_0 to N_2. Unfortunately, as we had discussed on page 1277, the resulting object is in general not a (smooth) submanifold of the smooth manifold $\mathbb{R}^m$.

Fortunately we had already developed all the tools to deal with the situation. More precisely, in the Collar-for-Submanifold Proposition 56.3 we saw that we can modify the cobordisms V and W to obtain cobordisms V' and W' that are products on $[0,\epsilon]\times\mathbb{R}^m$ and $[1-\epsilon,1]\times\mathbb{R}^m$ for a suitable $\epsilon\in(0,\frac{1}{2})$.



Now we can stick V' and W' together to obtain the desired cobordism X. ■

The previous lemma allows us to make the following definition:

Definition 300.16. We denote by $\Omega_k^{\text{th}}(\mathbb{R}^m)$ the set of thickened cobordism classes of thickened k-dimensional submanifolds of $\mathbb{R}^m$. In light of the subsequent the Thickened

Cobordism Group Proposition 300.17 we refer to $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ as a thickened cobordism group.

The definition of $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ is reminiscent of the definition of the cobordism group Ω_k that we gave in Definition 280.8. The group structure on Ω_k that we had introduced in the Cobordism Group Lemma 280.7 is given by "disjoint union". We will now see that $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ also admits a group structure that is defined quite similarly.

Proposition 300.17. (Thickened Cobordism Group Proposition) Let $k \in \mathbb{N}_0$ and let $m \in \mathbb{N}_{\geq 2}$. Let (M, g) and (N, h) be two thickened k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r, s \in \mathbb{R}_{>0}$ with $g(\overline{B}^{m-k} \times M) \subseteq B_r^m(0)$ and $h(\overline{B}^{m-k} \times N) \subseteq B_s^m(0)$. Furthermore we pick two orientation-preserving smooth embeddings $\varphi\colon \overline{B}_r^m(0) \to \mathbb{R}^m$ and $\psi\colon \overline{B}_s^m(0) \to \mathbb{R}^m$ with disjoint images. We define

$$[(M, g)] + [(N, h)] \coloneqq [(\varphi(M) \sqcup \psi(N), \varphi_*(g) \sqcup \psi_*(h))] \in \Omega_k^{\mathrm{th}}(\mathbb{R}^m).$$

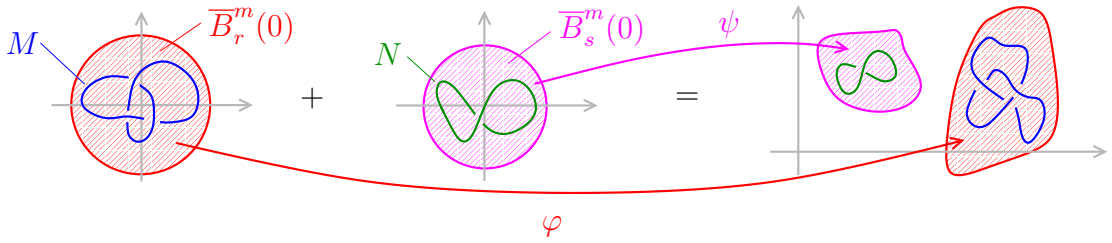
(The definition of the operation "+" is illustrated in the figure below.)

(1) The definition of "+" on $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ is well-defined and commutative.

(2) The operation "+" defines a group structure on $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$. Here the following two statements hold:

  (a) The neutral element is represented by the empty submanifold.[3314]

  (b) Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M, g)]$ is given by $[(\rho(M), \rho_*(g))]$.



Proof.

(1) It follows from the Smooth Ball Embedding Theorem 45.10 and the Diffeotopy–Cobordism Lemma 300.14 that "+" is well-defined and commutative. Note that here we use that $m \geq 2$.

(2) It is basically obvious that "+" is associative and that the empty submanifold is a neutral element. Thus it remains to prove the statement regarding the inverse.

First note that it follows from the Diffeotopy–Cobordism Lemma 300.14 and the Hyperplane–Reflection Lemma 41.14 that without loss of generality we can assume that the reflection $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ is the reflection in the $(x_m = 0)$-hyperplane. In the following let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a thickened k-dimensional submanifold of $\mathbb{R}^m$. We set $\epsilon = \frac{1}{4}$. It follows immediately from the discussion on page 5785 that we can assume that $g(\overline{B}^{m-k} \times M)$ is contained in the ball $\overline{B}_\epsilon^m(\underbrace{(0, \ldots, 0, 2\epsilon)}_{\in \mathbb{R}^m})$. We consider

[3314] It is worth remembering that the empty set is a submanifold of any dimension.

the map

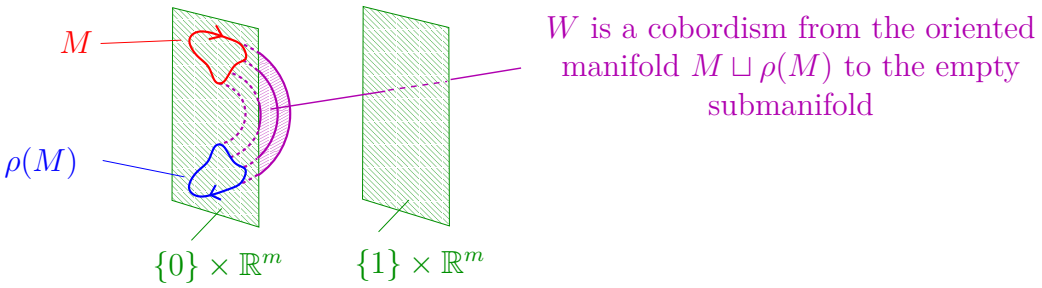
$$\begin{array}{rcl} \varphi\colon \overline{B}^{m-k} \times (M \times [0,\pi]) & \to & [0,1] \times \mathbb{R}^m \\ (v,(P,\gamma)) & \mapsto & \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & \mathrm{id}_{m-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ g(v,P) \end{pmatrix}}_{\in \mathbb{R}^{m+1}}. \end{array}$$

Using that $g(\overline{B}^{m-k} \times M)$ is contained $\overline{B}^m_\epsilon((0,\dots,0,2\epsilon))$ we can make the following three observations:

(a) The image of φ does indeed lie in $[0,1] \times \mathbb{R}^m$.

(b) The restriction to $\{0\} \times M \times [0,\pi]$ is a proper smooth embedding. Therefore we can now consider the compact orientable proper $(k+1)$-dimensional submanifold $W = \varphi(\{0\} \times M \times [0,\pi])$ of $[0,1] \times \mathbb{R}^m$.

(c) The map φ gives rise to a thickening for W.



It is now basically clear that we have constructed a thickened cobordism from the thickened submanifold $(M \sqcup \rho(M), g \sqcup \rho_*(g))$ to the empty submanifold. (We refer to the figure below for an illustration.) But by the definition of "+" this implies that $(\rho(M), \rho_*(g))$ represents the inverse of (M,g). ■

Evidently our challenge now is to compute the groups $\Omega^{\mathrm{th}}_k(\mathbb{R}^m)$ for suitable values of m and k. In general this might be quite difficult, but the following proposition gives a reasonably amusing calculation.

Proposition 300.18. ($\Omega^{\mathrm{th}}_0(\mathbb{R}^m)$-Proposition) Let $m \geq 2$. The map[3315]

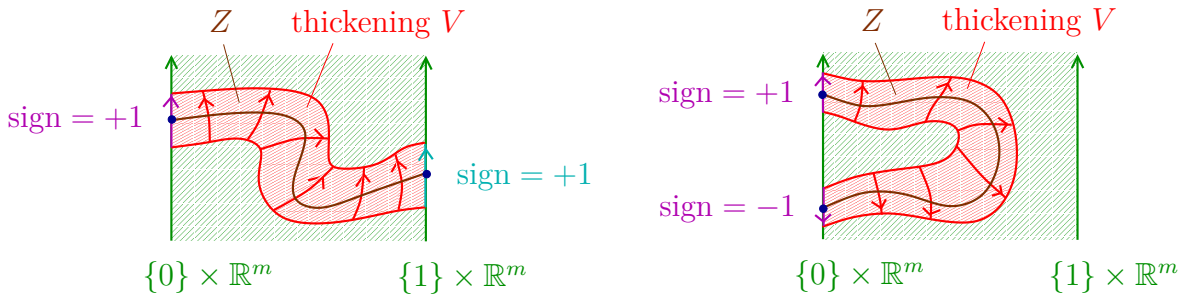
$$\begin{array}{rcl} \varphi\colon \Omega^{\mathrm{th}}_0(\mathbb{R}^m) & \to & \mathbb{Z} \\ [(\{P_1\},g_1) \sqcup \dots \sqcup (\{P_k\},g_k)] & \mapsto & \sum_{i=1}^k \mathrm{sign}(g_i) \end{array}$$

is well-defined and it is an isomorphism.

Sketch of proof. First we need to show that $\varphi\colon \Omega^{\mathrm{th}}_0(\mathbb{R}^m) \to \mathbb{Z}$ is well-defined. We point out that it suffices to show that for every *connected* thickened 1-dimensional cobordism $(Z,V) \subseteq [0,1] \times \mathbb{R}^m$ between two 0-dimensional thickened submanifolds (N_0,g_0) and (N_1,g_1) we have the equality $\varphi([(N_0,g_0)]) = \varphi([(N_1,g_1)])$. This statement follows from the Smooth 1-Manifold Classification Theorem 28.8 and some orientation considerations. Since writing down the precise arguments regarding the orientations will only obfuscate the argument we

[3315] We refer to Definition 300.7 for the definition of the sign of a thickening of a point.

refer to the figure below for convincing pictures, and we leave it to the reader to fill in the details.



Next note that the map $\varphi\colon \Omega_0^{\text{th}}(\mathbb{R}^m) \to \mathbb{Z}$ is, basically by definition, a homomorphism. Finally note that it is clear that φ is an epimorphism. Thus it remains to show that φ is a monomorphism. A straightforward induction argument shows that it suffices to prove the following claim:

Claim. For points $P, Q \in \mathbb{R}^m$ and thickenings g and h of P and Q with $\operatorname{sign}(g) = -\operatorname{sign}(h)$ we have $[(P,g)] + [(Q,h)] = 0 \in \Omega_0^{\text{th}}(\mathbb{R}^m)$.

Proof. In the following let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. By the discussion on page 5785 we can assume, after possibly applying a translation, that $Q = \rho(P)$. Since ρ is orientation-reversing we see that $\operatorname{sign}(\rho_*(g)) = -\operatorname{sign}(g) = \operatorname{sign}(h)$. Now we calculate that

$$[(P,g)] + [(Q,h)] \;=\; [(P,g)] + [(\rho(P),h)] \underset{\uparrow}{=} [(P,g)] + [(\rho(P),\rho_*(g))] \underset{\uparrow}{=} 0 \;\in \Omega_0^{\text{th}}(\mathbb{R}^m).$$

by $\operatorname{sign}(h) = \operatorname{sign}(\rho_*(g))$, by the Smooth Ball Embedding Theorem 45.10 and by the Diffeotopy–Cobordism Lemma 300.14

by the Thickened Cobordism Group Proposition 300.17 (2b) ∎

Considering the discussion in Section 300.3, the following theorem does not come as a big surprise. It justifies our interest in the groups $\Omega_k^{\text{th}}(\mathbb{R}^m)$.

Theorem 300.19. (Thom-Pontryagin Theorem) Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. The map

$$\begin{array}{rcl} \Psi\colon \Omega_k^{\text{th}}(\mathbb{R}^{n+k}) & \to & \pi_{n+k}(S^n, *) \\ [(N,g)] & \mapsto & [\rho_{(N,g)}] \end{array}$$

is well-defined and it is an isomorphism.

Sketch of proof. Fortunately we have already done basically all the work:

(1) First we argue that Ψ is well-defined. Thus suppose that we are given thickened k-dimensional submanifolds (M,g) and (N,h) of $\mathbb{R}^{n+k}$ that are cobordant. We pick a thickened cobordism $(W, F\colon \overline{B}^n \times W \to [0,1]\times\mathbb{R}^{n+k})$ in $[0,1]\times\mathbb{R}^{n+k}$. The corresponding map

$$\rho_{(W,F)}\colon [0,1] \times \mathbb{R}^{n+k} \;\to\; S^n$$

has the property that it sends a neighborhood of $[0,1] \times \{\infty\}$ to $*$. Therefore it follows easily from the Homotopy Combination Lemma 18.8 (1) that the following map is continuous:

$$\begin{array}{rcl} S^{n+k} \times [0,1] = (\mathbb{R}^{n+k} \cup \{\infty\}) \times [0,1] & \to & S^n \\ (P,t) & \mapsto & \begin{cases} \rho_{(W,F)}(t,P), & \text{if } P \in \mathbb{R}^{n+k}, \\ *, & \text{if } P = \infty. \end{cases} \end{array}$$

It follows from the above that this map is in fact a homotopy rel $*$ between the map $\rho_{(M,g)}\colon (S^{n+k},*)\to(S^n,*)$ and the map $\rho_{(N,h)}\colon (S^{n+k},*)\to(S^n,*)$. In particular we have $[\rho_{(M,g)}]=[\rho_{(N,h)}]\in\pi_{n+k}(S^n,*)$.

(2) It follows immediately from the Thom-Pontryagin Additivity Lemma 300.10 that Ψ is a homomorphism.

(3) The statement of the Homotopy Class-Realization Proposition 300.11 is equivalent to saying that Ψ is an epimorphism.

(4) Finally it remains to show that Ψ is in fact a monomorphism. So suppose that we are given two thickened k-dimensional submanifolds (M,g) and (N,h) of $\mathbb{R}^{n+k}$ such that $[\rho_{(M,g)}]=[\rho_{(N,h)}]\in\pi_{n+k}(S^n,*)$, i.e. such that there exists a homotopy $F\colon S^{n+k}\times[0,1]\to S^n$ rel $*$ from $\rho_{(M,g)}$ to $\rho_{(N,h)}$.

The proof of the Homotopy Class-Realization Proposition 300.11 can be modified without too many problems to show that the homotopy F gives rise to a thickened cobordism between (M,g) and (N,h). We leave it to the reader to fill in the details. See also [**Kos1993**, Theorem 9.5.5] or [**Bre1993**, Chapter 16] for similar claims stated with more authority. ■

From our above results we obtain the following corollary. Of course we proved this result earlier in the $\pi_n(S^n)$–Proposition 167.10, using very different methods.

Corollary 300.20. ($\pi_n(S^n)$-via-Thom-Pontryagin Corollary) For any $m\geq 2$ the map

$$\begin{array}{ccc}\mathbb{Z} & \to & \pi_m(S^m,*)\\ n & \mapsto & n\cdot[\mathrm{id}_{S^m}]\end{array}$$

is an isomorphism.

Proof. Let $P\in\mathbb{R}^m$ and let g be a thickening for the submanifold $\{P\}$. We consider the two maps

$$\begin{array}{ccc}\mathbb{Z} & \to & \Omega_0^{\mathrm{th}}(\mathbb{R}^m)\\ n & \mapsto & n\cdot[(\{0\},g\text{ with }\mathrm{sign}(g)=1)]\end{array}\quad\text{and}\quad\begin{array}{ccc}\Omega_0^{\mathrm{th}}(\mathbb{R}^m) & \to & \pi_m(S^m,*)\\ [(N,g)] & \mapsto & [\rho_{(N,g)}].\end{array}$$

The first map is an isomorphism by the $\Omega_0^{\mathrm{th}}(\mathbb{R}^m)$-Proposition 300.18 and the second map is an isomorphism by the Thom-Pontryagin Theorem 300.19, applied to $k=0$ and $m\geq 2$. Thus the composition of the two maps is an isomorphism. Finally it follows from the Point-in-Ball-Thickening Lemma 300.4 that the composition of these two maps sends n to $n\cdot[\mathrm{id}_{S^m}]$. ■

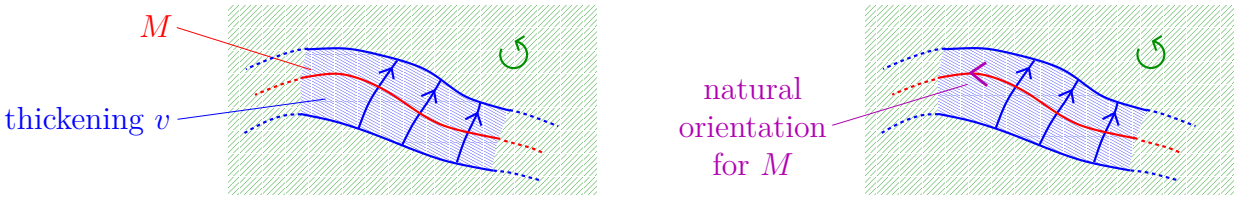
300.4. **The calculation of the thickened cobordism group $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$.** Surely the most interesting calculation of a higher homotopy group that we have performed so far is that in the $\pi_n(S^3)=\pi_n(S^2)$–Theorem 239.12 (2) we saw $\pi_3(S^2)\cong\mathbb{Z}$. Our next goal is to give a new proof of this fact using the Thom-Pontryagin Theorem 300.19. Thus we need to show that $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)\cong\mathbb{Z}$.

We start out with the following slightly subtle definition.

Definition 300.21. Let $m\in\mathbb{N}$.

(1) Let $(M,v\colon \overline{B}^{m-k}\times\mathbb{R}^m)$ be a thickened $(m-k)$-dimensional submanifold of $\mathbb{R}^m$. The natural orientation of M is the unique orientation that turns v into an orientation-preserving map.[3316]

(2) Similarly we define the natural orientation of a thickened cobordism.



This definition leads us to the following convention.

Convention 300.22. In this section we equip any thickened submanifold and any thickened cobordism with the natural orientation.

Before we write down an explicit isomorphism we introduce the following definition which generalizes the definition of the self-linking number of a thickened knot that we introduced in Definition 283.3.

Definition 300.23.

(1) Let $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ be two disjoint oriented links. We define the linking number of K and L as

$$\operatorname{lk}(K, L) \;:=\; \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{lk}(K_i, L_j).$$

(2) Let (L, g) be a thickened link. We write $L' = g(\{1\} \times L)$. We define the self-linking number of (L, g) as $\operatorname{slk}(L, g) := \operatorname{lk}(L, L')$.[3317]

The following elementary lemma is at times useful for determining self-linking numbers of thickened links.

Lemma 300.24. (Self-Linking Number Lemma) If $(L, g) = (L_1 \sqcup \cdots \sqcup L_m, g_1 \sqcup \cdots \sqcup g_m)$ is a thickened link, then

$$\operatorname{slk}(L, g) \;:=\; \sum_{i \neq j} \operatorname{lk}(L_i, L_j) \;+\; \sum_{i=1}^{m} \operatorname{slk}(L_i, g_i).$$

Proof. We will prove the lemma in Exercise 300.2. ■

Now we can give the promised isomorphism, which in particular gives us an affirmative answer to Question 134.25.

Theorem 300.25. (Cobordism Group–$\Omega_1^{\text{th}}(\mathbb{R}^3)$–Theorem) The map

$$\begin{array}{rcl} \Psi \colon \Omega_1^{\text{th}}(\mathbb{R}^3) & \to & \mathbb{Z} \\ [(L, g)] & \mapsto & \text{self-linking number } \operatorname{slk}(L, g) \end{array}$$

is well-defined and it is an isomorphism. In particular we have $\pi_3(S^2) \cong \mathbb{Z}$.

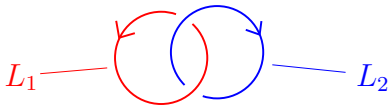
Examples.

[3316] Recall that M is assumed to be orientable, but a priori it does not come with an orientation. It is not difficult to see that there exists a unique orientation on M such that, if we equip $\overline{B}^{m-k} \times M$ with the product orientation, v becomes orientation-preserving.

[3317] Here we use the above convention to view L as an *oriented* submanifold of $\mathbb{R}^3$. Furthermore we equip $L' = g(\{1\} \times L)$ with the obvious corresponding orientation.

(1) We consider the trivial knot K with a thickening g of self-linking number $+1$. By the Cobordism Group–$\Omega_1^{\text{th}}(\mathbb{R}^3)$–Theorem 300.25 the thickened knot (K, g) defines a generator of $\Omega_1^{\text{th}}(\mathbb{R}^3)$. It follows from the isomorphism provided by the Thom-Pontryagin Theorem 300.19 that $\rho_{(K,g)} \colon S^3 \to S^2$ represents a generator of $\pi_3(S^2, *)$. In the Trivial Knot–Thom Pontryagin Map Lemma 300.9 we saw that $\rho_{(K,g)}$ represents the same element of $\pi_3(S^2, *)$ as the Hopf map $H \colon S^3 \to S^2$. Thus we see once again that the Hopf map represents a generator of $\pi_3(S^2)$.

(2) We consider the Hopf link $L = L_1 \sqcup L_2$ with the orientations shown in the figure below.



We equip $L = L_1 \sqcup L_2$ with a "trivial thickening" $g_1 \sqcup g_2$, i.e. where the thickening g_i of each component L_i has self-linking number 0. We see that

$$\operatorname{slk}(L_1 \sqcup L_2, g_1 \sqcup g_2) \underset{\substack{\uparrow \\ \text{by the Self-Linking Number} \\ \text{Lemma 300.24}}}{=} \operatorname{lk}(L_1, L_2) + \operatorname{lk}(L_2, L_1) + \overbrace{\operatorname{slk}(L_1, g_1)}^{=0} + \overbrace{\operatorname{slk}(L_2, g_2)}^{=0} \underset{\substack{\uparrow \\ \text{pick your favorite method} \\ \text{from Section 284.1}}}{=} 1 + 1 = 2.$$

This shows that the Hopf link, perhaps slightly counter-intuitively, gives rise to *twice* the generator of $\pi_3(S^2, *)$. □

Since we already have an independent proof of the Cobordism Group–$\Omega_1^{\text{th}}(\mathbb{R}^3)$–Theorem 300.25 we will take the liberty to cut a few corners in the new proof of the theorem. The new proof of the Cobordism Group–$\Omega_1^{\text{th}}(\mathbb{R}^3)$–Theorem 300.25 will build on the following four lemmas. The first two lemmas generalize the Seifert Surface Existence Theorem 121.2 and the Linking Number-via–Surfaces Proposition 285.7 from knots to links. They are perhaps interesting in their own right.

Lemma 300.26. (Link Seifert Surface Lemma) Given any oriented link $L \subseteq S^3$ there exists a compact oriented connected surface $F \subseteq S^3$ such that $\partial F = L$.[3318]

Proof of Lemma 300.26. Let $L \subseteq S^3$ be an oriented link. A modest generalization of the Seifert Surface Existence Theorem 121.2 shows that there exists a compact oriented 2-dimensional smooth manifold $G \subseteq S^3$ with $\partial G = L$. In Exercise 300.5 we will show that we can turn G into a compact oriented *connected* surface F with $\partial F = L$.[3319] ■

Lemma 300.27. (Linking Number-via–Surfaces Lemma) If $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ are two oriented links, then the following statements hold:

(1) There exist compact oriented proper 2-dimensional submanifolds S and T of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$ such that S and T are transverse.

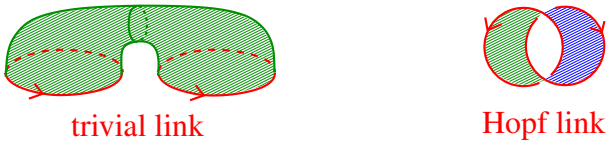
[3318] Here the equality $\partial F = L$ is understood to be an equality of *oriented* submanifolds.

[3319] Frankly it can be slightly annoying to give a completely rigorous proof that such a surface F exists. This is one of the main reasons why we made the caveat above that we are going to cut a few corners in the argument.

(2) Let S, T be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$. If S and T are transverse, then[3320]

$$\mathrm{lk}(K, L) \;=\; S \cdot_{\overline{B}^4} T.$$

Example. As in Definition 111.22 we consider the Hopf link that is illustrated in the figure below on the left.



On the right of the figure below we show a compact oriented connected surface whose boundary equals the Hopf link. □

Proof of Lemma 300.27.

(1) The proof of this statement is virtually the same as the proof of the Linking Number-via–Surfaces Proposition 285.7, we just need to replace the Seifert Surface Existence Theorem 121.2 by the Link Seifert Surface Lemma 300.26.
(2) We leave it to the reader to verify that the proof of the Linking Number-via–Surfaces Proposition 285.7 (2) can be modified to deal not only with knots but also with links.

■

The following two lemmas that we need for the proof of the Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25 are arguably more technical.

Lemma 300.28. (Self Linking Number Suffices Lemma) Let K be an oriented *knot* in $\mathbb{R}^3$ and let g and h be two thickenings for K in $\mathbb{R}^3$. If $\mathrm{slk}(K, g) = \mathrm{slk}(K, h)$, then $[(K, g)] = [(K, h)] \in \Omega_1^{\mathrm{th}}(\mathbb{R}^3)$.

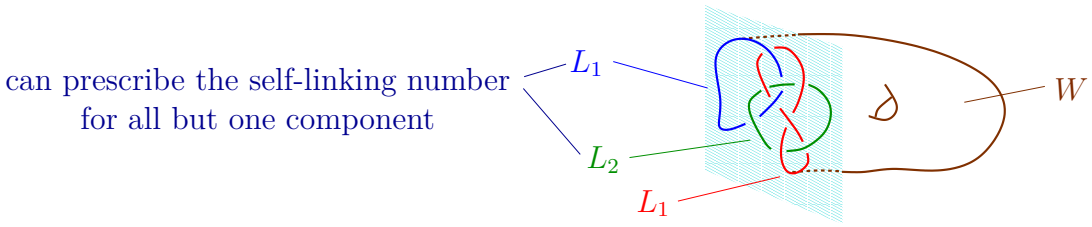
Proof of Lemma 300.28. Since $\mathrm{slk}(K, g) = \mathrm{slk}(K, h)$ it follows from the Knot Thickening–Self Linking Lemma 284.11 that there exists a smooth isotopy $F\colon (\overline{B}^2 \times K) \times [0, 1] \to \mathbb{R}^3$ between g and h which has compact support and which has the property that $F(0, k, t) = k$ for all $k \in K$ and all $t \in [0, 1]$. We consider $W = [0, 1] \times K$ with the thickening given by

$$\begin{array}{rcl} j\colon \overline{B}^2 \times ([0, 1] \times K) & \to & [0, 1] \times \mathbb{R}^3 \\ (v, (t, x)) & \mapsto & (t, F((v, x), t)). \end{array}$$

The thickened smooth manifold (W, j) is a thickened cobordism between (K, g) and (K, h). ■

Lemma 300.29. (Self Linking Number-Extension Lemma) Let W be a compact oriented connected proper 2-dimensional submanifold of $\mathbb{R}^3 \times [0, 1)$ with boundary components $L_1, \ldots, L_m$. Given $s_1, \ldots, s_{m-1} \in \mathbb{Z}$ there exists a thickening h for W such that for $i = 1, \ldots, m-1$ we have $\mathrm{slk}(L_i, h|_{L_i}) = s_i$.

[3320]Here $S \cdot_{\overline{B}^4} T$ denotes the algebraic intersection number of the oriented transverse surfaces S and T in the oriented smooth manifold $\overline{B}^4$. We refer to Definition 220.5 for the definition of the algebraic intersection number.



Proof. Evidently we can assume that $m \geq 1$. In this case W is a compact oriented connected proper 2-dimensional submanifold with non-empty boundary. Thus by the Product Tubular Neighborhood–Existence Theorem 58.4 (2) there exists a thickening $f\colon \overline{B}^2 \times W \to \mathbb{R}^3 \times [0,1)$ for W. For $i = 1, \ldots, m-1$ we write $r_i \coloneqq \operatorname{slk}(L_i, g|_{L_i})$.

Note that by the Surface-with-Boundary–Retraction Lemma 107.9 there exists a continuous map $\varphi\colon W \to L_1 \vee \cdots \vee L_{m-1}$[3321] that is the identity on each L_i. We pick a continuous map $\psi\colon L_1 \vee \cdots \vee L_{m-1} \to \mathrm{SO}(2) =_{\mathrm{i}} S^1$ with the following property: under an orientation-preserving identification $L_i = S^1$ the restriction of ψ to L_i is given by $z \mapsto z^{s_i - r_i}$. Now we consider the thickening

$$\begin{aligned} h\colon \overline{B}^2 \times W &\to W \\ (v, P) &\mapsto f(\overbrace{(\psi \circ \varphi)(P)}^{\in \mathrm{SO}(2)} \cdot v, P)). \\ &\qquad \underbrace{\qquad\qquad}_{\in \overline{B}^2 \times W} \end{aligned}$$

By the Knot Thickening–Self Linking Lemma 284.11 (1) we know that for any $i \in \{1, \ldots, m-1\}$ we added $s_i - r_i$ to the self-linking number of L_i. Thus the thickening h has all the desired properties. ∎

Now we are in a position to provide the proof of the Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25.

Sketch of proof of the Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25.

(1) First we need to show that $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is well-defined. In the proof of this fact we borrow some ideas from the proof of the Linking Number-via–Surfaces Proposition 285.7 (2). So let (W, F) be a thickened cobordism between (K, g) and (L, h). We write $W' = F(\{1\} \times W)$, $K' = g(\{1\} \times K)$ and $L' = h(\{1\} \times L)$. By the Linking Number-via–Surfaces Lemma 300.27 (1) there exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse. Similarly there exist T, T' for L and L'.

We can make the identification $S^4 = (-\overline{B}^4) \cup [0,1] \times S^3 \cup \overline{B}^4$[3322] (See the figure below for an illustration.) Using the Gluing Submanifolds Proposition 56.12 we can arrange that both $(-S) \cup W \cup T$ and $(-S') \cup W' \cup T'$ are oriented submanifolds of the

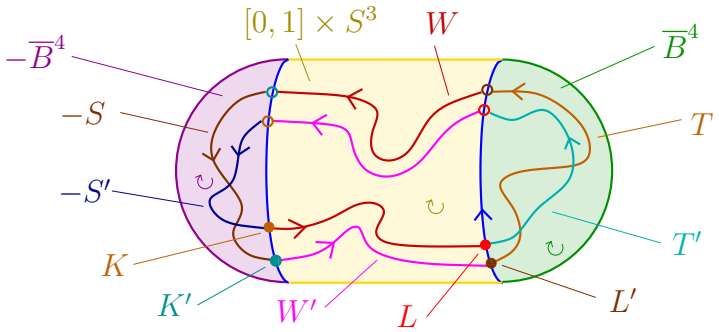
[3321] Here, as a reminder, $L_1 \vee \cdots \vee L_{m-1}$ denotes the wedge of the topological spaces $L_1, \ldots, L_{m-1}$.

[3322] If the reader does not feel like making this identification, then we point out that we only need that the right-hand side is a smooth manifold and that the second homology vanishes.

smooth manifold S^4. We calculate that

$$\begin{aligned} -\operatorname{slk}(K,g)+\operatorname{slk}(L,h) &\overset{\text{by definition of slk}}{=} -\operatorname{lk}(K,K')+\operatorname{lk}(L,L') \overset{\text{by the Linking Number-via–Surfaces Lemma 300.27 (2)}}{=} (-S)\cdot_{\overline{B}^4} S' + T\cdot_{\overline{B}^4} T' \\ &= (-S)\cdot_{-\overline{B}^4}(-S') + T\cdot_{\overline{B}^4} T' \overset{\text{since the only intersections appear between } S,S' \text{ and } T,T', \text{ here we use that } W \text{ and } W' \text{ are disjoint}}{=} (-S\cup W\cup T)\cdot_{S^4}(-S'\cup W'\cup T') \overset{\text{by the Maps-Cup Product–Intersection Number Theorem 221.16 and since } \mathrm{H}_2(S^4;\mathbb{Z})=0}{=} 0. \end{aligned}$$

We have thus shown that the self-linking numbers of (K,g) and (L,h) agree. This shows that the map $\Psi\colon \Omega_1^{\text{th}}(\mathbb{R}^3)\to\mathbb{Z}$ is indeed well-defined.



(2) It follows easily from Exercise 284.3 that Ψ is a homomorphism.

(3) By the Knot Thickening–Self Linking Lemma 284.11 there exists a knot in $\mathbb{R}^3$ with a thickening in $\mathbb{R}^3$ with self-linking number one, thus Ψ is an epimorphism.

(4) It remains to show that $\Psi\colon \Omega_1^{\text{th}}(\mathbb{R}^3)\to\mathbb{Z}$ is a monomorphism. So suppose that we are given a thickened m-component link $(L,g)=\bigsqcup_{i=1}^m (L_i,g_i)$ with $\Psi([(L,g)])=0$.

Note that by the Link Seifert Surface Lemma 300.26 there exists a compact oriented connected surface W in $\mathbb{R}^3\times[0,1)$ whose boundary equals $L\subseteq\mathbb{R}^3=\mathbb{R}^3\times\{0\}$. Next note that by the Self Linking Number-Extension Lemma 300.29 we can equip W with a thickening h such that for $i=1,\dots,m-1$ we have $\operatorname{slk}(L_i,h|_{L_i})=\operatorname{slk}(L_i,g_i)$.

Claim. We also have $\operatorname{slk}(L_m,h|_{L_m})=\operatorname{slk}(L_m,g_m)$.

Proof. Note that (W,h) is a thickened cobordism between $(L,h|_L)$ and the empty link. In particular we obtain from (1) that $\operatorname{slk}(L,h|_L)=0$. By our hypothesis on (L,g) we also have $\operatorname{slk}(L,g)=0$. We had already arranged that for $i=1,\dots,m-1$ we have $\operatorname{slk}(L_i,h|_{L_i})=\operatorname{slk}(L_i,g_i)$. It now follows immediately from the Self-Linking Number Lemma 300.24 that $\operatorname{slk}(L_m,h|_{L_m})=\operatorname{slk}(L_m,g_m)$. ⊞

Now we are done, since

$$[(L,g)] \overset{\text{by the Self Linking Number Suffices Lemma 300.28}}{=} [(L,h|_g)] \overset{\text{since } (W,h) \text{ is a null-cobordism for } (L,h|_g)}{=} 0 \in \Omega_1^{\text{th}}(\mathbb{R}^3).$$

■

300.5. **The linking number.** In the next section we will show that there exists an epimorphism $\Omega_{2n-1}^{\text{th}}(\mathbb{R}^{4n-1})\to\mathbb{Z}$. The key to doing so is the notion of the linking number which we discuss in this section.

Notation 300.30. Let $n \in \mathbb{N}$.

(1) Given a closed oriented m-dimensional submanifold K of S^{2m+1} we denote by
$$\beta_K \colon \mathrm{H}_m(K;\mathbb{Z}) \ \to \ \mathrm{H}^m(S^{2m+1} \setminus K;\mathbb{Z})$$
the inverse of the isomorphism given by the Sphere–Alexander Duality Theorem **??**.

(2) Given a compact oriented proper $(m+1)$-dimensional submanifold W of $[0,1] \times S^{2m+1}$ we denote by
$$\beta_W \colon \mathrm{H}_{m+1}(W, \partial W;\mathbb{Z}) \ \to \ \mathrm{H}^m(([0,1] \times S^{2m+1}) \setminus W;\mathbb{Z})$$
the inverse of the isomorphism given by the Sphere–Alexander Duality Theorem **??**.

With this notation we can introduce the following definition:

Definition 300.31. Let $m \in \mathbb{N}$ and let K and L be two disjoint closed oriented m-dimensional submanifolds of S^{2m+1}. We define the linking number of K and L as
$$\mathrm{lk}(K,L) \ := \ \langle \beta_K([K]), [L] \rangle \ \in \ \mathbb{Z}$$
where we view the fundamental class $[L]$ of L as a homology class in $\mathrm{H}_m(S^{2m+1} \setminus K;\mathbb{Z})$.

Remark. The vigilant reader will not have failed to notice that in the 3-dimensional setting it is a priori not clear that the above definition of the linking number agrees with the definition given in Definition 114.1 and Definition 284.1. In Exercise 300.6 we will prove that in the 3-dimensional setting the two competing notions of linking numbers agree. □

The following proposition is a generalization of the Linking Number-Symmetry Proposition 284.9.

Proposition 300.32. (Linking Number-Symmetry Proposition) Let $m \in \mathbb{N}$. Given any two disjoint closed oriented m-dimensional submanifolds K and L of S^{2m+1} we have the following equality:
$$\mathrm{lk}(K,L) \ = \ (-1)^{m+1} \cdot \mathrm{lk}(L,K).$$

Proof. We leave it to the reader to modify the proof of the Linking Number-Symmetry Proposition 284.9 to obtain the desired statement. ■

300.6. **The Hopf invariant and the thickened cobordism group $\Omega_{2n-1}{}^{\mathrm{th}}(\mathbb{R}^{4n-1})$.** In the $\pi_{4n-1}(S^{2n})$–Infinite Theorem 216.23 we showed that given any $n \in \mathbb{N}$ there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$. By the Thom-Pontryagin Theorem 300.19 this statement is equivalent to saying that there exists an epimorphism $\Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$.

First of all we work with the same convention as in the last section.

Convention 300.33. In this section we equip any thickened submanifold and any thickened cobordism with the natural orientation that we introduced in Definition 300.21.

Proposition 300.34. (Linking Number–Homomorphism Lemma) Let $n \in \mathbb{N}$ and let $* \in \overline{B}^n$. The map
$$\begin{array}{rcl} \mathrm{Hopf} \colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{2n-1}) & \to & \mathbb{Z} \\ [(N, g \colon \overline{B}^n \times N \to \mathbb{R}^{2n-1})] & \mapsto & \mathrm{lk}(N, g(* \times N)) \end{array}$$
is a well-defined homomorphism.

Definition 300.35. Let $n \in \mathbb{N}$ and let $* \in \overline{B}^n$. We refer to the homomorphism

$$\begin{array}{rcl} \mathrm{Hopf}\colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{2n-1}) & \to & \mathbb{Z} \\ [(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{2n-1})] & \mapsto & \mathrm{lk}(N, g(* \times N)) \end{array}$$

from the Linking Number–Homomorphism Proposition 300.34 as the Hopf homomorphism.

Remark. Let $n \in \mathbb{N}$. We recall that in Definition 216.10 we used the cup product to define the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{4n-1}(S^{2n}, *) \to \mathbb{Z}$$

which is, as shown in the Hopf Invariant–Homomorphism Proposition 216.14, in fact a homomorphism. The audacious reader can now attempt to show that, under the above isomorphism $\Omega^{\mathrm{th}}_{4n-1}(\mathbb{R}^{4n-1}) \xrightarrow{\cong} \pi_{4n-1}(S^{2n}, *)$, the Hopf invariant and the homomorphism $\mathrm{Hopf}\colon \Omega^{\mathrm{th}}_{4n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$ from the Linking Number–Homomorphism Proposition 300.34 agree up to sign.[3323] □

Proof. In Exercise 300.7 we will verify that Hopf is a homomorphism. Now let us actually show that Hopf is well-defined. Thus suppose that we are given thickened $(2n-1)$-dimensional submanifolds (N_0, g_0) and (N_1, g_1) of $\mathbb{R}^{2n-1}$ that are cobordant via a thickened cobordism (W, G). We denote by $i_0\colon \{0\} \times S^{2n-1} \to [0,1] \times S^{2n-1}$ the inclusion. Similarly we define i_1. We also use i_0, i_1 in a rather obvious way for restrictions of the above inclusion maps to suitable subsets. It follows immediately from the Sphere–Alexander Duality Theorem **??** that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(W, \partial W; \mathbb{Z}) & \xrightarrow[\beta_W]{\cong} & \mathrm{H}^{2n-1}(([0,1] \times S^{2n-1}) \setminus W; \mathbb{Z}) \\ \downarrow \partial & & \downarrow \begin{matrix} i_0^* \\ i_1^* \end{matrix} \\ \begin{matrix} \mathrm{H}_{2n-1}(\{0\} \times N_0; \mathbb{Z}) \\ \mathrm{H}_{2n-1}(\{1\} \times N_1; \mathbb{Z}) \end{matrix} & \xrightarrow[\cong]{\begin{pmatrix} \beta_{N_0} & 0 \\ 0 & \beta_{N_1} \end{pmatrix}} & \begin{matrix} \mathrm{H}^{2n-1}(\{0\} \times (S^{2n-1} \setminus N_0); \mathbb{Z}) \\ \mathrm{H}^{2n-1}(\{1\} \times (S^{2n-1} \setminus N_1); \mathbb{Z}) \end{matrix} \end{array}$$

where all horizontal maps are isomorphisms and where the left-hand vertical map is the connecting homomorphism of the long exact sequence in homology of the pair $(W, \partial W)$.

It follows that

$$\begin{array}{rcl} \mathrm{lk}(N_0, g_0(* \times N_0)) & = & \langle \beta_{N_0}([N_0]), [g_0(* \times N_0)] \rangle \\ & = & \langle \beta_W([W]), (i_0)_*([g_0(* \times N_0)]) \rangle \\ & = & \mathrm{lk}(N_1, g_1(* \times N_1)). \end{array} \quad \begin{array}{rcl} & & \\ & \overset{\downarrow}{=} & \langle i_0^*(\beta_W([W])), [g_0(* \times N_0)] \rangle \\ & \overset{\uparrow}{=} & \langle \beta_W([W]), (i_1)_*([g_1(* \times N_1)]) \rangle \\ & & \end{array}$$

follows from the above diagram and the fact that by Fundamental Class-of-∂-Proposition 194.13 we know that $[\partial W] = [\{0\} \times N_0] + [\{1\} \times N_1]$

by the Kronecker Pairing Natural Lemma 199.9 (1)

it follows from $\partial(G(* \times W)) = g_0(* \times N_0) \cup g_1(* \times N_1)$ together with the Topological Kernel-of-Boundary Inclusion Corollary 194.14 that $(i_0)_*([g_0(* \times N_0)]) = (i_1)_*([g_1(* \times N_1)])$

same argument backwards

We have thus shown that Hopf is indeed well-defined. ■

[3323] Attempting to determine the correct sign with confidence might have negative effects on one's mental health.

Of course an invariant is only interesting if we can show that it assumes non-trivial values. If $n = 1$, then we already know[3324] that the trivial knot K together with the thickening g of self-linking number one satisfies $\mathrm{Hopf}([(N,g)]) = 1$. In particular we see that $\mathrm{Hopf}\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is an epimorphism.

Lemma 300.36. (Hopf Map–Vanishing Lemma) For any $n \in \mathbb{N}$ the Hopf invariant $\Omega_n^{\mathrm{th}}(\mathbb{R}^{4n+1}) \to \mathbb{Z}$ is the zero map.

Proof. Let $(N, g\colon \overline{B}^{n+1} \times N \to \mathbb{R}^{4n+1})$ be a thickened n-dimensional submanifold of $\mathbb{R}^{4n+1}$. We calculate that

$$\overbrace{\mathrm{lk}(N, g(*\times N))}^{=\mathrm{Hopf}([(N,g)])} \underset{\substack{\uparrow\\ \text{by the Linking Number}\\ \text{Symmetry Proposition 300.32}}}{=} (-1)^{\dim(N)+1} \cdot \mathrm{lk}(g(*\times N), N) \underset{\substack{\uparrow\\ \text{since } N \sqcup g(*\times N) \text{ and } g(*\times N)\sqcup N\\ \text{are isotopic}}}{=} -\overbrace{\mathrm{lk}(N, g(*\times N))}^{=\mathrm{Hopf}([(N,g)])}.$$

This calculation shows that $\mathrm{Hopf}([(N,g)]) = 0$. ∎

Recall that the combination of the Hopf Invariant–Homomorphism Proposition 216.14 and the Bidegree–Realization Proposition 216.22 shows that given any $n \in \mathbb{N}$ there exists a continuous map $f\colon S^{4n-1} \to S^{2n}$ such that the Hopf invariant equals 2. The following proposition that can be viewed as an analog of this fact.

Proposition 300.37. (Even Hopf Invariant Proposition) For any $n \in \mathbb{N}$ there exists a thickened $(2n-1)$-dimensional submanifold (N, g) in $\mathbb{R}^{4n-1}$ with $\mathrm{Hopf}([(N,g)]) = 2$.

Sketch of proof. We consider the "trivial $(2n-1)$-dimensional knot"

$$K \;=\; S^{2n-1} \times \{0\} \;=\; \{(z_1, \dots, z_{2n}, 0, \dots, 0) \in \mathbb{R}^{4n-1} \mid z_1^2 + \cdots + z_{2n}^2 = 1\}$$

together with the map[3325]

$$\begin{array}{rcl} g\colon \overline{B}^{2n} \times K & \to & \mathbb{R}^{4n-1} \\ \big((x, \underbrace{(y_1, \dots, y_{2n-1})}_{=:y\in\mathbb{R}^{2n-1}}), \underbrace{(z,0)}_{\in K}\big) & \mapsto & (z + \frac{1}{4}x\cdot z, y). \end{array}$$

One can easily verify that, with a suitable orientation for K, the pair (K, g) is a thickened $(2n-1)$-dimensional submanifold of $\mathbb{R}^{4n-1}$. It is not particularly difficult to show that $[g(*\times K)] = 0 \in \mathrm{H}_{2n-1}(\mathbb{R}^{4n-1} \setminus K)$. Unfortunately this implies that $\mathrm{lk}(K, g(*\times K)) = 0$.

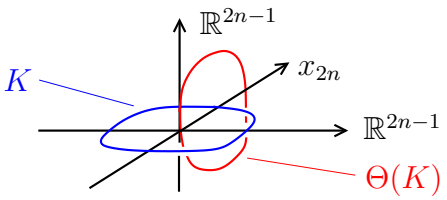
The first idea would be to modify the thickening g in a similar vein as in the Knot Thickening–Self Linking Lemma 284.11. This might work in principle, but at least your author does not know how to make this work in general.[3326] Instead we will recycle the example from page 5792 where we saw that the Hopf link gives rise to a non-trivial element in $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$. More precisely, we consider the diffeomorphism

$$\begin{array}{rcl} \Theta\colon \mathbb{R}^{4n-1} & \to & \mathbb{R}^{4n-1} \\ (\underbrace{x_1, \dots, x_{2n}}_{=x}, \underbrace{y_1, \dots, y_{2n-1}}_{=y}) & \mapsto & (y,x) + (0, \dots, 0, \underset{\substack{\uparrow\\ 2n\text{-th coordinate}}}{1}, 0, \dots, 0). \end{array}$$

[3324] Here we assume for convenience that we already know that the two different definitions of the linking pairing in the 3-dimensional setting agree.

[3325] This map g is basically just the "obvious" thickening of $S^{2n-1} \subseteq \mathbb{R}^{2n}$ extended to higher dimensions.

[3326] Also the remark after the proof might temper one's optimism.



Similar to the discussion on page 5792 one can now fairly easily show (voluntary exercise!) that $\mathrm{Hopf}([K \sqcup \Theta(K), g \sqcup \Theta_*(g)]) = \pm 2$. Since Hopf is a homomorphism we see that there exists a thickened $(2n-1)$-dimensional submanifold (N, g) in $\mathbb{R}^{4n-1}$ such that $\mathrm{Hopf}([(N, g)]) = 2$. ■

Remark. In the Adams Theorem 237.12 we saw that, unless $n = 2, 4, 8$, there does not exist a continuous map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$. If we believe for a second that the Hopf invariant corresponds to the above homomorphisms Hopf, then this shows that the Even Hopf Invariant Proposition 300.37 is optimal for $n \notin \{1, 2, 4\}$. □

In summary, the Thom-Pontryagin Theorem 300.19 together with the Linking Number–Homomorphism Proposition 300.34 and the Even Hopf Invariant Proposition 300.37 give us the following result, which is precisely the content of the $\pi_{4n-1}(S^{2n})$–Infinite Theorem 216.23.

Proposition 300.38. ($\pi_{4n-1}(S^{2n})$–Infinite Proposition) Given any *even* $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.

300.7. Exercises for Chapter 300.

Exercise 300.1. We consider the map

$$\begin{aligned} f\colon (S^1 \times \overline{B}^2) \cup_{S^1 \times S^1} (\overline{B}^2 \times S^1) &\to \overline{B}^2/S^1 \\ P &\mapsto \begin{cases} [Q], & \text{if } P = [(Q, z)] \text{ with } Q \in \overline{B}^2 \text{ and } z \in S^1, \\ [S^1], & \text{if } P = [(z, Q)] \text{ with } z \in S^1 \text{ and } Q \in \overline{B}^2. \end{cases} \end{aligned}$$

Show that f is null-homotopic.

Exercise 300.2. Let $(L = L_1 \sqcup \cdots \sqcup L_m, g = g_1 \sqcup \cdots \sqcup g_m)$ be a thickened link. Show that

$$\mathrm{slk}(L, g) \coloneqq \sum_{i \neq j} \mathrm{lk}(L_i, L_j) + \sum_{i=1}^{m} \mathrm{slk}(L_i, g_i).$$

Exercise 300.3. Let $m \geq 2$. The Thom-Pontryagin Theorem 300.19 implies in particular that $\Omega^{\mathrm{th}}_{m-1}(\mathbb{R}^m) \cong \pi_1(S^m) = 0$. This implies that every thickened submanifold of $\mathbb{R}^m$ of codimension-one is null-cobordant. Prove this statement without making use of the Thom-Pontryagin Theorem 300.19.
Hint. Use the Codimension Zero Submanifold Bounds Lemma 234.4.

Exercise 300.4. We consider the Hopf link H as shown in the figure below. Draw an explicit oriented compact connected surface whose boundary equals the oriented link H.
Remark. Note that the orientation of one of the components is different from the Hopf link that we considered above in the figure on page 5793.

Hopf link 

Exercise 300.5. Let $L \subseteq S^3$ be an oriented link and let $G \subseteq S^3$ be a compact oriented 2-dimensional smooth manifold with $\partial G = L$. We assume that G has no closed components.

(a) Show that $S^3 \setminus G$ is path-connected.

(b) Give an outline of an argument why there exists a compact oriented *connected* surface F with $\partial F = L$.

Exercise 300.6. Let K and J be two oriented disjoint knots in S^3. Show that the two definitions of the linking number $\mathrm{lk}(K, J)$ given in Definition 284.1 and Definition 300.31 agree.

Exercise 300.7. Show that the map

$$\begin{array}{rcl} \Psi\colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) & \to & \mathbb{Z} \\ [(N, g\colon \overline{B}^{2n} \times N \to \mathbb{R}^{4n-1})] & \mapsto & \mathrm{lk}(N, g(* \times N)) \end{array}$$

considered in the Linking Number–Homomorphism Proposition 300.34 is a homomorphism.

Exercise 300.8. By the Thom-Pontryagin Theorem 300.19 we know that for any $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} \Psi\colon \Omega^{\mathrm{th}}_{k}(\mathbb{R}^{k+1}) & \to & \pi_{k+1}(S^1, *) \\ [(N, g)] & \mapsto & [\rho_{(N,g)}] \end{array}$$

is an isomorphism. By the discussion on page 2587 we know that $\pi_{k+1}(S^1) = 0$. Give a direct proof that $\Omega^{\mathrm{th}}_{k}(\mathbb{R}^{k+1}) = 0$.

Exercise 300.9. In Section 300.6 we had just seen that given any even $n \in \mathbb{N}$ there exists an epimorphism $\Omega^{\mathrm{th}}_{n-1}(\mathbb{R}^{2n-1}) \to \mathbb{Z}$. What about the group $\Omega^{\mathrm{th}}_{n-1}(\mathbb{R}^{2n-1})$ if n is odd? Can we use linking numbers to define a homomorphism to $\mathbb{Z}$? Can we use them to find an epimorphism from $\Omega^{\mathrm{th}}_{n-1}(\mathbb{R}^{2n-1})$ to $\mathbb{Z}$?

CHAPTER 301

Thom-Pontryagin Theory II: The Freudenthal Suspension Theorem

In this chapter we continue with the exploration of the Thom-Pontryagin theory. The main goal will be to prove the Freudenthal Suspension Theorem 301.12.

This chapter is organized as follows:

- In Section 301.1 we will introduce the framed cobordism groups $\Omega_k^{\text{fr}}(\mathbb{R}^n)$.
- In Section 301.2 we will state and prove the Freudenthal Suspension Theorem 301.12.
- In Section 301.3 we will introduce the framed internal surgery.
- In Section 301.4 we will prove an addendum to the Freudenthal Suspension Theorem 301.12.
- In Section 301.5 we will provide exercises for Chapter 301.

In the next Chapter 302 we will use the Thom-Pontryagin theory to calculate several homotopy groups of spheres.

301.1. **The framed cobordism group.** In Chapter 300 we introduced the thickened cobordism groups $\Omega_k^{\text{th}}(\mathbb{R}^{n+k})$ and we showed that they are isomorphic to $\pi_{n+k}(S^n,*)$. Furthermore we computed that $\Omega_0^{\text{th}}(\mathbb{R}^n)\cong\mathbb{Z}$ and $\Omega_1^{\text{th}}(\mathbb{R}^3)\cong\mathbb{Z}$. Finally we showed that given any $n\in\mathbb{N}$ there exists an epimorphism $\Omega_{2n-1}^{\text{th}}(\mathbb{R}^{4n-1})\to\mathbb{Z}$. These calculations gave us new proofs for the $\pi_n(S^n)$–Proposition 167.10, the $\pi_n(S^3)=\pi_n(S^2)$–Theorem 239.12 (2) and the $\pi_{4n-1}(S^{2n})$–Infinite Theorem 216.23.

Even though these new proofs were very pleasant, it is now time for some new results. In this section we will prove the Freudenthal Suspension Theorem and we will calculate the homotopy groups $\pi_{n+1}(S^n)$ for $n\geq 3$. For the subsequent calculations it is better though to work with the "framed cobordism groups" $\Omega_k^{\text{fr}}(\mathbb{R}^n)$ that we introduce in this section. As we will see, the framed cobordism groups are isomorphic to the thickened cobordism group, but at times they are easier to calculate since they allow us to reduce many issues to linear algebra.

First we introduce the following definitions in linear algebra.

Definition 301.1. Given $m,k\in\mathbb{N}_0$ we write

$$\begin{aligned}\text{GL}(m,k) &:= \text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^m\\ \text{O}(m,k) &:= \text{set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^m.\end{aligned}$$

We adopt the following conventions:

(1) Given $v\in\text{GL}(m,k)$ we denote by $v_1,\dots,v_k$ the corresponding column vectors.
(2) Throughout we view both $\text{GL}(m,k)$ and $\text{O}(m,k)$ as subsets of the smooth manifold $\text{M}(m\times k,\mathbb{R})=\mathbb{R}^{mk}$.

Now we can give the topological definition we are really interested in:

Definition 301.2. Let $m \in \mathbb{N}_0$ and let N be an m-dimensional submanifold of $\mathbb{R}^m$. (Later on we will only be interested in $N = \mathbb{R}^m$, $N = [0,1] \times \mathbb{R}^{m-1}$ and the case that $N = \overline{B}_r^m(P)$ is a closed m-dimensional ball.)

(1) Let M be a compact orientable proper k-dimensional submanifold of N.
   (a) Let $P \in M$. We say that $v = (v_1, \ldots, v_{m-k}) \in \mathrm{GL}(m, m-k)$ is a frame of M at P if the vectors $v_1, \ldots, v_{m-k}$, together with $\mathrm{T}_P M$[3327] spans $\mathbb{R}^m$.
   (b) A framing for M is a smooth[3328] map $v\colon M \to \mathrm{GL}(m, m-k)$ such that for each $P \in M$ the vectors $v_1(P), \ldots, v_{m-k}(P)$ are a frame of M at P.
   (c) We say that a framing $v = (v_1, \ldots, v_{m-k})$ is perpendicular if for each $P \in M$ the vectors $v_1(P), \ldots, v_{m-k}(P)$ lie in the orthogonal complement of $\mathrm{T}_P M$, i.e. if they lie in the vector space

$$(\mathrm{T}_P M)^\perp \;=\; \{v \in \mathbb{R}^m \mid \langle v, w\rangle = 0 \text{ for all } w \in \mathrm{T}_P M\}.$$

   If the framing is perpendicular and if all the vectors are orthonormal, then we say that the framing is orthonormal .

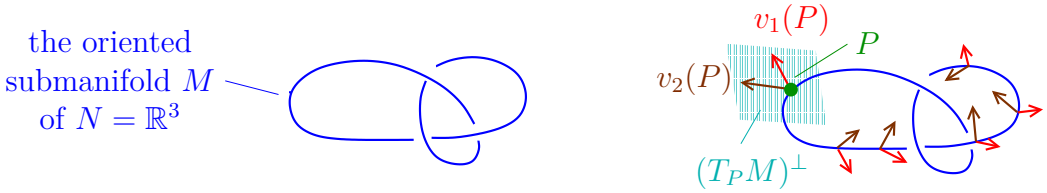
(2) A framed k-dimensional submanifold of N is a pair (M, v) where M is a compact orientable proper k-dimensional submanifold of N and v is a framing for M.

(3) Let (M, v) be a framed k-dimensional submanifold of N and let $\varphi\colon N \to W$ be a smooth embedding of N into a smooth manifold of the same dimension. We consider the map

$$\begin{array}{rcl} \varphi_*(v)\colon \varphi(M) & \to & \mathrm{GL}(m, m-k) \\ P & \mapsto & (\varphi_*(v_1(\varphi^{-1}(P))), \ldots, \varphi_*(v_{m-k}(\varphi^{-1}(P)))). \end{array}$$

   The pair $(\varphi(M), \varphi_*(v))$ is a framed submanifold of W.

Some of the above definitions are illustrated in the figure below.



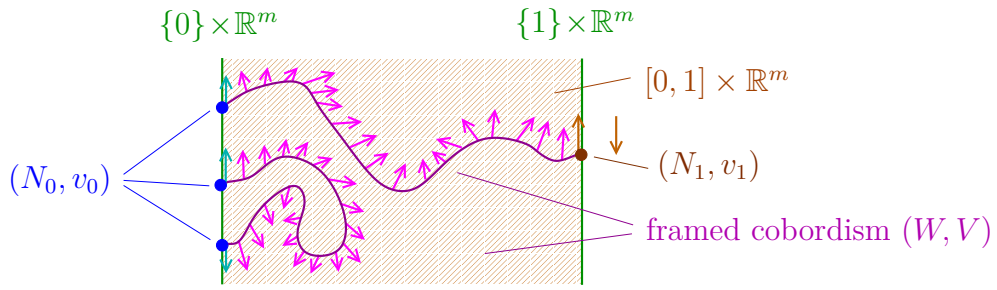
Definition 301.3. Let $k, m \in \mathbb{N}_0$ and let (N_0, v_0) and (N_1, v_1) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. A framed cobordism between (N_0, v_0) and (N_1, v_1) is a framed $(k+1)$-dimensional submanifold (W, V) of $[0,1] \times \mathbb{R}^m$ such that the following three conditions are satisfied:

(1) $\partial W = (\{0\} \times N_0) \cup (\{1\} \times N_1)$,
(2) $V|_{\{0\}\times N_0} = v_0$ under the identification $\{0\} \times \mathbb{R}^m = \mathbb{R}^m$,
(3) $V|_{\{1\}\times N_1} = v_1$ under the identification $\{1\} \times \mathbb{R}^m = \mathbb{R}^m$,

If such a framed cobordism exists, then we say that (N_0, v_0) and (N_1, v_1) are cobordant. The definition of a framed cobordism is illustrated in the figure below.

[3327] Here we use the isomorphism from the Tangent Space-Isomorphism Proposition 29.24 to identify $\mathrm{T}_P M$ with the vector subspace $\widetilde{\mathrm{T}}_P M$ of $\mathbb{R}^m$.

[3328] Since we view $\mathrm{GL}(m, m-k)$ as a subset of the smooth manifold $\mathrm{M}(m \times (m-k), \mathbb{R}) =_{\mathrm{i}} \mathbb{R}^{m\cdot(m-k)}$ it makes sense to say that a map $M \to \mathrm{GL}(m, m-k)$ is smooth.



Basically the same proof as in the Thickened Submanifold–Cobordism–Equivalence Relation Lemma 300.15 gives us the following lemma:

Lemma 301.4. (Framed Submanifold–Cobordism–Equivalence Relation Lemma) The property of "being cobordant" is an equivalence relation on the set of framed k-dimensional submanifolds of $\mathbb{R}^m$.

This lemma leads us, not surprisingly, to the following definition:

Definition 301.5. We denote by $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ the set of cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^m$. We refer to $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ as a framed cobordism group.

The following proposition, which is the analog of the Thickened Cobordism Group Proposition 300.17, justifies the name "framed cobordism group".

Proposition 301.6. (Framed Cobordism Group Proposition) Let $k \in \mathbb{N}$ and let $m \in \mathbb{N}_{\geq 2}$. Let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r, s \in \mathbb{R}_{>0}$ with $M \subseteq B_r^m(0)$ and $N \subseteq B_s^m(0)$. Furthermore we pick two orientation-preserving smooth embeddings $\varphi\colon \overline{B}_r^m(0) \to \mathbb{R}^m$ and $\psi\colon \overline{B}_s^m(0) \to \mathbb{R}^m$ with disjoint images. We define

$$[(M, v)] + [(N, w)] \coloneqq [(\varphi(M) \sqcup \psi(N), \varphi_*(v) \sqcup \psi_*(w))] \in \Omega_k^{\mathrm{fr}}(\mathbb{R}^m).$$

This operation "$+$" has the following properties:

(1) The definition of "$+$" on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ is well-defined and commutative.
(2) The operation "$+$" defines a group structure on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$. The following statements hold:
   (a) The neutral element is represented by the empty submanifold.
   (b) Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M, v)]$ is given by $[(\rho(M), \rho_*(v))]$.

Proof. The proof is basically identical to the proof of the Thickened Cobordism Group Proposition 300.17. We leave it to the reader to carry out the minor adjustments that are necessary to make the transition from thickened cobordisms groups to framed cobordism groups. ■

Lemma 301.7. (Orthonormal Framing Cobordism Lemma) In the following let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \ldots, m\}$.

(1) Every framed k-dimensional submanifold (M, v) of $\mathbb{R}^m$ is cobordant to (M, w) where w is an orthonormal framing.

(2) Let (W, V) be a framed cobordism between two framed k-dimensional submanifolds (M_0, v_0) and (M_1, v_1) of $\mathbb{R}^m$. If v_0 and v_1 are orthonormal, then there exist an orthonormal framing V' for W such that (W, V') is a framed cobordism between (M_0, v_0) and (M_1, v_1).

Proof. We prove Statement (1) of the lemma. The proof of Statement (2) is a mild variation on the proof of Statement (1). Thus let (M, u) be a framed k-dimensional submanifold of $\mathbb{R}^m$. Using the fact that being cobordant is transitive, the proof of the lemma can be broken up into two stages:

(1) First we show that there exists a perpendicular framing v such that (M, u) is cobordant to (M, v).
(2) Then we show that there exists an orthonormal framing w such that (M, v) is cobordant to (M, w).

We carry out the two steps:

(1) Let $(M, u\colon M \to \mathrm{GL}(m, m-k))$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. Given $P \in M$ we write $U_P M \coloneqq \mathrm{span}\{u_1(P), \dots, u_k(P)\}$ and we write $Z_P M \coloneqq (\mathrm{T}_P M)^\perp$. For each $P \in M$ the decomposition $\mathbb{R}^m = \mathrm{T}_P M \oplus Z_P M$ defines a projection of $\mathbb{R}^m$ onto $Z_P M$. We denote the restriction of this projection to $U_P M$ by $\rho_P\colon U_P M \to Z_P M$. By definition of a framing this map is an isomorphism. We consider the product cobordism $W = [0,1] \times M$ equipped with the framing[3329]

$$\begin{array}{rcl} V\colon W = [0,1] \times M & \to & \mathrm{GL}(m, m-k) \\ (t, P) & \mapsto & (u(P) \cdot (1-t) + \rho_P(u(P)) \cdot t). \end{array}$$

Basically by definition this is a framed cobordism from (M, u) to (M, v) where $v = V_1$ is a perpendicular framing.

(2) By the Matrix Groups–Smooth Deformation Retract Proposition 41.15 (1) there exists a smooth deformation retraction $R\colon \mathrm{GL}(m, m-k) \times [0,1] \to \mathrm{O}(m, m-k)$ from $\mathrm{GL}(m, m-k)$ to $\mathrm{O}(m, m-k)$ which has the following extra property:
(∗) For any $v = (v_1, \dots, v_{m-k}) \in \mathrm{GL}(m, m-k)$ and any $t \in [0,1]$ we have

$$\text{span of } R_t(v_1), \dots, R_t(v_{m-k}) \;=\; \text{span of } v_1, \dots, v_{m-k}.$$

Now let (M, v) be a framed k-dimensional submanifold. We consider the product cobordism $W = [0,1] \times M$ equipped with the framing

$$\begin{array}{rcl} V\colon W = [0,1] \times M & \to & \mathrm{GL}(m, m-k) \\ (t, P) & \mapsto & R_t(v(P)). \end{array}$$

It follows from the fact that R is smooth and that (∗) holds that this is a framed cobordism from (M, v) to (M, w) where $w = V_1$ is an orthonormal framing. ∎

The following proposition shows that the two types of cobordism groups that we introduced in this and the last chapter are in fact isomorphic.

[3329] It takes a little bit of thought to show that this is indeed a smooth map.

Proposition 301.8. (Framed-Thickened Cobordism Proposition) Let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \ldots, m\}$. The map

$$\begin{array}{rcl} \Phi\colon \Omega_k^{\mathrm{th}}(\mathbb{R}^m) & \to & \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) \\ [(N, g\colon \overline{B}^{m-k} \times N)] & \mapsto & \left[\begin{array}{l} N \text{ equipped with the framing that associates to } P \in N \\ \text{the frame } \mathrm{D}f_{(0,P)}(\mathfrak{e}_1), \ldots, \mathrm{D}f_{(0,P)}(\mathfrak{e}_{m-k}) \end{array}\right] \end{array}$$

is well-defined and it is an isomorphism.

Convention 301.9. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. We use the isomorphisms from the Thom-Pontryagin Theorem 300.19 and the Framed-Thickened Cobordism Proposition 301.8 to make the identifications

$$\pi_{n+k}(S^n, *) \ =_{\mathrm{i}}\ \Omega_k^{\mathrm{th}}(\mathbb{R}^{n+k}) \ =_{\mathrm{i}}\ \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}).$$

Sketch of proof.

(1) It follows basically from the definitions that the framing associated to a thickened submanifold is indeed a framing. Applying the same procedure to a thickened cobordism between thickened submanifolds we obtain a framed cobordism between framed submanifolds. This shows that Φ is well-defined.

(2) It follows immediately from the definitions of the group structures that Φ is a homomorphism.

(3) We sketch the construction of the inverse to Φ. Let $(N, \{v_1, \ldots, v_{m-k}\})$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. We consider the map

$$\begin{array}{rcl} g(v)\colon \mathbb{R}^{m-k} \times N & \to & \mathbb{R}^m \\ ((t_1, \ldots, t_{m-k}), P) & \mapsto & P + t_1 \cdot v_1(P) + \cdots + t_{m-k} \cdot v_{m-k}(P). \end{array}$$

Note that the same argument as in the proof of the Tubular Neighborhood–Existence Theorem 57.11 shows that there exists an $\epsilon > 0$ such that $g(v)$ restricts to a smooth embedding $\overline{B}_\epsilon^k(0) \times N \to \mathbb{R}^m$. After rescaling the ball we thus get a thickening $\overline{B}^k \times N \to \mathbb{R}^m$ of N. It is straightforward to see that the class $[(N, g(v)] \in \Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ does not depend on the choice of $\epsilon > 0$. Since the same approach also works for cobordisms we obtain a well-defined map

$$\Psi\colon \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) \ \to\ \Omega_k^{\mathrm{th}}(\mathbb{R}^m).$$

It remains to show that Φ and Ψ are inverses of one another. It follows basically immediately from the definitions and the Orthonormal Framing Cobordism Lemma 301.7 (1) that $\Phi \circ \Psi$ is the identity on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$. The fact that $\Psi \circ \Phi$ is the identity on $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ follows fairly immediately from the uniqueness of tubular neighborhoods, see the Tubular Neighborhood–Uniqueness Theorem 57.12, together with the Linear $\overline{B}^k$-Bundle-Isomorphism Lemma 58.6. ■

Definition 301.10. Let N be an orientable k-dimensional submanifold of $\mathbb{R}^m$ and let $n \geq m$.

(1) We write $N_n \coloneqq N \times \{0\} \subseteq \mathbb{R}^n$.

Now suppose that we are given a framing $v\colon N \to \mathrm{GL}(m, m-k)$ for N.

(2) We define

$$\begin{array}{rcl} v_n\colon N_n & \to & \mathrm{GL}(n, n-k) \\ (P,0) & \mapsto & \begin{pmatrix} v(P) & 0 \\ 0 & \mathrm{id}_{n-m} \end{pmatrix}. \end{array}$$

It is clear that v_n is a framing for N_n. Note that $N_m = N$ and $v_m = v$.

(3) We refer to (N_n, v_n) as the $(n-m)$-fold suspension of (N, v). In particular we refer to (N_{m+1}, v_{m+1}) as the suspension of (N, v).

The same notation applies in an obvious way to (framed) submanifolds of $[0,1] \times \mathbb{R}^m$.

The following lemma justifies the name "suspension" that we had used in the above definition.

Lemma 301.11. (Framed Suspension Lemma) Let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \ldots, m\}$.

(1) The map

$$\begin{array}{rcl} \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) & \to & \Omega_k^{\mathrm{fr}}(\mathbb{R}^{m+1}) \\ [(N,v)] & \mapsto & [\text{the suspension of } (N,v)] \end{array}$$

is well-defined and it is a homomorphism. We refer to it as the *suspension homomorphism*.

(2) Under the identification from the convention on page 5805 the suspension homomorphisms on $\pi_{n+k}(S^n, *)$ and $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k})$ agree up to sign,[3330] in other words, the following diagram commutes up to sign:

$$\begin{array}{ccc} \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) & \xrightarrow{\text{suspension}} & \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1}) \\ \downarrow \cong & & \downarrow \cong \\ \pi_{n+k}(S^n, *) & \xrightarrow{\text{suspension}} & \pi_{n+k+1}(S^{n+1}, *). \end{array}$$

Proof.

(1) Statement (1) is almost obvious.

(2) We leave it to the reader to go through all the definitions to verify this statement. ■

301.2. **The Freudenthal Suspension Theorem.** In this section we will prove the following theorem that was first proved in 1937 by Hans Freudenthal [**Freu1937**, p. 300].

Theorem 301.12. (Freudenthal Suspension Theorem) Let $n, k \in \mathbb{N}_0$. We suppose that $n + k \geq 2$. The suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ from the Suspension Homomorphism Lemma 134.27 is an isomorphism if $n > k+1$ and it is an epimorphism if $n = k+1$.

Remark.

(1) The Freudenthal Suspension Theorem 301.12 can be reformulated as follows: Given $n \in \mathbb{N}$ the suspension homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$ is an isomorphism if $m < 2n-1$ and that it is an epimorphism if $m = 2n-1$. We consider two situations in slightly more detail:

(a) The Freudenthal Suspension Theorem 301.12 says that the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is an epimorphism and it says that for any $n \geq 3$ the

[3330] To be more precise, they might agree on the nose, but we will not spend valuable time on figuring out the sign.

suspension homomorphism $\pi_{n+1}(S^n) \to \pi_{n+2}(S^{n+1})$ is an isomorphism. This special case will play an important role shortly.

(b) The Freudenthal Suspension Theorem 301.12 almost recovers the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 (2) and the Octonionic Hopf Fiber Bundle–Homotopy Group Theorem 298.14 (2). Recall that these theorems stated that for any $m \in \{0, \dots, 5\}$ we have $\pi_m(S^3) \cong \pi_{m+1}(S^4)$ and that for any $m \in \{0, \dots, 13\}$ we have $\pi_m(S^7) \cong \pi_{m+1}(S^8)$.

(2) Let $n \in \mathbb{N}_0$. Recall that on page 556 we saw that the suspension $\Sigma(S^n)$ is homeomorphic to S^{n+1}. Thus we see that the Freudenthal Suspension Theorem 301.12 is a consequence of the following stronger statement: Let X be an $(n-1)$-connected space and let $* \in X$ be a good base point.[3331] In this setting the suspension homomorphism $\pi_{n+k}(X, *) \to \pi_{n+k+1}(\Sigma(X), *)$ is an isomorphism for all $n > k+1$ and it is an epimorphism for $n = k+1$. This statement is proved in most textbooks on homotopy theory, see e.g. [**GrayB1975**, Theorem 16.34], [**WhdG1978**, Theorem VII.7.13] or [**DaK2001**, Theorem 8.7 and Chapter 10.3]. If X is a CW-complex, then the statement is also proved in [**Hat2002**, Corollary 4.24]. The proof of [**Hat2002**, Corollary 4.24] is actually a corollary to the Blakers-Massey Theorem 296.29 that we had stated but not proved.□

It follows from the Framed Suspension Lemma 301.11 that the Freudenthal Suspension Theorem 301.12 is equivalent to the following theorem.

Theorem 301.13. (Framed Freudenthal Suspension Theorem) Let $n, k \in \mathbb{N}$.

(1) The suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism whenever $n \geq k+1$.

(2) The suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism whenever $n > k+1$.

The basic behind the proof of the Framed Freudenthal Suspension Theorem 301.13 (1) is quite simple: given a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$ we want to use the "room provided by a large enough n" to show that it is cobordant to a framed submanifold coming from $\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\}$. This idea is for the most part carried out in the following two lemmas.

Lemma 301.14. (Rotate-into-Hyperplane Lemma) Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{n+k+1}$ with $N \subseteq \mathbb{R}^{n+k} \times \{0\}$.

Proof. Let $(M, v\colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. We point out that by the Orthonormal Framing Cobordism Lemma 301.7 (1) we can assume that v is an orthonormal framing.

It follows from $n \geq k+1$ and the Generic Vector Lemma 37.10 that there exists a vector $\nu \in \mathbb{R}^{n+k+1}$ which satisfies the following condition:

(∗) the vector ν does not lie in any tangent space $\mathrm{T}_P M$[3332] and ν is not secant to M, i.e. for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to ν.

[3331] We refer to Definition 99.8 for the definition of a good point.

[3332] Here and throughout this proof we use the Tangent Space-Isomorphism Proposition 29.24 to view the tangent spaces of submanifolds of some $\mathbb{R}^m$ as vector subspaces of $\mathbb{R}^m$.

By the Orthogonal Action Lemma 2.37 (4) there exists a matrix $A \in \mathrm{SO}(n+k+1)$ with $A \cdot \nu = \mathfrak{e}_{n+k+1}$. By the Diffeotopies-via-Matrices Lemma 41.12 there exists a diffeotopy of $\mathbb{R}^{n+k+1}$ from $\mathrm{id}_{\mathbb{R}^{n+k+1}}$ to the diffeomorphism given by multiplying by A. Therefore, by the obvious analog of the Diffeotopy–Cobordism Lemma 300.14 for the framed cobordism group, we can assume without loss of generality that $\nu = \mathfrak{e}_{n+k+1}$.

Claim 1. The map

$$\begin{array}{rcl} \Psi\colon [0,1]\times M & \to & [0,1]\times\mathbb{R}^{n+k+1} \\ (t, \underbrace{(x_1,\dots,x_{n+k+1})}_{\in M}) & \mapsto & (t, \underbrace{x_1,\dots,x_{n+k}, (1-t)\cdot x_{n+k+1}}_{\in\mathbb{R}^{n+k+1}}) \end{array}$$

is a proper smooth embedding.

Proof. It follows from the fact that for any two distinct points $x, y \in M$ the difference $x-y$ is not parallel to $\nu = \mathfrak{e}_{n+k+1}$ that the map is injective. Furthermore note that the map Ψ is the composition of the inclusion $i\colon [0,1]\times M \to [0,1]\times\mathbb{R}^{n+k+1}$ with the map

$$\begin{array}{rcl} \Theta\colon [0,1]\times\mathbb{R}^{n+k+1} & \to & [0,1]\times\mathbb{R}^{n+k+1} \\ (t,(x_1,\dots,x_{n+k+1})) & \mapsto & (t,x_1,\dots,x_{n+k},(1-t)\cdot x_{n+k+1}). \end{array}$$

A straightforward calculation shows that

$$D\Theta_{(t,x_1,\dots,x_{n+k+1})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix}.$$

It follows immediately from this calculation and the fact that $\mathfrak{e}_{n+k+1}$ is not contained in any tangent space $\mathrm{T}_P M$ that $\Psi = \Theta \circ i$ is an immersion. Together with the injectivity of Θ, the fact that M is compact and the ever-popular the Compact-Hausdorff Proposition 4.23 (2) we obtain that $\Psi = \Theta \circ i$ is in fact a smooth embedding. It is also clear that the smooth embedding is proper. ⊞

We set $W \coloneqq \Psi([0,1]\times M)$. By the claim together with the Smooth Embedding Theorem 30.15 (2) we know that W is a proper $(k+1)$-dimensional submanifold of $[0,1]\times\mathbb{R}^{n+k+1}$. Basically by definition W is a cobordism between $M \subseteq \mathbb{R}^{n+k+1}$ and some submanifold $N \subseteq \mathbb{R}^{n+k}\times\{0\}$.

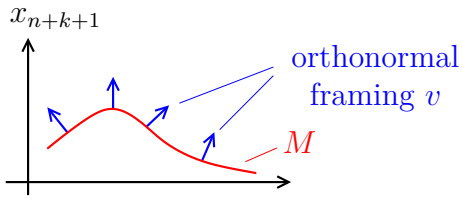
It remains to extend the orthonormal framing $v\colon M \to \mathrm{GL}(n+k+1, n+1)$ of M to a framing of W. In fact, we obtain a framing of $W = \Psi([0,1]\times M)$ by using the "same vectors" as for M. More precisely, we consider the map

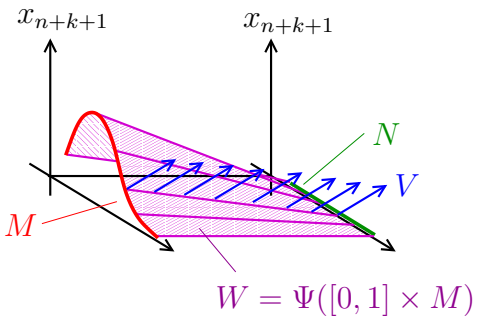
$$\begin{array}{rcl} V\colon W = \Psi([0,1]\times M) & \to & \mathrm{GL}(n+k+2, n+1) \\ \Psi(t,P) & \mapsto & \begin{pmatrix} 0 \\ v(P) \end{pmatrix}. \end{array}$$

Evidently under the identification $\{0\}\times\mathbb{R}^{n+k+1} = \mathbb{R}^{n+k+1}$ we have $V|_M = v$. Thus it remains to show that V is indeed a framing. A short moment's thought shows that V is smooth. The definition of W and V are illustrated in the figure below.
Therefore it suffices to prove the following claim:

Claim 2. For any $t \in [0,1]$ and any $P \in M$ the matrix $V(\Psi(t,P)) \in \mathrm{GL}(n+k+2, n+1)$ is a frame for $W = \Psi([0,1]\times M)$ at $\Psi(t,P)$.

Proof. Let $t \in [0,1]$ and let $P \in M$. We point out that the columns of $V(\Psi(t,P))$ span an $(n+1)$-dimensional vector subspace of $\mathbb{R}^{n+k+2}$. Since $\mathrm{T}_{\Phi(t,P)}(W)$ is a vector subspace





of the complementary dimension $k+1$ it suffices to show that the intersection of these two vector subspaces is trivial.

We note that

$$\underbrace{\mathrm{T}_{\Phi(t,P)}(W) \cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subseteq \mathbb{R}^{1+n+k+1}} \overset{\text{by definition of } W = \Psi([0,1]\times M) = (\Theta \circ i)([0,1]\times M) \text{ and by the calculation of } D\Theta}{=} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix} \cdot \begin{pmatrix} \mathbb{R} \\ \oplus \\ \mathrm{T}_P M \end{pmatrix} \cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subseteq \mathbb{R}^{1+n+k+1}}$$

$$\cong \underbrace{\begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot \mathrm{T}_P(M) \cap \begin{array}{c}\text{span of the}\\ \text{columns of } v(P)\end{array}}_{\subseteq \mathbb{R}^{n+k+1}} = \begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot \mathrm{T}_P(M) \cap (\mathrm{T}_P M)^\perp.$$

since the first row of V is zero we can "ignore" the first coordinate

since v is an orthonormal framing

Now let $u \in \mathrm{T}_P M$ be a non-zero vector. We write $u = \begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}$ with $u' \in \mathbb{R}^{n+k}$ and $u_{n+k+1} \in \mathbb{R}$. Note that u' is non-zero since we had arranged that $\mathfrak{e}_{n+k+1} \notin \mathrm{T}_P M$. Now we calculate that

$$\left\langle \begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}, \underbrace{\begin{pmatrix} u' \\ u_{n+k+1} \end{pmatrix}}_{=u \in \mathrm{T}_P M} \right\rangle = \langle u', u' \rangle + (1-t) \cdot u_{n+k+1}^2 \underset{\text{since } u' \neq 0 \text{ and since } (1-t) \geq 0}{>} 0.$$

But this shows that $\begin{pmatrix} \mathrm{id}_{n+k} & 0 \\ 0 & 1-t \end{pmatrix} \cdot u$ is not orthogonal to $\mathrm{T}_P M$, in other words, this vector does not lie in $(\mathrm{T}_P M)^\perp$. ■

Lemma 301.15. (Rotate Framing Lemma) Let (M, v) be a framed k-dimensional submanifold in $\mathbb{R}^{n+k} \times \{0\} \subseteq \mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there exists a framed cobordism to $(M, w = (w_1, \ldots, w_{n+1}))$ where we have $w_{n+1}(P) = \mathfrak{e}_{n+k+1}$ for all $P \in M$.

Proof. Let $(M, u = (u_1, \ldots, u_{n+1}) \colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold that is contained in $\mathbb{R}^{n+k} \times \{0\} \subseteq \mathbb{R}^{n+k+1}$. By the Orthonormal Framing Cobordism Lemma 301.7 (1) and the Framed Submanifold–Cobordism–Equivalence Relation Lemma 301.4 we can assume that u is orthonormal. We write $e \coloneqq \mathfrak{e}_{n+k+1}$.

Claim 1. There exists an orthonormal framing $v = (v_1, \dots, v_{n+1})$ for M such that (M, u) is cobordant to (M, v) and such that for every $P \in M$ we have $v_{n+1}(P) \neq -e$.

Proof. Let $x \in M$. Since M is contained in $\mathbb{R}^{n+k} \times \{0\}$ we know that e is orthogonal to $\mathrm{T}_x M$. Therefore we can uniquely write $e = \sum_{i=1}^{n+1} \lambda_i(x) \cdot u_i(P)$. Since $\|e\| = 1$ and since u is an orthonormal framing we actually have $\sum_{i=1}^{n+1} \lambda_i^2(P) = 1$. Thus we obtain a map

$$\begin{array}{rcl} \lambda\colon M & \to & S^n \\ x & \mapsto & (\lambda_1(x), \dots, \lambda_{n+1}(x)) \end{array}$$

which is fairly easily seen to be smooth. If $(0, \dots, 0, -1) \notin \lambda(M)$, then we can just take $u = v$ and we are done.[3333] If not we proceed as follows. Since $\dim(M) = k < n$ it follows from the Full Measure-Density Proposition 37.4 that there exists a $\nu \in S^n$ with $\nu \notin \lambda(M)$. By the Orthogonal Action Lemma 2.37 (4) there exists a matrix $A \in \mathrm{SO}(n+1)$ with $\nu = A \cdot (-e) \in S^n$. By the Matrix Group-$\pi_0$-Proposition 4.54 and the Smooth Path-Connectivity Proposition 25.32 there exists a smooth path $\gamma\colon [0,1] \to \mathrm{SO}(n+1)$ from id to A. The desired framed cobordism is now given by the smooth manifold $[0,1] \times M$ together with the framing[3334]

$$\begin{array}{rcl} [0,1] \times M & \to & \mathrm{GL}(n+k+2, n+1) \\ (t, x) & \mapsto & \underbrace{\begin{pmatrix} 0 \\ u(x) \end{pmatrix}}_{\in \mathrm{GL}(n+k+2, n+1)} \cdot \gamma(t). \end{array}$$

⊞

As a final step we want to "rotate v_{n+1} into e". To do this smoothly for every point we need to introduce two maps. Given $u \in \mathbb{R}^{n+1} \setminus \{0\}$ we write

$$\underbrace{\begin{array}{rcl} T_u\colon \mathbb{R}^{n+k+1} & \to & \mathbb{R}^{n+k+1} \\ y & \to & 2 \cdot \langle y, u \rangle \cdot u - y \end{array}}_{\text{reflection in the line } \mathbb{R} \cdot u} \quad \text{and} \quad \underbrace{\begin{array}{rcl} R_u\colon \mathbb{R}^{n+k+1} & \to & \mathbb{R}^{n+k+1} \\ y & \mapsto & T_e(T_u(y)). \end{array}}_{\text{orientation-preserving diffeomorphism}}$$

We will need the following elementary observation from linear algebra.

Claim 2. Let $w \in \mathbb{R}^{n+k+1}$ with $\|w\| = 1$. Let $u = \frac{w+e}{\|w+e\|}$ be the normalized midpoint between w and e. Then $R_u(w) = e$.

Proof. We have

$$R_u(w) \;=\; T_e(T_u(w)) \;=\; T_e\Big(2 \cdot \Big\langle w, \tfrac{w+e}{\|w+e\|} \Big\rangle \cdot \tfrac{w+e}{\|w+e\|} - w\Big) \underset{\uparrow}{=} T_e(e) \;=\; e.$$

short calculation using $\langle w, w \rangle = \langle e, e \rangle$

⊞

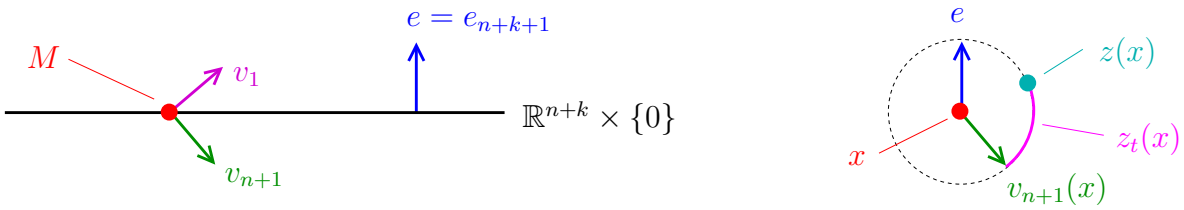
The above claim now allows us to come to the denouement of the argument. More precisely, given $t \in [0,1]$ and $x \in M$ we write

$$z(x) \;\coloneqq\; \underbrace{\tfrac{v_{n+1}(x)+e}{\|v_{n+1}(x)+e\|}}_{\text{note that } v_{n+1}(x) + e \neq 0} \qquad \text{and} \qquad z_t(x) \;=\; \underbrace{\tfrac{(1-t)\cdot e + t \cdot z(x)}{\|(1-t)\cdot e + t \cdot z(x)\|}}_{\text{note that } t \cdot e + (1\text{-}t) \cdot z(x) \neq 0}.$$

[3333] Indeed, if $v_{n+1}(x) = -e$, then $e = -v_{n+1}(x)$, i.e. $\lambda(x) = (0, \dots, 0, -1)$.

[3334] That this framing has the correct properties can be seen as follows. Let v be the new framing at $\{1\} \times M$. If we denote by $\lambda_u, \lambda_v\colon M \to S^n$ the corresponding maps, then one easily sees that for each $x \in M$ we have $\lambda_v(x) = A^{-1} \cdot \lambda_u(x)$. Thus the fact that ν does not lie in the image of λ_u implies that $-e = A^{-1}\nu$ does not lie in the image of λ_v.

Note that $z_0 = e$ and $R_{z_0} = \mathrm{id}$. Furthermore note that $z_1 = z(x)$.



Thus we obtain from the above claim that $R_{z_1(x)}(v_{n+1}(x)) = e$. Finally we consider the framed cobordism that is given by the submanifold $Z = [0,1] \times M$ of $[0,1] \times \mathbb{R}^{n+k+1}$ together with the framing

$$\begin{array}{rcl} W \colon Z = [0,1] \times M & \to & \mathrm{GL}(n+k+2, n+1) \\ (t,x) & \mapsto & \begin{pmatrix} 0 \\ R_{z_t(x)} \circ v(x) \end{pmatrix}. \end{array}$$

We leave it to the reader to verify that the map W is indeed a framing for the cobordism $Z = [0,1] \times M \subseteq [0,1] \times \mathbb{R}^{n+k+1}$. The discussion above shows that $(Z = [0,1] \times M, W)$ is a framed cobordism from (M, v) to a framed smooth manifold $(M, w = w_1, \ldots, w_n, e)$. ■

Now we turn to the actual proof of the Framed Freudenthal Suspension Theorem 301.13 and thus of the Freudenthal Suspension Theorem 301.12.

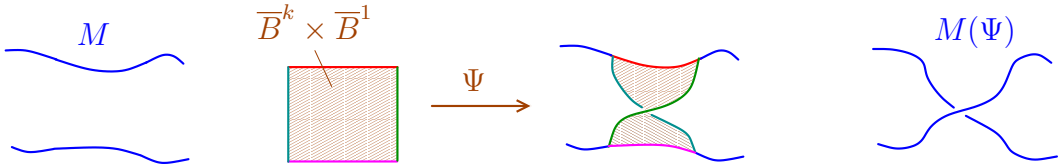
Proof. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$.

(1) Suppose that $n \geq k+1$. We want to show that the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism. Thus let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. It follows immediately from the Rotate-into-Hyperplane Lemma 301.14 and the Rotate Framing Lemma 301.15 that (M, v) is cobordant to a k-dimensional framed submanifold (N, w) such that $N \subseteq \mathbb{R}^{n+k} \times \{0\}$ and such that $w_{n+1}(P) = \mathfrak{e}_{n+k+1}$ for all $P \in N$. But this means that $[(M, v])] = [(N, w)]$ lies in the image of the suspension homomorphism. Thus the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism.

(2) Suppose that $n > k+1$. We want to show that the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. Therefore let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^{n+k}$. Suppose that there exists a framed cobordism (W, V) in $[0,1] \times \mathbb{R}^{n+k+1}$ between the suspensions of (M, v) and (N, w). Since $n > k+1$, an argument as in the Rotate-into-Hyperplane Lemma 301.14 and the Rotate Framing Lemma 301.15 shows that we can turn (W, V) into a cobordism (W', V') with $W' \subseteq [0,1] \times \mathbb{R}^{n+k} \times \{0\}$ and such that $V'(P) = \mathfrak{e}_{n+k+1}$ for all $P \in W'$. But this shows that (M, v) and (N, w) are already cobordant. Thus we have shown that for $n > k+1$ the suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. ■

301.3. **Framed internal surgery.** We start out with the following general definition that is somewhat similar to the construction of the internal connected sum that we gave in Section 187.7.

Definition 301.16. Let $k, n \in \mathbb{N}_0$ and let M be a k-dimensional submanifold of $\mathbb{R}^{n+k}$. Furthermore let $\Psi\colon \overline{B}^k \times \overline{B}^1 \to \mathbb{R}^{n+k}$ be a smooth embedding with the following properties:
(1) $M \cap \Psi(\overline{B}^k \times \overline{B}^1) = \Psi(\overline{B}^k \times S^0)$,
(2) M intersects $\Psi(S^k \times \overline{B}^1)$ transversally.
We refer to
$$M(\Psi) \coloneqq \underbrace{\Psi(S^k \times \overline{B}^1) \cup (M \setminus \Psi(B^k \times S^0))}_{\text{with corner } \Psi(S^{k-1} \times S^0) \text{ rounded as in Section 187.7}}$$
as the internal surgery along Ψ.



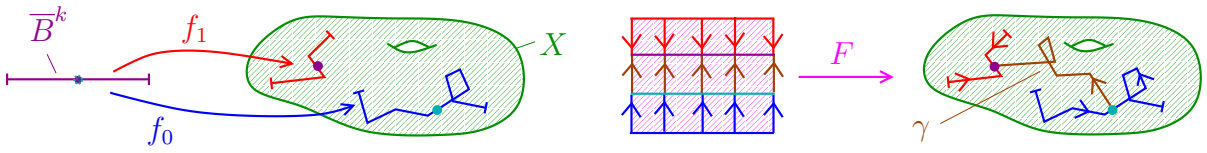
Remark. Note that $M(\Psi)$, viewed as a smooth manifold in its own right, is diffeomorphic to the surgery of M along $\Psi|_{\overline{B}^k \times S^0}$ as introduced in **??**. □

Proposition 301.17. (Internal Surgery–Framed Cobordism Proposition) Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k}$. Note that the framing and the orientation induced an orientation on M. Let $\Psi\colon \overline{B}^k \times \overline{B}^1 \to \mathbb{R}^{n+k}$ be a smooth embedding with the following properties:
(1) $M \cap \Psi(\overline{B}^k \times \overline{B}^1) = \Psi(\overline{B}^k \times S^0)$,
(2) M intersects $\Psi(S^k \times \overline{B}^1)$ transversally,
(3) the map
$$\Psi\colon \underbrace{\overline{B}^k \times S^0}_{\text{product orientation}} \to M$$
is orientation-preserving.
Then $M(\Psi)$ admits a framing w such that (M, v) and $(M(\Psi), w)$ are framed cobordant.

The proof of the Internal Surgery–Framed Cobordism Proposition 301.17 makes use of the following lemma, which is basically the same statement as Exercise 18.1.

Lemma 301.18. (Maps-from-Ball-Homotopic Lemma) Let X be a topological space and let $f_0, f_1\colon \overline{B}^k \to X$ be two continuous maps. If X is path-connected, then there exists a continuous map $F\colon \overline{B}^k \times [0,1] \to X$ with $F_0 = f_0$ and with $F_1 = f_1$.

Proof of the Maps-from-Ball-Homotopic Lemma 301.18. Since X is path-connected we see that there exists a continuous map $\gamma\colon [\frac{1}{3}, \frac{2}{3}] \to X$ with $\gamma(\frac{1}{3}) = f_0(0)$ and with $\gamma(\frac{2}{3}) = f_1(0)$. Now we consider the map
$$\begin{aligned} F\colon \overline{B}^k \times [0,1] &\to X \\ (x,t) &\mapsto \begin{cases} f_0(x \cdot (1-3t)), & \text{if } t \in [0, \frac{1}{3}], \\ \gamma(t), & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ f_1(x \cdot (3 \cdot t - 2)), & \text{if } t \in [\frac{2}{3}, 1]. \end{cases} \end{aligned}$$
It follows from the Homotopy Combination Lemma 18.8 (1) that F is continuous. It is clear that F has all the desired properties. ■



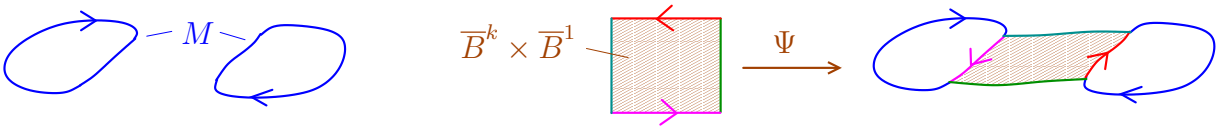
Sketch of a proof of Proposition 301.17. Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k}$ and let $\Psi\colon \overline{B}^k \times \overline{B}^1 \to \mathbb{R}^{n+k}$ be a smooth embedding with the following properties:

(1) $M \cap \Psi(\overline{B}^k \times \overline{B}^1) = \Psi(\overline{B}^k \times S^0)$,
(2) M intersects $\Psi(S^k \times \overline{B}^1)$ transversally,
(3) the map

$$\Psi\colon \underbrace{\overline{B}^k \times S^0}_{\text{product orientation}} \to M$$

is orientation-preserving.

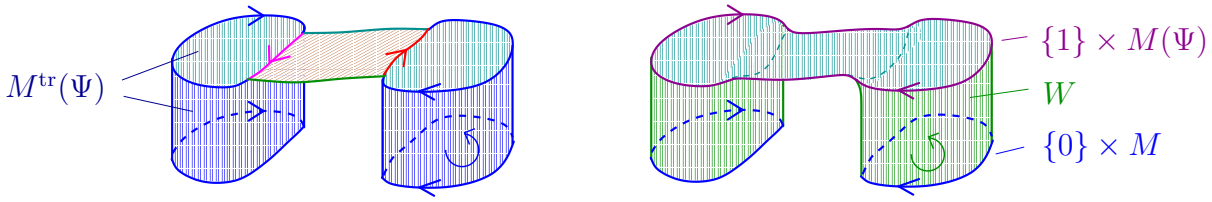
Note that by the Orthonormal Framing Cobordism Lemma 301.7 we can assume that v is an orthonormal framing.



Similar to the discussion in **??** and the proof of the Trace-of-Surgery Proposition **??** we consider the trace of the surgery

$$M^{\mathrm{tr}}(\Psi) \coloneqq ([0,1] \times M) \cup (\{1\} \times \Psi(\overline{B}^k \times \overline{B}^1)) \subseteq [0,1] \times \mathbb{R}^{n+k}.$$

In the Trace-of-Surgery Proposition **??** we saw that $M^{\mathrm{tr}}(\Psi)$ is a $(k+1)$-dimensional smooth manifold and we saw that we can equip $M^{\mathrm{tr}}(\Psi)$ with the orientation coming from the product orientation on $[0,1] \times M$. If we view $M^{\mathrm{tr}}(\Psi)$ as an oriented abstract smooth manifold, then its boundary is $(\{1\} \times M(\Psi)) \sqcup -(\{0\} \times M)$. Similar to Definition 35.11 and the Push-In Lemma 35.12 we can push $\Psi(B^k \times \overline{B}^1) \subseteq M^{\mathrm{tr}}(\Psi)$ into $(0,1) \times \mathbb{R}^{n+k}$ while keeping $\Psi(S^{k-1} \times \overline{B}^1)$ fixed. After rounding corners we obtain the desired cobordism.[3335]



For simplicity we now write $W = ([0,1] \times M) \cup \Psi(\overline{B}^k \times \overline{B}^1)$. Note that $\Psi(\overline{B}^k \times \overline{B}^1)$ is contractible and compact. Therefore it follows from the Vector Bundles-Homotopies Proposition 267.3 (2) that the normal vector bundle of $\Psi(\overline{B}^k \times \overline{B}^1)$ is trivializable. We pick a trivialization. Once we pick a trivialization an orthonormal framing is the same as a map $\Psi(\overline{B}^k \times \overline{B}^1) \to \mathrm{O}(n)$. Since $\Psi(\overline{B}^k \times \{0\})$ is connected we can as well assume that the map $\Psi(\overline{B}^k \times \{0\}) \to \mathrm{O}(n)$ takes values in $\mathrm{SO}(n)$. It follows from our hypothesis (3) that the map $\Psi(\overline{B}^k \times \{1\}) \to \mathrm{O}(n)$ also takes values in $\mathrm{SO}(n)$.

[3335] As the reader will have noticed, at this stage the author swiped some technicalities under a big carpet. This is way why we refer to this only as a "sketch of proof".

By the Matrix Group-π_0-Proposition 4.54 we know that $\mathrm{SO}(n)$ is path-connected. Therefore it follows from the Maps-from-Ball-Homotopic Lemma 301.18 that the framing $\Psi(\overline{B}^k \times \{0,1\}) \to \mathrm{O}(n)$ extends to a framing $\Psi(\overline{B}^k \times [0,1]) \to \mathrm{O}(n)$. ■

301.4. **Suspension and the Hopf invariant.** The following theorem was also proved in 1937 by Hans Freudenthal [**Freu1937**, Satz I].

Theorem 301.19. (Freudenthal Suspension Theorem II) Let $n \in \mathbb{N}$. As above let $\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ be the suspension homomorphism. With this notation we have

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \;=\; \ker\Big(\; \pi_{2n+1}(S^{n+1}) \;\underset{\uparrow}{\overset{\cong}{\leftarrow}}\; \Omega^{\mathrm{th}}_n(\mathbb{R}^{2n+1}) \;\underset{\uparrow}{\xrightarrow{\mathrm{Hopf}}}\; \mathbb{Z} \;\Big)$$

isomorphism from the Thom-Pontryagin Theorem 300.19 — Hopf map from Definition 300.35

Before we turn to the proof of the Freudenthal Suspension Theorem 301.19 let us state the following corollary:

Corollary 301.20. (Freudenthal-Epimorphism Corollary) For every $n \in \mathbb{N}$ the suspension homomorphism $\quad \Sigma\colon \pi_{4n}(S^{2n}) \;\to\; \pi_{4n+1}(S^{2n+1})$

is an epimorphism.

Remark. In the $\pi_{n+2}(S^n)$-Theorem 302.28, when we show that $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$, we will make essential use of the fact that by the Freudenthal-Epimorphism Corollary 301.20 we know that the suspension homomorphism $\Sigma\colon \pi_4(S^2) \to \pi_5(S^3)$ is an epimorphism. □

Proof of Corollary 301.20. This corollary follows immediately from the Freudenthal Suspension Theorem 301.19 and the fact, shown in the Hopf Map–Vanishing Lemma 300.36, that the Hopf invariant $\quad \mathrm{Hopf}\colon \Omega^{\mathrm{th}}_{2n}(\mathbb{R}^{4n+1}) \;\to\; \mathbb{Z}$

is the zero map. ■

We now turn to the proof of the Freudenthal Suspension Theorem 301.19]. We follow the argument given in [**Put**, Chapter 9]. Let $n \in \mathbb{N}$. We need to show that

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \;=\; \ker(\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

We prove the "$\subseteq$"-inclusion and "$\supseteq$"-inclusion separately.

Proof of the Freudenthal Suspension Theorem 301.19 "$\subseteq$"-inclusion. Let $n \in \mathbb{N}$. We need to show that

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \;\subseteq\; \ker(\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

First let us show the "$\subseteq$"-inclusion. So let (M, v) be a framed n-dimensional submanifold of $\mathbb{R}^{2n}$. We write $\widehat{M} \coloneqq M \times \{0\}$ and we denote by $\widehat{v}$ the framing on $\widehat{M}$ that is given by $P \mapsto (v(P), \mathfrak{e}_{2n+1})$. Then

$$\mathrm{Hopf}(\Sigma([M,v])) \;=\; \mathrm{Hopf}(\widehat{M}, \widehat{v}) \;=\; \mathrm{lk}(\widehat{M}, \widehat{M} + \mathfrak{e}_{n+1}) \;\underset{\uparrow}{=}\; 0.$$

since $[\widehat{M} + \mathfrak{e}_{n+1}] = 0 \in \mathrm{H}_n(\mathbb{R}^{2n} + \mathfrak{e}_{n+1})$ and since $\mathbb{R}^{2n} + \mathfrak{e}_{n+1} \subseteq \mathbb{R}^{2n+1} \setminus \widehat{M}$

■

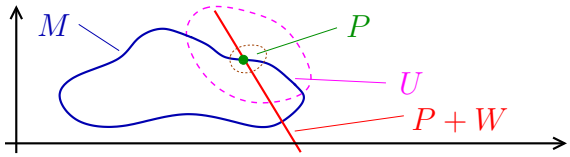
The proof of the reverse inclusion is much more interesting. Our proof relies on several preparations. We start out with the following proposition, that is mildly interesting in its own right.

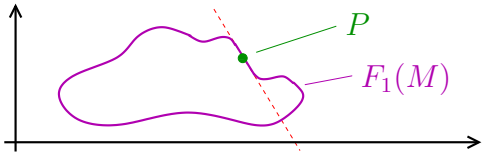
Proposition 301.21. (Cobordant-to-Lower Codimension Proposition) Let M be an oriented n-dimensional submanifold of $\mathbb{R}^{2n+1}$. There exists an oriented cobordism from M to an oriented k-dimensional submanifold N of $\mathbb{R}^{2n+1}$ with $N \subseteq \mathbb{R}^{2n} \times \{0\}$.

Our proof of the Cobordant-to-Lower Codimension Proposition 301.21 relies on the Flattening Lemma 41.23, which we now recall for the reader's convenience.

Lemma 41.23. (Flattening Lemma) Let $M \subseteq \mathbb{R}^m$ be a smooth submanifold, let $P \in M$, let $W \subseteq \mathbb{R}^m$ be a vector subspace and let $\varphi\colon \mathbb{R}^m \to W$ be a homomorphism such that $\varphi\colon \widetilde{\mathrm{T}}_P M \to W$ is a monomorphism. Given any open neighborhood U of P there exists a smooth isotopy $F\colon M \times [0,1] \to \mathbb{R}^m$ rel $(M \setminus U) \cup \{P\}$ and an open neighborhood $\widetilde{U}$ of P with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) $\widetilde{\mathrm{T}}_P F_1(M) = \varphi(\widetilde{\mathrm{T}}_P M)$,
(3) $F_1(\widetilde{U}) \subseteq P + W$,
(4) for each $x \in M$ and $t \in [0,1]$ the point $F(x,t)$ lies on the Euclidean segment between x and $\varphi(x) + P$.





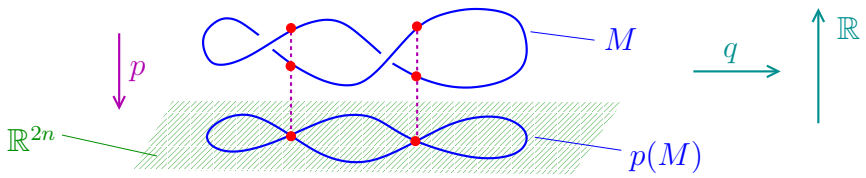
Proof. We consider the projections

$$\begin{array}{rcl} p\colon M & \to & \mathbb{R}^{2n} \\ (x_1, \ldots, x_{2n}, x_{2n+1}) & \to & (x_1, \ldots, x_{2n}) \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon M & \to & \mathbb{R}^{2n} \\ (x_1, \ldots, x_{2n}, x_{2n+1}) & \to & x_{2n+1}. \end{array}$$

By the same argument as in the proof of the Link Generic Projection Lemma 112.12 we see that, after possibly applying a rotation to M, we can assume that the projection $p\colon M \to \mathbb{R}^{2n}$ has the following properties:

(1) the map $p\colon M \to \mathbb{R}^{2n}$ is an immersion,
(2) whenever $x \neq y$ in M satisfy $p(x) = p(y)$, then $\mathrm{D}p_x(\mathrm{T}_x M) \cap \mathrm{D}p_y(\mathrm{T}_y M) = \{0\}$,
(3) for every $z \in \mathbb{R}^{2n}$ the preimage $p^{-1}(z)$ contains at most two points in M.

As in the Diagram–Double Point Lemma 112.3 we see that it follows from the first two properties that there exist only finitely many $z_1, \ldots, z_m \in \mathbb{R}^{2n}$ whose preimage consists of more than one point.



Claim. There exists a family of disjoint orientation-preserving smooth embeddings $\Psi_i\colon \overline{B}^n \times \{0,1\} \to M$, $i = 1,\ldots,m$ with disjoint images such that the restriction of p to $M(\Psi_1,\ldots,\Psi_m)$ is an injective immersion.

Proof. To simplify we deal in the following with the case $m = 1$. The general case is done in almost the same way, we just need to consider all points $z_1,\ldots,z_m$ separately. Thus suppose that we have $z \in \mathbb{R}^{2n}$ such that $p^{-1}(z)$ consists of two points x and y.

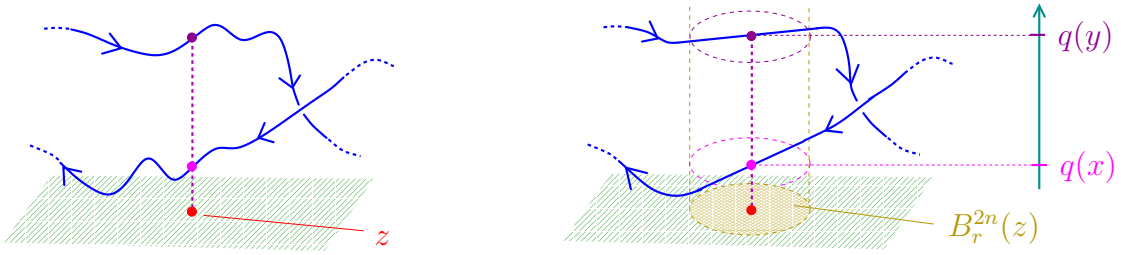
We apply the Flattening Lemma 41.23 to x, y and the homomorphism $p\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$. Thus, after a smooth isotopy, we can assume that locally around x, y is contained in some $\mathbb{R}^{2n} \times \{*\}$.

Next we apply the Flattening Lemma 41.23 to x, y, to the above open neighborhoods and to the orthogonal projection onto the physical tangent spaces of these points.

We have now arranged that there exists an $r > 0$ such that[3336]

$$p^{-1}(\overline{B}^{2n}_{r_1}(z)) \;=\; (\overline{B}^{2n}_{r_1}(z_1) \cap ((\widetilde{\mathrm{T}}M_x + x) \times \{q(x)\}) \cup ((\widetilde{\mathrm{T}}M_{x_y} + y) \times \{q(y)\}).$$

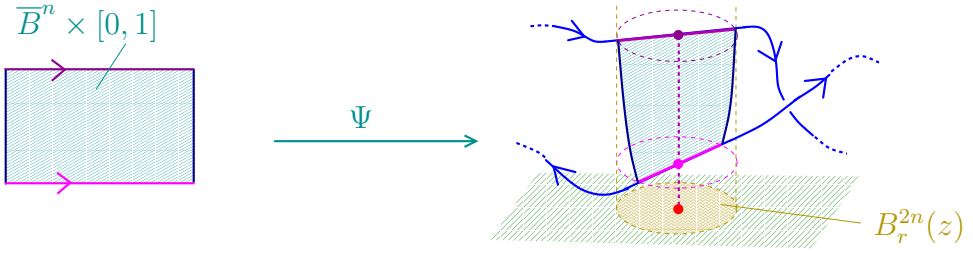
We consider the subspaces $V \coloneqq \mathrm{D}p_x(\widetilde{\mathrm{T}}_x M)$ and $W \coloneqq \mathrm{D}p_x(\widetilde{\mathrm{T}}_x M)$. These are n-dimen-



sional vector spaces of $\mathbb{R}^{2n}$ with $V \cap W = \{0\}$. We pick a *negative* basis $v_1,\ldots,v_n$ and we pick a *positive* basis $w_1,\ldots,w_n$ for W. Finally we consider the map

$$\begin{array}{rcl} \Psi\colon \overline{B}^n \times [0,1] & \to & \mathbb{R}^{2n+1} \\ (\underbrace{(u_1,\ldots,u_m)}_{=u}, t) & \mapsto & \Big(P + \underbrace{\sum_{j=1}^{n} u_j \cdot (v_j \cdot \cos(\tfrac{\pi}{2}\cdot t) + w_j \cdot \sin(\tfrac{\pi}{2}\cdot t))}_{\text{ball spanned by } v_j \cdot \cos(\frac{\pi}{2}\cdot t) + w_j \cdot \sin(\frac{\pi}{2}\cdot t)}, q(x)\cdot(1-t) + t\cdot q(y)\Big). \end{array}$$

Note that each $\Psi\colon \overline{B}^n \times [0,1] \to \mathbb{R}^{2n+1}$ is a smooth embedding.



Also note that the restriction of p to $\Psi(S^{n-1} \times [0,1])$ is injective. This follows from the observation that it is injective, almost by default, on each $\Psi(S^{n-1} \times \{t\})$ and the fact that t can be obtained from consider the plane spanned by say v_1 and W_1. ∎

[3336] If we were dealing with points $z_1,\ldots,z_m$ then we would of course pick $r_1,\ldots,r_m \in \mathbb{R}_{>0}$ such that the closed balls $\overline{B}^{2n}_{r_1}(z_1),\ldots,\overline{B}^{2n}_{r_1}(z_m)$ are disjoint.

The following corollary is the analog of the Rotate-into-Hyperplane Lemma 301.14. Note that the corollary has nothing to do with the Hopf invariant.

Corollary 301.22. (Framed Cobordant-to-Lower Codimension Corollary) Let (M, v) be a framed n-dimensional submanifold of $\mathbb{R}^{2n+1}$. There exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{2n+1}$ with $N \subseteq \mathbb{R}^{2n} \times \{0\}$.

Proof. We equip M with orientation coming from the framing. In the proof of the Cobordant-to-Lower Codimension Proposition 301.21 we constructed a cobordism from M to a k-dimensional submanifold $N \subseteq \mathbb{R}^{2n} \times \{0\}$. By doing surgeries along It follows from the Internal Surgery–Framed Cobordism Proposition 301.17. ■

Proof of the Freudenthal Suspension Theorem 301.19 "$\subseteq$"-inclusion. Let $n \in \mathbb{N}$. We need to show that

$$\mathrm{Im}(\Sigma\colon \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})) \ \subseteq \ \ker(\mathrm{Hopf}\colon \pi_{2n+1}(S^{n+1}) \to \mathbb{Z}).$$

Let (M, v) be a framed $2n$-dimensional submanifold in $\mathbb{R}^{4n+1}$ with Hopf invariant zero. We need to show that there exists a framed $2n$-dimensional submanifold $(\widetilde{M}, \widetilde{v})$ in $\mathbb{R}^{4n}$ such that there exists a framed cobordism from the suspension of $(\widetilde{M}, \widetilde{v})$ to (M, v).

By the Framed Cobordant-to-Lower Codimension Corollary 301.22 we know that there exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{2n+1}$ with $N \subseteq \mathbb{R}^{2n} \times \{0\}$.

As in the Rotate Framing Lemma 301.15 we need to show that w is equivalent to a framing such that the last vector is $\mathfrak{e}_{2n+1}$.

Here we are supposed to use that the Hopf invariant is zero.

Putman argues roughly as follows. Since $N \subseteq \mathbb{R}^{2n} \times \{0\}$ we know that $\mathfrak{e}_{2n+1}$ is orthogonal to all tangent spaces of N. Next recall that $w\colon N \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1}$ is an orthonormal framing. This means that for each $x \in N$ there exists a $g(x) \in S^{n+1}$ with $w(g(x)) = \mathfrak{e}_{2n+1}$. This defines a map $g\colon N \to S^n$. The degree of this map is supposedly the Hopf invariant of (N, w).

Since we assume that the Hopf invariant is zero it follows from the Hopf Degree Theorem that the map $N \to S^n$ is null-homotopic. Now proceed as in page 8 of Putman ■

301.5. Exercise for Chapter 301.

Exercise 301.1. In the proof of Proposition 302.6 we used the Freudenthal Suspension Theorem 301.12 to show that the map

$$\begin{array}{ccc} \mathbb{Z} & \to & \Omega^{\mathrm{fr}}_1(\mathbb{R}^4) \\ [k] & \mapsto & k \cdot [(\mathrm{C}_4, h_4)] \end{array}$$

is an epimorphism. Use Exercise **??** to give an alternative argument, which does not rely on the Freudenthal Suspension Theorem 301.12.

CHAPTER 302

Thom-Pontryagin Theory III: The group $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$

In this chapter we will calculate the framed cobordism groups $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ and $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ "by hand" for $n \geq 3$. By the Thom-Pontryagin Theorem 300.19 and the Framed-Thickened Cobordism Proposition 301.8 this means that we calculate the homotopy groups $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ for $n \geq 3$ By now there might be more modern and efficient ways for computing these homotopy groups. But the geometric approach using Thom-Pontryagin theorem is still very charming, in particular it lets us use many techniques we have developed so far, and it gives us an opportunity to introduce a few new tricks. In particular we will get to introduce quadratic forms and the Arf invariant, which are purely algebraic objects, and are of great interest in the own right.

This chapter is organized as follows:

- In Section 302.1 we calculate the framed cobordism group $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n \geq 3$.
- In Section 302.2 we will calculate the framed cobordism group $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 2$.
- In Section 302.3 we will introduce the Arf invariant of quadratic forms.
- In Section 302.4 we will discuss several further results obtained using the Thom-Pontryagin approach.
- In Section 302.6 we will provide exercises for Chapter 302.

In the next Chapter 303 we will summarize our results on homotopy groups of spheres.

302.1. **The calculation of the framed cobordism groups $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$.** In the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12 (2) (and alternatively in the Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25) we saw that $\pi_3(S^2) \cong \mathbb{Z}$ and we saw that the Hopf map represents a generator of $\pi_3(S^2)$. Furthermore, by the Freudenthal Suspension Theorem 301.12 we know that for every $n \geq 3$ the suspension homomorphism $\pi_3(S^2) \to \pi_{n+1}(S^n)$ is an epimorphism. Thus we know that $\pi_{n+1}(S^n)$ is isomorphic to a quotient of $\mathbb{Z}$ and that $\pi_{n+1}(S^n)$ is generated by the $(n-2)$-fold suspension of the Hopf map. The goal of this section is to prove the following theorem which determines this quotient.

Theorem 302.1. ($\pi_{n+1}(S^n)$–Theorem) For every $n \geq 3$ there exists an isomorphism $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark.

(1) In the Steenrod Operations-π_n-Theorem 236.19 we already saw that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \neq 0$. The proof of the $\pi_{n+1}(S^n)$–Theorem 302.1 below is independent of the Steenrod Operations-π_n-Theorem 236.19. In particular we will not make use of Steenrod operations.

(2) By the Thom-Pontryagin Theorem 300.19 and the Framed-Thickened Cobordism Proposition 301.8 the theorem is equivalent to showing that for every $n \geq 3$ we have an isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}/2\mathbb{Z}$. This is exactly what we will do below. The approach

that we take below to calculate $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ differs, to the best of my knowledge, from the other proofs in the literature. We refer to [**Pon1959**, Chapter 14] for Pontryagin's original proof, which makes for challenging reading for the modern reader. A third, quite readable approach is presented in [**Put**].

(3) There are various alternative proofs of the $\pi_{n+1}(S^n)$–Theorem 302.1. The first proof was given by Hans Freudenthal [**Freu1937**, p. 301] in 1937. Another proof is given in [**Hat**, p. 549] using "spectral sequences".

(4) The discussion preceding the theorem shows that the suspension of the Hopf map defines a non-trivial element in $\pi_4(S^3)$. This gives in particular an affirmative answer to Question 134.28. □

As we had just mentioned above, proving the $\pi_{n+1}(S^n)$–Theorem 302.1 is equivalent to showing that for every $n \geq 3$ there exists an isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}/2\mathbb{Z}$. We will perform this calculation in the remainder of this section.

We start out our calculation of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$ with the following definition:

Definition 302.2.

(1) We refer to $C := S^1$ as the standard circle in $\mathbb{R}^2$.

(2) We refer to the framing of $C \subseteq \mathbb{R}^2$ that is given by

$$\begin{array}{ccc} t\colon C & \to & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

as the trivial framing t of C.

(3) Let $n \in \mathbb{N}_{\geq 2}$. We adopt the Definition 301.10 to define the standard standard circle C_n in $\mathbb{R}^n$. Furthermore, using the Definition 301.10 we define the trivial framing t_n for C_n.

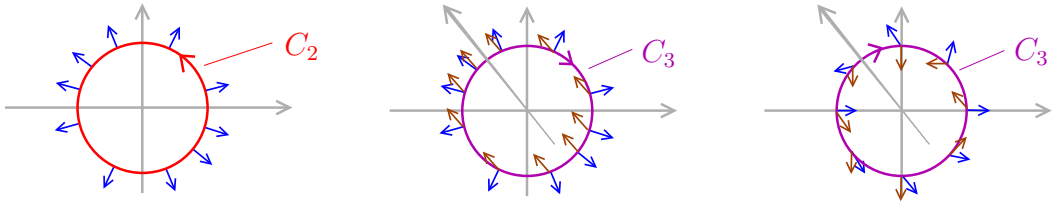
(4) We refer to the framing of $\mathrm{C}_3 \subseteq \mathbb{R}^3$ that is given by the map[3337]

$$\begin{array}{ccc} h\colon \mathrm{C}_3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} & \mapsto & t_3 \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & 0 \\ \sin(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \end{array}$$

↑ don't overlook the t_3

as the Hopf framing h of C_3.

(5) Let $n \in \mathbb{N}_{\geq 3}$. As in (2) we use the Definition 301.10 to define the Hopf framing h_n of C_n. These definitions are illustrated in the figure below.



trivial framing t_2 trivial framing t_3 Hopf framing h_3

[3337] We leave the task of showing that this map is smooth to the ever-ready reader.

Throughout this section we also use the following notation:

Notation 302.3.

(1) Given $g \in \mathbb{N}_0$ we denote by $\Sigma_{g,1}$ the surface of genus g minus one open disk. We pick once and for all an orientation for $\Sigma_{g,1}$.

(2) For any $n \in \mathbb{N}_{\geq 2}$ we denote by

$$\begin{array}{rcl} \rho\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, x_2, x_3, \ldots, x_n) & \mapsto & (x_1, -x_2, x_3, \ldots, x_n) \end{array}$$

the reflection in the $(x_2 = 0)$-hyperplane.

(3) Given $n \in \mathbb{N}_0$ we make the identification $\mathbb{R}^n = \{0\} \times \mathbb{R}^n \subseteq [0,1] \times \mathbb{R}^n$.

The following two lemmas justify the names "trivial framing" and "Hopf framing".

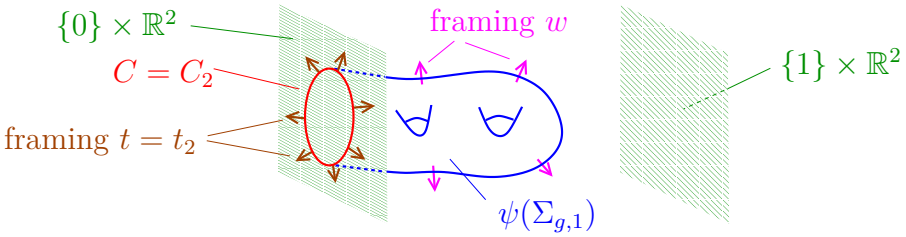
Lemma 302.4. (Trivial Framing Lemma)

(1) Given any $g \in \mathbb{N}_0$ there exists a proper smooth embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ and a framing w for $\psi(\Sigma_{g,1})$ such that $\partial\psi(\Sigma_{g,1}) = \mathrm{C}_2$ and such that the restriction of w to C_2 equals t.

(2) For every $n \geq 2$ we have $[(C_n, t_n)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

Sketch of proof.

(1) By now the seasoned reader should be fairly convinced that there exists a proper smooth embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ with $\psi(\partial\Sigma_{g,1}) = \mathrm{C}_2$ and such that the smooth embedding is a "product near the boundary". More precisely, such that for some $\epsilon \in (0,1)$ we have the equality $\psi(\Sigma_{g,1}) \cap ([0,\epsilon] \times \mathbb{R}^2) = [0,\epsilon] \times \mathrm{C}_2$. Since $\psi(\Sigma_{g,1})$ is an oriented connected submanifold of $[0,1] \times \mathbb{R}^2$ of codimension two we see that $\psi(\Sigma_{g,1})$ has precisely two normal fields of length one.



It follows almost immediately from the fact that $\psi(\Sigma_{g,1})$ is a product near $\{0\} \times \mathbb{R}^2$ that one of the above two normal vectors fields agrees with t on $\{0\} \times S^1$.

(2) It follows from (1) that $[(\mathrm{C}_2, t_2)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^2)$. From the Framed Suspension Lemma 301.11 we obtain that $[(C_n, t_n)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$ for any $n \in \mathbb{N}_{\geq 2}$. ■

Lemma 302.5. (Hopf Framing Lemma)

(1) Under the isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^3) \cong \pi_3(S^2, *)$ the element $[(\mathrm{C}_3, h_3)]$ corresponds, up to a sign, to the element represented by the Hopf map $S^3 \to S^2$.

(2) Let $n \geq 4$. Under the isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \cong \pi_n(S^{n-1}, *)$ the element $[(C_n, h_n)]$ corresponds, up to a sign, to the element represented by the $(n-3)$-fold suspension of the Hopf map $S^3 \to S^2$.

Proof. Statement (1) follows easily from the Trivial Knot–Thom Pontryagin Map Lemma 300.9. Statement (2) is a straightforward consequence of the Framed Suspension Lemma 301.11 (2). ■

The following proposition is the first key ingredient in the calculation of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$ for $n \geq 4$.

Proposition 302.6. (Epimorphism-onto-$\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition) Let $n \in \mathbb{N}_{\geq 4}$. The map

$$\begin{array}{rcl} \mathbb{Z}/2\mathbb{Z} & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is well-defined and it is an epimorphism.

Our proof of the Epimorphism-onto-$\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition 302.6 requires one definition and two lemmas.

Definition 302.7. Let $n \in \mathbb{N}_{\geq 2}$ and let $\gamma\colon S^1 \to \mathbb{R}^n$ be a smooth embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$ and let $\alpha\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be a smooth map. We define

$$\begin{array}{rcl} f[\alpha]\colon \gamma(S^1) & \to & \mathrm{GL}(n, n-1) \\ \gamma(z) & \mapsto & \underbrace{f(\gamma(z))}_{\in \mathrm{GL}(n,n-1)} \cdot \underbrace{\alpha(z)}_{\in \mathrm{GL}(n-1,\mathbb{R})}. \end{array}$$

It follows easily from the fact that α takes values in $\mathrm{GL}_+(n-1, \mathbb{R})$ that $f[\alpha]$ is again a framing of $\gamma(S^1)$.

The first lemma that goes into the proof of the Epimorphism-onto-$\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition 302.6 is really just a crucial example masquerading as a lemma.

Lemma 302.8. (Act-on-Framing Lemma) Let $n \in \mathbb{N}_{\geq 3}$ and let $\gamma\colon S^1 \to C_n$ be the obvious diffeomorphism. We consider the loop

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1, \mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

As above let t_n be the trivial framing for C_n and let h_n be the Hopf framing for C_n. We have the following two equalities of framings of C_n:

(1) We have $t_n[\alpha] = h_n$.
(2) We have $t_n[\overline{\alpha}] = \rho_*(h_n)$.[3338]

Proof. Statement (1) follows immediately from the definitions. We turn to the proof of Statement (2). As we will see, the proof of Statement (2) is given by a slightly tedious calculation. To simplify the notation we only deal with the case $n = 3$.

[3338]Here $\overline{\alpha}$ is the "inverse of the loop α", i.e. it is the loop given by $\overline{\alpha}(\exp(\mathrm{i}\varphi)) = \alpha(\exp(-\mathrm{i}\varphi))$. As above $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in the $(x_2 = 0)$-hyperplane. Finally the framing $\rho_*(h_n)$ is defined as in Definition 301.2.

At a given point $P = \gamma(\exp(\mathrm{i}\varphi)) = \begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix} \in \mathrm{C}_3$ we have

$$\begin{aligned} t_n[\overline{\alpha}]\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix} &= t_n\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix} \cdot \underbrace{\alpha(\exp(-\mathrm{i}\varphi))}_{=\overline{\alpha}(\exp(\mathrm{i}\varphi))} = \begin{pmatrix}\cos(\varphi) & 0\\ \sin(\varphi) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(-\varphi) & -\sin(-\varphi)\\ \sin(-\varphi) & \cos(-\varphi)\end{pmatrix} \\ &= \rho\left(\begin{pmatrix}\cos(-\varphi) & 0\\ \sin(-\varphi) & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\cos(-\varphi) & -\sin(-\varphi)\\ \sin(-\varphi) & \cos(-\varphi)\end{pmatrix}\right) = \rho\left(h_n\begin{pmatrix}\cos(-\varphi)\\ \sin(-\varphi)\\ 0\end{pmatrix}\right) \\ &\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ &\text{since } \rho \text{ is reflection in the second coordinate} \qquad \text{Definition 302.2 of } h_n \\ &\text{together with the symmetries of sin and cos} \\ &= \rho\left(h_n\left(\rho^{-1}\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}\right)\right) = \rho_*(h_n)\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}. \\ &\uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ &\text{definition of } \rho \qquad\qquad\qquad \rho_*(h_n) \text{ from Definition 301.2} \end{aligned}$$

This concludes the mind-numbing verification of Statement (2). ■

Lemma 302.9. (Homotopic-Framing Change Lemma) Let $n \in \mathbb{N}_{\geq 2}$. Let $\gamma\colon S^1 \to \mathbb{R}^n$ be a smooth embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$. Finally let $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be two smooth maps. If α and β are smoothly homotopic, then

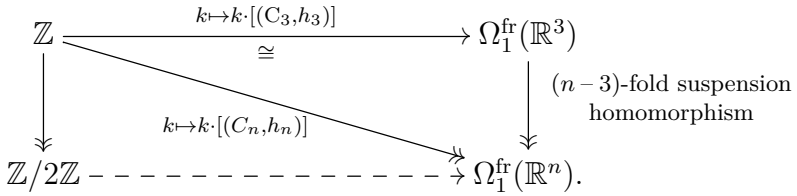
$$[(\gamma(S^1), f[\alpha])] \;=\; [(\gamma(S^1), f[\beta])] \;\in\; \Omega_1^{\mathrm{fr}}(\mathbb{R}^n).$$

Proof of Lemma 302.9. We write $M = \gamma(S^1)$. Since $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ are smoothly homotopic we can pick a smooth homotopy $H\colon S^1 \times [0,1] \to \mathrm{GL}_+(n-1, \mathbb{R})$ with $H_0 = \alpha$ and $H_1 = \beta$. We consider the product cobordism $[0,1] \times M$ equipped with the framing

$$\begin{aligned} V\colon [0,1] \times M &\to \mathrm{GL}(n+1, n) \\ (t, \gamma(z)) &\mapsto \begin{pmatrix}1 & 0\\ 0 & f(\gamma(z)) \cdot H(z,t)\end{pmatrix} \end{aligned}$$

It is clear that $([0,1] \times M, V)$ is a framed cobordism between $(\gamma(S^1), f[\alpha])$ and $(\gamma(S^1), f[\beta])$. ■

Proof of Proposition 302.6. Let $n \in \mathbb{N}_{\geq 4}$. We consider the following diagram:



We make the following observations:

(1) It follows quite easily from the Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25, the Hopf Framing Lemma 302.5 and the Framed-Thickened Cobordism Proposition 301.8 that the top horizontal map is an isomorphism.

(2) The right vertical map is given by the $(n-3)$-fold application of the suspension homomorphism from Definition 301.10. Note that the right vertical map is an epimorphism by the Framed Freudenthal Suspension Theorem 301.13.
(3) The top right triangle commutes by definition of the suspension homomorphism and by definition of (C_n, h_n).

The above diagram and the above discussion show that it remains to prove the following claim:

Claim. The element $2 \cdot [(C_n, h_n)]$ is trivial in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

Proof. We consider the loop

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1,\mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

We have the following equalities in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$:

$$2 \cdot [(C_n, h_n)] \underset{\uparrow}{=} [(C_n, h_n)] + [(C_n, t_n[\alpha])] \underset{\uparrow}{=} [(C_n, h_n)] + [(C_n, t_n[\overline{\alpha}])]$$

by the Act-on-Framing Lemma 302.8 (1)

since $n \geq 4$ we know by the Matrix Group-π_1-Proposition 299.12 (1) that $\pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$, this implies that $[\alpha] \cdot [\alpha] = 0 \in \pi_1(\mathrm{GL}_+(n-1,\mathbb{R}))$, this implies by the π_1–Group Proposition 88.12 that α is path-homotopic to $\overline{\alpha}$, in fact, by the Whitney Approximation Theorem 36.1 we see that α is actually smoothly homotopic to $\overline{\alpha}$, by the Homotopic-Framing Change Lemma 302.9 this gives us the equality $[(C_n, t_n[\alpha])] = [(C_n, t_n[\overline{\alpha}])]$

$$\underset{\uparrow}{=} [(C_n, h_n)] + [(\rho_*(\mathrm{C}_n), \rho_*(h_n))] \underset{\uparrow}{=} [(C_n, h_n)] + (-[(C_n, h_n)]) = 0.$$

by the Act-on-Framing Lemma 302.8 (2)

by the Framed Cobordism Group Proposition 301.6

■

Let $n \geq 4$. Note that by the Epimorphism-onto-$\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition 302.6 we now know that there exists an epimorphism $\mathbb{Z}/2\mathbb{Z} \to \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \cong \pi_n(S^{n-1})$. In the Steenrod Operations-π_n-Theorem 236.19 we showed that $\pi_{n+1}(S^n)$ is non-trivial. Therefore it follows that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$, which means we have proved the $\pi_{n+1}(S^n)$–Theorem 302.1. The readers who studied Steenrod operations *and* who happen to be tired of Thom-Pontryagin theory can thus move on to next section.

For all others we provide a proof of the $\pi_{n+1}(S^n)$–Theorem 302.1 that does not rely on the Steenrod Operations-π_n-Theorem 236.19. More precisely, we prove the following proposition which is the main technical result of this section.

Proposition 302.10. ($\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition) For any $n \in \mathbb{N}_{\geq 4}$ the map

$$\begin{array}{rcl} \mathbb{Z}/2\mathbb{Z} & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is an isomorphism.

As we had mentioned many times, the $\pi_{n+1}(S^n)$–Theorem 302.1 is an immediate consequence of the Thom-Pontryagin Theorem 300.19 and the Framed-Thickened Cobordism Proposition 301.8 together with the $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition 302.10.

Proof. Let $n \in \mathbb{N}_{\geq 4}$. By the Epimorphism-onto-$\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition 302.6 it remains to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

We start out with a more general discussion. Given any $m \in \mathbb{N}_{\geq 2}$ we make the identification $\mathrm{C}_m = \{0\} \times \mathrm{C}_m \subseteq [0,1] \times \mathbb{R}^m$. The following claim applies to all dimensions.

Claim. Let $m \in \mathbb{N}_{\geq 2}$. Let F be a compact orientable connected proper 2-dimensional smooth manifold in $[0,1] \times \mathbb{R}^m$ with $\partial F = \mathrm{C}_m$. At most one of the two framings t_m and h_m can be extended to a framing of F.

Proof. Suppose that there exist framings T and H on F with $T|_{\mathrm{C}_m} = t_m$ and $H|_{\mathrm{C}_m} = h_m$. It follows from the Orthonormal Framing Cobordism Lemma 301.7 that we can assume that T and H are orthonormal. We consider the map[3339]

$$\begin{array}{rcl} \Psi\colon F & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ P & \mapsto & \text{the unique matrix } \Psi(P) \in \mathrm{GL}(m-1,\mathbb{R}) \text{ with } T(P)\cdot\Psi(P) = H(P). \end{array}$$

We denote by $i\colon \mathrm{C}_m \to F$ the inclusion map and we denote by $\gamma\colon S^1 \to \mathrm{C}_m$ the usual diffeomorphism given by $\gamma(\exp(\mathrm{i}\varphi)) = (\cos(\varphi), \sin(\varphi), 0, \ldots, 0)$. We make the following key observation: it follows immediately from the definitions and the Act-on-Framing Lemma 302.8 that the map $\Psi \circ i \circ \gamma\colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ is precisely, on the nose, given by the map

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{m-3} \end{pmatrix}. \end{array}$$

Now we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(\partial F;\mathbb{Z}) & \xleftarrow{\gamma_*} & \mathrm{H}_1(S^1;\mathbb{Z}) & \xleftarrow[\cong]{\text{Hurewicz isom.}} & \pi_1(S^1,1) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ \mathrm{H}_1(F;\mathbb{Z}) & \xrightarrow{\Psi_*} & \mathrm{H}_1(\mathrm{GL}(m-1,\mathbb{R})) & \xleftarrow[\text{Hurewicz isom.}]{\cong} & \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id}). \end{array}$$

We make the following observations:

(1) The left-hand square commutes by the above equality $\alpha = \Psi \circ i \circ \gamma$.
(2) The two horizontal maps to the right are the Hurewicz homomorphisms that we introduced in Definition 166.5. The Hurewicz homomorphisms are isomorphisms by the Hurewicz Theorem 166.10 together with the fact that the fundamental groups on the right are both abelian.
(3) The right-hand square commutes by the naturality of the Hurewicz homomorphisms, see the Hurewicz Homomorphism Proposition 166.3 (3).
(4) By the Matrix Group-π_1-Proposition 299.12 (1) and (2) we know that the above loop $\alpha\colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ represents a non-trivial element in $\pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$. This implies that the right vertical map $\alpha_*\colon \pi_1(S^1,1) \to \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$ is non-trivial.
(5) It follows from (1) – (4) that the left vertical map is also non-trivial. But this contradicts the Surface-with-One Boundary Component-H_*-Lemma 155.23 (1). ⊞

[3339] Let $P \in F$. The fact that T and H are orthonormal means that the columns of $T(P)$ and $H(P)$ are bases for the same vector space, namely $(\mathrm{T}_P F)^\perp$. Therefore there exists a matrix $\Psi(P) \in \mathrm{GL}(m-1,\mathbb{R})$ such that $T(P)\cdot\Psi(P) = H(P)$. We leave it to the reader to verify that the map $P \mapsto \Psi(P)$ is continuous.

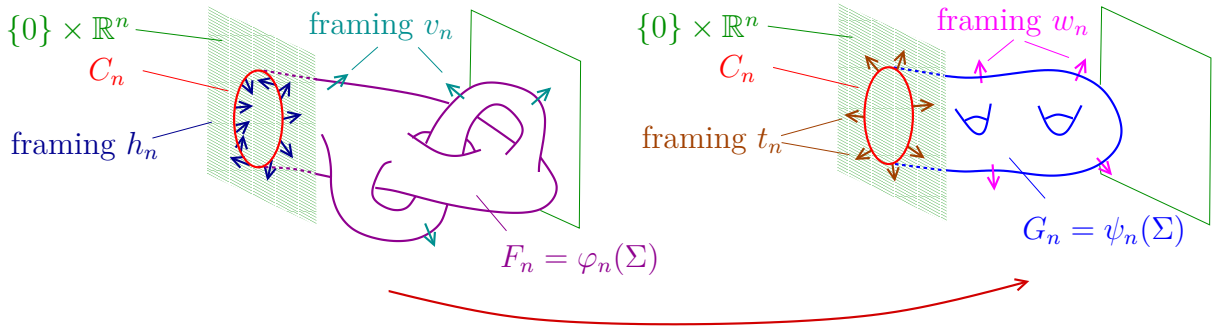
Now we start out with the actual proof of the theorem. Thus let $n \in \mathbb{N}_{\geq 4}$. As we pointed out above, we need to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. Suppose that (C_n, h_n) represents in fact the trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. This means that there exists a framed cobordism $(F, v) = (F_n, v_n)$ in $[0,1] \times \mathbb{R}^n$ from (C_n, h_n) to the empty smooth manifold. In other words, $F = F_n$ is a compact orientable proper 2-dimensional submanifold of $[0,1] \times \mathbb{R}^n$ with $\partial F_n = C_n$ and $v = v_n$ is a framing with $v|_{\mathrm{C}_n} = h_n$. By throwing away all closed components of F_n we can assume that F_n is connected.

By the Surface Classification Theorem 101.16 we know that there exists a $g \in \mathbb{N}_0$ such that F is diffeomorphic to $\Sigma \coloneqq \Sigma_{g,1}$. In other words, there exists a proper smooth embedding $\varphi\colon \Sigma \to [0,1] \times \mathbb{R}^n$ with $\varphi(\Sigma) = F_n$. Given any $m \geq n$ we denote by $\varphi_m\colon \Sigma \to [0,1] \times \mathbb{R}^m$ the smooth embedding that is given by the composition of the smooth embedding φ with the obvious smooth embedding $[0,1] \times \mathbb{R}^n \to [0,1] \times \mathbb{R}^m$.

By the Trivial Framing Lemma 302.4 (1) we know that there exists a proper smooth embedding $\psi\colon \Sigma \to [0,1] \times \mathbb{R}^2$ and a framing $w_2 = w$ for $G_2 = G \coloneqq \psi(\Sigma)$ such that $\partial\psi(\Sigma) = \mathrm{C}_2$ and such that the restriction of $w_2 = w$ to C_2 equals t_2. Given $m \geq 2$ we denote by $\psi_m\colon \Sigma \to [0,1] \times \mathbb{R}^m$ the smooth embedding that is given by the composition of the smooth embedding ψ with the obvious smooth embedding $[0,1] \times \mathbb{R}^2 \to [0,1] \times \mathbb{R}^m$.

Now we consider the two proper smooth embeddings $\psi_n, \varphi_n\colon \Sigma \to [0,1] \times \mathbb{R}^n$. We can arrange[3340] that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

It follows from Proposition **??** (2) and the Isotopy Extension Theorem 44.2 that there exists an $m \geq n$ and a diffeomorphism Ψ of the smooth manifold $[0,1] \times \mathbb{R}^m$ such that $\Psi(F_m) = G_m$ and such that the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity. We refer to the figure below for an illustration.



Since (F_m, v_m) is an extension of (C_m, h_m) and since the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity we see that $(\Psi(F_m), \Psi_*(v_m)) = (G_m, \Psi_*(v_m))$ is an extension of (C_m, h_m). On

[3340]This sentence contains a slight technical nuisance which we cleverly hide in a footnote. First of all, it should be pretty clear by now that Σ admits an orientation-reversing self-diffeomorphism. Thus, after possibly precomposing φ_n with such a diffeomorphism we can assume that $\psi_n^{-1} \circ \varphi_n$ is orientation-preserving. Next note that it follows from the Diffeotopies-of-S^1-Proposition 42.8 and the Diffeomorphism Group-of-S^1-Corollary 42.15 that if C and D are two oriented smooth manifolds that are diffeomorphic to S^1, then any two orientation-preserving diffeomorphisms $C \to D$ are diffeotopic. In our context this implies that, using the Smooth Collar Neighborhood Theorem 35.6, we can then arrange, after possibly precomposing φ_n by another diffeomorphism, that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

the other hand we saw (G_m, w_m) is an extension of (C_m, t_m). Thus both h_m and t_m can be extended over G_m. But this contradicts the above claim. ■

In summary we have the following proposition:

Proposition 302.11.

(1) The map

$$\begin{array}{ccccc} & & \text{given by the Framed-Thickened} & & \\ & & \text{Cobordism Proposition 301.8} & & \\ & & \downarrow & & \\ \Omega_1^{\mathrm{fr}}(\mathbb{R}^3) & \xleftarrow{\cong} & \Omega_1^{\mathrm{th}}(\mathbb{R}^3) & \to & \mathbb{Z} \\ & & [(C,g)] & \mapsto & \text{self-linking number } \mathrm{slk}(C,g) \end{array}$$

is an isomorphism.

(2) For $n \geq 4$ the map

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is an isomorphism.

For $n \in \mathbb{N}_{\geq 3}$ we denote by

$$\chi\colon \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \to \mathbb{Z}/2\mathbb{Z}$$

the corresponding epimorphism.

(3) Given any $n \geq 3$ the following diagram commutes:

$$\begin{array}{ccc} \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) & \xrightarrow{\chi} & \\ \downarrow & & \mathbb{Z}/2\mathbb{Z}. \\ \Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) & \xrightarrow{\chi} & \end{array}$$

Proof.

(1) This statement is the content of Cobordism Group–$\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$–Theorem 300.25.

(2) This statement is precisely the content of the $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$-Proposition 302.10.

(3) This statement follows immediately from the Framed Freudenthal Suspension Theorem 301.13. ■

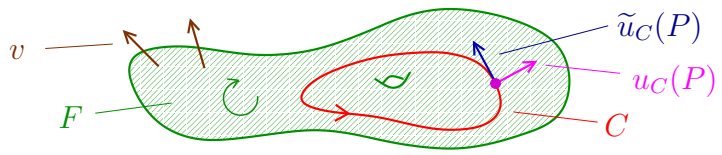
302.2. The calculation of the framed cobordism groups $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n \geq 3$ I.

++++++++ can we also do $\pi_4(S^2)$??

Definition 302.12. Let $n \in \mathbb{N}$. Let $(F, v\colon F \to \mathrm{GL}(n+2, n))$ be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^{n+2}$ and let C be a closed oriented 1-dimensional submanifold of F. Given $P \in C$ we write

$$\begin{array}{rcl} \widetilde{u}_C(P) & \coloneqq & \text{the unique positive tangent vector of } \widetilde{\mathrm{T}}_P C \text{ of length 1,} \\ u_C(P) & \coloneqq & \text{the unique vector in } (\widetilde{\mathrm{T}}_P C)^{\perp} \subseteq \widetilde{\mathrm{T}}_P F \text{ of length 1 that has the property} \\ & & \text{that } (\widetilde{u}_C(P), u_C(P)) \text{ form a positive basis for } \widetilde{\mathrm{T}}_P F. \end{array}$$

Note that (u_C, v) defines a framing for $C \subseteq \mathbb{R}^{n+2}$, in particular $(C, (u_C, v))$ defines an element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+2})$.



Lemma 302.13. Let C and D be two closed oriented 1-dimensional submanifolds of a closed oriented framed 2-dimensional submanifold (F,v) of $\mathbb{R}^{n+2}$. If $[C]=[D]\in \mathrm{H}_1(F;\mathbb{Z})$, then $\chi([(C,(u_C,v))])=\chi([(D,(u_D,v))])$.

Proof. Note that it follows from our hypothesis that $[C]=[D]\in \mathrm{H}_1(F;\mathbb{Z})$ together with the Cobounding Submanifolds Corollary 234.12 that there exists a compact oriented proper 2-dimensional submanifold $W\subseteq[0,1]\times F$ such that $\partial W=(\{0\}\times C)\cup(\{1\}\times -D)$. Similar to the above we consider for a point $P\in W$ the vector

$$u_W(P) \quad \coloneqq \quad \text{the unique vector in } (\widetilde{\mathrm{T}}_P W)^{\perp}\subseteq \widetilde{\mathrm{T}}_P([0,1]\times F) \text{ of length 1 that has the property that the orientation of } W \text{ at } P \text{ together with } u_2(M) \text{ form a positive basis for } \widetilde{\mathrm{T}}_P F.$$

One can now easily verify that $(W,(u_W,v))$, viewed as a framed submanifold of $[0,1]\times\mathbb{R}^{n+2}$ is a framed cobordism between (C,v_C) and (D,v_D). This implies that $\chi([(C,(u_C,v))])=\chi([(D,(u_D,v))])$. ■

Definition 302.14. It follows from Lemma 302.13 that we obtain a well-defined map

$$\begin{array}{rccc} q_{(F,v)}\colon & \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) & \to & \mathbb{Z}/2\mathbb{Z} \\ & \sigma & \mapsto & \chi([(\underset{\uparrow}{C},(u_C,v))]) \end{array}$$

closed oriented 1-dimensional submanifold with $[C]=\sigma$, this always exists by the Represent-by-1-Manifold Corollary 187.19

Notation 302.15. Let (F,v) be an oriented framed 2-dimensional submanifold of $\mathbb{R}^3$. By the Product Tubular Neighborhood–Existence Theorem 58.4 (3) together with the Tubular Map Lemma 58.3 there exists an orientation-preserving smooth embedding $\Theta\colon F\times[-1,1]\to\mathbb{R}^3$ such that $F(0,x)=x$ for all $x\in F$. Given a submanifold $C\subseteq F$ we write $C_-\coloneqq\Theta(C\times\{-1\})$ and we write $C_+\coloneqq\Theta(C\times\{1\})$.[3341] [3342]

The following lemma gives a fairly explicit approach to calculating $q_{(F,v)}$.

Lemma 302.16. Let (F,v) be an oriented framed 2-dimensional submanifold of $\mathbb{R}^3$. Given any closed oriented 1-dimensional submanifold C of F we have

$$q_{(F,v)}([C]) \quad = \quad \begin{cases} \mathrm{lk}(C,C_+), & \text{if for every } P\in F \text{ the orientation of } \widetilde{\mathrm{T}}_P F \text{ followed by } v(P) \text{ is positive} \\ \mathrm{lk}(C,C_+), & \text{if for every } P\in F \text{ the orientation of } \widetilde{\mathrm{T}}_P F \text{ followed by } v(P) \text{ is negative.} \end{cases}$$

Sketch of proof. Let C be a closed oriented 1-dimensional submanifold C of F. We denote by $\widetilde{u}_C, u_C\colon C\to\mathbb{R}^3$ the vector fields that we introduced on Definition 302.12. Recall that for each $P\in C$ the vectors $\{\widetilde{u}_C(P),u_C(P)\}$ form a positive basis for $\widetilde{\mathrm{T}}_P F$.

First assume that for every $P\in F$ the orientation of $\widetilde{\mathrm{T}}_P F$ followed by $v(P)$ is a positive basis of $\mathbb{R}^3$. In this case we see that

$$q_{(F,v)}([C]) \quad = \quad \chi(C,(u_C,v)) \quad \underset{\uparrow}{=} \quad \mathrm{lk}(C,\text{push of } C \text{ into } v\text{-direction}) \quad \underset{\uparrow}{=} \quad \mathrm{lk}(C,C_+).$$

by the Framed-Thickened Cobordism Proposition 301.8 since the curves are homotopic outside of C

Proposition 300.34 ■

[3341] It follows from the Tubular Map–Uniqueness Theorem 58.5 that the definitions of C_- and C_+ do not really depend on the choice of Θ.

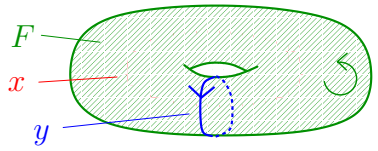
[3342] If C is oriented, then we equip the submanifolds $C_\pm$ with the obvious corresponding orientation.

On might hope that $q_{(F,v)}\colon \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism. In fact Pontryagin [**Pon1938**] initially thought so. But it turns out that the situation is more complicated, namely it turns out that the following proposition holds.

Proposition 302.17. The map $q_{(F,v)}\colon \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ has the property, that for any choice of $x, y \in \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z})$ we have

$$q(x+y) \;=\; q(x) + q(y) + Q_F(x,y)$$

where $Q_F\colon \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ denotes the intersection form that we introduced in Definition 226.33.



Remark. An alternative proof of Proposition 302.17 is given in [**Put**, Lemma 10.2]. □

The proof of Proposition 302.17 makes use of the following lemma, which will also be of independent interest to us.

Lemma 302.18. Let (F,v) be an oriented framed 2-dimensional submanifold of $\mathbb{R}^{n+2}$.

(1) If $G\colon \mathbb{R}^{n+2} \times [0,1] \to \mathbb{R}^{n+2}$ is a diffeotopy with $G_0 = \mathrm{id}$ to a diffeomorphism g, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\;q_{(F,v)}\;} & \\ \Big\downarrow{\scriptstyle g_*} & & \mathbb{Z}/2\mathbb{Z}. \\ \mathrm{H}_1(g(F);\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\;q_{(g(F),g(v))}\;} & \end{array}$$

(2) We denote by $i\colon \mathbb{R}^{n+2} \to \mathbb{R}^{n+3}$ the inclusion. We equip $i(F)$ with the framing $\widetilde{v}$ that we introduced in the definition of the suspension in Definition 301.10. With this notation the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\;q_{(F,v)}\;} & \\ \Big\downarrow{\scriptstyle i_*} & & \mathbb{Z}/2\mathbb{Z}. \\ \mathrm{H}_1(i(F);\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\;q_{(i(F),\widetilde{v})}\;} & \end{array}$$

Proof of Lemma 302.18. Statement (1) is basically obvious. Statement (2) follows easily from Proposition 302.11 (3). ■

Proof. It follows from Lemma 302.18 (2) and ?? that it suffices to consider the case $n = 1$.

Now let $F \subseteq \mathbb{R}^3$ be an oriented framed 2-dimensional submanifold of $\mathbb{R}^3$ and let $x, y \in \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z})$. We pick closed oriented 1-dimensional submanifolds X and Y with $[X] = x$ and $[Y] = y$. Furthermore we pick a closed oriented 1-dimensional submanifold Z with $[Z] = x + y$. We equip X, Y and Z with the corresponding framing. Then

$$q(x+y) \;=\; \mathrm{slk}(Z) \;=\; \mathrm{lk}(Z,Z^+) \;=\; \mathrm{lk}(X,Z^+) + \mathrm{lk}(Y,Z^+) \;=\; \mathrm{lk}(X,X^+) + \mathrm{lk}(X,Y^+) + \mathrm{lk}(Y,X^+) +$$

■

Proposition 302.17 shows that the map $q\colon \mathrm{H}_1(F;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ has a peculiar behavior that we had not yet encountered. In the next, purely algebraic section, we will see that q is a non-degenerate quadratic form over $\mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$ and we will give a complete classification of non-degenerate quadratic forms over $\mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$. This will then allow us to proceed with our actual topological problem. +++++++ need intersection form non-singular +++++++++

302.3. Interlude: Quadratic forms and the Arf invariant.

Definition 302.19. Let $\mathbb{F}$ be a field.

(1) A quadratic form over $\mathbb{F}$ is a pair (V,q) where V is a finite-dimensional $\mathbb{F}$-vector space and $q\colon V \to \mathbb{F}$ is a map which satisfies the following two conditions:
   (a) For any $x \in V$ and any $\lambda \in \mathbb{F}$ we have $q(\lambda \cdot x) = \lambda^2 \cdot q(v)$.
   (b) The map
   $$\begin{array}{rcl} V \times V & \to & \mathbb{F} \\ (x,y) & \mapsto & q(x+y) - q(x) - q(y) \end{array}$$
   is a symmetric form over $\mathbb{F}$ in the sense of Definition 224.2.
   We refer to the symmetric form in (b) as the symmetric form associated to q.

(2) We say that two quadratic forms (V,p) and (W,q) over $\mathbb{F}$ are isometric if there exists an isomorphism $\varphi\colon V \to W$ such that for all $v \in V$ we have $p(v) = q(\varphi(v))$.

Examples.

(1) Let $\mathbb{F}$ be a field, let V be a finite-dimensional vector space and let $\langle\,,\,\rangle\colon V \times V \to \mathbb{F}$ be a form. A straightforward calculation shows that $q(v) \coloneqq \langle v,v\rangle$ defines a quadratic form. Indeed, it is clear, that (a) is satisfied, and for (b) we calculate that for any $x,y \in V$ we have
$$q(x+y) - q(x) - q(y) \;=\; \langle x+y, x+y\rangle - \langle x,x\rangle - \langle y,y\rangle \;=\; 2{\cdot}\langle x,y\rangle.$$
   We refer to q as the quadratic form associated to $\langle\,,\,\rangle$.

(2) If $\mathbb{F}$ is a field of characteristic $\neq 2$, then basically the same calculation as in (1) shows that q is the quadratic form associated to the form $(x,y) \mapsto \frac{1}{2}(q(x+y) - q(x) - q(y))$.

(3) Let (F,v) be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^n$. It follows from Proposition 302.17 and the $\mathbb{F}_2$-Intersection Form Proposition 226.34 that the map $q_{(F,v)}\colon \mathrm{H}_1(F;\mathbb{F}_2) \to \mathbb{F}_2$ that we introduced in Definition 302.14 is a quadratic form over $\mathbb{F}_2$.

The above discussion shows that for fields of characteristic $\neq 2$ the classification of quadratic forms is essentially the same as the classification of forms. In the following we only consider the field $\mathbb{F}_2$.

Definition 302.20. Let (V,q) be a quadratic form over $\mathbb{F}_2$. We say that q is non-degenerate if the associated symmetric form is non-singular in the sense of Definition 224.3.

Example. We write $\mathcal{W} \coloneqq \mathbb{F}_2^2$ and $x \coloneqq (1,0)$ and $y \coloneqq (0,1)$. One can easily verify that
$$\begin{array}{rcl} \langle\,,\,\rangle\colon \mathcal{W} \times \mathcal{W} & \to & \mathbb{F}_2 \\ (v,w) & \mapsto & \langle v,w\rangle = v^{\mathrm{T}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w \end{array}$$

is the unique non-singular symmetric form on $\mathcal{W}$. Next we consider the following two maps:

$$\begin{array}{rcl} \sigma_0\colon \mathcal{W} & \to & \mathbb{F}_2 \\ 0, x, y & \mapsto & 0 \\ x+y & \mapsto & 1 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \sigma_1\colon \mathcal{W} & \to & \mathbb{F}_2 \\ 0 & \mapsto & 0 \\ x, y, x+y & \mapsto & 1. \end{array}$$

One can easily verify by hand that both maps define a quadratic form on $\mathcal{W}$ such that the associated symmetric form is the above form $\langle\,,\,\rangle$. In particular both quadratic forms are non-degenerate. It is also straightforward to verify that these two maps are in fact the only two non-degenerate quadratic forms on $\mathcal{W}$. Finally note that σ_0 and σ_1 are not isometric for the rather silly reason that σ_0 sends three elements of $\mathcal{W}$ to 0, whereas σ_1 only sends one element of $\mathcal{W}$ to 0. □

In the following our goal is to classify non-singular symmetric forms and non-degenerate quadratic forms over $\mathbb{F}_2$. It turns out that symmetric forms of $\mathbb{F}_2$ are not very interesting. In fact the following proposition holds:

Proposition 302.21. Let V be a finite-dimensional vector space over $\mathbb{F}_2$.

(1) Every non-singular $++++$ even $++++$ symmetric form $\langle\,,\,\rangle\colon V\times V\to\mathbb{F}_2$ admits a *symplectic basis*, i.e. there exists a basis $a_1,\dots,a_k,b_1,\dots,b_k$ for V such that the following conditions are satisfied:
   (a) For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$.
   (b) For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,b_j\rangle=\delta_{ij}$.

(2) Any two non non-singular symmetric forms over V are isometric.

Proof.

(1) The proposition follows immediately from the Non-Singular Forms-over-$\mathbb{F}_2$-Proposition 226.37. ■

Now let us turn to the study of non-degenerate quadratic forms over $\mathbb{F}_2$. We already encountered the non-isometric quadratic forms $\sigma_0,\sigma_1\colon V\to\mathbb{F}_2$. We obtain more quadratic forms by taking direct sums. But the following lemma says that this way we will produce some isometric quadratic forms.

Lemma 302.22.

(1) The quadratic forms $\sigma_0\oplus\sigma_0$ and $\sigma_1\oplus\sigma_1$ on $\mathcal{W}\oplus\mathcal{W}$ are isometric.

(2) Let $m,n\in\mathbb{N}_0$. The quadratic form $m\cdot\sigma_0\oplus n\cdot\sigma_1$ is isometric to $(m+n)\cdot\sigma_0$ if n is even, and it is isometric to $(m+n-1)\cdot\sigma_0\oplus\sigma_1$ if n is odd.

(3) Any non-degenerate quadratic form (V,q) over $\mathbb{F}_2$ is isometric to a quadratic form $k\cdot\sigma_0$ or to a quadratic form $k\cdot\sigma_0\oplus\sigma_1$. (Note that we do not claim that this is an exclusive "or".)

Proof.

(1) Let x_1,y_1,x_2,y_2 be the obvious basis of $\mathcal{W}\oplus\mathcal{W}=\mathbb{F}_2^2\oplus\mathbb{F}_2^2$. We consider the isomorphism $\varphi\colon\mathcal{W}\oplus\mathcal{W}\to\mathcal{W}\oplus\mathcal{W}$ that is determined by

$$\begin{array}{rcl} x_1 & \mapsto & x_1+x_2 \\ y_1 & \mapsto & y_1+x_2 \end{array} \qquad \begin{array}{rcl} x_2 & \mapsto & x_2+y_2+x_1+y_1 \\ y_2 & \mapsto & y_2+x_1+y_1. \end{array}$$

It is straightforward, albeit not particularly entertaining, to verify that φ defines an isometry from $(\mathcal{W}\oplus\mathcal{W},\sigma_0\oplus\sigma_0)$ to $(\mathcal{W}\oplus\mathcal{W},\sigma_1\oplus\sigma_1)$.

(2) This statement follows immediately from (1).
(3) Let (V,q) be a quadratic form. We denote by $\langle\,,\,\rangle$ the associated symmetric form. By Proposition 302.21 we know that there exists a symplectic basis $a_1,\dots,a_m,b_1,\dots,b_m$ for $(U,\langle\,,\,\rangle)$. Given $i=1,\dots,m$ we denote by V_i the span of $\{a_i,b_i\}$ and write $q_i \coloneqq q|_{V_i}$. It follows immediately from the fact that the V_i are orthogonal with respect to $\langle\,,\,\rangle$ that (V,q) is isometric to $(V_i,q_1)\oplus\cdots\oplus(V_m,q_m)$. Note that each (V_i,q_i) is a non-degenerate quadratic form over a 2-dimensional vector space. It follows from the above discussion that each (V_i,q_i) is isometric to either $(\mathcal{W},\sigma_0)$ or $(\mathcal{W},\sigma_1)$. The desired statement now follows from (2). ■

If we want to classify quadratic forms over $\mathbb{F}_2$ we will need to figure out whether or not $m\cdot\sigma_0$ and $(m-1)\cdot\sigma_0\oplus\sigma_1$ are isometric. Surprisingly the silly approach that worked for $m=1$, still works in general. More precisely, let us first introduce the following notation:

Notation 302.23. Given a quadratic form (V,q) we set

$$N(q) \;\coloneqq\; \{v\in V\,|\,q(v)=0\} \quad\text{and}\quad P(q) \;\coloneqq\; \{v\in W\,|\,q(v)=1\}.$$

Now we can formulate the following lemma:

Lemma 302.24. Given any $m\in\mathbb{N}$ we have

$$\begin{array}{rcl} \#N(m\cdot\sigma_0) & > & \#P(m\cdot\sigma_0) \quad\text{and}\\ \#N((m-1)\cdot\sigma_0\oplus\sigma_1) & < & \#P((m-1)\cdot\sigma_0\oplus\sigma_1).\end{array}$$

Proof. Given a quadratic form (V,q) over $\mathbb{F}_2$ we set

$$r(q) \;\coloneqq\; \#\{v\in V\,|\,q(v)=0\} \;-\; \#\{v\in V\,|\,q(v)=1\}.$$

The following claim contains the key calculation that we need to complete the proof of the lemma.

Claim. Let (V,q) be a quadratic form.
(1) We have $\#N(\sigma_0\oplus q)=3\cdot\#N(q)+\#P(q)$ and $\#P(\sigma_0\oplus q)=3\cdot\#P(q)+\#N(q)$.
(2) We have $r(\sigma_0\oplus q)=2\cdot r(q)$.

Proof. Note that (2) is an immediate consequence of (1) and the simple observation that $r(q)=\#N(q)-\#P(q)$. Thus it remains to prove (1). Note that given $v\oplus w\in V\oplus\mathcal{W}$ we see that

$$\begin{aligned}(\sigma_0\oplus q)(v\oplus w)=\sigma_0(v)+q(w) \;&=\; \begin{cases} 0, & \text{if } v\in N(\sigma_0) \text{ and } w\in N(q) \text{ or } v\in P(\sigma_0) \text{ and } w\in P(q)\\ 0, & \text{if } v\in N(\sigma_0) \text{ and } u\in P(q) \text{ or } v\in P(\sigma_0) \text{ and } w\in N(q).\end{cases}\\ &=\; \begin{cases} 0, & \text{if } v\in\{0,x,y\} \text{ and } w\in N(q) \text{ or } v=x+y \text{ and } w\in P(q)\\ 0, & \text{if } v\in\{0,x,y\} \text{ and } w\in P(q) \text{ or } v=x+y \text{ and } w\in N(q).\end{cases}\end{aligned}$$

It follows immediately that $P(q\oplus\sigma_0)=3\cdot P(q)+N(q)$ and that $N(q\oplus\sigma_0)=3\cdot N(q)+P(q)$. ⊞

Now iteratively applying the claim we see that

$$\begin{array}{rcl} r(m\cdot\sigma_0) & = & 2\cdot r((m-1)\cdot\sigma_0) \;=\; \dots \;=\; 2^{m-1}\cdot r(\sigma_0) \;=\; 2^{m-1}\cdot 2 \;>\; 0 \quad\text{and}\\ r((m-1)\cdot\sigma_0\oplus\sigma_1) & = & 2\cdot r((m-1)\cdot\sigma_0\oplus\sigma_1) \;=\; \dots \;=\; 2^{m-1}\cdot r(\sigma_1) \;=\; 2^{m-1}\cdot(-2) \;<\; 0.\end{array}$$

This calculation immediately implies the desired result. ■

This leads us to the following definition:

Definition 302.25. Let (V, q) be a quadratic form over $\mathbb{F}_2$. We define its Arf invariant $\mathrm{Arf}(q) \in \mathbb{F}_2$ as follows:

$$\mathrm{Arf}(q) \coloneqq \begin{cases} 0, & \text{if } \#N(q) > \#P(q), \\ 1, & \text{if } \#P(q) > \#N(q). \end{cases}$$

Remark.

(1) The Arf invariant was introduced in 1941 in a slightly different form, which we will explain below, by Cahit Arf [**Arf1941**].[3343]

(2) The Arf invariant is sometimes jokingly referred to as the *democratic invariant*: every element $u \in V$ gets to vote, and the majority decides that the value of the Arf invariant should be. □

The following theorem now gives us the complete classification of non-degenerate quadratic forms over $\mathbb{F}_2$.

Theorem 302.26.

(1) Let (V, q) be a non-degenerate quadratic form. We set $m \coloneqq \dim(V)$.
   (a) (V, q) is isometric to $m \cdot \sigma_0$ if and only if $\mathrm{Arf}(q) = 0$.
   (b) (V, q) is isometric to $(m-1) \cdot \sigma_0 \oplus \sigma_1$ if and only if $\mathrm{Arf}(q) = 1$.

(2) Two non-degenerate quadratic forms (V_1, q_1) and (V_2, q_2) over $\mathbb{F}_2$ are isometric if and only if $\dim(V_1) = \dim(V_2)$ and $\mathrm{Arf}(q_1) = \mathrm{Arf}(q_2)$.

(3) For any two non-degenerate quadratic forms (V_1, q_1) and (V_2, q_2) we have

$$\mathrm{Arf}(q_1 \oplus q_2) \;=\; \mathrm{Arf}(q_1) + \mathrm{Arf}(q_2).$$

Proof.

(1) This statement is an immediate consequence of Lemmas 302.22 and 302.24.

(2) This statement follows immediately from (1).

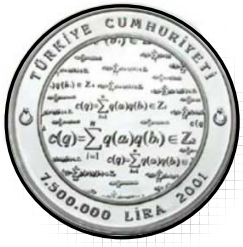
(3) This statement follows easily from (1) and Lemma 302.22 (1). ■

The following proposition gives a neat calculation of the Arf invariant. In the next section we will use this proposition in our calculation of the Arf invariant.

Proposition 302.27. Let (V, q) be a non-degenerate quadratic form over $\mathbb{F}_2$. Given any symplectic basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ of the associated symmetric form we have the following equality:

$$\mathrm{Arf}(q) \;=\; \sum_{i=1}^{n} q(a_i) \cdot q(b_i) \;\in\; \mathbb{F}_2.$$







[3343] To the best of my knowledge Cahit Arf and Alan Turing are the only two mathematicians of the 20th century who made it on a banknote.

Remark.

(1) The reinterpretation of the Arf invariant given in Proposition 302.27 is actually the original definition given by Cahit Arf, see also the figure above.
(2) This reinterpretation of the Arf invariant has the great advantage that it also makes sense for quadratic forms over any field of characteristic 2. In this more general setting it is not entirely trivial to see that one actually obtains a well-defined invariant. We refer to [**Dye1978**], [**Kapy1969**, Theorem 27] and [**CohnP2003**, Theorem 8.11.2] for details. □

Proof. Let (V,q) be a non-degenerate quadratic form over $\mathbb{F}_2$ and let $a_1,\dots,a_n,b_1,\dots,b_n$ be a symplectic basis of the associated symmetric form. As in the proof of Lemma 302.22 we denote by V_i the span of $\{a_i,b_i\}$, we write $q_i := q|_{V_i}$ and we point out that (V,q) is isometric to $(V_1,q_1)\oplus\dots\oplus(V_n,q_n)$.

Note that each (V_i,q_i) is a non-degenerate quadratic form over a 2-dimensional vector space. It follows from the above discussion that each (V_i,q_i) is isometric to either $(\mathcal{W},\sigma_0)$ or $(\mathcal{W},\sigma_1)$. Now we perform the following calculation:

$$\sum_{i=1}^n q(a_i)\cdot q(b_i) \;=\; \sum_{i=1}^n q_i(a_i)\cdot q(b_i) \;=\; \sum_{i=1}^n \begin{cases} 0, & \text{if } q_i \text{ is isometric to } \sigma_0 \\ 1, & \text{if } q_i \text{ is isometric to } \sigma_1 \end{cases} \;=\; \sum_{i=1}^n \mathrm{Arf}(q_i)$$

$$\uparrow \text{ by inspection of the } \sigma_i \qquad\qquad \uparrow \text{ by inspection of the } \sigma_i$$

$$= \mathrm{Arf}(q_1\oplus\dots\oplus q_n) \;=\; \mathrm{Arf}(q).$$

$$\uparrow \text{ by Theorem 302.26 (3)}$$

■

This concludes our quick introduction to the Arf invariant. In the next section we will use the Arf invariant to calculate the framed cobordism groups $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n\geq 3$. Note though that the Arf invariant appears in many other settings of topology, e.g. in knot theory [**Pra2007**, Chapter 6.2] and surgery theory [**Browd1972**].

302.4. **Further results obtained using the Thom-Pontryagin approach.** In this final section we summarize a few more results that have been proved using the Thom-Pontryagin approach. First of all, using the Thom-Pontryagin method Pontryagin [**Pon1950b**] proved in 1950 the following theorem.

Theorem 302.28. ($\pi_{n+2}(S^n)$-Theorem) For any $n\geq 2$ we have $\pi_{n+2}(S^n)\cong\mathbb{Z}/2\mathbb{Z}$.

Remark.

(1) Recall that by the $\pi_n(S^3)=\pi_n(S^2)$–Theorem 239.12 (1) we have for every $n\geq 3$ we have an isomorphism $\pi_n(S^3)\cong\pi_n(S^2)$. Furthermore we know by the $\pi_{n+1}(S^n)$–Theorem 302.1 that for every $n\geq 3$ we have $\pi_{n+1}(S^n)\cong\mathbb{Z}/2\mathbb{Z}$. Putting these two results together shows that $\pi_4(S^2)\cong\mathbb{Z}/2\mathbb{Z}$.
(2) In the Double Hopf Map Theorem 237.2 we used Steenrod operations to show that $\pi_{n+2}(S^n)\neq 0$ for all $n\geq 2$.
(3) By the Freudenthal Suspension Theorem 301.12 we know that the suspension homomorphism $\pi_5(S^3)\to\pi_6(S^4)$ is an epimorphism and we know that for $n>k+1$ the suspension homomorphism $\pi_{n+2}(S^n)\to\pi_{n+3}(S^{n+1})$ if an isomorphism for $n\geq 5$.
(4) Unfortunately (1), (2) and (3) are *not* strong enough to give an alternative proof of the $\pi_{n+2}(S^n)$-Theorem 302.28. □

Proof. Let $n \geq 2$. First note that we have the following isomorphisms and epimorphisms:

$$\begin{array}{ccccccccccc} & & & & \text{by the } \pi_n(S^3)=\pi_n(S^2)\text{–Theorem 239.12 (1)} & & & & & & \\ & & & & \downarrow & & & & & & \\ \mathbb{Z}/2\mathbb{Z} & \cong & \pi_4(S^3) & \cong & \pi_4(S^2) & \twoheadrightarrow & \pi_5(S^3) & \twoheadrightarrow & \pi_6(S^4) & \cong & \pi_{n+2}(S^n). \\ & \uparrow & & & \uparrow & & \uparrow & & \uparrow & \uparrow & \\ & \text{the } \pi_{n+1}(S^n)\text{–Theorem 302.1} & & & \text{by the Freudenthal-Epimorphism Corollary 301.20} & & & & \text{Freudenthal Suspension Theorem 301.12} & & \end{array}$$

Now there are two ways to proceed:

(1) In the Double Hopf Map Theorem 237.2 we used Steenrod operations to show that $\pi_{n+2}(S^n) \neq 0$ for all $n \geq 2$.

Either of these two statements shows that the maps sequence of maps are all necessarily isomorphisms. ■

Remark.

(1) As mentioned before, the theorem was proved by Lev Pontryagin [**Pon1950b, Pon1959**]. See also [**Scor2005**, p. 233] and [**WX2010**, Chapter 2.5] for outlines of the proof. The most readable account of the Thom-Pontryagin approach to proving the theorem is surely [**Put**].

(2) At the same time as Pontryagin two very different proofs that $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$ were given by George W. Whitehead [**WhdG1950**] and Jean-Pierre Serre [**Ser1953**, Proposition 10]. We also refer to [**Hat**, Theorem 1.40 (a)] for a proof using spectral sequences and Steenrod operations. Alternatively see also [**Hu1959**, Theorem XI.15.2].

Finally Rokhlin [**Rok1952, Rok1986a**] used the Thom-Pontryagin approach to prove the following theorem:

Theorem 302.29. ($\pi_{n+3}(S^n)$-Theorem)
(1) We have[3344] $\pi_5(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$.
(2) For any $n \geq 5$ we have $\pi_{n+3}(S^n) \cong \mathbb{Z}/24\mathbb{Z}$.

Remark.

(1) In the Steenrod Operations-π_n-Theorem 236.19 we used Steenrod operations to show the weaker statement that $\pi_{n+3}(S^n) \neq 0$ for all $n \geq 2$.

(2) Note that the $\pi_{n+3}(S^n)$-Theorem 302.29 (1) gives a negative answer to Question 216.24 (2). □

Proof.

(a) The theorem was proved by Vladimir Rokhlin [**Rok1952**][**Rok1986a**, p. 21] in 1952, using the above Thom-Pontryagin approach.[3345]

(b) A somewhat different proof is sketched in [**WX2010**, Chapter 2.6]. Some of the results can also be found in [**Ser1953**, Proposition 10]. Other proofs are given in [**Hat**, Theorem 1.40 (b)] and [**Hu1959**, Chapter XI.16]. ■

[3344] In fact the first isomorphism $\pi_5(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ is a consequence of the $\pi_n(S^3)=\pi_n(S^2)$–Theorem 239.12 and the $\pi_{n+2}(S^n)$-Theorem 302.28.

[3345] In fact Rokhlin [**Rok1986b**] initially thought that for $n \geq 5$ the groups $\pi_{n+3}(S^n)$ are isomorphic to $\mathbb{Z}/12\mathbb{Z}$. The key to fixing the mistake is Rokhlin's Theorem 228.7 regarding signatures of certain 4-dimensional smooth manifolds.

The results of the last two chapters summarize most (or all?) of the calculations of homotopy groups of spheres that have been obtained using the Thom-Pontryagin approach.

302.5. **Ring structure.** the smash product defines a map

$$\begin{array}{rcl} \pi_{m+k}(S^k) \times \pi_{n+l}(S^l) & \to & \pi_{m+n+k+l}(S^{k+l}) \\ [f\colon S^{m+k} \to S^k], [g\colon S^{n+l} \to S^l] & \mapsto & [f \wedge g\colon \underbrace{S^{m+k} \wedge S^{n+l}}_{=_{\mathrm{i}} S^{m+n+k+l}} \to \underbrace{S^k \wedge S^l}_{=_{\mathrm{i}} S^{k+l}}]. \end{array}$$

Note that this turns the stable homotopy groups of spheres into a graded ring. The Thom-Pontryagin construction preserves this ring structure.

302.6. **Exercises for Chapter 302.**

Exercise 302.1. Let $n \in \mathbb{N}_{\geq 2}$. We consider the loops

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n,\mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{\alpha}\colon S^1 & \to & \mathrm{GL}_+(n,\mathbb{R}) \\ z = \exp(\mathrm{i}\varphi) & \mapsto & \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Recall that in the proof of the Homotopic-Framing Change Lemma 302.9 we used the fact, proved in the Matrix Group-π_1-Proposition 299.12 (1), that $\pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$, to show that α is path-homotopic to $\overline{\alpha}$. Give a direct proof that the loop α is path-homotopic to the loop $\overline{\alpha}$.

CHAPTER 303

Summary of homotopy groups of spheres

The results from last chapter conclude our calculations of homotopy groups of spheres. Throughout the course we employed a long and varied list of tools to compute homotopy groups of spheres:

(1) Covering theory, see page 2587.
(2) The Cellular Approximation Theorem 141.2, see the Sphere–Low-π_n-Proposition 134.19.
(3) The Simplicial Approximation Theorem 175.15, see the Sphere-π_n-Countable Corollary 175.17 and Exercise 175.6.
(4) Homology theory and the Hurewicz Theorem 167.9, see the $\pi_n(S^n)$–Proposition 167.10.
(5) The cup product, seethe $\pi_{4n-1}(S^{2n})$–Infinite Theorem 216.23.
(6) Steenrod operations, see the Steenrod Operations-π_n-Theorem 236.19 and the Double Hopf Map Theorem 237.2.
(7) The long exact sequence of a fibration, see the $\pi_n(S^3)=\pi_n(S^2)$–Theorem 239.12.
(8) The Thom-Pontryagin Theory, see the Freudenthal Suspension Theorem 301.12 and the $\pi_{n+1}(S^n)$–Theorem 302.1.

In the following we summarize the results we have obtained on homotopy groups of spheres:

(1) On page 2587 we saw that $\pi_n(S^1) = 0$ for $n \geq 2$.
(2) The Sphere–Low-π_n-Proposition 134.19, or alternatively Exercise 175.6, says that for any $n \in \mathbb{N}$ we have $\pi_1(S^n) = \cdots = \pi_{n-1}(S^n) = 0$.
(3) In the Sphere-π_n-Countable Corollary 175.17 we showed that the homotopy groups of spheres are countable.
(4) By the $\pi_n(S^n)$–Proposition 167.10 (or alternatively by the $\pi_n(S^n)$-via-Thom-Pontryagin Corollary 300.20) we know that for any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$.
(5) By the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12 (1) we know that for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. In particular, together with (4) we know that $\pi_3(S^2) \cong \mathbb{Z}$. An alternative proof of the fact that $\pi_3(S^2) \cong \mathbb{Z}$ is given by the combination of the Thom-Pontryagin Theorem 300.19 and the Cobordism Group–$\Omega_1^{\text{th}}(\mathbb{R}^3)$–Theorem 300.25.
(6) In the $\pi_{4n-1}(S^{2n})$–Infinite Theorem 216.23, and alternatively in the $\pi_{4n-1}(S^{2n})$–Infinite Proposition 300.38 we saw that given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \twoheadrightarrow \mathbb{Z}$.
(7) By the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$. Furthermore, by the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.

(8) By the Freudenthal Suspension Theorem 301.12 we know that for $n = k + 1$ we have an epimorphism $\pi_{n+k}(S^n) \twoheadrightarrow \pi_{n+k+1}(S^{n+1})$ and we know that for $n > k + 1$ we have an isomorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$.
(9) By the Hopf Fiber Bundle–Homotopy Group Theorem 298.11 (2) we know that $\pi_6(S^4) \cong \pi_5(S^3)$ and by the Octonionic Hopf Fiber Bundle–Homotopy Group Theorem 298.14 (3) we know that $\pi_{14}(S^8) \cong \pi_{13}(S^7)$.
(10) In the Steenrod Operations-π_n-Theorem 236.19 we proved the following three statements:
  (a) for any $n \geq 2$ the group $\pi_{n+1}(S^n)$ is non-trivial,
  (b) for any $n \geq 4$ the group $\pi_{n+3}(S^n)$ is non-trivial,
  (c) for any $n \geq 8$ the group $\pi_{n+7}(S^n)$ is non-trivial.
(11) By the $\pi_{n+1}(S^n)$–Theorem 302.1 we know that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. This is a refinement of statement (10a). Note that it follows from this fact together with (5) that $\pi_4(S^2) \cong \mathbb{Z}_2$.
(12) In the Double Hopf Map Theorem 237.2 we proved the following three statements:
  (a) for any $n \geq 2$ the group $\pi_{n+2}(S^n)$ is non-trivial,
  (b) for any $n \geq 4$ the group $\pi_{n+6}(S^n)$ is non-trivial,
  (c) for any $n \geq 8$ the group $\pi_{n+14}(S^n)$ is non-trivial.

We recall that Serre's Theorem 169.11 says that all homotopy groups of spheres are finitely generated abelian groups. In the discussion above we saw that some of the homotopy groups of spheres are infinite.

The following theorem can be summarized as saying that we have picked up all elements of infinite order. In particular the theorem gives a complete answer to Question 216.25. This theorem can also be viewed as a strengthening of Serre's Theorem 169.11. In fact, the strengthening is also due to Jean-Pierre Serre.

Theorem 303.1. **($\pi_*(S^n)$-Finiteness Theorem)**
(1) If n is odd, then $\pi_k(S^n)$ is finite for $k \geq n + 1$.
(2) If n is even, then the following two statements hold:
  (a) The group $\pi_k(S^n)$ is finite for $k \geq n + 1$ and $k \neq 2n - 1$.
  (b) The group $\pi_{2n-1}(S^n)$ is isomorphic to the direct sum of $\mathbb{Z}$ with a finite group.

Proof. The theorem was first proved by Serre [**Ser1953**] using "spectral sequences". Expositions of the proof are also given in [**Hat**, Theorem 1.21] and [**Spa1995**, p. 515f]. ■

The following table, obtained from [**Tod1962**, Chapter XIV], shows the low-dimensional homotopy groups of low-dimensional spheres. The entries colored in blue have been proved

in these lecture notes.[3346]

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$

Note that we showed in the $\pi_n(S^3)=\pi_n(S^2)$–Theorem 239.12 (1) that the second and the third row, starting with π_3, are the same.

Recall that the Freudenthal Suspension Theorem 301.12 says that for any $n,k\in\mathbb{N}_0$ with $n>k+1$ the suspension homomorphism $\pi_{n+k}(S^n)\to\pi_{n+k+1}(S^{n+1})$ is an isomorphism. Thus we see that the groups $\pi_{n+k}(S^n)$ "stabilize" as $n\to\infty$. It is therefore perhaps more instructive to shift in the above diagram the i-th row leftward by $i-1$ terms. This way the diagonal lines become vertical. In other words we consider the following diagram where π_{n+k} is short for $\pi_{n+k}(S^n)$.

n	π_{n+0}	π_{n+1}	π_{n+2}	π_{n+3}	π_{n+4}	π_{n+5}	π_{n+6}	π_{n+7}	π_{n+8}	π_{n+9}	π_{n+10}
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$
4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$
5	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
6	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_9$
7	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_2$
8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_8^2\times\mathbb{Z}_2\times\mathbb{Z}_3^2$
9	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_6$
10	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2^3$	$\mathbb{Z}_4\times\mathbb{Z}_6$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}_3$

Here for convenience we colored the "diagonal" $\pi_n(S^{2n})$ in red, since the Freudenthal Suspension Theorem 301.12 implies that below the diagonal the groups on each column are the same.

We conclude this discussion of the precise homotopy groups of spheres with the following sobering fact:

Fact 303.2. There is no sphere S^n with $n\geq 2$ for which we[3347] know the isomorphism types of all homotopy groups.

As we had mentioned above, the Freudenthal Suspension Theorem 301.12 says that homotopy groups of spheres "stabilize". This leads us to the following definition:

[3346] To save space we use the shorthand notation $\mathbb{Z}_n:=\mathbb{Z}/n\mathbb{Z}$.

[3347] Here "we" is defined as members of our species.

Definition 303.3. Let $k \in \mathbb{N}_0$. We refer to[3348]

$$\pi_k^s \coloneqq \operatorname{colim}_{\longrightarrow} \pi_{n+k}(S^n)$$

as the k-th stable homotopy group of spheres.[3349]

Note that it follows from the $\pi_*(S^n)$-Finiteness Theorem 303.1 that for any $k \geq 1$ the k-th stable homotopy group π_k^s is finite. The following table obtained from [**Wiki**] gives the stable homotopy groups up to $k = 63$. It seems like 64 is the first dimension for which we do not know the corresponding stable homotopy groups.

n	0	1	2	3	4	5	6	7
π^s_{0+n}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{8\cdot 3}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16\cdot 3\cdot 5}$
π^s_{8+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_{8\cdot 9\cdot 7}$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{32\cdot 3\cdot 5}$
π^s_{16+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\,\mathbb{Z}_2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3\cdot 11}$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3}\,\mathbb{Z}_{16\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{24+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_{64\cdot 3\cdot 5\cdot 17}$
π^s_{32+n}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_4\,\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 27\cdot 7\cdot 19}$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_{2\cdot 3\cdot 5}$	$\mathbb{Z}_2^5\,\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 25\cdot 11}$
π^s_{40+n}	$\mathbb{Z}_2^5\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_{8\cdot 3\cdot 23}$	$\mathbb{Z}_8$	$\mathbb{Z}_2^3\,\mathbb{Z}_{16\cdot 9\cdot 5}$	$\mathbb{Z}_2^4\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_4\,\mathbb{Z}_3\,\mathbb{Z}_{32\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{48+n}	$\mathbb{Z}_4\,\mathbb{Z}_2^4$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_4\,\mathbb{Z}_2$	$\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 5\cdot 29}$
π^s_{56+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 9\cdot 7\cdot 11\cdot 31}$	$\mathbb{Z}_4$	0	$\mathbb{Z}_2^2\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_{128\cdot 3\cdot 5\cdot 17}$

This concludes our discussion of the homotopy groups of spheres. We refer to [**May1999b, WX2010, ?, Wiki**] and [**Hat2002**, Chapter 4] for a more detailed discussion.

Exercises for Chapter 303.

Exercise 303.1.

(a) Show that $\pi_4(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}$.
(b) Show that $\pi_n(\mathbb{H}\mathrm{P}^\infty)$ is finite for $n \neq 4$.
(c) Show that $\pi_5(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}_2$.

Remark. Use the Hopf-Bundle Proposition 298.10 together with the results of this chapter.

[3348] Given $k \in \mathbb{N}_0$ the groups $\{\pi_{n+k}(S^n)\}_{n\in\mathbb{N}}$ together with the suspension homomorphisms form a direct system. Thus it makes sense to consider the direct colimit.

[3349] Note that for any n, k with $n > k + 1$ the Freudenthal Suspension Theorem 301.12, together with the Direct Colimit–Stabilization Lemma 104.5 and the Cofinal-Isomorphism Lemma 209.13, says that the natural map $\pi_{n+k}(S^n) \to \pi_k^s$ is an isomorphism.

Part XXXIV

CW-complexes and Eilenberg-Maclane Spaces

In this part of the lecture notes we will prove several essential theorems about CW-complexes, and we will use these theorems to construct new and interesting examples of CW-complex with various properties.

This part of the lecture notes is organized as follows:

- In Chapter 304 we will prove the CW-Approximation Theorem 304.14 which says that for every topological space X there exists a CW-complex Y and a weak homotopy equivalence $f\colon X \to Y$, i.e. a continuous map which induces isomorphisms on all homotopy groups.
- In Chapter 305 we will prove the Whitehead Theorem 305.1 which says that any weak homotopy equivalence map between two CW-complexes is actually a homotopy equivalence. The Whitehead Theorem 305.1 has many applications, in particular it allows us to complete the classification of lens spaces up to homotopy equivalence.
- In Chapter 306 we will introduce generalized lens spaces. As the name suggests, these generalize the usual lens spaces $L(p,q)$. They also generalize the odd-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{2d-1}$ and the infinite-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{\infty}$.
- In Chapter 307, given an (abelian) group π and given $n \in \mathbb{N}$ we will introduce the concept of an Eilenberg-Maclane space $\mathrm{K}(\pi, n)$. This is a connected CW-complex together with an isomorphism $\pi_n(K) \to \pi$ and such that all other homotopy groups are zero. For example we will see that the generalized lens spaces from the previous chapter are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}/m\mathbb{Z}, 1)$. We will see that given any (abelian) group π and given any $n \in \mathbb{N}$ an Eilenberg-Maclane Space $\mathrm{K}(\pi, n)$ always exists and we will see, using the Whitehead Theorem 305.1, that they are unique up to homotopy equivalences.

  Eilenberg-Maclane spaces have many applications. In particular we will prove in the Hurewicz-plus-One Theorem 307.34 an extension to the earlier Hurewicz Theorem 167.9, namely we will show that if $n > 1$ and if X is an $(n-1)$-connected topological space, then the Hurewicz homomorphism $\Phi_{(X)}\colon \pi_{n+1}(X, x_0) \to \mathrm{H}_{n+1}(X)$ is an epimorphism.
- In Chapter 308 we see that, given an abelian group G and $n \in \mathbb{N}$, the homotopy classes of continuous maps $X \to \mathrm{K}(G, n)$ form an interesting abelian group.

More aspects of Eilenberg-Maclane spaces will be discussed at some point in the future.

CHAPTER 304

The CW-Approximation Theorem

In the following chapters we will work a lot with CW-complexes. In this chapter we will first prove a hotchpotch of results on CW-complexes. Afterwards we will use these results to prove the CW-Approximation Theorem 304.14 which often allows us to replace a topological space X by a CW-complex that is "weakly homotopy equivalent" to X. The study of "weak homotopy equivalences" will also play an essential role in the next chapter, when we prove the Whitehead Theorem 305.1.

This chapter is organized as follows:

- In Section 304.1 we will learn how to modify CW-complexes.
- In Section 304.2 we will study weak homotopy equivalences and we will state and prove the CW-Approximation Theorem 304.14.
- In Section 304.3 we will provide exercises for Chapter 304.

In the next Chapter 305 we will state and prove the Whitehead Theorem 305.1.

304.1. **Modifying CW-complexes.** As mentioned in the beginning of the chapter, CW-complexes will play a major role in this and the subsequent chapters. Thus, before we start out with proving the enticing theorems mentioned above, we intend to improve our understanding of CW-complexes.

The following, fairly self-evident convention will be in use in the coming chapters.

Convention 304.1.

(1) Given $n \in \mathbb{N}_0$ we write $* = (0, \ldots, 0, 1) \in S^n$. We view S^n as a CW-complex with one 0-cell $\{*\}$ and one n-cell.

(2) We always form any wedge of spheres $\bigvee_{i \in I} S_i^{n_i}$ by identifying the points $*$. By the CW-Wedge Lemma 139.11 we can view the wedge $\bigvee_{i \in I} S_i^{n_i}$ as a CW-complex in the obvious way, i.e. we have one 0-cell $\{*\}$ and for each $i \in I$ we have one n_i-cell.

Before we do anything else we will introduce the following definition which will accompany us in the near future.

Definition 304.2. Let $f\colon X \to Y$ be a continuous map between topological spaces.

(1) We say that the map $f\colon X \to Y$ is 0-connected if $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a surjection, in other words, each path-component of Y contains a point in $f(X)$.

(2) Let $k \in \mathbb{N}$. We say that the map $f\colon X \to Y$ is k-connected if the following two conditions are satisfied:

(a) the map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a bijection,

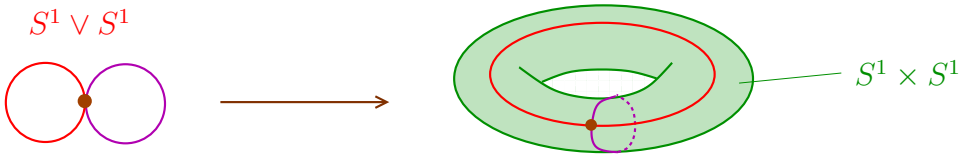
(b) for every $x_0 \in X$ the induced map

$$f_*\colon \pi_i(X, x_0) \to \pi_i(Y, f(x_0)) \text{ is an } \begin{cases} \text{isomorphism for } i = 1, \dots, k-1, \\ \text{epimorphism for } i = k. \end{cases}$$

Remark. Let $f\colon X \to Y$ be a continuous map between path-connected topological spaces and let $k \in \mathbb{N}$. It follows from the Change-of-Base Point Proposition 134.12 that to verify that f is k-connected it suffices to check the Condition (2b) for a single base point $x_0 \in X$. □

Examples.

(1) The map $f\colon S^1 \vee S^1 \to S^1 \times S^1$ that is illustrated in the figure below is easily seen to be 1-connected, but it is not 2-connected.

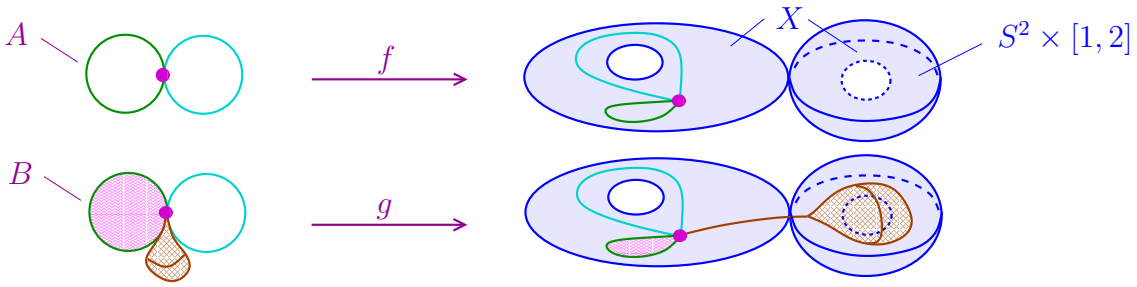


(2) Let X be a CW-complex and let $k \in \mathbb{N}_0$. The CW-Skeleton-π_n Corollary 296.20 says that the inclusion $X^k \to X$ of the k-skeleton[3350] is a k-connected map. In fact, the previous example is of such type if we consider $X = S^1 \times S^1$ with the usual CW-structure, where the 1-skeleton is precisely $S^1 \vee S^1$. □

The following is the main technical result which underpins many of the later results.

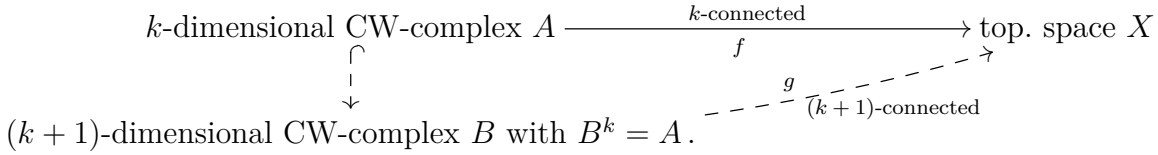
Proposition 304.3. (Higher Connected Map Proposition) Let $k \in \mathbb{N}_0$, let A be a CW-complex and let $f\colon A \to X$ be a continuous map to a topological space. If the map $f\colon A \to X$ is k-connected, then there exists CW-complex B and a continuous map $g\colon B \to X$ such that the following statements hold:

(1) A is a subcomplex of B,
(2) the only extra cells of B are $(k+1)$-dimensional, in particular the k-skeleton of B equals the k-skeleton of A,
(3) the restriction of g to A equals f,
(4) the map $g\colon B \to X$ is $(k+1)$-connected.



[3350]Since we did not work with CW-complexes for a while we remind the reader that for a CW-complex X and $l \in \mathbb{N}_0$ we denote by X^l the l-skeleton i.e. the union of all cells of dimension $\leq l$.

Remark. In most applications of the Higher Connected Map Proposition 304.3 we start out with a k-dimensional CW-complex. In this case we obtain a $(k+1)$-dimensional CW-complex such that k-skeleton equals A and such that we can complete the map $f\colon A \to X$ to the following diagram:



□

Before we turn to the proof of the Higher Connected Map Proposition 304.3 let us first formulate and prove the following corollary:

Corollary 304.4. (Extend-to-k-Connected Map Corollary) Let A be a CW-complex and let $f\colon A \to X$ be a continuous map to a topological space. There exists a CW-complex B and a continuous map $g\colon B \to X$ such that the following statements hold:

(1) A is a subcomplex of B.
(2) The restriction of g to A equals f.
(3) The map $g\colon B \to X$ is k-connected for every $k \in \mathbb{N}_0$.

Proof of Corollary 304.4. Let A be a CW-complex and let $f\colon A \to X$ be a continuous map to a topological space.

Let $\{A_i\}_{i\in I}$ be the family of components of A. By the Component–Subcomplex Lemma 136.14 and the discussion on page 2679 we know that each A_i is a CW-complex and we know that A is the disjoint union (as a CW-complex) of the subcomplexes A_i). Furthermore we know by the CW-Disjoint Union Lemma 139.8 that the disjoint union of a family $\{B_i\}_{i\in I}$ of CW-complexes is again a CW-complex. It follows from these considerations that it suffices to deal with the case that A is 0-connected.

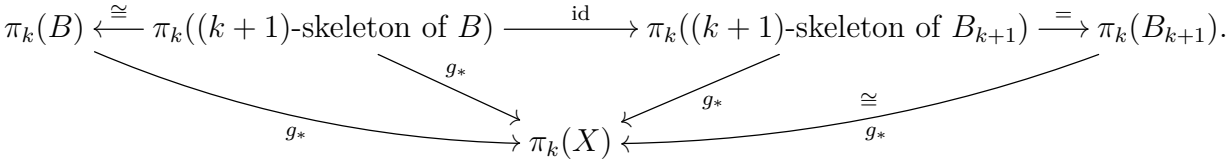
By the Image-Preimage-Component Lemma 4.50 (2) we know that $f(A)$ is contained in a path-component of X. Thus we can also assume that X is 0-connected.

We set $B_0 \coloneqq A$ and $g_0 \coloneqq f$. We apply the Higher Connected Map Proposition 304.3 to B_0 and g_0 and iterate this procedure to obtain a sequence of CW-complexes $\{B_n\}_{n\in\mathbb{N}_0}$ and continuous maps $\{g_n\colon B_n \to X\}_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}$ the following statements hold:

(a) A is a subcomplex of B_n and B_{n-1} is a subcomplex of B_n.
(b) The only cells of B_n that do not lie in B_{n-1} are n-dimensional.
(c) The restriction of g_n to A equals f.
(d) The map $g_n\colon B_n \to X$ is n-connected.

We set $B \coloneqq \operatorname{colim} B_n$ and we equip with the CW-structure given by the Direct Colimit–CW Lemma 139.13. Furthermore we denote by $g\colon B \to X$ the map that is uniquely determined by the property that for each $n \in \mathbb{N}_0$ we have $g|_{B_n} = g_n$. We claim that (B, g) have the desired properties. In fact (1) and (2) are clear by construction. Now let $k \in \mathbb{N}_0$. For a

fixed base point in B_{k+1} we consider the following commutative diagram:



Note that by construction the $(k+1)$-skeleton of B equals the $(k+1)$-skeleton of B_{k+1}, thus the middle map is indeed the identity. It follows from the CW-Skeleton-π_n-Proposition 141.3 (1) that the outer horizontal maps are isomorphisms. Finally note that $g|_{B^{k+1}} = g_{k+1}$, which induces an isomorphism on π_k. Thus the left diagonal map is, as promised, an isomorphism. Keeping in mind the remark on page 5844 we see that the map $g\colon B \to X$ is k-connected for every $k \in \mathbb{N}_0$. ■

Now we turn to the proof of the Higher Connected Map Proposition 304.3. The proof relies on the following analog of the Generating Set–Characterization Lemma 100.3.

Lemma 304.5. (Abelian Generating Set–Characterization Lemma) Let G be an abelian group and let $S \subseteq G$ be a subset. The following two statements are equivalent:
(1) S is a generating set of G.
(2) Any subgroup H of G that contains S is already equal to G.

Proof of Lemma 304.5. We proved the lemma in Exercise 97.8. ■

Proof of the Higher Connected Map Proposition 304.3. To simplify the discussion we only deal with the case that A and X are 0-connected. We leave it to the reader to handle the case that at least one of A and X is not 0-connected. In particular we will not deal with the case $k = 0$.

So in the following, let $k \in \mathbb{N}$, let A be a 0-connected CW-complex and let $f\colon A \to X$ be a k-connected map to a path-connected topological space. We pick a base point a_0 in the 0-skeleton of A. We write $x_0 = f(a_0)$. We note that in general the map $f\colon A \to X$ fails to be $(k+1)$-connected for the following two reasons:[3351]

(a) the map $\pi_k(A, a_0) \to \pi_k(X, x_0)$ is not a monomorphism,
(b) the map $\pi_{k+1}(A, a_0) \to \pi_{k+1}(X, x_0)$ is not an epimorphism.

Both issues will be fixed by attaching $(k+1)$-cells to A and by extending f to the $(k+1)$-cells in a suitable way. More precisely, we construct the CW-complex B and a continuous map $g\colon B \to X$ as follows:

(a) As usual we denote by A^k the k-skeleton of A. Note that by the CW-Skeleton-π_n-Proposition 141.3 the map $\pi_k(A^k, x_0) \to \pi_k(A, a_0)$ is an epimorphism. This implies that we can pick a family $\{\varphi_i\colon (S^k, *) \to (A^k, a_0)\}_{i\in I}$ of continuous maps that *go into the k-skeleton of A*, and that represent a generating set for $\ker(f_*\colon \pi_k(A, a_0) \to \pi_k(X, x_0))$.[3352] Note that by definition each map $f \circ \varphi_i\colon (S^k, *) \to (X, x_0)$ represents the trivial element in $\pi_k(X, x_0)$. Thus it follows from the Trivial-in-π_n-Lemma 134.10 that for each $i \in I$ there exists a continuous map $F_i\colon \overline{B}^{k+1} \to X$ such that $F_i|_{S^k} = f \circ \varphi_i$.

[3351] Here and throughout the proof we will implicitly use the remark on page 5844 regarding k-connected maps.
[3352] A generating set for the kernel is for example given by taking all elements in the kernel.

(b) Next we pick a family $\{\psi_j \colon (S^{k+1}, *) \to (X, x_0)\}_{j \in J}$ of continuous maps that represent a generating set for $\pi_{k+1}(X, x_0)$.

(c) We set $B = \left(A \sqcup \bigsqcup_{i \in I} \overline{B}_i^{k+1}\right) / \sim \ \vee \ \bigvee_{j \in J} S_j^{k+1}$ with $\varphi_i(s) \sim s$ for $s \in S_i^k = \partial \overline{B}_i^{k+1}$.

Note that B is a CW-complex in the following sense:

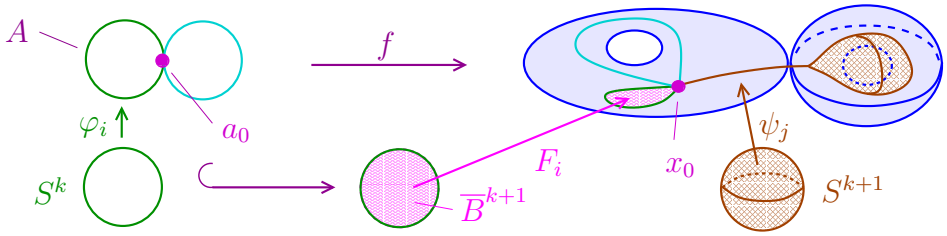
(i) We start out with the CW-complex that is given by the k-skeleton of A.

(ii) We attach one $(k+1)$-cell for each $i \in I$ with attaching map that is given by φ_i.

(iii) Afterwards we form the wedge with $\bigvee_{j \in J} S_j^{k+1}$. In other words, we attach one $(k+1)$-cell for each $j \in J$ where the attaching map is given by sending all points in $S_j^k = \partial \overline{B}_j^{k+1}$ to a_0.[3353]

(iv) Finally we attach the cells of A of dimension $\geq k+1$ as usual to the above.

(d) We extend the original map $f \colon A \to X$ to a map $g \colon B \to X$ by defining it to be F_i on each $\overline{B}_i^{k+1}$ and by defining it to be ψ_j on each S_j^{k+1}. It follows easily from the CW-Complex Pasting Lemma 136.15 that this map $g \colon B \to X$ is indeed continuous.



By construction A is a subcomplex of B and the only extra cells of B are $(k+1)$-dimensional. Furthermore, by construction the restriction of g to A equals f. Thus it remains to show that the map $g \colon B \to X$ is $(k+1)$-connected. We denote by $K \coloneqq A^k = B^k$ the k-skeleton of A which equals by construction the k-skeleton of B.

(a) First we show that the map $g \colon B \to X$ is k-connected. To do so we note that for any $r \in \mathbb{N}_0$ we have the following commutative diagram:

$$\begin{array}{ccc} \pi_r(A, a_0) & \xrightarrow{\ i_*\ } & \pi_r(B, a_0) \\ & {\scriptstyle i_*}\nwarrow \quad \nearrow{\scriptstyle i_*} & \\ & \pi_r(K, x_0) & \\ & {\scriptstyle f_*}\swarrow \ \downarrow {\scriptstyle f_*} \ \searrow{\scriptstyle g_*} & \\ & \pi_r(X, x_0) & \end{array}$$

where i always denotes an inclusion map. Since K is the k-skeleton of A and B we obtain from the CW-Skeleton-π_n Corollary 296.20 that the maps $K \to A$ and $K \to B$ are k-connected. By hypothesis the map $f \colon A \to X$ is k-connected. It follows almost immediately from this discussion and the commutative diagram that $g \colon B \to X$ is also k-connected. Note that this discussion also implies that the inclusion $A \to B$ is k-connected.

[3353] Here we implicitly use the Convention 304.1 to view the wedge of spheres, and thus also B, as a CW-complex.

(b) Next we show that $g_*\colon \pi_k(B,a_0) \to \pi_k(X,x_0)$ is a monomorphism. Similar to (a) we consider the following commutative diagram:

$$\begin{array}{ccc} \pi_k(A,a_0) & \xrightarrow{\quad i_* \quad} & \pi_k(B,a_0) \\ & {\scriptstyle f_*}\searrow \quad \swarrow{\scriptstyle g_*} & \\ & \pi_k(X,x_0). & \end{array}$$

As we pointed out in (a), we know that the top map $i_*\colon \pi_k(A,a_0) \to \pi_k(B,a_0)$ is an epimorphism. Thus it suffices to show that

$$\ker(f_*\colon \pi_k(A,a_0) \to \pi_k(X,x_0)) \;\subseteq\; \ker(i_*\colon \pi_k(A,a_0) \to \pi_k(B,a_0)).$$

But this inclusion holds basically by definition of B. More precisely, it follows from the definition of B that for each $i \in I$ the map $\varphi_i\colon (S^k,*) \to (A,a_0)$ extends to a map $(\overline{B}^{k+1},*) \to (B,a_0)$. But by the Trivial-in-$\pi_n$-Lemma 134.10 we know that this implies that the maps $\varphi_i\colon (S^k,*) \to (A,a_0)$ represent elements in the kernel of i_*. On the other hand, by definition, these maps represent a generating set for $\ker(f_*\colon \pi_k(A,a_0) \to \pi_k(X,x_0))$. The desired inclusion of kernels is now a consequence of the Abelian Generating Set–Characterization Lemma 304.5.

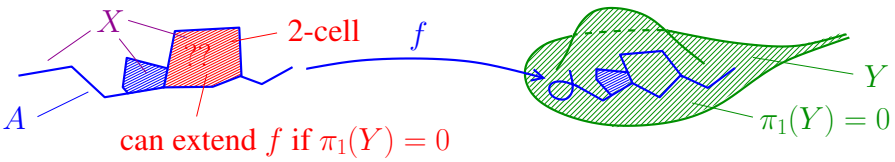
(c) It remains to show that the map $g_*\colon \pi_{k+1}(B,a_0) \to \pi_{k+1}(X,x_0)$ is an epimorphism. In fact this is true almost by definition of B. More precisely, given $j \in J$ we have $g_*([\mathrm{id}_{S^{k+1}_j}] = [g|_{S^{k+1}_j}] = [\psi_j]$. This shows that the classes $\{[\psi_j]\}_{j\in J}$ lie in the image of g_*. But since these classes generate $\pi_{k+1}(X,x_0)$ we obtain from the Abelian Generating Set–Characterization Lemma 304.5 that g_* is an epimorphism. ∎

The following lemma gives a simple but useful criterion for extending maps from a subcomplex to the whole CW-complex.

Lemma 304.6. (Map Extension Criterion) Let (X,A) be a pair of CW-complexes and let $f\colon A \to Y$ be a continuous map to a 0-connected topological space. Suppose the following condition holds:

(∗) For every $n \in \mathbb{N}$ such that $X \setminus A$ contains an open n-cell we have the equality[3354] $\pi_{n-1}(Y) = 0$.

Then f can be extended to a map $\widetilde{f}\colon X \to Y$.



Proof. We start out with the following claim:

Claim. Given any $n \in \mathbb{Z}_{\geq -1}$ we can extend $f\colon A \to Y$ to a map $A \cup X^n \to Y$.

Proof. We prove the claim by induction on $n \in \mathbb{Z}_{\geq -1}$. For $n = -1$ there is nothing to prove. So suppose the statement holds for some $n \in \mathbb{Z}_{n\geq -1}$. By the slightest of all abuses we denote the given extension $A \cup X^n \to Z$ again by f. We denote by $\{\Phi_i\colon S^n \to X^n\}_{i\in I}$ the family of the attaching maps of all $(n+1)$-cells of X that are not contained in A. If

[3354] Since Y is path-connected we know by the Change-of-Base Point Proposition 134.12 that we do not need to worry about base points.

$I = \varnothing$, then there is nothing to do. On the other hand if $I \neq \varnothing$, then we know by the hypothesis (∗) that $\pi_n(Y) = 0$. It follows from the Trivial-in-π_n-Lemma 134.10 that for each $i \in I$ there exists a continuous map $g_i \colon \overline{B}^{n+1} \to Y$ with $g_i|_{S^n} = f \circ \Phi_i \colon S^n \to Y$. We consider the following map:

$$\begin{array}{rcl} A \cup X^{n+1} = \left((A \cup X^n) \sqcup \bigsqcup\limits_{i \in I} \overline{B}_i^{n+1} \right) / \sim & \to & Y \\ [P] & \mapsto & \begin{cases} f(P), & \text{if } P \in A \cup X^n, \\ g_i(P), & \text{if } P \in \overline{B}_i^{n+1}. \end{cases} \end{array}$$

It follows from $g_i|_{S^n} = f \circ \Phi_i$ that this map is well-defined. Furthermore we obtain from the Topological-Quotient Proposition 8.2 (1b) that the map is continuous. ⊞

The sequence of maps from the claim defines a map $\widetilde{f} \colon X = A \cup X = \bigcup\limits_{n \in \mathbb{N}_0} (A \cup X^n) \to Y$. Since the map is continuous on each $A \cup X^n$ we obtain from the CW-Complex Pasting Lemma 136.15 that $\widetilde{f}$ is also continuous. It now clear that this map $\widetilde{f}$ is the desired extension. ■

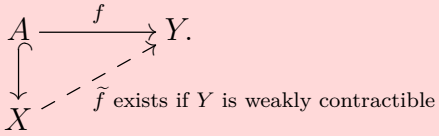
We conclude this section with the formulation of a special case the Map Extension Criterion Lemma 304.6.

Definition 304.7. We say that a topological space Y is weakly contractible if for all $n \in \mathbb{N}_0$ we have[3355] $\pi_n(Y) = 0$.

Example. On page 2643 we showed that $\pi_n(S^\infty) = 0$ for all $n \in \mathbb{N}$. In other words, we showed that the infinite-dimensional sphere S^∞ is weakly contractible. □

The following corollary is an immediate consequence of the Map Extension Criterion Lemma 304.6.

Corollary 304.8. (Weakly Contractible Domain Corollary) Let (X, A) be a pair of CW-complexes and let $f \colon A \to Y$ be a continuous map to a topological space. If Y is weakly contractible, then f can be extended to a map $\widetilde{f} \colon X \to Y$. In other words, we have the following situation



304.2. **Weak homotopy equivalences and the CW-Approximation Theorem.** The definition of a k-connected map between topological spaces naturally leads to the following definition which we had encountered once before in Definition 178.4.

Definition 304.9. We say that a map $f \colon X \to Y$ between topological spaces is a weak homotopy equivalence[3356] if f is k-connected for all $k \in \mathbb{N}_0$.

The following lemma gives in particular a practical approach to showing that a given map is a weak homotopy equivalence.

[3355] Recall that by Definition 4.43 the statement $\pi_0(Y) = 0$ means that Y is *non-empty* and path-connected.
[3356] In Exercise 304.5 we will see the name "weak homotopy *equivalence*" is not entirely deserved.

Lemma 304.10. (Weak Homotopy Equivalence Criteria Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces.

(1) If f is a homotopy equivalence, then it is also a weak homotopy equivalence.
(2) If $g\colon X \to Y$ is a continuous map that is homotopic to f, then

$$f \text{ is a weak homotopy equivalence} \iff g \text{ is a weak homotopy equivalence.}$$

(3) If X and Y are 0-connected, then the following holds:

$$f \text{ is a weak homotopy equivalence} \iff \begin{array}{c}\text{there exists an } x_0 \in X \text{ such that}\\ \text{for every } k \in \mathbb{N} \text{ the induced map}\\ f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))\\ \text{is an isomorphism.}\end{array}$$

Proof.

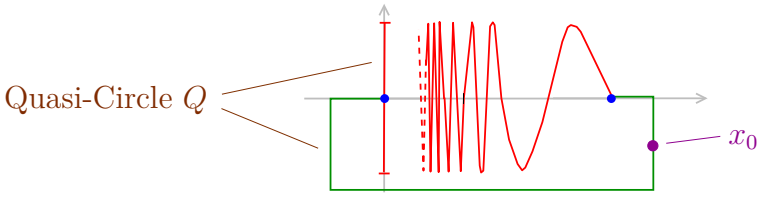
(1) This statement is an immediate consequence of the Homotopy Equivalence-π_n-Proposition 134.17 (1).
(2) This statement follows immediately from the Homotopy-π_n-Proposition 134.16.
(3) This statement follows easily from the definitions, the fact that X and Y are path-connected and the Change-of-Base Point Proposition 169.5 (3). ■

As we just saw in the Weak Homotopy Equivalence Criteria Lemma 304.10, a homotopy equivalence is also a weak homotopy equivalence. But the following example shows that in general the converse does not hold.

Example. Let

$$S \coloneqq \{(0, y) \,|\, y \in [-1, 1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \subseteq \mathbb{R}^2$$

be the "Topologist's Sine Curve" from opage 195. We turn the Topologist's Sine curve into the *Quasi-Circle* Q by connecting the point $(0, 0)$ with $(\frac{1}{\pi}, 0)$ via the green curve shown in the figure below.



We make the following observations:

(1) One can easily show that the quasi-circle is path-connected, i.e. $\pi_0(Q) = 0$.
(2) In Exercise 4.56 we showed that the Topologist's Sine Curve is not path-connected. As we saw in Exercise 134.9, this fact easily implies that all homotopy groups $\pi_n(Q)$, $n \geq 1$ are zero.
(3) Pick $x_0 \in Q$. It follows from (1) and (2) that the inclusion $i\colon \{x_0\} \to Q$ is a weak homotopy equivalence.
(4) In Exercise 19.25 we showed that Q is not contractible.[3357] By the Contractability Criterion Lemma 19.20 this implies that the inclusion i is not a homotopy equivalence. □

[3357]The argument we provided in Exercise 19.25 is rather ad hoc. There are several other approaches. For example one can use the Čech cohomology groups $\check{\mathrm{H}}(X; \mathbb{Z})$ of a topological space X, which are for example

We conclude this discussion of weak homotopies with the following proposition which relates weak homotopy equivalences to induced maps on homology groups.

Proposition 304.11. (Weak Homotopy-vs-Homology Isomorphisms Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces.

(1) If f is a weak homotopy equivalence, then for any abelian group G and any $n \in \mathbb{N}_0$ the induced maps $f_*\colon \mathrm{H}_n(X;G) \to \mathrm{H}_n(Y;G)$ and $f^*\colon \mathrm{H}^n(Y;G) \to \mathrm{H}^n(X;G)$ are isomorphisms.

(2) If X and Y are simply connected and if $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism for every $n \geq 2$, then f is a weak homotopy equivalence.

In fact we can basically generalize (2) to the following less readable statement:

(2′) Suppose the following conditions are satisfied:

(a) X and Y are both path-connected, both are locally path-connected and both are semi-locally simply connected. (As discussed in the Universal Covering Proposition 127.20 this means that both topological spaces admit universal coverings.)

(b) For some $x \in X$ the induced map $f_*\colon \pi_1(X,x) \to \pi_1(Y,f(x))$ is an isomorphism.

We write $y = f(x)$ and we denote by $p\colon (\widetilde{X},\widetilde{x}) \to (X,x))$ and $q\colon (\widetilde{Y},\widetilde{y}) \to (Y,y)$ the universal coverings of (X,x) and (Y,y). By (a) and the Map Lifting Criterion 127.5 we know that the map f lifts to a map $\widetilde{f}\colon (\widetilde{X},\widetilde{x}) \to (\widetilde{Y},\widetilde{y})$. We also demand the following:

(c) The map $\widetilde{f}_*\colon \mathrm{H}_n(\widetilde{X}) \to \mathrm{H}_n(\widetilde{Y})$ is an isomorphism for every $n \geq 2$.

Then f is a weak homotopy equivalence.

Proof.

(1) Let $f\colon X \to Y$ be a weak homotopy equivalence. It follows easily from the Path-Component–H_*–Lemma 143.21 and the Path-Component-H^*-Lemma 198.8 that we can assume, without loss of generality, that X and Y are path-connected. Let $x_0 \in X$. By our hypothesis we know that the map $f_*\colon \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$ is an isomorphism for all $n \in \mathbb{N}_0$. It follows from the Homology–Homotopy-Isomorphism Theorem 296.26 (1), which in turn is a consequence of the Relative Hurewicz Theorem 296.25, that the induced maps $f_*\colon \mathrm{H}_n(X;\mathbb{Z}) \to \mathrm{H}_n(Y;\mathbb{Z})$ are isomorphisms for all $n \in \mathbb{N}_0$. The desired statement is now a consequence of the Also-Iso-on-Cohomology Proposition 200.2.

(2) This statement is just a reformulation of the Homology–Homotopy-Isomorphism Theorem 296.26 (2).

(3) By (b) together with the Weak Homotopy Equivalence Criteria Lemma 304.10 it remains to show that $f_*\colon \pi_n(X,x)) \to \pi_n(Y,y)$ is an isomorphism for every $n \geq 2$. Now let $n \geq 2$. We consider the following diagram:

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defined in [**Dol1980**, Chapter VIII.6]. Note that

$$\begin{array}{ccccccccccc} \check{\mathrm{H}}(Q;\mathbb{Z}) & \cong & \widetilde{\mathrm{H}}_0(\mathbb{R}^2 \setminus Q;\mathbb{Z}) & \cong & \mathbb{Z} & \neq & 0 & \cong & \mathrm{H}^1(\{*\};\mathbb{Z}) & \cong & \check{\mathrm{H}}^1(\{*\};\mathbb{Z}). \\ \uparrow & & \uparrow & & & & & & \uparrow & & \\ \text{by the Duality Theorem} & & \text{by Exercise 51.11 we know that} & & & & & & \text{by Proposition VIII.6.12} & & \\ \text{[\textbf{Dol1980}, Theorem VIII.7.2]} & & \mathbb{R}^2 \setminus Q \text{ has two path-components} & & & & & & \text{in [\textbf{Dol1980}]} & & \end{array}$$

By [**Dol1980**, Proposition VIII.6.6] the Čech cohomology groups are homotopy invariants. Thus the above argument shows that Q and $\{*\}$ are not homotopy equivalent. A more direct calculation of $\check{\mathrm{H}}(Q;\mathbb{Z}) \cong \mathbb{Z}$ is provided in [**MillH2021**, Chapter 34].

$$\begin{array}{ccc} \pi_n(\widetilde{X},\widetilde{x}) & \xrightarrow{\widetilde{f}_*} & \pi_n(\widetilde{Y},\widetilde{y}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \pi_n(X,x) & \xrightarrow{f_*} & \pi_n(Y,y). \end{array}$$

We make the following observations:

(a) The diagram commutes by definition of $\widetilde{f}$.

(b) It follows from the Covering-π_n-Isomorphism Proposition 134.22 that the vertical maps are isomorphisms.

(c) By (2) and by our hypothesis (c) we know that $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is a weak homotopy equivalence. This implies that the top horizontal map is an isomorphism.

It follows from the above that the bottom horizontal map is an isomorphism. ■

In the remainder of this section we will study CW-approximations of topological spaces.

Definition 304.12. Let X be a topological space. A CW-approximation is a CW-complex Z together with a weak homotopy equivalence $f\colon Z \to X$.

Example. Let X be the quasi-circle. As we had just discussed, the inclusion of the CW-complex $\{x_0\}$, consisting of a single 0-cell, into the quasi-circle X is a CW-approximation of the quasi-circle X. □

Before we state a general theorem we turn to the delicate topic of products of CW-complexes. More precisely, if X and Y are CW-complex with only countably many cells, then it follows from the CW-Complex Product Proposition 139.18 that the product $X \times Y$ has a product CW-structure. But, as we pointed out in the Product-not-Product Lemma 139.19, in general this is not true for arbitrary CW-complexes. This leads us to the following lemma.

Lemma 304.13. (Product–CW Approximation Lemma) Let X and Y be two CW-complexes. Let $\mathcal{P}$ be the usual product topology on $X \times Y$ and let $\mathcal{CWP}$ be the CW-product topology that we introduced in Definition 139.15. With this notation the identity map $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{P})$ is a CW-approximation.

Proof. In the CW-Category–Product Proposition 139.16 we saw that $(X \times Y, \mathcal{CWP})$ admits a CW-structure. By the CW-Complex Product Proposition 139.18 we know that the identity map $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{P})$ is continuous. It follows from the CW-Product–Invariants Proposition 139.20 (2) that the map $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{P})$ is a weak homotopy equivalence. ■

The following theorem says that CW-approximations always exist.

Theorem 304.14. (CW-Approximation Theorem)

(1) Every topological space X admits a CW-approximation $f\colon Z \to X$.

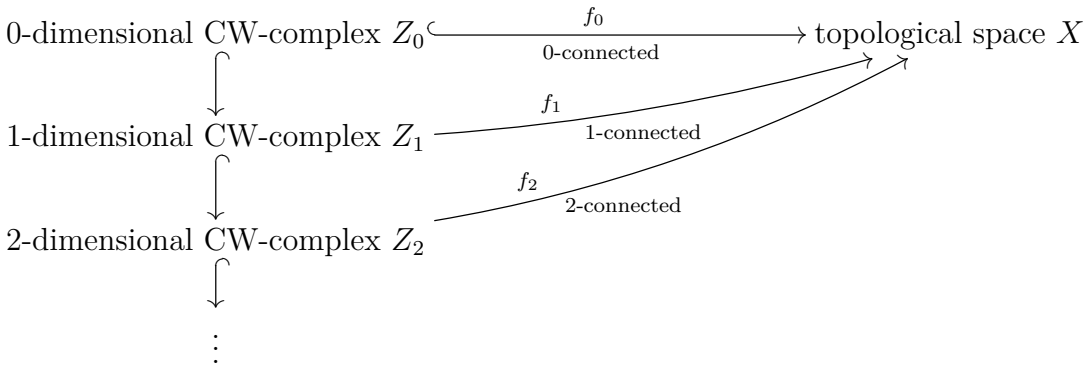
(2) Let $k \in \mathbb{N}_0$. If X is a k-connected space, then there exists a CW-approximation $f\colon Z \to X$ such that Z has a single 0-cell and such that Z has no cells in dimensions $1, \ldots, k$.

In both cases, given a point $x_0 \in X$ we can find f and $z_0 \in Z^0$ such that $f(z_0) = x_0$.

Proof of the CW-Approximation Theorem 304.14. Let X be a topological space.

(1) We need to show that X admits a CW-approximation. By considering each path-component of X separately we can assume that X is path-connected.

Thus let X be a path-connected topological space. We pick a point $x_0 \in X$. We set $Z_0 = \{x_0\}$ and we define $f_0 \colon Z_0 \to X$ as the inclusion map. By iteratively appealing to the Higher Connected Map Proposition 304.3 we obtain the following commutative diagram:

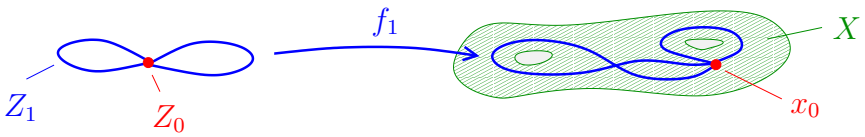


such that the $(k-1)$-skeleton of each Z_k equals Z_{k-1}. We set $Z = \bigcup_{k \in \mathbb{N}_0} Z_k$ and we define $f \colon Z \to X$ in the obvious way. Note that by construction Z is a CW-complex such that for each $k \in \mathbb{N}_0$ the k-skeleton is given by Z_k. Furthermore it follows from the CW-Complex Pasting Lemma 136.15 that f is continuous.

It remains to show that $f \colon Z \to X$ is indeed a weak homotopy equivalence. So let $k \in \mathbb{N}_0$. We consider the following commutative diagram:

$$\begin{array}{ccc} \pi_k(Z_{k+1}, x_0) & \xrightarrow{\quad \iota_* \quad} & \pi_k(Z, x_0) \\ & {\scriptstyle f_{k+1*}} \searrow \quad \swarrow {\scriptstyle f_*} & \\ & \pi_k(X, x_0) & \end{array}$$

where $\iota \colon Z_{k+1} \to Z$ denotes the inclusion map. The left-hand diagonal map is an isomorphism by construction of $f_{k+1} \colon Z_{k+1} \to X$. Furthermore the top horizontal map is an isomorphism by the CW-Skeleton-π_n-Proposition 141.3 and the fact that Z_{k+1} is the $(k+1)$-skeleton of Z. We deduce from the commutative diagram that the right-hand diagonal map is also an isomorphism.



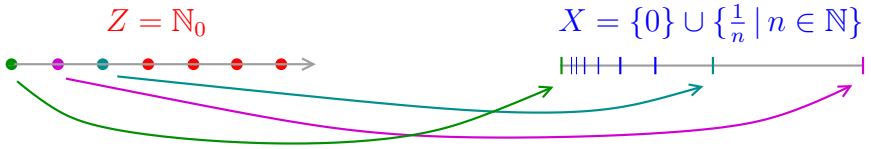
(2) Let $k \in \mathbb{N}_0$. If X is k-connected, then in the proof given in (1) we can start out with the CW-complexes $\{x_0\} = Z_0 = Z_1 = \cdots = Z_k$ and where $f_0 = f_1 = \cdots = f_k$ are the inclusion maps. Only beginning with f_{k+1} do we appeal to the Higher Connected Map Proposition 304.3 to provide the further CW-complexes Z_j and continuous maps $f_j \colon Z_j \to X$. For the remainder of the proof of (2) we now proceed as in (1). Since the k-skeleton of $Z = \bigcup_{k \in \mathbb{N}_0} Z_k$ equals $Z_k = \{x_0\}$ we see that Z has the desired properties. ■

Examples.

(1) We consider the topological space $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ equipped with the subspace topology coming from $\mathbb{R}$. Note that X is not a CW-complex.[3358] Furthermore we consider $Z = \mathbb{N}_0 \subseteq \mathbb{R}$ with the discrete topology. This is a 0-dimensional CW-complex and the map

$$\begin{array}{rcl} f\colon Z = \mathbb{N}_0 & \to & X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R} \\ n & \mapsto & \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n}, & \text{otherwise} \end{cases} \end{array}$$

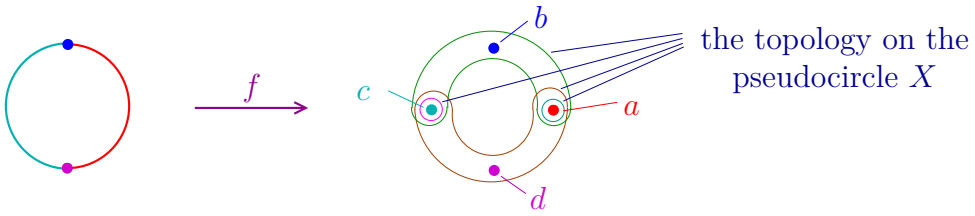
is easily seen[3359] to be a weak homotopy equivalence. Thus $f\colon Z \to X$ is a CW-approximation. We refer to the figure below for an illustration.



(2) We consider the pseudocircle that we introduced on page 172. Recall that the pseudocircle is the set $X = \{a, b, c, d\}$ where the topology is given by the set

$$\mathcal{T} \coloneqq \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

On page 172 we gave an explicit continuous map $f\colon S^1 \to X$ that is also illustrated in the figure below.



In Exercise 93.1 we showed that $f_*\colon \pi_1(S^1, 1) \to \pi_1(X, a)$ is an isomorphism. A slight variation on the definition of f also shows that X is path-connected. Furthermore, by the discussion on page 2587 and by Exercise 134.6 we know that we have $\pi_n(S^1, 1) = \pi_n(X, a) = 0$ for $n \geq 2$. Thus we obtain from the Weak Homotopy Equivalence Criteria Lemma 304.10 that f is a weak homotopy equivalence. Since S^1 is a CW-complex we see that the map $f\colon S^1 \to X$ is a CW-approximation for the pseudocircle X. □

304.3. Exercises for Chapter 304.

Exercise 304.1.

(a) Give an explicit CW-approximation for the Cantor set from Definition 3.45.
(b) Give an explicit CW-approximation for the Topologist's Sine Curve, as defined on page 195.

Exercise 304.2. If X is a compact connected topological space. Does X admit a CW-approximation by a finite CW-complex?

[3358] Why is it not a CW-complex?
[3359] What is so easy about this?

Exercise 304.3.

(1) Let M be a connected non-empty n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^n \to M$ be a smooth embedding. Show that there exists a homeomorphism $\varphi\colon M/\varphi(S^{n-1}) \to M \vee S^n$ such that the map $M \to M/\varphi(S^{n-1}) \xrightarrow{\varphi} M \vee S^n \to M$ is homotopic to the identity.

(2) We consider the map

$$\Theta\colon S^1 \times S^2 \xrightarrow{\text{projection}} (S^1 \times S^2) \vee S^3 \xrightarrow{\text{as in (1)}} (S^1 \times S^2) \vee S^3 \to S^1 \times S^2.$$

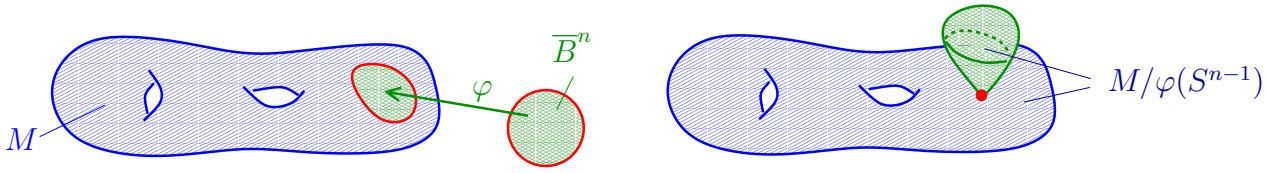
given by id on $S^1 \times S^2$ and by the Hopf map $S^3 \to S^2$ on S^3

Show the following statements:

(a) Θ induces the identity on all homology groups.

(b) Θ induces the identity on π_1 and π_2.

(c) Θ does not induce the identity on π_3.



Exercise 304.4. We consider the set $\mathbb{Q}$ with the discrete topology $\mathcal{D}$ and with the subspace topology $\mathcal{S}$ coming from $\mathbb{Q} \subseteq \mathbb{R}$.

(a) Show that there exists a weak homotopy equivalence $(\mathbb{Q}, \mathcal{D}) \to (\mathbb{Q}, \mathcal{S})$.

(b) Show that there is no homotopy equivalence $(\mathbb{Q}, \mathcal{D}) \to (\mathbb{Q}, \mathcal{S})$.

Exercise 304.5. Inspired by Definition 304.9 we say that two topological space X and Y are *weakly homotopy equivalent* if there exists weak homotopy equivalence $f\colon X \to Y$. Show that being "weakly homotopy equivalent" is not an equivalence relation on the class of topological spaces.

Remark. You could look at Exercise 304.4.

CHAPTER 305

The Whitehead Theorem

In this chapter we will prove the Whitehead Theorem 305.1 which says that if a map $f\colon X \to Y$ between CW-complexes is a weak homotopy equivalence, i.e. if it induces an isomorphism of all homotopy groups, then f is in fact a homotopy equivalence. Afterwards we will study several applications of the Whitehead Theorem 305.1. In particular we will be able to complete the classification of lens spaces up to homotopy equivalence.

This chapter is organized as follows:

- In Section 305.1 we will state and prove the Whitehead Theorem 305.1.
- In Section 305.2 we will use the Whitehead Theorem 305.1 to give the homotopy classification of lens spaces.
- In Section 305.3 we will study topological spaces dominated by CW-complexes.
- In Section 305.4 we will consider highly connected CW-complexes
- In Section 305.5 we will consider elementary collapses and elementary expansions of CW-complexes.
- In Section 305.6 we will provide exercises for Chapter 305.

In the next Chapter 306 we will discuss generalized lens spaces.

305.1. **The Whitehead Theorem.** Usually, in algebraic topology one goes from topological situations to algebra using various functors. For example, any continuous map between (pointed) topological spaces gives rise to a map between homology groups and homotopy groups. Unfortunately almost invariably in the passage from topology to algebra some information gets lost. For example there are many non-homeomorphic topological spaces with isomorphic homology groups.

In the Homotopy Equivalence-π_n-Proposition 134.17 (1) we saw that a homotopy equivalence between topological spaces gives rise to isomorphisms of higher homotopy groups. The following theorem shows that, rather amazingly, in this particular case the converse holds, at least if we restrict ourselves to maps between CW-complexes.

Theorem 305.1. (Whitehead Theorem)[3360] Let $f\colon X \to Y$ be a continuous map between two connected CW-complexes.[3361]

(1) If $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

In fact, there is a more refined version dealing with base points:

(2) Let $x_0 \in X^0$ and $y_0 \in Y^0$ be base points in the 0-skeleta of X respectively Y. If $f\colon X \to Y$ is a weak homotopy equivalence with $f(x_0) = y_0$, then there exists a homotopy equivalence between the pointed topological spaces (X, x_0) and (Y, y_0).

For inclusion maps we have the following very useful variation on the Whitehead Theorem 305.1.

Proposition 305.2. (Whitehead Deformation Retract Proposition) Let Y be a connected CW-complex.

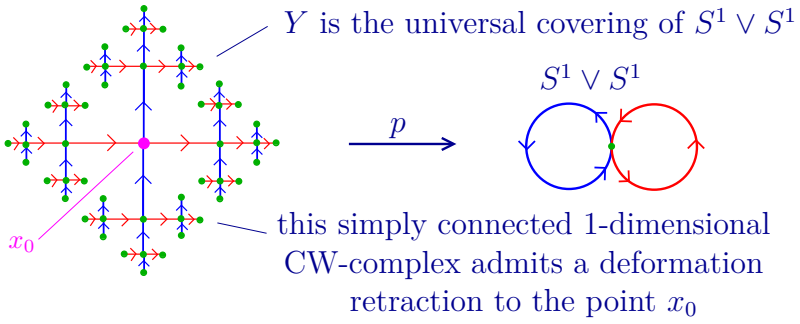
(1) Let X be a connected subcomplex and let $x_0 \in X$ be a point. We denote by $i\colon X \to Y$ the inclusion map. If the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are isomorphisms for all $n \in \mathbb{N}$, then X is a deformation retract of Y.

(2) If Y is weakly contractible, then for every point $y_0 \in Y$ the one-point subset $\{y_0\}$ is a deformation retract of Y. In particular Y is contractible.

Examples.

(1) Let Y be a simply connected 1-dimensional CW-complex and let x_0 be a point in the 0-skeleton of Y. By the 1-Complex-π_n-Lemma 138.2 (or alternatively the 1-Complex-π_n-Lemma 167.14) we know that $\pi_n(Y, x_0) = 0$ for all $n \in \mathbb{N}$. It follows from the Whitehead Deformation Retract Proposition 305.2 (2) that Y admits a deformation retraction to $\{x_0\}$, in particular Y is contractible.

For example, let Y be the universal covering of $S^1 \vee S^1$. We view $S^1 \vee S^1$ as a CW-complex in the obvious way. By the CW-Complex Covering Proposition 138.1 we can equip Y with the structure of a 1-dimensional CW-complex. By definition of a universal covering we know that Y is simply connected. It follows from the above discussion that any point x_0 in the 0-skeleton of Y is a deformation retract. We refer to the figure below for an illustration.



(2) On page 5849 we pointed out that S^∞ is weakly contractible. Thus it follows from the Whitehead Deformation Retract Proposition 305.2 (2) that S^∞ is contractible.[3362]

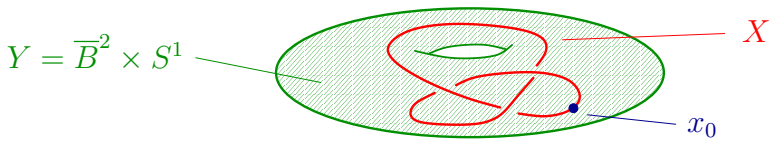
(3) We consider the knotted curve X in the solid torus $Y = \overline{B}^2 \times S^1$ that is shown in the figure below.

It follows from the Smooth Manifold-Simplicial Structure Theorem 179.5 (3) together with the Simplicial-Implies-CW Lemma 174.51 that we can equip Y with a CW-structure such that X is a subcomplex. It should be pretty clear that the inclusion-induced map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism. Furthermore, since X and Y

[3360] As we pointed out on page 5686, the name "Whitehead Theorem" is sometimes also used for the Homology–Homotopy-Isomorphism Theorem 296.26. Our naming convention agrees with [**Hat2002**, Theorem 4.5] and [**Rot1988**, p. 370].

[3361] Note that we do not assume that the map f is cellular.

[3362] In fact in the S^∞–Contractibility Lemma 104.25 we gave an explicit homotopy from the identity of S^∞ to a constant map. By the Contractability Criterion Lemma 19.20 this implies that S^∞ is contractible.



are homotopy equivalent to S^1 we see that the higher homotopy groups of X and Y vanish. In particular for every $n \in \mathbb{N}_{\geq 2}$ the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are isomorphisms. In other words we see that all the hypotheses of the Whitehead Deformation Retract Proposition 305.2 are satisfied. This shows that X is a retract of Y. This answers the question that we had posed on page 1762. □

Remark. One might be tempted to interpret the Whitehead Theorem 305.1 as saying that if two CW-complexes have isomorphic homotopy groups, then they are homotopy equivalent. But this statement is false.

(1) For example let us consider the lens spaces $L(5,1)$ and $L(5,2)$. Recall that both are quotients of S^3 by some action of $\mathbb{Z}/5\mathbb{Z}$ and both are closed 3-dimensional smooth manifolds. Note that both lens spaces are CW-complexes, this follows either from the very general Smooth Manifold-CW-Theorem 179.9 or more directly from the later Generalized Lens Space-CW-Proposition 306.6. On page 2587 we saw that the homotopy groups of these two lens spaces are isomorphic. On the other hand we seen on page **??** that $L(5,1)$ and $L(5,2)$ are not homotopy equivalent.

(2) We consider the topological spaces S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$. Evidently S^2 and S^3 are CW-complexes. On page 2628 we saw that $\mathbb{C}\mathrm{P}^\infty$ is a CW-complex. It follows from the CW-Complex Product Proposition 139.18 that $S^3 \times \mathbb{C}\mathrm{P}^\infty$ is also a CW-complex. In Exercise 298.12 we saw that S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ have isomorphic homotopic groups.[3363] On the other hand it follows immediately from the discussion on page 2955 and the Topological Künneth Theorem 164.6 that the homology groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are (dramatically) different. Thus the two CW-complexes are not homotopy equivalent. □

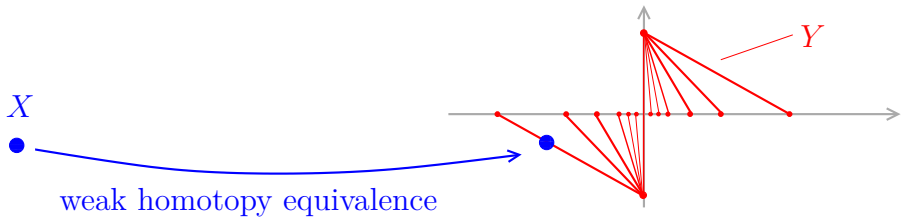
Remark. The analogous statement of the Whitehead Theorem 305.1, respectively the Whitehead Deformation Retract Proposition 305.2, does not hold if we drop the hypothesis that X and Y are CW-complexes, in other words, the statement does not hold for general topological spaces. We can provide two examples:

(1) On page 5850 we already discussed the fact that the Quasi-Circle is weakly contractible but not contractible.

(2) Since weird topological spaces are so much fun, let us consider another example. More precisely, let us consider the following subset of $\mathbb{R}^2$:

$$\begin{aligned} Y \coloneqq \text{Euclidean segment from } (0,1) \text{ to } (0,-1) &\cup \bigcup_{n\in\mathbb{N}} \text{Euclidean segment from } (0,1) \text{ to } (\tfrac{1}{n},0) \\ &\cup \bigcup_{n\in\mathbb{N}} \text{Euclidean segment from } (0,-1) \text{ to } (-\tfrac{1}{n},0). \end{aligned}$$

[3363] In fact this follows easily from the fact that $\pi_k(S^2) = \pi_k(S^3)$ for $k \geq 3$ (see the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12) and the fact that $\pi_k(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}$ for $k = 2$ and $\pi_k(\mathbb{C}\mathrm{P}^\infty) = 0$ for $k \neq 2$ (see the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 (2)).



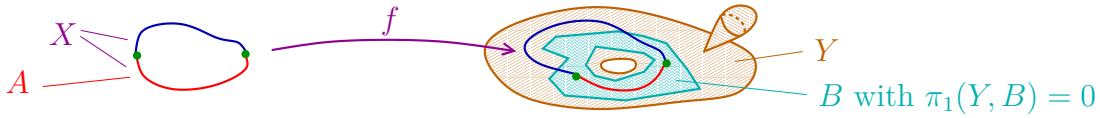
We equip Y with the subspace topology. In Exercise 305.4 (a) we will see that the homotopy groups of Y are trivial, i.e. Y is weakly contractible. On the other hand we show in Exercise 305.4 (b) that Y is not contractible. The readers who are less inclined towards doing the exercises are referred to [**Mau1980**, p. 301] instead.[3364]

(3) In Exercise 305.13 we will see that the long line $\mathbb{L}$, which we introduced in Definition 28.19, is weakly contractible but not contractible.

(4) On [**Barm2011**, p. 54] it is shown that there exists a finite topological space that is weakly contractible but not contractible. □

The proof of the Whitehead Theorem 305.1 relies on the following lemma:

Lemma 305.3. (Compression Lemma) Let (X, A) be a pair of CW-complexes and let (Y, B) be a pair of topological spaces with $B \neq \varnothing$. We make the following assumption: For every $n \in \mathbb{N}_0$ such that $X \setminus A$ has at least one open n-cell we assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every continuous map $f\colon (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.



Proof. We start out with the following claim:

Claim. Let $k \in \mathbb{N}$. Let $f\colon (X, A) \to (Y, B)$ be a continuous map. If $f(X^{k-1}) \subseteq B$, then there exists a homotopy rel $A \cup X^{k-1}$ from f to a map g with $g(X^k) \subseteq B$.

Proof. If there are no open k-cells in $X \setminus A$, then we have $X^k \subseteq X^{k-1} \cup A$, in particular we already have $f(X^k) \subseteq B$, and we are thus done. Now suppose that there exists at least one open k-cell in $X \setminus A$. By our hypothesis this implies that $\pi_k(Y, B, y_0) = 0$ for every $y_0 \in B$. Let $\{\Phi_i\colon \overline{B}^k \to X\}_{i \in I}$ be a full set of characteristic maps corresponding to the open$k$-cells in $X \setminus A$. Given $i \in I$ we denote by $\phi_i\colon S^{k-1} \to X$ the corresponding attaching map. It follows from $f(X^{k-1}) \subseteq B$, from $\pi_k(Y, B, y_0) = 0$ and the Relative Homotopy Group–Vanishing Proposition 296.18 (2)⇒(1) that there exists a homotopy F_i rel S^{k-1} from the map $f \circ \Phi_i\colon \overline{B}^k \to Y$ to a map whose image is contained in B. It follows from the Homotopy Pushout Lemma 18.13 (or alternatively the , applied to $A \cup X^k = (A \cup X^{k-1}) \cup_{\bigsqcup_{i \in I} \phi_i} \left(\bigsqcup_{i \in I} \overline{B}_i^k \right)$ that these homotopies on all such cells, together with the stationary homotopy on $A \cup X^{k-1}$, give rise to a homotopy H on $A \cup X^k$ rel $A \cup X^{k-1}$ with $\mathrm{H}_0|_{A \cup X^k} = f$ and with $\mathrm{H}_1(X^k) \subseteq B$. By the Homotopy Extension Theorem 140.2 we can extend this homotopy H to a homotopy $\widetilde{H}\colon X \times [0, 1] \to Y$ with $\widetilde{\mathrm{H}}_0 = f$. This homotopy has all the desired properties. ⊞

[3364] This shows in particular that Y is not a CW-complex. Is there a more direct way for showing that Y is not a CW-complex?

First suppose that X is a finite-dimensional CW-complex. It is clear that the hypothesis of the claim holds for $k = -1$ since, by definition, $X^{-1} = \varnothing$. Now we can apply the claim iteratively finitely many times. Combining these finitely many homotopies together as in the Homotopy Combination Lemma 18.8 (2) we obtain the desired homotopy.

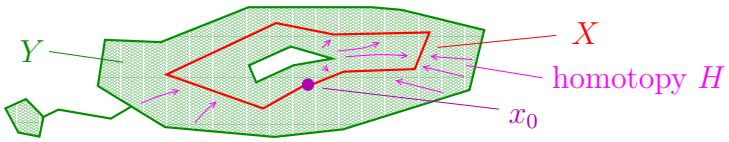
In the case that X is infinite-dimensional we obtain an infinite sequence of homotopies. We can combine these homotopies using the rather tricky Infinite Homotopies–CW–Lemma 140.19 and we obtain again the desired homotopy. ■

Now we are already in a position to provide the proof of the Whitehead Deformation Retract Proposition 305.2.

Proof of the Whitehead Deformation Retract Proposition 305.2. Let Y be a connected CW-complex.

(1) Let X be a connected subcomplex and let $x_0 \in X$. We denote by $i\colon X \to Y$ the inclusion map. We suppose that the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are isomorphisms for all $n \in \mathbb{N}$. We need to show that X is a deformation retract of Y.

Recall that by our hypothesis Y is connected. It follows from the CW-Complex–Local Properties Proposition 137.8 (2) together with the Locally Path-Connected Lemma 4.57 that X is path-connected. Thus it follows from the Change-of-Base Point Proposition 169.5 that the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are in fact isomorphisms for all $x_0 \in X$. Next we note that it follows from our hypothesis and the long exact sequence of homotopy groups given by the Homotopy Groups-LES-Proposition 296.10 that all relative homotopy groups $\pi_n(Y, X, x_0)$ are trivial for all $x_0 \in X$. This observation allows us apply the Compression Lemma 305.3 to the identity map $(Y, X) \to (Y, X)$. From the Compression Lemma 305.3 we obtain a homotopy H rel X from id_Y to a map whose image lies in X. But this means, by definition, that H is a deformation retraction from Y to X.



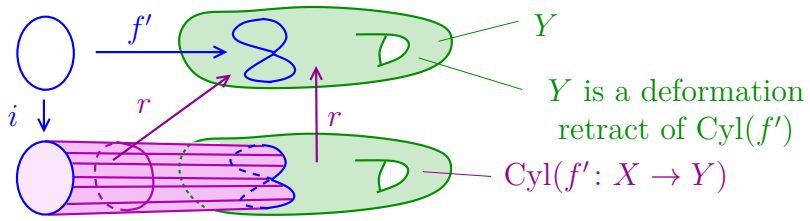
(2) We assume that Y is weakly contractible. Let $y_0 \in Y$. We denote by $i\colon \{y_0\} \to Y$ the inclusion. Since Y is weakly contractible we see that for each $n \in \mathbb{N}$ the map $i_*\colon \pi_n(\{y_0\}, y_0) \to \pi_n(Y, y_0)$ is an isomorphism, for the silly reason, that the groups on both sides are trivial. It follows from (1) that $\{y_0\}$ is a deformation retract of Y. The elementary Point Deformation Retract Lemma 19.19 now tells us that Y is contractible. ■

We will reduce the proof of the Whitehead Theorem 305.1 to the Whitehead Deformation Retract Proposition 305.2. The trick hereby is to use the notion of a mapping cylinder that we introduced in Definition 21.1. For the reader's convenience we recall the definition:

Definition 305.4. Given a continuous map $f\colon X \to Y$ between topological spaces we define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f) \;:=\; \mathrm{Cyl}(f\colon X \to Y) \;:=\; ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

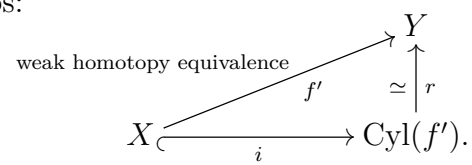
Now we can provide the actual proof of the Whitehead Theorem 305.1.



Proof of the Whitehead Theorem 305.1. Let X and Y be two connected CW-complexes. Let $x_0 \in X$ be a base point. Furthermore let $f\colon X \to Y$ be a continuous map such that the induced maps $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are isomorphisms for all $n \in \mathbb{N}$.

(1) By the Cellular Approximation Theorem 141.2 the map f is homotopic to a cellular map f'. We make the following two observations:
   (i) By the Homotopy-Equivalence Basics Lemma 19.14 (3) the map f is a homotopy equivalence if and only if f' is a homotopy equivalence.
   (ii) By the Weak Homotopy Equivalence Criteria Lemma 304.10 (2) the map f is a weak homotopy equivalence if and only if f' is a weak homotopy equivalence.

   As usual we denote by Cyl(f') the mapping cylinder of $f'\colon X \to Y$. We consider the following diagram of maps:



   We make the following comments and observations:[3365]
   (a) The map $r\colon \mathrm{Cyl}(f') \to Y$ is the obvious map that is given by the identity on Y and given by sending $[(x,t)]$ to $f'(x)$.
   (b) The map $i\colon X \to \mathrm{Cyl}(f')$ is the obvious inclusion given by $x \mapsto [(x,0)]$.
   (c) It is clear that the above diagram commutes.
   (d) By (ii) and our hypothesis the map $f'\colon X \to Y$ is a weak homotopy equivalence.
   (e) Recall that by the Mapping Cylinder–Homotopy Lemma 21.5 (2) we know that the map $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. In particular we obtain from the Weak Homotopy Equivalence Criteria Lemma 304.10 (1) that r is a weak homotopy equivalence.
   (f) It follows from (c), (d) and (e) that i is a weak homotopy equivalence.
   (g) Since $f'\colon X \to Y$ is cellular we can use the CW-Complex Mapping Cylinder+Cone Proposition 139.25 to equip the mapping cylinder Cyl($f'\colon X \to Y$) with a natural CW-structure such that $X \times \{0\} = i(X)$ is a subcomplex.
   (h) It follows immediately from (f) and (g) together with the Whitehead Deformation Retract Proposition 305.2 that the map $i\colon X \to \mathrm{Cyl}(f')$ is a homotopy equivalence.
   (i) We obtain from (e) and (h) and the Homotopy-Equivalence Basics Lemma 19.14 (1) that f is indeed a homotopy equivalence.

[3365] Some of the arguments below overlap with the proof of the Mapping Cylinder–Homotopy Lemma 21.5 (2). The reason for not appealing to the Mapping Cylinder–Homotopy Lemma 21.5 (2) is that in Statement (2) of the Whitehead Theorem 305.1 we also want to do the case rel a fixed point.

(2) Now suppose that we are given $x_0 \in X^0$ and $y_0 \in Y^0$ such that $f(x_0) = y_0$. Since f is cellular on the subcomplex $\{x_0\}$ we obtain from the Cellular Approximation Theorem 141.2 that the map f is homotopic *rel* x_0 to a cellular map f'. Note that we still have $f'(x_0) = y_0$. Using this input it is not difficult to convince oneself that the argument in (1) can be modified to give the desired homotopy rel $\{x_0\}$. We leave it to the punctilious reader to go through the details. ■

In the remainder of this chapter we will discuss many applications of the Whitehead Theorem 305.1 to various problems.

305.2. **Homotopy classification of lens spaces.** One of the recurring themes of these notes is the classification of lens spaces. Using the Whitehead Theorem 305.1 we can now finish the classification of lens spaces up to homotopy equivalence. It is worth recalling Definition 91.4, so that we are all on the same sheet.

Definition 305.5. We consider the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ as a submanifold of $\mathbb{C}^2$. Let $p \in \mathbb{N}$. Furthermore let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. The map

$$\begin{array}{rcl} \mathbb{Z}/p\mathbb{Z} \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z_1, z_2)) & \mapsto & (\exp(2\pi\,\mathrm{i}k/p) \cdot z_1, \exp(2\pi\,\mathrm{i}kq/p) \cdot z_2) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}/p\mathbb{Z}$ on S^3. We refer to $L(p, q) := S^3/(\mathbb{Z}/p\mathbb{Z})$ as a lens space.[3366]

Recall that in the Lens Space-Linking Pairing Corollary **??** we showed that if two lens spaces $L(p, q_1)$ and $L(p, q_2)$ are homotopy equivalent, then there exists an $\epsilon \in \{-1, 1\}$ and an $r \in \mathbb{Z}$, coprime to p, such that $q_1 = \epsilon \cdot r^2 \cdot q_2 \bmod p$. We can now prove the converse of this result.

Proposition 305.6. (Lens Space-Homotopy Equivalence Proposition) Let $p \in \mathbb{N}$ and let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(p, q_1) = \gcd(p, q_2) = 1$. If there exists an $m \in \mathbb{Z}$ with $q_1 \equiv \pm m^2 \cdot q_2 \bmod p$, then the lens spaces $L(p, q_1)$ and $L(p, q_2)$ are homotopy equivalent.

Remark. This proposition was first shown by John Whitehead [**WhdJ1941b**, Theorem 10], extending earlier work of Marcel Rueff [**?**, Satz VII]. A direct proof, that does not make use of the Whitehead Theorem 305.1, is also given in [**CoheM1973**, Chapter 29]. □

Before we turn to the proof of the Lens Space-Homotopy Equivalence Proposition 305.6 let us remind the reader that we have now proved the following theorem, which for convenience we had stated already at an earlier occasion.

Theorem ??. (Lens Space-Homotopy Classification Theorem) Let $p \in \mathbb{N}$. If $q, \widetilde{q} \in \mathbb{Z}$ are coprime to p, then

$$\begin{array}{c} \text{the lens spaces } L(p, q) \text{ and } L(p, \widetilde{q}) \\ \text{are homotopy equivalent} \end{array} \iff \begin{array}{c} \text{there exists an } r \in \mathbb{Z}, \text{ coprime to } p, \\ \text{such that } q \equiv \pm r^2 \cdot \widetilde{q} \bmod p. \end{array}$$

Remark. The classification of lens spaces up to homeomorphism will have to wait till the Lens Space-Homeomorphism Classification Theorem **??**. □

[3366] As we mentioned in Definition 91.4, it follows from the Manifold–Quotient Proposition 25.39 and the Orientation-Action Proposition 31.44 that $L(p, q)$ is a closed 3-dimensional smooth manifold that comes with a natural orientation.

Proof. The"$\Rightarrow$"-direction is precisely the statement of the Lens Space-Linking Pairing Corollary **??** and the"$\Leftarrow$"-direction is the content of the Lens Space-Homotopy Equivalence Proposition 305.6. ■

The proof of the Lens Space-Homotopy Equivalence Proposition 305.6 requires several preparations. The key idea is to make use of the recently acquired Whitehead Theorem 305.1. In fact the Whitehead Theorem 305.1 is the key ingredient in the proof of the following lemma.

Lemma 305.7. (Covered-by-Sphere Homotopy Lemma) Let M and N be closed oriented n-dimensional smooth manifolds such that the universal covering in both cases is S^n. Let $g\colon M \to N$ be a continuous map. If $\deg(g) = \pm 1$ and if $g_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism, then g is a homotopy equivalence.

Proof. First note that by the Smooth Manifold-CW-Theorem 179.9 we know that M and N admit CW-structures.[3367] Thus, by the Whitehead Theorem 305.1, it remains to show that $g\colon M \to N$ is a weak homotopy equivalence.

Let $p\colon S^n \to M$ and $q\colon S^n \to N$ be the universal coverings. It follows from the Map Lifting Criterion 127.5 that there exists a continuous map $\widetilde{g}\colon S^n \to S^n$ such that the following diagram commutes:

$$\begin{array}{ccc} S^n & \xrightarrow{\widetilde{g}} & S^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{g} & N. \end{array}$$

It follows from our hypothesis that $g_*\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism together with the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 $(2')$ that it suffices to prove the following claim:

Claim. For each $k \in \mathbb{N}_0$ the map $\widetilde{g}_*\colon \mathrm{H}_k(S^n) \to \mathrm{H}_k(S^n)$ is an isomorphism.

Proof. Evidently we only have to consider the case $k = n$. It follows from the above simple-minded commutative diagram, the Degree Multiplicativity Theorem 46.7, the hypothesis that $\deg(g) = \pm 1$ and the observation that by the Covering Map-Degree Proposition 186.9 (or alternatively the Covering Map-Degree Proposition 195.7) we have $\deg(p) = \#\pi_1(M) = \#\pi_1(N) = \deg(q)$, that $\deg(\widetilde{g}) = \pm 1$. But this implies that $\widetilde{g}_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is an isomorphism. ■

It follows from the Covered-by-Sphere Homotopy Lemma 305.7 that, in the setting of the Lens Space-Homotopy Equivalence Proposition 305.6, it suffices to show that there exists a map $L(p, q_1) \to L(p, q_2)$ of degree ± 1. The following lemma is the first step towards the construction of such a map.

[3367] Alternatively it follows from the Topological Manifold-Simplicial Complex Retract Theorem 189.13 together with the brand-new CW-Complex Domination Proposition 305.11 or from the Smooth Manifold-CW-Homotopy Type Proposition 254.4 that M and N are homotopy equivalent to CW-complexes, which is good enough for our purpose.

Lemma 305.8. (Lens Space–Degree Realization Lemma) Let $p \, in \mathbb{N}$ and let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(p, q_1) = \gcd(p, q_2) = 1$. Given any $t_1, t_2 \in \mathbb{Z}$ with $q_1 \cdot t_1 \equiv q_2 \cdot t_2 \bmod p$ the map

$$\begin{array}{rccc} f\colon & L(p,q_1) & \to & L(p,q_2) \\ & [(z,w)] & \mapsto & [(z^{t_2}, w^{t_1})] \end{array}$$

is well-defined, it is smooth and it has degree $t_1 \cdot t_2$.

Proof. First we show that the map $f\colon L(p,q_1) \to L(p,q_2)$ is well-defined. To do so we introduce two pieces of notation. We write $\xi := \exp(2\pi \mathrm{i}/p)$ and for $i = 1, 2$ we denote by $\sim_i$ the equivalence relation on S^3 that is defined by the action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 with parameter q_i. In other words

$$(z,w) \sim_i (\widetilde{z}, \widetilde{w}) \iff \text{there exists a } k \in \mathbb{Z} \text{ with } (z,w) = (\xi^k \cdot \widetilde{z}, \xi^{k \cdot q_i} \cdot \widetilde{w}).$$

We need to show that if we are given $(z,w), (\widetilde{z}, \widetilde{w}) \in S^3$ with $(z,w) \sim_1 (\widetilde{z}, \widetilde{w})$, then $(f(z), f(w)) \sim_2 (f(\widetilde{z}), f(\widetilde{w}))$. This follows from the following sequence of implications:

$$\begin{array}{lll} (z,w) \sim_1 (\widetilde{z}, \widetilde{w}) & \implies (z,w) = (\xi^k \cdot \widetilde{z}, \xi^{k \cdot q_1} \cdot \widetilde{w}) & \text{(definition of } \sim_1) \\ & \implies (z^{t_2}, w^{t_1}) = (\xi^{k \cdot t_2} \cdot \widetilde{z}^{t_2}, \xi^{k \cdot q_1 \cdot t_1} \cdot \widetilde{w}^{t_1}) & \text{(since take same powers)} \\ & \implies (z^{t_2}, w^{t_1}) = (\xi^{k t_2} \cdot \widetilde{z}^{t_2}, \xi^{k t_2 \cdot q_2} \cdot \widetilde{w}^{t_1}) & \text{(since } q_1 \cdot t_1 \equiv q_2 \cdot t_2 \bmod p) \\ & \implies (z^{t_2}, w^{t_1}) \sim_2 (\widetilde{z}^{t_2}, \widetilde{w}^{t_1}) & \text{(definition of } \sim_2) \\ & \implies (f(z), f(w)) \sim_2 (f(\widetilde{z}), f(\widetilde{w})) & \text{(definition of } f). \end{array}$$

We have thus shown that the map $f\colon L(p,q_1) \to L(p,q_2)$ is well-defined. Next note that it follows easily from the Smooth Map-Descends-to-Quotient Corollary 25.41 that f is smooth. It remains to calculate the degree of the map. We consider the following diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{(z,w) \mapsto (z^{t_1}, w^{t_2})} & S^3 \\ {\scriptstyle (z,w) \mapsto [(z,w)]_1}\downarrow & & \downarrow{\scriptstyle (z,w) \mapsto [(z,w)]_2} \\ L(p,q_1) & \xrightarrow{\quad f \quad} & L(p,q_2). \end{array}$$

We make the following observations:

(1) It is clear that the diagram commutes.
(2) The vertical maps are covering maps of degree d. It follows from the Covering Map-Degree Proposition 186.9 (or alternatively the Covering Map-Degree Proposition 195.7) that the vertical also have degree p in terms of fundamental classes.
(3) One can easily show, using the definition of the degree in Definition 46.1, that the degree of top horizontal map equals $t_1 \cdot t_2$.

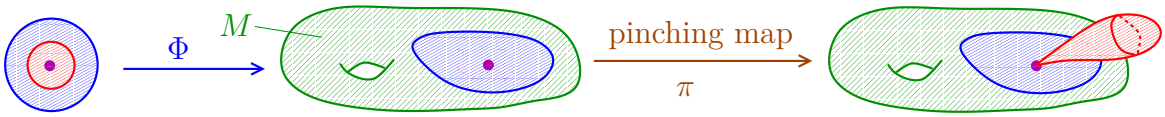
It follows from these three observations together with the Degree Multiplicativity Theorem 46.7 that the degree of the bottom horizontal map f also equals $t_1 \cdot t_2$. ∎

Lemma 305.9. (Pinching Lemma) Let M be a closed oriented connected non-empty n-dimensional smooth manifold. We pick an orientation-preserving smooth embedding

$\Phi\colon \overline{B}^n_2 \to M$. Let $\Theta\colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 8.10. We refer to the map[33683369]

$$\pi\colon M \to M \vee S^n$$
$$P \mapsto \begin{cases} \Phi(2(t-1)\cdot x), & \text{if } P = \Phi(t\cdot x) \text{ for } t \in [1,2] \text{ and } x \in S^{n-1}, \\ \Theta(y), & \text{if } P = \Phi(y) \text{ with } y \in \overline{B}^n, \\ P, & \text{otherwise.} \end{cases}$$

as the pinching map. This map has the property that the maps $M \xrightarrow{\pi} M \vee S^n \to M$ and $M \xrightarrow{\pi} M \vee S^n \to S^n$ both have degree 1.



Proof. This statement can be verified easily using the Local-Global Degree Proposition 186.8. We leave it to the reader to fill in the details. ■

Now we have assembled all the tools that we need for provide the proof of the Lens Space-Homotopy Equivalence Proposition 305.6.

Proof of Proposition 305.6. First let $p\, in \mathbb{N}$. Furthermore let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(p, q_1) = \gcd(p, q_2) = 1$. We suppose that there exists an $m \in \mathbb{Z}$ and an $\epsilon \in \{-1, 1\}$ with $q_1 \equiv \epsilon \cdot q_2 \cdot m^2 \bmod p$. We need to show that $L(p, q_1)$ and $L(p, q_2)$ are homotopy equivalent. By the Covered-by-Sphere Homotopy Lemma 305.7 it suffices to show that there exists a continuous map $g\colon L(p, q_1) \to L(p, q_2)$ of degree ϵ. We will construct such g in several steps:

(1) Since $\gcd(p, q_1) = \gcd(p, q_2) = 1$ we can pick $s_1, s_2 \in \mathbb{Z}$ with $s_i \cdot q_i \equiv 1 \bmod p$.
(2) By the Lens Space–Degree Realization Lemma 305.8, applied to $t_i = q_2 m \cdot s_i$, there exists a continuous map $f\colon L(p, q_1) \to L(p, q_2)$ of degree $(q_2 m)^2 \cdot s_1 \cdot s_2$.
(3) Note that

$$(q_2 m)^2 \cdot s_1 \cdot s_2 \equiv q_2 \cdot m^2 \cdot s_1 \cdot q_2 \cdot s_2 \underset{\substack{\uparrow \\ \text{since } q_1 \equiv \epsilon \cdot q_2 \cdot m^2 \bmod p}}{\equiv} \epsilon \cdot q_1 \cdot s_1 \cdot q_2 \cdot s_2 \equiv \epsilon \bmod p.$$

This calculation shows that there exists a $d \in \mathbb{Z}$ with $(q_2 m)^2 \cdot s_1 \cdot s_2 = \epsilon + d \cdot p$.
(4) By the Degree Realization Proposition 46.19 there exists a continuous map $h\colon S^3 \to S^3$ of degree $-d$.

We now consider the following map

$$L(p, q_1) \xrightarrow{\pi} L(p, q_1) \vee S^3 \xrightarrow{f \vee h} L(p, q_2) \vee S^3 \underset{\substack{\uparrow \\ \text{given by id on } L(p,q_2) \text{ and} \\ \text{given by the covering} \\ \text{map } S^3 \to L(p,q_2) \text{ on } S^3}}{\to} L(p, q_2)$$

[3367]It follows from the Pasting Proposition 4.7 together with the Compact Image Lemma 4.15 and the Compact-Closed Lemma 3.39 that π is continuous.
[3367]In Exercise 185.18 we introduced the pinching map $M\#N \to M \vee N$. Our pinching map is homotopic to the map $M \cong M\#S^n \to (M\#S^n) \vee S^n \to M \vee S^n$.

It follows from the Pinching Lemma 305.9, the choice of f and h and the Covering Map-Degree Proposition 186.9 (or alternatively the Covering Map-Degree Proposition 195.7) that the induced maps on homology groups are given as follows:

$$\underbrace{\mathrm{H}_3(L(p,q_1))}_{\mathbb{Z}\cdot[L(p,q_1)]} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \underbrace{\mathrm{H}_3(L(p,q_1)\vee S^3)}_{\mathbb{Z}\cdot[L(p,q_1)]\oplus\mathbb{Z}\cdot[S^3]} \xrightarrow{\begin{pmatrix}(q_2m)^2\cdot s_1\cdot s_2 & 0\\ 0 & -d\end{pmatrix}} \underbrace{\mathrm{H}_3(L(p,q_2)\vee S^3)}_{\mathbb{Z}\cdot[L(p,q_2)]\oplus\mathbb{Z}\cdot[S^3]} \xrightarrow{(1\ p)} \underbrace{\mathrm{H}_3(L(p,q_2))}_{\mathbb{Z}\cdot[L(p,q_2)]}.$$

In summary we see that the map $L(p,q_1)\to L(p,q_2)$ has degree $(q_2m)^2\cdot s_1\cdot s_2 - d\cdot p = \epsilon$. ■

305.3. **Topological spaces dominated by CW-complexes.** We move on to another application of the Whitehead Theorem 305.1. To do so, let us recall Definition 190.1:

Definition 305.10. We say that a topological space X is dominated by a topological space Y if there exist continuous maps $i\colon X\to Y$ and $r\colon Y\to X$ such that $r\circ i$ is homotopic to the identity id_X.

The goal of this section is to prove the following result which we already mentioned in Section 190.1.

Proposition 305.11. (CW-Complex Domination Proposition) Let X be a topological space. If X is dominated by a CW-complex, then X is homotopy equivalent to a CW-complex.

Remark.

(1) An alternative proof for the CW-Complex Domination Proposition 305.11, that does not use the Whitehead Theorem 305.1, but that uses a mapping telescope construction similar to the telescope construction from Definition 140.20, is given in [**Hat2002**, Proposition A.11].
(2) In Section 190.1 we stated, without proofs but with detailed references, several generalizations of refinements of the CW-Complex Domination Proposition 305.11. □

The following simply definition will prove convenient in the proof of the CW-Complex Domination Proposition 305.11.

Definition 305.12. Let $f\colon X\to Y$ be a continuous map between topological spaces. A continuous map $g\colon Y\to X$ is called a homotopy left inverse of f if $g\circ f$ is homotopic to id_X. Furthermore, a map $h\colon Y\to X$ is called a homotopy right inverse of f if $f\circ h$ is homotopic to id_Y.

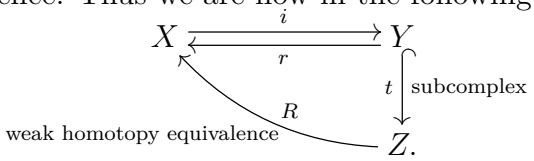
Lemma 305.13. (Left-Right Homotopy Inverse Lemma) Let $f\colon X\to Y$ be a continuous map between topological spaces. If f admits a homotopy left inverse g and a homotopy right inverse h, then f is in fact a homotopy equivalence and both g and h are homotopy inverses of f.

Proof of the Left-Right Homotopy Inverse Lemma 305.13. As in the Homotopy Equivalence-Reformulation Lemma 19.13 we see that homotopy left/right inverses correspond to left/right inverses in the homotopy category $\mathcal{HomTop}$. Thus the statement follows immediately from the Inverse Uniqueness Lemma 5.10 (1). ■

Now it is fairly straightforward to provide a proof of the CW-Complex Domination Proposition 305.11.

Proof of the CW-Complex Domination Proposition 305.11. Let X be a topological space that is dominated by a CW-complex Y. Recall that this means that there exist continuous maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X. Without loss of generality we can and will assume that X and Y are path-connected. We pick a base point $y_0 \in Y$ and we write $x_0 := r(x_0)$. Since $r \circ i$ is homotopic to the identity we obtain from the Homotopy-π_n-Proposition 134.16 that the map $r_* \circ i_* = (r \circ i)_*\colon \pi_j(X, x_0) \to \pi_j(X, x_0)$ is an isomorphism for all $j \in \mathbb{N}_0$. In particular we see that $r_*\colon \pi_j(Y, y_0) \to \pi_j(X, x_0)$ is an epimorphism for all $j \in \mathbb{N}_0$.

By the Extend-to-k-Connected Map Corollary 304.4 there exists a CW-complex Z which contains Y as a subcomplex and a continuous map $R\colon Z \to X$ that extends r and that is a weak homotopy equivalence. Thus we are now in the following setting:



Here denote by $t\colon X \to Z$ the inclusion map. (Note though that we do not claim that the diagram commutes.) It remains to prove the following claim:

Claim. The map $t \circ i\colon X \to Z$ is a homotopy equivalence.

Proof. Since $R \circ (t \circ i) = (R \circ t) \circ i = r \circ i$ is homotopic to id_X we know that R is homotopy left inverse of $t \circ i$. Furthermore, we obtain from the Homotopy-π_n-Proposition 134.16 that $R \circ (t \circ i)$ is a weak homotopy equivalence.[3368] Since R is a weak homotopy equivalence we see that $t \circ i$ is also a weak homotopy equivalence. Thus we see that $(t \circ i) \circ R\colon Z \to Z$ is a weak homotopy equivalence. Since Z is a CW-complex it follows from the Whitehead Theorem 305.1 that the map $(t \circ i) \circ R$ is a homotopy equivalence. This implies that there exists a continuous map $k\colon Z \to Z$ such that $(t \circ i) \circ R \circ k$ is homotopic to id_Z. We have thus shown that $R \circ k$ is a homotopy right inverse for $t \circ i$. Since we already pointed out above that $t \circ i$ admits is a homotopy left inverse we obtain from the Left-Right Homotopy Inverse Lemma 305.13 that $t \circ i$ is a homotopy equivalence. ∎

305.4. **Highly connected CW-complexes.** In this section we want to discuss the following natural question.

Question 305.14. Let X be a CW-complex which is n-connected for some $n \in \mathbb{N}_0$. Is X homotopy equivalent to a CW-complex Y which has one 0-cell and no cells in dimensions $1, \dots, n$?

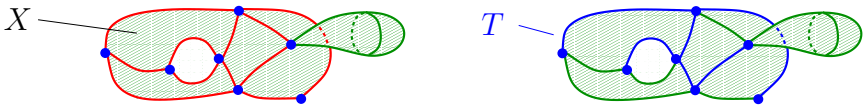
It turns out that we already dealt with the not entirely trivial case, that $n = 0$. For the convenience of the reader we recall the CW-Complex–Spanning Tree Proposition 140.5. Its precise statement gives us one possible approach to answering Question 305.14.

Proposition 140.5. (CW-Complex–Spanning Tree Proposition) Let X be a 0-connected CW-complex.

(1) There exists a contractible 1-dimensional subcomplex T which contains all 0-cells.[3369]

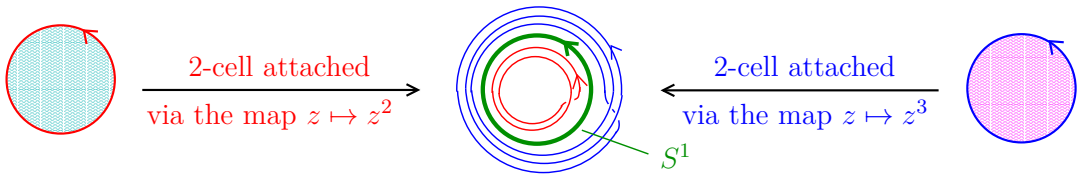
[3368] Throughout the proof we use the characterization of weak homotopy equivalences given by the Weak Homotopy Equivalence Criteria Lemma 304.10 (3).

(2) Given any T as in (1) the quotient X/T is a CW-complex with precisely one 0-cell. Furthermore the projection $X \to X/T$ is a homotopy equivalence.



The CW-Complex–Spanning Tree Proposition 140.5 suggests the following approach to answering Question 305.14: Given an n-connected CW-complex X we need to find a contractible subcomplex S that contains the n-skeleton X^n. The catch is, as the following example shows, in general such a subcomplex just does not exist.

Example. Let X be the 2-dimensional CW-complex corresponding to the presentation $\langle x \,|\, x^2 = e, x^3 = e \rangle$.[3370] This CW-complex is constructed by attaching two 2-cells to S^1 via the attaching maps $z \mapsto z^2$ and $z \mapsto z^3$.



We have the following isomorphisms:

$$\pi_1(X) \underset{\text{by the Presentation–CW Lemma 138.5}}{\cong} \langle x \,|\, x^2 = e, x^3 = e \rangle \underset{\text{since } \langle x \rangle \cong \mathbb{Z}}{\cong} \mathbb{Z}/\text{subgroup generated by 2 and 3} \underset{\text{since 2 and 3 are coprime}}{=} 0.$$

We have thus shown that X is 1-connected. In Exercise 305.8 we will see that X does not admit a contractible subcomplex S which contains the 1-skeleton of X and which has the property that X/S is homotopy equivalent to X. □

In light of the example it comes perhaps as a surprise that Question 305.14 actually does have an affirmative answer:

Proposition 305.15. (No-Low Cells Proposition) Let X be a CW-complex, let $x_0 \in X$ and let $n \in \mathbb{N}_0$. If X is n-connected, then there exists a CW-complex Y which has one 0-cell y_0 and no cells in dimensions $1, \ldots, n$ such that the pair (X, x_0) is homotopy equivalent to the pair (Y, y_0).

Proof. Let X be an n-connected CW-complex and let $x_0 \in X$. By the CW-Approximation Theorem 304.14 (2) there exists a CW-complex Y which has a single 0-cell $\{y_0\}$ and no cells in dimensions $1, \ldots, n$ and a continuous map $f\colon (Y, y_0) \to (X, x_0)$ which is a weak homotopy equivalence. Thus we can apply the Whitehead Theorem 305.1 (2) to the map f and we see that f is in fact a homotopy equivalence of pointed topological spaces. Well, that was quick. ■

Remark. The construction of the CW-complex Y in the proof of the No-Low Cells Proposition 305.15 is quite wasteful. In particular, without any refinement of the construction, it will almost invariably provide an infinite-dimensional CW-complex Y. It turns

[3369] Such a subcomplex is basically what we called a spanning tree in Definition 11.23.

[3370] Recall that in Construction 138.4 we showed that any finite presentation $\langle g_1, \ldots, r_k \,|\, r_1, \ldots, r_l \rangle$ naturally gives rise to a CW-complex with one 0-cell, k 1-cells and l 2-cells.

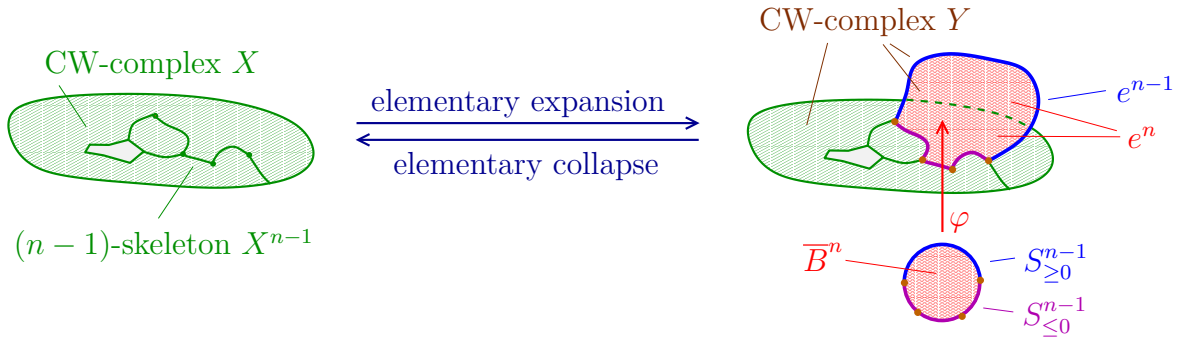
out that often one can do better. More precisely, in [**Hat2002**, Proposition 4.C.1] the following statement is shown: Given a finite n-connected CW-complex X there exists a CW-approximation $f\colon Y \to X$ where Y is a finite CW–complex with a single 0-cell and such that Y has no cells in dimensions $1, \dots, n$. □

305.5. **Elementary collapses.** It is not always easy to determine whether two topological spaces are homotopy equivalent. The following definition is useful for systematizing the search for homotopy equivalences between CW-complexes.

Definition 305.16. Let X be a CW-complex. We say that a CW-complex Y is an elementary expansion of X if the following holds:

(0) X is a subcomplex of Y.
(1) $Y = X \cup e^{n-1} \cup e^n$ where e^{n-1} is an $(n-1)$-cell that is not contained in X and e^n is an n-cell that is not contained in X.
(2) There exists a continuous map $\varphi\colon \overline{B}^n \to Y$ with the following properties:
  (a) The map φ is a characteristic map for the n-cell e^n.
  (b) The map[3371] $\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}_{\geq 0} \xrightarrow{\varphi} Y$ is a characteristic map for the $(n-1)$-cell e^{n-1}.
  (c) We have $\varphi(S^{n-1}_{\leq 0}) \subseteq X^{n-1}$.

Conversely we also say that the CW-complex X is an elementary collapse of Y . We refer to the figure below for an illustration.



We continue with another definition:

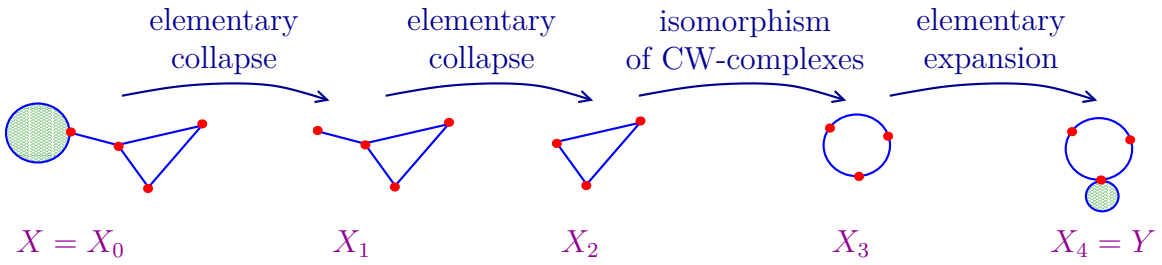
Definition 305.17. Let X and Y be CW-complexes. We say that X and Y are simple homotopy equivalent if there exists a sequence $X = X_0, X_1, \dots, X_k = Y$ such that for $i = 1, \dots, k$ one of the following three statements holds:

(1) X_i is an elementary expansion of X_{i-1}, or
(2) X_i is an elementary collapse of X_{i-1}, or
(3) X_i is isomorphic, as a CW-complex, to X_{i-1}.

Remark. In the Whitehead–Simple Homotopy Characterization Theorem 349.23 we will give a characterization of simple homotopy equivalences in terms of "Whitehead torsion". □

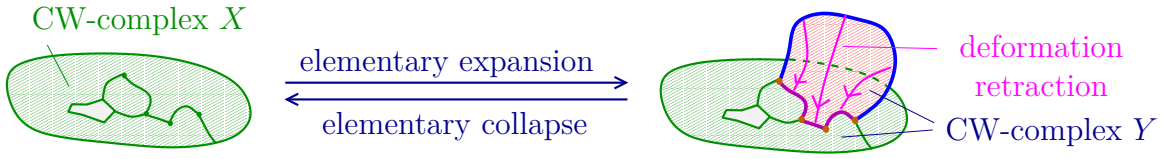
The following lemma justifies the name "simple homotopy equivalent".

[3371] Here we use the explicit homeomorphism $\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}_{\geq 0}$ from page 183.



Lemma 305.18. (Simple Homotopy Equivalence Lemma) Let X and Y be CW-complexes. If X and Y are simple homotopy equivalent, then they are homotopy equivalent.

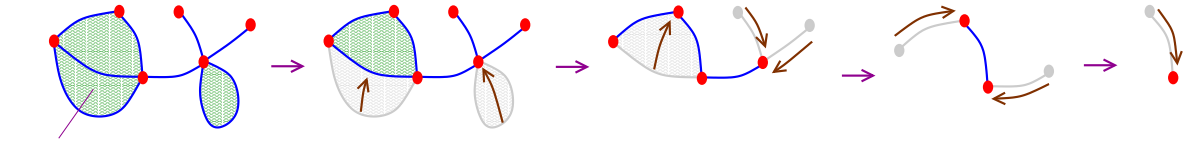
Proof. By the Homotopy-Equivalence Basics Lemma 19.14 the notion of being homotopy equivalent is an equivalence relation. Thus it suffices to show that if a CW-complex X is an elementary collapse of a CW-complex Y, then X is homotopy equivalent to Y.



In fact, by the Deformation Retract–Homotopy Equivalence Lemma 19.10 it suffices to show that X is a deformation retract of Y. We saw on page 523 that there exists a deformation retraction from $\overline{B}^n$ to $S^{n-1}_{\leq 0}$. It follows from the Homotopy Pushout Lemma 18.13 that this deformation retraction, together with the stationary homotopy on X, defines a deformation retraction from Y to X. ■

It is also interesting to consider an interesting special case.

Definition 305.19. A finite CW-complex is called collapsible if there exists a sequence of elementary collapses that abuts in a CW-complex consisting of a single point.



The following lemma is an immediate consequence of the Simple Homotopy Equivalence Lemma 305.18.

Lemma 305.20. (Collapsible-implies-Contractible Lemma) Let X be a CW-complex. If X is collapsible, then X is contractible.

It is natural to ask whether the converses of the Simple Homotopy Equivalence Lemma 305.18 and the Collapsible-implies-Contractible 305.20 hold. More precisely, we have the following questions.

Question 305.21.

(1) Let X be a finite CW-complex. If X is homotopy equivalent to a point, does it follow that X is collapsible?
(2) Let X and Y be finite CW-complexes.
  (a) If X and Y are homeomorphic, does it follow that X and Y are simple homotopy equivalent?
  (b) If X and Y are homotopy equivalent, does it follow that X and Y are simple homotopy equivalent?

In the remainder of this chapter we will discuss Question 305.21 (1). We will answer Question 305.21 (2) when we discuss Chapman's Theorem **??** and the Homotopy Equivalent-Not Simple Homotopy Equivalent Proposition **??**. To address Question 305.21 (1) it is useful to formulate the following corollary to the Whitehead Theorem 305.1.

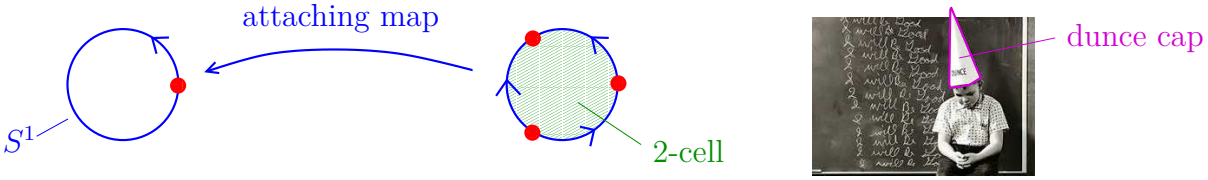
Corollary 305.22. (CW-Complex-Contractible Corollary) Let X be a CW-complex. If X is simply connected and if we have $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for all $n \geq 2$, then X is contractible, i.e. X is homotopy equivalent to a point.

Proof. We let $Y = \{*\}$ be the topological space that consists of a single point and let $f\colon X \to Y = \{*\}$ be the only map there is. It follows immediately from the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 that f is a weak homotopy equivalence. Thus the desired statement follows from the Whitehead Theorem 305.1 (1). ■

In the following we will use the CW-Complex-Contractible Corollary 305.22 to give not one, but two examples of CW-complexes that give a negative answer to Question 305.21 (1).

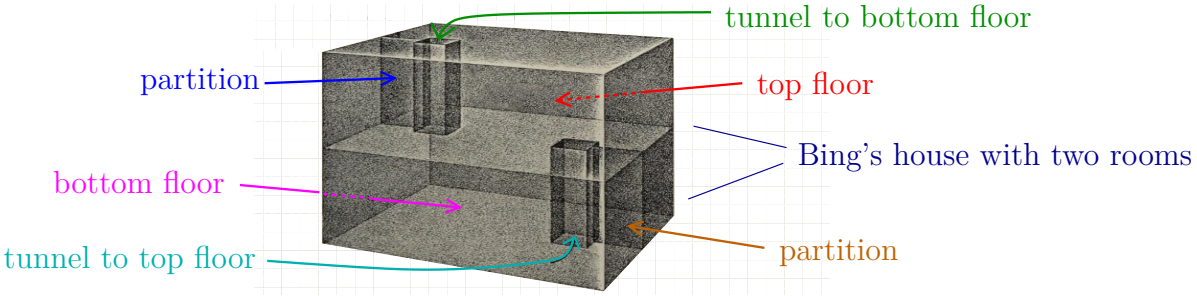
Examples.

(1) We equip the circle S^1 with the usual CW-structure with one 0-cell $1 \in S^1$ and one 1-cell. Let $\gamma = \mathrm{id}\colon S^1 \to S^1$ be the identity map. The *dunce cap* is defined as the CW-complex X that is given by attaching one 2-cell to S^1 where the attaching map $S^1 \to S^1$ is given by the loop $\gamma * \overline{\gamma} * \gamma$,[3372] i.e. by the loop that is given by the composition of γ, $\overline{\gamma}$ and γ. Using the Cell Attachment-π_1-Proposition 135.5 (2) it is straightforward to show that X is simply connected. Furthermore it follows almost from the Cellular Boundary–Degree Proposition 155.15 that the cellular homology, and thus by the Singular-Cellular-H_*-Proposition 155.8 also the singular homology, vanishes in dimensions ≥ 2. It follows from the CW-Complex-Contractible Corollary 305.22 that the dunce cap is contractible.[3373] On the other hand it is clear that the dunce cap is not collapsible since there is only one cell in dimensions 0, 1 and 2 and no two of them can be matched to allow for an elementary collapse.

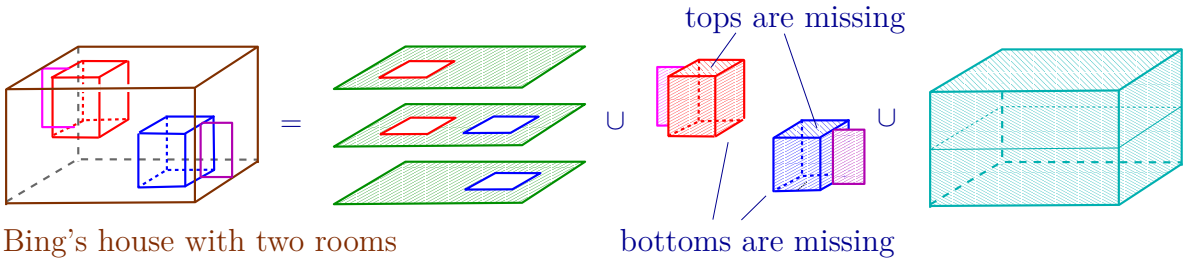


[3372] The conscientious reader will notice that we had encountered this topological space in Exercise 140.2.
[3373] This statement can also be obtained from the Homotopy Equivalence-Mapping Cone Proposition 22.25 or the Homotopy Equivalent Cell Attachment Proposition 135.13. In principle one could also prove this "by hand", but I am pretty sure that few people will enjoy doing this exercise.

(2) It is also possible to give an example of a CW-complex that can be visualized as a subspace of $\mathbb{R}^3$. More precisely, consider the topological space X that is shown in the two figures.



It is commonly referred to as *Bing's house with two rooms*.[3374] We equip X with the fairly obvious CW-structure. In Exercise 305.10 we will see that X is simply connected and that $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for $n \geq 2$.



Thus, as in the previous example we can appeal to the CW-Complex-Contractible Corollary 305.22 to conclude that the house with two rooms is contractible.[3375] We leave the arduous task of showing that X is not collapsible to the reader. □

Let X be the dunce cap. In 1963 Christopher Zeeman [**Zee1963a**, Theorem 1] showed that $X \times [0,1]$, equipped with the product CW-structure, is in fact collapsible. This result prompted Zeeman in 1963 to propose the following conjecture [**Zee1963a**, Conjecture 1].

Conjecture 305.23. (Zeeman Conjecture) If X is a finite contractible 2-dimensional CW-complex, then $X \times [0,1]$ is collapsible.

Like many other conjectures on CW-complexes, this conjecture sounds deceptively simple and elementary. In fact Zeeman was fully aware of the fact that the conjecture is almost certainly very difficult since he showed [**Zee1963a**, Theorem 2] that a proof of the Zeeman Conjecture 305.23 would lead to a new proof of the Poincaré Conjecture in dimension three, which was finally proved in 2003 by Perelman using a heavy dose of analysis, see the Poincaré 3D-Conjecture Theorem **??**. Furthermore, Sergey Matveev [**Matv1987**] (see also [**Kupe2017a**, Proposition 4.2] and [**Barm2011**, Chapter 11]) showed that the Andrews-Curtis Conjecture 100.34 is a special case of the Zeeman Conjecture 305.23. We refer to [**MR1993, Matv2007, Kupe2017a**] for more on the Zeeman Conjecture.

[3374] This example was given by R. H. Bing [**Bin1964**, p. 109] in 1964. In fact the picture we show in the first figure below is a slightly modified version of Bing's original drawing.

[3375] A more direct argument for showing that Bing's house with two rooms is contractible is given in Exercise 305.16 or alternatively on [**Hat2002**, p. 4].

305.6. Exercises for Chapter 305.

Exercise 305.1. Let $p \in \mathbb{N}$ and let $q_1, \ldots, q_n \in \mathbb{Z}$ be integers that are coprime to p. We consider the following action of $\mathbb{Z}/p\mathbb{Z}$ on $S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, |z_1|^2 + \cdots + |z_n|^2 = 1\}$:

$$\begin{array}{rcl} \mathbb{Z}/p\mathbb{Z} \times S^{2n-1} & \to & S^{2n-1} \\ ([k], (z_1, \ldots, z_n)) & \mapsto & (z_1 \cdot e^{2\pi \mathrm{i} k \cdot q_1/p}, \ldots, z_n \cdot e^{2\pi \mathrm{i} k \cdot q_n/p}). \end{array}$$

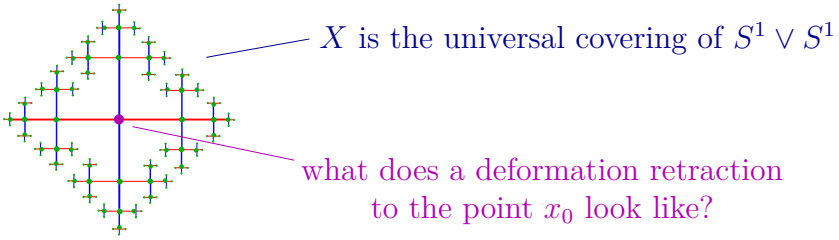
One can easily verify that this action is smooth, orientation-preserving and discrete action. Thus it follows from the Manifold–Quotient Proposition 25.39 and the Orientation-Action Proposition 31.44 that the quotient $L(p; q_1, \ldots, q_n) \coloneqq S^{2n-1}/(\mathbb{Z}/p\mathbb{Z})$ is a closed orientable smooth manifold of dimension $2n-1$.

Let $q_1, \ldots, q_n$ and $\widetilde{q}_1, \ldots, \widetilde{q}_n$ be integers that are coprime to p. Show that if there exists an $m \in \mathbb{Z}$ with $q_1 \cdot \ldots \cdot q_n \equiv m^n \cdot \widetilde{q}_1 \cdot \ldots \cdot \widetilde{q}_n$, then $L(p; q_1, \ldots, q_n)$ and $L(p; \widetilde{q}_1, \ldots, \widetilde{q}_n)$ are homotopy equivalent.

Remark. For $n = 2$ this is just a slight reformulation of the Lens Space-Homotopy Equivalence Proposition 305.6.

Exercise 305.2. Let $X = \mathbb{R}$. We equip X with the CW-structure where the 0-skeleton is given by $\mathbb{Z}$. The proof of the Whitehead Deformation Retract Proposition 305.2 provides a fairly explicit deformation retraction from $X = \mathbb{R}$ to $\{0\}$. What does this deformation retraction look like?

Exercise 305.3. Let X be the universal covering of $S^1 \vee S^1$. On page 5857 we saw that X admits a deformation retraction to $x_0 \in X$. What does an explicit deformation retraction of X to the point x_0 look like? We refer to the figure below for an illustration.



Exercise 305.4. We consider the following subset of $\mathbb{R}^2$:

$$\begin{aligned} X \;\coloneqq\; & \text{Euclidean segment from } (0,1) \text{ to } (0,-1) \\ & \cup \bigcup_{n \in \mathbb{N}} \text{Euclidean segment from } (0,1) \text{ to } (\tfrac{1}{n}, 0) \\ & \cup \bigcup_{n \in \mathbb{N}} \text{Euclidean segment from } (0,-1) \text{ to } (-\tfrac{1}{n}, 0). \end{aligned}$$

We equip X with the subspace topology coming from $\mathbb{R}^2$. We refer to the figure on page 5859 for an illustration.

(a) Show that all of the homotopy groups of X are trivial.
(b) Show that X is not contractible.
You might want to make use of the Contractability Criterion Lemma 19.20.

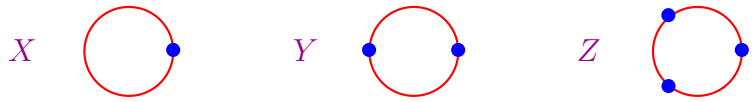
Exercise 305.5. Let X and Y be two CW-complexes of dimension at most n. Furthermore let $f\colon X \to Y$ be a continuous map that is $(n+1)$-connected. Show that f is a homotopy equivalence.

Exercise 305.6. Let $f\colon X \to Y$ be a continuous map between path-connected CW-complexes. Let $x_0 \in X$. We assume that $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is the trivial map for every $k \in \mathbb{N}$. Does it follow that f is homotopic to a constant map?

Exercise 305.7. Let X be a connected CW-complex and let $k \leq l$ be natural numbers. We suppose that $\pi_i(X) = \mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = k, \dots, l$. Does it follow that X is homotopy equivalent to a CW-complex which has no cells in dimensions $k, \dots, l$?
Remark. If $k = 1$, then this question is precisely Question 305.14.

Exercise 305.8. Let X be the 2-dimensional CW-complex corresponding to the presentation $\langle x \,|\, x^2 = e, x^3 = e\rangle$. We refer to the figure on page 5868 for an illustration. Show that X does not admit a contractible subcomplex S which contains the 1-skeleton of X and which has the property that X/S is homotopy equivalent to X.
Hint. Use the long exact sequence coming from the Quotient–LES Theorem 149.17.

Exercise 305.9. Show that the three CW-complexes shown in the figure below are simple homotopy equivalent.
Remark. It is possibly less confusing to first consider Y and Z.



Exercise 305.10. Let X be Bing's house with two rooms, i.e. X is the CW-complex that we defined in the figure on page 5872.

(a) Show that X is simply connected.
(b) Show that $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for $n \geq 2$.
Hint. Using the CW-Complex-H_*-Proposition 155.10 (3) and using Euler characteristics will cut down the work considerably.
(c) Show that X is not collapsible.

Exercise 305.11. Let $n \in \mathbb{N}$ and furthermore let $f, g\colon S^n \to S^n$ be two continuous maps. Show that if $f_* = g_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$, then f and g are homotopic.

Exercise 305.12. Let $T = S^1 \times S^1$ be the 2-dimensional torus, equipped with the usual CW-structure. We attach two 2-cells along $S^1 \times \{0\}$ and along $\{0\} \times S^1$ and we obtain a CW-complex X.

(a) Show that X that is homotopy equivalent to S^2.
(b) Show that there is no embedding $S^2 \to X$.
Remark. You might want to use Exercise 170.6.

Exercise 305.13. Show that the long line $\mathbb{L}$, which we introduced in Definition 28.19, is weakly contractible but not contractible.
Hint. Use the Long Ray Properties Proposition 28.21.

Exercise 305.14.

(a) Show that all homotopy groups of $\mathbb{R}\mathrm{P}^2 \times S^3$ and $S^2 \times \mathbb{R}\mathrm{P}^3$ are isomorphic.

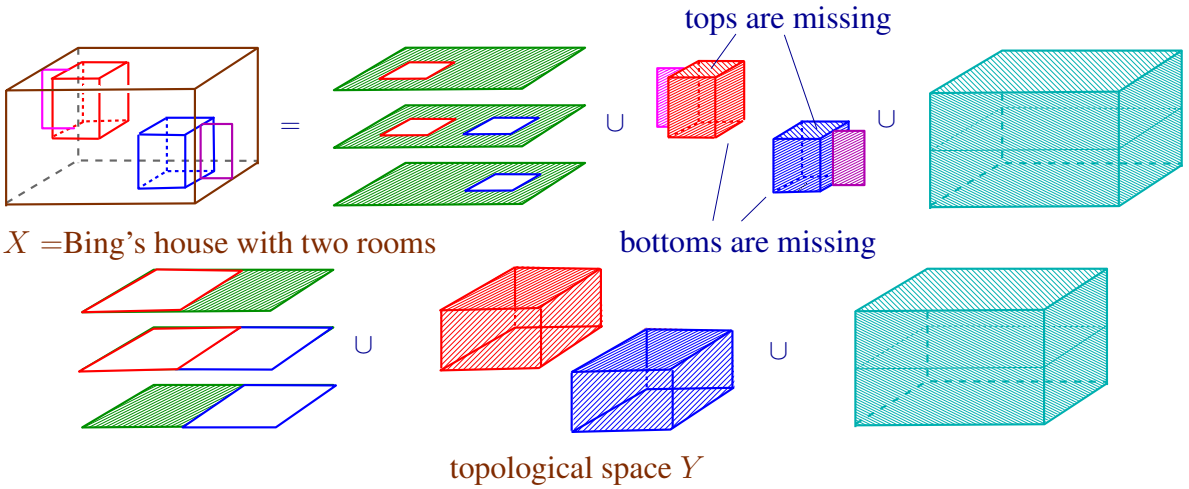
(b) Are $\mathbb{R}\mathrm{P}^2 \times S^3$ and $S^2 \times \mathbb{R}\mathrm{P}^3$ homotopy equivalent?

Exercise 305.15. Let X be a non-empty CW-complex such that all homology groups of X are zero. Show that there exists a $k \in \mathbb{N}_0$ such that the k-fold suspension of X is contractible.

Exercise 305.16. Let X be Bing's house with two rooms from page 5872 and that is also shown below.

(a) Show that X is homotopy equivalent to the topological space Y that is shown below.
Hint. Find a subspace Z of R^3 such that X and Y are both deformation retracts of Z.

(b) Show that X is contractible.



CHAPTER 306

Generalized lens spaces

In this chapter, given $m \in \mathbb{N}$ and $r_1, \dots, r_d \in \mathbb{Z}$ that are coprime to m we will introduce the generalized lens spaces $L(m|r_1, \dots, r_d) = S^{2d-1}/(\mathbb{Z}/m\mathbb{Z})$, which generalize the "classical" lens spaces $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ from Definition 91.4 and which generalize the odd-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{2d-1}$. We will also introduce the infinite-dimensional generalized lens spaces $L(m|r_1, r_2, \dots)$ which generalize the infinite-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{\infty}$. These generalized lens spaces will be an interesting source of examples in the next Chapter 307, when we will introduce the Eilenberg-Maclane Spaces.

This short chapter is organized as follows:

- In Section 306.1 we will introduce (infinite-dimensional) generalized lens spaces and we will discuss basic properties.
- In Section 306.2 we will see that generalized lens spaces have rather simple CW-structures.
- In Section 306.3 we will calculate several invariants of generalized lens spaces.
- In Section 306.4 we will provide exercises for Chapter 306.

In the next Chapter 307 we will introduce the Eilenberg-Maclane Spaces.

306.1. **Definition of generalized lens spaces.** We start out with the following lemma, which is a generalization of the Lens Space-Discrete Action Lemma 91.3:

Lemma 306.1. (Generalized Lens Space-Discrete Action Lemma) In the following let $m \in \mathbb{Z} \setminus \{0\}$, let $d \in \mathbb{N}_0$, and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m. We set $\xi \coloneqq \exp(2\pi \mathrm{i}/m)$.

(1) (a) The map[3376]

$$\begin{array}{rcl} \mathbb{Z}/m\mathbb{Z} \times S^{2d-1} & \to & S^{2d-1} \\ ([k], \underbrace{(z_1, \dots, z_d)}_{\in \mathbb{C}^{2d}}) & \mapsto & (\xi^{r_1 \cdot k} \cdot z_1, \dots, \xi^{r_d \cdot k} \cdot z_d) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}/m\mathbb{Z}$ on S^{2d-1}.

(b) The natural projection $p\colon S^{2d-1} \to L(m|r_1, \dots, r_d) \coloneqq S^{2d-1}/(\mathbb{Z}/m\mathbb{Z})$ is an m-fold covering.

(2) The quotient space $L(m|r_1, \dots, r_d) = S^{2d-1}/(\mathbb{Z}/m\mathbb{Z})$ is a closed $(2d-1)$-dimensional topological manifold and it admits a unique structure of an oriented smooth manifold such that the natural projection $p\colon S^{2d-1} \to L(m|r_1, \dots, r_d)$ is an orientation-preserving local diffeomorphism.

Proof. The proof of Statements (1a) and (2) is essentially identical to the proof of the Lens Space-Discrete Action Lemma 91.3. Finally note that Statement (1b) follows from Statement (1a) together with the Action–Covering Proposition 91.11. ∎

The following definition is an analog of the definition of lens spaces in Definition 91.4:

Definition 306.2. Let $m \in \mathbb{Z} \setminus \{0\}$, let $d \in \mathbb{N}_0$, and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m. We refer to

$$L(m|r_1, \dots, r_d) \coloneqq S^3/(\mathbb{Z}/m\mathbb{Z}),$$

with the action from the Generalized Lens Space-Discrete Action Lemma 306.1, as a generalized lens space. We always view $L(m|r_1, \dots, r_d)$ as an oriented $(2d-1)$-dimensional smooth manifold.

Example.

(1) Let $p \, in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We denote by $L(p, q)$ the lens space as defined on Definition 91.4. It follows immediately from the definitions that $L(p, q) = L(p|1, q)$.
(2) It follows immediately from the Projective Space-Sphere Quotient Lemma 9.13 (1) that for any $d \in \mathbb{N}_0$ the generalized lens space $L(2|\underbrace{1, \dots, 1}_{d \text{ times}})$ is diffeomorphic to $\mathbb{R}\mathrm{P}^{2d-1}$. □

On page 338 we saw that for any $m < n$ there exists an embedding $\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$. The following lemma generalizes this observation:

Lemma 306.3. (Generalized Lens Space–Embedding Lemma) Let $m \in \mathbb{Z} \setminus \{0\}$, let $d \in \mathbb{N}_0$ and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m. For any $c \leq d$ the map

$$\begin{array}{rccl} \iota \colon & L(m|r_1, \dots, r_c) & \to & L(m|r_1, \dots, r_d) \\ & [(z_1, \dots, z_c)] & \mapsto & [(z_1, \dots, z_c, 0, \dots, 0)] \end{array}$$

is an embedding.

Proof. The proof of the lemma is essentially identical to the argument on page 338: First one can easily verify that the map ι is injective. Next note that it follows from the Quotient-to-Quotient Lemma 8.6 that the map ι is continuous. By the Generalized Lens Space-Discrete Action Lemma 306.1 (2) we know that the left-hand side is compact and that the right-hand side is Hausdorff. It thus follows from the Compact-Hausdorff Proposition 4.23 that the map ι is indeed an embedding. ■

Recall that in Definition 104.21 we introduced the infinite-dimensional real projective space $\mathbb{R}\mathrm{P}^\infty \coloneqq \varinjlim \mathbb{R}\mathrm{P}^i$. We now have a similar definition in the context of generalized lens spaces:

Definition 306.4. Let $m \in \mathbb{Z} \setminus \{0\}$ and let $r_1, r_2, \dots \in \mathbb{Z}$ be integers, each of which is coprime to m. We consider the direct sequence

$$L(m|r_1) \xrightarrow{[x] \mapsto [(x, 0_{\mathbb{C}})]} L(m|r_1, r_2) \xrightarrow{[x] \mapsto [(x, 0_{\mathbb{C}})]} L(m|r_1, r_2, r_3) \xrightarrow{[x] \mapsto [(x, 0_{\mathbb{C}})]} \dots$$

of embeddings that are provided by the Generalized Lens Space–Embedding Lemma 306.3. We refer to

$$L(m|r_1, r_2, \dots) \coloneqq \varinjlim L(m|r_1, \dots, r_i)$$

as an infinite-dimensional generalized lens space.

Example. It follows almost immediately from the Projective Space-Sphere Quotient Lemma 9.13 (1) that the infinite-dimensional generalized lens space $L(2|1, 1, \dots)$ is homeomorphic to the infinite-dimensional real projective space $\mathbb{R}\mathrm{P}^\infty$. □

The following lemma, which is a mild generalization of the Infinite Projective Space-Sphere Quotient Lemma 104.22, gives us a reinterpretation of infinite-dimensional generalized lens spaces:

Lemma 306.5. (Infinite Lens Space-Sphere Quotient Lemma) Let $m \in \mathbb{Z} \setminus \{0\}$ and let $r_1, r_2, \cdots \in \mathbb{Z}$ be integers, each of which is coprime to m.

(1) The map

$$\begin{array}{rcl} \mathbb{Z}/m\mathbb{Z} \times S^\infty & \to & S^\infty \\ ([k], (z_1, z_2, \dots)) & \mapsto & (\xi^{r_1 \cdot k} \cdot z_1, \xi^{r_2 \cdot k} \cdot z_2, \dots) \end{array}$$

defines an action of the group $\mathbb{Z}/m\mathbb{Z}$ on S^∞ and this action is continuous and discrete.

(2) There exists a homeomorphism $S^\infty/(\mathbb{Z}/m\mathbb{Z}) \to L(m|r_1, r_2, \dots)$.

Proof. The proof is almost identical to the proof of the Infinite Projective Space-Sphere Quotient Lemma 104.22. ■

306.2. **CW-structures on generalized lens spaces.** Recall that on page 2628 we showed that for any $n \in \mathbb{N}_0$ the real projective space $\mathbb{R}\mathrm{P}^n$ admits a CW-structure with a single cell in dimension $0, \dots, n$. Furthermore recall that on page 2628 we showed that $\mathbb{R}\mathrm{P}^\infty = \varinjlim \mathbb{R}\mathrm{P}^i$ admits a CW-structure with precisely one cell in each dimension. Also recall that in the proof of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18 we determined the cellular chain complex of $\mathbb{R}\mathrm{P}^n$.

We can now prove analogous statements for generalized lens spaces:

Proposition 306.6. (Generalized Lens Space-CW-Proposition) Let $m \in \mathbb{Z} \setminus \{0\}$.

(1) Let $d \in \mathbb{N}_0$ and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m.

(a) The quotient space $L(m|r_1, \dots, r_d) \coloneqq S^{2d-1}/(\mathbb{Z}/m\mathbb{Z})$ admits a CW-structure which has precisely one cell of each dimension $0, \dots, 2d-1$, and no other cells and such that for any $c \in \{0, \dots, d\}$ the $(2c-1)$-skeleton is given by $\iota(L(m|r_1, \dots, r_c))$.

(b) The cellular chain complex corresponding to the CW-structure from (a) is isomorphic to the chain complex

$$0 \to \underset{\substack{\uparrow \\ \text{degree } 2d-1}}{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot m} \cdots \to \underset{\substack{\uparrow \\ \text{degree } 2}}{\mathbb{Z}} \xrightarrow{\cdot m} \underset{\substack{\uparrow \\ \text{degree } 1}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\uparrow \\ \text{degree } 0}}{\mathbb{Z}} \to 0.$$

(2) Let $r_1, r_2, \cdots \in \mathbb{Z}$ be integers, each of which is coprime to m.

(a) The infinite-dimensional generalized lens spaces $L(m|r_1, r_2, \dots)$ admits a CW-structure which has precisely one cell of each dimension and which has the property that for any $d \in \mathbb{N}_0$ the $(2d-1)$-skeleton is given by $\iota(L(m|r_1, \dots, r_d))$.

(b) The cellular chain complex corresponding to the CW-structure from (a) is isomorphic to the chain complex

$$\cdots \to \underset{\substack{\uparrow \\ \text{degree } 3}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\uparrow \\ \text{degree } 2}}{\mathbb{Z}} \xrightarrow{\cdot m} \underset{\substack{\uparrow \\ \text{degree } 1}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\uparrow \\ \text{degree } 0}}{\mathbb{Z}} \to 0.$$

Proof. First note that Statement (2a) follows immediately from the definition of a CW-complex, together with the definition

$$L(m|r_1, r_2, \dots) = \varinjlim L(m|r_1, \dots, r_i)$$

and together with Statement (1a).[3377] Similarly Statement (2b) follows easily from Statement (1b).

We turn to the proof of Statement (1). We prove the proposition by induction on $d \in \mathbb{N}_0$. The case $d = 0$ is trivial since in this case $L(m|) = \varnothing$. Now we assume that the desired statement holds for some $d-1$. Next let $m \in \mathbb{Z} \setminus \{0\}$ and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m. To simplify the notation we use the embedding ι from the Generalized Lens Space–Embedding Lemma 306.3 to view $L(m|r_1, \dots, r_{d-1})$ as a subspace of $L(m|r_1, \dots, r_d)$.

By induction there exists a CW-structure $X^0 \subseteq \dots \subseteq X^{2d-3}$ on $L(m|r_1, \dots, r_{d-1})$ with the desired properties. Next we consider the following subspace of $L(m|r_1, \dots, r_d)$:

$$X^{2d-2} := \{[\ \underbrace{(\sin(\varphi)\cdot v, \cos(\varphi)\cdot e^{2\pi i j/m})}_{\text{point in } \mathbb{R}^{2d-2}\times\mathbb{C} = \mathbb{R}^{2d-2}\times\mathbb{R}^2 \text{ of length } 1}\ ] \,|\, v \in S^{2d-3}, \varphi \in [0, \tfrac{\pi}{2}] \text{ and } j \in \{0, \dots, m-1\}.$$

Our goal is to show that $X^0 \subseteq \dots \subseteq X^{2d-3} \subseteq X^{2d-2} \subseteq X^{2d-1} := L(m|r_1, \dots, r_d)$ is a CW-structure for $L(m|r_1, \dots, r_d)$ and we need to show that the corresponding cellular chain complex is of the desired form.

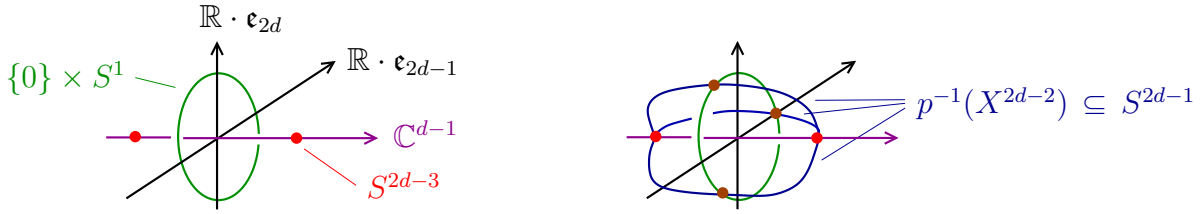


illustration for "$d = \frac{3}{2}$"

In fact these goals follow from the following two claims together with the Cellular Boundary–Degree Proposition 155.15.

Claim 1. We consider $S^{2d-3} \times [0, \frac{\pi}{2}]$ with the equivalence relation "$\sim$" that is given by $(v_0, 0) \sim (v_1, 0)$ for all $v_0, v_1 \in S^{2d-3}$. We consider the map

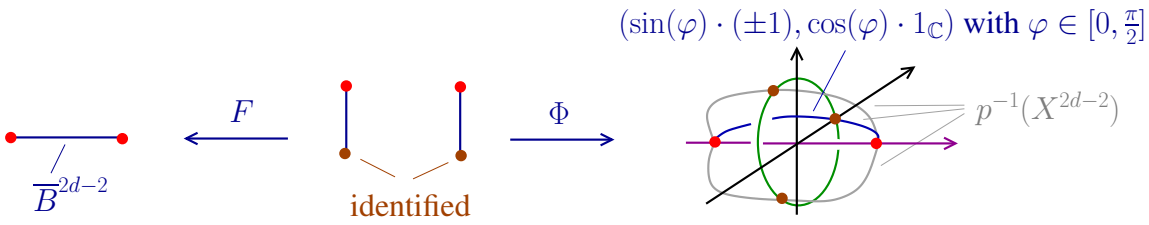
$$\begin{aligned} F\colon (S^{2d-3}\times[0,\tfrac{\pi}{2}])/\sim &\to \overline{B}^{2d-2} \\ [(v,\varphi)] &\mapsto \sin(\varphi)\cdot v \end{aligned}$$

and we consider the map

$$\begin{aligned} \Phi\colon (S^{2d-3}\times[0,\tfrac{\pi}{2}])/\sim &\to X^{2d-2} \\ [(v,\varphi)] &\mapsto [(\sin(\varphi)\cdot v, \cos(\varphi)\cdot 1_{\mathbb{C}})]. \end{aligned}$$

(1) The map F is a homeomorphism.
(2) (a) The map $\varphi := (\Phi \circ F^{-1})|_{S^{2d-3}}$ takes values in X^{2d-3}.
(b) The identity map on X^{2d-3} together with the map $\Phi \circ F^{-1}$ defines a homeomorphism
$$\alpha\colon X^{2d-3} \cup_\varphi \overline{B}^{2d-2} \to X^{2d-2}.$$
(3) The map $\varphi_*\colon \underbrace{\mathrm{H}_{2d-3}(S^{2d-3})}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{H}_{2d-3}(X^{2d-3}/X^{2d-4})}_{\cong\mathbb{Z}}$ is given by multiplication by $\pm m$.

[3377] For peace of mind one could also appeal to the Direct Colimit–CW Lemma 139.13, but this is not really necessary.



Proof.

(1) It is elementary to verify that the map F is a bijection. Using the Compact-Hausdorff Proposition 4.23 one can easily verify that F is a homeomorphism.

(2) Statement (a) follows easily from the observation that $F^{-1}(S^{2d-3})$ are precisely the points that are represented by points of the form $(v, \frac{\pi}{2}) \in S^{2d-2} \times [0, \frac{\pi}{2}]$. We now turn to the proof of Statement (b). Let

$$[(\sin(\varphi)\cdot v, \cos(\varphi)\cdot e^{2\pi \mathrm{i} j/m})] \;\in\; X^{2d-2} \setminus X^{2d-1}.$$

Since r_d is coprime to m we see that there exists a unique $k \in \{0, \ldots, m-1\}$ such that $k \cdot r_d \equiv -j \bmod m$, i.e. such that $e^{2\pi \mathrm{i} k r_d/m} \cdot e^{2\pi \mathrm{i} j/d} = 1_{\mathbb{C}}$. It follows from the definition of the $\mathbb{Z}/m\mathbb{Z}$-action on S^{2d-1} that $(\sin(\varphi)\cdot v, \cos(\varphi)\cdot e^{2\pi \mathrm{i} j/m})$ is equivalent to a unique point of the form $(\sin(t)\cdot w, \cos(t)\cdot 1_{\mathbb{C}})$. Using this observation it is now elementary to verify that the map $\alpha\colon X^{2d-3} \cup_{\varphi} \overline{B}^{2d-2} \to X^{2d-2}$ is a bijection.

Next note that it follows from the Topological-Pushout Proposition 8.12 (1) that the map α is continuous. By the Generalized Lens Space-Discrete Action Lemma 306.1 (2) together with the Pushout-Properties Proposition 8.14 (1) we know that the left-hand side is compact and that the right-hand side is Hausdorff. It thus follows from the Compact-Hausdorff Proposition 4.23 that α is indeed a homeomorphism.

(3) We consider the following diagram:

$$\begin{array}{ccc} & \mathrm{H}_{2d-3}(S^{2d-3}) & \\ {\scriptstyle p_*}\swarrow & & \searrow{\scriptstyle \varphi_*} \\ \mathrm{H}_{2d-3}(X^{2d-3}) & \xrightarrow{\;\text{я}\;} & \mathrm{H}_{2d-3}(X^{2d-3}, X^{2d-4}) \\ {\scriptstyle \text{я}}\searrow{\scriptstyle \cong} & & {\scriptstyle \cong}\swarrow \\ & \mathrm{H}_{2d-3}(X^{2d-3}, X^{2d-3}\setminus\{P\}). & \end{array}$$

The desired statement follows from the following clarifications and observations:

(a) Let $P \in X^{2d-3} \setminus X^{2d-4}$ be the center of the unique open $(2d-3)$-cell.

(b) The map $p\colon S^{2d-3} \to X^{2d-3} = L(m|r_1, \ldots, r_{d-1}) = S^{2d-3}/\sim$ is the natural projection.

(c) The maps я are the natural maps from absolute to relative homology.

(d) The right-bottom diagonal map is induced by the inclusion $X^{2d-4} \to X^{2d-3}\setminus\{P\})$.

(e) By the Generalized Lens Space-Discrete Action Lemma 306.1 (1b) we know that the natural projection $p\colon S^{2d-3} \to L(m|r_1, \ldots, r_{d-1})$ is an m-fold covering. It thus follows from the Covering Map-Degree Proposition 195.7 that the induced map $p_*\colon \mathrm{H}_{2d-3}(S^{2d-3}) \to \mathrm{H}_{2d-3}(L(m|r_1, \ldots, r_{d-1}))$ is given by multiplication by $\pm m$.

(f) Since $X^{2d-3} = L(m|r_1, \dots, r_{d-1})$ is a closed orientable connected non-empty $(2d-3)$-dimensional smooth manifold we obtain from the Orientation-Top-H_*-Theorem 193.3 that the left bottom diagonal map $я$ is an isomorphism.

(g) Note that we obtain X^{2d-3} from X^{2d-4} by attaching a single $(2d-3)$-cell. Since S^{2d-4} is a deformation retract of $\overline{B}^{2d-3} \setminus \{0\}$ we obtain from the Homotopy Pushout Lemma 18.13 that X^{2d-4} is a deformation retract of $X^{2d-3} \setminus \{P\}$. It now follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the right bottom diagonal map is an isomorphism.

(h) The diagram commutes by the functoriality of homology.

(i) Follows from (f)–(h) that the middle horizontal map is an isomorphism. ⊞

We are done with the proof of the proposition once we have proved the following claim:

Claim 2. We consider $S^{2d-3} \times [0, \frac{\pi}{2}] \times [0,1]$ with the equivalence relation "$\sim$" that is given by $(v_0, 0, t) \sim (v_1, 0, t)$ for $v_0, v_1 \in S^{2d-3}$. We consider the map

$$\begin{array}{rcccl} G\colon (S^{2d-3} \times [0, \frac{\pi}{2}] \times [0,1])/\sim & \to & \overline{B}^{2d-2} \times [0,1] & \xrightarrow{\cong} & \overline{B}^{2d-1} \\ [(v, \varphi, t)] & \mapsto & (\sin(\varphi) \cdot v, t) & & \end{array}$$

where the second map is the homeomorphism given by the Convex-to-Ball Proposition 4.28. We also consider the map

$$\begin{array}{rcl} \Psi\colon (S^{2d-3} \times [0, \frac{\pi}{2}] \times [0,1])/\sim & \to & X^{2d-1} \\ [(v, \varphi, t)] & \mapsto & [(\sin(\varphi) \cdot v, \underbrace{\cos(\varphi) \cdot e^{2\pi \mathrm{i} t/m}}_{\in \mathbb{C}})]. \end{array}$$

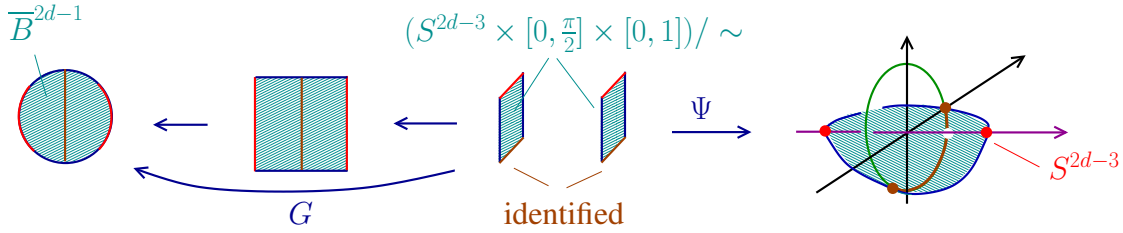
(1) The map G is a homeomorphism.

(2) (a) The map $\psi := (\Psi \circ G^{-1})|_{S^{2d-2}}$ takes values X^{2d-2}.

(b) The identity map on X^{2d-2} together with the map $\Psi \circ G^{-1}$ defines a homeomorphism

$$\beta\colon X^{2d-2} \cup_\psi \overline{B}^{2d-1} \to L(m|r_1, \dots, r_d).$$

(3) The map $\psi_*\colon \underbrace{H_{2d-2}(S^{2d-2})}_{\cong \mathbb{Z}} \to \underbrace{H_{2d-2}(X^{2d-2}/X^{2d-3})}_{\cong \mathbb{Z}}$ is the zero map.



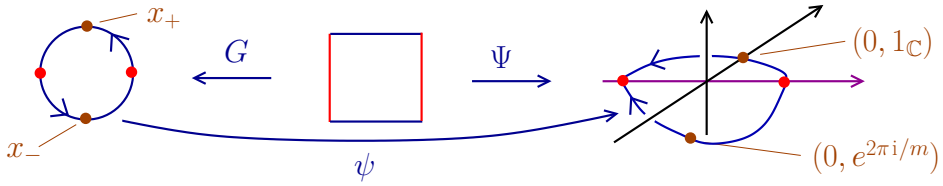
Proof.

(1) As in Claim 1 (1) we see that the first map in the definition of G is a homeomorphism. Since the second map is also a homeomorphism we have shown that G is indeed a homeomorphism.

(2) First note that $G^{-1}(S^{2d-2})$ are the points that are represented by points of the form $(v, \frac{\pi}{2}, t)$ or $(v, \varphi, 0)$ or $(v, \varphi, 1)$. All these points get sent under Ψ to X^{2d-2}. We now turn to the proof of Statement (b): Similar to Claim 1 we leave it to the reader to verify that the given map $\beta\colon X^{2d-2} \cup_\psi \overline{B}^{2d-1} \to L(m|r_1, \dots, r_d)$ is a bijection. Next

note that it follows from the Topological-Pushout Proposition 8.12 (1) that the map β is continuous. By the Generalized Lens Space-Discrete Action Lemma 306.1 (2) together with the Pushout-Properties Proposition 8.14 (1) we know that the left-hand side is compact and that the right-hand side is Hausdorff. It thus follows from the Compact-Hausdorff Proposition 4.23 that the map β is a homeomorphism.

(3) We consider the point $y := [(0, 1_{\mathbb{C}})] = [(0, e^{2\pi \mathrm{i}/m})] \in X^{2d-2}/X^{2d-3}$. Note that preimage $\psi^{-1}(y)$ consists of the two points $x_\pm := (0, \dots, 0, \pm 1)$.



We leave it to the reader to verify that, for suitable orientations of S^{2d-2} and of $X^{2d-2}/X^{2d-3} \cong S^{2d-3}$ the map ψ is orientation-preserving at x_+ and orientation-reversing at x_-. It thus follows from the Degree-via-Local Orientations Proposition 152.11 that the induced map $\psi_*\colon \mathrm{H}_{2d-2}(S^{2d-2}) \to \mathrm{H}_{2d-2}(X^{2d-2}/X^{2d-3})$ is the zero map. ■

306.3. **Invariants of generalized lens spaces.** In this section we turn to the calculation of various invariants of (infinite-dimensional) generalized lens spaces.

We start out with the following proposition, which generalizes the Lens Space-H_*-Lemma 183.21 and which is a close relative of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18:

Proposition 306.7. (Generalized Lens Space-H_*-Proposition) Let $m \in \mathbb{Z} \setminus \{0\}$.

(1) Let $d \in \mathbb{N}_0$ and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m. For any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(L(m|r_1, \dots, r_d)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/m\mathbb{Z}, & \text{if } k \text{ is odd and } k < 2d-1, \\ 0, & \text{if } k \text{ is even and } k > 0, \\ 0, & \text{if } k > 2d-1, \\ \mathbb{Z}, & \text{if } k = 2d-1. \end{cases}$$

(2) Let $r_1, r_2, \dots \mathbb{Z}$ be integers, each of which is coprime to m. For any $k \in \mathbb{N}_0$ we have We have

$$\mathrm{H}_k(L(m|r_1, r_2, \dots)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/m\mathbb{Z}, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Proof. Both statements follow almost immediately from the (Generalized Lens Space-CW-Proposition 306.6 together with the Singular-Cellular-H_*-Proposition 155.8. Alternatively, as in the proof of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 155.18, one can deduce Statement (2) from Statement (1), using the Exhaustions-H_*-Proposition 154.2. ■

In the final result of this chapter we calculate the homotopy groups of infinite-dimensional generalized lens spaces:

Proposition 306.8. (Generalized Lens Space-π_*-Proposition) Let $m \in \mathbb{Z} \setminus \{0\}$ and let $r_1, r_2, \dots \mathbb{Z}$ be integers, each of which is coprime to m.

(1) We have an isomorphism $\pi_1(L(m|r_1, r_2, \dots) \cong \mathbb{Z}/m\mathbb{Z}$.
(2) For any $k \in \mathbb{N}_{\geq 2}$ we have $\pi_k(L(m|r_1, r_2, \dots) = 0$.

Proof. [3378]

(a) It follows immediately from the Infinite Lens Space-Sphere Quotient Lemma 306.5, the Action–π_1-Theorem 93.1 and the discussion on page 2643 that we have an isomorphism $\pi_1(L(m|r_1, r_2, \dots)) \cong \pi_1(S^\infty/(\mathbb{Z}/m\mathbb{Z})) \cong \mathbb{Z}/m\mathbb{Z}$.
(b) Let $n \geq 2$. We calculate that

$$\pi_k(L(m|r_1, r_2, \dots)) \overset{\text{follows from the Infinite Lens Space-Sphere Quotient Lemma 306.5}}{\cong} \pi_k(S^\infty/(\mathbb{Z}/m\mathbb{Z})) \overset{\text{follows from the Action–Covering Proposition 91.11, the Covering-}\pi_n\text{-Isomorphism Proposition 134.22 and from the fact that } n \geq 2}{\cong} \pi_k(S^\infty) \overset{\text{see page 2643}}{=} 0.$$

■

On page **??** we saw, using the linking pairing, that lens spaces $L(5, 1)$ and $L(5, 2)$ with isomorphic fundamental groups are not necessarily homotopy equivalent. Unfortunately we cannot define a linking pairing on the topological spaces $L(m|r_1, r_2, \dots)$. Thus the following question arises:

Question 306.9. Let $m \in \mathbb{N}$. Does the homotopy type of the infinite-dimensional generalized lens spaces $L(m|r_1, r_2, \dots)$ depend on the choice of the parameters r_i?

We will answer this question on page 5899.

306.4. Exercises for Chapter 306.

Exercise 306.1. Classify the 1-dimensional generalized lens spaces $L(1|p)$ up to diffeomorphism.

Exercise 306.2. Let $m \in \mathbb{Z} \setminus \{0\}$ and let $r_1, r_2, \dots, r_d \in \mathbb{Z}$ be integers, each of which is coprime to m.

(a) Compute the homology groups $\mathrm{H}_k(L(m|r_1, \dots, r_d); \mathbb{Z}/m\mathbb{Z})$.
(b) Compute the cohomology groups $\mathrm{H}^k(L(m|r_1, \dots, r_d); \mathbb{Z}/m\mathbb{Z})$.
(c) Determine the cohomology ring $(\mathrm{H}^*(L(m|r_1, \dots, r_d); \mathbb{Z}/m\mathbb{Z}), \cup)$.
Hint. Imitate the proof of the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 213.19.

[3378] As on page 2643 one could also prove the proposition using the CW-Direct Colimit-π_n-Proposition 136.25.

CHAPTER 307

Eilenberg-Maclane Spaces

In this chapter we introduce Eilenberg-Maclane spaces. Even though the definition is quite straightforward, these topological spaces will henceforth play an essential role.

This chapter is organized as follows:

- In Section 307.1 we will introduce Eilenberg-Maclane Spaces and we will consider some basic examples.
- In Section 307.2 we will learn how to kill higher homotopy groups of topological spaces.
- In Section 307.3 we will prove the existence of Eilenberg-Maclane Spaces.
- In Section 307.4 we will show that Eilenberg-Maclane Spaces are unique up to homotopy.
- In Section 307.5 we will provide some first applications of Eilenberg-Maclane spaces.
- In Section 307.6 we will give a not entirely serious application of the Eilenberg–Maclane spaces to the social choice problem.
- In Section 307.7 we will prove an extension of the Hurewicz Theorem 167.9.
- In Section 307.8 we will provide exercises for Chapter 307.

In the next Chapter 308 we will see that Eilenberg-Maclane spaces give rise to invariants of topological spaces.

307.1. **The definition and basic examples of Eilenberg-Maclane Spaces.** We start out by recalling Definition 173.2:

Definition 307.1. Let π be an abelian group and let $n \in \mathbb{N}$. We say that a non-empty topological space X is a Moore space of type $\mathrm{M}(\pi, n)$ if the following conditions are satisfied:

(1) X admits a CW-structure.
(2) For any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(X) = 0$.
(3) $\mathrm{H}_n(X) \cong \pi$.
(4) If $n \geq 2$, then we demand that X is simply connected.

For the reader's convenience we recall the main result that we proved regarding Moore spaces:

Proposition 173.3. (Moore Space Existence Proposition) Given any $n \in \mathbb{N}$ and given any abelian group π there exists an $(n+1)$-dimensional CW-complex that has no cells in dimensions $1, \ldots, n-1$ and that is a Moore space of type $\mathrm{M}(\pi, n)$.

In light of the topics discussed in this chapter it is perhaps helpful to give a quick recap of the proof of the Moore Space Existence Proposition 173.3.

Sketch of proof. Since π is abelian it follows immediately from the Length 1–Free Resolution Lemma 162.9 that there exists a free resolution of π of the following form:

$$0 \to \mathbb{Z}^{(A)} \xrightarrow{\varphi} \mathbb{Z}^{(B)} \xrightarrow{\rho} \pi \to 0.$$

We let $X \coloneqq \bigvee_{a\in A} S^n_a$ and we let $Y \coloneqq \bigvee_{b\in B} S^n_b$. We equip X and Y with the obvious CW-structures and we can make the identifications $\mathrm{H}_n(X) = \mathbb{Z}^{(A)}$ and $\mathrm{H}_n(Y) = \mathbb{Z}^{(B)}$. We construct a cellular map $h\colon X \to Y$ such that the induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ agrees with φ under the above identifications. The mapping cone $\mathrm{Cone}(h\colon X \to Y)$ then has the desired properties. ■

The definition of an Eilenberg-Maclane space is almost identical to the definition of a Moore space, we just swap homology groups for homotopy groups:

Definition 307.2. Let π be a group and let $n \in \mathbb{N}$. Let X be a topological space.

(1) We say that X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if the following conditions hold:[3379]
   (a) $\pi_n(X) \cong \pi$.[3380]
   (b) For any $i \in \mathbb{N}_0$ with $i \neq n$ we have $\pi_i(X) = 0$.
   (c) X admits a CW-structure.

(2) Let $x_0 \in X$. We say that (X, x_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ and if X admits a CW-structure such that x_0 lies in the 0-skeleton of X.

Remark.

(1) Our definition of an Eilenberg-Maclane space agrees with the definition in [**DaK2001**, p. 177]. Conversely in [**WhdG1978**, p. 239], [**Spa1995**, p. 424], [**Bre1993**, p. 488] and [**Hat2002**, p. 365] it does not say (at least explicitly) that an Eilenberg-Maclane space needs to admit a CW-structure.

(2) If a topological space (or CW-complex) X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then in the literature one often just says that X *is a* $\mathrm{K}(\pi, n)$. We will refrain from using this language. □

It is perhaps initially not clear why the concept of an Eilenberg-Maclane space is useful, but at the very least it is a fun exercise to go through our long list of topological spaces that we have studied and to determine which ones are Eilenberg-Maclane spaces. This leads us to the following lemma, which is an expanded version of the earlier Eilenberg–Maclane Example Lemma 307.3:

Lemma 307.3. (Eilenberg–Maclane Example Lemma)

(0) An Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^m, n)$ is given by any contractible CW-complex, e.g. by any open or closed ball.

(1) An Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^k, 1)$ is given by $(S^1)^k$.

(2) (a) An Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$ is given by $\mathbb{R}\mathrm{P}^\infty = \operatorname{colim}_{\longrightarrow} \mathbb{R}\mathrm{P}^i$.
   (b) Let $m \in \mathbb{N}_{\geq 2}$ and let $r_1, r_2, \dots$ integers, each of which is coprime to m. The infinite-dimensional generalized lens space $L(m|r_1, r_2, \dots)$ from Definition 306.4 is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/m\mathbb{Z}, 1)$.

[3379] Note that for $n = 1$ we already gave Definition 257.9.

[3380] It is perhaps worth recalling that $\pi_0(X) = 0$ means, by Definition 4.43, that X is non-empty and path-connected. In particular, by the Change-of-Base Point Proposition 169.5 (2) the isomorphism type of the homotopy groups $\pi_n(X)$, $n \in \mathbb{N}$, does not depend on the choice of the base point.

(3) An Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(\Sigma_g), 1)$ is given by Σ_g.
(4) An Eilenberg-Maclane space of type $\mathrm{K}(\langle x_1, \ldots, x_k\rangle, 1)$ is given by the wedge of k copies of S^1.
(5) An Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$ is given by $\mathbb{C}\mathrm{P}^\infty = \varinjlim \mathbb{C}\mathrm{P}^i$.



type $\mathrm{K}(\mathbb{Z}, 1)$ type $\mathrm{K}(\mathbb{Z}^2, 1)$ type $\mathrm{K}(\pi_1(\Sigma_2), 1)$ type $\mathrm{K}(\langle x, y\rangle, 1)$

Proof.

(0) We start with the trivial group $\{e\}$. Let $n \in \mathbb{N}$. It follows from the Homotopy Equivalence-π_n-Proposition 134.17 (2) that any contractible CW-complex is an Eilenberg-Maclane space of type $\mathrm{K}(\{e\}, n)$. Thus we see in particular that for any $k \in \mathbb{N}_0$ the open ball B^k and the closed ball $\overline{B}^k$ are Eilenberg-Maclane spaces of type $\mathrm{K}(\{e\}, n)$.
(1) Let $k \in \mathbb{N}_0$. It follows from the Torus-π_1-Proposition 90.14, the discussion on page 2587, the CW-Category–Product Proposition 139.16 and the CW-Complex Product Proposition 139.18 that the k-dimensional torus $(S^1)^k$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^k, 1)$.
(2) (a) We consider the infinite-dimensional real projective space $\mathbb{R}\mathrm{P}^\infty = \varinjlim \mathbb{R}\mathrm{P}^i$. Recall that on page 2628 we showed that $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure. Furthermore, on page 2643 we saw, using the CW-Direct Colimit-π_n-Proposition 136.25, that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \cong \varinjlim \pi_1(\mathbb{R}\mathrm{P}^i) \cong \mathbb{Z}/2\mathbb{Z}$ and finally we saw that for $n \geq 2$ we have $\pi_n(\mathbb{R}\mathrm{P}^\infty) \cong \varinjlim \pi_n(\mathbb{R}\mathrm{P}^i) \cong \varinjlim \pi_n(S^i) = 0$. This shows that $\mathbb{R}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$.
(b) We consider an infinite-dimensional generalized lens space $L(m|r_1, r_2, \ldots)$. It follows from the Generalized Lens Space-CW-Proposition 306.6 (2) that this topological space admits a CW-structure. It now follows from the Generalized Lens Space-π_*-Proposition 306.8 that $L(m|r_1, r_2, \ldots)$ has the right fundamental group and vanishing higher homotopy groups.
(3) Let $g \in \mathbb{N}_{\geq 2}$ and let Σ_g be a surface of genus g. On page 2587 we saw that $\pi_i(\Sigma_g) = 0$ for $i \geq 2$. By the discussion on page 2626 we know that Σ_g admits a CW-structure. Thus we see that Σ_g is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(\Sigma_g), 1)$.
(4) Let $k \in \mathbb{N}$ and let $F = \langle x_1, \ldots, x_k\rangle$ be the free group on k generators. It follows from the Wedge-of-Circles-π_1-Lemma 99.11, the 1-Complex-π_n-Lemma 138.2 and the discussion on page 2625 that the wedge $\bigvee_{i=1}^k S^1$ of k circles is an Eilenberg-Maclane space of type $\mathrm{K}(F, 1)$.
(5) Recall that by the discussion on page 2628 we know that $\mathbb{C}\mathrm{P}^\infty$ admits a CW-structure. It follows from the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 (2) that the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. ■

Remark.

(1) One might initially hope that perhaps $\mathbb{H}\mathrm{P}^\infty$ could be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 4)$. First note that by the Quaternion-Projective Space Lemma 215.3 (3b) we

know that $\mathbb{H}\mathrm{P}^\infty$ is indeed a CW-complex. Furthermore, in Exercise 303.1 we obtained the following results:
(a) There exists an isomorphism $\pi_4(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}$.
(b) The group $\pi_n(\mathbb{H}\mathrm{P}^\infty)$ is finite for $n \neq 4$, in particular $\pi_n(\mathbb{H}\mathrm{P}^\infty) \otimes \mathbb{Q} = 0$ for $n \neq 4$.
(c) There exists an isomorphism $\pi_5(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}/2\mathbb{Z}$.
So we see that "rationally" $\mathbb{H}\mathrm{P}^\infty$ looks like an Eilenberg-Maclane space, but it is just not an Eilenberg-Maclane space.

(2) In the following chapters we will give many examples of "perfectly reasonable" topological spaces that are Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. For $n \geq 2$ it seems like $\mathbb{C}\mathrm{P}^\infty$ is basically[3381] the only Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ for some non-trivial group π which appears "in nature". □

The following proposition allows us to construct more Eilenberg-Maclane spaces out of given ones.

Proposition 307.4. (Eilenberg-Maclane Construction Proposition)
(1) Let X be an Eilenberg-Maclane space of type $\mathrm{K}(A, n)$ and let Y be an Eilenberg-Maclane space of type $\mathrm{K}(B, n)$.
  (a) Let $\mathcal{CWP}$ be the CW-product topology on $X \times Y$ that we introduced in Definition 139.15. The topological space $(X \times Y, \mathcal{CWP})$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.
  (b) Assume that one of the following conditions holds:
    • X *or* Y has finitely many cells.
    • X *and* Y both have countably many cells.
  Then the product $X \times Y$, with the usual product topology, is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

The remaining two statements only apply to the case $n = 1$.

(2) Let X be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$. We pick a base point $x_0 \in X$ and we pick an isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi$. Let Γ be a subgroup of π. We denote by $\widetilde{X}$ the covering space of X corresponding to the subgroup[3382] $\varphi^{-1}(\Gamma) \subseteq \pi_1(X, x_0)$. Then $\widetilde{X}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\Gamma, n)$.

(3) Let X be an Eilenberg-Maclane space of type $\mathrm{K}(A, 1)$ and let Y be an Eilenberg-Maclane space of type $\mathrm{K}(B, 1)$. We pick CW-structures for X and Y. Then given any $x_0 \in X^0$ and $y_0 \in Y^0$ the wedge $(X, x_0) \vee (Y, y_0)$ is an Eilenberg-Maclane space of type $\mathrm{K}(A * B, 1)$.

Remark. The restriction in the Eilenberg-Maclane Construction Proposition 307.4 (1b) on the CW-complexes X and Y is a major nuisance. Even though this condition on the number of cells is not an issue for the Eilenberg-Maclane spaces that we have considered above,

[3381] The slightly hesitant "basically" refers to the fact that out of $\mathbb{C}\mathrm{P}^\infty$ one can build a few other examples of Eilenberg-Maclane spaces, e.g. one can consider $\mathbb{C}\mathrm{P}^\infty \times \overline{B}^k$ or, as we will see shortly, the product $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^2, 2)$.

[3382] Since X is a CW-complex we know from the CW-Complex–Local Properties Proposition 137.8 (2) that X is locally simply connected. Therefore it follows from the Covering Existence Theorem 127.10 that the covering exists and it follows from the Covering Unique Existence Theorem 127.17 that the covering corresponding to the subgroup $\varphi^{-1}(\Gamma) \subseteq \pi_1(X, x_0)$ is essentially unique.

the "canonical" Eilenberg-Maclane spaces that we will construct shortly do *not* satisfy this condition. Thus one should apply the Eilenberg-Maclane Construction Proposition 307.4 (1b) with great caution. □

Proof.

(1) First recall that in the CW-Category–Product Proposition 139.16 we saw that the topological space $(X \times Y, \mathcal{CWP})$ admits a CW-structure. We pick $x \in X$ and $y \in Y$. Given any $k \in \mathbb{N}_0$ we have the following isomorphisms

$$\pi_k((X\times Y,\mathcal{CWP}),(x,y)) \underset{\uparrow}{\overset{\cong}{\to}} \pi_k(X\times Y,(x,y)) \underset{\uparrow}{\overset{\cong}{\to}} \pi_k(X,x)\times\pi_k(Y,y) \overset{\cong}{\to} \begin{cases} A\times B, & \text{if } k=n,\\ 0, & \text{otherwise.}\end{cases}$$

by the Product–CW Approximation Lemma 304.13

by the Product-π_1-Proposition 90.13 and by the Product-π_n-Proposition 134.18

We have thus shown that the CW-complex $(X \times Y, \mathcal{CWP})$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

If X *or* Y has finitely many cells or if X *and* Y both have countably many cells, then we know by the Countable CW-Complex Product Proposition 139.23[3383] that the CW-product topology $\mathcal{CWP}$ equals the usual product topology on $X \times Y$. Thus in this case $X \times Y$ with the product topology is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

(2) First note that we know by the CW-Complex Covering Proposition 138.1 that $\widetilde{X}$ is a CW-complex. It now follows immediately from the Covering-π_n-Isomorphism Proposition 134.22 that $\widetilde{X}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\Gamma, n)$.

(3) Let $x \in X^0$ and $y \in Y^0$ be points in the 0-skeleta of X and Y. It follows from the CW-Complex–Hausdorff Proposition 136.6 together with the CW-Complex–Local Properties Proposition 137.8 (1) that the two points $x \in X$ and $y \in Y$ are good. Therefore it follows from the Wedge-π_1-Proposition 99.10 that we have isomorphisms $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y) \cong A * B$. Furthermore we obtain basically immediately from Exercise 167.23 that all higher homotopy groups of $X \vee Y$ vanish. Finally we know by the CW-Wedge Lemma 139.11 that the wedge $(X, x_0) \vee (Y, y_0)$ admits a CW-structure. ■

The following lemma summarizes a few basic facts about Eilenberg-Maclane spaces.

Lemma 307.5. (Eilenberg-Maclane–Basics Lemma) Let $n \in \mathbb{N}$ and let π be a group. If X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then given any abelian group G the following statements hold:

(1) We have $\mathrm{H}_0(X; G) \cong G$ and $\mathrm{H}^0(X; G) \cong G$.
(2) For $i = 1, \ldots, n-1$ we have $\mathrm{H}_i(X; G) = 0$ and $\mathrm{H}^i(X; G) = 0$.
(3) Given any $x_0 \in X$ the Hurewicz homomorphism $\pi_n(X, x_0)_{\mathrm{ab}} \to \mathrm{H}_n(X; \mathbb{Z})$ is an isomorphism. In particular $\mathrm{H}_n(X; \mathbb{Z}) \cong \pi_{\mathrm{ab}}$.
(4) The evaluation map $\mathrm{H}^n(X; G) \to \mathrm{Hom}(\mathrm{H}_n(X; \mathbb{Z}), G)$ is an isomorphism.

[3383] If X and Y are both finite, we can also apply the CW-Complex Product Proposition 139.18.

Proof. The statements follow easily from the Hurewicz Theorems 166.10 and 167.9 and from the Universal Coefficient Theorems 162.16 and 201.23. The experienced reader will have no troubles with filling in the details. ■

307.2. **Killing higher homotopy groups.** In the subsequent section we will give a general construction of Eilenberg-Maclane spaces. To streamline the discussion we outsource a few preparations to the present section.

First we point out that the following, fairly self-evident convention will be in effect in the coming chapters.

Convention 307.6.

(1) Given $n \in \mathbb{N}_0$ we write $* = (0, \dots, 0, 1) \in S^n$. We equip S^n with the canonical CW-structure from page 2625, i.e. we view S^n as a CW-complex with one 0-cell $\{*\}$ and one n-cell.

(2) We always form any wedge of spheres $\bigvee_{i\in I} S_i^{n_i}$ by identifying the points $* \in S_i^{n_i}$. By the CW-Wedge Lemma 139.11 we can and will view the wedge $\bigvee_{i\in I} S_i^{n_i}$ as a CW-complex in the obvious way, i.e. we have one 0-cell $\{*\}$ and for each $i \in I$ we have one n_i-cell.

(3) By a slight abuse of notation we sometimes say X is a k-dimensional CW-complex if the highest dimension of a cell is $\leq k$.

We start out with the following calculation.

Proposition 307.7. (Wedge-of-Spheres-π_n-Proposition) Let I be a set, let $n \in \mathbb{N}$ and for each $i \in I$ let S_i^n be a copy of S^n. For each $i \in I$ we denote by $\varphi_i \colon S^n \to S_i^n$ the obvious homeomorphism. By a slight abuse of notation we denote by φ_i also the composition of φ_i with the canonical inclusion map $S_i^n \to \bigvee_{i\in I} S_i^n$.

(1) For any $i \in \{1, \dots, n-1\}$ we have $\pi_i\Big(\bigvee_{i\in I} S_i^n, *\Big) = 0$.

(2) If $n = 1$, then the homomorphism[3384]

$$\begin{array}{rcl} \overbrace{\langle \{x_i\}_{i\in I} \rangle}^{\text{free group}} & \to & \pi_1\Big(\bigvee_{i\in I} S_i^1, *\Big) \\ x_i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

(3) If $n \geq 2$, then the homomorphism[3385]

$$\begin{array}{rcl} \overbrace{\mathbb{Z}^{(I)} = \bigoplus_{i\in I} \mathbb{Z} \cdot i}^{\text{free abelian group}} & \to & \pi_n\Big(\bigvee_{i\in I} S_i^n, *\Big) \\ i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

[3384] By the Free Group–Existence Lemma 98.9 the given data does indeed determine a unique homomorphism.

[3385] Since $n \geq 2$ we know by the π_n-Abelian Proposition 134.11 that the right-hand side is an abelian group. Thus it follows from the Free Abelian Group–Existence Lemma 97.8 that the given data does indeed determine a unique homomorphism.

Proof.

(1) As we pointed out above, we can view $\bigvee_{i\in I} S_i^n$ as a CW-complex with one 0-cell and no other cells outside of dimension n. It follows from the CW-Skeleton-π_n-Proposition 141.3 that all homotopy groups in dimension $1, \dots, n-1$ vanish.

(2) This statement is precisely the content of the Wedge-of-Circles–π_1-Lemma 99.11.

(3) Let $n \in \mathbb{N}_{\geq 2}$. We consider the maps

$$\mathbb{Z}^{(I)} = \bigoplus_{i\in I} \mathbb{Z}\cdot i \xrightarrow[\uparrow]{\cong} \bigoplus_{i\in I} \mathrm{H}_n(S_i^n;\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n\Big(\bigvee_{i\in I} S_i^n;\mathbb{Z}\Big) \xleftarrow[\uparrow]{\cong} \pi_n\Big(\bigvee_{i\in I} S_i^n, *\Big).$$

given by $i \mapsto [S_i^n]$ — Wedge-H$_*$-Proposition 149.16 — by the Hurewicz Theorem 167.9 which we can apply by (1)

It is straightforward to verify that this composition of isomorphisms is precisely the isomorphism under consideration. ∎

We continue with the following two propositions which basically says that for a given CW-complex one can "kill" all homotopy groups beyond a chosen degree.

Proposition 307.8. (Kill-Higher-π_*-Proposition) Let $k \in \mathbb{N}$, let X be a connected CW-complex[3386] of dimension $\leq k+1$ and let $x_0 \in X$ be a base point.

(1) There exists a $(k+2)$-dimensional CW-complex Y with the following properties:
   (a) the $(k+1)$-skeleton of Y agrees with X,
   (b) for $i = 1, \dots, k$ the inclusion-induced map $\pi_i(X, x_0) \to \pi_i(Y, x_0)$ is an isomorphism, and
   (c) we have $\pi_{k+1}(Y, x_0) = 0$.

(2) The CW-complex Y in (1) can be chosen canonically.[3387]

(3) If X has countably many cells, then Y as in (1) can be chosen to have countably many cells.[3388]

Proof.

(1) We pick a set $\{\varphi_j \colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of continuous maps that represent a generating set for the group $\pi_{k+1}(X, x_0)$. We set

$$Y = \Big(X \sqcup \bigsqcup_{j\in J} \overline{B}_j^{k+2}\Big)/\sim \qquad \text{with } \varphi_j(s) \sim s \text{ for } s \in S_j^{k+1} = \partial\overline{B}_j^{k+2}.$$

Put differently, Y is the $(k+2)$-dimensional CW-complex that is obtained from X by attaching one $(k+2)$-cell for each $j \in J$ with attaching map that is given by φ_j. We denote by $i \colon X \to Y$ the inclusion map. Now we verify that Y has all the desired qualities.

   (a) By construction Y is a $(k+2)$-dimensional CW-complex such that the $(k+1)$-skeleton equals X.

   (b) It follows immediately from the CW-Skeleton-π_n-Proposition 141.3 and (a) that the inclusion-induced map $i_* \colon \pi_s(X, x_0) \to \pi_s(Y, x_0)$ is an isomorphism for $s = 1, \dots, k$ and that the map $i_* \colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(Y, x_0)$ is an epimorphism.

[3387] Here "canonical" is shorthand for saying that in the proof of the proposition we provide an explicit construction of Y does not depend on any choices.

[3388] Note though that we do *not* claim that we can find a canonical Y with countably many cells.

(c) Let $j \in J$. We have the following simple but useful commutative diagram:

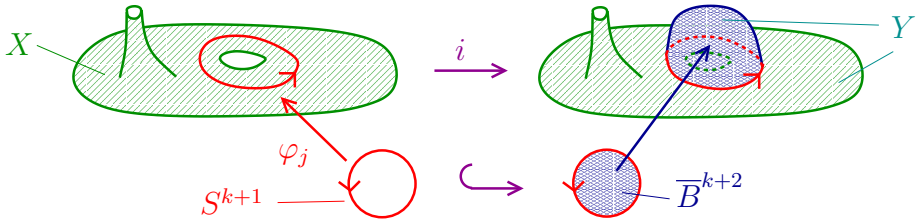
$$\begin{array}{ccccc} S^{k+1} & & \xrightarrow{\quad\varphi_j\quad} & & X \\ \downarrow & & & & \downarrow i \\ \overline{B}^{k+2} & \xrightarrow{\text{id}} \overline{B}_j^{k+2} & \hookrightarrow & X \sqcup \bigsqcup_{j\in J} \overline{B}_j^{k+2} & \twoheadrightarrow Y. \end{array}$$

In other words, the map $i \circ \varphi_j \colon S^{k+1} \to Y$ extends to a map $\overline{B}^{k+2} \to Y$. By the Trivial-in-π_n-Lemma 134.10 this implies that $i_*([\varphi_j]) = [i \circ \varphi_j] = 0 \in \pi_{k+1}(Y, x_0)$.

(d) Since $\{[\varphi_j]\}_{j\in J}$ is by choice a generating set for $\pi_{k+1}(X, x_0)$ we obtain from (c) that $i_* \colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(Y, x_0)$ has trivial image.

(e) It follows from (b) and (d) that $\pi_{k+1}(Y, x_0) = 0$.

We have thus verified that Y has all the desired properties.



(2) It is easy to come up with a canonical Y: in the construction provided in (1) we just need to consider the set of all continuous maps $(S^{k+1}, *) \to (X, x_0)$. Evidently this set is canonical and evidently it represents a generating set for $\pi_{k+1}(X, x_0)$. The resulting CW-complex Y is the canonical CW-complex we desired.

(3) Finally suppose that X has countably many cells. It follows from the rather non-trivial CW-Complex–π_n-Countable Proposition 190.8 that $\pi_{k+1}(X, x_0)$ is countable. Therefore we can pick a countable family $\{\varphi_j \colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of continuous maps that represent a generating set for the group $\pi_{k+1}(X, x_0)$. The resulting CW-complex Y evidently has only countably many cells. ■

We conclude this technical section with the following proposition that is mostly just an iterated application of the Kill-Higher-π_*-Proposition 307.8.

Proposition 307.9. (Kill-All Higher-π_n-Proposition) Let $k \in \mathbb{N}$, let Y be a connected $(k+1)$-dimensional CW-complex and let $y_0 \in Y$.

(1) There exists a CW-complex Z with the following three properties:
  (a) the $(k+1)$-skeleton of Z equals Y,
  (b) we have $\pi_i(Z) = 0$ for $i \geq k+1$,
  (c) for every $i \in \{0, \ldots, k\}$ the inclusion-induced map $\pi_i(Y, y_0) \xrightarrow{\cong} \pi_i(Z, y_0)$ is an isomorphism.

(2) The CW-complex Z in (1) can be chosen canonically.

(3) If Y has countably many cells, then Z as in (1) can be chosen to have countably many cells.

Proof. Let $k \in \mathbb{N}$ and let Y be a connected $(k+1)$-dimensional CW-complex. Let $y_0 \in Y$.

(1) We define $Z_{k+1} = Y$. For $i = k+1, k+2, \dots$ we iteratively apply the Kill-Higher-π_*-Proposition 307.8 to Z_i to obtain a sequence of CW-complexes $Z_{k+1}, Z_{k+2}, \dots$ with the following properties:
   (i) each Z_i is an i-dimensional CW-complex,
   (ii) for each $i \geq k+1$ the i-skeleton of Z_{i+1} is given by Z_i,
   (iii) for each $i \geq k+1$ we have $\pi_i(Z_{i+1}, y_0) = 0$.
   We set $Z := \operatorname{colim}_{i \geq k+1} Z_i = \bigcup_{i \geq k+1} Z_i$ and we equip it with the obvious CW-structure such that for each $i \geq k+1$ the i-skeleton of Z is given by Z_i. We claim that Z has the desired three properties:
   (a) It follows from (ii) that the $(k+1)$-skeleton of Z equals Y.
   (b) Let $i \geq k+1$. We have

$$\pi_i(Z, y_0) \overset{\cong}{\leftarrow} \pi_i((i+1)\text{-skeleton } Z^{i+1} \text{ of } Z, y_0) = \pi_i(Z_{i+1}, y_0) = 0.$$

   by the CW-Skeleton-π_n Proposition 141.3; since $Z^{i+1} = Z_{i+1}$; by (iii)

   (c) This statement follows immediately from (a) and the CW-Skeleton-π_n-Proposition 141.3.

(2) This statement follows immediately from the Kill-Higher-π_*-Proposition 307.8 (2).

(3) This statement follows immediately from the Kill-Higher-π_*-Proposition 307.8 (3). ■

307.3. **Existence of Eilenberg-Maclane Spaces.** Using the examples from the first section and using the Eilenberg-Maclane Construction Proposition 307.4 we can "by hand" construct Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$ for many groups π. One can ask for which pairs (π, n) there exist corresponding Eilenberg-Maclane spaces. There is one obvious restriction: since homotopy groups $\pi_n(X)$ with $n \geq 2$ are abelian, see the π_n-Abelian Proposition 134.11, we see that for $n \geq 2$ we can only hope to find an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if π is abelian. Perhaps somewhat surprisingly this is the only restriction on the existence of Eilenberg-Maclane spaces.

Theorem 307.10. (Eilenberg-Maclane Existence Theorem) Let $n \in \mathbb{N}$ and let π be a group. If $n \geq 2$, then we demand that π is abelian. There exists a *canonical* triple $(X, x_0, \varphi\colon \pi_n(X, x_0) \to \pi)$ with the following properties:
(1) X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$.
(2) X is a CW-complex with one 0-cell $\{x_0\}$ and with no cells in dimensions $1, \dots, n-1$.
(3) $\varphi\colon \pi_n(X, x_0) \to \pi$ is an isomorphism.

Definition 307.11. We refer to the canonical CW-complex from the Eilenberg-Maclane Existence Theorem 307.10 as the canonical Eilenberg-Maclane space $\mathrm{K}(\pi, n)$.

Remark. As we will see shortly, the canonical Eilenberg-Maclane spaces that we construct in the proof of the Eilenberg-Maclane Existence Theorem 307.10 are monstrously large. More precisely, the CW-complex of type $\mathrm{K}(\pi, n)$ has uncountably many cells in each dimension $\geq n+1$. The cautionary remark from page 5888 about taking products of Eilenberg-Maclane spaces thus applies to the canonical Eilenberg-Maclane spaces. □

After the discussion of the previous section the approach to producing an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ is presumably pretty clear:

(1) Start out with a CW-complex with one 0-cell, no cells in dimension $1, \ldots, n-1$ and "sufficiently" many n-cells to be able to "generate" the group π.
(2) Attach $(n+1)$-cells to chop down the n-th homotopy group to obtain the desired group π.
(3) Attach cells of dimension $\geq n+2$ to kill all homotopy groups in dimensions $\geq n+1$.

We basically know how to perform (1) and (3), so it remains to do (2) carefully. The combination of steps (1) and (2) is performed in the following proposition.

Proposition 307.12. (Homotopy Group-Realization Proposition)
(1) Let π be any group. There exists a canonical triple $(Y, x_0, \varphi\colon \pi_1(Y, x_0) \to \pi)$ with the following properties:
  (a) Y is a 2-dimensional CW-complex with one 0-cell $\{x_0\}$,
  (b) $\varphi\colon \pi_1(Y, x_0) \to \pi$ is an isomorphism.
(2) Let π be an abelian group π and let $n \in \mathbb{N}_{\geq 2}$. There exists a canonical triple $(Y, x_0, \varphi\colon \pi_n(Y, x_0) \to \pi)$ with the following properties:
  (a) Y is an $(n+1)$-dimensional CW-complex with one 0-cell $\{x_0\}$ and no cells in dimensions $1, \ldots, n-1$,
  (b) $\varphi\colon \pi_n(Y, x_0) \to \pi$ is an isomorphism.

Remark. The first result is related to the Presentation–CW Lemma 138.5 which says that given any *finitely presented* group π there exists a *finite* connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$. □

Before we provide the proof of the Homotopy Group-Realization Proposition 307.12 we convince ourselves of the fact that this proposition, together with the Kill-All Higher-π_n-Proposition 307.9, does indeed imply the Eilenberg-Maclane Existence Theorem 307.10.

Proof of the Eilenberg-Maclane Existence Theorem 307.10. In the following we discuss the case $n \geq 2$. The proof for $n = 1$ is almost identical. Thus let $n \in \mathbb{N}_{\geq 2}$ and let π be an abelian group. By the Homotopy Group-Realization Proposition 307.12 (2) there exists a canonical triple $(Y, x_0, \varphi\colon \pi_n(Y, x_0) \to \pi)$ with the following properties:

(1) Y is an $(n+1)$-dimensional CW-complex with one 0-cell $\{0\}$ and no cells in dimensions $1, \ldots, n-1$,
(2) $\varphi\colon \pi_n(Y, x_0) \to \pi$ is an isomorphism.

We plug this data into the Kill-All Higher-π_n-Proposition 307.9 and we obtain a canonical CW-complex Z with the following three properties:

(3) The $(n+1)$-skeleton of Z equals Y (here we use that Y equals its own $(n+1)$-skeleton.
(4) We have $\pi_j(Z) = 0$ for $j \geq n+1$.
(5) The inclusion $i\colon Y \to Z$ induces an isomorphism $\pi_n(Y, x_0) \xrightarrow{\cong} \pi_n(Z, x_0)$.

Note that it follows from (1) and (3) together with the CW-Skeleton-π_n-Proposition 141.3 that the following holds:

(6) We have $\pi_j(Z, x_0) = 0$ for $j = 1, \ldots, n-1$.

It follows immediately from all of the above statements that the canonical triple

$$\big(Z, x_0, \varphi \circ i_*^{-1} \colon \pi_n(Z, x_0) \xleftarrow[\cong]{i_*} \pi_1(Y, x_0) \xrightarrow[\cong]{\varphi} \pi\big)$$

has all the desired properties. ∎

Now we turn to the proof of the Homotopy Group-Realization Proposition 307.12. The proof naturally splits into the two cases $n = 1$ and $n \geq 2$.

Proof of the Homotopy Group-Realization Proposition 307.12 (1). Let π be a group. Given $g \in \pi$ we write $S^1_g \coloneqq S^1 \times \{g\}$, i.e. S^1_g is a copy of S^1. Let $* = (0,1) \in S^1$ be the usual base point of S^1. We consider the wedge $X \coloneqq \bigvee_{g \in \pi} S^1_g$ which is given by identifying all the points $(*, g) \in S^1_g$. We denote by $x_0 \in X$ the wedge point. We apply the Convention 307.6 to view X as a CW-complex in the obvious way.

By the Wedge-of-Spheres-π_n-Proposition 307.7 (2) we have the standard identification $\langle \{x_g\}_{g \in \pi} \rangle =_{\mathrm{i}} \pi_1(X, x_0)$. Thus, by the Free Group–Existence Lemma 98.9, there exists a unique homomorphism $\psi \colon \pi_1(X, x_0) =_{\mathrm{i}} \langle \{x_g\}_{g \in \pi} \rangle \to \pi$ such that $\psi(x_g) = g$ for every $g \in \pi$. This homomorphism is, by construction, an epimorphism. We set $\Gamma = \ker(\psi)$.

Next let $\{\alpha_j \colon (S^1, *) \to (X, x_0)\}_{j \in J}$ be the family of all loops that represent an element in $\Gamma = \ker(\psi \colon \pi_1(X, x_0) \to \pi)$. We set

$$Y \;=\; \Big(X \sqcup \bigsqcup_{j \in J} \overline{B}^2_j\Big) / \sim \qquad \text{with } \alpha_j(s) \sim s \text{ for } s \in S^1_j = \partial \overline{B}^2_j.$$

Put differently, Y is the 2-dimensional CW-complex that is obtained from X by attaching one 2-cell for each $j \in J$ with attaching map that is given by α_j. By construction Y is a 2-dimensional CW-complex such that the 1-skeleton equals X. We denote by $i \colon X \to Y$ the inclusion map.

Claim. The map $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, x_0)$ descends to a map

$$\varphi \colon \pi_1(X, x_0)/\Gamma \;\to\; \pi_1(Y, x_0)$$

and this map is an isomorphism.

Proof. The proof naturally breaks into several parts.

(1) Let $j \in J$. The same elementary argument as in the proof of the Kill-Higher-π_*-Proposition 307.8 shows that the map $i \circ \alpha_j \colon S^1 \to Y$ extends to a map $\overline{B}^2 \to Y$. By the Trivial-in-π_1-Lemma 88.28 this observation implies that $i_*([\alpha_j]) = [i \circ \alpha_j] = 0 \in \pi_1(X, x_0)$. Since $\{[\alpha_j]\}_{j \in J}$ contains every single element of Γ we see that $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ descends to a map

$$\varphi \colon \pi_1(X, x_0)/\Gamma \;\to\; \pi_1(Y, x_0).$$

(2) Note that it follows from $X = Y^1$ and the CW-Skeleton-π_n-Proposition 141.3 that the map $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an epimorphism. This implies that φ is also an epimorphism.

(3) We still need to show that the map $\varphi \colon \pi_1(X, x_0)/\Gamma \to \pi_1(Y, y_0)$ is a monomorphism. At this stage this is equivalent to showing that $\ker(i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)) = \Gamma$. Thus let $g \in \ker(i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0))$. We start out with the following subclaim.

Subclaim. There exists a finite subset $\{i_1, \ldots, i_m\} \subseteq I$ such that g lies in the kernel of the inclusion induced map $\pi_1(X, x_0) \to \pi_1\big(\big(X \sqcup (\overline{B}^2_{i_1} \cup \cdots \cup \overline{B}^2_{i_m})\big)/\sim, x_0\big)$.

Proof of subclaim. We pick a loop $\gamma \colon [0,1] \to X$ that represents g. By hypothesis γ is null-homotopic in Y. By definition of being null-homotopic this means that there

exists a path-homotopy $H\colon [0,1]\times[0,1]\to Y$ from the path γ to the constant path. By the Compact Image Lemma 4.15 we know that $K \coloneqq H([0,1]\times[0,1])$ is a compact subset of X. We have thus shown that there exists a compact subset K of Y such that g lies in the kernel of the inclusion-induced map $\pi_1(X,x_0)\to\pi_1(X\cup K,x_0)$. The claim now follows easily from the Contained-in-Finite Subcomplex Lemma 136.18 (2). □

Let $I=\{i_1,\dots,i_m\}$ be such a finite subset. It follows from the Cell Attachment-π_1-Proposition 135.5 (2), applied altogether m times, together with a variation on the Quotient Presentation–Lemma 100.18 (see also Exercise 100.8), that we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(X,x_0) & \longrightarrow & \pi_1\big(\big(X\sqcup\big(\overline{B}^2_{i_1}\cup\cdots\cup\overline{B}^2_{i_m}\big)\big)/\sim,x_0\big) \\ & \searrow & \downarrow \\ & & \pi_1(X,x_0)/\langle\langle[\alpha_{i_1}],\dots,[\alpha_{i_m}]\rangle\rangle. \end{array}$$

Thus we see that $g\in\langle\langle[\alpha_{i_1}],\dots,[\alpha_{i_m}]\rangle\rangle$. Since $[\alpha_{i_1}],\dots,[\alpha_{i_m}]$ lie by definition in Γ we see that $g\in\Gamma$.

We are now basically done. Indeed, the CW-complex Y is 2-dimensional, it has a single 0-cell x_0 and it is canonical. Furthermore the map $\psi\colon\pi_1(X,x_0)\to\pi$ induces an isomorphism $\overline{\psi}\colon\pi_1(X,x_0)/\Gamma\to\pi$. Thus we see that the canonical triple

$$\Big(Y,x_0,\pi_1(Y,x_0)\underset{\cong}{\overset{\varphi}{\leftarrow}}\pi_1(X,x_0)/\Gamma\underset{\cong}{\overset{\overline{\psi}}{\rightarrow}}\pi\Big)$$

has all the desired properties. ■

Now we turn to the proof of the Homotopy Group-Realization Proposition 307.12 (2).

Proof of the Homotopy Group-Realization Proposition 307.12 (2). Let π be an abelian group and let $n\geq 2$. If we do not care about a canonical construction, then we could just use the Moore Space Existence Proposition 173.3 to obtain an $(n+1)$-dimensional CW-complex Y with one 0-cell and no cells in dimensions $1,\dots,n-1$ and with $\mathrm{H}_n(Y;\mathbb{Z})\cong\pi$. By the Hurewicz Theorem 167.9 this implies that $\pi_n(Y)\cong\pi$. Unfortunately this construction is not canonical.

Since we are really eager to have canonical constructions we need to provide a different argument. Evidently, in the following discussion, we will try for as long as possible to roll over ideas from the proof of the Homotopy Group-Realization Proposition 307.12 (1).

For each $g\in\pi$ we write $S^n_g\coloneqq S^n\times\{g\}$. We consider $*=(0,\dots,0,1)\in S^n$. As in (1) we consider the wedge $X\coloneqq\bigvee_{g\in\pi}S^n_g$ and we view X canonically as a CW-complex. We denote by $x_0\in X$ the wedge point.

By the Wedge-of-Spheres-π_n-Proposition 307.7 (3) we have the standard identification $\pi_n(X,*)=_{\mathrm{i}}\mathbb{Z}^{(\pi)}$. Since π is an abelian group we know from the Free Abelian Group–Existence Lemma 97.8 that there exists a unique homomorphism $\psi\colon\pi_n(X,*)=_{\mathrm{i}}\mathbb{Z}^{(\pi)}\to\pi$ with $\psi(g)=g$ for every $g\in\pi$. This homomorphism is, by construction, an epimorphism. We set $\Gamma=\ker(\psi)$.

Next let $\{\alpha_j\colon(S^n,*)\to(X,x_0)\}_{j\in J}$ be the family of all continuous maps that represent an element in $\Gamma=\ker(\psi\colon\pi_n(X,*)\to\pi)$. As in the proof of (1) we set

$$Y\;=\;\Big(X\sqcup\bigsqcup_{j\in J}\overline{B}^{n+1}_j\Big)/\sim\qquad\text{with }\alpha_j(s)\sim s\text{ for }s\in S^n_j=\partial\overline{B}^n_j.$$

By construction Y is an $(n+1)$-dimensional CW-complex such that the n-skeleton equals X. We denote by $i\colon X \to Y$ the inclusion map.

Claim. The map $i_*\colon \pi_n(X,x_0) \to \pi_n(Y,x_0)$ descends to a map

$$\varphi\colon \pi_n(X,x_0)/\Gamma \;\to\; \pi_n(Y,x_0)$$

and this map is an isomorphism.

Proof. We consider the following diagram:

$$\begin{array}{ccccccc}
 & & \Gamma & \hookrightarrow & \pi_n(X,x_0) & \xrightarrow{i_*} & \pi_n(Y,x_0) \longrightarrow 0 \\
 & & {\scriptstyle a_j\mapsto[\alpha_j]}\uparrow\!\uparrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{j\in J}\mathbb{Z}\cdot a_j & & \xrightarrow{a_j\mapsto[\alpha_j]} & & \pi_n(X,x_0) & \xrightarrow{i_*} & \pi_n(Y,x_0) \longrightarrow 0 \\
\downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\bigoplus_{j\in J}\mathbb{Z}\cdot a_j & & \xrightarrow{a_j\mapsto\alpha_{j*}([S^n])} & & \mathrm{H}_n(X;\mathbb{Z}) & \xrightarrow{i_*} & \mathrm{H}_n(Y;\mathbb{Z}) \longrightarrow 0.
\end{array}$$

We make the following clarification and observations:

(1) The squares at the top commute basically by definition. Note that the vertical map to the top left is an epimorphism by definition of the family $\{\alpha_j\}_{j\in J}$.

(2) The vertical maps between the second and the third row, which are located in the middle and to the right, are the Hurewicz homomorphisms from Definition 167.2.

(3) It follows from the naturality of the Hurewicz homomorphism, see the Hurewicz Homomorphism Proposition 167.3 (3), that the square to the bottom right commutes.

(4) It follows basically immediately from the definition of the Hurewicz homomorphism that the square to the bottom left commutes.

(5) By construction we know that X and Y have precisely one 0-cell and no cells in dimension $1,\ldots,n-1$. This implies by the CW-Skeleton-π_n-Proposition 141.3 that X and Y are $(n-1)$-connected. Now we obtain from the Hurewicz Theorem 167.9 that the vertical maps in the middle and to the right are both isomorphisms.

(6) It follows from a modest generalization of the Cell Attachment-H_*-Proposition 153.12 that the bottom horizontal sequence is exact.[3389]

It follows from all the information assembled above that the top horizontal sequence is also exact. But this fact is just a reformulation of the statement that we are supposed to prove in the claim. ⊞

As in the proof of (1) we are now basically done. Indeed, the CW-complex Y is $(n+1)$-dimensional, it has a single 0-cell x_0, it has no cells in dimensions $1,\ldots,n$ and it is canonical. Furthermore the map $\psi\colon \pi_n(X,x_0) \to \pi$ induces an isomorphism $\overline{\psi}\colon \pi_n(X,x_0)/\Gamma \to \pi$. Thus we see that the canonical triple

$$\left(Y, x_0, \pi_n(Y,x_0) \underset{\cong}{\xleftarrow{\varphi}} \pi_n(X,x_0)/\Gamma \underset{\cong}{\xrightarrow{\overline{\psi}}} \pi\right)$$

has all the desired properties. ■

We have now seen that given any group there exists a CW-complex that is an Eilenberg-Maclane space. But it is pretty clear that the construction provided by the proof of the

[3389] The Cell Attachment-H_*-Proposition 153.12 was formulated for attaching a single $(n+1)$-cell, but basically the same argument also works for attaching an arbitrary family of $(n+1)$-cells.

Eilenberg-Maclane Existence Theorem 307.10 is pretty wasteful, except for the trivial group the construction provides an infinite-dimensional CW-complex. In many cases, e.g. for the groups $\pi = \mathbb{Z}^m$, there are, as we have seen in the Eilenberg–Maclane Example Lemma 307.3 (1), much simpler, finite-dimensional Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. This point towards a common tension:

(1) On the one hand it is nice to have "small constructions" with few cells. For example it is much easier to visualize finite CW-complexes and it is usually much easier to calculate invariants for finite CW-complexes.
(2) On the other hand it is nice to have canonical constructions which do not depend on any choices. But such constructions are almost invariably humongous.

This situation is comparable to the discussion of free resolutions of abelian groups. On the one hand, for actual calculations one prefers "small" free resolutions. On the hand, say to give unambiguous definitions of Tor-groups and Ext-groups one needs to work with the natural free resolution from Definition 162.2 which is outrageously wasteful.

It is natural to ask whether "small" groups also admit "small" Eilenberg-Maclane spaces. For example the following question arises:

Question 307.13. Let $m \in \mathbb{N}_{\geq 2}$. Can we find a finite-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/m\mathbb{Z}, 1)$?

In some instances one can also attempt to perform the three-step-process from page 5893 with more care to provide "smaller" Eilenberg-Maclane spaces. In the remainder of this section we want to discuss the construction of the Eilenberg-Maclane spaces $\mathrm{K}(\mathbb{Z}, n)$. For $n = 1$ everything that can be said had been said in the Eilenberg–Maclane Example Lemma 307.3 (1).

We will now construct an explicit Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$ using the above procedure:

(1) We start with $X_2 \coloneqq \mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$. We have $\pi_1(X_2 = \mathbb{C}\mathrm{P}^1) = 0$ and $\pi_2(X_2 = \mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}$.
(2) By the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 we know that $\pi_3(\mathbb{C}\mathrm{P}^1) = [p]$ where $p\colon S^3 \to \mathbb{C}\mathrm{P}^1$ denotes the Hopf map from Definition 298.9.
(3) We set $X_4 \coloneqq \overline{B}^4 \cup_{p\colon S^3 \to \mathbb{C}\mathrm{P}^1} \mathbb{C}\mathrm{P}^1$.
(4) By the Mapping Cone-of-Hopf Map Lemma 216.5 (1) we know that X_4 is homeomorphic to $\mathbb{C}\mathrm{P}^2$.[3390]

[3390]For the convenience of the reader we repeat the argument. For the convenience of the author we prove a slight generalization of the Mapping Cone-of-Hopf Map Lemma 216.5 (1). Thus let $n \in \mathbb{N}$. We consider the maps

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^{2n+1}) & \to & \mathbb{C}\mathrm{P}^{n+1} \\ [(z_0,\dots,z_n),r] & \mapsto & [(r\cdot z_0,\dots,r\cdot z_n,1-r)] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^{n+1} \\ [z_0:\cdots:z_n] & \mapsto & [z_0:\cdots:z_n:0]. \end{array}$$

These maps are continuous by the Topological-Quotient Proposition 8.2 (1b) and the Topological-Quotient Proposition 8.2 (1a). It follows immediately from the definitions that these two maps define a map

$$\underbrace{\mathrm{Cone}(p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n)}_{=(\mathrm{Cone}(S^{2n+1}) \cup \mathbb{C}\mathrm{P}^n)/\sim} \to \mathbb{C}\mathrm{P}^{n+1}$$

which is easily seen to be a bijection. It follows from the Topological-Quotient Proposition 8.2 (1b) that this map is continuous. Furthermore, using the Compact-Hausdorff Proposition 4.23 (3) and the Projective Spaces-Compact+Hausdorff Proposition 9.15 one sees that the map is in fact a homeomorphism.

(5) By the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 we know that $\pi_3(\mathbb{C}\mathrm{P}^2) = 0$ and[3391] $\pi_4(\mathbb{C}\mathrm{P}^2) = 0$. Furthermore it tells us that $\pi_5(\mathbb{C}\mathrm{P}^2) = [p]$ where $p\colon S^5 \to \mathbb{C}\mathrm{P}^2$ denotes the Hopf map from Definition 298.9.
(6) Similar to (3) we consider $X_6 \coloneqq \overline{B}^6 \cup_{p\colon S^3 \to \mathbb{C}\mathrm{P}^1} \mathbb{C}\mathrm{P}^1$.
(7) As in (4), using Footnote 3390, we see that X_6 is homeomorphic to $\mathbb{C}\mathrm{P}^3$.

Iterating this process we obtain the sequence $\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^4, \dots$ and we end up with the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$. This shows, once again, that $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. A similar discussion, shows how to construct an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$ starting with $X_2 = \mathbb{R}\mathrm{P}^2$, by imitating the above approach one once again ends up with $\mathbb{R}\mathrm{P}^\infty$.

For $n \geq 3$ there seem to be no straightforward models for $\mathrm{K}(\mathbb{Z}, n)$. But the following lemma gives us at least a little bit of extra control, which will play a crucial role at a later point.

Lemma 307.14. ($\mathrm{K}(\mathbb{Z}, n)$-Model Lemma) Given any $n \in \mathbb{N}$ there exists an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ with precisely one 0-cell, precisely one n-cell, no cell of dimension $n+1$ and only countably many cells in dimensions $\geq n+2$.

Remark. Let $n \in \mathbb{N}$. Note that the canonical Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ that we constructed in the proof of the Eilenberg-Maclane Existence Theorem 307.10 is *not* of the form as described in the $\mathrm{K}(\mathbb{Z}, n)$-Model Lemma 307.14. We will discuss this in more detail in Exercise 307.4. □

Proof. Let $n \in \mathbb{N}$. We consider $Y = S^n$. Recall that by the $\pi_n(S^n)$–Proposition 167.10 we know that $\pi_n(S^n) \cong \mathbb{Z}$. By the Kill-All Higher-π_n-Proposition 307.9 (3) there exists a CW-complex Z with the following four properties:

(1) The $(n+1)$-skeleton of Z equals $Y = S^n$, in particular Z has only one 0-cell, no cells in dimension $1, \dots, n-1$, one cell in dimension n and no cell in dimension $n+1$.
(2) We have $\pi_i(Z) = 0$ for $i \geq n+1$.
(3) For every $i \in \{1, \dots, n\}$ and every base point $x_0 \in S^n$ the inclusion-induced map $\pi_i(Y, x_0) \xrightarrow{\cong} \pi_i(Z, x_0)$ is an isomorphism.
(4) The CW-complex Z has only countably many cells.

By the Sphere–Low-π_n-Proposition 134.19 we know that $\pi_i(Y, x_0) = 0$ for $i = 1, \dots, n-1$. Thus Z is the desired Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$. ■

307.4. **Uniqueness of Eilenberg-Maclane Spaces.** Next we turn to the question to what degree Eilenberg-Maclane spaces of a given type are unique. Evidently the homeomorphism type is not unique, since given an Eilenberg-Maclane space X we obtain Eilenberg-Maclane spaces of the same type by considering say the products $X \times \overline{B}^m$. Nonetheless the following theorem says that the homotopy type of Eilenberg-Maclane spaces of a given type are in fact unique:

[3391] Here we are lucky, we did everything to kill $\pi_3(\mathbb{C}\mathrm{P}^2)$, but without any effort on our side we also managed to dispose of $\pi_4(\mathbb{C}\mathrm{P}^2)$.

Theorem 307.15. (Eilenberg-Maclane Uniqueness Theorem) Let $n \in \mathbb{N}$ and let π be an (abelian) group[3392]. The following two statements hold:

(1) Any two Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ are homotopy equivalent.[3393]
(2) If (X, x_0) and (Y, y_0) are two pointed Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $x_0 \in X^0$ and $y_0 \in Y^0$ and if $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism, then there exists a homotopy equivalence $f\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces with $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$.

Before we turn to the proof of the Eilenberg-Maclane Uniqueness Theorem 307.15 it is worth, and fun, to discuss several immediate consequences thereof:

Example. The Eilenberg-Maclane Uniqueness Theorem 307.15 settles Question 306.9 in a perhaps unexpected way: Given $m \in \mathbb{N}$ the homotopy type of the topological spaces $L(m|r_1, r_2, \dots)$ does *not* depend on the choice of the parameters r_i, since all of these topological spaces are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}/m\mathbb{Z}, 1)$, but these are therefore homotopy equivalent by the Eilenberg-Maclane Uniqueness Theorem 307.15.. □

Corollary 307.16. (Eilenberg-Maclane-H_*-Corollary) Let π be a group and let $n \in \mathbb{N}$. If X and Y are two Eilenberg-Maclane spaces of the type $\mathrm{K}(\pi, n)$, then for any $i \in \mathbb{N}$ and any abelian group G we have an isomorphism $\mathrm{H}_i(X; G) \cong \mathrm{H}_i(Y; G)$.

Proof. The corollary which is an immediate consequence of the Eilenberg-Maclane Uniqueness Theorem 307.15, the Homotopy Equivalence-H_*-Proposition 144.14 (2) and the discussion on page 3078. ■

Perhaps somewhat surprisingly we can use the Eilenberg-Maclane-H_*-Corollary 307.16 to give a negative answer to Question 307.13 for $m = 2$.

Corollary 307.17. (Infinite Eilenberg-Maclane Space Corollary) There is no finite-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$.

Proof. Let K be an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$. Given any $i \in \mathbb{N}$ we calculate that

$$\begin{array}{ccccc} \mathrm{H}_i(K; \mathbb{Z}/2\mathbb{Z}) & \overset{\text{by the Eilenberg-Maclane-H}_* \text{ Corollary 307.16}}{\cong} & \mathrm{H}_i(\text{any Eilenberg-Maclane space of type } \mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z}) & & \\ & \underset{\text{by the Eilenberg–Maclane Example Lemma 307.3 (2)}}{\cong} & \mathrm{H}_i(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) & \underset{\text{by the } \mathbb{R}\mathrm{P}^n\text{-H}_*\text{-Proposition 160.14}}{\cong} & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

It follows from the discussion on page 3081 that K is of dimension $> i$ for all $i \in \mathbb{N}$, in other words, K is infinite-dimensional. ■

The following proposition basically says that it is easy to construct maps from CW-complexes to Eilenberg-Maclane spaces. This statement is the key to proving the Eilenberg-Maclane Uniqueness Theorem 307.15. As we will see shortly, the proposition is of interest in its own right.

[3392]Whenever we deal with Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ it is understood throughout these lecture notes that π is abelian.
[3393]We mentioned this statement already earlier in the Eilenberg-Maclane Uniqueness Theorem 257.13.

Proposition 307.18. (Maps-to-K(π,n)-Proposition) Let $n \in \mathbb{N}$ and let π be an (abelian) group. Furthermore let Y be an Eilenberg-Maclane space of type $\mathrm{K}(\pi,n)$ and let $y_0 \in Y$. Finally let X be a CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. If X is $(n-1)$-connected, then the following two statements hold:

(1) Let $\varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$ be a homomorphism. There exists a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$.

(2) Suppose we are given continuous maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$. If $f_{0*} = f_{1*}\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$, then there exists a homotopy rel x_0 from f_0 to f_1.

We write

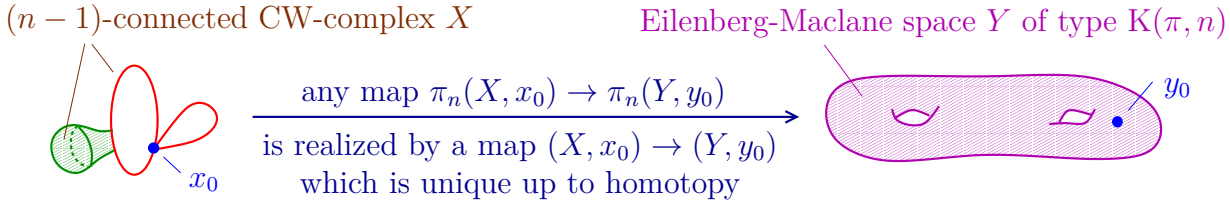
$$\langle (X,x_0),(Y,y_0)\rangle \coloneqq \{\text{maps } (X,x_0) \to (Y,y_0)\}/\sim \qquad \text{where } f \sim g \text{ if there exists a homotopy rel } \{x_0\} \text{ between } f \text{ and } g.$$

With this notation Statements (1) and (2) can be summarized to obtain the following statement:

(3) The map

$$\begin{array}{ccc} \langle (X,x_0),(Y,y_0)\rangle & \to & \mathrm{Hom}(\pi_n(X,x_0),\pi_n(Y,y_0)) \\ [f] & \mapsto & [f_*] \end{array}$$

is a bijection.



Remark.

(1) The CW–Complex Pointed Homotopy Proposition 140.14, the Sphere Pointed-Homotopy Corollary 140.15 and the Abelian-π_1–Drop Base Point Proposition 308.42 provide us with criteria under which the natural map $\langle (X,x_0),(Y,y_0)\rangle \to [X,Y]$ is a bijection.

(2) In the Maps-to-S^1-Proposition 233.4, with $n = 1$ and $Y = S^1$, we proved a "smooth manifold version" of the Maps-to-K(π,n)-Proposition 307.18. □

Proof of Theorem 307.15 assuming Proposition 307.18. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Statement (1) is basically an immediate consequence of Statement (2). Therefore it now remains to prove Statement (2). Thus let (X,x_0) and (Y,y_0) be two pointed Eilenberg-Maclane spaces of type $\mathrm{K}(\pi,n)$ where $x_0 \in X^0$ and $y_0 \in Y^0$. Furthermore let $\varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$ be an isomorphism. Since X is, as an Eilenberg-Maclane space of type $\mathrm{K}(\pi,n)$, $(n-1)$-connected, we can appeal to the Maps-to-K(π,n)-Proposition 307.18 (1) to obtain a map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X,x_0) \to \pi_n(Y,y_0)$. It suffices to prove the following claim:

Claim. The map f is a homotopy equivalence $(X,x_0) \to (Y,y_0)$ of pointed topological spaces.

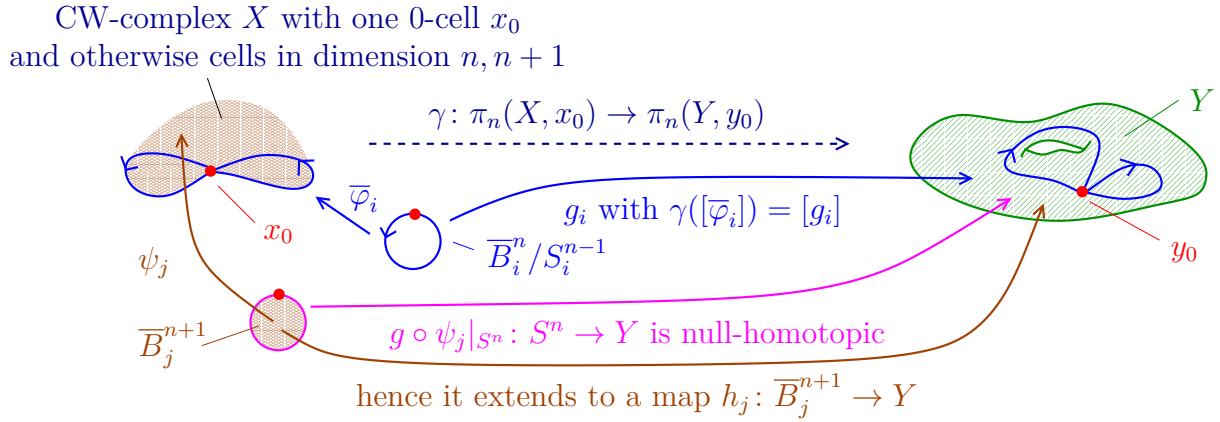
Proof. As above we obtain from the Maps-to-K(π,n)-Proposition 307.18 (1) the existence of a continuous map $g\colon Y \to X$ with $g(y_0) = x_0$ such that $g_* = \varphi^{-1}\colon \pi_n(Y,y_0) \to \pi_n(X,x_0)$. Note that

$$(g \circ f)_* \;=\; \varphi^{-1} \circ \varphi \;=\; \mathrm{id}_{\pi_n(X,x_0)} \;=\; (\mathrm{id}_X)_*$$

and note that $(g \circ f)(x_0) = x_0 = (\mathrm{id}_X)(x_0)$. Thus it follows from the Maps-to-K(π, n)-Proposition 307.18 (2) that $g \circ f$ is homotopic rel x_0 to id_X. Basically the same argument shows that $f \circ g$ is homotopic rel y_0 to id_Y. But this means that g is indeed the desired homotopy inverse. ■

At this point it remains to prove the Maps-to-K(π, n)-Proposition 307.18. The proof of the proposition relies on the following two lemmas. The following lemma is the key ingredient in the proof of the Maps-to-K(π, n)-Proposition 307.18.

Lemma 307.19. (Realize Map-on-π_n-Lemma) Let $n \in \mathbb{N}$. Let X be a CW-complex that consists of a single 0-cell x_0 and that otherwise contains only cells in dimensions n and $n+1$. Furthermore let (Y, y_0) be a pointed topological space and let $\gamma\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be a homomorphism. There exists a continuous map $h\colon (X, x_0) \to (Y, y_0)$ such that $h_* = \gamma$.



Proof of Lemma 307.19. We proceed in the following steps.

(1) We note that by our hypothesis we have $X^{n-1} = \{x_0\}$.

(2) Let $\{\varphi_i\colon (\overline{B}^n, S^{n-1}) \to (X^n, X^{n-1} = \{x_0\})\}_{i \in I}$ be a full set of characteristic maps of the n-cells of the CW-complex X. Note that since $X^{n-1} = \{x_0\}$ each map φ_i descends to a map $\overline{\varphi}_i\colon \overline{B}^n/S^{n-1} \to X^n$.

(3) It follows from $X^{n-1} = \{x_0\}$ and the Skeleton-Quotient Lemma 139.12 that the maps $\overline{\varphi}_i$ induce a homeomorphism $\Phi\colon \bigvee_{i \in I} \overline{B}^n_i/S^{n-1}_i \to X^n$. By a slight abuse of notation we denote by x_0 the wedge point of the wedge.

(4) Let $i \in I$. Note that the map $\overline{\varphi}_i$ defines an element in $\pi_n(X, x_0)$. We pick a continuous map $g_i\colon \overline{B}^n/S^{n-1} \to (Y, y_0)$ with $\gamma([\overline{\varphi}_i]) = [g_i]$.

(5) We consider the map

$$\begin{array}{rccl} g\colon \bigvee\limits_{i \in I} \overline{B}^n_i/S^{n-1}_i & \to & Y & \\ [x] & \mapsto & g_i([x]) & \text{if } x \in \overline{B}^n_i. \end{array}$$

(6) We consider the following diagram:

$$\begin{array}{ccccc} \pi_n(X^n, x_0) & \xleftarrow[\cong]{\Phi_*} & \pi_n\Big(\bigvee\limits_{i \in I} \overline{B}^n_i/S^{n-1}_i, x_0\Big) & \xrightarrow{g_*} & \pi_n(Y, y_0). \\ & {\scriptstyle i_*}\searrow & & \nearrow{\scriptstyle \gamma} & \\ & & \pi_n(X, x_0) & & \end{array}$$

where the left diagonal map is induced by the inclusion $i\colon X^n \to X$.

(7) It follows immediately from the construction of g that if we send any of the classes $[\mathrm{id}\colon \overline{B}^n \to \overline{B}_i] \in \pi_n(X^n, x)$ on a journey through the diagram, then we obtain the same result. But by the Wedge-of-Spheres-π_n-Proposition 307.7 (3) the classes $[\mathrm{id}\colon \overline{B}^n \to \overline{B}_i]$, $i \in I$, form a generating set for $\pi_n\Big(\bigvee_{i\in I} \overline{B}^n_i / S^{n-1}_i, x_0\Big)$. Thus it follows that the diagram commutes.

(8) Let $* = (0, \dots, 0, 1) \in S^n \subseteq \overline{B}^{n+1}$ the usual base point. Let $\{\psi_j\colon \overline{B}^{n+1} \to X^{n+1}\}_{j\in J}$ be a full set of characteristic maps of the $(n+1)$-cells of X. It follows from the Orthogonal Action Lemma 2.37 that we can arrange that for every $j \in J$ we have $\psi_j(*) = x_0$

(9) Let $j \in J$. Note that

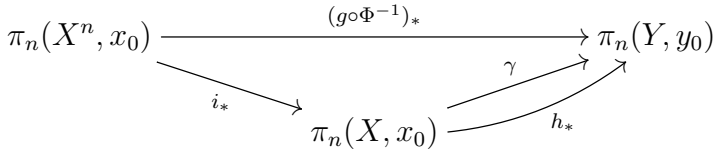
$$[g\circ\Phi^{-1}\circ\psi_j|_{S^n}] \;=\; (g_*\circ\Phi_*^{-1})(\overbrace{[\psi_j|_{S^n}]}^{\in\pi_n(X^n,x_0)}) \;\underset{\text{by the above commutative diagram}}{=}\; \gamma(\overbrace{[\psi_j|_{S^n}]}^{\in\pi_n(X,x_0)}) \;\underset{\text{since } \psi_j|_{S^n} \text{ extends to a map } \overline{B}^{n+1}\to X \text{ we have } [\psi_j|_{S^n}] = 0 \in \pi_n(X,x_0) \text{ by the Trivial-in-}\pi_n\text{-Lemma 134.10}}{=}\; \gamma([0]) \;\underset{\text{since } \gamma \text{ is a homomorphism}}{=}\; 0 \in \pi_n(Y, y_0).$$

Thus we obtain from the Trivial-in-π_n-Lemma 134.10 that there exists a continuous map $h_j\colon \overline{B}^{n+1} \to Y$ such that $h_j|_{S^n} = g \circ \Phi^{-1} \circ (\psi_j|_{S^n})$.

(10) Since for each $j \in J$ we have $h_j|_{S^n} = g \circ \Phi^{-1} \circ (\psi_j|_{S^n})$ we can use the maps h_j to extend $g \circ \Phi^{-1}$ to a map

$$h\colon X = \Big(X^n \cup \bigsqcup_{j\in J} \overline{B}^{n+1}_j\Big)/\sim \;\to\; Y \qquad \text{where } \psi_j(x) \sim x \text{ for any } x \in S^n_j.$$

(11) We obtain the following diagram:



(12) We already saw that the diagram commutes for γ. Since $h|_{X^n} = g \circ \Phi^{-1}$ we see that the diagram commutes also for the map h_*.

(13) The left diagonal map $i_*\colon \pi_n(X^n, x_0) \to \pi_n(X, x_0)$ is an epimorphism by the CW-Skeleton-π_n-Proposition 141.3.

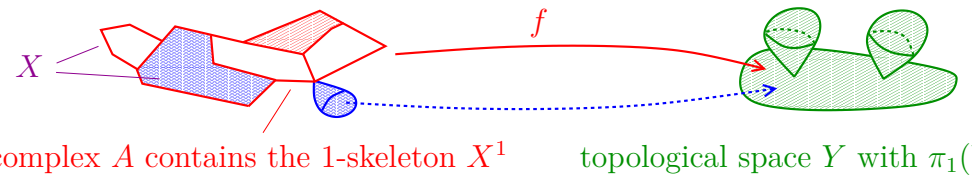
(14) It follows from (12) and (13) that the maps γ and h_* from $\pi_n(X, x_0)$ to $\pi_n(Y, y_0)$ agree. ∎

The second lemma is just a convenient special case of the Map Extension Criterion Lemma 304.6.

Lemma 307.20. (Extend Map-over-Cell Lemma) Let X be a CW-complex, let $n \in \mathbb{N}$ and let $f\colon A \to Y$ be a continuous map from a subcomplex $A \subseteq X$ to a path-connected topological space with $X^n \subseteq A$.

(1) If $\pi_n(Y) = 0$, then we can extend f to a map $f\colon A \cup X^{n+1} \to Y$.

(2) If $\pi_i(Y) = 0$ for every $i \geq n$, then we can extend f to a map $f\colon X \to Y$.



We break the proof of the Maps-to-K(π, n)-Proposition 307.18 into two parts. In the first part we deal with the situation that the $(n-1)$-skeleton consists just of a point. Afterwards we will discuss how the general case can be reduced to this special case.

Proof of the Maps-to-K(π, n)-Proposition 307.18 if $X^{n-1} = \{x_0\}$. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Furthermore let Y be an Eilenberg-Maclane space of type K(π, n) and let $y_0 \in Y$. Finally let X be a CW-complex and let $n \in \mathbb{N}$. We assume that $X^{n-1} = \{x_0\}$.

(1) Let $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be a homomorphism. We need to show that there exists a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. We denote by $i\colon X^{n+1} \to X$ the inclusion map. By the CW-Skeleton-π_n-Proposition 141.3 we know that $i_*\colon \pi_n(X^{n+1}, x_0) \to \pi_n(X, x_0)$ is an isomorphism. The following diagram contains all the maps of interest to us:

$$\begin{array}{ccc} \pi_n(X^{n+1}, x_0) & \overset{h_*}{\searrow} & \\ {\scriptstyle i_*}\downarrow{\scriptstyle \cong} & \overset{f_*}{\searrow} & \\ \pi_n(X, x_0) & \xrightarrow[\varphi]{} & \pi_n(Y, y_0). \end{array}$$

Next note that since $X^{n-1} = \{x_0\}$ we can apply the Realize Map-on-π_n-Lemma 307.19 to the CW-complex X^{n+1}. Therefore we obtain from the Realize Map-on-π_n-Lemma 307.19 that there exists a continuous map $h\colon X^{n+1} \to Y$ with $h(x_0) = y_0$ such that $h_* = \varphi \circ i_*\colon \pi_n(X^{n+1}, x_0) \to \pi_n(Y, y_0)$. Since Y is an Eilenberg-Maclane space of type K(π, n) we know that $\pi_j(Y) = 0$ for $j \geq n+1$. Thus we obtain from the Extend Map-over-Cell Lemma 307.20 (2) that we can extend $h\colon X^{n+1} \to Y$ to a map $f\colon X \to Y$. It remains to show that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. This is indeed the case since

$$f_* \underset{\substack{\uparrow \\ \text{since } i_* \text{ is an} \\ \text{isomorphism}}}{=} f_* \circ i_* \circ i_*^{-1} = (f \circ i)_* \circ i_*^{-1} \underset{\substack{\uparrow \\ \text{since } h \text{ extends } f}}{=} h_* \circ i_*^{-1} \underset{\substack{\uparrow \\ \text{since } h_* = \varphi \circ i_*}}{=} (\varphi \circ i_*) \circ i_*^{-1} = \varphi.$$

(2) Suppose we are given continuous maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$ and such that $f_{0*} = f_{1*}\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. We need to show that there exists a homotopy rel x_0 from f_0 to f_1. We consider the product $X \times [0, 1]$ and we equip it with the product CW-structure coming from the CW-Complex Product Proposition 139.18. Given $k \in \mathbb{N}_0$ we write

$$Z^k = (X \times \{0\}) \cup \underbrace{(X^{k-1} \times [0, 1])}_{k\text{-dimensional}} \cup (X \times \{1\}) \subseteq X \times [0, 1].$$

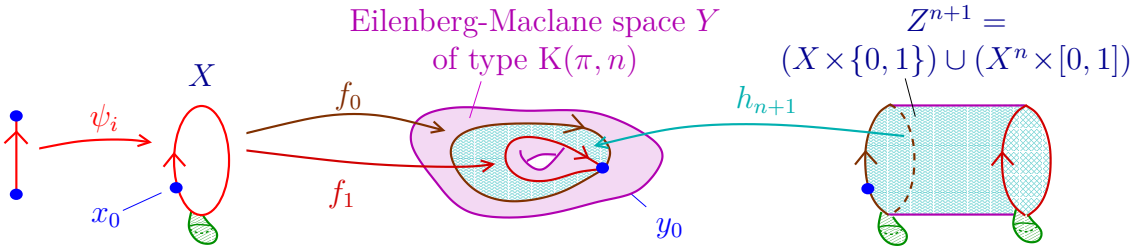
The idea is define the homotopy on $X \times [0,1] = \bigcup_{n \in \mathbb{N}_0} Z^n$ by breaking the task into three steps.

(a) First we note that the fact that $X^0 = \cdots = X^{n-1}$ implies that $Z^1 = \cdots = Z^n$. Now let $h\colon Z^1 = Z^n \to Y$ be the map that is given by f_0 on $X \times \{0\}$, f_1 on $X \times \{1\}$ and the constant map c_{x_0} on $\{x_0\} \times [0,1]$.

(b) We denote by $\{\varphi_i\colon \overline{B}_i^n \to X\}_{i \in I}$ the characteristic maps of the n-cells of X. Since $X^{n-1} = \{x_0\}$ we see that each φ_i descends to a map $\psi_i\colon \overline{B}_i^n / S_i^{n-1} \to X$. We denote by $* = [S_i^{n-1}]$ the obvious base point of the sphere $\overline{B}_i^n / S_i^{n-1}$. Now recall that by hypothesis we have $f_{0*} = f_{1*}\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. This implies that for every $i \in I$ we have $[f_0 \circ \psi_i] = [f_1 \circ \psi_i] \in \pi_n(Y, y_0)$. In other words, for every $i \in I$ there exists a continuous map $F_i\colon \overline{B}_i^n / S_i^{n-1} \times [0,1] \to Y$ with $F_{i0} = f_0 \circ \psi_i$, $F_{i1} = f_1 \circ \psi_i$ and $F_i(*, t) = y_0$ for all $t \in [0,1]$. We define

$$\begin{array}{rccl} h_{n+1}\colon & Z^{n+1} & \to & Y \\ & P & \mapsto & \begin{cases} h_0(P), & \text{if } P \in Z^1 = Z^n, \\ F_i(Q,t), & \text{if } P = (\varphi_i(Q), t) \text{ with } Q \in \overline{B}_i^n. \end{cases} \end{array}$$

It follows immediately from the properties of the F_i that this map is well-defined. Furthermore, it follows almost immediately from the Characteristic Maps Proposition 136.9 (2) that h_{n+1} is continuous.



(c) Finally we are in the following situation:

$$\begin{array}{ccc} Z^{n+1} & \xrightarrow{h_{n+1}} & Y \\ \downarrow & \nearrow_{H} & \\ X \times [0,1]. & & \end{array}$$

By definition of the CW-structure on $X \times [0,1]$ we know that the $(n+1)$-skeleton of $X \times [0,1]$ is contained in Z^{n+1}. In other words, any cell of $X \times [0,1]$ that is not contained in Z^{n+1} is of dimension $\geq n+2$. Since Y is an Eilenberg-Maclane space of type K(π, n) we know that $\pi_k(Y, y_0) = 0$ for $k \geq n+1$. Thus we obtain from the Extend Map-over-Cell Lemma 307.20 (2) that there exists a continuous map $H\colon X \times [0,1] \to Y$ which makes the above diagram commute. Evidently this is the desired homotopy between f_0 and f_1.

(3) The last statement is just a reformulation of Statement (1) together with Statement (2). ∎

Proof of the Maps-to-K(π, n)-Proposition 307.18 for the general case. Let X be a CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. We suppose that X is $(n-1)$-connected. By the No-Low Cells Proposition 305.15 we know that there exists a homotopy equivalence $f\colon (X, x_0) \to (\widetilde{X}, \widetilde{x}_0)$ where $\widetilde{X}$ is a CW-complex that has one

0-cell $\{\widetilde{x}_0\}$ and no cells in dimensions $1, \ldots, n$. By the Homotopy Equivalence-π_n-Proposition 134.17 (1) we know that $f_*\colon \pi_n(X, x_0) \to \pi_n(\widetilde{X}, \widetilde{x}_0)$ is an isomorphism. By the above argument we know that the desired conclusions hold for $(\widetilde{X}, \widetilde{x}_0)$. It is now fairly elementary to see that the statements of the Maps-to-K(π, n)-Proposition 307.18 also hold for (X, x_0). ■

Before we move on we prove an important addendum to the Homotopy-π_1-Proposition 90.8. To motivate its relevance we recall an earlier result: If $f_0, f_1\colon (X, x_0) \to (Y, y_0)$ are two continuous maps between pointed topological spaces, then by the Homotopy-π_1-Proposition 90.8 the following two statements hold:

(1) If f_0 and f_1 are homotopic rel x_0, then $f_{0*} = f_{1*}$.
(2) If f_0 and f_1 are homotopic (with no constraints on the base point), then there exists an $\alpha \in \pi_1(Y, y_0)$ such that $f_{0*}(z) = \alpha \cdot f_{1*}(z) \cdot \alpha^{-1}$ for all $z \in \pi_1(Y, y_0)$.

One can view the Maps-to-K(π, n)-Proposition 307.18 (2) as a partial converse to (1). The following proposition is the analogous partial converse to (2).

Proposition 307.21. (Map-to-K$(\pi, 1)$-Homotopy Lemma) Let π be a group, let Y be an Eilenberg-Maclane space of type K$(\pi, 1)$ and let $y_0 \in Y$. Furthermore let X be a 0-connected CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. Suppose we are given two continuous maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$. If there exists an $\alpha \in \pi_1(Y, y_0)$ such that $f_{0*}(z) = \alpha \cdot f_{1*}(z) \cdot \alpha^{-1}$ for all $z \in \pi_1(Y, y_0)$, then there exists a homotopy[3394] from f_0 to f_1.

Proof. The proof of the proposition is similar to the proof of the Maps-to-K(π, n)-Proposition 307.18 (2). In the following provide a sketch of the argument. We leave it to the reader to fill in the details. First of all we point out that by the No-Low Cells Proposition 305.15 we can assume that X has a single 0-cell $\{x_0\}$. Now we consider the product $X \times [0, 1]$ and we equip it with the product CW-structure coming from the CW-Complex Product Proposition 139.18. Given $k \in \mathbb{N}_0$ we write

$$Z^k \;=\; (X \times \{0\}) \cup \underbrace{(X^{k-1} \times [0, 1])}_{k\text{-dimensional}} \cup (X \times \{1\}) \;\subseteq\; X \times [0, 1].$$

As above we define the homotopy on $X \times [0, 1] = \bigcup\limits_{n \in \mathbb{N}_0} Z^n$ by breaking the task into three friendly steps.

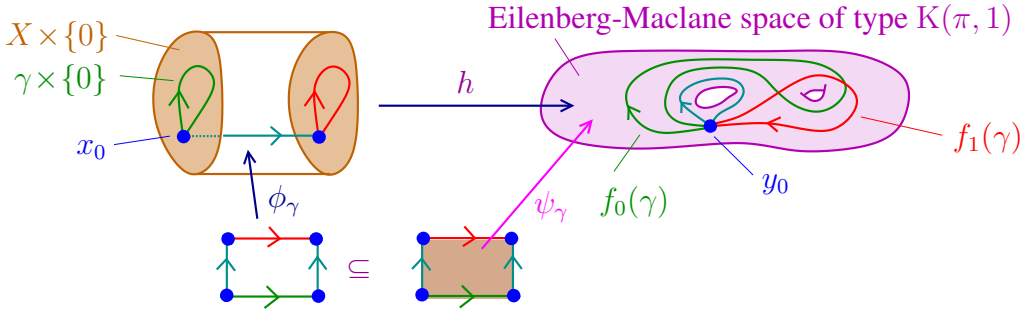
(1) First we pick a continuous map $\beta\colon [0, 1] \to Y$ that represents $\alpha \in \pi_1(Y, y_0)$. We denote by $h\colon Z^1 = Z^n \to Y$ the continuous map that is given by f_0 on $X \times \{0\}$, f_1 on $X \times \{1\}$ and the map β on $\{x_0\} \times [0, 1] = [0, 1]$.

[3394] Note that in contrast to the Maps-to-K(π, n)-Proposition 307.18 (2) it does *not* say "rel x_0".

(2) Given a 1-cell γ of X we denote by $\phi_\gamma\colon S^1 = \partial([0,1]^2) \to Z^1$ the attaching map of the product cell $\gamma \times [0,1]$. We have

$$\begin{array}{rcl} h_*([\phi_\gamma]) & \overset{\text{by a generous interpretation of the notation}}{=} & [h|_{\gamma\times 0}]\cdot[h|_{\{x_0\}\times[0,1]}]\cdot\overline{[h|_{\gamma\times 1}]}\cdot\overline{[h|_{\{x_0\}\times[0,1]}]} \overset{\text{by definition of } h}{=} [f_{0*}(\gamma)]\cdot[\beta]\cdot[f_{1*}(\gamma)]^{-1}\cdot[\beta]^{-1} \\ & \overset{\text{by hypothesis and choice of } \beta}{=} & \alpha\cdot[f_{1*}(\gamma)]\cdot\alpha^{-1}\cdot\alpha\cdot[f_{1*}(\gamma)]^{-1}\cdot\alpha^{-1} = e \in \pi_1(Y, y_0). \end{array}$$

It follows from the Trivial-in-π_1-Lemma 88.28 that the map $h \circ \phi_\gamma\colon \partial([0,1]^2) \to Y$ extends to a map $\psi_\gamma\colon [0,1]^2 \to Y$. We use this map to extend h over the 2-cell $\gamma \times [0,1]$. If we do this for all 1-cells of X we obtain the desired extension of h to a map $Z^2 \to Y$.



(3) Finally note that we use the Extend Map-over-Cell Lemma 307.20 (2) and the fact that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$ to extend the map h to a homotopy $F\colon X \times [0,1] \to Y$. ∎

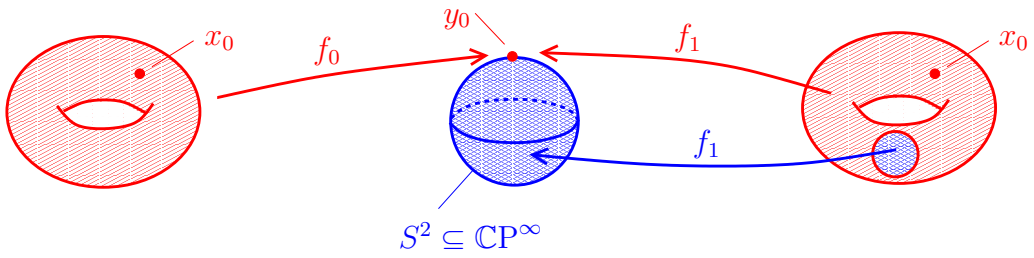
We conclude this section with a discussion of the statement of the Maps-to-$\mathrm{K}(\pi,n)$-Proposition 307.18. It is not unreasonable to ask whether one really needs in the Maps-to-$\mathrm{K}(\pi,n)$-Proposition 307.18 the hypothesis that X is $(n-1)$-connected. But we will now see that in neither of the two parts can we drop this condition.

(1) Recall that by the $\pi_n(S^3) = \pi_n(S^2)$–Theorem 239.12 we know that there exists an isomorphism $\varphi\colon \pi_3(S^2,*) \xrightarrow{\cong} \mathbb{Z}$. Now let (Y, y_0) be any pointed Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z},3)$. At a very first glance one might hope that there exists a continuous map $f\colon (S^2,*) \to (Y, y_0)$ such that $f_* = \varphi\colon \pi_3(S^2,*) \to \pi_3(Y,y_0) =_{\mathrm{i}} \mathbb{Z}$. But this is not the case. Indeed, since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z},2)$ we know that $\pi_2(Y,y_0) = 0$. This means that any continuous map $f\colon (S^2,*) \to (Y,y_0)$ is homotopic to the constant map. In particular any such map induces the zero map on π_3. Thus we cannot realize φ by a map $(S^2,*) \to (Y,y_0)$.

(2) By the Eilenberg–Maclane Example Lemma 307.3 (5) we know that the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z},2)$. By the discussion on page 340 we can view $S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$ as a subset of $\mathbb{C}\mathrm{P}^\infty$. We denote by $y_0 \in S^2$ the North Pole. We pick some point x_0 on the torus $S^1 \times S^1$. We consider the following two maps from the torus $S^1 \times S^1 \to S^2 \subseteq \mathbb{C}\mathrm{P}^\infty$:

(a) The map f_0 is the constant map that sends all points to y_0.

(b) The map $f_1\colon S^1 \times S^1 \to \overline{B}^2/S^1 =_{\mathrm{i}} S^2$ is a continuous map that sends the points of a given embedded disk to $\overline{B}^2$ and sends all other points to the single point $[S^1] = y_0$.



Now we claim that we have $f_{0*} = f_{1*} \colon \pi_2(S^1 \times S^1, *) \to \pi_2(\mathbb{C}\mathrm{P}^\infty, *)$, but that f_0 and f_1 are, contrary to what one might expect, *not* homotopic. To prove this claim we recall that on page 2587 we saw that $\pi_2(S^1 \times S^1) = 0$. In particular both maps induce the same map on π_2. Nonetheless the two maps are not homotopic. This can be seen by studying the induced map on H_2. It is clear that f_0 induces the trivial map on H_2. On the other hand we saw in the proof of the Degree Realization Proposition 46.19 that f_1 induces an isomorphism $\mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}_2(S^2; \mathbb{Z})$. By the discussion on page 2956 we know that the inclusion-induced map $\mathrm{H}_2(S^2; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ is an isomorphism. Therefore we see that the map $f_{1*} \colon \mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^\infty)$ is non-trivial. Thus we obtain from the Homotopy-H_*-Proposition 144.12 that the maps f_0 and f_1 are in fact not homotopic.

At least in terms of invariants any Eilenberg-Maclane space $\mathrm{K}(\pi, n)$ is a particularly simple topological space. Therefore it seems not totally unreasonable to hope that given a CW-complex X one can perhaps classify the maps from X to $\mathrm{K}(\pi, n)$, say up to homotopy. The above examples show that the maps are *not* classified by $\mathrm{Hom}(\pi_n(X), \pi)$. But perhaps there is some other way? Let us record this as a question:

Question 307.22. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Let X be a CW-complex. Can we classify the maps $X \to \mathrm{K}(\pi, n)$ up to homotopy?

307.5. **First applications of Eilenberg-Maclane spaces.** We start out by recalling Definition 233.1:

Definition 307.23. Let G be a group, let Y be a topological space and let $y_0 \in Y$ be a point with an identification $\pi_1(Y, y_0) \xrightarrow{=} G$. Given another topological space X and $x_0 \in X$ we say that a homomorphism $\varphi \colon \pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is realized by a continuous map $g \colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

Next we recall the following question that we had formulated a while ago.

Question 233.3 and Question 233.5. Let X be a 0-connected CW-complex.[3395]

(0) Can every homomorphism $\pi_1(X) \to \mathbb{Z} =_{\mathrm{i}} \pi_1(S^1)$ be realized by a continuous map $X \to S^1$?

(1) Let $g \geq 1$ and let Σ_g be the surface of genus $g \geq 1$. Can every homomorphism $\pi_1(X) \to \pi_1(\Sigma_g)$ be realized by a continuous map $X \to \Sigma_g$?

(2) Can every homomorphism $\pi_1(X) \to \pi_1\left(\bigvee_{i=1}^{k} S^1\right) = \langle x_1, \ldots, x_k \rangle$ be realized by a continuous map $X \to \bigvee_{i=1}^{k} S^1$?
(3) Can every homomorphism $\pi_1(X) \to \mathbb{Z}/2\mathbb{Z} =_{\mathrm{i}} \pi_1(\mathbb{R}\mathrm{P}^2)$ be realized by a continuous map $X \to \mathbb{R}\mathrm{P}^2$?

In the Maps-to-S^1-Proposition 233.4 we used de Rham cohomology to give an affirmative answer to (0) if X is a smooth manifold. Furthermore in Exercise 213.8 we gave, by considering the isomorphism $\pi_1(\mathbb{R}\mathrm{P}^3) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}$, a negative answer to (3). Fortunately we can now give a complete answer to (0), (1) and (2).

Proposition 307.24. (Map-out-of-CW Proposition)
(a) The answer to (0), (1) and (2) is "yes". Furthermore, if in the formulation of (3) we replace $\mathbb{R}\mathrm{P}^2$ by $\mathbb{R}\mathrm{P}^\infty$, then the answer to (3) is also yes.
(b) If X is a smooth manifold, then the following two statements hold:
(א) In (0) and (1) we can find smooth realizations.
(ב) In (2) we can find a realization $f\colon X \to \bigvee_{i=1}^{k} S_i^1$ which has the property that the restriction of f to the complement of $f^{-1}(\text{wedge point})$ is smooth.

Proof.
(a) The desired statement follows immediately from the Maps-to-K(π, n)-Proposition 307.18 and the observation that by the Eilenberg–Maclane Example Lemma 307.3 we know that S^1, $\bigvee_{i=1}^{k} S^1$, Σ_g and $\mathbb{R}\mathrm{P}^\infty$ are Eilenberg-Maclane spaces.
(b) (א) This statement follows from the Whitney Approximation Theorem 36.1.
(ב) As we pointed out on page 913, this statement is a consequence of the Smooth-on-Sphere Proposition 36.10. ■

Using the Map-out-of-CW Proposition 307.24 we can formulate a pleasant proposition. To formulate this result let us introduce two definitions, one from group theory and one from the theory of smooth manifolds:

Definition 307.25. Let π be a group. The corank of π is defined as

$$\mathrm{corank}(\pi) \;:=\; \max\{n \in \mathbb{N}_0 \cup \infty \,|\, \text{there exists an epimorphism from } \pi \text{ onto a free group rank } n\}.$$

We continue with the following definition that we first learned about in Exercise 155.31:

Definition 307.26. Let M be a compact orientable connected non-empty smooth manifold. We define its *cut number* $\mathrm{cut}(M)$ as the maximal number of pairwise disjoint proper orientable codimension-one submanifolds $F_1, \ldots, F_c$ such that $M \setminus (F_1 \cup \cdots \cup F_c)$ is still connected.

Now we can formulate our promised result:

[3395] If M is a smooth manifold, then we saw in the Smooth Manifold-Simplicial Structure Theorem 179.5 and the Simplicial-Implies-CW Lemma 174.51 that M admits a CW-structure.

Proposition 307.27. (Manifold–π_1–Free Group Epimorphism Proposition) If M is a compact orientable connected n-dimensional smooth manifold, then the following two statements are equivalent:

(1) There exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k \rangle$.
(2) There exist disjoint connected non-empty $(n-1)$-dimensional submanifolds $F_1, \dots, F_k$ of M such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected.

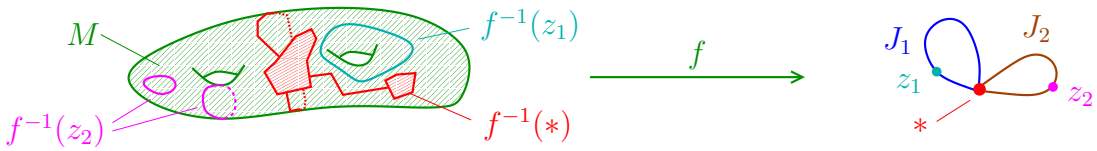
Sketch of a proof. Let M be a compact orientable connected n-dimensional smooth manifold. It follows from Definition 307.25 Let π be a group. The corank and Definition 307.26 that we need to prove the following claim:

Claim. For each $k \in \mathbb{N}_0$ the following two statements are equivalent:

(1) There exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k \rangle$.
(2) There exist disjoint connected non-empty $(n-1)$-dimensional submanifolds $F_1, \dots, F_k$ of M such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected.

Proof. First note that the "(2)⇒(1)" implication is precisely the content of the Epimorphism-onto-Free Group Proposition 234.14. Thus it remains to prove the "(1)⇒(2)" implication.

Therefore suppose that there exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k \rangle$. We consider the wedge $\bigvee_{i=1}^{k} S_i^1$. We denote by $*$ the wedge point and for $i = 1, \dots, k$ we write $J_i := S_i^1 \setminus \{*\}$. By the Map-out-of-CW Proposition 307.24 (b) we can realize this epimorphism by a map $f \colon M \to \bigvee_{i=1}^{k} S_i^1$ which is smooth on $M \setminus f^{-1}(*)$. Let $i \in \{1, \dots, k\}$. It follows from Sard's Theorem 38.1 that there exists a regular value $z_i \in J_i$ of f and also of $f|_{\partial M \setminus f^{-1}(*)}$.



Note that it follows from the Preimage-of-Point-in-Ball Lemma 36.11 that the preimage $G_i := f^{-1}(z_i)$ is an orientable proper $(n-1)$-dimensional submanifold of M. In Exercise 307.3 we will prove the following two statements:

(1) each G_i is non-empty,
(2) for each $i \in \{1, \dots, k\}$ we can find a component F_i of G_i such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected. ■

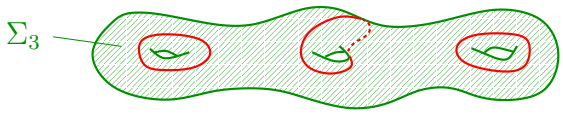
Now is a good time to recall the following question:

Question 234.15. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.

(1) What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \dots, F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected?
(2) What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of l generators?

We make the following easy observations:

(1) As we already observed on page 4515, it follows immediately from peeking at the figure below that $k(g) \geq g$.
(2) It follows from the Presentation–Homomorphism Lemma 100.13 that we have an epimorphism



there exist three curves such that the complement is still connected

$$\overbrace{\langle x_1, y_1, \ldots, x_g, y_g \,|\, x_1 y_1 x_1^{-1} y_1^{-1} \cdot \ldots \cdot x_g y_g x_g^{-1} y_g^{-1} \rangle}^{\cong \pi_1(\Sigma_g) \text{ by the Surface-}\pi_1\text{-Proposition 101.1}} \quad \begin{array}{rcl} \to & \langle x_1, \ldots, x_g \rangle \\ x_i & \mapsto & x_i \\ y_i & \mapsto & e \end{array}$$

from $\pi_1(\Sigma_g)$ onto a free group on g generators.

More interestingly, by the Manifold–π_1–Free Group Epimorphism Proposition 307.27 we now know that the two questions actually give the same answer. Even better, we can now give a complete answer:

Proposition 307.28. (Surface–Free Group Epimorphism Proposition) Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. If there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group on k generators, then $k \leq g$.

Proof. Let $g \in \mathbb{N}$, let Σ_g be the surface of genus g and let $* \in \Sigma_g$ be a base point. Let $\varphi\colon \pi_1(\Sigma_g, *) \to F_k$ be an epimorphism onto a free group on k generators. We start out with the following preparations:

(1) By the Eilenberg–Maclane Example Lemma 307.3 (4) we know that $X := \bigvee_{i=1}^{k} S^1$ is an Eilenberg-Maclane space of type $\mathrm{K}(F_k, 1)$. We pick $x_0 \in X$ and as usual we make the identification $\pi_1(X, x_0) = F_k$.
(2) It follows from the Maps-to-K(π, n)-Proposition 307.18 that there exists a continuous map $f\colon (\Sigma_g, *) \to (X, x_0)$ with $f_* = \varphi\colon \pi_1(\Sigma_g, *) \to \pi_1(X, x_0) = F_k$. Note that the map $f_*\colon \pi_1(\Sigma_g, *) \to \pi_1(X, x_0)$ is an epimorphism.
(3) The Hurewicz Theorem 166.10, together with the Abelianization-Epimorphism Lemma 98.17, implies that the induced map $f_*\colon \mathrm{H}_1(\Sigma_g; \mathbb{Z}) \to \mathrm{H}_1(X; \mathbb{Z})$ is an epimorphism.
(4) In general, if $\alpha\colon A \to B$ is an epimorphism between two groups, then elementary algebra tells us that the induced map $\mathrm{Hom}(B, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z})$ is a monomorphism. From this elementary observation, together with (3), we obtain that the induced map $f^*\colon \mathrm{Hom}(\mathrm{H}_1(X; \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_1(\Sigma_g; \mathbb{Z}), \mathbb{Z})$ is a monomorphism.
(5) It follows immediately from (4) together with the H^1-Hom-Proposition 201.24 that the induced map $f^*\colon \mathrm{H}^1(X; \mathbb{Z}) \to \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ is also a monomorphism.
(6) Since X is a 1-dimensional CW-complex we obtain almost immediately from the Singular-Cellular-H^*-Isomorphism Proposition 199.5 that $\mathrm{H}^2(X; \mathbb{Z}) = 0$. In particular the cup product on $\mathrm{H}^1(X; \mathbb{Z})$ vanishes.

Now we consider the form

$$\begin{array}{rcl} \varphi\colon \mathrm{H}^1(\Sigma_g; \mathbb{Z}) \times \mathrm{H}^1(\Sigma_g; \mathbb{Z}) & \to & \mathbb{Z} \\ (\alpha, \beta) & \mapsto & \langle \alpha \cup \beta, [\Sigma_g] \rangle. \end{array}$$

Given $U \subseteq \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ we write as usual

$$U^\perp = \{v \in \mathrm{H}^1(\Sigma_g; \mathbb{Z}) \,|\, \varphi(u, v) = 0 \text{ for all } u \in U\}.$$

Now we see that

$$\begin{array}{ccccc} & & \text{since } i^* \text{ is a} & & \text{by the Manifold-Non Singularity Proposition 213.3} \\ \text{since } X = \bigvee_{i=1}^{k} S^1 & & \text{monomorphism} & & \text{we know that } \varphi \text{ is non-singular,} \\ \downarrow & & \downarrow & & \text{hence we obtain the equality from Exercise 224.6} \\ k \;=\; \operatorname{rank}(\mathrm{H}^1(X;\mathbb{Z})) & & = \operatorname{rank}(i^*\mathrm{H}^1(X;\mathbb{Z})) & & \downarrow \\ & & & & = \operatorname{rank}(\mathrm{H}^1(\Sigma_g;\mathbb{Z})) - \operatorname{rank}\left((i^*\mathrm{H}^1(X;\mathbb{Z}))^{\perp}\right) \\ \leq \operatorname{rank}(\mathrm{H}^1(\Sigma_g;\mathbb{Z})) - \operatorname{rank}(i^*\mathrm{H}^1(X;\mathbb{Z})) & & = & & 2g-k. \\ \uparrow & & & & \uparrow \\ \text{it follows from (6) and the} & & & & \text{as above we have } \operatorname{rank}(i^*(\mathrm{H}^1(X;\mathbb{Z}))) = k \text{ and by the} \\ \text{Cup Product-Naturality Lemma 203.9} & & & & \text{discussion on page 3831 we know that } \operatorname{rank}(\mathrm{H}^1(\Sigma_g;\mathbb{Z})) = 2g \\ \text{that } i^*\mathrm{H}^1(X;\mathbb{Z}) \subseteq (i^*\mathrm{H}^1(X;\mathbb{Z}))^{\perp} & & & & \end{array}$$

Thus we see that $k \leq 2g - k$, i.e. we see that $k \leq g$. ■

307.6. **Eilenberg–Maclane spaces and the social choice problem.** In this section we give a fun, but perhaps not entirely serious application of Eilenberg-Maclane spaces. Let us recall Definition 168.1:

Definition 307.29. Let X be a topological space and let $n \in \mathbb{N}$. A social choice of type n is a map

$$f\colon X^n \;=\; \underbrace{X \times \cdots \times X}_{n \text{ times}} \;\to\; X$$

such that the following three axioms are satisfied:

(a) The map f is continuous.
(b) For every $x \in X$ we have $f(x, \ldots, x) = x$.
(c) For every permutation $\sigma \in S_n$ and any $(x_1, \ldots, x_n) \in X^n$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \;=\; f(x_1, \ldots, x_n).$$

We say that a topological space is social, if for every $n \in \mathbb{N}$ it admits a social choice of type n.

We refer to Chapter 168 for a discussion why social topological spaces are interesting. Now let X be a topological space. The combination of the Induced Social Homomorphism Lemma 168.7 and the Social Homomorphism Lemma 168.9 shows that if X is social, then for every $k \in \mathbb{N}$ the homotopy group $\pi_k(X)$ is abelian and it is strongly divisible, i.e. given any $n \in \mathbb{N}$ the map $g \mapsto n \cdot g$ is an isomorphism. Furthermore, the Social Choice Proposition 168.3 together with the Whitehead Theorem 305.1 show that if X is in fact a CW-complex, with the property that in any dimension it has only finitely many cells, then X is contractible. Back in Chapter 168 this discussion lead us to the following question.

Question 168.11. Does there exist a non-empty CW-complex that is social but that is not contractible?

It turns out that we can now formulate a positive answer to Question 168.11.

Theorem 307.30. (Social Choice–Eilenberg Maclane Theorem)

(1) Any CW-complex that is homotopy equivalent to the product of a family of Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Q}, k_i)$, $i \in I$ is social.
(2) Conversely, if X is a connected social CW-complex, then it is homotopy equivalent to the product of a family of Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Q}, k_i)$, $i \in I$.

Proof. This theorem is proved in [**EGH1962**, p. 90] and [**Weinb2004**, Theorem 1.3], see also [**Eckm2004**, Theorems 4 and 7]. For the most part the proofs do not go much

beyond what we have covered so far, except that they build on a result of Alexander Grothendieck [**Groth1957**, Chapter V], see also [**HZ1974**, p. 32] and [**Hirz1969**, p. 254]. ■

Remark. As always there are various variations and generalizations of the (Social Choice–Eilenberg Maclane Theorem 307.30. For example by [**Weinb2004**, Example 1.4] any Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1)$, has the property that given any *odd* $n \in \mathbb{N}$ it admits a social choice of type n. By the Eilenberg–Maclane Example Lemma 307.3 (2) this observation applies in particular to $\mathbb{R}\mathrm{P}^\infty$. □

307.7. **An extension of the Hurewicz Theorem 167.9.** In this last section we will use Eilenberg-Maclane spaces in a surprising way to obtain an addendum to the Hurewicz Theorem 167.9. To formulate the theorem it is convenient to recall Definition 257.14:

Definition 307.31. Given a group π and $n \in \mathbb{N}_0$ we define $\mathrm{H}_n(\pi) := \mathrm{H}_n(\mathrm{K}(\pi, 1))$ where $\mathrm{K}(\pi, 1)$ denotes the canonical Eilenberg-Maclane space introduced in the Eilenberg-Maclane Existence Theorem 307.10. We refer to $\mathrm{H}_n(\pi)$ as the n-th homology group of the group π.

Remark. The concept of the homology groups $\mathrm{H}_n(\pi)$ of a group π will play an important role in this section. Afterwards it takes a well-earned rest before it will take center stage in Chapter 317. □

The following proposition is identical to the earlier Group H_*-via-$\mathrm{K}(\pi, 1)$-Proposition 257.15:

Proposition 307.32. (Group H_*-via-$\mathrm{K}(\pi, 1)$-Proposition) Let π be a group. For any Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$ and any $n \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_n(\pi) \;\cong\; \mathrm{H}_n(X).$$

Proof. The proposition is an immediate consequence of the Eilenberg-Maclane-H_*-Corollary 307.16. ■

Example. Let $m \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. We calculate that

$$\mathrm{H}_n(\text{the group } \mathbb{Z}^m) \overset{\text{by the Group } \mathrm{H}_*\text{-via-K}(\pi,1) \text{ Proposition 307.32}}{\cong} \mathrm{H}_n(\text{any Eilenberg-Maclane space } \mathrm{K}(\mathbb{Z}^m, 1)) \overset{\text{by the Eilenberg–Maclane Example Lemma 307.3 (1) we know that } (S^1)^m \text{ is an Eilenberg-Maclane space of type } \mathrm{K}(\mathbb{Z}^m,1)}{\cong} \mathrm{H}_n((S^1)^m) \overset{\text{see page 3141}}{\cong} \mathbb{Z}^{\binom{m}{n}}.$$

□

We continue with the following lemma:

Lemma 307.33. (H_2-of-Group Lemma) Let (X, x_0) be a pointed 0-connected space. There exists a natural map

$$\eta\colon \mathrm{H}_2(X) \;\to\; \mathrm{H}_2(\pi_1(X, x_0))$$

which has the following property: whenever (Y, y_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(X, x_0), 1)$ and $f\colon (X, x_0) \to (Y, y_0)$ is a continuous map of pointed topological spaces that induces an isomorphism of fundamental groups, then there exists an

isomorphism $\mu\colon \mathrm{H}_2(Y) \to \mathrm{H}_2(\pi_1(X,x_0))$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_2(X) & \xrightarrow{\quad\eta\quad} & \mathrm{H}_2(\pi_1(X,x_0)) \\ & {\scriptstyle f_*}\searrow \quad \nearrow{\scriptstyle \cong}\ {\scriptstyle \mu} & \\ & \mathrm{H}_2(Y). & \end{array}$$

Proof. Let (X,x_0) be a pointed 0-connected space. We write $\pi = \pi_1(X,x_0)$. We denote by $(Z,z_0,\varphi\colon \pi_1(Z,z_0) \to \pi)$ the canonical triple introduced in the Eilenberg-Maclane Existence Theorem 307.10. By the CW-Approximation Theorem 304.14 there exists a CW-approximation $f\colon (Y,y_0) \to (X,x_0)$. Note that by the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 we know that f induces isomorphisms on all homology groups.

By the Maps-to-K(π,n)-Proposition 307.18 (1) we know that there exists a continuous map $g\colon (Y,y_0) \to (Z,z_0)$ such that $g_* = \varphi^{-1} \circ f_*\colon \pi_1(Y,y_0) \to \pi_1(Z,z_0)$. We set

$$\eta \coloneqq g_* \circ (f_*)^{-1}\colon \mathrm{H}_2(X) \ \to\ \mathrm{H}_2(Z) =: \mathrm{H}_2(\pi_1(X,x_0)).$$

Using the Maps-to-K(π,n)-Proposition 307.18 (1) it is not difficult to verify that η has the desired properties. ■

Now we can formulate the following theorem:

Theorem 307.34. (Hurewicz-plus-One Theorem) Let $n \in \mathbb{N}$ and let X be an $(n-1)$-connected space and let $x_0 \in X$.

(1) If $n = 1$, then the sequence

$$\pi_2(X,x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_2(X) \xrightarrow{\eta} \mathrm{H}_2(\pi_1(X,x_0)) \ \to\ 0$$

is exact. This implies in particular that the cokernel of the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_2(X,x_0) \to \mathrm{H}_2(X)$ is isomorphic to $\mathrm{H}_2(\pi_1(X))$.

(2) If $n > 1$, then the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{n+1}(X,x_0) \to \mathrm{H}_{n+1}(X)$ is an epimorphism.

Remark.

(1) We refer to [**WhdG1978**, Theorem V.7.9] for a generalization of the above theorem.

(2) Let $n > 1$. Furthermore let X be an $(n-1)$-connected space. Note that by the above Hurewicz-plus-One Theorem 307.34 (2) we now know that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{n+1}(X,x_0) \to \mathrm{H}_{n+1}(X)$ is an epimorphism. For $n \geq 3$ the kernel of the Hurewicz homomorphism is discussed in [**Hilt1953**, p.105] and [**WhdJ1950a**, p. 72]. Furthermore, for the more tricky case $n = 2$ the kernel of the Hurewicz homomorphism is determined in [**WhdJ1950a**, pages 72, 76 and Theorem 20]. □

Example. Let X be a simply connected topological space and let $x_0 \in X$. By the Hurewicz-plus-One Theorem 307.34, applied to the case $n = 2$, the Hurewicz homomorphism $\pi_3(X,x_0) \to \mathrm{H}_3(X)$ is in fact an epimorphism. It is natural to wonder whether the Hurewicz-plus-One Theorem 307.34 can be improved upon. For example it is a priori not unreasonable to ask whether the Hurewicz homomorphism $\pi_k(X,x_0) \to \mathrm{H}_k(X)$ is an epimorphism for every $k \in \mathbb{N}_{\geq 3}$. But it turns out, we have a counterexample at hand, namely $X = \mathbb{C}\mathrm{P}^\infty$. By the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 we know that $\mathbb{C}\mathrm{P}^\infty$ is simply connected and that $\pi_k(\mathbb{C}\mathrm{P}^\infty) = 0$ for $k \geq 3$. But by the discussion on page 2955 we

know that $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}$ for every even k. In particular the Hurewicz homomorphism $\pi_4(\mathbb{C}\mathrm{P}^\infty) \to \mathrm{H}_4(\mathbb{C}\mathrm{P}^\infty)$ is not an epimorphism. □

The proof of the Hurewicz-plus-One Theorem 307.34 crucially relies on the following lemma:

Lemma 307.35. (Eilenberg-Maclane-H_{n+1}-Lemma) Let π be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. If Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then $\mathrm{H}_{n+1}(Y) = 0$.

Examples.

(1) The statement of the Eilenberg-Maclane-H_{n+1}-Lemma 307.35 is somewhat surprising since it is emphatically wrong for $n = 1$. For example by the Eilenberg–Maclane Example Lemma 307.3 (1) we know that $S^1 \times S^1$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^2, 1)$. On the other hand we have $\mathrm{H}_2(S^1 \times S^1) \neq 0$.

(2) As a reality check, let us consider the only Eilenberg-Maclane space for $n \geq 2$ that we understand, namely $\mathbb{C}\mathrm{P}^\infty$ which Eilenberg–Maclane Example Lemma 307.3 (5) is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. By the calculation on page 2955 we do know that $\mathrm{H}_3(\mathbb{C}\mathrm{P}^\infty)$ is indeed trivial. □

Proof of the Eilenberg-Maclane-H_{n+1}-Lemma 307.35. Let $n \in \mathbb{N}_{\geq 2}$ and let π be an abelian group. First note that it follows from the Eilenberg-Maclane-H_*-Corollary 307.16 that it suffices to show that there exists a single Eilenberg-Maclane space Z of type $\mathrm{K}(\pi, n)$ with $\mathrm{H}_{n+1}(Z; \mathbb{Z}) = 0$.

By the Moore Space Existence Proposition 173.3 we know that there exists an $(n+1)$-dimensional CW-complex that is a Moore space Y of type $\mathrm{K}(\pi, n)$. Recall that this means that

(1) Y is simply connected,
(2) $\mathrm{H}_n(Y) \cong \pi$,
(3) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(Y) = 0$.

We pick a base point $y_0 \in Y$. By the Hurewicz Corollary 167.13 we now know that (1), (2) and (3) imply that

(4) $\pi_i(Y, y_0) = 0$ for $i = 1, \ldots, n-1$ and that $\pi_n(Y, y_0) \cong \pi$.

By the Kill-All Higher-π_n-Proposition 307.9 there exists a CW-complex Z with the following properties:

(5) The $(n+1)$-skeleton of Z equals Y.[3396]
(6) We have $\pi_i(Z) = 0$ for $i \geq n+1$.
(7) For $i = 1, \ldots, n$ the inclusion $Y \to Z$ induces an isomorphism $\pi_i(Y, y_0) \xrightarrow{\cong} \pi_i(Z, y_0)$.

It follows from (4), (6) and (7) that Z is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$.

Since Y is the $(n+1)$-skeleton of the CW-complex Z we obtain from the CW-Skeleton-H_*-Lemma 155.2 (4) that the map $\mathrm{H}_{n+1}(Y) \to \mathrm{H}_{n+1}(Z)$ is an epimorphism. Together with (3) this implies that $\mathrm{H}_{n+1}(Z) = 0$.[3397] We have thus found the desired Eilenberg-Maclane space Z. ■

[3396] Here we use that Y is its own $(n+1)$-skeleton.

[3397] The reader will not have failed to notice that this proof shares some ideas with the proof of the $\mathrm{K}(\mathbb{Z}, n)$-Model Lemma 307.14.

Now we turn to the actual proof of the Hurewicz-plus-One Theorem 307.34. For convenience we break the proof into three parts:

(1) First we prove the Hurewicz-plus-One Theorem 307.34 (1) for 2-dimensional CW-complexes.
(2) Then we prove the Hurewicz-plus-One Theorem 307.34 (2) for $(n+1)$-dimensional CW-complexes.
(3) Finally we deal with the general case of the Hurewicz-plus-One Theorem 307.34.

Proof of Theorem 307.34 (1) for 2-dimensional CW-complexes. Let X be a 2-dimensional CW-complex that is 0-connected. Let $x_0 \in X$. By the Kill-All Higher-π_n-Proposition 307.9 there exists a CW-complex Y with the following properties:

(1) The 2-skeleton of Y equals X.
(2) We have $\pi_i(Y, x_0) = 0$ for $i \geq 2$.
(3) The inclusion map $i\colon X \to Y$ induces an isomorphism $i_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, x_0)$.

It follows from (1), (2) and (3) that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(X, x_0), 1)$. Next we consider the following diagram:

$$\begin{array}{ccccccccccc}
\pi_3(Y,X,x_0) & \xrightarrow{\partial} & \pi_2(X,x_0) & \longrightarrow & \overbrace{\pi_2(Y,x_0)}^{=0} & \longrightarrow & \pi_2(Y,X,x_0) & \xrightarrow{\partial} & \pi_1(X,x_0) & \xrightarrow[\cong]{i_*} & \pi_1(Y,x_0) \\
\downarrow{\scriptstyle \Phi_{(Y,X,x_0)}} & & \downarrow{\scriptstyle \Phi_{(X,x_0)}} & & \downarrow{\scriptstyle \Phi_{(Y,x_0)}} & & \downarrow{\scriptstyle \Phi_{(Y,X,x_0)}} & & & & \\
\mathrm{H}_3(Y,X) & \xrightarrow{\partial} & \mathrm{H}_2(X) & \twoheadrightarrow & \mathrm{H}_2(Y) & \longrightarrow & \underbrace{\mathrm{H}_2(Y,X)}_{=0}. & & & &
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}_2(X) & \xrightarrow{\cong} & \mathrm{H}_2(Y) \\
\downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle \mu} & \\
\mathrm{H}_2(\pi_1(X,x_0)) & &
\end{array}$$

We make the following clarifications and observations:

(a) The horizontal sequences are the long exact sequences of homology and homotopy groups of the pair (Y, X), given by the LES-of-Pair Proposition 146.7 and by the Homotopy Groups-LES-Proposition 296.10.
(b) The vertical maps are the (relative) Hurewicz homomorphisms that we introduced in Definition 167.2 and Definition 296.21
(c) The diagram commutes by the Relative Hurewicz Homomorphism Lemma 296.22 (3).
(d) It follows from (2), (3) and the lower end of the upper long exact sequence that $\pi_j(Y, X) = 0$ for $j = 0, 1$. In other words, the pair (Y, X) is 2-connected.
(e) It follows from (d) and the Relative Hurewicz Theorem 296.25 that the left-hand vertical map is an epimorphism.[3398]
(f) The map η and the isomorphism μ are given by the H_2-of-Group Lemma 307.33. In particular these maps turn the lower triangle into a commutative triangle.
(g) Since X is the 2-skeleton of Y we obtain from the CW-Skeleton-H_*-Lemma 155.2 (3) and (4) and the long exact sequence of the pair (Y, X) that $\mathrm{H}_2(Y, X) = 0$.

[3398]People with shaky memories, like your author, might be surprised that it does not say that the map is an isomorphism. The point is that the Relative Hurewicz Theorem 296.25 shows that the Hurewicz homomorphism $\pi_3(Y, X, x_0)' \to \mathrm{H}_3(Y, X)$ is an isomorphism where $\pi_3(Y, X, x_0)'$ is the quotient of $\pi_3(Y, X, x_0)$ by a suitable action of $\pi_1(X, x_0)$. Thus the Hurewicz homomorphism $\pi_3(Y, X, x_0) \to \mathrm{H}_3(Y, X)$ is an epimorphism, but in general it is not an isomorphism.

From this data one can easily deduce that the sequence

$$\pi_2(X,x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_2(X) \xrightarrow{\eta} \mathrm{H}_2(\pi_1(X,x_0)) \to 0$$

is exact. ■

Proof of Theorem 307.34 (2) for $(n+1)$-dimensional CW-complexes. Let $n \in \mathbb{N}_{\geq 2}$ and let X be an $(n+1)$-dimensional CW-complex that is $(n-1)$-connected. Let $x_0 \in X$. By the Kill-All Higher-π_n-Proposition 307.9 there exists a CW-complex Y with the following properties:

(1) The $(n+1)$-skeleton of Y equals X.
(2) We have $\pi_i(Y,x_0) = 0$ for $i \geq n+1$.
(3) For every $j \in \{1,\dots,n\}$ the inclusion map $i\colon X \to Y$ induces a corresponding isomorphism $i_*\colon \pi_j(X,x_0) \xrightarrow{\cong} \pi_j(Y,x_0)$.

It follows from (1), (2) and (3) that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_n(X,x_0),n)$. Next we consider the following diagram:[3399]

$$\begin{array}{ccccccccccc}
\pi_{n+2}(Y,X) & \xrightarrow{\partial} & \pi_{n+1}(X) & \longrightarrow & \overbrace{\pi_{n+1}(Y)}^{=0} & \longrightarrow & \pi_{n+1}(Y,X) & \xrightarrow{\partial} & \pi_n(X) & \xrightarrow[\cong]{i_*} & \pi_n(Y) \longrightarrow \dots \\
\cong\downarrow{\scriptstyle \Phi_{(X,Y)}} & & \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_Y} & & & & & & \\
\mathrm{H}_{n+2}(Y,X) & \xrightarrow{\partial} & \mathrm{H}_{n+1}(X) & \longrightarrow & \underbrace{\mathrm{H}_{n+1}(Y)}_{=0 \text{ by Lemma 307.35}}. & & & & & &
\end{array}$$

We make the following clarifications and observations:

(a) As in the previous proof, the horizontal sequences are long exact sequences of the pair (Y,X). Furthermore the vertical maps are the (relative) Hurewicz homomorphisms and the diagram commutes by the Relative Hurewicz Homomorphism Lemma 296.22 (3).
(b) It follows from (2), (3) and the lower end of the upper long exact sequence that $\pi_j(Y,X) = 0$ for $j = 0,\dots,n+1$. In other words, the pair (Y,X) is $(n+1)$-connected.
(c) It follows immediately from (a) and the Relative Hurewicz Theorem 296.25, together with the fact that $\pi_1(X,x_0) = 0$, that the left-hand vertical map is an isomorphism.
(d) Since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_n(X,x_0),n)$ and since $n \geq 2$ we obtain from the Eilenberg-Maclane-H_{n+1}-Lemma 307.35 that $\mathrm{H}_{n+1}(Y) = 0$.

It follows from the above that the second vertical map $\pi_{n+1}(X) \to \mathrm{H}_{n+1}(X)$ is, as promised, an epimorphism. ■

Proof of the general the Hurewicz-plus-One Theorem 307.34. First we consider the case that $n \geq 2$. In the following let X be an $(n-1)$-connected space. Let $x_0 \in X$. By the CW-Approximation Theorem 304.14 there exists a CW-complex Y together with a weak homotopy equivalence $f\colon Y \to X$. Since f is a weak homotopy equivalence we see that Y itself is $(n-1)$-connected. We denote by $i\colon Y^{n+1} \to Y$ the inclusion of the $(n+1)$-skeleton Y^{n+1} into Y. We pick $y_1 \in Y^{n+1}$ and we write $x_1 = f(y_1)$. Since X is in particular

[3399] For evident space reasons we drop the base points from the discussion.

path-connected we can pick a path γ from x_0 to x_1. We consider the following diagram:

$$\begin{array}{ccccccc}
\pi_{n+1}(Y^{n+1}, y_1) & \xrightarrow{i_*} & \pi_{n+1}(Y, y_1) & \xrightarrow[\cong]{f_*} & \pi_{n+1}(X, x_1) & \xrightarrow[\cong]{\gamma_*} & \pi_{n+1}(X, x_0) \\
\downarrow{\scriptstyle \Phi_{(Y^{n+1}, y_1)}} & & \downarrow{\scriptstyle \Phi_{(Y, y_1)}} & & \downarrow{\scriptstyle \Phi_{(X, x_1)}} & & \downarrow{\scriptstyle \Phi_{(X, x_0)}} \\
\mathrm{H}_{n+1}(Y^{n+1}) & \xrightarrow{i_*} & \mathrm{H}_{n+1}(Y) & \xrightarrow[\cong]{f_*} & \mathrm{H}_{n+1}(X) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n+1}(X).
\end{array}$$

We make the following clarifications and observations:

(1) The vertical maps are the Hurewicz homomorphisms.
(2) The horizontal map to the top right is the isomorphism that is induced by the path γ_*, see the Change-of-Base Point Proposition 134.12.
(3) The two squares to the left commute by the Hurewicz Homomorphism Proposition 166.3.
(4) The square to the right commutes by the Hurewicz Homomorphism Proposition 167.3 (2).
(5) We know from the above proof of the Hurewicz-plus-One Theorem 307.34 for $(n+1)$-dimensional CW-complexes, and the fact that Y^{n+1} is $(n-1)$-connected, that the left vertical map is an epimorphism.
(6) By the CW-Skeleton-H_*-Lemma 155.2 (4) we know that the bottom left horizontal map is an epimorphism.
(7) Since f is a weak homotopy equivalence we know that the top middle horizontal map is an isomorphism. Furthermore we obtain from the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 that the bottom middle horizontal map is also an isomorphism.
(8) It follows from all the information assembled above that the right vertical map is an epimorphism.

The proof for $n = 1$ is quite similar, with a few minor extra complications. We leave it to the reader to fill in the details. ■

The Hurewicz-plus-One Theorem 307.34 (2) says that given a pointed 0-connected space (X, x_0) there is a curious interplay between $\pi_2(X, x_0)$, $\mathrm{H}_2(X)$ and the second homology of $\pi_1(X, x_0)$. We want to get a better intuition for what is happening by considering two corollaries to the Hurewicz-plus-One Theorem 307.34 (2).

Corollary 307.36. (Hurewicz-2D-Epimorphism Corollary) Let (X, x_0) be some pointed 0-connected space. If we have an isomorphism $\pi_1(X, x_0) \cong \mathbb{Z}$, then the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to \mathrm{H}_2(X)$ is an epimorphism.

Proof. In the discussion on page 5912 we saw that $\mathrm{H}_2(\pi_1(X, x_0)) \cong \mathrm{H}_2(\mathbb{Z}) \cong \mathrm{H}_2(S^1) = 0$. Thus it follows immediately from the Hurewicz-plus-One Theorem 307.34 (2) that the Hurewicz homomorphism $\pi_2(X, x_0) \to \mathrm{H}_2(X)$ is an epimorphism. ■

Remark. The proof of the Hurewicz-2D-Epimorphism Corollary 307.36 is in many ways beyond reproach, it is clear and rigorous. The only catch is that it is not very geometric and we do not really get a good understanding why the statement is true. To rectify this we will sketch an alternative, quite geometric, proof for the Hurewicz-2D-Epimorphism Corollary 307.36 in Exercise 307.10. □

We move on to the second corollary to the Hurewicz-plus-One Theorem 307.34 (2).

Corollary 307.37. (Hurewicz Cokernel Corollary) If X is a 0-connected space such that $\pi_1(X) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}_{\geq 2}$, then for any $x_0 \in X$ we have

$$\operatorname{coker}(\Phi_{(X,x_0)} \colon \pi_2(X,x_0) \to \mathrm{H}_2(X)) \;\underset{\uparrow}{\cong}\; \mathrm{H}_2(\pi_1(X)) \;\cong\; \mathrm{H}_2(\mathbb{Z}^m) \;\underset{\uparrow}{\cong}\; \mathbb{Z}^{\binom{m}{2}}.$$

by the Hurewicz-plus-One Theorem 307.34 (1) — by the example on page 5912

In particular we see that $\mathrm{H}_2(X)$ is non-trivial.

Proof. As we indicated in the statement, the corollary follows immediately from the Hurewicz-plus-One Theorem 307.34 (2). ■

In the following we want to get a better understanding how this "extra" homology in $\mathrm{H}_2(X)$ arises. To simplify the discussion let us consider the case $m = 2$. Thus let (X, x_0) be a pointed 0-connected space with $\pi_1(X, x_0) \cong \mathbb{Z}^2$. We proceed as follows:

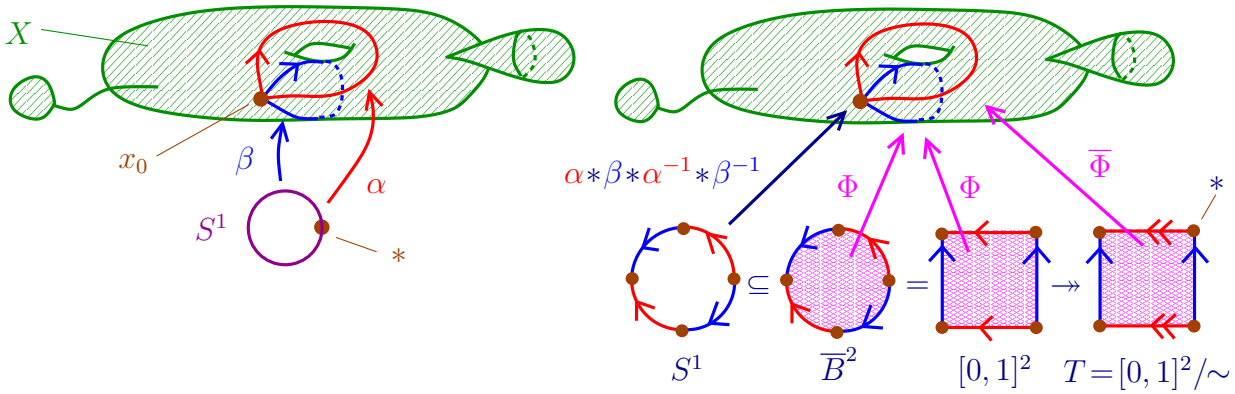
(1) We pick two loops $\alpha, \beta \colon (S^1, *) \to (X, x_0)$ that represent a generating set for the group $\pi_1(X, x_0) \cong \mathbb{Z}^2$.

We consider the loop $\varphi = \alpha * \beta * \overline{\alpha} * \overline{\beta} \colon (S^1, *) \to (X, x_0)$. Since $\pi_1(X, x_0)$ is abelian we see that the loop φ represents the trivial element in $\pi_1(X, x_0)$. By the Trivial-in-π_1-Lemma 88.28 this means that we can extend the loop to a map $\Phi \colon \overline{B}^2 \to X$.

(2) We make the "obvious" identification[3400] $\overline{B}^2 = [0,1]^2$ which identifies the "quartercircles" on S^1 with the four sides of the square.

(3) Let $\sim$ be the usual equivalence relation on $[0,1]^2$ that is given by $(x, 0) \sim (x, 1)$ and $(y, 0) \sim (y, 1)$. The quotient $T := [0,1]^2/\sim$ is the 2-dimensional torus. We equip T with the base point $*$ that is represented by any of the vertices.

(4) The map $\Phi \colon \overline{B}^2 = [0,1]^2 \to X$ descends to a map $\overline{\Phi} \colon T := [0,1]^2/\sim \to X$.



This approach feels like we are creating surfaces ex nihilo. But the following lemma shows that miraculously this approach works.

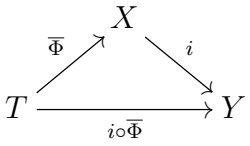
Lemma 307.38. (Hurewicz Cokernel Realization Lemma) We continue with the above notation. The map

[3400] The identification is given by rotation by $\frac{\pi}{4}$ followed by "outward" stretching as in the proof of the Convex-to-Ball Proposition 4.28.

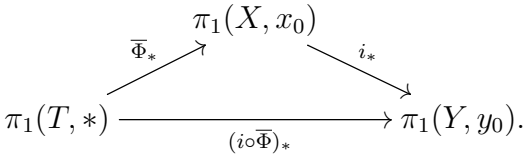
$$\begin{array}{rcl} \mu\colon \overbrace{\mathrm{H}_2(T)}^{=\mathbb{Z}\cdot[T]} & \to & \mathrm{coker}(\pi_2(X,x_0)\to \mathrm{H}_2(X)) \\ \sigma & \mapsto & [\,\overline{\Phi}_*(\sigma)] \end{array}$$

is an isomorphism.

Sketch of proof. As in the proof of the Hurewicz-plus-One Theorem 307.34 (2) we can assume that X is a 3-dimensional CW-complex. Furthermore, by the proof of the Hurewicz-plus-One Theorem 307.34 (2) we know that there exists an inclusion $i\colon X \to Y$ of X into an Eilenberg-Maclane space Y of type $\mathrm{K}(\mathbb{Z}^2,1)$ such that the inclusion-induced map $i_*\colon \pi_1(X,x_0) \to \pi_1(Y,x_0)$ is an isomorphism. We consider the following simple minded diagram



and we consider the induced diagram



We make the following observations:

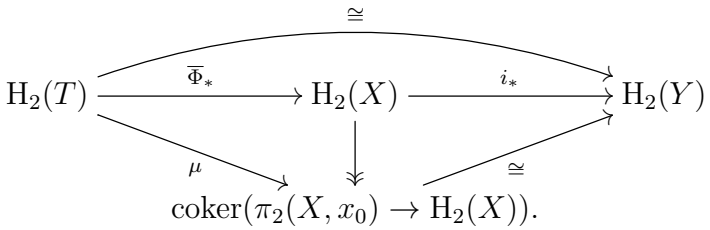
(1) The map $\overline{\Phi}_*\colon \pi_1(T,*) \to \pi_1(X,x_0)$ is by design an epimorphism. Since both groups are isomorphic to $\mathbb{Z}^2$ we obtain from the Epi-Automorphism Lemma 97.27 that $\overline{\Phi}_*$ is in fact an isomorphism.

(2) As we mentioned above, the map $i_*\colon \pi_1(X,x_0) \to \pi_1(Y,x_0)$ is an isomorphism.

(3) By (1) and (2) we know that $(i\circ\overline{\Phi})_* = i_*\circ\overline{\Phi}_*\colon \pi_1(T,*) \to \pi_1(Y,x_0)$ is an isomorphism. By the Eilenberg–Maclane Example Lemma 307.3 (1) and by the above we know that both T and Y are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}^2,1)$. Thus we obtain from the Whitehead Theorem 305.1 that $i\circ\overline{\Phi}\colon T \to Y$ is in fact a homotopy equivalence.

(4) Since $i\circ\overline{\Phi}$ is a homotopy equivalence we obtain from the Homotopy Equivalence-H_*-Proposition 144.14 (1a) that the induced map $(i\circ\overline{\Phi})_* = i_*\circ\overline{\Phi}_*\colon \mathrm{H}_2(T) \to \mathrm{H}_2(Y)$ is an isomorphism.

It remains to consider the following diagram:



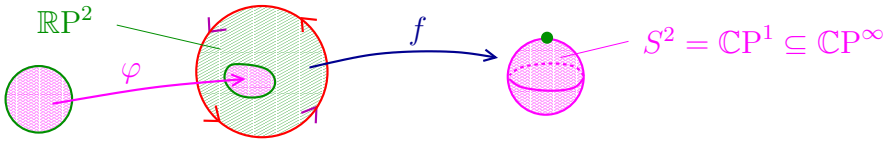
By the above discussion we know that the composition of the top two horizontal maps is an isomorphism and by the Hurewicz-plus-One Theorem 307.34, together with the H_2-of-Group Lemma 307.33, we know that the diagonal map to the right is an isomorphism. It follows that the diagonal map to the left is also an isomorphism. ■

307.8. **Exercises for Chapter 307.**

Exercise 307.1. We consider the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ where for $x \in S^1$ we have $x \sim -x$. Let $\varphi\colon \overline{B}^2 \to \mathbb{R}\mathrm{P}^2$ be a smooth embedding. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^2 & \to & \overline{B}^2/S^1 =_{\mathrm{i}} S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \subseteq \mathbb{C}\mathrm{P}^\infty \\ P & \mapsto & \begin{cases} [Q], & \text{if } P = \varphi(Q) \text{ for some } Q \in \overline{B}^2, \\ [S^1], & \text{otherwise.} \end{cases} \end{array}$$

Is the map f homotopic to the constant map?



Exercise 307.2. Let X be a connected k-dimensional CW-complex and let $k \in \mathbb{N}$.

(a) Does there exist a CW-complex Y with $Y^k = X$ and $\pi_k(Y) = 0$?
(b) Does there exist a CW-complex Y with $Y^k = X$ and $\mathrm{H}_k(Y) = 0$?

Exercise 307.3. Finish the proof of the Manifold–π_1–Free Group Epimorphism Proposition 307.27.
Hint. Use the Codimension One-H^1-Lemma 234.10.

Exercise 307.4. Let $n \in \mathbb{N}$. In the $\mathrm{K}(\mathbb{Z}, n)$-Model Lemma 307.14 we showed that there exists an Eilenberg-Maclane space X of type $\mathrm{K}(\mathbb{Z}, n)$ with precisely one 0-cell, precisely one n-cell and no cell of dimension $n+1$. What is the minimal dimension in which the canonical Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ has more cells than X?

Exercise 307.5. Let $n \in \mathbb{N}$. We consider $n \cdot \mathbb{R}\mathrm{P}^2$, i.e. we consider the connected sum of n copies of $\mathbb{R}\mathrm{P}^2$.

(a) What is the maximal number $k \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \ldots, F_k$ on $n \cdot \mathbb{R}\mathrm{P}^2$ such that $n \cdot \mathbb{R}\mathrm{P}^2 \setminus (F_1 \cup \cdots \cup F_k)$ is connected?
(b) What is the maximal $l \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(n \cdot \mathbb{R}\mathrm{P}^2)$ onto a free group of l generators?

Remark. By the Surface-π_1-Proposition 101.5 we know that there exists an isomorphism

$$\pi_1(n \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x_1, \ldots, x_n \mid x_1^2 \cdot \ldots \cdot x_n^2 \rangle.$$

Remark. This exercise is basically a variation on Question 234.15.

Exercise 307.6. In the Cup Product–Square Zero Lemma 203.21 we proved the following statement:

(∗) Let X be a topological space. For any $\phi \in \mathrm{H}^1(X; \mathbb{Z})$ we have $\phi \cup \phi = 0$.

Our goal is to give an alternative proof of this statement:

(a) Show that (∗) holds for $X = S^1$.
(b) Let X be a CW-complex. Use (a) and the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 307.18 to provide a proof of (∗) for CW-complexes.
(c) Use the CW-Approximation Theorem 304.14 and the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 to show, using (b), that (∗) holds for all topological spaces.

Exercise 307.7. Show that there exists an abelian group π and Moore spaces X and Y of type $\mathrm{M}(\pi,1)$ that are not homotopy equivalent.
Hint. You could use the fundamental group to distinguish X and Y.
Remark. In Exercise 307.8 we will deal with Moore spaces of type $\mathrm{M}(\pi,n)$ with $n \geq 2$.

Exercise 307.8. Let π be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. In this exercise we will show that any two Moore spaces of type $\mathrm{M}(\pi,n)$ are in fact homotopy equivalent. First note that by the Moore Space Existence Proposition 173.3 there exists an $(n+1)$-dimensional CW-complex X that has no cells in dimensions $1,\dots,n-1$ and that is a Moore space of type $\mathrm{M}(\pi,n)$. It suffices to show that X is homotopy equivalent to any other Moore space Y of type $\mathrm{M}(\pi,n)$. This task is broken up into two steps:

(a) Use the Realize Map-on-π_n-Lemma 307.19 to show that there exists a continuous map $f\colon X \to Y$ that induces an isomorphism $f_*\colon \pi_n(X) \to \pi_n(Y)$.
(b) Use the Hurewicz Homomorphism Proposition 167.3, the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 and the Whitehead Theorem 305.1 (1) to show that f is a homotopy equivalence.

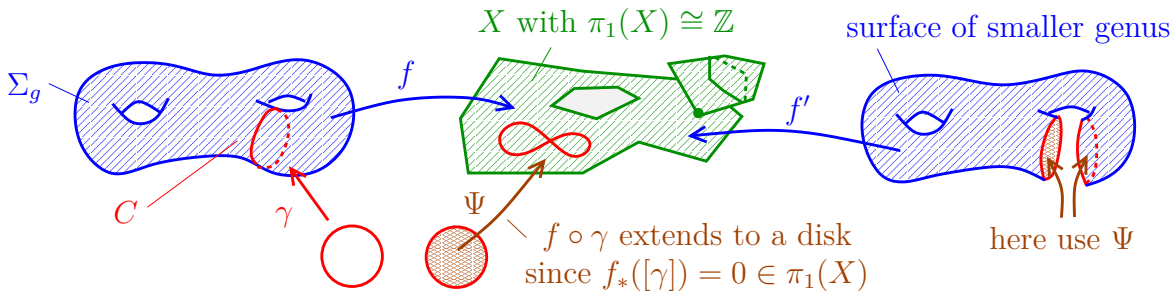
Remark. Together with Exercise 307.7 this gives a complete answer to Question 173.6.

Exercise 307.9. Let $n \in \mathbb{N}$. We set $M = (S^1)^n$. Let $\varphi\colon \pi_1(M) \to \mathbb{Z}^n$ be an isomorphism. Show that it is not possible to extend φ over a bounding $(n+1)$-dimensional smooth manifold. More precisely, show that there is no pair $(W, \psi\colon \pi_1(W) \to \mathbb{Z}^n)$ where W is a compact orientable connected smooth manifold with $\partial W = M$ and where ψ is a homomorphism that satisfies $\varphi = \psi \circ i_*$ where $i\colon M \to W$ denotes the inclusion.
Remark. By the Smooth Manifold-CW-Theorem 179.9 we know that any compact smooth manifold W admits a CW-structure.

Exercise 307.10. Let X be a 0-connected space with $\pi_1(X) \cong \mathbb{Z}$. Recall that it follows from the Hurewicz-2D-Epimorphism Corollary 307.36 that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X;\mathbb{Z})$ is an epimorphism. The goal of this exercise is to sketch an alternative geometric proof. Thus let $\varphi \in \mathrm{H}_2(X;\mathbb{Z})$. By the H_2-via-Surface Proposition 187.4 there exists a surface Σ_g of some genus $g \in \mathbb{N}_0$ and a continuous map $f\colon \Sigma_g \to X$ such that $f_*([\Sigma_g]) = \varphi \in \mathrm{H}_2(X;\mathbb{Z})$. If $g = 0$, then we are done. So suppose that $g \geq 1$. We proceed with the following steps:

(a) Show that there exists a curve $C \subseteq \Sigma_g$ with $[C] \neq 0 \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ but such that $f_*([C]) = \pi_1(X)$.
Remark. You might want to use the Represent-by-Connected 2-Manifold Corollary 187.22 at some point.
(b) Get inspired by the figure below.
(c) Complete the proof of the statement.

Exercise 307.11. Let π be a group, let $\varphi\colon \pi \to \pi$ be an endomorphism and let K be an Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$. By the Maps-to-$\mathrm{K}(\pi,n)$-Proposition 307.18 we know that there exists a continuous map $f\colon (\mathrm{K}(\pi,1),*) \to (\mathrm{K}(\pi,1))$ such that $f_* = \varphi$. Now suppose that φ is an isomorphism. Can always find such a map f which is furthermore a homeomorphism?



Exercise 307.12. Let $K \subseteq S^3$ be a knot and let $\Phi\colon \overline{B}^2 \times K \to S^3$ be a thickening with self-linking number 0. We write $X_K = S^3 \setminus \Phi(B^2 \times K)$.

(a) Show that there exists a continuous map $f\colon X_K \to S^1$ that induces an isomorphism of homology groups.
(b) Show that there exists a continuous map $f\colon X_K \to S^1$ such that $f \circ \Phi\colon \partial S^1 \times K \to S^1$ is just the projection onto the first factor.
(c) Show that there exists a continuous map $f\colon X_K \to \overline{B}^2 \times S^1$ that induces an isomorphism of homology groups and that restricts to a homeomorphism $\partial X_K \to S^1 \times S^1$.

Hint. Use the Maps-to-K(π, n)-Proposition 307.18 and use the Smooth Collar Neighborhood Theorem 35.6 to improve the map from (a) to (b) and from (b) to (c).

Exercise 307.13. Let $F_n \coloneqq \langle x_1, \dots, x_n \rangle$ the free group on n generators $x_1, \dots, x_n$ and let $\varphi\colon F_n \to F_n$ be an isomorphism. We consider the semidirect product $\pi \coloneqq F_n \rtimes \mathbb{Z}$ from Definition 93.11. Show that there exists a 2-dimensional Eilenberg-Maclane space of type K$(\pi, 1)$.
Hint. Note that $X \coloneqq \bigvee_{i=1}^{n} S_i^1$ is an Eilenberg-Maclane space of type K$(F_n, 1)$. By the Maps-to-K(π, n)-Proposition 307.18 there exists a continuous map $f\colon (X, *) \to (X, *)$ with $f_* = \varphi$. Now consider the mapping torus $\operatorname{Tor}(X, f)$. Use the Mapping Torus-π_1-Proposition 93.13. It might also be useful to consider Mapping Torus Covering Lemma 93.10.

Exercise 307.14. Let X be a connected aspherical CW-complex, let $U \subseteq X$ be a connected open subset and let $x_0 \in U$. We suppose that $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is trivial. Show that the inclusion $\iota\colon U \to X$ is homotopic to a constant map.
Hint. Use the universal covering of X.

Exercise 307.15. Let $K \subseteq S^3$ be a knot. By the Product Tubular Neighborhood–Existence Theorem 58.4 (2) there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0, P) = P$ for all $P \in K$. As in Definition 108.30 we refer to $X_K \coloneqq S^3 \setminus \Phi(B^2 \times K)$ as the *exterior of* K.

(a) Recall that by the Knot Complement-H$_*$-Proposition 185.29 we know that $\mathrm{H}_1(X_K) \cong \mathbb{Z}$. Show that there exists a continuous map $f\colon X_K \to S^1$ that induces an isomorphism on H_1.
(b) Show that there exists a continuous map $f\colon X_K \to \overline{B}^2 \times S^1$ of degree ± 1, i.e. a continuous map such that $f_*\colon \mathrm{H}_3(X_K, \partial X_K) \to \mathrm{H}_3(\overline{B}^2 \times S^1, S^1 \times S^1)$ is an isomorphism.
Hint. Make use of the Smooth Collar Neighborhood Theorem 35.6.

Exercise 307.16. Let $\{r_k\}_{k\in\mathbb{N}}$ be a sequence of natural numbers. We consider the topological space

$$X \coloneqq ((S^1 \times [0,1]) \times \mathbb{Z})/\sim \quad \text{where } ((z,1),k) \sim ((z^{r_{k-1}},0),k-1) \text{ for } z \in S^1 \text{ and } k \in \mathbb{N}.$$

(a) Show, say using the CW-Complex Product Proposition 139.18 and the CW-Gluing Proposition 139.9, that X is a 2-dimensional CW-complex.

(b) Let Π be the direct colimit of the direct system of abelian groups $\{\mathbb{Z}\}_{k\in\mathbb{N}}$ where for $k \leq l$ we consider the homomorphism that is given by multiplication by $r_k \cdot \ldots \cdot r_{l-1}$. Show that X is an Eilenberg-Maclane space of type $\mathrm{K}(\Pi, 1)$.
Hint. What can you say about the subcomplexes where k just runs between 1 and some $n \in \mathbb{N}$?

(c) Suppose that $r_k = k$. Show that X is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Q}, 1)$.
Remark. This statement follows from (a) and (b) together with Exercise 104.4.

(d) Suppose that $r_k = 2$. Show that X is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}[\frac{1}{2}], 1)$.

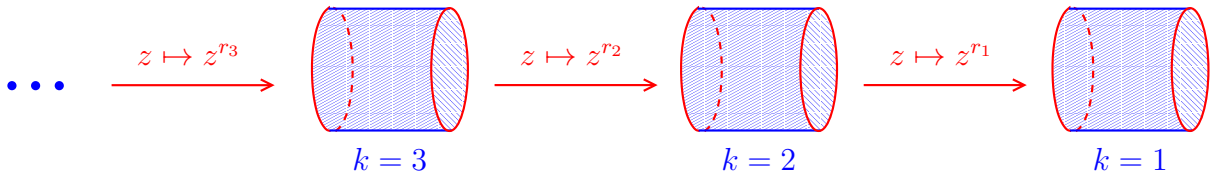
Given a group π we define its *geometric dimension*

$$\mathrm{gd}(\pi) \coloneqq \text{minimal dimension of an Eilenberg-Maclane space of type } \mathrm{K}(\pi,1) \in \mathbb{N}_0 \cup \{\infty\}.$$

the Eilenberg-Maclane Construction Proposition 307.4 (1) shows that we always have $\mathrm{gd}(A \times B) \leq \mathrm{gd}(A) + \mathrm{gd}(B)$. The following exercises shows that equality does not always hold.

(e) Show that $\mathrm{gd}(\mathbb{Q}) = 2$.

(f) Show that $\mathrm{gd}(\mathbb{Q} \times \mathbb{Q}) \leq 3$.
Hint. Write $\mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2$ as the direct colimit of a direct system of endomorphisms of $\mathbb{Z}^2$.



Exercise 307.17. Let $g, h \in \mathbb{N}$. We consider surfaces Σ_g and Σ_h of genus g and h. Let $x \in \Sigma_g$ and $y \in \Sigma_h$. Finally let $\varphi\colon \pi_1(\Sigma_g, x) \to \pi_1(\Sigma_h, y)$ be a monomorphism.

(a) Show that there exists a continuous map $f\colon (\Sigma_g, x) \to (\Sigma_h, y)$ with $f_* = \varphi$.

(b) Show that there exists a covering $p\colon \widetilde{\Sigma} \to \Sigma_h$ and a continuous map $\widetilde{f}\colon \Sigma_g \to \widetilde{\Sigma}$ such that $\widetilde{f}$ induces an isomorphism of fundamental groups.

(c) Show that $\widetilde{f}\colon \Sigma_g \to \widetilde{\Sigma}$ is a homotopy equivalence.

(d) Show that $f_*\colon \Sigma_g \to \Sigma_h$ has non-zero degree.

(e) Show that $g \geq h$.

Exercise 307.18. Let X be a connected non-empty topological manifold with cyclic fundamental group. Let $p\colon \widetilde{X} \to X$ be the universal covering. Show that $p_*\colon \mathrm{H}_2(\widetilde{X}) \to \mathrm{H}_2(X)$ is an epimorphism.

CHAPTER 308

Invariants obtained through Eilenberg-Maclane spaces

In this chapter we will see how one can use Eilenberg-Maclane spaces to define contravariant functors from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups. In this chapter and in **??** we will relate these functors to the cohomology functors.
Before we jump into the chapter we remind the reader that given topological spaces X and Y we denote by Y^X the set of all continuous maps from X to Y. Throughout this and the coming chapters we always equip Y^X with the compact-open topology that we introduced in Definition 16.2.

This chapter is organized as follows:

- In Section 308.1 we will recall the definition of the homotopy category of topological spaces.
- In Section 308.2 we will introduce the homotopy category of pointed topological spaces.
- In Section 308.3 we will see that Eilenberg-Maclane spaces can be used to define algebraic invariants of topological spaces
- In Section 308.4 we will give an alternative description of the group structure of the algebraic invariants defined in Section 308.3.
- In Section 308.5 we will discuss the functoriality of these new invariants.
- In Section 308.6 we will perform some basic calculations of these new invariants.
- In Section 308.7 we will discuss product structures on these new invariants.
- In Section 308.8 we will provide exercises for Chapter 308.

308.1. **The homotopy category of topological spaces.** In this short section we recall some definitions and results regarding topological spaces and the homotopy category of topological spaces. This section partly serves as a warm-up for the discussion of pointed topological spaces in the following sections.

First we recall Definition 19.11:

Definition 308.1. Given two topological spaces X and Y we define

$$[X,Y] := \underbrace{\{\text{continuous maps } X \to Y\}}_{=Y^X}/\sim \quad \text{where } f \sim g \text{ if there exists a homotopy between } f \text{ and } g.$$

We also recall Definition 19.12:

Definition 308.2. The homotopy category $\mathcal{H}$ is defined to be the category where the objects are topological spaces and where the set of morphisms between two topological spaces X and Y is given by the set $[X,Y]$.[3401]

The following lemma is basically an immediate consequence of the definitions.

[3401]The composition of $[f\colon X \to Y]$ and $[g\colon Y \to Z]$ is evidently defined to be $[g \circ f\colon X \to Z]$, by the Homotopy Composition Lemma 18.10 (2) this definition does not depend on the choice of representatives.

Lemma 308.3. (Homotopy Equivalence–Reformulation Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The corresponding morphism $[f] \in \mathrm{Mor}_{\mathcal{H}}(X,Y)$ is invertible in the sense of Definition 5.9 if and only if f is a homotopy equivalence.

Proof. Note that by definition $[f] \in \mathrm{Mor}_{\mathcal{H}}(X,Y)$ is invertible if and only if there exists $[g] \in \mathrm{Mor}_{\mathcal{H}}(Y,X)$ with $[f \circ g] = [\mathrm{id}_Y]$ and $[g \circ f] = [\mathrm{id}_X]$. Now the statement of the lemma is indeed clear. ■

In the following we show how some constructions of topological spaces interact with the homotopy category $\mathcal{H}$. First we have following rather dull lemma.

Lemma 308.4. (Homotopy-into-Product Lemma) Let X, Y and Z be topological spaces. The map

$$\begin{array}{rcccl} \chi\colon [X,Y] & \times & [X,Z] & \to & [X, Y \times Z] \\ ([f\colon X \to Y], [g\colon X \to Z]) & & & \mapsto & [f \times g\colon X \to Y \times Z] \end{array}$$

is well-defined.

Proof. Nothing can possibly go wrong in any attempt to prove this lemma. ■

Lemma 308.5. (Loop Space–Suspension Homotopy Functor Lemma) The maps

$$\begin{array}{rcl} X & \mapsto & \text{free loop space } \Omega(X) \coloneqq \{\varphi \in X^{[0,1]} \mid \varphi(0) = \varphi(1)\} \\ [f\colon X \to Y] & \mapsto & \left[\begin{array}{ccc} \Omega(f)\colon \Omega(X) & \to & \Omega(Y) \\ (\varphi\colon [0,1] \to X) & \mapsto & (f \circ \varphi\colon [0,1] \to Y) \end{array}\right] \end{array}$$

and

$$\begin{array}{rcl} X & \mapsto & \text{suspension } \Sigma(X) \coloneqq (X \times [-1,1])/\sim \\ [f\colon X \to Y] & \mapsto & \left[\begin{array}{ccc} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}\right] \end{array}$$

both define covariant functors from $\mathcal{H}$ to itself.

Proof. There are only two statements one needs to think about: given a continuous map $f\colon X \to Y$ one needs to show that $\Omega(f)$ and $\Sigma(f)$ are continuous, and given homotopic maps $f, g\colon X \to Y$ one needs to show that the corresponding maps on the right-hand side are also homotopic. For the free loop spaces the statements follow from the Compact-Open Functoriality Lemma 16.5 (1a) and (1c). For suspensions these two statements follow immediately from the Suspension Functor Lemma 20.23 (1) and the Suspension–Homotopy Lemma 20.28 (1). ■

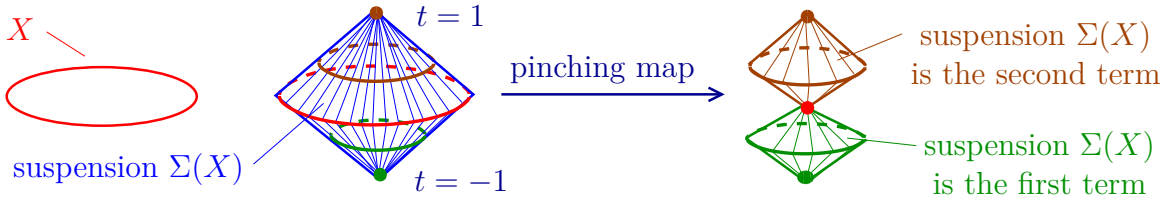
Just for kicks we also mention the following definition, which is reminiscent of the definition that we gave on page 2579. We will make use of this definition, with a slight twist, in the near future.

Definition 308.6. Let X be a topological space. We consider the wedge $\Sigma(X) \vee \Sigma(X)$ that is given by identifying the "South Pole $[X \times \{-1\}]$" on the left with the "North Pole

$[X \times \{1\}]$" on the right. We refer to the map

$$\begin{array}{rcl} \Sigma(X) & \to & \Sigma(X) \vee \Sigma(X) \\ [(x,t)] & \mapsto & \begin{cases} [(x, 2t+1)] \text{ in the first term}, & \text{if } t \in [-1,0], \\ [(x, 2t-1)] \text{ in the second term}, & \text{if } t \in [0,1]. \end{cases} \end{array}$$

as the pinching map.[3402]



In some instances the sets $[X, Y]$ have some extra structure. For example we have the following elementary but cute lemma.

Lemma 308.7. (Maps-to-Group-a-Group Lemma) Let X be a topological space and let G be an (abelian) topological group, e.g. we could take $G = S^1$. The set $[X, G]$ together with the multiplication map

$$\begin{array}{rcl} \nu\colon [X,G] \times [X,G] & \to & [X,G] \\ [f],[g] & \mapsto & \left[\begin{array}{ccc} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{array}\right] \end{array}$$

is an (abelian) group.

Remark. Let X be a topological space and let G be a topological group. In the Function Space–Topological Group Lemma 16.19 we showed, under the extra hypothesis that X is regionally compact, that the set G^X with the obvious multiplication map $G^X \times G^X \to G^X$ is in fact a topological group. We could equip $[X, G] = G^X/\sim$ with the quotient topology, but the discussion on page 5746 shows that in general the resulting multiplication map $\nu\colon [X,G] \times [X,G] \to [X,G]$ is no longer continuous. Thus in general we cannot view $[X, G]$ as a topological group. □

Proof. In fact the only thing one needs to verify is that the map ν is actually well-defined. But this follows easily from the fact that the group multiplication $G \times G \to G$ is continuous. We leave it to the reader to fill in the details. ■

There are several drawbacks to considering topological spaces which are not equipped with base points.

(1) Without a base point the homotopy groups $\pi_n(X)$ are at best defined up to isomorphism, but they are not defined "on the nose".
(2) The operation of taking wedges is not well-defined.

In the following section we will introduce the analog of the homotopy category for pointed topological spaces. This will take care of the above issues, but it will require some adjustments at other places.

[3402]It follows easily from the Topological-Quotient Proposition 8.2 (1b) and the Pasting Proposition 4.7 (2) that the pinching map is continuous.

308.2. **The homotopy category of pointed topological spaces.** In the latter part of this chapter it will be essential that we always work with pointed topological spaces. To prepare for this discussion we need to rephrase all the concepts and results from the previous section in the language of pointed topological spaces.

We start out with a few definitions. Some of them we have encountered before in Definition 140.11 and in the formulation of the Maps-to-K(π, n)-Proposition 307.18.

Definition 308.8.

(1) A pointed topological space is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a point. We refer to x_0 as the base point of the pointed topological space.
(2) A morphism $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces is a map $f\colon X \to Y$ with $f(x_0) = y_0$.
(3) We say that two continuous maps $f, g\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces are homotopic if there exists a homotopy $F\colon X \times [0,1] \to Y$ rel $\{x_0\}$ from f to g. We use this notion of "homotopic" to define the notion of a homotopy equivalence between pointed topological spaces.
(4) A pointed CW-complex is a pair (X, x_0) where X is a CW-complex and x_0 is a point in the 0-skeleton X^0 of X.
(5) Given two pointed topological spaces (X, x_0) and (Y, y_0) we write[3403]
$$\langle (X, x_0), (Y, y_0) \rangle := \{\text{maps } (X, x_0) \to (Y, y_0)\}/\sim \qquad \text{where } f \sim g \text{ if there exists a homotopy in the above sense between } f \text{ and } g.$$
Given a continuous map $f\colon (X, x_0) \to (Y, y_0)$ we denote its equivalence class by $[f]$ as usual.
(6) Let (X, x_0) and (Y, y_0) be two topological spaces. Let $0 \in \langle (X, x_0), (Y, y_0) \rangle$ be the element that is represented by constant map $X \to Y$ given by $x \mapsto y_0$.[3404] We refer to 0 as the trivial element.

The following lemma is the analog of the Homotopy Composition Lemma 18.10 (2).

Lemma 308.9. (Pointed Homotopy Composition Lemma) Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map
$$\begin{array}{ccc} \langle (X, x_0), (Y, y_0) \rangle \times \langle (Y, y_0), (Z, z_0) \rangle & \to & \langle (X, x_0), (Z, z_0) \rangle \\ ([f], [g]) & \mapsto & [g \circ f] \end{array}$$
is well-defined.

Proof. The proof of this lemma is almost identical to the proof of the Homotopy Composition Lemma 18.10 (2) and it is left to the insomniac reader. ∎

The Pointed Homotopy Composition Lemma 308.9 shows that the following definition makes sense.

[3403] The notation $\langle X, Y \rangle$ goes back to [**Hat2002**, p. 357], where it says "the notation $\langle \dots \rangle$ is intended to suggest pointed homotopy classes".
[3404] At the moment this notation might sound a little risqué. But it will turn out to be a useful notation.

Definition 308.10. The pointed homotopy category $\mathcal{PHomTop}$ is the category where the objects are pointed topological spaces and where the set of morphisms between two pointed topological spaces (X, x_0) and (Y, y_0) is given by the set $\langle (X, x_0), (Y, y_0) \rangle$.

The following lemma is the analog the Homotopy Equivalence–Reformulation of Lemma 308.3.

Lemma 308.11. (Pointed Homotopy Equivalence–Reformulation Lemma) Let (X, x_0) and (Y, y_0) be pointed topological spaces.

(1) Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map. The corresponding morphism[3405]

$$[f] \in \mathrm{Mor}_{\mathcal{PHomTop}}((X, x_0), (Y, y_0)) = \langle (X, x_0), (Y, y_0) \rangle$$

is invertible if and only if f is a homotopy equivalence.

(2) In the category $\mathcal{PHomTop}$ we have[3406]

$$\langle (X, x_0), (Y, y_0) \rangle_{\mathrm{inv}} = \text{all elements of } \langle (X, x_0), (Y, y_0) \rangle \text{ that are represented by a homotopy equivalence.}$$

Proof. Statement (1) is, as in the case of the Homotopy Equivalence–Reformulation Lemma 308.3, an immediate consequence of the definitions. Statement (2) is a reformulation of Statement (1). ■

In the following we discuss the operations of taking products, wedges, loop spaces and suspensions in the context of the pointed homotopy category. We start out with the following convention.

Convention 308.12. Given pointed topological spaces (X, x_0) and (Y, y_0) we equip the product $X \times Y$ with the point (x_0, y_0).

For completeness we mention the following uninspiring analog to the Homotopy-into-Product Lemma 308.4.

Lemma 308.13. (Pointed Homotopy-into-Product Lemma) Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map

$$\begin{array}{rcl} \chi\colon \langle (X, x_0), (Y, y_0) \rangle \times \langle (X, x_0), (Z, z_0) \rangle & \to & \langle (X, x_0), (Y \times Z, (y_0, z_0)) \rangle \\ ([f\colon X \to Y], [g\colon X \to Z]) & \mapsto & [f \times g\colon X \to Y \times Z] \end{array}$$

is well-defined and it is natural.

Proof. Conveniently enough the proof is identical to the proof of the Homotopy-into-Product Lemma 308.4. ■

As discussed in Definition 96.1, one advantage of working with pointed topological spaces is that we have an unambiguous wedge operation.

Definition 308.14. Given a family $\{(X_i, x_i)\}_{i \in I}$ of pointed topological spaces we define the wedge

$$\bigvee_{i \in I} (X_i, x_i) \coloneqq \Big(\bigcup_{i \in I} X_i \Big) / \bigsqcup_{i \in I} \{x_i\}.$$

We equip this wedge with the base point given by the wedge point.

[3405] We refer to Definition 5.9 for the definition of an invertible morphism.

[3406] In Definition 5.9 we defined $\mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ to be the set of all invertible morphisms.

Lemma 308.15. (Pointed Wedge-Homotopy Lemma) Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map

$$\begin{array}{rcl} \nu\colon \langle (X,x_0), (Z,z_0)\rangle \times \langle (Y,y_0), (Z,z_0)\rangle & \to & \langle (X,x_0) \vee (Y,y_0), (Z,z_0)\rangle \\ ([f\colon X \to Z], [g\colon Y \to Z]) & \mapsto & [f \vee g] \end{array}$$

is well-defined.

Proof. The statement of the lemma is, in contrast to the Pointed Homotopy Composition Lemma 308.9 and the Pointed Homotopy-into-Product 308.13, non-trivial. It is a reformulation of the Wedge–Homotopy Lemma 96.8 which traces its ancestry to the formidable Whitehead Quotient Theorem 17.6. ■

Now we would like to generalize the free loop space $\Omega(X)$ and the suspension $\Sigma(X)$ to pointed topological spaces. It is pretty clear what we do about loop spaces:

Definition 308.16. Given a pointed topological space (X, x_0) we consider the

$$\text{loop space } \Omega(X, x_0) \;:=\; \{f \in X^{[0,1]} \mid f(0) = f(1) = x_0\}$$

and we equip the loop space $\Omega(X, x_0)$ with the base point c_{x_0} which is just the constant path $[0, 1] \to X$, $t \mapsto x_0$. Often we will suppress the base point from the notation, but throughout the subsequent chapters we will always view $\Omega(X, x_0)$ as a pointed topological space.

For completeness we write down explicitly the following fairly obvious lemma which is the analog of Statement (1) of the Loop Space–Suspension Homotopy Functor Lemma 308.5.

Lemma 308.17. (Loop Space–Homotopy–Functor Lemma) The maps

$$\begin{array}{rcl} (X, x_0) & \mapsto & (\Omega(X, x_0), c_{x_0}) \\ [f\colon (X, x_0) \to (Y, y_0)] & \mapsto & \begin{bmatrix} \Omega(X, x_0) & \to & \Omega(Y, y_0) \\ (\gamma\colon [0,1] \to X) & \mapsto & (f \circ \gamma\colon [0,1] \to Y) \end{bmatrix} \end{array}$$

define a covariant functor from the category $\mathcal{PHomTop}$ to itself.

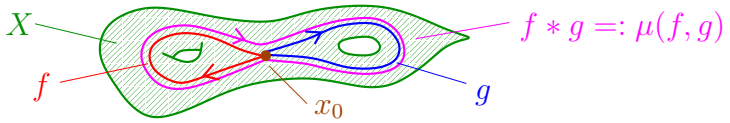
Proof. The lemma follows easily basically from the Compact-Open Functoriality Lemma 16.5 (1a) and (1c), one just needs to make a few minute modifications to factor in the base points. We leave this elementary task to the inexhaustible reader. ■

The loop spaces $\Omega(X, x_0)$ have one big advantage over the free loop spaces $\Omega(X)$. Given two loops in (X, x_0) we can form the product from Definition 88.7. As we had already hinted at in the Loop-Space–H Space Lemma 299.26, this multiplication turns $\Omega(X, x_0)$ into a group "up to homotopy". More precisely we have the following definition and lemma which hark back to the happy days when our biggest worry was to define the fundamental group of a pointed topological space.

Definition 308.18. Let (X, x_0) be a pointed topological space. We consider the "multiplication" map

$$\begin{array}{rcl} \mu\colon \Omega(X, x_0) \times \Omega(X, x_0) & \to & \Omega(X, x_0) \\ (f, g) & \mapsto & f * g = \begin{pmatrix} [0,1] & \to & X \\ t & \mapsto & \begin{cases} f(2t), & \text{if } t \in [0, \frac{1}{2}], \\ g(2t-1), & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{pmatrix}. \end{array}$$

For entertainment the definition of this map is illustrated in the figure below.



The following lemma summarizes many important properties of μ.

Lemma 308.19. (Loop Space–Multiplication Lemma) Let (X, x_0) be a pointed topological space. The multiplication map

$$\mu\colon \Omega(X,x_0)\times\Omega(X,x_0) \;\to\; \Omega(X,x_0)$$

has the following properties.

(1) The map $\mu\colon \Omega(X,x_0)\times\Omega(X,x_0)\to\Omega(X,x_0)$ is continuous[3407], it is base-point preserving and it is natural in (X,x_0).[3408]

(2) The two maps

$$\begin{array}{rcl}\lambda\colon \Omega(X,x_0) & \to & \Omega(X,x_0)\\ f & \mapsto & \mu(c_{x_0},f)\end{array} \quad\text{and}\quad \begin{array}{rcl}\rho\colon \Omega(X,x_0) & \to & \Omega(X,x_0)\\ f & \mapsto & \mu(f,c_{x_0})\end{array}$$

between pointed topological spaces are both homotopic to the identity.

(3) The map
$$\begin{array}{rcl}\eta\colon \Omega(X,x_0) & \to & \Omega(X,x_0)\\ f & \mapsto & \overline{f} \coloneqq \begin{pmatrix}[0,1] & \to & X\\ t & \mapsto & f(1-t)\end{pmatrix}\end{array}$$
is a homeomorphism and base point preserving.

(4) The map
$$\begin{array}{rcl}\Omega(X,x_0) & \to & \Omega(X,x_0)\\ f & \mapsto & \mu(f,\overline{f})\end{array}$$
between pointed topological spaces is homotopic to the constant map $f\mapsto c_{x_0}$.

(5) The two maps

$$\begin{array}{rcl}\Omega(X,x_0)\times\Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0)\\ (f,g,h) & \mapsto & \mu(\mu(f,g),h)\end{array}$$

and

$$\begin{array}{rcl}(f,g,h) & \mapsto & \mu(f,\mu(g,h))\end{array}$$

between pointed topological spaces are homotopic.

Remark.

(1) In many ways the Loop Space–Multiplication Lemma 308.19 is quite similar to the Path-Homotopy Product Proposition 88.10. In fact it is not difficult to formulate

[3407]Recall that $\Omega(X,x_0)$ is equipped with the compact-open topology and $\Omega(X,x_0)\times\Omega(X,x_0)$ is correspondingly equipped with the product topology. Furthermore we view $\Omega(X,x_0)$ and $\Omega(X,x_0)\times\Omega(X,x_0)$ as pointed topological spaces.

[3408]Here "natural" means that if $f\colon (X,x_0)\to(Y,y_0)$ is a continuous map between pointed topological spaces, then the following diagram commutes:

$$\begin{array}{ccc}\Omega(X,x_0)\times\Omega(X,x_0) & \xrightarrow{\mu} & \Omega(X,x_0)\\ \downarrow{\scriptstyle f_*\times f_*} & & \downarrow{\scriptstyle f_*}\\ \Omega(Y,y_0)\times\Omega(Y,y_0) & \xrightarrow{\mu} & \Omega(Y,y_0).\end{array}$$

and prove a result which combines both. As it is, the two results are stated at the appropriate level of generality for what they are aimed for.

(2) Let (X, x_0) be a pointed topological space. In the Loop Space–Multiplication Lemma 308.19 (2) we showed that the multiplication map $\Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$ is continuous. The fundamental group $\pi_1(X, x_0) := \Omega(X, x_0)/\sim$ is a quotient of $\Omega(X, x_0)$, hence it can be equipped with the quotient topology. But it turns out, see in particular the discussion on page 5746, that the induced multiplication map $\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$ is in general *not* continuous. □

Proof. Let (X, x_0) be a pointed topological space.

(1) The statement that μ is continuous is precisely the content of the Loop-Space–H Space Lemma 299.26 (1). The statements that μ is base-point preserving and natural follows basically immediately from the definitions.

(2) This statement is a reformulation of the Loop-Space–H Space Lemma 299.26 (2).

(3) Evidently the map $\eta\colon \Omega(X, x_0) \to \Omega(X, x_0)$ is base point preserving. It follows immediately from the Compact-Open Functoriality Lemma 16.5 (2a), applied to the homeomorphism $[0, 1] \to [0, 1]$ that is given by $t \mapsto 1-t$, that the map $\eta\colon \Omega(X, x_0) \to \Omega(X, x_0)$ is continuous. Since $\eta \circ \eta = \mathrm{id}$ we see that η is in fact a homeomorphism.

(4) We consider the map

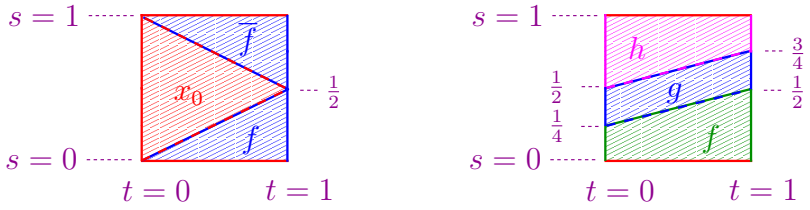
$$\begin{array}{rcl} H\colon \Omega(X,x_0)\times[0,1] & \to & \Omega(X,x_0) \\ (f,t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(2s\cdot t), & \text{if } s\in[0,\frac{1}{2}], \\ f(2t-2s\cdot t), & \text{if } s\in(\frac{1}{2},1] \end{cases} \end{array}\right). \end{array}$$

Again, using the argument in the proof of the Loop-Space–H Space Lemma 299.26 (1) we see that H is continuous. It is now clear that H is a homotopy between the constant map $f \mapsto c_{x_0}$ and the map $f \mapsto \mu(f, \overline{f})$. We refer to the figure below for an illustration.

(5) To simplify the notation we write $\Omega(X)$ instead of $\Omega(X, x_0)$. We consider the map

$$\begin{array}{rcl} H\colon (\Omega(X)\times\Omega(X)\times\Omega(X))\times[0,1] & \to & \Omega(X) \\ ((f,g,h),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{4s}{t+1}), & \text{if } s\in[0,\frac{t+1}{4}), \\ g(4s-t-1), & \text{if } s\in[\frac{t+1}{4},\frac{t+2}{4}), \\ h(\frac{4s-2-t}{2-t}), & \text{if } s\in[\frac{t+2}{4},1] \end{cases} \end{array}\right). \end{array}$$

The same argument as in the proof of the Loop-Space–H Space Lemma 299.26 (1) shows that the map H is continuous.

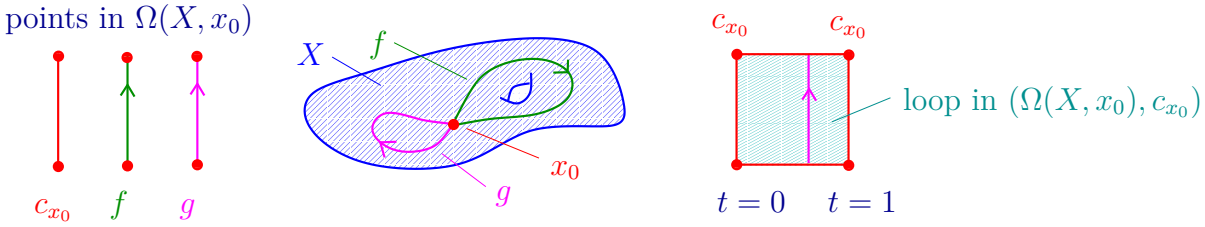


One can now easily verify that the map H defines a homotopy between the two maps $(f, g, h) \mapsto \mu(\mu(f, g), h)$ and $(f, g, h) \mapsto \mu(f, \mu(g, h))$. ■

The loop space construction is so much fun that one might as well apply it several times. This leads us to the following notation:

Notation 308.20. Let (X, x_0) be a pointed topological space. We define

$$\Omega^k(X,x_0) \;:=\; \Omega\cdots\Omega(\;\underbrace{\Omega(X,x_0)}_{\text{pointed topological space}}\;) \qquad \text{where we apply the } \Omega\text{-functor } k \text{ times.}$$



In the figure above we do our best at illustrating a loop in the pointed topological space $(\Omega(X,x_0),c_{x_0})$. What should become apparent is that such a loop is basically the same as a map $([0,1]^2,\partial([0,1]^2)) \to (X,x_0)$. The following proposition makes this idea precise.

Proposition 308.21. (Iterated Loop Space Proposition) Let (X,x_0) be a pointed topological space and let $k \in \mathbb{N}$. The map

$$\begin{array}{rcl} \overbrace{\{F \in X^{[0,1]^k} \mid F(\partial([0,1]^k)) = \{x_0\}\}}^{\text{subset of the topological space } X^{[0,1]^k}} & \to & \Omega^k(X,x_0) \\ F & \mapsto & (t_k \mapsto (t_{k-1} \mapsto \ldots (\underbrace{t_1 \mapsto F(t_1,\ldots,t_k)}_{\in\Omega(X,x_0)})\ldots)) \end{array}$$

is a homeomorphism.

Proof. To simplify the notation we only discuss the case $k=2$. The general case is treated the same way. We write $I=J=[0,1]$. We start out with the following subtle observation:

(∗) The inclusion $\Omega(X,x_0) \to X^I$ induces a map $\Omega(X,x_0)^J \to (X^I)^J$. By the Compact-Open Functoriality Lemma 16.5 (1b) this map is an embedding. This means that it is legitimate to view the topological space $\Omega(\Omega(X,x_0),c_{x_0})$ as a subspace of $(X^I)^J$.

This preamble implies that it suffices to show that the map

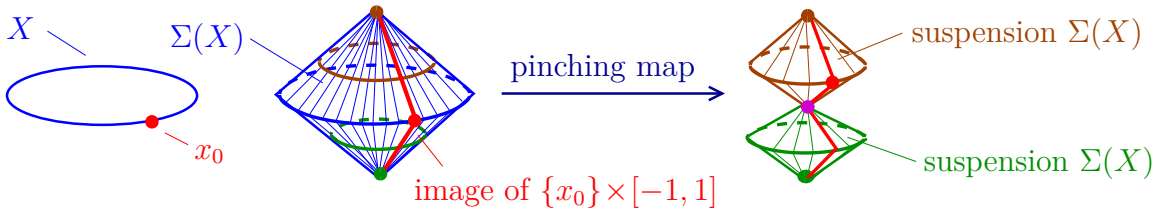
$$\begin{array}{rcl} \Phi\colon \{F \in X^{I\times J} \mid F(\partial(I\times J)) = \{x_0\}\} & \to & \Omega(\Omega(X,x_0),c_{x_0}) \subseteq (X^I)^J \\ F & \mapsto & \left(\begin{array}{ccc} J & \to & \Omega(X,x_0)\subseteq X^I \\ t & \mapsto & \left(\begin{array}{ccc} I & \to & X \\ s & \mapsto & F(s,t) \end{array}\right) \end{array}\right) \end{array}$$

is a homeomorphism. First note that the map Φ is, basically by definition, a bijection. Next note that Φ is a restriction of the map

$$\begin{array}{rcl} \Psi\colon X^{I\times J} & \to & (X^I)^J \\ (F\colon I\times J\to X) & \mapsto & \left(\begin{array}{ccc} J & \to & X^I \\ t & \mapsto & (s\mapsto F(s,t)) \end{array}\right). \end{array}$$

But it follows from the Exponential Proposition 16.12, and the fact that $I=J=[0,1]$ are regionally compact and Hausdorff, that the map Ψ is a homeomorphism. Thus we have shown that our original map Φ is also a homeomorphism. ∎

Now let us turn towards the suspension operation. Given a pointed topological space (X, x_0) ideally we would like to equip the suspension $\Sigma(X)$ with a base point in a "systematic" way. The obvious idea is to fix some $t \in [-1, 1]$ once and for all and to equip every suspension $\Sigma(X)$ with the base point represented by $[(x_0, t)]$. As we will see later on, we really really want the pinching map to be a continuous map of pointed topological spaces. But if one thinks about it, and if one keeps an eye on the figure below, one see that for every single choice of fixed t the pinching map (and note that for the wedge operation we also have to use our specified base points) will fail to be base point preserving.



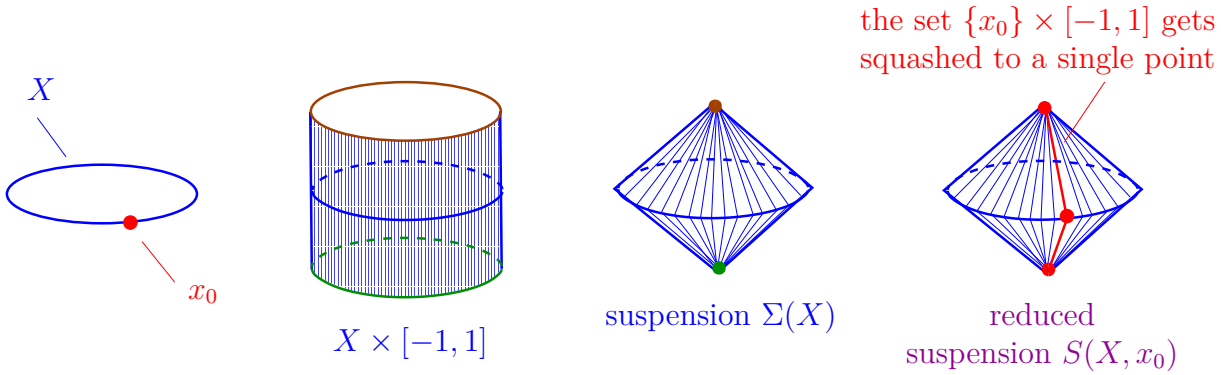
There is a rather "brutal" solution to the dilemma: We solve the conundrum of which $[(x_0, t)]$ to choose by declaring all these points to be the same. More precisely, we consider the following modified version of the suspension.

Definition 308.22. Given a pointed topological space (X, x_0) we defined the reduced suspension of (X, x_0) to be the topological space[3409]

$$S(X, x_0) \coloneqq \Sigma(X)/[\{x_0\} \times [-1, 1]] \underset{\uparrow}{=} (X \times [-1, 1])/(X \times \{-1, 1\} \cup \{x_0\} \times [-1, 1]).$$

strictly speaking we are using the Double-Quotient Lemma 8.7

We always equip $S(X, x_0)$ with the base point $s_{x_0} \coloneqq [\{x_0\} \times [-1, 1]]$. We refer to the figure below for an illustration.



The following lemma summarizes the key properties of the reduced suspension. The lemma can partly be viewed as the "pointed" version of the second part of the Loop Space–Suspension Homotopy Functor Lemma 308.5.

Lemma 308.23. (Reduced Suspension Lemma)

[3409]Note that given a topological space X we denote the usual suspension by $\Sigma(X)$ and given a pointed topological space we denote its reduced suspension by $S(X, x_0)$. Rather confusingly this notation is the opposite of the notation used in [**Hat2002**].

(1) The maps[3410]
$$\begin{array}{rcl} (X,x_0) & \mapsto & (S(X,x_0),s_{x_0}) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \left(\begin{array}{rcl} S(f)\colon S(X,x_0) & \to & S(Y,y_0) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}\right) \end{array}$$
define a covariant functor from the category $\mathcal{PHomTop}$ to itself.

(2) The natural projection $\Sigma(X)\to S(X,x_0)$ defines a natural transformation between the functors $(X,x_0)\mapsto\Sigma(X)$ and $(X,x_0)\mapsto S(X,x_0)$. In other words, in plain English, given a continuous map $f\colon (X,x_0)\to(Y,y_0)$ between pointed topological spaces the following diagram commutes:
$$\begin{array}{ccc} \Sigma(X) & \xrightarrow{\Sigma(f)} & \Sigma(Y) \\ \downarrow & & \downarrow \\ S(X,x_0) & \xrightarrow{S(f)} & S(Y,y_0). \end{array}$$

(3) Given any pointed topological space (X,x_0) the pinching map
$$\begin{array}{rcl} \omega\colon S(X,x_0) & \to & S(X,x_0)\vee S(X,x_0) \\ [(x,t)] & \mapsto & \begin{cases} [(x,2t+1)] \text{ in the first term}, & \text{if } t\in[-1,0], \\ [(x,2t-1)] \text{ in the second term}, & \text{if } t\in[0,1]. \end{cases} \end{array}$$
is continuous and base point preserving.

(4) If (X,x_0) is a pointed CW-complex, then the following two statements hold:
   (a) The reduced suspension $S(X,x_0)$ has a natural CW-structure such that the natural projection $\Sigma(X)\to S(X,x_0)$ is a cellular map and such that $\{s_{x_0}\}$ is a 0-cell. In particular $(S(X,x_0),s_{x_0})$ is a pointed CW-complex.
   (b) Given any $t\in[-1,1]$ the projection $\Sigma(X)\to S(X,x_0)$ is a homotopy equivalence of the pointed topological spaces $(\Sigma(X),[(x_0,t)])\to(S(X,x_0),s_{x_0})$.

Convention 308.24. Given a pointed CW-complex (X,x_0) we always equip the reduced suspension $S(X,x_0)$ with the CW-structure coming from the Reduced Suspension Lemma 308.23 (4a).

Proof.

(1) Statement (1) is proved basically the same way as we proved the Suspension Functor Lemma 20.23 and the Suspension–Homotopy Lemma 20.28 (1).

(2) This statement follows almost immediately from the definitions.

(3) This statement follows easily from the Topological-Quotient Proposition 8.2 (1b).

(4) The last statement is slightly more subtle. We proceed as follows:
   (a) By the CW-Complex Product Proposition 139.18 (2b) we can equip $X\times[-1,1]$ with the product CW-structure.
   (b) As in the CW-Complex Mapping Cylinder+Cone Proposition 139.24 we equip the suspension $\Sigma(X)=(X\times[-1,1])/\sim$ with the quotient CW-structure.
   (c) This CW-structure has the property that the image of $\{x_0\}\times[-1,1]$ in $\Sigma(X)$ is a subcomplex which we now denote by A.

[3410] It follows from the Topological-Quotient Proposition 8.2 (1b) that the map $S(f)$ is actually continuous.

(d) Note that it follows immediately from the CW-Quotient Lemma 139.10 that the quotient $\Sigma(X)/A = S(X,x_0)$ admits a natural CW-structure which has the convenient property that the projection $\Sigma(X) \to \Sigma(X)/A = S(X,x_0)$ is cellular and which has the property that s_{x_0} is a 0-cell.

Recall that by the Subcomplex Cofibration Theorem 22.12 we know that the inclusion map $i\colon A \to X$ is actually a cofibration. Since $[(x_0,t)]$ is a deformation retract of A we thus obtain from the Cofibration-Quotient Lemma 22.10 (3) that the natural projection $\Sigma(X) \to S(X,x_0)$ is a homotopy equivalence of the two pointed topological spaces $(\Sigma(X),[(x_0,t)]) \to (S(X,x_0),s_{x_0})$. ■

The following innocuous looking proposition is one of the main building stones for the subsequent discussion.

Proposition 308.25. (Suspension–Loop–Adjunction Proposition) Let (X,x_0) and (Y,y_0) be pointed topological spaces. The natural map

$$\begin{array}{rcl} \Upsilon\colon \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle & \to & \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle \\ \left[\begin{array}{rcl} f\colon S(X,x_0) & \to & Y \\ [(x,t)] & \mapsto & f([(x,t)]) \end{array}\right] & \mapsto & \left[\begin{array}{rcl} X & \to & \Omega(Y,y_0) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f([(x,2t-1)]) \end{array}\right) \end{array}\right] \end{array}$$

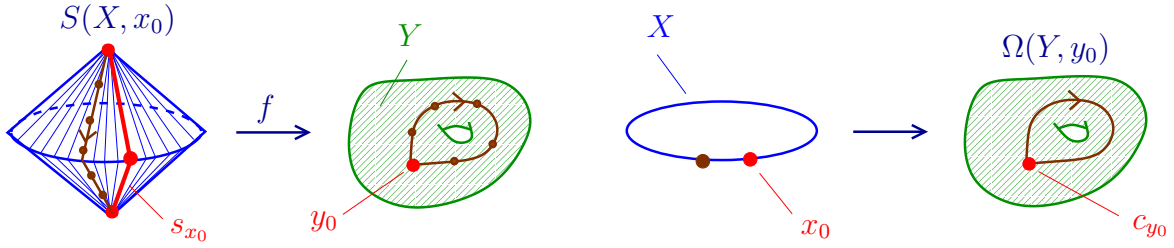
is a bijection. Furthermore this bijection is natural in (X,x_0) and (Y,y_0).[3411] In particular the functors[3412]

$$F\colon \mathcal{PHomTop} \to \mathcal{PHomTop} \qquad\qquad G\colon \mathcal{PHomTop} \to \mathcal{PHomTop}$$

which are defined by

$$\begin{array}{rcl} (X,x_0) & \mapsto & S(X,x_0) \\ [f] & \mapsto & [S(f)] \end{array} \quad \text{and} \quad \begin{array}{rcl} (X,x_0) & \mapsto & \Omega(X,x_0) \\ [f] & \mapsto & [\Omega(f)] \end{array}$$

are an adjoint pair in the sense of Definition 297.28 and Definition 297.28.



[3411] Put somewhat more precisely: If we are given a pointed topological space (Y,y_0), then the two maps $(X,x_0) \mapsto \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle$ and $(X,x_0) \mapsto \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle$ define contravariant functors from the category of pointed topological spaces to the category of sets. The map Υ defines a natural isomorphism between these two functors. The analogous statement holds with the roles of (X,x_0) and (Y,y_0) reversed.

[3412] Recall that we denote by $\mathcal{PHomTop}$ the pointed homotopy category that we introduced in Definition 308.10. In the language

Proof. Let (X, x_0) and (Y, y_0) be pointed topological spaces. First we consider the map

$$\overbrace{\{\text{set of continuous maps } X \times [-1,1] \to Y\}}^{=Y^{X\times[-1,1]}} \to \overbrace{\{\text{set of continuous maps } X \to Y^{[0,1]}\}}^{=(Y^{[0,1]})^X}$$
$$\begin{pmatrix} f\colon X\times[-1,1] & \to & Y \\ (x,t) & \mapsto & f(x,t) \end{pmatrix} \mapsto \begin{pmatrix} X & \to & Y^{[0,1]} \\ x & \mapsto & \begin{pmatrix} [0,1] & \to & Y \\ t & \mapsto & f(x,2t-1) \end{pmatrix} \end{pmatrix}.$$

It follows from the Compact-Open Exponential Proposition 16.10 and the fact that $[0,1] \cong [-1,1]$ is regionally compact that this map is well-defined[3413] and that it is a bijection. Using the Topological-Quotient Proposition 8.2 (1) it is not hard to see that the given map induces a bijection

$$\left\{ \begin{array}{c} \text{set of continuous maps} \\ (S(X,x_0), s_{x_0}) \to (Y, y_0) \end{array} \right\} \to \left\{ \begin{array}{c} \text{set of continuous maps} \\ (X, x_0) \to (\Omega(Y,y_0), c_{y_0}) \end{array} \right\}.$$

Basically the same approach also deals with homotopies. More precisely, it follows from the Compact-Open Exponential Proposition 16.10 and the fact that $[0,1] \cong [-1,1]$ is regionally compact that the "obvious" map

$$Y^{[-1,1]\times(X\times[0,1])} \to \left(Y^{[0,1]}\right)^{X\times[0,1]}$$

is a bijection. Together with the Homotopy Quotient Theorem 18.11 one can now easily show that this bijection induces a bijection between homotopies of continuous maps $(S(X,x_0), s_{x_0}) \to (Y, y_0)$ and homotopies of continuous maps $(X, x_0) \to (\Omega(Y,y_0), c_{y_0})$. In summary this shows that the map

$$\Upsilon\colon \langle (S(X,x_0), s_{x_0}), (Y, y_0)\rangle \to \langle (X, x_0), (\Omega(Y,y_0), c_{y_0})\rangle$$

is indeed a bijection. We leave it to the reader to fill in the details.

Finally we point out that it follows immediately from the definitions that Υ is natural in (X, x_0) and (Y, y_0). ∎

308.3. **Eilenberg-Maclane spaces give rise to invariants of topological spaces.** Ideally we prefer to work, for as long as possible, with CW-complexes. Alas not all interesting constructions of new topological spaces necessarily turn CW-complexes into CW-complexes. For example, if (X, x_0) is a pointed CW-complex it is not clear whether the loop space $\Omega(X, x_0)$ is again a CW-complex.[3414]

This leads us to the following definition:

Notation 308.26. We denote by $\mathcal{W}$ the class of all pointed topological spaces (X, x_0) such that there exists a pointed CW-complex (Y, y_0) which is homotopy equivalent to (Y, y_0).

The following theorem was first proved by Milnor in 1958.

Theorem 308.27. (Loop Space–CW Theorem) If $(X, x_0) \in \mathcal{W}$, then for any $k \in \mathbb{N}$ the iterated loop space $\Omega^k(X, x_0)$ also lies in $\mathcal{W}$.

[3413] Here "well-defined" means that the map $X \to Y^{[0,1]}$ that we associate to $f\colon X \times [-1,1] \to Y$ is indeed continuous.

[3414] I could not find an example where $\Omega(X, x_0)$ does not admit a CW-structure and I could not find a theorem that says that $\Omega(X, x_0)$ always admits a CW-structure.

Proof. If (X, x_0) is a pointed CW-complex, then [**Miln1959a**, Corollary 3] implies that the pointed topological space $(\Omega(X, x_0), c_{x_0})$ lies in $\mathcal{W}$. (Alternatively we refer to [**FrPi1990a**, Corollary 5.3.7] for a proof of a very similar statement.) More generally, if $(X, x_0) \in \mathcal{W}$, then it follows from the Compact-Open Functoriality Lemma 16.5 (2a) and the above that $(\Omega(X, x_0), c_{x_0})$ again lies in $\mathcal{W}$. Iterating this argument we also obtain the statement for every $k \geq 2$. ■

This leads us to the following definition:

Definition 308.28. Let $n \in \mathbb{N}$ and let G be an (abelian) group[3415].

(1) A weak pointed Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ is a pair $(X, x_0) \in \mathcal{W}$ such that for every $i \neq n$ we have $\pi_i(X, x_0) = 0$ and such that $\pi_n(X, x_0) \cong G$.
(2) A polarized Eilenberg-Maclane space[3416] of type $\mathrm{K}(G, n)$ is a triple (X, x_0, α) where (X, x_0) is a weak pointed Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ and α is an isomorphism $\pi_n(X, x_0) \xrightarrow{\cong} G$.

Examples.

(1) We equip S^1 with the base point $* = (1, 0)$. Let $\varphi\colon \pi_1(S^1, *) \xrightarrow{\cong} \mathbb{Z}$ be the standard isomorphism from page 2587. We see that $(S^1, *, \varphi)$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 1)$.
(2) We equip $\mathbb{C}\mathrm{P}^\infty$ with the base point $* = [1 : 0 : 0 \dots]$. Let $\psi\colon \pi_2(\mathbb{C}\mathrm{P}^\infty, *) \xrightarrow{\cong} \mathbb{Z}$ be the usual isomorphism from the $\pi_n(\mathbb{C}\mathrm{P}^\infty)$-Theorem 298.12 (2). We see that $(\mathbb{C}\mathrm{P}^\infty, *, \psi)$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. □

Now we turn to more general constructions of polarized Eilenberg-Maclane spaces.

Examples. Let $n \in \mathbb{N}$ and let G be an (abelian) group.

(1) We denote by $\mathrm{K}(G, n)$ the canonical Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ that was introduced in the Eilenberg-Maclane Existence Theorem 307.10. We denote its canonical base point by $*$. Furthermore we recall that the canonical Eilenberg-Maclane space $\mathrm{K}(G, n)$ comes with a canonical isomorphism $\varphi_G\colon \pi_n(\mathrm{K}(G, n), *) \xrightarrow{\varphi} G$.
(2) Let (X, x_0) be a pointed topological space. By the Loop Space-Homotopy Groups Corollary 297.22 we know that for any $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} \partial_{k+1}\colon \pi_{k+1}(X, x_0) & \to & \pi_k(\Omega(X, x_0), c_{x_0}) \\ \big[[0,1]^{k+1} \xrightarrow{f} X\big] & \mapsto & \left[\begin{array}{rcl} [0,1]^k & \to & \Omega(X, x_0) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(x,t) \end{array}\right) \end{array}\right] \end{array}$$

is a natural isomorphism. This observation, together with the Loop Space–CW Theorem 308.27, shows that if (X, x_0, α) is a polarized Eilenberg-Maclane space of type $\mathrm{K}(G, n+1)$, then $(\Omega(X, x_0), c_{x_0}, \alpha \circ \partial_{n+1}^{-1})$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(G, n)$.

[3415] As explained in Footnote 3392, whenever we deal with Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ it is understood that π is abelian.
[3416] Our language now allows for the slightly awkward possibility that a polarized Eilenberg-Maclane space is not an Eilenberg-Maclane space since the underlying topological space might not be a CW-complex. We are confident though that this will not become an issue.

The discussion above shows that given any $k \in \mathbb{N}_0$ we can and will view $\Omega^k(\mathrm{K}(G, n+k))$ as a polarized Eilenberg-Maclane of type $\mathrm{K}(G, n)$. □

The following lemma shows that in the based homotopy category all polarized Eilenberg-Maclane spaces of a given type are essentially the same.

Lemma 308.29. (Polarized Eilenberg-Maclane Space Lemma) Let (K, k_0, φ) and (L, l_0, ψ) be two polarized Eilenberg-Maclane spaces of the same type. There exists a unique class[3417] $\Xi = \Xi_{(K,k_0),(L,l_0)} \in \langle (K, k_0), (L, l_0) \rangle_{\mathrm{inv}}$ such that the corresponding induced map $\Xi_* \colon \pi_n(K, k_0) \to \pi_n(L, l_0)$ agrees with $\psi^{-1} \circ \varphi$. When there is no danger of confusion we just write $\Xi = \Xi_{(K,k_0),(L,l_0)}$.

Proof. Let $n \in \mathbb{N}$, let G be an (abelian) group and let (K, k_0, φ) and (L, l_0, ψ) be two polarized Eilenberg-Maclane spaces of type $\mathrm{K}(G, n)$. Since (K, k_0) and (L, l_0) lie in $\mathcal{W}$ we know that there exist pointed CW-complexes (X, x_0) and (Y, y_0) together with homotopy equivalences $\kappa \colon (X, x_0) \to (K, k_0)$ and $\lambda \colon (Y, y_0) \to (L, l_0)$ of pointed topological spaces. We consider the following diagram:

$$\begin{array}{ccc}
\langle (K, k_0), (L, l_0) \rangle_{\mathrm{inv}} & \xrightarrow{[f] \mapsto f_*} & \mathrm{Hom}(\pi_n(K, k_0), \pi_n(L, l_0)) \\
\downarrow{\scriptstyle \kappa^*} & & \downarrow{\scriptstyle \kappa^*} \\
\langle (X, x_0), (L, l_0) \rangle_{\mathrm{inv}} & \xrightarrow{[f] \mapsto f_*} & \mathrm{Hom}(\pi_n(X, x_0), \pi_n(L, l_0)) \\
\downarrow{\scriptstyle \lambda_*} & & \downarrow{\scriptstyle \lambda_*} \\
\langle (X, x_0), (Y, y_0) \rangle_{\mathrm{inv}} & \xrightarrow{[f] \mapsto f_*} & \mathrm{Hom}(\pi_n(X, x_0), \pi_n(Y, y_0)).
\end{array}$$

It follows almost immediately from the definitions that this diagram commutes. We obtain from the Pointed Homotopy Equivalence–Reformulation Lemma 308.11 and the discussion on page 240 that the vertical maps to the left are bijections. Furthermore we obtain from the Homotopy Equivalence-π_n-Proposition 134.17 (1) that the vertical maps to the right are bijections. It follows from the Eilenberg-Maclane Uniqueness Theorem 307.15 and the Maps-to-K(π, n)-Proposition 307.18 that any isomorphism in $\mathrm{Hom}(\pi_n(X, x_0), \pi_n(Y, y_0))$ is realized by a unique element in $\langle (X, x_0), (Y, y_0) \rangle_{\mathrm{inv}}$. By the diagram this information propagates to the top. ■

Notation 308.30. Let $n \in \mathbb{N}$, let G be an abelian group and let $k \in \mathbb{N}_0$. From the discussion on page 5937 and the Polarized Eilenberg-Maclane Space Lemma 308.29 we obtain a canonical element $\Xi \in \langle \mathrm{K}(G, n), \Omega^k \mathrm{K}(G, n+k) \rangle_{\mathrm{inv}}$. By the discussion on page 240 we know that the map

$$\begin{array}{rccl}
\Xi_* \colon \langle (X, x_0), & \overbrace{\mathrm{K}(G, n)}^{\substack{\text{viewed as a pointed} \\ \text{topological space}}} \rangle & \to & \langle (X, x_0), \overbrace{\Omega^k \mathrm{K}(G, n+k)}^{\substack{\text{viewed as a pointed} \\ \text{topological space}}} \rangle \\
& [g] & \mapsto & \Xi \circ [g]
\end{array}$$

is a bijection. Sometimes we use the bijection Ξ_* to identify the left-hand side with the right-hand side. We make the analogous identification if (X, x_0) is the target instead of the domain.

[3417] To lessen the burden on the reader's eyes we do not include φ and ψ in the notation.

We formulate the next example as a lemma. Since we will not directly make use of it we take a few liberties in its formulation.

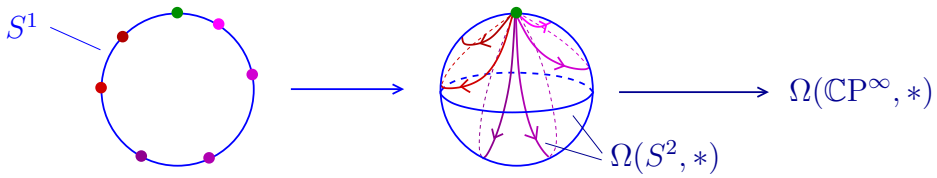
Lemma 308.31. (Loop Space-of-$\mathbb{C}\mathrm{P}^\infty$-Lemma) We equip S^1 with the usual base point $* = (1,0)$ and we equip $\mathbb{C}\mathrm{P}^\infty$ with the base point $[1:0:0\ldots]$. As on page 5937 we consider the polarized Eilenberg-Maclane spaces $(S^1, *, \varphi)$ of type $\mathrm{K}(\mathbb{Z},1)$ and we consider the polarized Eilenberg-Maclane space $(\mathbb{C}\mathrm{P}^\infty, *, \psi)$ of type $\mathrm{K}(\mathbb{Z},2)$. We have the following homeomorphisms and embeddings:

$$[0,1]^2/\partial([0,1]^2) \xrightarrow[\text{see page 189}]{\cong} \overline{B}^2/S^1 \xrightarrow[\text{from the Ball Quotient Sphere Lemma 8.10}]{\cong} S^2 \xrightarrow[\text{see page 340}]{\cong} \mathbb{C}\mathrm{P}^1 \xrightarrow[\text{embedding given by the Topological-Direct Colimit Proposition 104.10 (1a)}]{} \mathbb{C}\mathrm{P}^\infty.$$

We use these maps to view $[0,1]^2/\partial([0,1]^2)$ as a subset of $\mathbb{C}\mathrm{P}^\infty$.[3418] The map

$$\begin{array}{rcl} \Xi\colon S^1 & \to & \Omega(\mathbb{C}\mathrm{P}^\infty, *) \\ \exp(\mathrm{i}s) & \mapsto & \begin{pmatrix} [0,1] & \to & [0,1]^2/\partial([0,1]^2) \subseteq \mathbb{C}\mathrm{P}^\infty \\ t & \mapsto & [(s,t)] \end{pmatrix} \end{array}$$

represents the class $\Xi \in \langle (S^1,*), \Omega(\mathbb{C}\mathrm{P}^\infty,*)\rangle$ given by the Polarized Eilenberg-Maclane Space Lemma 308.29.



Proof. Let $i\colon [0,1]^2/\partial([0,1]^2) \to \mathbb{C}\mathrm{P}^\infty$ be the above inclusion map. Note that it follows immediately from the definition of $\Xi\colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ that the map Ξ factors through a map $\eta\colon S^1 \to \Omega([0,1]^2/\partial([0,1]^2), *)$. We consider the following diagram:

$$\begin{array}{ccc} \pi_2([0,1]^2/\partial([0,1]^2),*) & \xrightarrow[\cong]{\partial_2} & \pi_1(\Omega([0,1]^2/\partial([0,1]^2),*)) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \Omega(i)_*} \\ \pi_2(\mathbb{C}\mathrm{P}^\infty,*) & \xrightarrow[\cong]{\partial_2} & \pi_1(\Omega(\mathbb{C}\mathrm{P}^\infty,*)). \end{array} \qquad \begin{array}{c} \xleftarrow{\eta_*} \\ \pi_1(S^1,*) \\ \xleftarrow{\Xi_*} \end{array}$$

We make the following observations:

(1) The horizontal maps are defined in the Loop Space-Homotopy Groups Corollary 297.22. Furthermore note that the Loop Space-Homotopy Groups Corollary 297.22 tells us that the horizontal maps are isomorphisms and that the square commutes.
(2) The triangle commutes since $\Xi = \Omega(i) \circ \eta$.
(3) The left-hand vertical map is an isomorphism by the $\pi_n(S^n)$–Proposition 167.10.

[3418] Note that by the definition of the above maps the point $\partial([0,1]^2)$ corresponds to the point $* = [1\colon 0\colon 0\colon \ldots] \in \mathbb{C}\mathrm{P}^\infty$.

(4) By the $\pi_n(S^n)$–Proposition 167.10 we know that $\pi_2([0,1]^2/\partial([0,1]^2),*) = \mathbb{Z}\cdot \mathrm{id}_{[0,1]^2}$ and by the Circle-π_1-Theorem 88.22 (1) we know that $\pi_1(S^1,*) = \mathbb{Z}\cdot \mathrm{id}_{[0,1]}$.
(5) It follows easily from the definition of ∂_2, see the Loop Space-Homotopy Groups Corollary 297.22, that ∂_2 and η_* send their respective generators to the same element in the fundamental group $\pi_1(\Omega([0,1]^2/\partial([0,1]^2),*))$. Since all groups are isomorphic to $\mathbb{Z}$ and since ∂_2 is an isomorphism we see that η_* is an isomorphism.
(6) We obtain from the above data that Ξ_* is an isomorphism. It is now basically clear that Ξ represents the desired element in $\langle (S^1,*), \Omega(\mathbb{C}\mathrm{P}^\infty,*)\rangle$. ■

So far it seems like we have been rambling. But the following amazing definition puts us finally on an interesting track.

Definition 308.32. Let $n \in \mathbb{N}$, let G be an abelian group and let $X = (X, x_0)$ be a pointed topological space. To simplify the notation we suppress the base point x_0 throughout the subsequent discussion.[3419] We refer to the following map as the multiplication map on $\langle X, \mathrm{K}(G,n)\rangle$:

since the group G is abelian the Eilenberg-Maclane space $\Omega\,\mathrm{K}(G,n+1)$ exists for every n, therefore we can use the identification Ξ_* from page 5938

$$\big\langle X, \mathrm{K}(G,n)\big\rangle \times \big\langle X, \mathrm{K}(G,n)\big\rangle \underset{\downarrow}{=_{\mathrm{i}}} \big\langle X, \Omega\,\mathrm{K}(G,n+1)\big\rangle \times \big\langle X, \Omega\,\mathrm{K}(G,n+1)\big\rangle$$

$\downarrow$ map χ from the Pointed Homotopy-into-Product Lemma 308.13

$$\big\langle X, \Omega\,\mathrm{K}(G,n+1) \times \Omega\,\mathrm{K}(G,n+1)\big\rangle$$

$\downarrow \mu_*$ from Definition 308.18

$$\big\langle X, \Omega\,\mathrm{K}(G,n+1)\big\rangle \underset{\uparrow}{=_{\mathrm{i}}} \big\langle X, \mathrm{K}(G,n)\big\rangle.$$

again using the identification Ξ_* from page 5938

Remark.

(1) It is worth stressing what we just we did. In the definition of the multiplication on $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ we used two essential facts:
   (a) We could replace $\mathrm{K}(G,n)$ by the loop space $\Omega(\mathrm{K}(G,n+1))$.
   (b) The loop space $\Omega(\mathrm{K}(G,n+1))$ comes with a multiplication.
   Later on we will use these two ideas in a more general context.
(2) It is perhaps worth pointing out that in general topological spaces are *not* even homotopy equivalent to loop spaces. For example we know by the Loop-Space–H Space Lemma 299.26 together with the Sphere-H-Space Theorem 299.25, the discussion on page 5770 and Exercise 299.6 that S^n with $n \neq 1,3,7$ and $\mathbb{C}\mathrm{P}^n$ with $n \in \mathbb{N}$ are *not* homotopy equivalent to loop spaces. □

The following proposition says that the above definition is actually meaningful.

Proposition 308.33. (Maps-to-K(G,n)–Group Proposition) Let $n \in \mathbb{N}$, let G be an abelian group and let (X,x_0) be a pointed topological space. The above multiplication

[3419] It is though essential that all maps we consider are in fact base point preserving. In particular, recall that $\mathrm{K}(G,n)$ and $\Omega(\mathrm{K}(G,n))$ are pointed topological spaces.

turns the set $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ into a group. Furthermore the neutral element of the group $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ is given, not surprisingly, by 0.[3420]

Proof. Let $n \in \mathbb{N}$, let G be an abelian group and let (X,x_0) be a pointed topological space. Recall that we have the identification

$$\Xi_*\colon \langle (X,x_0), \mathrm{K}(G,n)\rangle \;=_{\mathrm{i}}\; \langle (X,x_0), \Omega\,\mathrm{K}(G,n+1)\rangle$$

from page 5938. Note that $\Xi_*(0) = 0$. Thus the proposition follows from the next lemma. ∎

Lemma 308.34. (Maps-to-Loop Space–Group Lemma) Let (K,k_0) be a pointed topological space. The map

$$\begin{array}{c} \langle (X,x_0), \Omega(K,k_0)\rangle \times \langle (X,x_0), \Omega(K,k_0)\rangle \\ \Big\downarrow \text{ map } \chi \text{ from the Pointed Homotopy-into-Product Lemma 308.13} \\ \langle (X,x_0), \Omega(K,k_0) \times \Omega(K,k_0)\rangle \\ \Big\downarrow \mu_* \\ \langle (X,x_0), \Omega(K,k_0)\rangle. \end{array}$$

defines a group structure. The group structure has the following properties:

(1) The neutral element is given by 0.

(2) We consider the map[3421]

$$\begin{array}{rcl} \eta\colon \Omega(K,k_0) & \to & \Omega(K,k_0) \\ f & \mapsto & \overline{f}. \end{array}$$

The inverse of $[f] \in \langle (X,x_0), \Omega(K,k_0)\rangle$ is given by $\eta_*([f])$.

Proof. As we will see shortly, this statement follows fairly easily from the Loop Space–Multiplication Lemma 308.19. In particular we will see that the statements have preciously little to do with (X,x_0). In an effort to lighten the notation we henceforth write X instead of the more accurate (X,x_0) and similarly we write K instead of (K,k_0). This time we denote by $c \in \Omega K$ the constant path at $k_0 \in K$. We consider the maps

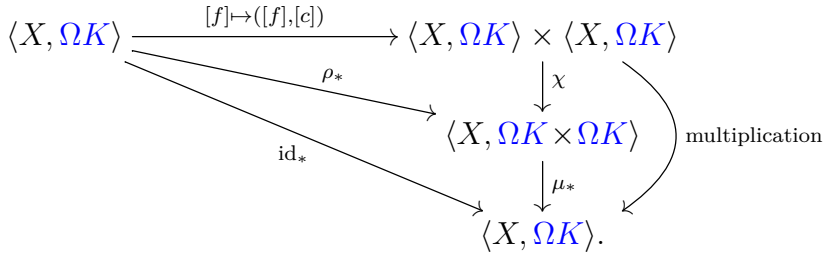
$$\begin{array}{rcl} \rho\colon \Omega K & \to & \Omega K \times \Omega K \\ f & \mapsto & (f,c) \end{array} \qquad \begin{array}{rcl} \lambda\colon \Omega K & \to & \Omega K \times \Omega K \\ f & \mapsto & (c,f) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \sigma\colon \Omega K & \to & \Omega K \times \Omega K \\ f & \mapsto & (f,\overline{f}). \end{array}$$

[3420]Recall that on Definition 308.8 we defined $0 \in \langle (X,x_0),(Y,y_0)\rangle$ to be the class that is represented by the constant map $X \to Y$, $x \mapsto y_0$.

[3421]In the Loop Space–Multiplication Lemma 308.19 (3) we showed that η is continuous.

Now we turn to the actual proofs of the various statements.

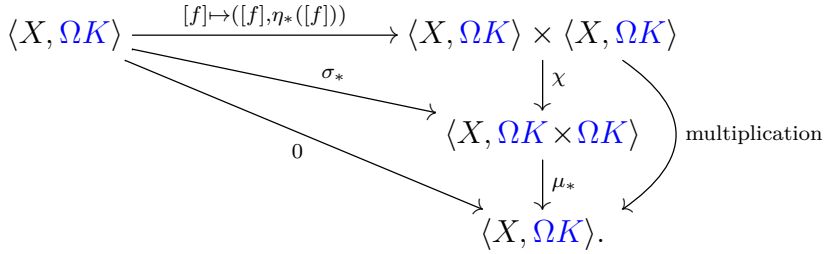
(1) Let us first show that $0 = [c]$ is a neutral element in $\langle X, \Omega K\rangle$. We consider the following diagram:



The upper triangle commutes basically by definition of the maps. Now let us turn to the lower triangle. It follows from the Loop Space–Multiplication Lemma 308.19 (2) that the map $\mu \circ \rho \colon \Omega K \to \Omega K$ is homotopic to the identity rel the base point. In other words, we have the equality $[\mu \circ \rho] = [\mathrm{id}] \in \langle \Omega K, \Omega K\rangle$. Thus it follows, if you want from the Fixed Domain-Covariant Functor Lemma 5.20, that the triangle does indeed commute.

Now we start out with $[f] \in \langle X, \Omega K\rangle$ at the top left. Since going right-down is precisely right-multiplication by $[c] = 0$ we see that 0 is indeed a right-neutral element with respect to the product structure. Basically the same argument, with ρ replaced by λ shows that 0 is also a left-neutral element.

(2) We consider the following diagram:



As in (1) we obtain from the Loop Space–Multiplication Lemma 308.19 (4) that the diagram commutes. It follows that $\eta_*([f])$ is a right inverse to $[f]$. The same argument shows that it is also a left inverse to $[f]$.

(3) Using the Loop Space–Multiplication Lemma 308.19 (5) one can show, with the same approach as above, that the product structure on $\langle X, \Omega K\rangle$ is associative. Filling in the details is the content of Exercise 308.3. ■

The reader of the previous proof might ask the following question:

Question 308.35. Why do we actually consider the sets $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ instead of the sets $\langle (X, x_0), \Omega\,\mathrm{K}(G, n+1)\rangle$? After all, we just showed that the group structure is really defined on the latter set.

The group structure on $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ is certainly rather mysterious. The following lemma hopefully elucidates the group structure in the case $\mathrm{K}(\mathbb{Z}, 1)$.

Lemma 308.36. (Loop Space-of-$\mathbb{C}\mathrm{P}^\infty$-Lemma) Let $\Xi\colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ be the homotopy equivalence defined in the Loop Space-of-$\mathbb{C}\mathrm{P}^\infty$- Lemma 308.31. Given any topological space X we have the following commutative diagram where the horizontal maps are bijections:

$$\begin{array}{ccc} [X, S^1] \times [X, S^1] & \xrightarrow{\Xi_* \times \Xi_*} & [X, \Omega\mathbb{C}\mathrm{P}^\infty] \times [X, \Omega\mathbb{C}\mathrm{P}^\infty] \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu_*} \\ [X, S^1] & \xrightarrow{\Xi_*} & [X, \Omega\mathbb{C}\mathrm{P}^\infty]. \end{array}$$

Here the left vertical map is given by the Maps-to-Group-a-Group Lemma 308.7.

Proof. The statement of the lemma has nothing to do with X. In fact, it suffices to show that the following map agrees up to homotopy:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\Xi \times \Xi} & (\Omega\mathbb{C}\mathrm{P}^\infty, *) \times (\Omega\mathbb{C}\mathrm{P}^\infty, *) \\ \downarrow{\scriptstyle (z,w) \mapsto z \cdot w} & & \downarrow{\scriptstyle \mu} \\ S^1 & \xrightarrow{\Xi} & \Omega(\mathbb{C}\mathrm{P}^\infty, *). \end{array}$$

We outsource this verification to Exercise 308.1. ■

Perhaps the first question that arises when we introduce a group is whether the group abelian. It turns out that in this case we can answer this question in the affirmative.

Proposition 308.37. (Maps-to-K(G,n)–Abelian Proposition) Let $n \in \mathbb{N}$ and let G be an abelian group. For every pointed topological space (X, x_0) the corresponding group $\langle (X, x_0), \mathrm{K}(G,n) \rangle$ is abelian.

Proof. Let $n \in \mathbb{N}$ and let G be an abelian group. We write $L = \mathrm{K}(G, n+1)$ and we write $M = \mathrm{K}(G, n+2)$. Finally let (X, x_0) be a pointed topological space. We make the following preparations.

(1) By the discussion on page 5937 we know that $\mathrm{K}(G,n)$, ΩL and $\Omega(\Omega M)$ are polarized Eilenberg-Maclane spaces of type $\mathrm{K}(G,n)$. The Polarized Eilenberg-Maclane Space Lemma 308.29 thus gives us homotopy equivalences $f\colon (\mathrm{K}(G,n), *) \to (\Omega L, *)$ and $g\colon (\Omega L, *) \to (\Omega(\Omega M), *)$.

(2) We define $\square M := \{f\colon [0,1]^2 \to M \mid f(\partial([0,1]^2)) = \{*\}\}$.
As always we equip $\square M$ with the compact-open topology. Furthermore let

$$\begin{array}{rcl} \Theta\colon \square M & \to & \Omega(\Omega M) \\ (f\colon [0,1]^2 \to M) & \mapsto & (t \mapsto (s \mapsto f(s,t))) \end{array}$$

be the *homeomorphism* from the Iterated Loop Space Proposition 308.21.

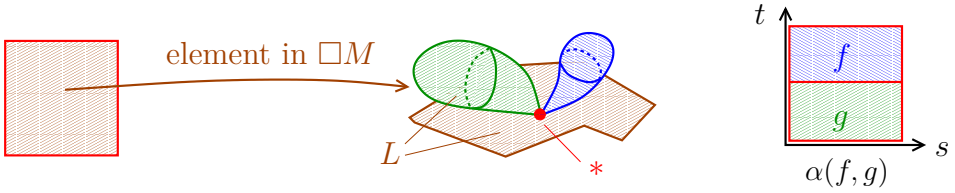
(3) We consider the map

$$\begin{array}{rcl} \alpha\colon \square M \times \square M & \to & \square M \\ (f,g) & \mapsto & \left(\begin{array}{rcl} [0,1]^2 & \to & M \\ (s,t) & \mapsto & \begin{cases} f(s,2t), & \text{if } t \in [0, \frac{1}{2}], \\ g(s,2t-1), & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array} \right). \end{array}$$

The same argument as in the proof of the Loop-Space–H Space Lemma 299.26 (1) shows that α is continuous.

(4) Since the pointed topological space (X, x_0) is really just a bystander we will abbreviate it to X.

Some of the above objects are illustrated in the figure below.



Claim. The following diagram commutes:

$$\begin{array}{ccccc}
\langle X, \mathrm{K}(G,n)\rangle \times \langle X, \mathrm{K}(G,n)\rangle & & \xrightarrow{\text{group multiplication}} & & \langle X, \mathrm{K}(G,n)\rangle \\
\downarrow{\scriptstyle (\Theta^{-1}\circ g\circ f)_*\times(\Theta^{-1}\circ g\circ f)_*} & & & & \downarrow{\scriptstyle (\Theta^{-1}\circ g\circ f)_*} \\
\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle.
\end{array}$$

Furthermore the vertical maps are bijections.

Proof. To show that the above diagram commutes we expand it into an even larger diagram. More precisely, we consider the following diagram:

$$\begin{array}{ccccc}
\langle X, \mathrm{K}(G,n)\rangle \times \langle X, \mathrm{K}(G,n)\rangle & & \xrightarrow{\text{group multiplication}} & & \langle X, \mathrm{K}(G,n)\rangle \\
\downarrow{\scriptstyle f_*\times f_*} & & & & \downarrow{\scriptstyle f_*} \\
\langle X, \Omega L\rangle \times \langle X, \Omega L\rangle & \xrightarrow{\chi} & \langle X, \Omega L \times \Omega L\rangle & \xrightarrow{\mu_*} & \langle X, \Omega L\rangle \\
\downarrow{\scriptstyle g_*\times g_*} & & \downarrow{\scriptstyle (g_*\times g_*)_*} & & \downarrow{\scriptstyle g_*} \\
\langle X, \Omega(\Omega M)\rangle \times \langle X, \Omega(\Omega M)\rangle & \xrightarrow{\chi} & \langle X, \Omega(\Omega M) \times \Omega(\Omega M)\rangle & \xrightarrow{\mu_*} & \langle X, \Omega(\Omega M)\rangle \\
\downarrow{\scriptstyle \Theta_*^{-1}\times\Theta_*^{-1}} & & \downarrow{\scriptstyle (\Theta_*^{-1}\times\Theta_*^{-1})_*} & & \downarrow{\scriptstyle \Theta_*^{-1}} \\
\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle.
\end{array}$$

We make the following observations:

(1) By the discussion on page 240 we know that the vertical maps between the first and second and between the second and the third row are bijections.
(2) Since Θ is a homeomorphism we see that the bottom vertical maps are also bijections.
(3) By definition of the group multiplication on $\langle X, \mathrm{K}(G,n)\rangle$ the top rectangle commutes.
(4) It follows immediately from the definition of χ that the two smaller rectangles on the left commute.
(5) It follows from the Loop Space–Multiplication Lemma 308.19 (1) that the middle right rectangle commutes.
(6) It follows immediately from the definition of the maps μ, α and Θ that

$$\alpha \circ (\Theta \times \Theta) \;=\; \Theta \circ \mu \colon \Omega(\Omega M) \times \Omega(\Omega M) \;\to\; \square M.$$

This equality implies that the lower right rectangle commutes. ⊞

The claim can be summarized as saying that it suffices to show that the bottom multiplication map on $\langle X, \square M\rangle$ is commutative. Next we consider the swap map

$$\begin{array}{rcl}\sigma\colon \square M \times \square M & \to & \square M \times \square M \\ (f,g) & \mapsto & (g,f).\end{array}$$

We obtain the following diagram:

$$\begin{array}{ccccc}\langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle \\ \downarrow{\scriptstyle (a,b)\mapsto(b,a)} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \langle X, \square M\rangle \times \langle X, \square M\rangle & \xrightarrow{\chi} & \langle X, \square M \times \square M\rangle & \xrightarrow{\alpha_*} & \langle X, \square M\rangle\end{array}$$

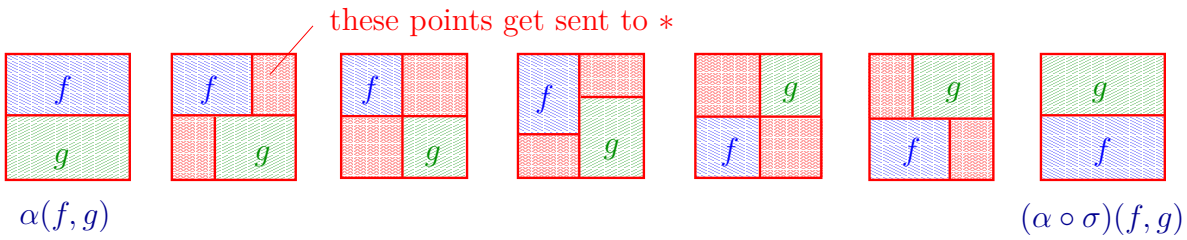
We make the following observations:

(1) The statement that the multiplication on $\langle X, \square M\rangle$ is commutative is, by definition, equivalent to the statement that the large rectangle commutes.
(2) It follows immediately from the definitions of the various maps that the smaller rectangle on the left commutes. Thus it suffices to show that the smaller rectangle on the right commutes.

In particular, since $\alpha_* \circ \sigma_* = (\alpha \circ \sigma)_*$ it suffices to prove the following claim:[3422]

Claim. The maps α and $\alpha\circ\sigma$ from the pointed topological space $\square M \times \square M$ to the pointed topological space $\square M$ are homotopic.

Proof. In the figure below we indicate how one can define a homotopy from α to $\alpha \circ \sigma$.



We leave it to the punctilious reader to write down a precise formula for a homotopy[3423]

$$(\square M \times \square M) \times [0,1] \ \to \ \square M.$$

As in the proof of the Loop Space–Multiplication Lemma 308.19 one can use the"⇒"-direction of the Compact-Open Exponential Proposition 16.10 to show that the map provided by the reader is actually continuous. ■

Remark. As the reader with a well-developed long-term memory will have noticed, the proof of the Maps-to-K(G,n)–Abelian Proposition 308.37 eventually is very similar to the proof of the π_n-Abelian Proposition 134.11. In particular the above figure looks almost the same as the figure on page 2581. □

308.4. **Alternative description of the group structure.** Once again let $n \in \mathbb{N}$, let G be an abelian group and let (X, x_0) be a pointed topological space. In this section we give

[3422]Thus once again X plays no role in the argument.
[3423]For example this could be done using the maps introduced in the proof of the Shrinking Lemma 134.8.

an alternative description of the group structure on

$$\langle (X,x_0), \mathrm{K}(G,n)\rangle \xrightarrow[\uparrow]{\Xi} \langle (X,x_0), \Omega\,\mathrm{K}(G,n+1)\rangle \xrightarrow[\uparrow]{\Upsilon} \langle S(X,x_0), \mathrm{K}(G,n+1)\rangle.$$

bijection from Notation 308.30 — bijection from the Suspension–Loop Adjunction Proposition 308.25

Before we state the relevant proposition we recall some of the objects that we have introduced recently:

(1) On page 5934 we introduced the pinching map

$$\omega\colon S(X,x_0) \;\to\; S(X,x_0)\vee S(X,x_0)$$

where $S(X,x_0)$ denotes the reduced suspension.

(2) In the Pointed Homotopy-into-Product Lemma 308.13and the Pointed Wedge-Homotopy Lemma 308.15 we introduced the maps

$$\begin{array}{rcl}\chi\colon \langle (X,x_0),(Y,y_0)\rangle \times \langle (X,x_0),(Z,z_0)\rangle & \to & \langle (X,x_0),(Y\times Z,(y_0,z_0))\rangle \\ ([f\colon X\to Y],[g\colon X\to Z]) & \mapsto & [f\times g\colon X\to Y\times Z]\end{array}$$

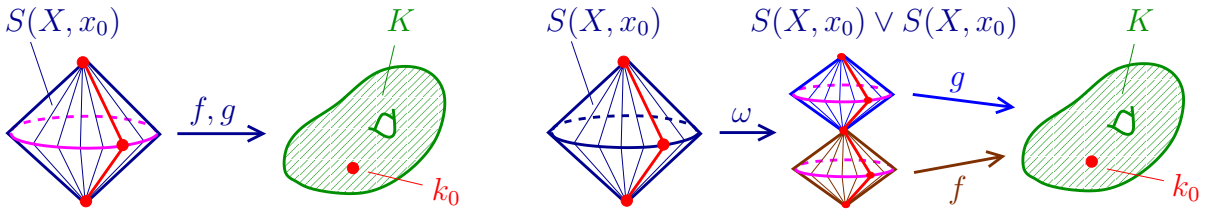
and

$$\begin{array}{rcl}\nu\colon \langle (X,x_0),(Z,z_0)\rangle \times \langle (Y,y_0),(Z,z_0)\rangle & \to & \langle (X,x_0)\vee (Y,y_0),(Z,z_0)\rangle \\ ([f\colon X\to Z],[g\colon Y\to Z]) & \mapsto & [f\vee g]\end{array}$$

The following proposition shows that instead of the map μ on the loop space we can also use the pinching map on the reduced suspension to define the group structure.

Proposition 308.38. (Pinching Map–Group Structure Proposition) Let (X,x_0) and (K,k_0) be pointed topological spaces. The following diagram commutes:

$$\begin{array}{ccc}\langle (X,x_0),\Omega(K,k_0)\rangle \times \langle (X,x_0),\Omega(K,k_0)\rangle & \xrightarrow{\Upsilon\times\Upsilon} & \langle S(X,x_0),(K,k_0)\rangle \times \langle S(X,x_0),(K,k_0)\rangle \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \nu} \\ \langle (X,x_0),\Omega(K,k_0)\times\Omega(K,k_0)\rangle & & \langle S(X,x_0)\vee S(X,x_0),(K,k_0)\rangle \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \omega^*} \\ \langle (X,x_0),\Omega(K,k_0)\rangle & \xrightarrow{\qquad\Upsilon\qquad} & \langle S(X,x_0),(K,k_0)\rangle.\end{array}$$



Remark. The multiplication map on $\langle S(X,x_0),(K,k_0)\rangle$ is arguably more geometric surely more appealing to some readers. Nonetheless, any attempt to prove that this multiplication map defines an (abelian) group structure promises to be notationally somewhat messy. □

Proof. Let $f,g\colon (X,x_0)\to\Omega(K,k_0)$ be two continuous maps. One can easily verify that

$$(\Upsilon\circ\mu_*\circ\chi)([f],[g]) = \left[\begin{array}{rcl} S(X,x_0) & \to & (K,k_0) \\ [(x,t)] & \mapsto & \begin{cases} f(t+1), & \text{if } t\in[-1,0], \\ g(t), & \text{if } t\in[0,1]\end{cases}\end{array}\right] = (\omega^*\circ\nu)(\Upsilon([f]),\Upsilon([g]).$$

This shows that the diagram does indeed commute. ■

308.5. **Functoriality.** Let $n \in \mathbb{N}$. In this section we will see, perhaps not surprisingly, that the abelian groups $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ are functorial in (X, x_0) and G. To make this precise we need to introduce the following piece of notation.

Definition 308.39. Let $n \in \mathbb{N}$ and let $\varphi\colon G \to H$ be a homomorphism between two (abelian) groups. We denote by $[\varphi] \in \langle \mathrm{K}(G, n), \mathrm{K}(H, n)\rangle$ the canonical homotopy class of continuous maps defined in the Maps-to-K(π, n)-Proposition 307.18.[3424]

We have the following lemma:

Lemma 308.40. (Maps-to-K(G, n)-Functor Lemma) Let $n \in \mathbb{N}$. The maps

$$\begin{array}{rcl} G & \mapsto & \mathrm{K}(G, n) \\ (\varphi\colon G \to H) & \mapsto & [\varphi] \in \langle \mathrm{K}(G, n), \mathrm{K}(H, n)\rangle \end{array}$$

define a covariant functor from the category of (abelian) groups to the pointed homotopy category.

Proof. The fact that the maps define a covariant functor is an immediate consequence of the uniqueness statement in the Maps-to-K(π, n)-Proposition 307.18. ■

If one thinks about this lemma for a second, then one realizes that this lemma is actually quite astounding. So far we spent almost all our time trying to define functors from topological spaces to (abelian) groups in the hope of being able to use algebra to understand topological spaces. But now we have functors which go the other way around. In principle this will allow us to use tools from topology to gain a better understanding of groups. In fact we already saw a glimpse of this phenomenon in the Surface–Free Group Epimorphism Proposition 307.28. Unfortunately at this very moment we cannot develop this theme. But in Chapter 317 this idea will come back with a vengeance.

After this short digression we can formulate the following proposition:

Proposition 308.41. (Pointed Maps-to-K(G, n)-Functor Proposition) Let $n \in \mathbb{N}$.

(1) Let G be an abelian group. The maps

$$\begin{array}{rcl} (X, x_0) & \mapsto & \langle (X, x_0), \mathrm{K}(G, n)\rangle \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & \left(\begin{array}{rcl} \langle (Y, y_0), \mathrm{K}(G, n)\rangle & \to & \langle (X, x_0), \mathrm{K}(G, n)\rangle \\ [\gamma] & \mapsto & [\gamma \circ f] \end{array} \right) \end{array}$$

define a contravariant functor from the category of pointed topological spaces to the category *AbGroup* of abelian groups.

(2) Let (X, x_0) be a pointed topological space. The maps

$$\begin{array}{rcl} G & \mapsto & \langle (X, x_0), \mathrm{K}(G, n)\rangle \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \langle (X, x_0), \mathrm{K}(G, n)\rangle & \to & \langle (X, x_0), \mathrm{K}(H, n)\rangle \\ [\gamma] & \mapsto & [\varphi \circ \gamma] \end{array} \right) \end{array}$$

[3424]More precisely, by the Maps-to-K(π, n)-Proposition 307.18 there exists a unique $[f] \in \langle \mathrm{K}(G, n), \mathrm{K}(H, n)\rangle$ such that the map $f_*\colon G = \pi_n(\mathrm{K}(G, n)) \to \pi_n(\mathrm{K}(H, n)) = H$ agrees with φ. Here we used that the canonical Eilenberg-Maclane spaces $\mathrm{K}(J, n)$ are pointed and that they come with an identification $\pi_n(\mathrm{K}(J, n)) = J$.

define a covariant functor from the category *AbGroup* of abelian groups to the category *AbGroup* of abelian groups.

Proof. Let $n \in \mathbb{N}$.

(1) We fix an abelian group G. First note that it follows from the Fixed Target-Contravariant Functor Lemma 5.23 that the assignment $(X, x_0) \to \langle (X, x_0), \mathrm{K}(G, n)\rangle$ is a contravariant functor from the category of pointed topological spaces to the category of sets. Thus it remains to show that this assignment defines actually a covariant functor to the category *AbGroup* of abelian groups. More precisely, it remains to show that for every continuous map $f \in \langle (X, x_0), (Y, y_0)\rangle$ the induced map $\langle (Y, y_0), \mathrm{K}(G, n)\rangle \to \langle (X, x_0), \mathrm{K}(G, n)\rangle$ is actually a group homomorphism. But that is basically obvious since the above definition of the group structure on $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ has nothing to do with X. We leave it to the skeptical reader to fill in the details.

(2) Let (X, x_0) be a pointed topological space. By the Fixed Domain-Covariant Functor Lemma 5.20 and the Maps-to-K(G, n)-Functor Lemma 308.40 we know that the assignment $G \mapsto \langle (X, x_0), \mathrm{K}(G, n)\rangle$ defines a covariant functor from the category *AbGroup* of abelian groups to the category of sets. As in (1) it remains to show that the assignment actually defines a covariant functor to the category *AbGroup* of abelian groups. This requires a little more effort than (1). The argument breaks up into two parts, namely we have to deal with the "horizontal" and the "vertical" aspects of Definition 308.32.

(a) Recall that in the Loop Space-Homotopy Groups Corollary 297.22 we gave an explicit *natural* isomorphism

$$\alpha_k \colon \pi_k(K, k_0) \xrightarrow{\cong} \pi_{k-1}(\Omega(K, k_0), c_{x_0}) \qquad \text{for every } k \in \mathbb{N}.$$

It follows from the naturality of these isomorphisms, and the precise definition of the maps Ξ in Notation 308.30, that for a given homomorphism $\varphi \colon G \to H$ of (abelian) groups we obtain a commutative diagram

$$\begin{array}{ccc} \langle (X, x_0), \mathrm{K}(G, n)\rangle & \xrightarrow[\Xi_*]{\mathrm{id}} & \langle (X, x_0), \Omega\,\mathrm{K}(G, n+1)\rangle \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \langle (X, x_0), \mathrm{K}(H, n)\rangle & \xrightarrow[\Xi_*]{\mathrm{id}} & \langle (X, x_0), \Omega\,\mathrm{K}(H, n+1)\rangle. \end{array}$$

This takes care of the "horizontal" part.

(b) It follows from the Pointed Homotopy-into-Product Lemma 308.13, the Loop Space–Multiplication Lemma 308.19 (1) and the Maps-to-K(G, n)-Functor Lemma 308.40 that the "vertical" part of Definition 308.32 is functorial in φ. ■

As the reader will know by now, base points are something of a nuisance. Fortunately in the, the CW–Complex Pointed Homotopy Proposition 140.14 and the Sphere Pointed-Homotopy Corollary 140.15 we already saw that on many occasions we can stop worrying about base points. The following proposition nicely complements the CW–Complex Pointed Homotopy Proposition 140.14.

Proposition 308.42. (Abelian-π_1–Drop Base Point Proposition) Let (X, x_0) and (Y, y_0) be two pointed CW-complexes. If Y is path-connected, if $\pi_1(Y, y_0)$ is abelian and if $\pi_i(Y, y_0) = 0$ for $i \geq 2$,[3425] then the natural map $\langle (X, x_0), (Y, y_0)\rangle \to [X, Y]$ is also a bijection.

Remark.

(1) For us the most important application of the Abelian-π_1–Drop Base Point Proposition 308.42 will be the case $Y = S^1$.
(2) We equip S^1, S^2 and $S^1 \vee S^2$ with the usual base point $*$. In Exercise 308.2 we will show that the map $[(S^2, *), (S^1 \vee S^2, *)] \to [S^2, S^1 \vee S^2]$ is not an injection. This shows that in the Abelian-π_1–Drop Base Point Proposition 308.42 we cannot just drop the hypothesis that $\pi_i(Y, y_0) = 0$ for $i \geq 2$. □

Proof. Let (Y, y_0) be a pointed connected CW-complex such that the fundamental group $\pi_1(Y, y_0)$ is abelian and such that $\pi_i(Y, y_0) = 0$ for $i \geq 2$. Note that by the CW–Complex Pointed Homotopy Proposition 140.14 (1) it remains to show that the map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$ is injective. Despite initial appearances, the argument will be quite different from the argument provided in the proof of the CW–Complex Pointed Homotopy Proposition 140.14 (1). So let $f_0, f_1 \colon (X, x_0) \to (Y, y_0)$ be two continuous maps that are homotopic as maps $X \to Y$. By the Homotopy-π_1-Proposition 90.8 we know that there exists a $\gamma \in \pi_1(Y, y_0)$ such that $f_{0*}(g) = \gamma \cdot f_{1*}(g) \cdot \gamma^{-1} \in \pi_1(Y, y_0)$ for every $g \in \pi_1(X, x_0)$. But since $\pi_1(Y, y_0)$ is abelian we see that $f_{0*} = f_{1*}$. Since by hypothesis we have $\pi_i(Y, y_0) = 0$ for $i \geq 2$ we can appeal to the Maps-to-K(π, n)-Proposition 307.18 (2) which now shows that there exists a homotopy from f_0 to f_1 rel x_0. ■

In the following discussion we will use the Pointed Maps-to-K(G, n)-Functor Proposition 308.41 (1) to dispose of the base points.

Definition 308.43. Let $n \in \mathbb{N}$ and let G be an abelian group.

(1) Let X be a connected CW-complex. We pick a point $x_0 \in X^0$. By the CW–Complex Pointed Homotopy Proposition 140.14 and the Abelian-π_1–Drop Base Point Proposition 308.42[3426] the natural map

$$\begin{array}{rccc} \Psi_{(X,x_0)} \colon & \langle (X, x_0), (\mathrm{K}(G, n), *) \rangle & \to & [X, \mathrm{K}(G, n)] \\ & [f] & \mapsto & [f] \end{array}$$

is a bijection. We use this bijection, together with the group structure on the left-hand side defined in the Maps-to-K(G, n)–Group Proposition 308.33, to turn the right-hand side $[X, \mathrm{K}(G, n)]$ into a group. In the Base Point–Independence Corollary 308.44 below we will see that this definition does not depend on the choice of x_0.

(2) Now let X be any CW-complex. Let $\{X_i\}_{i \in I}$ be the set of its components. We have

$$[X, \mathrm{K}(G, n)] \underset{\uparrow}{=} \prod_{i \in I} [X_i, \mathrm{K}(G, n)]$$

by the Topological-Coproduct Proposition 7.26 (1)

We define the group structure on each $[X_i, \mathrm{K}(G, n)]$ as in (1) and for $i \neq j$ we define the multiplication on $[X_i, \mathrm{K}(G, n)] \times [X_j, \mathrm{K}(G, n)]$ to be the trivial map. One easily verifies that this defines an abelian group structure on $[X, \mathrm{K}(G, n)]$.

[3425] In other words, (Y, y_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$ where $\pi \cong \pi_1(Y, y_0)$ is an abelian group.
[3426] Note that here we use that G is an abelian group, even for $n = 1$.

The following corollary to the Pointed Maps-to-K(G,n)-Functor Proposition 308.41 (1) shows that this definition of the group structure on $[X, \mathrm{K}(G,n)]$ does not depend on the choice of the base point $x_0 \in X^0$.

Corollary 308.44. (Base Point–Independence Corollary) Let $n \in \mathbb{N}$, let G be an abelian group and let X be a connected CW-complex. The definition of the group structure on $[X, \mathrm{K}(G,n)]$ does not depend on the choice of $x_0 \in X^0$.

Proof. Let $n \in \mathbb{N}$, let G be an abelian group and let X be a connected CW-complex. Now let $x_0, x_1 \in X$ be two points in the 0-skeleton of X. Since X is connected we know by the CW–Push Points Proposition 140.18 that there exists a map $f\colon X \to X$ with the following properties:

(a) The map f is homotopic to the identity.
(b) We have $f(x_0) = x_1$.
(c) The map f defines a homotopy equivalence $(X, x_0) \to (X, x_1)$ of pointed topological spaces.

We consider the following diagram:

$$\begin{array}{ccc} \langle (X,x_1), (\mathrm{K}(G,n), *)\rangle & \xrightarrow[[g]\mapsto[g\circ f]]{f^*} & \langle (X,x_0), (\mathrm{K}(G,n), *)\rangle \\ \Big\downarrow{\scriptstyle \Psi_{(X,x_1)}}\ {\scriptstyle [g]\mapsto[g]} & & \Big\downarrow{\scriptstyle \Psi_{(X,x_0)}}\ {\scriptstyle [g]\mapsto[g]} \\ [X, \mathrm{K}(G,n)] & \xrightarrow[[g]\mapsto[g\circ f]]{f^*} & [X, \mathrm{K}(G,n)]. \end{array}$$

We make the following observations:

(1) It follows basically immediately from the definitions that the diagram commutes.
(2) It follows from (c) and the Fixed Target-Contravariant Functor Lemma 5.23 and the Isomorphism Preservation Lemma 5.26 that the top horizontal map is a bijection.
(3) By the Pointed Maps-to-K(G,n)-Functor Proposition 308.41 (1) we know that top horizontal map is in fact an isomorphism.
(4) By (a) we know that the bottom map is in fact the identity.

It follows immediately from the above that it does not matter whether we use $\Psi_{(X,x_1)}$ or $\Psi_{(X,x_0)}$ to define the group structure on $[X, \mathrm{K}(G,n)]$. ■

308.6. **Basic calculations.** In this section we want to do a few bare-hands calculation of our new invariant $\langle (X,x_0), \mathrm{K}(G,n)\rangle$. Before we start with the calculations it is useful to introduce the following notation:

Notation 308.45. Let $n \in \mathbb{N}$ and let G be an abelian group. Furthermore let (X, x_0) be a pointed topological space. For the purpose of this section we write

$$h^n((X,x_0);G) \coloneqq \langle (X,x_0), \mathrm{K}(G,n)\rangle.$$

Before we state the second main calculation it is helpful to introduce the following convention.

Convention 308.46. Let G be an abelian group. We make the identifications

$$\mathrm{H}_1(\mathrm{K}(G,1);\mathbb{Z}) \underset{\text{by the Hurewicz Theorem 166.10}}{=_{\mathrm{i}}} \pi_1(\mathrm{K}(G,1),*)_{\mathrm{ab}} \underset{\text{by the Eilenberg-Maclane Existence Theorem 307.10}}{=_{\mathrm{i}}} G_{\mathrm{ab}} \underset{\text{since } G \text{ is abelian}}{=_{\mathrm{i}}} G.$$

Lemma 308.47. (h^1-Reinterpretation Lemma) Let G be an abelian group. Let X be a connected CW-complex and let $x_0 \in X^0$ be a point. The map

$$\begin{array}{rcl} h^1((X,x_0);G) & \to & \mathrm{Hom}(\mathrm{H}_1(X;\mathbb{Z}),G) \\ [f\colon (X,x_0)\to(\mathrm{K}(G,1),*)] & \to & (f_*\colon \mathrm{H}_1(X;\mathbb{Z})\to \mathrm{H}_1(\mathrm{K}(G,1);\mathbb{Z}) =_{\mathrm{i}} G) \end{array}$$

is a natural isomorphism.

Proof.

(1) It follows from the Maps-to-K(π,n)-Proposition 307.18 and the Hurewicz Theorem 166.10 that the given map is a bijection.
(2) It follows immediately from the definitions that the map is natural in X.
(3) It remains to show that the given map is a homomorphism. We write $Y = \mathrm{K}(G,2)$. We make the usual identification $\langle X, \mathrm{K}(G,1)\rangle = \langle X, \Omega Y\rangle$. Furthermore we use the natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(\mathrm{H}_1(X;\mathbb{Z}),G) & \to & \mathrm{Hom}(\pi_1(X,x_0),\pi_1(\Omega Y)) \\ \varphi & \mapsto & (\pi_1(X,x_0)\to \pi_1(X,x_0)_{\mathrm{ab}} = \mathrm{H}_1(X;\mathbb{Z}) \xrightarrow{\varphi} G = \pi_1(\Omega Y)) \end{array}$$

to identify the group on the left with the group on the right. Next we consider the following diagram:

$$\begin{array}{ccc} \langle X,\Omega Y\rangle \times \langle X,\Omega Y\rangle & \xrightarrow{([f],[g])\mapsto(f_*,g_*)} & \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)) \times \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)) \\ \downarrow{\scriptstyle ([f],[g])\mapsto[f\times g]} & & \downarrow{\scriptstyle (\varphi,\psi)\mapsto(g\mapsto(\varphi(g),\psi(g)))} \\ \langle X,\Omega Y\times\Omega Y\rangle & \longrightarrow & \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)\times\pi_1(\Omega Y)) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} \\ \langle X,\Omega Y\rangle & \longrightarrow & \mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y)) \end{array}$$

We make the following clarifications and observations:
(a) The upper rectangle commutes basically by definition.
(b) The lower square also commutes basically by definition. Slightly more precisely, let $[f] \in \langle X, \Omega Y\times\Omega Y\rangle$. We consider the maps $X \xrightarrow{f} \Omega Y\times\Omega Y \xrightarrow{\mu} \Omega Y$. By functoriality we have $(\mu\circ f)_* = \mu_*\circ f_*$. But this equality just means that in the lower square it does not matter whether we take the down-right route or the right-down route.
(c) It follows from the Binary Operations-on-π_1-Lemma 299.33 that the right vertical maps give precisely the group multiplication on $\mathrm{Hom}(\pi_1(X),\pi_1(\Omega Y))$.

This shows that the given map is indeed a group homomorphism. ■

308.7. **Product structures.** Let (X, x_0) be a pointed CW-complex and let $m, n \in \mathbb{N}$. Our goal in this section is to define a potentially interesting map

$$\langle (X, x_0), \mathrm{K}(\mathbb{Z}, m)\rangle \times \langle (X, x_0), \mathrm{K}(\mathbb{Z}, n)\rangle \ \to\ \langle (X, x_0), \mathrm{K}(\mathbb{Z}, m+n)\rangle.$$

The key to defining such a map is the following humble construction.

Definition 308.48. Given two pointed topological spaces (X, x_0) and (Y, y_0) the smash product is defined as

$$X \wedge Y \ :=\ (X \times Y)/(X \vee Y) \quad \text{where we use Exercise 96.8 to make the identification } X \vee Y := (X \times \{y_0\}) \times (\{x_0\} \times Y).$$

We equip $X \wedge Y$ with the base point given by $[X \vee Y]$.

In Exercise 96.8 we already showed that $S^m \wedge S^n$ is homeomorphic to S^{m+n}. For our purpose it is useful to have a canonical homeomorphism that we can all agree upon. This is given in the following lemma:

Lemma 308.49. (Smash-of-Spheres Lemma) Given $m, n \in \mathbb{N}$ there exists a canonical homeomorphism $S^m \wedge S^n \xrightarrow{\cong} S^{m+n}$.

Convention 308.50. We will use the canonical homeomorphism from the Smash-of-Spheres Lemma 308.49 to make the identification $S^m \wedge S^n =_{\mathrm{i}} S^{m+n}$.

Proof. We write $I = [0, 1]$. Given any $k \in \mathbb{N}_0$ we have the following identifications of pointed topological spaces:

$$(I^k/\partial I^k, [\partial I^k]) \ \underset{\uparrow}{\overset{\cong}{\to}}\ (\overline{B}^k/S^{k-1}, [S^{k-1}]) \ \underset{\uparrow}{\overset{\cong}{\to}}\ (S^k, *).$$

page 189 — homeomorphism from the Ball Quotient-Sphere Lemma 8.10

Let $m, n \in \mathbb{N}_0$. Using the above identification we obtain the following homeomorphism

$$S^m \wedge S^n \xrightarrow{\cong} I^m/\partial I^m \wedge I^n/\partial I^n \underset{[([x],[y])] \mapsto [(x,y)]}{\overset{\cong}{\downarrow}} (I^{m+n})/(I^m \times \partial I^n \cup \partial I^m \times I^n) = I^{m+n}/\partial I^{m+n} \xleftarrow{\cong} S^{m+n}.$$

homeomorphism by the Double-Quotient Lemma 8.7

We have thus given an explicit homeomorphism between $S^m \wedge S^n$ and S^{m+n}. ∎

The following lemma is basically just a mild reformulation of the $\mathrm{K}(\mathbb{Z}, n)$-Model Lemma 307.14.

Lemma 308.51. ($\mathbf{K}(\mathbb{Z}, n)$-Model Lemma) Let $n \in \mathbb{N}$. There exists a polarized Eilenberg-Maclane space $(Y_n, *, \varphi)$ of type $\mathrm{K}(\mathbb{Z}, n)$ with the following properties:

(1) The n-skeleton of Y_k equals S^n with the usual CW-structure defined on page 5889.
(2) The CW-complex Y_n has no cell of dimension $n+1$.
(3) The CW-complex Y_n has only countably many cells in dimensions $\geq n+2$.

Now we turn to the actual definition of the product structure.

Definition 308.52. Let $m, n \in \mathbb{N}$. In the following discussion all topological spaces are viewed as pointed objects.

(1) We consider the polarized Eilenberg-Maclane spaces Y_m and Y_n that we introduced in the $\mathrm{K}(\mathbb{Z}, n)$-Model Lemma 308.51. Recall that from the Polarized Eilenberg-Maclane Space Lemma 308.29 we obtain two canonical elements $\Xi_m \in \langle \mathrm{K}(\mathbb{Z}, m), Y_m \rangle_{\mathrm{inv}}$ and $\Xi_n \in \langle \mathrm{K}(\mathbb{Z}, n), Y_n \rangle_{\mathrm{inv}}$.

(2) Since Y_m and Y_n are CW-complexes with *countably* many cells it follows from the Countable CW-Complex Product Proposition 139.23 we can equip the product $Y_m \times Y_n$ of the topological spaces Y_m and Y_n with a natural CW-structure.

(3) We denote by $p\colon Y_m \times Y_n \to Y_m \wedge Y_n = (Y_m \times Y_n)/\sim$ the obvious projection.

(4) It follows from the explicit description of the CW-structure defined in the CW-Quotient Lemma 139.10 that $Y_m \wedge Y_n = (Y_m \times Y_n)/\sim$ has precisely one cell in dimension 0, no cells in dimension $1, \ldots, m+n-1$, one cell in dimension $m+n$ and no cell in dimension $m+n+1$. Furthermore it follows immediately from the definitions that the map
$$S^{m+n} \;=\; S^m \wedge S^n \;\to\; Y_m \wedge Y_n$$
defines a homeomorphism onto the $(m+n+1)$-skeleton of $Y_m \wedge Y_n$. In particular we obtain from the CW-Skeleton-π_n-Proposition 141.3 that Y_{m+n} is $(m+n-1)$-connected and we obtain that the map $\varphi\colon \mathbb{Z} = \pi_{m+n}(S^{m+n}) \to \pi_{m+n}(Y_m \wedge Y_n)$ is an isomorphism.

(5) Since $Y_m \times Y_n$ is an $(m+n-1)$-connected *CW-complex* we obtain from the Maps-to-$\mathrm{K}(\pi, n)$-Proposition 307.18 a unique class $\mu \in \langle Y_m \wedge Y_n, \mathrm{K}(\mathbb{Z}, m+n) \rangle$ that induces the preferred isomorphism $\varphi^{-1}\colon \pi_{m+n}(Y_m \wedge Y_n, *) \xrightarrow{\cong} \mathbb{Z}$.

After these preparations we consider the following diagram:

$$\begin{array}{ccccc}
\langle X, Y_m \rangle \times \langle X, Y_n \rangle & \xrightarrow{\chi} & \langle X, Y_m \times Y_n \rangle & \xrightarrow{p_*} & \langle X, Y_m \wedge Y_n \rangle \\
{\scriptstyle \Xi_{m*} \times \Xi_{n*}} \uparrow & & & & \searrow^{\mu_*} \\
\langle X, \mathrm{K}(\mathbb{Z}, m) \rangle \times \langle X, \mathrm{K}(\mathbb{Z}, n) \rangle & \overset{\text{definition of cup product } \cup}{- \to} & & & \langle X, \mathrm{K}(\mathbb{Z}, m+n) \rangle .
\end{array}$$

We refer to the bottom horizontal map, which we defined through the upper path of maps, as the cup product.

Theorem 308.53. (Cup Product-via-Maps Theorem) Let $X = (X, x_0)$ be a pointed CW-complex and let $m, n \in \mathbb{N}$. The cup product
$$\cup\colon \langle X, \mathrm{K}(\mathbb{Z}, m) \rangle \times \langle X, \mathrm{K}(\mathbb{Z}, n) \rangle \;\to\; \langle X, \mathrm{K}(\mathbb{Z}, m+n) \rangle$$
is bilinear. Furthermore the cup product is associative and graded commutative in the sense that for $\varphi \in \langle X, \mathrm{K}(\mathbb{Z}, m) \rangle$ and $\psi \in \langle X, \mathrm{K}(\mathbb{Z}, n) \rangle$ we have
$$\varphi \cup \psi \;=\; (-1)^{m \cdot n} \cdot \psi \cup \varphi .$$

Proof. We will not make use of this theorem, thus we will also not provide a proof. Instead we refer to [**Stro2011**, Theorem 21.61] for the proof. Note that [**Stro2011**, Theorem 21.61] does not explicitly say that X needs to be a CW-complex, but implicitly the discussion on [**Stro2011**, Chapter 21] seems to assume that the setting is a convenient category in the sense of Theorems 139.29 and 139.33. We will discuss this issue in more detail right after this proof. ■

The above definition of the cup product is in a way rather awkward, it is esthetically wrong to introduce the CW-complexes Y_m and Y_n. We conclude this section with a longish but hopefully quite interesting discussion which explains why we introduced the CW-complexes Y_m and Y_n and how we can get around introducing them.

Definition 308.54.

(1) Let A and B be CW-complexes. We denote by $A \otimes B$ the CW-complex that is given by equipping the set $A \times B$ with the CW-product topology from Definition 139.15 and by equipping this topological space with the CW-structure from the CW-Category–Product Proposition 139.16.

(2) Let (A, a_0) and (B, b_0) be pointed CW-complexes. We write

$$A \owedge B \coloneqq (A \otimes B)/((A \times \{b_0\}) \cup (\{a_0\} \times B)).$$

We equip $A \owedge B$ with the CW-structure given by the CW-structure on $A \otimes B$ and the quotient structure defined in the CW-Quotient Lemma 139.10.

(3) We denote by $\eta\colon A \otimes B \to A \times B$ the identity map. Recall that in the CW-Complex Product Proposition 139.18 (1) we showed that η is continuous.

Now let $X = (X, x_0)$ be a pointed topological space and let $m, n \in \mathbb{N}$. We introduce the following notation:

(1) Let $r \in \mathbb{N}$. We write $K_r = \mathrm{K}(\mathbb{Z}, r)$. We consider the polarized Eilenberg-Maclane spaces Y_r that we introduced in the $\mathrm{K}(\mathbb{Z}, n)$-Model Lemma 308.51 and we denote by $\Xi_r \in \langle K_r, Y_r\rangle_{\mathrm{inv}}$ the canonical element given by the Polarized Eilenberg-Maclane Space Lemma 308.29.

(2) Given topological spaces A and B we denote by $p\colon A \times B \to A \wedge B$ the natural projection. Similarly, given CW-complexes A and B we denote by $p\colon A \otimes B \to A \owedge B$ the natural projection.

We consider the following diagram:

$$\begin{array}{ccccccc}
\langle X, Y_m\rangle \times \langle X, Y_n\rangle & \xrightarrow{\times} & \langle X, Y_m \times Y_n\rangle & \xrightarrow{p_*} & \langle X, Y_m \wedge Y_n\rangle & & \\
\uparrow{\scriptstyle \Xi_{m*}\times\Xi_{n*}} & & & & & \searrow^{\mu_*} & \\
\langle X, K_m\rangle \times \langle X, K_n\rangle & \xrightarrow[(f,g)\mapsto f\times g]{\times} & \langle X, K_m \times K_n\rangle & \xrightarrow{p_*} & \langle X, K_m \wedge K_n\rangle & \xrightarrow{??} & \langle X, K_{m+n}\rangle. \\
\downarrow{\scriptstyle (f,g)\mapsto f\widetilde{\times}g} & & \uparrow{\scriptstyle \eta_*} & & & \nearrow_{\nu_*} & \\
\langle X, K_m \mathbin{\widetilde{\times}} K_n\rangle & \xrightarrow[\Theta_*]{} & \langle X, K_m \otimes K_n\rangle & \xrightarrow{p_*} & \langle X, K_m \owedge K_n\rangle & &
\end{array}$$

We make the following observations:

(1) Since K_m and K_n have uncountably many cells the product $K_m \times K_n$ is not a CW-complex.

(2) It follows from (1) that $K_m \wedge K_n$ has no obvious CW-structure. Thus there is no clear path to constructing a map $K_m \wedge K_n \to K_{m+n}$.[3427]

(3) It follows from (2) that the "direttissima" approach to defining the cup product by going straight from left to right runs into a major obstacle.

[3427] In fact it is also not immediately clear whether $K_m \wedge K_n$ is say $(m + n - 1)$-connected.

(4) Since Y_m and Y_n have countably many cells we know, by the Countable CW-Complex Product Proposition 139.23 together with the CW-Quotient Lemma 139.10 that $Y_m \times Y_n$ and $Y_m \wedge Y_n$ have natural CW-structures. As discussed above, we obtain from the Maps-to-K(π, n)-Proposition 307.18 a canonical class $\mu \in \langle Y_m \wedge Y_n, \mathrm{K}(\mathbb{Z}, m+n)\rangle$.

(5) One way of getting around (3) is to use (4), i.e. we take the "upper route" in the diagram. This is how in the above we defined the cup product.

(6) As for $Y_m \wedge Y_n$ one sees that $K_m \circledcirc K_n$ is $(m+n-1)$-connected and that we have a canonical map $\pi_{m+n}(K_m \circledcirc K_n) \to \mathbb{Z}$. Since $K_m \circledcirc K_n$ is a CW-complex we obtain from the Maps-to-K(π, n)-Proposition 307.18 a canonical element $\nu \in \langle K_m \circledcirc K_n, K_{m+n}\rangle$.

(7) Unfortunately the issue in (1) also means that the map $\eta\colon K_m \otimes K_n \to K_m \times K_n$ is not a homeomorphism. Thus there is no obvious way how we can take the lower route in the diagram.

Now we switch to the category of CGWH-spaces that we introduced in Definition 139.32. In particular we now assume that X is a CGWH-space. Recall that by the CGWH-Category Theorem 139.33 (2) we know that every CW-complex is a CGWH-space.

(8) Given two CGWH-spaces A and B we know by the CGWH-Category Theorem 139.33 that there exists a product *in the category of* CGWH-*spaces*. We denote this product by $A \mathbin{\widetilde{\times}} B$ and we denote by $p\colon A \mathbin{\widetilde{\times}} B \to A$ and $q\colon A \mathbin{\widetilde{\times}} B \to B$ the two projections maps.

(9) Let $f\colon X \to K_m$ and $g\colon X \to K_n$ be continuous maps. Since X, K_m and K_n are CGWH-spaces we know, by definition of the product in a category, that there exists a unique map $f \mathbin{\widetilde{\times}} g\colon X \to K_m \mathbin{\widetilde{\times}} K_n$ such that $f = p \circ (f \mathbin{\widetilde{\times}} g)$ and $g = q \circ (f \mathbin{\widetilde{\times}} g)$.

(10) Since K_m and K_n are CW-complexes we know by the CGWH-Category Theorem 139.33 (5) that the identity map $\Theta\colon K_m \mathbin{\widetilde{\times}} K_n \to K_m \otimes K_n$ is actually a homeomorphism.[3428]

(11) It follows from (9) and (10) that we have a "lower route" from $\langle X, K_m\rangle \times \langle X, K_n\rangle$ to $\langle X, K_{m+n}\rangle$. This gives us an approach to defining the cup product without introducing the slightly artificial Y_m and Y_n.

(12) It follows easily from the definitions that in the diagram the square to the lower left commutes. One can now see reasonably easily that the two definitions of the cup product agree.

The second approach to defining the cup product is much more natural than the initial approach. In particular, using a similar circle of ideas one can define a cup product

$$\langle X, \mathrm{K}(R,m)\rangle \times \langle X, \mathrm{K}(R,n)\rangle \to \langle X, \mathrm{K}(R, m+n)\rangle$$

for any commutative ring R. We refer to [**Stro2011**, Chapter 21.4] for details.

308.8. Exercises for Chapter 308.

Exercise 308.1. We consider the following diagram:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\Xi\times\Xi} & (\Omega\mathbb{C}\mathrm{P}^\infty, *) \times (\Omega\mathbb{C}\mathrm{P}^\infty, *) \\ \downarrow{\scriptstyle (z,w)\mapsto z\cdot w} & & \downarrow{\scriptstyle \mu} \\ S^1 & \xrightarrow{\Xi} & \Omega(\mathbb{C}\mathrm{P}^\infty, *). \end{array}$$

[3428] For our purpose we only need the weaker result that $\Theta\colon K_m \mathbin{\widetilde{\times}} K_n \to K_m \otimes K_n$ is continuous.

The map $\Xi\colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ is defined in the Loop Space-of-$\mathbb{C}\mathrm{P}^\infty$-Lemma 308.31 and the map μ from Definition 308.18. Show that the diagram commutes up to homotopy.

Exercise 308.2. We equip S^1, S^2 and $S^1 \vee S^2$ with the usual base points which in each case we call $*$. Show that the map $\langle (S^2, *), (S^1 \vee S^2, *)\rangle \to [S^2, S^1 \vee S^2]$ is not an injection.

Exercise 308.3. Let (X, x_0) and (K, k_0) be pointed topological spaces. Show that the multiplication on $\langle (X, x_0), (K, k_0)\rangle$ defined in the Maps-to-Loop Space–Group Lemma 308.34 is actually associative.

Exercise 308.4. Let (X, x_0) and $(K, *)$ be pointed topological spaces. We consider the double loop space $\Omega^2(K, *) = \Omega(\Omega(K, *))$. By the construction in Definition 308.32 we have a binary operation

$$\circ\colon \langle (X, x_0), \Omega(\Omega(K, *))\rangle \times \langle (X, x_0), \Omega(\Omega(K, *))\rangle \to \langle (X, x_0), \Omega(\Omega(K, *))\rangle.$$

In the Maps-to-K(G, n)–Group Proposition 308.33 we showed in particular that this binary operation is unital. Provide another binary unital operation "$\otimes$" on $\langle (X, x_0), \Omega(\Omega(K, *))\rangle$ such that the conditions of the Eckmann-Hilton Theorem 299.32 are satisfied.

Remark. Once we have shown that we can apply the Eckmann-Hilton Theorem 299.32 we see that "$\circ$" defines a group structure on $\langle (X, x_0), \Omega^2(K, *)\rangle$ and we obtain that this group is abelian. Thus this provides an alternative approach to proving the Maps-to-K(G, n)–Abelian Proposition 308.37 and to proving some of the statements of the Maps-to-K(G, n)–Group Proposition 308.33.

Remark. Coming up with an idea for defining "$\otimes$" is not difficult. Providing a complete proof that Condition (b) of the Eckmann-Hilton Theorem 299.32 is satisfied might be more painful. Especially once one starts wondering why certain maps are actually continuous.

Exercise 308.5. Let $n \in \mathbb{N}$ and let G be an abelian group. Let $k \in \mathbb{N}$. We denote by $*$ the usual base point of S^k. Compute the groups $\langle (S^k, *), \mathrm{K}(G, n)\rangle$.

Remark. Once one unravels the definitions one can fairly easily write down *bijections* to familiar groups. The only catch is that one really needs to show that these bijections are in fact group isomorphisms. The Eckmann-Hilton Theorem 299.32 might once again ride to the rescue.

Exercise 308.6. Let (X, x_0) be a pointed topological space. We equip S^1 with the base point $* = (0, 1)$. Show that the smash product $X \wedge S^1$ is homeomorphic to the reduced suspension $S(X, x_0)$.

Remark. Personally I like the Double-Quotient Lemma 8.7.

Exercise 308.7. To simplify the notation, given a pointed topological space (Y, y_0) we now write $SY = S(Y, y_0)$ and $\Omega Y = \Omega(Y, y_0)$.

(a) Let (Y, y_0) be a pointed topological space. Show that the set $\pi_0(\Omega Y)$ has a natural group structure.

(b) Let (X, x_0) be a pointed topological space.
   (1) Show that there exists a natural isomorphism from $\pi_0(\Omega(SX))$ to the free group on the set $\pi_0(X)$.
   (2) Let (X, x_0) be a pointed topological space. Show that there exists a natural isomorphism from $\pi_0(\Omega^2(S^2X))$ to the free *abelian* group on the set $\pi_0(X)$.
   (3) How does the sequence continue?

Part XXXV

Non-commutative algebra

Our next goal in these lecture notes will be to introduce (co-) homology with twisted coefficients and to study Reidemeister torsion. In both settings we start out with a topological space that is equipped with extra data, namely either a module over the group ring of the fundamental group, or a local coefficient system.

To prepare for these parts of the lecture notes we need to provide several chapters on (homological) algebra over rings. Later we will apply the concepts and results to the case where the ring is the group ring $\mathbb{Z}[\pi_1(X, x_0)]$ of the fundamental group of a based topological space (X, x_0). For many readers it might be more fun to start with the next Part XXXVI, and to return to this part of the lecture notes when the need arises.

This part of the lecture notes is organized as follows:

- In Chapter 309 we will state (and potentially recall) some definitions and results about modules over (possibly non-commutative) rings.
- In Chapter 310 we will discuss group rings and their modules.
- In Chapter 311 we will (re-) introduce the tensor product of abelian groups and we will introduce the tensor product of two rings.
- In Chapter 312 we will discuss the tensor product of modules over a rings.
- In Chapter 313 we will generalize some of the earlier results about chain complexes of abelian groups to chain complexes over rings.
- In Chapter 314 we will study the effect of tensoring a chain complex with a module.
- In Chapter 315 we will study the effect of taking the dual of a chain complex.
- In Chapter 316 we will discuss the universal coefficient spectral sequences for homology and cohomology.
- In Chapter 317 we will give a purely algebraic definition of the (co-) homology modules of a group.
- In Chapter 318 we will return to topology and we will see that topology can be used to calculate the group (co-) homology and topology of many interesting groups.

CHAPTER 309

Modules over rings

In the previous chapters we mostly worked with abelian groups or vector spaces. In future chapters we will also work with modules over possibly non-commutative rings. To prepare the reader for these future chapters we will summarize the (non-commutative) algebra involved in this and following chapters.

This chapter is organized as follows:

- In Section 309.1 we will recall the notion of a ring.
- In Section 309.2 we will introduce left and right-modules over rings.
- In Section 309.3 we will introduce products and direct sums of modules.
- In Section 309.4 we will introduce rings with involution.
- In Section 309.5 we will introduce free modules.
- In Section 309.6 we will introduce projective modules.
- In Section 309.7 we will discuss splittings of short exact sequences.
- In Section 309.8 we will study modules over skew fields and over PIDs.
- In Section 309.9 we will introduce bimodules.
- In Section 309.10 we will provide exercises for Chapter 309.

In the next Chapter 310 we will (re-) introduce group rings and we will discuss modules over group rings.

309.1. **Rings.** We start out with recalling the following concepts from Definition 2.6.

Definition 309.1.

(1) A ring[3429] is a triple $(R, +\colon R \times R \to R,\ \cdot\ \colon R \times R \to R)$ where $(R, +)$ is an abelian group and such that the following conditions are satisfied:
   (a) the multiplication "$\cdot$" is associative,
   (b) the multiplication "$\cdot$" is distributive,
   (c) there exists a multiplicatively neutral element $1 = 1_R$.

(2) A ring R is called commutative, if given any $r, s \in R$ we have $r \cdot s = s \cdot r$.

(3) A ring homomorphism between two rings $(R, +, \cdot)$ and $(S, +, \cdot)$ is a map $R \to S$ that preserves the additive and multiplicative structures and that sends 1_R to 1_S.

As in Definition 5.4 this leads us to the category $\mathcal{R}ing$ of rings.

Example.

(1) We refer to $R = \{0\}$ as the zero ring.
(2) We have of course the "standard" commutative rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

[3429] There is no generally accepted notion of what a "ring" is supposed to be, so it is advisable to always look up what an author means by a "ring". For example some authors add the condition that $0 \neq 1$, which we do not do.

(3) Let R be a commutative ring. We consider the polynomial ring $R[t_1, \ldots, t_m]$ and the Laurent polynomial ring $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.

(4) Let R be a ring and let $n \in \mathbb{N}_0$. Clearly the matrices $\mathrm{M}(n \times n, R)$, with the usual addition and multiplication of square matrices form a ring. Furthermore, if $\varphi\colon R \to S$ is a ring homomorphism, then the induced map

$$\begin{array}{ccc} \mathrm{M}(n \times n, R) & \to & \mathrm{M}(n \times n, S) \\ (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

is clearly also a ring homomorphism.

(5) By the Quaternion Lemma 171.3 we know that the quaternions $\mathbb{H} = (\mathbb{R}^4, +, \cdot)$ from Definition 171.2 are a non-commutative ring.

(6) Let $\{R_i\}_{i \in I}$ be a family of rings. The direct product $\prod_{i \in I} R_i$ has a natural ring structure that is given by "factorwise" addition and multiplication. One can easily verify that this is the categorical product, in the sense of Definition 6.1, in the category $\mathcal{R}ing$ of rings.

(7) Let X be a topological space. As in the Notation 10.19 we consider the set $C(X, \mathbb{R})$ of continuous maps from X to $\mathbb{R}$. Together with pointwise addition and multiplication the set $C(X, \mathbb{R})$ clearly forms a commutative ring.[3430] □

309.2. **Modules over rings.** Next we introduce the notion of a module. Since we do not assume that our rings are commutative we have to distinguish between R-left modules and R-right modules:

Definition 309.2. Let R be a ring.

(1) An R-left module is an abelian group A together with a map

$$\begin{array}{ccc} R \times A & \to & A \\ (r, a) & \mapsto & r \cdot a \end{array}$$

such that the following conditions are satisfied:

(a)	$r \cdot (a + b)$	$=$	$r \cdot a + r \cdot b$	for all $r \in R$ and $a, b \in A$,
(b)	$1 \cdot a$	$=$	a	for all $a \in A$,
(c)	$(r + s) \cdot a$	$=$	$r \cdot a + s \cdot a$	for all $r, s \in R$ and $a \in A$,
(d)	$s \cdot (r \cdot a)$	$=$	$(s \cdot r) \cdot a$	for all $r, s \in R$ and $a \in A$.

(2) An R-right module is an abelian group A together with a map

$$\begin{array}{ccc} R \times A & \to & A \\ (r, a) & \mapsto & a \cdot r \end{array}$$

such that the following conditions are satisfied:

(a)	$(a + b) \cdot r$	$=$	$a \cdot r + b \cdot r$	for all $r \in R$ and $a, b \in A$,
(b)	$a \cdot 1$	$=$	a	for all $a \in A$,
(c)	$a \cdot (r + s)$	$=$	$a \cdot r + a \cdot s$	for all $r, s \in R$ and $a \in A$,
(d)	$(a \cdot r) \cdot s$	$=$	$a \cdot (r \cdot s)$	for all $r, s \in R$ and $a \in A$.

[3430] In fact this is a subring of the direct product $\prod_{x \in X} \mathbb{R}$.

In both settings we have an obvious notion of a homomorphism, which we surely do not have to spell out.

Remark.

(1) If one is not used to dealing with R-left modules and R right-modules, then it is perhaps difficult to spot the essential difference. The fact that we write multiplication by R on the left respectively right is not essential, it is just a mnemonic device. What matter is the difference given by Condition (d). To make the difference clear we write it out in words:

Left R-module: multipl. by r followed by multipl. by s equals multipl. by $s \cdot r$.
Right R-module: multipl. by r followed by multipl. by s equals multipl. by $r \cdot s$.

In practice the notation is set up in such a way that it should be clear what is allowed and what is not.

(2) If R is a commutative ring, then any R-module in the usual sense satisfies both conditions (1d) and (2d). In other words, if R is a commutative ring, then any R-module is automatically an R-left module and also an R right-module. □

Definition 309.2 leads us to the following notation, which extends Notation 159.17:

Notation 309.3. Let R be a ring. We denote by $\mathcal{M}od_{R\text{-left}}$ the category of R-left modules and we denote by $\mathcal{M}od_{R\text{-right}}$ the category of R-right modules. If R is commutative, then we denote by $\mathcal{M}od_R$ the category of R-modules.

In this part of the lecture notes we will eventually encounter a large number of left and right modules. As of right now we only give a few simple examples.

Example.

(1) Let A be an abelian group. Given $n \in \mathbb{N}$ and $a \in A$ we define, as on page 116:

$$n \cdot a := \underbrace{a + \cdots + a}_{n \text{ times}} \quad \text{and} \quad -n \cdot a := \underbrace{-a - \cdots - a}_{n \text{ times}}.$$

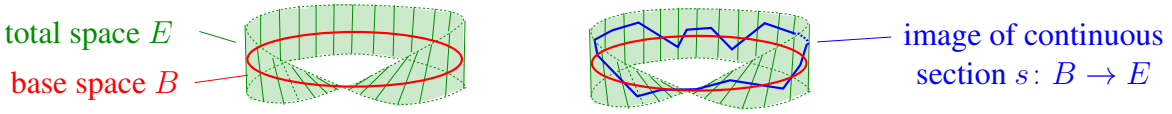
This defines a natural $\mathbb{Z}$-left module and a natural $\mathbb{Z}$-right module structure on A. Clearly this defines an isomorphism (in the sense of Definition 5.28) between the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups and the category ${}_{\mathbb{Z}}\mathcal{M}od$ of $\mathbb{Z}$-left modules (which equals the category of $\mathbb{Z}$-right modules.) Thus in the following we can, and will, view any abelian group as a $\mathbb{Z}$-left module and a $\mathbb{Z}$-right module.

(2) Let $p\colon E \to B$ be a real vector bundle over a topological space B.

(a) As in Definition 265.18 we define a section of $p\colon E \to B$ to be a map $s\colon B \to E$ such that $p \circ s = \mathrm{id}_B$. We set

$$\Gamma(p\colon E \to B) := \text{ set of continuous sections of } p\colon E \to B.$$

Similar to the Vector Bundle-Section Lemma 265.19 we see that $\Gamma(p\colon E \to B)$ is a module over the ring $C(B, \mathbb{R})$ "in the obvious way".



(b) If $q\colon F \to B$ is a second real vector bundle over the base space B, then any morphism $\Phi\colon E \to F$ of vector bundles over B (in the sense of Definition 265.2) clearly induces an isomorphism $\Gamma(p\colon E \to B) \to \Gamma(q\colon F \to B)$ of $C(B,\mathbb{R})$-modules.

This example of a module sounds a little weird, but we will see that these modules have some interesting features. □

Before we give our next, more standard example, let us introduce the following notation:

Notation 309.4. Let R be a ring and let $k,l \in \mathbb{N}_0$. Sometimes we use the short hand

$$R^{k\times l} := R^{\{1,\dots,k\}\times\{1,\dots,l\}} = \mathrm{M}(k\times l, R).$$

For $k=1$ elements of $R^{1\times l}$ are sometimes called row vectors and for $l=1$ elements of $R^{k\times 1}$ are sometimes called column vectors. Evidently we make the identification $R^{1\times 1} =_{\mathrm{i}} R$.

Example. Let R be a ring and let $k,l,m \in \mathbb{N}_0$. Recall that we have a matrix multiplication

$$R^{k\times l} \times R^{l\times m} \to R^{k\times m}.$$

This leads to several module structures:

(1) For $l=k$ and $m=1$ the map

$$\begin{aligned} \mathrm{M}(k\times k, R)\times R^{k\times 1} &\to R^{k\times 1} \\ (A,v) &\mapsto A\cdot v \end{aligned}$$

defines an $\mathrm{M}(k\times k,R)$-left module structure on the *column* vectors $R^{k\times 1}$. Furthermore for $k=1$ and $l=m$ the map

$$\begin{aligned} R^{1\times l}\times \mathrm{M}(l\times l, R) &\to R^{1\times l} \\ (A,v) &\mapsto v\cdot A \end{aligned}$$

defines an $\mathrm{M}(l\times l,R)$-right module structure on the *row* vectors $R^{1\times l}$.

(2) For $k=l=1$ the map

$$\begin{aligned} R^{1\times k}\times R &\to R^{1\times k} \\ (v,r) &\mapsto r\cdot v \end{aligned}$$

defines an R-left module structure on the *row* vectors $R^{k\times 1}$. Furthermore for $k=m=1$ the map

$$\begin{aligned} R\times R^{k\times 1} &\to R^{k\times 1} \\ (r,v) &\mapsto v\cdot r \end{aligned}$$

defines an R-right module structure on the *column* vectors $R^{1\times k}$.

The following, slightly confusing lemma, gives us all R-left module homomorphisms between the R-left modules $R^{1\times k}$ and $R^{1\times l}$:

Lemma 309.5. (Left Module Homomorphism–Matrix Lemma) Let R be a ring.

(1) Let $k,l \in \mathbb{N}_0$. We view $R^{1\times k}$ and $R^{1\times l}$ as *R-left* modules in the above way. If A is a $(k\times l)$–matrix over R, then the map

$$\begin{aligned} R^{1\times k} &\to R^{1\times l} \\ v &\mapsto v\cdot A \end{aligned}$$

is a homomorphism of *R-left* modules.

(2) Given any homomorphism $\varphi\colon R^{1\times k} \to R^{1\times l}$ of R-left modules there exists a $(k\times l)$–matrix over R such that φ is given by *right* multiplication by A.

Proof. Both statements are essentially clear. The reader who is unfamiliar with dealing with non-commutative rings should think for a second about the question, why left-multiplication by A is in general *not* a homomorphism of R-left modules. ■

In the remainder of the section we will show that left modules and right modules are two sides of (essentially) the same coin. To make sense of this we introduce the following notion:

Definition 309.6. Let R be a ring. We define the opposite ring R^{op} to be the same abelian group but where the multiplication is given by

$$r \underset{R^{\text{op}}}{\cdot} s := s \underset{R}{\cdot} r.$$

Evidently we have the equality $(R^{\text{op}})^{\text{op}} = R$ of rings. Furthermore note that $R \to R^{\text{op}}$ defines in an obvious way an isomorphism from the category $\mathcal{Ring}$ of rings to itself.

The following lemma now shows in particular that all results about left modules also apply to right modules.

Lemma 309.7. (Opposite Ring–Module Lemma) Let R be a ring.

(1) Every R-left module has a natural R^{op}-right module structure.
(2) We have an obvious isomorphism from the category $\mathcal{Mod}_{R\text{-left}}$ of R-left modules to the category $\mathcal{Mod}_{R^{\text{op}}\text{-left}}$ of R^{op}-right modules.

The same statement hold with the roles of left and right swapped.

Proof. Clearly this lemma follows immediately from the definitions. ■

309.3. **Products and direct sums of modules.** In this section we will provide the obvious construction of the product and the direct sum of a family of R-left modules. More precisely we have the following definition, which is very very similar to Definition 1.10:

Definition 309.8. Let R be a ring and let $\{X_i\}_{i\in I}$ be a family of R-left modules. As in Definition 1.10 we define the product of the X_i to be the R-left module

$$\prod_{i\in I} X_i := \left\{ f : I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\},$$

where the R-left module structure is given by addition and multiplication on each X_i.[3431] [3432]

The following lemma is the analog of the Product-of-Sets-and-Groups Lemma 6.4:

Lemma 309.9. (Product-of-Modules Lemma) Let R be a ring and let $\{X_i\}_{i\in I}$ be a family of R-left modules. The product, in the sense of Definition 6.1, in the category $\mathcal{Mod}_{R\text{-left}}$ of R-left modules is given by the R-left module

$$\prod_{i\in I} X_i := \left\{ f : I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\},$$

together with the obvious R-left module homomorphisms from $\prod_{i\in I} X_i$ to the X_j that are given by $f \mapsto f(j)$.

[3431] For example if we are given $f : I \to \bigcup_{i\in I} X_i$ as above, and if we are given an $r \in R$, then we define $r \cdot f$ via $(r \cdot f)(i) = r \cdot f(i)$.

[3432] Let $\{X_i\}_{i\in\varnothing}$ be the empty family of R-left modules. It follows from the discussion on page 98 that the product contains precisely one element, so the product is the trivial R-left module.

Proof. As in the case of the Product-of-Sets-and-Groups Lemma 6.4 the lemma follows easily from the definitions. ■

We continue with the following definition which is rather similar to Notation 6.9 and Definition 97.1.

Definition 309.10. Let R be a ring and let $\{X_i\}_{i\in I}$ be a family of R-left modules.

(1) We define the direct sum of the X_i to be the R-left module
$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \text{all but finitely many } x_i \text{ are trivial} \right\}.$$

(2) Given $i \in I$ we refer to
$$\begin{array}{rcl} \iota_t\colon X_i & \to & \bigoplus_{i\in I} X_i \\ x & \mapsto & \left(\begin{array}{lcl} I & \to & \bigcup_{i\in I} X_i \\ j & \mapsto & \begin{cases} x, & \text{if } j = i, \\ 0_{X_j}, & \text{if } j \neq i. \end{cases} \end{array} \right) \end{array}$$
as the natural homomorphism.

The following is the obvious analog of the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 and the Coproduct-of-Abelian Groups Lemma 97.2:

Lemma 309.11. (Coproduct-of-Modules Lemma) Let R be a ring and let $\{X_i\}_{i\in I}$ be a family of R-left modules.

(1) Let Z be an R-left module and let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of homomorphisms. The map
$$\begin{array}{rcl} \bigoplus_{i\in I} f_i\colon \bigoplus_{i\in I} X_i & \to & Z \\ (x_i)_{i\in I} & \mapsto & \sum_{\substack{i\in I \text{ with} \\ x_i \neq 0}} f_i(x_i) \end{array}$$
is the unique homomorphism such that for each $i \in I$ we have $f_i = \left(\bigoplus_{i\in I} f_i\right) \circ \circ \iota_i$.

(2) The coproduct, in the sense of Definition 6.5, in the category $\mathcal{M}\mathit{od}_{R\text{-left}}$ of R-left modules $\{X_i\}_{i\in I}$ is given by the R-left module $\bigoplus_{i\in I} X_i$ together with the above natural homomorphisms $\iota_j\colon X_j \to \bigoplus_{i\in I} X_i$.

Proof. The proof of the lemma is essentially the same as the proof of the Coproduct-of-Sets-and-Abelian Groups Lemma 6.8 (2) and of the Coproduct-of-Abelian Groups Lemma 97.2. But of course one can also prove it easily by hand. ■

Convention 309.12. Let R be a ring.

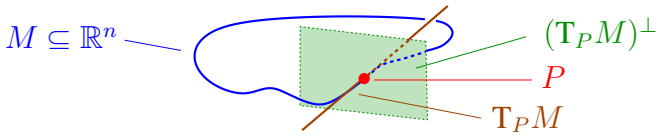
(1) Let $\{X_i\}_{i\in I}$ be a family of R-left modules. For $t \in I$ we use the natural monomorphism $\iota_t\colon X_t \to \bigoplus_{i\in I} X_i$ from the Coproduct-of-Modules Lemma 309.11 to view X_t as a submodule of the direct sum $\bigoplus_{i\in I} X_i$.

(2) Let Y be an R-left module and let $\{X_i\}_{i\in I}$ be a family of R-left submodules. The inclusions $X_i \to Y$ define a natural homomorphism

$$\bigoplus_{i\in I} X_i \;\to\; Y.$$

If this natural homomorphism is an isomorphism, then we refer to Y also as the direct sum of the X_i.

Example. Let $p\colon E \to B$ and $q\colon F \to B$ be two real vector bundles over a topological space B. We can form the direct sum $E \oplus F \to B$ of real vector bundles as in Definition 268.26.



by the discussion on page 5170 that $\mathrm{T}M \oplus (\mathrm{T}M)^\perp = M \times \mathbb{R}^n$

It follows easily from the definitions that we have a natural isomorphism

$$\Gamma(p\colon E \to B) \oplus \Gamma(p\colon F \to B) \;\xrightarrow{\cong}\; \Gamma(p\colon E \oplus F \to B)$$

of $C(B,\mathbb{R})$-modules. □

We continue with the following elementary lemma:

Lemma 309.13. (Module–Change-of-Ring–Functor Lemma) Let $\varphi\colon R \to S$ be a ring homomorphism.

(1) Let M be an S-left module. We denote by $\varphi^*M = {}_\varphi M$ the R-left module that is given by the map

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto \varphi(r) \cdot M. \end{aligned}$$

(2) The maps

$$\begin{aligned} M &\mapsto \varphi^* M \\ (f\colon A \to B) &\mapsto (f\colon {}_\varphi A \to {}_\varphi B) \end{aligned}$$

define a covariant functor from the category of the category $\mathcal{M}od_{S\text{-left}}$ of S-left modules to the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules. This functor preserves products and coproducts.

Similarly, for an S-right module M we obtain an R-right module M_φ.

Proof. The lemma follows easily from the definitions. ■

309.4. **Rings with involution.** In this short section we will introduce involutions on rings. As we will see, involutions allow us to turn left modules into right modules and vice versa. We start out with the following definition which we first encountered in Exercise 171.5:

Definition 309.14.

(1) Let R be a ring. An involution on R is a map $R \to R$, usually written $z \mapsto \overline{z}$, with the following properties:
   (a) $\overline{z+w} = \overline{z} + \overline{w}$ for all $z, w \in R$,
   (b) $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$ for all $z, w \in R$,
   (c) $\overline{\overline{z}} = z$ for all $z \in R$.

(2) A homomorphism between rings with involution is a ring homomorphism that respects the involutions.

Example.

(1) Every *commutative* ring admits of course the trivial involution that is given by the identity map.
(2) Unless we say something else we will view the complex numbers $\mathbb{C}$ as a ring with the usual involution $\overline{x+\mathrm{i}y} \coloneqq x-\mathrm{i}y$.
(3) We consider again the quaternions $\mathbb{H} = \mathbb{R}\cdot 1+\mathbb{R}\cdot \mathrm{i}+\mathbb{R}\cdot \mathrm{j}\oplus\mathbb{R}\cdot \mathrm{k}$ from Definition 171.2. It follows from the Quaternion Lemma 171.3 (3) that the quaternions form a non-commutative ring with an involution that is given by $\overline{a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k}} = a-b\mathrm{i}-c\mathrm{j}-d\mathrm{k}$.
(4) Let R be a ring (possibly with involution). We will view the ring of Laurent polynomials $R[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ as a ring that is equipped with the involution that is given by the involution on R and by $\overline{t_i} \coloneqq t_i^{-1}$.
(5) Let R be a commutative ring and let $n\in\mathbb{N}_0$. One can easily verify that the map
$$\begin{array}{rcl} \mathrm{M}(n\times n,R) & \to & \mathrm{M}(n\times n,R) \\ A & \mapsto & A^T \end{array}$$
is an involution on the ring $\mathrm{M}(n\times n,R)$. □

The following lemma gives us one application of involutions:

Lemma 309.15. (Opposite Ring–Involution Lemma) Let R be a ring that is equipped with an involution. Then the map
$$\begin{array}{rcl} \varphi\colon R & \to & R^{\mathrm{op}} \\ r & \mapsto & \overline{r} \end{array}$$
is an isomorphism of rings.

Proof. Of course the lemma follows easily from the definitions: First of all note that it follows from Definition 309.6 (c) that the map φ is a bijection. Next note that it follows from Definition 309.6 (a) that φ preserves addition. Finally let us have a quick look at multiplication: Let $r,s\in R$. Then we see that
$$\varphi(r\cdot_R s) \;=\; \overline{r\cdot_R s} \;\underset{\uparrow}{=}\; \overline{s}\cdot_R \overline{r} \;\underset{\uparrow}{=}\; \overline{r}\cdot_{R^{\mathrm{op}}}\overline{s} \;=\; \varphi(r)\cdot_{R^{\mathrm{op}}}\varphi(s).$$
Definition 309.14 Definition 309.6 (c)
■

The following lemma gives us one of the main uses of involutions: they can be used to turn left modules into right modules, and vice versa:

Lemma 309.16. (Ring-with-Involution–Module Lemma) Let R be a ring that is equipped with an involution.

(1) Let M be an R-left module. We consider the same abelian group M but this time with the right-multiplication that is given by $m\cdot r \coloneqq \overline{r}\cdot_M m$. This turns M into an R-right module which we denote by $\overline{M}$ (or sometimes more carefully by $\overline{M}^R$, if we have more ring structures to worry about). Thus for any $m\in\overline{M}$ and $r\in R$ we have $m\cdot_{\overline{M}} r \coloneqq \overline{r}\cdot_M m$.
(2) The maps
$$\begin{array}{rcl} M & \mapsto & \overline{M} \\ (f\colon M\to N) & \mapsto & (f\colon \overline{M}\to\overline{N}) \end{array}$$
define an isomorphism of categories from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_R$ of R-right modules.

Evidently the same argument holds with the roles of "left" and "right" flipped.

Proof. The lemma follows almost immediately from the Opposite Ring–Involution Lemma 309.15 together with the Opposite Ring–Module Lemma 309.7. Alternatively one can also say that all statements follow easily from the definitions. ■

309.5. **Free modules.** The following definition is modelled on the definition of a free abelian group that we gave in Definition 97.5.

Definition 309.17. Let R be a ring.

(1) Let M be an R-left module. We say that M is a free R-left module on a set $X \subseteq M$ if given any map $f\colon X \to N$ to an R-left module N there exists a unique homomorphism $\Xi_X\colon M \to N$ of R-left modules that makes the following diagram commute:

$$\begin{array}{ccc} X & \hookrightarrow & M \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\,\Xi_X} \\ & & N. \end{array}$$

(2) We say that an R-left module M is free if there exists a subset $X \subseteq M$ such that M is a free R-left module on the set X. We refer to X also as a basis for M.

In a totally analogous fashion we also introduce the corresponding notions for R-right modules.

Example.

(1) Let $p\colon E \to B$ be an n-dimensional real vector bundle over a topological space B. If p is trivializable in the sense of Definition 265.5, then it follows easily from the definitions and the discussion on page 5962 that $\Gamma(p\colon E \to B)$ is a free $C(B,\mathbb{R})$-module that admits a basis with n elements. In fact in Exercise 309.13 we will see that the converse to this statement holds: If $\Gamma(p\colon E \to B)$ is a free $C(B,\mathbb{R})$-module, then E is a trivializable vector bundle.

(2) Let R be a ring, let $n \in \mathbb{N}_0$ and let $i \in \{1,\dots,m\}$. Following Definition 2.10 we refer to

$$\mathfrak{e}_i \;:=\; (0,\dots,0,\underset{\substack{\uparrow\\ \text{the } i\text{-th entry}}}{1},0,\dots,0) \;\in\; R^{1\times m}$$

as the i-th standard basis vector. Clearly $\mathfrak{e}_1,\dots,\mathfrak{e}_m$ forms a basis of the R-left module $R^{1\times m}$. □

As always, it is easy to prove a suitable uniqueness of an object that is described by a universal property. More precisely, in our setting we have the following generalization of the Free Abelian Group–Uniqueness Lemma 97.6:

Lemma 309.18. (Free Module–Uniqueness Lemma) Let R be a ring.

(1) Let M be an R-left module that is free on a subset $X \subseteq M$ and let N be an R-left module that is free on a subset $Y \subseteq N$. Given any bijection $f\colon X \to Y$ there exists a unique isomorphism $\psi\colon M \to N$ of R-left modules such that $\psi(x) = f(x)$ for all $x \in X$.

(2) Let N be an R-left module that is free on a subset $Y \subseteq N$. Then the map
$$\Xi_Y \colon R^{(Y)} \to N$$
from Definition 309.17 is an isomorphism of R-left modules.

Proof.

(1) The proof of this statement is essentially identical to the proof of the Free Abelian Group–Uniqueness Lemma 97.6.

(2) This statement is an immediate consequence of the Free Module–Existence Lemma 309.20 together with Statement (1). ■

To deal with the existence of free modules we provide the following definition, which is an analog of Definition 97.7:

Definition 309.19. Let R be a ring and let X be a set.

(1) We define[3433]
$$R^{(X)} := \bigoplus_{x \in X} R = \begin{array}{c}\text{all maps from } X \text{ to } R \text{ which are non-zero} \\ \text{for only finitely many } x \in X.\end{array}$$
We equip $R^{(X)}$ with the obvious structure of an R-left module. We refer to $R^{(X)}$ as the free R-left module generated by X or as the free R-left module on basis X.

(2) Let $x \in X$. The map
$$\begin{array}{rcl} X & \to & R \\ y & \mapsto & \begin{cases} 1_R, & \text{if } y = x, \\ 0_R, & \text{otherwise.} \end{cases} \end{array}$$
lies in $R^{(X)}$ and it is also called x. Thus we can view X naturally as a subset of $R^{(X)}$.

The following lemma, which is a variation on the Free Abelian Group–Existence Lemma 97.8, justifies the above name:

Lemma 309.20. (Free Module–Existence Lemma) Let R be a ring and let X be a set. The R-left module $R^{(X)}$ is a free R-left module on the subset $X \subseteq R^{(X)}$.

Proof. The proof of the lemma is basically the same as the elementary proof of the analogous the Free Abelian Group–Existence Lemma 97.8. ■

Remark.

(1) It follows immediately from the definitions and the discussion on page 5961 that a free abelian group is basically the same as a free $\mathbb{Z}$-left module.

(2) In Section 312.8 we will study the question whether the cardinality of a basis of a free module is an invariant of the free module. □

For completeness' sake we state the following essentially trivial lemma:

Lemma 309.21. (Free Module–Natural Epimorphism Lemma) Let R be a ring. Given any R-left module M we have a natural epimorphism
$$\begin{array}{rcl} R^{(M)} & \to & M \\ m & \mapsto & m \end{array}$$

[3433]Note that, if one looks carefully at Definition 309.8 and Definition 309.10 one sees that the second equality is actually an equality of sets.

of R-modules.

Proof. The lemma is of course just an immediate consequence of the Free Module–Existence Lemma 309.20. ■

For completeness sake we provide the following definition, which is an analog of Definition 97.19:

Definition 309.22. Let R be a ring.

(1) Given an R-left module M we say that $X \subseteq M$ is a generating set if the natural R-left module homomorphism $\Xi_X : R^{(X)} \to M$ is an epimorphism.
(2) We call an R-left module finitely generated if it admits a finite generating set.

Finitely generated free modules have the nice feature that homomorphisms can be expressed by matrices. This leads us to the following variation on Definition 2.11:

Definition 309.23. Let R be a ring. Let V and W be two free R-left modules. We assume that V and W are equipped with finite ordered bases $\mathcal{B} = (b_1, \ldots, b_m)$ and $\mathcal{C} = (c_1, \ldots, c_n)$. Let $\varphi : V \to W$ be an R-left module homomorphism. We refer to the unique $(m \times n)$-matrix $[\varphi]^{\mathcal{C}}_{\mathcal{B}} = (p_{ij}) \in \mathrm{M}(m \times n, R)$ with[3434]

$$\varphi(b_i) = \sum_{j=1}^{n} p_{ij} \cdot c_j$$

as the matrix representing φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. Note that this is the unique matrix that makes the following diagram commute:

$$\begin{array}{ccc} R^{1\times m} & \xrightarrow{\xi \mapsto \xi \cdot [\varphi]^{\mathcal{C}}_{\mathcal{B}}} & R^{1\times n} \\ \cong \downarrow \Xi_{\mathcal{B}} & & \cong \downarrow \Xi_{\mathcal{C}} \\ V & \xrightarrow{\varphi} & W, \end{array}$$

where the vertical maps are the isomorphisms from the Free Module–Uniqueness Lemma 309.18 (2). We leave it to the reader to formulate the analogous definition for *R-right* modules.

The following lemma is a variation on the earlier Representing Matrix Lemma 2.12.

Lemma 309.24. (Representing Matrix Lemma) Let R be a ring.

(1) Let V be a free R-left module of finite rank. If $\mathcal{B}$ is a basis for V, then $\mathrm{id}^{\mathcal{B}}_{\mathcal{B}} = \mathrm{id}$.
(2) Let $\varphi : U \to V$ and $\psi : V \to W$ be two homomorphisms between free R-left modules that are equipped with bases $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$. Then[3435]

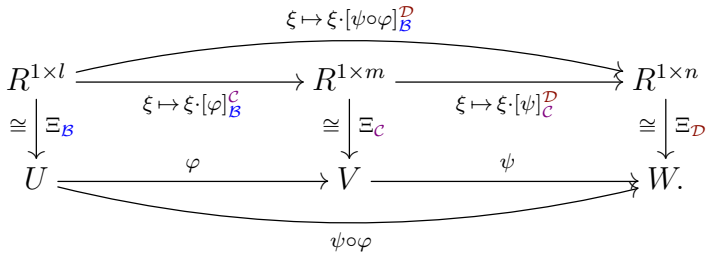
$$[\psi \circ \varphi]^{\mathcal{D}}_{\mathcal{B}} = [\varphi]^{\mathcal{C}}_{\mathcal{B}} \cdot [\psi]^{\mathcal{D}}_{\mathcal{C}}.$$

Proof.

[3434]Note that this matrix is the transpose of the matrix in Definition 2.11. The difference stems from the fact that we now allow non-commutative rings, and that we consider R-left module homomorphisms. As we pointed out in the Left Module Homomorphism–Matrix Lemma 309.5, these are necessarily given by *right* multiplication by a matrix.

[3435]Note that for right modules where we use column vectors and left multiplication we would, as was the case for the original Transition Matrix Lemma 31.2, get an equality $P_{\mathcal{B},\mathcal{D}} = P_{\mathcal{C},\mathcal{D}} \cdot P_{\mathcal{B},\mathcal{C}}$. Thus we see that the formulation for left modules is slightly neater.

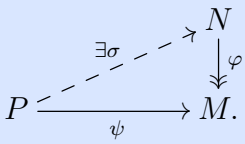
(1) This statement follows immediately from the definitions.
(2) This statement follows from the observation that the following diagram commutes:



■

309.6. **Projective modules.** The following definition might be familiar from a course in commutative algebra:

Definition 309.25. Let R be a ring. We say an R-left module P is projective if for every R-left module homomorphism $\psi\colon P \to M$ and every R-left module epimorphism $\varphi\colon N \to M$ there exists an R-left module homomorphism $\sigma\colon P \to N$ such that the following diagram commutes:



Lemma 309.26. (Free-is-Projective Lemma) Let R be a ring. Every R-left module that is free is also projective.

Proof. Let P be an R-left module that is free on a set $X \subseteq P$. Furthermore let $\psi\colon P \to M$ be an R-left module homomorphism and let $\varphi\colon N \to M$ be an R-left module epimorphism. Since φ is an epimorphism we see that for each $x \in X$ there exists an $n_x \in N$ with $\varphi(n_x) = \psi(x)$. By the existence statement of Definition 309.17 there exists a homomorphism $\sigma\colon P \to N$ such that for all $x \in X$ we have $\sigma(x) = n_x$.

It remains to prove the following claim:

Claim. We have the equality $\varphi \circ \sigma = \psi$.

Proof. Note that for every $x \in X$ we have $(\varphi \circ \sigma)(x) = \varphi(\sigma(x)) = \varphi(n_x) = \psi(x)$. It thus follows from the uniqueness statement of Definition 309.17 that $\varphi \circ \sigma = \psi$. ■

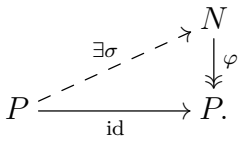
Example. In light of Free-is-Projective Lemma 309.26 one could wonder whether every projective module is already a free module. But this is not the case: For example let us consider the ring $R = \mathbb{Z}/6\mathbb{Z}$. We view $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}/6\mathbb{Z}$-module in the obvious way. In Exercise 309.6 we will show that the $\mathbb{Z}/6\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is projective but not free. □

In light of the Free-is-Projective Lemma 309.26 we can view the following lemma as a generalization of the Free Abelian Group–Splitting Lemma 97.11:

Lemma 309.27. (Projective Module–Splitting Lemma) Let R be a ring.
(1) Let $\varphi\colon N \to P$ be an epimorphism between R-left modules. If P is a projective R-left module, then there exists a right inverse $\sigma\colon N \to P$ of φ, this means that there exists a homomorphism $\sigma\colon P \to N$ with $\varphi \circ \sigma = \mathrm{id}_P$.
(2) If $\sigma\colon P \to N$ is a right inverse of φ, then $N = \ker(\varphi) \oplus \sigma(P)$.[3436]

Proof.

(1) The lemma follows immediately from applying Definition 309.25 to $M = P$ and $\psi = \mathrm{id}_P$. For the reader's convenience we write down the corresponding diagram:



(2) Let $\sigma\colon P \to N$ be a right inverse of φ. We start out with the following claim:

Claim. We have $\ker(\varphi) \cap \sigma(P) = \{0\}$.

Proof. If $g = \sigma(p) \in \ker(\varphi) \cap \sigma(P)$, then

$$g = \sigma(p) = (\sigma \circ \mathrm{id}_P)(p) \underset{\text{since } \varphi\circ\sigma = \mathrm{id}_P}{\uparrow=} \sigma(\varphi(\sigma(p))) \underset{\text{since } g = \sigma(p)}{\uparrow=} \sigma(\varphi(g)) \underset{\text{since } g \in \ker(\varphi)}{\uparrow=} \sigma(0) = 0.$$ ⊞

Note that it follows easily from the above claim that the natural homomorphism

$$\begin{aligned} \Theta\colon \ker(\varphi) \oplus \sigma(P) &\to N \\ g \oplus h &\mapsto g + h \end{aligned}$$

is a monomorphism. It remains to show that Θ is an epimorphism. Let $g \in N$. Note that

$$\varphi(g - \sigma(\varphi(g))) = \varphi(g) - \varphi(\sigma(\varphi(g))) \underset{\text{since } \varphi\circ\sigma = \mathrm{id}_P}{\uparrow=} \varphi(g) - \varphi(g) = 0.$$

This shows that $g - \sigma(\varphi(g)) \in \ker(\varphi)$. It follows that $g \in \mathrm{im}(\sigma) \oplus \ker(\varphi)$. ■

The following lemma gives a pleasant characterization of projective modules:

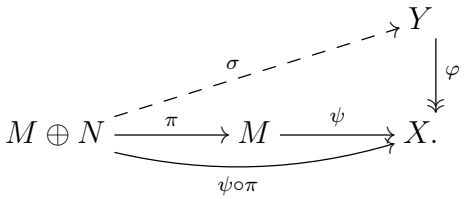
Lemma 309.28. (Projective Module–Characterization Lemma) Let R be a ring and let M be an R-left module. Then

$$M \text{ is projective} \iff M \text{ is a subsummand of free } R\text{-left module}$$

finitely generated

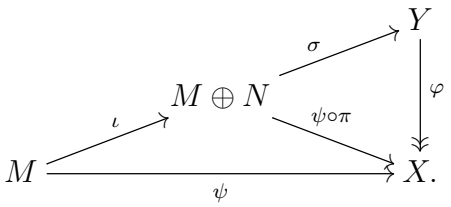
Proof. We start out with the proof of the "⇐"-direction. So we assume that there exists an R-left module N such that $M \oplus N$ is a free R-left module. We denote by $\iota\colon M \to M \oplus N$ the natural monomorphism and let $\pi\colon M \oplus N \to M$ be the natural projection.

Next let $\psi\colon M \to X$ be an R-left module homomorphism and let $\varphi\colon Y \to Z$ be an epimorphism between R-left modules. It follows from the Free-is-Projective Lemma 309.26 that there exists an R-left module homomorphism $\sigma\colon M \oplus N \to Y$ such that the following diagram commutes:



[3436]Here we use Convention 309.12.

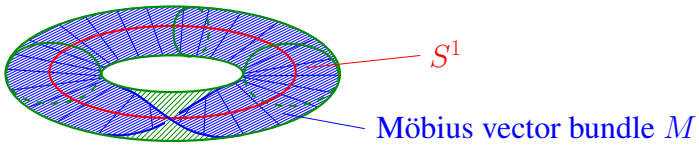
It follows immediately from $\pi \circ \iota = \mathrm{id}_M$ that the following diagram commutes:



The map $\sigma \circ \iota \colon M \to Y$ is thus the map we were searching for.

We turn to the proof of the "⇒"-direction. So let M be a projective R-left module. By the Free Module–Natural Epimorphism Lemma 309.21 we know that there exists a free R-left module P and an epimorphism $\varphi \colon P \to M$. The desired statement now follows from the Free-is-Projective Lemma 309.26 and the Projective Module–Splitting Lemma 309.27. ■

Example. We consider the Möbius vector bundle $p \colon M \to S^1$ from 5119 and we consider the corresponding $C(S^1, \mathbb{R})$-module $\Gamma(p \colon M \to S^1)$. Recall that the Möbius vector bundle is not trivializable. It thus follows from the discussion on page 5967 that $\Gamma(p \colon M \to S^1)$ is not a free $C(B, \mathbb{R})$-module.



We now see that we have the following isomorphisms of $C(S^1, \mathbb{R})$-modules:

$$\Gamma(M \to S^1) \oplus \Gamma(M \to S^1) \underset{\text{see page 5965}}{\cong} \Gamma(M \oplus M \to S^1) \underset{\substack{\text{on page 5171 we showed that } M \oplus M \text{ is a trivializable vector bundle}}}{\cong} \Gamma(\mathbb{R}^2 \times S^1 \to S^1) \underset{\text{see the discussion on page 5967}}{\cong} C(S^1, \mathbb{R})^2.$$

It follows from this discussion and the Projective Module–Characterization Lemma 309.28 that $\Gamma(p \colon M \to S^1)$ is a projective $C(S^1, \mathbb{R})$-module, even though it is not a free $C(S^1, \mathbb{R})$-module. [3437] □

For general culture we state the following theorem, even though we will not make use of it:

Theorem 309.29. (Quillen–Suslin Theorem) Let $m \in \mathbb{N}_0$. Let R either be the polynomial ring $\mathbb{Z}[t_1, \ldots, t_m]$ or the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. In this setting every finitely generated projective R-module is in fact a free R-module.

Proof. The theorem has its origin in a remark by Jean-Pierre Serre [**Ser1955**, p. 243]. It was proved independently by Daniel Quillen [**Quil1976**] and Andrei Suslin [**Sus1976**]. An account of the proof is given in [**Lam2006**, Corollary V.4.12]. An alternative proof is presented in [**Lang2002b**, Chapter XXI.3]. ■

[3437] In fact this example can be generalized: In Exercise 270.5 we showed that any real vector bundle $p \colon E \to B$ over a compact topological space B admits a real vector bundle $q \colon F \to B$ such that $E \oplus F \to B$ is trivializable. This observation, together with the Projective Module–Characterization Lemma 309.28, shows that the $C(B, \mathbb{R})$-module $\Gamma(E \to B)$ is projective. In fact it follows from [**Swa1962**, Theorem 2] and [**At1967**, p. 31] that for a compact topological space B the category of projective finitely generated $C(B, \mathbb{R})$-modules is equivalent to the category of finite-dimensional real vector bundles over B.

309.7. **Splitting sequences.** In Definition 145.4 we defined the notion of a splitting of a short exact sequence of (not necessarily abelian) groups. We now have the analog of modules.

Definition 309.30. Let R be a ring. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R-left modules splits, if there exists a splitting $s\colon C \to B$, i.e. an R-left module homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

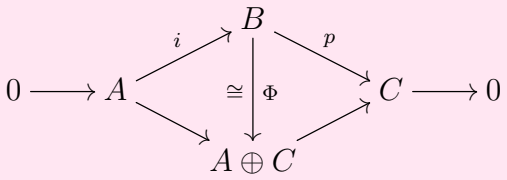
We continue with the following variation on the Splitting Lemma 145.5.

Lemma 309.31. (Module Splitting Lemma) Let R be a ring and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-left modules. The following three statements are equivalent:

(1) There exists a homomorphism of R-left modules $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.
(2) There exists an isomorphism $\Phi\colon B \to A \oplus C$ of R-left modules such that the following diagram commutes:



where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.
(3) The short exact sequence splits. (By the Projective Module–Splitting Lemma 309.27 this is the case if C is a projective R-module.)

Proof. The proof is a mild variation on the proof of the Splitting Lemma 145.5. We leave the details to the reader. ∎

309.8. **Modules over skew fields and PIDS.** In many ways free R-left modules are quite similar to free abelian groups. There is one major difference though. By the Free Abelian Subgroup Theorem 97.14 (1) we know that any submodule of a free $\mathbb{Z}$-module is itself a free $\mathbb{Z}$-module. The analogous statement is in general *not true* for free R-left modules, not even if R is commutative:

- We consider the ring $R = \mathbb{Z}/6\mathbb{Z}$. The subring $2 \cdot (\mathbb{Z}/6\mathbb{Z})$ is clearly not a free submodule.
- It takes a little more thought to find an example where the ring is a domain: This time we consider the polynomial ring $R = \mathbb{Z}[x]$. Furthermore we consider the ideal $I = (2, x)$ that is generated by 2 and x. It is a submodule of $\mathbb{Z}[x]$, which is of course a free $\mathbb{Z}[x]$-module in its own right. In Exercise 309.15 we will show that I, viewed as a $\mathbb{Z}[x]$-module, is *not free*.

In this section we will consider two special types of rings where this annoyance does not occur:

- In Section 309.8.1 we will consider skew fields.
- In Section 309.8.2 we will consider principal ideal domains.

309.8.1. **Skew fields.** We now introduce skew fields. In most applications the skew fields we use are in fact just ordinary fields. On the other hand, it costs little to generalize to skew fields and working with (possibly) non-commutative skew fields actually makes some of the future arguments clearer since it prevents us from using "accidents" in the commutative setting, e.g. the fact that over a commutative ring every left-module is automatically a right module.

Definition 309.32.

(1) A skew field is a ring $\mathbb{K}$ such that $0 \neq 1$ and such that given any $a \in \mathbb{K} \setminus \{0\}$ there exist $x, y \in \mathbb{K}$ with $a \cdot x = y \cdot a = 1$.[3438]
(2) A skew field that is commutative is called a field.
(3) If $\mathbb{K}$ is a skew field, then we refer to a $\mathbb{K}$-left module also as a $\mathbb{K}$-left vector space.

Example.

(1) Evidently all the "usual" fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are also skew fields.
(2) Here is an example that is a little more interesting: By the Quaternion Lemma 171.3 we know that the quaternions $\mathbb{H}$ form a skew field. □

The following propositions tells us that many results of modules over fields (i.e. commutative skew fields) from our first linear algebra course generalize in a suitable sense to modules over skew fields:

Proposition 309.33. (Module-over-Skew Field Proposition) Let $\mathbb{K}$ be a skew field.

(1) Every $\mathbb{K}$-left module M is a free $\mathbb{K}$-left module.
(2) (a) Any two bases for a $\mathbb{K}$-left module have the same cardinality.
    (b) Every finitely generated $\mathbb{K}$-left module M has a uniquely defined dimension, i.e. there exists a unique $d \in \mathbb{N}_0$ such that M is isomorphic to $\mathbb{K}^d$. As usual we write $\dim_{\mathbb{K}}(M) \in \mathbb{N}_0 := d$. For an infinitely generated $\mathbb{K}$-left vector space we sometimes write $\dim_{\mathbb{K}}(M) = \infty$.
(3) Every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbb{K}$-left modules splits and we have the equality $\dim_{\mathbb{K}}(B) = \dim_{\mathbb{K}}(A) + \dim_{\mathbb{K}}(C)$.
(4) Let $\varphi\colon M \to N$ be a $\mathbb{K}$-module homomorphism between two $\mathbb{K}$-left modules of the same finite-dimension. Then the following are equivalent:
    (a) φ is an isomorphism,
    (b) φ is a monomorphism,
    (c) φ is an epimorphism.

Proof. We leave it to the reader to dig out their favorite books on linear algebra and to verify that in the proofs of the corresponding statements over fields one never really needed to use that fields are commutative. The most obscure statement is potentially Statement (2a). The finite-dimensional case is dealt with in every book on linear algebra. The general case is sometimes referred to as Löwig's Theorem [**Löw**, p. 19]. The proof evidently makes use of the Axiom of Choice 1.1.

Finally note that Statement (4) is in fact a consequence of Statements (1), (2) and (3). ■

[3438] If such x, y exist, then as always $x = y$. Indeed, we have $x = 1 \cdot x = (y \cdot a) \cdot x = y \cdot (a \cdot x) = y \cdot 1 = y$.

309.8.2. **Principal ideal domains.** The following definition is surely familiar from an earlier algebra course:

Definition 309.34. A ring R is called a principal ideal domain (PID) if R is commutative[3439], if R is a domain and if every ideal of R is generated by a single element.

Examples.

(1) It follows from elementary algebra, e.g. the Module-over-Skew Field Proposition 309.33 (4), that every field is a PID.
(2) Every Euclidean domain (in the sense of Definition 100.4) is a PID, in particular $\mathbb{Z}$ is a PID and for every field $\mathbb{F}$ the polynomial ring $\mathbb{F}[t]$ and the Laurent polynomial ring $\mathbb{F}[t^{\pm 1}]$ are PIDs. □

The following theorem generalizes the Free Abelian Subgroup Theorem 97.14.

Theorem 309.35. (Free Module–PID Theorem) Let R be a PID.

(1) Every submodule of a free R-module is again free.[3440]
(2) Every projective R-module is a free R-module.

Proof.

(1) In the Free Abelian Subgroup Theorem 97.14 we proved the statement in the special case that $R = \mathbb{Z}$. The only property of $\mathbb{Z}$ we used is that every non-zero submodule of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. But the analogous statement also holds for any PID. Thus our original proof also works in the present setting. Alternatively, for readers who do not trust this author, this theorem is proved in [**Rot2002**, Theorem 8.9].
(2) This statement follows from (1) together with the Projective Module–Characterization Lemma 309.28. ■

We conclude this section with the following generalization of the Abelian Group-Classification Theorem 97.24:

Theorem 309.36. (PID–Module-Classification Theorem) Let R be a PID and let M be a finitely generated R-module. There exist $q_1, \dots, q_k \in R$, none of which is a unit, such that

$$M \cong \bigoplus_{i=1}^{k} R/q_i \cdot R$$

and such that for $i = 1, \dots, k-1$ we have $q_i | q_{i+1}$. Furthermore, $q_1, \dots, q_k$ are unique up to multiplication by a unit.

Proof. The proof of this theorem is similar to the proof of the Abelian Group-Classification Theorem 97.24. We refer to [**Row2006**, Theorems 2.54 and 2.61] for details. ■

309.9. **Bimodules.** Many of our future objects have two "compatible" modules structures at the same time. This leads us to the following definition:

[3439] One could also define a non-commutative analog of PIDs, but we will refrain from doing so.
[3440] Note that we do *not* demand that the free module is finitely generated.

Definition 309.37. Let R and S be two rings.

(1) An (R,S)-bimodule is an abelian group A together with an R-left module structure and an S-right module structure, such that for every $r \in R$, every $s \in S$ and every $a \in A$ we have $r \cdot (a \cdot s) = (r \cdot a) \cdot s$.[3441]

(2) An (R,S)-left left module is an abelian group A together with an R-left module structure and an S-left module structure, such that for every $r \in R$, every $s \in S$ and every $a \in A$ we have $r \cdot (s \cdot a) = s \cdot (r \cdot a)$. The same way we define an (R,S)-right right module.

In the above settings we have obvious notions of morphisms which lead to categories $\mathcal{M}od_{(R,S)\text{-bi}}$ and $\mathcal{M}od_{(R,S)\text{-left left}}$. Surely there is no need to introduce notation for these categories.

Examples.

(1) If R is a commutative ring, then any R-left module is automatically also an (R,R)-bimodule and an (R,R)-left left module.

(2) Let R be a ring. It follows easily from the discussion on page 5961 that any R-left module is also automatically an $(R,\mathbb{Z})$-bimodule and an $(R,\mathbb{Z})$-left left module. This simple observation will get used frequently.

(3) Let R be a ring. On page 5962 we saw that by the usual left and right multiplication we can view $R = R^{1\times 1}$ as an R-left module and an R-right module. Clearly these module structures define an (R,R)-bimodule structure on R.

(4) Let $\varphi\colon R \to S$ be a ring homomorphism. As in the Module–Change-of-Ring–Functor Lemma 309.13 we see that the maps

$$\begin{array}{ccc} R \times S & \to & S \\ (r,s) & \mapsto & \varphi(r)\cdot s \end{array} \quad \text{and} \quad \begin{array}{ccc} S \times R & \to & S \\ (s,r) & \mapsto & s\cdot\varphi(r) \end{array}$$

turn S into an R-left module ${}_\varphi S$ and into an R right-module S_φ, these satisfy the condition of a bimodule, thus we can view S as an (R,R)-bimodule ${}_\varphi S_\varphi$. By using these maps and multiplication in S we can also view S as an (R,S)-bimodule and an (S,R)-bimodule. The possibilities are endless! □

309.10. Exercises for Chapter 309.

Exercise 309.1. Let R be a ring. Show that $1 = 0$ if and only if R is the zero ring.

Exercise 309.2. Let R be a non-zero ring. Given $n \in \mathbb{N}_0$ we consider the matrix ring $\mathrm{M}(n \times n, R)$. Show that the matrix ring is non-commutative if $n \geq 2$.

Exercise 309.3. Let R be a ring with involution.

(a) Let V be an R-left module with basis $\mathcal{B}$. Show that $\mathcal{B}$ is also a basis for the R-right module $\overline{V}$.

(b) Let $f\colon V \to W$ be a homomorphism between R-left modules. Show that f is also a homomorphism between the R-right modules $\overline{V}$ and $\overline{W}$.

[3441] Note that this means that we can unambiguously talk about $r \cdot m \cdot s$.

(c) We suppose that we have a basis $\mathcal{B} = (b_1, \dots, b_m)$ for V and a basis $\mathcal{C} = (c_1, \dots, c_n)$ for W. Let $A = (a_{ij})$ be the $(m \times n)$-matrix that represents, in the sense of Definition 309.23, the homomorphism $f\colon V \to W$ of R-left modules with respect to these bases. Show that the $(n \times m)$-matrix $\overline{A}^{\mathrm{T}} = (\overline{a}_{ji})$ represents the homomorphism $f\colon \overline{V} \to \overline{W}$ of R-right modules.

Exercise 309.4. Let R be a ring and let M be an R-left module. Furthermore let $\mathcal{B} \subseteq M$ be a subset. We recall the following two concepts from linear algebra:

- We define $\operatorname{span}(\mathcal{B})$ as the smallest R-submodule of M that contains $\mathcal{B}$. We say $\mathcal{B}$ spans M if $\operatorname{span}(\mathcal{B}) = M$. One can easily verify that this is equivalent to saying that every element of M is an R-linear combination of elements of $\mathcal{B}$.
- We say $\mathcal{B}$ is *linearly independent* if for every choice of pairwise distinct $b_1, \dots, b_k$ in $\mathcal{B}$ and every choice of $r_1, \dots, r_k \in R$ we have

$$r_1 \cdot b_1 + \cdots + r_k \cdot b_k = 0 \implies r_1 = \cdots = r_k = 0.$$

Show that the following two statements are equivalent:

(i) $\mathcal{B}$ is a basis of M in the sense of Definition 309.17.
(ii) $\mathcal{B}$ spans M and $\mathcal{B}$ is linearly independent.

Remark. This exercise is very similar to Exercise 97.3.

Exercise 309.5. Is $\mathbb{Q}$ a projective $\mathbb{Z}$-module?

Exercise 309.6.

(a) Let R and S be rings. We consider the product ring $R \times S$. We can view R in an obvious way as an $(R \times S)$-left module. Show that this $(R \times S)$-left module is projective.
(b) We consider $R = \mathbb{Z}/6\mathbb{Z}$. Show that the $\mathbb{Z}/6\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is projective but not free.

Exercise 309.7. Let R be a commutative ring. Is every submodule of a projective R-module again projective?

Exercise 309.8. Let R be a ring and let M, N be two R-left modules. What is the "maximal algebraic structure" that we can give the set $\operatorname{Hom}_{R-\mathrm{left}}(M, N)$ of R-left module homomorphisms?

Exercise 309.9. Give an example of a ring R, a projective R-left module P and a free R-left module F such that $P \oplus F$ is free, but such that P itself is not free.

Exercise 309.10. Can you come up with a non-commutative skew field that is not isomorphic to the quaternions?

Exercise 309.11. Let $p\colon E \to B$ be a real vector bundle over a topological space B. We denote by $\Gamma(p\colon E \to B)$ the $C(B, \mathbb{R})$-module of continuous sections.

(a) Show that if B is compact, then the $C(B, \mathbb{R})$-module $\Gamma(p\colon E \to B)$ is finitely generated.
(b) Provide an example of a real vector bundle $p\colon E \to B$ such that the $C(B, \mathbb{R})$-module $\Gamma(p\colon E \to B)$ is not finitely generated.

Exercise 309.12. Let R be a ring and let $n \in \mathbb{N}_0$. We consider the ring $\mathrm{M}(n \times n, R)$ and as on page 5962 we consider the $\mathrm{M}(n \times n, R)$-left module $R^{n \times 1}$. What is the minimal cardinality of a generating set of this module?

Exercise 309.13. Let $p\colon E \to B$ be an n-dimensional real vector bundle over a topological space B.

(a) Show that if $\Gamma(p\colon E \to B)$ is a free $C(B,\mathbb{R})$-module, then any basis has precisely n elements.

(b) Show that if $\Gamma(p\colon E \to B)$ is a free $C(B,\mathbb{R})$-module, then p is trivializable in the sense of Definition 265.5.
Hint. Use the Trivializable–Vector Bundle Lemma 265.22.

Exercise 309.14. We consider the ring $R = \mathbb{Z}[x]$ of polynomials and we consider the ideal $I = (2, x)$ that is generated by 2 and x.

(a) Show that I is not a free $\mathbb{Z}[x]$-module.

(b) Show (without making use of the Quillen–Suslin Theorem 309.29) that I is not a projective $\mathbb{Z}[x]$-module.

Exercise 309.15. We consider the Laurent polynomial ring $R = \mathbb{Z}[x^{\pm 1}]$.

(a) Show that the ring $R = \mathbb{Z}[x^{\pm 1}]$ admits a submodule that is not free.

(b) Show that the ring $R = \mathbb{Z}[x^{\pm 1}]$ admits a submodule that is not projective.

Remark. Make use of Exercise 309.14.

Exercise 309.16. Let $\mathbb{K}$ be a skew field. Can you define a determinant for (2×2)-matrices over $\mathbb{K}$? More precisely, can you define a map

$$\det\colon \mathrm{M}(2\times 2,\mathbb{K}) \;\to\; \mathbb{K}$$

with the following three properties:

- For any two $A, B \in \mathrm{M}(2\times 2,\mathbb{K})$ we have $\det(A\cdot B) = \det(A)\cdot\det(B)$.
- A matrix A is invertible if and only if $\det(A) \neq 0$.
- For any diagonal matrix $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ we have $\det(D) = a\cdot b$.

Exercise 309.17. Recall that according to Definition 309.32 a *skew field* is a ring $\mathbb{K}$ such that $0 \neq 1$ and such that given any $a \in \mathbb{K}\setminus\{0\}$ there exist $x, y \in \mathbb{K}$ with $a\cdot x = y\cdot a = 1$. For which (if any) of the statements of the Module-over-Skew Field Proposition 309.33 do we need that $0 \neq 1$?

Exercise 309.18. Let R be a ring (possibly non-commutative) and let $A, B \in \mathrm{M}(n\times n, R)$ be two matrices.

(a) Show that in general we do not have $(A\cdot B)^{\mathrm{T}} = B^{\mathrm{T}}\cdot A^{\mathrm{T}}$.

(b) What extra hypotheses or structures on R do you need to get an equality of the form $(A\cdot B)^{\mathrm{T}} = B^{\mathrm{T}}\cdot A^{\mathrm{T}}$ (perhaps after making minor adjustments).

Exercise 309.19.

(a) Let R be a commutative ring. We consider the ring

$$\mathrm{UT}(2,R) \;=\; \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a,b,c \in R \right\}$$

of upper triangular matrices over R. Show that the opposite ring $\mathrm{UT}(2,R)^{\mathrm{op}}$ is isomorphic to a subring of $\mathrm{M}(2\times 2, R)$.

(b) We consider the abelian group

$$R = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$$

together with the (almost) obvious multiplication. Show that R has 11 left ideals and 12 right ideals. Conclude that R is not isomorphic to R^{op}.

Remark. Left and right ideals over "upper triangle matrix rings" as above are characterized in [**Lam1991**, Proposition 1.17].

Exercise 309.20. If R is a commutative ring, then any R-module also has automatically an (R, R)-bimodule structure. In this exercise we want to consider more interesting bimodules over commutative rings. We consider the tensor product $R \otimes R$ of *abelian groups*. It follows from the Tensor-with-Ring Lemma 159.18 that multiplication in the left respectively right factor defines two R-module structures on the abelian group $R \otimes R$.

(a) Show that these two R-module structures define an (R, R)-bimodule structure on the abelian group $R \otimes R$.

(b) Show that there exist commutative rings such that these two R-module structures on $R \otimes R$ are different.

CHAPTER 310

Group rings and their modules

In the previous Chapter 309 we studied left and right modules over a (possibly non-commutative) ring R. Our main example later on will be the group ring $R = \mathbb{Z}[\pi]$ where $\pi = \pi_1(X, x_0)$ is the fundamental group of some pointed topological space (X, x_0). In this chapter we recall the notion of a group ring and, more importantly, we will collect some examples and properties of modules over group rings.

In this chapter we will in particular present several results regarding representations of free groups, abelian groups and finite groups. Most of these we will not use later on. But the author feels that these results should be basic knowledge every working topologist is supposed to have.

This chapter is organized as follows:

- In Section 310.1 we will recall the concept of a group ring.
- In Section 310.2, given a group G and given a commutative ring, we will introduce R-representations of G and we will see that for all intents and purposes representations of G are the same as $(R, \mathbb{Z}[G])$-left left modules.
- In Section 310.3 we will study free modules over group rings.
- In Section 310.4 we will discuss R-representations over finitely generated free R-modules.
- In Section 310.5 we will see that every countable free group F admits a faithful representation $F \to \mathrm{SL}(2, \mathbb{Z})$.
- In Section 310.6 we will discuss the representation theory of abelian groups and of finite groups.
- In Section 310.7 we will provide exercises for Chapter 310.

In the next Chapter 311 we will (re-) introduce the tensor product of abelian groups and we will introduce the tensor product of two rings.

310.1. **Group rings.** We start out by recalling Definition 97.30 and Definition 286.6:

Definition 310.1. Let G be a group and let R be a commutative ring (in most cases $R = \mathbb{Z}$). We denote the free R-module generated by the set G by $R[G]$. Put differently with Definition 309.17 we have

$$R[G] \coloneqq \left\{ \sum_{i=1}^{m} r_i \cdot g_i \,\middle|\, r_1, \ldots, r_m \in R \text{ and } g_1, \ldots, g_m \in G \right\}.$$

We equip $R[G]$ with the multiplication that is given by

$$\left(\sum_{i=1}^{m} r_i \cdot g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \coloneqq \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_R s_j}_{\in R} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $R[G]$ as the group ring of G with R-coefficients. (For $R = \mathbb{Z}$ we refer to $\mathbb{Z}[G]$ as the group ring of G.)

The following lemma is a mild variation on the Group Ring Lemma 97.31:

Lemma 310.2. (Group Ring Lemma) Let G be a group and let R be a commutative ring.

(1) The group ring $R[G]$ is a ring in the sense of Definition 2.6. The multiplicatively neutral element is given by $1_R \cdot e$ where e denotes the trivial element in G.
(2) If R is not the zero ring, then the group ring $R[G]$ is commutative if and only if G is an abelian group.
(3) The assignment $\overline{g} := g^{-1}$ induces an involution, in the sense of Definition 309.14, on the ring $R[G]$. More precisely, the map

$$\begin{array}{rcl} R[G] & \to & \overline{R[G]} \\ \sum_{i=1}^{m} r_i \cdot g_i & \mapsto & \overline{\sum_{i=1}^{m} r_i \cdot g_i} := \sum_{i=1}^{m} r_i \cdot g_i^{-1} \end{array}$$

defines an involution on the ring $R[G]$.

Proof. The first statement of this lemma are already contained in the original Group Ring Lemma 97.31. The last statement follows easily from the definitions. ■

Statement (3) of the Group Ring Lemma 310.2 leads us to the following convention, which actually goes back to Definition 118.5:

Convention 310.3. Given a group G we will view the group ring $\mathbb{Z}[G]$ always as a ring with involution, where the involution is given by $\overline{g} := g^{-1}$.

Example. Let R be any commutative ring.

(1) We consider the infinite cyclic group $\langle t \rangle := \{\dots t^{-2}, t^{-1}, 1, t, t^2, \dots\}$. In this case the group ring $R[\langle t \rangle]$ is isomorphic to the ring $R[t^{\pm 1}]$ of Laurent polynomials. More precisely the "obvious" map

$$\begin{array}{rcl} R[t^{\pm 1}] & \to & R[\langle t \rangle] \\ \sum_{i=r}^{i=s} a_i \cdot t^i & \mapsto & \sum_{i=r}^{i=s} a_i \cdot t^i \end{array}$$

is clearly a natural isomorphism of rings with involution. We will use this natural ring isomorphism to make the identification $R[t^{\pm 1}] =_{\mathrm{i}} R[\langle t \rangle]$.
(2) If H is a free abelian group of rank m, then any choice of a basis $t_1, \dots, t_m$ for H gives in an obvious way rise to a natural isomorphism (as rings with involution) from the multivariable Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ to the group ring $R[H]$.
(3) Let R be a commutative ring and let $n \in \mathbb{N}$. One can easily verify that the map

$$\begin{array}{rcl} \mathbb{Z}[\langle t \,|\, t^n \rangle] & \to & \mathbb{Z}[t]/(t^n - 1) \\ \sum_{i=0}^{m} r_i \cdot t^i & \mapsto & \left[\sum_{i=0}^{m} r_i \cdot t^i\right] \end{array}$$

is a ring isomorphism. We will frequently use this isomorphism to go back and forth between these two rings. □

For completeness sake we recall the following lemma, which usually we will use in conjunction with the Module–Change-of-Ring–Functor Lemma 309.13:

Lemma 97.33. (Group Ring Functor Lemma) Let R be a commutative ring.

(1) The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{Group}) & \to & \mathrm{Ob}(\mathcal{Ring}) \\ G & \mapsto & R[G] \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Group}}(G,H) & \to & \mathrm{Mor}_{\mathcal{Ring}}(R[G],R[H]) \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \varphi_*\colon R[G] & \to & R[H] \\ \sum_{i=1}^m r_i \cdot g_i & \mapsto & \sum_{i=1}^m r_i \cdot \varphi(g_i) \end{array} \right) \end{array}$$

define a covariant functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{Ring}$ of rings.

(2) The group ring functor is a left adjoint functor in the sense of Definition 6.33. In particular we obtain from the Left Adjoint–Colimit Proposition 6.35 that the group ring functor preserves all colimits.

310.2. **Representations and modules over group rings.** The following definition, which can be viewed as a mild generalization of Definition 124.18 and Definition 124.21, might be familiar from an earlier course in algebra:

Definition 310.4. Let G be a group and let R be a ring.[3442]

(1) An R-representation of G consists of an R-left module M together with a group homomorphism $\alpha\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$.

(2) Let $\alpha\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$ and $\beta\colon G \to \mathrm{Aut}_{R\text{-left}}(N)$ be two R-representation of G. A morphism between these two R-representations of G is defined to be a homomorphism $\varphi\colon M \to N$ of R-left modules such that for any $g \in G$ and $m \in M$ we have the equality $\varphi(\alpha(g) \cdot m) = \beta(g) \cdot \varphi(m))$. In other words, we want the following diagram to commute:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle \alpha(g)\cdot} & & \downarrow{\scriptstyle \beta(g)\cdot} \\ M & \xrightarrow{\varphi} & N. \end{array}$$

(3) From (1) and (2) we obtain the category of R-representations $\mathcal{Rep}_R(G)$ of G.

(4) Given a family $\{\alpha_i\colon G \to \mathrm{Aut}_{R\text{-left}}(M_i)\}_{i\in I}$ of R-representations of G we define the direct sum of the R-representations as the R-module $\bigoplus_{i\in I} M_i$ with the obvious homomorphism $G \to \mathrm{Aut}_{R\text{-left}}\left(\bigoplus_{i\in I} M_i\right)$. One can easily verify that this is the coproduct (in the sense of Definition 6.5) in the category $\mathcal{Rep}_R(G)$.

The following lemma is a slight generalization of the Group Ring–Module Lemma 97.34:

Lemma 310.5. (Group Ring–Module Lemma) Let G be a group and let R be a ring.

(1) Let M be an R-left module. Suppose we are given an R-representation

$$\varphi\colon G \to \mathrm{Aut}_{R\text{-left}}(M),$$

then the map

$$\begin{array}{rcl} \mathbb{Z}[G] \times M & \to & M \\ \left(\sum_{i=1}^n a_i \cdot g_i, m\right) & \mapsto & \sum_{i=1}^n a_i \cdot \underbrace{\varphi(g_i)(m)}_{\in M} \end{array}$$

[3442] In most cases R will be a commutative ring. In these case we will not have to worry about left and right modules. But for the general theory we cannot avoid these issues.

defines a natural $(R, \mathbb{Z}[G])$-left left module structure on M.

(2) The assignment in (1) leads to an obvious isomorphism from the category $\mathcal{Rep}_R(G)$ of R-representations of G to the category of $(R, \mathbb{Z}[G])$-left left modules. Note that this isomorphism preserves by definition coproducts, so in our language, it preserves direct sums.

Proof.

(1) On a more formal level this statement can be rephrased as follows: We are given a group homomorphism $G \to \mathrm{Aut}_{R\text{-left}}(M)$. From the Group Ring–Universal Property Lemma 97.32 we obtain a corresponding ring homomorphism $\mathbb{Z}[G] \to \mathrm{End}_{R\text{-left}}(M)$. Since M is evidently an $\mathrm{End}_{R\text{-left}}(M)$-left module we get an induced $(R, \mathbb{Z}[G])$-left left module structure on M.

(2) This statement follows easily from the definitions. ■

The Group Ring–Module Lemma 310.5 leads us to the following notation:

Notation 310.6. Let G be a group, let R be a ring, let M be an R-left module and let $\varphi\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$ be a representation. We denote the corresponding $(R, \mathbb{Z}[G])$-left left module, which we obtain from the Group Ring–Module Lemma 310.5, by ${}_{\varphi}M$.

We camouflage our first example of a representation as a definition.

Definition 310.7. Let G be a group, let R be a ring and let M be an R-left module.

(1) We denote by $\mathrm{tr}\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$ the trivial homomorphism that is given by the assignment $g \mapsto \mathrm{id}_M$.

(2) From Notation 310.6 we obtain the corresponding $(R, \mathbb{Z}[G])$-left left module ${}_{\mathrm{tr}}M$. Note that the $\mathbb{Z}[G]$-left module structure is given by the trivial g-action $g \cdot m := m$ for each $g \in G$ and $m \in M$.

We move on to a particularly desirable type of representation:

Definition 310.8. Let G be a group, let R be a ring, let M be an R-left module and let $\varphi\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$ be a representation. We say the representation is faithful, if it is a monomorphism.

The following definition, which can be viewed as a slight generalization of the definition on page 2395, shows in particular that every finite group admits a faithful finite-dimensional representations over field:

Definition 310.9. Let G be a group and let R be a commutative ring.

(1) The representation
$$\begin{array}{rccl} \lambda_G^R\colon & G & \to & \mathrm{Aut}_R(R[G]) \\ & g & \mapsto & (f \mapsto g \cdot f) \end{array}$$
is called the left regular R-representation of G.

(2) The representation
$$\begin{array}{rccl} \rho_G^R\colon & G & \to & \mathrm{Aut}_R(R[G]) \\ & g & \mapsto & (f \mapsto f \cdot g^{-1}) \end{array}$$
is called the right regular R-representation of G.

If R is not the zero ring, then these two representations are clearly faithful.

310.3. **Free modules over group rings.** In Definition 2.36 we introduced the notion of an action of a group G on a set X. Since we are now carefully distinguishing between left and right module structures it is worth renaming the notion of an action as a "left action":

Definition 310.10. Let S be a set and let G be a group with trivial element e.

(1) A left action of G on S is a map

$$\begin{array}{rcl} G \times S & \to & S \\ (g,s) & \mapsto & g \cdot s \end{array}$$

with the following properties:

$$\begin{array}{rcll} e \cdot s & = & s, & \text{for all } s \in S, \\ \underset{\substack{\uparrow \\ \text{group action}}}{g} \cdot (\underset{\uparrow}{h} \cdot s) & = & (\underset{\substack{\uparrow \\ \text{multiplication in} \\ \text{the group } G}}{g \cdot_G h}) \cdot s, & \text{for all } s \in S \text{ and } g,h \in G. \end{array}$$

A right action is defined in the hopefully obvious way.

(2) The left action is called free, if $g \cdot s = s$ for some $s \in S$ implies that $g = e$.

(3) We say that G acts transitively, if for every s and t in S there exists a $g \in G$ with $g \cdot s = t$.

The following two lemmas connect G-left actions to $\mathbb{Z}[G]$-left modules:

Lemma 310.11. (Set Action–Group Ring Module Lemma) Let G be a group.

(1) Let S be a set together with a G-left action. The map

$$\begin{array}{rcl} G & \to & \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{(S)}) \\ g & \mapsto & \left(\begin{array}{rcl} \mathbb{Z}^{(S)} & \to & \mathbb{Z}^{(S)} \\ \left(\sum_{i=1}^{k} n_i \cdot s_i\right) & \mapsto & \sum_{i=1}^{k} n_i \cdot \underbrace{(g \cdot s_i)}_{\in S} \end{array} \right) \end{array}$$

is a group homomorphism. Via the Group Ring–Module Lemma 310.5 this turns $\mathbb{Z}^{(S)}$ into a $(\mathbb{Z}[G], \mathbb{Z})$-left left module.

(2) Let S, T be two sets with G-left actions and let $f\colon S \to T$ be a map. If f is G-equivariant, i.e. if for any $s \in S$ and $g \in G$ we have $f(g \cdot s) = g \cdot f(s)$, then the induced map $f_*\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is a homomorphism of $\mathbb{Z}[G]$-left modules.

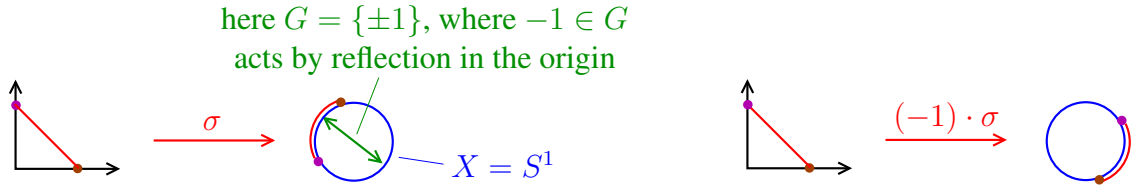
Proof. The lemma follows easily from the definitions. ■

Example. Later on we will consider the following situation: We have a topological space X together with a group G that acts continuously on the left on X.

(1) We consider the set $S_k(X)$ of singular k-simplices $\Delta_k \to X$.

(2) Since G acts continuously on X we obtain a left action of G on $S_k(X)$.

(3) As in Definition 143.11 we consider the k-th singular chain group $\mathrm{C}_k(X) := \mathbb{Z}^{(S_k(X))}$.



The Set Action–Group Ring Module Lemma 310.11 allows us to view $\mathrm{C}_k(X) = \mathbb{Z}^{(S_k(X))}$ as a $\mathbb{Z}[G]$-left module. This fact will become rather important later on. □

The next question is, when is the $\mathbb{Z}[G]$-left module from the Set Action–Group Ring Module Lemma 310.11 actually a free $\mathbb{Z}[G]$-left module? Before we can answer this question, let us introduce a new concept:

Definition 310.12. Suppose that a group G admits a left action on a set S. We say that $T \subseteq S$ is a complete set of representatives if given any $s \in S$ there exists a *unique* $t \in T$ such that $s = g \cdot t$.

Lemma 310.13. (Group Ring–Free Module Lemma) Let G be a group and let S be a set together with a G-left action.

(1) The G-left action on S admits a complete set of representatives.
(2) We suppose that the left action of G on S is free. If $T \subseteq S$ is a complete set of representatives, then $\mathbb{Z}^{(S)}$ is a free $\mathbb{Z}[G]$-left module with basis T.

Example. In this section figures are few and far between, so let us take this opportunity to look at one pictorial example:

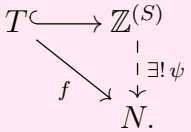


We consider the set $S = \mathbb{Z} \times \{A, B, C\}$ with the obvious $\mathbb{Z}$-action. This left action is free. There are many choices of complete sets of representatives. □

Proof.

(1) This statement follows easily from the Group Action-Equivalence Relation Lemma 2.38 and the Axiom of Choice 1.1. We refer to Exercise 1.4 for more details.
(2) Let $T \subseteq S$ be a complete set of representatives of the G-action on S. We want to show that $\mathbb{Z}^{(S)}$ is a free $\mathbb{Z}[G]$-left module with basis T. By Definition 309.17 we need to prove the following claim:

Claim. Let N be a $\mathbb{Z}[G]$-left module and let $f\colon T \to N$ be a map. There exists a unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to N$ of $\mathbb{Z}[G]$-left modules that makes the following diagram commute:



Proof. Recall that the hypothesis that the left action of G on S is free means that if $g \cdot s = s$ for some $s \in S$, then $g = e$. It follows easily from the axioms of a left action that this translates into the statement that given any $s, t \in S$ with $G \cdot s = G \cdot t$ there exists a *unique* $g \in G$ with $g \cdot s = t$.

We extend $f\colon T \to N$ to a map $\widetilde{f}\colon S \to N$ as follows:

$$\begin{array}{rcl} \widetilde{f}\colon S & \to & N \\ s & \mapsto & g \cdot f(t) \text{ where } G \cdot t = G \cdot s \text{ and where } g \in G \text{ is} \\ & & \qquad \text{the unique element with } g \cdot t = s. \end{array}$$

Next note that it follows from the Definition 309.17 of a free $\mathbb{Z}$-module that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to N$ of $\mathbb{Z}$-modules that makes the following diagram commute:

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{Z}^{(S)} \\ & \searrow^{\widetilde{f}} & \downarrow \psi \\ & & N. \end{array}$$

We need to show that ψ is in fact a homomorphism of $\mathbb{Z}[G]$-left modules. Note that one can easily verify that for any $s \in S$ and any $g \in G$ we have $\widetilde{f}(g \cdot s) = g \cdot \widetilde{f}(s)$. Since S is a generating set for $\mathbb{Z}^{(S)}$ as an abelian group we see that $\psi\colon \mathbb{Z}^{(S)} \to N$ is in fact a homomorphism of $\mathbb{Z}[G]$-left modules.

The proof of the fact that the $\mathbb{Z}[G]$-left module homomorphism ψ is unique can be reduced, with the same arguments as above, to the uniqueness of the homomorphism $\mathbb{Z}^{(S)} \to N$ of abelian groups. We leave it to the reader to fill in the details. ■

310.4. **Representations over free modules and matrix groups.** We start out with introducing some language:

Definition 310.14. Let G be a group and let R be a ring.

(1) By Definition 310.4 an R-representation of G consists of an R-left module M together with a homomorphism $\alpha\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$.
(2) If $R = \mathbb{Z}$, $R = \mathbb{Q}$, $R = \mathbb{R}$ or $R = \mathbb{C}$, then we also say integral, rational, real or complex representation.
(3) If R is a skew field and M is a d-dimensional $\mathbb{K}$-vector space, then we say that α is a d-dimensional representation. If $d < \infty$, then we call the representation finite-dimensional.

In the following we will mostly consider representations over finitely generated free R-left modules. This leads us to the following lemma which is a variation on the Representation Classification Lemma 124.19:

Lemma 310.15. (Free Module–Representation Classification Lemma) Let G be a group, let R be a ring and let $n \in \mathbb{N}_0$.

(1) (a) For the R-left module[3443] $R^{1\times n}$ from page 5962 we have the group isomorphism[3444]

$$\begin{array}{rcl} \mathrm{GL}(n,R) & \to & \mathrm{Aut}_{R\text{-left}}(R^{1\times n}) \\ A & \mapsto & \left(\begin{array}{ccc} R^{1\times n} & \to & R^{1\times n} \\ \underbrace{(v_1,\ldots,v_n)}_{\text{row vector}} & \mapsto & \underbrace{(v_1,\ldots,v_n)}_{\text{row vector}} \cdot A^{-1} \end{array} \right). \end{array}$$

(b) Let M be a free R-module of rank n and let $v_1, \dots, v_n$ be a basis of M. The map

$$\begin{array}{rcl} \mathrm{Aut}_{R\text{-left}}(M) & \to & \mathrm{GL}(n,R) \\ \varphi & \mapsto & \left(\begin{array}{l} \text{the matrix from Definition 309.23 that represents} \\ \text{the } R\text{-left module homomorphism } \varphi\colon M \to M \\ \text{with respect to the ordered basis } (v_1, \dots, v_n) \end{array} \right)^{-1} \end{array}$$

is an isomorphism. In fact, for $M = R^{1\times n}$ and the standard basis $(\mathfrak{e}_1, \dots, \mathfrak{e}_n)$ this map is the inverse to the map in (a).

(2) The isomorphisms from (1) define a natural bijection between the following two sets:
- The set of isomorphism classes of representations $G \to \mathrm{Aut}_{R\text{-left}}(M)$, where M is any free R-module of rank n.
- Conjugacy classes of homomorphisms $G \to \mathrm{GL}(n,R)$.

Proof. The lemma follows easily from the definitions. But just to show our good we will now verify that the given map $\Xi\colon \mathrm{GL}(n,R) \to \mathrm{Aut}_{R\text{-left}}(R^{1\times n})$ is a group homomorphism. So let $A, B \in \mathrm{GL}(n,R)$ and let $v \in R^{1\times n}$, then

$$\Xi(A\cdot B)(v) \;=\; v\cdot(A\cdot B)^{-1} \;=\; v\cdot B^{-1}\cdot A^{-1} \;=\; \Xi(A)(\Xi(B)(v)) \;=\; (\Xi(A)\circ\Xi(B))(v).$$

This shows that $\Xi(A\cdot B) = \Xi(A)\circ\Xi(B)$. We leave all the remaining details to the typing enthusiasm of the reader. ∎

The Free Module–Representation Classification Lemma 310.15 leads us to the following definition:

Convention 310.16. Let G be a group, let R be a ring, let $n \in \mathbb{N}_0$ and let $\varphi\colon G \to \mathrm{GL}(n,R)$ be a group homomorphism. As in the Free Module–Representation Classification Lemma 310.15 we consider the group isomorphism

$$\begin{array}{rcl} \Theta\colon \mathrm{GL}(n,R) & \to & \mathrm{Aut}_{R\text{-left}}(R^{1\times n}) \\ A & \mapsto & \left(\begin{array}{ccc} R^{1\times n} & \to & R^{1\times n} \\ (v_1,\dots,v_n) & \mapsto & (v_1,\dots,v_n)\cdot A^{-1} \end{array} \right). \end{array}$$

(1) Often we refer to the resulting homomorphism $\Theta\circ\varphi\colon G \to \mathrm{Aut}_{R\text{-left}}(R^{1\times n})$ by φ as well.

(2) Together with the Group Ring–Module Lemma 310.5 and with Notation 310.6 we obtain the corresponding $(R,\mathbb{Z}[G])$-left left module ${}_{\varphi}R^{1\times n}$.

(3) In light of (1) we refer to $\varphi\colon G \to \mathrm{GL}(n,R)$ also as a representation.

For the case $n = 1$ we will at times us a slightly different language:

Convention 310.17. Let G be a group, let R be a ring and let $\rho\colon \mathbb{Z}[G] \to R$ be a ring homomorphism. This defines a group homomorphism $\rho\colon G \to \mathrm{GL}(1,R)$. In this case the left-multiplication by g from Convention 310.16 is given by $g\cdot r := r \underset{R}{\cdot} \rho(g^{-1})$. We denote the corresponding $(R,\mathbb{Z}[G])$-left left module again by ${}_{\varphi}R$.

[3443] Psychologically it can help to think of $R^{1\times n}$ as row vectors.

[3444] The convention that we act on row vectors by right multiplication with A^{-1} might initially come as a surprise. But it is forced on us if we want to get an R-left module isomorphism. The case of a non-commutative ring R will occur naturally later on, for example in the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 323.5.

310.5. **Faithful representations of free groups.** The only goal of this section is to prove the following proposition:

Proposition 310.18. (Free Group–Faithful Representation Proposition) Let F be a countable free group. There exists a faithful representation $\varphi\colon F \to \mathrm{SL}(2,\mathbb{Z})$.[3445]

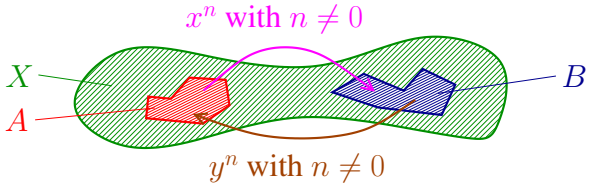
The proof of the Free Group–Faithful Representation Proposition 310.18 rests on the shoulders of the following cool little lemma, which has its origins in the work of Felix Klein [**Kle1882a**, p. 200] and which we first encountered as Exercise 98.10:

Lemma 310.19. (Ping Pong Lemma) Let G be a group and let $x, y \in G$. Suppose that there exists a left action of G on some set X and suppose that there exist two disjoint non-empty subsets A and B of X such that the following two conditions are satisfied:

(a) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $x^n \cdot A \subseteq B$, and

(b) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $y^n \cdot B \subseteq A$.

In this setting x and y form the basis of a free subgroup of G, i.e. the natural homomorphism $\varphi\colon \langle x, y\rangle \to G$ is a monomorphism.



Proof of the Ping Pong Proposition 310.19. In the following we denote by $\varphi\colon \langle x, y\rangle \to G$ the natural homomorphism. Let $g \in \langle x, y\rangle$ be non-trivial. We need to show that $\varphi(g) \in G$ is non-trivial. By Definition 98.1 we can represent g by a reduced word w in x and y. Note that there are now four possible cases:

(1) $w = x^{r_1} \cdot y^{s_1} \cdot \ldots \cdot x^{r_{m-1}} \cdot y^{s_{m-1}} \cdot x^{r_m}$ where $r_1, \ldots, r_m, s_1, \ldots, s_{m-1} \in \mathbb{Z} \setminus \{0\}$.

(2) $w = y^{s_1} \cdot x^{r_1} \cdot \ldots \cdot y^{s_{m-1}} \cdot x^{r_{m-1}} \cdot x^{y_m}$ where $r_1, \ldots, r_{m-1}, s_1, \ldots, s_{m-1}, s_m \in \mathbb{Z} \setminus \{0\}$.

(3) $w = x^{r_1} \cdot y^{s_1} \cdot \ldots \cdot x^{r_m} \cdot y^{s_m}$ where $r_1, \ldots, r_m, s_1, \ldots, s_m \in \mathbb{Z} \setminus \{0\}$.

(4) $w = y^{s_1} \cdot x^{r_1} \cdot \ldots \cdot y^{s_m} \cdot x^{r_m}$ where $r_1, \ldots, r_m, s_1, \ldots, s_m \in \mathbb{Z} \setminus \{0\}$.

Recall that we need to show that $\varphi(w) = \varphi(g)$ is non-trivial. We now consider the four cases separately:

(1) In the first case we obtain easily from Conditions (a) and (b) that $\varphi(w)(A) \subseteq B$ (This is the "ping pong part of the proof"). Since A is non-empty this implies that $\varphi(w) \in G$ acts non-trivially on X. Thus $\varphi(w)$ is non-trivial.

(2) This case is almost identical to Case (1).

(3) We now have $w = x^{r_1} \cdot y^{s_1} \cdot \ldots \cdot x^{r_m} \cdot y^{s_m}$ where $r_1, \ldots, r_m, s_1, \ldots, s_m \in \mathbb{Z} \setminus \{0\}$. Next consider $\widetilde{g} := x^{r_1} \cdot w \cdot x^{-r_1}$. Note that $\widetilde{g}$ is represented by the reduced word $\widetilde{w} = x^{2r_1} \cdot y^{s_1} \cdot \ldots \cdot x^{r_m} \cdot y^{s_m} \cdot x^{-r_1}$. By (1) we know that $\widetilde{g} = \widetilde{w}$ is non-trivial. But this implies that $g = x^{-r_1} \cdot \widetilde{g} \cdot x^{r_1}$ is non-trivial.

(4) This case is almost identical to Case (3). ∎

The Ping Pong Proposition 310.19 is the key ingredient in the proof of the following lemma:

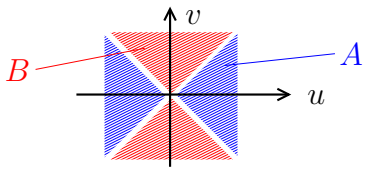
[3445] In other words, we claim that there exists a monomorphism $\varphi\colon F \to \mathrm{SL}(2,\mathbb{Z})$. But as polyglots we intend to speak the local patois that gets used in the different regions of mathematics.

Lemma 310.20. (Free Group–SL$(2,\mathbb{Z})$-Lemma) We consider the group $\mathrm{SL}(2,\mathbb{Z})$. The two matrices $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ form a basis for a free subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

Proof. We consider the set $X = \mathbb{R}^2$ together with the two subsets

$$\begin{aligned} A &= \{(u,v) \in \mathbb{R}^2 \,|\, |v| < |u|\}, \\ B &= \{(u,v) \in \mathbb{R}^2 \,|\, |v| > |u|\}. \end{aligned}$$

The group $G = \mathrm{SL}(2,\mathbb{Z})$ acts in an obvious way on $X = \mathbb{R}^2$.



By the Ping Pong Proposition 310.19 it suffices to prove the following claim:

Claim. (a) For every $n \in \mathbb{Z} \setminus \{0\}$ we have $x^n \cdot A \subseteq B$.
(b) For every $n \in \mathbb{Z} \setminus \{0\}$ we have $y^n \cdot B \subseteq A$.

Proof. Let $n \in \mathbb{Z} \setminus \{0\}$.

(a) Let $(u,v) \in A$, i.e. we have $(u,v) \in \mathbb{R}^2$ with $|u| > |v|$. Note that

$$x^n \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + 2n \cdot v \\ v \end{pmatrix} \in B.$$

it follows from the triangle inequality and $|u| < |v|$ that $|u + 2n \cdot v| > |v|$

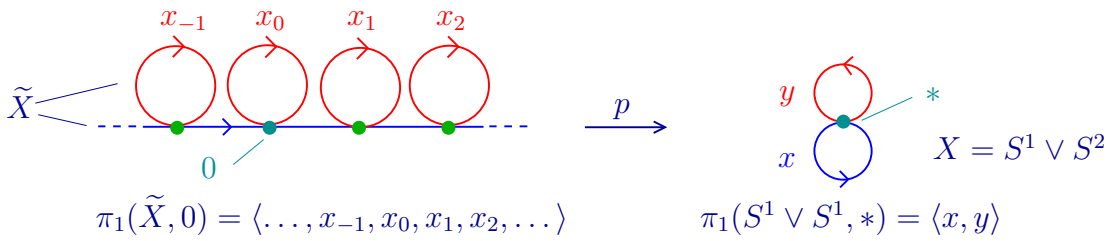
(b) The proof of this statement is almost identical to the proof of (a). ∎

Finally we can provide the proof of the Free Group–Faithful Representation Proposition 310.18:

Proof of the Free Group–Faithful Representation Proposition 310.18. Let F be a countable free group. By the Free Group–$\mathrm{SL}(2,\mathbb{Z})$-Lemma 310.20 it remains to prove the following claim:

Claim. The free group F admits a monomorphism to the group $\langle x, y \rangle$ on two generators.

Proof. This claim was proved on page 2053. Just for fun we now summarize the argument in the following figure:



In the figure we see a covering $p\colon \widetilde{X} \to X = S^1 \vee S^1$ of the wedge of two circles. By the Wedge-of-Circles-π_1-Lemma 99.11 we know that $\pi_1(X)$ is a free group on two generators

x and y. Furthermore by the Covering-π_1-Monomorphism Proposition 92.7 we know that the induced homomorphism $p_*\colon \pi_1(\widetilde{X},\widetilde{0}) \to \pi_1(X,*)$ is a monomorphism. Finally the discussion on page 2053 shows that $\pi_1(\widetilde{X},\widetilde{0})$ is a free group on $\mathbb{Z}$-many generators. ■

310.6. **Representation theory of abelian and finite groups.** In this last section we want to discuss some aspects of the representation theory of abelian groups and of finite groups.

For this discussion we will need the following definition:

Definition 310.21. Let G be a group and let R be a ring.

(1) A representation of $\alpha\colon G \to \mathrm{Aut}_{R\text{-left}}(M)$ is called reducible if $M = 0$ or if M admits an R-left module subsummand N with $0 \subsetneq N \subsetneq M$ such that G preserves N, i.e. such that for every $g \in G$ and $n \in N$ we have $\alpha(g)(n) \in N$. Otherwise we say that the representation is irreducible.

(2) A representation of G that is the direct sum of irreducible representations is called semisimple.

By the Group Ring–Module Lemma 310.5 these notions translate into equivalent notions about $(R,\mathbb{Z}[G])$-left left modules:

(1′) A $(R,\mathbb{Z}[G])$-left left module M is called reducible, if $M = 0$ or if M admits an R-left module subsummand N with $0 \subsetneq N \subsetneq M$ which is also a $\mathbb{Z}[G]$-left submodule. Otherwise we say that the $(R,\mathbb{Z}[G])$-left left module M is irreducible.

(2′) A $(R,\mathbb{Z}[G])$-left left module M that is the direct sum of irreducible $(R,\mathbb{Z}[G])$-left left modules is called semisimple.

Example.

(1) We consider the representation

$$\begin{array}{rcl} G = \mathbb{Z} & \to & \mathrm{GL}(2,\mathbb{C}) \\ n & \mapsto & \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \end{array}$$

This representation is reducible, since the subsummand $N = \mathbb{C} \times \{0\} \subseteq \mathbb{C}^2$ is preserved by G. In Exercise 310.14 we will show that the representation is not semisimple.

(2) Let G be a finite group and let R be a commutative ring. We consider the left regular R-representation $\lambda_G^R\colon G \to \mathrm{Aut}_R(R[G])$ from Definition 310.9. The R-submodule

$$N := R \cdot \underbrace{\sum_{g\in G} g}_{\substack{\text{defined since} \\ G \text{ is finite}}} \subseteq R[G]$$

is preserved by G. One can easily verify that N is a subsummand of $R[G]$. Thus, if G is a non-trivial group and if R is non-trivial, then the left regular R-representation λ_G^R of G is reducible. □

In the following two subsections we will consider the special cases which we mentioned above:

- In Section 310.6.1 we will discuss the representation theory of abelian groups.
- In Section 310.6.2 we will discuss the representation theory of finite groups.

310.6.1. **Representation theory of abelian groups.** The reader will not be surprised to hear that we will now consider representations of abelian groups:

Proposition 310.22. (Representations-of-Abelian Groups Proposition) If G is an abelian group, then every finite-dimensional complex representation $G \to \mathrm{Aut}_{\mathbb{C}}(M)$ of dimension ≥ 2 is reducible.

Proof. Let G be an abelian group and let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{C}}(M)$ be an m-dimensional complex representation with $m \geq 2$. Given $g \in G$ and given $\lambda \in \mathbb{C}$ we consider the λ-eigenspace

$$E_{g,\lambda} \;:=\; \{v \in M \,|\, \varphi(g)(v) = \lambda \cdot v\}$$

of the automorphism $\varphi(g)\colon M \to M$. We start out with two claims.

Claim 1. Given any $g \in G$ there exists a $\lambda \in \mathbb{C}$ with $E_{g,\lambda} \neq 0$.

Proof. This claim just follows from the standard linear algebra fact that every endomorphism of a finite-dimensional non-zero complex vector space has an eigenvalue. For the reader's convenience we recall the argument: After picking a basis for the complex vector space M we can describe $\varphi(g)$ by an $(m \times m)$-matrix A. Let

$$p(t) \;:=\; \det \underbrace{(t \cdot \mathrm{id}_m - A)}_{\substack{(m \times m)\text{-matrix} \\ \text{over } \mathbb{C}[t]}}$$

be the characteristic polynomial of A. It follows from the Polynomial Matrix–Determinant Lemma 123.11 that this is a polynomial with $\deg(p(t)) = m = \dim_{\mathbb{C}}(M) > 0$. By the Fundamental Theorem of Algebra 32.4, which we proved multiple times, the polynomial has a zero $\lambda \in \mathbb{C}$. This shows that $\lambda \cdot \mathrm{id}_M - \varphi(g)$ has non-zero kernel, which means that $E_{g,\lambda} \neq 0$. ⊞

Claim 2. Given any $g, h \in G$ and $\lambda \in \mathbb{C}$ we have $\varphi(h)(E_{g,\lambda}) \subseteq E_{g,\lambda}$.

Proof. Let $g \in G$ and let $v \in E_{g,\lambda}$. Next let $h \in G$. We need to show that $\varphi(h)(v) \in E_{g,\lambda}$. Indeed we see that

$$\begin{array}{ccccccc} & & \text{since } \varphi \text{ is a representation} & & \text{since } G \text{ is abelian} & & \text{since } \varphi \text{ is a representation} \\ & & \downarrow & & \downarrow & & \downarrow \\ \varphi(g)(\varphi(h)(v)) & = & \varphi(g \cdot h)(v) & = & \varphi(h \cdot g)(v) & = & \varphi(h)(\varphi(g)(v)) \\ & = & \varphi(h)(\lambda \cdot v) & = & \lambda \cdot \varphi(h)(v). & & \\ & & \uparrow & & \uparrow & & \\ & & \text{since } \lambda \in E_{g,\lambda} & & \text{since } \varphi(h) \in \mathrm{Aut}_{\mathbb{C}}(M) & & \end{array}$$

This shows that $\varphi(h)(v) \in E_{g,\lambda}$. ⊞

We now turn to the actual proof that the representation is reducible. We consider two cases:

- First we deal with the case that there exists a $g \in G$ and a scalar $\lambda \in \mathbb{C} \setminus \{0\}$ such that $0 < \dim_{\mathbb{C}}(E_{g,\lambda}) < \dim_{\mathbb{C}}(M)$. In this case it follows from Claim 2 that $E_{g,\lambda}$ is a vector subspace that is preserved by G. Thus we have shown that the representation is reducible in the sense of Definition 310.21.
- Now assume that for every $g \in G$ and every $\lambda \in \mathbb{C} \setminus \{0\}$ we have $\dim_{\mathbb{C}}(E_{g,\lambda}) = 0$ or $\dim_{\mathbb{C}}(E_{g,\lambda}) = \dim_{\mathbb{C}}(M)$. In this case it follows from Claim 1 that for each $g \in G$ there exists a $\lambda(g) \in \mathbb{C} \setminus \{0\}$ such that $\varphi(g)$ acts by multiplication by $\lambda(g)$. Thus we

see that each vector subspace is preserved by G. Since $\dim_{\mathbb{C}}(M) \geq 2$ we see that the representation is reducible. ■

Before we can state our next result we need to introduce some notation:

Notation 310.23. Let $n \in \mathbb{N}_0$ and let R be a ring. We denote by $\mathrm{UT}(n, R)$ the subgroup of $\mathrm{GL}(n, R)$ that is given by the upper triangular invertible $(n \times n)$-matrices with coefficients in R.

Proposition 310.24. (Abelian Group–Upper Triangular Representation Proposition) Let G be an abelian group and let $\varphi\colon G \to \mathrm{GL}(n, \mathbb{C})$ be a finite-dimensional complex representation. There exists a matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that for any $g \in G$ we have

$$A \cdot \varphi(g) \cdot A^{-1} \ \in \ \mathrm{UT}(n, \mathbb{C}).$$

Sketch of proof. This proposition can be proved easily by an induction argument on n and using the Free Module–Representation Classification Lemma 310.15. We leave it to the reader to provide the details. ■

Finally we specialize to the case of finite cyclic groups.

Proposition 310.25. (Cyclic Group–Representation Proposition) Let $n \in \mathbb{N}$. Given $z \in \mathbb{C}$ with $z^n = 1$ we consider the representation

$$\begin{array}{rcl} \varphi_z\colon \mathbb{Z}/n\mathbb{Z} & \to & \mathrm{U}(1) \\ [k] & \mapsto & z^k. \end{array}$$

(1) Every irreducible finite-dimensional complex representation of $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to one of the ρ_z.
(2) The complex left regular $\mathbb{C}$-representation $\lambda^{\mathbb{C}}_{\mathbb{Z}/n\mathbb{Z}}\colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[\mathbb{Z}/n\mathbb{Z}])$ from Definition 310.9 of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the direct sum of the representations $\varphi_{\exp(2\pi \mathrm{i} k/n)}$, $k = 0, \ldots, n-1$.

Proof. Let $n \in \mathbb{N}$.

(1) Let $\varphi\colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}_{\mathbb{C}}(M)$ be an irreducible finite-dimensional complex representation of $\mathbb{Z}/n\mathbb{Z}$. It follows from the Free Module–Representation Classification Lemma 310.15 that $\dim_{\mathbb{C}}(M) = 1$. Next note that it follows from the Free Module–Representation Classification Lemma 310.15 that φ is isomorphic to a representation of the form $\psi\colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(1, \mathbb{C})$. Since $\psi(1)^n = \psi(n \cdot 1) = \psi(0) = \mathrm{id}$ we see that $\psi = \varphi_z$ for some $z \in \mathbb{C}$ with $z^n = 1$.
(2) To keep the notation at bay we deal with the case $n = 3$. The general case is dealt with in the same way, just with more challenging latex issues. Thus we consider the left regular complex representation $\lambda^{\mathbb{C}}_{\mathbb{Z}/n\mathbb{Z}}\colon \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[\mathbb{Z}/3\mathbb{Z}])$. With respect to the basis given by $[0], [1], [2] \in \mathbb{Z}/3\mathbb{Z}$ of the $\mathbb{C}$-vector space $\mathbb{C}[\mathbb{Z}/3\mathbb{Z}]$ it is of the form

$$\rho(1) \ = \ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Next let $\xi = \exp(2\pi\mathrm{i}/3)$. We consider the Vandermonde matrix

$$P := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & (\xi^2)^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix} \quad \text{and we consider the diagonal matrix} \quad D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix}.$$

Note that by [**Came2008**, Proposition 4.27] the determinant of the Vandermonde matrix P is given by $\prod_{1\leq i<j\leq 3}(\xi^i - \xi^j)$, which is non-zero. Thus $P \in \mathrm{GL}(n,\mathbb{C})$. An elementary calculation shows that

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & (\xi^2)^2 \end{pmatrix}}_{=P} \cdot \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{=\rho(1)} = \begin{pmatrix} 1 & 1 & 1 \\ \xi & \xi^2 & 1 \\ \xi^2 & \xi^4 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix}}_{=D} \cdot \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & (\xi^2)^2 \end{pmatrix}}_{=P}.$$

This implies that $P \cdot \rho(1) \cdot P^{-1} = D$. This shows that the left regular complex representation $\lambda^{\mathbb{C}}_{\mathbb{Z}/n\mathbb{Z}} \colon \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[\mathbb{Z}/3\mathbb{Z}])$ is isomorphic to the representation that is given by

$$\begin{array}{ccc} \mathbb{Z}/n\mathbb{Z} & \to & \mathrm{GL}(n,\mathbb{C}) \\ k & \mapsto & D^k = \begin{pmatrix} 1^k & 0 & 0 \\ 0 & \xi^k & 0 \\ 0 & 0 & \xi^{2k} \end{pmatrix}. \end{array}$$

The desired conclusion follows easily from this observation. ■

310.6.2. **Representation theory of finite groups.** The following theorem has its origins in the work of Heinrich Maschke [**Masc1898, Masc1899**]:

Theorem 310.26. (Maschke's Theorem) Let G be a finite group and let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield. The following statements hold:

(1) Every finite-dimensional $\mathbb{K}$-representation is semisimple.
(2) Every irreducible finite-dimensional $\mathbb{K}$-representation is a subsummand of the left regular $\mathbb{K}$-representation $\lambda^{\mathbb{K}}_G \colon G \to \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}[G])$ from Definition 310.9.
(3) Every finitely generated $\mathbb{K}[G]$-left module is projective.
(4) Every finite-dimensional $\mathbb{C}$-representation $\varphi \colon G \to \mathrm{GL}(n,\mathbb{C})$ is conjugate to a unitary representation $G \to \mathrm{U}(n)$.

Example. Let G be a finite group and let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield. As we discussed page 5990 we have the $\mathbb{K}[G]$-left submodule $N = \mathbb{C} \cdot \sum_{g\in G} g$ of $\mathbb{K}[G]$. Note that it follows from Maschke's Theorem 310.26 that this is in fact a projective $\mathbb{K}[G]$-left module. If G is non-trivial this module is clearly not free, thus we obtain new examples of modules that are projective but not free. □

Proof. Let G be a finite group, let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield and let $\varphi \colon G \to \mathrm{GL}(n,\mathbb{K})$ be a finite-dimensional representation. To simplify the notation, given $g \in G$ and $v \in \mathbb{K}^n$ we henceforth write $g \cdot v := \varphi(g) \cdot v$. We consider the usual inner product

$$\begin{array}{ccc} \langle -,- \rangle \colon \mathbb{K}^n \times \mathbb{K}^n & \to & \mathbb{K} \\ (v,w) & \mapsto & v^{\mathrm{T}} \cdot \overline{w} \end{array}$$

on $\mathbb{K}^n$ which we first introduced on page 123.

Claim 1. Since G is finite it makes sense to consider the map

$$\begin{array}{rcl} \sigma\colon \mathbb{K}^n \times \mathbb{K}^n & \to & \mathbb{K} \\ (v,w) & \mapsto & \sum\limits_{g\in G} \langle g\cdot v, g\cdot w\rangle. \end{array}$$

This map has the following properties:

(a) σ is a sesquilinear hermitian form on $\mathbb{K}^n$ in the sense of Definition 125.16 and Definition 292.4.[3446]

(b) σ is positive-definite in the sense of Definition 2.27.

(c) The group G acts by isometries on $(\mathbb{K}^n, \sigma)$.

Proof. All the statements follow easily from the definitions. In the following we will, pars pro toto, show (b). So let $v \in \mathbb{K}^n \setminus \{0\}$ be a non-zero vector. Then we see that

$$\sigma(v,v) \;=\; \sum_{g\in G} \underbrace{\langle g\cdot v, g\cdot v\rangle}_{\substack{>0, \text{ since } \langle -,-\rangle \text{ is} \\ \text{positive definite}}} \;>\; 0.$$

⊞

Claim 2. Let $U \subseteq \mathbb{K}^n$ be a $\mathbb{K}[G]$-left submodule. Then there exists a $\mathbb{K}[G]$-left submodule V of $\mathbb{K}^n$ with $\mathbb{K}^n = U \oplus V$.

Proof. Let $U \subseteq \mathbb{K}^n$ be a $\mathbb{K}[G]$-left submodule. We set

$$V \;:=\; U^\perp \;:=\; \{v \in \mathbb{K}^n \mid \sigma(v,u) = 0 \text{ for all } u \in U\}.$$

It follows easily from Property (a) of σ that V is a $\mathbb{K}$-subvector space of $\mathbb{K}^n$. Next we claim that V is in fact a $\mathbb{K}[G]$-left submodule of $\mathbb{K}^n$. This can be seen as follows: Let $v \in V$ and let $g \in G$. We need to show that $g\cdot v \in U^\perp$. But this is indeed the case since for any $u \in U$ we have

$$\sigma(g\cdot v, u) \;=\; \sigma(g\cdot v, g\cdot g^{-1}\cdot u) \underset{\substack{\uparrow \\ \text{by Property (c) of } \sigma}}{=} \sigma(v, g^{-1}\cdot u) \underset{\substack{\uparrow \\ \text{since } g^{-1}\cdot u \in U \text{ and} \\ \text{since } v \in V = U^\perp}}{=} 0.$$

Finally note that it follows easily from the hypothesis that $\mathbb{K}$ is a subfield, and Property (c) that $\mathbb{K}^n = U \oplus V$. ⊞

Now we can turn to the proofs of the actual statements of the theorem.

(1) We need to show that every finite-dimensional $\mathbb{K}$-representation of G is semisimple. This follows easily from Claim 2 via an elementary induction argument which is cleverly left to the reader.

(2) Let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{K}}(M)$ be an irreducible finite-dimensional $\mathbb{K}$-representation. We pick a non-zero $v \in M$ and we consider the $\mathbb{K}[G]$-left module homomorphism

$$\begin{array}{rcl} \psi\colon \mathbb{K}[G] & \to & M \\ f & \mapsto & f\cdot v. \end{array}$$

Since $v \neq 0$ we see that $\psi(\mathbb{K}[G])$ is a non-zero $\mathbb{K}[G]$-left submodule of M. Since $\mathbb{K}$ is a field we see that $\psi(\mathbb{K}[G])$ is a subsummand of M. Finally, since the representation is irreducible we obtain from Definition 310.21 that $\psi(\mathbb{K}[G]) = M$, i.e. ψ is an epimorphism. Also note that $\ker(\psi)$ is a $\mathbb{K}[G]$-left submodule of $\mathbb{K}[G]$. It follows from Claim

[3446] Recall that "sesquilinear" means linear in the first entry, antilinear in the second entry.

2 that there exists a $\mathbb{K}[G]$-left submodule V of $\mathbb{K}[G]$ such that $\mathbb{K}[G] = \ker(\psi) \oplus V$. One can now easily verify that ψ restricts to an isomorphism $V \to M$ of $\mathbb{K}[G]$-left modules. Thus the $\mathbb{K}[G]$-left module M is indeed isomorphic to a subsummand of $\mathbb{K}[G]$.



(3) Let M be a finitely generated $\mathbb{K}[G]$-left module. Since M is finitely generated and since G is finite we see that $\dim_{\mathbb{K}}(M)$ is finite. It follows from (1) that M is the direct sum of finitely many irreducible $\mathbb{K}[G]$-left modules $P_1, \ldots, P_n$. By (2) we know that each P_i is a subsummand of $\mathbb{K}[G]$. It follows that M is a subsummand of the free $\mathbb{K}[G]$-left module $\mathbb{K}[G]^n$. It thus follows from the Projective Module–Characterization Lemma 309.28 that M is a projective $\mathbb{K}[G]$-left module.

(4) By Claim (1a) and (1b) we know that σ is a sesquilinear hermitian positive definite form on $\mathbb{C}^n$. Elementary linear algebra tells us that any sesquilinear hermitian positive definite form on $\mathbb{C}^n$ is isometric to the standard inner product.[3447] It thus follows that there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that for all $v, w \in \mathbb{C}^n$ we have $\langle P \cdot v, P \cdot w \rangle = \sigma(v, w)$. Next we consider the representation that is given by

$$\begin{array}{ccc} G & \to & \mathrm{GL}(n, \mathbb{C}) \\ g & \mapsto & P \cdot \varphi(g) \cdot P^{-1}. \end{array}$$

We now claim that this representation acts by isometries on $(\mathbb{K}^n, \langle -, - \rangle)$. Indeed, for any $v, w \in \mathbb{C}^n$ we have the following equalities:

$$\begin{array}{lll} \langle P \cdot \varphi(g) \cdot P^{-1} \cdot v, P \cdot \varphi(g) \cdot P^{-1} \cdot w \rangle & \overset{\text{by choice of } P}{\downarrow} & \\ & = \sigma(\varphi(g) \cdot P^{-1} \cdot v, \varphi(g) \cdot P^{-1} \cdot w) & \\ & = \sigma(P^{-1} \cdot v, P^{-1} \cdot w) & = \langle v, w \rangle. \\ & \overset{\text{by Claim (1c)}}{\uparrow} & \overset{\text{by choice of } P}{\uparrow} \end{array}$$

This implies that the representation $g \mapsto P \cdot \varphi(g) \cdot P^{-1}$ is unitary. ■

In the Representations-of-Abelian Groups Proposition 310.22 we learned a few basic facts about the representation theory of abelian groups and in Maschke's Theorem 310.26 we also learned something about the representation theory of finite groups. We can combine these facts to obtain results about the representation theory of finite abelian groups:

Proposition 310.27. (Finite Abelian Group–Representation Proposition) Every finite-dimensional complex representation of a finite abelian group is the direct sum of one-dimensional complex representations.

Proof. Let G be a finite abelian group. Since G is finite we obtain from Maschke's Theorem 310.26 that every finite-dimensional complex representation of G is semisimple, i.e. the representation is the direct sum of irreducible representations. The proposition now

[3447] Here we sketch a proof of this statement. In light of the Form–Matrix-Lemma 292.6 it suffices to prove the following statement: If $H \in \mathrm{M}(n \times n, \mathbb{C})$ is a hermitian matrix such that for any $v \in \mathbb{C}^n \setminus \{0\}$ we have $v^{\mathrm{T}} H \overline{v} > 0$, then there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P^{\mathrm{T}} H \overline{P} = \mathrm{id}_n$. This statement follows fortunately from the Signature-Eigenvalues Proposition 292.13 together with the Hermitian Matrices–Congruence Proposition 292.3. (This statement can be viewed as the hermitian analog of Sylvester's Theorem 226.12.)

follows from the Representations-of-Abelian Groups Proposition 310.22 which implies that every irreducible non-zero complex representation of an abelian group is necessarily one-dimensional. ■

We conclude this chapter with a theorem that we will neither use nor prove, but which is perhaps worth knowing:

Theorem 310.28. (Finite Group–Complex Regular Representation Theorem) Let G be a finite group. Let $V_1, \ldots, V_n$ be the isomorphism types of irreducible $\mathbb{C}$-representations of G. Then the left regular $\mathbb{C}$-representation $\mathbb{C}[G]$ is isomorphic to the direct sum $V_1^{\dim_{\mathbb{C}}(V_1)} \oplus \cdots \oplus V_n^{\dim_{\mathbb{C}}(V_n)}$.

Example. In the Cyclic Group–Representation Proposition 310.25 we verified this statement explicitly in the basic case that $G = \mathbb{Z}/n\mathbb{Z}$ is a finite cyclic group. □

Proof. This theorem is proved in [**Ser1977**, Corollary 1, p. 18]. ■

310.7. Exercises for Chapter 310.

Exercise 310.1. Let G be a group. Is the group ring $\mathbb{Z}[G]$ necessarily isomorphic to its opposite ring $\mathbb{Z}[G]^{\mathrm{op}}$, which we introduced in Definition 309.6?

Exercise 310.2. Let G be a group and let $H \subseteq G$ be a subgroup. We can view $\mathbb{Z}[G]$ as a $\mathbb{Z}[H]$-left module in the obvious way. Show that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-left module.

Exercise 310.3. Let G be a group and let $H \subseteq G$ be a subgroup. Let $\varphi\colon \mathbb{Z}[H] \to \mathbb{Z}[G]$ be the ring homomorphism that is induced by the inclusion $H \to G$. Show that if P is a projective $\mathbb{Z}[G]$-left module, then ${}_{\varphi}P$ is a projective $\mathbb{Z}[H]$-left module.
Remark. You could make use of Exercise 310.2.

Exercise 310.4. Let R be a ring and let $\mu\colon R \to R$ be a ring automorphism. We consider the abelian group

$$R_{\mu}[t^{\pm 1}] \ := \ \bigoplus_{n \in \mathbb{Z}} R \cdot t^n$$

where the multiplication is induced by the multiplication on R, the obvious multiplication on the powers of t and the rule that $t \cdot r := \mu(r) \cdot t$.

(a) Show that $R_{\mu}[t^{\pm 1}]$ is indeed a ring. Such a ring is called a *twisted Laurent polynomial ring over R*.

(b) Let π be a group which admits an epimorphism $\Phi\colon \pi \to \langle t \rangle$. Show that the group ring $\mathbb{Z}[\pi]$ can be written as a twisted Laurent polynomial ring.

Exercise 310.5. Decompose the left regular $\mathbb{C}$-representation $\mathbb{C}[S_3]$ of the permutation group S_3 into irreducible representations.

Exercise 310.6. We consider the permutation group S_3. Give an explicit monomorphism $S_3 \to \mathrm{GL}(2, \mathbb{Z})$

Exercise 310.7. For which $n \in \mathbb{N}$ is the group ring $\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ isomorphic to the subring $\mathbb{Q}(\exp(2\pi \mathrm{i}/n))$ of $\mathbb{C}$?

Exercise 310.8. What is the smallest dimension of a non-abelian complex representation of the permutation group S_4 that you can find?

Exercise 310.9. Let $n \in \mathbb{N}$.

(a) Show that there exists a faithful representation $\mathbb{Z}/n\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$.

(b) As on page 1815 we consider the dihedral group D_n which is given by the set $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ but where the multiplication is given by

$$(h,m)\cdot(h',m') \coloneqq (h\cdot(-1)^m\cdot h', m+m').$$

Show that there exists a faithful representation $D_n \to \mathrm{SL}(2,\mathbb{C})$.

(c) Decompose the left regular $\mathbb{C}$-representation $\mathbb{C}[D_n]$ into a direct sum of irreducible representations.

Exercise 310.10. Let G be a finite group. We consider the $\mathbb{Q}[G]$-module $A = \mathbb{Q}$ that is given by the trivial G-action on $\mathbb{Q}$.

(a) Provide an *explicit* monomorphism $\iota\colon A \to \mathbb{Q}[G]$ of $\mathbb{Q}[G]$-modules.

(b) Provide an *explicit* $\mathbb{Q}[G]$-submodule B of $\mathbb{Q}[G]$ such that $\mathbb{Q}[G]$ is isomorphic to the $\mathbb{Q}[G]$-module $A \oplus B$.

Exercise 310.11. We consider the left regular $\mathbb{Z}$-representation $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}])$. Is this representation semisimple?

Exercise 310.12. Let $n \in \mathbb{N}_0$.

- Let $\mathrm{UT}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$ be the subgroup of invertible upper triangular matrices.
- Let $\mathrm{D}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$ be the subgroup of invertible diagonal matrices.

Let $G \subseteq \mathrm{UT}(n,\mathbb{C})$ be a finite subgroup. Is G necessarily a subgroup of $\mathrm{D}(n,\mathbb{C})$?

Exercise 310.13. We consider the matrix group $\mathrm{SL}(2,\mathbb{Z})$ together with the two matrices $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Do these two matrices form a basis for a free subgroup of $\mathrm{SL}(2,\mathbb{Z})$?

Exercise 310.14. Show that the representation

$$\begin{array}{rcl} G = \mathbb{Z} & \to & \mathrm{GL}(2,\mathbb{C}) \\ n & \mapsto & \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \end{array}$$

is not semisimple.

Exercise 310.15. Let A be a finite abelian group.

(a) How many representations $A \to \mathrm{GL}(1,\mathbb{C})$ do exist?

(b) We consider the group ring $\mathbb{C}[A]$. Give an explicit/algorithmic decomposition of the left regular representation $\mathbb{C}[A]$ into 1-dimensional irreducible complex representations of A.

Exercise 310.16. We consider the group $(\mathbb{Q},+)$.

(a) Does $(\mathbb{Q},+)$ admit a faithful finite-dimensional representation over $\mathbb{R}$?

(b) Does $(\mathbb{Q},+)$ admit a faithful finite-dimensional representation over $\mathbb{Z}$?

Exercise 310.17. We consider the following homeomorphism:

$$\begin{array}{rcl} f\colon [0,1] & \to & [0,1] \\ t & \mapsto & \begin{cases} 4t, & \text{if } t \in [0,\frac{1}{5}], \\ \frac{4}{5} + \frac{1}{4}(t-\frac{1}{5}), & \text{if } t \in [\frac{1}{5},1]. \end{cases} \end{array}$$

(a)
- Show that for all $n \in \mathbb{N}$ we have $f^n([\frac{1}{5}, 1]) \subseteq [\frac{4}{5}, 1]$.
- Show that for all $n \in \mathbb{N}$ we have $f^{-n}([0, \frac{4}{5}]) \subseteq [0, \frac{1}{5}]$

(b) Given $t \in \mathbb{R}$ we write $\lfloor t \rfloor := \max\{k \in \mathbb{Z} \,|\, k \leq t\}$ and we consider the homeomorphism $\theta\colon \mathbb{R} \to \mathbb{R}$ that is given by $\theta(t) = t - \frac{1}{2}$. We consider the following two homeomorphisms of $\mathbb{R}$:

$$\begin{array}{rcl} \alpha\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \lfloor t \rfloor + f(t - \lfloor t \rfloor) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \theta \circ \alpha \circ \theta^{-1}. \end{array}$$

We also set

$$A := \bigcup_{k \in \mathbb{Z}} [k - \tfrac{1}{5}, k + \tfrac{1}{5}] \qquad \text{and} \qquad B := \theta(A).$$

Show that for any $n \in \mathbb{Z} \setminus \{0\}$ we have $\alpha^n(B) \subseteq A$ and $\beta^n(A) \subseteq B$.

(c) Use the Ping Pong Proposition 310.19 to show that the group $\mathrm{Homeo}(\mathbb{R})$ of self-homeomorphisms of the topological space $\mathbb{R}$ contains a free group on two generators.

(d) Show that for every non-empty 1-dimensional topological manifold of dimension ≥ 1 the group $\mathrm{Homeo}(X)$ contains a free group on two generators.

(e) Show that for every non-empty topological manifold the group $\mathrm{Homeo}(X)$ contains a free group on two generators.

Exercise 310.18. We consider the quaternion group Q_8 which is defined as the subgroup of the unit quaternions $\mathbb{H} \cap S^3$ that is generated by $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$. In other words, as a set the quaternion group is given by $Q_8 = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$.

(a) Show that Q_8 has eight elements.

(b) What are the dimensions of the irreducible complex representations of Q_8?

(c) Provide an explicit irreducible complex representation of Q_8 of dimension ≥ 2.

Remark. You can use the fact from [**MKS1976**, p. 10] that the map

$$\langle a, b \,|\, ab = b^{-1}a, ba = a^{-1}b \rangle \;\to\; Q_8$$

that is given by $a \mapsto \mathrm{i}$ and $b \mapsto \mathrm{j}$ is an isomorphism.

Exercise 310.19. Let $A, B \in \mathrm{GL}(n, \mathbb{C})$ be two diagonalizable matrices. Show that if A and B commute, then we can diagonalize A and B simultaneously, in other words, show that there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both diagonal.

CHAPTER 311

The tensor product of abelian groups and of rings

In this chapter we will first (re-) introduce the tensor product of two abelian groups. We will use this tensor product to define the tensor product of two rings. This notion in turn will allow us to put bimodules and left-left modules into the framework of left-modules.

This chapter is organized as follows:

- In Section 311.1 we will (re-) introduce the tensor product of two abelian groups.
- In Section 311.2 we will introduce the tensor product of two rings.
- In Section 311.3 we will use the tensor product of two rings to express bimodules and left-left modules as left modules over a suitable ring.
- In Section 311.4 we will provide exercises for Chapter 311.

In the next Chapter 312 we will study tensor products of modules over rings.

311.1. **The tensor product of abelian groups.** We start out by repeating the content of Definition 2.24 and Definition 159.1:

Definition 311.1. Let A, B and C be abelian groups. We say that a map $f\colon A \times B \to C$ is bilinear, if if for all $a, a' \in A$, $b, b' \in B$ we have

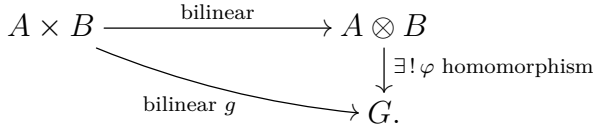
$$f(a+a', b) \;=\; f(a,b) + f(a',b) \quad \text{and} \quad f(a, b+b') \;=\; f(a,b) + f(a,b').$$

Example. Let A, B and C be abelian groups and let $f\colon A \times B \to C$ be a bilinear map. As on page 3051 we see that for all $a \in A$, $b \in B$ and $m \in \mathbb{Z}$ we have the equality

$$f(m \cdot a, b) \;=\; f(a, m \cdot b) \;=\; m \cdot f(a,b).$$

We will use this basic fact frequently. □

Our next goal is the following: Given abelian groups A and B we want to find "the universal bilinear map out of $A \times B$". More precisely, we want to find an abelian group, let's call it $A \otimes B$, together with a bilinear map $A \times B \to A \otimes B$ such that given any abelian group G and given any bilinear map $g\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ of abelian groups such that the following diagram commutes:



To achieve this goal we recall Definition 159.2:

Definition 311.2. Let A and B be abelian groups.

(1) We consider the free abelian group $\mathbb{Z}^{(A\times B)}$ that is generated by the set $A \times B$. We define the tensor product $A \otimes B$ to be the following abelian group:

$$A \otimes B \ := \ \mathbb{Z}^{(A\times B)}/N(A,B)$$

where $N(A,B)$ is the subgroup of $\mathbb{Z}^{(A\times B)}$ generated by the elements of the form

$$((a+a'),b) - ((a,b)+(a',b)) \qquad \text{and} \qquad (a,(b+b')) - ((a,b)+(a,b'))$$

with $a, a' \in A$ and $b, b' \in B$.

(2) For $a \in A$ and $b \in B$ we write

$$a \otimes b \ := \ [(a,b)] \quad \in \ \mathbb{Z}^{(A\times B)}/N(A,B) =: A \otimes B.$$

Such an element of $A \otimes B$ is sometimes called an elementary tensor.

The following lemma is identical to the Tensor Product–Basics Lemma 159.3:

Lemma 311.3. (Tensor Product–Basics Lemma) Let A and B be abelian groups.

(1) The tensor product $A \otimes B$ is an abelian group.

(2) Any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$.

Proof.

(1) This statement follows from the fact that $A \otimes B$ is defined as the quotient of the free abelian group $\mathbb{Z}^{(A\times B)}$ by a (necessarily normal) subgroup $N(A,B)$.

(2) It follows immediately from the definition of the tensor product that any element in the tensor product $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i)$, with $r_1, \ldots, r_n \in \mathbb{Z}$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$. Thus it remains to prove the following claim:

Claim. For any $a \in A$, $b \in B$ and $m \in \mathbb{Z}$ we have $m \cdot (a \otimes b) = (m \cdot a) \otimes b \in A \otimes B$.

Proof. Note that, essentially by definition of $N(A,B)$, the map $A \times B \to A \otimes B$ that is given by $(a,b) \mapsto a \otimes b$ is bilinear. The claim now follows from the discussion on page 5999. ∎

The following lemma is identical to the Tensor Product–Universal Property Lemma 159.4:

Lemma 311.4. (Tensor Product–Universal Property Lemma) Let A and B be abelian groups.

(1) The map

$$\begin{array}{rccc} f\colon & A \times B & \to & A \otimes B \\ & (a,b) & \mapsto & a \otimes b \end{array}$$

is bilinear. In particular the following statements hold:

(a) For any $a, a' \in A$ and any $b, b' \in B$ we have

$$\begin{array}{rcl} (a+a') \otimes b & = & a \otimes b + a' \otimes b \ \in \ A \otimes B \\ a \otimes (b+b') & = & a \otimes b + a \otimes b' \ \in \ A \otimes B. \end{array}$$

(b) For any $a \in A, b \in B$ and any $m \in \mathbb{Z}$ we have

$$(m \cdot a) \otimes b \ = \ a \otimes (m \cdot b) \ = \ m \cdot (a \otimes b).$$

(2) Given any abelian group G and given any bilinear map $g\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ of abelian groups such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow[\text{bilinear}]{(a,b)\mapsto a\otimes b} & A \otimes B \\ & \searrow^{\text{bilinear } g} & \downarrow \exists!\,\varphi \text{ homomorphism} \\ & & G. \end{array}$$

In fact this homomorphism is given by[3448]

$$\begin{array}{rcl} \varphi\colon A \otimes B & \to & G \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i). \end{array}$$

Example. Let us show, just for entertainment, that for $p, q \in \mathbb{N}$ with $\gcd(p,q) = 1$ we have $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = 0$. By the Tensor Product–Basics Lemma 311.3 (2) it suffices to show that each $a \otimes b = 0$. We start out with one observation: Since $\gcd(p,q) = 1$ there exists an $s \in \mathbb{Z}$ with $p \cdot s \equiv 1 \bmod q$. We now see that

$$a \otimes b = a \otimes 1 \cdot b = a \otimes p \cdot s \cdot b \underset{\uparrow}{=} a \cdot p \otimes s \cdot b = 0 \cdot 1 \otimes s \cdot b \underset{\uparrow}{=} 0 \cdot (1 \otimes s \cdot b) = 0.$$

by the Tensor Product–Universal Property Lemma 311.4 (1b)

by the Tensor Product–Universal Property Lemma 311.4 (1a) □

Proof. [3449] Let A and B be two abelian groups.

(1) We defined $N(A,B)$ exactly in such a way to ensure that the map

$$\begin{array}{rcl} f\colon A \times B & \to & A \otimes B := \mathbb{Z}^{(A\times B)}/N(A,B) \\ (a,b) & \mapsto & a \otimes b \end{array}$$

is bilinear. Statement (a) follows immediately from Definition 311.1 of "bilinear" and Statement (b) follows from the discussion on page 5999.

(2) Let G be an abelian group and let $g\colon A \times B \to G$ be a bilinear map.

Claim. The map

$$\begin{array}{rcl} A \otimes B & \to & G \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i) \end{array}$$

[3448] This universal property gives us the usual uniqueness property: Let H be an abelian group and let $g\colon A \times B \to H$ be a bilinear map. If this bilinear map satisfies the universal property of (2), then the standard argument (as in the proof of the Product Uniqueness Lemma 6.2) shows that there exists a unique *isomorphism* $\varphi\colon A \otimes B \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow[\text{bilinear}]{(a,b)\mapsto a\otimes b} & A \otimes B \\ & \searrow^{\text{bilinear } g} & \cong\downarrow \exists!\,\varphi \text{ isomorphism} \\ & & H. \end{array}$$

[3449] We provide the proof from page 3053 again, since in the next chapter we will need a very similar argument for a more general setting. So it is good to bring back to memory these arguments.

is a well-defined homomorphism and it is the unique homomorphism $\varphi\colon A \otimes B \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b)\mapsto a\otimes b} & A \otimes B \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & G. \end{array}$$

Proof. First note that it follows from the fact that $A \times B$ is a basis for the free abelian group $\mathbb{Z}^{(A\times B)}$ that there exists a *unique* homomorphism

$$\mathbb{Z}^{(A\times B)} \to G$$

which sends each $(a, b) \in A \times B$ to $g(a, b)$. Since g is bilinear we see that the subgroup $N(A, B)$ lies in the kernel of the above homomorphism. It follows from the Quotient Group Lemma 2.4 that the above homomorphism descends to a well-defined homomorphism

$$\begin{array}{rcl} \varphi\colon A \otimes B \coloneqq \mathbb{Z}^{(A\times B)}/N(A, B) & \to & G \\ \left[\sum_{i=1}^{n} (a_i, b_i)\right] = \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i). \end{array}$$

It is clear that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b)\mapsto a\otimes b} & A \otimes B \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & G. \end{array}$$

Finally note that it follows easily from the fact that $\mathbb{Z}^{(A\times B)} \to A \otimes B$ is an epimorphism that φ is uniquely determined. ∎

We also recall Definition 159.6:

Definition 311.5. Let $f\colon A \to A'$ and $g\colon B \to B'$ be homomorphisms between abelian groups. It follows from the Tensor Product–Universal Property Lemma 311.4 (1) that the map

$$\begin{array}{rcl} A \times B & \to & A' \otimes B' \\ (a, b) & \mapsto & f(a) \otimes g(b) \end{array}$$

is bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 311.4 (2) that the map

$$\begin{array}{rcl} f \otimes g\colon A \otimes B & \to & A' \otimes B' \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} f(a_i) \otimes g(b_i) \end{array}$$

is a well-defined homomorphism of abelian groups. We refer to $f \otimes g$ as the tensor product of the homomorphisms f and g.

The following lemma, which is identical to the Tensor–Map Composition Lemma 311.6, summarizes a few key properties of the above definition:

Lemma 311.6. (Tensor–Map Composition Lemma)
(1) Given any abelian groups A and B we have $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B}$.

(2) Let $f\colon A \to A'$, $f'\colon A' \to A''$, $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms between abelian groups. Then we have

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon A \otimes B \;\to\; A'' \otimes B''.$$

(3) Let $f, g\colon A \to B$ be homomorphisms between abelian groups and let G be an abelian group. Then

$$(f+g) \otimes \mathrm{id}_G \;=\; f \otimes \mathrm{id}_G + g \otimes \mathrm{id}_G\colon A \otimes G \;\to\; B \otimes G.$$

The same statement also holds if we swap the roles of the two sides.

We will not recall all the statements that we proved about tensor products of abelian groups. But as one sample result we recall the following lemma:

Lemma 159.9. (Tensor-with-$\mathbb{Z}$–Lemma) Let A be an abelian group. The maps

$$\begin{array}{rcl} \Theta\colon \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i \cdot a_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Omega\colon A & \to & \mathbb{Z} \otimes A \\ a & \mapsto & 1 \otimes a \end{array}$$

are well-defined homomorphisms, they are natural and they are inverses to one another. The analogous statements also hold if the $\mathbb{Z}$-term is on the right instead of the left.

311.2. **The tensor product of two rings.** The following lemma, which provides a solution to Exercise 159.8 and which is related to the Super Algebra–Tensor Product Lemma 208.11, shows that if we start out with two rings R and S, then the corresponding tensor product $R \otimes S$ is naturally a ring.

Lemma 311.7. (Tensor Product-of-Rings Lemma)

(1) Let R and S be two rings (not necessarily commutative). If we consider the underlying abelian groups then we can consider the corresponding tensor product $R \otimes S$. There exists a unique bilinear map

$$\cdot\colon R \otimes S \times R \otimes S \;\to\; R \otimes S$$

such that for all $r, \widetilde{r} \in R$ and all $s, \widetilde{s} \in S$ we have the following equality in $R \otimes S$:

$$(r \otimes s) \cdot (\widetilde{r} \otimes \widetilde{s}) \;=\; r \underset{R}{\cdot} \widetilde{r} \otimes s \underset{S}{\cdot} \widetilde{s}.$$

This multiplication defines a natural ring structure on the abelian group $R \otimes S$. If R and S are commutative, then so is the ring $R \otimes S$.

(2) Let $f\colon R \to R'$ and $g\colon S \to S'$ be homomorphisms between rings. From Definition 311.5 we get an induced homomorphism $f \otimes g\colon R \otimes S \to R' \otimes S'$ of abelian groups. This map is a ring homomorphism.

Definition 311.8. Let R and S be two rings. We refer to the ring $R \otimes S$ from the Tensor Product-of-Rings Lemma 311.7 as the tensor product of the rings R and S.[3450]

Example. Let R be a ring. It follows from the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that the natural map

$$\begin{array}{rcl} R & \to & R \otimes \mathbb{Z} \\ r & \mapsto & r \otimes 1 \end{array}$$

[3450]The name "tensor product" of the rings R and S is a little misleading, since this name does not reflect the categorical properties of $R \otimes S$. As we will see soon in the Ring–Coproduct Lemma 311.10, it would be more appropriate to call $R \otimes S$ the coproduct of R and S.

is an isomorphism of abelian groups. Considering the description of the multiplicative structure on the right hand side from the Tensor Product-of-Rings Lemma 311.7 we see that the above map is actually a natural isomorphism $R \to R \otimes \mathbb{Z}$ of rings. □

Proof.

(1) Let R and S be two rings. Given $r \in R$ we denote by $\lambda_r \colon R \to R$ the group homomorphism that is given by left-multiplication by r, similarly we define $\lambda_s \colon S \to S$. Let $w \in R \otimes S$. We want to define "right multiplication by w". To do so we consider the map

$$\begin{array}{ccc} R \times S & \to & R \otimes S \\ (r, s) & \mapsto & \underbrace{(\lambda_r \otimes \lambda_s)}_{\substack{\text{homomorphism} \\ R \otimes S \to R \otimes S \\ \text{from Definition 311.5}}} (w). \end{array}$$

Using the Tensor–Map Composition Lemma 311.6 it is straightforward to verify that this map is bilinear. It thus follows from the Tensor Product–Universal Property Lemma 311.4 (2) that there exists a unique homomorphism

$$\begin{array}{ccc} R \otimes S & \to & R \otimes S \\ v & \mapsto & v \cdot w \end{array}$$

that makes the following diagram commute:

$$\begin{array}{ccc} R \times S & \xrightarrow{(r,s) \mapsto r \otimes s} & R \otimes S \\ & \searrow^{(r,s) \mapsto (\lambda_r \otimes \lambda_s)(w)} & \downarrow{\scriptstyle v \mapsto v \cdot w} \\ & & R \otimes S. \end{array}$$

In summary we have defined a map

$$\begin{array}{ccc} R \otimes S \times R \otimes S & \to & R \otimes S \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

It follows from the Tensor Product–Basics Lemma 311.3 (2) that this map is uniquely determined by the values on elements of the form $(r \otimes s) \cdot (\widetilde{r} \otimes \widetilde{s})$.

We leave it to the reader to verify that this way we defined a natural ring structure on $R \otimes S$. Furthermore we also leave it to the reader to verify that if the rings R and S are commutative, then so is the ring $R \otimes S$.

(2) This statement follows easily from the defining property of the ring structure on tensor products together with the Tensor Product–Basics Lemma 311.3 (2). ■

Lemma 311.9. (Tensor Product-of-Rings–Modules Lemma) Let R and S be two wrings, let M be an R-left module and let N be an S-left module. There exists a unique natural $R \otimes S$-left module structure on $M \otimes N$ with the property that for any $r \in R$, $s \in S$, $m \in M$ and $n \in N$ we have

$$(r \otimes s) \cdot (m \otimes n) \;=\; (r \cdot m) \otimes (s \otimes n).$$

Furthermore the following two statements hold:

(1) If M is free on a set $X \subseteq M$ and if N is free on a set $Y \subseteq N$, then the $R \otimes S$-left module $M \otimes N$ is free on the set $\{(x \otimes y) \,|\, x \in M, y \in N\}$.

(2) If M is projective and N is projective, then $M \otimes N$ is a projective $R \otimes S$-module.

Sketch of proof. The first statements of the lemma follow easily from the definitions.

(1) This statement following from the following two observations: If X and Y consist of a single element, then the statement follows again easily from the definitions. The general case then follows from the Tensor–Direct Sum Lemma 312.15.
(2) This statement follows fairly easily from (1) together with Projective Module–Characterization Lemma 309.28. ■

For friends of category theory we conclude this section with the following lemma:

Lemma 311.10. (Ring–Coproduct Lemma) Let R and S be two commutative rings. We consider the tensor product ring $R \otimes S$ from the Tensor Product-of-Rings Lemma 311.7. We introduce two homomorphisms:

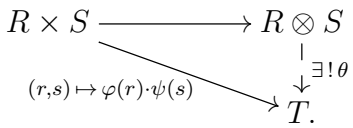
- We consider the ring homomorphism $i\colon R \xrightarrow{r \mapsto r \otimes 1} R \otimes \mathbb{Z} \xrightarrow{\mathrm{id}_R \otimes \iota} R \otimes S$ where $\iota\colon \mathbb{Z} \to S$ is the ring homomorphism that is given by $1 \mapsto 1_S$.
- Similarly we define $j\colon S \xrightarrow{s \mapsto 1 \otimes s} \mathbb{Z} \otimes S \xrightarrow{\iota \otimes \mathrm{id}_S} R \otimes S$.

The tensor product ring $R \otimes S$ together with the above ring homomorphisms $i\colon R \to R \otimes S$ and $j\colon S \to R \otimes S$ is the coproduct of R and S in the category *CommRing* of commutative rings.[3451]

Example.

(1) We consider the Laurent polynomial rings $\mathbb{Z}[x^{\pm 1}]$ and $\mathbb{Z}[y^{\pm 1}]$. We have the obvious ring homomorphisms $\mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and $\mathbb{Z}[y^{\pm 1}] \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. It follows from the Ring–Coproduct Lemma 311.10 that these induce a ring homomorphism
$$\mathbb{Z}[x^{\pm 1}] \otimes \mathbb{Z}[y^{\pm 1}] \ \to \ \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$$
with $x \otimes 1 \mapsto x$ and $y \otimes 1 \mapsto y$. In fact one can show quite easily that the right hand side is also a coproduct in the category *CommRing* of commutative rings. It thus follows from the Coproduct Uniqueness Lemma 6.6 that the above map is actually an isomorphism of rings.
(2) Let G be a group and let R be a commutative ring.
   (a) Similar to (1) we see that the obvious ring homomorphisms $\mathbb{Z}[G] \to R[G]$ and $R \to R[G]$ give rise to a ring isomorphism $R \otimes \mathbb{Z}[G] \to R[G]$.
   (b) Let A be a $\mathbb{Z}[G]$-left module. We consider $R \otimes_{\mathbb{Z}} A$. It follows from (a) together with the Tensor Product-of-Rings–Modules Lemma 311.9 that $R \otimes_{\mathbb{Z}} A$ is naturally a $R[G]$-left module. □

Proof. Let T be a commutative ring and let $\varphi\colon R \to T$ and $\psi\colon S \to T$ be two ring homomorphisms. We consider the following diagram:



[3451] By Definition 6.5 this means that for any ring T and any two ring homomorphisms $\varphi\colon R \to T$ and $\psi\colon S \to T$ there exists a unique ring homomorphism $\Theta\colon R \otimes S \to T$ such that $\Theta \circ i = \varphi$ and $\Theta \circ j = \psi$.

The left diagonal map is clearly bilinear. It thus follows from the Tensor Product–Universal Property Lemma 311.4 that we get a unique induced vertical homomorphism $\varphi\colon R\otimes S\to T$ of abelian groups that makes the diagram commute. It remains to prove the following claim:

Claim. The vertical map $\theta\colon R\otimes S\to T$ is a ring homomorphism.

Proof. It follows easily from the Tensor Product–Basics Lemma 311.3 (2) that one only needs to show that θ is multiplicative with respect to elementary tensors. Thus let $r\otimes s$ and $\widetilde{r}\otimes\widetilde{s}$ be two elementary tensors in $R\otimes S$. Then

$$\begin{aligned}
\theta((r\otimes s)\cdot(\widetilde{r}\otimes\widetilde{s}) &\overset{\text{see the Tensor Product-of-Rings Lemma 311.7}}{=} \theta((r\cdot\widetilde{r})\otimes(s\cdot\widetilde{s})) \\
&\overset{\text{since } \varphi \text{ and } \psi \text{ are ring homomorphisms}}{=} \varphi(r)\cdot\varphi(\widetilde{r})\cdot\psi(s)\cdot\psi(\widetilde{s}) \\
&= \theta(r\otimes s)\cdot\theta(\widetilde{r}\otimes\widetilde{s}).
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{by definition of } \theta}{=} \varphi(r\cdot\widetilde{r})\cdot\psi(s\cdot\widetilde{s}) \\
&\overset{\text{since } T \text{ is commutative}}{=} \varphi(r)\cdot\psi(s)\cdot\varphi(\widetilde{r})\cdot\psi(\widetilde{s})
\end{aligned}$$

■

311.3. **Bimodules and left-left modules as left modules.** We start out by recalling Definition 309.37:

Definition 311.11. Let R and S be two rings.

(1) An (R,S)-bimodule is an abelian group M together with an R-left module structure and an S-right module structure, such that for every $r\in R$, every $s\in S$ and every $m\in M$ we have $r\cdot(m\cdot s)=(r\cdot m)\cdot s$.[3452]

(2) An (R,S)-left left module is an abelian group M together with an R-left module structure and an S-left module structure, such that for every $r\in R$, every $s\in S$ and every $m\in M$ we have $r\cdot(s\cdot m)=s\cdot(r\cdot m)$.

In the above settings we have obvious notions of morphisms which lead to categories $\mathcal{M}od_{(R,S)\text{-bi}}$ and $\mathcal{M}od_{(R,S)\text{-left left}}$.

Our goal in this section is to show that bimodules and left left modules are essentially the same as left modules over a suitably chosen ring. At this point it is worth recalling the following definition:

Definition 309.6. Let R be a ring. We define the opposite ring R^{op} to be the same abelian group but where the multiplication is given by

$$r \underset{R^{\mathrm{op}}}{\cdot} s := s \underset{R}{\cdot} r.$$

Evidently we have the equality $(R^{\mathrm{op}})^{\mathrm{op}}=R$ of rings.

Now we can express bimodules as left modules:

Proposition 311.12. (Bimodule–Reinterpretation Proposition) Let S and S be two rings.

(1) (a) Let M be an (R,S)-bimodule. There exists a unique $R\otimes S^{\mathrm{op}}$-left module structure on the abelian group M such that for every $r\in R$, every $s\in S$ and every $m\in M$ we have the equality $(r\otimes s)\cdot m=r\cdot m\cdot s$.

[3452] Note that this means that we can unambiguously talk about $r\cdot m\cdot s$.

(b) Let M be an $R \otimes S^{\mathrm{op}}$-left module. There exists a unique (R,S)-bimodule structure on the abelian group M such that for every $r \in R$, every $s \in S$ and every $m \in M$ we have $r \cdot m \cdot s = (r \otimes s) \cdot m$.

(2) The maps in (1) define an isomorphism between the category $\mathcal{Mod}_{(R,S)\text{-bi}}$ of (R,S)-bimodules and the category $\mathcal{Mod}_{R \otimes S^{\mathrm{op}}\text{-left}}$ of $R \otimes S^{\mathrm{op}}$-left modules.

Proof. Let S and S be two rings.

(1) Let M be an abelian group. We start out with an observation:

(∗) Given any ring T a T-left ring structure on M is naturally the same as a ring homomorphism $\lambda\colon T \to \mathrm{End}(M)$.

Now we start with the actual proofs of the two statements:

(a) Let M be an (R,S)-bimodule. We need to define an $R \otimes S^{\mathrm{op}}$-left module structure on the abelian group M such that for every $r \in R$, every $s \in S$ and every $m \in M$ we have the equality $(r \otimes s) \cdot m = r \cdot m \cdot s$.

We start out by fixing $m \in M$. We want to define the $R \otimes S^{\mathrm{op}}$ left multiplication on m. To do so we consider the following diagram

$$\begin{array}{ccc} R \times S & \longrightarrow & R \otimes S \\ & \searrow^{(r,s) \mapsto r \cdot m \cdot s} & \downarrow \exists ! =: \lambda_m \\ & & M. \end{array}$$

Using Definition 311.11 (1) one sees easily that the diagonal map is bilinear. Thus we obtain from the Tensor Product–Universal Property Lemma 311.4 a unique homomorphism $\lambda_m\colon R \otimes S \to M$ of abelian groups that makes the diagram commute. In summary we get a homomorphism

$$\begin{array}{rcl} \lambda\colon R \otimes S & \to & \text{set of self-maps of } M \\ w & \mapsto & \begin{pmatrix} M & \to & M \\ m & \mapsto & \lambda_m(w) \end{pmatrix} \end{array}$$

of abelian groups. Note that by definition we have $\lambda(r \otimes s)(m) = r \cdot m \cdot s$.

The first thing that we have to verify is that this map takes values in $\mathrm{End}(M)$:

Claim 1. For each $w \in R \otimes S$ the map $m \mapsto \lambda_m(w)$ is a homomorphism of abelian groups.

Proof. Let $m, m' \in M$. Since $r \cdot (m + m') \cdot s = r \cdot m \cdot s + r \cdot m' \cdot s$ we see that the following diagram commutes:

$$\begin{array}{ccc} R \times S & \longrightarrow & R \otimes S \\ & \searrow^{(r,s) \mapsto r \cdot (m+m') \cdot s} & \downarrow \lambda_m + \lambda_{m'} \\ & & M. \end{array}$$

But this implies that $\lambda_{m+m'} = \lambda_m + \lambda_{m'}\colon R \otimes S \to M$. ⊞

By (∗) we still need to show that the map $\lambda\colon R \otimes S \to \mathrm{End}(M)$ is a ring homomorphism. We already point out that λ is a homomorphism of abelian groups. Thus we only need to worry about multiplication. By the Tensor Product-of-Rings Lemma 311.7 it remains to prove the following claim:

Claim 2. Let $r, r' \in R$ and $s, s' \in S$. We have the equality

$$\lambda(\underbrace{r \cdot r' \otimes s' \cdot s}_{\substack{\text{product of} \\ r \otimes s \text{ and } r' \otimes s' \\ \text{in the ring } R \otimes S^{\mathrm{op}}}}) = \lambda(r \otimes s) \circ \lambda(r' \otimes s').$$

Proof. This equality follows from the observation that for each $m \in M$ we have the following equalities:

$$\begin{aligned} \lambda(r \cdot r' \otimes s' \cdot s)(m) &\underset{\substack{\uparrow \\ \text{definition of } \lambda}}{=} (r \cdot r') \cdot m \cdot (s' \cdot s) \\ &\underset{\substack{\uparrow \\ \text{since } M \text{ is an } (R,S)\text{-bimodule,} \\ \text{note that here we implicitly use} \\ \text{that the } R\text{-left multiplication and} \\ \text{the } S\text{-right multiplication commute}}}{=} r \cdot (r' \cdot m \cdot s') \cdot s \\ &\underset{\substack{\uparrow \\ \text{definition of } \lambda}}{=} \lambda(r \otimes s)(\lambda(r' \otimes s')(m)). \end{aligned}$$

⊞

(b) Let M be an $R \otimes S^{\mathrm{op}}$-left module. As in the Ring–Coproduct Lemma 311.10 we consider the natural ring homomorphisms $i\colon R \to R \otimes S$ and $j\colon S^{\mathrm{op}} \to R \otimes S^{\mathrm{op}}$. These ring homomorphisms, together with the Module–Change-of-Ring–Functor Lemma 309.13 and the Opposite Ring–Module Lemma 309.7 allow us to view M as an R-left module $M = {}_iM$ and to view M also as an S-right module M_j.

Claim. These two module structures on the abelian group M are compatible in the sense of Definition 311.11 (1).

Proof. Let $r \in R$, $s \in S$ and $m \in M$. Then we have the following equalities:

$$\begin{aligned} i(r) \cdot (m \cdot j(s)) &= (r \otimes 1) \cdot ((1 \otimes s) \cdot m) \\ &\underset{\substack{\uparrow \\ \text{definition of the multiplication} \\ \text{in the ring } R \otimes S^{\mathrm{op}}}}{=} (r \cdot 1 \otimes s \cdot 1) \cdot m \\ &= ((1 \otimes s) \cdot (r \otimes 1)) \cdot m \\ &= (i(r) \cdot m) \cdot j(s). \end{aligned} \qquad \begin{aligned} &\underset{\substack{\downarrow \\ \text{since } M \text{ is an } R \otimes S^{\mathrm{op}}\text{-left module}}}{=} ((r \otimes 1) \cdot (1 \otimes s)) \cdot m \\ &= (1 \cdot r \otimes 1 \cdot s) \cdot m \\ &= (1 \otimes s) \cdot ((r \otimes 1) \cdot m) \end{aligned}$$

⊞

The desired statement now follows from the observation that in the proof of the claim we showed in particular that we always have $r \cdot m \cdot s = (r \otimes s) \cdot m$.

(2) It follows easily from the definitions that the two constructions in (1) are inverses of one another. ■

Not surprisingly we now turn our attention to left left modules:

Proposition 311.13. (Left Left Module–Reinterpretation Proposition) Let S and S be two rings.

(1) (a) Let M be an (R, S)-left left module. There exists a unique $R \otimes S$-left module structure on the abelian group M such that for every $r \in R$, every $s \in S$ and every $m \in M$ we have $(r \otimes s) \cdot m = r \cdot s \cdot m$.

(b) Let M be an $R \otimes S$-left module. There exists a unique (R, S)-left left module on the abelian group M such that for every $r \in R$, every $s \in S$ and every $m \in M$ we have $r \cdot s \cdot m = (r \otimes s) \cdot m$.

(2) The maps in (1) define an isomorphism between the category $\mathcal{M}od_{(R,S)\text{-left left}}$ of (R, S)-left left modules and the category $\mathcal{M}od_{R \otimes S\text{-left}}$ of $R \otimes S$-left modules.

Proof. First note that it follows easily from the Opposite Ring–Module Lemma 309.7 that an (R, S)-left left module is essentially the same as an (R, S^{op})-bimodule. Thus we see that this proposition follows almost immediately from the Bimodule–Reinterpretation Proposition 311.12 together with the observation that $(R^{\mathrm{op}})^{\mathrm{op}} = R$. ■

The above results allow us to prove the following variation on the Free Module–Existence Lemma 309.20:

Lemma 311.14. (Free Bimodule–Existence Lemma) Let R and S be rings and let X be a set.

(1) The (R, S^{op})-bimodule $(R \otimes S^{\mathrm{op}})^{(X)}$ is a free (R, S)-bimodule on the subset X of $(R \otimes S^{\mathrm{op}})^{(X)}$.

(2) The (R, S)-left left module $(R \otimes S)^{(X)}$ is a free (R, S)-left left module on the subset $X \subseteq (R \otimes S)^{(X)}$.

Proof.

(1) We set $T := R \otimes S^{\mathrm{op}}$. By the Free Module–Existence Lemma 309.20 we know that the T-left module $T^{(X)}$ is a free T-left module on the subset $X \subseteq T^{(X)}$. The statement now follows almost immediately from the Bimodule–Reinterpretation Proposition 311.12.

(2) This statement is proved the same way as (1), except that this time we appeal to the Left Left Module–Reinterpretation Proposition 311.13. ■

311.4. Exercises for Chapter 311.

Exercise 311.1. Let G and H be groups. Show that we have a natural ring isomorphism

$$\mathbb{Z}[G] \otimes \mathbb{Z}[H] \ \to \ \mathbb{Z}[G \times H].$$

Exercise 311.2.

(a) We consider the ring $\mathbb{Q}$. Show that the tensor ring $\mathbb{Q} \otimes \mathbb{Q}$ is isomorphic to the ring $\mathbb{Q}$.

(b) We consider the ring $\mathbb{R}$. Is the tensor ring $\mathbb{R} \otimes \mathbb{R}$ is isomorphic to the ring $\mathbb{R}$?

Exercise 311.3. Let R be a ring, let S be a commutative ring and let W be an (S, R)-bimodule.

(a) Show that, similar to the Tensor Product-of-Rings–Modules Lemma 311.9, we can equip $S \otimes_R W$ with a natural structure of an $(S \otimes R)$-right module.

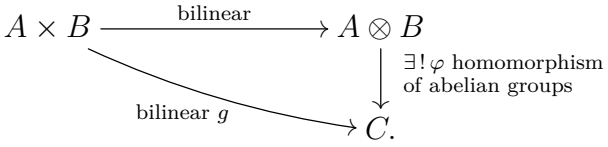
(b) Show that we have an $(S \otimes R)$-right module structure on W with the property that

$$w \cdot (s \otimes r) \ = \ s \cdot w \cdot r.$$

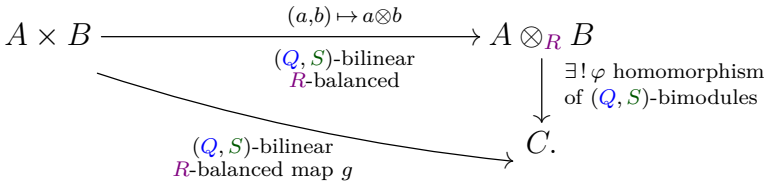
CHAPTER 312

The tensor product over rings

In Chapter 159 and Chapter 311 we introduced the tensor product $A \otimes B$ of two abelian groups. The tensor product was characterized by the following universal property: We have a bilinear map $A \times B \to A \otimes B$ such that given any abelian group C and given any bilinear map $g\colon A \times B \to C$ there exists a unique homomorphism $\varphi\colon A \otimes B \to C$ of abelian groups such that the following diagram commutes:



In this purely algebraic chapter we will introduce the tensor product of modules over a ring R. Evidently many features are similar to the tensor product of abelian groups. But there are several significant differences. Even the setup requires some care: We will start out with rings Q, R, S (not necessarily commutative), a (Q, R)-bimodule A and an (R, S)-bimodule B to define the tensor product $A \otimes_R B$ which is going to be a (Q, S)-bimodule which satisfies a universal property that can be summarized as follows:



This chapter is organized as follows:

- In Section 312.1 we will give a topological motivation for why one might want to study tensor products over rings.
- In Section 312.2 we will introduce the tensor product over rings.
- In Section 312.3 we will consider homomorphisms between tensor products.
- In Section 312.4 we will perform an important basic calculation of tensor products.
- In Section 312.5 we will prove several formal properties of the tensor product.
- In Section 312.6 we will see that the tensor functor and the hom-functor form an adjoint pair of functors.
- In Section 312.7 we will study the effect of tensoring *with* a ring.
- In Section 312.8 we will discuss the invariant basis number property.
- In Appendix 312.9 we will give an alternative construction of the tensor product.
- In Section 312.10 we will provide exercises for Chapter 312.

In the next Chapter 313 we will (re-) introduce some basic concepts of homological algebra over R-left modules.

As we will see, many definitions, statements and proofs look like they are copy-pasted from Chapter 159. But there are at times considerable differences and some proofs will turn out to be more tricky than before. As always, dealing with tensor products it is hard to distinguish what one has really understood from what one just got used to.

312.1. **Motivation.** In Chapter 159 and Chapter 311 we introduced the tensor product $A \otimes B$ of two abelian groups. The main application was the definition of homology with coefficients. This required the following input data:

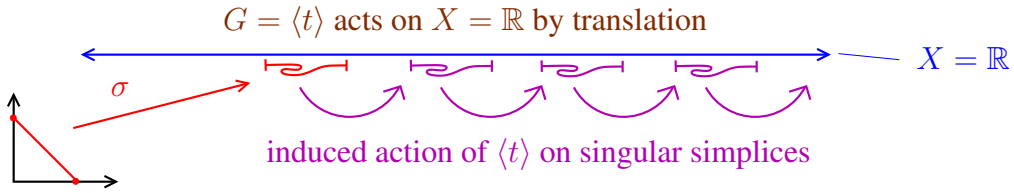
(a) A topological space X.
(b) An abelian group A.

In Chapter 160 we saw that the tensor product of abelian groups allows us to define the tensor chain complex $(\mathrm{C}_*(X) \otimes A, \partial_* \otimes \mathrm{id}_A)$ and the homology groups $\mathrm{H}_*(X;A) = \mathrm{H}_*(\mathrm{C}_*(X) \otimes A)$ of X with coefficients in the abelian group A.

In later chapters we will want to do something similar, but at a more fancy level. More precisely we will consider the following input data:

(a) A topological space X on which a group G acts continuously from the left.
(b) A $\mathbb{Z}[G]$-right module A.

First recall that on page 5984 we saw that the continuous G-left action on X turns $\mathrm{C}_*(X)$ into a $\mathbb{Z}[G]$-left module. It follows from this observation that the tensor product of modules allows us to define the tensor chain complex $(A \otimes_{\mathbb{Z}[G]} \mathrm{C}_*(X), \mathrm{id}_A \otimes \partial_*)$ and the homology groups $\mathrm{H}_*(A \otimes_{\mathbb{Z}[G]} \mathrm{C}_*(X))$ of X with coefficients in the $\mathbb{Z}[G]$-right module A. In fact, in many cases A will have more algebraic structure, for example A could be a real vector space together with a right G-action. In other words, A might be a $(\mathbb{R}, \mathbb{Z}[G])$-bimodule. So it will make sense to consider tensor products of bimodules.



312.2. **The tensor product over rings.** In Definition 159.2 we introduced the tensor product of two abelian groups. In this section we will introduce the tensor product of (bi-) modules over rings. We start out with the following definition which generalizes Definition 2.25 and Definition 311.1:

Definition 312.1. Let Q and S be rings, let A be a Q-left module and let B be an S-right module.[3453] Finally let C be a (Q,S)-bimodule. We say that a map $f\colon A \times B \to C$ is (Q,S)-bilinear, if for all $a, \widetilde{a} \in A$, $q, \widetilde{q} \in Q$, $b, \widetilde{b} \in B$ and $s, \widetilde{s} \in S$ we have the equalities

$$\begin{aligned} f(q \cdot a + \widetilde{q} \cdot \widetilde{a}, b) &= q \cdot f(a,b) + \widetilde{q} \cdot f(\widetilde{a}, b), \\ f(a, b \cdot s + \widetilde{b} \cdot \widetilde{s}) &= f(a,b) \cdot s + f(a, \widetilde{b}) \cdot \widetilde{s}. \end{aligned}$$

[3453] In many examples $Q = S = \mathbb{Z}$. In this setting we are just talking about bilinear maps in the sense of Definition 2.25 and Definition 311.1.

Example.

(1) If $Q = S = \mathbb{Z}$, then it follows from the discussion on page 5999 that a map $f\colon A\times B \to C$ is $(\mathbb{Z},\mathbb{Z})$-bilinear in the sense of Definition 312.1 if and only if it is bilinear in the sense of Definition 311.1.

(2) Let R be a commutative ring. We consider the case that $Q = S = R$. Let A, B and C be R-modules. We can then view these as R-left module, R-right module and (R,R)-bimodule. If $f\colon A \times B \to C$ is (R,R)-bilinear, then we see that for any $a \in A, b \in B$ and $r \in R$ we have

$$f(a\cdot r, b) \underset{\substack{\uparrow\\ \text{by Definition 312.1}\\ \text{of ``bilinear''}}}{=} r\cdot f(a,b) \underset{\substack{\uparrow\\ \text{since the ring } R\\ \text{is commutative}}}{=} f(a,b)\cdot r \underset{\substack{\uparrow\\ \text{by Definition 312.1}\\ \text{of ``bilinear''}}}{=} f(a, r\cdot b).$$

This shows that in this setting we can "push" the $r \in R$ "across the aisle". But we will see in Exercise 312.1 that if R is non-commutative, then this conclusion does not necessarily hold. □

The above example leads us to the following definition:

Definition 312.2. Let R be a ring, let A be an R-right module and let B be an R-left module. Finally let C be a set. We say that a map $f\colon A \times B \to C$ is R-balanced, if for all $a \in A$, $b \in B$ and $r \in R$ we have the equality

$$f(a\cdot r, b) \;=\; f(a, r\cdot b).$$

Example. Let R be a commutative ring. Furthermore let A, B and C be R-modules and let $f\colon A \times B \to C$ be a map. The discussion preceding Definition 312.2 shows that in the commutative setting an (R,R)-bilinear map is automatically R-balanced. But as we mentioned above, if R is non-commutative, then this conclusion does not necessarily hold. So if we like this property, we will have to ask for it. We will not get it for free by just asking for bilinearity. □

For whatever reason we now want to find the universal object for a (Q,S)-bilinear R-balanced map $A \times B \to C$. This honorable goal leads us to the following definition which is modelled on Definition 159.2 and Definition 268.11:

Definition 312.3. Let Q, R, S be rings, let A be a (Q,R)-bimodule and let B be an (R,S)-bimodule.[3454]

(1) As in the Free Bimodule–Existence Lemma 311.14 we consider the free (Q,S)-bimodule $(Q\otimes S)^{(A\times B)}$ that is generated by the set $A\times B$. We define the tensor product $A\otimes_R B$ to be the following (Q,S)-bimodule:[3455]

$$A\otimes_R B \;\coloneqq\; (Q\otimes S)^{(A\times B)}/N(A,B)$$

where $N(A,B)$ is the (Q,S)-bisubmodule that is generated by the elements of the form

(i)	$(q\cdot a + \widetilde{q}\cdot\widetilde{a}, b) - q\cdot(a,b) - \widetilde{q}\cdot(\widetilde{a},b)$	with $a,\widetilde{a}\in A$, $b\in B$ and $q,\widetilde{q}\in Q$,
(ii)	$(a, b\cdot s + \widetilde{b}\cdot\widetilde{s}) - (a,b)\cdot s - (a,\widetilde{b})\cdot\widetilde{s}$	with $a\in A$, $b,\widetilde{b}\in B$ and $s,\widetilde{s}\in S$,
$(\star)$	$(a\cdot r, b) - (a, r\cdot b)$	with $a\in A, b\in B$ and $r\in R$.

(2) For $a \in A$ and $b \in B$ we write[3456]

$$a \otimes b \;:=\; [(a,b)] \quad \in\; (Q \otimes S)^{(A\times B)}/N(A,B) =: A \otimes_R B.$$

Such an element of $A \otimes_R B$ is sometimes called an elementary tensor.

The following lemma is the analog of the Tensor Product–Basics Lemma 159.3:

Lemma 312.4. (Tensor Product–Basics Lemma) Let Q, R, S be rings, let A be a (Q,R)-bimodule and let B be an (R,S)-bimodule.

(1) The tensor product $A \otimes_R B$ is a (Q,S)-bimodule.
(2) Any $\varphi \in A \otimes_R B$ is of the form $\varphi = \sum_{i=1}^{n} a_i \otimes b_i$ with $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$.[3457]

Proof.

(1) This statement follows from the fact that $A \otimes_R B$ is defined as the quotient of the free (Q,S)-bimodule $(Q \otimes S)^{(A\times B)}$ by a (Q,S)-bisubmodule.
(2) It follows immediately from the definition of the tensor product, together with the notational convention of "formal linear combinations" that we introduced in Definition 97.7, that any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i) \cdot s_i$, with $r_1, \ldots, r_n \in R$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$ and $s_1, \ldots, s_n \in S$. Thus it remains to prove the following claim:

Claim. We always have $r \cdot (a \otimes b) \cdot s = (r \cdot a) \otimes (b \cdot s)$.

Proof. In fact it follows immediately from part (i) and (ii) of the definition of $N(A,B)$ that we have the equalities

$$\underbrace{[r \cdot (a,b) \cdot s]}_{= r\cdot(a\otimes b)\cdot s} = [(r \cdot a, b) \cdot s] = \underbrace{[(r \cdot a, b \cdot s)]}_{= (r\cdot a)\otimes(b\cdot s)} \in (Q \otimes S)^{(A\times B)}/N(A,B) = A \otimes_R B. \qquad \blacksquare$$

The following lemma is the analog of the Tensor Product–Universal Property Lemma 159.4:

Lemma 312.5. (Tensor Product–Universal Property Lemma) Let Q, R, S be rings, let A be a (Q,R)-bimodule and let B be an (R,S)-bimodule.

(1) The map

$$\begin{array}{ccc} A \times B & \to & A \otimes_R B \\ (a,b) & \mapsto & a \otimes b \end{array}$$

is (Q,S)-bilinear and R-balanced. In other words, the following two conditions hold: For all $a, \widetilde{a} \in A$, $q, \widetilde{q} \in Q$, $b, \widetilde{b} \in B$ and $s, \widetilde{s} \in S$ we have the equalities in $A \otimes_R B$:

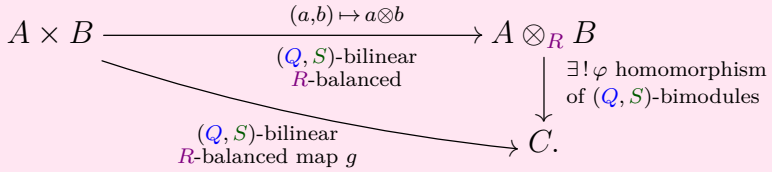
$$\text{(i)} \quad (q \cdot a + \widetilde{q} \cdot \widetilde{a}) \otimes b = q \cdot (a \otimes b) + \widetilde{q} \cdot (\widetilde{a} \otimes b)$$
$$\text{(ii)} \quad a \otimes (b \cdot s + \widetilde{b} \cdot \widetilde{s}) = (a \otimes b) \cdot s + (a \otimes \widetilde{b}) \cdot \widetilde{s}.$$

[3454] Strictly speaking the notation should remember all three rings. But we will see in the Tensor Product–Small Model Lemma 312.25 that Q and S are mostly immaterial.
[3455] In many cases $Q = S = \mathbb{Z}$. In this case the hypothesis can be simplified to saying that A is a right R-module and that B be a left R-module.
[3456] Note that in the notation we use the subscript R to denote the tensor product $A \otimes_R B$. To simplify the subsequent notation, given elements $a \in A$ and $b \in B$ we write $a \otimes b$ instead of the more logical $a \otimes b$.
[3457] Note that this description is in general by no means unique.

For any $a \in A, b \in B$ and $r \in R$ we have

$$(\star) \qquad (a \cdot r) \otimes b \;=\; a \otimes (r \cdot b).$$

(2) Given any (Q,S)-bimodule C and given any map $g\colon A \times B \to C$ which is (Q,S)-bilinear and R-balanced there exists a unique homomorphism $\varphi\colon A \otimes_R B \to C$ of (Q,S)-bimodules such that the following diagram commutes:[3458]



In fact this homomorphism is given by

$$\begin{array}{rcl} \varphi\colon A \otimes_R B & \to & C \\ \sum\limits_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum\limits_{i=1}^{n} g(a_i, b_i). \end{array}$$

Remark. We consider the case that $Q = R = S = \mathbb{Z}$: As we discussed on page 5961, an abelian group is essentially the same as a $\mathbb{Z}$-module, in particular, any abelian group is a $\mathbb{Z}$-left module, a $\mathbb{Z}$-right module and a $(\mathbb{Z},\mathbb{Z})$-bimodule. It follows immediately from the universal properties of tensor products that we formulated in the Tensor Product–Universal Property Lemma 311.4 and the Tensor Product–Universal Property Lemma 312.5 that we have a natural isomorphism

$$\underbrace{A \otimes_{\mathbb{Z}} B}_{\substack{\text{tensor product}\\ \text{of } \mathbb{Z}\text{-modules}\\ \text{from Definition 312.3}}} \xrightarrow{\cong} \underbrace{A \otimes B}_{\substack{\text{tensor product of}\\ \text{abelian groups}\\ \text{from Definition 311.2}}}$$

of abelian groups. □

Proof. [3459] Let Q, R, S be rings, let A be a (Q,R)-bimodule and let B be an (R,S)-bimodule.

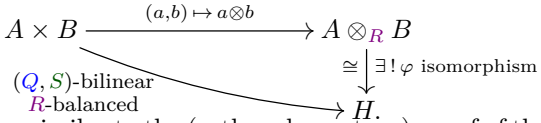
(1) Note that in Definition 312.3 we defined $N(A,B)$ exactly in such a way as to ensure that the map

$$\begin{array}{rcl} A \times B & \to & A \otimes_R B := (Q \otimes S)^{(A \times B)}/N(A,B) \\ (a,b) & \mapsto & a \otimes b \end{array}$$

is (Q,S)-bilinear and R-balanced.

(2) Let C be a (Q,S)-bimodule and let $g\colon A \times B \to C$ be a (Q,S)-bilinear map.

[3458] This universal property gives us the usual uniqueness property: Let H be a (Q,S)-bimodule and let $g\colon A \times B \to H$ be a (Q,S)-bilinear R-balanced map. If this (Q,S)-bilinear map satisfies the universal property of (2), then the standard argument (as in the proof of the Product Uniqueness Lemma 6.2) shows that there exists a unique *isomorphism* $\varphi\colon A \otimes_R B \to H$ of (Q,S)-bimodules such that the following diagram commutes:



[3459] The proof of this lemma is very similar to the (rather elementary) proof of the Tensor Product–Universal Property Lemma 312.5. For the reader's peace of mind we provide the proof again.

Claim. The map

$$\begin{array}{rcl} A \otimes_R B & \to & C \\ \sum_{i=1}^n a_i \otimes b_i & \mapsto & \sum_{i=1}^n g(a_i, b_i) \end{array}$$

is a well-defined homomorphism of (Q, S)-bimodules and it is the unique homomorphism $\varphi\colon A \otimes_R B \to C$ of (Q, S)-bimodules such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b) \mapsto a \otimes b} & A \otimes_R B \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & C. \end{array}$$

Proof. First note that it follows from the fact that $A \times B$ is a basis for the free (Q, S)-bimodule $(Q \otimes S)^{(A \times B)}$ that there exists a *unique* homomorphism

$$(Q \otimes S)^{(A \times B)} \to C$$

of (Q, S)-bimodules which sends each $(a, b) \in A \times B$ to $g(a, b)$. Since g is (Q, S)-bilinear we see that the submodule $N(A, B)$ lies in the kernel of the above homomorphism. It follows from the Quotient Group Lemma 2.4 that the above homomorphism descends to a well-defined homomorphism

$$\begin{array}{rcl} \varphi\colon A \otimes_R B := (Q \otimes S)^{(A \times B)}/N(A, B) & \to & C \\ \left[\sum_{i=1}^n (a_i, b_i)\right] = \sum_{i=1}^n a_i \otimes b_i & \mapsto & \sum_{i=1}^n g(a_i, b_i) \end{array}$$

of (Q, S)-bimodules. Next note that it is clear that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b) \mapsto a \otimes b} & A \otimes_R B \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & C. \end{array}$$

Finally note that it follows easily from the fact that $(Q \otimes S)^{(A \times B)} \to A \otimes_R B$ is an epimorphism that φ is uniquely determined. ∎

312.3. **Homomorphisms between tensor products.** The following definition is the analog of Definition 159.6 and Definition 311.5:

Definition 312.6. Let Q, R, S be rings. Let $f\colon A \to A'$ be a homomorphism of (Q, R)-bimodules and let $g\colon B \to B'$ be a homomorphism of (R, S)-bimodules. It follows from the Tensor Product–Universal Property Lemma 312.5 (1) that the map

$$\begin{array}{rcl} A \times B & \to & A' \otimes_R B' \\ (a, b) & \mapsto & f(a) \otimes g(b) \end{array}$$

is (Q, S)-bilinear and R-balanced. Thus it follows from the Tensor Product–Universal Property Lemma 312.5 (2) that the map

$$\begin{array}{rcl} f \otimes g\colon A \otimes_R B & \to & A' \otimes_R B' \\ \sum_{i=1}^n a_i \otimes b_i & \mapsto & \sum_{i=1}^n f(a_i) \otimes g(b_i) \end{array}$$

is a well-defined homomorphism of (Q, S)-bimodules. We refer to $f \otimes g$ as the tensor product of the homomorphisms f and g.

The following lemma is the analog of the Tensor–Map Composition Lemma 159.7:

Lemma 312.7. (Tensor–Map Composition Lemma) Let Q, R, S be rings.

(1) Given any (Q, R)-bimodule A and given any (R, S)-bimodule B we have the equality $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes_R B}$.

(2) Let $f\colon A \to A'$, $f'\colon A' \to A''$ be homomorphisms between (Q, R)-bimodules and let $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms between (R, S)-bimodules. Then

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon A \otimes_R B \;\to\; A'' \otimes_R B''.$$

(3) Let $f, g\colon A \to \widetilde{A}$ be homomorphisms between (Q, R)-bimodules and let B be an (R, S)-bimodule. Then

$$(f + g) \otimes \mathrm{id}_B \;=\; f \otimes \mathrm{id}_B + g \otimes \mathrm{id}_B\colon A \otimes_R B \;\to\; \widetilde{A} \otimes_R B.$$

The same statement also holds if we swap the roles of the two sides.

Proof. All the statements follow immediately from the above explicit description of the tensor product of homomorphisms in Definition 312.6. Alternatively one can also deduce the statements from the universal property stated in the Tensor Product–Universal Property Lemma 312.5 (2). ■

The following lemma is a generalization of the Tensor–Functor Lemma 159.8:

Lemma 312.8. (Tensor–Functor Lemma) Let Q, R, S be rings. Given any (R, S)-bimodule G the maps

$$\begin{array}{rcl} A & \mapsto & A \otimes G \\ (f\colon A \to B) & \mapsto & (f \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \end{array}$$

define a covariant functor from the category $\mathcal{M}\mathit{od}_{(Q,R)\text{-bi}}$ of (Q, R)-bimodules to the category $\mathcal{M}\mathit{od}_{(Q,S)\text{-bi}}$ of (Q, S)-bimodules. Of course an analogous statement holds if we tensor with G from the left.

Proof. This lemma follows immediately from the Tensor–Map Composition Lemma 312.7. ■

The following lemma shows that the tensor product is also functorial in the ring R:

Lemma 312.9. (Tensor Product-Ring-Functor Lemma) Let $Q, R, \widetilde{R}, S$ be rings and let $\varphi\colon R \to \widetilde{R}$ be a ring homomorphism. Furthermore let A be a $(Q, \widetilde{R})$-bimodule and let B be an $(\widetilde{R}, S)$-bimodule. As explained in the Module–Change-of-Ring–Functor Lemma 309.13 we obtain, via φ, an R-right module A_φ and an R-left module ${}_\varphi B$. With these conventions we obtain the following homomorphism of (Q, S)-bimodules:

$$\begin{array}{rcl} A_\varphi \otimes_R {}_\varphi B & \to & A \otimes_{\widetilde{R}} B \\ a \otimes b & \mapsto & a \otimes b. \end{array}$$

Proof. The map

$$\begin{array}{rcl} A_\varphi \times {}_\varphi B & \to & A \otimes_{\widetilde{R}} B \\ (a, b) & \mapsto & a \otimes b \end{array}$$

is clearly (Q, S)-bilinear and R-balanced. Therefore we get the desired homomorphism $A_\varphi \otimes_R {}_\varphi B \to A \otimes_{\widetilde{R}} B$ from the Tensor Product–Universal Property Lemma 312.5 (2). ■

312.4. **Calculation of a tensor product.** The next lemma (which is an analog of the Tensor-with-$\mathbb{F}$–Lemma 81.18 and of the Tensor-with-$\mathbb{Z}$–Lemma 159.9) gives us an important, albeit slightly dull, calculation of a tensor product:

Lemma 312.10. (Tensor-with-Base Ring Lemma) Let R be a ring. We view R as an (R, R)-bimodule in the usual way.

(1) Let S be a ring and let B be an (R, S)-bimodule. The maps

$$\begin{array}{rcl} \Theta\colon R \otimes_R B & \to & B \\ \sum_{i=1}^m r_i \otimes b_i & \mapsto & \sum_{i=1}^m r_i \cdot b_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Omega\colon B & \to & R \otimes_R B \\ b & \mapsto & 1_R \otimes b \end{array}$$

are well-defined and they are natural isomorphisms of (R, S)-bimodules that are inverses to one another.

(2) Let Q be a ring and let A be a (Q, R)-bimodule. The maps

$$\begin{array}{rcl} \Theta\colon A \otimes_R R & \to & A \\ \sum_{i=1}^m a_i \otimes r_i & \mapsto & \sum_{i=1}^m a_i \cdot r_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Omega\colon A & \to & A \otimes_R R \\ a & \mapsto & a \otimes 1_R \end{array}$$

are well-defined and they are natural isomorphisms of (Q, R)-bimodules that are inverses to one another.

Proof.

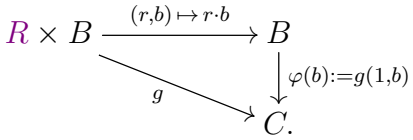
(1) We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} R \times B & \to & B \\ (r, b) & \mapsto & r \cdot b \end{array}$$

has the defining property from the Tensor Product–Universal Property Lemma 312.5 (2).

Proof. The map is clearly (R, S)-bilinear and R-balanced. Next, given any (R, S)-bilinear and R-balanced homomorphism $g\colon R \times B \to C$ we consider the homomorphism $\varphi\colon B \to C$ that is given by $\varphi(b) = g(1_R, b)$. In summary we have the following diagram:



We now see that the map g satisfies all conditions and that it is unique. ⊞

The desired statement now follows from the claim and the uniqueness statement for tensor products from Footnote 3458.

(2) The argument if the R-term is on the right is evidently almost identical. ■

312.5. **Formal properties of the tensor product.** The following lemma is an analog of the Tensor Commutative-Associative Lemma 159.11:

Lemma 312.11. (Tensor Commutative-Associative Lemma)

(1) The tensor product with a zero module is zero.

(2) Let Q, R and S be rings, let A be an (Q, R)-bimodule and let B be an (R, S)-bimodule. If R is a ring with involution, then there exists an isomorphism

$$\begin{array}{rcl} A \otimes_R B & \to & \overline{B} \otimes_R \overline{A} \\ a \otimes b & \mapsto & b \otimes a \end{array}$$

of (Q, S)-bimodules.

(3) Let Q, R, S, T be rings, let A be an (Q, R)-bimodule, let B be an (R, S)-bimodule and let C be an (S, T)-bimodule. The map[3460]

$$\begin{array}{rcl} (A \otimes_R B) \otimes_S C & \to & A \otimes_R (B \otimes_S C) \\ \sum_{j=1}^{n} (a_j \otimes b_j) \otimes c_j & \mapsto & \sum_{j=1}^{n} a_j \otimes (b_j \otimes c_j) \end{array}$$

is well-defined and it is a natural isomorphism of (Q, T)-bimodules.

Proof.

(1) This statement follows easily from the definitions.

(2) It follows almost immediately from the Tensor Product–Universal Property Lemma 312.5 that the map is well-defined and that the obvious inverse is also well-defined.

(3) [3461] We start out with the following claim:

Claim 1. There exists a unique homomorphism

$$\Phi \colon (A \otimes_R B) \otimes_S C \to A \otimes_R (B \otimes_S C)$$

of (Q, T)-bimodules such that we always have

$$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c).$$

Proof. In the following we view S as an (S, S)-bimodule in the usual way. Next note that it follows from the Tensor-with-Base Ring Lemma 312.10 (2) together with the Tensor–Map Composition Lemma 312.7 that we have an isomorphism

$$\begin{array}{rcl} \Theta \colon A \otimes_R B & \to & A \otimes_R (B \otimes_S S) \\ a \otimes b & \mapsto & a \otimes (b \otimes 1_S). \end{array}$$

of (Q, S)-bimodules. Next, given $c \in C$ we consider the S-left module homomorphism $\lambda_C \colon S \to C$ that is given by $\lambda_c(s) = s \cdot c$. Now we consider the map

$$\begin{array}{rcl} (A \otimes_R B) \times C & \to & A \otimes_R (B \otimes_S C) \\ ((a \otimes b), c) & \mapsto & \underbrace{((\mathrm{id}_A \otimes (\mathrm{id}_B \otimes \lambda_c))}_{\substack{\text{Definition 312.6} \\ \text{applied twice}}} \circ \Theta)(a \otimes b) = (a \otimes b) \otimes (\underbrace{1_S \cdot c}_{=c}). \end{array}$$

One can easily verify that this map is (Q, T)-bilinear and S-balanced. It follows from the Tensor Product–Universal Property Lemma 312.5 (2) that we get an induced homomorphism

$$\begin{array}{rcl} \Phi \colon (A \otimes_R B) \otimes_S C & \to & A \otimes_R (B \otimes_S C) \\ (a \otimes b) \otimes c & \mapsto & a \otimes (b \otimes c) \end{array}$$

of (Q, T)-bimodules. Clearly this map has the desired properties. ⊞

[3461] This proof is a little more tricky than the proof of the "classic" Commutative-Associative Lemma 159.11 (3).

Claim 2. There exists a unique homomorphism

$$\Psi\colon A \otimes_R (B \otimes_S C) \ \to\ (A \otimes_R B) \otimes_S C$$

of (Q, T)-bimodules such that we always have

$$a \otimes (b \otimes c) \ \mapsto\ (a \otimes b) \otimes c.$$

Proof. This claim is proved the same way as Claim 1. ⊞

Note that by the construction the homomorphisms $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on the generating sets $\{a \otimes (b \otimes c) \,|\, a \in A, b \in B, c \in C\}$ and $\{(a \otimes b) \otimes c) \,|\, a \in A, b \in B, c \in C\}$. We see that $\Phi \circ \Psi = \mathrm{id}_{A \otimes_R (B \otimes_S C)}$ and $\Psi \circ \Phi = \mathrm{id}_{(A \otimes_R B) \otimes_S C}$. This shows that Φ and Ψ are isomorphisms of (Q, T)-bimodules. Finally note that it follows easily from the definitions that Φ and Ψ are natural. ■

The following lemma gives us an application of the Tensor Commutative-Associative Lemma 312.11 (3) which will be used on several occasions:

Lemma 312.12. (Tensor Product–Cancel Ring Lemma) Let Q, R, S, T be rings and let $\varphi\colon S \to R$ be a ring homomorphism. Furthermore let A be a (Q, R)-bimodule and let B be a (S, T)-bimodule. As in the Module–Change-of-Ring–Functor Lemma 309.13 we use φ to view R as an (R, S)-bimodule R_φ and to view A as a (Q, S)-bimodule A_φ. The maps[3462]

$$\begin{array}{rcl} A \otimes_R (R_\varphi \otimes_S B) & \to & A_\varphi \otimes_S B \\ a \otimes (r \otimes b) & \mapsto & a \cdot r \otimes b \end{array} \quad \text{and} \quad \begin{array}{rcl} A_\varphi \otimes_S B & \to & A \otimes_R (R_\varphi \otimes_S B) \\ a \otimes b & \mapsto & a \otimes (1_R \otimes b) \end{array}$$

are well-defined (Q, T)-bimodule homomorphisms and they are and inverses of one another. In particular they are both isomorphisms of (Q, T)-bimodules.

Proof. We have the following isomorphisms of (Q, T)-bimodules:

$$\begin{array}{rcccccl} A \otimes_R (R_\varphi \otimes_S B) & \xrightarrow{\cong} & (A \otimes_R R_\varphi) \otimes_S B & \xrightarrow{\mathrm{id}} & (A \otimes_R R)_\varphi \otimes_S B & \xrightarrow{\cong} & A_\varphi \otimes_S B. \\ a \otimes (r \otimes b) & \mapsto & (a \otimes r) \otimes b & & & & \\ & & & & (a \otimes r) \otimes b & \mapsto & (a \cdot r) \otimes b \end{array}$$

Here the first isomorphism is given by the Tensor Commutative-Associative Lemma 312.11 (3), the second isomorphism follows immediately from the definition in the Module–Change-of-Ring–Functor Lemma 309.13 and the third isomorphism is given by the Tensor-with-Base Ring Lemma 312.10. ■

312.6. **The Hom-Tensor Adjunction.** The following lemma is a generalization of the Hom–Tensor Adjunction Lemma 159.13:

Lemma 312.13. (Hom–Tensor Adjunction Lemma) Let Q, R be rings and let M be a (Q, R)-bimodule. We consider the two functors[3463]

$$\begin{array}{rcl} F\colon \mathcal{M}od_{R\text{-left}} & \to & \mathcal{M}od_{Q\text{-left}} \\ X & \mapsto & M \otimes_R X \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{M}od_{Q\text{-left}} & \to & \mathcal{M}od_{R\text{-left}} \\ Y & \mapsto & \mathrm{Hom}_{Q\text{-left}}(M, Y). \end{array}$$

[3462] Here we view R as an (R, S)-bimodule, thus it follows immediately from Definition 312.3 that we can view $R \otimes_S B$ as an R-left module. This allows us to define the tensor product $A \otimes_R (R \otimes_S B)$.

For any R-left module X and any Q-left module Y the maps[3464]

$$\begin{array}{rcl} \mathrm{Hom}_{Q\text{-left}}(\overbrace{M \otimes_R X}^{=F(X)}, Y) & \to & \mathrm{Hom}_{R\text{-left}}(X, \overbrace{\mathrm{Hom}_{Q\text{-left}}(M, Y)}^{=G(Y)}) \\ (f\colon M \otimes_R X \to Y) & \mapsto & \left(\begin{array}{rcl} X & \to & \mathrm{Hom}_{Q\text{-left}}(M, Y) \\ x & \mapsto & \left(\begin{array}{rcl} M & \to & Y \\ m & \mapsto & f(m \otimes x) \end{array} \right) \end{array} \right) \end{array}$$

form a bijection that is natural in X and that it also natural in Y. In other words, the tensor functor is a left adjoint, in the sense of Definition 6.33, of the Hom-functor.

Proof. Using the Tensor Product–Universal Property Lemma 312.5 (2) one can easily show that this map is a natural bijection. This shows that F and G form an adjoint pair. ■

For completely formal reasons we now obtain the following generalization of the Tensor Product–Colimit Proposition 159.14:

Proposition 312.14. (Tensor Product–Colimit Proposition) Let Q, R, S be rings and let M be an (Q, R)-bimodule. The tensor functor

$$\begin{array}{rcl} F\colon \mathcal{M}od_{R\text{-left}} & \to & \mathcal{M}od_{Q\text{-left}} \\ X & \mapsto & M \otimes_R X \end{array}$$

preserves all colimits.

Proof. This proposition is an immediate consequence of the Hom–Tensor Adjunction Lemma 312.13 together with the Left Adjoint–Colimit Proposition 6.35. ■

On many occasions we will also make (implicit) use of the following lemma which is a generalization of the Tensor–Direct Sum Lemma 159.15:

Lemma 312.15. (Tensor–Direct Sum Lemma) Let Q, R, S be rings.

(1) Let $\{A_i\}_{i \in I}$ be a family of (Q, R)-bimodules and let B be an (R, S)-bimodule. Given $j \in I$ let $\iota_j\colon A_j \to \bigoplus_{i \in I} A_i$ be the natural homomorphism of (Q, R)-bimodules. Next note that it follows from the Coproduct-of-Modules Lemma 309.11 that the homomorphisms $\iota_j \otimes \mathrm{id}_B\colon A_j \otimes_R B \to \left(\bigoplus_{i \in I} A_i\right) \otimes_R B$ from Definition 312.6 give rise to a natural homomorphism

$$\begin{array}{rcl} \bigoplus_{i \in I} (A_i \otimes_R B) & \to & \left(\bigoplus_{i \in I} A_i\right) \otimes_R B \\ \sum_i a_i \otimes b & \mapsto & \left(\sum_i a_i\right) \otimes b \end{array}$$

of (Q, S)-bimodules. This homomorphism is a natural isomorphism of (Q, S)-bimodules.

(2) Let A be a (Q, R)-bimodule and let $\{B_i\}_{i \in I}$ be a family of (R, S)-bimodules. Similar to (1) we get a natural isomorphism

$$\bigoplus_{i \in I} (A \otimes_R B_i) \to A \otimes_R \left(\bigoplus_{i \in I} B_i\right)$$

of (Q, S)-bimodules.

[3464] Here we equip $\mathrm{Hom}_{Q\text{-left}}(M, Y)$ with the R-left module structure that is given by $(r \cdot \varphi)(m) := \varphi(m \cdot r)$. We will discuss this left-module structure in greater detail, later in the Hom–Module Lemma 315.2.

Sketch of proof.

(1) This lemma is an immediate consequence of the Tensor Product–Colimit Proposition 312.14 and the observation from Definition 6.5 that the coproduct is just a colimit.

But for the reader's convenience we will also give a down-to-earth proof of the lemma. We need to show that the above homomorphism

$$\Phi\colon \bigoplus_{i\in I}(A_i \otimes_R B) \;\to\; \Big(\bigoplus_{i\in I} A_i\Big) \otimes_R B$$

has an inverse. The key to doing so is the following claim:

Claim. There exists a unique map

$$\Big(\bigoplus_{j\in I} A_j\Big) \times B \;\to\; \bigoplus_{i\in I}(A_i \otimes_R B)$$

with $(a_j, b) \mapsto a_j \otimes b$ and this map is (Q, S)-bilinear and R-balanced.

Proof. We first fix $b \in B$. For each $j \in I$ we consider the map

$$\begin{array}{ccccc} A_j & \to & A_j \otimes_R B & \to & \bigoplus_{i\in I}(A_i \otimes_R B) \\ a & \mapsto & a \otimes b. & & \end{array}$$

This is clearly a Q-left module homomorphism. By the Coproduct-of-Modules Lemma 309.11 we get an induced homomorphism

$$\Theta_b\colon \bigoplus_{j\in I} A_j \;\to\; \bigoplus_{i\in I}(A_i \otimes_R B)$$

of Q-left modules. These maps Θ_b define a unique map

$$\begin{array}{ccc} \Big(\bigoplus_{j\in I} A_j\Big) \times B & \to & \bigoplus_{i\in I}(A_i \otimes_R B) \\ (a_j, b) & \mapsto & \Theta_b(a_j) = a_j \otimes b. \end{array}$$

One can easily verify that this map is (Q, S)-bilinear and R-balanced. ⊞

It follows from the claim and the Tensor Product–Universal Property Lemma 312.5 (2) that we get an induced homomorphism

$$\Psi\colon \Big(\bigoplus_{j\in I} A_j\Big) \otimes_R B \;\to\; \bigoplus_{i\in I}(A_i \otimes_R B)$$

of (Q, S)-bimodules. Note that the two homomorphisms $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are both the identity on the generating set $\Big\{a \otimes b \;\Big|\; a \in \bigcup_{i\in I} A_j \text{ and } b \in B\Big\}$. Thus we see that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. This implies that Φ is indeed an isomorphism of (Q, S)-bimodules. Finally note that one can easily verify that Φ is natural.

(2) The proof of this statement is evidently essentially the same as the proof of (1). ■

For completeness we state the following analog of the Tensor–Direct Colimit Lemma 159.16:

Lemma 312.16. (Tensor–Direct Colimit Lemma) Let Q, R, S be rings. Let $(I, \leq)$ be a directed set and let $(\{A_i\}_{i\in I}, \{\varphi_{ji}\}_{i\leq j})$ be a direct system of (Q, R)-bimodules (in the sense of Definition 104.2). Furthermore let B be an (R, S)-bimodule. Given $i \in I$ we consider the natural homomorphism $\iota_i\colon A_i \to \operatorname{colim} A_i$ of (Q, R)-bimodules and the natural homomorphisms $\iota_i \otimes \mathrm{id}_B\colon A_i \otimes_R B \to \big(\operatorname{colim} A_i\big) \otimes_R B$ of (Q, T)-bimodules from Definition 312.6. These induce a natural isomorphism

$$\operatorname{colim} (A_i \otimes_R B) \;\to\; \big(\operatorname{colim} A_i\big) \otimes_R B$$

of (Q, T)-bimodules.

Proof. This lemma is an immediate consequence of the Tensor Product–Colimit Proposition 312.14 and the observation from Definition 6.5 that the coproduct is just a colimit. Alternatively, as on page 3063, one can also prove the lemma directly "by hand". ■

312.7. **Tensoring with rings.** The following lemma is a generalization of the Tensor-with-Ring Lemma 159.18:

Lemma 312.17. (Tensor Product–Functorial Lemma) Let Q, R, S be rings.

(1) Given any (Q, R)-bimodule A the maps

$$\begin{array}{rcl} B & \mapsto & A \otimes_R B \\ (f\colon B \to B') & \mapsto & (\mathrm{id}_A \otimes f\colon A \otimes_R B \to A \otimes_R B') \end{array}$$

form a covariant functor from the category $\mathcal{M}od_{(R,S)\text{-bi}}$ of (R, S)-bimodules to the category $\mathcal{M}od_{(Q,S)\text{-bi}}$ of (Q, S)-bimodules.

(2) Given any (R, S)-bimodule B the maps

$$\begin{array}{rcl} A & \mapsto & A \otimes_R B \\ (f\colon A \to A') & \mapsto & (f \otimes \mathrm{id}_B\colon A \otimes_R B \to A' \otimes_R B) \end{array}$$

form a covariant functor from the category $\mathcal{M}od_{(Q,R)\text{-bi}}$ of (Q, R)-bimodules to the category $\mathcal{M}od_{(Q,S)\text{-bi}}$ of (Q, S)-bimodules.

Proof. This lemma is an almost immediate consequence of the Tensor Product–Basics Lemma 312.4 (1), the Tensor Product–Basics Lemma 312.4 and the Tensor–Map Composition Lemma 312.7. ■

Example.

(1) Let $Q = R = \mathbb{Z}$ and let S be some ring, e.g. $S = \mathbb{Q}$. We consider the $(\mathbb{Z}, S)$-bimodule S. In this setting the Tensor Product–Functorial Lemma 312.17 (2) just says that the maps

$$\begin{array}{rcl} A & \mapsto & A \otimes_{\mathbb{Z}} S \\ (f\colon A \to \widetilde{A}) & \mapsto & (f \otimes \mathrm{id}_S\colon A \otimes_{\mathbb{Z}} S \to \widetilde{A} \otimes_{\mathbb{Z}} S) \end{array}$$

define a covariant functor from the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups to the category $\mathcal{M}od_S$ of S-right modules (e.g. the category $\mathcal{V}ec_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces.) This is essentially the content of the Tensor-with-Ring Lemma 159.18.

(2) Let $Q = \mathbb{Z}$, $R = \mathbb{R}$ and $S = \mathbb{C}$. We consider the $(\mathbb{R}, \mathbb{C})$-bimodule $\mathbb{C}$. In this setting the Tensor Product–Functorial Lemma 312.17 (2) just says that the maps

$$\begin{array}{rcl} V & \mapsto & V \otimes_{\mathbb{R}} \mathbb{C} \\ (f\colon V \to W) & \mapsto & (f \otimes \mathrm{id}_{\mathbb{C}}\colon V \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}) \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces to the category $\mathcal{V}ec_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces. Note that this is precisely the content of the Tensor-with-$\mathbb{C}$-Lemma 268.13. □

The following is a generalization of the Free Module–Tensor-with-Ring Lemma 159.20:

Lemma 312.18. (Free Module–Tensor-with-Ring Lemma) Let R be a ring. Furthermore let $\varphi\colon R \to Q$ be a ring homomorphism. As in the Module–Change-of-Ring–Functor Lemma 309.13 we use this ring homomorphism to view Q_φ as a (Q, R)-bimodule.

(1) (a) Let F be a free R-left module with basis $\mathcal{B}$ in the sense of Definition 309.17. Then $Q_\varphi \otimes_R F$ is a free Q-left module with basis $\{1_Q \otimes b\}_{b\in\mathcal{B}}$.

(b) Let I be a set. There exists a unique isomorphism

$$\begin{array}{rccc} \Theta_I\colon & Q_\varphi \otimes_R R^{(I)} & \xrightarrow{\cong} & Q^{(I)} \\ & 1_Q \otimes i & \mapsto & i \end{array}$$

of Q-left modules. This isomorphism is natural in Q.

(2) Let A be an $(n\times m)$-matrix over R. With this notation the following diagram of Q-left module homomorphisms commutes:

$$\begin{array}{ccc} Q_\varphi \otimes_R R^{1\times n} & \xrightarrow[\cong]{\Theta_n} & Q^{1\times n} \\ \downarrow{\scriptstyle \mathrm{id}_Q \otimes (v\mapsto v\cdot A)} & & \downarrow{\scriptstyle w\mapsto w\cdot\varphi(A)} \\ Q_\varphi \otimes_R R^{1\times m} & \xrightarrow[\cong]{\Theta_m} & Q^{1\times m}. \end{array}$$

Proof.

(1) (a) We need to verify the condition from Definition 309.17:

Claim. Let N be an Q-left module and let $f\colon \{1_Q \otimes b\}_{b\in\mathcal{B}} \to N$ be a map. There exists a unique homomorphism $\Psi\colon Q_\varphi \otimes_R F \to N$ of Q-left modules such that for all $b \in \mathcal{B}$ we have $\Psi(1_Q \otimes b) = f(b)$.

Proof. Since $\mathcal{B}$ is a basis of the free R-left module F we obtain from Definition 309.17 a unique homomorphism $\Phi\colon F \to N$ of R-left modules with $\Phi(b) = f(1_Q \otimes b)$ for all $b \in B$. Next we consider the following diagram:

$$\begin{array}{ccc} Q \times F & \xrightarrow{(r,f)\mapsto r\otimes_R f} & Q_\varphi \otimes_R F \\ & \searrow^{(r,f)\mapsto r\cdot\Phi(f)} & \downarrow{\scriptstyle \exists!\Psi} \\ & & N. \end{array}$$

Clearly the diagonal map is bilinear. By the Tensor Product–Universal Property Lemma 312.5 we get a unique homomorphism $\Psi\colon Q_\varphi \otimes_R F \to N$ of abelian groups that makes the diagram commute. It follows easily from the uniqueness property that Ψ is actually Q-linear. Finally note that for each $b \in \mathcal{B}$ we have by definition $\Psi(1_Q \otimes b) = 1_Q \cdot \Phi(b) = \Phi(b)$. This gives us the existence statement.

It remains to show that Ψ is uniquely determined by the values on $\{1_Q \otimes b\}_{b\in\mathcal{B}}$. So suppose we have two homomorphisms $\Psi, \Psi'\colon Q_\varphi \otimes_R F \to N$ of Q-left modules such that for all $b \in \mathcal{B}$ we have $\Psi(1_Q \otimes b) = \Psi'(1_Q \otimes b)$. We consider the homomorphisms $\Phi\colon F \to N$ and $\Phi'\colon F \to N$ of R-left modules that are given by the precomposition of Ψ and Ψ' by $f \mapsto 1_Q \otimes f$. Note that Ψ and Ψ' are now the unique R-left module homomorphisms that make the above diagram commute for Φ and Φ'. Since $\mathcal{B}$ is a

[3463] Recall that in Footnote 3433 we saw that the to the left and right we do indeed have the identity.

[3464] Recall that, given a ring T and given $k \in \mathbb{N}_0$ we introduced on page 5962 the T-left module $T^{1\times k}$ of "row vectors".

basis of F we see that $\Phi = \Phi'$. But then it follows from the above characterization of Ψ and Ψ' that $\Psi = \Psi'$. ⊞

(b) This statement follows easily from (a). Alternatively one can see that the given map is the composition of the following isomorphisms of Q-left modules:

$$Q_\varphi \otimes_R R^{(I)} = Q_\varphi \otimes_R \left(\bigoplus_{i\in I} R\right) \xleftarrow[\uparrow]{\cong} \bigoplus_{i\in I}(Q_\varphi \otimes_R R) \xrightarrow[\uparrow]{\cong} \bigoplus_{i\in I} Q = Q^{(I)}.$$

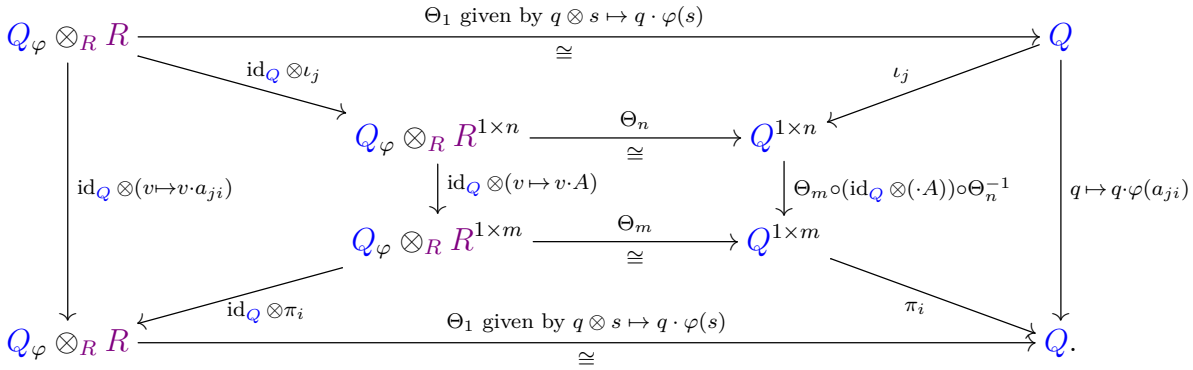
isomorphism from the Tensor–Direct Sum Lemma 312.15

isomorphism from the Tensor-with-Base Ring Lemma 312.10

(2) We start out with a simple observation: It follows easily from the Tensor Product–Basics Lemma 312.4 (1b) that given any $r \in R$ the following diagram commutes:

$$\begin{array}{ccc} Q_\varphi \otimes_R R & \xrightarrow{\Theta_1 \text{ given by } q\otimes s \mapsto q\cdot\varphi(s)} & Q \\ {\scriptstyle \mathrm{id}_Q \otimes (v\mapsto v\cdot r)}\downarrow & & \downarrow{\scriptstyle q\mapsto q\cdot\varphi(r)} \\ Q_\varphi \otimes_R R & \xrightarrow[\Theta_1 \text{ given by } q\otimes s \mapsto q\cdot\varphi(s)]{} & Q. \end{array}$$

Now let $A = (a_{ji}) \in \mathrm{M}(n\times m, R)$. Let $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. We denote by $\pi_i\colon R^{1\times m} \to R$ the projection onto the i-th term and we denote by $\iota_j\colon R \to R^{1\times n}$ the inclusion into the j-th coordinate. We consider the following diagram:



We make the following observations:

(a) The outer rectangle commutes by the discussion at the beginning of the proof.

(b) The small inner rectangle commutes by definition.

(c) The quadrilateral to the left commutes by the Tensor–Map Composition Lemma 312.7 (2).

(d) The quadrilaterals to the top and bottom commute by definition of Θ_m and Θ_n.

It follows from these observations and the fact that the horizontal maps are isomorphisms that the quadrilateral to the right commutes. But this implies precisely that the vertical map from $Q^{1\times n}$ to $Q^{1\times m}$ is given by right multiplication by the matrix $\varphi(A)$. ■

312.8. **Invariance basis number property.** In this last section we will consider the following natural definition:

Definition 312.19. We say that a ring R has the invariant basis number property if the following statement holds: Any two bases for a free R-left module have the same cardinality.[3465] A ring that has the invariant basis number property is called an IBN ring.

Example.

(1) The Module-over-Skew Field Proposition 309.33 (2a) says that every (skew) field is an IBN ring.
(2) The zero ring $R = \{0\}$ is *not* an IBN ring since for any set X the free R-left module $R^{(I)}$ is just once again the 0-module. □

The following lemma is the key to showing that many of our favorite rings are IBN rings:

Lemma 312.20. (Invariant Basis Number–Homomorphism) Let R be a ring. If R admits a ring homomorphism $\varphi\colon R \to S$ to an IBN ring S, then R is also an IBN ring.

Proof. [3466] Let R be a ring and let $\varphi\colon R \to S$ be a ring homomorphism to an IBN ring S. We need to show that R is an IBN ring.

It follows easily from the definitions that it suffices to prove the following claim:

Claim. Let X, Y be two sets. If there exists an isomorphism $\varphi\colon R^{(X)} \to R^{(Y)}$ of R-left modules, then X and Y have the same cardinality.

Proof. Let $\varphi\colon R^{(X)} \to R^{(Y)}$ be an isomorphism of R-left modules. Note that we have the following isomorphisms of S-left modules:

$$S^{(X)} \xleftarrow{\cong} S \otimes_R R^{(X)} \xrightarrow{\mathrm{id}_S \otimes \varphi} S \otimes_R R^{(Y)} \xrightarrow{\cong} S^{(Y)}.$$

(the middle arrow: by the Tensor–Map Composition Lemma 312.7, this is an isomorphism of S-left modules; the two outer isomorphisms: these two isomorphisms of S-left modules are provided by the Free Module–Tensor-with-Ring Lemma 312.18 (1))

Since S is an IBN ring we see that X and Y have the same cardinality. ■

Now we can easily prove the following convenient lemma:

Lemma 312.21. (Commutative Ring–Invariant Basis Number Lemma) Every *commutative* non-zero ring is an IBN ring.

Sketch of proof. Let R be a commutative non-zero ring. It follows from the Invariant Basis Number–Homomorphism 312.20 and the Module-over-Skew Field Proposition 309.33 (2a) that we just need to prove the following claim:

Claim. There exists a ring homomorphism from R to a field.

Proof. We first deal with the special case that R is a domain. In this case we have a ring monomorphism $R \to Q$ from R to its quotient field and we are done.

Now let R be any old commutative non-zero ring. Since R is non-zero we know, e.g. by [**Lang2002b**, Chapter II.2], that R admits a maximal ideal $\mathfrak{m} \subseteq R$. Thus we can consider

[3465] In particular this implies that if we have $m, n \in \mathbb{N}_0$ such that the R-left modules R^m and R^n are isomorphic, then $m = n$. Note though that in the definition we do not restrict ourselves to the case of finite bases.

[3466] The proof is very similar to the argument on page 3066.

the ring homomorphism $R \to R/\mathfrak{m}$. Since $\mathfrak{m}$ is maximal we know that $R/\mathfrak{m}$ is a field. Once again we are done. ■

Now we turn to the case of non-commutative ring. In this setting, for better or worse, not every ring is an IBN ring:

Lemma 312.22. (Non-IBN Lemma) There exists a non-commutative ring R such that the two R-left modules R and R^2 are isomorphic.

Proof. By the Opposite Ring–Module Lemma 309.7 it suffices to prove the following claim:

Claim. There exists a non-commutative ring S such that the two S-right modules S and S^2 are isomorphic.

Proof. We set $V := \mathbb{Z}^{(\mathbb{N})}$. Let $S = \mathrm{End}(V)$ be the ring of all endomorphisms of the abelian group V. By "Hilbert's hotel" we can pick an isomorphism $\epsilon : V \to V \oplus V$ of abelian groups. For $i = 1, 2$ we denote by $\pi_i : V \oplus V \to V$ the projection onto the i-th summand.

Next we consider the homomorphism

$$\begin{array}{rcccccc} \Theta : S & \to & \mathrm{Hom}(V, V \oplus V) & \xrightarrow{\pi_{1*} \oplus \pi_{2*}} & \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,V) & = & S \oplus S \\ \varphi & \mapsto & \epsilon \circ \varphi & & & & \\ & & \psi & \mapsto & (\pi_1 \circ \psi, \pi_2 \circ \psi) & & \end{array}$$

In other words, for each $\varphi \in S$ we have $\Theta(\varphi) = (\pi_1 \circ \epsilon \circ \varphi, \pi_2 \circ \epsilon \circ \varphi)$. Note that Θ is clearly an isomorphism of abelian groups. So it remains to show that this map is an S-right homomorphism. For any $\rho \in S$ and $\psi \in S$ we do indeed have

$$\begin{aligned} \Theta(\psi \cdot \rho) = \Theta(\psi \circ \rho) &= (\pi_1 \circ \epsilon \circ \psi \circ \rho, \pi_2 \circ \epsilon \circ \psi \circ \rho) \\ &= (\pi_1 \circ \epsilon \circ \psi, \pi_2 \circ \epsilon \circ \psi) \circ \rho \quad = \Theta(\psi) \circ \rho = \Theta(\psi) \cdot \rho. \end{aligned}$$ ■

Despite this setback we will now see that fortunately our favorite type of non-commutative rings, namely group rings, are IBN rings:

Lemma 312.23. (Group Ring–Invariance Basis Number Lemma) For every group G and every commutative ring R the group ring $R[G]$ is an IBN ring.

Proof. In the Commutative Ring–Invariant Basis Number Lemma 312.21 we just learned that the commutative ring R is an IBN ring. The present lemma thus follows from the Invariant Basis Number–Homomorphism 312.20 applied to the augmentation homomorphism

$$\begin{array}{ccc} R[G] & \to & R \\ \sum_{r=1}^{m} r_i \cdot g_i & \mapsto & \sum_{r=1}^{m} r_i \end{array}$$

from Definition 116.6. ■

312.9. **Appendix: An alternative construction of the tensor product.** In this appendix we will give an alternative construction of the tensor product of the tensor product $A \otimes_R B$. This alternative description has the advantage that it starts out with a smaller object.

Convention 312.24. Let A be a Q-left module and let B be an S-right module. We consider the tensor product $A \otimes B$ of abelian groups from Definition 311.2.

- Given $q \in Q$ we denote by $\lambda_q \colon A \to A$ the group homomorphism that is given by left multiplication by q. Note that we obtain from Definition 159.6 an induced group homomorphism $\lambda_q \otimes \mathrm{id}_B \colon A \otimes B \to A \otimes B$. It follows easily from the Tensor–Map Composition Lemma 159.7 that these maps λ_q defines a Q-left module structure on the abelian group $A \otimes B$.
- Similarly we can define for each $s \in S$ a right multiplication $\rho_s \colon A \otimes B \to A \otimes B$ to obtain an S-right module structure on the abelian group $A \otimes B$.

It follows from $(\lambda_q \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \rho_s) = \lambda_q \circ \rho_s = (\mathrm{id}_A \otimes \rho_s) \circ (\lambda_q \otimes \mathrm{id}_B)$ that the two module structures turn $A \otimes B$ into a (Q, S)-bimodule.

This observation allows us to give the following alternative construction of the tensor product of the tensor product $A \otimes_R B$:

Lemma 312.25. (Tensor Product–Small Model Lemma) Let Q, R, S be rings, let A be a (Q, R)-bimodule and let B be an (R, S)-bimodule. We consider

$$(A \otimes B)/\widetilde{N}(A, B)$$

where $\widetilde{N}(A, B)$ is the (Q, S)-submodule of the (Q, S)-module $A \otimes B$ that is generated by all elements of the form

$$(\star) \qquad (a \cdot r, b) - (a, r \cdot b) \quad \text{with } a \in A, b \in B \text{ and } r \in R.$$

The following two statements hold:

(1) The obvious map $A \times B \to (A \otimes B)/\widetilde{N}(A, B)$ is (Q, S)-bilinear and R-balanced.
(2) There exists a unique isomorphism $A \otimes_R B \to (A \otimes B)/\widetilde{N}(A, B)$ of (Q, S)-bimodules that makes the following diagram commute:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b) \mapsto a \otimes b} & A \otimes_R B \\ & \searrow & \cong \big\downarrow \Theta \\ & & (A \otimes B)/\widetilde{N}(A, B) \end{array}$$

Proof.

(1) It is clear that the given map is (Q, S)-bilinear and R-balanced.
(2) As we discussed in Footnote 3458 it follows easily from the universal property from the Tensor Product–Universal Property Lemma 312.5 (2) that it suffices to prove the following claim:

Claim. Given any (Q, S)-bimodule C and given any map $g \colon A \times B \to C$ which is (Q, S)-bilinear and which is also R-balanced there exists in fact a unique homomorphism $\varphi \colon (A \otimes B)/\widetilde{N}(A, B) \to C$ of (Q, S)-bimodules such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow[\substack{(Q,S)\text{-bilinear} \\ R\text{-balanced}}]{(a,b) \mapsto [(a,b)]} & (A \otimes B)/\widetilde{N}(A, B) \\ & \searrow^{\substack{(Q,S)\text{-bilinear} \\ R\text{-balanced map } g}} & \big\downarrow \exists!\, \varphi \text{ homomorphism of } (Q,S)\text{-bimodules} \\ & & C. \end{array}$$

Proof. Given $g\colon A \times B \to C$ we obtain from the Tensor Product–Universal Property Lemma 159.4 that there exists a unique group homomorphism $\psi\colon A \otimes B \to C$ that makes the diagram commute. It follows easily from the discussion in Convention 312.24 that this map is automatically (Q, S)-bilinear. Since g is R-balanced one sees that ψ factors through $\psi\colon A \otimes B \to C$. ■

312.10. Exercises for Chapter 312.

Exercise 312.1. Provide an example of a non-commutative ring R, (R, R)-bimodules A, B, C and a map $f\colon A \times B \to C$ with the following two properties:

- The map $f\colon A \times B \to C$ is (R, R)-bilinear in the sense of Definition 312.1.
- The map $f\colon A \times B \to C$ is *not* R-balanced in the sense of Definition 312.2.

Exercise 312.2. In the definition of the tensor product $A \otimes_R B$ that we gave in Definition 312.3 we started out with a right R-module A and a left R-module B. This played a role in the definition of $N(A, B)$ which contains in particular all elements of the form

$$(\star) \qquad a \cdot r \otimes b - a \otimes r \cdot b \qquad \text{with } a \in A, b \in B \text{ and } r \in R.$$

Suppose that we start out with two R-left modules A and B. We modify $(\star)$, namely we now consider all elements of the form

$$(\star') \qquad r \cdot a \otimes b - a \otimes r \cdot b \qquad \text{with } a \in A, b \in B \text{ and } r \in R.$$

With this definition it still makes sense to define $A \otimes B$ as a suitable quotient. Is this a reasonable definition? For example, is it still true that $R \otimes_R R \cong R$?

Exercise 312.3. Let R be a non-commutative ring, let A be an R-right module and let B be an R-left module. Can you define a natural R-module structure on $A \otimes_R B$?

Exercise 312.4. If R is a commutative ring, then any R-module also has automatically an (R, R)-bimodule structure. In this exercise we want to consider more interesting bimodules over commutative rings. We consider the tensor product $R \otimes R$ of *abelian groups*. It follows from the Tensor-with-Ring Lemma 159.18 that multiplication in the left respectively right factor defines two R-module structures on the abelian group $R \otimes R$.

(a) Show that these two R-module structures define an (R, R)-bimodule structure on the abelian group $R \otimes R$.

(b) Show that there exist commutative rings such that these two R-module structures on $R \otimes R$ are different.

Remark. This exercise is identical to Exercise 309.20.

Exercise 312.5. Let R be a *commutative* ring. Given two R-modules A and B we define the tensor product of A and B to be any pair (C, μ), where C is an R-module and $\mu\colon A \times B \to C$ is R-bilinear such that for any R-module D and any R-bilinear map $g\colon A \times B \to D$ there exists a unique homomorphism $\varphi\colon C \to D$ of R-modules such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{\mu} & C \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & D. \end{array}$$

Show that for a commutative ring the tensor product from the Tensor Product–Universal Property Lemma 312.5 applied to $Q = R$ and $S = \mathbb{Z}$ is a tensor product in the above sense.

Exercise 312.6. We consider the commutative ring $R = \mathbb{C}[t]$. Determine the tensor product of the $\mathbb{C}[t]$-modules $M = \mathbb{C}[t]/(t^2 - 1)$ and $N = \mathbb{C}[t]/(t^5 - 1)$.

Exercise 312.7. Does there exist a commutative ring R and an R-module M which is not free, but such that $M \otimes_R M$ is a free R-module?

Exercise 312.8. We consider the ring $\mathrm{M}(2 \times 2, \mathbb{C})$.

- As on page 5962 we view $\mathbb{C}^{1\times 2}$ ("row vectors") as a $\mathrm{M}(2 \times 2, \mathbb{C})$-right module.
- As on page 5962 we view $\mathbb{C}^{2\times 1}$ ("column vectors") as a $\mathrm{M}(2 \times 2, \mathbb{C})$-left module.

Determine the dimension of the complex vector space $\mathbb{C}^{1\times 2} \otimes_{\mathrm{M}(2\times 2,\mathbb{C})} \mathbb{C}^{2\times 1}$.

Exercise 312.9. Let A and B be abelian groups. The Tensor Commutative-Associative Lemma 159.11 (2) tells us that the map

$$\begin{array}{rcl} A \otimes B & \to & B \otimes A \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} b_i \otimes a_i \end{array}$$

is well-defined and that it is a natural isomorphism of abelian groups. Does this statement have an analog in the current setting of tensor products of modules?

Exercise 312.10. Let R be a commutative ring, let M be an R-module, let G be a group, let H be a subgroup of G and let $\varphi\colon H \to \mathrm{Aut}_R(M)$ be a representation. We view $\mathbb{Z}[G]$ as a $(\mathbb{Z}[G], \mathbb{Z}[H])$-bimodule in the obvious way. We consider the tensor product

$$\mathrm{Ind}_H^G M \;:=\; \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$$

which by the Tensor Product–Functorial Lemma 312.17 is a $\mathbb{Z}[G]$-left module. Thus we can view $\mathrm{Ind}_H^G M$ as an R-representation of G, which is called the *induced representation*.

(a) Show that if $R = \mathbb{K}$ is a field, then $\dim_{\mathbb{K}}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M) = [G : H] \cdot \dim_{\mathbb{K}}(M)$.

(b) According to Definition 310.8 a representation of a group is called *faithful* if it is a monomorphism. Show that if M is a faithful representation of H, then $\mathrm{Ind}_H^G M$ is a faithful representation of G.

Remark. We do not really need to assume that R is commutative, but it does make the discussion a little easier.

Exercise 312.11. We consider the polynomial ring $\mathbb{Q}[x]$. Given a polynomial $p(x) \in \mathbb{Q}[x]$ we view $\mathbb{Q}[x]/(p(x))$ as a $\mathbb{Q}[x]$-module in the obvious way. Show that the map

$$\begin{array}{rcl} \mathbb{Q} \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(p(x)) & \to & \mathbb{Q}/(p(1)) \\ n \otimes [f(x)] & \mapsto & n \cdot f(1) \end{array}$$

is an isomorphism.

Exercise 312.12. Let R be a ring, let $m \in R$ and let $\varphi\colon R \to S$ be a ring homomorphism such that $\varphi(m)$ is invertible in S. Show that $S \otimes_R R/mR = 0$.

Exercise 312.13. Let V be a $\mathbb{Q}[t^{\pm 1}]$-module that is finite-dimensional as a $\mathbb{Q}$-vector space. Show that $\mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} V = 0$.

Remark. Make use of Exercise 312.12.

Exercise 312.14. Let X be a topological space. Does the ring $C(X, \mathbb{R})$ of real valued functions have the invariant base property?

Exercise 312.15. We consider the permutation group S_3 and the group ring $\mathbb{Q}[S_3]$. We use sign homomorphism $S_3 \to \{\pm 1\}$ to view $\mathbb{Q}$ as a $\mathbb{Q}[S_3]$-right module and we consider the obvious permutation representation $S_3 \to \mathrm{GL}(3, \mathbb{Q})$ to view $\mathbb{Q}^3$ as a $\mathbb{Q}[S_3]$-left module. What is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Q}[S_3]} \mathbb{Q}^3$?

Exercise 312.16. Let R be a commutative ring and let V and W be finite-dimensional free R-modules that are equipped with bases $\mathcal{V} = (v_1, \ldots, v_m)$ and $\mathcal{W} = (w_1, \ldots, w_n)$. Let $f\colon V \to V$ and $g\colon W \to W$ be homomorphisms. We equip $V \otimes_R W$ with the basis $\mathcal{V} \otimes \mathcal{W} := \{v_i \otimes w_j\}_{i=1,\ldots,m, j=1,\ldots,n}$. Show that, with the hopefully obvious notation, we have the following equality:

$$\det(f \otimes g, \mathcal{V} \otimes \mathcal{W}) \;=\; \det(f, \mathcal{V})^n \cdot \det(g, \mathcal{W})^m.$$

Remark. Note that for a self-map the determinant does not depend on the choice of an ordering of the basis elements.

CHAPTER 313

Chain complexes over rings

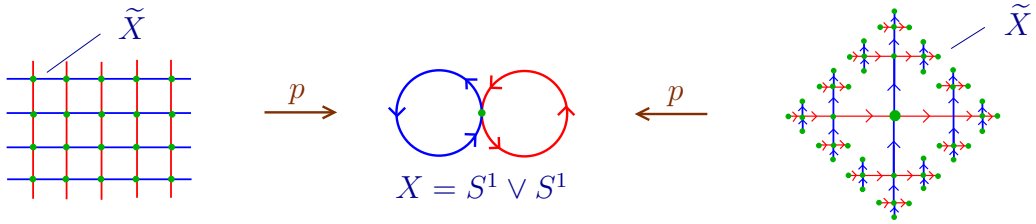
In this chapter we will generalize many concepts and results about chain complexes of abelian groups to chain complexes of modules. Most extensions are done in the obvious way and the chapter contains few surprises for the reader who is well-versed in homological algebra of abelian groups. The only concept which is really new is the concept of a "free imitation" of a chain complex which we will discuss in Section 313.6.

This chapter is organized as follows:

- In Section 313.1 we will give a short topological motivation why one might want to consider chain complexes over possibly non-commutative rings.
- In Section 313.2 we will introduce chain complexes over rings.
- In Section 313.3 we will study connecting homomorphisms.
- In Section 313.4 we will introduce and study projective resolutions.
- In Section 313.5 we will study quasi-isomorphisms.
- In Section 313.6 we will introduce free imitations of chain complexes.
- In Section 313.7 we will provide exercises for Chapter 313.

In the next Chapter 314 we will study the effect of tensoring a chain complex with a module.

313.1. **Motivation.** Let $p\colon \widetilde{X} \to X$ be a covering of a connected CW-complex with deck transformation group G. We equip $\widetilde{X}$ with the CW-structure from the CW-Complex Covering Proposition 138.1. We can consider the corresponding cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$. In the two examples below these chain groups are free abelian groups of infinite rank.



deck transformation group $G = \mathbb{Z}^2$

deck transformation group $G = \langle x, y \rangle$

We make the following observations:[3467]

- The deck transformation group G acts cellularly on $\widetilde{X}$. Thus we can view each cellular chain group $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ as a $\mathbb{Z}[G]$-left module. In fact in the above examples these are free $\mathbb{Z}[G]$-left modules of finite rank.

[3467]We will discuss this setup in much greater detail in Section 327.5. The current discussion is just supposed to serve as an amuse-bouche.

- Since the boundary maps are natural, we see that the boundary maps preserve the G-action.

Thus we see that in the above two examples each $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ is a chain complex of finitely generated free left $\mathbb{Z}[G]$-modules. This discussion explains hopefully why one might want to consider chain complexes over potentially non-commutative rings.

313.2. **Chain complexes over rings.** In this section we generalize the theory of chain complexes of abelian groups to R-left chain complexes. Basically everything we do in this section is self-explanatory. First we have the following utterly predictable generalizations of Definition 143.1, Definition 143.2 and Definition 143.3.

Definition 313.1. Let R be a ring.

(1) An R-left chain complex (C_*, ∂_*) is a sequence
$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$
of R-left module homomorphisms between R-left modules such that for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. As before we introduce the convention that $C_n = 0$ for $n \in \mathbb{Z}_{<0}$.

(2) An R-left chain map $f\colon C_* \to D_*$ between two R-left chain complexes consists of a family $\{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ of R-left module homomorphisms such that for each $n \in \mathbb{N}_0$ we have $f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n$.

From these definitions we obtain the category $\mathcal{ChCplx}_{R\text{-left}}$ of R-left chain complexes.

Definition 313.2. Let R be a ring. We define the n-th homology module of an R-left chain complex (C_*, ∂_*) as
$$\mathrm{H}_n(C) \;:=\; \ker(\partial_n\colon C_n \to C_{n-1}) \,/\, \mathrm{im}(\partial_{n+1}\colon C_{n+1} \to C_n).$$
Note that $\mathrm{H}_n(C)$ inherits the structure of an R-left module.

The following lemma is a very mild generalization of the Chain Complex-H_*-Functor Lemma 143.4:

Lemma 313.3. (Chain Complex-H_*-Functor Lemma) Let R be a ring.

(1) If $f\colon C_* \to D_*$ is a chain map between R-left chain complexes, then the map
$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$
is a well-defined homomorphism of R-left modules.

(2) Given $n \in \mathbb{N}_0$ the maps
$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon C_* \to D_*) & \mapsto & (f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)) \end{array}$$
define a covariant functor from the category $\mathcal{ChCplx}_{R\text{-left}}$ of R-left chain complexes to the category $\mathcal{Mod}_{R\text{-left}}$ of R-left modules.

Proof. As in the case of the Chain Complex-H_*-Functor Lemma 143.4 this lemma follows easily from the definitions. ∎

The following lemma is the obvious generalization of the Direct Sum-H_*-Lemma 143.7:

Lemma 313.4. (Direct Sum-H$_*$-Lemma) Let R be a ring and let $\{C_a\}_{a\in A}$ be a family of R-left chain complexes. For every $n \in \mathbb{N}_0$ the natural maps $C_a \to \bigoplus_{a\in A} C_a$ induce an isomorphism

$$\begin{array}{ccc} \bigoplus_{a\in A} \mathrm{H}_n(C_a) & \xrightarrow{\cong} & \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a\Big) \\ ([c_a])_{a\in A} & \mapsto & [(c_a)_{a\in A}] \end{array}$$

of R-left modules.

Proof. This lemma follows immediately from the Direct Sum-H$_*$-Lemma 143.7, since being an isomorphism has nothing to do with the R-module structure. ■

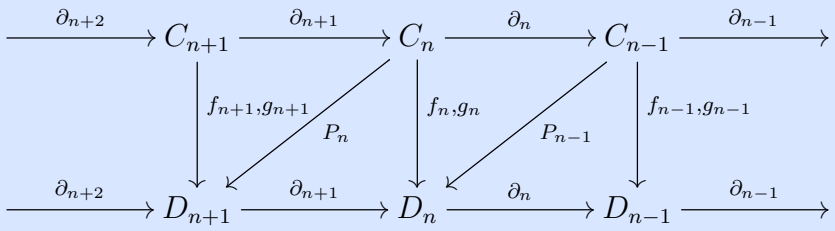
The following definition is a generalization of Definition 144.4:

Definition 313.5. Let R be a ring.

(1) Let $f, g\colon C_* \to D_*$ be chain maps between R-left chain complexes C_* and D_*. A chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g consists of a family $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ of R-left module homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$. For the reader's convenience we summarize the various homomorphisms in the following diagram:



(2) Let C_* and D_* be two R-left chain complexes.
   (a) A chain map $f\colon C_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to C_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.
   (b) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

Remark. As in the case of chain complexes of abelian groups one can easily show, see the Chain Homotopy Equivalence-Reformulation Lemma 144.9 and the remark on page 2769, that being homotopic and being chain homotopy equivalent is an equivalence relation. □

Example. It follows from the Chain Homotopy Equivalence-Reformulation Lemma 144.9 and the Isomorphism Lemma 5.12 that the relation "chain homotopy equivalence" of chain complexes is indeed an equivalence relation on the class of all chain complexes. □

The following lemma is the analog of the Chain Homotopy–H$_*$-Lemma 144.5:

Lemma 313.6. (Chain Homotopy–H$_*$-Lemma) Let R be a ring. Furthermore let C_* and D_* be two R-left chain complexes and let $f, g\colon C_* \to D_*$ be two chain maps. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ for all $n \in \mathbb{N}_0$.

Proof. This statement follows in fact immediately from the original Chain Homotopy–H_*-Lemma 144.5 since the statement has nothing to do with R-left module structures. ■

The following corollary is the analog of the Chain Homotopy Equivalence-H_*-Corollary 144.10:

Corollary 313.7. (Chain Homotopy Equivalence-H_*-Corollary) Let R be a ring. If $f\colon C_* \to D_*$ is a chain homotopy equivalence between two R-left chain complexes, then for each $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is an isomorphism of R-left modules.

Proof. The corollary follows easily from the Chain Homotopy–H_*-Lemma 313.6. ■

313.3. **Connecting homomorphisms.** The following is a slight generalization of Definition 145.8:

Definition 313.8. Let R be a ring and let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of R-left chain complexes. Let $n \in \mathbb{N}_0$. We define a homomorphism

$$\partial_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$$

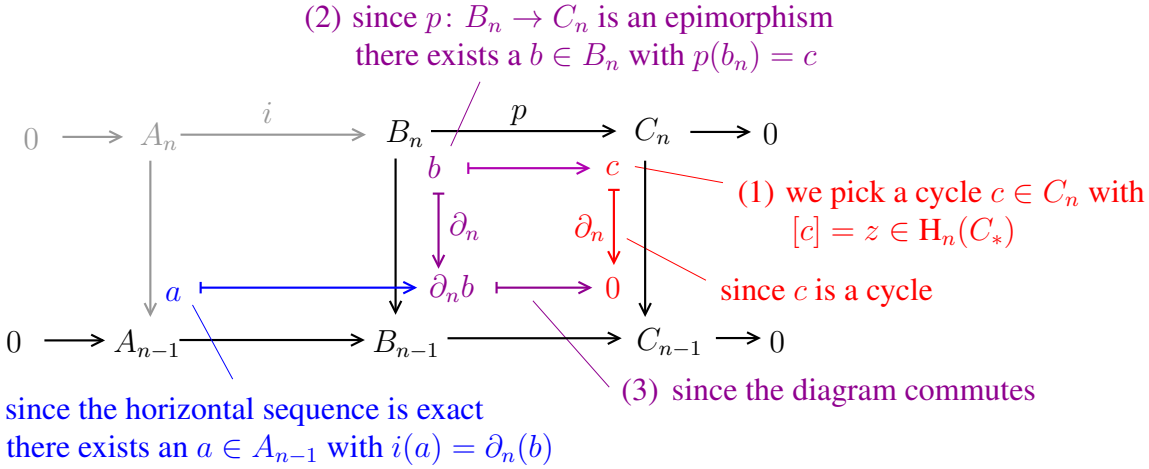
as follows:

(0) Let $z \in \mathrm{H}_n(C_*)$.
(1) We pick an n-cycle $c \in C_n$ with $[c] = z$.
(2) Since $0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$ is exact we know that $p\colon B_n \to C_n$ is an epimorphism. Thus there exists a $b \in B_n$ with $p(b) = c$.
(3) We have

$$p(\partial_n(b)) \underset{\text{since } p \text{ is a chain map}}{=} \partial_n(p(b)) = \partial_n(c) \underset{\text{since } c \text{ is a cycle}}{=} 0.$$

(4) Since $0 \to A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \to 0$ is exact it follows from (3) that there exists a unique $a \in A_{n-1}$ with $i(a) = \partial_n(b)$.
(5) We set $\partial_n(z) := [a] \in \mathrm{H}_{n-1}(A_*)$.

We refer to this map $\partial_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ as the connecting homomorphism.



The following lemma is the analog of the Connecting Homomorphism Lemma 145.9:

Lemma 313.9. (Connecting Homomorphism Lemma) Let R be a ring.

(1) Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of R-left chain complexes. The connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined and it is a homomorphism of R-left modules.

(2) The connecting homomorphism is natural.

Proof. The proof is basically the same as the proof of the Connecting Homomorphism Lemma 145.9. ■

The following proposition is the analog of the LES-Proposition 145.10.

Proposition 313.10. (LES Proposition) Let R be a ring and let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of R-left chain complexes. The sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$

is exact.

Proof. This proposition actually follows from the original LES-Proposition 145.10 since exactness has nothing to do with the module structures. ■

313.4. **Free and projective resolutions.** The following generalizes the notion of a (free) resolution of an abelian group that we gave in Definition 158.2.

Definition 313.11. Let R be a ring and let M be an R-left module.

(1) A (free, projective) resolution of an R-left module M is an exact sequence

$$\ldots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

where the P_i are (free, projective) R-left modules. We say that P_* is a resolution of length $n \in \mathbb{N}_0$ if $P_k = 0$ for all $k > n$.

(2) A weak resolution is the same as a resolution, except that we only demand that the above sequence is a chain complex, we no longer demand that the above sequence is exact.

In the following definition, which is modelled on Definition 162.2, we see in particular that every R-left module admits a natural free resolution.

Definition 313.12. Let R be a ring. We recall the following basic fact: If K is an R-left module, then it follows from the Free Module–Natural Epimorphism Lemma 309.21 that there exists a unique epimorphism $\alpha(K) \colon R^{(K)} \to K$ of R-left modules such that for each $k \in K$ we have $\alpha(K)(k) = k$. Now let M be any R-left module. We consider the sequence

$$R^{(K)} \xrightarrow{\alpha(K)} K \to 0.$$

The R-left module $R^{(K)}$ is free and $\alpha(K)$ is an epimorphism. In particular the sequence is exact. We set $K_1 := \ker\left(R^{(K)} \xrightarrow{\alpha(K)} K\right)$. Then

$$K_1 \hookrightarrow R^{(K)} \xrightarrow{\alpha(K)} K \to 0$$

is exact. Now we consider

$$R^{(K_1)} \xrightarrow{\alpha(K_1)} R^{(K)} \xrightarrow{\alpha(K)} K \to 0.$$

This sequence is exact and both $R^{(K_1)}$ and $R^{(K)}$ are free R-left modules. Next we set $K_2 := \ker\left(R^{(K_1)} \xrightarrow{\alpha(K)} K\right)$ and we iterate this procedure. This way we obtain a free resolution of the R-left module K. We refer to this free resolution of K as the natural free resolution of K.

The following definition is an extension (no pun intended) of Definition 158.3:

Definition 313.13. Let R be a ring and let $\alpha\colon H \to H'$ be a homomorphism between R-left modules. Furthermore let F_* be a resolution of H and let F'_* be a resolution of H'. An extension of α to the resolutions F_* and F'_* is a sequence of homomorphisms $\alpha_i\colon F_i \to F'_i$, $i \in \mathbb{N}_0$ of R-left modules such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0.
\end{array}$$

Note that the horizontal exact sequences on top and bottom are in particular chain complexes and that the maps α_* define a chain map of R-left module homomorphisms from one R-left chain complex to the other.

The following theorem is a straightforward generalization of the Homological Algebra–Fundamental Theorem 158.4.

Theorem 313.14. (Homological Algebra–Fundamental Theorem) Let R be a ring and let $\alpha\colon H \to H'$ be an R-left module homomorphism between R-left modules. Furthermore let P_* be a weak projective resolution of H and let P'_* be a resolution[3468] of H'.

(1) There exists an extension of α to the projective resolutions P_* and P'_*.
(2) Any two extensions α_* and α'_*, viewed as chain maps, are chain homotopic.

Proof. [3469] Let R be a ring. For the reader's convenience we first recall Definition 309.25: We say an R-left module P is *projective* if for every R-left module homomorphism $\psi\colon P \to M$ and every R-left module epimorphism $\varphi\colon N \to M$ there exists an R-left module homomorphism $\sigma\colon P \to N$ such that the following diagram commutes:

$$\begin{array}{ccc}
P & & \\
{\scriptstyle \exists\sigma}\downarrow & \searrow^{\psi} & \\
N & \xrightarrow{\varphi} & M.
\end{array}$$

This diagram can also be drawn as follows:

$$\begin{array}{ccc}
 & & P \\
 & {\scriptstyle \exists\sigma}\nearrow & \downarrow{\scriptstyle \psi} \\
N & \xrightarrow{\varphi} & M.
\end{array}$$

Now let P_* be a projective R-left resolution of H and let P'_* be a resolution of H'.

(1) We have to show that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of homomorphisms of R-left modules such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\ldots & \longrightarrow & P'_2 & \xrightarrow{f'_2} & P'_1 & \xrightarrow{f'_1} & P'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0.
\end{array}$$

[3468] Note that we only expect P_* to be a weak resolution and note that we do not demand that P_* is a P'_* is a projective (let alone free) resolution.
[3469] The proof is very similar to the proof of the Homological Algebra–Fundamental Theorem 158.4.

First we define the homomorphism $\alpha_0 \colon P_0 \to P'_0$. Since $f'_0 \colon P'_0 \to H'$ is an epimorphism and since P_0 is a projective R-left module we obtain directly from Definition 309.25 that there exists an R-left module homomorphism $\alpha_0 \colon P_0 \to P'_0$ that makes the right square commute.

Now we turn to the definition of the homomorphism $\alpha_1 \colon P_1 \to P'_1$. Since P_1 is projective it suffices to prove the following claim:

Claim 1. We have $\operatorname{im}(\alpha_0 \circ f_1) \subseteq \operatorname{im}(f'_1)$.

Proof. Since the diagram commutes and since the top sequence is a chain complex we see that $\operatorname{im}(\alpha_0 \circ f_1) \subseteq \ker(f'_0)$. Since the bottom sequence is exact we see that $\ker(f'_0) = \operatorname{im}(f'_1)$. ⊞

It is now clear how inductively we can construct the homomorphisms $\alpha_2, \alpha_3, \ldots$.

(2) Now let $\alpha_i, \alpha'_i \colon P_i \to P'_i, i \in \mathbb{N}_0$ be two extensions of α to the projective resolutions P_* and the resolution P'_*. We have to find a chain homotopy from the chain map α_i to the chain map α'_i. Note that the maps $\beta_i := \alpha_i - \alpha'_i$ are an extension of the zero map $\beta = \alpha - \alpha = 0 \colon H \to H'$. Therefore it suffices to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero chain map. As we will see, the construction of the chain homotopy is very similar to the construction of the homomorphisms α_i in the first part of the proof.

Recall that we have to find a sequence of homomorphisms λ_i, $i \in \mathbb{N}_0$ such that for every $i \in \mathbb{N}_0$ we have

$$f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i \;=\; \beta_i.$$

In the following discussion it is convenient to keep an eye on the following diagram that contains all the homomorphisms that we are given and that we need to construct:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & 0 & & \\
 & \swarrow^{\lambda_2} & \downarrow{\scriptstyle \beta_2} & \swarrow^{\lambda_1} & \downarrow{\scriptstyle \beta_1} & \swarrow^{\lambda_0} & \downarrow{\scriptstyle \beta_0} & \swarrow^{\lambda_{-1}} & & & \\
\ldots & \longrightarrow & P'_2 & \xrightarrow[f'_2]{} & P'_1 & \xrightarrow[f'_1]{} & P'_0 & \xrightarrow[f'_0]{} & 0. & &
\end{array}$$

We put $\lambda_{-1} = 0$. Next we need to find a homomorphism $\lambda_0 \colon P_0 \to P'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. Since P_0 is projective it suffices to prove the following claim:

Claim 2. We have $\operatorname{im}(\beta_0) \subseteq \operatorname{im}(f'_1)$.

Proof. Since the diagram commutes and since $\beta = 0$ we see that $\operatorname{im}(\beta_0) \subseteq \ker(f'_0)$. Since the bottom sequence is exact we know that $\ker(f'_0) = \operatorname{im}(f'_1)$. ⊞

Now we suppose that we have already found homomorphisms $\lambda_0, \ldots, \lambda_{i-1}$ which have the desired property that

$$(*) \qquad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} \;=\; \beta_{j-1} \quad \text{for } j = 0, \ldots, i.$$

Note that we need to find a homomorphism $\lambda_i \colon P_i \to P'_{i+1}$ of R-left modules such that $f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i = \beta_i$. Since P_i is projective it suffices to prove the following claim:

Claim 3. We have $\operatorname{im}(\beta_i - \lambda_{i-1} \circ f_i) \subseteq \operatorname{im}(f'_{i+1})$.

Proof. Since the bottom sequence is exact we know that $\operatorname{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $\operatorname{im}(\beta_i - \lambda_{i-1} \circ f_i) \subseteq \ker(f'_i)$. It turns out that indeed we have

$$\begin{array}{rcl} f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) & = & f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i \overset{\text{since the } \beta_i\text{'s are a chain map, i.e. since } f'_i \circ \beta_i = \beta_{i-1} \circ f_i}{=} \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\ & = & (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i \overset{\text{by } (*)}{=} \lambda_{i-2} \circ f_{i-1} \circ f_i \overset{\text{since } f_{i-1} \circ f_i = 0}{=} 0. \end{array}$$

■

The following corollary, which is a generalization of the Free Resolution–Chain Homotopic Corollary 158.5, says in particular that any two projective R-left resolutions of a given R-left module are chain homotopy equivalent.

Corollary 313.15. (Projective Resolution–Chain Homotopic Corollary) Let R be a ring and let

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} H \to 0$$

and

$$\ldots \to P'_2 \xrightarrow{f'_2} P'_1 \xrightarrow{f'_1} P'_0 \xrightarrow{f'_0} H \to 0$$

be two projective R-left resolutions of the *same* R-left module H. Let $\{\alpha_i \colon P_i \to P'_i\}_{i \in \mathbb{N}}$ be an extension of the identity map $\mathrm{id} \colon H \to H$ to the projective R-left resolutions P_* and P'_*. The maps α_* form a chain homotopy equivalence between the two R-left chain complexes P_* and P'_*.

Proof. Let $\{\alpha_i \colon P_i \to P'_i\}_{i \in \mathbb{N}}$ be an extension of the identity map $\mathrm{id} \colon H \to H$. We apply the Homological Algebra–Fundamental Theorem 313.14 (1) to the projective R-left resolutions P'_* and P_* (i.e. we swap the roles of the projective R-left resolutions) and we obtain homomorphisms $\{\beta_i \colon P'_i \to P_i\}_{i \in \mathbb{N}_0}$ that give us the lower half of the following commutative diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \ldots & \longrightarrow & P'_2 & \xrightarrow{f'_2} & P'_1 & \xrightarrow{f'_1} & P'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & H & \longrightarrow & 0. \end{array}$$

The maps $\beta_i \circ \alpha_i$ now provide us with the following commutative diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_2 \circ \alpha_2} & & \downarrow{\scriptstyle \beta_1 \circ \alpha_1} & & \downarrow{\scriptstyle \beta_0 \circ \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ \ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & H & \longrightarrow & 0. \end{array}$$

This shows that the maps $\beta_i \circ \alpha_i$ define an extension of $\mathrm{id} \colon H \to H$ to the projective R-left resolutions P_i and P_i. But the identity maps also define an extension. By the Homological Algebra–Fundamental Theorem 313.14 (2) the extensions $\{\beta_i \circ \alpha_i\}_{i \in \mathbb{N}_0}$ and $\{\mathrm{id}\}_{i \in \mathbb{N}_0}$ are chain homotopic. The same way one shows that the extensions $\{\alpha_i \circ \beta_i\}_{i \in \mathbb{N}_0}$ and $\{\mathrm{id}\}_{i \in \mathbb{N}_0}$ are chain homotopic. This shows that the maps α_i form a chain homotopy equivalence. ■

313.5. **Quasi-isomorphisms.** We start out with the following generalization of the Null-Homotopic Criterion Proposition 158.6:

Proposition 313.16. (Null-Homotopic Criterion Proposition) Let R be a ring and let (C_*, ∂_*) be a chain complex of free R-left modules. The following four statements are equivalent:

(a) The chain complex (C_*, ∂_*) is chain homotopy equivalent to the zero chain complex.
(b) The identity map and the zero map of C_* to itself are chain homotopic.
(c) There exists an automorphism of C_* that is chain homotopic to the zero map.
(d) For each $i \in \mathbb{N}_0$ there exists a homomorphism $\delta_i \colon C_i \to C_{i+1}$ of R-left modules such that for each i the map
$$\partial_{i+1} \circ \delta_i + \delta_{i-1} \circ \partial_i \colon C_i \to C_i$$
is an isomorphism of R-left modules.
(e) The chain complex (C_*, ∂_*) is acyclic in the sense of Definition 143.1, i.e. $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$.

Proof. The (a)⇔(b)-equivalence follows easily from the definitions. The (b)⇒(c)-implication is a tautology. The (c)⇒(d)-implication follows immediately from the definitions. The (d)⇒(c)-implication follows from the definitions and the observations that the maps $\partial_{i+1} \circ \delta_i + \delta_{i-1} \circ \partial_i$ clearly define a chain map. Next, the (c)⇒(e)-implication is a straightforward consequence of the Chain Homotopy–H_*-Lemma 313.6. Finally we move on to the most interesting part of the proof, namely the proof of the (e)⇒(b)-implication. Thus we assume that the chain complex (C_*, ∂_*) is acyclic. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0}\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0}\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0}\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0} & & \\
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0.
\end{array}$$

The hypothesis that $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$ means that C_* is a projective R-left resolution of the trivial group, i.e. the top and bottom are projective R-left resolutions. The zero map on the right now has two extensions:

(1) The identity maps $\mathrm{id} \colon C_i \to C_i$ define an extension.
(2) The zero maps $\mathrm{id} \colon C_i \to C_i$ define an extension.

By Statement (2) of the Homological Algebra–Fundamental Theorem 313.14 these two extensions are chain homotopic. But that is precisely Statement (b). ∎

We move on to the following definition which is essentially identical to Definition 158.7:

Definition 313.17. Let R be a ring and let $f \colon C_* \to D_*$ be a chain map between R-left chain complexes. We say that f is a quasi-isomorphism if for each $k \in \mathbb{N}_0$ the induced map $f_* \colon \mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.

We also have the following generalization of the Quasi Iso–Homotopy Equivalence Proposition 158.8:

Proposition 313.18. (Quasi Iso–Homotopy Equivalence Proposition) Let R be a ring and let $f \colon C_* \to C'_*$ be a chain map between chain complexes of projective R-left

modules. If f induces an isomorphism of all homology groups, then f is in fact a chain homotopy equivalence, i.e. there exists a chain map $f\colon C'_* \to C_*$ of R-left modules that is a chain homotopy inverse to f.

Proof. The proof of the current proposition is almost identical to the rather lengthy proof of the Quasi Iso–Homotopy Equivalence Proposition 158.8 which we gave in Section 158.3. We just need to replace the Homological Algebra–Fundamental Theorem 158.4 by the above Homological Algebra–Fundamental Theorem Rings 313.14 and we need to replace the Null-Homotopic Criterion Proposition 158.6 by the above Null-Homotopic Criterion Proposition 313.16. ■

The following corollary is a generalization of Two-Out-Of-Three-Homotopy Equivalence Corollary 158.9:

Corollary 313.19. (Two-Out-Of-Three-Homotopy Equivalence Corollary) Let R be a ring and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes of projective R-left modules. If two of the vertical maps are chain homotopy equivalences (in fact we only need that they are chain homotopy equivalences as abelian groups), then the third is also a chain homotopy equivalence (in fact as R-left modules).

Proof. It follows from the LES-Proposition 313.10 (1) and (2) that we get a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{\partial_{n+1}} & \mathrm{H}_n(A_*) & \longrightarrow & \mathrm{H}_n(B_*) & \longrightarrow & \mathrm{H}_n(C_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A_*) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \mathrm{H}_{n+1}(C'_*) & \xrightarrow{\partial_{n+1}} & \mathrm{H}_n(A'_*) & \longrightarrow & \mathrm{H}_n(B'_*) & \longrightarrow & \mathrm{H}_n(C'_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A'_*) & \longrightarrow & \ldots \end{array}$$

By our hypothesis, together with the Chain Homotopy Equivalence-H_*-Corollary 313.7, we know that always two out of three vertical maps are isomorphisms. But then it follows from the Five Lemma 145.11 that the third vertical map is also an isomorphism. The Quasi Iso–Homotopy Equivalence Proposition 313.18 implies that the corresponding map of chain complexes is also a chain homotopy equivalence. ■

Remark. It is also useful to know the results that do not generalize. In the proofs of the Chain Homotopy Equivalence Criterion 184.31 and the Lift-to-Chain Map-Proposition 158.21 we used the Free Abelian Subgroup Theorem 97.14 which says that subgroups of free abelian groups are free abelian. But in Exercise 309.15 we saw that in general a submodule of a projective module is not necessarily projective. Furthermore in the proof of the Finite-Rank Chain Complex Lemma 158.22 we used the Abelian Group-Classification Theorem 97.24. Thus the proofs of these results do not generalize in a one-to-one fashion to the setting of chain complexes over rings. □

313.6. **Free imitations of chain complexes.** Recall that in the definition of a chain complex we did not demand that the chain groups are free modules. In this section we will

see that we can replace any chain complex by a "similar looking" chain complex consisting of free modules. To make this precise we need the following definition:

Definition 313.20. Let R be a ring and let C_* be an R-left chain complex. A free imitation of C_* is a chain complex F_* of R-left modules together with a chain map $\varphi\colon F_* \to C_*$ of R-left chain complexes that has the following two properties:

(1) The chain map is a quasi-isomorphism $\varphi\colon F_* \to C_*$ in the sense of Definition 313.17, i.e. it induces an isomorphism on all homology groups.
(2) Each $\varphi_k\colon F_k \to C_k$ is an epimorphism.

We will now see that free imitations always exist and that they are unique up to chain homotopy equivalence. We start out with the existence statement:

Lemma 313.21. (Free Imitation Existence Lemma) Let R be a ring. Every R-left chain complex admits a free imitation.

Remark. For $R = \mathbb{Z}$ the statement of the Free Imitation Existence Lemma 313.21 was the content of Exercise 143.5. □

Proof. Let R be a ring and let (C_*, ∂_*) be an R-left chain complex. We need to show that there exists a chain complex $(F_*, \widetilde{\partial}_*)$ of free R-left modules together with a chain map $\varphi\colon F_* \to C_*$ that is a quasi-isomorphism and that is an epimorphism at each degree. Most of the work is done by formulating the following claim:

Claim. Suppose that we are in the following setting

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & F_k & \xrightarrow{\widetilde{\partial}_k} & F_{k-1} & \xrightarrow{\widetilde{\partial}_{k-1}} & \ldots & \xrightarrow{\widetilde{\partial}_2} & F_1 & \xrightarrow{\widetilde{\partial}_1} & F_0 & \xrightarrow{0} & 0 \\
 & & & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varphi_{k-1}} & & & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \\
\ldots & \longrightarrow & C_{k+1} & \xrightarrow[\partial_{k+1}]{} & C_k & \xrightarrow[\partial_k]{} & C_{k-1} & \xrightarrow[\partial_{k-1}]{} & \ldots & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 & \xrightarrow[0]{} & 0
\end{array}$$

where the following statements hold:

(1) The top sequence of maps is a chain complex of free R-left modules.
(2) The vertical maps define a chain map.
(3) For $i = 0, \ldots, k-1$ the map $\varphi_*\colon \mathrm{H}_i(F_*) \to \mathrm{H}_i(C_*)$ is an isomorphism.
(4) The map $\varphi_*\colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ is an epimorphism.
(5) Each vertical map is an epimorphism.

Then we can extend the above horizontal sequence by one term to the left such that the extensions of the above statements hold.

Proof. Fortunately formulating the claim takes (almost) longer than proving it. We set

$$\begin{aligned}
A &:= \{(f, c) \in \underbrace{\varphi_k^{-1}(\mathrm{im}(\partial_{k+1}\colon C_{k+1} \to C_k)) \cap \ker(\widetilde{\partial}_k)}_{\subseteq F_k} \times C_{k+1} \mid \varphi_k(f) = \partial_{k+1}(c)\} \\
B &:= \ker(\partial_{k+1}\colon C_{k+1} \to C_k).
\end{aligned}$$

Next we set $F_{k+1} := R^{(A)} \oplus R^{(B)}$. We consider the map

$$\widetilde{\partial}_{k+1}\colon F_{k+1} = R^{(A)} \oplus R^{(B)} \to F_k$$

that is induced by sending each $(f, c) \in A$ to $f \in F_k$ and by sending each $b \in B$ to 0. Furthermore we consider the map

$$\varphi_{k+1} \colon F_{k+1} = R^{(A)} \oplus R^{(B)} \to C_{k+1}$$

that is induced by sending each $(f, c) \in A$ to c and by sending each $b \in B$ to $b \in C_{k+1}$. We now claim that F_{k+1} together with the above maps gives the desired extension:

(a) Evidently F_{k+1} is a free R-left module. Note that it follows easily from the definitions that $\widetilde{\partial}_k \circ \widetilde{\partial}_{k+1} = 0$. This implies that we do indeed have a chain complex of free R-left modules.

(b) For $(f, c) \in A$ we have

$$(\partial_{k+1} \circ \varphi_{k+1})(f, c) \underset{\text{by definition of } \varphi_{k+1}}{=} \partial_{k+1}(c) \underset{\text{by definition of } A}{=} \varphi_k(f) \underset{\text{by definition of } \widetilde{\partial}_{k+1}}{=} (\varphi_k \circ \widetilde{\partial}_{k+1})(f, c).$$

Furthermore for $b \in B$ we have

$$(\partial_{k+1} \circ \varphi_{k+1})(b) = \partial_{k+1}(\varphi_{k+1}(b)) \underset{\text{by definition of } \varphi_{k+1}}{=} \partial_{k+1}(b) \overset{\text{since } b \in B = \ker(\partial_{k+1})}{=} 0 \underset{\text{by definition of } \widetilde{\partial}_{k+1}}{=} (\varphi_k \circ \widetilde{\partial}_{k+1})(b).$$

Thus the vertical maps form a chain map.

(c) We denote by F_* the chain complex starting at F_k and we denote by F'_* the chain complex starting at F_{k+1}. Recall that we assume that $\varphi_* \colon \mathrm{H}_k(F_*) \to \mathrm{H}_k(C_*)$ is an epimorphism. It follows almost immediately that $\varphi_* \colon \mathrm{H}_k(F'_*) \to \mathrm{H}_k(C_*)$ is also an epimorphism. It remains to show that $\varphi_* \colon \mathrm{H}_k(F'_*) \to \mathrm{H}_k(C_*)$ is also a monomorphism. Thus let $[\sigma] \in \mathrm{H}_k(F'_*)$ with $\varphi_*([\sigma]) = 0$. This means that $\varphi_k(\sigma) \in \mathrm{im}(\partial_{k+1} \colon C_{k+1} \to C_k)$. Also note that $\sigma \in \ker(\widetilde{\partial}_k)$. This shows that $\sigma \in \varphi_k^{-1}(\mathrm{im}(\partial_{k+1} \colon C_{k+1} \to C_k)) \cap \ker(\widetilde{\partial}_k)$. Evidently we can pick $c \in C_{k+1}$ with $\partial_{k+1}(c) = \varphi_k(\sigma)$. Note that $\widetilde{\partial}_{k+1}(\sigma, c) = \sigma$. In other words, we have shown that $\sigma \in \mathrm{im}(\widetilde{\partial}_{k+1} \colon F_{k+1} \to F_k)$. But this implies that $[\sigma] = 0 \in \mathrm{H}_k(F'_*)$.

(d) By construction the map $\varphi_{k+1} \colon \ker(\widetilde{\partial}_{k+1} \colon F_{k+1} \to F_k) \to \ker(\partial_{k+1} \colon C_{k+1} \to C_k)$ is an epimorphism. But this implies that $\varphi_* \colon \mathrm{H}_{k+1}(F_*) \to \mathrm{H}_{k+1}(C_*)$ is an epimorphism.

(e) Let $c \in C_{k+1}$. Since $\varphi_k \colon F_k \to C_k$ is an epimorphism there exists an $f \in F_k$ with $\varphi_k(f) = \partial_{k+1}(c)$. Note that $(f, c) \in A$ and note that $\varphi_{k+1}(f, c) = c$. Thus we have shown that $\varphi_{k+1} \colon F_{k+1} \to C_{k+1}$ is an epimorphism. ⊞

Now we start the iterative procedure with $k = -1$ by considering the chain complex consisting only of the zero group. By applying the claim iteratively we obtain the desired chain complex and the desired quasi-isomorphism consisting of epimorphisms. ■

Later on we will also require the following technical lemma:

Lemma 313.22. (Homomorphism Lifting Lemma) Let R be a ring, let C_* and D_* be two R-left chain complexes and let $\Theta \colon C \to D$ be a chain map of R-left chain complexes. We assume that we are given another two chain maps:

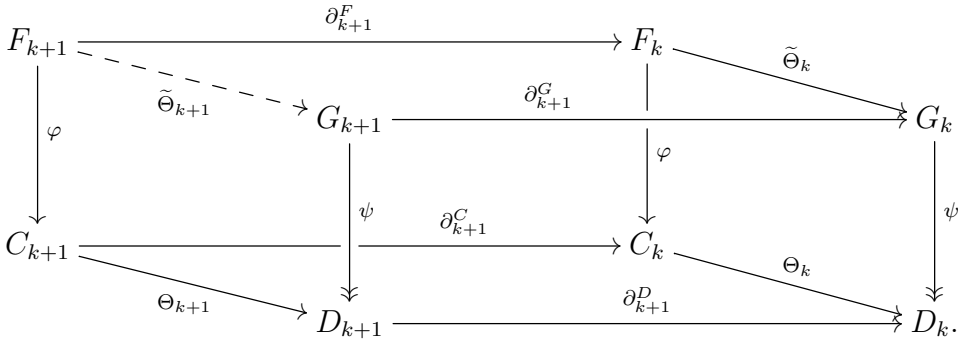
(1) Let $\varphi \colon F_* \to C_*$ be a chain map such that each F_k is a free R-left module.

(2) Let $\psi \colon G_* \to D_*$ be a quasi-isomorphism such that each $\psi \colon G_k \to D_k$ is an epimorphism.

Then there exists a chain map $\widetilde{\Theta}\colon F \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \overset{\widetilde{\Theta}}{\dashrightarrow} & G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\text{quasi-isomorphism } \psi} \\ C & \xrightarrow{\Theta} & D. \end{array}$$

Proof. We construct the chain map $\widetilde{\Theta}$ inductively. We define $\widetilde{\Theta}_{-1}\colon F_{-1} = 0 \to 0 = G_{-1}$ to be, evidently, the zero map. Now suppose we have already constructed maps $\widetilde{\Theta}_i\colon F_i \to G_i$, $i = 0, \ldots, k$ which form a (partial) chain map and such that the corresponding diagram commutes. Now let us consider the following diagram:



By hypothesis (1) we know that F_{k+1} is a free R-left module. We pick a basis B for F_{k+1}. It suffices to define $\widetilde{\Theta}_{k+1}(b) \in G_{k+1}$ for each $b \in B$ in such that a way that the above diagram commutes for b.
Thus let $b \in B$. We proceed as follows:

(a) By hypothesis (2) we know that $\psi\colon G_{k+1} \to D_{k+1}$ is an epimorphism. Therefore there exists a $g_{k+1} \in G_{k+1}$ with $\psi(g_{k+1}) = (\Theta_{k+1} \circ \varphi)(b)$.

(b) Note that the four sides of the hyperrectangle that do not involve the unknown dotted arrow commute. This implies that

$$\begin{aligned} (\psi \circ \widetilde{\Theta}_k \circ \partial^F_{k+1})(b) &= (\Theta_k \circ \varphi \circ \partial^F_{k+1})(b) &&= (\Theta_k \circ \partial^C_{k+1} \circ \varphi) \\ &= (\partial^D_{k+1} \circ \Theta_{k+1} \circ \varphi)(b) &&\underset{\text{by definition of } g_{k+1}}{=} (\partial^D_{k+1} \circ \psi)(g_{k+1}) \quad = \quad (\psi \circ \partial^G_{k+1})(g_{k+1}). \end{aligned}$$

(c) It follows in particular from (b) that $\psi((\widetilde{\Theta}_k \circ \partial^F_{k+1})(b))$ and $\psi(\partial^G_{k+1}(g_{k+1}))$ are homologous in D_*.

(d) By hypothesis (2) we know that $\mathrm{H}_k(G_*) \to \mathrm{H}_k(D_*)$ is a monomorphism. This observation together with (c) implies that $(\widetilde{\Theta}_k \circ \partial^F_{k+1})(b)$ and $\partial^G_{k+1}(g_{k+1})$ are homologous in G_*.

(e) By (d) there exists an $r_{k+1} \in G_{k+1}$ with $\partial^G_{k+1}(r_{k+1}) = \partial^G_{k+1}(g_{k+1}) - (\widetilde{\Theta}_k \circ \partial^F_{k+1})(b)$.

(f) Note that

$$\partial^D_k(\psi(r_{k+1})) \underset{\substack{\text{since } \psi \text{ is} \\ \text{a chain map}}}{=} \psi(\partial^G_k(r_{k+1})) \underset{\text{by (e)}}{=} \psi\big(\partial^G_{k+1}(g_{k+1}) - (\widetilde{\Theta}_k \circ \partial^F_{k+1})(b)\big) \underset{\text{by (b)}}{=} 0.$$

In other words, $\psi(r_{k+1}) \in D_{k+1}$ is a cycle in the chain complex D_*.

(g) By hypothesis (2) we know that $\mathrm{H}_{k+1}(G_*) \to \mathrm{H}_{k+1}(D_*)$ is an epimorphism. Note that it follows from this observation and (f) that there exists a *cycle* $s_{k+1} \in G_{k+1}$ with $\psi(s_{k+1}) = \psi(r_{k+1})$.

Claim. If we set $\widetilde{\Theta}_{k+1}(b) := g_{k+1} - r_{k+1} + s_{k+1}$, then the diagram commutes for b.

Proof.

- We first consider the upper parallelogram. We have the following equality in G_k:

$$(\partial^G_{k+1} \circ \widetilde{\Theta}_{k+1})(b) \underset{\substack{\uparrow \\ \text{by definition of } \widetilde{\Theta}_{k+1}(b)}}{=} \partial^G_{k+1}(g_{k+1} - r_{k+1} + s_{k+1}) \underset{\substack{\uparrow \\ \text{since } s_{k+1} \text{ is a cycle}}}{=} \partial^G_{k+1}(g_{k+1} - r_{k+1}) \underset{\substack{\uparrow \\ \text{by (e)}}}{=} \widetilde{\Theta}_k(\partial^F_{k+1}(b)).$$

- We consider the parallelogram to the left. We have the following equality in D_{k+1}:

$$(\psi \circ \widetilde{\Theta}_{k+1})(b) \underset{\substack{\uparrow \\ \text{by definition of } \widetilde{\Theta}_{k+1}(b)}}{=} \psi(g_{k+1} - r_{k+1} + s_{k+1}) \underset{\substack{\uparrow \\ \text{since } \psi(r_{k+1}) = \psi(s_{k+1})}}{=} \psi(g_{k+1}) \underset{\substack{\uparrow \\ \text{by (a)}}}{=} (\Theta_{k+1} \circ \varphi)(b).$$

■

We conclude this chapter with the following lemma, which we will not make use of:

Lemma 313.23. (Free Imitation Uniqueness Lemma) Let R be a ring and let (C_*, ∂_*) be an R-left chain complex. If $\varphi\colon F_* \to C_*$ and $\varphi'\colon F'_* \to C_*$ are two free imitations, then there exists a chain homotopy equivalence $\Theta\colon F_* \to F'_*$ of R-left chain complexes such that $\varphi' \circ \Theta = \varphi$.

Proof. By the Homomorphism Lifting Lemma 313.22 there exists a chain map $\widetilde{\Theta}\colon F \to P'$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \overset{\widetilde{\Theta}}{- - \to} & P' \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\text{quasi-isomorphism } \varphi'} \\ C & \xrightarrow{\text{id}} & C. \end{array}$$

Note that Θ is a quasi-isomorphism. Since F and P' are chain complexes of free R-left modules we obtain from the Quasi Iso–Homotopy Equivalence Proposition 313.18 that $\widetilde{\Theta}\colon F_* \to F'_*$ is in fact a chain homotopy equivalence. ■

313.7. Exercises for Chapter 313.

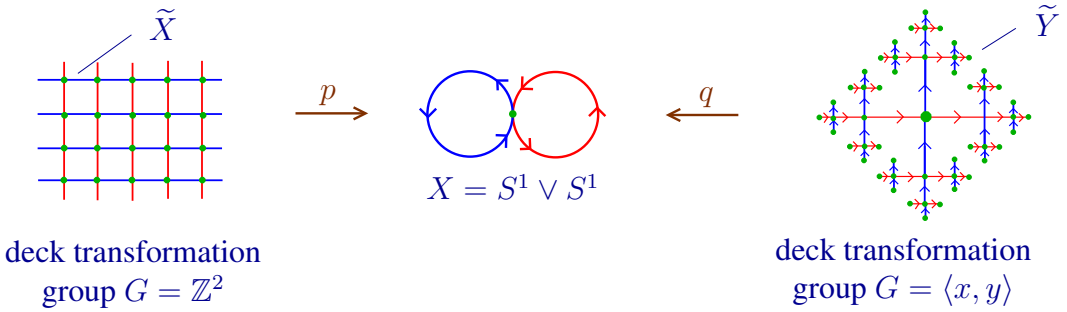
Exercise 313.1.

(a) In the figure below we see to the left a covering $p\colon \widetilde{X} \to X = S^1 \vee S^1$ with deck transformation group $G = \mathbb{Z}^2$. Describe the cellular chain complex $\mathrm{C}^{\mathrm{CW}}_*(\widetilde{X})$ as a $\mathbb{Z}[\mathbb{Z}^2]$-left module and compute the homology modules.

(b) In the figure below we see to the right a covering $p\colon \widetilde{Y} \to X = S^1 \vee S^1$ with deck transformation group $G =_{\mathrm{i}} \langle x, y \rangle$. Describe the cellular chain complex $\mathrm{C}^{\mathrm{CW}}_*(\widetilde{Y})$ as a $\mathbb{Z}[\langle x, y \rangle]$-left module and compute the homology modules.

Exercise 313.2. Let R and S be rings, let (C_*, ∂_*) be a chain complex of free R-left modules and let $\varphi\colon R \to S$ be a ring homomorphism. Show that if C_* is acyclic, then the chain complex $S \otimes_R C_*$ of S-left modules is also acyclic.

Hint. Use the Null-Homotopic Criterion Proposition 313.16.



Exercise 313.3. Let R be a ring and let $f\colon C_* \to D_*$ be a chain map between R-left chain complexes. Show that the boundary map of C_* define a chain complex structure on the R-left modules $\ker(C_k \to D_k)$ and show that the boundary map of D_* descends to a chain complex structure on the R-left modules $\operatorname{coker}(C_k \to D_k)$.

Exercise 313.4. We consider the group ring $R = \mathbb{Z}[\langle x \,|\, x^2\rangle]$ of the group $\langle x \,|\, x^2\rangle \cong \mathbb{Z}/2\mathbb{Z}$. We view $\widetilde{\mathbb{Z}}$ as module over $R = \mathbb{Z}[\langle x \,|\, x^2\rangle]$ where $x \in \langle x \,|\, x^2\rangle$ acts by multiplication by -1. We consider the chain complexes

$$\begin{array}{lcccccccc} C_* & = & 0 & \to & \mathbb{Z} & \xrightarrow{1 \mapsto (1+x)} & \mathbb{Z}[\langle x \,|\, x^2\rangle] & \to & 0 \\ D_* & = & & & 0 & \to & \widetilde{\mathbb{Z}} & \to & 0 \end{array}$$

of $R = \mathbb{Z}[\langle x \,|\, x^2\rangle]$-modules.

(a) Show that there exists a chain map $\varphi\colon C_* \to D_*$ which is a quasi-isomorphism of $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-modules

(b) Show that the chain complexes C_* and D_* are not chain homotopy equivalent over the ring $\mathbb{Z}[\langle x \,|\, x^2\rangle]$.

Remark. This exercise shows (similar to Exercise 313.4) that in the Quasi Iso–Homotopy Equivalence Proposition 313.18 we cannot drop the hypothesis that we are dealing with chain complexes of projective modules.

CHAPTER 314

Tensoring chain complexes and Tor-modules

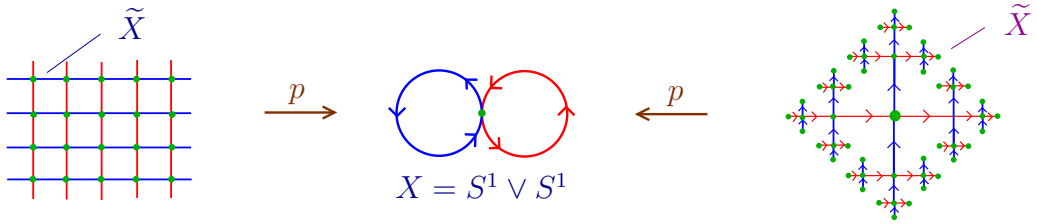
In the previous Chapter 313 we introduced R-left chain complex over rings. In this chapter we will study the effect of tensoring an R-left chain complex with an R-right module. This study will lead us to the definition of the Tor-modules. Finally, if the ring is a PID, then we can prove a universal coefficient theorem. The content of this chapter generalizes aspects that we dealt with in Chapter 160, Chapter 161 and Chapter 162.

This chapter is organized as follows:

- In Section 314.1 we will motivate why one might be interested in tensoring a chain complex with a module.
- In Section 314.2 we will (re-) introduce additive functors and we will apply them to R-left chain complexes.
- In Section 314.3 we will discuss what it means to take the tensor product of an R-left chain complex of modules with an R-right module.
- In Section 314.4 we will introduce exact functors.
- In Section 314.5 we will introduce Tor-modules and we will discuss several basic properties of torsion modules.
- In Section 314.6 we will perform several explicit calculations of torsion modules.
- In Section 314.7 we will state and prove the Universal Coefficient Theorem for PIDs.
- In Section 314.8 we will discuss splittings of the Universal Coefficient Theorem for PIDs.
- In Section 314.9 we will provide exercises for Chapter 314.

In the next Chapter 315 we will study the effect of taking the dual of an R-left chain complex.

314.1. **Motivation.** We return to the setting of Section 313.1. Thus let $p\colon \widetilde{X} \to X$ be a covering of a connected CW-complex with deck transformation group G. In Section 314.1 we saw that it is natural to view the cellular chain complex $\mathrm{C}_*^{\mathrm{tw}}(\widetilde{X})$ as a chain complex of $\mathbb{Z}[G]$-left modules.



The issue with this idea is that one ends up with modules over potentially nasty rings. Let us consider the two examples shown in the figure above:

- In the example to the left we consider the group ring $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. This ring is still quite reasonable.
- In the example to the right we consider the non-commutative ring $\mathbb{Z}[\langle x, y\rangle]$. This ring is an algebraic nightmare.

Thus it would be nice if one could move from chain complexes over a group ring $R = \mathbb{Z}[G]$ to a chain complex over a (commutative) ring Q, e.g. $Q = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ while keeping some of the information of the $\mathbb{Z}[G]$-left chain complex.

The key idea is to tensor a $\mathbb{Z}[G]$-left chain complex with a $(Q, \mathbb{Z}[G])$-bimodule to obtain a Q-left chain complex. This motivates the topic of this chapter: We will study what it means to tensor an R-left chain complex with a (Q, R)-bimodule and we will try to relate the homology modules of the original chain complex to the homology modules of the tensored chain complex.

314.2. **Additive functors applied to R-left chain complex.** The following definition is a generalization of Definition 160.1:

Definition 314.1. Let Q and R be rings. In the following we say that a covariant functor $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category of Q-left modules is additive if for any two R-left modules A and B the map

$$\begin{array}{rcl} \mathrm{Mor}_{R\text{-left}}(A, B) & \to & \mathrm{Mor}_{Q\text{-left}}(F(A), F(B)) \\ (\varphi\colon A \to B) & \mapsto & (F(\varphi)\colon F(A) \to F(B)) \end{array}$$

is a homomorphism of abelian groups.

The following lemma is a slight generalization of the Additive Functor–Basics Lemma 160.2:

Lemma 314.2. (Additive Functor–Basics Lemma) Let Q and R be rings. Furthermore let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be a covariant functor from the category $\mathcal{M}od_{R\text{-left}}$ to the category $\mathcal{M}od_{Q\text{-left}}$. If F is additive, then the following two statement holds:

(1) $F(\text{zero homomorphism}) = \text{zero homomorphism}$.
(2) $F(\text{zero module}) = \text{zero module}$.

Proof. Let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be an additive covariant functor.

(1) Let A, B be two R-left modules. Since F is an additive covariant functor we see that $F(0\colon A \to B) = F(0 + 0) = F(0) + F(0)\colon F(A) \to F(B)$. It follows that $F(0)$ is the zero homomorphism.
(2) Let A be a trivial R-left module. Then $\mathrm{id}_A = 0$. It follows from functoriality and (1) that $\mathrm{id}_{F(A)} = F(\mathrm{id}_A) = F(0) = 0$. This implies that $F(A)$ is the trivial Q-left module. ∎

The following lemma is a mild generalization of the Functor-applied-to-Chain Complex Lemma 160.3:

Lemma 314.3. (Functor-applied-to-Chain Complex Lemma) Let Q and R be rings. Let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be a covariant functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{Q\text{-left}}$ of Q-left modules.

(1) (a) If (C_*, ∂_*) is an R-left chain complex, then

$$\ldots \to F(C_2) \xrightarrow{F(\partial_2)} F(C_1) \xrightarrow{F(\partial_1)} F(C_0) \xrightarrow{F(\partial_0)} 0$$

is a chain complex of Q-left modules.

(b) If $\varphi_*\colon C_* \to D_*$ is a chain map, then $F(\varphi)_*\colon F(C)_* \to F(D)_*$ is also a chain map.

(2) The maps

$$\begin{aligned} (C_*, \partial_*) &\mapsto (F(C_*), F(\partial_*)) \\ (\varphi_*\colon C_* \to D_*) &\mapsto (F(\varphi)_*\colon F(C)_* \to F(D)_*) \end{aligned}$$

define a covariant functor from the category $\mathcal{C}h\mathcal{C}pl\mathcal{X}_{R\text{-left}}$ of R-left chain complexes to the category $\mathcal{C}h\mathcal{C}pl\mathcal{X}_{Q\text{-left}}$ of Q-left chain complexes.

(3) If $\varphi, \psi\colon C_* \to D_*$ are chain homotopic, then $F(\varphi), F(\psi)\colon F(C_*) \to F(D_*)$ are also chain homotopic.

Proof. Let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be an additive covariant functor.

(1) (a) Let (C_*, ∂_*) be an R-left chain complex. Given any $n \in \mathbb{N}_0$ we see that

$$\begin{array}{ccccc} F(\partial_{n-1}) \circ F(\partial_n) & = & F(\partial_{n-1} \circ \partial_n) & = & F(0) \quad = \quad 0. \\ & \uparrow & & \uparrow & \uparrow \\ & \text{since } F \text{ is a} & & \text{since } (C_*, \partial_*) \text{ is} & \text{by the Additive Functor} \\ & \text{covariant functor} & & \text{a chain complex} & \text{Basics Lemma 314.2 (1)} \end{array}$$

(b) Let $\varphi\colon (C_*, \partial_*^C) \to (D_*, \partial_*^D)$ be a chain map. Given any $n \in \mathbb{N}_0$ we have

$$\begin{array}{ccccc} F(\varphi_{n-1}) \circ F(\partial_n^C) & = & F(\varphi_{n-1} \circ \partial_n^C) & = & F(\partial_n^D \circ \varphi_n) \quad = \quad F(\partial_n^D) \circ F(\varphi_n). \\ & \uparrow & & \uparrow & \uparrow \\ & \text{since } F \text{ is a} & & \text{since } \varphi_* \text{ is a} & \text{since } F \text{ is at} \\ & \text{covariant functor} & & \text{chain map} & \text{covariant functor} \end{array}$$

(2) This statement follows easily from the definitions.

(3) Let $\varphi, \psi\colon C_* \to D_*$ be two chain maps between two R-left chain complexes (C_*, ∂_*^C) and (D_*, ∂_*^D). Furthermore let $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy between the chain maps φ and ψ. Given any $n \in \mathbb{N}_0$ we see that

$$\begin{array}{ccccc} & & & & \text{since } F \text{ is an additive functor} \\ & & & & \downarrow \\ F(\partial_{n+1}^D) \circ F(P_n) + F(P_{n-1}) \circ F(\partial_n^C) & = & F(\partial_{n+1}^D \circ P_n + P_{n-1} \circ \partial_n^C) & & \\ & = & F(\varphi_n - \psi_n) & = & F(\varphi_n) - F(\psi_n). \\ & \uparrow & & \uparrow & \\ & \text{since the } P_n \text{ form a chain} & & \text{since } F \text{ is} & \\ & \text{homotopy between } \varphi \text{ and } \psi & & \text{additive} & \end{array}$$

We have thus shown that the homomorphisms $F(P_n)\colon F(C_n) \to F(D_{n+1})$ form a chain homotopy between the chain maps $F(\varphi)$ and $F(\psi)$. ■

The following lemma is a generalization of the Tensor–Functor Lemma 160.4 and it is a slight refinement of the Tensor Product–Functorial Lemma 312.17:

Lemma 314.4. (Tensor–Functor Lemma) Let Q and R be rings and let M be a (Q, R)-bimodule. The maps

$$\begin{aligned} A &\mapsto M \otimes_R A \\ (f\colon A \to B) &\mapsto (\mathrm{id}_M \otimes f\colon M \otimes_R A \to M \otimes_R B) \end{aligned}$$

define an *additive* covariant functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{Q\text{-left}}$ of Q-left modules.

Proof.

(1) We verify the two properties of a covariant functor from Definition 5.18:
   (a) Let A be an R-left module. By the Tensor–Map Composition Lemma 312.7 (1) we have $\mathrm{id}_M \otimes \mathrm{id}_R = \mathrm{id}_{M\otimes_R A}$.
   (b) Let $f\colon A \to B$ and $g\colon B \to C$ be two homomorphisms between two R-left modules. It follows immediately from the Tensor–Map Composition Lemma 312.7 (2) that we have $(\mathrm{id}_M \otimes g) \circ (\mathrm{id}_M \otimes f) = \mathrm{id}_M \otimes (g \circ f)$. ∎

(2) It follows from the Tensor–Map Composition Lemma 312.7 (3) that the functor is additive.

314.3. **Tensoring R-left chain complex.** In this section we will apply the results from the previous section to tensor an R-left chain complex with a fixed (Q, RR)-bimodule. We start out with the following variation on Definition 160.5:

Definition 314.5. Let Q and R be rings. Let

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be an R-left chain complex in the sense of Definition 313.1. Let M be a (Q, R)-bimodule. It follows from the Tensor–Functor Lemma 314.4 together with the Functor-applied-to-Chain Complex Lemma 314.3 (1a) that the following sequence of homomorphisms

$$\ldots \to M \underset{R}{\otimes} C_n \xrightarrow{\mathrm{id}_M \otimes \partial_n} M \underset{R}{\otimes} C_{n-1} \xrightarrow{\mathrm{id}_M \otimes \partial_{n-1}} \ldots \xrightarrow{\mathrm{id}_M \otimes \partial_2} M \underset{R}{\otimes} C_1 \xrightarrow{\mathrm{id}_M \otimes \partial_1} M \underset{R}{\otimes} C_0 \to 0.$$

is a chain complex of Q-left modules. We refer to $\mathrm{H}_n(M \otimes_R C_*) = \mathrm{H}_n(M \otimes_R C_*, \mathrm{id}_M \otimes \partial_*)$ as the homology groups of C_* with M-coefficients.

The following lemma is a generalization of the Homology-with-Coefficients Lemma 160.6:

Lemma 314.6. (Homology-with-Coefficients Lemma) Let Q and R be rings. Let M be a (Q, R)-bimodule.

(1) (a) If $f\colon C_* \to D_*$ is a chain map between two R-left chain complexes C_* and D_*, then the maps $\mathrm{id}_M \otimes f\colon M \otimes_R C_* \to M \otimes_R D_*$ also form a chain map. Thus for each $n \in \mathbb{N}_0$ we get an induced homomorphism

$$(\mathrm{id}_M \otimes f)_*\colon \mathrm{H}_n(M \otimes_R C_*) \to \mathrm{H}_n(M \otimes_R D_*)$$

of R-left modules.

   (b) Given $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} C_* &\mapsto \mathrm{H}_n(M \otimes_R C_*) \\ (f\colon C_* \to D_*) &\mapsto (\underbrace{(\mathrm{id}_M \otimes f)_*}_{=:f_*}\colon \mathrm{H}_n(M \otimes_R C_*) \to \mathrm{H}_n(M \otimes_R D_*)) \end{aligned}$$

define a covariant functor from the category $\mathcal{C}h\mathcal{C}plx_{R\text{-left}}$ of R-left chain complexes to the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules.

(2) (a) Let $f, g\colon C_* \to D_*$ be chain maps between two R-left chain complexes (C_*, ∂_*) and (D_*, ∂_*). If $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ is a chain homotopy between f and g, then

the maps $\{\mathrm{id}_M \otimes P_n \colon M \otimes_R C_n \to M \otimes_R D_{n+1}\}_{n\in\mathbb{N}_0}$ form a chain homotopy between $\mathrm{id}_M \otimes f$ and $\mathrm{id}_M \otimes g$. It follows from the Chain Homotopy–H_*-Lemma 313.6 that we have

$$(\mathrm{id}_M \otimes f)_* = (\mathrm{id}_M \otimes g)_* \colon \mathrm{H}_n(M \otimes_R C_*) \to \mathrm{H}_n(M \otimes_R D_*).$$

(b) Let C_* and D_* be R-left chain complexes. If $f \colon C_* \to D_*$ is a chain homotopy equivalence, then $\mathrm{id}_M \otimes f \colon M \otimes_R C_* \to M \otimes_R D_*$ is also a chain homotopy equivalence. It follows immediately from the Chain Homotopy Equivalence-H_*-Corollary 313.7, that the induced map $(\mathrm{id}_M \otimes f)_* \colon \mathrm{H}_n(M \otimes_R C_*) \to \mathrm{H}_n(M \otimes_R D_*)$ is an isomorphism of Q-left chain complexes.

Proof. All statements follow almost immediately from the Tensor–Functor Lemma 314.4 together with the Functor-applied-to-Chain Complex Lemma 314.3. ■

The following proposition is a generalization of the LES-with-Coefficients Proposition 160.7:

Proposition 314.7. (LES-with-Coefficients Proposition) Let Q and R be rings. Let M be a (Q,R)-bimodule and let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of R-left chain complexes such that all chain modules C_* are projective R-left modules. We get natural connecting homomorphisms $\partial_k \colon \mathrm{H}_k(M \otimes_R C_*) \to \mathrm{H}_{k-1}(M \otimes_R A_*)$ of Q-left module homomorphisms and a long exact sequence

$$\to \mathrm{H}_k(M \otimes_R A_*) \xrightarrow{(\mathrm{id}_M \otimes i)_*} \mathrm{H}_k(B_* \otimes_R M) \xrightarrow{(\mathrm{id}_M \otimes p)_*} \mathrm{H}_k(M \otimes_R C_*) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(M \otimes_R A_*) \to$$

in homology with M-coefficients. Furthermore, if M is a commutative ring, then the connecting homomorphisms are in fact M-module homomorphisms.

The proof of the proposition requires some preparations. First let us state the following generalization of Definition 145.4:

Definition 314.8. Let R be a ring. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R-left modules splits, if there exists a splitting $s \colon C \to B$, i.e. if there exists a homomorphism $s \colon C \to B$ of R-left modules such that $p \circ s = \mathrm{id}_C$.

The following lemma is an analog of the Tensor-Split-SES Lemma 160.9:

Lemma 314.9. (Tensor-Split-SES Lemma) Let R be a ring, let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-left modules and let M be a (Q,R)-bimodule. If the short exact sequence splits (by the Projective Module–Splitting Lemma 309.27 this is the case if C is a projective Q-left module), then

$$0 \to M \otimes_R A \xrightarrow{\mathrm{id}_M \otimes i} M \otimes_R B \xrightarrow{\mathrm{id}_M \otimes p} M \otimes_R C \to 0$$

is a short exact sequence of Q-left modules and it also splits.

Proof of the Tensor-Split-SES Lemma 314.9. We start out with the following claim:

Claim. Let T be a ring and let $0 \to F \xrightarrow{\alpha} M \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between T-left modules. The following two statements are equivalent:

(1) The sequence is exact and the sequence splits.
(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma\colon M \to F$ and $\tau\colon H \to M$ of T-left modules such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_M$.

Proof. The (1)⇒(2) Statement follows almost immediately from the Module Splitting Lemma 309.31. Conversely the (2)⇒(1)-Statement can easily be verified by hand. ⊞

Let

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

be a short exact sequence of R-left modules that splits and let M be a (Q, R)-bimodule. We pick homomorphisms $\sigma\colon B \to A$ and $\tau\colon C \to B$ as in the claim. Note that it follows from the Tensor–Map Composition Lemma 312.7 (2) and (3) that we have the equalities

$$(\mathrm{id}_M \otimes p) \circ (\mathrm{id}_M \otimes \iota) = \mathrm{id}_M \otimes (p \circ \iota) = \mathrm{id}_M \otimes 0 = 0$$

and that the homomorphisms

$$\mathrm{id}_M \otimes \sigma\colon M \otimes_R B \to M \otimes_R A \quad \text{and} \quad \mathrm{id}_M \otimes \tau\colon M \otimes_R C \to M \otimes_R B$$

have the desired properties for the sequence

$$0 \;\to\; M \otimes_R A \;\xrightarrow{\mathrm{id}_M \otimes i}\; M \otimes_R B \;\xrightarrow{\mathrm{id}_M \otimes p}\; M \otimes_R C \;\to\; 0.$$

It follows from the claim that this sequence is exact and that it splits. ■

Proof of the LES-with-Coefficients Proposition 314.7. Let M be a (Q, R)-bimodule and let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of R-left chain complexes such that all chain modules C_* are projective R-left modules. Since each chain module C_* is a projective R-left module we obtain from the Projective Module–Splitting Lemma 309.27 that each of the above short exact sequences splits. It follows from this observation together with the Tensor-Split-SES Lemma 314.9 that

$$0 \to M \otimes_R A_* \to M \otimes_R B_* \to M \otimes_R C_* \to 0$$

is a short exact sequence of chain complexes of Q-left modules. Thus we see that the Connecting Homomorphism Lemma 313.9 and the LES Proposition 313.10 apply in the present context. ■

We conclude this section with the following analog of the Homology-Coefficients-Functor Lemma 160.11:

Lemma 314.10. (Homology-Coefficients-Functor Lemma) Let C_* be an R-left chain complex and let $\varphi\colon M \to N$ be a homomorphism of (Q, R)-bimodules. The maps

$$\varphi \otimes \mathrm{id}_{C_*}\colon M \otimes_R C_* \;\to\; N \otimes_R C_*$$

form a natural chain map. In particular for each $n \in \mathbb{N}_0$ we obtain a natural induced homomorphism

$$(\varphi \otimes \mathrm{id}_{C_*})_*\colon \mathrm{H}_n(M \otimes_R C_*) \;\to\; \mathrm{H}_n(N \otimes_R C_*)$$

of Q-left modules.

Proof. Given $k \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccc} M \otimes_R C_k & \xrightarrow{\varphi \otimes \mathrm{id}_{C_k}} & N \otimes_R C_k \\ {\scriptstyle \mathrm{id}_M \otimes \partial_k}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_N \otimes \partial_k} \\ M \otimes_R C_{k-1} & \xrightarrow{\varphi \otimes \mathrm{id}_{C_{k-1}}} & N \otimes_R C_{k-1}. \end{array}$$

It follows immediately from the Tensor–Map Composition Lemma 312.7 (2) that the diagram commutes. This means precisely that the horizontal maps are chain maps. The proof that the horizontal maps are natural follows by the same type of argument. ■

314.4. **Exact functors.** Next we state a slight variation on Definition 161.2:

Definition 314.11. Let Q and R be rings. Let $F \colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be an additive covariant functor.

(1) We say that F is left-exact, if for every exact sequence[3470]

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

of R-left modules the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C)$$

is also exact.

(2) We say that F is right-exact, if for every exact sequence

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of R-left modules the sequence

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0$$

is also exact.

(3) We say that F is exact if it is left-exact and right-exact.

The following lemma is a generalization of the Exact Functor Lemma 161.3:

Lemma 314.12. (Exact Functor Lemma) Let Q and R be two rings. Furthermore let $F \colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be an additive covariant functor. The following two statements are equivalent:

(1) F is exact.

(2) For every short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of R-left modules the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0$$

is also exact.

[3470] Note that we know by the Additive Functor–Basics Lemma 314.2 (2) that $F(0) \cong 0$.

Proof. The (1)⇒(2)-direction is clear. Now suppose that F satisfies (2). We first show that F is left-exact. Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ be an exact sequence of R-left modules. We consider the following little diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & \operatorname{im}(\psi) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & & \\
 & & & & & & \downarrow & & \\
 & & & & & & C/\operatorname{im}(\psi) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0. & &
\end{array}$$

We apply the functor F to this diagram. Since we assume that F satisfies condition (2) we see that the resulting top horizontal sequence and the resulting right vertical sequence are exact. It is not hard to see that this implies that the bottom horizontal sequence is also exact. The same type of argument also shows that F is right-exact. ■

The following lemma is a generalization of the Exact Functor–Homology Lemma 161.4.

Lemma 314.13. (Exact Functor–Homology Lemma) Let Q and R be rings and let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be an additive covariant functor and let (C_*, ∂_*) be an R-left chain complex. If F is exact, then for any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$F(\mathrm{H}_k(C_*, \partial_*)) \;\to\; \mathrm{H}_k(F(C_*), F(\partial_*))$$

of Q-left modules.

Proof. Let Q and R be two rings. Furthermore let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{Q\text{-left}}$ be an *exact* additive covariant functor. Furthermore let (C_*, ∂_*) be an R-left chain complex. Finally let $k \in \mathbb{N}_0$. In the following we denote by $\iota\colon \ker(\partial_k\colon C_k \to C_{k-1}) \to C_k$ the inclusion. Since $\operatorname{im}(\partial_{k+1}\colon C_{k+1} \to C_k) \subseteq \ker(\partial_k\colon C_k \to C_{k-1})$ we see that there exists a homomorphism $\widetilde{\partial}_{k+1}\colon C_{k+1} \to \ker(\partial_k\colon C_k \to C_{k-1})$ with $\partial_{k+1} = \iota \circ \widetilde{\partial}_{k+1}\colon C_{k+1} \to C_k$.

Now we consider the following diagram:

$$\begin{array}{ccccccc}
 & & C_{k+1} & & & & \\
 & & \downarrow{\scriptstyle \widetilde{\partial}_{k+1}} & \searrow^{\partial_{k+1}} & & & \\
0 & \longrightarrow & \ker(\partial_k\colon C_k \to C_{k-1}) & \xrightarrow[\iota]{} & C_k & \xrightarrow[\partial_k]{} & C_{k-1} \\
 & & \downarrow{\scriptstyle \pi} & & & & \\
 & & \mathrm{H}_k(C_*) & & & & \\
 & & \downarrow & & & & \\
 & & 0. & & & &
\end{array}$$

Note that the vertical and the horizontal sequence are exact. Since F is an additive covariant functor we obtain from the Additive Functor–Basics Lemma 314.2 the following

commutative diagram:

$$\begin{array}{ccccccccc} & & & & F(C_{k+1}) & & & & \\ & & & & \downarrow{\scriptstyle F(\widetilde{\partial}_{k+1})} & \searrow^{F(\partial_{k+1})} & & & \\ 0 & \longrightarrow & F(\ker(\partial_k\colon C_k \to C_{k-1})) & \xrightarrow[F(\iota)]{} & F(C_k) & \xrightarrow[F(\partial_k)]{} & F(C_{k-1}) & & \\ & & \downarrow{\scriptstyle F(\pi)} & & & & & & \\ & & F(\mathrm{H}_k(C_*)) & & & & & & \\ & & \downarrow & & & & & & \\ & & 0. & & & & & & \end{array}$$

Since F is exact we see that the vertical and the horizontal sequence are still exact. It follows from the exactness that the following maps are isomorphisms:

$$F(\mathrm{H}_k(C_*)) \xleftarrow{F(\pi)} \frac{F\Big(\ker(C_k \xrightarrow{\partial_k} C_{k-1})\Big)}{\mathrm{im}(F(\widetilde{\partial}_{k+1}))} \xrightarrow{F(\iota)} \frac{\ker\Big(F(C_k) \xrightarrow{F(\partial_k)} F(C_{k-1})\Big)}{\mathrm{im}\Big(F(C_{k+1}) \xrightarrow{F(\partial_{k+1})} F(C_k)\Big)} = \mathrm{H}_k(F(C_*)).$$

Since all the isomorphisms involved are natural we see also that the isomorphism from left to right is natural. ■

Example. The question arises, whether tensoring with a module is exact. But the experienced reader knows of course that this is in general just not the case. For example on page 3093 we already saw that for $Q = R = \mathbb{Z}$ taking the tensor product with any $M = \mathbb{Z}_k$ is *not* left exact. □

The following lemma is a generalization of the Tensor–Right-Exact Lemma 161.5.

Lemma 314.14. (Tensor–Right-Exact Lemma) Let Q and R be rings. For every (Q, R)-bimodule M the functor

$$\begin{array}{ccc} \mathcal{M}od_{R\text{-left}} & \to & \mathcal{M}od_{Q\text{-left}} \\ A & \mapsto & M \otimes_R A \end{array}$$

is right-exact.

Proof. As on page 3094 one can prove this lemma by hand. But it is more clever to prove the lemma using formal properties of the tensor product. Let Q and R be rings and let M be a (Q, R)-bimodule. By the Tensor Product–Colimit Proposition 312.14 we know that the functor

$$\begin{array}{ccc} \mathcal{M}od_{R\text{-left}} & \to & \mathcal{M}od_{Q\text{-left}} \\ A & \mapsto & M \otimes_R A \end{array}$$

preserves all colimits. Next recall that in Definition 6.23 we defined pushouts as colimits. Thus the desired statement follows from the following claim:

Claim. Let S be a ring. Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be homomorphisms of S-left modules groups. Then the following two statements are equivalent:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \text{ is exact} \iff \begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ 0 & \longrightarrow & C. \end{array} \text{ is a pushout diagram}$$

Proof. Similar to Exercise 145.5 we see that the claim follows from elementary algebra, which we are happy to leave to the reader. ∎

Using the Tensor–Right-Exact Lemma 314.14 we can now prove the following generalization of the Tensor-with-$\mathbb{Z}/k\mathbb{Z}$–Lemma 159.10:

Lemma 314.15. (Tensor-with-Cyclic–Lemma) Let R be a commutative ring, let $f \in R$ and let A be an R-left module. The maps

$$\begin{array}{rcl} \Theta\colon (R/f\cdot R)\otimes_R A & \to & A/f\cdot A \\ \sum_{i=1}^m [r_i]\otimes a_i & \mapsto & \left[\sum_{i=1}^m r_i\cdot a_i\right] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Omega\colon A/f\cdot A & \to & R/f\cdot R\otimes_R A \\ [a] & \mapsto & [1]\otimes a \end{array}$$

are well-defined, they are inverses of one another and they are natural isomorphisms.

Proof. One can easily verify that the two maps are well-defined natural homomorphisms. It is clear that $\Theta\circ\Omega = \mathrm{id}_{A/f\cdot A}$. So let us concentrate on the most interesting aspect of the proof, namely let us prove that $\Omega\colon A/f\cdot A \to (R/f\cdot R)\otimes_R A$ is an isomorphism. We consider the exact sequence

$$R \xrightarrow{a\mapsto f\cdot a} R \xrightarrow{r\mapsto [r]} R/f\cdot R \to 0.$$

Tensoring this exact sequence with A gives use the following diagram:

$$\begin{array}{ccccccc} R\otimes_R A & \xrightarrow{(a\mapsto a\cdot f)\otimes \mathrm{id}_A} & R\otimes_R A & \xrightarrow{r\otimes a\mapsto [r]\otimes a} & (R/f\cdot R)\otimes_R A & \longrightarrow & 0 \\ {\scriptstyle a\mapsto 1\otimes a}\uparrow{\scriptstyle \cong} & & {\scriptstyle a\mapsto 1\otimes a}\uparrow{\scriptstyle \cong} & & {\scriptstyle [a]\mapsto [1]\otimes a}\uparrow{\scriptstyle \Omega} & & \\ A & \xrightarrow{a\mapsto f\cdot a} & A & \xrightarrow{a\mapsto [a]} & A/f\cdot A & \longrightarrow & 0. \end{array}$$

We make the following observations:

(1) It follows from the Tensor–Right-Exact Lemma 314.14 that the top sequence is exact.
(2) The two vertical maps to the left are the natural isomorphisms given by the Tensor-with-Base Ring Lemma 312.10. It follows from the naturality that the square to the left commutes.
(3) It follows immediately from the definitions that the square to the right commutes.

It follows from the above and the Five Lemma 145.11 that the vertical map to the right is also an isomorphism. ∎

Before we state our next result we introduce the following notion which goes back to Jean-Pierre Serre.

Definition 314.16. Let R be a ring. An R-right module M is called flat if the covariant functor

$$A \mapsto M\otimes_R A$$

from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups is exact.

We can now formulate the following proposition, which is a variation on the Tensor-with-Torsion Free Proposition 161.6:[3471]

Proposition 314.17. (Tensor-with-Projective Proposition) Let R be a ring. Every projective R-right module is flat.

[3471] In Exercise 314.3 we will prove a statement that truly generalizes the Tensor-with-Torsion Free Proposition 161.6.

Proof. Let M be a projective right R-module. By the Exact Functor Lemma 314.12 it suffices to show that for every short exact sequence of R-left modules:

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0,$$

the "tensored" sequence is still exact. First we consider the special case that M is a free R-right module:

Claim 1. If M is a free R-right module, then the following sequence is exact:

$$0 \to M \otimes_R A \xrightarrow{\mathrm{id}_M \otimes \varphi} M \otimes_R B \xrightarrow{\mathrm{id}_M \otimes \psi} M \otimes_R C \to 0.$$

Proof. Since M is a free R-right module we know that there exists a set I and an isomorphism $\Xi \colon \bigoplus_{i \in I} R \to M$ of R-right modules. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i\in I} A & \xrightarrow{\mathrm{diag}(\varphi)_{i\in I}} & \bigoplus_{i\in I} B & \xrightarrow{\mathrm{diag}(\psi)_{i\in I}} & \bigoplus_{i\in I} C & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \bigoplus_{i\in I}(R \otimes_R A) & \xrightarrow{\mathrm{diag}(\mathrm{id}_R \otimes \varphi)_{i\in I}} & \bigoplus_{i\in I}(R \otimes_R B) & \xrightarrow{\mathrm{diag}(\mathrm{id}_R \otimes \psi)_{i\in I}} & \bigoplus_{i\in I}(R \otimes_R C) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \left(\bigoplus_{i\in I} R\right) \otimes_R A & \xrightarrow{\mathrm{id}_{R^{(I)}} \otimes \varphi} & \left(\bigoplus_{i\in I} R\right) \otimes_R B & \xrightarrow{\mathrm{id}_{R^{(I)}} \otimes \psi} & \left(\bigoplus_{i\in I} R\right) \otimes_R C & \longrightarrow & 0 \\
 & & \Xi\otimes\mathrm{id}_A \downarrow\cong & & \Xi\otimes\mathrm{id}_B \downarrow\cong & & \Xi\otimes\mathrm{id}_C \downarrow\cong & & \\
0 & \longrightarrow & M \otimes_R A & \xrightarrow{\mathrm{id}_M \otimes \varphi} & M \otimes_R B & \xrightarrow{\mathrm{id}_M \otimes \psi} & M \otimes_R C & \longrightarrow & 0.
\end{array}$$

The top vertical maps are the natural isomorphisms from the Tensor-with-Base Ring Lemma 312.10. Note that naturality implies that the top diagram commutes. The middle vertical maps are the natural isomorphisms from the Tensor–Direct Sum Lemma 312.15. Again naturality implies that the middle part of the diagram commutes. The bottom vertical maps are induced by the isomorphism Ξ, thus they are isomorphisms. It follows from the Tensor–Map Composition Lemma 312.7 that the bottom part of the diagram commutes.

Evidently the top sequence is exact. It follows from the above discussion that the bottom sequence is also exact. ⊞

We now deal with the general case:

Claim 2. If P is a projective R-right module, then the following sequence is exact:

$$0 \to P \otimes_R A \xrightarrow{\mathrm{id}_P \otimes \varphi} P \otimes_R B \xrightarrow{\mathrm{id}_P \otimes \psi} P \otimes_R C \to 0.$$

Proof. Let P be a projective R-right module It follows from the Projective Module–Characterization Lemma 309.28 that there exists an R-right module Q such that $P \oplus Q$ is a free R-right module. By Claim 1 we know that tensoring with $P \oplus Q$ is exact. By the Tensor–Direct Sum Lemma 312.15 we know that tensoring with a direct sum is distributive. It now follows easily that tensoring with the subsummands P and Q is also exact. ■

We continue with another instance of a flat module:

Proposition 314.18. (Tensor-with-Localization Proposition) Let R be a commutative domain and let $S \subseteq R \setminus \{0\}$ be a multiplicatively closed subset which contains 1.

(1) We consider the localization

$$RS^{-1} := R \times S/\sim$$

where $(r,s) \sim (r',s')$ precisely if $r \cdot s' = r' \cdot s$. This is clearly an equivalence relation and RS^{-1} is ring where for $\frac{r}{s} := [(r,s)]$ we have the usual addition and multiplication.

(2) The R-module RS^{-1} is flat.

Proof.

(1) This statement can easily verified by hand.

(2) It follows from the Tensor–Right-Exact Lemma 314.14 and the Exact Functor Lemma 314.12 that it suffices to show that the functor $- \otimes_R RS^{-1}$ preserves monomorphisms. We start out with two little claims:

Claim 1. Let C be an R-module. Any element of $C \otimes_R RS^{-1}$ is of the form $c \otimes \frac{r}{s}$.

Proof. Let $k \in C \otimes_R RS^{-1}$. By the Tensor Product–Basics Lemma 312.4 we know that we can write $k = c_1 \otimes \frac{r_1}{s_1} + \cdots + c_m \otimes \frac{r_m}{s_m}$ with the obvious notation. After forming a common denominator we can write $k = (c_1 + \cdots + c_m) \otimes \frac{r}{s}$. ⊞

Claim 2. Let C be an R-module. We have $c \otimes \frac{r}{s} = c' \otimes \frac{r'}{s'} \in C \otimes_R RS^{-1}$ if and only if $c \cdot rs' = c' \cdot r's$.

Proof. The "if"-direction follows easily from the definitions. For the "only if"-direction we note that one can easily verify that

$$(C \times RS^{-1})/\sim \qquad \text{with } (c, \tfrac{r}{s}) \sim (c', s\tfrac{r'}{s'}) \text{ if } c \cdot rs' = c' \cdot r's$$

satisfies the universal properties of the tensor product. It follows that the obvious map $(C \times RS^{-1})/\sim \to C \otimes_R RS^{-1}$ needs to be an isomorphism. This gives us the "only if"-direction of the claim. ⊞

Now we turn to the actual proof of the lemma. Thus let $f\colon A \to B$ be a monomorphism. We need to show that $f \otimes \mathrm{id}\colon A \otimes_R RS^{-1} \to B \otimes_R RS^{-1}$ is a monomorphism. Thus let k be in the kernel of $f \otimes \mathrm{id}$. By Claim 1 we can write $k = a \otimes \frac{r}{s}$ with $a \in A$, $r \in R$ and $s \in S$. We now see that $f(a) \otimes \frac{r}{s} = (f \otimes \mathrm{id})(a \otimes \frac{r}{s}) = 0 = 0 \otimes \frac{1}{r} \in B \otimes_R RS^{-1}$. It follows from Claim 2 that $f(a) \cdot r = 0 \in B$. Since $f(a \cdot r) = f(a) \cdot r$ and since f is a monomorphism we obtain that $a \cdot r = 0$. But then we have the desired equalities $a \otimes \frac{r}{s} = (a \cdot r) \otimes \frac{1}{s} = 0 \otimes \frac{r}{s} = 0 \in A \otimes_R RS^{-1}$. ■

We conclude this section with the following lemma which is a generalization of the Homology-Multiplication Lemma 161.7:

Lemma 314.19. (Homology-Multiplication Lemma) Let Q and R be rings. Furthermore let (C_*, ∂_*) be a chain complex of R-modules, let M be a (Q,R)-bimodule and let $n \in \mathbb{N}_0$. The map

$$\begin{aligned} \mu\colon M \otimes_R \mathrm{H}_n(C_*) &\to \mathrm{H}_n(M \otimes_R C_*) \\ \sum_{i=1}^{k} r_i \otimes [\sigma_i] &\mapsto \sum_{i=1}^{k} [r_i \otimes \sigma_i] \end{aligned}$$

is a well-defined natural homomorphism of left Q-modules.

Proof. It is straightforward to show that the map

$$\begin{array}{ccc} M \times \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(M \otimes_R C_*) \\ (m, [\sigma]) & \mapsto & [m \otimes \sigma] \end{array}$$

is well-defined, R-balanced and $(Q, \mathbb{Z})$-bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 312.5 (1) that the given map is well-defined. It follows easily from the definitions that the resulting homomorphism μ is natural. ■

314.5. **Tor-modules.** We start out with the following analog of Definition 162.4:

Definition 314.20. Let Q and R be rings. Let M be a (Q, R)-bimodule and let N be an R-left module. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0$$

the natural free resolution from Definition 313.12. We define the n-th M-torsion module of N as the Q-left module

$$\begin{aligned} \mathrm{Tor}_n^R(N, M) &:= \mathrm{H}_n(\underbrace{M \otimes_R F_*}_{\substack{\text{chain complex of} \\ Q\text{-left modules}}}) \\ &= \mathrm{H}_n(\ldots \to M \otimes_R F_2 \xrightarrow{\mathrm{id}_M \otimes f_2} M \otimes_R F_1 \xrightarrow{\mathrm{id}_M \otimes f_1} M \otimes_R F_0 \xrightarrow{0} 0). \end{aligned}$$

We will now see that the torsion modules $\mathrm{Tor}_n(N, M)$ are "covariantly functorial in both entries":

Definition 314.21. Let Q and R be rings.

(1) We fix a (Q, R)-bimodule M. Let $\beta\colon N \to N'$ be a homomorphism between two R-left modules. It follows from the Homological Algebra–Fundamental Theorem 313.14 applied to the natural free resolutions F_* and F'_* of N and N', that we get an extension $F_* \to F'_*$ of β to the free resolutions, and this extension is unique up to chain homotopy. It follows from the Chain Homotopy–H_*-Lemma 313.6, that β induces for each $n \in \mathbb{N}_0$ a well-defined homomorphism

$$\beta_*\colon \underbrace{\mathrm{Tor}_n^R(N, M)}_{=\mathrm{H}_n(M \otimes_R F_*)} \to \underbrace{\mathrm{Tor}_n^R(N', M)}_{=\mathrm{H}_n(M \otimes_R F'_*)}$$

of Q-left modules. One can now easily show that the maps

$$\begin{array}{ccc} N & \mapsto & \mathrm{Tor}_n^R(N, M) \\ (\beta\colon N \to N') & \mapsto & (\beta_*\colon \mathrm{Tor}_n^R(N, M) \to \mathrm{Tor}_n^R(N', M)) \end{array}$$

define a covariant functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{Q\text{-left}}$ of Q-left modules.

(2) We fix an R-left module N. Let $\alpha\colon M \to M'$ be a homomorphism between two (Q, R)-bimodules. Let F_* be the natural free resolution of N. Then $\mathrm{id}_M \otimes \alpha$ defines a chain map from $M \otimes_R F_*$ to $M' \otimes_R F_*$ and, as in the Homology-Coefficients-Functor

[3471] The notation $\mathrm{Tor}_n^R(N, M) := \mathrm{H}_n(M \otimes_R F_*)$ is a little unfortunate, but it forced on our if we want to stay consistent with Definition 162.4 and if we want to work with R-left chain complexes F_*, which in turn arise from the fact that chain complexes of coverings with deck group G are naturally $\mathbb{Z}[G]$-left modules.

Lemma 314.10, we get an induced homomorphism

$$\alpha_*\colon\ \mathrm{Tor}_n^R(N,M)\ \to\ \mathrm{Tor}_n^R(N,M')$$

of Q-left modules. One can now easily show that the maps

$$\begin{aligned} M &\mapsto \mathrm{Tor}_n^R(N,M) \\ (\alpha\colon M\to M') &\mapsto (\alpha_*\colon \mathrm{Tor}_n^R(N,M)\to \mathrm{Tor}_n^R(N,M')) \end{aligned}$$

define a covariant functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{Q\text{-left}}$ of Q-left modules.

The following proposition, which generalizes the Tor-Free Resolution Proposition 162.6, is the key tool for calculating torsion modules.

Proposition 314.22. (Tor-Projective Resolution Proposition) Let Q and R be rings, let M be a (Q,R)-bimodule and let N be an R-left module. Furthermore let

$$\ldots\ \to\ P_2\ \xrightarrow{f_2}\ P_1\ \xrightarrow{f_1}\ P_0\ \xrightarrow{f_0}\ N\ \to\ 0$$

be a projective resolution of the R-left module N. For every $n\in\mathbb{N}_0$ there exists an isomorphism

$$\mathrm{Tor}_n^R(N,M)\ \xrightarrow{\cong}\ \mathrm{H}_n(M\otimes_R P_*)$$

of Q-left modules which is natural in M.

Proof. Let M be a (Q,R)-bimodule, let N be an R-left module and let

$$\ldots\ \to\ P_2\ \xrightarrow{f_2}\ P_1\ \xrightarrow{f_1}\ P_0\ \xrightarrow{f_0}\ N\ \to\ 0$$

be a projective resolution of N. We want to show that there exists a natural isomorphism

$$\mathrm{Tor}_n^R(N,M)\ \xrightarrow{\cong}\ \mathrm{H}_n(M\otimes_R P_*).$$

We denote by (F'_*, f'_*) the natural free resolution of N from Definition 313.12. Since F'_* is a free resolution we obtain from the Homological Algebra–Fundamental Theorem 313.14 (1) that there exists an extension $\alpha_i, i\in\mathbb{N}$ of the identity map $\mathrm{id}_N\colon N\to N$ from the free resolutions F'_* to the projective resolution P_*. In other words, we obtain a commutative diagram of R-left modules the following form:

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}_N} & & \\ \ldots & \longrightarrow & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & N & \longrightarrow & 0. \end{array}$$

Since both resolutions are projective we obtain from the Projective Resolution–Chain Homotopic Corollary 313.15 that the maps α_* form a chain homotopy equivalence between the two F-left chain complexes F'_* and P_*. It follows from the Homology-with-Coefficients Lemma 314.6 (2) that $\mathrm{id}_M\otimes\alpha_*\colon M\otimes_R F'_*\to M\otimes_R P_*$ is also a chain homotopy equivalence. It follows from this observation together with the Chain Homotopy Equivalence-H_*-Corollary 313.7 that for every $n\in\mathbb{N}_0$ the map

$$\begin{aligned} \mathrm{id}_M\otimes\alpha_n\colon \overbrace{\mathrm{H}_n(M\otimes_R F'_*)}^{=:\mathrm{Tor}_n^R(N,M)} &\to \mathrm{H}_n(M\otimes_R P_*) \\ \left[\sum_{i=1}^k m_i\otimes c_i\right] &\mapsto \left[\sum_{i=1}^k m_i\otimes \alpha_n(c_i)\right] \end{aligned}$$

is an isomorphism of Q-left modules. Using the Homological Algebra–Fundamental Theorem 313.14 (2) one can easily show that this map does not depend on the choice of the extension α_*. Finally note that it follows easily from the Tensor–Map Composition Lemma 312.7 (2) that this isomorphism is natural in M. ■

We now turn to the calculation of the M-torsion modules. The following lemma, which is a generalization of the Tor_0–Lemma 162.7, shows that the 0-th M-torsion modules are objects that we are already familiar with:

Lemma 314.23. (Tor_0^R–Lemma) Let Q and R be rings. For any (Q, R)-bimodules M and any R-left module N there exists a natural isomorphism[3472]

$$\mathrm{Tor}_0^R(N, M) \xrightarrow{\cong} M \otimes_R N$$

of Q-left modules.

Proof. Let M be a (Q, R)-bimodule and let N be an R-left module. We consider the natural free resolution of N:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0$$

By the Tensor–Right-Exact Lemma 314.14 we know that the sequence

$$M \otimes_R F_1 \xrightarrow{\mathrm{id}_M \otimes f_1} M \otimes_R F_0 \xrightarrow{\mathrm{id}_M \otimes f_0} M \otimes_R N \to 0 \tag{$*$}$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned} \mathrm{Tor}_0(N, M) &\overset{\text{by Definition 314.20}}{=} \mathrm{H}_0\big(\cdots \to M \otimes_R F_1 \xrightarrow{\mathrm{id}_M \otimes f_1} M \otimes_R F_0 \to 0\big) \\ &= \mathrm{coker}\,\big(M \otimes_R F_1 \xrightarrow{\mathrm{id}_M \otimes f_1} M \otimes_R F_0\big) \xrightarrow[\text{since } (*) \text{ is exact we obtain from the Exactness Lemma 145.2 (3) that this map is an isomorphism}]{\mathrm{id}_M \otimes f_0} M \otimes_R N \end{aligned}$$

of Q-left modules. It follows easily from the Tensor–Map Composition Lemma 312.7 (2) that this isomorphism $\mathrm{Tor}_0^R(N, M) \to M \otimes_R N$ is natural in M. Furthermore, using the Homological Algebra–Fundamental Theorem 313.14 one can easily show that this isomorphism is natural in N. ■

The following lemma is a variation on the Tor-Properties Lemma 162.13:

Lemma 314.24. (Tor-Properties Lemma) Let Q and R be rings, let $M, \{M_j\}_{j \in J}$ be (Q, R)-bimodules and let N, $\{N_i\}_{i \in I}$ be R-left modules. Then the following holds:

(1) For any $k \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{Tor}_k^R\left(\bigoplus_i N_i, M\right) \cong \bigoplus_i \mathrm{Tor}_k^R(N_i, M)$.

(2) For any $k \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{Tor}_k^R\left(N, \bigoplus_j M_j\right) \cong \bigoplus_j \mathrm{Tor}_k^R(N, M_j)$.

(3) If M is a projective R-left module, then for each $k \in \mathbb{N}$ we have $\mathrm{Tor}_k^R(N, M) = 0$.

(4) If N is projective R-left module, then for each $k \in \mathbb{N}$ we have $\mathrm{Tor}_k^R(N, M) = 0$.

Proof.

[3472] Unfortunately various conventions that we picked collide to give us the ungainly result that $\mathrm{Tor}_0^R(N, M) \xrightarrow{\cong} M \otimes_R N$.

(1) For each $i \in I$ let F_*^i be the natural free resolution of N_i. One can easily verify that $\bigoplus_{i\in I} F_*^i$ is a free resolution of $\bigoplus_{i\in I} N_i$. For each $k \in \mathbb{N}_0$ we have the following natural isomorphisms of Q-left modules:

$$\begin{array}{ccccc}
\operatorname{Tor}_k^R\Big(\bigoplus_{i\in I} N_i, M\Big) & \overset{\text{by the Tor-Projective Resolution Proposition 314.22}}{\cong} & \operatorname{H}_k\Big(M \otimes_R \Big(\bigoplus_{i\in I} F_*^i\Big)\Big) & \overset{\text{by the Tensor–Direct Sum Lemma 312.15}}{\cong} & \operatorname{H}_k\Big(\bigoplus_{i\in I} (M \otimes_R F_*^i)\Big) \\
& \overset{\text{by the Direct Sum H}_*\text{-Lemma 313.4}}{\cong} & \bigoplus_{i\in I} \operatorname{H}_k(M \otimes_R F_*^i) & \overset{\text{by Definition 314.20}}{\cong} & \bigoplus_{i\in I} \operatorname{Tor}^R(N_i, M).
\end{array}$$

(2) Let F_* be the natural resolution for N. For each $k \in \mathbb{N}_0$ we have the following natural isomorphisms of Q-left modules:

$$\begin{array}{ccccc}
\operatorname{Tor}_k^R\Big(N, \bigoplus_{j\in J} M_j\Big) & \overset{\text{by Definition 314.20}}{=} & \operatorname{H}_k\Big(\bigoplus_{j\in J} (M_j \otimes_R F_*)\Big) & \overset{\text{by the Tensor–Direct Sum Lemma 312.15}}{\cong} & \operatorname{H}_k\Big(\bigoplus_{j\in J} (M_j \otimes_R F_*)\Big) \\
& \overset{\text{by the Direct Sum H}_*\text{-Lemma 313.4}}{\cong} & \bigoplus_{j\in J} \operatorname{H}_k(M_j \otimes_R F_*) & \overset{\text{by Definition 314.20}}{=} & \bigoplus_{j\in J} \operatorname{Tor}^R(N, M_j).
\end{array}$$

(3) We need to show that if M is a projective R-left module, then for any R-left module N and any $k \in \mathbb{N}$ we have $\operatorname{Tor}_k^R(N, M) = 0$. We consider the natural free resolution of N:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0.$$

We denote by F'_* the above sequence *with* N, i.e. $F'_i = F_i$ for $i \in \mathbb{N}_0$ and $F_{-1} = N$. Note that for each $k \in \mathbb{N}$ we have the following isomorphisms:

$$\begin{array}{ccccccc}
\operatorname{Tor}_k^R(N, M) & \overset{\text{by Definition 314.20}}{=} & \operatorname{H}_k(M \otimes_R F_*) & \overset{\text{since } k \geq 1}{=} & \operatorname{H}_k(M \otimes_R F'_*) & & \\
& & & \overset{\text{by the Exact Functor–Homology Lemma 314.13 and the Tensor-with-Projective Proposition 314.17}}{\cong} & M \otimes_R \operatorname{H}_k(F'_*) & \overset{\text{since } F'_* \text{ is exact}}{=} & M \otimes_R 0 \quad = \quad 0.
\end{array}$$

(4) Let N be a *projective* R-left module. Note that in this case $0 \to P_0 \coloneqq N \xrightarrow{\mathrm{id}} N \to 0$ is already a projective resolution, and it follows immediately from the Tor-Projective Resolution Proposition 314.22 that for every $k \in \mathbb{N}$ we have $\operatorname{Tor}_k^R(N, M) = 0$. ■

We conclude this section with the following little lemma:

Lemma 314.25. (Tor-Coefficient LES Lemma) Let Q and R be two rings. Furthermore let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of (Q, R)-bimodules. For any R-left module N we get a long exact sequence

$$\ldots \to \operatorname{Tor}_k^R(N, A) \xrightarrow{\alpha_*} \operatorname{Tor}_k^R(N, B) \xrightarrow{\beta_*} \operatorname{Tor}_k^R(N, C) \xrightarrow{\partial_k} \operatorname{Tor}_{k-1}^R(N, A) \xrightarrow{\alpha_*} \ldots$$

of Q-left modules.

Proof. We pick a free resolution F_* of the R-left module N. We obtain a sequence

$$0 \to A \otimes_R F_* \xrightarrow{\alpha\otimes\mathrm{id}} B \otimes_R F_* \xrightarrow{\beta\otimes\mathrm{id}} C \otimes_R F_* \to 0$$

of chain complexes of Q-left modules. Since the F_* are free R-left modules we obtain from the Tensor-with-Projective Proposition 314.17 that the above sequence is in fact exact.

The desired statement now follows from the LES-with-Coefficients Proposition 314.7. ■

314.6. **Calculations of torsion modules.** In Section 162.1 we saw that if the underlying ring is the ring $R = \mathbb{Z}$ of integers, then at least for finitely generated $\mathbb{Z}$-modules M and N we can calculate fairly easily all $\mathbb{Z}$-torsion modules $\mathrm{Tor}_n^{\mathbb{Z}}(M, N)$. In this section we will see that many of the results generalize from $\mathbb{Z}$-modules to modules over PIDs. But we will also see that the results do not generalize to non-PIDs.

314.6.1. **Torsion modules over PIDs.** We start out with the following generalization of the Length 1–Free Resolution Lemma 162.9:

Lemma 314.26. (Length 1–Free Resolution Lemma) Let R be a PID. Every R-module[3473] admits a free resolution of length ≤ 1.

Proof. Let R be a PID and let N be an R-module. Recall that we have the exact sequence

$$0 \to \ker(\alpha(N)\colon R^{(N)} \to N) \to R^{(N)} \xrightarrow{\alpha(N)} N \to 0$$

where $\alpha(N)$ is provided by the Free Module–Natural Epimorphism Lemma 309.21. The lemma is now a consequence of the observation that it follows from the Free Module–PID Theorem 309.35 that the R-module $\ker(\alpha(N)\colon R^{(N)} \to N)$, as a submodule the free R-module $R^{(N)}$, is itself a free R-module. ■

Now we can prove the following analog of the Higher Tor Lemma 162.10:

Lemma 314.27. (PID-Higher Tor Lemma) Let R be a PID. For any two R-modules N and M we have $\mathrm{Tor}_n^R(N, M) = 0$ for $n \geq 2$.

Proof. Let M and N be R-modules. By the Length 1–Free Resolution Lemma 314.26 there exists a free resolution of N of length 1, i.e. of the form

$$0 \to P_1 \to P_0 \to N \to 0.$$

It now follows that for $n \geq 2$ we have

$$\mathrm{Tor}_n^R(N, M) \underset{\text{by the Tor-Projective Resolution Proposition 314.22}}{\cong} \mathrm{H}_n(M \otimes_R P_*) \underset{\text{since } n \geq 2 \text{ implies that } P_n = 0}{=} 0.$$

■

Ideally one would like to calculate all Tor-modules of finitely generated modules over a PID. By the PID–Module-Classification Theorem 309.36 we know that every finitely generated module over a PID is a direct sum of cyclic modules. This fact, together with the PID-Higher Tor Lemma 314.27, the Tor_0^R–Lemma 314.23 and the Tor-Properties Lemma‘314.24 shows that it remains to prove the following generalizations of the Cyclic-Tensor-Cyclic Lemma 159.5 and of the $\mathbb{Z}/k\mathbb{Z}$-Tor-Lemma 162.12:

[3473]Recall that according to Definition 309.34 a PID is commutative. So we do not need to worry about left and right module structures.

Lemma 314.28. (Cyclic-Tensor-Cyclic Lemma) Let R be a ring and let $m, n \in R$. The map

$$\begin{array}{ccc} R/mR \otimes_R R/nR & \to & R/\gcd(m,n)R \\ \sum_{i=1}^{d} [a_i] \otimes [b_i] & \mapsto & \left[\sum_{i=1}^{d} a_i \cdot b_i\right] \end{array}$$

is well-defined and it is an isomorphism of R-modules.

Proof. We leave it to the reader to generalize the proof of the Cyclic-Tensor-Cyclic Lemma 159.5 to obtain the desired result. ∎

Lemma 314.29. (Cyclic-Tor-Lemma) Let R be a PID and let $k \in R$.

(1) Let S be a commutative ring and let M be an (R, S)-ring. We have a natural isomorphism
$$\ker\left(M \xrightarrow{r \mapsto r \cdot k} M\right) \xrightarrow{\cong} \operatorname{Tor}_1^R(R/kR, M)$$
of M-modules.

(2) Given any $m \in \mathbb{N}$ we have $\operatorname{Tor}_1^R(R/kR, R/mR) \cong R_{\gcd(k,m)}$.

Proof.

(1) We consider the free resolution $0 \to R \xrightarrow{r \mapsto r \cdot k} R \to R/kR \to 0$ of R/kR. Given any abelian group M this gives rise to the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overbrace{\ker\left(R \otimes_R M \xrightarrow{(n \mapsto n \cdot k) \otimes \mathrm{id}_M} R \otimes_R M\right)}^{\cong \operatorname{Tor}_1^R(R/kR, M)} & \longrightarrow & R \otimes_R M & \xrightarrow{(n \mapsto n \cdot k) \otimes \mathrm{id}_M} & R \otimes_R M \\ & & \uparrow & & \uparrow{\scriptstyle r \mapsto 1 \otimes r}\ \cong & & \uparrow{\scriptstyle r \mapsto 1 \otimes r}\ \cong \\ 0 & \longrightarrow & \ker(M \xrightarrow{r \mapsto r \cdot k} M) & \longrightarrow & M & \xrightarrow{r \mapsto r \cdot k} & M. \end{array}$$

We make the following observations:

(a) Since $0 \to R \xrightarrow{n \mapsto n \cdot k} R \to R/kR \to 0$ is a free resolution of R/kR we obtain from the Tor-Free Resolution Proposition 314.22 the isomorphism in the top left corner.

(b) The top horizontal sequence is exact by definition.

(c) The two vertical maps to the right are the natural isomorphisms given by the Tensor-with-Base Ring Lemma 312.10. It follows immediately from the Tensor Product–Universal Property Lemma 312.5 (1) that the square to the right commutes.

It now follows from a mild diagram chase that from the above we get an induced isomorphism $\ker\left(M \xrightarrow{m \mapsto m \cdot k} M\right) \xrightarrow{\cong} \operatorname{Tor}_1^R(R/kR, M)$. One can easily verify that this isomorphism is natural.

(2) We write $r = \gcd(m, k)$ and $s = m/r$. We see that

$$\begin{array}{ccccccc} \operatorname{Tor}_1^R(R/kR, R/mR) & \cong & \ker(R/mR \xrightarrow{r \mapsto r \cdot k} R/mR) & = & s \cdot R/mR & \cong & R_r. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by (1)} & & & & \text{follows from elementary algebra} & \end{array}$$

∎

314.6.2. **Torsion modules over non-PIDs.** In this subsection we will, just for practice, calculate some torsion modules over commutative rings that are not PIDs. More precisely, we will consider the rings $\mathbb{Z}[x, y]$ and $\mathbb{Z}[t]/(t^n - 1)$.

First recall that it follows from the Tor-Projective Resolution Proposition 314.22 that the key to calculating torsion modules $\operatorname{Tor}_n^R(N, M)$ are free resolutions of N. We start out

with the ring $R = \mathbb{Z}[x,y]$ and the module $N = {}_{\mathrm{tr}}\mathbb{Z}$, i.e. the $R = \mathbb{Z}[x,y]$-module that is given by the abelian group $\mathbb{Z}$ where x and y act trivially on $\mathbb{Z}$:

Lemma 314.30. (Free Resolution-$\mathbb{Z}[x,y]$-Lemma) Let $R = \mathbb{Z}[x,y]$ be the polynomial ring in two variables. The following sequence of $\mathbb{Z}[x,y]$-module homomorphisms

$$0 \to \mathbb{Z}[x,y] \xrightarrow{\begin{pmatrix} 1-y \\ x-1 \end{pmatrix}} \mathbb{Z}[x,y]^2 \xrightarrow{(x-1,y-1)} \mathbb{Z}[x,y] \xrightarrow{x\mapsto 1, y\mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0$$

is a free resolution of the $\mathbb{Z}[x,y]$-module ${}_{\mathrm{tr}}\mathbb{Z}$.

Proof. We outsource the proof to Exercise 314.1. ■

Example. In the following we want to calculate $\mathrm{Tor}_n^{\mathbb{Z}[x,y]}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z})$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y] & \xrightarrow{\mathrm{id}_{{}_{\mathrm{tr}}\mathbb{Z}} \otimes \begin{pmatrix} 1-y \\ x-1 \end{pmatrix}} & {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y]^2 & \xrightarrow{\mathrm{id}_{{}_{\mathrm{tr}}\mathbb{Z}} \otimes (x-1,y-1)} & {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y] & \to & 0 \\
 & & \cong \downarrow x\mapsto 1, y\mapsto 1 & & \cong \downarrow x\mapsto 1, y\mapsto 1 & & \cong \downarrow x\mapsto 1, y\mapsto 1 & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}^2 & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

It follows from the Free Module–Tensor-with-Ring Lemma 312.18 (1) and (2), applied to the ring homomorphism $\varphi\colon \mathbb{Z}[x,y] \xrightarrow{x,y\mapsto 1} \mathbb{Z}$, that the vertical maps are isomorphisms and that the diagram commutes. Thus we see that

$$\mathrm{Tor}_n^{\mathbb{Z}[x,y]}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z}) \underset{\uparrow}{\cong} \mathrm{H}_n(\text{above chain complex}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

by the Tor-Projective Resolution Proposition 314.22
and the Free Resolution-$\mathbb{Z}[x,y]$-Lemma 314.30

We have thus found an example where the second torsion module is non-zero. □

Next we turn to the ring $R = \mathbb{Z}[t]/(t^k - 1) \cong \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$ and the module $N = {}_{\mathrm{tr}}\mathbb{Z}$, i.e. the $R = \mathbb{Z}[t]/(t^k-1)$-module that is given by the abelian group $\mathbb{Z}$ where t acts trivially on $\mathbb{Z}$:

Lemma 314.31. (Free Resolution-$\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-Lemma) In the following let $k \in \mathbb{N}$ and let $R = \mathbb{Z}[t]/(t^k-1) \cong \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$ be the group ring of the group $\mathbb{Z}/k\mathbb{Z} = \langle t \mid t^k \rangle$. We denote by ${}_{\mathrm{tr}}\mathbb{Z}$ the $R = \mathbb{Z}[t]/(t^k-1)$ module that is given by the abelian group $\mathbb{Z}$ where t acts trivially on $\mathbb{Z}$. The following sequence of $\mathbb{Z}[t]/(t^k-1)$-module homomorphisms

$$\ldots \to \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(1+t+\cdots+t^{k-1})} \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k-1) \xrightarrow{t\mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0.$$

is a free resolution of the $\mathbb{Z}[t]/(t^k-1)$-module ${}_{\mathrm{tr}}\mathbb{Z}$.

Proof. We outsource the proof to Exercise 314.2. ■

Example. In the following we want to calculate $\mathrm{Tor}_n^{\mathbb{Z}[t]/(t^k-1)}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z})$. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\to {}_{\mathrm{tr}}\mathbb{Z} \underset{\mathbb{Z}[t]/(t^k-1)}{\otimes} \mathbb{Z}[t]/(t^k-1) & \xrightarrow{\cdot(1+t+\cdots+t^{k-1})} & {}_{\mathrm{tr}}\mathbb{Z} \underset{\mathbb{Z}[t]/(t^k-1)}{\otimes} \mathbb{Z}[t]/(t^k-1) & \xrightarrow{\cdot(t-1)} & {}_{\mathrm{tr}}\mathbb{Z} \underset{\mathbb{Z}[t]/(t^k-1)}{\otimes} \mathbb{Z}[t]/(t^k-1) & \to 0 \\
\cong \downarrow t\mapsto 1 & & \cong \downarrow t\mapsto 1 & & \cong \downarrow t\mapsto 1 & \\
\xrightarrow[0]{\cdot(1-1)} \mathbb{Z} & \xrightarrow[\cdot k]{\cdot(1+\cdots+1)} & \mathbb{Z} & \xrightarrow[0]{\cdot(1-1)} & \mathbb{Z} & \longrightarrow 0.
\end{array}$$

It follows from the Free Module–Tensor-with-Ring Lemma 312.18 (1) and (2), applied to the ring homomorphism $\varphi\colon \mathbb{Z}[t]/(t^n-1) \xrightarrow{t\mapsto 1} \mathbb{Z}$, that the vertical maps are isomorphisms and that the diagram commutes. Thus we see that

$$\mathrm{Tor}_n^{\mathbb{Z}[t]/(t^k-1)}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z}) \underset{\uparrow}{\cong} \mathrm{H}_n(\text{above chain complex}) \cong \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}/k\mathbb{Z}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even and} \neq 0. \end{cases}$$

by the Tor-Projective Resolution Proposition 314.22 and the Free Resolution-$\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-Lemma 314.31

We have thus found an example where all torsion modules of odd degree are non-zero. □

314.7. **The Universal Coefficient Theorem over PIDs.** In this section we will generalize the Algebraic Universal Coefficient Theorem 162.15 from the PID $\mathbb{Z}$ to general PIDs. To formulate this generalization let us recall the following concept that we first introduced in the Homology-Multiplication Lemma 314.19:

Definition 314.32. Let (C_*, ∂_*) be a chain complex of R-left modules and let M be a (Q,R)-bimodule. Given $n \in \mathbb{N}_0$, we denote by μ the natural Q-left module homomorphism

$$\begin{array}{rcl} \mu\colon M \otimes_R \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(M \otimes_R C_*) \\ \sum_{i=1}^m m_i \otimes [c_i] & \mapsto & \left[\sum_{i=1}^m m_i \otimes c_i\right]. \end{array}$$

The following theorem is now the promised generalization of the Algebraic Universal Coefficient Theorem 162.15:

Theorem 314.33. (Algebraic Universal Coefficient Theorem) Let R be a PID. Let (C_*, ∂_*) be a chain complex of free R-modules and let M be a (Q,R)-bimodule. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\nu\colon \mathrm{H}_n(M \otimes_R C_*) \to \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M)$ of Q-left modules such that the following sequence is exact:

$$0 \to M \otimes_R \mathrm{H}_n(C_*) \xrightarrow{\mu} \mathrm{H}_n(M \otimes_R C_*) \xrightarrow{\nu} \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M) \to 0.$$

Example. If R is not a PID, then there is no reason to expect that the Algebraic Universal Coefficient Theorem 314.33 should hold. For example let us consider the polynomial ring $R = \mathbb{Z}[x,y]$ in two variables. We consider the following chain complex of free $\mathbb{Z}[x,y]$-modules

$$C_* := 0 \to \mathbb{Z}[x,y] \xrightarrow{\binom{1-y}{x-1}} \mathbb{Z}[x,y]^2 \xrightarrow{(x-1, y-1)} \mathbb{Z}[x,y] \to 0.$$

Note that it follows from the Free Resolution-$\mathbb{Z}[x,y]$-Lemma 314.30 that

$$\mathrm{H}_n(C_*) \cong \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ 0, & \text{if } n \geq 1. \end{cases}$$

For the $\mathbb{Z}[x,y]$-module $M = {}_{\mathrm{tr}}\mathbb{Z}$ and $n = 2$ we now see that

$$0 \to \underbrace{{}_{\mathrm{tr}}\mathbb{Z} \otimes_R \mathrm{H}_2(C_*)}_{=0,\text{ by the above}} \xrightarrow{\mu} \underbrace{\mathrm{H}_2({}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[x,y]} C_*)}_{\neq 0,\text{ by the calculation on page 6064}} \xrightarrow{\nu} \underbrace{\mathrm{Tor}_1^{\mathbb{Z}[x,y]}(\mathrm{H}_1(C_*), {}_{\mathrm{tr}}\mathbb{Z})}_{=0,\text{ by the above}} \to 0.$$

This discussion shows that the sequence is *not* exact. Thus in this setting the statement of the Algebraic Universal Coefficient Theorem 314.33 fails miserably. But help is on the way: In the Homological UCSS Theorem 316.5 we will formulate a generalization of the Algebraic Universal Coefficient Theorem 314.33 for chain complexes over arbitrary rings.□

Proof. Let R be a PID and let (C_*, ∂_*) be a chain complex of free R-modules. For each $n \in \mathbb{N}_0$ we write as always

$$\begin{aligned} Z_n &:= \ker(\partial_n \colon C_n \to C_{n-1}) \\ B_n &:= \mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n). \end{aligned}$$

Since R is a PID and since C_n is by hypothesis a free R-module we obtain from the Free Module–PID Theorem 309.35 that B_n and Z_n are also free R-modules.

We consider the following diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \xhookrightarrow{\iota_n} & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \xhookrightarrow{\iota_{n-1}} & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. We can and will view the left and the right vertical columns as chain complexes of R-modules. In other words, the above diagram is a short exact sequence of chain complexes of R-modules.

Now let M be a (Q,R)-bimodule. We tensor the above short exact sequence of chain complexes of R-modules with M. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" chain complexes of Q-modules:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M \otimes_R Z_n & \xrightarrow{\mathrm{id}_M \otimes \iota_n} & M \otimes_R C_n & \xrightarrow{\mathrm{id}_M \otimes \partial_n} & M \otimes_R B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{id}_M \otimes \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & M \otimes_R Z_{n-1} & \xrightarrow{\mathrm{id}_M \otimes \iota_{n-1}} & M \otimes_R C_{n-1} & \xrightarrow{\mathrm{id}_M \otimes \partial_{n-1}} & M \otimes_R B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Proof.

(1) It follows from the Tensor–Functor Lemma 314.4 together with the Functor-applied-to-Chain Complex Lemma 314.3 (1a) that the vertical sequences are again chain complexes of R-modules. Thus the above is a sequence of "vertical" chain complexes of Q-modules.

(2) Recall that the groups B_n are free R-modules. Thus it follows from the Projective Module–Splitting Lemma 309.27 together with the Module Splitting Lemma 309.31 (1)⇒(3) that each of the horizontal sequences in the original diagram splits. It follows from the Tensor-Split-SES Lemma 314.9 that the horizontal sequences obtained from tensoring with M are still exact. ⊞

We apply the LES-Proposition 313.10 to the short exact sequence of chain complexes of Q-modules from the claim. We obtain a natural connecting homomorphism

$$\underbrace{M \otimes_R B_n}_{=\mathrm{H}_n(M\otimes_R B_*)} \to \underbrace{M \otimes_R Z_n}_{=\mathrm{H}_n(M\otimes_R Z_*)}$$

of Q-left modules. In fact, it follows rather easily from Definition 313.8 that the connecting homomorphism $M \otimes_R B_n \to M \otimes_R Z_n$ is the map that is given by $\mathrm{id}_M \otimes i_n \colon M \otimes_R B_n \to M \otimes_R Z_n$ where $i_n \colon B_n \to Z_n$ denotes the obvious inclusion map. By the LES-Proposition 313.10 this connecting homomorphism gives us the following long exact sequence

$$\ldots \to \underset{R}{M \otimes B_n} \xrightarrow{\mathrm{id}_M \otimes i_n} \underset{R}{M \otimes Z_n} \to \mathrm{H}_n(\underset{R}{M \otimes} C_*) \xrightarrow{\partial_n} \underset{R}{M \otimes B_{n-1}} \xrightarrow{\mathrm{id}_M \otimes i_{n-1}} \underset{R}{M \otimes Z_{n-1}} \to \ldots$$

Using the Break-into-SES Lemma 145.6 the above discussion implies that we obtain the following short exact sequence

$$0 \to \underbrace{\operatorname{coker}\left(\underset{R}{M \otimes B_n} \xrightarrow{\mathrm{id}_M \otimes i_n} \underset{R}{M \otimes Z_n}\right)}_{\text{need to show that } = M \otimes_R \mathrm{H}_n(C_*)} \to \mathrm{H}_n(\underset{R}{M \otimes} C_*) \to \underbrace{\ker\left(\underset{R}{M \otimes B_{n-1}} \xrightarrow{\mathrm{id}_M \otimes i_{n-1}} \underset{R}{M \otimes Z_{n-1}}\right)}_{\text{need to show that } = \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M)} \to 0.$$

We still have to show that the expressions left and right coincide with the desired expressions:

(1) We have natural isomorphisms

$$\operatorname{coker}\left(M \otimes_R B_n \xrightarrow{\mathrm{id}_M \otimes i_n} M \otimes_R Z_n\right) \xleftarrow[\uparrow]{\cong} \mathrm{Tor}_0^R(\mathrm{H}_n(C_*), M) \xrightarrow[\uparrow]{\cong} M \otimes_R \mathrm{H}_n(C_*)$$

by the Tor-Projective Resolution Proposition 314.22 since $0 \to B_n \xrightarrow{i_n} Z_n \to \mathrm{H}_n(C_*)$ is a free resolution of $\mathrm{H}_n(C_*)$

by the Tor_0^R–Lemma 314.23

One can easily verify that the resulting homomorphism $M \otimes_R \mathrm{H}_n(C_*) \to \mathrm{H}_n(M \otimes_R C_*)$ is the given homomorphism μ.

(2) Similarly to (1) we have a natural isomorphism

$$\ker\left(M \otimes_R B_{n-1} \xrightarrow{\mathrm{id}_M \otimes i_{n-1}} M \otimes_R Z_{n-1}\right) \xrightarrow[\uparrow]{\cong} \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M).$$

by the Tor-Projective Resolution Proposition 314.22 since $0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to \mathrm{H}_{n-1}(C_*)$ is a free resolution of $\mathrm{H}_{n-1}(C_*)$

In summary we have now finally shown the existence of the promised Q-left module homomorphism $\nu \colon \mathrm{H}_n(M \otimes_R C_*) \to \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M)$ and we have shown that μ and ν give rise to a short exact sequence. The statement that the homomorphism ν is natural follows easily from the definitions and the Homological Algebra–Fundamental Theorem 313.14 (2). We leave the verification of the details to the reader. ■

314.8. **Splittings of the Universal Coefficient Theorem.** We conclude this chapter with the following straightforward generalization of the Algebraic UCT-Splitting Proposition 162.17:

Proposition 314.34. (Algebraic UCT-Splitting Proposition) Let R be a PID, let (C_*, ∂_*) be a chain complex of free R-modules and let M be a (Q, R)-bimodule. For any $n \in \mathbb{N}_0$ the short exact sequence

$$0 \;\to\; M \otimes_R \mathrm{H}_n(C_*) \;\xrightarrow{\mu}\; \mathrm{H}_n(M \otimes_R C_*) \;\xrightarrow{\nu}\; \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M) \;\to\; 0$$

of the Algebraic Universal Coefficient Theorem 314.33 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(M \otimes_R C_*) \;\cong\; M \otimes_R \mathrm{H}_n(C_*) \oplus \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M)$$

of Q-left modules.

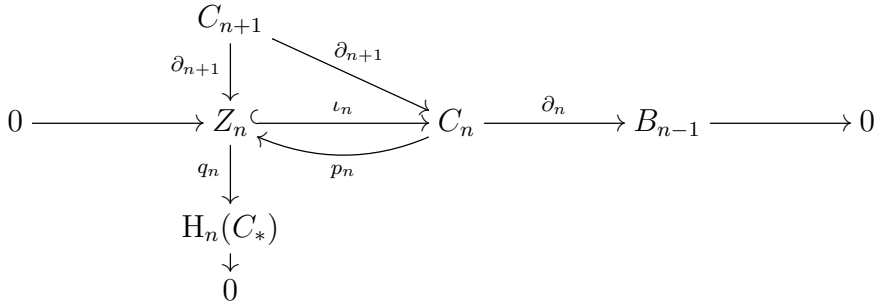
Proof. Let R be a PID and let (C_*, ∂_*) be a chain complex of free R-modules and let M be a (Q, R)-bimodule. According to the Module Splitting Lemma 309.31 (2)⇒(1) it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(M \otimes_R C_*) \to M \otimes_R \mathrm{H}_n(C_*)$ such that $\eta \circ \mu$ is the identity on $M \otimes_R \mathrm{H}_n(C_*)$. We summarize this problem as follows:

$$0 \longrightarrow M \otimes_R \mathrm{H}_n(C_*) \underset{\overset{\longleftarrow}{?\,\eta\,?}}{\xrightarrow{\mu}} \mathrm{H}_n(M \otimes_R C_*) \xrightarrow{\nu} \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_*), M) \longrightarrow 0.$$

For each $n \in \mathbb{N}_0$ we consider, as on page 6066, $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \xhookrightarrow{\iota_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

Note that $B_{n-1} \subseteq C_{n-1}$ is a submodule of a free R-module. Since R is by hypothesis a PID it follows from the Free Module–PID Theorem 309.35 that B_{n-1} is itself a free R-module. Therefore we can deduce from the Projective Module–Splitting Lemma 309.27 that the above short exact sequence splits. In particular, according to the Module Splitting Lemma 309.31 (1)⇒(2), for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. Again we summarize the setting as follows:



where q_n is just the natural projection.

Claim 1. The following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle q_{n+1}\circ p_{n+1}} & & \downarrow{\scriptstyle q_n \circ p_n} & & \downarrow{\scriptstyle q_{n-1}\circ p_{n-1}} & & \\
\cdots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{0} & \mathrm{H}_n(C_*) & \xrightarrow{0} & \mathrm{H}_{n-1}(C_*) & \longrightarrow & \cdots
\end{array}$$

Proof. The map down-right is evidently zero. On the other hand we also have

$$(q_n \circ p_n) \circ \partial_{n+1} \underset{\text{since } \iota_n \circ \partial_{n+1} = \partial_{n+1}}{\overset{\uparrow}{=}} q_n \circ p_n \circ \iota_n \circ \partial_{n+1} \underset{\text{since } p_n \circ \iota_n = \mathrm{id}_{Z_n}}{\overset{\uparrow}{=}} q_n \circ \partial_{n+1} \underset{\text{since } \mathrm{H}_n(C_*) = Z_n/\,\mathrm{im}(\partial_{n+1})}{\overset{\uparrow}{=}} 0.$$

The top and the bottom horizontal sequences of the claim are chain complexes of R-modules. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps. Now we tensor the top and the bottom horizontal sequences with M. From the Functor-applied-to-Chain Complex Lemma 314.3 together with the Tensor–Functor Lemma 314.4 we obtain the following diagram of chain complexes of Q-modules:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & M \otimes_R C_{n+1} & \xrightarrow{\mathrm{id}_M \otimes \partial_{n+1}} & M \otimes_R C_n & \xrightarrow{\mathrm{id}_M \otimes \partial_n} & M \otimes_R C_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \mathrm{id}_M \otimes (q_{n+1} \circ p_{n+1})} & & \downarrow{\scriptstyle \mathrm{id}_M \otimes (q_n \circ p_n)} & & \downarrow{\scriptstyle \mathrm{id}_M \otimes (q_{n-1} \circ p_{n-1})} & & \\
\ldots & \longrightarrow & M \otimes_R \mathrm{H}_{n+1}(C_*) & \xrightarrow{0} & M \otimes_R \mathrm{H}_n(C_*) & \xrightarrow{0} & M \otimes_R \mathrm{H}_{n-1}(C_*) & \longrightarrow & \ldots
\end{array}$$

The induced maps on the homology of these two chain complexes of Q-modules give us a homomorphism

$$\eta \colon \mathrm{H}_n(M \otimes_R C_*) \;\to\; \mathrm{H}_n(\text{lower sequence}) = M \otimes_R \mathrm{H}_n(C_*).$$

It remains to prove the following claim:

Claim 2. The composition $\eta \circ \mu$ is the identity on $M \otimes_R \mathrm{H}_n(C_*)$.

Proof. First let $m \otimes [\sigma] \in M \otimes_R \mathrm{H}_n(C_*)$ where $\sigma \in Z_n$. Then

$$(\eta \circ \mu)(m \otimes [\sigma]) \underset{\text{definition of } \mu}{=} \eta([m \otimes \sigma]) \underset{\text{definition of } \eta}{=} m \otimes (q_n(p_n(\sigma))) \underset{\text{since } p_n = \mathrm{id} \text{ on } Z_n}{=} m \otimes q_n(\sigma) \underset{\text{definition of } q_n}{=} m \otimes [\sigma].$$

The claim now follows from the observation that the Tensor Product–Basics Lemma 312.4 (2) tells us that the $m \otimes [\sigma]$ generate $M \otimes_R \mathrm{H}_n(C_*)$. ∎

314.9. Exercises for Chapter 314.

Exercise 314.1. Let $R = \mathbb{Z}[x, y]$ be the polynomial ring in two variables. We denote by ${}_{\mathrm{tr}}\mathbb{Z}$ the $R = \mathbb{Z}[x, y]$ module that is given by the abelian group $\mathbb{Z}$ where x and y act trivially on $\mathbb{Z}$. Show that the following sequence of $\mathbb{Z}[x, y]$-module homomorphisms

$$0 \;\to\; \mathbb{Z}[x, y] \xrightarrow{\binom{1-y}{x-1}} \mathbb{Z}[x, y]^2 \xrightarrow{(x-1, y-1)} \mathbb{Z}[x, y] \xrightarrow{x \mapsto 1, y \mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \;\to\; 0$$

is exact.

Exercise 314.2. Let $k \in \mathbb{N}$ and let $R = \mathbb{Z}[t]/(t^k - 1) \cong \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$ be the group ring of the group $\mathbb{Z}/k\mathbb{Z} = \langle t \mid t^k \rangle$. We denote by ${}_{\mathrm{tr}}\mathbb{Z}$ the $R = \mathbb{Z}[t]/(t^k - 1)$ module that is given by the abelian group $\mathbb{Z}$ where t acts trivially on $\mathbb{Z}$. Show that the following sequence is exact:

$$\ldots \to \mathbb{Z}[t]/(t^k - 1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k - 1) \xrightarrow{\cdot(1+t+\cdots+t^{k-1})} \mathbb{Z}[t]/(t^k - 1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k - 1) \xrightarrow{t \mapsto 1} \mathbb{Z} \to 0.$$

Remark. One possibility is the following: One can just use the obvious isomorphism $\mathbb{Z}[t]/(t^k - 1) \to \mathbb{Z}^k$ and write down explicit matrices for the homomorphisms that are given by $\cdot(t - 1)$ and $\cdot(1 + t + \cdots + t^{k-1})$.

Exercise 314.3. Let R be a ring and let P be a property of R-right modules. We say an R-right module M is *locally* P, if every finitely generated submodule of M has the property P.[3474]

(a) Show that if R is a PID, then every torsion-free R-module is locally free.
(b) Show that if G is a finite group, then every $\mathbb{Q}[G]$-module is locally projective.
(c) Show that if M is a (Q, R)-bimodule which as a right R-module is locally projective, then the covariant functor

$$A \mapsto M \otimes_R A$$

from the category $\mathcal{Mod}_{R\text{-left}}$ of R-left modules to the category $\mathcal{Mod}_{Q\text{-left}}$ of Q-left modules is exact.

Remark. As in the proof of the Tensor-with-Torsion Free Proposition 161.6 you should reduce this statement to the Tensor-with-Projective Proposition 314.17.

Exercise 314.4. Let R be a commutative ring. We say an R-module M is *flat over* R if the functor $N \mapsto R \otimes_R N$ is exact. Let R be a PID. Show that an R-module M is flat if and only if it is torsion-free.

Exercise 314.5. We consider the $\mathbb{Z}[t^{\pm 1}]$-chain complex

$$C_* \quad := \quad 0 \ \to \ \mathbb{Z}[t^{\pm 1}] \ \xrightarrow{\cdot(t-1)} \ \mathbb{Z}[t^{\pm 1}] \ \to \ 0.$$

Given $z \in \mathbb{C} \setminus \{0\}$ we consider the representation $\varphi_z\langle t\rangle \to \mathrm{GL}(1, \mathbb{C})$ that is given by $\varphi_z(t) = z$. This allows us to view $\mathbb{C}$ as a $\mathbb{Z}[t^{\pm 1}]$-module. Determine the homology groups of $\mathbb{C} \otimes_{\mathbb{Z}[t^{\pm 1}]} C_*$.

Exercise 314.6. Let V be a $\mathbb{C}[t^{\pm 1}]$-module that is finite dimensional as a $\mathbb{C}$-vector space. Furthermore let $z \in \mathbb{C} \setminus \{0\}$. We denote by ${}_z\mathbb{C}$ the $\mathbb{C}[t^{\pm 1}]$-module that is given by t acting by multiplication by z. Show that we have the following isomorphisms of $\mathbb{C}$-vector spaces:

$$\begin{aligned} {}_z\mathbb{C} \otimes_{\mathbb{C}[t^{\pm 1}]} V &\cong \{v \in V \,|\, t \cdot v = z \cdot v\} \\ \mathrm{Tor}_{\mathbb{C}[t^{\pm 1}]}(V, {}_z\mathbb{C}) &\cong \{v \in V \,|\, t \cdot v = z \cdot v\}. \end{aligned}$$

Hint. Use the PID–Module-Classification Theorem 309.36, the Cyclic-Tensor-Cyclic Lemma 314.28 and the Cyclic-Tor-Lemma 314.29.

Exercise 314.7. Let R be a ring and let $k \in \mathbb{N}_{\geq 2}$. Suppose that there exists an R-left module M and an R-right module N such that $\mathrm{Tor}_k^R(N, M) \neq 0$. Show that there exists a chain complex of free R-left modules such that the conclusion of the Algebraic Universal Coefficient Theorem 314.33 does not hold.

Exercise 314.8. We consider the ring $R = \mathbb{Z}[t]$. Give examples of two finitely generated $\mathbb{Z}[t]$-modules M and N such that $\mathrm{Tor}_2^{\mathbb{Z}[t]}(N, M) \neq 0$.

Exercise 314.9. We consider the ring $R = \mathbb{Z}[t]$.

- Let $\mathbb{Z}_{\mathrm{tr}}$ be the ring where t acts by multiplication by $+1$.
- Let $\mathbb{Z}_{\mathrm{tr}}^-$ be the ring where t acts by multiplication by -1.

Now perform the following calculations:

(a) Calculate the torsion modules $\mathrm{Tor}_n^{\mathbb{Z}[t]}(\mathbb{Z}_{\mathrm{tr}}, \mathbb{Z}_{\mathrm{tr}}^-)$.

[3474] This concept is closely related to the concept of a local property of a group, which we introduced in Exercise 100.24.

(b) Calculate the torsion modules $\mathrm{Tor}_n^{\mathbb{Z}[t]}(\mathbb{Z}_{\mathrm{tr}}^-, \mathbb{Z}_{\mathrm{tr}})$.

CHAPTER 315

Dualizing R-left chain complexes and Ext-modules

In Chapter 197 we introduced cochain complexes of abelian groups and in Chapter 201 we introduced Ext-modules and we formulated and proved the Universal Coefficient Theorem for Cohomology Groups 201.

In this chapter we will try to generalize the concepts and results from these earlier chapters from abelian groups to modules over rings. This chapter is organized as follows:

- In Section 315.1 we will study the group of homomorphisms between modules.
- In Section 315.2 we will introduce R-left cochain complexes.
- In Section 315.3 we will introduce duals of chain complexes.
- In Section 315.4 we will discuss the exactness of contravariant functors.
- In Section 315.5 we will introduce Ext-modules.
- In Section 315.6 we will perform several explicit calculations of Ext-modules.
- In Section 315.7 we will formulate and prove the Universal Coefficient Theorem for Cohomology modules over PIDs.
- In Section 315.8 we will provide exercises for Chapter 315.

315.1. **The group of homomorphisms between modules.** In this very straightforward section we will collect a few very elementary definitions and statements on homomorphisms between modules. We start out by introducing the following notation:

Notation 315.1. Let R be a ring. Given two R-left modules A and B we denote by $\mathrm{Hom}_{R\text{-left}}(A,B)$ the abelian group of all R-left module homomorphisms $A \to B$. Similarly we define $\mathrm{Hom}_{R\text{-right}}(A,B)$ in the obvious way.

Lemma 315.2. (Hom–Module Lemma) Let Q, R and S be rings, let M be an (R,S)-bimodule and let A be an (R,Q)-bimodule. The map

$$\begin{array}{ccc} \mathrm{Hom}_{R\text{-left}}(A,M) \times S & \to & \mathrm{Hom}_{R\text{-left}}(A,M) \\ (\varphi, s) & \mapsto & (a \mapsto \varphi(a) \cdot s) \end{array}$$

is a well-defined S-right module structure on the abelian group $\mathrm{Hom}_{R\text{-left}}(A,M)$.[3475] [3476] Furthermore the map

$$\begin{array}{ccc} Q \times \mathrm{Hom}_{R\text{-left}}(A,M) & \to & \mathrm{Hom}_{R\text{-left}}(A,M) \\ (q, \varphi) & \mapsto & (a \mapsto \varphi(a \cdot q)) \end{array}$$

is a well-defined Q-left module structure on the abelian group $\mathrm{Hom}_{R\text{-left}}(A,M)$. Together these two structures define a (Q,S)-left left module structure on $\mathrm{Hom}_{R\text{-left}}(A,M)$.

[3475] It is a good mental exercise to convince oneself that $\mathrm{Hom}(A,M)$ does not have an "obvious" R-left module structure.

[3476] Let R be a commutative ring. We can then view any R-left module M also an (R,R)-left left module. Thus we see that in this setting $\mathrm{Hom}(A,M)$ does have a natural R-left module structure.

Proof. The lemma follows easily from the definitions. But let us go through the most confusing part of the lemma, namely the observation that the Q-right module structure on A turns $\mathrm{Hom}_{R\text{-left}}(A, M)$ into a Q-left module. Indeed, given $q, \widetilde{q} \in Q$ and $\varphi \in \mathrm{Hom}_{R\text{-left}}(A, M)$ we see that for any $a \in A$ we have the following equalities in M:

$$((q \cdot \widetilde{q}) \cdot \varphi)(a) \;=\; \varphi(a \cdot (q \cdot \widetilde{q})) \;=\; \varphi((a \cdot q) \cdot \widetilde{q}) \;=\; ((\widetilde{q} \cdot \varphi)(a \cdot q) \;=\; q \cdot (\widetilde{q} \cdot \varphi))(a). \quad \blacksquare$$

We move straight to induced maps:

Definition 315.3. Let R and S be two rings, let M be an (R, S)-left left module and let $f\colon A \to B$ be a homomorphism of R-left modules. Then f induces a homomorphism

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}_{R\text{-left}}(B, M) & \to & \mathrm{Hom}_{R\text{-left}}(A, M) \\ (\varphi\colon B \to M) & \mapsto & \left(\begin{array}{rcl} \varphi \circ f\colon A & \to & M \\ a & \mapsto & \varphi(f(a)) \end{array} \right) \end{array}$$

of S-left modules.

On several occasions we will need the following lemma:

Lemma 315.4. (Homomorphism-Change Ring Lemma) Let R, S and T be rings. Let $\varphi\colon R \to S$ be a ring homomorphism, let C be an S-left module and let M be an (S, T)-left left module.

(1) The map

$$\begin{array}{rcl} \mathrm{Hom}_{R\text{-left}}(R \otimes_S C, {}_\varphi M) & \to & \mathrm{Hom}_{S\text{-left}}(C, M) \\ \gamma & \mapsto & (c \mapsto \gamma(1_R \otimes c)) \end{array}$$

is a well-defined natural homomorphism of T-left modules.

(2) If C is a free S-left module, then the above map is in fact an isomorphism.

Remark. In Exercise 315.1 we will see that in general the conclusion of the Homomorphism-Change Ring Lemma 315.4 (2) does not hold, if C is not a free S-left module. □

Proof. One can easily verify that the map is well-defined. If C is a free S-left module, then using the Free Module–Tensor-with-Ring Lemma 312.18 (1) and using the definition of a free module one can easily verify that the map is an isomorphism. ■

Next we recall Definition 197.3 for a contravariant functor to be additive:

Definition 315.5. Let R and S be rings. A contravariant functor $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{S\text{-left}}$ from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{S\text{-left}}$ of S-left modules is called additive if for any two R-left modules A and B the map

$$\begin{array}{rcl} \mathrm{Hom}_{R\text{-left}}(A, B) & \to & \mathrm{Hom}_{S\text{-left}}(F(A), F(B)) \\ \varphi & \mapsto & F(\varphi) \end{array}$$

is a group homomorphism.

With this definition we can now formulate the following generalization of the Hom–Contravariant Functor Lemma 197.4:

Lemma 315.6. (Hom–Contravariant Functor Lemma) Let R and S be rings.

(1) Given an (R, S)-left left module M the maps

$$\begin{array}{rcl} A & \mapsto & \mathrm{Hom}_{R\text{-left}}(A, M) \\ (f\colon A \to B) & \mapsto & (f^*\colon \mathrm{Hom}_{R\text{-left}}(B, M) \to \mathrm{Hom}_{R\text{-left}}(A, M)) \end{array}$$

define an additive *contravariant* functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{S\text{-left}}$ of S-left modules.

Proof. Both statements follow easily from the definitions. ■

We continue with the obvious analog of the Hom–Direct Sum-Product Lemma 197.6:

Lemma 315.7. (Hom–Direct Sum-Product Lemma) Let R and S be rings.

(1) Let M be an (R,S)-left left module. For any family $\{A_i\}_{i\in I}$ of R-left modules the natural map
$$\begin{aligned}\operatorname{Hom}_{R\text{-left}}\Big(\bigoplus_{i\in I} A_i, M\Big) &\to \prod_{i\in I} \operatorname{Hom}_{R\text{-left}}(A_i, M)\\ f &\mapsto \prod_{i\in I} f|_{A_i}\end{aligned}$$
is an isomorphism of S-left modules.

(2) For any R-left module and any family $\{M_i\}_{i\in I}$ of (R,S)-left left modules the natural map
$$\begin{aligned}\operatorname{Hom}_{R\text{-left}}\Big(A, \prod_{i\in I} M_i\Big) &\to \prod_{i\in I} \operatorname{Hom}_{R\text{-left}}(A, M_i)\\ f &\mapsto \prod_{i\in I} (\mathrm{pr}_i \circ f)\end{aligned}$$
(pr_i: natural projection to M_i)
is an isomorphism.[3477]

Proof.

(1) Note that for each $j \in I$ we have, by definition of a direct sum, a natural inclusion $\iota_j\colon A_j \to \bigoplus_{i\in I} A_i$. We get an induced map $\iota_j^*\colon \operatorname{Hom}_{R\text{-left}}\big(\bigoplus_{i\in I} A_i, M\big) \to \operatorname{Hom}_{R\text{-left}}(A_j, M)$. By definition of the product these homomorphisms give rise to a homomorphism
$$\operatorname{Hom}_{R\text{-left}}\Big(\bigoplus_{i\in I} A_i, M\Big) \to \prod_{j\in I} \operatorname{Hom}_{R\text{-left}}(A_j, M).$$
Using the Coproduct-of-Modules Lemma 309.11 one can easily verify that this map is an isomorphism.

(2) This statement follows easily from the definitions. ■

The following lemma is an analog of the Free Module–Tensor-with-Ring Lemma 312.18 and it is a generalization of the Free Module–Hom-to-Ring Lemma 197.7:

Lemma 315.8. (Free Module–Hom-to-Ring Lemma) Let $\varphi\colon R \to S$ be a ring homomorphism. We use φ to view S as an R-left module and as an (R,S)-bimodule.

(1) (a) Let V be an R-left module. Recall that by the Hom–Module Lemma 315.2 we can view $\operatorname{Hom}_{R\text{-left}}(V,S)$ as an S-right module. Now assume that V is a free R-left module with finite ordered basis $(v_1,\ldots,v_k)$. Then there exist unique $v_1^*,\ldots,v_k^* \in \operatorname{Hom}_{R\text{-left}}(V,S)$ with $v_i^*(v_j) = \delta_{ij}$, and $(v_1^*,\ldots,v_k^*)$ form an ordered basis of the S-right module $\operatorname{Hom}_{R\text{-left}}(V,S)$, called the dual basis.

(b) For each $k \in \mathbb{N}_0$ the map
$$\begin{aligned}\Theta_k\colon \operatorname{Hom}_{R\text{-left}}(R^k, S) &\to S^k\\ (f\colon R^k \to S) &\mapsto (f(\mathfrak{e}_1),\ldots,f(\mathfrak{e}_k))\end{aligned}$$

[3477] If I is finite, then the direct product is the same as the direct sum.

is an isomorphism of S-right modules.

(2) Let A be an $(m \times n)$-matrix over R. With this notation the following diagram of S-right module homomorphisms commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{R\text{-left}}(R^{1\times m}, S) & \xrightarrow[\cong]{\Theta_m} & S^{m\times 1} \\ \downarrow{\scriptstyle (v\mapsto v\cdot A)^*} & & \downarrow{\scriptstyle w\mapsto \varphi(A^{\mathrm{T}})\cdot w} \\ \mathrm{Hom}_{R\text{-left}}(R^{1\times n}, S) & \xrightarrow[\cong]{\Theta_n} & S^{n\times 1}. \end{array}$$

Proof.

(1) (a) It follows immediately from the definition of a basis that we have uniquely defined $v_1^*, \ldots, v_k^*$ that are easily seen to be linearly independent. By Exercise 309.4 it remains to show that they also span all of $\mathrm{Hom}_{R\text{-left}}(V, S)$. Thus let $\theta \in \mathrm{Hom}_{R\text{-left}}(V, S)$. Given any $i \in \{1, \ldots, k\}$ we set $s_i := \theta(v_i) \in S$. We claim that $\theta = v_1^* \cdot s_1 + \cdots + v_k^* \cdot s_k$. In fact for any $v \in V$ we can write $v = r_1 \cdot v_1 + \cdots + r_k \cdot v_k$ and we thus see that

$$\begin{aligned} \theta(v) &= \theta\Big(\sum_{i=1}^k r_i \cdot v_i\Big) &&= \sum_{i=1}^k r_i \cdot \theta(v_i) &&= \sum_{i=1}^k r_i \cdot s_i \\ &= \sum_{i=1}^k r_i \cdot \sum_{j=1}^k \underbrace{v_j^*(v_i) \cdot s_i}_{= v_j^*(v_i)\cdot s_j} &&= \Big(\sum_{j=1}^k v_j^* \cdot s_j\Big)\Big(\underbrace{\sum_{i=1}^k r_i \cdot v_i}_{= v}\Big) &&= \Big(\sum_{j=1}^k v_j^* \cdot s_j\Big)(v). \end{aligned}$$

(b) This statement follows easily from (a) and the fact that R^k is a free S-left module on the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_k$.

(2) We introduce some notation:

- We denote by $\widetilde{\mathfrak{e}}_1, \ldots, \widetilde{\mathfrak{e}}_k$ the standard S-basis of S^k.
- We denote by $\pi_i \colon R^m \to R$ the projection onto the i-th coordinate.
- Let $\Psi \colon R^m \to R^n$ be the map that is given by multiplication by $A = (a_{ij})$.

For $i \in \{1, \ldots, m\}$ we calculate that we have the following equality in S^n:

$$\begin{aligned} (\Theta_n \circ \Psi^* \circ \Theta_m^{-1})(\widetilde{\mathfrak{e}}_i) &\overset{\text{since } \Theta_m(\pi_i) = \widetilde{\mathfrak{e}}_i}{=} \Theta_n(\Psi^*(\pi_i)) \\ &\overset{\text{definition of } \Theta_n}{=} \sum_{j=1}^n (\Psi^*\pi_i)(\mathfrak{e}_j)\cdot\widetilde{\mathfrak{e}}_j \\ &\overset{\text{definition of } \Psi^*}{=} \sum_{j=1}^n \pi_i(\Psi(\mathfrak{e}_j))\cdot\widetilde{\mathfrak{e}}_j \\ &\overset{\text{definition of } \Psi}{=} \sum_{j=1}^n \pi_i\Big(\sum_{k=1}^n a_{kj}\cdot\mathfrak{e}_k\Big)\cdot\widetilde{\mathfrak{e}}_j \\ &\overset{\substack{\text{by definition of } \pi_i \text{ and since for } a\in R,\ v\in S^n \text{ we have } a\cdot v = \varphi(a)\cdot v}}{=} \sum_{j=1}^n \varphi(a_{ij})\cdot\widetilde{\mathfrak{e}}_j \\ &\overset{\text{definition of } A^{\mathrm{T}}}{=} \varphi(A^{\mathrm{T}})\cdot\widetilde{\mathfrak{e}}_i. \end{aligned}$$

■

The following lemma allows us to express some homomorphism modules in terms of duals and tensor products:

Lemma 315.9. (Hom-Dual-Tensor Lemma) Let R and S be rings, let F be a free S-left module of finite rank and let V be an (R, S)-left left module. The map

$$\begin{array}{rcl} \mathrm{Hom}_{S\text{-left}}(F, S) \otimes_S V & \to & \mathrm{Hom}_{S\text{-left}}(F, V) \\ \varphi \otimes v & \mapsto & (f \mapsto \varphi(f) \cdot v) \end{array}$$

is a natural isomorphism of R-left modules.

Proof. First note that it follows easily from the Tensor Product–Universal Property Lemma 320.5 that this map is well-defined. The fact that the map is an isomorphism is a fairly straightforward consequence of the Hom–Direct Sum-Product Lemma 315.7 (1) (here we use that F is a free S-left module of *finite* rank), the Free Module–Hom-to-Ring Lemma 315.8 and the Tensor–Direct Sum Lemma 312.15. We leave all details to the reader. ■

The following lemma is similar to the Tensor-Split-SES Lemma 314.9 and it is a generalization of the Hom-Split-SES Lemma 197.8:

Lemma 315.10. (Hom-Split-SES Lemma) Let R and S be rings and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-left modules and let M be an (R,S)-left left module. If the short exact sequence splits (by the Projective Module–Splitting Lemma 309.27 this is the case if C is a projective R-left module), then

$$0 \to \mathrm{Hom}_{R\text{-left}}(C,M) \xrightarrow{i^*} \mathrm{Hom}_{R\text{-left}}(B,M) \xrightarrow{p^*} \mathrm{Hom}_{R\text{-left}}(A,M) \to 0$$

is also a short exact sequence of S-left modules and it also splits as a short exact sequence of S-left modules.

Proof. We start out with the following claim which already appeared in the proof of the Tensor-Split-SES Lemma 314.9:

Claim. Let T be a ring and let $0 \to F \xrightarrow{\alpha} M \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between T-left modules. The following two statements are equivalent:

(1) The sequence is exact and the sequence splits.
(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma\colon M \to F$ and $\tau\colon H \to M$ of T-left modules such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_M$.

Proof. The (1)⇒(2) Statement follows almost immediately from the Module Splitting Lemma 309.31. Conversely the (2)⇒(1)-Statement can easily be verified by hand. ⊞

Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of R-left modules that splits and let M be an (R,S)-left left module. We pick homomorphisms $\sigma\colon B \to A$ and $\tau\colon C \to B$ of R-left module homomorphisms as in the claim. By the Hom–Contravariant Functor Lemma 315.6 we know that $D \mapsto \mathrm{Hom}_{R\text{-left}}(D,M)$ is an *additive contravariant* functor. It follows that $\alpha^* \circ \sigma^* = \mathrm{id}_{\mathrm{Hom}(A,M)}$, $\tau^* \circ \beta^* = \mathrm{id}_{\mathrm{Hom}(C,M)}$ and $\sigma^* \circ \alpha^* + \beta^* \circ \tau^* = \mathrm{id}_{\mathrm{Hom}(B,M)}$. Note that this shows that the two homomorphisms $\tau^*\colon \mathrm{Hom}_{R\text{-left}}(B,M) \to \mathrm{Hom}_{R\text{-left}}(C,M)$ and $\sigma^*\colon \mathrm{Hom}_{R\text{-left}}(B,M) \to \mathrm{Hom}_{R\text{-left}}(A,M)$ of S-left modules have the desired properties for the sequence

$$0 \to \mathrm{Hom}_{R\text{-left}}(M,C) \xrightarrow{p^*} \mathrm{Hom}_{R\text{-left}}(B,M) \xrightarrow{i^*} \mathrm{Hom}_{R\text{-left}}(A,M) \to 0.$$

It follows from the claim that this sequence is exact and that it splits. ■

315.2. **Cochain complexes.** We start out with the following generalization of Definition 197.9:

Definition 315.11. Let R be a ring.

(1) A R-left cochain complex is a sequence[3478]
$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \dots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$
of homomorphisms between R-left modules such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as coboundary maps.

(2) We define the n-th cohomology module of an R-left cochain complex (D^*, δ_*) to be the R-left module
$$\mathrm{H}^n(D^*) := \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

We continue with the following generalization of Definition 143.2:

Definition 315.12. Let R be a ring. A cochain map $f \colon C^* \to D^*$ between two R-left cochain complexes consists of a family $\{f_n \colon \mathrm{C}^n \to D^n\}_{n \in \mathbb{N}_0}$ of homomorphisms of R-left modules that commute with the coboundary maps, i.e. we demand that the following diagram commutes:
$$\begin{array}{ccccccccc}
\xleftarrow{\delta_{n+1}} & C^{n+1} & \xleftarrow{\delta_n} & C^n & \xleftarrow{\delta_{n-1}} & C^{n-1} & \xleftarrow{\delta_{n-2}} & \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\xleftarrow[\delta_{n+1}]{} & D^{n+1} & \xleftarrow[\delta_n]{} & D^n & \xleftarrow[\delta_{n-1}]{} & D^{n-1} & \xleftarrow[\delta_{n-2}]{} &
\end{array}$$

The above definition naturally leads us to the following definition:

Definition 315.13. Let R be a ring. We refer to category $\mathcal{CoChCplx}_{R\text{-left}}$ that is given by
$$\mathrm{Ob}(\mathcal{CoChCplx}_{R\text{-left}}) := \text{all } R\text{-left cochain chain complexes,}$$
and with morphisms
$$\mathrm{Mor}(C^*, D^*) := \text{all cochain maps from } C^* \text{ to } D^*.$$
with the obvious composition of morphism as the category of R-left cochain complexes.

The following lemma is an analog of the Chain Complex-H_*-Functor Lemma 313.3 and it is a generalization of the Cochain Complex-H^*-Functor Lemma 197.12:

Lemma 315.14. (Cochain Complex-H^*-Functor Lemma) Let R be a ring.

(1) If $f \colon C^* \to D^*$ is a cochain map between R-left cochain complexes C^* and D^*, then the map
$$\begin{array}{rcl} f_* \colon \mathrm{H}^n(C^*) & \to & \mathrm{H}^n(D^*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$
is a well-defined homomorphism of R-left homomorphisms.

(2) Given $n \in \mathbb{N}_0$ the maps
$$\begin{array}{rcl} C^* & \mapsto & \mathrm{H}^n(C^*) \\ (f \colon C^* \to D^*) & \mapsto & (f_* \colon \mathrm{H}^n(C^*) \to \mathrm{H}^n(D^*)) \end{array}$$
define a covariant functor from the category $\mathcal{CoChCplx}_{R\text{-left}}$ of R-left cochain complexes to the category $\mathcal{Mod}_{R\text{-left}}$ of R-left modules.

Proof. The proof is basically identical to the proof of the Cochain Complex-H^*-Lemma 77.4 and of the Chain Complex-H_*-Functor Lemma 143.4. ∎

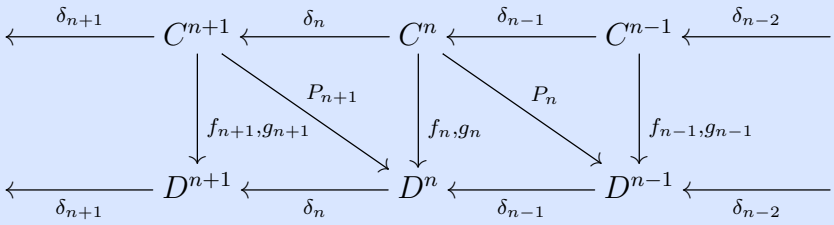
We continue with the following definition which is modelled on the Definitions 144.4 and (2).

Definition 315.15. Let R be a ring.

(1) Let $f, g\colon C^* \to D^*$ be two cochain maps between two R-left cochain complexes C^* and D^*. A cochain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n\colon \mathrm{C}^n \to D^{n-1}\}_{n\in\mathbb{N}_0}$ of R-left module homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n \;=\; f_n - g_n.$$

The maps are illustrated in the following diagram:



If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are cochain homotopic.

(2) A cochain map $f\colon C^* \to D^*$ between two R-left cochain complexes C^* and D^* is called a cochain homotopy equivalence if there exists a cochain map $g\colon D^* \to C^*$ such that $f \circ g$ is cochain homotopic to id_{C^*} and $g \circ f$ is cochain homotopic to id_{D^*}.

The following lemma is a generalization of the Cochain Homotopy-H*-Lemma 197.14:

Lemma 315.16. (Cochain Homotopy-H*-Lemma) Let R be a ring.

(1) If two cochain maps between R-left cochain complexes are cochain homotopic, then they induce the same map on cohomology modules.

(2) Every cochain homotopy equivalence induces an isomorphism of cohomology modules.

Proof. Both statements follow immediately from the Cochain Homotopy-H*-Lemma 197.14 since they have nothing to do with the R-left module structure. ■

The following is a generalization of Definition 197.15:

Definition 315.17. Let R be a ring and let

$$0 \;\to\; C^* \;\xrightarrow{f}\; B^* \;\xrightarrow{g}\; A^* \;\to\; 0$$

be a short exact sequence of R-left cochain complexes. This hypothesis means that for each $n \in \mathbb{N}_0$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n+1} & \xrightarrow{f_{n+1}} & B^{n+1} & \xrightarrow{g_{n+1}} & A^{n+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \\
0 & \longrightarrow & C^n & \xrightarrow{f_n} & B^n & \xrightarrow{g_n} & A^n & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are exact. Now we will define a connecting homomorphism

$$\delta_n\colon \mathrm{H}^n(A^*) \;\to\; \mathrm{H}^{n+1}(C^*).$$

The definition is almost verbatim the same as Definition 313.8 of the connecting homomorphism in homology. More precisely, we perform the following steps:

(1) Let $z \in \mathrm{H}^n(A)$.
(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) \coloneqq [\gamma] \in \mathrm{H}^{n+1}(C^*)$.

The following proposition is the obvious analog of the Connecting Homomorphism Lemma 313.9 and it is a generalization of the Cohomology–Connecting Homomorphism Lemma 197.16:

Lemma 315.18. (Cohomology–Connecting Homomorphism Lemma) Let R be a ring.

(1) The connecting homomorphisms are well-defined and they are homomorphisms of R-left modules.
(2) The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of R-left cochain complexes give rise to commuting diagrams of cohomology modules involving the connecting homomorphisms.

Proof. It follows from the Connecting Homomorphism Lemma 145.9 that the connecting homomorphism is well-defined and natural. The fact that the connecting homomorphism is an R-left module homomorphism follows easily from the definitions. ■

Not surprisingly we continue with the obvious generalization of the Cohomology–LES Proposition 197.17:

Proposition 315.19. (Cohomology–LES Proposition) Let R be a ring. If

$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$

is a short exact sequence of R-left cochain complexes, then

$$\ldots \to \mathrm{H}^n(C) \xrightarrow{f^*} \mathrm{H}^n(B) \xrightarrow{g^*} \mathrm{H}^n(A) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(C) \xrightarrow{f^*} \mathrm{H}^{n+1}(B) \xrightarrow{g^*} \ldots$$

forms an exact sequence.

Proof. This proposition actually follows immediately from the Cohomology–LES Proposition 197.17 since it has nothing to do with R-modules. ■

Finally the following lemma says that cohomology commutes with products. This lemma can be viewed as a cohomological analog of the Direct Sum-H_*-Lemma 313.4 and a generalization of the Product–H^*-Lemma 197.18:

Lemma 315.20. (Product–H^*-Lemma) Let R be a ring. If $\{C_a\}_{a \in A}$ is a family of R-left cochain complexes, then the natural projections $\prod_{a \in A} C_a \to C_b$, $b \in A$ (which come from the definition of a product) induce for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^n\Big(\prod_{a \in A} C_a\Big) \xrightarrow{\cong} \prod_{a \in A} \mathrm{H}^n(C_a)$$

of R-left modules.

Proof. Let $n \in \mathbb{N}_0$. Note that for each $b \in A$ the projection $\prod_{a \in A} C_a \to C_b$ induces a homomorphism

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \;\to\; \mathrm{H}^n(C_b).$$

It follows from the Definition 6.1 of a product that we get a well-defined homomorphism

$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \;\to\; \prod_{b\in A} \mathrm{H}^n(C_b)$$

of R-left modules. We leave it to the reader to verify that this map is in fact an isomorphism. ∎

315.3. **Dual R-left cochain complexes.** Now we will present one of the main sources of R-left cochain complexes:

Definition 315.21. Let R and S be rings. Let

$$(C_*, \partial_*)\colon \quad \to\; C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

be an R-left chain complex and let M be an (R,S)-left left module. For each $n \in \mathbb{N}_0$ we write

$$\delta_n \;:=\; \partial_{n+1}^*\colon \mathrm{Hom}_{R\text{-left}}(C_n, M) \;\to\; \mathrm{Hom}_{R\text{-left}}(C_{n+1}, M).$$

We obtain the following R-left cochain complex[3479]

$$\leftarrow \underset{R\text{-left}}{\mathrm{Hom}}(C_{n+1}, M) \xleftarrow{\delta_n} \underset{R\text{-left}}{\mathrm{Hom}}(C_n, M) \xleftarrow{\delta_{n-1}} \ldots \xleftarrow{\delta_1} \underset{R\text{-left}}{\mathrm{Hom}}(C_1, M) \xleftarrow{\delta_0} \underset{R\text{-left}}{\mathrm{Hom}}(C_0, M) \leftarrow 0.$$

We refer to it as R-left cochain complex dual to (C_*, ∂_*) and in the following we will denote it by $\mathrm{Hom}_{R\text{-left}}(C_*, M)$. We define the n-th cohomology module of the R-left chain complex (C_*, ∂_*) with M-coefficients as the cohomology of the R-left cochain complex $(\mathrm{Hom}_{R\text{-left}}(C_*, M), \delta_*)$, i.e. we define

$$\mathrm{H}^n(C; M) \;:=\; \frac{\ker\big(\delta_n\colon \underset{R\text{-left}}{\mathrm{Hom}}(C_n, M) \to \underset{R\text{-left}}{\mathrm{Hom}}(C_{n+1}, M)\big)}{\mathrm{im}\big(\delta_{n-1}\colon \underset{R\text{-left}}{\mathrm{Hom}}(C_{n-1}, M) \to \underset{R\text{-left}}{\mathrm{Hom}}(C_n, M)\big)}.$$

The following lemma is related in spirit to the Functor-applied-to-Chain Complex Lemma 314.3 and it is a generalization of the Dual Cochain Map Lemma 197.20:

Lemma 315.22. (Dual Cochain Map Lemma) Let R and S be rings and let M be an (R,S)-left left module.

(1) Let $f = \{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map between R-left chain complexes C_* and D_*, then the maps

$$\begin{aligned} f^*\colon \mathrm{Hom}_{R\text{-left}}(D_n, M) &\;\to\; \mathrm{Hom}_{R\text{-left}}(C_n, M) \\ (\varphi\colon D_n \to M) &\;\mapsto\; \underbrace{\big(C_n \xrightarrow{f_n} D_n \xrightarrow{\varphi} M\big)}_{=\varphi\circ f_n = f_n^*(\varphi)} \end{aligned}$$

form a cochain map.

(2) The maps

$$\begin{aligned} C_* &\;\mapsto\; \mathrm{Hom}_{R\text{-left}}(C_*, M) \\ f &\;\mapsto\; f^* \end{aligned}$$

define a *contravariant* functor from the category $\mathcal{ChCplx}$ of R-left chain complexes to the category $\mathcal{CoChCplx}_{S\text{-left}}$ of S-left cochain complex.

[3479] It follows from the Hom–Contravariant Functor Lemma 315.6 that $\delta_n \circ \delta_{n-1} = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0$, i.e. this is indeed an R-left cochain complex.

(3) (a) Let $f, g\colon C_* \to D_*$ be two chain maps. If they are chain homotopic, then the induced cochain maps $f^*, g^*\colon \mathrm{Hom}_{R\text{-left}}(D_*, M) \to \mathrm{Hom}_{R\text{-left}}(C_*, M)$ are cochain homotopic.
(b) Let $f\colon C_* \to D_*$ be a chain map. If f is a chain homotopy equivalence, then $f^*\colon \mathrm{Hom}_{R\text{-left}}(D_*, M) \to \mathrm{Hom}_{R\text{-left}}(C_*, M)$ is a cochain homotopy equivalence.

Proof.

(1) Let $f = \{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map. This means that for each $n \in \mathbb{N}_0$ we have

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n.$$

It follows from the Hom–Contravariant Functor Lemma 315.6 that

$$\underbrace{\partial_n^*}_{=\delta_{n-1}} \circ f_{n-1}^* \;=\; f_n^* \circ \underbrace{\partial_n^*}_{=\delta_{n-1}}.$$

We have thus shown that f^* is a cochain map.
(2) This statement follows easily from the definitions.
(3) (a) Let $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g. Recall that this means that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n.$$

We set $Q_n := P_{n-1}^*\colon \mathrm{Hom}_{R\text{-left}}(D_n, M) \to \mathrm{Hom}_{R\text{-left}}(C_{n-1}, M)$. By the Hom–Contravariant Functor Lemma 315.6 we know that $A \mapsto \mathrm{Hom}_{R\text{-left}}(A, M)$ is an *additive contravariant* functor. This implies that

$$\underbrace{P_n^*}_{=Q_{n+1}} \circ \underbrace{\partial_{n+1}^*}_{=\delta_n} + \underbrace{\partial_n^*}_{=\delta_{n-1}} \circ \underbrace{P_{n-1}^*}_{=Q_n} \;=\; g_n^* - f_n^*.$$

This shows that the maps $Q_n, n \in \mathbb{N}_0$ form a cochain homotopy.
(b) This statement follows easily from (a). ∎

The following lemma is a generalization of the Cohomology-Coefficients-Functor Lemma 197.21:

Lemma 315.23. (Cohomology-Coefficients-Functor Lemma) Let R and S be rings, let A and B be (R, S)-left left modules and let $f\colon A \to B$ be a homomorphism of (R, S)-left left modules.

(1) Given an R-left chain complex C_* and $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} A & \mapsto & \mathrm{H}^n(C_*; A) \\ (f\colon A \to B) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^n(C_*; A) & \to & \mathrm{H}^n(C_*; B) \\ [\varphi\colon C_n \to A] & \mapsto & [f \circ \varphi\colon C_n \to B] \end{array} \right) \end{array}$$

form a covariant functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{S\text{-left}}$ of S-left modules.
(2) The induced maps f_* define a natural transformation from the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; A)$ to the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; B)$.

Proof. Let C_* be an R-left chain complex and let $f\colon A \to B$ be a homomorphism of (R,A)-left left modules. One can easily verify that the maps

$$\begin{array}{rccc} f_*\colon & \mathrm{Hom}_{R\text{-left}}(C_n, A) & \to & \mathrm{Hom}_{R\text{-left}}(C_n, B) \\ & \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map. The two statements of the lemma now follow basically immediately from the definitions. ■

We conclude this section with the following generalization of the Kronecker Pairing Lemma 199.7 and the Ring–Kronecker Pairing Lemma 202.1:

Lemma 315.24. (Kronecker Pairing Lemma) Let R and S be rings, let C_* be an R-left chain complex and let M be an (R,S)-left left module.

(1) The map

$$\begin{array}{rccc} \langle\ ,\ \rangle\colon & \mathrm{H}^n(C;M) \times \mathrm{H}_n(C) & \to & M \\ & ([\varphi],[c]) & \mapsto & \langle [\varphi],[c]\rangle := \underset{\uparrow}{\varphi}(c) \end{array}$$

here $\varphi \in \mathrm{Hom}_{R\text{-left}}(C_n, M)$ and $c \in C_n$, so $\varphi(c) \in M$ is defined

is well-defined and $(S,\mathbb{Z})$-bilinear. We refer to this pairing as the Kronecker pairing.

(2) Let $f\colon C_* \to D_*$ be a chain map between two R-left chain complexes. For every choice of $c \in \mathrm{H}_n(C)$ and $\varphi \in \mathrm{H}^n(D;M)$ we have the following equality in M:

$$\langle f^*(\varphi), c\rangle \;=\; \langle \varphi, f_*(c)\rangle.$$

Proof. As in the case of the Kronecker Pairing Lemma 199.7 the lemma follows easily from the definitions. ■

315.4. **Contravariant functors and exactness.** In this section we will discuss when and in what sense contravariant are exact.

315.4.1. **Contravariant functors and exactness.** For the reader's convenience we recall Definition 201.1:

Definition 315.25. Let R and S be rings and let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{S\text{-left}}$ be an additive contravariant functor.

(1) We say that F is left-exact[3480] if for every exact sequence[3481]

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of R-left modules the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact.

(2) We say that F is right-exact, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

of R-left modules the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.
(3) We say that F is exact if it is left-exact and right-exact.

The following lemma is the analog of the Exact Functor–Homology Lemma 314.13 and it is a generalization of the Exact Functor–Cohomology Lemma 201.2:

Lemma 315.26. (Exact Functor–Cohomology Lemma) Let R and S be rings. Furthermore let $F\colon \mathcal{M}od_{R\text{-left}} \to \mathcal{M}od_{S\text{-left}}$ be an additive contravariant functor and let (C_*, ∂_*) be an R-left chain complex. If F is exact, then for any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{H}^k(\underbrace{F(C_*), F(\partial_*)}_{\substack{S\text{-left cochain}\\ \text{complex}}}) \to F(\mathrm{H}_k(\underbrace{C_*, \partial_*}_{\substack{R\text{-left cochain}\\ \text{complex}}})).$$

Proof. We leave it to the reader to modify the proof of the Exact Functor–Homology Lemma 314.13 to obtain our new lemma. ■

Example. On page 3851 we saw that the Hom-functor is in general not right-exact. More precisely, we showed that with $R = S = \mathbb{Z}$ and $M = \mathbb{Z}$ the functor $C \mapsto \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$ is not right-exact. □

The following lemma is an analog of the Tensor–Right-Exact Lemma 314.14 and it is a generalization of the Hom-Left-Exact Lemma 201.3:

Lemma 315.27. (Hom-Left-Exact Lemma) Let R and S be rings. For any (R, S)-left left module M the contravariant functor $A \mapsto \mathrm{Hom}_{R\text{-left}}(A, M)$ from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{R\text{-left}}$ of S-left modules is left-exact.

Proof. Let M be an (R, S)-left left module and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

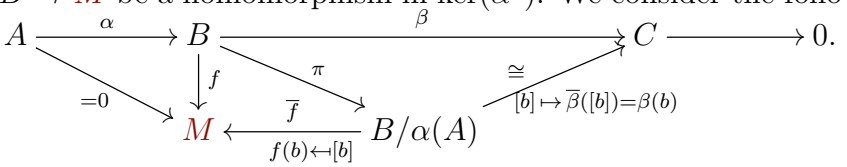
be an exact sequence of R-left modules. We need to show that the sequence

$$0 \longrightarrow \underset{R\text{-left}}{\mathrm{Hom}}(C, M) \xrightarrow{\beta^*} \underset{R\text{-left}}{\mathrm{Hom}}(B, M) \xrightarrow{\alpha^*} \underset{R\text{-left}}{\mathrm{Hom}}(A, M)$$

is also exact. It is clear that β^* is injective. Next note that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$, i.e. we have $\mathrm{im}(\beta^*) \subseteq \ker(\alpha^*)$. Thus it remains to prove the following claim:

Claim. We have $\ker(\alpha^*) \subseteq \mathrm{im}(\beta^*)$.

Proof. Let $f\colon B \to M$ be a homomorphism in $\ker(\alpha^*)$. We consider the following diagram:



[3480]Admittedly one could also refer to this definition as "right"-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.
[3481]As in the Additive Functor–Basics Lemma 160.2 we see that for an additive contravariant functor we have $F(0) = 0$.

We make the following observations:

(1) Since $f \in \ker(\alpha^*)$ we know that $\alpha^*(f) = 0$ which in turn implies that f vanishes on $\alpha(A)$. We denote by $\overline{f}\colon B/\alpha(A) \to M$ the resulting homomorphism given by $[b] \mapsto f(b)$.
(2) We denote by $\pi\colon B \to B/\alpha(A)$ the projection.
(3) It follows from the exactness of the original sequence and the Exactness Lemma 145.2 (3) that the homomorphism β induces an isomorphism $\overline{\beta}\colon B/\alpha(A) \to C$.
(4) By construction the above diagram commutes.
(5) Now we see that $f = \overline{f} \circ \overline{\beta}^{-1} \circ \beta$, i.e. we have $f = \beta^*(\overline{f} \circ \overline{\beta}^{-1})$. ■

315.4.2. **Injective modules.** In the Tensor-with-Projective Proposition 314.17 we showed that for any projective R-left module M the covariant functor $A \mapsto A \otimes_R M$ from the category $\mathit{Mod}_{R\text{-left}}$ of R-left modules to the category $\mathit{Mod}_{S\text{-left}}$ of S-left modules is exact. The question is, what condition on M do we have to impose so that $A \mapsto \mathrm{Hom}_{R\text{-left}}(A, M)$ is exact? It turns out that we have to replace the notion projective from Definition 309.25 by the notion injective. This leads us to the following generalization of Definition 201.4:

Definition 315.28. Let R be a ring. An R-left module M is called injective[3482] if for all R-left modules A and B, every injective homomorphism $\alpha\colon A \to B$ and every homomorphism $\mu\colon A \to M$ there exists a homomorphism $\nu\colon B \to M$ such that the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B \\ & & {\scriptstyle\mu}\downarrow & \swarrow_{\exists\nu} & \\ & & M. & & \end{array}$$

Example. Let $\mathbb{K}$ be a skew field. It follows easily from the Module-over-Skew Field Proposition 309.33 (3) that every $\mathbb{K}$-left vector space is an injective $\mathbb{K}$-vector field.

The following lemma is a generalization of the Hom-to-Injective Lemma 201.5:

Lemma 315.29. (Hom-to-Injective Lemma) Let R and S be rings and let M be an (R, S)-left left module. If M is an injective R-left module, then the contravariant functor $A \mapsto \mathrm{Hom}_{R\text{-left}}(A, M)$ from the category $\mathit{Mod}_{R\text{-left}}$ of R-left modules to the category $\mathit{Mod}_{S\text{-left}}$ of S-left modules is exact.

Proof. As in the Exact Functor Lemma 161.3 it suffices to show that for every short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of R-left modules the resulting sequence

$$0 \to \underset{R\text{-left}}{\mathrm{Hom}}(C, M) \xrightarrow{\beta^*} \underset{R\text{-left}}{\mathrm{Hom}}(B, M) \xrightarrow{\alpha^*} \underset{R\text{-left}}{\mathrm{Hom}}(A, M) \to 0$$

is also exact. Note that by the Hom-Left-Exact Lemma 315.27 it remains to prove that $\alpha^*\colon \mathrm{Hom}_{R\text{-left}}(B, M) \to \mathrm{Hom}_{R\text{-left}}(A, M)$ is an epimorphism. But this follows immediately from the fact that α is a monomorphism and the hypothesis that M is injective. ■

315.4.3. **Divisible modules.** The following definition generalizes Definition 201.6:

Definition 315.30. Let R be a ring. An R-left module M is called divisible if for every $m \in M$ and $r \in R \setminus \{0\}$ there exists an $n \in M$ with $r \cdot n = m$.

Examples.

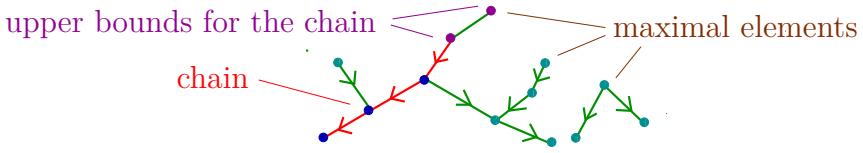
(1) The $\mathbb{Z}$-left modules $(\mathbb{Q},+),(\mathbb{R},+)$ and $(\mathbb{C},+)$ are clearly divisible.
(2) For every commutative domain R the corresponding quotient field $Q(R)$ is divisible as an R-ring.
(3) Every skew field $\mathbb{K}$ is clearly divisible as a $\mathbb{K}$-left module.
(4) Every quotient of a divisible module is easily seen to be divisible again. It thus follows from (2) that the quotient $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ and the quotient $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ are both divisible. □

The following proposition is a generalization of the Divisible Group-implies-Injective Proposition 201.7:

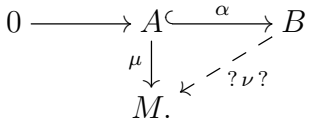
Proposition 315.31. (PID–Divisible-implies-Injective Proposition) Let R be a PID. Every R-left module that is divisible is also injective.

In the proof of the PID–Divisible-implies-Injective Proposition 315.31 we will need Zorn's Lemma which we now recall for the reader's convenience:

Lemma 1.29. (Zorn's Lemma) Suppose a partially ordered set $(P,\leq)$ has the property that every chain has an upper bound in P. If P is non-empty, then the set P contains at least one maximal element.

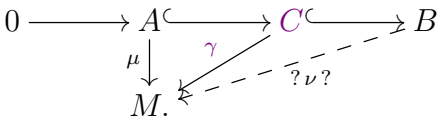


Proof of Proposition 315.31. Let M be an R-left module that is divisible. Suppose we are given a diagram



Since α is a monomorphism we can assume that A is in fact a submodule of B and that α is the inclusion map.

Let P be the set of all pairs (C,γ) where C is a submodule of B which contains A and where $\gamma\colon C\to M$ is a homomorphism such that $\gamma|_A=\mu$. Note that such pairs fit into the following diagram:



We define $(C_i,\gamma_i)\leq(C_j,\gamma_j)$ if $C_i\subseteq C_j$ and if $\gamma_j|_{C_i}=\gamma_i$. One can easily verify that this defines a partial order on P.

Claim 1. Every chain $(C_j,\gamma_j)_{j\in J}$ in P admits an upper bound.

Proof. Let $(C_j,\gamma_j)_{j\in J}$ be a chain in P. If we are dealing with the empty chain, then an upper bound is given by the pair $(A,\mu)\in P$. Now suppose that the chain $(C_j,\gamma_j)_{j\in J}$ in P

is non-empty. We get an upper bound by setting

$$C := \bigcup_{j\in J} C_j \qquad \text{and} \qquad \begin{array}{rccl} \gamma\colon C & \to & M \\ c & \mapsto & \gamma_j(c) \end{array}$$

where we pick $j \in J$ with $c \in C_j$

Since $(C_j, \gamma_j)_{j\in J}$ is a non-empty chain one can easily verify that C is in fact a submodule of B which contains A. Furthermore it follows easily from the definitions that γ is a homomorphism with $\gamma|_A = \mu$. ⊞

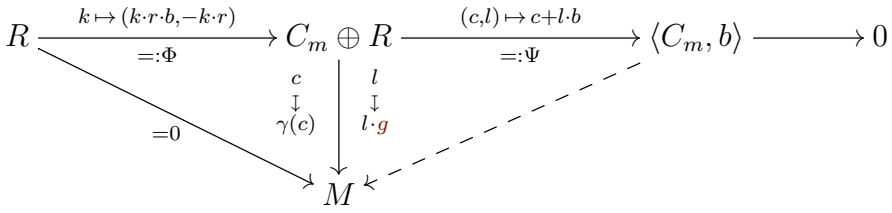
It follows from Claim 1 that we can apply Zorn's Lemma 1.29 to this partially ordered set $(P, \leq)$ and we obtain a maximal pair (C_m, γ_m). It remains to prove the following claim:

Claim 2. We have $C_m = B$.

Proof. Suppose that this is not the case. Then there exists a $b \in B \setminus C_m$. We denote by $\langle C_m, b\rangle$ the submodule of B that is generated by C_m and b, i.e. $\langle C_m, b\rangle$ is the smallest submodule of B that contains C_m and b. To obtain a contradiction it suffices to show that there exists a homomorphism $\delta\colon \langle C_m, b\rangle \to M$ that extends γ_m. We distinguish two cases:

(1) If there exists no $r \in R \setminus \{0\}$ such that $r \cdot b \in C_m$, then one can show easily that we have $C_m \oplus \langle b\rangle = \langle C_m, b\rangle$. We extend γ_m to $C_m \oplus \langle b\rangle = \langle C_m, b\rangle$ by setting $\delta(b) = 0$.

(2) We now suppose that there exists an $r \in R \setminus \{0\}$ such that $r \cdot b \in C_m$. We consider the ideal $I := \{r \in R \,|\, r \cdot b \in C_m\}$. By hypothesis this ideal is non-zero. Since R is a PID we can pick a generator r of that ideal. Since M is divisible there exists a $g \in M$ such that $r \cdot g = \gamma(r \cdot b)$. Next we consider the following little diagram:



We make the following observations:

(a) The horizontal sequence is exact: Indeed, it is clear that $\Psi \circ \Phi = 0$. Next note that the image of Ψ lies a priori in B, but it is straightforward to see that the image of Ψ is precisely $\langle C_m, b\rangle$. Finally note that it follows easily from our choice of r that we have $\ker(\Psi) \subseteq \mathrm{im}(\Phi)$.

(b) It follows from the Coproduct-of-Modules Lemma 309.11 that the vertical homomorphism is well-defined.

(c) One can easily verify that $\mathrm{im}(\Phi)$ is contained in the kernel of the vertical map.

(d) It follows from (a) and (c) that we get the desired extension $\langle C_m, b\rangle \to M$ of $\gamma\colon C_m \to M$.

This concludes the proof of the claim and thus of the lemma. ■

Later on one occasion we will need the following variation on the PID–Divisible-implies-Injective Proposition 315.31:

Proposition 315.32. (Torsion Free+Divisible-implies-Injective Proposition) Let R be a commutative ring. Any R-module that is divisible and torsion-free is injective.

Proof. This proposition is proved in [**Lam1999**, Proposition 3.25]. ■

Remark. Note that in the PID–Divisible-implies-Injective Proposition 315.31 one cannot drop the hypothesis that R is a PID and note that in the Torsion Free+Divisible-implies-Injective Proposition 315.32 we cannot drop the hypothesis that the module is torsion-free. For example it is pointed out in [**Lam1999**, p. 73] that for the ring $R = \mathbb{Z}[x]$ the module $M = \mathbb{Q}(x)/\mathbb{Z}[x]$ is divisible but not injective. □

315.4.4. **The evaluation homomorphism.** It follows from the Exact Functor–Cohomology Lemma 315.26 and the Hom-to-Injective Lemma 315.29 that for any R-left chain complex C_* and any injective R-left module M we have an isomorphism $\mathrm{H}^k(\mathrm{Hom}_{R\text{-left}}(C_*), M) \to \mathrm{Hom}_{R\text{-left}}(\mathrm{H}_k(C_*), M)$. Our next goal is to give an explicit isomorphism. This leads us to the following lemma, which is an analog of the Homology-Multiplication Lemma 314.19 and which generalizes the Evaluation Map Lemma 201.8:

Lemma 315.33. (Evaluation Map Lemma) Let R be a ring. Let (C_*, ∂_*) be an R-left chain complex and let M be an (R, S)-left left module. The map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C_*; M) & \to & \mathrm{Hom}_{R\text{-left}}(\mathrm{H}_n(C_*), M) \\ [\varphi\colon C_n \to M] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*) & \to & M \\ [c] & \mapsto & \underset{\substack{\uparrow \\ \text{Kronecker Pairing from the} \\ \text{Kronecker Pairing Lemma 315.24}}}{\langle [\varphi], [c] \rangle = \varphi(c)} \end{array} \right) \end{array}$$

is a well-defined natural homomorphism of S-left modules. It is called the evaluation homomorphism.

Proof. It follows from the Kronecker Pairing Lemma 315.24 (1) that this map is well-defined, that it is a homomorphism of S-left modules and that it is natural. ■

The following proposition is the cohomological analog of the Tensor-with-Projective Proposition 314.17 and it is a generalization of the Injective Coefficients Proposition 315.34:

Proposition 315.34. (Injective Coefficients Proposition) Let R be a ring, let (C_*, ∂_*) be an R-left chain complex, let M be an R-left module and let $n \in \mathbb{N}_0$. If M is an injective R-left module, then the map

$$\mathrm{ev}\colon \mathrm{H}^n(C_*; M) \to \mathrm{Hom}_{R\text{-left}}(\mathrm{H}_n(C_*), M)$$

from the Evaluation Map Lemma 315.33 is a natural isomorphism.

Proof. Similar to the proof of the Tensor-with-Projective Proposition 314.17 this proposition follows quite easily from the Exact Functor–Cohomology Lemma 315.26 and the Hom-to-Injective Lemma 315.29. We leave it to the reader to fill in the details. ■

315.5. **The Ext module.** The following definition is an analog of Definition 314.20 and a generalization of Definition 201.11:

Definition 315.35. Let R and S be rings. Let M be an (R, S)-left left module and let N be an R-left module. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0$$

the natural free resolution from Definition 313.12. We define the n-th M-Ext-module of N as the S-left module

$$\mathrm{Ext}^n_R(N, M) \;:=\; \mathrm{H}^n(\mathrm{Hom}_{R\text{-left}}(F_*, M)).$$

On page 6058 we saw that the modules $\mathrm{Tor}^R_n(N, M)$ are covariantly functorial in both M and N. Now we will see that the modules $\mathrm{Ext}^n_R(N, M)$ are contravariantly functorial in N and covariantly functorial in M:

Definition 315.36. Let R and S be rings.

(1) We fix an (R, S)-left left module M. Let $\alpha\colon N \to N'$ be a homomorphism between R-left modules. It follows from the Homological Algebra–Fundamental Theorem 313.14 together with the definition of the M-Ext modules that α induces a homomorphism

$$\alpha^*\colon \mathrm{Ext}^n_R(N', M) \;\to\; \mathrm{Ext}^n_R(N, M).$$

of S-left modules. It is now straightforward to show that the maps

$$\begin{aligned} N &\mapsto \mathrm{Ext}^n_R(N, M) \\ (\alpha\colon N \to N') &\mapsto (\alpha^*\colon \mathrm{Ext}^n_R(N', M) \to \mathrm{Ext}^n_R(N, M)) \end{aligned}$$

define a *contravariant* functor from the category $\mathcal{M}od_{R\text{-left}}$ of R-left modules to the category $\mathcal{M}od_{S\text{-left}}$ of S-left modules.

(2) We fix an R-left module N. If $\beta\colon M \to M'$ is a homomorphism between (R, S)-left left modules then one sees easily, using the Cohomology-Coefficients-Functor Lemma 315.23 (1), that β induces a homomorphism

$$\beta_*\colon \mathrm{Ext}^n_R(N, M) \;\to\; \mathrm{Ext}^n_R(N, M')$$

of S-left modules. Furthermore one can easily prove that the maps

$$\begin{aligned} M &\mapsto \mathrm{Ext}^n_R(N, M) \\ (\beta\colon M \to M') &\mapsto (\beta_*\colon \mathrm{Ext}^n_R(N, M) \to \mathrm{Ext}^n_R(N, M')) \end{aligned}$$

define a *covariant* functor from the category $\mathcal{M}od_{(R,S)\text{-left left}}$ of (R, S)-left left modules to the category $\mathcal{M}od_{S\text{-left}}$ of S-left modules.

The following proposition is the analog of the Tor-Free Resolution Proposition 314.22 and a generalization of the Ext-Projective Resolution Proposition 315.37:

Proposition 315.37. (Ext-Projective Resolution Proposition) Let R and S be rings, let M be an (R, S)-left left module and let N be an R-left module. Finally let

$$\ldots \;\to\; P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} N \;\to\; 0$$

be a projective resolution of N. For every $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{Ext}^n_R(N, M) \xrightarrow{\cong} \mathrm{H}^n(P_*, M)$$

of S-left modules which is natural in M.

Proof. The proof is almost the same as the proof of the Tor-Free Resolution Proposition 314.22. For the most part we just need to replace the Homology-with-Coefficients Lemma 314.6 (2) by the Dual Cochain Map Lemma 315.22 (3b) and at some point we need to make use of the Cochain Homotopy-H*-Lemma 315.16. We leave it to the reader to fill in the details. ■

In the Tor_0–Lemma 314.23 we saw that for all R-left modules M and N there exists a natural isomorphism $\mathrm{Tor}_0(N, M) \xrightarrow{\cong} N \otimes M$. The following lemma, which is a generalization of the Ext^0–Lemma 201.14, tells us that the Ext^0_R-modules are also classical objects:

Lemma 315.38. (Ext^0–Lemma) Let R and S be rings, let N be an R-left module and let M be an (R, S)-left left module. There exists a natural isomorphism

$$\mathrm{Ext}^0_R(N, M) \xrightarrow{\cong} \mathrm{Hom}_{R\text{-left}}(N, M)$$

of S-left modules.

Proof. We consider the natural free resolution of N:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0$$

We know from the Hom-Left-Exact Lemma 315.27 that the sequence

$$(*) \qquad 0 \to \mathrm{Hom}_{R\text{-left}}(N, M) \xrightarrow{f_0^*} \mathrm{Hom}_{R\text{-left}}(F_0, M) \xrightarrow{f_1^*} \mathrm{Hom}_{R\text{-left}}(F_1, M)$$

is exact. Now we obtain the following isomorphisms

$$\begin{aligned} \mathrm{Ext}^0_R(N, M) &\overset{\text{by Definition 315.35}}{=} \mathrm{H}^0\big(\cdots \leftarrow \mathrm{Hom}_{R\text{-left}}(F_1, M) \xleftarrow{f_1^*} \mathrm{Hom}_{R\text{-left}}(F_0, M) \leftarrow 0\big) \\ &= \ker\big(\mathrm{Hom}_{R\text{-left}}(F_1, M) \xleftarrow{f_1^*} \mathrm{Hom}_{R\text{-left}}(F_0, M)\big) \xrightarrow[\uparrow]{f_0^*} \mathrm{Hom}_{R\text{-left}}(N, M). \\ &\qquad\qquad \text{isomorphism since } (*) \text{ is exact} \end{aligned}$$

It is straightforward to verify that these isomorphisms are in fact natural. ∎

The following lemma plays the role of the Tor-Properties Lemma 314.24 and it is a generalization of the Ext-Properties Lemma 201.19:

Lemma 315.39. (Ext-Properties Lemma) Let R and S be rings, let M and $\{M_j\}_{j\in J}$ be (R, S)-left left modules and let N and $\{H_i\}_{i\in I}$ be R-left modules. The following statements hold:

(1) For any $k \in \mathbb{N}_0$ we have a natural isomorphism $\mathrm{Ext}^k_R\left(\bigoplus_{i\in I} H_i, M\right) \cong \prod_{i\in I} \mathrm{Ext}_R(H_i, M)$.
(2) For any $k \in \mathbb{N}$ we have a natural isomorphism $\mathrm{Ext}^k_R\left(N, \prod_{j\in J} M_j\right) \cong \prod_{j\in J} \mathrm{Ext}_R(N, M_j)$.
(3) If M is injective (e.g. if M is divisible), then for every $k \in \mathbb{N}$ we have $\mathrm{Ext}^k_R(N, M) = 0$.
(4) If N is a projective R-left module, then for every $k \in \mathbb{N}_0$ we have $\mathrm{Ext}^k_R(N, M) = 0$.

Proof.

(1) For each $i \in I$ we let F^i_* be the natural free resolution of H_i. One can easily verify that $\bigoplus_{i\in I} F^i_*$ is a free resolution of $\bigoplus_{i\in I} H_i$. For each $k \in \mathbb{N}_0$ we have the following natural

isomorphisms of S-left modules:

$$\begin{array}{ccc}
\operatorname{Ext}_R^k\Big(\bigoplus_{i\in I} H_i, M\Big) & \overset{\text{by the Ext-Free Resolution Proposition 315.37}}{\underset{\uparrow}{\overset{\downarrow}{\cong}}} & \mathrm{H}^k\Big(\operatorname{Hom}_{R\text{-left}}\Big(\bigoplus_{i\in I} F_*^i, M\Big)\Big) \\
& & \overset{\cong}{\underset{\uparrow}{\longleftarrow}} \prod_{i\in I} \mathrm{H}^k(\operatorname{Hom}_{R\text{-left}}(F_*^i, M)) \\
& & \text{by the Product–H}^* \text{ Lemma 315.20}
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}^k\Big(\operatorname{Hom}_{R\text{-left}}\Big(\bigoplus_{i\in I} F_*^i, M\Big)\Big) & \overset{\text{by the Direct Sum-Product Homomorphism Lemma 315.7 (1)}}{\underset{\uparrow}{\overset{\downarrow}{\cong}}} & \mathrm{H}^k\Big(\prod_{i\in I} \operatorname{Hom}_{R\text{-left}}(F_*^i, M)\Big) \\
& & = \prod_{i\in I} \operatorname{Ext}_R^k(H_i, M). \\
& & \text{by Definition 315.35}
\end{array}$$

(2) Let F_* be the natural resolution for N. For each $k \in \mathbb{N}_0$ we have the following natural isomorphisms of S-left modules:

$$\begin{array}{ccc}
\operatorname{Ext}_R^k\Big(N, \prod_{j\in J} M_j\Big) & \overset{\text{by Definition 315.35}}{\underset{\uparrow}{\overset{\downarrow}{=}}} & \mathrm{H}^k\Big(\operatorname{Hom}_{R\text{-left}}\Big(F_*, \prod_{j\in J} M_j\Big)\Big) \\
& & \overset{\cong}{\underset{\uparrow}{\longleftarrow}} \prod_{j\in J} \mathrm{H}^k(\operatorname{Hom}_{R\text{-left}}(F_*, M_j)) \\
& & \text{by the Product–H}^* \text{ Lemma 315.20}
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}^k\Big(\operatorname{Hom}_{R\text{-left}}\Big(F_*, \prod_{j\in J} M_j\Big)\Big) & \overset{\text{by the Direct Sum-Product Homomorphism Lemma 315.7 (2)}}{\underset{\uparrow}{\overset{\downarrow}{\cong}}} & \mathrm{H}^k\Big(\prod_{j\in J} \operatorname{Hom}_{R\text{-left}}(F_*, M_j)\Big) \\
& & = \prod_{j\in J} \operatorname{Ext}_R^k(N, M_j). \\
& & \text{by Definition 315.35}
\end{array}$$

(3) We need to show that if M is an injective R-left module, then for any R-left module N and every $k \in \mathbb{N}$ we have $\operatorname{Ext}_k^R(N, M) = 0$. To do so we consider the natural free resolution of N:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0.$$

We denote by F'_* the above sequence *with* N, i.e. $F'_i = F_i$ for $i \in \mathbb{N}_0$ and $F_{-1} = N$. Note that for each $k \in \mathbb{N}$ we have the following isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Ext}_R^k(N, M) & \overset{\text{by Definition 315.35}}{\underset{\uparrow}{=}} & \mathrm{H}^k(\operatorname{Hom}_{R\text{-left}}(F_*, M)) & \overset{\text{since } k\geq 1}{\underset{\uparrow}{=}} & \mathrm{H}^k(\operatorname{Hom}_{R\text{-left}}(F'_*, M)) \\
& \overset{\cong}{\underset{\uparrow}{}} & \operatorname{Hom}_{R\text{-left}}(\mathrm{H}_k(F_*), M) & \overset{\text{since } F'_* \text{ is exact}}{\underset{\uparrow}{=}} & \operatorname{Hom}_{R\text{-left}}(0, M) = 0. \\
& \text{by the Exact Functor–Cohomology Lemma 315.26 and the Hom-to-Injective Lemma 315.29} & & &
\end{array}$$

(4) Let N be a *projective* R-left module. Note that in this case $0 \to F_0 := N \xrightarrow{\mathrm{id}} N \to 0$ is already a free resolution. It follows immediately from the Ext-Projective Resolution Proposition 315.37 that for every $k \in \mathbb{N}$ we have $\operatorname{Ext}_R^k(N, M) = 0$. ∎

We conclude this section with the following analog of the Tor-Coefficient LES Lemma 314.25:

Lemma 315.40. (Ext-Coefficient LES Lemma) Let R and S be two rings. Furthermore let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of (R, S)-left left modules. For any R-left module N we get a long exact sequence

$$\ldots \to \operatorname{Ext}_R^k(N, A) \xrightarrow{\alpha_*} \operatorname{Ext}_R^k(N, B) \xrightarrow{\beta_*} \operatorname{Ext}_R^k(N, C) \xrightarrow{\delta_k} \operatorname{Ext}_R^{k+1}(N, A) \xrightarrow{\alpha_*} \ldots$$

of S-left modules.

Proof. We pick a free resolution F_* of the R-left module N. We obtain a sequence

$$0 \to \mathrm{Hom}_{R\text{-left}}(F_*, A) \xrightarrow{\alpha_*} \mathrm{Hom}_{R\text{-left}}(F_*, B) \xrightarrow{\beta_*} \mathrm{Hom}_{R\text{-left}}(F_*, C) \to 0$$

of cochain complexes of S-left modules. Since the F_* are free R-left modules one sees easily, say using the Hom–Direct Sum-Product Lemma 315.7, that the above sequence is in fact exact.

The desired statement now follows from the Cohomology–LES Proposition 315.19. ■

315.6. **Calculations of Ext-modules.** In Section 201.2 we saw that if the underlying ring is the ring $R = \mathbb{Z}$ of integers, then at least for finitely generated $\mathbb{Z}$-modules M and N we can calculate fairly easily all $\mathbb{Z}$-Ext modules $\mathrm{Ext}^{\mathbb{Z}}_n(M, N)$. In this section we will see that many of the results generalize from $\mathbb{Z}$-modules to modules over PIDs. But we will also see that the results do not generalize to non-PIDs.

315.6.1. **Ext-modules over PIDs.** We start out with the following lemma which is an analog of the PID-Higher Tor Lemma 314.27 and which is a generalization of the Higher Ext Lemma 201.16:

Lemma 315.41. (PID-Higher Ext Lemma) Let R be a PID. For every choice of R-left modules M and N we have $\mathrm{Ext}^n_R(N, M) = 0$ for $n \geq 2$.

Proof. By the Length 1–Free Resolution Lemma 314.26 we know that the R-left module N admits a free resolution of length 1:

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0.$$

It now follows that for $n \geq 2$ we have

$$\mathrm{Ext}^n_R(N, M) \underset{\text{by the Ext-Free Resolution Proposition 315.37}}{\cong} \mathrm{H}^n(\mathrm{Hom}(F_*, M)) \underset{\text{since } F_n = 0 \text{ for } n \geq 2}{=} 0.$$

■

Ideally one would like to calculate all Ext-modules of finitely generated modules over a PID. By the PID–Module-Classification Theorem 309.36 we know that every finitely generated module over a PID is a direct sum of cyclic modules. This fact, together with the PID-Higher Ext Lemma 315.41, the Ext^0–Lemma 315.38 and the Ext-Properties Lemma 315.39 shows that it remains to prove the following generalization of the $\mathbb{Z}/k\mathbb{Z}$-Ext-Lemma 201.18:

Lemma 315.42. (Cyclic-Ext-Lemma) Let R be a ring and let $k \in R$.

(1) For any R-left module M we have a natural isomorphism

$$\mathrm{Ext}^1_R(R/kR, M) \xrightarrow{\cong} M/k \cdot M.$$

(2) Given any $m \in \mathbb{N}$ we have $\mathrm{Ext}^1_R(R/kR, R/mR) \cong R/\gcd(k, m)R$.

Proof.

(1) We consider the free resolution $0 \to R \xrightarrow{r \mapsto r \cdot k} R \to R/kR \to 0$ of R/kR. Given any R-left module M this gives rise to the following diagram:

$$\begin{array}{ccccccccc} \underset{R\text{-left}}{\mathrm{Hom}}(R,M) & \xrightarrow{(r \mapsto r\cdot k)^*} & \underset{R\text{-left}}{\mathrm{Hom}}(R,M) & \longrightarrow & \overbrace{\mathrm{coker}\left(\underset{R\text{-left}}{\mathrm{Hom}}(R,M) \xrightarrow{(n \mapsto n\cdot k)^*} \underset{R\text{-left}}{\mathrm{Hom}}(R,M)\right)}^{\cong \mathrm{Ext}^1(R/kR,M)} & \longrightarrow & 0 \\ \varphi \mapsto \varphi(1) \downarrow \cong & & \varphi \mapsto \varphi(1) \downarrow \cong & & \downarrow & & \\ M & \xrightarrow{g \mapsto g\cdot k} & M & \longrightarrow & \mathrm{coker}(M \xrightarrow{g \mapsto g\cdot k} M) & \longrightarrow & 0. \end{array}$$

We make the following observations:

(a) Since $0 \to R \xrightarrow{r \mapsto r \cdot k} R \to R/kR \to 0$ is a free resolution of R/kR we obtain from the Ext-Projective Resolution Proposition 315.37 the isomorphism between the cokernel and the Ext-module that we show in the top right corner.

(b) The top horizontal sequence is exact by definition.

(c) The two vertical maps to the left are the natural isomorphisms. It follows immediately from the definitions that the square to the left commutes.

It now follows from a mild diagram chase that from the above we get an induced isomorphism $\mathrm{coker}\left(M \xrightarrow{g \mapsto g\cdot k} M\right) \xrightarrow{\cong} \mathrm{Ext}^1_R(R/kR, M)$. One can easily verify that this isomorphism is natural.

(2) By (1) we know that $\mathrm{Ext}^1_R(R/kR, R/mR) \cong \mathrm{coker}(R/mR \xrightarrow{h \mapsto h \cdot k} R/mR)$. Thus it suffices to prove the following claim:

Claim. The map

$$\begin{array}{rcl} \mathrm{coker}(R/mR \xrightarrow{h \mapsto h \cdot k} R/mR) & \to & R/\gcd(m,k)R \\ [a] & \mapsto & [a] \end{array}$$

is well-defined and it is an isomorphism.

Proof. Clearly any element of the form $[k \cdot b]$ gets sent to zero in $R/\gcd(m,k)R$, which implies that the map is well-defined. It is also clear that the map is a surjective. Finally, if we are given $a \in R$ with $[a] = 0 \in R/\gcd(m,k)R$, then $a = x \cdot m + y \cdot k$ for some $x, y \in R$. But then we see that $[a] = [x \cdot m + y \cdot k] = [y \cdot k] \in R/mR$ lies in the image of the map $R/mR \xrightarrow{h \mapsto h \cdot k} R/mR$. This shows that the map of the claim is also injective. ∎

315.6.2. **Ext-modules over non-PIDs.** In this subsection we will calculate some torsion Ext-modules over commutative rings that are not PIDs. More precisely, we will consider the rings $\mathbb{Z}[x,y]$ and $\mathbb{Z}[t]/(t^n - 1)$.

Example. We start out with the ring $R = \mathbb{Z}[x,y]$ and the module $N = {}_{\mathrm{tr}}\mathbb{Z}$, i.e. the $R = \mathbb{Z}[x,y]$-module that is given by the abelian group $\mathbb{Z}$ where x and y act trivially on $\mathbb{Z}$. By the Free Resolution-$\mathbb{Z}[x,y]$-Lemma 314.30 we know that the following sequence of $\mathbb{Z}[x,y]$-module homomorphisms

$$0 \to \mathbb{Z}[x,y] \xrightarrow{\binom{1-y}{x-1}} \mathbb{Z}[x,y]^2 \xrightarrow{(x-1, y-1)} \mathbb{Z}[x,y] \xrightarrow{x \mapsto 1, y \mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0$$

is a free resolution of the $\mathbb{Z}[x,y]$-module ${}_{\mathrm{tr}}\mathbb{Z}$. In the following we want to calculate $\mathrm{Ext}_n^{\mathbb{Z}[x,y]}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z})$. To do so we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \leftarrow & \underset{\mathbb{Z}[x,y]}{\mathrm{Hom}}(\mathbb{Z}[x,y], {}_{\mathrm{tr}}\mathbb{Z}) & \xleftarrow{\binom{1-y}{x-1}^*} & \underset{\mathbb{Z}[x,y]}{\mathrm{Hom}}(\mathbb{Z}[x,y]^2, {}_{\mathrm{tr}}\mathbb{Z}) & \xleftarrow{(x-1,y-1)^*} & \underset{\mathbb{Z}[x,y]}{\mathrm{Hom}}(\mathbb{Z}[x,y], {}_{\mathrm{tr}}\mathbb{Z}) & \leftarrow & 0 \\
 & & \cong\downarrow f\mapsto f(\mathfrak{e}_1) & & \cong\downarrow f\mapsto (f(\mathfrak{e}_1), f(\mathfrak{e}_2)) & & \cong\downarrow f\mapsto f(\mathfrak{e}_1) & & \\
0 & \leftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z}^2 & \xleftarrow{0} & \mathbb{Z} & \leftarrow & 0.
\end{array}$$

It follows from the Free Module–Hom-to-Ring Lemma 315.8 (1) and (2), applied to the ring homomorphism $\varphi\colon \mathbb{Z}[x,y] \xrightarrow{x,y\mapsto 1} \mathbb{Z}$, that the vertical maps are isomorphisms and that the diagram commutes. Thus we see that

$$\mathrm{Ext}_n^{\mathbb{Z}[x,y]}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z}) \underset{\substack{\uparrow\\ \text{by the Ext-Projective}\\ \text{Resolution Proposition 315.37}}}{\cong} \mathrm{H}^n(\text{above cochain complex}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have thus found an example where the second Ext-module is non-zero. □

We move straight to our second example:

Example. We consider the ring $R = \mathbb{Z}[t]/(t^k-1) \cong \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$ and the module $N = {}_{\mathrm{tr}}\mathbb{Z}$, i.e. the $R = \mathbb{Z}[t]/(t^k-1)$-module that is given by the abelian group $\mathbb{Z}$ where t acts trivially on $\mathbb{Z}$. By the Free Resolution-$\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-Lemma 314.31 the following sequence of $\mathbb{Z}[t]/(t^k-1)$-module homomorphisms

$$\ldots \to \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(1+t+\cdots+t^{k-1})} \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k-1) \xrightarrow{t\mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0.$$

is a free resolution of the Ext-module $\mathbb{Z}[t]/(t^k-1)$-module ${}_{\mathrm{tr}}\mathbb{Z}$. Our next goal is to calculate $\mathrm{Ext}_n^{\mathbb{Z}[t]/(t^k-1)}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z})$. To do so we consider the following diagram:

$$\begin{array}{ccccccccc}
 & \leftarrow & \underset{\mathbb{Z}[t]/(t^{k-1})}{\mathrm{Hom}}(\mathbb{Z}[t]/(t^k-1), {}_{\mathrm{tr}}\mathbb{Z}) & \xleftarrow{\cdot(1+\cdots+t^{k-1})^*} & \underset{\mathbb{Z}[t]/(t^{k-1})}{\mathrm{Hom}}(\mathbb{Z}[t]/(t^k-1), {}_{\mathrm{tr}}\mathbb{Z}) & \xleftarrow{\cdot(t-1)^*} & \underset{\mathbb{Z}[t]/(t^{k-1})}{\mathrm{Hom}}(\mathbb{Z}[t]/(t^k-1), {}_{\mathrm{tr}}\mathbb{Z}) & \leftarrow & 0 \\
 & & \cong\downarrow f\mapsto f(\mathfrak{e}_1) & & \cong\downarrow f\mapsto f(\mathfrak{e}_1) & & \cong\downarrow f\mapsto f(\mathfrak{e}_1) & & \\
\leftarrow & \xleftarrow{0} & \mathbb{Z} & \xleftarrow[\cdot(1+\cdots+1)]{\cdot k} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \leftarrow & 0.
\end{array}$$

It follows from the Free Module–Hom-to-Ring Lemma 315.8 (1) and (2), applied to the ring homomorphism $\varphi\colon \mathbb{Z}[t]/(t^n-1) \xrightarrow{t\mapsto 1} \mathbb{Z}$, that the vertical maps are isomorphisms and that the diagram commutes. Thus we see that

$$\mathrm{Ext}_n^{\mathbb{Z}[t]/(t^k-1)}({}_{\mathrm{tr}}\mathbb{Z}, {}_{\mathrm{tr}}\mathbb{Z}) \underset{\substack{\uparrow\\ \text{by the Ext-Projective}\\ \text{Resolution Proposition 315.37}}}{\cong} \mathrm{H}^n(\text{above cochain complex}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \text{ is odd}, \\ \mathbb{Z}/k\mathbb{Z}, & \text{if } n \text{ is even and} \neq 0. \end{cases}$$

We have thus found an example where all Ext-modules of even degree are non-zero. □

315.7. **The Universal Coefficient Theorem for Cohomology Groups over PIDs.** In this section we want to explore the connection between the cohomology modules of an

R-left chain complex (C_*, ∂_*) and the usual homology modules of the R-left chain complex. The key to relating homology modules to cohomology modules is given by the evaluation homomorphism which we introduced in the Evaluation Map Lemma 315.33:

Definition 315.43. Let R and S be rings, let (C_*, ∂_*) be an R-left chain complex and let M be an (R, S)-left left module. We refer to the S-left module homomorphism

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C_*; M) & \to & \mathrm{Hom}_{R\text{-left}}(\mathrm{H}_n(C_*), M) \\ [\varphi\colon C_n \to M] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*) & \to & M \\ [c] & \mapsto & \langle [\varphi], [c] \rangle = \varphi(c) \end{array} \right) \end{array}$$

as the evaluation homomorphism.

The following theorem is a generalization of the Algebraic Universal Coefficient-H*-Theorem 201.22 and also of the Universal Coefficient-H*–Field Theorem 202.2:

Theorem 315.44. (Algebraic Universal Coefficient-H*-Theorem) Let R be a PID and let S be a commutative ring. Let (C_*, ∂_*) be an R-chain complex of free R-modules and let M be an (R, S)-module. For each $n \in \mathbb{N}_0$ there exists a homomorphism

$$\epsilon\colon \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(C_*), M) \to \mathrm{H}^n(C_*; M)$$

of S-left modules which is natural in the R-chain complex C_* and natural in the (R, S)-module M and which has the property that the following sequence is exact:[3483]

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(C_*), M) \xrightarrow{\epsilon} \mathrm{H}^n(C_*; M) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_R(\mathrm{H}_n(C_*), M) \to 0.$$

This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^n(C_*; M) \cong \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(C_*), M) \oplus \mathrm{Hom}_R(\mathrm{H}_n(C_*), M).$$

The remainder of this section is occupied with the proof of the Algebraic Universal Coefficient-H*-Theorem 315.44. We break the proofs into two parts:

(1) First we show that the short exact sequences exist.
(2) Afterwards we show that the short exact sequences split.

Proof of the existence of short exact sequence. Let R be a PID and let (C_*, ∂_*) be an R-chain complex of free R-modules. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. Note that it follows from the Free Module–PID Theorem 309.35 that B_n and Z_n are free R-modules.

We consider the following short exact sequence of vertical R-chain complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Now let M be an (R, S)-module. We apply the functor $A \mapsto \mathrm{Hom}_R(A, M)$ to the above short exact sequence of R-chain complexes. This leads us to the following claim:

[3483] Recall that by Definition 309.34 a PID is commutative, so we do not have to distinguish between left and right module (homomorphism)s.

Claim. The following maps define a short exact sequence of "vertical" R-cochain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(B_{n-1}, M) & \xrightarrow{\partial_n^*} & \mathrm{Hom}_R(C_n, M) & \longrightarrow & \mathrm{Hom}_R(Z_n, M) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Hom}_R(B_{n-2}, M) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}_R(C_{n-1}, M) & \longrightarrow & \mathrm{Hom}_R(Z_{n-1}, M) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Dual Cochain Map Lemma 315.22 that this is a sequence of R-cochain complexes.

(2) Recall that the modules B_n are free R-modules, in particular they are projective R-modules. Note that it thus follows from the Projective Module–Splitting Lemma 309.27 that each of the horizontal sequences $0 \to Z_k \to C_k \to B_{k-1} \to 0$ in the original diagram splits. It follows from the Hom-Split-SES Lemma 315.10 that the above horizontal sequences of R-cochain complexes are still exact. ⊞

We apply the Cohomology–Connecting Homomorphism Lemma 315.18 to the short exact sequence of R-cochain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{\mathrm{Hom}_R(Z_n, M)}_{=\mathrm{H}^n(Z_*;M)} \to \underbrace{\mathrm{Hom}_R(B_n, M)}_{=\mathrm{H}^n(B_*;M)}.$$

In fact, it follows easily from Definition 315.17 that the connecting homomorphism $\mathrm{Hom}_R(Z_n, M) \to \mathrm{Hom}_R(B_n, M)$ is the map that is induced by the inclusion $i_n \colon B_n \to Z_n$. By the Cohomology–LES Proposition 315.19 this connecting homomorphism gives us the following long exact sequence:

$$\to \underset{R}{\mathrm{Hom}}(Z_{n-1}, M) \xrightarrow{i_n^*} \underset{R}{\mathrm{Hom}}(B_{n-1}, M) \to \mathrm{H}^n(C; M) \to \underset{R}{\mathrm{Hom}}(Z_n, M) \xrightarrow{d_{n-1}} \underset{R}{\mathrm{Hom}}(B_n, M) \to \dots$$

By the Break-into-SES Lemma 145.6 we obtain the following short exact sequence

$$\begin{aligned} 0 \to \mathrm{coker}(\underset{R}{\mathrm{Hom}}(Z_{n-1}, M) \to \underset{R}{\mathrm{Hom}}(B_{n-1}, M)) &\to \mathrm{H}^n(C; M) \\ &\to \ker(\underset{R}{\mathrm{Hom}}(Z_n, M) \to \underset{R}{\mathrm{Hom}}(B_n, M)) \to 0. \end{aligned}$$

We have to show that the expressions left and right coincide with the desired expressions. Both equalities follow immediately from the Ext-Projective Resolution Proposition 315.37 (together with the Ext^0–Lemma 315.38 for the right-hand term) and the observation that for each $l \in \mathbb{N}_0$ a free resolution of $\mathrm{H}_l(C)$ is given by $0 \to B_l \xrightarrow{i_l} Z_l \to \mathrm{H}_l(C_*)$. We have thus shown the existence of the desired short exact sequence

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(C_*), M) \xrightarrow{\epsilon} \mathrm{H}^n(C_*; M) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_R(\mathrm{H}_n(C_*), M) \to 0.$$

The statement that the homomorphism ϵ to the is natural follows easily from the definitions and the Homological Algebra–Fundamental Theorem 313.14 (2). We leave the verification of the details to the reader. ■

Proof that the short exact sequence splits. Let R be a PID and let (C_*, ∂_*) be an R-chain complex of free R-modules and let M be an (R, S)-module. By Definition 145.4 we need to construct an R-module homomorphism $\eta \colon \mathrm{Hom}_R(\mathrm{H}_n(C_*), M) \to \mathrm{H}^n(C_*; M)$

such that $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}_{R\text{-left}}(\mathrm{H}_n(C_*), M)$. We summarize this problem as follows:

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(C_*), M) \longrightarrow \mathrm{H}^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow[\mathrm{ev}]{\overset{?\,\eta\,?}{\longleftarrow\!\!\!

The induced maps on the cohomology of these two R-cochain complexes give us a homomorphism
$$\eta\colon \mathrm{H}^n(\text{lower sequence}) = \mathrm{Hom}_R(\mathrm{H}_n(C_*), M) \ \to\ \mathrm{H}_n(\mathrm{Hom}_R(C_*, M)).$$
It remains to prove the following claim:

Claim 2. The composition $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}_R(\mathrm{H}_n(C_*), M)$.

Proof. Let $\varphi\colon \mathrm{H}_n(C_*) \to M$ be a homomorphism. We need to show that $(\mathrm{ev} \circ \eta)(\varphi) = \varphi$ in $\mathrm{Hom}_R(\mathrm{H}_n(C_*), M)$. So let $[\sigma] \in \mathrm{H}_n(C_*)$ with $\sigma \in Z_n$. Then
$$\big((\mathrm{ev} \circ \eta)(\varphi)\big)([\sigma]) \underset{\text{definition of ev}}{\overset{\uparrow}{=}} \langle \overbrace{\eta(\varphi)}^{\in \mathrm{H}^n(C;M)}, [\sigma] \rangle \underset{\text{definition of } \eta \text{ and } \langle\,,\,\rangle}{\overset{\uparrow}{=}} (\overbrace{\varphi \circ q_n \circ p_n}^{\in \mathrm{Hom}_R(C_n, M)})(\sigma) \underset{\text{since } p_n = \mathrm{id} \text{ on } Z_n}{\overset{\uparrow}{=}} (\varphi \circ q_n)(\sigma) \underset{\text{since } q_n(\sigma) = [\sigma]}{\overset{\uparrow}{=}} \varphi([\sigma]).$$
■

315.8. Exercises for Chapter 315.

Exercise 315.1. We consider the polynomial ring $\mathbb{Q}[x]$. We consider the ring homomorphism $\varphi\colon \mathbb{Q} \to \mathbb{Q}[x]$ that is given by the inclusion and we consider the $\mathbb{Q}[x]$-modules $C := \mathbb{Q}[x]/(1-x)$ and $M := \mathbb{Q}[x]/(1+x)$. Show that the map
$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Q}[x]}, \varphi^* M) & \to & \mathrm{Hom}_{\mathbb{Q}[x]}(C, M) \\ \gamma & \mapsto & (c \mapsto \gamma(1_{\mathbb{Q}} \otimes c)) \end{array}$$
is not an isomorphism.
Remark. This shows that in the Homomorphism-Change Ring Lemma 315.4 (2) we cannot drop the hypothesis that the module C is a free $\mathbb{Q}[x]$-module.

Exercise 315.2. Let R and S be rings, let (C_*, ∂_*) be a chain complex of free R-left modules of finite rank and let $\varphi\colon R \to S$ be a ring homomorphism. We use φ to view S as an (R, S)-bimodule. Show that if C_* is acyclic, then the cochain complex $\mathrm{Hom}_{R\text{-left}}(C_*, R)$ of S-right modules is also acyclic.
Hint. Use the Null-Homotopic Criterion Proposition 313.16.
Remark. This exercise is similar to Exercise 313.2

Exercise 315.3. Let R be a ring and let $k \in \mathbb{N}_{\geq 2}$. Suppose that there exists an R-left module M and an R-right module N such that $\mathrm{Ext}^k_R(N, M) \neq 0$. Show that there exists a chain complex of free R-left modules such that the sequence from the Algebraic Universal Coefficient-H*-Theorem 315.44 is *not* exact.

Exercise 315.4. We consider the ring $R = \mathbb{Z}[t]$. Give examples of two finitely generated $\mathbb{Z}[t]$-modules M and N such that $\mathrm{Ext}^2_{\mathbb{Z}[t]}(N, M) \neq 0$.

Exercise 315.5. We consider the ring $R = \mathbb{Z}[t]$.

- Let $\mathbb{Z}_{\mathrm{tr}}$ be the ring where t acts by multiplication by $+1$.
- Let $\mathbb{Z}^-_{\mathrm{tr}}$ be the ring where t acts by multiplication by -1.

Now perform the following calculations:
(a) Calculate the Ext-modules $\mathrm{Ext}^n_{\mathbb{Z}[t]}(\mathbb{Z}_{\mathrm{tr}}, \mathbb{Z}^-_{\mathrm{tr}})$.
(b) Calculate the Ext-modules $\mathrm{Ext}^n_{\mathbb{Z}[t]}(\mathbb{Z}^-_{\mathrm{tr}}, \mathbb{Z}_{\mathrm{tr}})$.

CHAPTER 316

Universal coefficient spectral sequences

In the Algebraic Universal Coefficient Theorem 314.33 and the Algebraic Universal Coefficient-H*-Theorem 315.44 we saw how, given a chain complex C_* over a PID R and an R-module M, we can calculate the homology modules $\mathrm{H}_*(M \otimes_R C_*)$ and the cohomology modules $\mathrm{H}^n(\mathrm{Hom}_R(C_*, M))$ in terms of the homology modules of C_* and Tor-modules respectively Ext-modules. In this chapter we will generalize these results to rings that are not necessarily PIDs (in fact we will not even assume that the rings are commutative). These generalizations take the form of a "spectral sequence". More precisely, we will give a "user's guide" to spectral sequences and we will formulate the universal coefficient spectral sequences in homology and cohomology. We will not really make use of these spectral sequences, thus we will not provide a proof.

This chapter is organized as follows:

- In Section 316.1 we will provide a user's guide to homological spectral sequences.
- In Section 316.2 we will formulate the homological universal coefficient spectral sequence.
- In Section 316.3 we will provide a user's guide to cohomological spectral sequences.
- In Section 316.4 we will formulate the cohomological universal coefficient spectral sequence.
- In Section 316.5 we will provide exercises for Chapter 316.

In the next Chapter 317 we will give a purely algebraic definition of the (co-) homology modules of a group.

316.1. **User's guide to homological spectral sequences.** We start out with the following, initially rather intimidating definition, which is based on [**McCl2001**, p. 29]:

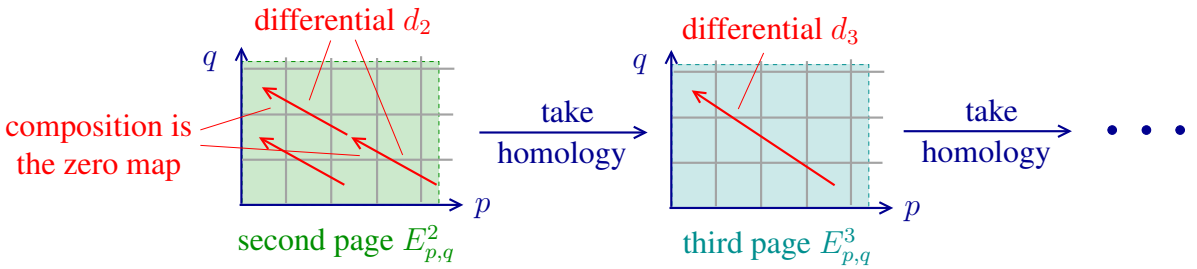
Definition 316.1. Let R be a ring. A homological first quadrant spectral sequence over R consists of the following pieces of data:

- For each $k \in \mathbb{N}_{\geq 2}$ a family $\{E^k_{p,q}\}_{p,q\in\mathbb{Z}}$ of R-left modules where $E^2_{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$.[3484]
- For each $k \in \mathbb{N}_{\geq 2}$ and $p, q \in \mathbb{Z}$ an R-left module homomorphism[3485] $d_k \colon E^k_{p,q} \to E^k_{p-k,q+k-1}$ (usually called differential) such that the composition of any two "consecutive" differentials d_k is the zero homomorphism.[3486]
- For each $k \in \mathbb{N}_{\geq 2}$ and $p, q \in \mathbb{Z}$ we have an isomorphism

$$E^{k+1}_{p,q} \xrightarrow{\cong} \frac{\ker(d_k \colon E^k_{p,q} \to E^k_{p-k,q+k-1})}{\mathrm{im}(d_k \colon E^k_{p+k,q-k+1} \to E^k_{p,q})}$$

  of R-left modules.[3487]

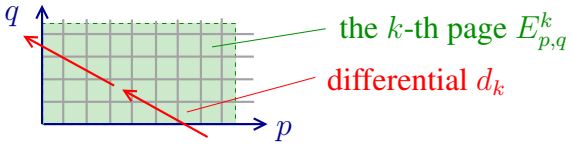
For a fixed $k \in \mathbb{N}_{\geq 2}$ the collection $\{E^k_{p,q}\}_{p,q\in\mathbb{Z}}$ of R-left modules is called the k-th page of the spectral sequence.



We continue with an addendum to Definition 316.1:

Definition 316.1. We continue with the above notation. Note that for each $p, q \in \mathbb{Z}$ there exists a $k \in \mathbb{N}_{\geq 2}$[3488] such that for $l \geq k$ the differential starting at $E^l_{p,q}$ ends up outside the first quadrant and that the differential to $E^l_{p,q}$ comes from outside the first quadrant. We then have $E^k_{p,q} = E^{k+1}_{p,q} = \dots$ and we write

$$E^\infty_{p,q} := E^k_{p,q}.$$



Example. Let $E^k_{p,q}$ be a homological first quadrant spectral sequence over a ring R. The following two observations follow easily from the definitions:

- For each $p \in \mathbb{N}_0$ we have natural inclusions $E^2_{p,0} \supseteq E^3_{p,0} \supseteq \dots \supseteq E^{p+1}_{p,0} = E^\infty_{p,0}$.
- For each $q \in \mathbb{N}_0$ we have natural epimorphisms $E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \dots \twoheadrightarrow E^{q-1}_{0,q} = E^\infty_{0,q}$. □

Next comes the most confusing definition in the context of spectral sequence. More precisely, we have the following definition, which is based on [**McCl2001**, p. 33]:

Definition 316.2. Let R be a ring. We suppose that we are given a homological spectral sequence as in Definition 316.1. We say that the spectral sequences converges to a family of R-left modules $\{H_k\}_{k\in\mathbb{N}_0}$ if for each $k \in \mathbb{N}_0$ we have a filtration

$$\{0\} \subseteq J_{0,k} \subseteq J_{1,k-1} \subseteq \dots \subseteq J_{k-1,k} \subseteq J_{k,0} = H_k$$

by R-left modules together with isomorphisms $J_{p,q}/J_{p-1,q+1} \xrightarrow{\cong} E^\infty_{p,q}$ of R-left modules. If the spectral sequence $E^*_{p,q}$ converges to H_k, then we write

$$E^2_{p,q} \implies H_k.$$

Example. Suppose we have a homological spectral sequence with $E^2_{p,q} \Rightarrow H_k$.

(1) (a) For each $k \in \mathbb{N}_0$ we clearly have a natural monomorphism $E^\infty_{0,k} \to H_k$.

[3484] In other words, the $E^2_{p,q}$-modules are zero outside of the first quadrant.

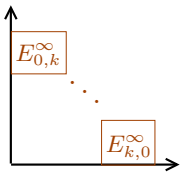
[3485] In other words, $d - K$ is a homomorphism of bidegree $(-k, k-1)$.

[3486] Note that the "total" degree $p+q$ gets lowered by the differential $d_k \colon E^k_{p,q} \to E^k_{p-k,q+k-1}$, this explains the name "homological" spectral sequence.

[3487] Usually this choice of isomorphism gets suppressed from the notation.

[3488] In fact one can take $k = \max\{p+1, q\}$.

$$\{0\} \subseteq \underbrace{J_{0,k}}_{\text{quotient } E^\infty_{0,k}} \subseteq \underbrace{J_{1,k-1}}_{\text{quotient } E^\infty_{1,k-1}} \subseteq \ldots \subseteq J_{k-1,1k} \subseteq \underbrace{J_{k,0}}_{\text{quotient } E^\infty_{k,0}} = H_k \quad \text{where}$$



(b) For each $k \in \mathbb{N}_0$ we clearly have a natural epimorphism $H_k \twoheadrightarrow E^\infty_{k,0}$.
These two homomorphisms are sometimes called *edge homomorphism*.

(2) Note that for $k = 0$ we clearly have $H_0 \cong E^\infty_{0,0} \cong E^2_{0,0}$.

(3) Note that for $k = 1$ we get a filtration

$$\{0\} \subseteq \underbrace{J_{0,1}}_{\text{quotient } E^\infty_{0,1}} \subseteq \underbrace{J_{1,0}}_{\text{quotient } E^\infty_{1,0}} = H_1$$

In other words, there exists a short exact sequence of the form

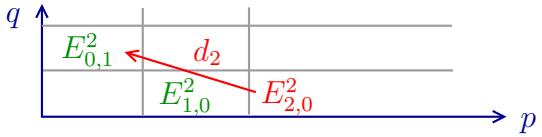
$$0 \to \underbrace{E^\infty_{0,1}}_{\cong J_{0,1}} \to H_1 \to \underbrace{E^\infty_{1,0}}_{\cong J_{1,0}/J_{0,1}} \to 0.$$

Note that the two homomorphisms to the left and right are precisely the edge homomorphisms from (1). □

To practice our definitions we now formulate and prove the following little lemma:

Lemma 316.3. (Four Term-Sequence Lemma) Suppose we are given a homological spectral sequence $E^2_{p,q}$ with $E^2_{p,q} \Rightarrow H_k$. We get an exact sequence of the form

$$E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0.$$



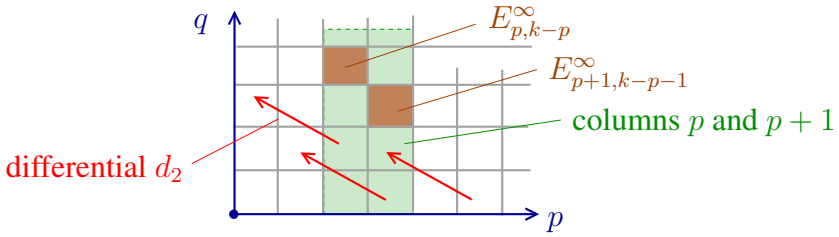
Proof. The lemma follows immediately from the above discussion and the observation that

$$E^\infty_{0,1} = E^3_{0,1} = \operatorname{coker}(d_2 \colon E^2_{2,0} \to E^2_{0,1}).$$ ■

We move on to the last definition of this section:

Definition 316.4. Let R be a ring. We suppose that we are given a homological spectral sequence as in Definition 316.1. We say the spectral sequence collapses at the k-th page if all differentials of degree $\geq k$ are zero. In this case it follows immediately from the definitions that for all $p, q \in \mathbb{Z}$ we have a natural isomorphism $E^\infty_{p,q} \cong E^k_{p,q}$.

Example. Let $E^2_{p,q}$ be a homological spectral sequence that is supported only in two consecutive columns p and $p+1$.
Clearly all differentials d_k with $k \geq 2$ either start or end up in a zero-module. Thus the spectral sequence collapses already at the second page. Similar to the above example we see that if this spectral sequences converges to H_k, then for each $k \in \mathbb{N}_0$ we have a short



exact sequence of the form

$$0 \to \underbrace{E^{\infty}_{p,k-p}}_{\cong J_{p,k-p}} \to H_k \to \underbrace{E^{\infty}_{p+1,k-p-1}}_{\cong J_{p+1,k-p-1}/J_{p,k-p}} \to 0.$$

□

316.2. **The homological universal coefficient spectral sequence.** We now have the tools to formulate the following theorem:

Theorem 316.5. (Homological UCSS Theorem) Let Q and R be two rings, let C_* be a chain complex of free R-left modules and finally let M be a (Q,R)-bimodule. There exists a natural homological first quadrant spectral sequence over the ring Q such that $E_2^{p,q} = \operatorname{Tor}_p^R(\mathrm{H}^q(C_*), M)$ and which converges to $\mathrm{H}_*(M \otimes_R C_*)$. In short hand, we have

$$\underbrace{\operatorname{Tor}_p^R(\mathrm{H}^q(C_*), M)}_{=E_2^{p,q}} \implies \mathrm{H}_*(M \otimes_R C_*).$$

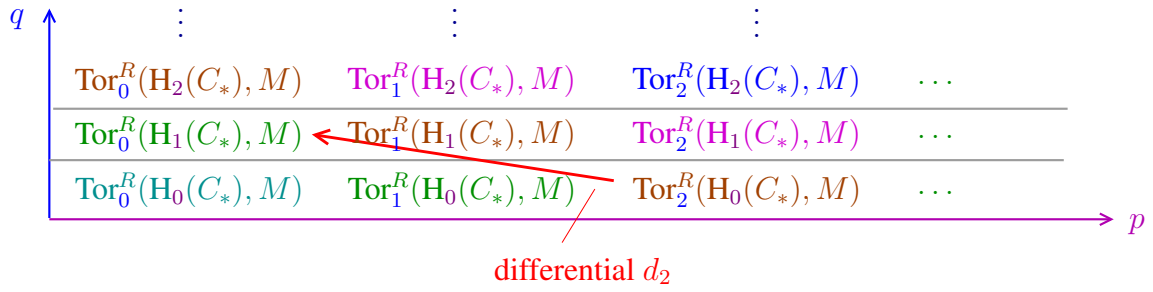
Furthermore, the edge homomorphism[3489]

$$M \otimes_R \mathrm{H}_k(C_*) =_{\mathrm{i}} \operatorname{Tor}_R^0(\mathrm{H}_k(C_*), M) \to \mathrm{H}_k(M \otimes_R C_*)$$

is given by the map

$$\begin{array}{rcl} \mu\colon M \otimes_R \mathrm{H}_k(C_*) & \to & \mathrm{H}_k(M \otimes_R C_*) \\ m \otimes [\varphi] & \mapsto & [m \otimes \varphi]. \end{array}$$

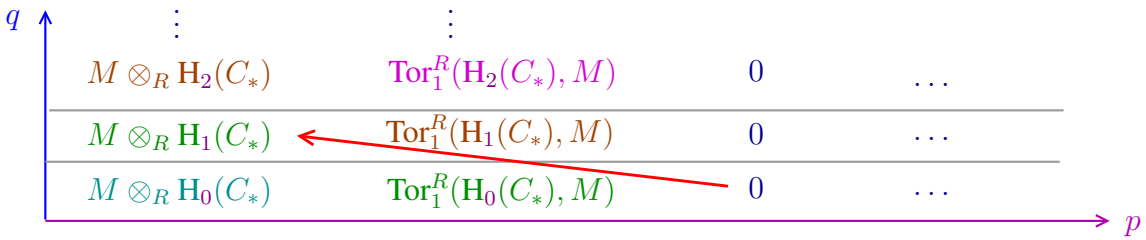
This spectral sequence is called the homological universal coefficient spectral sequence (UCSS).



Proof. This is a special case of the Künneth spectral sequence that is proved in [**Rot2009**, Theorem 10.90] or alternatively in [**McCl2001**, p. 56]. ■

[3489]Note that it follows from the discussion on page 6099 that if a homological first quadrant spectral sequence $E^2_{p,q}$ converges to H_k, then we have a natural homomorphism $E^2_{0,k} \to E^{\infty}_{0,k}$ and on page 6100 we saw that we have a natural epimorphism $E^{\infty}_{0,k} \to H_k$.

Example. We consider the case that R is a PID, e.g. $R = \mathbb{Z}$. In this case we know by the Higher Tor Lemma 314.27 that for all R-modules A and M we have $\operatorname{Tor}_p^R(A, M) = 0$ for $p \geq 2$. Furthermore we know by the Tor_0-Lemma 314.23 that we have a natural isomorphism $\operatorname{Tor}_0^R(A, M) \to M \otimes_R A$.



Thus we see that the spectral sequence from the Homological UCSS Theorem 316.5 is supported in the zeroth and the first column. As on page 6100 we see that the spectral sequence collapses at the second page, i.e. all differentials are the zero maps. It follows that the E_∞-terms are naturally isomorphic to the E_2-terms. In summary we see that the Homological UCSS Theorem 316.5 shows that for each $k \in \mathbb{N}_0$ there exists a natural homomorphism $\nu\colon \mathrm{H}_k(M \otimes_R C_*) \to \operatorname{Tor}_1^R(\mathrm{H}_{k-1}(C_*), M)$ of Q-modules such that the following sequence is exact:

$$0 \to \underbrace{M \otimes_R \mathrm{H}_k(C_*)}_{\cong E^\infty_{0,k}} \xrightarrow{\mu} \mathrm{H}_k(M \otimes_R C_*) \xrightarrow{\nu} \underbrace{\operatorname{Tor}_1^R(\mathrm{H}_{k-1}(C_*), M)}_{\cong E^\infty_{1,k-1}} \to 0.$$

In other words, we have rediscovered the Algebraic Universal Coefficient Theorem 162.15 and the Algebraic Universal Coefficient Theorem 314.33.[3490] □

316.3. **User's guide to cohomological spectral sequences.** In Definition 316.1 we introduced the notion of a homological first quadrant spectral sequence. We will now introduce the notion of a cohomological first quadrant spectral sequence. As always, the move from homology to cohomology is given by a confusing shift of direction of arrows. The following definition is based on [**McCl2001**, p. 4 and p. 29]:

Definition 316.6. Let R be a ring. A cohomological first quadrant spectral sequence over R consists of the following pieces of data:

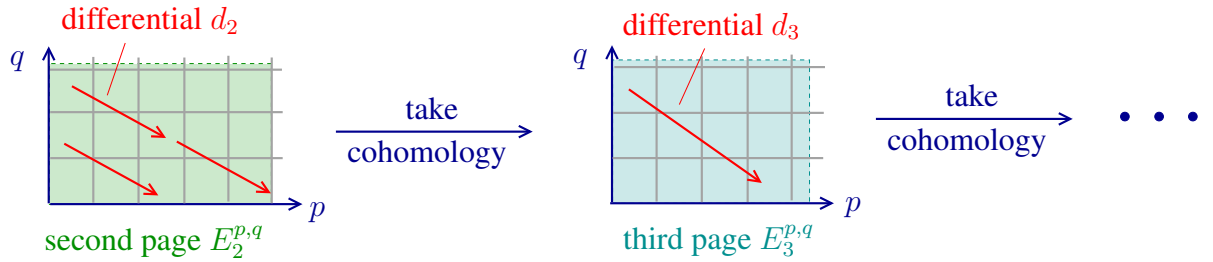
- For each $k \in \mathbb{N}_{\geq 2}$ a family $\{E_k^{p,q}\}_{p,q\in\mathbb{Z}}$ of R-left modules where $E_2^{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$.
- For each $k \in \mathbb{N}_{\geq 2}$ and $p, q \in \mathbb{Z}$ an R-left module homomorphism $d_k\colon E_k^{p,q} \to E_k^{p+k,q+1-k}$ (which is usually called a differential) such that the composition of any two "consecutive" differentials d_k is the zero homomorphism.[3491] [3492]
- For each $k \in \mathbb{N}_{\geq 2}$ and $p, q \in \mathbb{Z}$ we have an isomorphism

$$E_{k+1}^{p,q} \xrightarrow{\cong} \frac{\ker(d_k\colon E_k^{p,q} \to E_k^{p+k,q+1-k})}{\operatorname{im}(d_k\colon E_k^{p-k,q+k-1} \to E_k^{p,q})}$$

of R-left modules.

[3490] It is not so clear whether one can also recover the Algebraic UCT-Splitting Proposition 314.34 from this setup.

For a fixed $k \in \mathbb{N}_{\geq 2}$ the collection $\{E_k^{p,q}\}_{p,q\in\mathbb{Z}}$ of R-left modules is called the k-th page of the spectral sequence.



We continue with an addendum to Definition 316.6:

Definition 316.6. We continue with the above notation. Note that for each $p, q \in \mathbb{Z}$ there exists a $k \in \mathbb{N}_{\geq 2}$ such that for $l \geq k$ the differential starting at $E_l^{p,q}$ ends up outside the first quadrant and that the differential to $E_l^{p,q}$ comes from outside the first quadrant. We then have $E_k^{p,q} = E_{k+1}^{p,q} = \ldots$ and we write

$$E_\infty^{p,q} := E_k^{p,q}.$$

Example. Let $E_k^{p,q}$ be a cohomological first quadrant spectral sequence over a ring R. Similar to the discussion on page 6099 we can make the following observations:

- For each $p \in \mathbb{N}_0$ we have natural epimorphisms $E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{p-1}^{p,0} = E_\infty^{p,0}$.
- For each $q \in \mathbb{N}_0$ we have natural inclusions $E_2^{0,q} \supseteq E_3^{0,q} \supseteq \ldots \supseteq E_{q+1}^{0,q} = E_\infty^{0,q}$. □

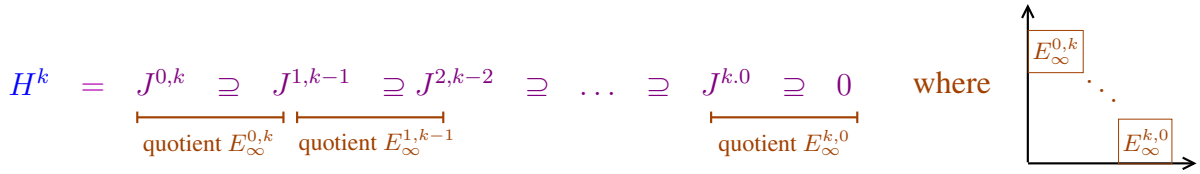
The following definition is the analog of Definition 316.2. The definition is based on [**McCl2001**, p. 5 and p. 33] and it is consistent with [**Hat**, p. 24]:

Definition 316.7. Let R be a ring. We suppose that we are given a cohomological spectral sequence as in Definition 316.6. We say that the spectral sequences converges to a family of R-left modules $\{H^k\}_{k\in\mathbb{N}_0}$ if for each $k \in \mathbb{N}_0$ we have a filtration

$$H^k = J^{0,k} \supseteq J^{1,k-1} \supseteq J^{2,k-2} \supseteq \ldots \supseteq J^{k,0} \supseteq = 0$$

by R-left modules together with isomorphisms $J^{p,q}/J^{p-1,q+1} \xrightarrow{\cong} E_\infty^{p,q}$ of R-left modules. If the spectral sequence $E_*^{p,q}$ converges to H^k, then we write

$$E_2^{p,q} \implies H^k.$$



Example. Suppose we have a cohomological spectral sequence with $E_2^{p,q} \Rightarrow H^k$.

(1) (a) For each $k \in \mathbb{N}_0$ we clearly have a natural monomorphism $E_\infty^{k,0} \hookrightarrow H^k$.
(b) For each $k \in \mathbb{N}_0$ we clearly have a natural epimorphism $H^k \twoheadrightarrow E_\infty^{0,k}$.
These two homomorphisms are sometimes called *edge homomorphism*.

[3491] In other words, $d - K$ is a homomorphism of bidegree $(k, 1-k)$.

[3492] Note that the "total" degree $p+q$ increases under the differential $d_k \colon E_k^{p,q} \to E_k^{p-k+1,q+k}$ by one, this explains the name "cohomological" spectral sequence.

(2) Note that for $k = 0$ we clearly have $H^0 \cong E_\infty^{0,0} \cong E_2^{0,0}$.
(3) For $k = 1$ we get a filtration

$$H^1 \quad = \quad J^{0,1} \quad \supseteq \quad J^{1,0} \quad \supseteq \quad \{0\}$$

where the quotient $J^{0,1}/J^{1,0} = E_\infty^{0,1}$ and the quotient $J^{1,0} = E_\infty^{1,0}$.

In other words, there exists a short exact sequence of the form

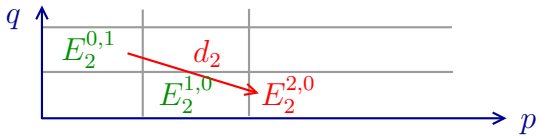
$$0 \to \underbrace{E_\infty^{1,0}}_{\cong J^{1,0}} \to H^1 \to \underbrace{E_\infty^{0,1}}_{\cong J^{0,1}/J^{1,0}} \to 0.$$

Note that the two homomorphisms are precisely the edge homomorphisms from (1). □

For future reference we record the following analog of the Four Term-Sequence Lemma 316.3:

Lemma 316.8. (Four Term-Sequence Lemma) Suppose we are given a homological spectral sequence $E_2^{p,q}$ with $E_2^{p,q} \Rightarrow H^k$. We get an exact sequence of the form

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}.$$



Proof. The lemma follows immediately from the above discussion and the observation that

$$E_\infty^{2,0} \quad = \quad E_3^{2,0} \quad = \quad \operatorname{coker}(d_2 \colon E_2^{0,1} \to E_2^{2,0}).$$

■

We conclude this section with the following analog to Definition 316.4:

Definition 316.9. Let R be a ring. We suppose that we are given a cohomological spectral sequence as in Definition 316.6. We say the spectral sequences collapses at the k-th page if all differentials of degree $\geq k$ are zero. In this case it follows immediately from the definitions that for all $p, q \in \mathbb{Z}$ we have a natural isomorphism $E_\infty^{p,q} \cong E_k^{p,q}$.

Example. Let $E_2^{p,q}$ be a cohomological spectral sequence that is supported only in two consecutive columns p and $p+1$. As on page 6100 we see that the spectral sequence collapses already at the second page. One can now easily verify that for each $k \in \mathbb{N}_0$ we have a short exact sequence of the form

$$0 \to \underbrace{E_\infty^{k-p+1,p-1}}_{\cong J_{p,k-p}} \to H^k \to \underbrace{E_\infty^{k-p,p}}_{\cong J_{p+1,k-p-1}/J_{p,k-p}} \to 0.$$

□

316.4. **The cohomological universal coefficient spectral sequence.** The following theorem is the cohomological analog of the Homological UCSS Theorem 316.5:

Theorem 316.10. (Cohomological UCSS Theorem) Let R and S be two rings, let C_* be a chain complex of free R-left modules and let M be an (R, S)-left left module. There exists a natural cohomological first quadrant spectral sequence over the ring S such

that $E_{p,q}^2 = \mathrm{Ext}_R^p(\mathrm{H}_q(C_*), M)$ and which converges to $\mathrm{H}^*(C_*; M)$. In short hand, we have

$$\underbrace{\mathrm{Ext}_R^p(\mathrm{H}_q(C_*), M)}_{=E_2^{p,q}} \implies \mathrm{H}^*(C_*; M).$$

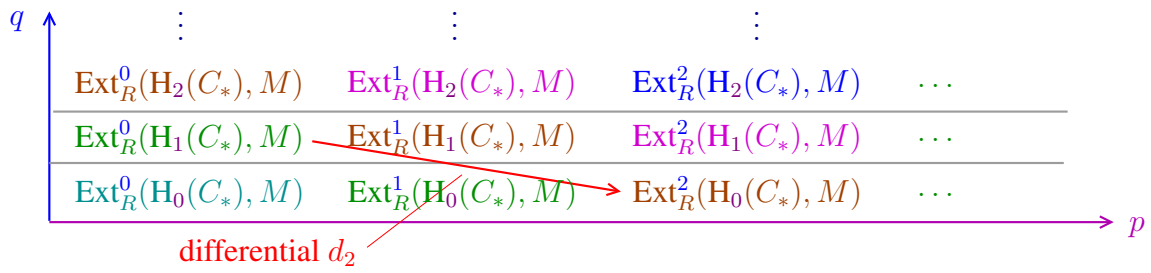
Furthermore, the edge homomorphism

$$\mathrm{H}^k(C_*; M) \;\to\; \mathrm{Ext}_R^0(\mathrm{H}_k(C_*), M) =_{\mathrm{i}} \mathrm{Hom}_R(\mathrm{H}_k(C_*), M)$$

is given by the evaluation map

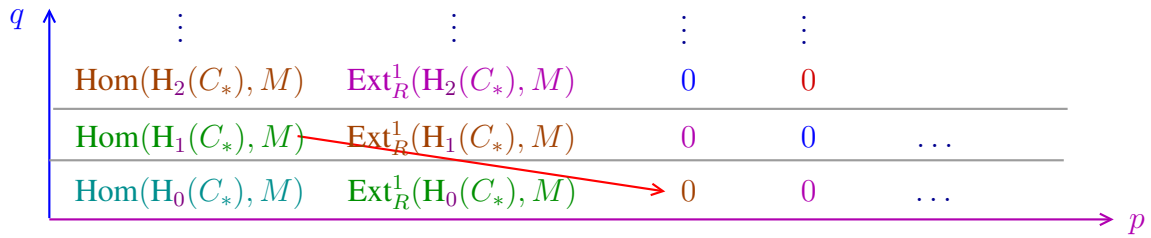
$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^k(C_*; M) & \to & \mathrm{Hom}_R(\mathrm{H}_k(C_*), M) \\ [\varphi\colon C_k \to M] & \mapsto & \begin{pmatrix} \mathrm{H}_k(C_*) & \to & M \\ [\sigma] & \mapsto & \varphi(\sigma) \end{pmatrix} \end{array}$$

This spectral sequence is called the cohomological universal coefficient spectral sequence (UCSS).



Proof. This theorem is proved in [**Levine1977**, p. 5].[3493] ■

Example. We consider again the special case that R is a PID. Thus let C_* be a chain complex of free R-modules and let M be an R-module. First note that we know by the Ext^0-Lemma 315.38 that we have a natural isomorphism $\mathrm{Ext}_R^0(M, N) \to \mathrm{Hom}_R(M, N)$. Furthermore recall that we know by the Higher Ext Lemma 315.41 that $\mathrm{Ext}_R^p(M, H) = 0$ for all $p \geq 2$.



We see that the Cohomological UCSS Theorem 316.10, together with the discussion on page 6104, shows that for each $k \in \mathbb{N}_0$ there exists a natural homomorphism

$$\epsilon\colon \mathrm{Ext}_R(\mathrm{H}_{k-1}(C_*), M) \;\to\; \mathrm{H}^k(C_*; M)$$

of S-modules such that the following sequence is exact:

$$0 \;\to\; \underbrace{\mathrm{Ext}_R(\mathrm{H}_{k-1}(C_*), M)}_{=E_\infty^{1,k-1}} \xrightarrow{\epsilon} \mathrm{H}^k(C_*; M) \xrightarrow{\mathrm{ev}} \underbrace{\mathrm{Hom}_{R\text{-left}}(\mathrm{H}_k(C_*), M)}_{=E_\infty^{0,k}} \;\to\; 0.$$

[3493] Note though that in [**Levine1977**] the roles of p and q are swapped.

In other words, we have rediscovered the Algebraic Universal Coefficient-H*-Theorem 201.22 and the Algebraic Universal Coefficient-H*-Theorem 315.44. □

Remark. We refer to [**Rot2009**, Chapter 10] and [**McCl2001**, Chapter 12.1] for many other related algebraic spectral sequences. Furthermore, in Chapter 341 we will discuss (alas once again without proofs) several other, more topologically inspired, spectral sequences. □

316.5. Exercises for Chapter 316.

Exercise 316.1. Suppose we have a homological spectral sequence $E^2_{p,q}$ that is supported in two successive *rows*. We suppose that $E^2_{p,q} = H_k$. What is the relationship between the modules $E^2_{p,q}$ and the modules H_k?

Exercise 316.2. Let R be a ring, let N be an R-left module and let F_* be a free resolution of N. Furthermore let M be an R-right module. What does the Homological UCSS Theorem 316.5 say in this context about $\mathrm{H}_*(M \otimes_R F_*)$?

Exercise 316.3. Formulate and prove the cohomological analog of the Four Term-Sequence Lemma 316.3.

Exercise 316.4. Let R be a ring. Suppose that we are given a homological spectral sequence with $E^2_{p,q} \Rightarrow H_k$ over R. Show that there exists an exact sequence of the form

$$H_2 \ \to \ E^2_{2,0} \ \xrightarrow{d_2} \ E^{0,1}_2 \ \to \ H_1 \ \to \ E^2_{1,0} \ \to \ 0.$$

Remark. This exact sequence is called the *five term exact sequence*.

Exercise 316.5. Let R be a ring. Suppose that we are given a cohomological spectral sequence with $E_2^{p,q} \Rightarrow H^k$ over R. Show that there exists an exact sequence of the form

$$0 \ \to \ E_2^{1,0} \ \to \ H^1 \ \to \ E_2^{0,1} \ \xrightarrow{d_2} \ E_2^{2,0} \ \to \ H^2.$$

Remark. This exact sequence is called the *five term exact sequence*.

Exercise 316.6. Let R be a ring, let N be an R-left module and let F_* be a free resolution of N. Furthermore let M be another R-left module. What does the Cohomological UCSS Theorem 316.10 say in this context about $\mathrm{H}^*(F_*; M)$?

CHAPTER 317

Group homology and cohomology

Let G be a group, possibly non-abelian. In Section 257.4 we sketched the definition of the homology groups $\mathrm{H}_n(G)$ of G and we proved a few basic results about these homology groups. In this chapter we will study the homology groups $\mathrm{H}_n(G)$ of a group G in much greater detail. In fact, we will consider a more general setup, where we consider the homology the module $\mathrm{H}_n(G; M)$ with coefficients given by any $\mathbb{Z}[G]$-left module M. As we will see, using our algebraic tools from the last chapters it will be surprisingly easy to define these invariants.

This chapter is organized as follows:

- In Section 317.1 we will give the (purely algebraic) definition of group homology.
- In Section 317.2 we will provide some basic calculations of group homology.
- In Section 317.3 we will formulate the Group-H_*-Künneth Theorem 317.10 which allows us to calculate the group homology of finite direct products of groups.
- In Section 317.4 we will provide the proof of the Group-H_*-Künneth Theorem 317.10.
- In Section 317.5 we will discuss the interplay between group homomorphisms and group homology. In particular we will see that group homology with trivial $\mathbb{Z}$-coefficients is covariantly functorial.
- In Section 317.6 we will introduce group cohomology.
- In Section 317.7 we will introduce the cohomological dimension of a group.
- In Section 317.8 we will provide exercises for Chapter 317.

In the next Chapter 318 we will see that topological tools can be used to calculate the homology groups of many interesting groups.

317.1. **Definition of group homology.** We start out by introducing the following notation that is modelled on Definition 310.7:

Definition 317.1. Let G be a group. We denote by ${}_{G,\mathrm{tr}}\mathbb{Z}$ the $\mathbb{Z}[G]$-left module where the underlying abelian group is given by $\mathbb{Z}$ and where each $g \in G$ acts trivially. Sometimes, when the group is clear from the contrast, we just write ${}_{\mathrm{tr}}\mathbb{Z}$.

To define group homology with coefficients we will also need the following notation that we introduced in the Ring-with-Involution–Module Lemma 309.16:

Notation 317.2. Let G be a group and let M be a $\mathbb{Z}[G]$-left module. We denote by $\overline{M}$ the $\mathbb{Z}[G]$-right module that is given by $m \underset{\overline{M}}{\cdot} g := g^{-1} \underset{M}{\cdot} m$. Note that if M is in fact an $(R, \mathbb{Z}[G])$-left left module for some ring R, then $\overline{M}$ is a $(R, \mathbb{Z}[G])$-bimodule.

Using ideas from Definition 314.20 we are already in a position to introduce group homology:

Definition 317.3. Let G be a group. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[G])$-left left module. As in Definition 317.1 let ${}_{G,\mathrm{tr}}\mathbb{Z}$ be the trivial $\mathbb{Z}[G]$-left module and as in Notation 317.2 we consider the $(R, \mathbb{Z}[G])$-bimodule $\overline{M}$.[3494] We define

$$\mathrm{H}_n(G; M) := \underbrace{\mathrm{Tor}_n^{\mathbb{Z}[G]}({}_{G,\mathrm{tr}}\mathbb{Z}, \overline{M})}_{\text{see Definition 314.20}}$$

which is an R-left module. We refer to $\mathrm{H}_n(G; M)$ as the n-th homology module of the group G with coefficients in M.

For the reader's convenience we spell out the same definition in slightly greater detail:

Definition 317.3. Let G be a group, let R be a ring and let M be an $(R, \mathbb{Z}[G])$-left left module. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} {}_{G,\mathrm{tr}}\mathbb{Z} \to 0$$

the natural free $\mathbb{Z}[G]$-left resolution from Definition 313.12. We set

$$\begin{aligned} \mathrm{H}_n(G; M) &\overset{\text{see above}}{\downarrow} := \mathrm{Tor}_n^{\mathbb{Z}[G]}({}_{G,\mathrm{tr}}\mathbb{Z}, \overline{M}) \overset{\text{see Definition 314.20}}{\downarrow} := \mathrm{H}_n(\overline{M} \otimes_{\mathbb{Z}[G]} F_*) \\ &= \mathrm{H}_n(\ldots \to \overline{M} \otimes_{\mathbb{Z}[G]} F_2 \xrightarrow{\mathrm{id}_{\overline{M}} \otimes f_2} \overline{M} \otimes_{\mathbb{Z}[G]} F_1 \xrightarrow{\mathrm{id}_{\overline{M}} \otimes f_1} \overline{M} \otimes_{\mathbb{Z}[G]} F_0 \xrightarrow{0} 0). \end{aligned}$$

By definition group homology depends on the group G together with the choice of a $\mathbb{Z}[G]$-left module. Often one cannot think of a really interesting $\mathbb{Z}[G]$-left module, but fortunately there is always one obvious choice, namely $\mathbb{Z}$ with the trivial G-action. This leads us to the following definition:

Definition 317.4. Let G be a group. Given $n \in \mathbb{N}_0$ we define the n-th homology group of G as

$$\mathrm{H}_n(G) := \mathrm{H}_n(G; {}_{G,\mathrm{tr}}\mathbb{Z}).$$

Remark. In Definition 257.14 we gave a different, topological, definition of $\mathrm{H}_n(G)$. In the Group H_*-via-$\mathrm{K}(G,1)$-Proposition 318.5 we will see that the two definitions of group homology from Definition 257.14 and Definition 317.4 are consistent. □

Example. Let G be a group. In Exercise 317.1 we will use the Tensor–Right-Exact Lemma 314.14 to show that we have a natural isomorphism $\mathrm{H}_0(G) \cong \mathbb{Z}$. □

317.2. **First calculations of group homology.** Our next goal is to calculate group homology for examples. To perform actual calculations it is usually convenient to use the following consequence of the Tor-Projective Resolution Proposition 314.22:

Proposition 317.5. (Group-H_*-Projective Resolution Proposition) Let G be a group, let R be a ring and let M be an $(R, \mathbb{Z}[G])$-left left module. Let

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} {}_{G,\mathrm{tr}}\mathbb{Z} \to 0$$

be a projective resolution of the trivial $\mathbb{Z}[G]$-left module ${}_{G,\mathrm{tr}}\mathbb{Z}$. For every $n \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_n(G; M) \xrightarrow{\cong} \mathrm{H}_n(M \otimes_{\mathbb{Z}[G]} P_*)$

[3494] Of course we could also have started out with an $(R, \mathbb{Z}[G])$-bimodule, but most "real life" examples tend to be $(R, \mathbb{Z}[G])$-left left modules.

of R-left modules which is natural in M.

Proof. This proposition follows immediately from Definition 317.3 and the Tor-Projective Resolution Proposition 314.22, which in turn is a straightforward consequence of the Homological Algebra–Fundamental Theorem 313.14. ■

Using the Group-H_*-Projective Resolution Proposition 317.5 we can now easily perform some calculations of group homology:

Lemma 317.6. (Group-H_*-of-Trivial Group Lemma) Let $G = \{e\}$ be the trivial group.

(1) A free resolution of ${}_{\{e\},\mathrm{tr}}\mathbb{Z}$ is given by $0 \to \mathbb{Z}[\{e\}] \to {}_{\{e\},\mathrm{tr}}\mathbb{Z} \to 0$.

(2) Let R be a ring and let M be an $(R, \mathbb{Z}[\{e\}])$-left left module. Then

$$\mathrm{H}_n(\{e\}; M) \cong \begin{cases} M, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

Proof.

(1) This statement follows easily from the definitions.

(2) Let M be an $(R, \mathbb{Z}[\{e\}])$-left left module. Then

$$\mathrm{H}_n(\{e\}; M) \underset{\substack{\uparrow \\ \text{by (1) and the Group-H}_*\text{-Projective} \\ \text{Resolution Proposition 317.5}}}{\cong} \mathrm{H}_n(M \otimes_{\mathbb{Z}[\{e\}]} \mathbb{Z}[\{e\}]) \underset{\substack{\uparrow \\ \text{by the Tensor-with-Base} \\ \text{Ring Lemma 312.10}}}{\cong} \begin{cases} M, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

■

We continue with the infinite cyclic group $\langle t \rangle$:

Lemma 317.7. (Group-H_*-of-$\mathbb{Z}$-Lemma) Let $G = \langle t \rangle \cong \mathbb{Z}$ be the infinite cyclic group.

(1) A free resolution of ${}_{\langle t \rangle,\mathrm{tr}}\mathbb{Z}$ is given by[3495]

$$0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{\cdot(t-1)} \mathbb{Z}[t^{\pm 1}] \xrightarrow{t \mapsto 1} {}_{\langle t \rangle,\mathrm{tr}}\mathbb{Z} \longrightarrow 0.$$

(2) Let R be a ring and let M be an $(R, \mathbb{Z}[t^{\pm 1}])$-left left module. Then

$$\mathrm{H}_n(\langle t \rangle; M) \cong \mathrm{H}_n(\mathbb{Z}; M) \cong \begin{cases} M/(t-1) \cdot M, & \text{if } n = 0, \\ \ker(\cdot(t-1) \colon M \to M), & \text{if } n = 1, \\ 0, & \text{else.} \end{cases}$$

In particular $$\mathrm{H}_n(\langle t \rangle) \cong \mathrm{H}_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, \\ 0, & \text{else.} \end{cases}$$

Proof.

(1) The only statement which needs a little bit of thought is the fact that any element in $\ker(\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z})$ is actually a multiple of $t - 1$. This can be seen as follows: If $p(t)$ lies in $\ker(\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z})$, then $p(1) = 0$. Using polynomial division one sees that $t - 1$ divides $p(t)$. In other words, $p(t) = (t-1) \cdot q(t)$ for some $q(t) \in \mathbb{Z}[t^{\pm 1}]$.

[3495] Here we make the usual identification $\mathbb{Z}[\langle t \rangle] = \mathbb{Z}[t^{\pm 1}]$.

(2) Let R be a ring and let M be an $(R, \mathbb{Z}[t^{\pm 1}])$-left left module. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{M} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[t^{\pm 1}] & \xrightarrow{\mathrm{id}_{\overline{M}} \otimes (\cdot(t-1))} & \overline{M} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[t^{\pm 1}] & \longrightarrow & 0. \\ & & \cong \downarrow m \otimes r \mapsto m \cdot r & & \cong \downarrow m \otimes r \mapsto m \cdot r & & \\ 0 & \longrightarrow & \overline{M} & \xrightarrow{\cdot(t-1)} & \overline{M} & \longrightarrow & 0. \end{array}$$

It follows from the Tensor-with-Base Ring Lemma 312.10 that the vertical maps are isomorphisms and it follows almost immediately from the definitions that the diagram commutes. All statements now follow easily from this observation together with the Group-H_*-Projective Resolution Proposition 317.5 applied to (1). ■

We move on to the free abelian group $\langle s, t \rangle_{\mathrm{ab}} \cong \mathbb{Z}^2$ of rank two:

Lemma 317.8. (Group-H_*-of-$\mathbb{Z}^2$-Lemma) In the following we consider the free abelian group $G = \langle s, t \rangle_{\mathrm{ab}} \cong \mathbb{Z}^2$ of rank 2.

(1) A free resolution of ${}_{\langle s,t \rangle_{\mathrm{ab}},\mathrm{tr}}\mathbb{Z}$ is given by[3496 3497 3498]

$$0 \to \mathbb{Z}[s^{\pm 1}, t^{\pm 1}] \xrightarrow{\begin{pmatrix} 1-s & t-1 \end{pmatrix}} \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]^{1 \times 2} \xrightarrow{\begin{pmatrix} t-1 \\ s-1 \end{pmatrix}} \mathbb{Z}[s^{\pm 1}, t^{\pm 1}] \xrightarrow{s \mapsto 1, t \mapsto 1} {}_{\langle s,t \rangle_{\mathrm{ab}},\mathrm{tr}}\mathbb{Z} \to 0.$$

(2) We have the following isomorphisms:

$$\mathrm{H}_n(\langle s, t \rangle_{\mathrm{ab}}) \cong \mathrm{H}_n(\mathbb{Z}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{else.} \end{cases}$$

Proof. We leave the elementary verification of this lemma to the reader. Note that soon, in the Group-H_*-of-$\mathbb{Z}^m$-Lemma 317.12, we will give an alternative calculation of the homology groups of any $\mathbb{Z}^m$. ■

Evidently at this point we could formulate a lemma regarding $\mathbb{Z}^3$, $\mathbb{Z}^4$ and so on. But we will soon get to know a general approach that will deal with all $\mathbb{Z}^n$ at once.

So instead let us move on to the finite cyclic groups. Fortunately, using the Group-H_*–Projective Resolution Proposition 317.5 together with the Free Resolution-$\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-Lemma 314.31 we can easily compute $\mathrm{H}_n(\mathbb{Z}/k\mathbb{Z})$:

Lemma 317.9. (Group-H_*-of-$\mathbb{Z}/k\mathbb{Z}$-Lemma) Let $k \in \mathbb{N}$. We consider the finite cyclic group $\langle t \,|\, t^k \rangle \cong \mathbb{Z}/k\mathbb{Z}$. Given $n \in \mathbb{N}_0$ we have isomorphisms[3499]

$$\mathrm{H}_n(\langle t \,|\, t^k \rangle) \cong \mathrm{H}_n(\mathbb{Z}/k\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}/k\mathbb{Z}, & \text{if } n \text{ is odd}, \\ 0, & \text{otherwise.} \end{cases}$$

[3496] This statement is very similar to the discussion in the Free Resolution-$\mathbb{Z}[x, y]$-Lemma 314.30.

[3497] Here we make the usual identification $\mathbb{Z}[\langle s, t \rangle_{\mathrm{ab}}] = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

[3498] Let R be a ring. Recall that in Notation 309.4 we introduced the set of row vectors $R^{1 \times k}$. On page 5962 we saw that the row vectors $R^{1 \times k}$ are naturally an R-left module. Furthermore we saw in the Left Module Homomorphism–Matrix Lemma 309.5 that right multiplication by a matrix defines a homomorphism of such R-left modules. Even though in our case $R = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ we could like to think of the modules as left modules.

Proof. Let $k \in \mathbb{N}$. Throughout the proof of this lemma we will make the usual identification $\mathbb{Z}[\langle t \,|\, t^k \rangle] \to \mathbb{Z}[t]/(t^k - 1)$. Next recall that in the Free Resolution-$\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-Lemma 314.31 we showed that

$$\ldots \to \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(1+t+\cdots+t^{n-1})} \mathbb{Z}[t]/(t^k-1) \xrightarrow{\cdot(t-1)} \mathbb{Z}[t]/(t^k-1) \xrightarrow{t\mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0$$

is a free resolution of ${}_{\mathrm{tr}}\mathbb{Z}$. Together with the Free Module–Tensor-with-Ring Lemma 312.18 (1) and (2) we obtain the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccccc}
\to \mathbb{Z} \underset{\mathbb{Z}[t]/(t^k-1)}{\otimes} \mathbb{Z}[t]/(t^k-1) & \xrightarrow{\cdot(1+t+\cdots+t^{n-1})} & \mathbb{Z} \underset{\mathbb{Z}[t]/(t^k-1)}{\otimes} \mathbb{Z}[t]/(t^k-1)g & \xrightarrow{\cdot(t-1)} & \mathbb{Z} \underset{\mathbb{Z}[t]/(t^k-1)}{\otimes} \mathbb{Z}[t]/(t^k-1) & \to 0 \\
\cong\downarrow t\mapsto 1 & & \cong\downarrow t\mapsto 1 & & \cong\downarrow t\mapsto 1 & \\
\xrightarrow[0]{\cdot(1-1)} \mathbb{Z} & \xrightarrow[\cdot k]{\cdot(1+\cdots+1)} \mathbb{Z} & \xrightarrow[0]{\cdot(1-1)} \mathbb{Z} & \longrightarrow & 0. &
\end{array}$$

By the Group-H_*-Projective Resolution Proposition 317.5 it remains to determine the homology groups of the bottom chain complex. But these are evidently the ones we wrote down in the proposition. ■

317.3. **Group homology and products.** Our next goal will be to develop the tools for calculating the homology groups of all finitely generated abelian groups. As we all know, it follows from the Abelian Group-Classification Theorem 97.24 that finitely generated abelian groups can be written as the direct product of finitely many cyclic groups. Since we already dealt with cyclic groups it remains to prove the following theorem, which is a more precise version of the Group-H_*-Künneth Theorem 257.16.

Theorem 317.10. (Group-H_*-Künneth Theorem) For any two groups A and B and any $n \in \mathbb{N}_0$ there exist natural homomorphisms $\times$ and ν which give us a short exact sequence[3500]

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(A) \otimes \mathrm{H}_q(B) \xrightarrow{\times} \mathrm{H}_n(A \times B) \xrightarrow{\nu} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(A), \mathrm{H}_q(B)) \to 0$$

of abelian groups. Furthermore this short exact sequence splits.

Proof. We outsource the proof of the Group-H_*-Künneth Theorem 317.10 to the next Section 317.4. ■

Example. Let $d \in \mathbb{N}$. We want to calculate the second homology group of $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. Given $n \in \mathbb{N}_0$ we see that

$$\begin{aligned}
\mathrm{H}_2(\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) &\overset{\text{by the Group-H}_*\text{-Künneth Theorem 317.10}}{\cong} \underbrace{\bigoplus_{p+q=2} \mathrm{H}_p(\mathbb{Z}/d\mathbb{Z}) \otimes \mathrm{H}_q(\mathbb{Z}/d\mathbb{Z})}_{\cong \mathbb{Z}/d\mathbb{Z}\text{, this follows from the Group-H}_*\text{-of-}\mathbb{Z}/k\mathbb{Z}\text{-Lemma 317.9 and the Cyclic-Tensor-Cyclic Lemma 159.5}} \oplus \underbrace{\bigoplus_{p+q=1} \mathrm{Tor}(\mathrm{H}_p(\mathbb{Z}/d\mathbb{Z}), \mathrm{H}_q(\mathbb{Z}/d\mathbb{Z}))}_{=0\text{, by the Tor-Properties Lemma 314.24}} \\
&\cong \mathbb{Z}/d\mathbb{Z}.
\end{aligned}$$

□

[3499] In fact this result follows also directly from the discussion on page 6064.

[3500] In the Group-H_*-Künneth Theorem **??** we will see that $\times$ is indeed a cross product in homology.

In the following we will consider some more consequences of the Group-H_*-Künneth Theorem 317.10, but which this time we will formulate as lemmas:

Lemma 317.11. (Group-H_*–Product-with-$\mathbb{Z}$-Lemma) Given any group G and given any $n \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{H}_n(G) \oplus \mathrm{H}_{n-1}(G) \xrightarrow{\cong} \mathrm{H}_n(G \times \langle t \rangle)$$

of abelian groups.

Proof. By the Group-H_*-Künneth Theorem 317.10 there exists a short exact sequence of the following form:

$$0 \to \underbrace{\bigoplus_{p+q=n} \mathrm{H}_p(G) \otimes \mathrm{H}_q(\langle t \rangle)}_{=0,\text{ unless } q=0,1} \xrightarrow{\times} \mathrm{H}_n(G \times \langle t \rangle) \xrightarrow{\nu} \underbrace{\bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(G), \mathrm{H}_q(\langle t \rangle))}_{=0} \to 0.$$

Note that by the Group-H_*-of-$\mathbb{Z}$-Lemma 317.7 we know that $\mathrm{H}_q(\langle t \rangle) \cong \mathbb{Z}$ for $q = 0, 1$ and $= 0$ otherwise. It follows from the Tensor-with-$\mathbb{Z}$–Lemma 159.9 that the left-hand side is isomorphic to $\mathrm{H}_n(G) \oplus \mathrm{H}_{n-1}(G)$ and it follows from the Tor-Properties Lemma 314.24 that the right-hand side is zero. ■

Not surprisingly we can now finally fulfill our promise from page 6110, namely we can now calculate the homology groups of all finitely generated free abelian groups:

Lemma 317.12. (Group-H_*-of-$\mathbb{Z}^m$-Lemma) Let $m \in \mathbb{N}_0$. We consider the free abelian group $\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}} \cong \mathbb{Z}^m$ of rank m. For any $n \in \mathbb{N}_0$ we have an isomorphism[3501]

$$\mathrm{H}_n(\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}) \cong \mathrm{H}_n(\mathbb{Z}^m) \cong \mathbb{Z}^{\binom{m}{n}}.$$

Proof. This lemma follows easily from an elementary induction argument that makes use of the Group-H_*-of-Trivial Group Lemma 317.6, the Group-H_*–Product-with-$\mathbb{Z}$-Lemma 317.11 and the recurrence formula from page 111. We leave it to the reader to work out the details of this routine proof. ■

317.4. **Proof of the Group-H_*-Künneth Theorem 317.10.** The proof of the Group-H_*-Künneth Theorem 317.10 will require some preparations. First of all we recall the following:

- If R and S are rings, then we can consider the tensor product ring $R \otimes S$ from the Tensor Product-of-Rings Lemma 311.7. Furthermore, if M is an R-left module and N is an S-left module, then the Tensor Product-of-Rings–Modules Lemma 311.9 gives us the following:
  - ▷ The abelian group $M \otimes N$ is naturally an $R \otimes S$-module.
  - ▷ If M and N are both free modules, then so is $M \otimes N$.
- Let $\Theta\colon R \to S$ be a ring homomorphism. If M is an S-left module, then it follows from the Module–Change-of-Ring–Functor Lemma 309.13 that we can view M also as an R-left module $\Theta^* M$.

After this reminder we can now formulate the following lemma:

[3501] We refer to Definition 1.43 for the definition of $\binom{m}{n}$.

Lemma 317.13. (Tensor Product-of-Group Rings Lemma) Let A and B be two groups. By the Tensor Product-of-Rings Lemma 311.7 we can view $\mathbb{Z}[A] \otimes \mathbb{Z}[B]$ as a ring. The map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}[A] \otimes \mathbb{Z}[B] & \to & \mathbb{Z}[A \times B] \\ m \cdot a \otimes n \cdot b & \mapsto & m \cdot n \cdot (a,b) \end{array}$$

defines a natural isomorphism of rings. This isomorphism has the following two properties:

(1) We have a natural isomorphism

$$\begin{array}{rcl} {}_{A,\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}} {}_{B,\mathrm{tr}}\mathbb{Z} & \cong & \Theta_*({}_{A\times B,\mathrm{tr}}\mathbb{Z}) \\ m \otimes n & \mapsto & m \cdot n \end{array}$$

of $\mathbb{Z}[A] \otimes \mathbb{Z}[B]$-left modules.

(2) We have a *natural* isomorphism $(\mathbb{Z} \otimes \mathbb{Z}[A]) \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{Z}[B]}) \to \mathbb{Z} \otimes_{\mathbb{Z}[A\times B]} \mathbb{Z}[A \times B]$ of abelian groups.

Sketch of proof of Lemma 317.13. First note that it follows easily from definition of a group ring and the Tensor–Direct Sum Lemma 312.15 that the given map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}[A] \otimes \mathbb{Z}[B] & \to & \mathbb{Z}[A \times B] \\ m \cdot a \otimes n \cdot b & \mapsto & m \cdot n \cdot (a,b) \end{array}$$

is an isomorphism of *abelian groups*. Next note that it follows from the Tensor Product-of-Rings Lemma 311.7 that the given map is in fact a *ring homomorphism*. Thus the given map does indeed define an isomorphism of rings. Finally note that it follows easily from the definitions that Statements (1) and (2) hold. ■

Proof of the Group-H$_*$-Künneth Theorem 317.10. Let A and B be two groups and let $n \in \mathbb{N}_0$. We consider the natural free resolutions

$$\underbrace{\ldots \to \mathbb{Z}[A]^{X_2} \xrightarrow{f_2} \mathbb{Z}[A]^{X_1} \xrightarrow{f_1} \mathbb{Z}[A]_0^{X_1} \xrightarrow{f_0}}_{=:F_*} {}_{A,\mathrm{tr}}\mathbb{Z} \to 0$$

$$\underbrace{\ldots \to \mathbb{Z}[B]^{Y_2} \xrightarrow{g_2} \mathbb{Z}[B]^{Y_1} \xrightarrow{g_1} \mathbb{Z}[B]_0^{Y_1} \xrightarrow{g_0}}_{=:G_*} {}_{B,\mathrm{tr}}\mathbb{Z} \to 0$$

from Definition 313.12. As in Definition 163.2 we consider the tensor product $F_* \otimes G_*$ of the chain complexes F_* and G_*. We make two observations:

(i) It follows from the Tensor Product-of-Rings–Modules Lemma 311.9 that we can view $F_* \otimes G_*$ as a chain complex of free $\mathbb{Z}[A] \otimes \mathbb{Z}[B]$-left modules.

(ii) We consider the ring isomorphism

$$\Theta\colon \mathbb{Z}[A] \otimes \mathbb{Z}[B] \to \mathbb{Z}[A \times B]$$

from the Tensor Product-of-Group Rings Lemma 317.13. It follows from (i) and the Tensor Product-of-Group Rings Lemma 317.13 that $(\Theta^{-1})^*(F_* \otimes G_*)$ is a chain complex of free $\mathbb{Z}[A \times B]$-left modules.

Claim. The groups $(\Theta^{-1})^*(F_* \otimes G_*)$ define a free resolution of the $\mathbb{Z}[A \times B]$-left module ${}_{A\times B,\mathrm{tr}}\mathbb{Z}$.

Proof. First note that F_* and G_* have trivial homology in degree ≥ 1 and that in degree 0 the maps f_0 and g_0 induce isomorphisms $f_{0*}\colon \mathrm{H}_0(F_*) \to {}_{A,\mathrm{tr}}\mathbb{Z}$ and $g_{0*}\colon \mathrm{H}_0(G_*) \to {}_{B,\mathrm{tr}}\mathbb{Z}$ of $\mathbb{Z}[A]$-left modules respectively $\mathbb{Z}[B]$-left modules. Next note that it follows from the

Algebraic–Künneth Theorem 163.6 that the tensor product, over $\mathbb{Z}$, of two chain complexes F_* and G_* has again trivial homology in degree ≥ 1 and that in degree 0 the homomorphism $f_{0*} \otimes g_{0*} \colon \mathrm{H}_0(F_* \otimes G_*) \to {}_{A,\mathrm{tr}}\mathbb{Z} \otimes {}_{B,\mathrm{tr}}\mathbb{Z}$ is an isomorphism of abelian groups.

Next we consider the following little diagram:

$$\begin{array}{ccc} \mathrm{H}_0(F_* \otimes G_*) & \xrightarrow[\cong]{f_{0*} \otimes g_{0*}} & {}_{A,\mathrm{tr}}\mathbb{Z} \otimes {}_{B,\mathrm{tr}}\mathbb{Z} \\ & \searrow^{[\sigma] \mapsto f_0(\sigma) \cdot g_0(\sigma)} & \cong \big\downarrow m \otimes n \mapsto m \cdot n \\ & & \Theta_*({}_{A \times B,\mathrm{tr}}\mathbb{Z}). \end{array}$$

We make the following observations:

- It follows from the above discussion that the horizontal map is an isomorphism of abelian groups. In fact it follows from the Tensor Product-of-Rings–Modules Lemma 311.9 that the map $f_{0*} \otimes g_{0*} \colon \mathrm{H}_0(F_* \otimes G_*) \to {}_{A,\mathrm{tr}}\mathbb{Z} \otimes {}_{B,\mathrm{tr}}\mathbb{Z}$ is an isomorphism of $\mathbb{Z}[A] \otimes \mathbb{Z}[B]$-left modules.
- The vertical map is the isomorphism of $\mathbb{Z}[A] \otimes \mathbb{Z}[B]$-left modules from the Tensor Product-of-Group Rings Lemma 317.13 (1).
- The diagram diagram clearly commutes.

It follows from this discussion that $(\Theta^{-1})^*(F_* \otimes G_*)$, together with the homomorphism $(F_* \otimes G_*) \to \mathbb{Z}$ that is given by $\sigma \mapsto f_0(\sigma) \cdot g_0(\sigma)$, is in fact a free resolution of ${}_{A \otimes B,\mathrm{tr}}\mathbb{Z}$ by free $\mathbb{Z}[A \otimes B]$-left modules. ⊞

Next note that it follows from the Tensor Product-of-Group Rings Lemma 317.13 (2) that we have a natural isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}[A \times B]} (F_* \otimes G_*) \cong (\mathbb{Z}_{\mathrm{tr}} \otimes_{\mathbb{Z}[A]} F_*) \otimes_{\mathbb{Z}} (\mathbb{Z}_{\mathrm{tr}} \otimes_{\mathbb{Z}[B]} G_*)$ of chain complexes. This discussion shows that the desired statement now follows from the Algebraic–Künneth Theorem 163.6 applied to the chain complexes $\mathbb{Z}_{\mathrm{tr}} \otimes_{\mathbb{Z}[A]} F_*$ and $\mathbb{Z}_{\mathrm{tr}} \otimes_{\mathbb{Z}[B]} G_*$. ■

317.5. **Group homology and induced maps.** As always our goal is not only to associate groups/abelian groups/modules to objects, but we really want to define a covariant functor. In the setting of group homology this requires some effort. The key to defining induced maps is the following definition:

Definition 317.14. Let $\varphi \colon G \to G'$ be a group homomorphism. We introduce the following notation:

- We denote by F_* the natural free $\mathbb{Z}[G]$-left resolution of the trivial $\mathbb{Z}[G]$-module ${}_{\mathrm{tr}}\mathbb{Z}$.
- We denote by F'_* the natural free $\mathbb{Z}[G']$-left resolution of the trivial $\mathbb{Z}[G']$-module ${}_{\mathrm{tr}}\mathbb{Z}$.

We consider the following diagram:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \xrightarrow{\partial_0} & {}_{G,\mathrm{tr}}\mathbb{Z} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f \mapsto 1 \otimes f} & & \big\downarrow{\scriptstyle f \mapsto 1 \otimes f} & & \big\downarrow{\scriptstyle f \mapsto 1 \otimes f} & & \big\downarrow{\scriptstyle n \mapsto 1 \otimes n} & & \\
\ldots & \longrightarrow & \mathbb{Z}[G'] \underset{\mathbb{Z}[G]}{\otimes} F_2 & \xrightarrow{\mathrm{id} \otimes \partial_2} & \mathbb{Z}[G'] \underset{\mathbb{Z}[G]}{\otimes} F_1 & \xrightarrow{\mathrm{id} \otimes \partial_1} & \mathbb{Z}[G'] \underset{\mathbb{Z}[G]}{\otimes} F_0 & \xrightarrow{\mathrm{id} \otimes \partial_0} & \mathbb{Z}[G'] \underset{\mathbb{Z}[G]}{\otimes} {}_{G,\mathrm{tr}}\mathbb{Z} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \alpha_2} & & \big\downarrow{\scriptstyle \alpha_1} & & \big\downarrow{\scriptstyle \alpha_0} & & \big\downarrow{\scriptstyle g' \otimes n \mapsto n} & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow{\partial'_2} & F'_1 & \xrightarrow{\partial'_1} & F'_0 & \xrightarrow{\partial'_0} & {}_{G,\mathrm{tr}}\mathbb{Z}' & \longrightarrow & 0.
\end{array}$$

We make the following observations:

(a) By hypothesis the top horizontal sequence is in particular a chain complex.
(b) It follows from the Functor-applied-to-Chain Complex Lemma 314.3 that the middle horizontal sequence is also a chain complex.[3502]
(c) It follows from the Free Module–Tensor-with-Ring Lemma 312.18 that the middle horizontal sequence is in fact a chain complex of *free* $\mathbb{Z}[G']$-left modules.
(d) By hypothesis the bottom sequence is exact.
(e) The map bottom right vertical map that is given by $g' \otimes n \mapsto n$ is a homomorphism of $\mathbb{Z}[G']$-left modules.

It follows from the above discussion and the Homological Algebra–Fundamental Theorem 313.14 (1) that we get the bottom vertical $\mathbb{Z}[G']$-left module homomorphisms which make the diagram commute and which are unique up to a chain homotopy. For each $n \in \mathbb{N}_0$ we define

$$\begin{array}{rcl} \varphi_*\colon F_n & \to & F'_n \\ \sigma & \mapsto & \alpha_n(1 \otimes \sigma). \end{array}$$

Since the diagram commutes we see that the maps φ_* form a chain map of R-left modules.

The following lemma shows that the above chain maps give rise to homomorphisms of homology modules:

Lemma 317.15. (Group-H$_*$–Induced Map Lemma) Let $\varphi\colon G \to G'$ be a group homomorphism. Furthermore let R be a ring, let M be $(R, \mathbb{Z}[G'])$-left left module. Note that φ induces a ring homomorphism $\mathbb{Z}[G] \to \mathbb{Z}[G']$. As in the Module–Change-of-Ring–Functor Lemma 309.13 we thus obtain the $(R, \mathbb{Z}[G'])$-left left module ${}_\varphi M$. For each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \varphi_*\colon \mathrm{H}_n(G; {}_\varphi M) & \to & \mathrm{H}_n(G'; M) \\ [m \otimes \sigma] & \mapsto & [m \otimes \underset{\uparrow}{\varphi_*}(\sigma)] \end{array}$$

see Definition 317.14

is a well-defined homomorphism of R-left modules.

Proof. As we discussed in Definition 317.14, it follows from the Homological Algebra–Fundamental Theorem 313.14 (2) that the chain map φ_* is well-defined up to a chain homotopy. It follows from this discussion together with the Functor-applied-to-Chain Complex Lemma 314.3, the Tensor–Functor Lemma 314.4 and the Chain Homotopy–H$_*$-Lemma 313.6 that the homomorphism $\varphi_*\colon \mathrm{H}_n(G; {}_\varphi M) \to \mathrm{H}_n(G'; M)$ does indeed not depend on any choices. ■

The statement of the Group-H$_*$–Induced Map Lemma 317.15 is cleanest if we consider the coefficient module ${}_{\mathrm{tr}}\mathbb{Z}$. This brings us to the following definition:

Definition 317.16. Let $\varphi\colon A \to B$ be a group homomorphism. Note that it follows immediately from the definitions that we have an equality ${}_\varphi({}_{B,\mathrm{tr}}\mathbb{Z}) = {}_{A,\mathrm{tr}}\mathbb{Z}$ of left $\mathbb{Z}[A]$-left modules. Thus the Group-H$_*$–Induced Map Lemma 317.15 gives us a well-defined induced homomorphism

$$\varphi_*\colon \mathrm{H}_n(A) \to \mathrm{H}_n(B)$$

of abelian groups.

[3502] We certainly do *not* claim that the middle horizontal sequence is exact.

We conclude this section with the following lemma, which can be viewed as the analog of the H_n–Functor Lemma 143.20:

Lemma 317.17. (Group-H_n–Functor Lemma) For each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} G & \mapsto & \mathrm{H}_n(G) \\ (\varphi\colon G \to H) & \mapsto & \underbrace{(\varphi_*\colon \mathrm{H}_n(G) \to \mathrm{H}_n(H))}_{\substack{\text{defined in the Group-H}_* \\ \text{Induced Map Lemma 317.15}}} \end{array}$$

define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

Sketch of proof. This lemma seems obvious, but if one looks at Definition 317.14 and writes down the statements that need to be proved, then one realizes that this takes more effort than one might initially realize. Eventually the proof of the lemma boils down to using the uniqueness statement of the Homological Algebra–Fundamental Theorem 313.14 (2) on several occasions. We leave it to the reader to fill in the details. ■

317.6. **Group cohomology.** If one can assign a homology module to an object, then surely one can also associate a cohomology module to the same object. For groups this is achieved by the following definition:

Definition 317.18. Let G be a group, let R be a ring and let M be an $(R, \mathbb{Z}[G])$-left left module. As in Definition 317.1 let ${}_{G,\mathrm{tr}}\mathbb{Z}$ be the trivial $\mathbb{Z}[G]$-left module. We define

$$\mathrm{H}^n(G; M) := \underbrace{\mathrm{Ext}^n_{\mathbb{Z}[G]}({}_{G,\mathrm{tr}}\mathbb{Z}, M)}_{\text{see Definition 315.35}}$$

which is an R-left module. We refer to $\mathrm{H}^n(G; M)$ as the n-th cohomology module of the group G with coefficients in M.

For the reader's convenience we spell out the same definition in slightly greater detail:

Definition 317.18. Let G be a group, let R be a ring and let M be an $(R, \mathbb{Z}[G])$-left left module. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} {}_{G,\mathrm{tr}}\mathbb{Z} \to 0$$

the natural free $\mathbb{Z}[G]$-left resolution from Definition 313.12. We set

$$\begin{array}{rcl} \mathrm{H}^n(G; M) & := & \mathrm{Ext}^n_{\mathbb{Z}[G]}({}_{G,\mathrm{tr}}\mathbb{Z}, M) \\ & := & \mathrm{H}^n\Big(\ldots \to \underset{\mathbb{Z}[G]\text{-left}}{\mathrm{Hom}}(F_2, M) \xleftarrow{f_2^*} \underset{\mathbb{Z}[G]\text{-left}}{\mathrm{Hom}}(F_1, M) \xleftarrow{f_1^*} \underset{\mathbb{Z}[G]\text{-left}}{\mathrm{Hom}}(F_0, M) \leftarrow 0\Big). \\ & & \uparrow \\ & & \text{see Definition 315.35} \end{array}$$

For the most part we will not really make use of group cohomology. Thus we will not work out the full theory. We just formulate the following obvious analog of the Group-H_*-Projective Resolution Proposition 317.5:

Proposition 317.19. (Group-H*-Projective Resolution Proposition) Let G be a group, let R be a ring and let M be an $(R, \mathbb{Z}[G])$-left left module. Let

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} {}_{G,\mathrm{tr}}\mathbb{Z} \to 0$$

be a projective resolution of the trivial $\mathbb{Z}[G]$-left module ${}_{G,\mathrm{tr}}\mathbb{Z}$. For every $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}^n(G; M) \xrightarrow{\cong} \mathrm{H}^n\Big(\ldots \to \underset{\mathbb{Z}[G]\text{-left}}{\mathrm{Hom}}(P_2, M) \xleftarrow{f_2^*} \underset{\mathbb{Z}[G]\text{-left}}{\mathrm{Hom}}(P_1, M) \xleftarrow{f_1^*} \underset{\mathbb{Z}[G]\text{-left}}{\mathrm{Hom}}(P_0, M) \leftarrow 0\Big).$$

of R-left modules which is natural in M.

Proof. This proposition follows immediately from Definition 317.18 and the Projective Resolution Proposition 315.37. ■

Example. There is surely no need to go through all examples again. But let us at least consider the case of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ for $k \in \mathbb{N}$. In this case we have

$$\mathrm{H}_n(\mathbb{Z}/k\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}/k\mathbb{Z}, & \text{if } n \text{ is odd}, \\ 0, & \text{otherwise} \end{cases} \qquad \mathrm{H}^n(\mathbb{Z}/k\mathbb{Z}; {}_{\mathrm{tr}}\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \text{ is odd}, \\ \mathbb{Z}/k\mathbb{Z}, & \text{if } n \text{ is even and} \neq 0. \end{cases}$$

by the Group-H$_*$-of-$\mathbb{Z}/k\mathbb{Z}$-Lemma 317.9 (or alternatively the discussion on page 6064)

follows from the calculation on page 6093 □

317.7. **Cohomological dimension.** The following definition is quite natural, even though the name is initially a little surprising:

Definition 317.20. Let G be a group. Its cohomological dimension $\mathrm{cd}(G)$ is defined as the minimal $n \in \mathbb{N}_0 \cup \{\infty\}$ such that ${}_{G,\mathrm{tr}}\mathbb{Z}$ admits a projective resolution by $\mathbb{Z}[G]$-left modules of length n.[3503]

The following proposition can in particular be used to give lower bounds on the cohomological dimension of a group:

Proposition 317.21. (H*-Dimension–Lower Bound–Proposition) Let G be a group of cohomological dimension $\mathrm{cd}(G)$. Furthermore let R be a ring, let M be an $(R, \mathbb{Z}[G])$-left left module and let $k \in \mathbb{N}_0$.

(1) If $k > \mathrm{cd}(G)$, then $\mathrm{H}_n(G; M) = 0$.
(2) If $k > \mathrm{cd}(G)$, then $\mathrm{H}^n(G; M) = 0$.

Proof. This proposition follows almost immediately from the Group-H$_*$-Projective Resolution Proposition 317.5 and the Group-H*-Projective Resolution Proposition 317.19. ■

Example.

(1) Let $k \in \mathbb{N}_{\geq 2}$. It follows immediately from the H*-Dimension–Lower Bound–Proposition 317.21 (1) together with the Group-H$_*$-of-$\mathbb{Z}/k\mathbb{Z}$-Lemma 317.9 that the cohomological dimension of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ is infinite.
(2) (a) Let A and B be two groups. Let F_* be a projective resolution of ${}_{A,\mathrm{tr}}\mathbb{Z}$ by $\mathbb{Z}[A]$-modules respectively and let G_* be a projective resolutions of ${}_{B,\mathrm{tr}}\mathbb{Z}$ by $\mathbb{Z}[B]$-modules. The argument in the *proof* of the Group-H$_*$-Künneth Theorem 317.10 together with the Tensor Product-of-Rings–Modules Lemma 311.9 (2) shows that $F_* \otimes G_*$ forms

[3503] Recall that in Definition 313.11 we said that a resolution P_* has length $n \in \mathbb{N}_0$ if $P_k = 0$ for all $k > n$.

a projective resolution of ${}_{A\times B,\mathrm{tr}}\mathbb{Z}$. From this discussion we obtain the inequality $\mathrm{cd}(A\times B)\leq \mathrm{cd}(A)+\mathrm{cd}(B)$.

(b) It follows almost immediately from (a), the Group-H_*-of-$\mathbb{Z}$-Lemma 317.7 (1) and the Group-H_*-of-$\mathbb{Z}^m$-Lemma 317.12 that for any $m\in\mathbb{N}_0$ we have $\mathrm{cd}(\mathbb{Z}^m)=m$. □

The following proposition is in particular a converse to the H^*-Dimension–Lower Bound–Proposition 317.21 (2) and it explains the name "cohomological dimension":

Proposition 317.22. (H^*-Dimension–Proposition) Let G be a group. The cohomological dimension $\mathrm{cd}(G)$ equals the maximal dimension $n\in\mathbb{N}_0\cup\{\infty\}$ for which there exists a left $\mathbb{Z}[G]$-module M such that $\mathrm{H}^n(G;M)\neq 0$.

Proof. This proposition is proved in [**BrownK1982**, Lemma VIII.2.1]. The proof is not particularly long, but since we will not need it, we have no qualms with not providing it. ■

As we will see shortly, the following proposition has surprisingly many interesting consequences:

Proposition 317.23. (H^*-Dimension–Subgroup Proposition) Let G be a group. For any subgroup H of G we have the inequality

$$\mathrm{cd}(H)\ \leq\ \mathrm{cd}(G).$$

Proof. Clearly it suffices to prove the following statement:

Statement. If

$$0\ \to\ P_n\ \xrightarrow{f_n}\ \ldots\ {}_2\ \xrightarrow{f_2}\ P_1\ \xrightarrow{f_1}\ P_0\ \xrightarrow{f_0}\ {}_{G,\mathrm{tr}}\mathbb{Z}\ \to\ 0$$

is a $\mathbb{Z}[G]$-projective resolution of length n of the trivial $\mathbb{Z}[G]$-left module ${}_{G,\mathrm{tr}}\mathbb{Z}$, then there exists also a $\mathbb{Z}[H]$-projective resolution of length n of the trivial $\mathbb{Z}[H]$-left module ${}_{H,\mathrm{tr}}\mathbb{Z}$.

By the Module–Change-of-Ring–Functor Lemma 309.13 we can view each P_i as a $\mathbb{Z}[H]$-left module. Thus we just need to show that each P_i is projective as a $\mathbb{Z}[H]$-module. In other words, we just need to prove the following claim:[3504]:

Claim. Let $\varphi\colon \mathbb{Z}[H]\to\mathbb{Z}[G]$ be the ring homomorphism that is induced by the inclusion $H\to G$. If P is a projective $\mathbb{Z}[G]$-left module, then ${}_\varphi P$ is a projective $\mathbb{Z}[H]$-left module.

Proof. Since P is a projective $\mathbb{Z}[G]$-left module we know by the Projective Module–Characterization Lemma 309.28 that there exists a $\mathbb{Z}[G]$-left module Q such that the $\mathbb{Z}[G]$-left module $P\oplus Q$ is isomorphic to $\mathbb{Z}[G]^{(X)}$ for some set X. Let S be a complete set of representatives of $H\backslash G$. Now we obtain the following isomorphisms of $\mathbb{Z}[H]$-left modules:

$$\begin{array}{ccccccccc}
{}_\varphi P\oplus {}_\varphi Q & \cong & {}_\varphi(P\oplus Q) & \cong & {}_\varphi\Big(\overbrace{\bigoplus_X \mathbb{Z}[G]}^{=\mathbb{Z}[G]^{(X)}}\Big) & \cong & \bigoplus_X {}_\varphi\mathbb{Z}[G] & \cong & \bigoplus_X\bigoplus_{H\backslash G}\mathbb{Z}[H].\\
\uparrow & & & & & & \uparrow & & \uparrow & & \\
\multicolumn{2}{c}{\text{by the Module–Change-of-Ring Functor Lemma 309.13 (2)}} & & & & & & & \multicolumn{2}{c}{\begin{array}{l}\text{since } {}_\varphi\mathbb{Z}[G] \text{ is a free}\\ \mathbb{Z}[H]\text{-left module with}\\ \text{basis } S\text{, see Exercise 310.2}\end{array}}
\end{array}$$

It thus follows from the Projective Module–Characterization Lemma 309.28 that ${}_\varphi P$ is indeed a projective $\mathbb{Z}[H]$-left module. ■

[3504] Note that the claim gives a solution to Exercise 310.3

We conclude this section with the following proposition which gives a very useful tool for showing that a group is torsion-free:

Proposition 317.24. (Torsion–H*-Dimension Proposition) Let G be a group. If G has a torsion element, i.e. if there exists a $g \neq e \in G$ and an $n \in \mathbb{N}$ with $g^n = e$, then $\mathrm{cd}(G) = \infty$. In other words, if $\mathrm{cd}(G) < \infty$, then G is torsion-free.

Example. Let G be a finite non-trivial group. It follows immediately from the Torsion–H*-Dimension Proposition 317.24 that $\mathrm{cd}(G) = \infty$. □

Proof. Let G be a group. We suppose that there exists an $e \neq g \in G$ and an $n \in \mathbb{N}$ with $g^n = e$. Let n be the minimal natural number with $g^n = e$. We now see that

$$\mathrm{cd}(G) \underset{\uparrow}{\geq} \mathrm{cd}(H) \underset{\uparrow}{=} \mathrm{cd}(\mathbb{Z}/n\mathbb{Z}) \underset{\uparrow}{=} \infty.$$

by the Cohomological Dimension–Subgroup Proposition 317.23 | by the choice of n | see the discussion on page 6117

■

317.8. Exercises for Chapter 317.

Exercise 317.1. Let G be a group.

(a) Show that we have a natural isomorphism $\mathrm{H}_0(G) \cong \mathbb{Z}$.
(b) Show that we have a natural isomorphism $\mathrm{H}^0(G; \mathbb{Z}) \cong \mathbb{Z}$.

Hint. Use the Tensor–Right-Exact Lemma 314.14 and use the Hom-Left-Exact Lemma 315.27.

Exercise 317.2.

(a) Compute the homology of the group $G = \langle t \rangle$ for the unique non-trivial representation $\langle t \rangle \to \mathrm{GL}(1, \mathbb{Z})$.
(b) Do the same as in (a) for cohomology.

Exercise 317.3. Let $k \in \mathbb{N}$. Compute the group cohomology groups $\mathrm{H}^*(\mathbb{Z}/k\mathbb{Z}; {}_{\mathrm{tr}}\mathbb{Z})$.

Exercise 317.4.

(a) Show that there exists a $(\mathbb{C}, \mathbb{Z}[\mathbb{Z}^2])$-left left module structure on $\mathbb{C}$ such that for all $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(\mathbb{Z}^2; M) = 0$.
(b) Show that there exists a $(\mathbb{C}, \mathbb{Z}[\mathbb{Z}^3])$-left left module structure on $\mathbb{C}$ such that for all $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(\mathbb{Z}^3; M) = 0$.

Exercise 317.5. Let $k \in \mathbb{N}_{\geq 2}$. Does there exist a $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$-left module M such that for all $i \in \mathbb{N}_0$ we have $\mathrm{H}_i(\mathbb{Z}/k\mathbb{Z}; M) \neq 0$?

Exercise 317.6. Let $k \in \mathbb{N}$. We consider the non-orientable surface $N_k = k \cdot \mathbb{R}\mathrm{P}^2$ of genus k. What are the possible isomorphism types of abelian subgroups of $\pi_1(N_k)$?

Exercise 317.7. Let M be a closed orientable connected non-empty aspherical n-dimensional topological manifold.

(a) Show that if Γ is a subgroup of infinite index of $\pi_1(M)$, then $\mathrm{H}_n(\Gamma) = 0$.

(b) Suppose that we have an isomorphism $\pi_1(M) \cong A *_G B$ such that the homomorphisms $G \to A$ and $G \to B$ are both monomorphisms. Show that if $\mathrm{H}_k(G) = 0$ for $k = n-1, n$, then one of the two groups A or B is trivial.

(c) Show that if M is homeomorphic to a connected sum $X \# Y$ of two closed smooth manifolds, then one of X or Y is a homotopy sphere.
Hint. Use Characterization (5) of the Homotopy Sphere-Characterization Proposition 210.18.

Exercise 317.8. Show that for every $k \in \mathbb{N}$ we have $\dim_{\mathbb{Q}}(\mathrm{H}_k(\mathbb{Z}^\infty; \mathbb{Q})) = \infty$.

Exercise 317.9. Apply the Group-H_*-Künneth Theorem 317.10 to the direct product $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and show that it is consistent with the calculations in the Group-H_*-of-$\mathbb{Z}/k\mathbb{Z}$-Lemma 317.9.

Exercise 317.10. We consider the free abelian group $G = \langle x, y, z \rangle_{\mathrm{ab}} \cong \mathbb{Z}^3$ of rank 3. We make the usual identification $\mathbb{Z}[\langle x, y, z \rangle_{\mathrm{ab}}] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

(a) Provide an explicit resolution of ${}_{\mathrm{tr}}\mathbb{Z}$ by finitely generated $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$-modules.

(b) Compute the homology groups of G just using your free resolution from (a).
Remark. Make sure it matches the calculation from the Group-H_*-of-$\mathbb{Z}^m$-Lemma 317.12.

Exercise 317.11. Let G be a finitely generated group with non-zero abelianization.

(a) Show that there exists a $(\mathbb{C}, \mathbb{Z}[G])$-left left module structure on $M = \mathbb{C}$ such that $\mathrm{H}_0(G; M) = 0$.

(b) Does the analogous statement in (a) also hold for $M = \mathbb{R}$?

(c) Does the analogous statement in (a) also hold for cohomology?

CHAPTER 318

Group (co-) homology and topology

Let G be a group. In the previous Chapter 317 we used purely algebraic tools to introduce the (co-) homology groups $\mathrm{H}_*(G)$ and $\mathrm{H}^*(G;\mathbb{Z})$. We managed to calculate these groups for finitely generated abelian groups. But we lacked the tools to tackle the more interesting case of non-abelian groups. In this section topology makes its triumphant comeback: More precisely we will see that topology can be used to calculate homology and cohomology of groups.

This chapter is organized as follows:

- In Section 318.1 we will see that topology can be used to give free resolutions of trivial $\mathbb{Z}[G]$-modules.
- In Section 318.2 we will use topological tools to calculate various homology groups of groups.
- In Section 318.3 we will discuss several applications of the topological approach to group homology.
- In Section 318.4 we will use group homology to give upper bounds on the deficiency of a finitely presented group.
- In Appendix 318.5 we will formulate the Eilenberg-Ganea Conjecture.
- In Section 318.6 we will provide exercises for Chapter 318.

318.1. **Free resolutions arising from topology.** We mark our return to topology by recalling Definition 126.11 and Definition 257.8:

Definition 318.1. A non-empty topological space X is called aspherical if it is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

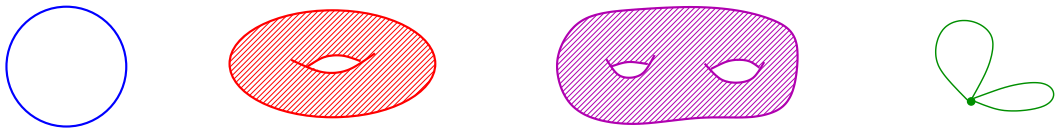
Next we recall Definition 257.9 and Definition 307.2:

Definition 318.2. Let G be a group and let X be a path-connected non-empty topological space. We say that X is an Eilenberg-Maclane space of type $\mathrm{K}(G,1)$ if the following three conditions hold:

(a) $\pi_1(X) \cong G$.
(b) X is aspherical.
(c) X admits a CW-structure.

Example. In the Eilenberg–Maclane Example Lemmas 257.10 and 307.3 we gave many examples of Eilenberg–Maclane spaces, some of which are illustrated below.
We will soon exploit these to perform calculations of group homology. □

The next propositions explains what Eilenberg-Maclane spaces got to do with group theory:



$\mathrm{K}(\mathbb{Z},1)$ $\mathrm{K}(\mathbb{Z}^2,1)$ $\mathrm{K}(\pi_1(\Sigma_2),1)$ $\mathrm{K}(\langle x,y\rangle,1)$

Proposition 318.3. (Free Resolution-via-K$(G,1)$-Proposition) Let G be a group, let X be a connected CW-complex and let $\varphi\colon \pi_1(X) \to G$ be an isomorphism. We denote by $p\colon \widetilde{X} \to X$ the universal covering of the CW-complex X.[3505]

(1) The group G acts naturally on $\widetilde{X}$ and each $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X})$ is a free $\mathbb{Z}[G]$-left module. In fact, if $\{c_i\}_{i\in I}$ is the set of k-cells of X and if for each $i \in I$ we pick a k-cell $\widetilde{c}_i$ of $\widetilde{X}$ with $p(\widetilde{c}_i) = c_i$, then the $\{\widetilde{c}_i\}_{i\in I}$ define a basis for $\mathrm{C}_k(\widetilde{X})$ as a free $\mathbb{Z}[G]$-left module.

(2) We now assume that X is an Eilenberg-Maclane space of type $\mathrm{K}(G,1)$.

(a) The cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ defines a free resolution of ${}_{G,\mathrm{tr}}\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules.[3506]

(b) We have $\mathrm{cd}(G) \leq \dim(X)$.[3507]

Proof. [3508] Let G be a group, let X be a connected CW-complex and let $\varphi\colon \pi_1(X) \to G$ be an isomorphism.

(1) The desired statement follows from the following observations:

- We use Convention 128.12 and the isomorphism φ to equip the universal covering $\widetilde{X}$ with a left G-action.
- We use the CW-Complex Covering Proposition 138.1 (1) to equip $\widetilde{X}$ with the corresponding natural CW-structure.
- It is clear from the definition of the CW-structure on $\widetilde{X}$ that the G-action on $\widetilde{X}$ preserves this CW-structure, in particular the G-action is given by cellular maps.
- It follows from the naturality of the boundary homomorphism and the Group Ring–Module Lemma 97.34 that we can view the cellular chain complex $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X})$ as a chain complex of $\mathbb{Z}[G]$-left modules.
- Let $\{c_i\}_{i\in I}$ be the set of k-cells of X. Furthermore suppose that for each $i \in I$ we have a k-cell $\widetilde{c}_i$ of $\widetilde{X}$ with $p(\widetilde{c}_i) = c_i$. It follows easily from the CW-Complex Covering Proposition 138.1 (2) and the Group Ring–Free Module Lemma 310.13 that the $\{\widetilde{c}_i\}_{i\in I}$ define a basis for $\mathrm{C}_k(\widetilde{X})$ as a free $\mathbb{Z}[G]$-left module.

[3505] In the CW-Complex–Local Properties Proposition 137.8 (2) we showed that CW-complexes are locally contractible. It thus follows from the Covering Existence Theorem 127.10 that the universal covering actually exists.

[3506] The argument of the proof will also show that the singular chain complex $\mathrm{C}_*(\widetilde{X})$ is also a free resolution of $\mathbb{Z}_{\mathrm{tr}}$ by free $\mathbb{Z}[G]$-modules. But for practical purposes it is much more useful to know that the cellular chain complex defines a free resolution, since the cellular chain complex is "significantly smaller" than the singular chain complex.

[3507] Recall that according to Definition 136.2 the dimension $\dim(X) \in \mathbb{Z}_{\geq -1} \cup \{\infty\}$ of a CW-complex is the largest n such that X contains an n-cell.

[3508] The following discussion has a significant overlap with the Alexander Module–CW Proposition 286.12.

(2) We now assume that X is an Eilenberg-Maclane space of type $\mathrm{K}(G,1)$. We consider the following sequence

$$\mathrm{C}_2^{\mathrm{CW}}(\widetilde{X}) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{CW}}(\widetilde{X}) \xrightarrow{\partial_2} \mathrm{C}_0^{\mathrm{CW}}(\widetilde{X}) \xrightarrow{\epsilon} {}_{G,\mathrm{tr}}\mathbb{Z} \to 0$$

where to the right we have the cellular augmentation map, which is clearly a homomorphism of $\mathbb{Z}[G]$-left modules. Next recall that by (1) we know that the $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X})$ are free $\mathbb{Z}[G]$-left modules.

Claim 1. The boundary maps are $\mathbb{Z}[G]$-left module homomorphisms.

Proof. This claim follows almost immediately from the naturality of the cellular boundary maps, see the Cellular Homology-Functor Lemma 155.7 and the observation that each deck transformation is a cellular self-map of $\widetilde{X}$. ⊞

We now see that it remains to prove the following claim:

Claim 2. The above sequence is exact.

Proof. Note that the above sequence is just the reduced cellular chain complex of the universal covering. Thus we see that for each $k \in \mathbb{Z}$ we have

$$\mathrm{H}_k(\text{above chain complex}) = \widetilde{\mathrm{H}}_k^{\mathrm{CW}}(\widetilde{X}) \underset{\text{follows from the Singular-Cellular-H}_* \text{ Proposition 155.8}}{\cong} \widetilde{\mathrm{H}}_k(\widetilde{X}) \underset{\text{follows from the Hurewicz Corollary 167.13, since } \widetilde{X} \text{ is aspherical}}{=} 0.$$

■

We formulate our first example of the Free Resolution-via-$\mathrm{K}(G,1)$-Proposition 318.3 as a lemma:

Lemma 318.4. (Free Resolution–Free Group Ring-Lemma) Let $m \in \mathbb{N}_0$. The following is a free resolution of the trivial $\mathbb{Z}[\langle x_1,\ldots,x_m\rangle]$-module ${}_{\mathrm{tr}}\mathbb{Z}$:[3509] [3510]

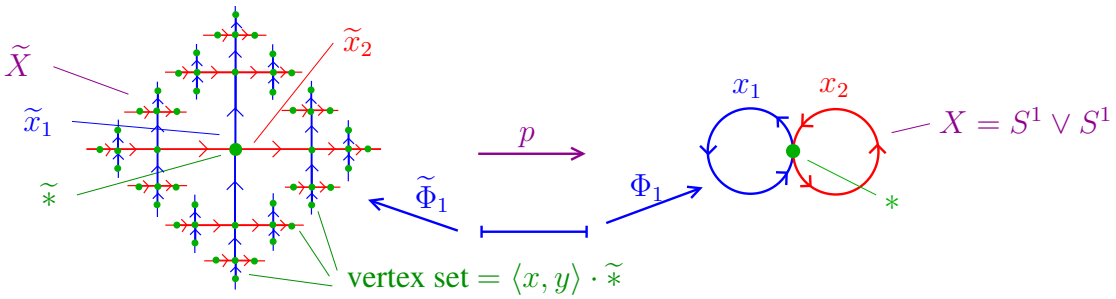
$$0 \to \mathbb{Z}[\langle x_1,\ldots,x_m\rangle]^{1\times m} \xrightarrow{\cdot\begin{pmatrix} x_1-1 \\ \vdots \\ x_m-1 \end{pmatrix}} \mathbb{Z}[\langle x_1,\ldots,x_m\rangle] \xrightarrow{x_i\mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0.$$

Sketch of proof. [3511] We consider the wedge $X = S^1 \vee \cdots \vee S^1$ of m circles where we always use $1 \in S^1$ as the wedge point. It follows from the discussion on page 2625 that X admits the structure of a 1-dimensional CW-complex with a single 0-cell (which is given by the wedge point $*$) and m 1-cells $x_1,\ldots,x_m$ which correspond to the m circles. For $i = 1,\ldots,m$ we denote by $\Phi\colon [-1,1] \to x_i$ be the obvious characteristic map that is given by going counterclockwise around the 1-cell x_i.

[3509] Let R be a ring. Recall that in Notation 309.4 we introduced the set of row vectors $R^{1\times k}$. On page 5962 we saw that the row vectors $R^{1\times k}$ are naturally an R-left module. Furthermore we saw in the Left Module Homomorphism–Matrix Lemma 309.5 that right multiplication by a matrix defines a homomorphism of such R-left modules.

[3510] This sequence is really quite remarkable: In the Group Ring–Invariance Basis Number Lemma 312.23 we learned that given any group G any free $\mathbb{Z}[G]$-left module has a well-defined rank. But we now see that this rank does not necessarily have the properties of a dimension as summarized in the Module-over-Skew Field Proposition 309.33, namely we just gave an example where a free $\mathbb{Z}[G]$-left module of rank 1 contains a free $\mathbb{Z}[G]$-left module of higher rank!

[3511] Note that the Free Resolution–Free Group Ring-Lemma 318.4 is purely algebraic. Nonetheless we will give a proof that makes use of topology. We leave it to the reader to come up with a purely algebraic proof.



Next we consider the homomorphism

$$\begin{array}{rcl} \varphi\colon \langle x_1, \ldots, x_m\rangle & \to & \pi_1(X, *) \\ x_i & \mapsto & [t \mapsto \Phi_i(2t-1)]. \end{array}$$

It follows from the Wedge-of-Circles-π_1-Lemma 99.11 that this map is an isomorphism. Furthermore note that it follows from the 1-Complex-π_n-Lemma 138.2 that all higher groups of X are zero, i.e. the topological space X is aspherical. We have thus shown that X is an aspherical 1-dimensional CW-complex and we exhibited an explicit isomorphism $\langle x_1, \ldots, x_m\rangle \to \pi_1(X, *)$.

Next we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}[\langle x_1, \ldots, x_m\rangle]^{1\times m} & \xrightarrow{\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_m - 1 \end{pmatrix}} & \mathbb{Z}[\langle x_1, \ldots, x_m\rangle] & \xrightarrow{x_i \mapsto 1} & {}_{\mathrm{tr}}\mathbb{Z} & \longrightarrow & 0 \\ & & \cong \downarrow{\scriptstyle \mathfrak{e}_i \mapsto \widetilde{x}_i} & & 1 \mapsto \widetilde{*} \downarrow \cong & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{C}_1^{\mathrm{CW}}(\widetilde{X}) & \xrightarrow{\partial_1} & \mathrm{C}_0^{\mathrm{CW}}(\widetilde{X}) & \xrightarrow{\epsilon} & {}_{\mathrm{tr}}\mathbb{Z} & \longrightarrow & 0. \end{array}$$

The lemma follows from the following clarifications and observations:

- We denote by $\widetilde{X}$ the universal covering of X.
- It follows from the Free Resolution-via-K$(G, 1)$-Proposition 318.3 (2) that the top horizontal sequence is exact
- We pick a lift $\widetilde{*}$ of the 0-cell $*$.
- For each $i \in \{1, \ldots, m\}$ let $\widetilde{\Phi}_i\colon [-1, 1] \to \widetilde{X}$ be the unique lift of characteristic map $\Phi_i[\colon [-1, 1] \to X$ with $\widetilde{\Phi}_i(-1) = \widetilde{*}$.
- For each $i \in \{1, \ldots, m\}$ we consider the 1-cell $\widetilde{x}_i := \widetilde{\Phi}_i([-1, 1])$ of $\widetilde{X}$.
- It follows easily from the Deck Transformation–Normal Subgroup Proposition 128.11 that $\partial_1(\widetilde{x}_i) = x_i \cdot \widetilde{*} - \widetilde{*} = (x_i - 1) \cdot \widetilde{*}$. This observation implies that the left square commutes. It follows immediately from the definitions that the right square commutes.
- It follows from the Free Resolution-via-K$(G, 1)$-Proposition 318.3 (1) that the vertical maps are isomorphisms of $\mathbb{Z}[\langle x_1, \ldots, x_m\rangle]$-left modules. ■

318.2. **Homology groups of groups via topology.** As we will now see, the Free Resolution-via-K$(G, 1)$-Proposition 318.3 allows us to use our topological expertise to calculate (co-) homology groups of groups. More precisely, we have the following proposition, which is closely related to the Group H_*-via-K$(G, 1)$-Proposition 257.15:

Proposition 318.5. (Group H_*-via-K$(G,1)$-Proposition) Let G be a group and let X be an Eilenberg-Maclane space of type $K(G,1)$.

(1) For each $n \in \mathbb{N}_0$ we have an isomorphism $H_n(G) \cong H_n(X)$.
(2) For each $n \in \mathbb{N}_0$ we have an isomorphism $H^n(G;\mathbb{Z}) \cong H^n(X;\mathbb{Z})$.

Remark.

(1) The Group H_*-via-K$(G,1)$-Proposition 318.5 shows that the two definitions of group homology from Definition 257.14 and Definition 317.4 are consistent.
(2) The Group H_*-via-K$(G,1)$-Proposition 318.5 only deals with group homology and group cohomology with the trivial ${}_{G,\mathrm{tr}}\mathbb{Z}$-coefficients. In Chapter 320 we will introduce twisted (co-) homology of CW-complexes. Later in the Group H_*-via-K$(G,1)$-Proposition **??** we will then relate group (co-) homology and twisted (co-) homology of Eilenberg-Maclane spaces with coefficients in any $\mathbb{Z}[G]$-left module.
(3) In this chapter we will mostly use the Group H_*-via-K$(G,1)$-Proposition 318.5 to perform calculations. But this result can also be used to prove theoretical statements. For example the earlier Group-H_*-Künneth Theorem 257.16 together with the Group H_*-via-K$(G,1)$-Proposition 318.5 gives a topological proof of the Group H_*-via-K$(G,1)$-Proposition 318.5. □

Proof. Let G be a group and let X be an Eilenberg-Maclane space of type $K(G,1)$. We denote by $\widetilde{X}$ the universal covering of the CW-complex X. It follows from the Free Resolution-via-K$(G,1)$-Proposition 318.3 (2) that $C_*^{\mathrm{CW}}(\widetilde{X})$ forms a free resolution of ${}_{G,\mathrm{tr}}\mathbb{Z}$.

(1) The desired statement follows easily from the above discussion, the Group-H_*-Projective Resolution Proposition 317.5 together with the Singular-Cellular-H_*-Proposition 155.8 and the following claim:

Claim. The maps
$$\begin{array}{rcl} {}_{G,\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[G]} C_*^{\mathrm{CW}}(\widetilde{X}) & \to & C_*^{\mathrm{CW}}(X) \\ n \otimes \widetilde{\sigma} & \mapsto & n \cdot p_*(\widetilde{\sigma}) \end{array}$$
define an isomorphism of chain complexes of abelian groups.

Proof. First note that it follows from the naturality of the cellular boundary map that the given map is indeed a chain map. Secondly note that it follows easily from the Free Resolution-via-K$(G,1)$-Proposition 318.3 (2) together with the Tensor–Direct Sum Lemma 312.15 and the Tensor-with-Base Ring Lemma 312.10 that the given map is an isomorphism of abelian groups. ⊞

(2) The desired statement follows easily from the above discussion, the Group-H^*-Projective Resolution Proposition 317.19 together with the Singular-Cellular-H^*-Isomorphism Proposition 199.5 and the following claim:

Claim. The maps
$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}}(C_*^{\mathrm{CW}}(X),\mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}[G]\text{-left}}(C_*^{\mathrm{CW}}(\widetilde{X}), {}_{G,\mathrm{tr}}) \\ \varphi & \mapsto & (\varphi \circ p_*) \end{array}$$
define an isomorphism of cochain complexes of abelian groups.

Proof. First note that it follows from the naturality of the cellular boundary map that the given map is indeed a cochain map. Next we consider the following little diagram

of chain maps:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_*^{\mathrm{CW}}(X),\mathbb{Z}) & \xrightarrow{\varphi\mapsto(\varphi\circ p_*)} & \mathrm{Hom}_{\mathbb{Z}[G]\text{-left}}(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}),\mathbb{Z}_{\mathrm{tr}}). \\ & {\scriptstyle \varphi\mapsto(n\otimes\sigma\mapsto n\cdot\varphi(p_*(\sigma)))} \searrow \quad \nearrow {\scriptstyle \gamma\mapsto(c\mapsto\gamma(1\otimes c))} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{\mathrm{tr}}\otimes_{\mathbb{Z}[G]}\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}),\mathbb{Z}) & \end{array}$$

The claim now follows from the following observations:

- It follows easily from the definitions that the diagram commutes.
- As in (1) it follows easily from the Free Resolution-via-K$(G,1)$-Proposition 318.3 (2) together with the Tensor–Direct Sum Lemma 312.15 and the Tensor-with-Base Ring Lemma 312.10 that the given left diagonal map is an isomorphism of abelian groups.
- It follows from the Homomorphism-Change Ring Lemma 315.4 that the right diagonal map is an isomorphism. ■

As a first application we prove the following, not entirely surprising, proposition:

Proposition 318.6. ($\mathrm{H}_1(G)$-Proposition) For any group G we have an isomorphism $\mathrm{H}_1(G)\cong G_{\mathrm{ab}}$.[3512 3513]

Proof. Let G be a group. By the Eilenberg-Maclane Existence Theorem 307.10 we know that there exists an Eilenberg-Maclane X of type $\mathrm{K}(G,1)$. We see that

$$\begin{array}{ccccccc} \mathrm{H}_1(G) & \cong & \mathrm{H}_1(X) & \cong & \pi_1(X)_{\mathrm{ab}} & \cong & G_{\mathrm{ab}}. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{this follows from the} & & \text{by the Hurewicz} & & \text{since } \pi_1(X)\cong G & & \\ \text{Group H}_*\text{-via-K}(G,1) & & \text{Theorem 166.10} & & & & \\ \text{Proposition 318.5} & & & & & & \end{array}$$

■

The following proposition summarizes some calculations of group homologies that we have unwittingly already performed in earlier chapters.

Proposition 318.7. (Group H_*–Topological Examples Proposition)

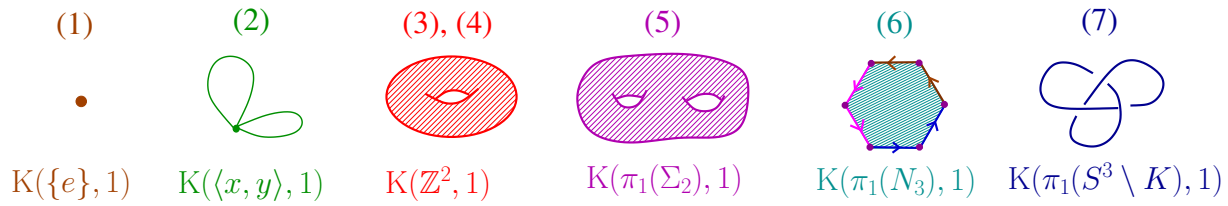
	group G	$\mathrm{K}(G,1)$	$\mathrm{cd}(G)$	$\mathrm{H}_0(G)$	$\mathrm{H}_1(G)$	$\mathrm{H}_2(G)$	$\mathrm{H}_3(G)$
(1)	trivial group	a point $\{P\}$	0	$\mathbb{Z}$	0	0	0
(2)	free group $\langle x_1,\dots,x_m\rangle$	$\bigvee_{i=1}^{m} S^1$	1	$\mathbb{Z}$	$\mathbb{Z}^m$	0	0
(3)	$\mathbb{Z}^2$	torus $S^1\times S^1$	2	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
(4)	$\mathbb{Z}^m$	torus $(S^1)^m$	m	$\mathbb{Z}$	$\mathbb{Z}^m$	$\mathbb{Z}^{\binom{m}{2}}$	$\dots$
(5)	$\pi_1(\text{surface }\Sigma_g), g\geq 1$	Σ_g	2	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}$	0
(6)	$\pi_1(k\cdot\mathbb{R}\mathrm{P}^2), k\geq 2$	$k\cdot\mathbb{R}\mathrm{P}^2$	2	$\mathbb{Z}$	$\mathbb{Z}^{k-1}\oplus\mathbb{Z}/2\mathbb{Z}$	0	0
(7)	$\pi_1(S\setminus\text{knot }K)$	$S^3\setminus K$	≤ 3	$\mathbb{Z}$	$\mathbb{Z}$	0	0

Similarly one can of course also calculate the cohomology groups of these groups.

Sketch of proof. Many of the statements of the proposition will be pretty clear to the reader, in fact many statements already make an appearance in the Eilenberg–Maclane

[3512] Here G_{ab} denotes the abelianization of G as introduced in Definition 98.15.

[3513] Evidently there exists also an algebraic proof of this statement. We leave it to the reader to figure out such an alternative proof.



Example Lemma 307.3. Nonetheless, for the readers convenience we will go through the motions and we will explain where the table comes from.

(1) This is basically clear.

(2) As we already mentioned in the proof of the Free Resolution–Free Group Ring-Lemma 318.4, it follows from the discussion on page 2625, the Wedge-of-Circles-π_1-Lemma 99.11 and the 1-Complex-π_n-Lemma 138.2 that the wedge of m circles is an aspherical 1-dimensional CW-complex whose fundamental group is isomorphic to $\langle x_1, \dots, x_m \rangle$. We computed the homology groups of the wedge of circles in the Wedge-H_*-Proposition 149.16.

(4) Recall that on page 2690 we equipped the m-dimensional torus $(S^1)^m$ with an m-dimensional CW-structure. It now follows from the discussion on the Torus-π_1-Proposition 90.14 and the discussion on page 2587 that $(S^1)^m$ is an Eilenberg-Maclane space of type $K(\mathbb{Z}^m, 1)$. We computed the homology groups of $(S^1)^m$ in Exercise 164.10.

(3) This is just a special case of (4).

(5) Let $g \in \mathbb{N}$. On page 2626 we equipped Σ_g with a 2-dimensional CW-structure and on page 2587 we saw that the surface of genus g is aspherical. We calculated the homology groups of the surface of genus g in the Surface-H_*-Proposition 155.17.

(6) Let $k \geq 2$. On page 2626 we equipped Σ_g with a 2-dimensional CW-structure. In Exercise 318.1 we will show that $N_k = k \cdot \mathbb{R}P^2$ is aspherical. We computed the homology groups of $N_k = k \cdot \mathbb{R}P^2$ in the Surface-H_*-Proposition 155.17.

(7) By the Knot Complement-π_n-Proposition **??** and the Aspherical Topological Manifold Criterion Proposition **??** we know that $S^3 \setminus K$ is aspherical. Furthermore, by the Smooth Manifold-CW-Theorem 179.9 we know that the 3-dimensional smooth manifold $S^3 \setminus K$ admits a 3-dimensional CW-structure. We computed the homology groups of $S^3 \setminus K$ in the Knot Complement-H_*-Proposition 185.29.

The above discussion shows that the given spaces are indeed Eilenberg-Maclane spaces. Together with the Group H_*-via-$K(G, 1)$-Proposition 318.5 we get from the above calculations the columns regarding the homology groups of the various groups.

Finally note that most of the column regarding the cohomological dimensions can be filled in easily by combining the above discussion with the Free Resolution-via-$K(G, 1)$-Proposition 318.3 and the H^*-Dimension–Lower Bound–Proposition 317.21. Last not least, the fact that for $k \geq 2$ we have $\mathrm{cd}(\pi_1(N_k)) = 2$ will be shown in Exercise 318.1. ■

Example. The Group H_*-via-$K(G, 1)$-Proposition 318.5 can at times also be used to go from algebra to topology. More precisely: Let $m \in \mathbb{Z} \setminus \{0\}$ and let $r_1, r_2, \dots \mathbb{Z}$ be integers, each of which is coprime to m. In the following we consider the infinite-dimensional lens space $L(m|r_1, r_2, \dots) = S^\infty/(\mathbb{Z}/m\mathbb{Z})$ from Definition 306.4. Recall that in the Eilenberg–Maclane Example Lemma 307.3 we showed that $L(m|r_1, r_2, \dots) = S^\infty/(\mathbb{Z}/m\mathbb{Z})$

is an Eilenberg-Maclane space $\mathrm{K}(\mathbb{Z}/m, 1)$. For each $k \in \mathbb{N}_0$ we now see that

$$\mathrm{H}_k(L(m|r_1, r_2, \dots)) \quad \underset{\text{by the Group } \mathrm{H}_*\text{-via-K}(G,1) \text{ Proposition 318.5}}{\cong} \quad \mathrm{H}_k(\mathbb{Z}/m\mathbb{Z}) \quad \underset{\text{by the Group-H}_*\text{-of-}\mathbb{Z}/k\mathbb{Z} \text{ Lemma 317.9}}{\cong} \quad \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}/m\mathbb{Z}, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

This gives a new, and depending on one's taste, simpler proof of the Generalized Lens Space-H_*-Proposition 306.7. □

318.3. **Applications of the topological approach to group homology.** Given $g \in \mathbb{N}$ let Σ_g be, as always, the surface of genus g. We recall the following two questions:

- In Question 101.6 we asked whether $\pi_1(\Sigma_g)$ is torsion-free.
- In Question 101.3 we asked whether $\pi_1(\Sigma_g)$ is a free group.

We answered the first question in the affirmative in the Surface-π_1-Properties Proposition 105.8 and the Surface Group–Torsion Free Corollary 130.8. Furthermore the reader might remember that in the Surface-π_1–Not Free Proposition 130.21 we already gave a proof of the Surface-π_1–Not Free Proposition 130.21 using a slightly indirect argument using Euler characteristics. (We also gave a proof in Exercise 116.11.)

Using group homology we now have an arguably much more systematic and conceptual approach to dealing with these two questions:

Proposition 318.8. (Surface Group–Torsion Free Proposition) Let $g \in \mathbb{N}$. The fundamental group of the surface of genus g is torsion-free.

Proof. Recall that in the Group H_*–Topological Examples Proposition 318.7 we showed that $\mathrm{cd}(\pi_1(\Sigma_g)) \leq 2$. It thus follows from the Torsion–H^*-Dimension Proposition 317.24 that $\pi_1(\Sigma_g)$ is torsion-free. ■

Proposition 318.9. (Surface-π_1–Not Free Proposition) Let $g \in \mathbb{N}$. The fundamental group of the surface of genus g is not isomorphic to any free group.

Proof. Recall that in the Group H_*–Topological Examples Proposition 318.7 we showed that $\mathrm{H}_2(\Sigma_g) \cong \mathbb{Z}$ and we showed that the second homology group of any free group is zero. It follows from the Group-H_*–Functor Lemma 317.17 that the isomorphism type of group homology is an isomorphism invariant of groups. Thus we see that $\pi_1(\Sigma_g)$ cannot be a free group. ■

Next we apply the same trick as in the proof of the Surface Group–Torsion Free Proposition 318.8 to knot groups:

Proposition 318.10. (Knot Group–Torsion Free Proposition) For any knot $K \subseteq S^3$ the group $\pi_1(S^3 \setminus K)$ is torsion-free.

Proof. First recall that in the Group H_*–Topological Examples Proposition 318.7 we saw that $\mathrm{cd}(\pi_1(S^3 \setminus K)) \leq 3$. It thus follows from the Torsion–H^*-Dimension Proposition 317.24 that $\pi_1(S^3 \setminus K)$ is torsion-free. ■

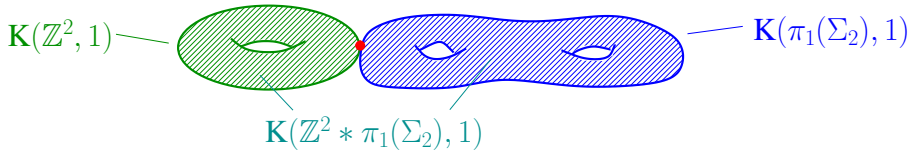
Finally we conclude this section with the following purely algebraic proposition that deals with free products of finitely many groups.

Proposition 318.11. (Group-H_*-Free Product Proposition) Let $G_1, \dots, G_m$ be groups. For $j \in \{1, \dots, m\}$ we denote by $i_j \colon G_j \to G_1 * \cdots * G_m$ the inclusion map. For each $n \in \mathbb{N}$ the map

$$\bigoplus_{j=1}^{m} i_{j*} \colon \bigoplus_{j=1}^{m} \mathrm{H}_n(G_j) \;\to\; \mathrm{H}_n(G_1 * \cdots * G_m)$$

is an isomorphism.

Proof. Surely one can give a purely algebraic proof of this proposition. But at least for a topologically trained person it seems more natural to attack this proposition using our tools from topology. By Exercise 167.23, the Wedge-π_1-Proposition 99.10 and the CW-Wedge Lemma 139.11 we know that $\mathrm{K}(G_1, 1) \vee \cdots \vee \mathrm{K}(G_m, 1)$ is an aspherical CW-complex whose fundamental group is isomorphic to the free product $G_1 * \cdots * G_m$.



We see that for each $n \in \mathbb{N}$ we have the following isomorphisms:

$$\bigoplus_{j=1}^{m} \mathrm{H}_n(G_j) \;\underset{\uparrow}{\cong}\; \bigoplus_{j=1}^{m} \mathrm{H}_n(\mathrm{K}(G_j, 1); \mathbb{Z}) \;\underset{\uparrow}{\cong}\; \mathrm{H}_n\Big(\bigvee_{j=1}^{m} \mathrm{K}(G_j, 1); \mathbb{Z} \Big) \;\underset{\uparrow}{\cong}\; \mathrm{H}_n(G_1 * \cdots * G_m).$$

by the Group H_*-via-$\mathrm{K}(\pi, 1)$ Proposition 318.5

by the Wedge-H_* Proposition 149.16

by the Group H_*-via-$\mathrm{K}(\pi, 1)$ Proposition 318.5 and the above discussion

One needs to work just a little harder to see that the isomorphism from left to right is the map indicated in the proposition. ∎

318.4. **Group homology gives upper bounds on the deficiency.** We conclude this chapter with a particularly nice application of group homology. Namely we will use group homology to give upper bounds on the deficiency of a group.[3514]

We start out by recalling the content of Definition 257.1 and Definition 257.2:

Definition 318.12.

(1) The deficiency of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ is defined to be $k - l$, i.e. the deficiency is the number of generators minus the number of relators.
(2) Let G be a finitely presented group. We define the deficiency $\mathrm{def}(G)$ of G as the maximal deficiency of a finite presentation for G.

Example. Let $m \in \mathbb{N}_0$. We consider the free abelian group $\mathbb{Z}^m$. In the Free Abelian Group-Presentation Lemma 100.14 we saw that there exists an isomorphism[3515]

$$\langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } i < j \in \{1, \dots, m\} \rangle \;\xrightarrow{\cong}\; \mathbb{Z}^m.$$

This shows that $\mathbb{Z}^m$ admits a presentation of deficiency $m - \frac{m(m-1)}{2}$. But can we do any better? Can we find presentations of larger deficiency? Stick around to get an answer. □

[3514] Note that the content of this section is largely copy-pasted from Section 257.5.

[3515] As usual, given a group G and $x, y \in G$ we denote by $[x, y] := xyx^{-1}y^{-1}$ the commutator of x and y.

Our goal is to use group homology to provide upper bounds on the deficiency of a group. To formulate this bound we need to recall Notation 252.9:

Notation 318.13. Let A be an abelian group. We write

$$q(A) \ := \ \text{minimal number of generators of the torsion subgroup Tor}\, A.$$

The following theorem is identical to the Deficiency-upper-Bound Theorem 257.18:

Theorem 318.14. (Deficiency-upper-Bound Theorem) For every finitely presented group G we have the inequality

$$\mathrm{def}(G) \ \leq \ \mathrm{rank}(\mathrm{H}_1(G)) - \mathrm{rank}(\mathrm{H}_2(G)) - q(\mathrm{H}_2(G)).$$

Example. We just saw that $\mathbb{Z}^m$ admits a presentation of deficiency $m - \frac{m(m-1)}{2}$. On the other hand using the calculations from the Group-H_*-of-$\mathbb{Z}^m$-Lemma 317.12 we see that

$$\mathrm{def}(\mathbb{Z}^m) \ \underset{\substack{\uparrow \\ \text{by the Deficiency-upper-Bound} \\ \text{Theorem 318.14}}}{\leq} \ \mathrm{rank}(\overbrace{\mathrm{H}_1(\mathbb{Z}^m)}^{\cong \mathbb{Z}^m}) - \mathrm{rank}(\overbrace{\mathrm{H}_2(\mathbb{Z}^m)}^{\cong \mathbb{Z}^{\frac{m\cdot(m-1)}{2}}}) - q(\overbrace{\mathrm{H}_2(\mathbb{Z}^m)}^{\cong \mathbb{Z}^{\frac{m\cdot(m-1)}{2}}}) \ = \ m - \frac{m\cdot(m-1)}{2}.$$

We have thus shown that the deficiency of $\mathbb{Z}^m$ equals $m - \frac{m\cdot(m-1)}{2}$. □

The proof of the Deficiency-upper-Bound Theorem 318.14 rests very much on the following earlier proposition:

Proposition 252.10. (Algebraic Morse Inequalities Proposition) Let C_* be a chain complex of finitely generated free abelian groups. Given any $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k} (-1)^j \cdot \mathrm{rank}(C_{k-j}) \ \geq \ q(\mathrm{H}_k(C_*)) + \sum_{j=0}^{k} (-1)^j \cdot \mathrm{rank}(\mathrm{H}_{k-j}(C_*)).$$

Proof of the Deficiency-upper-Bound Theorem 318.14. Let G be a finitely presented group. It follows from the Deficiency-upper Bound-Lemma 257.3 that we can pick a presentation $G = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ with $\mathrm{def}(G) = k - l$.

Claim. There exists an aspherical CW-complex X with a single 0-cell, k 1-cells and l 2-cells such that $\pi_1(X) \cong G$.

Proof. By the π_1-Realization Proposition 257.11 (1) we know that there exists a connected CW-complex X with a single 0-cell, k 1-cells and l 2-cells such that $\pi_1(X) \cong G$. The desired CW-complex is now constructed by attaching cells of index $3, 4, \ldots$. The details are worked out in the Kill-All Higher-π_n-Proposition 307.9. ⊞

We see that

$$\begin{array}{rcl}
& & \text{it follows from the Skeleton-H}_*\text{-Lemma 155.2 that} \\
& & \text{for the cellular chain complex } \mathrm{C}_*^{\mathrm{CW}}(X) \text{ we have} \\
& & \mathrm{rank}(\mathrm{C}_0^{\mathrm{CW}}(X)) = \#0\text{-cells} = 1 \\
& & \mathrm{rank}(\mathrm{C}_1^{\mathrm{CW}}(X)) = \#1\text{-cells} = k \\
& & \mathrm{rank}(\mathrm{C}_2^{\mathrm{CW}}(X)) = \#2\text{-cells} = l \\
& & \downarrow \\
\mathrm{def}(G) \;=\; k-l & = & 1 - \sum_{j=0}^{2} (-1)^j \cdot \mathrm{rank}(\mathrm{C}_{2-j}^{\mathrm{CW}}(X)) \\
& \leq & 1 - q(\mathrm{H}_2(X)) - \underbrace{\mathrm{rank}(\mathrm{H}_0(X))}_{=1} + \mathrm{rank}(\mathrm{H}_1(X)) - \mathrm{rank}(\mathrm{H}_2(X)) \\
& & \uparrow \\
& & \text{by the Algebraic Morse Inequalities Proposition 252.10, together} \\
& & \text{with the Singular-Cellular-H}_*\text{-Proposition 155.8} \\
& = & \mathrm{rank}(\mathrm{H}_1(G)) - \mathrm{rank}(\mathrm{H}_2(G)) - q(\mathrm{H}_2(G)). \\
& & \uparrow \\
& & \text{this equality follows from the} \\
& & \text{Group H}_*\text{-via-K}(G,1)\text{-Proposition 318.5}
\end{array}$$

■

318.5. **Appendix: The Eilenberg-Ganea Conjecture.** For the reader's convenience we recall Definition 317.20:

Definition 318.15. Let G be a group. We define its cohomological dimension $\mathrm{cd}(G)$ in $\mathbb{N}_0 \cup \{\infty\}$ as

$$\mathrm{cd}(G) \;:=\; \text{minimal length of a projective resolution of } {}_{G,\mathrm{tr}}\mathbb{Z} \text{ by } \mathbb{Z}[G]\text{-left modules.}$$

The discussion of this chapter leads us to the following definition:

Definition 318.16. Let G be a group. We define its geometric dimension $\mathrm{gd}(G)$ in $\mathbb{N}_0 \cup \{\infty\}$ as

$$\mathrm{gd}(G) \;:=\; \text{minimal dimension of an Eilenberg-Maclane space of type } \mathrm{K}(G,1).$$

Example. In Exercise 307.16 we showed that $\mathrm{gd}(\mathbb{Q}) = 2$ and that $\mathrm{gd}(\mathbb{Q} \times \mathbb{Q}) \leq 3$. Thus we see that in general the geometric dimension is not additive under direct products. □

After reading this chapter, the following proposition will not come as a surprise to the reader:

Proposition 318.17. (Geometric–Cohomological Dimension Inequality Proposition) For any group G we have $\mathrm{cd}(G) \leq \mathrm{gd}(G)$.

Proof. This proposition is an immediate consequence of the Free Resolution-via-K$(G,1)$-Proposition 318.3 (2b). ■

The real question is, whether the converse to the Geometric–Cohomological Dimension Inequality Proposition 318.17 holds. The following two theorems show that this is known to be the case in almost all dimensions:

Theorem 318.18. (Eilenberg–Ganea Theorem) Let G be a group.

(1) If $\mathrm{cd}(G) \geq 3$, then $\mathrm{gd}(G) = \mathrm{cd}(G)$.
(2) If $\mathrm{cd}(G) = 2$, then $\mathrm{gd}(G) \in \{2, 3\}$.

Proof. This theorem was formulated by Samuel Eilenberg and Tudor Ganea [**EG1957**, Theorem 1], without much of a proof. A textbook proof is given in [**BrownK1982**, Corollary VIII.7.2]. Alternatively it follows from Wall's Finiteness Theorem 331.15, which we will formulate (but not prove) later on. ■

Theorem 318.19. (Stallings-Swan Theorem) Let G be a group. If $\mathrm{cd}(G) = 1$, then $\mathrm{gd}(G) = 1$.

Proof. This theorem was proved by John Stallings [**Stal1968**, Theorem 6.8] (for finitely generated groups) and independently by Richard Swan [**Swa1969**, p. 585] (for arbitrary groups). A textbook proof can be found in [**DD1989**, Theorem IV.3.15]. ■

Example. It follows from the Stallings-Swan Theorem 318.19 and the above example that $\mathrm{cd}(\mathbb{Q}) = 2$ and that $\mathrm{cd}(\mathbb{Q} \times \mathbb{Q}) = 3$. Thus in general cohomological dimension is also not additive under direct products. □

The Eilenberg–Ganea Theorem 318.18 together with the Stallings-Swan Theorem 318.19 leads us to the following conjecture:

Conjecture 318.20. (Eilenberg-Ganea Conjecture) Let G be a group. If $\mathrm{cd}(G) = 2$, then $\mathrm{gd}(G) = 2$.

Remark. Later we will formulate Wall's D2-Conjecture 331.16, which can be viewed as a generalization of the Eilenberg-Ganea Conjecture 318.20.

For reasons that will be apparent in a second, we now mention the following conjecture which has its origin in [**WhdJ1941a**]:

Conjecture 318.21. (Whitehead Conjecture) If X is an aspherical 2-dimensional CW-complex, then any connected non-empty subcomplex of X is also aspherical.

We can now formulate the following theorem which is due to Mladen Bestvina and Noel Brady [**BeB1997**, Theorem 8.7].

Theorem 318.22. (Bestvina–Brady Theorem) There exists a finite counterexample to the Eilenberg-Ganea Conjecture 318.20 or there exists a counterexample to the Whitehead Conjecture 318.21.

318.6. Exercises for Chapter 318.

Exercise 318.1. Let $k \geq 2$. We consider $N_k = k \cdot \mathbb{R}\mathrm{P}^2$, the non-orientable surface of genus k, which agrees with the connected sum of k copies of $\mathbb{R}\mathrm{P}^2$.

(a) Show that $N_k = k \cdot \mathbb{R}\mathrm{P}^2$ is aspherical.
Hint. You could use the Covering-π_n-Isomorphism Proposition 134.22 and the Hurewicz Theorem 167.9.

(b) Find a $\mathbb{Z}[\pi_1(N_k)]$-left module M such that $\mathrm{H}_2(\pi_1(N_k); M) \neq 0$.
Hint. This is easier than it looks.

(c) Show that $\mathrm{cd}(\pi_1(N_k)) = 2$.

Exercise 318.2. For $k \in \mathbb{N}$ let N_k be the non-orientable surface of genus k which we introduced in Definition 9.19. For which $k \in \mathbb{N}$ is $\pi_1(N_k)$ torsion-free?

Exercise 318.3. Let $g \in \mathbb{N}_{\geq 2}$. We consider the corresponding surface Σ_g of genus g.

(a) Show that there exists no group homomorphism $\varphi\colon \mathbb{Z}^2 \to \pi_1(\Sigma_g)$ such that $\varphi(\mathbb{Z}^2)$ is a retract, i.e. such that there exists a group homomorphism $\psi\colon \pi_1(\Sigma_g) \to \mathbb{Z}^2$ with $\psi \circ \varphi = \mathrm{id}_{\mathbb{Z}^2}$.

(b) Show that there exists no monomorphism $\varphi\colon \mathbb{Z}^2 \to \pi_1(\Sigma_g)$.
Hint. Suppose that there exists such a group homomorphism. Construct an Eilenberg-Maclane space for $\mathbb{Z}^2$ by considering a covering of Σ_g.

Exercise 318.4. Given $k \in \mathbb{N}$ we denote by Σ_k the surface of genus k.

(a) Let $g < h$. Show that there is no injective homomorphism $\pi_1(\Sigma_g) \to \pi_1(\Sigma_h)$.
(b) Let $g \geq 2$. Use (a) to show that every abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.
Remark. We gave a rather different proof of that statement in the Surface-π_1-Properties Proposition 105.8.[3516]

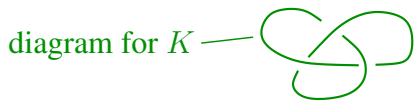
Exercise 318.5. Let $k \in \mathbb{N}$. We consider the non-orientable surface $N_k = k \cdot \mathbb{R}\mathrm{P}^2$ of genus k. What are the possible isomorphism types of abelian subgroups of $\pi_1(N_k)$?

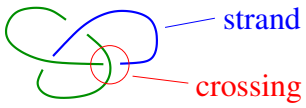
Exercise 318.6. Let M be a closed orientable connected non-empty aspherical n-dimensional topological manifold.

(a) Show that if Γ is a subgroup of infinite index of $\pi_1(M)$, then $\mathrm{H}_n(\Gamma) = 0$.
(b) Suppose that we have an isomorphism $\pi_1(M) \cong A *_G B$ such that the homomorphisms $G \to A$ and $G \to B$ are both monomorphisms. Show that if $\mathrm{H}_k(G) = 0$ for $k = n-1, n$, then one of the two groups A or B is trivial.
(c) Show that if M is homeomorphic to a connected sum $X \# Y$ of two closed smooth manifolds, then one of X or Y is a homotopy sphere.
Hint. Use Characterization (5) of the Homotopy Sphere-Characterization Proposition 210.18.

Exercise 318.7. Let $k, n \in \mathbb{N}_0$. Suppose that the group $\mathbb{Z}^k$ acts freely, smoothly, properly discontinuously, cocompactly on $\mathbb{R}^n$. Show that $n = k$.

Exercise 318.8. Let $K \subseteq S^3$ be a knot. Show that $S^3 \setminus K$ is homotopy equivalent to a 2-dimensional CW-complex that is embedded in S^3.
Hint. Start with a diagram of K and construct a CW-complex complex X with a single 0-cell, a 1-cell for each strand and a 2-cell for each crossing, such that X is a deformation retract for $\mathbb{R}^3 \setminus K$.





Exercise 318.9. Let G be a finitely generated group with non-zero abelianization.

(a) Show that there exists a $(\mathbb{C}, \mathbb{Z}[G])$-left left module structure on $M = \mathbb{C}$ such that $\mathrm{H}_0(G; M) = 0$.
(b) Does the analogous statement in (a) also hold for $M = \mathbb{R}$?
(c) Does the analogous statement in (a) also hold for cohomology?

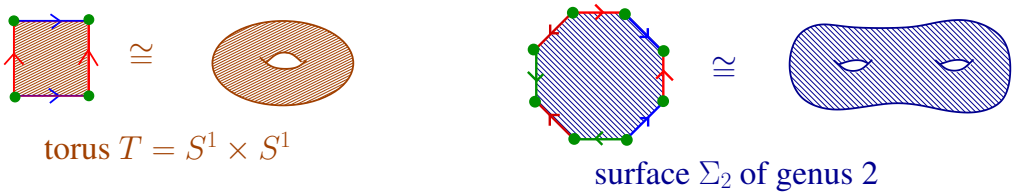
Exercise 318.10.

(a) We consider the 2-dimensional torus $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$ with the CW-structure that is shown below. We set $\pi := \pi_1(T) \cong \mathbb{Z}^2$. Determine explicitly the

[3516] Well, on reflection, perhaps the proofs are not all that dissimilar.

free resolution of ${}_{\pi,\mathrm{tr}}\mathbb{Z}$ that we obtain from the Free Resolution-via-K$(G,1)$-Proposition 318.3.

(b) We consider the surface Σ_2 of genus 2 with the CW-structure that is shown below. We set $\pi := \pi_1(\Sigma_2)$. Determine explicitly the free resolution of ${}_{\pi,\mathrm{tr}}\mathbb{Z}$ that we obtain from the Free Resolution-via-K$(G,1)$-Proposition 318.3.



Exercise 318.11. Let G be the fundamental group of surface Σ_2 of genus 2.

(a) Show that there exists a $(\mathbb{C}, \mathbb{Z}[G])$-left left module structure on $M = \mathbb{C}$ such that $\mathrm{H}_0(G; M) = \mathrm{H}_2(G; M) = 0$.

(b) In the setting of (a), what can you say about $\mathrm{H}_1(G; M)$?

Hint. Make use of Exercise 318.10 (b).

Exercise 318.12. Let $m, n \in \mathbb{N}$. What is the deficiency of the group $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$?

Exercise 318.13.

(a) Show that the deficiency of any finitely generated abelian group is at most one.

(b) Show that the only finitely generated abelian groups of deficiency 0 (up to isomorphism) are $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with $n \neq -1, 0, 1$ and $\mathbb{Z}^3$.

(c) Show that the only finitely generated abelian groups of deficiency 1 (up to isomorphism) are $\mathbb{Z}$ and $\mathbb{Z}^2$.

Part XXXVI

(Co-) homology with twisted coefficients

The goal of this part of the lecture notes is to introduce and study (co-) homology with twisted coefficients and (co-) homology with local coefficient systems. In both settings we start out with a topological space that is equipped with extra data, namely either a module over the group ring of the fundamental group, or a local coefficient system.

The two theories are quite similar and in many settings they give isomorphic results. Even though in most settings the theories of (co-) homology with twisted coefficients and of (co-) homology with local coefficient systems are isomorphic both are of interest and relevant:

(1) (Co-) homology with twisted coefficients requires the choice of a base point. This makes it less flexible when it comes to considering maps between topological spaces. But the definition of (co-) homology with twisted coefficients is closer to the theory that we have developed so far and makes it easier to apply results that we have proved earlier. Furthermore there is a cellular version of (co-) homology with twisted coefficients that is very useful when it comes to doing actual calculations.

(2) (Co-) homology with local coefficient systems does not need the choice of a base point. It works particularly well when one wants to study maps that are induced by maps between topological spaces. But the definitions are further from our known concepts and it requires more work to make sure that all the definitions actually make sense.

In the previous Part XXXV we laid the algebraic foundations for this part of the lecture notes. More precisely we discussed (homological) algebra over rings. In this part of the lecture notes we will apply these concepts and results to the case where the ring is the group ring $\mathbb{Z}[\pi_1(X, x_0)]$ of the fundamental group of a based topological space (X, x_0). This allows us define in a fairly painless fashion (co-) homology with twisted coefficients. Afterwards we introduce the (co-) homology with local coefficient systems. The latter concept is more elegant, but it requires more of an effort to make it work and for most applications it suffices to work with (co-) homology with twisted coefficients.

This part of the lecture notes is organized as follows:

- In Chapter 319 we will summarize some concepts and results from covering theory.
- In Chapter 320 we will introduce the twisted (co-) homology of any "reasonable" pointed topological space (X, x_0) together with a representation $\pi_1(X, x_0) \to \mathrm{Aut}_R(M)$.
- In Chapter 321 we will see that the twisted (co-) homology is in suitable sense functorial in the topological space and the coefficients.
- In Chapter 322 we will see that homotopy equivalent topological spaces have the "same" twisted (co-) homology modules.
- In Chapter 323 we will perform several basic calculations of twisted (co-) homology and we will consider a few examples. For example we will calculate 0-th twisted (co-) homology and we will consider twisted homology of mapping tori.
- In Chapter 324 we will use the results from Chapter 323 to introduce the fundamental class of non-orientable manifolds.
- In Chapter 325 we will introduce the notion of a "universally excisive triad" and we will use this concept to generalize our earlier concept of Mayer-Vietoris sequences from ordinary homology to twisted homology.
- In Chapter 326 we will introduce the Kronecker pairing that relates twisted homology to twisted cohomology.

- In Chapter 327 we will introduce twisted cellular (co-) homology.
- In Chapter 328 we will show that cellular and singular twisted (co-) homology are naturally isomorphic.
- In Chapter 329 we will present many calculations of twisted cellular (co-) homology.
- In Chapter 330 we will introduce the presentation complex of a presentation and we will use Fox calculus to calculate the corresponding the cellular chain complex of a universal covering.
- In Chapter 331 we will provide some theoretical applications of twisted cellular (co-) homology.
- In Chapter 332 we will introduce handle-complexes and handle-homology of handle complexes.
- In Chapter 333 we will introduce twisted handle (co-) homology.
- In Chapter 334 we will introduce cup and cap-products on twisted (co-) homology and we will prove several basic statements.
- In Chapter 335 we will generalize the Poincaré Duality Theorem 210.1 from singular (co-) homology to the twisted context.
- In Chapter 336 we will discuss a few applications of Poincaré Duality with twisted coefficients.
- In Chapter 337 we will use Poincaré Duality for twisted (co-) homology to introduce twisted intersection pairings of compact oriented topological manifolds.
- In Chapter 338 we will use the twisted intersection pairings to introduce the twisted signatures of smooth manifolds.
- In Chapter 339 we will (re-) introduce Alexander modules of knots and links.
- In Chapter 340 we will apply Poincaré duality to Alexander modules. This will allow us to introduce the Blanchfield pairing on the Alexander module of a knot.
- In **??** we will use our tools from twisted (co-) homology to (re-) introduce group homology and cohomology.
- In Chapter 341 we will discuss the (co-) homological Leray-Serre spectral sequence.
- In Chapter 342 we will introduce the notion of local coefficient systems over a topological space.
- In Chapter 343 we will introduce (co-) homology with local coefficient systems. We will see that this concept is essentially isomorphic to twisted (co-) homology.

CHAPTER 319

Reminder of covering theory

Since coverings will play such an essential role in this part of the lecture notes we will now dedicate a full chapter just to recalling some of the concepts and results from covering theory that we obtained in earlier chapters. This chapter contains essentially no proofs, its only purpose is to get the reader ready to read the following chapters. The reader who is already comfortable with covering theory is advised to go straight to Chapter 320.

This short chapter is organized as follows:

- In Section 319.1 we will recall the definition of covering spaces and we will recall several essential examples.
- In Section 319.2 we will see that suitable group actions give rise to coverings.
- In Section 319.3 we will recall an existence statement for coverings.
- In Section 319.4 we will recall a uniqueness statement for coverings.
- In Section 319.5 we will reintroduce the deck transformation group of a covering.
- In Section 319.6 we will provide exercises for Chapter 319.

In the next Chapter 320 we will use covering theory to introduce twisted (co-) homology.

319.1. **Definition and examples of coverings.** Since coverings will play such an essential role in this part of the lecture notes we will now recall Definition 91.8:

Definition 319.1. Let $p\colon \widetilde{X} \to X$ be a continuous map between topological spaces.

(1) An open subset $U \subseteq X$ is called uniformly covered, if $p^{-1}(U) \subseteq \widetilde{X}$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of $\widetilde{X}$ with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.

(2) We say that the map $p\colon \widetilde{X} \to X$ is a covering, if p is surjective and if for every $x \in X$ there exists an open neighborhood U of $x \in X$ which is uniformly covered.

(3) If X is connected, then we saw in the Covering–Preimage Lemma 91.9 that the cardinality of $p^{-1}(x)$ is independent of the choice of $x \in X$ and as in Definition 91.10 we refer to

$$\deg(p\colon \widetilde{X} \to X) \;:=\; \#p^{-1}(x)$$

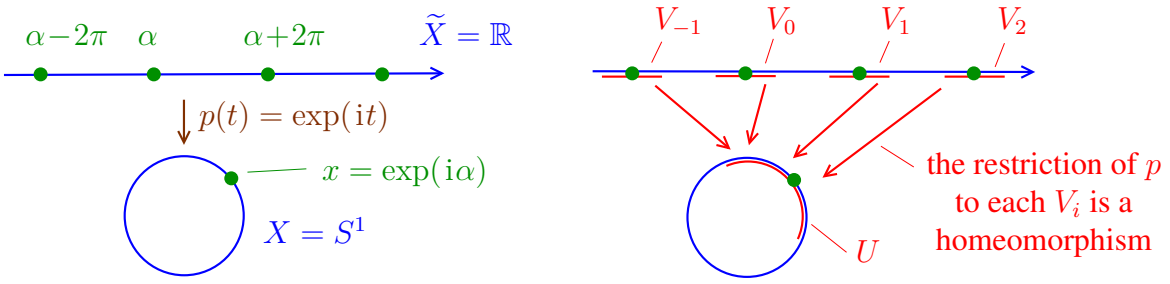
as the degree of the covering.

Example.

(1) On page 1781 we saw that the map

$$\begin{array}{rcl} p\colon \mathbb{R}^k & \to & (S^1)^k \\ (t_1,\ldots,t_k) & \mapsto & \big(\exp(\mathrm{i}t_1),\ldots,\exp(\mathrm{i}t_k)\big) \end{array}$$

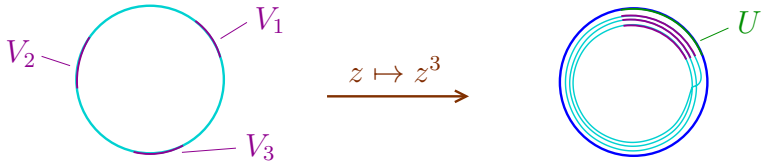
is a covering of infinite degree. The case $k = 1$ is illustrated in the following figure:



(2) As we discussed on page 1783, for any $m \in \mathbb{N}$ the map

$$\begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & z^m \end{array}$$

is a covering of degree m, i.e. it is an m-fold covering.



(3) On page 1783 we saw that there exists a 2-fold covering from the torus to the Klein bottle. □

The following proposition is one of the first non-trivial results in the theory of coverings:

Proposition 92.7. (Covering-π_1-Monomorphism Proposition) Let $p\colon \widetilde{X} \to X$ be a covering. Given any $\widetilde{x}_0 \in \widetilde{X}$ the induced map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, p(x_0))$ is a monomorphism.

Example. We consider the map $p\colon \widetilde{X} \to X$ that is shown in the figure below. One can easily verify that this map $p\colon \widetilde{X} \to X$ is a covering.



π_1 is a free group on four generators

π_1 is a free group on two generators

Next note that it follows from the Graph-π_1-Proposition 99.13 and the Wedge-of-Circles–π_1-Lemma 99.11 that $\pi_1(\widetilde{X})$ is a free group on four generators and that $\pi_1(X)$ is a free group on two generators. The Covering-π_1-Monomorphism Proposition 92.7 shows that the map $p_*\colon \pi_1(\widetilde{X}, x_0) \to \pi_1(X, p(x_0))$ is a monomorphism. Thus we have obtained a monomorphism from the free group on four generators into the free group on two generators. □

The following proposition gives us one of the key properties of coverings:

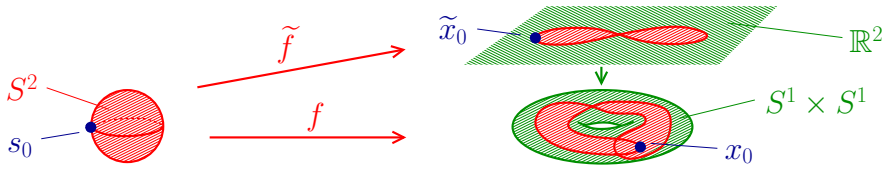
Proposition 127.5. (Map Lifting Criterion) Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z,z_0) \to (X,x_0)$ be a continuous map.

(1) If Z is locally path-connected, then we have the following equivalence of statements:

$$\begin{array}{ccc} & & (\widetilde{X},\widetilde{x}_0) \\ & \overset{\text{there exists a lift } \widetilde{f}}{\nearrow} & \downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (X,x_0) \end{array} \qquad \Longleftrightarrow \qquad \operatorname{im}(f_*) \subseteq \operatorname{im}(p_*).$$

(2) There exist at most one such lift, in other words, any two such lifts agree.

Example. Let $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ be a covering of pointed topological spaces, let $n \in \mathbb{N}_{\geq 2}$ and let $f\colon (S^n,s_0) \to (X,x_0)$ be a continuous map. The sphere S^n is path-connected and locally path-connected. By the Sphere-π_1-Proposition 88.24 the sphere S^n is simply connected if $n \geq 2$.



Therefore it follows from the Map Lifting Criterion 127.5 that there exists a uniquely determined lift $\widetilde{f}\colon (S^n,s_0) \to (\widetilde{X},\widetilde{x}_0)$ of the map $f\colon (S^n,s_0) \to (X,x_0)$. In the Covering-$\pi_n$-Isomorphism Proposition 134.22 we used similar ideas to show that for every $n \geq 2$ the map

$$p_*\colon \pi_n(\widetilde{X},\widetilde{x}_0) \;\to\; \pi_n(X,x_0)$$

is an isomorphism. As an example: If we apply this principle to the covering $\mathbb{R}^n \to (S^1)^n$ from page 6138 we see that for any $n \geq 2$ and all $k \in \mathbb{N}_0$ we have $\pi_n((S^1)^k) \cong \pi_n(\mathbb{R}^k) = 0$. □

We conclude this section with the following useful consequence of the Map Lifting Criterion 127.5:

Theorem 319.2. (Covering–Covering Theorem) Let B be a locally path-connected topological space, let $b_0 \in B$ and let $p\colon (X,x_0) \to (B,b_0)$ be a path-connected covering of pointed topological spaces. For every other path-connected covering $q\colon (Y,y_0) \to (B,b_0)$ with $p_*(\pi_1(X,x_0)) \subseteq q_*(\pi_1(Y,y_0))$ there exists a *unique* covering $r\colon (X,x_0) \to (Y,y_0)$ of pointed topological spaces such that the following diagram commutes:

$$\begin{array}{ccc} (X,x_0) & \overset{\exists!\, r}{\searrow} & \\ \downarrow p & & (Y,y_0) \\ (B,b_0). & \overset{q}{\swarrow} & \end{array}$$

Proof. It follows from the Map Lifting Criterion 127.5 that we have a unique continuous map $r\colon (X,x_0) \to (Y,y_0)$ that makes the diagram commute. It remains to show that this map is a covering map. This we showed in the proof of the Covering–Covering Theorem 319.2. ■

319.2. **Group actions and coverings.** In this section we will see that suitable actions of groups are a great source of coverings. We start out by recalling Definition 91.1:

Definition 319.3. Let G be a group that acts on a topological space Z.

(1) We say that G acts freely if $g \cdot z = z$ for some $g \in G$ and some $z \in Z$ implies that $g = e$.

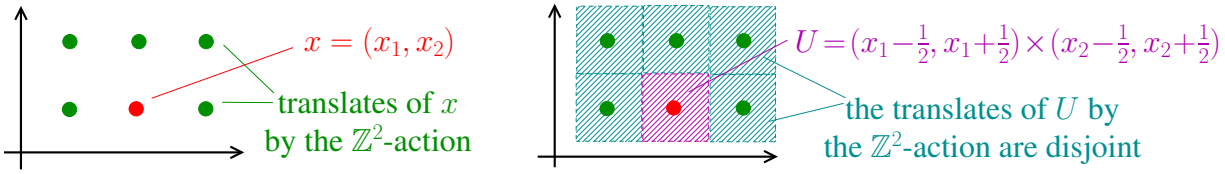
(2) We say that G acts discretely, if for each $z \in Z$ there exists an open neighborhood U of z so that
$$U \cap g{\cdot}U = \varnothing \quad \text{for all } g \neq e.$$

Clearly a discrete action is also free. In Exercise 319.1 we will see that the converse does not hold.

Next we recall some examples of discrete actions from page 1776:

Examples.

(1) The group $G = \mathbb{Z}^n$ acts on $Z = \mathbb{R}^n$ by addition. The figure below illustrates (hopefully) the argument why this action is discrete.



(2) The map
$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$
defines an action of the group $\{\pm 1\}$ on S^n. Again it is elementary to see that this action is discrete. □

The following proposition shows that discrete actions give rise to coverings:

Proposition 91.11. (Action–Covering Proposition) Let Z be a topological space together with a discrete and continuous action by a group G. The natural projection $p\colon Z \to Z/G$ is a covering where the degree is given by $\#G$.

Example. It follows easily from the Action–Covering Proposition 91.11 and the above example that the map
$$\begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x \\ x & \mapsto & [x] \end{array}$$
is a 2-fold covering from the sphere S^n to the real projective space $\mathbb{R}\mathrm{P}^n$. □

Sketch of proof. Let $p\colon Z \to Z/G$ be the projection. Let $[z] \in Z/G$. Let U be an open neighborhood of $z \in Z$ as in Definition 319.3. One can easily verify that $p(U)$ is uniformly covered. The details for the proof were worked out in the proof of the earlier Action–Covering Proposition 91.11. ■

We formulate our next example as a lemma:

Lemma 91.3. (Lens Space-Discrete Action Lemma) We consider the 3-dimensional sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^{1\times 2} \,|\, |z_1|^2 + |z_2|^2 = 1\}$. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. We set $\xi := \exp(2\pi\,\mathrm{i}/p)$.

(1) The map

$$\begin{array}{rcl} \mathbb{Z}/p\mathbb{Z} \times S^3 & \to & S^3 \\ (k+p\mathbb{Z}, (z_1, z_2)) & \mapsto & (\xi^k \cdot z_1, \xi^{q\cdot k} \cdot z_2) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}/p\mathbb{Z}$ on S^3.

(2) The quotient space $S^3/(\mathbb{Z}/p\mathbb{Z})$ is a closed 3-dimensional topological manifold that has a unique structure of an oriented smooth manifold such that the natural projection $S^3 \to S^3/(\mathbb{Z}/p\mathbb{Z})$ is a local diffeomorphism and orientation-preserving.

The Lens Space-Discrete Action Lemma 91.3 leads us to the following definition, which is essentially identical to Definition 91.4.

Definition 319.4. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. We refer to

$$L(p,q) \;:=\; S^3/(\mathbb{Z}/p\mathbb{Z})$$

(action from the Lens Space Discrete Action Lemma 91.3)

as a lens space.

Lens spaces are the simplest types of quotients of S^3. So it is natural that one would like to classify them up to diffeomorphism. The following lemma gives us the diffeomorphisms that one can write down explicitly without too much of an effort:

Lemma 91.5. (Lens Space-Diffeomorphism Lemma) Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$, then there exists a diffeomorphism $L(p,q) \to L(p,r)$.

The following question arises: Does the Lens Space-Diffeomorphism Lemma 91.5 account for all diffeomorphisms? Put differently, if $L(p,q)$ is diffeomorphic to $L(p',q')$, does it follow that $p = p'$ and $q \equiv \pm r^{\pm 1} \bmod p$? To address this question, the idea of course is to calculate suitable invariants of lens spaces. We will start out with the most basic invariant, namely the fundamental group.

To calculate the fundamental group of lens spaces $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ we recall the following theorem which on many occasions is an effective tool for calculating fundamental groups:

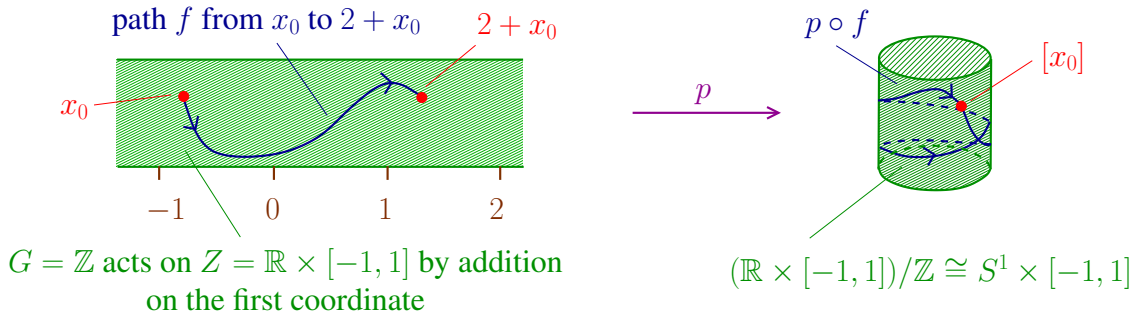
Theorem 319.5. (Action–π_1-Theorem) Let Z be a simply connected topological space and let G be a group that acts continuously and discretely on Z. We choose an $z_0 \in Z$. We denote by $p\colon Z \to Z/G$ the natural projection. The map

$$\begin{array}{rcl} \Theta_{x_0}\colon G & \to & \pi_1(Z/G, [z_0]) \\ g & \mapsto & [p \circ (\text{path in } Z \text{ from } z_0 \text{ to } g \cdot z_0)] \end{array}$$

is a well-defined isomorphism of groups.

Proof. This theorem is just a dumbed-down version of the Action–π_1-Theorem 93.1. ■

Example. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. It follows from the Action–π_1-Theorem 319.5 and the Lens Space-Discrete Action Lemma 91.3 that for the lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ we have an explicit isomorphism $\mathbb{Z}/p\mathbb{Z} \to \pi_1(L(p,q))$. This gives us in particular a partial answer to the above question: If $L(p,q)$ is diffeomorphic to $L(p',q')$, then $p = p'$. But it is not so clear how one can track down the parameter q. □



319.3. **Existence of coverings.** Recall that the Covering-π_1-Monomorphism Proposition 92.7 tells us that for any covering $p\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ between pointed topological spaces the induced homomorphism $p_*\colon\pi_1(\widetilde{X},\widetilde{x}_0)\to\pi_1(X,x_0)$ is a monomorphism. In particular a covering gives of (X,x_0) rise to a subgroup of $\pi_1(X,x_0)$. In this section and the next section we will see that (under reasonable hypotheses) all subgroups arise from coverings and that coverings are essentially determined by these subgroups.

The following definition is essentially the same as Definition 127.11:

Definition 319.6. Let X be a topological space, let $x_0\in X$ and let $\Gamma\subseteq\pi_1(X,x_0)$ be a subgroup.

(1) As in Definition 16.2 we denote by $X^{[0,1]}$ the set of all continuous maps $[0,1]\to X$.

(2) We consider the path space

$$P_{x_0} \;:=\; \{\gamma\in X^{[0,1]}\,|\,\gamma(0)=x_0\} \;=\; \{\text{all paths in } X \text{ with starting point } x_0\}.$$

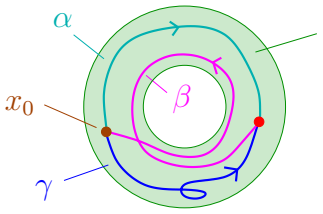
(3) For $u,v\in P_{x_0}$ we write

$$u\sim_\Gamma v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [\;\underbrace{u*\overline{v}}_{\text{loop in } (X,x_0)}\;]\in\Gamma.$$

(4) We write $\widetilde{X}_{x_0,\Gamma}:=P_{x_0}/\sim_\Gamma$ and $\widetilde{x}_0:=[\mathrm{const}_{\widetilde{x}_0}]$, i.e. $\widetilde{x}_0$ is the equivalence class of the constant path.

(5) We consider the evaluation map

$$\begin{array}{rcl} p\colon \widetilde{X}_{x_0,\Gamma}=P_{x_0}/\sim_\Gamma & \to & X \\ [v\colon[0,1]\to X] & \mapsto & v(1). \end{array}$$



X is an annulus

we consider $\Gamma=2\mathbb{Z}\subseteq\mathbb{Z}=_{\mathrm{i}}\pi_1(X,x_0)$,
then α and β are equivalent,
but α and γ are inequivalent

The following lemma, which is a slight reformulation of the CO-Topology Lemma 127.14, shows in particular that each subgroup gives rise to a covering:

Lemma 319.7. (CO-Topology Lemma) Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup.

(i) We equip $X^{[0,1]} = \text{set of all continuous maps } [0,1] \to X$ with the compact-open topology from Definition 16.2.
(ii) We equip $P_{x_0} = \{\gamma \in X^{[0,1]} \mid \gamma(0) = x_0\}$ with the subspace topology.
(iii) We equip $\widetilde{X}_{x_0,\Gamma} = P_{x_0}/\sim_\Gamma$ with the quotient topology.

If X is locally path-connected and semi-locally simply connected, then the following statements hold:

(1) The above evaluation map $p\colon \widetilde{X}_{x_0,\Gamma} \to X$ is continuous and open.
(2) The map $p\colon \widetilde{X}_{x_0,\Gamma} \to X$ is a covering.
(3) The topological space $\widetilde{X}_{x_0,\Gamma}$ is path-connected.
(4) We have $p_*(\pi_1(\widetilde{X}_{x_0,\Gamma}, \widetilde{x}_0)) = \Gamma$.

319.4. **Uniqueness of coverings.** Now that we know that, in reasonable settings, subgroups give rise to coverings, the following question arises: In what sense are these coverings unique? To make sense of the uniqueness statement we need to recall Definition 127.16:

Definition 319.8. Let $p\colon \widetilde{X} \to X$ and $q\colon \widehat{X} \to X$ be two coverings of a topological space X. We say that p and q are isomorphic, if there exists a homeomorphism $\Phi\colon \widetilde{X} \to \widehat{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow[\cong]{\Phi} & \widehat{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X. & \end{array}$$

Similarly we define an isomorphism of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi \mathrm{i} t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

are isomorphic. In fact the desired map $\Phi\colon \mathbb{R}/n\mathbb{Z} \to S^1$ is given by $[t] \mapsto \exp(2\pi \mathrm{i} \frac{t}{n})$. □

Theorem 319.9. (Covering Unique Existence Theorem) Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$ and let $\Gamma \subseteq \pi_1(X, x_0)$ be a subgroup. There exists, up to isomorphism, a *unique* path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of pointed topological spaces such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$.

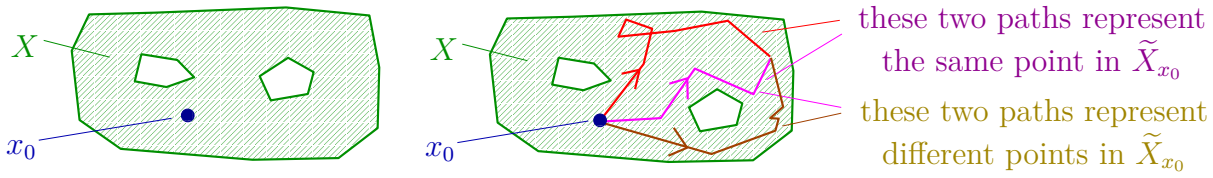
Proof. As we discussed in the proof of the original Covering Unique Existence Theorem 127.17, the theorem is a straightforward consequence of the CO-Topology Lemma 319.7 together with the Covering–Covering Theorem 319.2. ■

We move on to the type of covering that will be most relevant for us throughout the following chapters:

Definition 319.10. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$.

(1) As in Definition 127.21 we refer to the natural covering $p\colon \widetilde{X}_{x_0} := \widetilde{X}_{x_0,\{e\}} \to X$ from Definition 319.6 as the natural universal covering of (X, x_0). Furthermore we equip $\widetilde{X}_{x_0}$ with the natural base point $\widetilde{x}_0$ that is given by the constant path.

(2) We refer to any covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering of (X, x_0).



Example.

(1) The covering $p\colon \mathbb{R} \to S^1$ is *a* universal covering of S^1 but it is not *the* natural universal covering of S^1. For most purposes this difference is irrelevant.

(2) Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering and let $q\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ be another path-connected covering. Since the trivial group is contained in every other subgroup we obtain from the Covering–Covering Theorem 319.2. that there exists *unique* covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (\widehat{X}, \widehat{x}_0)$ of pointed topological spaces such that the following diagram commutes:

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{x}_0) & \overset{\exists!\, r}{\dashrightarrow} & (\widehat{X}, \widehat{x}_0) \\ {\scriptstyle p}\downarrow & \swarrow_{q} & \\ (X, x_0). & & \end{array}$$

This discussion justifies perhaps the name "universal covering". □

319.5. **The deck transformation group.** In this section we will recall the notion of a deck transformation and the notion of the deck transformation group, and we will recall some basic properties. We start out our recap with recalling Definition 128.1:

Definition 319.11. Let $p\colon \widetilde{X} \to X$ be a covering.

(1) A deck transformation is a homeomorphism $d\colon \widetilde{X} \to \widetilde{X}$ such that $p(d(x)) = p(x)$ for all $x \in \widetilde{X}$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow[\cong]{d} & \widetilde{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X. & \end{array}$$

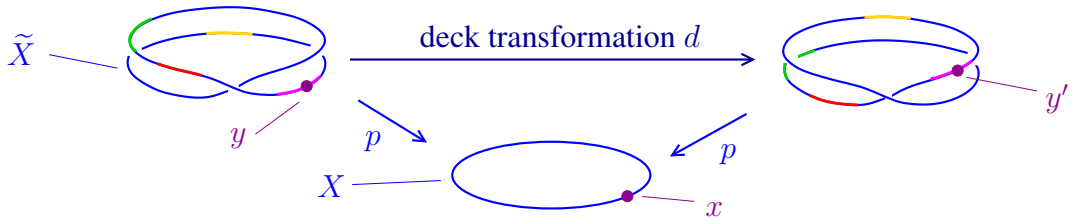
The group of deck transformations is called the deck transformation group of the covering $p\colon \widetilde{X} \to X$ and we denote it by $\mathrm{Deck}(p\colon \widetilde{X} \to X)$.

(2) We say that the covering is regular if for every $x \in X$ the deck transformation group $\mathrm{Deck}(p\colon \widetilde{X} \to X)$ acts transitively on $p^{-1}(x)$, i.e. if given any two points $y, y' \in \widetilde{X}$ with $p(y) = p(y') = x$ there exists a deck transformation d with $d(y) = y'$. Otherwise we say that the covering is irregular.

(3) Let P be a property of isomorphism types of groups, e.g. cyclic, free, nilpotent, solvable etc. We say that the covering $p\colon \widetilde{X} \to X$ is a P-covering if the covering is regular and if the deck transformation group has the property P.

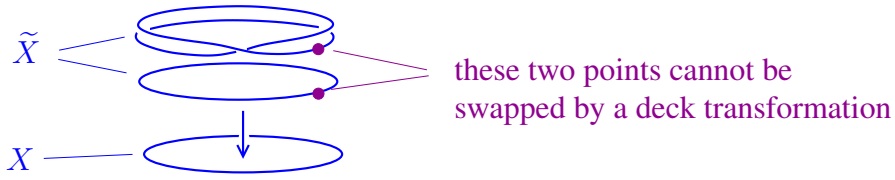
Example.

(1) In the figure below we see a 2-fold covering $p\colon \widetilde{X} \to X$ of S^1.



In the figure we also show a deck transformation. One can easily verify that this covering is regular and that the deck transformation group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In the language of Definition 319.11 (3) this is a 2-fold cyclic covering.

(2) In the figure below we show a 3-fold covering of S^1:



One can easily verify that the deck transformation group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and that the covering is irregular. □

In the following we will recall some basic statements about deck transformation groups:

Lemma 128.3. (Deck Group-Free Action Lemma) Given any covering $p\colon \widetilde{X} \to X$ of path-connected topological spaces and given any $x_0 \in X$ the deck transformation group $\mathrm{Deck}(p\colon \widetilde{X} \to X)$ acts freely on $p^{-1}(x_0)$.

Proof. This lemma follows easily from the uniqueness statement of the Map Lifting Criterion 127.5. We refer to the original proof of the Deck Group-Free Action Lemma 128.3 for a few more details. ■

The following simplified version of the Regular Covering-Characterization Proposition 128.13 gives us a criterion for a covering to be regular:

Proposition 128.13. (Regular Covering-Characterization Proposition) In the following let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering of pointed topological spaces that are path-connected and locally path-connected. Then the following statements are equivalent:

(1) The covering is regular.
(2) The subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

In the following we will mostly care about regular coverings. This brings us to the following proposition:

Proposition 319.12. (Deck Transformation Group Proposition) Let $\widetilde{X}$ and X be two path-connected and locally path-connected topological spaces. Furthermore suppose that we are given a regular covering $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ of pointed topological spaces. The map[3517]

$$\begin{array}{rcl} \Xi_{X,x_0}\colon \mathrm{Deck}(p\colon \widetilde{X}\to X) & \to & \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \\ d & \mapsto & \big[p\circ(\text{path in } \widetilde{X} \text{ from } \widetilde{x}_0 \text{ to } d(\widetilde{x}_0))\big] \end{array}$$

is a natural isomorphism where the inverse is given by

$$\begin{array}{rcl} \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) & \to & \mathrm{Deck}(p\colon \widetilde{X}\to X) \\ [f\colon [0,1]\to X] & \mapsto & \text{unique lift}^{3518} \text{ of } p\colon \widetilde{X}\to X \text{ to } \widetilde{p}\colon \widetilde{X}\to\widetilde{X} \\ & & \text{with } \widetilde{p}(\widetilde{x}_0)=\widetilde{f}(1), \text{ where } \widetilde{f}\colon [0,1]\to\widetilde{X} \\ & & \text{is the unique lift of } f \text{ with } \widetilde{f}(0)=\widetilde{x}_0 \end{array}$$

Proof. The fact that the map is Ξ_{X,x_0} is an isomorphism is just a special case of the Deck Transformation–Normalizer Proposition 128.9. The fact that the map is natural is made more precise, and proved, in the Deck-to-π_1-Functoriality Lemma 128.16. ■

The Deck Transformation Group Proposition 319.12 motivates the following convention:

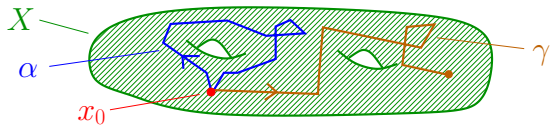
Convention 319.13. Let $p\colon (\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ be a *regular* covering of pointed topological spaces. We suppose that $\widetilde{X}$ and X are both path-connected and locally path-connected. The isomorphism

$$\begin{array}{rcl} \Xi_{X,x_0}\colon \mathrm{Deck}(p\colon \widetilde{X}\to X) & \to & \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \\ d & \mapsto & \big[p\circ(\text{path in } \widetilde{X} \text{ from } \widetilde{x}_0 \text{ to } d(\widetilde{x}_0))\big] \end{array}$$

from the above Deck Transformation Group Proposition 319.12 allows us to equip $\widetilde{X}$ with a natural left action by the group $\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$. More precisely, we equip $\widetilde{X}$ with the left action

$$\begin{array}{rcl} \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))\times\widetilde{X} & \to & \widetilde{X} \\ (g,y) & \mapsto & \underbrace{((\Xi_{X,x_0})^{-1})(g)}_{\in \mathrm{Deck}(p\colon \widetilde{X}\to X)}(y). \end{array}$$

Example. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $x_0\in X$. We denote by $p\colon (\widetilde{X}_{x_0},\widetilde{x}_0)\to(X,x_0)$ the corresponding natural universal covering from Definition 319.10. By the discussion on page 6147 we have an explicit natural $\pi_1(X,x_0)$-left action on $\widetilde{X}_{x_0}$.



the $\pi_1(X,x_0)$-action on $\widetilde{X}_{x_0}$ is given by $[\alpha]\cdot[\gamma] := [\alpha * \gamma]$

[3517] Since $p\colon \widetilde{X}\to X$ is a regular covering we know by the Regular Covering-Characterization Proposition 128.13 that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ is a normal subgroup of $\pi_1(X,x_0)$, thus $\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ is a group.
[3518] This unique lift is given by the Covering–Covering Theorem 319.2.

In Exercise 319.6 we will see that this left action is given by

$$\begin{array}{rcl} \pi_1(X,x_0)\times \widetilde{X}_{x_0} & \to & \widetilde{X}_{x_0} \\ ([\alpha],[\gamma]) & \mapsto & [\alpha * \gamma]. \end{array}$$

□

Remark. In the following chapters, when we discuss (co-) homology with twisted coefficients, we will only be interested in topological spaces that are locally contractible. Throughout these chapters we frequently and often subconsciously make use of the following facts:

(1) It follows from the Homotopy-Equivalence π_0-Lemma 19.17 and the Homotopy Equivalence-π_1-Proposition 90.9 that a topological space that is locally contractible is also locally path-connected and semi-locally simply connected.

(2) If a topological space X is a locally contractible, then it follows from the (1) and Locally Path-Connected Lemma 4.57 that X is path-connected if and only if X is connected. For no particular reason we usually write "path-connected and locally contractible" instead of the equivalent notion "locally contractible and connected".

(3) By the Manifold–Local Properties Lemma 23.8 every topological manifold is locally contractible and by the CW-Complex–Local Properties Proposition 137.8 (2) every CW-complex is locally contractible. □

319.6. Exercises for Chapter 319.

Exercise 319.1. Provide an example of a continuous action of a group G on a topological space that is free but not discrete.

Exercise 319.2. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. Compute the homology groups of the lens space $L(p,q)$.

Exercise 319.3. Let $p \in \mathbb{N}$.

(a) Let $A \in \mathrm{O}(4)$ be a matrix of order p such that the resulting action

$$\begin{array}{rcl} \mathbb{Z}/p\mathbb{Z}\times S^3 & \to & S^3 \\ ([k],v) & \mapsto & A^k \cdot v \end{array}$$

is discrete. Show that there exist $q_1, q_2 \in \mathbb{Z}$, both coprime to p, such that A is conjugate to the matrix

$$\begin{pmatrix} \exp(2\pi \mathrm{i} q_1/p) & 0 \\ 0 & \exp(2\pi \mathrm{i} q_2/p) \end{pmatrix}$$

in $\mathrm{U}(2) \subseteq \mathrm{SO}(4)$.

(b) We introduce two concepts:

- We say that two actions of a group G on a topological space X are *equivalent* if there exists an isomorphism $\Phi\colon G \to G$ and a homeomorphism $\Psi\colon X \to X$ such that for all $g \in G$ and $x \in X$ we have $g \cdot_1 \Psi(x) = \Psi(\Phi(g) \cdot_2 x)$.
- We say that an action of a group G on S^3 is *isometric*, if for each $g \in G$ the action is given by multiplication by an orthogonal matrix. In other words, the action is given by a group homomorphism $G \to \mathrm{O}(4)$.

Show that any discrete action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 is equivalent to one of the actions from the Lens Space-Discrete Action Lemma 91.3.

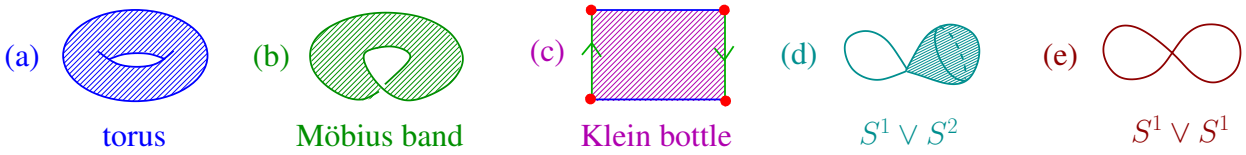
Remark. This exercise is identical to Exercise 93.13.

Exercise 319.4. We consider the wedge $X = S^1 \vee S^1$ of two circles. Provide an explicit example of a connected irregular 3-fold covering of $X = S^1 \vee S^1$.

Exercise 319.5. Draw figures for the universal coverings of the following topological spaces:

(a) The torus.
(b) The Möbius band.
(c) The Klein bottle.
(d) The wedge product $S^1 \vee S^2$.
(e) The wedge product $S^1 \times S^1$.

In each case illustrate the action by the deck transformation group.



Exercise 319.6. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. As in the CO-Topology Lemma 319.7 we consider the topological space

$$\widetilde{X}_{x_0} \;=\; \{\gamma\colon [0,1] \to X \,|\, \gamma(0) = x_0\}/\sim \qquad \begin{array}{l}\text{where } \alpha \sim \beta \text{ if } \alpha(1) = \beta(1) \text{ and} \\ \alpha * \beta \text{ is null-homotopic.}\end{array}$$

We denote by $p\colon \widetilde{X}_{x_0} \to X$ the covering map that is given by the evaluation map $[\gamma] \mapsto \gamma(1)$ In the Deck Transformation–Normalizer Proposition 128.9 we gave an explicit isomorphism $\mathrm{Deck}(p\colon \widetilde{X}_{x_0} \to X) \cong \pi_1(X, x_0)$. Show that the corresponding action of the fundamental group $\pi_1(X, x_0)$ on $\widetilde{X}_{x_0}$ is given by

$$\begin{array}{rcl}\pi_1(X, x_0) \times \widetilde{X}_{x_0} & \to & \widetilde{X}_{x_0} \\ ([\alpha], [\gamma]) & \mapsto & [\alpha * \gamma].\end{array}$$

Exercise 319.7. For which $d \in \mathbb{N}$ does the following statement hold: For every topological space X that path-connected, locally path-connected and semi-locally simply connected, every connected covering $p\colon \widetilde{X} \to X$ of degree d is regular?

Exercise 319.8. We consider the quaternion group Q_8 which is defined as the subgroup of the unit quaternions $\mathbb{H} \cap S^3$ that is generated by $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$. In other words, as a set the quaternion group is given by $Q_8 = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$. Show that Q_8 is not abelian but that nonetheless every subgroup is normal.

CHAPTER 320

Twisted (co-) homology

In the previous Chapter 319 we recalled some basic concepts of covering theory. We showed in particular that (under reasonable hypotheses) any pointed topological space (X, x_0) has a natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and that $\pi_1(X, x_0)$ acts freely on the natural universal covering $\widetilde{X}_{x_0}$. In this chapter we will first see that this implies that the chain groups $\mathrm{C}_*(\widetilde{X}_{x_0})$ are naturally $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

Since in general $\mathbb{Z}[\pi_1(X, x_0)]$ is a non-commutative ring the above observation leads us into the real of (homological) algebra over a possibly non-commutative ring which we already studied in the previous Part XXXV.

In this chapter we will recall some concepts non-commutative ring theory, with a focus on the group ring.

Next suppose that a pointed topological space (X, x_0) is equipped with a $\mathbb{Z}[\pi_1(X, x_0)]$-left module M. In this chapter we will use the general algebraic concepts to "marry" the $\mathbb{Z}[\pi_1(X, x_0)]$-left module chain complex $\mathrm{C}_*(\widetilde{X}_{x_0})$ with the $\mathbb{Z}[\pi_1(X, x_0)]$-left module M to define the twisted homology groups $\mathrm{H}^{\mathrm{tw}}_*(X; M) = \mathrm{H}_*(\overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*(\widetilde{X}_{x_0}))$ and the twisted cohomology groups $\mathrm{H}^*_{\mathrm{tw}}(X; M) = \mathrm{H}^*(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{x_0}), M))$. We will also study some basic properties of these twisted invariants.

This chapter is organized as follows:

- In Section 320.1 we will study the singular chain complex of coverings.
- In Section 320.2 we will recall some basic definitions and results on tensor product over rings.
- In Section 320.3 we will recall a few basic facts about modules over group rings.
- In Section 320.4 we will introduce twisted (co-) homology.
- In Section 320.5 we will calculate twisted (co-) homology with trivial coefficients.
- In Section 320.6 we will calculate some twisted homology modules of S^1.
- In Section 320.7 we will see that at times we can calculate twisted (co-) homology through intermediate coverings.
- In Section 320.8 we will give a summary of some of the various results on twisted (co-) homology that we will prove in these lecture notes.
- In Section 320.9 we will provide exercises for Chapter 320.

In the next Chapter 321 we will discuss the functoriality of twisted (co-) homology.

320.1. **The chain complex of coverings.** We start out by recalling Definition 97.30:

Definition 320.1. Let G be a group and let R be a commutative ring. We denote the free R-module generated by the set G by $R[G]$. Put differently with Definition 309.17 we

have

$$R[G] \coloneqq \left\{ \sum_{i=1}^{m} r_i \cdot g_i \,\middle|\, r_1, \ldots, r_m \in R \text{ and } g_1, \ldots, g_m \in G \right\}.$$

We equip $R[G]$ with the multiplication that is given by

$$\left(\sum_{i=1}^{m} r_i \cdot g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \coloneqq \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_R s_j}_{\in R} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $R[G]$ as the group ring of G with R-coefficients.

Next we recall the following lemma:

Lemma 310.11. (Set Action–Group Ring Module Lemma) Let G be a group and let S be a set together with a G-left action. The map

$$\begin{array}{ccc} G & \to & \mathrm{Aut}_R(R^{(S)}) \\ g & \mapsto & \left(\begin{array}{ccc} R^{(S)} & \to & R^{(S)} \\ \left(\sum_{i=1}^{k} n_i \cdot s_i \right) & \mapsto & \sum_{i=1}^{k} n_i \cdot \underbrace{(g \cdot s_i)}_{\in S} \end{array} \right) \end{array}$$

is a group homomorphism. Via the Group Ring–Module Lemma 310.5 this group homomorphism turns $R^{(S)}$ into an $(R, R[G])$-left left module.

Next we want to recall a criterion for when a module as in the Set Action–Group Ring Module Lemma 310.11 is actually a free module. To do so we need to recall the following definition:

Definition 310.12. Suppose that a group G admits a left action on a set S. We say that $T \subseteq S$ is a complete set of representatives if given any $s \in S$ there exists a *unique* $t \in T$ such that $s = g \cdot t$.

Now we can recall the following elementary lemma:

Lemma 310.13. (Group Ring–Free Module Lemma) Let G be a group and let S be a set together with a free G-left action. If $T \subseteq S$ is a complete set of representatives, then $R^{(S)}$ is a free $R[G]$-left module with basis T.

Example. We consider the set $S = \langle x \rangle \times \{A, B, C\}$ with the obvious $\langle x \rangle$-action.



This left action is free. It is clear that there are many choices of complete sets of representatives. This shows that for any commutative ring R the $R[x^{\pm 1}]$-module $R^{(S)}$ is a free $R[x^{\pm 1}]$-module of rank three. □

Using Definition 310.12 we can now give a description of the singular chain complex of a regular covering:

Lemma 320.2. (Regular Covering–Chain Complex Lemma) Let X be a topological space that is path-connected and locally contractible. Furthermore let $p\colon \widetilde{X} \to X$ be a *regular* covering and let $A \subseteq X$. We write $\widetilde{A} := p^{-1}(A)$. Finally we consider the deck transformation group $D := \mathrm{Deck}(p\colon \widetilde{X} \to X)$. Since a deck transformation commutes with the projection $p\colon \widetilde{X} \to X$ we see that the natural left action of D on $\widetilde{X}$ preserves $\widetilde{A}$. It follows from the Set Action–Group Ring Module Lemma 310.11 that we can view each $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ as a $\mathbb{Z}[D]$-left module.

(1) (a) For each singular simplex $\sigma\colon \Delta^k \to X$ there exists a singular simplex $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$ with $p \circ \widetilde{\sigma} = \sigma$.

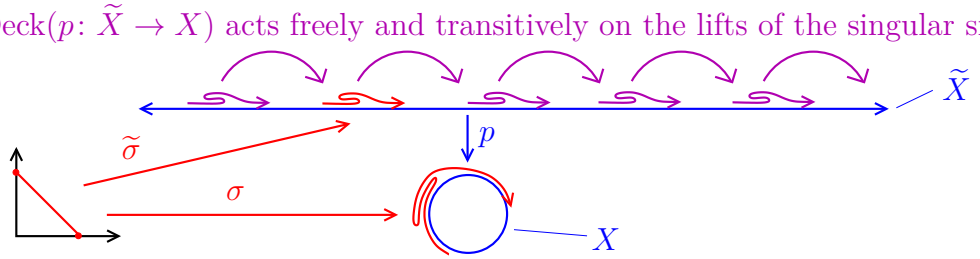
(b) For each singular simplex $\sigma\colon \Delta^k \to X$ that is not contained in A we pick a lift $\widetilde{\sigma}\colon \Delta^k \to X$.[3519] The set of singular simplices

$$\{\widetilde{\sigma}\colon \Delta^k \to \widetilde{X} \mid \sigma\colon \Delta^k \to X \text{ with } \sigma(\Delta^k) \not\subseteq A\}$$

is a basis for the $\mathbb{Z}[D]$-left module $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$. In particular $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ is a free $\mathbb{Z}[D]$-left module.

(2) The usual boundary map $\partial_k\colon \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to \mathrm{C}_{k-1}(\widetilde{X}, \widetilde{A})$ is a homomorphism of $\mathbb{Z}[D]$-left modules.

In summary we have shown that $(\mathrm{C}_*(\widetilde{X}, \widetilde{A}), \partial_*)$ is a chain complex of free $\mathbb{Z}[D]$-left modules.



Proof.

(1) (a) First recall that a covering is by Definition 91.8 surjective. Since Δ^k is evidently path-connected and locally path-connected we thus obtain from the Map Lifting Criterion 127.5 that we can lift any continuous map $\sigma\colon \Delta^k \to X$ to a continuous map $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$.

(b) Let $k \in \mathbb{N}_0$. We start out with some notation. Given a singular simplex $\sigma\colon \Delta^k \to X$ with $\sigma(\Delta^k) \not\subseteq A$ we consider

$$L(\sigma) \;=\; \{\text{all lifts of } \sigma \text{ to maps } \Delta^k \to \widetilde{X}\}.$$

We make two observations:

- It follows easily from our hypothesis that $p\colon \widetilde{X} \to X$ is a *regular* covering in the sense of Definition 319.11, together with the Deck Group-Free Action Lemma 128.3 and the Map Lifting Criterion 127.5, that the obvious action of D on $L(\sigma)$ is free and transitive.
- It follows easily from Definition 143.11 that as an abelian group $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ is generated by the union of all the $L(\sigma)$.

[3519] It follows from Statement (a) and the Axiom of Choice 1.1 that we can do so.

- If for each σ we pick precisely one element $\widetilde{\sigma} \in L(\sigma)$, then this family is clearly a complete set of representatives in the sense of Definition 310.12.

It follows from these two observations, together with the Group Ring–Free Module Lemma 310.13, that $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ is a free $\mathbb{Z}[D]$-left module and that a basis is given by a choice of a lift for each such σ.

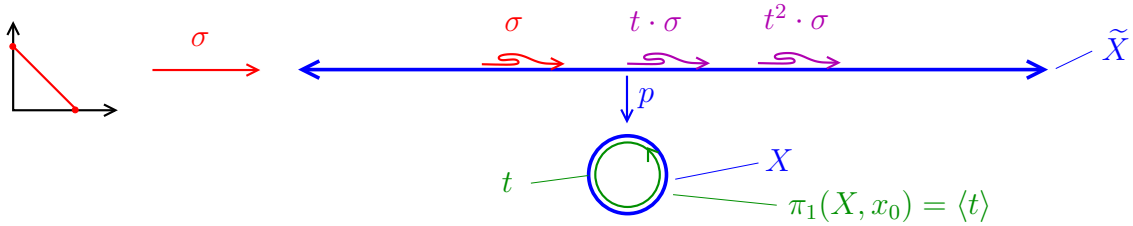
(2) By the C_*–Functor Lemma 143.16 we know that the boundary maps in singular chain complexes are natural, which implies in particular in our setting that the boundary map $\partial_k \colon \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to \mathrm{C}_{k-1}(\widetilde{X}, \widetilde{A})$ commutes with the D-action. It follows from the Set Action–Group Ring Module Lemma 310.11 (2) that this means that ∂_k is a homomorphism of $\mathbb{Z}[D]$-left modules. ■

The Regular Covering–Chain Complex Lemma 320.2 leads us to the following convention:

Convention 320.3. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and let $A \subseteq X$.

(1) Let $\Gamma \subseteq \pi_1(X, x_0)$ be a normal subgroup. If $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is any path-connected covering with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$[3520], then it follows from the convention on page 6147 together with the Regular Covering–Chain Complex Lemma 320.2 that we can view $\mathrm{C}_*(\widetilde{X}, \widetilde{A})$ as a chain complex of free $\mathbb{Z}[\pi_1(X, x_0)/\Gamma]$-left modules.

(2) In particular, if we consider the natural universal covering $\widetilde{X}_{x_0}$ from Definition 319.10, then we can view $\mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})$ as a chain complex of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.



320.2. **Tensor product over rings and homomorphism modules.** We just saw in Section 320.1 that for a regular covering $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ with deck transformation group $D \cong \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ the chain complex $\mathrm{C}_*(\widetilde{X})$ is a chain complex of free $\mathbb{Z}[D]$-left modules. In this section we recall two constructions of new modules out of given (bi-) modules :

- We can consider tensor products $M \otimes_R N$.
- We can consider the group of homomorphisms $\mathrm{Hom}_R(N, M)$.

320.2.1. **The tensor product.** In this subsection we will recall the construction of the tensor product over a (possibly non-commutative) ring. More precisely we start out by recalling Definition 312.3:

Definition 320.4. Let R, S, T be rings, let A be a (R, S)-bimodule and let B be an (S, T)-bimodule.[3521]

(1) As in the Free Bimodule–Existence Lemma 311.14 we consider the free (R, T)-bimodule $(R \otimes T)^{(A \times B)}$ that is generated by the set $A \times B$. We define the tensor product $A \otimes_S B$

[3520] It follows from the CO-Topology Lemma 319.7 that such a covering does indeed exist.

to be the following (R,T)-bimodule:[3522]

$$A \otimes_S B \;:=\; (R \otimes T)^{(A\times B)}/N(A,B)$$

where $N(A,B)$ is the (R,T)-bisubmodule that is generated by the elements of the form

(i) $(r\cdot a+\widetilde{r}\cdot\widetilde{a},b)-r\cdot(a,b)-\widetilde{r}\cdot(\widetilde{a},b)$ with $a,\widetilde{a}\in A$, $b\in B$ and $r,\widetilde{r}\in R$,

(ii) $(a,b\cdot t+\widetilde{b}\cdot\widetilde{t})-(a,b)\cdot t-(a,\widetilde{b})\cdot\widetilde{t}$ with $a\in A$, $b,\widetilde{b}\in B$ and $t,\widetilde{t}\in T$,

$(\star)$ $(a\cdot r,b)-(a,r\cdot b)$ with $a\in A, b\in B$ and $r\in S$.

(2) For $a\in A$ and $b\in B$ we write

$$a\otimes b \;:=\; [(a,b)] \quad\in\quad (R\otimes T)^{(A\times B)}/N(A,B) =: A\otimes_S B.$$

Such an element of $A\otimes_S B$ is sometimes called an elementary tensor.

The following lemma, which is essentially the same as the Tensor Product–Universal Property Lemma 312.5, gives us the universal property of the tensor product:

Lemma 320.5. (Tensor Product–Universal Property Lemma) Let R,S,T be rings, let A be a (R,S)-bimodule and let B be an (S,T)-bimodule.

(1) The map

$$\begin{array}{ccc} A\times B & \to & A\otimes_S B \\ (a,b) & \mapsto & a\otimes b \end{array}$$

is (R,T)-bilinear and S-balanced. In other words, the following conditions hold: For all $a,\widetilde{a}\in A$, $r,\widetilde{r}\in R$, $b,\widetilde{b}\in B$ and $t,\widetilde{t}\in T$ we have the equalities in $A\otimes_S B$:

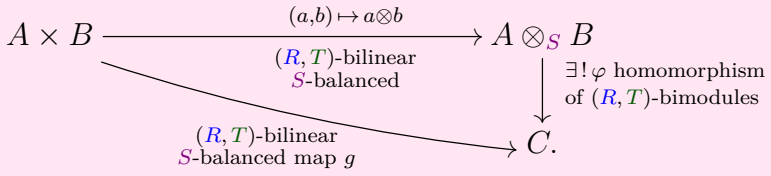
(i) $(r\cdot a+\widetilde{r}\cdot\widetilde{a})\otimes b = r\cdot(a\otimes b)+\widetilde{r}\cdot(\widetilde{a}\otimes b)$

(ii) $a\otimes(b\cdot t+\widetilde{b}\cdot\widetilde{t}) = (a\otimes b)\cdot t+(a\otimes\widetilde{b})\cdot\widetilde{t}$.

For any $a\in A, b\in B$ and $r\in S$ we have

$(\star)$ $(a\cdot r)\otimes b = a\otimes(r\cdot b)$.

(2) Given any (R,T)-bimodule C and given any map $g\colon A\times B\to C$ which is (R,T)-bilinear and S-balanced there exists a unique homomorphism $\varphi\colon A\otimes_S B\to C$ of (R,T)-bimodules such that the following diagram commutes:



Proof. This lemma combines the Tensor Product–Basics Lemma 312.4 together with the original Tensor Product–Universal Property Lemma 312.5. ∎

320.2.2. **The homomorphism group.** In this very short subsection we recall the second construction of new groups or modules out of given modules. More precisely, we have the following lemma, which is a slightly simplified version of the Hom–Module Lemma 315.2:

Lemma 320.6. (Hom–Module Lemma) Let S and R be rings, let M be an (R,S)-left left module and let A be an S-left module. The map

$$\begin{array}{rcl} \mathrm{Hom}_{S\text{-left}}(A,M)\times R & \to & \mathrm{Hom}_{S\text{-left}}(A,M) \\ (\varphi, r) & \mapsto & (a \mapsto r\cdot\varphi(a)) \end{array}$$

is a well-defined R-right module structure on the abelian group $\mathrm{Hom}_{S\text{-left}}(A,M)$.

320.3. **Modules over group rings.** In the previous section we recalled how one can construct new modules by taking tensor products of S-modules and by considering S-left homomorphisms. In our setting the ring S is going to be some group ring $S=\mathbb{Z}[G]$. Furthermore, as we saw in Section 320.1, in our topological setting we will obtain a $\mathbb{Z}[G]$-left module by considering the chain complex $\mathrm{C}_*(\widetilde{X})$ of a covering. But as we saw in the previous section, it takes two modules to tango, i.e. we need a second $\mathbb{Z}[G]$-left or right module to apply the constructions from the previous section.

First of all we recall the following notation from the Ring-with-Involution–Module Lemma 309.16 which allows us to turn any $\mathbb{Z}[G]$-left module into a $\mathbb{Z}[G]$-right module (and vice versa).

Notation 320.7. Let G be a group and let M be a $\mathbb{Z}[G]$-left module. We denote by $\overline{M}$ the $\mathbb{Z}[G]$-right module that is given by $m \underset{\overline{M}}{\cdot} g := g^{-1} \underset{M}{\cdot} m$. Note that if M is in fact an $(R,\mathbb{Z}[G])$-left left module, then $\overline{M}$ is clearly an $(R,\mathbb{Z}[G])$-bimodule.

Now that we know how to get $\mathbb{Z}[G]$-right modules out of $\mathbb{Z}[G]$-left modules, we just need good sources of $\mathbb{Z}[G]$-left modules. In fact the Group Ring–Module Lemma 310.5 leads us to the following notation which we encountered already as Notation 310.6:

Notation 320.8. Let G be a group, let R be a ring (not necessarily commutative), let M be an R-left module and let $\varphi\colon G\to \mathrm{Aut}_{R\text{-left}}(M)$ be a representation, i.e. a group homomorphism. By the Group Ring–Module Lemma 310.5 we know that the map

$$\begin{array}{rcl} \mathbb{Z}[G]\times M & \to & M \\ \left(\sum_{i=1}^n a_i\cdot g_i, m\right) & \mapsto & \sum_{i=1}^n a_i\cdot \underbrace{\varphi(g_i)(m)}_{\in M} \end{array}$$

defines a natural $(R,\mathbb{Z}[G])$-left left module structure on M. We denote the corresponding $(R,\mathbb{Z}[G])$-left left module by ${}_{\varphi}M$.

In practice most homomorphisms to automorphism groups arise from the following convention which we already encountered as Convention 310.17:

Convention 320.9. Let G be a group, let R be a ring, let $n\in\mathbb{N}_0$ and let $\varphi\colon G\to \mathrm{GL}(n,R)$ be a group homomorphism. As in the Free Module–Representation Classification Lemma 310.15 we consider the group isomorphism[3523 3524]

$$\begin{array}{rcl} \Theta\colon \mathrm{GL}(n,R) & \to & \mathrm{Aut}_{R\text{-left}}(R^{1\times n}) \\ A & \mapsto & \left(\begin{array}{rcl} R^{1\times n} & \to & R^{1\times n} \\ (v_1,\ldots,v_n) & \mapsto & (v_1,\ldots,v_n)\cdot A^{-1} \end{array}\right). \end{array}$$

(1) Often we refer to the homomorphism $\Theta\circ\varphi\colon G\to \mathrm{Aut}_{R\text{-left}}(R^{1\times n})$ by φ as well.

(2) Together with the Group Ring–Module Lemma 310.5 and with Notation 320.8 we obtain the corresponding $(R, \mathbb{Z}[G])$-left left module ${}_{\varphi}R^{1\times n}$.

Example. Let G be a group, let R be a ring, let $n \in \mathbb{N}_0$ and let $\varphi\colon G \to \mathrm{GL}(n, R)$ be a group homomorphism. From Convention 320.9 we obtain an $(R, \mathbb{Z}[G])$-left left module ${}_{\varphi}R^{1\times n}$. The definition of the left $\mathbb{Z}[G]$-structure seems a little awkward. But note that for the $(R, \mathbb{Z}[G])$-bimodule $\overline{{}_{\varphi}R^{1\times n}}$ we obtain the following module structure: Given $g \in G$ and $v \in R^{1\times n}$ we have the equalities

$$v \cdot g \underset{\text{see Notation 320.7}}{\overset{\uparrow}{=}} g^{-1} \cdot v \underset{\text{see Convention 320.9}}{\overset{\uparrow}{=}} v \cdot \varphi(g^{-1})^{-1} = v \cdot \varphi(g)$$

which suddenly does not look that bad after all. □

We conclude this section with the following piece of notation:

Notation 320.10. Let $\varphi\colon R \to S$ be a ring homomorphism. Let M be an S-left module. As in the Module–Change-of-Ring–Functor Lemma 309.13 we denote by $\varphi^*M = {}_{\varphi}M$ the R-left module that is given by

$$\begin{array}{rcl} R \times M & \to & M \\ (r, m) & \mapsto & \varphi(r) \cdot M. \end{array}$$

320.4. **Definition of twisted (co-) homology.** The Regular Covering–Chain Complex Lemma 320.2 together with the algebraic tools from Section 320.2 leads us to the following definition:

Definition 320.11. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Furthermore let $A \subseteq X$. Finally let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X_0, x_0)])$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering from Definition 319.10 and we write $\widetilde{A} := p^{-1}(A)$. Given $k \in \mathbb{N}_0$ we define the R-left modules [3525] [3526]

$$\begin{array}{lcll} \mathrm{C}_k^{\mathrm{tw}}(X, A; M) & := & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \text{(twisted chain complex)} \\ \mathrm{C}^k_{\mathrm{tw}}(X, A; M) & := & \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), M) & \text{(twisted cochain complex).} \end{array}$$

By the Regular Covering–Chain Complex Lemma 320.2 (2) we know that the boundary maps are $\mathbb{Z}[\pi_1(X, x_0)]$-left module homomorphisms. Thus we obtain from Definition 314.5 and the discussion in Definition 315.21 that $\mathrm{id} \otimes \partial$ and ∂^* define (co-) boundary maps. Thus we can define the R-left modules

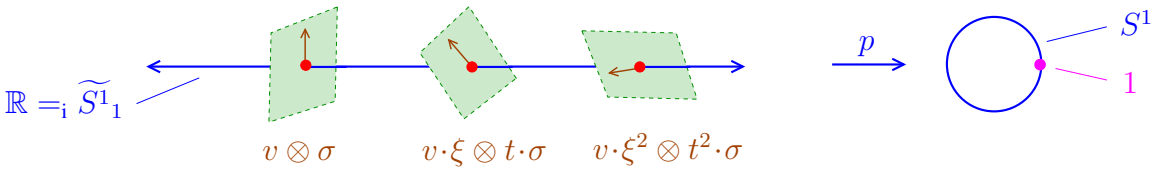
$$\begin{array}{lcll} \mathrm{H}_k^{\mathrm{tw}}(X, A; M) & := & \mathrm{H}_k\big(\overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})\big) & \text{(twisted homology)} \\ \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & := & \mathrm{H}^k\big(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A}), M)\big) & \text{(twisted cohomology).} \end{array}$$

As always, if $A = \varnothing$, then we drop it from the notation.

Illustration. Why is "twisted homology" called "twisted homology"? First of all, "twisted homology" is short for "homology with twisted coefficients". In the following we will discuss

[3523] As on page 5962 we think of $R^{1\times n}$ as the R-left module that is given by n-dimensional row vectors.
[3524] This definition might look rather awkward. But the point is, that we do not assume that R is commutative, and thus only *right* multiplication by a matrix commutes with left multiplication by R.
[3525] Here $\overline{M}$ is the $\mathbb{Z}[\pi_1(X, x_0)]$-right module from Notation 320.7.
[3526] It follows from the Tensor Product–Universal Property Lemma 320.5 and the Hom–Module Lemma 320.6 that $\mathrm{C}_k^{\mathrm{tw}}(X, A; M)$ and $\mathrm{C}^k_{\mathrm{tw}}(X, A; M)$ are both R-left modules.

an example which might justify the name "twisted coefficients". To do so we consider the topological space $X = S^1$ with the representation $\varphi\colon \pi_1(S^1) = \langle t \rangle \to \mathrm{GL}(1, \mathbb{C})$ that is given by $t \mapsto \xi$.



In this setting we see that for any 0-simplex σ and $v \in \mathbb{C}$ we have $v \otimes \sigma = v \cdot \xi \otimes t \cdot v = \dots$. As we see in the figure above, it looks like the $v \cdot \xi^k \otimes t^k \cdot v$ "twist" around the x-axis. □

320.5. **Twisted (co-) homology with trivial coefficients.** In the previous section we introduced twisted (co-) homology. The first question that arises is the following: Is "classical" (co-) homology a special case of twisted (co-) homology?

In this section we will answer this question. We now have the following setup: Throughout this section let (X, A) be a pair of topological spaces and let M be an abelian group.

- In Definition 160.12 we introduced the singular homology groups
$$\mathrm{H}_k(X, A; M) \;:=\; \mathrm{H}_k(\mathrm{C}_*(X, A) \otimes_{\mathbb{Z}} M)$$
with coefficients in M.
- In Definition 198.1 we introduced the singular cohomology groups
$$\mathrm{H}^k(X, A; M) \;:=\; \mathrm{H}^k(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_*(X, A), M).$$

If M has an R-left module structure over a ring R, then it follows immediately from the definitions, see also the discussion on page 3078, that both $\mathrm{H}_k(X, A; M)$ and $\mathrm{H}^k(X, A; M)$ can be equipped with a natural R-left module structure in an obvious way.

To see how these objects fit into the setting of twisted invariants we need to recall Definition 310.7:

Definition 320.12. Let G be a group and let R be a ring. Given an R-left module M we denote by ${}_{\mathrm{tr}}M$ the $(R, \mathbb{Z}[G])$-left left module where the $\mathbb{Z}[G]$-left module structure is given by the trivial g-action $g \cdot m := m$ for each $g \in G$ and $m \in M$.

The following lemma now shows that these two earlier types of (co-) homology groups are basically special cases of the present more general setting.

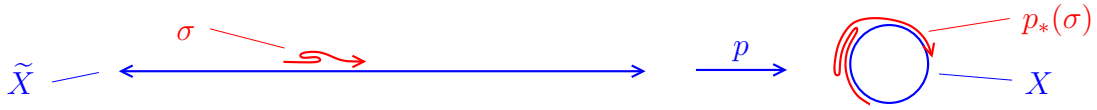
Lemma 320.13. (Twisted–Trivial Coefficients Lemma) Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and let $A \subseteq X$. In the following we write $\pi := \pi_1(X, x_0)$ and we denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We write $\widetilde{A} := p^{-1}(A)$. Furthermore let R be a ring and let M be an R-left module. As in Definition 320.12 we consider the $(R, Z[\pi_1(X, x_0)])$-left left module ${}_{\mathrm{tr}}M$. The maps

$$\overbrace{{}_{\mathrm{tr}}\overline{M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}M)} \to \mathrm{C}_k(X, A) \underset{\mathbb{Z}}{\otimes} M \quad \& \quad \mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_k(X, A), M) \to \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), {}_{\mathrm{tr}}M)}^{=\mathrm{C}^k_{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}M)}$$
$$m \otimes \sigma \mapsto p_*(\sigma) \otimes m \qquad \varphi \mapsto \varphi \circ p_*$$

are natural[3527] isomorphisms of R-left chain complexes. Thus they give rise to natural isomorphisms

$$T\colon \underbrace{\mathrm{H}_k^{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}M)}_{\substack{\text{twisted homology}\\ \text{from Definition 320.11}}} \xrightarrow{\cong} \underbrace{\mathrm{H}_k(X,A;M)}_{\text{see Definition 160.12}} \quad \text{and} \quad T\colon \underbrace{\mathrm{H}^k(X,A;M)}_{\text{see ??}} \xrightarrow{\cong} \underbrace{\mathrm{H}^k_{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}M)}_{\substack{\text{twisted cohomology}\\ \text{from Definition 320.11}}}$$

of R-left modules.



Proof.

(a) First we deal with homology. We start out with the following claim:

Claim. The given map $\overline{{}_{\mathrm{tr}}M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}) \to \mathrm{C}_k(X,A) \otimes_{\mathbb{Z}} M$ is well-defined.

Proof. First note that for any choice of $m \in M$, of a singular simplex σ and some $g \in \pi$ we have $p_*(g\cdot\sigma)\otimes m = p_*(\sigma)\otimes m = p_*(\sigma)\otimes(m\cdot g)$. It thus follows from the Tensor Product–Universal Property Lemma 320.5 that the given map is well-defined. ⊞

Next note that one can easily verify that the maps from a chain map. It remains to show that the chain maps are isomorphisms. To do so we break our map into the composition of three maps:

$$\begin{aligned}
\overline{{}_{\mathrm{tr}}M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}) &\to \overline{{}_{\mathrm{tr}}M} \underset{\mathbb{Z}}{\otimes} ({}_{\mathrm{tr}}\mathbb{Z} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A})) \to M \underset{\mathbb{Z}}{\otimes} \mathrm{C}_k(X,A) \to \mathrm{C}_k(X,A) \underset{\mathbb{Z}}{\otimes} M\\
m\otimes\sigma &\mapsto m\otimes(1\otimes\sigma)\\
m\otimes(n\otimes\sigma) &\mapsto m\otimes n\cdot p_*(\sigma)\\
m\otimes\sigma &\mapsto \sigma\otimes m
\end{aligned}$$

The desired statements follows from the following observations:

- The first map is an isomorphism since by the Tensor Product–Cancel Ring Lemma 312.12 we know that one can always cancel a term $\otimes_R R$.
- It follows easily from the Regular Covering–Chain Complex Lemma 320.2, together with the Free Module–Tensor-with-Ring Lemma 312.18 (1), that the second map is an isomorphism.
- It follows from the Tensor Commutative-Associative Lemma 159.11 that the third map is an isomorphism.

(b) Now we consider the statements about the cochain complexes. First note that one can can easily verify that the map $\varphi\circ p_*$ is indeed $\mathbb{Z}[\pi]$-left linear. Next note that it follows easily from the definitions that the map between the Hom-modules form a cochain map. It remains to show that these maps are isomorphisms. Using the Homomorphism-Change Ring Lemma 315.4 one can easily verify that the map on the "Hom-level" is also an isomorphism. We leave it to the reader to provide the details. ■

320.6. **Twisted homology of a circle.** We do not want to conclude this chapter without having performed at least one calculation of twisted homology with a non-trivial coefficient module. So let us calculate some twisted homology modules of the circle:

[3527] We leave it to the reader to figure out what "natural" might mean in this context.

Proposition 320.14. (Twisted-H_*-of-S^1-Proposition) We make the usual identification $\pi_1(S^1,1) =_{\mathrm{i}} \langle t\rangle$. Let $\xi \in \mathbb{C}\setminus\{0\}$. We consider the 1-dimensional complex representation $\varphi\colon \pi_1(S^1,1) =_{\mathrm{i}} \langle t\rangle \to \mathrm{GL}(1,\mathbb{C})$ that is given by $\varphi(t) = (\xi)$.

(1) If $\xi = 1$, then we obtain the usual homology groups of S^1.
(2) If $\xi \neq 1$, then all twisted homology groups $\mathrm{H}_*(S^1;{}_\varphi\mathbb{C})$ are zero.

The proof of the Twisted-H_*-of-S^1-Proposition 320.14 makes use of the following lemma:

Lemma 320.15. (Group Ring–Modification Lemma)

(1) Let R be a ring and let S be a commutative ring, let W be an (S,R)-bimodule and let C be an R-left module. We consider the tensor product $S\otimes_{\mathbb{Z}} R$ of the rings R and S from Definition 311.8. By the Tensor Product-of-Rings–Modules Lemma 311.9 we can view $S\otimes_{\mathbb{Z}} C$ as an $S\otimes_{\mathbb{Z}} R$-left module. Furthermore we saw in Exercise 311.3 that we have a natural $S\otimes_{\mathbb{Z}} R$-right module structure on W with the property that $w\cdot(s\otimes r) = s\cdot w\cdot r$. With these conventions we have well-defined isomorphisms

$$\begin{array}{rcl} W\otimes_R C & \to & W\otimes_{S\otimes_{\mathbb{Z}} R}(S\otimes_{\mathbb{Z}} C) \\ w\otimes c & \mapsto & w\otimes(1_S\otimes c)\end{array} \quad\text{and}\quad \begin{array}{rcl} W\otimes_{S\otimes_{\mathbb{Z}} R}(S\otimes_{\mathbb{Z}} C) & \to & W\otimes_R C \\ w\otimes s\otimes c & \mapsto & (s\cdot w)\otimes c\end{array}$$

of S-modules.

(2) Let G be a group, let S be a commutative ring, let W be an $(S,\mathbb{Z}[G])$-bimodule and let C be a $\mathbb{Z}[G]$-left module. Then we have well-defined isomorphisms

$$\begin{array}{rcl} W\otimes_{\mathbb{Z}[G]} C & \to & W\otimes_{S[G]}(S\otimes_{\mathbb{Z}} C) \\ w\otimes c & \mapsto & w\otimes(1_S\otimes c)\end{array} \quad\text{and}\quad \begin{array}{rcl} W\otimes_{S[G]}(S\otimes_{\mathbb{Z}} C) & \to & W\otimes_{\mathbb{Z}[G]} C \\ w\otimes s\otimes c & \mapsto & (s\cdot w)\otimes c\end{array}$$

of S-modules.

Sketch of proof of Lemma 320.15.

(1) Using the Tensor Product–Universal Property Lemma 320.5 one can easily verify that both maps are well-defined. It is also straightforward to check that the two maps are inverses of one another.

(2) This statement follows from (1), applied to $R = \mathbb{Z}[G]$, and the observation from page 6005 that the map

$$\begin{array}{rcl} S\otimes_{\mathbb{Z}}\mathbb{Z}[G] & \to & S[G] \\ s\otimes g & \mapsto & s\cdot g\end{array}$$

is a natural isomorphism of rings. ■

Proof of Proposition 320.14. [3528] Let $\xi\in\mathbb{C}\setminus\{0\}$. We see that for each $k\in\mathbb{N}_0$ we have the following isomorphisms:

$$\mathrm{H}_k(S^1;{}_\varphi\mathbb{C}) \underset{\text{see Definition 320.11}}{\cong} \mathrm{H}_*({}_{\overline{\varphi}}\overline{\mathbb{C}}\otimes_{\mathbb{Z}[t^{\pm1}]}\mathrm{C}_*(\widetilde{S^1})) \underset{\text{by the Group Ring–Modification Lemma 320.15 (2)}}{\cong} \mathrm{H}_k({}_{\overline{\varphi}}\overline{\mathbb{C}}\otimes_{\mathbb{C}[t^{\pm1}]}(\mathbb{C}\otimes_{\mathbb{Z}}\mathrm{C}_*(\widetilde{S^1})))$$

$$\underset{\text{by the Algebraic Universal Coefficient Theorem 314.33, here we use that } \mathbb{C}[t^{\pm1}] \text{ is a PID}}{\cong} {}_{\overline{\varphi}}\overline{\mathbb{C}}\otimes_{\mathbb{C}[t^{\pm1}]}\mathrm{H}_k(\mathbb{C}\otimes_{\mathbb{Z}}\mathrm{C}_*(\widetilde{S^1}))\oplus \mathrm{Tor}_{\mathbb{C}[t^{\pm1}]}(\mathrm{H}_{k-1}(\mathbb{C}\otimes_{\mathbb{Z}}\mathrm{C}_*(\widetilde{S^1})),{}_{\overline{\varphi}}\overline{\mathbb{C}}).$$

[3528] The case $\xi = 1$ is of course taken care of by the Twisted–Trivial Coefficients Lemma 320.13. But nonetheless we will deal with the case of any $\xi\in\mathbb{C}\setminus\{0\}$.

To continue we need to calculate the $\mathbb{C}[t^{\pm 1}]$-modules $\mathrm{H}_k(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\widetilde{S^1}))$:

Claim 1. For $l \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_l(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\widetilde{S^1})) \cong \begin{cases} 0, & \text{if } l \geq 1, \\ \mathbb{C}[t^{\pm 1}]/(t-1)\mathbb{C}[t^{\pm 1}], & \text{if } l = 0 \end{cases}$$

of $\mathbb{C}[t^{\pm 1}]$-modules.

Proof. Note that the natural universal covering $\widetilde{S^1}$ of S^1 is homeomorphic to $\mathbb{R}$. Thus we see that as $\mathbb{C}$-modules we have isomorphisms

$$\mathrm{H}_l(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\widetilde{S^1})) \underset{\text{since } \widetilde{S^1} \cong \mathbb{R}}{\uparrow\cong} \mathrm{H}_l(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\mathbb{R})) \underset{\text{essentially by Definition 160.12}}{\uparrow\cong} \overbrace{\mathrm{H}_l(\mathbb{R}; \mathbb{C})}^{\substack{\text{usual homology} \\ \text{with } \mathbb{C}\text{-coefficients} \\ \text{of the space } \mathbb{R}}} \underset{\text{since } \mathbb{R} \text{ is contractible}}{\uparrow\cong} \begin{cases} 0, & \text{if } l \geq 1, \\ \mathbb{C}, & \text{if } l = 0. \end{cases}$$

Furthermore, similar to the discussion on page 5475, one can easily see that any automorphism acts trivially on $\mathrm{H}_0(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\widetilde{S^1}))$. In our setting this means that any t^k acts trivial on $\mathrm{H}_0(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\widetilde{S^1}))$. Thus, as a $\mathbb{C}[t^{\pm 1}]$-module, we have an isomorphism $\mathrm{H}_l(\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{C}_*(\widetilde{S^1})) \cong \mathbb{C}[t^{\pm 1}]/(t-1)\mathbb{C}[t^{\pm 1}]$. ⊞

Next let us rewrite the $\mathbb{C}[t^{\pm 1}]$-module ${}_{\varphi}\overline{\mathbb{C}}$:

Claim 2. We have an isomorphism $\mathbb{C}[t^{\pm 1}]/(t-\xi^{-1})\mathbb{C}[t^{\pm 1}] \to {}_{\varphi}\overline{\mathbb{C}}$ of $\mathbb{C}[t^{\pm 1}]$-modules.

Proof. One can easily verify that the homomorphism $\mathbb{C}[t^{\pm 1}] \to {}_{\varphi}\overline{\mathbb{C}}$ that is given by $t \mapsto \xi^{-1}$ descends to the promised isomorphism. ⊞

The lemma now follows from the following purely algebraic claim:

Claim 3.

(1) The ring $\mathbb{C}[t^{\pm 1}]$ is a PID.
(2) Let R be a PID and let $a, b \in R$. Then there exist isomorphisms

$$R/aR \otimes_R R/bR \cong R/\gcd(a,b)R \cong \mathrm{Tor}_R(R/aR, R/bR)$$

of R-modules.
(3) Let $\xi, \zeta \in \mathbb{C} \setminus \{0\}$. Then

$$\gcd_{\mathbb{C}[t^{\pm 1}]}(t-\xi, t-\zeta) = \begin{cases} 1, & \text{if } \xi \neq \zeta, \\ t-\xi, & \text{if } \xi = \zeta. \end{cases}$$

Proof.

(1) We proved this statement on page 5975.
(2) These two statements follows from the Cyclic-Tensor-Cyclic Lemma 314.28 and the Cyclic-Tor-Lemma 314.29.
(3) This statement follows from elementary algebra. ■

320.7. **Replacing the universal covering.** In Definition 320.11 of twisted (co-) homology we work with the universal covering. In some settings the following propositions allows us to work with alternative universal coverings and "smaller coverings". At least in principle this can simplify life quite a bit:

Proposition 320.16. (Replace Universal Covering Proposition) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering. Let $A \subseteq X$. We write $\widetilde{A} \coloneqq p^{-1}(A)$. We set $\pi \coloneqq \pi_1(X, x_0)$. Let $\Gamma \subseteq \pi$ a normal subgroup. We denote by $\rho\colon \pi \to \pi/\Gamma$ the projection and we denote by $\rho\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi/\Gamma]$ also the corresponding ring homomorphism. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi/\Gamma])$-left left module. Let $q\colon (Y, y_0) \to (X, x_0)$ be any path-connected regular covering with $q_*(\pi_1(Y, y_0)) = \Gamma$. We set $B \coloneqq q^{-1}(A)$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (Y, y_0)$ $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (Y_\Gamma, y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow{\quad r \quad} & Y_\Gamma. \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

The natural maps[3529] [3530]

$$\begin{array}{rcl} \overbrace{\overline{\rho^* M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{\mathrm{C}_k^{\mathrm{tw}}(X, A; \rho^* M)} & \to & \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k(Y, B) \\ m \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & m \otimes (r \circ \sigma\colon \Delta^k \to Y) \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}[\pi/\Gamma]\text{-left}}(\mathrm{C}_k(Y, B), M) & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), \rho^* M)}^{= \mathrm{C}_{\mathrm{tw}}^k(X, A; \rho^* M)} \\ \varphi & \mapsto & (\sigma \mapsto \varphi(r \circ \sigma)) \end{array}$$

are isomorphisms of (co-) chain complexes of R-left modules. In particular we get induced isomorphisms of (co-) homology modules.

Example. We consider the two extreme cases of Γ the trivial subgroup and the whole group:

- For $\Gamma = \{e\}$ the Replace Universal Covering Proposition 320.16 tells us that we can use any universal covering to calculate the isomorphism type of twisted (co-) homology. In other words, we do need to work with the natural universal covering from Definition 319.10.
- The case $\Gamma = \pi_1(X, x_0)$ can only occur if M is the trivial $\mathbb{Z}[\pi]$-module. In this case the conclusion of the Replace Universal Covering Proposition 320.16 gives us essentially the same statement as the Twisted–Trivial Coefficients Lemma 320.13. □

Proof. We denote by $\rho\colon \pi \to \pi/\Gamma$ the projection and we also denote by $\rho\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi/\Gamma]$ the corresponding ring homomorphism.

Claim. The map

$$\begin{array}{rcl} \mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \to & \mathrm{C}_k(Y, B) \\ g \otimes \sigma & \mapsto & \rho(g) \cdot (r \circ \sigma) \end{array}$$

is well-defined and it is an isomorphism of $\mathbb{Z}[\Gamma\backslash\pi]$-left modules.

[3529] Note that our hypothesis that $Y \to X$ is a regular covering is, by the Regular Covering-Characterization Proposition 128.13, equivalent to the statement that $\Gamma \coloneqq q_*(\pi_1(Y, y_0))$ is a normal subgroup of $\pi = \pi_1(X, x_0)$

[3530] As we discussed on page 6147 we have a natural left action of $\pi/\Gamma = \pi_1(X, x_0)/q_*(\pi_1(Y, y_0))$ on Y.

Proof. First note that it follows from the naturality statement of the Deck Transformation Group Proposition 319.12 that for any $g \in \pi_1(X, x_0)$ and any $w \in \widetilde{X}_{x_0}$ we have the equality $r(g \cdot w) = \rho(g) \cdot r(w)$. It now follows easily from the Tensor Product–Universal Property Lemma 320.5 that the map is well-defined.

Given any singular simplex $\sigma\colon \Delta^k \to X$ that is not contained in A we pick a lift $\widetilde{\sigma}\colon \Delta \to \widetilde{X}$. In the Regular Covering–Chain Complex Lemma 320.2 we showed that these lifts form a basis of the free $\mathbb{Z}[\pi]$-left module $\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})$. Furthermore by the Regular Covering–Chain Complex Lemma 320.2 we also know that the singular simplices $r \circ \widetilde{\sigma}$ form a basis of the free $\mathbb{Z}[\pi/\Gamma]$-left module $\mathrm{C}_k(Y, B)$. It follows from this observation, together with the Free Module–Tensor-with-Ring Lemma 312.18 (1), that the map sends a basis of the free $\mathbb{Z}[\pi/\Gamma]$-left module $\mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})$ to a basis of the free $\mathbb{Z}[\pi/\Gamma]$-left module $\mathrm{C}_k(Y, B)$. Thus we see that the map is indeed an isomorphism. ⊞

Note that the first map we are supposed to study equals the composition of the following maps:

$$\begin{array}{rcccl} \overline{\rho^* M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} (\mathbb{Z}[\pi/\Gamma] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k(Y, B) \\ m \otimes \sigma & \mapsto & m \otimes (1 \otimes \sigma) & & \\ & & m \otimes (n \otimes \sigma) & \mapsto & m \otimes r_*(n \cdot \sigma). \end{array}$$

It follows from the Tensor Product–Cancel Ring Lemma 312.12 that the map to the left is well-defined and an isomorphism. It follows from the claim that the map to the right is an isomorphism. In summary, we are done.

Statement (2) regarding the cochain complexes can be proved, using the Homomorphism-Change Ring Lemma 315.4, in a very similar fashion. We leave the details to the reader. ■

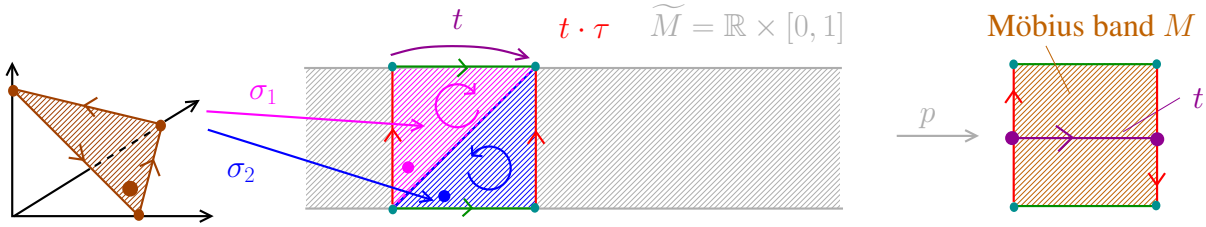
320.8. **Outlook.** By the Twisted–Trivial Coefficients Lemma 320.13 we now know that twisted (co-) homology is for all intents and purposes a generalization of "ordinary" homology and cohomology. It it thus natural to ask, in what sense the earlier results on "untwisted" (co-) homology extend to twisted (co-) homology. This question, and many other questions will be discussed in the following chapters. More precisely, we will do the following:

- In the Change-of-Base Point Proposition 321.3 we will see that twisted (co-) homology is "functorial in the base point".
- In the Twisted Homology-Functoriality Proposition 321.9 we will discuss in what sense twisted (co-) homology is functorial in the underlying topological space.
- In the Twisted-LES-Proposition 321.10 we will see that a triple of topological spaces gives rise to a long exact sequence in twisted (co-) homology.
- In the Functorial-in-Coefficients Lemma 321.13 we will see that twisted (co-) homology is functorial in the coefficient module.
- In the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3 we will discuss to what degree homotopic maps induce the same maps on twisted (co-) homology.
- In the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1 we will compare twisted (co-) homology for homotopy equivalent topological spaces.
- In the Twisted H_0-Proposition 323.3 we will calculate 0-th twisted homology and cohomology.

- (a) In the Covering-H_*^{tw}-Proposition 323.5 we will see that in some situations twisted homology is the same as twisted homology of a covering.
  (b) In the Finite Covering-H^*_{tw}-Proposition 323.13 we will see that in some situations twisted cohomology is the same as twisted cohomology of a *finite* covering.
- In the Twisted Excision Theorem 325.4 we will formulate an excision theorem for twisted (co-) homology.
- In the Twisted Mayer–Vietoris Theorem 325.12 we will formulate and prove a twisted analog of the classical Mayer–Vietoris theorems.
- In the Singular-Cellular Twisted Isomorphism Theorem 328.1 we will see that singular twisted (co-) homology is naturally isomorphic to cellular twisted (co-) homology. This result will finally open the door to performing many calculations of twisted (co-) homology.
- In the Singular-Handle Twisted Isomorphism Theorem 333.4 we will see that singular twisted (co-) homology is naturally isomorphic to handle twisted (co-) homology.
- In Chapter 334 we will introduce the cup product and the cap product on twisted (co-) homology.
- In the Twisted Poincaré Duality Theorem 335.1 we will state and prove Poincaré Duality for (co-) homology with twisted coefficients.
- In Chapter 337 we will introduce twisted intersection pairings on compact orientable even-dimensional topological manifolds.

320.9. Exercises for Chapter 320.

Exercise 320.1. We consider the Möbius band $M := ([0,1]\times[0,1])/(0,x)\sim(1,1-x)$ together with the usual universal covering $p\colon \mathbb{R}\times[0,1]\to M$. Next let $\varphi\colon \pi_1(M)\to \mathrm{GL}(1,\mathbb{Z})$ be the representation which sends a generator of $\pi_1(M)=_{\mathrm{i}}\langle t\rangle$ to -1. In the figure below we also show two singular simplices $\sigma_i\colon \Delta^2\to\mathbb{R}\times[0,1]$. Show that there exist non-zero $f_1,f_2\in\mathbb{Z}$ such that $f_1\otimes\sigma_1+f_2\otimes\sigma_2$ is a cycle in ${}_\varphi\mathbb{Z}\otimes_{\mathbb{Z}[t^{\pm1}]}\mathrm{C}_*(\widetilde{M})$.



Exercise 320.2. Let (X,x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\varphi\colon \pi_1(X,x_0)\to\mathrm{GL}(1,\mathbb{C})$ be a non-trivial representation. Show that $\mathrm{H}_0^{\mathrm{tw}}(X;{}_\varphi\mathbb{C})=0$.
Hint. You need to show that each $v\otimes P$ with $v\in\mathbb{C}$ and $P\in X$ is trivial in $\mathrm{H}_0^{\mathrm{tw}}(X;{}_\varphi\mathbb{C})$.

Exercise 320.3. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Show that if X is contractible, then all twisted homology and cohomology groups of degree ≥ 1 are zero.

Exercise 320.4. We want to calculate some twisted homology modules of $S^1 \vee S^2$ with wedge point $*$. We make the usual identification $\pi_1(S^1 \vee S^2, *) =_{\mathrm{i}} \langle t \rangle$. Let $z \in \mathbb{C} \setminus \{0\}$. We consider the representation $\varphi\colon \pi_1(S^1 \vee S^2, *) =_{\mathrm{i}} \langle t \rangle \to \mathrm{GL}(1, \mathbb{C})$ that is given by $\varphi(t) = (z)$. For each $k \in \mathbb{N}_0$ compute $\mathrm{H}_k^{\mathrm{tw}}(S^1 \vee S^2; {}_{\varphi}\mathbb{C})$.

Exercise 320.5. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and which has the property that the universal covering $\widetilde{X}_{x_0}$ is contractible. We write $\pi := \pi_1(X, x_0)$, Let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module.

(a) What is the conclusion about $\mathrm{H}_k^{\mathrm{tw}}(X; M) := \mathrm{H}_k^{\mathrm{tw}}(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0}))$ that you obtain from the Homological UCSS Theorem 316.5 in this setup?

(b) What is the conclusion $\mathrm{H}^k_{\mathrm{tw}}(X; M) := \mathrm{H}^k(\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{x_0}), M))$ that you obtain from the Cohomological UCSS Theorem 316.10 in this setup?

Exercise 320.6. We want to calculate some twisted homology modules of $S^1 \times S^2$. We pick a base point $x_0 \in X$. We make the usual identification $\pi_1(S^1 \times S^2, x_0) =_{\mathrm{i}} \langle t \rangle$. Let $z \in \mathbb{C} \setminus \{0\}$. We consider the representation $\varphi\colon \pi_1(S^1 \times S^2, *) =_{\mathrm{i}} \langle t \rangle \to \mathrm{GL}(1, \mathbb{C})$ that is given by $\varphi(t) = (z)$. For each $k \in \mathbb{N}_0$ compute $\mathrm{H}_k^{\mathrm{tw}}(S^1 \times S^2; {}_{\varphi}\mathbb{C})$.

Exercise 320.7. Let $n \in \mathbb{N}_{\geq 2}$ and let $\rho\colon S^n \to S^n$ be the map that is given by $P \mapsto -P$. We consider the mapping torus $\mathrm{Tor}(S^n, \rho) := (S^n \times [0,1])/(x,0) \sim (\rho(x), 1)$ from Definition 21.19. Since $n \geq 2$ we know by the Mapping Torus-π_1-Proposition 93.13 that the natural projection $p\colon \mathrm{Tor}(S^n, \rho) \xrightarrow{[(x,t)] \mapsto \exp(2\pi i t)} S^1$ induces an isomorphism of fundamental groups. We make the usual identification $\pi_1(S^1, 1) =_{\mathrm{i}} \langle t \rangle$. Let $z \in \mathbb{C} \setminus \{0\}$. We consider the representation $\varphi\colon \pi_1(\mathrm{Tor}(S^n, \rho)) \to \mathrm{GL}(1, \mathbb{C})$ that is given by p_* and $t \mapsto (z)$. For each $k \in \mathbb{N}_0$ compute $\mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(S^n, \rho); {}_{\varphi}\mathbb{C})$.

Exercise 320.8. We make the usual identification $\pi_1(S^1, 1) =_{\mathrm{i}} \langle t \rangle$. We consider the 1-dimensional complex representation $\varphi\colon \pi_1(S^1, 1) =_{\mathrm{i}} \langle t \rangle \to \mathrm{GL}(1, \mathbb{Q}(t))$ that is given by $\varphi(t) = (t)$. Compute all twisted homology groups $\mathrm{H}_*(S^1; {}_{\varphi}\mathbb{Q}(t))$.

Exercise 320.9. Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{Z})$ be a representation. Given $k \in \mathbb{Z}$ we also consider the representation $\varphi_k\colon \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}/k\mathbb{Z})$ where the last map is given by applying the natural projection $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ to each entry. Are the isomorphism types of the twisted homology groups $\mathrm{H}_*(X; {}_{\varphi_k}(\mathbb{Z}/k\mathbb{Z})^n)$ determined by the isomorphism types of the twisted homology groups $\mathrm{H}_*(X; {}_{\varphi}\mathbb{Z}^n)$?

CHAPTER 321

Functoriality of twisted (co-) homology

In the previous Chapter 320 we introduced twisted (co-) homology. In this chapter we will study to what degree twisted (co-) homology is functorial in the (pointed) topological space and in the coefficient modules.

This chapter is organized as follows:

- In Section 321.1 we will discuss to what degree twisted (co-) homology depends on the choice of the base point.
- In Section 321.2 we will study the functoriality of twisted (co-) homology in the topological space.
- In Section 321.3 we will discuss the long exact sequence of a pair (X, A) in twisted homology.
- In Section 321.4 we will see that twisted (co-) homology is functorial in the coefficients.
- In Section 321.5 we will provide exercises for Chapter 321.

In the next Chapter 322 we will study the interplay between twisted (co-) homology and homotopies.

321.1. **Dependence on the base point.** (Co-) homology with twisted coefficients depends in two ways on the choice of a base point x_0:

(1) We use the base point to define the fundamental group $\pi_1(X, x_0)$ which in turn we need to define the corresponding group ring $\mathbb{Z}[\pi_1(X, x_0)]$.
(2) In Definition 319.10 we need the base point to define the natural universal covering $\widetilde{X}_{x_0}$.

To discuss this dependence of the base point it is necessary to introduce the following notation:

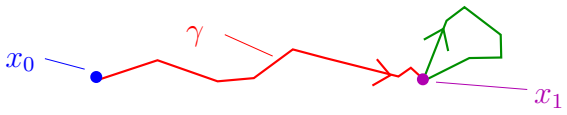
Notation 321.1. Let X be a topological space and let $\gamma\colon [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. Finally let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. In the Change-of-Base Point Proposition 88.14 we showed that the map

$$\begin{array}{rcl} \gamma^*\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\alpha] & \mapsto & [\gamma * \alpha * \overline{\gamma}] \end{array}$$

is an isomorphism and that it has the following properties:

(1) The induced map γ^* only depends on the path-homotopy class of γ.
(2) (a) If γ is a constant path, then $\gamma^* = \mathrm{id}$.
    (b) If $\delta\colon [0,1] \to X$ is a path from x_1 to another point x_2, then

$$(\gamma * \delta)^* = \gamma^* \circ \delta^*\colon \pi_1(X, x_2) \to \pi_1(X, x_0).$$



the path γ from x_0 to x_1 induces an isomorphism $\gamma^*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$

Notation 321.2. Let X be a topological space that is path-connected and locally contractible. Let $\gamma\colon [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. In the Natural Covering-Base Point Lemma 127.19 we showed that the map

$$\begin{array}{rcl} \gamma^*\colon \widetilde{X}_{x_1} & \to & \widetilde{X}_{x_0} \\ [\beta] & \mapsto & [\gamma * \beta] \end{array}$$

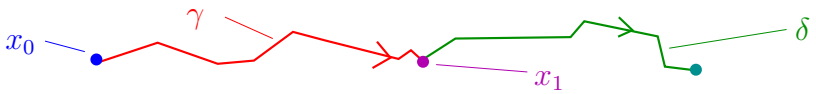
is a homeomorphism and that it has the following properties:

(1) The induced map γ^* only depends on the path-homotopy class of γ.

(2) (a) If γ is a constant path, then $p^* = \mathrm{id}$.

(b) If $\delta\colon [0,1] \to X$ is a path from x_1 to another point x_2, then

$$(\gamma * \delta)^* \;=\; \gamma^* \circ \delta^*\colon \pi_1(X,x_2) \;\to\; \pi_1(X,x_0).$$



The following proposition describes how we can go from one base point to another.

Proposition 321.3. (Change-of-Base Point Proposition) Let X be a topological space that is path-connected and locally contractible and let $A \subseteq X$. Let $\gamma\colon [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. For $i = 0,1$ we denote by $p_i\colon \widetilde{X}_{x_i} \to X$ the natural universal covering and we write $\widetilde{A}_i := p_i^{-1}(A)$. Finally let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module. We consider the induced group isomorphism $\gamma^*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$ from Notation 321.1. The isomorphism γ^* gives rise to a ring homomorphism $\gamma^*\colon \mathbb{Z}[\pi_1(X,x_1)] \to \mathbb{Z}[\pi_1(X,x_0)]$. Thus by Notation 320.10 we can consider the $\mathbb{Z}[\pi_1(X,x_1)]$-left module ${}_{\gamma^*}M$. Furthermore we consider the homeomorphism $\gamma^*\colon \widetilde{X}_{x_1} \to \widetilde{X}_{x_0}$ from Notation 321.2.

(1) (a) With the above notation the maps

$$\begin{array}{rcl} \gamma^*\colon \overline{{}_{\gamma^*}M} \otimes_{\mathbb{Z}[\pi_1(X,x_1)]} \mathrm{C}_k(\widetilde{X}_{x_1}, \widetilde{A}_1) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}_0) \\ m \otimes \sigma & \mapsto & m \otimes \gamma^*(\sigma) \end{array}$$

form an isomorphism of R-left chain complexes and the maps

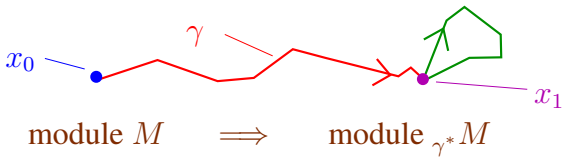
$$\begin{array}{rcl} \gamma^*\colon \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_1)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{X}_{x_1}, \widetilde{A}_1), {}_{\gamma^*}M) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}_0), M) \\ \varphi & \mapsto & \varphi \circ \gamma^* \end{array}$$

form an isomorphism of R-left module cochain complexes. In particular we get induced isomorphisms on (co-) homology with twisted coefficients.

(b) If $\gamma_0, \gamma_1\colon [0,1] \to X$ are path-homotopic[3531], then $\gamma_0^* = \gamma_1^*$, in other words, the maps γ^* only depend on the path-homotopy class of γ.

(2) (a) If γ is the constant path, then $\gamma^* = \mathrm{id}$.

(b) Given any two paths $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ with $\alpha(1) = \beta(0)$ we have the equality $(\alpha * \beta)^* = \beta^* \circ \alpha^*$.



the path γ from x_0 to x_1
induces an isomorphism
$\gamma^*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$

Proof.

(1) (a′) We start out with the following claim:

Claim. The maps

$$\begin{array}{rcl} \gamma^*\colon \overline{{}_{\gamma^*}M} \otimes_{\mathbb{Z}[\pi_1(X,x_1)]} \mathrm{C}_k(\widetilde{X}_{x_1}, \widetilde{A}_1) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}_0) \\ m \otimes \sigma & \mapsto & m \otimes \gamma^*(\sigma) \end{array}$$

are well-defined and they form a chain map.

Proof. To do so we need to consider the corresponding $\mathbb{Z}[\pi_1(X,x_0)]$-right module $\overline{M}$ and the corresponding $\mathbb{Z}[\pi_1(X,x_1)]$-right module $\overline{{}_{\gamma^*}M}$. We start out with an observation: Namely, note that given any $h \in \pi_1(X,x_1)$ and given any $m \in M$ we have the following equalities

$$m \underset{\overline{{}_{\gamma^*}M}}{\cdot} h \;=\; h^{-1} \underset{{}_{\gamma^*}M}{\cdot} m \;=\; \gamma^*(h^{-1}) \underset{M}{\cdot} m \;=\; m \underset{\overline{M}}{\cdot} \gamma^*(h).$$

Now we turn to the actual proof of the claim. First we consider the map γ^* on the chain level. Given any $m \in M$, given any singular simplex $\sigma\colon \Delta^k \to \widetilde{X}_{x_1}$ and given any $h \in \pi_1(X,x_1)$ we have the following equalities:

$$\begin{array}{rcl} \gamma^*\big(m \underset{\overline{{}_{\gamma^*}M}}{\cdot} h \otimes \sigma\big) \;=\; m \underset{\overline{{}_{\gamma^*}M}}{\cdot} h \otimes \gamma^*(\sigma) & \overset{\text{by the above observation}}{\downarrow =} & m \underset{\overline{M}}{\cdot} \gamma^*(h) \otimes \sigma \;=\; m \otimes \gamma^*(h) \cdot \gamma^*(\sigma) \\ & = & m \otimes \gamma^*(h \cdot \sigma) \;=\; \gamma^*(m \otimes h \cdot \sigma). \end{array}$$

This calculation, together with the Tensor Product–Universal Property Lemma 320.5, shows that the map on the chain level is well-defined. Furthermore it follows easily from the definitions that these maps form a chain map. ⊞
Similarly to above claim one can show easily that the second set of maps form a cochain map.

(b) If γ_0 and γ_1 are path-homotopic, then it follows the Change-of-Base Point Proposition 88.14 that $\gamma_0^* = \gamma_1^*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$ and it follows from the Natural Covering-Base Point Lemma 127.19 that $\gamma_0^* = \gamma_1^*\colon \widetilde{X}_{x_1} \to \widetilde{X}_{x_0}$.

(2) This statement follows easily from the definitions together with the Change-of-Base Point Proposition 88.14 and the Natural Covering-Base Point Lemma 127.19.

(1) (a″) It follows easily from (1b) and (2) that for a path γ the inverse to γ^* is given by $\overline{\gamma}^*$, where $\overline{\gamma}(t) := \gamma(1-t)$ is the inverse path. This shows that the maps γ^* are isomorphisms of R-left chain complexes and also of R-left cochain complexes. ■

Remark.

[3531]Recall that according to Definition 87.3 two paths $\gamma_0, \gamma_1\colon [0,1] \to X$ are called *path-homotopic* if γ_0 and γ_1 are homotopic rel $\{0,1\}$.

(a) In a rather loose sense one could say that for a fixed topological space X the Change-of-Base Point says that twisted homology is a functor over the fundamental groupoid from Exercise 88.16.

(b) We suppose that we are in the setting of the Change-of-Base Point Proposition 321.3 with $x_0 = x_1$. If M is a $\mathbb{Z}[\pi_1(X, x_0)]$-left module that has the property that the map $\pi_1(X, x_0) \to \mathrm{Aut}(M)$ factors through the abelianization of $\pi_1(X, x_0)$, then one can easily see that for any loop γ in (X, x_0) we have an equality $\gamma^* M = M$ of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. It follows from the Change-of-Base Point Proposition 321.3 (1) and (2) that the map

$$\begin{array}{rcl} \pi_1(X, x_0) \times \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \to & \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \\ ([\gamma], h) & \mapsto & \overline{\gamma}^*(h) \end{array}$$

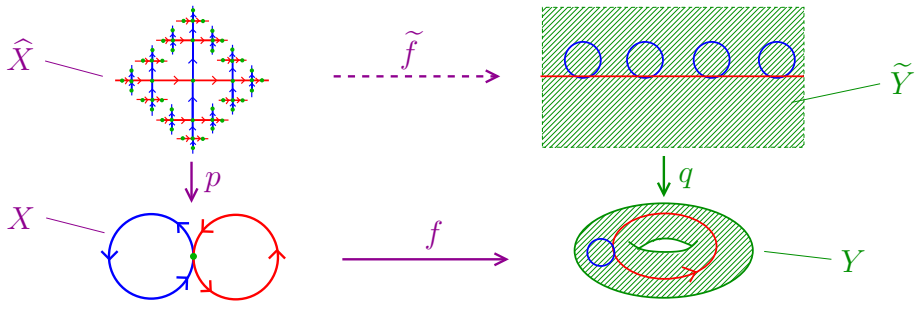
defines a left action by $\pi_1(X, x_0)$ on the group $\mathrm{H}^k_{\mathrm{tw}}(X, A; M)$. In Exercise 321.1 and Exercise 329.11 we will see that this left action is in general, e.g. for $S^1 \vee S^2$ or the surface of genus 2, non-trivial. □

321.2. **Functoriality of twisted (co-) homology.** In this section we will consider the question in what sense (co-) homology with twisted coefficients is functorial in the underlying topological space. The formulations of the statements require some care, since we have to deal with maps between *pointed* topological spaces and since we also have to worry about the module structures we are dealing with. We start out with a discussion of induced maps between covering spaces:

Definition 321.4. Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map between pointed topological spaces. We assume that X is locally path-connected and path-connected. Let $p\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ and $q\colon (\widehat{Y}, \widehat{y}_0) \to (Y, y_0)$ be two path-connected coverings with[3532] $f_*(p_*(\pi_1(\widehat{X}, \widehat{x}_0))) \subseteq q_*(\pi_1(\widehat{Y}, \widehat{y}_0))$. By the Map Lifting Criterion 127.5 together with the Lift Uniqueness Proposition 92.2, applied to the map $f \circ p\colon (\widehat{X}, \widehat{x}_0) \to (Y, y_0)$ and the covering $q\colon (\widehat{Y}, \widehat{y}_0) \to (Y, y_0)$, there exists a unique continuous map $\widetilde{f}\colon (\widehat{X}, \widehat{x}_0) \to (\widehat{Y}, \widehat{y}_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\widehat{X}, \widehat{x}_0) & \xrightarrow{\widetilde{f}} & (\widehat{Y}, \widehat{y}_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ (X, x_0) & \xrightarrow{f} & (Y, y_0) \end{array} .$$

We refer to $\widetilde{f}$ as the induced map.



[3532] Note that this condition is trivially satisfied if $\pi_1(\widehat{X}, x_0)$ is trivial, i.e. if $\widehat{X} \to X$ is a universal covering.

Example. Let $f\colon (X,x_0) \to (Y,y_0)$ be a continuous map between pointed topological spaces that are path-connected and locally contractible. We obtain an induced map

$$\widetilde{f}\colon \widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$$

between the natural universal coverings from Definition 319.10. It follows almost immediately from the definitions that it is given by

$$\underbrace{[\alpha\colon [0,1] \to X]}_{\in \widetilde{X}_{x_0}} \mapsto \underbrace{[f \circ \alpha\colon [0,1] \to Y]}_{\in \widetilde{Y}_{y_0}}.$$

□

Lemma 321.5. (Lift–Functoriality Lemma) In the following let $f\colon (X,x_0) \to (Y,y_0)$ and $g\colon (Y,y_0) \to (Z,z_0)$ be two continuous maps between pointed topological spaces. We assume that X and Y are locally path-connected and path-connected. Furthermore let $p\colon (\widehat{X},\widehat{x}_0) \to (X,x_0)$, $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$ and $r\colon (\widehat{Z},\widehat{z}_0) \to (Z,z_0)$ be coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0))) \subseteq q_*(\pi_1(\widehat{Y},\widehat{y}_0))$ and $g_*(q_*(\pi_1(\widehat{Y},\widehat{y}_0))) \subseteq r_*(\pi_1(\widehat{Z},\widehat{z}_0))$. Then we have an equality of maps

$$\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}\colon (\widehat{X},\widehat{x}_0) \to (\widehat{Z},\widehat{z}_0).$$

Proof. We will prove the lemma in Exercise 321.2. ■

Next we want to consider the effect on twisted homology. To formulate the main result we will need the following definition:

Definition 321.6. Let $f\colon (X,x_0) \to (Y,y_0)$ be a continuous map between pointed topological spaces. Furthermore let R be a ring and let M be an $(R,\mathbb{Z}[\pi_1(Y,y_0)])$-left left module. Inspired by Notation 320.10 we denote by f^*M the $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left module that is induced by the ring homomorphism $f_*\colon \mathbb{Z}[\pi_1(X,x_0)] \to \mathbb{Z}[\pi_1(Y,y_0)]$.

Proposition 321.7. (Chain Complex-Equivariance Proposition) In the following let $f\colon (X,x_0) \to (Y,y_0)$ be a continuous map between pointed topological spaces. We assume that X is locally path-connected and path-connected. Let $p\colon (\widehat{X},\widehat{x}_0) \to (X,x_0)$ and $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$ be two *regular* coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0))) \subseteq q_*(\pi_1(\widehat{Y},\widehat{y}_0))$. We consider the induced map $\widetilde{f}\colon (\widehat{X},\widehat{x}_0) \to (\widehat{Y},\widehat{y}_0)$ from Definition 321.4.

(1) Suppose that we are given subsets $A \subseteq X$ and $B \subseteq Y$ with $f(A) \subseteq B$. We write $\widehat{A} := p^{-1}(A)$ and $\widehat{B} := q^{-1}(B)$. The induced map $\widetilde{f}$ has the property that given any $Q \in \widehat{X}_{x_0}$ and given any $g \in \pi_1(X,x_0)/\operatorname{im}(p_*)$ we have

$$\widetilde{f}(g \cdot Q) = \underbrace{f_*(g)}_{\in \pi_1(Y,y_0)/\operatorname{im}(q_*)} \cdot \widetilde{f}(Q).$$

(2) Let R be a ring and let M be an $(R,\mathbb{Z}[\pi_1(Y,y_0)])$-left left module. The maps

$$\begin{aligned} \overline{f^*M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)/\operatorname{im}(p_*)]} \mathrm{C}_k(\widehat{X},\widehat{A}) &\to \overline{M} \otimes_{\mathbb{Z}[\pi_1(Y,y_0)/\operatorname{im}(q_*)]} \mathrm{C}_k(\widehat{Y},\widehat{B}) \\ m \otimes (\sigma\colon \Delta^k \to \widehat{X}) &\mapsto m \otimes (\widetilde{f} \circ \sigma) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}[\pi_1(Y,y_0)/\operatorname{im}(q_*)]\text{-left}}(\mathrm{C}_k(\widehat{Y},\widehat{B}),M) &\to \operatorname{Hom}_{\mathbb{Z}[\pi_1(X,x_0)/\operatorname{im}(p_*)]\text{-left}}(\mathrm{C}_k(\widehat{X},\widehat{A}),f^*M) \\ \varphi &\mapsto (\sigma \mapsto \varphi(\widetilde{f} \circ \sigma)) \end{aligned}$$

are well-defined R-left module homomorphisms and they form (co-) chain maps.

Proof.

(1) This statement is a slight variation on the Lift–Action Proposition 128.18.
(2) This statement follows easily from (1), the Tensor–Map Composition Lemma 312.7 and the Tensor Product-Ring-Functor Lemma 312.9, respectively the Hom–Contravariant Functor Lemma 315.6 and the Hom-Ring-Functor Lemma 309.13. ■

The Chain Complex-Equivariance Proposition 321.7 leads us to the following definition:

Definition 321.8. Let $f\colon (X,x_0)\to(Y,y_0)$ be a continuous map between pointed topological spaces that are path-connected and locally contractible. We write $\pi_X \coloneqq \pi_1(X,x_0)$ and $\pi_Y \coloneqq \pi_1(Y,y_0)$. Furthermore let R be a ring and let M be an $(R,\mathbb{Z}[\pi_Y])$-left left module. We denote by $p\colon \widetilde{X}_{x_0}\to X$ and $q\colon \widetilde{Y}_{y_0}\to Y$ the natural universal coverings. Suppose that we are given subsets $A\subseteq X$ and $B\subseteq Y$ with $f(A)\subseteq B$. We write $\widetilde{A}\coloneqq p^{-1}(A)$ and $\widetilde{B}\coloneqq q^{-1}(B)$. We denote by $\widetilde{f}\colon \widetilde{X}_{x_0}\to\widetilde{Y}_{y_0}$ the map from Definition 321.4 that is induced by $f\colon X\to Y$. Using the Chain Complex-Equivariance Proposition 321.7 we define the R-left module homomorphisms

$$\begin{aligned} f_*\colon \overbrace{\overline{f^*M}\otimes_{\mathbb{Z}[\pi_X]}\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;f^*M)} &\to \overbrace{\overline{M}\otimes_{\mathbb{Z}[\pi_Y]}\mathrm{C}_k(\widetilde{Y}_{y_0},\widetilde{B})}^{=\mathrm{C}_k^{\mathrm{tw}}(Y,B;M)} \\ m\otimes(\sigma\colon\Delta^k\to\widetilde{X}_{x_0}) &\mapsto m\otimes(\widetilde{f}\circ\sigma\colon\Delta^k\to\widetilde{Y}_{y_0}) \end{aligned}$$

and

$$\begin{aligned} f^*\colon \overbrace{\operatorname{Hom}_{\mathbb{Z}[\pi_Y]\text{-left}}(\mathrm{C}_k(\widetilde{Y}_{y_0},\widetilde{B}),M)}^{=\mathrm{C}_{\mathrm{tw}}^k(Y,B;M)} &\to \overbrace{\operatorname{Hom}_{\mathbb{Z}[\pi_X]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}),f^*M)}^{=\mathrm{C}_{\mathrm{tw}}^k(X,A;f^*M)} \\ \varphi &\mapsto (\sigma\mapsto\varphi(r\circ\sigma)) \end{aligned}$$

and we define the corresponding induced R-left module homomorphisms

$$\begin{aligned} f_*\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;f^*M) &\to \mathrm{H}_k^{\mathrm{tw}}(Y,B;M) \\ f^*\colon \mathrm{H}_{\mathrm{tw}}^k(Y,B;M) &\to \mathrm{H}_{\mathrm{tw}}^k(X,A;f^*M). \end{aligned}$$

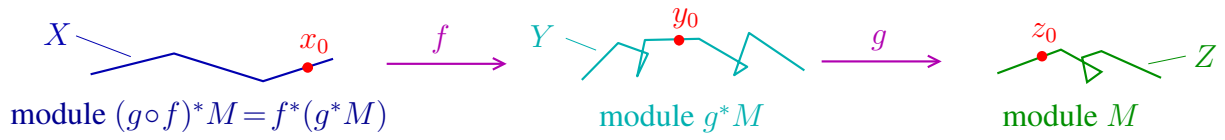
The following proposition says that (co-) homology with twisted coefficients is (co-) variantly functorial, at least if we take care of the choice of coefficients we are dealing with.

Proposition 321.9. (Twisted Homology-Functoriality Proposition) Let (Z,z_0) be a pointed topological space that is path-connected and locally contractible and let $C\subseteq Z$. Furthermore let R be a ring and let M be an $(R,\mathbb{Z}[\pi_1(Z,z_0)])$-left left module.

(1) We have $(\mathrm{id}_Z)^*M = M$ and the induced maps $\mathrm{id}_*\colon \mathrm{C}_k^{\mathrm{tw}}(Z,C;\mathrm{id}_Z^*M)\to\mathrm{C}_k^{\mathrm{tw}}(Z,C;M)$ and $\mathrm{id}^*\colon \mathrm{C}_k^{\mathrm{tw}}(Z;M)\to\mathrm{C}_k(Z;\mathrm{id}_Z^*M)$ are the identity.
(2) Let X and Y be topological spaces that are path-connected and locally contractible and let $A\subseteq X$ and $B\subseteq Y$. Let $f\colon(X,x_0)\to(Y,y_0)$ and $g\colon(Y,Y_0)\to(Z,Z_0)$ be continuous maps between pointed topological spaces. We assume that $f(A)\subseteq B$ and $g(B)\subseteq C$. Then $(g\circ f)^*M = f^*(g^*M)$ as $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and

$$\begin{aligned} (g\circ f)_* = g_*\circ f_*\colon \mathrm{C}_k^{\mathrm{tw}}(X,A;(g\circ f)^*M) &\to \mathrm{C}_k^{\mathrm{tw}}(Z,C;M) \\ (g\circ f)^* = f^*\circ g^*\colon \mathrm{C}_{\mathrm{tw}}^k(Z,C;M) &\to \mathrm{C}_{\mathrm{tw}}^k(X,A;(g\circ f)^*M). \end{aligned}$$

In both cases the equalities on maps between (co-) chain complexes induce of course equalities of maps between (co-) homology groups.



Proof. The proposition follows easily from the Lift–Functoriality Lemma 321.5. We leave the details to the reader. ■

In summary we see that induced maps between twisted homology modules are tricky objects. In practice it is usually quite hard to calculate them rigorously.

321.3. **The long exact sequence in twisted homology.** By the LES-of-Triple Proposition 146.5 and the LES-of-Triple Proposition 198.13 we know that a triple (X, A, B) of topological spaces gives rise to a long exact sequence of (relative) singular (co-) homology groups. We now have an analogous statement in the present setting:

Proposition 321.10. (Twisted-LES-Proposition) Let X be a topological space that is path-connected and locally contractible, let B be subspace of X that is also path-connected and locally contractible, let $A \subseteq B$ be a further subspace and let $b_0 \in B$. We denote by $i\colon B \to X$ the inclusion. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, b_0)])$-left left module. Given any $n \in \mathbb{N}_0$ there exists a natural R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X, B; M) \;\to\; \mathrm{H}_{n-1}^{\mathrm{tw}}(B, A; i^*M)$$

such that the following sequence is exact

$$\ldots \xrightarrow{\partial_{n+1}} \mathrm{H}_n^{\mathrm{tw}}(B, A; i^*M) \xrightarrow{i_*} \mathrm{H}_n^{\mathrm{tw}}(X, A; M) \to \underset{\substack{\uparrow\\ \text{induced by the natural}\\ \text{map } (X,A)\to(X,B)}}{\mathrm{H}_n^{\mathrm{tw}}(X, B; M)} \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{\mathrm{tw}}(B, A; i^*M) \to \ldots$$

An analogous statement also holds for cohomology groups with twisted coefficients.

The proof of the Twisted-LES-Proposition 321.10 makes use of the following technical lemma, which we will also need on several other occasions:

Lemma 321.11. (Change-of-Covering Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $q\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let $B \subseteq X$ be a subspace with $x_0 \in B$. We assume that B is also path-connected and locally contractible. We write $\widetilde{B} := q^{-1}(B)$ and we denote by $p\colon \widehat{B}_{x_0} \to B$ the natural universal covering. Finally let $A \subseteq B$. We write $\widehat{A} := p^{-1}(A)$ and $\widetilde{A} := q^{-1}(A)$. We denote by $i\colon B \to X$ the inclusion map and we denote by $\widetilde{i}\colon \widehat{B}_{x_0} \to \widetilde{X}_{x_0}$ the induced map from Definition 321.4.

(1) The maps

$$\begin{array}{ccc} \overbrace{\overline{i^*M} \otimes_{\mathbb{Z}[\pi_1(B,x_0)]} \mathrm{C}_k(\widehat{B}_{x_0}, \widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(B,A;i^*M)} & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \\ m \otimes \sigma & \mapsto & m \otimes \widetilde{i}_*(\sigma) \end{array}$$

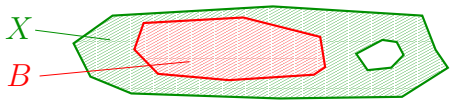
are well-defined and they are natural isomorphisms of R-left chain complexes.

(2) For each $k \in \mathbb{N}_0$ the map

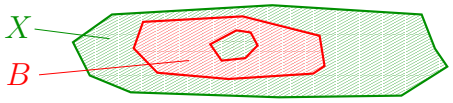
$$\overbrace{\overline{i^*M} \underset{\mathbb{Z}[\pi_1(B,x_0)]}{\otimes} \mathrm{C}_k(\widehat{B}_{x_0}, \widehat{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(B,A;i^*M)} \to \overline{M} \underset{\mathbb{Z}[\pi_1(X,x_0)]}{\otimes} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \xrightarrow{\mathrm{id}\otimes\text{inclusion}} \overbrace{\overline{M} \underset{\mathbb{Z}[\pi_1(X,x_0)]}{\otimes} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;M)}$$

$\uparrow$ isomorphism from (1)

is on the nose the map $i_*\colon \mathrm{C}_k^{\mathrm{tw}}(B, A; i^*M) \to \mathrm{C}_k^{\mathrm{tw}}(X, A; M)$ that we introduced in Definition 321.8.



$q\colon \widetilde{B} \to B$ is an infinite covering, but in this case $\widehat{B} = B$



we have $\widetilde{B} = B$, but in this setting $p\colon \widehat{B} \to B$ is an infinite covering

Remark. We will develop the theme of the Change-of-Covering Lemma 321.11 further in the eagerly awaited sequel, the Change-of-Covering Lemma 325.7. □

Proof of the Change-of-Covering Lemma 321.11.

(1) To simplify the notation a little bit we assume that $A = \varnothing$. Furthermore we write $\pi_B \coloneqq \pi_1(B, x_0)$ and we write $\pi_X \coloneqq \pi_1(X, x_0)$. Let $\widetilde{i}\colon \widehat{B}_{x_0} \to \widetilde{X}_{x_0}$ be the induced map from Definition 321.4. Note that it follows easily from the definitions that $\widetilde{i}(\widehat{B}) \subseteq \widetilde{B}$. In the Chain Complex-Equivariance Proposition 321.7 we saw that the map

$$\overline{i^*M} \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) \to \overline{M} \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k(\widetilde{X}_{x_0})$$

is well-defined. The current map under consideration is exactly the same map, except that it follows from $\widetilde{i}(\widehat{B}) \subseteq \widetilde{B}$ that the map we are really interested in actually takes values in $\overline{M} \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k(\widetilde{B})$. This shows that the map in our proposition is well-defined.

Claim. In the following we will use the inclusion induced map $\pi_B \to \pi_X$ to view $\mathbb{Z}[\pi_X]$ as a $\mathbb{Z}[\pi_B]$-right module. The map

$$\begin{aligned} \mathbb{Z}[\pi_X] \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) &\to \mathrm{C}_k(\widetilde{B}) \\ g \otimes \sigma &\mapsto g \cdot (\widetilde{i} \circ \sigma) \end{aligned}$$

is well-defined and it is an isomorphism of $\mathbb{Z}[\pi_X]$-left modules.

Proof. First note that it follows easily from the Tensor Product–Universal Property Lemma 320.5 that the map is well-defined. Next, given any singular k-simplex σ in B we pick a lift $\widetilde{\sigma}$ to $\widehat{B}_{x_0}$. In the Regular Covering–Chain Complex Lemma 320.2 we showed that these lifts form a basis of the free $\mathbb{Z}[\pi_B]$-left module $\mathrm{C}_k(\widehat{B}_{x_0})$. Furthermore by the Regular Covering–Chain Complex Lemma 320.2 we also know that the singular simplices $\widetilde{i}_* \circ \widetilde{\sigma}$ form a basis of the free $\mathbb{Z}[\pi_X]$-left module $\mathrm{C}_k(\widetilde{B})$. It follows from this observation, together with the Free Module–Tensor-with-Ring Lemma 312.18 (1), that the map sends a basis of the free $\mathbb{Z}[\pi_X]$-left module $\mathbb{Z}[\pi_X] \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0})$ to a basis of the free $\mathbb{Z}[\pi_X]$-left module $\mathrm{C}_k(\widetilde{B})$. Thus we see that the map is indeed an isomorphism of $\mathbb{Z}[\pi_X]$-left modules. ⊞

Note that the map we are supposed to study equals the composition of the following maps:

$$\begin{array}{rcccl} \overline{i^*M} \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0}) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_X]} (\mathbb{Z}[\pi_X] \otimes_{\mathbb{Z}[\pi_B]} \mathrm{C}_k(\widehat{B}_{x_0})) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k(\widetilde{B}) \\ m \otimes \sigma & \mapsto & m \otimes (1 \otimes \sigma) & & \\ & & m \otimes (g \otimes \sigma) & \mapsto & m \otimes g \cdot (\widetilde{i}_* \circ \sigma). \end{array}$$

It follows from the Tensor Product–Cancel Ring Lemma 312.12 that the map to the left is well-defined and an isomorphism of R-left modules. It follows from the claim that the map to the right is also an isomorphism of R-left modules.

(2) This statement follows immediately from the definitions. ■

Proof of the Twisted-LES-Proposition 321.10. In the following let $p\colon \widehat{B}_{b_0} \to B$ and $q\colon \widetilde{X}_{b_0} \to X$ be the natural universal coverings. We write $\widehat{A} \coloneqq p^{-1}(A)$, $\widetilde{A} \coloneqq q^{-1}(A)$ and $\widetilde{B} \coloneqq q^{-1}(B)$. By Definition 321.4 the inclusion map $i\colon B \to X$ induces a continuous map $\widetilde{i}\colon \widehat{B} \to \widetilde{X}$. Evidently $\widetilde{i}(\widehat{B}) \subseteq \widetilde{B}$.

Claim. The maps

$$0 \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{X}, \widetilde{B}) \to 0$$

form a short exact sequence of R-left modules.

Proof. [3533] First let us consider the sequence

$$0 \to \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \to \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to \mathrm{C}_k(\widetilde{X}, \widetilde{B}) \to 0.$$

Note that by the Regular Covering–Chain Complex Lemma 320.2 we know that this is sequence of $\mathbb{Z}[\pi_1(X, x_0)]$-left homomorphisms between free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. Furthermore, as in the proof of the LES-of-Triple Proposition 146.5, we obtain from the third isomorphism theorem that the sequence is exact. Since the module $\mathrm{C}_k(\widetilde{X}, \widetilde{A})$ to the right is a free $\mathbb{Z}[\pi_1(X, x_0)]$-left module we obtain from the Tensor-Split-SES Lemma 314.9 that

$$0 \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{X}, \widetilde{A}) \to \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{X}, \widetilde{B}) \to 0$$

is indeed a short exact sequence of R-left modules. ⊞

From the purely algebraic LES Lemma 313.9 we get a connecting homomorphism and a long exact sequence of the homology modules. We obtain the desired connecting homomorphism and the desired long exact sequence by replacing the homology modules of the chain complex $\overline{M} \otimes_{\mathbb{Z}[\pi_1(X,b_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A})$ by the homology groups $\mathrm{H}_k^{\mathrm{tw}}(B, A; i^*M)$ using the isomorphism that is provided by the Change-of-Covering Lemma 321.11 (1). Note that the Change-of-Covering Lemma 321.11 (2) implies that the resulting connecting homomorphism does indeed give rise to the desired long exact sequence.

Finally note that the cohomological analog is proved in a very similar fashion. We leave the proof to the exercise-deprived reader. ■

321.4. **Functoriality in the coefficients.** Recall that it follows from the Homology-Coefficients-Functor Lemma 160.11 and the Cohomology-Coefficients-Functor 197.21 that singular (co-) homology are both covariantly functorial in the coefficients. In this section we

[3533] The following argument essentially appeared already in the proof of the LES-with-Coefficients Proposition 314.7.

will see that this feature generalizes to our more general setting. We start out with the following definition:

Definition 321.12. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering. Let $A \subseteq X$. We write $\widetilde{A} := p^{-1}(A)$. Let R be a ring. Furthermore let M and N be $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules and let $\Theta\colon M \to N$ be a homomorphism of $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules. It follows from the Homology-Coefficients-Functor Lemma 314.10 and the Cohomology-Coefficients-Functor Lemma 315.23 that the R-left module homomorphisms

$$\begin{array}{rcl} \Theta_*\colon \overbrace{M \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;M)} & \to & \overbrace{N \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;N)} \\ m \otimes \sigma & \mapsto & \Theta(m) \otimes \sigma \end{array}$$

$$\begin{array}{rcl} \Theta_*\colon \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), M)}^{=\mathrm{C}^k_{\mathrm{tw}}(X,A;M)} & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), N)}^{=\mathrm{C}^k_{\mathrm{tw}}(X,A;N)} \\ \gamma & \mapsto & \Theta \circ \gamma \end{array}$$

are chain maps and thus induce maps on twisted (co-) homology which we denote by Θ_* as well.

Example. Here is a "real life" example that we will make use of later on: Let (X, x_0) be a pointed topological space as in Definition 321.12 with fundamental group $\pi := \pi_1(X, x_0)$. Let $\varphi\colon \pi \to \{\pm 1\} = \mathrm{GL}(1, \mathbb{Z})$ be a homomorphism. Then the projection $\Theta\colon {}_{\varphi}\mathbb{Z} \to {}_{\mathrm{tr}}\mathbb{F}_2$ is a homomorphism of $\mathbb{Z}[\pi]$-left modules. Thus we obtain from Definition 321.12 together with the Twisted–Trivial Coefficients Lemma 320.13 induced natural homomorphisms $\mathrm{H}_k^{\mathrm{tw}}(X, A; {}_{\varphi}\mathbb{Z}) \to \mathrm{H}_k(X, A; \mathbb{F}_2)$. □

The following lemma says that (co-) homology with twisted coefficients is indeed covariantly functorial in the $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module:

Lemma 321.13. (Functorial-in-Coefficients Lemma) We suppose that we are in the setting of the above Definition 321.12.

(1) For $\mathrm{id}\colon M \to M$ we have $\mathrm{id}_* = \mathrm{id}$ on the (co)-chain complexes and on the (co)-homology modules.
(2) For $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module homomorphisms $\Phi\colon L \to M$ and $\Psi\colon M \to N$ we have $(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*$ on the (co)-chain complexes and on the (co)-homology modules.

Proof. Both statements are essentially trivial. ■

Using functoriality of (co-) homology with twisted coefficients we can easily state and prove the following little, surprisingly useful, lemma.

Lemma 321.14. (Direct Sum-of-Coefficients Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let $A \subseteq X$. Let R be a ring and let M and N be $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules. We consider the direct sum

$M \oplus N$ from the Coproduct-of-Modules Lemma 309.11, which is again an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. We denote by $i\colon M \to M \oplus N$ and $j\colon N \to M \oplus N$ the natural inclusions and we denote by $p\colon M \oplus N \to M$ and $q\colon M \oplus N \to N$ the natural projections. The maps

$$\begin{aligned} i_* \oplus j_*\colon \mathrm{C}_k^{\mathrm{tw}}(X, A; M) \oplus \mathrm{C}_k^{\mathrm{tw}}(X, A; N) &\to \mathrm{C}_k^{\mathrm{tw}}(X, A; M \oplus N) \\ i_* \oplus j_*\colon \mathrm{C}_{\mathrm{tw}}^k(X, A; M) \oplus \mathrm{C}_{\mathrm{tw}}^k(X, A; N) &\to \mathrm{C}_{\mathrm{tw}}^k(X, A; M \oplus N) \end{aligned}$$

from Definition 321.12 are isomorphisms of R-left module (co)-chain complexes where the inverses are given by

$$\begin{aligned} (p_*, q_*)\colon \mathrm{C}_k^{\mathrm{tw}}(X, A; M \oplus N) &\to \mathrm{C}_k^{\mathrm{tw}}(X, A; M) \oplus \mathrm{C}_k^{\mathrm{tw}}(X, A; N) \\ (p_*, q_*)\colon \mathrm{C}_{\mathrm{tw}}^k(X, A; M \oplus N) &\to \mathrm{C}_{\mathrm{tw}}^k(X, A; M) \oplus \mathrm{C}_{\mathrm{tw}}^k(X, A; N). \end{aligned}$$

In particular the above maps induce isomorphisms on (co-) homology with twisted coefficients.

Proof. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering and as usual we write $\widetilde{A} := p^{-1}(A)$. To simplify the notation we now write $\pi := \pi_1(X, x_0)$.

(1) We first deal with the twisted chain complex. It follows immediately from the Tensor–Direct Sum Lemma 312.15 that

$$i_* \oplus j_*\colon \overbrace{\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;M)} \oplus \overbrace{\overline{N} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;N)} \to \overbrace{\overline{(M \oplus N)} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;M\oplus N)}$$

is an isomorphism of R-left modules. Furthermore using the Functorial-in-Coefficients Lemma 321.13 one can easily check that the composition $(p_*, q_*) \circ (i_* \oplus j_*)$ is the identity. Thus (p_*, q_*) is also an isomorphism.

(2) Now we turn to the twisted cochain complexes. Proof is almost identical to the proof of (1), we just need to replace the Tensor–Direct Sum Lemma 312.15 by the Hom–Direct Sum-Product Lemma 315.7. ∎

We conclude this section with the following twisted analog of **??** and **??**:

Lemma 321.15. (Twisted Bockstein Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let $A \subseteq X$. Furthermore let R be a ring, let L, M and N be $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules and let

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

be a short exact sequence of $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules.

(1) We have a long exact sequence of R-left modules

$$\ldots \xrightarrow{\beta_{k+1}} \mathrm{H}_k^{\mathrm{tw}}(X, A; L) \xrightarrow{\varphi_*} \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \xrightarrow{\psi_*} \mathrm{H}_k^{\mathrm{tw}}(X, A; N) \xrightarrow{\beta_k} \mathrm{H}_{k-1}^{\mathrm{tw}}(X, A; L) todots$$

(2) We have a long exact sequence of R-left modules

$$\ldots \xrightarrow{\beta_k} \mathrm{H}_{\mathrm{tw}}^k(X, A; L) \xrightarrow{\varphi_*} \mathrm{H}_{\mathrm{tw}}^k(X, A; M) \xrightarrow{\psi_*} \mathrm{H}_{\mathrm{tw}}^k(X, A; N) \xrightarrow{\beta_{k+1}} \mathrm{H}_{\mathrm{tw}}^{k+1}(X, A; L) todots$$

As in **??** and **??** we refer to the connecting homomorphisms β_k as Bockstein homomorphisms. Finally the long exact sequence is called the Bockstein sequence in (co-) homology with twisted coefficients.

Example. Let R be a ring and let π be a group.. We suppose that we are given a representation of the form

$$\begin{array}{rcl}\varphi\colon \pi & \to & \mathrm{GL}(m+n,R)\\ g & \mapsto & \begin{pmatrix}\alpha(g) & \theta(g)\\ 0 & \beta(g)\end{pmatrix}.\end{array}$$

One can easily verify that in this setting the maps

$$0 \to {}_{\alpha}R^{1\times m} \xrightarrow{x\mapsto(x,0)} {}_{\varphi}R^{1\times(m+n)} \xrightarrow{(x,y)\mapsto y} {}_{\beta}R^{1\times n} \to 0$$

form a short exact sequence of $(R,\mathbb{Z}[\pi])$-left left modules. □

Proof. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and as usual we write $\widetilde{A} := p^{-1}(A)$. To simplify the notation we again write $\pi := \pi_1(X,x_0)$.

(1) It follows from the Regular Covering–Chain Complex Lemma 320.2, the Free-is-Projective Lemma 309.26 and the Tensor-with-Projective Proposition 314.17 that

$$0 \to \overline{L} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{\widetilde{x}_0},\widetilde{A}) \xrightarrow{\varphi\otimes\mathrm{id}} \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{\widetilde{x}_0},\widetilde{A}) \xrightarrow{\psi\otimes\mathrm{id}} \overline{N} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{\widetilde{x}_0},\widetilde{A}) \to 0$$

is a short exact sequence of chain complexes of R-left modules.

(2) It follows from the Regular Covering–Chain Complex Lemma 320.2 and Definition 309.17 that

$$0 \to \operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{\widetilde{x}_0},\widetilde{A}),L) \xrightarrow{\varphi^*} \operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{\widetilde{x}_0},\widetilde{A}),M) \xrightarrow{\psi_*} \operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_{\widetilde{x}_0},\widetilde{A}),N) \to 0$$

is a short exact sequence of chain complexes of R-left modules.

The desired result is now a consequence of the LES Proposition 313.10. ■

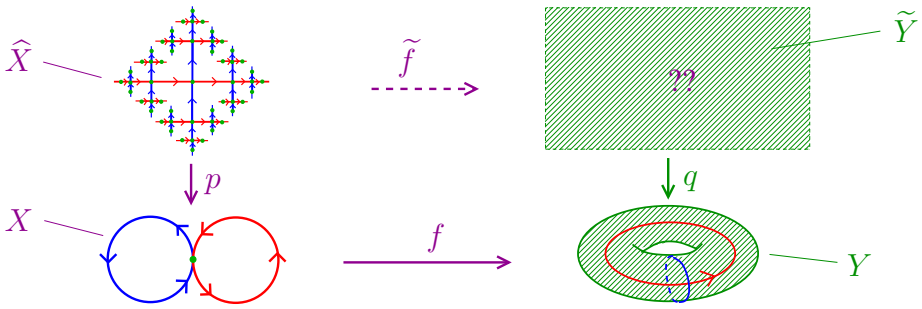
321.5. Exercises for Chapter 321.

Exercise 321.1. We consider the topological space $X = S^1 \vee S^2$ and we take as base point the wedge point x_0. Note that $\pi_1(S^1 \vee S^2, x_0) \cong \pi_1(S^1,x_0)$ is abelian. As we pointed out on page 6168, it follows from the Change-of-Base Point Proposition 321.3 that $\pi_1(X,x_0)$ acts on any twisted homology module $\mathrm{H}_k^{\mathrm{tw}}(X;M)$ and on any twisted cohomology module $\mathrm{H}^k_{\mathrm{tw}}(X;M)$. Given any example of a $\mathbb{Z}[\pi_1(X,x_0)]$-left module for which this left action is non-trivial.

Exercise 321.2. Let $f\colon (X,x_0) \to (Y,y_0)$ and $g\colon (Y,y_0) \to (Z,z_0)$ be continuous maps between pointed topological spaces. We assume that X and Y are locally path-connected and path-connected. Furthermore let $p\colon (\widehat{X},\widehat{x}_0) \to (X,x_0)$, $q\colon (\widehat{Y},\widehat{y}_0) \to (Y,y_0)$ and $r\colon (\widehat{Z},\widehat{z}_0) \to (Z,z_0)$ be three coverings with $f_*(p_*(\pi_1(\widehat{X},\widehat{x}_0))) \subseteq \pi_1(\widehat{Y},\widehat{y}_0)$ and with $g_*(q_*(\pi_1(\widehat{Y},\widehat{y}_0))) \subseteq \pi_1(\widehat{Z},\widehat{z}_0)$. In the following we consider the induced maps $\widetilde{f}\colon (\widehat{X},\widehat{x}_0) \to (\widehat{Y},\widehat{y}_0)$, $\widetilde{g}\colon (\widehat{Y},\widehat{y}_0) \to (\widehat{Z},\widehat{z}_0)$ and $\widetilde{g\circ f}\colon (\widehat{X},\widehat{x}_0) \to (\widehat{Z},\widehat{z}_0)$ that we introduced in Definition 321.4. Show that we have an equality of maps

$$\widetilde{g\circ f} \;=\; \widetilde{g}\circ\widetilde{f}\colon (\widehat{X},\widehat{x}_0) \;\to\; (\widehat{Z},\widehat{z}_0).$$

Exercise 321.3. In the figure below we consider the universal coverings $p\colon \widehat{X} \to X = S^1 \vee S^1$ and $q\colon \widetilde{Y} = \mathbb{R}^2 \to Y = S^1 \times S^1$ and we consider a continuous map $f\colon X \to Y$. Illustrate a continuous map $\widetilde{f}\colon \widehat{X} \to \widetilde{Y}$ that makes the diagram commute.



Exercise 321.4. We consider the group $\langle x \,|\, x^2\rangle$, which is nothing but the group $\mathbb{Z}/2\mathbb{Z}$ written multiplicatively. We denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left module that is given by $x \cdot n := -n$.

(a) Show that the sequence

$$0 \;\to\; {}_{\mathrm{tr}}\mathbb{Z} \;\xrightarrow{n \mapsto n\cdot(1+x)}\; \mathbb{Z}[\langle x \,|\, x^2\rangle] \;\xrightarrow{x \mapsto -1}\; \widetilde{\mathbb{Z}} \;\to\; 0$$

of $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left modules is exact and show that it does not split.

(b) Show that if in (a) we replace "$\mathbb{Z}$" throughout by "$\mathbb{Q}$", then the short exact sequence of $\mathbb{Q}[\langle x \,|\, x^2\rangle]$-left modules splits.

(c) Does there exist an exact sequence of $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left modules of the form

$$0 \;\to\; \widetilde{\mathbb{Z}} \;\to\; \mathbb{Z}[\langle x \,|\, x^2\rangle] \;\to\; {}_{\mathrm{tr}}\mathbb{Z} \;\to\; 0?$$

Exercise 321.5. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Furthermore suppose we are given a ring R and a representation of the form

$$\begin{array}{rcl} \varphi\colon \pi_1(X, x_0) & \to & \mathrm{GL}(m+n, R) \\ g & \mapsto & \begin{pmatrix} \alpha(g) & \theta(g) \\ 0 & \beta(g) \end{pmatrix}. \end{array}$$

Show that $\mathrm{H}_k^{\mathrm{tw}}(X; {}_{\varphi}R^{m+n}) = 0$ if and only if $\mathrm{H}_k^{\mathrm{tw}}(X; {}_{\alpha}R^{m}) = 0$ and $\mathrm{H}_k^{\mathrm{tw}}(X; {}_{\beta}R^{n}) = 0$.

Exercise 321.6. We make the usual identification $\pi_1(S^1, 1) =_{\mathrm{i}} \langle t\rangle$. Let $A \in \mathrm{GL}(n, \mathbb{C})$. We consider the n-dimensional complex representation $\varphi\colon \pi_1(S^1, 1) =_{\mathrm{i}} \langle t\rangle \to \mathrm{GL}(n, \mathbb{C})$ that is given by $\varphi(t) = A$. Show that the following two statements are equivalent:

(א) All twisted homology groups of S^1 with ${}_{\varphi}\mathbb{C}^n$-coefficients vanish.

(ב) 1 is not an eigenvalue of A.

Hint. Use the Twisted-H_*-of-S^1-Proposition 320.14 and use results from this chapter.

Exercise 321.7. We consider the circle $X = S^1$ and the subspace $A := \{P, Q\}$ of S^1 that consists of two distinct points on S^1. Let $\xi \in \mathbb{C} \setminus \{0\}$ and let $\varphi\colon \pi_1(S^1, 1) \to \mathrm{GL}(1, \mathbb{C})$ be a representation that sends a generator of $\pi_1(S^1, 1)$ to (ξ). Calculate the relative twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(S^1, \{P, Q\}; {}_{\varphi}\mathbb{C})$.

Exercise 321.8. Let G be a finite group. We consider the $\mathbb{Q}[G]$-module $A = {}_{\mathrm{tr}}\mathbb{Q}$ that is given by the trivial G-action on $\mathbb{Q}$.

(a) Provide an *explicit* monomorphism $\iota\colon A \to \mathbb{Q}[G]$ of $\mathbb{Q}[G]$-modules.

(b) Provide an *explicit* $\mathbb{Q}[G]$-submodule B of $\mathbb{Q}[G]$ such that $\mathbb{Q}[G]$ is isomorphic to the $\mathbb{Q}[G]$-module $A \oplus B$.

Remark. This exercise is identical to Exercise 310.10.

Exercise 321.9. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a finite group G. We use φ to view $\mathbb{Q}[G]$ as a $(\mathbb{Q}, \mathbb{Z}[\pi_1(X, x_0)])$-left left module ${}_{\varphi}\mathbb{Q}[G]$. Show that for any $k \in \mathbb{N}_0$ there exists an epimorphism

$$\mathrm{H}_k^{\mathrm{tw}}(X; {}_{\varphi}\mathbb{Q}[G]) \;\to\; \mathrm{H}_k(X; \mathbb{Q}).$$

Hint. Make use of Exercise 321.8.

Exercise 321.10. Let π be a group and let $\varphi\colon \pi \to \{\pm 1\} = \mathrm{GL}(1, \mathbb{Z})$ be a homomorphism. Clearly the projection $\Theta\colon {}_{\varphi}\mathbb{Z} \to {}_{\mathrm{tr}}\mathbb{F}_2$ is a homomorphism of $\mathbb{Z}[\pi]$-left modules. Is the kernel a trivial $\mathbb{Z}[\pi]$-left module?

Exercise 321.11. Provide an instance where the Bockstein homomorphism from the Twisted Bockstein Lemma 321.15 is non-zero.
Hint. You could even just consider modules with trivial $\pi_1(X, x_0)$-action.

CHAPTER 322

Twisted (co-) homology and homotopies

In Chapter 320 we introduced twisted (co-) homology and in the previous Chapter 321 we showed that twisted (co-) homology is in a sense functorial in the (pointed) topological space. In this chapter we will see to what degree homotopic maps induce the same map on twisted (co-) homology and we will see that in a sense homotopy equivalent topological spaces have the "same" twisted (co-) homology modules.

This chapter is organized as follows:

- In Section 322.1 we will state and discuss the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1 which says, in a nutshell, that homotopy equivalent topological spaces have isomorphic twisted (co-) homology modules.
- In Section 322.2 we will see that homotopic maps induce essentially the same maps on twisted (co-) homology.
- In Section 322.3 we will provide a proof of the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3.
- In Section 322.4 we will see how twisted homology behaves under deformation retracts.
- In Section 322.5 we will finally prove the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1.
- In Section 322.6 we will provide exercises for Chapter 322.

In the next Chapter 323 we will provide several calculations and examples of twisted (co-) homology.

322.1. **Statement of the main result.** The main goal of this chapter is to prove the following proposition, which can be viewed as the rightful heir to the Homotopy Equivalence-H_*-Proposition 144.14 and the Homotopy Equivalence-H^*-Proposition 198.12:

Proposition 322.1. (Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition Let X and Y be topological spaces that are both path-connected and locally contractible. We assume that X and Y are homotopy equivalent. Given any $x_0 \in X$ and $y_0 \in Y$ there exists an isomorphism $\Phi\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ such that for any ring R and any $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module M the following statements hold:

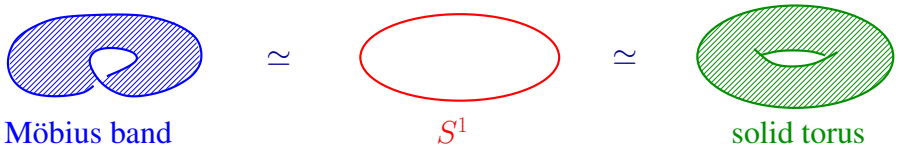
(1) The twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}(X; M)$ and $\mathrm{H}_k^{\mathrm{tw}}(Y; \Phi^* M)$ are isomorphic R-left modules.[3534]

(2) The twisted cohomology modules $\mathrm{H}^k_{\mathrm{tw}}(X; M)$ and $\mathrm{H}^k_{\mathrm{tw}}(Y; \Phi^* M)$ are isomorphic R-left modules.

Along the way we will prove several more refined and more technical statements, which can also be of interest. But for most purposes it is enough to know the above proposition.

[3534]Here $\Phi^* M$ is the $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module given by Notation 320.10.

Example. Let (X, x_0) be a topological space that is both path-connected and locally contractible and that is homotopy equivalent to the circle S^1. For example, X could be a Möbius band or a solid torus $\overline{B}^n \times S^1$.



Now let $\gamma\colon \pi_1(X, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a non-trivial 1-dimensional complex representation. Let $\Phi\colon \pi_1(S^1, 1) \to \pi_1(X, x_0)$ be an isomorphism as in the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1. We then see that

$$\mathrm{H}_*^{\mathrm{tw}}(X; {}_\gamma\mathbb{C}) \underset{\text{by the Homotopy Equivalence } \mathrm{H}_*^{\mathrm{tw}}\text{-Proposition 322.1}}{\cong} \mathrm{H}_*^{\mathrm{tw}}(S^1; \Phi^*({}_\gamma\mathbb{C})) \underset{\text{one can easily see that } \Phi^*({}_\gamma\mathbb{C}) = {}_{\gamma\circ\Phi}\mathbb{C}}{=} \mathrm{H}_*^{\mathrm{tw}}(S^1; {}_{\gamma\circ\Phi}\mathbb{C}) \underset{\text{by the Twisted-H}_*\text{-of-}S^1 \text{ Proposition 320.14, since } \gamma\circ\Phi \text{ is non-trivial}}{\cong} 0.$$

□

The remainder of this chapter is dedicated to the proof of the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1. In the next Chapter 323 we will finally provide several interesting calculations of twisted (co-) homology.

322.2. **Homotopic maps and (co-) homology with twisted coefficients.** In the Homotopy-H_*-Proposition 144.12 and in the Homotopy-H^*-Proposition 198.11 we saw that homotopic maps induce the same map on (co-) homology. A similar statement also holds in the present context, but, as we will see, one needs to be careful about the twistings because of base point issues.

First let us recall Definition 19.24:

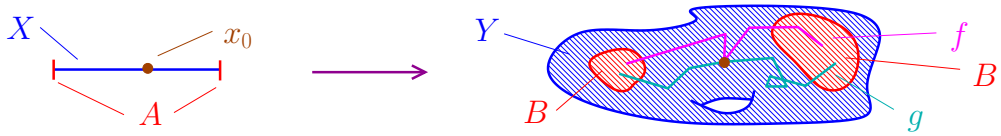
Definition 322.2. Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subsets. Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps of pairs of topological spaces.

(1) A homotopy between f and g is a continuous map

$$H\colon X \times [0,1] \to Y$$

such that the following two conditions are satisfied:
   (a) $H_0 = f$ and $H_1 = g$, and
   (b) $H(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.

(2) Let $x_0 \in X$. We say that a map $F\colon X \times [0,1] \to Y$ is a homotopy rel x_0 if for all $s, t \in [0,1]$ we have $F(x_0, s) = F(x_0, t)$.



$f, g\colon (X, A) \to (Y, B)$ are homotopic rel x_0

The following proposition is now an analog of the Homotopy-H_*-Proposition 144.12 and the Homotopy-H^*-Proposition 198.11:

Proposition 322.3. (Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition) Let X and Y be topological spaces that are both path-connected and locally contractible. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Let $f_0, f_1 \colon (X, A) \to (Y, B)$ be two continuous maps of pairs of topological spaces. Let $x_0 \in X$ and $y_0 \in Y$ with $f_0(x_0) = y_0$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module. Let $F \colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. We denote by $\gamma \colon [0,1] \to Y$ the path that is given by $t \mapsto F(x_0, t)$. We have the following equality of $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules :

$$\overset{\substack{\text{notation from the Change-of-Base}\\ \text{Point Proposition 321.3}\\ \downarrow}}{f_1^*({}_{\gamma^*}M)} \;=\; f_0^*M.$$

Furthermore the maps

$$\begin{array}{ccccc} \mathrm{C}_k^{\mathrm{tw}}(X, A; f_0^*M) & \xrightarrow{f_{0*}} & \mathrm{C}_k^{\mathrm{tw}}(Y, B; M) & & \\ \mathrm{C}_k^{\mathrm{tw}}(X, A; \underbrace{f_0^*M}_{=f_1^*({}_{\gamma^*}M)}) & \xrightarrow{f_{1*}} & \mathrm{C}_k^{\mathrm{tw}}(X, A; {}_{\gamma^*}M) & \xrightarrow{\gamma_*} & \mathrm{C}_k^{\mathrm{tw}}(Y, B; M) \\ & & & \uparrow & \\ & & & \substack{\text{isomorphism given by the}\\ \text{Change-of-Base Point}\\ \text{Proposition 321.3}} & \end{array}$$

are chain homotopic as chain maps between R-left chain complexes. An analogous statement also holds for cochain complexes.

Proof. We outsource the proof of the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3 to the next Section 322.3. ■

As the reader will surely have spotted, the statement becomes a little neater if γ happens to be the constant path. This leads us to the following useful special case of the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3:

Corollary 322.4. (Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Same Map Corollary) Let X and Y be topological spaces that are both path-connected and locally contractible. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module. Let $f, g \colon (X, A) \to (Y, B)$ be two continuous maps of pairs of topological spaces with $f(x_0) = g(x_0) = y_0$. If f and g are homotopic rel x_0, then by the Homotopy-π_1-Proposition 90.8 we know that $f_* = g_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, thus $f^*M = g^*M$, and we see that

$$f_*, g_* \colon \mathrm{C}_k^{\mathrm{tw}}(X, A; f^*M) \;\to\; \mathrm{C}_k^{\mathrm{tw}}(Y, B; M) \;\text{ and }\; f_*, g_* \colon \mathrm{C}_{\mathrm{tw}}^k(Y, B; M) \;\to\; \mathrm{C}_{\mathrm{tw}}^k(X, A; f^*M)$$

are (co-) chain homotopic. In particular the induced maps

$$f_*, g_* \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M) \;\to\; \mathrm{H}_k^{\mathrm{tw}}(Y, B; M) \;\text{ and }\; f_*, g_* \colon \mathrm{H}_{\mathrm{tw}}^k(Y, B; M) \;\to\; \mathrm{H}_{\mathrm{tw}}^k(X, A; f^*M)$$

are equal.

Proof of Corollary 322.4 assuming Proposition 322.3. The proposition follows immediately from the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3 and the observation that under the given hypotheses the path γ in the statement of the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3 is just the constant path. ■

322.3. **Proof of the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3.** The following classical lemma will play an essential role in the proof of the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3.

Lemma 322.5. (Homotopy–Chain Homotopy Lemma) Let V and W be topological spaces and let $\widetilde{F}\colon V \times [0,1] \to W$ be a homotopy. There exists a sequence of singular chains[3535] $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$, $n \in \mathbb{N}_0$ such that the homomorphisms

$$\begin{aligned} P_n\colon \mathrm{C}_n(V) &\to \mathrm{C}_{n+1}(W) \\ (\sigma\colon \Delta^n \to V) &\mapsto (\widetilde{F} \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{aligned}$$

satisfy

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = \widetilde{F}_{0*} - \widetilde{F}_{1*}$$

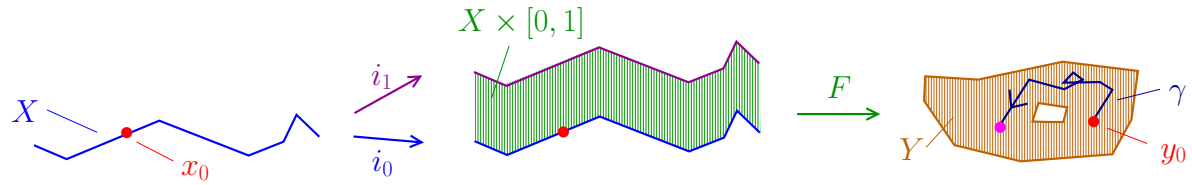
as maps from $\mathrm{C}_n(V)$ to $\mathrm{C}_n(W)$.

Proof. We proved this lemma on page 2775 as part of the proof of the Homotopy-H_*-Proposition 144.12. ∎

Proof of Proposition 322.3. We start out with two caveats:

- To simplify the notation we assume that $A = B = \varnothing$. Mathematically this extra hypothesis makes no real difference.
- In the following we will only prove the statement regarding chain homotopies between chain complexes. We leave it to the reader to adapt the argument below to produce a cochain homotopy between the given cochain maps.

Without further ado, let $f_0, f_1\colon X \to Y$ be two continuous maps of topological spaces. Let $x_0 \in X$ and $y_0 \in Y$ with $f_0(x_0) = y_0$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module. Finally let $F\colon X \times [0,1] \to Y$ be a homotopy between f_0 and f_1. We denote by $i_0, i_1\colon X \to X \times [0,1]$ the natural embeddings. Note that by Definition 322.2 of a homotopy we have $F \circ i_0 = f_0$, $F \circ i_1 = f_1$. We denote by $\gamma\colon [0,1] \to X$ the path that is given by $t \mapsto F(x_0, t)$.



Claim 1. We have the following equality of $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules.

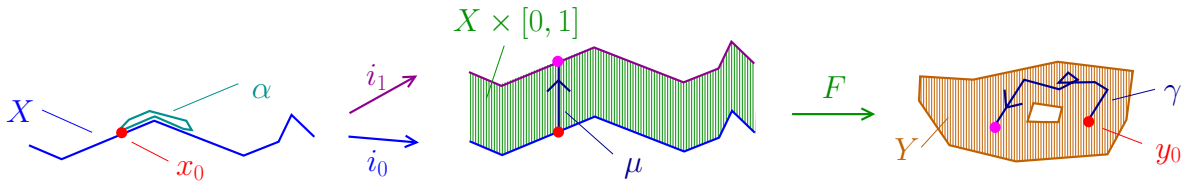
$$f_1^*({}_{\gamma^*}M) = f_0^* M.$$

Proof. We need to show that the maps

$$\begin{aligned} &\pi_1(X, x_0) \xrightarrow{f_{0*}} \pi_1(Y, y_0) \\ \text{and} \quad &\pi_1(X, x_0) \xrightarrow{f_{1*}} \pi_1(X, f_1(x_0)) \xrightarrow{\gamma^*} \pi_1(Y, y_0) \end{aligned}$$

agree. Thus let $\alpha\colon [0,1] \to X$ be a loop in x_0. We denote by $\mu\colon [0,1] \to X \times [0,1]$ the path that is given by $t \mapsto (x_0, t)$. Note that $\gamma = F \circ \mu$.
Then we have the following equalities:

[3535] In Notation 144.16 we give an explicit description of such Ω_n. But the precise definition of the Ω_n is completely irrelevant for what follows.



there is an explicit path-homotopy between $i_0 \circ \alpha$ and $\mu * i_1 \circ \alpha * \overline{\mu}$

$$\begin{array}{rcl} f_{0*}([\alpha]) = [f_0 \circ \alpha] & = & [F \circ (i_0 \circ \alpha)] \\ & = & [(F \circ \mu) * (F \circ i_1 * \alpha) * (F \circ \overline{\mu})] \end{array} \quad \begin{array}{rcl} & \downarrow & \\ & = & [F \circ (\mu * i_1 \circ \alpha * \overline{\mu})] \\ & = & [\gamma] * f_{1*}([\alpha]) * [\overline{\gamma}] \end{array} \quad \boxplus$$

As usual we denote by $p\colon \widetilde{X}_{x_0} \to X$ and $q\colon \widetilde{Y}_{y_0} \to Y$ the natural universal coverings from Definition 319.10. One can easily see that $p \times \mathrm{id}_{[0,1]}\colon \widetilde{X}_{x_0} \times [0,1] \to X \times [0,1]$ is a universal covering.[3536] Thus it follows from the Lift-to-Universal Coverings Proposition 127.26 that there exists a unique continuous map $\widetilde{F}\colon \widetilde{X}_{x_0} \times [0,1] \to \widetilde{Y}_{y_0}$ with $\widetilde{F}(\widetilde{x}_0, 0) = \widetilde{y}_0$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} \times [0,1] & \overset{\widetilde{F}}{\dashrightarrow} & \widetilde{Y}_{y_0} \\ {\scriptstyle p \times \mathrm{id}_{[0,1]}}\downarrow & & \downarrow{\scriptstyle q} \\ X \times [0,1] & \xrightarrow{\quad F \quad} & Y. \end{array}$$

Again we denote by $i_0, i_1\colon \widetilde{X}_{x_0} \to \widetilde{X}_{x_0} \times [0,1]$ the natural embeddings. Below we will need the following subtle claim:

Claim 2. We write $y_1 = f_1(x_0) \in Y$. We denote by $q\colon \widetilde{Y}_{y_1} \to Y$ the corresponding natural universal covering. The following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow[\text{from Definition 321.4}]{\text{induced by } f_1} & \widetilde{Y}_{y_1} \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \gamma^*} \\ \widetilde{X}_{x_0} \times [0,1] & \xrightarrow{\quad \widetilde{F} \quad} & \widetilde{Y}_{y_0}. \end{array}$$

Proof. We start with two observations:

(a) By inspection one sees that both maps from $\widetilde{X}_{x_0}$ to $\widetilde{Y}_{y_1}$ send the natural base point of $\widetilde{X}_{x_0}$ to the point of $\widetilde{Y}_{y_0}$ that is represented by the path γ.

(b) Both maps are lifts of the map $f_1 \circ p\colon \widetilde{X}_{x_0} \to Y$ to a map $\widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$.

Since $\widetilde{X}_{x_0}$ is path-connected we obtain from the Lift Uniqueness Proposition 92.2 that the two maps from $\widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$ agree. But this implies that the diagram commutes. $\boxplus$

Now we can turn to the construction of the desired chain homotopy. In the following let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$, $n \in \mathbb{N}_0$, be the sequence of singular chains from the Homotopy–Chain Homotopy Lemma 322.5. By the Homotopy–Chain Homotopy Lemma 322.5 we know that the maps

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(\widetilde{X}_{x_0}) & \to & \mathrm{C}_{n+1}(\widetilde{Y}_{y_0}) \\ (\sigma\colon \Delta^n \to \widetilde{X}_{x_0}) & \mapsto & (\widetilde{F} \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n) \end{array}$$

[3536] But note that this is *not* the natural universal covering of $X \times [0,1]$.

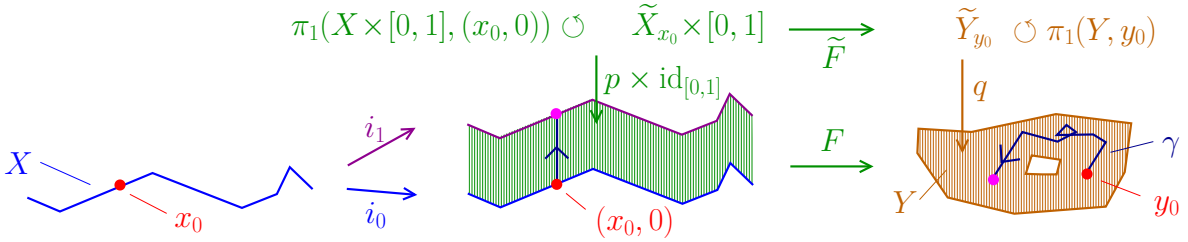
satisfy

$$(*) \qquad \partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n \;=\; (\widetilde{F}\circ i_0)_* - (\widetilde{F}\circ i_1)_*$$

as maps from $\mathrm{C}_n(\widetilde{X}_{x_0})$ to $\mathrm{C}_n(\widetilde{Y}_{y_0})$.

Claim 3. Given any singular simplex $\sigma\colon \Delta^n \to \widetilde{X}_{x_0}$ and given any $g \in \pi_1(X, x_0)$ we have the following equality in $\mathrm{C}_{n+1}(\widetilde{X}_0)$:

$$P_n(g\cdot\sigma) \;=\; \underbrace{(F\circ i_0)_*(g)}_{\in\pi_1(Y,y_0)} \cdot P_n(\sigma).$$



Proof. First note that by the Lift–Action Proposition 128.18 the map $\widetilde{F}\colon \widetilde{X}_{x_0}\times[0,1]\to\widetilde{Y}$ has the following property: Given any $\widetilde{z}\in\widetilde{X}_{x_0}\times[0,1]$ and given any $h\in\pi_1(X\times[0,1],(x_0,0))$ we have

$$\widetilde{F}(h\cdot\widetilde{z}) \;=\; \underbrace{F_*(h)}_{\in\pi_1(Y,y_0)} \cdot \widetilde{F}(\widetilde{z}).$$

Now let $\sigma\colon\Delta^n\to\widetilde{X}_{x_0}$ be a singular simplex and let $g\in\pi_1(X,x_0)$. Then

$$\begin{array}{rcl@{\qquad}rcl}
 & & \text{by definition of } P_n & & & \text{by the Lift–Action Proposition 128.18}\\
P_n(g\cdot\sigma) & \overset{\downarrow}{=} & (\widetilde{F}\circ((g\cdot\sigma\times\mathrm{id}_{[0,1]})))_*(\Omega_n) & & \overset{\downarrow}{=} & (\widetilde{F}\circ(i_{0*}(g)\cdot(\sigma\times\mathrm{id}_{[0,1]})))_*(\Omega_n)\\
 & = & (F_*(i_{0*}(g))\cdot(\widetilde{F}\circ(\sigma\times\mathrm{id}_{[0,1]})))_*(\Omega_n) & & = & (F\circ i_0)_*(g)\cdot P_n(\sigma).\\
 & \overset{\uparrow}{} & & & \overset{\uparrow}{} & \\
 & & \text{by the above equality applied to } h=i_{0*}(g)\in\pi_1(X\times[0,1],(x_0,0)) & & & \text{by definition of } P_n
\end{array}$$

⊞

It follows easily from Claim 3 together with the Tensor Product–Universal Property Lemma 320.5 (1) and (2) that we get an induced R-left module homomorphism

$$\mathrm{id}\otimes P_n\colon \overline{(F\circ i_0)^*M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_n(\widetilde{X}_{x_0}) \;\to\; \overline{M}\otimes_{\mathbb{Z}[\pi_1(Y,y_0)]}\mathrm{C}_n(\widetilde{Y}_{y_0}).$$

It follows from $(*)$ that these maps have the property that

$$(\mathrm{id}\otimes\partial_{n+1})\circ(\mathrm{id}\otimes P_n)+(\mathrm{id}\otimes P_{n-1})\circ(\mathrm{id}\otimes\partial_n) \;=\; \mathrm{id}\otimes(\widetilde{F}\circ i_0)_* - \mathrm{id}\otimes(\widetilde{F}\circ i_1)_*$$

as maps from $\overline{(F\circ i_0)^*M}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_n(\widetilde{X}_{x_0})$ to $\overline{M}\otimes_{\mathbb{Z}[\pi_1(Y,y_0)]}\mathrm{C}_n(\widetilde{Y}_{y_0})$. In other words, we have found a chain homotopy between the maps $\mathrm{id}\otimes(\widetilde{F}\circ i_0)_*$ and $\mathrm{id}\otimes(\widetilde{F}\circ i_1)_*$. It remains to show that these two maps are the chain maps we are supposed to study:

- It follows immediately from the definitions that $\mathrm{id}\otimes(\widetilde{F}\circ i_0)_* = f_{0*}$.
- It follows easily from Claim 1 and 2 that $\mathrm{id}\otimes(\widetilde{F}\circ i_1)_* = \gamma_*\circ f_{1*}$. ■

322.4. **Homotopic equivalences and (co-) homology with twisted coefficients.** In the Homotopy Equivalence-H_*-Proposition 144.14 (2) and the Homotopy Equivalence-H^*-Proposition 198.12 (2) we saw that homotopy equivalent topological spaces have isomorphic

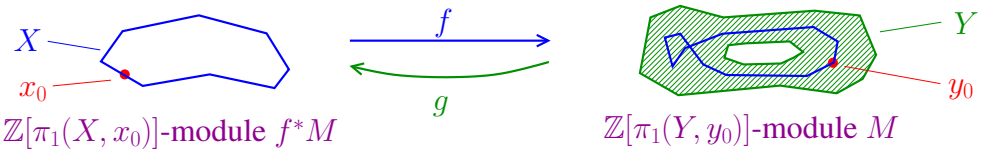
(co-) homology groups. Now we have similar statements in the context of (co-) homology with twisted coefficients. But as usual some care needs to be taken regarding base points and modules involved.

We start out with the following, slightly restrictive proposition.

Proposition 322.6. (Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition) In the following let (X, x_0) and (Y, y_0) be pointed topological spaces that are both path-connected and locally contractible. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let $f\colon (X, A) \to (Y, B)$ be a continuous map with $f(x_0) = y_0$. We suppose that there exists a continuous map $g\colon (Y, B) \to (X, A)$ such that $g \circ f\colon (X, A) \to (X, A)$ is homotopic rel x_0 to $\mathrm{id}_X\colon (X, A) \to (X, A)$ and such that $f \circ g\colon (Y, B) \to (Y, B)$ is homotopic rel y_0 to $\mathrm{id}_Y\colon (Y, A) \to (Y, B)$[3537]. Then f induces isomorphisms on twisted (co-) homology, i.e. given any $\mathbb{Z}[\pi_1(Y, y_0)]$-left module M the induced maps

$$\begin{aligned} f_*\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M) &\to \mathrm{H}_k^{\mathrm{tw}}(Y, B; M) \\ f^*\colon \mathrm{H}_{\mathrm{tw}}^k(Y, B; M) &\to \mathrm{H}_{\mathrm{tw}}^k(X, A; f^*M) \end{aligned}$$

are isomorphisms.



Proof. First note that by hypothesis there exists a homotopy $S\colon X \times [0, 1] \to X$ rel x_0 with $S_0 = g \circ f\colon (X, A) \to (X, A)$ and $S_1 = \mathrm{id}_X\colon (X, A) \to (X, A)$ and there exists a homotopy $T\colon Y \times [0, 1] \to Y$ rel y_0 with $T_0 = f \circ g\colon (Y, B) \to (Y, B)$ and $T_1 = \mathrm{id}_Y\colon (Y, B) \to (Y, B)$. It follows from the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Same Map Corollary 322.4 and the Twisted Homology-Functoriality Proposition 321.9 that the composition of the following two maps

$$\mathrm{H}_k^{\mathrm{tw}}(X, A; \overbrace{f^*(g^*(f^*M))}^{=(g\circ f)^*f^*M=\mathrm{id}^*\, f^*M=f^*M}) \xrightarrow{f_*} \mathrm{H}_k^{\mathrm{tw}}(Y, B; \overbrace{g^*(f^*M)}^{=(f\circ g)^*M=\mathrm{id}^*\, M=M}) \xrightarrow{g_*} \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M)$$

is the identity and that the composition of the following two maps

$$\mathrm{H}_k^{\mathrm{tw}}(Y, B; \underbrace{g^*(f^*M)}_{=(f\circ g)^*M=\mathrm{id}^*\, M=M}) \xrightarrow{g_*} \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M) \xrightarrow{f_*} \mathrm{H}_k^{\mathrm{tw}}(Y, B; M)$$

is also the identity. This shows that f_* is an isomorphism. Basically the same argument also works for twisted cohomology instead of twisted homology.[3538] ∎

The following corollary considers an important special case of a homotopy equivalence:

Corollary 322.7. (Deformation Retract-$\mathrm{H}_*^{\mathrm{tw}}$-Corollary) Let T be a topological space and let $A \subseteq V \subseteq T$ be subspaces. We assume that T and V are both path-connected and

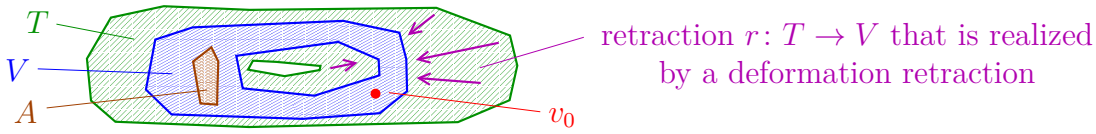
[3537] The fact that we require our homotopies to be homotopies rel x_0 and y_0 is in practice rather awkward and restrictive. This is why in Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1 we will formulated a result with a more user-friendly (but less precise) conclusions.

[3538] Note that this argument is formally the same as the argument that we used in the proof of the Homotopy Equivalence-H_*-Proposition 144.14.

locally contractible. Furthermore let $r\colon T \to V$ be a retraction. We assume that r is realized by a deformation retraction $F\colon T \times [0,1] \to T$, i.e. according to Definition 19.3 we assume that there exists a homotopy $F\colon T \times [0,1] \to T$ rel V with $F_0 = \mathrm{id}_T$ and with $F_1 = r$. For any $v_0 \in V$ and any $\mathbb{Z}[\pi_1(V,v_0)]$-left module M the induced maps

$$\begin{array}{rcl} r_*\colon \mathrm{H}_k^{\mathrm{tw}}(T,A;r^*M) & \to & \mathrm{H}_k^{\mathrm{tw}}(V,A;M) \\ r^*\colon \mathrm{H}^k_{\mathrm{tw}}(V,A;M) & \to & \mathrm{H}^k_{\mathrm{tw}}(T,A;r^*M) \end{array}$$

are isomorphisms.



Proof. We denote by $i\colon V \to T$ the inclusion. Note that $r \circ i = \mathrm{id}_V$. The desired statement follows immediately from the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.6 since the homotopy $F\colon T \times [0,1] \to T$ satisfies $F_0 = \mathrm{id}_T$ and $F_1 = r = i \circ r$ and fixes the base point v_0, and since for $r \circ i$ we can use the stationary homotopy, which of course also fixes the base point v_0. ■

322.5. **Proof of the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1.** Using the Deformation Retract-$\mathrm{H}_*^{\mathrm{tw}}$-Corollary 322.7 we can now prove the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1, which we now repeat for the reader's convenience:

Proposition 322.1. (Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition Let X and Y be topological spaces that are both path-connected and locally contractible. We assume that X and Y are homotopy equivalent. Given any $x_0 \in X$ and $y_0 \in Y$ there exists an isomorphism $\Phi\colon \pi_1(Y,y_0) \to \pi_1(X,x_0)$ such that for any ring R and any $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left module M the following statements hold:

(1) The twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}(X;M)$ and $\mathrm{H}_k^{\mathrm{tw}}(Y;\Phi^*M)$ are isomorphic R-left modules.

(2) The twisted cohomology modules $\mathrm{H}^k_{\mathrm{tw}}(X;M)$ and $\mathrm{H}^k_{\mathrm{tw}}(Y;\Phi^*M)$ are isomorphic R-left modules.

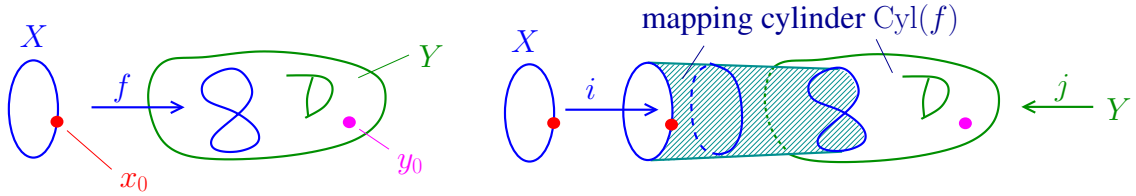
Proof. Let $f\colon X \to Y$ be a homotopy equivalence. As in Definition 21.1 we consider the mapping cylinder

$$\mathrm{Cyl}(f) := \mathrm{Cyl}(f\colon X \to Y) := ((X \times [0,1]) \sqcup Y)/\sim \quad \text{where } (x,1) \sim f(x) \text{ for all } x \in X.$$

In the Mapping Cylinder–Embedding Lemma 21.4 we saw that the two maps

$$\begin{array}{rcl} i\colon X & \to & \mathrm{Cyl}(f\colon X \to Y) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon Y & \to & \mathrm{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

are closed embeddings.

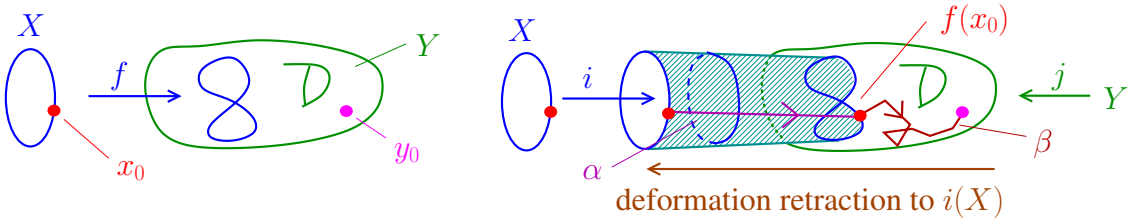


We continue with two observations:

(i) In the Mapping Cylinder–Homotopy Lemma 21.5 we showed that $j(Y)$ is always, regardless of the choice of f, a deformation retract of $\mathrm{Cyl}(f)$.

(ii) Since $f\colon X \to Y$ is by hypothesis a homotopy equivalence we now know by the Homotopy Equivalence–Characterization Corollary 22.20 that $i(X)$ is a deformation retract of $\mathrm{Cyl}(f)$. We denote by $r\colon \mathrm{Cyl}(f) \to i(X)$ the corresponding retraction.

Let $\alpha\colon [0,1] \to \mathrm{Cyl}(f)$ be the path that is given by $t \mapsto [(x_0, t)]$ and let $\beta\colon [0,1] \to Y$ be any path from $f(x_0)$ to y_0.



We see that[3539]

$$\mathrm{H}_k^{\mathrm{tw}}(X;M) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{tw}}(i(X);(i^{-1})^*M) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Cyl}(f);\beta^{**}\alpha^{**}r^*(i^{-1})^*M) \xleftarrow{\cong} \mathrm{H}_k^{\mathrm{tw}}(Y;j^*\beta^{**}\alpha^{**}r^*(i^{-1})^*M).$$

isomorphism given by the homeomorphism $i\colon X \to i(X)$

isomorphism by the Change-of-Base Point Proposition 321.3

isomorphism by the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.6

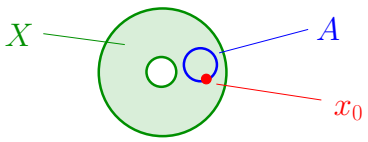
isomorphism by the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.6 and since $j\colon Y \to j(Y)$ is a homeomorphism

The desired statement now follows for the isomorphism $\Phi\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ that is induced by the homotopy equivalences i and j and the base changes given by α and β.

This concludes the arguments regarding twisted homology modules. The argument regarding twisted cohomology modules is very similar. We leave the details to the reader. ■

322.6. Exercises for Chapter 322.

Exercise 322.1. We consider the annulus X together with the subspace $A \subseteq X$ that are shown in the figure below. Let $\xi \in \mathbb{C} \setminus \{0\}$ and let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(1,\mathbb{C})$ be a representation that sends a generator of $\pi_1(X, x_0)$ to (ξ). Calculate the relative twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(X, A; {}_\varphi\mathbb{C})$.

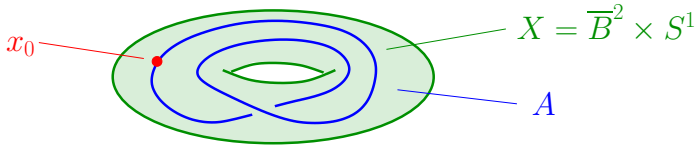


[3539] In Exercise 21.6 we saw, using in particular an argument as in the proof of the Mapping Cylinder–Homotopy Lemma 21.5 (1), that if X and Y are locally contractible, then the mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$ is also locally contractible. Thus it makes sense to consider the twisted homology modules of $\mathrm{Cyl}(f)$.

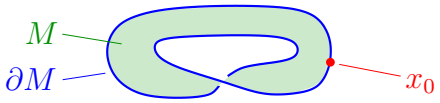
Exercise 322.2. We consider the solid torus $X = \overline{B}^2 \times S^1$ together with the subspace $A \subseteq X$ that are shown in the figure below. Let $\xi \in \mathbb{C} \setminus \{0\}$ and let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a representation that sends a generator of $\pi_1(X, x_0)$ to (ξ).

(a) Calculate the relative twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(X, A; {}_\varphi\mathbb{C})$.

(b) Does there exist a choice of ξ such that $\mathrm{H}_2^{\mathrm{tw}}(X, A; {}_\varphi\mathbb{C}) \neq 0$?



Exercise 322.3. We consider the Möbius band M with a base point $x_0 \in \partial M$. Furthermore let $\xi \in \mathbb{C} \setminus \{0\}$ and let $\varphi\colon \pi_1(M, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a representation that sends a generator of $\pi_1(M, x_0)$ to (ξ). Calculate the relative twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(M, \partial M; {}_\varphi\mathbb{C})$.



CHAPTER 323

Calculations and examples of twisted (co-) homology

In the previous Chapters 320 to 322 we introduced twisted (co-) homology and we proved several general statements. But the reader might have noticed that the last chapters contained few actual calculations. In this chapter we will collect some fairly low hanging fruit to calculate twisted (co-) homology in some instances.

This chapter is organized as follows:

- In Section 323.1 we will study (co-) homology with twisted coefficients in degree 0.
- In Section 323.2 we will express the homology of covering spaces as twisted homology.
- In Section 323.3 we will formulate and discuss the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8 which allows us to calculate certain twisted homology modules of mapping tori.
- In Section 323.4 we will use the main result of Section 323.2 to provide a proof of the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8.
- In Section 323.5 we will perform some more calculations of twisted homology of mapping tori.
- In Section 323.6 we will relate the homology groups of infinite and finite cyclic coverings.
- In Section 323.7 we will discuss the interplay between twisted cohomology and cohomology of finite-index coverings.
- In Section 323.8 we will calculate twisted (co-) homology groups of lens spaces.
- In Section 323.9 we will reinterpret the transfer map in the language of twisted homology.
- In Appendix 323.10 we give a newish proof of the Borsuk-Ulam Theorem 47.12.
- In Section 323.11 we will provide exercises for Chapter 323.

In the next Chapter 324 we will use the results of this chapter to introduce the fundamental class of non-orientable manifolds.

323.1. **(Co-) homology with twisted coefficients in degree 0.** In the H_0-Proposition 143.25, on page 3079 and in the H^0-Proposition 198.9 we saw that it is straightforward to calculate singular (co-) homology in degree 0. As we will see, it is also pretty straightforward to calculate singular (co-) homology with twisted coefficients in degree 0. To state the corresponding result we need the following definition:

Definition 323.1. Let π be a group, let R be a ring and let M be an $(R, \mathbb{Z}[\pi])$-left left module. We define

$$\begin{array}{lcll} M_\pi & \coloneqq & M \Big/ \left\{ \sum_{i=1}^{k} (g_i \cdot v_i - v_i) \,\middle|\, g_i \in \pi \text{ and } v_i \in M \right\} & \pi\text{-coinvariants of } M \\ M^\pi & \coloneqq & \{v \in M \,|\, g \cdot v = v \text{ for all } g \in \pi\} & \pi\text{-invariants of } M. \end{array}$$

Note that it follows easily from Definition 309.37 that M_π and M^π are both naturally R-left modules.

Examples. Let π be a group.

(1) Let R be a ring and let M be an R-left module on which π acts trivially, i.e. we consider the module ${}_{\mathrm{tr}}M$ from Definition 320.12. It follows immediately from the definitions that the R-module $({}_{\mathrm{tr}}M)_\pi$ is naturally isomorphic to M and furthermore that $({}_{\mathrm{tr}}M)^\pi = M$ as R-left modules.

(2) We consider the case $M = \mathbb{Z}[\pi]$ and $R = \mathbb{Z}$. We equip M with the obvious $\mathbb{Z}[\pi]$-left module structure. In Exercise 323.1 we will show that

$$\begin{array}{rcl} \mathbb{Z}[\pi]_\pi & \to & \mathbb{Z} \\ \left[\sum_{i=1}^k r_i \cdot g_i\right] & \mapsto & \sum_{i=1}^k r_i \end{array}$$

is an isomorphism of abelian groups and we will see that

$$\mathbb{Z}[\pi]^\pi \;=\; \begin{cases} 0, & \text{if } \pi \text{ is infinite,} \\ \mathbb{Z} \cdot \sum_{g\in\pi} g, & \text{if } \pi \text{ is finite.} \end{cases}$$

□

Before we can state our calculation of twisted (co-) homology in degree 0 it is convenient to recall the following notation from page **??**:

Notation 323.2. Let X be a topological space. Given a point $P \in X$ we denote by P also the singular 0-simplex $\Delta^0 \to X$ that is given by sending the single point in Δ^0 to P.

The following lemma is the analog of the H_0-Proposition 143.25, the discussion on page 2798 and 3079 and the H^0-Proposition 198.9:

Proposition 323.3. (Twisted H_0-Proposition) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi])$-left left module.

(1) We consider the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and we denote by $\widetilde{x}_0$ the natural base point of $\widetilde{X}_{x_0}$ from Definition 319.10. The augmentation map

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X; M) & \to & M_\pi \\ \left[\sum_{i=1}^k m_i \otimes \sigma_i\right] & \mapsto & \left[\sum_{i=1}^k m_i\right] \end{array}$$

and the evaluation map

$$\begin{array}{rcl} \mathrm{ev}_X\colon \mathrm{H}^0_{\mathrm{tw}}(X; M) & \to & M^\pi \\ (\varphi\colon \mathrm{C}_0(\widetilde{X}_{x_0}) \to M) & \mapsto & \varphi(\widetilde{x}_0) \end{array}$$

here we use Notation 323.2

are natural[3540] isomorphisms of R-left modules.

(2) For any non-empty subspace $A \subseteq X$ we have

$$\mathrm{H}_0^{\mathrm{tw}}(X, A; M) \;=\; 0 \qquad \text{and} \qquad \mathrm{H}^0_{\mathrm{tw}}(X, A; M) \;=\; 0.$$

[3540] Let us explain what we mean by "natural". Thus let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module. In this setting one can easily verify that the following two diagrams commute:

$$\begin{array}{ccc} \mathrm{H}_0^{\mathrm{tw}}(X; M) & \xrightarrow{\epsilon_X} & M_{\pi_1(X,x_0)} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle [m]\mapsto[m]} \\ \mathrm{H}_0^{\mathrm{tw}}(Y; M) & \xrightarrow{\epsilon_Y} & M_{\pi_1(Y,x_0)} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{H}^0_{\mathrm{tw}}(X; M) & \xrightarrow{\mathrm{ev}_X} & M^{\pi_1(X,x_0)} \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle m\mapsto m} \\ \mathrm{H}^0_{\mathrm{tw}}(Y; M) & \xrightarrow{\mathrm{ev}_Y} & M^{\pi_1(Y,x_0)} \end{array}$$

Note that the maps f_* and f^* are given by Definition 321.8.

Example.

(1) We suppose that we are in the setting of the Twisted H_0-Proposition 323.3. If π acts trivially on M, i.e. if we consider the module ${}_{\mathrm{tr}}M$, then it follows from the Twisted H_0-Proposition 323.3 together with the discussion from Item (1) that $\mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}M)$ and $\mathrm{H}^0_{\mathrm{tw}}(X;{}_{\mathrm{tr}}M)$ are both naturally isomorphic to the R-left module M. This result is consistent with the fact, shown in the Twisted–Trivial Coefficients Lemma 320.13, that $\mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}M) \cong \mathrm{H}_0(X;M)$ and $\mathrm{H}^0_{\mathrm{tw}}(X;{}_{\mathrm{tr}}M) \cong \mathrm{H}^0(X;M)$ and our earlier results in the H_0-Proposition 143.25, the H^0-Proposition 198.9 and on page 3079.

(2) Let X be a topological space as above and let $\pi_1(X,x_0) \to \mathrm{GL}(1,\mathbb{C})$ be a 1-dimensional representation. It follows easily from the Twisted H_0-Proposition 323.3 that

$$\mathrm{H}_0^{\mathrm{tw}}(X;{}_{\alpha}\mathbb{C}) \cong \begin{cases} \mathbb{C}, & \text{if } \alpha \text{ is trivial,} \\ 0, & \text{if } \alpha \text{ is non-trivial.} \end{cases}$$

A similar result also holds for twisted cohomology. □

Proof. In the following we will deal with twisted 0-th homology. We will prove the statement regarding twisted 0-th cohomology in Exercise 323.7.

(1) We start out with the following claim:

Claim 1.

(a) The map

$$\begin{array}{rcl} \epsilon_X\colon \overbrace{\overline{M}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_0(\widetilde{X}_{x_0})}^{=\mathrm{C}_0^{\mathrm{tw}}(X;M)} & \to & M_\pi \\ \sum\limits_{i=1}^{k} m_i\otimes\sigma_i & \mapsto & \sum\limits_{i=1}^{k} m_i \end{array}$$

is well-defined.

(b) The map

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) & \to & M_\pi \\ \left[\sum\limits_{i=1}^{k} m_i\otimes\sigma_i\right] & \mapsto & \left[\sum\limits_{i=1}^{k} m_i\right] \end{array}$$

is well-defined.

Proof.

(a) Note that for any $m\in M$, $\sigma\in\mathrm{C}_0(\widetilde{X}_{x_0})$ and $g\in\pi$ we have the following equality in M_π:

$$\epsilon_X(m \underset{\overline{M}}{\cdot} g\otimes\sigma) \underset{\uparrow}{=} \epsilon_X(g^{-1} \underset{M}{\cdot} m\otimes\sigma) = [g^{-1} \underset{M}{\cdot} m] \underset{\uparrow}{=} [m] = \epsilon_X(m\otimes g\cdot\sigma).$$

see Notation 320.7 for the definition of $\overline{M}$

since we have the equality $g^{-1}\cdot m - m = (g^{-1}-1)\cdot m = 0 \in M_\pi$

It follows easily from this little calculation together with the Tensor Product–Universal Property Lemma 320.5 (2) that the given map $\overline{M}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_0(\widetilde{X}_{x_0}) \to M_\pi$ is well-defined.

(b) We need to show that the map $\epsilon_X\colon \mathrm{C}_0^{\mathrm{tw}}(X;M)\to M_\pi$ from (a) vanishes on the image of the boundary map $\partial_1\colon \mathrm{C}_1^{\mathrm{tw}}(X;M)\to\mathrm{C}_0^{\mathrm{tw}}(X;M)$. Thus let $m\in M$ and let $\sigma\colon\Delta^1\to\widetilde{X}_{x_0}$ be a singular 1-simplex. Then

$$\epsilon_X(\partial_1(m\otimes\sigma)) \underset{\uparrow}{=} \epsilon_X(m\otimes\partial_1(\sigma)) \underset{\uparrow}{=} \epsilon_X(m\otimes\sigma(0,1) - m\otimes\sigma(1,0)) = m-m = 0.$$

see the Tensoring Chain Maps Lemma 163.4

see page 2751

This shows that $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M)\to M_\pi$ is indeed well-defined. ⊞

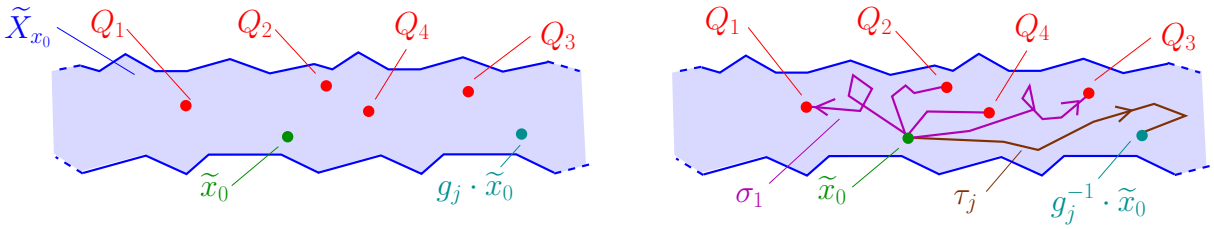
Next note that it follows basically immediately from the fact that X is non-empty that the augmentation map $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) \to M_\pi$ is an epimorphism. It remains to show that the augmentation map is a monomorphism. Thus suppose that we are given $\sum_{i=1}^{k} m_i \otimes Q_i \in \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_0(\widetilde{X}_{x_0}) = \mathrm{C}_0^{\mathrm{tw}}(X;M)$ which represents an element in the kernel of ϵ_X. By Definition 323.1 of M_π and by definition of ϵ_X this means that for $j = 1, \dots, l$ there exist $g_j \in \pi$ and $n_j \in M$ such that the following equality holds in M:

$$\underbrace{\sum_{i=1}^{k} m_i}_{\epsilon_X\left(\sum_{i=1}^{k} m_i \otimes Q_i\right)} = \sum_{j=1}^{l} (g_j \cdot n_j - n_j).$$

We start out as in the proof of the H_0-Proposition 143.25: By the CO-Topology Lemma 319.7 we know that $\widetilde{X}_{x_0}$ is path-connected. Thus there exist singular 1-simplices $\sigma_1, \dots, \sigma_k\colon \Delta^1 \to \widetilde{X}_{x_0}$ with $\partial_1(\sigma_i) = Q_i - \widetilde{x}_0$. We now perform the following calculation:

$$\begin{aligned}
\partial_1\Big(\sum_{i=1}^{k} m_i \otimes \sigma_i\Big) &= \sum_{i=1}^{k} (m_i \otimes Q_i) - \Big(\sum_{i=1}^{k} m_i\Big) \otimes \widetilde{x}_0 \\
&\quad \uparrow \text{ follows from the definition of } \partial_1 \text{ and the bilinearity of } \otimes \\
&= \sum_{i=1}^{k} (m_i \otimes Q_i) - \Big(\sum_{j=1}^{l} (\underbrace{g_j \cdot n_j}_{=n_j \cdot g_j^{-1}} - n_j)\Big) \otimes \widetilde{x}_0 \\
&\quad \uparrow \text{ by the above equality} \\
&= \sum_{i=1}^{k} (m_i \otimes Q_i) - \sum_{j=1}^{l} n_j \otimes (g_j^{-1} \cdot \widetilde{x}_0 - \widetilde{x}_0). \\
&\quad \uparrow \text{ since } \otimes \text{ is balanced in } \pi_1(X, x_0)
\end{aligned}$$

Next we pick singular 1-simplices $\tau_1, \dots, \tau_l\colon \Delta^1 \to \widetilde{X}_{x_0}$ such that for each $j \in \{1, \dots, l\}$ we have $\partial_1(\tau_j) = g_j^{-1} \cdot \widetilde{x}_0 - \widetilde{x}_0$.



Continuing with the above calculation we see that

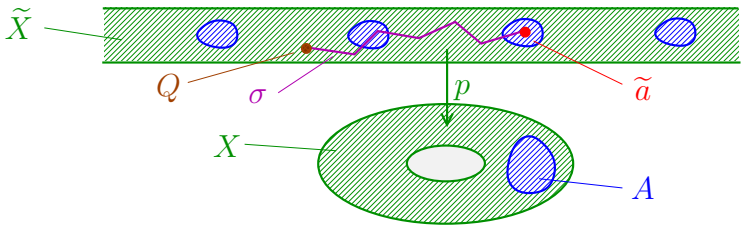
$$\partial_1\Big(\underbrace{\sum_{i=1}^{k} m_i \otimes \sigma_i}_{=:\alpha}\Big) = \sum_{i=1}^{k} (m_i \otimes Q_i) - \sum_{j=1}^{l} n_j \otimes \underbrace{(g_j^{-1} \cdot \widetilde{x}_0 - \widetilde{x}_0)}_{=\partial_1(\tau_j)} = \sum_{i=1}^{k} (m_i \otimes Q_i) - \partial_1\Big(\underbrace{\sum_{j=1}^{l} n_j \otimes \tau_j}_{=:\beta}\Big).$$

It follows from this calculation that $\sum_{i=1}^{k} m_i \otimes Q_i$ is the boundary of $\alpha + \beta$, i.e. it represents the zero element in $\mathrm{H}_0^{\mathrm{tw}}(X;M)$. In other words, we have shown that the evaluation map $\epsilon_X\colon \mathrm{H}_0^{\mathrm{tw}}(X;M) \to M_\pi$ is a monomorphism.

(2) Now let $A \subseteq M$ be a non-empty subspace. We need to show that $\mathrm{H}_0^{\mathrm{tw}}(X, A; M) = 0$. In other words, we have to prove the following claim:

Claim. For each $Q \in \widetilde{X}_0$ and each $m \in M$ we have $[m \otimes Q] = 0 \in \mathrm{H}_0^{\mathrm{tw}}(X, A; M)$.

Proof. Let $Q \in \widetilde{X}_0$ and let $m \in M$. Since A is non-empty we can pick a point $\widetilde{a} \in \widetilde{A} = p^{-1}(A) \subseteq \widetilde{X}_{x_0}$. Since $\widetilde{X}_{x_0}$ is path-connected there exists a singular 1-simplex $\sigma\colon \Delta^1 \to \widetilde{X}_{x_0}$ with $\partial_1(\sigma) = Q - \widetilde{a}$.



We now see that we have the following equality in $\mathrm{H}_0^{\mathrm{tw}}(X; M)$:

$$[m \otimes Q] \quad = \quad [m \otimes Q - m \otimes \underbrace{\partial_1(\sigma)}_{=Q-\widetilde{a}}] \quad = \quad [\quad \underbrace{m \otimes \widetilde{a}}_{=0 \text{ in } \mathrm{C}_0^{\mathrm{tw}}(X,A;M)} \quad] \quad = \quad 0.$$

■

323.2. **Twisted homology and homology of covering spaces.** In this section we will see that for certain twistings we can calculate twisted homology modules in terms of ordinary homology of covering spaces. In practice this will turn out to be one of the most useful tricks for calculating twisted homology groups. Before we can state the main result we need to introduce the following notation:

Notation 323.4. Let S be a *commutative* ring, let π be a group and let $\Gamma \subseteq \pi$ be a subgroup.

- We denote by $S[\Gamma\backslash\pi]$ the free S-module that is generated by the set $\Gamma\backslash\pi$.
- We equip the free S-module $S[\Gamma\backslash\pi]$ with the $\mathbb{Z}[\pi]$-left module structure that is given by $h \cdot (\Gamma g) := \Gamma g h^{-1}$. This turns $S[\Gamma\backslash\pi]$ into an $(S, \mathbb{Z}[\pi])$-left left module.[3541]

Note that if $\Gamma \subseteq \pi$ is normal, then $S[\Gamma\backslash\pi]$ is just the group ring of the group $\Gamma\backslash\pi$ from Definition 320.1. It then follows easily from the definitions that the left multiplication by $S[\Gamma\backslash\pi]$ and the above left-multiplication by $\mathbb{Z}[\pi]$ turn $S[\Gamma\backslash\pi]$ into an $(S[\Gamma\backslash\pi], \mathbb{Z}[\pi])$-left left module.[3542]

Now we can formulate the main result of this section:

Proposition 323.5. (Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition) Let X be a topological space that is path-connected and locally contractible. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering which we equip with the natural base point $\widetilde{x}_0$. Let $A \subseteq X$. We write $\widetilde{A} := p^{-1}(A)$. Furthermore let $\Gamma \subseteq \pi$ be a subgroup and let S be a *commutative* ring. We equip the free S-module $S[\Gamma\backslash\pi]$ with the $(S, \mathbb{Z}[\pi])$-left left module structure from Notation 323.4. Let $q\colon (Y_\Gamma, y_0) \to (X, x_0)$ be a covering of pointed topological spaces with $q_*(\pi_1(Y_\Gamma, y_0)) = \Gamma$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to$

[3541] Note that the corresponding $\mathbb{Z}[\pi]$-right module structure from Notation 320.7 on $\overline{S[\Gamma\backslash\pi]}$ is actually the obvious one given by right multiplication.

[3542] If $\Gamma \subseteq \pi$ is normal, then we have the natural homomorphism $\pi \to \mathrm{GL}(1, \mathbb{Z}[\Gamma\backslash\pi])$. The corresponding $\mathbb{Z}[\pi]$-left module structure on $\mathbb{Z}[\Gamma\backslash\pi]$ from Convention 320.9 is precisely the one we just introduced.

(Y_Γ, y_0) such that the following diagram commutes:

$$\begin{array}{ccccc} \widetilde{X}_{x_0} & & \xrightarrow{\quad r \quad} & & Y_\Gamma. \\ & \searrow^{p} & & \swarrow_{q} & \\ & & X & & \end{array}$$

Finally we write $A_\Gamma := q^{-1}(A)$.

(1) The maps

$$\begin{array}{rcl} \Omega\colon \overbrace{S[\Gamma\backslash\pi] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X,A;S[\Gamma\backslash\pi])} & \to & \overbrace{\mathrm{C}_k(Y_\Gamma, A_\Gamma) \otimes_{\mathbb{Z}} S}^{=\mathrm{C}_k(Y_\Gamma,A_\Gamma;S)} \\ s \cdot \Gamma g \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & (r \circ (g \cdot \sigma\colon \Delta^k \to \widetilde{X}_{x_0})) \otimes s \end{array}$$

form an isomorphism of chain complexes of S-modules. In particular for each $k \in \mathbb{N}_0$ these maps induce an isomorphism

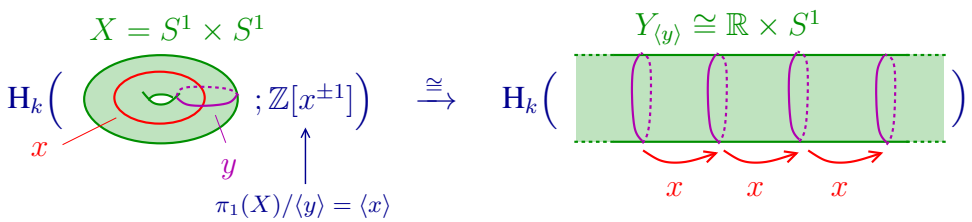
$$\Omega_*\colon \underbrace{\mathrm{H}_k^{\mathrm{tw}}(X, A; S[\Gamma\backslash\pi])}_{\text{twisted homology}} \to \underbrace{\mathrm{H}_k(Y_\Gamma, A_\Gamma; S)}_{\substack{\text{singular homology} \\ \text{with } S\text{-coefficients}}}$$

of S-modules. These homomorphisms are furthermore compatible with the connecting homomorphisms from the LES-of-Triple Proposition 146.5 and the Twisted-LES-Proposition 321.10.

(2) We now suppose that Γ is a *normal* subgroup of π. As we mentioned in Notation 323.4 this allows us to view $S[\Gamma\backslash\pi]$ as an $(S[\Gamma\backslash\pi], \mathbb{Z}[\pi])$-left left module. Furthermore we equip Y_Γ with the $\Gamma\backslash\pi$-left action given by the Deck Transformation Group Proposition 319.12. By Notation 320.8 this allows us to view $\mathrm{C}_k(Y_\Gamma, A_\Gamma; S)$ and $\mathrm{H}_k(Y_\Gamma, A_\Gamma; S)$ as a $\mathbb{Z}[\Gamma\backslash\pi]$-left module. With these actions the above maps

$$\begin{array}{rcl} \Omega\colon \mathrm{C}_k^{\mathrm{tw}}(X, A; S[\Gamma\backslash\pi]) & \to & \mathrm{C}_k(Y_\Gamma, A_\Gamma; S) \\ \Omega_*\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; S[\Gamma\backslash\pi]) & \to & \mathrm{H}_k(Y_\Gamma, A_\Gamma; S) \end{array}$$

are natural isomorphisms of $S[\Gamma\backslash\pi]$-left modules.[3543 3544]



Remark. Note that for once the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 only concerns homology. We will discuss cohomology later in Section 323.7. □

Example.

(0) The case $\Gamma = \pi$ of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 is essentially equivalent to the homological content of the Twisted–Trivial Coefficients Lemma 320.13.

(1) Let X be a topological space that is path-connected and locally contractible. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. If we apply the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 to the trivial subgroup $\Gamma = \{e\}$, then we see that for every $k \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_k^{\mathrm{tw}}(X; \mathbb{Z}[\pi_1(X, x_0)]) \;\cong\; \mathrm{H}_k(\text{universal covering of } X)$$

of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. For example, if for some $n \in \mathbb{N}$ we consider the n-dimensional torus $X = (S^1)^n$, then

$$\mathrm{H}_k^{\mathrm{tw}}((S^1)^n; \mathbb{Z}[\pi_1(X)]) \underset{\substack{\uparrow \\ \text{by the Covering-H}_*^{\mathrm{tw}} \\ \text{Proposition 323.5}}}{\cong} \mathrm{H}_k(\overbrace{\text{universal covering of } (S^1)^n}^{\cong \mathbb{R}^n, \text{ see page 2465}}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{else.} \end{cases}$$

[3542] Again the adjective "natural" needs to be turned into a meaningful statement. For this let $\Gamma' \subseteq \Gamma'' \subseteq \pi$ be two subgroups, which are both normal in π. Then we get a homomorphism $\Theta\colon S[\Gamma'\backslash\pi] \to S[\Gamma''\backslash\pi]$ of $\mathbb{Z}[\pi]$-left modules and from the Covering–Covering Theorem 319.2 we get a covering map $r\colon Y_{\Gamma'} \to Y_{\Gamma''}$. Then one can easily verify that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X, A; S[\Gamma'\backslash\pi]) & \xrightarrow{\Omega} & \mathrm{H}_k(Y_{\Gamma'}; S) \\ \downarrow{\scriptstyle \Theta_*} & & \downarrow{\scriptstyle r_*} \\ \mathrm{H}_k^{\mathrm{tw}}(X, A; S[\Gamma''\backslash\pi]) & \xrightarrow{\Omega} & \mathrm{H}_k(Y_{\Gamma''}; S) \end{array}$$

where the vertical homomorphism to the left is given by Definition 321.12 and where the vertical homomorphism to the right is given by Definition 321.8. Of course all details are left to the reader. Similarly we leave it to the reader to show that the isomorphisms Ω are in a suitable sense natural in (X, A). We leave it to the reader to formulate and prove such a naturality statement.

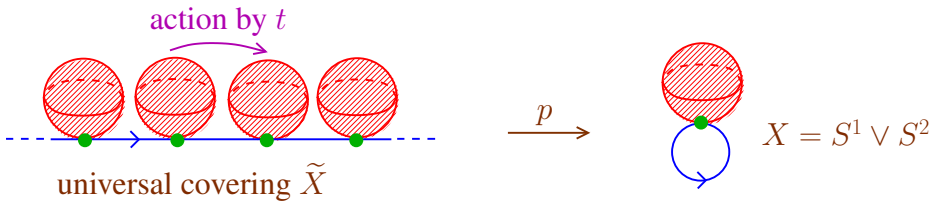
[3543] In practice one often uses the following alternative point of view: Let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a group G. We denote by $Y_{\ker(\varphi} \to X$ a covering corresponding to the subgroup $\ker(\varphi) \subseteq \pi_1(X, x_0)$. Then we obtain a natural isomorphism

$$\Omega_*\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; S[G]) \xrightarrow{\cong} \mathrm{H}_k(Y_{\ker(\varphi)}, A_{\ker(\varphi)}; S)$$

of $S[G]$-left modules.

[3544] Everybody who tries to write up the story of twisted (co-) homology groups and modules will notice that sooner or later it is necessary to make choices about the type of actions that are considered. Whichever choice one takes, this choice will sooner or later haunt the author at some other point. Our choices are designed in such a way that we get the clean statement in the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 about $S[\Gamma\backslash\pi]$-left modules. In particular this choice lead us to the popular topic of left-left modules.

(2) Once again we consider the covering $\widetilde{X} \to X = S^1 \vee S^2$ that is shown below.

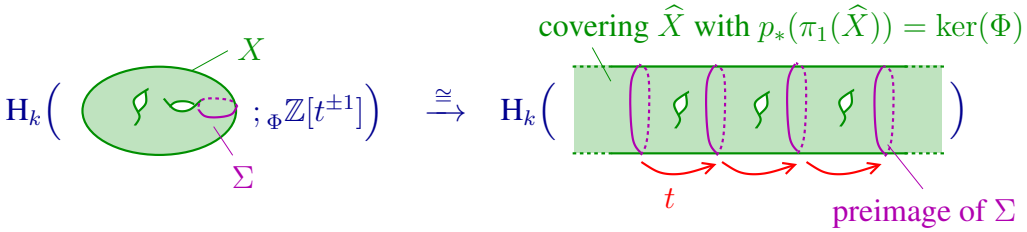


Let $\varphi\colon \pi_1(S^1 \vee S^2, *) \to \langle t \rangle$ be the standard isomorphism. We now see that

$$\mathrm{H}_k^{\mathrm{tw}}(S^1 \vee S^2; {}_{\varphi}\mathbb{Z}[t^{\pm 1}]) \;\cong\; \mathrm{H}_k(\widetilde{X}) \;\cong\; \begin{cases} \mathbb{Z}[t^{\pm 1}]/(t-1)\mathbb{Z}[t^{\pm 1}], & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, 2, \\ \mathbb{Z}[t^{\pm 1}], & \text{if } k = 2. \end{cases}$$

$\uparrow$ by the Covering-$\mathrm{H}_*^{\mathrm{tw}}$ Proposition 323.5 $\qquad$ $\uparrow$ see page 5475

(3) Let X be a compact smooth manifold and let $\Phi \in \mathrm{H}^1(X;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(X), \langle t \rangle)$ be a primitive class. Let Σ be an oriented proper codimension-one submanifold of X such that $\mathrm{PD}_X([\Sigma]) = \Phi$.[3545]



In the Cyclic-Covering Proposition 234.21 we used Σ to construct an explicit covering $\widehat{X}$ of X (with an explicit action by $\langle t \rangle$) that corresponds to $\ker(\Phi)$. It now follows from the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 that one can calculate $\mathrm{H}_k(X; {}_{\Phi}\mathbb{Z}[t^{\pm 1}])$ using that covering.

We will discuss more examples for the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 in Section 323.3 and Section 323.8. □

Proof. Let X be a topological space that is path-connected and locally contractible. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. Furthermore let $\Gamma \subseteq \pi$ be a subgroup. To simplify the notation we will just deal with the case $A = \varnothing$.

(1) We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} \Omega\colon \overbrace{S[\Gamma\backslash\pi] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0})}^{=\mathrm{C}_k^{\mathrm{tw}}(X;S[\Gamma\backslash\pi])} & \to & \overbrace{\mathrm{C}_k(Y_\Gamma) \otimes S}^{=\mathrm{C}_k(Y_\Gamma;S)} \\ s \cdot \Gamma g \otimes (\sigma\colon \Delta^k \to Y_\Gamma) & \mapsto & r \circ (g \cdot \sigma\colon \Delta^k \to \widetilde{X}_{x_0}) \otimes s \end{array}$$

$\uparrow$ where $q\colon \widetilde{X}_{x_0} \to Y_\Gamma$

is well-defined.

[3545] It follows from the Codimension-One Submanifold Existence Proposition 234.6 that such Σ always exists.

Proof. We start out by fixing some notation: Given $k \in \pi$ we denote by $d_k \colon \widetilde{X}_{x_0} \to \widetilde{X}_{x_0}$ the corresponding deck transformation that is given by the Deck Transformation Group Proposition 319.12. Now we turn to the actual proof of the claim: First we show that the map is independent of the representative $g \in \pi$. Indeed, if $h \in \Gamma$, then

$$q \circ (hg \cdot \sigma) \;=\; r \circ (d_h \circ d_g \circ \sigma) \;=\; (r \circ d_h) \circ d_g \circ \sigma \;\underset{\uparrow}{=}\; r \circ d_g \circ \sigma \;=\; r \circ g \cdot \sigma.$$

it follows from the Deck Transformation Group Proposition 319.12 and and $h \in \Gamma$ that $q \circ d_h = q$

Next let us show, using the Tensor Product–Universal Property Lemma 320.5, that the map Ω respects the tensor product $\overline{S[\Gamma\backslash\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0})$. Thus let $\Gamma f \in \Gamma\backslash\pi$ and let $\sigma \in \mathrm{C}_k(\widetilde{X}_{x_0})$. Then given any $g \in \pi$ we have

$$\Omega((s\Gamma f) \cdot g \otimes \sigma) \;\underset{\uparrow}{=}\; \Omega(s\Gamma f \cdot g \otimes \sigma) \;\overset{\downarrow}{=}\; r \circ (f \cdot g) \cdot \sigma \otimes s \;=\; r \circ f \cdot (g \cdot \sigma) \otimes s.$$

definition of Ω

this follows from our definition of the $\mathbb{Z}[\pi]$-left module structure on $S[\Gamma\backslash\pi]$ and by Notation 323.4 and Notation 320.7 for the $\mathbb{Z}[\pi]$-right module $\overline{S[\Gamma\backslash\pi]}$

$$= \Omega(\Gamma f \otimes g \cdot \sigma). \qquad \boxplus$$

Recall that in the C_*–Functor Lemma 143.16 we showed that the boundary maps ∂ are natural. It is now straightforward to verify that the maps Ω are chain maps. It remains to show that the maps Ω are isomorphisms. To do so we consider the homomorphism

$$\begin{array}{rcl} \overbrace{\mathrm{C}_k(Y_\Gamma) \otimes S}^{=\mathrm{C}_k(Y_\Gamma;S)} & \to & \overbrace{\overline{S[\Gamma\backslash\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0})}^{=\mathrm{C}_k^{\mathrm{tw}}(X;S[\Gamma\backslash\pi])} \\ (\sigma \colon \Delta^k \to Y_\Gamma) \otimes s & \mapsto & s \cdot e \otimes \text{any lift of } \sigma \text{ to a map } \Delta^k \to \widetilde{X}_{x_0}. \end{array}$$

It follows easily from the Universal Covering–Quotient Proposition 128.17 that this map is well-defined. It is straightforward to verify that this map is an inverse to Ω.

Finally note that it follows easily from the definitions that these homomorphisms Ω are in fact compatible with the connecting homomorphisms from the LES-of-Triple Proposition 146.5 and the Twisted-LES-Proposition 321.10.

(2) We now assume that $\Gamma \subseteq \pi$ is a normal subgroup. We only need to show that Ω is $S[\Gamma\backslash\pi]$-left linear. Thus let $t \cdot \Gamma h \in S[\Gamma\backslash\pi]$. Furthermore let $s \in S$, $\Gamma g \in \Gamma\backslash\pi$ and let $\sigma \in \mathrm{C}_k(\widetilde{X}_{x_0})$. Then we have the following equalities $\mathrm{C}_k(Y_\Gamma; S) = \mathrm{C}_k(Y_\Gamma) \otimes_{\mathbb{Z}} S$:

$$\begin{array}{rcll} \Omega(t \cdot \Gamma h \cdot (s \cdot \Gamma g \otimes \sigma)) & \overset{\downarrow}{=} & \Omega(ts \cdot \Gamma hg \otimes \sigma) & \overset{\downarrow}{=} \quad (r \circ h \cdot g \cdot \sigma) \otimes ts \\ & = & (r_*(h) \cdot (r \circ g \cdot \sigma)) \otimes ts & = \quad t \cdot \Gamma h \cdot \Omega(s\Gamma g \otimes \sigma). \\ & \underset{\uparrow}{=} & & \underset{\uparrow}{=} \end{array}$$

by the definition of the $S[\Gamma\backslash\pi]$-left module structure on $S[\Gamma\backslash\pi]$

by definition of Ω

follows from the naturality of the map in the Deck Transformation Group Proposition 319.12 which is spelled out in the Deck-to-π_1-Functoriality Lemma 128.16

by definition of Ω and since $r_*(h) = \Gamma h$

■

323.3. **Twisted homology of mapping tori.** [3546] For our next example let us recall Definition 21.19:

Definition 323.6. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])\,/\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X,f). We adopt the following language:

(1) We refer to the map

$$\begin{array}{rccc} p\colon \mathrm{Tor}(X,f) = (X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

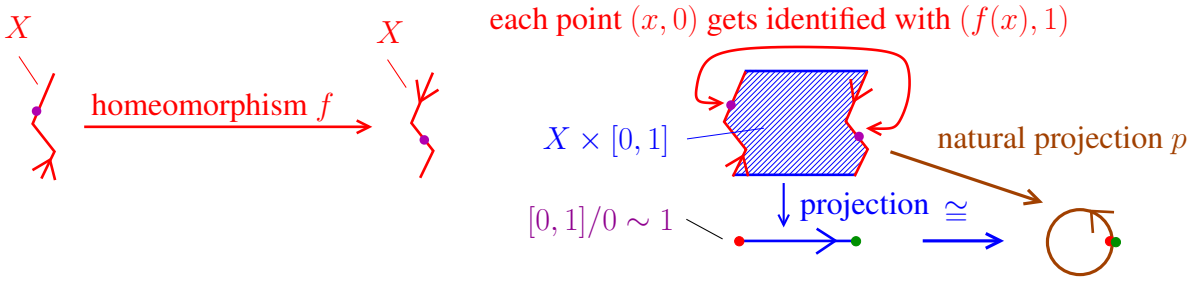
as the natural projection onto S^1. If X is connected, then given any $x\in X$ we refer to the map

$$\Phi\colon \pi_1(\mathrm{Tor}(X,f),[(x,0)]) \xrightarrow{p_*} \pi_1(S^1,1) \xrightarrow[\uparrow]{\cong} \langle t\rangle$$

unique isomorphism that sends the loop $s\mapsto\exp(2\pi\,\mathrm{i}s)$ to t

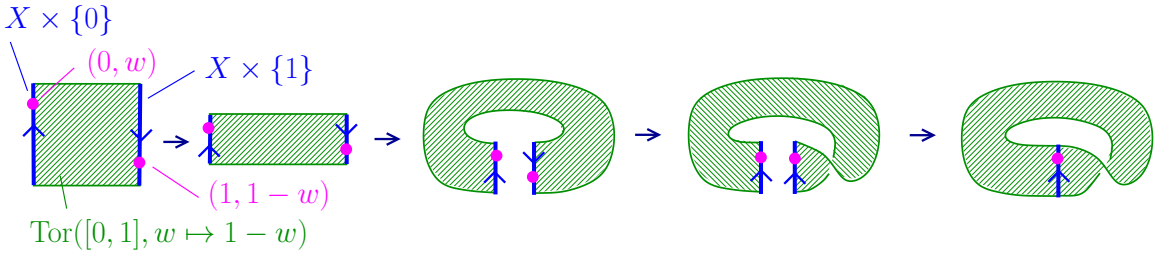
as the natural epimorphism.

(2) If f is a homeomorphism, then we refer to X as the fiber of the mapping torus and we refer to f as the monodromy of the mapping torus.



Example.

(1) We consider the topological space $[0,1]$ and the self-homeomorphism $f\colon[0,1]\to[0,1]$ that is given by $f(w)=1-w$. It follows immediately from the definitions that the corresponding mapping torus $\mathrm{Tor}([0,1],f)$ is precisely the Möbius band as defined on page 310.



[3546] This section has a significant overlap with Section 286.3. To keep this part of the lecture notes somewhat self-contained we duplicate some of the earlier examples and results.

(2) Let $X = S^1$ and let $f\colon S^1 \to S^1$ be the homeomorphism that is given by $f(z) = \overline{z}$. On page 575 we pointed out that the mapping torus $\mathrm{Tor}(S^1, f)$ is homeomorphic to the Klein bottle from page 312.

(3) Let X be a topological space. In the Mapping Torus-of-id Lemma 21.22 we showed that the map

$$\begin{array}{rccc} \Phi\colon & \mathrm{Tor}(X, \mathrm{id}_X) & \to & X \times S^1 \\ & [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a natural homeomorphism. □

To state the main result of this section it is convenient to introduce the following notation:

Notation 323.7. Let R be a commutative ring, let A be an R-module and let $\Theta \in \mathrm{Aut}_R(A)$. We denote by $(A, t = \Theta)$ the $R[t^{\pm 1}]$-module where the underlying R-module is given by A and where t acts by application by Θ, thus for $n \in \mathbb{N}$ the power t^n acts by application of the automorphism $\Theta^n = \Theta \circ \cdots \circ \Theta$ and t^{-n} acts by application of the automorphism $\Theta^{-n} = \Theta^{-1} \circ \cdots \circ \Theta^{-1}$

In the Mapping Torus-H_*-Proposition 153.6 we calculated the ordinary homology groups of mapping tori. In the following proposition we will calculate a certain type of twisted homology modules of mapping tori:

Proposition 323.8. (Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $f\colon X \to X$ be a homeomorphism. We denote by $\Phi\colon \pi_1(\mathrm{Tor}(X,f), [(x_0,0)]) \to \langle t \rangle$ the corresponding natural epimorphism from Definition 323.6. Given any commutative ring R and given any $k \in \mathbb{N}_0$ we have a natural isomorphism[3547]

$$\underbrace{(\mathrm{H}_k(X;R), t = f_*)}_{\substack{\text{the } R[t^{\pm 1}]\text{-module} \\ \text{from Notation 323.7}}} \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X,f); \underbrace{{}_\Phi R[t^{\pm 1}]}_{\substack{\uparrow \\ \text{see Notation 320.8}}})$$

of $R[t^{\pm 1}]$-modules.

Proof. We hide the proof of the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8 in Section 323.4. ■

323.4. **Proof of the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8.** The key to proving the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8 is the following proposition:

Proposition 286.5. (Mapping Torus-Covering Proposition) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible and let $f\colon X \to X$ be a homeomorphism. We consider the natural epimorphism $\Phi\colon \pi_1(\mathrm{Tor}(X,f), [(x_0,0)]) \to \mathbb{Z}$ from Definition 323.6. There exists a covering

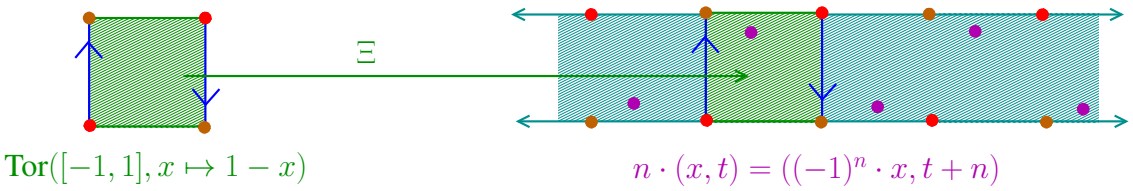
$$p\colon X \times \mathbb{R} \to \mathrm{Tor}(X,f)$$

[3547]Note that one can easily verify that $\mathrm{Tor}(X,f)$ is also path-connected and locally contractible. So the twisted homology is actually defined.

corresponding to $\ker(\Phi)$ such that the $\mathbb{Z}$-action from Convention 286.3 is given by the action

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x, s)) & \mapsto & (f^n(x), s + n). \end{array}$$

Example. We consider the case that $X = [-1, 1]$ and $f(x) = -x$. The corresponding $\mathbb{Z}$-action on $[-1, 1] \times \mathbb{R}$ is given by $n \cdot (x, t) := ((-1)^n \cdot x, t + n)$.



In the figure we attempt to illustrate the mapping torus $\mathrm{Tor}([-1, 1], x \mapsto -x)$ and we illustrate $[-1, 1] \times \mathbb{R}$ with the $\mathbb{Z}$-action. □

Sketch of a proof of Proposition 286.5. As we pointed out in the Mapping Torus-of-Homeomorphism Lemma 93.8 one can easily verify that the following two statements hold:

(1) The map

$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x, t)) & \mapsto & (f^n(x), t + n) \end{array}$$

defines a continuous, proper and discrete action of the group $\mathbb{Z}$ on $X \times \mathbb{R}$.

(2) The map $\Theta\colon \mathrm{Tor}(X, f) = (X \times [0, 1])/\sim \to (X \times \mathbb{R})/\mathbb{Z}$, $[(x, t)] \mapsto [(x, t)]$ is a homeomorphism.

Using these two facts one can easily verify that the map

$$\begin{array}{rcl} X \times \mathbb{R} & \to & \mathrm{Tor}(X, f) \\ (x, t) & \mapsto & \Theta^{-1}([(x, t)]) \end{array}$$

has all the desired properties. ■

Proof of the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8. Let (X, x_0) be a pointed topological space that is path-connected and locally contractible, let $f\colon X \to X$ be a homeomorphism and let R be a commutative ring. It follows almost immediately from the Mapping Torus-Covering Proposition 286.5 together with the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 that we have a natural isomorphism

$$\Omega\colon \mathrm{H}_k(X; {}_\Phi R[t^{\pm 1}]) \xrightarrow{\cong} (\mathrm{H}_k(X \times \mathbb{R}; R), t = \mu_*)$$

of $R[t^{\pm 1}]$-modules where $\mu\colon X \times \mathbb{R} \to X \times \mathbb{R}$ is given by $(x, s) \mapsto (f(x), s + 1)$. The proposition now follows from the following claim:

Claim. We denote by $p\colon X \times \mathbb{R} \to X$ the projection. The induced map $p_*\colon \mathrm{H}_k(X \times \mathbb{R}; R) \to \mathrm{H}_k(X; R)$ defines an isomorphism between the $R[t^{\pm 1}]$-modules $(\mathrm{H}_k(X \times \mathbb{R}; R), t = \mu_*)$ and $(\mathrm{H}_k(X; R), t = f_*)$.

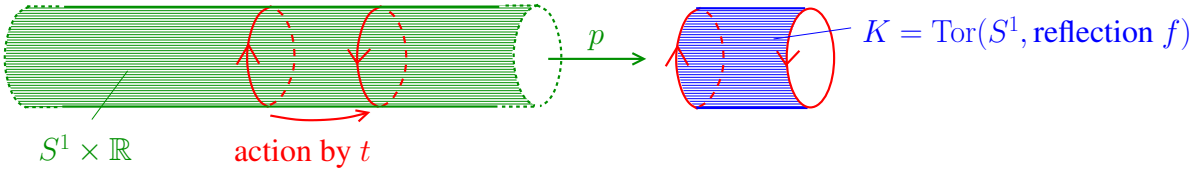
Proof. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_k(X \times \mathbb{R}; R) & \xrightarrow{\mu_*} & \mathrm{H}_k(X \times \mathbb{R}; R) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \mathrm{H}_k(X; R) & \xrightarrow{f_*} & \mathrm{H}_k(X; R). \end{array}$$

It follows from $p \circ \mu = f \circ p$ that the diagram commutes. Furthermore p is evidently a homotopy equivalence. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the two vertical maps are isomorphisms. The desired statement is an immediate consequence of these two observations and of the functoriality of homology. ■

323.5. **Calculations of twisted homology of mapping tori.** In this section we will discuss some examples and variations on the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8. We start out with an example:

Example. We consider the Klein bottle K, which as on page 6199 we think of as the mapping torus $K := \mathrm{Tor}(S^1, f)$, where $f(z) = \overline{z}$ is the reflection in the real axis. Just for fun we mention that in this case the covering of the proof of the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8 is given by a map $p\colon S^1 \times \mathbb{R} \to K := \mathrm{Tor}(S^1, f)$ where the t^n-action on $S^1 \times K$ is given by the map $t^n \cdot (x, s) \mapsto (x + n, (-1)^n \cdot s)$.



We want to use the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$- Proposition 323.8 to calculate the homology groups $\mathrm{H}_k^{\mathrm{tw}}(K; {}_\Phi\mathbb{Z}[t^{\pm 1}])$. We start out with the following observations:

(0) It follows from the H_0-Proposition 143.25 (2) that $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity.
(1) By the Sphere–Degree–Matrix Lemma 151.7 we know that $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is multiplication by -1.
(2) By the Sphere-H_*-Proposition 149.4 we have $\mathrm{H}_k(S^1) = 0$ for $k \geq 2$.

It follows that for $k \in \mathbb{N}$ we have the following isomorphisms of $\mathbb{Z}[t^{\pm 1}]$-modules:

$$\mathrm{H}_k^{\mathrm{tw}}(K; {}_\Phi\mathbb{Z}[t^{\pm 1}]) \cong \begin{cases} (\mathbb{Z}, t = \mathrm{id}), & \text{if } k = 0, \\ (\mathbb{Z}, t = -1), & \text{if } k = 1, \\ 0, & \text{if } k \geq 2 \end{cases} \cong \begin{cases} \mathbb{Z}[t^{\pm 1}]/(t-1)\mathbb{Z}[t^{\pm 1}], & \text{if } k = 0, \\ \mathbb{Z}[t^{\pm 1}]/(t+1)\mathbb{Z}[t^{\pm 1}], & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}$$

$\uparrow$ follows from the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.8 together with (0), (1) and (2)

$\uparrow$ see Exercise 323.14

□

We continue with the following calculation:

Proposition 323.9. (Mapping Torus–Vanishing-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition) Let X be a topological space that is path-connected and locally contractible. We assume that all homology modules $\mathrm{H}_k(X; \mathbb{Q})$ are finite dimensional[3548]. Furthermore let $f\colon X \to X$ be a homeomorphism. We pick $x \in X$. We denote by $\Phi\colon \pi_1(\mathrm{Tor}(X, f), [(x, 0)]) \to \langle t \rangle$ the corresponding natural epimorphism. Given any $k \in \mathbb{N}_0$ we see that

$$\mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X, f); \underbrace{{}_\Phi\mathbb{Q}(t)}) = 0.$$

Remark. The calculation of the Mapping Torus–Vanishing-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.9 is initially a little disappointing, but later in the Mapping Torus–Reidemeister Torsion-Proposition **??** this will turn out to be a blessing in disguise. □

[3548]This condition is for example satisfied if X is a compact topological manifold or a finite CW-complex.

In the proof of the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.9 we will need the following lemma which can be quite useful:

Lemma 323.10. (Switch-to-Localization Lemma) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Let S be a ring and let $R \subseteq S$ be a subring. Finally let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(n, R)$ be a representation. Note that we can also view φ as a representation $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(n, S)$. If S is flat[3549] over R[3550], then for any subspace $A \subseteq X$ we have a natural isomorphism

$$\begin{array}{rcl} S \otimes_R \mathrm{H}_*^{\mathrm{tw}}(X, A; {}_\varphi R^{1\times n}) & \to & \mathrm{H}_*^{\mathrm{tw}}(X, A; {}_\varphi S^{1\times n}) \\ s \otimes (v \otimes \sigma) & \mapsto & (s \cdot v) \otimes \sigma \end{array}$$

of S-left modules.

Proof of Lemma 323.10. We write $\pi := \pi_1(X, x_0)$ and we denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. As always we write $\widetilde{A} := p^{-1}(A)$. First note that it follows from the Free Module–Tensor-with-Ring Lemma 159.20 that the map

$$\begin{array}{rcl} S \otimes_R R^{1\times n} & \to & S^{1\times n} \\ s \otimes v & \mapsto & s \cdot v \end{array}$$

is an isomorphism of S-left modules. It follows easily from Convention 320.9 of the $\mathbb{Z}[\pi]$-left module structures that the above map is even an isomorphism of $(S, \mathbb{Z}[\pi])$-left left modules. Next we consider the map

$$\begin{array}{rcl} S \otimes_R \overbrace{\overline{{}_\varphi R^{1\times n}} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_*^{\mathrm{tw}}(X,A;{}_\varphi R^{1\times n})} & \to & \overbrace{\overline{{}_\varphi S^{1\times n}} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_*^{\mathrm{tw}}(X,A;{}_\varphi S^{1\times n})} \\ s \otimes (v \otimes \sigma) & \mapsto & (s \cdot v) \otimes \sigma. \end{array}$$

It follows from the above discussion together with the Tensor Commutative-Associative Lemma 312.11 (3) that this map is an isomorphism of S-left modules. The desired statement now follows easily from the Exact Functor–Homology Lemma 314.13 and our hypothesis that S is flat over R. ∎

Proof of Proposition 323.9. Let (X, x) be a pointed topological space that is path-connected and locally contractible and such that all homology modules $\mathrm{H}_k(X; \mathbb{Q})$ are finite dimensional. Furthermore let $f\colon X \to X$ be a homeomorphism. As usual we denote by $\Phi\colon \pi_1(\mathrm{Tor}(X, f), [(x, 0)]) \to \langle t \rangle$ the natural epimorphism. Given $k \in \mathbb{N}_0$ we see that

$$\begin{array}{rcl} \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X, f); \underbrace{{}_\Phi \mathbb{Q}(t)}) & & \\ & \overset{\downarrow}{\cong} & \mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X, f); {}_\Phi \mathbb{Q}[t^{\pm 1}]) \\ & \overset{\uparrow}{\cong} & \mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} (\mathrm{H}_k(X; \mathbb{Q}), t = f_*). \end{array}$$

follows from the Switch-to-Localization Lemma 323.10, since $\mathbb{Q}(t)$ is a localization of $\mathbb{Q}[t^{\pm 1}]$, and thus flat over $\mathbb{Q}[t^{\pm 1}]$

by the Mapping Torus–$\mathrm{H}_*^{\mathrm{tw}}$ Proposition 323.8

[3549] Recall that according to Definition 314.16 this means that tensoring with S over R is exact.

[3549] By the Tensor-with-Localization Proposition 314.18 we know that any localization of R is flat over R.

By hypothesis $\mathrm{H}_k(X;\mathbb{Q})$ is finite-dimensional. Thus the desired statement follows from the following elementary claim:[3550]

Claim. Let V be a $\mathbb{Q}[t^{\pm 1}]$-module that is finite-dimensional as a $\mathbb{Q}$-vector space. Then $\mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} V = 0$.

Proof. Since V is in particular finitely generated over the PID $\mathbb{Q}[t^{\pm 1}]$ we obtain from the PID–Module-Classification Theorem 309.36 that V is isomorphic to the direct sum of cyclic modules $\mathbb{Q}[t^{\pm 1}]/p_i(t) \cdot \mathbb{Q}[t^{\pm 1}]$, $i = 1, \dots, k$, where $p_1(t), \dots, p_k(t) \in \mathbb{Q}[t^{\pm 1}] \setminus \{0\}$. We now see that

$$\begin{array}{rcccl} \mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} V & \cong & \mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} \bigoplus_{i=1}^{k} \mathbb{Q}[t^{\pm 1}]/p_i(t) \cdot \mathbb{Q}[t^{\pm 1}] & & \\ & \cong & \bigoplus_{i=1}^{k} (\mathbb{Q}(t) \otimes_{\mathbb{Q}[t^{\pm 1}]} \mathbb{Q}[t^{\pm 1}]/p_i(t) \cdot \mathbb{Q}[t^{\pm 1}]) & = & 0. \\ & \uparrow & & \uparrow & \\ & \text{by the Tensor–Direct Sum Lemma 312.15} & & \text{see Exercise 312.12, since the } p_i(t) \neq 0 & \end{array}$$

■

We conclude this section with the following variation on the above results:

Proposition 323.11. (Mapping Torus–$\mathbb{C}$-Twisted Homology Proposition) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. We assume that each homology group $\mathrm{H}_k(X;\mathbb{C})$ is finite-dimensional. Furthermore let $f\colon X \to X$ be a homeomorphism. As usual we denote by $\Phi\colon \pi_1(\mathrm{Tor}(X,f), [(x_0,0)]) \to \langle t \rangle$ the natural epimorphism from Definition 323.6. Given $z \in \mathbb{C} \setminus \{0\}$ we consider the representation $\Phi_z\colon \pi_1(\mathrm{Tor}(X,f), [(x,0)]) \to \mathrm{GL}(1,\mathbb{C})$ that is given by the composition of Φ with the homomorphism $\langle t \rangle \xrightarrow{t \to (z)} \mathrm{GL}(1,\mathbb{C})$. Given any $k \in \mathbb{N}_0$ we have a short exact sequence

$$\begin{array}{rcl} 0 \;\to\; \{v \in \mathrm{H}_k(X;\mathbb{C}) \,|\, f(v) = z \cdot v\} & \to & \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X,f);{}_{\Phi_z}\mathbb{C}) \\ & \to & \{v \in \mathrm{H}_{k-1}(X;\mathbb{C}) \,|\, f(v) = z \cdot v\} \;\to\; 0 \end{array}$$

of $\mathbb{C}$-vector spaces.

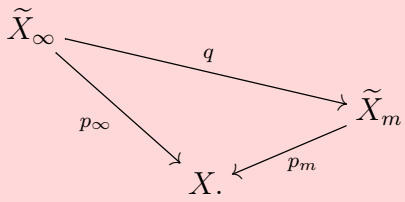
Proof. We will prove this proposition in Exercise 323.16. ■

323.6. **Homology of cyclic coverings.** As an application of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 we now prove the following proposition, which is essentially the same as the Infinite–Finite Cyclic Covering Proposition 289.23:

Proposition 323.12. (Infinite–Finite Cyclic Covering Proposition) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $\Phi\colon \pi_1(X,x_0) \to \langle t \rangle$ be an epimorphism and let $m \in \mathbb{N}$. In the following we denote by $\Phi_m\colon \pi_1(X,x_0) \to \langle t \,|\, t^m \rangle$ the epimorphism that is given by the composition of Φ with the natural projection $\langle t \rangle \to \langle t \,|\, t^m \rangle$. We introduce the following notation:

- Let $p_\infty\colon \widetilde{X}_\infty \to X$ be the covering corresponding to $\ker(\Phi\colon \pi_1(X,x_0) \to \langle t \rangle)$.
- Let $p_m\colon \widetilde{X}_m \to X$ be the covering corresponding to $\ker(\Phi_m\colon \pi_1(X,x_0) \to \langle t \,|\, t^m \rangle)$.
- We denote by $q\colon \widetilde{X}_\infty \to \widetilde{X}_m$ the natural covering with $p_m \circ q = p_\infty$ that is given by the Covering–Covering Theorem 319.2. In summary, we have the following commutative diagram of coverings:

[3550] Note that the claim is just the content of Exercise 312.13.

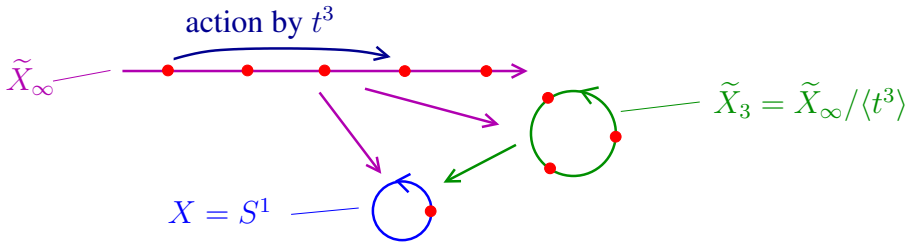


• We equip $\widetilde{X}_\infty$, and thus also $\mathrm{H}_k(\widetilde{X}_\infty)$, with the $\langle t\rangle$-action from Convention 286.3.

For every commutative ring S there exists a long exact sequence of the form[3551]

$$\ldots \to \mathrm{H}_k(\widetilde{X}_\infty; S) \xrightarrow{\cdot(t^m-1)} \mathrm{H}_k(\widetilde{X}_\infty; S) \xrightarrow{q_*} \mathrm{H}_k(\widetilde{X}_m; S) \to \mathrm{H}_{k-1}(\widetilde{X}_\infty; S) \to \ldots$$

Furthermore, for any subspace A there exists a corresponding long exact sequence of relative homology groups.



Example.

(1) A typical application of the Infinite–Finite Cyclic Covering Proposition 323.12 is provided by mapping tori together with the natural epimorphism $\pi_1(\mathrm{Tor}(X, f)) \to \langle t\rangle$.

(2) We refer to Section 289.7 for applications of the Infinite–Finite Cyclic Covering Proposition 323.12 in the setting of knot complements. □

Proof. To simplify the notation we only deal with the case that $A = \varnothing$. In this setting we consider the following diagram:

$$\begin{array}{ccccccccc}
\to \mathrm{H}_k^{\mathrm{tw}}(X; S[t^{\pm1}]) & \xrightarrow{\cdot(t^m-1)} & \mathrm{H}_k^{\mathrm{tw}}(X; S[t^{\pm1}]) & \xrightarrow{\pi_*} & \mathrm{H}_k^{\mathrm{tw}}(X; S[t^{\pm1}]/(t^m-1)) & \xrightarrow{\beta_k} & \mathrm{H}_k^{\mathrm{tw}}(X; S[t^{\pm1}]) \to \\
\Omega\downarrow\cong & & \Omega\downarrow\cong & & \Omega\downarrow\cong & & \Omega\downarrow\cong & \\
\to \mathrm{H}_k(\widetilde{X}_\infty; S) & \xrightarrow{\cdot(t^m-1)} & \mathrm{H}_k(\widetilde{X}_\infty; S) & \xrightarrow{q_*} & \mathrm{H}_k(\widetilde{X}_m; S) & \xrightarrow{=:\partial_k} & \mathrm{H}_{k-1}(\widetilde{X}_\infty; S) \to &
\end{array}$$

The proposition follows from the following clarifications and observations:

(1) The top horizontal sequence is the long exact sequence from the Twisted Bockstein Lemma 321.15 corresponding to the short exact sequence

$$0 \to S[t^{\pm1}] \xrightarrow{\cdot(t^m-1)} S[t^{\pm1}] \xrightarrow{\pi} S[t^{\pm1}]/(t^m-1) \to 0$$

of $S[t^{\pm1}]$-modules.

(2) The vertical isomorphisms Ω are provided by the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5.

(3) It follows from the naturality of the vertical isomorphisms (see Footnote 3542) that the first and the second square commute.

(4) We define the homomorphisms ∂_k such that the third square commutes. ■

[3551] In fact, more generally, for any subspace $A \subseteq X$ there exists a long exact sequence of the form

$$\to \mathrm{H}_k(\widetilde{X}_\infty, p_\infty^{-1}(A); S) \xrightarrow{\cdot(t^m-1)} \mathrm{H}_k(\widetilde{X}_\infty, p_\infty^{-1}(A); S) \xrightarrow{q_*} \mathrm{H}_k(\widetilde{X}_m, p_m^{-1}(A); S) \to \mathrm{H}_{k-1}(\widetilde{X}_\infty, p_\infty^{-1}(A); S) \to$$

323.7. **Twisted cohomology and cohomology of finite-index coverings.** The Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 is very appealing since for many twistings it gives us a very nice intuition for what homology with twisted coefficients means. Note though that the analogous statement for *cohomology* with twisted coefficients does in general not hold:

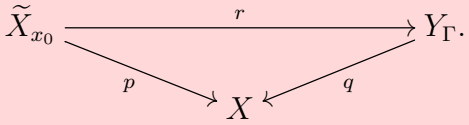
Example. We consider $X = S^1$ with the base point $x_0 = 1 \in S^1$. We write $\pi := \pi_1(S^1, 1)$. Then we have the following isomorphisms of abelian groups:

$$\mathrm{H}^0_{\mathrm{tw}}(S^1; \mathbb{Z}[\pi]) \underset{\text{follows from the Twisted } \mathrm{H}_0\text{-Proposition 323.3}}{\overset{\cong}{\uparrow}} \mathbb{Z}[\pi]^\pi \underset{\text{see the discussion on page 6190 here we use that } \pi \text{ is infinite}}{\overset{=}{\uparrow}} 0 \neq \mathbb{Z} \cong \mathrm{H}^0(\mathbb{R}; \mathbb{Z}) = \mathrm{H}^0(\text{universal covering of } S^1; \mathbb{Z}).$$

Later, in Exercises 329.2 and 329.5 we will see more examples where the twisted cohomology modules $\mathrm{H}^k_{\mathrm{tw}}(X; \mathbb{Z}[\Gamma\backslash\pi])$ differ dramatically from the cohomology groups of the covering of X corresponding of the subgroup Γ. □

It turns out that this nasty issue arises only for subgroups of *infinite index*. In fact for *finite index* normal subgroups we do have the following cohomological version of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5:

Proposition 323.13. (Finite Covering-$\mathrm{H}^*_{\mathrm{tw}}$-Proposition) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. We write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering which we equip with the natural base point $\widetilde{x}_0$. Let $A \subseteq X$. We write $\widetilde{A} := p^{-1}(A)$. Furthermore let $\Gamma \subseteq \pi$ be a subgroup and let S be a *commutative* ring. We equip the free S-module $S[\Gamma\backslash\pi]$ with the $(S, \mathbb{Z}[\pi])$-left left module structure from Notation 323.4. Let $q\colon (Y_\Gamma, y_0) \to (X, x_0)$ be a covering of pointed topological spaces with $q_*(\pi_1(Y_\Gamma, y_0)) = \Gamma$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (Y_\Gamma, y_0)$ such that the following diagram commutes:



Finally we write $A_\Gamma := q^{-1}(A)$.

(1) (a) The maps

$$\underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}\left(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), S[\Gamma\backslash\pi]\right) \xrightarrow{\Phi} \underset{\mathbb{Z}}{\mathrm{Hom}}(\mathrm{C}_k(Y_\Gamma, A_\Gamma), S)$$

$$\gamma \mapsto \begin{pmatrix} \mathrm{C}_k(Y_\Gamma, A_\Gamma) & \to & S \\ [\sigma\colon \Delta^k \to Y_\Gamma] & \mapsto & \begin{array}{c}\text{pick a lift } \widetilde{\sigma}\colon \Delta^k \to \widetilde{X}_{x_0} \text{ and} \\ \text{take the } \Gamma = \Gamma e\text{-coefficient} \\ \text{of } \gamma(\widetilde{\sigma}) \in S[\Gamma\backslash\pi]\end{array} \end{pmatrix}$$

are well-defined S-module homomorphisms and they form a cochain map.

(b) If $\Gamma \subseteq \pi$ is a *normal* subgroup *of finite index*, then the maps[3552]

$$\begin{array}{ccc} \underset{\mathbb{Z}}{\mathrm{Hom}}(\mathrm{C}_k(Y_\Gamma, A_\Gamma), S) & \xrightarrow{\Psi} & \underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}), S[\Gamma\backslash\pi]) \\ \gamma & \mapsto & \left(\begin{array}{ccc} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A}) & \to & S[\Gamma\backslash\pi] \\ [\sigma\colon \Delta^k \to \widetilde{X}_{x_0}] & \mapsto & \underbrace{\sum_{\Gamma g \in \Gamma\backslash\pi} \gamma(\Gamma g \cdot (\Delta^k \xrightarrow{r\circ\sigma} Y_\Gamma)) \cdot \Gamma g}_{\in S[\Gamma\backslash\pi] \text{ since } \Gamma \text{ is a subgroup of finite index}} \end{array} \right) \end{array}$$

are well-defined S-module homomorphisms and they form a cochain map.

If Γ is a *normal* subgroup of *finite index*, then the maps Φ and Ψ are both defined and they are inverses of one another. In particular we see that in this setting the maps Φ induce for each $k \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^k_{\mathrm{tw}}(X, A; S[\Gamma\backslash\pi]) \;\to\; \mathrm{H}^k(Y_\Gamma, A_\Gamma; S)$$

of S-modules.

(2) We again suppose that Γ is a *normal* subgroup of π of *finite* index. Thus we can consider the group $\Gamma\backslash\pi$ and the group ring $S[\Gamma\backslash\pi]$. We introduce two $S[\Gamma\backslash\pi]$-left module structures:

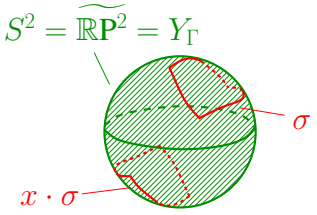
- As in Notation 323.4 we see that $S[\Gamma\backslash\pi]$-becomes an $(S[\Gamma\backslash\pi], \mathbb{Z}[\pi])$-left left module. In particular $\mathrm{H}^k_{\mathrm{tw}}(X; S[\Gamma\backslash\pi])$ becomes an $S[\Gamma\backslash\pi]$-left module.
- Since Γ is a normal subgroup we can equip Y_Γ with the $\Gamma\backslash\pi$-left action given by the Deck Transformation Group Proposition 319.12. This allows us to view $\mathrm{H}^k(Y_\Gamma, A_\Gamma; S)$ as an $S[\Gamma\backslash\pi]$-left module by setting $\Gamma g \cdot \varphi := (\Gamma g^{-1})^*(\varphi)$.[3553] [3554]

With these two conventions the above maps Φ and Ψ are $S[\Gamma\backslash\pi]$-left module isomorphisms.

Example.

(1) The case $\Gamma = \pi$ of the Finite Covering-$\mathrm{H}^*_{\mathrm{tw}}$-Proposition 323.13 is essentially equivalent to the cohomological content of the Twisted–Trivial Coefficients Lemma 320.13.

(2) Let $n \in \mathbb{N}_{\geq 2}$. We consider the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$. By the discussion on page 1799 we know that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z} =_{\mathrm{i}} \langle x \,|\, x^2 \rangle$.



given $[\gamma] \in \mathrm{H}^2(S^2; S) = \mathrm{H}^2(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_2(S^2), S))$ we have
$\Psi([\gamma]) \in \mathrm{H}^2_{\mathrm{tw}}(X; S[\Gamma\backslash\pi]) = \mathrm{H}^2\Big(\underset{\mathbb{Z}[\langle x|x^2\rangle]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_2(\widetilde{\mathbb{R}\mathrm{P}^2}), S[\langle x|x^2\rangle]\Big)$
where $\Psi(\gamma)(\sigma) = 1 \cdot \gamma(\sigma) + x \cdot \gamma(x \cdot \sigma) \;\in\; S[\langle x \,|\, x^2]\rangle]$

[3552] Since $\Gamma \subseteq \pi$ is normal we obtain from the Deck Transformation Group Proposition 319.12 a $\Gamma\backslash\pi$-left action on Y_Γ. Thus it makes sense to consider the singular simplex $\Gamma g \cdot (r \circ \sigma\colon \Delta^k \to Y_\Gamma)$.

[3553] Note that for a cochain $\varphi \in \mathrm{C}^k(Y_\Gamma, A_\Gamma; S)$ the cochain $(\Gamma g^{-1}\varphi)^*$ is given by $(\Gamma g^{-1}\varphi)^*(\sigma) = \varphi(\Gamma g^{-1} \cdot \sigma)$.

[3554] Since cohomology is contravariant this does indeed define a $S[\Gamma\backslash\pi]$-left module structure on $\mathrm{H}^k_{\mathrm{tw}}(Y_\Gamma, A_\Gamma; S)$.

We now see that we have the following isomorphisms of $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-modules:[3555]

$$\begin{array}{ccccc} \mathrm{H}^k_{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]) & \cong & \mathrm{H}^k(S^n;\mathbb{Z}) & \cong & \begin{cases} (\mathbb{Z}, x=(-1)^{n+1}), & \text{if } k=n, \\ 0, & \text{if } k\neq 0,n, \\ (\mathbb{Z}, x=1), & \text{if } k=0. \end{cases} \\ \uparrow & & \uparrow & & \\ \text{by the Finite Covering-}\mathrm{H}^*_{\mathrm{tw}}\text{ Proposition 323.13} & & \text{follows from various standard facts in particular for } k=n \text{ we use the Dual Fundamental Class–Degree Lemma 201.28} & & \end{array}$$

We will consider more examples in Section 323.8. □

Proof. In the following, to simplify the notation, we will just assume that $A=\varnothing$.

(1) As with so many of these propositions, as long as the definitions are right, the proofs are rather dull and essentially elementary. Thus we leave the verification of most aspects of (1a) and (1b) to the reader. But for peace of mind let us verify that the map $\Psi(\gamma)\colon \mathrm{C}_k(\widetilde{X},\widetilde{A}) \to S[\Gamma\backslash\pi]$ in (1b) is indeed a homomorphism of $\mathbb{Z}[\pi]$-left modules. So we need to verify the following claim:

Claim 1. We now assume that $\Gamma \subseteq \pi$ is a normal subgroup of finite index. For any $\gamma \in \mathrm{Hom}_S(\mathrm{C}_k(Y_\Gamma), S)$, any $h \in \pi$ and any chain $\sigma \in \mathrm{C}_k(\widetilde{X}_{x_0})$ we have the equality $\Psi(\gamma)(h\cdot\sigma) = h\cdot\Psi(\gamma)(\sigma)$.

Proof. The claim follows from the following calculation in $S[\Gamma\backslash\pi]$:

$$\begin{array}{rcll} \Psi(\gamma)(h\cdot\sigma) & \overset{\text{by definition of } \Psi}{=} & \sum_{\Gamma g\in\Gamma\backslash\pi} \gamma(\Gamma g\cdot(r\circ h\cdot\sigma))\cdot\Gamma g & \overset{\text{by the naturality statement of the Deck Transformation Group Proposition 319.12 we have for } x\in\widetilde{X} \text{ that } r(h\cdot x)=\Gamma h\cdot r(x)}{=} \sum_{\Gamma g\in\Gamma\backslash\pi} \gamma(\Gamma gh\cdot(r\circ\sigma))\cdot\Gamma g \\ & = & \sum_{\Gamma g\in\Gamma\backslash\pi} \gamma(\Gamma gh\cdot(r\circ\sigma))\cdot\Gamma gh\cdot h^{-1} & = \sum_{\Gamma k\in\Gamma\backslash\pi} \gamma(\Gamma k\cdot(r\circ\sigma))\cdot\Gamma k\cdot h^{-1} \\ & & & \uparrow \text{ since } \Gamma g\mapsto\Gamma g\cdot h \text{ is a bijection of } \Gamma\backslash\pi \\ & = & \Psi(\gamma)(\sigma)\cdot h^{-1} & = h\cdot\Psi(\gamma)(\sigma). \\ & \uparrow \text{ by definition of } \Psi & & \uparrow \text{ by Notation 323.4 of the } \mathbb{Z}[\pi]\text{-left module structure on } S[\Gamma\backslash\pi] \end{array}$$ ⊞

Next we turn to the proof of the last statement of (1):

Claim 2. Let Γ be a normal subgroup of *finite* index. Then $\Psi\circ\Phi = \mathrm{id}$ and $\Phi\circ\Psi = \mathrm{id}$.

[3555]To the right we use the hopefully suggestive notation where $(\mathbb{Z}, x=\epsilon)$ means the $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-module where the underlying abelian group is $\mathbb{Z}$ and where x acts by multiplication by ϵ.

Proof. Let us verify that $\Psi \circ \Phi = \mathrm{id}$. Thus let $\mu \in \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}), S[\Gamma\backslash\pi])$ and let $\sigma \in \mathrm{C}_k(\widetilde{X}_{x_0})$. Then we have the following equalities in $S[\Gamma\backslash\pi]$:

$$\begin{aligned}((\Psi\circ\Phi)(\mu))(\sigma) &\overset{\text{by definition of }\Psi}{=} \sum_{\Gamma g\in\Gamma\backslash\pi} \Phi(\mu)(\Gamma g\cdot(r\circ\sigma\colon \Delta^k\to Y_\Gamma))\cdot\Gamma g\\ &= \sum_{\Gamma g\in\Gamma\backslash\pi} (\Gamma e\text{-coefficient of } \mu(g\cdot\sigma))\cdot\Gamma g\\ &\overset{\substack{\text{by definition of }\Phi\text{ and since it follows from the naturality statement}\\ \text{of the Deck Transformation Group Proposition 319.12 that}\\ g\cdot\sigma\text{ is a lift of }\Gamma g\cdot(r\circ\sigma)\colon\Delta^k\to Y_\Gamma}}{=} \sum_{\Gamma g\in\Gamma\backslash\pi} (\Gamma g\text{-coefficient of } \mu(\sigma))\cdot\Gamma g \qquad \overset{\substack{\text{follows from elementary}\\ \text{algebra}}}{=} \mu(\sigma).\end{aligned}$$

since μ is $\mathbb{Z}[\pi]$-left linear we obtain from Notation 323.4 that $\mu(g\cdot\sigma)=\mu(\sigma)\cdot g^{-1}$

The verification that $\Phi\circ\Psi=\mathrm{id}$ is very similar and it is left to the reader. ⊞

(2) Again we assume that $\Gamma\subseteq\pi$ is a normal subgroup of finite index. Since Φ and Ψ are inverses of one another it suffices to show that Ψ is an $S[\Gamma\backslash\pi]$-left module homomorphism. Thus we need to prove the following claim:

Claim 3. Given any $\gamma\in\mathrm{Hom}_S(\mathrm{C}_k(Y_\Gamma,A_\Gamma),S)$ and given any $\Gamma h\in\Gamma\backslash\pi$ we have the equality
$$\Psi(\Gamma h\cdot\gamma)=\Gamma h\cdot\Psi(\gamma)$$
in $\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k(\widetilde{X}_{x_0}),S[\Gamma\backslash\pi])$.

Proof. Let $\sigma\in\mathrm{C}_k(\widetilde{X}_{x_0})$. We see that we have the following equalities in $S[\Gamma\backslash\pi]$:

$$\begin{aligned}\underbrace{\Psi(\Gamma h\cdot\gamma)(\sigma)}_{=\Psi((\Gamma h^{-1})^*\gamma)(\sigma)} &\overset{\text{by Definition of }\Psi}{=} \sum_{\Gamma g\in\Gamma\backslash\pi}((\Gamma h^{-1})^*\gamma)(\Gamma g\cdot(q\circ\sigma))\cdot\Gamma g \overset{\text{by definition of }(\Gamma h^{-1})^*\gamma}{=} \sum_{\Gamma g\in\Gamma\backslash\pi}\gamma(\Gamma h^{-1}g\cdot(q\circ\sigma))\cdot\Gamma g\\ &= \sum_{\Gamma g\in\Gamma\backslash\pi}\gamma(\Gamma h^{-1}g\cdot(q\circ\sigma))\cdot\Gamma h\cdot h^{-1}g \overset{\substack{\text{substitution } k=h^{-1}g\text{, we also use that}\\ \Gamma h\text{ commutes with elements of }S}}{=} \Gamma h\cdot\sum_{\Gamma k\in\Gamma\backslash\pi}\gamma(\Gamma k\cdot(q\circ\sigma))\cdot\Gamma k\\ &\overset{\text{definition of }\Psi}{=} \Gamma h\cdot\Psi(\gamma)(\sigma).\end{aligned}$$

■

323.8. **Twisted (co-) homology groups of lens spaces.** We turn to our old nemesis, the lens spaces $L(p,q)=S^3/(\mathbb{Z}/p\mathbb{Z})$ from Definition 91.4 and Definition 319.4. So far these have have evaded all our attempts at classifying them. Recall that in the Lens Space-π_1-Proposition 93.2 we saw that $\pi_1(L(p,q))\cong\mathbb{Z}/p\mathbb{Z}$. This shows that $\pi_1(L(p,q))$ has at least some non-trivial representations. But since $\pi_1(L(p,q))$ is abelian we know by the Finite Abelian Group–Representation Proposition 310.27 that every finite-dimensional complex representation is the direct sum of one-dimensional complex representations. Thus by the Direct Sum-of-Coefficients Lemma 321.14 we only need to understand invariants corresponding to one-dimensional complex representations.

For trivial one-dimensional representations we know by the Twisted–Trivial Coefficients Lemma 320.13 that we only recover classical invariants. Thus in the following lemma we restrict ourselves to non-trivial one-dimensional complex representations:

Proposition 323.14. (Lens Space–Twisted Invariants Proposition) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We consider the corresponding lens space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$. Let $\rho\colon \pi_1(L(p,q)) \to \mathrm{GL}(1,\mathbb{C})$ be a non-trivial representation. We consider the corresponding $\mathbb{Z}[\pi_1(L(p,q))]$-module ${}_\rho\mathbb{C}$ from Notation 320.8.

(1) For every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k^{\mathrm{tw}}(L(p,q); {}_\rho\mathbb{C}) = 0$.

(2) For every $k \in \mathbb{N}_0$ we have $\mathrm{H}^k_{\mathrm{tw}}(L(p,q); {}_\rho\mathbb{C}) = 0$.

Remark.

(1) The Lens Space–Twisted Invariants Proposition 323.14 is at first glance quite disappointing since it shows that (co-) homology with twisted complex coefficients cannot be used to distinguish lens spaces. But in **??** we will see, counterintuitively, that this calculation is in fact a big step towards solving the conundrum of the classification of lens spaces.

(2) The eternal optimist might hope to use twisted (co-) homology with other coefficients to distinguish lens spaces. But by the Homotopy Equivalence-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 322.1 we know that the (co-) homology groups with twisted coefficients that can occur, only depend on the homotopy type of lens spaces. But by the Lens Space-Homotopy Classification Theorem **??** there exist lens spaces that are homotopy equivalent albeit not homeomorphic. □

Proof. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick a base point $x_0 \in L(p,q)$. By the discussion on page 6142 we can pick an identification $\pi_1(L(p,q), x_0) =_{\mathrm{i}} \mathbb{Z}/p\mathbb{Z}$. Given $z \in \mathbb{C}$ with $z^p = 1$ we consider the representation

$$\begin{array}{rccc} \rho_z\colon & \mathbb{Z}/p\mathbb{Z} & \to & \mathrm{U}(1) \\ & [k] & \mapsto & z^k. \end{array}$$

By the Cyclic Group–Representation Proposition 310.25 we know the following two facts:

(a) The left regular complex representation $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$ of the cyclic group $\pi_1(L(p,q), x_0) = \mathbb{Z}/p\mathbb{Z}$ which we introduced in Definition 310.9 is isomorphic to the direct sum of the representations $\rho_{\exp(2\pi \mathrm{i} k/p)}$, $k = 0, \ldots, p-1$.

(b) Every non-trivial irreducible representation of $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to one of the ρ_z with $z \neq 1$.

We now turn to the actual calculations:

(1) We first deal with the twisted homology modules. We see that

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(L(p,q);{}_{\mathrm{tr}}\mathbb{C}) \oplus \bigoplus_{\substack{z^p=1,\\ z\neq 1}} \mathrm{H}_k^{\mathrm{tw}}(L(p,q);{}_{\rho_z}\mathbb{C}) & \overset{\text{follows from (1) and the Direct Sum-of-Coefficients Lemma 321.14 together, with the observation that we have } {}_{\mathrm{tr}}\mathbb{C} = \mathbb{C}_{\rho_1}}{\cong} & \mathrm{H}_k^{\mathrm{tw}}(L(p,q);\bigoplus_{z^p=1}{}_{\rho_z}\mathbb{C}) \\ & \overset{\text{follows from (a)}}{\cong} & \mathrm{H}_k^{\mathrm{tw}}(L(p,q);\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]) \overset{\text{by the Covering-H}_*^{\mathrm{tw}}\text{-Proposition 323.5, since } S^3 \text{ is a universal covering of } L(p,q)}{\cong} \mathrm{H}_k(S^3;\mathbb{C}). \end{array}$$

Note that it follows from (b) and the above calculation that it suffices to show that $\mathrm{H}_k(L(p,q);{}_{\mathrm{tr}}\mathbb{C}) \cong \mathrm{H}_k(S^3;\mathbb{C})$ and that both are finite-dimensional complex vector spaces. Indeed we have

$$\mathrm{H}_k^{\mathrm{tw}}(L(p,q);{}_{\mathrm{tr}}\mathbb{C}) \overset{\text{by the Twisted-with-Trivial Coefficients Lemma 320.13}}{\cong} \mathrm{H}_k(L(p,q);\mathbb{C}) \overset{\text{follows from the Universal Coefficient Theorem 162.16 together with the Tor-Properties Lemma 162.13 and the calculations of } \mathrm{H}_*(L(p,q)) \text{ and } \mathrm{H}_*(S^3) \text{ in the Sphere-H}_*\text{-Proposition 149.4 and the Lens Space-H}_*\text{-Lemma 183.21}}{\cong} \begin{cases} \mathbb{C}, & \text{if } k=0,3 \\ 0, & \text{else} \end{cases} \cong \mathrm{H}_k(S^3;\mathbb{C}).$$

(2) The argument for calculating the twisted cohomology modules is very similar, the main difference is that we need to replace the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 by the Finite Covering-$\mathrm{H}_{\mathrm{tw}}^*$-Proposition 323.13. Fortunately we can do so, since $S^3 \to L(p,q)$ is a *finite* covering. ∎

323.9. **The transfer map.** We start out by recalling Definition 165.1:

Definition 323.15. Let $p\colon Y \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected there exist precisely k different lifts of σ to Y, in other words, there exist precisely k different singular simplices $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to Y$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \dots, k$. We write

$$p^*(\sigma) \coloneqq \sum_{i=1}^{k} \widetilde{\sigma}_i \in \mathrm{C}_n(Y).$$

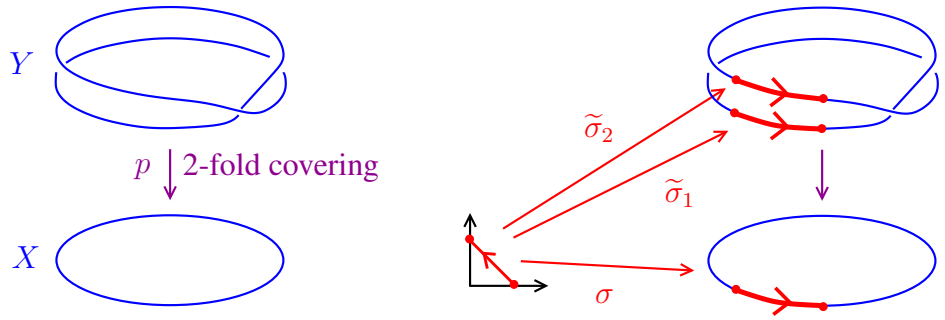
Given any commutative ring R we refer to the R-module homomorphism

$$\begin{array}{rcl} p^*\colon \mathrm{C}_n(X;R) & \to & \mathrm{C}_n(Y;R) \\ \sum_{i=1}^{m} \sigma_i \otimes r_i & \mapsto & \sum_{i=1}^{m} p^*(\sigma_i) \otimes r_i \end{array}$$

as the transfer map of the covering. In the Transfer Map Lemma 165.2 we showed that the transfer maps form a natural chain map. Thus, as in Definition 165.3 we see that p^* induces natural R-module homomorphism

$$\begin{array}{rcl} p^*\colon \mathrm{H}_*(X;R) & \to & \mathrm{H}_*(Y;R) \\ p_*\colon \mathrm{H}^*(Y;R) & \to & \mathrm{H}^*(X;R) \end{array}$$

which we also call transfer maps.[3556]



We can now reinterpret the above transfer map on homology in terms of twisted (co-) homology:

Proposition 323.16. (Transfer Map-H_*^{tw}-Proposition) Let $p\colon (Y,y_0)\to(X,x_0)$ be a *finite* covering of pointed topological spaces that are path-connected and locally contractible. We write $\pi := \pi_1(X,x_0)$ and $\Gamma := p_*(\pi_1(Y,y_0))$. Furthermore let R be a commutative ring.

(1) The map

$$\begin{array}{rccl} \tau\colon & {}_{\mathrm{tr}}R & \to & R[\Gamma\backslash\pi] \\ & r & \mapsto & \sum\limits_{\Gamma g\in\Gamma\backslash\pi} r\cdot\Gamma g \end{array}$$

is a homomorphism of $(R,\mathbb{Z}[\pi])$-left left modules.

(2) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X;{}_{\mathrm{tr}}R) & \xrightarrow[\text{from Definition 321.12}]{\tau_*} & \mathrm{H}_k^{\mathrm{tw}}(X;R[\Gamma\backslash\Gamma]) \\ \cong\downarrow\Omega & & \cong\downarrow\Omega \\ \mathrm{H}_k(X;R) & \xrightarrow{p^*} & \mathrm{H}_k(Y;R). \end{array}$$

Here the vertical isomorphisms Ω are given by the Covering-H_*^{tw}-Proposition 323.5. A similar result also holds for (twisted) cohomology.

Proof. We will not really make use of this proposition. Thus we do not feel bad about outsourcing this proposition to Exercise 323.16. ■

We conclude this section with the following proposition, which is essentially the same as the Double Covering-LES-Proposition 165.5:

Proposition 323.17. (Double Covering-LES-Proposition) Let $p\colon (Y,y_0)\to(X,x_0)$ be a 2-fold covering of pointed topological spaces that are both path-connected and locally contractible. There exist natural homomorphisms $\partial_n\colon \mathrm{H}_n(X;\mathbb{F}_2)\to\mathrm{H}_{n-1}(X;\mathbb{F}_2)$ such that the following sequence is exact:

$$\ldots\longrightarrow \mathrm{H}_n(X;\mathbb{F}_2)\xrightarrow{p^*}\mathrm{H}_n(Y;\mathbb{F}_2)\xrightarrow{p_*}\mathrm{H}_n(X;\mathbb{F}_2)\xrightarrow{\partial_n}\mathrm{H}_{n-1}(X;\mathbb{F}_2)\longrightarrow\ldots.$$

[3556]Of course we can also define relative versions of these maps. To simplify the discussion we leave task of generalizing the current discussion to the relative setting to the reader.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}_n^{\mathrm{tw}}(X;{}_{\mathrm{tr}}\mathbb{F}_2) & \xrightarrow{\iota_*} & \mathrm{H}_n^{\mathrm{tw}}(X;\mathbb{F}_2[\langle x \mid x^2\rangle]) & \xrightarrow{\pi_*} & \mathrm{H}_n^{\mathrm{tw}}(X;{}_{\mathrm{tr}}\mathbb{F}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(X;{}_{\mathrm{tr}}\mathbb{F}_2) & \longrightarrow & \dots \\
 & & \Omega\downarrow\cong & & \Omega\downarrow\cong & & \Omega\downarrow\cong & & \Omega\downarrow\cong & & \\
\dots & \longrightarrow & \mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow{p^*} & \mathrm{H}_n(Y;\mathbb{F}_2) & \xrightarrow{p_*} & \mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow{=:\partial_n} & \mathrm{H}_{n-1}(X;\mathbb{F}_2) & \longrightarrow & \dots.
\end{array}$$

The proposition follows from the following clarifications and observations:

(1) First note that one can easily verify that

$$0 \;\to\; {}_{\mathrm{tr}}\mathbb{F}_2 \xrightarrow[=:\iota]{1\mapsto(1+x)} \mathbb{F}_2[\langle x \mid x^2\rangle] \xrightarrow[=:\pi]{1,x\mapsto 1} {}_{\mathrm{tr}}\mathbb{F}_2 \;\to\; 0$$

is a short exact sequence of $\mathbb{Z}[\langle x \mid x^2\rangle]$-modules. The top sequence is the corresponding long exact sequence from the Twisted Bockstein Lemma 321.15.

(2) The vertical isomorphisms Ω are provided by the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5.

(3) It follows from the Transfer Map-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.16 that the first square commutes. Furthermore it follows from the naturality of the vertical isomorphisms (see Footnote 3542) that the second square commutes

(4) We define the bottom connecting homomorphisms ∂_k such that the third square commutes. ∎

Example. Let $n \in \mathbb{N}$. As on page 3171 we will now see that we can use the Double Covering-LES-Proposition 165.5 to calculate the homology groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ just using the following two basic facts:

(a) By the Sphere-H_*-Proposition 160.13 we know that

$$\mathrm{H}_k(S^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k=0 \text{ or } k=n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since $\mathbb{R}\mathrm{P}^n$ is an n-dimensional CW-complex we know from the discussion on page 3081 that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$.

In the following discussion we write $\mathbb{F} = \mathbb{F}_2$ for space reasons. Applying the Double Covering-LES-Proposition 323.17 to the obvious 2-fold covering $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ we obtain the following long exact sequence:

$$\underbrace{\mathrm{H}_{n+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=0 \text{ by (b)}} \xrightarrow{\partial_{n+1}} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F} \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0 \text{ by (a)}} \to$$

$$\dots \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_1(S^n;\mathbb{F})}_{=0 \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_1} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_0(S^n;\mathbb{F})}_{=\mathbb{F} \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \to 0.$$

Now we can make the following observations:

(1) Considering the bottom part of the long exact sequence we see that $\mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$.

(2) If we work our way upward through the long exact sequence we see that the connecting homomorphisms give us isomorphisms

$$\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial_{n-1}} \mathrm{H}_{n-2}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial_{n-2}} \dots \xrightarrow[\cong]{\partial_2} \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial_1} \mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \;\cong\; \mathbb{F}.$$

(3) Now that we know that $\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$ let us look again at the upper end of the long exact sequence:

$$0 \longrightarrow \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial_n} \underbrace{\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0} \longrightarrow$$

It follows easily from staring at this data that the maps $p^*\colon \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \to \mathrm{H}_n(S^n;\mathbb{F})$ and $\partial_n\colon \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \to \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})$ are also isomorphisms.

In particular we have shown that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$ □

323.10. **Appendix: The Borsuk-Ulam Theorem.** Our goal in this appendix is to use the results from the last Section 323.9 to give a proof of the following theorem:

Theorem 47.12. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.

Example. We consider the continuous map

$$\begin{array}{rcl} \text{earth} =_{\mathrm{i}} S^2 & \to & \mathbb{R}^2 \\ x & \mapsto & (\text{temperature at } x, \text{air pressure at } x). \end{array}$$

It follows from the Borsuk-Ulam Theorem 47.12, applied to the above function $S^2 \to \mathbb{R}^2$, that on earth there exists a pair of antipodal points where the temperature and the air pressure agree. □

The proof of the Borsuk-Ulam Theorem 47.12 we will give now is essentially the same as the proof in Appendix 323.10. Before we can get to the proof we should recall Definition 151.3:

Definition 323.18. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be a continuous map. By the Sphere-H_*-Proposition 149.4 we know that $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$. Thus the induced homomorphism $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$ is given by multiplication by some integer, which we call the degree $\deg(f)$ of f.

For the reader's convenience we recall some basic properties of the degree:

Lemma 151.4. (Sphere–Degree Lemma) Let $n \in \mathbb{N}_0$ and let $f, g\colon S^n \to S^n$ be two continuous maps.

(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$.[3557]

The key ingredient in the proof of the Borsuk-Ulam Theorem 47.12 is the following proposition, which is identical to the earlier Odd Map–Odd Degree Proposition 165.6:

Proposition 323.19. (Odd Map–Odd Degree Proposition) Let $f\colon S^n \to S^n$ be a continuous map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.

Proof of the Odd Map–Odd Degree Proposition 323.19. We start out with a very general claim:

Claim. Let $f\colon S^n \to S^n$ be a continuous map. For every $k \in \mathbb{N}$ the induced map

$$f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z}) \;\to\; \mathrm{H}_n(S^n;\mathbb{Z}/k\mathbb{Z})$$

is given by multiplication by $\deg(f)$.

Proof. As on page 3174 one can deduce the claim easily from the Universal Coefficient Theorem 162.16. ⊞

Next let $f\colon S^n \to S^n$ be a continuous map such that we have $f(-x) = -f(x)$ for all $x \in S^n$. We need to show that $\deg(f)$ is odd. By the above claim it suffices to show that $f_*\colon \mathrm{H}_n(S^n;\mathbb{F}_2) \to \mathrm{H}_n(S^n;\mathbb{F}_2)$ is an isomorphism.

We start out with the observation that it follows from the Quotient-to-Quotient Lemma 8.6 and our hypothesis $f(-x) = -f(x)$ that the map $f\colon S^n \to S^n$ descends to a well-defined continuous map

$$\begin{aligned} \overline{f}\colon \mathbb{R}\mathrm{P}^n &\to \mathbb{R}\mathrm{P}^n \\ [x] &\mapsto [f(x)]. \end{aligned}$$

Note that this map gives rise to the following diagram:

$$\begin{array}{ccc} S^n & \xrightarrow{\;p\;} & \mathbb{R}\mathrm{P}^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ S^n & \xrightarrow{\;p\;} & \mathbb{R}\mathrm{P}^n \end{array}$$

of continuous maps. Note that the following statements hold:

(a) The diagram commutes.
(b) $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ is a 2-fold covering.
(c) For each $x \in \mathbb{R}\mathrm{P}^n$ the map $f|_{p^{-1}(x)}\colon p^{-1}(x) \to p^{-1}(\overline{f}(x))$ is a bijection. (Note that here we use that $f(-x) = -f(x)$).

Now we consider the following diagram:

$$\begin{array}{ccccccccccc} \mathrm{H}_n(S^n;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_2} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_1} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} \\ \mathrm{H}_n(S^n;\mathbb{F}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_2} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) & \xrightarrow{\partial_1} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2). \end{array}$$

We make the following clarifications and observations:

(1) The map p^* is the natural transfer map from Definition 323.15. It follows from (a)–(c) above that the naturality statement applies, i.e. the square to the left commutes.
(2) The maps $\partial_n, \dots, \partial_1$ are the natural connecting homomorphisms from the Double Covering-LES-Proposition 323.17. It follows from (a)–(c) above that the naturality statement applies, i.e. the squares to the right commutes.
(3) It follows from the discussion on page 6212 that all the horizontal maps are isomorphisms.
(4) It follows from the obvious generalization of the H_0-Proposition 143.25, see also the discussion on page 3079, that the right vertical map is an isomorphism.

It follows from the above that the left vertical map is an isomorphism. But that is exactly what we had wanted to prove. ■

Now we finally turn to the proof of the Borsuk-Ulam Theorem 47.12.

Proof of the Borsuk-Ulam Theorem 47.12. Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. We suppose that there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. Put differently, we suppose that for every $x \in S^n$ we have $f(x) \neq f(-x)$. We consider the composition of the following two continuous maps

$$\begin{array}{rccccc} \Phi\colon S^{n-1} \times [0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\ & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}}_{\text{defined since } f(x) \neq f(-x)} \end{array}$$

Now we see that

$$\text{odd number} \overset{\text{by definition we have } \Phi(-x,0) = -\Phi(x,0), \text{ thus it follows from the Odd Map–Odd Degree Proposition 323.19 that } \deg(\Phi_0) \text{ is odd}}{=} \deg(\Phi_0) \underset{\text{by the Sphere–Degree Lemma 151.4 (3) since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{=} \deg(\Phi_1) \underset{\text{by definition of } \Phi_1}{=} \deg(\text{constant map}) \overset{\text{by the Sphere–Degree Lemma 151.4 (2)}}{=} 0.$$

We have thus obtained a contradiction. ∎

323.11. Exercises for Chapter 323.

Exercise 323.1. Let π be a group. We equip $M = \mathbb{Z}[\pi]$ with the obvious $\mathbb{Z}[\pi]$-left module structure. We consider the π-coinvariants $\mathbb{Z}[\pi]_\pi$ and the π-invariants $\mathbb{Z}[\pi]^\pi$ from Definition 323.1.

(a) Show that the following map is an isomorphism:

$$\begin{array}{rcl} \mathbb{Z}[\pi]_\pi & \to & \mathbb{Z} \\ \left[\sum\limits_{i=1}^k r_i \cdot g_i\right] & \mapsto & \sum\limits_{i=1}^k r_i. \end{array}$$

(b) Show that

$$\mathbb{Z}[\pi]^\pi = \begin{cases} 0, & \text{if } \pi \text{ is infinite,} \\ \mathbb{Z} \cdot \sum\limits_{g \in \pi} g, & \text{if } \pi \text{ is finite.} \end{cases}$$

Exercise 323.2. Let (X, x_0) be a topological space that is path-connected and locally contractible and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$ be a representation. Show that if φ is irreducible in the sense of Definition 310.21, then $\mathrm{H}_0^{\mathrm{tw}}(X; {}_\varphi\mathbb{C}^n) = 0$ and $\mathrm{H}^0_{\mathrm{tw}}(X; {}_\varphi\mathbb{C}^n) = 0$.

Exercise 323.3. Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map between pointed topological spaces that are path-connected and locally contractible. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module.

(a) Show that the induced map

$$f_*\colon \mathrm{H}_0^{\mathrm{tw}}(X; f^*M) \to \mathrm{H}_0^{\mathrm{tw}}(Y; M)$$

from Definition 321.8 is an epimorphism.

(b) Show that the induced map

$$f^*\colon \mathrm{H}^0_{\mathrm{tw}}(Y;M) \;\to\; \mathrm{H}^0_{\mathrm{tw}}(X;f^*M)$$

from Definition 321.8 is a monomorphism.

Exercise 323.4. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X,x_0)$. Let R be a ring and let M be an $(R,\mathbb{Z}[\pi])$-left left module. Use the naturality statement of the Twisted H_0-Proposition 323.3 (1), the Twisted-LES-Proposition 321.10 and Exercise 323.3 to show that for any non-empty subspace $A \subseteq X$ which is path-connected and locally contractible we have

$$\mathrm{H}^{\mathrm{tw}}_0(X,A;M) \;=\; 0.$$

Remark. This gives a longer, but perhaps more conceptual proof of an important special case of the Twisted H_0-Proposition 323.3 (2).

Exercise 323.5. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let $\mathbb{F}$ be a ring and let M be an $(\mathbb{F},\mathbb{Z}[\pi_1(X,x_0)])$-left left module.

(a) We assume that M is finite-dimensional as an $\mathbb{F}$-vector space. Show that the $\mathbb{F}$-vector spaces $\mathrm{H}^{\mathrm{tw}}_0(X;M)$ and $\mathrm{H}^0_{\mathrm{tw}}(X;M)$ are isomorphic.

(b) Show that in (a) we cannot drop the hypothesis that $\mathbb{F}$ is finite-dimensional.

Exercise 323.6.

(a) Let G be a finite group that acts on the left on some $M := \mathbb{Q}^n$. Show that M^G and M_G are isomorphic.
Hint. Show that the map

$$\begin{array}{rcl} M & \to & M \\ m & \mapsto & \frac{1}{|G|} \cdot \sum\limits_{g\in G} g\cdot m \end{array}$$

induces an isomorphism $M_G \to M^G$.

(b) Let G be a finite group that acts on the left on some $M := \mathbb{Z}^n$. Show that M^G and M_G are in general not isomorphic.

Exercise 323.7. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X,x_0)$. Let R be a ring and let M be an $(R,\mathbb{Z}[\pi])$-left left module. We consider the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and we denote by $\widetilde{x}_0$ the natural base point of $\widetilde{X}_{x_0}$.

(a) Show that the evaluation map

$$\begin{array}{rcl} \mathrm{ev}_X\colon \mathrm{H}^0_{\mathrm{tw}}(X;M) & \to & M^\pi = \{m\in M \mid g\cdot m = m \text{ for all } g\in\pi\} \\ (\varphi\colon \mathrm{C}_0(\widetilde{X}_{x_0}) \to M)] & \mapsto & \varphi(\widetilde{x}_0) \end{array}$$

from the Twisted H_0-Proposition 323.3 is well-defined, in the sense that it does indeed take values in M^π.

(b) Show that the map

$$\begin{array}{rcl} \Theta\colon M^\pi & \to & \mathrm{H}^0_{\mathrm{tw}}(X;M) \\ m & \mapsto & (\widetilde{x} \to m) \end{array}$$

is well-defined, in the sense that it does take values in $\mathrm{H}^0_{\mathrm{tw}}(X;M)$.

(c) Show that the maps from (a) and (b) are inverses of one another.

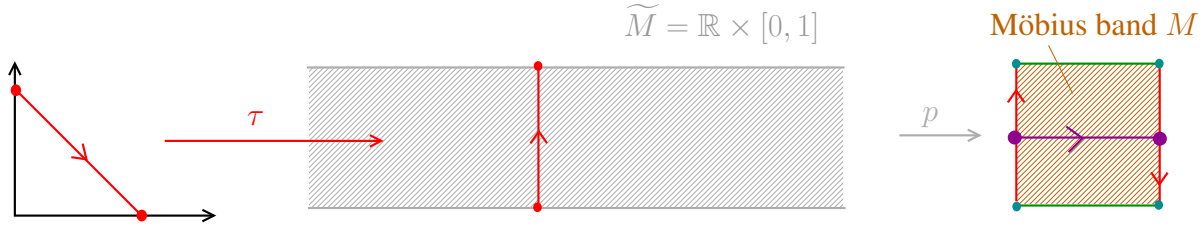
Exercise 323.8. We consider the Möbius band $M := ([0,1] \times [0,1])/(0,y) \sim (1,1-y)$ together with the usual universal covering $p\colon \mathbb{R} \times [0,1] \to M$. Let $\varphi\colon \pi_1(M) \to \mathrm{GL}(1,\mathbb{Z})$ be the unique non-trivial representation. We also consider the long exact sequence

$$\ldots \to \mathrm{H}_1^{\mathrm{tw}}(M, \partial M; {}_\varphi\mathbb{Z}) \xrightarrow{\partial_1} \mathrm{H}_0^{\mathrm{tw}}(\partial M; {}_\varphi\mathbb{Z}) \to \mathrm{H}_0^{\mathrm{tw}}(M; {}_\varphi\mathbb{Z}) \to \ldots.$$

We consider the cycle $\tau\colon \Delta^1 \to \mathbb{R} \times [0,1]$ that is given by $(t, 1-t) \mapsto (0,t)$.

(a) Show that $[1 \otimes \tau]$ defines an element of $\mathrm{H}_1^{\mathrm{tw}}(M, \partial M; {}_\varphi\mathbb{Z})$.

(b) Show that $\mathrm{H}_0^{\mathrm{tw}}(\partial M; {}_\varphi\mathbb{Z}) \cong \mathbb{Z}$ and show that $\partial_1([1 \otimes \tau])$ is twice a generator.



Exercise 323.9. Let X be a *compact* topological space that is path-connected and locally contractible. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. Furthermore let $\Gamma \subseteq \pi$ be a normal subgroup and let $q\colon (Y_\Gamma, y_0) \to (X, x_0)$ be a covering of pointed topological spaces with $p_*(\pi_1(Y_\Gamma, y_0)) = \Gamma$. Show that there exists an isomorphism

$$\mathrm{H}^k_{\mathrm{tw}}(X; \mathbb{Z}[\Gamma\backslash\pi]) \xrightarrow{\cong} \mathrm{H}^k_{\mathrm{c}}(Y_\Gamma; S)$$

between twisted cohomology and cohomology with *compact support*, as introduced in Chapter 209.

Remark. This result is also proved in [**BrownK1982**, Proposition VIII.7.5]

Exercise 323.10. Let X be a topological space that is path-connected and locally contractible. Furthermore let $\varphi\colon \pi_1(X, x_0) \to G$ be a group homomorphism. We do not assume that φ is a monomorphism nor that it is an epimorphism. How can you express $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\varphi\mathbb{Z}[G])$ in terms of "ordinary" homology of a suitable topological space?

Exercise 323.11. Let X be a topological space that is path-connected and locally contractible and let $p\colon (Y, y_0) \to (X, x_0)$ be a path-connected finite covering of degree $d \in \mathbb{N}$. Let R be a ring and let $\psi\colon \pi_1(Y, y_0) \to \mathrm{GL}(n, R)$ be a representation. Show that there exists a representation $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(d \cdot n, R)$ such that for each $k \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_k^{\mathrm{tw}}(Y; {}_\psi R^{1\times n}) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{tw}}(Y; {}_\psi R^{1\times d\cdot n})$$

of R-left modules.

Remark. The special case that ψ is a trivial one-dimensional representation is essentially dealt with in the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5.

Exercise 323.12.

(a) Let M be a closed orientable connected non-empty n-dimensional topological manifold and let $x_0 \in M$. Furthermore let G be a finite group, let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism and let $\psi\colon G \to \mathrm{GL}(k, \mathbb{C})$ be an irreducible non-trivial representation. Show that $\mathrm{H}_n(M; {}_{\varphi\circ\psi}\mathbb{C}^k) = 0$.

Hint. Use Maschke's Theorem 310.26 together with the Manifold-Top-H_*-Theorem 193.1.

(b) In (a), is did we write "orientable" for convenience, or is this really necessary?

Exercise 323.13. Let (X, x_0) be a topological space that is path-connected and locally contractible. Given $n \in \mathbb{N}$ we denote by $\tau\colon \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$ the trivial representation. Furthermore let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$ be another representation.

(a) Show that $\dim_{\mathbb{C}}(\mathrm{H}_0(X; {}_{\tau}\mathbb{C}^n)) \geq \dim_{\mathbb{C}}(\mathrm{H}_0(X; {}_{\varphi}\mathbb{C}^n))$ and show that equality holds if and only if φ is the trivial representation τ.

(b) Give an example where $\dim_{\mathbb{C}}(\mathrm{H}_k(X; {}_{\tau}\mathbb{C}^n)) < \dim_{\mathbb{C}}(\mathrm{H}_k(X; {}_{\varphi}\mathbb{C}^n))$.
Hint. Make use of the Mapping Torus–$\mathbb{C}$-Twisted Homology Proposition 323.11.

Exercise 323.14. In this exercise we use Notation 323.7.

(a) Show that the $\mathbb{Z}[t^{\pm 1}]$-modules $(\mathbb{Z}, t = +1)$ and $\mathbb{Z}[t^{\pm 1}]/(t-1)\mathbb{Z}[t^{\pm 1}]$ are isomorphic.

(b) Show that the $\mathbb{Z}[t^{\pm 1}]$-modules $(\mathbb{Z}, t = -1)$ and $\mathbb{Z}[t^{\pm 1}]/(t+1)\mathbb{Z}[t^{\pm 1}]$ are isomorphic.

Exercise 323.15. Give an example of a topological space X that is path-connected and locally contractible together with a self-homeomorphism $f\colon X \to X$ such that for some $k \in \mathbb{N}$ we have

$$\mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X, f); \underbrace{{}_{\Phi}\mathbb{Q}(t)}) \neq 0.$$

Remark. By the Mapping Torus–Vanishing-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.9 you need to pick X such that $\mathrm{H}_k(X; \mathbb{Q})$ is not finite dimensional. What is the simplest such example?

Exercise 323.16.

(a) Let V be a $\mathbb{C}[t^{\pm 1}]$-module that is finite dimensional as a $\mathbb{C}$-vector space. Furthermore let $z \in \mathbb{C} \setminus \{0\}$. We denote by ${}_z\mathbb{C}$ the $\mathbb{C}[t^{\pm 1}]$-module that is given by $\mathbb{C}$ where t acts by multiplication by z. Show that we have the following isomorphisms of $\mathbb{C}$-vector spaces:

$$\begin{aligned} {}_z\mathbb{C} \otimes_{\mathbb{C}[t^{\pm 1}]} V &\cong \{v \in V \,|\, t \cdot v = z \cdot v\} \\ \mathrm{Tor}_{\mathbb{C}[t^{\pm 1}]}(V, {}_z\mathbb{C}) &\cong \{v \in V \,|\, t \cdot v = z \cdot v\}. \end{aligned}$$

Hint. Use the PID–Module-Classification Theorem 309.36, the Cyclic-Tensor-Cyclic Lemma 314.28 and the Cyclic-Tor-Lemma 314.29.
Remark. This exercise is identical to Exercise 314.6.

(b) Let X be a topological space that is path-connected and locally contractible. We assume that each $\mathrm{H}_k(X; \mathbb{C})$ is finite-dimensional. Furthermore let $f\colon X \to X$ be a homeomorphism. We pick an $x \in X$. As usual we denote by $\Phi\colon \pi_1(\mathrm{Tor}(X, f), [(x, 0)]) \to \langle t \rangle$ the natural epimorphism from Definition 323.6. Given $z \in \mathbb{C} \setminus \{0\}$ we consider the representation $\Phi_z\colon \pi_1(\mathrm{Tor}(X, f), [(x, 0)]) \to \mathrm{GL}(1, \mathbb{C})$ that is given by the composition of Φ with the $\langle t \rangle \xrightarrow{t \to (z)} \mathrm{GL}(1, \mathbb{C})$. Show that given any $k \in \mathbb{N}_0$ we have a short exact sequence

$$\begin{aligned} 0 \to \{v \in \mathrm{H}_k(X; \mathbb{C}) \,|\, f(v) = z \cdot v\} &\to \mathrm{H}_k^{\mathrm{tw}}(\mathrm{Tor}(X, f); {}_{\Phi_z}\mathbb{C}) \\ &\to \{v \in \mathrm{H}_{k-1}(X; \mathbb{C}) \,|\, f(v) = z \cdot v\} \to 0 \end{aligned}$$

of $\mathbb{C}$-vector spaces.
Hint. Borrow ideas from the proof of the Twisted-H_*-of-S^1-Proposition 320.14 and make use of the Algebraic Universal Coefficient Theorem 314.33 (here you rejoice in the fact that $\mathbb{C}[t^{\pm 1}]$ is a PID.)

Exercise 323.17. We consider $X = S^1$. We make the usual identification $\pi_1(S^1, 1) =_{\mathrm{i}} \langle t \rangle$. Find a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of $\mathbb{Z}[t^{\pm 1}]$-modules such that the

corresponding sequence

$$0 \to \mathrm{H}_1^{\mathrm{tw}}(A;\mathbb{Z}[t^{\pm 1}]) \xrightarrow{\alpha_*} \mathrm{H}_1^{\mathrm{tw}}(B;\mathbb{Z}[t^{\pm 1}]) \xrightarrow{\beta_*} \mathrm{H}_1^{\mathrm{tw}}(C;\mathbb{Z}[t^{\pm 1}]) \to 0$$

is *not* exact.

Exercise 323.18. Let $n \in \mathbb{N}_{\geq 2}$. By the Real Projective Space–π_1-Proposition 93.3 we know that we have an isomorphism $\pi_1(\mathbb{R}\mathrm{P}^n, *) \cong \mathbb{Z}_2$. Thus there exists a unique non-trivial representation $\varphi\colon \pi_1(\mathbb{R}\mathrm{P}^n, *) \to \mathrm{GL}(1,\mathbb{C})$.

(a) Compute the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n;{}_\varphi\mathbb{C})$.

(b) Compute the twisted cohomology modules $\mathrm{H}^*_{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n;{}_\varphi\mathbb{C})$.

CHAPTER 324

The fundamental class of non-orientable manifolds

In the previous Chapter 323 we learned about several tricks for calculating twisted homology modules. In this chapter we will use these tricks to show that a certain twisted top-dimensional homology group of compact *non-orientable* topological manifolds is non-trivial. This allows us to generalize the notion of a fundamental class of compact topological manifolds from the oriented setting to the non-orientable setting.

This chapter is organized as follows:

- In Section 324.1 we will recall the definition of the fundamental class of a compact oriented topological manifold.
- In Section 324.2 we will (re-)introduce the orientation character of a topological manifold.
- In Section 324.3 we will formulate the Twisted Top Homology Proposition 324.8 which allows us to introduce the fundamental class of compact non-orientable topological manifolds.
- In Section 324.4 we will provide a proof of the Twisted Top Homology Proposition 324.8.
- In Section 324.5 we will provide exercises for Chapter 324.

In the next Chapter 325 we will introduce universally excisive triads and Mayer-Vietoris sequences.

324.1. **The fundamental class of oriented topological manifolds.** We start out by reminding the reader of the following theorem:

Theorem 324.1. (Manifold-Top-H_*-Theorem) Let X be a compact connected non-empty n-dimensional topological manifold. The following three statements hold:

(1) For every $i > n$ we have $\mathrm{H}_i(X, \partial X; \mathbb{Z}) = 0$.

(2) We have the following isomorphisms:

$$\mathrm{H}_n(X, \partial X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } X \text{ is orientable,} \\ 0, & \text{if } X \text{ is non-orientable.} \end{cases}$$

(3) If X is orientable, then for every $Q \in X \setminus \partial X$ the natural map

$$\mathrm{H}_n(X, \partial X; \mathbb{Z}) \to \mathrm{H}_n(X, X \setminus \{Q\}; \mathbb{Z})$$

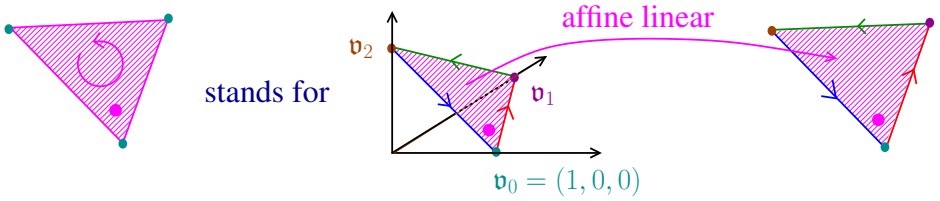
is an isomorphism.

Proof. This theorem is just a simplified version of the original Manifold-Top-H_*-Theorem 193.1. ■

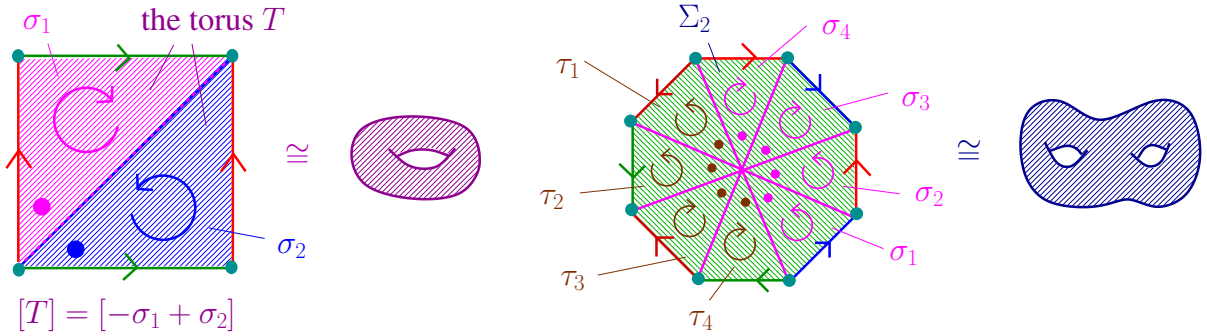
Next we recall Definition 194.3:

Definition 324.2. Let X be a compact n-dimensional topological manifold. Furthermore let $\{\alpha_Q\}_{Q\in X\setminus\partial X}$ be an $\mathbb{Z}$-orientation for X in the sense of Definition 191.17.[3558] Note that it follows easily from the Manifold-Top-H_*-Theorem 324.1 that there exists a unique homology class $[X]\in \mathrm{H}_n(X,\partial X;\mathbb{Z})$ such that for all points $Q\in X\setminus\partial X$ the image of the homology class $[X]$ under the natural map $\mathrm{H}_n(X,\partial X;\mathbb{Z})\to \mathrm{H}_n(X,X\setminus\{Q\};\mathbb{Z})$ equals α_Q. We refer to $[X]$ as the fundamental class of the $\mathbb{Z}$-oriented topological manifold X.

Example. We recall some examples of fundamental classes of 2-dimensional topological manifolds from page 3755. In the description of these examples we will use the notation from the figure below to indicate a singular simplex.



In the figure below we now show various cycles.



It follows from the Fundamental Class-Criterion 194.8 that these cycles represent the fundamental classes of the torus T and the surface Σ_2 of genus 2. □

324.2. **The orientation covering and the orientation character.** The following proposition says that to any non-orientable topological manifold we can assign an oriented topological manifold:

Proposition 324.3. (Orientation Covering–Properties-Proposition)

(1) To each topological manifold X we can naturally associate an oriented topological manifold $\widetilde{X}^{\mathrm{or}}$ together with a 2-fold covering $p\colon \widetilde{X}^{\mathrm{or}}\to X$.

(2) Let X be a connected topological manifold and let $p\colon \widetilde{X}^{\mathrm{or}}\to X$ be the 2-fold covering from (1).

  (a) If X is orientable, then $\widetilde{X}^{\mathrm{or}}$ is homeomorphic (as an oriented topological manifold) to $X\sqcup -X$.

[3558] Recall that by the Smooth-Topological Orientation Proposition 191.25 we know that for a smooth manifold an orientation in terms of local homology groups is essentially the same as an orientation in terms of tangent spaces.

(b) If X is non-orientable, then $\widetilde{X}^{\mathrm{or}}$ is connected.

We refer to $\widetilde{X}^{\mathrm{or}}$ as the orientation covering of X.

Proof. The key idea, at least if $\partial X = \varnothing$, is to define $\widetilde{X}^{\mathrm{or}}$ as the set

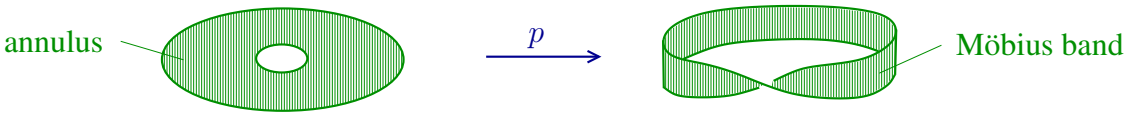
$$\widetilde{X}^{\mathrm{or}} \;=\; \{(x,\omega)\,|\,x \in X \text{ and } \omega \text{ is an orientation at } x\}$$

and to equip this set $\widetilde{X}^{\mathrm{or}}$ with a suitable topology. The full details (and also the more precise statements) are worked out in the much more detailed Orientation Covering–Properties-Proposition 191.29.[3559] ■

Next we recall Definition 191.30, which is the topological analog of Definition 95.9:

Definition 324.4. Let X be a non-orientable connected topological manifold. We refer to the connected 2-fold covering $p\colon \widetilde{X} \to X$ from the Orientation Covering–Properties-Proposition 324.3 as the orientation covering of X.

Example. As we discussed on page 3700, it follows easily from the uniqueness statement of the Orientation Covering–Properties-Proposition 191.29 (4) that the orientation covering of the Möbius band is homeomorphic to the annulus.



Similarly one sees that the orientation covering of the Klein bottle is homeomorphic to the torus and that for every even $n \in \mathbb{N}$ the orientation covering of the real projective space $\mathbb{R}\mathrm{P}^n$ is homeomorphic to S^n. □

Next we recall Definition 191.31, which in turn is the topological analog of Definition 95.15:

Definition 324.5. Let X be a connected topological manifold and let $x_0 \in X$.

(1) If X is non-orientable, then we consider the orientation covering $p\colon \widetilde{X}^{\mathrm{or}} \to X$ and we pick $\widetilde{x}_0 \in \widetilde{X}^{\mathrm{or}}$ with $p(\widetilde{x}_0) = x_0$. We then refer to the epimorphism

$$w_1^{x_0}(X) := w_1^{x_0}\colon \pi_1(X,x_0) \;\to\; \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X}^{\mathrm{or}},\widetilde{x}_0)) =_{\mathrm{i}} \mathbb{Z}/2\mathbb{Z},$$

as the orientation character of X. Note that it follows from the Orientation Character Proposition 95.14 (1b) that this definition does not depend on the choice of the point $\widetilde{x}_0$.

(2) If X is orientable, then we define the orientation character of X to be the trivial homomorphism $w_1\colon \pi_1(X,x_0) \to \mathbb{Z}/2\mathbb{Z}$.

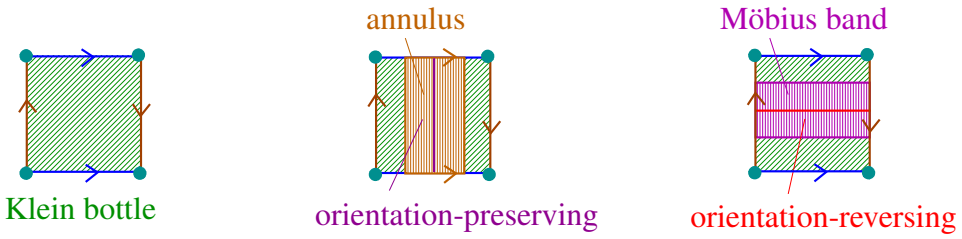
The following definition is almost identical to Definition 95.18:

Definition 324.6. Let X be a connected topological manifold and let $\gamma\colon [0,1] \to X$ be a loop in a point x_0. We say that γ is orientation-preserving if $w_1^{x_0}(\gamma) = 0$ otherwise we say that γ is orientation-reversing.

[3559] For smooth manifolds the analogous result is proved in the Orientation Covering–Existence Proposition 95.8 and the Orientation Covering–Properties-Proposition 95.10.

Example. Let Σ be a 2-dimensional smooth manifold and let $\gamma \subseteq \Sigma$ be a smoothly embedded loop. The argument of page 1853 gives us the following convenient rule:

- If the tubular neighborhood of γ is an annulus, then γ is orientation-preserving.
- If the tubular neighborhood of γ is a Möbius band, then γ is orientation-reversing.



A similar discussion also holds in higher dimensions. □

324.3. **Twisted homology of non-orientable topological manifolds.** Our next goal is to construct a fundamental class for non-orientable topological manifolds. To do so we need to introduce the following notation:

Notation 324.7. We denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left module that is given by the unique isomorphism $\langle x \,|\, x^2\rangle \to \{\pm 1\}$, in other words, it is given by $x \cdot n := -n$.

Now we can formulate the main result of this chapter:

Proposition 324.8. (Twisted Top Homology Proposition) Let (X, x_0) be a compact *non-orientable* path-connected non-empty n-dimensional topological manifold. We denote by $\widetilde{X}^{\mathrm{or}} \to X$ the natural path-connected 2-fold covering from the Orientation Covering–Properties-Proposition 324.3. We denote by

$$w_1 \colon \pi_1(X, x_0) \;\to\; \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}^{\mathrm{or}}, \widetilde{x}_0)) \underset{\uparrow}{=_{\mathrm{i}}} \langle x \,|\, x^2\rangle$$

unique isomorphism

the orientation-character that we introduced in Definition 324.5. The following three statements hold:

(1) There exists an isomorphism $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}}) \cong \mathbb{Z}$.
(2) There exists a natural[3560] generator of $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}}) \cong \mathbb{Z}$.
(3) For each $Q \in X \setminus \partial X$ the map $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}}) \to \mathrm{H}_n^{\mathrm{tw}}(X, X \setminus \{Q\}; w_1^*\widetilde{\mathbb{Z}})$ is an isomorphism.

Remark. A statement similar to the Twisted Top Homology Proposition 324.8 is proved in [**Hat2002**, p. 327]. □

Proof. We hide the proof of the proposition in Section 324.4. ■

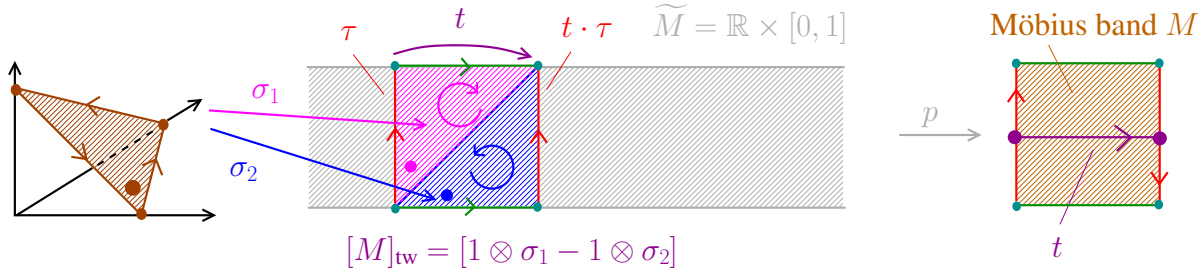
The Twisted Top Homology Proposition 324.8 shows, that for a compact non-orientable path-connected non-empty n-dimensional topological manifold we have a naturally defined generator which has, in a slightly different setup, the defining property of a fundamental class from Definition 324.2. This motivates the following definition:

[3560] The precise natural choice will be made precise in the proof of the proposition.

Definition 324.9. Let X be a compact non-orientable path-connected non-empty n-dimensional topological manifold. We refer to the natural generator of $\mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}})$ as the twisted fundamental class $[X]_{\mathrm{tw}}$ of X. Sometimes, when we want a uniform notation, we also just write $[X]$.

Remark. The main application of the twisted fundamental class is that it allows us to formulate the Non-Orientable–Poincaré Duality Theorem 335.15. □

Example. We consider the Möbius band $M := ([0,1] \times [0,1])/(0,x) \sim (1, 1-x)$ together with the usual universal covering $p\colon \mathbb{R} \times [0,1] \to M$. We make the standard identification $\pi_1(M) =_{\mathrm{i}} \langle t \rangle$. As on page 6223 we see that for the orientation-character w_1 we have $w_1(t) = -1$. In the figure below we also show two singular simplices $\sigma_i\colon \Delta^2 \to \mathbb{R} \times [0,1]$ and a singular 1-simplex $\tau\colon \Delta^1 \to \mathbb{R} \times [0,1]$.



Note that in $w_1^*\widetilde{\mathbb{Z}} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathrm{C}_2(\mathbb{R} \times [0,1], \mathbb{R} \times \{0,1\})$ we have the following equalities:

$$\begin{aligned} \partial_2(1 \otimes \sigma_1 - 1 \otimes \sigma_2) &= 1 \otimes \partial_2(\sigma_1) - 1 \otimes \partial_2(\sigma_2) \underset{\text{all other contributions cancel}}{=} 1 \otimes \tau + 1 \otimes t \cdot \tau \\ &\underset{\text{by Definition 320.4 of the tensor product}}{=} 1 \otimes \tau + 1 \cdot (-1)^{w_1(t)} \otimes \tau \underset{\text{since } w_1(t) = 1}{=} 1 \otimes \tau + (-1) \otimes \tau = 0. \end{aligned}$$

This shows that $1 \otimes \sigma_1 - 1 \otimes \sigma_2$ is a cycle in the twisted chain complex. Using the Twisted Top Homology Proposition 324.8 one can now easily verify (say using a point Q that lies in the interior of one of the simplices) that this cycle represents the twisted fundamental class (at least up to a sign). □

324.4. **Proof of the Twisted Top Homology Proposition 324.8.** The proof of the Twisted Top Homology Proposition 324.8 relies on the following lemma:

Lemma 321.15. (Twisted Bockstein Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let $A \subseteq X$. Furthermore let R be a ring, let L, M and N be $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules and let

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

be a short exact sequence of $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules. Then we get a long exact sequence of twisted (co-) homology groups:

$$\dots \xrightarrow{\beta_{k+1}} \mathrm{H}_k^{\mathrm{tw}}(X, A; L) \xrightarrow{\varphi_*} \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \xrightarrow{\psi_*} \mathrm{H}_k^{\mathrm{tw}}(X, A; N) \xrightarrow[\uparrow]{\beta_k} \mathrm{H}_{k-1}^{\mathrm{tw}}(X, A; L) \to \dots$$

called the Bockstein homomorphism

of R-left modules.

Example. Our idea now is to apply the Twisted Bockstein Lemma 321.15 to what is arguably the simplest non-trivial setup: More precisely we consider the group $\langle x \,|\, x^2 \rangle$, which is nothing but the group $\mathbb{Z}/2\mathbb{Z}$ written multiplicatively. As in Notation 324.7 we denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2 \rangle]$-left module that is given by the unique isomorphism $\langle x \,|\, x^2 \rangle \to \{\pm 1\}$. In Exercise 321.4 we showed that

$$0 \to \widetilde{\mathbb{Z}} \xrightarrow{n \mapsto n \cdot (1-x)} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{x \mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0$$

and

$$0 \to {}_{\mathrm{tr}}\mathbb{Z} \xrightarrow{n \mapsto n \cdot (1+x)} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{x \mapsto -1} \widetilde{\mathbb{Z}} \to 0$$

are short exact sequences of $\mathbb{Z}[\langle x \,|\, x^2 \rangle]$-left modules. □

Proof of the Twisted Top Homology Proposition 324.8. Let (X, x_0) be a compact *non-orientable* path-connected non-empty n-dimensional topological manifold. In this proof we denote by $\widetilde{X} \to X$ the natural path-connected 2-fold covering from the Orientation Covering–Properties-Proposition 324.3. Recall that $\widetilde{X}$ is a compact n-dimensional topological manifold that comes with a natural orientation.

As in the previous example we consider the following short exact sequence of $\mathbb{Z}[\langle x \,|\, x^2 \rangle]$-modules:

$$0 \to \widetilde{\mathbb{Z}} \xrightarrow{n \mapsto n \cdot (1-x)} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow[=:\gamma]{x \mapsto 1} {}_{\mathrm{tr}}\mathbb{Z} \to 0.$$

see Definition 310.7

Via the homomorphism $w_1 \colon \pi_1(X, x_0) \to \langle x \,|\, x^2 \rangle$ and Notation 320.8 we obtain a corresponding short exact sequence of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. Note that $w_1^* \, {}_{\mathrm{tr}}\mathbb{Z} = {}_{\mathrm{tr}}\mathbb{Z}$. Next we consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_{n+1}^{\mathrm{tw}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{Z}) & \xrightarrow{\beta_{n+1}} & \mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}}) & \to & \mathrm{H}_n^{\mathrm{tw}}(X, \partial X; w_1^*\mathbb{Z}[\langle x \,|\, x^2 \rangle]) & \to & \mathrm{H}_n^{\mathrm{tw}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{Z}) \\
\cong \downarrow T & & & & \cong \downarrow \Omega & & \cong \downarrow T \\
\underbrace{\mathrm{H}_{n+1}(X, \partial X; \mathbb{Z})}_{=0} & & & & \underbrace{\mathrm{H}_n(\widetilde{X}; \mathbb{Z})}_{=\mathbb{Z} \cdot [\widetilde{X}]} & & \underbrace{\mathrm{H}_n(X, \partial X; \mathbb{Z})}_{=0}
\end{array}$$

We make the following clarifications and observations:

(a) The top horizontal sequence is the Bockstein sequence from the Twisted Bockstein Lemma 321.15 corresponding to the above short exact sequence of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

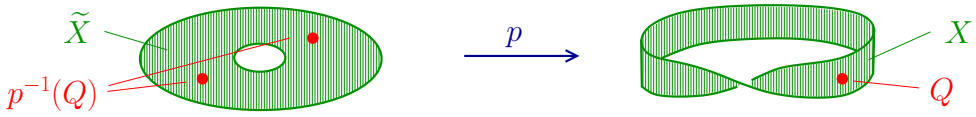
(b) The vertical map in the first and fourth column are the natural isomorphisms T provided by the Twisted–Trivial Coefficients Lemma 320.13.

(c) Since $\dim(X) = n$ we obtain immediately from the Manifold-Top-H_*-Theorem 324.1 (1) that $\mathrm{H}_{n+1}(X, \partial X; \mathbb{Z}) = 0$.

(d) Since the topological manifold X is non-orientable we obtain from the Manifold-Top-H_*-Theorem 324.1 (2) that $\mathrm{H}_n(X, \partial X; \mathbb{Z}) = 0$.
(e) The vertical map in the third column is the natural isomorphism Ω that is given by the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 323.5.
(f) Since $\widetilde{X}$ is path-connected, non-empty and naturally oriented we know by the Fundamental Class-Proposition 194.2 (2) that $\mathrm{H}_n(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z}) = \mathbb{Z} \cdot [\widetilde{X}]$.

The above discussion shows that $\mathrm{H}^{\mathrm{tw}}_n(X, \partial X; w_1^*\widetilde{\mathbb{Z}})$ is isomorphic to $\mathbb{Z}$ and we see that the natural isomorphisms and the natural generator $[\widetilde{X}]$ of $\mathrm{H}_n(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z})$ give rise to a natural generator $[X]_{\mathrm{tw}}$ of $\mathrm{H}^{\mathrm{tw}}_n(X, \partial X; w_1^*\widetilde{\mathbb{Z}})$. This concludes the proof of Statements (1) and (2).

It remains to prove Statement (3). Thus let $Q \in X \setminus \partial X$.



We consider the following diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}^{\mathrm{tw}}_n(X, \partial X; \widetilde{\mathbb{Z}})}^{=\mathbb{Z}\cdot[X]_{\mathrm{tw}}} & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_n(X, \partial X; \mathbb{Z}[\langle x \,|\, x^2\rangle]) & & \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & \overset{\cong}{\searrow}{\scriptstyle \Omega} & \\
 & & & \mathrm{H}_n(\widetilde{X}, \partial\widetilde{X}) & \\
 & & & \downarrow{\scriptstyle i_*} & \\
 & & & \mathrm{H}_n(\widetilde{X}, \widetilde{X}\setminus p^{-1}(Q)) & \xrightarrow{p_*} \mathrm{H}_n(X, X\setminus\{Q\}) \\
 & & & \overset{\cong}{\nearrow}{\scriptstyle \Omega} & \overset{\cong}{\nearrow}{\scriptstyle T} \\
\mathrm{H}^{\mathrm{tw}}_n(X, X\setminus\{Q\}; \widetilde{\mathbb{Z}}) & \hookrightarrow & \mathrm{H}^{\mathrm{tw}}_n(X, X\setminus\{Q\}; \mathbb{Z}[\langle x \,|\, x^2\rangle]) & \xrightarrow{\gamma} & \mathrm{H}^{\mathrm{tw}}_n(X, X\setminus\{Q\}; {}_{\mathrm{tr}}\mathbb{Z}).
\end{array}$$

$i_*([X]_{\mathrm{tw}})$

We make the following clarifications and observations:

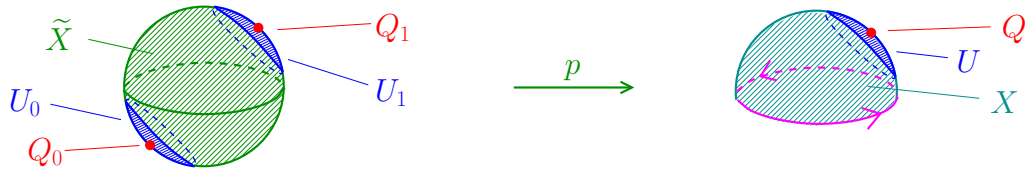
(a) As before we see that the horizontal map at the top left is an isomorphism and that the horizontal map at the bottom left is a monomorphism.
(b) All vertical maps are induced by the obvious inclusions.
(c) The rectangle to the left commutes by naturality of i_*. The middle quadrilateral commutes by the naturality of Ω. We leave it to the reader to show, by inspecting the explicit maps, that the quadrilateral to the right commutes.
(d) The bottom sequence is exact.

We want to show that $i_*([X]_{\mathrm{tw}})$ is a generator of $\mathrm{H}^{\mathrm{tw}}_n(X, X\setminus\{Q\}; \widetilde{\mathbb{Z}})$. By the above diagram and by definition of $[X]_{\mathrm{tw}}$ it suffices to prove the following claim:

Claim. The image of the fundamental class $[\widetilde{X}] \in \mathrm{H}_n(\widetilde{X}, \partial\widetilde{X})$ in $\mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus p^{-1}(Q))$ is a generator of the kernel of p_*.

Proof. Let $\tau\colon \widetilde{X} \to \widetilde{X}$ be the orientation-reversing deck transformation of the covering $p\colon \widetilde{X} \to X$. We pick a point $Q_0 \in p^{-1}(Q)$ and we write $Q_1 := \tau(Q_1)$. We pick an open

neighborhood U_0 around Q_0 such that U_0 and $U_1 \coloneqq \tau(U_0)$ are disjoint. We set $\widetilde{U} \coloneqq U_0 \cup U_1$ and we set $U \coloneqq p(\widetilde{U})$.



We consider the following little diagram:

$$\begin{array}{ccccc}
 & & & & \mathrm{H}_n(\widetilde{X}, \partial\widetilde{X}) \\
 & & & & \downarrow{\scriptstyle [\widetilde{X}]\, i_*} \\
\ker(p_*) & \hookrightarrow & \mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus p^{-1}(Q)) & \xrightarrow{p_*} & \mathrm{H}_n(X, X \setminus \{Q\}) \\
 & & \downarrow{\scriptstyle \mu_X} & & \\
 & & \mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus \{Q_0\}) & & \\
 & & \uparrow{\scriptstyle \cong} & & \\
 & & \mathrm{H}_n(\widetilde{U}, \widetilde{U} \setminus \{Q_0\}) & & \\
 & & \uparrow{\scriptstyle \mu_U} & & \\
\underbrace{\{\sigma \oplus -\tau_*\sigma \mid \sigma \in \mathrm{H}_n(U_0, U_0 \setminus \{Q_0\})\}}_{\subseteq \mathrm{H}_n(U_0, U_0 \setminus \{Q_0\}) \oplus \mathrm{H}_n(U_1, U_1 \setminus \{Q_1\})} & \longrightarrow & \mathrm{H}_n(\widetilde{U}, \widetilde{U} \setminus p^{-1}(Q)) & \xrightarrow{p_*} & \mathrm{H}_n(U, U \setminus \{Q\}).
\end{array}$$

We make the following observations:

(1) The three vertical maps that are decorated by "$\cong$" are induced by inclusions and they are isomorphisms by the "classical" Excision Theorem 148.2.
(2) The maps μ_X and μ_U are induced by the obvious map of pairs of topological spaces.
(3) The diagram commutes by functoriality of homology.
(4) Since $\mathrm{H}_n(\widetilde{U}, \widetilde{U} \setminus p^{-1}(Q)) = \mathrm{H}_n(U_0, U_0 \setminus p^{-1}(Q_0)) \oplus \mathrm{H}_n(U_1, U_1 \setminus p^{-1}(Q_1))$ one can easily verify that the bottom horizontal sequence is exact.
(5) It follows easily from (4) that the restriction of μ_U to $\ker(p_*)$ is an isomorphism.
(6) It follows from (3) and (5) that the restriction of μ_X to $\ker(p_*)$ is an isomorphism.
(7) By Definition 324.2 of a fundamental class we know that the image of $[\widetilde{X}]$ under the map $\mu_X \circ i_*$ in $\mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus \{Q_0\})$ is a generator.
(8) It follows from (6) and (7) that $i_*([\widetilde{X}])$ is indeed a generator of $\ker(p_*)$. ■

324.5. Exercises for Chapter 324.

Exercise 324.1. Does there exist a closed orientable connected non-empty smooth manifold that is *not* diffeomorphic to the orientation covering of a non-orientable smooth manifold?

Exercise 324.2. Let (X, x_0) be a compact *non-orientable* path-connected non-empty n-dimensional topological manifold. We denote by $\widetilde{X}^{\mathrm{or}} \to X$ the natural path-connected 2-fold covering from the Orientation Covering–Properties-Proposition 324.3. As before let $w_1 \colon \pi_1(X, x_0) \to \pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}^{\mathrm{or}}, \widetilde{x}_0)) =_{\mathrm{i}} \langle x \,|\, x^2 \rangle$ be the orientation character. Determine the twisted cohomology module $\mathrm{H}^n_{\mathrm{tw}}(X, \partial X; w_1^*\widetilde{\mathbb{Z}})$.

Exercise 324.3. We consider the real projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} S^2/x \sim -x =_{\mathrm{i}} \overline{B}^2/\sim$ where $x \sim -x$ for all $x \in S^1$. Determine a singular cycle in $w_1^*\widetilde{\mathbb{Z}} \otimes_{\mathbb{Z}[\pi_1(\mathbb{R}\mathrm{P}^2)]} \mathrm{C}_*(\mathbb{R}\mathrm{P}^2)$ that represents the fundamental class $[\mathbb{R}\mathrm{P}^n]_{\mathrm{tw}} \in \mathrm{H}_2(\mathbb{R}\mathrm{P}^n; w_1^*\widetilde{\mathbb{Z}})$.

Exercise 324.4. Let $f \colon M \to N$ be a local homeomorphism between closed connected non-empty n-dimensional topological manifolds. It follows from the w_1-Functorial Lemma 191.33 and Definition 321.8 that we get an induced homomorphism

$$\underbrace{f_* \colon \mathrm{H}_n(M; w_1(M)^*\widetilde{\mathbb{Z}})}_{=\mathbb{Z}\cdot[M]} \to \underbrace{\mathrm{H}_n(N; w_1(N)^*\widetilde{\mathbb{Z}})}_{=\mathbb{Z}\cdot[N]}$$

which allows us to define the degree $\deg(f \colon M \to N) \in \mathbb{Z}$ of f.

(a) Does there exist such a map f between non-orientable topological manifolds of degree $\neq \pm 1$?

(b) Does there exist such a map f from an orientable topological manifold to a non-orientable topological manifold of degree $\neq \pm 1$?

(c) Does there exist such a map f from an non-orientable topological manifold to an orientable topological manifold of degree $\neq \pm 1$?

Exercise 324.5. As in the above Exercise 324.4 we consider the degree $\deg(f \colon M \to N) \in \mathbb{Z}$ of a local homeomorphism $f \colon M \to N$ between two closed connected non-empty n-dimensional topological manifolds. For every $n \in \mathbb{N}$ determine the degree of the natural projection $f \colon S^n \to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$.

Exercise 324.6. Let (X, x_0) be a pointed compact connected n-dimensional topological manifold. We write $\pi := \pi_1(X, x_0)$. As on page 6174 we consider the natural epimorphism $\Theta \colon w_1^*\widetilde{\mathbb{Z}} \to {}_{\mathrm{tr}}\mathbb{F}_2$ of $\mathbb{Z}[\pi]$-left modules. Show that the induced map

$$\Theta_* \colon \mathrm{H}^{\mathrm{tw}}_n(X, \partial X; w_1^*\widetilde{\mathbb{Z}}) \to \mathrm{H}_n(X, \partial X; \mathbb{F}_2)$$

is an epimorphism.

Hint. Consider the characterizations of the (twisted) fundamental class.

CHAPTER 325

Universally excisive triads and Mayer-Vietoris sequences

In this, unfortunately rather technical, chapter we will introduce the concept of a universally excisive triad, which is a variation on the notion of an excisive triad that we introduced in Definition 158.17. Excisive triads played roles on two different occasions:

(1) They allowed us to formulate Mayer-Vietoris type theorems and
(2) they allowed us to define relative cup and cap products.

Universally excisive triads will play the same role in the setting of (co-) homology with twisted coefficients.

In this chapter we will also introduce internal twisted (co-) homology. This slightly clunky concept allows us to work with disconnected spaces and will come in handy on several occasions.

This chapter is organized as follows:

- In Section 325.1 we will introduce universally excisive triads.
- In Section 325.2 we will state and prove Excision Theorems for (co-) homology with twisted coefficients.
- In Section 325.3 we will introduce internal twisted (co-) homology.
- In Section 325.4 we will formulate and prove Mayer-Vietoris Theorems for (co-) homology with twisted coefficients.
- In Section 325.5 we will provide exercises for Chapter 325.

In the next Chapter 326 we will introduce the Kronecker pairing that relates twisted homology to twisted cohomology.

325.1. **Universally excisive triads.** We start out by recalling Definition 158.17:

Definition 325.1.

(1) A triad of topological spaces is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.
(2) A morphism $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ of triads is a continuous map $f\colon X \to Y$ with $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$.
(3) Given a triad (X, A_1, A_2) of topological spaces we consider the following subgroup of $\mathrm{C}_n(A_1 \cup A_2)$:

$$\mathrm{C}_n^{\{A_1,A_2\}}(A_1 \cup A_2) := \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \in \mathrm{C}_n(A_1 \cup A_2) \;\middle|\; \begin{array}{l} \text{for each } j \text{ there exists an } i \in \{1,2\} \\ \text{such that the image of } \sigma_j \text{ lies in } A_i \end{array} \right\}.$$

The usual boundary map on the groups $\mathrm{C}_*(A_1 \cup A_2)$ restricts to a boundary map on $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$, so we can view $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$ as a chain complex. We denote the corresponding homology groups by $\mathrm{H}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$.

(4) We say that a triad (X, A_1, A_2) is excisive if the inclusion map
$$\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{C}_*(A_1 \cup A_2)$$
is a chain homotopy equivalence.

To these familiar definitions we now add a new one:

(5) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering from Definition 319.10. We say that a triad (X, A_1, A_2) is universally excisive if the triad $(\widetilde{X}_{x_0}, p^{-1}(A_1), p^{-1}(A_2))$ is excisive.

Lemma 325.2. (Universally Excisive Properties Lemma)
Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let (X, A_1, A_2) be a universally excisive triad. We write $\widetilde{A}_1 := p^{-1}(A_1)$ and $\widetilde{A}_2 := p^{-1}(A_2)$. The following three statements hold:

(1) The left action by $\pi_1(X, x_0)$ on $\widetilde{X}_{x_0}$ from Convention 319.13 turns $\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2)$ into a free $\mathbb{Z}[\pi_1(X, x_0)]$-left module.

(2) The inclusion map
$$\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2) \;\to\; \mathrm{C}_*(\widetilde{A}_1 \cup \widetilde{A}_2)$$
is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left module chain complexes.

(3) The triad (X, A_1, A_2) is also excisive.

Proof.

(1) The same argument as in the Regular Covering–Chain Complex Lemma 320.2 shows that $\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2)$ is a free $\mathbb{Z}[\pi_1(X, x_0)]$-left module. For example a basis is given by picking a lift for each singular simplex that is contained in A_1 or in A_2.

(2) Since (X, A_1, A_2) is by hypothesis universally excisive we know that the natural map $\mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2) \to \mathrm{C}_*(\widetilde{A}_1 \cup \widetilde{A}_2)$ is a chain homotopy equivalence. By the super classical Chain Homotopy Equivalence-H_*-Corollary 313.7 this implies that the map induces an isomorphism of homology groups, i.e. it is a quasi-isomorphism. By the Regular Covering–Chain Complex Lemma 320.2 and (1) we know that we have in fact a quasi-isomorphism of chain complexes of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. Thus it follows from the Quasi Iso–Homotopy Equivalence Proposition 313.18 that the map is in fact a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left module chain complexes.

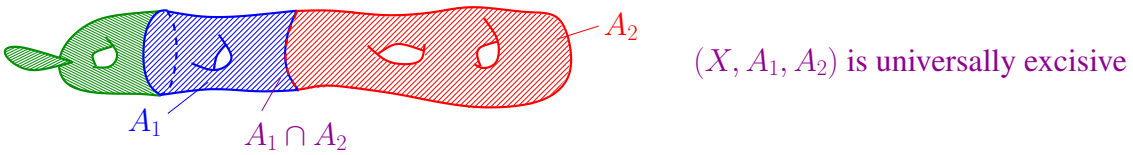
(3) We consider the following diagram:
$$\begin{array}{ccc}
{}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*^{\{\widetilde{A}_1,\widetilde{A}_2\}}(\widetilde{A}_1 \cup \widetilde{A}_2) & \longrightarrow & {}_{\mathrm{tr}}\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*(\widetilde{A}_1 \cup \widetilde{A}_2) \\
\cong \downarrow n\otimes\sigma \mapsto n\cdot p_*(\sigma) & & \cong \downarrow n\otimes\sigma \mapsto n\cdot p_*(\sigma) \\
\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) & \longrightarrow & \mathrm{C}_*(A_1 \cup A_2).
\end{array}$$
Here the horizontal maps are induced by the inclusions and the vertical maps are basically the natural isomorphisms from the Twisted–Trivial Coefficients Lemma 320.13. It is clear that the diagram commutes. It follows from (1) and (2) together with the

Homology-with-Coefficients Lemma 314.6 that the top horizontal map is a chain homotopy equivalence. But then the bottom map is also a chain homotopy equivalence. That is exactly what we needed to show. ■

In the Excisive Triad-Proposition 158.20 we gave many convenient examples of excisive triads. The following proposition now says that these triads from the Excisive Triad-Proposition 158.20 are not only excisive but they are - how exciting - even universally excisive.

Proposition 325.3. (Universally Excisive Triad Proposition) Let (X, A_1, A_2) be a triad of topological spaces. We assume that X is path-connected and locally contractible. The triad is universally excisive if one of the following conditions is satisfied:

(0) (a) $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.
(b) $A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$.
(1) (a) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$.
(b) A_1 and A_2 are both open subsets of $A_1 \cup A_2$.
(2) $A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2 (this setting is illustrated in the figure below),
(3) X is a CW-complex and A_1 and A_2 are both subcomplexes.



Proof. Let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We write $\widetilde{A}_i := p^{-1}(A_i)$. By the Excisive Triad-Proposition 158.20 it suffices to prove the following claim:

Claim. If (X, A_1, A_2) satisfies one of the Conditions (0)–(3), then so does $(\widetilde{X}, \widetilde{A}_1, \widetilde{A}_2)$.

Proof. For Conditions (0a), (0b) and (2) it is clear that the conclusion holds. For Condition (1a) note that for a covering we say in Exercise 91.18 that taking the preimage and taking the interior commutes. With this observation Condition (1a) is also clear.

Now let us consider Condition (2). By the Manifold Covering Proposition 95.4 we know that any path-connected covering of a topological manifold is again a topological manifold. Since being a submanifold is a local condition we see that $(\widetilde{X}, \widetilde{A}_1, \widetilde{A}_2)$ also satisfies Condition (2).

Finally we turn to Condition (3). First note that by the CW-Complex Covering Proposition 138.1 we know that $\widetilde{X}$ has a natural CW-structure. Furthermore note that it follows easily from the Subcomplex–Characterization Proposition 136.12, that $\widetilde{A}_1$ and $\widetilde{A}_2$ are subcomplexes of $\widetilde{X}$. ■

325.2. **Excision for (co-) homology with twisted coefficients.** As we have seen, excision theorems play an essential role when dealing with ordinary (co-) homology groups. The reader will thus be greatly relieved to hear that analogous statements also hold in

the twisted context. More precisely we have the following analog of the Excision Theorems 148.1, 148.2, the Manifold–Excision Theorem 148.3, the CW-Excision Theorem 148.4 and the Excision-in-H*-Theorem 200.1:

Theorem 325.4. (Twisted Excision Theorem) Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module and let $n \in \mathbb{N}_0$.

(1) (a) Let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. If $x_0 \in X \setminus Z$ and if $X \setminus Z$ is path-connected and locally contractible, then the inclusion $i\colon (X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism

$$i^*\colon \mathrm{H}_n^{\mathrm{tw}}(X \setminus Z, A \setminus Z; i^*M) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{tw}}(X, A; M).$$

(b) Let K be a subset of X and let U be a neighborhood of the closure of K. If $x_0 \in U$ and if U is path-connected and locally contractible, then the inclusion $i\colon (U, U \setminus K) \to (X, X \setminus K)$ induces an isomorphism

$$i^*\colon \mathrm{H}_n^{\mathrm{tw}}(U, U \setminus K; i^*M) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{tw}}(X, X \setminus K; M).$$

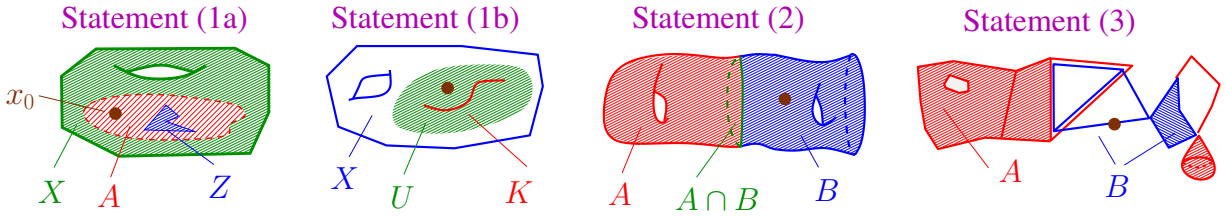
(2) We suppose that X is a topological manifold. Let $A, B \subseteq X$ be two codimension-zero submanifolds such that the following statements hold:
(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.
If $x_0 \in B$ and if B is connected, then the inclusion $i\colon (B, A \cap B) \to (X, A)$ induces an isomorphism

$$i_*\colon \mathrm{H}_n^{\mathrm{tw}}(B, A \cap B; i^*M) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{tw}}(X, B; M).$$

(3) We suppose that X is a CW-complex and that A and B are two subcomplexes of X with $X = A \cup B$. If $x_0 \in B$ and if B is connected, then the inclusion $i\colon (B, A \cap B) \to (X, A)$ induces an isomorphism

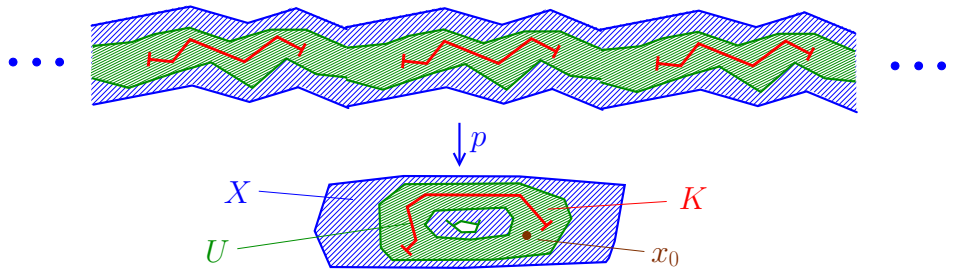
$$i_*\colon \mathrm{H}_n^{\mathrm{tw}}(B, A \cap B; i^*M) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{tw}}(X, B; M).$$

Analogous statements also hold for twisted cohomology.



Proof.

(1) (b) Let K be a subset of X and let U be a neighborhood of the closure of K in X. We suppose that $x_0 \in U$ and that U is path-connected and locally contractible. We denote by $i\colon (U, U \setminus K) \to (X, X \setminus K)$ the inclusion. Furthermore we denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and we write $\widetilde{U} := p^{-1}(U)$ and $\widetilde{K} := p^{-1}(K)$.



Claim. The inclusion-induced map

$$i_*\colon \mathrm{C}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \;\to\; \mathrm{C}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K})$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

Proof. We make the following two observations:

(i) First note that

$$\text{closure of } \widetilde{K} \underset{\text{see Exercise 91.18}}{=} p^{-1}(\text{closure of } K) \underset{\text{by hypothesis}}{\subseteq} p^{-1}(\text{interior of } U) \underset{\text{see Exercise 91.18}}{=} \text{interior of } \widetilde{U}.$$

Thus it follows from the original Excision Theorem 148.2 that for every $n \in \mathbb{N}_0$ the inclusion i induces an isomorphism

$$i_*\colon \mathrm{H}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \;\xrightarrow{\cong}\; \mathrm{H}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K}).$$

(ii) Next we consider the map

$$i_*\colon \mathrm{C}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \;\to\; \mathrm{C}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K}).$$

In the Regular Covering–Chain Complex Lemma 320.2 we saw that both sides are free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. By definition of the module structure the map i_* is actually a homomorphism between two free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

The claim now follows from (i) and (ii) combined with Quasi Iso–Homotopy Equivalence Proposition 313.18. ⊞

Now let M be a $\mathbb{Z}[\pi_1(X, x_0)]$-left module. It follows immediately from the claim together with the Homology-with-Coefficients Lemma 314.6 (2b) and the Chain Homotopy Equivalence-H_*-Corollary 313.7 that the map

$$\mathrm{id}_{\overline{M}} \otimes i_*\colon \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{U}, \widetilde{U} \setminus \widetilde{K}) \;\to\; \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{X}, \widetilde{X} \setminus \widetilde{K})$$

induces an R-left isomorphism of homology modules. It follows from this observation together with the Change-of-Covering Lemma 321.11 (1) and (2) that the map

$$i_*\colon \mathrm{H}_n^{\mathrm{tw}}(U, U \setminus K; i^*M) \;\to\; \mathrm{H}_n^{\mathrm{tw}}(X, X \setminus K; M)$$

is an R-left isomorphism. The statement regarding cohomology groups is proved in a very similar fashion, using the Dual Cochain Map Lemma 315.22 and the Cochain Homotopy-H^*-Lemma 315.16. We leave the details to the reader.

(a) This statement follows from Statement (b) by setting $U = X \setminus Z$ and $K = X \setminus A$. Again we leave the few details to the reader.

(2) As in the proof of the Manifold–Excision Theorem 148.3 one can use the Topological Collar Neighborhood Theorem 35.14 to deduce this statement from (1a). The key to

doing so is the Deformation Retract-$\mathrm{H}_*^{\mathrm{tw}}$-Corollary 322.7. We leave the details to the reader.

(3) As in the proof of the CW-Excision Theorem 148.4 one can use the Subcomplex-Neighborhood Proposition 137.9 to deduce this statement from (1a). Again all details are cheerfully left to the reader.

We leave it to the reader to modify the arguments to deal with twisted cohomology. ■

325.3. **Internal twisted (co-) homology.** In this section we introduce what we call internal twisted (co-) homology. This concept is rather technical, but it can be very useful when one wants to consider the relationship between the twisted invariants of a topological space X and of (possibly disconnected) subspaces of X. In particular we will use this concept to prove a first Mayer–Vietoris style theorem and the concept will play a key role in our proof of the Twisted Poincaré Duality Theorem 335.1.

Definition 325.5. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering.

(1) Given any subset $A \subseteq B \subseteq X$ we write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$ and we set

$$\begin{aligned} \mathrm{C}_k^{X,M}(B, A) &:= \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A}) \\ \mathrm{C}_{X,M}^k(B, A) &:= \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{B}, \widetilde{A}), M). \end{aligned}$$

These are (co-) chain complexes in the obvious way and we obtain the corresponding (co-) homology modules $\mathrm{H}_k^{X,M}(B, A)$ and $\mathrm{H}_{X,M}^k(B, A)$. We refer to these as internal (co-) homology modules with twisted coefficients.

(2) Given two pairs (B_1, A_1) and (B_2, A_2) of subspaces of X with $B_1 \subseteq B_2$ and with $A_1 \subseteq A_2$ we obtain, with the hopefully obvious notation, an inclusion-induced chain map $\mathrm{C}_k(\widetilde{B_1}, \widetilde{A_1}) \to \mathrm{C}_k(\widetilde{B_2}, \widetilde{A_2})$ which is a $\mathbb{Z}[\pi_1(X, x_0)]$-left homomorphism. Thus we get induced maps

$$\begin{aligned} i_*\colon \mathrm{H}_k^{X,M}(B_1, A_1) &\to \mathrm{H}_k^{X,M}(B_2, A_2) \\ i^*\colon \mathrm{H}_{X,M}^k(B_2, A_2) &\to \mathrm{H}_{X,M}^k(B_1, A_1) \end{aligned}$$

of R-left modules. Evidently internal (co-) homology is functorial under inclusions.

Note that in the setting of internal (co-) homology with twisted coefficients the subspaces are not required to be path-connected. This leads us to the following lemma:

Lemma 325.6. (Internal Homology–Components Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let $B \subseteq X$ be a subspace with path-components $\{B_j\}_{j\in J}$. Given $j \in J$ we denote by $i_j\colon B_j \to B$ the inclusion map. For each $n \in \mathbb{N}_0$ the maps

$$\bigoplus_{j\in J} i_{j*}\colon \bigoplus_{j\in J} \mathrm{H}_n^{X,M}(B_j) \to \mathrm{H}_n^{X,M}(B) \quad \text{and} \quad \prod_{j\in J} i_j^*\colon \mathrm{H}_n^{X,M}(B) \to \prod_{j\in J} \mathrm{H}_n^{X,M}(B_j)$$

are isomorphisms.

Proof. This lemma follows easily from the Path-Component–H_*–Lemma 143.21. We leave it to the reader to fill in the details. ■

As the reader will have noticed, the Change-of-Covering Lemma 321.11 relates internal (co-) homology with twisted coefficients to ordinary (co-) homology with twisted coefficients. For the reader's convenience we state the Change-of-Covering Lemma 321.11 with our new language:

Lemma 325.7. (Change-of-Covering Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $q\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let $B \subseteq X$ be a subspace with $x_0 \in B$. We assume that B is also path-connected and locally contractible. We write $\widetilde{B} := q^{-1}(B)$ and we denote by $p\colon \widehat{B}_{x_0} \to B$ the natural universal covering. Finally let $A \subseteq B$. We write $\widehat{A} := p^{-1}(A)$ and $\widetilde{A} := q^{-1}(A)$. We denote by the $i\colon B \to X$ the inclusion map and we denote by $\widetilde{i}\colon \widehat{B}_{x_0} \to \widetilde{X}_{x_0}$ the induced map from Definition 321.4.

(1) The maps

$$\begin{array}{rccl} \Omega^X_{B,A}\colon & \overbrace{\overline{i^*M} \otimes_{\mathbb{Z}[\pi_1(B,x_0)]} \mathrm{C}_k(\widehat{B}_{x_0}, \widehat{A})}^{=\mathrm{C}^{\mathrm{tw}}_k(B,A;i^*M)} & \to & \overbrace{\overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}, \widetilde{A})}^{=\mathrm{C}^{X,M}_k(B,A)} \\ & m \otimes \sigma & \mapsto & m \otimes \widetilde{i}_*(\sigma) \end{array}$$

are well-defined and they are natural isomorphisms of R-left chain complexes. Similarly the maps

$$\begin{array}{rccl} \Omega^X_{B,A}\colon & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{B}, \widetilde{A}), M)}^{=\mathrm{C}^k_{X,M}(B,A)} & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_1(B,x_0)]\text{-left}}(\mathrm{C}_k(\widehat{B}_{x_0}, \widehat{A}), i^*M)}^{=\mathrm{C}^k_{\mathrm{tw}}(B,A;i^*M)} \\ & \varphi & \mapsto & (\sigma \mapsto \varphi(\widetilde{i}_*(\sigma))) \end{array}$$

are well-defined (co-) chain maps and they are natural isomorphisms of R-left module cochain complexes. When $A = \varnothing$, then we just write Ω^X_B.

(2) The maps Ω from (1) are natural in the following sense. Suppose we have subspaces $S \subseteq T \subseteq X$ which are both path-connected and locally contractible and such that $x_0 \in S$. Furthermore suppose we have $A \subseteq S$ and $B \subseteq T$ with $A \subseteq B$. We denote by $i\colon S \to T$ and by $j\colon T \to X$ the inclusion. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^{\mathrm{tw}}_k(S, A; i^*j^*M) & \xrightarrow{\quad i_* \quad} & \mathrm{H}^{\mathrm{tw}}_k(T, B; j^*M) \\ \Omega^X_S \big\downarrow \cong & & \Omega^X_T \big\downarrow \cong \\ \mathrm{H}^{X,M}_k(S, A) & \xrightarrow{\quad i_* \quad} & \mathrm{H}^{X,M}_k(T, B). \end{array}$$

The analogous statements for cohomology modules also holds.

Proof. Statement (1) is just the content of the Change-of-Covering Lemma 321.11 (1). Statement (2) is new. Thus let us concentrate on the proof of Statement (2). We denote by $s\colon \overline{S}_{x_0} \to S$, $t\colon \widehat{T}_{x_0} \to T$ and $p\colon \widetilde{X}_{x_0} \to X$ the natural universal coverings.

We write $k = i \circ j\colon S \to X$. By the discussion in Definition 321.4 we get induced maps $\widetilde{i}\colon \overline{S}_{x_0} \to \widehat{T}_{x_0}$, $\widetilde{j}\colon \widehat{T}_{x_0} \to \widetilde{X}_{x_0}$ and $\widetilde{k}\colon \overline{S}_{x_0} \to \widetilde{X}_{x_0}$. One can easily verify that $\widetilde{k} = \widetilde{j} \circ \widetilde{i}$. Statement (2) follows in a straightforward fashion from this observation. ∎

The following proposition is a refinement of the Change-of-Base Point Proposition 321.3 in our present context:

Proposition 325.8. (Change-of-Base Point Proposition) Let X be a topological space that is path-connected and locally contractible and let $A \subseteq B \subseteq X$. Furthermore let $\gamma\colon [0,1] \to X$ be a path from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. For $i = 0, 1$ we denote by $p_i\colon \widetilde{X}_{x_i} \to X$ the natural universal covering and we write $\widetilde{A}_i := p_i^{-1}(A)$ and $\widetilde{B}_i := p_i^{-1}(B)$. Finally let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. As in the Change-of-Base Point Proposition 88.14 (1) we consider the isomorphism

$$\begin{array}{rcl} \gamma^*\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\alpha] & \mapsto & [\gamma * \alpha * \overline{\gamma}]. \end{array}$$

The isomorphism γ^* gives rise to a ring homomorphism $\gamma^*\colon \mathbb{Z}[\pi_1(X, x_1)] \to \mathbb{Z}[\pi_1(X, x_0)]$. Thus, by the Module–Change-of-Ring–Functor Lemma 309.13 we can consider the $\mathbb{Z}[\pi_1(X, x_1)]$-left module ${}_{\gamma^*}M$. Furthermore, as in the Natural Covering-Base Point Lemma 127.19 we consider the homeomorphism

$$\begin{array}{rcl} \gamma^*\colon \widetilde{X}_{x_1} & \to & \widetilde{X}_{x_0} \\ [\beta] & \mapsto & [\gamma * \beta]. \end{array}$$

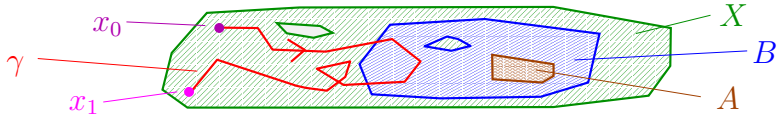
With this notation the maps

$$\begin{array}{rcl} \gamma^*\colon \overline{{}_{\gamma^*}M} \otimes_{\mathbb{Z}[\pi_1(X,x_1)]} \mathrm{C}_k(\widetilde{B}_1, \widetilde{A}_1) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{B}_0, \widetilde{A}_0) \\ m \otimes \sigma & \mapsto & m \otimes \gamma^*(\sigma) \end{array}$$

form an isomorphism of R-left chain complexes and the maps

$$\begin{array}{rcl} \gamma^*\colon \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_1)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{B}_1, \widetilde{A}_1), {}_{\gamma^*}M) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]-\mathrm{left}}(\mathrm{C}_k(\widetilde{B}_0, \widetilde{A}_0), M) \\ \varphi & \mapsto & \varphi \circ \gamma^* \end{array}$$

form a natural isomorphism of R-left module cochain complexes.



Proof. The proof is basically identical to the proof of the original Change-of-Base Point Proposition 321.3. ■

Many results on ordinary twisted (co-) homology have fairly obvious analogs in the context of internal twisted (co-) homology. For example we have the following variation on the Twisted-LES-Proposition 321.10:

Proposition 325.9. (Internal Twisted-LES-Proposition) Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, b_0)])$-left left module. Furthermore let A, B and C be subspaces of X. Given any $n \in \mathbb{N}_0$ there exists a natural R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{X,M}(C, B) \to \mathrm{H}_{n-1}^{X,M}(B, A)$$

such that the following sequence is exact[3561]

$$\ldots \xrightarrow{\partial_{n+1}} \mathrm{H}_n^{X,M}(B, A) \xrightarrow{i_*} \mathrm{H}_n^{X,M}(C, A) \xrightarrow{j_*} \mathrm{H}_n^{X,M}(C, B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(B, A) \to \ldots$$

An analogous statement also holds for internal cohomology groups with twisted coefficients.

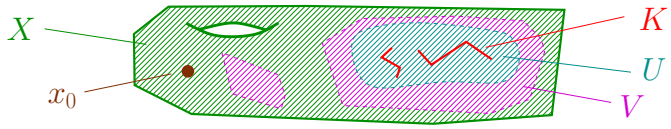
[3561]Here we denote by $i\colon (B, A) \to (C, A)$ and $j\colon (C, A) \to (C, B)$ the natural maps.

Proof. We essentially proved this statement in the proof of the Twisted-LES-Proposition 321.10. In fact one could argue, that the Twisted-LES-Proposition 321.10 is a consequence of the Internal Twisted-LES-Proposition 325.9. ■

Finally we also have the following variation on the Twisted Excision Theorem 325.4.

Theorem 325.10. (Internal Twisted Excision Theorem) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Suppose we are given subsets $K \subseteq U \subseteq V$ where U is a neighborhood of the closure of K in V. The inclusion $i\colon (U, U \setminus K) \to (V, V \setminus K)$ induces for each $n \in \mathbb{N}_0$ isomorphisms

$$\begin{aligned} i_*\colon \mathrm{H}_n^{X,M}(U, U \setminus K) &\xrightarrow{\cong} \mathrm{H}_n^{X,M}(V, V \setminus K) \\ i^*\colon \mathrm{H}^n_{X,M}(V, V \setminus K) &\xrightarrow{\cong} \mathrm{H}^n_{X,M}(U, U \setminus K). \end{aligned}$$



Proof. The proof is similar to the proof of the Excision Theorem 325.4. We leave it to the reader to make the necessary changes. ■

325.4. **Mayer-Vietoris Theorems for (co-) homology with twisted coefficients.** Working with internal twisted cohomology one can easily formulate Mayer–Vietoris type theorems:

Theorem 325.11. (Internal–Mayer–Vietoris Theorem) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let (X, A, B) be a universally excisive triad with $X = A \cup B$. For every $n \in \mathbb{N}_0$ there exists a natural connecting R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{X,M}(A \cup B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(A \cap B)$$

such that the following sequence

$$\cdots \to \mathrm{H}_n^{X,M}(A \cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n^{X,M}(A \cup B) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{X,M}(A \cap B) \to \dots$$

of R-left module homomorphisms is exact.

Proof. As usual we denote by $p\colon \widetilde{X} \to X$ the natural universal covering and we write $\pi := \pi_1(X, x_0)$. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$. Note that $\widetilde{A} \cap \widetilde{B} = p^{-1}(A \cap B)$ and $\widetilde{A} \cup \widetilde{B} = p^{-1}(A \cup B)$. In the proof of the original Mayer–Vietoris Theorem 153.2 we showed that the sequence

$$0 \to \mathrm{C}_*(\widetilde{A} \cap \widetilde{B}) \xrightarrow{(i_{\widetilde{A}\cap\widetilde{B}}, -i_{\widetilde{A}\cap\widetilde{B}})} \mathrm{C}_*(\widetilde{A}) \oplus \mathrm{C}_*(\widetilde{B}) \xrightarrow{i_{\widetilde{A}} \oplus i_{\widetilde{B}}} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \to 0$$

of chain maps of $\mathbb{Z}[\pi]$-left chain complexes is exact. By the Universally Excisive Properties Lemma 325.2 (1) we know that $\mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})$ is a free $\mathbb{Z}[\pi]$-left module. It follows from this observation together with the Tensor-Split-SES Lemma 314.9 and the Tensor–Direct Sum Lemma 312.15 that the sequence

$$0 \to \overline{M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*(\widetilde{A}\cap\widetilde{B}) \xrightarrow{(i_{\widetilde{A}\cap\widetilde{B}}, -i_{\widetilde{A}\cap\widetilde{B}})} \overline{M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*(\widetilde{A}) \oplus \overline{M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*(\widetilde{B}) \xrightarrow{i_{\widetilde{A}} \oplus i_{\widetilde{B}}} \overline{M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A}\cup\widetilde{B}) \to 0$$

of R-left module chain maps is also exact. We deduce from the Connecting Homomorphism Lemma 313.9 together with the LES Proposition 313.10 that we get natural connecting homomorphisms of R-left modules which give rise to a long exact sequence of R-left module homomorphisms.

Recall that we assume that (X, A, B) is in fact universally excisive with $X = A \cup B$. It follows from the Universally Excisive Properties Lemma 325.2 (2) that the inclusion map

$$\mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \;\to\; \mathrm{C}_*(\widetilde{A} \cup \widetilde{B})$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi]$-left module chain complexes. Hence for every $n \in \mathbb{N}_0$ it induces an isomorphism

$$\Xi\colon \mathrm{H}_n\big(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})\big) \;\xrightarrow{\cong}\; \underbrace{\mathrm{H}_n\big(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{A} \cup \widetilde{B})\big)}_{=\mathrm{H}_n^{X,M}(A\cup B)}.$$

The idea now is to use Ξ to obtain the desired long exact sequence. To do so we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) & \xrightarrow{\mathrm{id}_{\overline{M}} \otimes i_{\widetilde{A}} + \mathrm{id}_{\overline{M}} \otimes i_{\widetilde{B}}} & \mathrm{H}_n\big(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})\big) & \xrightarrow{\partial_n} & \mathrm{H}_n^{X,M}(A \cap B) \\
& \searrow^{i_A^X \oplus i_B^X} & \cong \big\downarrow \Xi & \nearrow_{=:\partial_n} & \\
& & \mathrm{H}_n^{X,M}(A \cup B). & &
\end{array}$$

The crucial observation is that one can easily verify that the triangle to the left commutes. Thus going "via the bottom" we do indeed obtain the desired connecting homomorphism that gives rise to the promised long exact sequence. ■

The Internal Mayer–Vietoris Theorem 325.11 for Internal Twisted Homology, as it is formulated, will play a major role in the proof of the Twisted Poincaré Duality Theorem 335.1. But the alert reader will have noticed that the Internal Mayer–Vietoris Theorem 325.11 for Internal Twisted Homology is not all that useful for performing calculations since it is formulated in terms of the unwieldy internal homology modules. Of course, by the Change-of-Covering Lemma 325.7 together with the Internal Homology–Components Lemma 325.6, we know that each such internal homology module is isomorphic to the direct sum of an appropriate twisted homology module for each of the components. The catch is, that at this point it becomes increasingly unclear what are the "inclusion-induced maps".

The following theorem gives us a clean statement if all subspaces involved are path-connected:

Theorem 325.12. (Twisted Mayer–Vietoris Theorem) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let $A, B \subseteq X$ be subspaces such that the following conditions are satisfied:

(1) (X, A, B) is a universally excisive triad in the sense of Definition 325.1,
(2) we have $X = A \cup B$,
(3) A, B and $A \cap B$ are path-connected and locally contractible,
(4) $x_0 \in A \cap B$.

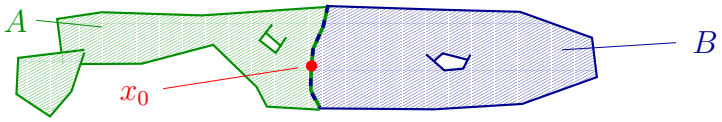
There exists a natural connecting R-left module homomorphism[3562]

$$\partial_n \colon \mathrm{H}_n^{\mathrm{tw}}(X;M) \ \to \ \mathrm{H}_{n-1}^{\mathrm{tw}}(A\cap B; i_{A\cap B}^* M)$$

such that the following sequence

$$\to \mathrm{H}_n^{\mathrm{tw}}(A\cap B; i_{A\cap B}^* M) \xrightarrow{(i_{A\cap B}^A, -i_{A\cap B}^B)} \mathrm{H}_n^{\mathrm{tw}}(A; i_A^* M) \oplus \mathrm{H}_n^{\mathrm{tw}}(B; i_B^* M) \xrightarrow{i_A^X \oplus i_B^X} \mathrm{H}_n^{\mathrm{tw}}(X;M) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \mathrm{H}_{n-1}^{\mathrm{tw}}(A\cap B; i_{A\cap B}^* M) \to \dots$$

of R-left module homomorphisms is exact.[3563]



Proof. By the Internal Mayer–Vietoris Theorem 325.11 for Internal Twisted homology there exists a natural connecting R-left module homomorphism

$$\partial_n \colon \mathrm{H}_n^{X,M}(X) \ \xrightarrow{\partial_n} \ \mathrm{H}_{n-1}^{X,M}(A\cap B)$$

such that the following sequence is exact:

$$\cdots \to \ \mathrm{H}_n^{X,M}(A\cap B) \xrightarrow{(i_{A\cap B}^A, -i_{A\cap B}^B)} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) \xrightarrow{i_A^X \oplus i_B^X} \mathrm{H}_n^{X,M}(X) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \ \mathrm{H}_{n-1}^{X,M}(A\cap B) \to \dots$$

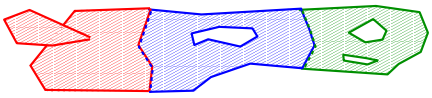
The Change-of-Covering Lemma 325.7 II (1) gives us isomorphisms

$$\Omega_{A\cap B}^X \colon \mathrm{H}_k^{\mathrm{tw}}(A\cap B; i_{A\cap B}^* M) \ \to \ \mathrm{H}_k^{X,M}(A\cap B),$$

and similarly for A and B. After making all these replacements we obtain a long exact sequence with the right modules at the right place. It remains to show that the maps in the resulting diagram are the maps we promised. But this is an immediate consequence of the Change-of-Covering Lemma 325.7 II (2). ■

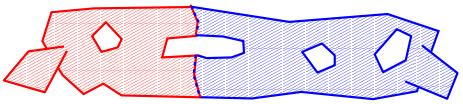
The most restrictive extra condition that we impose in the above Twisted Mayer–Vietoris Theorem 325.12 compared to the traditional Mayer–Vietoris Theorem 153.2 is that we now demand that $A\cap B$ is path-connected. If $A\cap B$ is not path-connected then sometimes we can get around this issue by applying the Twisted Mayer–Vietoris Theorem 325.12 iteratively while moving the base point. But this approach does not always work.

decomposition of a topological space X



can iteratively apply Mayer-Vietoris for connected intersection

decomposition of a topological space X



cannot iteratively apply Mayer-Vietoris for connected intersection

The following theorem will allows us to get rid of the hypothesis in the Twisted Mayer–Vietoris Theorem 325.12 that the intersection $A\cap B$ needs to be path-connected. The price we pay for this flexibility will another layer of notation which obfuscates the statement even further.

[3563]Here $A\cap B$, A, B and X are all equipped with the base point x_0. The various maps denoted by i are the obvious inclusion (induced) maps, with our choice of base point, all maps are base point preserving.

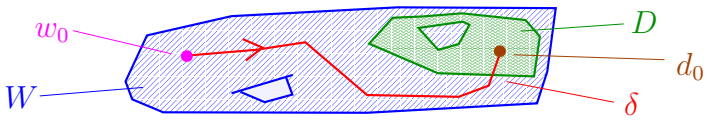
Notation 325.13. Let W be a topological space that is path-connected and locally contractible and let $w_0 \in W$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(W, w_0)])$-left left module. Let $D \subseteq W$ be a subspace that is path-connected and locally contractible, let $d_0 \in D$ and let $\delta\colon [0,1] \to W$ be a path from w_0 to d_0. We denote by $i\colon D \to W$ the inclusion and we denote by i^δ the chain map

$$\mathrm{C}_k^{\mathrm{tw}}(D; i^*\delta^{**}M) \xrightarrow[\uparrow]{i_*} \mathrm{C}_k^{\mathrm{tw}}(W; \delta^{**}M) \xrightarrow[\uparrow]{\delta_*} \mathrm{C}_k^{\mathrm{tw}}(W; M)$$

map that is induced by the inclusion $i\colon (D, d_0) \to (W, w_0)$, see Definition 321.8

isomorphism given by the Change-of-Base Point Proposition 321.3

of R-left chain complexes. Furthermore we denote by $i^\delta\colon \mathrm{H}_k^{\mathrm{tw}}(D; i^*\delta^{**}M) \to \mathrm{H}_k^{\mathrm{tw}}(W; M)$ the induced map. Finally we simplify $\delta^{**}M$ and $i^*\delta^{**}M$ to δ_*M.



Theorem 325.14. (Twisted Mayer–Vietoris Theorem) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let $A, B \subseteq X$ be subspaces such that the following conditions are satisfied:

(1) (X, A, B) is a universally excisive triad,
(2) we have $X = A \cup B$,
(3) A and B are path-connected,
(4) A, B and $A \cap B$ are locally contractible,
(5) $A \cap B$ consists of finitely many path-components $C_1, \ldots, C_k$.

We pick $a_0 \in A$, $b_0 \in B$, we pick $c_j \in C_j$, $j = 1, \ldots, k$, we pick a path $\alpha\colon [0,1] \to X$ from x_0 to a_0, we pick a path $\beta\colon [0,1] \to X$ from x_0 to b_0, we pick paths $\gamma_{j,A}\colon [0,1] \to A$ from a_0 to c_j and we pick paths $\gamma_{j,B}\colon [0,1] \to B$ from b_0 to c_j. There exist natural R-left module homomorphisms

$$\partial_n\colon \mathrm{H}_n^{\mathrm{tw}}(X; M) \to \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_*M)$$

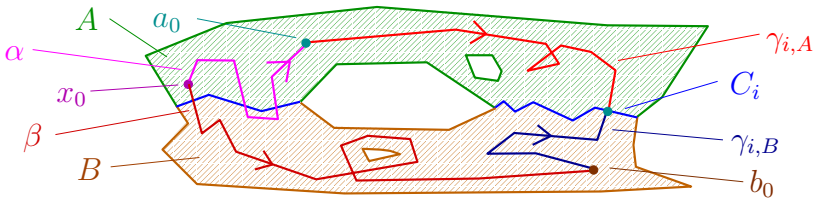
and natural homomorphisms[3564]

$$\Psi_j\colon \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_*M) \to \mathrm{H}_n^{\mathrm{tw}}(B; \beta_*M)$$

such that the following sequence

$$\to \bigoplus_{j=1}^{k} \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_*M) \xrightarrow{\begin{pmatrix} \oplus_j i^{\gamma_{j,A}}, -\oplus_j \Psi_j \end{pmatrix}} \begin{matrix} \mathrm{H}_n^{\mathrm{tw}}(A; \alpha_*M) \\ \mathrm{H}_n^{\mathrm{tw}}(B; \beta_*M) \end{matrix} \xrightarrow{i^\alpha \oplus i^\beta} \mathrm{H}_n^{\mathrm{tw}}(X; M) \xrightarrow{\partial_n}$$
$$\xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_*M) \to \ldots$$

of R-left module homomorphisms is exact.



Remark.

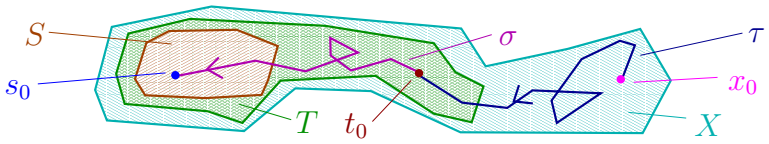
(1) At this stage it should be clear to the reader that one does not necessarily need to assume that A and B are path-connected. With even more messy notation one can formulate a generalization to the non path-connected case. We leave the task of texing this to the reader.

(2) In practice it can be very hard to compute honestly and rigorously the involved homomorphisms. □

For the proof of the above Twisted Mayer–Vietoris Theorem 325.14 we will need the following technical lemma.

Auxiliary Lemma 325.15. Let X be a topological space and let $S \subseteq T \subseteq X$ be subspaces. We assume that S, T and X are path-connected and locally contractible. Let $s_0 \in S$, $t_0 \in T$ and $x_0 \in X$ be points and let $\sigma\colon [0,1] \to T$ be a path from t_0 to s_0 and let $\tau\colon [0,1] \to X$ be a path from x_0 to t_0. We denote by $i\colon S \to T$ and $j\colon T \to X$ the inclusions. Finally let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module. With all this notation, together with Notation 325.13 and the notation from the Change-of-Covering Lemma 325.7 the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_n^{X,M}(S) & \xrightarrow{(\tau*\sigma)_*} & \mathrm{H}_n^{X,(\tau*\sigma)_*M}(S) & \xrightarrow{\Omega_S^X} & \mathrm{H}_n^{\mathrm{tw}}(S;(\tau*\sigma)_*M) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i^\sigma} & & \downarrow{\scriptstyle i_*} \\ \mathrm{H}_n^{X,M}(T) & \xrightarrow{\tau_*} & \mathrm{H}_n^{X,\tau_*M}(T) & \xrightarrow{\Omega_T^X} & \mathrm{H}_n^{\mathrm{tw}}(T;\tau_*M). \end{array}$$



Proof of Lemma 325.15. We leave it to the reader to verify this lemma. ■

Proof of the Twisted Mayer–Vietoris Theorem 325.14. It follows from the Internal Mayer–Vietoris Theorem 325.11 together with the Internal Homology–Components Lemma 325.6 that there exists a natural R-left module homomorphism

$$\partial_n\colon \mathrm{H}_n^{tw}(X;M) \xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{X,M}(C_j)$$

[3564]These homomorphisms are described in the proof of the theorem. There does not seem to be a snappy description of these maps.

such that the following sequence

$$\cdots \to \bigoplus_{j=1}^{k} \mathrm{H}_n^{X,M}(C_j) \xrightarrow{(\oplus i_{C_j}, -\oplus i_{C_j})} \mathrm{H}_n^{X,M}(A) \oplus \mathrm{H}_n^{X,M}(B) \xrightarrow{i_A^X \oplus i_B^X} \mathrm{H}_n^{\mathrm{tw}}(M) \xrightarrow{\partial_n} \xrightarrow{\partial_n} \bigoplus_{j=1}^{k} \mathrm{H}_{n-1}^{X,M}(C_j) \to \dots$$

of R-left module homomorphisms is exact. As before we want to replace these internal homology modules by honest-to-G*d twisted homology modules. But this time we are in a sea of troubles since we surrendered all control over base points. Nonetheless, we will give it a try. We have the isomorphisms

$$\begin{array}{lccccc}
\text{(i)} & \mathrm{H}_n^{X,M}(C_j) & \xrightarrow{(\alpha * \gamma_{j,A})_*} & \mathrm{H}_n^{X,(\alpha*\gamma_{j,A})_*M}(C_j) & \xrightarrow{\Omega^X_{C_j}} & \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_* M) \\
\text{(ii)} & \mathrm{H}_n^{X,M}(A) & \xrightarrow{\alpha_*} & \mathrm{H}_n^{X,\alpha_*M}(A) & \xrightarrow{\Omega^X_{A}} & \mathrm{H}_n^{\mathrm{tw}}(A; \alpha_* M) \\
\text{(iii)} & \mathrm{H}_n^{X,M}(B) & \xrightarrow{\beta_*} & \mathrm{H}_n^{X,\beta_*M}(B) & \xrightarrow{\Omega^X_{B}} & \mathrm{H}_n^{\mathrm{tw}}(B; \beta_* M).
\end{array}$$

Here the isomorphisms to the left are provided by the Change-of-Base Point Proposition 325.8 and the isomorphisms to the right are provided by the Change-of-Covering Lemma 325.7. Fortunately the above auxiliary Lemma 325.15 sails to our rescue. It implies that in most cases, if we use the above isomorphisms to replace our previous inclusion-induced maps, we get again inclusion-induced maps. For example for each $j \in \{1, \dots, k\}$ we get that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n^{X,M}(C_j) & \xrightarrow[\cong]{(\alpha*\gamma_{j,A})_*} & \mathrm{H}_n^{X,(\alpha*\gamma_{j,A})^*M}(C_j) & \xrightarrow[\cong]{\Omega^X_{C_j}} & \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,A})_* M) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i^{\gamma_{j,A}}} & & \downarrow{\scriptstyle i} \\
\mathrm{H}_n^{X,M}(A) & \xrightarrow[\cong]{\alpha_*} & \mathrm{H}_n^{X,\alpha_*M}(A) & \xrightarrow[\cong]{\Omega^X_{A}} & \mathrm{H}_n^{\mathrm{tw}}(A; \alpha_* M).
\end{array}$$

A similar, in fact much simpler logic, applies to i_A and i_B. But the inclusion from C_j to B gets screwed up, since the paths do not match up. In fact the map Ψ_j is the following composition of maps, for which this particular author cannot find a pleasant shortcut:

$$\begin{array}{ccccc}
\mathrm{H}_n^{X,M}(C_j) & \xrightarrow[\cong]{(\alpha*\gamma_{j,B})_*} & \mathrm{H}_n^{X,(\alpha*\gamma_{j,B})^*M}(C_j) & \xrightarrow[\cong]{\Omega^X_{C_j}} & \mathrm{H}_n^{\mathrm{tw}}(C_j; (\alpha * \gamma_{j,B})_* M) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle =:\Psi_j} \\
\mathrm{H}_n^{X,M}(B) & \xrightarrow[\cong]{\beta_*} & \mathrm{H}_n^{X,\beta_*M}(B) & \xrightarrow[\cong]{\Omega^X_{B}} & \mathrm{H}_n^{\mathrm{tw}}(B; \beta_* M).
\end{array}$$

■

325.5. **Exercises for Chapter 325.**

Exercise 325.1. Use the Mayer–Vietoris theorem to show that given any lens space and given any non-trivial one-dimensional complex representation all twisted homology modules are zero.

Hint. Use the Lens Space–Heegaard Splitting Proposition **??**.

Exercise 325.2. Let $T = S^1 \times S^1$ be a 2-dimensional torus, let $x_0 \in T = S^1 \times S^1$ be base point and let $\varphi\colon \pi_1(T, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a non-trivial representation. Calculate the twisted homology modules of $\mathrm{H}_*^{\mathrm{tw}}(T; {}_\varphi\mathbb{C})$.

CHAPTER 326

The Kronecker pairing for twisted homology and cohomology

Let R be a commutative ring. In "untwisted" (co-) homology theory the Kronecker pairing

$$\begin{aligned} \langle\ ,\ \rangle\colon \quad \overbrace{\mathrm{H}^n(C;R)}^{=\mathrm{H}^n(\mathrm{Hom}(C_*,R))} \ \times\ \overbrace{\mathrm{H}_n(C;R)}^{=\mathrm{H}_n(C_*\otimes R)} \ &\to\ R \\ \Big([\varphi],\Big[\sum_{i=1}^{k} c_i\otimes a_i\Big]\Big) \ &\mapsto\ \sum_{i=1}^{k}\varphi(c_i)\cdot a_i. \end{aligned}$$

from the Ring–Kronecker Pairing Lemma 202.1 was a useful tool for relating homology to cohomology. In this chapter we will introduce an analog and we will see that at least in some settings it can be used to calculate twisted cohomology in terms of twisted homology.

This chapter is organized as follows:

- In Section 326.1 we will introduce the Kronecker pairing for twisted homology and cohomology.
- In Section 326.2 we will study non-singular pairings over possibly non-commutative rings.
- In Section 326.3 we will formulate the Twisted Evaluation–Isomorphism Proposition 326.9 which says that in some settings a dual of twisted homology is isomorphic to twisted cohomology.
- In Section 326.4 we will provide a proof of the Twisted Evaluation–Isomorphism Proposition 326.9.
- In Section 326.5 we will provide exercises for Chapter 326.

In the next Chapter 327 we will introduce twisted cellular (co-) homology which can be a convenient tool for calculating twisted (co-) homology.

326.1. **The Kronecker pairing for twisted homology and cohomology.** We start out by recalling the first half of Definition 323.1:

Definition 326.1. Let π be a group, let R be a ring and let M be an $(R,\mathbb{Z}[\pi])$-left left module. We define

$$M_\pi \ := \ M \Big/ \Big\{ \sum_{i=1}^{k} (g_i\cdot v_i - v_i) \ \Big|\ g_i\in\pi \text{ and } v_i\in M \Big\} \qquad \pi\text{-coinvariants of } M.$$

We continue with the following definition which is inspired by Definition 199.8 and by the Ring–Kronecker Pairing Lemma 202.1:

Definition 326.2. Let $\mathbb{Z}[\pi]$ be a ring, let C_* be a chain complex of $\mathbb{Z}[\pi]$-left modules and let M,N be two $\mathbb{Z}[\pi]$-left modules. We first make two preparations:

- We equip $M\otimes_{\mathbb{Z}} N$ with the $\mathbb{Z}[\pi]$-left module structure that is defined by the condition that for each $g\in\pi$ we have $g\cdot(m\otimes n)=(g\cdot m)\otimes(g\cdot n)$.

- We consider the π-coinvariants $(M \otimes_{\mathbb{Z}} N)_\pi$ from Definition 326.1.

For each $k \in \mathbb{N}_0$ we now consider the map[3565]

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_k, M) \times (\overline{N} \otimes_{\mathbb{Z}[\pi]} C_k) & \to & (M \otimes_{\mathbb{Z}} N)_\pi \\ (\varphi, n \otimes \sigma) & \mapsto & [n \otimes \varphi(\sigma)]. \end{array}$$

As in the proof of the Kronecker Pairing Lemma 199.7 one can easily verify that this map descends to a map on (co-) homology groups, thus we get a map

$$\mathrm{H}^k(\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, M)) \times \mathrm{H}_k(\overline{N} \otimes_{\mathbb{Z}[\pi]} C_*) \to (M \otimes_{\mathbb{Z}} N)_\pi$$

which is called the Kronecker pairing.

This purely algebraic notion evidently also leads to a pairing in topology:

Definition 326.3. Let (X, x_0) be a pointed topological space that is path-connected and locally contractible and let $A \subseteq X$. We write $\pi := \pi_1(X, x_0)$. Let R be a ring and let M and N be two $(R, \mathbb{Z}[\pi])$-left left modules. For each $k \in \mathbb{N}_0$ we obtain from Definition 326.2 a pairing

$$\langle -, - \rangle_{X,A} \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_k(X, A; N) \to (M \otimes_{\mathbb{Z}} N)_\pi$$

which is also called a Kronecker pairing.

Remark. Later in Definition 334.23 we will introduce a cap product on twisted (co-) homology which, in light of the Twisted H_0-Proposition 323.3, can be viewed as a generalization of Definition 326.3. □

326.2. **Non-singular pairings.** In this section we will generalize a few definitions about pairings over commutative rings to pairings over general rings. We will deal with several rings with involution, to keep track of these rings and their involutions we introduce the following notation:

Notation 326.4. Let R be a ring with involution.

- Let M be an R-left module. We denote by $\overline{M}^R$ the R-right module from the Ring-with-Involution–Module Lemma 309.16.
- Let M be an R-right module. We denote by ${}^R\overline{M}$ the R-left module from the Ring-with-Involution–Module Lemma 309.16.

We should mention the following convention that was established in the Hom–Module Lemma 315.2:

Convention 326.5. Let Q, R and S be rings and let M be a (Q, R)-left left module and let N be an (R, S)-left left module. We equip $\mathrm{Hom}_{R\text{-left}}(M, R)$ with the natural Q-right-module structure and the natural S-left module structure. Again we use the obvious variations on this convention.

After these pieces of notation we can now move on to the definitions that really interest us. First of all, given a commutative ring we introduced in Definition 125.16 the notion of an

[3565]Note that for each $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_k, M)$, $n \in N, \sigma \in C_k$ and each $g \in \pi$ we have the following equalities in $(M \otimes_{\mathbb{Z}} N)_\pi$:

$$(n \cdot g) \otimes \varphi(\sigma) = (g^{-1} \cdot n) \otimes \varphi(\sigma) = g \cdot (g^{-1} \cdot n) \otimes g \cdot \varphi(\sigma) = n \otimes g \cdot \varphi(\sigma) = n \otimes \varphi(g \cdot \varphi).$$

It now follows from the Tensor Product–Universal Property Lemma 320.5 that this map is well-defined, i.e. we get a well-defined map out of the tensor product $\overline{N} \otimes_{\mathbb{Z}[\pi]} C_k$.

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R-sesquilinear pairing. We now extend this notion in the obvious way to non-commutative rings:

Definition 326.6. Let R be a ring with (possibly trivial) involution and let V and W be two R-left modules. Furthermore let Z be an R-bimodule. We say that a pairing $\varphi\colon V \times W \to Z$ is R-sesquilinear if it is R-linear in the first entry and R-antilinear in the second entry. In other words, we demand that for each $v \in V$ and $w \in W$ the following maps

$$\begin{array}{ccc} V & \to & Z \\ v & \mapsto & \varphi(v,w) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W & \to & {}^{R}\overline{Z} \\ w & \mapsto & \varphi(v,w) \end{array}$$

are homomorphisms of R-left modules.

Next, given a commutative ring we introduced in Definition 213.1 the notion of a *non-singular* R-bilinear pairing. We now extend this notion, again in the almost obvious way, to the more general setting of R-sesquilinear pairings over arbitrary rings with involution:

Definition 326.7. Let R be a ring with (a possibly trivial) involution and let V and W be two R-left modules and let Z be an R-bimodule. We use the involution to define the (R,R)-left left module ${}^{R}\overline{Z}$. We say that an R-sesquilinear pairing $\varphi\colon V \times W \to Z$ is non-singular if the induced maps[3566] [3567] [3568]

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_{R\text{-left}}(W, {}^{R}\overline{Z}) \\ v & \mapsto & \left(\begin{array}{ccc} W & \to & {}^{R}\overline{Z} \\ w & \mapsto & \varphi(v,w) \end{array} \right) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W & \to & {}^{R}\overline{\mathrm{Hom}_{R\text{-left}}(V, Z)} \\ w & \mapsto & \left(\begin{array}{ccc} V & \to & Z \\ v & \mapsto & \varphi(v,w) \end{array} \right) \end{array}$$

are isomorphisms of R-left modules. Otherwise we say that the pairing is singular

Lemma 326.8. (Dual-π-Lemma) Let R be a ring with involution, let π be a group and let V, W be two $(R, \mathbb{Z}[\pi])$-left left modules. Furthermore let Z be an R-bimodule. Let $\varphi\colon V \times W \to Z$ be an R-sesquilinear form. We suppose that π respects the form, i.e. we suppose that for each $v \in V$, $w \in W$ and $g \in \pi$ we have $\varphi(g \cdot v, g \cdot w) = \varphi(v,w)$.

(1) The map[3569]

$$\begin{array}{rccc} \nu\colon V & \to & \mathrm{Hom}_{R\text{-left}}(\overline{W}^{\mathbb{Z}[\pi]}, {}^{R}\overline{Z}) \\ v & \mapsto & w \mapsto \varphi(v,w) \end{array}$$

is a homomorphism of $(R, \mathbb{Z}[\pi])$-left left modules.

(2) If φ is non-singular, then the map in (1) is an isomorphism of $(R, \mathbb{Z}[\pi])$-left left modules.

Proof.

[3566] It follows from Definition 326.6 that the maps do indeed take values in $\mathrm{Hom}_{R\text{-left}}(W, {}^{R}\overline{Z})$ and $\mathrm{Hom}_{R\text{-left}}(V, Z)$.

[3567] We equip the set of homomorphisms with the R-modules structures from Convention 326.5.

[3568] If the pairing $\varphi\colon V \times W \to R$ is (anti-) Hermitian, then one can easily verify that if one map is an isomorphism (respectively monomorphism), then so is the other.

[3569] We leave it to the reader to verify that all the conventions play out well.

(1) We need to show that the map ν also respects the $\mathbb{Z}[\pi]$-left module structure. Thus let $g \in \pi$ and $v \in V$. Then given any $w \in W$ we have

$$\begin{array}{rcl} \nu(g \cdot v)(w) & = & \varphi(g \cdot v, w) \\ & & \overset{\text{since } \pi \text{ respects } \varphi}{\downarrow} \\ & & = \quad (\nu(v) \cdot g)(w) \\ & & \underset{\text{see the convention from the Hom–Module Lemma 315.2}}{\uparrow} \end{array} \begin{array}{l} \overset{\downarrow}{=} \ \varphi(, g^{-1} \cdot w) \\ \\ = \ (g \cdot \nu(v))(w). \\ \underset{\text{see Notation 320.7}}{\uparrow} \end{array} \begin{array}{l} = \ \nu(v)(g^{-1} \cdot w) \\ \\ \\ \end{array}$$

(2) This statement follows from (1) and the definition of a non-singular form. ■

326.3. **Dual of twisted homology.** The following proposition is the main result of this chapter:

Proposition 326.9. (Twisted Evaluation–Isomorphism Proposition) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let $\mathbb{K}$ be a skew field with involution and let V and W be two $(\mathbb{K}, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules. Finally suppose that we are given a $\mathbb{Z}[\pi_1(X, x_0)]$-left linear map $\Theta \colon V \otimes_{\mathbb{Z}} W \to \mathbb{K}$ such that the corresponding pairing

$$\begin{array}{rcl} \varphi_\Theta \colon V \times W & \to & \mathbb{K} \\ (v, w) & \mapsto & \Theta(v \otimes w) \end{array}$$

is $\mathbb{K}$-sesquilinear and non-singular. Then for any $A \subseteq X$ the $\mathbb{K}$-left linear map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{tw}}(X, A; V) & \to & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}^{\mathrm{tw}}_k(X, A; W), \mathbb{K}) \\ & & \overset{\mathbb{K}\text{-left module via } k \cdot m := m \cdot_{\mathbb{K}} \bar{k}}{\downarrow} \\ \varphi & \mapsto & (\mu \mapsto \Theta(\langle \varphi, \mu \rangle)) \end{array}$$

is an isomorphism of $\mathbb{K}$-left modules.[3570]

Proof. The proof of the proposition will be given in Section 326.4. ■

Example.

(1) Let $\mathbb{K}$ be a field (i.e. a commutative skew field) with the trivial $\pi_1(X, x_0)$-left action. We then view ${}_{\mathrm{tr}}\mathbb{K}$ as $(\mathbb{K}, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. We consider the obvious non-singular $\mathbb{K}$-bilinear pairing $\varphi \colon \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ that is given by $\varphi(v, w) = v \cdot w$. The Twisted Evaluation–Isomorphism Proposition 326.9 then gives us an explicit isomorphism

$$\mathrm{H}^k_{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}\mathbb{K}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{K}}(\mathrm{H}^{\mathrm{tw}}_k(X, A; {}_{\mathrm{tr}}\mathbb{K})).$$

Using the Twisted–Trivial Coefficients Lemma 320.13 one sees that this is essentially the same as the isomorphism from the classical Universal Coefficient-H*–Field Theorem 202.2.

[3570] For this statement the $\mathbb{K}$-left module structure on $\mathrm{H}^k_{\mathrm{tw}}(X, A; V)$ comes from the given $\mathbb{K}$-left module structure on V. Furthermore the $\mathbb{K}$-left module structure on $\mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}^{\mathrm{tw}}_k(X, A; V), \mathbb{K})$ is given by the obvious $\mathbb{K}$-left module structure on $\mathbb{K}$ that is given by left multiplication.

(2) We consider the circle S^1 together with any non-trivial 1-dimensional complex representation $\varphi\colon \pi_1(S^1,1)\to \mathrm{GL}(1,\mathbb{C})$. We now see that for each $i\in\mathbb{N}_0$ we have

$$\begin{array}{ccccc} \mathrm{H}^*(S^1;{}_\varphi\mathbb{C}) & \cong & \mathrm{H}_*(S^1;{}_\varphi\mathbb{C}) & \cong & 0. \\ & \uparrow & & \uparrow & \\ & \text{by the Twisted Evaluation} & & \text{by the Twisted-H}_*\text{-of-}S^1 & \\ & \text{Isomorphism Proposition 326.9} & & \text{Proposition 320.14} & \end{array}$$

The main application of the Twisted Evaluation–Isomorphism Proposition 326.9 will have to wait till the Twisted Intersection Pairing–Non-Singularity Proposition 337.14. □

326.4. **Proof of the Twisted Evaluation–Isomorphism Proposition 326.9.** Not surprisingly this section is completely dedicated to the proof of the Twisted Evaluation–Isomorphism Proposition 326.9. For the proof we will need the following slight generalization of the Evaluation-Isomorphism Theorem 77.18:

Theorem 326.10. (Evaluation-Isomorphism Theorem) Let $\mathbb{K}$ be a skew field and let (D_*,∂_*) be a chain complex of $\mathbb{K}$-left vector spaces. We view $\mathbb{K}$ as $(\mathbb{K},\mathbb{K})$-bimodule in the obvious way. The map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{K}}\colon \mathrm{H}^k(\mathrm{Hom}_{\mathbb{K}\text{-left}}(D_*,\mathbb{K})) & \to & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}_k(D_*),\mathbb{K}). \\ [\varphi\colon D_k\to\mathbb{K}] & \mapsto & \left(\begin{array}{ccc} \mathrm{H}_k(D_*) & \to & \mathbb{K} \\ [v] & \mapsto & \varphi(v) \end{array}\right) \end{array}$$

is well-defined, it is natural, and it is an isomorphism of $\mathbb{K}$-right vector spaces.

Proof of the Evaluation-Isomorphism Theorem 326.10. The proof is essentially identical to the original Evaluation-Isomorphism Theorem 77.18. Alternatively the theorem follows from the discussion on page 6084, the Hom-to-Injective Lemma 315.29 and the Injective Coefficients Proposition 315.34. ■

We will also need the following variation on the Hom–Tensor Adjunction Lemma 312.13:

Lemma 326.11. (Hom–Tensor Adjunction Lemma) Let Q,R,S be rings and let W be a (Q,R)-bimodule. We consider the two functors[3571]

$$\begin{array}{rcl} F\colon \mathit{Mod}_{R\text{-left}} & \to & \mathit{Mod}_{Q\text{-left}} \\ C & \mapsto & W\otimes_R C \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathit{Mod}_{Q\text{-left}} & \to & \mathit{Mod}_{R\text{-left}} \\ K & \mapsto & \mathrm{Hom}_{Q\text{-left}}(W,K). \end{array}$$

For any R-left module C and any Q-left module K the maps[3572]

$$\begin{array}{rcl} \mathrm{Hom}_{Q\text{-left}}(\overbrace{W\otimes_R C}^{=F(C)},K) & \to & \mathrm{Hom}_{R\text{-left}}(C,\overbrace{\mathrm{Hom}_{Q\text{-left}}(W,K)}^{=G(K)}) \\ (f\colon W\otimes_R C\to K) & \mapsto & \left(\begin{array}{ccc} C & \to & \mathrm{Hom}_{Q\text{-left}}(W,K) \\ x & \mapsto & \left(\begin{array}{ccc} W & \to & K \\ m & \mapsto & f(m\otimes x) \end{array}\right) \end{array}\right) \end{array}$$

form a bijection that is natural in C and that it also natural in K. In other words, the tensor functor is a left adjoint, in the sense of Definition 6.33, of the Hom-functor.

Proof. The lemma is just a mild generalization of the original Hom–Tensor Adjunction Lemma 312.13. Again the statement follows almost immediately from the definitions. ■

Proof of the Twisted Evaluation–Isomorphism Proposition 326.9. Let (X,x_0) be a pointed topological space that is path-connected and locally contractible and let $A\subseteq X$.

We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and we write $\widetilde{A} := p^{-1}(X)$. Let $\mathbb{K}$ be a skew field with involution and let V and W be two $(\mathbb{K}, \mathbb{Z}[\pi_1(X,x_0)])$-left left modules. Finally let $\Theta\colon V \otimes_{\mathbb{Z}} W \to {}_{\mathrm{tr}}\mathbb{K}$ be a $\mathbb{Z}[\pi_1(X,x_0)]$-left linear such that the corresponding pairing $\varphi_\Theta\colon V \times W \to \mathbb{K}$ is $\mathbb{K}$-sesquilinear and non-singular. By Definition 326.7 this implies that the map

$$\begin{array}{rcl} \Xi\colon V & \to & \mathrm{Hom}_{R\text{-right}}(\overline{W}, R) \\ v & \mapsto & \begin{pmatrix} \overline{W} & \to & R \\ w & \mapsto & \varphi(v,w) \end{pmatrix} \end{array}$$

is an isomorphism of R-left modules. To simplify the notation we assume that $A = \varnothing$. We write $\pi := \pi_1(X,x_0)$ and we write $C_* := \mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A})$. We want to show that the map

$$\begin{array}{rcl} \overbrace{\mathrm{H}^k(\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, V))}^{=\mathrm{H}^k_{\mathrm{tw}}(X,A;V)} & \to & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overbrace{\mathrm{H}_k(\overline{W} \otimes_{\mathbb{Z}[\pi]} C_*)}^{=\mathrm{H}^{\mathrm{tw}}_k(X,A;W)}, \mathbb{K}) \\ \varphi & \mapsto & (\mu \mapsto \Theta_*(\langle \varphi, \mu \rangle)) \end{array}$$

is an isomorphism of $\mathbb{K}$-left modules. The hope is to reduce this statement to the Evaluation-Isomorphism Theorem 326.10 applied to the $\mathbb{K}$-left chain complex $D_* := \overline{W} \otimes_{\mathbb{Z}[\pi]} C_*)$. The idea is that we can rewrite the above cohomology module (using the fact that Θ is non-singular) to get it into the desired shape.

Note hat it follows from the Hom–Tensor Adjunction Lemma 326.11 together with the Dual-π-Lemma 326.8 that the following two maps are defined and isomorphisms of $\mathbb{K}$-left modules

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{W}_* \otimes_{\mathbb{Z}[\pi]} C_*, \mathbb{K}) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, \mathrm{Hom}_{\mathbb{K}\text{-left}}(\overline{W}, \mathbb{K})) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C_*, \overline{V}) \\ (\varphi\colon \overline{W} \otimes_{\mathbb{Z}[\pi]} C_* \to \mathbb{K}) & \mapsto & (c \mapsto (w \mapsto \varphi(c \otimes w))) & & \\ & & \psi & \mapsto & (c \mapsto \Xi^{-1}(\psi(c))) \end{array}$$

Since these isomorphisms are natural we see that they give us in fact isomorphisms of cochain complexes.

Claim 3. Given any $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathrm{C}_k(\widetilde{X}), V)$ and $w \otimes \sigma \in \overline{W} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X})$ we have

$$\epsilon_X(\Theta_*(\varphi \cap_\otimes (w \otimes \sigma))) = \Xi(\varphi)(w \otimes \sigma).$$

Proof. Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathrm{C}_k(\widetilde{X}), V)$ and let $w \otimes \sigma \in \overline{W} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X})$. Then

$$\begin{array}{rcl} \epsilon_X(\Theta_*(\varphi \cap_\otimes (w \otimes \sigma))) & \overset{\text{see Definition 334.23 of } \cap_\otimes}{=} & \epsilon_X(\Theta_*(\varphi(\sigma) \otimes w \otimes (\overbrace{\sigma \circ [\mathfrak{v}_k]}^{\text{0-simplex}}))) \\ & = & \epsilon_X(\Theta(\varphi(\sigma)) \otimes w) \otimes (\sigma \circ [\mathfrak{v}_k]) \\ & \overset{\text{Definition 321.12 of } \Theta_*}{=} & \Theta(\varphi(\sigma) \otimes w) \overset{\text{by Definition 337.10 of } \epsilon_X}{=} \Xi(\varphi)(w \otimes \sigma). \end{array}$$

∎

We can now finally conclude the argument. To do so we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(X; V) & \xrightarrow{\varphi \mapsto (w \otimes \sigma \mapsto \epsilon_X(\Theta_*(\varphi \cap_\otimes w \otimes \sigma)))} & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}_k(X;V), \mathbb{K}) \\ \cong \downarrow \Xi & & \downarrow \mathrm{id} \\ \mathrm{H}^k(\overline{W} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0}); \mathbb{K}) & \xrightarrow[\substack{\text{isomorphism by the} \\ \text{Evaluation-Isomorphism} \\ \text{Theorem 326.10}}]{\varphi \mapsto (\mu \mapsto \varphi(\mu))} & \mathrm{Hom}_{\mathbb{K}\text{-left}}(\mathrm{H}_k(\overline{W} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0})), \mathbb{K}). \end{array}$$

We make the following observations:

(1) By Claim 2 we know that the vertical map Ξ to the left is an isomorphism.
(2) By Claim 3 we know that the diagram commutes.
(3) Since $\mathbb{K}$ is a skew field we obtain from the Evaluation-Isomorphism Theorem 326.10, applied to the $\mathbb{K}$-left chain complex $D_* := \overline{W} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0})$, that the bottom horizontal map is an isomorphism.

It follows from the above that the top horizontal map is indeed an isomorphism. ■

326.5. Exercises for Chapter 326.

Exercise 326.1. Let X be a finite connected CW-complex and let $x_0 \in X$. Let $\mathbb{F}$ be a field and let $\varphi\colon \pi_1(X, x_0) \to \mathrm{SL}(2, \mathbb{F})$ be a representation. Show that for any $k \in \mathbb{N}_0$ we have an isomorphism between $\mathrm{H}^k_{\mathrm{tw}}(X; {}_\varphi\mathbb{F}^2)$ and $\mathrm{H}^{\mathrm{tw}}_k(X; {}_\varphi\mathbb{F}^2)$.
Hint. Find a suitable pairing $\mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$.

Exercise 326.2. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let $\varphi\colon \pi_1(X, x_0) \to \langle t \rangle$ be an epimorphism and let M be $\mathbb{Z}[t^{\pm 1}]$-module.

(a) Show that there exists a natural epimorphism

$$\mathrm{H}^1_{\mathrm{tw}}(Y; M) \;\to\; \mathrm{Hom}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}^{\mathrm{tw}}_1(X, \mathbb{Z}[t^{\pm 1}]), M).$$

Hint. Make use of the Homological UCSS Theorem 316.5.
(b) Show that the map in (a) is in general not an isomorphism.

CHAPTER 327

Twisted cellular (co-) homology

One of the most popular tools for calculating "*untwisted*" singular (co-) homology groups is to calculate them via cellular (co-) homology. In this chapter we will see that in the *twisted* setting this is also possible.

This chapter is organized as follows:

- In Section 327.1 we will (re-) introduce the category of CW-complexes.
- In Section 327.2 we will consider coverings of CW-complexes.
- In Section 327.3 we will remind the reader of the definition of the cellular chain complex of a CW-complex.
- In Section 327.4 we will discuss the cellular chain complex of coverings.
- In Section 327.5 we will introduce twisted cellular (co-) homology.
- In Section 327.6 we will discuss the functoriality of twisted cellular (co-) homology.
- In Section 327.7 we will formulate the Singular-Cellular Twisted Isomorphism Theorem 328.1 which says that cellular twisted (co-) homology is naturally isomorphic to singular twisted (co-) homology. The proof will have to wait till the next Chapter 328.
- In Section 327.8 we will give one sample calculation, namely we will use twisted cellular homology to recalculate twisted homology of S^1. Later in Chapter 329 we will present many more calculations of twisted cellular (co-) homology.
- In Section 327.9 we will provide exercises for Chapter 327.

In the next Chapter 328 we will prove the Singular-Cellular Twisted Isomorphism Theorem 328.1.

327.1. **The category of CW-complexes.** In this and the following chapters we will consider CW-complexes (and the related handle-complexes) in great detail. For the reader's convenience we recall Definition 136.1:

Definition 327.1. A CW-complex X is a topological space X together with a filtration

$$\varnothing = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \ldots \subseteq X$$

by subspaces which has the following properties:

(CW1) We have $X = \operatorname{colim} X^n$, as described in the Topological-Direct Colimit Proposition 104.10. In other words, as a set we have $X = \bigcup\limits_{n\in\mathbb{N}_0} X^n$ and $U \subseteq X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is open in X^n.

(CW2) For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism

$$\Xi_n \colon X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup\limits_{i\in I} S^{n-1} \,:\, \bigsqcup\limits_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

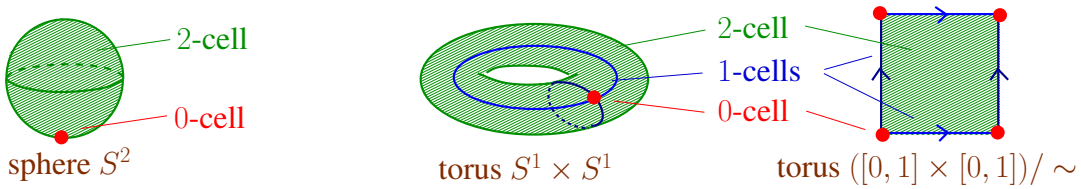
A CW-structure on a topological space X is a filtration as above.

Before we provide examples of CW-complexes let us recall Definition 136.2:

Definition 327.2. Using the notation from above we introduce the following language:

(1) Given $n \in \mathbb{Z}_{\geq -1}$ we refer to X^n as the n-skeleton of X.
(2) The components of $X^n \setminus X^{n-1}$ are called the open n-cells of X.
(3) For each open n-cell σ of X we refer to its closure $\overline{\sigma}$ in X as an n-cell of X.
(4) We set
$$\dim(X) \;:=\; \sup\{n \in \mathbb{Z}_{\geq -1} \,|\, X^{n-1} \neq X^n\} \;\in\; \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$
(5) We say that the CW-complex is finite if it has only finitely many cells.

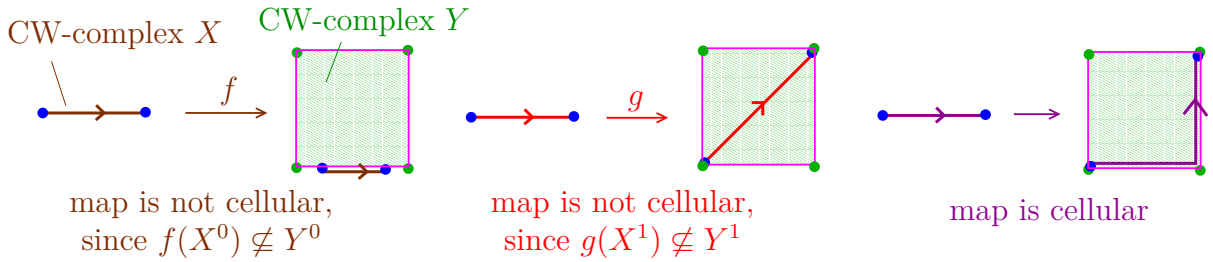
Example. In the figure below we see a CW-structure for the sphere S^2 and we see a CW-structure for the torus $S^1 \times S^1$.



More generally one can equip any S^n with a CW-structure consisting of one 0-cell and one extra n-cell. □

As any type of objects, CW-complexes really would like to form a category. Thus we should remind the reader of the concept of a cellular map:

Definition 327.3. As in Definition 139.2 we say that a continuous map $g\colon X \to Y$ between two CW-complexes is cellular if for each $k \in \mathbb{N}_0$ we have $g(X^k) \subseteq Y^k$.



Next we recall the essentially self-explanatory Definition 139.3:

Definition 327.4. The category $\mathcal{CW}$ of CW-complexes is defined as the category $\mathcal{CW}$ that is given by
$$\begin{aligned} \mathrm{Ob}(\mathcal{CW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y. \end{aligned}$$

For our later purposes we also need to recall Definition 136.7:

Definition 327.5. Let X be a CW-complex

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X. Furthermore we denote by $\mathrm{Cell}(X) := \bigcup_{n \in \mathbb{N}_0} \mathrm{Cell}_n(X)$ the set of all cells of X.

(2) Let $c \in \mathrm{Cell}_n(X)$ be an n-cell of X.
   (a) An attaching map for c is a continuous map $\varphi\colon S^{n-1} \to X^{n-1}$ such that there exists a homeomorphism
$$X^{n-1} \cup_{X^{n-1} \leftarrow S^{n-1} :\varphi} \overline{B}^n \;\to\; \underbrace{X^{n-1} \cup c}_{\text{subspace of } X^n}$$
   which is the identity on X^{n-1}.
   (b) A characteristic map for c is a continuous map $\Phi\colon \overline{B}^n \to X$ such that the map
$$X^{n-1} \cup_{X^{n-1} \leftarrow S^{n-1} :\Phi|_{S^{n-1}}} \overline{B}^n \;\to\; X^{n-1} \cup c$$
   induced by the identity on X^{n-1} and by the map Φ on $\overline{B}^n$ is well-defined and it is a homeomorphism.
(3) A full set of attaching maps for the n-cells of the CW-complex X is defined as a family of continuous maps $\{\varphi_c\colon S^{\dim(c)-1} \to X^{\dim(c)-1}\}_{c \in \mathrm{Cell}_n(X)}$ such that each φ_c is an attaching map for the cell c. Similarly we define a full set of characteristic maps for the n-cells of X.

327.2. **Coverings of CW-complexes.** The main goal of this chapter is to introduce and study twisted cellular (co-) homology of CW-complexes. This requires a good understanding of coverings of CW-complexes. This is surely a good moment to recall the CW-Complex Covering Proposition 138.1:

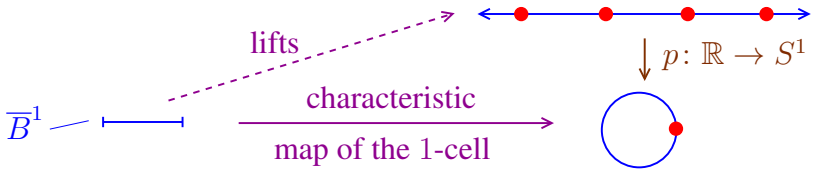
Proposition 327.6. (CW-Complex Covering Proposition) Let $p\colon \widetilde{X} \to X$ be a connected covering of a non-empty CW-complex X.

(1) The filtration
$$\varnothing = p^{-1}(X^{-1}) \subseteq p^{-1}(X^0) \subseteq p^{-1}(X^1) \subseteq \ldots \subseteq \widetilde{X}$$
defines a CW-structure for $\widetilde{X}$ which has the property that for any $k \in \mathbb{N}_0$ we have
$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$
(2) (a) Each characteristic map $\Phi\colon \overline{B}^n \to X^n$ admits precisely $[\widetilde{X} : X]$ lifts to $\widetilde{X}$.
   (b) Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. The set of all lifts of all these characteristic maps is a full set of characteristic maps for the CW-complex $\widetilde{X}$.

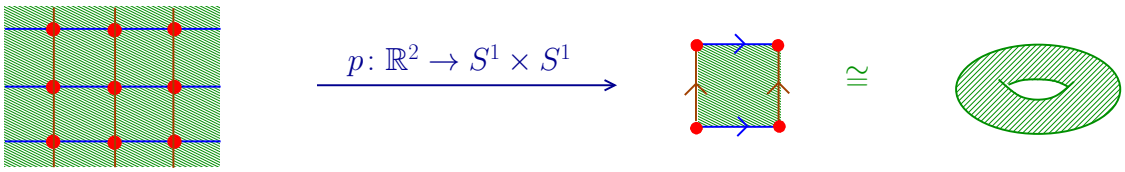
Proof. This is really just the CW-Complex Covering Proposition 138.1 which for consistency of this part of the lecture notes we equipped with a new number. ■

Examples. We recall some examples from page 2668:

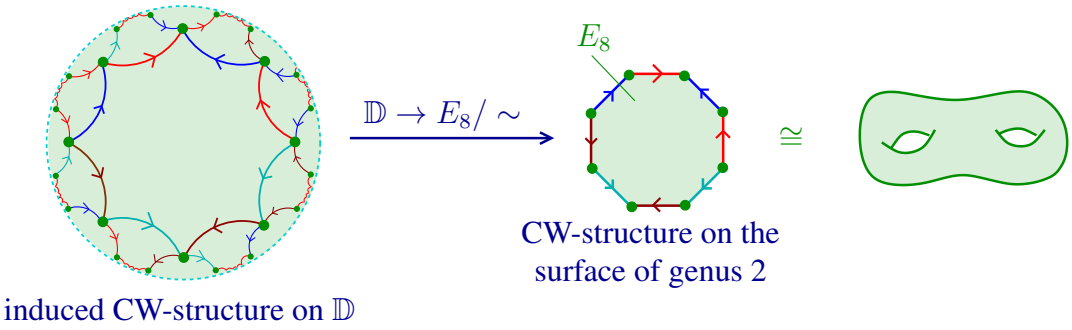
(1) We equip S^1 with the CW-structure $\varnothing \subseteq \{1\} \subseteq S^1$ consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. It follows from the CW-Complex Covering Proposition 327.6, that a CW-structure on $\mathbb{R}$ is given by $\varnothing \subseteq p^{-1}(\{1\}) \subseteq S^1$, i.e. by $\varnothing \subseteq \mathbb{Z} \subseteq \mathbb{R}$.



(2) As above we equip $S^1 \times S^1 =_{\mathrm{i}} ([0,1]\times[0,1])/\sim$ with the CW-structure with one 0-cell, two 1-cells and one 2-cell. We consider the universal covering $\mathbb{R}^2 \to S^1 \times S^1$. The resulting CW-structure on $\mathbb{R}^2$ is illustrated in the figure below.



(3) We consider the surface of genus 2, which by the definition on page 342, is given as $X := E_8/\sim$, i.e. it is defined as the quotient of the octagon E_8 by identifying pairs of sides. As we discussed on page 2626, we can equip $X = E_8/\sim$ with a CW-structure with a single 0-cell, four 1-cells and one 2-cell.



Next we want to understand the universal covering of $X = E_8/\sim$. To do so we consider the open disk $\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}$. In the proof of Hadamard's Theorem 132.14 we constructed an explicit covering map $\mathbb{D} \to H_8/\sim \cong E_8/\sim$. We obtain from the CW-Complex Covering Proposition 327.6 the CW-structure on $\mathbb{D} = \widetilde{X}$ that is shown in the figure above. □

327.3. **The cellular chain complex.** Before we introduce twisted cellular (co-) homology let us remind the reader of the definition of the ordinary cellular chain complex:

Notation 327.7. Let X be a CW-complex.

(1) (a) As in Definition 155.5 we set

$$\mathrm{C}_n^{\mathrm{CW}}(X) \;:=\; \mathrm{H}_n(X^n, X^{n-1}).$$

(b) As in Definition 155.3 we denote by $\partial_n^{\mathrm{han}}$ the cellular boundary map that is given by

$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{\text{я}_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$$
$$=:\partial_n^{\mathrm{han}}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where $\text{я}_{n-1}\colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map introduced in Definition 146.6.

(c) The corresponding homology groups are the cellular homology groups $\mathrm{H}_n^{\mathrm{CW}}(X)$.

(2) Given a subcomplex A of X we consider the relative chain complex
$$\mathrm{C}_*^{\mathrm{CW}}(X,A) \coloneqq \operatorname{coker}\big(i_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)\big),$$
and we refer to the corresponding homology groups as the relative cellular homology groups $\mathrm{H}_*^{\mathrm{CW}}(X,A)$.

The following lemma is just a reformulation of the Cellular Chain Complex–Basis Lemma 155.13:

Lemma 327.8. (Cellular Chain Complex–Basis Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$. Furthermore let $\{\Phi_c\colon \overline{B}^n \to X^n\}_{c\in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.[3573] For each $c \in \mathrm{Cell}_n(X)$ we consider the homology class[3574] $\Phi_{c*}([\overline{B}^n]) \in \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X^n, X^{n-1})$. Then $\{\Phi_{c*}([\overline{B}^n])\}_{c\in \mathrm{Cell}_n(X)}$ is a basis for the cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X^n, X^{n-1})$.

Proof. This lemma is an immediate consequence of the CW-Skeleton-H_*-Lemma 155.2 (1). ∎

There are three reasons why we were and are interested in cellular chain complexes:

(1) Cellular chain complexes are incomparably smaller than the unwieldy singular chain complexes.
(2) In the Cellular Boundary–Degree Proposition 155.15 we gave a practical way for determining the boundary maps in the cellular chain complex.
(3) In the Singular-Cellular-H_*-Proposition 155.8 and the Singular-Cellular-H^*-Isomorphism Proposition 199.5 we saw that given a CW-complex X the singular (co-) homology groups and the cellular (co-) homology groups are naturally isomorphic.

Our goal in and the following chapters is to repeat this success story in the present context of twisted (co-) homology.

327.4. **The cellular chain complex of coverings.** To extend cellular homology to the twisted setting we need to make a few preparations. We start out by introducing some notation:

Notation 327.9. Let X be a CW-complex and let A be a subcomplex.

(1) By the CW-Complex–Local Properties Proposition 137.8 (2) we know that X is locally contractible. Thus, if X is path-connected, then given any $x_0 \in X$ we know that the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ exists.
(2) If $q\colon Y \to X$ is any covering, e.g. the natural universal covering, then it follows from the CW-Complex Covering Proposition 327.6 that we can equip Y with a natural CW-structure which has the property, that each deck transformation of $q\colon Y \to X$ acts cellularly. If $A \subseteq X$ is a subcomplex, then it follows from the Subcomplex–Characterization Proposition 136.12 that $p^{-1}(A)$ is a subcomplex of Y.

We continue with the following analog of the Regular Covering–Chain Complex Lemma 320.2:

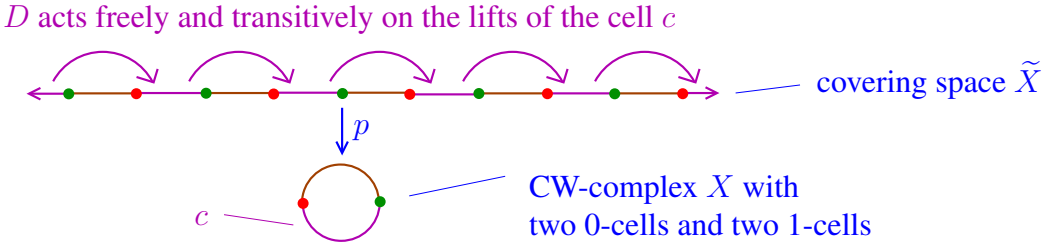
[3573] As we explain in Footnote 1413, it follows easily from the definitions that such a full set of characteristic maps always exists.
[3574] Here $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ denotes the standard fundamental class which we introduced in Definition 155.11.

Lemma 327.10. (Regular Covering–Cellular Chain Complex Lemma) Let X be a connected CW-complex and let $A \subseteq X$ be a subcomplex. Let $p\colon \widetilde{X} \to X$ be a regular covering. We write $\widetilde{A} \coloneqq p^{-1}(A)$. Finally we denote by $D \coloneqq \mathrm{Deck}(p\colon \widetilde{X} \to X)$ the corresponding deck transformation group.

(1) (a) Let $k \in \mathbb{N}_0$. For each k-cell σ in X that is not contained in A we pick a single cell $\widetilde{\sigma} \subseteq \widetilde{X}$ with $p(\widetilde{\sigma}) = \sigma$. These cells $\widetilde{\sigma}$ (together with a choice of a characteristic map) give rise to a basis for $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ as a free $\mathbb{Z}[D]$-left module.

(b) Each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ is a free $\mathbb{Z}[D]$-left module where the cardinality of the basis is given by the cardinality of the set of k-cells of the CW-complex X that are not contained in A.[3575] [3576]

(2) The cellular boundary maps $\partial_k\colon \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A}) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ are $\mathbb{Z}[D]$-left module homomorphisms.



Sketch of proof.

(1) (a) This statement follows easily from the CW-Complex Covering Proposition 327.6 together with the Deck Group-Free Action Lemma 128.3, the Cellular Chain Complex–Basis Lemma 327.8 and the Group Ring–Free Module Lemma 310.13.

(b) This statement is just a slightly more reader friendly version of Statement (a).

(2) This statement follows almost immediately from the naturality of the cellular boundary maps, see the Cellular Homology-Functor Lemma 155.7 and the fact, observed above, that each deck transformation is a cellular self-map of $\widetilde{X}$. ■

327.5. **Twisted cellular (co-) homology.** We can now introduce the cellular analogs of the twisted (co-) homology groups that we introduced in Definition 320.11. All we need to do is to replace the singular chain complex $\mathrm{C}_*(\widetilde{X}_{x_0})$ of the natural universal covering by the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0})$ of the universal covering:

Definition 327.11. Let X be a connected CW-complex, let $x_0 \in X$ and let $A \subseteq X$ be a subcomplex. We write $\pi \coloneqq \pi_1(X, x_0)$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X_0, x_0)])$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and we write $\widetilde{A} \coloneqq p^{-1}(A)$. Given $k \in \mathbb{N}_0$ we define the R-left modules

$$\mathrm{C}^{\mathrm{tw,CW}}_k(X, A; M) \coloneqq \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}) \qquad \text{(twisted cellular chain complex)}$$
$$\mathrm{C}^k_{\mathrm{tw,CW}}(X, A; M) \coloneqq \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}), M) \qquad \text{(twisted cellular cochain complex).}$$

[3575] This statement is already contained in the Free Resolution-via-K$(G, 1)$-Proposition 318.3 (1).

[3576] If X is compact, then it follows from the CW-Complex Compactness Corollary 136.17 that each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$ is in fact a finitely generated free $\mathbb{Z}[D]$-left module.

By the Regular Covering–Cellular Chain Complex Lemma 327.10 (2) we know that the boundary maps of $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A})$ are $\mathbb{Z}[\pi]$-left module homomorphisms. Thus we see that $\mathrm{id} \otimes \partial_*$ and ∂^* induce (co-) chain maps. Thus we can define the R-left modules

$$\begin{aligned} \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M) &:= \mathrm{H}_k\big(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A})\big) && \text{(twisted cellular homology)} \\ \mathrm{H}^k_{\mathrm{tw,CW}}(X, A; M) &:= \mathrm{H}^k\big(\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}), M)\big) && \text{(twisted cellular cohomology).} \end{aligned}$$

As always, if $A = \varnothing$, then we drop it from the notation.

Example. Clearly we have the obvious analog of the Twisted–Trivial Coefficients Lemma 320.13, i.e. cellular (co-) homology with trivial ${}_{\mathrm{tr}}\mathbb{Z}$-coefficients is easily shown to be naturally isomorphic to ordinary cellular (co-) homology. □

We will a first example twisted cellular (co-) homology in Section 328.4. We will discuss many more examples later in Chapter 329, once we have shown that the cellular and singular flavors of twisted (co-) homology are naturally isomorphic.

327.6. **Functoriality of twisted cellular (co-) homology.** In Chapter 321 we showed that twisted (co-) homology is functorial (in a suitable sense) in the base point, the underlying topological space and the coefficients. We will not repeat all the notions and results in the cellular context, except that we want to show that cellular maps induce maps on twisted cellular (co-) homology. More precisely, we have the following variation on Definition 321.8:

Definition 327.12. Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map between pointed connected CW-complexes. We write $\pi_X := \pi_1(X, x_0)$, $\pi_Y := \pi_1(Y, y_0)$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_Y])$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ and $q\colon \widetilde{Y}_{y_0} \to Y$ the natural universal coverings. Suppose that we are given subcomplexes $A \subseteq X$ and $B \subseteq Y$ with $f(A) \subseteq B$. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := q^{-1}(B)$. We denote by $\widetilde{f}\colon \widetilde{X}_{x_0} \to \widetilde{Y}_{y_0}$ the map from Definition 321.4 that is induced by $f\colon X \to Y$. It follows easily from the fact that f is cellular that $\widetilde{f}$ is also cellular. Using the cellular analog of the Chain Complex-Equivariance Proposition 321.7 we define the R-left module homomorphisms

$$\begin{aligned} f_*\colon \overbrace{\overline{f^*M} \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw,CW}}(X,A;f^*M)} &\to \overbrace{\overline{M} \otimes_{\mathbb{Z}[\pi_Y]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{Y}_{y_0}, \widetilde{B})}^{=\mathrm{C}_k^{\mathrm{tw,CW}}(Y,B;M)} \\ m \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) &\mapsto m \otimes (\widetilde{f} \circ \sigma\colon \Delta^k \to \widetilde{Y}_{y_0}) \end{aligned}$$

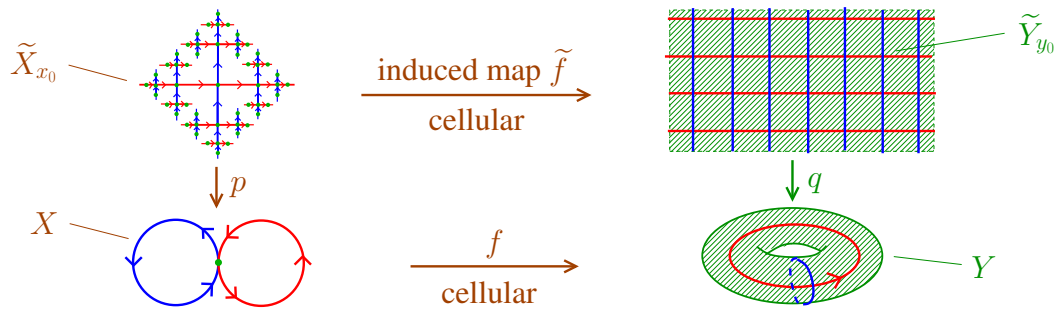
and

$$\begin{aligned} f^*\colon \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_Y]\text{-left}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{Y}_{y_0}, \widetilde{B}), M)}^{=\mathrm{C}^k_{\mathrm{tw,CW}}(Y,B;M)} &\to \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi_X]\text{-left}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}), f^*M)}^{=\mathrm{C}^k_{\mathrm{tw,CW}}(X,A;f^*M)} \\ \varphi &\mapsto (\sigma \mapsto \varphi(r \circ \sigma)) \end{aligned}$$

and we define the corresponding induced R-left module homomorphisms

$$\begin{aligned} f_*\colon \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; f^*M) &\to \mathrm{H}_k^{\mathrm{tw,CW}}(Y, B; M) \\ f^*\colon \mathrm{H}^k_{\mathrm{tw,CW}}(Y, B; M) &\to \mathrm{H}^k_{\mathrm{tw,CW}}(X, A; f^*M). \end{aligned}$$

We leave it to the reader to formulate and prove the obvious analog of the Twisted Homology-Functoriality Proposition 321.9.



327.7. **The interaction between twisted cellular and singular (co-) homology.** In the Singular-Cellular-H_*-Proposition 155.8 and the Singular-Cellular-H^*-Isomorphism Proposition 199.5 we saw that given a CW-complex X the singular (co-) homology groups and the cellular (co-) homology groups are naturally isomorphic. The following theorem gives us an analogous statement for twisted (co-) homology:

Theorem 328.1. (Singular-Cellular Twisted Isomorphism Theorem) Let X be a connected CW-complex, let $A \subseteq X$ be a subcomplex, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Then there exist natural isomorphisms

(1) $$\Phi_{(X,A)} \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M)$$

(2) $$\Phi_{(X,A)} \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \to \mathrm{H}^k_{\mathrm{tw,CW}}(X, A; M)$$

of R-left modules. Here natural means two statements:

(a) If $f \colon (X, A) \to (Y, B)$ is a cellular map between pairs of CW-complexes, if X and Y are connected, if $x_0 \in X$, if R is a ring, and if M is an $(R, \mathbb{Z}[\pi_1(Y, f(x_0))])$-left left module, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X, A; f^*M) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; f^*M) \\ \downarrow{\scriptstyle f_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}_k^{\mathrm{tw}}(Y, B; M) & \xrightarrow{\Phi_{(Y,B)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(Y, B; M) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

(b) Let M and N be $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules and let $\varphi \colon M \to N$ be a homomorphism of $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X, A; M) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M) \\ \downarrow{\scriptstyle \varphi_*} & & {\scriptstyle \varphi_*}\downarrow \\ \mathrm{H}_k^{\mathrm{tw}}(X, A; N) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; N) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

Proof. As the number already indicates, we will prove the Singular-Cellular Twisted Isomorphism Theorem 328.1 in the next Chapter 328. ■

327.8. **Twisted homology of S^1.** In Chapter 329 we will explain in great detail how in practice one might be able to determine twisted cellular chain complexes. But we do not

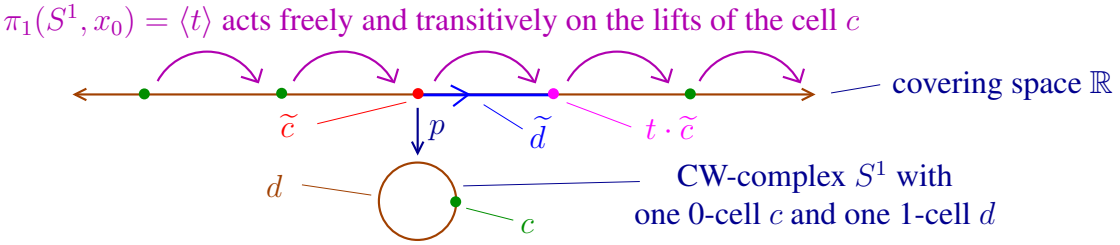
want to conclude this chapter without having provided at least one example. So, as a proof of concept, we will now prove the following proposition, which is a slight generalization of the earlier Twisted-H_*-of-S^1-Proposition 320.14:

Proposition 327.13. (Twisted-H_*-of-S^1-Proposition) We make the usual identification $\pi_1(S^1,1) =_{\mathrm{i}} \langle t\rangle$. Let R be a commutative ring and let $\varphi\colon \pi_1(S^1,1) =_{\mathrm{i}} \langle t\rangle \to \mathrm{GL}(1,R)$ be a one-dimensional representation. Then

$$\begin{aligned}\mathrm{H}_0^{\mathrm{tw}}(S^1;{}_\varphi R) &\cong \operatorname{coker}(\cdot(\varphi(t)-1)\colon R\to R)\\ \mathrm{H}_1^{\mathrm{tw}}(S^1;{}_\varphi R) &\cong \ker(\cdot(\varphi(t)-1)\colon R\to R).\end{aligned}$$

In particular, if R is a field and if $\varphi(t)\neq 1$, then both twisted homology groups vanish.

Proof. We consider the topological space S^1 which we equip with the CW-structure that consists of a single 0-cell $c=\{1\}$ and a single 1-cell d. Let $\varphi\colon \pi_1(S^1,1)\to\langle t\rangle$ be the usual isomorphism. We denote by $p\colon \mathbb{R}\to S^1$ the universal covering that is given by $p(t)=\exp(2\pi\,\mathrm{i}t)$. We pick the lift $\widetilde{c}=\{0\}\subseteq\mathbb{R}$ and as a lift of the 1-cell d we pick the 1-cell $\widetilde{d}$ with characteristic map $\Phi\colon[-1,1]\to\mathbb{R}$ that is given by $\Phi(s)=\frac{1}{2}(1+s)$.



With respect to these lifts the cellular chain $\mathrm{C}_*^{\mathrm{CW}}(\mathbb{R})$ is easily seen to be given by

$$0\ \to\ \mathbb{Z}[t^{\pm1}]\cdot\widetilde{d}\ \xrightarrow{\widetilde{d}\mapsto(t-1)\cdot\widetilde{c}}\ \mathbb{Z}[t^{\pm1}]\cdot\widetilde{c}\ \to\ 0.$$

We now see that[3577]

$$\begin{aligned}\mathrm{H}_i^{\mathrm{tw}}(S^1;{}_\varphi R) &\overset{\text{by the Singular-Cellular Twisted Isomorphism Theorem 328.1}}{\cong} \mathrm{H}_i^{\mathrm{CW,tw}}(S^1;{}_\varphi R)\\ &\cong \mathrm{H}_i\big(0\to {}_\varphi R\underset{\mathbb{Z}[t^{\pm1}]}{\otimes}\mathbb{Z}[t^{\pm1}]\cdot\widetilde{d}\xrightarrow{\mathrm{id}\otimes(t-1)}{}_\varphi R\underset{\mathbb{Z}[t^{\pm1}]}{\otimes}\mathbb{Z}[t^{\pm1}]\cdot\widetilde{c}\to 0\big)\\ &\overset{\text{by the above discussion}}{\cong} \mathrm{H}_i(0\to R\xrightarrow{\cdot(\varphi(t)-1)}R)\to 0).\\ &\overset{\text{by the Free Module Tensor-with-Ring Lemma 312.18}}{\cong}\end{aligned}$$

■

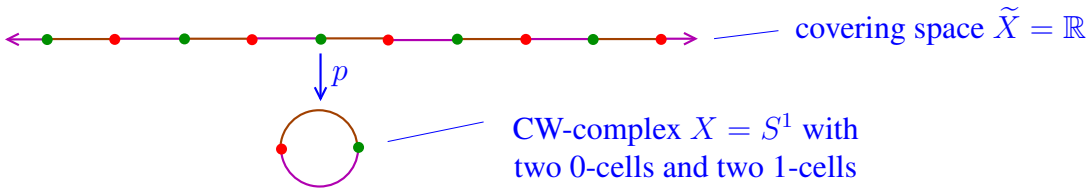
327.9. Exercises for Chapter 327.

[3577] We know by the Replace Universal Covering Proposition 320.16 that we can determine the isomorphism type of (co-) homology with twisted coefficients using any universal covering, i.e. for the isomorphism type we do not have to work with the natural universal covering. This logic also applies to twisted cellular (co-) homology. We will use this little observation on numerous occasions.

Exercise 327.1. In the figure below we show $X = S^1$ with a CW-structure that consists of two 0-cells and two 1-cells.

(a) Describe the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ with respect to a suitable choice of lifts and characteristic maps.

(b) Given a 1-dimensional representation $\varphi\colon \pi_1(S^1) \to \mathrm{GL}(1,\mathbb{C})$ describe the twisted cellular chain complex ${}_{\varphi}\overline{\mathbb{C}} \otimes_{\mathbb{Z}[\pi_1(S^1)]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ and calculate its homology groups.



Exercise 327.2. We make the usual identification $\pi_1(S^1,1) =_{\mathrm{i}} \langle t \rangle$. Let $A \in \mathrm{GL}(n,\mathbb{C})$. We consider the n-dimensional complex representation $\varphi\colon \pi_1(S^1,1) =_{\mathrm{i}} \langle t \rangle \to \mathrm{GL}(n,\mathbb{C})$ that is given by $\varphi(t) = A$.

(a) Compute the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(S^1; {}_{\varphi}\mathbb{C}^n)$ in terms of linear algebraic data of the matrix A.

(b) Compute the twisted cohomology modules $\mathrm{H}^*_{\mathrm{tw}}(S^1; {}_{\varphi}\mathbb{C}^n)$ in terms of linear algebraic data of the matrix A.

Remark. This exercise is an extension of Exercise 321.6.

Exercise 327.3. Let $m \in \mathbb{N}$ and let X be a CW-complex with the following three properties:

- X has a single cell in each dimension $0, \dots, n$ and no other cells.
- $\pi_1(X) \cong \mathbb{Z}/m\mathbb{Z}$.
- The usual homology groups of the universal covering of $\widetilde{X}$ are isomorphic to the usual homology groups of S^n.

Now we can formulate the two exercises:

(a) Show that if $m = 2$, then the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ viewed as a chain complex of free $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$-modules is isomorphic to the following chain complex:

$$0 \to \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{1+(-1)^{n+1}\cdot x} \dots \to \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{1+x} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{1-x} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \to 0.$$

(b) Does the above conclusion also hold for $m = 3$ or $m = 5$?

Exercise 327.4. Let X be a connected non-empty CW-complex. By the CW-Complex–Spanning Tree Proposition 140.5 (2) we know that X is homotopy equivalent to a CW-complex with a single 0-cell. Use this feature to give a new proof of the Twisted H_0-Proposition 323.3 in our current context. More precisely, let $x_0 \in X$ be the point in the 0-cell, let R be a ring and let M be an $(R, \mathbb{Z}[\pi])$-left left module, where $\pi := \pi_1(X, x_0)$.

(a) Show that we have an isomorphism

$$\mathrm{H}_0^{\mathrm{CW}}(X; M) \;\cong\; M_\pi \;:=\; M \Big/ \Big\{ \sum_{i=1}^{k} (g_i \cdot m_i - m_i) \,\Big|\, g_i \in \pi \text{ and } m_i \in M \Big\}.$$

(b) Show that we have an isomorphism

$$\mathrm{H}^0_{\mathrm{CW}}(X; M) \;\cong\; M^\pi \;:=\; \{m \in M \,|\, g \cdot m = m \text{ for all } g \in \pi\}.$$

CHAPTER 328

The relationship between twisted cellular and singular (co-) homology

In the previous Chapter 327 we introduced twisted cellular (co-) homology and we performed many calculations. In this chapter we will prove that twisted cellular and singular (co-) homology are naturally isomorphic.

This chapter is organized as follows:

- In Section 328.1 we will state and discuss the main result of this chapter, namely the theorem, which says that twisted cellular and singular (co-) homology are naturally isomorphic.
- In Section 328.2 we will (re-)introduce the intermediate chain complex.
- In Section 328.3 we will recall some concepts and results from algebra which we will need.
- In Section 328.4 we will provide a proof of the Singular-Cellular Twisted Isomorphism Theorem 328.1.
- In Appendix 328.5 we will present an idea for an alternative proof of the Singular-Cellular Twisted Isomorphism Theorem 328.1.
- In Section 328.6 we will provide exercises for Chapter 328.

In the next Chapter 329 we will present many calculations of twisted cellular (co-) homology.

328.1. **The interaction between twisted cellular and singular (co-) homology.** The only goal of this chapter is to prove the following theorem, which we already formulated in Section 327.7:

Theorem 328.1. (Singular-Cellular Twisted Isomorphism Theorem)[3578] Let X be a connected CW-complex, let $A \subseteq X$ be a subcomplex, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Then there exist natural isomorphisms

$$\Phi_{(X,A)} \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M) \tag{1}$$

$$\Phi_{(X,A)} \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \to \mathrm{H}^k_{\mathrm{tw,CW}}(X, A; M) \tag{2}$$

of R-left modules. Here natural means two statements:

(a) If $f \colon (X, A) \to (Y, B)$ is a cellular map between pairs of CW-complexes, if X and Y are connected, if $x_0 \in X$, if R is a ring, and if M is an $(R, \mathbb{Z}[\pi_1(Y, f(x_0))])$-left left

module, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X,A;f^*M) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X,A;f^*M) \\ \downarrow{\scriptstyle f_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}_k^{\mathrm{tw}}(Y,B;M) & \xrightarrow{\Phi_{(Y,B)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(Y,B;M) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

(b) Let M and N be $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left modules and let $\varphi\colon M\to N$ be a homomorphism of $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left modules. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X,A;M) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X,A;M) \\ \downarrow{\scriptstyle \varphi_*} & & {\scriptstyle \varphi_*}\downarrow \\ \mathrm{H}_k^{\mathrm{tw}}(X,A;N) & \xrightarrow{\Phi_{(X,A)}} & \mathrm{H}_k^{\mathrm{tw,CW}}(X,A;N) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

Example. Let X be a CW-complex, let $x_0\in X$ and let M be a $\mathbb{Z}[\pi_1(X,x_0)]$-left module. If X has no k-cells, then it follows almost immediately from the Singular-Cellular Twisted Isomorphism Theorem 328.1 that $\mathrm{H}_k^{\mathrm{tw}}(X;M)=0$ and $\mathrm{H}^k_{\mathrm{tw}}(X;M)=0$. □

Remark.

(1) The proof we give of the Singular-Cellular Twisted Isomorphism Theorem 328.1 is based on the proof given in [**Lü1998**, Lemma 4.2] and [**Lü1989**, p. 263], see also [**Wall1966a**, Lemma 1]. An alternative proof is given in [**KwS2018**, Theorem 4.1]. It seems like the naturality statement was not dealt with in great detail in the earlier references.

(2) Let K be an *ordered* abstract simplicial complex. By the Simplicial-Singular Chain Complex Theorem 177.12 the maps

$$\begin{array}{rcl} \Theta_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\underset{\uparrow}{\Phi_s^{\leq}}\colon \Delta^n\to |K|) \end{array}$$

Definition 174.18

are a natural chain homotopy equivalence. Note that naturality implies that if we work with the universal covering of an ordered simplicial complex, then the above map is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(K)]$-left module chain complexes, and thus induces an isomorphism between "twisted simplicial" (co-) homology and twisted singular (co-) homology groups. The details will be filled in in Exercise 328.1. □

328.2. **The intermediate chain complex.** We gave two rather different proofs that "untwisted" singular homology is naturally isomorphic to "untwisted" cellular homology:

(1) In the proof of the Singular-Cellular-H_*-Proposition 155.8 we used a quick diagram chase to prove the desired result.

[3578]By the CW-Complex–Local Properties Proposition 137.8 (2) we know that every CW-complex is locally contractible. This implies that twisted singular (co-) homology of a connected CW-complex is actually defined.

(2) In the proof of the Singular-Cellular Chain Complex Proposition 158.16 we introduced an "intermediate chain complex" and used it to show that the cellular chain complex and the singular chain complex are chain homotopy equivalent.

The first approach certainly does not generalize to the twisted setting. Thus we will attempt to make the second approach work. The main new concept of the second proof was that in Definition 158.13 we introduced a suitable intermediate chain complex $C_*^{CW,int}(W)$, which allowed us to relate the singular chain complex $C_*(W)$ to the cellular chain complex $C_*^{CW,int}(W)$. In the following we will recall and expand on this concept:

Definition 328.2. Let W be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we define the intermediate cellular chain complex

$$C_n^{CW,int}(W) := \ker\big(C_n(W^n) \xrightarrow{\partial_n} C_{n-1}(W^n) \xrightarrow{я} C_{n-1}(W^n, W^{n-1})\big).$$

In the CW-Intermediate-Singular-Lemma 158.14 we saw that $C_*^{CW,int}(W)$ is a subcomplex of $C_*(W)$ and that $C_*^{CW,int}(W)$ is covariantly functorial under cellular maps.

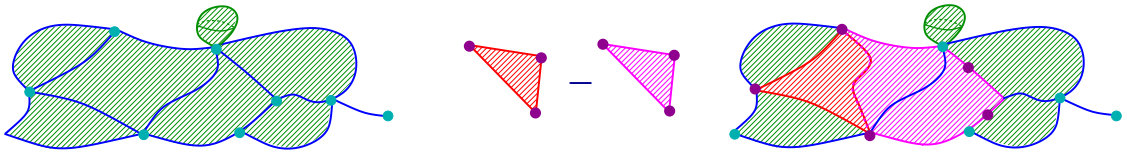
(2) Let B be a subcomplex of W. We denote by $i\colon B \to W$ the inclusion. It follows easily from the definitions that the induced map $i_*\colon C_n^{CW,int}(B) \to C_n^{CW,int}(W)$ is a monomorphism. We set

$$C_n^{CW,int}(W,B) := \operatorname{coker}\big(i_*\colon C_n^{CW,int}(B) \to C_n^{CW,int}(W)\big).$$

(3) Given any subcomplex B of W we have natural chain maps

$$\iota\colon C_*^{CW,int}(W,B) \hookrightarrow C_*(W,B) \qquad \text{and} \qquad \pi\colon C_*^{CW,int}(W,B) \twoheadrightarrow C_*^{CW}(W,B)$$

which are induced by the obvious inclusion and the natural projection.



CW-complex W element of $C_2^{CW,int}(W)$

We continue with the following mild generalization of the CW-Intermediate-Singular Proposition 158.15:

Proposition 328.3. (Relative CW-Intermediate-Singular Proposition) Let W be a CW-complex and let B be a subcomplex of W. For each $n \in \mathbb{N}_0$ the natural maps

(1) $\iota_*\colon H_n(C_*^{CW,int}(W,B)) \to H_n(C_*(W,B))$

(2) $\pi_*\colon H_n(C_*^{CW,int}(W,B)) \to H_n(C_*^{CW}(W,B))$

from Definition 328.2 are isomorphisms.

Proof. In the CW-Intermediate-Singular Proposition 158.15 we proved both statements in the absolute case, i.e. if $B = \varnothing$. Using generic little tricks, which we know from Exercise 145.14 and the proof of the Two-Out-Of-Three-Homotopy Equivalence Corollary 158.9, we will now deduce the relative case from the absolute case. To deal with Statement (1)

let us consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*^{\mathrm{CW,int}}(B) & \longrightarrow & \mathrm{C}_*^{\mathrm{CW,int}}(W) & \longrightarrow & \mathrm{C}_*^{\mathrm{CW,int}}(W,B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \\ 0 & \longrightarrow & \mathrm{C}_*(B) & \longrightarrow & \mathrm{C}_*(W) & \longrightarrow & \mathrm{C}_*(W,B) & \longrightarrow & 0. \end{array}$$

It follows easily from the definitions that the diagram commutes and that the horizontal sequences are short exact sequences of chain complexes. Therefore, by the LES-Proposition 313.10, we get a commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc} \dots \to \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(B)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(W)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(W,B)) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathrm{C}_*^{\mathrm{CW,int}}(B)) \to \dots \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ \dots \longrightarrow \mathrm{H}_n(\mathrm{C}_*(B)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*(W)) & \longrightarrow & \mathrm{H}_n(\mathrm{C}_*(W,B)) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathrm{C}_*(B)) \longrightarrow \dots. \end{array}$$

By the CW-Intermediate-Singular Proposition 158.15 (1) we know that the two vertical maps relating the absolute cases are isomorphisms. It follows from the Five Lemma 145.11 that the vertical maps dealing with the relative case are also isomorphisms. This concludes the proof of Statement (1).

We turn to the proof of Statement (2). Using basically the same approach as above we can deduce from the CW-Intermediate-Singular Proposition 158.15 (2) that the maps $\pi_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW,int}}(W,B)) \to \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(W,B))$ are isomorphisms. ■

328.3. **Algebraic ingredients.** The idea now is quite simple: We could like to apply the Relative CW-Intermediate-Singular Proposition 328.3 to universal coverings and we would like to argue that a chain map that induces isomorphisms on homology is already a chain homotopy equivalence. Looking back at our algebra chapters we find the following result:

Proposition 313.18. (Quasi Iso–Homotopy Equivalence Proposition) Let R be a ring and let $f\colon C_* \to C'_*$ be a chain map between chain complexes of projective R-left modules. If f is a quasi-isomorphism[3579], then f is in fact a chain homotopy equivalence, i.e. there exists a chain map $f\colon C'_* \to C_*$ of R-left modules that is a chain homotopy inverse to f.

Now let X be a connected CW-complex with universal covering $\widetilde{X}$ and fundamental group $\pi := \pi_1(X, x_0)$. We already discussed that the singular chain complex $\mathrm{C}_*(\widetilde{X})$ and the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ are both chain complexes of free (in particular projective) $\mathbb{Z}[\pi]$-left modules. The intermediate chain complex $\mathrm{C}_n^{\mathrm{CW,int}}(\widetilde{X})$ consists of $\mathbb{Z}[\pi]$-left submodules of $\mathrm{C}_*(\widetilde{X})$. But the author does not see a reason why these should be projective, let alone free $\mathbb{Z}[\pi]$-modules.

At this point we should recall Definition 313.20:

Definition 328.4. Let R be a ring and let C_* be an R-left chain complex. A free imitation of C_* is a chain complex F_* of R-left modules together with a quasi-isomorphism $\varphi\colon F_* \to C_*$.

We also need to recall the following lemma:

[3579]Recall that according to Definition 313.17 a *quasi-isomorphism* is a chain map that induces isomorphisms on homology groups.

Lemma 313.21. (Free Imitation Existence Lemma) Let R be a ring. Every R-left chain complex admits a free imitation.

328.4. **Proof of the Singular-Cellular Twisted Isomorphism Theorem 328.1.** We now turn the actual proof of the Singular-Cellular Twisted Isomorphism Theorem 328.1. We break it up into several steps:

(i) First we construct the isomorphisms $\Phi_{(X,A)}\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X,A;M)$ in twisted homology.
(ii) Secondly we show that the isomorphisms are natural.
(iii) Thirdly we quickly outline how to modify (i) and (ii) to prove the corresponding statements for twisted cohomology.

Proof of Theorem 328.1 (1) – Construction of the Isomorphism. Let X be a connected CW-complex, let $A \subseteq X$ be a subcomplex, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module. We write $\pi := \pi_1(X,x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering and we write $\widetilde{A} := p^{-1}(A)$.

We equip $\widetilde{X}$ with the CW-structure given by the CW-Complex Covering Proposition 327.6. By the Deck Transformation Group Proposition 319.12 we have a natural isomorphism $\pi \cong \mathrm{Deck}(p\colon \widetilde{X} \to X)$. Thus, via the Regular Covering–Cellular Chain Complex Lemma 327.10, we can view $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A})$ as a chain complex of free $\mathbb{Z}[\pi]$-left modules.

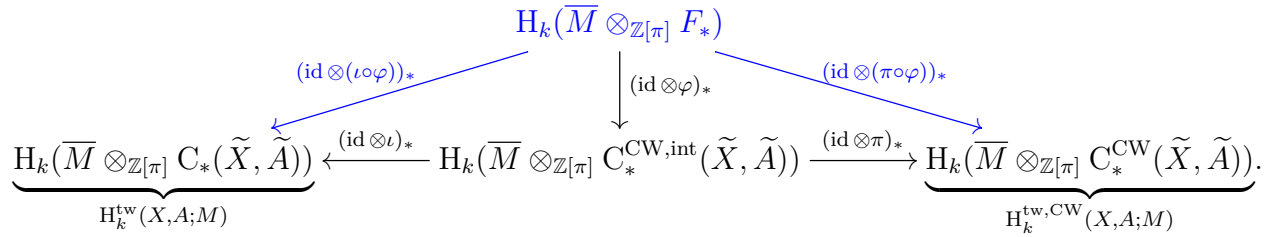
Note that in our present setting the Free Imitation Existence Lemma 313.21 says that there exists a chain complex (F_*,∂_*) of *free* $\mathbb{Z}[\pi]$-left modules together with a quasi-isomorphism $\varphi\colon F_* \to \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X},\widetilde{A})$ of $\mathbb{Z}[\pi]$-left modules. Now we consider the following little diagram:

$$\begin{array}{ccccc} & & F_* & & \\ & \swarrow^{\iota\circ\varphi} & \downarrow{\scriptstyle \varphi} & \searrow^{\pi\circ\varphi} & \\ \mathrm{C}_*(\widetilde{X},\widetilde{A}) & \xleftarrow{\iota} & \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X},\widetilde{A}) & \xrightarrow{\pi} & \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A}). \end{array}$$

Note that the action of π on $\widetilde{X}$ turns all three chain complexes at the bottom into chain complexes of $\mathbb{Z}[\pi]$-left modules and the naturality of ι and π implies that these maps commute with the π-action, in other words, the horizontal maps ι and π are homomorphisms of $\mathbb{Z}[\pi]$-left modules. It follows from the above together with the Relative CW-Intermediate-Singular Proposition 328.3 that $i\circ\varphi\colon F_* \to \mathrm{C}_*(\widetilde{X},\widetilde{A})$ and $\pi\circ\varphi\colon F_* \to \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A})$ are quasi-isomorphisms between chain complexes of free $\mathbb{Z}[\pi]$-left modules. Thus it follows from the Quasi Iso–Homotopy Equivalence Proposition 313.18 that $i\circ\varphi$ and $\pi\circ\varphi$ are chain homotopy equivalences of $\mathbb{Z}[\pi]$-left module chain complexes.[3580] Next we consider

[3580]Recall that it follows from the Relative CW-Intermediate-Singular Proposition 328.3 that the chain maps $\iota\colon \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X},\widetilde{A}) \to \mathrm{C}_*(\widetilde{X},\widetilde{A})$ and $\pi\colon \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X},\widetilde{A}) \to \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{A})$ are quasi-isomorphisms. But there is no reason to believe that $\mathrm{C}_n^{\mathrm{CW,int}}(\widetilde{X},\widetilde{A})$ is a *free* (or even just projective) $\mathbb{Z}[\pi]$-left module. Thus there is no reason why these chain maps should be chain homotopy equivalences of $\mathbb{Z}[\pi]$-left chain complexes of $\mathbb{Z}[\pi]$-left module chain complexes. The case $\pi = \{e\}$ is much easier, since by the Free Abelian Subgroup Theorem 97.14 we know that any subgroup of a free abelian group is again free abelian.

the induced maps



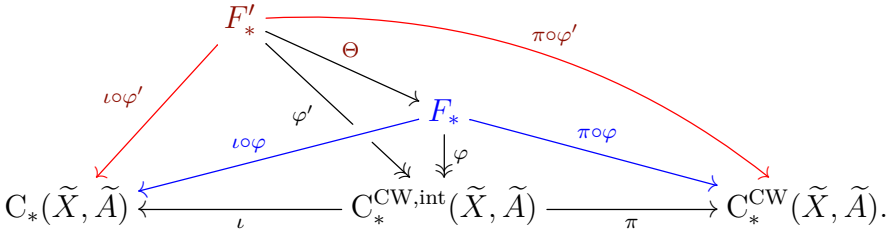
Since $\iota \circ \varphi$ and $\pi \circ \varphi$ are both chain homotopy equivalence we obtain from the Homology-with-Coefficients Lemma 314.6 (2b) and the Chain Homotopy Equivalence-H_*-Corollary 313.7 that the diagonal maps are isomorphisms of R-left modules. Therefore we can now define

$$\Phi_{(X,A)} \coloneqq (\mathrm{id} \otimes (\pi \circ \varphi))_* \circ (\mathrm{id} \otimes (\iota \circ \varphi))_*^{-1} \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \ \to \ \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M).$$

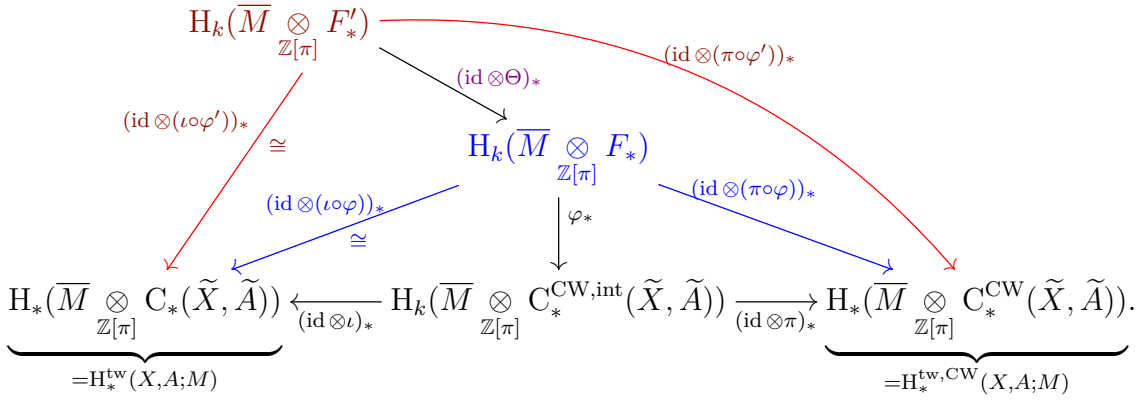
As we just discussed, the above diagonal maps are isomorphisms of R-left modules. Thus we see that $\Phi_{(X,A)} \colon \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \to \mathrm{H}_k^{\mathrm{tw,CW}}(X, A; M)$ is indeed an isomorphism of R-left modules. ∎

Proof of Theorem 328.1 (1) – Well-definedness. We now want to show that the isomorphism we just constructed does not depend on the choice of the free imitation (F_*, φ_*) of $\mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X}, \widetilde{A})$.

Thus let (F'_*, φ'_*) be another free imitation of $\mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X}, \widetilde{A})$. It follows from the Homomorphism Lifting Lemma 313.22 that there exists a chain map $\Theta \colon F_* \to F'_*$ with $\varphi = \Theta \circ \varphi'$. We now consider the following diagram:



Note that the diagram commutes by construction and the choice of Θ. Tensoring with M we obtain the following commutative diagram:



Since the diagram commutes we see that (F_*, φ) and (P'_*, φ') lead to the same definition of the isomorphism $\Phi_{(X,A)} \colon \ = \mathrm{H}_*^{\mathrm{tw}}(X, A; M) \to \mathrm{H}_*^{\mathrm{tw,CW}}(X, A; M)$. ∎

Proof of Theorem 328.1 (1) – Naturality. To unburden the notation from unnecessary baggage we will prove naturality only for absolute twisted homology. The argument for the relative case is basically the same, one just needs to carry a little bit more data around.

Thus let $f\colon X \to Y$ be a cellular map between two connected CW-complexes. Let $x_0 \in X$, let R be a ring, and let M be an $(R, \mathbb{Z}[\pi_1(Y, f(x_0))])$-left left module. We write $\pi_X \coloneqq \pi_1(X, x_0)$ and $\pi_Y \coloneqq \pi_1(Y, f(x_0))$. We denote by $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ the natural universal coverings corresponding to the base points x_0 and $f(x_0)$. Furthermore we denote by $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{y}_0 \in \widetilde{Y}$ the natural base points. In the following we denote by $\widetilde{f}\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0) \to (\widetilde{Y}, \widetilde{y}_0)$ the induced map from Definition 321.4.

From the Free Imitation Existence Lemma 313.21 we obtain the following:

- A chain complex (F_{X*}, ∂_*) of free $\mathbb{Z}[\pi_X]$-left modules together with a quasi-isomorphism $\varphi_X\colon F_{X*} \to \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X})$ of $\mathbb{Z}[\pi_X]$-left chain complexes.
- A chain complex (F_{Y*}, ∂_*) of free $\mathbb{Z}[\pi_Y]$-left modules together with a quasi-isomorphism $\varphi_Y\colon F_{Y*} \to \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{Y})$ of $\mathbb{Z}[\pi_Y]$-left chain complexes, which is furthermore an epimorphism.

Claim 1. There exists a chain map

$$\Theta\colon \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X \;\to\; F_Y$$

of $\mathbb{Z}[\pi_Y]$-left chain complexes such that the following diagram commutes:

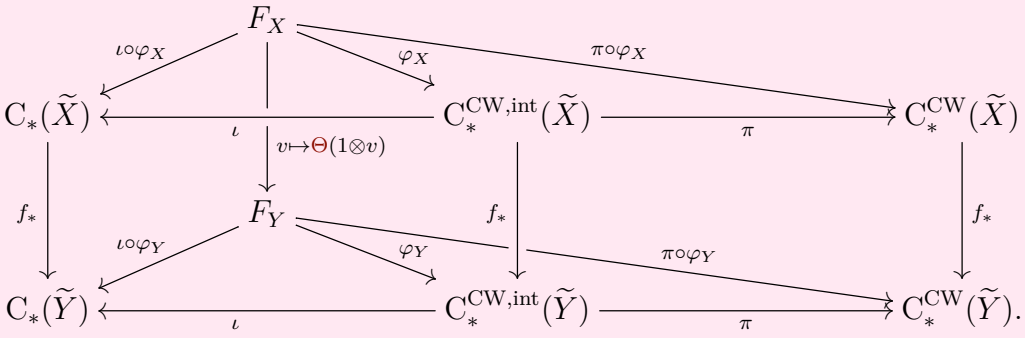
$$\begin{array}{ccc} \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X & \xrightarrow{\mathrm{id} \otimes \varphi_X} & \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{X}) \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle g\otimes\sigma \mapsto g\cdot(f_*\sigma)} \\ F_Y & \xrightarrow{\varphi_Y} & \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{Y}). \end{array}$$

Proof. We make the following three observations:

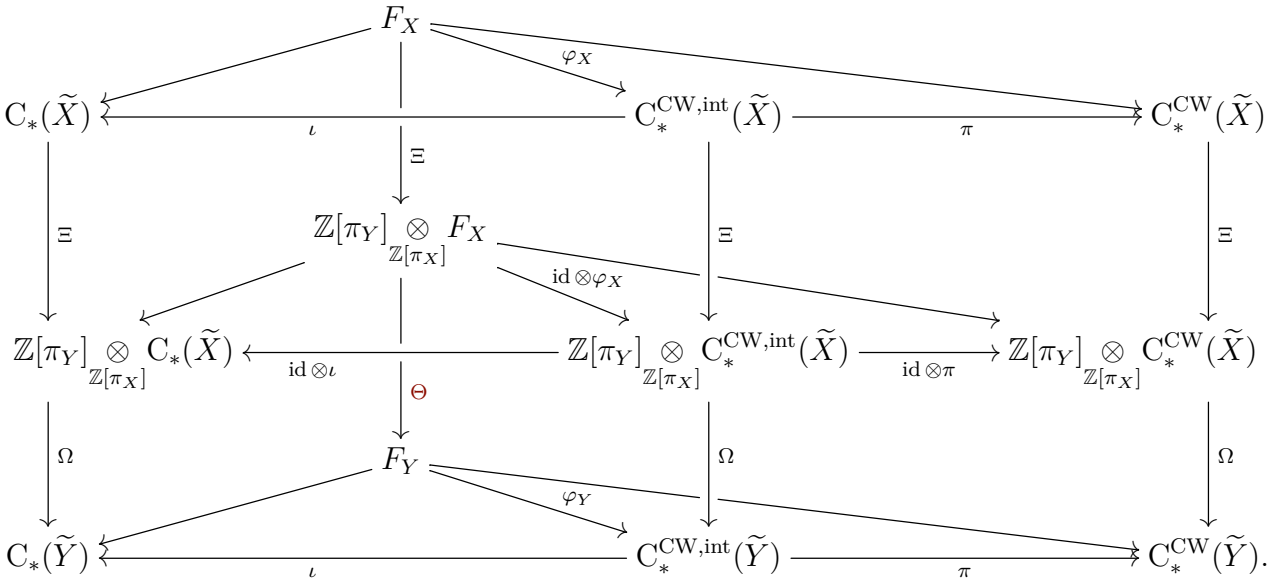
- It follows easily from the Chain Complex-Equivariance Proposition 321.7 that the right vertical map is a chain map of $\mathbb{Z}[\pi_Y]$-left chain complexes.
- By the Free Module–Tensor-with-Ring Lemma 312.18 (1) we know that $\mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X$ is a chain complex of *free* $\mathbb{Z}[\pi_Y]$-left modules.
- By construction we know that $\varphi_Y\colon F_{Y*} \to \mathrm{C}_*^{\mathrm{CW,int}}(\widetilde{Y})$ is a quasi-isomorphism.

It follows from these observations together with the Homomorphism Lifting Lemma 313.22 that there exists a chain map $\Theta\colon \mathbb{Z}[\pi_Y] \otimes_{\mathbb{Z}[\pi_X]} F_X \to F_Y$ with the desired properties. ⊞

Claim 2. The following diagram commutes:



Proof. We consider the following even more excessive diagram:



We make the following clarifications:

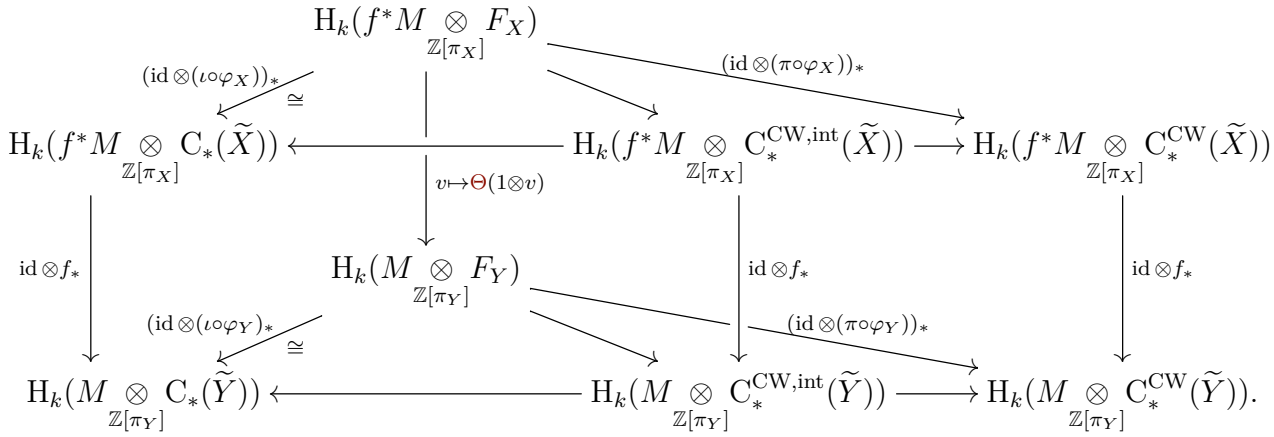
- The undecorated diagonal maps are, as before, given by the composition of the corresponding two other maps. For example the top left diagonal map is $\iota \circ \varphi_X$.
- The curved maps $\widetilde{f}_*$ are induced by the cellular map $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$.
- The top vertical maps Ξ are the purely algebraic maps given by $\sigma \mapsto 1 \otimes \sigma$.
- The bottom vertical maps Ω are all given by $g \otimes \sigma \mapsto g \cdot f_*(\sigma)$.

Now we want to argue that the diagram commutes:

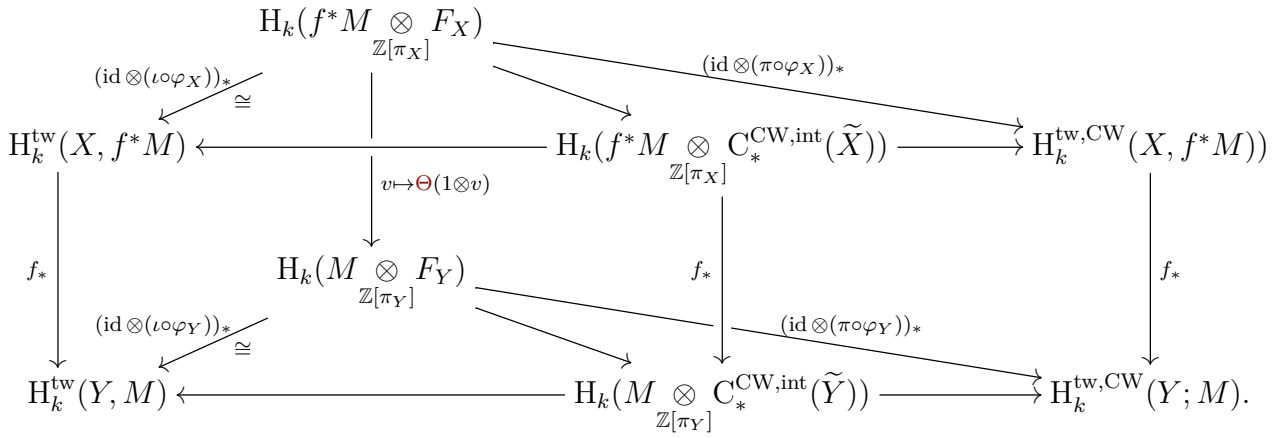
- The top part of the diagram commutes basically by definition.
- It follows immediately from the definitions that the two bottom rectangles commute.
- The bottom center parallelogram involving Θ commutes by the choice of Θ. The two other bottom parallelograms commute since they are defined in terms of the other maps.

In summary we see that the big diagram commutes. Finally note that the composition $\Omega \circ \Xi$ in each case equals, basically by definition, the map f_*. This final observation implies that the commutativity of the big diagram gives us also the commutativity of the diagram given in the claim. ⊞

Next note that it follows easily from the claim that the following diagram also commutes:



In other words, the following diagram commutes:



By definition of Φ_X and Φ_Y this just means that the following diagram does indeed commute, fulfilling our deepest wishes:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(X; f^*M) & \xrightarrow{\Phi_X} & \mathrm{H}_k^{\mathrm{tw,CW}}(X; f^*M) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_k^{\mathrm{tw}}(Y; M) & \xrightarrow{\Phi_Y} & \mathrm{H}_k^{\mathrm{tw,CW}}(Y; M). \end{array}$$

■

It remains to prove the statement of the Singular-Cellular Twisted Isomorphism Theorem 328.1 regarding twisted cohomology.

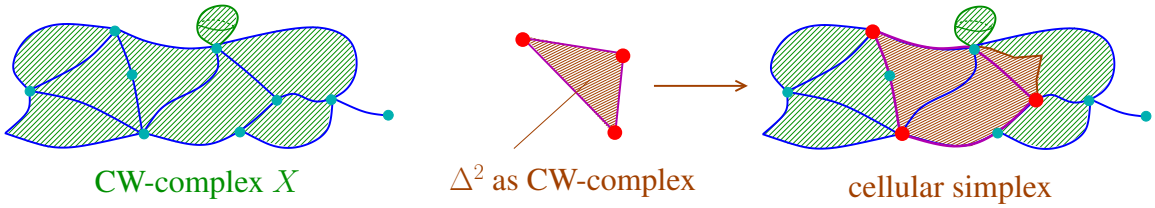
Proof of Theorem 328.1 (2). The proof of the statement regarding twisted cohomology is evidently almost identical to the proof for twisted homology. All one needs to do is to replace the Homology-with-Coefficients Lemma 314.6 (2b) by the Dual Cochain Map Lemma 315.22. We leave it to the reader to fill in the details. ■

328.5. **Appendix: Cellular simplices.** In this appendix we sketch a different approach to proving the Singular-Cellular Twisted Isomorphism Theorem 328.1. In the following let X be a CW-complex. There is a different candidate for an intermediate chain complex,

namely the *chain complex of cellular simplices*, i.e. the chain complex

$$\mathrm{C}_k^{\mathrm{cs}}(X) \;:=\; \text{subcomplex of } \mathrm{C}_k(X) \text{ generated by all cellular simplices } \Delta^k \to X,$$

where we equip Δ^k with the "obvious" CW-structure. Note that if X is equipped with a free left action by a group G, then $\mathrm{C}_k^{\mathrm{cs}}(X)$ is in fact a *free* $\mathbb{Z}[G]$-left module.



Again we have two natural chain maps

$$\mathrm{C}_k^{\mathrm{cs}}(X) \;\to\; \mathrm{C}_k(X) \qquad \text{and} \qquad \mathrm{C}_k^{\mathrm{cs}}(X) \to \mathrm{C}_k^{\mathrm{CW}}(X).$$

Note that if X is equipped with a free G-left action, then it follows from the naturality of these two chain maps that these are in fact homomorphisms of *free* $\mathbb{Z}[G]$-left modules. So it remains to show that these two are quasi-isomorphisms. For the left-hand side this can be done using [**Löh2019**, Lemma 4.5.7]. We leave it to the reader to fill in the details, to type it up and to send the write-up to the author.

328.6. Exercises for Chapter 328.

Exercise 328.1. Let K be an *ordered* connected non-empty abstract simplicial complex. Use the Simplicial-Singular Chain Complex Theorem 177.12 and some of the algebraic facts from this chapter to show that twisted simplicial (co-) homology is naturally isomorphic to twisted singular (co-) homology.
Remark. This exercise is much easier than it looks, since most technical issues that arise for cellular chain complexes go away, once we consider simplicial chain complexes.

Exercise 328.2. Let (M, x_0) be a pointed compact connected smooth manifold and let $\alpha\colon \pi_1(M, x_0) \to \mathrm{GL}(n, \mathbb{Z})$ be a representation. Show that every twisted homology group $\mathrm{H}_k^{\mathrm{tw}}(M; {}_\alpha\mathbb{Z}^n)$ is a finitely generated abelian group.

Exercise 328.3. Let X be a connected CW-complex and let M be a $\mathbb{Z}[\pi_1(X)]$-left module. Furthermore let $k \in \mathbb{N}_{\geq 2}$. We use the projection $X \times S^k \to X$ to view M also as a $\mathbb{Z}[\pi_1(X \times S^k)]$-left module. Compute the twisted homology groups $\mathrm{H}_*^{\mathrm{tw}}(X \times S^k; M)$ in terms of the twisted homology groups $\mathrm{H}_*^{\mathrm{tw}}(X \times S^k; M)$.

Exercise 328.4. Let $n \in \mathbb{N}$. We consider the corresponding real projective space $\mathbb{R}\mathrm{P}^n$. We write $\pi := \pi_1(\mathbb{R}\mathrm{P}^n)$. In the following let $\varphi\colon \pi \to \mathrm{GL}(1, \mathbb{Z})$ be the unique non-trivial representation. Note that the natural projection $\mathbb{Z} \to \mathbb{F}_2$ induces a homomorphism ${}_\varphi\mathbb{Z} \to {}_{\mathrm{tr}}\mathbb{F}_2$ of $\mathbb{Z}[\pi]$-left modules. For which $k \in \{0, \dots, n\}$ is the induced homomorphism $\mathrm{H}_k^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_\varphi) \to \mathrm{H}_k^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_{\mathrm{tr}}\mathbb{F}_2) =_{\mathrm{i}} \mathrm{H}_k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$ an epimorphism?

CHAPTER 329

Calculating twisted cellular homology

In Chapter 327 we introduced twisted cellular (co-) homology and in Chapter 328 we proved that twisted cellular and singular (co-) homology are naturally isomorphic. In this chapter we will provide many calculations of twisted cellular chain complexes and their (co-) homology groups.

This chapter is organized as follows:

- In Section 329.1 we will see that often one can calculate twisted cellular (co-) homology via intermediate coverings or alternative universal coverings.
- In Section 329.2 we will remind the reader how one can calculate the "usual" cellular chain complex of a CW-complex.
- In Section 329.3 we will make several general observations for how to determine twisted cellular (co-) chain complexes.
- In Section 329.4 we will provide several basic calculations of twisted cellular chain complexes and twisted cellular homology modules.
- In Section 329.5 we will formulate the Lens Space-CW-Proposition 329.15 which gives us a cellular chain complex of the universal covering of any lens space $L(p,q)$.
- In Section 329.6 we will provide a proof of the Lens Space-CW-Proposition 329.15.
- In Section 329.7 we will provide exercises for Chapter 329.

In the next Chapter 330 we will introduce presentation complexes and we will see that we can calculate the cellular chain complex of their universal coverings via Fox calculus.

329.1. **Twisted cellular (co-) homology via intermediate coverings.** The following proposition, which is an analog of the Replace Universal Covering Proposition 320.16, says in a nutshell that to calculate twisted cellular (co-) homology one can use alternative universal coverings and possibly small coverings:

Proposition 329.1. (Replace Universal Covering Proposition) Let X be a path-connected CW-complex and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering. Let $A \subseteq X$ be subcomplex. We write $\widetilde{A} := p^{-1}(A)$. We set $\pi := \pi_1(X, x_0)$. Let $\Gamma \subseteq \pi$ a normal subgroup. We denote by $\rho\colon \pi \to \pi/\Gamma$ the projection and we denote by $\rho\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi/\Gamma]$ also the corresponding ring homomorphism. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi/\Gamma])$-left left module. Let $q\colon (Y, y_0) \to (X, x_0)$ be any path-connected regular covering with $q_*(\pi_1(Y, y_0)) = \Gamma$. We set $B := q^{-1}(A)$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering

$r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (Y_\Gamma, y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow{\quad r \quad} & Y_\Gamma. \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

The natural maps

$$\begin{array}{rcl} \overbrace{\overline{\rho^*M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A})}^{\mathrm{C}_k^{\mathrm{CW,tw}}(X,A;\rho^*M)} & \to & \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k^{\mathrm{CW}}(Y, B) \\ m \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & m \otimes (r \circ \sigma\colon \Delta^k \to Y) \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}[\pi/\Gamma]\text{-left}}(\mathrm{C}_k^{\mathrm{CW}}(Y, B), M) & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{A}), \rho^*M)}^{=\mathrm{C}^k_{\mathrm{CW,tw}}(X,A;\rho^*M)} \\ \varphi & \mapsto & (\sigma \mapsto \varphi(r \circ \sigma)) \end{array}$$

are isomorphisms of (co-) chain complexes of R-left modules. In particular we get induced isomorphisms of (co-) homology modules.

Example.

- For $\Gamma = \{e\}$ the Replace Universal Covering Proposition 329.1 tells us that we can use any universal covering to calculate the isomorphism type of the twisted cellular (co-) chain complex and twisted cellular (co-) homology. Thus we do need to work with the natural universal covering from Definition 319.10. This feature will get used all the time.
- The case $\Gamma = \pi_1(X, x_0)$ can only occur if M is the trivial $\mathbb{Z}[\pi]$-module. In this case the Replace Universal Covering Proposition 329.1 shows that untwisted cellular homology equals twisted cellular homology with trivial coefficients.
- Let π be a group and let Γ be a normal subgroup of π. We denote by $\rho\colon \pi \to \pi/\Gamma$ the projection. Let M be a $\mathbb{Z}[\pi]$-left module. If every $g \in \Gamma$ acts trivially on M, then there exists an obvious $\mathbb{Z}[\pi/\Gamma]$-left module $\widetilde{M}$ such that $\rho^*\widetilde{M} = M$. In practice that is often how the Replace Universal Covering Proposition 320.16 gets applied. □

Proof. As in the case of the proof of the original Replace Universal Covering Proposition 320.16 the desired statements follow easily from the definitions. We leave it to the reader to make the necessary changes to the earlier proof. ■

329.2. **Calculating cellular chain complexes.** Before we turn to the calculation of the cellular chain complex of coverings let us first recall how one can calculate the cellular chain complex of a CW-complex W.

To state our earlier result we should recall Definition 164.17 and Definition 155.12:

Definition 329.2.

(1) As in Definition 164.17 we define a charted CW-complex as a pair consisting of a CW-complex W together with a full set $\{\Phi_c\colon \overline{B}^{\dim(c)} \to W^{\dim(c)}\}_{c \in \mathrm{Cell}(W)}$ of characteristic maps of the cells of W.

(2) Let W be a charted CW-complex as in (1). For an n-cell α we write

$$\Xi_\alpha \;\coloneqq\; \Phi_{\alpha *}([\overline{B}^n]) \;\in\; \mathrm{H}_n(W^n, W^{n-1}) \;=\; \mathrm{C}_n^{\mathrm{CW}}(W).$$

The above definition allows us to specify a basis for cellular chain complexes:

Lemma 329.3. (Cellular Chain Complex–Basis Lemma) Suppose that we are given a charted CW-complex $(W, \{\Phi_c\colon \overline{B}^{\dim(c)} \to W^{\dim(c)}\}_{c\in \mathrm{Cell}(W)})$. For every $n \in \mathbb{N}_0$ the set $\{\Xi_c\}_{c\in \mathrm{Cell}_n(W)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(W) = \mathrm{H}_n(W^n, W^{n-1})$.

Proof. This lemma is just a reformulation of the Cellular Chain Complex–Basis Lemma 327.8. ■

The following proposition reduces the calculation of the boundary maps in the cellular chain complex of a CW-complex to the calculation of degrees of continuous self-maps of spheres. This reduction is a great help, since in Chapters 151 and 152 we developed many tools for calculating such self-maps.

Proposition 329.4. (Cellular Boundary–Degree Proposition) Let W be a charted CW-complex and let $n \in \mathbb{N}$. Given an n-cell α and an $(n-1)$-cell β we consider the following continuous self-map of S^{n-1}:

$$\begin{array}{ccccccc} S^{n-1} & \xrightarrow{\Phi_\alpha|_{S^{n-1}}} & W^{n-1} & \xrightarrow{q} & W^{n-1}/W^{n-2} & & \\ & & & & \cong\uparrow & & \\ & & & & \bigvee_{j\in \mathrm{Cell}_{n-1}(W)} \overline{B}^{n-1}/S^{n-2} & \xrightarrow{p_\beta} & \overline{B}^{n-1}/S^{n-2} \xrightarrow[\cong]{g} S^{n-1} \end{array}$$

where we adopt the following notation:

- For $c \in \mathrm{Cell}(W)$ we denote by Φ_c the corresponding characteristic map.
- $q\colon W^{n-1} \to W^{n-1}/W^{n-2}$ is the natural projection.
- Let $j \in \mathrm{Cell}_{n-1}(W)$. Since $\Phi_j(S^{n-2}) \subseteq W^{n-2}$ we see that the characteristic map $\Phi_j\colon \overline{B}^{n-1} \to W^{n-1}$ descends to a continuous map $\overline{\Phi}_j\colon \overline{B}^{n-1}/S^{n-2} \to W^{n-1}/W^{n-2}$.
- $\bigvee_{j\in \mathrm{Cell}_{n-1}(W)} \overline{\Phi}_j\colon \bigvee_{j\in \mathrm{Cell}_{n-1}(W)} \overline{B}^{n-1}/S^{n-2} \to W^{n-1}/W^{n-2}$ is the natural map provided by the Wedge Proposition 96.2.
- $p_\beta\colon \bigvee_{j\in \mathrm{Cell}_n(W)} \overline{B}^{n-1}/S^{n-2} \to \overline{B}^{n-1}/S^{n-2}$ is the natural map that is induced by the identity on the β-copy of $\overline{B}^{n-1}/S^{n-2}$ and the constant map on all other copies of $\overline{B}^{n-1}/S^{n-2}$.
- $g\colon \overline{B}^{n-1}/S^{n-2} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from the Ball Quotient-Sphere Lemma 8.10.

For any n-cell α we have

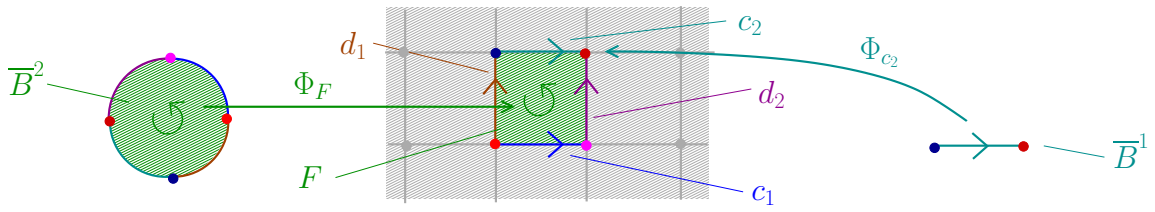
$$\partial_n(\Xi_\alpha) \;=\; \sum_{\beta\in \mathrm{Cell}_{n-1}(W)} \deg(\Omega_{\beta,\alpha}) \cdot \Xi_\beta \;\in\; \mathrm{C}_{n-1}^{\mathrm{CW}}(W)$$

where

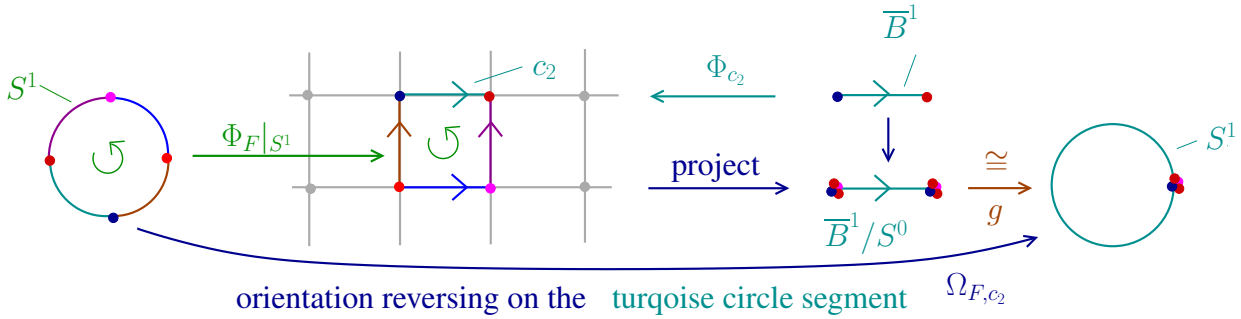
$$\Omega_{\beta,\alpha} \;:=\; g \circ p_\beta \circ \Big(\bigvee_{j\in \mathrm{Cell}_{n-1}(W)} \overline{\Phi}_j\Big)^{-1} \circ q \circ \Phi_\alpha|_{S^{n-1}}\colon S^{n-1} \;\to\; S^{n-1}.$$

Proof. The proposition is essentially the same as the Cellular Boundary–Degree Proposition 155.15. ■

Example. We consider the CW-complex $W = \mathbb{R}^2$ that is shown below.
We also show a characteristic map for a 2-cell F and for several 1-cells. Our next goal is the calculation of $\partial_2(\Xi_F) \in \mathrm{C}_1^{\mathrm{CW}}(W)$. It follows from the Cellular Boundary–Degree



Proposition 329.4. that the Ξ_{c_2}-coefficient of $\partial_2(\Xi_F)$ is given by the degree of the map $\Omega_{F,c_2}\colon S^1 \to S^1$ that is shown in the figure below:



Using the Degree-via-Local Orientations Proposition 152.11 one can easily verify that the degree is -1. If one proceeds like this for all 1-cells we see that

$$\partial_2(\Xi_F) \;=\; \Xi_{c_1} + \Xi_{d_2} - \Xi_{c_2} - \Xi_{d_1}. \qquad \square$$

329.3. **Calculating twisted cellular chain complexes.** In this section we will make several general comments about calculating twisted cellular chain complexes. In the subsequent sections these methods will be applied to various examples. These calculations will not only be helpful for calculating twisted (co-) homology, the description of the twisted cellular chain complexes will also come in handy when we discuss Reidemeister torsion.

The setup is as follows: We consider a regular covering $p\colon (\widetilde{Y},\widetilde{y}_0) \to (X,x_0)$ between pointed connected CW-complexes. We write $\pi := \pi_1(X,x_0)$ and $\Gamma := p_*(\pi_1(\widetilde{Y},\widetilde{y}_0))$. We use the base point $\widetilde{y}_0$ to define a left action of π/Γ on $\widetilde{Y}_0$. Furthermore let R be a ring and let $\varphi\colon \pi/\Gamma \to \mathrm{GL}(n,R)$ be a representation, i.e. a group homomorphism. The calculation of

(a) the twisted cellular chain complex $\overline{{}_\varphi R^{1\times n}} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X})$ and of
(b) the twisted cellular cochain complex $\mathrm{Hom}_{\mathbb{Z}[\pi/\Gamma]\text{-left}}(\mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}), {}_\varphi R^{1\times n})$

breaks up into two parts:

(1) We need to describe the cellular $\mathrm{C}^{\mathrm{CW}}_*(\widetilde{X})$ as a chain complex of free $\mathbb{Z}[\pi/\Gamma]$-left modules.
(2) (a) We need to describe the effect of tensoring $\mathrm{C}^{\mathrm{tw}}_*(\widetilde{X})$ with $\overline{{}_\varphi R^{1\times n}}$ and
    (b) we need to describe the cochain complex obtained by considering the $\mathbb{Z}[\pi/\Gamma]$-left homomorphisms from $\mathrm{C}^{\mathrm{tw}}_*(\widetilde{X})$ to $R^{1\times n}$.

In the following two subsections we explain how one can address these issues.

[3580]Note that the definition of the π/Γ action depends on the choice of the point $\widetilde{y}_0$ in the preimage of x_0.

329.3.1. **The chain complex of a covering.** Let $p\colon (\widetilde{Y},\widetilde{y}_0) \to (X,x_0)$ be a regular covering of connected CW-complexes. We write $\pi := \pi_1(X,x_0)$ and $\Gamma := p_*(\pi_1(\widetilde{Y},\widetilde{y}_0))$. Of course there is no general principle how one can calculate the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{Y})$ as a chain complex of free $\mathbb{Z}[\pi/\Gamma]$-left modules. But in the following we present some general steps which often will lead to success:

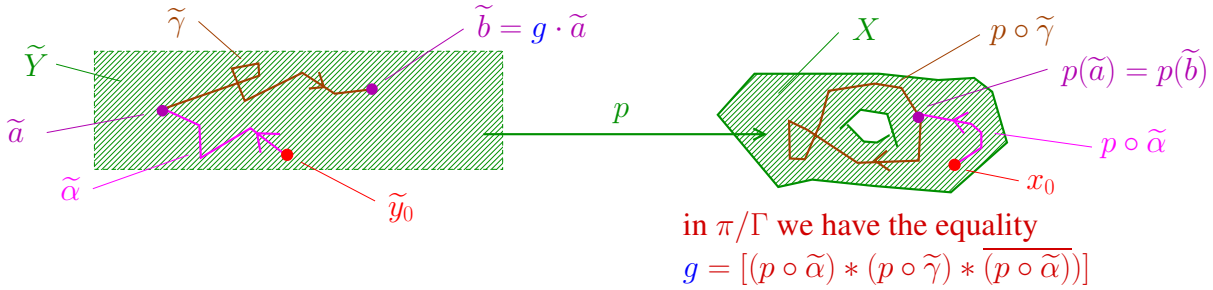
(1) It follows from the Replace Universal Covering Proposition 329.1 that we can calculate the isomorphism type of the twisted cellular chain complexes using any universal covering (or potentially even smaller) covering. So one first needs to find a convenient CW-structure for X and a convenient description of the appropriate covering. In the following we assume that we picked a CW-structure for X with a single 0-cell x_0.[3581]
(2) We need to pick lifts of characteristic maps of the various cells of X. For a 1-cell it is usually convenient to pick a lift of the characteristic map that starts at the base point $\widetilde{y}_0$ of $\widetilde{Y}$.
(3) The boundaries of the lifts can be determined by the Cellular Boundary–Degree Proposition 329.4 together with the Deck Transformation Group Proposition 319.12.

We now assume that we are in the setting of the Replace Universal Covering Proposition 329.1. Then we need to understand the action of the group π/Γ on the covering $\widetilde{Y}$. In real life examples this goal can be achieved by the following observation:

Observation 329.5. Let $p\colon (\widetilde{Y},\widetilde{y}_0) \to (X,x_0)$ be a regular covering between two topological spaces that are path-connected and locally contractible. We write $\pi := \pi_1(X,x_0)$ and $\Gamma := p_*(\pi_1(\widetilde{Y},\widetilde{y}_0))$. We equip $\widetilde{Y}$ with the π/Γ-left action from the Deck Transformation Group Proposition 319.12 that corresponds to the base point $\widetilde{y}_0 \in \widetilde{Y}$. Now let $\widetilde{a},\widetilde{b}$ be two points in $\widetilde{Y}$ with $p(\widetilde{a}) = p(\widetilde{b})$. We pick a path $\widetilde{\alpha}\colon [0,1] \to \widetilde{Y}$ from $\widetilde{y}_0$ to $\widetilde{a}$ and we pick a path $\widetilde{\gamma}\colon [0,1] \to \widetilde{Y}$ from $\widetilde{a}$ to $\widetilde{b}$. By Exercise 128.7 we know that with

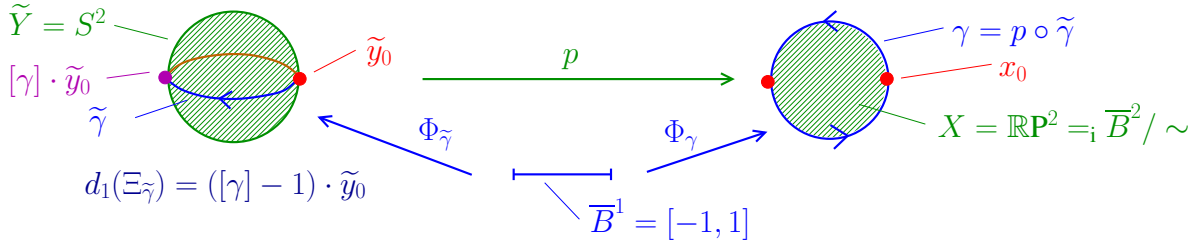
$$g \;:=\; [\underbrace{(p\circ\widetilde{\alpha}) * (p\circ\widetilde{\gamma}) * \overline{(p\circ\widetilde{\alpha})}}_{\text{loop in } (X,x_0)}] \;\in\; \pi/\Gamma$$

we have $g\cdot[\widetilde{a}] = [\widetilde{b}]$. If $\widetilde{a} = \widetilde{y}_0$, then we can of course take $\widetilde{\alpha}$ to be the constant path and we see that $g = [p\circ\widetilde{\gamma}] \in \pi/\Gamma$ has already the desired property.



[3581] We point out that it follows from the CW-Complex–Spanning Tree Proposition 140.5 (2) that every connected non-empty CW-complex is homotopy equivalent to a CW-complex Y with a single 0-cell. By the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition refprop:homotopy-equivalent-twisted-coefficients-2 it suffices to calculate the twisted invariants of Y.

Example. A typical example is given as follows: Let $p\colon \widetilde{Y} \to X$ be a covering of a CW-complex with a single 0-cell x_0. Let γ be a 1-cell with characteristic map $\Phi_\gamma\colon [-1,1] \to X$. Note that by an obvious reparametrization Φ_γ defines an element in $\pi_1(X, x_0)$ which we denote by $[\gamma]$. Let $\widetilde{y}_0$ be a point in the preimage of x_0 and let $\Phi_{\widetilde{\gamma}}\colon [-1,1] \to \widetilde{Y}$ be the lift of Φ_γ such that $\Phi_{\widetilde{\gamma}}(-1) = \widetilde{y}_0$.



We then see that

$$\partial_1(\Xi_{\widetilde{\gamma}}) \underset{\text{by the Cellular First Boundary Map Lemma 155.14}}{=} \widetilde{\gamma}(1) - \widetilde{\gamma}(-1) \underset{\text{by Observation 329.5, applied to the points } \widetilde{a} := \gamma(-1) = \widetilde{y}_0 \text{ and } \widetilde{b} := \gamma(1) \text{ and to the constant path } \widetilde{\alpha}}{=} [\gamma] \cdot \widetilde{y}_0 - \widetilde{y}_0 = ([\gamma] - 1) \cdot \widetilde{y}_0.$$

□

329.3.2. **Tensoring a chain complex.** Next we recall Notation 124.5:

Notation 329.6. Let π be a group, let R be a ring and let $\gamma\colon \pi \to \mathrm{GL}(k, R)$ be a homomorphism.

(1) In the Group Ring-to-Matrix Ring Lemma 124.2 we showed that the map

$$\begin{aligned} \gamma_*\colon \mathbb{Z}[\pi] &\to \mathrm{M}(k \times k, R) \\ \sum_{i=1}^{s} \underbrace{a_i \cdot g_i}_{\text{where } a_i \in \mathbb{Z} \text{ and } g_i \in \pi} &\mapsto \sum_{i=1}^{s} a_i \cdot \gamma(g_i) \end{aligned}$$

is a ring homomorphism.

(2) Given $m, n \in \mathbb{N}$ we denote by γ_* also the map

$$\begin{aligned} \gamma_*\colon \mathrm{M}(m \times n, \mathbb{Z}[\pi]) &\to \mathrm{M}((m \cdot k) \times (n \cdot k), R) \\ (a_{ij}) &\mapsto (\gamma_*(a_{ij})). \end{aligned}$$

we replace the entry $a_{ij} \in \mathbb{Z}[\pi]$ by the $(k \times k)$-matrix $\gamma_*(a_{ij})$ over R

Note that it follows from the Group Ring-to-Matrix Ring Lemma 124.2, the Matrix Multiplication–Change-of-Ring Lemma 124.3 and the Matrix-of-Matrices Lemma 124.4 that the above map $\gamma_*\colon \mathrm{M}(m \times n, \mathbb{Z}[\pi]) \to \mathrm{M}((m \cdot k) \times (n \cdot k), R)$ is a ring homomorphism.

We continue by recalling Definition 309.23:

Definition 329.7. Let S be a ring and let V and W be two free S-left modules. We assume that V and W are equipped with finite ordered bases $\mathcal{B} = (b_1, \ldots, b_m)$ and $\mathcal{C} = (c_1, \ldots, c_n)$. Let $\varphi\colon V \to W$ be an S-left module homomorphism. We refer to the unique $(m \times n)$-matrix

$P = (p_{ij}) \in \mathrm{M}(m \times n, S)$ with

$$\varphi(b_i) \;=\; \sum_{j=1}^{n} c_j \cdot p_{ij}$$

as the matrix representing φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. Note that this is the unique matrix that makes the following diagram of S-left module homomorphisms commute:

$$\begin{array}{ccc} S^{1\times m} & \xrightarrow{\xi \mapsto \xi\cdot P} & S^{1\times n} \\ \cong\downarrow{\scriptstyle \mathfrak{e}_i \mapsto b_i} & & \cong\downarrow{\scriptstyle \mathfrak{e}_i \mapsto c_i} \\ V & \xrightarrow{\varphi} & W. \end{array}$$

The following lemma can be viewed as a variation on the Free Module–Tensor-with-Ring Lemma 312.18:

Lemma 329.8. (Tensor–Matrix Lemma) Let π be a group, let R be a ring and let $\alpha\colon \pi \to \mathrm{GL}(k, R)$ be a representation. As in Convention 320.9 we use φ to view $R^{1\times k}$ as an $(R, \mathbb{Z}[\pi])$-left left module and as in Notation 320.7 we can thus view $\overline{R^{1\times k}}$ as an $(R, \mathbb{Z}[\pi])$-bimodule.

(1) Let B be a free $\mathbb{Z}[\pi]$-left module with basis $\mathcal{B}$ in the sense of Definition 309.17. Then $\overline{R^{1\times k}} \otimes_{\mathbb{Z}[\pi]} B$ is a free R-left module with basis $\{\mathfrak{e}_i \otimes b\}_{i\in\{1,\dots,k\}, b\in\mathcal{B}}$.[3582]

(2) Let B and C be two $\mathbb{Z}[\pi]$-left modules with ordered bases $\mathcal{B} = (b_1, \dots, b_m)$ for B and $\mathcal{C} = (c_1, \dots, c_n)$ for C. Furthermore let $\varphi\colon B \to C$ be a $\mathbb{Z}[\pi]$-left module homomorphism and let A be the $(m \times n)$-matrix that represents φ with respect to the given basis. As in Definition 329.7 we denote by $\varphi_*(A) \in \mathrm{M}(m\cdot k \times n\cdot k, R)$ the matrix that is given by replacing each entry a_{ij} by the $(k \times k)$-matrix $\varphi(a_{ij})$. Then the following diagram of R-left module homomorphisms commutes:

$$\begin{array}{ccc} R^{1\times m\cdot k} & \xrightarrow{v\mapsto v\cdot\varphi_*(A)} & R^{1\times n\cdot k} \\ \downarrow\cong & & \downarrow\cong \\ \overline{R^{1\times k}} \otimes_{\mathbb{Z}[\pi]} B & \xrightarrow{\mathrm{id}\otimes(v\mapsto v\cdot A)} & \overline{R^{1\times k}} \otimes_{\mathbb{Z}[\pi]} C. \end{array}$$

Here the vertical isomorphisms are given by the bases from (1).

Proof.

(1) This statement follows easily from the Free Module–Tensor-with-Ring Lemma 312.18 together with the Tensor–Direct Sum Lemma 312.15.

(2) The difficulty lies of course not in the proof, but in the formulation of the lemma, to make sure that all conventions (in particular the convention from 3582) match. We thus leave the verification of this statement to the reader. ∎

In the following we also want to understand the effect of applying the Hom-functor. At this point it is convenient to recall the following little lemma:

Lemma 315.9 (Hom-Dual-Tensor Lemma) Let R and S be rings (later S will be a group ring $\mathbb{Z}[\pi]$), let F be a free S-left module of finite rank and let V be an (R, S)-left

[3582] If $\mathcal{B} = (b_1, \dots, b_l)$ is a finite ordered basis, then we equip the basis $\{\mathfrak{e}_i \otimes b_j\}_{i=1,\dots,k, j=1,\dots,l}$ with the "anti-lexicographic" ordering $\mathfrak{e}_1 \otimes b_1, \dots, \mathfrak{e}_k \otimes b_1, \dots, \mathfrak{e}_1 \otimes b_l, \dots, \mathfrak{e}_k \otimes b_l$.

left module. The map[3583]

$$\begin{array}{rcl} \mathrm{Hom}_{S\text{-left}}(F,S)\otimes_S V & \to & \mathrm{Hom}_{S\text{-left}}(F,V) \\ \varphi\otimes v & \mapsto & (f\mapsto \varphi(f)\cdot v) \end{array}$$

is a natural isomorphism of R-left modules.

Now we can formulate the lemma we were really aiming for:

Lemma 329.9. (Hom–Matrix Lemma) Let π be a group. Furthermore let R be a ring and let $\alpha\colon \pi\to \mathrm{GL}(k,R)$ be a representation. As in Convention 320.9 we use φ to view $R^{1\times k}$ as an $(R,\mathbb{Z}[\pi])$-left left module.

(1) Let B be a free $\mathbb{Z}[\pi]$-left module with finite ordered basis $\mathcal{B}=(b_1,\dots,b_m)$. Let $\mathcal{B}^*=(b_1^*,\dots,b_m^*)$ be the ordered dual basis of the $\mathbb{Z}[\pi]$-*right* module $\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(B,\mathbb{Z}[\pi])$ from the Free Module–Hom-to-Ring Lemma 315.8 (1). Note that it follows from Exercise 309.3 (a) that $b_1^*,\dots,b_m^*$ also form a basis of the $\mathbb{Z}[\pi]$-left module $\overline{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(B,\mathbb{Z}[\pi])}$. Next we use the natural isomorphism from the Hom-Dual-Tensor Lemma 315.9 and the Tensor Commutative-Associative Lemma 312.11 (2) to make the identification

$$\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(B,R^{1\times k}) \xrightarrow{\cong} \overline{R^{1\times k}}\otimes_{\mathbb{Z}[\pi]}\overline{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(B,\mathbb{Z}[\pi])}$$

of R-left modules. Thus we see, as in the Tensor–Matrix Lemma 329.8, that the ordered basis $\mathcal{B}(b_1,\dots,b_m)$ of B together with the standard basis of $R^{1\times k}$ give rise to an ordered basis for $\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(B,R^{1\times k})$ with $m\cdot k$ elements.

(2) Let B and C be two $\mathbb{Z}[\pi]$-left modules with ordered bases $\mathcal{B}=(b_1,\dots,b_m)$ for B and $\mathcal{C}=(c_1,\dots,c_n)$ for C. Furthermore let $\varphi\colon B\to C$ be a $\mathbb{Z}[\pi]$-left module homomorphism and let $A=(a_{ij})$ be the $(m\times n)$-matrix that represents φ with respect to the given basis. As in Definition 329.7 we denote by $\varphi_*(\overline{A}^{\mathrm{T}})\in \mathrm{M}(n\times k\times m\times k,R)$ the matrix that is given by replacing each entry $\overline{a}_{ji}$ of $\overline{A}^{\mathrm{T}}$ by the $(k\times k)$-matrix $\varphi(\overline{a}_{ji})$. Then the following diagram of R-left module homomorphisms commutes:

$$\begin{array}{ccc} R^{1\times n\cdot k} & \xrightarrow{v\mapsto v\cdot\varphi_*(\overline{A}^{\mathrm{T}})} & R^{1\times m\cdot k} \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C,R^{1\times k}) & \xrightarrow{\gamma\mapsto\gamma\circ\varphi} & \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(B,R^{1\times k}). \end{array}$$

Here the vertical isomorphisms are given by the ordered bases from (1).

Sketch of proof.

(1) The reader will have noticed that the statement already contains the justification.

(2) This statement follows easily from (1) together with the Free Module–Hom-to-Ring Lemma 315.8 (2), Exercise 309.3 and the Tensor–Matrix Lemma 329.8 (2). ∎

329.4. **First examples of twisted cellular complexes.** In this section we will give several calculations of cellular twisted chain complexes and the corresponding twisted homology[3584] groups. We formulate the examples as lemmas, since we will make use of many

[3583]Here we view S as an (S,S)-bimodule in the usual way. This allows us to view $\mathrm{Hom}_{S\text{-left}}(F,S)$ as an S-right module.

[3584]In all cases one could also use the Hom–Matrix Lemma 329.9 to determine the twisted cohomology groups, but since conciseness is our top priority, we will refrain from doing so.

of the calculations later on, once we have introduced Reidemeister torsion. We start out with the simplest somewhat non-trivial case:

Lemma 329.10. (Circle–Twisted Cellular Chain Complex Lemma) In the following we consider S^1 together with the base point $x_0 = (1,0)$. We make the usual identifications $\pi := \pi_1(S^1, x_0) =_{\mathrm{i}} \langle x \rangle$ and $\mathbb{Z}[\pi] =_{\mathrm{i}} \mathbb{Z}[x^{\pm 1}]$.

(1) For the standard CW-structure on S^1 the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{S^1}_{x_0})$ of free $\mathbb{Z}[x^{\pm 1}]$-modules is isomorphic to the chain complex[3585 3586]

$$0 \to \mathbb{Z}[x^{\pm 1}] \xrightarrow{(x-1)} \mathbb{Z}[x^{\pm 1}] \to 0.$$

(2) Let R be a ring and let $\varphi\colon \langle x \rangle \to \mathrm{GL}(n, R)$ be a representation.

(a) The twisted cellular chain complex ${}_{\varphi}R^{1\times n} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{S^1}_{x_0})$ is isomorphic to the chain complex

$$0 \to R^{1\times n} \xrightarrow{\varphi(x)-\mathrm{id}_n} R^{1\times n} \to 0.$$

(b) If R is a field and if $\varphi\colon \pi_1(S^1, x_0) \to \mathrm{GL}(1, R)$ is a non-trivial one-dimensional representation, then the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(S^1; {}_{\varphi}R)$ are all zero.

Proof.

(1) This statement was proved, implicitly, on page 6259.

(2) (a) The statement about the twisted cellular chain complex follows from (1) together with the Tensor–Matrix Lemma 329.8.

(b) Now we assume that R is a field and we assume that we are given a non-trivial one-dimensional representation $\pi_1(T, t_0) \to \mathrm{GL}(1, R)$. This implies that $\varphi(x) - 1$ is invertible in R. Clearly this implies that the twisted chain complex is acyclic. By the Singular-Cellular Twisted Isomorphism Theorem 328.1 together with the Replace Universal Covering Proposition 329.1 this implies that the twisted singular homology modules are all zero. ∎

We continue with a slightly more interesting calculation:

Lemma 329.11. (Torus–Twisted Cellular Chain Complex Lemma) In the following we consider the 2-dimensional torus $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1]\times[0,1])/\sim$ together with the base point $t_0 = [(0,0)]$. We make the usual identifications $\pi := \pi_1(S^1 \times S^1, t_0) =_{\mathrm{i}} \langle x, y \,|\, [x,y] \rangle$ and $\mathbb{Z}[\pi] =_{\mathrm{i}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

(1) For the standard CW-structure on $S^1 \times S^1$ the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{T}_{t_0})$ of free $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-modules is isomorphic to the chain complex[3587 3588]

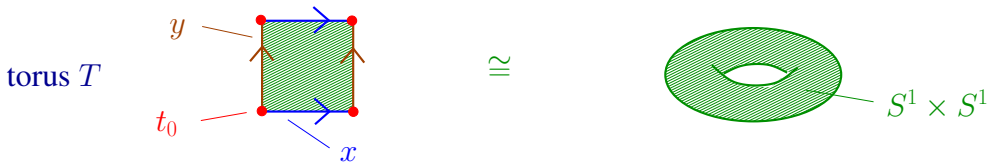
$$0 \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \xrightarrow{\begin{pmatrix} 1-y & x-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{1\times 2} \xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \to 0.$$

(2) Let R be a ring and let $\varphi\colon \langle x, y \,|\, [x,y] \rangle \to \mathrm{GL}(n, R)$ be a representation.

(a) The twisted cellular chain complex ${}_{\varphi}R^{1\times n} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{T})$ is isomorphic to the chain complex

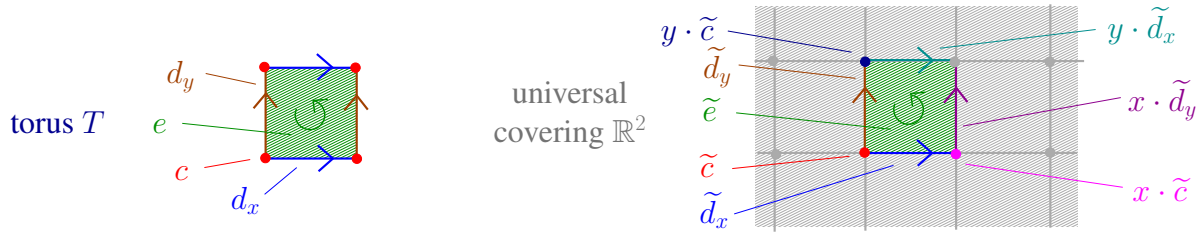
$$0 \to R^{1\times n} \xrightarrow{\begin{pmatrix} \mathrm{id}_n - \varphi(y) & \varphi(x) - \mathrm{id}_n \end{pmatrix}} R^{1\times 2n} \xrightarrow{\begin{pmatrix} \varphi(x) - \mathrm{id}_n \\ \varphi(y) - \mathrm{id}_n \end{pmatrix}} R^{1\times n} \to 0.$$

(b) If R is a field and if $\varphi\colon \pi_1(T, t_0) \to \mathrm{GL}(1, R)$ is a non-trivial one-dimensional representation, then the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(T; {}_{\varphi}R)$ are all zero.



Proof.

(1) We equip T with the usual CW-structure that we introduced on page 6252 which has one 0-cell c, two 1-cells d_x and d_y and one 2-cell e. We equip each cell with the (fairly) obvious characteristic map. Next recall that we have a universal covering $p\colon \mathbb{R}^2 \to T$. The resulting CW-structure on $\mathbb{R}^2$, and lifts of c, d_x, d_y and e are shown in the following figure.



Using the Cellular Boundary–Degree Proposition 329.4 and Observation 329.5 we see, as on page 6274, that $\partial_2(\widetilde{e}) = (1-y)\cdot\widetilde{d}_x + (x-1)\cdot\widetilde{d}_y$ and as on page 6276 we see that $\partial_1(\widetilde{d}_x) = (x-1)\cdot\widetilde{c}$ and $\partial_1(\widetilde{d}_y) = (y-1)\cdot\widetilde{c}$. Thus we see that the cellular chain complex $\mathrm{C}_2^{\mathrm{CW}}(\widetilde{T})$ is of the form

$$0 \;\to\; \mathbb{Z}[x^{\pm1}, y^{\pm1}] \xrightarrow{\begin{pmatrix}1-y & x-1\end{pmatrix}} \mathbb{Z}[x^{\pm1}, y^{\pm1}]^2 \xrightarrow{\begin{pmatrix}x-1\\ y-1\end{pmatrix}} \mathbb{Z}[x^{\pm1}, y^{\pm1}] \;\to\; 0.$$

The desired statement follows from this calculation together with the Replace Universal Covering Proposition 329.1.

(2) (a) The statement about the twisted cellular chain complex follows from (1) together with the Tensor–Matrix Lemma 329.8.

(b) Now we assume that R is a field and we assume that we are given a one-dimensional representation $\pi_1(T, t_0) \to \mathrm{GL}(1, R)$. Without loss of generality we assume that $\varphi(x) \neq 1$. This implies that $\varphi(x) - 1$ is invertible in R. But now it follows from elementary linear algebra that the twisted chain complex is acyclic. By the Singular-Cellular Twisted Isomorphism Theorem 328.1 together with the Replace Universal Covering Proposition 329.1 this implies that the twisted singular homology modules are all zero. ■

Recall that according to the discussion on page 343 the torus is just the surface of genus 1. Recall that more generally on page 6254 we defined the surface of genus g as the quotient of the $4g$-gon E_{4g} by a suitable identification of pairs of sides. The following lemma can thus be viewed as a generalization of the Torus–Twisted Cellular Chain Complex Lemma 329.11:

[3587]Note that it follows from the subtle sign in the boundary map ∂_2 that this is indeed a chain complex, as it should be.

[3588]Note that this chain complex appeared already in the Group-H_*-of-$\mathbb{Z}^2$-Lemma 317.8.

Lemma 329.12. (Surface–Twisted Cellular Chain Complex Lemma) Let $g \in \mathbb{N}$. In the following we consider the surface $\Sigma_g = E_{4g}/\sim$ where we use the Surface-π_1-Proposition 101.1 to make the identification $\pi := \pi_1(\Sigma_g) = \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$.

(1) For the standard CW-structure on $\Sigma_g = E_{4g}/\sim$ the cellular chain complex on $\widetilde{\Sigma}_g$ is isomorphic to a chain complex of the form

$$0 \to \mathbb{Z}[\pi] \xrightarrow{\begin{pmatrix}(a_1 \cdot y_1 \cdot a_1^{-1} - 1)\cdot b_1 & (1 - a_2 \cdot x_1 \cdot a_2^{-1})\cdot b_2 & \ldots\end{pmatrix}} \mathbb{Z}[\pi]^{1\times 2g} \xrightarrow{\begin{pmatrix}(x_1-1)\cdot c_1\\(y_1-1)\cdot c_2\\ \vdots \\ (y_g-1)\cdot c_{2g}\end{pmatrix}} \mathbb{Z}[\pi] \to 0$$

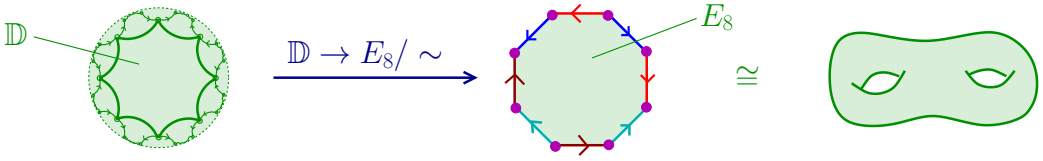
where $a_1, \ldots, a_{2g}, b_1, \ldots, b_{2g}, c_1, \ldots, c_{2g} \in \pi$.

(2) If $\mathbb{F}$ is a field and if $\varphi\colon \pi_1(\Sigma_g, x_0) \to \mathrm{GL}(1, \mathbb{F})$ is a non-trivial 1-dimensional representation, then

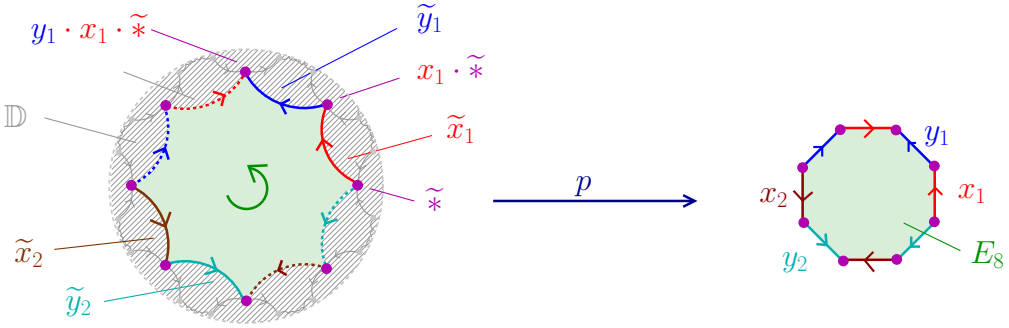
$$\mathrm{H}_i^{\mathrm{tw}}(\Sigma_g;{}_\varphi\mathbb{F}) \cong \begin{cases} 0, & \text{if } i \neq 1, \\ \mathbb{C}^{2g-2}, & \text{if } i = 1. \end{cases}$$

Proof. In the following we deal with the case $g = 2$. The other cases are treated in a very similar fashion.

(1) As we discussed on page 2626, we can equip $X = E_8/\sim$ with a CW-structure with a single 0-cell, four 1-cells and one 2-cell. On page 6254 we discussed the universal covering $\mathbb{D} = B^2$ of $X = E_8/\sim$. We obtain from the CW-Complex Covering Proposition 327.6 the CW-structure on $\mathbb{D} = \widetilde{X}$ that is shown in the figure below:



Next we pick lifts of cells. There is not really a canonical way of picking lifts, so we need to make some slightly random choices. In the following we pick the lifts that are shown in the figure below:



We leave it to the reader to figure out, using Observation 329.5, what the group elements $a_1, \ldots, a_{2g}, b_1, \ldots, b_{2g}, c_1, \ldots, c_{2g} \in \pi$ are.[3589]

[3589] In principle you have the reality check that the composition of the two boundary matrices needs to be zero. But it can be a little tricky to verify this, since we work over the group ring of the non-abelian group π.

(2) The statement about the homology groups follows easily from (1) together with the Singular-Cellular Twisted Isomorphism Theorem 328.1. ■

We move straight to our next example:

Lemma 329.13. ($\mathbb{R}\mathrm{P}^n$–Twisted Cellular Chain Complex Lemma) Let $n \in \mathbb{N}_{\geq 2}$. We consider the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$. By the Real Projective Space–π_1-Proposition 93.3 we can make an identification $\pi_1(\mathbb{R}\mathrm{P}^n) =_{\mathrm{i}} \langle x \,|\, x^2 \rangle$.

(1) For the standard CW-structure on $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$ the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{\mathbb{R}\mathrm{P}^n})$ is isomorphic to the chain complex

$$0 \to \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{1+(-1)^n \cdot x} \ldots \to \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{1+x} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \xrightarrow{1-x} \mathbb{Z}[\langle x \,|\, x^2 \rangle] \to 0.$$

(2) Let $\mathbb{F}$ be a field and let $\varphi\colon \langle x \,|\, x^2 \rangle \to \mathrm{GL}(1, \mathbb{F})$ be a non-trivial 1-dimensional representation. Then

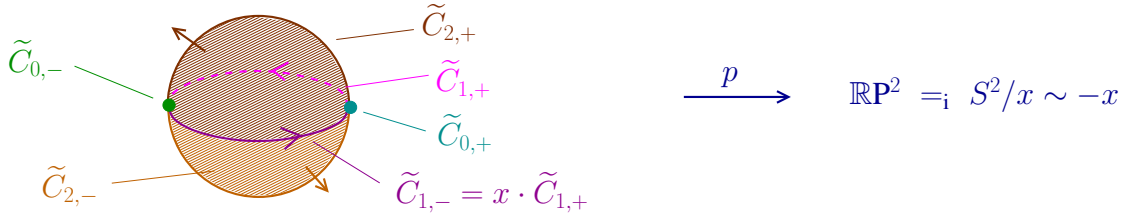
$$\mathrm{H}_k^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n;{}_\varphi\mathbb{F}) \cong \begin{cases} \mathbb{F}, & \text{if } k = n \text{ and } n \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $n \in \mathbb{N}_{\geq 2}$. We consider the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/P \sim -P$.

(1) Since $n \geq 2$ we know that the natural projection $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ is actually a universal covering. We equip $\mathbb{R}\mathrm{P}^n$ with the CW-structure that we introduced on page 2628, which has one cell in each of the dimensions $0, \ldots, n$. The corresponding CW-structure on S^n has precisely two cells in each of the dimensions $0, \ldots, n$. In fact the open cells are precisely given by

$$\begin{aligned} \widetilde{C}_{k,+} &:= \{(x_0, \ldots, x_k, 0, \ldots, 0) \in S^n \,|\, x_k > 0\} \\ \text{and} \quad \widetilde{C}_{k,-} &:= \{(x_0, \ldots, x_k, 0, \ldots, 0) \in S^n \,|\, x_k < 0\} \end{aligned}$$

for $k = 0, \ldots, n$.



With an appropriate choice of characteristic maps we see that for each $k \in \{1, \ldots, n\}$ we have

$$\partial \widetilde{C}_{k,+} \underset{\uparrow}{=} \widetilde{C}_{k-1,+} + (-1)^k \cdot \widetilde{C}_{k-1,-} \underset{\uparrow}{=} (1 + (-1)^k \cdot x) \cdot \widetilde{C}_{k-1,+}.$$

follows from the Cellular Boundary Degree Proposition 329.4 — follows easily from Observation 329.5

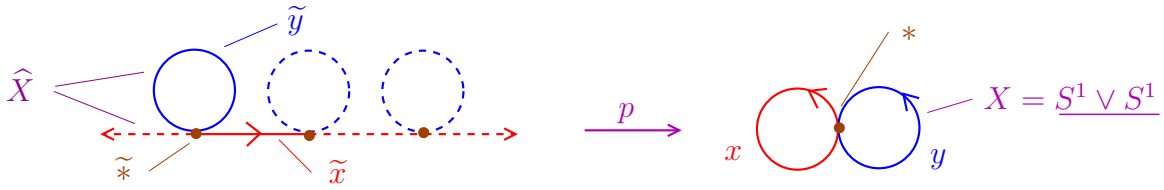
This gives us precisely the promised result.

(2) The statement about the twisted homology groups follows again easily from (1) together with the Singular-Cellular Twisted Isomorphism Theorem 328.1. ■

We conclude this section with an example, which for once is not dressed up as a lemma:

Example. As a proof of concept we will now show how one can use the Replace Universal Covering Proposition 329.1 to work with a "smaller" covering space. In the following we consider the CW-complex $X = S^1 \vee S^1$ with wedge point $*$. We make the usual identification

$\pi_1(X,*) =_{\mathrm{i}} \langle x,y\rangle$ from the Wedge-of-Circles-π_1-Lemma 99.11. Let $\xi \in \mathbb{C}\setminus\{0,1\}$. We consider the representation $\varphi\colon \pi_1(X,*) =_{\mathrm{i}} \langle x,y\rangle \to \langle x\rangle \xrightarrow{x\mapsto(\xi)} \mathrm{GL}(1,\mathbb{C})$. We denote by $\widehat{X}$ the covering of $X = S^1\vee S^1$ corresponding to the kernel of the natural epimorphism $\pi_1(X,*) =_{\mathrm{i}} \langle x,y\rangle \to \langle x\rangle$.



We pick the lifts of the cells $*, x, y$ that we show in the figure above. Following the discussion on page 6276 one can now easily verify that the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widehat{X})$ is isomorphic to the chain complex

$$0 \to \mathbb{Z}[x^{\pm1}]^{1\times 2} \xrightarrow{(1-x\ \ 0)} \mathbb{Z}[x^{\pm1}] \to 0.$$

We now see that we have the following isomorphisms:

$$\mathrm{H}_k^{\mathrm{tw}}(S^1\vee S^1;{}_\varphi\mathbb{C}) \cong \mathrm{H}_k(\overline{{}_\varphi\mathbb{C}}\otimes_{\mathbb{Z}[x^{\pm1}]}\mathrm{C}_k^{\mathrm{CW}}(\widehat{X}))$$

$\uparrow$ by the Replace Universal Covering Proposition 329.1

$$\cong \mathrm{H}_k(0\to\mathbb{C}^{1\times 2}\xrightarrow{(1-\xi\ \ 0)}\mathbb{C}\to 0) \cong \begin{cases}\mathbb{C}, & \text{if } k=1,\\ 0, & \text{if } k=0.\end{cases}$$

$\uparrow$ follows from the above together with the Tensor–Matrix Lemma 329.8; $\uparrow$ since $\xi\neq 1$

□

329.5. **The cellular chain complex of the universal coverings of lens spaces.** In this section we want to calculate the twisted cellular chain complexes of lens spaces. To formulate our result it is convenient to recall Definition 91.4 and Definition 319.4:

Definition 329.14. Let $p\in\mathbb{N}$ and let $q\in\mathbb{Z}$ with $\gcd(p,q)=1$.

(1) We write $\xi := e^{2\pi\mathrm{i}i/p}$. We consider the action

$$\begin{array}{rcl}\mathbb{Z}/p\mathbb{Z}\times S^3 & \to & S^3\\ (k+p\mathbb{Z},(z_1,z_2)) & \mapsto & (\xi^k\cdot z_1,\xi^{q\cdot k}\cdot z_2)\end{array}$$

We consider the corresponding lens space $L(p,q) := S^3/(\mathbb{Z}/p\mathbb{Z})$.

(2) We equip $L(p,q)$ with the base point $* := [(1,0)]$.

(3) It follows easily from the Action–π_1-Theorem 319.5 that the map

$$\Theta_{x_0}\colon \langle x\,|\,x^p\rangle \to \pi_1(L(p,q),*),\quad x\mapsto \left[\begin{array}{rcl}[0,1] & \to & L(p,q)\\ t & \mapsto & [(e^{2\pi\mathrm{i}t/p},0)]\end{array}\right]$$

is an isomorphism. In the following we will use this isomorphism to make the identification $\pi_1(L(p,q),*) =_{\mathrm{i}} \langle x\,|\,x^p\rangle$.

Example. Note that $L(2,1)$ is by definition the same as $S^3/x\sim -x$, which is of course homeomorphic to the real projective space $\mathbb{R}\mathrm{P}^3$. □

We can now formulate the main result of this section:

Proposition 329.15. (Lens Space-CW-Proposition) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. We consider the lens space $L(p,q) \coloneqq S^3/(\mathbb{Z}/p\mathbb{Z})$ and we use the isomorphism from Definition 329.14 to make the identification $\pi_1(L(p,q),*) =_{\mathrm{i}} \langle x \,|\, x^p \rangle$.

(a) The quotient space $L(p,q) = S^3/(\mathbb{Z}/p\mathbb{Z})$ admits a CW-structure which has precisely one cell of each dimension $0,1,2,3$, and no other cells.

(b) We equip $L(p,q)$ with the CW-structure from (a). The corresponding cellular chain complex of the universal covering $L(p,q) = S^3$ is isomorphic to the chain complex[3590]

$$0 \to \mathbb{Z}[\langle x \,|\, x^p \rangle] \xrightarrow{\cdot(1-x^r)} \mathbb{Z}[\langle x \,|\, x^p \rangle] \xrightarrow{\cdot(1+x+\cdots+x^{p-1})} \mathbb{Z}[\langle x \,|\, x^p \rangle] \xrightarrow{\cdot(x-1)} \mathbb{Z}[\langle x \,|\, x^p \rangle] \to 0$$

where $r \cdot q \equiv 1 \mod p$.

Proof. We outsource the proof of the Lens Space-CW-Proposition 329.15 to the next Section 329.6. ■

Example. As we just mentioned, the lens space $L(2,1)$ is just another avatar of $\mathbb{R}\mathrm{P}^3$. This matches the observation that for $L(2,1)$ the conclusion of the Lens Space-CW-Proposition 329.15 agrees with the conclusion of the $\mathbb{R}\mathrm{P}^n$–Twisted Cellular Chain Complex Lemma 329.13 for $n = 3$. □

The Lens Space-CW-Proposition 329.15 allows us to give a new, more systematic proof of the following proposition:

Proposition 323.14. (Lens Space–Twisted Invariants Proposition) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. Let $\rho\colon \pi_1(L(p,q)) \to \mathrm{GL}(1,\mathbb{C})$ be a non-trivial representation. We consider the corresponding $\mathbb{Z}[\pi_1(L(p,q))]$-module ${}_\rho\mathbb{C}$ from Notation 320.8.

(1) For every $k \in \mathbb{N}_0$ we have $\mathrm{H}^{\mathrm{tw}}_k(L(p,q); {}_\rho\mathbb{C}) = 0$.

(2) For every $k \in \mathbb{N}_0$ we have $\mathrm{H}^k_{\mathrm{tw}}(L(p,q); {}_\rho\mathbb{C}) = 0$.

Proof. This proposition follows easily from the Lens Space-CW-Proposition 329.15 together with the Tensor–Matrix Lemma 329.8 and the Hom–Matrix Lemma 329.9. We leave it to the reader to fill in the few missing details. ■

329.6. **Proof of the Lens Space-CW-Proposition 329.15.** The proof of the Lens Space-CW-Proposition 329.15 naturally breaks into two parts:

- First we will construct an explicit CW-structure on $L(p,q)$. More precisely, we will construct a CW-structure on S^3 which descends to a CW-structure on $L(p,q)$.
- Afterwards we will calculate the corresponding cellular chain complex of the universal covering of the lens space $L(p,q)$.

[3590] Note that in the Lens Space-CW-Proposition 329.15 we do finally see the parameter q! The only question is, how do extract it from the chain complex? In other words, what invariants of chain complexes will detect the q?

Construction of the CW-structure. [3591] Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. In the following we will construct a CW-structure $\widetilde{X}^0 \subseteq \widetilde{X}^1 \subseteq \widetilde{X}^2 \subseteq \widetilde{X}^3$ on S^3 which descends to a CW-structure on $L(p,q)$.[3592] We do so as follows:

(0) We set $\widetilde{X}^0 := \{(z,0) \,|\, z \in S^1 \text{ with } z^p = 1\}$.

(1) We set $\widetilde{X}^1 := S^1 \times \{0\} \subseteq S^3$.

(2) For $j = 0, \dots, p-1$ we consider the following subspace of S^3:[3593]

$$\widetilde{X}^2_j := \{ \underbrace{(\sin(\gamma) \cdot v, \cos(\gamma) \cdot e^{2\pi \mathrm{i} j/p})}_{\text{point in } \mathbb{R}^2 \times \mathbb{C} = \mathbb{R}^2 \times \mathbb{R}^2 \text{ of length } 1} \,|\, v \in S^1 \text{ and } \gamma \in [0, \tfrac{\pi}{2}]\}.$$

Furthermore we set $\widetilde{X}^2 := \widetilde{X}^2_0 \cup \dots \cup \widetilde{X}^2_{p-1}$.

(3) We set $\widetilde{X}^3 := S^3$.

Our goal is to show that the filtration $\widetilde{X}^0 \subseteq \widetilde{X}^1 \subseteq \widetilde{X}^2 \subseteq \widetilde{X}^3 = S^3$ is a CW-structure on S^3 that descends to a CW-structure for $L(p,q)$.[3594]

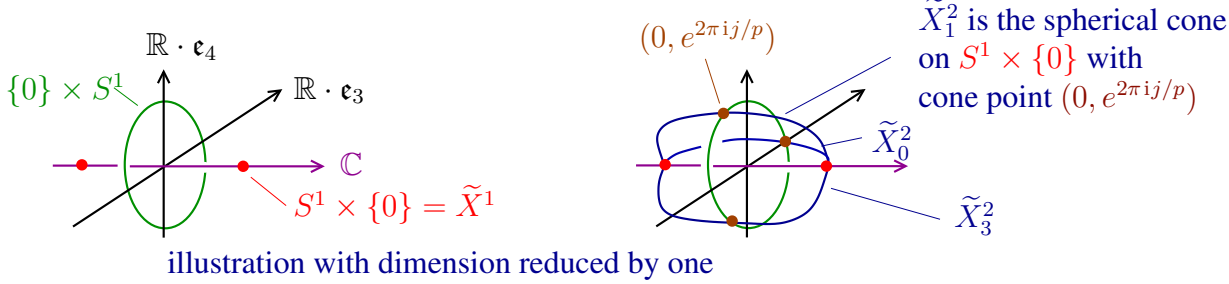


illustration with dimension reduced by one

Claim 1. For $j = 0, \dots, p-1$ let $\widetilde{\alpha}_j \colon \overline{B}^1 \to \widetilde{X}^1 = S^1 \times \{0\}$ be the characteristic map that is given by $\widetilde{\alpha}_j(t) = (e^{\pi i (t+1)/p}, 0)$. These characteristic maps form a complete set of characteristic maps for the 1-skeleton $\widetilde{X}^1$.

Proof. This claim is essentially trivial. We just formulated the claim to be consistent in our discussion of the various skeleta. ⊞

Claim 2. We consider $S^1 \times [0, \frac{\pi}{2}]$ together with the equivalence relation "$\sim$" that is given by $(v_0, 0) \sim (v_1, 0)$ for all $v_0, v_1 \in S^1$. We consider the map

$$\begin{aligned} F \colon (S^1 \times [0, \tfrac{\pi}{2}])/\sim &\to \overline{B}^2 \\ [(v, \varphi)] &\mapsto \sin(\varphi) \cdot v \end{aligned}$$

and for $j = 0, \dots, p-1$ we consider the map

$$\begin{aligned} \Phi \colon (S^1 \times [0, \tfrac{\pi}{2}])/\sim &\to \widetilde{X}^2 \\ [(v, \varphi)] &\mapsto (\sin(\varphi) \cdot v, \cos(\varphi) \cdot e^{2\pi \mathrm{i} j/p}). \end{aligned}$$

[3591] The construction of the CW-structure is closely related to the proof of the Generalized Lens Space-CW-Proposition 306.6.

[3592] In the case $p = 2$, $q = 1$ we have $L(p,q) =_{\mathrm{i}} \mathbb{R}\mathrm{P}^3$. One can easily verify that in this setting the CW-structure we are about to construct is equivalent to the usual CW-structure from page 2628, which we also considered in the proof of the $\mathbb{R}\mathrm{P}^n$–Twisted Cellular Chain Complex Lemma 329.13.

[3593] Visually speaking $\widetilde{X}^2_j$ is the "spherical cone" of $\widetilde{X}^1 = S^1 \times \{0\}$ with cone point $(0, e^{2\pi \mathrm{i} j/p})$.

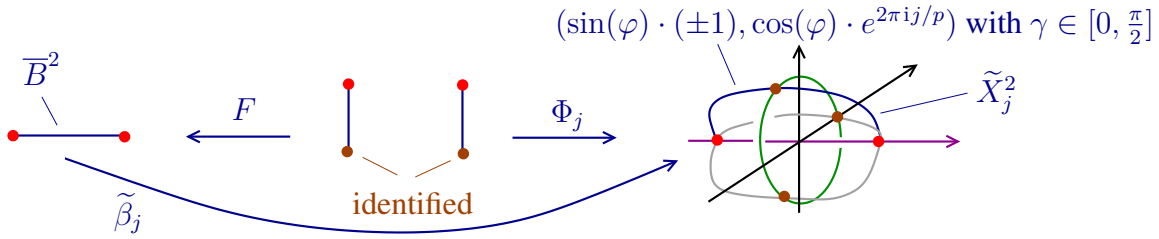
[3594] Note that curiously enough this CW-structure on S^3 does *not* depend on the choice of q.

(1) The map F is a homeomorphism.
(2) (a) Each attaching map $\varphi_j := (\Phi \circ F^{-1})|_{S^1} \colon S^1 \to \widetilde{X}^1$ is an orientation-preserving diffeomorphism.
(b) For each $j \in \{0, \ldots, p-1\}$ the identity map on the 1-skeleton $\widetilde{X}^1$ together with the characteristic map $\widetilde{\beta}_j := \Phi_j \circ F^{-1} \colon \overline{B}^2 \to \widetilde{X}^2$ defines a homeomorphism

$$\widetilde{X}^1 \cup_{\varphi_j} \overline{B}^2 \to \widetilde{X}^2_j.$$

Furthermore, the identity map on $\widetilde{X}^1$ together with all the characteristic maps $\widetilde{\beta}_j \colon \overline{B}^2 \to \widetilde{X}^2$, $j = 0, \ldots, p-1$ defines a homeomorphism

$$\widetilde{X}^1 \cup_{\varphi_0} \overline{B}^2 \cdots \cup_{\varphi_{p-1}} \overline{B}^2 \;\to\; \widetilde{X}^2.$$



Proof.

(1) It is elementary to verify that the map F is a bijection. Using the Compact-Hausdorff Proposition 4.23 one can easily verify that F is a homeomorphism.
(2) (a) This statement follows easily from the observation that $F^{-1}(S^1)$ are precisely the points that are represented by points of the form $(v, \frac{\pi}{2}) \in S^1 \times [0, \frac{\pi}{2}]$.
(b) It follows easily from the definitions that the given map $\widetilde{\beta}_j \colon \widetilde{X}^1 \cup_{\varphi_j} \overline{B}^2 \to \widetilde{X}^2_j$ is a bijection and it follows from the Topological-Pushout Proposition 8.12 (1) that the map $\widetilde{\beta}_j$ is continuous. By the Pushout-Properties Proposition 8.14 (1) we know that the left-hand side is compact and clearly the right hand side is Hausdorff. It thus follows from the Compact-Hausdorff Proposition 4.23 that the map $\widetilde{\beta}_j$ is indeed a homeomorphism. The same way one can show that second map, involving all 2-cells, is a homeomorphism. ⊞

Claim 3. We consider $S^1 \times [0, \frac{\pi}{2}] \times [0,1]$ with the equivalence relation "$\sim$" that is given by $(v_0, 0, t) \sim (v_1, 0, t)$ for $v_0, v_1 \in S^1$. We consider the map

$$\begin{array}{rcccc} G \colon (S^1 \times [0, \frac{\pi}{2}] \times [0,1])/\sim & \to & \overline{B}^2 \times [0,1] & \xrightarrow{\cong} & \overline{B}^3 \\ [(v, \varphi, t)] & \mapsto & (\sin(\varphi)\cdot v, t) & & \end{array}$$

where the second map is the homeomorphism given by the Convex-to-Ball Proposition 4.28. For $j = 0, \ldots, p-1$ we consider the map

$$\begin{array}{rcl} \Psi_j \colon (S^1 \times [0, \frac{\pi}{2}] \times [0,1])/\sim & \to & \widetilde{X}^3 \\ [(v, \varphi, t)] & \mapsto & (\sin(\varphi)\cdot v, \underbrace{\cos(\varphi)\cdot e^{2\pi \mathrm{i}(j+t)/p}}_{\in \mathbb{C}}). \end{array}$$

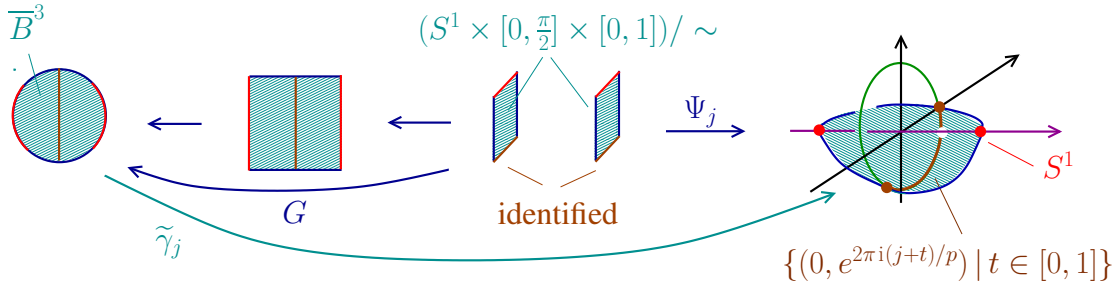
(1) The map G is a homeomorphism.
(2) (a) For $j \in \{0, \ldots, p-1\}$ the attaching map $\psi_j := (\Psi \circ G^{-1})|_{S^2}$ to $\widetilde{X}^2_j \cup \widetilde{X}^2_{j+1 \bmod p}$ is a homeomorphism.

(b) For each $j \in \{0, \dots, p-1\}$ the identity map on $\widetilde{X}^2$ together with the characteristic map $\widetilde{\gamma}_j := \Psi_j \circ G^{-1} \colon \overline{B}^3 \to \widetilde{X}^3$ defines an embedding

$$\widetilde{X}^2 \cup_{\psi_j} \overline{B}^3 \;\to\; S^3.$$

Furthermore, the identity map on $\widetilde{X}^2$ together with all the characteristic maps $\widetilde{\gamma}_0, \dots, \widetilde{\gamma}_{p-1}$ defines a homeomorphism

$$\widetilde{X}^2 \cup_{\psi_0} \overline{B}^3 \cdots \cup_{\psi_{p-1}} \overline{B}^3 \;\to\; \underbrace{\widetilde{X}^3}_{=S^3}.$$



Proof.

(1) As in Claim 2 (1) we see that the first map in the definition of G is a homeomorphism. Since the second map is also a homeomorphism we have shown that G is indeed a homeomorphism.

(2) (a) First note that $G^{-1}(S^2)$ are the points that are represented by points of the form $(v, \frac{\pi}{2}, t)$ or $(v, \varphi, 0)$ or $(v, \varphi, 1)$. The image of the points $(v, \varphi, 0)$ equals $\widetilde{X}^2_j$ and the image of the points $(v, \varphi, 0)$ equals $\widetilde{X}^2_{j+1 \bmod p}$. It is now elementary to verify the promised statement.

(b) This statement follows, similar to Claim 2 (2b), easily from the definitions and the Compact-Hausdorff Proposition 4.23. We leave all details to the reader. ⊞

Claim 4. The above CW-structure on S^3 descends to a CW-structure on $L(p,q)$ with precisely one cell in each dimension $0, 1, 2, 3$.

Proof. Let $\pi \colon S^3 \to L(p,q)$ be the natural projection. We write $X^i := \pi(\widetilde{X}^i)$. It is elementary to see that the three maps $\alpha := \pi \circ \widetilde{\alpha}_0 \colon \overline{B}^1 \to X^1$, $\beta := \pi \circ \widetilde{\beta}_0 \colon \overline{B}^2 \to X^2$ and $\gamma := \pi \circ \widetilde{\gamma}_0 \colon \overline{B}^3 \to X^3$ define characteristic maps in such a way that the filtration $X^0 \subseteq X^1 \subseteq X^2 \subseteq X^3 = L(p,q)$ forms a CW-complex. We leave the verification of the details to the reader. ■

Calculation of the chain complex. We continue with the notation from the construction of the CW-structure on $L(p,q)$. Our goal now is to determine the cellular chain complex of the universal covering of $L(p,q)$, i.e. of S^3. First of all we need to pick lifts of the cells:

(0) We consider the 0-cell $\widetilde{*} := (1, 0)$.

(1) We consider the 1-cell $\widetilde{a} := \widetilde{\alpha}_0(\overline{B}^1)$.

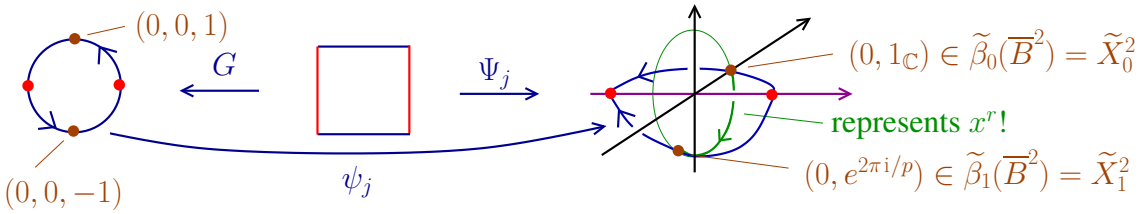
(2) We consider the 2-cell $\widetilde{b} := \widetilde{\beta}_0(\overline{B}^2)$.

(3) We consider the 3-cell $\widetilde{c} := \widetilde{\gamma}_0(\overline{B}^3)$.

Now we turn to the calculation of the boundary maps:

(1) It follows immediately from the discussion on page 6276 that $\partial_1(\widetilde{a}) = x\cdot\widetilde{*} - \widetilde{*} = (x-1)\cdot\widetilde{*}$.
(2) It follows easily from Claim 2 (2a) that $\partial_2(\widetilde{b}) = (1 + x + \cdots + x^{p-1}) \cdot \widetilde{a}$.
(3) We turn to the most interesting calculation, namely the calculation of $\partial_3(\widetilde{c})$. We make a few observations:
   (a) First note that it follows easily from Claim 3 (2a) that $\partial_3(\widetilde{c})$ is a linear combination of the 2-cells $\widetilde{X}^2_0 = \widetilde{\beta}_0(\overline{B}^2)$ and $\widetilde{X}^2_1 = \widetilde{\beta}_1(\overline{B}^2)$.
   (b)
   - We consider the point $y_0 = (0, 1_{\mathbb{C}}) = \widetilde{\beta}_0(0)$ in the interior of $\widetilde{X}^2_0$. The only point in the preimage of y_0 is $(0, 0, 1)$. Note that the map ψ is an orientation-preserving local diffeomorphism at $(0, 0, 1)$.
   - We consider the point $y_1 = (0, e^{2\pi \mathrm{i}/p}) = \widetilde{\beta}_1(0)$ in the interior of $\widetilde{X}^2_1$. The preimage of y_1 is $(0, 0, -1)$. Note that the map ψ is an orientation-reversing local diffeomorphism at $(0, -, -1)$.

   (c) It follows from (b) and the Degree-via-Local Orientations Proposition 152.11 that $\partial_3(\widetilde{c}) = \widetilde{\beta}_0(\overline{B}^2) - \widetilde{\beta}_1(\overline{B}^2)$.
   (d) The points $(0, 1_{\mathbb{C}}) = \widetilde{\beta}_0(0)$ and $(0, e^{2\pi \mathrm{i}/p}) = \widetilde{\beta}_1(0)$ are connected by the path $f\colon [0, 1] \to S^3$ that is given by $t \mapsto (0, e^{2\pi \mathrm{i}t/p})$ which travels along S^3 in the second coordinate.



Now let $r \in \mathbb{Z}$ with $q \cdot r \equiv 1 \mod p$. We write $\xi = e^{2\pi \mathrm{i}/p}$. Note that with our definition of the group action we have

$$x^r \cdot \widetilde{\beta}_0(0) \;=\; x^r \cdot (0, 1) \;\underset{\text{see Definition 329.14}}{\uparrow}{=}\; (\xi^r \cdot 0, \xi^{r\cdot q} \cdot 1 \;\underset{\text{since } q\cdot r \equiv 1 \mod p}{\uparrow}{=}\; (0, e^{2\pi \mathrm{i}/p}) \;=\; \widetilde{\beta}_1(0).$$

This shows that $\widetilde{\beta}_1(\overline{B}^3) = x^r \cdot \widetilde{\beta}_0(\overline{B}^3)$.[3595] It follows from the Group Action–Deck Transformation-π_1-Lemma 128.10 (which tells us that our various conventions are consistent) that $\partial_3(\widetilde{c}) = \widetilde{b} - x^r \cdot \widetilde{b} = (1 - x^r) \cdot \widetilde{b}$. ∎

329.7. Exercises for Chapter 329.

Exercise 329.1. Let $\varphi\colon \pi_1(S^1) \to \mathrm{GL}(1, \mathbb{Z})$ be the homomorphism that sends a generator to (-1).

(a) Compute $\mathrm{H}^{\mathrm{tw}}_i(S^1; {}_\varphi\mathbb{Z})$ for $i = 0, 1$.
(b) Compute $\mathrm{H}^i_{\mathrm{tw}}(S^1; {}_\varphi\mathbb{Z})$ for $i = 0, 1$.

Exercise 329.2. Give an example of a connected non-empty CW-complex X with fundamental group $\pi := \pi_1(X)$ and a $k \in \mathbb{N}$ such that $\mathrm{H}^k_{\mathrm{tw}}(X; \mathbb{Z}[\pi]) \neq 0$ but such that the k-th

[3595] One could also argue using Observation 329.5, but this time one needs to be careful about the various starting points.

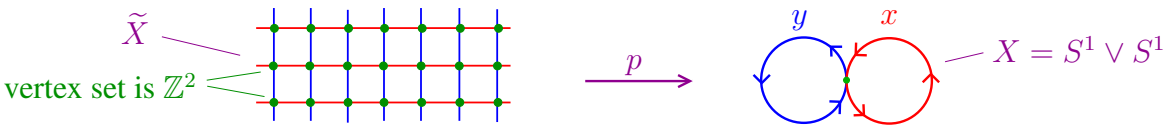
cohomology group of the universal covering of X is zero.
Remark. This shows again that the cohomological analog of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 does not hold.

Exercise 329.3. As on page 5468 we consider the wedge $S^1 \vee S^1$ with the "obvious" epimorphism $\Phi\colon \mathrm{H}_1(S^1 \vee S^1) \to \langle x, y\rangle_{\mathrm{ab}}$.

(a) Find a resolution for $\mathrm{H}_1^{\mathrm{tw}}(S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_\Phi)$.

(b) Determine $\mathrm{ord}_{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]}(\mathrm{H}_1^\Phi(S^1 \vee S^1; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]))$.

Remark. This exercise is closely related to Exercise 329.3. But now we should (??) be more efficient at doing the calculation.



Exercise 329.4. Let $n \in \mathbb{N}_{\geq 2}$. By the Real Projective Space–π_1-Proposition 93.3 we know that $\pi_1(\mathbb{R}\mathrm{P}^n, *) \cong \mathbb{Z}_2$. Given any $k \in \mathbb{N}$ we consider the representation $\varphi\colon \pi_1(\mathbb{R}\mathrm{P}^n, *) \to \mathrm{GL}(1, \mathbb{Z}/k\mathbb{Z})$ that sends the non-trivial element of $\pi_1(\mathbb{R}\mathrm{P}^n, *)$ to -1.

(a) Compute the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_\varphi\mathbb{Z}/k\mathbb{Z})$.

(b) Compute the twisted cohomology modules $\mathrm{H}^*_{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_\varphi\mathbb{Z}/k\mathbb{Z})$.

Exercise 329.5.

(a) Give an example of a finite connected non-empty CW-complex X with fundamental group $\pi := \pi_1(X)$ and universal covering $\widetilde{X}$ and of a $k \in \mathbb{N}$ such that $\mathrm{H}^k_{\mathrm{tw}}(X; \mathbb{Z}[\pi])$ is countable and such that $\mathrm{H}^k(\widetilde{X}; \mathbb{Z})$ is uncountable.

(b) Can the opposite effect to (a) also occur?

Remark. This shows again that the cohomological analog of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 does not hold.

Exercise 329.6. Given $g \in \mathbb{N}$ and $k \in \mathbb{N}$ we denote by $\Sigma_{g,k}$ the surface of genus g minus k open disks. Do you have enough information to calculate the twisted homology groups for any non-trivial 1-dimensional complex representation $\pi_1(\Sigma_{g,k}) \to \mathrm{GL}(1, \mathbb{C})$?

Exercise 329.7. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Furthermore suppose we are given a ring R and a representation of the form

$$\begin{array}{rcl} \varphi\colon \pi_1(X, x_0) & \to & \mathrm{GL}(m+n, R) \\ g & \mapsto & \begin{pmatrix} \alpha(g) & \theta(g) \\ 0 & \beta(g) \end{pmatrix}. \end{array}$$

(a) Use the Twisted Bockstein Lemma 321.15 to show that if $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\alpha R^m) = 0$ and $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\beta R^n) = 0$, then $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\varphi R^{m+n}) = 0$.

(b) Show that in general $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\varphi R^{m+n})$ is *not* isomorphic to the direct sum of $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\alpha R^m)$ and $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\beta R^n)$. .

Exercise 329.8. We consider the 2-dimensional torus $S^1 \times S^1$ together with the homeomorphism $f\colon S^1 \times S^1 \to S^1 \times S^1$ that is given by $f(x, y) = (y, x)$. We consider the corresponding

mapping torus $\mathrm{Tor}(S^1 \times S^1, f)$ from Definition 21.19 together with the natural map

$$\begin{array}{rccl} p\colon \mathrm{Tor}(S^1 \times S^1, f) := (S^1 \times S^1 \times [0,1])/(x,0) \sim (f(x),1)) & \to & S^1 \\ [(x,s)] & \mapsto & \exp(2\pi \mathrm{i} s). \end{array}$$

For $\xi \in \mathbb{C} \setminus \{0\}$ we consider the representation

$$\alpha\colon \pi_1(\mathrm{Tor}(S^1 \times S^1, f)) \xrightarrow{p_*} \pi_1(S^1,1) =_{\mathrm{i}} \langle x \rangle \xrightarrow{x \mapsto (\xi)} \mathrm{GL}(1,\mathbb{C}).$$

Compute $\mathrm{H}_*^{\mathrm{tw}}(\mathrm{Tor}(S^1 \times S^1, f); {}_\alpha\mathbb{C})$.
Hint. Equip $\mathrm{Tor}(S^1 \times S^1, f)$ with a CW-structure.

Exercise 329.9. Let

$$K \;=\; ([0,1] \times [0,1])/\sim \qquad \text{where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y)$$

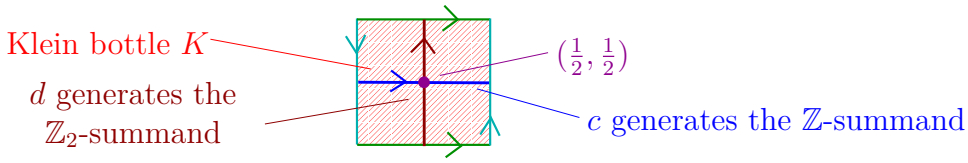
be the Klein bottle as defined on page 312. We consider the two singular 1-cycles

$$\begin{array}{rcl} c\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} d\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array}$$

In the Klein Bottle-H_*-Lemma 153.10 we saw that $\mathrm{H}_1(K) = \mathbb{Z} \cdot [c] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [d]$.

(a) Describe the chain complex of the covering of the Klein bottle corresponding to the commutator subgroup $[\pi_1(K), \pi_1(K)]$, i.e. corresponding to the kernel of the natural epimorphism $\pi_1(K) \to \pi_1(K)_{\mathrm{ab}} =_{\mathrm{i}} \mathrm{H}_1(K)$.

(b) Let
$$\alpha\colon \pi_1(K; [(\tfrac{1}{2},\tfrac{1}{2})]) \to \mathrm{H}_1(K) \to \mathrm{GL}(1,\mathbb{C})$$
(Hurewicz homomorphism from Definition 166.5)
be a 1-dimensional representation. We consider the corresponding $(\mathbb{C}, \mathbb{Z}[\pi_1(K; [(\frac{1}{2},\frac{1}{2})])])$-left left module ${}_\alpha\mathbb{C}$ from Convention 320.9. Use (a) together with the Replace Universal Covering Proposition 329.1 to compute the twisted homology modules $\mathrm{H}_i^{\mathrm{tw}}(K; {}_\alpha\mathbb{C})$ in terms of $\alpha([c])$ and $\alpha([d])$.



Exercise 329.10. Let K be the Klein bottle. Does there exist a 1-dimensional complex representation $\varphi\colon \pi_1(K) \to \mathrm{GL}(1,\mathbb{C})$ such that

(a) ... $\mathrm{H}_0^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) = 0$ and $\mathrm{H}_2^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) = 0$?
(b) ... $\mathrm{H}_0^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) = 0$ and $\mathrm{H}_2^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) \neq 0$?
(c) ... $\mathrm{H}_0^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) \neq 0$ and $\mathrm{H}_2^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) = 0$?
(d) ... $\mathrm{H}_0^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) \neq 0$ and $\mathrm{H}_2^{\mathrm{tw}}(K; {}_\varphi\mathbb{C}) \neq 0$?

Exercise 329.11. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring and let $\varphi\colon \pi_{\mathrm{ab}} \to \mathrm{GL}(d,R)$ be a representation of the abelianization of π. Using the natural projection $\pi \to \pi_{\mathrm{ab}}$ we can use φ to view $R^{1\times d}$ as a $\mathbb{Z}[\pi]$-left module ${}_\varphi R^{1\times d}$ and we can thus consider the twisted homology modules $\mathrm{H}_k^{\mathrm{CW}}(X; {}_\varphi R^{1\times d})$. Next let $\gamma\colon [0,1] \to X$ be

a loop. On page 6168 we pointed out that the map γ^* from the Change-of-Base Point Proposition 321.3 induces an automorphism of $\mathrm{H}_k^{\mathrm{CW}}(X;{}_\varphi R^{1\times d})$ of R-left modules. Provide an example of a representation of $X = \Sigma_2$ the surface of genus 2, where the automorphism corresponding to γ is non-trivial.

Exercise 329.12. Let X be a path-connected finite CW-complex and let $x_0 \in X$. We consider the product topological space $X \times S^1$. We denote by $p\colon X \times S^1 \to S^1$ the projection. Furthermore let $\Phi\colon \pi_1(S^1,1) \to \langle t\rangle$ be the isomorphism that is given by the Circle-π_1-Theorem 88.22. Given $z \in S^1$ we consider the representation

$$\rho_z\colon \pi_1(X\times S^1,(x_0,1)) \xrightarrow{p_*} \pi_1(S^1,1) \xrightarrow{\Phi} \langle t\rangle \xrightarrow{t^n\mapsto z^n} \mathrm{GL}(1,\mathbb{C}).$$

We consider the corresponding $(\mathbb{C},\mathbb{Z}[\pi_1(X\times S^1,(x_0,1))])$-left left module ${}_{\rho_z}\mathbb{C}$ from Convention 320.9. Show that if $z \neq 1$, then for all $k \in \mathbb{N}_0$ we have $\mathrm{H}_k^{\mathrm{tw}}(X\times S^1;{}_{\rho_z}\mathbb{C}) = 0$.
Hint. Consider the product CW-structure on $X \times S^1$ as defined in the CW-Complex Product Proposition 139.18 and the CW-Category–Product Proposition 139.16.

Exercise 329.13. Let $p\colon M \to F$ be an $(S^1,\mathrm{O}(2))$-bundle over a closed connected 2-dimensional smooth manifold. We denote by $t \in \pi_1(M)$ the element that is generated by the S^1-fiber. Let $\varphi\colon \pi_1(M) \to \mathrm{GL}(1,\mathbb{C})$ be a representation such that $\varphi(t) \neq 1$. Calculate the twisted homology groups $\mathrm{H}_*^{\mathrm{tw}}(M;{}_\varphi\mathbb{C})$.
Hint. Use Exercise 329.12 and use the Twisted Mayer–Vietoris Theorem 325.12.

Exercise 329.14. Let X be a path-connected finite CW-complex and let $x_0 \in X$ be a point in the 0-skeleton. We consider the wedge product $X \vee S^1$ that is given by identifying $x_0 \in X$ with $(1,0) \in S^1$. We denote by $p\colon X \vee S^1 \to S^1$ the natural map. Furthermore let $\Phi\colon \pi_1(S^1,1) \to \langle t\rangle$ be the isomorphism that is given by the Circle-π_1-Theorem 88.22. Given $z \in S^1$ we consider the representation

$$\rho_z\colon \pi_1(X\vee S^1,(x_0,1)) \xrightarrow{p_*} \pi_1(S^1,1) \xrightarrow{\Phi} \langle t\rangle \xrightarrow{t^n\mapsto z^n} \mathrm{GL}(1,\mathbb{C}).$$

We consider the corresponding $(\mathbb{C},\mathbb{Z}[\pi_1(X\vee S^1,(x_0,1))])$-left left module ${}_{\rho_z}\mathbb{C}$ from Convention 320.9. Determine the twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}(X\vee S^1;{}_{\rho_z}\mathbb{C})$ in terms of k, z and the usual homology groups of X.
Hint. Using the CW-Wedge Lemma 139.11 you can equip $X \vee S^1$ with a CW-structure.

Exercise 329.15. Let $k \in \mathbb{N}$. We denote by $N_k =_{\mathrm{i}} k\cdot\mathbb{R}\mathrm{P}^2$ the non-orientable surface of genus k. Compute the twisted homology and cohomology of N_k with respect to all non-dimensional 1-dimensional representations $\varphi\colon \pi_1(N_k) \to \mathrm{GL}(1,\mathbb{Z})$. Does the result depend on the representation φ?

Exercise 329.16. We consider the 3-dimensional torus $(S^1)^3 = [0,1]^3/\sim$. We make the usual identification $\pi_1((S^1)^3) =_{\mathrm{i}} \mathbb{Z}^3$.

(a) We equip $(S^1)^3$ with the usual CW-structure. Describe the cellular chain complex of the universal covering of $(S^1)^3$ as a chain complex of $\mathbb{Z}[\mathbb{Z}^3]$-modules.
(b) Show that for every non-trivial representation $\alpha\colon \mathbb{Z}^3 \to \mathrm{GL}(1,\mathbb{C})$ the twisted homology modules of $(S^1)^3$ are all zero.

Intrepid readers are encouraged to generalize the statements from the 3-dimensional setting to tori of arbitrary dimension.

Exercise 329.17. Let $n \in \mathbb{N}$. We consider the n-dimensional torus $(S^1)^n$.

(a) Compute a cellular chain complex of the universal covering of $(S^1)^n$.

(b) Let $\alpha\colon \pi_1((S^1)^n, *) \to \mathrm{GL}(1, \mathbb{C})$ be a one-dimensional representation. Determine the isomorphism types of the twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}((S^1)^n; {}_\alpha\mathbb{C})$ as far as possible.

Exercise 329.18. We consider the topological space $X = S^1$ with the base point $1 \in S^1$. Let R be a ring.

(a) Show that for any $(R, \mathbb{Z}[\pi_1(S^1, 1)])$-left left module and any $i \in \{0, 1\}$ we have an isomorphism

$$\mathrm{H}^i_{\mathrm{tw}}(S^1; M) \;\to\; \mathrm{H}^{\mathrm{tw}}_{1-i}(S^1; M)$$

of R-left modules.

Remark. This slightly mystifying result foreshadows the Twisted Poincaré Duality Theorem 335.1.

(b) Show that in general $\mathrm{H}^1_{\mathrm{tw}}(S^1; M)$ and $\mathrm{H}^{\mathrm{tw}}_1(S^1; M)$ are not isomorphic.

Exercise 329.19. Let $n \in \mathbb{N}_{\geq 2}$. We consider the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x$. We denote by $\varphi\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \{\pm 1\}$ the unique non-trivial representation.

(a) Compute the ordinary homology groups $\mathrm{H}_k^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_{\mathrm{tr}}\mathbb{Z})$.

(b) Compute the ordinary cohomology groups $\mathrm{H}^k_{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_{\mathrm{tr}}\mathbb{Z})$.

(c) Compute the twisted homology groups $\mathrm{H}_k^{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_\varphi\mathbb{Z})$.

(d) Compute the twisted cohomology groups $\mathrm{H}^k_{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_\varphi\mathbb{Z})$.

Exercise 329.20. For courageous readers:

(a) Can you formulate a Poincaré Duality Theorem for twisted (co-) homology of closed orientable smooth manifolds?

(b) Can you formulate a Poincaré Duality Theorem for twisted (co-) homology of closed non-orientable smooth manifolds?

Hint. Consider the calculations from Exercise 329.1, Exercise 329.18 and Exercise 329.19.

Exercise 329.21. For even more courageous readers:

(a) Can you formulate and prove a Poincaré Duality Theorem for twisted (co-) homology of closed orientable 2-dimensional smooth manifolds?

(b) Can you formulate and prove a Poincaré Duality Theorem for twisted (co-) homology of closed non-orientable 2-dimensional smooth manifolds?

CHAPTER 330

Presentation complexes and Fox calculus

In this chapter we will introduce presentation complexes and we will calculate the cellular chain complexes of their universal coverings using Fox calculus.

This chapter is organized as follows:

- In Section 330.1 we will introduce word-presentations.
- In Section 330.2 we will associate to each word-presentation its presentation complex.
- In Section 330.3 we will (re-) introduce Fox calculus.
- In Section 330.4 we will see how Fox calculus can be used to calculate the cellular chain complexes of universal coverings of presentation complexes.
- In Appendix 330.5 we will discuss the presentation complexes of one relator presentations.
- In Section 330.6 we will provide exercises for Chapter 330.

In the next Chapter 331 we will present some theoretical applications of twisted cellular (co-) homology.

330.1. **Word-presentations.** We start out with the following slight variation on Definition 100.15:

Definition 330.1. Let X be a set.

(1) According to Definition 100.1 a word in X is a finite sequence $(x_1^{\epsilon_1}, \dots, x_k^{\epsilon_k})$ with $x_1, \dots, x_k \in X$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. As we discussed in Definition 100.1, a word naturally defines an element in the free group $\langle X \rangle$ that is generated by X. For the most part in our notation we will not distinguish a word from the group element it represents.

(2) A word-presentation[3596] is a pair (X, R) consisting of a set X together with a set R of words in X. We call the elements of X the generators and we call the elements of R the relators of the word-presentation.

(3) (a) Given a word-presentation (X, R) we define the group[3597]

$$\langle X \mid R \rangle \coloneqq \langle X \rangle / \langle\langle R \rangle\rangle.$$

For brevity sometimes we blur the distinction between a word-presentation and the corresponding group in our notation, for example, we might write "let $\pi = \langle X \mid R \rangle$ be a word-presentation".

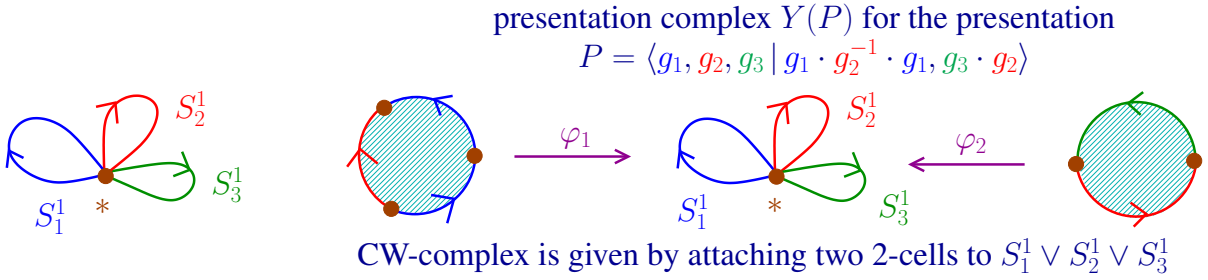
(b) If $X = \{x_1, \dots, x_k\}$ and $R = \{r_1, \dots, r_l\}$, then we use the convenient shorthand

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \coloneqq \langle \{x_1, \dots, x_k\} \,|\, \{r_1, \dots, r_l\} \rangle.$$

330.2. **Presentation complexes.** In the following we will now see that a word-presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ gives rise to a CW-complex $Y(P)$. The basic idea is to define $Y(P)$ as a CW-complex with one 0-cell $*$, k 1-cells (one for each generator) and l 2-cells (one for each relation), where the attaching map φ_i for the i-th 2-cell is the obvious map that is given by tracing out the word r_i. In the following definition (which is essentially the same as Construction 138.4) we turn this idea into precise mathematics:

Definition 330.2. Given a word-presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ we denote by $Y(P)$ the corresponding presentation complex which is defined as follows:

- We consider the wedge $\bigvee_{i=1}^{k} (S^1_i, 1)$ of k copies of S^1 along $1 \in S^1$. We view this wedge as a CW-complex in the obvious way, namely with one 0-cell $*$ for the wedge point and k 1-cells.
- For $i = 1, \dots, k$ let $\Phi_i \colon [0,1] \to S^1_i$ be the obvious loop. For the purpose of this definition we write $\Phi_i^{-1} := \overline{\Phi_i}$ for the reverse loop.
- Given a word $r = (x_{i_1}^{\epsilon_1}, \dots, x_{i_s}^{\epsilon_s})$ in $\{x_1, \dots, x_k\}$ we consider the corresponding loop $\psi(r) \colon S^1 \to \bigvee_{i=1}^{k} S^1_i$ that is defined by $\Phi_{i_1}^{\epsilon_1} * \cdots * \Phi_{i_s}^{\epsilon_s}$. We now attach l 2-cells to $\bigvee_{i=1}^{k} S^1_i$ with the attaching maps $\psi(r_1), \dots, \psi(r_l)$.



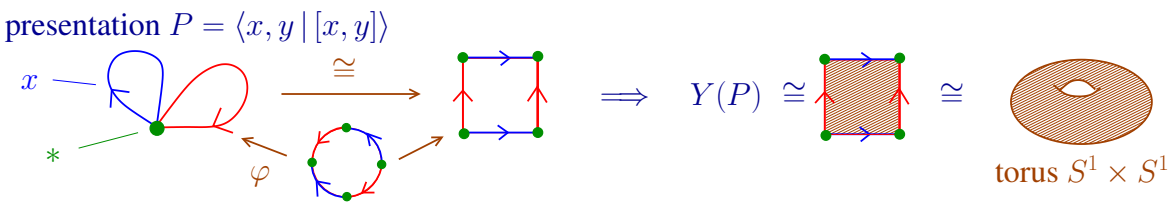
Remark. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a word presentation. Note that it follows easily from the Homotopic Attaching Map Lemma 135.12 together with the Iterative Cell-Attachment Lemma 135.10 that the homotopy type of the presentation complex only depends on the group elements which the words $r_1, \dots, r_l$ represent in the free group $\langle X \rangle$. □

Example.

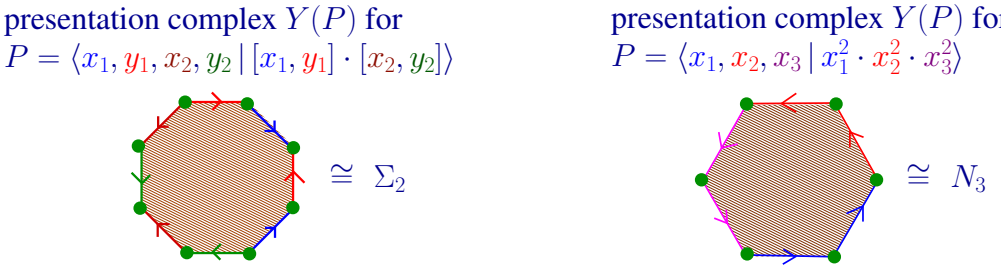
(1) We consider the word-presentation $P = \langle x, y \,|\, [x,y] \rangle = \langle x, y \,|\, xyx^{-1}y^{-1} \rangle$. In the figure below we illustrate the construction of the corresponding presentation complex $Y(P)$. If we compare the above figure with the discussion on page 6280 we see that $Y(P)$ is just the standard CW-structure for the 2-dimensional torus $S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$.

[3596] In Definition 100.15 we defined a *presentation* as a pair (X, R) consisting of a set X together with a subset $R \subseteq \langle X \rangle$. Since a word in X represents an element of $\langle X \rangle$ (but not vice versa) we see that a word-presentation determines a presentation, but not vice versa. In most settings the difference between a word-presentation and a presentation is immaterial. But on page 6295 we will see that in this section it is really better to distinguish between these two notions.

[3597] Here, given a subset S of a group G we denote, as in Definition 98.20, by $\langle\langle S \rangle\rangle \subseteq G$ the smallest normal subgroup of G that contains S.

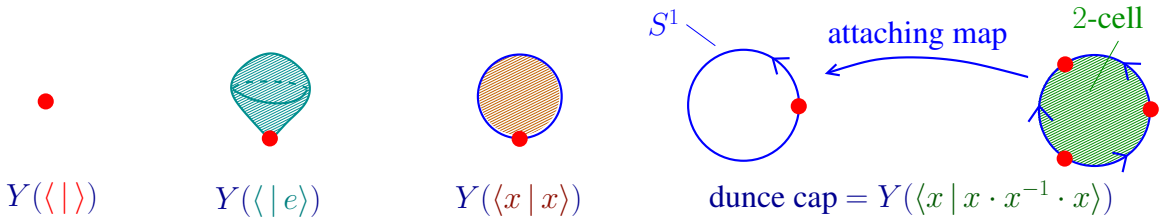


(2) Let $g \in \mathbb{N}$. Similarly to (1) one sees that the presentation complex corresponding to the word-presentation $\langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1,y_1] \cdot \ldots \cdot [x_g,y_g] \rangle$ is homeomorphic to the surface Σ_g of genus g.



One also sees the same way that for any $k \in \mathbb{N}$ the presentation complex corresponding to the word-presentation $\langle x_1, \ldots, x_k, \,|\, x_1^2 \cdot \ldots \cdot x_k^2 \rangle$ is homeomorphic to the non-orientable surface $N_k \cong k \cdot \mathbb{R}\mathrm{P}^2$ of genus k.

(3) For the trivial group we can consider several word-presentations, e.g. $\langle \,|\, \rangle$, $\langle \,|\, e \rangle$, $\langle x \,|\, x \rangle$ and $\langle x \,|\, x \cdot x^{-1} \cdot x \rangle$. In the figure we consider the corresponding presentation-complexes. Note that in the last case we get the so-called *dunce cap*.



Since $x \cdot x^{-1} \cdot x = x \in \langle x \rangle$ we obtain from the remark on page 6294 that the dunce cap is homotopy equivalent to the presentation complex of $\langle x \,|\, x \rangle$ which in turn is homeomorphic to $\overline{B}^2$. This shows that the dunce cap is contractible. What makes the dunce cap interesting is the observation from page 5871 that even though it is contractible, it is not collapsible in the sense of Definition 305.19. Loosely speaking this means that we cannot just find a straightforward deformation retraction by collapsing 2-cells.

(4) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p In the proof of the Lens Space-CW-Proposition 329.15 we constructed a CW-structure on S^3 which descended to a CW-structure on $L(p,q)$. One can easily verify that the 2-skeleton of the CW-structure on S^3 is homeomorphic to the universal covering of the presentation complex of $\langle x \,|\, x^p \rangle$. □

We conclude this section with the following unsurprising lemma:

Lemma 330.3. (Presentation Complex-π_1-Lemma) Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a finite word-presentation. For $i = 1, \dots, m$ we denote by $\Phi_i \colon [-1,1] \to \bigvee_{i=1}^{k} S^1$ the characteristic map that is given by going counterclockwise around the i-th copy of S^1. The homomorphism

$$\begin{array}{rcl} \langle x_1, \dots, x_k \rangle & \to & \pi_1(Y(P), *) \\ x_i & \mapsto & [t \mapsto \Phi_i(2t-1)] \end{array}$$

descends to an isomorphism

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \;\to\; \pi_1(Y(P), *).$$

Proof. This lemma is just a slight reformulation of the Presentation–CW Lemma 138.5. As we saw before, this lemma follows easily from the Cell Attachment-π_1-Proposition 135.5. ■

330.3. **Fox calculus.** Let P be a finite word-presentation with corresponding presentation-complex $Y(P)$. We denote by π the corresponding group. Recall that, by the Regular Covering–Cellular Chain Complex Lemma 327.10 we should view $C_*^{\mathrm{tw}}(Y(P))$ as a chain complex of free $\mathbb{Z}[\pi]$-left modules. Our goal is to describe this chain complex explicitly. To formulate our calculation we need to recall the following proposition:

Proposition 330.4. (Fox Derivative Proposition) Let $\langle x_1, \dots, x_k \rangle$ be the free group on generators $x_1, \dots, x_k$. For $i \in \{1, \dots, k\}$ there exists a unique map

$$\frac{\partial}{\partial x_i} \colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \;\to\; \mathbb{Z}[\langle x_1, \dots, x_k \rangle],$$

called the i-th Fox derivative, with the following properties:

(1) For any $i, j \in \{1, \dots, k\}$ we have $\frac{\partial}{\partial x_i} x_j = \delta_{ij}$.

(2) For any $u, v \in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$ we have

$$\frac{\partial}{\partial x_i}(u \cdot v) \;=\; \frac{\partial}{\partial x_i}(u) + u \cdot \frac{\partial}{\partial x_i}(v) \qquad \text{(Leibniz Rule)}.$$

(3) The map $\frac{\partial}{\partial x_i}$ is a homomorphism of abelian groups.

The Fox derivatives have furthermore the following properties:

(4) If $e \in \langle x_1, \dots, x_k \rangle$ is the trivial element, then

$$\frac{\partial}{\partial x_i} e \;=\; 0.$$

(5) For any $i, j \in \{1, \dots, k\}$ we have

$$\frac{\partial}{\partial x_i} x_j^{-1} \;=\; \begin{cases} -x_i^{-1}, & \text{if } i = j, \\ 0, & \text{else.} \end{cases}$$

Example. We consider the free group $\langle x, y \rangle$. By applying the various properties of Fox derivatives from the Fox Derivative Proposition 330.4 we can perform the following calculation in $\mathbb{Z}[\langle x, y \rangle]$:

$$\begin{array}{lll} \frac{\partial}{\partial x}(xyx^{-1}y^{-1}) & = \frac{\partial}{\partial x} x + x \cdot \frac{\partial}{\partial x}(yx^{-1}y^{-1}) & = 1 + x \cdot (\frac{\partial}{\partial x} y + y \cdot \frac{\partial}{\partial x}(x^{-1}y^{-1})) \\ & \quad \uparrow \text{ by (2)} & \quad \uparrow \text{ by (1) and (2)} \\ & = 1 + x \cdot y \cdot (\frac{\partial}{\partial x} x^{-1} + x^{-1} \cdot \frac{\partial}{\partial x} y^{-1}) & = 1 - x \cdot y \cdot x^{-1}. \\ & \quad \uparrow \text{ by (1) and (2)} & \quad \uparrow \text{ by (5)} \end{array}$$

Similarly one sees that $\frac{\partial}{\partial y} xyx^{-1}y^{-1} = x - xyx^{-1}y^{-1}$. □

Sketch of proof. The basic idea of the proof is quite straightforward: First one can easily verify that (4) and (5) are a consequence of (1), (2) and (3). Next, as in the example one sees that the rules (1)–(5) allow us to determine the value of the Fox derivative for each word in $x_1, \dots, x_k$: One just needs to read the word "from left to right" and iteratively apply the Leibniz rule together with (1) and (5). The details are worked out in the proofs of the Fox Derivative Proposition 116.3 and the Fox Derivative Properties Lemma 116.5. ■

330.4. **Fox calculus and chain complexes.** Before we can state the main result of this section we should provide one more definition, which happens to be almost identical to Definition 116.8:

Definition 330.5.

(1) Given any finite presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ we denote by

$$\mu = \mu_P \colon \langle x_1, \dots, x_k \rangle \to P$$

the natural projection.

(2) Let π be a group. A finite word-presentation for the group π consists of a finite word-presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ together with an isomorphism

$$\gamma \colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \xrightarrow{\cong} \pi.$$

Usually we suppress the isomorphism γ from the notation.

Now we can finally provide the main result of this section, which was first formulated and proved by Ralph Fox [**Fox1953, CF1977**]:

Proposition 330.6. (CW-Boundary Maps-via-Fox Calculus Proposition) In the following let $P = \langle x_1, \dots, x_k | r_1, \dots, r_l \rangle$ be a finite word-presentation for a group π. Then there exists a lift of the cells of $Y(P)$ to the universal covering $\widetilde{Y(P)}$ such that with respect to the corresponding bases, which we obtain from the Regular Covering–Cellular Chain Complex Lemma 327.10, the based cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{Y(P)})$ is isomorphic to the chain complex

$$0 \to \mathbb{Z}[\pi]^{1\times l} \xrightarrow{v \mapsto v \cdot \begin{pmatrix} \mu(\frac{\partial r_1}{\partial x_1}) & \dots & \mu(\frac{\partial r_1}{\partial x_k}) \\ \vdots & & \vdots \\ \mu(\frac{\partial r_l}{\partial x_1}) & \dots & \mu(\frac{\partial r_l}{\partial x_k}) \end{pmatrix}} \mathbb{Z}[\pi]^{1\times k} \xrightarrow{v \mapsto v \cdot \begin{pmatrix} \mu(x_1)-1 \\ \vdots \\ \mu(x_k)-1 \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

Example. Let $X = S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$ with the usual CW-structure. On page 6294 we discussed that X is homeomorphic to the presentation complex $Y(P)$ with $P = \langle x, y \,|\, [x,y] \rangle = \langle x, y \,|\, xyx^{-1}y^{-1} \rangle$. Next note that we have the following equalities in $\mathbb{Z}[\langle x, y \,|\, xyx^{-1}y^{-1} \rangle]$:

$$\begin{array}{ccccc} \mu(\frac{\partial}{\partial x}(xyx^{-1}y^{-1})) & = & \mu(1 - x \cdot y \cdot x^{-1}) & = & 1 - y. \\ & & \uparrow & & \uparrow \\ & & \text{see page 6296} & & \text{since } xyx^{-1}y^{-1} = e \\ & & \downarrow & & \downarrow \\ \mu(\frac{\partial}{\partial y}(xyx^{-1}y^{-1})) & = & \mu(x - xyx^{-1}y^{-1}) & = & x - 1. \end{array}$$

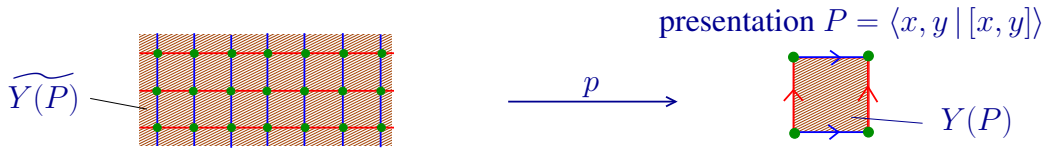
From this calculation together with the CW-Boundary Maps-via-Fox Calculus Proposition 330.6 we now obtain the following cellular chain complex for the universal covering of

the torus $X = S^1 \times S^1$:

$$0 \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \xrightarrow{\begin{pmatrix} 1-y & x-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{1\times 2} \xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \to 0.$$

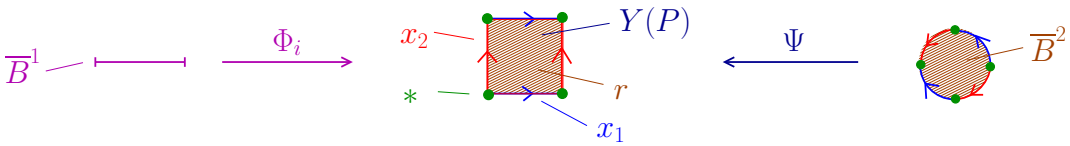
This is essentially the same chain complex that we found in the Torus–Twisted Cellular Chain Complex Lemma 329.11. □

Proof. [3598] Let $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a finite word-presentation for a group π. Note that throughout this proof we will follow the usual convention that in the notation we will not distinguish between elements of $\langle x_1, \ldots, x_k \rangle$ and elements of the group $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$. In particular we will ignore μ in our notation. We consider the presentation complex $Y(P)$ from Definition 330.2 with universal covering $\widetilde{Y(P)}$.



We introduce the following notation:

(1) For $i = 1, \ldots, k$ we denote by $\Phi_i \colon [-1, 1] \to \bigvee_{i=1}^{k} S^1$ the characteristic map that is given by going counterclockwise around the i-th copy of S^1.
(2) For $j = 1, \ldots, l$ we denote by $\Psi_j \colon \overline{B}^2 \to Y(P)$ the characteristic map with $\Psi_j(1) = *$.



Next we consider the following diagram:

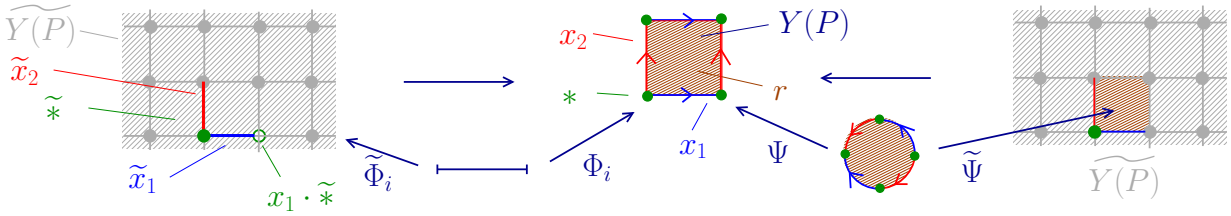
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}[\pi]^{1\times l} & \xrightarrow{\cdot \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial r_l}{\partial x_1} & \cdots & \frac{\partial r_l}{\partial x_k} \end{pmatrix}} & \mathbb{Z}[\pi]^{1\times k} & \xrightarrow{\cdot \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_k - 1 \end{pmatrix}} & \mathbb{Z}[\pi] & \longrightarrow & 0 \\
 & & \cong \downarrow \mathfrak{e}_j \mapsto \widetilde{r}_j & & \cong \downarrow \mathfrak{e}_i \mapsto \widetilde{x}_i & & 1 \mapsto \widetilde{*} \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{C}_2^{\mathrm{CW}}(\widetilde{Y(P)}) & \xrightarrow{\partial_2} & \mathrm{C}_1^{\mathrm{CW}}(\widetilde{Y(P)}) & \xrightarrow{\partial_1} & \mathrm{C}_0^{\mathrm{CW}}(\widetilde{Y(P)}) & \longrightarrow & 0.
\end{array}$$

We make the following clarifications and observations:

- We pick a lift $\widetilde{*}$ of the 0-cell $*$.
- For each $i \in \{1, \ldots, k\}$ let $\widetilde{\Phi}_i \colon \overline{B}^1 \to \widetilde{Y(P)}$ be the unique lift of characteristic map $\Phi_i \colon \overline{B}^1 \to Y(P)$ with $\widetilde{\Phi}_i(-1) = *$. We write $\widetilde{x}_i := \widetilde{\Phi}_i([-1, 1]) \subseteq \widetilde{Y(P)}$.
- For each $j \in \{1, \ldots, l\}$ let $\widetilde{\Psi}_j \colon \overline{B}^2 \to \widetilde{Y(P)}$ be the unique lift of characteristic map $\Phi_j \colon \overline{B}^2 \to Y(P)$ with $\widetilde{\Phi}_j(1, 0) = *$. We write $\widetilde{r}_j := \widetilde{\Phi}_j(\overline{B}^2) \subseteq \widetilde{Y(P)}$.

[3598] Note that the case that P has no relations is closely related to the proof of the Free Resolution–Free Group Ring-Lemma 318.4.

- It follows from the Regular Covering–Cellular Chain Complex Lemma 327.10 that in the above diagram the vertical maps are all isomorphisms of $\mathbb{Z}[\pi]$-left modules.
- It follows immediately from the discussion on page 6276 that we have the equalities $\partial_1(\widetilde{x}_i) = x_i \cdot \widetilde{*} - \widetilde{*} = (x_i - 1) \cdot \widetilde{*}$.



At this point it remains to prove the following claim:

Claim. For each $j \in \{1, \ldots, l\}$ we have

$$\partial_2(\widetilde{r}_j) \;=\; \sum_{i=1}^{k} \cdot \frac{\partial r_j}{\partial x_i} \cdot \widetilde{x}_i.$$

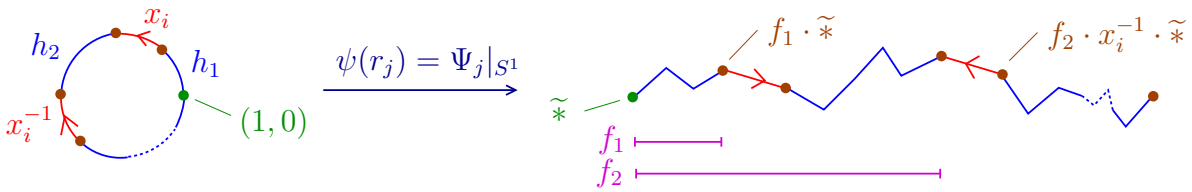
Proof. Let $j \in \{1, \ldots, l\}$ and let $i \in \{1, \ldots, k\}$. We need to show that the $\widetilde{x}_i$-coordinate of $\partial_2(\widetilde{r}_j)$ is given by $\frac{\partial r_j}{\partial x_i}$. We write

$$r_j \;=\; g_1 \cdot {x_i}^{\epsilon_1} \cdot g_2 \cdot {x_i}^{\epsilon_2} \cdot g_3 \cdot \ldots \cdot g_m \cdot {x_i}^{\epsilon_m} \cdot g_{m+1}$$

where $g_1, \ldots, g_{m+1}$ are (possibly empty) words in $x_1, \ldots, \widehat{x}_i, \ldots, x_k$ and where the exponents $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$. Next, for $h = 1, \ldots, m$ we write $f_h := g_1 \cdot {x_i}^{\epsilon_1} \cdot g_2 \cdot {x_i}^{\epsilon_2} \cdot g_3 \cdot \ldots \cdot g_h$. In other words, f_h is the word "to the left" of the h-th occurrence of x_i. It follows easily from the Fox Derivative Proposition 330.4 that

$$\frac{\partial r_j}{\partial x_i} \;=\; \sum_{h=1}^{m} \begin{cases} f_h, & \text{if } \epsilon_h = 1, \\ -f_h \cdot {x_i}^{-1}, & \text{if } \epsilon_h = -1. \end{cases}$$

We claim that it follows from the Cellular Boundary–Degree Proposition 329.4 and the Observation 329.5 that this is also just the $\widetilde{x}_i$-coordinate of $\partial_2(\widetilde{r}_j)$. We will illustrate this calculation in the following figure:



We leave it to the reader to write down the completely formal, lean-ready proof. ∎

Remark. Let $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ be a presentation for a group π. In the CW-Boundary Maps-via-Fox Calculus Proposition 330.6 we described the boundary maps of a chain complex in terms of matrices. Since we are dealing with a chain complex we see that the composition of the two boundary maps is zero. Since our matrices correspond to the linear maps that are given by right multiplication we see that the matrix

$$\begin{pmatrix} \mu(\frac{\partial r_1}{\partial x_1}) & \ldots & \mu(\frac{\partial r_1}{\partial x_k}) \\ \vdots & & \vdots \\ \mu(\frac{\partial r_l}{\partial x_1}) & \ldots & \mu(\frac{\partial r_l}{\partial x_k}) \end{pmatrix} \cdot \begin{pmatrix} \mu(x_1) - 1 \\ \vdots \\ \mu(x_k) - 1 \end{pmatrix}$$

is the zero matrix over $\mathbb{Z}[\pi]$. This is consistent with the purely algebraic Fundamental Theorem-of-Fox Calculus 116.7 which says that for any $u \in \langle x_1, \ldots, x_k \rangle$ we have the following equality

$$(*) \qquad u - 1 = \sum_{i=1}^{k} \frac{\partial u}{\partial x_i} \cdot (x_i - 1)$$

in $\mathbb{Z}[\langle x_1, \ldots, x_k \rangle]$. In fact (*), applied to $u = r_1, \ldots, r_k$ implies the fact that the above product of matrices is zero over $\mathbb{Z}[\pi]$. □

330.5. **Appendix: Presentation complexes for one relator presentations.** In this appendix we will use Fox calculus to sketch an argument why the presentation complexes of suitable one-relator presentations are aspherical. We will not make use of this result.

To formulate this result we need the following definition, which is inspired by Definition 98.1:

Definition 330.7. Let X be a set and let $r = (x_1^{\epsilon_1}, \ldots, x_m^{\epsilon_m})$ be a word in X.

(1) We say that the word is reduced if for all $j \in \{1, \ldots, m-1\}$ we have either $x_i \neq x_{i+1}$ or $\epsilon_i \neq -\epsilon_{i+1}$.
(2) We say that the word is cyclically reduced if either $x_1 \neq x_m$ or $\epsilon_1 \neq -\epsilon_m$.

Example.

(1) We consider the free group $\langle x, y \rangle$ and the words $v = x \cdot y \cdot x$ and $w = x \cdot y \cdot x^{-1}$, both are reduced, the former is also cyclically reduced but the latter is not cyclically reduced.
(2) Let $g \in \mathbb{N}$. We consider the free group $\langle x_1, y_1, \ldots, x_g, y_g \rangle$. It is clear that the word $r = [x_1, y_1] \cdot \ldots \cdot [x_g, y_g]$ in $\langle x_1, y_1, \ldots, x_g, y_g \rangle$ is reduced and cyclically reduced. □

Next we recall Definition 307.2 and Definition 318.2:

Definition 330.8. Let G be a group and let X be a path-connected non-empty topological space. We say that X is an Eilenberg-Maclane space of type $\mathrm{K}(G, 1)$ if the following three conditions hold:

(a) $\pi_1(X) \cong G$.
(b) X is aspherical.
(c) X admits a CW-structure.

Now we can formulate the main result of this appendix:

Theorem 330.9. (One Relator–Presentation Complex Theorem) In the following let $P = \langle x_1, \ldots, x_k \,|\, r \rangle$ be a word-presentation with a single, non-empty relator r. We denote by π the corresponding group. We assume that r is reduced and cyclically reduced. We pick a maximal $n \in \mathbb{N}$ such that $r = s^n$ for some word s.

(1) If $n = 1$, then the presentation-complex $Y(P)$ is aspherical, in particular it is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$.
(2) If $n > 1$, then the presentation-complex $Y(P)$ is not aspherical.

Remark. The One Relator–Presentation Complex Theorem 330.9 illustrates that one-relator groups are well understood by the standards of group theory. Many more properties of one-relator groups are discussed in [**MKS1976**, Chapter 4.4]. □

Example.

(1) It follows from the One Relator–Presentation Complex Theorem 330.9 (1) and the discussion on page 6295 that for each $g \in \mathbb{N}$ the surface Σ_g of genus g is aspherical and that it is an Eilenberg-Maclane space.

(2) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. It follows easily from the One Relator–Presentation Complex Theorem 330.9 that the presentation complex of $\langle x, y \,|\, x^p \cdot y^q \rangle$ is aspherical. In Exercise 330.6 we will see that this presentation complex is homotopy equivalent to the complement of the (p, q) torus knot $T(p, q)$. This shows that the knot complement $S^3 \setminus T(p, q)$ is aspherical and that it is an Eilenberg-Maclane space.



(3) We consider the presentation $\langle x \,|\, x^2 \rangle$. The One Relator–Presentation Complex Theorem 330.9 (2) tells us that the corresponding presentation complex is not aspherical. But that does not come as a big surprise, since it is pretty clear that the corresponding presentation complex is homeomorphic to $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} S^2/x \sim -x$. □

Sketch of proof.

(1) First we assume that $n = 1$, i.e. we assume that r is not a proper power. We need to show that $Y(P)$ is aspherical. Note that (similar to Exercise 167.21) it follows easily from the Hurewicz Theorem 167.9, the fact that $Y(P)$ is a 2-dimensional CW-complex and the Singular-Cellular-H_*-Proposition 155.8 that it suffices to prove the following claim:

Claim. We have $\mathrm{H}_2(\widetilde{Y(P)}) = 0$.

Proof. Note that it follows from the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5, the Singular-Cellular Twisted Isomorphism Theorem 328.1 and the CW-Boundary Maps-via-Fox Calculus Proposition 330.6 that $\mathrm{H}_2(\widetilde{Y(P)})$ is isomorphic to second homology of the following chain complex:

$$0 \to \mathbb{Z}[\pi] \xrightarrow{v \mapsto v \cdot \begin{pmatrix} \mu(\frac{\partial r}{\partial x_1}) & \dots & \mu(\frac{\partial r}{\partial x_k}) \end{pmatrix}} \mathbb{Z}[\pi]^{1 \times k} \xrightarrow{v \mapsto v \cdot \begin{pmatrix} \mu(x_1) - 1 \\ \vdots \\ \mu(x_k) - 1 \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

We see that it remains to prove the purely group-theoretic fact that the homomorphism to the left is injective. This fact is shown in [**Put3**, Chapter 7].

(2) We now assume that $n > 1$. In [**MKS1976**, Theorem 4.12] it is shown that the group element $s \in \langle x_1, \dots, x_k \,|\, s^n \rangle$ is non-trivial. Clearly s^n represents the trivial element. Thus we have shown that $\langle x_1, \dots, x_k \,|\, s^n \rangle$ has torsion. By the Torsion–H^*-Dimension Proposition 317.24 together with the Free Resolution-via-K$(G, 1)$-Proposition 318.3 (2b) this implies that the 2-dimensional CW-complex $Y(P)$ cannot be aspherical. ■

330.6. Exercises for Chapter 330.

Exercise 330.1. For which $n \in \mathbb{Z}$ is the presentation complex of $\langle x \,|\, x^n \rangle$ a topological manifold?

Exercise 330.2.

(a) We consider the presentation $\langle x \mid \overbrace{x^2}\rangle$. Illustrate the presentation complex $Y(P)$ from Definition 330.2 and show that $Y(P)$ is homeomorphic to a subspace of $\mathbb{R}^3$.

(b) We consider the presentation $\langle x \mid \overbrace{x^3}\rangle$. Illustrate the presentation complex $Y(P)$ from Definition 330.2 and show that $Y(P)$ is homeomorphic to a subspace of $\mathbb{R}^3$.

Exercise 330.3. If we are given a word-presentation of the trivial group, is the corresponding presentation complex necessarily contractible?

Exercise 330.4. Let $k \in \mathbb{N}$. We consider the corresponding non-orientable surface N_k of genus k. Show that N_k is homeomorphic to the presentation complex of a suitable presentation.

Exercise 330.5. Let $m \in \mathbb{N}_{\geq 3}$. Show directly (without making use of the One Relator–Presentation Complex Theorem 330.9) that the presentation complex of $\langle x \,|\, x^m \rangle$ is not aspherical.

Exercise 330.6. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We consider the torus knot $T(p,q)$ from Definition 111.13. Show that the knot complement $S^3 \setminus T(p,q)$ is homotopy equivalent to the presentation complex of $\langle x, y \,|\, x^p \cdot y^q \rangle$.
Hint. Consider the description of $S^3 \setminus T(p,q)$ in the Torus Knot-π_1-Proposition 111.18.
Remark. This exercise is meant for the intrepid reader.

Exercise 330.7. Use Fox calculus to reprove the Surface–Twisted Cellular Chain Complex Lemma 329.12

Exercise 330.8. Let K be the Klein bottle. Use Fox calculus to recalculate the 1-dimensional twisted homology groups $\mathrm{H}_*^{\mathrm{tw}}(K; {}_{\varphi}\mathbb{C})$.

Exercise 330.9. Let X be a connected CW-complex and let $x_0 \in X^0$ be a point in the 0-skeleton. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module.

(a) We denote by X^2 the 2-skeleton of X. It follows from the Kill-All Higher-π_n-Proposition 307.9 that there exists an Eilenberg-Maclane space $Z = \mathrm{K}(\pi, 1)$ such that the 2-skeleta agree, i.e. such that $Z^2 = X^2$. In particular we obtain an identification $\pi_1(Z, x_0) = \pi_1(X, x_0)$. Show that we have an isomorphism $\mathrm{H}_1^{\mathrm{tw}}(X; M) \cong \mathrm{H}_1^{\mathrm{tw}}(Z; M)$ of R-left modules.

(b) Let $\langle x_1, \ldots, x_l \,|\, r_1, \ldots, r_k \rangle$ be a presentation for the group $\pi_1(X, x_0)$. Show that one can calculate $\mathrm{H}_1^{\mathrm{tw}}(X; M)$ using Fox calculus.
Remark. You should reduce the calculation to the CW-Boundary Maps-via-Fox Calculus Proposition 330.6.

Remark. This exercise shows that one can calculate twisted first homology in terms of the fundamental group. This observation can be viewed as a variation on the Hurewicz Theorem 166.10.

CHAPTER 331

Applications of twisted cellular (co-) homology

In Chapter 327 we introduced twisted cellular (co-) homology and in Chapter 328 we showed that it is naturally isomorphic to twisted singular (co-) homology. The twisted cellular chain complex is by definition "much smaller" than the twisted singular chain complex. In the previous Chapter 329 we used this feature to perform many calculations. In this chapter we will make use of this feature to show that many twisted (co-) homology modules are finitely generated and we will use this feature to relate twisted (co-) homology to the Euler characteristic.

This chapter is organized as follows:

- In Section 331.1 we will use twisted cellular (co-) homology to see that frequently twisted (co-) homology is given by finitely generated modules.
- In Section 331.2 we will provide a proof of the Twisted Invariants–Finite Generation Proposition 331.6.
- In Section 331.3 we will relate twisted (co-) homology to the Euler characteristic.
- In Section 331.4 we will provide a proof of the Twisted Homology-χ-Proposition 331.12.
- In Appendix 331.5 we will discuss Wall's Finiteness Theorem 331.15.
- In Section 331.6 we will provide exercises for Chapter 331.

In the next Chapter 332 we will introduce the notion of a handle-complex, which is the manifold analog of a CW-complex. Afterwards, in Chapter 333 we will use handle-complexes to introduce twisted handle (co-) homology.

331.1. **Finitely generated twisted (co-) homology modules.** We start out by introducing a slight extension of Definition 287.1:

Definition 331.1. Let R be a ring and let M be an R-left module. We say M is finitely generated if there exists a subset $X \subseteq M$ such that the natural homomorphism[3599] $R^{(X)} \to M$ of R-left modules is an epimorphism.

We start out with two observations:

(1) It follows easily from the Singular-Cellular-H_*-Proposition 155.8 that the singular homology groups of a finite CW-complex are finitely generated.

(2) In the Smooth Manifolds-Invariants Proposition 179.10 and in the Topological Manifolds-Invariants Proposition 189.14 we used (1) to show that the ordinary homology groups of compact smooth and topological manifolds are finitely generated.

In this section we want to generalize these results to (co-) homology with twisted coefficients. To control the "size" of (co-) homology of topological spaces we need control over the rings involved and we need some control over the topological spaces involved. Let us

[3599] As in Definition 309.19 we denote by $R^{(X)}$ the free R-left module that is generated by the set X.

start out with the following definition which is the obvious generalization of Definition 287.2 from commutative rings to arbitrary rings (see also [**Lang2002b**, p. 186] or [**Lam1991**, p. 20]):

Definition 331.2. Let R be a ring. If every R-left submodule of R is finitely generated, then we call R left Noetherian. If R is commutative, then we also say Noetherian instead of left Noetherian.[3600]

The following proposition, which is a slight generalization of the earlier Noetherian Rings Proposition 287.3, shows that many rings that are close to our heart are (left) Noetherian:

Proposition 331.3. (Noetherian Rings Proposition)

(1) (a) Every skew field is left Noetherian.
(b) Every principal ideal domain (PID), in particular the ring $\mathbb{Z}$, is Noetherian.[3601]
(2) If A is a commutative Noetherian ring, then the polynomial ring $A[t]$ is Noetherian.
(3) Let A be a commutative Noetherian ring. If $S \subseteq A$ is a multiplicatively closed subset, then the ring $S^{-1}A$, which is given by the localization of A at S, is also Noetherian.
(4) If A is a Noetherian ring (not necessarily commutative) and if $I \subseteq A$ is an ideal, then the ring A/I is also Noetherian.

Proof.

(1) (a) Clearly every submodule of a skew field is either zero or the whole skew field.
(b) Note that every submodule of a ring is in particular an ideal. Thus it follows immediately from Definition 309.34 that every PID is Noetherian.
(2) If A is commutative and Noetherian, then it follows from [**Lang2002b**, Theorem IV.4.1] that $A[t]$ is Noetherian.
(3) This statement is [**Lang2002b**, Proposition X.1.6].
(4) This statement is an elementary exercise in algebra. ■

For the reader's convenience we also recall the Laurent Polynomial Ring–Noetherian Proposition 287.4 and its proof:

Proposition 331.4. (Laurent Polynomial Ring–Noetherian Proposition) Let A be a commutative Noetherian ring (e.g. $A = \mathbb{Z}$ or A is field).

(1) Every multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ is Noetherian.
(2) For any finitely generated abelian group H the group ring $A[H]$ is Noetherian.

Proof. Let A be a Noetherian ring.

(1) It follows from the Noetherian Rings Proposition 331.3 (2) applied altogether m times, that $A[t_1, \ldots, t_m]$ is Noetherian. Note that we obtain the multivariable Laurent polynomial ring $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ by inverting the multiplicatively closed subset that is given by all monomials in $t_1, \ldots, t_k$. It thus follows from the Noetherian Rings Proposition 331.3 (3) that $A[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ is Noetherian.
(2) Let H be a finitely generated abelian group. By the Abelian Group-Classification Theorem 97.24 we know that H is isomorphic to $\mathbb{Z}^m \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_n\mathbb{Z}$ for some $m, n \in \mathbb{N}_0$ and some $k_1, \ldots, k_n \in \mathbb{N}$. One can easily see that the group ring $A[H]$ is

[3600] For commutative rings we introduced this concept in Definition 287.2.

[3601] We refer to Definition 309.34 for our definition of a PID. Note that in our definition a PID is automatically commutative.

isomorphic to $A[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1, \ldots, y_n]/(y_1^{k_1}, \ldots, y_n^{k_n})$. Similar to (1) it thus follows from the Noetherian Rings Proposition 331.3 (2), (3) and (4) that the group ring $A[H]$ is Noetherian. ■

The following proposition, which is a generalization of the earlier Noetherian Ring–Submodule Proposition 287.5, states the key property of left Noetherian rings that we are interested in:

Proposition 331.5. (Noetherian Ring–Submodule Proposition) Let R be a left Noetherian ring. If P is a finitely generated R-left module, then every R-left submodule of P is also finitely generated.

Proof. This proposition is proved in [**Lam1991**, Proposition 1.21]. The commutative case is also proved in [**Lang2002b**, Proposition X.1.1]. ■

The following proposition explains why we are suddenly interested in left Noetherian rings:

Proposition 331.6. (Twisted Invariants–Finite Generation Proposition) We assume that we are in the following setup:

- Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $A \subseteq X$ be a subspace such that each component of A is also locally contractible. We assume that X and A both have the property that they are homotopy equivalent to a retract of a finite CW-complex.
- Let R be a left Noetherian ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module that is finitely generated as an R-left module.

Then for any $k \in \mathbb{N}_0$ the twisted (co-) homology modules $\mathrm{H}_k^{\mathrm{tw}}(X, A; M)$ and $\mathrm{H}_{\mathrm{tw}}^k(X, A; M)$ are finitely generated R-left modules.

Proof. We dedicate the next Section 331.2 to the proof. ■

The following question arises: Does the Twisted Invariants–Finite Generation Proposition 331.6 hold in our favorite setup, namely does it hold for a compact manifold X and where A is perhaps a union of components of ∂X? Fortunately the following theorem tells us that this is indeed the case:

Theorem 331.7. (Manifold–CW Complex Theorem)

(1) Every non-empty compact n-dimensional smooth manifold is homotopy equivalent to a finite n-dimensional CW-complex.
(2) Every compact topological manifold is homotopy equivalent to a retract of a finite CW-complex.

Proof.

(1) Let M be a compact smooth manifold.
  (a) By the Morse Function-Existence Theorem 249.26 we know that M admits a Morse function. In the Morse Function–Handle Decomposition Theorem 250.6 we showed that this implies that M admits a handle decomposition. In the Smooth Manifold-CW-Homotopy Type Proposition 254.4 we showed that any handle decomposition on M leads to a finite CW-complex X that is homotopy equivalent to M and such that the number of k-handles equals the number of k-cells.
  (b) In the Smooth Manifold-CW-Theorem 179.9 we sketched a proof that M admits a simplicial structure, in particular that it admits a CW-structure.

(2) Let M be a compact topological manifold. Using Borsuk's Theorem 189.7 we showed in the Topological Manifold-Simplicial Complex Retract Theorem 189.13 that M is a retract of a finite simplicial complex, in particular we see that M is a retract of a finite CW-complex. ■

331.2. **Proof of the Twisted Invariants–Finite Generation Proposition 331.6.** Before we can provide the proof of the Twisted Invariants–Finite Generation Proposition 331.6 we need to get the following elementary lemma, which can be viewed as a generalization of the Finite Generation Lemma 97.20, out of the way:

Lemma 331.8. (Finite Generation Lemma) Let R be a ring and let $\varphi\colon A \to B$ be a homomorphism of R-left modules.

(1) If φ is an epimorphism and if A is finitely generated, then so is B.
(2) If φ is a monomorphism, if B is finitely generated and if R is left Noetherian, then A is also finitely generated.
(3) Let
$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$
be a short exact sequence of R-left modules. If two of these modules are finitely generated and if R is left Noetherian, then the third module is also finitely generated.

Proof of the Finite Generation Lemma 331.8.

(1) This statement is clear.
(2) This statement is just a reformulation of the Noetherian Ring–Submodule Proposition 331.5.
(3) By (1) and (2) it only remains to deal with the case that A and C are finitely generated R-left modules with finite generating sets $X \subseteq A$ and $Z \subseteq C$. For each $z \in Z$ we pick $\widetilde{z} \in B$ with $\beta(\widetilde{z}) = z$. It is quite elementary to see that $\alpha(A) \cup \{\widetilde{z} \,|\, z \in Z\}$ is a generating set for the R-left module B. The details will be filled in in Exercise 331.4. ■

We will need one more elementary algebraic lemma:

Lemma 331.9. (Tensor Product–Finite Generation Lemma) Let R and S be rings, let M be an (R, S)-bimodule that is finitely generated as an R-left module and let C be a free S-left module with a finite basis. Then $M \otimes_S C$ is a finitely generated R-left module.

Proof of the Tensor Product–Finite Generation Lemma 331.9. First of all, by hypothesis we know that C is isomorphic, as an S-left module, to a direct sum of finitely many copies of S. It now follows from the Tensor–Direct Sum Lemma 312.15 that $M \otimes_S C$ is isomorphic, as an R-left module, to a direct sum of finitely many copies of M. Since M is finitely generated we see that $M \otimes_S C$ is a finitely generated R-left module. ■

Proof of the Twisted Invariants–Finite Generation Proposition 331.6. In the following we only prove the equality regarding homology with twisted coefficients. The proof for cohomology with twisted coefficients is very similar. The details will be worked out in Exercise 331.5.

We proceed in three steps:

(a) First we deal with the case that X is a finite CW-complex and $A = \varnothing$.
(b) Then we deal with the case that X is homotopy equivalent to a retract of a finite CW-complex.

(c) Finally we deal with the relative setting.

So let us deal with these three steps:

(a) Let (X, x_0) be a pointed connected CW-complex topological space, let R be a left Noetherian ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module that is finitely generated as an R-left module.

We see that we have the following isomorphisms of R-left modules:

$$\begin{array}{ccccc} \mathrm{H}_k^{\mathrm{tw}}(X; M) & \cong & \mathrm{H}_k^{\mathrm{tw,CW}}(X; M) & = & \mathrm{H}_k(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})) \\ & \uparrow & & \uparrow & \\ & \text{follows from the Singular-Cellular} & & \text{Definition 327.11} & \\ & \text{Twisted Isomorphism Theorem 328.1} & & & \end{array}$$

We make a few observations:

(i) By the Regular Covering–Cellular Chain Complex Lemma 327.10 (1) we know that each $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X})$ is a free $\mathbb{Z}[\pi]$-left module. Since X is assumed to be compact we obtain from the Regular Covering–Cellular Chain Complex Lemma 327.10 (1b) and the CW-Complex Compactness Corollary 136.17 that each $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X})$ is a finitely generated free $\mathbb{Z}[\pi]$-left module.

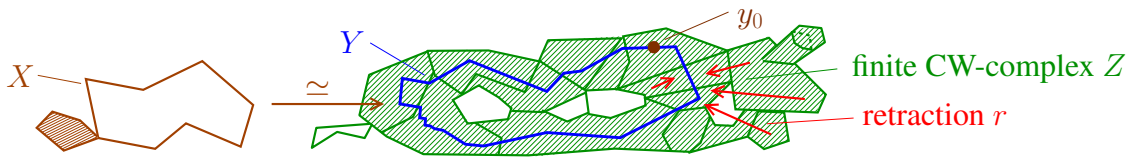
(ii) Since M is a finitely generated R-left module we obtain from (i) and the Finite Generation Lemma 331.8 that each $\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$ is a finitely generated R-left module.

(iii) It follows from (ii) and the Finite Generation Lemma 331.8 (2) that each

$$\ker\left(\mathrm{id} \otimes \partial_k \colon \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}) \to \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X})\right)$$

is also a finitely generated R-left module. Thus we see that $\mathrm{H}_k^{\mathrm{tw}}(X; M)$ is a quotient of a finitely generated R-left module. It follows from the Finite Generation Lemma 331.8 (1) that $\mathrm{H}_k^{\mathrm{tw}}(X; M)$ is also a finitely generated R-left module.

(b) We now suppose that X is homotopy equivalent to a retract Y of a finite CW-complex Z. First we show that Y has the desired result. So let $y_0 \in Y$ and let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module that is finitely generated as an R-left module. We denote by $i \colon Y \to Z$ the inclusion and we denote by $r \colon Z \to Y$ the retraction.



We consider the following little diagram:

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{tw}}(Y; \overbrace{(r \circ i)^* M}^{=\mathrm{id}^* M = M}) & \xrightarrow{(r \circ i)_* = \mathrm{id}_* = \mathrm{id}} & \mathrm{H}_k^{\mathrm{tw}}(Y; M) \\ & {\scriptstyle i_*} \searrow \quad \nearrow {\scriptstyle r_*} & \\ & \mathrm{H}_k^{\mathrm{tw}}(Z; r^* M) & \end{array}$$

Note that it follows from the Twisted Homology-Functoriality Proposition 321.9 that the diagram commutes. Thus we see that $\mathrm{H}_k^{\mathrm{tw}}(Y; M)$ is isomorphic to a quotient of $\mathrm{H}_k^{\mathrm{tw}}(Z; r^* M)$, which by (a) we know to be a finitely generated R-left module. It thus

follows again from the elementary Finite Generation Lemma 331.8 (1) that $\mathrm{H}_k^{\mathrm{tw}}(Y;M)$ itself is a finitely generated R-left module.

Finally note that it follows from the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1 that the twisted homology modules of X are also finitely generated.

(c) Finally let (X,x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $A \subseteq X$ be a subspace such that each component of A is also locally contractible. We assume that X and A are both homotopy equivalent to a retract of a finite CW-complex. Note that this implies in particular that A consists of finitely many components $A_1,\dots,A_m$.

Claim. There exists a long exact sequence of R-left modules of the form

$$\cdots \to \bigoplus_{i=1}^{m} \mathrm{H}_k^{\mathrm{tw}}(A_i;M) \;\to\; \mathrm{H}_k^{\mathrm{tw}}(X;M) \;\to\; \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \xrightarrow{\partial_k} \dots$$

Proof. This claim follows easily from the Internal Twisted-LES-Proposition 325.9 together with the Internal Homology–Components Lemma 325.6 and the Change-of-Covering Lemma 321.11. We leave it to the reader to fill in the details. ⊞

Now note that it follows from (b) that in the above long exact sequence the "absolute" homology modules are all finitely generated. It now follows easily from the Finite Generation Lemma 331.8 that the relative homology modules are also finitely generated. ■

331.3. **Twisted (co-) homology and the Euler characteristic.** We start out with a slight extension of Definition 156.1:

Definition 331.10. Given a finite CW-complex X and a subcomplex A of X we define its Euler characteristic $\chi(X,A)$ as follows:

$$\chi(X,A) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells of } X \text{ that are not contained in } A.$$

Evidently we write $\chi(X) := \chi(X,\varnothing)$.

The argument of the Homology-χ-Proposition 156.2 shows that given a finite CW-complex X and a subcomplex A we have the following relation between the Euler characteristic (X,A) and its homology groups $\mathrm{H}_n(X,A)$:

$$\sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{rank}(\mathrm{H}_n(X,A)) \;=\; \chi(X,A).$$

Note that the left hand side of the above equality is also defined for many topological spaces that are not CW-complexes. This thought lead us to Definition 156.6, which we now recall:

Definition 331.11. Let X be a topological space and let $A \subseteq X$ be a subspace. If $\sum_{n\in\mathbb{N}_0} \mathrm{rank}(\mathrm{H}_n(X,A))$ is finite, then we define the Euler characteristic of (X,A) to be

$$\chi(X,A) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{rank}(\mathrm{H}_n(X,A)).$$

Again we write $\chi(X) := \chi(X,\varnothing)$.

Example. Let X be a compact topological manifold. It follows from the Topological Manifold–Boundary Proposition 23.15 (1) and (2), the Twisted Invariants–Finite Generation

Proposition 331.6 and the Twisted–Trivial Coefficients Lemma 320.13 [3602] that the Euler characteristics $\chi(X)$ and $\chi(X, \partial X)$ are defined. □

We can now generalize the above results to twisted homology for suitable coefficients:

Proposition 331.12. (Twisted Homology-χ-Proposition) We assume that we are in the following setup:

- Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $A \subseteq X$ be a subspace such that each component of A is also locally contractible. We assume that X and A are both homotopy equivalent to a finite CW-complex.
- Let $\mathbb{K}$ be a skew field and let M be a $(\mathbb{K}, \mathbb{Z}[\pi_1(X, x_0)])$-left left module which is finite-dimensional as a $\mathbb{K}$-left vector space.

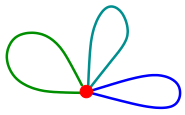
Then the following two equalities hold:

$$\begin{aligned} \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X, A; M)) &= \dim_{\mathbb{K}}(M) \cdot \chi(X, A) \\ \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}^n_{\mathrm{tw}}(X, A; M)) &= \dim_{\mathbb{K}}(M) \cdot \chi(X, A). \end{aligned}$$

Proof. If X is a finite CW-complex and $A \subseteq X$ is a subcomplex, then the proof of the Twisted Homology-χ-Proposition 331.12 is similar to the previous, similar sounding results, like the Homology-χ-Proposition 160.15. In the general setting the argument does become a little technical. Thus we put the proof of the Twisted Homology-χ-Proposition 331.12 into a separate section, namely Section 331.4. ■

Example. As a proof of concept we provide one example. Thus let $X = S^1 \vee \cdots \vee S^1$ be the wedge of k circles, let $x_0 \in X$ and finally let $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a one-dimensional non-trivial representation. We consider the $(\mathbb{C}, \mathbb{Z}[\pi_1(X, x_0)])$-left left module ${}_{\varphi}\mathbb{C}$ from Notation 320.8. We view X as a CW-complex in the obvious way.

$X = S^1 \vee S^1 \vee S^1$
with $\chi(X) = -2$



$$\begin{aligned} \mathrm{H}_2^{\mathrm{tw}}(S^1 \vee S^1 \vee S^1; {}_{\varphi}\mathbb{C}) &= 0 \\ \mathrm{H}_1^{\mathrm{tw}}(S^1 \vee S^1 \vee S^1; {}_{\varphi}\mathbb{C}) &\cong \mathbb{C}^2 \\ \mathrm{H}_0^{\mathrm{tw}}(S^1 \vee S^1 \vee S^1; {}_{\varphi}\mathbb{C}) &= 0 \end{aligned}$$

We make the following observations:

(2) Since X has no cells of dimension ≥ 2 we obtain from the discussion on page 6262 that $\mathrm{H}_k^{\mathrm{tw}}(X; {}_{\varphi}\mathbb{C}) = 0$ for $k \geq 2$.

(0) Since φ is a one-dimensional non-trivial representation we obtain easily from the Twisted H_0-Proposition 323.3 that $\mathrm{H}_0^{\mathrm{tw}}(X; {}_{\varphi}\mathbb{C}) = 0$.

(1) Clearly the Euler characteristic of X equals $1 - k$. Thus we obtain from the above calculations together with the Twisted Homology-χ-Proposition 331.12 that we have an isomorphism $\mathrm{H}_1^{\mathrm{tw}}(X; {}_{\varphi}\mathbb{C}) \cong \mathbb{C}^{k-1}$.

This of course generalizes the calculation from page 6283. □

Again the question arises, whether the Twisted Homology-χ-Proposition 331.12 applies to compact topological manifolds. Going back to the Manifold–CW Complex Theorem 331.7

[3602] Alternatively, as we already pointed out in Definition 190.22, one could also appeal to the Topological Manifolds-Invariants Proposition 189.14 (2) and (3), together with the Relative-H_*-Finite Generation Lemma 179.11.

we see that it is strong enough to take care of compact *smooth* manifolds. But it is not strong enough to satisfy the hypothesis of the Twisted Homology-χ-Proposition 331.12 for compact *topological* manifolds. Thus we will need the following strengthened version of the Manifold–CW Complex Theorem 331.7:

Theorem 331.13. (Manifold–CW Complex Theorem)
(1) Every compact smooth manifold is homotopy equivalent to a finite CW-complex.
(2′) Every compact topological manifold is homotopy equivalent to a finite CW-complex.

Proof.

(2′) Let M be a compact topological manifold. Using Borsuk's Theorem 189.7 and the Finite CW-Homotopy Type Theorem 190.10 (4)⇒(3) we proved the Compact Manifold–Homotopy Type Corollary 190.12 which tells us that M is indeed homotopy equivalent to a finite CW-complex. Note though that the situation is not entirely satisfactory, since we did not even come close to proving the Finite CW-Homotopy Type Theorem 190.10. ■

331.4. **Proof of the Twisted Homology-χ-Proposition 331.12.** We start out with the following purely algebraic lemma which is a variation on the Chain Complex–H_*–χ-Lemma 156.4:

Lemma 331.14. (Chain Complex–H_*–χ-Lemma) Let $\mathbb{K}$ be a skew field and let

$$C_* \;:=\; 0 \;\to\; C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots\; C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

be a chain complex of finite-dimensional $\mathbb{K}$-left vector spaces. The following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{K}}(C_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{K}}(H_n(C_*)).$$

Proof of the Chain Complex–H_*–χ-Lemma 331.14. The proof of the lemma is essentially identical to the proof the Chain Complex–H_*–χ-Lemma 156.4, except that we need to replace the Rank Additivity Proposition 97.18 by the Module-over-Skew Field Proposition 309.33 (4). ■

Proof of the Twisted Homology-χ-Proposition 331.12. In the following we only prove the equality regarding homology with twisted coefficients. The proof for cohomology with twisted coefficients is very similar. Once again we leave it to the reader to make the necessary changes. For example one will have to replace the Free Module–Tensor-with-Ring Lemma 312.18 by the Free Module–Hom-to-Ring Lemma 315.8.

As in the proof of the Twisted Invariants–Finite Generation Proposition 331.6 we proceed in three steps:

(a) First we deal with the case that X is a finite CW-complex and $A = \varnothing$.
(b) Then we deal with the case that X is homotopy equivalent to a finite CW-complex and that $A = \varnothing$.
(c) Finally we deal with the relative setting.

We now go through these three steps.

(a) First we assume that X itself is a finite CW-complex. In this case the proof of the Twisted Homology-χ-Proposition 331.12 is almost identical to the proof of the

Homology-χ-Proposition 156.2. But for fun, let us go through the motions. We introduce some notation: We write $\pi := \pi_1(X, x_0)$, $k := \dim(X)$ and as usual we denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We perform the following calculation:

$$\begin{aligned}
\dim_{\mathbb{K}}(M)\cdot\chi(X) &\overset{\text{by Definition 331.10}}{=} \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{K}}(M) \cdot \#n\text{-cells} \\
&= \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw,CW}}(X; M)) \\
&\overset{\text{follows from the Chain Complex–H}_*\text{–}\chi\text{-Lemma 331.14}}{=} \\
&\overset{\text{follows from the Cellular Chain Complex–Free Module Lemma 327.10 (1b), together with the Tensor–Direct Sum Lemma 312.15 and the Free Module–Tensor-with-Ring Lemma 312.18 (1)}}{=} \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{K}}(\overline{M} \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_n^{\mathrm{CW}}(\widetilde{X}_{x_0})) \\
&\overset{\text{by the Singular-Cellular Twisted Isomorphism Theorem 328.1}}{=} \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X; M)).
\end{aligned}$$

(b) Now we assume that X is homotopy equivalent to a finite CW-complex Y. We pick a base point $y_0 \in Y$. By the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1 there exists a $(\mathbb{K}, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module, which as a $\mathbb{K}$-left vector space is isomorphic to M, such that for all $n \in \mathbb{N}_0$ the $\mathbb{K}$-left vector spaces $\mathrm{H}_n^{\mathrm{tw}}(X; M)$ and $\mathrm{H}_n^{\mathrm{tw}}(Y; N)$ are isomorphic. Now we see that

$$\begin{aligned}
\sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X; M)) &\overset{\text{by the choice of } N}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(Y; N)) \\
&= \dim_{\mathbb{K}}(N) \cdot \chi(Y) \\
&\overset{\text{by (a), since } Y \text{ is a finite CW-complex}}{=} \\
&= \dim_{\mathbb{K}}(M) \cdot \chi(X). \\
&\overset{\text{since } M \text{ and } N \text{ are isomorphic } \mathbb{K}\text{-left vector spaces}}{=} \\
&= \dim_{\mathbb{K}}(N) \cdot \chi(X) \\
&\overset{\text{follows from the Homotopy Type-}\chi\text{-Lemma 156.7}}{=}
\end{aligned}$$

(c) We now deal with the general setting. Thus let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $A \subseteq X$ be a subspace such that each component of A is locally contractible. We assume that X and A are both homotopy equivalent to a finite CW-complex. This implies in particular that A has finitely many components $A_1, \ldots, A_m$.

As on page 6308 we see that there exists a long exact sequence of $\mathbb{K}$-left modules of the form

$$\cdots \to \bigoplus_{i=1}^{m} \mathrm{H}_k^{\mathrm{tw}}(A_i; M) \to \mathrm{H}_k^{\mathrm{tw}}(X; M) \to \mathrm{H}_k^{\mathrm{tw}}(X, A; M) \xrightarrow{\partial_k} \ldots$$

We now see that

$$\begin{aligned}
\sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X, A; M)) &\overset{\text{follows easily from the above long exact sequence together with the Chain Complex–H}_*\text{–}\chi\text{-Lemma 331.14}}{=} \\
&= \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(X; M)) - \sum_{i=1}^{m} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{K}}(\mathrm{H}_n^{\mathrm{tw}}(A_i; M)) \\
&= \dim_{\mathbb{K}}(M) \cdot \chi(X) - \sum_{i=1}^{m} \dim_{\mathbb{K}}(M) \cdot \chi(A_i) \\
&\overset{\text{follows from (b)}}{=} \\
&= \dim_{\mathbb{K}}(M) \cdot \chi(X, A). \\
&\overset{\text{by the Relative-}\chi\text{-Lemma 156.8}}{=}
\end{aligned}$$

■

331.5. **Appendix: Wall's Finiteness Theorem.** Note that it follows from the Singular-Cellular Twisted Isomorphism Theorem 328.1 together with the Homotopy Equivalence-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 322.1 and the discussion on page 6262 that if a connected topological space X is homotopy equivalent to an n-dimensional CW-complex, then all twisted (co-) homology groups of X of dimension $> n$ vanish. It is natural to ask whether the converse to this statement holds. It turns out that in almost all dimensions this is indeed the case. More precisely we have the following topological analog of the H^*-Dimension–Proposition 317.22:

Theorem 331.15. (Wall's Finiteness Theorem) Let X be a (finite) connected non-empty CW-complex with universal covering $\widetilde{X}$. We assume that there exists an $r \in \mathbb{N}_0$ such that the following conditions are satisfied:

- For all $s > r$ we have $\mathrm{H}_s(\widetilde{X}) = 0$.
- For all $s > r$ all twisted cohomology modules $\mathrm{H}^s_{\mathrm{tw}}(X; M)$ vanish.

If $r \geq 3$ or if $r = 1$, then X is homotopy equivalent to a (finite) *r-dimensional* CW-complex.

Proof. For $r \geq 3$ the theorem was proved by C.T.C. Wall [**Wall1966a**, Theorem 5 and Corollary 5.1] (see also [**Wall1965a**, Theorem E] and [**Var1989**, Chapter 6.2.5]). Now we turn to the case that $r = 1$. In this case the hypotheses, together with the Hurewicz Theorem 167.9 imply in particular that X is aspherical, i.e. X is an Eilenberg-Maclane space. Thus the case $r = 1$ is equivalent to the Stallings-Swan Theorem 318.19, in other words, the case $r = 1$ was proved by John Stallings [**Stal1968**, Theorem 6.8] (for finite CW-complexes) and independently by Richard Swan [**Swa1969**] (for arbitrary CW-complexes). ■

Similar to the discussion on Section 318.5 we see that the case $r = 2$ is an open problem. More precisely we have the following conjecture, which is closely related to the Eilenberg-Ganea Conjecture 318.20:

Conjecture 331.16. (Wall's D2-Conjecture) Let X be a finite connected non-empty CW-complex with universal covering $\widetilde{X}$. We assume that the following two conditions hold:

- For all $s > 2$ we have $\mathrm{H}_s(\widetilde{X}) = 0$.
- For all $s > 2$ all twisted cohomology modules $\mathrm{H}^s_{\mathrm{tw}}(X; M)$ vanish.

Then X is homotopy equivalent to a finite 2-dimensional CW-complex.

Remark. Note that the Bestvina–Brady Theorem 318.22 implies that Wall's D2-Conjecture 331.16 is not compatible with the Whitehead Conjecture 318.21. □

331.6. **Exercises for Chapter 331.**

Exercise 331.1. Show that there exists a connected finite-dimensional CW-complex X such that all rational homology groups $\mathrm{H}_k(X; \mathbb{Q})$ are finite-dimensional, but such that there exists a representation $\varphi\colon \pi_1(X, x_0) \to \mathrm{GL}(d, \mathbb{Q})$, such that at least one of the corresponding twisted homology modules $\mathrm{H}^{\mathrm{tw}}_k(X; {}_\varphi\mathbb{Q}^d)$ is infinite-dimensional.
Hint. You could pick a suitable mapping torus.

Exercise 331.2. Let $\pi = \langle x, y \rangle$ be the free group on two generators. Is the group ring $\mathbb{Z}[\pi]$ Noetherian?

Exercise 331.3. Let G be any finite group. Show that the group ring $\mathbb{Z}[G]$ is Noetherian.

Exercise 331.4. Let R be a left Noetherian ring. Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-left modules and if two are finitely generated, then so is the third.

Exercise 331.5.

(a) Formulate and prove a "Hom"-analog of the Tensor Product–Finite Generation Lemma 331.9.

(b) Use (a) to prove the cohomological statement of the Twisted Invariants–Finite Generation Proposition 331.6.

Exercise 331.6. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and let $\varphi\colon \pi_1(X, x_0) \to \langle t \rangle$ be an epimorphism.

(a) Show that for each commutative ring S there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_{k+1}(X; S) \to \mathrm{H}_k^{\mathrm{tw}}(X; {}_{\varphi}S[t^{\pm 1}]) \xrightarrow{(t-1)\cdot} \mathrm{H}_k^{\mathrm{tw}}(X; {}_{\varphi}S[t^{\pm 1}]) \to \mathrm{H}_k(X; S) \to \ldots$$

(b) Suppose that X is finite CW-complex which satisfies $\mathrm{H}_*(X; \mathbb{Q}) \cong \mathrm{H}_*(S^1; \mathbb{Q})$. Show that each $\mathbb{Q}[t^{\pm 1}]$-module $\mathrm{H}_k^{\mathrm{tw}}(X; {}_{\varphi}\mathbb{Q}[t^{\pm 1}])$ is a torsion $\mathbb{Q}[t^{\pm 1}]$-module.

Exercise 331.7. Let M be a closed orientable connected non-empty 4-dimensional smooth manifold with an isomorphism $\alpha\colon \pi_1(M) \to \langle t \rangle$. We assume that M has precisely one cell in dimension $0, 1, 2, 3, 4$.

(a) How many of the ordinary homology groups with $\mathbb{C}$-coefficients can you calculate?

(b) We consider the non-trivial representation $\varphi\colon \pi_1(M) \to \mathrm{GL}(1, \mathbb{C})$ that is induced by α and by $t \mapsto (-1)$. How many of the twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}(M; {}_{\varphi}\mathbb{C})$ can you calculate?

CHAPTER 332

Handle-complexes

In Chapter 327 we introduced twisted cellular (co-) homology. Our next goal is to introduce twisted handle (co-) homology. But to make sense of this we first need to introduce the notion of a handle-complex which is the love child of a CW-complex and a handle decomposition of a compact smooth manifold. This chapter is completely dedicated to the introduction of handle-complexes and the corresponding (untwisted) handle chain complexes and handle homology groups. Only in the next Chapter 333 will we return to twisted chain complexes.

Many of the concepts and results in this chapter are generalizations of earlier concepts and results from the setting of a compact smooth manifold with a (necessarily finite) handle decomposition to the more general notion of a handle-structure on a (possibly non-compact) smooth manifold.

This chapter is organized as follows:

- In Section 332.1 we will recall the notion of a handle attachment.
- In Section 332.2 we will introduce handle-complexes.
- In Section 332.3 we will introduce the handle chain complex.
- In Section 332.4 we will recall the notion of the intersection numbers between oriented submanifolds.
- In Section 332.5 we will see how one can calculate the boundary maps in the handle chain complex using algebraic intersection numbers.
- In Section 332.6 we will see that for certain handle complexes we can calculate the handle complex of the universal covering via Fox calculus.
- In Section 332.7 we will provide exercises for Chapter 332.

In the next Chapter 333 we will introduce twisted handle (co-) homology.

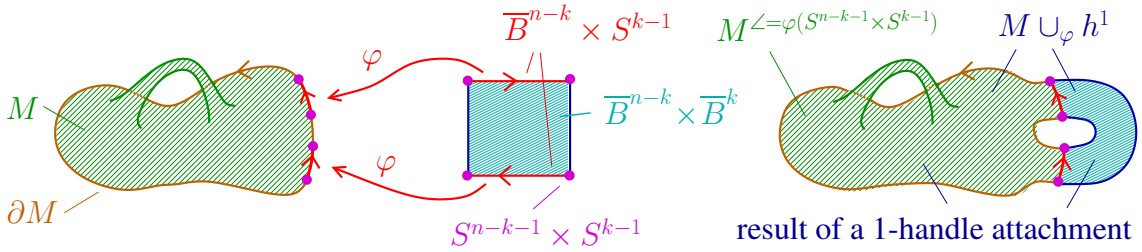
332.1. **Handle attachments and handle-structures.** We start out by recalling parts of Definition 246.4:

Definition 332.1. Let M be an n-dimensional smooth manifold. Suppose we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$.[3603]

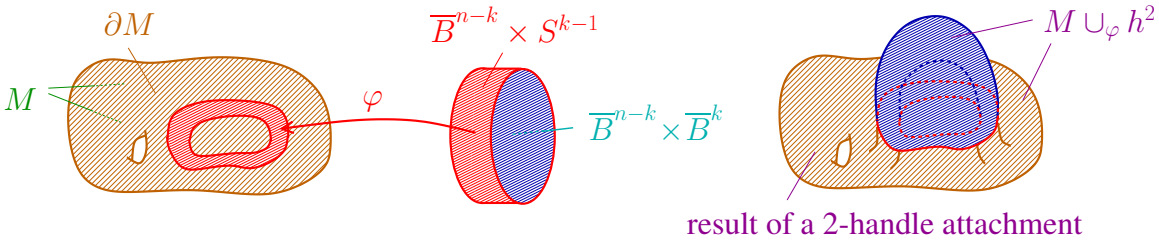
(1) We say that[3604]
$$M \cup_\varphi h^k \;:=\; M \cup_{\partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \left(\overline{B}^{n-k} \times \overline{B}^k\right)$$
is obtained from M by attaching the k-handle $\overline{B}^{n-k} \times \overline{B}^k$ along the attaching map φ. In particular we refer to the image of $\overline{B}^{n-k} \times \overline{B}^k$ in $M \cup_\varphi h^k$ as a k-handle or as a handle of index k.

(2) Let W be a smooth manifold with submanifold M. We say W is obtained through a k-handle attachment of M if there exists an attaching map $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ and a diffeomorphism $M \cup_\varphi h^k \to W$ which is the identity on M.

Example. In the figure below we consider the case $n = 2$ and $k = 1$, i.e. we show a 2-dimensional smooth manifold together with a 1-handle attachment:



In the figure below we consider the case $n = 3$ and $k = 2$, i.e. we show a 3-dimensional smooth manifold together with a 2-handle attachment:



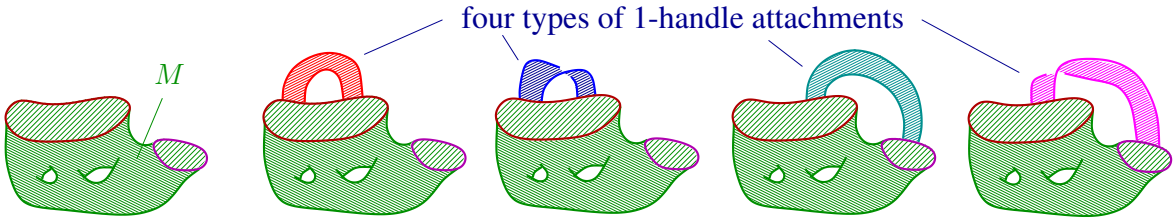
In the next example we will discuss handles more systematically. □

Example. Let M be a compact n-dimensional smooth manifold. Ideally one would like to classify the possible handle attachments. We repeat a few basic observations from 4748:

(0) First we consider the extreme case of attaching a 0-handle. It follows basically from the definitions and the fact that $S^{-1} = \varnothing$ that attaching a 0-handle is the same as taking the disjoint union with $\overline{B}^n$.



(1) We turn to attachments of 1-handles. The figure below illustrates the fact that in general the effect of attaching a 1-handle depends on the choice of the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$.

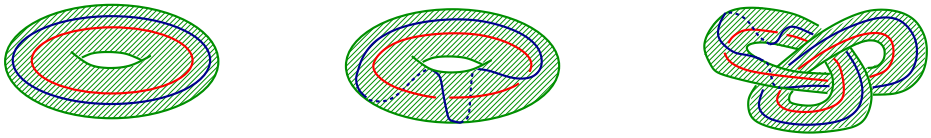


(2) In general there is no straightforward classification of all possibilities to attach a k-handle with $k \geq 2$. For example we consider the smooth manifold $M = \overline{B}^4$ with boundary $\partial\overline{B}^4 = S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. In the figure below we show three different smooth embeddings $\overline{B}^2 \times S^1 \to \mathbb{R}^3 \subseteq S^3$. More precisely, we show the images in green, and we

[3603] In Definition 246.2 we referred to such a smooth embedding as a *thickened* $(k-1)$*-sphere*.

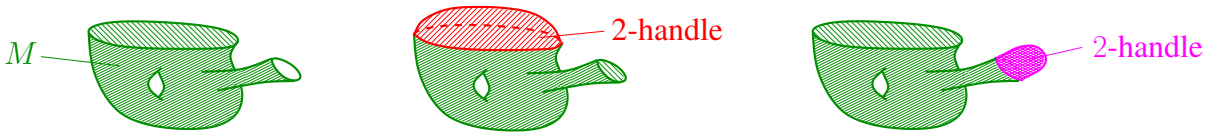
[3604] In the Handle Attachment Proposition 246.3 we studied many properties of $M \cup_\varphi h^k$. In particular we discussed in what sense it is again a smooth manifold.

show in red the image of $\{(0,0)\} \times S^1$ and we show in blue the image of $\{(1,0)\} \times S^1$. Note that the two maps to the left have the same image, but the maps are different.



Any such thickened sphere describes a 2-handle attachment to $\overline{B}^4$. There is no reason to believe that all these 2-handle attachments can be classified easily.

(n) Finally we turn to the other extreme, namely to attaching an n-handle to an n-dimensional smooth manifold. Note that any smooth embedding $\varphi\colon \overline{B}^0 \times S^{n-1} \to \partial M$ is a diffeomorphism onto a component of ∂M.

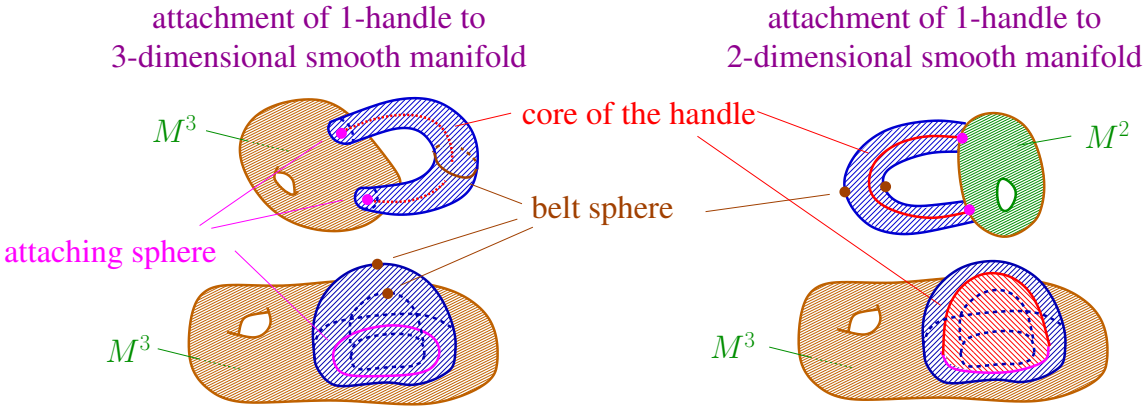


Thus an n-handle attachment is essentially the same as gluing an n-ball to a boundary component of M that is diffeomorphic to S^{n-1} via the diffeomorphism φ. □

Next we recall more concepts from Definition 246.4:

Definition 332.2. Let M be an n-dimensional smooth manifold. Suppose we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$.

(1) We refer to the image of $\{0\} \times S^{k-1}$ in ∂M as the attaching sphere of the handle.
(2) We refer to the image of $\{0\} \times \overline{B}^k$ in $M \cup_\varphi h^k$ as the core of the handle.
(3) We refer to the image of $S^{n-k-1} \times \{0\}$ in $\partial(M \cup_\varphi h^k)$ as the belt sphere of the handle.



In the following we will also want to study non-compact smooth manifolds (e.g. universal coverings of compact smooth manifolds). At that stage we will want to attach infinitely many handles at once. Some care needs to be taken to make sense of this concept:

Definition 332.3. Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold. Furthermore let $k \in \{0, \ldots, n\}$ and let $\{\varphi_i\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M\}_{i \in I}$ be a family of smooth

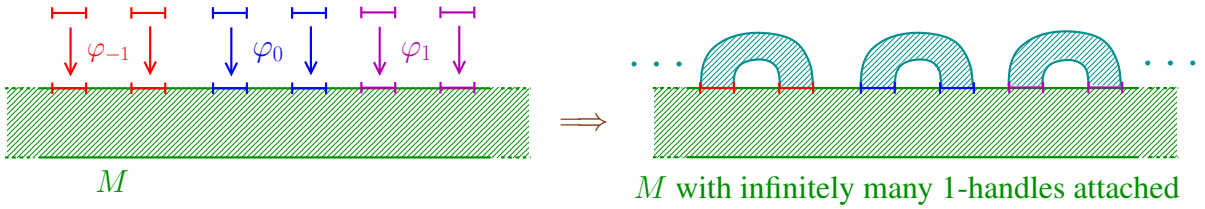
embeddings such that the resulting map

$$\bigcup_{i\in I}\varphi_i\colon \bigcup_{i\in I}(\overline{B}^{n-k}\times S^{k-1})\to\partial M$$

is a closed[3605] smooth embedding. As in the Handle Attachment Proposition 246.3[3606] we see that

$$M\cup_{M\leftarrow \bigsqcup_{i\in I}\overline{B}^{n-k}\times S^{k-1}\,:\bigsqcup_{i\in I}\varphi_i}\bigsqcup_{i\in I}\overline{B}^{n-k}\times\overline{B}^{k}$$

is again a smooth manifold.



332.2. **Handle-complexes.** In Definition 136.1 and Definition 327.1 we introduced the notion of a CW-complex. We now provide an analog in the setting of smooth manifolds:

Definition 332.4. A handle-complex M rel A is an n-dimensional smooth manifold M together with a subspace $A\subseteq\partial M$ which is a union of components of ∂M and together with a filtration

$$M^{-1}\subseteq M^0\subseteq M^1\subseteq\ldots\subseteq M$$

by closed subspaces, each of which has the structure of an n-dimensional smooth manifold[3607] such that the following two conditions are satisfied:

(a) There exists a diffeomorphism $\Phi\colon[0,1]\times A\xrightarrow{\cong}M^{-1}$ such that for each $a\in A$ we have $\Phi(0,a)=a$.

(b) For every choice of $k\in\{0,\ldots,n-1\}$ there exists a set I together with a family $\{\varphi_i\colon\overline{B}^{n-k}\times S^{k-1}\to\partial M^{k-1}\backslash(\{0\}\times A)\}_{i\in I}$ of smooth embeddings as in Definition 332.3 and a diffeomorphism

$$M^{k-1}\cup_{M^{k-1}\leftarrow\bigsqcup_{i\in I}\overline{B}^{n-k}\times S^{k-1}\,:\bigsqcup_{i\in I}\varphi_i}\bigsqcup_{i\in I}\overline{B}^{n-k}\times\overline{B}^{k}\ \to\ M^k$$

which is the inclusion on M^{k-1}.

Definition 332.5. Let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. A handle-structure on M rel A is a handle-complex as in Definition 332.4.

Before we provide examples let us introduce the following notions, which are inspired by Definition 327.2:

[3605]Recall that according to Definition 4.12 a map between topological spaces is called *closed* if the image of each closed subset is again a closed subset.

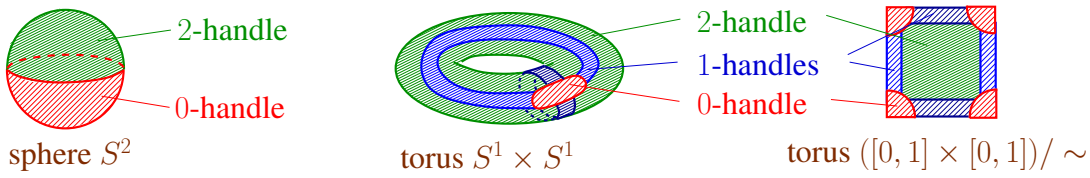
[3606]In the proof of the Handle Attachment Proposition 246.3 we made use of various results, like the Smooth Collar Neighborhood Theorem 59.7, the Introducing Corner Proposition 61.4 and the Double Collar Theorem 61.2. These earlier results generalize to smooth embeddings with closed images, thus these results generalize to the current setting, where we allow possibly infinitely many handles.

[3607]Note that we do *not* demand that the M^k are smooth submanifolds of M.

Definition 332.6. Using the notation from above we introduce the following language:
(1) Given $k \in \{-1, \ldots, n\}$ we refer to M^k as the k-skeleton of M.
(2) The components of $M^k \setminus M^{k-1}$ are called the open k-handles of M.
(3) For each open k-handle σ of M we refer to its closure $\overline{\sigma}$ in M as an k-handle of M.
(4) We say that the handle-complex is finite if it has only finitely many handles.

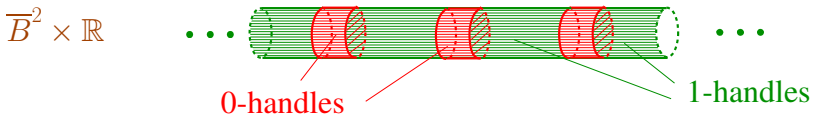
Example.

(1) Similar to the discussion on page 6252 we consider in the figure below the sphere S^n and the torus $S^1 \times S^1$, but this time with a handle-structure instead of a CW-structure.

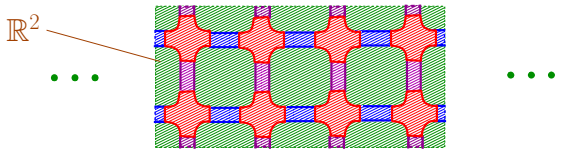


(2) Let M be a compact smooth manifold and let A be a union of components of ∂M. Any standard handle decomposition for M rel A, in the sense of Definition 247.1 and Definition 247.6, clearly gives rise to a handle-structure on M rel A as above.

(3) In contrast to earlier we now allow non-compact smooth manifolds. For example, as we illustrate in the figure below, the infinite tube $\overline{B}^{n-1} \times \mathbb{R}$ has a handle-structure consisting of $\mathbb{Z}$-many 0-handles and $\mathbb{Z}$-many 1-handles:

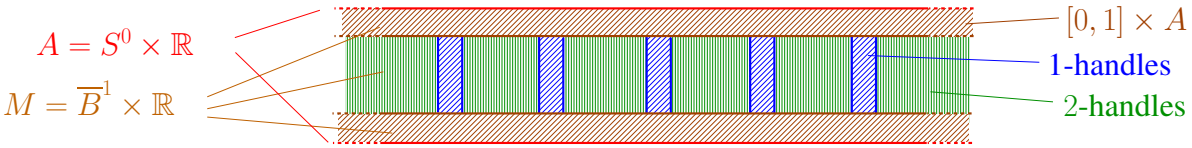


Furthermore for $\mathbb{R}^2$ the following figure gives us an infinite handle-structure:



Taking products we obtain in a similar fashion handle-structures for all smooth manifolds of the form $\overline{B}^m \times \mathbb{R}^n$.

(4) Finally we want to provide an example of a non-compact handle-complex M together rel some non-empty A: More precisely, we now consider $M = \overline{B}^1 \times \mathbb{R}$ with $A = S^0 \times \mathbb{R}$.



The figure above illustrates that in this case we have a handle-structure rel A with $\mathbb{Z}$-many 1-handles and $\mathbb{Z}$-many 2-handles. □

Remark. We do not claim that every smooth manifold admits a handle-structure. But we have two existence results which are good enough for our purposes:

- By the Morse Function-Existence Theorem 249.26 we know that for every compact smooth M, together with a union of boundary components $A \subseteq \partial M$, there exist a Morse

function rel A which by the Morse Function–Handle Decomposition Theorem 250.6 gives rise to a handle decomposition of M rel A.
- In the Handle-Complex Covering Proposition 333.1 we will see that every covering of a handle-complex is again naturally a handle-complex. □

Finally, to state our next results it is convenient to introduce the following analog of Definition 136.7 and Definition 327.5:

Definition 332.7. Let M be an n-dimensional handle-complex rel some $A \subseteq \partial M$.

(1) Given $k \in \{0, \dots, n\}$ we denote by $\mathrm{Handle}_k(M)$ the set of k-handles of M. Furthermore we denote by $\mathrm{Handle}(M) := \bigcup_{k=0}^{n} \mathrm{Handle}_k(M)$ the set of all handles of M.

(2) A full set of characteristic maps for the k-handles of the handle-complex M is defined as a family $\{\Phi_h \colon \overline{B}^{n-k} \times \overline{B}^k \to M^k\}_{h \in \mathrm{Handle}_k(M)}$ of smooth embeddings[3608] with the following properties:
- For each k-handle h we have $\Phi_h(\overline{B}^{n-k} \times S^{k-1}) \subseteq \partial M^{k-1}$.
- For each k-handle h we have $\Phi_h(\overline{B}^{n-k} \times \overline{B}^k) = h$.
- The map
$$M^{k-1} \cup_{M^{k-1} \leftarrow \bigsqcup_{i \in I} \overline{B}^{n-k} \times S^{k-1} \,:\, \bigsqcup_{i \in I} \Phi_h|_{\overline{B}^{n-k} \times S^{k-1}}} \bigsqcup_{i \in I} \overline{B}^{n-k} \times \overline{B}^k \;\to\; M^k$$
which is induced by the identity on M^{k-1} and by the maps Φ_h is well-defined and a diffeomorphism.

Note that once we have a full set of characteristic maps, then we can use Definition 332.2 to speak of the core, belt sphere and attaching sphere of each handle.

332.3. **The handle chain complex.** Next we slightly generalize Definition 252.3 of a handle chain complex:[3609]

Notation 332.8. Let M be a handle-complex rel $A \subseteq \partial M$.

(1) Given $k \in \mathbb{N}_0$ we denote as in Definition 332.6 by M^k the k-skeleton of M.

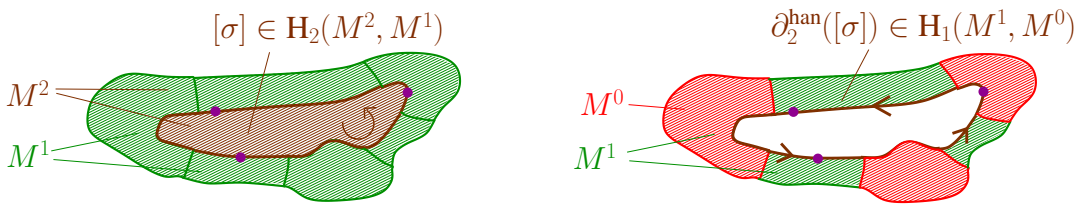
(2) Similar to Definition 252.3 we set
$$\mathrm{C}_k^{\mathrm{han}}(M, A) \;:=\; \mathrm{H}_n(M^n, M^{n-1}).$$

(3) Given $k \in \mathbb{N}_0$ we denote by $\partial_k^{\mathrm{han}}$ the map
$$\mathrm{H}_k(M^k, M^{k-1}) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(M^{k-1}, A) \xrightarrow{j_{k-1}} \mathrm{H}_{k-1}(M^{k-1}, M^{k-2})$$
$$\text{with } \partial_k = j_{k-1} \circ \partial_k$$
where the first map is the connecting homomorphism for the triple (M^k, M^{k-1}, A). The argument of the Handle Boundary Map Lemma 252.2 (which is almost the same as the argument of the Cellular Boundary Map Lemma 155.4) shows that $\partial_k^{\mathrm{han}} \circ \partial_{k+1}^{\mathrm{han}}$ is always the zero map. In other words, $(\mathrm{H}_*^{\mathrm{han}}(M, A), \partial_*^{\mathrm{han}})$ is a chain complex, which we refer to as the handle chain complex.

(4) The corresponding homology groups are the handle homology groups $\mathrm{H}_n^{\mathrm{han}}(M, A)$.

[3608] Recall that the smooth structure of the M^j is part of the data of a handle-complex, so the notion of a smooth embedding does indeed make sense.

[3609] Clearly the following definition is the handle analog of Definition 252.3 and Notation 327.7.

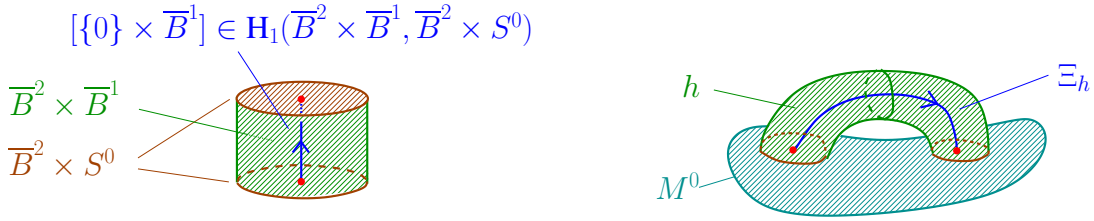


Our next goal is to describe a basis for the handle chain complex. To do so we introduce the following almost self-explanatory notation:

Notation 332.9. Let $n \in \mathbb{N}_0$ and let $k \in \{0, \dots, n\}$.

- We denote by $[\overline{B}^k] \in \mathrm{H}_k(\overline{B}^k, S^{k-1})$ the standard fundamental class which we introduced in Definition 155.11.
- We denote by $[\{0\} \times \overline{B}^k] \in \mathrm{H}_n(\overline{B}^{n-k} \times \overline{B}^k, \overline{B}^{n-k} \times S^{k-1})$ the image of $[\overline{B}^k]$ under the map in homology that is induced by the inclusion $\overline{B}^k \xrightarrow{x \mapsto (0,x)} \overline{B}^{n-k} \times \overline{B}^k$.
- Let M be a handle-complex rel some $A \subseteq \partial M$. Given a k-handle h with characteristic map $\Phi_h \colon \overline{B}^{n-k} \times \overline{B}^k \to M$ we write, similar to Definition 329.2,

$$\Xi_h \;:=\; \Phi_{h*}([\{0\} \times \overline{B}^k] \;\in\; \mathrm{H}_k(M^k, M^{k-1}) \;=:\; \mathrm{C}_k^{\mathrm{han}}(M, A).$$



The following lemma is an analog of the Cellular Chain Complex–Basis Lemma 327.8:

Lemma 332.10. (Handle Chain Complex–Basis Lemma) Let $n \in \mathbb{N}_0$ and let M be an n-dimensional handle complex rel some $A \subseteq \partial M$. Furthermore let $k \in \{0, \dots, n\}$ and let $\{\Phi_h \colon \overline{B}^{n-k} \times \overline{B}^k \to M^k\}_{c \in \mathrm{Handle}_k(M)}$ be a full set of characteristic maps for the k-handles of M rel A in the sense of Definition 332.7. For each $h \in \mathrm{Handle}_k(M)$ we consider, as in Notation 332.9, the homology class $\Xi_h := \Phi_{h*}([\{0\} \times \overline{B}^k]) \in \mathrm{H}_k(M^k, M^{k-1}) =: \mathrm{C}_k^{\mathrm{han}}(M, A)$. The family $\{\Xi_h\}_{h \in \mathrm{Handle}_k(M)}$ is a basis for the handle chain group $\mathrm{C}_k^{\mathrm{han}}(M, A)$.

Remark. Loosely speaking the Handle Chain Complex–Basis Lemma 332.10 says that the cores of the handles form a basis of the handle chain complex. □

Proof. In the case of a finite handle decomposition this lemma is just a special case of the Handle-Skeleton-H_*-Lemma 252.1. The infinite case is proved in almost the same way (e.g. similar to the CW-Skeleton-H_*-Lemma 155.2. . We leave it to the reader to fill in the details. ■

332.4. **Intersection numbers.** The reader might remember that in the Handle Chain Complex-Boundary Map Proposition 252.6 we showed that we can calculate the boundary maps in the handle chain complex of a compact smooth manifold by considering intersection numbers between attaching spheres and belt spheres.

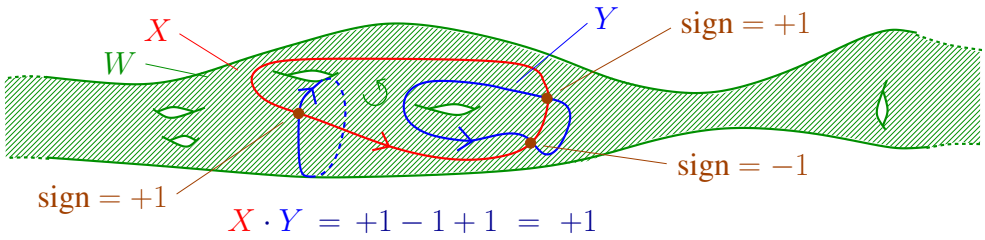
To explain this result and to generalize this result we should first talk a little bit about intersection numbers. The general setup for intersection numbers is as follows: Let W be

an oriented m-dimensional smooth manifold (possibly non-compact) and let $X \subseteq W$ be a closed oriented k-dimensional submanifold and let $Y \subseteq W$ be a closed oriented submanifold of the complementary dimension $m-k$. In Definition 224.29 we defined the corresponding algebraic intersection number $X \cdot Y \in \mathbb{Z}$. For the purpose of this chapter the precise definition is largely irrelevant. In practice all one needs is the following theorem which is a slight reformulation of the Cup Product–Intersection Number Theorem 224.30:

Theorem 332.11. (Cup Product–Intersection Number Theorem) Let W be an oriented m-dimensional smooth manifold (possibly non-compact) and let $X \subseteq W$ be a closed oriented k-dimensional submanifold and let $Y \subseteq W$ be a closed oriented submanifold of the complementary dimension $m-k$. If X and Y are transverse in the sense of Definition 220.3, then

$$X \cdot Y \;=\; \sum_{P \in X \cap Y} \operatorname{sign}(P)$$

where $\operatorname{sign}(P)$ is defined as in Definition 220.5. (The example below might help with reverse engineering the definition of the sign of a transverse intersection point.)



332.5. **The boundary maps in the handle chain complex.** In the Cellular Boundary–Degree Proposition 329.4 we learned about a fairly practical way to calculate the boundary maps in the cellular chain complex. In this section we will devise an even nicer tool for calculating the boundary maps in a given handle chain complex. More precisely, we have the following proposition, which is a mild generalization of the Handle Chain Complex-Boundary Map Proposition 252.6:

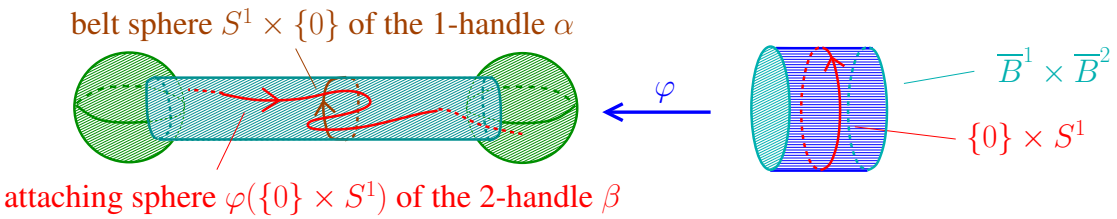
Proposition 332.12. (Handle Chain Complex-Boundary Map Proposition) Let M be an oriented n-dimensional handle-complex rel some union A of components of ∂M. For each $k \in \{0, \dots, n\}$ let $\{\Phi_h \colon \overline{B}^{n-k} \times \overline{B}^{k} \to \partial M^{k-1}\}_{h \in \mathrm{Handle}_k(M)}$ be a full set of characteristic maps in the sense of Definition 332.7 such that each Φ_h is orientation-preserving. Given a k-handle h we write, as in Notation 332.9, $\Xi_h := \Phi_{h*}([\{0\} \times \overline{B}^{k}] \in \mathrm{C}_k^{\mathrm{han}}(M, A)$. By the Handle Chain Complex–Basis Lemma 332.10 we know that $\{\Xi_h\}_{h \in \mathrm{Handle}_k(M)}$ is a basis for $\mathrm{C}_k^{\mathrm{han}}(M, A)$. Given any k-handle β and given a $(k-1)$-handle α we now set[3610]

$$d_{\beta,\alpha} \;:=\; \begin{array}{c}\text{algebraic intersection number in } \partial M^{k-1} \text{ of} \\ \text{the attaching sphere of the } k\text{-handle } \beta \\ \text{with the belt sphere of the } (k-1)\text{-handle } \alpha.\end{array}$$

With this notation we have the following equality in $\mathrm{C}_{k-1}^{\mathrm{han}}(M, A)$:

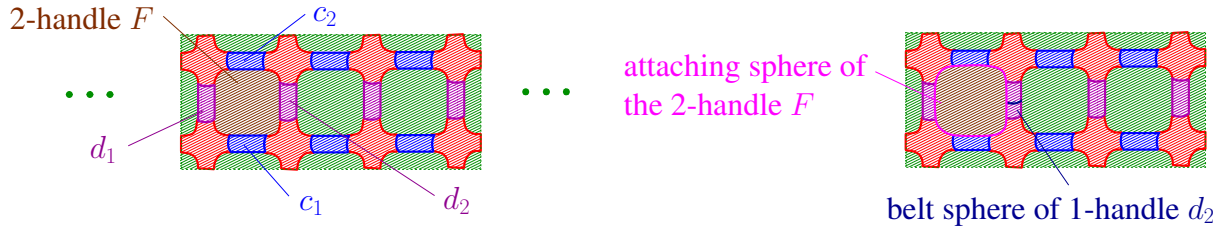
$$\partial_k^{\mathrm{han}}(\Xi_\beta) \;=\; \sum_{\alpha \in \mathrm{Handle}_{k-1}(M)} d_{\beta,\alpha} \cdot \Xi_\beta.$$

[3610]For the right-hand side note that ∂M^{k-1} is an oriented $(n-1)$-dimensional smooth manifold, the attaching sphere of the k-handle β is an oriented $(k-1)$-dimensional sphere in ∂M^{k-1} and the belt



Sketch of proof. As in the CW-Complex-Finiteness Theorem 136.16 one sees that the k-handle β is contained in the union W of finitely many handles. Using the Non-Compact Intersection Number Lemma 224.28 one can now reduce the calculation to the case W, which we dealt with in Handle Chain Complex-Boundary Map Proposition 252.6. Clearly there are several i's to dot and a few t's to cross, but there will be no major problems in filling in the details. ■

Example. We consider the 3-dimensional smooth manifold $\overline{B}^1 \times \mathbb{R}^2$ with the handle-structure that is shown in the figure below.



In the above figure of $\overline{B}^1 \times \mathbb{R}^2$ we consider the 2-handle F and several 1-handles. Our goal is the calculation of the boundary of 2-handle F, i.e. for $\Xi_F := \Phi_{F*}([\{0\} \times \overline{B}^2]) \in \mathrm{C}_2^{\mathrm{han}}(\mathbb{R}^2)$ we want to determine $\partial_2^{\mathrm{han}}(\Xi_F) \in \mathrm{C}_1^{\mathrm{han}}(\mathbb{R}^2)$. In fact, somewhat similar to the discussion on page 6274 it follows easily from the Handle Chain Complex-Boundary Map Proposition 332.12 that

$$\partial_2^{\mathrm{han}}(\Xi_{F*}) \;=\; \Xi_{c_1} + \Xi_{d_2} - \Xi_{c_2} - \Xi_{d_1}.$$ □

332.6. **Handle-complexes and Fox calculus.** We conclude this chapter with the following proposition, which is a manifold analog of the CW-Boundary Maps-via-Fox Calculus Proposition 330.6:

Proposition 332.13. (Handle-Boundary Maps-via-Fox Calculus Proposition) Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold that is equipped with a standard handle decomposition of the following form:

(0) We have a single 0-handle.
(1) We have k 1-handles $x_1, \ldots, x_k$.
(2) We have l 2-handles $r_1, \ldots, r_l$ such that the attaching sphere of each r_i is transverse to the belt spheres of the 1-handles $x_1, \ldots, x_k$.
(3) We have no handles of index ≥ 3.

From the Handle Attachment-π_1-Proposition 246.13 and Exercise 246.9 we see that each r_i defines naturally an element of $\langle x_1, \ldots, x_k \rangle$ (which we denote again by r_i) and we obtain

---

sphere of the $(k-1)$-handle α is an oriented $(n-k-2)$-dimensional sphere in ∂M^{k-1}. So, according to Definition 224.29 it makes sense to consider in ∂M^{k-1} the algebraic intersection number of the two spheres.

a natural isomorphism

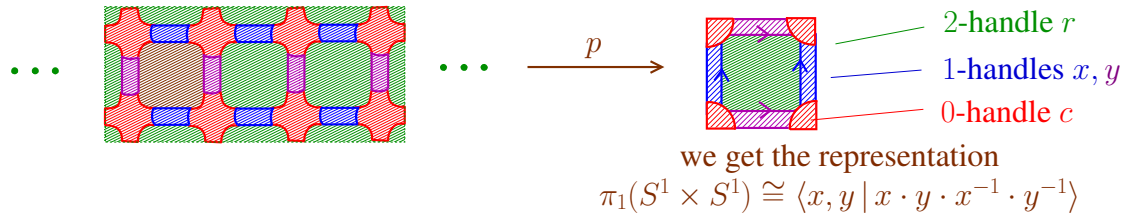
$$\mu\colon \langle x_1,\dots,x_k | r_1,\dots,r_l\rangle \;\to\; \pi_1(M).$$

Then there exists a lift of the handles of M to the universal covering $\widetilde{M}$ such that with respect to the corresponding bases, which we obtain from the Regular Covering–Handle Chain Complex Lemma 333.2, the based handle chain complex $\mathrm{C}_*^{\mathrm{han}}(\widetilde{M})$ is isomorphic to the chain complex

$$0 \to \mathbb{Z}[\pi_1(M)]^{1\times l} \xrightarrow{v\mapsto v\cdot \begin{pmatrix} \mu(\frac{\partial r_1}{\partial x_1}) & \dots & \mu(\frac{\partial r_1}{\partial x_k}) \\ \vdots & & \vdots \\ \mu(\frac{\partial r_l}{\partial x_1}) & \dots & \mu(\frac{\partial r_l}{\partial x_k}) \end{pmatrix}} \mathbb{Z}[\pi_1(M)]^{1\times k} \xrightarrow{v\mapsto v\cdot \begin{pmatrix} \mu(x_1)-1 \\ \vdots \\ \mu(x_k)-1 \end{pmatrix}} \mathbb{Z}[\pi_1(M)] \to 0.$$

Proof. The proof is very similar to the proof of the CW-Boundary Maps-via-Fox Calculus Proposition 330.6. We leave it to the reader to make the necessary adjustments. ■

Example. Let $n \in \mathbb{N}_{\geq 2}$. In the following we consider the n-dimensional smooth manifold $M = \overline{B}^{n-2} \times S^1 \times S^1$ with the usual handle decomposition that is given by one 0-handle c, two 1-handles x, y and one 2-handle r.



Note that this handle decomposition is of the form as described in the Handle-Boundary Maps-via-Fox Calculus Proposition 332.13. Thus, similar to the discussion on page 6297, we see that the handle chain complex for the universal covering of $\overline{B}^{n-2} \times S^1 \times S^1$ is isomorphic to the following chain complex:

$$0 \;\to\; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \xrightarrow{(1-y \quad x-1)} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{1\times 2} \xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \;\to\; 0. \qquad \square$$

332.7. Exercises for Chapter 332.

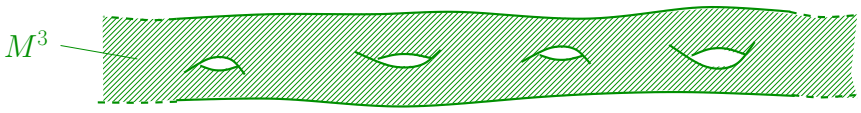
Exercise 332.1. Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold. Furthermore let $k \in \{0,\dots,n\}$ and let $\{\varphi_i\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M\}_{i\in I}$ be a family of smooth embeddings. In Definition 332.3 we had the hypothesis that the resulting map

$$\bigcup_{i\in I} \varphi_i\colon \bigcup_{i\in I} (\overline{B}^{n-k} \times S^{k-1}) \to \partial M$$

should be a closed smooth embedding. What goes wrong, if we drop this hypothesis?

Exercise 332.2. In the figure below we show a non-compact 3-dimensional smooth manifold M.

(a) Describe a handle-structure for M.
(b) Determine the untwisted handle chain complex for M.



Exercise 332.3. Let $n \in \mathbb{N}$. Describe a handle-structure for $M := \overline{B}^n \times \mathbb{R}$ rel ∂M.

Exercise 332.4. Let $K \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be a knot that is given by a diagram $D \subseteq \mathbb{R}^2$ in the sense of Definition 112.6. We assume that the diagram has m strands and m crossings. We consider the knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$, where $\tau\colon \overline{B}^2 \times K \to S^3$ is a tubular map for K. Show that X_K admits a handle-structure with 0-handle, m 1-handles and $(m-1)$-handles.
Hint. We can think of K as a knot in $\mathbb{R}^2 \times [0,1]$. Take the 0-handle to be "the ball below K" and pick a 1-handle for each strand.
Remark. In this exercise surely some corners in rigor will have to be cut.

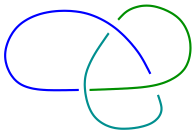
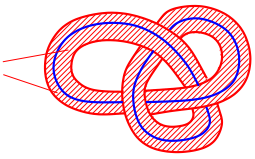


diagram for knot K with 3 strands

tubular neighborhood



CHAPTER 333

Twisted handle (co-) homology

In Chapter 327 we introduced cellular (co-) homology, which turned out to be a rousing success. The hope is to repeat this success story in the context of smooth manifolds. As a first step we introduced in the last Chapter 332 the notion of a handle-complex, the handle-chain complex and "ordinary" handle homology. In this chapter we will introduce twisted handle (co-) homology. As we will see, this concept is very a useful tool when one wants to calculate twisted (co-) homology of compact smooth manifolds. This handle chain complexes will also play a major role in the study of Reidemeister torsion.

This chapter is organized as follows:

- In Section 333.1 we will study coverings of handle-complexes.
- In Section 333.2 we will study the handle chain complex of coverings.
- In Section 333.3 we will introduce twisted handle (co-) homology.
- In Section 333.4 we will discuss the interaction between twisted handle and singular (co-) homology.
- In Section 333.5 we will show in a basic example how one can calculate handle chain complexes of universal coverings and twisted handle homology.
- In Section 333.6 we will provide exercises for Chapter 333.

333.1. **Coverings of handle-complexes.** We start out with the following analog of the CW-Complex Covering Proposition 327.6:

Proposition 333.1. (Handle-Complex Covering Proposition) Let M be a smooth manifold and let A be a union of components of ∂M. We assume that M is equipped with a handle-structure rel A in the sense of Definition 332.5. Next let $p\colon \widetilde{M} \to M$ be a covering. We set $\widetilde{A} := p^{-1}(A)$.

(1) The filtration
$$\varnothing = p^{-1}(M^{-1}) \subseteq p^{-1}(M^0) \subseteq p^{-1}(M^1) \subseteq \ldots \subseteq \widetilde{M}$$
defines a handle-structure for $\widetilde{M}$ rel $\widetilde{A}$ which has the property that for any $k \in \mathbb{N}_0$ we have
$$\#k\text{-handles of } \widetilde{M} = \big[\widetilde{M} : M\big] \cdot \#k\text{-handles of } M.$$

(2) (a) Let $\{\Phi_h\colon \overline{B}^{n-\mathrm{index}(h)} \times \overline{B}^{\mathrm{index}(h)} \to M\}_{h\in \mathrm{Handle}(M)}$ be a full set of characteristic maps for the handle-complex M rel A in the sense of Definition 332.7. The set of all lifts of all these characteristic maps to $\widetilde{M}$ is a full set of characteristic maps for the handle-complex $\widetilde{M}$ rel $\widetilde{A}$.

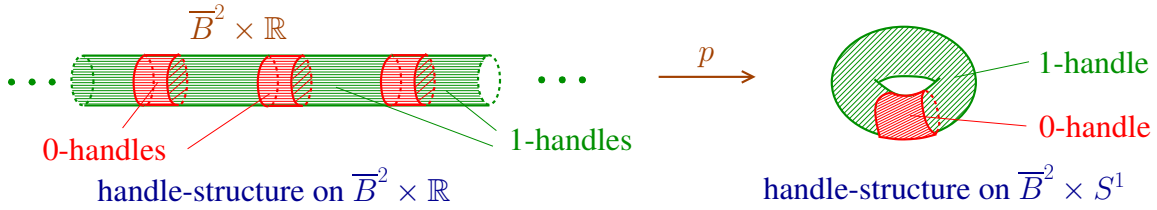
(b) Each characteristic map $\Phi_h\colon \overline{B}^{n-k} \times \overline{B}^k \to M$ admits precisely $[\widetilde{M} : M]$ lifts to the covering $\widetilde{M}$.

Sketch of proof. The proof of this proposition is similar to the proof of the CW-Complex Covering Proposition 138.1. In fact it it a little easier, since we do not have to worry about

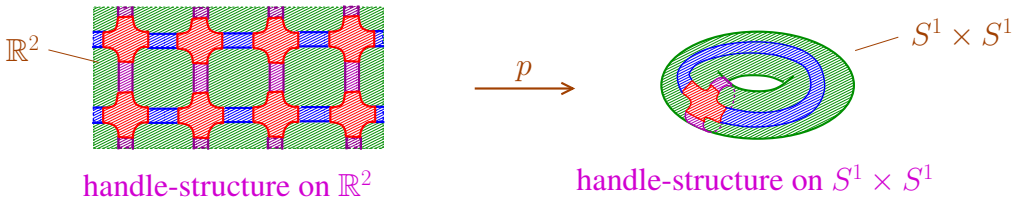
the limit topology. We leave it to the reader to make the necessary adjustments and to fill in the details. ■

Examples.

(1) Let $n \in \mathbb{N}_0$. We equip the solid torus $\overline{B}^n \times S^1$ with the handle decomposition from the Ball–Handle Attachment Lemma 246.9 consisting of one 0-handle and one 1-handle. We denote by $p\colon \overline{B}^n \times \mathbb{R} \to \overline{B}^n \times S^1$, $(z,t) \mapsto (z, \exp(2\pi\,\mathrm{i}t))$ the universal covering of $\overline{B}^n \times S^1$. It follows easily from the Handle-Complex Covering Proposition 333.1 that we have a handle-structure on $\overline{B}^n \times \mathbb{R}$ with $\mathbb{Z}$-many 0-handles and $\mathbb{Z}$-many 1-handles.



(2) As on page 6318 we equip the torus $S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$ with the handle-structure with one 0-handle, two 1-handles and one 2-handle.



We consider the universal covering $\mathbb{R}^2 \to S^1 \times S^1$. The resulting handle-structure on $\mathbb{R}^2$ is illustrated in the figure above. □

333.2. **The handle chain complex of coverings.** Before we can introduce twisted handle homology we should better get a better understanding of the handle chain complex of coverings. Not surprisingly we have the following analog of the Regular Covering–Chain Complex Lemma 320.2 and of the Regular Covering–Cellular Chain Complex Lemma 327.10:

Lemma 333.2. (Regular Covering–Handle Chain Complex Lemma) Let M be a path-connected handle-complex rel $A \subseteq \partial M$. Let $p\colon \widetilde{M} \to M$ be a regular covering. We write $\widetilde{A} := p^{-1}(A)$ and we denote by $D := \mathrm{Deck}(p\colon \widetilde{M} \to M)$ the corresponding deck transformation group.

(1) (a) Let $k \in \mathbb{N}_0$. For each open k-handle σ of M we pick a single open handle $\widetilde{\sigma} \subseteq \widetilde{M}$ with $p(\widetilde{\sigma}) = \sigma$. These handles $\widetilde{\sigma}$ (together with a choice of a characteristic map) give rise to a basis for $\mathrm{C}_k^{\mathrm{han}}(\widetilde{M}, \widetilde{A})$ as a free $\mathbb{Z}[D]$-left module.

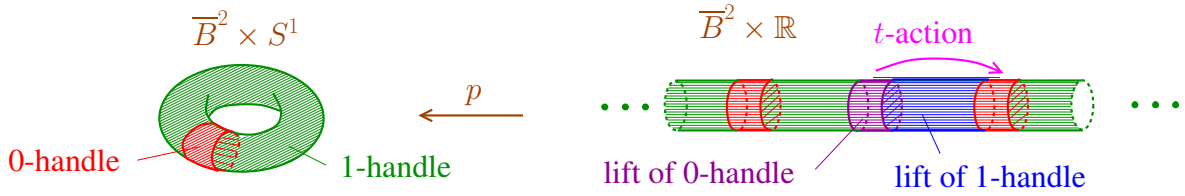
(b) Each handle chain group $\mathrm{C}_k^{\mathrm{han}}(\widetilde{M}, \widetilde{A})$ is a free $\mathbb{Z}[D]$-left module where the cardinality of the basis is given by the cardinality of the set of k-handles of the handle-complex M.

(2) The handle boundary maps $\partial_k^{\mathrm{han}}\colon \mathrm{C}_k^{\mathrm{han}}(\widetilde{M}, \widetilde{A}) \to \mathrm{C}_{k-1}^{\mathrm{han}}(\widetilde{M}, \widetilde{A})$ are $\mathbb{Z}[D]$-left module homomorphisms.

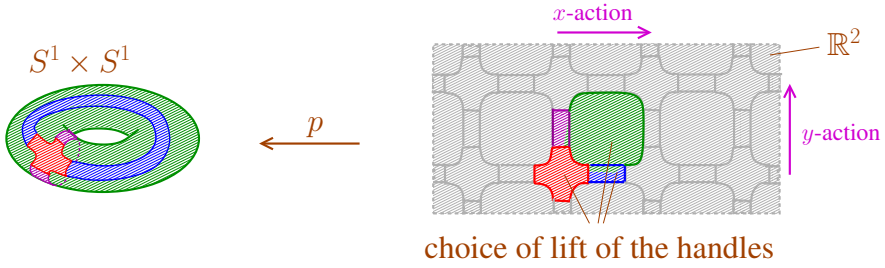
Proof. The proof is almost the same as the proof of the Regular Covering–Cellular Chain Complex Lemma 327.10. We just need to replace the CW-Complex Covering Proposition 327.6 by the Handle-Complex Covering Proposition 333.1 and we need to replace the Cellular Chain Complex–Basis Lemma 327.8 by the Handle Chain Complex–Basis Lemma 332.10. ■

Example. We continue with the examples from page 6326.

(1) Let $n \in \mathbb{N}_0$. As above we equip the solid torus $\overline{B}^n \times S^1$ with a handle decomposition consisting of a single 0-handle and a single 1-handle. In the figure below we show a lift of these handles. By the Regular Covering–Handle Chain Complex Lemma 333.2 these form a basis of $\mathrm{C}_*^{\mathrm{han}}(\overline{B}^n \times S^1)$ as $\mathbb{Z}[t^{\pm 1}]$-left modules.



(2) In the figure below we see an analogous picture for the 2-dimensional torus and its universal covering $\mathbb{R}^2$.



Similar to (1) we see that the lifts give rise to a basis of the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(\mathbb{R}^2)$ as $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-left modules. □

333.3. **Twisted handle (co-) homology.** We can now introduce the handle analogs of the twisted (co-) singular and handle homology groups that we introduced in Definition 320.11 and Definition 327.11:

Definition 333.3. Let M be a path-connected handle-complex rel $A \subseteq \partial M$ and let $x_0 \in M$. We write $\pi := \pi_1(M, x_0)$. Furthermore let R be a ring and let V be an $(R, \mathbb{Z}[\pi_1(M_0, x_0)])$-left left module. We denote by $p\colon \widetilde{M}_{x_0} \to M$ the natural universal covering and we write $\widetilde{A} := p^{-1}(A)$. Given $k \in \mathbb{N}_0$ we define the R-left modules[3611]

$$\mathrm{C}_k^{\mathrm{tw,han}}(M, A; V) := \overline{V} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{han}}(\widetilde{M}_{x_0}, \widetilde{A}) \qquad \text{(twisted handle chain complex)}$$
$$\mathrm{C}^k_{\mathrm{tw,han}}(M, A; V) := \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k^{\mathrm{han}}(\widetilde{M}_{x_0}, \widetilde{A}), V) \qquad \text{(twisted handle cochain complex).}$$

By the Regular Covering–Handle Chain Complex Lemma 333.2 (2) we know that the boundary maps of $\mathrm{C}_k^{\mathrm{han}}(\widetilde{M}_{x_0}, \widetilde{A})$ are $\mathbb{Z}[\pi]$-left module homomorphisms. Thus we see that $\mathrm{id} \otimes \partial_*^{\mathrm{han}}$ and ∂^* induce (co-) chain maps. Thus we can define the R-left modules

$$\mathrm{H}_k^{\mathrm{tw,han}}(M, A; V) := \mathrm{H}_k\big(\overline{V} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{han}}(\widetilde{M}_{x_0}, \widetilde{A})\big) \qquad \text{(twisted handle homology)}$$
$$\mathrm{H}^k_{\mathrm{tw,han}}(M, A; V) := \mathrm{H}^k\big(\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*^{\mathrm{han}}(\widetilde{M}_{x_0}, \widetilde{A}), V)\big) \qquad \text{(twisted handle cohomology).}$$

As always, if $A = \varnothing$, then we drop it from the notation.

Example. We have the obvious analog of the Twisted–Trivial Coefficients Lemma 320.13, i.e. handle homology with trivial ${}_{\text{tr}}\mathbb{Z}$-coefficients is easily shown to be naturally isomorphic to ordinary handle homology. □

333.4. **The interaction between twisted handle and singular (co-) homology.** The following theorem is the analog of the Singular-Cellular Twisted Isomorphism Theorem 328.1:

Theorem 333.4. (Singular-Handle Twisted Isomorphism Theorem) Let M be a connected handle-complex rel $A \subseteq \partial M$ and let $x_0 \in M$. Furthermore let R be a ring and let V be an $(R, \mathbb{Z}[\pi_1(M, x_0)])$-left left module. Then there exist natural isomorphisms

$$\begin{aligned} &(1) && \Phi_{(M,A)} \colon \mathrm{H}_k^{\text{tw}}(M, A; V) \ \to \ \mathrm{H}_k^{\text{tw,han}}(M, A; V) \\ &(2) && \Phi_{(M,A)} \colon \mathrm{H}^k_{\text{tw}}(M, A; V) \ \to \ \mathrm{H}^k_{\text{tw,han}}(M, A; V) \end{aligned}$$

of R-left modules. Here natural means the following two statements:

(a) If $f \colon (M, A) \to (N, B)$ is a handle map between pairs of handle-complexes (i.e. a map which sends the k-skeleton into the k-skeleton), if M and N are connected, if $x_0 \in M$, if R is a ring, and if V is an $(R, \mathbb{Z}[\pi_1(N, f(x_0))])$-left left module, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\text{tw}}(M, A; f^*V) & \xrightarrow{\Phi_{(M,A)}} & \mathrm{H}_k^{\text{tw,han}}(M, A; f^*V) \\ \downarrow{\scriptstyle f_*} & & {\scriptstyle f_*}\downarrow \\ \mathrm{H}_k^{\text{tw}}(N, B; V) & \xrightarrow{\Phi_{(N,B)}} & \mathrm{H}_k^{\text{tw,han}}(N, B; V) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

(b) Let V and W be $(R, \mathbb{Z}[\pi_1(M, x_0)])$-left left modules and let $\varphi \colon V \to W$ be a homomorphism of $(R, \mathbb{Z}[\pi_1(M, x_0)])$-left left modules. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k^{\text{tw}}(M, A; V) & \xrightarrow{\Phi_{(M,A)}} & \mathrm{H}_k^{\text{tw,han}}(M, A; V) \\ \downarrow{\scriptstyle \varphi_*} & & {\scriptstyle \varphi_*}\downarrow \\ \mathrm{H}_k^{\text{tw}}(M, A; W) & \xrightarrow{\Phi_{(M,A)}} & \mathrm{H}_k^{\text{tw,han}}(M, A; W) \end{array}$$

and the corresponding diagram for twisted cohomology also commutes.

The key to proving the Singular-Handle Twisted Isomorphism Theorem 333.4 is the following definition, which is an analog of Definition 158.13 and Definition 328.2:

Definition 333.5. Let M be a handle-complex rel some union A of components of ∂M.

(1) Given $n \in \mathbb{N}_0$ we define the intermediate handle chain complex

$$\mathrm{C}_n^{\text{han,int}}(M, A) \ := \ \ker\big(\mathrm{C}_n(M^n) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(M^n) \xrightarrow{\text{я}} \mathrm{C}_{n-1}(M^n, M^{n-1})\big).$$

The argument of the CW-Intermediate-Singular-Lemma 158.14 shows that $\mathrm{C}_*^{\text{han,int}}(M)$ is a subcomplex of $\mathrm{C}_*(M)$.

[3611] As usual we follow Convention 319.13, i.e. we use the Deck Transformation Group Proposition 319.12 to define an action of $\pi_1(X, x_0)$ on $\widetilde{M}_{x_0}$.

(2) We have natural chain maps

$$\iota\colon \mathrm{C}_*^{\mathrm{han,int}}(M,A) \hookrightarrow \mathrm{C}_*(M,A) \qquad \text{and} \qquad \pi\colon \mathrm{C}_*^{\mathrm{han,int}}(M,A) \twoheadrightarrow \mathrm{C}_*^{\mathrm{han}}(M,A)$$

which are induced by the obvious inclusion and the natural projection.

We continue with the following analog of the CW-Intermediate-Singular Proposition 158.15 and of the Relative CW-Intermediate-Singular Proposition 328.3:

Proposition 333.6. (Handle-Intermediate-Singular Proposition) Let M be a handle-complex rel some union A of components of ∂M. For each $n \in \mathbb{N}_0$ the natural maps

(1) $\iota_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{han,int}}(M,A)) \to \mathrm{H}_n(\mathrm{C}_*(M,A))$

(2) $\pi_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{han,int}}(M,A)) \to \mathrm{H}_n(\mathrm{C}_*^{\mathrm{han}}(M,A))$

are isomorphisms.

Proof of Proposition 333.6. The proof is almost identical to the proof of the CW-Intermediate-Singular Proposition 158.15 which we gave in Section 158.5. We just need to replace the CW-Skeleton-H_*-Lemma 155.2 by the Handle Chain Complex–Basis Lemma 332.10. ■

Sketch of proof of Theorem 333.4. The proof of the Singular-Handle Twisted Isomorphism Theorem 333.4 follows evidently the same logic as the proof of the Singular-Cellular Twisted Isomorphism Theorem 328.1. In the following we will just outline the existence of an isomorphism. We will not repeat the argument why the isomorphism we construct below is well-defined and natural.

In the following let M be a connected handle-complex rel $A \subseteq \partial M$, let $x_0 \in M$, let R be a ring and let V be an $(R, \mathbb{Z}[\pi_1(M,x_0)])$-left left module. We write $\pi := \pi_1(M,x_0)$. We denote by $p\colon \widetilde{M}_{x_0} \to M$ the natural universal covering and we write $\widetilde{A} := p^{-1}(A)$.

We equip $\widetilde{M}$ with the handle-structure rel $\widetilde{A}$ given by the Handle-Complex Covering Proposition 333.1. By the Deck Transformation Group Proposition 319.12 we have a natural isomorphism $\pi \cong \mathrm{Deck}(p\colon \widetilde{M} \to M)$. Thus, via the Regular Covering–Handle Chain Complex Lemma 333.2, we can view $\mathrm{C}_*^{\mathrm{han}}(\widetilde{M},\widetilde{A})$ as a chain complex of free $\mathbb{Z}[\pi]$-left modules.

The Free Imitation Existence Lemma 313.21 implies that there exists a chain complex (F_*,∂_*) of *free* $\mathbb{Z}[\pi]$-left modules together with a quasi-isomorphism $\varphi\colon F_* \to \mathrm{C}_*^{\mathrm{han,int}}(\widetilde{M},\widetilde{A})$ of $\mathbb{Z}[\pi]$-left modules. Now we consider the following diagram:

$$\begin{array}{ccccc} & & F_* & & \\ & \swarrow^{\iota\circ\varphi} & \downarrow{\scriptstyle\varphi} & \searrow^{\pi\circ\varphi} & \\ \mathrm{C}_*(\widetilde{M},\widetilde{A}) & \xleftarrow{\ \iota\ } & \mathrm{C}_*^{\mathrm{han,int}}(\widetilde{M},\widetilde{A}) & \xrightarrow{\ \pi\ } & \mathrm{C}_*^{\mathrm{han}}(\widetilde{M},\widetilde{A}). \end{array}$$

Note that the action of π on $\widetilde{M}$ turns all three chain complexes at the bottom into chain complexes of $\mathbb{Z}[\pi]$-left modules and the naturality of ι and π implies that these maps commute with the π-action, in other words, the horizontal maps ι and π are homomorphisms of $\mathbb{Z}[\pi]$-left modules. It follows from the above together with the Handle-Intermediate-Singular Proposition 333.6 that $i \circ \varphi\colon F_* \to \mathrm{C}_*(\widetilde{M},\widetilde{A})$ and $\pi \circ \varphi\colon F_* \to \mathrm{C}_*^{\mathrm{han}}(\widetilde{M},\widetilde{A})$ are quasi-isomorphisms between chain complexes of free $\mathbb{Z}[\pi]$-left modules. Thus it follows from the Quasi Iso–Homotopy Equivalence Proposition 313.18 that $i\circ\varphi$ and $\pi\circ\varphi$ are chain

homotopy equivalences of $\mathbb{Z}[\pi]$-left module chain complexes. Next we consider the induced maps

$$\begin{array}{ccccc} & & \mathrm{H}_k(V\otimes_{\mathbb{Z}[\pi]} F_*) & & \\ & \swarrow^{(\mathrm{id}\otimes(\iota\circ\varphi))_*} & \downarrow{\scriptstyle (\mathrm{id}\otimes\varphi)_*} & \searrow^{(\mathrm{id}\otimes(\pi\circ\varphi))_*} & \\ \underbrace{\mathrm{H}_k(V\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{M},\widetilde{A}))}_{\mathrm{H}_k^{\mathrm{tw}}(M,A;V)} & \xleftarrow{(\mathrm{id}\otimes\iota)_*} & \mathrm{H}_k(V\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*^{\mathrm{han,int}}(\widetilde{M},\widetilde{A})) & \xrightarrow{(\mathrm{id}\otimes\pi)_*} & \underbrace{\mathrm{H}_k(V\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*^{\mathrm{han}}(\widetilde{M},\widetilde{A}))}_{\mathrm{H}_k^{\mathrm{tw,han}}(M,A;V)}. \end{array}$$

Since $\iota\circ\varphi$ and $\pi\circ\varphi$ are both chain homotopy equivalence we obtain from the Homology-with-Coefficients Lemma 314.6 (2b) and the Chain Homotopy Equivalence-H_*-Corollary 313.7 that the diagonal maps are isomorphisms of R-left modules. Therefore we can now define

$$\Phi_{(M,A)} \coloneqq (\mathrm{id}\otimes(\pi\circ\varphi))_*\circ(\mathrm{id}\otimes(\iota\circ\varphi))_*^{-1}\colon \mathrm{H}_k^{\mathrm{tw}}(M,A;V) \;\to\; \mathrm{H}_k^{\mathrm{tw,han}}(M,A;V).$$

As we just discussed, the above diagonal maps are isomorphisms of R-left modules. Thus we see that $\Phi_{(M,A)}\colon \mathrm{H}_k^{\mathrm{tw}}(M,A;V)\to\mathrm{H}_k^{\mathrm{tw,han}}(M,A;V)$ is indeed an isomorphism of R-left modules. ∎

333.5. **Calculating twisted handle homology.** In Chapter 329 we learned various results and tricks which allowed us to calculate the twisted cellular chain complexes in many settings. Many of the results and ideas have obvious analogs in the handle context, and surely there is no need to write down all the little tricks for a second time.

We only formulate the following analog of the Replace Universal Covering Proposition 329.1, even though it is something of a mouthful:

Proposition 333.7. (Replace Universal Covering Proposition) Let M be a path-connected handle-complex rel $A\subseteq\partial M$ and let $x_0\in M$. We denote by $p\colon \widetilde{M}_{x_0}\to M$ the corresponding natural universal covering. We write $\widetilde{A}\coloneqq p^{-1}(A)$. We set $\pi\coloneqq\pi_1(M,x_0)$. Let $\Gamma\subseteq\pi$ a normal subgroup. We denote by $\rho\colon\pi\to\pi/\Gamma$ the projection and we denote by $\rho\colon\mathbb{Z}[\pi]\to\mathbb{Z}[\pi/\Gamma]$ also the corresponding ring homomorphism. Let R be a ring and let V be an $(R,\mathbb{Z}[\pi/\Gamma])$-left left module. Let $q\colon(Y,y_0)\to(M,x_0)$ be any path-connected regular covering with $q_*(\pi_1(Y,y_0))=\Gamma$. We set $B\coloneqq q^{-1}(A)$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon(\widetilde{M}_{x_0},\widetilde{x}_0)\to(Y_\Gamma,y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M}_{x_0} & \xrightarrow{\quad r\quad} & Y_\Gamma. \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & M & \end{array}$$

The natural maps

$$\begin{array}{rcl} \overbrace{\overline{\rho^*V}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_k^{\mathrm{han}}(\widetilde{M}_{x_0},\widetilde{A})}^{\mathrm{C}_k^{\mathrm{han,tw}}(M,A;\rho^*V)} & \to & \overline{V}\otimes_{\mathbb{Z}[\pi/\Gamma]}\mathrm{C}_k^{\mathrm{han}}(Y,B) \\ v\otimes(\sigma\colon\Delta^k\to\widetilde{M}_{x_0}) & \mapsto & v\otimes(r\circ\sigma\colon\Delta^k\to Y) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}[\pi/\Gamma]\text{-left}}(\mathrm{C}_k^{\mathrm{han}}(Y,B),V) & \to & \overbrace{\mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_k^{\mathrm{han}}(\widetilde{M}_{x_0},\widetilde{A}),\rho^*V)}^{=\mathrm{C}^k_{\mathrm{han,tw}}(M,A;\rho^*V)} \\ \varphi & \mapsto & (\sigma \mapsto \varphi(r\circ\sigma)) \end{array}$$

are isomorphisms of (co-) chain complexes of R-left modules. In particular we get induced isomorphisms of (co-) homology modules.

Proof. As in the case of the proof of the original Replace Universal Covering Propositions 320.16 and 329.1 the desired statements follow easily from the definitions. We leave it to the reader to make the necessary changes to the earlier proofs. ■

To illustrate how one can calculate twisted handle chain complexes in practice, let us consider the solid torus $\overline{B}^n \times S^1$ and also (once again) the torus $S^1 \times S^1$. First of all we have the following lemma, which is the handle analog of the Circle–Twisted Cellular Chain Complex Lemma 329.10:

Lemma 333.8. (Solid Torus–Twisted Handle Chain Complex Lemma) Let $n \in \mathbb{N}_0$. We consider the solid torus $\overline{B}^n \times S^1$ together with the base point $x_0 = (0, 1_{\mathbb{C}})$. We make the usual identifications $\pi := \pi_1(\overline{B}^n \times S^1, x_0) =_{\mathrm{i}} \langle x\rangle$ and $\mathbb{Z}[\pi] =_{\mathrm{i}} \mathbb{Z}[x^{\pm 1}]$.

(1) For the standard handle-structure on $\overline{B}^n \times S^1$ the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(\widetilde{\overline{B}^n \times S^1})$ of free $\mathbb{Z}[x^{\pm 1}]$-modules is isomorphic to the chain complex[3612 3613]

$$0 \;\to\; \mathbb{Z}[x^{\pm 1}] \;\xrightarrow{(x-1)}\; \mathbb{Z}[x^{\pm 1}] \;\to\; 0.$$

(2) Let R be a ring and let $\varphi\colon \langle x\rangle \to \mathrm{GL}(n,R)$ be a representation.

(a) The twisted handle chain complex $\overline{{}_\varphi R^{1\times n}} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{han}}(\widetilde{\overline{B}^n \times S^1})$ is isomorphic to the chain complex

$$0 \;\to\; R^{1\times n} \xrightarrow{\varphi(x)-\mathrm{id}_n} R^{1\times n} \;\to\; 0.$$

(b) If R is a field and if $\varphi\colon \pi_1(\overline{B}^n \times S^1, x_0) \to \mathrm{GL}(1,R)$ is a non-trivial one-dimensional representation, then the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(\overline{B}^n \times S^1; {}_\varphi R)$ are all zero.

Proof. The proof is very similar to the Circle–Twisted Cellular Chain Complex Lemma 329.10. We leave it to the reader to fill in the details. Note that for the next, slightly more interesting lemma, we provide full details. The reader should have no troubles with downgrading the proof of Torus–Twisted Handle Chain Complex Lemma 333.9 to a proof of the current lemma. ■

The next lemma is the handle analog of the Torus–Twisted Cellular Chain Complex Lemma 329.11:

Lemma 333.9. (Torus–Twisted Handle Chain Complex Lemma) In the following we consider the 2-dimensional torus $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1]\times[0,1])/\sim$ together with the base point $t_0 = [(0,0)]$. We make the usual identifications $\pi := \pi_1(S^1 \times S^1, t_0) =_{\mathrm{i}} \langle x, y \,|\, [x,y]\rangle$ and $\mathbb{Z}[\pi] =_{\mathrm{i}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

(1) For the standard handle-structure on $S^1 \times S^1$ the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(\widetilde{T}_{t_0})$ of free $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-modules is isomorphic to the chain complex

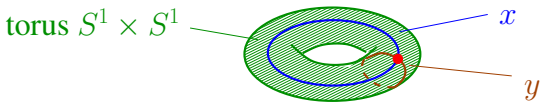
$$0 \;\to\; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \;\xrightarrow{\begin{pmatrix} 1-y & x-1 \end{pmatrix}}\; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{1\times 2} \;\xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}}\; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \;\to\; 0.$$

(2) Let R be a ring and let $\varphi\colon \langle x,y \,|\, [x,y]\rangle \to \mathrm{GL}(n,R)$ be a representation.
  (a) The twisted handle chain complex ${}_{\varphi}\overline{R^{1\times n}} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{han}}(\widetilde{T}_{t_0})$ is isomorphic to the chain complex

$$0 \;\to\; R^{1\times n} \xrightarrow{\begin{pmatrix}\mathrm{id}_n - \varphi(y) & \varphi(x) - \mathrm{id}_n\end{pmatrix}} R^{1\times 2n} \xrightarrow{\begin{pmatrix}\varphi(x)-\mathrm{id}_n\\ \varphi(y)-\mathrm{id}_n\end{pmatrix}} R^{1\times n} \;\to\; 0.$$

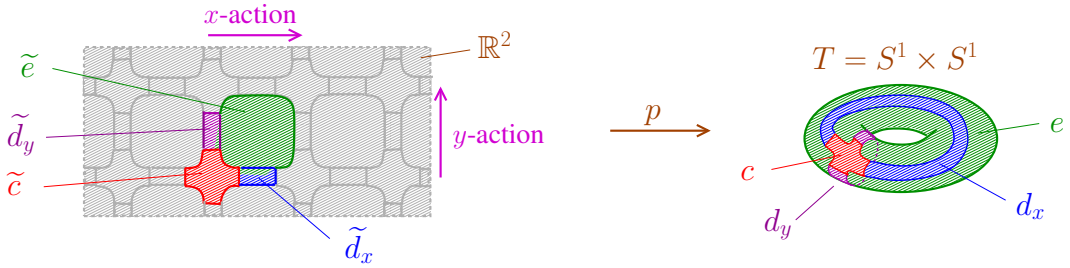
  (b) If R is a field and if $\varphi\colon \pi_1(T,t_0) \to \mathrm{GL}(1,R)$ is a non-trivial one-dimensional representation, then the twisted homology modules $\mathrm{H}_*^{\mathrm{tw}}(T;{}_{\varphi}R)$ are all zero.

Finally we point out that the same results also hold for the "thickened" tori $\overline{B}^k \times S^1 \times S^1$.



Proof.

(1) We equip T with the usual handle-structure that we introduced on page 6252 which has one 0-handle c, two 1-handles d_x and d_y and one 2-handle e. We equip each handle with the (fairly) obvious characteristic map. Next recall that we have a universal covering $p\colon \mathbb{R}^2 \to T$. The resulting handle-structure on $\mathbb{R}^2$, and lifts of c, d_x, d_y and e are shown in the following figure.



By the Regular Covering–Handle Chain Complex Lemma 333.2 we know that these lifts give rise to a basis for the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(\mathbb{R}^2)$ as a chain complex of $\mathbb{Z}[x^{\pm1},y^{\pm}]$-modules. Next, using the Handle Chain Complex-Boundary Map Proposition 332.12 and Observation 329.5 we see, as on page 6322, that we have an equality $\partial_2^{\mathrm{han}}(\widetilde{e}) = (1-y)\cdot\widetilde{d}_x + (x-1)\cdot\widetilde{d}_y$ and as on page 6276 we see that $\partial_1^{\mathrm{han}}(\widetilde{d}_x) = (x-1)\cdot\widetilde{c}$ and $\partial_1^{\mathrm{han}}(\widetilde{d}_y) = (y-1)\cdot\widetilde{c}$. Thus we see that the handle chain complex $\mathrm{C}_2^{\mathrm{han}}(\widetilde{T})$ is of the form

$$0 \;\to\; \mathbb{Z}[x^{\pm1},y^{\pm1}] \xrightarrow{\begin{pmatrix}1-y & x-1\end{pmatrix}} \mathbb{Z}[x^{\pm1},y^{\pm1}]^2 \xrightarrow{\begin{pmatrix}x-1\\ y-1\end{pmatrix}} \mathbb{Z}[x^{\pm1},y^{\pm1}] \;\to\; 0.$$

The desired statement follows from this calculation together with the Replace Universal Covering Proposition 333.7.

(2) (a) The statement about the twisted handle chain complex follows from (1) together with the Tensor–Matrix Lemma 329.8.
  (b) Now we assume that R is a field and we assume that we are given a one-dimensional representation $\pi_1(T,t_0) \to \mathrm{GL}(1,R)$. Without loss of generality we can assume that $\varphi(x) \neq 1$. This implies that $\varphi(x) - 1$ is invertible in R. But now it follows from

elementary linear algebra that the twisted chain complex is acyclic. By the Singular-Handle Twisted Isomorphism Theorem 333.4 together with the Replace Universal Covering Proposition 333.7 this implies that the twisted singular homology modules are all zero. ■

For $\overline{B}^k \times S^1 \times S^1$ the argument is essentially the same, we just have to equip $\overline{B}^k \times S^1 \times S^1$ with the product handle decomposition coming from the Product–Handle Decomposition Proposition 251.5.

333.6. Exercises for Chapter 333.

Exercise 333.1. Formulate and prove the handle analog of the $\mathbb{R}\mathrm{P}^n$–Twisted Cellular Chain Complex Lemma 329.13.

Exercise 333.2.

(a) Equip the Klein bottle K with a handle-structure.
(b) Describe the handle chain complex of the universal covering $\widetilde{K}$ of K.

Exercise 333.3. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \quad \text{and we consider the orientation-reversing diffeomorphism} \quad \begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

Let $\theta\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$ be the usual diffeomorphism that is given by $[(x,y)] \mapsto (e^{2\pi \mathrm{i} x}, e^{2\pi \mathrm{i} y})$. We set

$$M := ((\overline{B}^2 \times S^1) \sqcup (\overline{B}^2 \times S^1))/\sim \qquad \text{where } (x,y) \sim (\theta \circ \varphi \circ \theta^{-1})(x,y) \text{ for } x, y \in S^1 \times S^1.$$

By the Gluing-Smooth Manifolds-Proposition 54.5 we know that M is a closed 3-dimensional smooth manifold.

(a) Show that M admits a handle decomposition with a unique handle in dimensions 0,1,2 and 3.
(b) Determine the corresponding handle chain complex of $L(p,q)$.
(c) Show that for every $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{han}}(M) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n = 1. \end{cases}$$

(d) Determine the handle chain complex of the universal covering of $L(p,q)$.
(e) Determine the twisted homology groups of $L(p,q)$ corresponding to non-trivial one-dimensional complex representations.

Remark. In the Lens Space–Heegaard Splitting Proposition **??** we showed that M is diffeomorphic to the lens space $L(p,q)$.
Remark. This exercise has a significant overlap with Exercise 252.10.

Exercise 333.4.

(a) Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. We equip Σ_g with the usual handle decomposition which has one 0-handle, 2g 1-handles and one 2-handle. Describe the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(\widetilde{\Sigma}_g)$ of the universal covering.

(b) Let $k \in \mathbb{N}$ and let N_k be the non-orientable surface of genus k. We equip N_k with the usual handle decomposition which has one 0-handle, k 1-handles and one 2-handle. Describe the handle chain complex $C_*^{han}(\widetilde{N}_k)$ of the universal covering. You should be especially careful when it comes to signs of the second boundary map.

Exercise 333.5. Let M be a connected non-empty n-dimensional smooth manifold. By the Minimal 0-Handle Proposition 247.13 we know that M admits a handle decomposition with a single 0-handle. Use this feature to give a new proof of the Twisted H_0-Proposition 323.3 in our current context. More precisely, let $x_0 \in M$ be a point in the 0-handle, let R be a ring and let V be an $(R, \mathbb{Z}[\pi])$-left left module, where $\pi := \pi_1(M, x_0)$.

(a) Show that we have an isomorphism

$$H_0^{tw}(M;V) \;\cong\; V_\pi \;:=\; V \Big/ \left\{ \sum_{i=1}^{k} (g_i \cdot v_i - v_i) \,\middle|\, g_i \in \pi \text{ and } v_i \in V \right\}$$

of R-left modules.

(b) Show that we have an isomorphism

$$H_{tw}^0(M;V) \;\cong\; V^\pi \;:=\; \{v \in V \mid g \cdot v = v \text{ for all } g \in \pi\}$$

of R-left modules.

Exercise 333.6. Let M be a closed connected non-empty n-dimensional smooth manifold. By the Minimal n-Handle Proposition 248.4 we know that M admits a handle decomposition with a single n-handle. Let $x_0 \in M$ be a point in the n-handle, let R be a ring and let V be an $(R, \mathbb{Z}[\pi])$-left left module, where $\pi := \pi_1(M, x_0)$.

(a) We assume that M is orientable. Calculate $H_n^{tw}(M;V)$ and $H_{tw}^n(M;V)$.
(b) We assume that M is non-orientable. Calculate $H_n^{tw}(M;V)$ and $H_{tw}^n(M;V)$.
(c) Bonus question: Compare your results with Exercise 333.5.

Hint. You might want to think about the dual handle decomposition from the Dual Handle Proposition 248.1.
Remark. As in Exercise 333.4 you should be very careful about the signs that appear in the boundary map of the handle chain complex, since these depend on the choices of orientations (or of characteristic maps) of the handles.

Exercise 333.7. Let $P = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ be a word-presentation. In Definition 330.2 we defined the corresponding presentation complex $Y(P)$ with one 0-cell, k 1-cells and l 2-cells.

(a) For which $n \in \mathbb{N}$ can you associate to P an n-dimensional handle complex $H(P)$ with one 0-handle, k 1-handles and l 2-handles such that $H(P)$ is homotopy equivalent to $Y(P)$?
(b) Can one again calculate the handle chain complex of the universal covering of such $H(P)$ by Fox calculus? In other words, does a handle analog of the CW-Boundary Maps-via-Fox Calculus Proposition 330.6 hold?

Exercise 333.8. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a word-presentation. In Definition 330.2 we defined the corresponding presentation complex $Y(P)$ as a CW-complex with one 0-cell, k 1-cells and l 2-cells.

(a) For which $n \in \mathbb{N}$ can you associate to P an n-dimensional handle complex $H(P)$ with one 0-handle, k 1-handles and l 2-handles such that $H(P)$ is homotopy equivalent to $Y(P)$?

(b) Show that we can calculate the handle chain complex of the universal covering of a handle presentation complex in terms of Fox calculus. In other words, prove the handle analog of the CW-Boundary Maps-via-Fox Calculus Proposition 330.6 hold.

Exercise 333.9. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We consider the torus knot $T := T(p, q)$ from Definition 111.13. Recall that it has the feature that it lies on the standard solid torus in $\mathbb{R}^3$.

(a) We pick a tubular map $\tau \colon \overline{B}^2 \times T \to S^3$ and we consider the corresponding knot exterior $X_T := S^3 \setminus \tau(B^2 \times T)$. Show that the exterior X_K admits a handle decomposition with one 0-handle, one 1-handle and one 2-handle.

(b) Recall that in the Torus Knot-π_1-Proposition 111.18 we gave a fairly explicit isomorphism $\langle x, y \,|\, x^p \cdot y^q \rangle \to \pi_1(S^3 \setminus T)$. Show that we can think of X_T as a handle presentation complex of $\langle x, y \,|\, x^p \cdot y^q \rangle$.
Hint. Take a look at the proof of the Torus Knot-π_1-Proposition 111.18.

(c) Use (b) and Exercise 333.8 to determine the handle chain complex of the universal covering of the handle-complex $Y(P)$.

Remark. This exercise has an overlap with Exercise 247.15 and it is related to Exercise 330.6.



CHAPTER 334

The cup and cap product on twisted (co-) homology

Let X be a topological space and let R be a commutative ring. In Definition 203.8 and Definition 206.5 we introduced the "classical" cup and cap product as R-bilinear maps

$$\mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) \xrightarrow{\cup} \mathrm{H}^{k+l}(X;R) \qquad \text{and} \qquad \mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) \xrightarrow{\frown} \mathrm{H}_{l-k}(X;R).$$

In this chapter we will recall the definitions and we will then generalize these concepts to cup products and cap products on twisted (co-) homology. These notions will be essential in the formulation of the Poincaré Duality Theorem for twisted coefficients which we will discuss in the next Chapter 335.

This chapter is organized as follows:

- In Section 334.1 we will recall the definition and a few key properties of the classical cup product.
- In Section 334.2 we will make a few algebraic preparations which we will need to define the cup product on twisted cohomology.
- In Section 334.3 we will introduce the cup product on twisted cohomology.
- In Section 334.4 we will recall the definition and a few key properties of the classical cap product.
- In Section 334.5 we will introduce the cap product on twisted (co-) homology.
- In Appendix 334.6 we will study the interaction between various types of cup and cap product on finite coverings.
- In Section 334.7 we will provide exercises for Chapter 334.

In the next Chapter 335 we will state and prove the Twisted Poincaré Duality Theorem 335.1.

334.1. **The classical cup product.** Before we discuss the cup and cap product on (twisted) (co-) homology we should better recall Definition 143.8:

Notation 334.1. Let $n \in \mathbb{N}_0$. For $i = 0, \ldots, n$ we write

$$\mathfrak{v}_i \coloneqq (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}.$$

Furthermore for $a_0, \ldots, a_k \in \Delta^n$ we consider the continuous map

$$\begin{aligned} [a_0, \ldots, a_k] \colon \Delta^k &\to \Delta^n \\ \sum_{j=0}^{k} \lambda_j \cdot \mathfrak{v}_j &\mapsto \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

For example, the map $[\mathfrak{v}_0, \ldots, \mathfrak{v}_k] \colon \Delta^k \to \Delta^n$ is the inclusion in the first coordinates and the map $[\mathfrak{v}_{n-k}, \ldots, \mathfrak{v}_n] \colon \Delta^k \to \Delta^n$ is the inclusion in the last coordinates.

Now we can recall Definition 203.11 of the cup product on "untwisted" cohomology:

Definition 334.2. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^k(X;R)$ and $\psi \in \mathrm{C}^l(X;R)$ be cochains. We define the cup product of φ and ψ to be the $(k+l)$-cochain

$$\varphi \cup \psi \colon \mathrm{C}_{k+l}(X) \to R$$

that is given by

$$(\sigma \colon \Delta^{k+l} \to X) \mapsto \underbrace{\varphi(\underbrace{\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]}_{\text{map } \Delta^k \to X})}_{\in R} \cdot_R \underbrace{\psi(\underbrace{\sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_{k+l}]}_{\text{map } \Delta^l \to X})}_{\in R}.$$

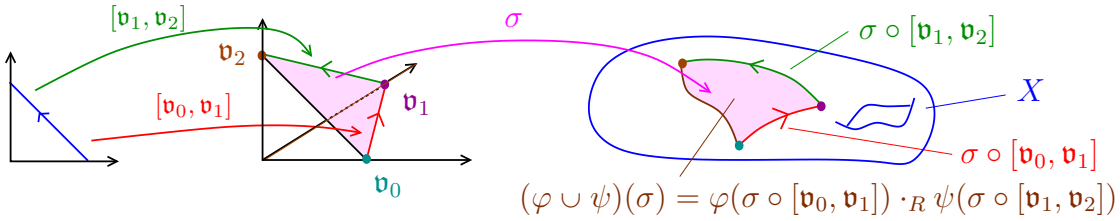
The cup product thus defines a map

$$\cup \colon \mathrm{C}^k(X;R) \times \mathrm{C}^l(X;R) \to \mathrm{C}^{k+l}(X;R).$$

As we saw in the Cup Product–Definition Lemma 203.13, it follows from the Cup Product–Coboundary Lemma 203.12 that this map on the cochain level descends to a well-defined R-bilinear map

$$\cup \colon \mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) \to \mathrm{H}^{k+l}(X;R)$$

on cohomology.



Recall that in Chapter 204 we developed the theory of cup products. In the following we summarize some of the earlier discussion: First of all, in Definition 204.7 we generalized the cup product to relative cohomology for "excisive triads". Furthermore we proved many properties of the cup product. For example we showed in the Cup Product-Naturality Lemma 204.8 that the cup product is natural in the topological space and in the Cup Product-Coefficients Lemma 204.9 we showed that the cup product is natural in the underlying ring.

We also proved the following, slightly tricky proposition:

Proposition 204.10. (Cup Product-Commutativity Proposition) Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. For any cohomological classes $\varphi \in \mathrm{H}^k(X, A; R)$ and $\psi \in \mathrm{H}^l(X, B; R)$ we have

$$\varphi \cup \psi = (-1)^{kl} \cdot \psi \cup \varphi \in \mathrm{H}^{k+l}(X, A \cup B; R).$$

334.2. **Algebraic preparations.** Our next goal is to adapt the above Definition 334.2 to the twisted setting. But note that if we start out with cohomology with twisted coefficients, then it is a priori not entirely clear what coefficients the target of the cup product should have. To deal with this issue we first need to introduce a little more algebra, which we will do in this section.

First we recall the Tensor Product-of-Rings Lemma 311.7 in a slightly simplified form:

Lemma 334.3. (Tensor Product-of-Rings Lemma) Let R and S be rings. The abelian group $R \otimes_{\mathbb{Z}} S$ together with the multiplication that is defined by

$$\left(\sum_{i=1}^{k} r_i \otimes s_i\right) \cdot \left(\sum_{j=1}^{l} \widetilde{r}_j \otimes \widetilde{s}_j\right) \coloneqq \sum_{i=1}^{k} \sum_{j=1}^{l} r_i \cdot \widetilde{r}_j \otimes s_i \cdot \widetilde{s}_j$$

is a ring. If R and S are commutative, then so $R \otimes_{\mathbb{Z}} S$.

Convention 334.4. Let R and S be rings. We will always use the Tensor Product-of-Rings Lemma 334.3 to view $R \otimes_{\mathbb{Z}} S$ as a ring.

Next we consider module structures on tensor products:

Lemma 334.5. (Tensor Product-of-Rings–Modules Lemma) Let R and S be rings. If M is an R-left module and N is an S-left module, then

$$\left(\sum_{i=1}^{k} r_i \otimes s_i\right) \cdot \left(\sum_{j=1}^{l} m_j \otimes n_j\right) \coloneqq \sum_{i=1}^{k} \sum_{j=1}^{l} r_i \cdot m_j \otimes s_i \cdot n_j$$

defines an $R \otimes_{\mathbb{Z}} S$-left module structure on the abelian group $M \otimes_{\mathbb{Z}} N$.

Proof. This statement is just a special case of the Tensor Product-of-Rings–Modules Lemma 311.9. ■

Finally we point out a special feature of modules over group rings:

Lemma 334.6. (Group Ring–Tensor Module Lemma) Let G be a group.

(1) Let M be a $\mathbb{Z}[G]$-left module and let N be a $\mathbb{Z}[G]$-left module. The map

$$\begin{array}{rcl} G \times M \otimes_{\mathbb{Z}} N & \to & M \otimes_{\mathbb{Z}} N \\ (g, m \otimes n) & \mapsto & g \cdot (m \otimes n) \coloneqq g \cdot m \otimes g \cdot n \end{array}$$

defines a representation of G. In particular it defines a $\mathbb{Z}[G]$-left module structure on the abelian group $M \otimes_{\mathbb{Z}} N$.

(2) Let R and S be rings. If M is an $(R, \mathbb{Z}[G])$-left left module and N is an $(S, \mathbb{Z}[G])$-left left module, then the module structures given in the Tensor Product-of-Rings–Modules Lemma 334.5 and (1) define an $(R \otimes_{\mathbb{Z}} S, \mathbb{Z}[G])$-left left module structure on $M \otimes_{\mathbb{Z}} N$.

Proof.

(1) This statement follows easily from the Tensor–Map Composition Lemma 311.6 together with Notation 320.8.

(2) This statement follows easily from the definitions. ■

We conclude this section with the following convention:

Convention 334.7. Let R and S be rings, let G be a group, let M be an $(R, \mathbb{Z}[G])$-left left module and let N be an $(S, \mathbb{Z}[G])$-left left module. We will always equip $M \otimes_{\mathbb{Z}} N$ with the $(R \otimes_{\mathbb{Z}} S, \mathbb{Z}[G])$-left left module structure that we defined in the Tensor Product-of-Rings–Modules Lemma 334.5 together with the Group Ring–Tensor Module Lemma 334.6.

334.3. The cup product on twisted cohomology.

With the algebraic preparations from the last section we are now fully equipped so that we can adapt the definition of the cup product on singular cohomology from Definition 334.2, to the twisted setting:

This section is organized as follows:

- In Section 334.3.1 we will introduce the cup product on "absolute" twisted cohomology.

- In Section 334.3.2 we will generalize the cup product to "relative" twisted cohomology.
- In Section 334.3.3 we will discuss some properties of the cup product on twisted cohomology.

334.3.1. **The cup product on absolute twisted cohomology.** Here is the definition of the cup product on twisted cochains:

Definition 334.8. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering from Definition 319.10. Furthermore let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $k,l \in \mathbb{N}_0$. Finally let $\varphi \in \mathrm{C}^k_{\mathrm{tw}}(X;M)$ and $\psi \in \mathrm{C}^l_{\mathrm{tw}}(X;N)$ be cochains. We define the cup product of φ and ψ to be the map

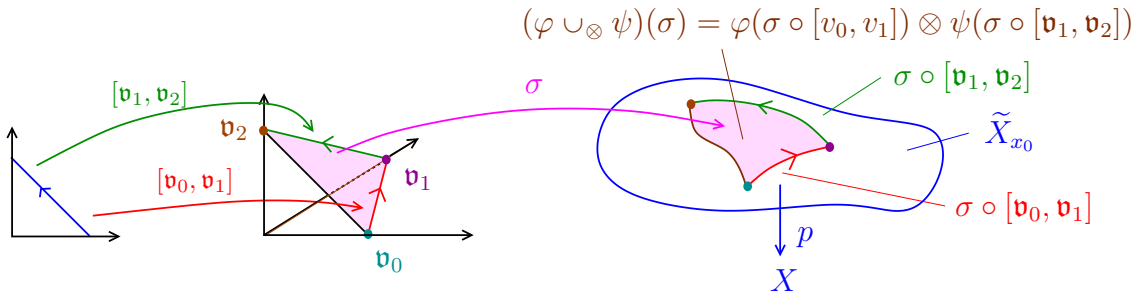
$$\varphi \cup_\otimes \psi\colon \mathrm{C}_{k+l}(\widetilde{X}_{x_0}) \;\to\; M \otimes_{\mathbb{Z}} N$$

that is given by

$$(\sigma\colon \Delta^{k+l} \to \widetilde{X}_{x_0}) \;\mapsto\; \underbrace{\varphi(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_k])}_{\in M} \otimes \underbrace{\psi(\sigma\circ[\mathfrak{v}_k,\ldots,\mathfrak{v}_{k+l}])}_{\in N}.$$

Note that the above map is evidently a $\mathbb{Z}[\pi_1(X,x_0)]$-left module homomorphism[3614], thus $\varphi \cup_\otimes \psi$ is an element in $\mathrm{C}^{k+l}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N)$. In other words, the cup product defines a map

$$\cup_\otimes\colon \mathrm{C}^k_{\mathrm{tw}}(X;M) \times \mathrm{C}^l_{\mathrm{tw}}(X;N) \;\to\; \mathrm{C}^{k+l}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N).$$



Evidently we want to show that the cup product from Definition 334.8 descends to cohomology. The key for doing so is the following lemma:

Lemma 334.9. (Twisted Cup Product–Coboundary Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Furthermore let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $k,l \in \mathbb{N}_0$. Given any cochains $\varphi \in \mathrm{C}^k_{\mathrm{tw}}(X;M)$ and $\psi \in \mathrm{C}^l_{\mathrm{tw}}(X;N)$ we have the equality

$$\delta_{k+l}(\varphi \cup_\otimes \psi) \;=\; \delta_k(\varphi) \cup_\otimes \psi \;+ (-1)^k \cdot \varphi \cup_\otimes \delta_l(\psi) \;\in\; \mathrm{C}^{k+l+1}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N).$$

Proof. The proof of this statement is basically the same as the proof of the Cup Product–Coboundary Lemma 203.12. Throughout that long and unpleasant calculation we just need to replace multiplication "$\cdot$" in R by the map $(a,b) \mapsto a \otimes b$. It will not come as a surprise to the reader that the details are left to the reader. ∎

Lemma 334.10. (Twisted Cup Product–Definition Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. The cup product

[3614]Here we use Convention 334.7 to view $M \otimes_{\mathbb{Z}} N$ as a $\mathbb{Z}[\pi_1(X,x_0)]$-left module.

$\cup_\otimes$ on cochains from Definition 334.8 descends to a well-defined map

$$\cup_\otimes \colon \mathrm{H}^k_{\mathrm{tw}}(X;M) \times \mathrm{H}^l_{\mathrm{tw}}(X;N) \;\to\; \mathrm{H}^{k+l}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N)$$

on cohomology with twisted coefficients.

Proof. As we saw already in the corresponding Cup Product–Definition Lemma 203.13, this statement follows easily from the Twisted Cup Product–Coboundary Lemma 334.9. ■

Now we can formulate the following variation on Definition 203.14:

Definition 334.11. We refer to the pairing

$$\cup_\otimes \colon \mathrm{H}^k_{\mathrm{tw}}(X;M) \times \mathrm{H}^l_{\mathrm{tw}}(X;N) \;\to\; \mathrm{H}^{k+l}_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N).$$

from the Twisted Cup Product–Definition Lemma 334.10 also as the cup product.

Example. If M and N are trivial $\mathbb{Z}[\pi]$-modules, then, using the explicit isomorphism from the Twisted–Trivial Coefficients Lemma 320.13, one can easily see that the Definition 334.11 of $\cup_\otimes$ is equivalent to **??** of $\cup_\otimes$. □

334.3.2. **The cup product on relative twisted cohomology.** In the classical setting we first introduced in Definition 203.11 the cup product on absolute untwisted cohomology. Afterwards, in Chapter 204 we gave the slightly more tricky definition of the cup product on relative cohomology.

We now follow the same route. In other words, we now turn to the somewhat technical definition of the relative cup product with twisted coefficients. As the reader surely remembers, even in the untwisted setting developing the relative cup product took some effort. In the following we will go through the key steps again and make appropriate modifications where necessary.

Definition 334.12. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let $A, B \subseteq X$ be subspaces. We write $\widetilde{A} \coloneqq p^{-1}(A)$ and $\widetilde{B} \coloneqq p^{-1}(B)$. As in Definition 325.1 we write

$$\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \;\coloneqq\; \left\{ \sum_{j=1}^{k} n_j \cdot \sigma_j \in \mathrm{C}_n(\widetilde{A} \cup \widetilde{B}) \;\middle|\; \begin{array}{c} \text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } \widetilde{A} \text{ or it lies in } \widetilde{B} \end{array} \right\}$$

Given a ring R and given an $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module M we set

$$\mathrm{C}^n_{\mathrm{tw}}(X, \{A,B\}; M) \;\coloneqq\; \ker\Big(\mathrm{C}^n_{\mathrm{tw}}(X;M) \underset{\substack{\uparrow \\ \text{we restrict a homomorphism } \mathrm{C}_n(\widetilde{X}_{x_0}) \to M \text{ to} \\ \text{the submodule } \mathrm{C}_n^{\{A,B\}}(\widetilde{A} \cup \widetilde{B})}}{\to} \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}\Big(\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}), M\Big)\Big).$$

This is again an R-left cochain complex and we denote the corresponding cohomology modules by $\mathrm{H}^*_{\mathrm{tw}}(X, \{A,B\}; M)$.

Before we continue we also need to introduce the following convention which is a slight variation on a convention that we had introduced on page 3815:

Identification 334.13. Let R be a ring. Given an R-left module C, an R-left submodule D of C and an R-left module M we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{R\text{-left}}(C/D, M) & \xrightarrow{\cong} & \{f\colon C \xrightarrow{\text{homomorphism}} M \mid f \text{ vanishes on } D\} \\ (\varphi\colon C/D \to M) & \mapsto & (C \to C/D \xrightarrow{\varphi} M). \end{array}$$

We will use this isomorphism to identify the modules left and right.

The following lemma is an analog of the Excisive Triad–Invariants Lemma 204.4:

Lemma 334.14. (Excisive Triad–Twisted Cohomology Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let W be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Furthermore let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces in the sense of Definition 325.1. By Identifiation 334.13 we have R-left module homomorphisms

$$\mathrm{C}^*_{\mathrm{tw}}(X, A \cup B; W) \quad \to \quad \mathrm{C}^*_{\mathrm{tw}}(X, \{A, B\}; W)$$

which form a chain. This chain map induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^n_{\mathrm{tw}}(X, A \cup B; W) \quad \xrightarrow{\cong} \quad \mathrm{H}^n_{\mathrm{tw}}(X, \{A, B\}; W)$$

of R-left modules.

Proof. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We write $\widetilde{A} := p^{-1}(A)$ and $\widetilde{B} := p^{-1}(B)$. We have the following commutative diagram of short exact sequences of chain complexes of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) & \longrightarrow & \mathrm{C}_*(\widetilde{X}_{x_0}) & \longrightarrow & \dfrac{\mathrm{C}_*(\widetilde{X}_{x_0})}{\mathrm{C}_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=\mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_*(\widetilde{A} \cup \widetilde{B}) & \longrightarrow & \mathrm{C}_*(\widetilde{X}_{x_0}) & \longrightarrow & \mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A} \cup \widetilde{B}) & \longrightarrow & 0. \end{array}$$

It follows immediately from the Regular Covering–Chain Complex Lemma 320.2 that all but possibly the top right module are free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. But the argument of the Regular Covering–Chain Complex Lemma 320.2 shows that the top right module is also free $\mathbb{Z}[\pi_1(X, x_0)]$-left module, for example a basis is given by taking for each singular simplex in X, that is neither contained in A nor in B, a lift to $\widetilde{X}_{x_0}$. Next note that by our hypothesis the triad (X, A, B) is universally excisive. It follows from Definition 325.1 and the Universally Excisive Properties Lemma 325.2 that this means that the left vertical map is a chain homotopy equivalence of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. The middle vertical map is the identity. It now follows from the Two-Out-Of-Three-Homotopy Equivalence Corollary 313.19 that the right vertical map is also a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules.

Note that by Identifiation 334.13 we have a natural identification

$$\mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}\left(\frac{\mathrm{C}_n(\widetilde{X}_{x_0})}{\mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B})}, W\right) \xrightarrow{\cong} \mathrm{C}^n_{\mathrm{tw}}(X, \{A, B\}; W).$$

The lemma follows from the above vertical chain homotopy equivalence, the above observation and the Dual Cochain Map Lemma 315.22. ■

The Excisive Triad–Twisted Cohomology Lemma 334.14 allows us to introduce the following definition:

Definition 334.15. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let M and N be $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. If (X, A, B) is a universally excisive triad of topological spaces, then we refer to the composition

$$\mathrm{H}^k_{\mathrm{tw}}(X, A; M) \times \mathrm{H}^l_{\mathrm{tw}}(X, B; N) \underset{\uparrow}{\to} \mathrm{H}^{k+l}_{\mathrm{tw}}(X, \{A, B\}; M \underset{\mathbb{Z}}{\otimes} N) \underset{\uparrow}{\xleftarrow{\cong}} \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A \cup B; M \underset{\mathbb{Z}}{\otimes} N)$$

as in the Excisive Triad–Lemma 204.4 (2b) we see that the formula for $\cup_\otimes$ from from Definition 334.8 defines such a map

isomorphism by the Excisive Triad–Twisted Cohomology Lemma 334.14

again as a cup product (or sometimes as a relative cup product) and we denote it as above by the "$\cup_\otimes$"-symbol.

Remark. As in the case of the original cup product, see the discussion on page 3927, one can easily show that the relative cup product

$$\cup_\otimes \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \times \mathrm{H}^l_{\mathrm{tw}}(X, B; N) \ \to \ \mathrm{H}^{\mathrm{tw}}_{k+l}(X, A \cup B; M \otimes_{\mathbb{Z}} N)$$

is given by the explicit formula from Definition 334.8, if $A = \varnothing$, or $B = \varnothing$ or $A = B$. □

334.3.3. **Properties of the cup product on twisted cohomology.** In this subsection we will collect some essential facts about the cup product on twisted cohomology.

Lemma 334.16. (Cup Product–Tensor Module Homomorphism Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R and S be rings, let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module and let N be an $(S, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Furthermore let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces. The relative cup product

$$\cup_\otimes \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^l_{\mathrm{tw}}(X, B; N) \ \to \ \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A \cup B; M \otimes_{\mathbb{Z}} N)$$

is a homomorphism of $R \otimes_{\mathbb{Z}} S$-left modules.[3615]

Proof. This lemma follows immediately from the definitions and the (Tensor Product-of-Rings–Modules Lemma 334.5. ■

We continue with a frequently used variation on the cup product:

Definition 334.17. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring with (possibly trivial) involution and let $\varphi \colon \mathbb{Z}[\pi] \to R$ be a homomorphism of rings with involution.[3616] We recall the following notation:

- According to Convention 320.9 we denote by ${}_\varphi R$ the $(R, \mathbb{Z}[\pi])$-left left module, where $g \in \pi$ acts on $r \in R$ via $g \cdot r := r \cdot_R \varphi(g^{-1})$.
- As in Definition 310.7 we consider the $\mathbb{Z}[\pi]$-left module ${}_{\mathrm{tr}} R$ that is given by the trivial π-action.

[3615] We can use the Tensor Product-of-Rings–Modules Lemma 334.5 to view $\mathrm{H}^k_{\mathrm{tw}}(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^l_{\mathrm{tw}}(X, B; N)$ and $M \otimes_{\mathbb{Z}} N$ as $R \otimes_{\mathbb{Z}} S$-left modules. Thus both sides have a natural $R \otimes_{\mathbb{Z}} S$-left module structure.

We define the multiplication map

$$\begin{aligned}\mu\colon {}_{\varphi}R\otimes_{\mathbb{Z}}{}_{\varphi}R &\to {}_{\mathrm{tr}}R\\ r\otimes s &\mapsto r\cdot\overline{s}.\end{aligned}$$

Note that this is a homomorphism of $\mathbb{Z}[\pi]$-left modules[3617] and of (R,R)-left left modules[3618]. Next let $A,B\subseteq X$ be subspaces such that (X,A,B) is a universally excisive triad of topological spaces. Then we define[3619]

$$\begin{aligned}\cup_R\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;{}_{\varphi}R)\times\mathrm{H}^l_{\mathrm{tw}}(X,B;{}_{\varphi}R) &\to \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B;{}_{\mathrm{tr}}R)\\ (\alpha;\beta) &\mapsto \mu_*\big(\underbrace{\alpha\cup_{\otimes}\beta}_{\in\mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B;{}_{\varphi}R\otimes_{\mathbb{Z}}{}_{\varphi}R)}\big).\end{aligned}$$

Note that the fact that μ is (R,R)-left left linear implies, by our module structures, that this pairing is sesquilinear over R.

The following lemma shows that we recover our classical cup product on "untwisted homology" in all relevant cases from the twisted cup product:

Lemma 334.18. (Classical-Twisted Cup Product Lemma) Let X be a topological space that is path-connected and locally contractible, let $x_0\in X$ and let $A,B\subseteq X$ be subspaces such that (X,A,B) is a universally excisive triad of topological spaces. Let R be a commutative ring which we equip with the trivial involution. We consider the ring homomorphism $\varphi\colon\mathbb{Z}[\pi]\xrightarrow{g\mapsto 1_R}R$. We make two observations:

- The map φ is a homomorphism between rings with involution. Thus the cup product $\cup_R$ is defined.
- The resulting $\mathbb{Z}[\pi]$-module ${}_{\varphi}R$ is precisely the trivial $\mathbb{Z}[\pi_1(X,x_0)]$-left module ${}_{\mathrm{tr}}R$.

In this setting the following diagram commutes:[3620]

$$\begin{array}{ccc}\mathrm{H}^k(X,A;R)\times\mathrm{H}^l(X,B;R) & \xrightarrow[\text{see Definition 203.11}]{\cup} & \mathrm{H}^{k+l}(X,A\cup B;R)\\ \cong\big\downarrow T \quad T\big\downarrow\cong & & T\big\downarrow\cong\\ \mathrm{H}^k_{\mathrm{tw}}(X,A;{}_{\mathrm{tr}}R)\times\mathrm{H}^l_{\mathrm{tw}}(X,B;{}_{\mathrm{tr}}R) & \xrightarrow[\text{see Definition 334.17}]{\cup_R} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B;{}_{\mathrm{tr}}R).\end{array}$$

Here the vertical maps are the natural isomorphisms T from the Twisted–Trivial Coefficients Lemma 320.13.

[3616]As we mentioned on page 5981, we will always view the group ring $\mathbb{Z}[\pi]$ as equipped with the involution $\overline{g}:=g^{-1}$. Note that any ring homomorphism with $g\mapsto 1_R$ for every $g\in\pi$, is necessarily a homomorphism of rings with involution.

[3617]Indeed, given $g\in\pi$ and $r,s\in R$ we have the following equalities:

$$\mu(g\cdot(r\otimes s))=\mu((r\cdot g^{-1})\otimes(s\cdot g^{-1}))=(r\cdot g^{-1})\cdot\overline{(s\cdot g^{-1})}=r\cdot g^{-1}\cdot g\cdot\overline{s}=r\cdot\overline{s}=g\underset{{}_{\mathrm{tr}}R}{\cdot}r\cdot\overline{s}.$$

[3618]We use the two obvious left multiplications by R on ${}_{\varphi}R\otimes_{\mathbb{Z}}{}_{\varphi}R$ to view ${}_{\varphi}R\otimes_{\mathbb{Z}}{}_{\varphi}R$ as an (R,R)-left left module. Furthermore we use the usual left multiplication by R, and the right multiplication together with the involution, to view R as an (R,R)-left left module. With these R-module structures it is clear that the map μ is a homomorphism of (R,R)-left left modules.

[3619]Recall that in Definition 321.12 we saw that a homomorphism of $\mathbb{Z}[\pi]$-left modules induces a map on twisted cohomology. We apply this fact to the $\mathbb{Z}[\pi]$-left module homomorphism $\mu\colon {}_{\varphi}R\underset{\mathbb{Z}}{\otimes}{}_{\varphi}R\to{}_{\mathrm{tr}}R$.

[3620]By the Universally Excisive Properties Lemma 325.2 we know that universally excisive triads are also excisive, this shows that the bottom relative cup product is indeed defined.

Proof. The proof follows easily from the definitions and the explicit description of all of the maps involved. ■

Next we state some properties of the cup product on twisted cohomology. First of all we have the following proposition, which can be viewed as an analog of the "classical" Cup Product-Commutativity Proposition 203.20:

Proposition 334.19. (Twisted Cup Product-Commutativity Proposition) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let M and N be $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. We denote by

$$\begin{array}{rcl} \tau\colon M_{\otimes_{\mathbb{Z}}} N & \to & N \otimes_{\mathbb{Z}} M \\ \sum_{i=1}^{k} m_i \otimes n_i & \mapsto & \sum_{i=1}^{k} n_i \otimes m_i \end{array}$$

the isomorphism that swaps the two factors. Note that it follows easily from Convention 334.7 that this is in fact a $\mathbb{Z}[\pi_1(X, x_0)]$-left module homomorphism. Given any subset $A \subseteq X$ and given any $\varphi \in \mathrm{H}^k_{\mathrm{tw}}(X, A; M)$ and $\psi \in \mathrm{H}^l_{\mathrm{tw}}(X, A; N)$ the following equality holds:

$$\tau(\underbrace{\varphi \cup_{\otimes} \psi}_{\in M \otimes_{\mathbb{Z}} N}) = (-1)^{k \cdot l} \cdot \psi \cup_{\otimes} \varphi \in \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A, N \otimes_{\mathbb{Z}} M).$$

Proof. We leave it to the reader to verify that the not entirely trivial proof of the Cup Product-Commutativity Proposition 203.20 does indeed generalize to the present setting. Again we mostly have to replace multiplication "$\cdot$" in a commutative ring by the map $(a, b) \mapsto a \otimes b$. ■

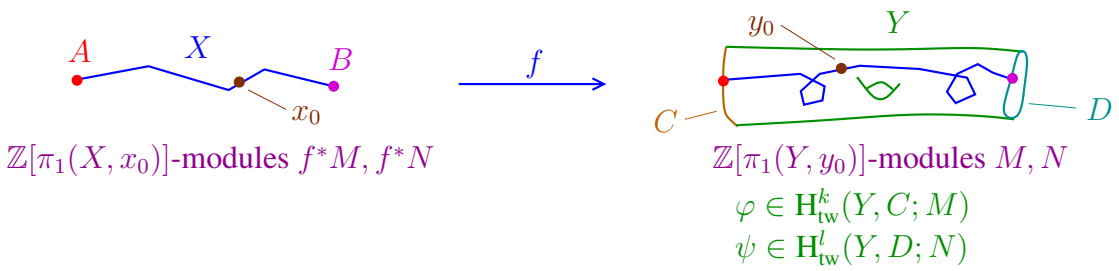
The following proposition, which is the analog of the Cup Product-Naturality Lemma 204.8, shows that the twisted cup product is natural in the topological space:

Proposition 334.20. (Twisted Cup Product-Naturality Proposition) Let X and Y be topological spaces that are path-connected and locally contractible and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let $A, B \subseteq X$ and $C, D \subseteq Y$ be subspaces such that (X, A, B) and (Y, C, D) are universally excisive triads of topological spaces. and let R and S be rings. Let M be an $(R, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module and let N be an $(S, \mathbb{Z}[\pi_1(Y, y_0)])$-left left module. Let $f\colon (X, A, B) \to (Y, C, D)$ be a continuous map with $f(x_0) = y_0$. For all $k, l \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(Y, C; M) \times \mathrm{H}^l_{\mathrm{tw}}(Y, D; N) & \xrightarrow{\cup_{\otimes}} & \mathrm{H}^{k+l}_{\mathrm{tw}}(Y, C \cup D; M \otimes_{\mathbb{Z}} N) \\ \downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^k_{\mathrm{tw}}(X, A; f^*M) \times \mathrm{H}^l_{\mathrm{tw}}(X, B; f^*N) & \xrightarrow{\cup_{\otimes}} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A \cup B; \underbrace{f^*M \otimes_{\mathbb{Z}} f^*N}_{=f^*(M \otimes_{\mathbb{Z}} N)}). \end{array}$$

Proof. As in the proof of the classical Cup Product-Naturality Lemma 204.8 one sees that the statements follow easily from the definitions. ■

Finally the following lemma, which is an analog of the Cup Product-Coefficients Lemma 204.9, says that the cup product is natural in the coefficients:



Lemma 334.21. (Twisted Cup Product-Coefficients Lemma) Let X and Y be topological spaces that are path-connected and locally contractible and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let (X,A,B) and (Y,C,D) be universally excisive triads of topological spaces and let R and S be rings. If $\varphi\colon M \to M'$ is a homomorphism between two $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left modules and if $\psi\colon N \to N'$ is a homomorphism between two $(S,\mathbb{Z}[\pi_1(X,x_0)])$-left left modules, then for all $k,l \in \mathbb{N}_0$ the following diagram commutes:[3621]

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X,A;M) \times \mathrm{H}^l_{\mathrm{tw}}(X,B;N) & \xrightarrow{\cup_\otimes} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B;M\otimes_{\mathbb{Z}} N) \\ \downarrow{\scriptstyle \varphi_*} \quad \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle (\varphi\otimes\psi)_*} \\ \mathrm{H}^k_{\mathrm{tw}}(X,A;M') \times \mathrm{H}^l_{\mathrm{tw}}(X,B;N') & \xrightarrow{\cup_\otimes} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X,A\cup B;M'\otimes_{\mathbb{Z}} N'). \end{array}$$

Here the vertical maps are the maps from Definition 321.12.

Proof. As in the proof of the classical Cup Product-Coefficients Lemma 204.9 one sees, once again, that the statements follow easily from the definitions. ∎

334.4. **The classical cap product.** Before we get to introduce the cap product on twisted (co-) homology we should recall Definition 206.2 and Definition 206.5 of the untwisted cap product:

Definition 334.22. Let X be a topological space and let R be a commutative ring. Given a cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma\colon \Delta^l \to X$ with $k \leq l$ we define the cap product as[3622]

$$\varphi \cap \sigma \;:=\; \underbrace{\sigma \circ [\mathfrak{v}_k,\ldots,\mathfrak{v}_l]}_{\text{map } \Delta^{l-k} \to X} \otimes \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0,\ldots,\mathfrak{v}_k])}_{\in R} \;\in\; \mathrm{C}_{l-k}(X;R) := \mathrm{C}_{l-k}(X) \otimes R.$$

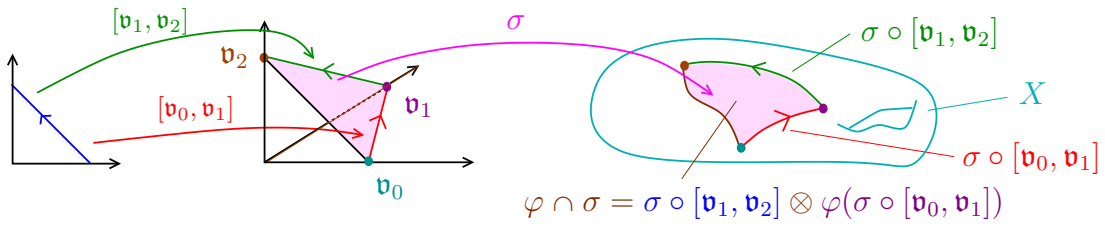
For $k > l$ we define the cap product to be the zero map. Since $\mathrm{C}_l(X)$ is the free abelian group on the set of singular l-simplices we get an induced map

$$\begin{array}{rcl} \cap\colon \mathrm{C}^k(X;R) \times \mathrm{C}_l(X;R) & \to & \mathrm{C}_{l-k}(X;R) \\ \left(\varphi, \sum\limits_{i=1}^d \sigma_i \otimes r_i\right) & \mapsto & \sum\limits_{i=1}^d (\varphi \cap \sigma_i) \otimes r_i. \end{array}$$

As we saw in the proof of the Cap Product–Definition Lemma 206.4, it follows from the Cap Product–Boundary Lemma 206.3 that the above map descends to a well-defined R-bilinear map

$$\cap\colon \mathrm{H}^k(X;R) \times \mathrm{H}_l(X) \;\to\; \mathrm{H}_{l-k}(X;R).$$

[3621] Note that $\varphi \otimes \psi\colon M \otimes_{\mathbb{Z}} N \to M' \otimes_{\mathbb{Z}} N'$ is easily sense to be a homomorphism of $\mathbb{Z}[\pi]$-left modules, where we equip both sides with the $\mathbb{Z}[\pi]$-left module structures from Convention 334.7.



Recall that in Chapter 206 we developed the theory of the cap product. First of all, in Definition 206.10 we extended the above definition of the cap product to relative (co-) homology of excisive triads. Furthermore in the Cap Product-Naturality Lemma 206.13 we showed that the cap product is natural in the topological space. We conclude this recollection with the following lemma which relates the classical cup product to the classical cap product:

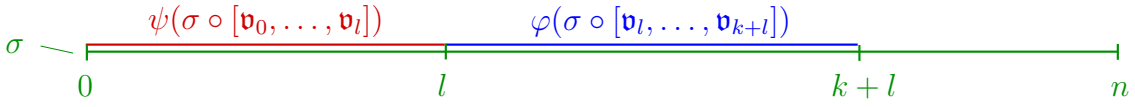
Lemma 206.15. (Cup-Cap Lemma) Let X be a topological space and let R be a commutative ring. Let $A\subseteq X$ be a subset and let $C,D\in\{\varnothing,A\}$. For any $\varphi\in\mathrm{H}^k(X,C;R)$, any $\psi\in\mathrm{H}^l(X,D;R)$ and any $\sigma\in\mathrm{H}_n(X,C\cup D;R)$ we have

$$\varphi\cap(\underbrace{\psi\cap\sigma}_{\in\mathrm{H}_{n-l}(X,C;R)})\;=\;(\underbrace{\psi\cup\varphi}_{\in\mathrm{H}^{k+l}(X,C\cup D;R)})\cap\sigma\quad\in\mathrm{H}_{n-k-l}(X;R).$$

Proof. Let X be a topological space and let R be a commutative ring. The lemma follows easily from the corresponding claim on the (co-) chain level:

Claim. Let $\varphi\in\mathrm{C}^k(X;R)$ and let $\psi\in\mathrm{C}^l(X;R)$. Given any singular n-simplex $\sigma\colon\Delta^n\to X$ we have

$$\varphi\cap(\psi\cap\sigma)\;=\;(\psi\cup\varphi)\cap\sigma\;\in\;\mathrm{C}_{n-k-l}(X;R).$$



Proof. If $k+l>n$, then both sides are by definition 0. So suppose that $k+l\leq n$. We then see that

$$\begin{aligned}\varphi\cap(\psi\cap\sigma)&=\varphi\cap(\sigma\circ[\mathfrak{v}_l,\ldots,\mathfrak{v}_n]\otimes\psi(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_l]))\\&=(\sigma\circ[\mathfrak{v}_{l+k},\ldots,\mathfrak{v}_n]\otimes\varphi(\sigma\circ[\mathfrak{v}_l,\ldots,\mathfrak{v}_{l+k}]))\cdot_R\psi(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_l])\\&=F\sigma\circ[\mathfrak{v}_{l+k},\ldots,\mathfrak{v}_n]\otimes\underbrace{\psi(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_l])\cdot_R\varphi(\sigma\circ[\mathfrak{v}_l,\ldots,\mathfrak{v}_{l+k}])}_{=(\psi\cup\varphi)(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_{l+k}])}\\&\quad\uparrow\\&\quad\text{since }R\text{ is commutative}^{3623}\\&=(\psi\cup\varphi)\cap\sigma.\end{aligned}$$

■

334.5. **The cap product on twisted (co-) homology.** In this section we want to generalize Definition 334.22 from singular (co-) homology to the twisted setting. This section is organized as follows:

- In Section 334.5.1 we will introduce the cap product on "absolute" twisted (co-) homology.
- In Section 334.5.2 we will generalize the cap product to "relative" twisted (co-) homology.

[3623]Note that this shows that in more general settings some extra care will be needed to the get the statement right.

- In Section 334.5.3 we will perform an explicit calculation of a non-trivial twisted cap product in degree ≥ 1.
- In Section 334.5.4 we will discuss some properties of the cap product on twisted (co-) homology and we will discuss its relationship to the cup product.

334.5.1. **The cap product on absolute twisted (co-) homology.** We jump into the fray with the following definition:

Definition 334.23. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $k,l \in \mathbb{N}_0$. Given any $\varphi \in \mathrm{C}^k_{\mathrm{tw}}(X;M)$, given any singular simplex $\sigma\colon \Delta^l \to X$ with $k \leq l$ and given any $n \in N$ we define the cap product as[3624]

$$\varphi \cap_{\otimes} (n \otimes \sigma) \coloneqq \big(\underbrace{\varphi(\sigma \circ [\mathfrak{v}_0,\ldots,\mathfrak{v}_k])}_{\in M} \otimes n\big) \otimes \underbrace{\sigma \circ [\mathfrak{v}_k,\ldots,\mathfrak{v}_l]}_{\text{map } \Delta^{l-k} \to \widetilde{X}_{x_0}} \in \underbrace{(\overline{M \underset{\mathbb{Z}}{\otimes} N}) \underset{\mathbb{Z}[\pi]}{\otimes} \mathrm{C}_{l-k}(\widetilde{X}_{x_0})}_{=\mathrm{C}^{\mathrm{tw}}_{l-k}(X;M\otimes_{\mathbb{Z}}N)}.$$

(We will show in the Twisted Cap Product–Boundary Lemma 334.25 (1) below that this definition makes sense.) It now follows easily from the Regular Covering–Chain Complex Lemma 320.2 together with the Tensor–Direct Sum Lemma 312.15 that we can extend this definition linearly to $\mathrm{C}^{\mathrm{tw}}_k(X;N) = \overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_l(\widetilde{X}_{x_0})$ and we obtain the cap product

$$\cap_{\otimes}\colon \mathrm{C}^k_{\mathrm{tw}}(X;M) \times \mathrm{C}^{\mathrm{tw}}_l(X;N) \to \mathrm{C}^{\mathrm{tw}}_{l-k}(X;M \otimes_{\mathbb{Z}} N).$$

For $k > l$ we define the cap product to be the zero map.

The following lemma is the analog of the Cap Product–Boundary Lemma 206.3:

Lemma 334.24. (Twisted Cap Product–Boundary Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let M and N be $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and let $k,l \in \mathbb{N}_0$.

(1) The above Definition 334.23 of the cap product on the (co-) chain level makes sense, in the sense that we did indeed define a map out of tensor product over $\mathbb{Z}[\pi_1(X,x_0)]$.
(2) Given $\varphi \in \mathrm{C}^k_{\mathrm{tw}}(X;M)$, given a singular simplex $\sigma\colon \Delta^l \to \widetilde{X}_{x_0}$ with $k \leq l$ and given $n \in N$ we have the following equality in $\mathrm{C}^{\mathrm{tw}}_{l-k-1}(X;M \otimes_{\mathbb{Z}} N)$:

$$\partial_{l-k}(\varphi \cap_{\otimes} (n \otimes \sigma)) = (-1)^k \cdot (-\delta_k(\varphi) \cap_{\otimes} (n \otimes \sigma) + \varphi \cap_{\otimes} (n \otimes \partial_l(\sigma))).$$

Proof. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi = \pi_1(X,x_0)$ and we denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let M and N be $\mathbb{Z}[\pi]$-left modules and let $k,l \in \mathbb{N}_0$ with $k \leq l$. We start out with the following claim:

[3624]The definition does not look entirely consistent with Definition 334.22. The reason is that for ordinary singular homology with coefficients in an abelian group G we considered the tensor product $\mathrm{C}_k(X) \otimes G$ where the coefficients are on the right. But in twisted homology we work with the coefficients on the left. The author is reduced to offering his most sincerest apologies for this inconsistency.

Claim. Given $\varphi \in \mathrm{C}^k_{\mathrm{tw}}(X;M)$, given a singular simplex $\sigma\colon \Delta^l \to \widetilde{X}_{x_0}$ with $k \leqq l$, given $n \in N$ and given $g \in \pi$ we have the following equality in $(\overline{M \otimes_{\mathbb{Z}} N}) \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_{l-k}(\widetilde{\widetilde{X}}_{x_0})$:

$$\big(\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]) \otimes n{\cdot}g\big) \otimes \sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_l] = \big(\varphi((g{\cdot}\sigma) \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k]) \otimes n\big) \otimes (g{\cdot}\sigma) \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_l].$$

Proof. The claim follows from a careful little calculation in $(\overline{M \otimes_{\mathbb{Z}} N}) \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_{l-k}(\widetilde{X}_{x_0})$:

$$\begin{aligned}
\big(\varphi(\sigma \circ [\mathfrak{v}_0 \ldots, \mathfrak{v}_k]) \otimes n \cdot g\big) \otimes \sigma \circ [\mathfrak{v}_k \ldots, \mathfrak{v}_l] &= \big(\varphi(\sigma \circ [\mathfrak{v}_0 \ldots, \mathfrak{v}_k]) \cdot g^{-1} \cdot g \otimes n \cdot g\big) \otimes \sigma \circ [\mathfrak{v}_k \ldots, \mathfrak{v}_l] \\
&\underset{\uparrow}{=} \big(\varphi(\sigma \circ [\mathfrak{v}_0 \ldots, \mathfrak{v}_k]) \cdot g^{-1} \otimes n\big) \otimes g \cdot (\sigma \circ [\mathfrak{v}_k \ldots, \mathfrak{v}_l])
\end{aligned}$$

by Definition 312.3 of the tensor product over $\mathbb{Z}[\pi]$ and by Notation 320.7 of the $\mathbb{Z}[\pi]$-right module structure on $\overline{M \otimes_{\mathbb{Z}} N}$

$$\underset{\uparrow}{=} \big(\varphi((g \cdot \sigma) \circ [\mathfrak{v}_0 \ldots, \mathfrak{v}_k]) \otimes n\big) \otimes (g \cdot \sigma) \circ [\mathfrak{v}_k \ldots, \mathfrak{v}_l].$$

since $(g \cdot \sigma)([\mathfrak{v}_i, \ldots, \mathfrak{v}_j]) = g \cdot (\sigma \circ [\mathfrak{v}_i, \ldots, \mathfrak{v}_j])$, since φ is a $\mathbb{Z}[\pi]$-left module homomorphism, and by Notation 320.7 of the $\mathbb{Z}[\pi]$-right module structure on $\overline{M}$ ⊞

Now we turn to the proofs of the two statements of the lemma:

(1) It follows from the claim together with the Tensor Product–Universal Property Lemma 320.5 that we did indeed define a map out of $N \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*(\widetilde{X}_{x_0})$.

(2) The proof of this statement is almost identical to the proof of the Cap Product–Boundary Lemma 206.3. In fact, as we already explained in the proof of the Tensor–Cup Product Lemma 334.9, we just need to replace multiplication "$\cdot$" in R by the tensor product $\otimes$. ■

The following lemma is the analog of the Cap Product–Definition Lemma 206.4 in the untwisted case:

Lemma 334.25. (Twisted Cap Product Lemma) We suppose that we are in the setting of Definition 334.23. The cap product on the (co-) chain level descends to a map on the (co-) homological level.

Proof. As in the proof of the Cap Product–Definition Lemma 206.4 we see that this lemma is a straightforward consequence of the Twisted Cap Product–Boundary Lemma 334.24. ■

We can now give the twisted analog of Definition 334.22:

Definition 334.26. We refer to the pairing

$$\cap_{\otimes}\colon \mathrm{H}^k_{\mathrm{tw}}(X;M) \times \mathrm{H}^{\mathrm{tw}}_l(X;N) \;\to\; \mathrm{H}^{\mathrm{tw}}_{l-k}(X;M \otimes_{\mathbb{Z}} N).$$

from the Twisted Cap Product Lemma 334.25 as the cap product.

334.5.2. **The cap product on relative twisted (co-) homology.** Before we discuss properties of the cap product with twisted coefficients let us immediately move on to the slightly more tricky definition of the relative cap product on twisted (co-) homology. As in the classical setting this requires a little bit of an effort. We start out with the following definition:

Definition 334.27. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Let R be a commutative ring and let N be an $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module. Let (X,A,B)

be a triad of topological spaces. We write $\widetilde{A} \coloneqq p^{-1}(A)$ and $\widetilde{B} \coloneqq p^{-1}(B)$. We write

$$\mathrm{C}_n^{\mathrm{tw}}(X, \{A, B\}; N) \;\coloneqq\; \mathrm{coker}\left(\overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}) \to \overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{X}_{x_0})\right).$$

As always the boundary map on the chain complex $\overline{N} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_n(\widetilde{X}_{x_0})$ descends to a boundary map on $\mathrm{C}_n^{\mathrm{tw}}(X, \{A, B\}; N)$ which we can thus view as an R-left chain complex. We denote the corresponding homology modules by $\mathrm{H}_*^{\mathrm{tw}}(X, \{A, B\}; N)$.

The following lemma is the analog of the "untwisted" Excisive Triad–Invariants Lemma 204.4:

Lemma 334.28. (Universally Excisive Triad–Invariants Lemma) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a commutative ring and let N be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. If $A, B \subseteq X$ are subspaces such that (X, A, B) is a universally excisive triad of topological spaces, then the obvious homomorphism

$$\mathrm{C}_*^{\mathrm{tw}}(X, \{A, B\}; N) \quad \to \quad \mathrm{C}_*^{\mathrm{tw}}(X, A \cup B; N)$$

of R-left modules induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathrm{tw}}(X, \{A, B\}; N) \quad \xrightarrow{\cong} \quad \mathrm{H}_n^{\mathrm{tw}}(X, A \cup B; N)$$

of R-left modules.

Proof. In the proof of the Excisive Triad–Twisted Cohomology Lemma 334.14 we already showed that the map

$$\mathrm{C}_*(\widetilde{X}, \{\widetilde{A}, \widetilde{B}\}) \;\to\; \mathrm{C}_*(\widetilde{X}, \widetilde{A} \cup \widetilde{B})$$

is a chain homotopy equivalence of free $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. The lemma now follows from the Homology-with-Coefficients Lemma 314.6 (2b) together with the Chain Homotopy Equivalence-H_*-Corollary 313.7. ■

The following definition can be viewed as the twisted analog of Definition 206.10:

Definition 334.29. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Furthermore let M and N be $\mathbb{Z}[\pi_1(X, x_0)]$-left modules and let $k, l \in \mathbb{N}_0$. Finally suppose that we are given subspaces $A, B \subseteq X$ such that (X, A, B) is a universally excisive triad. We refer to the map

$$\begin{array}{ccc} & & \text{isomorphism given by the Universally Excisive Triad–Invariants Lemma 334.28} \\ & & \downarrow \\ \mathrm{H}_{\mathrm{tw}}^k(X, A; M) \times \mathrm{H}_l^{\mathrm{tw}}(X, A \cup B; N) & \xleftarrow{\cong} & \mathrm{H}_{\mathrm{tw}}^k(X, A; M) \times \mathrm{H}_l^{\mathrm{tw}}(X, \{A, B\}; N) \\ & \xrightarrow{\cap_\otimes} & \mathrm{H}_{l-k}^{\mathrm{tw}}(X, B; M \otimes_{\mathbb{Z}} N) \\ & & \uparrow \\ & & \text{the argument of the Triad–Cap Product Lemma 206.9 shows that the same explicit map as in Definition 334.23 descends to a well-defined map} \end{array}$$

as the cap product (sometimes we refer to it as the relative cap product) that we also denote by "$\cap_\otimes$". For $k > l$ we define the cap product to be the zero map.

Remark. As in the case of the original cap product, see the discussion on page 3958, one can easily show that the relative cap product

$$\cap_\otimes \colon \mathrm{H}_{\mathrm{tw}}^k(X, A; M) \times \mathrm{H}_l^{\mathrm{tw}}(X, A \cup B; N) \;\to\; \mathrm{H}_{l-k}^{\mathrm{tw}}(X, B; M \otimes_{\mathbb{Z}} N)$$

is given by the explicit formula from Definition 334.23, if $A = \varnothing$, or $B = \varnothing$ or $A = B$. □

334.5.3. **An example of the twisted cap product.** The only goal in this section is to perform one explicit calculation of a twisted cap product in dimension ≥ 1:

Example. We consider the Möbius band $M := ([0,1]\times[0,1])/(0,y)\sim(1,1-y)$ together with the usual universal covering $p\colon \mathbb{R}\times[0,1]\to M$. Let $\varphi\colon \pi_1(M)\to \mathrm{GL}(1,\mathbb{Z})$ be the unique non-trivial representation. We would like to calculate the twisted cap product

$$\mathrm{H}^1_{\mathrm{tw}}(M;{}_{\mathrm{tr}}\mathbb{Z})\times \mathrm{H}^{\mathrm{tw}}_2(M,\partial M;{}_{\varphi}\mathbb{Z}) \xrightarrow{\cap_\otimes} \mathrm{H}^{\mathrm{tw}}_1(M,\partial M;{}_{\mathrm{tr}}\mathbb{Z}\otimes_{\mathbb{Z}}{}_{\varphi}\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \mathrm{H}^{\mathrm{tw}}_1(M,\partial M;{}_{\varphi}\mathbb{Z}).$$

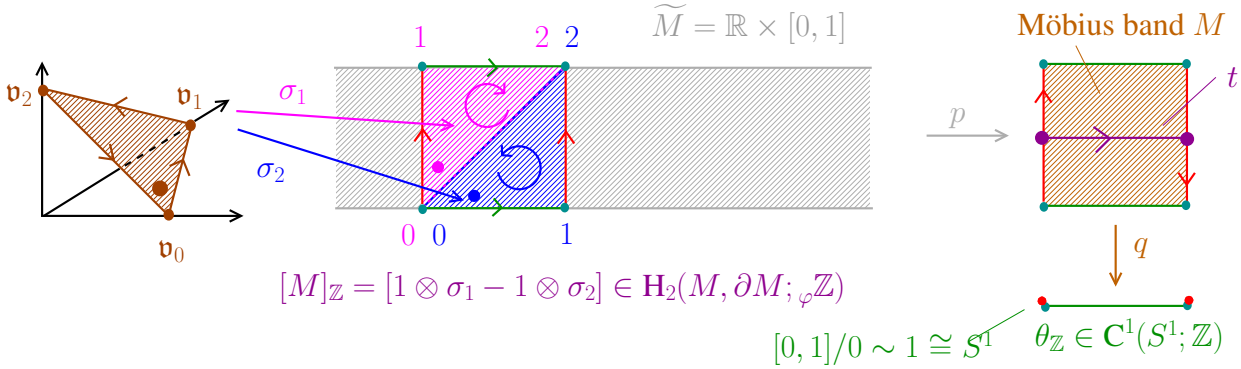
induced by the multiplication isomorphism $\mu\colon {}_{\mathrm{tr}}\mathbb{Z}\otimes_{\mathbb{Z}}{}_{\varphi}\mathbb{Z}\to{}_{\varphi}\mathbb{Z}$

More precisely, we consider the following (co-) homology classes:

(a) Let $q\colon M\to[0,1]/0\sim 1=_{\mathrm{i}} S^1$ be the projection. Recall that this map is a homotopy equivalence. Next let $\theta_{\mathbb{Z}}\in \mathrm{C}^1(S^1;\mathbb{Z})$ be the singular cocycle from page 3813. For the purpose of this discussion we just need to know the following:
   (i) For a constant map $c\colon\Delta^1\to S^1$ we have $\theta_{\mathbb{Z}}(c)=0$.
   (ii) For the standard cycle $\tau\colon\Delta^1\to S^1$ that is given by $(1-t,t)\mapsto e^{2\pi i t}$ we have $\theta_{\mathbb{Z}}(\tau)=1$.
   (iii) By the discussion on page 3835 we know that $\mathrm{H}^1(S^1;\mathbb{Z})=\mathbb{Z}\cdot[\theta_{\mathbb{Z}}]$.
   We consider the cohomology class $[q^*(\theta_{\mathbb{Z}})]\in \mathrm{H}^1(M;\mathbb{Z})$. From the Twisted–Trivial Coefficients Lemma 320.13 we then obtain $T([q^*(\theta_{\mathbb{Z}})])\in \mathrm{H}^1_{\mathrm{tw}}(M;{}_{\mathrm{tr}}\mathbb{Z})$.

(b) Similar to page 6224 let $[M]_{\mathbb{Z}}=[1\otimes\sigma_1-1\otimes\sigma_2]\in \mathrm{H}^{\mathrm{tw}}_2(M,\partial M;{}_{\varphi}\mathbb{Z})$ be the fundamental class of the Möbius band that is illustrated in the figure below.

Our goal is to calculate $\mu_*(T(q^*(\theta_{\mathbb{Z}}))\cap_\otimes[M]_{\mathbb{Z}})\in \mathrm{H}^{\mathrm{tw}}_1(M,\partial M;{}_{\varphi}\mathbb{Z})$.



We perform the following calculation in $\mathrm{H}^{\mathrm{tw}}_1(M,\partial M;{}_{\varphi}\mathbb{Z})$:

$$\begin{aligned}
\mu([T(p^*(\theta_{\mathbb{Z}}))]\cap_\otimes[M]_{\mathbb{Z}}) &\overset{\text{by the definition of } T \text{ in the Twisted–Trivial Coefficients Lemma 320.13}}{=} [\mu(q^*(p^*(\theta_{\mathbb{Z}}))\cap_\otimes(1\otimes\sigma_1)-q^*(p^*(\theta_{\mathbb{Z}}))\cap_\otimes(1\otimes\sigma_2))]\\
&\overset{\text{see Definition 334.23}}{=} [\underbrace{(q\circ p)^*(\theta_{\mathbb{Z}}))(\sigma_1\circ[\mathfrak{v}_0,\mathfrak{v}_1])}_{=\theta_{\mathbb{Z}}(q\circ p\circ\sigma_1\circ[\mathfrak{v}_0,\mathfrak{v}_1])=0,\text{ by (i)}}\otimes(\sigma_1\circ[\mathfrak{v}_1,\mathfrak{v}_2]-)\underbrace{(q\circ p)^*(\theta_{\mathbb{Z}}))(\sigma_2\circ[\mathfrak{v}_0,\mathfrak{v}_1])}_{=\theta_{\mathbb{Z}}(q\circ p\circ\sigma_2\circ[\mathfrak{v}_0,\mathfrak{v}_1])=1,\text{ by (ii)}}\otimes(\sigma_2\circ[\mathfrak{v}_1,\mathfrak{v}_2])]\\
&= -[1\otimes(\sigma_2\circ[\mathfrak{v}_1,\mathfrak{v}_2])].
\end{aligned}$$

In Exercise 334.5 we will see that $[1\otimes(\sigma_2\circ[\mathfrak{v}_1,\mathfrak{v}_2])]\in \mathrm{H}^{\mathrm{tw}}_1(M,\partial M;{}_{\varphi}\mathbb{Z})$ is non-zero, in fact we will see that it is a generator of the twisted homology group. □

334.5.4. **Properties of the cap product on twisted (co-) homology.** In this final subsection we will collect some essential facts about the cap product on twisted (co-) homology. First of all we have the following lemma, which is the analog of the Cup Product–Tensor Module Homomorphism Lemma 334.16:

Lemma 334.30. (Cap Product–Tensor Module Homomorphism) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R and S be rings, let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module and let N be an $(S, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Furthermore let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces. The relative cap product

$$\cap_{\otimes} \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; N) \to \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; M \otimes_{\mathbb{Z}} N)$$

is a homomorphism of $R \otimes_{\mathbb{Z}} S$-left modules.

Proof. This lemma follows immediately from the definitions and the Tensor Product-of-Rings–Modules Lemma 334.5. ■

In Definition 334.17 we defined the twisted cup product with coefficients in a ring. We have the following analogous definition in the context of the cap product:

Definition 334.31. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring with involution and let $\varphi \colon \mathbb{Z}[\pi] \to R$ be a homomorphism of rings with involution. As in Definition 334.17 we consider the multiplication map

$$\begin{aligned} \mu \colon {}_{\varphi}R \otimes_{\mathbb{Z}} {}_{\varphi}R &\to {}_{\mathrm{tr}}R \\ r \otimes s &\mapsto r \cdot \overline{s}. \end{aligned}$$

By Footnote 3617 and Footnote 3618 we know that this is a homomorphism of $\mathbb{Z}[\pi]$-left modules and of (R, R)-left left modules. Next let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces. Then we define

$$\begin{aligned} \cap_R \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; {}_{\varphi}R) \times \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; {}_{\varphi}R) &\to \mathrm{H}^{l-k}_{\mathrm{tw}}(X, B; {}_{\mathrm{tr}}R) \\ (\alpha, \beta) &\mapsto \mu_*\Big(\underbrace{\alpha \cap_{\otimes} \beta}_{\in \mathrm{H}^{l-k}_{\mathrm{tw}}(X, B; {}_{\varphi}R \otimes_{\mathbb{Z}} {}_{\varphi}R)} \Big). \end{aligned}$$

The following, rather unsurprising analog of the Classical-Twisted Cup Product Lemma 334.18 says that we can recover our classical cap product in all relevant cases from the twisted cap product:

Lemma 334.32. (Classical-Twisted Cap Product Lemma) Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces. Let R be a commutative ring. In this setting the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}R) & \times & \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; {}_{\mathrm{tr}}R) & \xrightarrow[\text{see Definition 334.31}]{\cap_R} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; {}_{\mathrm{tr}}R) \\ {\scriptstyle T}\uparrow\cong & & {\scriptstyle T}\downarrow\cong & & {\scriptstyle T}\downarrow\cong \\ \mathrm{H}^k(X, A; R) & \times & \mathrm{H}_l(X, A \cup B; R) & \xrightarrow[\text{see Definition 206.2}]{\cap} & \mathrm{H}_{l-k}(X, B; R). \end{array}$$

Here the vertical maps are again the natural isomorphisms T from the Twisted–Trivial Coefficients Lemma 320.13.

Proof. Again the lemma follows easily from plugging in all the definitions. ■

The following lemma is the twisted analog of the Cap Product-Kronecker Lemma 206.12:

Lemma 334.33. (Twisted–Cap Product-Kronecker Lemma) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible and let $A \subseteq X$. We write $\pi := \pi_1(X, x_0)$. Let R be a ring and let M and N be two $(R, \mathbb{Z}[\pi])$-left left modules. The composition

$$\mathrm{H}^k_{\mathrm{tw}}(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_k(X, A; N) \xrightarrow{\cap_\otimes} \mathrm{H}^{\mathrm{tw}}_0(X; M \otimes_{\mathbb{Z}} N) \xrightarrow[\substack{\text{from the Twisted } \mathrm{H}_0 \\ \text{Proposition 323.3}}]{\text{augmentation } \epsilon_X} (M \otimes_{\mathbb{Z}} N)_\pi$$

($\uparrow$ co-invariants as introduced in Definition 323.1)

is precisely the Kronecker pairing $\langle\ ,\ \rangle_{(X,A)}$ from Definition 326.3. In other words, for all $\varphi \in \mathrm{H}^k_{\mathrm{tw}}(X, A; M)$ and $\sigma \in \mathrm{H}^{\mathrm{tw}}_k(X, A; N)$ we have the following equality in $(M \otimes_{\mathbb{Z}} N)_\pi$:

$$\epsilon_X(\varphi \cap_\otimes \sigma) \ = \ \langle \varphi, \sigma \rangle_{(X,A)}.$$

Proof. The lemma follows immediately from plugging in the definitions. ■

Next we want to introduce an important variation on the cap product, which allows us - perhaps counterintuitively - to cap a twisted cohomology class with an untwisted ordinary homology class:

Definition 334.34. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$ and let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces. Furthermore let R be a ring, let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module and let $k, l \in \mathbb{N}_0$. As always we consider the natural isomorphism $T\colon \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; {}_{\mathrm{tr}}\mathbb{Z}) \to \mathrm{H}_l(X, A \cup B)$ from the Twisted–Trivial Coefficients Lemma 320.13 (together with the remark on page 3077) and we consider the natural isomorphism

$$\begin{aligned} \mu\colon M \otimes_{\mathbb{Z}} {}_{\mathrm{tr}}\mathbb{Z} &\to M \\ m \otimes n &\mapsto n \cdot m \end{aligned}$$

of R-left modules from the Free Module–Tensor-with-Ring Lemma 312.18 (1). We define

$$\begin{aligned} \cap_{\mathbb{Z}}\colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \times \mathrm{H}_l(X, A \cup B) &\to \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; M) \\ (\varphi, \sigma) &\mapsto \mu_*\big(\underbrace{\varphi \cap_\otimes T^{-1}(\sigma)}_{\in \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; M \otimes_{\mathbb{Z}} {}_{\mathrm{tr}}\mathbb{Z})} \big). \end{aligned}$$

Finally note that it follows easily from the Cap Product–Tensor Module Lemma 334.30 that given any $\sigma \in \mathrm{H}_l(X, A \cup B)$ the map

$$\begin{aligned} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) &\to \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; M) \\ \varphi &\mapsto \varphi \cap_{\mathbb{Z}} \sigma \end{aligned}$$

is an R-left module homomorphism.

Example. Let (X, x_0) be a compact oriented n-dimensional pointed topological manifold with fundamental class $[X] \in \mathrm{H}_k(X, \partial X)$. Furthermore let R be a ring, let M be an

$(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module and let $k \in \mathbb{N}_0$. From Definition 334.34 we obtain the R-left module homomorphism

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, \partial X; M) & \to & \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M) \\ \varphi & \mapsto & \varphi \cap_{\mathbb{Z}} [X]. \end{array}$$

The study of this map will be the main topic of the next Chapter 335. □

Next, the following proposition is essentially a generalization of the Cap Product-Naturality Lemma 206.13 and it can be viewed as a sibling of the Twisted Cup Product-Naturality Proposition 334.20:

Proposition 334.35. (Twisted Cap Product-Naturality Proposition) Let X and Y be topological spaces that are path-connected and locally contractible and let $x_0 \in X$ and $y_0 \in Y$. Furthermore let $A, B \subseteq X$ and $C, D \subseteq Y$ be subspaces such that (X, A, B) and (Y, C, D) are universally excisive triads of topological spaces. Let $f\colon (X, A, B) \to (Y, C, D)$ be a continuous map with $f(x_0) = y_0$ and let $k, l \in \mathbb{N}_0$.

(1) Let M and N be $\mathbb{Z}[\pi_1(Y, y_0)]$-left modules. The following diagram commutes:

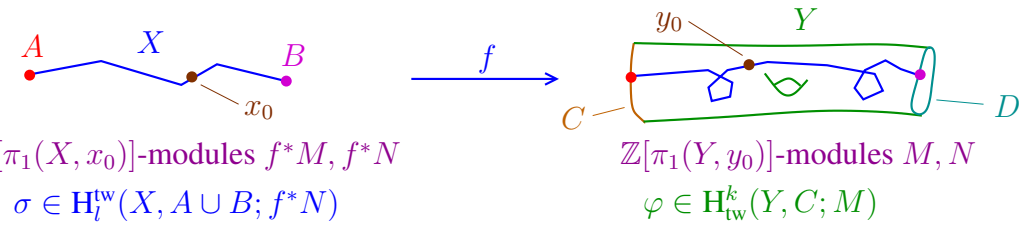
$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(Y, C; M) \times \mathrm{H}^{\mathrm{tw}}_l(Y, C \cup D; N) & \xrightarrow{\cap_\otimes} & \mathrm{H}^{\mathrm{tw}}_{l-k}(Y, D; M \otimes_{\mathbb{Z}} N) \\ \downarrow{\scriptstyle f^*} \quad \uparrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f_*} \\ \mathrm{H}^k_{\mathrm{tw}}(X, A; f^*M) \times \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; f^*N) & \xrightarrow{\cap_\otimes} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; \underbrace{f^*M \otimes_{\mathbb{Z}} f^*N}_{=f^*(M \otimes_{\mathbb{Z}} N)}). \end{array}$$

More precisely, for any $\varphi \in \mathrm{H}^k_{\mathrm{tw}}(Y, C; M)$ and any $\sigma \in \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; f^*N)$ we have the following equality in $\mathrm{H}^{\mathrm{tw}}_{l-k}(Y, D; M \otimes_{\mathbb{Z}} N)$:

$$\varphi \cap_\otimes f_*(\sigma) \;=\; f_*(f^*(\varphi) \cap_\otimes \sigma).$$

(2) Let M be a $\mathbb{Z}[\pi_1(Y, y_0)]$-left module. The following diagram commutes:[3625]

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(Y, C; M) \times \mathrm{H}_l(Y, C \cup D) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(Y, D; M) \\ \downarrow{\scriptstyle f^*} \quad \uparrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f_*} \\ \mathrm{H}^k_{\mathrm{tw}}(X, A; f^*M) \times \mathrm{H}_l(X, A \cup B) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; f^*M). \end{array}$$



Proof. As in the Cap Product-Naturality Lemma 206.13 one sees that all statements follow easily from the definitions. ■

[3625] Statement (2) is essentially Statement (1) with $N = {}_{\mathrm{tr}}\mathbb{Z}$. Since this special case will be needed frequently we decided to formulate it separately.

For completeness sake we state the following analog of the Twisted Cup Product-Coefficients Lemma 334.21:

Lemma 334.36. (Twisted Cap Product-Coefficients Lemma) Let (X, x_0) be a topological space that is path-connected and locally contractible. Furthermore let $A, B \subseteq X$ be subspaces such that (X, A, B) is a universally excisive triad of topological spaces. If $\varphi\colon M \to M'$ and $\psi\colon N \to N'$ are homomorphisms of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules, then for all $k, l \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; N) & \xrightarrow{\cap_\otimes} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; M \otimes_{\mathbb{Z}} N) \\ \downarrow{\scriptstyle \varphi_*} \quad \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle (\varphi\otimes\psi)_*} \\ \mathrm{H}^k_{\mathrm{tw}}(X, A; M') \times \mathrm{H}^{\mathrm{tw}}_l(X, A \cup B; N') & \xrightarrow{\cap_\otimes} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X, B; M' \otimes_{\mathbb{Z}} N'). \end{array}$$

Here the vertical maps are the maps from Definition 321.12.

Proof. Not surprisingly again the author claims that the statements follow easily from the definitions. ∎

The following lemma is the generalization of the above Cup-Cap Lemma 206.15 to the present context:

Lemma 334.37. (Twisted Cup-Cap Lemma) Let X be a topological space that is path-connected and locally contractible. Let $x_0 \in X$ and let L, M and N be $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. Furthermore let $A \subseteq X$ be a subset and let $C, D \in \{\varnothing, A\}$. Finally we denote by $\tau\colon L \otimes_{\mathbb{Z}} M \to M \otimes_{\mathbb{Z}} L$ the natural isomorphism.

(1) Given any $\varphi \in \mathrm{H}^k_{\mathrm{tw}}(X, C; L)$, $\psi \in \mathrm{H}^l_{\mathrm{tw}}(X, D; M)$ and $\sigma \in \mathrm{H}^{\mathrm{tw}}_d(X, C \cup D; N)$ we have[3626]

$$\varphi \cap_\otimes (\psi \cap_\otimes \sigma) \;=\; \tau_*(\psi \cup_\otimes \varphi) \cap_\otimes \sigma \;\in\; \mathrm{H}^{\mathrm{tw}}_{d-k-l}(X; L \otimes_{\mathbb{Z}} M \otimes_{\mathbb{Z}} N).$$

(2) If $\sigma \in \mathrm{H}_d(X, C \cup D)$, then

$$\varphi \cap_\otimes (\psi \cap_{\mathbb{Z}} \sigma) \;=\; \tau_*(\psi \cup_\otimes \varphi) \cap_{\mathbb{Z}} \sigma \;\in\; \mathrm{H}^{\mathrm{tw}}_{d-k-l}(X; L \otimes_{\mathbb{Z}} M).$$

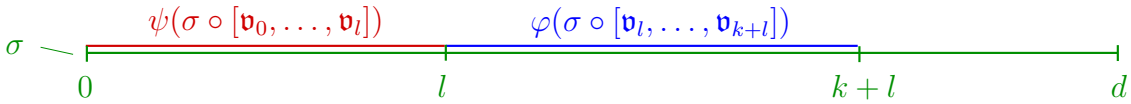
Proof.

(1) As in the case of the Cup-Cap Lemma 206.15 this statement follows almost immediately from the following claim:

Claim. Let $\varphi \in \mathrm{C}^k_{\mathrm{tw}}(X; L)$ and let $\psi \in \mathrm{C}^l_{\mathrm{tw}}(X; M)$. Given any singular d-simplex $\sigma\colon \Delta^d \to \widetilde{X}_{x_0}$ and given any $n \in N$ we have the following equality

$$\varphi \cap_\otimes (\psi \cap_\otimes (n \otimes \sigma)) \;=\; \tau(\psi \cup_\otimes \varphi) \cap_\otimes (n \otimes \sigma)$$

in $(L \otimes_{\mathbb{Z}} M \otimes_{\mathbb{Z}} N) \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_{d-k-l}(\widetilde{X}_{x_0})$.



Proof. If $k + l > d$, then both sides are by definition 0. So suppose that $k + l \leq d$. We then see that

[3626] As we discussed on page 3060, for tensor products over $\mathbb{Z}$ we decided - with a slightly guilty conscience - to drop parentheses.

$$\begin{aligned}
\varphi \cap_\otimes (\psi \cap_\otimes (n \otimes \sigma)) &\overset{\text{Definition 334.23}}{=} \varphi \cap_\otimes (\psi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_l]) \otimes n \otimes (\sigma \circ [\mathfrak{v}_l, \dots, \mathfrak{v}_d])) \overset{\text{Definition 334.23}}{=} \\
&= \varphi(\sigma \circ [\mathfrak{v}_l, \dots, \mathfrak{v}_{l+k}]) \otimes \psi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_l]) \otimes n \otimes (\sigma \circ [\mathfrak{v}_{l+k}, \dots, \mathfrak{v}_d]) \\
&= \tau(\psi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_l]) \otimes \varphi(\sigma \circ [\mathfrak{v}_l, \dots, \mathfrak{v}_{l+k}])) \otimes n \otimes (\sigma \circ [\mathfrak{v}_{l+k}, \dots, \mathfrak{v}_d]) \\
&\qquad \overset{\text{definition of } \tau}{\uparrow} \\
&= \tau_*(\psi \cup_\otimes \varphi) \cap_\otimes (n \otimes \sigma). \\
&\overset{\text{Definition 334.23}}{\uparrow}
\end{aligned}$$

(2) This statement is essentially Statement (1) applied to $N = {}_{\mathrm{tr}}\mathbb{Z}$. ■

334.6. **Appendix: Cup and Cap product on finite coverings.** We conclude this chapter with a reinterpretation of some cup and cap products in the context of finite coverings. To make sense of our result we first provide a slightly simplified version of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5:

Proposition 334.38. (Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition) Let X be a topological space that is path-connected and locally contractible. We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering which we equip with the natural base point $\widetilde{x}_0$. Let $A \subseteq X$. We write $\widetilde{A} := p^{-1}(A)$. Furthermore let $\varphi\colon \pi \to G$ be an epimorphism onto a group and let S be a *commutative* ring. We equip the free S-module $S[G]$ with the $(S, \mathbb{Z}[\pi])$-left left module structure from Notation 323.4. Next let $q\colon (Y_\varphi, y_0) \to (X, x_0)$ be a covering of pointed topological spaces such that $q_*(\pi_1(Y_\varphi, y_0)) = \ker(\varphi)$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (Y_\varphi, y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{X}_{x_0} & \xrightarrow{\quad r \quad} & Y_\varphi. \\
& {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\
& X &
\end{array}$$

Finally we write $A_\varphi := q^{-1}(A)$. The maps

$$\begin{aligned}
\Omega\colon \overbrace{S[G] \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}, \widetilde{A})}^{=\mathrm{C}_k^{\mathrm{tw}}(X, A; {}_\varphi S[G])} &\to \overbrace{\mathrm{C}_k(Y_\varphi, A_\varphi) \otimes_{\mathbb{Z}} S}^{=\mathrm{C}_k(Y_\varphi, A_\varphi; S)} \\
s \cdot g \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) &\mapsto (r \circ (g \cdot \sigma\colon \Delta^k \to \widetilde{X}_{x_0})) \otimes s
\end{aligned}$$

form an isomorphism of chain complexes of $S[G]$-left-modules. In particular for each $k \in \mathbb{N}_0$ these maps induce an isomorphism

$$\Omega_*\colon \underbrace{\mathrm{H}_k^{\mathrm{tw}}(X, A; {}_\varphi S[G])}_{\text{twisted homology}} \to \underbrace{\mathrm{H}_k(Y_\varphi, A_\varphi; S)}_{\substack{\text{singular homology} \\ \text{with } S\text{-coefficients}}}$$

of $S[G]$-left modules.

We continue with the following simplified version of the Finite Covering-$\mathrm{H}_{\mathrm{tw}}^*$-Proposition 323.13:

Proposition 334.39. (Finite Covering-$\mathrm{H}_{\mathrm{tw}}^*$-Proposition) We assume that we are in the same setup as in the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 334.38, except that now we demand

that G is *finite*. Then the maps

$$\begin{aligned}\Phi\colon \underset{\mathbb{Z}[\pi]\text{-left}}{\operatorname{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}),{}_\varphi S[G]) &\to \underset{\mathbb{Z}}{\operatorname{Hom}}(\mathrm{C}_k(Y_\varphi,A_\varphi),S)\\ \gamma &\mapsto \begin{pmatrix} \mathrm{C}_k(Y_\varphi,A_\varphi) & \to & S\\ [\sigma\colon \Delta^k\to Y_\varphi] & \mapsto & \begin{array}{c}\text{pick a lift } \widetilde{\sigma}\colon\Delta^k\to\widetilde{X}_{x_0} \text{ and}\\ \text{take the } e\text{-coefficient}\\ \text{of } \gamma(\widetilde{\sigma})\in S[G]\end{array}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\Psi\colon \underset{\mathbb{Z}}{\operatorname{Hom}}(\mathrm{C}_k(Y_\varphi,A_\varphi),S) &\to \underset{\mathbb{Z}[\pi]\text{-left}}{\operatorname{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}),{}_\varphi S[G])\\ \gamma &\mapsto \begin{pmatrix} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}) & \to & S[G]\\ [\sigma\colon \Delta^k\to \widetilde{X}_{x_0}] & \mapsto & \underbrace{\sum_{g\in G}\gamma(g\cdot(\Delta^k\xrightarrow{r\circ\sigma}Y_\varphi))\cdot g}_{\in S[G]\text{ since }G\text{ is a finite group}}\end{pmatrix}\end{aligned}$$

are well-defined isomorphisms of $S[G]$-left cochain complexes.

We can now formulate the first new result of this appendix:

Lemma 334.40. (Finite Covering–Cap Product Lemma) Let (X,x_0) be a topological space that is path-connected and locally contractible. Furthermore let $\varphi\colon \pi_1(X,x_0)\to G$ be an epimorphism onto a finite group and let $q\colon (Y_\varphi,y_0)\to(X,x_0)$ be a path-connected covering with $q_*(\pi_1(Y_\varphi,y_0))=\ker(\varphi)$. For every $A\subseteq X$ the following diagram commutes:

$$\begin{array}{ccc}\mathrm{H}^k_{\mathrm{tw}}(X,A;{}_\varphi\mathbb{Z}[G])\times\mathrm{H}_l(X,A) & \xrightarrow{\quad\cap_{\mathbb{Z}}\quad} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X;{}_\varphi\mathbb{Z}[G])\\ \cong\big\downarrow\Phi_* \quad \big\downarrow q^* & & \cong\big\downarrow\Omega_*\\ \mathrm{H}^k(Y_\varphi,B;\mathbb{Z})\times\mathrm{H}_l(Y_\varphi,B) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{l-k}(Y_\varphi).\end{array}$$

We make the following clarifications:

- We write $B:=q^{-1}(A)$.
- The map $q^*\colon \mathrm{H}_l(X,A)\to\mathrm{H}_l(Y_\varphi,B)$ is the transfer map from Definition 165.3 (here we use in particular that G is finite.)
- The map $\cap_{\mathbb{Z}}$ is the cap product from Definition 334.34.
- The map $\cap$ at the bottom is the cap product from Definition 206.10.
- The map $\Phi_*\colon \mathrm{H}^k_{\mathrm{tw}}(X,A;\mathbb{Z}[G])\to\mathrm{H}^k(Y_\varphi,B;\mathbb{Z})$ is the isomorphism from the Finite Covering-$\mathrm{H}^*_{\mathrm{tw}}$-Proposition 334.39.
- The map $\Omega_*\colon \mathrm{H}^{\mathrm{tw}}_{l-k}(X;{}_\varphi\mathbb{Z}[G])\to\mathrm{H}_{l-k}(Y_\varphi)$ is the isomorphism from the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 334.38.

Proof. We make a few preparations:

- We write $Y:=Y_\varphi$.
- The Replace Universal Covering Proposition 320.16 allows us to replace the natural universal covering of X by the covering $q\colon Y\to X$.
- It suffices to show that the diagram commutes on the (co-) chain level.
- By the observation on page 6349 it suffices to deal with the case $A=\varnothing$.

It follows from these preparations that it suffices to prove the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}[G]\text{-left}}(\overset{\gamma}{\mathrm{C}_k}(Y), {}_{\varphi}\mathbb{Z}[G]) \times \mathrm{C}_l(\overset{\sigma}{X}) & \xrightarrow{\quad\cap_{\mathbb{Z}}\quad} & \overline{{}_{\varphi}\mathbb{Z}[G]} \otimes_{\mathbb{Z}[G]} \mathrm{C}_{l-k}(Y) \\ \cong \big\downarrow \Phi \qquad \big\downarrow q^* & & \cong \big\downarrow \Omega \\ \mathrm{Hom}(\mathrm{C}_k(Y), \mathbb{Z}) \times \mathrm{C}_l(Y) & \xrightarrow{\quad\cap\quad} & \mathrm{C}_{l-k}(Y). \end{array}$$

Proof. Let $\gamma \in \mathrm{Hom}_{\mathbb{Z}[G]\text{-left}}(\mathrm{C}_k(Y), {}_{\varphi}\mathbb{Z}[G])$ and let $\sigma\colon \Delta^l \to X$ be a singular simplex. Let $\widetilde{\sigma}\colon \Delta^l \to Y$ be an arbitrary lift of $\sigma\colon \Delta^l \to X$ to the covering Y. We introduce some notation:

- We write $\widetilde{\sigma}' := \widetilde{\sigma} \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]\colon \Delta^k \to Y$ and $\widetilde{\sigma}'' := \widetilde{\sigma} \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l]\colon \Delta^{l-k} \to Y$.
- Given $g \in G$ we denote by a_g the g-coefficient of $\gamma(\widetilde{\sigma}') \in \mathbb{Z}[G]$. In other words, we have the equality $\gamma(\widetilde{\sigma}') = \sum\limits_{g\in G} a_g \cdot g$.

We start with out an initial calculation in $\mathbb{Z}[G]$:

$$\begin{array}{rcl} \sum\limits_{g\in G} \underbrace{\Phi(\gamma)(g\cdot\widetilde{\sigma}')}_{\in\mathbb{Z}} \cdot g & \overset{\text{definition of } \Phi}{=} & \sum\limits_{g\in G} (e\text{-coefficient of } \gamma(g\cdot\widetilde{\sigma}'))\cdot g \\ & & \overset{\substack{\text{since } \gamma \text{ is } \mathbb{Z}[G]\text{-left linear, and by definition of the } \mathbb{Z}[G]\text{-left module structure on } {}_{\varphi}\mathbb{Z}[G]}}{=} \sum\limits_{g\in G} (e\text{-coefficient of } \gamma(\widetilde{\sigma}')\cdot g^{-1})\cdot g \\ \overset{\substack{\uparrow \\ \text{by the definition of the } a_h}}{=} \sum\limits_{g\in G} (e\text{-coefficient of } \sum\limits_{h\in G} a_h\cdot h\cdot g^{-1})\cdot g & = & \overset{\substack{\uparrow \\ \text{follows from elementary algebra}}}{\sum\limits_{h\in G} a_h\cdot h.} \end{array}$$

We now see that we have the following equalities in $\mathrm{C}_{l-k}(Y)$:

$$\begin{array}{rcl} \Omega(\gamma\cap_{\mathbb{Z}}\sigma) & \overset{\substack{\text{Definition 334.34} \\ \downarrow}}{=} & \Omega(\gamma(\widetilde{\sigma}')\otimes\widetilde{\sigma}'') \\ & = & \sum\limits_{g\in G} \Phi(\gamma)(g\cdot\widetilde{\sigma}')\cdot(g\cdot\widetilde{\sigma}'') \\ & & \overset{\substack{\uparrow \\ \text{definition of } \cap}}{=} \end{array} \quad \begin{array}{rcl} & \overset{\substack{\text{by definition of } \Omega \text{ and} \\ \text{the choice of the } a_g \\ \downarrow}}{=} & \Big(\sum\limits_{g\in G} a_g\cdot g\Big)\cdot\widetilde{\sigma}'' \\ & = & \Phi(\gamma)\cap\sum\limits_{g\in G} g\cdot\widetilde{\sigma} \\ & & \overset{\substack{\uparrow \\ \text{Definition 165.3 of the} \\ \text{transfer map } q^*}}{=} \end{array} \quad \begin{array}{rcl} & \overset{\substack{\text{follows from the} \\ \text{above calculation} \\ \downarrow}}{=} & \Big(\sum\limits_{g\in G} \Phi(\gamma)(g\cdot\widetilde{\sigma}')\cdot g\Big)\cdot\widetilde{\sigma}'' \\ & = & \Phi(\gamma)\cap q^*(\sigma). \end{array}$$

■

We turn to the second new result of this appendix:

Lemma 334.41. (Finite Covering–Cup Product Lemma) Let (X, x_0) be a pointed topological space that is path-connected and locally contractible and let $A \subseteq X$. Furthermore let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a finite group. Finally let $q\colon (Y_\varphi, y_0) \to (X, x_0)$ be a path-connected covering with $q_*(\pi_1(Y_\varphi, y_0)) = \ker(\varphi)$. We

write $B := q^{-1}(A)$. The following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^k(Y_\varphi, B; \mathbb{Z}) & \times & \mathrm{H}^l(Y_\varphi, B; \mathbb{Z}) & \xrightarrow{(\widetilde{\mu},\widetilde{\nu}) \mapsto \sum_{g\in G} (\widetilde{\mu} \cup g\cdot\widetilde{\nu})(\sigma)\otimes g} & \mathrm{H}^{k+l}(Y_\varphi, B; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[G] \\
\cong \big\downarrow \Psi_* & & \cong \big\downarrow \Psi_* & & \big\downarrow (q^*)^* \otimes \mathrm{id}_{\mathbb{Z}[G]} \\
 & & & & \mathrm{H}^{k+l}(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[G] \\
 & & & & \big\downarrow [\sigma]\otimes g \mapsto [\sigma\otimes g] \\
 & & & & \mathrm{H}^{k+l}(X, A; \mathbb{Z}[G]) \\
 & & & & \cong \big\downarrow T \\
\mathrm{H}^k_{\mathrm{tw}}(X, A; \mathbb{Z}[G]) & \times & \mathrm{H}^l_{\mathrm{tw}}(X, A; \mathbb{Z}[G]) & \xrightarrow{\cup} \mathrm{H}^k_{\mathrm{tw}}(X, A; \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \xrightarrow{\mu_*} & \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}\mathbb{Z}[G]).
\end{array}$$

$$\mathrm{H}^k_{\mathrm{tw}}(X, A; \mathbb{Z}[G]) \times \mathrm{H}^l_{\mathrm{tw}}(X, A; \mathbb{Z}[G]) \xrightarrow{\cup_{\mathbb{Z}[G]}} \mathrm{H}^{k+l}_{\mathrm{tw}}(X, A; {}_{\mathrm{tr}}\mathbb{Z}[G]).$$

We make the following clarifications:

- The map $\Psi_* \colon \mathrm{H}^r(Y_\varphi, B; \mathbb{Z}) \to \mathrm{H}^r_{\mathrm{tw}}(X, A; \mathbb{Z}[G])$ is the isomorphism from the Finite Covering-$\mathrm{H}^*_{\mathrm{tw}}$-Proposition 334.39.
- We denote by $\mu \colon \mathbb{Z}[G] \otimes \mathbb{Z}[G] \to \mathbb{Z}[G]$ the map that is given by $(r, s) \mapsto r \cdot \overline{s}$.
- The map $q^* \colon \mathrm{C}_k(X, A) \to \mathrm{C}_k(Y_\varphi, B)$ is the transfer map from Definition 165.3. Note that this induces a cochain map $(q^*)^*$ on cochains and a map on cohomology.
- The map T is provided by the Twisted–Trivial Coefficients Lemma 320.13.

Proof. We will consider the statement on the (co-) chain level. Thus it suffices to prove the following claim:

Claim. Given any $\widetilde{\mu} \in \mathrm{C}^r(Y_\varphi; \mathbb{Z})$ and $\widetilde{\nu} \in \mathrm{C}^s(Y_\varphi; \mathbb{Z})$ we have the equality

$$\Psi(\widetilde{\mu}) \ \cup_{\mathbb{Z}[G]} \ \Psi(\widetilde{\nu}) \ = \ q^*\Big(\sum_{h\in G} \big(\underbrace{\widetilde{\mu} \cup h \cdot \widetilde{\nu}}_{\in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_{r+s}(Y_\varphi), \mathbb{Z})} \big) \cdot h \Big)$$

in $\mathrm{Hom}_{\mathbb{Z}[G]\text{-left}}(\mathrm{C}_{r+s}(Y_\varphi), {}_{\mathrm{tr}}\mathbb{Z}[G])$.

Proof. Let $\widetilde{\mu} \in \mathrm{C}^r(Y_\varphi; \mathbb{Z})$ and let $\widetilde{\nu} \in \mathrm{C}^s(Y_\varphi; \mathbb{Z})$. We perform the following calculation in $\mathrm{Hom}_{\mathbb{Z}[G]\text{-left}}(\mathrm{C}_{k+l}(Y); {}_{\mathrm{tr}}\mathbb{Z}[G])$:

$$\begin{array}{lll}
 & \text{by definition of } \Psi & \text{by Definition 334.17 of } \cup_{\mathbb{Z}[G]} \\
\Psi(\widetilde{\mu}) \cup_{\mathbb{Z}[G]} \Psi(\widetilde{\nu}) \ \overset{\downarrow}{=} \ \Big(\sum_{g\in G} g^*\widetilde{\mu} \cdot g \Big) \cup_{\mathbb{Z}[G]} \Big(\sum_{h\in G} h^*\widetilde{\nu} \cdot h \Big) & & \overset{\downarrow}{=} \ \sum_{g\in G} \sum_{h\in G} (g^*\widetilde{\mu} \cup h^*\widetilde{\nu}) \cdot gh^{-1} \\
\overset{\uparrow}{=} \ \sum_{g\in G} \sum_{h\in G} g^* \cdot (\widetilde{\mu} \cup \underbrace{(g^{-1})^* h^*}_{=(hg^{-1})^*}(\widetilde{\nu})) \cdot gh^{-1} & & \overset{\uparrow}{=} \ \sum_{g\in G} g^*\Big(\sum_{k\in G} (\widetilde{\mu} \cup (k^{-1})^*\widetilde{\nu}) \cdot k \Big) \\
\text{since by the Cup Product–Naturality Lemma 203.9 we know that } g^*\alpha \cup g^*\beta = g^*(\alpha \cup \beta) & & \text{we rename } k = gh^{-1}, \\
\overset{\uparrow}{=} \ \sum_{g\in G} g^*\Big(\sum_{k\in G} (\widetilde{\mu} \cup k \cdot \widetilde{\nu}) \cdot k \Big) & & \overset{\uparrow}{=} \ q^*\Big(\sum_{k\in G} (\widetilde{\mu} \cup k \cdot \widetilde{\nu}) \cdot k \Big) \\
\text{by definition, see the Covering-}\mathrm{H}^{\mathrm{tw}}_*\text{-Proposition 323.5, we have } h \cdot \widetilde{\nu} = (h^{-1})^*\widetilde{\nu} & & \text{by definition of the transfer map}
\end{array}$$

■

334.7. Exercises for Chapter 334.

Exercise 334.1. Let π be a group, let R be a commutative ring and let M and N be two $(R, \mathbb{Z}[\pi])$-left left modules. Similar to Convention 334.7 we can view $M \otimes_R N$ as an $(R, \mathbb{Z}[\pi])$-left left module. Show that if M is a free R-module of rank 1, then there exists an $(R, \mathbb{Z}[\pi])$-left left module N such that the $\mathbb{Z}[\pi]$-left module $M \otimes_R N$ is isomorphic to the trivial $\mathbb{Z}[\pi]$-module ${}_{\mathrm{tr}}R$.

Exercise 334.2. Let (X, x_0) be a pointed topological space that is path-connected and locally contractible. Furthermore let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a finite group. We denote by $Y \to X$ the covering corresponding to $\ker(\varphi)$. We now consider the following diagram

$$\begin{array}{ccc} \mathrm{H}^k(X;\mathbb{Z}[G]) \times \mathrm{H}^l(X;\mathbb{Z}[G]) & \xrightarrow{\cup_{\mathbb{Z}[G]}} & \mathrm{H}^{k+l}(X;\mathbb{Z}[G]) \\ \cong\downarrow\Omega \quad \Omega\downarrow\cong & & \cong\downarrow\Omega \\ \mathrm{H}^k(Y;\mathbb{Z}) \times \mathrm{H}^l(Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(Y;\mathbb{Z}) \end{array}$$

Here the vertical maps are the isomorphisms from the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 323.5. Does this diagram commute?

Exercise 334.3. We consider the wedge $X = S^1 \vee S^1 \vee S^2$ with wedge point x_0. We write $\pi := \pi_1(X, x_0)$. Show that for any $\mathbb{Z}[\pi]$-left modules M and N the twisted cup product $\cup_{\otimes}\colon \mathrm{H}^1_{\mathrm{tw}}(X; M) \times \mathrm{H}^1_{\mathrm{tw}}(X; N) \to \mathrm{H}^2_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} N)$ is the zero map.

Exercise 334.4. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring with involution and let $\varphi\colon \mathbb{Z}[\pi] \to R$ be a homomorphism of rings with involution. In the Twisted H_0-Proposition 323.3 we calculated twisted (co-) homology in degree zero. Express the cup product $\cup_R$ and the cap product $\cap_R$ in degree zero with respect to these calculations.

Exercise 334.5. We consider the Möbius band $M := ([0,1] \times [0,1])/(0, y) \sim (1, 1-y)$ together with the usual universal covering $p\colon \mathbb{R} \times [0,1] \to M$. Let $\varphi\colon \pi_1(M) \to \mathrm{GL}(1, \mathbb{Z})$ be the unique non-trivial representation. We also consider the long exact sequence

$$\ldots \to \mathrm{H}^{\mathrm{tw}}_1(M, \partial M; {}_{\varphi}\mathbb{Z}) \xrightarrow{\partial_1} \mathrm{H}^{\mathrm{tw}}_0(\partial M; {}_{\varphi}\mathbb{Z}) \to \mathrm{H}^{\mathrm{tw}}_0(M; {}_{\varphi}\mathbb{Z}) \to \ldots.$$

(a) We consider the cycle $\tau\colon \Delta^1 \to \mathbb{R} \times [0,1]$ that is given by $(t, 1-t) \mapsto (0, t)$. Note that $1 \otimes \tau$ defines an element of $\mathrm{H}^{\mathrm{tw}}_1(M, \partial M; {}_{\varphi}\mathbb{Z})$. Show that $\mathrm{H}^{\mathrm{tw}}_0(\partial M; {}_{\varphi}\mathbb{Z}) \cong \mathbb{Z}$ and show that $\partial_1([1 \otimes \tau])$ is twice a generator.

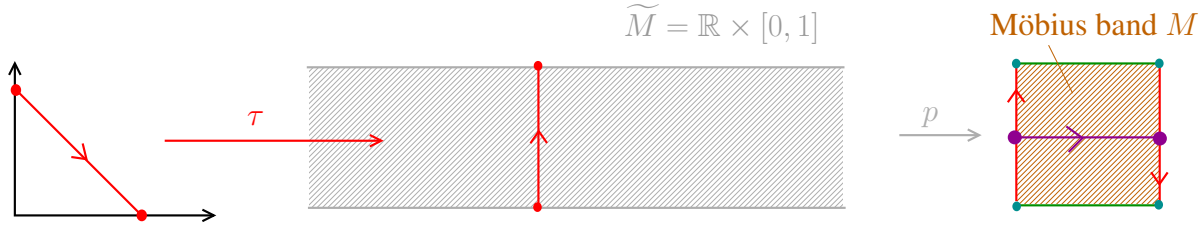
(b) Show that $\mathrm{H}^{\mathrm{tw}}_1(M, \partial M; {}_{\varphi}\mathbb{Z}) = \mathbb{Z} \cdot [1 \otimes \tau]$.

(c) Show that the map

$$\begin{array}{ccc} \mathrm{H}^1(M;\mathbb{Z}) & \to & \mathrm{H}_1(M, \partial M; {}_{\varphi}\mathbb{Z}) \\ \varphi & \mapsto & \mu(\varphi \cap_{\mathbb{Z}} [M]_{\mathbb{Z}}) \end{array}$$

from page 6350 is an isomorphism.

Remark. This exercise has a significant overlap with Exercise 323.8.


$\widetilde{M} = \mathbb{R} \times [0,1]$
Möbius band M
τ
p

CHAPTER 335

Poincaré Duality with twisted coefficients

In the previous Chapter 334 we introduced the cup product and the cap product on twisted (co-) homology. In this chapter we will use the cap product to state and prove the Poincaré Duality Theorem 335.1 with twisted coefficients. As the name suggests, this theorem is a twisted generalization of the untwisted Poincaré Duality Theorem 210.1.

This chapter is organized as follows:

- In Section 335.1 we will state the Twisted Poincaré Duality Theorem 335.1.
- In Section 335.2 we will prove the Twisted Poincaré Duality Theorem 335.1 for closed topological manifolds.
- In Section 335.3 we will prove the Twisted Poincaré Duality Theorem 335.1 for topological manifolds with boundary.
- In Section 335.4 we will formulate a Poincaré Duality Theorem for non-orientable topological manifolds.
- In Section 335.5 we will provide exercises for Chapter 335.

In the next Chapter 336 we will discuss some applications of Poincaré Duality with twisted coefficients.

335.1. **Statement of the Poincaré Duality Theorem with twisted coefficients.** The Poincaré Duality Theorem 210.1 was surely the most important result so far that we proved regarding the topology of compact topological manifolds. Now we have all the tools to formulate and prove a significant generalization of the Poincaré Duality Theorem 210.1:

Theorem 335.1. (Twisted Poincaré Duality Theorem) Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As usual we denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class from Definition 194.3. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For every ring R and every $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module M and each $k \in \mathbb{N}_0$ the map

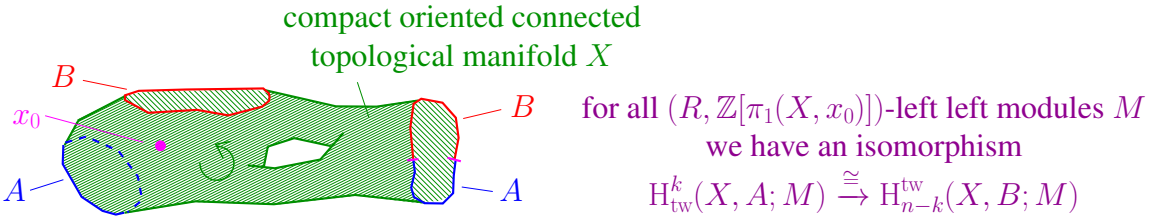
$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \varphi & \mapsto & \varphi \underset{\uparrow}{\cap_{\mathbb{Z}}} [X] \end{array}$$

see Definition 334.34

is an isomorphism of R-left modules.

Remark.

(1) It follows from the Classical-Twisted Cap Product Lemma 334.32 that for the trivial coefficient module $M = {}_{\mathrm{tr}}\mathbb{Z}$ from Definition 320.12 we recover the $R = \mathbb{Z}$-statement of the Poincaré Duality Theorem 210.1.



(2) The Twisted Poincaré Duality Theorem 335.1 gets commonly used in the literature. But proofs in the literature are few and far between. The proof we provide is a "twisted version" of our proof of the original Poincaré Duality Theorem 210.1. Another "twisted version" of a classical proof is provided in [**FNOP2025**] and in [**Sun2017**]. Somewhat different approaches are given in [**KwS2018**] and [**Wall1970**, Chapter 2].

(3) For compact smooth manifolds there are two alternative approaches one can take to obtain some form of Poincaré Duality for twisted invariants:

(a) Let M be a compact smooth manifold and let A be a union of boundary components of M. Using handle decompositions of M rel A we will give an alternative proof for some of the statements in the Twisted Poincaré Duality Theorem **??**.

(b) Let M be a closed oriented smooth manifold. By the Smooth Manifold-Simplicial Structure Theorem 179.5 we know that M admits a smooth simplicial structure. In Chapter 188 we used this fact to provide for such M an alternative proof of the Poincaré Duality Theorem 210.1. Using suitable equivariant versions of the chain homotopy equivalence used in the proof of the Simplicial Poincaré Duality Theorem 188.5 one can surely provide an alternative proof of the Twisted Poincaré Duality Theorem 335.1 for closed oriented smooth manifolds. Our well established taste for brevity prevents us from carrying out this program. □

The Twisted Poincaré Duality Theorem 335.1 leads us to the following notation:

Definition 335.2. The isomorphism of the Twisted Poincaré Duality Theorem 335.1 is called the Poincaré Duality isomorphism. We denote it and its inverse both by PD_X, i.e. with the notation of the Twisted Poincaré Duality Theorem 335.1 we have the isomorphism

$$\begin{array}{rcl} \mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

of R-left modules together with the inverse

$$\mathrm{PD}_X \colon \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \xrightarrow{\cong} \mathrm{H}^k_{\mathrm{tw}}(X, A; M).$$

We will prove the Twisted Poincaré Duality Theorem 335.1 in the following sections. We will try to stick as closely as possible to the earlier proof of the untwisted Poincaré Duality Theorem 210.1, and we will try to outline where the differences lie.

335.2. **The proof of Theorem 335.1 for topological manifolds with empty boundary.** In this section we will prove the Twisted Poincaré Duality Theorem 335.1 for closed topological manifolds. More precisely, we will prove the following theorem:

Theorem 335.3. (Closed Manifold–Twisted PD Theorem) Let X be a closed oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As usual we denote by $[X] \in \mathrm{H}_n(X)$ the fundamental class. For each $k \in \mathbb{N}_0$, every ring R and every $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module M the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism of R-left modules.

As in the proof of the Poincaré Theorem 210.1 we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$. To do so we need a version of Poincaré Duality for every topological manifold (not necessarily compact) with empty boundary. Formulating this generalized Poincaré Duality Theorem requires some preparations.

335.2.1. **Twisted cohomology with compact support.** In the proof of Poincaré Theorem 210.1 we used cohomology with compact support. But in fact we did not work with the actual Definition 209.3 of cohomology with compact support. Instead we worked with a reformulation that was given in the $\mathrm{H}^n_{\mathrm{c}}$-via-Direct Colimits Proposition 209.15. We use this reformulation to define cohomology with compact support and twisted coefficients.

Definition 335.4.

(1) Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. We equip $\mathcal{K}(X)$ with the partial order that is given by inclusion.

(2) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Using the functoriality of cohomology with twisted coefficients (see Definition 321.8) we can define the cohomology of X with compact support and twisted coefficient corresponding to M to be the following R-left module:

$$\mathrm{H}^k_{\mathrm{tw},c}(X; M) \coloneqq \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^k_{\mathrm{tw}}(X, X \setminus K; M).$$

The following notation is inspired by Notation 211.2:

Notation 335.5. Let X be an n-dimensional topological manifold with empty boundary. If X is equipped with an orientation $\{\mu_x \in \mathrm{H}_n(X, X \setminus \{x\})\}_{x \in X}$ in the sense of Definition 191.17, then we saw in the Homology-Section Theorem 193.6 (i) that given any compact subset K of X there exists a unique homology class $\mu^X_K \in \mathrm{H}_n(X, X \setminus K)$ such that for every $x \in K$ the image of μ^X_K under the map $\mathrm{H}_n(X, X \setminus K) \to \mathrm{H}_n(X, X \setminus \{x\})$ equals the given orientation μ_x at x. If X is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ^X_K.



We continue with the following twisted analog of the Cap–Non Compact Manifold Lemma 211.3:

Lemma 335.6. (Cap–Non Compact Manifold Lemma) Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$ and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. For any $k \in \mathbb{N}_0$ there exists a unique R-left module homomorphism

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw,c}}(X; M) \ \to \ \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

such that for each compact subset $K \subseteq X$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, X \setminus K; M) & \xrightarrow{\ \cap_{\mathbb{Z}} \mu_K\ } & \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M). \\ & \searrow \quad \nearrow_{\mathrm{PD}_X} & \\ & \mathrm{H}^k_{\mathrm{tw,c}}(X; M) & \end{array}$$

Proof. The proof is basically identical to the proof of the Cap–Non Compact Manifold Lemma 211.3, we only need to replace the Cap Product-Naturality Lemma 206.13 by the Twisted Cap Product-Naturality Proposition 334.35 (2). ■

335.2.2. **Poincaré Duality for possibly non-compact manifolds.** We will now actually prove a generalization of the Closed Manifold–Twisted PD Theorem 335.3 where we replace the hypothesis that X is closed by the hypothesis that the boundary is empty. This more general version of Poincaré Duality will not only be useful for the divide-et-impera approach to the proof of the Closed Manifold–Twisted PD Theorem 335.3, but it will also be of interest when we deal with Poincaré Duality for topological manifolds with non-empty boundary:

Theorem 335.7. (Twisted PD–Non-Compact Theorem) Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$. Let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. For each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw},c}(X; M) \ \xrightarrow{\cong} \ \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

from the Cap–Non Compact Manifold Lemma 335.6 is an isomorphism of R-left modules.

335.2.3. **Poincaré Duality for balls.** As a warm up for the proof of the Twisted PD–Non-Compact Theorem 335.7 we will deal with the case $X = B^n$ by hand.[3627] More precisely, we will prove the following lemma, which can be viewed as an extension of the PD-for-Ball Lemma 211.7:

Lemma 335.8. (Twisted PD-for-Ball Lemma) Let X be an oriented topological manifold that is homeomorphic to the open ball B^n, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. For each $k \in \mathbb{N}_0$ the map

$$\mathrm{PD}_X \colon \mathrm{H}^k_{\mathrm{tw},c}(X; M) \ \xrightarrow{\cong} \ \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

is an isomorphism of R-left modules.

Proof. Since X is homeomorphic to the open ball B^n we see that $\pi_1(X, x_0)$ is the trivial group, which implies that M is the trivial $\mathbb{Z}[\pi_1(X, x_0)]$-module ${}_{\mathrm{tr}}M$ from Definition 320.12.

[3627] We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation induced from the standard orientation of $\mathbb{R}^n$, see the Topological Manifold-Induced Orientation Lemma 192.1.

We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw},c}(X;{}_{\mathrm{tr}}M) & \xrightarrow{\cap_{\mathbb{Z}}\mu_X} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X;{}_{\mathrm{tr}}M) \\ {\scriptstyle T}\uparrow\cong & & \cong\downarrow{\scriptstyle T} \\ \mathrm{H}^k_c(X;M) & \xrightarrow{\cap_{\mathbb{Z}}\mu_X} & \mathrm{H}_{n-k}(X;M). \end{array}$$

We make the following clarifications and observations:

(1) By the Twisted–Trivial Coefficients Lemma 320.13 we have a natural isomorphism
$$T\colon \mathrm{H}^{\mathrm{tw}}_{n-k}(X;{}_{\mathrm{tr}}M) \ \to\ \mathrm{H}_{n-k}(X;M).$$
This gives us the vertical isomorphism on the right.

(2) Again by the Twisted–Trivial Coefficients Lemma 320.13 we know that for each compact subspace $K \subseteq X$ we have a natural isomorphism
$$T\colon \mathrm{H}^k(X, X\setminus K;M) \ \to\ \mathrm{H}^k_{\mathrm{tw}}(X, X\setminus K;{}_{\mathrm{tr}}M).$$
By the naturality of these isomorphisms, these isomorphisms induce the vertical isomorphism T on the left.

(3) It follows easily from the definitions, similar to the Classical-Twisted Cap Product Lemma 334.32, that the diagram commutes.

(4) In the PD-for-Ball Lemma 211.7 we showed, using an explicit calculation, that the bottom horizontal map is an isomorphism for $X = B^n$. But since X is homeomorphic to B^n it follows almost immediately that the bottom horizontal map is also an isomorphism in our case.

It follows from the above discussion that the top horizontal map is also an isomorphism. ■

335.2.4. **Internal cohomology with twisted coefficients.** One of the key ideas in the proof of the Poincaré Duality Theorem 210.1, and in the intermediate Poincaré Duality–Non-Compact Theorem 211.4, was to reduce the general case to the rather trivial case of an open ball that we just dealt with. To reduce the general case to the trivial case we needed to show that if Poincaré Duality holds on two open subsets U and V of a given topological manifold X, then it also holds in the union $U \cup V$ of the two open subsets. To show this result we had to state and prove in the Poincaré Duality–Mayer Vietoris Proposition 211.8 a Mayer–Vietoris type result that relates U, V to $U \cup V$ and also the intersection $U \cap V$. The technical nuisance that arises in our current setting is that there is no way that we can assume that the intersection $U \cap V$ is again path-connected, thus we cannot talk about twisted (co-) homology of $U \cap V$. This means that we have to revert to working with internal twisted (co-) homology that we introduced in Section 325.3. For the reader's convenience we recall the definitions:

Definition 335.9. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering.

(1) Given any subspaces $S \subseteq T \subseteq X$ we write $\widetilde{S} \coloneqq p^{-1}(S)$ and $\widetilde{T} \coloneqq p^{-1}(T)$ and we set
$$\begin{aligned} \mathrm{C}^{X,M}_k(T,S) &\coloneqq \overline{M} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_k(\widetilde{T},\widetilde{S}) \\ \mathrm{C}^k_{X,M}(T,S) &\coloneqq \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x_0)]\text{-left}}(\mathrm{C}_k(\widetilde{T},\widetilde{S}),M). \end{aligned}$$

These are (co-) chain complexes in the obvious way and we obtain the corresponding (co-) homology modules $\mathrm{H}_k^{X,M}(T,S)$ and $\mathrm{H}^k_{X,M}(T,S)$. We refer to these as internal (co-) homology modules with twisted coefficients.

(2) Given two pairs (T_1,S_1) and (T_2,S_2) of subspaces of X with $T_1 \subseteq T_2$ and also with $S_1 \subseteq S_2$ we obtain, with the hopefully obvious notation, an inclusion-induced chain map $\mathrm{C}_k(\widetilde{T}_1,\widetilde{S}_1) \to \mathrm{C}_k(\widetilde{T}_2,\widetilde{S}_2)$ which is a $\mathbb{Z}[\pi_1(X,x_0)]$-left homomorphism. Thus we get induced R-left module homomorphisms

$$\begin{aligned} i_*\colon \mathrm{H}_k^{X,M}(T_1,S_1) &\to \mathrm{H}_k^{X,M}(T_2,S_2) \\ i^*\colon \mathrm{H}^k_{X,M}(T_2,S_2) &\to \mathrm{H}^k_{X,M}(T_1,S_1). \end{aligned}$$

Evidently internal (co-) homology is functorial under inclusions.

We add some new definitions:

(3) Let (T,S) be a pair of subspaces of X. As in Definition 334.34 we define a cap product

$$\cap_{\mathbb{Z}}\colon \mathrm{H}^k_{X,M}(T,S) \times \mathrm{H}_n(T,S) \to \mathrm{H}_{n-k}^{X,M}(T).$$

(4) Given $S \subseteq X$ we can use (2) to define internal cohomology with compact support and twisted coefficients

$$\mathrm{H}^k_{X,M,c}(S) := \operatorname*{colim}_{K\in\mathcal{K}(S)} \mathrm{H}^k_{X,M}(S, S\setminus K).$$

The next lemma allows us to go back and forth between internal twisted (co-) homology and usual twisted (co-) homology:

Lemma 335.10. (Internal Twisted Homology Lemma) Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let R be a ring and let M be an $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left module.

(1) Let (T,S) be a pair of subspaces of X with $x_0 \in T$. We assume that T is path-connected and locally contractible. We denote by $i\colon T \to X$ the inclusion. We have isomorphisms

$$\begin{aligned} \Omega^X_{T,S}\colon \mathrm{H}_k^{\mathrm{tw}}(T,S;i^*M) &\to \mathrm{H}_k^{X,M}(T,S) \\ \Omega^X_{T,S}\colon \mathrm{H}^k_{X,M}(T,S) &\to \mathrm{H}^k_{\mathrm{tw}}(T,S;i^*M) \end{aligned}$$

that are natural under inclusions.

(2) The following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(T,S;i^*M) \times \mathrm{H}_n(T,S) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{n-k}(T;i^*M) \\ \cong\uparrow \Omega^X_{T,S} \quad \downarrow \mathrm{id} & & \cong\uparrow \Omega^X_T \\ \mathrm{H}^k_{X,M}(T,S) \times \mathrm{H}_n(T,S) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k}(T). \end{array}$$

Proof. Statement (1) is a simplified version of the Change-of-Covering Lemma 325.7. Statement (2) follows easily from the definitions. ∎

We will need the following reformulation of the Cap–Non Compact Manifold Lemma 335.6.

Lemma 335.11. (Cap–Non Compact Manifold Lemma) Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$, let $x_0 \in X$, let R be a ring and let M be an $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left module. For any open subset $U \subseteq X$ and any

$k \in \mathbb{N}_0$ there exists a unique homomorphism

$$\mathrm{PD}_U \colon \mathrm{H}^k_{X,M,c}(U) \to \mathrm{H}^{X,M}_{n-k}(U)$$

of R-left modules such that for each compact subset $K \subseteq U$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{X,M}(U, U \setminus K) & \xrightarrow{\cap_{\mathbb{Z}} \mu_K} & \mathrm{H}^{X,M}_{n-k}(U). \\ & \searrow \quad \nearrow_{\mathrm{PD}_U} & \\ & \mathrm{H}^k_{X,M,c}(U) & \end{array}$$

Proof. The proof of the lemma is basically the same as the proof of the analogous statements in the Cap–Non Compact Manifold Lemma 211.3 (2) and the Cap–Non Compact Manifold Lemma 335.6. Alternatively one sees that the lemma follows immediately from the Cap–Non Compact Manifold Lemma 335.6 together with the Internal Twisted Homology Lemma 335.10. ■

Before we can formulate our next results we need to introduce some notation:

Notation 335.12. Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module.

(1) Given $A \subseteq X$ we denote by $i_* \colon \mathrm{H}^{X,M}_k(A) \to \mathrm{H}^{X,M}_k(X)$ and $i^* \colon \mathrm{H}^{X,M}_k(X) \to \mathrm{H}^{X,M}_k(A)$ the inclusion-induced maps.

(2) Suppose that X is Hausdorff and that U is an open subset of X with $x_0 \in U$. Then we denote by $i \colon \mathrm{H}^k_{X,M,c}(U) \to \mathrm{H}^k_{X,M,c}(X)$ the map that is defined by

$$\overbrace{\operatorname*{colim}_{K \in \mathcal{K}(U)} \mathrm{H}^k_{X,M}(U, U \setminus K)}^{=\mathrm{H}^k_{X,M,c}(U)} \xleftarrow{\cong} \operatorname*{colim}_{K \in \mathcal{K}(U)} \mathrm{H}^k_{X,M}(X, X \setminus K) \to \overbrace{\operatorname*{colim}_{K \in \mathcal{K}(X)} \mathrm{H}^k_{X,M}(X, X \setminus K)}^{=\mathrm{H}^k_{X,M,c}(X)}.$$

isomorphism by the Internal Twisted Excision Theorem 325.10[3628]

As in the Covariant Cohomology Lemma 209.17 we see that the above homomorphisms $\mathrm{H}^k_{X,M,c}(U) \to \mathrm{H}^k_{X,M,c}(X)$ are covariantly functorial with respect to compositions of inclusions.

335.2.5. **Mayer–Vietoris sequences and Poincaré Duality.** Now we can state the following generalization of the Poincaré Duality–Mayer Vietoris Proposition 211.8, which will be a key technical ingredient to the proof of the Twisted PD–Non-Compact Theorem 335.7:

Proposition 335.13. (Twisted PD–Mayer Vietoris Proposition) Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$. Let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let U and V be open

[3628] If one looks carefully one sees that at this point we want to argue that compact subspaces are closed subsets. This follows from our hypothesis that X is Hausdorff together with the Compact-Closed Lemma 3.39 (2).

subsets of X with $X = U \cup V$. Then there exists a diagram

$$\begin{array}{ccccccccc}
\dots \to \mathrm{H}^k_{X,M,\mathrm{c}}(U \cap V) & \xrightarrow{(i,-i)} & \mathrm{H}^k_{X,M,\mathrm{c}}(U) \oplus \mathrm{H}^k_{X,M,\mathrm{c}}(V) & \xrightarrow{i\oplus i} & \overbrace{\mathrm{H}^k_{X,M,\mathrm{c}}(X)}^{=\mathrm{H}_{\mathrm{tw},\mathrm{c}}(X;M)} & \to & \mathrm{H}^{k+1}_{X,M,\mathrm{c}}(U \cap V) \to \dots \\
\downarrow{\scriptstyle \mathrm{PD}_{U\cap V}} & & \downarrow{\scriptstyle \mathrm{PD}_U \oplus \mathrm{PD}_V} & & \downarrow{\scriptstyle \mathrm{PD}_X} & & \downarrow{\scriptstyle \mathrm{PD}_{U\cap V}} \\
\dots \to \mathrm{H}^{X,M}_{n-k}(U \cap V) & \xrightarrow{(i,-i)} & \mathrm{H}^{X,M}_{n-k}(U) \oplus \mathrm{H}^{X,M}_{n-k}(V) & \xrightarrow{i\oplus i} & \underbrace{\mathrm{H}^{X,M}_{n-k}(X)}_{=\mathrm{H}^{\mathrm{tw}}_{n-k}(X;M)} & \to & \mathrm{H}^{X,M}_{n-k-1}(U \cap V) \to \dots
\end{array}$$

with the following properties:

(1) the first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$,
(2) the lower exact sequence is the long exact Mayer–Vietoris sequence for $X = U \cup V$ that we constructed in the Internal Mayer–Vietoris Theorem 325.11,
(3) the upper horizontal sequence is also exact.

The proof of the Twisted PD–Mayer Vietoris Proposition 335.13 requires several preparations. The reader might want to put their finger on the fast-forward button to page 6373. First of all we will need the following "inverted" Mayer–Vietoris Theorem, which is the twisted analog of the Inverted Mayer–Vietoris Theorem 211.11:

Theorem 335.14. (Inverted Mayer–Vietoris Theorem) Let X be a topological space that is path-connected and locally contractible, let $x_0 \in X$, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Furthermore let A and B be open subsets of X.

(1) For every $n \in \mathbb{N}_0$ and every cochain $\varphi \in \mathrm{C}^n_{X,M}(X, A \cup B)$ there exist $\varphi_A \in \mathrm{C}^n_{X,M}(X, A)$ and $\varphi_B \in \mathrm{C}^n_{X,M}(X, B)$ such that $\varphi_A + \varphi_B = \varphi$.
(2) For all $n \in \mathbb{N}_0$ there exists a unique homomorphism
$$\delta_n \colon \mathrm{H}^n_{X,M}(X, A \cap B) \ \to \ \mathrm{H}^{n+1}_{X,M}(X, A \cup B)$$
of R-left modules which has the following property: For every $\varphi_A \in \mathrm{C}^n_{X,M}(X, A)$ and $\varphi_B \in \mathrm{C}^n_{X,M}(X, B)$ such that $\varphi_A + \varphi_B$ is a cocycle in $\mathrm{C}^n_{X,M}(X, A \cap B)$ we have
$$\delta_n([\varphi_A + \varphi_B]) \ = \ [\delta\varphi_A] \ = \ -[\delta\varphi_B] \ \in \ \mathrm{H}^{n+1}_{X,M}(X, \{A, B\}) = \mathrm{H}^{n+1}_{X,M}(X, A \cup B).$$
(3) The sequence
$$\dots \to \mathrm{H}^n_{X,M}(X, A \cup B) \xrightarrow{(i^*,-i^*)} \mathrm{H}^n_{X,M}(X, A) \oplus \mathrm{H}^n_{X,M}(X, B) \xrightarrow{i^*\oplus i^*} \mathrm{H}^n_{X,M}(X, A \cap B) \xrightarrow{\delta_n} \mathrm{H}^{n+1}_{X,M}(X, A \cup B) \to \dots$$
is exact.

Proof of Theorem 335.14. We start out with the following two observations:

(1) Similar to the discussion in Definition 334.12 we set
$$\mathrm{C}^k_{X,M}(X, \{A, B\}) \ := \ \begin{array}{l}\text{all } \mathbb{Z}[\pi_1(X, x_0)]\text{-left homomorphisms } \mathrm{C}_k(\widetilde{X}_{x_0}) \to M \\ \qquad \text{that vanish on } \mathrm{C}_k(\widetilde{A}) \text{ and on } \mathrm{C}_k(\widetilde{B}).\end{array}$$
This is again a cochain complex and we denote the corresponding cohomology modules by $\mathrm{H}^k_{X,M}(X, \{A, B\})$. It follows easily from the Regular Covering–Chain Complex Lemma 320.2 together with the Hom-Split-SES Lemma 315.10 that the following sequence

of cochain maps between cochain complexes is exact:

$$0 \to \mathrm{C}^*_{X,M}(X,\{A,B\}) \xrightarrow{(i^*,-i^*)} \mathrm{C}^*_{X,M}(X,A) \oplus \mathrm{C}^*_{X,M}(X,B) \xrightarrow{i^*\oplus i^*} \mathrm{C}^*_{X,M}(X,A\cap B) \to 0.$$

(2) Since the subsets A and B are open we know from the Universally Excisive Triad Proposition 325.3 (1b) that the triad (X,A,B) is universally excisive. Therefore it follows from the argument of the Excisive Triad–Twisted Cohomology Lemma 334.14 that the obvious map $\mathrm{C}^*_{X,M}(X,A\cup B) \to \mathrm{C}^*_{X,M}(X,\{A,B\})$ induces for every $n \in \mathbb{N}_0$ an isomorphism $\mathrm{H}^n_{X,M}(X,A\cup B) \xrightarrow{\cong} \mathrm{H}^n_{X,M}(X,\{A,B\})$.

With these two observations it is straightforward to modify the proof of the Inverted Mayer–Vietoris Theorem 211.11 and of the Mayer–Vietoris Theorem 153.2 to obtain the theorem. We leave it to the reader to fill in the details. ■

In the proof of the Twisted PD–Mayer Vietoris Proposition 335.13 we will also need to recycle the following two technical lemmas.

Lemma 211.12. (Compact Subset Decomposition Lemma) Let X be a topological manifold and let U,V be open subsets such that $X = U \cup V$. Then given any compact subset $C \subseteq X$ there exist compact subsets $K \subseteq U$ and $L \subseteq V$ such that $C = K \cup L$.



Lemma 211.13. (Fundamental Cycle Decomposition Lemma) Let X be a compact oriented n-dimensional topological manifold and let U,V be two open subsets such that $X = U \cup V$. Furthermore let $K \subseteq U$ and $L \subseteq V$ be compact subsets. Then there exist singular n-chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:

(a) $\mu^X_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$ in $\mathrm{H}_n(X, X\setminus(K\cup L))$

(b) $\mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V, (U\cap V)\setminus(K\cap L))$

(c) $\mu^U_K = [\alpha_{U\setminus L} + \alpha_{U\cap V}]$ in $\mathrm{H}_n(U, U\setminus K)$.

We ware now finally ready to provide a proof of the Twisted PD–Mayer Vietoris Proposition 335.13:

Proof of the Twisted PD–Mayer Vietoris Proposition 335.13. Let X be an oriented path-connected n-dimensional topological manifold with $\partial X = \varnothing$. Let $x_0 \in X$, let R be a ring and let M be an $(R,\mathbb{Z}[\pi_1(X,x_0)])$-left left module. We start with two preliminaries:

(1) Given any two subsets $A \subseteq B$ and $i \in \mathbb{N}_0$ we write $\mathrm{H}^i_{X,M}(B|A) \coloneqq \mathrm{H}^i_{X,M}(B, B\setminus A)$.

(2) If U is an open subset of X and $K \subseteq U$ is compact, then, by Definition 335.9, we obtain a map

$$-\cap_{\mathbb{Z}} \mu^U_K \colon \mathrm{H}^k_{X,M}(U|K) \to \mathrm{H}^{X,M}_{n-k}(U).$$

Now let U and V be open subsets with $X = U \cup V$. Furthermore let $K \subseteq U$ and $L \subseteq V$ be compact subsets. We consider the following diagram:

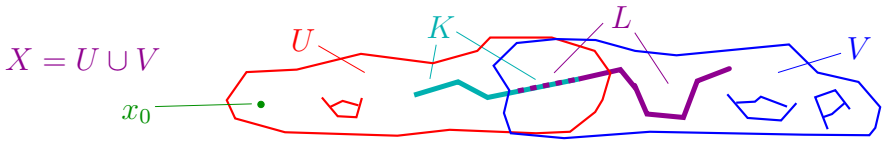
$$\begin{array}{ccccccccc}
.. \to \mathrm{H}^k_{X,M}(X|K\cap L) & \to & \mathrm{H}^k_{X,M}(X|K)\oplus\mathrm{H}^k_{X,M}(X|L) & \longrightarrow & \mathrm{H}^k_{X,M}(X|K\cup L) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}_{X,M}(X|K\cap L) & \to .. \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & \\
\mathrm{H}^k_{X,M}(U\cap V|K\cap L) & & \mathrm{H}^k_{X,M}(U|K)\oplus\mathrm{H}^k_{X,M}(V|L) & & \downarrow\cap_{\mathbb{Z}}\mu^X_{K\cup L} & & \mathrm{H}^{k+1}_{X,M}(U\cap V|K\cap L) & \\
\downarrow\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L} & & \downarrow\cap_{\mathbb{Z}}\mu^U_K\oplus\cap_{\mathbb{Z}}\mu^V_L & & & & \downarrow\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L} & \\
.. \longrightarrow \mathrm{H}^{X,M}_{n-k}(U\cap V) & \longrightarrow & \mathrm{H}^{X,M}_{n-k}(U)\oplus\mathrm{H}^{X,M}_{n-k}(V) & \longrightarrow & \mathrm{H}^{X,M}_{n-k}(X) & \xrightarrow{\partial_{n-k}} & \mathrm{H}^{X,M}_{n-k-1}(U\cap V) & \longrightarrow ..
\end{array}$$

which we now explain in more detail:

(1) We write $A \coloneqq X \setminus K$ and $B \coloneqq X \setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology modules from the Inverted Mayer–Vietoris Theorem 335.14.
(2) The lower horizontal sequence is the long exact sequence coming from the Internal Mayer–Vietoris Theorem 325.11, corresponding to the decomposition $X = U \cup V$.
(3) The vertical maps on the top are all induced by inclusions of pairs of topological spaces, they are isomorphisms by the Internal Twisted Excision Theorem 325.10.

The following is the key claim of the proof:

Claim. The first two rectangles of the above diagram commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.



Before we prove the claim we show that it implies the lemma. For each module appearing in the above diagram we take the direct colimit over the directed set[3629]

$$\mathcal{K}(U,V) \;=\; \{(K,L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

and we consider the induced maps of direct colimits that were introduced in Definition 104.9. We make the following remarks regarding this new diagram:

(1) Using the Cofinal-Isomorphism Lemma 209.13 one can show that the modules that we obtain taking the direct colimit over $\mathcal{K}(U,V)$ are naturally isomorphic to the corresponding cohomology modules with compact support. We carry out the argument for two modules:

$$\operatorname*{colim}_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k_{X,M}(U\cap V|K\cap L) \;=\; \operatorname*{colim}_{C\in\mathcal{K}(U\cap V)} \mathrm{H}^k_{X,M}(U\cap V|C) \;=\; \mathrm{H}^k_{X,M,c}(U\cap V).$$

we apply the Cofinal-Isomorphism Lemma 209.13 to the cofinal morphism of directed sets that is given by $(K,L) \mapsto K \cap L$

and

[3629] We view $\mathcal{K}(U,V)$ as a directed set given by the relation $(K,L) \leq (K',L')$ if $K \subseteq K'$ and $L \subseteq L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

$$\operatorname*{colim}_{(K,L)\in\mathcal{K}(U,V)} \mathrm{H}^k_{X,M}(X|K\cup L) \;=\; \operatorname*{colim}_{C\in\mathcal{K}(X)} \mathrm{H}^k_{X,M}(X|C) \;=\; \mathrm{H}^k_{X,M,c}(X).$$

we apply the Cofinal-Isomorphism Lemma 209.13 to the morphism of directed sets given by $(K,L)\mapsto K\cup L$, this morphism is cofinal by the Compact Subset Decomposition Lemma 211.12

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct colimits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in the Cap–Non Compact Manifold Lemma 335.11 one can easily verify that the direct colimit of the diagram does indeed give rise to the above diagram specified in the statement of the lemma.

(4) The Direct Colimit-Exactness Proposition 154.6 says that the direct colimit of exact sequences is again exact.

All these remarks put together show that the direct colimit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the proposition.

Proof. Now we turn to the actual proof of the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of the Twisted Cap Product-Naturality Proposition 334.35 (2). The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in internal (co-) homology.

Subclaim 1. The first rectangle of the diagram on page 6370 commutes.

So we need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(X, X\setminus(K\cap L)) & \xrightarrow{(i,-i)} & \mathrm{H}^k_{X,M}(X, X\setminus K)\oplus \mathrm{H}^k_{X,M}(X, X\setminus L) \\
\downarrow\cong & \searrow^{(i,-i)} & \downarrow\cong \\
\mathrm{H}^k_{X,M}(U\cap V, (U\cap V)\setminus(K\cap L)) & & \mathrm{H}^k_{X,M}(U, U\setminus K)\oplus \mathrm{H}^k_{X,M}(V, V\setminus L) \\
\downarrow\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L} & & \downarrow\cap_{\mathbb{Z}}\mu^{U}_{K}\oplus\cap_{\mathbb{Z}}\mu^{V}_{L} \\
\mathrm{H}^{X,M}_{n-k}(U\cap V) & \xrightarrow{(i,-i)} & \mathrm{H}^{X,M}_{n-k}(U)\oplus \mathrm{H}^{X,M}_{n-k}(V).
\end{array}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand summands. So let $\phi\in \mathrm{H}^k_{X,M}(X, X\setminus(K\cap L))$. We consider the following diagram of maps between pairs of topological spaces:

$$\begin{array}{ccc}
(U\cap V, (U\cap V)\setminus(K\cap L)) & \xrightarrow{=:\alpha} & (U, U\setminus(K\cap L)) \\
\downarrow f\circ\alpha & \swarrow^{f} & \uparrow{=:\beta} \\
(X, X\setminus(K\cap L)) & \xleftarrow[f\circ\beta]{} & (U, U\setminus K).
\end{array}$$

With this notation we need to show that

$$\alpha_*\big((f\circ\alpha)^*(\phi)\cap_{\mathbb{Z}}\mu^{U\cap V}_{K\cap L}\big) \;=\; (f\circ\beta)^*(\phi)\cap_{\mathbb{Z}}\mu^{U}_{K}.$$

But this is indeed the case, since

$$\alpha_*\big(\underbrace{(f\circ\alpha)^*}_{=\alpha^*\circ f^*}(\phi)\cap_{\mathbb{Z}}\mu_{K\cap L}^{U\cap V}\big) \underset{\uparrow}{=} f^*(\phi)\cap_{\mathbb{Z}}\alpha_*(\mu_{K\cap L}^{U\cap V}) \underset{\uparrow}{=} f^*(\phi)\cap_{\mathbb{Z}}\mu_{K\cap L}^{U}$$

by the Twisted Cap Product Naturality Proposition 334.35 (2)

by functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 193.6 we have $\alpha_*(\mu_{K\cap L}^{U\cap V})=\mu_{K\cap L}^{U}$

$$\underset{\uparrow}{=}\ \beta_*\big((f\circ\beta)^*(\phi)\cap_{\mathbb{Z}}\mu_K^U\big) \underset{\uparrow}{=} (f\circ\beta)^*(\phi)\cap_{\mathbb{Z}}\mu_K^U$$

same argument backwards with the roles of α and β exchanged

since β is on the first coordinate the identity

□

Subclaim 2. The second rectangle of the diagram on page 6370 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. Thus we leave it to the reader to fill in the details. □

Subclaim 3. The third rectangle of the diagram on page 6370 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{array}{ccccc}
\mathrm{H}^k_{X,M}(X, X\setminus K\cup L) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}_{X,M}(X, X\setminus(K\cap L)) & \xrightarrow{\cong} & \mathrm{H}^{k+1}_{X,M}(U\cap V, (U\cap V)\setminus(K\cap L)) \\
\downarrow{\scriptstyle \cap_{\mathbb{Z}}\mu_{K\cup L}} & & & & \downarrow{\scriptstyle \cap_{\mathbb{Z}}\mu_{K\cap L}} \\
\mathrm{H}^{X,M}_{n-k}(X) & & \xrightarrow{\qquad\partial_{n-k}\qquad} & & \mathrm{H}^{X,M}_{n-k-1}(U\cap V).
\end{array}$$

So let $[\varphi]\in \mathrm{H}^k_{X,M}(X, X\setminus(K\cup L)) = \mathrm{H}^k_{X,M}(X, A\cap B)$. We start out with the following preparations:

(1) By the Inverted Mayer–Vietoris Theorem 335.14 (1) we can find $\varphi_A\in \mathrm{C}^k_{X,M}(X,A)$ and $\varphi_B\in \mathrm{C}^k_{X,M}(X,B)$ such that $\varphi=\varphi_A+\varphi_B$.

(2) By the Fundamental Cycle Decomposition Lemma 211.13 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

(2a) $\mu_{K\cup L} = [\alpha_{U\setminus L}+\alpha_{U\cap V}+\alpha_{V\setminus K}]$ in $\mathrm{H}_n(X, X\setminus(K\cup L))$

(2b) $\mu_{K\cap L}^{U\cap V} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V, (U\cap V)\setminus(K\cap L))$

(2c) $\mu_K^U = [\alpha_{U\setminus L}+\alpha_{U\cap V}]$ in $\mathrm{H}_n(U, U\setminus K)$.

After these preparations we obtain the following equalities in $\mathrm{H}^{X,M}_{n-k-1}(U \cap V)$:[3630]

$$\begin{array}{l}
\qquad\qquad\qquad

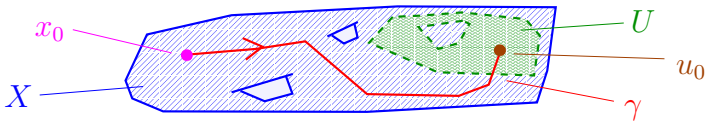
from the Cap–Non Compact Manifold Lemma 335.11 is an isomorphism of R-left modules.

Our goal is to show that Poincaré Duality holds for all manifolds. We try to follow the logic of the proof of Theorem 211.4. But now we need an extra initial claim:

Claim 0. Let X be a manifold and let $U \subseteq X$ be an open subset. If each path-component of U, viewed as a manifold in its own right, satisfies Poincaré Duality, then U satisfies Internal Poincaré Duality within X.

Proof. Let X be a path-connected manifold.

(1) First let $U \subseteq X$ be a path-connected open subset that, if viewed as n-dimensional topological manifold in its own right, satisfies Poincaré Duality. We need to show that U also satisfies Internal Poincaré Duality within X. Let $x_0 \in X$. We pick $u_0 \in U$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to u_0. We denote by $i\colon U \to X$ the inclusion.



Next we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^k_{\mathrm{tw},c}(U; i^*\gamma_* M) & \xleftarrow[\cong]{\Omega^X_U} & \mathrm{H}^k_{X,\gamma_* M,c}(U) & \xrightarrow[\cong]{\gamma_*} & \mathrm{H}^k_{X,M,c}(U) \\
\cong \downarrow \mathrm{PD}_U & & \downarrow \mathrm{PD}_U & & \downarrow \mathrm{PD}_U \\
\mathrm{H}^{\mathrm{tw}}_{n-k}(U; i^*\gamma_* M) & \xrightarrow[\cong]{\Omega^X_U} & \mathrm{H}^{X,\gamma_* M}_{n-k}(U) & \xrightarrow[\cong]{\gamma_*} & \mathrm{H}^{X,M}_{n-k}(U).
\end{array}$$

Here the horizontal maps to the left are given by the Internal Twisted Homology Lemma 335.10. The horizontal maps to the right are isomorphisms that are defined as in the Change-of-Base Point Proposition 325.8. It follows easily from the Internal Twisted Homology Lemma 335.10 (2) that the square to the left commutes. Furthermore it is straightforward to verify that the square to the right commutes. By hypothesis the vertical map to the left is an isomorphism. It follows from the above that the vertical map to the right is also an isomorphism.

(2) Let $\{U_i\}_{i\in I}$ be the (possibly infinite) family of path components of U. We suppose that each U_i satisfies Poincaré Duality viewed as a manifold. By (1) we know that each U_i also satisfies Internal Poincaré Duality within X. As in the Cup Product–Disjoint Union Lemma 204.16 and the Cap Product–Disjoint Union Lemma 206.14 one can show easily, using the Internal Homology–Components Lemma 325.6 and using Exercise 335.1, that $U = \bigsqcup_{i\in I} U_i$ also satisfies Internal Poincaré Duality within X. ⊞

Claim 1. Let X be a manifold.

(A) If Poincaré Duality holds for X equipped with an orientation, then it also holds for any other orientation of X.

(B) If X is homeomorphic to a manifold for which Poincaré Duality holds, then Poincaré Duality also holds for X.

(C) Suppose that X is the union of two open subsets U and V. If Poincaré Duality holds for each path-component of U, V and $U \cap V$, then Poincaré Duality also holds for X.
(D) If X is the union of a nested sequence $U_1 \subseteq U_2 \subseteq \dots$ of open subsets such that Poincaré Duality holds for each path-component of each U_i, then Poincaré Duality also holds for X.

Proof. So let X be a manifold.

(A) Recall that by our convention in this proof every manifold is path-connected. By the Number-of-Orientations Lemma 191.18 the oriented path-connected topological manifold X admits only one other orientation, namely the reverse orientation $-X$. It follows from $[-X] = -[X]$ that Poincaré Duality also holds for $-X$.

(B) If we have an orientation-preserving homeomorphism, then this statement follows immediately from the Twisted Cup Product-Naturality Proposition 334.20. For a non-orientation-preserving homeomorphism this statement follows from (A) together with the Twisted Cup Product-Naturality Proposition 334.20.

(C) We suppose that Poincaré Duality holds for each path-component of U, V and $U \cap V$. It follows from Claim 0 that Internal Poincaré Duality holds for U, V and $U \cap V$ within X. The combination of the Twisted PD–Mayer Vietoris Proposition 335.13 and the Five Lemma 145.11 now shows that also each PD_X is an isomorphism, i.e. Poincaré Duality also holds for $X = U \cup V$.

(D) Let $k \in \mathbb{N}_0$. We suppose that each path-component of each $U_i \subseteq X$ satisfies Poincaré Duality. We need to show that X also satisfies Poincaré Duality. Let $x_0 \in X$ and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. By hypothesis and Claim 0 we know that for every $i \in \mathbb{N}$ the map $\mathrm{PD}_{U_i} \colon \mathrm{H}^k_{X,M,\mathrm{c}}(U_i) \to \mathrm{H}^{X,M}_{n-k}(U_i)$ is an isomorphism. Using the pretty obvious variation on the Twisted Cap Product-Naturality Proposition 334.35 and using the Cap–Non Compact Manifold Lemma 335.11 it is straightforward to verify that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{X,M,c}(U_i) & \xrightarrow{\mathrm{PD}_{U_i}} & \mathrm{H}^{X,M}_{n-k}(U_i) \\ \downarrow & & \downarrow \\ \mathrm{H}^k_{X,M,c}(U_j) & \xrightarrow{\mathrm{PD}_{U_j}} & \mathrm{H}^{X,M}_{n-k}(U_j). \end{array}$$

We obtain isomorphisms

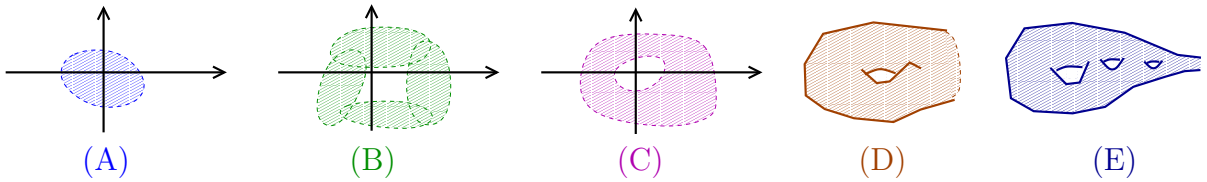
$$\overbrace{\mathrm{H}^k_{X,M,\mathrm{c}}(X)}^{=\mathrm{H}^k_{\mathrm{tw},\mathrm{c}}(X;M)} \underset{\text{by a straightforward generalization of the Exhaustion } \mathrm{H}^*_{\mathrm{c}}\text{-Proposition 209.18}}{=} \varinjlim \mathrm{H}^k_{X,M,\mathrm{c}}(U_i) \underset{\text{isomorphism by our hypothesis and since the above diagram commutes}}{\xrightarrow{\varinjlim \mathrm{PD}_{U_i}}} \varinjlim \mathrm{H}^{X,M}_{n-k}(U_i) \underset{\text{by a straightforward generalization of the Exhaustion-}\mathrm{H}_* \text{ Proposition 154.2}}{=} \overbrace{\mathrm{H}^{X,M}_{n-k}(X)}^{=\mathrm{H}^{\mathrm{tw}}_{n-k}(X;M)}.$$

We leave it to the reader to verify that the map from left to right is indeed given by PD_X. ⊞

The following claim completes the proof of the theorem.

Claim 2.

(A) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.
(B) For every $m \in \mathbb{N}$ Poincaré Duality holds for every path-connected sum of $\mathbb{R}^n$ that is the union of m open bounded convex subsets of $\mathbb{R}^n$.
(C) Poincaré Duality holds for every path-connected open subset of $\mathbb{R}^n$.
(D) For every $m \in \mathbb{N}$ Poincaré Duality holds for all manifolds that admit an atlas with m charts.
(E) Poincaré Duality holds for all manifolds.



Proof. [3632]

(A) Let X be an open bounded convex subset of $\mathbb{R}^n$. If X is empty, then there is nothing to show. Now assume that X is non-empty. By the Convex-to-Ball Proposition 4.28 (1) there exists a homeomorphism $X \to B^n$. It follows from the PD-for-Ball Lemma 335.8 and Claim 1 (B) that Poincaré Duality also holds for X.

(B) We prove the claim by induction on m. By (A) we know that the statement holds for $m = 1$. Now suppose that it holds for all path-connected subsets of $\mathbb{R}^n$ that are unions of $m-1$ open bounded convex subsets of $\mathbb{R}^n$. Let X be a path-connected subset of $\mathbb{R}^n$ that is the union of m open bounded convex subsets $U_1, \dots, U_m$. We set $V = U_1 \cup \dots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for each path-component of $V \cap U_m = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$, since each path-component is the union of $(m-1)$ open bounded convex subsets. It follows from Claim 1 (C) that Poincaré Duality also holds for

$$X \;=\; V \cup U_m \;=\; (U_1 \cup \dots \cup U_{m-1}) \cup U_m.$$

(C) Let X be an arbitrary path-connected open subset of $\mathbb{R}^n$. By the Rational Open Balls Lemma 231.7 we can write X as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup_{j \le i} B_j$. Thus we obtain from (B) that Poincaré Duality holds for each path-component of each $U_i \in \mathbb{N}$. But then it follows from Claim 1 (D) that Poincaré Duality also holds for X.

(D) We prove this statement by induction on m. The case $m = 1$ is taken care of by (C) and Claim 1 (B). Now suppose Poincaré Duality holds for all manifolds that admit an atlas with $m-1$ charts. Let X be a manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\dots,m}$ with m charts. We write $U = U_1 \cup \dots \cup U_{m-1}$ and $V = U_m$. Both are open subsets of a manifold. Note that U and V admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$ admits an atlas with $m-1$ charts. It follows easily from induction that Poincaré Duality holds for each path-component of U, V and $U \cap V$. Thus it follows from Claim 1 (C) that Poincaré Duality also holds for X itself.

[3632] The proof is basically the same as the proof of Claim 2 on page 4057 of the untwisted case.

(E) Finally let X be any manifold. By the Countable Cover Lemma 13.4 we know that X admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j\in\mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup_{j\leq i} U_j$. It follows easily from (D) that Poincaré Duality holds for each path-component of each X_i. It follows from Claim 1 (D) that Poincaré Duality also holds for X. ■

335.3. **The proof of Poincaré Duality for topological manifolds with boundary.** Now we will finally prove the general statement of the Twisted Poincaré Duality Theorem 335.1. For the reader's convenience we recall the statement.

Theorem 335.1. (Twisted Poincaré Duality Theorem) Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As usual we denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$, every ring R and every $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module M the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism of R-left modules.

In the following we will prove the Twisted Poincaré Duality Theorem 335.1 using the Twisted PD–Non-Compact Theorem 335.7 and formal arguments that have basically nothing to do with twisted coefficients. The proofs below are very very similar to the arguments for the untwisted case that we provided in Section 211.3. For peace of mind let us nonetheless go through the arguments.

Proof of Theorem 335.1 for $A = \partial X$ and $B = \varnothing$. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As usual we denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class. Let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. We want to show that for each $k \in \mathbb{N}_0$ the map

$$- \cap_{\mathbb{Z}} [X] \colon \mathrm{H}^k_{\mathrm{tw}}(X, \partial X; M) \to \mathrm{H}^{\mathrm{tw}}_{n-k}(X; M)$$

is an isomorphism of R-left modules. To do so we first introduce the following objects:

(a) From the Topological Collar Neighborhood Theorem 35.14 we obtain a collar neighborhood $[-1, 0] \times \partial X$ of $\partial X = \{0\} \times X$. We refer to it as the "internal collar".

(b) We set

$$W := X \cup ([0, 1) \times \partial X)/\sim \qquad \text{with } x \sim (x, 0) \text{ for every } x \in \partial X.$$

Sometimes the subspace $[0, 1) \times \partial X$ is called an *external collar of* X. Furthermore, given any $t \in [-1, 1)$ we write

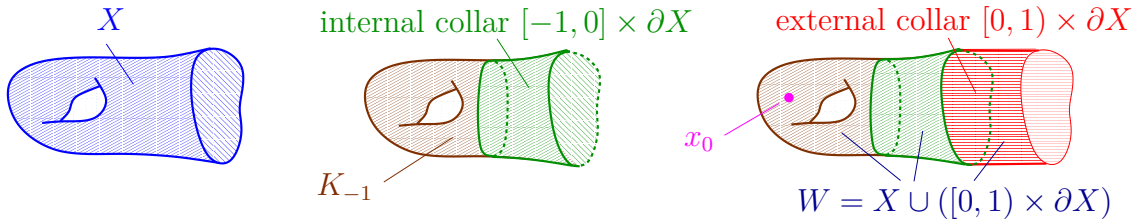
$$K_t := (X \setminus ([-1, 0] \times \partial X)) \cup ([-1, t] \times \partial X)/\sim .$$

(c) We denote by $i \colon X \to W$ the obvious map. By the Topological Pushout-Maps Lemma 8.16 (3) and (4) this map is a closed embedding. We will use this to view X as a closed subset of W. Note that it follows easily from the Homotopy Pushout Lemma 18.13 that $X = i(X)$ is a deformation retract of W.

(d) We denote by $i \colon (X, \partial X) \to (W, W \setminus K_{-1})$ the inclusion map.

(e) We equip W with the base point x_0.

(f) By the Homotopy Equivalence-π_1-Proposition 90.9 we know that the inclusion induced map $i_*\colon \pi_1(X, x_0) \to \pi_1(W, x_0)$ is an isomorphism. This allows us to consider the $\mathbb{Z}[\pi_1(W, x_0)]$-left module $(i_*)^{-1}M$.



By elementary arguments and using the Unique Type-of-Chart Proposition 150.11 (i) one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary.
(B) W admits an orientation such that X is a submanifold and such that $X \to W$ is orientation-preserving.
(C) Every K_t is a compact subset of W and given any compact subset K of W there exists a $t \in [-1, 1)$ with $K \subseteq K_t$.

We obtain the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^k_{\mathrm{tw}}(X, \partial X; M) & \xleftarrow{i^*} & \mathrm{H}^k_{\mathrm{tw}}(W, W \setminus K_{-1}; (i_*)^{-1}M) & \longrightarrow & \mathrm{H}^k_{\mathrm{tw,c}}(W; (i_*)^{-1}M) \\
\downarrow{\scriptstyle \cap_{\mathbb{Z}}[X]} & & \downarrow{\scriptstyle \cap_{\mathbb{Z}}\mu_{K_t}} & & \downarrow{\scriptstyle \mathrm{PD}_W} \\
\mathrm{H}^{\mathrm{tw}}_{n-k}(X; M) & \xrightarrow{i_*} & \mathrm{H}^{\mathrm{tw}}_{n-k}(W; (i_*)^{-1}M) & \xrightarrow{\mathrm{id}} & \mathrm{H}^{\mathrm{tw}}_{n-k}(W; (i_*)^{-1}M).
\end{array}$$

It remains to prove the following claim:

Claim. The following three statements hold:
(1) the diagram commutes,
(2) the right vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

Proof.

(1) The right-hand square commutes by Definition 335.2 of PD_W. Using (B) one can show easily, using the Twisted Cap Product-Naturality Proposition 334.35, that the left-hand square commutes.
(2) The Twisted PD–Non-Compact Theorem 335.7 implies that the right vertical map is an isomorphism.
(3) It follows easily from the discussion in (c) together with the Deformation Retract-$\mathrm{H}^{\mathrm{tw}}_*$-Corollary 322.7 that the left horizontal maps are isomorphisms. Since the bottom right map is the identity it remains to show that the top right horizontal map is an

isomorphism. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(W, W \setminus K_{-1}; (i_*)^{-1}M) & \longrightarrow & \overbrace{\varinjlim_{K \in \mathcal{K}(W)} \mathrm{H}^k_{\mathrm{tw}}(W, W \setminus K; (i_*)^{-1}M)}^{=\mathrm{H}^k_{\mathrm{tw,c}}(W;(i_*)^{-1}M)}. \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t \in [-1,1)} \mathrm{H}^k_{\mathrm{tw}}(W, W \setminus K_t; (i_*)^{-1}M) & \end{array}$$

We make the following two observations:

- By (C) every compact subset of W is contained in some K_t. Therefore it follows from the Cofinal-Isomorphism Lemma 209.13 that the right diagonal map is an isomorphism.
- It follows easily from the Homotopy Equivalence-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 322.6 (2) and the Direct Colimit–Stabilization Lemma 104.5 that the left diagonal map is an isomorphism.

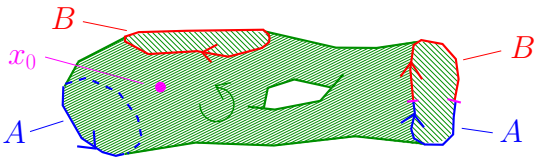
These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim and it concludes the proof of the Twisted Poincaré Duality Theorem 335.1 in the special case that $A = \partial X$ and $B = \varnothing$. ■

Proof of Theorem 335.1 for the general case. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. As usual we denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Finally suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. We want to show that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X, A; M) & \to & \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) \\ \varphi & \mapsto & \varphi \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism.



We equip ∂X and then also B with the orientations coming from the Boundary Orientation Proposition 192.3 and the Topological Manifold-Induced Orientation Lemma 192.1. As before we work again with internal twisted (co-) homology. We consider the following

diagram:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{H}^k_{X,M}(X,\partial X) & \longrightarrow & \overbrace{\mathrm{H}^k_{X,M}(X,A)}^{=\mathrm{H}^k_{\mathrm{tw}}(X,A;M)} & \longrightarrow & \mathrm{H}^k_{X,M}(\partial X,A) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}_{X,M}(X,\partial X) \longrightarrow \dots \\
 & & \cong\downarrow -\cap_{\mathbb{Z}}[X] & & \downarrow -\cap_{\mathbb{Z}}[X] & & \cong\downarrow i^* & & \cong\downarrow -\cap_{\mathbb{Z}}[X] \\
 & & & & & & \mathrm{H}^k_{X,M}(B,\partial B) & & \\
 & & & & & & \cong\downarrow -\cap_{\mathbb{Z}}[B] & & \\
\dots & \longrightarrow & \mathrm{H}^{X,M}_{n-k}(X) & \longrightarrow & \underbrace{\mathrm{H}^{X,M}_{n-k}(X,B)}_{=\mathrm{H}^{\mathrm{tw}}_{n-k}(X,B;M)} & \xrightarrow{\partial_{n-k}} & \mathrm{H}^{X,M}_{n-k-1}(B) & \longrightarrow & \mathrm{H}^{X,M}_{n-k-1}(X) \longrightarrow \dots
\end{array}$$

We make the following observations:

(1) The top and bottom horizontal sequences are the long exact sequence in internal (co-) homology with twisted coefficients of the triple $(X,\partial X,A)$ and the pair (X,B), see the Internal Twisted-LES-Proposition 325.9.

(2) The above vertical map $i^*\colon \mathrm{H}^k_{X,M}(\partial X,A) \to \mathrm{H}^k_{X,M}(B,\partial B)$ is induced by the inclusion $i\colon (B,\partial B) \to (\partial X,A)$. It is an isomorphism by the Universally Excisive Triad Proposition 325.3 (2) together with the obvious twisted version of the Excisive Characterization Lemma 158.19 (3).

(3) The other maps decorated with "$\cong$" are isomorphisms by the special case of the Twisted Poincaré Duality Theorem 335.1 that we have already dealt with.

By the Five Lemma 145.11 it remains to prove the following claim:

Claim. The above diagram commutes up to signs, more precisely:

(1) the first rectangle commutes,

(2) the second rectangle commutes up to the sign $(-1)^k$,

(3) the third rectangle commutes.

Proof. We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(X,\partial X) \times \mathrm{H}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k}(X) \\
\mathrm{id}\uparrow \quad \downarrow\mathrm{id} & & \downarrow \\
\mathrm{H}^k_{X,M}(X,\partial X) \times \mathrm{H}^{X,M}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}_{n-k}(X,B) \\
\downarrow \quad \uparrow= & & \mathrm{id}\uparrow \\
\mathrm{H}^k_{X,M}(X,A) \times \mathrm{H}^{X,M}_n(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}_{n-k}(X,B).
\end{array}$$

We can convince ourselves that the diagram commutes by applying the Twisted Cap Product-Naturality Proposition 334.35 (2) to the maps $(X,\partial X,\varnothing) \to (X,\partial X,B)$ and $(X,A,B) \to (X,\partial X,B)$ of universally excisive triads. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $j\colon (B,\partial B) \to (\partial X,A)$ the inclusion map and we denote by $\partial_n\colon \mathrm{H}_n(X,\partial X) \to \mathrm{H}_{n-1}(\partial X,A)$ the connecting homomorphism in the long exact sequence

of the triple $(X, \partial X, A)$. It follows from the Fundamental Class-of-∂-Proposition 194.13, the Codimension Zero-Fundamental Class Lemma 194.10 and the naturality of the connecting homomorphism that the following equality holds

$$(*) \qquad j_*([B]) \;=\; \partial_n([X]) \in \mathrm{H}_{n-1}(\partial X, A).$$

Now we continue with the proof of the claim.

(2) We apply the twisted analog of the Cap Product-Triad-Connecting Homomorphism Lemma 206.20 to the universally excisive triad (X, A, B). (Recall that $A \cup B = \partial X$ and $A \cap B = \partial B$.) We obtain that for any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(X, A) \times \mathrm{H}_n(X, \partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k}(X, B) \\
{\scriptstyle i^*}\downarrow \quad {\scriptstyle \partial_n}\downarrow & & \\
\mathrm{H}^k_{X,M}(\partial X, A) \times \mathrm{H}_{n-1}(\partial X, A) & & \downarrow{\scriptstyle \partial_{n=k}} \\
{\scriptstyle j^*}\downarrow\cong \quad {\scriptstyle (j_*)^{-1}}\downarrow\cong & & \\
\mathrm{H}^k_{X,M}(B, \partial B) \times \mathrm{H}_{n-1}(B, \partial B) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(B).
\end{array}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^k$.

(3) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k_{X,M}(B, \partial B) \times \mathrm{H}_{n-1}(B, \partial B) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(B) \\
{\scriptstyle i^*}\uparrow \quad \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^k_{X,M}(\partial X, A) \times \mathrm{H}_{n-1}(\partial X, A) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(\partial X) \\
{\scriptstyle \delta_k}\downarrow \quad \uparrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^{k+1}_{X,M}(X, \partial X) \times \mathrm{H}_n(X, \partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{X,M}_{n-k-1}(X).
\end{array}$$

The upper part of this diagram commutes by the Twisted Cap Product-Naturality Proposition 334.35 (2) applied to the map $(B, \partial B, \varnothing) \to (\partial X, A, \varnothing)$ between universally excisive triads. The lower part commutes by the twisted analog of the Cap Product-Triple-Connecting Homomorphism Lemma 206.19 applied to the triple $(X, \partial X, A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim (see page 6380) commutes.

This concludes the proof of the claim and thus also of the Twisted Poincaré Duality Theorem 335.1. ■

335.4. **Appendix: Poincaré duality for non-orientable topological manifolds.** We conclude this chapter with the following generalization on the Twisted Poincaré Duality Theorem 335.1:

Theorem 335.15. (Non-Orientable–Poincaré Duality Theorem) Let X be a compact path-connected n-dimensional topological manifold and let $x_0 \in X$. We denote by

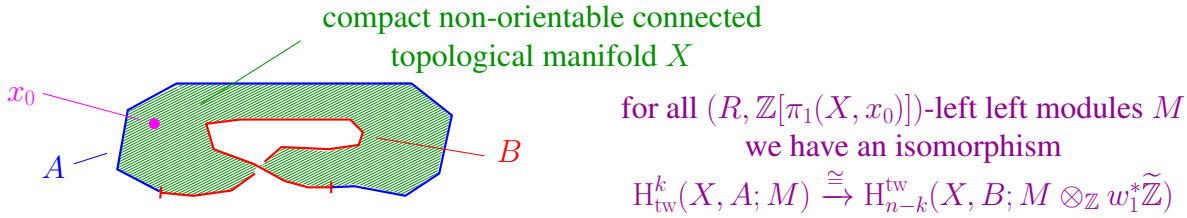
$$w_1 \colon \pi_1(X,x_0) \;\to\; \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \;=_{\mathrm{i}}\; \langle x \,|\, x^2\rangle$$

($=_{\mathrm{i}}$: unique isomorphism)

the orientation-character that we introduced in Definition 191.31 and Definition 324.5 and we denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}[\langle x \,|\, x^2\rangle]$-left module that is given by $x \cdot n := -n$. We denote by $[X] \in \mathrm{H}_n(X,\partial X; w_1^*\widetilde{\mathbb{Z}})$ the twisted fundamental class that we introduced in Definition 324.2 (in the orientable case) and in Definition 324.9 (in the non-orientable case). Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ every ring R and every $(R, \mathbb{Z}[\pi_1(X,x_0)])$-left left module M the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X,A;M) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{n-k}(X,B; M \otimes_{\mathbb{Z}} w_1^*\widetilde{\mathbb{Z}}) \\ \sigma & \mapsto & \sigma \cap_{\mathbb{Z}} [X] \end{array}$$

is an isomorphism of R-left modules.



Remark. For a trivial $\mathbb{Z}[\pi]$-module the statement of the Non-Orientable–Poincaré Duality Theorem 335.15 can also be found in [**DaK2001**, Theorem 5.7], arguably without too much of a proof. □

Proof. The proof of the Non-Orientable–Poincaré Duality Theorem 335.15 is in principle quite similar to the proof of the Twisted Poincaré Duality Theorem 335.1. To make this clearer we make two observations:

(1) First recall that for the oriented case, by Definition 194.3, the defining property of the fundamental class $[X] \in \mathrm{H}_n(X,\partial X)$ is the fact that for each $Q \in X \setminus \partial X$ the map $[X] \in \mathrm{H}_n(X, X \setminus \{Q\})$ is an isomorphism and the image of $[X]$ is a generator.

(2) Similarly the key property of the twisted fundamental class $[X] \in \mathrm{H}^{\mathrm{tw}}_n(X,\partial X; w_1^*\widetilde{\mathbb{Z}})$, which we proved in the Twisted Top Homology Proposition 324.8, is the fact that for each $Q \in X \setminus \partial X$ the map $\mathrm{H}^{\mathrm{tw}}_n(X,\partial X; w_1^*\widetilde{\mathbb{Z}}) \to \mathrm{H}^{\mathrm{tw}}_n(X, X\setminus\{Q\}; w_1^*\widetilde{\mathbb{Z}})$ is an isomorphism and the image of $[X]$ is a generator.

The last input one needs is that by the w_1-Functorial Lemma 191.33 we know that the orientation character w_1 is functorial under local homeomorphisms. With these observations the proof for the oriented case, which we provided above, should generalize without any major problems to the non-orientable case. Like every author that is known to this author we will not work out the details of the proof of this Poincaré Duality Theorem. We hope that the next generation of authors will step up and finally provide a proof of this well-known folklore theorem. ■

335.5. Exercises for Chapter 335.

Exercise 335.1. Let X be a topological space that is Hausdorff, path-connected and locally contractible and let $x_0 \in X$. Let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Let $B \subseteq X$ be a subspace with path-components $\{B_j\}_{j\in J}$. We suppose that each B_j is open in B. Given $j \in J$ we denote by $i_j \colon B_j \to B$ the inclusion map. Show for each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{j\in J} i_{j*} \colon \bigoplus_{j\in J} \mathrm{H}^n_{X,M,\mathrm{c}}(B_j) \to \mathrm{H}^n_{X,M,\mathrm{c}}(B)$$

is an isomorphism.

Remark. This is an analog of Exercise 209.17.

Exercise 335.2. Let Σ_g be the surface of genus g and let $x_0 \in X$.

(a) Let $\varphi \colon \pi_1(\Sigma_g, x_0) \to \mathrm{GL}(1, \mathbb{C})$ be a one-dimensional non-trivial representation. We consider the corresponding $(\mathbb{C}, \mathbb{Z}[\pi_1(\Sigma_g, x_0)])$-left left module ${}_\varphi\mathbb{C}$ from Convention 320.9. Use the Twisted Poincaré Duality Theorem 335.1 together with the Twisted H_0-Proposition 323.3 and the Twisted Homology-χ-Proposition 331.12 to calculate the twisted homology modules $\mathrm{H}^{\mathrm{tw}}_k(\Sigma_g; {}_\varphi\mathbb{C})$ in a pain-free way.

(b) Can you repeat the success story of (a) if we replace $\mathbb{C}$ by a finite field $\mathbb{F}_p$?

(c) Can you repeat the success story of (a) if we replace $\mathbb{C}$ by $\mathbb{Z}$?

Exercise 335.3. Let M be a closed orientable connected non-empty 4-dimensional smooth manifold with an isomorphism $\alpha \colon \pi_1(M) \to \langle t \rangle$. We assume that M has a CW-structure with precisely one cell in dimension $0, 1, 2, 3, 4$. Let $z \in \mathbb{C} \setminus \{0, 1\}$. We consider the representation $\varphi \colon \pi_1(M) \to \mathrm{GL}(1, \mathbb{C})$ that is induced by α and $t \mapsto (z)$. How many of the homology modules $\mathrm{H}^{\mathrm{tw}}_k(M; {}_\varphi\mathbb{C})$ can you calculate?

Exercise 335.4. Let K be the Klein bottle and let $\varphi \colon \pi_1(K) \to \mathrm{GL}(1, \mathbb{Z})$ be a representation.

(a) Use a CW-decomposition of K to calculate the relative twisted homology modules $\mathrm{H}^{\mathrm{tw}}_*(K; {}_\varphi\mathbb{Z})$.

(b) Calculate the relative twisted cohomology modules $\mathrm{H}^*_{\mathrm{tw}}(K; {}_\varphi\mathbb{Z} \otimes_{\mathbb{Z}} w_1^*\widetilde{\mathbb{Z}})$.

Remark. Your calculations hopefully verify the conclusion of the Non-Orientable–Poincaré Duality Theorem 335.15 in this context.

Remark. This exercise has a significant overlap with Exercise 322.3

Exercise 335.5. Let $n \in \mathbb{N}_{\geq 2}$. By the Real Projective Space–π_1-Proposition 93.3 we know that $\pi_1(\mathbb{R}\mathrm{P}^n, *) \cong \mathbb{Z}_2$. This implies that there exists a unique non-trivial representation $\varphi \colon \pi_1(\mathbb{R}\mathrm{P}^n, *) \to \mathrm{GL}(1, \mathbb{Z})$.

(a) Compute the twisted homology modules $\mathrm{H}^{\mathrm{tw}}_*(\mathbb{R}\mathrm{P}^n; {}_\varphi\mathbb{Z})$.

(b) Compute the twisted cohomology modules $\mathrm{H}^*_{\mathrm{tw}}(\mathbb{R}\mathrm{P}^n; {}_\varphi\mathbb{Z})$.

Verify that the Non-Orientable–Poincaré Duality Theorem 335.15 holds in this context.

Remark. Recall that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is even. Also note that this exercise has a significant overlap with Exercise 329.19.

Exercise 335.6. Let (X, x_0) be a pointed compact connected n-dimensional topological manifold with non-empty boundary.

(a) Show that for any $\mathbb{Z}[\pi_1(X, x_0)]$-left module M we have $\mathrm{H}^{\mathrm{tw}}_n(X; M) = 0$.

(b) Show that for any $\mathbb{Z}[\pi_1(X, x_0)]$-left module M we have $\mathrm{H}^n_{\mathrm{tw}}(X; M) = 0$.

Exercise 335.7. Let M be a compact connected non-empty n-dimensional topological manifold and let G be an abelian group. Show that if M is non-orientable, then for the untwisted cohomology group $\mathrm{H}^n(M, \partial M; G)$ we have an isomorphism $\mathrm{H}^n(M, \partial M; G) \cong G/2 \cdot G$.
Hint. Use the Non-Orientable–Poincaré Duality Theorem 335.15.
Remark. This gives a new proof for some of the statements of the Top Cohomology Theorem 201.26.

Exercise 335.8.

(a) Let π be a group and let $\varphi\colon \pi \to \mathrm{GL}(1, \mathbb{Z})$ be a homomorphism. We consider the $\mathbb{Z}[\pi]$-module ${}_{\varphi}\mathbb{Z}$, we consider the $\mathbb{Z}[\pi]$-module ${}_{\varphi}\mathbb{Z} \otimes_{\mathbb{Z}} {}_{\varphi}\mathbb{Z}$ from Convention 334.7 and we consider the trivial $\mathbb{Z}[\pi]$-module ${}_{\mathrm{tr}}\mathbb{Z}$. Show that the $\mathbb{Z}[\pi]$-modules ${}_{\varphi}\mathbb{Z} \otimes_{\mathbb{Z}} {}_{\varphi}\mathbb{Z}$ and ${}_{\mathrm{tr}}\mathbb{Z}$ are isomorphic.

(b) Let X be a compact *non-orientable* path-connected n-dimensional topological manifold and let $x_0 \in X$. We suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. Show that for each $k \in \mathbb{N}_0$, every ring R and every $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module M there exists an isomorphism

$$\mathrm{H}^k_{\mathrm{tw}}(X, A; M \otimes_{\mathbb{Z}} w_1^* \widetilde{\mathbb{Z}}) \xrightarrow{\cong} \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M)$$

of R-left modules.
Hint. Reduce this statement to the Non-Orientable–Poincaré Duality Theorem 335.15.

Exercise 335.9. Let M be a compact odd-dimensional topological manifold (not necessarily orientable). Use the Non-Orientable–Poincaré Duality Theorem 335.15 to show that

$$\chi(\partial M) \;=\; 2 \cdot \chi(M).$$

CHAPTER 336

Applications of Poincaré Duality with twisted coefficients

In the previous Chapter 335 we proved the Poincaré Duality Theorem 335.1 with twisted coefficients. In this chapter we will summarize several applications of this theorem. In particular we will collect some low hanging fruit about high-dimensional twisted homology and cohomology modules. We will use these calculations, together with several statements (only some of which we can prove in these lecture notes), to provide more information on the homotopy type of compact topological manifolds.

This chapter is organized as follows:

- In Section 336.1 we will apply the Twisted Poincaré Duality Theorem 335.1 to obtain some results on the top twisted (co-) homology modules of a topological manifold.
- In Section 336.2 we will use the calculations from Section 336.1 (and some more fancy input) to prove some statements about the homotopy type of topological manifolds.
- In Section 336.3 we will provide exercises for Chapter 336.

In the next Chapter 337 we will introduce twisted intersection pairings of oriented even-dimensional topological manifolds.

336.1. **Top twisted homology.** We start out with the following two theorems which are the (perhaps not particularly surprising) twisted analogs of some of the statements of the Manifold-Top-H_*-Theorem 193.1 and the Manifold-Top-H^*-Theorem 201.26:

Theorem 336.1. (Twisted Manifold-Top-H_*-Theorem) Let X be a compact path-connected n-dimensional topological manifold and let $x_0 \in X$. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. Finally let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module.

(1) For every $i > n$ we have $\mathrm{H}_i^{\mathrm{tw}}(X, A; M) = 0$.

(2) We have an isomorphism[3633]

$$\mathrm{H}_n^{\mathrm{tw}}(X, A; M) \cong \begin{cases} M^{\pi_1(X,x_0)}, & \text{if } A = \partial X, \\ 0, & \text{otherwise} \end{cases}$$

of R-left modules.

[3633] We recall Definition 323.1: If R is a ring, π is a group and M is an $(R, \mathbb{Z}[\pi])$-left left module, then we define the R-left module of π-invariants

$$M^\pi := \{v \in M \mid g \cdot v = v \text{ for all } g \in \pi\}.$$

Theorem 336.2. (Twisted Manifold-Top-H*-Theorem) With the same hypothesis as in the Twisted Manifold-Top-H$_*$-Theorem 336.1 the following statements hold:

(1) For every $i > n$ we have $\mathrm{H}^i_{\mathrm{tw}}(X, A; M) = 0$.

(2) We have an isomorphism[3634]

$$\mathrm{H}^n_{\mathrm{tw}}(X, A; M) \cong \begin{cases} M_{\pi_1(X,x_0)}, & \text{if } A = \partial X, \\ 0, & \text{otherwise} \end{cases}$$

of R-left modules.

Proof of the Twisted Manifold-Top-H$_*$-Theorem 336.1. [3635] We write $\pi := \pi_1(X, x_0)$. Given $i \in \mathbb{N}_{\geq n}$ we calculate

$$\mathrm{H}^{\mathrm{tw}}_i(X, A; M) \overset{\cong}{\longleftarrow} \mathrm{H}^{n-i}_{\mathrm{tw}}(X, B; M) \cong \begin{cases} 0, & \text{if } i > n, \\ M^\pi, & \text{if } i = n \text{ and } B = \varnothing \text{ (i.e. if } A = \partial X), \\ 0, & \text{if } i = n \text{ and } B \neq \varnothing \text{ (i.e. if } A \neq \partial X). \end{cases}$$

$\uparrow$ by the Non-orientable Poincaré Duality Theorem 335.15

$\uparrow$ for $n > i$ this follows since cohomology in negative degrees is zero and for $n = i$ this follows from the Twisted H$_0$-Proposition 323.3

■

Proof of the Twisted Manifold-Top-H*-Theorem 336.2. As in the previous proof we write $\pi := \pi_1(X, x_0)$. Given $i \in \mathbb{N}_{\geq n}$ we calculate

$$\mathrm{H}^i_{\mathrm{tw}}(X, A; M) \overset{\cong}{\longrightarrow} \mathrm{H}^{\mathrm{tw}}_{n-i}(X, B; M) \cong \begin{cases} 0, & \text{if } i > n, \\ M_\pi, & \text{if } i = n \text{ and } B = \varnothing \text{ (i.e. if } A = \partial X), \\ 0, & \text{if } i = n \text{ and } B \neq \varnothing \text{ (i.e. if } A \neq \partial X). \end{cases}$$

$\uparrow$ by the Non-Orientable Poincaré Duality Theorem 335.15

$\uparrow$ since homology in negative degrees is zero and by the Twisted H$_0$-Proposition 323.3

■

336.2. **The homotopy type of topological manifolds.** We first recall some facts and statements about smooth and topological manifolds:

- As we mentioned recently in the Manifold–CW Complex Theorem 331.7, by the Smooth Manifold-CW-Homotopy Type Proposition 254.4 we know that every compact n-dimensional smooth manifold M is homotopy equivalent to a finite n-dimensional CW-complex.[3636] Furthermore, if M is a compact connected n-dimensional smooth manifold with non-empty boundary, then it follows from the Minimal n-Handle Proposition 248.4 together with the Handle Decomposition–CW-Complex Theorem 254.1 that M is homotopy equivalent to a CW-complex of dimension $n - 1$.
- The situation for topological manifolds is unfortunately less satisfactory: As we mentioned on page 6310, the Compact Manifold–Homotopy Type Corollary 190.12 tells us

[3634] We recall Definition 323.1: If R is a ring, π is a group and M is an $(R, \mathbb{Z}[\pi])$-left left module, then we define the R-left module of π-coinvariants

$$M_\pi := M \Big/ \left\{ \sum_{i=1}^{k} (g_i \cdot v_i - v_i) \,\middle|\, g_i \in \pi \text{ and } v_i \in M \right\}.$$

[3635] For smooth manifolds Statement (1) is also an immediate consequence of the Singular-Handle Twisted Isomorphism Theorem 333.4 and the existence of handle decomposition.

[3636] In fact, in the Smooth Manifold-CW-Theorem 179.9 we even sketched a proof that M admits a simplicial structure, in particular that it admits a CW-structure.

that every compact topological manifold is homotopy equivalent to a finite CW-complex. But this result did not make any statements about the dimensions of the CW-complex.

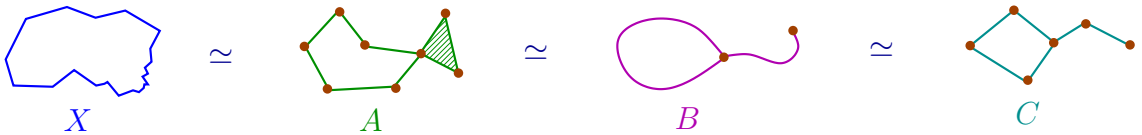
Using the Twisted Manifold-Top-H*-Theorem 336.2 we can now prove the following refinement of the Compact Manifold–Homotopy Type Corollary 190.12:

Theorem 336.3. (Manifold–Simplicial Complex Homotopy Type Theorem)

(1) Every closed non-empty n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional simplicial complex.
(2) Every compact connected n-dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.

Sketch of proof. First note that the 1-dimensional case follows immediately from the classification result proved in the Topological 1-Manifold Classification Theorem 28.1. Furthermore note that it follows from Radó's Theorem 190.15 that every 2-dimensional topological manifold admits a smooth structure and that it follows from Moise's Theorem 190.16 that the analogous statement also holds in dimension three. Thus we see that the 2-dimensional and the 3-dimensional cases are dealt with by the aforementioned smooth results from the Manifold–CW Complex Theorem 331.7, the Minimal n-Handle Proposition 248.4 and the Handle Decomposition–CW-Complex Theorem 254.1. Thus we can now assume that $n \geq 4$.

(1) Let X be a closed non-empty n-dimensional topological manifold. Clearly we can assume that the topological manifold is connected.
  (a) By the Compact Manifold–Homotopy Type Corollary 190.12 (which alas we did not prove) we know that X is homotopy equivalent to a finite simplicial complex A. By the Simplicial-Implies-CW Lemma 174.51 we can view A also as a finite CW-complex
  (b) It follows from the Twisted Manifold-Top-H*-Theorem 336.2 (1) that all twisted homology and cohomology modules of X in degree $> n$ vanish.
  (c) It follows from (b) and the Homotopy Equivalence-H$^{\text{tw}}_*$-Proposition 322.1 that all twisted homology and cohomology modules of A in degree $> n$ vanish.
  (d) Since $n \geq 3$ we obtain from (c) and Wall's Finiteness Theorem 331.15 (which unfortunately we did not prove either) that A is homotopy equivalent to a compact n-dimensional CW-complex B.
  (e) It follows from the Finite CW-Homotopy Type Theorem 190.10 (2)⇔(3) that B is homotopy equivalent to a finite n-dimensional simplicial complex C.
  (f) By the Homotopy-Equivalence Basics Lemma 19.14 (2) we know that being homotopy equivalent is an equivalence relation. Thus the above discussion shows that X is homotopy equivalent to the finite n-dimensional simplicial complex C.



(2) If X is compact orientable connected n-dimensional topological manifold *with non-empty boundary*, then we can run the same argument as in (1), except that we now use the Twisted Manifold-Top-H*-Theorem 336.2 (2) to conclude that all twisted homology and cohomology modules of X in degree $> n-1$ vanish. Since $n-1 \geq 3$ we can

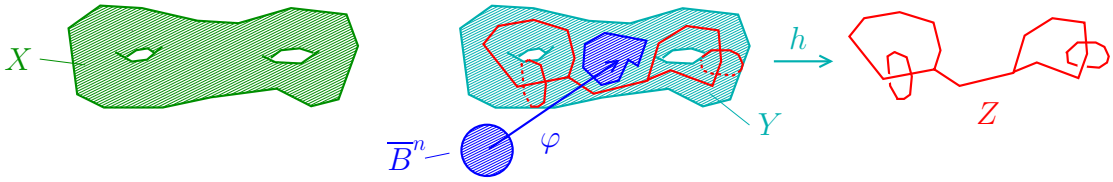
still appeal to Wall's Finiteness Theorem 331.15 and we obtain that X is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex. ■

As we already saw on page 3664, the Manifold–Simplicial Complex Homotopy Type Theorem 336.3 allows us to prove the following theorem:

Proposition 190.13. (Single Zero-and-Top Cell Proposition) Every closed connected non-empty n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional CW-complex which has a single 0-cell and which has a single n-cell.

Proof. Let X be a closed connected non-empty n-dimensional topological manifold. In the following we assume that $n \geq 2$. (The case $n = 1$ can be dealt with either by slightly adjusting the argument below or alternatively by appealing to the Topological 1-Manifold Classification Theorem 28.1.) Clearly we can pick an embedding $\varphi\colon \overline{B}^n \to X$ such that $\varphi(\overline{B}^n)$ is a submanifold. It follows from the Codimension-Zero Submanifold Proposition 150.16 that $Y \coloneqq X \setminus \varphi(B^n)$ is a compact non-empty n-dimensional topological manifold with $\partial Y = \varphi(S^{n-1})$. Since $n \geq 2$ we know that $\varphi(S^{n-1})$ is connected. Since X is also connected it follows from Exercise 23.9 that Y is also connected.

Recall that by the Manifold–Simplicial Complex Homotopy Type Theorem 336.3 we know that there exists a finite $(n-1)$-dimensional CW-complex Z and a homotopy equivalence $h\colon Y \to Z$.



We continue with the following observations:

(1) It follows from the Homotopy Equivalent Cell Attachment Proposition 135.13 that the cell attachments $Y \cup_{Y \leftarrow S^{n-1}:\varphi|_{S^{n-1}}} \overline{B}^n$ and $Z \cup_{Z \leftarrow S^{n-1}:h\circ\varphi|_{S^{n-1}}} \overline{B}^n$ are homotopy equivalent.
(2) By the Regluing Lemma 8.17 we know that $Y \cup_{Y \leftarrow S^{n-1}:\varphi|_{S^{n-1}}} \overline{B}^n$ is homeomorphic to X.
(3) By the CW-Gluing Proposition 139.9 we know that $Z \cup_{Z \leftarrow S^{n-1}:h\circ\varphi|_{S^{n-1}}} \overline{B}^n$ admits a CW-structure with a single n-cell and no cells of dimension $> n$.
(4) Finally note that it follows from the CW-Complex–Spanning Tree Proposition 140.5 that we can in fact find an n-dimensional CW-complex that is homotopy equivalent to Y (and thus to X) that not only has the feature, that it contains only a single n-cell, but that also has the feature that it contains only a single 0-cell. ■

336.3. Exercises for Chapter 336.

Exercise 336.1. Let (X, x_0) be a topological space that is path-connected and locally contractible, let R be a ring and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Furthermore let $X_0 \subseteq X_1 \subseteq \dots$ be a sequence of subspaces of X such that $x_0 \in X$.

(a) Under what conditions do we have a theorem of the form

$$\mathrm{H}_k^{\mathrm{tw}}(X; M) \cong \operatorname*{colim}_{\longrightarrow} \mathrm{H}_k^{\mathrm{tw}}(X; M)\,?$$

(b) Provide a proof of your statement in (a).

Exercise 336.2. Let X be a non-compact oriented path-connected n-dimensional smooth manifold with $\partial X = \varnothing$ and let $x_0 \in X$. Show that for any $\mathbb{Z}[\pi_1(X, x_0)]$-left module we have $\mathrm{H}_n^{\mathrm{tw}}(X; M) = 0$.

Remark. You could make use of Exercise 336.1 and of the Compact Submanifold Exhaustion Proposition 38.2.

CHAPTER 337

Twisted intersection pairings

In Chapter 335 we proved the Poincaré Duality Theorem 335.1 with twisted coefficients. In this chapter we will use Poincaré Duality to introduce and discuss the twisted intersection pairings, which are the twisted cousins of the intersection pairings that we introduced in Definition 225.3.

This chapter is organized as follows:

- In Section 337.1 we will recall the definition and basic properties of the untwisted twisted intersection pairings.
- In Section 337.2 we will prepare the algebraic scene for the definition of twisted intersection pairings.
- In Section 337.3 we will introduce twisted intersection pairings.
- In Section 337.4 we will see that the twisted intersection pairing is functorial in the coefficients.
- In Section 337.5 we will see that the twisted intersection pairings are often non-singular.
- In Section 337.6 we will discuss the relationship between twisted intersection pairings and intersection pairings of finite coverings.
- In Section 337.7 we will provide exercises for Chapter 337.

In the next Chapter 338 we will use the twisted intersection pairings from this chapter to introduce the twisted signatures of smooth manifolds.

337.1. **The classical untwisted twisted intersection pairings.** We start out by recalling some aspects of Definition 213.1:

Definition 337.1. Let A, B, C be abelian groups. A pairing on $A \times B$ is a map

$$\varphi\colon A \times B \ \to \ C$$

such that for all $a, a' \in A$ and $b, b' \in B$ we have $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$ and $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$.

Before we can recall the definition of the "classical" intersection pairing, we should recall the following definition from page 3079:

Definition 337.2. Given a topological space X and an abelian group G we consider the augmentation homomorphism

$$\begin{aligned} \epsilon_X\colon \mathrm{H}_0(X;G) \ &\to \ G \\ \Big[\sum_{i=1}^{k} (\sigma_i\colon \Delta^0 \to X) \otimes g_i\Big] \ &\mapsto \ \sum_{i=1}^{k} g_i. \end{aligned}$$

On page 3079 we also pointed out that this map is an isomorphism if X is path-connected and non-empty.

The following definition is a slight modification of Definition 225.3:[3637]

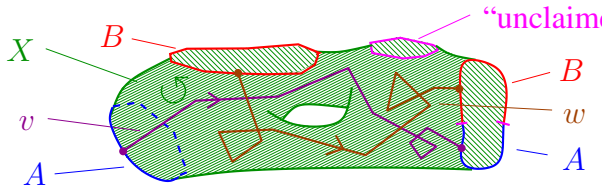
Definition 337.3. Let X be a compact oriented n-dimensional topological manifold and let R be a commutative ring. We denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class from Definition 194.3 or alternatively the more recent Definition 324.2:

(a) Suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. We write $A^\dagger := \partial X \setminus \mathring{A}$ and $B^\dagger := \partial X \setminus \mathring{B}$. For each $k \in \mathbb{N}_0$ we refer to the pairing[3638 3639 3640]

$$\begin{aligned} Q^R_{X,A,B} \colon \mathrm{H}_k(X, A; R) \times \mathrm{H}_{n-k}(X, B; R) &\to R \\ (v, w) &\mapsto \epsilon_X\big(\big(\underbrace{\mathrm{PD}_X(v)}_{\in \mathrm{H}^{n-k}(X, A^\dagger; R)} \cup \underbrace{\mathrm{PD}_X(w)}_{\in \mathrm{H}^{k}(X, B^\dagger; R)} \big) \cap [X]\big) \end{aligned}$$

as a intersection pairing of X. Clearly this is a pairing in the sense of Definition 337.1.

(b) For convenience we set $Q^R_X := Q^R_{X,\emptyset,\emptyset}$.



$$Q^{\mathbb{Q}}_{X,A,B}(v, w) = +1 - 1 = 0$$

To formulate properties of intersection pairings we need some basic algebraic notions:

Definition 337.4. Let R be a commutative ring and let V, W and S be R-modules.

(1) A pairing $\varphi \colon V \times W \to S$ is called R-bilinear if it is R-linear in V and if it is R-linear in W.

(2) A pairing $\varphi \colon V \times V \to S$ is called symmetric if for all $v, v' \in V$ we have the equality $\varphi(v, v') = \varphi(v', v)$.

The following proposition is a variation on the Intersection Form Proposition 224.13:

[3637] The difference to Definition 225.3 is that now we allow arbitrary commutative rings, that for $R = \mathbb{Z}$ we no longer consider homology mod torsion, and that we no longer call it an "asymmetric" intersection pairing.

[3638] Note that by the Codimension-Zero Submanifold Proposition 150.16 we know that $A^\dagger$ and $B^\dagger$ are also compact submanifolds of ∂X with $\partial A^\dagger = \partial A = A \cap A^\dagger$ and $\partial B^\dagger = \partial B = B \cap B^\dagger$. Furthermore note that it follows from $A \cap B = \partial A = \partial B$ that $\partial X = A^\dagger \cup B^\dagger$ and that $\partial X = (A^\dagger \setminus \partial A^\dagger) \cup (B^\dagger \setminus \partial B^\dagger) \cup C$ where each component of C is a component of $\partial A^\dagger$ and a component of $\partial B^\dagger$. Thus it follows from a slight generalization of the Excisive Triad-Proposition 158.20 (2), see also Exercise 158.8, that $(X, A^\dagger, B^\dagger)$ is an excisive triad. In particular it follows, see Definition 204.7, that the cup product is actually defined.

[3639] Here by PD_X we denote the Poincaré Duality isomorphism from Definition 210.2.

[3640] Recall that the Cap Product-Kronecker Lemma 206.12 tells us that that for any $\varphi \in \mathrm{H}^k(X, A; R)$ and any $\sigma \in \mathrm{H}_k(X, A; R)$ we have $\epsilon_X(\varphi \cap \sigma) = \langle \varphi, \sigma \rangle_X$. Using this observation we see that we also have

$$Q^R_{X,A,B}(v, w) \;=\; \big\langle \mathrm{PD}_X(v) \cup \mathrm{PD}_X(w), [X] \big\rangle_X$$

which is closer to the language from Definition 225.3.

Proposition 337.5. (Intersection Form Proposition) Let X be a compact oriented n-dimensional topological manifold and let R be a commutative ring.

(1) Suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the intersection pairing $Q^R_{X,A,B} \colon \mathrm{H}_k(X, A; R) \times \mathrm{H}_{n-k}(X, B; R) \to R$ is R-bilinear.

(2) We suppose that $n = 2k$ for some $k \in \mathbb{N}_0$.
   (a) If k is even, then $Q^R_X \colon \mathrm{H}_k(X; R) \times \mathrm{H}_k(X; R) \to R$ is symmetric.
   (b) If k is odd, then $Q^R_X \colon \mathrm{H}_k(X; R) \times \mathrm{H}_k(X; R) \to R$ is anti-symmetric.

Proof. As in the proof of the Intersection Form Proposition 224.13 we see that this proposition is a straightforward consequence of the bilinearity of the cup product and the Cup Product Commutativity Proposition 204.10. ■

Next we recall one more feature of Definition 213.1:

Definition 337.6. Let R be a commutative ring and let V, W and S be R-modules. We say that an R-bilinear pairing $\langle\ ,\ \rangle \colon V \times W \to S$ is non-singular if the induced maps

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W, S) \\ v & \mapsto & \begin{pmatrix} W & \to & S \\ w & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V, S) \\ w & \mapsto & \begin{pmatrix} V & \to & S \\ v & \mapsto & \langle v, w\rangle \end{pmatrix} \end{array}$$

are isomorphisms.

We continue with the following analog of the Manifold-Non Singularity Proposition 213.3, of the Intersection Form–Non Singularity Proposition 224.14 and of the Asymmetric Intersection Pairing–Non Singularity Proposition 225.4:

Proposition 337.7. (Intersection Form–Non Singularity Proposition) Let X be a compact oriented n-dimensional topological manifold. Furthermore suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For any field $\mathbb{F}$ and any $k \in \mathbb{N}_0$ the intersection pairing

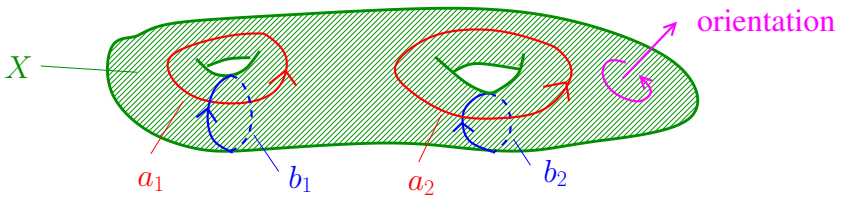
$$Q^{\mathbb{F}}_{X,A,B} \colon \mathrm{H}_k(X, A; \mathbb{F}) \times \mathrm{H}_{n-k}(X, B; \mathbb{F}) \ \to\ \mathbb{F}$$

is non-singular.

Example. If X is a *closed* oriented n-dimensional topological manifold, then the Intersection Form–Non Singularity Proposition 337.7 tells us that for any field $\mathbb{F}$ and any $k \in \mathbb{N}_0$ the intersection pairing

$$Q^{\mathbb{F}}_X \colon \mathrm{H}_k(X; \mathbb{F}) \times \mathrm{H}_{n-k}(X; \mathbb{F}) \ \to\ \mathbb{F}$$

is non-singular.



$$Q^{\mathbb{Q}}_X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In the figure above we recall the calculation of the intersection pairing of the surface of genus 2 over the field $\mathbb{Q}$. Using the Form–Matrix-Lemma 224.5 (2) we see that it is indeed non-singular. □

Proof. Let $k \in \mathbb{N}_0$. First we point out that it follows from the fact that the Poincaré Duality map is an isomorphism that it suffices to show that the pairing

$$\begin{array}{rcl} \mathrm{H}^{n-k}(X, B; \mathbb{F}) \times \mathrm{H}^k(X, A; \mathbb{F}) & \to & \mathbb{F} \\ (\varphi, \psi) & \mapsto & \epsilon_X((\varphi \cup \psi) \cap [X]) \end{array}$$

is non-singular. Next note that it follows easily from the Cup Product-Commutativity Proposition 203.20 and Definition 337.6 that it suffices to show that for any decomposition $X = A \cup B$ as in the proposition the map

$$\begin{array}{rcl} \Omega\colon \mathrm{H}^{n-k}(X, B; \mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F}), \mathbb{F}) \\ \psi & \mapsto & (\varphi \mapsto \epsilon_X((\varphi \cup \psi) \cap [X])) \end{array}$$

is an isomorphism. Similar to the discussion on pages 4082 and 4332 we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^{n-k}(X, B; \mathbb{F}) & \xrightarrow[\psi \mapsto (\varphi \mapsto \epsilon_X((\varphi \cup \psi) \cap [X]))]{\Omega} & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F}), \mathbb{F}). \\ & {\scriptstyle \mathrm{ev}} \searrow \quad \nearrow {\scriptstyle (\cap [X])^*} & \\ & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_{n-k}(X, B; \mathbb{F}), \mathbb{F}) & \end{array}$$

The proposition follows from the following clarifications and observations:

(1) The map ev is the evaluation map $\varphi \mapsto (\sigma \mapsto \langle \varphi, \sigma \rangle) = \epsilon_X(\varphi \cap \sigma)$ from the Universal Coefficient-H*–Field Theorem 202.2. Since $\mathbb{F}$ is a field the theorem tells us that the evaluation map ev is an isomorphism.

(2) The right diagonal is the dual homomorphism corresponding to the homomorphism $\mathrm{H}^k(X, A; \mathbb{F}) \xrightarrow{\cap [X]} \mathrm{H}_{n-k}(X, B; \mathbb{F})$. It follows from the Poincaré Duality Theorem 210.1 that this map is an isomorphism.

(3) It remains to show that the diagram commutes. Note that for any $\psi \in \mathrm{H}^{n-k}(X, B; \mathbb{F})$ and any $\varphi \in \mathrm{H}^k(X, A; \mathbb{F})$ the following equalities hold in $\mathbb{F}$:

$$\Omega(\psi)(\varphi) \underset{\text{by definition of } \Omega}{=} \epsilon_X((\varphi \cup \psi) \cap [X]) \underset{\text{by the Cup-Cap Lemma 206.15}}{=} \epsilon_X(\psi \cap (\varphi \cap [X])) \underset{\text{by the Cap Product Kronecker Lemma 206.12 and the definition of ev}}{=} \mathrm{ev}(\psi)(\varphi \cap [X]).$$

■

337.2. **Algebraic preparations.** Our next major goal is to introduce the twisted analogs of the intersection pairings from Definition 337.3. But before we get to this point it is worth discussing some algebraic notions which are generalizations and analogs of the concepts introduced in Definition 213.1 and Section 337.1.

We start out with a basic observation:

Observation 337.8. Let A, B, Z be abelian groups. The Tensor Product–Universal Property Lemma 312.5 shows in particular that the maps

$$\begin{array}{rcl} \operatorname{Hom}(A \otimes_{\mathbb{Z}} B, Z) & \to & \left\{ \begin{array}{c} \text{set of pairings } A \times B \to Z \text{ in} \\ \text{the sense of Definition 337.1} \end{array} \right\} \\ (\Theta\colon A \otimes_{\mathbb{Z}} B \to Z) & \mapsto & \left(\begin{array}{rcl} \varphi_\Theta\colon A \times B & \to & Z \\ (a,b) & \mapsto & \Theta(a \otimes b) \end{array} \right) \end{array}$$

$$\left(\begin{array}{rcl} \Theta_\varphi\colon A \otimes_{\mathbb{Z}} B & \to & Z \\ \sum_{i=1}^{k} a_i \otimes b_i & \mapsto & \sum_{i=1}^{k} \varphi(a_i, b_i) \end{array} \right) \quad \leftarrow\!\shortmid \quad (\varphi\colon A \times B \to Z)$$

are well-defined and inverses of one another. In particular both are bijections. For many technical purposes the view point on the left is cleaner, but for psychological reasons it is often better to take the point of the view from the right. Also (as we will see shortly) many such objects originally arise on the right.

Our next goal is to summarize how, in the bijection of Observation 337.8, various properties translate from the left to right and back. To be more precise, we will now formulate in a dictionary of properties for the two points of view:

Dictionary 337.9. In the dictionary below, in rows (i)–(iv), we will have the following setups:

(i) In row (i) we just have abelian groups A, B and Z.

(ii) In row (ii) we assume that π is a group and that A and B are $\mathbb{Z}[\pi]$-left modules. We equip $A \otimes_{\mathbb{Z}} B$ with the $\mathbb{Z}[\pi]$-left module structure from Convention 334.7. Furthermore, as usual we denote by ${}_{\mathrm{tr}}Z$ the $\mathbb{Z}[\pi]$-left module with the trivial π-action.

(iii) In row (iii) we assume that we are given a ring R with involution in the sense of Definition 309.14. Furthermore we now suppose that A, B are R-left modules and that Z is an (R, R)-bimodule.
- We use the Tensor Product-of-Rings–Modules Lemma 334.5 to view $A \otimes_{\mathbb{Z}} B$ as an $R \otimes R$-left module.
- We use the (R, R)-bimodule structure on Z and the involution on R, together with the Opposite Ring–Involution Lemma 309.15 and the Bimodule–Reinterpretation Proposition 311.12, to view Z as an $R \otimes R$-left module, where $(r \otimes s) \cdot z = r \cdot z \cdot \overline{s}$.

(iv) In row (iv) we assume the following:
- We have $A = B$.
- We now assume that Z is equipped with an involution, i.e. we assume that there exists a map
$$\begin{array}{rcl} \rho\colon Z & \to & Z \\ z & \mapsto & \overline{z} \end{array}$$
which, similar to Definition 309.14, has the following two properties:
(a) $\overline{z + w} = \overline{z} + \overline{w}$ for all $z, w \in Z$,
(b) $\overline{\overline{z}} = z$ for all $z \in Z$.
- We denote by $\tau\colon A \otimes_{\mathbb{Z}} A \to A \otimes_{\mathbb{Z}} A$ the isomorphism of abelian groups that is given by $a \otimes a' \mapsto a' \otimes a$.

After these preparations we can now formulate the following dictionary:[3641]

	$\Theta = \Theta_\varphi \colon A \otimes_{\mathbb{Z}} B \to Z$	$\iff$	map $\varphi = \varphi_\Theta \colon A \times B \to Z$
(i)	$\Theta \colon A \otimes_{\mathbb{Z}} B \to Z$ is a homomorphism of $\mathbb{Z}$-modules	$\iff$	φ is $\mathbb{Z}$-bilinear, i.e. a pairing
(ii)	$\Theta \colon A \otimes_{\mathbb{Z}} B \to {}_{\mathrm{tr}}Z$ is a homomorphism of $\mathbb{Z}[\pi]$-left modules	$\iff$	φ is preserved by the π-action, i.e. $\varphi(g \cdot a, g \cdot b) = \varphi(a, b)$
(iii)	Θ is a homomorphism of $(R \otimes R)$-modules	$\iff$	φ is R-sesquilinear[3648], i.e. $\varphi(r \cdot a, s \cdot b) = r \cdot \varphi(a, b) \cdot \overline{s}$
(iv)	$\Theta = \rho \circ \Theta \circ \tau \colon A \otimes_{\mathbb{Z}} A \to Z$	$\iff$	φ is hermitian[3655], i.e. for all $a, a' \in A$ we have $\varphi(a', a) = \overline{\varphi(a, a')}$,
(iv′)	$\Theta = -\rho \circ \Theta \circ \tau \colon A \otimes_{\mathbb{Z}} A \to Z$	$\iff$	φ is anti-hermitian, i.e. for all $a, a' \in A$ we have $\varphi(a', a) = -\overline{\varphi(a, a')}$.

In the following we will expect fluency of the reader in using Dictionary 337.9, i.e. we will use Dictionary 337.9 to go back and forth without explicitly referring to it.

Examples. Let π be a group and let $S \subseteq \mathbb{C}$ be a subring that is closed under the complex involution (in most cases $S = \mathbb{Z}, S = \mathbb{R}$ or $S = \mathbb{C}$).

(1) We consider the ring $R = S$. Let $\alpha \colon \pi \to \mathrm{U}_S(d) := \{A \in \mathrm{M}(d \times d, S) \mid A \cdot \overline{A}^{\mathrm{T}} = \mathrm{id}\}$ be a unitary representation. From Convention 320.9 we obtain the corresponding $(S, \mathbb{Z}[\pi])$-left left module ${}_\varphi S^{1\times d}$. We equip the abelian group S with the usual involution. We consider the standard inner product[3662]

$$\begin{array}{rccc} \langle -, - \rangle \colon & {}_\varphi S^{1\times d} \times {}_\varphi S^{1\times d} & \to & S \\ & (v, w) & \mapsto & v \cdot \overline{w}^{\mathrm{T}}. \end{array}$$

First note that this pairing is clearly S-sesquilinear. Next note that since α is a unitary representation we see that the corresponding left action of π on ${}_\varphi S^{1\times d}$ preserves $\langle -, - \rangle$.[3663] Finally note this pairing $\langle -, - \rangle$ is clearly hermitian.

(2) Let G be a group and let $\varphi \colon \pi \to G$ be a homomorphism. Note that this defines a homomorphism $\varphi \colon \pi \to \mathrm{Aut}(S[G])$. By Convention 320.9 this defines a $\mathbb{Z}[\pi]$-left

[3659] In each case it follows easily from the definitions that the above bijection between homomorphisms $A \otimes_{\mathbb{Z}} B \to Z$ and pairings $A \times B \to Z$ gives a bijection between the entries in the rows shown below.
[3660] If the ring R has the trivial involution, then we usually say "bilinear" instead of "sesquilinear".
[3661] If the ring R has the trivial involution, then we usually say "symmetric" instead of "hermitian".
[3662] Recall that we view think of elements of $S^{1\times d}$ as *row* vectors, so given $v, w \in S^{1\times d}$ it makes sense to consider $v \cdot \overline{w}^{\mathrm{T}} \in S$.
[3663] Indeed: Let $v, w \in S^{1\times d}$ and let $g \in \pi$. We then see that

$$\begin{aligned} \langle g \cdot v, g \cdot w \rangle &= \langle v \cdot \varphi(g^{-1}), w \cdot \varphi(g^{-1}) \rangle = v \cdot \varphi(g^{-1}) \cdot \overline{(w \cdot \varphi(g^{-1}))}^{\mathrm{T}} \\ &= v \cdot \underbrace{\varphi(g^{-1}) \cdot \overline{\varphi(g^{-1})}^{\mathrm{T}}}_{= \mathrm{id}, \text{ since } \varphi(g^{-1}) \in \mathrm{U}_S(d)} \cdot \overline{w}^{\mathrm{T}} = \langle v, w \rangle. \end{aligned}$$

module structure ${}_{\varphi}S[G]$ on the abelian group $S[G]$ which is given as follows: For $g \in \pi$ and $f \in S[G]$ we have $g \cdot f := f \cdot \varphi(g^{-1})$.

(a) We consider the ring $R = S[G]$, which we view as a ring with the usual involution that is given by the complex involution on $S \subseteq \mathbb{C}$ and by $\overline{g} := g^{-1}$. Note that we can view the abelian group $S[G]$ as an $(S[G], \mathbb{Z}[\pi])$-left left module. We also consider the abelian group $S[G]$ as an $(S[G], S[G])$-bimodule in the obvious way. We equip the abelian group $S[G]$ with the above involution. Similar to (1) we consider map

$$\begin{array}{rcl} \langle -,- \rangle \colon S[G] \times S[G] & \to & S[G] \\ (r,s) & \mapsto & \langle r,s \rangle := r \cdot \overline{s}. \end{array}$$

Note that this pairing is $S[G]$-sesquilinear, that it is preserved by the π-action[3664] and note that it is clearly hermitian.

(b) We consider the ring $R = S \subseteq \mathbb{C}$. We view $S[G]$ as an S-module in the obvious way. We consider the abelian group S which we again equip with the involution coming from complex involution. Clearly the pairing

$$\begin{array}{rcl} \chi \colon S[G] \times S[G] & \to & S \\ (r,s) & \mapsto & e_G\text{-coefficient of } r \cdot \overline{s} \text{ in } S[G] \end{array}$$

is S-sesquilinear, it is preserved by the π-action and it is hermitian. □

Remark. The above examples (1) and (2b) have an interesting overlap: Let π be a group and let $S \subseteq \mathbb{C}$ be a subring that is closed under the complex involution. Furthermore let $\varphi \colon \pi \to G$ be an epimorphism onto a finite group with d elements $g_1, \dots, g_d$ We get an isomorphism $\gamma \colon S^{1\times d} \to S[G]$ of S-modules that is given by $\mathfrak{e}_i \mapsto g_i$. We make the following observations:

(i) The π-action on $S[G]$ now translates into an action on $S^{1\times d}$ which is given by permutation matrices. In particular the matrices are unitary.

(ii) The pairing
$$\begin{array}{rcl} S[G] \times S[G] & \to & S[G] \\ (r,s) & \mapsto & \langle r,s \rangle := r \cdot \overline{s} \end{array}$$
clearly has the property that the g_i form an orthonormal basis. Since $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ is an orthonormal basis of $(S^{1\times d}, \langle -,- \rangle$ it is now clear that the isomorphism γ defines an isometry $(S^{1\times d}, \langle -,- \rangle) \to (S[G], \langle -,- \rangle)$.

This discussion shows that for G finite the above Example (2b) can be viewed as a special case of Example (1). □

Examples. Let π be a group. We consider a few more examples of pairings:

(3) Let R be a commutative ring with the trivial involution. The pairing
$$\begin{array}{rcl} {}_{\mathrm{tr}}R \times {}_{\mathrm{tr}}R & \to & {}_{\mathrm{tr}}R \\ (r,s) & \mapsto & r \cdot_R s \end{array}$$
is preserved by the π-action and it is clearly R-bilinear and symmetric.

(4) Let R be any commutative ring with the trivial involution. Let $\alpha \colon \pi \to \mathrm{SL}(2,R)$ be a representation. We consider the corresponding $(R, \mathbb{Z}[\pi])$-left left module ${}_{\alpha}S^{1\times 2}$ from

[3664] Indeed, given $r, s \in S[G]$ and $g \in \pi$ we have

$$\begin{aligned} \langle g \cdot r, g \cdot s \rangle = \langle r \cdot \varphi(g^{-1}), s \cdot \varphi(g^{-1}) \rangle &= (r \cdot \varphi(g^{-1})) \cdot \overline{(s \cdot \varphi(g^{-1}))} \\ &= (r \cdot \varphi(g^{-1})) \cdot \overline{\varphi(g^{-1})} \cdot \overline{s} = r \cdot \varphi(g^{-1}) \cdot \varphi(g) \cdot \overline{s} = r \cdot \overline{s} = \langle r,s \rangle. \end{aligned}$$

Convention 320.9. The pairing

$$\begin{array}{rcl} \det\colon R^{1\times 2}\times R^{1\times 2} & \to & R \\ (v,w) & \mapsto & \det(v\ w) \end{array}$$

is clearly R-bilinear. Since α takes values in $\mathrm{SL}(2,R)$ it follows from a straightforward argument that the corresponding left action of π on $R^{1\times 2}$ preserves the pairing det.[3665] Finally note that det is clearly an anti-symmetric pairing.

(5) Let $\varphi\colon \pi\to\langle t\rangle$ be a group homomorphism. We consider the corresponding homomorphism $\varphi\colon \pi\to \mathrm{GL}(1,\mathbb{Q}(t))$, where $\mathbb{Q}(t)$ is the quotient field of the group ring $\mathbb{Z}[t^{\pm 1}]$. We consider the ring $R=\mathbb{Z}[t^{\pm 1}]$, the $(\mathbb{Z}[t^{\pm 1}],\mathbb{Z}[\pi])$-left left module $\mathbb{Z}[t^{\pm 1}]$ and we equip $\mathbb{Q}(t)$ with the obvious $(\mathbb{Z}[t^{\pm 1}],\mathbb{Z}[t^{\pm 1}])$ bimodule structure. Finally we equip the abelian group $\mathbb{Q}(t)$ with the involution that is induced by $t\mapsto t^{-1}$. Then the map

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm 1}]\times\mathbb{Z}[t^{\pm 1}] & \to & \mathbb{Q}(t) \\ (r,s) & \mapsto & r\cdot\overline{s} \end{array}$$

defines a hermitian $\mathbb{Z}[t^{\pm 1}]$-sesquilinear pairing that is preserved by the π-action. □

337.3. **Definition of twisted intersection pairings.** With this section we return to topology. First of all we recall the following definition, which essentially generalizes Definition 337.2, and which has its origins in the Twisted H_0-Proposition 323.3:

Definition 337.10. Let X be a topological space that is path-connected and locally contractible and let $x_0\in X$. Let S be a ring and let Z be an S-left module. We consider the $\mathbb{Z}[\pi_1(X,x_0)]$-left module ${}_{\mathrm{tr}}Z$ that is given by the trivial $\pi_1(X,x_0)$-action. We refer to the map

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0(\overline{{}_{\mathrm{tr}}Z}\otimes_{\mathbb{Z}[\pi_1(X,x_0)]}\mathrm{C}_*(\widetilde{X}_{x_0})) =: \mathrm{H}_0^{\mathrm{tw}}(X;{}_{\mathrm{tr}}Z)) & \to & Z \\ \left[\sum_{i=1}^k z_i\otimes\sigma_i\right] & \mapsto & \sum_{i=1}^k z_i \end{array}$$

as the augmentation map. Note that it follows from the Twisted H_0-Proposition 323.3 that this map is well-defined and that it is an isomorphism of S-left modules.

Now we can finally introduce twisted intersection pairings. More precisely, we have the following twisted variation on Definition 337.3:

Definition 337.11. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0\in X$. We write $\pi=\pi_1(X,x_0)$. We denote by $[X]\in\mathrm{H}_n(X,\partial X)$ the fundamental class. Furthermore let M be a $\mathbb{Z}[\pi_1(X,x_0)]$-left module, let Z be an abelian group and let $\Theta\colon M\otimes_{\mathbb{Z}} M\to {}_{\mathrm{tr}}Z$ be a homomorphism of $\mathbb{Z}[\pi]$-left modules.[3666]

(1) Suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A\cap B=\partial A=\partial B$. We write $A^\dagger:=\partial X\setminus\mathring{A}$ and $B^\dagger:=\partial X\setminus\mathring{B}$. For each

[3665] Indeed, we have the following equalities:

$$\det(v\cdot A\ \ w\cdot A)\underset{\substack{\uparrow\\ \text{by definition}\\ \text{of matrix}\\ \text{multiplication}}}{=}\det((v\ \ w)\cdot A)\underset{\substack{\uparrow\\ \text{multiplicativity}\\ \text{of the determinant}}}{=}\det(v\ \ w)\cdot\underbrace{\det(A)}_{=1}\underset{\substack{\uparrow\\ \text{since } A\in\mathrm{SL}(2,R)}}{=}\det(v\ \ w).$$

$k \in \mathbb{N}_0$ we refer to the map[3667] [3668]

$$\begin{array}{rcl} \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X,B;M) & \xrightarrow{Q_{X,A,B}^{M,\Theta}} & Z \\ (v,w) & \mapsto & \epsilon_X\big(\Theta_*\big(\big(\underbrace{\mathrm{PD}_X(v)}_{\in \mathrm{H}_{\mathrm{tw}}^{n-k}(X,A^\dagger;M)} \cup_\otimes \underbrace{\mathrm{PD}_X(w)}_{\in \mathrm{H}_{\mathrm{tw}}^{k}(X,B^\dagger;M)}\big) \cap_{\mathbb{Z}} \underbrace{[X]}_{\in \mathrm{H}_n(X,\partial X)}\big)\big). \end{array}$$

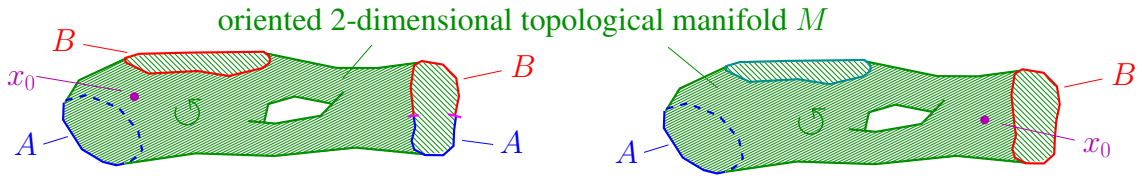
$\in \mathrm{H}_{\mathrm{tw}}^{n}(X,\partial X;M\otimes_{\mathbb{Z}} M)$

$\in \mathrm{H}_{\mathrm{tw}}^{n}(X,\partial X;{}_{\mathrm{tr}}Z)$

$\in \mathrm{H}_0^{\mathrm{tw}}(X,\partial X;{}_{\mathrm{tr}}Z)$

as a twisted intersection pairing.

(2) We write $Q_X^{M,\Theta} := Q_{X,\varnothing,\varnothing}^{M,\Theta}$.



twisted intersection pairing $Q_{X,A,B}^{M,\Theta}\colon \mathrm{H}_k^{\mathrm{tw}}(X,A;M) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X,B;M) \to Z$

Remark. In this remark we mostly stick with the setting of Definition 337.11. But now let R be a commutative ring. If we work with the trivial ${}_{\mathrm{tr}}R$-coefficients and if as on page 6396 we consider the multiplication map $\mu\colon R \times R \to R$, $(r,s) \mapsto r \cdot_R s$, then it follows easily from the Classical-Twisted Cup Product Lemma 334.18 and the Classical-Twisted Cap Product Lemma 334.32 that the twisted intersection pairing $Q_{X,A,B}^{R,\mu}$ from Definition 337.11 is isomorphic to the intersection pairing $Q_{X,A,B}^{R}$ from Definition 337.3. □

Ideally we want the twisted intersection pairings to have some nice properties. The twisted intersection pairing is built out of Poincaré Duality, the cup product, the map Θ, the cap product and the evaluation map. Almost all of these maps have all kinds of nice properties. The bottleneck is the map $\Theta_*\colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}Z$. So far we only demand that it is a homomorphism of $\mathbb{Z}[\pi]$-left modules. If we want $Q_{X,A,B}^{M,\Theta}$ to have more properties, then we have to demand more from Θ. As we will see in a second, properties of Θ as in Section 337.2 frequently translate into corresponding properties of the twisted intersection pairings.

For example, we have the following proposition which is an analog of the Intersection Form Proposition 224.13 and of the Intersection Form Proposition 337.5:

[3666] Using the dictionary from Dictionary 337.9 we can reformulate the hypothesis: We suppose that we are given a pairing $\varphi\colon M \times M \to Z$ that is preserved by the π-action, i.e. such that given any $v,w \in M$ and given any $g \in \pi$ we have $\varphi(g\cdot v, g\cdot w) = \varphi(v,w)$. We then consider the corresponding homomorphism $\Theta_\varphi\colon M \otimes_{\mathbb{Z}} M \to Z$ which is given by $\Theta_\varphi(v\otimes w) = \varphi(v,w)$.

[3667] Note that it follows from the Universally Excisive Triad Proposition 325.3 (2) and (1a) that (X,A,B) is a universally excisive triad. In particular it follows, see Definition 334.15, that the cup product is actually defined.

[3668] As we discussed above, Θ gives rise to a homomorphism $\Theta\colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}R$ of $\mathbb{Z}[\pi_1(X,x_0)]$-left modules.

Proposition 337.12. (Twisted Intersection Pairing Proposition) Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. Let R be a ring with involution and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Finally let Z be an abelian group and let $\Theta\colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}Z$ be a $\mathbb{Z}[\pi]$-left module homomorphism. Then the following statements hold:

(1) If Z is an (R,R)-bimodule and if $\Theta\colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}Z$ is a homomorphism of $(R \otimes_{\mathbb{Z}} R)$-left modules[3669], then the pairing

$$\begin{aligned} Q^{M,\Theta}_{X,A,B}\colon \mathrm{H}^{\mathrm{tw}}_k(X, A; M) \times \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) &\to Z \\ (v, w) &\mapsto \epsilon_X(\Theta_*((\mathrm{PD}_X(v) \cup_{\otimes} \mathrm{PD}_X(w)) \cap_{\mathbb{Z}} [X]) \end{aligned}$$

is R-sesquilinear.

(2) We now suppose that $n = 2k$ for some $k \in \mathbb{N}_0$. Furthermore we suppose that Z is equipped with an involution as in Dictionary 337.9 and that $\Theta\colon M \otimes_{\mathbb{Z}} M \to Z$ is hermitian.
   (a) If k is even, then $Q^{M,\Theta}_X\colon \mathrm{H}^{\mathrm{tw}}_k(X; M) \times \mathrm{H}^{\mathrm{tw}}_k(X; M) \to Z$ is hermitian.
   (b) If k is odd, then $Q^{M,\Theta}_X\colon \mathrm{H}^{\mathrm{tw}}_k(X; M) \times \mathrm{H}^{\mathrm{tw}}_k(X; M) \to Z$ is anti-hermitian.
   If Θ is anti-Hermitian, then the above two statements flip in the obvious way.

Proof. We start out with the following observations:

(i) As we mentioned on page 6362, the Poincaré Duality isomorphism is an isomorphism of R-left modules.

(ii) It follows from (i) and the Cup Product–Tensor Module Homomorphism Lemma 334.16 that the map

$$\begin{aligned} \mathrm{H}^{\mathrm{tw}}_k(X, A; M) \otimes_{\mathbb{Z}} \mathrm{H}^{\mathrm{tw}}_{n-k}(X, B; M) &\to \mathrm{H}^n_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} M) \\ (v, w) &\mapsto \mathrm{PD}_X(v) \cup_{\otimes} \mathrm{PD}_X(w) \end{aligned}$$

is a homomorphism of $R \otimes_{\mathbb{Z}} R$-left modules.

(iii) As we mentioned in Definition 337.10, by the Twisted H_0-Proposition 323.3 we know that the augmentation isomorphism $\mathrm{H}^n_{\mathrm{tw}}(X; M \otimes_{\mathbb{Z}} M) \to M \otimes_{\mathbb{Z}} M$ from Definition 337.10 is an isomorphism of $R \otimes_{\mathbb{Z}} R$-left modules.

Now we turn to the proof of the proposition:

(1) We now assume that Z is an (R,R)-bimodule and that $\Theta\colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}Z$ is a homomorphism of $(R \otimes_{\mathbb{Z}} R)$-left modules. It now follows almost immediately from (i), (ii), (iii) and the discussion in Dictionary 337.9 that the pairing $Q^{M,\Theta}_{X,A,B}$ is R-sesquilinear.

(2) We now suppose that $n = 2k$ and that $\Theta\colon M \otimes_{\mathbb{Z}} M \to Z$ is Hermitian. First note that it follows from the Twisted Cup Product-Commutativity Proposition 334.19 that given $\varphi, \psi \in \mathrm{H}^k_{\mathrm{tw}}(X; M)$ we have the following equality in $\mathrm{H}^{2k}_{\mathrm{tw}}(X; M)$:

$$\varphi \cup_{\otimes} \psi = (-1)^{k \cdot k} \cdot \underset{\substack{\uparrow \\ \tau\colon M \otimes_{\mathbb{Z}} M \text{ swaps} \\ \text{the two factors}}}{\tau}(\psi \cup_{\otimes} \varphi).$$

[3669] As in Dictionary 337.9 we use the (R,R)-bimodule structure on Z and the involution on R to view Z as an $(R \otimes_{\mathbb{Z}} R)$-left module where $(r \otimes s) \cdot z = r \cdot z \cdot \overline{s}$.

The promised statement now follows easily from the fact that $(-1)^{k\cdot k} = (-1)^k$, Definition 337.11 and Dictionary 337.9 (iv). ■

Examples. We now want to apply the algebraic examples on page 6395 to define various types of intersection pairings on topological manifolds. Throughout the following discussion let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$.

(1) Let $S \subseteq \mathbb{C}$ be a subring that is closed under the involution.
   (a) Let $\alpha\colon \pi \to \mathrm{U}_S(d) := \{A \in \mathrm{M}(d\times d, S) \mid A \cdot \overline{A}^{\mathrm{T}} = \mathrm{id}\}$ be a unitary representation. From the discussion on page 6395 we obtain the S-sesquilinear, hermitian pairing
   $$\begin{aligned}\langle -,-\rangle\colon {}_\varphi S^{1\times d} \times {}_\varphi S^{1\times d} &\to S\\ (v,w) &\mapsto v\cdot \overline{w}^{\mathrm{T}}\end{aligned}$$
   which is preserved by the left action of π. It follows from the Twisted Intersection Pairing Proposition 337.12 that for each $k \in \mathbb{N}_0$ the corresponding intersection pairing
   $$Q_{X,A,B}^{{}_\varphi S^{1\times d},\langle -,-\rangle}\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; {}_\varphi S^{1\times d}) \times \mathrm{H}_k^{\mathrm{tw}}(X, B; {}_\varphi S^{1\times d}) \to S$$
   is S-sesquilinear and that it is (anti-) hermitian if $n = 2k$ and $A = B = \varnothing$.
   (b) Let $\alpha\colon \pi \to \mathrm{U}(1)$ be the trivial 1-dimensional unitary representation. By the Twisted–Trivial Coefficients Lemma 320.13 we know that $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\alpha\mathbb{C})$ is naturally isomorphic to $\mathrm{H}_k(X;\mathbb{C})$. But note that the twisted intersection form $Q_X^{\mathbb{C},\langle -,-\rangle}$ is *not* isomorphic to the intersection pairing $Q_X^{\mathbb{C}}$ from Definition 337.3, since for the former we use the $\mathbb{C}$-sesquilinear map $(z,w) \mapsto \langle z, w\rangle = z\cdot\overline{w}$ whereas for the latter we use the $\mathbb{C}$-bilinear multiplication map $(z,w) \mapsto z \cdot_{\mathbb{C}} w$. For most purposes the former pairing is better, since it is sesquilinear.[3670]

(2) Let G be a group and let $\varphi\colon \pi \to G$ be a homomorphism. Recall that this defines a homomorphism $\varphi\colon \pi \to \mathrm{Aut}(S[G])$ that is given by $g \mapsto (p \mapsto p\cdot\varphi(g^{-1}))$.
   (a) We consider the ring $R = S[G]$, which we view as a ring with the usual involution that is given by $\overline{g} := g^{-1}$. Note that we can view $S[G]$ as an $(S[G], \mathbb{Z}[\pi])$-left left module. We view the abelian group $S[G]$ as an $(S[G], S[G])$-bimodule in the obvious way and we equip the abelian group $S[G]$ with the usual involution. By the discussion on page 6396 we have the $S[G]$-sesquilinear hermitian pairing $\langle -,-\rangle\colon S[G]\times S[G] \to S[G]$, that is preserved by the π-action. It follows from the Twisted Intersection Pairing Proposition 337.12 that for each $k \in \mathbb{N}_0$ the corresponding intersection pairing
   $$Q_{X,A,B}^{S[G],\langle -,-\rangle}\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; {}_\varphi S[G]) \times \mathrm{H}_k^{\mathrm{tw}}(X, B; {}_\varphi S[G]) \to S[G]$$
   is $S[G]$-sesquilinear and that it is (anti-) hermitian if $n = 2k$ and $A = B = \varnothing$. We refer to this as an equivariant intersection pairing.
   (b) By the discussion on page 6396 we know that the pairing
   $$\begin{aligned}\chi\colon S[G]\times S[G] &\to S\\ (r,s) &\mapsto e_G\text{-coefficient of } r\cdot\overline{s} \text{ in } S[G]\end{aligned}$$

[3670] We will see in Exercise 337.2 that bilinear forms over $\mathbb{C}$ are not very interesting.

is S-sesquilinear, hermitian and that it is preserved by the π-action. It follows from the Twisted Intersection Pairing Proposition 337.12 that for each $k \in \mathbb{N}_0$ the corresponding intersection pairing

$$Q^{S[G],\chi}_{X,A,B}\colon \mathrm{H}^{\mathrm{tw}}_k(X,A;{}_{\varphi}S[G]) \times \mathrm{H}^{\mathrm{tw}}_k(X,B;{}_{\varphi}S[G]) \;\to\; S$$

is S-sesquilinear and that it is (anti-) hermitian if $n = 2k$ and $A = B = \varnothing$. □

337.4. **Functoriality of the twisted intersection pairing.** We conclude this section with the following proposition which says that the twisted intersection pairing is functorial in the values Z:

Proposition 337.13. (Twisted Intersection Pairing–Functorial Proposition) In the following let (X, x_0) be a pointed compact oriented path-connected n-dimensional topological manifold. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. Let M, M' be two $\mathbb{Z}[\pi_1(X,x_0)]$-left modules, let Z, Z' be two abelian groups and let $\Theta\colon M \otimes_{\mathbb{Z}} M \to Z$ and $\Theta'\colon M' \otimes_{\mathbb{Z}} M' \to Z'$ be two $\mathbb{Z}[\pi_1(X,x_0)]$-left module homomorphisms. Finally suppose that we are given a homomorphism $\varphi\colon M \to M'$ of $\mathbb{Z}[\pi_1(X,x_0)]$-left modules and that we are given a homomorphism $\psi\colon Z \to Z'$ of abelian groups such that the following diagram commutes:

$$\begin{array}{ccc} M \otimes_{\mathbb{Z}} M & \xrightarrow{\quad\Theta\quad} & Z \\ \downarrow{\scriptstyle \varphi\otimes\varphi} & & \downarrow{\scriptstyle \psi} \\ M' \otimes_{\mathbb{Z}} M' & \xrightarrow{\quad\Theta'\quad} & Z'. \end{array}$$

Then for all $v \in \mathrm{H}^{\mathrm{tw}}_k(X,A;M)$ and $w \in \mathrm{H}^{\mathrm{tw}}_{n-k}(X,B;M)$ we have

$$Q^{M',\Theta'}_{X,A,B}(\underset{\uparrow}{\varphi_*}(v), \underset{\uparrow}{\varphi_*}(w)) \;=\; \psi(Q^{M,\Theta}_{X,A,B}(v,w)).$$

here φ_* is the induced map $\varphi_*\colon \mathrm{H}^{\mathrm{tw}}_l(X,C;M) \to \mathrm{H}^{\mathrm{tw}}_l(X,C;M')$

Proof. It follows easily from Definition 337.11 that it suffices to prove the following claim:

Claim. For all $\alpha \in \mathrm{H}^{n-k}_{\mathrm{tw}}(X,B;M)$ and $\beta \in \mathrm{H}^{k}_{\mathrm{tw}}(X,A;M)$ we have

$$\epsilon_X(\Theta'_*(\varphi_*(\alpha) \cup_{\otimes} \varphi_*(\beta)) \cap_{\mathbb{Z}} [X]) \;=\; \psi(\epsilon_X(\Theta_*(\alpha \cup_{\otimes} \beta) \cap_{\mathbb{Z}} [X])).$$

Proof. This statement follows easily from the definitions, the naturality of the cup product, the naturality of the cap product and the naturality of the augmentation map. ■

Example. Let X be a compact oriented path-connected n-dimensional topological manifold and let $x_0 \in X$. We suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. We write $\pi := \pi_1(X, x_0)$. Finally let $S \subseteq \mathbb{C}$ be a subfield that is closed under the usual involution.

(1) Let $\gamma\colon \pi \to G$ be a homomorphism to some group G. As always we view ${}_{\gamma}S[G]$ as a $\mathbb{Z}[\pi]$-left module by $g \cdot p := p \cdot \gamma(g^{-1})$. It follows from the discussion on page 6396 that we can now consider the following two pairings that are preserved by the π-action:

$$\begin{array}{rcl} {}_{\gamma}S[G] \times {}_{\gamma}S[G] & \to & S[G] \\ (r,s) & \mapsto & \langle r,s\rangle := r \cdot \overline{s} \end{array} \quad \text{and} \quad \begin{array}{rcl} \chi\colon {}_{\gamma}S[G] \times {}_{\gamma}S[G] & \to & S \\ (r,s) & \mapsto & e_G\text{-coefficient of } r \cdot \overline{s}. \end{array}$$

The map $\varphi = \mathrm{id}_{S[G]}$ and the map

$$\begin{aligned} \psi\colon S[G] &\to S \\ p &\mapsto e_G\text{-coefficient of } p \end{aligned}$$

have the properties of the Twisted Intersection Pairing–Functorial Proposition 337.13. Thus we see that for any $v \in \mathrm{H}_k^{\mathrm{tw}}(X, A; {}_\gamma S[G])$ and $w \in \mathrm{H}_{n-k}^{\mathrm{tw}}(X, A; {}_\gamma S[G])$ we have

$$Q_{X,A,B}^{S[G],\chi}(v, w) \;=\; \psi(Q_{X,A,B}^{S[G],\langle -,-\rangle}(v, w)) \;\in\; S.$$

In Exercise 337.6 we will see how conversely one can obtain the intersection pairing $Q_{X,A,B}^{S[G],\langle -,-\rangle}$ from the intersection pairing $Q_{X,A,B}^{S[G],\chi}(v, w)$.

(2) Let $\gamma\colon \pi \to G$ be a homomorphism to a finite group with d elements $g_1, \dots, g_d$. As on page 6396 we see that we obtain a corresponding unitary representation $\gamma\colon \pi \to \mathrm{U}_S(d)$. It follows easily from the discussion on page 6396 together with the Twisted Intersection Pairing–Functorial Proposition 337.13 that the intersection pairing

$$Q_{X,A,B}^{S[G],\chi}\colon \mathrm{H}_k(X, A; {}_\gamma S[G]) \times \mathrm{H}_k(X, A; {}_\gamma S[G]) \;\to\; S$$

is isomorphic to the intersection pairing

$$Q_{X,A,B}^{S^d,\langle -,-\rangle}\colon \mathrm{H}_k(X, A; {}_\gamma S^d) \times \mathrm{H}_k(X, A; {}_\gamma S^d) \;\to\; S. \qquad \square$$

337.5. **Non-singular twisted intersection pairings.** The following proposition is the analog of the Intersection Form–Non Singularity Proposition 224.14 and of the Intersection Form–Non Singularity Proposition 337.7:

Proposition 337.14. (Twisted Intersection Pairing–Non-Singularity Proposition) Let X be a compact oriented path-connected non-empty n-dimensional topological manifold and let $x_0 \in X$. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. Let $\mathbb{K}$ be a skew field with involution and let V be a $(\mathbb{K}, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Finally suppose that we are given a homomorphism $\Theta\colon V \otimes_{\mathbb{Z}} V \to {}_{\mathrm{tr}}\mathbb{K}$ of $\mathbb{Z}[\pi]$-left module homomorphism such that the corresponding form

$$\begin{aligned} \varphi_\Theta\colon V \otimes_{\mathbb{Z}} V &\to \mathbb{K} \\ (v, w) &\mapsto \Theta(v \otimes w) \end{aligned}$$

is $\mathbb{K}$-sesquilinear and non-singular. If $A \cup B = \partial X$, then the twisted intersection pairing

$$Q_{X,A,B}^{V,\Theta}\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; V) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X, B; V) \;\to\; \mathbb{K}$$

is non-singular.

Proof. Note that it follows from the hypotheses $A \cap B = \partial A = \partial B$ and $A \cup B = \partial X$ that the Twisted Poincaré Duality Theorem 335.1 gives us the Poincaré Duality isomorphism

$$\mathrm{PD}\colon \mathrm{H}_{n-k}^{\mathrm{tw}}(X, A; V) \;\to\; \mathrm{H}_{\mathrm{tw}}^k(X, B; V)$$

of $\mathbb{K}$-left vector spaces. In the following we denote by

$$\begin{aligned} \tau\colon V \otimes_{\mathbb{Z}} V &\to V \otimes_{\mathbb{Z}} V \\ v \otimes w &\mapsto w \otimes v \end{aligned}$$

the natural isomorphism of $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. We continue with the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^{\mathrm{tw}}_{n-k}(X,A;V)\times\mathrm{H}^{\mathrm{tw}}_{k}(X,B;V) & \xrightarrow{(v,w)\mapsto \mathrm{PD}(v)\cup_{\otimes}\mathrm{PD}(w)} & \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;V\otimes_{\mathbb{Z}}V) & \xrightarrow{\Theta_*} & \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;{}_{\mathrm{tr}}\mathbb{K}) \\
\downarrow \mathrm{id} \quad \downarrow \mathrm{id} & & \downarrow (-1)^{(n-k)\cdot k} & & \downarrow (-1)^{(n-k)\cdot k} \\
\mathrm{H}^{\mathrm{tw}}_{n-k}(X,A;V)\times\mathrm{H}^{\mathrm{tw}}_{k}(X,B;V) & \xrightarrow{(v,w)\mapsto \mathrm{PD}(w)\cup_{\otimes}\mathrm{PD}(v)} & \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;V\otimes_{\mathbb{Z}}V) & \xrightarrow{\Theta_*} & \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;{}_{\mathrm{tr}}\mathbb{K}) \\
\downarrow \mathrm{PD} \quad \downarrow \mathrm{id} & & \downarrow \tau_* & & \downarrow -\cap_{\mathbb{Z}}[X] \\
\mathrm{H}^{k}_{\mathrm{tw}}(X,B;V)\times\mathrm{H}^{\mathrm{tw}}_{k}(X,B;V) & & \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;V\otimes_{\mathbb{Z}}V) & & \\
 & & \downarrow -\cap_{\mathbb{Z}}[X] & & \\
 & \xrightarrow{\cap_{\otimes}} & \mathrm{H}^{\mathrm{tw}}_{0}(X;V\otimes_{\mathbb{Z}}V) & \xrightarrow{\Theta_*} & \mathrm{H}^{\mathrm{tw}}_{0}(X;{}_{\mathrm{tr}}\mathbb{K}).
\end{array}$$

Proof. We consider the various squares separately.

(1) It follows from the Twisted Cup Product-Commutativity Proposition 334.19 that the top left square commutes.

(2) It is essentially clear that the top right square commutes.

(3) We turn to the bottom left square. Thus let $v \in \mathrm{H}^{\mathrm{tw}}_{n-k}(X,A;V)$ and $w \in \mathrm{H}^{\mathrm{tw}}_{k}(X,B;V)$. Then we have the following equality in $\mathrm{H}^{\mathrm{tw}}_{0}(X;V\otimes_{\mathbb{Z}}V)$:

$$\tau_*(\mathrm{PD}(w)\cup_{\otimes}\mathrm{PD}(v))\cap_{\mathbb{Z}}[X] \underset{\text{by the Twisted Cup-Cap Lemma 334.37}}{=} \mathrm{PD}(v)\cap_{\otimes}(\mathrm{PD}(w)\cap_{\mathbb{Z}}[X]) \underset{\text{by Definition 335.2 of PD}(w)}{=} \mathrm{PD}(v)\cap_{\otimes} w.$$

(4) Finally we turn to the rectangle to the bottom right. Note that it follows easily from the Twisted Cap Product-Coefficients Lemma 334.36 that for each $k,l \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^{k}_{\mathrm{tw}}(X,\partial X;V\otimes_{\mathbb{Z}}V)\times\mathrm{H}_{l}(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X;V\otimes_{\mathbb{Z}}V) \\
\downarrow \Theta_* \quad \downarrow \mathrm{id} & & \downarrow \Theta_* \\
\mathrm{H}^{k}_{\mathrm{tw}}(X,\partial X;{}_{\mathrm{tr}}\mathbb{K})\times\mathrm{H}_{l}(X,\partial X) & \xrightarrow{\cap_{\mathbb{Z}}} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X;{}_{\mathrm{tr}}\mathbb{K}).
\end{array}$$

It is now clear that the rectangle to the right commutes. ⊞

Next we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^{\mathrm{tw}}_{n-k}(X,A;V)\times\mathrm{H}^{\mathrm{tw}}_{k}(X,B;V) & & \xrightarrow{(v,w)\mapsto\Theta_*(\mathrm{PD}(v))\cup_{\otimes}\mathrm{PD}(w)} & \mathrm{H}^{n}_{\mathrm{tw}}(X,\partial X;{}_{\mathrm{tr}}\mathbb{K}) & \xrightarrow{\epsilon_X(-\cap_{\mathbb{Z}}[X])} \mathbb{K} \\
\cong\downarrow \mathrm{PD} \quad \downarrow \mathrm{id} & \overset{Q^{V,\Theta}_{X,A,B}}{\nearrow} & & \downarrow -\cap_{\mathbb{Z}}[X] & \downarrow \mathrm{id} \\
\mathrm{H}^{k}_{\mathrm{tw}}(X,B;V)\times\mathrm{H}^{\mathrm{tw}}_{k}(X,B;V) & & \xrightarrow{(\varphi,w)\mapsto(-1)^{(n-k)\cdot k}\cdot\Theta_*(\varphi\cap_{\otimes}w)} & \mathrm{H}^{\mathrm{tw}}_{0}(X;{}_{\mathrm{tr}}\mathbb{K}) & \xrightarrow{\epsilon_X} \mathbb{K}.
\end{array}$$

non-singular by the Twisted Evaluation–Isomorphism Proposition 326.9

We make the following observations:

(1) The top sequence is by Definition 337.11 the intersection pairing $Q^{V,\Theta}_{X,A,B}$.
(2) It follows from Claim 1 that the square to the left commutes.
(3) The square to the right commutes by fiat.
(4) As we mentioned before, by the Twisted Poincaré Duality Theorem 335.1 the vertical map PD is an isomorphism of $\mathbb{K}$-left modules.
(5) It follows from the Twisted Evaluation–Isomorphism Proposition 326.9 together with the Twisted–Cap Product-Kronecker Lemma 334.33 and our hypothesis on Θ that the $\mathbb{K}$-sesquilinear pairing at the bottom is non-singular.

All these statements combined show that the $\mathbb{K}$-sesquilinear intersection pairing $Q^{V,\Theta}_{X,A,B}$ is indeed non-singular. ∎

337.6. **Twisted invariants and finite coverings.** Let X be a closed oriented path-connected n-dimensional topological manifold and let $x_0 \in X$ and let $\varphi\colon \pi_1(X,x_0) \to G$ be an epimorphism onto a *finite* group G. Furthermore let $S \subseteq \mathbb{C}$ be a subring that is closed under the complex involution. It follows from the discussion on page 6396 that we can consider the following two pairings:

$$\begin{array}{rccc} \langle -,-\rangle\colon & {}_{\varphi}S[G] \times {}_{\varphi}S[G] & \to & S[G] \\ & (r,s) & \mapsto & r\cdot\overline{s} \end{array} \quad \text{and} \quad \begin{array}{rccc} \chi\colon & {}_{\varphi}S[G] \times {}_{\varphi}S[G] & \to & S \\ & (r,s) & \mapsto & e\text{-coefficient of } r\cdot\overline{s}. \end{array}$$

The first pairing is $S[G]$-sesquilinear and invariant under the $\pi_1(X,x_0)$-action. The later pairing is S-bilinear and invariant under the $\pi_1(X,x_0)$-action. In the following two subsections we will see that the corresponding twisted intersection pairings $Q_X^{S[G],\langle -,-\rangle}$ and $Q_X^{S[G],\chi}$ are determined by the intersection pairing of the covering of X corresponding to $\ker(\varphi)$.

337.6.1. **The $S[G]$-valued intersection pairing.** In this subsection we will discuss of the $S[G]$-valued intersection pairing. It is now convenient to recall the following definition:

Definition 337.15. Let X be a compact oriented path-connected n-dimensional topological manifold, let $x_0 \in X$ and let $\varphi\colon \pi_1(X,x_0) \to G$ be an epimorphism onto a *finite* group G. Furthermore let $S \subseteq \mathbb{C}$ be a subring that is closed under the complex involution. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. In the following let $p\colon (\widehat{X}_\varphi, y_0) \to (X,x_0)$ be any covering corresponding to the subgroup $\ker(\varphi)$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (\widehat{X}_\varphi, y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow{\quad r \quad} & (\widehat{X}_\varphi, x_0). \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

Finally let $A \subseteq X$. We set $\widetilde{A} := p^{-1}(A)$ and $\widehat{A} := q^{-1}(A)$. We consider the isomorphism

$$\begin{array}{rccc} \Omega\colon \mathrm{C}_k^{\mathrm{tw}}(X,A;S[G]) = \overline{{}_{\varphi}S[G]} \otimes_{S[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0},\widetilde{A}) & \to & \mathrm{C}_k(\widehat{X}_\varphi,\widehat{A}) \\ g \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & r \circ (g\cdot\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) \end{array}$$

of $S[G]$-left modules from the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5. These maps are chain maps, thus they induce isomorphisms on homology which are also called Ω.

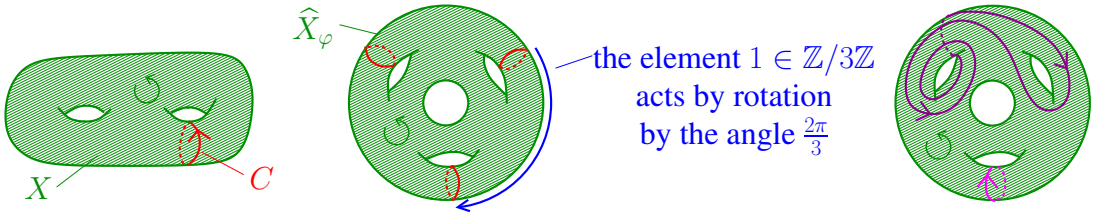
Proposition 337.16. (DI–$S[G]$-Intersection Pairing–Finite Covering-Proposition) Let X be a compact oriented path-connected n-dimensional topological manifold, let $x_0 \in X$ and let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a *finite* group G. Furthermore let $S \subseteq \mathbb{C}$ be a subring that is closed under the complex involution. In the following let $q\colon (\widehat{X}_\varphi, y_0) \to (X, x_0)$ be any covering corresponding to the subgroup $\ker(\varphi)$. Finally suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. We write $\widehat{A} := q^{-1}(A)$ and $\widehat{B} := q^{-1}(B)$. Given any $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X, A; S[G])$ and given any $\nu \in \mathrm{H}_{n-k}^{\mathrm{tw}}(X, B; S[G])$ we have the following equality in $S[G]$:[3671]

$$Q_{X,A,B}^{S[G],\langle -,-\rangle}(\mu, \nu) \;=\; \sum_{g \in G} Q^S_{\widehat{X}_\varphi, \widehat{A}, \widehat{B}}\big(\underbrace{\Omega(\mu)}_{\in \mathrm{H}_k^{\mathrm{tw}}(\widehat{X}_\varphi, \widehat{B})} , g \cdot \underbrace{\Omega(\nu)}_{\in \mathrm{H}_{n-k}^{\mathrm{tw}}(\widehat{X}_\varphi, \widehat{A}, \widehat{B})} \big) \cdot g.$$

Example. Let X be the surface of genus 2 and let $C \subseteq X$ be a non-separating closed orientable path-connected 1-dimensional submanifold. The homology class $[C] \in \mathrm{H}_1(X; \mathbb{Z})$ determines, via the isomorphisms

$$\mathrm{H}_1(X; \mathbb{Z}) \xrightarrow[\cong]{\mathrm{PD}} \mathrm{H}^1(X; \mathbb{Z}) \;\to\; \mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \;=\; \mathrm{Hom}(\pi_1(X), \mathbb{Z}),$$

an epimorphism $\varphi\colon \pi_1(X) \to \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$. In the Cyclic-Covering Proposition 234.21 we saw how one can explicitly construct the covering $\widehat{X}_\varphi$ corresponding to $\ker(\varphi)$.



Thus the Twisted Intersection Pairing–Finite Covering-Proposition 337.16 allows us to compute the twisted intersection pairing $Q_X^{\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}],\langle -,-\rangle}$ by considering the usual intersection pairing $Q_{\widehat{X}_\varphi}$ on the explicit covering $\widehat{X}_\varphi$. □

Proof. In the following we use the Replace Universal Covering Proposition 320.16 to replace the universal covering of X by the covering $q\colon \widehat{X}_\varphi \to X$. More precisely we will use the isomorphism

$$\Omega\colon \overline{{}_\varphi S[G]} \otimes_{S[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}) \;\to\; \overline{{}_\varphi S[G]} \otimes_{S[G]} \mathrm{C}_k(\widehat{X}_\varphi)$$

to identify these two chain complexes. With this identification we obtain the isomorphism

$$\begin{array}{rcl} \Omega\colon \overline{{}_\varphi S[G]} \otimes_{S[G]} \mathrm{C}_k(\widehat{X}_\varphi) & \to & \mathrm{C}_k(\widehat{X}_\varphi) \\ g \otimes (\sigma\colon \Delta^k \to \widehat{X}_\varphi) & \mapsto & (g \cdot \sigma\colon \Delta^k \to \widehat{X}_\varphi) \end{array}$$

[3671] It is a good reality check, see Exercise 337.5 to verify that the map on the right-hand side is indeed $S[G]$-sesquilinear.

of $S[G]$-left modules from the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 323.5. Furthermore, since G is finite we obtain from the Finite Covering-$\mathrm{H}^*_{\mathrm{tw}}$-Proposition 323.13 that the maps

$$\begin{aligned}\Phi\colon \overbrace{\mathrm{C}^k_{\mathrm{tw}}(X;S[G])}^{=_{\mathrm{i}}\mathrm{Hom}_{S[G]\text{-left}}\,\mathrm{C}_k(\widehat{X}_\varphi,S[G])} &\to \mathrm{C}^k(\widehat{X}_\varphi;S)\\ \varphi &\mapsto (\sigma\mapsto e_G\text{-coefficient of }\varphi(\sigma))\end{aligned}$$

and

$$\begin{aligned}\Psi\colon \mathrm{C}^k(\widehat{X}_\varphi;S) &\to \overbrace{\mathrm{C}^k_{\mathrm{tw}}(X;S[G])}^{=_{\mathrm{i}}S[G]\otimes_{S[G]}\mathrm{C}_k(\widehat{X}_\varphi)}\\ \gamma &\mapsto \sum_{g\in G} g^*(\gamma)\cdot g\end{aligned}$$

are well-defined $S[G]$-left homomorphisms and inverses of one another. (Similarly we also have relative statements.) Note that Ω, Φ and Ψ are (co-) chain maps so they induce eponymous maps on (co-) homology.

Claim 1. Given any $\mu \in \mathrm{H}^{\mathrm{tw}}_k(X;S[G])$ we have an equality

$$\mathrm{PD}^{S[G]}_X(\mu) \;=\; \Psi_*(\mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\mu)))$$

in $\mathrm{H}^{n-k}_{\mathrm{tw}}(X,\partial X;S[G])$.

Proof. First note that it follows from the Finite Covering–Cap Product Lemma 334.40 that the following diagram commutes:

$$\begin{array}{ccc}\mathrm{H}^k_{\mathrm{tw}}(X,\partial X;{}_\varphi S[G])\times \mathrm{H}_l(X,\partial X) & \xrightarrow{\quad\cap_S\quad} & \mathrm{H}^{\mathrm{tw}}_{l-k}(X;{}_\varphi S[G])\\ \cong\downarrow \Phi_* \qquad \downarrow p^* & & \cong\downarrow \Omega_*\\ \mathrm{H}^k(\widehat{X}_\varphi,\partial\widehat{X}_\varphi)\times \mathrm{H}^{\mathrm{lf}}_l(\widehat{X}_\varphi,\partial\widehat{Y}_\varphi) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{l-k}(\widehat{X}_\varphi),\end{array}$$

where $p^*\colon \mathrm{H}_k(X,\partial X)\to \mathrm{H}_k(\widehat{X}_\varphi,\partial\widehat{X}_\varphi)$ is the transfer map from Definition 165.3. Now we see that we have following equalities in $\mathrm{H}_k(\widehat{X}_\varphi,\partial\widehat{X}_\varphi)$:

$$\begin{array}{lcll}\Phi_*(\mathrm{PD}^{S[G]}_X(\mu))\cap[\widehat{X}_\varphi] & & \text{follows from the Fundamental Class-Covering Proposition 194.11} & \text{follows from the above commutative diagram}\\ & \downarrow & & \downarrow\\ & = & \Phi_*(\mathrm{PD}^{S[G]}_X(\mu))\cap p^*([X]) & = \quad \Omega_*(\mathrm{PD}^{S[G]}_X(\mu)\cap_S[X])\\ & = & \Omega_*(\mu) & = \quad \mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\mu))\cap[\widehat{X}_\varphi].\\ & \uparrow & & \uparrow\\ & & \text{by Definition 335.2 of } \mathrm{PD}^{S[G]}_X & \text{by Definition 335.2 of } \mathrm{PD}_{\widehat{X}_\varphi}\end{array}$$

By the Poincaré Duality Theorem 210.1 we know that capping with $[\widehat{X}_\varphi]$ is an isomorphism. Thus we see that $\Phi_*(\mathrm{PD}^{S[G]}_X(\mu)) = \mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\mu)) \in \mathrm{H}^{n-k}(\widehat{X}_\varphi,\partial\widehat{X}_\varphi;S)$. Since Ψ is the inverse of Φ we have thus obtained the desired result. ⊞

Claim 2. Given any $\widetilde{\mu}\in \mathrm{H}^r(\widehat{X}_\varphi;S)$ and $\widetilde{\nu}\in \mathrm{H}^s(\widehat{X}_\varphi;S)$ we have the following equality in $\mathrm{H}^{r+s}_{\mathrm{tw}}(X;{}_{\mathrm{tr}}S[G]) =_{\mathrm{i}} \mathrm{H}^{r+s}(X;S[G])$:

$$\Psi(\widetilde{\mu})\;\cup_{S[G]}\;\Psi(\widetilde{\nu}) \;=\; q^*\Big(\sum_{g\in G}\underbrace{\widetilde{\mu}\cup g\cdot\widetilde{\nu}}_{\in \mathrm{H}^{r+s}(Y_\varphi);S)}\Big)\cdot g\Big)$$

Proof. This claim is just a special case of the more general Finite Covering–Cup Product Lemma 334.41. ⊞

We can now finally turn to the actual the proof of the proposition. Thus let $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X; S[G])$ and $\nu \in \mathrm{H}_{n-k}^{\mathrm{tw}}(X; S[G])$ we perform the following calculation in $S[G]$:

$$
\begin{aligned}
Q_X^{S[G],\langle -,-\rangle}(\mu,\nu) &\overset{\text{by Definition 337.11 of } Q_X^{S[G],\langle -,-\rangle}}{=} \epsilon_X\big(\big(\mathrm{PD}_X^{S[G]}(\mu) \cup_{S[G]} \mathrm{PD}_X^{S[G]}(\nu)\big) \cap_S [X]\big) \\
&= \epsilon_X\big(\big(\underbrace{\Psi(\mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\mu)))}_{=:\widetilde{\mu}} \cup_{S[G]} \underbrace{\Psi(\mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\nu)))}_{=:\widetilde{\nu}}\big) \cap [X]\big) \\
&\overset{\text{by Claim 1}}{=} \epsilon_X((\Psi(\widetilde{\mu}) \cup_{S[G]} \Psi(\widetilde{\nu})) \cap [X]) \\
&= \epsilon_{\widehat{X}_\varphi}\Big(\sum_{g\in G}((\widetilde{\mu} \cup g\cdot\widetilde{\nu}) \cap \underbrace{q^*([X])}_{=[\widehat{X}_\varphi]}\Big) \cdot g \\
&\overset{\substack{\text{follows easily from Claim 2 and the} \\ \text{definitions of } \epsilon_X \text{ and } \epsilon_{\widehat{X}_\varphi}}}{=} \sum_{g\in G} \epsilon_{\widehat{X}}\big(\mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\mu)) \cup g\cdot \mathrm{PD}^S_{\widehat{X}_\varphi}(\Omega(\nu))) \cap_S [X]\big) \cdot g \\
&\overset{\substack{\text{by definition of } \widetilde{\mu} \text{ and } \widetilde{\nu}\text{, we also use the} \\ \text{Fundamental Class-Covering Proposition 194.11} \\ \text{which tells us that } q^*([X]) = [\widehat{X}_\varphi]}}{=} \sum_{g\in G} Q_{\widehat{X}_\varphi}(\Omega(\mu), g\cdot\Omega(\nu)) \cdot g. \\
&\overset{\text{by Definition 337.3 of } Q_{\widehat{X}_\varphi}}{=}
\end{aligned}
$$

■

337.6.2. **The S-valued intersection pairing.** We continue with the S-valued intersection pairing. The following proposition shows that intersection pairings of finite coverings can be viewed as special cases of twisted intersection pairings:

Proposition 337.17. (Twisted S-Intersection Pairing–Finite Covering-Proposition) Let X be a compact oriented path-connected n-dimensional topological manifold, let $x_0 \in X$ and let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a *finite* group G. Furthermore suppose that we are given compact $(n-1)$-dimensional submanifolds A and B of ∂X such that $A \cap B = \partial A = \partial B$. We consider the twisted intersection pairing

$$Q^{S[G],\chi}_{X,A,B}\colon \mathrm{H}_k^{\mathrm{tw}}(X, A; S[G]) \times \mathrm{H}_{n-k}^{\mathrm{tw}}(X, B; S[G]) \;\to\; S.$$

In the following we denote by $p\colon \widehat{X}_\varphi \to X$ any covering corresponding to the subgroup $\ker(\varphi)$. We write $\widehat{A} := q^{-1}(A)$ and $\widehat{B} := q^{-1}(B)$. Let $\Omega\colon \mathrm{H}_k^{\mathrm{tw}}(X; S[G]) \to \mathrm{H}_k(\widehat{X}_\varphi)$ be the isomorphism from the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5. Furthermore let $S \subseteq \mathbb{C}$ be a subring that is closed under the complex involution. Let $k \in \mathbb{N}_0$. Given any $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X; S[G])$ and given any $\nu \in \mathrm{H}_{n-k}^{\mathrm{tw}}(X; S[G])$ we have the following equality in S:

$$Q^{S[G],\chi}_{X,A,B}(\mu,\nu) \;=\; Q^S_{\widehat{X}_\varphi,\widehat{A},\widehat{B}}\big(\underbrace{\Omega(\mu)}_{\in \mathrm{H}_k^{\mathrm{tw}}(\widehat{X}_\varphi,\widehat{A})}, \underbrace{\Omega(\nu)}_{\in \mathrm{H}_{n-k}^{\mathrm{tw}}(\widehat{X}_\varphi,\widehat{B})}\big).$$

Proof. We consider the rather dull map

$$\begin{aligned} \zeta\colon S[G] &\to S \\ f &\mapsto \text{the } e\text{-coefficient of } f. \end{aligned}$$

Given any $\mu \in \mathrm{H}_k^{\mathrm{tw}}(X, A; S[G])$ and given any $\nu \in \mathrm{H}_{n-k}^{\mathrm{tw}}(X, B; S[G])$ we now see that we have the following equalities in S:

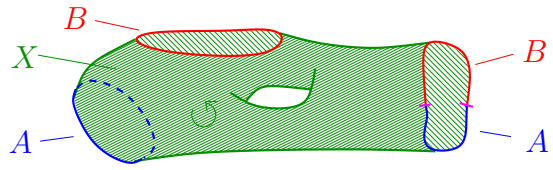
$$Q^{S[G],\chi}_{X,A,B}(\mu,\nu) \;\underset{\text{see page 6401}}{\overset{\uparrow}{=}}\; \zeta\big(Q_X^{S[G],\langle -,-\rangle}(\mu,\nu)\big) \;\underset{\substack{\text{by the DI–}S[G]\text{-Intersection Pairing}\\ \text{Finite Covering-Proposition 337.16}}}{\overset{\uparrow}{=}}\; \zeta\Big(\sum_{g\in G} Q^S_{\widehat{X}_\varphi,\widehat{A},\widehat{B}}(\Omega(\mu), g\cdot\Omega(\nu))\cdot g\Big)$$
$$= Q^S_{\widehat{X}_\varphi,\widehat{A},\widehat{B}}(\Omega(\mu),\Omega(\nu)). \qquad \blacksquare$$

337.7. Exercises for Chapter 337.

Exercise 337.1. We consider the oriented 2-dimensional smooth manifold X together with the decomposition $\partial X = A \cup B$ that is shown below. By the Intersection Form–Non Singularity Proposition 337.7 we know that that the intersection form

$$Q^{\mathbb{Q}}_{X,A,B}\colon \mathrm{H}_1(X, A; \mathbb{Q}) \times \mathrm{H}_1(X, B; \mathbb{Q}) \;\to\; \mathbb{Q}$$

is non-singular. Find explicit bases for $\mathrm{H}_1(X, A; \mathbb{Q})$ and $\mathrm{H}_1(X, B; \mathbb{Q})$, compute the corresponding intersection matrix and verify that this matrix is indeed invertible.



the intersection pairing
$Q^{\mathbb{Q}}_{X,A,B}\colon \mathrm{H}_1(X, A; \mathbb{Q}) \times \mathrm{H}_1(X, B; \mathbb{Q}) \to \mathbb{Q}$
is non-singular

Exercise 337.2. We consider $\mathbb{C}$ with the trivial involution.

(a) Can you classify $\mathbb{C}$-bilinear symmetric pairings over finite-dimensional $\mathbb{C}$-vector spaces?
(b) Can you classify $\mathbb{C}$-bilinear anti-symmetric pairings over finite-dimensional $\mathbb{C}$-vector spaces?

Exercise 337.3. Let $\alpha\colon \pi \to \mathrm{SL}(2,\mathbb{C})$ be a representation. On page 6396 we saw that the pairing

$$\begin{aligned}\det\colon \mathbb{C}^{1\times 2} \times \mathbb{C}^{1\times 2} &\to \mathbb{C}\\ (v, w) &\mapsto \det(v\ w)\end{aligned}$$

is $\mathbb{C}$-bilinear, anti-symmetric and we see that it is preserved by the π-action. Is this pairing non-singular?

Exercise 337.4. We return to the setting of the Twisted Intersection Pairing Proposition 337.12. Thus let X be a compact oriented path-connected $4n$-dimensional topological manifold and let $x_0 \in X$. Let R be a ring with involution and let M be an $(R, \mathbb{Z}[\pi_1(X, x_0)])$-left left module. Finally let Z be an abelian group and let $\Theta\colon M \otimes_{\mathbb{Z}} M \to {}_{\mathrm{tr}}Z$ be a $\mathbb{Z}[\pi]$-left module homomorphism such that the corresponding pairing is hermitian. By the Twisted Intersection Pairing Proposition 337.12 we know that for $A = \varnothing$ the twisted intersection pairing

$$Q^{M,\Theta}_{X,A,A}\colon \mathrm{H}^{\mathrm{tw}}_{2n}(X, A; M) \;\times\; \mathrm{H}^{\mathrm{tw}}_{2n}(X, A; M) \to Z$$

is hermitian. Are there cases where for $A \neq \varnothing$ we also get a hermitian pairing?

Exercise 337.5. Let G be a finite group, let M be a $\mathbb{Z}[G]$-left module and let $\Theta\colon M \times M \to \mathbb{Z}$ be a pairing that is preserved by the G-action. We assume that for every $r, s \in M$ there

exist only finitely many $g \in G$ with $\Theta(r, g \cdot s) \neq 0$. (This hypothesis is of course satisfied if G is finite.) This hypothesis allows us to consider the map

$$\begin{array}{rcl} M \times M & \to & \mathbb{Z}[G] \\ (r,s) & \mapsto & \sum\limits_{g \in G} \Theta(r, g \cdot s) \cdot g \ \in \ \mathbb{Z}[G]. \end{array}$$

Show that this map is $\mathbb{Z}[G]$-sesquilinear.

Exercise 337.6. Let G be a finite group. Furthermore let M be a $\mathbb{Z}[G]$-left module and let $\Theta\colon M \times M \to \mathbb{Z}[G]$ be a $\mathbb{Z}[G]$-sesquilinear pairing. We consider the map

$$\begin{array}{rcl} \mathbb{Z}[G] & \to & \mathbb{Z} \\ p & \mapsto & e_G\text{-coefficient of } p. \end{array}$$

From this we obtain the $\mathbb{Z}$-bilinear pairing $(\chi \circ \Theta)\colon M \times M \to \mathbb{Z}$. Show that for all $v, w \in M$ we have

$$\Theta(v,w) \ = \ \sum_{g \in G} (\chi \circ \Theta)(v, g \cdot w) \cdot g.$$

Exercise 337.7. Let Σ be the surface of genus 3 and let $\varphi\colon \pi_1(\Sigma) \to \mathrm{GL}(1, \mathbb{C})$ be a non-trivial representation. Find a matrix that represents the twisted intersection form $Q_\Sigma^{\varphi, \langle -, - \rangle}$. *Hint.* What do you know about antisymmetric forms?

twisted intersection pairing via intersection points and paths

CHAPTER 338

The twisted signatures of manifolds

In the previous Chapter 337 we used the Poincaré Duality Theorem 335.1 to introduce twisted intersection pairings. In this chapter we will use the twisted intersection pairings to define twisted signatures of compact oriented $4n$-dimensional topological manifolds. We will state and discuss two major theorems, namely the Atiyah–Singer Theorem 338.10 and the Atiyah–Patodi–Singer Theorem 338.11, which unfortunately we will not be able to prove.

This chapter is organized as follows:

- In Section 338.1 we will recall the definition of the signature of a compact oriented $4n$-dimensional topological manifold.
- In Section 338.2 we will introduce twisted signatures of $4k$-dimensional topological manifolds.
- In Section 338.3 we will discuss the Atiyah-Patodi-Singer invariant of odd-dimensional smooth manifolds.
- In Section 338.4 we will provide exercises for Chapter 338.

In the next Chapter 339 we will use twisted homology to (re-) introduce the Alexander module of oriented links in S^3.

338.1. **The (untwisted) signature of a manifold.** In this section we will recall the definition of the signature of a compact oriented $4k$-dimensional topological manifold.

338.1.1. **Algebraic interlude: The signature of a Hermitian form.** [3672] To define the signature of a manifold we should first recall the definition of the signature in linear algebra. To do so we recall some concepts from Definition 2.29, Definition 224.2 and Definition 292.4:

Definition 338.1. Let $\mathbb{F} \subseteq \mathbb{C}$ be a subring that is closed under complex conjugation. Furthermore let V be an $\mathbb{F}$-vector space.

(1) A pairing $\varphi\colon V \times V \to \mathbb{F}$ is called $\mathbb{F}$-sesquilinear if for all $\lambda \in \mathbb{F}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \overline{\lambda}$.

(2) A pairing $\varphi\colon V \times V \to \mathbb{F}$ is called Hermitian if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

An $\mathbb{F}$-sesquilinear Hermitian pairing $\varphi\colon V \times V \to \mathbb{F}$ over a finite-dimensional $\mathbb{F}$-vector space is called a Hermitian form.

The following definition is a variation on Definition 226.6:

[3672] Most of the material of this subsection is already contained in Section 226.2 and Section 292.1.

Definition 338.2. Let $\mathbb{F} \subseteq \mathbb{C}$ be a subring that is closed under complex conjugation. Furthermore let (V, φ) be a Hermitian form over $\mathbb{F}$.

(1) We say that φ is positive-definite if $\varphi(v,v) \in \mathbb{R}_{>0}$ for all $v \neq 0$.[3673]
(2) We say that φ is negative-definite if $\varphi(v,v) \in \mathbb{R}_{<0}$ for all $v \neq 0$.

Next we recall Notation 226.7:

Notation 338.3. Let $\mathbb{F} \subseteq \mathbb{C}$ be a subring that is closed under complex conjugation. Furthermore let $(V, \varphi\colon V \times V \to \mathbb{F})$ be a Hermitian form over $\mathbb{F}$. Given a vector subspace U of the vector space V we denote by $\varphi|_U$ the Hermitian form that is given by restricting the map $\varphi\colon V \times V \to \mathbb{F}$ to $U \times U \to \mathbb{F}$.

The following definition is modelled on Definition 226.8 and Definition 292.10:

Definition 338.4. Let $\mathbb{F} \subseteq \mathbb{C}$ be a subring that is closed under complex conjugation. Furthermore let (V, φ) be a Hermitian form over $\mathbb{F}$. We define

$$\begin{aligned} b^+(\varphi) &:= \text{maximal dimension of a vector subspace } U \text{ of } V \text{ s.t. } \varphi|_U \text{ is positive-definite,} \\ b^-(\varphi) &:= \text{maximal dimension of a vector subspace } U \text{ of } V \text{ s.t. } \varphi|_U \text{ is negative-definite.} \end{aligned}$$

Furthermore we define the signature of φ as

$$\operatorname{sign}(\varphi) := b^+(\varphi) - b^-(\varphi).$$

Example. Let $\mathbb{F} \subseteq \mathbb{C}$ be a subring that is closed under complex conjugation and let (V, φ) be a Hermitian form over $\mathbb{F}$. In practice the signature can be calculated as follows: We pick a basis $v_1, \ldots, v_n$ for V and consider the corresponding matrix $A = (\varphi(v_i, v_j))_{i,j=1,\ldots,n}$. It then follows from the Signature-Eigenvalues Proposition 292.13 that

$$\operatorname{sign}(V,\varphi) = \begin{array}{l} \text{number of positive eigenvalues of } A \text{ in } \mathbb{R} \\ - \text{ number of negative eigenvalues of } A \text{ in } \mathbb{R}. \end{array}$$ □

338.1.2. **The signature of a topological manifold.** The algebraic tools of the last section now allows us to introduce the following definition:

Definition 338.5. Let $k \geq 1$ and let X be a compact oriented $4k$-dimensional topological manifold. We define the signature of X to be

$$\operatorname{sign}(X) := \operatorname{sign}(\underbrace{\overset{\text{see Definition 337.3}}{\downarrow} Q_X^{\mathbb{R}}\colon \mathrm{H}_{2k}(X;\mathbb{R}) \times \mathrm{H}_{2k}(X;\mathbb{R}) \to \mathbb{R}}_{\substack{\text{by the Intersection Form Proposition 337.5 t} \\ \text{this form is symmetric over } \mathbb{R}, \\ \text{hence it is in particular hermitian}}}).$$

Remark. It follows almost immediately from the Intersection Form Reinterpretation Lemma 227.2 that this definition of the signature of a manifold agrees with the original Definition 226.15. □

We start out by a slightly simplified version of the Manifold–Orientation–Signature Lemma 226.16, which reminds us in particular of the fact that the signature depends on the choice of an orientation:

[3673] Note that it follows from the hypothesis that φ is Hermitian that for any $v \in V$ we have $\varphi(v,v) = \overline{\varphi(v,v)}$, i.e. we have $\varphi(v,v) \in \mathbb{R}$.

Lemma 338.6. (Manifold–Orientation–Signature Lemma) Let $k \in \mathbb{N}$ and let X and Y be two compact oriented connected $4k$-dimensional topological manifolds. If $f\colon X \to Y$ is a homeomorphism, then

$$\operatorname{sign}(X) \;=\; \begin{cases} \operatorname{sign}(Y), & \text{if } f \text{ is orientation-preserving,} \\ -\operatorname{sign}(Y), & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Sketch of proof. The lemma follows easily from the naturality of the cup product and the cap product and from the observation (which is essentially the content of the Fundamental Class–Opposite Orientation Lemma 194.4) that an orientation-reversing homeomorphism flips the sign of the fundamental class. ■

The signature has many rather remarkable properties. In particular (similar to the Euler characteristic) it behaves surprisingly well under various constructions of smooth manifolds. First of all we have the following product formula:

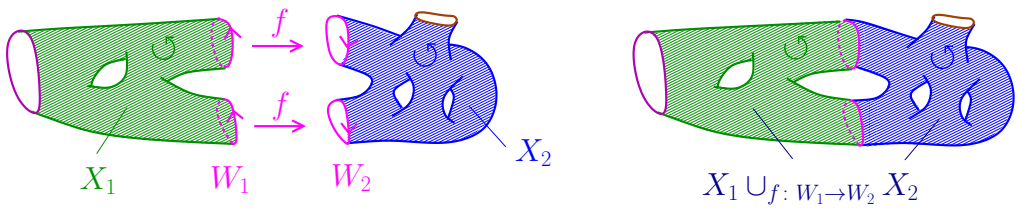
Theorem 227.4. (Signature–Product Theorem) For any two compact oriented topological manifolds X and Y we have

$$\operatorname{sign}(X \times Y) \;=\; \operatorname{sign}(X) \cdot \operatorname{sign}(Y).$$

We continue with the following much more interesting gluing result:

Theorem 227.11. (Novikov Additivity Theorem) Let X_1 and X_2 be two compact oriented $4k$-dimensional topological manifolds, let W_1 be a union of components of ∂X_1 and let W_2 be a union of components of ∂X_2. Finally let $f\colon W_1 \to W_2$ be an orientation-reversing homeomorphism. Then[3674]

$$\operatorname{sign}\Big(\underbrace{X_1 \cup_{f\colon W_1 \to W_2} X_2}_{\substack{\text{by the Gluing-Topological Manifolds-Proposition 54.3 this is} \\ \text{an oriented topological manifold}}}\Big) \;=\; \operatorname{sign}(X_1) + \operatorname{sign}(X_2).$$



We conclude our summary of earlier results with one of the most amazing results in manifold topology:

Theorem 281.16. (Signature–Covering Theorem) Let $n \in \mathbb{N}_0$. Let M be a closed oriented $4n$-dimensional smooth manifold and let $q\colon \widetilde{M} \to M$ be a finite covering. Then

$$\operatorname{sign}(\widetilde{M}) \;=\; [\widetilde{M} : M] \cdot \operatorname{sign}(M).$$

Sketch of proof. By the Hirzebruch Signature Theorem 281.14 we can express the signature of a closed oriented smooth manifold N in terms of the Pontryagin numbers of N from Definition 281.5, which are in turn defined in terms of the Pontryagin classes $p_i(\mathrm{T}N) \in \mathrm{H}^{4i}(N;\mathbb{Z})$ of the tangent bundle $\mathrm{T}N$ from Definition 276.4. Given a covering

[3674] As always we equip ∂X_i with the boundary orientation from the Boundary Orientation Lemma 31.38.

$p\colon \widetilde{M} \to M$ the tangent bundle $\mathrm{T}\widetilde{M}$ is the pullback of $\mathrm{T}M$. We also recall the following facts:

- By Pontryagin-Proposition 276.5 we know that the Pontryagin classes are natural.
- The cup product is natural.
- By the Fundamental Class-Covering Proposition 194.11 we know that we have the equality $q_*([\widetilde{M}]) = [\widetilde{M} : M] \cdot [M] \in \mathrm{H}_{4n}(M)$.

Using these observations one sees quite easily that the Pontryagin numbers are multiplicative under finite coverings. ■

338.2. **Twisted signatures.** We now turn to the twisted analogs of the above untwisted signatures. We start out with the following variation on Definition 338.5:

Definition 338.7. Let $k \geq 1$, let (X, x_0) be a pointed compact connected oriented $4k$-dimensional topological manifold and let $\varphi\colon \pi_1(X, x_0) \to \mathrm{U}(d)$ be a unitary representation. We define the twisted signature of X and φ to be [3675]

$$\mathrm{sign}(X, \varphi) \;:=\; \mathrm{sign}\Big(\underbrace{\overbrace{Q_X^{\mathbb{C}^{1\times n}, \langle -,- \rangle}}^{\text{see Definition 337.11}}\colon \mathrm{H}^{\mathrm{tw}}_{2k}(X; {}_\varphi\mathbb{C}^{1\times n}) \times \mathrm{H}^{\mathrm{tw}}_{2k}(X; {}_\varphi\mathbb{C}^{1\times n}) \to \mathbb{C}}_{\text{by the discussion on page 6400 we know that this pairing is hermitian}}\Big).$$

Example. Let (X, x_0) be a pointed compact connected oriented $4k$-dimensional topological manifold and let $\varphi\colon \pi_1(X, x_0) \to G$ be an epimorphism onto a finite group of order d. We make a few observations and preparations:

- As we saw on page 6402, the homomorphism $\varphi\colon \pi_1(X, x_0) \to G$ defines a unitary representation $\pi_1(X, x_0) \to \mathrm{U}(d)$ with $d = |G|$ which, for convenience, we also denote by φ.
- Let X_φ be any covering corresponding to $\ker(\varphi)$.

We now see that

$$\mathrm{sign}(X, \varphi) \underset{\text{follows from the discussion on page 6402}}{\overset{\uparrow}{=}} \mathrm{sign}\left(Q_X^{\mathbb{C}[G], \langle -,- \rangle}\colon \mathrm{H}^{\mathrm{tw}}_{2k}(X; {}_\varphi\mathbb{C}[G]) \times \mathrm{H}^{\mathrm{tw}}_{2k}(X; {}_\varphi\mathbb{C}[G]) \to \mathbb{C}\right) \underset{\text{by the Twisted } \mathbb{Z}\text{-Intersection Pairing Finite Covering-Proposition 337.17}}{\overset{\uparrow}{=}} \mathrm{sign}(X_\varphi).$$

Thus we have shown that in this case the twisted signature is just the signature of a finite covering. □

The following lemma is the obvious analog of the Manifold–Orientation–Signature Lemma 338.6:

Lemma 338.8. (Manifold–Orientation–Twisted Signature Lemma) Let $k \in \mathbb{N}$ and let X and Y be two compact oriented connected $4k$-dimensional topological manifolds. If we are given a homeomorphism $f\colon (X, x_0) \to (N, y_0)$, then for any unitary representation

[3675] Implicitly use that it follows from the Twisted Invariants–Finite Generation Proposition 331.6 together with the Manifold–CW Complex Theorem 331.7 that $\mathrm{H}^{\mathrm{tw}}_{2k}(X; {}_\varphi\mathbb{C}^{1\times n})$ is a finite-dimensional complex vector space.

$\varphi\colon \pi_1(Y, y_0) \to \mathrm{U}(d)$ we have

$$\mathrm{sign}(X, \varphi \circ f_*) \;=\; \begin{cases} \mathrm{sign}(Y, \varphi), & \text{if } f \text{ is orientation-preserving,} \\ -\,\mathrm{sign}(Y, \varphi), & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

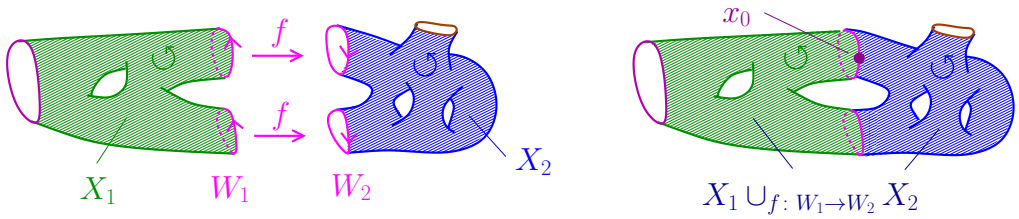
Sketch of proof. This time the lemma follows easily from the naturality of the twisted cup product and of the twisted cap product and again from the observation (which is essentially the content of the Fundamental Class–Opposite Orientation Lemma 194.4) that an orientation-reversing homeomorphism flips the sign of the fundamental class. We leave it to the reader to fill in all the missing details. ■

It turns out that the twisted signature shares many interesting features with the ordinary signature. In particular we have the following generalization of the Novikov Additivity Theorem 227.11:

Theorem 338.9. (Twisted Novikov Additivity Theorem) Let X_1 and X_2 be two compact oriented connected $4k$-dimensional topological manifolds, let W_1 be a union of components of ∂X_1 and let W_2 be a non-empty union of components of ∂X_2. Finally let $f\colon W_1 \to W_2$ be an orientation-reversing homeomorphism. We consider $X := X_1 \cup_f X_2$ is connected. Let $x_0 \in X_1 \cap X_2$ and let $\varphi\colon \pi_1(X, x_0) \to \mathrm{U}(d)$ be a unitary representation. Then

$$\mathrm{sign}(X, \varphi) \;=\; \mathrm{sign}(X_1, \varphi_1) + \mathrm{sign}(X_2, \varphi_2)$$

where for $i = 1, 2$ we denote by φ_i the obvious representation that is given by the composition of the inclusion induced homomorphism $\pi_1(X_i, x_0) \to \pi_1(X, x_0)$ with the original representation $\varphi\colon \pi_1(X, x_0) \to \mathrm{U}(d)$.



Proof. We leave the heroic task of generalizing the proof of the Novikov Additivity Theorem 227.11 to the twisted setting to the reader. ■

Next recall that in the Signature–Covering Theorem 281.16 we showed that the signature of closed oriented smooth manifolds is multiplicative under finite coverings. By the above example one can interpret this result as saying that the twisted signature for suitable d-dimensional unitary representations is just d times the ordinary signature. It turns out that this statement holds, amazingly, for all d-dimensional unitary representations. More precisely, the following theorem holds:

Theorem 338.10. (Atiyah–Singer Theorem) Let $n \in \mathbb{N}_0$. Let X be a *closed* oriented $4k$-dimensional smooth manifold and let $\varphi\colon \pi_1(X) \to \mathrm{U}(d)$ be a representation. Then

$$\mathrm{sign}(X, \varphi) \;=\; d \cdot \mathrm{sign}(X).$$

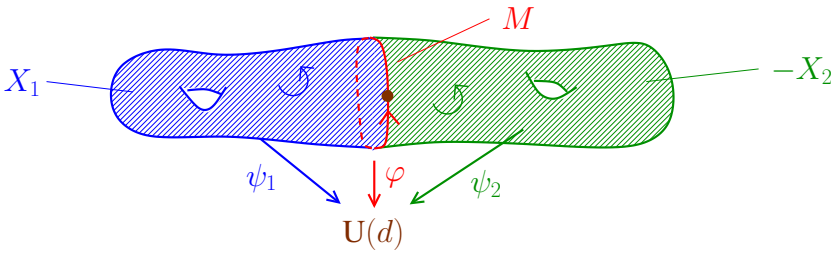
Proof. This theorem was proved by Michael Atiyah and Isadore Singer [**AtS1963**, p. 427] and [**AtS1968**, Chapter 6] as a consequence of the more general Atiyah–Singer Index Theorem. ■

338.3. **The Atiyah-Patodi-Singer invariant.** The Atiyah–Singer Theorem 338.10 might initially appear as a disappointment since it tells us that twisted signatures of closed smooth manifolds do not contain any information that was not already available from the untwisted invariants. But it turns out that the combination of the Atiyah–Singer Theorem 338.10 and the Twisted Novikov Additivity Theorem 338.9 leads to an interesting invariant for odd-dimensional smooth manifolds.

Thus let M be a closed oriented connected smooth manifold M of dimension $4k-1$ and let $\varphi\colon \pi_1(M,x_0) \to \mathrm{U}(d)$ be a unitary representation. Suppose that for $i=1,2$ we have a compact oriented smooth manifold X_i of dimension $4k$ together with a representation $\psi\colon \pi_1(X_i,x_0) \to \mathrm{U}(d)$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_1(M,x_0) & \xrightarrow{\text{induced by the inclusion}} & \pi_1(X_i,x_0) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi_i} & \\ & \mathrm{U}(d). & \end{array}$$

Note that it follows from the Manifold–Seifert-van Kampen Theorem 102.1 that there exists a unique representation $\psi\colon \pi_1(X_1 \cup_M \cup(-X_2),x_0) \to \mathrm{U}(d)$ that restricts to ψ_i on $\pi_1(X_i,x_0)$.



From the Twisted Novikov Additivity Theorem 338.9 together with the Manifold–Orientation–Twisted Signature Lemma 338.8 we obtain the following two equalities:

$$\begin{array}{lrcl} \text{(I)} & \mathrm{sign}(X_1 \cup_M \cup(-X_2),\psi) & = & \mathrm{sign}(X_1,\psi_1) - \mathrm{sign}(X_2,\psi_2), \\ \text{(II)} & \mathrm{sign}(X_1 \cup_M \cup(-X_2)) & = & \mathrm{sign}(X_1) - \mathrm{sign}(X_2). \end{array}$$

Since $X_1 \cup_M (-X_2)$ is a closed oriented smooth manifold we obtain from the Atiyah–Singer Theorem 338.10 that $\mathrm{sign}(X_1 \cup_M \cup(-X_2),\psi) = d \cdot \mathrm{sign}(X_1 \cup_M \cup(-X_2))$. In other words, we see that (I) $- d \cdot$ (II) $= 0$. This tells us that

$$\mathrm{sign}(X_1,\psi_1) - d \cdot \mathrm{sign}(X_1) \;=\; \mathrm{sign}(X_2,\psi_1) - d \cdot \mathrm{sign}(X_2).$$

In other words, if we can extend $\varphi\colon \pi_1(M,x_0) \to \mathrm{U}(d)$ to a compact oriented smooth manifold X with $\partial X = M$, then the value of $\mathrm{sign}(X,\psi) - d \cdot \mathrm{sign}(X)$ does not depend on the choice of (X,ψ)! In other words, we have essentially found an invariant of closed oriented $(4k-1)$-dimensional smooth manifolds together with a unitary representation.

The following theorem turns this idea into a somewhat more general concept:

Theorem 338.11. (Atiyah–Patodi–Singer Theorem) To each closed oriented connected *odd-dimensional* smooth manifold M together with a unitary representation $\varphi\colon \pi_1(M,x_0) \to$

$\mathrm{U}(d)$ one can associate a real number $\rho(M,\varphi)\in\mathbb{R}$, called the Atiyah-Patodi-Singer ρ-invariant, such that the following statements hold:

(1) We have $\rho(-M,\varphi)=-\rho(M,\varphi)$.

(2) If M is of dimension $2k-1$ and if there exists a compact oriented smooth manifold X of dimension $2k$ with $\partial X=M$ together with a representation $\psi\colon\pi_1(X,x_0)\to\mathrm{U}(d)$ such that the following diagram commutes

$$\begin{array}{ccc}\pi_1(M,x_0) & \xrightarrow{\text{induced by the inclusion}} & \pi_1(X,x_0)\\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathrm{U}(d), & \end{array}$$

then[3676]
$$\rho(M,\varphi) \;=\; \mathrm{sign}(X,\psi)-d\cdot\mathrm{sign}(X).$$

(3) Let $\varphi\colon\pi_1(S^1,1)\to\mathrm{U}(1)$ be the representation that sends the standard generator of $\pi_1(S^1)$ to $\exp(2\pi\,\mathrm{i}t)$ with $t\in[0,1)$. Then
$$\rho(S^1,\varphi) \;=\; \begin{cases}0, & \text{if } t=0,\\ 1-2t, & \text{if } t\in(0,1).\end{cases}$$

(4) Let N be a closed oriented connected smooth manifold of *even* dimension $2n$. Furthermore let $\beta\colon\pi_1(N)\to\mathrm{U}(n)$ be a unitary representation. For the odd-dimensional smooth manifold $N\times M$ and the representation $\beta\otimes\alpha\colon\pi_1(N\times M)\to\mathrm{U}(e\cdot d)$ we have[3677]
$$\rho(N\times M,\beta\otimes\alpha) \;=\; (-1)^{n\cdot k}\cdot\mathrm{sign}(N,\beta)\cdot\rho(M,\alpha).$$

Proof. The existence of such an invariant is given by [**APS1975**, Theorem 2.4]. The calculation for S^1 is given in [**APS1975**, p. 418]. The other properties are implicit in [**APS1975**] (see also [**NeumW1979**, Theorem 1.2]). ■

We conclude this chapter with a sample calculation of the Atiyah-Patodi-Singer ρ-invariant:

Theorem 338.12. (Levine-Tristram–ρ-Theorem) Let $K\subseteq S^3$ be an oriented knot. By the Knot Thickening–Self Linking Lemma 284.11 there exists an orientation-preserving smooth embedding $\tau\colon\overline{B}^2\times S^1\to S^3$ with the following two properties:[3678]

- The smooth embedding $\tau\colon\overline{B}^2\times S^1\to S^3$ restricts to an orientation-preserving diffeomorphism $\{0\}\times S^1\to K$.
- We have $\mathrm{lk}(K,\tau(\{*\}\times K)=0$.

We consider the knot exterior $X_K:=S^3\setminus(\tau(B^2)\times K)$ and we consider the closed smooth manifold[3679]
$$N_K \;:=\; X_K\cup_{X_K\leftarrow S^1\times S^1:\tau}(\overline{B}^2\times S^1).$$

By the Dehn Surgery–Invariants Proposition **??** we know that $\mathrm{H}_1(N_K)=\mathbb{Z}\cdot[\mu_K]$. Next let $\Phi\colon\pi_1(N_K)\to\langle t\rangle$ be the unique epimorphism with $\Phi(\mu_K)=t$. Given any $z\in S^1$ we consider the representation $\rho_z\colon\pi_1(N_K)\to\mathrm{U}(1)$ that is given by the composition of Φ with

[3677] If n is odd, then the (twisted) intersection of (N,β) is anti-hermitian. It is my understanding that in this case the signature is meant to be zero. But the reader should check the literature before citing this result.

[3677] Here we consider the action of $(h,g)\in\pi_1(N)\times\pi_1(M)=_{\mathrm{i}}\pi_1(N\times M)$ on $\mathbb{C}^e\otimes_{\mathbb{C}}\mathbb{C}^d\cong\mathbb{C}^{e\cdot d}$ that is given by $(h,g)\cdot(w\otimes v)=(h\cdot w)\otimes(g\cdot v)$.

$t \mapsto (z)$. Then

$$\rho(N_K, \rho_z) \;=\; \sigma_K(z).$$

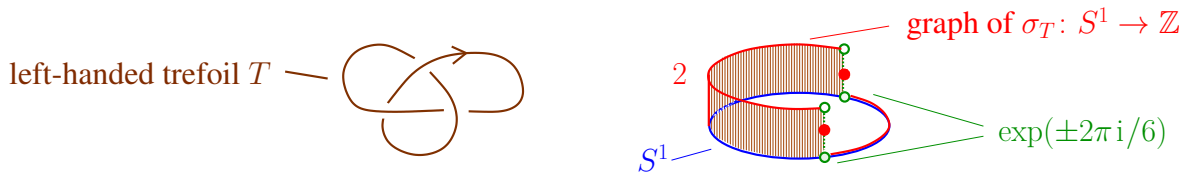
Levine-Tristram signature of K from Definition 292.21

Proof. This theorem follows fairly easily from [**Lit1984**, p. 330]. The idea is to use a Seifert surface F for K to construct, by pushing F into the 4-ball, a suitable 4-dimensional smooth manifold that cobounds N_K and to then apply the Atiyah–Patodi–Singer Theorem 338.11. ■

Example. We consider the left-handed trefoil $T \subseteq S^3$. On page 5606 we showed that for $z = \exp(2\pi\,\mathrm{i}t)$ with $t \in [0,1]$ we have

$$\sigma_T(z) \;=\; \begin{cases} 2, & \text{if } t \in (\frac{1}{6}, \frac{5}{6}), \\ 1, & \text{if } t = \frac{1}{6} \text{ or } t = \frac{5}{6}, \\ 0, & \text{otherwise.} \end{cases}$$

We summarize this calculation in the following figure:



By the Levine-Tristram–ρ-Theorem 338.12 this translate into a calculation of the Atiyah–Patodi–Singer ρ-invariant. □

338.4. Exercises for Chapter 338.

Exercise 338.1. Let M_1 and M_2 be two compact oriented connected non-empty $4k$-dimensional smooth manifolds and let $\varphi_1\colon \pi_1(M_1, x_1) \to \mathrm{U}(d)$ and $\varphi_2\colon \pi_1(M_2, x_2) \to \mathrm{U}(d)$ be two unitary representations. We consider the connected sum $M_1 \# M_2$. Show that for a suitable representation $\varphi\colon \pi_1(M, x_0) \to \mathrm{U}(d)$ we have

$$\mathrm{sign}(M, \varphi) \;=\; \mathrm{sign}(M_1, \varphi_1) + \mathrm{sign}(M_2, \varphi_2).$$

[3678] In the language of **??** this means that τ is a thickening of K of self-linking number zero.
[3679] In **??** this is called the 0-Dehn surgery along K.

CHAPTER 339

The Alexander module of links via twisted homology

This chapter is completely dedicated to a single example of twisted homology, namely the Alexander module (and the resulting Alexander polynomial $\Delta_L(t_1, \dots, t_m)$) of an oriented m-component link in $L \subseteq S^3$. We had encountered these two concepts already in Chapter 286, Chapter 287 and Chapter 288 using a slightly more naive approach to defining the Alexander module.

At first glance we thus do not get any new information about links that was not already accessible earlier. But as we will see in this and the following chapter, our general theory of twisted invariants gives us some nice features:

- Many of the earlier ad hoc proofs on Alexander modules and polynomials now follow more easily from the general theory of twisted homology modules.
- We will be able to show that Alexander modules of knots are always torsion over $\mathbb{Z}[t^{\pm 1}]$. In other words, we will show that the Alexander polynomial $\Delta_K(t)$ of a knot is always non-zero.
- We will see that the Alexander polynomial of a link is always symmetric.
- Given an oriented knot $K \subseteq S^3$ we will introduce the Blanchfield pairing
$$\mathrm{Bl}_K \colon \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \ \to \ \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$
which is a variation on the linking pairing on rational homology spheres.

This chapter is organized as follows:

- In Section 339.1 we will recall the notion of the order of a finitely generated module over a Noetherian UFD.
- In Section 339.2 we will (re-) introduce Alexander modules and polynomials of topological spaces that are equipped with a homomorphism to a free abelian group.
- In Section 339.3 we will (re-) introduce Alexanders module of oriented links.
- In Section 339.4 we will use the Alexander modules and orders to (re-) introduce Alexander polynomials of oriented links.
- In Section 339.5 we will see that at times one can lift the acyclicity of chain complexes to more interesting representations.
- In Section 339.6 we will use the results from Section 339.5 to show that Alexander modules of knots are torsion, in other words, that the Alexander polynomial of any knot $K \subseteq S^3$ is non-zero.
- In Section 339.7 we will introduce the notion of link concordance and we will use Alexander modules to show that some links are not concordant.
- In Section 339.8 we will provide exercises for Chapter 339.

In the next Chapter 340 we will apply Poincaré duality to Alexander modules. This will allow us to show that Alexander polynomials of links are symmetric and this will allow us to introduce the Blanchfield pairing on the Alexander module of a knot.

339.1. **Orders of modules.** We start out with a purely algebraic section, namely we will remind ourselves of the concept of the order of a finitely generated module over a Noetherian UFD. We start out with the following slight variation on Definition 287.6:

Definition 339.1. Let R be a commutative ring and let M be an R-module. A finite presentation for M is a finite exact sequence of homomorphisms of R-modules of the form

$$R^{1\times k} \xrightarrow{v\mapsto v\cdot A} R^{1\times l} \xrightarrow{\psi} M \to 0$$

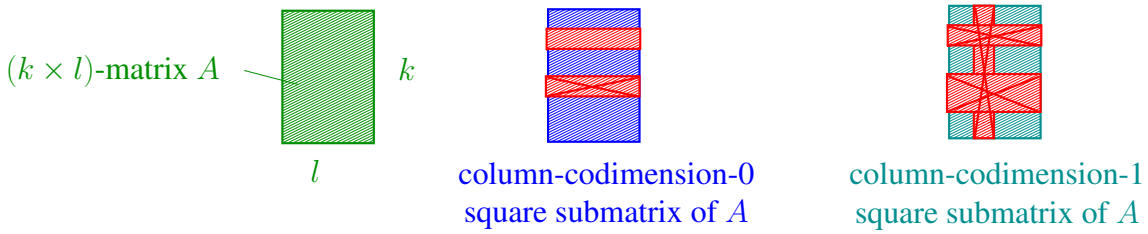
where $A \in \mathrm{M}(k\times l, R)$. Any such matrix A is called a presentation matrix for M.[3680]

The following definition is a variation on Definition 287.16:

Definition 339.2. Let R be a ring and let $A \in \mathrm{M}(k\times l, R)$ be a matrix.

(1) Let $i \in \{0,\dots,l\}$ with $i+k-l \geq 0$. We refer to any $(l-i)\times(l-i)$-matrix that is obtained from A by crossing out i columns and $(i+k-l)$ rows as a column-codimension-i submatrix of A.[3681]

(2) We now suppose that R is a commutative ring. We refer to the ideal generated by the determinants of all square submatrices of column-codimension $\leq i$ of A as the i-th elementary ideal $E_i(A) \subseteq R$ of A.



We also recall Definition 287.18 (1):

Definition 339.3. Let R be a commutative Noetherian ring[3682] and let M be a finitely generated R-module. Since R is Noetherian it follows from the Presentation Matrix–Existence Proposition 287.7 that M admits a presentation matrix A. Let $k \in \mathbb{N}_0$. We refer to

$$E_k(M) \coloneqq E_k(A) \subseteq R$$

as the k-th elementary ideal of the R-module M. It follows from the Module–Elementary Ideal Proposition 287.17 that the ideal $E_k(M)$ is well-defined, i.e. it is independent of the choice of the presentation matrix A.

Before we can recall part (2) of Definition 287.18 we should first recall Notation 287.15:

[3680] In contrast to Definition 287.6 we now work with row vectors and right-multiplication by matrices. Thus a presentation matrix in the sense of the current definition is the transpose of a presentation matrix in the sense of Definition 287.6.

[3681] Since we are now working with transposes of the matrices from Definition 287.16 we see that with respect to the original Definition 287.16 the roles of rows and columns got swapped.

[3682] We refer to Definition 287.2 or Definition 331.2 for the definition of a Noetherian ring.

Notation 339.4. Let R be a ring. Given $p, q \in R$ we write $p \doteq q$ if there exists a unit $\epsilon \in R$ such that $p = \epsilon \cdot q$.

Now we can recall Definition 287.18 (2):

Definition 339.5. Let R be a Noetherian UFD and let M be a finitely generated R-module. We refer to

$$\operatorname{ord}_R(M) \;:=\; \underset{\substack{\uparrow \\ \text{see Definition 116.20}}}{\gcd_R(E_0(M))} \;\in\; R$$

as the order of the R-module M. It follows from the UFD-GCD Lemma 287.14 that the order is well-defined up to "$\doteq$", i.e. up to multiplication by a unit in R.

Example. Let R be a Noetherian UFD and let $M = \{0\}$ be the 0-module. In this case the (0×0)-matrix is a presentation matrix, its determinant is equal to 1. Thus we have shown that $\operatorname{ord}_R(\{0\}) = 1$. □

To formulate our next two lemmas we need to introduce the following two algebraic objects:

Definition 339.6. Let R be a commutative ring. For any R-module M we refer to

$$\operatorname{Tor}_R(M) \;:=\; \{m \in M \mid \text{ there exists an } r \in R \setminus \{0\} \text{ with } r \cdot m = 0\}$$

as the R-torsion submodule of M[3683]. We say M is R-torsion if $\operatorname{Tor}_R(M) = M$.

Definition 339.7. Let R be a commutative domain.

(1) We denote by $Q(R)$ the quotient field of R.
(2) Let M be an R-module. We define its rank as[3684]

$$\operatorname{rank}_R(M) \;:=\; \dim_{Q(R)}\Big(\underbrace{Q(R) \otimes_R M}_{\substack{\text{vector space over} \\ \text{the field } Q(R)}}\Big).$$

We can now formulate our promised lemma:

Lemma 339.8. (Module Torsion Lemma) Let R be a Noetherian UFD and let M be a finitely generated R-module. The following statements are equivalent:

(1) $Q(R) \otimes_R M = 0$, i.e. $\operatorname{rank}_R(M) = 0$.
(2) $\operatorname{ord}_R(M) \neq 0$.
(3) $M = \operatorname{Tor}_R(M)$, i.e. M is R-torsion.

Sketch of proof. Let A be a $(k \times l)$-presentation matrix over R for the R-module M.

Claim. We have a commutative diagram

$$\begin{array}{ccccccc}
R^{1\times k} & \xrightarrow{v \mapsto v\cdot A} & R^{1\times l} & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle m\mapsto 1\otimes m} & & \\
Q(R)^{1\times k} & \xrightarrow{v \mapsto v\cdot A} & Q(R)^{1\times l} & \longrightarrow & Q(R)\otimes_R M & \longrightarrow & 0
\end{array}$$

[3683] Since R is commutative one can easily verify that this is indeed a submodule.
[3684] It follows from the Rank-via-Dimensions Proposition 161.10 that the rank of an abelian group A (as introduced in Definition 97.15) equals the above rank of A viewed as a $\mathbb{Z}$-module.

where the top and bottom sequences are both exact.

Proof. We consider the following expanded diagram:

$$\begin{array}{ccccccc}
R^{1\times k} & \xrightarrow{v\mapsto v\cdot A} & R^{1\times l} & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle r\mapsto 1\otimes r} & & \downarrow{\scriptstyle r\mapsto 1\otimes r} & & \downarrow{\scriptstyle m\mapsto 1\otimes m} & & \\
Q(R)\otimes_R R^{1\times k} & \xrightarrow{\mathrm{id}_{Q(R)}\otimes(v\mapsto v\cdot A)} & Q(R)\otimes_R R^{1\times l} & \longrightarrow & Q(R)\otimes_R M & \longrightarrow & 0 \\
\cong\downarrow{\scriptstyle q\otimes r\mapsto q\cdot r} & & \cong\downarrow{\scriptstyle q\otimes r\mapsto q\cdot r} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
Q(R)^{1\times k} & \xrightarrow{v\mapsto v\cdot A} & Q(R)^{1\times l} & \longrightarrow & Q(R)\otimes_R M & \longrightarrow & 0.
\end{array}$$

We make the following observations:

- The top sequence is exact since A is a presentation matrix for M.
- By the Tensor–Right-Exact Lemma 314.14 we know that taking the tensor product is right exact in the sense of Definition 314.11. It follows that the middle sequence is exact.
- From the Free Module–Tensor-with-Ring Lemma 312.18 we obtain the bottom vertical isomorphisms.
- One can easily verify that the diagram commutes and that the composition of the vertical maps are precisely the maps in the claim. ⊞

We turn to the equivalence of statements of the lemma:
(1)⇒(2). If $Q(R)\otimes_R M = 0$, then it follows from the above exact sequence that A, viewed as a matrix over $Q(R)$ has rank l. This implies by elementary linear algebra that there exists an $(l\times l)$-submatrix A with non-zero determinant $\det(A)$. Note that by Definition 287.16 we have $\det(A)\in E_0(M)$. Thus $\det(A)\mid \mathrm{ord}_R(M)$. This implies that $\mathrm{ord}_R(M)\neq 0$.
(2)⇒(1). If $\mathrm{ord}_R(M)\neq 0$, then it follows easily from the definitions that there exists an $(l\times l)$-submatrix A with non-zero determinant. But this implies that A, viewed as a matrix over $Q(R)$ has rank l. By the exactness of the bottom sequence of the above diagram in the claim this implies that $Q(R)\otimes_R M = 0$.
(3)⇒(1). In Exercise 339.1 we will see that for any R-torsion module M the tensor product $Q(R)\otimes_R M$ is zero.
(1)⇒(3). In Exercise 339.2 we will see that any R-module M with $Q(R)\otimes_R M = 0$ is actually R-torsion. ■

Almost for free the Module Torsion Lemma 339.8 provides us with the following lemma:

Lemma 339.9. (Acyclicity Lemma) Let R be a Noetherian UFD and let C_* be a chain complex of finitely generated free R-modules. For each $i\in\mathbb{N}_0$ the following statements are equivalent:
(0) We have $\mathrm{H}_i(Q(R)\otimes_R C_*) = 0$, i.e. $Q(R)\otimes_R C_*$ is acyclic.
(1) We have $Q(R)\otimes_R \mathrm{H}_i(C_*) = 0$, i.e. $\mathrm{rank}_R(\mathrm{H}_i(C_*)) = 0$.
(2) We have $\mathrm{ord}_R(\mathrm{H}_i(C_*))\neq 0$.
(3) We have $\mathrm{H}_i(C_*) = \mathrm{Tor}_R(\mathrm{H}_i(C_*))$, i.e. $\mathrm{H}_i(C_*)$ is R-torsion.

Proof. Note that the equivalence of (1), (2) and (3) follows immediately from Module Torsion Lemma 339.8. Finally the following claim takes care of the equivalence of (0) and (1):

Claim. The $Q(R)$-modules $\mathrm{H}_*(Q(R)\otimes_R C_*)$ and $Q(R)\otimes_R \mathrm{H}_*(C_*)$ are isomorphic.

Proof. The claim is an immediate consequence of the Tensor-with-Localization Proposition 314.18 (which says that tensoring with localizations is exact) together with the Exact Functor–Homology Lemma 314.13. ■

For the reader's convenience we recall the following multiplicativity statement for orders:

Proposition 287.21. (Order–Exactness Proposition) Let R be a Noetherian UFD.

(1) For any short exact sequence $0 \to K \to L \to M \to 0$ of finitely generated R-modules we have
$$\operatorname{ord}_R(L) \doteq \operatorname{ord}_R(K) \cdot \operatorname{ord}_R(M).$$
(2) For any two finitely generated R-modules M and N we have
$$\operatorname{ord}_R(M \oplus N) \doteq \operatorname{ord}_R(M) \cdot \operatorname{ord}_R(N).$$

We conclude this recapitulation of the order of a module with the following elementary lemma:

Lemma 287.19. (Order–Functorial Lemma) Let R be a Noetherian UFD and let M be a finitely generated R-module.

(1) If N is an R-module that is isomorphic to M, then $\operatorname{ord}_R(M) \doteq \operatorname{ord}_R(N)$.
(2) Let $\varphi\colon S \to R$ be an isomorphism of rings. If we use φ to view M as an S-module, then
$$\operatorname{ord}_R(R\text{-module } M) \doteq \varphi(\operatorname{ord}_S(S\text{-module } M)).$$

339.2. **Alexander modules and Alexander polynomial.** We now return to topology, more precisely we will (re-) introduce Alexander modules and Alexander polynomials. To economize on the ink we will adopt the following notation:

Notation 339.10. Let π be a group and let H be an abelian group.

- Given a homomorphism $\Phi\colon \pi \to H$ we also denote by Φ the corresponding homomorphism $\pi \to \mathrm{GL}(1, \mathbb{Z}[H])$.
- As in Convention 320.9 we use $\Phi\colon \pi \to \mathrm{GL}(1, \mathbb{Z}[H])$ to introduce the $(\mathbb{Z}[H], \mathbb{Z}[\pi])$-left left module ${}_{\Phi}\mathbb{Z}[H]$.

With this convention we can formulate the following definition:

Definition 339.11. Let (X, x_0) be a pointed topological space that is connected, path-connected and locally contractible. Furthermore let $A \subseteq X$ be a subspace. Finally let H be a free abelian group and let $\Phi\colon \pi_1(X, x_0) \to H$ be a homomorphism. As in Notation 339.10 we use Φ to introduce the $(\mathbb{Z}[H], \mathbb{Z}[\pi_1(X, x_0)])$-left left module ${}_{\Phi}\mathbb{Z}[H]$. Furthermore we consider the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and we set $\widetilde{A} := p^{-1}(A)$. Given $k \in \mathbb{N}_0$ we refer to the $\mathbb{Z}[H]$-module
$$\mathrm{H}_k^{\mathrm{tw}}(X, A; {}_{\Phi}\mathbb{Z}[H]) := \mathrm{H}_k(\overline{{}_{\Phi}\mathbb{Z}[H]} \otimes_{\mathbb{Z}[\pi_1(X, x_0)]} \mathrm{C}_*(\widetilde{X}_{x_0}, \widetilde{A}))$$
as the k-th Alexander module of (X, Φ).

The vigilant reader will have noticed that we now introduced Alexander modules twice, namely once in Definition 286.9 and once above in Definition 339.11. Fortunately the following proposition shows that the two definitions are essentially the same:

Proposition 339.12. (Alexander Modules-Agree Proposition) Let (X, x_0) be a pointed topological space that is connected, path-connected and locally contractible and

let $A \subseteq X$. Furthermore let H be a free abelian group and let $\Phi\colon \pi_1(X) \to H$ be an epimorphism. Finally let $x_0 \in X$ and let $p\colon \widehat{X}_\Phi \to X$ be any covering of X corresponding to the kernel of the homomorphism $\pi_1(X, x_0) \to \mathrm{H}_1(X) \xrightarrow{\Phi} H$. We write $\widehat{A} := p^{-1}(A)$. We recall the following:

- The group H acts via deck transformations on $\widehat{X}_\Phi$.
- The action of H turns $\mathrm{H}_k(\widehat{X}_\Phi, \widehat{A})$ into a module over the group ring $\mathbb{Z}[H]$.

Given any $k \in \mathbb{N}_0$ have an isomorphism

$$\underbrace{\mathrm{H}_k^{\mathrm{tw}}(X, A; {}_\Phi\mathbb{Z}[H])}_{\substack{\text{Alexander module}\\ \text{from Definition 339.11}}} \xrightarrow{\cong} \underbrace{\mathrm{H}_k(\widehat{X}_\Phi, \widehat{A})}_{\substack{\text{Alexander module}\\ \text{from Definition 286.9}}}$$

of $\mathbb{Z}[H]$-modules.

Proof. This proposition is just a special case of the much more general Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5. ■

The following proposition is a slight variation on the Alexander Module–Finite Presentation Proposition 287.9:

Proposition 339.13. (Alexander Module–Finite Presentation Proposition) Let (X, x_0) be a topological space that is path-connected and locally contractible and let $\Phi\colon \pi_1(X, x_0) \to H$ be a homomorphism to a finitely generated abelian group. If X is homotopy equivalent to a finite CW-complex, then each Alexander module $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])$ is a finitely generated $\mathbb{Z}[H]$-module.

Proof. First note that this proposition follows almost immediately from the Alexander Module–Finite Presentation Proposition 287.9 together with the Alexander Modules-Agree Proposition 339.12.

But now we can also give an alternative "off-the-shelves"-proof. More precisely, this proposition follows immediately from the Twisted Invariants–Finite Generation Proposition 331.6 together with the Laurent Polynomial Ring–Noetherian Proposition 331.4 and the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.1. ■

Similar to Definition 287.22 we can now (re-) introduce Alexander polynomials:

Definition 339.14. Let X be a topological space that is path-connected and locally contractible. We assume that X is homotopy equivalent to a finite CW-complex. Let $\Phi\colon \pi_1(X) \to H$ be a homomorphism to a finitely generated free abelian group. Given any $k \in \mathbb{N}_0$ we consider the Alexander module $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])$ from Definition 339.11. By the Alexander Module–Finite Presentation Proposition 339.13 we know that $\mathrm{H}_k^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])$ is a finitely generated $\mathbb{Z}[H]$-module. Since H is a finitely generated *free* abelian group we know by the Laurent Polynomial Ring–UFD Proposition 287.12 that $\mathbb{Z}[H]$ is a UFD. Thus we can use the order from Definition 339.3 to define the k-th Alexander polynomial of (X, Φ) as

$$\Delta_k^\Phi(X) \;:=\; \mathrm{ord}_{\mathbb{Z}[H]}(\mathrm{H}_k^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])) \;\in\; \mathbb{Z}[H].$$

Note that it follows from Definition 339.3 that $\Delta_k^\Phi(X)$ is well-defined up to "$\doteq$".

As a warm-up we start out with the most basic calculation:

Lemma 339.15. (Zeroth-Alexander Polynomial Lemma) Let X be a topological space that is path-connected and locally contractible. We assume that X is homotopy equivalent to a finite CW-complex. Let $\Phi\colon \mathrm{H}_1(X) \to H$ be an epimorphism onto a free abelian group of rank ≥ 1. Then

$$\Delta_0^\Phi(X) \;=\; \begin{cases} h-1, & \text{if } H = \langle h \rangle \text{ has rank 1,} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The lemma follows easily from the Twisted H_0-Proposition 323.3 and (similar to the proof of the Zeroth-Alexander Polynomial Lemma 287.23) some elementary algebraic observations. We leave the details of the proof to Exercise 339.5. ■

We conclude this section with the following lemma which gives a criterion for Alexander polynomials to be non-zero:

Lemma 339.16. (Alexander Module Torsion Lemma) Let X be a topological space that is path-connected and locally contractible. We assume that X is homotopy equivalent to a finite CW-complex. Let $\Phi\colon \pi_1(X) \to H$ be a homomorphism to a finitely generated free abelian group. We denote by $\mathbb{Q}(H)$ the quotient field of $\mathbb{Z}[H]$. Given $i \in \mathbb{N}_0$ the following statements are equivalent:

(0) $\mathrm{H}_i^{\mathrm{tw}}(X; {}_\Phi\mathbb{Q}(H)) = 0$.
(1) $\mathrm{H}_i^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H]) \otimes_{\mathbb{Z}[H]} \mathbb{Q}(H) = 0$, i.e. $\mathrm{rank}_{\mathbb{Z}[H]}(\mathrm{H}_i^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])) = 0$.
(2) $\mathrm{ord}(\mathrm{H}_i^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])) \neq 0$.
(3) $\mathrm{H}_i^{\mathrm{tw}}(X; {}_\Phi\mathbb{Z}[H])$ is $\mathbb{Z}[H]$-torsion.

Proof. This lemma follows almost immediately from the Acyclicity Lemma 339.9 together with the Replace Universal Covering Proposition 320.16. ■

339.3. **The Alexander modules of oriented links.** Similar to the discussion in Chapter 288 we now want to apply the above general theory of Alexander modules and polynomials to the setting of oriented links. This thought leads us to the following definition, which is mostly copy-pasted from Definition 288.1:

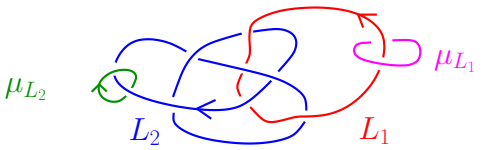
Definition 339.17. Let $L = L_1 \sqcup \cdots \sqcup L_m \subseteq S^3$ be an oriented m-component link in S^3. As usual we denote the meridians by $\mu_{L_1}, \ldots, \mu_{L_m}$.

(1) It follows from the Link Complement-H_*-Proposition 283.4 that there exists a unique isomorphism
$$\Phi_L\colon \mathrm{H}_1(S^3 \setminus L) \;\to\; \langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$$
such that for $i = 1, \ldots, m$ we have $\Phi_L([\mu_{L_i}]) = t_i$.
(2) Let $k \in \mathbb{N}_0$. By Definition 339.11 we can consider the corresponding multivariable Alexander module[3685]
$$\mathrm{H}_k(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]) \;:=\; \mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus L; {}_{\Phi_L}\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]).$$

It follows from the Alexander Modules-Agree Proposition 339.12 that the above definitions are essentially the same as the ones from Definition 288.1.

In the following we will recall the notion of the link exterior. The link exterior has the great advantage that it is a compact smooth manifold. On the other hand, it has the slight

[3685]The only content of this notation is that we drop the Φ_L from the notation, since for an oriented link it is clear which epimorphism to $\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$ we mean.



we have $\mathrm{H}_1(S^3 \setminus L) = \mathbb{Z} \cdot [\mu_{L_1}] \oplus \mathbb{Z} \cdot [\mu_{L_2}]$

disadvantage that it is not 100% well-defined, as it depends on the choice of a tubular map. But, as we will see, the diffeomorphism type of the link exterior is well-defined and for most purposes that is more than enough.

Without further ado let us recall the following definition:

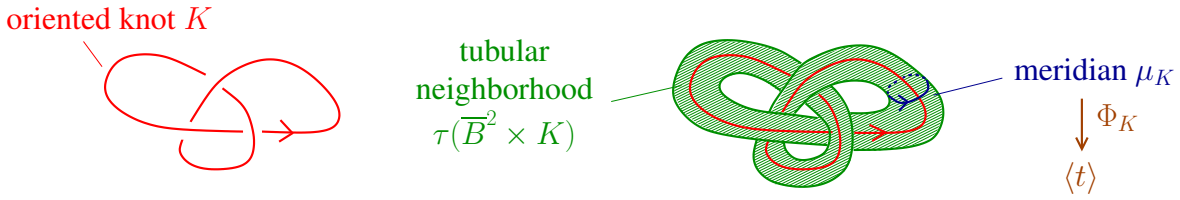
Definition 108.30. Let L be a link in S^3. By the Link Tubular Map Theorem 108.29 (1) there exists a tubular map $\tau\colon \overline{B}^2 \times L \to S^3$. We refer to $X_L := S^3 \setminus \tau(B^2 \times L)$ as the exterior of L.

Next we recall the following lemma:

Lemma 108.31. (Link Exterior Lemma) Let $L \subseteq S^3$ be an m-component link in S^3 and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map.

(1) The exterior $X_L := S^3 \setminus \tau(B^2 \times L)$ of L is a compact smooth submanifold of S^3 with $\partial X_L = \tau(S^1 \times L)$. In particular X_L inherits an orientation from S^3.
(2) The exterior X_L is well-defined up to an orientation-preserving diffeomorphism.
(3) If L and L' are smoothly isotopic links, then there exists an orientation-preserving diffeomorphism between the link exteriors X_L and $X_{L'}$.
(4) The link exterior X_L is a deformation retract of $S^3 \setminus L$, in particular the inclusion $X_L \to S^3 \setminus L$ is a homotopy equivalence and the inclusion induces an isomorphism of homology groups. Thus we obtain an isomorphism

$$\Phi_L\colon \mathrm{H}_1(X_L) \xrightarrow{\cong} \mathrm{H}_1(S^3 \setminus L) \xrightarrow{\Phi_L(\mu_{L_i})=t_i} \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}.$$



The following shows that the link complement and the link exterior lead to isomorphic Alexander modules:

Lemma 339.18. (Link Exterior–Alexander Module Lemma) Let L be an oriented m-component link. Let X_L be an exterior of L as in Definition 108.30. The inclusion $X_L \to S^3 \setminus L$ induces an isomorphism of Alexander modules.

Proof. This statement follows immediately from the Deformation Retract-$\mathrm{H}_*^{\mathrm{tw}}$-Corollary 322.7 together with the Link Exterior Lemma 108.31 (4). ■

The Link Exterior–Alexander Module Lemma 339.18 might look mildly disappointing, but it does play an important role, since we can now tap into the various results which we

proved about twisted homology of compact smooth manifolds. For example we can use link exteriors to prove the following proposition about the Alexander modules in degree 2:

Proposition 339.19. (Higher Alexander Module Proposition) In the following let $L \subseteq S^3$ be a non-empty oriented m-component link. The following statements hold:

(1) We have

$$\operatorname*{rank}_{\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]}(\mathrm{H}_2^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}])) = \operatorname*{rank}_{\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}])).$$

(2) If the rank in (1) is zero, then $\mathrm{H}_2^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]) = 0$.

Proof. We start out with a few preparations:

- We write $\Lambda := \mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ and we denote by $Q(\Lambda) := \mathbb{Q}(t_1,\dots,t_m)$ the quotient field of Λ.
- By the Link Exterior–Alexander Module Lemma 339.18 it suffices to prove the corresponding statements of the proposition for the link exterior instead of the link complement.
- As in the proof of the above Acyclicity Lemma 339.9 we see that for each $i \in \mathbb{N}_0$ we have the equality $\operatorname{rank}_\Lambda(\mathrm{H}_k^{\mathrm{tw}}(X_L;{}_{\Phi_L}\Lambda)) = \dim_{Q(\Lambda)}(\mathrm{H}_k^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda))$.

With these observations it is clear that the proposition follows from the following claim:

Claim.

(1) (a) For $i = 0$ and $i \geq 3$ we have $\mathrm{H}_i^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) = 0$.
(b) We have $\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) \cong \mathrm{H}_1^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda))$.
(2) If $\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) = 0$, then $\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}\Lambda) = 0$.

Proof. We first point out that it follows from $m \in \mathbb{N}$ and the Singular-Handle Twisted Isomorphism Theorem 333.4 that we can equip X_L with a handle decomposition with no 3-handle. We now turn to the proof of the claim:

(1) (a) Note that it follows from the Twisted H_0-Proposition 323.3 (1) and the fact that $m \geq 1$ that $\mathrm{H}_0^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) = 0$. Furthermore note that it follows from the above choice of a handle decomposition and the Singular-Handle Twisted Isomorphism Theorem 333.4 that for every $i \geq 3$ we have $\mathrm{H}_i^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) = 0$.[3686]

(b) We have the following equalities:

$$\begin{array}{ccccc} 0 & = & \chi(X_L) & = & \dim_{Q(\Lambda)}(\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda))) - \dim_{Q(\Lambda)}(\mathrm{H}_1^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda))). \\ & \uparrow & & \uparrow & \\ & \text{by the Link Exterior } \chi\text{-Lemma 156.17} & & \text{follows from the Twisted Homology-}\chi\text{-Proposition 331.12 and Statement (a)} & \end{array}$$

(2) We now suppose that $\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) = 0$. We consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}\Lambda) & \xhookrightarrow{\cong} & \mathrm{H}_2^{\mathrm{tw,han}}(X_L;{}_{\Phi_L}\Lambda) & \hookrightarrow & \mathrm{C}_2^{\mathrm{tw,han}}(X_L;{}_{\Phi_L}\Lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \underbrace{\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda))}_{=0} & \xrightarrow{\cong} & \mathrm{H}_2^{\mathrm{tw,han}}(X_L;{}_{\Phi_L}Q(\Lambda)) & \longrightarrow & \mathrm{C}_2^{\mathrm{tw,han}}(X_L;{}_{\Phi_L}Q(\Lambda)). \end{array}$$

[3686] Alternatively we could appeal to the Twisted Manifold-Top-H_*-Theorem 336.1.

We make the following clarifications and observations:

- The left horizontal maps are the natural isomorphisms from the Singular-Handle Twisted Isomorphism Theorem 333.4.
- Since X_L has no 3-handles we see that the right top horizontal map is a monomoprhism.
- The vertical maps are induced by the inclusion $\Lambda \to Q(\Lambda)$, see also Definition 321.12.
- It follows easily from the Free Module–Tensor-with-Ring Lemma 312.18 that the right hand vertical map is a monomoprhism.

Since $\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}Q(\Lambda)) = 0$ we thus see that $\mathrm{H}_2^{\mathrm{tw}}(X_L;{}_{\Phi_L}\Lambda) = 0$. ■

339.4. **The Alexander polynomials of oriented links.** As the reader surely is well-aware off by now, any discussion of Alexander modules gets followed by a definition of Alexander polynomials:

Definition 339.20. Let $L = L_1 \sqcup \cdots \sqcup L_m \subseteq S^3$ be an oriented m-component link in S^3. We define the multivariable Alexander polynomial of L as[3687] [3688]

$$\Delta_L(t_1,\dots,t_m) \underset{\text{see Definition 339.14}}{:=} \Delta_1^{\Phi_L}(S^3 \setminus L) \underset{\text{see Definition 339.3}}{:=} \mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L;\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]))$$

in $\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. We make the following observations:

- As in Definition 339.3 we see that the Alexander polynomial $\Delta_L(t_1,\dots,t_m)$ is well-defined up to "$\doteq$", i.e. it is well-defined up to a sign and up to multiplication by a monomial in $t_1,\dots,t_m$.
- It follows from the argument in the proof of the Link Isotopy–Alexander Polynomial Proposition 288.3 that $\Delta_L(t_1,\dots,t_m)$ is a smooth isotopy invariant of oriented links in S^3.
- It follows from the Alexander Modules-Agree Proposition 339.12 that the above definition of the Alexander polynomial leads to the same invariant as Definition 288.1.
- It follows from the Link Exterior–Alexander Module Lemma 339.18 that we could also have used the link exterior (instead of the link complement) to define the Alexander polynomial.

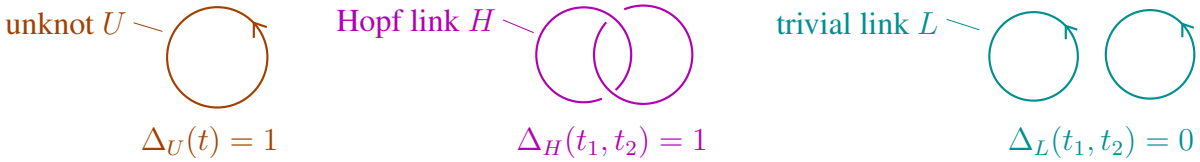
The following lemma is the analog of the Link Alexander Polynomial–Example Lemma 288.2:

[3687] Note that $S^3 \setminus L$ is an open subset of S^3, thus it is path-connected and locally contractible. Furthermore, by the Link Exterior Lemma 108.31 (4) we know that $S^3 \setminus L$ is homotopy equivalent to the link exterior, which in turn is a compact smooth manifold. By the Smooth Manifold-CW-Theorem 179.9 this discussion implies that $S^3 \setminus L$ is homotopy equivalent to a finite CW-complex. Thus it makes sense to study the order of the Alexander module.

[3688] In principle one could also consider $\mathrm{ord}_{\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]}(\mathrm{H}_2(S^3 \setminus L;\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]))$. But this invariant is not very interesting. In fact it follows from the Higher Alexander Module Proposition 339.19 together with the Alexander Module Torsion Lemma 339.16 that (with $\Lambda := \mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$) we have

$$\mathrm{ord}_\Lambda(\mathrm{H}_2^{\mathrm{tw}}(S^3 \setminus L;\Lambda)) = \begin{cases} 0, & \text{if } \Delta_L := \mathrm{ord}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L;\Lambda)) = 0, \\ 1, & \text{if } \Delta_L := \mathrm{ord}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L;\Lambda)) \neq 0. \end{cases}$$

Lemma 339.21. (Link Alexander Polynomial–Example Lemma)
(1) For the unknot U we have $\Delta_U(t) = 1$.
(2) For the Hopf link H we have $\Delta_H(t_1, t_2) = 1$.
(3) If $m \geq 2$, then for the trivial m-component link $L \subseteq S^3$ we have $\Delta_L(t_1, \ldots, t_m) = 0$.



Proof.

(1) Let $U \subseteq S^3$ be the unknot. By the Trivial Knot–Complement Proposition 111.11 we know that $S^3 \setminus U$ is diffeomorphic to the open solid torus $B^2 \times S^1$. We now see that

$$\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus U;{}_{\Phi_U}\mathbb{Z}[t^{\pm 1}]) \underset{\substack{\uparrow \\ \text{by the Homotopy-H}_*^{\mathrm{tw}} \\ \text{Proposition 322.3}}}{\cong} \mathrm{H}_1^{\mathrm{tw}}(S^1;\mathbb{Z}[t^{\pm 1}]) \underset{\substack{\uparrow \\ \text{by the Twisted-H}_*\text{-of-}S^1 \\ \text{Proposition 327.13}}}{=} 0.$$

By the discussion on page 6420 this implies that $\Delta_U(t) = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus U;{}_{\Phi_U}\mathbb{Z}[t^{\pm 1}])) \doteq 1$.

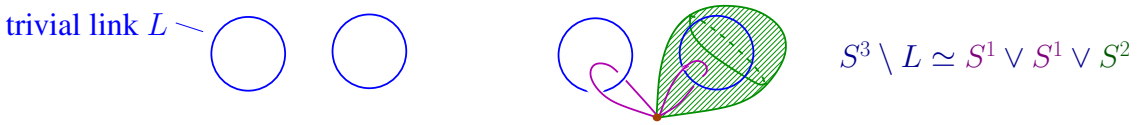
(2) Let $H = H_1 \cup H_2 \subseteq S^3$ be the Hopf link. By the Hopf Link-Complement-Proposition 111.23 we know that $S^3 \setminus H$ is diffeomorphic to $S^1 \times S^1 \times (-1, 1)$, which in turn admits a deformation retraction to $S^1 \times S^1$. Similar to (1) we now see that

$$\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus H;{}_{\Phi_H}\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]) \underset{\substack{\uparrow \\ \text{by the Homotopy-H}_*^{\mathrm{tw}} \\ \text{Proposition 322.3}}}{\cong} \mathrm{H}_1^{\mathrm{tw}}(S^1 \times S^1;\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]) \underset{\substack{\uparrow \\ \text{by the Torus–Twisted Cellular} \\ \text{Chain Complex Lemma 329.11}}}{=} 0.$$

Thus, as in (1) it now follows easily from the definitions that $\Delta_H(t_1, t_2) \doteq 1$.

(3) Let $m \in \mathbb{N}$ and let $L = L_1 \sqcup \cdots \sqcup L_m$ be the trivial m-component link. We start out with the following claim:

Claim. The complement $S^3 \setminus L$ is homotopy equivalent to the wedge of m copies of S^1 and $m - 1$ copies of S^2.



Proof. As in the proof of the Link Alexander Polynomial–Example Lemma 288.2 we outsource the proof of this statement to Exercise 288.2. Alternatively one can also prove the claim using the more general Manifold-with-Boundary-minus-Ball Proposition 247.14. ⊞

We need to show that $\Delta_L(t_1, \ldots, t_m) = 0$. By the Alexander Module Torsion Lemma 339.16 it suffices to show that $\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L;{}_{\Phi_L}\mathbb{Q}(t_1, \ldots, t_m)) \neq 0$. In fact we see that

$$\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L; {}_{\Phi_L}\mathbb{Q}(t_1,\dots,t_m)) \;\underset{\uparrow}{\cong}\; \mathrm{H}_1^{\mathrm{tw}}\Big(\Big(\bigvee_{i=1}^{m} S_i^1\Big) \vee S^2; {}_{\Phi_L}\mathbb{Q}(t_1,\dots,t_m)\Big)$$

by the claim together with the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$−Proposition 322.3

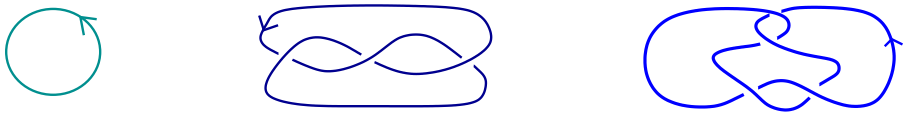
$$\underset{\uparrow}{\cong}\; \mathbb{Q}(t_1,\dots,t_m)^{m-1}.$$

follows quite easily from the Singular-Cellular Twisted Isomorphism Theorem 328.1

■

Example. The above Link Alexander Polynomial–Example Lemma 339.21 contains pretty much all examples of links where one can easily describe the complement. Most other calculations of Alexander polynomials require much more of an effort. But fortunately we already developed some techniques for calculating Alexander polynomials. More precisely, as we pointed out in Definition 339.20 the "current" Alexander polynomial" agrees with the Alexander polynomial from Definition 288.1. In the Alexander Module–Seifert Matrix Theorem 289.12 we saw that one can calculate these Alexander polynomials of knots via Seifert surfaces and Seifert matrices. This way we obtained on page 5530 the following values:

unknot U | trefoil T | figure-8 knot K



$\Delta_U(t) \doteq 1$ | $\Delta_T(t) \doteq 1 - t + t^2$ | $\Delta_K(t) \doteq 1 - 3t + t^2$

It follows from these calculations that the unknot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic. □

We conclude this section with the following analog of the Alexander Polynomial–Orientation Lemmas 117.6 and 288.4:

Lemma 339.22. (Alexander Polynomial–Orientation Lemma) Let L be an oriented m-component link and let $\epsilon_1,\dots,\epsilon_m \in \{-1,1\}$. We denote by $L^{(\epsilon_1,\dots,\epsilon_m)}$ the same link L but where we flipped the orientation of the i-th component precisely if $\epsilon_i = -1$. Then

$$\Delta_{L^{(\epsilon_1,\dots,\epsilon_m)}}(t_1,\dots,t_m) \;\doteq\; \Delta_L(t_1^{\epsilon_1},\dots,t_m^{\epsilon_m}).$$

Sketch of proof. The promised equality follows easily from the Meridian–Symmetries Lemma 109.3, the discussion in Definition 321.12 and the Order–Functorial Lemma 287.19. We leave it to the reader to fill in the details. ■

339.5. **Lifting acyclic representations.** Our next goal is to show that the Alexander polynomial of any knot in S^3 is always non-zero. In this section we will provide the tools for doing so. Later in this chapter we will use the same tools to prove a statement about link concordance.

339.5.1. **Algebra.** We start out with a remark: Let C_* be a finite chain complex of finitely generated free R-left modules and let $\varphi\colon R \to S$ be a ring homomorphism. If C_* is acyclic, then it follows easily from the Null-Homotopic Criterion Proposition 313.16 (see also Exercise 313.2) that $S \otimes_R C_*$ is also acyclic. The following proposition gives a criterion for going the other way round:

Proposition 339.23. (Acyclic Lift Proposition) Let C_* be a finite chain complex of finitely generated free R-left modules and let $\varphi\colon R \to S$ be a ring homomorphism, such that the following condition holds:

(∗) Whenever A is a matrix over R such that $\varphi(A)$ is invertible over S, then A is already invertible over R.

Then the following holds: If $S_\varphi \otimes_R C_*$ is acyclic over S, then C_* is already acyclic over R.

Proof. We start out by recalling the following Acyclicity Criterion:

(⋆) By the Null-Homotopic Criterion Proposition 313.16 we know that a chain complex D_* over a ring T is acyclic if and only if there exist homomorphisms $\delta_i\colon D_i \to D_{i+1}$ of T-left modules such that each $\partial_{i+1}\circ\delta_i + \delta_{i-1}\circ\partial_i\colon D_i \to D_i$ is an isomorphism of T-left modules.

We now turn to the actual proof of the proposition: We suppose that $S \otimes_R C_*$ is acyclic over S. We proceed as follows:

(a) By the Acyclicity Criterion (⋆) we can pick homomorphisms $s_i\colon S \otimes_R C_i \to S \otimes_R C_{i+1}$ of S-left modules such that each $(\mathrm{id}\otimes\partial_{i+1})\circ s_i + s_{i-1}\circ(\mathrm{id}\otimes\partial_i)\colon S \otimes_R C_i \to S \otimes_R C_i$ is an isomorphism of S-left modules.
(b) Since $\varphi\colon R \to S$ is surjective we obtain easily from the Free Module–Tensor-with-Ring Lemma 312.18 that for each i there exists a homomorphism $r_i\colon C_i \to C_{i+1}$ of R-left modules such that $s_i = \mathrm{id}_S \otimes r_i$.
(c) It follows easily from our hypothesis and the Free Module–Tensor-with-Ring Lemma 312.18 that each $\partial_{i+1}\circ\delta_i + \delta_{i-1}\circ\partial_i\colon C_i \to C_i$ is an isomorphism of R-left modules.
(d) It thus follows from the above Acyclicity Criterion (⋆) that C_* is acyclic over R. ■

We move on to the following consequence of the Acyclic Lift Proposition 339.23:

Proposition 339.24. (Free Abelian–Lift Acyclicity Proposition) Let H be a torsion-free abelian group and let C_* be a chain complex of finitely generated free $\mathbb{Z}[H]$-modules. We introduce the following notation:

- We denote by $\mathbb{Q}(H)$ the quotient field of $\mathbb{Z}[H]$.
- We denote by $\epsilon\colon \mathbb{Z}[H] \to \mathbb{Z}$ the homomorphism that is given by $h \mapsto 1$ for all $h \in H$. This allows us to view $\mathbb{Z}$ as a $\mathbb{Z}[H]$-module.

If $\mathbb{Z} \otimes_{\mathbb{Z}[H]} C_*$ is acyclic, then so is $\mathbb{Q}(H) \otimes_{\mathbb{Z}[H]} C_*$.

Proof. Let H be a torsion-free abelian group and let C_* be a chain complex of finitely generated free $\mathbb{Z}[H]$-module such that $\mathbb{Z} \otimes_{\mathbb{Z}[H]} C_*$ is acyclic. We first introduce some more notation:

- We write $S \coloneqq \{f \in \mathbb{Z}[H] \mid \epsilon(f) = \pm 1\}$ and we consider the ring $\mathbb{Z}[H]S^{-1}$ that is given by localization as defined in the Tensor-with-Localization Proposition 314.18.
- Note that we have two ring homomorphisms:

$$\begin{array}{rcl} \varphi\colon \mathbb{Z}[H]S^{-1} & \to & \mathbb{Z} \\ \frac{p}{q} & \mapsto & \epsilon(p) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \psi\colon \mathbb{Z}[H]S^{-1} & \to & \mathbb{Q}(H) \\ \frac{p}{q} & \mapsto & \frac{p}{q}. \end{array}$$

Now we turn to the actual proof of the proposition:

(a) Note that if we are given a square matrix A over $\mathbb{Z}[H]S^{-1}$ such that $\varphi(A)$ is invertible, then $\varphi(\det(A)) = \det(\varphi(A)) = \pm 1$. Thus $\det(A)$ is invertible in $\mathbb{Z}[H]S^{-1}$, which implies that A is invertible over $\mathbb{Z}[H]S^{-1}$.

(b) It follows from (a), our hypothesis and the Acyclic Lift Proposition 339.23 that the tensored chain complex $\mathbb{Z}[H]S^{-1} \otimes_{\mathbb{Z}[H]} C_*$ is acyclic.

(c) From (b) and the remark preceding the proposition it follows that the tensored chain complex $\mathbb{Q}(H) \otimes_{\mathbb{Z}[H]S^{-1}} (\mathbb{Z}[H]S^{-1} \otimes_{\mathbb{Z}[H]} C_*)$ is also acyclic.

(d) It follows from the Tensor Commutative-Associative Lemma 312.11 that the tensored chain complex $\mathbb{Q}(H) \otimes_{\mathbb{Z}[H]S^{-1}} (\mathbb{Z}[H]S^{-1} \otimes_{\mathbb{Z}[H]} C_*)$ is isomorphic to $\mathbb{Q}(H) \otimes_{\mathbb{Z}[H]} C_*$.

(e) It follows from (c) and (d) that the chain complex $\mathbb{Q}(H) \otimes_{\mathbb{Z}[H]} C_*$ is indeed acyclic. ■

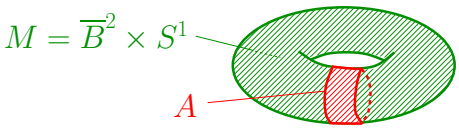
339.5.2. **Topology.** With the next proposition we return to topology:

Proposition 339.25. (Boundary-Inclusion Isomorphism Proposition) We assume that we are in one of the following two situations:

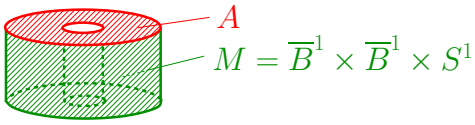
(1) Let M be a compact smooth manifold and let $A \subseteq \partial M$ be a compact codimension-zero submanifold of ∂M.

(2) Let M be a compact smooth manifold with corner $\angle M$. We assume that the corner $\angle M$ is two-sided in the sense of Definition 60.1. Furthermore let $A \subseteq \partial M$ be a smooth boundary in the sense of Definition 59.4.

Let H be a torsion-free abelian group and let $\Phi\colon \pi_1(M) \to H$ be a homomorphism. If $\mathrm{H}_*(M, A) = 0$, then $\mathrm{H}_*^{\mathrm{tw}}(M, A; {}_\Phi\mathbb{Q}(H)) = 0$.



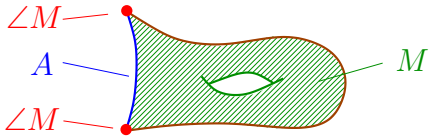
Setting (1)



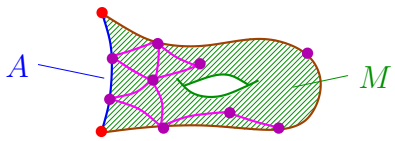
Setting (2)

Proof. We start out with the following claim:

Claim. In either of the two situations there exists a CW-structure on M such that A is a subcomplex.



A is a smooth boundary of ∂M



CW-structure on M such that A is a subcomplex

Proof. This claim follows immediately from the Boundary Submanifold–Subcomplex Theorem 179.8 together with the Simplicial-Implies-CW Lemma 174.51. ⊞

Now we turn to the actual proof of the proposition. We assume that $\mathrm{H}_*(M, A) = 0$. By the claim there exists a CW-structure on M such that A is a subcomplex. Note that it follows from the Twisted–Trivial Coefficients Lemma 320.13 together with the Singular-Cellular Twisted Isomorphism Theorem 328.1 that $\mathrm{H}_*^{\mathrm{tw,CW}}(M, A; {}_{\mathrm{tr}}\mathbb{Z}) \equiv 0$. Now we see

that

$$\mathrm{H}_*^{\mathrm{tw}}(M, A; {}_{\Phi}\mathbb{Q}(H)) \underset{\text{by the Singular-Cellular Twisted Isomorphism Theorem 328.1}}{\cong} \mathrm{H}_*^{\mathrm{tw,CW}}(M, A; {}_{\Phi}\mathbb{Q}(H)) \underset{\text{by the Free Abelian–Lift Acyclicity Proposition 339.24 applied to the cellular chain complex } \mathrm{C}_*^{\mathrm{CW}}(\widetilde{M}_\Phi, \widetilde{A}_\Phi)}{=} 0.$$

■

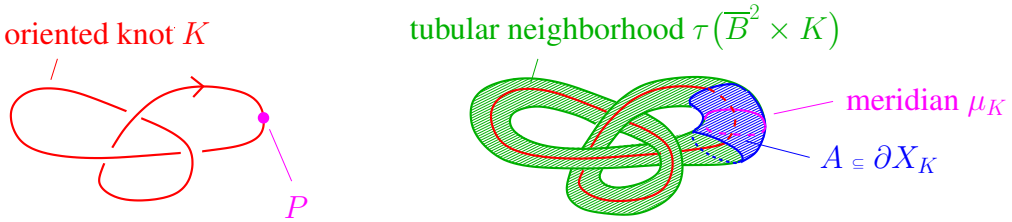
339.6. **Alexander modules of knots are torsion.** Now we can prove the promised result about Alexander modules and polynomials of knots:

Proposition 339.26. (Knot Alexander Module Proposition) Let $K \subseteq S^3$ be an oriented knot. The Alexander module $\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus K; {}_{\Phi_K}\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion. In other words, $\Delta_K(t) \neq 0$.

Proof. We give two very different proofs of the proposition:

(1) It follows from the Alexander Polynomial-at-1 Proposition 289.17 that the Alexander polynomial (with the original definition) of a knot is non-zero. It thus follows from the Alexander Module Torsion Lemma 339.16 together with the Alexander Modules-Agree Proposition 339.12 that the Alexander module $\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus K; {}_{\Phi_K}\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion.

(2) We start out with a few preparations:
- Let $\tau\colon \overline{B}^2 \times K \to S^3$ be a tubular map for K. As usual we consider the corresponding knot exterior $X_K := S^3 \setminus \tau(B^2 \times K)$.
- We pick $P \in K$. Recall that it follows from the Meridian Proposition 109.2 that $\mu_K := \tau(S^1 \times \{P\})$ is a meridian for K.
- We pick a closed non-trivial interval $I \subseteq K$ with $P \in I$ and we set $A := \tau(S^1 \times I)$.



We now start with the actual proof of the proposition:

(a) It follows easily from the Link Complement-H_*-Proposition 283.4 together with the Homotopy Equivalence-Relative-H_*-Corollary 146.8 that the inclusion induced maps $\mathrm{H}_k(A) \to \mathrm{H}_k(X_K)$ are all isomorphisms.

(b) It follows from (a) and the LES-of-Triple Proposition 146.5 that $\mathrm{H}_*(X_K, A) = 0$.

(c) It follows from (b) and the Boundary Submanifold–Subcomplex Theorem 179.8 that $\mathrm{H}_k^{\mathrm{tw}}(X_K, A; {}_{\Phi_K}\mathbb{Q}(t)) = 0$.

(d) We now see that

$$\mathrm{H}_k^{\mathrm{tw}}(X_K; {}_{\Phi_K}\mathbb{Q}(t)) \underset{\text{follows from (c) and the Twisted-LES Proposition 321.10}}{\cong} \mathrm{H}_k^{\mathrm{tw}}(A; {}_{\Phi_K}\mathbb{Q}(t)) \underset{\text{by the Homotopy Equivalence-H}_*^{\mathrm{tw}} \text{ Proposition 322.6}}{\cong} \mathrm{H}_k^{\mathrm{tw}}(\mu_K; {}_{\Phi_K}\mathbb{Q}(t)) \underset{\text{follows from the Twisted-H}_*\text{-of-}S^1 \text{ Proposition 327.13}}{\cong} 0.$$

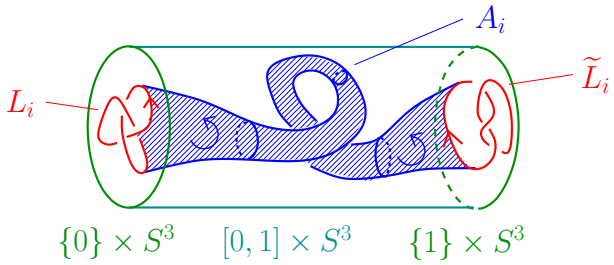
(e) It follows from (d), the Alexander Module Torsion Lemma 339.16 and the Alexander Modules-Agree Proposition 339.12 that $\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus K; {}_{\Phi_K}\mathbb{Z}[t^{\pm 1}])$ is indeed $\mathbb{Z}[t^{\pm 1}]$-torsion. ■

339.7. **Link concordance and Alexander modules.** We conclude this chapter with a short discussion of link concordance and Alexander modules. We start out with the following definition, which the diligent reader will already have encountered in Exercise 258.13:

Definition 339.27. Let L and $\widetilde{L}$ be two oriented m-component links in S^3.

(1) A smooth concordance between L and $\widetilde{L}$ is defined as a collection of disjoint properly smoothly embedded annuli $A_1, \dots, A_m \subseteq [0,1] \times S^3$ such that for each $i \in \{1, \dots, m\}$ we have $\partial A_i = (\{0\} \times (-L_i)) \cup (\{1\} \times \widetilde{L}_i)$.

(2) We say L and $\widetilde{L}$ are concordant if a smooth concordance exists.



The following proposition now says that if two links are concordant, then the ranks of their Alexander modules agree:

Proposition 339.28. (Link Concordance–Alexander Module Rank Proposition) Let L and $\widetilde{L}$ be two oriented m-components links. If L and $\widetilde{L}$ are smoothly concordant, then for each $k \in \mathbb{N}_0$ we have

$$\mathrm{rank}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) = \mathrm{rank}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus \widetilde{L}; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])).$$

In other words, we have

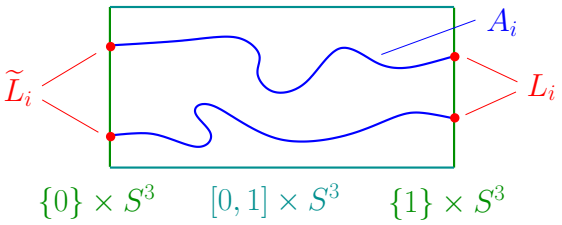
$$\dim_{\mathbb{Q}(t_1, \dots, t_m)}(\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Q}(t_1, \dots, t_m))) = \dim_{\mathbb{Q}(t_1, \dots, t_m)}(\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus \widetilde{L}; \mathbb{Q}(t_1, \dots, t_m))).$$

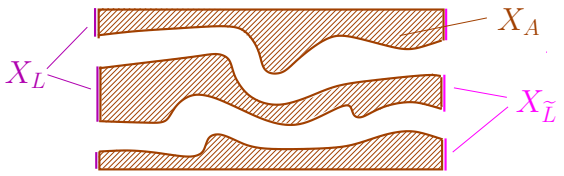
Example. Note that it follows from the Link Concordance–Alexander Module Rank Proposition 339.28, the Alexander Module Torsion Lemma 339.16 and the Link Alexander Polynomial–Example Lemma 339.21 that the Hopf link and the trivial 2-component link are not smoothly concordant. This should just be seen as a proof of concept, as we will see in Exercise 339.19 there are much more elementary ways to proving the same result. □

Proof. Let $A \subseteq [0,1] \times S^3$ be a link concordance between L and $\widetilde{L}$. We make the following observations and preparations:

(a) It follows from the Product Tubular Neighborhood–Existence Theorem 243.12 that we can pick a tubular map $\tau \colon \overline{B}^2 \times A \to [0,1] \times S^3$ for the proper smooth submanifold $A \subseteq [0,1] \times S^3$. We consider the exterior $X_A := ([0,1] \times S^3) \setminus \tau(B^2 \times A)$ of A. It follows from Definition 57.9, the Smooth Submanifold-with-Corner Lemma 26.16 and the discussion on pages 1336 and 1351 that X_A is a smooth manifold with corner $\angle X_A = \tau(S^1 \times \partial A)$ which is two-sided in the sense of Definition 60.1 and where $X_L \times \{0\}$ and $X_{\widetilde{L}} \times \{1\}$ are smooth boundaries in the sense of Definition 59.4.

(b) For convenience we identify X_L with $X_L \times \{0\} = X_A \cap (S^3 \times \{0\})$ and we also identify $X_{\widetilde{L}}$ with $X_{\widetilde{L}} \times \{1\} = X_A \cap (S^3 \times \{1\})$ in the obvious way.





(c) Note that

$$\mathrm{H}_*(X_A, X_L) \underset{\text{by the Alexander Duality Boundary Theorem 217.16}}{\cong} \mathrm{H}^*(A, \widetilde{L}; \mathbb{Z}) \underset{\text{since } A \text{ is an annulus}}{=} 0.$$

Similarly we see that $\mathrm{H}_*(X_A, X_{\widetilde{L}}) \cong \mathrm{H}^*(A, L; \mathbb{Z}) = 0$.

(d) We have the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(X_L) & \xrightarrow{\cong} & \mathrm{H}_1(X_A) & \xleftarrow{\cong} & \mathrm{H}_1(X_{\widetilde{L}}) \\ & {\scriptstyle \Phi_{\widetilde{L}},\ \mu_{L_i} \mapsto t_i} \searrow & \downarrow {\scriptstyle =:\Phi_A} & \swarrow {\scriptstyle \Phi_L,\ \mu_{\widetilde{L}_i} \mapsto t_i} & \\ & & \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}. & & \end{array}$$

It follows from (c) together with the LES-of-Pair Proposition 146.7 that the horizontal maps are both isomorphisms. We claim that the diagram commutes. This can be seen as follows: Recall that by the Meridian Proposition 109.2 the meridians of L and $\widetilde{L}$ are given by the tubular neighborhoods of L and $\widetilde{L}$. The tubular neighborhood of A thus shows that in X_A the meridians of L are smoothly isotopic to the corresponding meridians of $\widetilde{L}$. This shows that the above diagram commutes.

This discussion shows that we get an isomorphism $\Phi_A \colon \mathrm{H}_1(X_A) \to \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$ that makes the diagram commute.

(e) It follows from (c) together with the Boundary Submanifold–Subcomplex Theorem 179.8 that $\mathrm{H}_k^{\mathrm{tw}}(X_A, X_L; {}_{\Phi_A}\mathbb{Q}(t_1, \dots, t_m)) = 0$ and that $\mathrm{H}_k^{\mathrm{tw}}(X_A, X_{\widetilde{L}}; {}_{\Phi_A}\mathbb{Q}(t_1, \dots, t_m)) = 0$.

We now see that

$$\begin{aligned} \dim_{\mathbb{Q}(t_1,\dots,t_m)}(\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus L; {}_{\Phi_L}\mathbb{Q}(t_1, \dots, t_m))) &\underset{\text{by the Link Exterior–Alexander Module Lemma 339.18 and since } \Phi_A = \Phi_L}{\downarrow =} \dim_{\mathbb{Q}(t_1,\dots,t_m)}(\mathrm{H}_k^{\mathrm{tw}}(X_L; {}_{\Phi_A}\mathbb{Q}(t_1, \dots, t_m))) \\ &= \dim_{\mathbb{Q}(t_1,\dots,t_m)}(\mathrm{H}_k^{\mathrm{tw}}(X_A; \mathbb{Q}(t_1, \dots, t_m))) \\ &\underset{\text{both equalities follows from (e) and the Twisted-LES-Proposition 321.10}}{\uparrow} \\ &\underset{}{\downarrow =} \dim_{\mathbb{Q}(t_1,\dots,t_m)}(\mathrm{H}_k^{\mathrm{tw}}(X_{\widetilde{L}}; {}_{\Phi_A}\mathbb{Q}(t_1, \dots, t_m))) \\ &= \dim_{\mathbb{Q}(t_1,\dots,t_m)}(\mathrm{H}_k^{\mathrm{tw}}(S^3 \setminus \widetilde{L}; {}_{\Phi_{\widetilde{L}}}\mathbb{Q}(t_1, \dots, t_m))). \\ &\underset{\text{by the Link Exterior–Alexander Module Lemma 339.18}}{\uparrow} \end{aligned}$$

■

Later in the Link Concordance–Torsion Proposition **??** (or Exercise 340.5 for the intrepid reader) we will study what the Alexander polynomial can tell us about link concordance.

339.8. **Exercises for Chapter 339.**

Exercise 339.1. Let R be a domain with quotient field $Q(R)$. Show that for any R-torsion module M the tensor product $Q(R) \otimes_R M$ is zero.

Exercise 339.2. Let R be a domain with quotient field $Q(R)$. Show that any R-module M with $Q(R) \otimes_R M = 0$ is actually R-torsion.
Hint. Have a look at the proof of the Tensor-with-Localization Proposition 314.18.

Exercise 339.3. Let R be a Noetherian UFD and let M be a finitely generated R-module *which admits a finite square presentation matrix*. Furthermore let $S \subseteq R \setminus \{0\}$ be a multiplicatively closed subset that contains 1. We consider the localization RS^{-1} from the Tensor-with-Localization Proposition 314.18. Show that the following statements are equivalent:

(i) $RS^{-1} \otimes_R M = 0$.
(ii) $\operatorname{ord}_R(M) \in S$.
(iii) For every $m \in M$ there exists an $s \in S$ with $s \cdot m = 0$, i.e. M is S-torsion.

Remark. This exercise is a variation on the Module Torsion Lemma 339.8.

Exercise 339.4. Suppose that X is a connected CW-complex and that $\Phi\colon \mathrm{H}_1(X) \to \mathbb{Z}$ is an epimorphism. Does there necessarily exist a homeomorphism $f\colon X \to X$ such that $\Phi \circ f_* = -\Phi$?

Exercise 339.5. Let X be finite connected non-empty CW-complex.

(a) Let $\Phi\colon \pi_1(X) \to H$ be an epimorphism onto a free abelian group of rank ≥ 1. Use the Twisted H_0-Proposition 323.3 to show that

$$\underbrace{\operatorname{ord}_{\mathbb{Z}[H]}(\mathrm{H}_0^{\mathrm{tw}}(X;{}_\Phi\mathbb{Z}[H]))}_{=\Delta_0^\Phi(X)} \doteq \begin{cases} h-1, & \text{if } H = \langle h \rangle \text{ has rank 1,} \\ 1, & \text{otherwise.} \end{cases}$$

(b) How does the result in (a) change if we no longer assume that $\Phi\colon \pi_1(X) \to H$ is an epimorphism?

Exercise 339.6. We consider the torus $S^1 \times S^1$. Let H be a finitely generated free abelian group and let $\Phi\colon \pi_1(S^1 \times S^1) \to H$ be a non-trivial homomorphism.

(a) Show that

$$\underbrace{\operatorname{ord}_{\mathbb{Z}[H]}(\mathrm{H}_1^{\mathrm{tw}}(S^1 \times S^1;{}_\Phi\mathbb{Z}[H]))}_{=\Delta_1^\Phi(S^1\times S^1)} = \underbrace{\operatorname{ord}_{\mathbb{Z}[H]}(\mathrm{H}_0^{\mathrm{tw}}(S^1 \times S^1;{}_\Phi\mathbb{Z}[H]))}_{=\Delta_0^\Phi(S^1\times S^1)}.$$

(b) Show that $\operatorname{ord}_{\mathbb{Z}[H]}(\mathrm{H}_2^{\mathrm{tw}}(S^1 \times S^1;{}_\Phi\mathbb{Z}[H])) = 1$.

Remark. You could make use of the Torus–Twisted Cellular Chain Complex Lemma 329.11. Also note that (as in Exercise 339.5) the calculation in (a) depends on whether or not $\operatorname{im}(\Phi) \subseteq H$ is a free abelian group of rank one or two.

Exercise 339.7. Let X be a path-connected finite CW-complex and let $x_0 \in X$. Furthermore let $\varphi\colon \pi_1(X, x_0) \to \langle t \rangle$ be an epimorphism. We denote by $q\colon \widehat{X} \to X$ the natural covering of X corresponding to $\ker(\varphi)$. Show that if all rational homology groups $\mathrm{H}_k(\widehat{X};\mathbb{Q})$ are finite-dimensional, then the Euler characteristic $\chi(X)$ of X is zero.
Hint. Consider twisted homology of X with $\mathbb{Q}[t^{\pm 1}]$-coefficients and $\mathbb{Q}(t)$-coefficients and make use of the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 with $S = \mathbb{Q}$.

Exercise 339.8. Let K and L be two oriented knots in S^3. We consider the connected sum $K\#L$ from Definition 109.12. Use the Twisted Mayer–Vietoris Theorem 325.12 to show that we have an isomorphism

$$\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus K\#L; \mathbb{Z}[t^{\pm 1}]) \;\cong\; \mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) \oplus \mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t^{\pm 1}])$$

of $\mathbb{Z}[t^{\pm 1}]$-modules.

Exercise 339.9. Let $p, q \in \mathbb{Z}$ be two coprime natural numbers. We consider the corresponding torus knot $T(p,q)$ from Definition 111.13. Show that the Alexander module $\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus T(p,q); \mathbb{Z}[t^{\pm 1}])$ of the torus knot $T(p,q)$ is cyclic as a $\mathbb{Z}[t^{\pm 1}]$-module.
Hint. Consider the description of the complement $S^3 \setminus T(p,q)$ that we gave in the Torus Knot-π_1-Proposition 111.18.

Exercise 339.10. Let $K \subseteq S^3$ be an oriented knot and let $n \in \mathbb{N}$.

(a) Show that there exists a long exact sequence of the form

$$\to \mathrm{H}_n^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\cdot(1-t^k)} \mathrm{H}_n^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) \to \mathrm{H}_n^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]/(t^k - 1)) \xrightarrow{\partial_n} \to$$

Hint. Use the Twisted Bockstein Lemma 321.15.

(b) Show that multiplication by $t - 1$ is an isomorphism of the corresponding Alexander module $\mathrm{H}_n^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$.

(c) Show that the above also works with rational coefficients. Use this fact to give a new prove that $\mathrm{H}_n^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Q}(t)) = 0$.

Exercise 339.11. Let $K \subseteq S^3$ be an oriented knot. Note that we can use the rational Alexander module $\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}])$ to define the corresponding rational Alexander polynomial $\Delta_K^{\mathbb{Q}}(t) \in \mathbb{Q}[t^{\pm 1}]$ which is well-defined up to multiplication by $r \cdot t^n$ where $r \in \mathbb{Q} \setminus \{0\}$ and $n \in \mathbb{Z}$.

(a) Show that (up to the larger indeterminacy) we have $\Delta_K^{\mathbb{Q}}(t) \doteq \Delta_K(t)$.

(b) Show that $\deg(\Delta_K^{\mathbb{Q}}(t)) = \dim_{\mathbb{Q}}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Q}[t^{\pm 1}]))$.

(c) Let A be a finitely generated torsion-free abelian group. Show that the map $A \mapsto A \otimes \mathbb{Q}$ given by $a \mapsto a \otimes 1$ is injective.
Hint. Try to reduce the problem to the case that A is finitely generated.
Remark. This exercise already appeared as Exercise 159.15.

(d) Show that $\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) = 0$ if and only if $\Delta_K(t) \doteq 1$.
Hint. Recall that by the Alexander Module-$\mathbb{Z}$-Torsion Free Proposition 289.18 we know that $\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) = 0$ is torsion-free as an abelian group.

Remark. This exercise is essentially the same as Exercise 290.11.

Exercise 339.12. Let $L_1 \subseteq S^3$ be an oriented m_1-component link and let $L_2 \subseteq S^3$ be an oriented m_2-component link. We consider the split union $L = L_1 \sqcup L_2$ that we constructed in Definition 109.4. Express the Alexander module of L in terms of the Alexander modules of L_1 and L_2.
Remark. Note that these modules are initially rings over different rings.

Exercise 339.13. Let C_* be a chain complex of finitely generated free R-left modules and let $\varphi\colon R \to S$ be a ring homomorphism, such that the following condition holds:

(∗) Whenever A is a matrix over R such that $\varphi(A)$ is invertible over S, then A is already invertible over R.



Show that the following holds: If $\mathrm{H}_i(S_\varphi \otimes_R C_*) = 0$ for $i = 0, \dots, k$, then $\mathrm{H}_i(C_*) = 0$ for $i = 0, \dots, k$.
Remark. This exercise is a slight refinement of the Acyclic Lift Proposition 339.23.

Exercise 339.14. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an m-component link. Show that

$$\operatorname{rank}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) \;\leq\; m - 1.$$

Hint. Take the union of the m meridians and connect them via m arcs to a point. Call the resulting subspace M. Consider $\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L, M; \mathbb{Q}(t_1, \dots, t_m))$ and make use of Exercise 339.13.

Exercise 339.15. As in Exercise 121.11 we say that an m-component link $L = L_1 \sqcup \cdots \sqcup L_m$ is a *boundary link* if there exists a collection of m disjoint Seifert surfaces $F_1, \dots, F_m$ such that for $i = 1, \dots, m$ we have $\partial F_i = L_i$. Show that

$$\operatorname{rank}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) \;\geq\; m - 1.$$

Hint. First show that there exists a continuous map $f\colon S^3 \setminus L \to \bigvee_{i=1}^{m} S_i^1$ which induces an epimorphism of fundamental groups and isomorphism of homology groups. Then show that f induces an epimorphism of Alexander modules.
Remark 1. It follows from the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33 that we can also work with the link exterior instead of the link complement.
Remark 2. Together with Exercise 339.14 this shows that for an m-component boundary link we have $\operatorname{rank}_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\mathrm{H}_1^{\mathrm{tw}}(S^3 \setminus L; \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])) = m - 1$.

Exercise 339.16. Let $K \subseteq S^3$. We pick a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ with self-linking number zero and we pick a diffeomorphism $\gamma\colon S^1 \to K$. We consider the closed smooth manifold

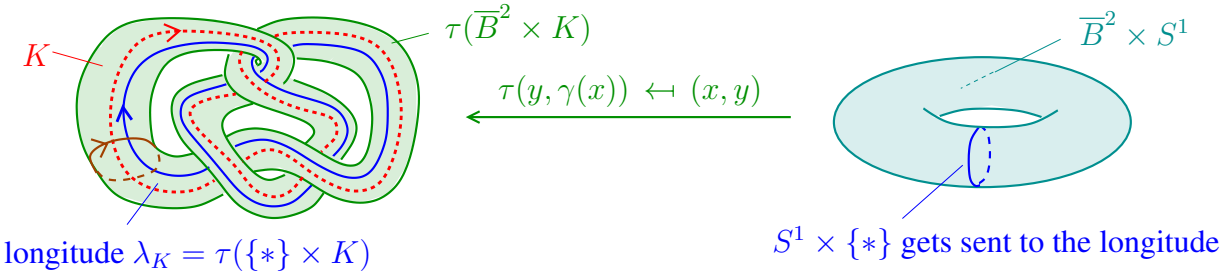
$$N_K \;:=\; S^3_0(K) \;:=\; \overbrace{(S^3 \setminus \tau(B^2 \times K))}^{\text{knot exterior } X_K} \cup_{\substack{\partial X_K \;\leftarrow\; S^1 \times S^1 \\ \tau(y, \gamma(x)) \;\leftarrow\; (x, y)}} (\overline{B}^2 \times S^1).$$

In the language of **??** this is the *0-Dehn surgery along* K.

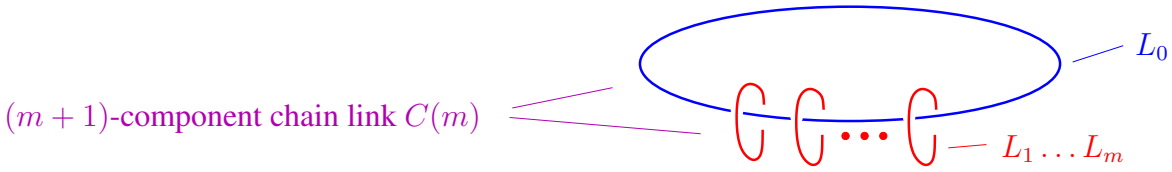
(a) Compute the ordinary homology groups of N_K and show that the inclusion induced map $\mathrm{H}_1(X_K) \to \mathrm{H}_1(N_K)$ is an isomorphism.
(b) Show that the inclusion induced map $\mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \to \mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Z}[t^{\pm 1}])$ is an isomorphism.
(c) Determine $\mathrm{H}_2^{\mathrm{tw}}(N_K; \mathbb{Z}[t^{\pm 1}])$.

Exercise 339.17. Let $m \in \mathbb{N}$. As in Exercise 111.7 we consider the $(m+1)$-component link $C(m) = L_0 \cup L_1 \cup \cdots \cup L_m$ that is shown in the figure below.

(a) Calculate the Alexander module of $C(m)$.
(b) Calculate the Alexander polynomial of $C(m)$.



Hint. Consider first the complement of L_0.
Remark. The link $C(m)$ is sometimes called a *chain link* or *key chain link*. Note that $C(0)$ is just a trivial knot and that $C(1)$ is smoothly isotopic to the Hopf link.



Exercise 339.18. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link. Use Exercise 330.9 and the CW-Boundary Maps-via-Fox Calculus Proposition 330.6 to show that the definition of the Alexander polynomial in Definition 117.2 and Definition 117.4 agrees with the definition from Definition 339.20.
Remark. This exercise might come across as slightly ambitious.

Exercise 339.19. Let L and $\widetilde{L}$ be two m-component links in S^3 Show that if L and $\widetilde{L}$ are smoothly concordant in the sense of Definition 339.27, then the linking numbers of L and $\widetilde{L}$ agree, i.e. for all $i \neq j \in \{1, \dots, m\}$ we have $\mathrm{lk}(L_i, L_j) = \mathrm{lk}(\widetilde{L}_i, \widetilde{L}_j)$.
Remark. This exercise is identical to Exercise 258.13.

CHAPTER 340

The symmetry of Alexander polynomials and the Blanchfield pairing

In the previous Chapter 339 we (re-) introduced Alexander modules of oriented links via twisted homology. In this chapter we will apply Poincaré Duality to Alexander modules. This approach leads to two new results:

- We will show that Alexander polynomials of links are always symmetric, i.e. we will show that for any oriented m-component link $L \subseteq S^3$ we have
$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$
- Given an oriented knot $K \subseteq S^3$ we will introduce the Blanchfield pairing
$$\mathrm{Bl}_K \colon \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$
which is a variation on the linking pairing on rational homology spheres. We will discuss several fun properties of the Blanchfield pairing and we will discuss how one can calculate it via Seifert matrices.

This chapter is organized as follows:

- In Section 340.1 we will, starting with the Alexander module of an oriented link, provide a sequence of altogether eight homomorphisms and we will discuss under which conditions these are actually isomorphisms.
- In Section 340.2 we will use the results from Section 340.1 to show that Alexander polynomials of links are symmetric.
- In Section 340.3 we will introduce the general notion of a twisted linking pairing.
- In Section 340.4 we will use the results from Section 340.1 and the discussion from Section 340.3 to introduce the Blanchfield pairing of an oriented link.
- In Section 340.5 we will discuss the relationship between the Blanchfield pairing of a knot and the twisted intersection form on 4-dimensional topological manifolds.
- In Section 340.6 we will provide exercises for Chapter 340.

In the next **??** we will use our tools from twisted (co-) homology to (re-) introduce group homology and cohomology.

340.1. **A sequence of homomorphisms.** Throughout this section we work with the following notation:

Notation 340.1.
(1) Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link.
(2) We pick a tubular map $\tau \colon \overline{B}^2 \times L \to S^3$ for L and we consider the corresponding link exterior $X_L := S^3 \setminus \tau(B^2 \times L)$.

(3) We pick a base point $x_0 \in X_L$. Sometimes we write $\pi := \pi_1(X_L, x_0)$.
(4) We consider the corresponding natural universal covering $\widetilde{X}_L := (X_L)_{x_0}$ from Definition 319.10.
(5) We write $\Lambda := \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and $Q(\Lambda) := \mathbb{Q}(t_1 \ \dots \ t_m)$.
(6) Let $\Phi_L \colon \pi_1(X_L, x_0) \to \langle t_1, \dots, t_m \rangle_{\text{ab}}$ be the usual epimorphism from Definition 339.17.
(7) We use Φ_L to view Λ, $Q(\Lambda)$ and $Q(\Lambda)/\Lambda$ as $(\Lambda, \mathbb{Z}[\pi_1(X, x_0)])$-left left modules.

In this section we will see that we have the following epic sequence of homomorphisms of Λ-modules:

$$\begin{aligned}
\mathrm{H}_1^{\text{tw}}(X_L; \Lambda) &\xrightarrow[\text{(I)}]{\text{я}} \mathrm{H}_1^{\text{tw}}(X_L, \partial X_L; \Lambda) \\
&\xrightarrow[\text{(II)}]{\text{PD}} \mathrm{H}^2_{\text{tw}}(X_L; \Lambda) \\
&\xleftarrow[\text{(III)}]{\beta} \mathrm{H}^1_{\text{tw}}(X_L; Q(\Lambda)/\Lambda) \\
&\xrightarrow[\text{(IV)}]{\text{id}} \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}} (\mathrm{C}_*(\widetilde{X}_L); Q(\Lambda)/\Lambda) \Big) \\
&\xrightarrow[\text{(V)}]{\Theta_*} \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}} (\mathrm{C}_*(\widetilde{X}_L), \mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]}, \overline{Q(\Lambda)/\Lambda}) \Big) \\
&\xleftarrow[\text{(VI)}]{\text{Adj}_*} \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L), \overline{Q(\Lambda)/\Lambda})) \\
&\xrightarrow[\text{(VII)}]{\text{ev}} \mathrm{Hom}_\Lambda(\underbrace{\mathrm{H}_1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L)))}_{=\mathrm{H}_1^{\text{tw}}(X_L;\Lambda)}, \overline{Q(\Lambda)/\Lambda}) \\
&\xrightarrow[\text{(VIII)}]{\cong} \overline{\mathrm{Hom}_\Lambda(\mathrm{H}_1^{\text{tw}}(X_L; \Lambda), Q(\Lambda)/\Lambda)} \\
&\xrightarrow[\text{(IX)}]{\beta} \overline{\mathrm{Ext}^1_\Lambda(\mathrm{H}_1^{\text{tw}}(X_L; \Lambda), \Lambda)}.
\end{aligned}$$

In the following nine subsections we will discuss each of the nine homomorphisms individually in greater detail and we will discuss whether (or under what hypotheses) each map is in fact an isomorphism. For quick access we list the subsections:

- In Section 340.1.1 we will discuss Homomorphism (I).
- In Section 340.1.2 we will discuss Homomorphism (II).
- In Section 340.1.3 we will discuss Homomorphism (III).
- In Section 340.1.4 we will discuss Homomorphism (IV).
- In Section 340.1.5 we will discuss Homomorphism (V).
- In Section 340.1.6 we will discuss Homomorphism (VI).
- In Section 340.1.7 we will discuss Homomorphism (VII).
- In Section 340.1.8 we will discuss Homomorphism (VIII).
- In Section 340.1.9 we will discuss Homomorphism (IX).

340.1.1. **Homomorphism (I).** Here is the description of the above homomorphism (I):

Notation 340.2. We denote by $\text{я} \colon \mathrm{H}_1^{\text{tw}}(X_L; \Lambda) \to \mathrm{H}_1^{\text{tw}}(X_L, \partial X_L; \Lambda)$ the homomorphism that is induced (in the sense of Definition 321.8) by the map $(X_L, \varnothing) \to (X_L, \partial X_L)$.

Lemma 340.3. (Homomorphism (I)-Lemma) Let $L \subseteq S^3$ be an oriented m-component link.

(1) We denote by $T_1, \dots, T_m$ the boundary tori of X_L. The homomorphism (I) from Notation 340.2 fits into an exact sequence of the form
$$\to \bigoplus_{i=1}^{m} \mathrm{H}_1^{\mathrm{tw}}(T_i;\Lambda) \to \mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda) \xrightarrow{\text{я}} \mathrm{H}_1^{\mathrm{tw}}(X_L,\partial X_L;\Lambda) \xrightarrow{\partial_1} \bigoplus_{i=1}^{m} \mathrm{H}_0^{\mathrm{tw}}(T_i;\Lambda) \to \dots$$
Note that we also have a similar exact sequence with $Q(\Lambda)$-coefficients.

(2) For a knot $K \subseteq S^3$ the homomorphism $\text{я}\colon \mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda) \to \mathrm{H}_1^{\mathrm{tw}}(X_K,\partial X_K;\Lambda)$ is an isomorphism.

Sketch of proof. First note that the long exact sequence comes from the Internal Twisted-LES-Proposition 325.9 together with the Change-of-Covering Lemma 325.7.

For the remainder of this proof we consider the case of a knot $K \subseteq S^3$. In the following let $p\colon \widehat{X}_K \to X_K$ be the covering corresponding to $\ker(\Phi_K\colon \pi_1(X_k,x_0) \to \mathbb{Z})$. It follows from the above discussion and the Covering-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 323.5 that we have the following commutative diagram
$$\begin{array}{ccccccccc}
\mathrm{H}_1^{\mathrm{tw}}(\partial X_K;\mathbb{Z}[t^{\pm1}]) & \to & \mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm1}]) & \xrightarrow{\text{я}} & \mathrm{H}_1^{\mathrm{tw}}(X_K,\partial X_K;\mathbb{Z}[t^{\pm1}]) & \xrightarrow{\partial_1} & \mathrm{H}_0^{\mathrm{tw}}(\partial X_K;\mathbb{Z}[t^{\pm1}]) & \to & \dots \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
\mathrm{H}_1(\partial\widehat{X}_K) & \longrightarrow & \mathrm{H}_1(\widehat{X}_K) & \xrightarrow{\text{я}} & \mathrm{H}_1(\widehat{X}_K,\partial\widehat{X}_K) & \xrightarrow{\partial_1} & \mathrm{H}_0(\partial\widehat{X}_K) & \longrightarrow & \dots
\end{array}$$
where all vertical maps are isomorphisms of $\mathbb{Z}[t^{\pm1}]$-modules. By going a little further to the right of the shown sequences, and using the Twisted H_0-Proposition 323.3 one sees that the connecting homomorphism ∂_1 is the zero homomorphism.

It remains to prove the following claim:

Claim. The inclusion induced map $\mathrm{H}_1(\partial\widehat{X}_K) \to \mathrm{H}_1(\widehat{X}_K)$ is the zero map.

Proof. Here is a sketch for this argument:

(a) It follows from the Meridian Proposition **??** and the Longitude Proposition 115.2 that we can make the identification $\partial X_K = \lambda_K \times \mu_K$ where $\Phi_K([\lambda_K]) = 0$.

(b) Using (a) one can now easily see that $\mathrm{H}_1(\partial\widehat{X}_K) = \mathbb{Z}\cdot[\widehat{\lambda}_K]$ where $\widehat{\lambda}_K$ is a lift of the longitude of K.

(c) It follows from the Seifert Surface Existence Theorem 121.2 together with the Tubular Neighborhood–Boundary-of-Submanifold Theorem 57.33 that the longitude bounds a surface $F \subseteq X_K$.

(d) As we discussed in Exercise 285.10 it follows easily from the Linking #-via-Intersection #-Proposition 285.5 that the restriction of the homomorphism $\Phi_K\colon \mathrm{H}_1(X_K) \to \mathbb{Z}$ to the map $\mathrm{H}_1(F) \xrightarrow{\text{inclusion induced}} \mathrm{H}_1(X_K) \to \mathbb{Z}$ is the zero map.[3689]

(e) It follows from (d) that F lifts to a surface $\widehat{F} \subseteq \widehat{X}_K$ with $\partial\widehat{F} = \widehat{\lambda}_K$.

(f) It follows from (e) and the Surface-with-One Boundary Component-H_*-Lemma 155.23 (2) that $[\widehat{\lambda}_K] = 0 \in \mathrm{H}_1(\widehat{X}_K)$.

We leave it to the reader to fill in the details. ∎

[3689] Alternatively this statement can also be deduced from the Link–Infinite Cyclic Covering Proposition 288.18.

340.1.2. **Homomorphism (II): The Poincaré duality isomorphism.** Here is the description of the homomorphism (II) from page 6440:

Notation 340.4. By the Twisted Poincaré Duality Theorem 335.1 we know that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{tw}}(X_L, \partial X_L; \Lambda) & \xrightarrow{\cong} & \mathrm{H}^{\mathrm{tw}}_{3-k}(X; \Lambda) \\ \varphi & \mapsto & \varphi \cap_{\mathbb{Z}} [X_L] \\ & & \uparrow \\ & & \text{see Definition 334.34} \end{array}$$

is an isomorphism of Λ-modules. As usual we denote by

$$\mathrm{PD}\colon \mathrm{H}^{\mathrm{tw}}_{3-k}(X; \Lambda) \ \to \ \mathrm{H}^k_{\mathrm{tw}}(X_L, \partial X_L; \Lambda)$$

the inverse of the above isomorphism.

For consistency in exposition we mention the obvious:

Lemma 340.5. (Homomorphism (II)-Lemma) For any oriented link $L \subseteq S^3$ the homomorphism

$$\mathrm{PD}\colon \mathrm{H}^{\mathrm{tw}}_1(X_L; \Lambda) \ \to \ \mathrm{H}^2_{\mathrm{tw}}(X_L, \partial X_L; \Lambda)$$

from Notation 340.4 is an isomorphism of Λ-modules.

Proof. As we just mentioned, this is just a special case of the Twisted Poincaré Duality Theorem 335.1. ■

340.1.3. **Homomorphism (III): The Bockstein homomorphism.** Here is the description of the homomorphism (III) from page 6440:

Notation 340.6. Note that we have a short exact sequence

$$0 \ \to \ \Lambda \ \xrightarrow{\varphi} \ Q(\Lambda) \ \xrightarrow{\psi} \ Q(\Lambda)/\Lambda \ \to \ 0$$

of $(\Lambda, \mathbb{Z}[\pi_1(X_L, x_0)])$-left left modules. From the Twisted Bockstein Lemma 321.15 we get a long exact sequence of Λ-modules

$$\ldots \ \to \ \mathrm{H}^1_{\mathrm{tw}}(X_L; Q(\Lambda)) \ \xrightarrow{\psi_*} \ \mathrm{H}^1_{\mathrm{tw}}(X_L; Q(\Lambda)/\Lambda) \ \xrightarrow{\beta_k} \ \mathrm{H}^2_{\mathrm{tw}}(X_L; \Lambda) \ \xrightarrow{\varphi_*} \ \mathrm{H}^2_{\mathrm{tw}}(X_L; Q(\Lambda)) \ \to \ \ldots$$

We refer to the connecting homomorphism β_k as the k-th Bockstein homomorphism.

Lemma 340.7. (Homomorphism (III)-Lemma) Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link. If $\Delta_L(t_1, \ldots, t_m) \neq 0$, then the Bockstein homomorphism

$$\beta\colon \mathrm{H}^1_{\mathrm{tw}}(X_L; Q(\Lambda)/\Lambda) \ \to \ \mathrm{H}^2_{\mathrm{tw}}(X_L; \Lambda)$$

from Notation 340.6 is an isomorphism.

Proof. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an oriented m-component link. We denote by $T_1, \ldots, T_m$ the boundary tori of X_L.

(a) Recall that we assume that $\Delta_L(t_1, \ldots, t_m) \neq 0$. Note that it follows from the Alexander Module Torsion Lemma 339.16 together with the Higher Alexander Module Proposition 339.19 and the humble Twisted H_0-Proposition 323.3 that under this hypothesis we have $\mathrm{H}^{\mathrm{tw}}_*(X_L; Q(\Lambda)) \equiv 0$.

(b) Since the restriction of Φ_L to each boundary torus T_i is non-trivial we obtain from the Torus–Twisted Cellular Chain Complex Lemma 329.11 that $\mathrm{H}^{\mathrm{tw}}_*(T_i; Q(\Lambda)) \equiv 0$.

(c) It follows from (a) and (b), together with the exact sequence from the Homomorphism (I)-Lemma 340.3 that $\mathrm{H}^{\mathrm{tw}}_*(X_L, \partial X_L; Q(\Lambda)) \equiv 0$.

(d) It follows from (c) together with the Twisted Poincaré Duality Theorem 335.1 that $\mathrm{H}^*_{\mathrm{tw}}(X_L; Q(\Lambda)) \equiv 0$.
(e) It follows from (d) and the Twisted Bockstein Lemma 321.15 that the Bockstein homomorphism $\beta_1 \colon \mathrm{H}^1_{\mathrm{tw}}(X_L; Q(\Lambda)/\Lambda) \to \mathrm{H}^2_{\mathrm{tw}}(X_L; \Lambda)$ is indeed an isomorphism. ■

340.1.4. **Homomorphism (IV): The identity.** Here is the rather dull description of the homomorphism (IV) from page 6440:

Notation 340.8. By Definition 320.11 we have the equality

$$\mathrm{H}^1_{\mathrm{tw}}(X_L; Q(\Lambda)/\Lambda) \;=\; \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_L), Q(\Lambda)/\Lambda)\Big).$$

That's all there is to say.

For consistency of exposition we write down the following tautology:

Lemma 340.9. (Homomorphism (IV)-Lemma) The identity is an isomorphism.

340.1.5. **Homomorphism (V): The non-singular form.** We start out by citing an earlier lemma:

Lemma 326.8. (Dual-π-Lemma) Let R be a ring with involution (in our applications we will take $R = \Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$), let π be a group and let V, W be two $(R, \mathbb{Z}[\pi])$-left left modules. Furthermore let Z be an R-bimodule. Let

$$\Theta \colon V \times W \;\to\; Z$$

be an R-sesquilinear form in the sense of Definition 326.6. We suppose that π respects the form, i.e. we suppose that for each $v \in V$, $w \in W$ and $g \in \pi$ we have $\Theta(g{\cdot}v, g{\cdot}w) = \Theta(v, w)$.
(1) The map[3690]

$$\begin{array}{rcl} \nu \colon V & \to & \operatorname*{Hom}_{R\text{-left}}(\overline{W}^{\mathbb{Z}[\pi]}, {}^{R}\overline{Z}) \\ v & \mapsto & w \mapsto \varphi(v, w) \end{array}$$

is a homomorphism of $(R, \mathbb{Z}[\pi])$-left left modules.
(2) If Θ is non-singular in the sense of Definition 326.7, then the map in (1) is an isomorphism.

We can now give the description of the homomorphism (V) from page 6440:

Notation 340.10. We consider the Λ-sesquilinear pairing

$$\begin{array}{rcl} \Theta \colon Q(\Lambda)/\Lambda \times \Lambda & \to & Q(\Lambda)/\Lambda \\ ([f], g) & \mapsto & [f \cdot \overline{g}] \end{array}$$

Note that the left-action by $\pi = \pi_1(X_L, x_0)$ respects the pairing.[3691] Thus we we obtain from the Dual-π-Lemma 326.8 a cochain map

$$\Theta_* \colon \operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_L), Q(\Lambda)/\Lambda) \;\to\; \operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}\big(\mathrm{C}_*(\widetilde{X}_L), \mathrm{Hom}_{\Lambda}(\overline{\Lambda}^{\mathbb{Z}[\pi]}, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})\big)$$

of Λ-cochain complexes and thus a homomorphism

$$\Theta_* \colon \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_L), Q(\Lambda)/\Lambda)\Big) \;\to\; \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}\big(\mathrm{C}_*(\widetilde{X}_L), \mathrm{Hom}_{\Lambda}(\overline{\Lambda}^{\mathbb{Z}[\pi]}, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})\big)\Big)$$

of Λ-modules.

[3690] Here we make use of Notation 326.4 and Convention 326.5.

We conclude this subsection with the following lemma:

Lemma 340.11. (Homomorphism (V)-Lemma) The homomorphism (V) from Notation 340.10 is an isomorphism.

Proof. By the Dual-π-Lemma 326.8 (2) we only need to show that the pairing

$$\begin{array}{rcl}\Theta\colon Q(\Lambda)/\Lambda \times \Lambda & \to & Q(\Lambda)/\Lambda \\ ([f],g) & \mapsto & [f\cdot\overline{g}]\end{array}$$

is non-singular in the sense of Definition 326.7. Recall that this means that we have to show that the maps

$$\begin{array}{rcl} Q(\Lambda)/\Lambda & \to & \mathrm{Hom}_\Lambda(\Lambda, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) \\ v & \mapsto & \begin{pmatrix} \Lambda & \to & {}^{\Lambda}\overline{Q(\Lambda)/\Lambda} \\ w & \mapsto & \varphi(v,w)\end{pmatrix}\end{array} \quad \text{and} \quad \begin{array}{rcl} \Lambda & \to & {}^{\Lambda}\overline{\mathrm{Hom}_\Lambda(Q(\Lambda)/\Lambda, Q(\Lambda)/\Lambda)} \\ w & \mapsto & \begin{pmatrix} Q(\Lambda)/\Lambda & \to & Q(\Lambda)/\Lambda \\ v & \mapsto & \varphi(v,w)\end{pmatrix}\end{array}$$

are isomorphisms of Λ-modules. In Exercise 340.4 we will see that this is indeed the case. ■

340.1.6. **Homomorphism (VI): Adjunction.** Before we turn to the description of the homomorphism (VI) it is advisable to consider a slightly more general setup. In fact it is worth citing the following lemma:

Lemma 312.13. (Hom–Tensor Adjunction Lemma) Let π be a group, let R be a ring and let W be an $(R,\mathbb{Z}[\pi])$-left left module. We denote by $\overline{W}^{\mathbb{Z}[\pi]}$ the corresponding $(R,\mathbb{Z}[\pi])$-bimodule. For any $\mathbb{Z}[\pi]$-left module C and any R-bimodule Z the maps[3692]

$$\begin{array}{rcl}\mathrm{Adj}\colon \underset{R\text{-left}}{\mathrm{Hom}}(\overline{W}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]} C, {}^{R}\overline{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}[\pi]\text{-left}}(C, \underset{R\text{-left}}{\mathrm{Hom}}(\overline{W}^{\mathbb{Z}[\pi]}, {}^{R}\overline{Z})) \\ (f\colon \overline{W}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]} C \to {}^{R}\overline{Z}) & \mapsto & \begin{pmatrix} C & \to & \underset{R\text{-left}}{\mathrm{Hom}}(\overline{W}^{\mathbb{Z}[\pi]}, {}^{R}\overline{Z}) \\ c & \mapsto & \begin{pmatrix} \overline{W}^{\mathbb{Z}[\pi]} & \to & {}^{R}\overline{Z} \\ w & \mapsto & f(w\otimes c)\end{pmatrix}\end{pmatrix}\end{array}$$

form an isomorphism of R-left modules.

Here is now the description of the homomorphism (VI) from page 6440:

Notation 340.12. We apply the Hom–Tensor Adjunction Lemma 312.13 to $R=\Lambda$, $W=\Lambda$ and $C_* = \mathrm{C}_*(\widetilde{X}_L)$ and we obtain a cochain map

$$\underset{\Lambda}{\mathrm{Hom}}(\overline{\Lambda}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) \xrightarrow{\mathrm{Adj}_*} \underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_*(\widetilde{X}_L), \underset{\Lambda}{\mathrm{Hom}}(\overline{\Lambda}^{\mathbb{Z}[\pi]}, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}))$$

of Λ-cochain complexes and thus a homomorphism

$$\mathrm{H}^1\Big(\underset{\Lambda}{\mathrm{Hom}}(\overline{\Lambda}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{X}_L), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})\Big) \xrightarrow{\mathrm{Adj}_*} \mathrm{H}^1\Big(\underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_*(\widetilde{X}_L), \underset{\Lambda}{\mathrm{Hom}}(\overline{\Lambda}^{\mathbb{Z}[\pi]}, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}))\Big)$$

of Λ-modules.

We conclude this subsection with the following lemma:

[3691] Note that according to Convention 320.9 the left-action of $g\in\pi=\pi_1(X_L,x_0)$ on $Q(\Lambda)/\Lambda$ and Λ is given by *right*-multiplication by $t^{-\Phi_K(g)}$. Thus we do not even need to use that Λ is abelian to see that the π-action respects the pairing.

[3692] Here and in the following discussion we again make use of Notation 326.4 and Convention 326.5.

Lemma 340.13. (Homomorphism (VI)-Lemma) The homomorphism (VI) from Notation 340.12 is an isomorphism.

Proof. As we discussed, this is just a special case of the Hom–Tensor Adjunction Lemma 312.13. ■

340.1.7. **Homomorphism (VII): The universal coefficient spectral sequence.** Again it is useful to first recall a more general lemma:

Lemma 315.33. (Evaluation Map Lemma) Let R be a ring. Let (D_*, ∂_*) be an R-left module chain complex and let Z be an (R,S)-left left module. The map

$$\begin{array}{rccl} \mathrm{ev}\colon & \overbrace{\mathrm{H}^n(\mathrm{Hom}_{R\text{-left}}(D_*, Z))}^{=\mathrm{H}^n(D_*;Z)} & \to & \mathrm{Hom}_{R\text{-left}}(\mathrm{H}_n(D_*), Z) \\ & [\varphi\colon D_n \to Z] & \mapsto & \begin{pmatrix} \mathrm{H}_n(D_*) & \to & Z \\ [d] & \mapsto & \varphi(d) \end{pmatrix} \end{array}$$

is a well-defined natural homomorphism of S-left modules. It is called the evaluation homomorphism.

Here is now the description of the homomorphism (VII) from page 6440:

Notation 340.14. We consider the ring $R = S = \Lambda$ and we consider the chain complex $D_* := \overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L)$ of Λ-modules. We use the involution on Λ to view $Q(\Lambda)/\Lambda$ as a (Λ,Λ)-left left module ${}^{\Lambda}\overline{Q(\Lambda)/\Lambda}$. We now consider the evaluation homomorphism

$$\mathrm{H}^1(\mathrm{Hom}_\Lambda(\underbrace{\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L)}_{=:D_*}, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_\Lambda(\underbrace{\mathrm{H}_1(\underbrace{\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L)}_{=:D_*})}_{=\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda)}, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})$$

of Λ-modules from the above Evaluation Map Lemma 315.33.

As usual we turn to the question whether the given homomorphism is an isomorphism. At turns out this time the situation is a little more complicated than in the previous homomorphisms:

Lemma 340.15. (Homomorphism (VII)-Lemma) Let $L \subseteq S^3$ be an oriented m-component link.

(1) We have an exact sequence of Λ-modules of the form[3693]

$$\begin{array}{rcl} 0 & \to & \mathrm{Ext}^2_\Lambda(\mathrm{H}_0^{\mathrm{tw}}(\mathrm{X}_L;\Lambda), {}^{\Lambda}\overline{\Lambda}) \to \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})) \\ & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(\mathrm{X}_L;\Lambda), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) \to \mathrm{Ext}^3_\Lambda(\mathrm{H}_0^{\mathrm{tw}}(\mathrm{X}_L;\Lambda), {}^{\Lambda}\overline{\Lambda}). \end{array}$$

(2) If $L = K$ is in fact a knot, then the evaluation map

$$\mathrm{ev}\colon \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_K), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})) \to \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(\mathrm{X}_K;\Lambda), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}))$$

is an isomorphism of Λ-modules.

For the proof of the Homomorphism (VII)-Lemma 340.15 we should recall the following theorem:

[3693] No, there is no "0" missing at the end.

Theorem 316.10. (Cohomological UCSS Theorem) Let R and S be two rings, let D_* be a chain complex of free R-left modules and let M be an (R,S)-left left module. There exists a natural cohomological first quadrant spectral sequence over the ring S with

$$\underbrace{\mathrm{Ext}_R^p(\mathrm{H}_q(D_*),M)}_{=E_2^{p,q}} \implies \mathrm{H}^*(D_*;M).$$

Furthermore, the edge homomorphism

$$\mathrm{H}^k(D_*;M) \twoheadrightarrow \underset{\text{see page 6103}}{\uparrow} \mathrm{Ext}_R^0(\mathrm{H}_k(D_*),M) =_{\mathrm{i}} \underset{\text{by the Ext}^0\text{–Lemma 315.38}}{\uparrow} \mathrm{Hom}_{R\text{-left}}(\mathrm{H}_k(D_*),M)$$

from page 6103 is given by the above evaluation map.

We will also need the following much less fancy input:

Lemma 340.16. (Ext-$Q(R)/R$-Lemma) Let R be a commutative domain with quotient field $Q(R)$ and let N be an R-module with $\mathrm{Hom}_R(N,Q(R))=0$. For every $k\in\mathbb{N}_0$ we have a natural isomorphism

$$\beta\colon \mathrm{Ext}_R^k(N;Q(R)/R) \xrightarrow{\cong} \mathrm{Ext}_R^{k+1}(N;R)$$

of R-modules.

Proof of the Ext-$Q(R)/R$-Lemma 340.16. In the following let R be a commutative domain and let N be an R-module with $\mathrm{Hom}_R(N,Q(R))=0$. We apply the Ext-Coefficient LES Lemma 315.40 to the short exact sequence $0\to R\xrightarrow{\varphi}Q(R)\xrightarrow{\psi}Q(R)/R\to 0$ of R-modules and we obtain a long exact sequence

$$\cdots\to \mathrm{Ext}_R^k(N,Q(R)) \xrightarrow{\psi_*} \mathrm{Ext}_R^k(N,Q(R)/R) \xrightarrow{\delta_k} \mathrm{Ext}_R^{k+1}(N,R) \xrightarrow{\varphi_*} \mathrm{Ext}_R^{k+1}(N,Q(R))\to\dots$$

of R-modules. Thus it remains to prove the following claim:

Claim. All Ext-modules $\mathrm{Ext}_R^k(N,Q(R))$ with $Q(R)$-coefficients vanish.

Proof. This argument has two parts:

- For $k=0$ this follows immediately from our hypothesis on N together with the Ext^0–Lemma 315.38.
- It follows from the Torsion Free+Divisible-implies-Injective Proposition 315.32 and the Ext-Properties Lemma 315.39 (3) that all the Ext-modules with $Q(R)$-coefficients of degree ≥ 1 vanish. ∎

Proof of the Homomorphism (VII)-Lemma 340.15 (1). Let $L\subseteq S^3$ be an oriented m-component link. We start out with the following claim:

Claim. We have an exact sequence of Λ-modules of the form

$$\begin{aligned} 0 &\to \mathrm{Ext}_\Lambda^1(\mathrm{H}_0^{\mathrm{tw}}(\mathrm{X}_L;\Lambda),{}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) \to \mathrm{H}_{\mathrm{tw}}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{X}_L),{}^{\Lambda}\overline{Q(\Lambda)/\Lambda})) \\ &\to \mathrm{Ext}_\Lambda^0(\mathrm{H}_1^{\mathrm{tw}}(\mathrm{X}_L;\Lambda),{}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) \to \mathrm{Ext}_\Lambda^2(\mathrm{H}_0^{\mathrm{tw}}(\mathrm{X}_L;\Lambda),{}^{\Lambda}\overline{Q(\Lambda)/\Lambda}). \end{aligned}$$

Proof. First note that it follows from the Cohomological UCSS Theorem 316.10, applied to the Λ-chain complex $D_*:=\overline{\Lambda}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{X}_L)$ and the Λ-module $Q(\Lambda)/\Lambda$, that there

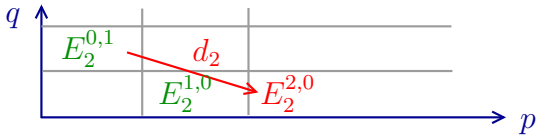
exists a natural cohomological first quadrant spectral sequence over the ring Λ with

$$\underbrace{\mathrm{Ext}^p_\Lambda(\overbrace{\mathrm{H}^{\mathrm{tw}}_q(\mathrm{X}_L;\Lambda)}^{=\mathrm{H}_q(D_*)}, {}^\Lambda\overline{Q(\Lambda)/\Lambda})}_{=E_2^{p,q}} \implies \mathrm{H}^*(D_*; {}^\Lambda\overline{Q(\Lambda)/\Lambda})$$

where the edge homomorphism

$$\mathrm{H}^k(D_*; {}^\Lambda\overline{Q(\Lambda)/\Lambda}) \twoheadrightarrow \mathrm{Ext}^0_\Lambda(\mathrm{H}_k(D_*), {}^\Lambda\overline{Q(\Lambda)/\Lambda}) =_{\mathrm{i}} \mathrm{Hom}_\Lambda(\mathrm{H}_k(D_*), {}^\Lambda\overline{Q(\Lambda)/\Lambda})$$

is given by the above evaluation map.



UCSS with $E_2^{p,q} = \mathrm{Ext}^p_\Lambda(\mathrm{H}^{\mathrm{tw}}_q(\mathrm{X}_L;\Lambda), {}^\Lambda\overline{Q(\Lambda)/\Lambda})$

Since we are only interested in the first cohomology we obtain from the Four Term-Sequence Lemma 316.8 that we have in fact an exact sequence of the form

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}.$$

These facts already give us the desired statement. ⊞

Next note that

$$\mathrm{Hom}_\Lambda(\mathrm{H}^{\mathrm{tw}}_0(X_L;\Lambda), Q(\Lambda)) \underset{\uparrow}{=} \mathrm{Hom}_\Lambda(\mathbb{Z}, Q(\Lambda)) = 0.$$

by the Twisted H_0-Proposition 323.3

It thus follows from the Ext-$Q(R)/R$-Lemma 340.16, applied to $N = \mathrm{H}^{\mathrm{tw}}_0(X_L,\Lambda)$, that for each $k \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{Ext}^k_\Lambda(\mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_L;\Lambda), Q(\Lambda)/\Lambda) \to \mathrm{Ext}^{k+1}_\Lambda(\mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_L;\Lambda),\Lambda)$$

of Λ-modules. Together with the above claim this gives us the desired conclusion. ■

Proof of the Homomorphism (VII)-Lemma 340.15 (2). Let $K \subseteq S^3$ be an oriented knot. By Statement (1) we have an exact sequence of Λ-modules of the form

$$\begin{aligned} 0 \to\ & \mathrm{Ext}^2_\Lambda(\mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_K;\Lambda), {}^\Lambda\overline{\Lambda}) \to \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_K), {}^\Lambda\overline{Q(\Lambda)/\Lambda})) \\ \xrightarrow{\mathrm{ev}}\ & \mathrm{Hom}_\Lambda(\mathrm{H}^{\mathrm{tw}}_1(\mathrm{X}_K;\Lambda), {}^\Lambda\overline{Q(\Lambda)/\Lambda}) \to \mathrm{Ext}^3_\Lambda(\mathrm{H}_0(\mathrm{X}_K;\Lambda), {}^\Lambda\overline{\Lambda}). \end{aligned}$$

Thus it remains to prove the following claim:

Claim. For any $k \geq 2$ and any Λ-module M we have

$$\mathrm{Ext}^k_\Lambda(\mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_K;\Lambda), M) = 0.$$

Proof. Note that with $\Lambda = \mathbb{Z}[t^{\pm 1}]$ we have

$$\mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_K;\Lambda) \underset{\uparrow}{\cong} (\Lambda)_\pi \underset{\uparrow}{=} \Lambda/\{(t^k-1)\cdot\lambda \,|\, k \in \mathbb{Z}, \lambda \in \Lambda\} \underset{\uparrow}{\cong} \Lambda/(t-1)\cdot\Lambda.$$

by the Twisted H_0-Proposition 323.3 — by Definition 323.1 — since $(t-1)|(t^k-1)$

This discussion shows that we have a free resolution

$$0 \to \Lambda \xrightarrow{\cdot(t-1)} \Lambda \to \mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_K;\Lambda) \to 0.$$

It now follows immediately from Definition 315.35 that for any $k \geq 2$ and any Λ-module M we have

$$\mathrm{Ext}^k_\Lambda(\mathrm{H}^{\mathrm{tw}}_0(\mathrm{X}_K;\Lambda), M) \;=\; 0.$$ ■

340.1.8. **Homomorphism (VIII).** Here is the description of the homomorphism (VIII) from page 6440:

Notation 340.17. We consider the Λ-bimodule $Q(\Lambda)/\Lambda$. Note that the usual involution on Λ induces an involution on $Q(\Lambda)$ and also on $Q(\Lambda)/\Lambda$. For any Λ-module M we clearly have an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_\Lambda(M, {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) & \to & {}^{\Lambda}\overline{\mathrm{Hom}_\Lambda(M, Q(\Lambda)/\Lambda)} \\ (\varphi\colon M \to Q(\Lambda)/\Lambda) & \mapsto & \begin{pmatrix} M & \to & Q(\Lambda)/\Lambda \\ m & \mapsto & \overline{\varphi(m)} \end{pmatrix} \end{array}$$

of Λ-modules.

For completeness sake we state the obvious:

Lemma 340.18. (Homomorphism (VIII)-Lemma) The homomorphism (VIII) from the above notation is an isomorphism.

Proof. This lemma is essentially obvious. ■

340.1.9. **Homomorphism (IX): The Bockstein homomorphism.** Here is the description of the homomorphism (IX) from page 6440:

Notation 340.19. Let $L \subseteq S^3$ be an oriented m-component link. The homomorphism (IX) is induced by the composition of the following two homomorphisms of Λ-modules:

$$\mathrm{Hom}_\Lambda(\mathrm{H}^{\mathrm{tw}}_1(X_L;\Lambda), Q(\Lambda)/\Lambda) \xleftarrow[\uparrow]{\cong} \mathrm{Ext}^0_\Lambda(\mathrm{H}^{\mathrm{tw}}_1(X_L;\Lambda),\Lambda) \xrightarrow[\uparrow]{\beta_1} \mathrm{Ext}^1_\Lambda(\mathrm{H}^{\mathrm{tw}}_1(X_L;\Lambda), Q(\Lambda)/\Lambda).$$

by the Ext^0–Lemma 315.38 — by the Ext-Coefficient LES Lemma 315.40

Lemma 340.20. (Homomorphism (IX)-Lemma) Let $L \subseteq S^3$ be an oriented m-component link. If $\Delta_L(t_1,\dots,t_m) \neq 0$, then the homomorphism (IX) from the above notation is an isomorphism.

Proof. First note that it follows from the Ext^0–Lemma 315.38 that the homomorphism to the left is an isomorphism. Secondly note that it follows immediately from our hypothesis on Δ_L, the Alexander Module Torsion Lemma 339.16 and the Ext-$Q(R)/R$-Lemma 340.16 that the homomorphism to the right is an isomorphism. ■

340.2. **Alexander polynomials are symmetric.** In this section we will use the results from the previous Section 340.1 to prove the following result:

Theorem 340.21. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link $L \subseteq S^3$ we have

$$\Delta_L(t_1,\dots,t_m) \;\doteq\; \Delta_L(t_1^{-1},\dots,t_m^{-1}).$$

Remark.

(1) For fibered links we proved a closely related statement in Exercise **??**.

(2) In the Alexander Polynomial–Symmetry Theorem 117.7 we also showed that the Alexander polynomial of an oriented link is symmetric. But at that point we used a different definition of the Alexander polynomial, and only in the Alexander Polynomials Agree Proposition **??** will we see that the different definitions of the Alexander polynomial match. □

The proof of the Alexander Polynomial–Symmetry Theorem 340.21 requires two results on orders of modules. The first one is elementary, the second one is significantly more subtle:

Lemma 340.22. (Order–Involution Lemma) Let H be a free abelian group and let M be a finitely generated $\mathbb{Z}[H]$-module. Then

$$\mathrm{ord}_{\mathbb{Z}[H]}\left(\overline{M}^{\mathbb{Z}[H]}\right) \;=\; \overline{\mathrm{ord}_{\mathbb{Z}[H]}(M)}.$$

Proof. This statement follows easily from the Order–Functorial Lemma 287.19 applied to the ring isomorphism $\mathbb{Z}[H] \to \mathbb{Z}[H]$ that is given by the usual involution. ■

We will also need the following more fancy result:

Proposition 340.23. (Order-Ext Proposition) Let R be a Noetherian UFD and let M be a finitely generated R-module which is R-torsion. Then for every $q \in \mathbb{N}_0$ we have

$$\mathrm{ord}_R(\mathrm{Ext}^q_R(M,R)) \;\doteq\; \begin{cases} \mathrm{ord}_R(M), & \text{if } q = 1, \\ 1, & \text{if } q \neq 1. \end{cases}$$

Sketch of proof of Order-Ext Proposition 340.23. We start out with a special case:

Claim. The statement of the proposition holds for any PID T.

Proof. The claim follows easily from the PID–Module-Classification Theorem 309.36, the Cyclic-Ext-Lemma 315.42 and the PID-Higher Ext Lemma 315.41. ⊞

Now let R be any Noetherian UFD. We denote by $\mathcal{P} \subseteq R$ the set of prime elements of R. Given $p \in \mathcal{P}$ we consider the corresponding prime ideal $(p) \subseteq R$ and we consider the localization $R_{(p)} := R(R \setminus (p))^{-1} = \{\frac{a}{b} \,|\, b \notin (p)\} = \{\frac{a}{b} \,|\, a, b \in R \text{ with } p \not| b\}$. This is called the local ring of R at (p). The proposition follows from the following observations:

(a) Since we localize along a prime ideal that is generated by a prime element we see that the local ring $R_{(p)}$ is in fact a discrete valuation ring (DVR) and in particular a PID.
(b) By the Tensor-with-Localization Proposition 314.18 we know that for any multiplicative subset $S \subseteq R \setminus \{0\}$ that contains 1 the corresponding localization RS^{-1} is an exact R-module.
(c) It follows quite easily from (b) that $\mathrm{Ext}^q_R(RS^{-1} \otimes_R M, R) \cong RS^{-1} \otimes_R \mathrm{Ext}^q_R(M,R)$.
(d) Given a ring T and given $f, g \in T$ we write (similar to Notation 339.4) $f \doteq_T g$ if there exists a unit $\epsilon \in T$ such that $f = \epsilon \cdot g$.
(e) Given a finitely generated R-module N and given a multiplicatively closed subset $S \subseteq R$ as above we obtain easily from (b) together with the Determinant–Ring Homomorphism Lemma 2.22 that $\mathrm{ord}_{RS^{-1}}(RS^{-1} \otimes_R N) \doteq_{RS^{-1}} \mathrm{ord}_R(N)$.
(f) Since we are dealing with a UFD we see that for $f, g \in R$ we have $f \doteq_{R_p} g$ for all $p \in \mathcal{P}$ if and only if $f \doteq_R g$.

We leave it to the reader to fill in the details for all of the above statements. We refer to [**DF2004**, Chapters 15 and 16] for more information on localizations of rings and modules that can help with filling in the details. ■

Proof of the Alexander Polynomial–Symmetry Theorem 340.21. Let $L \subseteq S^3$ be an oriented m-component link. We need to show that

$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$

If $\Delta_L(t_1, \dots, t_m) = 0$, then the statement is trivial. So henceforth we consider the case that $\Delta_L(t_1, \dots, t_m) \neq 0$. As on page 6440 we consider the following sequence of homomorphisms of Λ-modules:

$$\begin{aligned}
\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda) &\xrightarrow[\text{(I)}]{\text{я}} \mathrm{H}_1^{\mathrm{tw}}(X_L, \partial X_L;\Lambda)\\
&\xrightarrow[\text{(II)}]{\mathrm{PD}} \mathrm{H}^2_{\mathrm{tw}}(X_L;\Lambda)\\
&\xleftarrow[\text{(III)}]{\beta} \mathrm{H}^1_{\mathrm{tw}}(X_L; Q(\Lambda)/\Lambda)\\
&\xrightarrow[\text{(IV)}]{\mathrm{id}} \mathrm{H}^1_{\mathrm{tw}}\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_L); Q(\Lambda)/\Lambda)\Big)\\
&\xrightarrow[\text{(V)}]{\Theta_*} \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_L), \mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]}, \overline{Q(\Lambda)/\Lambda})\Big)\\
&\xleftarrow[\text{(VI)}]{\mathrm{Adj}_*} \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L), \overline{Q(\Lambda)/\Lambda}))\\
&\xrightarrow[\text{(VII)}]{\mathrm{ev}} \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda), \overline{Q(\Lambda)/\Lambda})\\
&\xrightarrow[\text{(VIII)}]{\cong} \overline{\mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda), Q(\Lambda)/\Lambda)}\\
&\xrightarrow[\text{(IX)}]{\beta} \overline{\mathrm{Ext}^1_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda), \Lambda)}.
\end{aligned}$$

Note that it follows from the Homomorphism (II)-Lemma 340.5, the Homomorphism (III)-Lemma 340.7, the Homomorphism (IV)-Lemma 340.9, the Homomorphism (V)-Lemma 340.11, the Homomorphism (VI)-Lemma 340.13, the Homomorphism (VIII)-Lemma 340.18 and the Homomorphism (IX)-Lemma 340.20 that the homomorphisms (II)-(VI), (VIII) and (IX) (which are also marked in red) are isomorphisms. These evidently show in particular that the corresponding Λ-modules have the same orders.

In fact, if L is a knot, then the Homomorphism (I)-Lemma 340.3 and the Homomorphism (VII)-Lemma 340.15 tell us that the remaining two homomorphisms are also isomorphisms. Thus for knots the following claim gives us already the desired result:

Claim 1. We have

$$\overline{\mathrm{ord}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda))} \doteq \mathrm{ord}_\Lambda(\overline{\mathrm{Ext}^1_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda), \Lambda)}).$$

Proof. This claim follows immediately from the Order–Involution Lemma 340.22 together with the Order-Ext Proposition 340.23. ⊞

The above discussion shows that we have already achieved our goal for knots. For links we have to work a little harder, since the above homomorphisms (I) and (VII) are not necessarily isomorphisms. But the following two claims show that at least the orders are the same. The following claim takes care of homomorphism (I):

Claim 2. If $m \geq 2$, then we have $\mathrm{ord}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda)) = \mathrm{ord}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_L, \partial X_L;\Lambda))$.

Proof. We denote by $T_1, \dots, T_m$ the boundary tori of X_L. As in the Homomorphism (I)-Lemma 340.3 we consider the following sequence:

$$\begin{array}{ccccccc} \dots & \to & \mathrm{H}_2^{\mathrm{tw}}(X_L;\Lambda) & \to & \mathrm{H}_2^{\mathrm{tw}}(X_L,\partial X_L;\Lambda) & \to \\ \to \bigoplus_{i=1}^{m} \mathrm{H}_1^{\mathrm{tw}}(T_i;\Lambda) & \to & \mathrm{H}_1^{\mathrm{tw}}(X_L;\Lambda) & \to & \mathrm{H}_1^{\mathrm{tw}}(X_L,\partial X_L;\Lambda) & \to \\ \to \bigoplus_{i=1}^{m} \mathrm{H}_0^{\mathrm{tw}}(T_i;\Lambda) & \to & \dots & & & \end{array}$$

We make the following observations:

- It follows from the Higher Alexander Module Proposition 339.19 that $\mathrm{H}_2^{\mathrm{tw}}(X_L;\Lambda) = 0$.
- It follows from a fairly elementary calculation, say using the Torus–Twisted Cellular Chain Complex Lemma 329.11, that for every $i \in \{1,\dots,m\}$ we have the equality $\mathrm{ord}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(T_i;\Lambda)) = \mathrm{ord}_\Lambda(\mathrm{H}_0^{\mathrm{tw}}(T_i;\Lambda))$. We refer to the reader's solution of Exercise 339.6 for more details.
- Using the Twisted Poincaré Duality Theorem 335.1 and the Cohomological UCSS Theorem 316.10 one can show that $\mathrm{ord}_\Lambda(\mathrm{H}_2^{\mathrm{tw}}(X_L,\partial X_L;\Lambda)) = 1$. The details will be worked out in Exercise 340.1.

The desired statement now follows from the Order–Exactness Proposition 287.21. ⊞

Finally the following claim takes care of homomorphism (VII):

Claim 3. We have

$$\mathrm{ord}_\Lambda(\mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})) = \mathrm{ord}_\Lambda(\mathrm{H}^1_{\mathrm{tw}}(X_L; {}^{\Lambda}\overline{Q(\Lambda)/\Lambda})).$$

Proof. By the Homomorphism (VII)-Lemma 340.15 we have an exact sequence of Λ-modules of the form

$$\begin{array}{rcl} 0 \to \mathrm{Ext}^2_\Lambda(\mathrm{H}_0^{\mathrm{tw}}(\mathrm{X}_L;\Lambda), {}^{\Lambda}\overline{\Lambda}) & \to & \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_L), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) \\ \xrightarrow{\mathrm{ev}} \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(\mathrm{X}_L;\Lambda), {}^{\Lambda}\overline{Q(\Lambda)/\Lambda}) & \to & \mathrm{Ext}^3_\Lambda(\mathrm{H}_0^{\mathrm{tw}}(\mathrm{X}_L;\Lambda), {}^{\Lambda}\overline{\Lambda}). \end{array}$$

Note that it follows from the Order-Ext Proposition 340.23 that the orders of the first and the last Ext-term are equal to 1. The desired equality is now a straightforward consequence of the Order–Exactness Proposition 287.21. ■

340.3. **Twisted linking pairings.** In this section we will introduce the rather general notion of a twisted linking pairing. In the next Section 340.4 we will then specialize to the setting of knots.

The following definition is a variation on the definition of a linking form in **??**:

Definition 340.24. Let $n \in \mathbb{N}_0$. Let X be a compact oriented connected non-empty $(2n+1)$-dimensional topological manifold. Furthermore let R be a ring with involution and let $\varphi\colon \pi \to \mathrm{GL}(1,R)$ be a representation such that for each $g \in \pi_1(X)$ we have $\varphi(g^{-1}) = \overline{\varphi(g)}$. Furthermore let $\psi\colon R \to S$ be a monomorphism of rings such that $\mathrm{H}_i^{\mathrm{tw}}(X; {}_{\psi\circ\varphi}S) = 0$ for $i = n, n+1$. Note that the map

$$\begin{array}{rcl} \Theta\colon S/R \otimes_{\mathbb{Z}} R & \to & S/R \\ ([v], w) & \mapsto & [v \cdot \overline{w}] \end{array}$$

is preserved by the π-action. We consider the Bockstein homomorphism

$$\beta\colon \mathrm{H}^n_{\mathrm{tw}}(X;R) \to \mathrm{H}^{n+1}_{\mathrm{tw}}(X;S/R)$$

that we introduced in the Twisted Bockstein Lemma 321.15. It follows from $\mathrm{H}^{\mathrm{tw}}_i(X;S)=0$ for $i=n,n+1$ that this map is an isomorphism. We refer to the map[3694]

$$\begin{aligned}\lambda^{\varphi}_X\colon \mathrm{H}^{\mathrm{tw}}_n(X;{}_{\varphi}R)\times \mathrm{H}^{\mathrm{tw}}_n(X;{}_{\varphi}R) &\to S/R\\ (v,w) &\mapsto \epsilon_X\Big(\Theta_*\Big(\underbrace{\underbrace{\beta^{-1}(\mathrm{PD}_X(v))}_{\in \mathrm{H}^n_{\mathrm{tw}}(X,\partial X;{}_{\varphi}(S/R))} \cup_{\otimes} \underbrace{\mathrm{PD}_X(w)}_{\in \mathrm{H}^{n+1}_{\mathrm{tw}}(X,\partial X;{}_{\varphi}R)}}_{\in \mathrm{H}^n_{\mathrm{tw}}(X,\partial X;{}_{\varphi}S/R\otimes_{\mathbb{Z}}{}_{\varphi}R)}\Big) \cap_{\mathbb{Z}} \underbrace{[X]}_{\in \mathrm{H}_n(X,\partial X)}\Big).\end{aligned}$$

$\in \mathrm{H}^n_{\mathrm{tw}}(X,\partial X;{}_{\mathrm{tr}}(S/R))$

as a twisted linking pairing λ_X of (X,φ).

Example. Let $n\in\mathbb{N}$ and let X be an $(2n+1)$-dimensional rational homology sphere. Furthermore let $R=\mathbb{Z}$, $S=\mathbb{Q}$ and let $\varphi\colon \pi_1(Y)\to \mathrm{GL}(1,\mathbb{Z})$ be the trivial representation. It follows quite easily from the definitions (and various results along the lines of the Twisted–Trivial Coefficients Lemma 320.13 and the Classical-Twisted Cup Product Lemma 334.18) that in this setting we recover the linking form $\lambda_X\colon \mathrm{H}_n(X)\times \mathrm{H}_n(X)\to \mathbb{Q}/\mathbb{Z}$ from **??**. □

The following proposition is a generalization of the Linking Pairing-Symmetry Proposition **??** and it is an analog of the Twisted Intersection Pairing–Non-Singularity Proposition 337.14 (2):

Proposition 340.25. (Twisted Linking Pairing-Hermitian Proposition) In the following let $n\in\mathbb{N}_0$. Let X be a compact oriented connected non-empty $(2n+1)$-dimensional topological manifold. Furthermore let R be a ring with involution and let $\varphi\colon \pi\to \mathrm{GL}(1,R)$ be a representation such that for each $g\in\pi_1(X)$ we have $\varphi(g^{-1})=\overline{\varphi(g)}$. Finally let $\psi\colon R\to S$ be a monomorphism of rings such that $\mathrm{H}^{\mathrm{tw}}_i(X;{}_{\psi\circ\varphi}S)=0$ for $i=n,n+1$. The twisted linking pairing

$$\lambda^{\varphi}_X\colon \mathrm{H}^{\mathrm{tw}}_n(X;{}_{\varphi}R)\times \mathrm{H}^{\mathrm{tw}}_n(X;{}_{\varphi}R) \to S/R$$

is sesquilinear and hermitian.

Proof. It follows fairly easily from the definitions, as in the Twisted Intersection Pairing Proposition 337.12, that the twisted linking pairing is R-sesquilinear. The fact that the twisted linking pairing is in fact hermitian is much more subtle. This follows from combining the ideas of the proofs of the Linking Pairing-Symmetry Proposition **??** and the Twisted Intersection Pairing–Non-Singularity Proposition 337.14 (2). We have no intention of working out the details. This might make a frightfully amusing bachelor thesis. ■

340.4. **The Blanchfield pairing of knots.** We now return to knot theory. Thus let $K\subseteq S^3$ be an oriented knot with corresponding homomorphism $\Phi_K\colon \pi_1(X_K)\to\langle t\rangle$. In Definition 339.11 we introduced the corresponding Alexander module $\mathrm{H}_1(X_K;{}_{\Phi_K}\mathbb{Z}[t^{\pm1}])$.

[3694]Here $\epsilon_X\colon \mathrm{H}^{\mathrm{tw}}_0(X;M)\to M_\pi$ is the augmentation homomorphism from the Twisted H_0-Proposition 323.3. Note that for the trivial π-action (which is what we are dealing with in our context), we have $M_\pi=M$, thus get an augmentation map $\epsilon_X\colon \mathrm{H}_0(X;{}_{\mathrm{tr}}M)\to M$.

Now we can use the machinery from the last section to equip the Alexander module with an extra structure:

Definition 340.26. Let $K \subseteq S^3$ be an oriented knot with corresponding homomorphism $\Phi_K \colon \pi_1(X_K) \to \langle t \rangle$. We consider the inclusion $\mathbb{Z}[t^{\pm 1}] \to \mathbb{Q}(t)$. We also consider the hermitian pairing

$$\begin{array}{rcl} \Theta \colon \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \times \mathbb{Z}[t^{\pm 1}] & \to & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ ([f], g) & \mapsto & [f \cdot \overline{g}]. \end{array}$$

By the Alexander Module Torsion Lemma 339.16 and the Knot Alexander Module Proposition 339.26 we know that $\mathrm{H}_*^{\mathrm{tw}}(X_K; \mathbb{Q}(t)) = 0$. It thus follows from Definition 340.24 that we obtain the hermitian pairing

$$\mathrm{Bl}_K \colon \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

which is called the Blanchfield pairing of K.[3695]

In the following we collect some more or less basic statements about the Blanchfield pairing:

Proposition 340.27. (Blanchfield Pairing–Hermitian Proposition) For any oriented knot $K \subseteq S^3$ the Blanchfield pairing

$$\mathrm{Bl}_K \colon \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

is sesquilinear and Hermitian over $\mathbb{Z}[t^{\pm 1}]$.

Proof. This proposition is just a special case of the Twisted Linking Pairing-Hermitian Proposition 340.25. ■

Next we continue with an analog of the Linking Pairing–Nonsingular Proposition **??**:

Proposition 340.28. (Blanchfield Pairing Non-Singular Proposition) For any oriented knot $K \subseteq S^3$ the Blanchfield pairing

$$\mathrm{Bl}_K \colon \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(X_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

is non-singular.

Proof. Let $K \subseteq S^3$ be an oriented knot. It follows from Definition 326.7 and the Blanchfield Pairing–Hermitian Proposition 340.27 that it suffices to prove the following statement:

Statement. The map

$$\begin{array}{rcl} \mathrm{H}_1^{\mathrm{tw}}(X_K; \Lambda) & \to & \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X; \Lambda), \overline{Q(\Lambda)/\Lambda}) \\ a & \mapsto & \mathrm{Bl}_K(a, b) \end{array}$$

is an isomorphism of Λ-modules.

The idea is to rewrite the homomorphism using the many homomorphisms that we have encountered this far. More precisely, as on page 6440 we consider the following sequence

[3695] The pairing was first introduced, using somewhat different language, by Richard Blanchfield [**Bla1957**].

of homomorphisms of Λ-modules:

$$\begin{aligned}
\mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda) &\xrightarrow[\text{(I)}]{\text{ я}} \mathrm{H}_1^{\mathrm{tw}}(X_K,\partial X_K;\Lambda)\\
&\xrightarrow[\text{(II)}]{\mathrm{PD}} \mathrm{H}^2_{\mathrm{tw}}(X_K;\Lambda)\\
&\xleftarrow[\text{(III)}]{\beta} \mathrm{H}^1_{\mathrm{tw}}(X_K;Q(\Lambda)/\Lambda)\\
&\xrightarrow[\text{(IV)}]{\mathrm{id}} \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_K);Q(\Lambda)/\Lambda)\Big)\\
&\xrightarrow[\text{(V)}]{\Theta_*} \mathrm{H}^1\Big(\operatorname*{Hom}_{\mathbb{Z}[\pi]\text{-left}}(\mathrm{C}_*(\widetilde{X}_K),\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]},\overline{Q(\Lambda)/\Lambda})\Big)\\
&\xleftarrow[\text{(VI)}]{\mathrm{Adj}_*} \mathrm{H}^1(\mathrm{Hom}_\Lambda(\overline{\Lambda}^{\mathbb{Z}[\pi]}\otimes_{\mathbb{Z}[\pi]}\mathrm{C}_*(\widetilde{X}_K),\overline{Q(\Lambda)/\Lambda}))\\
&\xrightarrow[\text{(VII)}]{\mathrm{ev}} \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda),\overline{Q(\Lambda)/\Lambda}).
\end{aligned}$$

Note that it follows from the Homomorphism (I)-Lemma 340.3 (here we use that K is a knot!), the Homomorphism (II)-Lemma 340.5, the Homomorphism (III)-Lemma 340.7, the Homomorphism (IV)-Lemma 340.9, the Homomorphism (V)-Lemma 340.11, the Homomorphism (VI)-Lemma 340.13 and the Homomorphism (VII)-Lemma 340.15 that all these homomorphisms are in fact isomorphisms. In summary we have constructed an explicit isomorphism

$$\Theta\colon \mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda) \;\to\; \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda),\overline{Q(\Lambda)/\Lambda}).$$

By the above discussion it suffices to prove the following claim:

Claim. The map

$$\begin{aligned}
\mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda) &\;\to\; \mathrm{Hom}_\Lambda(\mathrm{H}_1^{\mathrm{tw}}(X_K;\Lambda),\overline{Q(\Lambda)/\Lambda})\\
a &\;\mapsto\; \mathrm{Bl}_K(a,b)
\end{aligned}$$

is precisely the above isomorphism Θ.

Proof. The statement of this claim is quite similar to the the Linking Pairing–Reformulation Lemma **??**. In fact the proof of the claim is a quite similar to the proof of the Linking Pairing–Reformulation Lemma **??**, together with the Twisted Cup-Cap Lemma 334.37. Once again this author does not feel like working out the details. ■

Next let us recall how one can calculate the Alexander module of a knot $K\subseteq S^3$: Let A be a Seifert matrix for K of size k. In the Alexander Module–Seifert Matrix Theorem 289.12 we showed that $t\cdot A - A^{\mathrm{T}}\in \mathrm{M}(k\times k,\mathbb{Z}[t^{\pm1}])$ is a presentation matrix for the Alexander module $\mathrm{H}_1^{\mathrm{tw}}(S^3\setminus K;\mathbb{Z}[t^{\pm1}])\cong \mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm1}])$, i.e. we have an explicit isomorphism

$$\mathbb{Z}[t^{\pm1}]^k/(t\cdot A - A^{\mathrm{T}})\cdot\mathbb{Z}[t^{\pm1}]^k \;\xrightarrow{\cong}\; \mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm1}])$$

of $\mathbb{Z}[t^{\pm1}]$-modules. The following theorem now gives us a strengthened version of this result:

Theorem 340.29. (Blanchfield Pairing–Seifert Matrix Theorem) Let $K\subseteq S^3$ be an oriented knot and let A be a Seifert matrix for K of size k. The Blanchfield pairing

$$\mathrm{Bl}_K\colon \mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm1}])\times \mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm1}]) \;\to\; \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]$$

is isometric to the pairing

$$\begin{aligned}
\mathbb{Z}[t^{\pm1}]^k/(tA-A^{\mathrm{T}})\cdot\mathbb{Z}[t^{\pm1}]^k\times\mathbb{Z}[t^{\pm1}]^k/(tA-A^{\mathrm{T}})\cdot\mathbb{Z}[t^{\pm1}]^k &\;\to\; \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]\\
([v],[w]) &\;\mapsto\; v^{\mathrm{T}}\cdot(t-1)\cdot(A-tA^{\mathrm{T}})^{-1}\cdot\overline{w}.
\end{aligned}$$

Remark. First note that given any integral $(k \times k)$-matrix A with $\det(t \cdot A - A^{\mathrm{T}}) \neq 0$ we can consider the pairing

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm 1}]^k/(tA-A^{\mathrm{T}}) \cdot \mathbb{Z}[t^{\pm 1}]^k \times \mathbb{Z}[t^{\pm 1}]^k/(tA-A^{\mathrm{T}}) \cdot \mathbb{Z}[t^{\pm 1}]^k & \xrightarrow{\lambda(A)} & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ ([v],[w]) & \mapsto & v^{\mathrm{T}} \cdot (t-1) \cdot (A - tA^{\mathrm{T}})^{-1} \cdot \overline{w}. \end{array}$$

Next recall that in Definition 291.34 we introduced the notion of what it means for two square integral matrices to be S-equivalent. As we discussed in Exercise 291.4, it is rather elementary to show that if two such matrices A and B are S-equivalent, then the corresponding pairings $\lambda(A)$ and $\lambda(B)$ are isometric. More interestingly Hale Trotter [**Tro1973**, p. 179] showed the converse, namely given two such A and B the pairings $\lambda(A)$ and $\lambda(B)$ are isometric if and only if the matrices are S-equivalent. Thus the Blanchfield form of a knot K contains the same information as the S-equivalence class of a Seifert matrix. □

Proof. This theorem was first proved by Cherry Kearton [**Kea1975**, p. 159]. A detailed proof is given in [**FriP2017**, Theorem 1.3]. A generalization to links is given in [**CFT2018**, Theorem 1.2]. ■

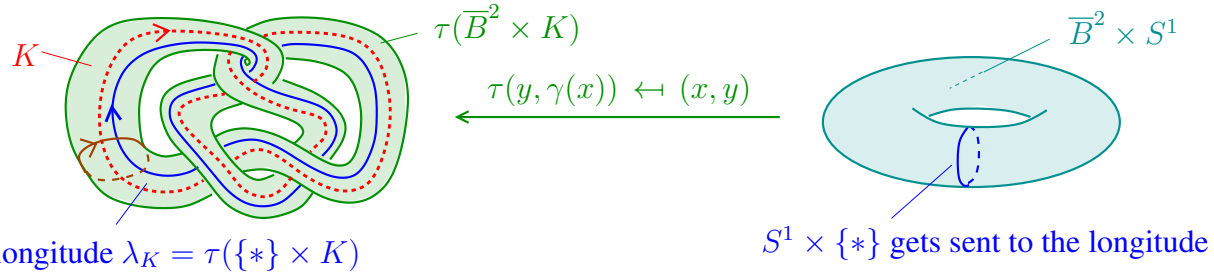
340.5. **The Blanchfield pairing and intersection forms.** In the Linking Pairing-via-Intersection Form Proposition **??** we saw that linking forms on rational homology spheres are related to intersection forms of bounding 4-dimensional topological manifolds. In this section we want to formulate a corresponding result for the Blanchfield pairing.

But the reader will surely have spotted that such a result does not make sense for the Blanchfield pairing of knots: The Blanchfield pairing is defined for knot exteriors, which have boundary, and which thus cannot be the boundary of a compact 4-dimensional topological manifolds. To deal with this issue we now introduce the 0-Dehn surgery along a knot K:

Definition 340.30. Let $K \subseteq S^3$. We pick a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ with self-linking number zero (i.e. such that $\mathrm{lk}(\tau(\{*\} \times K, K)) = 0$) and we pick a diffeomorphism $\gamma\colon S^1 \to K$. We consider the closed smooth manifold

$$N_K := S^3_0(K) := \overbrace{(S^3 \setminus \tau(B^2 \times K))}^{\text{knot exterior } X_K} \cup_{\substack{\partial X_K \leftarrow S^1 \times S^1 \\ \tau(y,\gamma(x)) \leftarrow (x,y)}} (\overline{B}^2 \times S^1).$$

In the language of **??** this is the 0-Dehn surgery along K.



The following proposition shows that the 0-Dehn surgery N_K along K is a "closed lookalike" of the knot exterior X_K:

Proposition 340.31. (0-Dehn Surgery–Blanchfield Pairing Proposition) Let K be an oriented knot. We pick a tubular map $\tau\colon \overline{B}^2 \times K \to S^3$ of self-linking number 0 which we use to define the knot exterior X_K and the 0-Dehn surgery N_K. The following statements hold:

(a) The inclusion induced map $\mathrm{H}_1(X_K) \to \mathrm{H}_1(N_K)$ is an isomorphism. This implies in particular that the homomorphism $\Phi_K\colon \mathrm{H}_1(X_K) \to \mathbb{Z}$ extends to $\mathrm{H}_1(N_K)$.

(b) The inclusion induced map $\mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm 1}]) \to \mathrm{H}_1^{\mathrm{tw}}(N_K;\mathbb{Z}[t^{\pm 1}])$ is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules.

(c) The inclusion induced map $\mathrm{H}_1^{\mathrm{tw}}(X_K;\mathbb{Z}[t^{\pm 1}]) \to \mathrm{H}_1^{\mathrm{tw}}(N_K;\mathbb{Z}[t^{\pm 1}])$ induces an isomorphism of twisted linking pairings, i.e. the Blanchfield pairing on K is isometric to the twisted linking pairing

$$\lambda_{N_K}\colon \mathrm{H}_1^{\mathrm{tw}}(N_K;\mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(N_K;\mathbb{Z}[t^{\pm 1}]) \ \to\ \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

Sketch of proof. The first two statements were hopefully proved in Exercise 339.16. The last statement follows, similar to the proof of the Intersection Form-Submanifold Proposition 225.9, fairly easily from (2) and the naturality of the cup product. All details are left to the reader. ■

We conclude this discussion of the Blanchfield pairing with the following proposition, which is the promised analog of the Linking Pairing-via-Intersection Form Proposition **??**:

Proposition 340.32. (Blanchfield Pairing-via-Intersection Form Proposition) Let $K \subseteq S^3$ be an oriented knot. As in Definition 340.30 we denote by N_K the 0-Dehn surgery on K. Next let W be a compact oriented connected 4-dimensional topological manifold with $\partial W = N_K$ such that $\pi_1(W) \cong \mathbb{Z}$ and such that the inclusion induced map $\mathrm{H}_1(N_K) \to \mathrm{H}_1(W)$ is an isomorphism. The following statements hold:

(1) The homology module $\mathrm{H}_2^{\mathrm{tw}}(W;\mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$-module.

(2) Let $V \in \mathrm{M}(n \times n, \mathbb{Z}[t^{\pm 1}])$ be a matrix that represents the equivariant intersection form

$$Q_W^{\mathbb{Z}[t^{\pm 1}]}\colon \mathrm{H}_2^{\mathrm{tw}}(W;\mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_2^{\mathrm{tw}}(W;\mathbb{Z}[t^{\pm 1}]) \ \to\ \mathbb{Z}[t^{\pm 1}]$$

from Definition 337.11. Then the twisted linking pairing

$$\mathrm{H}_1^{\mathrm{tw}}(N_K;\mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(N_K;\mathbb{Z}[t^{\pm 1}]) \ \to\ \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

is isometric to the pairing

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm 1}]^n/V \cdot \mathbb{Z}[t^{\pm 1}]^n \times \mathbb{Z}[t^{\pm 1}]^n/V \cdot \mathbb{Z}[t^{\pm 1}]^n & \to & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (v, w) & \mapsto & v^{\mathrm{T}} \cdot V^{-1} \cdot \overline{w}. \end{array}$$

Remark. The Blanchfield Pairing-via-Intersection Form Proposition 340.32, together with the 0-Dehn Surgery–Blanchfield Pairing Proposition) 340.31 and work of Ki Hyoung Ko [**Ko1989**] can be used to give an alternative proof of the Blanchfield Pairing–Seifert Matrix Theorem 340.29. □

Proof. Let $K \subseteq S^3$ be an oriented knot.

(1) It is proved in [**BorF2015**, Lemma 2.7] (building very much on [**Kaw1986**, Corollary 3.7]) that $\mathrm{H}_2^{\mathrm{tw}}(W;\mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$-module.

(2) The proof of this statement is formally very similar to the proof of the Linking Pairing-via-Intersection Form Proposition **??**. Once again the author lacks the appetite for providing the details. ■

340.6. Exercises for Chapter 340.

Exercise 340.1. Let $m \geq 2$ and let $L \subseteq S^3$ be an oriented m-component link with $\Delta_L(t_1, \dots, t_m) \neq 0$. Show that

$$\mathrm{ord}_\Lambda(\mathrm{H}_2^{\mathrm{tw}}(X_L, \partial X_L; \Lambda)) \;=\; 1.$$

Hint. As practiced in this chapter, use the Twisted Poincaré Duality Theorem 335.1 and the Cohomological UCSS Theorem 316.10.

Exercise 340.2. Let $m \in \mathbb{N}$. We make the usual identification $\pi_1((S^1)^m) = \mathbb{Z}^m$.

(a) Show that for any $\mathbb{Z}[\mathbb{Z}^m]$-module M and any $k \in \mathbb{N}_0$ we have an isomorphism

$$\mathrm{Ext}^k_{\mathbb{Z}[\mathbb{Z}^m]}({}_{\mathrm{tr}}\mathbb{Z}, M) \;\cong\; \mathrm{H}^k_{\mathrm{tw}}((S^1)^m; M)$$

of $\mathbb{Z}[\mathbb{Z}^m]$-modules.
Hint. Use $(S^1)^m$ to describe a free resolution of the $\mathbb{Z}[\mathbb{Z}^m]$-module ${}_{\mathrm{tr}}\mathbb{Z}$.

(b) Compute $\mathrm{H}^k_{\mathrm{tw}}((S^1)^m; \mathbb{Z}[\mathbb{Z}^m])$.

(c) Let $k \in \mathbb{N}$. Show that there exists a link L with $\mathrm{Ext}^k_{\mathbb{Z}[\mathbb{Z}^m]}(\mathrm{H}_0^{\mathrm{tw}}(X_L; \mathbb{Z}[\mathbb{Z}^m]), \mathbb{Z}[\mathbb{Z}^m]) \neq 0$.

Remark. This shows that in the Homomorphism (VII)-Lemma 340.15 (1) the first and the last entry can indeed be non-zero.

Exercise 340.3. Let $m \in \mathbb{N}$ and let $L \subseteq S^3$ be an m-component link with non-zero Alexander polynomial, i.e. such that $\Delta_L(t_1, \dots, t_m) \neq 0$. We write $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we consider the quotient field $Q(\Lambda) = \mathbb{Q}(t_1, \dots, t_m)$. From Definition 340.24 we obtain a Blanchfield pairing

$$\mathrm{Bl}_L \colon \mathrm{H}_1^{\mathrm{tw}}(X_L; \Lambda) \times \mathrm{H}_1^{\mathrm{tw}}(X_L; \Lambda) \;\to\; Q(\Lambda)/\Lambda$$

which is again sesquilinear and hermitian. For which $m \in N$ is the Blanchfield pairing non-singular?
Remark. You should first do Exercise 340.2.

Exercise 340.4. Let $m \in \mathbb{N}_0$. We set $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Show that the pairing

$$\begin{array}{rcl} \Theta \colon Q(\Lambda)/\Lambda \times \Lambda & \to & Q(\Lambda)/\Lambda \\ ([f], g) & \mapsto & [f \cdot \overline{g}] \end{array}$$

is non-singular in the sense of Definition 326.7. In other words, show that the maps

$$\begin{array}{rcl} Q(\Lambda)/\Lambda & \to & \mathrm{Hom}_\Lambda(\overline{\Lambda}, \overline{Q(\Lambda)/\Lambda}) \\ v & \mapsto & \begin{pmatrix} \overline{\Lambda} & \to & Q(\Lambda)/\Lambda \\ w & \mapsto & \varphi(v, w) \end{pmatrix} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \overline{\Lambda} & \to & \overline{\mathrm{Hom}_\Lambda(Q(\Lambda)/\Lambda, Q(\Lambda)/\Lambda)} \\ w & \mapsto & \begin{pmatrix} Q(\Lambda)/\Lambda & \to & Q(\Lambda)/\Lambda \\ v & \mapsto & \varphi(v, w) \end{pmatrix} \end{array}$$

are isomorphisms of Λ-modules.

Exercise 340.5. Let L and $\widetilde{L}$ be two oriented m-components links with $\Delta_L(t_1, \dots, t_m) \neq 0$ and $\Delta_{\widetilde{L}}(t_1, \dots, t_m) \neq 0$. We assume that they are smoothly concordant in the sense of

Definition 339.27. Show that there exist non-zero $f, g \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that

$$\Delta_L(t_1, \dots, t_m) \cdot f \cdot \overline{f} \doteq \Delta_{\widetilde{L}}(t_1, \dots, t_m) \cdot g \cdot \overline{g}.$$

Hint. Let $A \subseteq [0,1] \times S^3$ be a concordance. Similar to the proof of the Link Concordance–Alexander Module Rank Proposition 339.28 consider the long exact sequences of the two pairs (X_A, X_L) and $(X_A, X_{\widetilde{L}})$, but this time with $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$-coefficients. Apply the Twisted Poincaré Duality Theorem 335.1 and the Cohomological UCSS Theorem 316.10 and some of the other tricks of this chapter.

Exercise 340.6. Let $K_1, K_2 \subseteq S^3$ be two oriented knots. Show that the Blanchfield pairing of the connected sum $K_1 \# K_2$ is isometric to the orthogonal sum of the Blanchfield pairings of K_1 and K_2.

Exercise 340.7. Let $K \subseteq S^3$ be an oriented knot. We consider the mirror K^{mir}. Show that the Blanchfield pairing of K^{mir} is isometric to $-\mathrm{Bl}_K$.

Exercise 340.8. Let $K \subseteq S^3$ be a knot. As in Definition 293.15 we say that K is *smoothly slice* if K bounds a properly smoothly embedded disk $D \subseteq \overline{B}^4$. We pick a tubular map $\tau \colon \overline{B}^2 \times D \to \overline{B}^4$ and we consider the slice disk exterior $W_D := \overline{B}^4 \setminus \tau(B^2 \times D)$. We use τ also to define the knot exterior $X_K = S^3 \setminus \tau(B^2 \times K)$ and the 0-Dehn surgery N_K. Note that it follows from the Alexander Duality–Boundary Theorem 217.16 that the inclusion map $\mathrm{H}_1(X_K) \to \mathrm{H}_1(W_D)$ is an isomorphism. Thus it makes sense to consider twisted homology of W_D with $\mathbb{Z}[t^{\pm 1}]$-coefficients.

(a) Show that $\partial W_D = N_K$.

(b) Show that the twisted linking pairing

$$\lambda_{N_K} \colon \mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Z}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

vanishes on $\ker(\mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathrm{H}_1^{\mathrm{tw}}(W_D; \mathbb{Z}[t^{\pm 1}]))$.

(c) We consider the rational linking pairing

$$\lambda_{N_K}^{\mathbb{Q}} \colon \mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Q}[t^{\pm 1}]) \times \mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Q}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$$

that is defined with $\mathbb{Q}[t^{\pm 1}]$-coefficients instead of $\mathbb{Z}[t^{\pm 1}]$-coefficients. Show that in this setting $P := \ker(\mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Q}[t^{\pm 1}]) \to \mathrm{H}_1^{\mathrm{tw}}(W_D; \mathbb{Q}[t^{\pm 1}]))$ is a metabolizer of the rational linking pairing $\lambda_{N_K}^{\mathbb{Q}}$, i.e. show that we have

$$P = P^{\perp} := \{b \in \mathrm{H}_1^{\mathrm{tw}}(N_K; \mathbb{Q}[t^{\pm 1}]) \mid \lambda_{N_K}^{\mathbb{Q}}(a, b) = 0 \text{ for all } a \in P\}.$$

Exercise 340.9. As we discussed in Exercise 340.3 one can also define the Blanchfield pairing for links with non-zero Alexander polynomial. The main technical issue is, that there is no good way to express it in terms of closed 3-dimensional smooth manifolds (and thus not in terms of compact 4-dimensional smooth manifolds). This can be illustrated by the following exercise: Show that there exists an m-component link $L \subseteq S^3$ such that any gluing in of m solid tori to the boundary tori leads to a smooth manifold N_L such that the inclusion induced map $\mathrm{H}_1(X_L) \to \mathrm{H}_1(N_L)$ is not injective.

CHAPTER 341

The Leray-Serre Spectral sequences

In Chapter 316 we introduced the general notion of a (co-) homological spectral sequence. As particular examples we studied the (co-) homological universal coefficient spectral sequences (UCSS), which are purely algebraic gadgets.

In this chapter we will get to know spectral sequences that apply directly to topological spaces. More specifically, we will discuss the Leray-Serre spectral sequences in (co-) homology. These have the following structure: Given a fibration $F \to Y \to B$ these allow us to relate the (co-) homology of the total space Y of the fibration to twisted (co-) homology of B where the coefficients are given by the homology groups of the fiber F, viewed in an appropriate way as left-modules over $\mathbb{Z}[\pi_1(B)]$.

This chapter is organized as follows:

- In Section 341.1 we will remind the reader of a few features of fibrations.
- In Section 341.2 we will discuss the homological Leray-Serre Spectral Sequence.
- In Section 341.3 we will discuss the cohomological Leray-Serre Spectral Sequence.
- In Section 341.4 we will give a brief outlook for other applications of the Leray-Serre Spectral Sequences.
- In Section 341.5 we will provide exercises for Chapter 341.

In the next **??** we will discuss several other examples of spectral sequences.

Note that in this chapter we assume that the reader is familiar with the user's guide to (co-) homological spectral sequences that we provided in Section 316.1 and Section 316.3.

341.1. **Fibrations.** We start out by recalling the following definition:

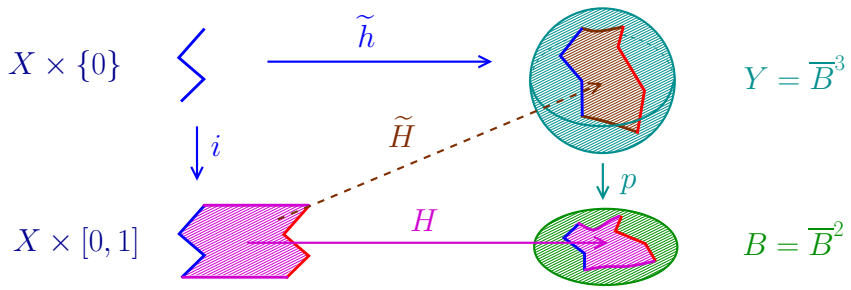
Definition 297.1. Let $p\colon Y \to B$ be a continuous map between topological spaces.

(1) Let X be a topological space. We say that the map $p\colon Y \to B$ has the homotopy lifting property with respect to X if given any continuous map $H\colon X \times [0,1] \to B$ and given any continuous map $\widetilde{h}\colon X \times \{0\} \to Y$ with $p \circ \widetilde{h} = H \circ i$ there exists a continuous map $\widetilde{H}\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\widetilde{h}} & Y \\ \downarrow{\scriptstyle i} & \overset{\widetilde{H}}{\nearrow}{\scriptstyle ??} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[H]{} & B. \end{array}$$

(2) We say that the map $p\colon Y \to B$ is a fibration if p has the homotopy lifting property with respect to all topological spaces.

(3) We say that the map $p\colon Y \to B$ is a Serre fibration if p has the homotopy lifting property with respect to all closed balls $\overline{B}^n$, $n \in \mathbb{N}_0$.



For our purposes the most important source of Serre fibrations is given by fiber bundles:

Proposition 298.3. (Fiber Bundle–Serre Fibration Proposition) Let F be a topological space. Any F-bundle $p\colon Y \to B$ in the sense of Definition 298.2 is a Serre fibration $F \to Y \to B$ in the sense of Definition 297.1.

To formulate the Leray-Serre spectral sequence we will need the following lemma:

Lemma 341.1. (Fibration–Monodromy Lemma) Let $Y \to B$ be a fibration and let $b_0 \in B$. We consider the corresponding fiber $F := p^{-1}(b_0)$. Let $\gamma\colon [0,1] \to B$ be a loop in b_0. We consider the map

$$\begin{array}{rcl} H\colon F \times [0,1] & \to & B \\ (f,t) & \mapsto & \gamma(t) \end{array} \qquad \text{together with the map} \qquad \begin{array}{rcl} \widetilde{h}\colon F \times \{0\} & \to & Y \\ (f,0) & \mapsto & f. \end{array}$$

It follows from Definition 297.1 that there exists a continuous map

$$\widetilde{H}\colon F \times [0,1] \to Y$$

such that the following diagram commutes:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\widetilde{h}} & Y \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \widetilde{H}} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[H]{} & B. \end{array}$$

For each $k \in \mathbb{N}_0$ and each abelian group G we consider the homomorphism

$$\mu(\gamma) := \widetilde{H}_{1*}\colon \mathrm{H}_k(F;G) \to \mathrm{H}_k(F;G).$$

This map has the following properties:

(1) The map $\mu(\gamma)$ only depends on the path-homotopy class of the loop γ.
(2) The map $\mu(\gamma)$ is an automorphism of $\mathrm{H}_k(F;G)$.
(3) The resulting map

$$\begin{array}{rcl} \mu\colon \pi_1(B,b_0) & \to & \mathrm{Aut}(\mathrm{H}_k(F;G)) \\ [\gamma] & \mapsto & \mu(\gamma) \end{array}$$

is a group homomorphism.

Proof. All statements follow easily from the Fiber Homotopy Proposition 297.8 together with the Homotopy Equivalence-H_*-Proposition 144.14. ■

The above lemma leads us to the following definition:

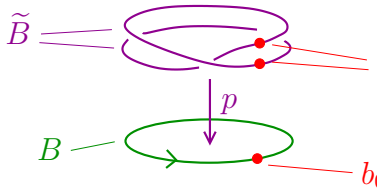
Definition 341.2. Let $Y \to B$ be a fibration and let $b_0 \in B$. Furthermore let G be an abelian group. We consider the corresponding fiber $F := p^{-1}(b_0)$. We refer to the homomorphism

$$\begin{array}{rcl} \mu\colon \pi_1(B,b_0) & \to & \mathrm{Aut}(\mathrm{H}_k(F;G)) \\ [\gamma] & \mapsto & \mu(\gamma) \end{array}$$

from the Fibration–Monodromy Lemma 341.1 as the homological monodromy of the fibration. Almost the same way we also define the cohomological monodromy of the fibration.

Example.

(1) Let $p\colon \widetilde{B} \to B$ be a covering of connected topological spaces. We pick a base point $\widetilde{b}_0 \in \widetilde{B}$ and we set $b_0 := p(\widetilde{b}_0)$. We set $F := p^{-1}(b_0)$. We also set $\pi := \pi_1(B,b_0)$ and $\Gamma := p_*(\pi_1(\widetilde{B}_{b_0},\widetilde{b}_0)) \subseteq \pi$. Elementary covering theory shows that the monodromy action of π on $\mathrm{H}_0(F;\mathbb{Z})$ from the Fibration–Monodromy Lemma 341.1 is equivalent to the standard left-action of π on $\mathbb{Z}[\Gamma\backslash\pi]$.[3696]



these two points get swapped by the action of $[\gamma] \in \pi_1(B,b_0)$ on $F = p^{-1}(b_0)$, thus the action of $\pi_1(B,b_0)$ on $\mathrm{H}_0(F;\mathbb{Z})$ is non-trivial

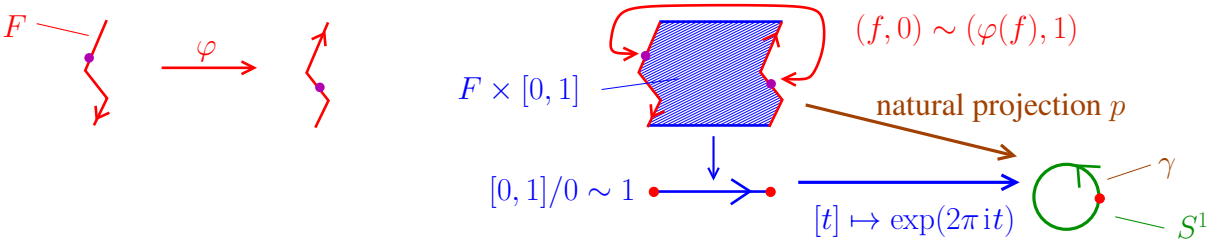
(2) Let F be a topological space and let $\varphi\colon F \to F$ be a homeomorphism. As in Definition 21.19 we consider the mapping torus

$$\mathrm{Tor}(F,\varphi) \;:=\; (F \times [0,1])\,/\sim \qquad \text{where } (f,0) \sim (\varphi(f),1) \text{ for all } f \in F.$$

We also consider the natural projection

$$\begin{array}{rcl} p\colon \mathrm{Tor}(F,\varphi) & \to & S^1 \\ [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

By the Mapping Torus–Bundle Lemma 238.14 we know that this defines an F-bundle over S^1.



Now let $\gamma \in \pi_1(S^1,1)$ be the standard generator that is given by $t \mapsto \exp(2\pi\,\mathrm{i}t)$. It follows easily from the definitions and from considering the map

$$\begin{array}{rcl} \widetilde{H}\colon F \times [0,1] & \to & \mathrm{Tor}(F,\varphi) \\ (f,t) & \mapsto & [(f,t)] \end{array}$$

that the homological monodromy is given by $\mu(\gamma) = (\varphi_*)^{-1}\colon \mathrm{H}_k(F) \to \mathrm{H}_k(F)$.

[3696] Which of course is given by $g \cdot (\Gamma h) := (\Gamma h) \cdot g^{-1}$.

341.2. **The Homological Leray-Serre Spectral Sequence.** Now we can formulate the first main result of this chapter:

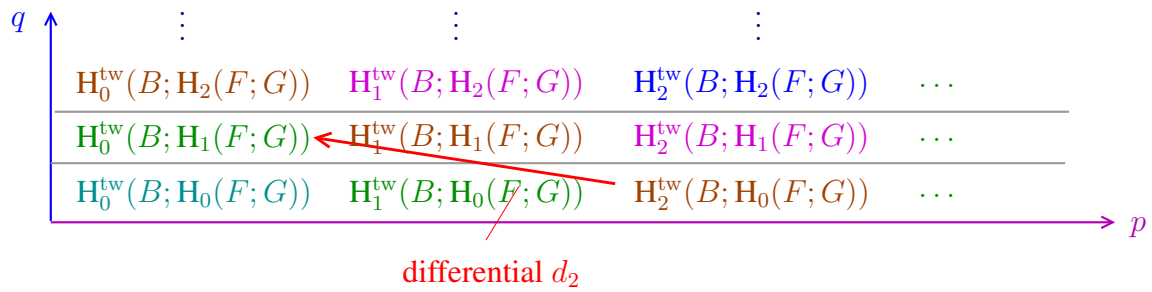
Theorem 341.3. (Homological Leray-Serre Spectral Sequence) Let G be an abelian group and let $Y \to B$ be a Serre fibration where B is non-empty and path-connected. We introduce the following notation:

- We pick $b_0 \in B$.
- We set $F := p^{-1}(b_0)$.
- For each $q \in \mathbb{N}_0$ we use the Fibration–Monodromy Lemma 341.1 to view $\mathrm{H}_q(F;G)$ as a $\mathbb{Z}[\pi_1(B,b_0)]$-left module.

There exists a natural[3697] first quadrant homological spectral sequence $(E^r_{*,*}, d_r)$ that converges to $\mathrm{H}_*(Y;G)$ with the following two properties:

(1) We have[3698]
$$E^2_{p,q} = \mathrm{H}^{\mathrm{tw}}_p(B;\mathrm{H}_q(F;G)).$$

(2) The k-th differential has bidegree $(-k, k-1)$.



Proof. This theorem goes back to the work of Jean-Pierre Serre [**Ser1950**] (see also [**NT2012**, Chapter 1]). The version as formulated was proved by Andreas Dress [**Dre1973**, Satz 1]. A proof for the theorem (with slightly stronger hypothesis) is also given in [**McCl2001**, Theorem 5.1 on page 134] and [**WhdG1978**, Theorem XIII.4.9]. Finally the case that $\pi_1(B,b_0)$ acts trivially on the homology of F is also dealt with in [**Hat**, Theorem 1.3] and in [**Swi1975**, Theorem 15.27]. ■

For completeness sake we also formulate the following property of the Homological Leray-Serre Spectral Sequence:

Proposition 341.4. (LSSS–Edge Homomorphism Proposition) Let G be an abelian group and let $F \xrightarrow{\iota} Y \xrightarrow{\pi} B$ be a Serre fibration where B is non-empty and path-connected

[3698]Here "natural" means that all maps involved are natural with respect to fiber-preserving continuous maps of fibrations.

[3698]Here $\mathrm{H}^{\mathrm{tw}}_p(B;\mathrm{H}_q(F;G))$ is the homology of the base space B with twisted coefficients in the $\mathbb{Z}[\pi_1(B)]$-left module $\mathrm{H}_q(F;G)$.

and such that the fiber F is also non-empty and path-connected. The spectral sequence from the Homological Leray-Serre Spectral Sequence 341.3 has the following two features:

(1) For each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(Y;G) & \xrightarrow{\quad\pi_*\quad} & \mathrm{H}_k(B;G) \\ \downarrow & & \cong\uparrow{\scriptstyle \epsilon} \\ E^\infty_{k,0} & \hookrightarrow & E^2_{k,0} = \mathrm{H}_k(B;\mathrm{H}_0(F;G)) \end{array}$$

where the left vertical map is given by the edge homomorphism from page 6100, the bottom horizontal map is given by the natural monomorphism from page 6099 and where the right vertical map is induced by the natural isomorphism $\epsilon_F\colon \mathrm{H}_0(F;G) \to G$ that is provided by the augmentation homomorphism.

(2) For each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_k(F;G) & \xrightarrow{\quad\iota_*\quad} & \mathrm{H}_k(Y;G) \\ \cong\uparrow{\scriptstyle \epsilon} & & \uparrow \\ E^2_{0,k} = \mathrm{H}_0(B;\mathrm{H}_k(F;G)) & \twoheadrightarrow & E^\infty_{0,k} \end{array}$$

where the left vertical map is induced by the aforementioned augmentation isomorphism $\epsilon_B\colon \mathrm{H}_0(B;H) \to H$, the bottom horizontal map is given by the natural epimorphism from page 6099 and where the right vertical map is given by the edge homomorphism from page 6100.

Proof. This result is proved in [**WhdG1978**, Theorem XIII.7.2 and Theorem XIII.7.6]. ■

Now we finally turn to examples of the Homological Leray-Serre Spectral Sequence:

Example. Let $p\colon \widetilde{B} \to B$ be a covering of connected topological spaces. We pick a base point $\widetilde{b}_0 \in \widetilde{B}$, we set $b_0 := p(\widetilde{b}_0)$ and we set $F := p^{-1}(b_0)$. We also set $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(\widetilde{B}_{b_0}, \widetilde{b}_0)) \subseteq \pi$. As we mentioned on page 6461, elementary covering theory shows that the action of π on $\mathrm{H}_0(F;\mathbb{Z})$ from the Fibration–Monodromy Lemma 341.1 is equivalent to the standard left-action of π on $\mathbb{Z}[\Gamma\backslash\pi]$. With this observation one sees that in this setting the Homological Leray-Serre Spectral Sequence 341.3 gives us back the isomorphism

$$\mathrm{H}_k(\widetilde{B};\mathbb{Z}) \;\cong\; \mathrm{H}^{\mathrm{tw}}_k(B;\mathbb{Z}[\Gamma\backslash\pi])$$

from the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 323.5. □

We formulate our next example as a proposition:

Proposition 341.5. (Mapping Torus-H_*-Proposition) Let F be a topological space and let $\varphi\colon F \to F$ be a homeomorphism. We denote by $i\colon F \to \mathrm{Tor}(F,\varphi)$ the natural map that is given by $f \mapsto [(f,0)]$. For every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n\colon \mathrm{H}_n(\mathrm{Tor}(F,\varphi)) \;\to\; \mathrm{H}_{n-1}(F)$$

such that the following sequence is exact[3699]

$$0 \to \mathrm{coker}(\mathrm{H}_n(F) \xrightarrow{\varphi_*-\mathrm{id}} \mathrm{H}_n(F)) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{Tor}(F,\varphi)) \xrightarrow{\partial_n} \ker(\mathrm{H}_{n-1}(F) \xrightarrow{\varphi_*-\mathrm{id}} \mathrm{H}_{n-1}(F)) \to 0.$$

Remark. The content of the above proposition is essentially equivalent to the earlier Mapping Torus-H_*-Proposition 153.6. □

Proof. We proceed in several steps:

(a) As above we consider the F-bundle map

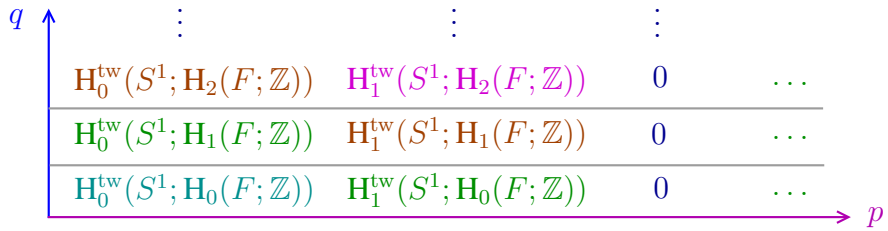
$$\begin{array}{rccc} p\colon & \mathrm{Tor}(F,\varphi) & \to & S^1 \\ & [(f,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

(b) From the Homological Leray-Serre Spectral Sequence 341.3, applied to (a), we obtain a natural first quadrant homological spectral sequence with $E_2^{p,q} = \mathrm{H}_p^{\mathrm{tw}}(S^1;\mathrm{H}_q(F;\mathbb{Z}))$ that converges to $\mathrm{H}_*(\mathrm{Tor}(F,\varphi);\mathbb{Z})$.

(c) Note that

$$\begin{array}{lcccccl} \mathrm{H}_0^{\mathrm{tw}}(S^1;\mathrm{H}_k(F;\mathbb{Z})) & \cong & \mathrm{coker}(\mathrm{H}_k(F) \xrightarrow{\varphi_*^{-1}-\mathrm{id}} \mathrm{H}_k(F)) & \cong & \mathrm{coker}(\mathrm{H}_k(F) \xrightarrow{\varphi_*-\mathrm{id}} \mathrm{H}_k(F)) \\ \mathrm{H}_1^{\mathrm{tw}}(S^1;\mathrm{H}_k(F;\mathbb{Z})) & \cong & \mathrm{ker}(\mathrm{H}_k(F) \xrightarrow{\varphi_*^{-1}-\mathrm{id}} \mathrm{H}_k(F)) & \cong & \mathrm{ker}(\mathrm{H}_k(F) \xrightarrow{\varphi_*-\mathrm{id}} \mathrm{H}_k(F)) \\ & \uparrow & & \uparrow & \\ & \text{this follows from the discussion on page 6461 and the argument of the Circle–Twisted Cellular Chain Complex Lemma 329.10} & & \text{since } \varphi_* \text{ is an isomorphism} & \end{array}$$

and all other twisted homology groups are zero.



(d) The above discussion shows that the spectral sequence is concentrated on two columns. It thus follows from an elementary argument, see page 6100, that for each $k \in \mathbb{N}_0$ we obtain a short exact sequence of the form

$$0 \to \underbrace{\mathrm{H}_0^{\mathrm{tw}}(S^1;\mathrm{H}_k(F;\mathbb{Z}))}_{\cong \mathrm{coker}(\varphi_*-\mathrm{id}\colon \mathrm{H}_k(F)\to\mathrm{H}_k(F))} \to \mathrm{H}_k(\mathrm{Tor}(F,\varphi);\mathbb{Z}) \to \underbrace{\mathrm{H}_1^{\mathrm{tw}}(S^1;\mathrm{H}_{k-1}(F;\mathbb{Z}))}_{\cong \ker(\varphi_*-\mathrm{id}\colon \mathrm{H}_k(F)\to\mathrm{H}_k(F))} \to 0.$$

(e) Finally note that it follows from the LSSS–Edge Homomorphism Proposition) 341.4 (2) that the resulting homomorphism $\mathrm{H}_k(F) \to \mathrm{H}_k(\mathrm{Tor}(F,\varphi);\mathbb{Z})$ in (d) is indeed induced by the natural inclusion i of F into the mapping torus. ■

We conclude this section with the following theorem, which is a generalization of the Manifold Fiber Bundle-χ-Theorem 250.15 and the Fiber Bundle-χ-Theorem 242.7:

Theorem 341.6. (Fibration-χ-Theorem) Let $F \to Y \to B$ be a fibration. We assume that the total homology of F with $\mathbb{F}_2$-coefficients is finite-dimensional. If B is homotopy equivalent to a finite CW-complex, then

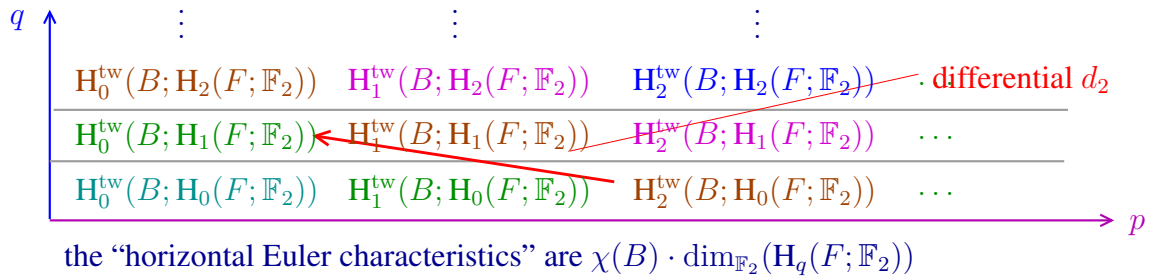
$$\chi(Y) \;=\; \chi(B)\cdot\chi(F).$$

Sketch of proof. By the Homological Leray-Serre Spectral Sequence Theorem 341.3 we know that there exists a homological spectral sequence which converges to $\mathrm{H}_*(Y;\mathbb{F}_2)$ and which has the following two features:

(1) we have $$E^2_{p,q} \;=\; \mathrm{H}^{\mathrm{tw}}_p(B;\mathrm{H}_q(F;\mathbb{F}_2))$$
(2) the k-th differential has bidegree $(-k,k-1)$.

We start out with the following claim:

Claim 1. For each $k \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ we have
$$\sum_{p\in\mathbb{N}_0}\sum_{q\in\mathbb{N}_0}(-1)^{p+q}\cdot\dim_{\mathbb{F}_2}(E^k_{p,q}) \;=\; \chi(B)\cdot\chi(F).$$



Proof.

(1) For $k=2$ this follows from the following calculation:
$$\begin{aligned}
\sum_{p\in\mathbb{N}_0}\sum_{q\in\mathbb{N}_0}(-1)^{p+q}\cdot\dim_{\mathbb{F}_2}(E^2_{p,q}) &\overset{\text{by the definition of } E_2^{p,q}}{\downarrow} \\
&= \sum_{p\in\mathbb{N}_0}\sum_{q\in\mathbb{N}_0}(-1)^{p+q}\cdot\dim_{\mathbb{F}_2}(\mathrm{H}^{\mathrm{tw}}_p(B;\mathrm{H}_q(F;\mathbb{F}_2))) \\
&= \sum_{q\in\mathbb{N}_0}(-1)^q\cdot\sum_{p\in\mathbb{N}_0}(-1)^p\cdot\dim_{\mathbb{F}_2}(\mathrm{H}^{\mathrm{tw}}_p(B;\mathrm{H}_q(F;\mathbb{F}_2))) \\
&= \sum_{q\in\mathbb{N}_0}(-1)^q\cdot\chi(B)\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_q(F;\mathbb{F}_2)) \\
&\overset{\text{by an argument as in the proof Twisted Homology-}\chi\text{-Proposition 331.12}}{\uparrow} \\
&\overset{}{\downarrow} \\
&= \chi(B)\cdot\chi(F).
\end{aligned}$$

(2) For $k=2,3,4,\ldots$ we prove the equality by induction. In (1) we just dealt with the base case $k=2$. Ihe induction step follows from the observation that our differentials have degree $(-k,k-1)$ (and which thus flips the parity of $p+q$), which implies by elementary linear algebra that the alternating sum of the dimensions is unchanged.
(3) The statement for $k=\infty$ follows from the observation from page 6099 that each infinity-term $E^\infty_{p,q}$ is isomorphic to some $E^k_{p,q}$ for sufficiently large k and the fact that we have only finitely many p,q with non-zero entries. ⊞

It remains to prove the following claim:

Claim 2. We have $$\sum_{p\in\mathbb{N}_0}\sum_{q\in\mathbb{N}_0}(-1)^{p+q}\cdot\dim_{\mathbb{F}_2}(E^\infty_{p,q}) \;=\; \chi(Y).$$

Proof. We have the following equalities:

$$\begin{aligned}\sum_{p\in\mathbb{N}_0}\sum_{q\in\mathbb{N}_0}(-1)^{p+q}\cdot\dim_{\mathbb{F}_2}(E^\infty_{p,q}) &= \sum_{k\in\mathbb{N}_0}(-1)^k\cdot\sum_{\substack{(p,q)\text{ with}\\ p+q=k}}\dim_{\mathbb{F}_2}(E^\infty_{p,q})\\ &= \sum_{k\in\mathbb{N}_0}(-1)^k\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_k(Y;\mathbb{F}_2)) \qquad = \chi(Y).\end{aligned}$$

$= \sum_{k\in\mathbb{N}_0}(-1)^k\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_k(Y;\mathbb{F}_2))$: see Definition 316.2.

$= \chi(Y)$: by the Homology-χ Proposition 160.15. ∎

341.3. **The Cohomological Leray-Serre Spectral Sequence.** The following theorem is the cohomological sibling of the Homological Leray-Serre Spectral Sequence 341.3:

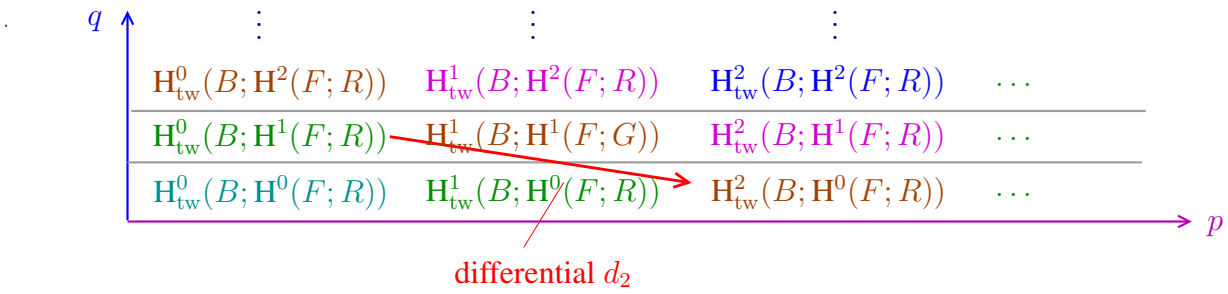
Theorem 341.7. (Cohomological Leray-Serre Spectral Sequence) Let R be an abelian group and let $Y \to B$ be a Serre fibration where B is non-empty and path-connected. We introduce the following notation:

- We pick $b_0 \in B$.
- We set $F := p^{-1}(b_0)$.
- For each $q \in \mathbb{N}_0$ we use the Fibration–Monodromy Lemma 341.1 to view $\mathrm{H}_q(F;R)$ as a $\mathbb{Z}[\pi_1(B,b_0)]$-left module.

There exists a natural first quadrant cohomological spectral sequence $(E_r^{*,*}, d_r)$ that converges to $\mathrm{H}^*(Y;R)$ with the following two properties:

(1) We have $E_2^{p,q} = \mathrm{H}^p_{\mathrm{tw}}(B;\mathrm{H}^q(F;R))$.

(2) The k-th differential has bidegree $(k, 1-k)$.



We continue with the above setup, but we now have an addendum that only makes sense in the cohomological setting:

Theorem 341.7. (Cohomological Leray-Serre Spectral Sequence) If we deal with a commutative ring R and if each $\mathrm{H}_k(F;R)$ is a trivial $\mathbb{Z}[\pi_1(B,b_0)]$-module, then for all $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ there exist bilinear pairings

$$\begin{aligned}E_r^{p,q}\times E_r^{s,t} &\to E_r^{p+s,q+t}\\ (a,b) &\mapsto a\cdot b\end{aligned}$$

such that the following conditions are satisfied:

(a) The product $E_2^{p,q}\times E_2^{s,t} \to E_2^{p+s,q+t}$

equals $(-1)^{qs}$-times the standard cup product[3700]

$$\underbrace{\mathrm{H}^p(B;\mathrm{H}^q(F;R))}_{=E_2^{p,q}} \times \underbrace{\mathrm{H}^s(B;\mathrm{H}^t(F;R))}_{=E_2^{s,t}} \to \underbrace{\mathrm{H}^{p+s}(B;\mathrm{H}^{q+t}(F;R))}_{=E_2^{p+s,q+t}}.$$

(b) (i) Each differential d_r is a derivation, i.e. for any $\varphi \in E_r^{p,q}$ and $\psi \in E_r^{s,t}$ we have

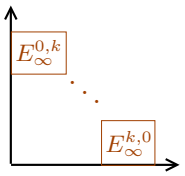
$$d_r(\varphi \cdot \psi) \;=\; d_r(\varphi) \cdot \psi + (-1)^{p+q} \cdot \varphi \cdot d_r(\psi).$$

(ii) Note that the condition from (i) implies that the product on the r-level induces a product on the $(r+1)$-level. This product is indeed the product on the $(r+1)$-level that is part of the structure.

(c) The following diagrams commute[3701]

$$\begin{array}{ccc}
\mathrm{H}^m(Y;R) \times \mathrm{H}^n(Y;R) & \xrightarrow{\cup} & \mathrm{H}^{m+n}(Y;R) \\
\uparrow \times \uparrow & & \uparrow \\
J^{p,m-p} \times J^{s,n-s} & \xrightarrow{\cup} & J^{p+s,m+n-p-s} \\
\downarrow \times \downarrow & & \downarrow \\
J^{p,m-p}/J^{p+1,m-(p+1)} \times J^{s,n-s}/J^{s+1,n-(s+1)} & \longrightarrow & J^{p+s,m+n-(p+s)}/J^{p+s+1,m+n-(p+s+1)} \\
\downarrow \cong \times \downarrow \cong & & \downarrow \cong \\
E_\infty^{p,m-p} \times E_\infty^{s,n-s} & \longrightarrow & E_\infty^{p+s,m+n-p-s}.
\end{array}$$

$$H^k \;=\; \underbrace{J^{0,k} \supseteq J^{1,k-1}}_{\text{quotient } E_\infty^{0,k}} \underbrace{\supseteq J^{2,k-2}}_{\text{quotient } E_\infty^{1,k-1}} \supseteq \ldots \supseteq \underbrace{J^{k,0} \supseteq 0}_{\text{quotient } E_\infty^{k,0}} \qquad \text{where}$$



Proof. The first part of the theorem is proved in [**McCl2001**, Theorem 5.2 on page 135] and [**WhdG1978**, Theorem XIII.4.9*], see also [**Dre1973**, p. 178]. Furthermore the case that $\pi_1(B, b_0)$ acts trivially on the cohomology of F is again also dealt with in [**Hat**, Theorem 1.14]. The same references also prove the statement about products. We refer to [**Swi1975**, p. 352] and [**Hat**, p. 24] for particularly detailed discussions of the multiplicative structures. ■

Before we provide an example, let us formulate the following analog of the LSSS–Edge Homomorphism Proposition 341.4:

Proposition 341.8. (LSSS–Edge Homomorphism Proposition) Let G be an abelian group and let $F \xrightarrow{\iota} Y \xrightarrow{\pi} B$ be a Serre fibration where B is non-empty and path-connected and

[3700] In this formulation of the cup product we use the ring structure on $\mathrm{H}^*(F;R)$ that is given by the cup product on cohomology.

[3701] Here the modules $J^{p,m}$ come from the definition of a convergent homological spectral sequence, see Definition 316.7. We summarize their role also in the figure shown below.

such that the fiber F is also non-empty and path-connected. The spectral sequence from the Cohomological Leray-Serre Spectral Sequence 341.7 has the following two features:

(1) For each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(B;G) & \xrightarrow{\quad\pi^*\quad} & \mathrm{H}^k(Y;G) \\ \cong\uparrow\epsilon & & \uparrow \\ E_2^{k,0} = \mathrm{H}^k(B;\mathrm{H}^0(F;G)) & \twoheadrightarrow & E_\infty^{k,0} \end{array}$$

where the left vertical map is induced by the natural isomorphism $\mathrm{ev}_F\colon \mathrm{H}^0(F;G) \to G$ that is provided by the evaluation homomorphism, the bottom horizontal map is given by the natural epimorphism from page 6103 and the right vertical map is given by the edge homomorphism from page 6103.

(2) For each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(Y;G) & \xrightarrow{\quad\iota^*\quad} & \mathrm{H}^k(F;G) \\ \downarrow & & \cong\uparrow\epsilon \\ E_\infty^{0,k} & \hookrightarrow & E_2^{0,k} = \mathrm{H}^0(B;\mathrm{H}^k(F;G)) \end{array}$$

where the left vertical map is given by the edge homomorphism from page 6103, the bottom horizontal map is given by the natural monomorphism from page 6103 and where the right vertical map is induced by the evaluation isomorphism $\mathrm{ev}_B\colon \mathrm{H}^0(B;H) \to H$.

Proof. This result is proved in [**WhdG1978**, Theorem XIII.7.2* and Theorem XIII.7.6*]. Implicitly the first result is also proved in [**Swi1975**, p. 355]. ■

As an application we get the following variation on the Gysin Sequence Proposition 245.9:

Proposition 341.9. (Gysin Sequence Proposition) Let $n \in \mathbb{N}$ and let $S^n \to Y \xrightarrow{p} B$ be a fibration over a path-connected topological space B with fiber S^n.

(1) We first assume that the fibration is "orientable" in the sense that the action of $\pi_1(B,b_0)$ on $\mathrm{H}^n(S^n;\mathbb{Z})$ is trivial. In this setting there exists a cohomology class $e_{\mathbb{Z}}(Y) \in \mathrm{H}^{n+1}(B;\mathbb{Z})$ and for each $m \in \mathbb{N}_0$ there exists a natural homomorphism

$$\xi_m\colon \mathrm{H}^m(Y;\mathbb{Z}) \to \mathrm{H}^{m-n}(B;\mathbb{Z})$$

such that the following sequence is exact:

$$\cdots \to \mathrm{H}^{k+n}(Y;\mathbb{Z}) \xrightarrow{\xi_{k+n-1}} \mathrm{H}^k(B;\mathbb{Z}) \xrightarrow{\varphi \mapsto e_{\mathbb{Z}}(Y)\cup\varphi} \mathrm{H}^{k+n+1}(B;\mathbb{Z}) \xrightarrow{p^*} \mathrm{H}^{k+n+1}(Y;\mathbb{Z}) \to \ldots$$

(2) Regardless of the action $\pi_1(B,b_0)$ on $\mathrm{H}^n(S^n;\mathbb{Z})$, we always have the $\mathbb{F}_2$-analog of Statement (1).

These two sequences are called Gysin sequences.

Remark. Let $n \in \mathbb{N}_0$ and let $p\colon Y \to B$ be an $(S^n, \mathrm{O}(n+1))$-bundle with a finite atlas. In Definition 245.8 we introduced the Euler class $e_{\mathbb{F}_2}(p\colon Y \to B) \in \mathrm{H}^{n+1}(B;\mathbb{F}_2)$. Furthermore, if we have an $(S^n, \mathrm{SO}(n+1))$-bundle, then we introduced the Euler class $e_{\mathbb{Z}}(p\colon Y \to B) \in \mathrm{H}^{n+1}(B;\mathbb{Z})$. Clearly these cohomology classes should be (perhaps up to sign) the cohomology classes from the Gysin Sequence Proposition 341.9. We leave it to people with more time, more energy and longer life-spans to verify this statement. □

Sketch of proof. In the following we deal with Statement (1). The proof of Statement (2) is evidently almost indistinguishable from the proof of Statement (1), we only need to make use of the observation that any group acting on $\mathrm{H}^n(S^n;\mathbb{F}_2) \cong \mathbb{F}_2$ acts automatically trivially, since $\mathbb{F}_2$ has no non-trivial automorphism.[3702]

We start out with the following key claim:

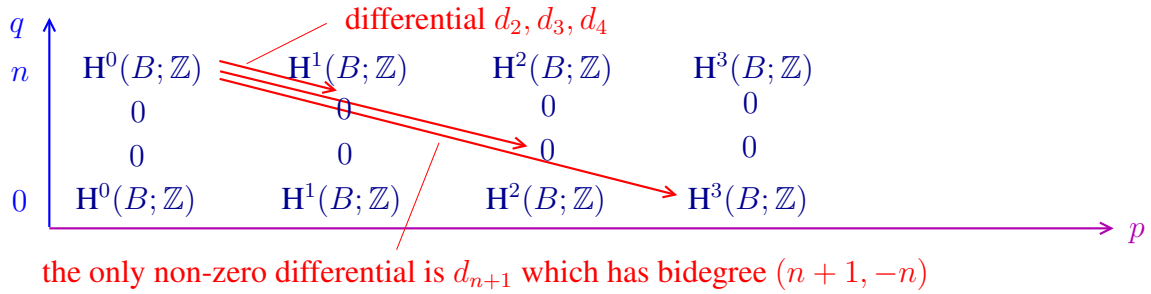
Claim. We have the following two exact sequences:

$$\begin{array}{lccccccccccc} \text{(I)} & 0 & \to & E_\infty^{p,n} & \to & E_2^{p,n} & \xrightarrow{d_{n+1}} & E_2^{p+n+1,0} & \to & E_\infty^{p+n+1,0} & \to & 0, \\ \text{(II)} & 0 & \to & E_\infty^{m,0} & \to & \mathrm{H}^m(Y;\mathbb{Z}) & \to & E_\infty^{m-n,n} & \to & 0. & & \end{array}$$

Proof. Since the fibration is "orientable" we can apply the Leray-Serre Spectral Sequence 341.7 for the ring $R = \mathbb{Z}$. Note that in this case we have

$$E_2^{p,q} = \mathrm{H}^p(B;\mathrm{H}^q(S^n;\mathbb{Z})) =_{\mathrm{i}} \begin{cases} \mathrm{H}^p(B;\mathbb{Z}), & \text{if } q = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

here we use the standard generators of $\mathrm{H}^0(S^n;\mathbb{Z})$ and $\mathrm{H}^n(S^n;\mathbb{Z})$ to make the identifications



Evidently all differential $d_k \colon E_k^{p,q} \to E_k^{p+k,q-k+1}$ must be zero, except for d_{n+1}. Next note that by Definition 316.6 we have

$$\begin{aligned} E_{n+2}^{p,n} &= \ker(d_{n+1} \colon E_{n+1}^{p,n} \to E_{n+1}^{p+n+1,0}), \\ E_{n+2}^{p,0} &= \operatorname{coker}(d_{n+1} \colon E_{n+1}^{p-n-1,n} \to E_{n+1}^{p,0}). \end{aligned}$$

Since all other differentials are zero we see that for any $p, q \in \mathbb{N}_0$ we have

$$E_{n+2}^{p,q} = E_{n+3}^{p,q} = E_{n+4}^{p,q} = \ldots = E_\infty^{p,q}.$$

This discussion gives us the exact sequences (I). By Definition 316.7 of a convergent cohomological spectral sequence they conveniently enough also give us the exact sequences (II). ⊞

We just learned that we have the following two exact sequences:

$$\begin{array}{lccccccccccc} \text{(I)} & 0 & \to & E_\infty^{p,n} & \xrightarrow{\alpha} & \overbrace{E_2^{p,n}}^{\cong \mathrm{H}^p(B;\mathbb{Z})} & \xrightarrow{d_{n+1}} & \overbrace{E_2^{p+n+1,0}}^{\cong \mathrm{H}^{p+n+1}(B;\mathbb{Z})} & \xrightarrow{\beta} & E_\infty^{p+n+1,0} & \to & 0, \\ \text{(II)} & 0 & \to & E_\infty^{m,0} & \xrightarrow{\beta'} & \mathrm{H}^m(Y;\mathbb{Z}) & \xrightarrow{\alpha'} & E_\infty^{m-n,n} & \to & 0. & & \end{array}$$

[3702] But note that here subtly we use that $n \in \mathbb{N}$, thus we exclude the case $n = 0$.

By "gluing" these two exact sequences together, and using that $E_2^{p,0} \cong E_2^{p,n} \cong \mathrm{H}^p(B;\mathbb{Z})$ we obtain a long exact sequence

$$\ldots \to \mathrm{H}^m(Y;\mathbb{Z}) \xrightarrow[=:\xi_m]{\alpha\circ\alpha'} \mathrm{H}^{m-n}(B;\mathbb{Z}) \xrightarrow{d_{n+1}} \mathrm{H}^{m+1}(B;\mathbb{Z}) \xrightarrow{\beta'\circ\beta} \mathrm{H}^{m+1}(Y;\mathbb{Z}) \to \ldots$$

The proposition now follows from the following two extra observations:

- It follows easily from the LSSS–Edge Homomorphism Proposition 341.8 (1) that the map $\mathrm{H}^{m+1}(B;\mathbb{Z}) \to \mathrm{H}^{m+1}(Y;\mathbb{Z})$ is actually given by p^*.
- We denote by $\mathrm{const}_1 \in \mathrm{C}^0(B;\mathbb{Z})$ the 0-cocycle that assigns to each singular 0-simplex the value 1. We set $e_{n+1} := d_{n+1}([\mathrm{const}]) \in \mathrm{H}^{n+1}(B;\mathbb{Z})$. Using the multiplicativity statement of the Cohomological Leray-Serre Spectral Sequence 341.7 one can show quite easily that the map $d_{n+1}\colon \mathrm{H}^{m-n}(B;\mathbb{Z}) \to \mathrm{H}^{m+1}(B;\mathbb{Z})$ is given by the cup product with $e_{n+1} = d_{n+1}([\mathrm{const}_1])$. We refer to [**Swi1975**, p. 356] for full details on this argument. ■

341.4. **More applications.** Unfortunately we do not have time and energy right now to provide more applications of the Leray-Serre Spectral sequences. But there are many others:

- In [**DaK2001**, Theorem 10.25] they get used to calculate the rational cohomology rings of the Eilenberg-Maclane spaces $\mathrm{K}(n,\mathbb{Z})$.
- In [**DaK2001**, Theorem 10.29] they get used to show that $\pi_4(S^2) \cong \mathbb{Z}/2\mathbb{Z}$.

We refer to [**DaK2001**, Chapter 10], [**Hat**, Chapter 1], [**Swi1975**, Chapter 15] and [**Spa1995**, Chapter 9] for the above and many more applications.

341.5. **Exercises for Chapter 341.**

Exercise 341.1. Let $n \in \mathbb{N}$. By the Matrix Group-Fiber Proposition 299.8 the maps

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1) & \to & S^{4n+3} \\ (v_1 \ldots v_{n+1}) & \mapsto & v_{n+1} \end{array}$$

are bundle maps with fibers given by $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ respectively.

(a) Explore the Homological Leray-Serre Spectral Sequence 341.3 for $\mathrm{O}(2)$.
(b) Explore the Homological Leray-Serre Spectral Sequence 341.3 for $\mathrm{U}(2)$.
(c) Explore the Homological Leray-Serre Spectral Sequence 341.3 for $\mathrm{Sp}(2)$.

Exercise 341.2. Let F and B be topological spaces. We consider the trivial fibration $p\colon F\times B \to B$. Compare the conclusion of the Künneth Theorem 164.15 with the conclusion of the Homological Leray-Serre Spectral Sequence 341.3.

Exercise 341.3. What are the isomorphism types of the homology groups of total spaces of S^{11}-bundles over S^{13}?

Exercise 341.4. We consider the following bundles:

(i) The product bundle $S^2 \times S^2 \to S^2$.
(ii) The S^2-bundle $\mathbb{C}\mathrm{P}^2\# - \mathbb{C}\mathrm{P}^2 \to S^2$ from the Projective Space–Connected Sum Proposition 238.10

Note that the total spaces have isomorphic (co-) homology groups but note that the cup products differ. Explore the multiplicative aspect of the Cohomological Leray-Serre Spectral Sequence Theorem 341.7 for these two bundles.

Exercise 341.5. Let $n \in \mathbb{N}$ and let $S^n \to Y \xrightarrow{p} B$ be a fibration over a path-connected topological space B with fiber S^n. Use the Homological Leray-Serre Spectral Sequence 341.3 to formulate and prove a homological analog of the Gysin Sequence Proposition 341.9.

Exercise 341.6. Let $n \in \mathbb{N}_{\geq 2}$ and let $F \xrightarrow{\iota} Y \to S^n$ be a fibration. Show that there exists the following:

- for each $m \in \mathbb{N}_0$ a natural homomorphisms
$$\xi_m \colon \mathrm{H}_m(Y) \quad \to \quad \mathrm{H}_{m-n}(F)$$
- for each $m \in \mathbb{N}_0$ a natural homomorphisms
$$\zeta_m \colon \mathrm{H}_m(F) \quad \to \quad \mathrm{H}_{m+n-1}(F)$$

such that the following sequence is exact:

$$\ldots \to \mathrm{H}_k(F) \xrightarrow{iota_*} \mathrm{H}_k(Y) \xrightarrow{\xi_k} \mathrm{H}_{k-n}(F) \xrightarrow{\zeta_{k-n}} \mathrm{H}_{k-1}(F) \xrightarrow{\iota_*} \mathrm{H}_k(Y) \to \ldots$$

This sequence is called the *Wang sequence*.

Exercise 341.7. Let $p \colon \widetilde{B} \to B$ be a covering. What conclusion do we obtain in this context from the Cohomological Leray-Serre Spectral Sequence 341.7? How does the conclusion relate to the Statement of the Finite Covering-$\mathrm{H}^*_{\mathrm{tw}}$-Proposition 323.13?

CHAPTER 342

Local coefficients

In the last many chapters we introduced and discussed the main properties of (co-) homology with twisted coefficients. It was fairly easy to develop this theory with off-the-shelves methods and it works quite well when we want to study a single topological space.

But (co-) homology with twisted coefficients has various awkward features: It depends on the choice of a base point, it is only defined for a connected topological space and it is very difficult to study maps between different topological spaces. In the next chapter we will introduce (co-) homology with local coefficients. This concept rectifies many of the issues we had with (co-) homology with twisted coefficients (admittedly at creating new headaches). But before we can talk about (co-) homology with local coefficients we need to introduce the notion of a local coefficient system over a topological space. In a nutshell a local coefficient system over a topological space X is a bundle $C \to X$ such that each fiber is a group (equipped with the discrete topology).

In this chapter we will develop this concept in greater details.

This chapter is organized as follows:

- In Section 342.1 we will introduce local coefficient systems.
- In Section 342.2 we will relate local coefficient systems to modules over group rings,
- In Section 342.3 we will study pullbacks of local coefficient systems.
- In Section 342.4 we will study the effect of homotopies on local coefficient systems.
- In Section 342.5 we will introduce the tensor products of local coefficient systems.
- In Section 342.6 we will provide exercises for Chapter 342.

In the next Chapter 343 we will use the machinery from this chapter to introduce (co-) homology with local coefficients.

342.1. Local coefficient system.

342.1.1. Definition of local coefficient systems.

The following definition gives us the most important new concept of this chapter:

Definition 342.1. Let X be a topological space and let G be an abelian group. A G-local coefficient system on X consists of the following:

(1) a topological space C,
(2) a continuous map $p\colon C \to X$,
(3) a group structure on each fiber $C_x := p^{-1}(x)$, with $x \in X$,

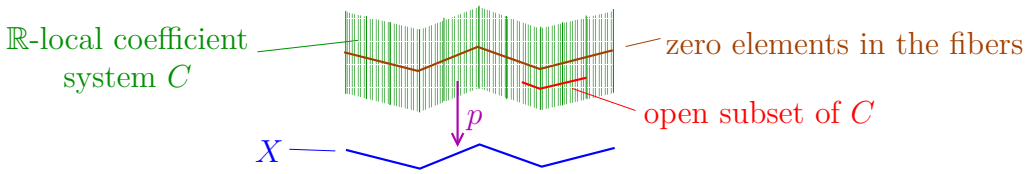
such that the following condition is satisfied: Given any $x \in X$ there exists an open neighborhood U of x and a homeomorphism $f\colon p^{-1}(U) \to G \times U$, where we equip G with the discrete topology, such that the following two conditions are satisfied:[3703]

(1) The following diagram commutes:

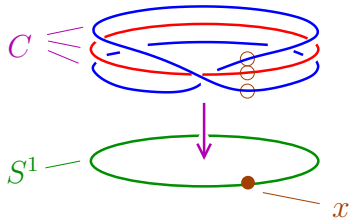
$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{f} & G \times U \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle (g,x)\mapsto x} & \\ & U. & \end{array}$$

(2) For each $x \in U$ the map $C_x = p^{-1}(x) \to G \times \{x\} \xrightarrow{(g,x)\mapsto g} G$ is an isomorphism of groups.[3704]

Given a ring R we define in a very similar way an R-left module local coefficient system on X.[3705] If the group or the module are irrelevant, then we just talk of a local coefficient system.



Example. In the figure below we show a bundle $p\colon C \to S^1$.



in each fiber we have the following group structure:
red is the trivial element,
the sum of two different blue elements is zero,
the sum of the same blue element is the other blue element

If we interpret everything with good will, then we see that this is a $\mathbb{Z}/3\mathbb{Z}$-local system over the circle S^1. □

Our first general example of a local coefficient system gets an immediate upgrade to a definition:

Definition 342.2. Let X be a topological space and let G be an abelian group. We equip G with the discrete topology and we equip $G \times X$ with the product topology. We denote by $p\colon G \times X \to X$ the natural projection. It is clear that p defines a G-local coefficient system on X. We refer to it as the trivial G-local coefficient system and we denote it by ${}_{\mathrm{tr}}G$.

[3703] Later we will call such a map f a *local trivialization*.

[3704] A G-local coefficient system is in particular a G-bundle (where G is equipped with the discrete topology) in the sense of Definition 238.1. As an extra feature each fiber has a group structure and we have locally coherent isomorphisms to G.

[3705] In this chapter we will formulate many results just for local coefficient systems over a group, but the notions and results have obvious analogs for local coefficient systems of rings. We leave it to the reader to make this precise at each given moment.

The following definition shows that a local coefficient system gives rise to many other local coefficient systems:

Definition 342.3. Let X be a topological space and let G be an abelian group. Furthermore let $p\colon C \to X$ be a G-local coefficient system. Given any subspace $U \subseteq X$ we refer to

$$p|_U\colon p^{-1}(U) \;\to\; X$$

as the restriction of $p\colon C \to X$ to U. It is straightforward to verify that this is again a G-local coefficient system.

The following lemma can be viewed as an analog of the Vector Bundle-Section Lemma 265.19.

Lemma 342.4. (Local Coefficient System-Lift Lemma) Let $p\colon C \to X$ be a local coefficient system and let W be a topological space.

(1) Let $f\colon W \to X$ be a continuous map. The map

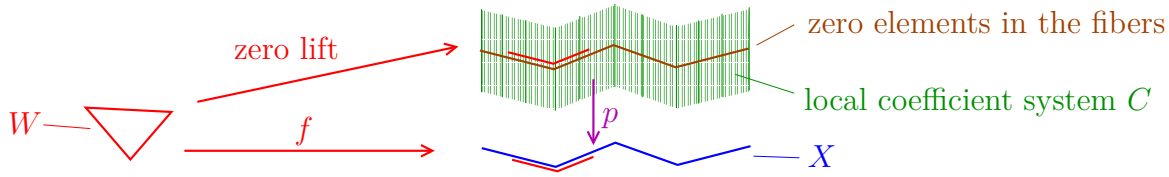
$$\begin{array}{rcl} \widetilde{f}\colon W & \to & C \\ w & \mapsto & \text{the unique zero element in } p^{-1}(f(w)) \end{array}$$

is continuous. We refer to it as the *zero-lift of* f.

(2) Let $f, g\colon W \to C$ be two continuous maps with $p \circ f = p \circ g$. Then the maps

$$\begin{array}{rcl} W & \to & C \\ w & \mapsto & \underbrace{f(w) + g(w)}_{\substack{\text{addition in the abelian} \\ \text{group } p^{-1}(f(w))}} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} W & \to & C \\ w & \mapsto & -f(w) \end{array}$$

are also continuous.



Proof. Note that by the Pasting Proposition 4.7 we have to verify continuity only locally. But locally the statement follows easily from the Definition 342.1 of a local coefficient system. ∎

We conclude this subsection with the following lemma, which connects local coefficient systems to earlier concepts:

Lemma 342.5. (Local System–Covering Lemma) Let X be a topological space and let G be an abelian group. Every G-local coefficient system $p\colon C \to X$ is a covering map, in the sense of Definition 91.8.

Proof. This lemma is a special case of the Covering-as-Bundle Lemma 238.2 (1). For the reader's convenience let us recall the argument: Since groups are by definition non-empty we see that $p\colon C \to X$ is surjective. Now let $x \in X$. By definition there exists an open neighborhood U of x and a homeomorphism $f\colon p^{-1}(U) \to G \times U$, where we equip G with

the discrete topology, such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{f} & G \times U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle (g,x)\mapsto x} & \\ & U. & \end{array}$$

It follows now easily from Exercise 7.20 that U is uniformly covered in the sense of Definition 91.8. ■

342.1.2. **Morphisms of local coefficient systems.** The following definition is a variation on Definition 265.2 of a bundle map of vector bundles:

Definition 342.6. Let G and H be abelian groups.

(1) Let X and Y be topological spaces, let $p\colon C \to X$ be a G-local coefficient system and let $q\colon D \to X$ be an H-local coefficient system. A morphism from the G-local coefficient system $p\colon C \to Y$ to the H-local coefficient system $q\colon D \to Y$ is a pair of continuous maps $\Phi\colon C \to D$ and $\varphi\colon X \to Y$ such that the following two conditions are satisfied:
   (a) The following diagram commutes:
$$\begin{array}{ccc} C & \xrightarrow{\Phi} & D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi} & Y. \end{array}$$
   (b) For each $x \in X$ the map $f\colon C_x = p^{-1}(x) \to q^{-1}(\varphi(x)) = D_{\varphi(x)}$ is a homomorphism of abelian groups.

(2) Let X be a topological space, let $p\colon C \to X$ be a G-local coefficient system and let $q\colon D \to X$ be an H-local coefficient system. A morphism over X from the G-local coefficient system $p\colon C \to X$ to the H-local coefficient system $q\colon D \to X$ is a morphism as in (1) but with $\varphi = \mathrm{id}_X$. This notion turns the G-local coefficient systems over a given topological space X into a category $\mathcal{LocalSys}_G(X)$. In particular it also allows us to say what it means for two local coefficient systems over a topological space X to be isomorphic.

Remark. Note that the analogy between Definition 265.2 and Definition 342.6 is not complete. In the former we only allow maps that are fiberwise isomorphisms of vector spaces, whereas in the latter we allow any group homomorphism. □

342.1.3. **Trivializations.** The following definition is the analog of Definition 265.5 of a trivialization of a vector bundle:

Definition 342.7. Let $p\colon C \to X$ be a G-local coefficient system.

(1) We say that p is trivializable if p is isomorphic over X to the trivial G-local coefficient system ${}_{\mathrm{tr}}G = G \times X$.

(2) Any choice of an isomorphism $C \to {}_{\mathrm{tr}}G = G \times X$ over X is called a trivialization of the G-local coefficient system.

Next we have the following analog of the Vector Bundle–Action-on-Trivializations Proposition 266.1:

Proposition 342.8. (Trivialization–Classification Proposition) Let $p\colon C \to X$ be a G-local coefficient system over a topological space X and let $\Phi\colon C \to {}_{\mathrm{tr}}G = G \times X$ be a trivialization. We denote by $q\colon G \times X \to G$ the projection. We equip $\mathrm{Aut}(G)$ with the discrete topology.

(1) If $f\colon X \to \mathrm{Aut}(G)$ is a continuous map[3706], then the map

$$\begin{array}{rcl} C & \to & G \times X \\ c & \mapsto & (\underbrace{f(p(c))}_{\in \mathrm{Aut}(G)}(\underbrace{q(\Phi(c))}_{\in G}, p(c)) \end{array}$$

is also a trivialization.

(2) Every trivialization of $p\colon C \to X$ is of the form described in (1).

Proof. The proof is similar to the rather terse proof of the Vector Bundle–Action-on-Trivializations Proposition 266.1. ∎

342.2. **Local coefficient systems and modules over group rings.** We start out with a very general construction of local coefficient systems:[3707]

Proposition 342.9. (Local System-via-Representation Proposition) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering from Definition 127.21. Furthermore let M be a $\mathbb{Z}[\pi]$-left module, i.e. let M be an abelian group and suppose that we are given a group homomorphism $\varphi\colon \pi \to \mathrm{Aut}(M)$. As in Notation 320.7 we denote by $\overline{M}$ the $\mathbb{Z}[G]$-right module that is given by $m \underset{\overline{M}}{\cdot} g := g^{-1} \underset{M}{\cdot} m$. With slightly risqué notation we set[3708]

$${}_{\varphi}M := \overline{M} \times_{\pi} \widetilde{X}_{x_0} := (\overline{M} \times \widetilde{X}_{x_0})/\sim$$

where $(m \cdot g, x) \sim (m, g \cdot x)$ for all $m \in M$, $g \in G$ and $x \in X$. We equip $\overline{M}$ with the discrete topology, we equip $\overline{M} \times \widetilde{X}_{x_0}$ with the product topology and finally we equip ${}_{\varphi}M := (\overline{M} \times \widetilde{X}_{x_0})/\sim$ with the quotient topology.

(1) The map

$$\begin{array}{rcl} q\colon \overline{M} \times_{\pi} \widetilde{X}_{x_0} & \to & X \\ [(m, x)] & \mapsto & p(x) \end{array}$$

is continuous.

(2) Let $x \in X$. We pick a path $\mu\colon [0,1] \to X$ from x_0 to x. The map[3709]

$$\begin{array}{rcl} q^{-1}(x) & \to & M \\ [(m, [\alpha])] & \mapsto & \varphi([\underbrace{\mu * \overline{\alpha}}_{\text{loop in } x_0}])(m) \end{array}$$

$\uparrow$ each point in $\widetilde{X}_{x_0}$ is represented by a path from x_0 to x

[3706] Since $\mathrm{Aut}(G)$ is discrete this is (by Exercise 4.3) equivalent to saying that $f\colon X \to \mathrm{Aut}(G)$ is locally constant.

[3707] In fact the construction below was pioneered in Exercise **??**.

is well-defined[3710] and a bijection. We use the abelian group structure on M and this bijection to define a group structure on $q^{-1}(x)$. This group structure does not depend on the choice of μ.

All this data combined defines an M-local coefficient system ${}_{\varphi}M$ on X. Furthermore, if R is a ring and if we are given a homomorphism $\varphi\colon \pi \to \mathrm{Aut}_{R\text{-left}}(M)$, then we obtain a local coefficient system of R-left modules.

Proof. Let $\varphi\colon \pi_1(X,x_0) \to \mathrm{Aut}(M)$ be a representation.

(1) It follows immediately from the Topological-Product Proposition 7.2 (1a) and the Topological-Quotient Proposition 8.2 (1b) that the given map $q\colon M \times_{\pi_1(X,x_0)} \widetilde{X}_{x_0} \to X$ is continuous.

(2) Let $x \in X$. Evidently we have $q^{-1}(x) = M \times_{\pi_1(X,x_0)} p^{-1}(x)$.

(a) We pick a path $\mu\colon [0,1] \to X$ from x to x_0. We consider the map

$$\begin{array}{rcl} F\colon q^{-1}(x) = M \times_{\pi_1(X,x_0)} p^{-1}(x) & \to & M \\ [(m,\alpha)] & \mapsto & \varphi([\mu * \overline{\alpha}])(m). \end{array}$$

We need to show that the map F is well-defined. Thus let $[\gamma] \in \pi_1(X,x_0)$. Then

$$\begin{array}{rclcl} F(\varphi([\gamma]^{-1})\cdot v, [\alpha]) & = & \varphi(\mu * \overline{\alpha})\cdot\varphi([\gamma]^{-1})\cdot v & = & \varphi(\mu * \overline{\alpha})\cdot\varphi([\overline{\gamma}])\cdot v \\ & = & \varphi(\mu * \overline{\alpha} * \overline{\gamma})\cdot v & = & \varphi(\mu * \overline{\gamma * \alpha})\cdot v \\ & = & F([(v,[\gamma * \alpha])]) & = & F([(v,[\gamma]\cdot[\alpha])]). \end{array}$$

(b) By the Regular Covering-Characterization Proposition 128.13 and the Deck Group-Free Action Lemma 128.3 we know that the action of $\pi_1(X,x_0)$ on $p^{-1}(x)$ is free and transitive. It follows easily that the above map F is a bijection.

(c) We need to show that the resulting group (or module) structure on $q^{-1}(x)$ does not depend on the choice of μ. Thus let $\nu\colon [0,1] \to X$ be another path from x to x_0. We see that

$$\varphi([\nu * \overline{\alpha}])(m) \;=\; \varphi([\nu * \overline{\mu} * \mu * \overline{\alpha}])(m) \;=\; \varphi([\nu * \overline{\mu}]) \cdot \varphi([\mu * \overline{\alpha}])(m).$$

Thus we see that the maps $[(m,\alpha)] \mapsto \varphi([\nu * \overline{\alpha}])(m)$ and $[(m,\alpha)] \mapsto \varphi([\mu * \overline{\alpha}])(m)$ differ by multiplication by the automorphism $\varphi([\nu * \overline{\mu}])$. Thus both induce the same group (respectively module) structure on $q^{-1}(x)$.

It remains to show that the above data defines a local coefficient system over X. Thus let $x \in X$. We pick an open neighborhood U of x such that $p^{-1}(U)$ is uniformly covered in the sense of Definition 91.8. Furthermore we pick a path $\mu\colon [0,1] \to X$ from x_0 to x. Similar to the above we consider the map

$$\begin{array}{rcl} F\colon q^{-1}(U) = M \times_{\pi_1(X,x_0)} p^{-1}(U) & \to & M \times U \\ [(v,[\alpha])] & \mapsto & (\varphi(\mu * \overline{\alpha})\cdot v, [\alpha]). \end{array}$$

Using our discussion above one can easily verify that this map is a trivialization. ∎

Example.

[3708] The notation $\overline{M} \times_{\pi} \widetilde{X}_{x_0}$ is motivated by the definition and notation for the tensor product, as given in Definition 312.3.

[3709] Recall that according to Definition 127.21 the natural universal covering $\widetilde{X}_{x_0}$ is defined as a set of certain equivalence classes of path starting at x_0.

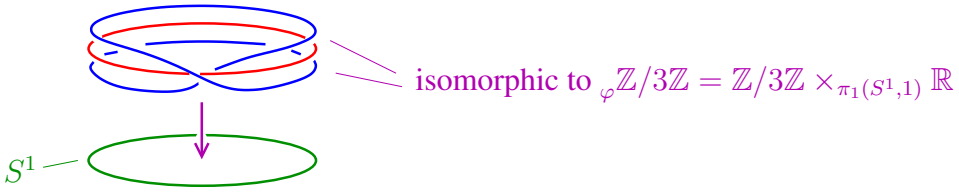
[3710] This statement has an overlap with the Fiber Homotopy Proposition 297.8.

(1) Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. Furthermore let G be an abelian group. Let $\mathrm{tr}\colon \pi_1(X, x_0) \to \mathrm{Aut}(G)$ be the trivial homomorphism. One can easily verify that the map

$$\begin{array}{rcl} {}_{\mathrm{tr}}G = \overline{G} \times_{\pi_1(X,x_0)} \widetilde{X}_{x_0} & \to & {}_{\mathrm{tr}}G = G \times X \\ (v, \widetilde{x}) & \mapsto & (v, p(\widetilde{x})) \end{array}$$

is an isomorphism of local coefficient systems, in other words, it is a trivialization of the G-local coefficient system ${}_{\mathrm{tr}}G$.

(2) Let $X = S^1$, $C = \mathbb{Z}/3\mathbb{Z}$ and let $\varphi\colon \pi_1(X) \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z}) = \{\pm 1\}$ be the unique non-trivial homomorphism.



We leave it to the meticulous reader to verify that the corresponding $\mathbb{Z}/3\mathbb{Z}$-local system ${}_{\varphi}\mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \times_{\pi_1(S^1,1)} \mathbb{R} \to S^1$ is isomorphic over S^1 to the $\mathbb{Z}/3\mathbb{Z}$-local system that is shown above. □

It turns out, that the converse to the Local System-via-Representation Proposition 342.9 also holds:

Proposition 342.10. (Local System–Classification Proposition) Let X be a topological space that is path-connected and locally contractible and let $p\colon C \to X$ be a G-local coefficient system. Let $x_0 \in X$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. There exists a $\mathbb{Z}[\pi_1(X, x_0)]$-left module structure on G such that the corresponding local coefficient system is isomorphic over X to the given G-local coefficient system $p\colon C \to X$.

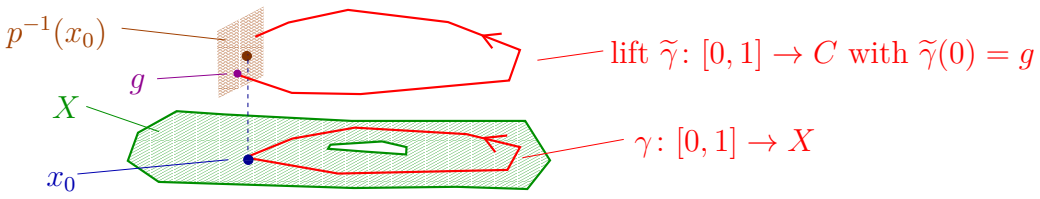
Proof. Let x_0. Since $p\colon C \to X$ is a G-local coefficient system we know that the fiber $p^{-1}(x_0)$ is isomorphic to G. For the remainder of this proof we fix an identification $p^{-1}(x_0) =_{\mathrm{i}} G$.

Claim. The map

$$\begin{array}{rcl} \varphi\colon \pi_1(X, x_0) & \to & \mathrm{Aut}(p^{-1}(x_0)) \\ [\gamma] & \mapsto & \left(\begin{array}{rcl} p^{-1}(x_0) & \to & p^{-1}(x_0) \\ & & \text{by the Local System–Covering Lemma 342.5 and} \\ & & \text{the Path-Lifting Proposition 92.6 (1) there exists} \\ g & \mapsto & \text{a unique lift } \widetilde{\gamma}\colon [0,1] \to C \text{ of } \gamma\colon [0,1] \to X, \\ & & \text{with } \gamma(0) = g, \text{ we set } \varphi([\gamma])(g) := \widetilde{\gamma}(1) \end{array} \right) \end{array}$$

is well-defined.

Proof. It follows immediately from the Path-Lifting Proposition 92.6 (2) that the definition does not depend on the choice of the representative $\gamma\colon [0,1] \to X$ of the given element in $\pi_1(X, x_0)$. Furthermore it follows easily from the Local Coefficient System-Lift Lemma 342.4 that given $[\gamma] \in \pi_1(X, x_0)$ the corresponding map $\varphi([\gamma])\colon p^{-1}(x_0) \to p^{-1}(x_0)$ is indeed a group homomorphism, i.e. it defines an element in the endomorphism monoid



$\mathrm{End}(p^{-1}(x_0))$. But one can also easily show that φ is a homomorphism of monoids, which then also implies that each $\varphi(g)$ is actually an automorphism of $p^{-1}(x_0)$. ⊞
Now we need to show that the G-local coefficient system that is defined by φ is isomorphic to the G-local coefficient system we started out with. We consider the map

$$\begin{array}{rcl} \Theta\colon p^{-1}(x_0) \times \widetilde{X}_{x_0} & \to & C \\ (g, [\gamma]) & \mapsto & \begin{array}{c} \text{by the Local System–Covering Lemma 342.5 and} \\ \text{the Path-Lifting Proposition 92.6 (1) there exists a unique} \\ \text{lift } \widetilde{\gamma}\colon [0,1] \to C \text{ of } \gamma\colon [0,1] \to X \text{ with } \gamma(0) = g, \\ \text{we set } \Theta(g, [\gamma]) \coloneqq \widetilde{\gamma}(1). \end{array} \end{array}$$

It follows again almost immediately from the Path-Lifting Proposition 92.6 (2) that the definition does not depend on the choice of the representative $\gamma\colon [0,1] \to X$ of the given point in $\widetilde{X}_{x_0}$. It is straightforward to see that the map Θ descends to a map

$$p^{-1}(x) \times_{\pi_1(X,x_0)} \widetilde{X}_{x_0} \to C.$$

This map commutes with the projections to X and once again one can easily verify that it is an isomorphism on the fibers. Finally note that it follows easily from the Definition 342.1 of a local coefficient system that this implies that the map is in fact an isomorphism of local coefficient systems. ■

342.3. **Pullbacks of local coefficient systems.** The following lemma is the analog of the Bundle–Pullback Proposition 240.21 and the Vector Bundle–Pullback Proposition 265.36:

Proposition 342.11. (Local Coefficient System–Pullback Proposition) Let G be an abelian group. Let $p\colon C \to X$ be a G-local coefficient system and let $h\colon Y \to X$ be a continuous map. We consider the corresponding pullback diagram

$$\begin{array}{ccc} h^*C & \xrightarrow{(c,y)\mapsto y} & C \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{h} & X. \end{array}$$

(1) The map

$$\begin{array}{rcl} q\colon h^*C \coloneqq \{(y,c) \mid y \in Y \text{ and } c \in p^{-1}(h(y))\} & \to & Y \\ (y,c) & \mapsto & y \end{array}$$

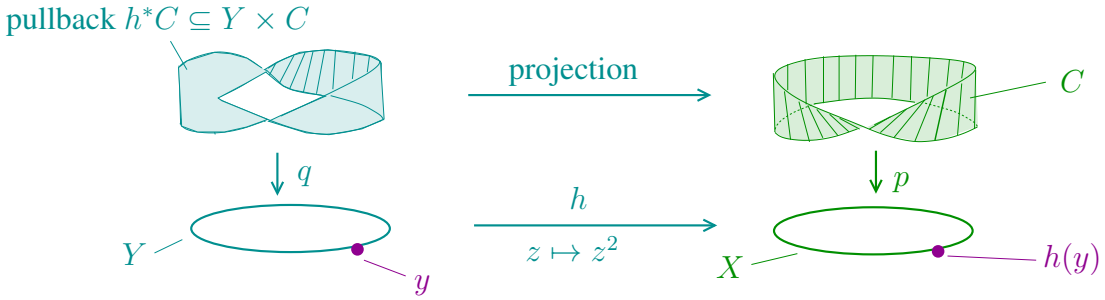
is also a G-local coefficient system where for each $y \in Y$ the abelian group structure on $q^{-1}(y) = \{y\} \times q^{-1}(g(y)) \subseteq Y \times C$ is the obvious abelian group structure coming from the group $q^{-1}(g(y))$. Furthermore the maps h and H define a morphism of G-local coefficient systems.
(2) If we are given a morphism

$$\begin{array}{ccc} D & \xrightarrow{H} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{h} & X \end{array}$$

of G-local coefficient systems which is fiberwise an isomorphism, then the map

$$\begin{array}{rccc} T\colon & D & \to & h^*C \\ & d & \mapsto & (r(d), H(d)) \end{array}$$

is a natural isomorphism of G-local coefficient systems over C.



Proof. The proof is almost identical to the straightforward proofs of the Bundle–Pullback Proposition 240.21 and the Vector Bundle–Pullback Proposition 265.36. ■

Example. Let G be an abelian group.

(1) Let $q\colon C \to X$ be a G-local coefficient system, let $Y \subseteq X$ be a subspace and let $\iota\colon Y \to X$ be the inclusion. As on page 4647 one can easily verify that the maps

$$\begin{array}{rccc} & \iota^*(C) & \to & q^{-1}(Y) \\ & (y, c) & \mapsto & c \end{array} \qquad \text{and} \qquad \begin{array}{rccc} & q^{-1}(Y) & \to & \iota^*(C) \\ & c & \mapsto & (q(c), c) \end{array}$$

give natural isomorphisms over Y between the pullback G-local coefficient system $\iota^*(C)$ and the restriction G-local coefficient system $q\colon q^{-1}(Y) \to Y$, as defined on page 5120.

(2) Let $q\colon G \times X = {}_{\mathrm{tr}}G \to X$ be the trivial G-local coefficient system and let $\varphi\colon Y \to X$ be a continuous map. The map

$$\begin{array}{rccc} & \varphi^*\,{}_{\mathrm{tr}}G & \to & {}_{\mathrm{tr}}G \\ & (y, (g, x)) & \mapsto & (g, y) \end{array}$$

is a natural isomorphism over Y from the pullback of the trivial G-local coefficient system on Y to the trivial G-local coefficient system on Y. □

The next lemma says that pullbacks of local coefficient systems are essentially contravariantly functorial. The lemma is the analog of the Bundle-Pullback–Functorial Lemma 240.24 and the Vector Bundle-Pullback–Functorial Lemma 265.39:

Lemma 342.12. (Local Coefficient System-Pullback–Functorial Lemma)

(1) Let $q\colon C \to X$ be a G-local coefficient system. Then

$$\begin{array}{rccc} & \mathrm{id}_X^*\, C & \to & C \\ & (x, c) & \mapsto & c \end{array}$$

is a natural isomorphism over X from the pullback G-local coefficient system $\mathrm{id}_G^* C \to C$ to the original G-local coefficient system $q\colon C \to X$.

(2) Let $q\colon C \to X$ be a G-local coefficient system. If $\varphi\colon Z \to Y$ and $\psi\colon Y \to X$ are continuous maps between topological spaces, then

$$\begin{array}{ccc} \varphi^*(\psi^* F) & \to & (\psi \circ \varphi)^* F \\ (z,(y,c)) & \mapsto & (z,c) \end{array}$$

is a natural isomorphism over Z from the double pullback G-local coefficient system $\varphi^*(\psi^* F) \to Z$ to the pullback G-local coefficient system $(\psi \circ \varphi)^* F \to Z$.

Proof. The proof is almost identical to the straightforward proofs of the Bundle-Pullback–Functorial Lemma 240.24 and the Vector Bundle-Pullback–Functorial Lemma 265.39. ■

Lemma 342.13. (Local Coefficient System-Pullback-Functor Lemma) Let G be an abelian group. For any continuous map $h\colon Y \to X$ between topological spaces the maps

$$\begin{array}{ccc} C & & h^*C \\ \downarrow{\scriptstyle p} & \mapsto & \downarrow{\scriptstyle q} \\ X & & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array} \quad \mapsto \quad \begin{array}{ccc} h^*C & \xrightarrow{(y,c)\mapsto(y,\varphi(c))} & h^*C' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & Y & \end{array}$$

define a covariant functor from the category $\mathcal{LocalSys}_G(X)$ of G-local coefficient systems over X to the category $\mathcal{LocalSys}_G(C)$ of G-local coefficient systems over Y.

Proof. As in the case of the Bundle-Pullback-Functor Lemma 240.23 the lemma follows easily from the definition: In the proof of the Bundle-Pullback-Functor Lemma 240.23 we just need to replace Bundle–Pullback Proposition 240.21 by the Local Coefficient System–Pullback Proposition 342.11. ■

Example. Let G be an abelian group.

(1) It follows from the Local Coefficient System-Pullback-Functor Lemma 342.13 together with the Isomorphism Preservation Lemma 5.26 that the pullbacks of isomorphic G-local coefficient systems are again isomorphic.

(2) It follows from (1) and the fact that the pullback of a trivial G-local coefficient system is naturally trivializable (see page 4647) that the pullback of a trivializable G-local coefficient system is again a trivializable G-local coefficient system. □

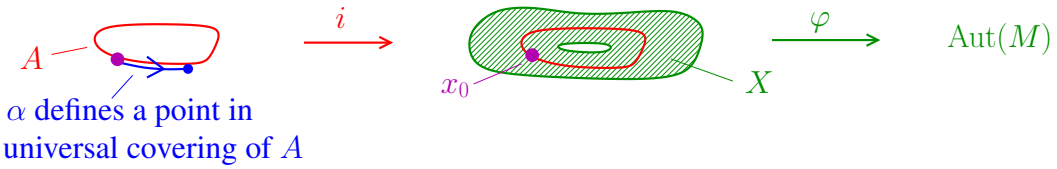
We conclude this section with the following lemma:

Lemma 342.14. (Representation–Restriction Lemma) Let X be a topological space that is path-connected and locally contractible. Let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Furthermore let M be a $\mathbb{Z}[\pi]$-left module, i.e. let M be an abelian group and suppose that we are given a group homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathrm{Aut}(M)$. Let $A \subseteq X$ be a subspace that is also path-connected and locally contractible and that contains x_0. We denote by $i\colon A \to X$ the inclusion map and we write $\Gamma := \pi_1(A, a_0)$. We equip A with the

representation $\varphi \circ i_* \colon \Gamma = \pi_1(A, x_0) \to \mathrm{Aut}(M)$. Then the map

$$\begin{array}{ccc} {}_{\varphi\circ i_*}M & \to & i^*{}_{\varphi}M \\ (m, [\alpha]) & \mapsto & (\alpha(1), \underbrace{[(m, [\alpha])]}_{\in M \times_\pi \widetilde{X}_{x_0}}) \\ \uparrow & & \\ \alpha \text{ is a path in } A & & \\ \text{with } \alpha(0) = x_0 & & \end{array}$$

is a natural isomorphism over A from the M-local coefficient system ${}_{\varphi\circ i_*}M \to A$ corresponding to the representation $\varphi \circ i_* \colon \pi_1(A, x_0) \to \mathrm{Aut}(M)$ to the pullback M-local coefficient system $i^*{}_{\varphi}M \to A$.



Proof. We consider the following heroic map:

$$\begin{array}{ccc} i^*{}_{\varphi}M & \to & {}_{\varphi\circ i_*}M \\ (a, [(v, [\gamma])]) & \mapsto & [(\varphi([\gamma * \overline{\alpha}])(v), \alpha)] \\ \uparrow & & \uparrow \\ \gamma \text{ is a path in } M & & \text{we pick a path } \alpha \text{ in } A \\ \text{with } \gamma(0) = x_0 & & \text{with } \alpha(0) = x_0 \text{ and } \alpha(1) = a \end{array}$$

We leave it to the reader to show that the given continuous maps ${}_{\varphi\circ i_*}M \to i^*{}_{\varphi}M$ and $i^*{}_{\varphi}M \to {}_{\varphi\circ i_*}M$ are well-defined, that they are morphisms of M-local coefficient systems and that they are inverses of one another. ∎

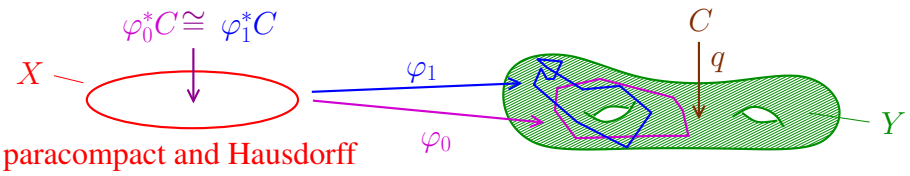
342.4. **Local coefficient systems and homotopies.** In the Pullback-Homotopy Proposition 241.5 and the Bundles-Homotopies Proposition 241.10 (and similarly in the Vector Bundle–Pullback-Homotopy Proposition 267.1 and the Vector Bundles-Homotopies Proposition 267.3) we studied pullbacks of (vector) bundles under homotopic maps. The following two propositions are now the analogs of these two propositions in our present context:

Proposition 342.15. (Local System–Pullback-Homotopy Proposition) Let G be an abelian group and let $q \colon C \to Y$ be a G-local coefficient system. Furthermore let X be a topological space[3711] and let $\varphi_0, \varphi_1 \colon X \to Y$ be two continuous maps.

(1) Every homotopy $\Phi \colon X \times [0, 1] \to Y$ between φ_0 and φ_1 naturally defines an isomorphism between the G-local coefficient systems $\varphi_0^* C \to X$ and $\varphi_1^* C \to X$.
(2) In particular, if the two maps φ_0 and φ_1 are homotopic, then the G-local coefficient systems $\varphi_0^* C \to X$ and $\varphi_1^* C \to X$ are isomorphic over X.

Proposition 342.16. (Local System–Homotopies Proposition) Let G be an abelian group and let X be a topological space. If X is contractible, then every G-local coefficient system over X is trivializable.

[3711] Note that in contrast to the Pullback-Homotopy Proposition 241.5 and the Vector Bundle–Pullback-Homotopy Proposition 267.1 we do *not* have to demand that X is paracompact and Hausdorff. In practice this makes no difference, but it shows that the proof of the Local System–Pullback-Homotopy Proposition 342.15 for once differs from the proofs of its siblings.



The proofs of these two propositions will require the remainder of this section. In the next Section 342.5 we will discuss tensor products of local coefficient systems.

In preparation for the proofs of the two propositions we first need to introduce some objects and we need to state and prove two lemmas:

Definition 342.17. Let G be an abelian group.

(1) If $q\colon C \to X$ is a G-local coefficient system and if Y is a topological space, then the product topological space $C \times Y$ together with the map

$$\begin{aligned} q \times \mathrm{id}_Y\colon C \times Y &\to X \times Y \\ (c, y) &\mapsto (q(c), y) \end{aligned}$$

defines a G-local coefficient system in the obvious way

(2) Let $p\colon C \to X \times [0,1]$ be a G-local coefficient system. Given $t \in [0,1]$ we consider the G-local coefficient system

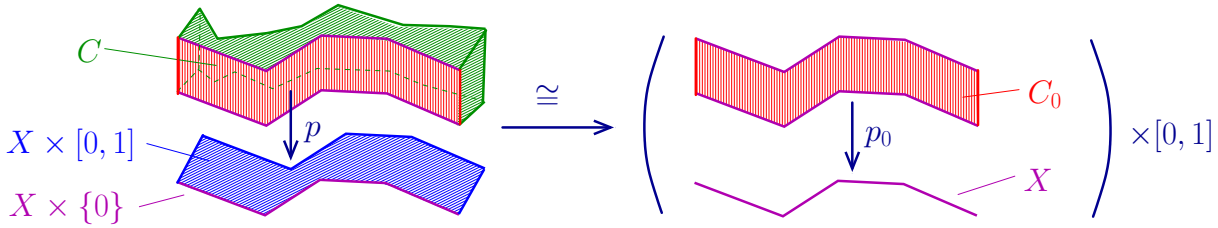
$$p_t\colon C_t \coloneqq p^{-1}(X \times \{t\}) \xrightarrow{p} X \times \{t\} \xrightarrow{(x,t)\mapsto x} X.$$

Lemma 342.18. (Coefficient System-on-Product Lemma) Let X be a topological space and let $p\colon C \to X \times [0,1]$ be a G-local coefficient system. As above we consider the G-local coefficient system $p_0\colon C_0 \coloneqq p_0^{-1}(X \times \{0\}) \to X$. There exists a *unique*[3712] isomorphism of G-local coefficient systems

$$\begin{array}{ccc} C & \xrightarrow[\cong]{\Phi} & C_0 \times [0,1] \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{diag}(p_0, \mathrm{id}_{[0,1]})} & \\ & X \times [0,1] & \end{array}$$

over $X \times [0,1]$ such that for every $x \in X$ and $v \in p^{-1}(x,0)$ we have

$$\Phi(c) \;=\; (c, 0).$$



Proof of Lemma 342.18. We start out with the following claim:

[3712] Here we use that G-local coefficient systems, with their discrete fibers, are much more rigid than vector bundles or other bundles. This greatly simplifies the discussion at this point.

Claim 1. Let $p\colon C \to Y \times [a,c]$ be a G-local coefficient system and let $b \in [a,c]$. If the restrictions of p to $Y \times [a,b]$ and to $Y \times [b,c]$ are both trivializable, then so is the restriction of p to $Y \times [a,c]$.

Proof. Since the restrictions of p to $Y \times [a,b]$ and to $Y \times [b,c]$ are both trivializable we can pick trivializations $\Phi\colon p^{-1}(Y \times [a,b]) \to G \times Y \times [a,b]$ and $\Psi\colon p^{-1}(Y \times [b,c]) \to G \times Y \times [b,c]$. We denote by $q\colon G \times Y \times [a,c] \to G$ the projection. Note that it follows almost immediately from the Trivialization–Classification Proposition 342.8 that there exists a continuous map $\Theta\colon Y \times \{b\} \to \mathrm{Aut}(G)$ such that for all points $v \in p^{-1}(Y \times \{b\})$ we have the equality $q(\Phi(v)) = \Theta(p(v)) \cdot q(\Psi(v)) \in G$. It is straightforward to verify that

$$\begin{array}{rcl} p^{-1}(Y \times [a,c]) & \to & G \times Y \times [a,c] \\ v & \mapsto & \begin{cases} \Phi(v), & \text{if } p(v) \in Y \times [a,b], \\ (p(v), \Theta(p(y,b)) \cdot q(\Psi(v))), & \text{if } p(v) = (y,t) \in Y \times [b,c] \end{cases} \end{array}$$

is an isomorphism of G-local coefficient systems. ⊞

Now we turn to the actual setting of the lemma. Thus let $p\colon C \to X \times [0,1]$ be a G-local coefficient system over a topological space X.

Claim 2. For every $x \in X$ there exists an open neighborhood U_x of $x \in X$ and a trivialization

$$\Theta_x\colon p^{-1}(U_x \times [0,1]) \to G \times U_x \times [0,1]$$

of the restriction of the G-local coefficient system $p\colon E \to X \times [0,1]$ to $U_x \times [0,1]$.

Proof. Let $x \in X$. By Definition 342.1 of a local coefficient system and by compactness of $[0,1]$ there exist neighborhoods $V_1, \dots, V_m$ of x and $0 = t_0 < \cdots < t_m = 1$ such that the restriction of $p\colon E \to X \times [0,1]$ to each $V_i \times [t_{i-1}, t_i]$ is trivializable. It follows from Claim 1 that the restriction of p to $(V_1 \cap \cdots \cap V_m) \times [0,1]$ is also trivializable. Thus $U_x \coloneqq V_1 \cap \cdots \cap V_m$ admits such a trivialization Φ_x. ⊞

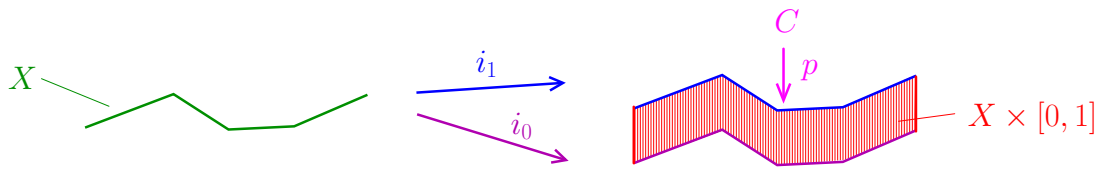
We denote by $r\colon X \times [0,1] \to X$ and $s\colon X \times [0,1] \to [0,1]$ the natural projections. Now let $x \in X$. We define

$$\begin{array}{rcl} \Phi_{U_x}\colon p^{-1}(U_x \times [0,1]) & \to & p^{-1}(U_x \times \{0\}) \times [0,1] \\ v & \mapsto & \begin{array}{c} \text{we write } \Theta_x(v) = (g,y,t) \in G \times U_x \times [0,1] \text{ and} \\ \text{we set } \Phi_{U_x}(v) = (\Theta_x(y,0,g), t). \end{array} \end{array}$$

One can easily verify that this map is continuous and that it is an isomorphism on the fibers. Let $y \in X$. It follows from the Trivialization–Classification Proposition 342.8 together with the Connected-to-Discrete Lemma 4.37 applied to the map $\{y\} \times [0,1] \to p^{-1}(\{(y,0)\})$ that this map does not depend on the choice of $x \in X$ with $y \in U_x$. Thus it follows from the Pasting Proposition 4.7 that these maps Φ_x define a continuous map $\Phi\colon C \to C_0 \times [0,1]$ that is an isomorphism of local coefficient systems. Basically by definition we have $\Phi(v) = (v,0)$ for every $x \in X$ and $v \in p^{-1}(x,0)$. ■

The following lemma is somewhat similar to the Bundle-over-Product Lemma 241.6 and the Vector Bundle-over-Product Lemma 267.2:

Lemma 342.19. (Local Coefficients-over-Product Lemma) Let X be a topological space. We denote by $i_0\colon X \xrightarrow{x \mapsto (x,0)} X \times [0,1]$ and $i_1\colon X \xrightarrow{x \mapsto (x,1)} X \times [0,1]$ the natural embeddings. Given any G-local coefficient system $p\colon C \to X \times [0,1]$ the pullbacks i_0^*C and i_1^*C are naturally isomorphic over X.



Proof of Lemma 342.19. Let $\Phi\colon C \to C_0 \times [0,1]$ be the unique isomorphism of G-local coefficient systems over $X \times [0,1]$ from the Coefficient System-on-Product Lemma 342.18. Note that for $t \in [0,1]$ we have

$$i_t^*(C_0 \times [0,1]) \;=\; \{(x,v,t) \in X \times C_0 \times [0,1] \,|\, v \in p^{-1}(x,0)\}.$$

Then we see that

$$\begin{array}{ccccccc} i_0^*C & \cong & i_0^*(C_0 \times [0,1]) & \cong & i_1^*(C_0 \times [0,1]) & \cong & i_1^*C. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{via the isomorphism } \Phi & & \text{a natural isomorphism is} & & \text{via the isomorphism } \Phi^{-1} & \\ & & & \text{given by } (x,v,0) \mapsto (x,v,1) & & & \end{array}$$

■

The same way as we deduced the Vector Bundle–Pullback-Homotopy Proposition 267.1 and the Vector Bundles-Homotopies Proposition 267.3 from the Vector Bundle-over-Product Lemma 267.2 we can now deduce the Local System–Pullback-Homotopy Proposition 342.15 and the Local System–Homotopies Proposition 342.16 from the Local Coefficients-over-Product Lemma 342.19:

Proof of the Local System–Pullback-Homotopy Proposition 342.15. Let G be an abelian group, let $q\colon C \to Y$ be a G-local coefficient system, let X be a topological space and let $H\colon X \times [0,1] \to Y$ be a homotopy between two maps $f_0, f_1\colon X \to Y$. We denote by $i_0, i_1\colon X \to X \times [0,1]$ the two obvious inclusions. Now we see that we have the following natural isomorphisms of G-local coefficient systems over X:

$$\begin{array}{ccccccccccc} & & & & \text{follows from the Local} & & & & & & \\ & & & & \text{Coefficients-over-Product} & & & & & & \\ & & & & \text{Lemma 342.19} & & & & & & \\ \text{since } f_0 = H \circ i_0 & & & & \downarrow & & & & & & \text{since } f_1 = H \circ i_1 \\ f_0^*C & = & (H \circ i_0)^*C & \cong & i_0^*(H^*C) & \cong & i_1^*(H^*C) & \cong & (H \circ i_1)^*C & = & f_1^*C. \\ & & & & \uparrow & & & & \uparrow & & \\ & & & & \text{by the Local Coefficient System-Pullback} & & & & & & \\ & & & & \text{Functorial Lemma 342.12 (2)} & & & & & & \end{array}$$

■

Proof of the Local System–Homotopies Proposition 342.16. Let $C \to X$ be a G-local coefficient system over a topological space that is contractible. We pick $x_0 \in X$. By the Contractability Criterion Lemma 19.20 the inclusion map $i\colon \{x_0\} \to X$ is a homotopy equivalence. We denote by $p\colon X \to \{x_0\}$ the unique map. Note that $p \circ i$ is homotopic to id_X. Thus we have the following isomorphisms of G-local coefficient systems over X:

$$\begin{array}{ccccccc} & & & & \text{by the Local System–Pullback} & & \\ & & & & \text{Homotopy Proposition 342.15} & & \\ & & & & \downarrow & & \\ C & \cong & (\mathrm{id}_X)^*C & \cong & (i \circ p)^*C & \cong & p^*(i^*C). \\ & \uparrow & & & & \uparrow & \\ & \text{follows from the Local Coefficient System-Pullback} & & & & & \\ & \text{Functorial Lemma 342.12 (1) and (2)} & & & & & \end{array}$$

Note that i^*C is a local coefficient system on a point, hence trivializable. Thus C is isomorphic to the pullback of a trivializable local coefficient system. Thus we obtain from the Local Coefficient System-Pullback-Functor Lemma 342.13 (see also the discussion on page 6481) that C is trivializable. ■

342.5. **Tensor products of local coefficient systems.** The reader will have noticed that G-local coefficient systems are somewhat similar in spirit to vector bundles. In Section 268.4 we saw that many well-known constructions in linear algebra also work for vector bundles. For example we saw how to construct the direct sum and the tensor product of two vector bundles. Similarly we can now construct new local coefficient systems out of given ones.

We will not develop the full theory, since right now we do not need it. But we will provide the definition of the tensor product of two local systems, since we will need it in the next chapter, when we discuss the cup product on cohomology with local coefficients. More precisely, we have the following definition, which is modelled on the Definition 268.29 of the tensor product of vector bundles:

Definition 342.20. Let X be a topological space, let $p\colon C \to X$ be a G-local coefficient system and let $q\colon D \to X$ be an H-local coefficient system. We consider the set

$$C \otimes_{\mathbb{Z}} D \;:=\; \bigsqcup_{x\in X} \underbrace{p^{-1}(x) \otimes_{\mathbb{Z}} q^{-1}(x)}_{\substack{\text{abelian group as}\\ \text{defined explicitly}\\ \text{in Definition 159.2}}} \quad \text{with the obvious map } r\colon C \otimes_{\mathbb{Z}} D \to X.$$

We equip $C \otimes_{\mathbb{Z}} D$ with the smallest topology that has the property that for any open subset $U \subseteq X$ and any two trivializations $\Phi\colon p^{-1}(U) \to G \times U$ and $\Psi\colon q^{-1}(U) \to U \times H$ the map

$$\begin{array}{ccc} r^{-1}(U) & \to & U \times \overbrace{(G \otimes_{\mathbb{Z}} H)}^{\text{discrete topology}} \\ c_x \otimes d_x & \mapsto & (x, \Phi_x(c_x) \otimes \Psi_x(d_x)) \end{array}$$

is a homeomorphism. One can now easily show that $r\colon C \otimes_{\mathbb{Z}} D \to X$ is a $G \otimes_{\mathbb{Z}} H$-local coefficient system. It is called the tensor product of the local coefficient systems C and D. We proceed similarly for the tensor product of two local coefficient systems that are modules over a fixed ring R.

Lemma 342.21. (Local Coefficients–Tensor Product Lemma) Let X be a topological space, let G be an abelian group and let $p\colon C \to X$ be a G-local coefficient system.

(1) The map

$$\begin{array}{ccc} \mu\colon C \otimes_{\mathbb{Z}\ {}_{\mathrm{tr}}\mathbb{Z}} & \to & C \\ (c \otimes n) & \mapsto & n \cdot c \end{array}$$

is a natural isomorphism over X of G-local coefficient systems.[3713]

(2) If $q\colon D \to X$ is another local coefficient system, then

$$\begin{array}{ccc} \tau\colon C \otimes_{\mathbb{Z}} D & \to & D \otimes_{\mathbb{Z}} C \\ c \otimes d & \mapsto & d \otimes c \end{array}$$

is a natural isomorphism over X of local coefficient systems.

[3713]Note that if G and G' are isomorphic abelian groups, then a G-local coefficient system is the same as a G'-local coefficient system. In particular every $G \otimes_{\mathbb{Z}} \mathbb{Z}$-local coefficient system is also a G-local coefficient system.

Proof. We leave the rather uninspiring verification of both statements to the reader. ■

342.6. Exercises for Chapter 342.

Exercise 342.1. Show that every $\mathbb{Z}/2\mathbb{Z}$-local coefficient system over every topological space is trivializable.

Exercise 342.2. Let X be a topological space that is path-connected and locally contractible.

(a) Show that there exists a covariant functor from the category of n-dimensional real vector bundles over X to the category of $\mathbb{R}^n$-vector bundle local systems over X, where the assignment is given by "discretizing the fiber".
(b) Is the functor in (a) surjective on the set of isomorphism classes?
(c) Is the functor in (a) injective on the set of isomorphism classes?

Exercise 342.3. Let X be a topological space that is path-connected and locally contractible and let $x_0 \in X$. We write $\pi := \pi_1(X, x_0)$. Let M and N be abelian groups and let $\varphi\colon \pi \to \mathrm{Aut}(M)$ and $\psi\colon \pi \to \mathrm{Aut}(N)$ be representations. We consider the corresponding $\mathbb{Z}[\pi_1(X, x_0)]$-left modules M and N and we consider the corresponding $\mathbb{Z}[\pi_1(X, x_0)]$-left module $M \otimes_{\mathbb{Z}} N$ that is given by the representation

$$\begin{array}{rcl} \varphi \otimes \psi\colon \pi & \to & \mathrm{Aut}(M \otimes_{\mathbb{Z}} N) \\ g & \mapsto & ((m \otimes n) \mapsto \varphi(g)(m) \otimes \psi(g)(n)). \end{array}$$

As we saw in the Local System-via-Representation Proposition 342.9 these modules give rise to local coefficient systems that we denote by ${}_{\varphi}M$, ${}_{\varphi}N$ and ${}_{\varphi\otimes\psi}M \otimes_{\mathbb{Z}} N$. Show that the tensor product of the local coefficient systems ${}_{\varphi}M$ and ${}_{\varphi}N$ is isomorphic to the local system ${}_{\varphi\otimes\psi}(M \otimes_{\mathbb{Z}} N)$.

Exercise 342.4. Let X be a connected non-empty n-dimensional topological manifold with $\partial X = \varnothing$. We consider the set

$$C \;:=\; \bigsqcup_{x \in X} \mathrm{H}_n(X, X \setminus \{x\}).$$

Given an open subset $U \subseteq X$ and an element $\varphi \in \mathrm{H}_n(X, X \setminus U)$ we consider

$$M(U, \varphi) \;:=\; \bigsqcup_{x \in u} \text{image of } \varphi \text{ in } \mathrm{H}_n(X, X \setminus \{x\}).$$

(a) Show that the subsets $M(U, \varphi)$ of C have the basis property that we introduced in Definition 3.50.
(b) We equip C with the topology that is defined by the sets $M(U, \varphi)$ and the Topology-from-Basis Lemma 3.51. Show that the natural projection $C \to X$ is a local coefficient system for X.
(c) We pick a base point $x_0 \in X$. Let $w_1(\pi_1(X, x_0)) \to \mathbb{Z}/2\mathbb{Z}$ be the orientation-character from Definition 191.31. As in Exercise 268.15 we see, using the Local System-via-Representation Proposition 342.9, that this orientation character gives rise to a local coefficient system ${}_{w_1}\mathbb{Z}$. Show that this local coefficient system ${}_{w_1}\mathbb{Z}$ is isomorphic to the local coefficient system from (b).

Remark. This local coefficient system is called the *orientation local coefficient system*.

CHAPTER 343

(Co-) homology with local coefficients

In the last Chapter 342 we introduced the rather dry topic of local coefficient systems over a topological space. In this chapter we will use local coefficient systems to introduce (co-) homology with local coefficients. As we will see, this theory is closely related to the theory of twisted (co-) homology.

This chapter is organized as follows:

- In Section 343.1 we will introduce homology with local coefficients.
- In Section 343.2 we will introduce cohomology with local coefficients.
- In Section 343.3 we will introduce the cup product for cohomology with local coefficients.
- In Section 343.4 we will introduce the cap product for (co-) homology with local coefficients.
- In Section 343.5 we will study the interaction between twisted (co-) homology and (co-) homology with local coefficients.
- In Section 343.6 we will discuss Poincaré Duality for (co-) homology with local coefficients.
- In Section 343.7 we will provide exercises for Chapter 343.

Before the reader heads into this chapter, one warning: Since we will not make use of the results of this chapter we will have an abundance of sketches of proofs. Working out the details could easily be a separate book in its own right.

343.1. **Homology with local coefficients.** In this section we will develop the theory of homology with local coefficients.

343.1.1. **Definition of homology with local coefficients.** In the following we give the slightly tricky definition of singular simplices with coefficients in a local system:

Definition 343.1. Let X be a topological space and let G be an abelian group. Furthermore let $p\colon E \to X$ be a G-local coefficient system in the sense of Definition 342.1.[3714 3715]

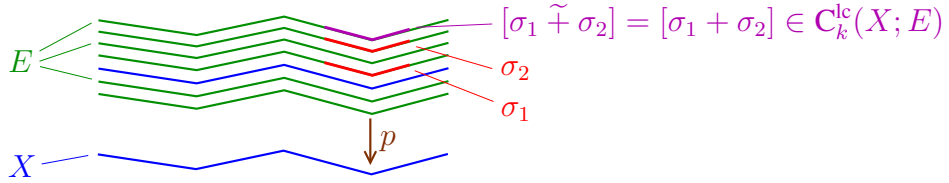
(1) Given two singular k-simplices $\sigma_1, \sigma_2\colon \Delta^k \to E$ with $p \circ \sigma_1 = p \circ \sigma_2$ we define[3716 3717]

$$\begin{array}{rccc} \sigma_1 \widetilde{+} \sigma_2\colon \Delta^k & \to & E \\ x & \mapsto & \underbrace{\sigma_1(x) + \sigma_2(x)}_{\text{sum in the fiber } E_x}. \end{array}$$

(2) As usual we denote by $\mathrm{C}_k(E)$ the set of singular k-chains in the topological space E.

- For the purpose of this definition we consider the subgroup $N_k(E)$ of $\mathrm{C}_k(E)$ which is generated by all elements of the form $(\sigma_1 + \sigma_2) - (\sigma_1 \widetilde{+} \sigma_2)$ where σ_1, σ_2 are any two singular k-simplices in E with $p \circ \sigma_1 = p \circ \sigma_2$.
- Now we set[3718]
$$\mathrm{C}_k^{\mathrm{lc}}(X; E) \coloneqq \mathrm{C}_k(E)/N_k(E).$$

(3) Given a singular k-simplex $\sigma\colon \Delta^k \to E$ we refer to the corresponding element in $\mathrm{C}_k^{\mathrm{lc}}(X;E)$ as a singular k-simplex with coefficients in E.
(4) We refer to an element of $\mathrm{C}_k^{\mathrm{lc}}(X;E)$ as a singular k-chain with coefficients in E.



Convention 343.2. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. To preserve sanity in our notation we will usually not distinguish between a singular k-simplex with coefficients in E and the element it represents in $\mathrm{C}_k^{\mathrm{lc}}(X;E)$.

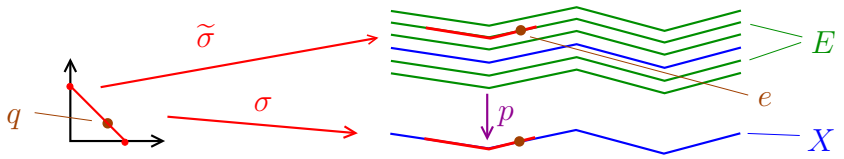
Example. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system corresponding to the trivial group. Let $\sigma\colon \Delta^k \to E$ be a singular k-complex with coefficients in E. Note that in $\mathrm{C}_k^{\mathrm{lc}}(X;E)$ we have the following equalities:

$$\sigma + \sigma \underset{\text{follows from Definition 343.1 (2)}}{=} \sigma \mathbin{\widetilde{+}} \sigma = \sigma \underset{\text{by Definition 343.1 (1) and since fiberwise we are dealing with the trivial group}}{=} \sigma.$$

This implies of course that $[\sigma] = 0 \in \mathrm{C}_k^{\mathrm{lc}}(X;E)$. This shows that $\mathrm{C}_k^{\mathrm{lc}}(X;E) = 0$. □

the following elementary lemma gets used frequently:

Lemma 343.3. (Unique Simplex Lifting Lemma) Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Furthermore let $\sigma\colon \Delta^n \to X$ be a singular simplex, let $q \in \Delta^n$ and let $e \in E$ such that $p(e) = \sigma(q)$. There exists a unique singular simplex $\widetilde{\sigma}\colon \Delta^n \to E$ with $p \circ \widetilde{\sigma} = \sigma\colon \Delta^n \to X$ and $\widetilde{\sigma}(q) = e$.



Proof. By the Local System–Covering Lemma 342.5 we know that $p\colon E \to X$ is a covering map. Thus the statement follows from general covering theory. More precisely, it follows from the Map Lifting Criterion 127.5 together with the observation that Δ^n is simply connected and locally path-connected. ■

In Definition 143.13 we introduced the boundary maps in the ordinary singular chain complexes. Since $\mathrm{C}_k^{\mathrm{lc}}(X;E)$ is a quotient of $\mathrm{C}_k(E)$ we do not need to be very creative when it comes to defining a boundary map on $\mathrm{C}_k^{\mathrm{lc}}(X;E)$:

[3716] By the Local Coefficient System-Lift Lemma 342.4 we know that $\sigma_1 \mathbin{\widetilde{+}} \sigma_2\colon \Delta^k \to E$ is continuous.
[3717] Since G is abelian we see that $\sigma_1 \mathbin{\widetilde{+}} \sigma_2 = \sigma_2 \mathbin{\widetilde{+}} \sigma_1$. It is a fun exercise to figure out where this actually gets used.
[3718] In plain English this means that for any two singular k-simplices σ_1, σ_2 in E with $p \circ \sigma_1 = p \circ \sigma_2$ we have $[\sigma_1] + [\sigma_2] = [\sigma_1 \mathbin{\widetilde{+}} \sigma_2] \in \mathrm{C}_k^{\mathrm{lc}}(X;E)$.

Definition 343.4. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. We refer to[3719]

$$\begin{array}{ccc} \partial_n\colon \mathrm{C}_k^{\mathrm{lc}}(X;E) & \to & \mathrm{C}_{k-1}^{\mathrm{lc}}(X;E) \\ \Big[\underbrace{\sum_{i=1}^{k} a_i \cdot (\sigma_i\colon \Delta^n \to E)}_{\in \mathrm{C}_k(E)}\Big] & \mapsto & \Big[\sum_{i=1}^{k} \underbrace{a_i \cdot \partial_k(\sigma_i)}_{\in \mathrm{C}_{k-1}(E)}\Big] \end{array}$$

as the k-th boundary map.

By the Double Boundary Proposition 143.14 we know that each $\partial_{k-1}\circ\partial_k\colon \mathrm{C}_k(E) \to \mathrm{C}_{k-2}(E)$ is the zero map. It follows immediately that each $\partial_{k-1}\circ\partial_k\colon \mathrm{C}_k^{\mathrm{lc}}(X;E) \to \mathrm{C}_{k-2}^{\mathrm{lc}}(X;E)$ is also the zero map. In other words, $(\mathrm{C}_*^{\mathrm{lc}}(X;E), \partial_*)$ is a chain complex. Thus we can define the corresponding homology groups:

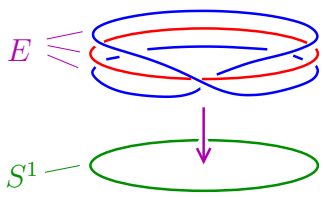
Definition 343.5. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Given $n \in \mathbb{N}_0$ we define the n-th singular homology group of X with E-coefficients as the quotient group

$$\mathrm{H}_n^{\mathrm{lc}}(X;E) \;:=\; \frac{\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{lc}}(X;E) \to \mathrm{C}_{n-1}^{\mathrm{lc}}(X;E))}{\mathrm{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{lc}}(X;E) \to \mathrm{C}_n^{\mathrm{lc}}(X;E))}.$$

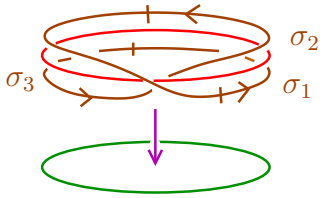
Often we refer this type of a homology groups as homology with local coefficients.

Remark. The literature contains many different, but essentially equivalent, definitions of homology groups with coefficients in a local coefficient system. We refer to [**KwS2018**, p. 284], [**DaK2001**, Chapter 5.3], [**WhdG1978**, Chapter VII.2] for slightly different definitions. Our definition is basically the same as the definition given in [**Hat2002**, Chapter 3.H]. There seem to be only few sources that study homology with local coefficients in detail, the most thorough source seems to be [**WhdG1978**, Chapter VII]. □

Example. We consider again the $\mathbb{Z}/3\mathbb{Z}$-local system $p\colon E \to S^1$ from page 6473.



$\mathbb{Z}/3\mathbb{Z}$-local system over S^1



$\sigma_1 + \sigma_2 + \sigma_3$ is a cycle in $\mathrm{C}_1^{\mathrm{lc}}(S^1;E)$

It follows easily from the definitions that the singular 1-chain $\sigma_1 + \sigma_2 + \sigma_3$ with coefficients in E, which is shown above, is in fact a cycle, i.e. it defines an element of $\mathrm{H}_1^{\mathrm{lc}}(S^1;E)$. □

343.1.2. **Functoriality of homology with local coefficients.** From ordinary homology we are used to the fact that maps between topological spaces induce homomorphisms between homology groups. A similar statements holds for homology with local coefficients, but, as in the case of homology with twisted coefficients that we studied on Definition 321.8, some care needs to be taken to work with matching local coefficient systems.

[3719] Let $i_j^k\colon \Delta^{k-1} \to \Delta^k$ be the face maps from Definition 143.12. Let $\sigma_1, \sigma_2\colon \Delta^k \to X$ be two singular simplices with $p\circ\sigma_1 = p\circ\sigma_2$. It follows easily from the definitions that $i_j^k \circ \sigma_1 \mathbin{\widetilde{+}} i_j^k \circ \sigma_2 = i_j^k \circ (\sigma_1 \mathbin{\widetilde{+}} \sigma_2)$. It follows from this observation that the usual boundary map ∂_n descends to a map $\mathrm{C}_k^{\mathrm{lc}}(X;E) \to \mathrm{C}_{k-1}^{\mathrm{lc}}(X;E)$.

Definition 343.6. Let G be an abelian group, let $f\colon X \to Y$ be a continuous map between topological spaces and let $p\colon E \to Y$ be a G-local coefficient system on Y. We consider the pullback G-local coefficient system $f^*E \to X$ from Definition 342.6. We denote by $q\colon f^*E \to E$ the projection. We define

$$\begin{array}{rcl} f_*\colon \mathrm{C}_k^{\mathrm{lc}}(X; f^*E) & \to & \mathrm{C}_k^{\mathrm{lc}}(Y; E) \\ [\sigma\colon \Delta^k \to f^*E] & \mapsto & [q \circ \sigma\colon \Delta^k \to E]. \end{array}$$

One can easily verify that these maps are well-defined and that they form a chain map. Thus we obtain induced maps

$$f_*\colon \mathrm{H}_k^{\mathrm{lc}}(X; f^*E) \to \mathrm{H}_k^{\mathrm{lc}}(Y; E).$$

The previous definition now also allows us to introduce relative homology with local coefficients:

Definition 343.7. Let G be an abelian group.

(1) Let X be a topological space, let $A \subseteq X$ and let $p\colon E \to X$ be a G-local coefficient system. We denote by $i\colon A \to X$ the inclusion. We set

$$\mathrm{C}_n^{\mathrm{lc}}(X, A; E) \coloneqq \operatorname{coker}(i_*\colon \mathrm{C}_n^{\mathrm{lc}}(A; i^*E) \to \mathrm{C}_n^{\mathrm{lc}}(X; E)).$$

It follows from elementary considerations (see e.g. Exercise 313.3) that the chain map on $\mathrm{C}_n^{\mathrm{lc}}(X; E)$ descends to a chain map on $\mathrm{C}_n^{\mathrm{lc}}(X, A; E)$. Thus we can set

$$\mathrm{H}_n^{\mathrm{lc}}(X, A; E) \coloneqq \mathrm{H}_n(\mathrm{C}_*^{\mathrm{lc}}(X, A; E)).$$

(2) If we are given a continuous map $f\colon (X, A) \to (Y, B)$ of pairs of topological spaces and if we are given a G-local coefficient system E on Y, then as before we get induced chain maps

$$f_*\colon \mathrm{C}_k^{\mathrm{lc}}(X, A; f^*E) \to \mathrm{C}_k^{\mathrm{lc}}(Y, B; E)$$

and thus induced homomorphisms

$$f_*\colon \mathrm{H}_k^{\mathrm{lc}}(X, A; f^*E) \to \mathrm{H}_k^{\mathrm{lc}}(Y, B; E).$$

The following lemma, which can be viewed as a generalization of the Homology-Coefficients-Functor Lemma 314.10 and an analog of the Functorial-in-Coefficients Lemma 321.13 says that singular homology with local coefficients is covariantly functorial in the local coefficient system.

Lemma 343.8. (Functorial-in-Coefficients Lemma) Let G be an abelian group, let X be a topological space and let $A \subseteq X$.

(1) Let $\varphi\colon E \to F$ be a morphism over X from a G-local coefficient system $p\colon E \to X$ to an H-local coefficient system $q\colon F \to X$. The maps

$$\begin{array}{rcl} \varphi_*\colon \mathrm{C}_k^{\mathrm{lc}}(X, A; E) & \to & \mathrm{C}_k^{\mathrm{lc}}(X, A; F) \\ [\sigma\colon \Delta^k \to E] & \mapsto & [\varphi \circ \sigma\colon \Delta^k \to F] \end{array}$$

are a well-defined chain map and thus they induce homomorphisms

$$\varphi_*\colon \mathrm{H}_k^{\mathrm{lc}}(X, A; E) \to \mathrm{H}_k^{\mathrm{lc}}(X, A; F).$$

(2) With the obvious interpretations we have $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ and $\mathrm{id}_* = \mathrm{id}$.

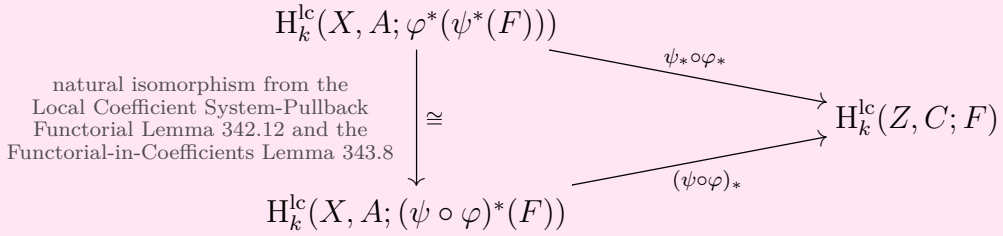
Proof. We leave the elementary proof of the lemma to the reader. ∎

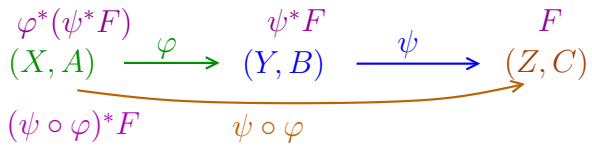
The following lemma shows that homology with local coefficients is "almost" covariantly functorial in the topological space:

Lemma 343.9. (LC-Homology Functoriality Lemma) Let G be an abelian group, let $p\colon F \to Z$ be a G-local coefficient system on a topological space Z and let $C \subseteq Z$.

(1) The natural isomorphism between the G-local coefficient systems $\mathrm{id}^* F$ and F that we gave in the Local Coefficient System-Pullback–Functorial Lemma 342.12 (1) induces an isomorphism $\mathrm{id}_*\colon \mathrm{H}_k^{\mathrm{lc}}(Z, C; \mathrm{id}^* F) \to \mathrm{H}_k^{\mathrm{lc}}(Z, C; F)$.

(2) Let $\varphi\colon (X, A) \to (Y, B)$ and $\psi\colon (Y, B) \to (Z, C)$ be continuous maps between pairs of topological spaces. We have the following commutative diagram:





Remark. The LC-Homology Functoriality Lemma 343.9 plays the same role as the Twisted Homology-Functoriality Proposition 321.9 did for homology with twisted coefficients. Note that in this context the statement for homology with twisted coefficients is much neater, since we do not need the awkward vertical map to go from one coefficient system to another. On the other hand, when we are dealing with twisted coefficients we have to constantly worry about base points. □

Proof. All the statements follow immediately from the definitions. We leave it to the reader to provide the details. ■

We formulate our first example of homology with local coefficients as a lemma. It says (similar to the Twisted–Trivial Coefficients Lemma 320.13) that homology with local coefficients in a trivial G-local coefficient system is basically the same as our old friend, namely singular homology with G-coefficients.

Lemma 343.10. (Trivial Local Coefficients Lemma) Let X be a topological space, let $A \subseteq X$ and let G be an abelian group. We denote by $p\colon {}_{\mathrm{tr}}G := G \times X \to X$ the corresponding trivial G-local coefficient system that we introduced in Definition 342.2.

(1) The maps

$$\begin{array}{rccl} T\colon & \mathrm{C}_k(X, A) \otimes_{\mathbb{Z}} G & \to & \mathrm{C}_k^{\mathrm{lc}}(X, A; {}_{\mathrm{tr}}G) = \mathrm{C}_k^{\mathrm{lc}}(X, A; G \times X) \\ & \sigma \otimes g & \mapsto & \text{the lift } \sigma \to G \times X \text{ given by } x \mapsto (g, \sigma(x)) \end{array}$$

define a chain isomorphism that is natural in the sense that given a homomorphism $\varphi\colon G \to H$ between abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_k(X,A;G) & \xrightarrow{\varphi_*} & \mathrm{C}_k(X,A;H) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathrm{C}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G) & \xrightarrow{\varphi_*} & \mathrm{C}_k^{\mathrm{lc}}(X,A;H\,{}_{\mathrm{tr}}H) \end{array}$$

and it is natural in the sense that given any continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{C}_k(X,A;G) & & \xrightarrow{f_*} & & \mathrm{C}_k(Y,B;G) \\ {\scriptstyle T}\downarrow & & & & \downarrow{\scriptstyle T} \\ \mathrm{C}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G) & \xrightarrow[\text{page 6480}]{\text{isomorphism from}} & \mathrm{C}_k^{\mathrm{lc}}(X,A;f^*({}_{\mathrm{tr}}G)) & \xrightarrow{f_*} & \mathrm{C}_k^{\mathrm{lc}}(Y,B;{}_{\mathrm{tr}}G). \end{array}$$

(2) The natural chain isomorphisms from (1) give rise to isomorphisms

$$\mathrm{H}_k(X,A;G) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{lc}}(X,A;{}_{\mathrm{tr}}G)$$

that are natural in the above sense.

Proof. We denote by $p\colon G\times X \to X$ and $q\colon G\times X \to G$ the two projections. We consider the map

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{lc}}(X,A;G\times X) & \to & \mathrm{C}_k(X,A)\otimes_{\mathbb{Z}} G \\ [\sigma\colon \Delta^k \to G\times X] & \mapsto & \sigma\otimes g, \text{where } g \text{ is the unique } g\in G \text{ with } q(\sigma(\Delta^k)) = \{g\}. \end{array}$$

We leave it to the reader to show that the two given maps are chain maps, that they are natural in the stated sense and that they are inverses of one another. ■

343.1.3. **Homology with local coefficients and homotopies.** In the Homotopy-H_*-Proposition 144.12 and the Homotopy-H_*-Proposition 146.11 we showed that homotopic maps induce chain homotopic maps on (relative) singular chain complexes. In the Homotopy-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.3 we saw that an analogous statement also holds for (co-) homology with twisted coefficients. The following proposition is the "local coefficient system" analog.

Proposition 343.11. (Homotopy-$\mathrm{H}_*^{\mathrm{lc}}$-Proposition) Let G be an abelian group, let $h_0, h_1\colon (X,A) \to (Y,B)$ be two continuous maps between pairs of topological spaces and let $p\colon F \to Y$ be a G-local coefficient system on Y. If $H\colon X\times[0,1] \to Y$ is a homotopy between h_0 and h_1, then we obtain from the Local System–Pullback-Homotopy Proposition 342.15 an explicit isomorphism Θ_H between the G-local coefficient systems h_0^*F and h_1^*F. If for each $t\in[0,1]$ we have $H_t(A)\subseteq B$, then the maps

$$\begin{array}{rcl} (h_0)_*\colon \mathrm{C}_k^{\mathrm{lc}}(X,A;h_0^*F) & \to & \mathrm{C}_k^{\mathrm{lc}}(Y,B;F) \\ \text{and} \quad (h_1)_*\circ(\Theta_H)_*\colon \mathrm{C}_k^{\mathrm{lc}}(X,A;h_0^*F) & \to & \mathrm{C}_k^{\mathrm{lc}}(X,A;h_1^*F) \to \mathrm{C}_k^{\mathrm{lc}}(Y,B;F) \end{array}$$

are chain homotopic, in particular they induce the same map on homology.

The following proposition is the ever-so-slightly imprecise sibling of the Homotopy Equivalence-$\mathrm{H}_*^{\mathrm{tw}}$-Proposition 322.6:

[3719] By the Unique Simplex Lifting Lemma 343.3 we know that such a lift exists and that it is unique.

Proposition 343.12. (Homotopy Equivalence-H_*^{lc}-Proposition) Homotopy equivalences induce isomorphisms on homology groups with local coefficients.

Proof. We leave it to the reader to make the statement of this proposition precise and to prove it. ■

In the Homotopy-H_*-Proposition 144.12 we showed the analogous statement for ordinary singular homology. The idea for the proof of the Homotopy-H_*^{lc}-Proposition 343.11 is that, by carefully reading the proof of the Homotopy-H_*-Proposition 144.12, we can modify the proof of the Homotopy-H_*-Proposition 144.12 to give us the statement we are presently interested in. It is thus convenient to recall the main concepts behind the proof of the Homotopy Equivalence-H_*-Proposition 144.14:

Lemma 322.5. (Homotopy–Chain Homotopy Lemma) Let W and F be topological spaces and let $\widetilde{H}\colon W \times [0,1] \to F$ be a homotopy. Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$, $n \in \mathbb{N}_0$, be the sequence of singular chains from Notation 144.16.[3720] The homomorphisms

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(W) & \to & \mathrm{C}_{n+1}(F) \\ (\sigma\colon \Delta^n \to W) & \mapsto & (\widetilde{F} \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{array}$$

satisfy

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \widetilde{H}_{0*} - \widetilde{H}_{1*}$$

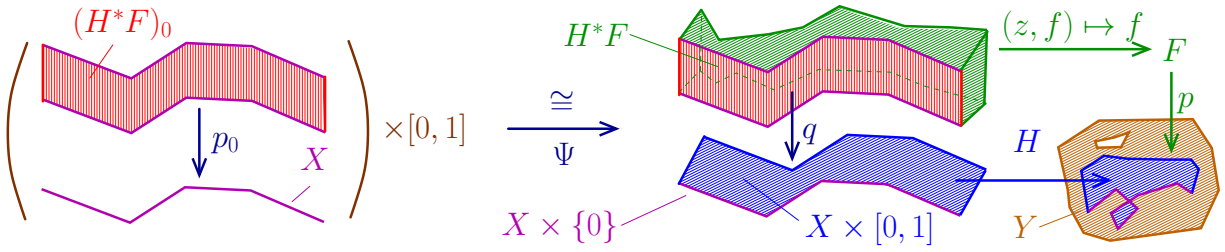
as maps from $\mathrm{C}_n(W)$ to $\mathrm{C}_n(F)$.

Proof of Proposition 343.11. To simplify the notation we assume that $A = B = \varnothing$. Mathematically this extra hypothesis makes no real difference. Thus let $h_0, h_1\colon X \to Y$ be two continuous maps between topological spaces and let $p\colon F \to Y$ be a G-local coefficient system on Y. Furthermore let $H\colon X \times [0,1] \to Y$ be a homotopy between h_0 and h_1. We consider the G-local coefficient system $q\colon H^*F \to X \times [0,1]$. As in Definition 342.17 we consider for any $t \in [0,1]$ the G-local coefficient system

$$\begin{array}{rcl} q_t\colon (H^*F)_t \coloneqq q^{-1}(X \times \{t\}) & \to & X \\ v & \mapsto & \text{unique } x \in X \text{ with } q(v) = (x,t). \end{array}$$

Recall that by the Coefficient System-on-Product Lemma 342.18 there exists a unique isomorphism of G-local coefficient systems $\Psi\colon (H^*F)_0 \times [0,1] \to H^*F$ over $X \times [0,1]$ such that for every $x \in X$ and $f \in q^{-1}(x,0)$ we have $\Psi(f,0) = f$. Next we consider the map

$$\begin{array}{rcccl} \widetilde{H}\colon (H^*F)_0 \times [0,1] & \to & H^*F & \to & F \\ (f,t) & \mapsto & \Psi(f,t) & & \\ & & (z,f) & \mapsto & f. \end{array}$$



[3720] The precise definition of the Ω_n is completely irrelevant for what follows, all we need is, that the conclusion of the lemma holds.

Claim. The explicit chain homotopy

$$P_n\colon \mathrm{C}_n((H^*F)_0) \;\to\; \mathrm{C}_{n+1}(F)$$

from the Homotopy–Chain Homotopy Lemma 322.5 between $\widetilde{H}_{0*}\colon \mathrm{C}_n((H^*F)_0) \to \mathrm{C}_n(F)$ and $\widetilde{H}_{1*}\colon \mathrm{C}_n((H^*F)_0) \to \mathrm{C}_n(F)$ descends to a chain homotopy

$$Q_n\colon \underbrace{\mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0)}_{=\mathrm{C}_n((H^*F)_0)/\sim} \;\to\; \underbrace{\mathrm{C}^{\mathrm{lc}}_{n+1}(Y;F)}_{=\mathrm{C}_{n+1}(F)/\sim}$$

between

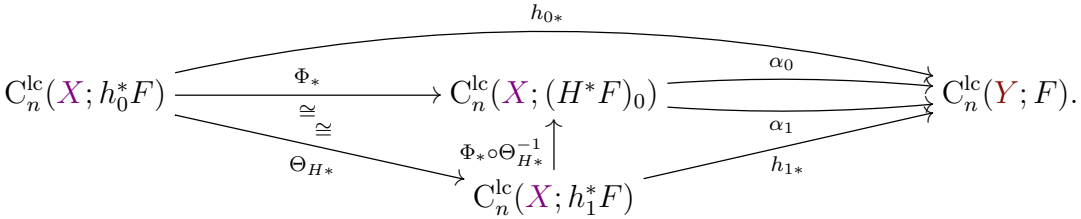
$$\begin{array}{rcl} \alpha_0\colon \mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0) & \to & \mathrm{C}^{\mathrm{lc}}_n(Y;F) \\ [\sigma] & \mapsto & [\widetilde{H}_{0*}(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_1\colon \mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0) & \to & \mathrm{C}^{\mathrm{lc}}_n(Y;F) \\ [\sigma] & \mapsto & [\widetilde{H}_{1*}(\sigma)]. \end{array}$$

Proof. We need to show that the equivalence relation that we introduced in Definition 343.1 is respected. Thus let $\sigma_1, \sigma_2\colon X \to (H^*F)_0$ be two singular n-simplices with $p\circ\sigma_1 = p\circ\sigma_2$. It follows easily from the explicit definitions of the maps P_n in the Homotopy–Chain Homotopy Lemma 322.5 that $P_n(\sigma_1 \mathbin{\widetilde{+}} \sigma_2) = P_n(\sigma_1) \mathbin{\widetilde{+}} P_n(\sigma_2)$. ⊞

Before we come to the denouement of the argument we need to introduce the following isomorphism of local coefficient systems over X:

$$\begin{array}{rcl} \Phi\colon h_0^*F & \to & (H^*F)_0 \\ (x,f) & \mapsto & ((x,0),f). \end{array}$$

Finally we consider the following diagram:



We leave it to the reader to verify that this diagram commutes. By the claim the two maps $\alpha_0, \alpha_1\colon \mathrm{C}^{\mathrm{lc}}_n(X;(H^*F)_0) \to \mathrm{C}^{\mathrm{lc}}_n(Y;F)$ are chain homotopic. Since the diagram commutes we see that the two maps $\mathrm{C}^{\mathrm{lc}}_n(X;h_0^*F) \to \mathrm{C}^{\mathrm{lc}}_n(Y;F)$ are also chain homotopic. But that is what we had to show. ■

343.1.4. **Homology with local coefficients and coverings.** We conclude this section with the following analog of the Covering-$\mathrm{H}^{\mathrm{tw}}_*$-Proposition 323.5:

Proposition 343.13. (Covering-$\mathrm{H}^{\mathrm{lc}}_*$-Proposition) Let X be a topological space that is path-connected and locally contractible. We pick $x_0 \in X$ and we write $\pi \coloneqq \pi_1(X,x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the corresponding natural universal covering from Definition 319.10 which we equip with the natural base point $\widetilde{x}_0$. Furthermore let $\Gamma \subseteq \pi_1(X,x_0)$ be a subgroup[3721]. Let $p\colon (Y_\Gamma, y_0) \to (X,x_0)$ be a covering of pointed topological spaces with $p_*(\pi_1(Y_\Gamma,y_0)) = \Gamma$. It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $q\colon \widetilde{X}_{x_0} \to Y_\Gamma$ with $q(\widetilde{x}_0) = y_0$. We equip the abelian group $\mathbb{Z}[\Gamma\backslash\pi]$ with the $\mathbb{Z}[\pi]$-left module structure φ that is given by $\varphi(g)\cdot\Gamma h \coloneqq \Gamma h\cdot g^{-1}$. As in the Local

System-via-Representation Proposition 342.9 we consider the local coefficient system

$$\begin{array}{rcl} {}_{\varphi}\mathbb{Z}[\Gamma\backslash\pi] \coloneqq \mathbb{Z}[\Gamma\backslash\pi] \times_{\pi} \widetilde{X}_{x_0} & \to & X \\ [(m,\widetilde{x})] & \mapsto & p(\widetilde{x}). \end{array}$$

(1) The maps

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{lc}}(X; {}_{\varphi}\mathbb{Z}[\Gamma\backslash\pi]) & \to & \mathrm{C}_k(Y_\Gamma) \\ (\sigma\colon \Delta^k \to \mathbb{Z}[\Gamma\backslash\pi] \times_{\pi} \widetilde{X}_{x_0}) & \mapsto & \begin{array}{c} \text{pick any lift } (\sigma_1,\sigma_2)\colon \Delta^k \to \mathbb{Z}[\Gamma\backslash\pi] \times \widetilde{X}_{x_0} \text{ of} \\ \text{the map } \sigma \text{ and then consider the} \\ \text{map } \underbrace{\sigma_1(\Delta^k)}_{\in\Gamma} \cdot q \circ \sigma_2\colon \Delta^k \to Y_\Gamma \end{array} \end{array}$$

form an isomorphism of chain complexes.

(2) If Γ is normal in π, then we equip $\mathbb{Z}[\Gamma\backslash\pi]$ with the obvious $\mathbb{Z}[\Gamma\backslash\pi]$-left module structure and we equip Y_Γ with the $\Gamma\backslash\pi$-left action given by the Deck Transformation Group Proposition 319.12. In this setting the above isomorphism is an isomorphism of $\mathbb{Z}[\Gamma\backslash\pi]$-left modules.

(3) For each $k \in \mathbb{N}_0$ the above maps induce an isomorphism

$$\mathrm{H}_k^{\mathrm{lc}}(X; {}_{\varphi}\mathbb{Z}[\Gamma\backslash\pi]) \to \mathrm{H}_k(Y_\Gamma).$$

Proof. We can write down explicit inverse maps. More precisely we consider the map

$$\begin{array}{rcl} \mathrm{C}_k(Y_\Gamma) & \to & \mathrm{C}_k^{\mathrm{lc}}(X; {}_{\varphi}\mathbb{Z}[\Gamma\backslash\pi]) \\ (\sigma\colon \Delta^k \to Y_\Gamma) & \mapsto & \begin{array}{c} \text{pick any lift } \widetilde{\sigma}\colon \Delta^k \to \widetilde{X}_{x_0} \text{ and consider the map} \\ \Delta^k \to \mathbb{Z}[\Gamma\backslash\pi] \times \widetilde{X}_{x_0} \text{ that is given by } x \mapsto [(1,\widetilde{\sigma}(x))]. \end{array} \end{array}$$

We leave it to the reader to show that the maps are both chain maps and that they are inverses of one another. ■

Example. In this example we consider the circle S^1 together with the usual representation $\varphi\colon \pi_1(S^1,1) \to \mathrm{GL}(1,\mathbb{Z}[t^{\pm 1}])$ that sends the standard generator of $\pi_1(S^1,1)$ to (t). We also consider the corresponding coefficient system ${}_{\varphi}\mathbb{Z}[t^{\pm 1}]$ from the Local System-via-Representation Proposition 342.9. Finally we consider the universal covering $p\colon \mathbb{R} \to S^1$ that is given by $p(t) = \exp(2\pi\,\mathrm{i}t)$. We now see that for each $k \in \mathbb{N}_0$ we have the following isomorphisms of $\mathbb{Z}[t^{\pm 1}]$-modules:

$$\mathrm{H}_k^{\mathrm{lc}}(S^1; {}_{\varphi}\mathbb{Z}[t^{\pm 1}]) \underset{\substack{\uparrow \\ \text{by the Covering-H}_*^{\mathrm{lc}} \\ \text{Proposition 343.13}}}{\cong} \mathrm{H}_k^{\mathrm{lc}}(\mathbb{R};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{otherwise} \end{cases}$$

where t acts trivially on $\mathbb{Z}$. □

343.1.5. **Excision with local coefficients.** Our next goal is to formulate the Excision Theorem and the Mayer–Vietoris Theorem for homology with local coefficients.

We start out by introducing the following space-saving convention:

[3721] Note that we do not demand that the subgroup is normal.

Convention 343.14. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Given any subspace $A \subseteq X$ we just write $E := E|_A$ for the local coefficient system on A that we obtain by restriction.

We we can formulate the following variation on Definition 158.18:

Definition 343.15.

(1) A triad of topological spaces is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.

(2) Given a triad (X, A_1, A_2) of topological spaces and given a local coefficient system $p\colon E \to X$ we consider the following subgroup of $\mathrm{C}_n^{\mathrm{lc}}(A_1 \cup A_2; E)$:

$$\mathrm{C}_n^{\mathrm{lc},\{A_1,A_2\}}(A_1 \cup A_2; E) := \left\{ \sum_{j=1}^{k} \sigma_j \;\middle|\; \begin{array}{c} \text{for each } j \text{ there exists an } i \in \{1,2\} \\ \text{such that } \sigma_j \text{ lies in } \mathrm{C}_n^{\mathrm{lc}}(A_i; E) \end{array} \right\}.$$

Clearly the usual boundary map on the groups $\mathrm{C}_*^{\mathrm{lc}}(A_1 \cup A_2; E)$ restricts to a boundary map on $\mathrm{C}_*^{\mathrm{lc},\{A_1,A_2\}}(A_1 \cup A_2; E)$, so we can view $\mathrm{C}_*^{\mathrm{lc},\{A_1,A_2\}}(A_1 \cup A_2; E)$ as a chain complex.

(3) We say that a triad (X, A_1, A_2) is excisive with respect to local coefficient systems if for every local coefficient system $p\colon E \to X$ the inclusion map

$$\mathrm{C}_*^{\mathrm{lc},\{A_1,A_2\}}(A_1 \cup A_2; E) \;\to\; \mathrm{C}_*^{\mathrm{lc}}(A_1 \cup A_2; E)$$

is a chain homotopy equivalence.

In the Excisive Triad-Proposition 158.20 and the Universally Excisive Triad Proposition 325.3 we gave many convenient examples of (universally) excisive triads. The following proposition now says that these triads are also excisive with respect to local coefficients:

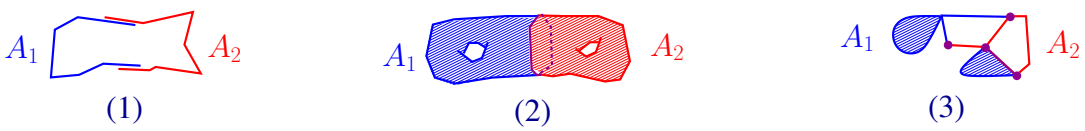
Proposition 343.16. (LC-Excisive Triad Proposition) Let (X, A_1, A_2) be a triad of topological spaces. The triad is excisive with respect to local coefficient systems if one of the following conditions is satisfied:

(0) (a) $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.
(b) $A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$.

(1) (a) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$.
(b) A_1 and A_2 are both open subsets of $A_1 \cup A_2$.

(2) $A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2.

(3) X is a CW-complex and A_1 and A_2 are both subcomplexes.



Proof. Statements (0a) and (0b)| are trivial. Statement (1b) is just a special case of Statement (1a). So let us now deal with Statement (1a).

Let $p\colon E \to X$ be a local coefficient system over X. For each $n \in \mathbb{N}_0$ we denote by

$$u_n\colon \mathrm{C}_n(E) \;\to\; \mathrm{C}_n(E)$$

the subdivision map that we introduced on page 2825. We recall the following statements:

(i) In the Chain Subdivision Lemma 148.11 we saw that the maps u_n form a chain map.
(ii) In the Subdivision–Natural Lemma 148.10 we saw that the maps u_n are natural.
(iii) In the Subdivision–Chain Homotopy-Lemma 148.12 we saw the u_n are chain homotopic to the identity. Even though we did not say so explicitly, the proof of the Subdivision–Chain Homotopy-Lemma 148.12 shows that we can find a natural chain homotopy inverse $v_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ to u_n.

The constructions of u_n and v_n show that they preserve $\widetilde{+}$, i.e. they descend to maps $\mathrm{C}_n^{\mathrm{lc}}(X;E) \to \mathrm{C}_n^{\mathrm{lc}}(X;E)$. We leave it to the reader to pick up the threads and complete the proof for Statement (1a).

As in the proof of the Excisive Triad-Proposition 158.20 the remaining Statements (2) and (3) can be reduced to the earlier case. We leave the nasty task of making this work to the reader. ■

We are now ready for the following analog of the Twisted Excision Theorem 325.4:[3722]

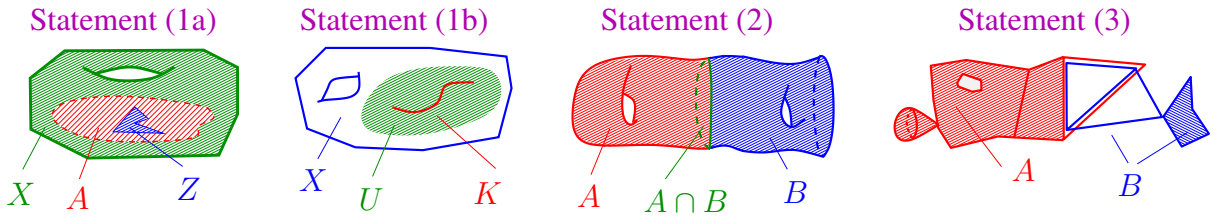
Theorem 343.17. (Local Coefficients-Excision Theorem) Let X be a topological space, let $E \to X$ be a local coefficient system module and let $n \in \mathbb{N}_0$.

(1) (a) Let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. The inclusion $i\colon (X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism
$$i^*\colon \mathrm{H}_n^{\mathrm{lc}}(X \setminus Z, A \setminus Z; E) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{lc}}(X, A; E).$$
(b) Let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $i\colon (U, U \setminus K) \to (X, X \setminus K)$ induces an isomorphism
$$i^*\colon \mathrm{H}_n^{\mathrm{lc}}(U, U \setminus K; E) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{lc}}(X, X \setminus K; E).$$
(2) We suppose that X is a topological manifold. Let $A, B \subseteq X$ be two codimension-zero submanifolds such that the following statements hold:
(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.
The inclusion $i\colon (B, A \cap B) \to (X, A)$ induces an isomorphism
$$i_*\colon \mathrm{H}_n^{\mathrm{lc}}(B, A \cap B; E) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{lc}}(X, B; E).$$
(3) We suppose that X is a CW-complex and that A and B are two subcomplexes of X with $X = A \cup B$. The inclusion $i\colon (B, A \cap B) \to (X, A)$ induces an isomorphism
$$i_*\colon \mathrm{H}_n^{\mathrm{lc}}(B, A \cap B; E) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{lc}}(X, B; E).$$

Proof. This theorem can be deduced quite easily from the LC-Excisive Triad Proposition 343.16. Again all details are left to the reader. ■

343.1.6. **The Mayer–Vietoris sequence in homology with local coefficients.** The following theorem is essentially a generalization of the Mayer–Vietoris Theorem 153.2 and

[3722] In fact the statement of the Local Coefficients-Excision Theorem 343.17 is much neater than the statement of the Twisted Excision Theorem 325.4 since we do not need to worry about base points and connectedness.



it is the "local coefficients analog" of the Internal–Mayer–Vietoris Theorem 325.11 and the Twisted Mayer–Vietoris Theorem 325.12:

Theorem 343.18. (Mayer–Vietoris Theorem) Let X be a topological space and let $A, B \subseteq X$ be subspaces such that one of the following three conditions holds:

(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subseteq X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.

For any local coefficient system $E \to X$ and any $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n \colon \mathrm{H}^{\mathrm{lc}}_n(X;E) \to \mathrm{H}^{\mathrm{lc}}_{n-1}(A \cap B;E)$$

such that the following sequence is exact:

$$\begin{aligned} \ldots \to \mathrm{H}^{\mathrm{lc}}_n(A\cap B;E) &\xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}^{\mathrm{lc}}_n(A;E) \oplus \mathrm{H}^{\mathrm{lc}}_n(B;E) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}^{\mathrm{lc}}_n(X;E) \xrightarrow{\partial_n} \\ \xrightarrow{\partial_n} \mathrm{H}^{\mathrm{lc}}_{n-1}(A\cap B;E) &\to \ldots \end{aligned}$$

Remark. The statement of the Mayer–Vietoris Theorem 343.18 is quite clean and attractive. The catch is, that in practice it can be very hard to figure out what the various inclusion induced maps really are. □

Proof. As in the proof of the original Mayer–Vietoris Theorem 153.2 one can reduce this statement to the Local Coefficients-Excision Theorem 343.17. We leave the hard slog of filling in the details and writing up a proof to the reader. ■

343.2. **Cohomology with local coefficients.** In this section we will (surprise!) develop the theory of cohomology with local coefficients.

343.2.1. **Definition of cohomology with local coefficients.** In Definition 198.1 we defined singular cohomology by a purely formal method, namely by taking the dual of the singular chain complex. A similar, purely formal, approach allowed us in Definition 320.11 to define cohomology with twisted coefficients. Perhaps somewhat surprisingly singular cohomology with local coefficients is defined independently of the chain complex with local coefficients. First let us introduce the following notation:

Notation 343.19. Given a topological space Y and given $n \in \mathbb{N}_0$ we set

$$\mathrm{S}_n(Y) \;:=\; \text{the set of all singular } n\text{-simplices in } Y,$$

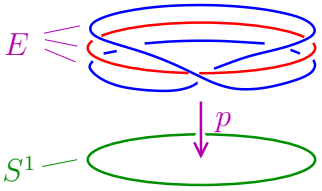
in other words, $\mathrm{S}_n(Y)$ is the set of all continuous maps $\Delta^n \to Y$.

Definition 343.20. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system.

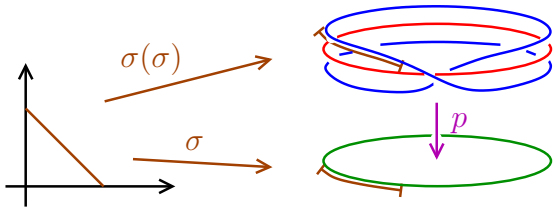
(1) An n-cochain with respect to the local coefficient system $p\colon E \to X$ is a map

$$\varphi\colon \mathrm{S}_n(X) \to \mathrm{S}_n(E)$$

such that for every singular simplex $\sigma\colon \Delta^n \to X$ the corresponding singular simplex $\varphi(\sigma)\colon \Delta^n \to E$ is a lift of σ, i.e. we have $p \circ \varphi(\sigma) = \sigma$.



$\mathbb{Z}/3\mathbb{Z}$-local system over S^1



Definition 343.20.

(2) Given two n-cochains $\varphi, \psi\colon \mathrm{S}_n(X) \to \mathrm{S}_n(E)$ we define

$$\begin{array}{rcl} \varphi+\psi\colon \mathrm{S}_n(X) & \to & \mathrm{S}_n(E) \\ \sigma & \mapsto & \Bigg(\begin{array}{rcl} \Delta^n & \to & E \\ x & \mapsto & \underbrace{(\varphi(\sigma)(x) + (\psi(\sigma)(x)}_{\text{addition in the group } p^{-1}(x)} \end{array} \Bigg). \end{array}$$

continuous by the Local Coefficient System-Lift Lemma 342.4

One can easily verify that this addition turns the set of cochains into an abelian group, where the trivial element is the cochain that sends every simplex to the zero-lift.

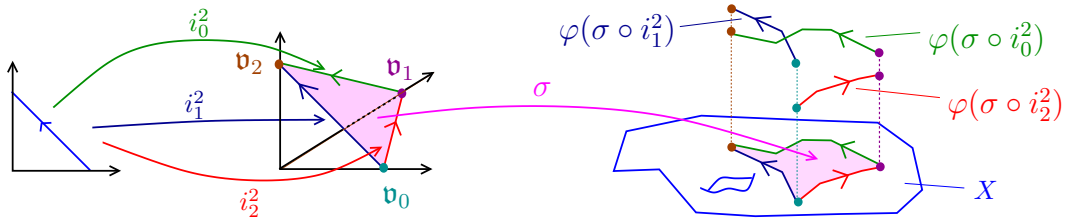
(3) We denote by $\mathrm{C}^n_{\mathrm{lc}}(X;E)$ the group of all n-cochains.

(4) Given $n \in \mathbb{N}_0$ we define the coboundary[3723]

$$\delta^n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) \to \mathrm{C}^{n+1}_{\mathrm{lc}}(X;E)$$

$$\varphi \mapsto \Bigg(\begin{array}{rcl} \mathrm{S}_{n+1}(X) & \to & \mathrm{S}_{n+1}(E) \\ \sigma & \mapsto & \sum_{j=0}^{n+1} (-1)^j \cdot \text{ the unique lift}^{3724}\ \widetilde{\sigma}\colon \Delta^{n+1} \to E \text{ such that } \varphi(\sigma \circ i^{n+1}_j) \text{ and } \widetilde{\sigma} \circ i^n_j \text{ agree} \end{array} \Bigg).$$

It follows easily from the definitions that δ^n is a homomorphism.



[3722] We define the sum of two lifts and the inverse of a lift by the usual fiberwise operations.

[3723] By the Unique Simplex Lifting Lemma 343.3 we know that such a lift exists and that it is unique.

[3724] Here $i^{n+1}_j\colon \Delta^n \to \Delta^{n+1}$ is the j-th face map as introduced in Definition 143.12.

Recall that one of the most fundamental facts in algebraic topology, proved in the Double Boundary Proposition 143.14, is the fact that the composition of two boundary maps $\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ and $\partial_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-2}(X)$ is the zero map. The following proposition now says that the analogous statement also holds for the above coboundary maps:

Proposition 343.21. (Double Coboundary Proposition) Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. For any $n \in \mathbb{N}_0$ the composition

$$\delta^{n+1} \circ \delta^n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) \;\to\; \mathrm{C}^{n+2}_{\mathrm{lc}}(X;E)$$

is the zero map.

Proof.

We start out with the following observation: If we are dealing with a trivial local coefficient system, then the Double Coboundary Proposition 343.21 boils down very quickly to the setting for the usual cochains. For a general local coefficient system the situation is initially much more puzzling, since we cannot compare the various fibers in a coherent way. What saves the day is the observation that we only have to work with a singular simplex $\sigma\colon \Delta^{n+2} \to X$ at a time, and that the pullback of the local coefficient system E to Δ^{n+2} is trivializable, since Δ^{n+2} is contractible. Thus we can deduce the desired statement after all from the classical statement.

We start out with several preparations:

(a) Let $\mu\colon D \to X$ be a continuous map between topological spaces and let $p\colon E \to X$ be a G-local coefficient system on X. We define

$$\begin{array}{rcl} \mu^*\colon \mathrm{C}^k_{\mathrm{lc}}(X;E) & \to & \mathrm{C}^k_{\mathrm{lc}}(D;\mu^*E) \\ \psi & \mapsto & \left(\begin{array}{rcl} \mathrm{S}_k(D) & \to & \mathrm{S}_k(\mu^*E) \\ (\sigma\colon \Delta^k \to D) & \mapsto & \left(\begin{array}{rcl} \Delta^k & \to & \mu^*E \\ z & \mapsto & (\sigma(z), \underbrace{\underbrace{\psi(\mu \circ \sigma)}_{\in \mathrm{S}_k(X)}(z)}_{\in \mathrm{S}_k(E)}) \end{array} \right) \end{array} \right) \end{array}$$

We make the following two observations:

- It follows easily from the definitions that $\delta_k \circ \mu^* = \mu^* \circ \delta_k$.
- We denote by $q\colon \mu^*E \to E$ the projection. If $\sigma\colon \Delta^k \to D$ is a singular simplex, then it follows immediately from the various definitions that we have the equality $q \circ \mu^*(\psi)(\sigma) = \psi(\mu \circ \sigma)\colon D \to E$.

(b) Let $\Theta\colon A \to B$ be a morphism from a G-local coefficient system $p\colon A \to D$ to an H-local coefficient system $q\colon B \to D$. We consider the maps

$$\begin{array}{rcl} \Theta_*\colon \mathrm{C}^k_{\mathrm{lc}}(D;A) & \to & \mathrm{C}^k_{\mathrm{lc}}(D;B) \\ (\psi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(A)) & \to & (\Theta_* \circ \psi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(B)). \end{array}$$

It follows easily from the definitions that $\delta_k \circ \Theta^* = \Theta^* \circ \delta_k$. If Θ is an isomorphism of local coefficient systems, then Θ_* is evidently also an isomorphism.

(c) By the Local System–Homotopies Proposition 342.16 we know that every G-local coefficient system over a topological space D which is contractible, is trivializable, i.e. it is isomorphic to ${}_{\mathrm{tr}}G$.

(d) One can show easily, say using the Δ^n-Homeomorphism Lemma 143.9 (1) and using the discussion on page 531, that each Δ^k is contractible.

(e) Let D be a topological space, let G be an abelian group and let $p\colon G \times D \to D$ be the trivial G-local coefficient system. One can easily verify that the maps

$$\begin{aligned} \Xi\colon\ & \mathrm{C}^k(D;G) \to \mathrm{C}^k_{\mathrm{lc}}(D;{}_{\mathrm{tr}}G) \\ (\mathrm{C}_k(D) \xrightarrow{\psi} G) \mapsto & \begin{pmatrix} \mathrm{S}_k(D) \to \mathrm{S}_k(G\times D) \\ \sigma \mapsto \text{the lift } \sigma\colon \Delta^k \to G\times D \text{ given by } x \mapsto (\sigma(x), \psi(\sigma)) \end{pmatrix} \end{aligned}$$

define an isomorphism of cochain groups such that the following diagram commutes:

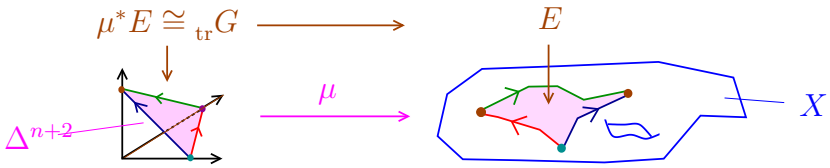
$$\begin{array}{ccc} \mathrm{C}^k(D;G) & \xrightarrow{\quad\Xi\quad} & \mathrm{C}^k_{\mathrm{lc}}(D;{}_{\mathrm{tr}}G) \\ \downarrow{\scriptstyle (\partial_{k+1})^*} & & \downarrow{\scriptstyle \delta_k} \\ \mathrm{C}^{k+1}(D;G) & \xrightarrow{\quad\Xi\quad} & \mathrm{C}^{k+1}_{\mathrm{lc}}(D;{}_{\mathrm{tr}}G), \end{array}$$

here $\partial_k\colon \mathrm{C}_{k+1}(D;G) \to \mathrm{C}_k(D;G)$ is the usual boundary map in the singular chain complex and the dual $\partial_k^*\colon \mathrm{C}^k(D;G) \to \mathrm{C}^{k+1}(D;G)$ is the usual coboundary map in the singular cochain complex.

Now we turn to the actual proof of the proposition. Thus let X be a topological space, let $p\colon E \to X$ be a local coefficient system and let $n \in \mathbb{N}_0$. Recall that we need to show that the composition $\delta^{n+1} \circ \delta^n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E) \to \mathrm{C}^{n+2}_{\mathrm{lc}}(X;E)$ is the zero map. In other words, we need to prove the following claim:

Claim. Let $\mu\colon \Delta^{n+2} \to X$ be a singular simplex. Then $(\delta_{n+1}\circ\delta_n)(\varphi)(\mu) = 0 \in \mathrm{C}^{n+2}_{\mathrm{lc}}(X;E)$.

Proof. Let $\mu\colon \Delta^{n+2} \to X$ be a singular simplex.



By the above we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{C}^n_{\mathrm{lc}}(X;E) & \xrightarrow{\mu^*} & \mathrm{C}^n_{\mathrm{lc}}(\Delta^{n+2};\mu^*E) & \xrightarrow[\cong]{\Theta_*} & \mathrm{C}^n_{\mathrm{lc}}(\Delta^{n+2};{}_{\mathrm{tr}}G) & \xleftarrow[\cong]{\Xi} & \mathrm{C}^n(\Delta^{n+2};G) \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \partial^*_{n+1}} \\ \mathrm{C}^{n+1}_{\mathrm{lc}}(X;E) & \xrightarrow{\mu^*} & \mathrm{C}^{n+1}_{\mathrm{lc}}(\Delta^{n+2};\mu^*E) & \xrightarrow[\cong]{\Theta_*} & \mathrm{C}^{n+1}_{\mathrm{lc}}(\Delta^{n+2};{}_{\mathrm{tr}}G) & \xleftarrow[\cong]{\Xi} & \mathrm{C}^{n+1}(\Delta^{n+2};G) \\ \downarrow{\scriptstyle \delta_{n+1}} & & \downarrow{\scriptstyle \delta_{n+1}} & & \downarrow{\scriptstyle \delta_{n+1}} & & \downarrow{\scriptstyle \partial^*_{n+2}} \\ \mathrm{C}^{n+2}_{\mathrm{lc}}(X;E) & \xrightarrow{\mu^*} & \mathrm{C}^{n+2}_{\mathrm{lc}}(\Delta^{n+2};\mu^*E) & \xrightarrow[\cong]{\Theta_*} & \mathrm{C}^{n+2}_{\mathrm{lc}}(\Delta^{n+2};{}_{\mathrm{tr}}G) & \xleftarrow[\cong]{\Xi} & \mathrm{C}^{n+2}(\Delta^{n+2};G). \end{array}$$

Recall that

$$\partial^*_{n+1} \circ \partial^*_{n+2} \underset{\text{since } \mathrm{Hom}_{\mathbb{Z}}(-,G) \text{ is contravariant}}{\uparrow =} (\partial_{n+2} \circ \partial_{n+1})^* \underset{\text{by the Double Boundary Proposition 143.14}}{\uparrow =} 0^* = 0.$$

Thus we see that the two composition of the two vertical maps in the fourth column, and thus also the third and the second column, is zero. In summary we see that we have the

following equality of lifts of $\mu\colon \Delta^{n+2} \to X$ to a map $\Delta^{n+2} \to E$:

$$\begin{array}{ccccccc} (\delta(\delta(\varphi))(\mu)) & = & (\delta\circ\delta)(\varphi)(\mu) & = & (\delta\circ\delta)(\varphi)(\mu\circ \mathrm{id}_{\Delta^{n+2}}) & \overset{\text{follows from the discussion in (a), applied to } \sigma=\mathrm{id}_{\Delta^{n+2}} \text{ and } \psi=(\delta\circ\delta)(\varphi) \text{ where } q\colon \mu^*E\to E \text{ denotes the projection}}{=} & q\circ\mu^*((\delta\circ\delta)(\varphi))(\mathrm{id}_{\Delta^{n+2}}) \\ & & & & \overset{\text{since the squares to the left commute}}{=} q\circ((\delta\circ\delta)(\mu^*\varphi))(\mathrm{id}_{\Delta^{n+2}}) & \overset{\text{since the composition of the vertical maps in the second column is zero}}{=} & q(\text{zero-lift of } \mathrm{id}_{\Delta^{n+2}} \text{ to } \mu^*E) \\ & & & & \overset{\text{follows easily from the definition of } \mu^*E \text{ and the definition of } q}{=} \text{zero-lift of } \mu\colon \Delta^{n+2}\to X \text{ to } E. & & \end{array}$$

■

The Double Coboundary Proposition 343.21 allows us to introduce the following definition:

Definition 343.22. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Given $n \in \mathbb{N}_0$ we define the *n-th singular cohomology group of X with E-coefficients* as the quotient group

$$\mathrm{H}^n_{\mathrm{lc}}(X;E) \coloneqq \frac{\ker(\delta_n\colon \mathrm{C}^n_{\mathrm{lc}}(X;E)\to \mathrm{C}^{n+1}_{\mathrm{lc}}(X;E))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}^{n-1}_{\mathrm{lc}}(X;E)\to \mathrm{C}^n_{\mathrm{lc}}(X;E))}.$$

343.2.2. **Functoriality of cohomology with local coefficients.** In Definition 343.6 we saw that continuous maps between topological spaces induce homomorphisms between homology groups with local coefficients. We now have the analogous statement for cohomology with local coefficients:

Definition 343.23. Let $f\colon X\to Y$ be a continuous map between topological spaces and let $p\colon E\to Y$ be a G-local coefficient system on Y. We define

$$\begin{array}{rcl} f^*\colon \mathrm{C}^n_{\mathrm{lc}}(Y;E) & \to & \mathrm{C}^n_{\mathrm{lc}}(X;f^*E) \\ (\varphi\colon \mathrm{S}_n(Y)\to \mathrm{S}_n(E)) & \mapsto & \left(\begin{array}{rcl} \mathrm{S}_n(X) & \to & \mathrm{S}_n(f^*E) \\ (\sigma\colon \Delta^n\to X) & \mapsto & \left(\begin{array}{rcl} \Delta^n & \to & f^*E \\ z & \mapsto & (\sigma(z), \underbrace{\underbrace{\varphi(\underbrace{f\circ\sigma}_{\in \mathrm{S}_n(Y)})}_{\in \mathrm{S}_n(E)}(z)}_{\in f^*E}) \end{array}\right) \end{array}\right) \end{array}$$

One can easily verify that these maps are well-defined and that they form a chain map. Thus we obtain induced homomorphisms

$$f^*\colon \mathrm{H}^n_{\mathrm{lc}}(Y;E) \to \mathrm{H}^n_{\mathrm{lc}}(X;f^*E).$$

The previous definition now also allows us to introduce relative cohomology with local coefficients:

Definition 343.24. Let X be a topological space, let $A\subseteq X$, let G be an abelian group and let $p\colon E\to X$ be a G-local coefficient system. We denote by $i\colon A\to X$ the inclusion. We set

$$\mathrm{C}^n_{\mathrm{lc}}(X,A;E) \coloneqq \ker(i^*\colon \mathrm{C}^{\mathrm{lc}}_n(X;E)\to \mathrm{C}^n_{\mathrm{lc}}(A;i^*E)).$$

One can easily show that the cochain map on $\mathrm{C}^*_{\mathrm{lc}}(X;E)$ descends to a cochain map on $\mathrm{C}^*_{\mathrm{lc}}(X,A;E)$. Thus we can set

$$\mathrm{H}^n_{\mathrm{lc}}(X,A;E) \coloneqq \mathrm{H}^n(\mathrm{C}^*_{\mathrm{lc}}(X,A;E)).$$

In the Cohomology-Coefficients-Functor Lemma 197.21 we saw that singular homology is covariantly functorial in the coefficients. The following lemma generalizes this statement to the setting of cohomology with local coefficients:

Lemma 343.25. Let X be a topological space and let $A \subseteq X$.

(1) Let $\Theta\colon E \to F$ be a morphism from a G-local coefficient system $p\colon E \to X$ to an H-local coefficient system $q\colon F \to X$. The maps

$$\begin{array}{rcl} \Theta_*\colon \mathrm{C}^k_{\mathrm{lc}}(X, A; E) & \to & \mathrm{C}^k_{\mathrm{lc}}(X, A; F) \\ (\varphi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(E)) & \mapsto & (\Theta_* \circ \varphi\colon \mathrm{S}_k(X) \to \mathrm{S}_k(F)) \end{array}$$

form a well-defined cochain map and thus they induce maps

$$\varphi_*\colon \mathrm{H}^k_{\mathrm{lc}}(X, A; E) \ \to \ \mathrm{H}^k_{\mathrm{lc}}(X, A; F).$$

(2) With the obvious interpretations we have $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ and $\mathrm{id}_* = \mathrm{id}$.

Proof. Once again we leave the elementary proof of the lemma to the reader. ■

The following lemma is the cohomological analog of the Trivial Local Coefficients Lemma 343.10.

Lemma 343.26. (Trivial Local Coefficients Lemma) Let X be a topological space, let $A \subseteq X$, let G be an abelian group and let $p\colon G \times X \to X$ be the trivial G-local coefficient system from Definition 342.2. The maps

$$\begin{array}{rcl} \mathrm{C}^k(X, A; G) & \to & \mathrm{C}^k_{\mathrm{lc}}(X, A; {}_{\mathrm{tr}}G) \\ (\psi\colon \mathrm{C}_k(X, A) \to G) & \mapsto & \left(\begin{array}{c} \mathrm{S}_k(X) \to \mathrm{S}_k(G \times X) \\ \sigma \mapsto \text{the lift } \sigma\colon \Delta^k \to G \times X \text{ given by } x \mapsto (\psi(\sigma), \sigma(x)) \end{array} \right) \end{array}$$

define an isomorphism of cochain complexes that is natural in the sense of the Trivial Local Coefficients Lemma 343.10. Furthermore, as for homology groups, these natural chain isomorphisms give rise to a natural isomorphism

$$\mathrm{H}^k(X, A; G) \ \xrightarrow{\cong} \ \mathrm{H}^k_{\mathrm{lc}}(X, A; G_{\mathrm{tr}})$$

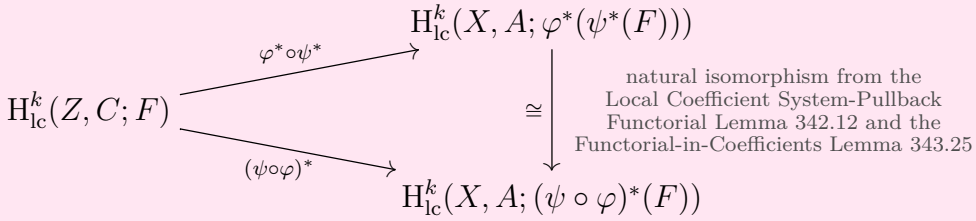
Proof. The lemma follows easily from the definitions. The less said, the clearer. ■

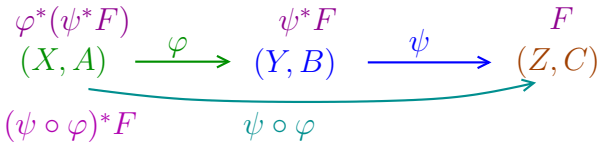
In the H^n–Functor Lemma 198.3 we saw that singular cohomology is contravariantly functorial and in the Twisted Homology-Functoriality Proposition 321.9 we saw that cohomology with twisted coefficients is in a suitable sense contravariantly functorial. In the LC-Homology Functoriality Lemma 343.9 we showed that homology with local coefficients is close to being a covariant functor. In a similar vein we now have the following lemma:

Lemma 343.27. (LC-Cohomology Functoriality) Let $p\colon F \to Z$ be a G-local coefficient system on a topological space Z and let $C \subseteq Z$.

(1) The natural isomorphism between the G-local coefficient systems $\mathrm{id}^* F$ and F that we gave in the Local Coefficient System-Pullback–Functorial Lemma 342.12 (1) induces an isomorphism $\mathrm{id}^*\colon \mathrm{H}^k_{\mathrm{lc}}(Z, C; F) \to \mathrm{H}^k_{\mathrm{lc}}(Z, C; \mathrm{id}^* F)$.

(2) Let $\varphi\colon (X,A) \to (Y,B)$ and $\psi\colon (Y,B) \to (Z,C)$ be continuous maps between topological spaces. We have the following commutative diagram:





Proof. As so often, the lemma follows almost immediately from the definitions. ∎

343.3. **Cup product with local coefficients.** Recall that given a topological space X and given a commutative ring R we introduced in Definition 203.11 the cup product $\cup\colon \mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) \to \mathrm{H}^{k+l}(X;R)$. The cup product, together with its sibling the cap product, gave us many of the most interesting applications of cohomology groups. It is thus natural to try to introduce a cup product on cohomology with local coefficients.

First we recall Definition 143.8:

Notation 343.28. For $i = 0,\dots,n$ we write

$$\mathfrak{v}_i \coloneqq (0,\dots,1,0,\dots,0) \in \mathbb{R}^{n+1}.$$

Furthermore for $a_0,\dots,a_k \in \Delta^n$ we consider the map

$$\begin{array}{rcl} [a_0,\dots,a_k]\colon \Delta^k & \to & \Delta^n \\ \sum_{j=0}^{k} \lambda_j \cdot \mathfrak{v}_j & \mapsto & \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{array}$$

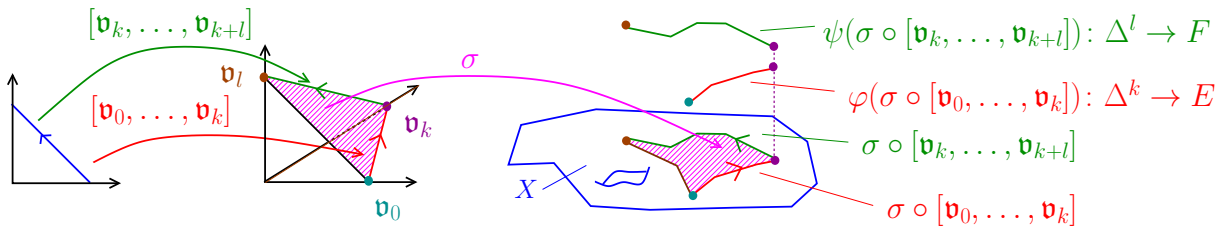
Now we can provide the definition of the cup product on cohomology with local coefficients:

Definition 343.29. Let X be a topological space, let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. We consider the $G \otimes_{\mathbb{Z}} H$-local coefficient system $E \otimes_{\mathbb{Z}} F$ from Definition 342.20. Given $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X;E)$ and $\psi \in \mathrm{C}^l_{\mathrm{lc}}(X;F)$ we define $\varphi \cup_\otimes \psi \in \mathrm{C}^{k+l}_{\mathrm{lc}}(X;E \otimes_{\mathbb{Z}} F)$ to be the following map:[3725]

$$\begin{array}{rcl} \varphi \cup_\otimes \psi\colon \mathrm{S}_{k+l}(X) & \to & \mathrm{S}_{k+l}(E \otimes_{\mathbb{Z}} F) \\ (\sigma\colon \Delta^{k+l} \to X) & \mapsto & \text{the unique lift of } \sigma \text{ to a map } \widetilde{\sigma}\colon \Delta^{k+l} \to E \otimes_{\mathbb{Z}} F \text{ such that} \\ & & \widetilde{\sigma}(\mathfrak{v}_k) = \underbrace{(\varphi(\sigma \circ [\mathfrak{v}_0,\dots,\mathfrak{v}_k]))}_{\Delta^k \to E}(\mathfrak{v}_k) \otimes \underbrace{(\psi(\sigma \circ [\mathfrak{v}_k,\dots,\mathfrak{v}_{k+l}]))}_{\Delta^l \to F}(\mathfrak{v}_0). \end{array}$$

As for the cup product on singular homology we would like to show that the above definition on the cochain level descends to a map on cohomology. This leads us to the following analog of the Cup Product–Coboundary Lemma 203.12:

[3725] Implicitly we are using once again the Unique Simplex Lifting Lemma 343.3.



Lemma 343.30. (Cup Product–Coboundary Lemma) Let X be a topological space and let $p\colon E \to X$ and $q\colon F \to X$ be local coefficient systems. Given any $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X; E)$ and given any $\psi \in \mathrm{C}^l_{\mathrm{lc}}(X; F)$ we have

$$\delta_{k+l}(\varphi \cup_\otimes \psi) \;=\; \delta_k(\varphi) \cup_\otimes \psi + (-1)^k \cdot \varphi \cup_\otimes \delta_l(\psi) \;\in\; \mathrm{C}^{k+l+1}_{\mathrm{lc}}(X; E \otimes_{\mathbb{Z}} F).$$

Proof. In the proof of the Double Coboundary Proposition 343.21 we used a very general approach to reducing proofs on (co-) homology with local coefficients to the well-known case of ordinary singular (co-) homology. Entirely the same approach allows us to reduce the present setting to the original Cup Product–Coboundary Lemma 203.12. ■

The experienced reader knows that Cup Product–Coboundary Lemma 343.30 begets the following lemma:

Lemma 343.31. (Cup Product–Definition Lemma) Let X be a topological space, let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. The map

$$\begin{array}{rcl} \cup\colon \mathrm{H}^k_{\mathrm{lc}}(X; E) \times \mathrm{H}^l_{\mathrm{lc}}(X; F) & \to & \mathrm{H}^{k+l}_{\mathrm{lc}}(X; E \otimes_{\mathbb{Z}} F) \\ ([\varphi], [\psi])] & \mapsto & [\varphi] \cup_\otimes [\psi] \coloneqq [\varphi \cup_\otimes \psi] \end{array}$$

is well-defined.

Proof. The proof of the Cup Product–Definition Lemma 343.31 is basically the same as the rather elementary proof of the Cup Product–Definition Lemma 203.13. We just need to replace the Cup Product–Coboundary Lemma 203.12 by the Cup Product–Coboundary Lemma 343.30. ■

Definition 343.32. We refer to the pairing of the Cup Product–Definition Lemma 343.31 as the cup product on cohomology with local coefficients.

Proposition 343.33. (Cup Product-Commutativity Proposition) Let X be a topological space and let $p\colon E \to X$ and $q\colon F \to X$ be local coefficient systems. Let $\tau\colon E \otimes_{\mathbb{Z}} F \to F \otimes_{\mathbb{Z}} E$ be the natural isomorphism from the Local Coefficients–Tensor Product Lemma 342.21 (2). Then given any $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X; E)$ and given any $\psi \in \mathrm{C}^l_{\mathrm{lc}}(X; F)$ we have

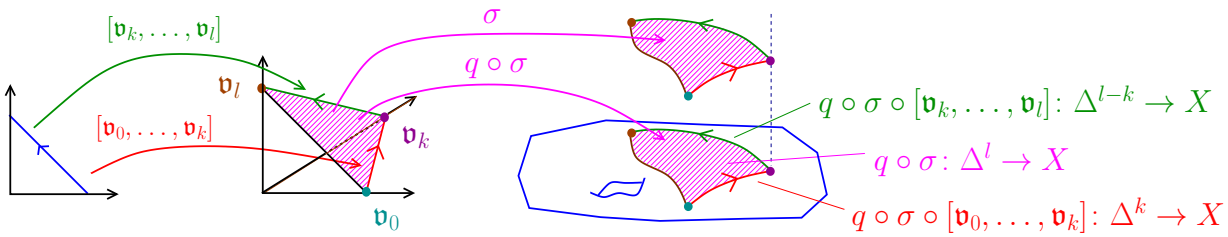
$$\varphi \cup \psi \;=\; (-1)^{kl} \cdot \tau_*(\psi \cup \varphi).$$

Proof. Recall that we proved the Cup Product–Coboundary Lemma 343.30 by modifying the "classical" proof of the Cup Product–Coboundary Lemma 203.12 in a suitable way. The same approach can also be applied to obtain a proof of the Cup Product-Commutativity Proposition 343.33 by suitably modifying the proof of the Cup Product-Commutativity Proposition 203.20. We leave the unpleasant task of turning this slogan into a proper proof to the reader. ■

343.4. **Cap product with local coefficients.** By now the experienced reader knows that the cup product never fails to show up without its buddy, the cap product:

Definition 343.34. Let X be a topological space and let $p\colon E \to X$ be a G-local coefficient system and let $q\colon F \to X$ be an H-local coefficient system. We consider the $G \otimes_{\mathbb{Z}} H$-local coefficient system $E \otimes_{\mathbb{Z}} F$ from Definition 342.20. Given $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X;E)$ and given a singular simplex $\sigma\colon \Delta^l \to F$ we define $\varphi \cap_{\otimes} \sigma\colon \Delta^{k-l} \to E \otimes_{\mathbb{Z}} F$ to be the following map:[3726]

$$\varphi \cap_{\otimes} \sigma \;:=\; \text{the unique lift of } q \circ \sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l]\colon \Delta^{l-k} \to X \text{ to a map } \widetilde{\mu}\colon \Delta^{l-k} \to E \otimes_{\mathbb{Z}} F$$
$$\text{such that } \widetilde{\mu}(\mathfrak{v}_0) = \underbrace{(\varphi(q \circ \sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]))}_{\Delta^k \to E}(\mathfrak{v}_k) \otimes (\underbrace{\sigma}_{\Delta^k \to F})(\mathfrak{v}_k)$$



Lemma 343.35. (Cap Product–Boundary Lemma) Let X be a topological space and let $p\colon E \to X$ and $q\colon F \to X$ be local coefficient systems.

(1) The map $\cap_{\otimes}$ descends to a well-defined map

$$\cap_{\otimes}\colon \mathrm{C}^k_{\mathrm{lc}}(X;E) \times \mathrm{C}^{\mathrm{lc}}_l(X;F) \;\to\; \mathrm{C}^{\mathrm{lc}}_{l-k}(X; E \otimes_{\mathbb{Z}} F).$$

(2) Given any $\varphi \in \mathrm{C}^k_{\mathrm{lc}}(X;E)$ and given any $\sigma \in \mathrm{C}^{\mathrm{lc}}_l(X;F)$ we have

$$\partial_{l-k}(\varphi \cap_{\otimes} \sigma) \;=\; (-1)^k \cdot (-\delta_k(\varphi) \cap_{\otimes} \sigma + \varphi \cap_{\otimes} \partial_l(\sigma)) \;\in\; \mathrm{C}_{l-k-1}(X; E \otimes_{\mathbb{Z}} F).$$

(3) The map $\cap_{\otimes}$ descends to a well-defined map

$$\cap_{\otimes}\colon \mathrm{H}^k_{\mathrm{lc}}(X;E) \times \mathrm{H}^{\mathrm{lc}}_l(X;F) \;\to\; \mathrm{H}^{\mathrm{lc}}_{l-k}(X; E \otimes_{\mathbb{Z}} F).$$

Sketch of proof.

(1) One can easily show that if $\sigma_1, \sigma_2\colon X \to F$ are two simplices with $q \circ \sigma_1 = q \circ \sigma_2$, then $\varphi \cap_{\otimes} (\sigma_1 \widetilde{+} \sigma_2) = \varphi \cap_{\otimes} \sigma_1 \widetilde{+} \varphi \cap_{\otimes} \sigma_2$. From this observation it follows that we do indeed get a well-defined map $\mathrm{C}^k_{\mathrm{lc}}(X;E) \times \mathrm{C}^{\mathrm{lc}}_l(X;F) \to \mathrm{C}^{\mathrm{lc}}_{l-k}(X; E \otimes_{\mathbb{Z}} F)$.

(2) Using the approach of the proof of the Double Coboundary Proposition 343.21 one can reduce the proof of Statement (2) to the Cap Product–Boundary Lemma 206.3. We leave the details to the reader.

(3) This statement is a straightforward consequence of Statement (2). ∎

The following variation of the cap product is the analog of Definition 334.34 of $\cap_{\mathbb{Z}}$:

Definition 343.36. Let X be a topological space and let $p\colon E \to X$ be a local coefficient system. Let $\varphi \in \mathrm{H}^k_{\mathrm{lc}}(X;E)$ and let $\sigma \in \mathrm{H}_l(X;\mathbb{Z})$. We denote by $T\colon \mathrm{H}_l(X;\mathbb{Z}) \to \mathrm{H}^{\mathrm{lc}}_l(X; {}_{\mathrm{tr}}\mathbb{Z})$ the natural isomorphism from the Trivial Local Coefficients Lemma 343.10 and we denote by $\mu\colon E \otimes_{\mathbb{Z}} \mathbb{Z} \to E$ the natural isomorphism of local coefficient systems from the Local

[3726] Implicitly we are using once again the Unique Simplex Lifting Lemma 343.3.

Coefficients–Tensor Product Lemma 342.21 (1). We define

$$\varphi \cap_{\mathbb{Z}} \sigma \;:=\; \mu_*(\underbrace{\varphi \cap_{\otimes} T(\sigma)}_{\in \mathrm{H}^{\mathrm{lc}}_{l-k}(X;E\otimes_{\mathbb{Z}\mathrm{tr}}\mathbb{Z})}) \;\in\; \mathrm{H}^{\mathrm{lc}}_{l-k}(X;E).$$

343.5. **The interaction between twisted (co-) homology and (co-) homology with local coefficients.** Recall that in the Local System-via-Representation Proposition 342.9 and the Local System–Classification Proposition 342.10 we saw that for a path-connected and locally contractible space we have a natural bijection between the class of representations and the class of local systems. We will now see that we also have a natural isomorphism between the corresponding twisted homology and homology with local coefficients:

Theorem 343.37. (Twisted Homology–LC-Homology Theorem) Let (X, x_0) be a pointed topological space that is path-connected space and locally contractible. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi])$-left left module. We introduce the following notation:

- We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering.
- We denote by $\varphi\colon \pi \to \mathrm{Aut}_R(M)$ the representation where $\varphi(g)$ is precisely left-multiplication by g on M.
- As in the Local System-via-Representation Proposition 342.9 we consider the R-local coefficient system
$$_{\varphi}M \;:=\; \overline{M} \times_{\pi} \widetilde{X}_{x_0} \;:=\; (\overline{M} \times \widetilde{X}_{x_0})/\sim .$$
- We denote by $\mathrm{pr}\colon \overline{M} \times \widetilde{X}_{x_0} \to \overline{M} \times_{\pi} \widetilde{X}_{x_0} = (\overline{M} \times \widetilde{X}_{x_0})/\sim$ the natural projection.

The map

$$\begin{array}{rcl} \Phi\colon \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}) & \to & \mathrm{C}^{\mathrm{lc}}_k(X;{}_{\varphi}M) = \mathrm{C}_k(\overbrace{(\overline{M} \times \widetilde{X}_{x_0})/\sim}^{{}_{\varphi}M = \overline{M}\times_{\pi}\widetilde{X}_{x_0}})/N_k((\overline{M} \times \widetilde{X}_{x_0})/\sim) \\ m \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & \left[\mathrm{pr} \circ \begin{pmatrix} \Delta^k & \to & M \times \widetilde{X}_{x_0} \\ z & \mapsto & (m, \sigma(z)) \end{pmatrix}\right] \end{array}$$

is a natural isomorphism between chain complexes of R-left modules. These maps thus define a natural isomorphism

$$\Phi_*\colon \underbrace{\mathrm{H}_k(\overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0}))}_{=\mathrm{H}^{\mathrm{tw}}_k(X;M),\text{ see Definition 320.11}} \xrightarrow{\cong} \underbrace{\mathrm{H}_k(\mathrm{C}^{\mathrm{lc}}_*(X;{}_{\varphi}M))}_{=\mathrm{H}^{\mathrm{lc}}_k(X;{}_{\varphi}M),\text{ see Definition 343.5}}$$

of R-left modules from homology with twisted coefficients to homology with local coefficients.

Proof. We leave it to the reader to verify that Φ is well-defined. For an inverse we consider the map

$$\begin{array}{rcl} \Psi\colon \mathrm{C}^{\mathrm{lc}}_k(X;{}_{\varphi}M) = \mathrm{C}_k((\overline{M} \times \widetilde{X}_{x_0})/\sim)//N_k((\overline{M} \times \widetilde{X}_{x_0})/\sim) & \to & \overline{M} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k(\widetilde{X}_{x_0}) \\ (\sigma\colon \Delta^k \to M \times_{\pi} \widetilde{X}_{x_0}) & \mapsto & \mathrm{pr}_M(\widetilde{\sigma})(*) \otimes (\mathrm{pr}_{\widetilde{X}_{x_0}} \circ \widetilde{\sigma}) \end{array}$$

where $\widetilde{\sigma}\colon \Delta^k \to M \times \widetilde{X}_{x_0}$ is a lift of $\sigma\colon \Delta^k \to M \times_{\pi} \widetilde{X}_{x_0}$ and $* \in \Delta^k$ is a point

Again we leave it to the reader to verify that this map Ψ is well-defined. Finally note that it is elementary to verify that both maps are natural chain maps and that they are inverses of one another. ■

We move breathlessly to the cohomological analog of the above theorem:

Theorem 343.38. (Twisted Cohomology–LC-Cohomology Theorem) Let (X, x_0) be a pointed topological space that is path-connected space and locally contractible. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be a ring and let M be an $(R, \mathbb{Z}[\pi])$-left left module. We introduce the following notation:

- We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering.
- We denote by $\varphi\colon \pi \to \mathrm{Aut}_R(M)$ the representation where $\varphi(g)$ is precisely left-multiplication by g on M.
- As in the Local System-via-Representation Proposition 342.9 we consider the R-local coefficient system
$$_\varphi M \;:=\; \overline{M} \times_\pi \widetilde{X}_{x_0} \;:=\; (\overline{M} \times \widetilde{X}_{x_0})/\!\sim .$$
- We denote by $\mathrm{pr}\colon \overline{M} \times \widetilde{X}_{x_0} \to \overline{M} \times_\pi \widetilde{X}_{x_0} = (\overline{M} \times \widetilde{X}_{x_0})/\!\sim$ the natural projection.

The map

$$\begin{array}{rcl} \Phi\colon \underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0}), M) & \to & \mathrm{C}^k_{\mathrm{lc}}(X; {}_\varphi M) \\ (\theta\colon \mathrm{C}_k(\widetilde{X}_{x_0}) \to M) & \mapsto & \left(\begin{array}{rcl} \mathrm{S}_k(X) & \to & \mathrm{S}_k({}_\varphi M) = \mathrm{S}_k(M \times_\pi \widetilde{X}_{x_0}) \\ \sigma & \mapsto & (\theta(\widetilde{\sigma}) \otimes \widetilde{\sigma}) \end{array} \right) \end{array}$$

where $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}_{x_0}$ is a random lift of $\sigma\colon \Delta^k \to X$

is a natural isomorphism between cochain complexes of R-left modules. These maps define a natural isomorphism

$$\Phi_*\colon \underbrace{\mathrm{H}^k(\underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_*(\widetilde{X}_{x_0}), M)}_{= \mathrm{H}^k_{\mathrm{tw}}(X; M), \text{ see Definition 320.11}} \xrightarrow{\cong} \underbrace{\mathrm{H}_k(\mathrm{C}_*((\overline{M} \times \widetilde{X}_{x_0})/\!\sim)}_{= \mathrm{H}^k_{\mathrm{lc}}(X; {}_\varphi M), \text{ see Definition 343.22}}$$

of R-left modules from cohomology with twisted coefficients to cohomology with local coefficients.

Proof. We leave it to the reader to verify that the map Φ is well-defined. For an inverse we consider the map

$$\begin{array}{rcl} \Psi\colon \mathrm{C}^k_{\mathrm{lc}}(X; {}_\varphi M) & \to & \underset{\mathbb{Z}[\pi]\text{-left}}{\mathrm{Hom}}(\mathrm{C}_k(\widetilde{X}_{x_0}), M) \\ (\theta\colon \mathrm{S}_k(X) \to \mathrm{S}_k({}_\varphi M) = \mathrm{S}_k(M \times_\pi \widetilde{X}_{x_0})) & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_k(\widetilde{X}_{x_0}) & \to & M \\ \widetilde{\sigma} & \mapsto & m \end{array} \right) \end{array}$$

where $m \in M$ is chosen such that $\theta(p \circ \widetilde{\sigma})$ equals the map $\mathrm{pr} \circ (z \mapsto (m, \widetilde{\sigma}(z)))$

Not surprisingly we leave it to the reader to verify that Ψ is well-defined, that Φ and Ψ are natural chain maps and that they are inverses of one another. ■

We conclude this section with the following curt theorem:

Theorem 343.39. (Twisted–Local Coefficients Cup–Cap Theorem) The isomorphisms from the Twisted Homology–LC-Homology Theorem 343.37 and the Twisted Cohomology–LC-Cohomology Theorem 343.38 have the following properties:

(1) They descend to isomorphisms on relative (co-) homology.
(2) For trivial coefficients they commute with the natural isomorphisms to untwisted singular (co-) homology.
(3) They respect all connecting homomorphisms.
(4) They respect all cup products.
(5) They respect all cap products.

Proof. At this point the reader surely no longer expects precise statements let alone proofs. ■

343.6. **Poincaré Duality with local coefficients.** We conclude this chapter with the formulation of the Poincaré Duality Theorem for (co-) homology with local coefficients. We start out with the following piece of notation:

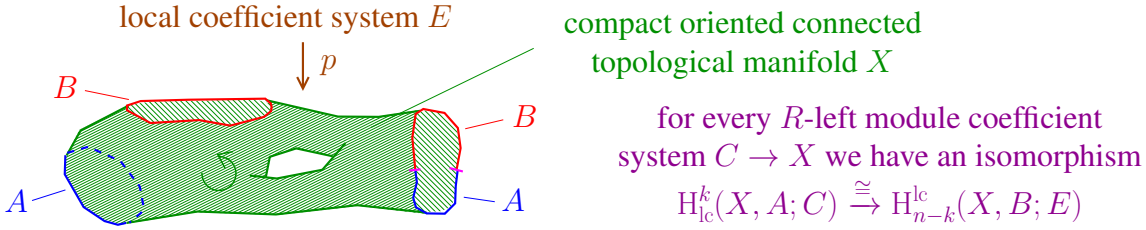
Notation 343.40. Let X be a compact oriented non-empty n-dimensional topological manifold. As usual we denote by $[X] \in \mathrm{H}_n(X, \partial X)$ the fundamental class from Definition 194.3. In the following we denote by $[X] \in \mathrm{H}_n^{\mathrm{lc}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{Z})$ also the image of the fundamental class $\mathrm{H}_n(X, \partial X; \mathbb{Z})$ under the natural isomorphism $\mathrm{H}_n(X, \partial X; \mathbb{Z}) \to \mathrm{H}_n^{\mathrm{lc}}(X, \partial X; {}_{\mathrm{tr}}\mathbb{Z})$ from the Trivial Local Coefficients Lemma 343.10.

Now we are ready to formulate the main result:

Theorem 343.41. (LC-Poincaré Duality Theorem) Let X be a compact oriented path-connected n-dimensional topological manifold. Suppose that we are given a decomposition $\partial X = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂X such that $A \cap B = \partial A = \partial B$. For every ring R and every local coefficient system $p\colon E \to X$ of R-left modules and each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{lc}}^k(X, A; E) & \xrightarrow{\cong} & \mathrm{H}_{n-k}^{\mathrm{lc}}(X, B; E) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

is an isomorphism of R-left modules.

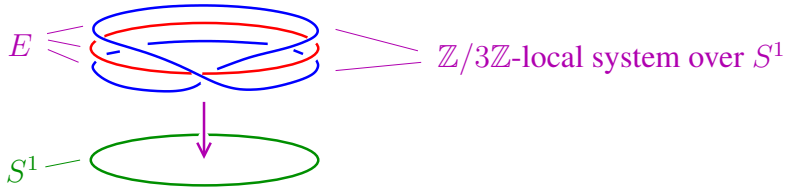


Proof. The theorem follows immediately from the Twisted Poincaré Duality Theorem 335.1 together with the Twisted Homology–LC-Homology Theorem 343.37, the Twisted Cohomology–LC-Cohomology Theorem 343.38 and the Twisted–Local Coefficients Cup–Cap Theorem 343.39. ■

343.7. **Exercises for Chapter 343.**

Exercise 343.1. In this exercise we consider again this chapter's favorite $\mathbb{Z}/3\mathbb{Z}$-local coefficient system $p\colon E \to S^1$.

(a) Calculate $\mathrm{H}_*^{\mathrm{lc}}(S^1; E)$.
(b) Calculate $\mathrm{H}^*_{\mathrm{lc}}(S^1; E)$.
(c) Do the reality check and check whether your calculations are consistent with the LC-Poincaré Duality Theorem 343.41.



Exercise 343.2. Let X be a topological space and let $0 \to A \to B \to C \to 0$ be a short exact sequence of local coefficient systems over X. Show that this give rise to short exact sequences of (co-) homology groups with local coefficients.

Part XXXVII

Reidemeister torsion

In this part of the lecture notes we will introduce the Reidemeister torsion of a finite CW-complex or compact smooth manifold that is equipped with a representation such that the corresponding twisted homology is acyclic. The exposition is inspired by [**DMK1967, Miln1966, Tur1986, Tur2001**].

- In Chapter 344 we will associate to each ring R an abelian group $\mathrm{K}_1(R)$ and we will use this construction to associate to each group G its Whitehead group $\mathrm{Wh}(G)$.
- In Chapter 345 we will introduce the torsion of a finite acyclic based chain complex.
- In Chapter 346 we will show that torsion is in a suitable sense natural.
- In Chapter 347 we will get to know several approaches to calculating torsion.

CHAPTER 344

The K_1-group and the Whitehead group

In this chapter we will associate to each ring (not necessarily commutative) an abelian group $K_1(R)$ which for each $d \in \mathbb{N}_0$ comes with a natural homomorphism $GL(d, R) \to K_1(R)$. This natural homomorphism $GL(d, R) \to K_1(R)$ can be viewed as a generalization and a refinement of the usual determinant over a commutative ring.

This chapter is organized as follows:

- In Section 344.1 we will associate to each ring R an abelian group $K_1(R)$.
- In Section 344.2 we will perform some basic calculations in $K_1(R)$.
- In Section 344.3 we will discuss the interplay between elementary matrices and commutators. This will give us a reinterpretation of the group $K_1(R)$.
- In Section 344.4 we will calculate the abelian group $K_1(R)$ for Euclidean domains and we will state Morita Invariance Theorem.
- In Section 344.5 we will associate to each group G the Whitehead group $Wh(G)$, which is an abelian group.
- In Appendix 344.6 we will give a short survey on various calculations of the Whitehead group $Wh(G)$.
- In Section 344.7 we will provide exercises for Chapter 344.

In the next Chapter 345 we will use $K_1(R)$ to introduce torsion of based chain complexes.

344.1. **Definition of $K_1(R)$.** In this section we will associate to any ring R (not necessarily commutative) an abelian group $K_1(R)$. To make sure we are all on the same page let us first recall Definition 2.6:

Definition 344.1.

(1) A ring is a triple $(R, +, \cdot)$ consisting of a set R and two maps $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$) such that $(R, +)$ is an abelian group and such that the following conditions are satisfied:
   (a) The multiplication "$\cdot$" is associative.
   (b) The multiplication "$\cdot$" is distributive.
   (c) There exists a multiplicatively neutral element $1 = 1_R$.

(2) A ring homomorphism between two rings $(R, +, \cdot)$ and $(S, +, \cdot)$ is a map $R \to S$ that preserves the additive and multiplicative structures and that sends 1_R to 1_S.

We continue with the following variation on Definition 299.17:

Definition 344.2. Let R be a ring. We define[3727]

$$GL(R) := \left\{ A \in \underbrace{M(\mathbb{N} \times \mathbb{N}, R)}_{\text{see Notation 2.14}} \,\middle|\, \begin{array}{c}\text{there exists a } d \in \mathbb{N} \text{ and} \\ \text{a } B \in GL(d, R) \text{ with}\end{array} \quad A = \begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} \right\}.$$

We endow $\mathrm{GL}(R)$ with the obvious matrix multiplication (which actually makes sense) and one can easily verify that this turns $\mathrm{GL}(R)$ into a group.

The following lemma gives us a reinterpretation of $\mathrm{GL}(R)$ via a universal property:

Lemma 344.3. (Invertible Matrix–Direct Colimit Lemma) Let R be a ring. The sequence of group homomorphisms[3728]

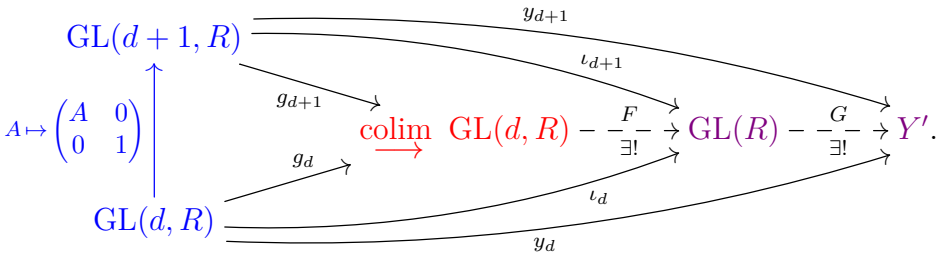
$$\begin{array}{ccc} \mathrm{GL}(d,R) & \to & \mathrm{GL}(d+1,R) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

with $d = 0, 1, 2, \ldots$ forms a direct system in the sense of Definition 104.2 in the category *Group* of groups. The obvious homomorphisms $\iota_d \colon \mathrm{GL}(d,R) \to \mathrm{GL}(R)$ for $d \in \mathbb{N}_0$ induce an isomorphism[3729]

$$\underbrace{\varinjlim \ \mathrm{GL}(d,R)}_{\substack{\text{direct colimit in the} \\ \text{sense of Definition 104.4}}} \ \to \ \mathrm{GL}(R)$$

of groups.

Sketch of proof. In the following diagram we summarize the notion of a direct system and a direct colimit in our context:



It is now clear that we have a natural homomorphism $\varinjlim \mathrm{GL}(d,R) \to \mathrm{GL}(R)$. Furthermore, using the same elementary argument as on page 2026 one can easily see that $\mathrm{GL}(R)$ also satisfies the universal property of a direct limit, which then implies that the map F is an isomorphism. ∎

The above group $\mathrm{GL}(R)$ is evidently highly non-commutative. To get a grip on it we will now consider the abelianization of that group. To introduce the abelianization let us recall the content of Definition 98.13 and Definition 98.15:

Definition 344.4. Let G be a group.

(1) Given $x, y \in G$ we refer to $[x,y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$ as the commutator of x and y.

(2) We refer to

$$[G,G] := \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \ldots, x_n, y_n \in G \right\}$$

as the commutator subgroup of G.

[3727]The more formal definition would be that we take all $A = (a_{ij}) \in \mathrm{M}(\mathbb{N} \times \mathbb{N}, R)$ for which there exists a $d \in \mathbb{N}_0$ such that for all $i, j > d$ we have $a_{ij} = \delta_{ij}$.

[3728]Recall that on page 119 we saw that $\mathrm{GL}(0,R)$ is the trivial group with a single element.

[3729]Recall that in the Algebraic–Direct Colimit Existence Proposition 104.7 we showed that in the category *Group* of groups the direct colimit always exists.

(3) By the Commutator Subgroup Lemma 98.14 (1) we know that $[G, G]$ is a normal subgroup of G. We refer to
$$G_{\text{ab}} := G/[G, G]$$
as the abelianization of G. By the Commutator Subgroup Lemma 98.14 (2) we know that G_{ab} is an abelian group.

The notion of an abelianization allows us to introduce the following concept:

Definition 344.5. Let R be a ring.

(1) The Whitehead group $\mathrm{K}_1(R)$ of R is defined as the abelian group
$$\mathrm{K}_1(R) := \mathrm{GL}(R)_{\text{ab}} := \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)].$$

(2) Let $(-1) \in \mathrm{K}_1(R)$ be the element that is represented by $\mathrm{diag}(-1, 1, 1, \ldots)$. Note that this defines an element of order at most two. The reduced Whitehead group is defined as[3730]
$$\overline{\mathrm{K}}_1(R) := \mathrm{K}_1(R)/\text{subgroup of order} \leq 2 \text{ generated by } (-1).$$

Remark. Let R be a ring. By definition $\mathrm{K}_1(R)$ and $\overline{\mathrm{K}}_1(R)$ are abelian groups. Following [**Tur2001**] we use multiplicative notation for these groups. This in contrast to the conventions say in [**Miln1966**] and [**CoheM1973**] which both use additive notation. □

The categorically minded reader will not be surprised by the next lemma:

Lemma 344.6. (K$_1$-Functor Lemma) The maps
$$\begin{array}{rcl} R & \mapsto & \mathrm{K}_1(R) \\ (\varphi\colon R \to S) & \mapsto & \left(\begin{array}{rcl} \varphi_*\colon \mathrm{K}_1(R) & \to & \mathrm{K}_1(S) \\ [(a_{ij})] & \mapsto & [(\varphi(a_{ij}))] \end{array} \right) \end{array}$$
define a covariant functor from the category *Ring* of rings to the category *AbGroup* of abelian groups. The analogous statement also holds for the reduced Whitehead group.

Proof. The lemma follows easily from the various definitions and the Abelianization Functor Proposition 98.16. ■

We turn to the last lemma of this section which only applies to commutative rings. To formulate the lemma it is convenient to one piece of notation from Definition 2.6:

Definition 344.7. Given a ring R we denote by $R^\times$ the multiplicative group of multiplicative units in R.

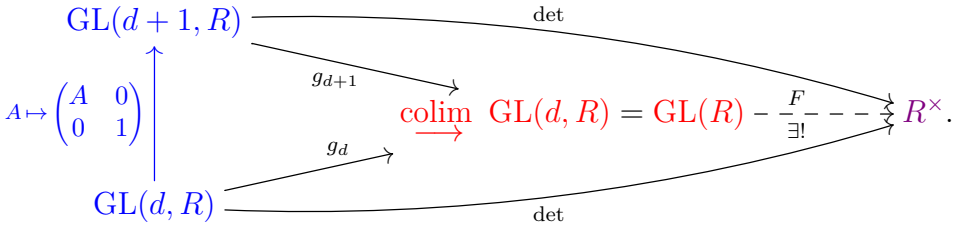
Now we can formulate our lemma:

Lemma 344.8. (K$_1$-Determinant Lemma) If R is a *commutative* ring, then there exists a unique homomorphism
$$\det\colon \mathrm{K}_1(R) \to R^\times$$
with the property that for every $A \in \mathrm{GL}(d, R)$ we have
$$\det\left[\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix}\right] = \det(A).$$
This homomorphism is natural in R.

[3730] Since $\mathrm{K}_1(R)$ is an abelian group we see that any subgroup is normal. It thus follows that $\overline{\mathrm{K}}_1(R)$ is indeed naturally a group.

The obvious question that arises from the K$_1$-Determinant Lemma 344.8 is, under what circumstances is the homomorphism $\det\colon \mathrm{K}_1(R) \to R^\times$ an isomorphism? We will address this question in the next section.

Proof. Recall that we know by the Invertible Matrix–Direct Colimit Lemma 344.3 that $\mathrm{GL}(R)$ is the direct colimit of the $\mathrm{GL}(d, R)$. We consider the following diagram:



Clearly the outer triangle commutes for all $d \in \mathbb{N}_0$. Thus we obtain by Definition 104.4 of a direct colimit an induced homomorphism $\det\colon \mathrm{GL}(R) \to R^\times$. Since R is commutative we know that $R^\times$ is an abelian group, thus we obtain from the Commutator Subgroup Lemma 98.14 (3) an induced homomorphism $\det\colon \mathrm{K}_1(R) \to R^\times$. ■

344.2. **Calculations in $\mathrm{K}_1(R)$.** In this section we will perform some basic calculations in the Whitehead groups $\mathrm{K}_1(R)$. For our purpose it is convenient to adopt the following convention:

Convention 344.9. Let R be a ring.

(1) Recall that for every $d \in \mathbb{N}_0$ there exists a natural homomorphism $\mathrm{GL}(d, R) \to \mathrm{K}_1(R)$, in particular there exists a homomorphism from the units of R into $\mathrm{K}_1(R)$. By abuse of notation we usually denote the image of $A \in \mathrm{GL}(d, R)$ in $\mathrm{K}_1(R)$ by A as well. With this convention we have $\overline{\mathrm{K}}_1(R) = \mathrm{K}_1(R)/\{(\pm 1)\}$.

(2) We proceed in a similar fashion for the reduced Whitehead group $\overline{\mathrm{K}}_1(R)$.

We will use Convention 344.9 to formulate the following lemma, which will turn out to be extremely useful for performing calculations in $\mathrm{K}_1(R)$:

Lemma 344.10. (Product-in-K$_1$-Lemma) Let R be a ring. For any choice of matrices $A \in \mathrm{GL}(k, R)$, $B \in \mathrm{GL}(l, R)$, $X \in \mathrm{M}(k \times l, R)$ and $Y \in \mathrm{M}(l \times k, R)$ we have (with Convention 344.9) the following equality in $\mathrm{K}_1(R)$:[3731]

$$\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = A \cdot B.$$

Proof. For the purpose of this proof we introduce the following notation: Given $d \in \mathbb{N}$ and given $i \neq j \in \{1, \ldots, d\}$ we denote by E_{ij} the $d \times d$-matrix which is zero except for the (i, j)-entry which equals 1.

Claim 1. If $d \geq 3$, then for any $i \neq j \in \{1, \ldots, d\}$ and any $a \in R$ the matrix $\mathrm{id} + a \cdot E_{ij}$ is a commutator in $\mathrm{GL}(d, R)$.

[3731] Note that after "adding ones on the diagonal" we can assume that $k = l$, i.e. we can make sense of $A \cdot B$.

Proof. For $i \neq j \neq k$ and $a \in R$ one can easily verify that[3732]

$$\begin{aligned} \mathrm{id} + a \cdot E_{ik} &= (\mathrm{id} + a \cdot E_{ij}) \cdot (\mathrm{id} + E_{jk}) \cdot (\mathrm{id} - a \cdot E_{ij}) \cdot (\mathrm{id} - E_{jk}) \\ &= (\mathrm{id} + a \cdot E_{ij}) \cdot (\mathrm{id} + E_{jk}) \cdot (\mathrm{id} + a \cdot E_{ij})^{-1} \cdot (\mathrm{id} + E_{jk})^{-1}. \end{aligned}$$

Thus we have shown that the matrix $\mathrm{id} + a \cdot E_{ij}$ is a commutator in $\mathrm{GL}(d, R)$. ⊞

Claim 2. Any matrix of the form

$$\begin{pmatrix} \mathrm{id}_k & X_{k\times l} \\ 0 & \mathrm{id}_l \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \mathrm{id}_k & 0 \\ Y_{l\times k} & \mathrm{id}_l \end{pmatrix}$$

represents the trivial element in $\mathrm{K}_1(R)$.

Proof. We first deal with the matrix to left. Thus let $X = (x_{ij})$ be a $(k \times l)$-matrix over the ring R. Clearly

$$X = \prod_{i=1}^{k} \prod_{j=k+1}^{k+l} \underbrace{(\mathrm{id} + x_{ij} \cdot E_{ij})}_{\text{trivial by Claim 1}}.$$

By Claim 1 we know that each of factors represents the trivial element in $\mathrm{K}_1(R)$. But then the product is also trivial. The same way we deal with the second matrix in the statement of the claim. ⊞

Claim 3. For any $A \in \mathrm{GL}(k, R)$, $B \in \mathrm{GL}(l, R)$, $X \in \mathrm{M}(k \times l, R)$ and $Y \in \mathrm{M}(l \times k, R)$ we have the following equalities in $\mathrm{K}_1(R)$:

$$\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}.$$

Proof. First note that

$$\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \cdot \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{id} & 0 \\ Y \cdot A^{-1} & \mathrm{id} \end{pmatrix}}_{\substack{\text{trivial in } \mathrm{K}_1(R), \text{ by} \\ \text{Claim 2}}} \cdot$$

The proof of the second equality in the claim is of course almost the same. ⊞
It remains to prove the following claim:

Claim 4. For any two matrices $A \in \mathrm{GL}(k, R)$ and $B \in \mathrm{GL}(l, R)$ we have the following equality in $\mathrm{K}_1(R)$:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = A \cdot B.$$

Proof. We start out with a special case. Namely, note that for any $X \in \mathrm{GL}(k, R)$ we have

$$\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{pmatrix}}_{\text{trivial by Claim 2}} \cdot \underbrace{\begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - X^{-1} & \mathrm{id} \end{pmatrix}}_{\text{trivial by Claim 2}} \cdot \underbrace{\begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ 0 & \mathrm{id} \end{pmatrix}}_{\text{trivial by Claim 2}} \cdot \underbrace{\begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - X & \mathrm{id} \end{pmatrix}}_{\text{trivial by Claim 2}}.$$

By Claim 2 we know that the right hand side is trivial in $\mathrm{K}_1(R)$.

[3732] In a more reader friendly notation we have

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now suppose that we are given $A \in \mathrm{GL}(k,R)$ and $B \in \mathrm{GL}(l,R)$. As we mentioned in Footnote 3731, after "adding ones on the diagonal" we can assume that $k = l$. Then we obtain the following equalities in $\mathrm{K}_1(R)$:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \underbrace{\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}}_{\text{trivial by the above discussion}} = \begin{pmatrix} A \cdot B & 0 \\ 0 & \mathrm{id} \end{pmatrix} \underset{\text{see Convention 344.9}}{=} A \cdot B.$$

■

344.3. **Elementary matrices and commutators.** As it stands we have little understanding for the definition of the Whitehead group $\mathrm{K}_1(R)$. To gain more understanding we want to give an alternative description of the commutator subgroup of $\mathrm{GL}(R)$.

We start out with the following definition form linear algebra which we first introduced in the Matrix Group-Generating Set Lemma 100.5:

Definition 344.11. Let R be a ring. The elementary matrices are the matrices of the following form:

$$\underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(I)}}, \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(II)}} \quad \text{and} \quad \underbrace{\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}}_{\text{(III)}},$$

with u a unit in R and $r \in R$.

We have the following elementary calculation in $\mathrm{K}_1(R)$:

Lemma 344.12. (Row Swapping Lemma) Let R be a ring.

(1) Ever elementary matrix of type (II) represents in $\mathrm{K}_1(R)$ the same element as (-1).
(2) Ever elementary matrix of type (II) represents in $\overline{\mathrm{K}}_1(R)$ the trivial element.

Proof.

(1) One can easily verify that all elementary matrices of type (II) are conjugate. Since $\mathrm{K}_1(R)$ is abelian it suffices to prove the following claim:

Claim. The matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represent the same element in $\mathrm{K}_1(R)$.

Proof. We need to show that the product $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents the trivial element in $\mathrm{K}_1(R)$. Now note that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

It follows from the Product-in-K_1-Lemma 344.10 that each of the matrices to the right hand side represents the trivial element in $\mathrm{K}_1(R)$. ⊞

(2) This statement follows immediately from (1). ■

We move on to the following basic definition:

Definition 344.13. Let R be a ring. We denote by $\mathrm{E}(R)$ the subgroup of $\mathrm{GL}(R)$ that is generated by all elementary matrices of Type (III).

The following proposition, which was first observed in [**WhdJ1950b**], gives a convenient description of the commutator subgroup of $\mathrm{GL}(R)$:

Proposition 344.14. (GL(R)–Commutator Subgroup Proposition) Given any ring R we have the equality

$$\mathrm{E}(R) \;=\; [\mathrm{GL}(R), \mathrm{GL}(R)].$$

In particular we have

$$\mathrm{K}_1(R) \;=\; \mathrm{GL}(R)/\mathrm{E}(R).$$

Proof. We need to show that $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.

("$\subseteq$") First note that it follows from the Product-in-K$_1$-Lemma 344.10 that every elementary matrix of type (III) represents the trivial element in $\mathrm{K}_1(R)$. Since the commutator subgroup is in particular a subgroup we obtain that $\mathrm{E}(R) \subseteq [\mathrm{GL}(R), \mathrm{GL}(R)]$.

("$\supseteq$") In the following we will borrow some ideas from the proof of the Product-in-K$_1$-Lemma 344.10. We start out with the following claim:

Claim 1. Any matrix of the form

$$\begin{pmatrix} \mathrm{id}_k & X_{k\times l} \\ 0 & \mathrm{id}_l \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \mathrm{id}_k & 0 \\ Y_{l\times k} & \mathrm{id}_l \end{pmatrix}$$

lies in $\mathrm{E}(R)$.

Proof. We first deal with the matrix to left. Thus let $X = (x_{ij})$ be a $(k \times l)$-matrix over R. As on page 6519 we see that

$$X = \prod_{i=1}^{k} \prod_{j=k+1}^{k+l} \underbrace{(\mathrm{id} + x_{ij} \cdot E_{ij})}_{\text{matrix of type (III)}} \;\in\; \mathrm{E}(R).$$

The same way we deal with the second matrix in the statement of the claim. ⊞

Claim 2. Any matrix of the form

$$\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}$$

lies in $\mathrm{E}(R)$.

Proof. As on page 6519 we see that for any $X \in \mathrm{GL}(d, R)$ we have

$$\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{pmatrix}}_{\in \mathrm{E}(R), \text{ by Claim 1}} \cdot \underbrace{\begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - X^{-1} & \mathrm{id} \end{pmatrix}}_{\in \mathrm{E}(R), \text{ by Claim 1}} \cdot \underbrace{\begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ 0 & \mathrm{id} \end{pmatrix}}_{\in \mathrm{E}(R), \text{ by Claim 1}} \cdot \underbrace{\begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - X & \mathrm{id} \end{pmatrix}}_{\in \mathrm{E}(R), \text{ by Claim 1}} \cdot$$

⊞

Finally we turn to the proof of the inclusion $[\mathrm{GL}(R), \mathrm{GL}(R)] \subseteq \mathrm{E}(R)$. Thus let $X, Y \in \mathrm{GL}(R)$. Essentially by Definition 344.2 of $\mathrm{GL}(R)$ we can represent X and Y by finite matrices of the same size. Now note that

$$\begin{pmatrix} XYX^{-1}Y^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} (YX)^{-1} & 0 \\ 0 & YX \end{pmatrix}.$$

It follows from Claim 2 that all the matrices to the right lie in $E(R)$, thus the matrix to the left also lies in $E(R)$. ■

344.4. **The Whitehead group K_1 of Euclidean domains and Morita invariance.** The following proposition gives in particular a solution to Exercise 100.20:

Proposition 344.15. (K_1-Euclidean Domain Proposition) Let R be a commutative Euclidean domain in the sense of Definition 100.4 (in our applications we will have $R = \mathbb{Z}$ or R could be a field). The natural determinant homomorphism

$$\begin{array}{rcl} \det\colon K_1(R) & \to & R^\times \\ [A] & \mapsto & \det(A) \end{array}$$

from the K_1-Determinant Lemma 344.8 is a natural isomorphism.

Convention 344.16. Let R be a commutative Euclidean domain. We will use the isomorphism from the K_1-Euclidean Domain Proposition 344.15 to make the identification $K_1(R) =_{\mathrm{i}} R^\times$.

Example. If we apply the K_1-Euclidean Domain Proposition 344.15 to the Euclidean Domain $R = \mathbb{Z}$, then we see that the determinant induces an isomorphism $K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\}$. It follows easily that for the reduced Whitehead group we have $\overline{K}_1(\mathbb{Z}) = \{e\}$. □

Proof. Let R be a commutative Euclidean domain. It is clear that the map

$$\begin{array}{rcl} \det\colon \mathrm{GL}(R) & \to & R^\times \\ \left[\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix}\right] & \mapsto & \det(A) \end{array}$$

is an epimorphism. It remains to show that the map is a monomorphism. This in turn clearly follows from the following claim:

Claim. Any element in $K_1(R)$ is represented by a (1×1)-matrix.

Proof. So suppose that we are given an element in $K_1(R)$. We pick a matrix $A \subseteq \mathrm{GL}(d, R)$ which represents the given element of $K_1(R)$. Since R is a commutative Euclidean domain it follows from the Matrix Group-Generating Set Lemma 100.5 that the set of elementary matrices form a generating set for the group $\mathrm{GL}(d, R)$, i.e. A is a product of elementary matrices.[3733] The claim now follows easily from the following observations:

(I) It follows from iteratively applying the Product-in-K_1-Lemma 344.10 that any diagonal matrix is equivalent in $K_1(R)$ to a (1×1)-matrix.

(II) It follows from the Row Swapping 344.12 that any elementary matrix of type (II) is equivalent in $K_1(R)$ to a diagonal matrix.

(III) It follows from the Product-in-K_1-Lemma 344.10 that any elementary matrix of type (III) is trivial in $K_1(R)$. ■

Remark.

(1) The proof of the K_1-Euclidean Domain Proposition 344.15 shows that the "size" of $K_1(R)$ can be viewed as a measure of how far the elementary matrices of types (I), (II) and (III) are from generating all invertible matrices.

[3733] Here we secretly use that the inverse of an elementary matrix is again an elementary matrix

(2) Some careless authors might think that in the statement of the K_1-Euclidean Domain Proposition 344.15 one can replace Euclidean rings by PIDs. But it follows from work of Hyman Bass [**Bas1973**, p. 63] (see also [**GrayD1981**, Corollary 2] together with [**Len1981**, Theorem 0.4] or alternatively from [**Isc1980**]) that there exist PIDs where the determinant homomorphism is not an isomorphism.

(3) Calculating $K_1(R)$ for a given ring R is in general, evidently, rather hard. We refer to [**Rosb1994**, Chapter 2] for more calculations and results. □

One can wonder whether there are other (perhaps even non-commutative) rings with trivial Whitehead group K_1. For general culture we mention the following theorem, which together with the K_1-Euclidean Domain Proposition 344.15 does indeed give examples of non-commutative rings with trivial Whitehead group K_1:

Theorem 344.17. (Morita Invariance Theorem) Let R be a ring and let $k \in \mathbb{N}$. We consider the ring $M(k \times k, R)$. Note that we get a well-defined homomorphism

$$\begin{array}{rcl} K_1(M(k \times k, R) & \to & K_1(R) \\ [(n \times n)\text{-matrix } A \text{ over } M(k \times k, R)] & \mapsto & \Big[\underbrace{\begin{array}{c}\text{the matrix } A \text{ now viewed as a} \\ (nk \times nk)\text{-matrix over } R\end{array}}_{\substack{\text{defined in the Matrix-of-Matrices} \\ \text{Lemma 124.4}}}\Big]. \end{array}$$

This map is in fact an isomorphism.

Sketch of proof. First note that it follows easily from the Matrix-of-Matrices Lemma 124.4 that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} GL(n, M(k \times k, R) & \to & GL(nk, R) \\ [(n \times n)\text{-matrix } A \text{ over } M(k \times k, R)] & \mapsto & \left[\begin{array}{c}\text{the matrix } A \text{ now viewed as a} \\ (nk \times nk)\text{-matrix over } R\end{array}\right] \end{array}$$

is well-defined and that it is an isomorphism of groups. The desired statement now follows from taking the direct colimits, from applying the Cofinal-Isomorphism Lemma 209.13 and from the Abelianization Functor Proposition 98.16 (2).[3734] ■

In the K_1-Euclidean Domain Proposition 344.15 we saw that it was relatively straightforward to calculate the group K_1 for fields. It seems natural that for skew fields one should get a similarly satisfactory answer, and fortunately this is indeed the case:

Proposition 344.18. (K_1-Skew Field Proposition) Let $\mathbb{K}$ be a skew field. The natural homomorphism $\mathbb{K}^\times \to K_1(\mathbb{K})$ from Convention 344.9 induces an isomorphism

$$\begin{array}{rcl} (\mathbb{K}^\times)_{ab} & \to & K_1(\mathbb{K}) \\ [k] & \mapsto & [(k)] \end{array}$$

Proof. This proposition is proved in [**Sil1981**, p. 133] or alternatively in [**Rosb1994**, Corollary 2.2.6]. ■

We conclude this section with a neat definition, even though we will not make use of it:

[3734] The same story is also told in [**Miln1968b**, p. 361] with more authority. We also refer to [**Weib2013**, Proposition III.1.6.4] and [**Rosb1994**, Theorem 1.2.3 and Exercise 2.1.8] for alternative proofs.

Definition 344.19. Let $\mathbb{K}$ be a skew field. The inverse of the isomorphism of the K_1-Skew Field Proposition 344.18 is usually written as

$$\det\colon K_1(\mathbb{K}) \ \to \ (\mathbb{K}^\times)_{ab}.$$

This map, and the resulting maps

$$GL(d,R) \ \to \ K_1(\mathbb{K}) \ \xrightarrow{\det} \ \mathbb{K}^\times$$

are usually called the Dieudonné determinant. We refer to [**Dra1983**, Chapter 20] for more information on the Dieudonné determinant.

344.5. **The Whitehead group.** In Definition 344.5 we just associated to each ring R (not necessarily commutative) the two abelian groups $K_1(R)$ and $\overline{K}_1(R) = K_1(R)/\{\pm 1\}$. Next recall that to each group G we can associate its group ring $\mathbb{Z}[G]$. Thus it makes sense to associate to its group G the abelian group $K_1(\mathbb{Z}[G])$. Also note that we have a natural group homomorphism

$$\begin{array}{rcl} \iota\colon G & \to & K_1(\mathbb{Z}[G]) \\ g & \mapsto & [(g)]. \end{array}$$

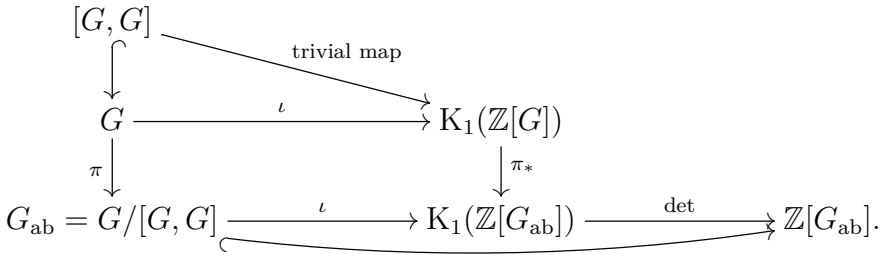
Since $K_1(\mathbb{Z}[G])$ is abelian we see that the commutator subgroup $[G,G]$ of G lies necessarily in the kernel of $\iota\colon G \to K_1(\mathbb{Z}[G])$. The following lemma makes the more interesting statement, that the kernel of ι actually equals the commutator subgroup of G:

Lemma 344.20. (G-to-K_1-Lemma) Let G be a group. The kernel of the natural map

$$\begin{array}{rcl} \iota\colon G & \to & K_1(\mathbb{Z}[G]) \\ g & \mapsto & [(g)] \end{array}$$

is given by the commutator subgroup $[G,G]$ of G.

Proof. We denote by $\pi\colon G \to G_{ab} := G/[G,G]$ the natural epimorphism onto the abelianization of G. Note that we have the following diagram:



We make the following clarifications and observations:

(a) The vertical map is clearly exact.
(b) Since $K_1(\mathbb{Z}[G])$ is abelian we see that the diagonal map is the trivial map.
(c) The diagram commutes by the naturality of ι.
(d) Since $\mathbb{Z}[G_{ab}]$ is a commutative ring we have the determinant homomorphism from the K_1-Determinant Lemma 344.8.
(e) The composition of the two bottom horizontal maps is clearly injective.

The desired statement now follows from a minor diagram chase. ∎

The idea now is to associate to a group G a Whitehead group that measures features of matrices over the group ring that do not come directly from the group itself. This thought leads us to the following definition:

Definition 344.21. Let G be a group and let

$$\begin{array}{rcl} \iota\colon G & \to & \overline{\mathrm{K}}_1(\mathbb{Z}[G]) \\ g & \mapsto & [(g)] \end{array}$$

be the natural homomorphism. We refer to

$$\mathrm{Wh}(G) \;:=\; \overline{\mathrm{K}}_1(\mathbb{Z}[G])/\iota(G)$$

as the Whitehead group of G.

Not surprisingly the Whitehead group also gives us a functor:

Lemma 344.22. (Wh-Functor Lemma) The maps

$$\begin{array}{rcl} G & \mapsto & \mathrm{Wh}(G) \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \varphi_*\colon \mathrm{Wh}(G) & \to & \mathrm{Wh}(H) \\ [(a_{ij})] & \mapsto & [(\varphi(a_{ij}))] \end{array} \right) \end{array}$$

define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

Proof. The lemma follows easily from the definitions, the Group Ring Functor Lemma 97.33 and the K_1-Functor Lemma 344.6. ∎

We now provide our first example, in fact it is the only example that we can calculate easily by hand:

Lemma 344.23. (Whitehead Group-of-Trivial Group Lemma) The Whitehead group of the trivial group is trivial.

Proof. The group ring $\mathbb{Z}[\{e\}]$ of the trivial group is of course isomorphic to the ring $\mathbb{Z}$. On page 6522 we showed that $\overline{\mathrm{K}}_1(\mathbb{Z})$ is the trivial group. It follows almost immediately that $\mathrm{Wh}(\{e\}) = \overline{\mathrm{K}}_1(\mathbb{Z}[\{e\}])/\iota(\{e\}) \cong \overline{\mathrm{K}}_1(\mathbb{Z})/\iota(\{e\})$ is the trivial group. ∎

It is not unreasonable to ask whether the Whitehead group can ever be non-trivial. In fact in the next section we will see that frequently it turns out to the trivial group. But the next lemma shows that the Whitehead group of a finite cyclic group can be non-trivial, in fact even infinite:

Lemma 344.24. (Non-Trivial Whitehead Group Lemma) Let $G = \langle x | x^5 = 1 \rangle$ be the cyclic group of order 5. Then the (1×1)-matrix

$$(x + x^{-1} - 1)$$

represents an element of infinite order in $\mathrm{Wh}(G)$.

Proof. First note that we have the following identity in the group ring $\mathbb{Z}[\langle x | x^5 = 1 \rangle]$:

$$\begin{array}{rcl} (x + x^{-1} - 1) \cdot (x^2 + x^{-2} - 1) & = & (x^3 + x^{-1} - x) + (x + x^{-3} - x^{-1}) + (-x^2 - x^{-2} + 1) \\ & = & x^3 + x^{-3} - x^2 - x^{-2} + 1 = 1. \end{array}$$

This shows that the (1×1)-matrix $(x + x^{-1} - 1)$ defines an element in $\mathrm{Wh}(G)$. Next we set $\xi := \exp(2\pi \mathrm{i}/5) \in \mathbb{C}$. We consider the ring homomorphism

$$\begin{array}{rcl} \varphi\colon \mathbb{Z}[\langle x | x^5 = 1\rangle] & \to & \mathbb{C} \\ \sum_{j=0}^{4} a_j \cdot x^j & \mapsto & \sum_{j=0}^{4} a_j \cdot \xi^j. \end{array}$$

It follows easily from the K$_1$-Functor Lemma 344.6 that φ induces a homomorphism

$$\varphi_*\colon \mathrm{Wh}(\langle x | x^5 = 1\rangle) \to \underbrace{\overline{\mathrm{K}}_1(\mathbb{C})/\{\xi^k \mid k \in \mathbb{Z}\}}_{=_{\mathrm{i}}\mathbb{C}^\times/\{\xi^k \mid k\in\mathbb{Z}\}}.$$

Thus it remains to prove the following claim:

Claim. By the K$_1$-Determinant Lemma 344.8 it remains to show that $\xi + \xi^{-1} - 1$ has infinite order in the multiplicative group $\mathbb{C}^\times/\{\xi^k \mid k \in \mathbb{Z}\}$.

Proof. First note that the map

$$\begin{array}{rcl} \mathbb{C}^\times/\{\xi^k \mid k \in \mathbb{Z}\} & \to & (\mathbb{R}_{>0}, \cdot) \\ [z] & \mapsto & |z| \end{array}$$

is a well-defined homomorphism. The claim now follows from the observation that $|\xi + \xi^{-1} - 1| = |2\cos(\frac{2\pi}{5} - 1| \neq 1$. ∎

We conclude this section with the following lemma which we will almost certainly use on many occasions, without noticing it:

Lemma 344.25. (Whitehead Group–Conjugation Lemma) Let G be a group and let $g \in G$. Let $\varphi\colon G \to G$ be be the automorphism that is given by conjugation by $g \in G$. Then the induced map $\varphi_*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(G)$ from the Wh-Functor Lemma 344.22 is the identity map.

Proof. Let $g \in G$. We denote by $\varphi\colon G \to G$ the homomorphism that is given by conjugation by g, i.e. it is given by $h \mapsto ghg^{-1}$. Now let $A \in \mathrm{Wh}(G)$. By definition we can represent A by an eponymous matrix $A \in \mathrm{GL}(d, \mathbb{Z}[G])$. Now now that we have the following equalities in $\mathrm{Wh}(G)$:

$$\varphi_*([A]) = [\varphi(A)] \underset{\substack{\uparrow \\ \text{follows from} \\ \text{elementary algebra}}}{=} [(g \cdot \mathrm{id}_d) \cdot A \cdot (g \cdot \mathrm{id}_d)^{-1}] = [g \cdot \mathrm{id}_d] \cdot [A] \cdot [g \cdot \mathrm{id}_d]^{-1} \underset{\substack{\uparrow \\ \text{since } \mathrm{Wh}(G) \text{ is} \\ \text{an abelian group}}}{=} [A].$$

∎

344.6. **Appendix: A short survey on the Whitehead group.** In this last section we will give a short survey on various facts about the Whitehead group. Unfortunately we will not be able to prove any of the statements of this appendix.

In the Non-Trivial Whitehead Group Lemma 344.24 we showed that the Whitehead group of the cyclic group $\mathbb{Z}/5\mathbb{Z}$ is non-trivial. Rather amazingly we, as humanity, completely understand the Whitehead group of any cyclic group:

Theorem 344.26. (Whitehead Group-of-Cyclic Group Theorem) Let $n \in \mathbb{Z}$. The Whitehead group $\mathrm{Wh}(\mathbb{Z}/n\mathbb{Z})$ is a free abelian group of rank $\lfloor \frac{n}{2} \rfloor + 1 - \delta(q)$ where $\delta(q)$ is the number of divisors of q.

Proof. The ranks of the groups were first determined by Hyman Bass [**Bas1964**, Corollary 20.3]. The fact that they are actually free abelian groups is proved in [**ADOS1987**, Theorem 3.3] and in [**Oli1988**, Theorem 5.6]. ■

Example.

(1) It follows from the Whitehead Group-of-Cyclic Group Theorem 344.26 that $\mathrm{Wh}(\mathbb{Z}/5\mathbb{Z})$ is a free group of rank $\lfloor \frac{5}{2} \rfloor + 1 - \delta(5) = 2 + 1 - 2 = 1$.

(2) It follows easily from the Whitehead Group-of-Cyclic Group Theorem 344.26 that for $n \in \mathbb{N}$ we have $\mathrm{Wh}(\mathbb{Z}/n\mathbb{Z}) = 0$ if and only if $n \in \{1, 2, 3, 4, 6\}$. One would think that for small n one should be able to prove this statement "by hand". But even the proof for the case $n = 2$ is surprisingly involved, see [**Rosb1994**, Theorem 2.4.3].

(3) One can easily convince oneself that it is a consequence of the Whitehead Group-of-Cyclic Group Theorem 344.26 that for two groups G and H there is no naive formula relating the Whitehead group $\mathrm{Wh}(G \times H)$ of the direct product $G \times H$ to the Whitehead groups $\mathrm{Wh}(G)$ and $\mathrm{Wh}(H)$. □

Remark. Let G be a finite group. By [**Bas1964**, Corollary 20.3] and [**BMS1967**, Proposition 4.14] (see also [**CoheM1973**, Corollary 11.2]) we know that $\mathrm{Wh}(G)$ is a finitely generated abelian group and we know that its rank can be computed from the representation theory of G. But calculating the torsion subgroups of Whitehead groups of finite groups is very hard. We refer to [**Stein1978**] and [**Oli1988**] for more information and calculations on Whitehead groups of finite groups. □

The Whitehead groups of finite cyclic groups are surprisingly large and complicated. Fortunately life is much easier for the infinite cyclic group $\mathbb{Z}$. In fact even more generally, for all free abelian groups we get an equally surprisingly simple answer:

Theorem 344.27. (Whitehead Group-of-Free Abelian Theorem) The Whitehead group of any free abelian group is trivial.

Proof. First note that it follows from the K_1-Euclidean Domain Proposition 344.15 (applied to the Euclidean Domain $\mathbb{Z}[\{e\}] \cong \mathbb{Z}$) that the Whitehead group of the trivial group is trivial. Higman [**Hig1940**] (see also [**CoheM1973**, p. 42]) showed that the Whitehead group of the infinite cyclic group is trivial (a textbook proof is also given in [**CoheM1973**, p. 42]). This was extended by Bass–Heller–Swan [**BHS1964**] to all free abelian groups. ■

We just discussed that Whitehead groups do not behave well with respect to taking direct products. Fortunately they are much better behaved under free products:

Theorem 344.28. (Whitehead Group–Free Product Theorem) Let G and G' be two groups. The natural monomorphisms $G \to G * G'$ and $G' \to G * G'$ induce an isomorphism

$$\mathrm{Wh}(G) \oplus \mathrm{Wh}(G') \xrightarrow{\cong} \mathrm{Wh}(G * G').$$

Proof. This theorem was proved by Stallings [**Stal1965c**]. ■

The above results provide us with the following theorem:

Theorem 344.29. (Whitehead Group-of-Free Group Theorem) The Whitehead group of any free group is trivial.

Proof. It follows from the Whitehead Group-of-Free Abelian Theorem 344.27 together with the Whitehead Group–Free Product Theorem 344.28 that the Whitehead group of any finitely generated free group is trivial.

Finally let G be an infinitely generated free group. By the Free Subgroup Proposition 138.6 we know that any (finitely generated) subgroup of G is a free group. It now follows from Exercise 344.9 that the Whitehead group of G is trivial. ■

The above results suggests that perhaps Whitehead groups behave very differently for torsion-free groups and for groups with torsion. In fact the following conjecture was formulated many years ago.

Conjecture 344.30. The Whitehead group is trivial for any torsion-free group.

Remark.

(1) Tom Farrell and Lowell Jones [**FJ1986**, Corollary 1] proved Conjecture 344.30 for many special cases, for example they showed it for fundamental groups of hyperbolic manifolds. Together with the Whitehead Group-of-Free Abelian Theorem 344.27 and the Surface–Möbius Structure Proposition 132.4 this implies that the Whitehead group of fundamental groups of closed orientable surfaces vanishes.

(2) As is explained in [**Lü2025**], the above Conjecture 344.30 can can be viewed as a special case of the much more general Farrell–Jones Conjecture. The original Conjecture 344.30 and the Farrell–Jones Conjecture have been established in many cases. For example, Arthur Bartels, Wolfgang Lück and Holger Reich [**BLR2008**] showed that it holds for all word-hyperbolic groups. □

For future reference we also record the following theorem:

Theorem 344.31. (Whitehead Group-3D Theorem) The Whitehead group of the fundamental group of any compact, orientable, non-spherical prime[3735] 3-dimensional smooth manifold is trivial.

Proof. This theorem follows from the Geometrization Theorem **??** together with the aforementioned work of Farrell–Jones [**FJ1986**, Corollary 1], work of Friedhelm Waldhausen [**Wald1978**, Theorem 17.5], Tom Farrell–W-Chung Hsiang [**FaH1981**] and Steven Plotnick [**Pl1980**]. ■

344.7. Exercises for Chapter 344.

Exercise 344.1. We consider the permutation group S_5 and the group ring $\mathbb{Z}[S_5]$. Is the reduced Whitehead group $\overline{\mathrm{K}}_1(S_5)$ non-trivial?

Exercise 344.2. Let R be a Euclidean Domain and let $n \in \mathbb{N}_{\geq 3}$. Show that the natural map

$$\begin{array}{rcl} \det\colon \mathrm{GL}(n,R)_{\mathrm{ab}} & \to & R^\times \\ A & \mapsto & \det(A) \end{array}$$

is an isomorphism.

[3735] Let M be a connected 3-dimensional smooth manifold. According to **??** we say that M is *prime* if M is not diffeomorphic to S^3 and if M cannot be written as a non-trivial connected sum of two manifolds, i.e. if M is diffeomorphic to a connected sum $M_1 \# M_2$, then M_1 or M_2 is diffeomorphic to S^3.

Exercise 344.3. Recall that in Exercise **??** we showed that we have an isomorphism

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \;\to\; \mathrm{SL}(2,\mathbb{Z})$$

where the homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ are the unique monomorphisms. Use this fact to determine the abelianization of $\mathrm{SL}(2,\mathbb{Z})$.
Remark. This shows that in Exercise 344.2 we do indeed have to assume that $n \geq 3$. Also note that this exercise is identical to Exercise 100.21.

Exercise 344.4. We consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Show that A is a commutator in $\mathrm{GL}(2,\mathbb{R})$.
(b) Show that A is *not* a commutator in $\mathrm{GL}(2,\mathbb{Z})$.
Hint. Use mod 2 reduction.

Exercise 344.5. Let R be a ring. Use the Eckmann-Hilton Theorem 299.32 to show that for any $A \in \mathrm{GL}(k,R)$ and $B \in \mathrm{GL}(l,R)$ we have the following equality in $\mathrm{K}_1(R)$:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;=\; A \cdot B.$$

Exercise 344.6. Let R be a ring with involution (in the sense of Definition 125.15). Show that the map

$$\begin{array}{ccc} \mathrm{K}_1(R) & \to & \mathrm{K}_1(R) \\ [A] & \mapsto & [\overline{A}^{\mathrm{T}}] \end{array}$$

is a well-defined automorphism of the group $\mathrm{K}_1(R)$.

Exercise 344.7. We consider the Gaussian integers $\mathbb{Z}[\mathrm{i}]$. Show that $\mathrm{K}_1(\mathbb{Z}[\mathrm{i}]) \cong \{\pm 1, \pm \mathrm{i}\}$.

Exercise 344.8. We consider the quaternions $\mathbb{H} \cong \mathbb{R}^4$ from Definition 171.2. As in the formulation of the Morita Invariance Theorem 344.17 we get for each $d \in \mathbb{N}_0$ an induced homomorphism $\varphi_d \colon \mathrm{GL}(d,\mathbb{H}) \to \mathrm{GL}(4d,\mathbb{R})$ which then define a homomorphism $\varphi \colon \mathrm{K}_1(\mathbb{H}) \to \mathrm{K}_1(\mathbb{R})$.

(a) Show that every element in $\mathrm{K}_1(\mathbb{H})$ is represented by a (1×1)-matrix.
(b) Show that every element in $S^3 = \{z \in \mathbb{H}^3 \,|\, |z| = 1\}$ is a commutator.
Hint. By the Quaternion Units Proposition 171.5 we know that the group S^3 of unit quaternions is isomorphic to the special unitary group $\mathrm{SU}(2)$. In $\mathrm{SU}(2)$ calculate the commutator of $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and use the fact that unitary matrices are diagonalizable.
(c) Show that the map

$$\mathrm{K}_1(\mathbb{H}) \xrightarrow{\varphi} \mathrm{K}_1(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times \xrightarrow{x \mapsto |x|} \mathbb{R}_{>0}$$

is an isomorphism.

Exercise 344.9.

(a) Let G be a group. Show that if for every finitely generated subgroup of G the Whitehead group vanishes, then the Whitehead group of G is also trivial.
(b) Show that the Whitehead group of any torsion-free abelian group (e.g. of $(\mathbb{Q},+)$) vanishes.

Exercise 344.10. Let $(I, \leq)$ be a directed set.

(a) Let $\{R_i\}_{i \in I}$ be a directed system of rings in the sense of Definition 104.2. Show that the natural homomorphisms $R_i \to \operatorname{colim} R_i$ induces an isomorphism

$$\operatorname*{colim} \mathrm{K}_1(R_i) \xrightarrow{\cong} \mathrm{K}_1\Big(\operatorname*{colim} R_i\Big).$$

(b) Let $\{G_i\}_{i \in I}$ be a directed system of groups. Show that the natural homomorphisms $G_i \to \operatorname{colim} G_i$ induces an isomorphism

$$\operatorname*{colim} \mathrm{Wh}(G_i) \xrightarrow{\cong} \mathrm{Wh}\Big(\operatorname*{colim} G_i\Big).$$

Remark. Note that this exercise, together with Exercise 104.7, gives a new more systematic approach to Exercise 344.9.

Exercise 344.11. Let R be a ring. For $i = 1, \ldots, m$ let $A_i \in \mathrm{GL}(k_i, R)$ be invertible matrices. Furthermore suppose that for each $1 \leq i < j \leq m$ we are given $B_{ij} \in \mathrm{M}(k_i \times k_j, R)$. Show that in $\mathrm{K}_1(R)$ we have

$$\begin{pmatrix} A_1 & B_{12} & \ldots & B_{1m} \\ 0 & \ddots & & \vdots \\ 0 & 0 & A_{m-1} & B_{m-1,m} \\ 0 & 0 & 0 & A_m \end{pmatrix} = A_1 \cdot \ldots \cdot A_m.$$

Exercise 344.12. Let $G = \langle x, y \mid y^2 = 1 \rangle$. We consider $a = 1 - y$ and $b = x(1 + y) \in \mathbb{Z}[G]$.

(a) Show that $ab \neq 0$ and show that $ba = 0$.

(b) Show that $(1 - ab)$ is trivial in $\mathrm{Wh}(G)$.

Hint. Show that the matrix $\begin{pmatrix} 1 - ab & 0 \\ 0 & 1 \end{pmatrix}$ is a commutator of two matrices X and Y.

This shows sometimes one needs to stabilize to show that elements in $\mathrm{Wh}(G)$ are trivial.

Exercise 344.13. Let $n \in \mathbb{N}$. Use the Whitehead Group-of-Cyclic Group Theorem 344.26 to show that $\mathrm{Wh}(\mathbb{Z}/n\mathbb{Z})$ is trivial if and only if $n \in \{1, 2, 3, 4, 6\}$.

CHAPTER 345

Torsion of based chain complexes

In this purely algebraic chapter we will introduce the torsion $\tau(C_*, \mathcal{C}_*) \in \mathrm{K}_1(R)$ of a based acyclic chain complex $(C_*, \mathcal{C}_*)$ over some ring R. This algebraic notion will later be the key to introducing Reidemeister torsion of suitable topological spaces.

This chapter is organized as follows:

- In Section 345.1 we will collect some basic facts about bases and transition matrices.
- In Section 345.2 we will discuss basings of filtered modules.
- In Section 345.3 we will define torsion of acyclic based chain complexes under a slight technical hypothesis.
- In Section 345.4 we will discuss stably free modules.
- In Section 345.3 we will drop the technical hypothesis in the definition of torsion.
- In Section 345.6 we will see that torsion is multiplicative under short exact sequences.
- In Section 345.7 we will provide exercises for Chapter 345.

In the next Chapter 346 we will see that torsion is functorial.

345.1. **Bases and transition matrices.** We start out by recalling Definition 309.17:

Definition 345.1. Let R be a ring.

(1) Let M be an R-left module. We say that M is a free R-left module on a set $\mathcal{B} \subseteq M$ if given any map $f\colon \mathcal{B} \to N$ to an R-left module N there exists a unique homomorphism $\Xi_{\mathcal{B}}\colon M \to N$ of R-left modules that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{B} & \hookrightarrow & M \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\, \Xi_{\mathcal{B}}} \\ & & N. \end{array}$$

(2) We say that an R-left module M is free if there exists a subset $\mathcal{B} \subseteq M$ such that M is a free R-left module on the set $\mathcal{B}$. We refer to $\mathcal{B}$ also as a basis for M.

Next we recall the following notation from page 5962:

Notation 345.2. For a ring R and $n \in \mathbb{N}_0$ we consider the set $R^{1\times n}$ of row vectors. We will view $R^{1\times n}$ as an R-left module by the obvious left multiplication. We make two observations:

- Note that $R^{1\times n}$ is a free R-left module on the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$.
- As discussed in the Left Module Homomorphism–Matrix Lemma 309.5, R-left homomorphisms $R^{1\times m} \to R^{1\times n}$ are given by right multiplication by a matrix in $\mathrm{M}(m \times n, R)$.

Convention 345.3. Let R be a ring and let $n \in \mathbb{N}_0$. Unless we say something else we will equip the R-module $R^{1\times n}$ with the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$.

One might initially think that two finite bases of a free R-left module necessarily need to have the same cardinality. But in the Non-IBN Lemma 312.22 we got disabused of this naive idea. This mildly shocking fact lead us to Definition 312.19:

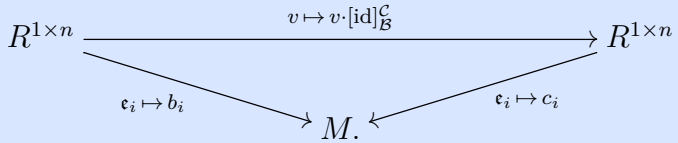
Definition 345.4. We say that a ring R has the invariant basis number property if the following statement holds: Any two bases for a free R-left module have the same cardinality. A ring that has the invariant basis number property is called an IBN ring.

Example. Even though we know by the Non-IBN Lemma 312.22 that there exist rings that are not IBN rings, these tend to be rather exotic. Mercifully almost all rings that we will care about are IBN rings. For example we already know the following:

(1) In the Commutative Ring–Invariant Basis Number Lemma 312.21 we saw that every *commutative* non-zero ring is an IBN ring.
(2) In the Group Ring–Invariance Basis Number Lemma 312.23 we showed, using (1), that for every group G and every commutative ring R the group ring $R[G]$ is an IBN ring.□

We continue with the following variation on Definition 31.1:[3736]

Definition 345.5. Let R be an IBN ring and let M be a free R-left module of finite rank. If $\mathcal{B} = (b_1, \dots, b_n)$ and $\mathcal{C} = (c_1, \dots, c_n)$ are bases[3737] for M, then there exists a unique matrix[3738] [3739] $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}} = (p_{ij}) \in \mathrm{M}(n \times n, R)$ such that given any $i \in \{1, \dots, n\}$ we have $b_i = \sum_{j=1}^{n} p_{ji} \cdot c_j$. We refer to $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}$ as the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Note that this matrix is the unique matrix that makes the following diagram commute:



The following lemma is a variation on the Transition Matrix Lemma 31.2:[3740]

Lemma 345.6. (Transition Matrix Lemma) Let R be an IBN ring and let M be a free R-left module of finite rank.

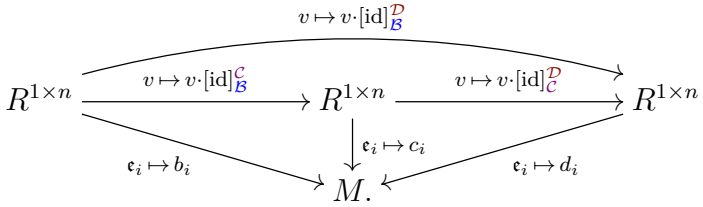
(1) If $\mathcal{B}$ is a basis for M, then $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{B}} = \mathrm{id}$.
(2) If $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are bases of M, then $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{D}} = [\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}} \cdot [\mathrm{id}]_{\mathcal{C}}^{\mathcal{D}}$.[3741]
(3) If $\mathcal{B}$ and $\mathcal{C}$ are two bases of M, then $[\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}} = ([\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}})^{-1}$.

[3736] Note that in contrast to Definition 31.1 we now work with left modules.
[3737] Since R is an IBN ring we know that the two bases have the same cardinality.
[3738] In the language of Definition 309.23 the matrix $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}$ is the matrix representing the identity map with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.
[3739] Note that since we work with row vectors we end up with the transpose of the matrix from Definition 31.1.
[3740] Since we now work with R-left modules the statements differ slightly from the Transition Matrix Lemma 31.2.
[3741] Note that for right modules where we use column vectors and left multiplication we would, as was the case for the original Transition Matrix Lemma 31.2, get an equality $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{D}} = [\mathrm{id}]_{\mathcal{C}}^{\mathcal{D}} \cdot [\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}$. Thus we see that the formulation for left modules is slightly neater.

Proof. [3742]

(1) This statement follows immediately from the definitions.

(2) This statement follows from the observation that the following diagram commutes:



(3) This statement follows almost immediately from (1) and (2). ■

We move on to the following convenient notation:

Definition 345.7. Let R be an IBN ring and let M be a free R-left module of finite rank.

(1) Given two bases $\mathcal{B}$ and $\mathcal{C}$ of M we write[3743]

$$[\mathcal{B}/\mathcal{C}] \;:=\; [[\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}}] \;\in\; \overline{\mathrm{K}}_1(R).$$

It follows immediately from the Transition Matrix Lemma 345.6 that for bases $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of M we have the following equalities:

$$\begin{aligned} [\mathcal{B}/\mathcal{B}] &= 1, \\ [\mathcal{C}/\mathcal{B}] &= [\mathcal{B}/\mathcal{C}]^{-1}, \\ [\mathcal{B}/\mathcal{D}] &= [\mathcal{B}/\mathcal{C}]\cdot[\mathcal{C}/\mathcal{D}]. \end{aligned}$$

(2) We say that two bases $\mathcal{B}$ and $\mathcal{C}$ of M are equivalent if $[\mathcal{B}/\mathcal{C}]$ is trivial in $\overline{\mathrm{K}}_1(R)$.[3744]

Example. Let R be a Euclidean Domain (e.g. R could be a field). It follows from the K_1-Euclidean Domain Proposition 344.15 that two bases are equivalent if and only if the determinant of the base change matrix equals ± 1. □

We conclude this section with the following definition:

Definition 345.8. Let R be an IBN ring.

(1) A basing for a free R-left module of finite rank is an equivalence class of bases in the sense of Definition 345.7.[3745]

(2) A based R-left module is a free R-left module of finite rank together with a basing.

345.2. **Basings and filtrations.** We start out with the following construction:

[3742] This lemma can be viewed as a special case of the Representing Matrix Lemma 309.24. For the reader's convenience we provide again a proof.

[3743] Throughout the remainder of these lecture notes we will invariably work with the reduced Whitehead group $\overline{\mathrm{K}}_1(R) = \mathrm{K}_1(R)/\{\pm 1\}$ from Definition 344.5. The key advantage of $\overline{\mathrm{K}}_1(R)$ over the Whitehead group $\mathrm{K}_1(R)$ is that it follows easily from the Row Swapping Lemma 344.12 that in $\overline{\mathrm{K}}_1(R)$ we do not have to distinguish between ordered and unordered bases.

[3744] It follows from the discussion in (1) that this is indeed an equivalence relation on the set of bases of M.

[3745] As in many other instances we will frequently not distinguish in the notation between a basis and the equivalence class (i.e. the basing) it represents.

Lemma 345.9. (Basis-Filtration Construction Lemma) Let R be an IBN ring and let $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n$ be a sequence of R-left modules. We assume that each V_i/V_{i-1} is a free R-left module that is equipped with a basing $\mathcal{B}_i$. We do the following:

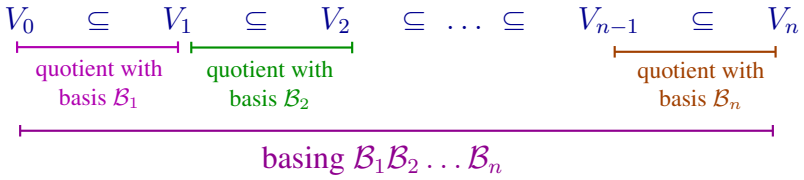
- For each $i \in \{1, \ldots, n\}$ we pick a representative basis $\{b_i^1, \ldots, b_i^{m_i}\}$ for $\mathcal{B}_i$.
- For each $b_i^j \in V_i/V_{i-1}$ we pick a lift $\widetilde{b}_i^j \in V_i \subseteq V_n$.

The following statements hold:

(1) The vectors $\widetilde{b}_1^1, \ldots, \widetilde{b}_1^{m_1}, \ldots, \widetilde{b}_k^1, \ldots, \widetilde{b}_k^{m_k}$ form a basis for V_n/V_0. In particular V_n/V_0 is a free R-left module of finite rank.

(2) The equivalence class of the above basis does not depend on any of the choices.

Notation 345.10. Let R be an IBN ring and let $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n$ be a sequence of R-left modules. We assume that each V_i/V_{i-1} is a free R-left module that is equipped with a basing $\mathcal{B}_i$. We denote by $\mathcal{B}_1 \ldots \mathcal{B}_n$ the basing for V_n/V_0 from the Basis-Filtration Construction Lemma 345.9.



Sketch of proof. It is straightforward to verify that the given vectors define a basis for the quotient R-left module V_n/V_0. Furthermore note that an elementary (but potentially confusing) argument shows that any other choice of a representative basis for $\mathcal{B}_i$ and another choice of lifts leads to a transition matrix of the form

$$\begin{pmatrix} X_1 & * & * \\ 0 & \ddots & 0 \\ 0 & 0 & X_n \end{pmatrix}$$

such that $X_1, \ldots, X_n$ are square matrices that represent the trivial element in $\overline{\mathrm{K}}_1(R)$. It now follows from the Product-in-$\overline{\mathrm{K}}_1$-Lemma 344.10 that this upper triangular matrix does indeed represent the trivial element in $\overline{\mathrm{K}}_1(R)$. ■

It is worth discussing a frequently used variation on the above setup:

Definition 345.11. Let R be a ring. Suppose that we are given a short exact sequence

$$0 \to A \to B \to C \to 0$$

of R-left modules.

(a) Suppose that we are given a basing $\mathcal{A}$ for A and a basis $\mathcal{C}$ for C. Under the natural isomorphisms $A/\{0\} \cong A$ and $B/A \cong C$ we obtain from Notation 345.10 a basing $\mathcal{AC}$ for B.[3746]

(b) We say that basings $\mathcal{A}$ for A, $\mathcal{B}$ for B and $\mathcal{C}$ for C are compatible if $\mathcal{AC} = \mathcal{B}$.[3747]

[3746] This construction is similar to the construction from the Short Exact Sequence–Orientation Lemma 31.10.

[3747] This notion is similar to the compatibility notion from Definition 31.11.

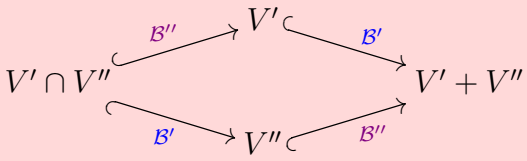
The following proposition shows in particular that the above construction is in a suitable sense associative and commutative.

Proposition 345.12. (Basis-Filtration Proposition) Let R be an IBN ring. Furthermore let $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n$ be a filtration of an R-left module V_n. Suppose that for each V_i/V_{i-1} we have a basing $\mathcal{B}_i$.

(1) Let $k \in \{1, \ldots, n\}$. We do the following:
- From the filtration $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k$ we obtain a basing $\mathcal{B}_1 \ldots \mathcal{B}_k$ for V_k/V_0.
- From the filtration $V_k \subseteq V_{k+1} \subseteq \ldots \subseteq V_n$ we obtain a basing $\mathcal{B}_{k+1} \ldots \mathcal{B}_n$ for V_n/V_k.
- We consider the filtration $V_1 \subseteq V_k \subseteq V_n$. Using (a) and (b) we obtain a basing $(\mathcal{B}_1 \ldots \mathcal{B}_k)(\mathcal{B}_{k+1} \ldots \mathcal{B}_n)$ for V_n/V_0.

Then $$(\mathcal{B}_1 \ldots \mathcal{B}_k)(\mathcal{B}_{k+1} \ldots \mathcal{B}_n) \;=\; \mathcal{B}_1 \ldots \mathcal{B}_n.$$

(2) Let V be an R-left module and let V', V'' be two submodules such that $(V' + V'')/V'$ is free with basis $\mathcal{B}'$ and $(V' + V'')/V''$ is free with basis $\mathcal{B}''$. Then we obtain the following two filtrations:



Note that elementary linear algebra tells us that we have two natural inclusion induced isomorphisms $V'/(V' \cap V'') \to (V' + V'')/V''$ and $V'/(V' \cap V'') \to (V' + V'')/V''$. By a slight abuse of notation we can now view $\mathcal{B}'$ and $\mathcal{B}''$ as bases of the modules to the left. With this notation we have the following equality of equivalence classes of bases for $(V' + V'')/(V' \cap V'')$:

$$\mathcal{B}'\mathcal{B}'' \;=\; \mathcal{B}''\mathcal{B}'.$$

(3) Suppose that for each V_i/V_{i-1} we have an alternative basings $\mathcal{A}_i$. Then we obtain the following equality in $\overline{\mathrm{K}}_1(R)$:

$$[\mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_n/\mathcal{B}_1\mathcal{B}_2 \ldots \mathcal{B}_n] \;=\; \prod_{i=1}^{n} [\mathcal{A}_i/\mathcal{B}_i].$$

Proof.

(1) First note that it follows easily from the definitions that

$$[(\mathcal{B}_1 \ldots \mathcal{B}_k)(\mathcal{B}_{k+1} \ldots \mathcal{B}_n)/\mathcal{B}_1 \ldots \mathcal{B}_n] \;=\; \begin{pmatrix} \mathrm{id} & * \\ 0 & \mathrm{id} \end{pmatrix}$$

which, by the Product-in-$\overline{\mathrm{K}}_1$-Lemma 344.10, represents the trivial element in $\overline{\mathrm{K}}_1(R)$.

(2) The base change matrix is clearly of the form

$$\begin{pmatrix} 0 & \mathrm{id}_k \\ \mathrm{id}_k & 0 \end{pmatrix}$$

which is easily seen to correspond to the trivial element of $\overline{\mathrm{K}}_1(R)$.

(3) Similar to (1) one sees that the transition matrix is going to be of the form

$$[\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n/\mathcal{B}_1\mathcal{B}_2\dots\mathcal{B}_n] \;=\; \begin{pmatrix} \mathcal{A}_1/\mathcal{B}_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mathcal{A}_n/\mathcal{B}_n \end{pmatrix}.$$

The desired statement now follows from the Product-in-$\overline{\mathrm{K}}_1$-Lemma 344.10. ∎

345.3. **Definition of algebraic torsion I.** In this section we finally get to introduce torsion of a based acyclic chain complex. We start out by (re-) introducing the following standard notation for chain complexes:

Notation 345.13. Let R be a ring and let (C_*, ∂_*) be a chain complex of R-left modules. As usual we write for each $i \in \mathbb{N}_0$:

$$\begin{array}{rcl} Z_i & := & \ker(\partial_i \colon C_i \to C_{i-1}), \\ B_i & := & \mathrm{im}(\partial_{i+1} \colon C_{i+1} \to C_i), \\ H_i & := & Z_i/B_i. \end{array}$$

We make the following observations which we will need at times:

(i) For each $i \in \mathbb{N}_0$ we have a short exact sequence $0 \to Z_i \hookrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$.
(ii) For each $i \in \mathbb{N}_0$ we have a short exact sequence $0 \to B_i \hookrightarrow Z_i \twoheadrightarrow H_i \to 0$.
(iii) Note that for each i we have inclusions

$$0 \;\subseteq\; B_i \;\subseteq\; Z_i \;\subseteq\; C_i$$

and that the boundary map induces a natural isomorphism $C_i/Z_i \xrightarrow{\cong} B_{i-1}$.

$$\underset{\text{quotient}\atop B_i/\{0\}\cong B_i}{0 \;\subseteq\; B_i} \;\subseteq\; \underset{\text{quotient}\atop Z_i/B_i = H_i}{Z_i} \;\subseteq\; \underset{\text{quotient}\atop C_i/Z_i \xrightarrow{\cong} B_{i-1}}{C_i}$$

Definition 345.14. Let R be a ring. We say that a chain complex (C_*, ∂_*) of R-left chain complex has length n if $C_i = 0$ for all $i > n$.

Now we are finally ready to introduce torsion:

Lemma 345.15. (Torsion–Well-Defined Lemma) Let R be an IBN ring. Furthermore let (C_*, ∂_*) be a chain complex of R-left modules of length n such that each homology module $H_i = \mathrm{H}_i(C_*)$ is a free R-left module.[3748] We make the following technical hypothesis:
($\star$) We assume that each $B_i = \mathrm{im}(\partial_{i+1} \colon C_{i+1} \to C_i)$ is also a free R-left module.[3749]
We also suppose that we are given the following data:

- For each $i \in \mathbb{N}_0$ we are given a basing $\mathcal{C}_i$ of C_i.
- For each $i \in \mathbb{N}_0$ we are given a basing $\mathcal{H}_i$ of $H_i = \mathrm{H}_i(C_*)$.

Then each B_i is a free R-left module *of finite rank*. For each $i \in \mathbb{N}_0$ we pick a basis $\mathcal{B}_i$ of the free R-left module B_i. Note that these bases, applied to the filtration $0 \subseteq B_i \subseteq Z_i \subseteq C_i$ and the natural isomorphism $C_i/Z_i \xrightarrow{\cong} B_{i-1}$, give by Notation 345.10 rise to a basis $\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}$ of C_i. Then

$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}, \{\mathcal{H}_i\}_{i\in\mathbb{N}_0}) \;:=\; \prod_{i=0}^{n} [\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^i} \in \overline{\mathrm{K}}_1(R).$$

is independent of the choice of the bases $\mathcal{B}_i$.

Definition 345.16. We continue with the setting of the Torsion–Well-Defined Lemma 345.15. In particular we assume that (⋆) holds.[3750] We refer to

$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}, \{\mathcal{H}_i\}_{i\in\mathbb{N}_0}) := \prod_{i=0}^{n} [\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}} \in \overline{\mathrm{K}}_1(R).$$

as the torsion[3751] of $(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}, \{\mathcal{H}_i\}_{i\in\mathbb{N}_0})$. If (C_*, ∂_*) is acyclic, then we just write $\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0})$.

Remark.

(1) The concept of the torsion of an acyclic based chain complex goes back to the work of Kurt Reidemeister [**Rei1935**]. The term "torsion" was coined by his student Wolfgang Franz [**Fraz1935**]. On page 6538 and more generally on page 6570 we will discuss a special case of torsion, that might explain where the name "torsion" comes from.

(2) In our definition of torsion we follow the convention from [**Tur2001**] and [**Nic2003**], which is the multiplicative inverse of the torsion as it is introduced in [**Miln1966**] and in [**CoheM1973**]. We prefer the convention from [**Tur2001, Nic2003**] since for knots and links it leads to cleaner statements. □

Proof. Let R be an IBN ring and let (C_*, ∂_*) be a chain complex of R-left modules of length n such that each homology module $H_i = \mathrm{H}_i(C_*)$ is a free R-left module. We make the following technical hypothesis:

(⋆) We assume that each $B_i = \mathrm{im}(\partial_{i+1}\colon C_{i+1} \to C_i)$ is also a free R-left module.

We also suppose that we are given the following data:

- For each $i \in \mathbb{N}_0$ we are given a finite basis $\mathcal{C}_i$ of C_i.
- For each $i \in \mathbb{N}_0$ we are given a finite basis $\mathcal{H}_i$ of $H_i = \mathrm{H}_i(C_*)$.

We start out with the following claim:

Claim. Each B_i is a free R-left module *of finite rank*.

Proof. As in Notation 345.13 we point out the following:

(i) For each $i \in \mathbb{N}_0$ we have a short exact sequence $0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0$.

(ii) For each $i \in \mathbb{N}_0$ we have a short exact sequence $0 \to B_i \to Z_i \to H_i \to 0$.

Note that by hypothesis all the modules H_i and B_{i-1} on the right are by hypothesis free R-left modules. It thus follows from the Module Splitting Lemma 309.31 that we have the following statements:

(i′) For each $i \in \mathbb{N}_0$ we have an isomorphism $C_i/B_{i-1} \cong Z_i$.

(ii′) For each $i \in \mathbb{N}_0$ we have an isomorphism $Z_i/H_i \cong B_i$.

We now turn to the actual proof of the claim. First note that it follows from (i′) and our hypothesis on the C_i that each Z_i (as a quotient of a finitely generated module) is finitely

[3748] In most of our examples the chain complex C_* will be acyclic, so the homology modules will be free with a basis given by the empty set.

[3749] By the Free Module–PID Theorem 309.35 we know that this condition is automatically satisfied if R is a PID.

[3750] In Definition 345.26 we will explain how to dispose of this restriction.

[3751] Sometimes this is also called Reidemeister torsion.

generated. It then follows from (ii′), by the same logic, that each B_i is indeed finitely generated. ⊞

Now suppose that for each i we have bases $\mathcal{B}_i$ and $\widetilde{\mathcal{B}}_i$ of the free R-left module B_i. We then see that we have the following equalities in $\overline{\mathrm{K}}_1(R)$:

$$\begin{aligned}\prod_{i=0}^{n}[\widetilde{\mathcal{B}}_i\mathcal{H}_i\widetilde{\mathcal{B}}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}} &\overset{\text{by the Transition Matrix Lemma 345.6 (2)}}{=} \prod_{i=0}^{n}\left([\widetilde{\mathcal{B}}_i\mathcal{H}_i\widetilde{\mathcal{B}}_{i-1}/\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}]\cdot[\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]\right)^{(-1)^{i+1}}\\ &= \prod_{i=0}^{n}[\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}}\cdot\underbrace{\prod_{i=0}^{n}[\widetilde{\mathcal{B}}_i/\mathcal{B}_i]^{(-1)^{i+1}}\cdot\prod_{i=0}^{n}[\widetilde{\mathcal{B}}_{i-1}/\mathcal{B}_{i-1}])^{(-1)^{i+1}}}_{=1}.\end{aligned}$$

by the Basis-Filtration Proposition 345.12 (3) ■

For future reference we record the simplest example of torsion in the following lemma:

Lemma 345.17. (Elementary Torsion Lemma) Let R be an IBN ring. Furthermore let C_k and C_{k-1} be two R-modules of rank n that are equipped with bases $\mathcal{C}_k$ and $\mathcal{C}_{k-1}$. Finally let $\partial_k\colon C_k\to C_{k-1}$ be an isomorphism. We denote by A the square matrix over R which represents (in the sense of Definition 309.23) ∂_k with respect to the given bases. Then we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau(0\to C_k\xrightarrow{\partial_k}C_{k-1}\to 0,\{\mathcal{C}_k,\mathcal{C}_{k-1}\}) \;=\; [A]^{(-1)^k}.$$

Remark. Let R be a commutative ring. The Elementary Torsion Lemma 345.17 together with the K_1-Determinant Lemma 344.8 says in particular that the torsion of a chain complex of based R-left modules can be viewed as a generalization and refinement of the determinant of an isomorphism between two based R-left modules. □

Proof. So suppose we are given bases $\mathcal{C}_k=\{v_1,\dots,v_n\}$ and $\mathcal{C}_{k-1}=\{u_1,\dots,u_n\}$. From Definition 309.23 we obtain the following diagram:

$$\begin{array}{ccc} C_k & \xrightarrow{\quad\partial_k\quad} & C_{k-1}\\ {\scriptstyle \mathfrak{e}_i\mapsto v_i}\uparrow & & \uparrow{\scriptstyle \mathfrak{e}_i\mapsto u_i}\\ R^{1\times n} & \xrightarrow{\quad v\mapsto v\cdot A\quad} & R^{1\times n}.\end{array}$$

We equip $B_{k-1}=\mathrm{im}(\partial_k\colon C_k\to C_{k-1})$ with the basis $\mathcal{B}_{k-1}=\{\partial_k(v_1),\dots,\partial_k(v_n)\}$. We pick the obvious lift of these vectors to C_k, namely we pick the lifts $v_1,\dots,v_n$. We now see that we have the following equalities in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*) \underset{\uparrow}{=} [\overbrace{\mathcal{B}_{k-1}}^{=\mathcal{C}_k}/\mathcal{C}_k]^{(-1)^{k+1}}\cdot[\mathcal{B}_{k-1}/\mathcal{C}_{k-1}]^{(-1)^k} \underset{\uparrow}{=} [\mathcal{B}_{k-1}/\mathcal{C}_{k-1}]^{(-1)^k} \underset{\uparrow}{=} [A]^{(-1)^k}.$$

by Definition 345.16; since $v_i\in C_k$ is a lift of $\partial_k(v_i)$ we see that $[\mathcal{B}_{k-1}/\mathcal{C}_k]=\mathrm{id}$; follows from comparing Definition 345.5 with Definition 309.23 ■

Example. Let A be an $(n\times n)$-matrix over $\mathbb{Z}$ with $\det(A)\neq 0$. It follows from the Elementary Torsion Lemma 345.17 that we have the following equalities in $\overline{K}_1(\mathbb{Q})=_{\mathrm{i}}$

$\mathbb{Q}^\times/\{\pm 1\}$:

$$\tau\big(0 \to \mathbb{Q}^n \xrightarrow{A} \mathbb{Q}^n \to 0, \text{standard basis of } \mathbb{Q}^n\big) \underset{\uparrow}{=} \det(A) = |\det(A)|.$$

by the Elementary Torsion Lemma 345.17

By the Finite Matrix-Quotient Lemma 97.23 we know that $|\det(A)|$ equals the number of elements in the finite abelian group $\mathbb{Z}^n/A\mathbb{Z}^n$. This explains, perhaps, the name torsion for the invariant $\tau(C_*, \mathcal{C}_*)$. □

345.4. **Stably free modules.** In the formulation of the Torsion–Well-Defined Lemma 345.15 we added the technical hypothesis (⋆) that the images B_i need to be free R-left modules. In this section we will explain how to get rid of this hypothesis.

The key to dealing with this conundrum is the following definition:

Definition 345.18. Let R be an IBN ring. We say an R-left module M is stably free if there exists a free R-left module F of finite rank such that the direct sum $M \oplus F$ is a stably free R-module of finite rank.

Lemma 345.19. (Short Exact Sequence–Stably Free) Let R be a ring.

(1) Every stably free R-left module is projective in the sense of Definition 309.25.

(2) Let
$$0 \to X \to Y \to Z \to 0$$
be a short exact sequence of R-left modules. If Y and Z are stably free, then X is also stably free.

Proof. Let R be a ring.

(1) By Definition 345.18 a stably free module is in particular a subsummand of a free modul. It thus follows immediately from the Projective Module–Characterization Lemma 309.28 that every stably free R-left module is also projective.

(2) Suppose that we are given a short exact sequence $0 \to X \to Y \to Z \to 0$ of R-left modules such that Y and Z are stably free. By (1) we know that Z is a projective R-left module. It thus follows from the Projective Module–Splitting Lemma 309.27 that the above sequence splits. By the Module Splitting Lemma 309.31 this implies in particular that there exists an isomorphism $X \oplus Z \cong Y$ of R-left modules. Since Y and Z are stably free R-left modules we know by Definition 345.18 that there exist free R-left modules F_Y and F_Z such that $Y \oplus F_Y$ is a free R-left module and such that $Z \oplus F_Z$ is a free R-left module. But now we see that
$$X \oplus \underbrace{Z \oplus F_Z}_{\text{free}} \oplus \underbrace{F_Y}_{\text{free}} \cong \underbrace{Y \oplus F_Y}_{\text{free}} \oplus \underbrace{Z}_{\text{free}}$$
as R-left modules. Thus we have shown that X is stably free. ■

The following lemma is the key to generalizing our notion of torsion to more general contexts:

Lemma 345.20. (Image–Stably Free) Let R be a ring and let (C_*, ∂_*) be a chain complex of free R-left modules such that each homology module $\mathrm{H}_i(C_*)$ is also free. Then all the R-left modules
$$B_i := \operatorname{im}(\partial_{i+1} \colon C_{i+1} \to C_i)$$

are stably free R-left modules.

Proof. We will prove an even stronger statement, namely we will show that all the R-left modules

$$\begin{array}{rcl} Z_i & := & \ker(\partial_i \colon C_i \to C_{i-1}), \\ B_i & := & \operatorname{im}(\partial_{i+1} \colon C_{i+1} \to C_i) \end{array}$$

are stably free R-left modules. We consider the short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & Z_i & \to & \underbrace{C_i}_{\text{free}} & \to & B_{i-1} & \to & 0 \\ 0 & \to & B_i & \to & Z_i & \to & \underbrace{\mathrm{H}_i(C_*)}_{\text{free}} & \to & 0 \end{array}$$

from Notation 345.13. Note that $B_{-1} = \{0\}$ is stably free. It thus follows from the Short Exact Sequence–Stably Free 345.19, applied to the above two short exact sequences, and an elementary induction argument that all the B_i and Z_i are stably free R-left modules. ■

Now that we have seen that many of our favorite modules are stably free, let us develop the theory of stably free modules:

Definition 345.21. Let R be an IBN ring.

(a) Let M be a stably free R-left module. By Definition 345.18 this means that there exist $s, t \in \mathbb{N}_0$ such that $M \oplus R^s$ is isomorphic to R^t. We define $\operatorname{rank}(M) := t - s$. It follows easily from the fact that R is an IBN ring and the additivity of ranks that this rank is well-defined.

(b) A stable basis for a stably free R-module M of rank r is a basis $(b_1, \dots, b_{r+t})$ for some direct sum $M \oplus R^t$.

The following definition is a variation on Definition 345.7:

Definition 345.22. Let R be an IBN ring.

(1) Let $\mathcal{B} = (b_1, \dots, b_{r+t})$ and $\mathcal{C} = (c_1, \dots, c_{r+u})$ be two stable bases for some R-left module M. We define $[\mathcal{B}/\mathcal{C}] \in \overline{\mathrm{K}}_1(R)$ as follows:

- We pick some natural number $v \geq \max\{t, u\}$. We extend the basis $\mathcal{B} = (b_1, \dots, b_{r+t})$ of $M \oplus R^t \xrightarrow{\cong} M \oplus R^t \oplus \{0\} \subseteq M \oplus R^v$ to a basis $\widetilde{\mathcal{B}}$ of $M \oplus R^v$ by adding $(0, \mathfrak{e}_i)$ for $i = t+1, \dots, v$. Similarly we define a basis $\widetilde{\mathcal{C}}$ of $M \oplus R^v$.
- We set $[\mathcal{B}/\mathcal{C}] := [\widetilde{\mathcal{B}}/\widetilde{\mathcal{C}}] \in \overline{\mathrm{K}}_1(R)$. It is elementary to show that this definition does not depend on the choice of $v \in \mathbb{N}_0$.
- As in Definition 345.7 it follows easily from the Transition Matrix Lemma 345.6 that for stable bases $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of M we have the following equalities:

$$\begin{array}{rcl} [\mathcal{B}/\mathcal{B}] & = & 1, \\ [\mathcal{C}/\mathcal{B}] & = & [\mathcal{B}/\mathcal{C}]^{-1}, \\ [\mathcal{B}/\mathcal{D}] & = & [\mathcal{B}/\mathcal{C}] \cdot [\mathcal{C}/\mathcal{D}]. \end{array}$$

(2) We say that two stable bases $\mathcal{B}$ and $\mathcal{C}$ of M are equivalent if $[\mathcal{B}/\mathcal{C}]$ is trivial in $\overline{\mathrm{K}}_1(R)$.

345.5. **Definition of algebraic torsion II.** The goal of this section is to define torsion without the artificial hypothesis $(\star)$. We start out with a variation on Notation 345.10:

Notation 345.23. Let R be an IBN ring and let $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n$ be a sequence of R-left modules. We assume that each V_i/V_{i-1} is a stably free R-left module that is equipped with a stable basing $\mathcal{B}_i$. For each i we pick an eponymous basis $\mathcal{B}_i$ for some $V_i/V_{i-1} \oplus R^{d_i} \cong R^{s_i}$. For $i = 0, \ldots, n$ we set $W_i := V_i \oplus R^{d_1} \oplus \cdots \oplus R^{d_i} \oplus 0 \cdots \oplus 0$. From Notation 345.10 we now obtain a basis $\mathcal{B}_1 \ldots \mathcal{B}_n$ for the quotient W_n/W_0. Since we have natural isomorphisms $W_n/W_0 \cong V_n/V_0 \cong R^{d_1} \oplus \cdots \oplus R^{d_n}$ we see that we have constructed a stable basis for V_n/V_0. As before we denote the corresponding stable basing for V_n/V_0 by $\mathcal{B}_1 \ldots \mathcal{B}_n$. As in the Basis-Filtration Construction Lemma 345.9 one can easily verify that this basing is well-defined.

We continue with the following analog of Definition 345.11:

Definition 345.24. Let R be a ring. Suppose that we are given a short exact sequence

$$0 \to A \to B \to C \to 0$$

of R-left modules.

(a) Suppose that we are given a stable basing $\mathcal{A}$ for A and a stable basing $\mathcal{C}$ for C. Under the natural isomorphisms $A/\{0\} \cong A$ and $B/A \cong C$ we obtain from Notation 345.23 a stable basing $\mathcal{AC}$ for B.

(b) We say that stable basings $\mathcal{A}$ for A, $\mathcal{B}$ for B and $\mathcal{C}$ for C are compatible if $\mathcal{AC} = \mathcal{B}$.

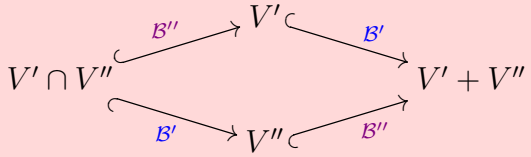
For completeness sake we state the following generalization of the Basis-Filtration Proposition 345.12:

Proposition 345.25. (Basis-Filtration Proposition) Let R be an IBN ring. Furthermore let $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n$ be a filtration of an R-left module V_n. We suppose that for each V_i/V_{i-1} we have a stable basing $\mathcal{B}_i$.

(1) Let $k \in \{1, \ldots, n\}$. We do the following:
- From the filtration $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k$ we obtain a stable basing $\mathcal{B}_1 \ldots \mathcal{B}_k$ for V_k/V_0.
- From the filtration $V_k \subseteq V_{k+1} \subseteq \ldots \subseteq V_n$ we obtain a stable basing $\mathcal{B}_{k+1} \ldots \mathcal{B}_n$ for V_n/V_k.
- We consider the filtration $V_1 \subseteq V_k \subseteq V_n$. Using (a) and (b) we obtain a stable basing $(\mathcal{B}_1 \ldots \mathcal{B}_k)(\mathcal{B}_{k+1} \ldots \mathcal{B}_n)$ for V_n/V_0.

Then $$(\mathcal{B}_1 \ldots \mathcal{B}_k)(\mathcal{B}_{k+1} \ldots \mathcal{B}_n) = \mathcal{B}_1 \ldots \mathcal{B}_n.$$

(2) Let V be an R-left module and let V', V'' be two submodules such that $(V' + V'')/V'$ is stably free with stable basis $\mathcal{B}'$ and such that $(V' + V'')/V''$ is stably free with stable basis $\mathcal{B}''$. Then we obtain the following two filtrations:



Note that in this setting we again have two natural inclusion induced isomorphisms $V'/(V' \cap V'') \to (V' + V'')/V''$ and $V'/(V' \cap V'') \to (V' + V'')/V''$. By a slight abuse of notation we can now view $\mathcal{B}'$ and $\mathcal{B}''$ as stable bases of the modules to the left. With this notation we have the following equality of equivalence classes of stable bases for

$(V' + V'')/(V' \cap V'')$:

$$\mathcal{B}'\mathcal{B}'' \;=\; \mathcal{B}''\mathcal{B}'.$$

(3) Suppose that for each V_i/V_{i-1} we have an alternative stable basing $\mathcal{A}_i$. Then we obtain the following equality in $\overline{\mathrm{K}}_1(R)$:

$$[\mathcal{A}_1\mathcal{A}_2\dots\mathcal{A}_n/\mathcal{B}_1\mathcal{B}_2\dots\mathcal{B}_n] \;=\; \prod_{i=1}^{n}[\mathcal{A}_i/\mathcal{B}_i].$$

Proof. The proof of this proposition is a mild generalization of the proof of the Basis-Filtration Proposition 345.12. We leave it to the reader to make the necessary changes. ■

We can now finally provide the definition of torsion without any annoying technical hypotheses, more precisely, we can now formulate the following extension of Definition 345.16:

Definition 345.26. Let R be an IBN ring and let (C_*, ∂_*) be a chain complex of R-left modules of length n such that each homology module $H_i = \mathrm{H}_i(C_*)$ is a free R-left module of finite rank. We also suppose that we are given the following data:

- For each $i \in \mathbb{N}_0$ we are given a finite basis $\mathcal{C}_i$ of C_i.
- For each $i \in \mathbb{N}_0$ we are given a finite basis $\mathcal{H}_i$ of $H_i = \mathrm{H}_i(C_*)$.

It follows from the Image–Stably Free 345.20 and the argument of the Torsion–Well-Defined Lemma 345.15 that for each $i \in \{0,\dots,n\}$ we can pick a stable basis $\mathcal{B}_i$ of the stably free R-left module B_i. We set[3752]

$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}, \{\mathcal{H}_i\}_{i\in\mathbb{N}_0}) \;:=\; \prod_{i=0}^{n}[\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}} \in \overline{\mathrm{K}}_1(R).$$

Similar to the Torsion–Well-Defined Lemma 345.15 one can show that this definition does not depend on the choice of the stable bases of the B_i.[3753] Finally, if C_* is acyclic, then we drop the $\mathcal{H}_i$ from the notation.

345.6. **Properties of torsion.** In this section we will collect some properties of torsion. We start out with the following lemma, which is essentially trivial:

Lemma 345.27. (Torsion–Isomorphism Lemma) Let R be an IBN ring and let C_* be a chain complex of finitely generated free R-left modules such that all homology modules are finitely generated free R-left modules. Furthermore let $\{\mathcal{C}_i\}_{i\in\mathbb{N}_0}$ be bases for the chain modules and let $\{\mathcal{H}_i\}_{i\in\mathbb{N}_0}$ be bases for the homology modules. Finally let $\varphi\colon C_* \to C'_*$ be an isomorphism between chain complexes of R-left modules. Then we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*, \{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}, \{\mathcal{H}'_i\}_{i\in\mathbb{N}_0}) \;=\; \tau(C'_*, \{\varphi(\mathcal{C}_i)\}_{i\in\mathbb{N}_0}, \{\varphi(\mathcal{H}_i)\}_{i\in\mathbb{N}_0}).$$

Proof. The lemma follows immediately from the definitions and the observation that the representing matrices are utterly unchanged. ■

We continue with the following lemma, which tells us in what sense torsion depends on the various choices of bases:

[3752] Note that $[\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]$ makes prefect sense, even if the stable bases involved do not have the same rank.

[3753] All that we really need to do is that we need to replace the Basis-Filtration Proposition 345.12 (3) by the Basis-Filtration Proposition 345.25 (3).

Lemma 345.28. (Torsion–Change-of-Basis Lemma) Let R be an IBN ring and let C_* be a chain complex of finitely generated free R-left modules such that all homology modules are finitely generated free R-left modules. Furthermore let $\{\mathcal{C}_i\}_{i\in\mathbb{N}_0}$ and $\{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}$ be two choices of bases for the chain modules and let $\{\mathcal{H}_i\}_{i\in\mathbb{N}_0}$ and $\{\mathcal{H}'_i\}_{i\in\mathbb{N}_0}$ be two choices of bases for the homology modules. Then we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*,\{\mathcal{C}'_i\}_{i\in\mathbb{N}_0},\{\mathcal{H}'_i\}_{i\in\mathbb{N}_0}) \;=\; \tau(C_*,\{\mathcal{C}_i\}_{i\in\mathbb{N}_0},\{\mathcal{H}_i\}_{i\in\mathbb{N}_0})\cdot\prod_i[\mathcal{C}'_i/\mathcal{C}_i]^{(-1)^i}\cdot\prod_i[\mathcal{H}'_i/\mathcal{H}_i]^{(-1)^{i+1}}.$$

Proof. As in Definition 345.26 we can pick for each B_i a stable basis $\mathcal{B}_i$. The lemma now follows from Definition 345.26 and the following equalities in $\overline{\mathrm{K}}_1(R)$:

$$\begin{aligned}
\overbrace{\prod_i[\mathcal{B}_i\mathcal{H}'_i\mathcal{B}_{i-1}/\mathcal{C}'_i]^{(-1)^{i+1}}}^{=\tau(C_*,\{\mathcal{C}'_i\}_{i\in\mathbb{N}_0},\{\mathcal{H}'_i\}_{i\in\mathbb{N}_0})} &\overset{\text{by the Transition Matrix Lemma 345.6 (2)}}{=} \prod_i\left([\mathcal{B}_i\mathcal{H}'_i\mathcal{B}_{i-1}/\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}]^{(-1)^{i+1}}\cdot[\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}}\cdot\underbrace{[\mathcal{C}_i/\mathcal{C}'_i]}_{=[\mathcal{C}'_i/\mathcal{C}_i]^{-1}}{}^{(-1)^{i+1}}\right)\\
&= \prod_i[\mathcal{H}'_i/\mathcal{H}_i]^{(-1)^{i+1}}\cdot\underbrace{\prod_i[\mathcal{B}_i\mathcal{H}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}}}_{=\tau(C_*,\{\mathcal{C}_i\}_{i\in\mathbb{N}_0},\{\mathcal{H}_i\}_{i\in\mathbb{N}_0})}\cdot\prod_i[\mathcal{C}'_i/\mathcal{C}_i]^{(-1)^i}\\
&\overset{\text{by the Basis-Filtration Proposition 345.25 (2) and and the Transition Matrix Lemma 345.6 (3)}}{\uparrow}
\end{aligned}$$

We have thus proved the desired equality. ∎

The following lemma also gets used frequently (often subconsciously):

Lemma 345.29. (Torsion–Direct Sum Lemma) Let R be an IBN ring and let $(C_*,\mathcal{C}_*)$, $(D_*,\mathcal{D}_*)$ be two based chain complexes of finitely generated free R-left modules where both chain complexes have finite length. Then we can form the direct sum $C_*\oplus D_*$ of the chain complexes and we see that

$$\tau(C_*\oplus D_*,\{\mathcal{C}_*\oplus\mathcal{D}_*\}) \;=\; \tau(C_*,\mathcal{C}_*)\cdot\tau(D_*,\mathcal{D}_*).$$

Proof. This lemma follows easily from the definitions and the Product-in-K_1-Lemma 344.10. ∎

In the remainder of this section we will prove a generalization of the Torsion–Direct Sum Lemma 345.29, namely we will see that torsion is multiplicative under short exact sequences. This multiplicativity is arguably the most important tools for calculating torsion. To increase readability we first deal with the case that we will mostly use, namely the case that the chain complexes involved are all acyclic:

Theorem 345.30. (SES-Torsion Theorem) Let R be an IBN ring and let

$$0 \;\to\; C'_* \;\xrightarrow{f}\; C_* \;\xrightarrow{p}\; C''_* \;\to\; 0$$

be a short exact sequence of based chain complexes of finitely generated free R-left modules, such that for each $i\in\mathbb{N}_0$ the bases $\mathcal{C}'_i,\mathcal{C}_i$ and $\mathcal{C}''_i$ are compatible in the sense of Definition 345.11. If two out of the three chain complexes are acyclic, then so is the third, and we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*,\{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \;=\; \tau(C'_*,\{\mathcal{C}'_i\}_{i\in\mathbb{N}_0})\cdot\tau(C''_*,\{\mathcal{C}''_i\}_{i\in\mathbb{N}_0}).$$

Proof. Let R be an IBN ring and let

$$0 \to C'_* \xrightarrow{f} C_* \xrightarrow{p} C''_* \to 0$$

be a short exact sequence of based chain complexes of finitely generated free R-left modules, such that for each $i \in \mathbb{N}_0$ the bases $\mathcal{C}'_i, \mathcal{C}_i$ and $\mathcal{C}''_i$ are compatible and such that two out of the three chain complexes are acyclic. We start out with two initial remarks and observations:

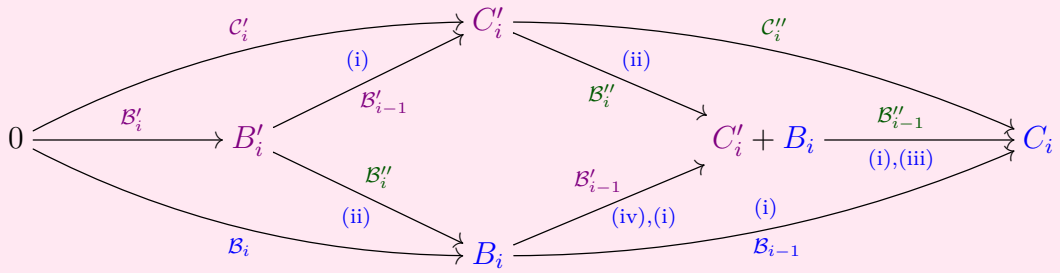
- To simplify the notation a little bit we identify each C'_i with its image in C_i, i.e. we view C'_i as a submodule of C_i.
- Recall that it follows from the LES Proposition 313.10 that the short exact sequence of chain complexes gives rise to a long exact sequence of homology modules. It follows immediately that if two out of the three chain complexes are acyclic, then so is the third. Thus in the following we assume that all chain complexes are acyclic.

Inspired by Notation 345.13 we introduce the following almost self-explanatory notation:

$$\begin{aligned} B'_i &:= \operatorname{im}(\partial'_{i+1} \colon C'_{i+1} \to C'_i), \\ B_i &:= \operatorname{im}(\partial_{i+1} \colon C_{i+1} \to C_i), \\ B''_i &:= \operatorname{im}(\partial''_{i+1} \colon C''_{i+1} \to C''_i). \end{aligned}$$

We start out with the following claim:

Claim. We consider the following diagram of submodules:



We make the following observations and clarifications:

(1) We have $C'_i \cap B_i = B'_i$.

(2) We have the following natural isomorphisms which are all induced by the various maps f_i, p_i and the boundary maps:

(i) The boundary maps induce isomorphisms $B'_i/C'_i \to B'_{i-1}$ and $B''_i/C''_i \to B''_{i-1}$.

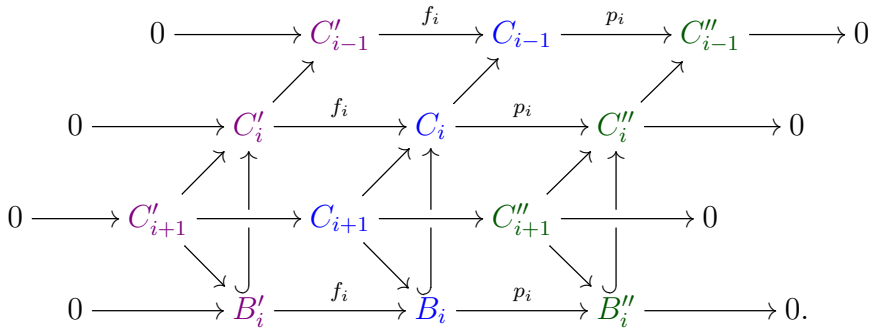
(ii) $B_i/B'_i \xrightarrow{\cong} (C'_i + B_i)/C'_i \xleftarrow{\cong} B_i/(C'_i \cap B_i) \xrightarrow{\cong} B''_i$.

(iii) $C_i/(C'_i + B_i) \xrightarrow{\cong} C''_i/B''_i$.

(iv) $(C'_i + B_i)/B_i \xleftarrow{\cong} C'_i/(C'_i \cap B_i) \underset{(1)}{=} C'_i/B'_i$.

We can use the above isomorphisms to equip each inclusion with the given basis, as indicated in the above diagram.

Proof. The statements follow easily from the various definitions and a little diagram chase using the following diagram:



Note that the diagonal sequences are by assumption exact and the first three horizontal sequences also also exact. We leave it to the reader to fill in the details. ⊞

For each i we pick stable bases $\mathcal{B}'_i$ of B'_i and $\mathcal{B}''_i$ of B''_i. By Claim (2-ii) we can and will equip each B_i with the stable basis $\mathcal{B}_i := \mathcal{B}'_i\mathcal{B}''_i$ from Definition 345.24. With these choices we see that we have the following equalities in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \overset{\text{by Definition 345.26}}{=} \prod_i [\mathcal{B}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}} \overset{\text{by definition of } \mathcal{B}_i \text{ and since } \mathcal{C}_i \text{ is compatible with } \mathcal{C}'_i \text{ and } \mathcal{C}''_i}{=} \prod_i [\mathcal{B}'_i\mathcal{B}''_i\mathcal{B}'_{i-1}\mathcal{B}''_{i-1}/\mathcal{C}'_i\mathcal{C}''_i]^{(-1)^{i+1}}$$

$$\overset{\text{by the Basis-Filtration Proposition 345.25 (1)}}{=} \prod_i [\mathcal{B}'_i(\mathcal{B}''_i\mathcal{B}'_{i-1})\mathcal{B}''_{i-1}/\mathcal{C}'_i\mathcal{C}''_i]^{(-1)^{i+1}} \overset{\text{by the Basis-Filtration Proposition 345.25 (2) together with Claim (1)}}{=} \prod_i [\mathcal{B}'_i(\mathcal{B}'_{i-1}\mathcal{B}''_i)\mathcal{B}''_{i-1}/\mathcal{C}'_i\mathcal{C}''_i]^{(-1)^{i+1}}$$

$$\overset{\text{by the Basis-Filtration Proposition 345.25 (1)}}{=} \prod_i [(\mathcal{B}'_i\mathcal{B}'_{i-1})(\mathcal{B}''_i\mathcal{B}''_{i-1})/\mathcal{C}'_i\mathcal{C}''_i]^{(-1)^{i+1}} \overset{\text{by the Basis-Filtration Proposition 345.25 (3)}}{=} \prod_i [\mathcal{B}'_i\mathcal{B}'_{i-1}/\mathcal{C}'_i]^{(-1)^{i+1}} \cdot \prod_i [\mathcal{B}''_i\mathcal{B}''_{i-1}/\mathcal{C}''_i]^{(-1)^{i+1}}$$

$$\overset{\text{by Definition 345.16}}{=} \tau(C'_*, \{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}) \cdot \tau(C''_*, \{\mathcal{C}''_i\}_{i\in\mathbb{N}_0}).$$

■

The following theorem generalizes the SES-Torsion Theorem 345.30 to the setting where the chain complexes are not necessarily acyclic:

Theorem 345.31. (SES-Torsion Theorem) Let R be an IBN ring and let

$$0 \to C'_* \xrightarrow{f} C_* \xrightarrow{p} C''_* \to 0$$

be a short exact sequence of based chain complexes of finitely generated free R-left modules, such that for each i the bases $\mathcal{C}'_i, \mathcal{C}_i$ and $\mathcal{C}''_i$ are compatible. We assume that furthermore we are given bases $\{\mathcal{H}'_i\}_{i\in\mathbb{N}_0}$, $\{\mathcal{H}_i\}_{i\in\mathbb{N}_0}$ and $\{\mathcal{H}''_i\}_{i\in\mathbb{N}_0}$ of the various homology modules. By the LES Proposition 313.10 the homology modules form a long exact sequence (in particular

a chain complex) which we can now view as a based chain complex:

$$\overset{\mathcal{H}'_1}{\mathrm{H}_1(C'_*)} \xrightarrow{f_*} \overset{\mathcal{H}_1}{\mathrm{H}_1(C_*)} \xrightarrow{p_*} \overset{\mathcal{H}''_1}{\mathrm{H}_1(C''_*)} \xrightarrow{\partial_1} \overset{\mathcal{H}'_0}{\mathrm{H}_0(C'_*)} \xrightarrow{f_*} \underset{\text{degree 1}}{\overset{\mathcal{H}_0}{\mathrm{H}_0(C_*)}} \xrightarrow{p_*} \underset{\text{degree 0}}{\overset{\mathcal{H}''_0}{\mathrm{H}_0(C''_*)}} \to 0.$$

We have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*, \{\mathcal{C}_i\}, \{\mathcal{H}_i\}) \;=\; \tau(C'_*, \{\mathcal{C}_i{}'\}, \{\mathcal{H}_i{}'\}) \cdot \tau(C''_*, \{\mathcal{C}''_i\}, \{\mathcal{H}''_i\}) \cdot \tau\left(\begin{array}{c}\text{the above based}\\ \text{homology sequence}\end{array}\right).$$

Proof. We will not make use of this theorem, thus we will only provide a sketch. First of all we set

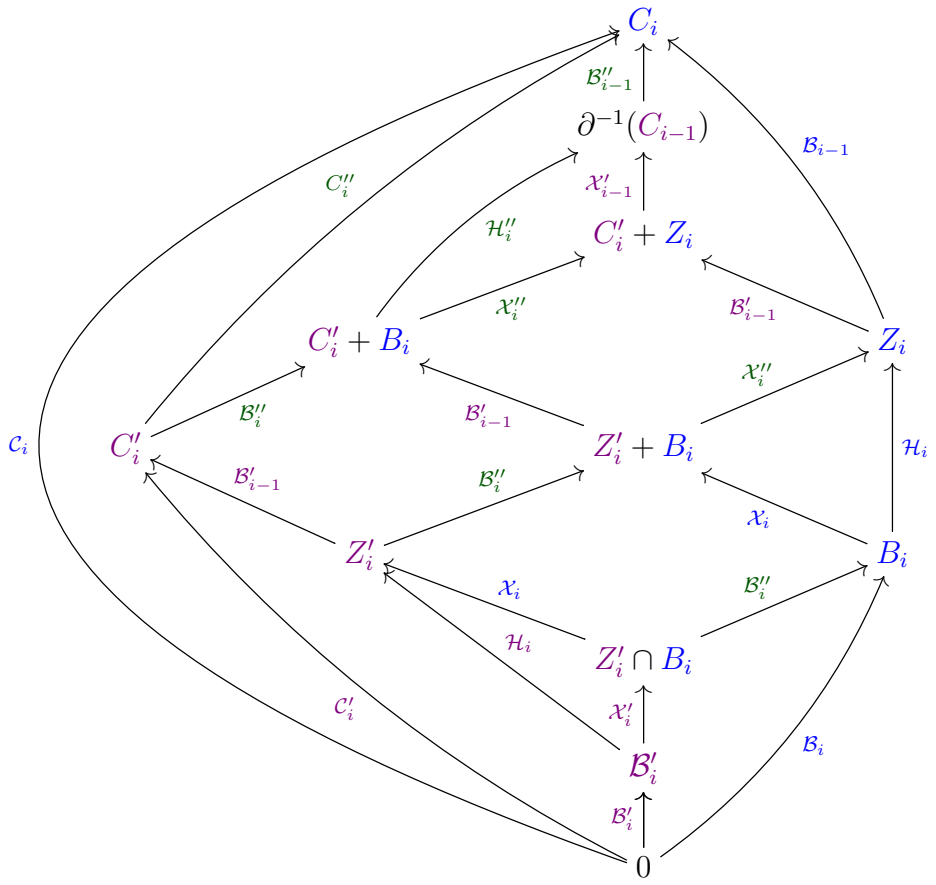
$$\begin{aligned} X'_i &:= \ker(f_*\colon H'_i \to H_i)\\ X_i &:= \ker(p_*\colon H_i \to H''_i)\\ X''_i &:= \ker(\partial_i\colon H''_i \to H'_{i-1}). \end{aligned}$$

It follows from the exactness of the long exact sequence in homology that we have short exact sequences

$$\begin{aligned} 0 &\to X'_i \to H'_i \to X_i \to 0\\ 0 &\to X_i \to H_i \to X''_i \to 0\\ 0 &\to X''_i \to H''_i \to X'_{i-1} \to 0. \end{aligned}$$

Now note that it follows from these three sequences and an elementary induction argument using the Short Exact Sequence–Stably Free 345.19 that the modules X'_i, X_i and X''_i are all stably free R-left modules. Thus we can choose stable bases $\mathcal{X}'_i$, $\mathcal{X}_i$ and $\mathcal{X}''_i$ of the above

modules. Next we consider the following diagram of inclusions:



We leave it to the reader to show that the given sets do indeed give bases for the various quotients under natural isomorphisms. A proof of the theorem can now be assembled using the above diagram and the Basis-Filtration Proposition 345.25. More details can be found in [**Miln1968b**, Theorem 3.2]. ■

345.7. **Exercises for Chapter 345.**

Exercise 345.1. Let $\mathbb{F}$ be a field and let $x, y \in \mathbb{F}$. We consider the following chain complex

$$0 \longrightarrow \mathbb{F} \xrightarrow{(-y \;\; x)} \mathbb{F}^2 \xrightarrow{\binom{x}{y}} \mathbb{F} \longrightarrow 0$$

that is equipped with the obvious bases.

(a) Complete the sentence: The chain complex is acyclic if and only if ??

(b) We now suppose that the above chain complex is in fact acyclic. Compute the torsion in $\overline{\mathrm{K}}_1(\mathbb{F}) = \mathbb{F}^\times/\{\pm 1\}$.

Exercise 345.2. Let R be an IBN ring. Furthermore let $0 \to C_m \to \cdots \to C_0 \to 0$ be a chain complex of finitely generated free R-left modules. If the chain complex is acyclic, then

$$\sum_{i=0}^{m} (-1)^i \cdot \mathrm{rank}_R(C_i) \;=\; 0.$$

Exercise 345.3. Let F be a field. Furthermore:

- Let A be a $(k \times k)$-matrix over $\mathbb{F}$,
- let B be a $(k \times l)$-matrix over $\mathbb{F}$,
- let C be a $(k \times l)$-matrix over $\mathbb{F}$, and
- let D be an $(l \times l)$-matrix over $\mathbb{F}$.

We assume that

$$(A \;\; B)\begin{pmatrix} C \\ D \end{pmatrix} \;=\; 0.$$

Compute the torsion of the base chain complex

$$0 \longrightarrow \mathbb{F}^{1\times k} \xrightarrow{(A\;\;B)} \mathbb{F}^{1\times k} \oplus \mathbb{F}^{1\times l} \xrightarrow{\binom{C}{D}} \mathbb{F}^{1\times l} \longrightarrow 0.$$

Hint. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{F}^{1\times l} & \xrightarrow{D} & \mathbb{F}^{1\times l} & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow{\scriptstyle y\mapsto(0,y)} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{F}^{1\times k} & \xrightarrow{(A\;\;B)} & \mathbb{F}^{1\times k}\oplus\mathbb{F}^{1\times l} & \xrightarrow{\binom{C}{D}} & \mathbb{F}^{1\times l} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (x,y)\mapsto,x} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{F}^{1\times k} & \xrightarrow{A} & \mathbb{F}^{1\times k} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Exercise 345.4. Let R be a ring and let C_* and D_* be two finite-length based acyclic chain complexes of free R-left modules of finite rank. If C_* and D_* are chain homotopic, does it follow that the torsions agree?

Exercise 345.5. Let $k, l \in \mathbb{N}_0$. We consider

$$M \;:=\; \{(P,Q) \in \mathrm{M}(k \times (k+l), \mathbb{C}) \times \mathrm{M}((k+l) \times l), \mathbb{C}) \,|\, P \cdot Q = 0\}.$$

In other words, these are precisely the matrices such that the following sequence is a chain complex:

$$0 \;\to\; \mathbb{C}^{1\times k} \xrightarrow{v\mapsto v\cdot P} \mathbb{C}^{1\times(k+l)} \xrightarrow{v\mapsto v\cdot Q} \mathbb{C}^{1\times l} \;\to\; 0.$$

We consider the subset

$$A \;:=\; \{(P,Q) \in A \,|\, \mathrm{im}(P) = \ker(Q)\}.$$

In other words, A corresponds to the acyclic chain complexes. Show that A has (in a suitable sense) full measure in M.

Remark. This shows that a generic chain complex over $\mathbb{C}$ of length two with zero Euler characteristic is acyclic. Of course a similar statement holds for all finite length chain complexes of Euler characteristic zero.

Exercise 345.6. Let R be a commutative ring and let $(C_*, \mathcal{C}_*)$, $(D_*, \mathcal{D}_*)$ be two based chain complexes of finitely generated free R-modules where both chain complexes have finite length. Note that it makes sense to consider the Euler characteristics $\chi(C_*)$ and $\chi(D_*)$ that are defined as the alternating sums of the ranks of the chain modules. Formulate and prove a statement of the form

$$\tau(C_* \otimes_R D_*, \{\mathcal{C}_* \otimes \mathcal{D}_*\}) \;=\; \tau(C_*, \mathcal{C}_*)^{\chi(D_*)} \cdot \tau(D_*, \mathcal{D}_*)^{\chi(C_*)}.$$

Remark. Use Exercise 312.16. The current exercise might turn out to be somewhat heroic.

CHAPTER 346

Naturality of torsion

In the previous Chapter 345 we learned how to associate to a based acyclic chain complex $(C_*, \mathcal{C}_*)$ of R-left modules its torsion $\tau(C_*, \mathcal{C}_*) \in \overline{\mathrm{K}}_1(R)$. There are two standard ways to get new based chain complexes out of $(C_*, \mathcal{C}_*)$:

(1) Given a ring homomorphism $\varphi\colon R \to S$ we can consider the tensor product $S_\varphi \otimes_R C_*$.
(2) Given a ring homomorphism $\varphi\colon R \to S$ we can consider the dual (co-) chain complex $\mathrm{Hom}_{R\text{-left}}(C_*, {}_\varphi S)$.

In this chapter we will see how these two operations affect torsion. Along the way we will introduce an alternative definition of torsion via chain contractions.

This chapter is organized as follows:

- In Section 346.1 we will introduce chain contractions and we will see that they give an alternative approach to defining the torsion of acyclic based chain complexes.
- In Section 346.2 we will provide a proof of the Torsion–Chain Contraction Proposition 346.6.
- In Section 346.3 we will use the Torsion–Chain Contraction Proposition 346.6 to see that torsion is functorial in a suitable sense.
- In Section 346.4 we will use the Torsion–Chain Contraction Proposition 346.6 to calculate the torsion of dual chain complexes.
- In Section 346.5 we will provide exercises for Chapter 346.

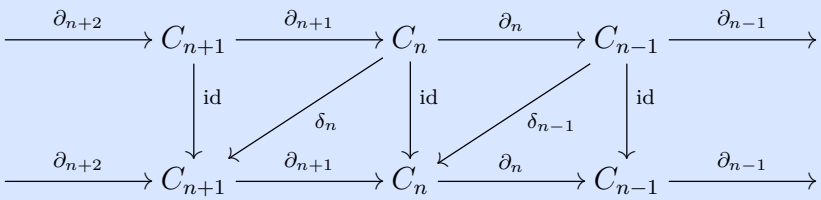
In the next Chapter 347 we will get to know two approaches to calculating torsion.

346.1. **Chain contractions.** In this section we will give a reinterpretation of torsion of an *acyclic* based chain complex. We start out with the following definition:

Definition 346.1. Let R be a ring and let (C_*, ∂_*) be a chain complex of R-left modules. A chain contraction for (C_*, ∂_*) is a chain homotopy in the sense of Definition 313.5 from the identity chain map $C_* \to C_*$ to the zero chain map $C_* \to C_*$. In other words, a chain contraction is a family $\{\delta_n\colon C_n \to C_{n+1}\}_{n\in\mathbb{N}_0}$ of R-left module homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ \delta_n + \delta_{n-1} \circ \partial_n \;=\; \mathrm{id}_{C_n}\,.$$

We summarize the various homomorphisms in the following diagram:



The same way one also defines a cochain contraction for a cochain complex.

Example. Let R be a ring and let $0 \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \to 0$ be a chain complex consisting of a single non-zero homomorphism. It follows easily from the definitions that this chain complex has a chain contraction precisely if ∂_{n+1} is an isomorphism (i.e. if the chain complex is acyclic), and in that case $\partial_{n+1}^{-1}\colon C_n \to C_{n+1}$ is easily seen to be the only possible chain contraction. □

The following proposition gives a convenient criterion for a chain complex to be acyclic:

Proposition 346.2. (Acyclic–Chain Contraction Proposition) Let R be a ring and let (C_*, ∂_*) be a chain complex of free R-left modules. The following two statements are equivalent:

(i) The chain complex is acyclic.
(ii) The chain complex admits a chain contraction.

Proof. This proposition is equivalent to the (b)⇔(e)-statement of the Null-Homotopic Criterion Proposition 313.16. ■

The idea now is to express torsion in terms of chain contractions. To do so it is convenient to recall Definition 309.23:

Definition 346.3. Let R be a ring. Let V and W be two free R-left modules that are equipped with finite ordered bases $\mathcal{B} = (b_1, \ldots, b_m)$ and $\mathcal{C} = (c_1, \ldots, c_n)$. Let $\varphi\colon V \to W$ be an R-left module homomorphism. In this setting there exists a unique $(m \times n)$-matrix $[\varphi]_{\mathcal{B}}^{\mathcal{C}} \in \mathrm{M}(m \times n, R)$ that makes the following diagram commute:

$$\begin{array}{ccc} R^{1\times m} & \xrightarrow{\xi \mapsto \xi\cdot[\varphi]_{\mathcal{B}}^{\mathcal{C}}} & R^{1\times n} \\ \cong \downarrow \mathfrak{e}_i \mapsto b_i & & \cong \downarrow \mathfrak{e}_i \mapsto c_i \\ V & \xrightarrow{\varphi} & W. \end{array}$$

We move on to some new notation:

Notation 346.4. Let R be a ring and let (C_*, ∂_*) be a chain complex of R-left modules.

- We set $$C_{\mathrm{even}} := \bigoplus_{n \text{ even}} C_n \qquad \text{and} \qquad C_{\mathrm{odd}} := \bigoplus_{n \text{ odd}} C_n.$$
- Note that the boundary maps ∂_* define a homomorphism $\partial_{\mathrm{even}}\colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ of R-left modules and that a chain contraction δ_* defines a homomorphism $\delta_{\mathrm{even}}\colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ of R-left modules. Of course we can also swap the roles of odd and even.
- If each C_i is equipped with a basis $\mathcal{C}_i$, then we obtain a basis $\mathcal{C}_{\mathrm{even}}$ for C_{even} and we obtain a basis $\mathcal{C}_{\mathrm{odd}}$ for C_{odd} in an obvious way.

We continue with the following somewhat technical lemma:

Lemma 346.5. (Boundary–Chain Contraction Lemma) Let R be a ring and let (C_*, ∂_*) be an *acyclic* chain complex of finitely generated free R-left modules and such that (C_*, ∂_*) has finite length. Furthermore let $\delta\colon C_i \to C_{i+1}$ be a chain contraction for (C_*, ∂_*).

(1) The maps $\partial_{\mathrm{even}} + \delta_{\mathrm{even}}\colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ and $\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}\colon C_{\mathrm{odd}} \to C_{\mathrm{even}}$ are both isomorphisms.

(2) We now assume that R is an IBN ring and that each C_i is equipped with a basis $\mathcal{C}_i$ Then we have the following equality in $\overline{\mathrm{K}}_1(R)$:
$$[\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \;=\; \big([\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}_{\mathrm{even}}}\big)^{-1}.$$

Proof. Let R be an IBN ring and let (C_*, ∂_*) be an *acyclic* chain complex of finitely generated free R-left modules of length n such that each C_i is equipped with a basis $\mathcal{C}_i$. After possibly adding a zero group we can assume that n is even. Finally let $\{\delta_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}_0}$ be a chain contraction. We need to prove the following claim:

Claim. The homomorphisms $\partial_{\mathrm{even}} + \delta_{\mathrm{even}} \colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ and $\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}} \colon C_{\mathrm{odd}} \to C_{\mathrm{even}}$ are both isomorphisms and we have the following equality in $\overline{\mathrm{K}}_1(R)$:
$$[\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \;=\; \big([\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}_{\mathrm{even}}}\big)^{-1}.$$

Proof. First note that
$$(\partial_{\mathrm{even}} + \delta_{\mathrm{even}}) \circ (\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}) \overset{\text{by the clever FOIL method}}{=} \underbrace{\partial_{\mathrm{even}} \circ \partial_{\mathrm{odd}}}_{=0} + \underbrace{\partial_{\mathrm{even}} \circ \delta_{\mathrm{odd}} + \delta_{\mathrm{even}} \circ \partial_{\mathrm{odd}}}_{=\,\mathrm{id},\ \text{since } \delta \text{ is a chain contraction, see Definition 346.1}} + \delta_{\mathrm{even}} \circ \delta_{\mathrm{odd}}$$
$$= \mathrm{id} + \delta_{\mathrm{even}} \circ \delta_{\mathrm{odd}}.$$
We thus see that the matrix representing $(\partial_{\mathrm{even}} + \delta_{\mathrm{even}}) \circ (\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}) \colon C_{\mathrm{odd}} \to C_{\mathrm{odd}}$ is of the form
$$\begin{array}{c} \\ C_1 \\ C_3 \\ C_5 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_3 & C_5 & \dots & \dots \end{array} \\ \begin{pmatrix} \mathrm{id} & \delta_3 \circ \delta_2 & 0 & \dots & \dots \\ 0 & \mathrm{id} & \delta_5 \circ \delta_4 & 0 & 0 \\ 0 & 0 & \ddots & \delta_{n-1} \circ \delta_n & 0 \\ 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \end{array}$$
The above matrix is clearly invertible. In a very similar fashion one sees that the composition $(\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}) \circ (\partial_{\mathrm{even}} + \delta_{\mathrm{even}})$ is also invertible. It now follows from an elementary argument, see Exercise 5.1, that the two homomorphisms $\partial_{\mathrm{even}} + \delta_{\mathrm{even}} \colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ and $\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}} \colon C_{\mathrm{odd}} \to C_{\mathrm{even}}$ are both isomorphisms. Furthermore, by the Product-in-K_1-Lemma 344.10 we know that the above matrix represents the trivial element in $\overline{\mathrm{K}}_1(R)$. It thus follows from the above equalities that in $\overline{\mathrm{K}}_1(R)$ we have the promised equality
$$[\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \;=\; \big([\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}_{\mathrm{even}}}\big)^{-1}. \qquad \blacksquare$$

Now we turn to the following much more interesting proposition, which gives us an alternative approach to torsion of *acyclic* chain complexes:

Proposition 346.6. (Torsion–Chain Contraction Proposition) Let R be an IBN ring and let (C_*, ∂_*) be an *acyclic* chain complex of R-left modules and such that (C_*, ∂_*) has finite length. We also suppose that for each $i \in \mathbb{N}_0$ we are given a finite basis $\mathcal{C}_i$ of C_i. For any chain contraction δ of the chain complex C_* we have the following equality in $\overline{\mathrm{K}}_1(R)$:[3754 3755]
$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \;=\; [\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}}.$$

Proof. We will dedicate the next Section 346.2 to the proof of the proposition. ■

346.2. **Proof of the Torsion–Chain Contraction Proposition 346.6.** The idea for the proof of the Torsion–Chain Contraction Proposition 346.6 is to take the right hand side of the Torsion–Chain Contraction Proposition 346.6 as an alternative definition of an invariant, to explore its properties, and to then show that it agrees with torsion.

Lemma 346.7. (Alternative Torsion–Well-Defined Lemma) Let R be an IBN ring and let $(C_*, \mathcal{C}_*)$ be a based *acyclic* chain complex of R-left modules such that (C_*, ∂_*) has finite length. For any two chain contractions δ and $\widetilde{\delta}$ we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$[\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \;=\; [\partial_{\mathrm{even}} + \widetilde{\delta}_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}}.$$

Proof. As usual we set $B_i := \mathrm{im}(\partial_{i+1}\colon C_{i+1} \to C_i)$. In the following we assume that the B_i are free R-left modules. The general case can be dealt with in an analogous way by stabilizing, we refer to [**CoheM1973**, p. 49] for details.

Claim 1. For any chain contraction $\widehat{\delta}$ and any $i \in \mathbb{N}_0$ we have $C_i = B_i \oplus \widehat{\delta}_{i-1}(B_{i-1})$.

Proof. Since C_i is acyclic we know, essentially by definition, that we have a short exact sequence

$$0 \;\to\; B_i \;\hookrightarrow\; C_i \;\xrightarrow{\partial_i}\; B_{i-1} \;\to\; 0.$$

By the Module Splitting Lemma 309.31 it suffices to show that $\widehat{\delta}_{i-1}$ is a right-inverse of ∂_i. But this indeed the case sind for any $b \in B_{i-1}$ we have

$$(\partial_i \circ \widehat{\delta}_{i-1})(b) \;\underset{\substack{\uparrow\\ \text{since } \partial_{i-1}=0 \text{ on } B_{i-1}}}{=}\; (\partial_i \circ \widehat{\delta}_{i-1} + \widehat{\delta}_{i-2} \circ \partial_{i-1})(b) \;\underset{\substack{\uparrow\\ \text{Definition 346.1}}}{=}\; \mathrm{id}(b) \;=\; b.$$

⊞

Claim 2. Let $i \in \mathbb{N}_0$.

(1) The map $\delta_{i-1} \circ \partial_i + \partial_{i+1} \circ \widetilde{\delta}_i\colon C_i \to C_i$ is of the form[3756]

$$B_i \oplus \widetilde{\delta}_{i-1}(B_{i-1}) \;\xrightarrow{\begin{pmatrix} \mathrm{id} & \mu \\ 0 & \mathrm{id} \end{pmatrix}}\; B_i \oplus \widetilde{\delta}_{i-1}(B_{i-1}).$$

(2) The map $\delta_{i-1} \circ \partial_i + \partial_{i+1} \circ \widetilde{\delta}_i\colon C_i \to C_i$ is an isomorphism and it represents the trivial element in $\overline{\mathrm{K}}_1(R)$.

Proof. First note that Statement (2) follows immediately from Statement (1) together with the Product-in-K_1-Lemma 344.10. So we now concentrate on the proof of Statement (1).

[3754] Note that if R is an IBN-ring and if we are given a chain complex of finite length consisting of free R-left modules of finite rank, then it follows from Exercise 345.2 (or alternatively the Acyclic–Vanishing-χ-Lemma 348.6) that $\mathrm{rank}(C_{\mathrm{even}}) = \mathrm{rank}(C_{\mathrm{odd}})$.

[3755] By the Boundary–Chain Contraction Lemma 346.5 we know that the maps $\partial_{\mathrm{even}} + \delta_{\mathrm{even}}\colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ and $\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}\colon C_{\mathrm{odd}} \to C_{\mathrm{even}}$ are both isomorphisms, so (together with the given bases) they do indeed define elements of $\overline{\mathrm{K}}_1(R)$. Furthermore we showed that we have $[\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} = \big([\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}_{\mathrm{even}}}\big)^{-1}$ in $\overline{\mathrm{K}}_1(R)$. Thus we also obtain the equality

$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \;=\; \big([\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}_{\mathrm{even}}}\big)^{-1}$$

at no extra expense.

[3756] Note that it follows from Claim 1 that $C_i = B_i \oplus \widetilde{\delta}_{i-1}(B_{i-1})$.

First note that for any $b_i \in B_i$ we have

$$(\delta_{i-1}\circ\partial_i + \partial_{i+1}\circ\widetilde{\delta}_i)(b_i) \underset{\substack{\uparrow\\ \text{since } \partial_i(b_i)=0}}{=} (\widetilde{\delta}_{i-1}\circ\partial_i + \partial_{i+1}\circ\widetilde{\delta}_i)(b_i) = \mathrm{id}(b_i) \underset{\substack{\uparrow\\ \text{Definition 346.1}}}{=} b_i.$$

Furthermore, for any $\widetilde{\delta}_{i-1}(b_{i-1}) \in \widetilde{\delta}_{i-1}(B_{i-1})$ we see that

$$\begin{aligned}
(\delta_{i-1}\circ\partial_i + \partial_{i+1}\circ\widetilde{\delta}_i)(\widetilde{\delta}_{i-1}(b_{i-1})) &\underset{\substack{\downarrow\\ \text{since } \widetilde{\delta} \text{ is a chain contraction}}}{=} (\delta_{i-1}\circ\partial_i)(\widetilde{\delta}_{i-1}(b_{i-1})) + (\mathrm{id} - \widetilde{\delta}_{i-1}\circ\partial_i)(\widetilde{\delta}_{i-1}(b_{i-1}))\\
&= (\delta_{i-1}\circ\partial_i)(\widetilde{\delta}_{i-1}(b_{i-1})) + \widetilde{\delta}_{i-1}(b_{i-1}) - \widetilde{\delta}_{i-1}\circ \underbrace{(\partial_i\circ\widetilde{\delta}_{i-1})(b_{i-1})}_{=(\mathrm{id}-\widetilde{\delta}\circ\partial)(b_{i-1})=b_{i-1}}\\
&= (\delta_{i-1}\circ\partial_i)(\widetilde{\delta}_{i-1}(b_{i-1})) + \widetilde{\delta}_{i-1}(b_{i-1}) - \widetilde{\delta}_{i-1}(b_{i-1})\\
&= (\delta_{i-1}\circ\partial_i)(\widetilde{\delta}_{i-1}(b_{i-1}))\\
&= \widetilde{\delta}_{i-1}(b_{i-1}) + \underbrace{(\delta_{i-1}\circ\partial_i)(\widetilde{\delta}_{i-1}(b_{i-1})) - \widetilde{\delta}_{i-1}(b_{i-1})}_{\text{need to show that this lies in } B_i}.
\end{aligned}$$

So we need to show that for any $v = \widetilde{\delta}_{i-1}(b_{i-1})$ we have $(\delta_{i-1}\circ\partial_i)(v) - v \in B_i$. Since C_* is exact it suffices to show that the boundary of this element is zero. The following calculation shows that this is indeed the case:

$$\partial_i((\delta_{i-1}\circ\partial_i)(v) - v) \underset{\substack{\uparrow\\ \text{since } \delta \text{ is a chain contraction}}}{=} \partial_i((\mathrm{id} - (\partial_{i+1}\circ\delta_i))(v) - v) \underset{\substack{\uparrow\\ \text{since } \partial\circ\partial_{i+1}=0}}{=} 0.$$

⊞

For notational convenience we pick an $n \in \mathbb{N}_0$ such that $C_i = 0$ for all $i \geq 2n$. Then we see that we have the following equalities in $\overline{\mathrm{K}}_1(R)$:

$$\begin{aligned}
[\partial_{\text{even}} + \delta_{\text{even}}]\cdot[\partial_{\text{even}} + \widetilde{\delta}_{\text{even}}]^{-1} &\underset{\substack{\downarrow\\ \text{by the Boundary–Chain Contraction Lemma 346.5}}}{=} [\partial_{\text{even}} + \delta_{\text{even}}]\cdot[\partial_{\text{odd}} + \widetilde{\delta}_{\text{odd}}]\\
&= [(\partial_{\text{even}} + \delta_{\text{even}})\circ(\partial_{\text{odd}} + \widetilde{\delta}_{\text{odd}})]\\
&\underset{\substack{\uparrow\\ \text{here we use that } \partial_{\text{even}}\circ\partial_{\text{odd}}=0}}{=} [\partial_{\text{even}}\circ\widetilde{\delta}_{\text{odd}} + \delta_{\text{even}}\circ\partial_{\text{odd}} + \delta_{\text{even}}\circ\widetilde{\delta}_{\text{odd}}]\\
&= \begin{array}{c} \\ C_1\\ C_3\\ C_5\\ \vdots \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} C_1 & C_3 & C_5 & \dots & \dots \end{array}\\ \begin{pmatrix} \delta\circ\partial + \partial\circ\widetilde{\delta} & \delta\circ\widetilde{\delta} & 0 & \dots & \dots\\ 0 & \delta\circ\partial + \partial\circ\widetilde{\delta} & \delta\circ\widetilde{\delta} & 0 & 0\\ 0 & 0 & \ddots & \delta\circ\widetilde{\delta} & 0\\ 0 & 0 & 0 & \delta\circ\partial + \partial\circ\widetilde{\delta} & 0\\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \end{array}\\
&\underset{\substack{\uparrow\\ \text{by the Product-in-K}_1\text{-Lemma 344.10}}}{=} \prod_{i=1}^{n}[\delta_{2i-2}\circ\partial_{2i-1} + \partial_{2i}\circ\widetilde{\delta}_{2i-1}] \underset{\substack{\uparrow\\ \text{by Claim 2}}}{=} \prod_{i=1}^{n} 1 = 1.
\end{aligned}$$

Thus we have proved the desired equality $[\partial_{\text{even}} + \delta_{\text{even}}] = [\partial_{\text{even}} + \widetilde{\delta}_{\text{even}}] \in \overline{\mathrm{K}}_1(R)$. ■

The Alternative Torsion–Well-Defined Lemma 346.7 allows us to introduce the following definition:

Definition 346.8. Let R be an IBN ring and let $(C_*, \mathcal{C}_*)$ be a based *acyclic* chain complex of R-left modules of finite length. By the Acyclic–Chain Contraction Proposition 346.2 we can pick a chain contraction δ. We set

$$\tau^{\mathrm{alt}}(C_*, \mathcal{C}_*) \;:=\; [\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{C_{\mathrm{even}}}^{C_{\mathrm{odd}}} \;\in\; \overline{\mathrm{K}}_1(R).$$

It follows from the Alternative Torsion–Well-Defined Lemma 346.7 that this definition does not depend on the choice of the chain contraction.

Example. Let R be an IBN ring and let C_k and C_{k-1} be two finitely generated free R-left modules that are equipped with bases $\mathcal{C}_k$ and $\mathcal{C}_{k-1}$. Finally let $\partial_k \colon C_k \to C_{k-1}$ be an isomorphism. We denote by A the square matrix over R which represents ∂_k with respect to the given bases. It follows easily from the discussion on page 6550 and the Boundary–Chain Contraction Lemma 346.5 (2) that we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau^{\mathrm{alt}}(0 \to C_k \xrightarrow{\partial_k} C_{k-1} \to 0, \{\mathcal{C}_k, \mathcal{C}_{k-1}\}) \;=\; [A]^{(-1)^k}.$$

This observation is consistent with the Elementary Torsion Lemma 345.17. □

The following lemma is the analog of the Torsion–Change-of-Basis Lemma 345.28:

Lemma 346.9. (Alternative Torsion–Change-of-Basis Lemma) Let R be an IBN ring and let C_* be a chain complex of finitely generated free R-left modules such that (C_*, ∂_*) has finite length. Furthermore let $\{\mathcal{C}_i\}_{i\in\mathbb{N}_0}$ and $\{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}$ be two choices of bases for the chain modules. Then we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau^{\mathrm{alt}}(C_*, \{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}) \;=\; \tau^{\mathrm{alt}}(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \cdot \prod_i [\mathcal{C}'_i/\mathcal{C}_i]^{(-1)^i}.$$

Proof of Lemma 346.9. We perform the following calculation in $\overline{\mathrm{K}}_1(R)$:

$$\begin{aligned}
\tau^{\mathrm{alt}}(C_*, \{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}) &\overset{\text{see Definition 346.8}}{=} [\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}'_{\mathrm{even}}}^{\mathcal{C}'_{\mathrm{odd}}} \\
&\overset{\text{follows from the Representing Matrix Lemma 309.24}}{=} [\mathrm{id}]_{\mathcal{C}'_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{even}}} \cdot [\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \cdot [\mathrm{id}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}'_{\mathrm{odd}}} \\
&\overset{\text{follows from Definition 345.7 which says that } [\mathcal{X}/\mathcal{Y}] = [\mathrm{id}]_{\mathcal{X}}^{\mathcal{Y}} \text{ and that } [\mathcal{X}/\mathcal{Y}] = [\mathcal{Y}/\mathcal{X}]^{-1}}{=} [\mathcal{C}'_{\mathrm{even}}/\mathcal{C}_{\mathrm{even}}] \cdot [\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \cdot [\mathcal{C}'_{\mathrm{odd}}/\mathcal{C}_{\mathrm{odd}}]^{-1} \\
&= [\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}} \cdot \prod_{i \text{ even}} [\mathcal{C}'_i/\mathcal{C}_i] \cdot \prod_{i \text{ odd}} [\mathcal{C}'_i/\mathcal{C}_i]^{(-1)} \\
&\overset{\text{see Definition 346.8}}{=} \tau^{\mathrm{alt}}(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \cdot \prod_i [\mathcal{C}'_i/\mathcal{C}_i]^{(-1)^i}.
\end{aligned}$$

■

Now we can finally provide a proof of the Torsion–Chain Contraction Proposition 346.6:

Proof of Proposition 346.6. Let R be an IBN ring and let (C_*, ∂_*) be an *acyclic* chain complex of R-left modules of finite length such that each C_i is equipped with a finite basis. Furthermore let δ be a chain contraction. By Definition 346.8 we need need to show we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \;=\; \tau^{\mathrm{alt}}(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}).$$

To do so we introduce some extra notation:

- By the Image–Stably Free 345.20 we know that each B_i is stably free. Thus we can find a free R-left module F_i such that $B_i \oplus F_i$ is a free R-left module. We equip F_i with some basis.
- We set $G_i := F_{i-1}$ which we equip with the same basis as F_{i-1}.
- For $i \in \mathbb{N}_0$ we consider the based chain complex $T_i := 0 \to G_{i+1} \xrightarrow{\mathrm{id}} F_i \to 0$.
- We consider the chain complex $C' := C \oplus T_0 \oplus \cdots \oplus T_n$. We use the basis $\mathcal{C}_i$ and the fixed bases for F_i (and the corresponding bases for the G_{i-1}) to equip each chain module of C'_i with a basis $\mathcal{C}'_i$.

The following claim shows that henceforth we can work with the chain complex C'_* instead of the chain complex C_*:

Claim 1.
(1) We have $\tau(C'_*, \mathcal{C}'_*) = \tau(C_*, \mathcal{C}_*)$.
(2) We have $\tau^{\mathrm{alt}}(C'_*, \mathcal{C}'_*) = \tau^{\mathrm{alt}}(C_*, \mathcal{C}_*)$.

Proof.

(1) This statement follows from the Torsion–Direct Sum Lemma 345.29 together with the observation that each $\tau(T_i)$ clearly equals 1.
(2) We consider the map $\delta' \colon C'_* \to C'_*$ that is given by δ on C_i and that is given by the identity map $F_i \to G_{i+1}$. It follows easily from the fact that δ is a chain contraction together with the example on page 6550 that δ' is a chain contraction. Note that clearly $\partial'_{\mathrm{even}} + \delta'_{\mathrm{even}} \colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ differs from $\partial_{\mathrm{even}} + \delta_{\mathrm{even}} \colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$ by the block sum with the identity, which does not change the corresponding element in $\overline{\mathrm{K}}_1(R)$. ⊞

For each $i \in \mathbb{N}_0$ we do the following:

- We pick a basis $\mathcal{B}_i$ for the free R-left module $B_i \oplus F_i$.
- We pick a right inverse $\sigma_{i-1} \colon B_{i-1} \to C_i$, i.e. we pick a homomorphism of R-left modules with $\partial_i \circ \sigma_{i-1} = \mathrm{id}_{B_i}$.
- The map $\sigma_{i-1} \colon B_{i-1} \to C_i$ together with the identity map $F_{i-1} \to G_i$ defines a homomorphism $\Delta_{i-1} \colon B_{i-1} \oplus F_{i-1} \to C'_i$ of R-left modules.
- Since C'_* is acyclic we see that $C_i = B_i \oplus F_i \oplus \Delta_{i-1}(B_{i-1} \oplus F_{i-1})$.

We can now calculate the alternative torsion with respect to the basis $\{\mathcal{B}_i \Delta_{i-1}(\mathcal{B}_{i-1})\}$:

Claim 2. With respect to the bases $\{\mathcal{B}_i \Delta_{i-1}(\mathcal{B}_{i-1})\}$ of the C'_i we have the equality

$$\tau^{\mathrm{alt}}(C', \{\mathcal{B}_i \Delta_{i-1}(\mathcal{B}_{i-1})\}_i) \;=\; 1 \;\in\; \overline{\mathrm{K}}_1(R).$$

Proof. We consider the map

$$\delta_i \colon C'_i = B_i \oplus F_i \oplus \Delta_{i-1}(B_{i-1} \oplus F_{i-1}) \;\to\; C'_{i+1}$$

that is given by Δ_i on $B_i \oplus F_i$ and by the zero map on $\Delta_{i-1}(B_{i-1} \oplus F_{i-1})$. One can easily verify (see Exercise 346.1) that these maps form a chain contraction. Furthermore note that it follows easily from the definitions that for each i we have

$$[\partial_{2i} \oplus \delta_{2i}]^{\mathcal{B}_{\mathrm{odd}} \Delta(\mathcal{B}_{\mathrm{even}})}_{\mathcal{B}_{\mathrm{even}} \Delta(\mathcal{B}_{\mathrm{odd}})} = \begin{array}{c} \\ B_{2i+1} \\ \Delta B_{2i} \end{array} \begin{array}{c} \begin{array}{cccc} B_{2i} & \Delta B_{2i+1} & & \end{array} \\ \begin{pmatrix} 0 & 0 & \mathrm{id} & 0 \\ 0 & \mathrm{id} & 0 & 0 \end{pmatrix} \end{array}$$

It now follows easily from Definition 346.8 that $\tau^{\mathrm{alt}}(C', \{\mathcal{B}_i\Delta_{i-1}(\mathcal{B}_{i-1})\}_i) = 1 \in \overline{\mathrm{K}}_1(R)$. ⊞
By Claim 1 it remains to prove the following claim:

Claim 3. We have $\tau^{\mathrm{alt}}(C'_*, \mathcal{C}_*) = \tau(C'_*, \mathcal{C}_*)$.

Proof. We have the following equalities:

$$\tau^{\mathrm{alt}}(C'_*, \{\mathcal{C}'_i\}) \underset{\substack{\uparrow\\ \text{by the Alternative Torsion Change-of-Basis Lemma 346.9 and since } [\mathcal{X}/\mathcal{Y}] = [\mathcal{Y}/\mathcal{X}]^{-1}}}{=} \overbrace{\tau^{\mathrm{alt}}(C'_*, \{\mathcal{B}_i\Delta_{i-1}(\mathcal{B}_{i-1})\})}^{=1, \text{ by Claim 2}} \cdot \prod_{i\in\mathbb{N}_0} (-1)^{i+1} \cdot [\mathcal{B}_i\Delta_{i-1}(\mathcal{B}_{i-1})/\mathcal{C}'_i] \underset{\substack{\uparrow\\ \text{by Definition 345.26}}}{=} \tau(C'_*, \mathcal{C}'_*).$$

■

346.3. **Naturality of torsion.** At times for simpleminded authors it is hard to figure out whether a construction is functorial or natural. For the purpose of this chapter it is perhaps helpful to recall the following basic discussion, which makes sense of the statement that "the determinant is natural":

Example. Let $\mathcal{CRing}$ be the category of commutative rings and let $\mathcal{Group}$ be the category of groups. Let $n \in \mathbb{N}$. We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{CRing} & \to & \mathcal{Group} \\ R & \mapsto & \mathrm{GL}(n, R) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{CRing} & \to & \mathcal{Group} \\ R & \mapsto & R^\times := \underset{\substack{\uparrow\\ \text{see Definition 2.6}}}{\text{units in } R}. \end{array}$$

For each commutative ring R we consider the group homomorphism

$$\begin{array}{rcl} \mathrm{GL}(n, R) & \to & R^\times \\ A & \mapsto & \det(A). \end{array}$$

Note that for any homomorphism $\varphi\colon R \to S$ between commutative rings and any invertible matrix $A \in \mathrm{GL}(n, R)$ the following two statements hold:
(1) $\varphi(A) \in \mathrm{GL}(n, S)$, i.e. $\varphi(A)$ is again invertible.
(2) We have the following equality in $S^\times$:

$$\det(\varphi(A)) = \varphi(\det(A)).$$

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{GL}(n, R) & \xrightarrow{\det} & R^\times \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{GL}(n, S) & \xrightarrow{\det} & S^\times. \end{array}$$

As on page 242 we can reinterpret this statement as saying that "taking the determinant" is "natural in R", i.e. it defines a natural transformations from F to G. Since torsion is a generalization of the determinant it is now not surprising to hear that we will soon see that torsion is also natural. □

To formulate the naturality of torsion we need the following piece of notation:

Notation 346.10. Let R be a ring and let F be a free R-left module with basis $\mathcal{B}$. Furthermore let $\varphi\colon R \to S$ be a ring homomorphism. It follows almost immediately from

the Free Module–Tensor-with-Ring Lemma 312.18 (1) that $S \otimes_R M$ is a free S-left module on the set $1_S \otimes \mathcal{B} := \{1_S \otimes b\}_{b \in \mathcal{B}}$.

Now we can state our naturality statement:

Theorem 346.11. (Torsion–Natural Theorem) Let R and S be two IBN rings. Let $(C_*, \{\mathcal{C}_i\}_{i \in \mathbb{N}_0})$ be an acyclic based chain complex over a ring R and let $\varphi\colon R \to S$ be a ring homomorphism. The following statements hold:

(1) The chain complex $S_\varphi \otimes_R C_*$ is also acyclic.

(2) We have the following equality in $\overline{\mathrm{K}}_1(S)$:

$$\tau(S_\varphi \otimes_R C_*, \{1_S \otimes \mathcal{C}_i\}_{i \in \mathbb{N}_0}) \;=\; \underset{\substack{\uparrow \\ \text{homomorphism from the} \\ \overline{\mathrm{K}}_1\text{-Functor Lemma 344.6}}}{\varphi_*}(\tau(C_*, \{\mathcal{C}_i\}_{i \in \mathbb{N}_0})).$$

Proof. Let R and S be IBN rings and let $\varphi\colon R \to S$ be a ring homomorphism. Furthermore let $(C_*, \{\mathcal{C}_i\}_{i \in \mathbb{N}_0})$ be an acyclic based chain complex.

(1) Since C_* is an acyclic chain complex of free R-left modules we know by the Acyclic–Chain Contraction Proposition 346.2 (i)⇒(ii) that the chain complex (C_*, ∂_*) admits a chain contraction $\{\delta_i\colon C_i \to C_{i+1}\}_{i \in \mathbb{N}_0}$ consisting of R-left module homomorphisms.

It follows from almost the same argument as in the Homology-with-Coefficients Lemma 314.6 (3) (together with the Tensor–Functor Lemma 314.4) that the maps

$$\mathrm{id}_S \otimes \delta_i\colon S_\varphi \otimes_R C_i \;\to\; S_\varphi \otimes_R C_{i+1}$$

with $i \in \mathbb{N}_0$ are a chain contraction for the chain complex $S_\varphi \otimes_R C_*$ consisting of S-left module homomorphisms. By the Acyclic–Chain Contraction Proposition 346.2 (i)⇒(ii) this implies in particular that the chain complex $S_\varphi \otimes_R C_*$ is acyclic.

(2) We continue with the notation from (1). Now note that

$$\begin{aligned} \tau(S_\varphi \otimes_R C_*, \{1_S \otimes \mathcal{C}_i\}_{i \in \mathbb{N}_0}) &\underset{\substack{\downarrow \\ \text{by the Torsion–Chain Contraction Proposition 346.6}}}{=} [\mathrm{id}_S \otimes \partial_{\mathrm{even}} + \mathrm{id}_S \otimes \delta_{\mathrm{even}}]_{1_S \otimes_R \mathcal{C}_{\mathrm{even}}}^{1_S \otimes_R \mathcal{C}_{\mathrm{odd}}} \\ &\underset{\substack{\uparrow \\ \text{follows easily from the Free Module–Tensor-with-Ring Lemma 312.18 (2)}}}{=} \varphi_*\big([\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}}\big) \underset{\substack{\uparrow \\ \text{by the Torsion–Chain Contraction Proposition 346.6}}}{=} \varphi_*(\tau(C_*, \{\mathcal{C}_i\}_{i \in \mathbb{N}_0})). \end{aligned}$$

■

Remark. We now want to make super precise in what sense torsion is natural. We first fix some notation. Let R be a ring.

- We denote by $\mathrm{BCC}^{\mathrm{iso}}(R)$ the set of all isomorphism classes of finite length chain complexes of R-left modules where each chain module is equipped with a basing.
- Note that $\mathrm{BCC}^{\mathrm{iso}}(R)$ forms a monoid under the direct sum operation.
- We denote by $\mathcal{M}\mathit{onoid}$ the category of monoids.

We consider the covariant functors

$$\begin{aligned} F\colon \mathcal{C}\mathit{Ring} &\to \mathcal{M}\mathit{onoid} \\ R &\mapsto \mathrm{BCC}^{\mathrm{iso}}(R) \\ (R \xrightarrow{\varphi} S) &\mapsto ((C_*, \mathcal{C}_*) \mapsto (S_\varphi \otimes_R C_*, 1_S \otimes_R \mathcal{C}_*)) \end{aligned}$$

and

$$\begin{aligned} G\colon \mathcal{C}\mathit{Ring} &\to \mathcal{M}\mathit{onoid} \\ R &\mapsto \overline{\mathrm{K}}_1(R). \end{aligned}$$

The Torsion–Natural Theorem 346.11 (2) can be reformulated as saying that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{BCC}^{\mathrm{iso}}(R) & \xrightarrow{\quad\tau\quad} & \overline{\mathrm{K}}_1(R) \\ \Big\downarrow{\scriptstyle (C_*,\mathcal{C}_*)\mapsto(S_\varphi\otimes_R C_*,1_S\otimes\mathcal{C}_*)} & & \Big\downarrow{\scriptstyle \varphi_*} \\ \mathrm{BCC}^{\mathrm{iso}}(S) & \xrightarrow{\quad\tau\quad} & \overline{\mathrm{K}}_1(S). \end{array}$$

In fact by the Torsion–Direct Sum Lemma 345.29 we know that the horizontal maps are monoid homomorphisms. In other words, torsion defines a natural transformation from F to G. □

346.4. **Torsion of dual chain complexes.** In this section we will define the dual of a (based acyclic) chain complex and compute its torsion. This discussion will run somewhat in parallel to the previous section.

Convention 346.12. Let R be a ring and let V be an R-left module. Furthermore let S be a ring with involution (in the sense of Definition 125.15) and let $\varphi\colon R\to S$ be a ring homomorphism.

- As in the Module–Change-of-Ring–Functor Lemma 309.13 we use φ to obtain the R-left module ${}_\varphi S$. In fact, together with the right multiplication by S we can view ${}_\varphi S$ as an (R,S)-bimodule.
- As in the Hom–Module Lemmas 315.2 and 320.6 we can use the S-right module structure on ${}_\varphi S$ to view $\mathrm{Hom}_{R\text{-left}}(V,{}_\varphi S)$ as an S-right module.
- We use the involution on S to view $\mathrm{Hom}_{R\text{-left}}(V,{}_\varphi S)$ as an S-left module..[3757]

The following notation plays the role of Notation 346.10:

Notation 346.13. Let R be a ring with involution and let V be an R-left module that is equipped with a basis $\mathcal{B}=(v_1,\ldots,v_k)$. Next let S be a ring with involution and let $\varphi\colon R\to S$ be a ring homomorphism. By the Free Module–Hom-to-Ring Lemma 315.8 (1) we can equip the S-left module $\mathrm{Hom}_{R\text{-left}}(V,{}_\varphi S)$ from Convention 346.12 with the dual basis $\mathcal{B}^*:=(v_1^*,\ldots,v_k^*)$ that is determined by $v_i^*(v_j)=\delta_{ij}$.

Next we turn to chain complexes:

Definition 346.14. Let R be a ring with involution and let

$$0\ \to\ C_m\ \xrightarrow{\partial_m}\ C_{m-1}\ \to\ \ldots\ \to\ C_1\ \xrightarrow{\partial_1}\ C_0\ \to\ 0$$

be a chain complex of R-left modules. Next let S be a ring with involution and let $\varphi\colon R\to S$ be a ring homomorphism. For each i we write $C_i^\dagger:=\mathrm{Hom}_{R\text{-left}}(C_{m-i},{}_\varphi S)$. We use Notation 346.13 to view $C_i^\dagger$ as an S-left module. We also consider the S-left module homomorphism $\partial_i{}^\dagger:=\partial_{m-i+1}^*\colon C_i^\dagger\to C_{i-1}^\dagger$. We refer to

$$0\ \to\ C_m^\dagger\ \xrightarrow{\partial_m^\dagger}\ C_{m-1}^\dagger\ \to\ \ldots\ \to\ C_1^\dagger\ \xrightarrow{\partial_1^\dagger}\ C_0^\dagger\ \to\ 0$$

as the dual chain complex. It follows from the functoriality of the Hom-functor and Convention 346.12 that this is a chain complex of S-left modules.

We have the following analog of the Torsion–Natural Theorem 346.11:

[3757] In other words, given $\varphi\colon V\to S$ and given $s\in S$ we define $s\cdot\varphi$ via $(s\cdot\varphi)(v):=\varphi(v)\cdot\overline{s}$.

Theorem 346.15. (Dual Chain Complex-Torsion Theorem) Let R be an IBN ring with involution and let

$$0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \ldots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be an acyclic chain complex of free R-left modules such that each C_i is equipped with a basis $\mathcal{C}_i$. Next let S be a ring with involution and let $\varphi\colon R \to S$ be a ring homomorphism. For each $i \in \{0, \ldots, m\}$ we consider the free S-left module $C_i^\dagger := \mathrm{Hom}_{R\text{-left}}(C_{m-i}, {}_\varphi S)$ from Notation 346.13. We then consider the dual chain complex

$$0 \to C_m^\dagger \xrightarrow{\partial_m^\dagger} C_{m-1}^\dagger \to \ldots \to C_1^\dagger \xrightarrow{\partial_1^\dagger} C_0^\dagger \to 0$$

from Definition 346.14.

(1) The dual chain complex $(C_*^\dagger, \partial_*^\dagger)$ is also acyclic.

(2) By the Free Module–Hom-to-Ring Lemma 315.8 we can equip $C_i^\dagger$ with the dual basis from Notation 346.13. We have the following equality in $\overline{\mathrm{K}}_1(R)$:[3758]

$$\tau(C^\dagger, \{\mathcal{C}_i^\dagger\}_{i\in\mathbb{N}_0}) \;=\; \varphi_*(\overline{\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0})}^{\mathrm{T}})^{(-1)^{m+1}}.$$

Proof. Let R and S be IBN rings and let $\varphi\colon R \to S$ be a ring homomorphism. Furthermore let $(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0})$ be an acyclic based chain complex.

(1) Since C_* is an acyclic chain complex of free R-left modules we know by the Acyclic–Chain Contraction Proposition 346.2 (i)⇒(ii) that the chain complex (C_*, ∂_*) admits a chain contraction $\{\delta_i\colon C_i \to C_{i+1}\}_{i\in\mathbb{N}_0}$.

It follows from almost the same argument as in the Dual Cochain Map Lemma 315.22 (using the Hom–Contravariant Functor Lemma 315.6) that the maps

$$\delta_i^*\colon \mathrm{Hom}_{R\text{-left}}(C_{i-1}, {}_\varphi S) \;\to\; \mathrm{Hom}_{R\text{-left}}(C_i, {}_\varphi S)$$

are a cochain contraction for the cochain complex $\mathrm{Hom}_{R\text{-left}}(C_*, {}_\varphi S)$. By the Acyclic–Chain Contraction Proposition 346.2 (i)⇒(ii) this implies in particular that the cochain complex $\mathrm{Hom}_{R\text{-left}}(C_*, {}_\varphi S)$ is acyclic. Evidently this implies that the dual chain complex $(C_i^\dagger, \partial_*^\dagger) := (\mathrm{Hom}_{R\text{-left}}(C_{m-i}, {}_\varphi S), \delta_{m-i}^*)$ is acyclic.

(2) We continue with the notation from (1). First we assume that m is even. In this case we see that

$$\begin{aligned}
\tau(C^\dagger, \{\mathcal{C}_i^\dagger\}) &\overset{\text{follows from the Torsion Chain Contraction Proposition 346.6}}{=} [\partial^\dagger_{\mathrm{even}} + \delta^\dagger_{\mathrm{even}}]_{\mathcal{C}^\dagger_{\mathrm{even}}}^{\mathcal{C}^\dagger_{\mathrm{odd}}} \\
&\overset{\text{since } m \text{ is even, we see that for an even index } i \text{ of } \partial_i^\dagger \text{ the index } m-i+1 \text{ of } \delta_i^* \text{ is now odd}}{=} [\partial^*_{\mathrm{odd}} + \delta^*_{\mathrm{odd}}]_{\mathcal{C}^*_{\mathrm{odd}}}^{\mathcal{C}^*_{\mathrm{even}}} \\
&\overset{\text{follows easily from the Free Module–Hom-to-Ring}}{=} \varphi_*\big(\big(\overline{[\partial_{\mathrm{odd}} + \delta_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{odd}}}^{\mathcal{C}_{\mathrm{even}}}}^{\mathrm{T}}\big)^{-1}\big) \\
&\overset{\text{by the Boundary–Chain Contraction Lemma 346.5}}{=} \varphi_*\big(\big(\overline{[\partial_{\mathrm{even}} + \delta_{\mathrm{even}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}}}^{\mathrm{T}}\big)^{-1}\big) \\
&\overset{\text{by the Torsion–Chain Contraction Proposition 346.6 and since } m \text{ is even}}{=} \varphi_*(\overline{\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0})}^{\mathrm{T}})^{(-1)^{m+1}}.
\end{aligned}$$

[3758] In Exercise 344.6 we saw that the map $[A] \mapsto [\overline{A}^{\mathrm{T}}]$ defines an automorphism of $\overline{\mathrm{K}}_1(R)$.

We now assume that m is odd. In this case we have the following variation on the above calculation:

$$\begin{array}{lll} & \text{follows from the Torsion Chain Contraction Proposition 346.6} & \text{since } m \text{ is odd, we see that for an even index } i \text{ of } \partial_i^\dagger \text{ the index } m-i+1 \text{ of } \partial_i^* \text{ is again even} \\ \tau(C^\dagger, \{\mathcal{C}_i^\dagger\}) & \downarrow & \downarrow \\ & = \quad [\partial^\dagger_{\text{even}} + \delta^\dagger_{\text{even}}]^{\mathcal{C}^\dagger_{\text{odd}}}_{\mathcal{C}^\dagger_{\text{even}}} & = \quad [\partial^*_{\text{even}} + \delta^*_{\text{even}}]^{\mathcal{C}^*_{\text{odd}}}_{\mathcal{C}^*_{\text{even}}} \\ & = \quad \varphi_*\big(\big(\overline{[\partial_{\text{even}} + \delta_{\text{even}}]^{\mathcal{C}_{\text{odd}}}_{\mathcal{C}_{\text{even}}}}\big)^{\mathrm{T}}\big) & = \quad \varphi_*\big(\overline{\tau(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0})}^{\mathrm{T}}\big)^{(-1)^{m+1}}. \\ & \uparrow & \uparrow \\ & \text{follows easily from the Free Module–Hom-to-Ring} & \text{by the Torsion–Chain Contraction Proposition 346.6 and since } m \text{ is odd} \end{array}$$

■

346.5. Exercises for Chapter 346.

Exercise 346.1. Let R be a ring and let C_* be a chain complex of R-left modules such that each $B_i := \mathrm{im}(\partial_{i+1}\colon C_{i+1} \to C_i)$ is a free R-left module. For each $i \in \mathbb{N}$ we pick a right inverse $\sigma_i\colon B_i \to C_{i+1}$, i.e. we pick a homomorphism of R-left modules with $\partial_{i+1} \circ \sigma_i = \mathrm{id}_{B_i}$. Show that the maps

$$C_i = B_i \oplus \sigma_{i-1}(B_{i-1}) \quad \to \quad C_{i+1} = B_{i+1} \oplus \sigma_i(B_i)$$

that are given by σ_i on B_i and by 0 on the second summand, form a chain contraction.

Exercise 346.2. Let R be an IBN ring and let $(C_*, \mathcal{C}_*)$, $(D_*, \mathcal{D}_*)$ be two based chain complexes of finitely generated free R-left modules where both chain complexes have finite length. Show that

$$\tau^{\mathrm{alt}}(C_* \oplus D_*, \{\mathcal{C}_* \oplus \mathcal{D}_*\}) \;=\; \tau^{\mathrm{alt}}(C_*, \mathcal{C}_*) \cdot \tau^{\mathrm{alt}}(D_*, \mathcal{D}_*).$$

Remark. This is evidently the analog of the Torsion–Direct Sum Lemma 345.29.

Exercise 346.3. Let R be an IBN ring and let

$$0 \;\to\; C'_* \;\xrightarrow{f}\; C_* \;\xrightarrow{p}\; C''_* \;\to\; 0$$

be a short exact sequence of based chain complexes of finitely generated free R-left modules, such that for each $i \in \mathbb{N}_0$ the bases $\mathcal{C}'_i, \mathcal{C}_i$ and $\mathcal{C}''_i$ are compatible in the sense of Definition 345.11. We assume hat all three chain complexes are acyclic. Show that we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau^{\mathrm{alt}}(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0}) \;=\; \tau^{\mathrm{alt}}(C'_*, \{\mathcal{C}'_i\}_{i\in\mathbb{N}_0}) \cdot \tau^{\mathrm{alt}}(C''_*, \{\mathcal{C}''_i\}_{i\in\mathbb{N}_0}).$$

Remark. This is evidently the analog of the SES-Torsion Theorem 345.30.

CHAPTER 347

Calculations of torsion

In Chapter 345 we introduced torsion of a based chain complex. In Chapter 346 we showed that torsion is in a suitable sense natural. In this chapter we will turn to the more practical task of calculating torsion. In fact will get to know two ways how to calculate torsion, or, alternatively, how to relate torsion to other invariants.

This chapter is organized as follows:

- In Section 347.1 we will see that frequently one can calculate torsion using "τ-chains".
- In Section 347.2 we will see that in some settings Reidemeister torsion can be calculated in terms of the orders of modules that we initially introduced in Definition 287.18.
- In Section 347.3 we will provide exercises for Chapter 347.

347.1. **Torsion via τ-chains.** In this section we will present a way to calculate torsion that was first developed by Vladimir Turaev [**Tur1986**]. We will follow the exposition in [**Tur2001**, Section 2.1].

Definition 347.1. Let $\mathbb{F}$ be a field and let

$$0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \dots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex of $\mathbb{F}$-vector spaces where each C_i is equipped with a finite basis $\mathcal{C}_i$. For each $i \in \{0, \dots, m\}$ we denote by A_i the matrix corresponding to the boundary map and the given basis.[3759]

(1) A matrix chain for the chain complex C_* is a collection of sets $\alpha = (\alpha_m, \dots, \alpha_1, \alpha_0)$ where $\alpha_i \subseteq \{1, 2, \dots, \dim_{\mathbb{F}}(C_i)\}$ and $\alpha_0 = \varnothing$.

(2) Given a matrix chain α we denote by $S_i(\alpha)$ the submatrix of A_i that we obtain from A_i by the following procedure:
   - We keep precisely the rows in α_{i+1} and
   - we cross out the columns in α_i.[3760]

(3) A matrix chain is called a τ-chain if all the matrices $S_i(\alpha)$ are square matrices.

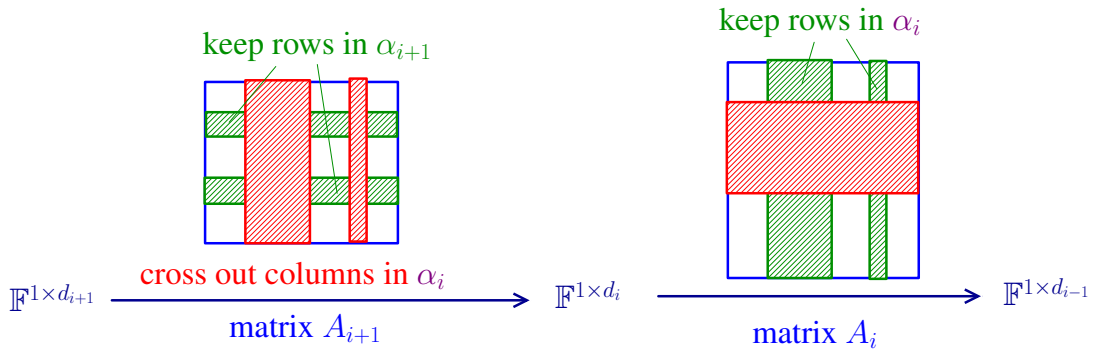
(4) A τ-chain is called non-singular if all of the $S_i(\alpha)$ are invertible.

The following chapter is one of the two main results of this chapter:

[3759] We still follow the convention from Definition 309.23 which says that we represent homomorphisms by *right* multiplication by matrices. So in the current setting this leads to commutative diagrams of the form

$$\begin{array}{ccc} \mathbb{F}^{1\times d_i} & \xrightarrow{\xi \mapsto \xi \cdot A_i} & \mathbb{F}^{1\times d_{i-1}} \\ \cong \downarrow \Xi_{\mathcal{C}_i} & & \cong \downarrow \Xi_{\mathcal{C}_{i-1}} \\ C_i & \xrightarrow{\partial_i} & C_{i-1}. \end{array}$$

[3760] Alternatively one could say that one keeps the columns in the complement of α_i.



Theorem 347.2. (Torsion–τ-Chain Theorem) Let $\mathbb{F}$ be a field and let

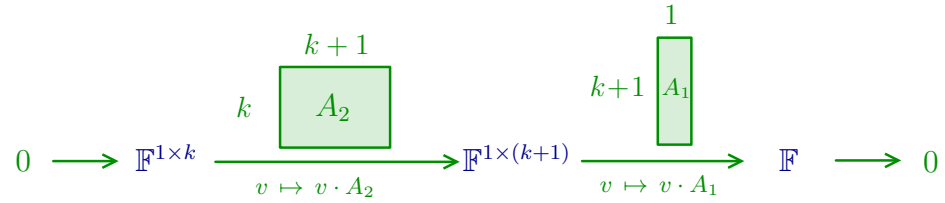
$$0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \ldots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex of $\mathbb{F}$-vector spaces where each C_i is equipped with a finite basis $\mathcal{C}_i$. Let $\alpha = (\alpha_m, \ldots, \alpha_1, \alpha_0)$ be a τ-chain for C such that for all *odd*[3761] $i \in \mathbb{N}_0$ we have $\det(S_i(\alpha)) \neq 0$. Then the τ-chain is non-singular, the chain complex is acyclic and we have the following equality in $\overline{\mathrm{K}}_1(\mathbb{F}) =_{\mathrm{i}} \mathbb{F}^\times/\{\pm 1\}$:

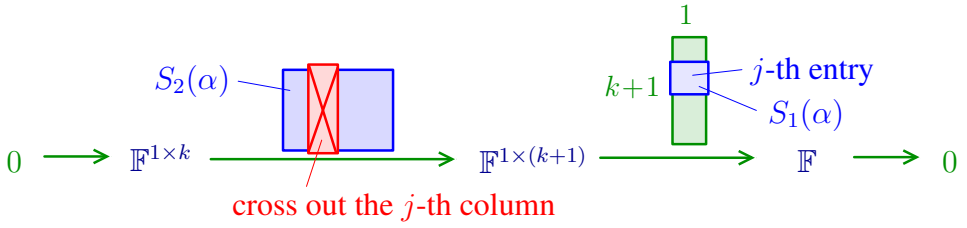
$$\underbrace{\tau(C_*, \mathcal{C}_*)}_{\text{see Definition 345.16}} = \prod_{i=1}^{m} \det(S_i(\alpha))^{(-1)^i}.$$

Example. Let $\mathbb{F}$ be a field.

(1) We consider an acyclic chain complex of the following form:[3762]



Since the chain complex is acyclic we know that there exists a j-th entry in A_1 which is non-zero. We now consider the τ-chain $\alpha = ((\{1, \ldots, k\}, \{j\}, \varnothing)$. We end up with the following situation:



Note that by construction $S_1(\alpha) = (j\text{-th entry of } A_1)$ is non-zero. Since we are working over a field we see that the (1×1)-matrix $S_1(\alpha)$ is in fact invertible. It follows from

[3761] The same statement of course also holds if for all *even* $i \in \mathbb{N}_0$ we have $\det(S_i(\alpha)) \neq 0$.

[3762] It goes without saying that all vector spaces involved are equipped with the standard basis.

the Torsion–τ-Chain Theorem 347.2 that $S_2(\alpha)$ is also invertible and we see that

$$\tau\left(\begin{array}{c}\text{the above based}\\ \text{chain complex}\end{array}\right) \overset{\text{by the Torsion–}\tau\text{-Chain Theorem 347.2}}{\underset{\downarrow}{=}} \prod_{i=0}^{1} \det(S_i(\alpha))^{(-1)^i} = \frac{\det(S_2(\alpha))}{\det(S_1(\alpha))} = \frac{\det(A_2 \text{ with } j\text{-th column crossed out})}{j\text{-th entry of } A_1}.$$

(2) To make the above example more explicit we now consider a chain complex of the form

$$0 \longrightarrow \mathbb{F} \xrightarrow[A_2]{\begin{pmatrix}-y & x\end{pmatrix}} \mathbb{F}^{1\times 2} \xrightarrow[A_1]{\begin{pmatrix}x\\ y\end{pmatrix}} \mathbb{F} \longrightarrow 0$$

where $\mathbb{F}$ is a field and $x, y \in \mathbb{F}$. Clearly the chain complex is acyclic if and only if one of x, y is non-zero. If $x \neq 0$, then we see that

$$\tau(\text{above based chain complex}) \overset{\text{we apply the formula in (1) to } j=1}{\underset{\uparrow}{=}} \frac{\det(A_2 \text{ with } j\text{-th column crossed out})}{j\text{-th entry of } A_1} = \frac{x}{x} = 1.$$

Alternatively, if $y \neq 0$, then we see that

$$\tau(\text{above based chain complex}) \overset{\text{we apply the formula in (1) to } j=2}{\underset{\uparrow}{=}} \frac{\det(A_2 \text{ with } j\text{-th column crossed out})}{j\text{-th entry of } A_1} = \frac{-y}{y} = -1.$$

So we do not get the same element of $\mathbb{F}^\times$ on the nose. But that is ok since we only promised that we obtain the same element up to multiplication by -1. □

In the proof of the Torsion–τ-Chain Theorem 347.2 we will make use of an elementary lemma from linear algebra. To formulate this lemma we need the following notion from linear algebra:

Definition 347.3. Let $\mathbb{F}$ be a field and let A be a $(k \times l)$-matrix over $\mathbb{F}$. We define

$$\operatorname{rank}(A) := \dim_{\mathbb{F}}\left(\operatorname{im}\left(\mathbb{F}^{1\times k} \xrightarrow{v\mapsto v\cdot A} \mathbb{F}^{1\times l}\right)\right).$$

Now we can formulate the promised lemma:[3763]

Auxiliary Lemma 347.4. Let $\mathbb{F}$ be a field. Suppose we have the following type of matrices:

- A is an $(k \times l)$-matrix over $\mathbb{F}$.
- B is an $(k \times m)$-matrix over $\mathbb{F}$.
- C is an $(l \times n)$-matrix over $\mathbb{F}$.
- D is an $(m \times n)$-matrix over $\mathbb{F}$.

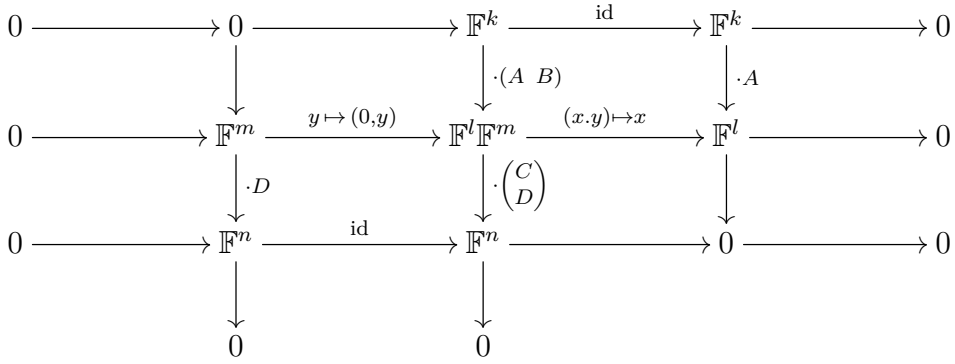
If

$$\begin{pmatrix}A & B\end{pmatrix} \cdot \begin{pmatrix}C\\ D\end{pmatrix} = 0,$$

[3763]Note that the lemma gives in particular a solution to Exercise 145.21.

and if A has rank l, then $\operatorname{rank}(D) = \operatorname{rank}\begin{pmatrix} C \\ D \end{pmatrix}$.

Proof. We consider the following diagram:



We make the following observations:

- By hypothesis the vertical sequences are chain complexes.
- The horizontal sequences are clearly exact.
- It is also clear that the diagram commutes.

These observations imply that we have a short exact sequence of chain complexes at our hand. It thus follows from the LES-Proposition 145.10 that we get a long exact sequence of homology groups. The lower end of this long exact sequence is clearly of the form

$$\operatorname{coker}(A) \xrightarrow{\partial_1} \operatorname{coker}(D) \to \operatorname{coker}\begin{pmatrix} C \\ D \end{pmatrix} \to 0.$$

By hypothesis the $(k \times l)$-matrix A has rank l. Thus the cokernel to the left is zero. It now follows that the cokernels of D and $\begin{pmatrix} C \\ D \end{pmatrix}$ have the same dimension. But since both maps end up in $\mathbb{F}^n$ we see that the two matrices do indeed have the same rank. ∎

Proof of the Torsion–τ-Chain Theorem 347.2. [3764] Let $\mathbb{F}$ be a field and let

$$0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \ldots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex of $\mathbb{F}$-vector spaces where each C_i is equipped with a finite basis $\mathcal{C}_i$. Let $\alpha = (\alpha_m, \ldots, \alpha_1, \alpha_0)$ be a τ-chain for C such that for all *odd* $i \in \mathbb{N}_0$ we have $\det(S_i(\alpha)) \neq 0$. We start out with a few preparations:

- For each i we write $d_i := \dim_{\mathbb{F}}(C_i)$.
- Note that after a permutation of the bases vectors of the C_i we can arrange that each α_i consists of the last $|\alpha_i|$ entries of $\{1, \ldots, d_i\}$.[3765]
- Throughout the proof we write $S_i := S_i(\alpha)$.
- For each $i \in \{0, \ldots, m\}$ we set $\gamma_i := \operatorname{rank}(A_i) := \dim_{\mathbb{F}}(\operatorname{im}(\partial_i))$.

Claim 1. For each $i \in \mathbb{N}_0$ we have

$$\gamma_i = \dim_{\mathbb{F}}(C_i) - \dim_{\mathbb{F}}(C_{i-1}) + \cdots + (-1)^i \cdot \dim_{\mathbb{F}}(C_0).$$

[3764] An alternative approach to proving the theorem is sketched in Exercise 347.2.

[3765] Note that reordering changes the torsion up to a sign, but this gets eaten up by the fact that we work in $\overline{\mathrm{K}}_1(\mathbb{F}) =_{\mathrm{i}} \mathbb{F}^\times/\{\pm 1\}$.

Proof. It follows from the hypothesis that C_* is acyclic and the Break-into-SES Lemma 145.6 that $0 \to \operatorname{im}(\partial_i) \to C_i \xrightarrow{\partial_i} C_{i-1} \to \ldots \to C_0 \to 0$ is exact. It thus follows from the Chain Complex–H_*–χ-Lemma 156.4 that the alternating sum of the dimensions of this chain complex is zero. By Definition 347.3 this gives us the statement in the claim. ⊞

Claim 2. Each $S_i = S_i(\alpha)$ is a $(\gamma_i \times \gamma_i)$-matrix.

Proof. By Definition 347.1 we know that for each j the matrix $S_j = S_j(\alpha)$ has $|\alpha_{j+1}|$ rows and $\dim_{\mathbb{F}}(C_j) - |\alpha_j|$ columns. Since we are given a τ-chain (in the sense of Definition 347.1) we know that each S_j is a square matrix. Thus we also know that $|\alpha_{j+1}| = \dim_{\mathbb{F}}(C_j) - |\alpha_j|$. Using this fact iteratively we see that

$$\begin{aligned} |\alpha_{i+1}| &= \dim_{\mathbb{F}}(C_i) - |\alpha_i| \\ &= \dim_{\mathbb{F}}(C_i) - \dim_{\mathbb{F}}(C_{i-1}) + |\alpha_{i-1}| \\ &= \ldots \\ &= \dim_{\mathbb{F}}(C_i) - \dim_{\mathbb{F}}(C_{i-1}) + \cdots + (-1)^i \cdot \dim_{\mathbb{F}}(C_0) \pm \overbrace{|\alpha_0|}^{=0} \\ &\underset{\uparrow}{=} \gamma_i. \end{aligned}$$

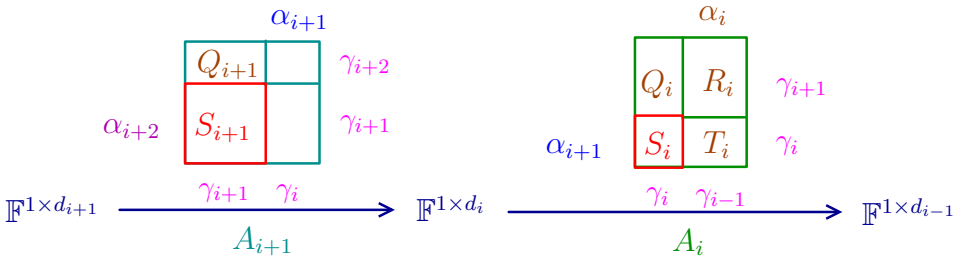
by Claim 1 and since by Definition 347.1 we have $\alpha_0 = \varnothing$

⊞

Claim 3. Each $S_i = S_i(\alpha)$ is invertible.

Proof. By hypothesis we already know this statement for all odd i. So it remains to consider the case that i is even. Thus throughout the proof of this claim let i be an even natural number.

In the following we consider the submatrices Q_j, R_j, T_j of A_j as defined in the figure below.



Subclaim 1. The $(\gamma_i \times d_{i-1})$-matrix $(S_i \quad T_i)$ has rank $\gamma_i = \operatorname{rank}(A_i)$.

Proof of subclaim. We make two observations:

- Since we are dealing with a chain complex we know that $\partial_i \circ \partial_{i+1} = 0$, which implies that $A_{i+1} \cdot A_i = 0$.
- By hypothesis we know that the $(\gamma_{i_1} \times \gamma_{i+1})$-matrix S_{i+1} is invertible. This implies that the $(d_{i+1} \times \gamma_{i+1})$-matrix $\begin{pmatrix} Q_{i+1} \\ S_{i+1} \end{pmatrix}$ has rank γ_{i+1}.

It thus follows from Auxiliary Lemma 347.4 that the rank of the $(\gamma_i \times d_i)$-matrix $(S_i \quad T_i)$ equals the rank of A_i, namely γ_i. □

Subclaim 2. The $(d_i \times \gamma_i)$-matrix $\begin{pmatrix} Q_i \\ S_i \end{pmatrix}$ has rank $\gamma_i = \operatorname{rank}(A_i)$.

Proof of subclaim. This statement is proved in the same fashion as Subclaim 1, except that this time we use that $A_i \cdot A_{i-1} = 0$ and that S_{i-1} is invertible. In fact one can reduce the proof of this subclaim to the setting of Subclaim 2 by looking at the transposes of all the matrices. We leave it to the reader to fill in the details. □

It follows from the two subclaims together with some elementary linear algebra (see Exercise 347.1) that $\operatorname{rank}(S_i) = \operatorname{rank}(A_i) = \gamma_i$. By Claim 2 we know that S_i is a $(\gamma_i \times \gamma_i)$-matrix. Since we work over a field we thus see that S_i is invertible. ⊞

Claim 4. The chain complex C_* is acyclic.

Proof. We start out with two observations:

(1) For each $i \in \mathbb{N}_0$ we have the (in-) equalities

$$\operatorname{rank}(A_i) \;\geq\; \operatorname{rank}(S_i(\alpha)) \;\underset{\text{since the } \tau\text{-chain is non-singular and since } S_i(\alpha) \text{ is a square matrix of size } |\alpha_i|}{\uparrow =}\; |\alpha_i| \;\overset{\text{see Claim 1 and 2}}{\downarrow =}\; \dim_{\mathbb{F}}(C_i) - \dim_{\mathbb{F}}(C_i) + \cdots + (-1)^i \cdot \dim_{\mathbb{F}}(C_0).$$

(2) Let $i \in \mathbb{N}_0$. As in Notation 345.13 we see that it follows easily from the filtration $0 \subseteq B_i \subseteq Z_i \subseteq C_i$ that we have the equality

$$\underbrace{\dim_{\mathbb{F}}(B_i)}_{=\operatorname{rank}(A_i)} + \dim_{\mathbb{F}}(\mathrm{H}_i(C_*)) + \underbrace{\dim_{\mathbb{F}}(B_{i-1})}_{=\operatorname{rank}(A_{i-1})} \;=\; \dim_{\mathbb{F}}(C_i).$$

We extract two bits of information:

(a) We have the inequality $\operatorname{rank}(A_i) + \operatorname{rank}(A_{i-1}) \leq \dim_{\mathbb{F}}(C_i)$.

(b) Since $\mathbb{F}$ is a field we see that equality holds in (a) if and only if $\mathrm{H}_i(C_*) = 0$.

One can easily show that (1) is compatible with (2a) if and only if for all i we have $\operatorname{rank}(A_i) + \operatorname{rank}(A_{i-1}) = \dim_{\mathbb{F}}(C_i)$. But by (2b) this implies that the chain complex C_* is acyclic. ⊞

It remains to prove the following claim:

Claim 5. We have the following equality in $\overline{\mathrm{K}}_1(\mathbb{F}) =_{\mathrm{i}} \mathbb{F}^\times/\{\pm 1\}$:

$$\tau(C_*) \;=\; \prod_{i=1}^{m} \det(S_i(\alpha))^{(-1)^i}.$$

Proof. Let $i \in \mathbb{N}_0$. By Claim 2 and 3 we know that $\operatorname{rank}(S_i) = \gamma_i = \operatorname{rank}(A_i)$. Thus we see that the j-rows of A_i with $j \in \alpha_{i+1}$ form a basis $\mathcal{B}_i$ of $B_i = \operatorname{im}(\partial_{i+1}\colon C_{i+1} \to C_i)$. As a lift $\widetilde{\mathcal{B}}_i$ of $\mathcal{B}_i$ to C_{i+1} we can and will take the last γ_i vectors of the basis $\mathcal{C}_{i+1}$. Now we see that

$$\tau(C_*, \mathcal{C}_*) \;\underset{\text{by Definition 345.16}}{\uparrow =}\; \prod_{i=1}^{m} [\mathcal{B}_i \widetilde{\mathcal{B}}_{i-1}/\mathcal{C}_i]^{(-1)^{i+1}} \;\underset{\text{this follows easily from our choices, the degree shift comes from } B_{i+1} = \operatorname{im}(\partial_{i+2})}{\uparrow =}\; \prod_{i=1}^{m} \det \begin{pmatrix} S_i & 0 \\ 0 & \mathrm{id}_{|\alpha_i|} \end{pmatrix}^{(-1)^{i+2}} \;=\; \prod_{i=1}^{m} \det(S_i(\alpha))^{(-1)^i}.$$

■

We conclude this section with the following partial converse to the Torsion–τ-Chain Theorem 347.2:

Lemma 347.5. (Non Degenerate τ-Chain Existence Lemma) Let $\mathbb{F}$ be a field and let C_* be a based chain complex of finite length consisting of finite-dimensional $\mathbb{F}$-vector spaces. The following two statements are equivalent:

(i) The chain complex is acyclic.
(ii) There exists a non-singular τ-chain.

Proof. Let $\mathbb{F}$ be a field and let C_* be a based chain complex of length m consisting of finite-dimensional $\mathbb{F}$-vector spaces. As before we denote by A_i the matrices representing the boundary maps.

We first prove the (i)⇒(ii)-direction. Thus we assume that the chain complex is acyclic. We proceed as follows:

(a) For each i we set $\gamma_i := \operatorname{rank}(A_i)$.
(b) As on page 6566 we see that the hypothesis that C_* is acyclic implies that we have the equality $\gamma_i + \gamma_{i-1} = \dim_{\mathbb{F}}(C_i)$.
(c) Clearly we can choose for each *odd* i a square submatrix S_i of A_i of rank $\gamma_i = \operatorname{rank}(A_i)$.
(d) This choice in (c) gives us sets α_i for $i = m, m-1, \ldots, 1, 0$ such that for each odd i we have $S_i(\alpha) = S_i$, which by construction are invertible. In particular $S_i(\alpha)$ is a square matrix of size $|\alpha_i|$. Together with (c) we get $|\alpha_i| = \gamma_i$.
(e) These α_i also define the matrices $S_i(\alpha)$ for i *even*.

It remains to show that that the $S_i(\alpha)$ for i even are invertible. Note that it follows from Definition 347.1 and (b) that for each even i we have

$$\begin{aligned} \text{number of rows of } S_i(\alpha) &= |\alpha_{i+1}| = \dim_{\mathbb{F}}(C_{i+1}) - |\alpha_{i+1}| = \dim_{\mathbb{F}}(C_{i+1}) - \gamma_{i+1} = \gamma_i \\ \text{number of columns of } S_i(\alpha) &= \dim_{\mathbb{F}}(C_i) - |\alpha_{i-1}| = \dim_{\mathbb{F}}(C_i) - \gamma_{i-1} = \gamma_i. \end{aligned}$$

Thus the even $S_i(\alpha)$ are also square matrices. It now follows from the Torsion–τ-Chain Theorem 347.2 that the even $S_i(\alpha)$ are also invertible.

Finally note that the (ii)⇒(i)-direction is an immediate consequence of the Torsion–τ-Chain Theorem 347.2. ∎

347.2. **Torsion and orders.** In this section we will see that in many settings one can calculate torsions in terms of the orders of modules from Definition 287.18. This result cuts both ways: One can use it to calculate torsions, or, in many settings one can use this result to translate results about torsions (which, as we saw, are quite well-behaved) into results about orders (which are in general more tricky to deal with). Put differently, at times questions about orders (which can be quite tricky) can be replaced by much easier questions about torsions.

347.2.1. **Orders of modules.** In this subsection (which is largely a carbon-copy of Section 339.1) we will remind ourselves of the concept of an order of a finitely generated module over a Noetherian UFD. We start out with the following definition which is a slight variation on Definition 287.6 and which equals Definition 339.1:

Definition 347.6. Let R be a commutative ring and let M be an R-module. A finite presentation for M is a finite exact sequence of homomorphisms of R-modules of the form

$$R^{1\times k} \xrightarrow{v\mapsto v\cdot A} R^{1\times l} \xrightarrow{\psi} M \to 0$$

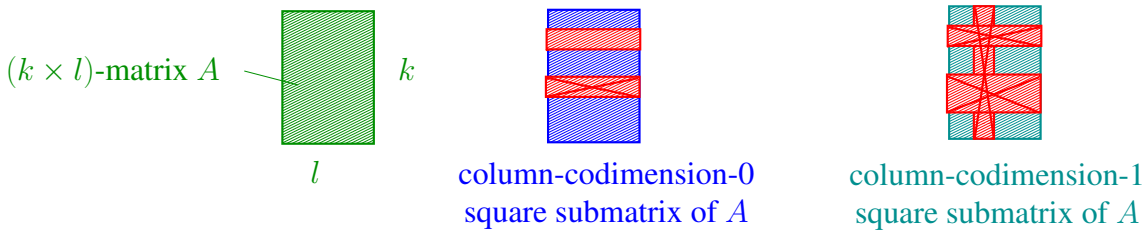
where $A \in \mathrm{M}(k \times l, R)$. Any such matrix A is called a presentation matrix for M.[3766]

The following definition is a variation on Definition 287.16:

Definition 347.7. Let R be a ring and let $A \in \mathrm{M}(k \times l, R)$ be a matrix.

(1) Let $i \in \{0, \ldots, l\}$ with $i + k - l \geq 0$. We refer to any $(l - i) \times (l - i)$-matrix that is obtained from A by crossing out i columns and $(i + k - l)$ rows as a column-codimension-i submatrix of A.[3767]

(2) We now suppose that R is a commutative ring. We refer to the ideal generated by the determinants of all square submatrices of column-codimension $\leq i$ of A as the i-th elementary ideal $E_i(A) \subseteq R$ of A.



We also recall Definition 287.18 (1):

Definition 347.8. Let R be a Noetherian ring and let M be a finitely generated R-module. Since R is Noetherian it follows from the Presentation Matrix–Existence Proposition 287.7 that M admits a presentation matrix A. Let $k \in \mathbb{N}_0$. We refer to

$$E_k(M) \;\coloneqq\; E_k(A) \;\subseteq\; R$$

as the k-th elementary ideal of the R-module M. It follows from the Module–Elementary Ideal Proposition 287.17 that the ideal $E_k(M)$ is well-defined, i.e. it is independent of the choice of the presentation matrix A.

Before we can recall part (2) of Definition 287.18 we should first recall Notation 287.15:

Notation 347.9. Let R be a ring. Given $p, q \in R$ we write $p \doteq q$ if there exists a unit $\epsilon \in R$ such that $p = \epsilon \cdot q$.

Now we can recall Definition 287.18 (2):

Definition 347.10. Let R be a Noetherian UFD and let M be a finitely generated R-module. We refer to

$$\mathrm{ord}_R(M) \;\coloneqq\; \underset{\text{see Definition 116.20}}{\gcd_R}(E_0(M)) \;\in\; R$$

as the order of the R-module M. It follows from the UFD-GCD Lemma 287.14 that the order is well-defined up to "$\doteq$", i.e. up to multiplication by a unit in R.

To formulate our next lemma we will need the following almost self-explanatory definition:

[3766] As in Definition 339.1 (but in contrast to Definition 287.6) we work with row vectors and right-multiplication by matrices. Thus a presentation matrix in the sense of the current definition is the transpose of a presentation matrix in the sense of Definition 287.6.

[3767] Since we are now working with transposes of the matrices from Definition 287.16 we see that with respect to the original Definition 287.16 the roles of rows and columns got swapped.

Definition 347.11. Let R be a commutative ring. For any R-module M we refer to

$$\mathrm{Tor}_R(M) \;:=\; \{m \in M \mid \text{ there exists an } r \in R \setminus \{0\} \text{ with } r \cdot m = 0\}$$

as the R-torsion submodule of M[3768]. We say M is R-torsion if $\mathrm{Tor}_R(M) = M$.

Next we recall the following lemma:

Lemma 339.8. (Module Torsion Lemma) Let R be a Noetherian UFD and let M be a finitely generated R-module. We denote by $Q(R)$ the quotient field of R. The following statements are equivalent:

(1) $Q(R) \otimes_R M = 0$, i.e. $\mathrm{rank}_R(M) = 0$.
(2) $\mathrm{ord}_R(M) \neq 0$.
(3) $M = \mathrm{Tor}_R(M)$, i.e. M is R-torsion.

It is also worth recalling the following consequence of the Module Torsion Lemma 339.8:

Lemma 339.9. (Acyclicity Lemma) Let R be a Noetherian UFD and let C_* be a chain complex of finitely generated free R-modules. For each $i \in \mathbb{N}_0$ the following statements are equivalent:

(0) We have $\mathrm{H}_i(Q(R) \otimes_R C_*) = 0$, i.e. $Q(R) \otimes_R C_*$ is acyclic.
(1) We have $Q(R) \otimes_R \mathrm{H}_i(C_*) = 0$, i.e. $\mathrm{rank}_R(\mathrm{H}_i(C_*)) = 0$.
(2) We have $\mathrm{ord}_R(\mathrm{H}_i(C_*)) \neq 0$.
(3) We have $\mathrm{H}_i(C_*) = \mathrm{Tor}_R(\mathrm{H}_i(C_*))$, i.e. each $\mathrm{H}_i(C_*)$ is R-torsion.

Proof. Since the proof is so short, we might as well repeat it: Note that the equivalence of (1), (2) and (3) follows immediately from Module Torsion Lemma 339.8. Finally the following claim takes care of the equivalence of (0) and (1):

Claim. The $Q(R)$-modules $\mathrm{H}_*(Q(R) \otimes_R C_*)$ and $Q(R) \otimes_R \mathrm{H}_*(C_*)$ are isomorphic.

Proof. The claim is an immediate consequence of the Tensor-with-Localization Proposition 314.18 (which says that tensoring with localizations is exact) together with the Exact Functor–Homology Lemma 314.13. ∎

347.2.2. **The relationship between torsion and orders.** The following theorem was first proved by Vladimir Turaev [**Tur1986**], extending earlier work by John Milnor [**Miln1968b**, p. 388]:

Proposition 347.12. (Torsion–Order Proposition) Let R be a Noetherian UFD. We denote by $Q(R)$ the quotient field of R. Let C_* be a finite chain complex of finitely generated free R-modules. We equip the free R-modules C_i with any basis $\mathcal{C}_i$. As in Notation 346.10 we see that it follows from the Free Module–Tensor-with-Ring Lemma 312.18 that these bases gives rise to a basis $1_{Q(R)} \otimes \mathcal{C}_i$ for each $Q(R) \otimes_R C_i$. If $Q(R) \otimes_R C_*$ is acyclic[3769], then we have the following equality in $Q(R)^\times$ up to "$\doteq$":[3770] [3771]

$$\tau(Q(R) \otimes_R C_*, 1_{Q(R)} \otimes \mathcal{C}_*) \;\doteq\; \prod_{i \in \mathbb{N}_0} \mathrm{ord}_R(\mathrm{H}_i(C_*))^{(-1)^{i+1}}.$$

[3768] Since R is commutative one can easily verify that this is indeed a submodule.

Example. Let $0 \to C_n \to \cdots \to C_0 \to 0$ be a chain complex of finitely generated free abelian groups such that all homology groups are finite. Note that it follows from the Acyclicity Lemma 339.9 (3)⇒(0) that $\mathbb{Q} \otimes_{\mathbb{Z}} C_*$ is acyclic. We equip each C_i with a basis $\mathcal{C}_i$. Then we have the following equality in $\mathbb{Q}^\times$:

$$\tau(\mathbb{Q} \otimes_{\mathbb{Z}} C_*, 1_{\mathbb{Q}} \otimes \mathcal{C}_*) \underset{\substack{\uparrow \\ \text{by the Torsion–Order} \\ \text{Proposition 347.12}}}{=} \pm \prod_{i \in \mathbb{N}_0} \operatorname{ord}_{\mathbb{Z}}(\mathrm{H}_i(C_*))^{(-1)^{i+1}} \underset{\substack{\uparrow \\ \text{by the Finite Matrix} \\ \text{Quotient Lemma 97.23}}}{=} \pm \prod_{i \in \mathbb{N}_0} |\mathrm{H}_i(C_*)|^{(-1)^{i+1}}.$$

In this setting $\tau(\mathbb{Q} \otimes_{\mathbb{Z}} C_*, 1_{\mathbb{Q}} \otimes \mathcal{C}_*)$ is thus the alternating product of the orders of the homology groups (which are all torsion). This observation might (?) explain where the name "torsion" comes from. □

The proof of the Torsion–Order Proposition 347.12 rests on two lemmas. The first lemma is quite elementary:

Lemma 347.13. (Order-Ideal-Multiplicative Lemma) Let R be a UFD.
(1) For any ideal $I \subseteq R$ and any $r \in R$ we have $\gcd(r \cdot I) \doteq r \cdot \gcd(I)$.
(2) For any two ideals $I, J \subseteq R$ we have[3772] $\gcd(I \cdot J) \doteq \gcd(I) \cdot \gcd(J)$.

Proof.
(1) This statement follows easily from the definitions.
(2) Let I, J be ideals. First note that it follows easily from the fact that we are dealing with a UFD that there exist two ideals I' and J' such that $I = \gcd(I) \cdot I'$ and such $J = \gcd(I) \cdot J'$. By (1) we know that $\gcd(I') = 1$ and $\gcd(J') = 1$ and it follows from (1) (together with the identity $(r \cdot I) \cdot (s \cdot J) = (rs) \cdot (I \cdot J)$ for arbitrary $r, s \in R$) that $\gcd(I \cdot J) = \gcd(I) \cdot \gcd(J') \cdot \gcd(I' \cdot J')$. Thus it suffices to show that $\gcd(I' \cdot J') = 1$. In other words, we need to show that for every prime element p of R we have $p \nmid \gcd(I' \cdot J')$. So let p be a prime element of R. Since $\gcd(I') = 1$ we know that there exists an $x \in I'$ with $p \nmid x$ and since $\gcd(J') = 1$ we know that there exists a $y \in J'$ with $p \nmid y$. Since p is a prime element we know that $p \nmid x \cdot y$. Since $x \cdot y \in I' \cdot J'$ we see that $p \nmid \gcd(I' \cdot J')$. ■

The second lemma we need is more subtle:

Lemma 347.14. (Order–Torsion Submodule Lemma) Let R be a Noetherian UFD and let M be a finitely generated R-module. We set $\operatorname{rank}_R(M) := \dim_{Q(R)}(Q(R) \otimes_R M)$. Then for any $k \in \mathbb{N}_0$ we have

$$\gcd_R(E_k(M)) \doteq \begin{cases} 0, & \text{if } k < \operatorname{rank}_R(M), \\ \gcd_R(E_{k-\operatorname{rank}_R(M)})(\operatorname{Tor}_R(M)), & \text{if } k \geq \operatorname{rank}_R(M). \end{cases}$$

[3769] In practice it might be easier to check that one of the equivalent conditions of the Acyclicity Lemma 339.9 is satisfied.
[3770] Since we work over the field $Q(R)$ we use the K_1-Euclidean Domain Proposition 344.15 to make the identification $\mathrm{K}_1(Q(R)) =_{\mathrm{i}} Q(R)^\times$. Thus we will view $\tau(Q(R) \otimes_R C_*, 1_{Q(R)} \otimes \mathcal{C}_*)$ as an element of $Q(R)^\times$.
[3771] Note that it follows easily from the Torsion–Change-of-Basis Lemma 345.28 that the left-hand side is well-defined "up to $\doteq$".
[3772] Recall that the product of two ideals I and J is defined as the ideal that is generated by all products $x \cdot y$ with $x \in I$ and $y \in J$.

Proof. In the naive setting that $M \cong \mathrm{Tor}_R(M) \oplus R^d$ one can easily prove the promised statement by hand (in fact we will do so in Exercise 347.5). The general case requires more algebraic dexterity. A proof of the general case is provided in [**Hill2012**, Theorem 3.4] or alternatively in [**Bla1957**, Lemma 4.10]. ■

Proof of the Torsion–Order Proposition 347.12. Let R be a Noetherian UFD and let $(C_*, \mathcal{C}_*)$ be a based chain complex of length m of finitely generated free R-modules. We assume that the tensored chain complex $Q(R) \otimes_R C_*$ is acyclic. This implies in particular that the torsion $\tau(Q(R) \otimes_R C_*, 1_{Q(R)} \otimes \mathcal{C}_*) \in Q(R)^\times$ is defined.

For each $i \in \{0, \dots, m\}$ we make the following preparations:

(i) We set $d_i := \dim_{Q(R)}(Q(R) \otimes C_i) = \mathrm{rank}_R(C_i)$.
(ii) We denote by A_i the $(d_i \times d_{i-1})$-matrix over R representing $\partial_i \colon C_i \to C_{i-1}$ with respect to the given bases.
(iii) We can view the matrix A_i as a matrix over the field $Q(R)$. Thus it makes sense to use Definition 347.3 to set $\gamma_i := \mathrm{rank}_R(A_i) := \dim_{Q(R)}(\mathrm{im}(A_i))$.
(iv) As on page 6567 we see that the hypothesis that $Q(R) \otimes_R C_*$ is acyclic implies that we have the equality $\gamma_i + \gamma_{i-1} = d_i$.
(v) Note that it follows from (iii) and (iv) that $d_i - \gamma_i = \dim_{Q(R)}(\mathrm{coker}(A_i))$.
(vi) We write $I_i := E_{d_i - \gamma_i}(A_i)$.

Claim 1. For each $i \in \{0, \dots, m\}$ we have

$$\mathrm{ord}_R(\mathrm{Tor}_R(\mathrm{H}_{i-1}(C_*))) \doteq \gcd(I_i).$$

Proof. Let $i \in \{0, \dots, m\}$. We start out with two observations:

(a) We consider the exact sequence

$$C_i \xrightarrow{\partial_i} C_{i-1} \twoheadrightarrow \mathrm{coker}(\partial_i) \to 0.$$

Since each C_j is a free R-module we obtain that A_i is a presentation matrix for $\mathrm{coker}(\partial_i)$.

(b) We consider the following little commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_i & \xrightarrow{\mathrm{id}} & C_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} & & \downarrow \\
0 & \longrightarrow & Z_{i-1} & \hookrightarrow & C_{i-1} & \xrightarrow{\partial_{i-1}} & C_{i-1} \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & \mathrm{H}_{i-1}(C_*) & \longrightarrow & \mathrm{coker}(\partial_i) & \xrightarrow{\partial_{i-1}} & C_{i-2} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
\end{array}$$

Note that the top and bottom horizontal sequence are exact and that the vertical sequences are exact. It now follows from a mild diagram chase (see Exercise 347.6) that we have the following short exact sequence:

$$0 \to \mathrm{H}_{i-1}(C_*) \to \mathrm{coker}(\partial_i) \xrightarrow{\partial_{i-1}} C_{i-1}.$$

Since C_{i-1} is a free R-module we now see that $\mathrm{Tor}_R(\mathrm{H}_{i-1}(C)) = \mathrm{Tor}_R(\mathrm{coker}(\partial_i))$.

We now see that

$$\gcd(I_i) \overset{\text{definition of } I_i}{\doteq} \gcd(E_{d_i-\gamma_i}(A_i)) \overset{\text{follows from (a)}}{\doteq} \gcd_R(E_{d_i-\gamma_i}(\operatorname{coker}(\partial_i))) \overset{\text{by the Order–Torsion Submodule Lemma 347.14 and since } d_i-\gamma_i=\dim_{Q(R)}(\operatorname{coker}(A_i))}{\doteq} \gcd_R(E_0(\operatorname{Tor}_R(\operatorname{coker}(\partial_i)))) \overset{\text{follows from (b) and Definition 287.18}}{\doteq} \operatorname{ord}_R(\operatorname{Tor}_R(\mathrm{H}_{i-1}(C_*))).$$

⊞

Claim 2. For all $i = 0, 1, \ldots, m$ we have $\operatorname{Tor}_R(\mathrm{H}_i(C_*)) = \mathrm{H}_i(C_*)$.

Proof. By hypothesis we know that $Q(R) \otimes_R C_*$ is acyclic. The claim thus follows from the Acyclicity Lemma 339.9 (0)⇒(3). ⊞

Claim 3. We write $\tau(Q(R) \otimes_R C_*, 1_{Q(R)} \otimes \mathcal{C}_*) =: \frac{x}{y}$ for $x, y \in R$. We have the following equality of ideals in R:

$$x \cdot \prod_{i \text{ odd}} I_i \;=\; y \cdot \prod_{i \text{ even}} I_i.$$

Proof. It follows from elementary algebra that for each odd i we can choose a square submatrix S_i of A_i of size $\gamma_i = \operatorname{rank}_R(A_i)$ such that $\det(S_i) \neq 0$. As we saw in the proof of the Torsion–τ-Chain Theorem 347.2 and of the Non Degenerate τ-Chain Existence Lemma 347.5 this choice determines a τ-chain for C_*, in particular we also get submatrices S_i of A_i for all even i. We obtain the following inclusion in $Q(R)$:

$$\tau(Q(R) \otimes_R C_*) \cdot \prod_{i \text{ odd}} \det(S_i) \overset{\text{by the Torsion–}\tau\text{-Chain Theorem 347.2}}{=} \pm \prod_{i \text{ even}} \det(S_i) \overset{\text{since } S_i \text{ is a } (\gamma_i \times \gamma_i)\text{-matrix we see by definition that } \det(S_i) \in I_i = E_{d_i-\gamma_i}(A_i)}{\in} \prod_{i \text{ even}} I_i.$$

Since by Definition 287.16 we know that for each i the $\operatorname{rank}_R(A_i)$-minors of A_i generate I_i we obtain that

$$\tau(Q(R) \otimes_R C_*) \cdot \prod_{i \text{ odd}} I_i \;\subseteq\; \prod_{i \text{ even}} I_i.$$

We now apply the same logic to the shifted chain complex $Q(R) \otimes_R C_{*+1}$ and we obtain from the above discussion that we have the following inclusion in $Q(R)$

$$\underbrace{\tau(Q(R) \otimes_R C_{*+1})}_{=\tau(Q(R)\otimes_R C_*)^{-1}} \cdot \prod_{i \text{ even}} I_i \;\subseteq\; \prod_{i \text{ odd}} I_i.$$

These two inclusions give us the desired result. ⊞

We now see that

$$x \cdot \prod_{i \text{ odd}} \operatorname{ord}_R(\mathrm{H}_{i-1}(C_*)) \overset{\text{both equalities follow from the Order-Ideal-Multiplicative Lemma 347.13 together with the above Claim 1 and Claim 2}}{=} x \cdot \operatorname{ord}_R\left(\prod_{i \text{ odd}} I_i\right) \overset{\text{by Claim 3}}{=} y \cdot \operatorname{ord}_R\left(\prod_{i \text{ even}} I_i\right) = y \cdot \prod_{i \text{ even}} \operatorname{ord}_R(\mathrm{H}_{i-1}(C_*)).$$

Solving for $\frac{x}{y}$ we now see that we have the following equalities in $Q(R)^\times$:

$$\tau(Q(R)\otimes_R C_*, 1_{Q(R)}\otimes \mathcal{C}_*) \underset{\substack{\uparrow\\ \text{choice of}\\ x \text{ and } y}}{=} \frac{x}{y} \underset{\substack{\uparrow\\ \text{by the above}\\ \text{discussion}}}{=} \Big(\prod_{i \text{ odd}} \operatorname{ord}_R(\mathrm{H}_{i-1}(C_*))\Big)^{-1} \cdot \prod_{i \text{ even}} \operatorname{ord}_R(\mathrm{H}_{i-1}(C_*))$$
$$= \prod_{j\in\mathbb{N}_0} \operatorname{ord}_R(\mathrm{H}_j(C_*))^{(-1)^{j+1}}. \qquad \blacksquare$$

347.3. **Exercises for Chapter 347.**

Exercise 347.1. Let $\mathbb{F}$ be a field and let

$$A = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$$

be a block matrix over $\mathbb{F}$ where S is a $(d\times d)$-matrix. Show that if $\operatorname{rank}(A) = d$, if $\operatorname{rank}(S \;\; T) = d$ and if $\operatorname{rank}\begin{pmatrix} Q \\ S \end{pmatrix} = d$, then the $(d\times d)$-matrix S is invertible.

Exercise 347.2. Use the SES-Torsion Theorem 345.30 and induction on the length of the chain complex to give an alternative proof of the Torsion–τ-Chain Theorem 347.2.
Hint. Consider Exercise 345.3.

Exercise 347.3. Use the Torsion–τ-Chain Theorem 347.2 to give an alternative proof of the Torsion–Natural Theorem 346.11 for homomorphisms $\varphi\colon \mathbb{F}\to\mathbb{G}$ between fields.

Exercise 347.4. Use the Torsion–τ-Chain Theorem 347.2 to give an alternative proof of the Dual Chain Complex-Torsion Theorem 346.15 for homomorphisms $\varphi\colon \mathbb{F}\to\mathbb{G}$ between fields.

Exercise 347.5. Let R be a Noetherian UFD. Let T be a torsion R-module and let $d\in\mathbb{N}_0$. Show "by hand" that for any $k\in\mathbb{N}_0$ we have

$$\gcd_R(E_k(T\oplus R^d)) \doteq \begin{cases} 0, & \text{if } k<d,\\ \gcd_R(E_{k-d}(\operatorname{Tor}_R(T))), & \text{if } k\geq d. \end{cases}$$

Exercise 347.6. Let R be a ring. We consider the following commutative diagram of homomorphisms between R-left modules:

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0. \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

We assume that the three horizontal sequences are exact and that the two left vertical sequences are exact. Show that the right vertical sequence is also exact.
Remark. This exercise is almost a carbon copy of Exercise 145.15.

CHAPTER 348

Reidemeister torsion of CW-complexes

In this chapter we will use the algebraic machinery from the previous chapters to introduce Reidemeister torsion of a finite CW-complex that is equipped with a representation such that all twisted homology modules vanish.

We will also perform a few basic calculations and we will state several basic properties of Reidemeister torsion. In particular we will see that Reidemeister torsion has some uncanny formal similarities with the Euler characteristic.

This chapter is organized as follows:

- In Section 348.1 we will introduce Reidemeister torsion $\tau(X;\alpha)$ of a CW–complex X that is equipped with a representation $\alpha\colon \pi_1(X) \to \mathrm{GL}(d,R)$ such that the corresponding twisted homology modules $\mathrm{H}_k^{\mathrm{tw}}(X;{}_\alpha R^d)$ all vanish.
- In Section 348.2 we will provide a proof of the Twisted Torsion–Well-Defined Proposition 348.5.
- In Section 348.3 we will calculate Reidemeister torsions for some CW-complexes.
- In Section 348.4 we will discuss some basic formal properties of Reidemeister torsion.
- In Section 348.5 we will discuss some more interesting topological properties of Reidemeister torsion. In particular we will formulate and prove a Mayer–Vietoris-style decomposition formula for Reidemeister torsion of CW-complexes.
- In Section 348.6 we will provide exercises for Chapter 348.

In the next **??** we will see that Reidemeister torsion is an invariant under *simple* homotopy equivalences. We will also cite Chapman's Theorem which says that homeomorphisms are simple homotopy equivalences. This implies that Reidemeister torsion is a homeomorphism invariant.

348.1. **Reidemeister torsion of a CW–complex.** Before we can introduce Reidemeister torsion we should recall and introduce a few basic concepts:

Notation 348.1. Let X be a connected CW-complex and let $Y \subseteq X$ be a subcomplex. We denote by $\mathrm{Cell}_k(X \setminus Y)$ the set of all k-cells of X that are not contained in Y.

Next we remind the reader of the following lemma:

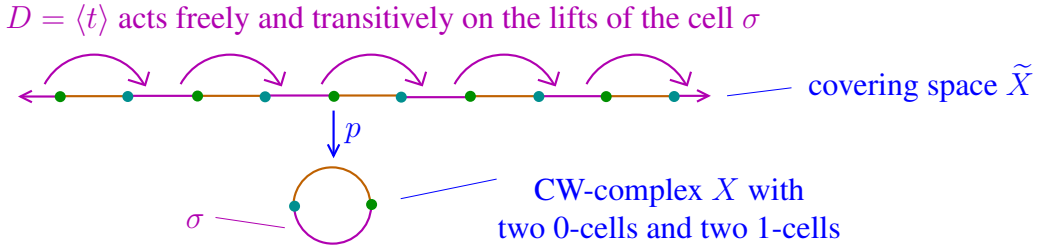
Lemma 348.2. (Regular Covering–Cellular Chain Complex Lemma) Let X be a connected CW-complex and let $Y \subseteq X$ be a subcomplex. Let $p\colon \widetilde{X} \to X$ be a regular covering. We write $\widetilde{Y} \coloneqq p^{-1}(Y)$. Finally we denote by $D \coloneqq \mathrm{Deck}(p\colon \widetilde{X} \to X)$ the corresponding deck transformation group.

(1) (a) Let $k \in \mathbb{N}_0$. For each $\sigma \in \mathrm{Cell}_k(X \setminus Y)$ we pick a single cell $\widetilde{\sigma} \subseteq \widetilde{X}$ with $p(\widetilde{\sigma}) = \sigma$. These cells $\widetilde{\sigma}$ (together with a choice of a characteristic map) form a basis for $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X},\widetilde{Y})$ as a free $\mathbb{Z}[D]$-left module.

(b) Each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X},\widetilde{Y})$ is a free $\mathbb{Z}[D]$-left module where the cardinality of the basis is given by the cardinality of the set $\mathrm{Cell}_k(X \setminus Y)$.

(2) The cellular boundary maps $\partial_k \colon \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X},\widetilde{Y}) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\widetilde{X},\widetilde{Y})$ are $\mathbb{Z}[D]$-left module homomorphisms.

Finally let $x_0 \in X$ and $x_0 \in p^{-1}(x_0)$. As we pointed out in Convention 319.13, we have an natural isomorphism $\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},x_0)) \xrightarrow{\cong} D$. Thus in the above discussion we replace the above deck transformation group D by $\pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},x_0))$.



Proof. The lemma is just a slight reformulation of the original Regular Covering–Cellular Chain Complex Lemma 327.10. ∎

We continue with one new piece of notation:

Notation 348.3. Let R be an IBN ring, let V be a finitely generated free R-left module and let $\varphi \colon V \to V$ be an automorphism of R-left modules. We pick a basis $\mathcal{V}$ for V and we set

$$[\varphi] \quad := \quad \underbrace{[\varphi]_{\mathcal{V}}^{\mathcal{V}}}_{\text{see Definition 345.5}} \quad \in \mathrm{K}_1(R).$$

It follows easily from the Representing Matrix Lemma 309.24 and the fact that $\mathrm{K}_1(R)$ is abelian that this definition does not depend on the choice of the basis $\mathcal{V}$ of V.[3773] Furthermore, using the Representing Matrix Lemma 309.24 one can also easily verify that the map

$$\begin{array}{ccc} \mathrm{Aut}_R(V) & \to & \mathrm{K}_1(R) \\ \varphi & \mapsto & [\varphi] \end{array}$$

is actually a group homomorphism.

We are now in a position to define Reidemeister torsion of a CW-complex:

Definition 348.4. We assume that we are given the following *data*:

(a) Let (X,x_0) be a pointed connected finite CW–complex and let $Y \subseteq X$ be a subcomplex.

(b) Let R be an IBN ring and let V be a free R-left module of finite rank. Furthermore let $\alpha \colon \pi_1(X,x_0) \to \mathrm{Aut}_{R\text{-left}}(V)$ be a representation.

Furthermore we make the following *choices*:

(c) We pick a basis $e_1,\ldots,e_d$ for V.

[3773] Indeed: Let $\mathcal{V}$ and $\mathcal{W}$ be two bases of V. Then

$$[\varphi]_{\mathcal{W}}^{\mathcal{W}} \;=\; [\mathrm{id}_V \circ \varphi \circ \mathrm{id}_V]_{\mathcal{V}}^{\mathcal{W}} \underset{\uparrow}{=} [\mathrm{id}_V]_{\mathcal{V}}^{\mathcal{W}} \cdot [\varphi]_{\mathcal{V}}^{\mathcal{V}} \cdot [\mathrm{id}_V]_{\mathcal{W}}^{\mathcal{V}} \underset{\uparrow}{=} [\mathrm{id}_V]_{\mathcal{V}}^{\mathcal{W}} \cdot [\varphi]_{\mathcal{V}}^{\mathcal{V}} \cdot ([\mathrm{id}_V]_{\mathcal{V}}^{\mathcal{W}})^{-1}.$$

follows from the Representing Matrix Lemma 309.24 (1) and (2)

Since $\mathrm{K}_1(R)$ is abelian we see that $[\varphi]_{\mathcal{W}}^{\mathcal{W}} = [\varphi]_{\mathcal{V}}^{\mathcal{V}} \in \mathrm{K}_1(R)$.

(d) We pick a normal subgroup Γ of $\pi_1(X,x_0)$ with $\Gamma \subseteq \ker(\alpha)$. Furthermore we pick a regular covering $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ with $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = \Gamma$.[3774]

(e) For each $k \in \mathbb{N}_0$ and each $\sigma \in \mathrm{Cell}_k(X\backslash Y)$ we do the following: We pick a characteristic map for σ and we pick a lift $\widetilde{\sigma}$ to the covering $\widetilde{X}$.

We now do the following:

- We write $\pi := \pi_1(X,x_0)$.
- We write $\widetilde{Y} := p^{-1}(Y)$.
- Since $\Gamma \subseteq \ker(\alpha)$ we obtain an induced group homomorphism $\pi/\Gamma \to \mathrm{Aut}_{R\text{-left}}(V)$ which we again denote by α.
- As in Notation 310.6 we use the homomorphism α to define the $(R,\mathbb{Z}[\pi/\Gamma])$-left left module ${}_{\alpha}V$. Furthermore as in Notation 320.7 we use the involution on $\mathbb{Z}[\pi]$ to define the $(R,\mathbb{Z}[\pi/\Gamma])$-bimodule $\overline{{}_{\alpha}V}$.
- By the Regular Covering–Cellular Chain Complex Lemma 348.2 we know that the choices of (e) give us a basis of each $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X},\widetilde{Y})$ as a free $\mathbb{Z}[\pi/\Gamma]$-left module.
- It follows easily from the Tensor–Direct Sum Lemma 312.15 and the Tensor-with-Base Ring Lemma 312.10 that each $\overline{{}_{\alpha}V} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X},\widetilde{Y})$ is a free R-left module where a basis is given by $\{e_j \otimes \widetilde{\sigma}\}_{\substack{j\in\{1,\dots,d\}\\ \sigma\in\mathrm{Cell}_k(X\setminus Y)}}$.

If the chain complex $(\overline{{}_{\alpha}V} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X},\widetilde{Y}),\partial_*)$ is acyclic[3775], then we define the Reidemeister torsion of $(X,Y;\alpha)$ as[3776] [3777]

$$\tau(X,Y;\alpha) \;:=\; \tau\Big(\overline{{}_{\alpha}V}\otimes_{\mathbb{Z}[\pi/\Gamma]}\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{Y}),\{e_j\otimes\widetilde{\sigma}\}_{\substack{j\in\{1,\dots,d\}\\ \sigma\in\mathrm{Cell}_k(X\setminus Y)}}\Big) \;\in\; \overline{\mathrm{K}}_1(R)/\alpha(\pi).$$

Not surprisingly, for $Y=\varnothing$ we write $\tau(X;\alpha) := \tau(X;\varnothing;\alpha)$.

The following proposition shows that the Reidemeister torsion $\tau(X,Y;\alpha)$ of Definition 348.4 does not depend on the choices (c), (d) and (e):

Proposition 348.5. (Twisted Torsion–Well-Defined Proposition) Let (X,x_0) be a pointed connected finite CW–complex and let $Y \subseteq X$ be a subcomplex. In the following we write $\pi := \pi_1(X,x_0)$. Furthermore let R be an IBN ring and let V be a finitely generated free R-left module. Finally let $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(V)$ be a representation. We assume that $\mathrm{H}_*^{\mathrm{tw}}(X,Y;{}_{\alpha}V) \equiv 0$. We do the following:

(c) We pick a basis $e_1,\dots,e_d$ for V.

(d) We pick a subgroup $\Gamma \subseteq \ker(\alpha)$ and we pick a regular covering $p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ with $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = \Gamma$.

[3774] If we are out of ideas we can always take $\Gamma = \{e\}$ together with the natural universal covering $\widetilde{X}_{x_0} \to X$ from Definition 319.10.

[3775] Note that it follows from the Replace Universal Covering Proposition 329.1 together with the Singular-Cellular Twisted Isomorphism Theorem 328.1 that the chain complex $\overline{{}_{\alpha}V} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X},\widetilde{Y})$ is acyclic if and only if the twisted singular homology modules $\mathrm{H}_*(\overline{{}_{\alpha}V} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*(\widetilde{X}_{x_0},\widetilde{Y})) =: \mathrm{H}_*^{\mathrm{tw}}(X,Y;{}_{\alpha}V)$ from Definition 320.11 all vanish.

[3776] In the Twisted Torsion–Well-Defined Proposition 348.5 we will see that this invariant does indeed only depend on the data (X,Y) and α, i.e. it does not depend on the choices made in (c), (d) and (e).

[3777] Note that we use Notation 348.3 so each $g \in \pi$ defines an element $\alpha(g) \in \overline{\mathrm{K}}_1(R)$ and so what we obtain the subgroup $\alpha(\pi)$ of $\overline{\mathrm{K}}_1(R)$.

(e) For each cell $\sigma \in \mathrm{Cell}(X \setminus Y)$ we pick a characteristic map for σ and we pick a lift $\widetilde{\sigma}$ to the covering $\widetilde{X}$.

Then the torsion

$$\tau(X, Y; \alpha) \;=\; \tau\Big({}_{\alpha}\overline{V} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}, \widetilde{Y}), \{e_j \otimes \widetilde{\sigma}\}_{\substack{j \in \{1,\dots,d\} \\ \sigma \in \mathrm{Cell}_k(X \setminus Y)}}\Big)$$

from Definition 348.4, viewed as an element in $\overline{\mathrm{K}}_1(R)/\alpha(\pi)$ is independent of the choices (c), (d) and (e), i.e. it is an invariant of the CW-complex pair (X, Y) together with the representation α.

The proof of the Twisted Torsion–Well-Defined Proposition 348.5 will require all of the next Section 348.2. In the subsequent Section 348.3 we will perform a few first calculations of Reidemeister torsion.

348.2. **Proof of the Twisted Torsion–Well-Defined Proposition 348.5.** In the proof of the Twisted Torsion–Well-Defined Proposition 348.5 we will need the following variation on the Twisted Homology-χ-Proposition 331.12, which might also be of independent interest:

Lemma 348.6. (Acyclic–Vanishing-χ-Lemma) Let R be an IBN ring. Furthermore let $0 \to C_m \to \cdots \to C_0 \to 0$ be a chain complex of finitely generated free R-left modules. If the chain complex is acyclic, then

$$\sum_{i=0}^{m} (-1)^i \cdot \mathrm{rank}_R(C_i) \;=\; 0.$$

Proof of the Acyclic–Vanishing-χ-Lemma 348.6. [3778] We make the following preparations:

(a) As always we write $B_i := \mathrm{im}(\partial_{i+1} \colon C_{i+1} \to C_i)$.

(b) Since C_* is acyclic we know (essentially by definition) that for each i we have a short exact sequence $0 \to B_i \hookrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$.

(c) By the Image–Stably Free Lemma 345.20 we know that each B_j is a stably free R-left module.

(d) By Definition 345.21 we know that we can associate to each stable R-left module M a well-defined rank which we denote by $\mathrm{rank}_R(M)$.

(e) By the Short Exact Sequence–Stably Free 345.19 we know that a short exact sequence of stably free R-left modules splits. It now follows easily that the rank is additive, in the sense that for any short exact sequence $0 \to A \to B \to C \to 0$ of stably free R-left modules we have $\mathrm{rank}_R(B) = \mathrm{rank}_R(A) + \mathrm{rank}_R(C)$.

We now see that

$$\sum_{i=0}^{m} (-1)^i \cdot \mathrm{rank}_R(C_i) \;\underset{\substack{\uparrow \\ \text{follows from (b) and (e)}}}{=}\; \sum_{i=0}^{m} (-1)^i \cdot (\mathrm{rank}_R(B_i) - \mathrm{rank}_R(B_{i-1})) \;\underset{\substack{\uparrow \\ \text{all summands cancel}}}{=}\; 0.$$

■

We turn to the second algebraic ingredient for the proof of the Twisted Torsion–Well-Defined Proposition 348.5:

[3778] Note that this lemma gives a solution to Exercise 345.2.

Lemma 348.7. (Tensor Product–Matrix Lemma) Let R be an IBN ring, let π be a group and let V be a free R-left module of rank d. Next let W be a free $\mathbb{Z}[\pi]$-left module of rank e. Furthermore let $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(V)$ be a representation. We consider the R-left module ${}_{\alpha}\overline{V} \otimes_{\mathbb{Z}[\pi]} W$. Finally let $\mathcal{V}, \mathcal{V}'$ be bases of V as an R-left module and let $\mathcal{W}, \mathcal{W}'$ be bases of W as a $\mathbb{Z}[\pi]$-left module. Then using Definition 345.7 we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$[(\mathcal{V} \otimes \mathcal{W})/(\mathcal{V}' \otimes \mathcal{W}')] \;=\; [\mathcal{V}/\mathcal{V}']^e \cdot [\alpha([\mathcal{W}/\mathcal{W}'])]$$

where $\alpha([\mathcal{W}/\mathcal{W}'])$ is the $(de \times de)$-matrix over R that we obtain as follows:

(i) We first consider the $(e \times e)$-base change matrix $[\mathcal{W}/\mathcal{W}']$ over $\mathbb{Z}[\pi]$.

(ii) Then, as in Notation 329.6, we consider the $(de \times de)$-matrix over R that we obtain by applying $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(V)$ to the entries of the matrix obtained in (i).

Proof of the Tensor Product–Matrix Lemma 348.7. The lemma follows easily from the definitions. We leave all details to the reader. ■

Proof of the Twisted Torsion–Well-Defined Proposition 348.5. In the following let (X, x_0) be a pointed connected finite CW–complex, let R be an IBN ring, let V be a free R-left module and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{Aut}_{R\text{-left}}(V)$ be a representation. (To unburden the notation we assume throughout the proof that $Y = \varnothing$.) We need to show that the definition of Reidemeister torsion is independent of the following:

(c) The choice of basis of V.

(d) The choice of the subgroup Γ and corresponding covering $\widetilde{X} \to X$.

(e) The choice of lifts of the cells and the choice of the characteristic maps.

We will do so in the following three claims:

Claim C. The torsion[3779] $\tau\big({}_{\alpha}\overline{V} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}), \{e_j \otimes \widetilde{\sigma}^k_i\}\big)$ does not depend on the choice of the basis $e_1, \dots, e_d$ of V.

Proof. Suppose we are given two bases $\mathcal{E} = \{e_1, \dots, e_d\}$ and $\mathcal{F} = \{f_1, \dots, f_d\}$ for V. We now see that

$$\tau\big({}_{\alpha}\overline{V} \underset{\mathbb{Z}[\pi/\Gamma]}{\otimes} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}), \{f_j \otimes \widetilde{\sigma}^k_i\}\big) \underset{\uparrow}{=} \tau\big({}_{\alpha}\overline{V} \underset{\mathbb{Z}[\pi/\Gamma]}{\otimes} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}), \{e_j \otimes \widetilde{\sigma}^k_i\}\big) \cdot \dot{\prod_{k \in \mathbb{N}_0}} [\mathcal{F}/\mathcal{E}]^{(-1)^k \cdot \#\mathrm{Cell}_k(X)}$$

follows easily from the Torsion Change-of-Basis Lemma 345.28 together with the Tensor Product–Matrix Lemma 348.7

$$\underset{\uparrow}{=} \tau\big({}_{\alpha}\overline{V} \underset{\mathbb{Z}[\pi/\Gamma]}{\otimes} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}), \{e_j \otimes \widetilde{\sigma}^k_i\}\big) \cdot [\mathcal{F}/\mathcal{E}]^{\chi(X)}$$

follows from elementary algebra and Definition 156.1 of the Euler characteristic

$$\underset{\uparrow}{=} \tau\big({}_{\alpha}\overline{V} \underset{\mathbb{Z}[\pi/\Gamma]}{\otimes} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}), \{e_j \otimes \widetilde{\sigma}^k_i\}\big).$$

we have $\chi(X) = 0$, this follows from the hypothesis that the chain complex is acyclic together with the Acyclic–Vanishing-χ-Lemma 348.6 ⊞

Claim D. The torsion does not depend on the choice of Γ and the regular covering

[3779] The notation $\{e_j \otimes \widetilde{\sigma}^k_i\}$ is the hopefully acceptable shorthand notation for $\{e_j \otimes \widetilde{\sigma}\}_{\substack{j \in \{1,\dots,d\} \\ \sigma \in \mathrm{Cell}_k(X)}}$.

Proof. It suffices to show that the natural universal covering gives rise to the same invariant as any other covering. We introduce the following notation:

- We denote by $p\colon \widetilde{X}_{x_0} \to X$ the universal covering.
- We set $\pi := \pi_1(X, x_0)$. Let Γ be a normal subgroup of π with $\Gamma \subseteq \ker(\alpha)$ and let $q\colon (\widehat{X}, \widehat{x}_0) \to (X, x_0)$ be a regular covering with $q_*(\pi_1(\widehat{X}, \widehat{x}_0)) = \Gamma$.
- It follows from the Covering–Covering Theorem 319.2 that there exists a unique covering $r\colon (\widetilde{X}_{x_0}, \widetilde{x}_0) \to (\widehat{X}, \widehat{x}_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X}_{x_0} & \xrightarrow{\quad r \quad} & \widehat{X}. \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

- We pick characteristic maps of the cells and we pick lifts of the cells to $\widetilde{X}_{x_0}$. The images of these lifts under the map $r\colon \widetilde{X}_{x_0} \to \widehat{X}$ then also give us lifts of the cells to $\widehat{X}$.

As in the Replace Universal Covering Proposition 329.1 we see that the maps

$$\begin{array}{rcl} \mathrm{id} \otimes r_*\colon {}_{\alpha}\overline{V} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}_{x_0}) & \to & {}_{\alpha}\overline{V} \otimes_{\mathbb{Z}[\pi/\Gamma]} \mathrm{C}_k^{\mathrm{CW}}(\widehat{X}) \\ v \otimes (\sigma\colon \Delta^k \to \widetilde{X}_{x_0}) & \mapsto & v \otimes (r \circ \sigma\colon \Delta^k \to \widehat{X}) \end{array}$$

form an isomorphism of R-left chain complexes. Basically by construction these maps are in fact an isomorphism of *based* of R-left chain complexes. In particular we get induced isomorphisms of homology modules. Furthermore, if these vanish, then we obtain from the Torsion–Isomorphism Lemma 345.27 that the torsions agree. ⊞

Claim E.

(i) Torsion does not depend on the choice of the characteristic maps of the cells of X.
(ii) Torsion does not depend on the choice of the lifts of the cells of X.

Proof. If we pick a different characteristic map of a cell σ or if we pick a different lift of the cell σ, then we replace the basis vectors $\{e_i \otimes \widetilde{\sigma}\}_{i=1,\ldots,n}$ by $\{\epsilon \cdot e_i \otimes g \cdot \widetilde{\sigma}\}_{i=1,\ldots,n}$ for some $\epsilon \in \{-1, 1\}$ and for some $g \in \pi/\Gamma$, and all other basis vectors stay unchanged. It follows easily from the Torsion–Change-of-Basis Lemma 345.28 that the corresponding Reidemeister torsion, viewed as an element in $\overline{\mathrm{K}}_1(R)/\alpha(\pi) = \mathrm{K}_1(R)/\pm\alpha(\pi)$ is unchanged. ■

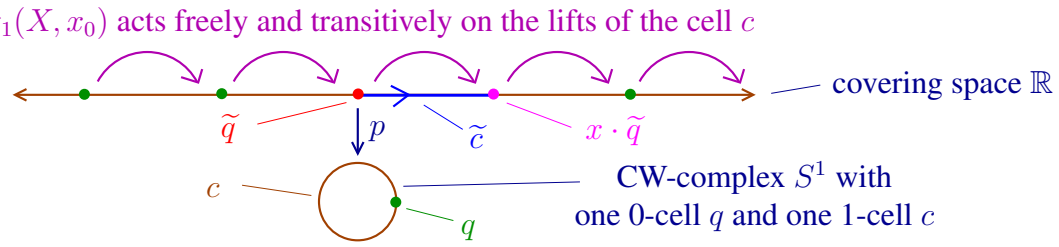
348.3. **Examples of Reidemeister torsion.** We start out with the simplest calculation of Reidemeister torsion:

Lemma 348.8. (Circle Torsion Lemma) We consider the circle S^1 with the standard CW-complex structure given by one 0-cell and one 1-cell. We make the usual identification $\pi_1(S^1, 1) =_{\mathrm{i}} \langle x \rangle$. Let $\alpha\colon \pi_1(S^1) \to \mathrm{GL}(1, \mathbb{F})$ be a one-dimensional representation over a field $\mathbb{F}$. Then $\mathrm{H}_*^{\mathrm{tw}}(S^1; {}_{\alpha}\mathbb{F}) = 0$ if and only if α is non-trivial. If that is the case, then we have the following equality in $\overline{\mathrm{K}}_1(\mathbb{F})/\alpha(\pi_1(S^1, 1)) =_{\mathrm{i}} \mathbb{F}^\times/\pm\langle\alpha(x)\rangle$:

$$\tau(S^1; \alpha) \;=\; (1 - \alpha(x))^{-1}.$$

Proof. We equip S^1 with the usual CW-structure with one 0-cell $q = \{1\}$ and one 1-cell $c = S^1$. We denote by $p\colon \mathbb{R} \to S^1$ the universal covering. In the following we summarize the calculation of the twisted chain complex ${}_{\alpha}\overline{\mathbb{F}} \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathrm{C}_*^{\mathrm{CW}}(\mathbb{R})$ from the Circle–Twisted Cellular Chain Complex Lemma 329.10:

First of all we pick the obvious lift $\widetilde{q} = \{0\}$ of q and we pick the lift $\widetilde{c} = [0,1]$ of c to $\mathbb{R}$. Then $\partial_1^{\mathrm{CW}}(\widetilde{c}) = x \cdot \widetilde{q} - \widetilde{q}$.



As in the Twisted-H_*-of-S^1-Proposition 327.13 we see that with respect to the bases $\widetilde{c}$ and $\widetilde{q}$ the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\mathbb{R})$ is thus given by

$$0 \;\to\; \mathbb{Z}[x^{\pm 1}] \;\xrightarrow{(x-1)}\; \mathbb{Z}[x^{\pm 1}] \;\to\; 0.$$

If $\alpha(x) \neq 1$, then we see that

$$\begin{aligned}
\tau(S^1;\alpha) \;&\underset{\text{by Definition 348.4 and the above calculation of } \mathrm{C}_*^{\mathrm{CW}}(\mathbb{R})}{=} \tau\big(0 \to {}_{\alpha}\overline{\mathbb{F}} \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathbb{Z}[x^{\pm 1}] \xrightarrow{\mathrm{id}\otimes(1-x)} {}_{\alpha}\overline{\mathbb{F}} \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathbb{Z}[x^{\pm 1}] \to 0\big) \\
&\underset{\text{by the Free Module–Tensor-with-Ring Lemma 312.18}}{=} \tau\big(0 \to \mathbb{F} \xrightarrow{(1-\alpha(x))} \mathbb{F} \to 0\big) \;\underset{\text{by the Elementary Torsion Lemma 345.17}}{=}\; (1-\alpha(x))^{-1}.
\end{aligned}$$

Finally, if $\alpha(x) = 1$, then the above chain complex is clearly not acyclic. ■

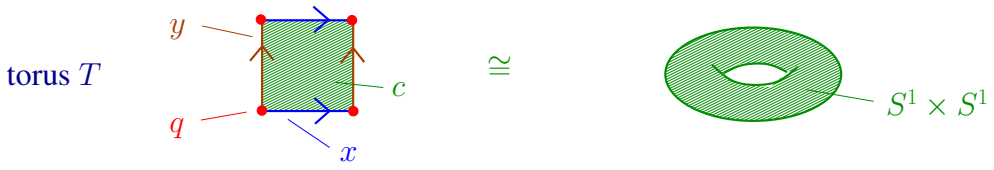
We move onto our second example which, surprisingly, says that Reidemeister torsion of a 2-dimensional torus is always trivial:

Lemma 348.9. (2-Torus Torsion Lemma) We consider $S^1 \times S^1$ with the standard CW-structure from page 2625. Let $\alpha\colon \pi_1(S^1 \times S^1) \to \mathrm{GL}(1,\mathbb{F})$ be a one-dimensional representation over a field $\mathbb{F}$. Then $\mathrm{H}_*^{\mathrm{tw}}(S^1 \times S^1; {}_{\alpha}\mathbb{F}) = 0$ if and only if α is non-trivial. If this is the case, then we have the following equality in $\overline{\mathrm{K}}_1(\mathbb{F})/\alpha(\pi_1(S^1 \times S^1)) =_{\mathrm{i}} \mathbb{F}^\times/\pm\alpha(\pi_1(S^1 \times S^1))$:

$$\tau(S^1 \times S^1;\alpha) \;=\; 1.$$

Remark. The Circle Torsion Lemma 348.8 together with the 2-Torus Torsion Lemma 348.9 shows that there is no naive product formula for torsion. □

Proof. We equip $T \coloneqq S^1 \times S^1$ with the usual CW-structure with one 0-cell q, two 1-cells x and y and one 2-cell c.



We denote by $\widetilde{T}$ the universal covering of T. In the following we will summarize the calculation of the twisted chain complex ${}_{\alpha}\overline{\mathbb{F}} \otimes_{\mathbb{Z}[\pi_1(S^1\times S^1)]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{T})$ from the Torus–Twisted Cellular Chain Complex Lemma 329.11:

First of all, we showed that for an appropriate choice of lifts of q, x, y and c to the universal covering the based chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{T})$ is isomorphic to

$$0 \;\to\; \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \xrightarrow{\begin{pmatrix} y-1 & 1-x \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{1\times 2} \xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \;\to\; 0.$$

Using the Free Module–Tensor-with-Ring Lemma 312.18 we see that the based cellular chain complex ${}_{\alpha}\overline{\mathbb{F}} \otimes_{\mathbb{Z}[\pi_1(T)]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{T})$ is isomorphic to

$$0 \;\to\; \mathbb{F} \xrightarrow{\begin{pmatrix} \alpha(y)-1 & 1-\alpha(x) \end{pmatrix}} \mathbb{F}^{1\times 2} \xrightarrow{\begin{pmatrix} 1-\alpha(x) \\ 1-\alpha(y) \end{pmatrix}} \mathbb{F} \;\to\; 0.$$

If α is trivial, then this chain complex is not acyclic. Now suppose that α is non-trivial. It follows immediately from the discussion on page 6563 that in this case the torsion is trivial. ∎

348.4. **Formal properties of Reidemeister torsion.** Before we discuss properties of Reidemeister torsion it is convenient to introduce an ever-so-slight extension of Definition 348.4. To formulate this extension we need the following notation:

Notation 348.10.

(1) Let R be an IBN ring. We consider the set $\overline{\mathrm{K}}_1(R) \sqcup \{0\}$ where $\star$ is one extra element. We endow this set with a multiplication that is the usual group structure on $\overline{\mathrm{K}}_1(R)$ and that has the property that the product of $\star$ with anything else is again $\star$.

(2) Given a ring homomorphism $\varphi\colon R \to S$ between IBN rings we get an obvious induced map $\overline{\mathrm{K}}_1(R) \sqcup \{0\} \to \overline{\mathrm{K}}_1(S) \sqcup \{0\}$ which sends $\star$ to $\star$ and which preserves multiplication.

Now we can formulate the promised extension of Definition 348.4:

Definition 348.11. We assume that we are given the following:

(a) Let (X, x_0) be a pointed connected finite CW–complex and let $Y \subseteq X$ be a subcomplex. We set $\pi := \pi_1(X, x_0)$.

(b) Let R be an IBN ring and let V be a free R-left module of finite rank. Furthermore let $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(V)$ be a representation.

We now do the following: If some $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\alpha}V)$ is non-zero, then we set

$$\tau(X, Y; \alpha) \;:=\; 0.$$

Otherwise we set, as in Definition 348.11:

$$\tau(X, Y; \alpha) \;:=\; \tau\Big({}_{\alpha}\overline{V} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{Y}), \{e_j \otimes \widetilde{\sigma}\}_{\substack{j \in \{1,\dots,d\} \\ \sigma \in \mathrm{Cell}_k(X \setminus Y)}}\Big) \;\in\; \overline{\mathrm{K}}_1(R)/\alpha(\pi).$$

For $Y = \varnothing$ we write $\tau(X; \alpha) := \tau(X; \varnothing; \alpha) \in \overline{\mathrm{K}}_1(R) \sqcup \{0\}$.

Now we turn to the two lemmas of this section. First of all we have the following utterly predictable lemma:

Lemma 348.12. (Torsion–Isomorphic Representation Lemma) Let (X, x_0) be a pointed finite connected CW-complex and let $Y \subseteq X$ be a subcomplex. We write $\pi := \pi_1(X, x_0)$. Furthermore let R be an IBN ring, let V and W be finitely generated free R-left modules and let $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(V)$ and $\beta\colon \pi \to \mathrm{Aut}_{R\text{-left}}(W)$ be two representations. If

α and β are isomorphic in the sense of Definition 310.4, then[3780]

$$\tau(X,Y;\alpha) \;=\; \tau(X,Y;\beta) \;\in\; \overline{\mathrm{K}}_1(R)/\alpha(\pi) \sqcup \{0\}.$$

Example.

(1) It follows from the Torsion–Isomorphic Representation Lemma 348.12 that when it comes to Reidemeister torsions it suffices, for all intents and purposes, to consider representations where $V = R^d$, i.e. it suffices to consider representations of the form $\pi_1(X, x_0) \to \mathrm{GL}(d, R)$.

(2) Let X be a finite connected non-empty CW-complex, let $Y \subseteq X$ be a subcomplex and let $\alpha, \beta\colon \pi_1(X, x_0) \to \mathrm{GL}(d, R)$ be two representations over an IBN ring R. If the representations α and β are conjugate, i.e. if there exists an $A \in \mathrm{GL}(d, R)$ such that $\alpha(g) = A \cdot \beta(g) \cdot A^{-1}$ for all $g \in \pi_1(X, x_0)$, then we know by the Free Module–Representation Classification Lemma 310.15 and the corresponding representations are isomorphic. It now follows from the Torsion–Isomorphic Representation Lemma 348.12 that the corresponding Reidemeister torsions agree. □

Proof. Let $\alpha\colon \pi \to \mathrm{Aut}_{R\text{-left}}(V)$ and $\beta\colon \pi \to \mathrm{Aut}_{R\text{-left}}(W)$ be two representations and let $\Theta\colon V \to W$ be an isomorphism of R-left modules such that for every $g \in \pi$ we have $\beta(g) = \Theta \circ \alpha(g) \circ \Theta^{-1}\colon W \to W$.

First note that the map $\Theta\colon {}_\alpha V \to {}_\beta W$ is clearly an isomorphism of $(R, \mathbb{Z}[\pi])$-left left modules. It therefore follows immediately from the Functorial-in-Coefficients Lemma 321.13 that $\mathrm{H}^{\mathrm{tw}}_*(X, Y; {}_\alpha V) = 0$ if and only if $\mathrm{H}^{\mathrm{tw}}_*(X, Y; {}_\beta W) = 0$. In particular $\tau(X, Y; \alpha) = 0$ if and only if $\tau(X, Y; \beta) = \mathit{star}$.

We now assume that all these twisted homology modules vanish. We do the following:

(c) We pick a basis $v_1, \ldots, v_d$ of V and we equip W with the basis $\Theta(v_1), \ldots, \Theta(v_d)$.

(d) We consider the natural universal covering $p\colon \widetilde{X}_{x_0} \to X$ and we write $\widetilde{Y} := p^{-1}(Y)$.

(e) For each cell $\sigma \in \mathrm{Cell}(X \setminus Y)$ we pick a characteristic map for σ and we pick a lift $\widetilde{\sigma}$ to the covering $\widetilde{X}$.

One can now easily verify that (for appropriate choices of bases) the map

$$\begin{array}{rcl} \Theta \otimes \mathrm{id}\colon \overline{{}_\alpha V} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}_{x_0}, \widetilde{Y}) & \to & \overline{{}_\beta W} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}_{x_0}, \widetilde{Y}) \\ v \otimes \mu & \mapsto & \Theta(v) \otimes \mu \end{array}$$

is an isomorphism of based R-left modules. By the Torsion–Isomorphism Lemma 345.27 this implies that the torsions agree. ■

We turn to the last lemma of this section, which shows in particular that Reidemeister torsion is natural in a suitable sense:

Lemma 348.13. (Naturality-of-CW-Torsion Lemma) Let (X, x_0) be a finite pointed connected CW-complex and let $Y \subseteq X$ be a subcomplex. Furthermore let R be an IBN ring and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(d, R)$ be a representation. Finally let $\varphi\colon R \to S$ be a

[3779] In plain English the two representations are isomorphic if there exists an isomorphism $\Theta\colon V \to W$ of R-left modules such that for any $g \in \pi$ we have $\beta(g) = \Theta \circ \alpha(g) \circ \Theta^{-1}$ as automorphisms of W.

[3780] This statement includes the statement that $\mathrm{H}^{\mathrm{tw}}_*(X, Y; {}_\alpha V) \neq 0$ if and only if $\mathrm{H}^{\mathrm{tw}}_*(X, Y; {}_\beta W) \neq 0$. In fact, in this case we know by the Functorial-in-Coefficients Lemma 321.13 a stronger statement holds: both modules are in fact isomorphic.

ring homomorphism to another IBN ring. Note that we obtain an induced representation $\varphi_*(\alpha)\colon \pi_1(X,x_0) \to \mathrm{GL}(d,S)$. If $\tau(X,Y;\alpha) \neq 0$, then we have the following equality in $\overline{\mathrm{K}}_1(S)/\varphi(\alpha(\pi_1(X,x_0))) \sqcup \{0\}$:[3781]

$$\tau(X,Y;\varphi_*(\alpha)) \;=\; \varphi(\tau(X,Y;\alpha)).$$

Proof. We write $\pi := \pi_1(X,x_0)$. We denote by $p\colon \widetilde{X}_{x_0} \to X$ the natural universal covering. We write $\widetilde{Y} := p^{-1}(Y)$. Note that from the Tensor Commutative-Associative Lemma 312.11 together with the Free Module–Tensor-with-Ring Lemma 312.18 we obtain the following isomorphism

$$\begin{array}{rcl} S \otimes_R (\overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}_{x_0},\widetilde{Y})) & \to & \overline{{}_{\varphi_*(\alpha)} S^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}^{\mathrm{CW}}_*(\widetilde{X}_{x_0},\widetilde{Y})) \\ s \otimes (v \otimes \sigma) & \mapsto & (s \cdot v) \otimes \sigma. \end{array}$$

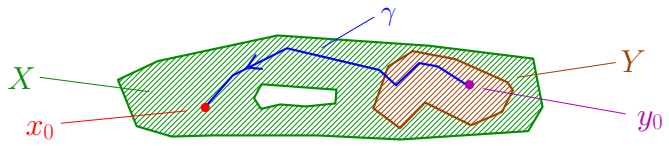
of chain-complexes of S-left modules. In fact this is clearly an isomorphism of based chain-complexes of S-left modules. Thus the lemma follows from the Torsion–Natural Theorem 346.11. ■

348.5. **Topological properties of Reidemeister torsion.** We start out by introducing the following lemma:

Definition 348.14. Let (X,x_0) be a pointed path-connected space, let R be a ring, let V be an R-left module and let $\alpha\colon \pi_1(X,x_0) \to \mathrm{Aut}_{R\text{-left}}(V)$ be a representation. Next let $Y \subseteq X$ be a subspace and let $y_0 \in Y$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from y_0 to x_0. We get an induced representation

$$\pi_1(Y,y_0) \xrightarrow{i_*} \pi_1(X,y_0) \xrightarrow{\gamma^*} \pi_1(X,x_0) \xrightarrow{\alpha} \mathrm{Aut}_{R\text{-left}}(V)$$

where the first homomorphism is induced by the inclusion $i\colon Y \to X$ and where the second homomorphism is induced by the path γ, as in the Change-of-Base Point Proposition 88.14. We denote the induced representation $\pi_1(Y,y_0) \to \pi_1(X,x_0) \to \mathrm{Aut}_{R\text{-left}}(V)$ by $\gamma^*(\alpha)$.



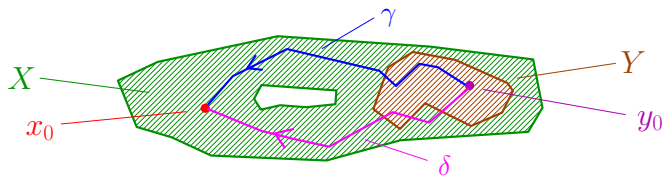
Lemma 348.15. (Torsion–Base Point Lemma) We continue with the notation from Definition 348.14, except that now we demand that X is a CW-complex, that $Y \subseteq X$ is a subcomplex and that R is an IBN ring. For any two paths $\gamma, \delta\colon [0,1] \to X$ from y_0 to x_0 and any subcomplex A of Y we have the following equality in $\overline{\mathrm{K}}_1(R) \sqcup \{0\}$:

$$\tau(Y,A,\gamma^*\alpha) \;=\; \tau(Y,A,\delta^*\alpha).$$

Remark. The Torsion–Base Point Lemma 348.15 implies that we do not have to worry at all about base points and paths connecting base points. This feature is even quite useful for the case $Y = X$. □

Proof. It follows easily from the Change-of-Base Point Proposition 88.14 that $\gamma^*\alpha$ and $\delta^*\alpha$ are conjugate by $[\overline{\gamma} * \delta] \in \pi_1(Y,y_0)$.
The lemma now follows from the Torsion–Isomorphic Representation Lemma 348.12. ■

[3781] This statement includes the statement that if $\mathrm{H}^{\mathrm{tw}}_*(X,Y;{}_\alpha R^d) \equiv 0$, then $\mathrm{H}^{\mathrm{tw}}_*(X,Y;{}_{\varphi_*(\alpha)} S^d) \equiv 0$.



Convention 348.16. We continue with the notation from Definition 348.14. We use the Torsion–Base Point Lemma 348.15 as an excuse to drop the path γ from the notation, i.e. for the representation

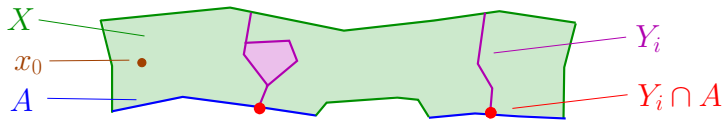
$$\pi_1(Y,y_0) \xrightarrow{i_*} \pi_1(X,y_0) \xrightarrow{\gamma^*} \pi_1(X,x_0) \xrightarrow{\alpha} \mathrm{Aut}_{R\text{-left}}(V)$$

we will usually just write α again.

On numerous occasions we will need the following lemma:

Lemma 348.17. (Subcomplex–Torsion Lemma) Let (X,x_0) be a pointed connected finite CW-complex and let $A \subseteq X$ be a subcomplex. Furthermore let R be an IBN ring and let $\alpha\colon \pi_1(X,x_0) \to \mathrm{GL}(d,R)$ be a representation. Finally let $Y \subseteq X$ be a subcomplex with components $Y_1,\ldots,Y_n$. We denote by $p\colon \widetilde{X} \to X$ any universal covering of X. As usual we write $\widetilde{Y} := p^{-1}(Y)$ and we write $\widetilde{A} := p^{-1}(A)$. Then we have the following equality in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(X,x_0) \sqcup \{0\}$:

$$\prod_{i=1}^{n} \tau(Y_i, Y_i \cap A; \alpha) \;=\; \tau\big(\overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{Y}, \widetilde{Y} \cap \widetilde{A})\big).$$



Proof. We start out with three preliminary remarks:

- To simplify the notation we only deal with the case that $A = \varnothing$.
- It follows easily from Exercise 345.6 that without loss of generality we can assume that Y is actually path-connected.
- It follows from the Torsion–Base Point Lemma 348.15 that we can pick the same base point x_0 for X and Y.

Next we introduce some notation:

- We denote by $\widehat{Y}_{x_0}$ the natural universal covering of Y.
- We denote by $i\colon Y \to X$ the inclusion map and we denote by $\widetilde{i}\colon \widehat{Y}_{x_0} \to \widetilde{Y}$ the induced map from Definition 321.4.

As in the Change-of-Covering Lemma 321.11 it is straightforward to verify that for appropriate choices of bases the maps

$$\begin{array}{ccc} \overline{{}_{\alpha\circ i_*} R^d} \otimes_{\mathbb{Z}[\pi_1(Y,x_0)]} \mathrm{C}_*^{\mathrm{CW}}(\widehat{Y}_{x_0}) & \to & \overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi_1(X,x_0)]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{Y}) \\ v \otimes \sigma & \mapsto & v \otimes i_*(\sigma) \end{array}$$

are isomorphisms of chain complexes of based R-left modules. The lemma now follows from the Torsion–Isomorphism Lemma 345.27. ∎

The following proposition can be viewed as an analog of the LES-of-Pair Proposition 146.7, the LES-of-Triple Proposition 146.5, the Twisted-LES-Proposition 321.10 and of the Relative-χ-Lemma 156.8:

Proposition 348.18. (CW Triple–Torsion Proposition) Let (X, x_0) be a pointed connected CW–complex, let $Y \subseteq X$ be a subcomplex with components $Y_1, \dots, Y_m$ and let $Z \subseteq Y$ be a subcomplex (in most, but not all applications $Z = \varnothing$). Let R be an IBN ring and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(d, R)$ be a representation. We suppose that two out of the following three statements hold:

(i) All $\mathrm{H}_*^{\mathrm{tw}}(Y_i, Y_i \cap Z; {}_\alpha R^d) \equiv 0$, in other words all $\tau(Y_i, Y_i \cap Z; \alpha) \neq 0$,
(ii) $\mathrm{H}_*^{\mathrm{tw}}(X, Y; {}_\alpha R^d) \equiv 0$, in other words $\tau(X, Y; \alpha) \neq 0$,
(iii) $\mathrm{H}_*^{\mathrm{tw}}(X, Z; {}_\alpha R^d) \equiv 0$, in other words $\tau(X, Z; \alpha) \neq 0$.

Then all three statements hold and in this setting we then have the following equality in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(X, x_0)) \sqcup \{0\}$:

$$\tau(X, Z; \alpha) \;=\; \tau(X, Y; \alpha) \cdot \prod_{i=1}^{m} \tau(Y_i, Y_i \cap Z; \alpha).$$

Proof. First of all note that the statement about the homology modules follows from the Twisted-LES-Proposition 321.10. Next we turn to the discussion of the Reidemeister torsions. We write $\pi := \pi_1(X, x_0)$. Let $p\colon \widetilde{X} \to X$ be a universal covering of X. We write $\widetilde{Y} := p^{-1}(Y)$ and $\widetilde{Z} := p^{-1}(Z)$. As in the proof of the Twisted-LES-Proposition 321.10 we see that the maps

$$0 \to \overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{Y}, \widetilde{Z}) \to \overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{Z}) \to \overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{Y}) \to 0$$

form a short exact sequence of R-left modules. Next for each $k \in \mathbb{N}_0$ we do the following:

(a) We pick characteristic maps and lifts of the k-cells of $Y \setminus Z$ to $\widetilde{Y}$. This endows the $\mathbb{Z}[\pi]$-left module $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{Y}, \widetilde{Z})$ with a basis.
(b) We pick characteristic maps and lifts of the k-cells of $X \setminus Y$ to $\widetilde{X}$. This endows the $\mathbb{Z}[\pi]$-left module $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{Y})$ with a basis.
(c) The choices of (a) and (b) endow the $\mathbb{Z}[\pi]$-left module $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{X}, \widetilde{Z})$ with a basis.

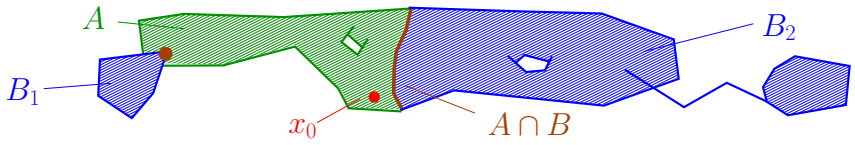
It is now essentially clear that the bases in the above short exact sequence are compatible in the sense of Definition 345.11. The promised statement now follows easily from the SES-Torsion Theorem 345.30 together with the Subcomplex–Torsion Lemma 348.17. ■

The following proposition can be viewed as an analog of the Twisted Excision Theorem 325.4:

Proposition 348.19. (Excision–Torsion Proposition) Let (X, x_0) be a pointed connected CW–complex and let $X = A \cup B$ be a decomposition into subcomplexes where B has the components $B_1, \dots, B_k$. Let R be an IBN ring and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(d, R)$ be a representation. Then we have the following equality in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(X, x_0)) \sqcup \{0\}$:

$$\tau(X, A; \alpha) \;=\; \prod_{i=1}^{k} \tau(B_i, B_i \cap A; \alpha).$$

Proof. Let $p\colon \widetilde{X}_{x_0} \to X$ be the natural universal covering. As always we denote the preimage of any $C \subseteq X$ by $\widetilde{C}$. It follows easily from the Regular Covering–Cellular Chain



Complex Lemma 348.2 that the inclusions induce an isomorphism

$$\mathrm{C}_*^{\mathrm{CW}}(\widetilde{B}, \widetilde{B \cap A}) \;\to\; \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}, \widetilde{A})$$

of based $\mathbb{Z}[\pi_1(X, x_0)]$-left modules. The desired statement now follows easily from the Torsion–Isomorphism Lemma 345.27 together with the Subcomplex–Torsion Lemma 348.17. ■

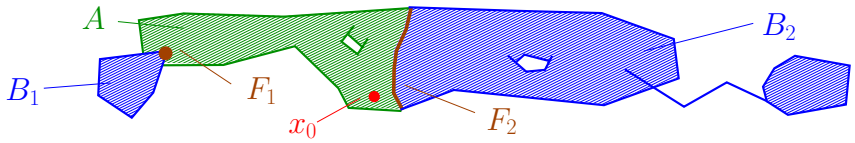
The following proposition plays the role of the Mayer–Vietoris sequence in (twisted) homology and it can be viewed as an analog of the Union-χ-Proposition 156.10:

Proposition 348.20. (Decomposition-CW Torsion Theorem) Let $X = A \cup_F B$ be a decomposition of a finite connected CW-complex into two subcomplexes A and B. We set $F = A \cap B$. We introduce the following notation:

- We denote by $A_1, \dots, A_a$ the components of A.
- We denote by $B_1, \dots, B_b$ the components of B.
- We denote the components of F by $F_1, \dots, F_f$.

Let $x_0 \in X$ and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(d, R)$ be a representation over an IBN ring R such that each $\mathrm{H}_*^{\mathrm{tw}}(F_i; {}_\alpha R^d) \equiv 0$. Then we have the following equality in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(X, x_0)) \sqcup \{0\}$:

$$\tau(X; \alpha) \;=\; \prod_{i=1}^{a} \tau(A_i; \alpha) \cdot \prod_{j=1}^{b} \tau(B_j; \alpha) \cdot \prod_{k=1}^{f} \tau(F_k; \alpha)^{-1}.$$



Proof. The proposition follows from the following equalities in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(X, x_0)) \sqcup \{0\}$:

$$\begin{aligned}
\tau(X; \alpha) &\overset{\text{by the CW Triple–Torsion Proposition 348.18}}{=} \prod_{i=1}^{a} \tau(A_i; \alpha) \cdot \tau(X, A, \alpha) \\
&= \prod_{i=1}^{a} \tau(A_i; \alpha) \cdot \prod_{j=1}^{b} \tau(B_j; \alpha) \cdot \dot{\prod_{\substack{k \in \{1,\dots,f\},\\ \text{with } F_k \subseteq B \cap F}}} \tau(F_k; \alpha)^{-1} \\
&\underset{\text{by the CW Triple–Torsion Proposition 348.18}}{=} \prod_{i=1}^{a} \tau(A_i; \alpha) \cdot \prod_{j=1}^{b} \tau(B_j; \alpha) \cdot \prod_{k=1}^{f} \tau(F_k; \alpha)^{-1}.
\end{aligned}$$

$$\overset{\text{by the Excision–Torsion Proposition 348.19}}{=} \prod_{i=1}^{a} \tau(A_i; \alpha) \cdot \prod_{j=1}^{b} \tau(B_j, B_j \cap F; \alpha)$$

■

Summarizing, in this chapter we defined an invariant for a CW-complex together with a representation. The question that now arises is, to what degree does the invariant depend

on the precise structure of the CW-complex? Does it, as most invariants in algebraic topology, depend only on the homeomorphism type or the homotopy type? To answer this question we need to study the torsion in more detail. This will happen in the next **??**.

348.6. Exercises for Chapter 348.

Exercise 348.1. Let (X, x_0) be a pointed connected CW-complex and let $Y \subseteq X$ be a subspace. Furthermore suppose we are given a ring R and a representation of the form

$$\begin{array}{rcl} \varphi\colon \pi_1(X, x_0) & \to & \mathrm{GL}(m+n, R) \\ g & \mapsto & \begin{pmatrix} \alpha(g) & \theta(g) \\ 0 & \beta(g) \end{pmatrix}. \end{array}$$

(a) Show that in general the twisted homology module $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\varphi}R^{m+n})$ is *not* isomorphic to the direct sum of the R-left modules $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\alpha}R^{m})$ and $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\beta}R^{n})$.
Hint. Consider $X = S^1$.
(b) Show if both twisted homology module $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\alpha}R^{m}) = 0$ and $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\beta}R^{n}) = 0$, then $\mathrm{H}_k^{\mathrm{tw}}(X, Y; {}_{\varphi}R^{m+n}) = 0$.

Remark. This exercise is closely related to Exercise 329.7.

Exercise 348.2. Let (X, x_0) be a pointed connected CW-complex and let $Y \subseteq X$ be a subcomplex. Furthermore suppose we are given an IBN ring R and a representation of the form

$$\begin{array}{rcl} \varphi\colon \pi_1(X, x_0) & \to & \mathrm{GL}(m+n, R) \\ g & \mapsto & \begin{pmatrix} \alpha(g) & \theta(g) \\ 0 & \beta(g) \end{pmatrix}. \end{array}$$

We suppose that one of $\tau(X, Y; \alpha) \neq 0$ or $\tau(X, Y; \beta) \neq 0$. Show that

$$\tau(X, Y; \varphi) \;=\; \tau(X, Y; \alpha) \cdot \tau(X, Y; \beta) \;\in\; \overline{\mathrm{K}}_1(R) \sqcup \{0\}.$$

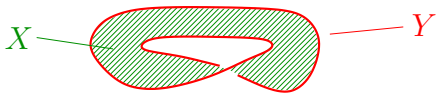
Remark. This exercise can be viewed as a continuation of Exercise 348.1.

Exercise 348.3. We consider the circle S^1. Let $\alpha\colon \pi_1(S^1, 1) \to \mathrm{GL}(1, \mathbb{F})$ be a non-trivial one-dimensional representation over a field $\mathbb{F}$. In the Circle Torsion Lemma 348.8 we equipped S^1 with the standard CW-complex structure and we showed that with respect to this CW-structure we have $\tau(S^1; \alpha) = (1 - \alpha(x))^{-1} \in \mathbb{F}^{\times}/\langle \alpha(x) \rangle$ where $x \in \pi_1(S^1, 1)$ is a generator.

(a) Classify all CW-structures on S^1.
(b) Show that for all CW-structures on S^1 we obtain the same result for Reidemeister torsion.

Exercise 348.4. Let X be the Möbius band and let $Y = \partial M$. We consider X with the *trivial* representation $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(1, \mathbb{Q})$.

(a) Describe a CW-structure for X such that Y is a subcomplex.
(b) Determine $\tau(X, Y; \alpha)$.



Exercise 348.5. Let $n \in \mathbb{N}_0$. Provide $\overline{B}^n \times S^1$ with an explicit CW-structure (e.g. the product CW-structure) and compute the Reidemeister torsion of $\overline{B}^n \times S^1$ for any 1-dimensional representation $\pi_1(\overline{B}^n \times S^1) \to \mathrm{GL}(1, \mathbb{F})$ over a field $\mathbb{F}$.

Exercise 348.6. Let X be a finite connected non-empty CW-complex. We equip S^1 with the standard CW-structure and we equip $X \times S^1$ with the product CW-structure. We denote by $\pi\colon X \times S^1 \to S^1$ the projection. Let $\pi_1(X \times S^1) \xrightarrow{\pi_*} \pi_1(S^1) \to \mathrm{GL}(1, \mathbb{F})$ be a non-trivial 1-dimensional representation. Compute the Reidemeister torsion of $X \times S^1$.
Remark. This exercise is related to Exercise 329.12 and Exercise 348.5.

Exercise 348.7. Let (X, x_0) be a pointed finite CW-complex. Furthermore let R be an IBN ring and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(d, R)$ a representation. We denote by $\pi\colon X \times [0,1] \to X$ the projection. Show that $\tau(X \times [0,1], X; \alpha \circ \pi_*) = 1 \in \overline{\mathrm{K}}_1(R)/\alpha(\pi_1(X, x_0))$.

Exercise 348.8. We consider the Klein bottle K with the CW-structure that is shown in the figure below.

(a) In the figure we also show a base point $*$ and two loops x and y in $*$. Express the fundamental group $\pi_1(K, *)$ in terms of x and y.
(b) Determine all 1-dimensional representations α for which $\mathrm{H}_*^{\mathrm{tw}}(K; {}_\alpha\mathbb{C}) = 0$.
(c) For all representations from (b) determine the corresponding Reidemeister torsion.

Remark. This exercise has an overlap with Exercise 329.9.



Exercise 348.9. We consider the Klein bottle K with the CW-structure and the subcomplexes X and Y that are shown in the figure below. We also consider the *trivial* representation $\pi_1(K) \to \mathrm{GL}(1, \mathbb{Q})$.

(a) Determine $\tau(K, X; \alpha)$.
(b) Determine $\tau(K, Y; \alpha)$.



Exercise 348.10. Let (X, x_0) be a pointed connected finite CW–complex and let $Y \subseteq X$ be a subcomplex. Furthermore let $\mathbb{F}$ be a field and let $\alpha\colon \pi_1(X, x_0) \to \mathrm{SL}(n, \mathbb{F})$ be a representation such that $\mathrm{H}_*^{\mathrm{tw}}(X, Y; {}_\alpha\mathbb{F}^n) \equiv 0$. Show that if n is even, then we get a well-defined torsion $\tau(X, Y; \alpha) \in \mathbb{F}^\times$, i.e. we get a well-defined torsion without any indeterminacy.

Exercise 348.11. Let $n_{\geq 3} \in \mathbb{N}$ odd and let $\alpha\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \mathrm{GL}(1, \mathbb{C})$ be the unique non-trivial representation. We equip $\mathbb{R}\mathrm{P}^n$ with the standard CW-structure from page 2628. Calculate $\tau(\mathbb{R}\mathrm{P}^n; \alpha)$.
Remark. Make use of the $\mathbb{R}\mathrm{P}^n$–Twisted Cellular Chain Complex Lemma 329.13.

Exercise 348.12. We consider the torus $T = S^1 \times S^1$ together with the homeomorphism $f\colon S^1 \times S^1 \to S^1 \times S^1$ that is given by $f(x,y) = (y,x)$. We consider the corresponding mapping torus $X := \mathrm{Tor}(S^1 \times S^1, f)$ from Definition 21.19 together with the natural map

$$\begin{array}{rcl} p\colon \mathrm{Tor}(T,f) := (T \times [0,1])/(x,0) \sim (f(x),1)) & \to & S^1 \\ [(x,s)] & \mapsto & \exp(2\pi \mathrm{i} s). \end{array}$$

For $\xi \in \mathbb{C} \setminus \{0\}$ we consider the representation

$$\alpha\colon \pi_1(\mathrm{Tor}(T,f)) \xrightarrow{p_*} \pi_1(S^1,1) =_{\mathrm{i}} \langle x \rangle \xrightarrow{x \mapsto (\xi)} \mathrm{GL}(1,\mathbb{C}).$$

Compute $\tau(X;\alpha)$ for a suitable CW-structure.[3782]
Remark. This exercise is related to Exercise 329.8.

[3782]One could for example take the CW-structure that arises from the standard CW-structure on T and the CW-Complex Mapping Torus Proposition 139.26.

CHAPTER 349

Whitehead torsion and simple homotopy equivalences

In the previous Chapter 348 we introduced Reidemeister torsion of finite CW-complexes. Note that a priori Reidemeister torsion depends on the precise CW-structure (and a priori not just on the underlying topological). In particular it is a priori not clear at all in what sense Reidemeister torsion is a homeomorphism invariant. In this chapter we will address this question.

In this chapter we will first introduce Whitehead torsion of a homotopy equivalence. A homotopy equivalence with trivial Whitehead torsion is called *simple*. We will then see that Reidemeister torsion is in fact invariant under simple homotopy equivalences. In the next **??** we will state Chapman's Theorem which says that finite CW-complexes that are homeomorphic are in fact simple homotopy equivalent. This then shows that Reidemeister torsion is in a certain sense a homeomorphism invariant.

This chapter is organized as follows:

- In Section 349.1 we will introduce Whitehead torsion of a homotopy equivalence.
- In Section 349.2 we will see that Whitehead torsion controls to what degree Reidemeister torsion is a homotopy invariant.
- In Section 349.3 we will introduce the notion of a simple homotopy equivalence.
- In Section 349.4 we will introduce the notion of a subdivision of a CW-complex. We will see that subdivisions are in particular simple homotopy equivalences.
- In Section 349.5 we will (re-) introduce elementary expansions and collapses. We will see that these are again simple homotopy equivalences and we will point out that in a suitable sense these are in fact all simple homotopy equivalences.
- In Section 349.6 we will provide exercises for Chapter 349.

In the next **??** we will state Chapman's Theorem which, as we just mentioned, says that finite CW-complexes that are homeomorphic are simple homotopy equivalent.

349.1. **Whitehead torsion of a homotopy equivalence.** In this section we will introduce the Whitehead torsion of a cellular homotopy equivalence. As we will see, the Whitehead torsion will tell us that not all homotopy equivalences are created equal.

We will proceed in two steps:

- In Section 349.1.1 we will introduce the Whitehead torsion of a deformation retract.
- In Section 349.1.2 we will introduce Whitehead torsion of a *cellular* homotopy equivalence.

349.1.1. **Whitehead torsion of a deformation retract.** We start out by recalling Definition 344.21:

Definition 349.1. Given a group G we refer to

$$\mathrm{Wh}(G) \coloneqq \overline{\mathrm{K}}_1(\mathbb{Z}[G])/G$$

as the Whitehead group of G.

Remark. Let X be a path-connected topological space and let $x_0, x_1 \in X$ be two base points. By the Change-of-Base Point Proposition 88.14 we know that any path γ from x_0 to x_1 induces an isomorphism $\gamma^*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$, and we know that this isomorphism is up to conjugation well-defined, i.e. independent of the choice of the path. It follows from this discussion and the Whitehead Group–Conjugation Lemma 344.25 that we actually have a natural isomorphism $\mathrm{Wh}(\pi_1(X, x_1)) \to \mathrm{Wh}(\pi_1(X, x_0))$ which does not depend on the choice of a path connecting x_0 to x_1. This shows that for the Whitehead group of the fundamental group of a path-connected topological space we do not really have to worry about base point issues. □

Now we can introduce the Whitehead torsion of a deformation retract:

Definition 349.2. Let Z be a connected CW-complex and let X be a subcomplex of Z which is a deformation retract of Z. We pick a base point $x_0 \in X$ and we write $\pi \coloneqq \pi_1(Z, x_0)$. As usual let $p\colon \widetilde{Z} \to Z$ be the natural universal covering. We write $\widetilde{X} \coloneqq p^{-1}(X)$. We start out with a few preparations:

(a) Since X is a deformation retract of Z we know that there exists a deformation retraction $R\colon Z \times [0,1] \to Z$ from Z to X in the sense of Definition 19.3. As in the proof of the Homotopy–Chain Homotopy Lemma 322.5 we see that it follows easily from the Lift-to-Universal Coverings Proposition 127.26 that the deformation retraction $R\colon Z \times [0,1] \to Z$ lifts to a deformation retraction $\widetilde{R}\colon \widetilde{Z} \times [0,1] \to \widetilde{Z}$ from the universal covering $\widetilde{Z}$ to $\widetilde{X}$.

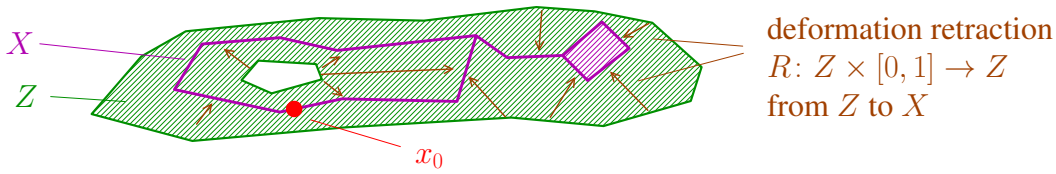
(b) It follows from (a) together with the Homotopy Equivalence-H_*-Proposition 144.14 and the Singular-Cellular-H_*-Proposition 155.8 that the chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{Z}, \widetilde{X})$ is acyclic.

(c) For each k-cell $\sigma \in \mathrm{Cell}_k(Z \setminus X)$ we pick a characteristic map and we pick a lift to the natural universal covering $\widetilde{Z}$. It follows from the Regular Covering–Cellular Chain Complex Lemma 348.2 that these choices give rise to a basis of $\mathrm{C}_k^{\mathrm{CW}}(\widetilde{Z}, \widetilde{X})$ as a free $\mathbb{Z}[\pi]$-left module.

We now suppose that Z is a finite CW-complex. It follows from the above that we can define the Whitehead torsion $\tau(Z, X)$ as[3783 3784]

$$\tau(Z, X) \coloneqq \tau\underbrace{\big(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{Z}, \widetilde{X}), \{\widetilde{\sigma}\}_{\sigma \in \mathrm{Cell}_k(Z \setminus X)}\big)}_{\substack{\text{based acyclic chain complex} \\ \text{of } \mathbb{Z}[\pi]\text{-left modules}}} \in \underbrace{\overline{\mathrm{K}}_1(\mathbb{Z}[\pi])/\pi}_{=:\mathrm{Wh}(\pi)}.$$

As in the Twisted Torsion–Well-Defined Proposition 348.5 we see that $\tau(Z, X)$ is indeed well-defined as an element of $\overline{\mathrm{K}}_1(\mathbb{Z}[\pi])/\pi = \mathrm{Wh}(\pi)$, i.e. it does not depend on the choices in (c).



The following proposition shows that in our current context anything that can happen theoretically (i.e. algebraically), can also happen in practice (i.e. topologically):

Proposition 349.3. (Whitehead Torsion–Realization Proposition) Let (X, x_0) be a pointed finite connected CW-complex and let $A \in \mathrm{Wh}(\pi_1(X, x_0))$. Then there exists a finite CW-complex Z that contains X as a subcomplex, such that X is a deformation retract of Z and such that

$$\iota_*^{-1}(\tau(Z, X)) \;=\; A \;\in\; \mathrm{Wh}(\pi_1(X, x_0)),$$

where $\iota_*\colon \pi_1(X, x_0) \to \pi_1(Z, x_0)$ is the inclusion induced isomorphism.

Example. Recall that in the Non-Trivial Whitehead Group Lemma 344.24 we showed that that there exist groups G with non-trivial Whitehead group, e.g. $G = \langle x \,|\, x^5 \rangle$. It follows from the Whitehead Torsion–Realization Proposition 349.3 that there exists a pair (Z, X) with non-trivial Whitehead torsion. □

Proof. Let X be a finite connected CW-complex with base point x_0 in the 0-skeleton of X. We write $\pi = \pi_1(X, x_0)$. Furthermore suppose that we are given an element of $\mathrm{Wh}(\pi)$. By Definition 344.21 we can represent this element by some matrix $A = (a_{ij}) \in \mathrm{GL}(k, \mathbb{Z}[\pi])$. We pick an $n \in \mathbb{N}_{\geq 2}$. We construct the CW-complex Z in two stages:

(1) First we consider the wedge

$$Y \;:=\; X \vee \bigvee_{j=1}^{k} S_j^n$$

where we form the wedge by identifying $x_0 \in X$ with the various North Poles of the spheres. We make a few observations:

- It follows from $n \geq 2$ and the CW-Seifert–van Kampen Theorem 137.10 that the inclusion $\iota\colon X \to Y$ induces an isomorphism of fundamental groups. Thus we can identify π with $\pi_1(Y, x_0)$.
- We equip each S_j^n with the standard CW-structure from page 2625 which consists of a single 0-cell, namely the North Pole and a single n-cell.
- It follows from the CW-Gluing Proposition 139.9 that we can equip Y with a CW-structure such that X is a subcomplex and such that we have k extra n-cells.

(2) For $j = 1, \ldots, k$ we denote by $[S_j^n] \in \pi_n(Y, x_0)$ the element represented by the obvious map $S^n \to S_j^n \to Y$. Next recall that in Definition 169.3 and the Homotopy Group–Module Lemma 169.8 we introduced an action of $\pi = \pi_1(Y, x_0)$ on $\pi_n(Y, x_0)$ and we used this to define a $\mathbb{Z}[\pi]$-left module structure on the abelian group $\pi_n(Y, x_0)$. Given

[3783] It follows easily from the Tensor-with-Base Ring Lemma 312.10 that we have a natural isomorphism $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}, \widetilde{Z}) \xrightarrow{\cong}{}_{\mathrm{id}\overline{\mathbb{Z}[\pi]}} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}, \widetilde{Z})$ of based free $\mathbb{Z}[\pi]$-left modules. Thus we obtain from the Torsion–Isomorphism Lemma 345.27 that $\tau(X, Z) = \tau(X, Z; \pi \xrightarrow{\mathrm{id}} \mathbb{Z}[\pi])$.

[3784] It follows easily from the Tensor-with-Base Ring Lemma 312.10 together with the above discussion about Whitehead groups that for the most part we can ignore base points.

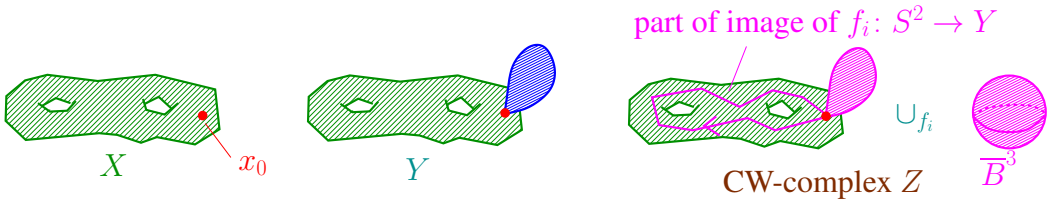
$i \in \{1, \dots, k\}$ we now consider

$$\alpha_i \;:=\; \sum_{j=1}^{k} a_{ij} \cdot [S_j^n] \;\in\; \pi_n(Y, x_0).$$

Next note that it follows from the Cellular Approximation Theorem 141.2 that we can realize α_i by a cellular map $f_i \colon S^n \to Y$. Finally note that it follows from the CW-Gluing Proposition 139.9 that we can equip

$$Z \;:=\; Y \cup_{Y \leftarrow S^n : f_1} \overline{B}^{n+1} \cdots \cup_{Y \leftarrow S^n : f_k} \overline{B}^{n+1}$$

with a CW-structure such that Y is a subcomplex and such that we have k extra $(n+1)$-cells.



In the following discussion we will verify that the pair (Z, X) has, at least for an appropriate n, the desired properties.

As usual we denote by $p \colon \widetilde{Z} \to Z$ the universal covering and we write $\widetilde{X} := p^{-1}(X)$.

Claim 1. There exist lifts $\widetilde{f}_1, \dots, \widetilde{f}_k$ of the new $(n+1)$-cells and lifts $\widetilde{e}_1, \dots, \widetilde{e}_k$ of the extra n-cells to $\widetilde{Z}$ such that we obtain the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[\pi]^{1\times k} & \xrightarrow{v \mapsto v \cdot A} & \mathbb{Z}[\pi]^{1\times k} & \longrightarrow & 0 \longrightarrow \dots \\
 & & & & \downarrow{\scriptstyle \mathfrak{e}_i \mapsto \widetilde{f}_i} & & \downarrow{\scriptstyle \mathfrak{e}_i \mapsto \widetilde{e}_i} & & \\
\dots & \longrightarrow & 0 & \longrightarrow & \mathrm{C}^{\mathrm{CW}}_{n+1}(\widetilde{Z}, \widetilde{X}) & \xrightarrow{\partial^{\mathrm{CW}}_{n+1}} & \mathrm{C}^{\mathrm{CW}}_{n}(\widetilde{Z}, \widetilde{X}) & \longrightarrow & 0 \longrightarrow \dots
\end{array}$$

Proof. We pick a lift $\widetilde{x}_0$ of the wedge point x_0. For each n-cell e_i we pick the unique lift $\widetilde{e}_i$ which gets attached to $\widetilde{x}_0$. For each $(n+1)$-cell f_i we pick the unique lift $\widetilde{f}_i$ which attaches the North Pole to $\widetilde{x}_0$. We leave it to the reader to show, say using the Cellular Boundary–Degree Proposition 329.4 and Observation 329.5, that we get the promised boundary matrix. ⊞

Claim 2. The subcomplex X is a deformation retract of Z.

Proof. We start out with a few observations:

(a) It follows from Claim 1 and the fact that A is invertible that $\mathrm{C}^{\mathrm{CW}}_*(\widetilde{Z}, \widetilde{X})$ is acyclic.

(b) It follows from (a), the Singular-Cellular-H_*-Proposition 155.8 and the long exact sequence in homology that the inclusion $\widetilde{Z} \to \widetilde{X}$ induces an isomorphism on all homology groups.

(c) It follows from (b), and the fact that $n \geq 2$, together with the Weak Homotopy-vs-Homology Isomorphisms Proposition 304.11 (2c′) (which is a consequence of the Relative Hurewicz Theorem 296.25) that the inclusion $\iota \colon X \to Z$ is a weak homotopy equivalence, i.e. all induced maps $\iota_* \colon \pi_j(X, x_0) \to \pi_j(Z, x_0)$ are isomorphisms.

(d) Since X is a subcomplex of Z we obtain from (c) and the Whitehead Deformation Retract Proposition 305.2 (1) that X is a deformation retract of Z. ⊞

The following claim shows that if we start out with an odd n, then we have in fact achieved our goal:

Claim 3. We have $\iota_*^{-1}(\tau(Z,X)) = [A]^{(-1)^{n+1}} \in \mathrm{Wh}(\pi)$.

Proof. This claim follows immediately from Claim 1 together with the Elementary Torsion Lemma 345.17. ■

349.1.2. **Whitehead torsion of a cellular homotopy equivalence.** In this subsection we will use Definition 349.2 to introduce the Whitehead torsion of a cellular homotopy equivalence. The key to doing so is to relate homotopy equivalences to deformation retracts. At this point it is worth recalling Definition 21.1:

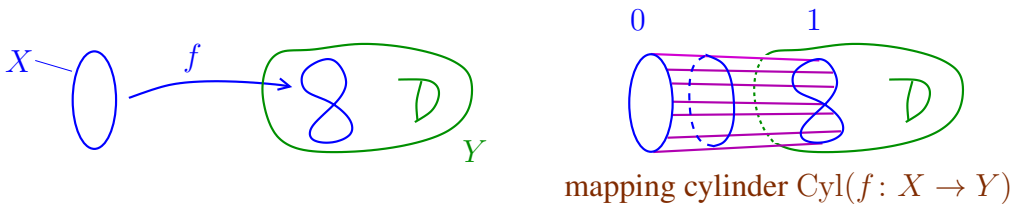
Definition 349.4. Let $f\colon X \to Y$ be a continuous map between topological spaces. We define the corresponding mapping cylinder to be the topological space

$$\mathrm{Cyl}(f) \coloneqq \mathrm{Cyl}(f\colon X \to Y) \coloneqq ((X \times [0,1]) \sqcup Y)/\sim \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

By the Mapping Cylinder–Embedding Lemma 21.4 we know that the natural maps

$$\begin{array}{ccc} X & \to & \mathrm{Cyl}(f) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \to & \mathrm{Cyl}(f) \\ y & \mapsto & [y] \end{array}$$

are closed embeddings. We will use these closed embeddings to identify X with $X \times 0$ and to identify Y with the image of Y in $\mathrm{Cyl}(f)$.



mapping cylinder $\mathrm{Cyl}(f\colon X \to Y)$

Recall that we know by the Mapping Cylinder–Homotopy Lemma 21.5 that Y is always deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$. More interesting is the question, when is X a deformation retract of $\mathrm{Cyl}(f\colon X \to Y)$? The following proposition from elementary homotopy theory gives a complete answer to this question:

Proposition 349.5. (Homotopy Equivalence–Characterization Proposition) In the following let $f\colon X \to Y$ be a continuous map between topological spaces. The following statements are equivalent:

(1) $f\colon X \to Y$ is a homotopy equivalence.
(2) The inclusion $X \to \mathrm{Cyl}(f)$ is a homotopy equivalence.
(3) X is a deformation retract of $\mathrm{Cyl}(f)$.

Proof. The equivalence of Statements (1) and (2) is the content of the Mapping Cylinder–Homotopy Lemma 21.5 (2b) and the equivalence of Statements (2) and (3) is the content of the Homotopy Equivalence–Characterization Corollary 22.20. ■

The Homotopy Equivalence–Characterization Proposition 349.5 allows us to introduce the following definition:

Definition 349.6. Let $f\colon X \to Y$ be a *cellular* map between finite connected CW–complexes. We equip $\mathrm{Cyl}(f)$ with the structure of a CW-complex from the CW-Complex Mapping Cylinder+Cone Proposition 139.25.[3785] If $f\colon X \to Y$ is a homotopy equivalence then we know by the Homotopy Equivalence–Characterization Proposition 349.5 that we can view X as a deformation retract of $\mathrm{Cyl}(f)$. Using Definition 349.2 we can thus, for any $y_0 \in Y$, define the Whitehead torsion of f as[3786 3787]

$$\tau(f) \;:=\; p_*(\tau(\mathrm{Cyl}(f), X)) \;\in\; \mathrm{Wh}(\pi_1(Y, y_0))$$

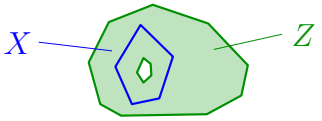
where $p\colon \mathrm{Cyl}(f) \to Y$ is the natural deformation retraction.

Remark. Later in **??** we will define the Whitehead torsion for any homotopy equivalence, i.e. we will drop the hypothesis that the Whitehead torsion is cellular. But for the purpose of this chapter we restrict ourselves to cellular homotopy equivalences. □

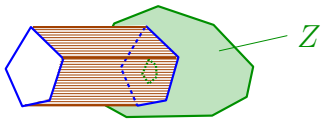
The following lemma shows that for a deformation retract Definition 349.2 and Definition 349.6 lead to the same result:

Lemma 349.7. (Torsion-of-Deformation Retraction Lemma) Let Z be a finite connected non-empty CW-complex and let $X \subseteq Z$ be a subcomplex that is a deformation retract of Z. We denote by $\iota\colon X \to Z$ the inclusion. Then for any $z_0 \in Z$ we have the following equality in $\mathrm{Wh}(\pi_1(Z, z_0))$:

$$\underbrace{\mathrm{Wh}(\iota\colon X \to Z)}_{\text{Definition 349.6}} \;=\; \underbrace{\mathrm{Wh}(Z, X).}_{\text{Definition 349.2}}$$



X is a deformation retract of Z



mapping cylinder $\mathrm{Cyl}(\iota\colon X \to Z)$

Before we provide the proof of the Torsion-of-Deformation Retraction Lemma 349.7 let us give a first straightforward application:

Lemma 349.8. (Torsion-of-id-Lemma) For any finite connected non-empty CW-complex X the Whitehead torsion of the identity $\mathrm{id}\colon X \to X$ is trivial.

Proof of the Torsion-of-id-Lemma 349.8 assuming Lemma 349.7. Let X be any finite connected non-empty CW-complex. For any $x_0 \in X$ we have the following equality in $\mathrm{Wh}(\pi_1(X, x_0))$:

$$\mathrm{Wh}(\mathrm{id}\colon X \to X) \underset{\substack{\uparrow\\ \text{follows from the above}\\ \text{Torsion-of-Deformation}\\ \text{Retraction Lemma 349.7}}}{=} \mathrm{Wh}(X, X) \underset{\substack{\uparrow\\ \text{Definition 349.2}}}{=} \tau(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X}_{x_0}, \widetilde{X}_{x_0})) \underset{\substack{\uparrow\\ \text{since the relative}\\ \text{chain complex}\\ \text{is trivial}}}{=} 1.$$

■

[3785] Loosely speaking the cells of $\mathrm{Cyl}(f)$ are given by all the cells of X and Y, together with the product cells $e \times (0,1)$, where e is a cell in X.

[3786] We follow the convention of [**CoheM1973**, p. 72] and [**Miln1968b**, p. 382] to view $\tau(f)$ as an element of the Whitehead group of $\pi_1(Y, y_0)$.

[3787] As in Footnote 3783 we see that it follows easily from the Tensor-with-Base Ring Lemma 312.10 that that $\tau(X, Y) = p_*(\tau(\mathrm{Cyl}(f), X; \pi \xrightarrow{\mathrm{id}} \mathbb{Z}[\pi]))$.

Proof of the Torsion-of-Deformation Retraction Lemma 349.7. Let Z be a finite connected non-empty CW-complex and let $X \subseteq Z$ be a subcomplex that is a deformation retract of Z. We denote by $\iota\colon X \to Z$ the inclusion. We pick a base point $x_0 \in X$, we write $\pi = \pi_1(Z, x_0)$ and we denote by $\widetilde{X} \to X$ the natural universal covering of X. We make the obvious identifications $\pi =_{\mathrm{i}} \pi_1(X, x_0) =_{\mathrm{i}} \pi_1(X \times [0,1], (x_0, 0))$ and $\pi =_{\mathrm{i}} \pi_1(\mathrm{Cyl}(\iota), x_0)$. We start out with the following claim:

Claim. We have $\tau(X \times [0,1], X \times \{0\}) = 1$.

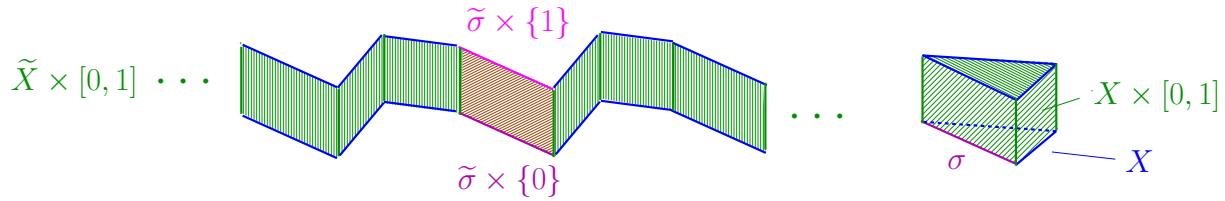
Proof. [3788] For each cell σ of X we pick a lift $\widetilde{\sigma}$ to $\widetilde{X}$. We lift the cells $\sigma \times \{1\}$ and $\sigma \times (0,1)$ in the obvious way. We write

$$C_i \;\coloneqq\; \mathrm{C}_i^{\mathrm{CW}}(\widetilde{X} \times [0,1], \widetilde{X} \times \{0\}).$$

Note that it follows from the Regular Covering–Cellular Chain Complex Lemma 348.2 that the cells $\widetilde{\sigma} \times \{1\}$ and $\widetilde{\sigma} \times (0,1)$ give rise to a basis of the free $\mathbb{Z}[\pi]$-left module C_i. We now set

$$\begin{aligned} D_i &\coloneqq \text{free } \mathbb{Z}[\pi]\text{-left submodule of } C_i \text{ spanned by } \{\widetilde{\sigma} \times \{1\} \mid \sigma \text{ is an } i\text{-cell of } X\}, \\ E_i &\coloneqq \text{free } \mathbb{Z}[\pi]\text{-left submodule of } C_i \text{ spanned by } \{\widetilde{\sigma} \times (0,1) \mid \sigma \text{ is an } (i-1)\text{-cell of } X\}. \end{aligned}$$

Note that by the above discussion we have $C_i = D_i \oplus E_i$.

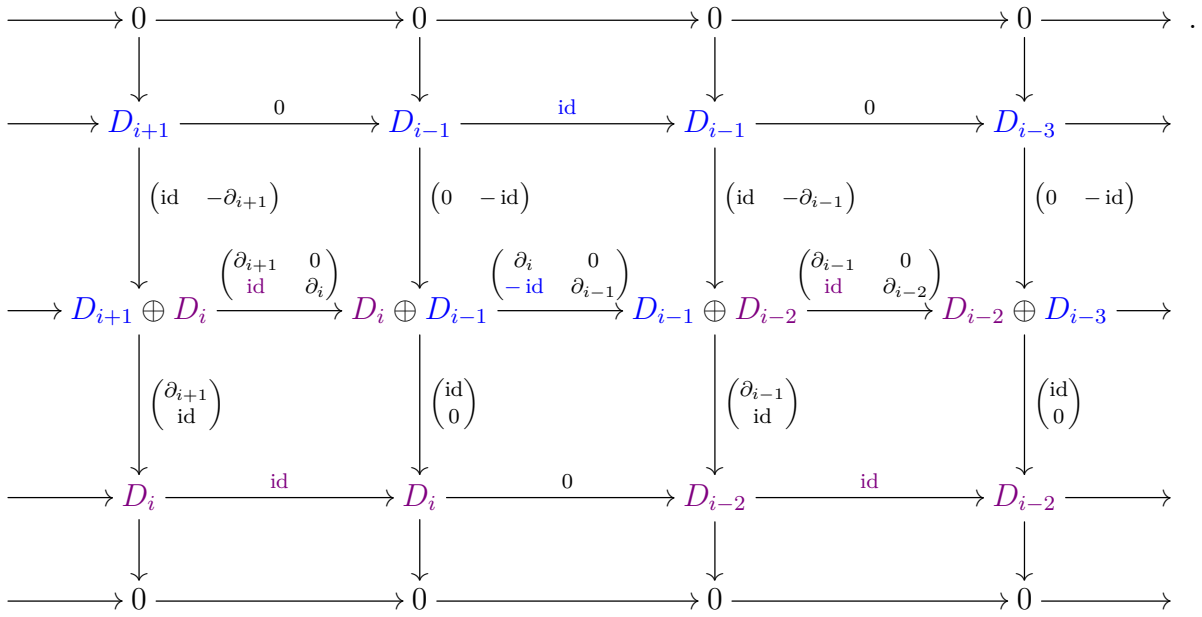


Using the natural isomorphism $E_i \xrightarrow{\sigma \times (0,1) \;\mapsto\; \sigma} D_{i-1}$ one can now easily see that the based chain complex $C_i = D_i \oplus E_i$ is isomorphic to the based chain complex

$$\ldots \to D_{i+1} \oplus D_i \xrightarrow{\begin{pmatrix} \partial_{i+1} & 0 \\ \mathrm{id} & \partial_i \end{pmatrix}} D_i \oplus D_{i-1} \xrightarrow{\begin{pmatrix} \partial_i & 0 \\ -\mathrm{id} & \partial_{i-1} \end{pmatrix}} D_{i-1} \oplus D_{i-2} \xrightarrow{\begin{pmatrix} \partial_{i-1} & 0 \\ \mathrm{id} & \partial_{i-2} \end{pmatrix}} D_{i-2} \oplus D_{i-3} \to \ldots$$

[3788] Note that this calculation essentially solves Exercise 348.7.

We now consider the diagram:



One can easily verify that the diagram commutes, that the vertical sequences are exact and that the horizontal sequences are chain complexes. In other words, we have a short exact sequence of chain complexes. It thus follows from the SES-Torsion Theorem 345.30 that the torsion of the based chain complex $C_i = D_i \oplus D_{i-1}$ equals the products of the torsions of the based chain complexes on the top and at the bottom. But, using the Elementary Torsion Lemma 345.17 together with the Torsion–Direct Sum Lemma 345.29 one can easily verify that these two torsions are trivial. ⊞

We introduce one last piece of notation: We denote by $\alpha\colon \pi \to \mathbb{Z}[\pi]$ the obvious map. Now we can turn to the actual proof of the lemma. More precisely, we perform the following calculation in $\mathrm{Wh}(\pi)$:

$$\tau(\iota\colon X \to Z) \overset{\text{by Definition 349.6 and Footnote 3787}}{=} \tau(\mathrm{Cyl}(\iota), X\times\{0\}; \alpha) \overset{\text{by the CW Triple–Torsion Proposition 348.18}}{=} \tau(\mathrm{Cyl}(\iota), X\times[0,1]; \alpha) \cdot \underbrace{\tau(X\times[0,1], X\times\{0\}; \alpha)}_{=1 \text{ by the claim}}$$

$$= \tau(Z, X; \alpha)\cdot 1 \overset{\text{by the Excision–Torsion Proposition 348.19 and the claim}}{=} \tau(Z,X) \overset{\text{by Definition 349.2 and Footnote 3783}}{=}$$

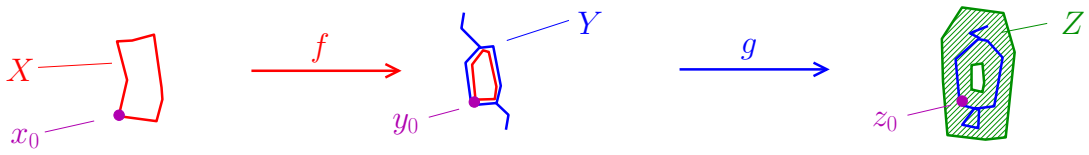
■

We conclude this discussion of the Whitehead torsion with the following lemma:

Lemma 349.9. (Whitehead Torsion–Additivity Lemma) Let X, Y, Z be finite connected CW-complexes and let $f\colon X \to Y$ and $g\colon Y \to Z$ be two cellular homotopy equivalences. Given $x_0 \in X$ we write $y_0 := f(x_0)$ and $z_0 := g(f(x_0))$. Then we have the following equality in $\mathrm{Wh}(\pi_1(Z, z_0))$:

$$\tau(g \circ f) \;=\; \tau(g) \cdot g_*(\tau(f)).$$

Proof. A proof is given in [**CoheM1973**, p. 84]. ■



349.2. **Whitehead torsion and Reidemeister torsion.** In this section we will relate Whitehead torsion to Reidemeister torsion. To formulate our main result we will need the following definition:

Definition 349.10. Let G be a group, let R be an IBN-ring and let $\alpha\colon G \to \mathrm{GL}(d,R)$ be a homomorphism. From the Group Ring–Universal Property Lemma 97.32 we obtain an induced ring homomorphism $\alpha\colon \mathbb{Z}[G] \to \mathrm{M}(d\times d,R)$. We obtain a group homomorphism

$$\begin{array}{rcccc} \alpha_*\colon \mathrm{K}_1(\mathbb{Z}[G]) & \xrightarrow{\varphi} & \mathrm{K}_1(\mathrm{M}(d\times d,R)) & \to & \mathrm{K}_1(R) \\ [(a_{ij})] & \mapsto & [(\alpha(a_{ij})] & & \\ & & \left[\begin{array}{c}(k\times k)\text{-matrix } (a_{ij}) \text{ with}\\ \text{entries in } \mathrm{M}(d\times d,R)\end{array}\right] & \mapsto & \left[\begin{array}{c}(kd\times kd)\text{-matrix } \alpha(a_{ij})\\ \text{with entries in } R\end{array}\right]. \end{array}$$

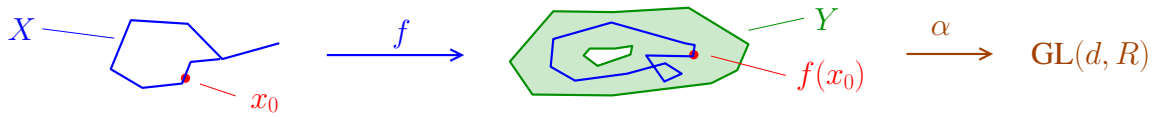
Here the first map map is induced by the ring homomorphism $\alpha\colon \mathbb{Z}[G] \to \mathrm{M}(d\times d,R)$ and the second map is the same map as in the Morita Invariance Theorem 344.17.[3789] Similarly we also get a homomorphism $\alpha_*\colon \overline{\mathrm{K}}_1(\mathbb{Z}[G]) \to \overline{\mathrm{K}}_1(R)$. Note that the above group homomorphism induces a group homomorphism

$$\alpha_*\colon \overbrace{\overline{\mathrm{Wh}}(G)}^{=\mathrm{K}_1(\mathbb{Z}[G])/G} \to \overline{\mathrm{K}}_1(R)/\alpha(G).$$

The following theorem shows that Whitehead torsion controls to what degree Reidemeister torsion is a homotopy invariant:

Theorem 349.11. (Torsion–Whitehead Torsion Theorem) Let $f\colon X \to Y$ be a homotopy equivalence between finite connected non-empty CW-complexes, let $x_0 \in X$ and let $\alpha\colon \pi_1(Y,f(x_0)) \to \mathrm{GL}(d,R)$ be a representation. We have the following equality in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(Y,y_0)) \sqcup \{0\}$:

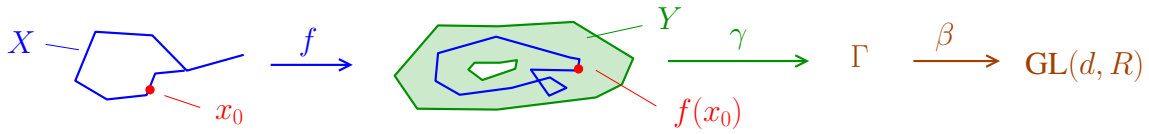
$$\tau(Y;\alpha) \;=\; \tau(X;\alpha\circ f_*)\cdot \alpha_*\big(\underbrace{\tau(f)}_{\in \mathrm{Wh}(\pi_1(Y,x_0))}\big).$$



Example. Let $f\colon X \to Y$ be a homotopy equivalence between finite connected non-empty CW-complexes and let $x_0 \in X$. We write $\pi = \pi_1(Y,f(x_0))$. Next let $\gamma\colon \pi \to \Gamma$ be a homomorphism to some group Γ and let $\beta\colon \Gamma \to \mathrm{GL}(d,R)$ be a representation over some IBN ring R.

It follows easily from the explicit Definition 349.10 that for any $A \in \overline{\mathrm{K}}_1(\mathbb{Z}[\pi])$ we have the equality $(\beta\circ\gamma)_*(A) = \beta_*([\gamma(A)]) \in \overline{\mathrm{K}}_1(R)$. It thus follows from the Torsion–Whitehead

[3789]The Morita Invariance Theorem 344.17 tells us that the map $\mathrm{K}_1(\mathrm{M}(d\times d,R)) \to \mathrm{K}_1(R)$ is an isomorphism. In our setting this is a nice-to-have-fact, but we will not make use of this fact.



Torsion Theorem 349.11 that we have the following equality in $\overline{\mathrm{K}}_1(R)/\beta(\pi_1(Y,y_0)) \sqcup \{0\}$:

$$\tau(Y;\beta\circ\gamma) \;=\; \tau(X;\beta\circ\gamma\circ f_*)\cdot(\beta\circ\gamma)_*(\underbrace{\gamma(\tau(f))}_{\in \mathrm{Wh}(\Gamma)}).$$

The point is, that $\gamma(\tau(f)) \in \mathrm{Wh}(\Gamma)$. In particular if $\mathrm{Wh}(\Gamma))$ is trivial (and in Appendix 344.6 we saw many instances where that is the case, e.g. if Γ is a torsion-free abelian group), then we obtain that $\tau(Y;\beta\circ\gamma) = \tau(X;\beta\circ\gamma\circ f_*) \in \overline{\mathrm{K}}_1(R)/\beta\circ\gamma(\pi)$. □

In the proof of the Torsion–Whitehead Torsion Theorem 349.11 we will need the following variation on the Torsion–Natural Theorem 346.11:

Lemma 349.12. (Torsion–Group Ring-Natural Lemma) Let π be a group and let $(C_*, \{\mathcal{C}_i\}_{i\in\mathbb{N}_0})$ be an acyclic based chain complex over the ring $\mathbb{Z}[\pi]$. Let R be an IBN ring and let $\alpha\colon \pi \to \mathrm{GL}(d,R)$ be a homomorphism. Then the chain complex ${}_{\alpha}\overline{R^d} \otimes_{\mathbb{Z}[\pi]} C_*$ of R-left modules is also acyclic and we have the following equality in $\overline{\mathrm{K}}_1(R)$:

$$\tau\big({}_{\alpha}\overline{R^d} \otimes_{\mathbb{Z}[\pi]} C_*, \{\mathfrak{e}_1,\dots,\mathfrak{e}_d\}\otimes\{\mathcal{C}_i\}_{i\in\mathbb{N}_0}\big) \;=\; \underset{\uparrow}{\alpha_*}(\tau(C_*,\{\mathcal{C}_i\}_{i\in\mathbb{N}_0})).$$

see Definition 349.10

Proof of the Torsion–Group Ring-Natural Lemma 349.12. [3790] We start out with a few preparations:

- Since C_* is by hypothesis an acyclic chain complex of free $\mathbb{Z}[\pi]$-left modules we know by the Acyclic–Chain Contraction Proposition 346.2 (i)⇒(ii) that the chain complex (C_*,∂_*) admits a chain contraction $\{\delta_i\colon C_i \to C_{i+1}\}_{i\in\mathbb{N}_0}$ consisting of $\mathbb{Z}[\pi]$-left module homomorphisms.
- Since the tensor functor $A \mapsto {}_{\alpha}\overline{R^d} \otimes_R A$ is an additive functor from the category of $\mathbb{Z}[\pi]$-left modules to the category of R-left modules we see easily that the "tensored" family $\{\mathrm{id}\otimes\delta_i\colon {}_{\alpha}\overline{R^d}\otimes_R C_i \to {}_{\alpha}\overline{R^d}\otimes_R C_{i+1}\}_{i\in\mathbb{N}_0}$ of R-left module homomorphisms is a chain contraction of R-left modules.
- We denote by $\mathcal{E} = \{\mathfrak{e}_1,\dots,\mathfrak{e}_d\}$ the standard basis of R^d.

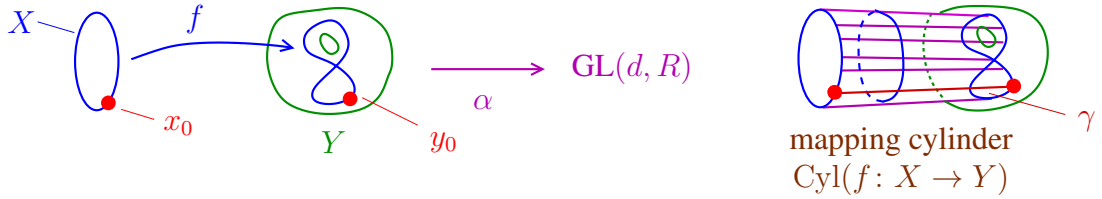
We now see that we have the following equalities in $\overline{\mathrm{K}}_1(R)$:

$$\begin{aligned}
\tau\big({}_{\alpha}\overline{R^d}\otimes_{\mathbb{Z}[\pi]} C_*, \mathcal{E}\otimes\{\mathcal{C}_i\}_{i\in\mathbb{N}_0}\big) &\underset{\text{by the Torsion–Chain Contraction Proposition 346.6}}{\downarrow} \\
&= \big[\mathrm{id}\otimes\partial + \mathrm{id}\otimes\delta\colon {}_{\alpha}\overline{R^d}\otimes_{\mathbb{Z}[\pi]} C_{\mathrm{even}} \to {}_{\alpha}\overline{R^d}\otimes_{\mathbb{Z}[\pi]} C_{\mathrm{odd}}\big]_{\mathcal{E}\otimes\mathcal{C}_{\mathrm{even}}}^{\mathcal{E}\otimes\mathcal{C}_{\mathrm{odd}}} \\
&= \underset{\uparrow}{\alpha_*}([\partial+\delta\colon C_{\mathrm{even}} \to C_{\mathrm{odd}}]_{\mathcal{C}_{\mathrm{even}}}^{\mathcal{C}_{\mathrm{odd}}}) \\
&\underset{\text{by the Tensor Product–Matrix Lemma 348.7}}{\uparrow} \\
&= \underset{\uparrow}{\alpha_*}(\tau(C_*,\{\mathcal{C}_i\}_{i\in\mathbb{N}_0})). \\
&\underset{\text{by the Torsion–Chain Contraction Proposition 346.6}}{\uparrow}
\end{aligned}$$

[3790] As we will see the proof of the lemma is very similar to the proof of the Torsion–Natural Theorem 346.11.

■

Proof of the Torsion–Whitehead Torsion Theorem 349.11. Let $f\colon X \to Y$ be a cellular homotopy equivalence between two finite connected non-empty CW-complexes. Furthermore let $x_0 \in X$. We set $y_0 := f(x_0)$. Finally let $\alpha\colon \pi_1(Y, y_0) \to \mathrm{GL}(d, R)$ be a representation. We denote by $\gamma\colon [0,1] \to \mathrm{Cyl}(f)$ the "horizontal" path $t \mapsto [(t, x_0)]$ from $x_0 = [(x_0, 0)]$ to $y_0 = f(x_0) = [(x_0, 1)]$.



Note that this path γ, the map f and the inclusions induce the following diagram of isomorphisms:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow[\cong]{f_*} & \pi_1(Y, y_0) \\ \downarrow\cong & & \downarrow\cong \\ \pi_1(\mathrm{Cyl}(f), x_0) & \xrightarrow[\cong]{\gamma^*} & \pi_1(\mathrm{Cyl}(f), y_0). \end{array}$$

Clearly the diagram commutes. Thus we can use the isomorphisms to identify all the groups coherently. In particular all induced homomorphisms to $\mathrm{GL}(d, R)$ will be called α.

Next note that it follows from the CW Triple–Torsion Proposition 348.18 together with Convention 348.16 that we have the following two equalities in $\overline{\mathrm{K}}_1(R)/\alpha(\pi) \sqcup \{0\}$:

$$\begin{aligned} \tau(\mathrm{Cyl}(f); \alpha) &= \tau(\mathrm{Cyl}(f), X; \alpha) \cdot \tau(X; \alpha) \\ \tau(\mathrm{Cyl}(f); \alpha) &= \tau(\mathrm{Cyl}(f), Y; \alpha) \cdot \tau(Y; \alpha). \end{aligned}$$

The desired statement now follows from the following two claims:

Claim 1. We have $\tau(\mathrm{Cyl}(f), X; \alpha) = \alpha_*(\tau(f)) \in \overline{\mathrm{K}}_1(R)$.

Proof. The claim follows from the following two equalities:

$$\tau(\mathrm{Cyl}(f\colon X \to Y), X; \alpha) \underset{\substack{\uparrow \\ \text{by the Torsion–Group} \\ \text{Ring-Natural Lemma 349.12}}}{=} \alpha_*(\tau(\mathrm{Cyl}(f\colon X \to Y))) \underset{\substack{\uparrow \\ \text{Definition 349.6}}}{=} \alpha_*(\tau(f)).$$

⊞

Claim 3. We have $\tau(\mathrm{Cyl}(f), Y; \alpha) = 1 \in \overline{\mathrm{K}}_1(R)$.

Proof. First note that it follows easily from the Regular Covering–Cellular Chain Complex Lemma 348.2 that the natural map $\widetilde{X} \times [0,1] \to \mathrm{Cyl}(f)$ induces an isomorphism

$$\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X} \times [0,1], \widetilde{X} \times \{1\}) \xrightarrow{\cong} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{\mathrm{Cyl}(f)}, \widetilde{Y})$$

of based chain complexes of $\mathbb{Z}[\pi]$-left modules. We now see that

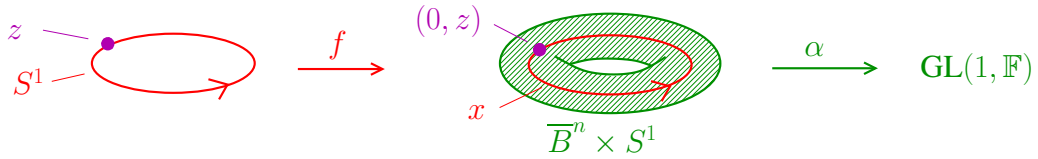
$$\begin{aligned}
\tau(\mathrm{Cyl}(f), Y; \alpha) &\overset{\text{by Definition 348.4}}{=} \tau\big(\overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{\mathrm{Cyl}(f)}, \widetilde{Y})\big) \\
&\overset{\text{by the above isomorphism of based chain complexes}}{=} \tau\big(\overline{{}_\alpha R^d} \otimes_{\mathbb{Z}[\pi]} \mathrm{C}_*^{\mathrm{CW}}(\widetilde{X} \times [0,1], \widetilde{X} \times \{1\})\big)\big) \\
&\overset{\text{by the Torsion–Group Ring-Natural Lemma 349.12}}{=} \alpha_*\big(\tau\big(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X} \times [0,1], \widetilde{X} \times \{1\})\big)\big) \\
&\overset{\text{see the claim on page 6596}}{=} \alpha_*(1) = 1 \in \overline{\mathrm{K}}_1(R). \qquad \blacksquare
\end{aligned}$$

As an example for the Torsion–Whitehead Torsion Theorem 349.11, and as a proof of concept, we now calculate the torsion of solid tori:

Lemma 349.13. (Solid Torus–Torsion Lemma) Let $n \in \mathbb{N}_0$. We consider the solid torus $\overline{B}^n \times S^1$. Let $z \in S^1$. Let $\alpha\colon \pi_1(\overline{B}^n \times S^1, (0,z)) \to \mathrm{GL}(1,\mathbb{F})$ be a one-dimensional representation over a field $\mathbb{F}$. Then $\mathrm{H}_*^{\mathrm{tw}}(\overline{B}^n \times S^1; {}_\alpha\mathbb{F}) \equiv 0$ if and only if the representation is non-trivial. If that is the case, then for any generator x of $\pi_1(\overline{B}^n \times S^1, (0,z)) \cong \mathbb{Z}$ we obtain the following equality in $\overline{\mathrm{K}}_1(\mathbb{F})/\alpha(\pi_1(\overline{B}^n \times S^1)) =_{\mathrm{i}} \mathbb{F}^\times/\pm\langle\alpha(x)\rangle$:[3791]

$$\tau(\overline{B}^n \times S^1; \alpha) = (1-\alpha(x))^{-1}.$$

In particular the torsion does not depend on the CW-structure on $\overline{B}^n \times S^1$.



Proof. Let $f\colon S^1 \to \overline{B}^n \times S^1$ be the standard embedding that is given by $f(a) = (0,a)$. Clearly f is a homotopy equivalence. Thus we see that we have the following equalities in $\overline{\mathrm{K}}_1(\mathbb{F})/\alpha(\pi_1(\overline{B}^n \times S^1)) =_{\mathrm{i}} \mathbb{F}^\times/\pm\langle\alpha(x)\rangle$:

$$\begin{aligned}
\tau(\overline{B}^n \times S^1; \alpha) &\overset{\text{by the Torsion–Whitehead Torsion Theorem 349.11}}{=} \tau(S^1; \alpha \circ f_*) \cdot \alpha_*(\tau(f)) \\
&\overset{\text{by the Circle Torsion Lemma 348.8}}{=} (1-\alpha(x))^{-1} \cdot \alpha_*(\overbrace{\tau(f)}^{\in \mathrm{Wh}(\pi_1(\overline{B}^n \times S^1))}) \\
&\overset{\text{since } \pi_1(\overline{B}^n \times S^1) \cong \mathbb{Z} \text{ it follows from the Whitehead Group-of-Free Abelian Theorem 344.27 that } \mathrm{Wh}(\pi_1(\overline{B}^n \times S^1)) = \{e\}}{=} (1-\alpha(x))^{-1}. \qquad \blacksquare
\end{aligned}$$

349.3. **Simple homotopy equivalences.** The discussion from the last section, in particular the Torsion–Whitehead Torsion Theorem 349.11, naturally lead us to the following definition:

Definition 349.14. Let X and Y be two finite connected non-empty CW-complexes.

(1) Let $f\colon X \to Y$ be cellular homotopy equivalence.[3792] We say f is a simple homotopy equivalence if the Whitehead torsion $\tau(f)$ is trivial.

[3791] In Exercise 348.5 we calculated this torsion in a different way.

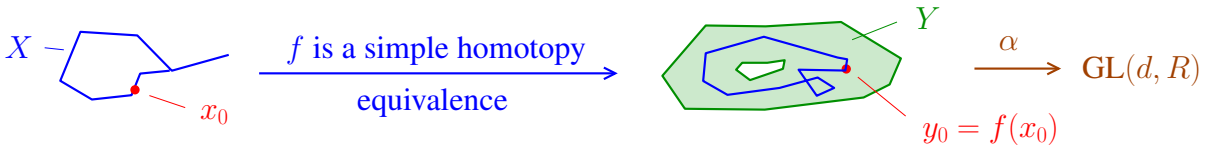
(2) We say X and Y are simple homotopy equivalent if there exists a cellular simple homotopy equivalence $f\colon X \to Y$.[3793]

Remark. Note that it follows from the Whitehead Torsion–Additivity Lemma 349.9 together with the Torsion-of-id-Lemma 349.8 that being simple homotopy equivalent is an equivalence relation. □

We can now pick up the following proposition for free:

Proposition 349.15. (Torsion–Simple Homotopy Equivalence Proposition) In the following let $f\colon X \to Y$ be a simple cellular homotopy equivalence between two finite connected non-empty CW–complexes. Let $x_0 \in X$. We write $y_0 := f(x_0)$. Furthermore let $\alpha\colon \pi_1(Y, y_0) \to \mathrm{GL}(d, R)$ be a representation over an IBN ring R. Then we have the following equality in $\overline{\mathrm{K}}_1(R)/\alpha(\pi_1(Y, y_0)) \sqcup \{0\}$:

$$\tau(X; \alpha \circ f_*) \;=\; \tau(Y; \alpha).$$



Proof. This proposition follows immediately from the Torsion–Whitehead Torsion Theorem 349.11. ■

The following question naturally arises: Which homotopy equivalences are simple? We will give some answers:

- Similar to the discussion on page 6592 we see that it follows from the Whitehead Torsion–Realization Proposition 349.3 together with the Non-Trivial Whitehead Group Lemma 344.24 and the Torsion-of-Deformation Retraction Lemma 349.7 that there exist cellular homotopy equivalences that are not simple.
- Obviously, if the Whitehead group $\mathrm{Wh}(\pi_1(Y, y_0))$ is trivial, then any cellular homotopy equivalence $f\colon X \to Y$ is simple. In the Whitehead Group-of-Free Abelian Theorem 344.27 and the Whitehead Group-of-Free Group Theorem 344.29 we saw that this is frequently the case. In fact Conjecture 344.30 proposes that for any torsion-free group π the corresponding Whitehead group $\mathrm{Wh}(\pi)$ is necessarily trivial.
- In Section 349.4 we will introduce the notion of a subdivision of a CW-complex. We will see that subdivisions lead to simple homotopy equivalences.
- In Section 349.5 we will give a more geometric interpretation of what it means for a homotopy equivalence to be simple.

349.4. **Subdivisions of CW-complexes.** In this section we will introduce the notion of a subdivision of a CW-complex. We will also see that these subdivisions give rise to simple cellular homotopy equivalences.

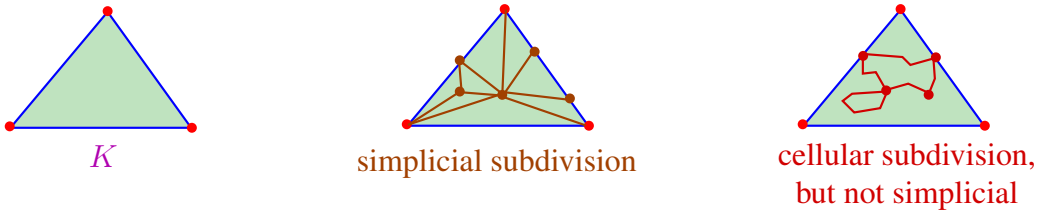
[3792] Just to forestall any confusion: By a *cellular homotopy equivalence* we mean a continuous map that is cellular and that is a homotopy equivalence.

[3793] We will see in the Whitehead–Simple Homotopy Characterization Theorem 349.23 that this definition is consistent with Definition 305.17.

Definition 349.16. Let X be a CW-complex. We say another CW-complex Y is a cellular subdivision of X if they have the same underlying topological space and if each k-cell of Y is contained in an l-cell of X with $l \geq k$.



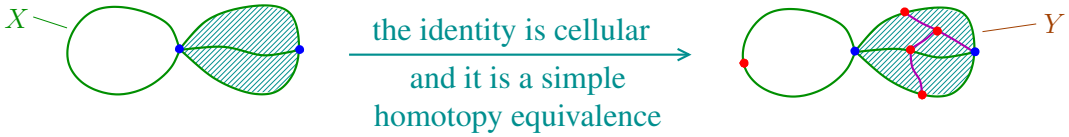
Example. Let $K = (V, S)$ be a simplicial complex. In Definition 180.1 we introduced the notion of a simplicial subdivision of a K. It follows easily from the definitions that a simplicial subdivision is in particular a cellular subdivision in the sense of Definition 349.16.



On the other hand cellular subdivisions are usually not simplicial subdivisions. □

The following proposition can be viewed as a generalization of the Torsion-of-id-Lemma 349.8:

Proposition 349.17. (Subdivision–Simple Homotopy Proposition) Let X be a finite connected non-empty CW-complex. If Y is a cellular subdivision of X, then the identity map $X \to Y$ is cellular (in the sense of Definition 139.2) and it is a simple homotopy equivalence.



Proof. [3794] We prove the proposition by induction on the number of cells of X. If X has just one cell, then that is a 0-cell, and hence its only subdivision is X itself.

We now assume that the statement holds if X has $\leq k$ cells. Next let X be a CW-complex with $k+1$ cells. We do the following:

- We pick $x_0 \in X$ and we write $\pi := \pi_1(X, x_0)$.
- We pick a cell e of maximal dimension n of X. Note that we can write $X = X' \cup e$ where X' is a subcomplex of X with one cell less.[3795]
- One can easily verify that the CW-structure Y endows X' with a CW-structure Y'.
- We denote by $p\colon \widetilde{X} \to X$ the universal covering of X.
- We denote by $\iota\colon \mathbb{Z} \to \mathbb{Z}[\pi]$ the unique ring homomorphism.

[3794] A somewhat different proof of the proposition is given in [**CoheM1973**, p. 81].

[3795] Note that we cannot assume that e itself defines in any way a subcomplex.

Claim. There exists a chain complex D_* over $\mathbb{Z}$ consisting of finitely generated free $\mathbb{Z}$-modules and a short exact sequence of chain complexes of $\mathbb{Z}[\pi]$-left modules of the form

$$0 \to \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(\mathrm{id}\colon X' \to Y'), X') \to \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(\mathrm{id}\colon X \to Y), X) \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} D_* \to 0$$

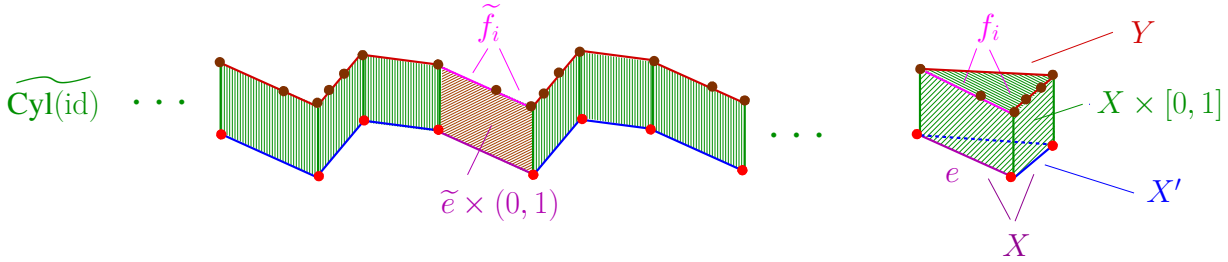
such that the bases are compatible.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_*^{\mathrm{CW}}(X') & \longrightarrow & \mathrm{C}_*^{\mathrm{CW}}(X) & \longrightarrow & \dfrac{\mathrm{C}_*^{\mathrm{CW}}(X)}{\mathrm{C}_*^{\mathrm{CW}}(X')} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(X' \xrightarrow{\mathrm{id}} Y')) & \longrightarrow & \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(X \xrightarrow{\mathrm{id}} Y)) & \longrightarrow & \dfrac{\mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(X \xrightarrow{\mathrm{id}} Y)}{\mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(X' \xrightarrow{\mathrm{id}} Y'))} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(X' \xrightarrow{\mathrm{id}} Y'), X') & \longrightarrow & \mathrm{C}_*^{\mathrm{CW}}(\mathrm{Cyl}(X \xrightarrow{\mathrm{id}} Y), X) & \longrightarrow & E_* & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here E_* is defined as the quotient of the vertical sequence to the right. Clearly the vertical sequences and the two top horizontal sequences are exact. It thus follows from the "Nine Lemma", see Exercise 145.16, that the bottom sequence is also exact.

Now we pick a lift $\widetilde{e}$ of e to $\widetilde{X}$. For each cell f_i of Y, which is not contained in Y', we pick a lift $\widetilde{f}_i$ that is contained in $\widetilde{e}$.



Recall that the cells $\widetilde{e} \times (0,1)$ and $\widetilde{f}_i$ form a basis of E_* as a free $\mathbb{Z}[\pi]$-left module. Also note that one can easily verify that the boundary matrix only has values in $\mathbb{Z}$. Thus there exists a acyclic chain complex D_* over $\mathbb{Z}$ consisting of finitely generated free $\mathbb{Z}$-modules such that $\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} D_* \cong E_*$. ⊞

We now see that

$$\begin{aligned}
\tau(\mathrm{id}\colon X \to Y) &\overset{\text{Definition 349.6}}{\underset{\downarrow}{=}} \tau(C_*^{\mathrm{CW}}(\mathrm{Cyl}(\mathrm{id}\colon X \to Y), X)) \\
&\underset{\uparrow}{=} \tau(C_*^{\mathrm{CW}}(\mathrm{Cyl}(\mathrm{id}\colon X' \to Y'), X')) \cdot \tau(\mathbb{Z}[\pi] \otimes_{\mathbb{Z}} D_*) \\
&\quad \text{by the claim together with the SES-Torsion Theorem 345.30} \\
&\underset{\uparrow}{=} \tau(C_*^{\mathrm{CW}}(\mathrm{Cyl}(\mathrm{id}\colon X' \to Y'), X')) \cdot \iota_*(\underbrace{\tau(D_*)}_{\in \overline{K}_1(\mathbb{Z})}) \\
&\quad \text{by the Torsion–Natural Theorem 346.11} \\
&\underset{\uparrow}{=} \tau(C_*^{\mathrm{CW}}(\mathrm{Cyl}(\mathrm{id}\colon X' \to Y'), X')) \qquad = \tau(\mathrm{id}\colon X' \to Y'). \\
&\quad \text{since by the } K_1\text{-Euclidean Domain Proposition 344.15 we know that } \overline{K}_1(\mathbb{Z}) = \{1\} \qquad \underset{\uparrow}{\text{Definition 349.6}}
\end{aligned}$$

■

The following question arises naturally:

Question 349.18. Given two CW-complex structures X and Y for the same compact non-empty connected topological space, can we always find subdivisions X' of X and Y' of Y such that X' and Y' are isomorphic as CW-complexes?[3796]

A positive answer to Question 349.18, together with the Subdivision–Simple Homotopy Proposition 349.17, would imply that Reidemeister torsion is a homeomorphism invariant. It turns out that the answer to Question 349.18 is, for better or worse, no:

Theorem 349.19. (Subdivision–Non Existence Theorem) There exist CW-structures for the same compact non-empty connected topological space which do not become isomorphic after subdivisions.

Proof. The theorem was first proved by John Milnor [**Miln1961**, Theorems 1 and 2] in 1961. Alternative proofs are given in [**CoheM1973**, p. 84], [**Stal1965b**, p. 252] and [**Ran1996**, Chapter 2]. ■

We will discuss Question 349.18, the closely related Hauptvermutung 177.6 and a more precise version of the Subdivision–Non Existence Theorem 349.19 later in **??**.

349.5. **Topologically simple homotopy equivalences.** The goal of this section is to give a more topological interpretation of what it means for a cellular homotopy equivalence to be simple. We will not really make use of the concepts and results of this section, so this section can also be skipped.

First we recall Definition 305.16:

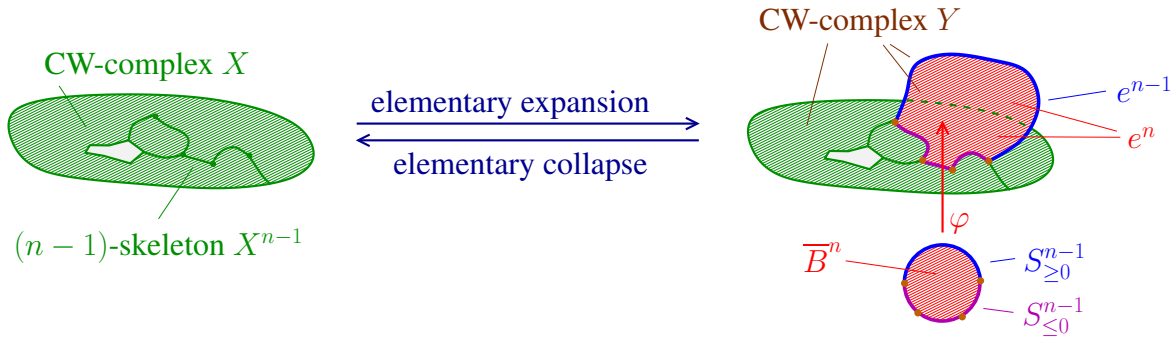
Definition 349.20. Let X be a CW-complex. We say that a CW-complex Y is an elementary expansion of X if the following holds:

(0) X is a subcomplex of Y.
(1) Y has precisely two cells that are not contained in X, namely one $(n-1)$-cell e^{n-1} and one n-cell e^n.

[3796] As we pointed out on page 2677, a map $f\colon X \to Y$ between two CW-complexes is an isomorphism in the category $\mathcal{CW}$ of CW-complexes from Definition 139.3 if and only if f is a homeomorphism which for each $k \in \mathbb{N}_0$ restricts to a homeomorphism $X^k \to Y^k$ between the k-skeleta.

(2) There exists a continuous map $\varphi\colon \overline{B}^n \to Y$ with the following properties:
   (a) The map φ is a characteristic map for the n-cell e^n.
   (b) The map $\overline{B}^{n-1} \xrightarrow[\cong]{x \mapsto (x, \sqrt{1-\|x\|^2})} S^{n-1}_{\geq 0} \xrightarrow{\varphi} Y$ is a characteristic map for the $(n-1)$-cell e^{n-1}.
   (c) We have $\varphi(S^{n-1}_{\leq 0}) \subseteq X^{n-1}$.

Conversely we also say that the CW-complex X is an elementary collapse of Y .

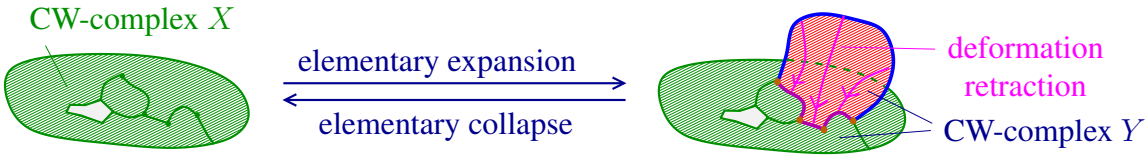


The following lemma gives a refinement of the Simple Homotopy Equivalence Lemma 305.18:

Lemma 349.21. (Elementary Expansion–Simple Lemma) Let X and Y be CW-complexes such that Y is an elementary expansion of X.
(1) There exists a deformation retraction from Y to X, in particular the inclusion $X \to Y$ is a homotopy equivalence.
(2) The Whitehead torsion of the inclusion $X \to Y$ is trivial.

Proof. Let X and Y be CW-complexes such that Y is an elementary expansion of X. We denote the two extra cells by e^{n-1} and e^n.

(1) We saw on page 523 that there exists a deformation retraction from $\overline{B}^n$ to $S^{n-1}_{\leq 0}$. It follows from the Homotopy Pushout Lemma 18.13 that this deformation retraction, together with the stationary homotopy on X, defines a deformation retraction from Y to X.



(2) Let $\iota\colon X \to Y$ be the inclusion. We write $\pi := \pi_1(Y, y_0)$ for some base point $y_0 \in Y$. We denote by $p\colon \widetilde{Y} \to Y$ the natural universal covering of X and we write $\widetilde{X} := p^{-1}(X)$. We pick a lift $\widetilde{e}^n$ of the cell e^n and we pick the unique lift $\widetilde{e}^{n-1}$ of e^{n-1} which is contained

in $\widetilde{e}^n$. Then we see that

$$\begin{aligned}
\tau(\iota) &= \tau(Y, X) && = \tau(\mathrm{C}_*^{\mathrm{CW}}(\widetilde{Y}, \widetilde{X}), \{\widetilde{e}^n, \widetilde{e}^{n-1}\}) \\
&\uparrow \text{ by the Torsion-of-Deformation Retraction Lemma 349.7} && \uparrow \text{ see Definition 349.2} \\
&= \tau(0 \to \mathbb{Z}[\pi] \cdot \widetilde{e}^n \xrightarrow{\cdot e_\pi} \mathbb{Z}[\pi] \cdot \widetilde{e}^{n-1} \to 0) && = (e_\pi) \;=\; 1 \;\in \mathrm{Wh}(\pi). \\
&\uparrow \text{ by the Regular Covering–Cellular Chain Complex Lemma 348.2} && \uparrow \text{ by the Elementary Torsion Lemma 345.17 and the choice of the lift } \widetilde{e}^{n-1}
\end{aligned}$$

■

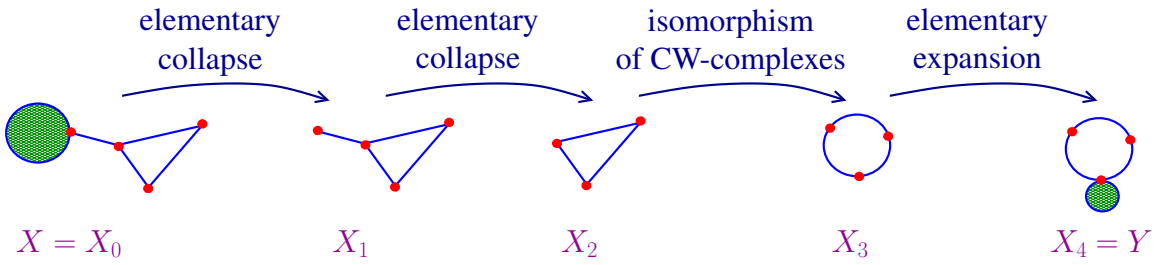
The above discussion leads us to the following definition:

Definition 349.22.

(1) Let Y be an elementary expansion of a CW-complex X. We refer to the inclusion $X \to Y$ as a simple inclusion and for any deformation retraction $F\colon Y \times [0,1] \to Y$ from Y to X we refer to the map $F_1\colon Y \to X$ as a simple retraction.

(2) Let X and Y be CW-complexes. We say a map $f\colon X \to Y$ is a topologically simple homotopy equivalence if there exists a sequence $X = X_0, X_1, \ldots, X_k = Y$ of CW-complexes and continuous maps $f_i\colon X_i \to X_{i+1}$ with the following two properties:
- Each f_i is a simple inclusion or a simple retraction or an isomorphism of CW-complexes.[3797]
- We have $f = f_{k-1} \circ \cdots \circ f_0\colon X = X_0 \to X_k = Y$.

(3) We say that two CW-complexes are topologically simple homotopy equivalent if there exists a topologically simple homotopy equivalence from X to Y.[3798]



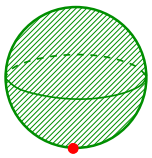
Theorem 349.23. (Whitehead–Simple Homotopy Characterization Theorem) Let X and Y be finite CW-complexes and let $f\colon X \to Y$ be a cellular map. The following two statements are equivalent:

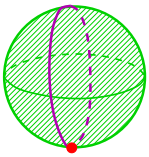
(i) The map $f\colon X \to Y$ is a topologically simple homotopy equivalence.
(ii) The map $f\colon X \to Y$ is a simple homotopy equivalence.

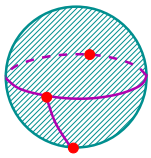
Example. Note that it follows from the Whitehead–Simple Homotopy Characterization Theorem 349.23 together with the Whitehead Group-of-Trivial Group Lemma 344.23 that any two CW-structures on a simply connected CW-complex are topologically simple homotopy equivalent.

[3797] This condition requires in particular that for each $i \in \{0, \ldots, k-1\}$ either X_{i+1} is an elementary expansion of X_i or X_{i+1} is an elementary collapse of X_i or X_{i+1} is isomorphic as a CW-complex to X_i.

[3798] This notion is equivalent to the notion of being "simple homotopy equivalent" from Definition 305.17. The Whitehead–Simple Homotopy Characterization Theorem 349.23 implies that all these different notions of being (topologically) simple homotopy equivalent are actually the same.


CW-structure X on S^2


CW-structure Y on S^2


CW-structure Z on S^2

For example it follows from this discussion that the above three CW-structures on S^2 are all topological simple homotopy equivalent. It is initially not so clear how to see this "by hand". We will deal with this issue in Exercise 349.5. □

Proof. The (i)⇒(ii)-direction is a straightforward consequence of the Elementary Expansion–Simple Lemma 349.21 together with the Homotopy-Equivalence Basics Lemma 19.14 and the observation from page 6602 that being a simple homotopy equivalence is an equivalence relation.

The much harder (ii)⇒(i)-direction was first proved by John Whitehead [**WhdJ1950b**, p. 1] in 1950. A textbook proof can be found in [**CoheM1973**, (22.2)]. We will not make use of this statement. ■

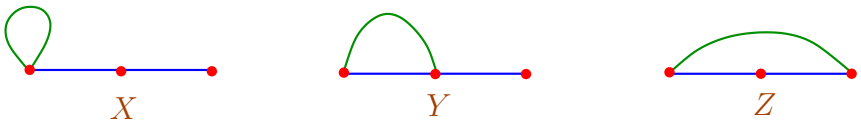
349.6. Exercises for Chapter 349.

Exercise 349.1. Equip the solid torus $\overline{B}^2 \times S^1$ with the product CW-structure. Show that this CW-complex can be turned into S^1 via elementary collapses.

Exercise 349.2. Show "by hand" that the two CW-complexes shown in the figure below are topologically simple homotopy equivalent.



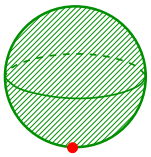
Exercise 349.3. Show "by hand" that the three CW-complexes shown in the figure below are topologically simple homotopy equivalent.

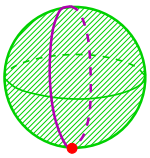


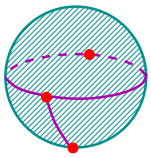
Exercise 349.4. Show "by hand" that the three CW-complexes shown in the figure below are topologically simple homotopy equivalent.



Exercise 349.5. We consider the 2-dimensional sphere together with the three CW-structures that are shown in the figure below. Show "by hand" that these three CW-structures on S^2 are topologically simple homotopy equivalent.


CW-structure X on S^2


CW-structure Y on S^2


CW-structure Z on S^2

Exercise 349.6. Let X and Y be finite connected non-empty CW-complex of dimension n. We assume that X and Y are simple homotopy equivalent. Are they topologically simple homotopy equivalent via a sequence of CW-complexes where each of them has dimension $\leq n$?

Exercise 349.7. Let $K \subseteq S^3$ be an oriented knot. Let $\tau\colon \overline{B}^2 \times K \to S^3$ be a tubular map for K. We consider the knot exterior $X_K = S^3 \setminus \tau(B^2 \times K)$ and the meridian $\mu_K = \tau(S^1 \times \{*\})$. Compute the Reidemeister torsion of $\tau(X_K, \mu_K; {}_{\mathrm{tr}}\mathbb{Q})$.

Exercise 349.8. Let (X, x_0) and (Y, y_0) be two pointed finite connected CW-complexes. Furthermore let $\pi_1(Y, y_0) \to \mathrm{GL}(d, R)$ be a representation over an IBN ring R such that $\tau(Y; \alpha) \neq 0$. We denote by $\pi\colon X \times Y \to Y$ the projection. We equip $X \times Y$ with the product CW-structure from the CW-Complex Product Proposition 139.18. Show that

$$\tau(X \times Y; \alpha \circ \pi_*) \;=\; \tau(Y; \alpha)^{\chi(X)} \;\in\; \overline{\mathrm{K}}_1(R).$$

Hint. You could do induction on the number of cells of X.

Exercise 349.9. Let (X, x_0) be a pointed finite connected CW-complex. Given any map $f\colon X \to X$ we can consider the mapping torus $\mathrm{Tor}(X, f)$ from Definition 21.19 together with the natural map

$$\begin{array}{rccc} p\colon \mathrm{Tor}(X,f) := (T \times [0,1])/(x,0) \sim (f(x),1)) & \to & S^1 \\ [(x,s)] & \mapsto & \exp(2\pi \mathrm{i} s) \end{array}$$

from Definition 21.19. We denote by $\Phi\colon \pi_1(\mathrm{Tor}(X, f), (x_0, 0)) \to \pi_1(S^1, 1) = \langle x \rangle$ the natural epimorphism from Definition 286.4. By a slight abuse of notation we denote by Φ also the corresponding representation $\Phi\colon \pi_1(\mathrm{Tor}(X, f), (x_0, 0)) \to \mathrm{GL}(1, \mathbb{Q}(t))$.

(a) We suppose that f is cellular. Show that there exist bimonic Laurent polynomials $p, q \in \mathbb{Z}[t^{\pm 1}]$ (i.e. the highest and lowest non-zero coefficient is ± 1) such that

$$\tau(\mathrm{Tor}(X, f); \Phi) \;=\; \frac{p}{q} \;\in\; \overline{\mathrm{K}}_1(\mathbb{Q}(t)) =_{\mathrm{i}} \mathbb{Q}(t)^\times / \{\pm t^k\}.$$

Remark. You could work with the CW-structure that arises from the given CW-structure on X and the CW-Complex Mapping Torus Proposition 139.26.

(b) We now just assume that f is a homeomorphism and we suppose that we have some CW-structure on $\mathrm{Tor}(X, f)$. We can thus consider the Reidemeister torsion $\tau(\mathrm{Tor}(X, f); \Phi)$.[3799] Does the conclusion from (a) still hold?

Exercise 349.10. Let X be a finite connected non-empty CW-complex. By the Spanning Tree–Existence Proposition 11.26 and the CW-Complex–Spanning Tree Proposition 140.5

[3799] Note that it follows from the discussion on page 6598 together with the fact that the Whitehead group of $\langle t \rangle$ is trivial (see the Whitehead Group-of-Free Abelian Theorem 344.27) that the torsion does *not* depend on the choice of the CW-structure on the mapping torus $\mathrm{Tor}(X, f)$.

we know that X admits a spanning tree T, that the quotient X/T has a CW-structure with a single 0-cell and that the projection $\pi\colon X \to X/T$ is a homotopy equivalence. Show that the homotopy equivalence is in fact a simple homotopy equivalence.
Hint. Use the Whitehead Group-of-Trivial Group Lemma 344.23.

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